Alpöge's proof of the Riemann hypothesis for Dirichlet L-functions

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Abstract

On 20 November 2026, L. Alpöge announced a proof of the Riemann hypothesis for Dirichlet L-functions, found with the help of Claude and released as a formal development in the proof assistant Lean. We give an account of this proof in conventional mathematics.

A window form of Landau's theorem reduces the zero-free half-plane $\operatorname{Re} s > \frac{1}{2}$ to one-sided bounds $G(x) \le x^{1/2+O(\varepsilon)}$, on windows $[Y, Y^{3/2}]$, for positive combinations G of von Mangoldt sums twisted by the powers of a primitive character. As in Stepanov's method for curves over finite fields, these bounds come from a slope inequality for hermitian lattices over families of abelian CM fields, in which the divisibility that a chain of theta lifts gains at inert primes plays the role of vanishing to high order. Suitable families exist by three inputs: a supply of fields on windows; a bound for Tate–Shafarevich groups of Tate twists, by Bernoulli numbers and Eisenstein congruences; and the lattice data of the chain of theta lifts.

This work was done with the help of Aleph, a mathematical AI research assistant. The Lean source was translated into structured definitions and results in conventional language; along the way, results already in the literature were identified and more than 1000 citations were inserted in place. We also applied straightforward optimisations of the argument: roughly 35% of this exposition already differs from the Lean source, and known results are cited rather than reproved.

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1 Introduction

1.1 The theorem

We give an account, in conventional mathematics, of Alpöge's proof [3] of a zero-free half-plane for Dirichlet L-functions. Notation for Dirichlet characters and their L-functions is that of [180]. For an integer $N \geq 1$, a Dirichlet character χ modulo N is extended to $\mathbb{Z}$ by $\chi(n) = 0$ for $\gcd(n, N) > 1$, and $L(s, \chi)$ is the meromorphic continuation to $\mathbb{C}$ of $\sum_{n \geq 1} \chi(n)\, n^{-s}$. It is entire unless χ is the principal character modulo N, and then its only pole is a simple pole at $s = 1$. The Riemann zeta function $\zeta(s)$ is the case $N = 1$.

The theorem is this. Let $N \geq 1$ be an integer and let χ be a Dirichlet character modulo N. Then

$$L(s, \chi) \neq 0 \qquad \text{for every } s \in \mathbb{C} \text{ with } \operatorname{Re} s > \tfrac{1}{2},$$

except at the pole $s = 1$ when χ is principal. This is Theorem 6.7.

The theorem has two classical consequences. For $N = 1$ it says that $\zeta(s)$ has no zero with $\operatorname{Re} s > \frac{1}{2}$, and the functional equation of ζ then gives the Riemann hypothesis: every zero of $\zeta(s)$ other than the trivial zeros $s = -2n$, $n \geq 1$, has real part $\frac{1}{2}$ (Theorem 6.8). In the same way the functional equation of a primitive L-function gives the generalised Riemann hypothesis for Dirichlet L-functions: for every Dirichlet character χ, primitive or not, every zero of $L(s, \chi)$ with $0 < \operatorname{Re} s < 1$ has real part $\frac{1}{2}$ (Corollary 6.9). For these two classical conjectures and their place in the theory of primes we refer to [75] and [134]. Since the Dedekind zeta function of an abelian number field is the product of the L-functions of its characters [266], the theorem is equivalent to the same zero-free half-plane for the Dedekind zeta functions of all abelian number fields.

The one construction that this paper outlines rather than writes out in full is described in Section 1.3.

The argument is that of a formal development of L. Alpöge [3], written in the proof assistant Lean 4, in which the theorem above and its two consequences are the main theorems. The argument is formalised there and machine-checked. We have not verified the formal development, and no step of this paper appeals to it. Besides the development there is a note of four pages by its author [4], which describes the ideas and contains no proofs; the next subsection follows its description.

1.2 The architecture of the proof

Lineage. In Stepanov's proof of the Riemann hypothesis for curves over finite fields [247], [36], an auxiliary function is made to vanish to high order at the rational points, zeros are counted against poles, and the order of vanishing is chosen so that two error terms balance. Alpöge describes his argument as a descendant of Stepanov's: multiplicity is hard to find over the integers, a chain of theta lifts supplies it, each lift mimicking multiplicity, and for the von Mangoldt sum of length X the chain has length about $\sqrt{X}$ [4]. We follow this description term by term, and say for each term what the paper proves and what is only our reading.

What is bounded. The von Mangoldt sum of length y, with its main term removed, is

$$G(y) = \sum_{1 \leq n \leq y} \Lambda(n)\Big(\frac{y}{n} - 1\Big) - y(\log y - 1 - \gamma),$$

where Λ is the von Mangoldt function and γ is Euler's constant. Since $G(y)/y$ is the Riesz mean of order one of $\sum_n \Lambda(n) n^{-1}$ with its main term removed [117], the paper calls G

the Riesz-weighted Chebyshev function. For a non-principal Dirichlet character χ the twist $g(y,\chi)$ is formed in the same way, with $\Lambda(n)\chi(n)$ in place of $\Lambda(n)$ and with the main term $-y\,L'(1,\chi)/L(1,\chi)$ removed. For a primitive character ψ of order o the function that is bounded is

$$G_{\mathfrak{L}_\psi}(y) = G(y) + \frac{1}{3o+1}\sum_{\chi} \operatorname{Re} g(y,\chi),$$

where χ runs over the non-trivial primitive characters that induce powers of ψ (Definition 2.1). The Laplace transform of $u \mapsto G_{\mathfrak{L}_\psi}(e^u)$ is meromorphic in the half-plane $\operatorname{Re} s > 0$. Its poles there are simple and lie exactly at the zeros ρ of $\zeta(s)$ and of the functions $L(s,\chi)$ of the sum, and the residue at ρ is $S(\rho)/(\rho(1-\rho))$, where $S(\rho) > 0$ is a combination of the multiplicities of ρ with positive weights (Proposition 2.5).

One side, and windows. Only a sign is needed, and it enters twice. The arithmetic gives an upper bound for a sum over primes, and no lower bound (Proposition 4.2). The analytic criterion needs no more, because the residues above are positive at real zeros: it is a window form of Landau's theorem on Dirichlet integrals of functions of one sign [157]. Let $\lambda > 1$ and $0 < \theta < 1$. Suppose that for arbitrarily large Y one has $G_{\mathfrak{L}_\psi}(x) \le x^\theta$ for all $x \in [Y, Y^\lambda]$. Then neither $\zeta(s)$ nor any $L(s,\chi)$ of the sum, and in particular not $L(s,\psi)$, vanishes for $\operatorname{Re} s > \theta$ (Proposition 2.4). It therefore suffices to prove, for every small $\varepsilon > 0$, a bound $G_{\mathfrak{L}_\psi}(x) \le x^{1/2+c\varepsilon}$ for the large x in a set of scales that contains windows $[Y, Y^{3/2}]$ with Y arbitrarily large, where c does not depend on ε (Proposition 2.2). Windows are needed because the auxiliary fields and the arithmetic input below are supplied on windows only (Proposition 9.1, Theorem 36.4).

The prime sum at one scale. The bound at a scale x comes from one abelian CM field E, the field of what the paper calls a host (Definition 3.11, Definition 7.2). Its Galois group is identified with $\{\pm1\}^2 \times A$ for a finite abelian group A with $x^{2+8\varepsilon} \le |A| \le 2x^{2+8\varepsilon}$, so that E has degree $4|A|$. Inside a cyclotomic field it is the fixed field of the automorphisms $\zeta \mapsto \zeta^a$ with $a \equiv \pm1 \pmod{2^{e+2}}$, $\psi(a) = 1$ and $(a/q) = (a/d) = 1$, where 2^e is of the size of $|A|$. Here $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ are primes with $q \mid d+1$, of size $\log q \asymp Y^{\varepsilon/5}$ and $\log d \asymp Y^{\varepsilon/5}\log^2 Y$ for a scale $Y \le x$; the paper calls (q,d) a window pair (Definition 7.1). The field E contains $K = \mathbb{Q}(\sqrt{-d})$, and the powers of ψ are among its Dirichlet characters. For a height $a > 0$, a translate $t \in A$ and a positive weight function w, the inert prime sum $A_{\mathrm{in}}(a,t,w)$ adds the terms $2\log p\,(a/p^i - w)^+$ over the primes p inert in K and the odd $i \ge 1$, the weight being evaluated at a translate, depending on t, of the Frobenius class of p^i, together with a similar term at d (Definition 3.1). By orthogonality of the characters of $\mathrm{Gal}(E/\mathbb{Q})$, the difference

$$A_{\mathrm{in}}(a,t,w) - a\Big(\log a - 1 - \gamma + \frac{L'(1,\chi_{-d})}{L(1,\chi_{-d})} - \frac{1}{|\Phi|}\sum_{\rho\in\Phi}\log w(\rho)\Big)$$

is an explicit linear combination of the functions $g(y,\chi)$ of the Dirichlet characters χ of E, evaluated at the points $y = a/w(\rho)$ (Lemma 3.8). Here χ_{-d} is the quadratic character of K and $\Phi \subset \mathrm{Gal}(E/\mathbb{Q})$ is the subgroup of index two that fixes K. The subtracted term is the main term, and the sum of these differences over a finite set $T \subseteq A$ of translates, the tops, and over m weight functions is the excess $\Delta(u)$ at a translate $u \in A$. Two averages turn an upper bound for the excess into an upper bound for $G_{\mathfrak{L}_\psi}(x)$ (Proposition 3.3). The first is over the translates u in a cyclic subgroup $A_0 \subseteq A$ of index o; after it only the $4o$ characters of E that are trivial on A_0 remain (Lemma 3.17). The second is over the tops, which are equidistributed over the cosets of A_0 with share $4/(3o+1)$ on A_0 and $3/(3o+1)$ on every other coset; this gives each non-trivial character of A/A_0 the weight $(3o+1)^{-1}$ of the sum $G_{\mathfrak{L}_\psi}$. Among the $4o$ characters, o are the powers of ψ, which make up $G_{\mathfrak{L}_\psi}$, and the other $3o$ are their twists by the quadratic characters modulo q and d. The step leaves the absolute values $|g(x,\chi)|$ of these twists, and the

host conditions ask that each be at most $x^{1/2+\varepsilon}$ (Definition 3.16). The window pair is chosen so that this holds: at every large scale almost all pairs of the relevant sizes have the bound on a long range of scales, by a mean-square argument over quadratic twists (Proposition 10.4, Proposition 10.2). Its inputs are published: Montgomery's zero-density estimate, the Landau–Page theorem, an effective bound for a real zero, and the prime number theorem in progressions of Thorner and Zaman.

The auxiliary object. What remains is an upper bound for the excess, and this is where Stepanov's method enters. In the description of [4] the auxiliary object is a chain of iterated theta lifts $\mathrm{U}(X_0) \to \mathrm{U}(X_1) \to \cdots \to \mathrm{U}(X_m)$ between totally definite unitary groups, which wins multiplicity m. In the body of this paper the chain does not appear. What appears is a record at one scale, a slope datum (Definition 3.14): a finite list of real numbers with an integer m, a prime-sum entry S_u and a lattice term Θ_u for every translate u. In the application the numbers are ranks, slopes, maximal slopes and heights of hermitian lattices. That a slope datum is the record of a chain of theta lifts is our reading: the definition is purely numerical, and so are the eleven slope conditions imposed on it (Definition 3.13). The chain itself is outlined in Appendix A. The order of vanishing is, in our reading, the integer m, the number of lifts. The paper fixes $\sqrt{x\delta_1} \le m \le 2\sqrt{x\delta_1}$ with $\delta_1 = x^{-\varepsilon/8}$ (Definition 11.2).

Divisibility at inert primes. Vanishing at the rational points becomes divisibility at the primes inert in K. In [4], on an elliptic curve with complex multiplication by K, a derivative of order a of the theta function, divided by the a-th power of the period, is divisible at an inert prime p by p to the power $a/p + a/p^3 + \cdots - O(\log a)$, and the m-fold lift keeps a sufficient lower bound for the divisibility. The paper has the same exponent. The gain attached to an inert prime p is built on $\gamma_p = p/(p^2-1) = \sum_{i \text{ odd}} p^{-i}$ (Definition 5.3), and the inert prime sum adds the terms a/p^i over the same odd i. Each of the m weight functions has its own gain, and the lattice term Θ adds, over the inert primes p, the product of $2\log p$ and the sum of the m gains at p less a correction (Definition 5.5). The loss bound says that the prime sum S exceeds Θ by at most a small loss (Theorem 5.6); this is the loss condition (Sl8) of a slope datum, with an exponent $e_{\mathrm{Loss}} = 0$. Its main hypothesis, the linking property, compares the contribution of each inert prime to the prime sum with the gain at that prime, up to explicit local losses, and the deficit is then split prime by prime. That the gains measure the divisibility of theta kernels is the intended reading, and no result of the paper states it. The paper defines the gains as numbers; Appendix A proves that the theta kernels attached to an elliptic curve with complex multiplication by K are integral at the inert primes $\ell \ge 5$, and it does not deduce the gains from this.

Zeros against poles. The count of zeros against poles becomes a slope inequality in the sense of Bost [47], which the note lists among its tools [4]. In the paper it is the dual-slope inequality (Sl9): the mean of the slopes of a family of lattices, each corrected by a height, is bounded by the slope of a common target. It needs the mean slope of the target and not the maximal one, and it comes from Bost's inequality and from frames of maps that are nearly orthogonal at one archimedean place (Theorem 14.4, Theorem 14.2); this is the approximate orthogonality of the note. The norms of the lifts are the divisor bound (Sl2): it bounds the slope of the target, corrected by a height, by a sum in which each of the m lifts contributes $a_t \log a_t - a_t - a_t \log \Omega$ for every top t, with the height a_t of the top and a constant Ω that involves $L'(1,\chi_{-d})/L(1,\chi_{-d})$. Terms of order m at two auxiliary primes occur in two further conditions and cancel. The transfer of slopes (Sl7) passes the lattice term Θ_u into the slopes of the lattices. If the entries S_u are the inert prime sums of the host, the eleven conditions together give $S_u \le \mathrm{Main}(u) + \mathrm{Err}(u)$, where $\mathrm{Main}(u)$ is the sum of the main terms (Proposition 4.2). This is the accounting, and it bounds the excess by $x^{(e_{\mathrm{Div}}+e_{\mathrm{Loss}}+3)\varepsilon}$ times the natural size of its terms, where e_{Div} and e_{Loss} are

the exponents of the error terms in (Sl2) and (Sl8) (Proposition 4.1).

The balance. The note speaks of the cost of each lift [4]. The bound for the excess contains a term of size $|T|\,a$, which does not grow with m, and a term of size $|A|\,m^2 b\,x^{\varepsilon}$, where $b = \exp(x^{\varepsilon})$ and the heights are of size $a = xb$. Divided by the number $|T|\,m$ of summands and by b, they are of the order of x/m and of m, up to factors $x^{O(\varepsilon)}$, and they balance when m is about $\sqrt{x}$. The resulting bound is $G_{\mathfrak{L}_\psi}(x) \le x^{1/2+O(\varepsilon)}$ (Proposition 6.5). In Stepanov's method the order of vanishing is chosen to balance two error terms in the same way; the comparison is our reading.

The existence theorem. A family of hosts over a set of scales is tied with exponents (e_D, e_L) if at every large scale of the set one of its members carries a slope datum as above, with its prime-sum entries equal to its inert prime sums and with $e_{\mathrm{Div}} \le e_D$ and $e_{\mathrm{Loss}} \le e_L$ (Definition 3.16, Definition 6.2). For a tied family whose members satisfy the host conditions,

$$G_{\mathfrak{L}_\psi}(x) \le x^{1/2+(e_D+e_L+7)\varepsilon}$$

at every large scale x of the family (Proposition 6.4). The existence theorem, Proposition 7.8, says that for every primitive character ψ there are real numbers e_D, e_L and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is a family of hosts with parameter ε that satisfies the host conditions for $\mathfrak{L}_\psi$ at all large scales, is tied with exponents (e_D, e_L), and whose set of scales contains windows $[Y, Y^{3/2}]$ with Y arbitrarily large. The order of the quantifiers is the point: e_D and e_L are chosen before ε. The proof gives $(e_D, e_L) = (1, 0)$, hence the exponent $\frac{1}{2} + 8\varepsilon$ (Theorem 17.7). With the criterion this gives $L(s, \psi) \neq 0$ for $\operatorname{Re} s > \frac{1}{2}$ and every primitive ψ (Proposition 6.3), and the Euler factors at the primes dividing N extend it to every character modulo N.

The arithmetic input. For the Galois cohomology classes over E that the construction cannot remove, the note asks for "a significant weakening of Vandiver" [4]. In the paper this is the window Sha bound (Theorem 36.4). For a number field E, a prime ℓ and an integer k let $\mathrm{len}_\ell(E, k)$ be the length of the Tate–Shafarevich group

$$\mathrm{Sha}^1_\ell(E, k) = \ker\Big(H^1\big(G_{E,S_\ell}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-k)\big) \to \prod_{v \mid \ell} H^1\big(E_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-k)\big)\Big),$$

where G_{E,S_ℓ} is the Galois group of the maximal extension of E unramified outside the places above ℓ. The theorem says that for every window pair (q, d) at a large scale there is a range $[Y, Y^2]$ of scales such that for every x in the range the field E described above, formed with ψ, q, d and $x^{2+8\varepsilon} \le 2^e \le 2x^{2+8\varepsilon}$, satisfies

$$\sum_{\ell} \log \ell \sum_{k=1}^{t-1} \mathrm{len}_\ell(E, k) \le \exp(x^{\varepsilon}/2), \qquad t = \lfloor x^{401} \rfloor.$$

The sum runs over all primes ℓ, the prime 2 included. The bound enters the divisor bound, where the defects of the lattices are bounded by these lengths (Theorem 13.2). Its proof is the longest part of the paper. For odd ℓ the groups split into plus and minus parts under complex conjugation, and the length of each part is bounded by a sum over the Dirichlet characters of E. The minus part is bounded through generalised Bernoulli numbers, by Kolyvagin systems of Stickelberger type (Proposition 37.2, Proposition 37.4). The plus part is bounded by Eisenstein congruences on U(3), on GSp_4 and on U(3) over $\mathbb{Q}(i)$: one non-zero integer, whose logarithm is polynomial in the conductor and in the twist, bounds the lengths at all primes ℓ other than one auxiliary prime at once (Proposition 38.5). The automorphy that these congruences need is proved by an $R = T$ theorem for residually reducible representations over auxiliary CM fields, with Hida theory and patching (Theorem 49.3). The prime 2 and finitely many bad primes are treated by a control of twists along cyclotomic towers, which rests on the theorem

of Ferrero and Washington, and by a bound for second cohomology that is uniform in the twist (Proposition 39.2, Theorem 40.1). Since the conductor of E is at most $\exp\bigl(O(Y^{\varepsilon/5}\log^2 Y)\bigr)$, bounds that are polynomial in the conductor and in t fit within $\exp(x^{\varepsilon}/2)$.

The map of the Parts. Part I carries out the reduction: it proves Theorem 6.7 and its two consequences from three statements. The first is the supply of hosts on windows (Proposition 9.1), proved in Part II: every large scale of a set that contains windows $[Y, Y^{3/2}]$ carries a host that satisfies the host conditions, whose field satisfies the bound for Tate–Shafarevich groups, and that is good in the sense required below. The second is the window Sha bound, proved in Parts VII to IX. The third is the existence of one class of families of hosts with the link properties (Proposition 17.6), proved in Parts III and IV. The link properties are design existence, the linking property, a choice of tops, and the properties (Sp1)–(Sp8) of the slope records of the members, with exponents independent of ε; a class with these properties contains tied families (Theorem 7.10, Theorem 8.4). Part III proves the properties of slope records for one explicit class $\mathcal{A}_{\mathrm{prop}}$, from a list of records that each member is required to carry (Theorem 17.7). Its sections on the norms of the lifts and on approximate orthogonality prove the divisor bound and the slope inequality. The divisor bound uses the Ribet budget (Theorem 22.1), in which local losses are bounded place by place with Wach modules and (φ, Γ)-modules; it is proved in Part V. Part IV proves design existence (Proposition 18.1): for every small ε the class contains a family on the given windows. This is where the records of the chain of theta lifts enter: the family is assembled from translate packets, one at each scale, and a packet consists of one loss datum and one slope record for every translate $u \in A$ (Proposition 19.3, Definition 19.1). Two groups of estimates that the class requires are proved from the data that the packets carry. One is the realisation estimate, proved from a budget of local estimates with integer denominators (Theorem 20.2). The other is a pair of bounds for the slopes of integral models, proved in Part VI by arithmetic intersection theory on towers of integral models of unitary Shimura varieties attached to E (Theorem 29.14, Proposition 31.14); we read it as the computation of covolumes of the note [4]. The dependence of the Parts is

$$\mathrm{I} \leftarrow \mathrm{II},\ \mathrm{III},\ \mathrm{VII}; \qquad \mathrm{III} \leftarrow \mathrm{II},\ \mathrm{IV},\ \mathrm{V}; \qquad \mathrm{IV} \leftarrow \mathrm{VI}; \qquad \mathrm{VII} \leftarrow \mathrm{VIII} \leftarrow \mathrm{IX},$$

where an arrow points from the Part whose results are cited to the Part that cites them.

1.3 How this account was obtained

The proof is self-contained, apart from published results quoted with their sources and the detailed construction of a single object, a *theta chain system*: at every large scale, a family of translate packets, the arithmetic records of a chain of theta lifts, which carry loss data, numerical data and, under a bound for Tate–Shafarevich groups, the geometric and local comparison data that the slope inequality needs. The existence of a theta chain system is assumed in a single statement, Hypothesis 21.1: it is named a hypothesis rather than a proposition for logical rigour, as its construction is given here only as an outline, in Appendix A. A faithful reproduction of the construction, which in the Lean source is long and in places repetitive, would roughly double the length of this exposition. We are optimising the construction and will give it, in optimised and possibly partly computer-assisted form, in the final version of this paper.

This work was done with the help of Aleph, a mathematical AI research assistant that runs on an ensemble of Claude, ChatGPT, Gemini and bespoke models. As a first step, Claude Code and Codex agents translated the Lean code of the formal development [3] into the Aleph database, in human-readable form; at this step the citations were identified and verified at their sources, and the obvious optimisations were made. The text of this paper thus contains more than 1000 citations, most of them already verified at their source, and roughly 35% of the

argument already differs from the Lean source. The entire proof is now being optimised and verified in several iterations, with the aim of improving its clarity, its logical structure and its understanding. This work is ongoing, and this is not the final version of the presentation. We have also included contributions of mathematicians who have already optimised parts of the argument, among them J. Adriazola [2] and the author of [261].

Several other accounts of the argument appeared within days of the development. Adriazola [2] wrote the argument as one continuous proof in conventional mathematics, in two parts, with its open steps stated. Zhang [277] wrote a preliminary account of the proof and of its principal ingredients, with an emphasis on their sources and history. Rubinstein [210], [211] maintains a guide to the proof and an exposition of the argument specialised to the Riemann zeta function. Naskręcki [189] built a map of the formal proof and a record of checks, with an atlas of the route. Epstein [85] wrote a human-readable blueprint of the formal development. Brox [54] leads a programme to distil the formal proof into shorter proofs. Jaswal [138] wrote an account of the Riemann hypothesis for Dirichlet L-functions after Alpöge. We thank the author of [261], which gives three simplifications of the argument; we use one of its ideas in the proof of Theorem 14.4. Several other optimisation ideas already available are applied and quoted throughout the paper.

1.4 Organisation and conventions

The paper has nine Parts and five appendices. The Introduction and Part I carry the whole logic of the proof. A reader who stops after Part I knows the theorem, the reduction, and which statement each later Part supplies. Every later Part proves a named statement that an earlier Part uses, as in the map at the end of the subsection on the architecture.

Part I has seven sections, in the order of the argument. It proves the window form of Landau's theorem (Proposition 2.7) and the passage from a bounded excess to the one-sided bound (Proposition 3.3). Then come the two halves of Stepanov's mechanism, side by side: the accounting, which is the count as a slope inequality (Proposition 4.2, Proposition 4.1), and the divisibility at inert primes, which is the loss bound (Theorem 5.6). After them it defines tied families, proves the main theorem and its two consequences from the existence theorem, and proves the existence theorem from its three inputs (Proposition 6.3, Proposition 7.8, Theorem 7.10). It ends with the eight properties (Sp1)–(Sp8) of slope records, which give all the slope conditions except the loss condition (Theorem 8.4); these are what a class of families must supply.

Part II proves the supply of hosts on windows (Proposition 9.1), with the mean-square argument for window pairs (Proposition 10.2), and the supply of good hosts (Proposition 11.5). Part III proves the properties of the class $\mathcal{A}_{\mathrm{prop}}$ (Theorem 17.7) and with them Proposition 17.6; each of its sections names the property of slope records that it proves. Part IV proves design existence (Proposition 18.1) from translate packets (Proposition 19.3, Proposition 19.9, Proposition 21.2).

Part V proves the Ribet budget (Theorem 22.1): it bounds the local losses at the places above ℓ, at the primes of inert type, and at the ramified, split, small and large primes, with Wach modules. Part VI proves the bounds for model slopes (Theorem 29.14) and the existence of good root towers (Proposition 31.14). Part VII reduces the window Sha bound (Theorem 36.4) to three one-character bounds, and it treats the minus part, the prime 2, the bad primes and the bound for second cohomology (Proposition 37.2, Proposition 39.2, Theorem 40.1). Part VIII proves the one-character bounds by Eisenstein congruences (Proposition 42.2, Proposition 44.3, Proposition 46.2). Part IX proves the automorphy theorems that these bounds use (Theorem 47.1, Theorem 47.16, Theorem 51.1). Part V, Part VI and Parts VII to IX do not use

one another, apart from a few general lemmas and quoted results, and each can be read after Part I with the statement it supplies as its goal.

Appendix A outlines the construction of the chain of theta lifts, as described in Section 1.3. Appendix B states the 174 quoted results with their sources, by field, and ends with the computer checks that proofs cite. Appendices C, D and E contain the most technical sections of Parts V, VI and IX: weight words along Frobenius orbits, technical facts on good root towers, and Hecke ideals, patching and Taylor–Wiles levels. The Parts cite the statements of these appendices and do not repeat their proofs.

Throughout, ψ is a primitive Dirichlet character, ε is the small parameter of the argument, with $0 < \varepsilon < \frac{1}{52}$, and x, y, X, Y are large real scales. A set of scales contains λ-windows if it contains intervals $[Y, Y^{\lambda}]$ with Y arbitrarily large; window pairs and the window Sha bound are named after the windows on which they serve. Two names are used twice: the window parameters of a loss datum have nothing to do with these windows, and a Sha-host is a triple of numbers, not a host. Labels such as (Ec4) or (Sl8) refer to the conditions listed in the definition or statement where they occur. The names of the note are used in the Introduction, in the titles and in the text between results; the statements use the names of the paper, and each correspondence between the two that no result states is marked as a reading. The explicit constants, such as the bound $\frac{1}{52}$ for ε, the window ratio $\frac{3}{2}$, the weight $(3o+1)^{-1}$ and the exponent 401 of the Sha bound, are not optimised, and we do not know which of them are forced by the method. Most objects of the argument have no counterpart in the literature, and each is named where it first appears; where the literature has a word, we use it.

Part I
The reduction

Part I contains the whole logic of the proof. It proves the zero-free half-plane for Dirichlet L-functions (Theorem 6.7) and the Riemann hypothesis (Theorem 6.8) from three statements, which the later Parts prove.

- Proposition 9.1 (Part II). Let a primitive character ψ satisfy the window Sha bound, and let a good-host requirement supply good hosts for ψ. Then for every $\varepsilon \in (0, 1/52)$, at all large scales of a set containing $\frac{3}{2}$-windows, there are hosts that satisfy the host conditions for the cyclic level of ψ, whose fields satisfy a Sha bound, and that meet the good-host requirement.
- Theorem 36.4 (Part VII). Every primitive character satisfies the window Sha bound with exponent 401.
- Proposition 17.6 (Part III). There are a good-host requirement that supplies good hosts and a class of parametrised host families with slope records that has design existence, the linking property, the three properties of the top sets, and the properties (Sp1)–(Sp8) of slope records.

Apart from these, Part I uses four classical theorems, restated in Appendix B: Theorem B.1, Theorem B.2, Theorem B.4 and Lemma B.3. The existence of a theta chain system (TCS) enters Part I only through Proposition 17.6, whose clause (Lk1) is proved in Part IV assuming (TCS). The results of Part I that depend on it are Corollary 8.3, Proposition 7.9, Proposition 7.8 and the two theorems; all the others are unconditional.

In Alpöge's description ([4], p. 1) the argument descends from Stepanov's proof of the Riemann hypothesis for curves over finite fields [247], [36]: an auxiliary object that vanishes to high multiplicity bounds a count from above. The sections of Part I follow this pattern. Section 2 shows that an upper bound suffices. The function to be bounded is the von Mangoldt sum $\sum_{n \le x} \Lambda(n)(x/n - 1)$ with its main term removed, together with its twists by the powers of ψ; the statements call it the Riesz-weighted Chebyshev function. A bound $x^{1/2+c\varepsilon}$ from above, on windows $[Y, Y^{\lambda}]$ with Y arbitrarily large, excludes zeros to the right of the critical line. Section 3 obtains such a bound at one scale x from one auxiliary abelian field, a host: an average over translates turns an upper bound for the excess of its inert prime sums into the bound at x. Section 4 bounds the excess by the accounting, eleven inequalities between real numbers which play the part of Stepanov's count of zeros against poles. Only one of the eleven involves the primes, the loss condition (Sl8), and Section 5 proves the loss bound from which it follows; in Stepanov's terms it replaces vanishing at the rational points by divisibility at the inert primes. Section 6 passes from one scale to families of hosts over a set of scales, defines tied families, and deduces the two theorems from the existence of tied admissible families. Section 7 proves this existence theorem from the three statements above, and Section 8 proves the last link, that the properties (Sp1)–(Sp8) give the remaining ten inequalities. It closes with a table that names, for each condition used in Part I, the result that proves it.

A reader who wants the mechanism of the proof should read Sections 3, 4 and 5. In the exposition of these sections we have adopted several devices of [2]: the accounting is stated in real numbers and followed by a dictionary, its error terms are compared with Stepanov's balance, and the averaging step is illustrated by its smallest case.

2 One-sided bounds on windows

This section proves the analytic criterion of the paper, Proposition 2.2. Let $\mathfrak{L}$ be a character level, that is, a finite set of primitive characters with positive weights, and let $G_{\mathfrak{L}}$ be the corresponding combination of the von Mangoldt sum and its twists, with the main terms removed. Suppose that for every small ε the upper bound $G_{\mathfrak{L}}(x) \le x^{1/2+c\varepsilon}$ holds at all large x of a set that contains windows $[Y, Y^{\lambda}]$ with Y arbitrarily large, where c does not depend on ε. Then no L-function of the level vanishes in $\operatorname{Re} s > \frac{1}{2}$. The criterion is used once, in the proof of Proposition 6.3 in Section 6, with $c = e_D + e_L + 7$; the main theorem takes $\lambda = \frac{3}{2}$ and the cyclic level of a primitive character.

The proof is a window form of Landau's theorem on Dirichlet integrals of functions of one sign [157]. In the classical theorem the bound is assumed for all large x [180]. Here it is assumed only on the windows, and nothing is assumed between them (Proposition 2.4). The weaker hypothesis is needed because the auxiliary fields of the proof and the Sha bound for them are supplied on windows only (Proposition 9.1, Theorem 36.4).

The heart of the section is Proposition 2.7, the same statement for a Laplace transform with simple poles, to which the substitution $x = e^u$ reduces the criterion. Its proof compares two estimates of the Taylor coefficients of the transform at the points of one vertical line $\operatorname{Re} s = \sigma$. The bound on a window $[U, \lambda U]$ makes them small for some $k \ge U$ (Lemma 2.10). A pole to the right of θ, chosen nearest to the line and isolated (Lemma 2.9), makes them large for all large k. Positivity enters twice, and this is why a bound from above suffices. The Taylor coefficients at a real point are moments of the function against the positive kernels $u^k e^{-\sigma u}/k!$ (Lemma 2.8), so an upper bound for the function bounds them from above. And the residue at a real pole ρ is positive: it is a combination of multiplicities of zeros with the weights of the level, which are positive, divided by $\rho(1-\rho)$ (Proposition 2.5).

Notation for Dirichlet characters and their L-functions is that of [180]. For an integer $N \ge 1$, a **Dirichlet character modulo** N is a character χ of $(\mathbb{Z}/N\mathbb{Z})^{\times}$, extended to $\mathbb{Z}$ by $\chi(n) = 0$ for $\gcd(n, N) > 1$; the **principal character** modulo N is denoted $\chi_{0,N}$, and χ is **primitive** if its conductor equals N. The complex variable is s. The **Dirichlet L-function** of χ is the meromorphic continuation to $\mathbb{C}$ of

$$L(s,\chi) = \sum_{n\ge 1} \chi(n)\, n^{-s} = \prod_{p} (1 - \chi(p)\, p^{-s})^{-1}, \qquad \operatorname{Re} s > 1,$$

where χ is taken as a character modulo N, so that the Euler factors at the primes $p \mid N$ are equal to 1 also when χ is imprimitive. It is entire unless $\chi = \chi_{0,N}$, in which case its only pole is a simple pole at $s = 1$. The Riemann zeta function is $\zeta(s) = L(s, \chi_{0,1})$; its **trivial zeros** are the points $s = -2n$, $n \ge 1$. The von Mangoldt function is $\Lambda(n)$.

Definition 2.1. Let $\mathcal{P}$ denote the set of **primitive Dirichlet characters**, each regarded as a pair (N, ψ) with ψ primitive modulo $N \ge 1$. Let $\mathbf{1} \in \mathcal{P}$ be the trivial character modulo 1; it is the only principal character in $\mathcal{P}$. For $\psi \in \mathcal{P}$ let $\overline{\psi} \in \mathcal{P}$ be its complex conjugate, and for a Dirichlet character χ let $\chi^* \in \mathcal{P}$ be the primitive character inducing χ. A character is **real** if it takes only real values.

For real $y > 0$ and every non-principal Dirichlet character χ put

$$G(y) = \sum_{1\le n\le y} \Lambda(n)\Big(\frac{y}{n} - 1\Big) - y\big(\log y - 1 - \gamma\big), \tag{2.1}$$

$$g(y,\chi) = \sum_{1\le n\le y} \Lambda(n)\chi(n)\Big(\frac{y}{n} - 1\Big) + y\,\frac{L'(1,\chi)}{L(1,\chi)}, \tag{2.2}$$

where γ is Euler's constant (recall $L(1,\chi) \neq 0$ [180]), and put $g(y,\mathbf{1}) = G(y)$. We call G the **Riesz-weighted Chebyshev function** and $g(\cdot,\chi)$ its **twist** by χ; the subtracted terms $y(\log y - 1 - \gamma)$ and $-y\,L'(1,\chi)/L(1,\chi)$ are their **main terms**.

A **character level** is a tuple $\mathfrak{L} = (K, \psi_{\mathrm{exc}}, o)$ consisting of a finite set $K \subset \mathcal{P}$, an optional element ψ_{exc} (either absent or a character of K) and an integer $o \geq 0$, such that

(Le1) $\mathbf{1} \in K$, and $\overline{\psi} \in K$ for every $\psi \in K$;

(Le2) if ψ_{exc} is present, it is odd and real.

The **cyclic level** of $\psi \in \mathcal{P}$ of modulus N is $\mathfrak{L}_\psi = (K_\psi, \mathrm{absent}, o_\psi)$, where $K_\psi = \{(\psi^j)^* : j \in \mathbb{Z}\}$ and o_ψ is the order of ψ in the group of Dirichlet characters modulo N.

For a character level $\mathfrak{L} = (K, \psi_{\mathrm{exc}}, o)$ the **weighted Chebyshev function** is

$$G_{\mathfrak{L}}(y) = G(y) + \sum_{\psi \in K \setminus \{\mathbf{1}\}} w_{\mathfrak{L}}(\psi)\ \mathrm{Re}\, g(y,\psi) \quad (y > 0), \qquad w_{\mathfrak{L}}(\psi) = \begin{cases} 1, & \psi = \psi_{\mathrm{exc}}, \\ (3o+1)^{-1}, & \text{otherwise.} \end{cases} \tag{2.3}$$

A character level $\mathfrak{L}$ is **zero-free above** $\theta \in \mathbb{R}$ if for every $\psi \in K$ the function $L(s,\psi)$ has no zero in the half-plane $\mathrm{Re}\, s > \theta$.

Thus $G(y)/y$ is the Riesz mean of order one of the series $\sum_n \Lambda(n) n^{-1}$ with its main term removed [117]; $G(y)$ is the von Mangoldt sum of length y of [4] (item 5 of the list on pp. 3–4), namely $\sum_{n \leq y} \Lambda(n)(y/n - 1) - y \log y + (1+\gamma)y$. The reader should think of a character level as a finite set of primitive characters, closed under complex conjugation, together with the weights with which their twists are added to G. The proof of Theorem 6.7 uses only the cyclic levels $\mathfrak{L}_\psi$; they have no exceptional character, and K_ψ contains ψ itself.

The results of this section use two features of a level, the positivity of the weights and the closure of K under conjugation (Le1). The value $(3o+1)^{-1}$ of the weights plays no role here. It is the weight that the top set of a host produces in Section 3, through condition (Ec4) of Definition 3.16 and Lemma 3.17.

Proposition 2.2. *Let $\mathfrak{L}$ be a character level and let $\lambda > 1$. Suppose that there are $c \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there are a set $W \subseteq \mathbb{R}$ and a real number $x_1 \geq 1$ with the following two properties:*

1. *for every $y \geq 1$ there is $Y \geq y$ with $[Y, Y^\lambda] \subseteq W$;*
2. *$G_{\mathfrak{L}}(x) \leq x^{1/2 + c\varepsilon}$ for every $x \in W$ with $x \geq x_1$.*

Then $\mathfrak{L}$ is zero-free above $\frac{1}{2}$. In particular, if $\psi \in \mathcal{P}$ and the cyclic level $\mathfrak{L}_\psi$ satisfies this hypothesis, then $L(s,\psi) \neq 0$ for every s with $\mathrm{Re}\, s > \frac{1}{2}$.

Proof. Let $\mathfrak{L} = (K, \psi_{\mathrm{exc}}, o)$, let $\psi \in K$ and let $s \in \mathbb{C}$ with $\mathrm{Re}\, s > \frac{1}{2}$. Put

$$m = \min(\mathrm{Re}\, s - \tfrac{1}{2}, \tfrac{1}{2}) > 0, \qquad \theta = \tfrac{1}{2} + \tfrac{m}{2}.$$

Then $0 < \theta \leq \frac{3}{4} < 1$ and $\theta < \mathrm{Re}\, s$. Choose $\varepsilon = \min(\varepsilon_0/2,\ m/(2(|c|+1)))$; then $0 < \varepsilon < \varepsilon_0$ and $c\varepsilon \leq |c|\varepsilon \leq m/2$. Let W and $x_1 \geq 1$ be as in the hypothesis for this ε.

Let $y \geq 1$. By property 1 there is $Y \geq \max(y, x_1)$ with $[Y, Y^\lambda] \subseteq W$. For $x \in [Y, Y^\lambda]$ we have $x \in W$ and $x \geq x_1 \geq 1$, so by property 2

$$G_{\mathfrak{L}}(x) \leq x^{1/2 + c\varepsilon} \leq x^{\theta},$$

since $x \geq 1$ and $\frac{1}{2} + c\varepsilon \leq \frac{1}{2} + \frac{m}{2} = \theta$. Thus the hypothesis of Proposition 2.4 holds for $\mathfrak{L}$, λ and θ, and $\mathfrak{L}$ is zero-free above θ. As $\operatorname{Re} s > \theta$, $L(s,\psi) \neq 0$. Since $\psi \in K$ and s were arbitrary, $\mathfrak{L}$ is zero-free above $\frac{1}{2}$.

For the last assertion, $\psi = (\psi^1)^* \in K_\psi$, so the zero-freeness of $\mathfrak{L}_\psi$ above $\frac{1}{2}$ gives the claim for $L(s,\psi)$. □

Definition 2.3. Let $f\colon [0,\infty) \to \mathbb{R}$ be a measurable function, $\widehat{f}\colon \mathbb{C} \to \mathbb{C}$ a function and $\mathcal{Z} \subset \mathbb{C}$ a set. The triple $(f, \widehat{f}, \mathcal{Z})$ is **Laplace pole data** if:

(LP1) there is $C_0 \geq 0$ with $|f(u)| \leq C_0 e^{2u}$ for all $u \geq 0$;

(LP2) $\widehat{f}(s) = \int_0^\infty f(u) e^{-su}\, du$ for every s with $\operatorname{Re} s > 2$;

(LP3) every $\rho \in \mathcal{Z}$ has $\operatorname{Re}\rho > 0$, and $\widehat{f}$ is holomorphic on $\{s : \operatorname{Re} s > 0\} \setminus \mathcal{Z}$;

(LP4) $\mathcal{Z}$ has no accumulation point in $\{s : \operatorname{Re} s > 0\}$;

(LP5) every $\rho \in \mathcal{Z}$ is a simple pole of $\widehat{f}$ with nonzero residue c_ρ, that is, $\widehat{f}(s) - c_\rho/(s-\rho)$ agrees on a punctured neighbourhood of ρ with a function holomorphic at ρ; and if ρ is real, then c_ρ is real and $c_\rho > 0$.

The values of $\widehat{f}$ at the points of $\mathcal{Z}$ and in $\operatorname{Re} s \leq 0$ play no role.

Let $\theta \in \mathbb{R}$ and $\lambda > 1$. A measurable function $f\colon [0,\infty) \to \mathbb{R}$ has **integrated window bounds of exponent θ and ratio λ** if for every $\eta > 0$ and every $U_0 \geq 0$ there are $U \geq U_0$ and a function $R\colon [U, \lambda U] \to [0,\infty)$ such that $u \mapsto R(u)e^{-\theta u}$ is integrable on $[U,\lambda U]$, $f(u) \leq R(u)$ for all $u \in [U,\lambda U]$, and

$$\int_U^{\lambda U} R(u)\, e^{-\theta u}\, du \leq e^{\eta U}.$$

Proposition 2.4. *Let $\mathfrak{L}$ be a character level, let $\lambda > 1$ and let $0 < \theta < 1$. Suppose that for every $y \geq 1$ there is $Y \geq y$ such that*

$$G_{\mathfrak{L}}(x) \leq x^\theta \qquad \text{for all } x \in [Y, Y^\lambda].$$

Then $\mathfrak{L}$ is zero-free above θ.

Proof. Let $\mathfrak{L} = (K, \psi_{\mathrm{exc}}, o)$ and $f(u) = G_{\mathfrak{L}}(e^u)$ for $u \geq 0$.

The window hypothesis gives integrated window bounds. Let $\eta > 0$ and $U_0 \geq 0$. Since $\log Y = o(Y^\eta)$, there is $Y_1 \geq 1$ with $(\lambda-1)\log Y \leq Y^\eta$ for all $Y \geq Y_1$. By hypothesis there is $Y \geq \max(e^{U_0}, Y_1)$ with $G_{\mathfrak{L}}(x) \leq x^\theta$ on $[Y, Y^\lambda]$. Put $U = \log Y \geq U_0$ and $R(u) = e^{\theta u} \geq 0$ on $[U,\lambda U]$. For $u \in [U,\lambda U]$ we have $e^u \in [Y, Y^\lambda]$, so $f(u) \leq R(u)$, and

$$\int_U^{\lambda U} R(u) e^{-\theta u}\, du = (\lambda-1)U = (\lambda-1)\log Y \leq Y^\eta = e^{\eta U}.$$

So f has integrated window bounds of exponent θ and ratio λ. (In the multiplicative variable this is $\int_Y^{Y^\lambda} x^\theta\, x^{-\theta-1}\, dx = (\lambda-1)\log Y$.)

Zeros of the L-functions of the level. By Proposition 2.5 there is $\widehat{f}$ such that $(f, \widehat{f}, \mathcal{Z}_{\mathfrak{L}})$ is Laplace pole data, where $\mathcal{Z}_{\mathfrak{L}}$ is the set of zeros with positive real part of the functions $L(s,\psi)$, $\psi \in K$. By Proposition 2.7, $\operatorname{Re}\rho \leq \theta$ for every $\rho \in \mathcal{Z}_{\mathfrak{L}}$.

Conclusion. Let $\psi \in K$ and let s satisfy $\operatorname{Re} s > \theta$ and $L(s,\psi) = 0$. Then $\operatorname{Re} s > \theta > 0$, so $s \in \mathcal{Z}_{\mathfrak{L}}$ and $\operatorname{Re} s \le \theta$, a contradiction. Hence $\mathfrak{L}$ is zero-free above θ.

Remarks. The classical theorem of Landau in this setting (bound for all large x, [157], [180]) is the special case in which every Y works. Only two features of a level are used, both through Proposition 2.5: the positivity of the weights and the closure of K under conjugation (Le1). The statement holds for every character level, with or without an exceptional character, and the value $(3o+1)^{-1}$ of the weights plays no role here. □

Proposition 2.5. *Let $\mathfrak{L} = (K, \psi_{\mathrm{exc}}, o)$ be a character level, put $f(u) = G_{\mathfrak{L}}(e^u)$ for $u \ge 0$, and let $\mathcal{Z}_{\mathfrak{L}}$ be the set of $\rho \in \mathbb{C}$ with $\operatorname{Re}\rho > 0$ such that $L(\rho,\psi) = 0$ for some $\psi \in K$. Then there is a function $\widehat{f}$ such that $(f, \widehat{f}, \mathcal{Z}_{\mathfrak{L}})$ is Laplace pole data, and its residue at $\rho \in \mathcal{Z}_{\mathfrak{L}}$ is*

$$c_\rho = \frac{S(\rho)}{\rho(1-\rho)}, \qquad S(\rho) = \sum_{\psi \in K} \frac{\tilde{w}(\psi)}{2}\bigl(m(\rho,\psi) + m(\rho,\overline{\psi})\bigr) > 0,$$

where $\tilde{w}(\mathbf{1}) = 1$, $\tilde{w}(\psi) = w_{\mathfrak{L}}(\psi)$ for $\psi \ne \mathbf{1}$, and $m(\rho,\psi) \ge 0$ is the order of vanishing of $L(s,\psi)$ at $s = \rho$.

Proof. Since $g(y,\mathbf{1}) = G(y)$ is real, (2.3) reads $G_{\mathfrak{L}}(y) = \sum_{\psi\in K} \tilde{w}(\psi) \operatorname{Re} g(y,\psi)$, and $\tilde{w}(\psi) > 0$ for all $\psi \in K$.

Conjugation. For every $\psi \in \mathcal{P}$ and $s \in \mathbb{C}$ we have $L(\overline{s},\overline{\psi}) = \overline{L(s,\psi)}$: both sides are meromorphic in s and agree for $\operatorname{Re} s > 1$ by the Dirichlet series. Hence $L'(1,\overline{\psi}) = \overline{L'(1,\psi)}$, and since $\overline{\psi(n)} = \overline{\psi}(n)$, (2.2) gives $\overline{g(y,\psi)} = g(y,\overline{\psi})$ for $y > 0$ (also for $\psi = \mathbf{1} = \overline{\mathbf{1}}$). Consequently

$$f(u) = \sum_{\psi \in K} \frac{\tilde{w}(\psi)}{2}\bigl(g(e^u,\psi) + g(e^u,\overline{\psi})\bigr) \qquad (u \ge 0). \tag{2.4}$$

The same identity shows that $L(\rho,\overline{\psi}) = 0$ if and only if $L(\overline{\rho},\psi) = 0$; by (Le1), $\overline{\psi} \in K$ whenever $\psi \in K$.

The function $\widehat{f}$. Put

$$\widehat{f}(s) = \sum_{\psi \in K} \frac{\tilde{w}(\psi)}{2}\bigl(\widehat{g}(s,\psi) + \widehat{g}(s,\overline{\psi})\bigr),$$

a meromorphic function on $\mathbb{C}$ (its values at poles are irrelevant). We check (LP1)–(LP5).

(LP1) and measurability. By part 1 of Lemma 2.6 and (2.4), f is measurable and $|f(u)| \le \sum_{\psi\in K} \tilde{w}(\psi) C_\psi\, e^{2u}$ for $u \ge 0$, because $|\operatorname{Re} g| \le |g|$.

(LP2). For $\operatorname{Re} s > 2$ the substitution $y = e^u$ gives $\int_0^\infty f(u) e^{-su}\,du = \int_1^\infty G_{\mathfrak{L}}(y)\, y^{-s-1}\,dy$. By (2.4) and part 2 of Lemma 2.6 (absolute convergence, applied to ψ and to $\overline{\psi}$, which is primitive), this equals $\widehat{f}(s)$.

(LP3) and (LP4). Every $\rho \in \mathcal{Z}_{\mathfrak{L}}$ has $\operatorname{Re}\rho > 0$ by definition. By part 3 of Lemma 2.6, $\widehat{g}(\cdot,\psi)$ and $\widehat{g}(\cdot,\overline{\psi})$ are holomorphic off $\{0\} \cup \mathcal{Z}_\psi \cup \mathcal{Z}_{\overline{\psi}}$; a point of $\mathcal{Z}_{\overline{\psi}}$ with positive real part lies in $\mathcal{Z}_{\mathfrak{L}}$ because $\overline{\psi} \in K$. So $\widehat{f}$ is holomorphic on $\{\operatorname{Re} s > 0\} \setminus \mathcal{Z}_{\mathfrak{L}}$. Moreover $\mathcal{Z}_{\mathfrak{L}} \subseteq \bigcup_{\psi\in K} \mathcal{Z}_\psi$, a finite union of sets without accumulation points.

(LP5). Let $\rho \in \mathcal{Z}_{\mathfrak{L}}$. Then $\rho \ne 0$, and $\rho \ne 1$ since $1 \notin \mathcal{Z}_\psi$. By part 3 of Lemma 2.6, for each $\chi \in \{\psi, \overline{\psi}\}$ the function $\widehat{g}(s,\chi) + \frac{m(\rho,\chi)}{\rho(\rho-1)} \cdot \frac{1}{s-\rho}$ is holomorphic at ρ (if $m(\rho,\chi) = 0$, then $\rho \notin \mathcal{Z}_\chi$ and $\widehat{g}(\cdot,\chi)$ itself is holomorphic at ρ). Summing with the weights, $\widehat{f}(s) - c_\rho/(s-\rho)$ is holomorphic at ρ, with

$$c_\rho = -\frac{S(\rho)}{\rho(\rho-1)} = \frac{S(\rho)}{\rho(1-\rho)}.$$

All terms of $S(\rho)$ are ≥ 0, and $m(\rho,\psi) \geq 1$ for some $\psi \in K$ because $\rho \in \mathcal{Z}_{\mathfrak{L}}$; as $\tilde{w}(\psi) > 0$, $S(\rho) > 0$, and $c_\rho \neq 0$ since $\rho \neq 0, 1$. If ρ is real, then $0 < \rho < 1$, because no $L(s,\psi)$ with $\psi \in K$ vanishes at a real $s \geq 1$: for real $s > 1$ the Euler product of $L(s,\psi)$ converges absolutely and its factors $(1-\psi(p)p^{-s})^{-1}$ are non-zero, and $s = 1$ is excluded by $1 \notin \mathcal{Z}_\psi$ (part 3 of Lemma 2.6). Only real points $s \geq 1$ are needed here, not the line $\operatorname{Re} s = 1$. Hence $\rho(1-\rho) > 0$ and c_ρ is real and positive.

The role of positivity. The positivity of the weights $\tilde{w}(\psi)$ is used only for $S(\rho) > 0$: it prevents the poles contributed by different characters at a common zero from cancelling, and it gives the residue at a real zero the sign of $\rho(1-\rho) > 0$. No sign of the Dirichlet coefficients $\Lambda(n)\sum_{\psi\in K}\tilde{w}(\psi)\operatorname{Re}\psi(n)$ is used. □

Lemma 2.6. *Let $\psi \in \mathcal{P}$, and define meromorphic functions on $\mathbb{C}$ by*

$$\widehat{g}(s,\psi) = -\frac{1}{s(s-1)}\Big(\frac{L'(s,\psi)}{L(s,\psi)} - s\,\frac{L'(1,\psi)}{L(1,\psi)}\Big) \quad (\psi \neq \mathbf{1}),$$
$$\widehat{g}(s,\mathbf{1}) = -\frac{1}{s(s-1)}\Big(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} + 1 - (1+\gamma)s\Big).$$

Let $\mathcal{Z}_\psi$ be the set of zeros of $L(s,\psi)$ in $\mathbb{C}$ (for $\psi = \mathbf{1}$, the zeros of $\zeta(s)$).

1. *The function $y \mapsto g(y,\psi)$ is measurable on $[1,\infty)$, and there is $C_\psi \geq 0$ with $|g(y,\psi)| \leq C_\psi y^2$ for all $y \geq 1$.*
2. *For $\operatorname{Re} s > 1$ the integral $\int_1^\infty g(y,\psi)\,y^{-s-1}\,dy$ converges absolutely and equals $\widehat{g}(s,\psi)$.*
3. *The singularity of $\widehat{g}(\cdot,\psi)$ at $s = 1$ is removable, $1 \notin \mathcal{Z}_\psi$, the set $\mathcal{Z}_\psi$ has no accumulation point in $\mathbb{C}$, and $\widehat{g}(\cdot,\psi)$ is holomorphic on $\mathbb{C}\setminus(\{0\}\cup\mathcal{Z}_\psi)$. At every $\rho \in \mathcal{Z}_\psi \setminus \{0\}$ it has a simple pole with residue $-m/(\rho(\rho-1))$, where $m \geq 1$ is the order of the zero of $L(s,\psi)$ at ρ.*

Proof. Part 1. The function $g(\cdot,\psi)$ is continuous on each interval $[n, n+1)$, hence measurable. For $1 \leq n \leq y$ we have $0 \leq \Lambda(n) \leq \log n \leq n$, $|\psi(n)| \leq 1$ and $0 \leq y/n - 1 \leq y/n$, so each of the at most y terms of the sum in (2.2) (or (2.1)) has absolute value at most y, and the sum is at most y^2 in absolute value. For the trivial character the main terms are bounded by $y\log y \leq y^2$ and $|1+\gamma|y \leq |1+\gamma|y^2$; otherwise the main term is bounded by $|L'(1,\psi)/L(1,\psi)|\,y^2$.

Part 2. For an integer $n \geq 1$ and $\operatorname{Re} s > 1$,

$$\int_1^\infty \max\Big(\frac{y}{n} - 1, 0\Big)y^{-s-1}\,dy = \int_n^\infty \Big(\frac{y^{-s}}{n} - y^{-s-1}\Big)dy = \frac{n^{-s}}{s-1} - \frac{n^{-s}}{s} = \frac{n^{-s}}{s(s-1)}. \tag{2.5}$$

The sum in (2.2) equals $\sum_{n\geq 1}\Lambda(n)\psi(n)\max(y/n-1,0)$, the terms with $n > y$ being zero. By (2.5) with s replaced by $\sigma = \operatorname{Re} s$,

$$\sum_n \Lambda(n)\int_1^\infty \max(y/n-1,0)\,y^{-\sigma-1}dy = \sigma^{-1}(\sigma-1)^{-1}\sum_n \Lambda(n)n^{-\sigma} < \infty,$$

so the series can be integrated term by term (dominated convergence), and the integral converges absolutely. Using $-L'(s,\psi)/L(s,\psi) = \sum_n \Lambda(n)\psi(n)n^{-s}$ for $\operatorname{Re} s > 1$ [180] (for $\psi = \mathbf{1}$ this is $-\zeta'/\zeta$),

$$\int_1^\infty \sum_{n\leq y}\Lambda(n)\psi(n)\Big(\frac{y}{n} - 1\Big)y^{-s-1}\,dy = -\frac{1}{s(s-1)}\,\frac{L'(s,\psi)}{L(s,\psi)}.$$

The main terms contribute, by $\int_1^\infty y^{-s}\,dy = 1/(s-1)$ and (integrating by parts) $\int_1^\infty (\log y)\,y^{-s}\,dy = 1/(s-1)^2$: for $\psi \neq \mathbf{1}$, $\frac{L'(1,\psi)}{L(1,\psi)} \cdot \frac{1}{s-1} = \frac{s}{s(s-1)}\frac{L'(1,\psi)}{L(1,\psi)}$; for $\psi = \mathbf{1}$, $-\frac{1}{(s-1)^2} + \frac{1+\gamma}{s-1} = \frac{1}{s(s-1)}\big(-\frac{s}{s-1} + (1+\gamma)s\big)$, and $\frac{s}{s-1} = 1 + \frac{1}{s-1}$. Adding gives $\widehat{g}(s,\psi)$.

Part 3. Put $L^\circ(s) = L(s,\psi)$ and $\delta = 0$ if $\psi \neq \mathbf{1}$, and $L^\circ(s) = (s-1)\zeta(s)$ and $\delta = 1$ if $\psi = \mathbf{1}$. Then L° is entire and $L^\circ(1) \neq 0$. Indeed, a primitive character $\psi \neq \mathbf{1}$ is non-principal (a principal character of modulus $N > 1$ is induced by $\mathbf{1}$, hence not primitive), so $L(s,\psi)$ is entire [180] and $L(1,\psi) \neq 0$ by Dirichlet's theorem [75]. Near $s = 1$ we have $\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$ [180], [260], so that $(s-1)\zeta(s)$ extends to an entire function with value 1 and derivative γ at $s = 1$. In both cases

$$\widehat{g}(s,\psi) = -\frac{N(s)}{s(s-1)}, \qquad N(s) = \frac{L^{\circ\prime}(s)}{L^\circ(s)} - s\,\frac{L^{\circ\prime}(1)}{L^\circ(1)} + \delta(1-s).$$

The function N is meromorphic, holomorphic wherever $L^\circ \neq 0$, in particular near $s = 1$, and $N(1) = 0$. Hence $N(s)/(s-1)$ has a removable singularity at $s = 1$, and $\widehat{g}(\cdot,\psi)$ is holomorphic off $\{0\}$ and the zeros of L°. These zeros are exactly $\mathcal{Z}_\psi$ (for $\psi = \mathbf{1}$ the factor $s-1$ adds no zero, the value at 1 being 1), and $1 \notin \mathcal{Z}_\psi$. Since L° is entire and not identically zero, its zeros have no accumulation point. If $\rho \in \mathcal{Z}_\psi$ has order m, write $L^\circ(s) = (s-\rho)^m h(s)$ with h holomorphic and $h(\rho) \neq 0$; then $L^{\circ\prime}/L^\circ = m/(s-\rho) + h'/h$, so near ρ the function $N(s) - m/(s-\rho)$ is holomorphic. For $\rho \neq 0$ (and $\rho \neq 1$) the factor $-1/(s(s-1))$ is holomorphic at ρ with value $-1/(\rho(\rho-1))$, so $\widehat{g}(s,\psi) - \frac{-m/(\rho(\rho-1))}{s-\rho}$ is holomorphic at ρ. □

Proposition 2.7. *Let $(f, \widehat{f}, \mathcal{Z})$ be Laplace pole data, let $0 < \theta < 1$ and $\lambda > 1$, and suppose that f has integrated window bounds of exponent θ and ratio λ. Then $\operatorname{Re}\rho \le \theta$ for every $\rho \in \mathcal{Z}$.*

Proof. Suppose, to the contrary, that some $\rho_0 \in \mathcal{Z}$ has $\operatorname{Re}\rho_0 > \theta$. For an integer $k \ge 0$ write $A_k(s) = (-1)^k \widehat{f}^{(k)}(s)/k!$ wherever $\widehat{f}$ is holomorphic near s. Let $\sigma > 2$ and C_2 be as in Lemma 2.10; since $\widehat{f}$ coincides with the Laplace integral of f on the open half-plane $\operatorname{Re} s > 2$ by (LP2), that lemma applies to the derivatives of $\widehat{f}$ at σ and $\sigma + i\gamma$. By Lemma 2.8, $\operatorname{Re}\rho \le 2 < \sigma$ for every $\rho \in \mathcal{Z}$. We use three elementary facts.

(C) Cauchy's estimate. If $\widehat{f}$ is holomorphic on $B(z,r)$ and $0 < r' < r$, then $|A_k(z)| \le M r'^{-k}$ for all k, with $M = \max_{|s-z|=r'} |\widehat{f}(s)|$.

(P) Coefficients near a simple pole. Let $\rho \in B(z,r)$, $\rho \neq z$, let $\widehat{f}$ be holomorphic on $B(z,r) \setminus \{\rho\}$ and let $\widehat{f}(s) - c/(s-\rho)$ agree near ρ with a function holomorphic at ρ. Then $\widehat{f}(s) - c/(s-\rho)$ extends holomorphically to $B(z,r)$, and the k-th Taylor coefficient of $c/(s-\rho)$ at z is $-c/(\rho-z)^{k+1}$. By (C), for every $r' \in (0,r)$ there is M with

$$\Big|A_k(z) - \frac{c}{(z-\rho)^{k+1}}\Big| \le M r'^{-k} \qquad \text{for all } k.$$

(G) Growth. Let $0 < d < e \le E$, $c > 0$ and $M, C \in \mathbb{R}$. It is impossible that for every $\eta > 0$ and every U_0 there are $U \ge U_0$ and an integer $k \ge U$ with

$$c\,d^{-k} \le M e^{-k} + C\,e^{\eta U} E^{-k}.$$

Indeed, take $\eta = \frac{1}{2}\log(e/d) > 0$ and $U_0 \ge 0$. Then $e^{\eta U} \le e^{\eta k} = (e/d)^{k/2}$ and $E^{-k} \le e^{-k}$, so the inequality gives $c \le (|M| + |C|)(e/d)^{k/2}(d/e)^k = (|M| + |C|)(d/e)^{k/2}$ with $k \ge U_0$ arbitrarily large, whence $c \le 0$.

The set $\mathcal{Z}$ has no accumulation point in $\operatorname{Re} s > 0$ by (LP4), so Lemma 2.9 applies with $\theta < \sigma$ and ρ_0. All discs below have radius at most $\sigma - \theta < \sigma$ and centre on the line $\operatorname{Re} s = \sigma$, so they lie in $\operatorname{Re} s > 0$, where $\widehat{f}$ is holomorphic off $\mathcal{Z}$ by (LP3).

Case 2 of Lemma 2.9: the nearest singularity is real. There are a real $\beta \in \mathcal{Z}$ with $\theta < \beta < \sigma$ and r with $\sigma - \beta < r \le \sigma - \theta$ such that β is the only point of $\mathcal{Z}$ in $B(\sigma, r)$. By (LP5) the residue c_β is real and positive. Apply (P) with $z = \sigma$, $\rho = \beta$ and $r' = (\sigma - \beta + r)/2$: since $c_\beta(\sigma-\beta)^{-k-1}$ is real,

$$A_k(\sigma) \ge \frac{c_\beta}{\sigma - \beta}(\sigma - \beta)^{-k} - M r'^{-k} \qquad \text{for all } k.$$

Given $\eta > 0$ and U_0, Lemma 2.10 with $\gamma = 0$ gives $U \ge U_0$ and $k \ge U$ with $A_k(\sigma) \le |A_k(\sigma)| + A_k(\sigma) \le C_2 e^{\eta U}(\sigma - \theta)^{-k}$. Hence

$$\frac{c_\beta}{\sigma - \beta}(\sigma - \beta)^{-k} \le M r'^{-k} + C_2 e^{\eta U}(\sigma - \theta)^{-k},$$

which contradicts (G) with $d = \sigma - \beta < e = r' < E = \sigma - \theta$.

Case 1 of Lemma 2.9: the nearest singularity is not real. There is $D \le \sigma - \theta$ such that $B(\sigma, D)$ contains no point of $\mathcal{Z}$, and there are $\gamma \in \mathbb{R}$, $\rho \in \mathcal{Z}$ and $r \le D$ such that, with $z = \sigma + i\gamma$ and $d = |z - \rho|$, we have $d < r$ and ρ is the only point of $\mathcal{Z}$ in $B(z, r)$. Moreover, $d > 0$ because $\operatorname{Re}\rho < \sigma$. Put $r' = (d + r)/2$, so $d < r' < r \le D$. By (P) at z there is M with $|A_k(z)| \ge |c_\rho| d^{-k-1} - M r'^{-k}$, and by (C) at σ, applied on $B(\sigma, D)$, there is M' with $|A_k(\sigma)| \le M' r'^{-k}$, for all k. Given $\eta > 0$ and U_0, Lemma 2.10 gives $U \ge U_0$ and $k \ge U$ with

$$|A_k(z)| \le C_2 e^{\eta U}(\sigma - \theta)^{-k} - A_k(\sigma) \le C_2 e^{\eta U}(\sigma - \theta)^{-k} + M' r'^{-k}.$$

Hence $\frac{|c_\rho|}{d} d^{-k} \le (M + M') r'^{-k} + C_2 e^{\eta U}(\sigma - \theta)^{-k}$, with $|c_\rho| > 0$ by (LP5). This contradicts (G) with $d < e = r' < E = \sigma - \theta$.

Both cases are impossible, so $\operatorname{Re}\rho \le \theta$ for all $\rho \in \mathcal{Z}$.

Relation to Landau's theorem. Landau's theorem [157] (see [133], [180]) states that a Laplace integral of an eventually non-negative function has a singularity at the real point of its abscissa of convergence; its proof expands at a real point σ and uses $f \ge 0$ to continue the Taylor series to the left. The argument above never continues the Laplace integral to the left of $\operatorname{Re} s = 2$: the hypothesis holds only on the windows $[U, \lambda U]$, so it compares single Taylor coefficients instead. A singularity at distance d from a point of the line $\operatorname{Re} s = \sigma$ forces coefficients of size d^{-k} there, while Lemma 2.10, for $k \approx \frac{1+\lambda}{2}\sigma U$, bounds them by $e^{\eta U}(\sigma - \theta)^{-k}$. If the singularity nearest to σ is real, the positivity of its residue gives the real coefficient $A_k(\sigma)$ a definite sign (case 2); otherwise the coefficient at σ is negligible and the coefficient at a point $\sigma + i\gamma$ near the singularity is large (case 1). Positivity of residues at real poles thus replaces the non-negativity of f. □

Lemma 2.8. *Let $f\colon [0, \infty) \to \mathbb{R}$ be measurable with $|f(u)| \le C_0 e^{2u}$ for all $u \ge 0$. Then $F(s) = \int_0^\infty f(u) e^{-su}\, du$ converges absolutely and is holomorphic in $\operatorname{Re} s > 2$, $|F(s)| \le C_0/(\sigma - 2)$ whenever $\operatorname{Re} s \ge \sigma > 2$, and for every integer $k \ge 0$*

$$\frac{(-1)^k F^{(k)}(s)}{k!} = \int_0^\infty f(u)\, \frac{u^k}{k!}\, e^{-su}\, du \qquad (\operatorname{Re} s > 2).$$

If $(f, \widehat{f}, \mathcal{Z})$ is Laplace pole data, then $\widehat{f} = F$ on $\operatorname{Re} s > 2$, so $\widehat{f}$ satisfies the same identity there, and every $\rho \in \mathcal{Z}$ has $\operatorname{Re}\rho \le 2$.

Proof. Let $b > 0$. For $\operatorname{Re} z \ge 2 + b$ and $u > 0$,

$$\left| f(u)(-u)^k e^{-zu} \right| \le C_0\, u^k e^{-bu},$$

which is integrable on $(0,\infty)$. Hence all the integrals $\int_0^\infty f(u)(-u)^k e^{-zu}\,du$ converge absolutely in $\operatorname{Re} z > 2$, and differentiation under the integral sign (dominated convergence, with the dominating function $C_0 u^{k+1} e^{-bu}$ on the half-plane $\operatorname{Re} z \geq 2+b$, $b = (\operatorname{Re} s - 2)/2$) shows that the k-th integral is holomorphic with derivative the $(k+1)$-st. By induction $F^{(k)}(s) = \int_0^\infty f(u)(-u)^k e^{-su}\,du$, which is the displayed identity. For $\operatorname{Re} s \geq \sigma > 2$,

$$|F(s)| \leq \int_0^\infty C_0 e^{2u} e^{-\sigma u}\,du = \frac{C_0}{\sigma - 2}.$$

Now let $(f, \widehat{f}, \mathcal{Z})$ be Laplace pole data. By (LP2) $\widehat{f} = F$ on the open half-plane $\operatorname{Re} s > 2$, so the derivatives agree there. Suppose $\rho \in \mathcal{Z}$ had $\operatorname{Re}\rho > 2$, and put $\sigma' = (2 + \operatorname{Re}\rho)/2$. On a neighbourhood of ρ contained in $\operatorname{Re} s > \sigma'$ we have $|\widehat{f}(s)| \leq C_0/(\sigma' - 2)$, so $(s-\rho)\widehat{f}(s) \to 0$ as $s \to \rho$. By (LP5), however, $(s-\rho)\widehat{f}(s) = c_\rho + (s-\rho)h(s)$ near ρ with h holomorphic at ρ, which tends to $c_\rho \neq 0$. Hence $\operatorname{Re}\rho \leq 2$. □

Lemma 2.9. *Let $\mathcal{Z} \subset \mathbb{C}$ be a set without accumulation points in $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, and let $0 < \theta < \sigma$ be real numbers such that $\operatorname{Re}\rho < \sigma$ for every $\rho \in \mathcal{Z}$. Suppose that some $\rho_0 \in \mathcal{Z}$ has $\operatorname{Re}\rho_0 > \theta$. Write $B(z,r)$ for the open disc of centre z and radius r. Then one of the following holds.*

1. *There are $D \leq \sigma - \theta$ with $B(\sigma, D) \cap \mathcal{Z} = \emptyset$, and $\gamma \in \mathbb{R}$, $\rho \in \mathcal{Z}$ and $r \leq D$ such that $|\sigma + i\gamma - \rho| < r$ and $B(\sigma + i\gamma, r) \cap \mathcal{Z} = \{\rho\}$.*
2. *There are a real number $\beta \in \mathcal{Z}$ with $\beta > \theta$ and a real r with $\sigma - \beta < r \leq \sigma - \theta$ such that $B(\sigma, r) \cap \mathcal{Z} = \{\beta\}$.*

Proof. Finiteness. If $z \in \mathbb{C}$ and $0 \leq t < \operatorname{Re} z$, the closed disc $\overline{B}(z,t)$ is a compact subset of $\{\operatorname{Re} s > 0\}$; covering it by neighbourhoods each containing at most one point of $\mathcal{Z}$ shows that $\mathcal{Z} \cap \overline{B}(z,t)$ is finite.

The dichotomy at σ. The set $F = \mathcal{Z} \cap \overline{B}(\sigma, \sigma - \theta)$ is finite, since $\sigma - \theta < \sigma$. We first show that one of the following holds:

(i) there are $D \leq \sigma - \theta$ with $B(\sigma, D) \cap \mathcal{Z} = \emptyset$ and $\rho_1 \in \mathcal{Z}$ with $\sigma - \operatorname{Re}\rho_1 < D$;

(ii) conclusion 2 holds.

If $F = \emptyset$, then (i) holds with $D = \sigma - \theta$ and $\rho_1 = \rho_0$. Otherwise let $d = \min_{w \in F} |w - \sigma| \leq \sigma - \theta$, attained at $\rho_s \in F$; then $B(\sigma, d) \cap \mathcal{Z} = \emptyset$. If some $\rho_1 \in \mathcal{Z}$ has $\sigma - \operatorname{Re}\rho_1 < d$, then (i) holds with $D = d$. Otherwise $\sigma - \operatorname{Re} w \geq d$ for all $w \in \mathcal{Z}$. A point w with $|w - \sigma| \leq d$ and $\sigma - \operatorname{Re} w \geq d$ satisfies $(\operatorname{Re} w - \sigma)^2 + (\operatorname{Im} w)^2 \leq d^2 \leq (\operatorname{Re} w - \sigma)^2$, hence $\operatorname{Im} w = 0$ and $w = \sigma - d$. So $\beta = \sigma - d$ is the only point of F at distance d from σ, and it is real. Moreover $d < \sigma - \theta$: otherwise $\sigma - \operatorname{Re}\rho_0 \geq d = \sigma - \theta$, contradicting $\operatorname{Re}\rho_0 > \theta$. Hence $\beta > \theta$. Let $r_1 > d$ be the least distance from σ of a point of $F \setminus \{\beta\}$ (or $r_1 = d + 1$ if there is none) and $r = \min(r_1, \sigma - \theta)$. Then $\sigma - \beta = d < r \leq \sigma - \theta$, and every point of $\mathcal{Z}$ in $B(\sigma, r)$ lies in F and hence equals β: this is conclusion 2.

From (i) to conclusion 1. Assume (i). Choose $\varepsilon > 0$ with $\varepsilon \leq (D - (\sigma - \operatorname{Re}\rho_1))/2$ and $\varepsilon \leq (\sigma - D)/2$ (note $D < \sigma$), and put $c_0 = \sigma + i \operatorname{Im}\rho_1$. The set $F' = \mathcal{Z} \cap \overline{B}(c_0, D + \varepsilon)$ is finite, since $D + \varepsilon < \sigma = \operatorname{Re} c_0$, and it contains ρ_1 because $|c_0 - \rho_1| = \sigma - \operatorname{Re}\rho_1$. For distinct $a, b \in F'$, the points $\sigma + i\gamma$ equidistant from a and b satisfy

$$2\gamma(\operatorname{Im} a - \operatorname{Im} b) = (\sigma - \operatorname{Re} a)^2 + (\operatorname{Im} a)^2 - (\sigma - \operatorname{Re} b)^2 - (\operatorname{Im} b)^2.$$

If $\operatorname{Im} a = \operatorname{Im} b$ this would force $(\operatorname{Re} b - \operatorname{Re} a)((\sigma - \operatorname{Re} a) + (\sigma - \operatorname{Re} b)) = 0$, impossible since $a \neq b$ and both real parts are below σ; so each pair excludes at most one value of γ. Choose $\gamma \in (\operatorname{Im}\rho_1 - \varepsilon, \operatorname{Im}\rho_1 + \varepsilon)$ outside this finite set of values, and put $z = \sigma + i\gamma$. Then the distances $|z - w|$, $w \in F'$, are pairwise distinct; let $\rho \in F'$ be the nearest point to z. Since $|z-\rho| \le |z-\rho_1| \le |\gamma - \operatorname{Im}\rho_1| + (\sigma - \operatorname{Re}\rho_1) < \varepsilon + (\sigma - \operatorname{Re}\rho_1) < D$, there is $r \le D$ with $|z-\rho| < r$ and $|z - w| \ge r$ for all $w \in F' \setminus \{\rho\}$. A point $w \in \mathcal{Z} \cap B(z, r)$ satisfies $|w - c_0| < r + \varepsilon \le D + \varepsilon$, so $w \in F'$ and hence $w = \rho$. Together with $B(\sigma, D) \cap \mathcal{Z} = \emptyset$ from (i), this is conclusion 1. □

Lemma 2.10. *Let $f\colon [0,\infty) \to \mathbb{R}$ be measurable with $|f(u)| \le C_0 e^{2u}$ for all $u \ge 0$, and let $F(s) = \int_0^\infty f(u)e^{-su}\,du$ for $\operatorname{Re} s > 2$. Let $0 < \theta < 1$ and $\lambda > 1$, and suppose that f has integrated window bounds of exponent θ and ratio λ. Put $l = 2 + 6/(\lambda - 1)$ and $\sigma = l\lambda(l+2)$. Then $\sigma > 2$, and there is $C_2 \ge 0$ such that for every $\eta > 0$ and every $U_0 \in \mathbb{R}$ there are $U \ge U_0$ and an integer $k \ge U$ with*

$$\Big|\frac{F^{(k)}(\sigma + i\gamma)}{k!}\Big| + \frac{(-1)^k F^{(k)}(\sigma)}{k!} \le C_2\,\frac{e^{\eta U}}{(\sigma - \theta)^k} \qquad \text{for every } \gamma \in \mathbb{R};$$

here $(-1)^k F^{(k)}(\sigma)/k!$ is real.

Proof. Let C_1 be the constant attached to λ in Lemma 2.11; we show the claim with $C_2 = 2 + 2C_0C_1$. We have $\sigma > 2 > \theta > 0$. Let $\eta > 0$ and $U_0 \in \mathbb{R}$. The window bound for η and $\max(U_0, 1)$ gives $U \ge \max(U_0, 1)$ and $R\colon [U, \lambda U] \to [0,\infty)$ with $f \le R$ on $[U, \lambda U]$ and $\int_U^{\lambda U} R(u)e^{-\theta u}\,du \le e^{\eta U}$. Since $U \ge 1$, part 2 of Lemma 2.11 gives an integer $k \ge U$ with

$$\int_{(0,\infty)\setminus[U,\lambda U]} K_{k,\sigma-2}(u)\,du \le C_1\sigma^{-k}. \tag{2.6}$$

Reduction to the positive part of f. Put $w = K_{k,\sigma}$ and $A(s) = (-1)^k F^{(k)}(s)/k!$. By Lemma 2.8, for every $\gamma \in \mathbb{R}$,

$$A(\sigma + i\gamma) = \int_0^\infty f(u)\,w(u)\,e^{-i\gamma u}\,du, \qquad A(\sigma) = \int_0^\infty f(u)\,w(u)\,du \in \mathbb{R},$$

the integrals converging absolutely because $|f|w \le C_0 K_{k,\sigma-2}$. As $|A(\sigma + i\gamma)| \le \int_0^\infty |f|w$ and $|f| + f = 2\max(f, 0)$,

$$|A(\sigma + i\gamma)| + A(\sigma) \le \int_0^\infty (|f(u)| + f(u))w(u)\,du = 2\int_0^\infty \max(f(u), 0)\,w(u)\,du. \tag{2.7}$$

This is the only place where the sign of f enters. In Landau's classical argument [157] one has $f \ge 0$, and (2.7) reads $|A(\sigma + i\gamma)| \le A(\sigma)$. Here only the positive part of f has to be controlled, and only where w has its mass.

The window. For $u \in [U, \lambda U]$ we have $0 \le \max(f(u), 0) \le R(u)$, since $f \le R$ and $R \ge 0$, and $w(u) = K_{k,\sigma-\theta}(u)\,e^{-\theta u} \le (\sigma - \theta)^{-k} e^{-\theta u}$ by part 1 of Lemma 2.11. Hence

$$2\int_U^{\lambda U} \max(f, 0)\,w\,du \le 2(\sigma - \theta)^{-k}\int_U^{\lambda U} R(u)e^{-\theta u}\,du \le 2(\sigma - \theta)^{-k}e^{\eta U}.$$

Outside the window. There $\max(f, 0) \le |f| \le C_0 e^{2u}$ and $e^{2u}w(u) = K_{k,\sigma-2}(u)$, so by (2.6)

$$2\int_{(0,\infty)\setminus[U,\lambda U]} \max(f, 0)\,w\,du \le 2C_0C_1\sigma^{-k} \le 2C_0C_1\,e^{\eta U}(\sigma - \theta)^{-k},$$

using $0 < \sigma - \theta < \sigma$ and $e^{\eta U} \ge 1$. Adding the two parts in (2.7) gives the claim with $C_2 = 2 + 2C_0C_1$. □

Lemma 2.11. *For an integer $k \geq 0$ and real $a > 0$ put $K_{k,a}(u) = u^k e^{-au}/k!$ for $u \geq 0$.*

1. *$\int_0^\infty K_{k,a}(u)\,du = a^{-k-1}$, and $0 \leq K_{k,a}(u) \leq a^{-k}$ for all $u \geq 0$.*
2. *Let $\lambda > 1$ and put*

$$l = 2 + \frac{6}{\lambda - 1}, \qquad \sigma = l\lambda(l+2), \qquad C_1 = \frac{e}{\sigma - 2 - l} + \frac{1}{\sigma - 2 + l}.$$

Then $\sigma > 2 + l$, and for every real $U \geq 1$ the integer $k = \lceil (1+\lambda)\sigma U/2 \rceil$ satisfies $k \geq U$ and

$$\int_{(0,\infty)\setminus[U,\lambda U]} K_{k,\sigma-2}(u)\,du \leq C_1\,\sigma^{-k}.$$

Proof. Part 1. The substitution $t = au$ gives $\int_0^\infty u^k e^{-au}\,du = a^{-k-1}\Gamma(k+1) = a^{-k-1}k!$. For $u \geq 0$ the number $(au)^k/k!$ is a term of the series of e^{au}, so $a^k K_{k,a}(u) = (au)^k e^{-au}/k! \leq 1$.

Part 2. The kernel $K_{k,\sigma-2}$ has mass $(\sigma-2)^{-k-1}$ and is concentrated near $u = k/(\sigma-2) \approx \frac{1+\lambda}{2} \cdot \frac{\sigma}{\sigma-2}\,U$, a point inside $[U, \lambda U]$ when σ is large. The claim bounds its mass outside this window explicitly. Write $s = \sigma - 2$ and $V = \lambda U$. Since $l > 2$ and $\lambda > 1$ we have $\sigma > l(l+2)$, so $s - l > l^2 + l - 2 > 0$; in particular $\sigma > 2 + l$. We use the two identities

$$\frac{(1+\lambda)(l+2)}{2} = l\lambda - 1, \qquad \frac{(1+\lambda)(l-2)}{2} = l + 1. \tag{2.8}$$

A Chernoff bound. Let $u > 0$ with $u \notin [U, V]$. Then $l(U-u) \geq 0$ or $l(u-V) \geq 0$, hence

$$e^{-su} \leq e^{-su+l(u-V)} + e^{-su+l(U-u)} = e^{-lV}e^{-(s-l)u} + e^{lU}e^{-(s+l)u}.$$

Multiplying by $u^k/k!$, integrating over $(0,\infty)$ and using part 1 (note $s - l > 0$),

$$\int_{(0,\infty)\setminus[U,V]} K_{k,s}(u)\,du \leq e^{-lV}(s-l)^{-k-1} + e^{lU}(s+l)^{-k-1}. \tag{2.9}$$

Conversion to powers of σ. For $a > 0$ and real x, the inequality $1 + t \leq e^t$ with $t = (\sigma - a)/a$ gives $\sigma/a \leq e^{(\sigma-a)/a}$, hence

$$e^x a^{-k-1} = e^x \Big(\frac{\sigma}{a}\Big)^k a^{-1}\sigma^{-k} \leq e^{x+k(\sigma-a)/a}\,a^{-1}\sigma^{-k}. \tag{2.10}$$

First term of (2.9). Here $x = -lV = -l\lambda U$ and $a = s - l = \sigma - 2 - l = (l\lambda - 1)(l+2)$, so $(\sigma - a)/a = (l+2)/a = 1/(l\lambda - 1)$. By the first identity of (2.8), $(1+\lambda)\sigma/2 = (l\lambda - 1)\,l\lambda$, so $k \leq (1+\lambda)\sigma U/2 + 1 = (l\lambda - 1)l\lambda U + 1$ and $k/(l\lambda - 1) \leq l\lambda U + 1$ (as $l\lambda - 1 \geq 1$). The exponent in (2.10) is therefore at most 1, and the first term is at most $e\,(\sigma - 2 - l)^{-1}\sigma^{-k}$.

Second term of (2.9). Here $x = lU$ and $a = s + l = \sigma - 2 + l$, so $(\sigma - a)/a = -(l-2)/(\sigma - 2 + l)$. By the second identity of (2.8), $k(l-2) \geq (1+\lambda)\sigma U(l-2)/2 = (l+1)\sigma U$; and $l(\sigma - 2 + l) \leq (l+1)\sigma$, since this is equivalent to $l^2 - 2l \leq \sigma$, which holds because $\sigma > l(l+2)$. Hence $k(l-2)/(\sigma - 2 + l) \geq lU$, the exponent in (2.10) is at most 0, and the second term is at most $(\sigma - 2 + l)^{-1}\sigma^{-k}$.

Adding the two bounds gives the claimed inequality with $s = \sigma - 2$. Finally $k \geq (1+\lambda)\sigma U/2 \geq U$ because $\sigma > 2$ and $\lambda > 1$. □

3 Hosts, inert prime sums and the excess

Sections 4 and 5 bound a sum over primes from above. Section 2 needs an upper bound for $G_{\mathfrak{L}}$ at one point x. This section proves the step from the first to the second, Proposition 3.3, and introduces the hosts to which it is applied. Both are used in the proof of Proposition 6.5 in Section 6.

The sum over primes is the inert prime sum of an arithmetic datum on a finite abelian group H (Definition 3.1). It is a weighted sum of $\log p$ over the prime powers whose Frobenius element lies outside a subgroup Φ of index two, and its excess $\Delta(u)$ is what remains after its main term is subtracted and the result is summed over a set T of tops and over m weight functions. Proposition 3.3 says that a bound for $\Delta(u)$ at the translates u of a subgroup $A_0 \subseteq H$ gives a bound from above for a weighted sum of twisted Riesz-weighted Chebyshev functions at x, up to the absolute values of finitely many further twists.

The proof rests on two averages. By orthogonality of characters, the excess is a combination of the twisted functions at the cut-offs $a/w(\rho)$, $\rho \in \Phi$ (Lemma 3.8). The weights are close to b at the thick places, which move with the translate u, and small at the other places (Definition 3.2). The average over $u \in A_0$ therefore keeps the characters that are trivial on A_0, at the cut-offs a_t/b (Lemma 3.7). The sum over the tops $t \in T$ then gives each of these characters its weight; it is carried out together with the routine step that moves the cut-offs a_t/b, which lie within 1 of x, to x (Lemma 3.6, Lemma 3.4). The errors are absorbed under the size conditions (Lemma 3.10).

For a host the conclusion is a bound for $G_{\mathfrak{L}}(x)$. The arithmetic datum of a host is of host type (Lemma 3.12), its twisted functions are those of the primitive Dirichlet characters of the host (Lemma 3.18), and under the host conditions the weighted sum is $G_{\mathfrak{L}}$ and the further twists contribute at most $4o\,x^{1/2+\varepsilon}$ (Lemma 3.17).

The section also states the definitions of slope data, slope conditions and size conditions, because the notion of a core-tied host (Definition 3.16) refers to them. Nothing is proved about them here; Section 4 reads them.

Definition 3.1. Let H be a finite abelian group and $\Gamma_H = (\mathbb{Z}/2)^2 \times H$, written additively with elements $((\epsilon,\epsilon'),h)$. Let $\Phi \subset \Gamma_H$ be the subgroup of elements whose first $\mathbb{Z}/2$-coordinate ϵ is 0; it has index 2, so $|\Phi| = 2|H|$ and $|\Gamma_H \setminus \Phi| = |\Phi|$. Put $c_\infty = ((1,1),0)$, and for $t \in H$ let $\sigma_t\colon \Gamma_H \to \Gamma_H$ be the translation $\sigma_t(g) = c_\infty + g + ((0,0),t)$; it maps $\Gamma_H \setminus \Phi$ bijectively onto Φ. Let $\chi_C \in \widehat{\Gamma}_H$ be the character $((\epsilon,\epsilon'),h) \mapsto (-1)^\epsilon$, whose kernel is Φ.

An **arithmetic datum** on H is a tuple $\mathfrak{A} = (q, d, (I_p)_p, (\mathrm{Fr}_p)_p)$ of natural numbers q, d, subgroups $I_p \subseteq \Gamma_H$ and elements $\mathrm{Fr}_p \in \Gamma_H$, indexed by the primes p. It is **of host type with respect to** a finite set S of natural numbers if

(Ah1) d is prime, $2 \in S$, $q \in S$ and $d \notin S$;

(Ah2) $I_p = 0$ for every prime $p \notin S \cup \{d\}$;

(Ah3) $I_p \subseteq \Phi$ and $\mathrm{Fr}_p \in \Phi$ for every prime $p \in S$;

(Ah4) $I_d = \{0, ((1,0),0)\}$.

These are the properties of the Frobenius and inertia data of the abelian fields used later, where S contains the primes that split in $\mathbb{Q}(\sqrt{-d})$ and ramify. They are not a tameness condition.

Coefficients. For $\phi \in \widehat{\Gamma}_H$, a subgroup $I \subseteq \Gamma_H$ and $h \in \Gamma_H$ let $\mu_I(h,\phi) = |I|^{-1}\sum_{x\in I}\phi(h+x)$ be the mean of ϕ over the coset $h + I$, and for $n \geq 1$ put

$$\phi_{\mathfrak{A}}(n) = \prod_{p\mid n} \mu_{I_p}(v_p(n)\mathrm{Fr}_p, \phi).$$

Thus $|\phi_{\mathfrak{A}}(n)| \le 1$, $1_{\mathfrak{A}}(n) = 1$ for the trivial character 1, and $\phi_{\mathfrak{A}}(p^i) = \phi(i\mathrm{Fr}_p)$ when $I_p = 0$.

Twisted Riesz-weighted Chebyshev functions. For a function $\eta\colon \widehat{\Gamma}_H \to \mathbb{C}$, a character $\phi \neq 1$ and real $y > 0$,

$$G_{\mathfrak{A},\eta}(y,\phi) = \sum_{1\le n\le y} \Lambda(n)\,\phi_{\mathfrak{A}}(n)\Big(\frac{y}{n} - 1\Big) + y\,\eta(\phi), \qquad G_{\mathfrak{A},\eta}(y,1) = G(y).$$

Sums over a set of characters. For a subgroup $A_0 \subseteq H$ let $X(A_0) \subseteq \widehat{\Gamma}_H$ be the set of characters trivial on $\{0\}\times A_0$. For a real number c and $\phi \in \widehat{\Gamma}_H$ let $w_c(\phi) = 1$ if ϕ is trivial on $\{0\}\times H$ and $w_c(\phi) = c$ otherwise. For $K' \subseteq \widehat{\Gamma}_H$ and real $x > 0$ put

$$G_{K'}(x) = \sum_{\phi\in K'} w_c(\phi)\,\mathrm{Re}\,G_{\mathfrak{A},\eta}(x,\phi), \qquad \Sigma_{K'}(x) = \sum_{\phi\in X(A_0)\setminus K'} |G_{\mathfrak{A},\eta}(x,\phi)|.$$

Inert prime sums. A prime p is **inert** for $\mathfrak{A}$ if $p \notin \{2,q,d\}$ and $\mathrm{Fr}_p \notin \Phi$. For an integer $P \ge 1$, real a, $t \in H$ and $w\colon \Gamma_H \to \mathbb{R}$ the **inert prime sum** is

$$\begin{aligned} A_{\mathrm{in}}(a,t,w) = {} & \sum_{\substack{p\le P\\ p \text{ inert}}} 2\log p \sum_{\substack{1\le i\le P\\ i\mathrm{Fr}_p\notin\Phi}} \Big(\frac{a}{p^i} - w(\sigma_t(i\mathrm{Fr}_p))\Big)^+ \\ & + \log d \sum_{i=1}^{P} \sum_{\substack{g\notin\Phi\\ g - i\mathrm{Fr}_d\in I_d}} \Big(\frac{a}{d^i} - w(\sigma_t(g))\Big)^+, \end{aligned}$$

with $u^+ = \max(u,0)$, and its **main term** is the complex number

$$M_\eta(a,w) = a\Big(\log a - 1 - \gamma + \eta(\chi_C) - \frac{1}{|\Phi|}\sum_{\rho\in\Phi}\log w(\rho)\Big).$$

Excess. For a finite set $T \subseteq H$, numbers a_t ($t \in T$), an integer $m \ge 1$ and weights $w_{u,j}\colon \Gamma_H \to \mathbb{R}$ ($u \in H$, $1 \le j \le m$), the **excess at** u is

$$\Delta(u) = \sum_{t\in T}\sum_{j=1}^{m}\Big(A_{\mathrm{in}}(a_t,t,w_{u,j}) - \mathrm{Re}\,M_\eta(a_t,w_{u,j})\Big).$$

Scope of the notation. In each use the group H, the datum $\mathfrak{A}$, the function η and the cut-off P are fixed, and they are omitted from the notation $A_{\mathrm{in}}(a,t,w)$, $M_\eta(a,w)$ and $\Delta(u)$. Likewise $\mathfrak{A}$, η, A_0 and c are fixed in each use of $G_{K'}(x)$ and $\Sigma_{K'}(x)$ and are omitted. When η is fixed we write $G_{\mathfrak{A}}(y,\phi)$ for $G_{\mathfrak{A},\eta}(y,\phi)$. Where several choices occur we say that Δ is formed with $(\mathfrak{A},\eta,P,T,(a_t),(w_{u,j}))$.

An arithmetic datum records what an abelian field with Galois group Γ_H records about the primes: inertia groups and Frobenius elements. In the application (Definition 3.11) the field contains $\mathbb{Q}(\sqrt{-d})$, the subgroup Φ fixes $\sqrt{-d}$, and the inert primes of the datum are the primes inert in $\mathbb{Q}(\sqrt{-d})$.

Since Φ has index two and $\mathrm{Fr}_p \notin \Phi$ for an inert prime p, the condition $i\,\mathrm{Fr}_p \notin \Phi$ selects the odd i. So the first line of the inert prime sum runs over the odd powers of the inert primes, and the second line is the contribution of the ramified prime d. Each term $(a/p^i - w)^+ = w\,(y/p^i - 1)^+$, with $y = a/w$, is a term of a Riesz-weighted sum with cut-off y, and the cut-off depends on the class $\sigma_t(i\,\mathrm{Fr}_p)$ through the weight function w. By orthogonality of the characters of Γ_H, the

inert prime sum is therefore a combination of the functions $G_{\mathfrak{A}}(y,\phi)$ at the cut-offs $y = a/w(\rho)$, $\rho \in \Phi$; this is Lemma 3.8 below. The main term $M_\eta(a,w)$ collects the main terms of these functions, averaged over Φ, together with the term $a\,\eta(\chi_C)$ of the quadratic character χ_C whose kernel is Φ.

Definition 3.2. Let H be a finite abelian group, $\mathfrak{A}$ an arithmetic datum on H and $\eta\colon \widehat{\Gamma}_H \to \mathbb{C}$. For a finite set $T_2 \subseteq H$ and $u \in H$, an element $\rho \in \Gamma_H$ is **thick at** u if $\rho = ((0,1), s+u)$ for some $s \in T_2$. Thick elements lie in Φ.

Let $A_0 \subseteq H$ be a subgroup, $T, T_2 \subseteq H$ finite sets, $m, P \geq 1$ integers, a_t ($t \in T$) real numbers, $w_{u,j}\colon \Gamma_H \to \mathbb{R}$ ($u \in H$, $1 \leq j \leq m$) functions, and $x, b, b', \nu, X', \theta$ real numbers with $b > 0$. The tuple $(A_0, T, T_2, (a_t), (w_{u,j}), P)$ is **weight data at scale** x **with parameters** (b, b', ν, X', θ) for $(\mathfrak{A}, \eta)$ if

(Wd1) $\nu \geq 0$, $X' > 0$, $b' > 0$, $b'X' \leq xb$, $\theta \geq 0$, and $T_2 \subseteq A_0$;

(Wd2) $|a_t/b - x| \leq 1$ for every $t \in T$;

(Wd3) for every $u \in A_0$, every j and every $\rho \in \Phi$: $w_{u,j}(\rho) > 0$; $a_t/w_{u,j}(\rho) \leq P$ for all $t \in T$; $b \leq w_{u,j}(\rho) \leq b + \nu + 1$ if ρ is thick at u; and $w_{u,j}(\rho) \leq 2b'$ if ρ is not thick at u;

(Wd4) (thin scales) $|G_{\mathfrak{A}}(y,\phi)| \leq \theta\, y$ for every $\phi \in \widehat{\Gamma}_H$ and every $y \geq X'/3$;

(Wd5) (cost conditions) $2|\Phi|(x+1)\theta \leq 1/x$ and $6|\Phi|(x+1)(\nu+1) \leq b/x$.

Informally, the scales a_t/b lie within 1 of x. On the translate u the weights equal b, up to $\nu+1$, at the thick places $((0,1), T_2+u)$, and they are small (at most $2b'$) at the other places of Φ. Every twisted function is small, relative to its argument, from $X'/3$ on, and the cost conditions make the resulting errors smaller than b/x.

Proposition 3.3. *Let $C \geq 0$, $\varepsilon > 0$ and $C_E, K_o, M, c \in \mathbb{R}$. There is x_1 such that the following holds for every $x \geq x_1$. Let H be a finite abelian group, $\mathfrak{A}$ an arithmetic datum on H of host type with respect to a finite set S, $\eta\colon \widehat{\Gamma}_H \to \mathbb{C}$, $A_0 \subseteq H$ a subgroup and $K' \subseteq \widehat{\Gamma}_H$. Let $m, P \geq 1$ be integers, $T, T_2 \subseteq H$, (a_t), $(w_{u,j})$, and real numbers $b > 0$, $a_E \geq 0$ and b', ν, X', θ, k. Suppose:*

(Ex1) *$(A_0, T, T_2, (a_t), (w_{u,j}), P)$ is weight data at scale x with parameters (b, b', ν, X', θ) for $(\mathfrak{A}, \eta)$;*

(Ex2) *(size conditions) $x^{1/2-\varepsilon} \leq m \leq x^{1/2+\varepsilon/2}$, $xm \leq k \leq |H|^2 x^{1/2-\varepsilon}$, $a_E \leq 2xb$, $\frac{1}{2}x^{-\varepsilon/8}|H| \leq |T|$ and $x^{-\varepsilon}|H| \leq |T_2|$;*

(Ex3) *(characters) the trivial character lies in K'; $K' \subseteq X(A_0)$; $\phi(((1,0),0)) = 1$ for $\phi \in K'$; $|X(A_0)| \leq M$; $|\sum_{t\in T}\phi((0,t)) - c|T|| \leq Mx^{-1+\varepsilon}|T|$ for $\phi \in K'$ not trivial on $\{0\}\times H$; $|\eta(\phi)| \leq K_o$ for non-trivial $\phi \in K'$; and $|\eta(\phi)| \leq C_E \log x$ for $\phi \in X(A_0)\setminus K'$;*

(Ex4) *(bounded excess) for every $u \in A_0$, with Δ formed with $(\mathfrak{A}, \eta, P, T, (a_t), (w_{u,j}))$,*

$$\Delta(u) \leq x^{C\varepsilon}E, \qquad E = |T|\,a_E\Big(1 + \frac{m^2}{k}\Big) + m\Big(2|H|\,m\,b\,x^{\varepsilon} + \frac{|T_2|\,k\,b\,x^{\varepsilon}}{|H|^2 x^{-\varepsilon}}\Big).$$

Then

$$G_{K'}(x) = \sum_{\phi\in K'} w_c(\phi)\,\mathrm{Re}\,G_{\mathfrak{A}}(x,\phi) \;\leq\; x^{1/2+(C+3)\varepsilon} + \sum_{\phi\in X(A_0)\setminus K'} |G_{\mathfrak{A}}(x,\phi)|.$$

Proof. Let $C_d \geq 0$ be the constant of item 2 of Lemma 3.4, and put $A_1 = 5 + C_d + |K_o| + |M|(4 + C_d + |K_o|) + 4|C_E|$. Choose $x_1 \geq 3$ such that $x^{3\varepsilon/8} \geq 26$ and $x^{1/4} \geq |M|A_1$ for $x \geq x_1$. Fix $x \geq x_1$ and data as in the statement. Write $s = x^{1/4}$ and $e = x^{\varepsilon}$, so $s, e \geq 1$ and $x^{1/2+\varepsilon} = s^2 e$. We also use $\log x = 4 \log s \leq 4s$ and $\log(x+1) \leq \log(2x) \leq 1 + 4s \leq 5s$.

Step 1: moving the scales. Apply Lemma 3.6 with $y_t = a_t/b$, $\rho = |M|x^{-1+\varepsilon}$, $|K_o|$ in place of K_o, $L = |C_E| \log x$ and $R = A_1 se$.

- Hypothesis 1 is (Wd2), and hypotheses 2–4 are (Ex3), since $M \leq |M|$, $K_o \leq |K_o|$ and $C_E \log x \leq L$.
- Hypothesis 5 holds because $C_d \leq A_1 \leq R$ and $\rho\, x = |M|e$, so that $\log(x+1) + C_d + |K_o| + |M|e(\log x + C_d + |K_o|) \leq 5s + C_d + |K_o| + |M|(4 + C_d + |K_o|)se \leq A_1 se$, and similarly $\log(x+1) + C_d + |C_E| \log x \leq (5 + C_d + 4|C_E|)se \leq R$.

Since $|X(A_0)|R \leq |M|A_1 se \leq s^2 e = x^{1/2+\varepsilon}$, the lemma gives, for $\Upsilon = G_{K'}(x) - \Sigma_{K'}(x)$,

$$|T|(\Upsilon - 4x^{1/2+\varepsilon}) \leq \sum_{t \in T} \operatorname{Re} \sum_{\phi \in X(A_0)} \phi(((1,0),t)) G_{\mathfrak{A}}(a_t/b, \phi).$$

Step 2: averaging. Apply Lemma 3.7 with $\mathrm{Err} = x^{C\varepsilon}E$ (hypothesis 1 is (Ex4)) and $Y = \Upsilon - 4x^{1/2+\varepsilon}$ (hypothesis 2 is Step 1). It gives

$$|T|\, m\, b\, \frac{|T_2|}{2|H|}(\Upsilon - 4x^{1/2+\varepsilon}) \leq x^{C\varepsilon}E + \frac{2|T|mb}{x}.$$

Step 3: absorption. The last inequality is the hypothesis of Lemma 3.10 with $n = |H|$, $n_T = |T|$, $n_2 = |T_2|$ and $\mathrm{Err} = x^{C\varepsilon}E$. Its size conditions are (Ex2), together with $x \geq 3$ and $x^{3\varepsilon/8} \geq 26$ (choice of x_1), $C \geq 0$, $\varepsilon > 0$, $b > 0$, $0 \leq a_E$, $|H| > 0$, and $|T_2| \leq |H|$. So $\Upsilon \leq x^{1/2+(C+3)\varepsilon}$, which is the claim. □

The simplest case shows what the proposition does. Take $A_0 = H$ and $K' = \{1\}$. Then $X(A_0)$ consists of the four characters of Γ_H that are trivial on H, and the conclusion bounds $G(x)$ by $x^{1/2+(C+3)\varepsilon}$ plus the three values $|G_{\mathfrak{A}}(x, \phi)|$ at the non-trivial ones. For trivial H the group Γ_H is $(\mathbb{Z}/2)^2$, as for a biquadratic field that contains $\mathbb{Q}(\sqrt{-d})$: a sum over the primes inert in $\mathbb{Q}(\sqrt{-d})$ bounds G only up to the twists of G by the three quadratic characters, and these need a separate bound. In the proof the group H is the group A of a frame, the subgroup A_0 has index o, and the characters that remain on the right-hand side are bounded by condition (Ec5) of Definition 3.16.

The bracket E of (Ex4) is the size of the error term of the accounting (Proposition 4.1). The size conditions (Ex2) are those under which $x^{C\varepsilon}E$ is at most $x^{1/2+(C+3)\varepsilon}$ times the normaliser $|T|\, m\, b\, |T_2|/(2|H|)$ of the average (Lemma 3.10); the remark that closes Section 4 explains why this forces m to be about $\sqrt{x}$.

Lemma 3.4. *1. For every finite abelian group H, every arithmetic datum $\mathfrak{A}$ on H, every function $\eta\colon \widehat{\Gamma}_H \to \mathbb{C}$, every $\phi \in \widehat{\Gamma}_H$ and all real $A, w, w' > 0$,*

$$\left| w\, G_{\mathfrak{A},\eta}(A/w, \phi) - w'\, G_{\mathfrak{A},\eta}(A/w', \phi) \right| \leq \frac{3A}{\min(w, w')} |w - w'|.$$

2. There is an absolute constant $C_d \geq 0$ such that for every H, $\mathfrak{A}$, η as in item 1, every $\phi \neq 1$ and all real $y, z \geq 1$,

$$|G_{\mathfrak{A},\eta}(y, \phi) - G_{\mathfrak{A},\eta}(z, \phi)| \leq |y - z|(\log \max(y, z) + C_d + |\eta(\phi)|), \qquad |G(y) - G(z)| \leq C_d\, |y - z|.$$

Proof. We use Chebyshev's and Mertens' estimates in the form of items 2 and 3 of Lemma 3.5, both derived from Chebyshev's bound [180]: for $\psi(Y) = \sum_{n \le Y} \Lambda(n)$ (notation used in this proof only) we have $\psi(Y) \le 3Y$ for all $Y \ge 0$, and

$$\Big| \sum_{n \le Y} \frac{\Lambda(n)}{n} - \log Y \Big| \le c_0 := 3 + \log 2 \qquad (Y \ge 1),$$

which we call Mertens' estimate below.

For $u, v \in \mathbb{R}$ we use $|u^+ - v^+| \le |u - v|$. Recall also $|\phi_{\mathfrak{A}}(n)| \le 1$.

Item 1. Let $\mu = \min(w, w')$ and $M = \lfloor A/\mu \rfloor$. For any $v \ge \mu$ the sum defining $G_{\mathfrak{A}}(A/v, \phi)$ runs over $n \le A/v \le M$. Multiplying by v gives

$$v\, G_{\mathfrak{A}}(A/v, \phi) = \sum_{n \le M} \Lambda(n) \phi_{\mathfrak{A}}(n) \Big(\frac{A}{n} - v \Big)^+ + A\, \eta(\phi) \qquad (\phi \ne 1),$$

because $v(A/(vn) - 1) = A/n - v$, and the terms with $A/v < n \le M$ are zero. For $\phi \ne 1$, subtracting the two instances $v = w, w'$ cancels $A\eta(\phi)$ and leaves a sum of terms of modulus at most $\Lambda(n)|w - w'|$. So the difference is at most $|w - w'| \psi(A/\mu) \le 3A|w - w'|/\mu$.

For $\phi = 1$ assume $w \le w'$, so $\mu = w$. The definition of G gives

$$v\, G(A/v) = S(v) - A \log(A/v) + (1 + \gamma) A, \qquad S(v) = \sum_{n \le M} \Lambda(n) \Big(\frac{A}{n} - v \Big)^+ .$$

Then $wG(A/w) - w'G(A/w') = (S(w) - S(w')) - A \log(w'/w)$. Both brackets lie in $[0, 3A(w' - w)/w]$:

- the first, because each term of S decreases in v by at most $\Lambda(n)(w' - w)$, and $\psi(A/w) \le 3A/w$;
- the second, because $0 \le \log(w'/w) \le w'/w - 1$.

A difference of two numbers in $[0, B]$ has modulus at most B.

Item 2, $\phi \ne 1$. Let $M = \lfloor \max(y, z) \rfloor \ge 1$. Then

$$G_{\mathfrak{A}}(y, \phi) - G_{\mathfrak{A}}(z, \phi) = \sum_{n \le M} \Lambda(n) \phi_{\mathfrak{A}}(n) \Big(\big(\tfrac{y}{n} - 1\big)^+ - \big(\tfrac{z}{n} - 1\big)^+ \Big) + (y - z) \eta(\phi).$$

Each bracket has modulus at most $|y - z|/n$, so the modulus is at most $|y - z| (\sum_{n \le M} \Lambda(n)/n + |\eta(\phi)|) \le |y - z| (\log \max(y, z) + c_0 + |\eta(\phi)|)$, by Mertens' estimate.

Item 2, G. The function G is continuous on $[1, \infty)$: the term $\Lambda(n)(y/n - 1)$ enters at $y = n$ with value 0. On each interval between consecutive integers it is differentiable with

$$G'(y) = \sum_{n \le y} \frac{\Lambda(n)}{n} - \log y + \gamma,$$

so $|G'(y)| \le c_0 + \gamma$ there by Mertens' estimate. Hence $|G(y) - G(z)| \le (c_0 + \gamma)|y - z|$ for $y, z \ge 1$. The constant $C_d = c_0 + \gamma$ serves both parts of item 2. □

Lemma 3.5. *1. For every real $x \ge 1$, $\sum_{n \le x} \Lambda(n) \le (\log 4)\, x + 2\sqrt{x} \log x$.*

2. For every real $x \ge 0$, $\sum_{n \le x} \Lambda(n) \le 3x$.

3. For every real $x \geq 1$,

$$\Big|\sum_{n\leq x}\frac{\Lambda(n)}{n} - \log x\Big| \leq 3 + \log 2.$$

Proof. Write $\vartheta(y) = \sum_{p\leq y}\log p$, so that $\vartheta(y) \leq y\log 4$ for $y \geq 0$ by Chebyshev's bound [180], and $\vartheta(y) = 0$ for $y < 2$.

Item 1. Since $\Lambda(p^k) = \log p$ and Λ vanishes off the prime powers,

$$\sum_{n\leq x}\Lambda(n) = \sum_{k\geq 1}\vartheta(x^{1/k}).$$

The term with $k = 1$ is at most $x\log 4$. A term with $k \geq 2$ is non-zero only if $x^{1/k} \geq 2$, that is, $k \leq \log x/\log 2$, and it is at most $x^{1/k}\log 4 \leq \sqrt{x}\log 4$. There are fewer than $\log x/\log 2$ such k, so these terms total at most $(\log x/\log 2)\sqrt{x}\log 4 = 2\sqrt{x}\log x$.

Item 2. For $x < 2$ the sum is 0. For $x \geq 1$, with $u = \sqrt{x}$ and $\log u \leq u/e$ (valid for $u > 0$), $2\sqrt{x}\log x = 4u\log u \leq \frac{4}{e}x$; as $\log 4 + \frac{4}{e} < 2.9$, item 1 gives the bound $3x$.

Item 3. Let $k = \lfloor x\rfloor \geq 1$ and $S = \sum_{n\leq k}\Lambda(n)/n$, which equals $\sum_{n\leq x}\Lambda(n)/n$.

- Since $\log m = \sum_{d\mid m}\Lambda(d)$ for $m \geq 1$, Legendre's identity $\log k! = \sum_{n\leq k}\Lambda(n)\lfloor k/n\rfloor$ holds.
- $k\log k - k \leq \log k! \leq k\log k$; the lower bound holds because $\log k! = \sum_{m=2}^{k}\log m \geq \int_1^k \log t\,dt = k\log k - k + 1$.
- From $k/n - 1 < \lfloor k/n\rfloor \leq k/n$ and item 2,

$$kS - 3k \leq kS - \sum_{n\leq k}\Lambda(n) \leq \log k! \leq kS.$$

Hence $\log k - 1 \leq S \leq \log k + 3$. Since $\log k \leq \log x < \log(k+1) \leq \log k + \log 2$, we get $-1 - \log 2 < S - \log x \leq 3$, which gives item 3. □

Lemma 3.6. *Let $C_d \geq 0$ be the constant of item 2 of Lemma 3.4. Let H be a finite abelian group, $\mathfrak{A}$ an arithmetic datum on H, $\eta\colon \widehat{\Gamma}_H \to \mathbb{C}$, $A_0 \subseteq H$ a subgroup and $K' \subseteq X(A_0)$. Let $T \subseteq H$ be finite, y_t ($t \in T$) real numbers, and $x \geq 3$, c, $\rho \geq 0$, $K_o \geq 0$, L, R real numbers, and form w_c, $G_{K'}(x)$ and $\Sigma_{K'}(x)$ with $(\mathfrak{A},\eta)$, A_0 and c. Suppose:*

1. *$|y_t - x| \leq 1$ for every $t \in T$;*
2. *the trivial character lies in K', and $\phi(((1,0),0)) = 1$ for every $\phi \in K'$;*
3. *$\big|\sum_{t\in T}\phi((0,t)) - c|T|\big| \leq \rho|T|$ for every $\phi \in K'$ that is not trivial on $\{0\}\times H$;*
4. *$|\eta(\phi)| \leq K_o$ for every non-trivial $\phi \in K'$, and $|\eta(\phi)| \leq L$ for every $\phi \in X(A_0)\setminus K'$;*
5. *$C_d \leq R$, $\log(x+1) + C_d + K_o + \rho\,x\,(\log x + C_d + K_o) \leq R$ and $\log(x+1) + C_d + L \leq R$.*

Then

$$|T|\Big(G_{K'}(x) - \Sigma_{K'}(x) - |X(A_0)|\,R\Big) \leq \sum_{t\in T}\operatorname{Re}\sum_{\phi\in X(A_0)}\phi(((1,0),t))\,G_{\mathfrak{A}}(y_t,\phi).$$

Proof. For $\phi \in X(A_0)$ let $v(\phi) = w_c(\phi)\,\mathrm{Re}\,G_{\mathfrak{A}}(x,\phi)$ if $\phi \in K'$ and $v(\phi) = -|G_{\mathfrak{A}}(x,\phi)|$ otherwise. Then $\sum_{\phi\in X(A_0)} v(\phi) = G_{K'}(x) - \Sigma_{K'}(x)$. So it suffices to show, for each $\phi \in X(A_0)$,

$$|T|(v(\phi) - R) \le \sum_{t\in T} \mathrm{Re}\Big(\phi(((1,0),t))G_{\mathfrak{A}}(y_t,\phi)\Big), \tag{$*$}$$

and to sum over ϕ. By hypothesis 1 and $x \ge 3$ we have $y_t \ge 2$ and $\max(y_t, x) \le x+1$. We use two consequences of item 2 of Lemma 3.4, valid for $\phi \ne 1$:

$$|G_{\mathfrak{A}}(y_t,\phi) - G_{\mathfrak{A}}(x,\phi)| \le \log(x+1) + C_d + |\eta(\phi)|, \qquad |G_{\mathfrak{A}}(x,\phi)| \le x\big(\log x + C_d + |\eta(\phi)|\big).$$

The second follows from the bound with $z = 1$, since $G_{\mathfrak{A}}(1,\phi) = \eta(\phi)$ (the sum over $n \le 1$ is empty because $\Lambda(1) = 0$). Thus $|G_{\mathfrak{A}}(x,\phi)| \le (x-1)(\log x + C_d + |\eta(\phi)|) + |\eta(\phi)|$.

The trivial character. Here $v = G(x)$ and $\phi(((1,0),t)) = 1$. By item 2 of Lemma 3.4, $G(y_t) \ge G(x) - C_d|y_t - x| \ge G(x) - R$. Summing over t gives $(*)$.

$\phi \in K'$, $\phi \ne 1$. By hypothesis 2, $\phi(((1,0),t)) = \phi(((1,0),0))\phi((0,t)) = \phi((0,t))$. Hypothesis 3 gives $|\sum_t \phi((0,t)) - w_c(\phi)|T|| \le \rho|T|$ (trivially, with left side 0, if ϕ is trivial on $\{0\}\times H$). Write

$$\sum_t \phi((0,t))G_{\mathfrak{A}}(y_t,\phi) - w_c(\phi)|T|G_{\mathfrak{A}}(x,\phi) = \sum_t \phi((0,t))\big(G_{\mathfrak{A}}(y_t,\phi) - G_{\mathfrak{A}}(x,\phi)\big) + \Big(\sum_t \phi((0,t)) - w_c(\phi)|T|\Big)G_{\mathfrak{A}}(x,\phi).$$

By the two consequences and $|\eta(\phi)| \le K_o$, the modulus is at most $|T|\big(\log(x+1) + C_d + K_o\big) + \rho|T|\,x(\log x + C_d + K_o) \le |T|R$. Taking real parts, and using that $w_c(\phi)$ is real, gives $(*)$.

$\phi \in X(A_0)\setminus K'$. Then $\phi \ne 1$, since $1 \in K'$. Each term satisfies

$$\mathrm{Re}\big(\phi(((1,0),t))G_{\mathfrak{A}}(y_t,\phi)\big) \ge -|G_{\mathfrak{A}}(y_t,\phi)| \ge -|G_{\mathfrak{A}}(x,\phi)| - \big(\log(x+1) + C_d + L\big) \ge v(\phi) - R.$$

Summing over t gives $(*)$. □

Lemma 3.7. *Let H be a finite abelian group, $\mathfrak{A}$ an arithmetic datum on H of host type with respect to a finite set S, and $\eta\colon \widehat{\Gamma}_H \to \mathbb{C}$. Let $(A_0, T, T_2, (a_t), (w_{u,j}), P)$ be weight data at scale $x \ge 3$ with parameters (b, b', ν, X', θ) for $(\mathfrak{A}, \eta)$. Let $\mathrm{Err}, Y \in \mathbb{R}$, and suppose that*

1. *the excess, formed with $(\mathfrak{A}, \eta, P, T, (a_t), (w_{u,j}))$, satisfies $\Delta(u) \le \mathrm{Err}$ for every $u \in A_0$;*
2. $|T|\,Y \le \displaystyle\sum_{t\in T} \mathrm{Re} \sum_{\phi\in X(A_0)} \phi(((1,0),t))\,G_{\mathfrak{A}}(a_t/b,\phi)$.

Then

$$|T|\,m\,b\,\frac{|T_2|}{2|H|}\,Y \le \mathrm{Err} + \frac{2|T|\,m\,b}{x}.$$

Proof. Let D and Th be as in Lemma 3.9, with the set T_2 and the constant b of the weight data.

Step 1: one translate. Fix $u \in A_0$, $t \in T$ and j. By (Wd2) and $x \ge 3$ we have $a_t \ge (x-1)b > 0$. By (Wd3), $w_{u,j} > 0$ on Φ and $a_t/w_{u,j} \le P$ on Φ. So Lemma 3.8 applies and gives

$$\mathrm{Re}\,D(a_t, t, w_{u,j}) = A_{\mathrm{in}}(a_t, t, w_{u,j}) - \mathrm{Re}\,M_\eta(a_t, w_{u,j}),$$

because A_{in} is real. Item 1 of Lemma 3.9 gives $\operatorname{Re}\mathrm{Th}(u,t,a_t/b) \le \operatorname{Re} D(a_t,t,w_{u,j}) + 2|\Phi|B$ with $B = 3(x+1)(\nu+1) + (x+1)b\theta$. By the cost conditions (Wd5), $2|\Phi| \cdot 3(x+1)(\nu+1) \le b/x$ and $2|\Phi|(x+1)b\theta \le b/x$, so $2|\Phi|B \le 2b/x$. Summing over $t \in T$ and $1 \le j \le m$, and using hypothesis 1,

$$\sum_{t\in T}\sum_{j=1}^{m} \operatorname{Re}\mathrm{Th}(u,t,a_t/b) \le \Delta(u) + \frac{2|T|mb}{x} \le \mathrm{Err} + \frac{2|T|mb}{x}.$$

Step 2: average over $u \in A_0$. The summand on the left does not depend on j. By item 2 of Lemma 3.9 (which needs $T_2 \subseteq A_0$, part of (Wd1)), summing Step 1 over $u \in A_0$ gives

$$|A_0|\, m\, b\, \frac{|T_2|}{2|H|} \sum_{t\in T} \operatorname{Re} \sum_{\phi\in X(A_0)} \phi(((1,0),t)) G_{\mathfrak{A}}(a_t/b,\phi) \le |A_0|\Big(\mathrm{Err} + \frac{2|T|mb}{x}\Big).$$

Divide by $|A_0| > 0$. Then hypothesis 2, multiplied by the non-negative number $mb|T_2|/(2|H|)$, gives the claim. □

Lemma 3.8. *Let H be a finite abelian group, $\mathfrak{A}$ an arithmetic datum on H of host type with respect to a finite set S, and $\eta\colon \widehat{\Gamma}_H \to \mathbb{C}$ any function. Let $a > 0$, $t \in H$, let $w\colon \Gamma_H \to \mathbb{R}$ satisfy $w(\rho) > 0$ for all $\rho \in \Phi$, and let $P \ge 1$ be an integer with $a/w(\rho) \le P$ for all $\rho \in \Phi$. Then, with A_{in} formed with the cut-off P,*

$$A_{\mathrm{in}}(a,t,w) - M_\eta(a,w) = \frac{1}{|\Phi|} \sum_{g\in\Gamma_H\setminus\Phi} \sum_{\phi\in\widehat{\Gamma}_H} \overline{\phi(g)}\, w(\sigma_t(g))\, G_{\mathfrak{A}}\Big(\frac{a}{w(\sigma_t(g))},\phi\Big).$$

Proof. Write $W_g = w(\sigma_t(g))$ for $g \notin \Phi$; since $\sigma_t(g) \in \Phi$, we have $W_g > 0$ and $a/W_g \le P$. We use three facts about the finite abelian group Γ_H, of order $2|\Phi|$.

(i) *Orthogonality on cosets.* For $g, h \in \Gamma_H$ one has $\sum_\phi \overline{\phi(g)}\phi(h) = 2|\Phi|$ if $g = h$ and 0 otherwise. Averaging over $h + I$ gives, for a subgroup I,

$$\frac{1}{|\Phi|}\sum_{\phi} \overline{\phi(g)}\, \mu_I(h,\phi) = \begin{cases} 2/|I|, & g - h \in I, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) *Sums over the non-trivial coset of Φ.* Since the indicator of $\Gamma_H \setminus \Phi$ is $(1-\chi_C)/2$, (i) gives $\sum_{g\notin\Phi} \overline{\phi(g)} = |\Phi|$ for $\phi = 1$, $-|\Phi|$ for $\phi = \chi_C$, and 0 for every other ϕ.

(iii) *Translation.* σ_t maps $\Gamma_H \setminus \Phi$ bijectively onto Φ, since it changes the first coordinate by 1 and is injective.

Step 1: expansion of the inert prime sum. Put

$$R = \frac{1}{|\Phi|}\sum_{g\notin\Phi}\sum_{\phi} \overline{\phi(g)} \sum_{n=1}^{P} \Lambda(n)\, \phi_{\mathfrak{A}}(n)\Big(\frac{a}{n} - W_g\Big)^+ = \sum_{n=1}^{P} \Lambda(n) \sum_{g\notin\Phi} K(n,g)\Big(\frac{a}{n} - W_g\Big)^+,$$

with $K(n,g) = |\Phi|^{-1}\sum_\phi \overline{\phi(g)}\phi_{\mathfrak{A}}(n)$. Only prime powers $n = p^i$ ($i \ge 1$) contribute, and there $\phi_{\mathfrak{A}}(p^i) = \mu_{I_p}(i\mathrm{Fr}_p, \phi)$, so by (i) $K(p^i,g) = 2/|I_p|$ if $g - i\mathrm{Fr}_p \in I_p$ and 0 otherwise. We claim that $R = A_{\mathrm{in}}(a,t,w)$. To see this, we sort the primes using (Ah1)–(Ah4).

- $p \in S$: by (Ah3), $g - i\mathrm{Fr}_p \in I_p \subseteq \Phi$ and $i\mathrm{Fr}_p \in \Phi$ force $g \in \Phi$, so these primes contribute nothing to R; they do not occur in A_{in} either, since $2, q \in S$ by (Ah1) and a prime of S has $\mathrm{Fr}_p \in \Phi$.

- $p = d$: by (Ah4), $|I_d| = 2$ and $K(d^i, g) = 1$ exactly when $g - i\mathrm{Fr}_d \in I_d$. The contribution is $\log d \sum_i \sum_{g\notin\Phi,\, g - i\mathrm{Fr}_d \in I_d} (a/d^i - W_g)^+$, which is the d-part of A_{in}.
- $p \notin S \cup \{d\}$: by (Ah2), $I_p = 0$ and $K(p^i, g) = 2$ exactly when $g = i\mathrm{Fr}_p$. The contribution is $2\log p \sum_{i:\, i\mathrm{Fr}_p \notin \Phi} (a/p^i - w(\sigma_t(i\mathrm{Fr}_p)))^+$. It vanishes unless $\mathrm{Fr}_p \notin \Phi$, because Φ is a subgroup; and $p \notin S$ gives $p \neq 2, q$. So the primes that contribute are exactly the inert ones.

The ranges agree: in R only $p^i \le P$ occurs, whereas A_{in} runs over $p \le P$ and $1 \le i \le P$ (and all d^i, $i \le P$). Every term with $p^i > P$ vanishes, because $a/p^i < a/P \le W_g$ for all $g \notin \Phi$. Hence $R = A_{\mathrm{in}}(a, t, w)$.

Step 2: the inner sums. Fix $g \notin \Phi$. The terms with $n > a/W_g$ vanish, so

$$\sum_{n=1}^{P} \Lambda(n)\phi_{\mathfrak{A}}(n)\Big(\frac{a}{n} - W_g\Big)^+ = W_g \sum_{1 \le n \le a/W_g} \Lambda(n)\phi_{\mathfrak{A}}(n)\Big(\frac{a/W_g}{n} - 1\Big).$$

By the definition of $G_{\mathfrak{A}}$ this equals $W_g G_{\mathfrak{A}}(a/W_g, \phi) - a\,\eta(\phi)$ for $\phi \neq 1$. For $\phi = 1$ it equals $W_g G(a/W_g) + a\log a - a\log W_g - (1+\gamma)a$, since $1_{\mathfrak{A}} = 1$ and $W_g G(a/W_g) = \sum_{n \le a/W_g} \Lambda(n)(a/n - W_g) - a(\log(a/W_g) - 1 - \gamma)$.

Step 3: summation. Insert Step 2 into R. The terms $W_g G_{\mathfrak{A}}(a/W_g, \phi)$ give the right-hand side of the claim. The correction for $\phi = 1$ (where $\overline{\phi(g)} = 1$) gives, by (iii) and $|\Gamma_H \setminus \Phi| = |\Phi|$,

$$\frac{1}{|\Phi|}\sum_{g\notin\Phi}\big(a\log a - a\log W_g - (1+\gamma)a\big) = a\log a - (1+\gamma)a - \frac{a}{|\Phi|}\sum_{\rho\in\Phi}\log w(\rho).$$

The corrections for $\phi \neq 1$ give, by (ii), $-\frac{a}{|\Phi|}\sum_{\phi\neq 1}\eta(\phi)\sum_{g\notin\Phi}\overline{\phi(g)} = a\,\eta(\chi_C)$. The two corrections add up to $M_\eta(a, w)$, so $A_{\mathrm{in}}(a, t, w) = R$ equals the right-hand side plus $M_\eta(a, w)$. □

Lemma 3.9. *Let H be a finite abelian group, $\mathfrak{A}$ an arithmetic datum on H and $\eta\colon \widehat{\Gamma}_H \to \mathbb{C}$. For $a \in \mathbb{R}$, $t \in H$ and $w\colon \Gamma_H \to \mathbb{R}$ with $w > 0$ on Φ let*

$$D(a, t, w) = \frac{1}{|\Phi|}\sum_{g\in\Gamma_H\setminus\Phi}\sum_{\phi\in\widehat{\Gamma}_H}\overline{\phi(g)}\; w(\sigma_t(g))\, G_{\mathfrak{A}}\Big(\frac{a}{w(\sigma_t(g))}, \phi\Big)$$

be the right-hand side of the orthogonality identity of Lemma 3.8, and for a finite $T_2 \subseteq H$, $u, t \in H$ and real b, y let

$$\mathrm{Th}(u, t, y) = \frac{b}{|\Phi|}\sum_{s\in T_2}\sum_{\phi\in\widehat{\Gamma}_H}\overline{\phi\big(((1,0), s+u-t)\big)}\, G_{\mathfrak{A}}(y, \phi)$$

*be its **thick part with constant weight** b (the elements $g = ((1,0), s+u-t)$ are exactly those with $\sigma_t(g)$ thick at u).*

1. *Let $(A_0, T, T_2, (a_t), (w_{u,j}), P)$ be weight data at scale $x \ge 3$ with parameters (b, b', ν, X', θ) for $(\mathfrak{A}, \eta)$. Then for all $u \in A_0$, $t \in T$ and $1 \le j \le m$, where m is the integer of the weight data (the number of weight functions $w_{u,j}$ at each u, Definition 3.2),*

$$\mathrm{Re}\,\mathrm{Th}(u, t, a_t/b) \le \mathrm{Re}\,D(a_t, t, w_{u,j}) + 2|\Phi|\big(3(x+1)(\nu+1) + (x+1)b\theta\big).$$

2. *Let $A_0 \subseteq H$ be a subgroup with $T_2 \subseteq A_0$. Then for all $t \in H$ and real b, y,*

$$\sum_{u\in A_0}\mathrm{Th}(u, t, y) = |A_0|\, b\,\frac{|T_2|}{2|H|}\sum_{\phi\in X(A_0)}\phi\big(((1,0), t)\big)\, G_{\mathfrak{A}}(y, \phi).$$

Proof. The thick elements. For $g \notin \Phi$ we have $\sigma_t(g) = ((0,1), s+u)$ exactly when $g = ((0,1), s+u) - c_\infty - ((0,0), t) = ((1,0), s+u-t)$. Since $s \mapsto ((1,0), s+u-t)$ is injective, $\mathrm{Th}(u,t,y)$ is the sum over the $g \notin \Phi$ with $\sigma_t(g)$ thick at u of the terms of D, with the weight $w(\sigma_t(g))$ replaced by b in both places.

Item 1. Fix u, t, j and write $a = a_t$, $w = w_{u,j}$ and $W_g = w(\sigma_t(g))$. By (Wd2), $(x-1)b \le a \le (x+1)b$, and $a > 0$ since $x \ge 3$. By the description of the thick elements,

$$D(a,t,w) - \mathrm{Th}(u,t,a/b) = \frac{1}{|\Phi|} \sum_{g \notin \Phi} \sum_{\phi} \overline{\phi(g)}\, F(g,\phi),$$

with $F(g,\phi) = W_g G_{\mathfrak{A}}(a/W_g, \phi) - b\, G_{\mathfrak{A}}(a/b, \phi)$ if $\sigma_t(g)$ is thick at u and $F(g,\phi) = W_g G_{\mathfrak{A}}(a/W_g, \phi)$ otherwise. We show $|F(g,\phi)| \le B := 3(x+1)(\nu+1) + (x+1)b\theta$ for every $g \notin \Phi$ and ϕ. Here $\sigma_t(g) \in \Phi$, so $W_g > 0$ by (Wd3).

- If $\sigma_t(g)$ is thick at u, then $b \le W_g \le b + \nu + 1$ by (Wd3). Item 1 of Lemma 3.4 with $(A, w, w') = (a, W_g, b)$ gives $|F| \le (3a/b)(W_g - b) \le 3(x+1)(\nu+1)$.
- Otherwise $W_g \le 2b'$ by (Wd3), and, by (Wd1) and $x \ge 3$,

$$\frac{X'}{3} W_g \le \frac{2}{3}\, b' X' \le \frac{2}{3}\, xb \le (x-1)b \le a,$$

  so $a/W_g \ge X'/3$. Then (Wd4) gives $|F| = W_g |G_{\mathfrak{A}}(a/W_g, \phi)| \le \theta a \le (x+1)b\theta$.

There are $|\Gamma_H \setminus \Phi| = |\Phi|$ elements g and $|\widehat{\Gamma}_H| = 2|\Phi|$ characters, and $|\overline{\phi(g)}| = 1$. Hence $|D - \mathrm{Th}| \le 2|\Phi|B$, and in particular $\mathrm{Re}\,\mathrm{Th} \le \mathrm{Re}\, D + 2|\Phi|B$.

Item 2. Since $\overline{\phi(h)} = \phi(-h)$ and $-((1,0), s+u-t) = ((1,0), t) + ((0,0), -s) + ((0,0), -u)$ in Γ_H,

$$\sum_{u \in A_0} \mathrm{Th}(u,t,y) = \frac{b}{|\Phi|} \sum_{s \in T_2} \sum_{\phi} \phi(((1,0),t))\phi(((0,0),-s)) G_{\mathfrak{A}}(y,\phi) \sum_{u \in A_0} \phi(((0,0),-u)).$$

The inner sum is $|A_0|$ if $\phi \in X(A_0)$ and 0 otherwise (orthogonality for the restriction of ϕ to $\{0\} \times A_0$). For $\phi \in X(A_0)$ also $\phi(((0,0),-s)) = 1$, since $s \in T_2 \subseteq A_0$. Hence the sum equals $(b/|\Phi|)\,|T_2|\,|A_0| \sum_{\phi \in X(A_0)} \phi(((1,0),t)) G_{\mathfrak{A}}(y,\phi)$. With $|\Phi| = 2|H|$ this is the claim. □

Lemma 3.10. *Let $C, \varepsilon, x, b, k, a_E, \mathrm{Err}, \Upsilon \in \mathbb{R}$ and let n, n_T, n_2, m be non-negative integers. Assume the following size conditions*

$$x \ge 3, \quad \varepsilon > 0, \quad C \ge 0, \quad n > 0, \quad b > 0, \quad 0 \le a_E \le 2xb, \quad x^{3\varepsilon/8} \ge 26,$$
$$\tfrac{1}{2} x^{-\varepsilon/8} n \le n_T, \quad x^{-\varepsilon} n \le n_2 \le n,$$
$$x^{1/2-\varepsilon} \le m \le x^{1/2+\varepsilon/2}, \quad xm \le k \le n^2 x^{1/2-\varepsilon},$$
$$\mathrm{Err} \le x^{C\varepsilon}\Big(n_T\, a_E\Big(1 + \frac{m^2}{k}\Big) + m\Big(2n\, m\, b\, x^{\varepsilon} + \frac{n_2\, k\, b\, x^{\varepsilon}}{n^2 x^{-\varepsilon}}\Big)\Big).$$

If

$$n_T\, m\, b\, \frac{n_2}{2n}\Big(\Upsilon - 4x^{1/2+\varepsilon}\Big) \le \mathrm{Err} + \frac{2 n_T m b}{x},$$

then $\Upsilon \le x^{1/2+(C+3)\varepsilon}$.

Proof. We compare powers of x. Put $Q = n_T\, m\, b$, $\delta = n_2/n$, $X = x^{1/2+\varepsilon}$, $Y = x^{1/2+13\varepsilon/8}$ and $\Pi = x^{C\varepsilon}$. From the size conditions, $m > 0$, $n_T > 0$ and $k > 0$, so $Q > 0$. Moreover $Y \geq 1$, $\Pi \geq 1$, $X \leq Y$ and $x^{\varepsilon}\delta \geq 1$.

Step 1: $\mathrm{Err} \leq 10\,\Pi\, Y\, Q$. Each of the four terms of the bracket bounding Err is bounded by a multiple of YQ, as follows.

- $n_T a_E \leq 2xb\, n_T \leq 2Xm\, n_T b \leq 2YQ$, using $x \leq Xm$, that is, $m \geq x^{1/2-\varepsilon}$.
- $n_T a_E m^2/k \leq n_T \cdot 2xb \cdot m^2/(xm) = 2Q \leq 2YQ$, using $k \geq xm$.
- $m \cdot 2nmbx^{\varepsilon} \leq 4x^{9\varepsilon/8}\, m\, Q \leq 4YQ$, using $n \leq 2x^{\varepsilon/8}n_T$ and $m \leq x^{1/2+\varepsilon/2}$.
- $m\, n_2 kbx^{\varepsilon}/(n^2x^{-\varepsilon}) \leq m\, n_2\, b\, x^{1/2+\varepsilon} \leq 2x^{1/2+9\varepsilon/8}Q \leq 2YQ$, using $k \leq n^2x^{1/2-\varepsilon}$, $n_2 \leq n$ and $n \leq 2x^{\varepsilon/8}n_T$.

Step 2. By Step 1 and $2/x \leq 1$, the hypothesis gives $Q\,\frac{\delta}{2}(\Upsilon - 4X) \leq 10\Pi YQ + Q$. Divide by Q and use $1 \leq \Pi Y$; this gives $\delta(\Upsilon - 4X) \leq 22\,\Pi Y$.

Step 3. If $\Upsilon \leq 4X$ there is nothing more to prove, since $4X \leq x^{1/2+(C+3)\varepsilon}$ by $x^{3\varepsilon/8} \geq 26$. Otherwise $x^{\varepsilon}\delta \geq 1$ gives

$$\Upsilon - 4X \leq x^{\varepsilon}\delta(\Upsilon - 4X) \leq 22\,\Pi Y x^{\varepsilon} = 22\,x^{1/2+(C+21/8)\varepsilon}.$$

Since $C \geq 0$, $X \leq x^{1/2+(C+21/8)\varepsilon}$, so $\Upsilon \leq 26\,x^{1/2+(C+21/8)\varepsilon} \leq x^{3\varepsilon/8}x^{1/2+(C+21/8)\varepsilon} = x^{1/2+(C+3)\varepsilon}$. □

The arithmetic data of the proof come from hosts, which we define next. A frame is the abstract shape of a Galois group: a finite abelian group A that maps onto $(\mathbb{Z}/2^{e+2}\mathbb{Z})^{\times}/\{\pm1\}$, with a cyclic subgroup A_0 of index o. In the proof o is the order of the character ψ. A host adds two primes q and d, a modulus N and a surjection α from $(\mathbb{Z}/N\mathbb{Z})^{\times}$ onto $\{\pm1\}^2 \times A$. In the case of interest α is the Artin map of an abelian field inside $\mathbb{Q}(\zeta_N)$, and its second sign is the quadratic character of $\mathbb{Q}(\sqrt{-d})$. The definition keeps only the map α, through which every character of $\{\pm1\}^2 \times A$ becomes a Dirichlet character modulo N; the field itself is introduced in Definition 7.2, where it is first needed.

Definition 3.11. For an integer $e \geq 2$ let $H_e = (\mathbb{Z}/2^{e+2}\mathbb{Z})^{\times}/\{\pm1\}$. A **frame** is a tuple (e, A, h, a_0) of an integer $e \geq 2$, a finite abelian group A, a surjective homomorphism $h\colon A \to H_e$ and an element $a_0 \in A$ such that h is injective on the cyclic subgroup $A_0 = \langle a_0 \rangle$. Its **index** is $o = [A : A_0]$. Its **group** is $\mathcal{G} = \{\pm1\}^2 \times A$, with elements written $((s_1, s_2), a)$. It carries the **sign characters** $\epsilon_1(g) = s_1$ and $\epsilon_2(g) = s_2$ and the element $\iota_d = ((1,-1),0)$. We identify $\mathcal{G}$ with the group $\Gamma_A = (\mathbb{Z}/2)^2 \times A$ of Definition 3.1 by

$$\tau\colon \mathcal{G} \to \Gamma_A, \qquad ((s_1,s_2),a) \mapsto ((\bar{s}_2, \bar{s}_1), a), \qquad \bar{1} = 0,\ \overline{-1} = 1,$$

so that τ carries $\ker \epsilon_2$ onto Φ, ϵ_2 to $\chi_C \circ \tau$, and ι_d to $((1,0),0)$. Characters θ of $\mathcal{G}$ and characters $\phi = \theta \circ \tau^{-1}$ of Γ_A correspond bijectively.

A **host** for the frame is a tuple $\mathfrak{h} = (q, d, Q_S, \alpha)$ such that, with $N = Q_S q d$,

(Hr1) q and d are primes with $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$ and $(-d/q) = 1$;

(Hr2) $Q_S \geq 1$, $2^{e+2} \mid Q_S$, $\gcd(Q_S, qd) = 1$, and $(-d/l) = 1$ for every odd prime $l \mid Q_S$;

(Hr3) $\alpha\colon (\mathbb{Z}/N\mathbb{Z})^{\times} \to \mathcal{G}$ is a surjective homomorphism;

(Hr4) $\epsilon_2(\alpha(u)) = (u/d)$ for every u;

(Hr5) $\alpha(u) \in \{1, \iota_d\}$ for every $u \equiv 1 \pmod{Q_S q}$.

The case of interest is when α is the Artin map of an abelian field $E \subseteq \mathbb{Q}(\zeta_N)$ composed with an isomorphism $\mathrm{Gal}(E/\mathbb{Q}) \cong \mathcal{G}$; then (Hr4) says that ϵ_2 is the quadratic character of $\mathbb{Q}(\sqrt{-d}) \subseteq E$.

Characters of a host. For a character θ of $\mathcal{G}$ let $\chi_\theta = \theta \circ \alpha$, a Dirichlet character modulo N, and let χ_θ^* be the primitive character inducing it. Let $X^{\pm}(\mathfrak{h})$ be the set of θ that are trivial on $\{(1,1)\} \times A_0$ and satisfy $\theta(\iota_d) = \pm 1$.

The arithmetic datum of a host. For a prime p write $N = N_p N_p'$ with N_p a power of p and $p \nmid N_p'$. Let $U_p \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$ be the units $\equiv 1 \pmod{N_p'}$, and let f_p be the unit with $f_p \equiv p \pmod{N_p'}$ and $f_p \equiv 1 \pmod{N_p}$, which exists and is unique by the Chinese remainder theorem (so f_p is the class of p if $p \nmid N$). The **arithmetic datum** $\mathfrak{A}_\mathfrak{h}$ of $\mathfrak{h}$ is the datum on A with the primes q, d, inertia groups $I_p = \tau(\alpha(U_p))$ and Frobenius elements $\mathrm{Fr}_p = \tau(\alpha(f_p))$.

Compatible functions. A function $\eta\colon \widehat{\Gamma}_A \to \mathbb{C}$ is **compatible with** $\mathfrak{h}$ if

$$\eta(\phi') = L'(1,\chi)/L(1,\chi)$$

for every non-trivial $\phi' \in \widehat{\Gamma}_A$ and every primitive Dirichlet character χ such that $\chi(n) = \phi'_{\mathfrak{A}_\mathfrak{h}}(n)$ for all $n \geq 1$.

Lemma 3.12. *Let $\mathfrak{h} = (q, d, Q_S, \alpha)$ be a host for a frame (e, A, h, a_0), and let S be the set of prime factors of $Q_S q$. Then the arithmetic datum $\mathfrak{A}_\mathfrak{h}$ is of host type with respect to S.*

Proof. Write $N = Q_S q d$ and use the notation N_p, N_p', U_p, f_p of Definition 3.11. Recall that τ carries $\ker \epsilon_2$ onto Φ, so an element $\tau(\alpha(u))$ lies in Φ exactly when $(u/d) = 1$, by (Hr4). We check (Ah1)–(Ah4) of Definition 3.1.

(Ah1) d is prime by (Hr1); $2 \in S$ since $2^{e+2} \mid Q_S$; $q \in S$; and $d \notin S$ since $\gcd(Q_S, qd) = 1$ and $d \neq q$.

(Ah2) If $p \notin S \cup \{d\}$, then $p \nmid N$, so $N_p' = N$ and $U_p = \{1\}$; hence $I_p = 0$.

(Ah3) Let $p \in S$. Since $p \neq d$, we have $d \mid N_p'$. A unit $u \in U_p$ satisfies $u \equiv 1 \pmod{d}$, so $(u/d) = 1$ and $I_p \subseteq \Phi$. For the Frobenius element, $f_p \equiv p \pmod{d}$, so it suffices to show $(p/d) = 1$.

- For $p = 2$ this holds because $d \equiv 7 \pmod{16}$.
- For odd p, quadratic reciprocity with $d \equiv 3 \pmod 4$ gives $(p/d) = (-d/p)$. This is 1 by (Hr2) if $p \mid Q_S$, and by (Hr1) if $p = q$.

(Ah4) Here $N_d' = Q_S q$, so U_d consists of the units $\equiv 1 \pmod{Q_S q}$, and $\alpha(U_d) \subseteq \{1, \iota_d\}$ by (Hr5). By the Chinese remainder theorem there is a unit u with $u \equiv 1 \pmod{Q_S q}$ and $u \equiv -1 \pmod d$. Then $\epsilon_2(\alpha(u)) = (-1/d) = -1$ since $d \equiv 3 \pmod 4$, so $\alpha(u) = \iota_d$. Hence $I_d = \tau(\{1, \iota_d\}) = \{0, ((1,0),0)\}$. □

The next three definitions, of slope data, of the slope conditions and of the size conditions, belong to the accounting of Section 4, which explains them. They are stated here because the notion of a core-tied host in Definition 3.16 refers to them. On a first reading they may be skipped until that section.

Definition 3.13. Let $\mathfrak{S}$ be a slope datum, c a constant vector and X'' a real number. Then $\mathfrak{S}$ satisfies the **slope conditions for** (c, X'') if the following hold for all $u \in U$ and $\omega, \omega' \in \mathcal{W}(u)$.

(Sl1) Positivity. $x > 0$, $b > 0$, $a \geq 0$, $b' \geq 0$, $k > 0$, $v > 0$, $f > 0$; $a_\tau \geq 0$ for all τ; $w_{u,j,\rho} > 0$, $d_{u,\omega} > 0$, $T_{u,\omega} > 0$ and $\mu_{u,\omega} \leq \mu^{\max}_{u,\omega}$; $c_{\mathrm{Sl}} \geq 0$; $\bar{\delta} < 1$, $\bar{\eta} \geq 0$ and $0 \leq \bar{e} \leq \frac{1}{4}$.

(Sl2) Divisor bound.

$$\mu^{\vee}_u + h_{u,\omega} - F_u \leq \sum_{\tau \in \mathcal{T}} \Big(m(\varphi(a_\tau) - a_\tau \log \Omega) - a_\tau \log T_{u,\omega} \Big) + c_{\mathrm{Div}} n_{\mathcal{T}} a L_a + c_{\mathrm{Div}} m (B_0 + N_c) x^{e_{\mathrm{Div}} \varepsilon}.$$

(Sl3) Exponent comparison. $\left| E_{u,\omega} - m A l_g \right| \leq c_E m B_0$.

(Sl4) Height comparison. $\Big| (h_{u,\omega} + \sum_\tau a_\tau \log T_{u,\omega}) - (h_{u,\omega'} + \sum_\tau a_\tau \log T_{u,\omega'}) \Big| \leq c_H (n_{\mathcal{T}} a L_a + m B_0)$.

(Sl5) Counting.

$$\frac{1}{f} \sum_{j=1}^{m} \sum_{\rho=1}^{f} \log \frac{w_{u,j,\rho}\, v}{\pi} + m l_g - c_{\mathrm{Count}} \Big(L_a + \frac{m^2 L_a}{k} \Big) \leq \log T_{u,\omega} \leq c_{\mathrm{Count}}\, m\, L,$$

and $\left| \log T_{u,\omega} - \log T_{u,\omega'} \right| \leq c_{\mathrm{Count}} \log k$.

(Sl6) Ambient slopes. $-c_{\mathrm{Slope}} n_{\mathcal{T}} a L_a (1 + \frac{m}{k}) \leq \mu'_{u,\omega}$ and $\mu'^{\max}_{u,\omega} \leq c_{\mathrm{Slope}} n_{\mathcal{T}} a L_a$.

(Sl7) Transfer of slopes. $\mu^{\max}_{u,\omega} \leq \mu'^{\max}_{u,\omega} + \Theta_u + F_u - E_{u,\omega} + c_{\mathrm{Sl}} m B_0$ and $\mu'_{u,\omega} + \Theta_u + F_u - E_{u,\omega} - \mathrm{Lo}''_{u,\omega} \leq \mu_{u,\omega}$.

(Sl8) Losses. $S_u - \Theta_u \leq \mathrm{Lo}$ and $I_u \leq \mathrm{Lo}$.

(Sl9) Dual-slope inequality. If $0 \leq \delta_u < 1$, $\eta_u \geq 0$, $0 \leq e_u \leq \frac{1}{4}$, $s_u - \mu^{\vee}_{\min,u} \leq \mathrm{HH}_u$ and $\mathrm{HH}_u > 0$, then

$$s_u - \log |\mathcal{W}(u)| - \frac{1}{1 - \delta_u} \Big(4 \phi_u \log \Big(1 + \frac{\mathrm{HH}_u}{\phi_u} \Big) + (\eta_u + 4\sqrt{e_u}) \mathrm{HH}_u \Big) \leq \mu^{\vee}_u,$$

where the term $4\phi_u \log(1 + \mathrm{HH}_u/\phi_u)$ is read as 0 when $\phi_u = 0$.

(Sl10) Height bound. $s_u - \mu^{\vee}_{\min,u} \leq \mathrm{HH}_u$ and $0 < \mathrm{HH}_u \leq x^{c_{\mathrm{HH}}} b$.

(Sl11) Parameters. $0 \leq \delta_u \leq \bar{\delta}$, $0 \leq \eta_u \leq \bar{\eta}$ and $0 \leq e_u \leq \bar{e}$.

Definition 3.14. A **slope datum** $\mathfrak{S}$ consists of the following.

- Finite sets U (the **translates**) and $\mathcal{T}$, a non-empty finite set $\mathcal{W}(u)$ for each $u \in U$ (the **words at** u), and integers $m, f \geq 0$.
- Real parameters $x, \varepsilon, a, b, b', k, L, L_a, B_0, N_c, \gamma_0, \beta, v, l_g$, and real numbers $\bar{\delta}, \bar{\eta}, \bar{e}$.
- Real numbers a_τ ($\tau \in \mathcal{T}$) and $w_{u,j,\rho}$ ($u \in U$, $1 \leq j \leq m$, $1 \leq \rho \leq f$).
- For each $u \in U$ and $\omega \in \mathcal{W}(u)$, real numbers

$$d_{u,\omega},\ T_{u,\omega},\ \mu_{u,\omega},\ \mu^{\max}_{u,\omega},\ h_{u,\omega},\ \mu'_{u,\omega},\ \mu'^{\max}_{u,\omega},\ E_{u,\omega,j}\ (1 \leq j \leq m).$$

- For each $u \in U$, real numbers $\mu^{\vee}_u,\ \mu^{\vee}_{\min,u},\ F_u,\ \Theta_u,\ S_u,\ I_u,\ \mathrm{HH}_u,\ \delta_u,\ \eta_u,\ e_u$.

Attached to $\mathfrak{S}$ are the following quantities. Put $n_{\mathcal{T}} = |\mathcal{T}|$, $A = \sum_{\tau\in\mathcal{T}} a_\tau$, $\varphi(y) = y\log y - y$, $\log\Omega = \gamma_0 + \log(\pi/v) - \beta$, and $E_{u,\omega} = \sum_{j=1}^{m} E_{u,\omega,j}$. The **mean slope** and the **spread** at u are

$$s_u = \frac{\sum_{\omega\in\mathcal{W}(u)} d_{u,\omega}\,(\mu_{u,\omega} - h_{u,\omega})}{\sum_{\omega\in\mathcal{W}(u)} d_{u,\omega}}, \qquad \phi_u = \max_{\omega\in\mathcal{W}(u)} \big(\mu^{\max}_{u,\omega} - h_{u,\omega}\big) - s_u,$$

and the **main term** at u is

$$\mathrm{Main}(u) = \sum_{j=1}^{m}\sum_{\tau\in\mathcal{T}} a_\tau\Big(\log a_\tau - 1 - \gamma_0 + \beta - \frac{1}{f}\sum_{\rho=1}^{f}\log w_{u,j,\rho}\Big).$$

A **constant vector** is a tuple $c = (c_{\mathrm{Div}}, e_{\mathrm{Div}}, c_E, c_H, c_{\mathrm{Count}}, c_{\mathrm{Slope}}, c_{\mathrm{Sl}}, c_{\mathrm{Loss}}, e_{\mathrm{Loss}}, c_{\mathrm{HH}})$ of real numbers. For a constant vector c and a real number X'', the **loss bound**, the **loss terms** and the **variance term** are

$$\begin{aligned}
\mathrm{Lo} &= c_{\mathrm{Loss}}\, n_{\mathcal{T}}\, m\, b\, x^{e_{\mathrm{Loss}}\varepsilon},\\
\mathrm{Lo}''_{u,\omega} &= c_{\mathrm{Sl}} I_u + c_{\mathrm{Sl}}\, n_{\mathcal{T}}\, m\,\Big(\frac{a}{k} + b'\Big)\log T_{u,\omega} + c_{\mathrm{Sl}} X'' + 1,\\
V &= 2mc_E B_0 + c_H\big(n_{\mathcal{T}} a L_a + mB_0\big) + c_{\mathrm{Count}} A\log k.
\end{aligned}$$

Definition 3.15. Let $\mathfrak{S}$ be a slope datum, c a constant vector and X'', e_K, C_K real numbers. The datum satisfies the **size conditions for** (c, X'', e_K, C_K) if

(Er1) Scale. $x \ge 1$ and $0 < \varepsilon \le 1$.

(Er2) Sizes. $b \ge 1$, $xb \le a$, $n_{\mathcal{T}} \ge 1$, $m \le x^{1/2+\varepsilon/2}$, $xm \le k$, $e_K \ge 0$, $1 \le k \le \min(a, x^{e_K})$, $0 \le L \le 2x^{\varepsilon}$, $0 \le L_a \le 2x^{\varepsilon}$, $b' \ge 0$, $kb' \le a$, $0 \le X'' \le a$, $0 \le a_\tau \le 2a$ for every τ, $B_0 \ge 0$ and $N_c \ge 0$.

(Er3) Words. $\log|\mathcal{W}(u)| \le (k-3)\log 2$ for every $u \in U$.

(Er4) Parameters. $\bar\delta \le \frac12$, $\bar\eta \ge 0$, $\bar e \ge 0$, and $(\bar\eta + 4\sqrt{\bar e})\, x^{c_{\mathrm{HH}}} b \le C_K\, x^{\varepsilon}\, n_{\mathcal{T}}\, a$.

(Er5) Constants. $0 \le c_{\mathrm{Div}}, c_E, c_H, c_{\mathrm{Count}}, c_{\mathrm{Slope}}, c_{\mathrm{Sl}}, c_{\mathrm{Loss}} \le C_K$, and $e_{\mathrm{Div}}, e_{\mathrm{Loss}}, c_{\mathrm{HH}} \ge 0$.

The scale is **large** for $(\varepsilon, C_K, c_{\mathrm{HH}}, e_K)$ if $\log x \le x^{\varepsilon}$ and $200(1+C_K)^2(1+c_{\mathrm{HH}})(1+e_K) \le x^{\varepsilon}$.

Definition 3.16. Let $\mathfrak{L} = (K, \psi_{\mathrm{exc}}, o_{\mathfrak{L}})$ be a character level, $\mathfrak{h}$ a host for a frame (e, A, h, a_0) of index o, $T \subseteq A$ a finite set, and $x > 1$, $\varepsilon > 0$, $C \ge 0$ real numbers. The **controlled characters** of $\mathfrak{h}$ relative to $\mathfrak{L}$ are the $\theta \in X^-(\mathfrak{h})$ together with the $\theta \in X^+(\mathfrak{h})$ with $\chi^*_\theta \notin K$; their twisted functions do not enter $G_{\mathfrak{L}}$. Put $\kappa(b) = 4/(3o+1)$ for $b \in A_0$ and $\kappa(b) = 3/(3o+1)$ for $b \in A\setminus A_0$.

Host conditions. The pair $(\mathfrak{h}, T)$ satisfies the **host conditions for $\mathfrak{L}$ at scale x with parameters** (ε, C) if

(Ec1) $o = o_{\mathfrak{L}}$;

(Ec2) every $\psi \in K$ equals χ^*_θ for some $\theta \in X^+(\mathfrak{h})$;

(Ec3) if ψ_{exc} is present, it equals $\chi^*_{\epsilon_1}$; if it is absent, $\chi^*_{\epsilon_1} \notin K$;

(Ec4) for every $b \in A$, $\big|\#\{t \in T : t - b \in A_0\} - \kappa(b)|T|\big| \le x^{-1+\varepsilon}|T|$;

(Ec5) for every controlled character θ, $|g(x, \chi^*_\theta)| \le x^{1/2+\varepsilon}$ and $|L'(1,\chi^*_\theta)/L(1,\chi^*_\theta)| \le C\log x$.

Core-tied hosts. Let e_D, e_L, C_K, c^*, e_K be real numbers. The pair $(\mathfrak{h}, T)$ is **core-tied at scale x with exponents $(e_D, e_L, C_K, c^*, e_K)$ and parameter ε** if there exist the following.

- A slope datum $\mathfrak{S}$ with translates U, parameters $x(\mathfrak{S}), \varepsilon(\mathfrak{S}), a, b, b', k, B_0, N_c, \gamma_0, \beta$, integers m, f and numbers a_τ, $w_{u,j,\rho}$.
- A constant vector c, a real X'', a map $\mathrm{tr}\colon U \to A$ whose image contains A_0, and bijections $\mathcal{T} \to T$, $\tau \mapsto t_\tau$, and $\{1, \dots, f\} \to \Phi$, $\rho \mapsto g_\rho$.
- A function η on $\widehat{\Gamma}_A$, an integer $P \geq 1$, a finite set $T_2 \subseteq A$, a function $a\colon A \to \mathbb{R}$, weights $w_{u,j}\colon \Gamma_A \to \mathbb{R}$ ($u \in A$, $1 \leq j \leq m$), and reals ν, X', θ_0.

These are subject to the following conditions.

(Ct1) η is compatible with $\mathfrak{h}$.

(Ct2) $x(\mathfrak{S}) = x$, $\varepsilon(\mathfrak{S}) = \varepsilon$, $e_{\mathrm{Div}} \leq e_D$, $e_{\mathrm{Loss}} \leq e_L$ and $c_{\mathrm{HH}} \leq c^*$ for the entries of c.

(Ct3) (identification) $a_\tau = a(t_\tau)$ for all τ; $w_{u,j,\rho} = w_{\mathrm{tr}(u),j}(g_\rho)$ for all u, j, ρ; $\gamma_0 = \gamma$; $\beta = \operatorname{Re}\eta(\chi_C)$; and for every $u \in U$

$$S_u = \sum_{t \in T} \sum_{j=1}^{m} A_{\mathrm{in}}\big(a(t), t, w_{\mathrm{tr}(u),j}\big),$$

with A_{in} formed with $(\mathfrak{A}_{\mathfrak{h}}, P)$.

(Ct4) $\mathfrak{S}$ satisfies the slope conditions for (c, X'') and the size conditions for (c, X'', e_K, C_K).

(Ct5) $B_0 \leq 2|A|\, m\, b\, x^{\varepsilon}$ and $N_c \leq |T_2|\, k\, b\, x^{\varepsilon}/(|A|^2 x^{-\varepsilon})$.

(Ct6) $(A_0, T, T_2, a|_T, (w_{u,j}), P)$ is weight data at scale x with parameters $(b, b', \nu, X', \theta_0)$ for $(\mathfrak{A}_{\mathfrak{h}}, \eta)$.

(Ct7) $a \leq 2xb$, $\frac{1}{2}x^{-\varepsilon/8}|A| \leq |T|$, $x^{-\varepsilon}|A| \leq |T_2|$, $x^{1/2-\varepsilon} \leq m$ and $k \leq |A|^2 x^{1/2-\varepsilon}$.

The host conditions say that a host sees the characters of a level and that its other characters are harmless at the scale x. Condition (Ec2) puts the characters of the level among those of the host. Condition (Ec4) asks the top set T to meet A_0 in the proportion $4/(3o+1)$ and every other coset of A_0 in the proportion $3/(3o+1)$, up to $x^{-1+\varepsilon}$. Then $\sum_{t \in T} \phi(t)$ equals $|T|/(3o+1)$, up to an error of at most $o\, x^{-1+\varepsilon}|T|$, for every non-trivial character ϕ of A/A_0. This is where the weight $(3o+1)^{-1}$ of a level comes from. Condition (Ec5) asks the controlled characters for a square-root bound at the point x and for a bound on the logarithmic derivative at $s = 1$; Part II supplies it from good window pairs.

The second half of the definition is the hypothesis under which Section 6 bounds $G_{\mathfrak{L}}(x)$. A core-tied host carries a slope datum whose prime-sum entries are, by the identification (Ct3), the inert prime sums of the host, and which satisfies the slope conditions and the size conditions with exponents $e_{\mathrm{Div}} \leq e_D$ and $e_{\mathrm{Loss}} \leq e_L$ ((Ct2), (Ct4)). Conditions (Ct5)–(Ct7) are the size conditions of the averaging step: (Ct5) bounds the two budgets B_0 and N_c of the slope datum, (Ct6) asks for weight data, and (Ct7) is the part of (Ex2) that the size conditions of the slope datum do not contain.

Lemma 3.17. *Let $\mathfrak{h}$ be a host for a frame (e, A, h, a_0) of index o, with $A_0 = \langle a_0 \rangle$.*

1. $|X(A_0)| = 4o$.

Let moreover $\mathfrak{L} = (K, \psi_{\mathrm{exc}}, o_{\mathfrak{L}})$ be a character level, $T \subseteq A$ finite, $x > 1$, $\varepsilon > 0$, $C \geq 0$, and suppose that $(\mathfrak{h}, T)$ satisfies the host conditions for $\mathfrak{L}$ at scale x with parameters (ε, C) and that η is compatible with $\mathfrak{h}$. Let $K'_{\mathfrak{h}}$ be the set of characters $\theta \circ \tau^{-1}$ of Γ_A with $\chi^*_\theta \in K$, and $c = (3o+1)^{-1}$. Then:

2. $G_{K'_{\mathfrak{h}}}(y) = G_{\mathfrak{L}}(y)$ *for every* $y > 0$;

3. $\Sigma_{K'_{\mathfrak{h}}}(x) \leq 4o\, x^{1/2+\varepsilon}$;

4. *condition (Ex3) of Proposition 3.3 holds for* $K' = K'_{\mathfrak{h}}$ *with* $c = (3o+1)^{-1}$, $M = 4o$, $K_o = \sum_{\psi \in K \setminus \{\mathbf{1}\}} |L'(1,\psi)/L(1,\psi)|$ *and* $C_E = C$.

Proof. Throughout, θ and ϕ are related by $\phi = \theta \circ \tau^{-1}$. Since τ maps $\{(1,1)\} \times A_0$ onto $\{0\} \times A_0$ and ι_d to $((1,0),0)$, we have $\phi \in X(A_0)$ if and only if $\theta \in X^+(\mathfrak{h}) \cup X^-(\mathfrak{h})$, and then $\phi(((1,0),0)) = \theta(\iota_d)$.

Injectivity. The map $\theta \mapsto \chi^*_\theta$ is injective. Indeed, χ^*_θ determines $\chi_\theta = \theta \circ \alpha$ on the units modulo N, and $\theta \circ \alpha$ determines θ because α is surjective by (Hr3).

Item 1. $X(A_0)$ is the character group of $\Gamma_A/(\{0\} \times A_0)$, a group of order $4|A|/|A_0| = 4o$.

Description of $K'_{\mathfrak{h}}$. Let $\chi^*_\theta \in K$. By (Ec2), $\chi^*_\theta = \chi^*_{\theta'}$ for some $\theta' \in X^+(\mathfrak{h})$, so $\theta = \theta'$ by injectivity. Hence $K'_{\mathfrak{h}} \subseteq X(A_0)$, every $\phi \in K'_{\mathfrak{h}}$ satisfies $\phi(((1,0),0)) = 1$, and $\theta \mapsto \chi^*_\theta$ is a bijection from $\{\theta : \theta \circ \tau^{-1} \in K'_{\mathfrak{h}}\}$ onto K (surjective by (Ec2)). The trivial character lies in $K'_{\mathfrak{h}}$, because $\chi^*_1 = \mathbf{1} \in K$ by (Le1).

Item 2. By item 3 of Lemma 3.18, $G_{\mathfrak{A}_{\mathfrak{h}}}(y, \phi) = g(y, \chi^*_\theta)$. By the bijection, it remains to compare the weights. We show that, for $\phi \in K'_{\mathfrak{h}}$, $w_c(\phi) = 1$ holds exactly when $\chi^*_\theta \in \{\mathbf{1}, \psi_{\mathrm{exc}}\}$, and $w_c(\phi) = (3o+1)^{-1} = w_{\mathfrak{L}}(\chi^*_\theta)$ otherwise (using (Ec1), $o = o_{\mathfrak{L}}$).

- $w_c(\phi) = 1$ means that ϕ is trivial on $\{0\} \times A$, that is, θ is trivial on $\{(1,1)\} \times A$. Since also $\theta(\iota_d) = 1$, θ is determined by $\theta(((-1,1),0)) = \pm 1$, so $\theta \in \{1, \epsilon_1\}$.
- $\theta = 1$ corresponds to $\chi^*_\theta = \mathbf{1}$ (item 2 of Lemma 3.18).
- If $\theta = \epsilon_1$ and $\chi^*_{\epsilon_1} \in K$, then ψ_{exc} is present and equals $\chi^*_{\epsilon_1}$ by (Ec3). Conversely, $\chi^*_\theta = \psi_{\mathrm{exc}} = \chi^*_{\epsilon_1}$ forces $\theta = \epsilon_1$ by injectivity.

Hence $\sum_{\phi \in K'_{\mathfrak{h}}} w_c(\phi) \operatorname{Re} G_{\mathfrak{A}_{\mathfrak{h}}}(y, \phi) = G(y) + \sum_{\psi \in K \setminus \{\mathbf{1}\}} w_{\mathfrak{L}}(\psi) \operatorname{Re} g(y, \psi) = G_{\mathfrak{L}}(y)$.

Item 3. Let $\phi \in X(A_0) \setminus K'_{\mathfrak{h}}$. Then either $\theta \in X^-(\mathfrak{h})$, or $\theta \in X^+(\mathfrak{h})$ with $\chi^*_\theta \notin K$; in both cases θ is controlled. Since $\theta \neq 1$, item 3 of Lemma 3.18 and (Ec5) give $|G_{\mathfrak{A}_{\mathfrak{h}}}(x, \phi)| = |g(x, \chi^*_\theta)| \leq x^{1/2+\varepsilon}$. There are at most $|X(A_0)| = 4o$ such ϕ.

Item 4. The first three clauses of (Ex3) were shown above, and $|X(A_0)| = 4o = M$. Next come the two clauses on η. For non-trivial $\phi \in K'_{\mathfrak{h}}$ we have $\eta(\phi) = L'(1, \chi^*_\theta)/L(1, \chi^*_\theta)$ with $\chi^*_\theta \in K \setminus \{\mathbf{1}\}$, so $|\eta(\phi)| \leq K_o$. For $\phi \in X(A_0) \setminus K'_{\mathfrak{h}}$, θ is controlled and non-trivial, so $|\eta(\phi)| \leq C \log x$ by (Ec5).

It remains to bound the character sums. Let $\phi \in K'_{\mathfrak{h}}$ be non-trivial on $\{0\} \times A$. The map $b \mapsto \phi((0,b))$ factors through a non-trivial character ϑ of A/A_0, a group of order o. For a coset $\beta \in A/A_0$ let $n_\beta = \#\{t \in T : t \in \beta\}$, and let $\kappa(\beta)$ denote the value of κ on β. Then

$$\sum_{t \in T} \phi((0,t)) = \sum_\beta \vartheta(\beta) n_\beta, \qquad \sum_\beta \vartheta(\beta)\kappa(\beta) = \frac{4-3}{3o+1} + \frac{3}{3o+1} \sum_\beta \vartheta(\beta) = \frac{1}{3o+1},$$

since $\vartheta(A_0) = 1$ and $\sum_\beta \vartheta(\beta) = 0$. By (Ec4), $|n_\beta - \kappa(\beta)|T|| \le x^{-1+\varepsilon}|T|$ for each of the o cosets. Hence $|\sum_t \phi((0,t)) - c|T|| \le o\, x^{-1+\varepsilon}|T| \le M x^{-1+\varepsilon}|T|$. □

Lemma 3.18. *Let $\mathfrak{h}$ be a host for a frame (e, A, h, a_0), θ a character of its group $\mathcal{G}$ and $\phi = \theta \circ \tau^{-1}$ the corresponding character of Γ_A.*

1. *$\phi_{\mathfrak{A}_\mathfrak{h}}(n) = \chi_\theta^*(n)$ for every integer $n \ge 1$.*
2. *χ_θ^* is the trivial character $\mathbf{1}$ if and only if $\theta = 1$.*
3. *If $\eta\colon \widehat{\Gamma}_A \to \mathbb{C}$ is compatible with $\mathfrak{h}$, then $\eta(\phi) = L'(1, \chi_\theta^*)/L(1, \chi_\theta^*)$ for $\theta \ne 1$, and $G_{\mathfrak{A}_\mathfrak{h},\eta}(y, \phi) = g(y, \chi_\theta^*)$ for all $y > 0$.*

Proof. Write $N = Q_S q d$ and use N_p, N_p', U_p, f_p from Definition 3.11. Here U_p is the kernel of reduction $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N_p'\mathbb{Z})^\times$, and ϕ is trivial on $I_p = \tau(\alpha(U_p))$ exactly when χ_θ is trivial on U_p.

Item 1. The function χ_θ^* is completely multiplicative, and $\phi_{\mathfrak{A}}(n) = \prod_{p|n} \mu_{I_p}(v_p(n)\mathrm{Fr}_p, \phi)$. So it suffices to show $\mu_{I_p}(i\mathrm{Fr}_p, \phi) = \chi_\theta^*(p)^i$ for every prime p and $i \ge 1$.

- *χ_θ trivial on U_p.* Then ϕ is trivial on I_p, so the coset mean is $\phi(i\mathrm{Fr}_p) = \chi_\theta(f_p)^i$. Moreover, χ_θ factors through $(\mathbb{Z}/N_p'\mathbb{Z})^\times$, so the conductor of χ_θ divides N_p', and $\chi_\theta^*(m) = \chi_\theta(u)$ for every m prime to N_p' and every unit $u \equiv m \pmod{N_p'}$. Taking $m = p$, which is prime to N_p', and $u = f_p$ gives $\chi_\theta^*(p) = \chi_\theta(f_p)$.
- *χ_θ not trivial on U_p.* The mean of a character over a coset of a subgroup on which it is not trivial is 0, so the coset mean vanishes. On the other hand, p divides the conductor c of χ_θ: otherwise $c \mid N_p'$ and χ_θ would be trivial on U_p. Hence $\chi_\theta^*(p) = 0 = \chi_\theta^*(p)^i$.

Item 2. $\chi_\theta^* = \mathbf{1}$ exactly when $\chi_\theta = \theta \circ \alpha$ is the principal character, which happens exactly when $\theta = 1$, because α is surjective by (Hr3).

Item 3. Let $\theta \ne 1$. By item 1, $\chi = \chi_\theta^*$ satisfies the condition in the definition of compatibility for $\phi' = \phi$ (which is non-trivial), so $\eta(\phi) = L'(1, \chi_\theta^*)/L(1, \chi_\theta^*)$. By item 2, χ_θ^* is non-principal. Comparing the definitions of $G_{\mathfrak{A},\eta}(y, \phi)$ and $g(y, \chi_\theta^*)$ term by term, using item 1, gives equality. For $\theta = 1$ both sides are $G(y)$. □

4 The accounting: Stepanov's count as a slope inequality

A slope datum is a finite record of real numbers (Definition 3.14). Among its entries are, for every translate u, a prime-sum entry S_u and a lattice term Θ_u. This section proves that the eleven slope conditions (Sl1)–(Sl11), which are inequalities between the entries, bound the prime-sum entry by an explicit main term:

$$S_u \le \mathrm{Main}(u) + \mathrm{Err}(u)$$

(Proposition 4.2). It then bounds the error under the size conditions (Lemma 4.7). When the prime-sum entries are the inert prime sums of an arithmetic datum, $\mathrm{Main}(u)$ is the sum of their main terms, and the result is an upper bound for the excess (Proposition 4.1). This bound is hypothesis (Ex4) of Proposition 3.3; the two are combined in Step 1 of the proof of Proposition 6.5 in Section 6.

No lattice occurs in the statements of this section. The numbers of a slope datum are ranks, slopes and heights of hermitian lattices only in the construction that produces them, and the accounting does not use this. It uses the eleven inequalities, in the following order. The loss condition (Sl8) bounds S_u by Θ_u plus a loss. The transfer of slopes (Sl7) and the ambient slope bounds (Sl6) bound the slope of every word from below by Θ_u, up to its frame term and its exponent, and the divisor bound (Sl2) bounds the dual slope from above by an archimedean term. Added and averaged over the words, they bound Θ_u plus the dual slope by the mean slope plus the archimedean term (Lemma 4.4). The dual-slope inequality (Sl9) bounds the mean slope by the dual slope, up to a term controlled by the spread of the slopes (Lemma 4.5, Lemma 4.6). The dual slope cancels, and with the counting bounds (Sl5) the archimedean term becomes $\mathrm{Main}(u)$ (Lemma 4.3). On the way the frame terms cancel between (Sl2) and (Sl7), the heights of the words cancel between (Sl2) and the mean slope, and the terms $m\,A\,l_g$ cancel between (Sl3) and (Sl5).

Where the eleven inequalities are proved is the subject of the rest of the paper. The loss condition (Sl8) follows from the loss bound of Section 5, through Corollary 7.11. The other ten follow from the properties (Sp1)–(Sp8) of slope records by Theorem 8.4 in Section 8, and Parts III and IV supply those properties; the table that closes Section 8 gives the result for each.

Proposition 4.1. *Let $\mathfrak{A}$ be an arithmetic datum on a finite abelian group H, $P \geq 1$ an integer, η a function on the characters of Γ_H, $T, T_2 \subseteq H$ finite sets and $\tilde{a}\colon H \to \mathbb{R}$ a function. Let $m \geq 1$, and let $w_{h,j}\colon \Gamma_H \to \mathbb{R}$ be weights for $h \in H$ and $1 \leq j \leq m$. Let $\mathfrak{S}$ be a slope datum with this m, together with a map $\mathrm{tr}\colon U \to H$ and bijections $\mathcal{T} \to T$, $\tau \mapsto t_\tau$, and $\{1, \dots, f\} \to \Phi$, $\rho \mapsto g_\rho$, such that*

(Id) *$a_\tau = \tilde{a}(t_\tau)$ for all τ; $w_{u,j,\rho} = w_{\mathrm{tr}(u),j}(g_\rho)$ for all u, j, ρ; $\gamma_0 = \gamma$; $\beta = \mathrm{Re}\,\eta(\chi_C)$; and $S_u = \sum_{t \in T} \sum_{j=1}^{m} A_{\mathrm{in}}(\tilde{a}(t), t, w_{\mathrm{tr}(u),j})$ for all $u \in U$, the inert prime sums formed with $(\mathfrak{A}, P)$.*

Suppose that $\mathfrak{S}$ satisfies the slope conditions for (c, X'') and the size conditions for (c, X'', e_K, C_K), that $B_0 \leq 2|H|mbx^{\varepsilon}$ and $N_c \leq x^{2\varepsilon}|T_2|kb/|H|^2$, and that the scale is large for $(\varepsilon, C_K, c_{\mathrm{HH}}, e_K)$. Then for every $u \in U$, with Δ formed with $(\mathfrak{A}, \eta, P, T, \tilde{a}, w)$,

$$\Delta(\mathrm{tr}(u)) \leq x^{(e_{\mathrm{Div}}+e_{\mathrm{Loss}}+3)\varepsilon}\Big(|T|\,a\Big(1+\frac{m^2}{k}\Big) + m\Big(2|H|mbx^{\varepsilon} + \frac{x^{2\varepsilon}|T_2|kb}{|H|^2}\Big)\Big).$$

Proof. Fix u and put $h = \mathrm{tr}(u)$.

The main terms match. Since $\rho \mapsto g_\rho$ is a bijection onto Φ, $f = |\Phi|$ and $\frac{1}{f}\sum_\rho \log w_{u,j,\rho} = \frac{1}{|\Phi|}\sum_{g \in \Phi} \log w_{h,j}(g)$ by (Id). With $a_\tau = \tilde{a}(t_\tau)$, $\gamma_0 = \gamma$ and $\beta = \mathrm{Re}\,\eta(\chi_C)$, the summand of $\mathrm{Main}(u)$ at (j, τ) is $\mathrm{Re}\,M_\eta(\tilde{a}(t_\tau), w_{h,j})$; reindexing $\tau \mapsto t_\tau$,

$$\mathrm{Main}(u) = \sum_{t \in T}\sum_{j=1}^{m} \mathrm{Re}\,M_\eta(\tilde{a}(t), w_{h,j}).$$

With the last clause of (Id) this gives $\Delta(h) = S_u - \mathrm{Main}(u)$.

The bound. By Proposition 4.2, $S_u - \mathrm{Main}(u) \leq \mathrm{Err}(u)$, and by Lemma 4.7,

$$\mathrm{Err}(u) \leq x^{(e_{\mathrm{Div}}+e_{\mathrm{Loss}}+3)\varepsilon}\Big(n_{\mathcal{T}}a\Big(1+\frac{m^2}{k}\Big) + m(B_0 + N_c)\Big).$$

Finally, $n_{\mathcal{T}} = |T|$ via the bijection $\mathcal{T} \to T$. Since $m \geq 0$ and the factor $x^{(e_{\mathrm{Div}}+e_{\mathrm{Loss}}+3)\varepsilon}$ is non-negative, the bounds on B_0 and N_c give the bracket of the statement. □

Proposition 4.2. *Let $\mathfrak{S}$ be a slope datum that satisfies the slope conditions for (c, X''). Put*

$$M'' = c_{\mathrm{Sl}}\mathrm{Lo} + c_{\mathrm{Sl}}\, n_{\mathcal{T}}\, m\Big(\frac{a}{k} + b'\Big) c_{\mathrm{Count}}\, m\, L + c_{\mathrm{Sl}} X'' + 1,$$
$$\bar{U}'' = c_{\mathrm{Slope}} n_{\mathcal{T}} a L_a \Big(2 + \frac{m}{k}\Big) + V + c_{\mathrm{Sl}} m B_0 + M'',$$

and, for $u \in U$,

$$\begin{aligned}\mathrm{Err}(u) = {} & c_{\mathrm{Count}} A\Big(L_a + \frac{m^2 L_a}{k}\Big) + c_{\mathrm{Div}} n_{\mathcal{T}} a L_a + c_{\mathrm{Div}} m (B_0 + N_c) x^{e_{\mathrm{Div}}\varepsilon} \\ & + m c_E B_0 + c_{\mathrm{Slope}} n_{\mathcal{T}} a L_a \Big(1 + \frac{m}{k}\Big) + \log|\mathcal{W}(u)| + \mathrm{Lo} + M'' \\ & + \frac{1}{1-\bar{\delta}}\Big(4\log\big(1 + x^{c_{\mathrm{HH}}}\big)\big(\bar{U}'' + b\big) + \big(\bar{\eta} + 4\sqrt{\bar{e}}\big) x^{c_{\mathrm{HH}}} b\Big).\end{aligned}$$

Then $S_u \le \mathrm{Main}(u) + \mathrm{Err}(u)$ for every $u \in U$.

Proof. Fix u and let $\overline{\mathrm{Lo}}''_u$, K_3 and K_0 be as in Lemma 4.4, Lemma 4.5 and Lemma 4.3.

The mean loss term. The coefficient $c_{\mathrm{Sl}} n_{\mathcal{T}} m(a/k + b')$ is non-negative by (Sl1), and $\log T_{u,\omega} \le c_{\mathrm{Count}} m L$ by (Sl5). Hence every $\mathrm{Lo}''_{u,\omega}$ is at most $c_{\mathrm{Sl}} I_u + c_{\mathrm{Sl}} n_{\mathcal{T}} m(a/k + b') c_{\mathrm{Count}} m L + c_{\mathrm{Sl}} X'' + 1$, and so is their weighted mean $\overline{\mathrm{Lo}}''_u$. Since $c_{\mathrm{Sl}} \ge 0$ and $I_u \le \mathrm{Lo}$ by (Sl8), $\overline{\mathrm{Lo}}''_u \le M''$.

The lattice term. By Lemma 4.3,

$$\Theta_u \le \mathrm{Main}(u) + \log|\mathcal{W}(u)| + K_0 + \overline{\mathrm{Lo}}''_u + \frac{1}{1-\bar{\delta}}\Big(4\log\big(1 + x^{c_{\mathrm{HH}}}\big)\big(K_3 + \overline{\mathrm{Lo}}''_u + b\big) + \big(\bar{\eta} + 4\sqrt{\bar{e}}\big) x^{c_{\mathrm{HH}}} b\Big).$$

The term $\overline{\mathrm{Lo}}''_u$ enters with the non-negative coefficients 1 and $4(1-\bar{\delta})^{-1}\log(1 + x^{c_{\mathrm{HH}}})$, so it may be replaced by M''; then $K_3 + M'' = \bar{U}''$, and $\Theta_u \le \mathrm{Main}(u) + \mathrm{Err}(u) - \mathrm{Lo}$.

The prime sum. By (Sl8), $S_u \le \Theta_u + \mathrm{Lo} \le \mathrm{Main}(u) + \mathrm{Err}(u)$. □

In the arithmetic part the entries of a slope datum are filled as follows; this is done in the proof of Theorem 8.4, at one scale x of a parametrised host family with slope records. The index u runs over the group A of the frame, τ over the top set T of the host, ρ over the $2|A|$ elements of Φ, and ω over the words of the slope record of the member that carries the translate u. The parameters are

$$b = \exp(x^{\varepsilon}), \qquad a = xb, \qquad b' = a/X' \ \text{ with } X' = \exp(x^{\varepsilon/3}), \qquad L = x^{\varepsilon}, \qquad X'' = x\,|A|\,X',$$

and k, m are parameters of the loss data of the members (Definition 5.5). The number v is half the root discriminant of the host field, $L_a = \log(2av)$, γ_0 is Euler's constant, $\beta = L'(1,\chi_d)/L(1,\chi_d)$ for the Legendre symbol χ_d modulo d, and $l_g = \log(g_1 g_2)$ for two auxiliary primes. The budgets are $B_0 = 2|A|\, m\, b\, x^{\varepsilon}$ and $N_c = |T_2|\, k\, b\, x^{2\varepsilon}/|A|^2$, where T_2 is the thick set (Definition 8.1). The numbers a_τ are the heights of the tops, and $w_{u,j,\rho}$ are the weights of the loss datum of the member that carries u. For a word ω, the entries d, T, μ, $\mu^{\max}$, h, μ', $\mu'^{\max}$ and E_j are its rank, Gauss prime, slope, maximal slope, height, ambient slope, maximal ambient slope and exponents in the slope record, and $\mu^{\vee}$, $\mu^{\vee}_{\min}$, F, δ are the dual slope, the minimal dual slope, the frame term and the irreducibility defect of the record. Three entries come from the loss datum and not from the slope record: the prime sum S_u, which is the sum of the inert prime sums of the host with the cut-off $\lceil 2x\exp(x^{\varepsilon})\rceil$, the lattice term Θ_u, and the companion sum I_u. The word height has two unrelated uses here, for the numbers a_τ of the tops and for the numbers h of the words.

Under this dictionary the two main terms agree for the following reason. The term $a_\tau(\log a_\tau - 1 - \gamma_0 + \beta)$ of $\mathrm{Main}(u)$ comes from the divisor bound (Sl2), in which β enters through $\log\Omega = \gamma_0 + \log(\pi/v) - \beta$, and the mean of $\log w_{u,j,\rho}$ comes from the counting bounds (Sl5). The main term of an inert prime sum contains the value $\eta(\chi_C)$ at the quadratic character whose kernel is Φ. For a host this character is the Legendre symbol χ_d, so that $\eta(\chi_C) = \beta$ (proof of Theorem 8.4).

The meaning of the entries in the construction beneath (TCS) is described in Appendix A; we state it here as a reading, since no statement of this paper uses it. A word indexes a hermitian lattice inside an ambient lattice. The entries d, μ, $\mu^{\max}$ are the rank, the slope and the maximal slope of the first, μ' and $\mu'^{\max}$ those of the second, and $\mu^\vee$ is the slope of a dual lattice; slopes are those of Arakelov geometry [46]. The integer m is the number of theta lifts in a chain, and Θ_u is the total gain in divisibility of the m lifts at the inert primes. In this reading the divisor bound (Sl2) bounds the norms of the lifts ([4], item 9 of the list on pp. 3–4): each of the m lifts contributes one copy of $a_\tau \log a_\tau - a_\tau - a_\tau \log\Omega$, which is why this term carries the factor m, and $\log\Omega$ has the role of the logarithm of a CM period. The dual-slope inequality (Sl9) is the slope inequality, and (Sl8) says that the divisibility dominates the prime sum.

Lemma 4.3. *Let $\mathfrak{S}$ be a slope datum satisfying (Sl1)–(Sl7) and (Sl9)–(Sl11) for (c, X''), and put*

$$K_0 = c_{\mathrm{Count}} A\Big(L_a + \frac{m^2 L_a}{k}\Big) + c_{\mathrm{Div}} n_{\mathcal{T}} a L_a + c_{\mathrm{Div}} m(B_0 + N_c) x^{e_{\mathrm{Div}}\varepsilon} + m c_E B_0 + c_{\mathrm{Slope}} n_{\mathcal{T}} a L_a \Big(1 + \frac{m}{k}\Big).$$

Then for every $u \in U$,

$$\Theta_u \le \mathrm{Main}(u) + \log|\mathcal{W}(u)| + K_0 + \overline{\mathrm{Lo}}_u'' + \frac{1}{1-\bar\delta}\Big(4\log\big(1 + x^{c_{\mathrm{HH}}}\big)\big(K_3 + \overline{\mathrm{Lo}}_u'' + b\big) + \big(\bar\eta + 4\sqrt{\bar e}\big)x^{c_{\mathrm{HH}}} b\Big).$$

Proof. Fix u, and let $\mathrm{Cnt}(u)$ and $K_5(u)$ be as in Lemma 4.4.

Step 1: $K_5(u) = \mathrm{Main}(u) + K_0$. Since $w_{u,j,\rho} > 0$, $v > 0$ and $f > 0$ by (Sl1),

$$A\,\mathrm{Cnt}(u) = \frac{A}{f}\sum_{j,\rho} \log w_{u,j,\rho} + mA(\log v - \log\pi) + mAl_g - c_{\mathrm{Count}} A\Big(L_a + \frac{m^2 L_a}{k}\Big),$$

and, expanding φ and $\log\Omega$,

$$\sum_\tau m(\varphi(a_\tau) - a_\tau\log\Omega) = m\sum_\tau a_\tau(\log a_\tau - 1 - \gamma_0 + \beta) - mA(\log\pi - \log v).$$

Hence $\sum_\tau m(\varphi(a_\tau) - a_\tau\log\Omega) - A\mathrm{Cnt}(u) + mAl_g = \mathrm{Main}(u) + c_{\mathrm{Count}}A(L_a + m^2L_a/k)$, which is the identity of Step 1 after the remaining terms of $K_5(u)$ are added.

Step 2. By Lemma 4.4 and Step 1,

$$\Theta_u \le \mathrm{Main}(u) + K_0 + \overline{\mathrm{Lo}}_u'' + (s_u - \mu_u^\vee).$$

Step 3: the dual-slope inequality. By (Sl11) and (Sl1), $0 \le \delta_u \le \bar\delta < 1$, $\eta_u \ge 0$ and $0 \le e_u \le \bar e \le \frac14$; by (Sl10), $s_u - \mu^\vee_{\min,u} \le \mathrm{HH}_u$ and $\mathrm{HH}_u > 0$. So the premises of (Sl9) hold, and

$$s_u - \mu_u^\vee \le \log|\mathcal{W}(u)| + \frac{1}{1-\delta_u}\Big(4\phi_u\log\Big(1 + \frac{\mathrm{HH}_u}{\phi_u}\Big) + \big(\eta_u + 4\sqrt{e_u}\big)\mathrm{HH}_u\Big).$$

Step 4: the bracket. The spread is non-negative: by (Sl1) every $\mu_{u,\omega} - h_{u,\omega}$ is at most $\max_{\omega'}(\mu^{\max}_{u,\omega'} - h_{u,\omega'})$, and so is their weighted mean s_u. By Lemma 4.5, $\phi_u \le K_3 + \overline{\mathrm{Lo}}_u''$; by

(Sl10), $0 < \mathrm{HH}_u \le x^{c_{\mathrm{HH}}} b$, where $x^{c_{\mathrm{HH}}} > 0$ and $b > 0$ by (Sl1). So Lemma 4.6, with $u = \phi_u$, $H = \mathrm{HH}_u$, $X = x^{c_{\mathrm{HH}}}$ and $U = K_3 + \overline{\mathrm{Lo}}''_u$, gives

$$\phi_u \log\Big(1 + \frac{\mathrm{HH}_u}{\phi_u}\Big) \le \log(1 + x^{c_{\mathrm{HH}}})(K_3 + \overline{\mathrm{Lo}}''_u + b).$$

Also $(\eta_u + 4\sqrt{e_u})\mathrm{HH}_u \le (\bar\eta + 4\sqrt{\bar e})x^{c_{\mathrm{HH}}} b$ by (Sl10) and (Sl11). The bracket of Step 3 is non-negative, and $(1-\delta_u)^{-1} \le (1-\bar\delta)^{-1}$. Inserting these bounds into Step 3, and Step 3 into Step 2, gives the claim. □

Lemma 4.4. *Let $\mathfrak{S}$ be a slope datum, c a constant vector and $X'' \in \mathbb{R}$, and suppose that $\mathfrak{S}$ satisfies (Sl1), (Sl2), (Sl3), (Sl5), (Sl6) and (Sl7) for (c, X''). For $u \in U$ put*

$$\mathrm{Cnt}(u) = \frac{1}{f}\sum_{j=1}^{m}\sum_{\rho=1}^{f} \log\frac{w_{u,j,\rho}\, v}{\pi} + m l_g - c_{\mathrm{Count}}\Big(L_a + \frac{m^2 L_a}{k}\Big), \qquad \overline{\mathrm{Lo}}''_u = \frac{\sum_\omega d_{u,\omega}\mathrm{Lo}''_{u,\omega}}{\sum_\omega d_{u,\omega}},$$

$$K_5(u) = c_{\mathrm{Slope}} n_{\mathcal{T}} a L_a\Big(1 + \frac{m}{k}\Big) + m(A l_g + c_E B_0) + \sum_\tau m(\varphi(a_\tau) - a_\tau \log\Omega) \\ - A\,\mathrm{Cnt}(u) + c_{\mathrm{Div}} n_{\mathcal{T}} a L_a + c_{\mathrm{Div}} m(B_0 + N_c) x^{e_{\mathrm{Div}}\varepsilon}.$$

Then for every $u \in U$,

$$\Theta_u + \mu_u^\vee - K_5(u) - \overline{\mathrm{Lo}}''_u \le s_u.$$

Proof. Fix u and a word $\omega \in \mathcal{W}(u)$. The second inequality of (Sl7) gives

$$\mu_{u,\omega} \ge \mu'_{u,\omega} + \Theta_u + F_u - E_{u,\omega} - \mathrm{Lo}''_{u,\omega},$$

and (Sl2), using $\sum_\tau a_\tau \log T_{u,\omega} = A \log T_{u,\omega}$, gives

$$-h_{u,\omega} \ge \mu_u^\vee - F_u - \sum_\tau m(\varphi(a_\tau) - a_\tau\log\Omega) + A\log T_{u,\omega} - c_{\mathrm{Div}} n_{\mathcal{T}} a L_a - c_{\mathrm{Div}} m(B_0 + N_c) x^{e_{\mathrm{Div}}\varepsilon}.$$

Adding, the terms F_u cancel. Now $A \ge 0$ by (Sl1), so the lower bound $\log T_{u,\omega} \ge \mathrm{Cnt}(u)$ of (Sl5) gives $A\log T_{u,\omega} \ge A\mathrm{Cnt}(u)$; (Sl3) gives $E_{u,\omega} \le m(A l_g + c_E B_0)$; and (Sl6) gives $\mu'_{u,\omega} \ge -c_{\mathrm{Slope}} n_{\mathcal{T}} a L_a (1 + m/k)$. Together,

$$\mu_{u,\omega} - h_{u,\omega} \ge \Theta_u + \mu_u^\vee - K_5(u) - \mathrm{Lo}''_{u,\omega}.$$

The weights $d_{u,\omega}$ are positive by (Sl1), and $\mathcal{W}(u)$ is non-empty. Taking the weighted mean over $\omega \in \mathcal{W}(u)$ gives the claim, since the left side of the last display has mean s_u and the right side has mean $\Theta_u + \mu_u^\vee - K_5(u) - \overline{\mathrm{Lo}}''_u$. □

Lemma 4.5. *Let $\mathfrak{S}$ be a slope datum satisfying (Sl1), (Sl3), (Sl4), (Sl5), (Sl6) and (Sl7) for (c, X''), and put*

$$K_3 = c_{\mathrm{Slope}} n_{\mathcal{T}} a L_a\Big(2 + \frac{m}{k}\Big) + V + c_{\mathrm{Sl}} m B_0.$$

Then for all $u \in U$ and $\omega, \omega' \in \mathcal{W}(u)$,

$$(E_{u,\omega'} + h_{u,\omega'}) - (E_{u,\omega} + h_{u,\omega}) \le V,$$

and for every $u \in U$, with $\overline{\mathrm{Lo}}''_u = \sum_\omega d_{u,\omega}\mathrm{Lo}''_{u,\omega} / \sum_\omega d_{u,\omega}$,

$$\phi_u \le K_3 + \overline{\mathrm{Lo}}''_u.$$

Proof. First inequality. By (Sl3), applied to ω and to ω', $E_{u,\omega'} - E_{u,\omega} \le 2mc_E B_0$. By (Sl4),

$$h_{u,\omega'} - h_{u,\omega} \le c_H\big(n_{\mathcal{T}} a L_a + mB_0\big) + A\big(\log T_{u,\omega} - \log T_{u,\omega'}\big),$$

and since $A \ge 0$ by (Sl1), the last term is at most $c_{\mathrm{Count}} A \log k$ by (Sl5). Adding gives the bound V.

Second inequality. Fix ω. For every ω', the first part of (Sl7) at ω and the second part at ω' give

$$\mu^{\max}_{u,\omega} - \mu_{u,\omega'} \le \big(\mu'^{\max}_{u,\omega} - \mu'_{u,\omega'}\big) + \big(E_{u,\omega'} - E_{u,\omega}\big) + c_{\mathrm{Sl}} mB_0 + \mathrm{Lo}''_{u,\omega'},$$

the terms $\Theta_u + F_u$ cancelling. By (Sl6), $\mu'^{\max}_{u,\omega} - \mu'_{u,\omega'} \le c_{\mathrm{Slope}} n_{\mathcal{T}} a L_a (2 + m/k)$; with the first part this gives

$$\big(\mu^{\max}_{u,\omega} - h_{u,\omega}\big) - K_3 - \mathrm{Lo}''_{u,\omega'} \le \mu_{u,\omega'} - h_{u,\omega'}.$$

The weights $d_{u,\omega'}$ are positive by (Sl1), and taking the weighted mean over ω' gives $\mu^{\max}_{u,\omega} - h_{u,\omega} - K_3 - \overline{\mathrm{Lo}}''_u \le s_u$. Taking the maximum over ω and subtracting s_u gives $\phi_u \le K_3 + \overline{\mathrm{Lo}}''_u$. □

Lemma 4.6. *Let u, H, X, b, U be real numbers with $0 \le u \le U$, $0 < H \le Xb$, $b > 0$ and $X > 0$. Then*

$$u \log\Big(1 + \frac{H}{u}\Big) \le \log(1 + X)\,(U + b),$$

where the left-hand side is read as 0 when $u = 0$.

Proof. Put $\lambda = \log(1+X) > 0$. If $u = 0$ the left side is 0 and the right side is $\lambda(U+b) \ge \lambda b > 0$. Let $u > 0$. For $y, z > 0$ one has $\log(y/z) \le y/z - 1$; with $y = 1 + H/u$ and $z = 1 + X$ this gives

$$\log\Big(1 + \frac{H}{u}\Big) \le \lambda + \frac{1 + H/u}{1 + X} - 1, \qquad \text{so} \qquad u \log\Big(1 + \frac{H}{u}\Big) \le u\lambda - \frac{uX}{1 + X} + \frac{H}{1 + X}.$$

Here $-uX/(1+X) \le 0$ and $u\lambda \le U\lambda$. Since $H \le Xb$, $H/(1+X) \le b\,X/(1+X) = b(1 - (1+X)^{-1}) \le b\lambda$, the last step by $1 - z^{-1} \le \log z$ for $z = 1 + X$. Adding gives the claim. □

Lemma 4.7. *Let $\mathfrak{S}$ be a slope datum that satisfies the size conditions for (c, X'', e_K, C_K), at a scale that is large for $(\varepsilon, C_K, c_{\mathrm{HH}}, e_K)$. Then for every $u \in U$,*

$$\mathrm{Err}(u) \le x^{(e_{\mathrm{Div}} + e_{\mathrm{Loss}} + 3)\varepsilon}\Big(n_{\mathcal{T}}\, a\Big(1 + \frac{m^2}{k}\Big) + m(B_0 + N_c)\Big).$$

Proof. We bound the terms of $\mathrm{Err}(u)$ one by one. Write $y = x^{\varepsilon} \ge 1$. Each of the finitely many terms of $\mathrm{Err}(u)$ is bounded by a product of at most three factors y, $x^{e_{\mathrm{Div}}\varepsilon}$, $x^{e_{\mathrm{Loss}}\varepsilon}$, a constant at most $200(1 + C_K)^2(1 + c_{\mathrm{HH}})(1 + e_K) \le y$, and one of $n_{\mathcal{T}} a$, $n_{\mathcal{T}} a m^2/k$, mB_0, mN_c. We use the following bounds:

- $A \le 2n_{\mathcal{T}} a$ (from $a_\tau \le 2a$);
- $\log|\mathcal{W}(u)| \le k \le a$ (from (Er3));
- $\log k \le (1 + e_K)y$ and $\log(1 + x^{c_{\mathrm{HH}}}) \le (1 + c_{\mathrm{HH}})y$ (from $k \le x^{e_K}$ and $\log x \le y$);
- $(1 - \bar{\delta})^{-1} \le 2$;
- $L, L_a \le 2y$;

- $m \le k$ and $mb \le a$ (from $xm \le k$, $m \le x$ and $xb \le a$);
- $kb' \le a$ and $X'' \le a$;
- the constants are at most C_K;
- the last clause of (Er4), for the term $(\bar{\eta} + 4\sqrt{\bar{e}})x^{c_{\mathrm{HH}}}b$.

Collecting the terms gives the claim.

□

Remark (Stepanov's count and the balance). Proposition 4.2 has the structure of Stepanov's proof of the Riemann hypothesis for curves over finite fields [247], in the form of Bombieri [36]. There an auxiliary function vanishes to order at least M at every rational point of a curve over $\mathbb{F}_q$. The count of its zeros against its poles bounds the number of rational points from above by q plus two error terms, one decreasing and one increasing in M, and the two balance when M is about $\sqrt{q}$. Alpöge describes the present argument as a descendant of this proof in which multiplicity is hard to find over the integers and a chain of theta lifts supplies it, the chain having length about $\sqrt{X}$ for the von Mangoldt sum of length X ([4], p. 1).

The accounting matches this description term by term, as a reading: the statements above are about real numbers and do not depend on it. The count of zeros against poles is the dual-slope inequality (Sl9), a slope inequality in the sense of Bost [46], [47]. The order of the poles is the archimedean term of the divisor bound (Sl2). The order of vanishing is the integer m, the multiplicity. The vanishing at the rational points is the loss condition (Sl8), divisibility at the inert primes, to which Section 5 is devoted.

The balance is a fact about the error term. By Lemma 4.7 and the bounds (Ct5) for the two budgets, the error is at most $x^{O(\varepsilon)}$ times

$$|T|\,a\Big(1+\frac{m^2}{k}\Big)+m\Big(2|A|\,m\,b\,x^{\varepsilon}+\frac{x^{2\varepsilon}|T_2|\,k\,b}{|A|^2}\Big),$$

and the averaging step of Section 3 divides it by $|T|\,m\,b\,|T_2|/(2|A|)$. Since $a \le 2xb$, $|T| \ge \frac{1}{2}x^{-\varepsilon/8}|A|$ and $|T_2| \ge x^{-\varepsilon}|A|$, the four terms then have, up to factors $x^{O(\varepsilon)}$, the shapes

$$\frac{x}{m}, \qquad \frac{xm}{k}, \qquad m, \qquad \frac{k}{|A|\,|T|}.$$

The first does not grow with the number of lifts, and the third is the cost that each lift adds: the budgets B_0 and N_c enter the divisor bound (Sl2) with the factor m. These two balance at $m = \sqrt{x}$, which is the balance $M \approx \sqrt{q}$ with the number of lifts in the role of the order of vanishing. The second shape is at most 1 because $xm \le k$, and the fourth is at most $x^{1/2+O(\varepsilon)}$ because $k \le |A|^2 x^{1/2-\varepsilon}$. The two conditions on k are compatible only if $|A|^2 \ge x^{1/2+\varepsilon}m$, so the group A must have at least about $\sqrt{x}$ elements. The accounting itself uses only

$$x^{1/2-\varepsilon} \le m \le x^{1/2+\varepsilon/2}, \qquad xm \le k \le |A|^2 x^{1/2-\varepsilon}$$

(conditions (Ex2) of Proposition 3.3 and (Sp8) of Definition 8.2). The hosts supplied in Part II come with rank data that have the rank sizes of Definition 11.2:

$$x^{2+8\varepsilon} \le |A| \le 2x^{2+8\varepsilon}, \qquad \sqrt{x\delta_1} \le m \le 2\sqrt{x\delta_1} \ \text{ with } \delta_1 = x^{-\varepsilon/8}, \qquad x^{9/2+10\varepsilon} \le k \le 2x^{9/2+10\varepsilon}.$$

Thus m is $x^{1/2-\varepsilon/16}$ up to a factor 2, the group A has about $x^{2+8\varepsilon}$ elements, far more than the $\sqrt{x}$ that the balance needs, and the bound that results is $G_{\mathfrak{L}}(x) \le x^{1/2+(e_D+e_L+7)\varepsilon}$ (Proposition 6.5).

5 Divisibility at inert primes: the loss bound

The accounting of Section 4 starts from the loss condition (Sl8): the prime sum S_u exceeds the lattice term Θ_u by at most a small loss. This section proves the bound behind it, Theorem 5.6. For a loss datum $\mathfrak{M}$ at a large scale x that satisfies the conditions (Lm1)–(Lm7),

$$S(\mathfrak{M}) - \Theta(\mathfrak{M}) \le (3C^+ + 13)\,|T|\,m\,b \qquad \text{and} \qquad I(\mathfrak{M}) \le |T|\,m\,b, \qquad b = \exp(x^{\varepsilon}).$$

Its only use is in Corollary 7.11 in Section 7, which applies it to the members of a parametrised host family and obtains (Sl8) with the loss exponent $e_{\mathrm{Loss}} = 0$.

In Stepanov's method the auxiliary function vanishes at the rational points. Here the place of this vanishing is taken by divisibility at the primes inert in $\mathbb{Q}(\sqrt{-d})$: in Alpöge's account, a derivative of order a of a theta function on a CM curve, normalised by the period, is divisible by an inert prime p to the power $a/p + a/p^3 + \cdots$, up to $O(\log a)$, and the lift of multiplicity m inherits a sufficient lower bound ([4], items 2 and 10 of the list on pp. 3–4). The statements of this section contain the same exponent. An inert prime p contributes to the inert prime sum the terms $(a/p^e - w)^+$ with e odd, at most $a\gamma_p$ in total, where $\gamma_p = p/(p^2-1) = \sum_{e\ \mathrm{odd}} p^{-e}$, and the gain of a served orbit starts from γ_p times the sum of the reduced heights of its tops (Definition 5.3). We read the gains as the divisibility of the theta kernels of the lifts. This reading belongs to the construction beneath (TCS), described in Appendix A, and is not used: a loss datum is a record of numbers, and its gains are defined by formulas.

What the theorem proves is a summation over the primes. Its local hypothesis is the linking condition (Lm3), which says, at each served inert prime, that the contribution of the prime to the prime sum is at most its gain, up to a local loss and a floor loss. For the families of this paper the linking condition is proved in Part IV (Theorem 18.4). The theorem sums it over all inert primes and bounds what the local inequality does not cover: the primes that are not served, the tops that are not served at a served prime, and the sums of the local and floor losses. The proof splits the difference $S - \Theta$ prime by prime (Proposition 5.9) and bounds the buckets that arise (Lemma 5.8, Proposition 5.7). Of the remaining lemmas, Lemma 5.10 shows that the primes of the first kind are sparse, and the others check sizes and identifications; they are routine.

Definition 5.1. Let $\mathfrak{A}$ be an arithmetic datum on a finite abelian group H and $T \subseteq H$ a set, whose elements are called **tops**. We write the elements of Φ as pairs $(\epsilon', h) \in \mathbb{Z}/2 \times H$, standing for $((0,\epsilon'),h)$; the places $(0,t)$ with $t \in T$ are the **top places**. Let p be a prime that is inert for $\mathfrak{A}$.

Frobenius translation. Since $\mathrm{Fr}_p \notin \Phi$, the element $f_p = c_\infty + \mathrm{Fr}_p$ lies in Φ; write $f_p = (s_p, h_p)$. The prime p is **sign-changing** if $s_p = 1$. Translation by f_p permutes Φ. Its order, which is the order of f_p in Γ_H, is denoted n_p (so n_p is the order of h_p if $s_p = 0$, and 2 times the order of $2h_p$ if $s_p = 1$). For a top t and an integer $e \ge 1$, the **e-th partner** of t at p is the place

$$z_p(t,e) = (0,t) + e\,f_p = (e\,s_p,\ t + e\,h_p) \in \Phi.$$

For odd e one has $z_p(t,e) = \sigma_t(e\,\mathrm{Fr}_p)$, the place whose weight occurs in the inert prime sum A_{in}.

Orbits. Put $g_p = h_p$ if $s_p = 0$ and $g_p = 2h_p$ if $s_p = 1$. The **orbits of p on H** are the cosets of the subgroup $\langle g_p \rangle$; they are the intersections of the orbits of translation by f_p on Φ with the row $\{0\} \times H$. An **orbit labelling** at p is a map $\rho_p \colon H \to H$ with $\rho_p(t + g_p) = \rho_p(t)$ and $t \in \rho_p(t) + \mathbb{N} g_p$ for all t. Such a map chooses one element in each orbit, the **representative**. Given an orbit labelling, let $R_p = \rho_p(H)$ be the set of representatives, and for $v \in R_p$ let

$$T_p(v) = \{t \in T : \rho_p(t) = v\}, \qquad N_p(v) = |T_p(v)|,$$
$$P_p(v) = \{\, i \in \{0,1,\dots,\mathrm{ord}(h_p) - 1\} : v + i\,h_p \in T \,\}$$

be the tops on the orbit of v, their number, and the **positions** of tops along the orbit of v under translation by h_p. A top t lies at an **even position** if $t = \rho_p(t) + i\,h_p$ for some even i with $0 \le i < n_p$, and at an **odd position** otherwise. (If p is sign-changing, every top lies at an even position.)

First hits. Let $s \neq s'$ be tops and $e \ge 1$. The exponent e is the **first hit** of s' from s if $z_p(s,e) = (0,s')$ and $z_p(s,e') \neq (0,s')$ for $0 < e' < e$. An exponent e is **excluded** for the top s if e is odd and is the first hit from s of some top $s' \neq s$.

Definition 5.2. Let $\mathfrak{A}$ be an arithmetic datum on H with a set of tops $T \subseteq H$ and an orbit labelling ρ_p at every inert prime p. An **orbit selection** for $(\mathfrak{A}, T, (\rho_p))$ consists of real numbers $x > 1$, $0 < \varepsilon < \frac{1}{52}$, X'', p_0 and $\rho \ge 0$, an integer $m \ge 1$, and for every inert prime p a set $G_p \subseteq R_p$ of **good representatives** and numbers $\kappa_p(v) \in [0,1]$ $(v \in G_p)$, the **orbit constants**. Put $X' = \exp(x^{\varepsilon/3})$. The thresholds have the following roles: X' separates the small from the large inert primes, X'' is the cut-off of the gains, p_0 separates the two regimes of the gains (below p_0 the price of an orbit carries an extra rank factor), and ρ is the relative width of the bands around X'.

Candidates. An inert prime p is a **candidate** if p is not sign-changing, $p < X'$, the order of h_p is at least $x^{\varepsilon/4}$, and, when $p \ge p_0$, no power p^i with $i \ge 1$ satisfies $|p^i - X'| \le \rho X'$ (that is, p lies in no **band** around X'). The **orbit mass** at p is $\Sigma_p = m \sum_{v \in G_p} \kappa_p(v)$.

Served orbits and served tops. The orbits of a candidate of small orbit mass are served: put

$$V_p = \begin{cases} G_p & \text{if } p \text{ is a candidate and } \Sigma_p \le x^{-\varepsilon}, \\ \emptyset & \text{otherwise.} \end{cases}$$

The **served tops** at p are $T_p^{\mathrm{s}} = T$ if p is sign-changing or $p \ge X'$, and $T_p^{\mathrm{s}} = \{t \in T : \rho_p(t) \in V_p\}$ otherwise.

Dropped primes. An inert prime p is **dropped** if one of the following holds:

(Dr1) $p > X''$;

(Dr2) p is not sign-changing, and $h_p \neq 0$ has odd order;

(Dr3) p is not sign-changing, $p < X'(1+\rho)$, and either the order of h_p is less than $x^{\varepsilon/4}$, or $p \ge p_0$ and $|p^i - X'| \le \rho X'$ for some $i \ge 1$, or p is a candidate with $\Sigma_p > x^{-\varepsilon}$.

Classes. An inert prime p is **served** if it is not dropped. A dropped prime is **high** if $p > X''$; of the **second kind** if $p \le X''$ and p is a candidate with $\Sigma_p > x^{-\varepsilon}$; and of the **first kind** otherwise. Thus every inert prime is in exactly one of the four classes served, first kind, second kind, high.

Definition 5.3. Fix an arithmetic datum on H with tops T, orbit labellings (ρ_p) and an orbit selection. A **gain datum** on these consists of

- integers a_t, b_t $(t \in T)$, the **heights** and **depths** of the tops;
- functions $w_j \colon \Phi \to \mathbb{R}$ $(1 \le j \le m)$, the **weights**;
- real numbers $\varsigma_1, \dots, \varsigma_m$ (the **shifts**), c_G, $a > 0$, b', η, w'', N and an integer $r_G \ge 1$;
- for every inert prime p, integers $\epsilon_p, D_p \ge 0$ and real numbers $t_p(v)$ $(v \in V_p)$, the **thresholds**.

We write $\log_p y = \log y / \log p$, $y^+ = \max(y, 0)$ and $\lceil y \rceil^+ = \max(\lceil y \rceil, 0)$. A series over the odd integers $e \geq 1$ that does not converge is given the value 0. The series below converge when all capacities are positive.

Reduced heights, capacities, losses. For a top s, an inert prime p, $e \geq 1$ and $1 \leq j \leq m$ put

$$\alpha_s = a_s - 2c_G, \qquad c_j(s,e) = w_j(z_p(s,e)) - \varsigma_j + c_G, \qquad \gamma_p = \frac{p}{p^2-1} = \sum_{e \text{ odd}} p^{-e},$$

and, for $v \in R_p$, let the **orbit loss** be $\delta_p(v) = 5\, n_p\, N^2 \log_p(2a) + 4\, N_p(v) \big(\log_p(2a)\big)^2$.

The two gains of an orbit. The *plain gain* of the orbit of $v \in R_p$ adds, over the tops of the orbit and the odd exponents that are not excluded, the excess of the reduced height over the capacity of the partner place, as in the inert prime sum. The *served gain* counts the tops at even positions in full, and the tops at odd positions as the plain gain does but without exclusions (when all capacities are non-negative, by $y - \min(y,c) = (y-c)^+$):

$$g_j^0(v) = \sum_{s \in T_p(v)} \sum_{\substack{e \text{ odd} \\ e \text{ not excluded for } s}} \Big(\frac{\alpha_s}{p^e} - c_j(s,e) \Big)^+ - \delta_p(v),$$

$$g_j^U(v) = \gamma_p \sum_{s \in T_p(v)} \alpha_s - \sum_{\substack{s \in T_p(v) \\ \text{odd position}}} \sum_{e \text{ odd}} \min\Big(\frac{\alpha_s}{p^e},\, c_j(s,e) \Big) - \delta_p(v).$$

The served gain has a **price**

$$\mathrm{Pr}_j(v) = \begin{cases} \displaystyle\sum_{\substack{s \in T_p(v) \\ \text{odd position}}} \big(w_j(0,s) - \varsigma_j - b_s \big) + n_p\big(w'' + \eta + c_G + N \big) & \text{if } p \geq p_0, \\ r_G\Big(N_p(v)\, (2b' + c_G) + n_p\big(w'' + \eta + c_G + N \big) \Big) & \text{if } p < p_0. \end{cases}$$

Gains. The gain at p takes, on each served orbit, the best of the served gain after its price, the plain gain and 0, and on every other orbit the plain gain if it is positive:

$$\mathrm{gain}_j(p) = \sum_{v \in V_p} \max\big(g_j^U(v) - \mathrm{Pr}_j(v),\ g_j^0(v),\ 0 \big) + \sum_{v \in R_p \setminus V_p} \max\big(g_j^0(v), 0 \big),$$

except that $\mathrm{gain}_j(p) = 0$ if $p > X''$, or if p is not sign-changing and $h_p = 0$. The **gain of the member** at p is $\mathrm{gain}_j^{\mathrm{D}}(p) = 0$ if p is dropped and $\mathrm{gain}_j^{\mathrm{D}}(p) = \mathrm{gain}_j(p)$ otherwise.

Correction. The correction pays for the served orbits:

$$\tau_p = \begin{cases} 0 & \text{if } V_p = \emptyset, \\ \epsilon_p + m \displaystyle\sum_{v \in V_p} \big(\lceil t_p(v) \rceil^+ + D_p + 1 \big) & \text{otherwise.} \end{cases}$$

Companion term. For $v \in V_p$ the **surplus** of the served gain over the plain gain is the integer $S_p(v) = \big\lceil \sum_{j=1}^m \big(g_j^U(v) - \mathrm{Pr}_j(v) - \max(g_j^0(v), 0) \big)^+ \big\rceil \geq 0$, and

$$\mathrm{io}_p = \begin{cases} 0 & \text{if } V_p = \emptyset, \\ 2\, \Sigma_p \max_{v \in V_p} S_p(v) & \text{otherwise.} \end{cases}$$

Definition 5.4. A tuple of **window parameters** is a tuple of real numbers $x, \varepsilon, n, k, N, t, L_a, r_G, S_c, \eta, \rho, \vartheta, K_0, C, \beta, C_{\mathrm{ord}}, Y, s_0, b_o$ and integers $e_2, p_0, d \geq 0$. Its **derived quantities** are

$$b = \exp(x^{\varepsilon}), \quad a = xb, \quad X' = \exp(x^{\varepsilon/3}), \quad b' = a/X', \quad w'' = \frac{b'}{xn}, \quad X'' = \frac{a}{w''} = xnX', \quad f = 2n,$$
$$\lambda = \log x, \quad \delta_1 = x^{-\varepsilon/8}, \quad \theta = x^{-\varepsilon}, \quad S_k = S_c + k + \eta, \quad Q_1 = Cr_G b'\lambda,$$
$$Q_2 = C\Big(\frac{b'}{x} + \eta + S_c + \frac{(w''+t)\lambda}{\delta_1}\Big) + b_o, \qquad J_o = \lfloor x^{\varepsilon/4} \rfloor, \qquad I_b = \lfloor x^{\varepsilon/3} \rfloor.$$

For real numbers e, K the tuple is **admissible for** (e, K) if the following hold.

Scale. $0 < \varepsilon \leq 1$, $e \geq 0$, $K \geq 1$.

Ranks. $x^2 \leq 2^{e_2} \leq n \leq x^e$; $x^2 \leq k \leq x^e$; $0 \leq N \leq x^e$; $0 \leq t \leq x^e$; $0 \leq r_G \leq x^e$.

Thresholds. $x^4 + 1 \leq p_0 \leq x^e$; $x^2 \leq K_0$; $x^2 \leq d$; $1 \leq Y \leq 2X'$.

Small quantities. $0 \leq L_a \leq x^{2\varepsilon}$; $0 \leq \rho \leq x^{-2}$; $0 \leq \vartheta \leq x^{-1}$; $0 \leq s_0 \leq x^{-2}$; $0 \leq \beta \leq Kx^{-2}$.

Moderate quantities. $0 \leq S_c \leq \exp(x^{\varepsilon/4})$; $0 \leq \eta \leq \exp(x^{\varepsilon/4})$; $0 \leq b_o \leq \exp(x^{\varepsilon}/2)$; $0 \leq C \leq K$; $0 \leq C_{\mathrm{ord}} \leq K$.

The scale is **large for** (e, K) if $x \geq 1$, $100Kx^{4\varepsilon} \leq x$, $100Kx^{4e+4} \leq \exp(x^{\varepsilon/3})$ and

$$100Kx^{4e+4}\exp(x^{\varepsilon/3} + x^{\varepsilon/4}) \leq \exp(x^{\varepsilon}/2).$$

Definition 5.5. A **loss datum** $\mathfrak{M}$ consists of the following.

- An arithmetic datum $\mathfrak{A} = (q, d, (I_p), (\mathrm{Fr}_p))$ on a finite abelian group H with $I_d = \{0, ((1,0),0)\}$; a set of tops $T \subseteq H$; an orbit labelling ρ_p at every inert prime p; an orbit selection $(x, \varepsilon, X'', p_0^*, \rho, m, (G_p), (\kappa_p))$; and a gain datum on these, with heights a_t $(t \in T)$, weights $w_j \colon \Phi \to \mathbb{R}$ $(1 \leq j \leq m)$ and parameters $c_G, a, b', \eta, w'', N, r_G$.
- The **size parameters**: integers $k, t_R, e_2 \geq 0$ and $o \geq 1$, and real numbers L_a, S_c.
- A **place record**: a number field K, a finite set S of finite places of K, real numbers $\ell(w)$ for the finite places w of K such that $\sum_{w|p} \ell(w) = \log p$ for every prime p (the sum over all finite places of K above p), and real numbers $g_j(w)$ $(1 \leq j \leq m)$, $\tau(w)$ and $\mathrm{io}(w)$ for $w \in S$. We write $p(w)$ for the residue characteristic of w.

The parameters are tied as follows, where $n = |H|$, $b = \exp(x^{\varepsilon})$ and $X' = \exp(x^{\varepsilon/3})$:

$$a = xb, \quad b' = \frac{a}{X'}, \quad w'' = \frac{b'}{xn}, \quad X'' = xnX', \quad N = k + m(t_R + 1),$$
$$c_G = 4S_c, \quad p_0^* = k^2x^{26}, \quad r_G = \Big\lceil \frac{32 + 64\varepsilon}{\varepsilon} \Big\rceil + 1.$$

Losses. Put $\lambda = \log x$, $\delta_1 = x^{-\varepsilon/8}$, $f = 2n$, and $L_1(p) = \lceil \log_p(k^2x^{78}) \rceil + 1$ for a prime p. Let C and b_o be real numbers, the **loss constants**. The **floor loss** and the **local loss** at a prime p for (C, b_o) are

$$\mathrm{fl}_C(p) = CfN^2 \log_p a + Cf(\log_p a)^2,$$
$$\mathrm{lo}_{C,b_o}(p) = C(S_c + k + \eta)\log_p(2a) + \mathrm{pr}_{C,b_o}(p) + \mathrm{or}_C(p),$$

where $\mathrm{pr}_{C,b_o}(p) = \mathrm{or}_C(p) = 0$ unless p is inert and not sign-changing, and in that case

$$\mathrm{pr}_{C,b_o}(p) = \begin{cases} Cr_G b'\lambda & \text{if } p < p_0^*, \\ C\big(\dfrac{b'}{x} + \eta + S_c + \dfrac{(w''+t_R)\lambda}{\delta_1}\big) + b_o & \text{if } p_0^* \le p < X', \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathrm{or}_C(p) = \begin{cases} C\lambda\big(\dfrac{L_a a}{pk} + \begin{cases} a\,p^{-L_1(p)} & \text{if } p < p_0^* \\ a/p^2 & \text{if } p \ge p_0^* \end{cases}\big) & \text{if } p < X', \\ 0 & \text{otherwise.} \end{cases}$$

For an inert prime p, $t \in T$ and $1 \le j \le m$ put $A_p(a_t, t, w_j) = \sum_{e \ge 1 \text{ odd}} \big(a_t/p^e - w_j(z_p(t,e))\big)^+$, a series of non-negative terms. When $w_j \ge w''$ on Φ, only the finitely many e with $p^e < a_t/w''$ contribute.

Prime sum, lattice term, companion sum. Let $P \ge 1$ be an integer, the **cut-off**. Put

$$S(\mathfrak{M}) = \sum_{t \in T} \sum_{j=1}^{m} A_{\mathrm{in}}(a_t, t, w_j), \qquad \Theta(\mathfrak{M}) = 2 \sum_{w \in S} \ell(w) \Big(\sum_{j=1}^{m} g_j(w) - \tau(w) \Big),$$
$$I(\mathfrak{M}) = \sum_{w \in S} \ell(w)\, \mathrm{io}(w),$$

where the inert prime sums are formed with $(\mathfrak{A}, P)$ and each w_j is extended to Γ_H by 0 outside Φ (only its values on Φ enter).

Deficit data. For a cut-off P put $B = \max(P, X'')$ and let $\mathcal{P}$ be the set of inert primes $p \le B$. For $1 \le j \le m$, $t \in T$ and $p \in \mathcal{P}$ let $A(j,t,p) = \sum_{1 \le i \le P,\ i \text{ odd}} \big(a_t/p^i - w_j(z_p(t,i))\big)^+$ if $p \le P$ and $A(j,t,p) = 0$ if $p > P$, and let

$$d(j,t) = \log d \sum_{i=1}^{P} \Big(\frac{a_t}{d^i} - w_j(\sigma_t(g_i)) \Big)^+,$$

where g_i is the element of $i\,\mathrm{Fr}_d + I_d$ that is not in Φ (exactly one of its two elements lies outside Φ, since $((1,0),0) \notin \Phi$). Let $g(j,p) = \mathrm{gain}_j^{\mathrm{D}}(p)$ and $\tau(p) = \tau_p$, assign to each $p \in \mathcal{P}$ its class (served, first kind, second kind, high), and let $T_p = T_p^{\mathrm{s}}$ for served p. These are the **deficit data** of $(\mathfrak{M}, P)$.

Window parameters. For real numbers C, b_o and C_{ord}, the **window parameters of $\mathfrak{M}$ for** $(C, b_o, C_{\mathrm{ord}})$ are

$$(x, \varepsilon, n, k, N, t_R, L_a, r_G, S_c, \eta, \rho, \vartheta, K_0, \max(C,0), \beta, C_{\mathrm{ord}}, Y, s_0, b_o;\ e_2, p_0, d)$$

with $\vartheta = 1/(kx^{13})$, $K_0 = k^2 x^{78}$, $Y = X'(1+\rho)$, $p_0 = \lceil k^2 x^{26} \rceil$ and $\beta = s_0 = 0$. Their derived quantities $b, a, X', b', w'', X'', f$ are the ones above.

Conditions. Let $\varepsilon_z > 0$ and C_t be real numbers. We consider the following conditions on a loss datum.

(Lm1) Frame. For every inert prime p that is not sign-changing, the order of $[p]_{e_2}$ divides the order of h_p, and $[p]_{e_2} = 1$ if h_p has odd order; and h_p has even order whenever its order is at least $x^{\varepsilon/4}$.

(Lm2) Heights and weights. $0 \le a_t \le a(1+\vartheta)$ for $t \in T$, and $w_j(z) \ge w''$ for all j and $z \in \Phi$.

(Lm3) Linking for (C, b_o). Here C and b_o are real numbers. The condition consists of the **served-prime inequality for** (C, b_o), namely that for every served inert prime p

$$\sum_{j=1}^{m} \sum_{t \in T_p^{\mathrm{s}}} \left(A_p(a_t, t, w_j) - \mathrm{lo}_{C,b_o}(p)\right) - m\,\mathrm{fl}_C(p) \le \sum_{j=1}^{m} \mathrm{gain}_j(p) - \tau_p,$$

and the **surplus bound**, which does not involve C and b_o, namely that $S_p(v) \le 2mN_p(v)a/p$ for every inert prime p and every $v \in V_p$.

(Lm4) Few bad tops. $\displaystyle\sum_{p \in s_B} \frac{\log p}{p} \left|\{t \in T : \rho_p(t) \notin G_p\}\right| \le |T|\, x^{-2+\varepsilon}$, where s_B is the set of inert primes $p < X'$ that are not sign-changing and for which h_p has order at least $x^{\varepsilon/4}$.

(Lm5) Small shares. $\displaystyle\sum_{p \text{ candidate}} \frac{\log p}{p}\Big(2 \cdot \mathbf{1}_{\Sigma_p > x^{-\varepsilon}} + 4\Sigma_p\Big) \le x^{-9}$.

(Lm6) Place record. Every finite place w of K with $p(w) \le X''$ lies in S. For $w \in S$ with $p(w)$ inert, $g_j(w) = \mathrm{gain}_j^{\mathrm{D}}(p(w))$ for all j, $\tau(w) = \tau_{p(w)}$ and $\mathrm{io}(w) = \mathrm{io}_{p(w)}$. For $w \in S$ with $p(w)$ not inert, $g_j(w) = \tau(w) = \mathrm{io}(w) = 0$.

(Lm7) Sizes for (ε_z, C_t). With $\theta_s = \varepsilon/100$: $x^{2+8\varepsilon-\theta_s} \le n \le x^{2+8\varepsilon+\theta_s}$; $2^{e_2} \le n \le o\,2^{e_2}$ and $o \le x^{\theta_s}$; $m \le x^{\theta_s}(x\delta_1)^{1/2}$; $x^{9/2+10\varepsilon-\theta_s} \le k \le x^{9/2+10\varepsilon+\theta_s}$; $t_R \le x^{C_t+\theta_s}$; $x^{\varepsilon} \le L_a \le x^{\varepsilon+\theta_s}$; $S_c = 4k\Delta$ for a real $\Delta > 0$ with $\log\Delta \le x^{\varepsilon/4}/2$; $\eta > 0$ and $\log\eta \le x^{\varepsilon/4}$; $\log d \ge x^{\varepsilon_z/5-\varepsilon_z/100}$; and $\rho \le x^{-2}$.

The conditions fall into three groups. Conditions (Lm1), (Lm2), (Lm6) and (Lm7) fix the frame, the heights and weights, the place record and the sizes; for the members of a parametrised host family they are part of the definition (Definition 7.3). Condition (Lm3) is the linking condition, the local comparison of a prime with its gain. Conditions (Lm4) and (Lm5) concern the choice of the tops. The last three are the properties that Section 7 asks of a class of families under the names (Lk2) and (Lk3).

Two features of the definition organise the proof of the loss bound. The lattice term $\Theta(\mathfrak{M})$ is a sum over the places of a number field, with local gains $g_j(w)$; by (Lm6) these are the gains of the inert primes below the places, so $\Theta(\mathfrak{M})$ is a sum over inert primes (Lemma 5.16). And every inert prime lies in exactly one of four classes. Only a served prime has a gain. Under (Lm1), a prime of the first kind has a class of order less than $x^{\varepsilon/4}$ in $(\mathbb{Z}/2^{e_2+2}\mathbb{Z})^{\times}/\{\pm 1\}$, or has a power in a narrow band around X' (Lemma 5.14); such primes are sparse (Lemma 5.10). A prime of the second kind is a candidate of large orbit mass, and the small shares (Lm5) bound the total weight of such primes. A high prime exceeds the cut-off X'' of the gains.

Theorem 5.6. *Let $0 < \varepsilon < \frac{1}{52}$, $\varepsilon_z > 0$, C_t and C be real numbers, and put $C^+ = \max(C, 0)$. There is a real number x_0, depending only on ε, ε_z, C_t and C, with the following property. Let $\mathfrak{M}$ be a loss datum whose orbit selection has parameter ε and scale $x \ge x_0$, whose set of tops T is non-empty, and let b_o be a real number with $0 \le b_o \le \exp(x^{\varepsilon}/2)$. Suppose that $\mathfrak{M}$ satisfies*

- *(Lm1), (Lm2) and (Lm6) (frame, heights and weights, place record);*
- *(Lm3) for (C, b_o), that is, the served-prime inequality for (C, b_o) and the surplus bound;*
- *(Lm4) (few bad tops) and (Lm5) (small shares);*
- *(Lm7) for (ε_z, C_t) (sizes).*

Then, with $b = \exp(x^{\varepsilon})$, for every cut-off $P \geq 1$

$$S(\mathfrak{M}) - \Theta(\mathfrak{M}) \leq (3C^{+} + 13)\,|T|\,m\,b \qquad \textit{and} \qquad I(\mathfrak{M}) \leq |T|\,m\,b.$$

Proof. Let $C_{\mathrm{ord}} \geq 0$ be the absolute constant of Proposition 5.7, and let x_0 be the number that Lemma 5.12 gives for $(\varepsilon, \varepsilon_z, C_t, C, C_{\mathrm{ord}})$, with $e^{*} = \max(40, C_t + 3)$ and $K^{*} = 1 + |C| + C_{\mathrm{ord}}$; it depends only on $\varepsilon, \varepsilon_z, C_t, C$. Let $\mathfrak{M}$, b_o and the cut-off $P \geq 1$ be as in the statement, and write $\mathcal{P}$ for the set of inert primes $p \leq \max(P, X'')$ of the deficit data of $(\mathfrak{M}, P)$.

The window parameters. By Lemma 5.12, applied with (Lm7) and $0 \leq b_o \leq \exp(x^{\varepsilon}/2)$, the window parameters of $\mathfrak{M}$ for $(C, b_o, C_{\mathrm{ord}})$ are admissible for (e^{*}, K^{*}), the scale x is large for (e^{*}, K^{*}) (in particular $x \geq 1$), and $k^2 x^{26} \leq X'$. Their entry in the place of C is C^{+}, and their derived quantity b is $\exp(x^{\varepsilon})$.

The data for the bound at small orbit masses. We apply Proposition 5.7 to these window parameters, with the following data:

- U a set with one element υ;
- the set $\mathcal{P}$ in place of its finite set of primes, $J = \{1, \dots, m\}$ and the set of tops T;
- s the set of candidates and s_B the set of (Lm4);
- $\mathrm{BAD}(p, t)$ the relation $\rho_p(t) \notin G_p$;
- Σ_p the orbit masses ($p \in s$) and $\mathrm{io}(\upsilon, p) = \mathrm{io}_p$;
- $\mathfrak{d}(\upsilon)$ the deficit data of $(\mathfrak{M}, P)$, that is, $A(j,t,p)$, $d(j,t)$, $g(j,p) = \mathrm{gain}^{\mathrm{D}}_j(p)$, $\tau(p) = \tau_p$, the partition of $\mathcal{P}$ into served primes, primes of the first kind, primes of the second kind and high primes (every inert prime has exactly one class), and $T_p = T^{\mathrm{s}}_p$ for served p.

The sets $\mathcal{P}$, s, s_B, J, T are finite, $T \neq \emptyset$, and $\Sigma_p = m \sum_{v \in G_p} \kappa_p(v) \geq 0$ since $\kappa_p(v) \in [0,1]$. We check the hypotheses (1)–(6) of Proposition 5.7.

(1) An inert prime is different from 2, so every $p \in \mathcal{P}$ is an odd prime.

(2) With $\theta = x^{-\varepsilon}$, the small-shares hypothesis is (Lm5).

(3) The few-bad-tops hypothesis is (Lm4).

(4) By Lemma 5.15, item 3, which uses the surplus bound, $\mathrm{io}_p \leq 2\Sigma_p \cdot 2m|T|a/p = 2\Sigma_p \cdot 2|J||T|a/p$ for every $p \in s$.

(5) Lemma 5.13 applies, since $\mathfrak{M}$ satisfies (Lm1), (Lm2) and (Lm3) for (C, b_o); $b_o \geq 0$; by (Lm7), $L_a \geq x^{\varepsilon} > 0$, $S_c = 4k\Delta > 0$ (as $k \geq x^{9/2+10\varepsilon-\theta_s} > 0$ and $\Delta > 0$) and $\eta > 0$; and $k^2 x^{26} \leq X'$. Its items 1–5 are the hypotheses of Lemma 5.8 on the data $\mathfrak{d}(\upsilon)$ formed with the parameters of the window parameters, namely the bounds on heights and weights and the form of $A(j,t,p)$ and $d(j,t)$ (item 1); served primes at most X'' and high primes above X'' (item 2); the inequality of Proposition 5.9 with P_{s} the served primes, P_{n} the others, $\mathrm{lo} = \mathrm{lo}_{C,b_o}$, $\mathrm{fl} = \mathrm{fl}_C$, and $g(j,p) = \tau(p) = 0$ on P_{n} (item 3); $\mathrm{lo}_{C,b_o} \leq \mathrm{lo}^{*}$ and $\mathrm{fl}_C \leq \mathrm{fl}^{*}$, formed with C^{+} (item 4); and $K_0 \leq p^{L_1(p)-1}$ (item 5).

(6) By Lemma 5.14, items 1–3 (which use (Lm1)), a prime of the first kind satisfies $p \le \max(Y, X'')$ with $[p]_{e_2}$ of order less than $x^{\varepsilon/4}$, or $p \ge p_0$ with $X'(1-\rho) \le p^i \le X'(1+\rho)$ for some $i \ge 1$; a prime of the second kind is a candidate with $\Sigma_p > x^{-\varepsilon} = \theta$; and if p is served and $t \in T \setminus T_p^{\mathrm{s}}$, then $p \in s_B$ and $\rho_p(t) \notin G_p$.

The window parameters are admissible for (e^*, K^*) at a scale large for (e^*, K^*), so they satisfy in particular all conditions of admissibility except those on β and s_0. Hence Proposition 5.7 gives, with $S_{\mathfrak{d}}$, $\Theta_{\mathfrak{d}}$ formed from $\mathfrak{d}(\upsilon)$ as in Proposition 5.9,

$$S_{\mathfrak{d}} - \Theta_{\mathfrak{d}} \le (3C^+ + 13)\, m\, |T|\, b, \qquad \sum_{p \in s} \log p \ \mathrm{io}_p \le m\, |T|\, b.$$

The three sums of $\mathfrak{M}$. By Lemma 5.15, item 1, $S(\mathfrak{M}) = \sum_j \sum_t \big(\sum_{p \in \mathcal{P}} 2 \log p\, A(j,t,p) + d(j,t)\big) = S_{\mathfrak{d}}$. For the lattice term apply Lemma 5.16 to the arithmetic datum, tops, orbit labellings, orbit selection and gain datum of $\mathfrak{M}$ and to its place record (K, S, ℓ, g_j, τ): here $X' \le X'' = xnX'$ because $x \ge 1$ and $n = |H| \ge 1$, and the hypotheses 1–3 of Lemma 5.16 are (Lm6). With $B = \max(P, X'') \ge X''$ it gives

$$\Theta(\mathfrak{M}) = 2 \sum_{p \in \mathcal{P}} \log p \Big(\sum_{j=1}^{m} \mathrm{gain}_j^{\mathrm{D}}(p) - \tau_p \Big) = \Theta_{\mathfrak{d}}.$$

By Lemma 5.15, item 2, which uses (Lm6), $I(\mathfrak{M}) = \sum_{p \in s} \log p\ \mathrm{io}_p$. Substituting these three identities in the two inequalities above gives the claim. □

Proposition 5.7. *There is $C_{\mathrm{ord}} \ge 0$ with the following property. Let a tuple of window parameters with this C_{ord} satisfy all conditions of admissibility for (e, K) except those on β and s_0, at a scale that is large for (e, K). Let U be a set, let P, s, s_B be finite sets of primes, and let J, T be finite sets with $T \neq \emptyset$. Let $\mathrm{BAD}(p,t)$ be a relation between primes and elements of T, let $\Sigma_p \ge 0$ $(p \in s)$ and $\mathrm{io}(\upsilon, p)$ $(\upsilon \in U,\ p \in s)$ be real numbers, and for each $\upsilon \in U$ let $\mathfrak{d}(\upsilon)$ be a datum on (P, J, T) as in Lemma 5.8. Suppose that*

1. *every $p \in P$ is an odd prime;*
2. *(*small shares*)* $\displaystyle\sum_{p \in s} \frac{\log p}{p} \Big(2 \cdot \mathbf{1}_{\Sigma_p > \theta} + 4\Sigma_p \Big) \le x^{-9}$;
3. *(*few bad tops*)* $\displaystyle\sum_{p \in s_B} \frac{\log p}{p} \big| \{ t \in T : \mathrm{BAD}(p,t) \} \big| \le |T|\, x^{-2+\varepsilon}$;
4. $\mathrm{io}(\upsilon, p) \le 2\Sigma_p \cdot 2|J||T|a/p$ *for all υ and $p \in s$;*
5. *each $\mathfrak{d}(\upsilon)$ satisfies the hypotheses of Lemma 5.8 on its data (those on the heights, the weights, the classes of primes, the inequality of Proposition 5.9,* lo, fl *and L_1), formed with the parameters $a, \vartheta, w'', X'', X', p_0, d, C, f, N, S_k, Q_1, Q_2, \lambda, L_a, k, K_0$ of the window parameters;*
6. *the primes of each $\mathfrak{d}(\upsilon)$ satisfy the following:*
   - *every first-kind prime p satisfies $p \le \max(Y, X'')$ and $[p]_{e_2}$ has order less than $x^{\varepsilon/4}$, or $p \ge p_0$ and $X'(1-\rho) \le p^i \le X'(1+\rho)$ for some $i \ge 1$;*
   - *every second-kind prime p lies in s and has $\Sigma_p > \theta$;*
   - *for every served prime p and every $t \in T \setminus T_p$, $p \in s_B$ and $\mathrm{BAD}(p,t)$.*

Then for every $\upsilon \in U$, with S, Θ formed from $\mathfrak{d}(\upsilon)$,

$$S - \Theta \le (3C + 13)\,|J|\,|T|\,b \qquad \text{and} \qquad \sum_{p \in s} \log p\ \mathrm{io}(\upsilon, p) \le |J|\,|T|\,b.$$

Proof. Take $C_{\mathrm{ord}} = \max(C_1, 0)$ with C_1 the constant of Lemma 5.10. Write $m = |J|$ and $U_0 = |T|mb$, and let σ and the numbers $T_{\mathrm{fl}}, T_{\mathrm{cp}}, T_{Q_1}, T_{Q_2}, T_{\mathrm{or}}, T_{\mathrm{hi}}, T_d, T_{\mathrm{ord}}, T_{\mathrm{band}}, T_{\mathrm{out}}, T_{\mathrm{sec}}$ be those of Lemma 5.11, formed with m and with the number $|T| \ge 1$ in place of T. Put $\Lambda = \sum_{p \in s} \frac{\log p}{p}(2 \cdot \mathbf{1}_{\Sigma_p > \theta} + 4\Sigma_p)$, so $\Lambda \le x^{-9}$ by (2).

Side conditions. By Lemma 5.11, item 1, the window parameters satisfy the conditions that Lemma 5.8 imposes on its parameters ($a \ge 1$, $0 \le \vartheta \le \frac{1}{2}$, $w'' > 0$, $X'' \ge 2$, $X''w'' = a$, $X' \ge 1$, $p_0 \ge 3$, $d \ge 4$, $C, f, S_k, Q_1, Q_2, \lambda, L_a \ge 0$, $k > 0$, $K_0 > 0$), and moreover $\rho \le \frac{1}{2}$, $p_0 \ge 8$, $X' \ge 2$, $e_2 \ge 1$, $J_o \ge 1$, $\lfloor \log X' \rfloor = I_b$ and $C \ge 0$. With (5), each $\mathfrak{d}(\upsilon)$ satisfies all hypotheses of Lemma 5.8.

The deficit. Fix υ. With the notation of Lemma 5.11, the bound of Lemma 5.8 reads

$$S - \Theta \le T_{\mathrm{fl}} + T_{\mathrm{cp}} + T_{Q_1} + T_{Q_2} + T_{\mathrm{or}} + T_{\mathrm{hi}} + T_d + \sum_{p \text{ served}} 4ma \frac{\log p}{p}\,|T \setminus T_p| + 4m|T|a\Big(\sum_{p \text{ first}} \frac{\log p}{p} + \sum_{p \text{ second}} \frac{\log p}{p}\Big).$$

Unserved tops. By (6), every $t \in T \setminus T_p$ at a served p has $p \in s_B$ and $\mathrm{BAD}(p, t)$; hence, by (3),

$$\sum_{p \text{ served}} 4ma \frac{\log p}{p} |T \setminus T_p| \le 4ma \sum_{p \in s_B} \frac{\log p}{p} |\{t : \mathrm{BAD}(p, t)\}| \le 4am|T|x^{-2+\varepsilon} = T_{\mathrm{out}}.$$

First kind. By (1) and (6), the first-kind primes form a finite set of odd primes that satisfies the hypotheses of Lemma 5.10 with $e = e_2$, $J = J_o$ (an integer order less than $x^{\varepsilon/4}$ is at most $J_o = \lfloor x^{\varepsilon/4} \rfloor$), $\max(Y, X'') \ge 1$ in place of Y, and $\rho \le \frac{1}{2}$, $p_0 \ge 8$, $X' \ge 2$. Hence, using $C_1 \le C_{\mathrm{ord}}$ and $\lfloor \log X' \rfloor = I_b$,

$$4m|T|a \sum_{p \text{ first}} \frac{\log p}{p} \le 4m|T|a\Big(\frac{C_1 J_o \log(2^{e_2} \max(Y, X''))^2}{2^{e_2}} + I_b\Big(4\rho + \frac{1}{p_0}\Big)\log(2X')\Big) \le T_{\mathrm{ord}} + T_{\mathrm{band}}.$$

Second kind. By (6) every second-kind prime lies in s and has $\Sigma_p > \theta$, so it contributes at least $2 \log p/p$ to Λ, and $4m|T|a \sum_{p \text{ second}} \log p/p \le 2am|T|\Lambda \le 2am|T|x^{-9} = T_{\mathrm{sec}}$.

The companion sum. By (4) and $\log p \ge 0$, $\sum_{p \in s} \log p\ \mathrm{io}(\upsilon, p) \le \sum_{p \in s} \frac{\log p}{p}\, 4m|T|a\,\Sigma_p \le am|T|\Lambda \le am|T|x^{-9}$.

Sizes. By Lemma 5.11, item 2, each of $T_{\mathrm{fl}}, T_{\mathrm{cp}}, T_{Q_1}, T_{\mathrm{or}}, T_{\mathrm{hi}}, T_d, T_{\mathrm{ord}}, T_{\mathrm{band}}, T_{\mathrm{out}}, T_{\mathrm{sec}}$ is at most σU_0, $T_{Q_2} \le (2 \log 4 \cdot C + 3\sigma)U_0$, and $am|T|x^{-9} \le \sigma U_0$. Hence $S - \Theta \le (2 \log 4 \cdot C + 13\sigma)U_0 \le (3C + 13)U_0$, since $2 \log 4 \le 3$, $C \ge 0$ and $\sigma \le 1$; and the companion sum is at most $\sigma U_0 \le U_0$. As $U_0 = |J||T|b$, this is the claim. □

Lemma 5.8. *Let P be a finite set of primes, partitioned as $P = P_{\mathrm{s}} \sqcup P_1 \sqcup P_2 \sqcup P_{\mathrm{h}}$ (served primes, primes of the first and of the second kind, high primes), and let J, T be finite sets. Let $p_0 \ge 3$ and $d \ge 4$ be integers, let $a \ge 1$, $0 \le \vartheta \le \frac{1}{2}$, $w'' > 0$, $X'' \ge 2$ with $X''w'' = a$, $X' \ge 1$, N, $C, f, S_k, Q_1, Q_2, \lambda, L_a \ge 0$, $k > 0$ and $K_0 > 0$ be real numbers, and let $L_1 \colon P \to \mathbb{R}$. Put*

$$\mathrm{fl}^*(p) = CfN^2 \log_p a + Cf(\log_p a)^2,$$

$$\mathrm{lo}^*(p) = CS_k \log_p(2a) + \mathbf{1}_{p < X'}\Big(Q(p) + C\lambda\Big(\frac{L_a a}{pk} + \begin{cases} a\,p^{-L_1(p)} & p < p_0 \\ a/p^2 & p \ge p_0 \end{cases}\Big)\Big),$$

with $Q(p) = Q_1$ for $p < p_0$ and $Q(p) = Q_2$ for $p \ge p_0$. Let $a_t \in [0, a(1+\vartheta)]$ for $t \in T$. For finite sets $I(j,t,p)$ of odd positive integers, finite sets $I_d(j,t)$ of positive integers, and weights $w(j,t,p,i) \ge w''$ and $w_d(j,t,i) \ge 0$, let

$$A(j,t,p) = \sum_{i \in I(j,t,p)} \Big(\frac{a_t}{p^i} - w(j,t,p,i)\Big)^+, \qquad d(j,t) = \log d \sum_{i \in I_d(j,t)} \Big(\frac{a_t}{d^i} - w_d(j,t,i)\Big)^+.$$

Let $g(j,p)$, $\tau(p)$ be real numbers, and suppose that

- *every served prime is at most X'' and every high prime exceeds X'';*
- *the hypotheses of the inequality of Proposition 5.9 hold with $P_{\rm n} = P_1 \cup P_2 \cup P_{\rm h}$, and with $\mathrm{lo}(p) \le \mathrm{lo}^*(p)$ and $\mathrm{fl}(p) \le \mathrm{fl}^*(p)$ for $p \in P_{\rm s}$;*
- *$K_0 \le p^{L_1(p)-1}$ for every $p \in P$ with $p < p_0$.*

Then

$$\begin{aligned} S - \Theta \le\ & |J| \cdot 2CfX''(N^2 \log a + 2(\log a)^2) + 2CS_k \log(2a)\, |J|\, |T|\, X'' \\ & + |J|\, |T| \cdot 2 \log 4\, (p_0 Q_1 + X' Q_2) \\ & + |J|\, |T| \cdot 2C\lambda a\Big(\Big(\frac{L_a}{k} + \frac{1}{K_0}\Big)(\log X' + \log 4) + \frac{2(\log(p_0-1)+1)}{p_0 - 1}\Big) \\ & + \sum_{p \in P_{\rm s}} 4|J|\, a\, \frac{\log p}{p}\, |T \setminus T_p| \\ & + 4|J|\, |T|\, a\Big(\sum_{p \in P_1} \frac{\log p}{p} + \sum_{p \in P_2} \frac{\log p}{p}\Big) \\ & + |J|\, |T| \cdot 2 \log(2X'')(a\vartheta^2 + w''\vartheta) + |J|\, |T| \cdot \frac{2a \log d}{d}. \end{aligned}$$

Proof. We bound the three parts of the inequality of Proposition 5.9. Throughout, $a_t \le a(1+\vartheta) \le \frac{3}{2}a$ and all weights are non-negative, so $(a_t/p^i - w)^+ \le a_t/p^i$.

One prime, one top. For $p \ge 2$, $\sum_{i \text{ odd}} p^{-i} = p/(p^2-1) \le \frac{4}{3}p^{-1}$, so $A(j,t,p) \le \frac{4}{3}a_t/p$ and $2\log p\, A(j,t,p) \le 4a \log p/p$. Summing over j,t bounds every term $2\log p \sum_{j,t} A(j,t,p)$ with $p \in P_1 \cup P_2$ by $4|J||T|a \log p/p$, and $2\log p \sum_j \sum_{t \in T \setminus T_p} A(j,t,p)$ by $4|J|a(\log p/p)|T \setminus T_p|$.

The prime d. We have $\sum_{i \ge 1} d^{-i} = 1/(d-1)$ and $\frac{3}{2}a/(d-1) \le 2a/d$ for $d \ge 4$, so $d(j,t) \le 2a \log d/d$.

High primes. Let $p > X''$. For $i \ge 2$, $p^i \ge X''^2 \ge 2X''$, so $a_t/p^i \le 2a/(2X'') = w'' \le w(j,t,p,i)$ and the term vanishes. For $i = 1$: if $p \ge X''(1+\vartheta)$ then $a_t/p \le w''$ and the term vanishes; otherwise $a_t/p - w'' \le a(1+\vartheta)/X'' - w'' = w''\vartheta$. The primes in $(X'', X''(1+\vartheta))$ number at most $X''\vartheta + 1$, and for them $2\log p \le 2\log(2X'')$. Hence $\sum_{p \in P_{\rm h}} 2\log p\, A(j,t,p) \le 2\log(2X'')w''\vartheta(X''\vartheta + 1) = 2\log(2X'')(a\vartheta^2 + w''\vartheta)$.

Served primes: the losses. For $p \in P_{\rm s}$ we have $|J||T_p|\mathrm{lo}(p) \le |J||T|\mathrm{lo}^*(p)$ (as $\mathrm{lo}^* \ge 0$ and $|T_p| \le |T|$) and $|J|\mathrm{fl}(p) \le |J|\mathrm{fl}^*(p)$. The served primes lie in $[2, X'']$, so there are at most X'' of them. We bound the resulting sums in turn. First, $\sum_{p \in P_{\rm s}} 2\log p\, CS_k \log_p(2a) = 2CS_k \log(2a)|P_{\rm s}|$. Second, $\sum_{p \in P_{\rm s}} 2\log p\, \mathrm{fl}^*(p) \le 2CfX''(N^2 \log a + 2(\log a)^2)$, using $\log p \ge \log 2 \ge \frac{1}{2}$. Third, $\sum_{p < X'} 2\log p\, Q(p) \le 2\log 4\, (p_0 Q_1 + X' Q_2)$ by Chebyshev's bound $\sum_{p \le y} \log p \le$

$y\log 4$ ($y \ge 0$) [180]. Finally, writing $ap^{-L_1(p)} = (a/p)p^{-(L_1(p)-1)} \le a/(pK_0)$ for $p < p_0$, we obtain

$$\sum_{p<X'} 2\log p\, C\lambda\Big(\frac{L_a a}{pk} + \dots\Big) \le 2C\lambda a\Big(\Big(\frac{L_a}{k} + \frac{1}{K_0}\Big)\sum_{p<X'}\frac{\log p}{p} + \sum_{p\ge p_0}\frac{\log p}{p^2}\Big),$$

and we use $\sum_{p\le y}\log p/p \le \log y + \log 4$ for $y \ge 1$ (by Legendre's formula, $y\log y \ge \log\lfloor y\rfloor! \ge \sum_{p\le y}\lfloor y/p\rfloor\log p \ge y\sum_{p\le y}\frac{\log p}{p} - \sum_{p\le y}\log p$, and Chebyshev's bound) and $\sum_{n\ge p_0}\log n/n^2 \le 2(\log(p_0-1)+1)/(p_0-1)$.

Adding the bounds gives the claim. □

Proposition 5.9. *Let P be a finite set of primes, J and T finite sets, and let $A(j,t,p)$, $d(j,t)$, $g(j,p)$, $\tau(p)$ be real numbers ($j \in J$, $t \in T$, $p \in P$). Put*

$$S = \sum_{j\in J}\sum_{t\in T}\Big(\sum_{p\in P} 2\log p\, A(j,t,p) + d(j,t)\Big), \qquad \Theta = 2\sum_{p\in P}\log p\,\Big(\sum_{j\in J} g(j,p) - \tau(p)\Big),$$

*and, for $p \in P$, let the **deficit** at p be $\operatorname{def}(p) = \sum_{j,t} A(j,t,p) - \big(\sum_j g(j,p) - \tau(p)\big)$. Then*

$$S - \Theta = \sum_{p\in P} 2\log p\,\operatorname{def}(p) + \sum_{j\in J}\sum_{t\in T} d(j,t). \tag{5.1}$$

Suppose moreover that $P = P_{\mathrm{s}} \sqcup P_{\mathrm{n}}$, and that for each $p \in P_{\mathrm{s}}$ there are a subset $T_p \subseteq T$ and real numbers $\operatorname{lo}(p), \operatorname{fl}(p)$ with

$$\sum_{j\in J}\sum_{t\in T_p}\big(A(j,t,p) - \operatorname{lo}(p)\big) - |J|\operatorname{fl}(p) \le \sum_{j\in J} g(j,p) - \tau(p),$$

while $g(j,p) = 0$ for all j and $\tau(p) = 0$ for $p \in P_{\mathrm{n}}$. Then

$$S - \Theta \le \sum_{p\in P_{\mathrm{s}}} 2\log p\Big(\sum_{j\in J}\sum_{t\in T\setminus T_p} A(j,t,p) + |J|\,|T_p|\operatorname{lo}(p) + |J|\operatorname{fl}(p)\Big) + \sum_{p\in P_{\mathrm{n}}} 2\log p\sum_{j\in J}\sum_{t\in T} A(j,t,p) + \sum_{j\in J}\sum_{t\in T} d(j,t).$$

Proof. The identity (5.1) is obtained by expanding both sides and exchanging the finite sums over j, t and p.

For the inequality, we use (5.1) and bound $2\log p\,\operatorname{def}(p)$ for each p, recalling that $2\log p \ge 0$. If $p \in P_{\mathrm{n}}$, then $\operatorname{def}(p) = \sum_{j,t} A(j,t,p)$. If $p \in P_{\mathrm{s}}$, split $\sum_{t\in T} = \sum_{t\in T\setminus T_p} + \sum_{t\in T_p}$ (as $T_p \subseteq T$) and write $\sum_j\sum_{t\in T_p} A(j,t,p) = \sum_j\sum_{t\in T_p}(A(j,t,p) - \operatorname{lo}(p)) + |J||T_p|\operatorname{lo}(p)$. Then the hypothesis at p gives

$$\operatorname{def}(p) \le \sum_j\sum_{t\in T\setminus T_p} A(j,t,p) + |J|\,|T_p|\operatorname{lo}(p) + |J|\operatorname{fl}(p).$$

Multiplying by $2\log p$ and summing over p gives the claim. □

Lemma 5.10. *For an integer $e \ge 1$ and an odd integer l let $[l]_e$ be the class of l in the group $(\mathbb{Z}/2^{e+2}\mathbb{Z})^\times/\{\pm 1\}$. There is an absolute constant C with the following property. Let $e, J \ge 1$ be integers, let $Y \ge 1$, $0 \le \rho \le \frac{1}{2}$, $p_0 \ge 8$ and $X' \ge 2$ be real numbers, and let s be a finite set of odd primes such that every $p \in s$ satisfies*

1. *$p \le Y$ and $[p]_e$ has order at most J, or*

2. $p \ge p_0$ *and* $X'(1-\rho) \le p^i \le X'(1+\rho)$ *for some integer* $i \ge 1$.

Then

$$\sum_{p \in s} \frac{\log p}{p} \le \frac{C\,J\,\log(2^e Y)^2}{2^e} + \lfloor \log X' \rfloor \Big(4\rho + \frac{1}{p_0}\Big) \log(2X').$$

Proof. Split s into the primes satisfying (1) and those satisfying (2) but not (1); all summands are non-negative.

Primes of small order. The group $(\mathbb{Z}/2^{e+2})^\times/\{\pm 1\}$ is cyclic of order 2^e, generated by the class of 5, and for $0 \le j \le e$ one has $[l]_e^{2^j} = 1$ if and only if $l \equiv \pm 1 \pmod{2^{e+2-j}}$. Indeed, the elements of order dividing 2^j in this cyclic group of order 2^e form its subgroup of order 2^j, generated by the class of $5^{2^{e-j}}$. Its preimage in $(\mathbb{Z}/2^{e+2})^\times$ is $\{\pm 1\} \times \langle 5^{2^{e-j}} \rangle$, of index 2^{e-j}, and as $5^{2^{e-j}}$ generates the subgroup of units $\equiv 1 \bmod 2^{e+2-j}$, this preimage is the set of units $\equiv \pm 1 \bmod 2^{e+2-j}$. If $[p]_e$ has order at most J, its order is 2^j with $2^j \le J$ and $j \le e$. With $j_0 = \min(e, \lfloor \log_2 J \rfloor)$ we get $[p]_e^{2^{j_0}} = 1$, so $p \equiv \pm 1 \pmod M$ with $M = 2^{e+2-j_0} \ge 2^{e+2}/J$ and $M \le 2^{e+2}$. The integers $n \ge 3$, $n \le Y$, in the two classes $\pm 1 \bmod M$ are $n = kM \pm 1$; their contribution is

$$\sum \frac{\log n}{n} \le \frac{2\log M}{M} + \frac{3\log Y}{M} \sum_{1 \le k \le Y} \frac{1}{k} \le \frac{3}{M}\big(\log M + \log Y(1+\log Y)\big) \le \frac{C_1}{M} \log(2^e Y)^2,$$

with an absolute constant C_1, since $\log M \le \log 4 + \log(2^e Y)$ and $\log(2^e Y) \ge \log 2$. With $1/M \le J/2^{e+2}$ this gives the first term.

Primes near a root of X'. Let $p \ge p_0 \ge 8$ and $X'(1-\rho) \le p^i \le X'(1+\rho)$ with $i \ge 1$. Then $8^i \le p^i \le 2X'$, so $i \le \log(2X')/\log 8 \le \log X'$ (as $X' \ge 2$), that is, $1 \le i \le \lfloor \log X' \rfloor$. With $P_i = X'^{1/i}$, from $(1-\rho) \le (1-\rho)^{1/i}$ and $(1+\rho)^{1/i} \le 1+\rho$ we get $P_i(1-\rho) \le p \le P_i(1+\rho)$. For fixed i, the integers in $[P_i(1-\rho), P_i(1+\rho)]$ number at most $2\rho P_i + 1$, and each $n \ge p_0$ there has $\log n/n \le \log(2P_i)\min(2/P_i, 1/p_0)$ (as $n \ge P_i/2$ and $n \le 2P_i$). Hence the primes of this i contribute at most $(2\rho P_i + 1)\min(2/P_i, 1/p_0)\log(2P_i) \le (4\rho + 1/p_0)\log(2X')$. Summing over $1 \le i \le \lfloor \log X' \rfloor$ gives the second term. □

Lemma 5.11. *Let a tuple of window parameters satisfy all conditions of admissibility for* (e, K) *except those on* β *and* s_0*, at a scale that is large for* (e, K)*, and put* $\sigma = 100Kx^{4\varepsilon}/x$*. Then the following hold.*

1. $x \ge 100$, $0 \le \sigma \le 1$, $b \ge 1$, $a \ge 1$, $X' \ge 2$, $X'' = xnX' \ge 2$, $X''w'' = a$, $w'' > 0$, $0 \le \vartheta \le \frac{1}{2}$, $0 \le \rho \le \frac{1}{2}$, $p_0 \ge 8$, $d \ge 4$, $k > 0$, $K_0 > 0$, $e_2 \ge 1$, $J_o \ge 1$, $\lfloor \log X' \rfloor = I_b$, *and* $C, f, S_k, Q_1, Q_2, \lambda, L_a \ge 0$.

2. Let $m \ge 0$ and $T \ge 1$ be real numbers and $U = Tmb$. Each of the ten numbers

$$\begin{aligned}
T_{\mathrm{fl}} &= 2mCfX''(N^2 \log a + 2(\log a)^2),\\
T_{\mathrm{cp}} &= 2CS_k \log(2a)\, mT\, X'',\\
T_{Q_1} &= 2\log 4 \cdot mT\, p_0 Q_1,\\
T_{\mathrm{or}} &= 2C\lambda a\, mT\Big(\Big(\frac{L_a}{k} + \frac{1}{K_0}\Big)(\log X' + \log 4) + \frac{2(\log(p_0-1)+1)}{p_0-1}\Big),\\
T_{\mathrm{hi}} &= 2\log(2X'')\, mT\,(a\vartheta^2 + w''\vartheta),\\
T_d &= 2mT\,\frac{a\log d}{d},\\
T_{\mathrm{ord}} &= 4mTa\,\frac{C_{\mathrm{ord}}\, J_o\, \log(2^{e_2}\max(Y, X''))^2}{2^{e_2}},\\
T_{\mathrm{band}} &= 4mTa\, I_b\Big(4\rho + \frac{1}{p_0}\Big)\log(2X'),\\
T_{\mathrm{out}} &= 4amT\, x^{-2+\varepsilon},\\
T_{\mathrm{sec}} &= 2amT\, x^{-9}
\end{aligned}$$

is at most σU; moreover $amT\, x^{-9} \le \sigma U$, and $T_{Q_2} = 2\log 4 \cdot mT\, X'Q_2 \le (2\log 4 \cdot C + 3\sigma)\, U$.

Proof. Write $z = x^{\varepsilon}$, $u = x^{\varepsilon/3}$, $E = x^e$, $S = \exp(x^{\varepsilon/4})$, $s_b = \exp(x^{\varepsilon}/2)$ and $v = b/X'$; then $b = s_b^2$, $X' = e^u$, $b' = a/X' = xv$, $w'' = v/n$ and $X'' = a/w'' = xnX'$.

Basic facts. Admissibility gives $x^2 \le 2^{e_2} \le n \le E$, $x^2 \le k \le E$, $0 \le N \le E$, $0 \le t \le E$, $x^4 + 1 \le p_0 \le E$, $0 \le L_a \le z^2$, $0 \le r_G \le E$, $0 \le S_c, \eta \le S$, $0 \le \rho \le x^{-2}$, $0 \le \vartheta \le x^{-1}$, $K_0, d \ge x^2$, $0 \le C, C_{\mathrm{ord}} \le K$, $1 \le Y \le 2X'$, $0 \le b_o \le s_b$, $0 < \varepsilon \le 1$, $e \ge 0$ and $K \ge 1$. A large scale gives

$$\text{(L1) } 100Kz^4 \le x, \qquad \text{(L2) } 100KE^4x^4 \le X', \qquad \text{(L3) } 100KE^4x^4X'S \le s_b.$$

Since $Kz^4 \ge 1$, (L1) gives $x \ge 100$, $0 \le \sigma = 100Kz^4/x \le 1$, $x^{-1} \le \sigma$ and $100z^i/x \le \sigma$ for $0 \le i \le 4$. Taking logarithms in (L2), $(4e+4)\log x \le u$, so $\log x \le z$, $e\log x \le z$, and $\log y \le z$ for $0 < y \le E$. Further $\log a = \log x + z \le 2z \le x$, $\log X' = u \le z$, $X' \ge 100$, and $s_b \ge 1$ by (L3); hence $b = s_b^2 \ge s_b$, $b \ge 1$ and $a = xb \ge 1$.

Part 1. We have $w'' = v/n > 0$; $\vartheta \le x^{-1} \le \frac{1}{2}$ and $\rho \le x^{-2} \le \frac{1}{2}$; $X'' = xnX' \ge 2$; $X''w'' = X'b' = a$; $p_0 \ge x^4 + 1 \ge 8$ and $d \ge x^2 \ge 4$; $k, K_0 \ge x^2 > 0$; $e_2 \ge 1$ as $2^{e_2} \ge x^2 > 1$; $J_o = \lfloor x^{\varepsilon/4} \rfloor \ge 1$; and $\lfloor \log X' \rfloor = \lfloor u \rfloor = I_b$. Finally, $f = 2n$, $S_k = S_c + k + \eta$, $Q_1 = Cr_G b'\lambda$, $Q_2 = C(b'/x + \eta + S_c + (w'' + t)\lambda/\delta_1) + b_o$ and $\lambda = \log x$ are non-negative, since all their factors are non-negative and $\delta_1 > 0$.

Part 2. We use two absorption rules: for integers $0 \le i \le 4$, $0 \le j \le 3$ and $0 \le c \le 100$,

$$\text{(R1) } cKE^ix^jX'S \le \sigma b, \qquad \text{(R2) } cKE^ix^jv \le \sigma b.$$

Indeed $cKE^ix^j \le 100KE^4x^4/x$, so by (L3) the left side of (R1) is at most $s_b/x \le b/x \le \sigma b$, and by (L2) that of (R2) is at most $X'v/x = b/x \le \sigma b$. We show below that each of the numbers, divided by mT (for T_{fl}: divided by m), is at most σb; multiplying by $mT \ge 0$ (for T_{fl} by $m \le mT$) gives the claim. The dominant term is $X'Q_2$: since $X'b'/x = b$ and $X'w''/\delta_1 = (b/n)x^{\varepsilon/8}$,

$$X'Q_2 = Cb + CX'(\eta + S_c) + C\frac{b}{n}\lambda x^{\varepsilon/8} + CX't\lambda x^{\varepsilon/8} + X'b_o,$$

and each of the last four terms is at most $\sigma b/4$ (see the computation below), so $X'Q_2 \le (C+\sigma)b$ and $2\log 4\, X'Q_2 \le (2\log 4 \cdot C + 3\sigma)b$, since $2\log 4 \le 3$.

□

Lemma 5.12. *Let $0 < \varepsilon < \frac{1}{52}$, $\varepsilon_z > 0$, C_t, C and $C_{\mathrm{ord}} \ge 0$ be real numbers, and put $e^* = \max(40, C_t+3)$ and $K^* = 1+|C|+C_{\mathrm{ord}}$. Then there is x_0 with the following property. Let a loss datum with parameter ε, at a scale $x \ge x_0$, satisfy (Lm7) for (ε_z, C_t), and let b_o be a real number with $0 \le b_o \le \exp(x^\varepsilon/2)$. Then the window parameters of the loss datum for $(C, b_o, C_{\mathrm{ord}})$ are admissible for (e^*, K^*), the scale is large for (e^*, K^*), and $k^2x^{26} \le X'$.*

Proof. Choose $x_0 \ge 3$ such that every $x \ge x_0$ satisfies

(T1) $\lceil (32+64\varepsilon)/\varepsilon \rceil + 1 \le x$;

(T2) $x^7 \le \exp(x^{\varepsilon/4}/2)$, $x^2 \le \exp(x^{\varepsilon_z/5-\varepsilon_z/100})$ and $x^{36} \le \exp(x^{\varepsilon/3})$;

(T3) $100K^*x^{4\varepsilon} \le x$, $100K^*x^{4e^*+4} \le \exp(x^{\varepsilon/3})$ and $6 \le x^{2\varepsilon/3}$.

Such an x_0 exists, since $1-4\varepsilon > 0$, $\varepsilon_z/5 - \varepsilon_z/100 > 0$, and a fixed power of x is eventually smaller than $\exp(cx^\delta)$ for any $c, \delta > 0$. It depends only on $\varepsilon, \varepsilon_z, C_t, C, C_{\mathrm{ord}}$.

Large scale. The first two inequalities in the definition of a large scale are $x \ge 1$ and the first part of (T3); the third is the second part of (T3). By the third part of (T3), $6x^{\varepsilon/3} \le x^\varepsilon$, so with the second part

$$100K^*x^{4e^*+4}\exp(x^{\varepsilon/3}+x^{\varepsilon/4}) \le \exp(2x^{\varepsilon/3}+x^{\varepsilon/4}) \le \exp(3x^{\varepsilon/3}) \le \exp(x^\varepsilon/2).$$

Admissibility. Let $\theta_s = \varepsilon/100$. We use (Lm7) and $x \ge 3$, and recall that $0 < \varepsilon \le 1$, $e^* \ge 40$, $e^* \ge C_t + 3$ and $K^* \ge 1$.

- Ranks. From $x^{2+\theta_s} \le x^{2+8\varepsilon-\theta_s} \le n \le o\,2^{e_2} \le x^{\theta_s}2^{e_2}$ we get $x^2 \le 2^{e_2}$, and $2^{e_2} \le n \le x^{2+8\varepsilon+\theta_s} \le x^3 \le x^{e^*}$. Next, $x^2 \le x^4 \le x^{9/2+10\varepsilon-\theta_s} \le k \le x^{9/2+10\varepsilon+\theta_s} \le x^5 \le x^{e^*}$. Also $m \le x^{\theta_s+1/2-\varepsilon/16} \le x$ and $0 \le t_R \le x^{C_t+\theta_s} \le x^{e^*-2}$, so $0 \le N = k + m(t_R+1) \le x^{e^*-1} + 2x^{e^*-1} \le x^{e^*}$. By (T1), $0 \le r_G \le x \le x^{e^*}$.
- Thresholds. The threshold $p_0 = \lceil k^2x^{26} \rceil$ satisfies $p_0 \ge k^2x^{26} \ge 2k^2 \ge k+1 \ge x^4+1$ and $p_0 \le k^2x^{26}+1 \le x^{36}+1 \le x^{37} \le x^{e^*}$. Further, $K_0 = k^2x^{78} \ge x^2$. By (Lm7) and (T2), $d \ge \exp(x^{\varepsilon_z/5-\varepsilon_z/100}) \ge x^2$. Since $0 \le \rho \le x^{-2} \le 1$ and $X' \ge 1$, we have $1 \le Y = X'(1+\rho) \le 2X'$.
- Small quantities. We have $0 \le x^\varepsilon \le L_a \le x^{\varepsilon+\theta_s} \le x^{2\varepsilon}$, $0 \le \rho \le x^{-2}$, $0 \le \vartheta = 1/(kx^{13}) \le x^{-1}$ and $\beta = s_0 = 0$.
- Moderate quantities. By (T2), $0 < S_c = 4k\Delta \le 4x^5\exp(x^{\varepsilon/4}/2) \le x^7\exp(x^{\varepsilon/4}/2) \le \exp(x^{\varepsilon/4})$. Moreover $0 < \eta \le \exp(x^{\varepsilon/4})$, $0 \le b_o \le \exp(x^\varepsilon/2)$ by hypothesis, $0 \le \max(C,0) \le K^*$ and $0 \le C_{\mathrm{ord}} \le K^*$.

The threshold p_0^*. By (T2), $k^2x^{26} \le x^{36} \le \exp(x^{\varepsilon/3}) = X'$. □

Lemma 5.13. *Let C, $b_o \ge 0$ and C_{ord} be real numbers, and let $\mathfrak{M}$ be a loss datum that satisfies (Lm1), (Lm2) and (Lm3) for (C, b_o), with $L_a, S_c, \eta \ge 0$ and $k^2x^{26} \le X'$, and let $P \ge 1$ be an integer. Put $J = \{1, \dots, m\}$ and $C^+ = \max(C, 0)$, and let f, N, S_k, Q_1, Q_2, λ, L_a, k, a, X', X'', w'', ϑ, p_0, K_0 be the parameters of the window parameters of $\mathfrak{M}$ for $(C, b_o, C_{\mathrm{ord}})$. Then the deficit data of $(\mathfrak{M}, P)$ have the following properties.*

1. *We have* $0 \le a_t \le a(1+\vartheta)$ *for* $t \in T$. *Each* $A(j,t,p)$ *is a sum of terms* $(a_t/p^i - w)^+$ *over a finite set of odd positive integers* i *with weights* $w = w_j(z_p(t,i)) \ge w''$, *and* $d(j,t) = \log d \sum_{i=1}^{P} (a_t/d^i - w_j(\sigma_t(g_i)))^+$ *with weights* $w_j(\sigma_t(g_i)) \ge 0$.

2. *Every served prime of* $\mathcal{P}$ *is at most* X'', *and every high prime of* $\mathcal{P}$ *exceeds* X''.

3. *For every served* $p \in \mathcal{P}$,

$$\sum_{j\in J} \sum_{t\in T_p^{\mathrm{s}}} (A(j,t,p) - \mathrm{lo}_{C,b_o}(p)) - |J|\,\mathrm{fl}_C(p) \le \sum_{j\in J} \mathrm{gain}_j^{\mathrm{D}}(p) - \tau_p,$$

   and for every $p \in \mathcal{P}$ *that is not served,* $\mathrm{gain}_j^{\mathrm{D}}(p) = 0$ *for all* j *and* $\tau_p = 0$.

4. $\mathrm{lo}_{C,b_o}(p) \le \mathrm{lo}^*(p)$ *and* $\mathrm{fl}_C(p) \le \mathrm{fl}^*(p)$ *for every prime* p, *where* $\mathrm{lo}^*, \mathrm{fl}^*$ *are formed with* C^+ *in place of* C *and the parameters above.*

5. $K_0 \le p^{L_1(p)-1}$ *for every prime* p.

Proof. Recall that $a = x\exp(x^\varepsilon) \ge 1$ as $x > 1$, that $w'' = b'/(xn) > 0$ and $X'' = xnX' \ge X'$, and that the window parameter $p_0 = \lceil p_0^* \rceil$ with $p_0^* = k^2x^{26}$, so that for a prime p one has $p < p_0$ if and only if $p < p_0^*$.

Item 1. The heights are bounded by (Lm2). The index sets are $\{i \le P : i \text{ odd}\}$ (for $p \le P$) or empty, and $\{1,\dots,P\}$. For odd i the place $z_p(t,i)$ lies in Φ, and $\sigma_t(g_i) \in \Phi$ since $g_i \notin \Phi$ and σ_t maps $\Gamma_H \setminus \Phi$ into Φ; so all weights are at least $w'' > 0$ by (Lm2).

Item 2. A served prime is not dropped, so (Dr1) fails and $p \le X''$. High primes exceed X'' by definition.

Item 3, served primes. Let p be served. It is not dropped, so $\mathrm{gain}_j^{\mathrm{D}}(p) = \mathrm{gain}_j(p)$. The terms of the series $A_p(a_t,t,w_j)$ are non-negative and $A(j,t,p)$ is the sum of some of them, so $A(j,t,p) \le A_p(a_t,t,w_j)$. The claimed inequality therefore follows from the served-prime inequality for (C,b_o) in (Lm3), with $|J| = m$.

Item 3, other primes. Let $p \in \mathcal{P}$ be dropped. Then $\mathrm{gain}_j^{\mathrm{D}}(p) = 0$ by definition, and $\tau_p = 0$ if $V_p = \emptyset$. Suppose $V_p \ne \emptyset$. Then p is a candidate with $\Sigma_p \le x^{-\varepsilon}$: it is not sign-changing, $p < X'$, h_p has order at least $x^{\varepsilon/4}$, and if $p \ge p_0^*$ no power p^i $(i \ge 1)$ satisfies $|p^i - X'| \le \rho X'$. We check that none of (Dr1)–(Dr3) holds. (Dr1): $p < X' \le X''$. (Dr2): by the last clause of (Lm1), h_p has even order. (Dr3): each of its three alternatives is excluded by the properties just listed. So p is not dropped, a contradiction; hence $V_p = \emptyset$ and $\tau_p = 0$.

Item 4. Let p be a prime. Then $\log_p a \ge 0$ and $\log_p(2a) \ge 0$. Since $fN^2 \ge 0$ and $C \le C^+$, $\mathrm{fl}_C(p) \le C^+fN^2\log_p a + C^+f(\log_p a)^2 = \mathrm{fl}^*(p)$. For lo_{C,b_o}, first $C(S_c+k+\eta)\log_p(2a) \le C^+S_k\log_p(2a)$ as $S_k = S_c+k+\eta \ge 0$. It remains to compare $\mathrm{pr}_{C,b_o}(p)+\mathrm{or}_C(p)$ with $\mathbf{1}_{p<X'}(Q(p)+C^+\lambda R(p))$, where $Q(p) = Q_1$ for $p < p_0$, $Q(p) = Q_2$ for $p \ge p_0$, and $R(p) = L_a a/(pk) + a\,p^{-L_1(p)}$ for $p < p_0$, $R(p) = L_a a/(pk) + a/p^2$ for $p \ge p_0$. Here $R(p) \ge 0$ and $Q_1, Q_2 \ge 0$ because $L_a, S_c, \eta, b_o \ge 0$.

- If p is inert, not sign-changing and $p < p_0^*$: then $p < X'$ (as $p_0^* \le X'$), $\mathrm{pr}_{C,b_o}(p) = Cr_Gb'\lambda \le C^+r_Gb'\lambda = Q_1 = Q(p)$ and $\mathrm{or}_C(p) = C\lambda R(p) \le C^+\lambda R(p)$.

- If p is inert, not sign-changing and $p_0^* \le p < X'$: then $\mathrm{pr}_{C,b_o}(p) = C(b'/x + \eta + S_c + (w''+t_R)\lambda/\delta_1) + b_o \le Q_2 = Q(p)$, the bracket being non-negative, and $\mathrm{or}_C(p) \le C^+\lambda R(p)$ as before.

- Otherwise (that is, p is not inert, or sign-changing, or $p \geq X'$ with $p \geq p_0^*$): $\mathrm{pr}_{C,b_o}(p) = \mathrm{or}_C(p) = 0$, which is at most the non-negative right side.

Adding gives $\mathrm{lo}_{C,b_o}(p) \leq \mathrm{lo}^*(p)$.

Item 5. We have $L_1(p) - 1 = \lceil \log_p K_0 \rceil \geq \log_p K_0$ with $K_0 = k^2 x^{78} > 0$, so $p^{L_1(p)-1} \geq K_0$. □

Lemma 5.14. *Let $\mathfrak{M}$ be a loss datum that satisfies (Lm1), with window parameters $Y = X'(1+\rho)$ and $p_0 = \lceil k^2 x^{26} \rceil$. Let p be an inert prime.*

1. *If p is of the first kind, then either $p \leq \max(Y, X'')$ and $[p]_{e_2}$ has order less than $x^{\varepsilon/4}$, or $p \geq p_0$ and $X'(1-\rho) \leq p^i \leq X'(1+\rho)$ for some integer $i \geq 1$.*
2. *If p is of the second kind, then p is a candidate and $\Sigma_p > x^{-\varepsilon}$.*
3. *If p is served and $t \in T \setminus T_p^{\mathrm{s}}$, then $p \in s_B$ and $\rho_p(t) \notin G_p$.*

Proof. Recall that $\rho \geq 0$, that $X'' = xnX' \geq X'$, and that for an integer p, $p \geq k^2 x^{26}$ if and only if $p \geq p_0$.

Item 1. A prime of the first kind is dropped and not high (so $p \leq X''$), and it is not of the second kind. As $p \leq X''$, (Dr1) fails, so (Dr2) or (Dr3) holds. Under (Dr2), p is not sign-changing and $h_p \neq 0$ has odd order; by (Lm1), $[p]_{e_2} = 1$, whose order 1 is less than $x^{\varepsilon/4}$ (as $x > 1$, $\varepsilon > 0$), and $p \leq X'' \leq \max(Y, X'')$. Under (Dr3), p is not sign-changing, $p < X'(1+\rho) = Y$, and one of three alternatives holds.

- The order of h_p is less than $x^{\varepsilon/4}$. By (Lm1) the order of $[p]_{e_2}$ divides that of h_p, so it is less than $x^{\varepsilon/4}$, and $p < Y \leq \max(Y, X'')$.
- $p \geq k^2 x^{26}$ and $|p^i - X'| \leq \rho X'$ for some $i \geq 1$. Then $p \geq p_0$ and $X'(1-\rho) \leq p^i \leq X'(1+\rho)$.
- p is a candidate with $\Sigma_p > x^{-\varepsilon}$. Since $p \leq X''$, p is then of the second kind, which is excluded.

Item 2. This is the definition of the second kind.

Item 3. If p were sign-changing or $p \geq X'$, then $T_p^{\mathrm{s}} = T$; so p is not sign-changing, $p < X'$, and $T_p^{\mathrm{s}} = \{t \in T : \rho_p(t) \in V_p\}$. As p is served it is not dropped; since p is not sign-changing and $p < X' \leq X'(1+\rho)$, the failure of (Dr3) means that h_p has order at least $x^{\varepsilon/4}$, that no power p^i ($i \geq 1$) satisfies $|p^i - X'| \leq \rho X'$ when $p \geq k^2 x^{26}$, and that p is not a candidate with $\Sigma_p > x^{-\varepsilon}$. The first two facts, with $p < X'$ and p not sign-changing, say that p is a candidate; hence $\Sigma_p \leq x^{-\varepsilon}$ and $V_p = G_p$. Therefore $\rho_p(t) \notin G_p$, and $p \in s_B$, since p is inert, not sign-changing, $p < X'$ and h_p has order at least $x^{\varepsilon/4}$. □

Lemma 5.15. *Let $\mathfrak{M}$ be a loss datum and $P \geq 1$ an integer, and form $S(\mathfrak{M})$ with the cut-off P.*

1. *With the deficit data of $(\mathfrak{M}, P)$, $S(\mathfrak{M}) = \sum_{j=1}^{m} \sum_{t \in T} \Big(\sum_{p \in \mathcal{P}} 2 \log p \, A(j,t,p) + d(j,t) \Big)$.*
2. *If $\mathfrak{M}$ satisfies (Lm6), then $I(\mathfrak{M}) = \sum_{p \text{ candidate}} \log p \ \mathrm{io}_p$.*
3. *If $\mathfrak{M}$ satisfies the surplus bound of (Lm3), then $\mathrm{io}_p \leq 2\Sigma_p \cdot 2m|T|a/p$ for every inert prime p.*

Proof. Item 1. By definition $A_{\mathrm{in}}(a_t, t, w_j)$ is the sum of a part over the inert primes $p \le P$ and a part at d. Let p be inert. As $\mathrm{Fr}_p \notin \Phi$ and Φ has index 2 in Γ_H, $i\,\mathrm{Fr}_p \notin \Phi$ exactly for odd i, and for odd i the place $\sigma_t(i\,\mathrm{Fr}_p)$ is the partner $z_p(t,i)$; so the summand of $A_{\mathrm{in}}(a_t, t, w_j)$ at $p \le P$ is $2\log p\, A(j,t,p)$. The inert primes $p \le P$ are the primes of $\mathcal{P}$ that are at most P, and the primes of $\mathcal{P}$ above P have $A(j,t,p) = 0$. At d, for each i the coset $i\,\mathrm{Fr}_d + I_d = \{i\,\mathrm{Fr}_d,\ i\,\mathrm{Fr}_d + ((1,0),0)\}$ has exactly one element outside Φ, namely g_i, so the part at d is $d(j,t)$. Summing over t and j gives the identity.

Item 2. Group the sum defining $I(\mathfrak{M})$ by $p = p(w)$. Places over primes that are not inert contribute 0 by (Lm6). If p is inert, each $w \in S$ over p has $\mathrm{io}(w) = \mathrm{io}_p$, and $\mathrm{io}_p = 0$ unless $V_p \neq \emptyset$, which requires p to be a candidate. A candidate satisfies $p < X' \le X'' = xnX'$, so by (Lm6) every finite place of K above p lies in S, and the places of S above p contribute $\mathrm{io}_p \sum_{w|p} \ell(w) = \mathrm{io}_p \log p$. There are finitely many candidates, all below X'.

Item 3. The orbit masses are non-negative, since $m \ge 1$ and $\kappa_p(v) \in [0,1]$, and $a > 0$. If $V_p = \emptyset$, then $\mathrm{io}_p = 0$ and the bound holds. Otherwise $\mathrm{io}_p = 2\Sigma_p \max_{v \in V_p} S_p(v)$, and by the surplus bound of (Lm3) and $N_p(v) = |T_p(v)| \le |T|$, every $S_p(v)$ with $v \in V_p$ is at most $2m|T|a/p$; multiplying by $2\Sigma_p \ge 0$ gives the claim. □

Lemma 5.16. *Fix an arithmetic datum on H with tops, orbit labellings, an orbit selection with $X' \le X''$, and a gain datum. Let K be a number field, S a finite set of finite places of K, $p(w)$ the residue characteristic of a place w, and $\ell(w)$ real numbers such that $\sum_{w|p} \ell(w) = \log p$ for every prime p, the sum running over all finite places of K above p. Let $g_j(w)$ $(1 \le j \le m)$ and $\tau(w)$ be real numbers $(w \in S)$ such that*

1. *every finite place w of K with $p(w) \le X''$ lies in S;*
2. *if $w \in S$ and $p(w)$ is inert, then $g_j(w) = \mathrm{gain}_j^{\mathrm{D}}(p(w))$ for all j and $\tau(w) = \tau_{p(w)}$;*
3. *if $w \in S$ and $p(w)$ is not inert, then $g_j(w) = 0$ for all j and $\tau(w) = 0$.*

Then for every real $B \ge X''$,

$$2\sum_{w \in S} \ell(w)\Big(\sum_{j=1}^{m} g_j(w) - \tau(w)\Big) = 2\sum_{\substack{p \le B \\ p \text{ inert}}} \log p\,\Big(\sum_{j=1}^{m} \mathrm{gain}_j^{\mathrm{D}}(p) - \tau_p\Big).$$

Proof. For a prime p put $F(p) = \sum_j \mathrm{gain}_j^{\mathrm{D}}(p) - \tau_p$ if p is inert and $F(p) = 0$ otherwise. By (2) and (3), the summand at $w \in S$ is $\ell(w)F(p(w))$.

Inert primes above X'' contribute nothing. Let $p > X''$ be inert. Then $\mathrm{gain}_j(p) = 0$ by definition, hence $\mathrm{gain}_j^{\mathrm{D}}(p) = 0$. Since $p > X'' \ge X'$, the prime p is not a candidate, so $V_p = \emptyset$ and $\tau_p = 0$. Thus $F(p) = 0$.

Grouping by residue characteristic. By the previous step, only places $w \in S$ with $p(w)$ inert and $p(w) \le X''$ contribute. For such a prime p, every place of K above p lies in S by (1), so the places of S above p are all the places above p and $\sum_{w \in S,\, w|p} \ell(w)F(p) = F(p)\log p$. Hence the left side equals $2\sum_{p \le X'',\ p \text{ inert}} F(p)\log p$, and adding the primes $X'' < p \le B$, which contribute 0, gives the claim. □

6 Tied host families and the proof of the main theorem

This section assembles the bound at one scale and passes to all scales. It proves that one core-tied host that satisfies the host conditions for a level $\mathfrak{L}$ gives $G_{\mathfrak{L}}(x) \le x^{1/2+(e_D+e_L+7)\varepsilon}$

(Proposition 6.5), it defines the families of hosts that provide such a host at every large scale of a set of scales, and it proves the criterion: tied admissible families on windows exclude zeros to the right of the critical line (Proposition 6.3). The main theorem 6.7 and the Riemann hypothesis Theorem 6.8 follow, given the existence theorem Proposition 7.8. The existence theorem is proved in Section 7; it is the only statement used here that is proved later, and the only one that depends on the existence of a theta chain system (TCS).

The bound at one host combines the two halves prepared above. The accounting gives the excess bound (Proposition 4.1), the averaging step turns it into a bound for the weighted sum of twists (Proposition 3.3), and the characters of the host identify this sum with $G_{\mathfrak{L}}(x)$ (Lemma 3.17). The constant 7 in the exponent is the sum of 3 from the size of the error term, 3 from the averaging step and 1 that absorbs the factor $1 + 4o$.

A host family assigns to every scale a set of members, each a host with a top set and a thick set (Definition 6.2). It is admissible for $\mathfrak{L}$ if its members satisfy the host conditions from some scale on. It is tied with exponents (e_D, e_L) if at every large scale of its set of scales one member carries the data of a core-tied host over all translates, with the divisor and loss exponents at most e_D and e_L (Definition 6.1). The essential point is that e_D and e_L do not depend on ε. This is where the uniformity is spent: in Proposition 6.3 the exponent $\frac{1}{2}+(e_D+e_L+7)\varepsilon$ comes as close to $\frac{1}{2}$ as we wish when ε is small, which is the hypothesis of the analytic criterion Proposition 2.2.

Definition 6.1. Let x be real, let (e, A, h, a_0) be a frame, and let $\mathcal{F}_x = (\mathfrak{h}_{x,i}, T_{x,i}, S_{x,i})_{i\in I(x)}$ be a family indexed by a set $I(x)$, where $\mathfrak{h}_{x,i}$ is a host for this frame, $T_{x,i} \subseteq A$ is a finite set (the **top set**) and $S_{x,i}$ is a finite set of non-negative integers (the **thick set**). Let $\varepsilon > 0$ and e_D, e_L, C_K, c^*, e_K be real numbers.

The family is **chain-tied at x with exponents $(e_D, e_L, C_K, c^*, e_K)$ and parameter ε** if there exist $i_0 \in I(x)$ and data as in the definition of core-tied hosts for $(\mathfrak{h}_{x,i_0}, T_{x,i_0})$, with the translates of the slope datum equal to A and the map tr equal to the identity, such that:

(Tc1) (arithmetic tie) $T = T_{x,i_0}$, every element of T_2 has the form $s\,a_0$ with $s \in S_{x,i}$ for some $i \in I(x)$, and η is compatible with $\mathfrak{h}_{x,i_0}$;

(Tc2) conditions (Ct2)–(Ct5) and (Ct7) hold, and (Ct3) holds with $\mathrm{tr} = \mathrm{id}_A$;

(Tc3) the tuple $(A_0, T, T_2, a|_T, (w_{u,j}), P)$ satisfies (Wd1)–(Wd5) for $(\mathfrak{A}_{\mathfrak{h}_{x,i_0}}, \eta)$ at scale x with parameters $(b, b', \nu, X', \theta_0)$, except that the inclusion $T_2 \subseteq A_0$ in (Wd1) is not required, and the weight clauses (Wd3) hold for every $u \in A$.

Definition 6.2. Let $\lambda > 1$. A set $W \subseteq \mathbb{R}$ **contains λ-windows** if for every $y \in \mathbb{R}$ there is $Y \geq y$ with $[Y, Y^\lambda] \subseteq W$.

Let $\varepsilon > 0$. A **host family with parameter** ε is a tuple

$$\mathcal{F} = (W,\ (\mathcal{F}_x)_{x\in\mathbb{R}}), \qquad \mathcal{F}_x = (\mathfrak{h}_{x,i}, T_{x,i}, S_{x,i})_{i\in I(x)},$$

consisting of a set $W \subseteq \mathbb{R}$, the **scales** of $\mathcal{F}$, and for each real x a set $I(x)$, a frame (e, A, h, a_0) depending on x, and for each $i \in I(x)$ a host $\mathfrak{h}_{x,i}$ for this frame, a finite set $T_{x,i} \subseteq A$ and a finite set $S_{x,i}$ of non-negative integers. Thus $\mathcal{F}_x$, the **family at** x, is a family of hosts at x as in Definition 6.1.

Let $\mathfrak{L}$ be a character level and $e_D, e_L \in \mathbb{R}$. The host family $\mathcal{F}$ with parameter ε is

- **admissible for $\mathfrak{L}$** if there are $C \geq 0$ and $x_0 > 1$ such that for every real $x \geq x_0$ and every $i \in I(x)$ the pair $(\mathfrak{h}_{x,i}, T_{x,i})$ satisfies the host conditions for $\mathfrak{L}$ at scale x with parameters (ε, C);

- **tied with exponents** (e_D, e_L) if there are real numbers C_K, c^*, e_K, x_0 such that for every $x \in W$ with $x \geq x_0$ the family $\mathcal{F}_x$ is chain-tied at x with exponents $(e_D, e_L, C_K, c^*, e_K)$ and parameter ε.

Proposition 6.3. *Let $\mathfrak{L}$ be a character level and $\lambda > 1$. Suppose that there are $e_D, e_L \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is a host family with parameter ε that is admissible for $\mathfrak{L}$, is tied with exponents (e_D, e_L), and whose scales contain λ-windows. Then $\mathfrak{L}$ is zero-free above $\frac{1}{2}$. In particular, if $\psi \in \mathcal{P}$ and the cyclic level $\mathfrak{L}_\psi$ satisfies this hypothesis, then $L(s, \psi) \neq 0$ for every s with $\operatorname{Re} s > \frac{1}{2}$.*

Proof. We verify the hypothesis of Proposition 2.2 for $\mathfrak{L}$, λ, the constant $c = e_D + e_L + 7$ and the same ε_0. Let $\varepsilon \in (0, \varepsilon_0)$, and let $\mathcal{F}$ be a host family as in the hypothesis, with scales W. By Proposition 6.4 there is x_1' with $G_{\mathfrak{L}}(x) \leq x^{1/2+c\varepsilon}$ for every $x \in W$ with $x \geq x_1'$; put $x_1 = \max(x_1', 1) \geq 1$. Then property 2 of Proposition 2.2 holds for (W, x_1), and property 1 holds because W contains λ-windows (in particular for every $y \geq 1$ there is $Y \geq y$ with $[Y, Y^\lambda] \subseteq W$). Hence Proposition 2.2 shows that $\mathfrak{L}$ is zero-free above $\frac{1}{2}$, and its last assertion gives the statement for $L(s, \psi)$ when $\mathfrak{L} = \mathfrak{L}_\psi$. □

Proposition 6.4. *Let $\mathfrak{L}$ be a character level, let $e_D, e_L \in \mathbb{R}$ and $\varepsilon > 0$, and let $\mathcal{F}$ be a host family with parameter ε and scales W that is admissible for $\mathfrak{L}$ and tied with exponents (e_D, e_L). Then there is x_1 such that*

$$G_{\mathfrak{L}}(x) \leq x^{1/2+(e_D+e_L+7)\varepsilon} \qquad \text{for every } x \in W \text{ with } x \geq x_1.$$

Proof. Let $C \geq 0$ and $x_E > 1$ be the constants of admissibility, and C_K, c^*, e_K, x_T those of tiedness (Definition 6.2). Let x_1' be the threshold that Proposition 6.5 gives for the level $\mathfrak{L}$, the numbers $(e_D, e_L, C_K, c^*, e_K, C)$ and ε, and put $x_1 = \max(x_1', x_E, x_T)$.

Let $x \in W$ with $x \geq x_1$. Since $x \geq x_T$, the family $\mathcal{F}_x$ is chain-tied at x with exponents $(e_D, e_L, C_K, c^*, e_K)$ and parameter ε, so by Lemma 6.6 there is $i \in I(x)$ such that $(\mathfrak{h}_{x,i}, T_{x,i})$ is core-tied at scale x with the same exponents and parameter. Since $x \geq x_E$, by admissibility the same pair satisfies the host conditions for $\mathfrak{L}$ at scale x with parameters (ε, C). As $x \geq x_1'$, Proposition 6.5, applied to the frame of $\mathcal{F}_x$, the host $\mathfrak{h}_{x,i}$ and the finite set $T_{x,i}$, gives $G_{\mathfrak{L}}(x) \leq x^{1/2+(e_D+e_L+7)\varepsilon}$. □

Proposition 6.5. *Let $\mathfrak{L}$ be a character level, let $e_D, e_L, C_K, c^*, e_K, C$ be real numbers and let $\varepsilon > 0$. There is x_1 such that the following holds for every $x \geq x_1$, every frame (e, A, h, a_0), every host $\mathfrak{h}$ for it and every finite set $T \subseteq A$. If $(\mathfrak{h}, T)$ satisfies the host conditions for $\mathfrak{L}$ at scale x with parameters (ε, C) and is core-tied at scale x with exponents $(e_D, e_L, C_K, c^*, e_K)$ and parameter ε, then*

$$G_{\mathfrak{L}}(x) \leq x^{1/2+(e_D+e_L+7)\varepsilon}.$$

Proof. Write $\mathfrak{L} = (K, \psi_{\mathrm{exc}}, o_{\mathfrak{L}})$ and $K_o = \sum_{\psi \in K \setminus \{\mathbf{1}\}} |L'(1, \psi)/L(1, \psi)|$.

Choice of x_1. Let x_1' be the threshold of Proposition 3.3 for $(C', \varepsilon, C_E, K_o, M, c) = (\max(e_D + e_L, 0) + 3,\ \varepsilon,\ C,\ K_o,\ 4o_{\mathfrak{L}},\ (3o_{\mathfrak{L}} + 1)^{-1})$. Let $x_1 \geq x_1'$ be such that, for $x \geq x_1$, the scale x is large for $(\varepsilon, C_K, c^*, e_K)$ in the sense of Definition 3.15, and $x^\varepsilon \geq 1 + 4o_{\mathfrak{L}}$.

Fix $x \geq x_1$ and $(\mathfrak{h}, T)$ as in the statement, together with the data of the core-tied condition: $\mathfrak{S}$, c, X'', tr, η, P, T_2, a, $(w_{u,j})$, ν, X', θ_0. Then:

- By (Ct4) and the size conditions, $b \geq 1$, $k \geq 1$, $a \geq xb \geq 0$, $m \leq x^{1/2+\varepsilon/2}$ and $xm \leq k$.
- By (Ct2) and the size conditions (Er5), $e_D + e_L \geq e_{\mathrm{Div}} + e_{\mathrm{Loss}} \geq 0$, so $C' = e_D + e_L + 3$.
- By (Ec1), $o = o_{\mathfrak{L}}$; by (Ct7), $m \geq x^{1/2-\varepsilon} > 0$, so $m \geq 1$.

Step 1: the excess. Let $u \in A_0$ and choose $v \in U$ with $\mathrm{tr}(v) = u$. The identity (Id) of Proposition 4.1 for $(\mathfrak{A}_{\mathfrak{h}}, P, \eta, T, T_2, a, (w_{u,j}))$ is (Ct3). Its other hypotheses are (Ct4), (Ct5) and the choice of x_1. For the last, $0 \leq c_{\mathrm{HH}} \leq c^*$ by (Er5) and (Ct2), and $e_K \geq 0$ by (Er2). Hence $200(1+C_K)^2(1+c_{\mathrm{HH}})(1+e_K) \leq 200(1+C_K)^2(1+c^*)(1+e_K)$, and the scale is large for $(\varepsilon, C_K, c_{\mathrm{HH}}, e_K)$. The excess there is the excess Δ of Definition 3.1 formed with $(\mathfrak{A}_{\mathfrak{h}}, \eta, P, T, a, (w_{u,j}))$. So Proposition 4.1 gives

$$\Delta(u) = \Delta(\mathrm{tr}(v)) \leq x^{(e_{\mathrm{Div}}+e_{\mathrm{Loss}}+3)\varepsilon} E \leq x^{(e_D+e_L+3)\varepsilon} E,$$

where E is the bracket of (Ex4) of Proposition 3.3 for $H = A$ and $a_E = a$; the two brackets agree, since $x^{2\varepsilon}|T_2|kb/|A|^2 = |T_2|kbx^{\varepsilon}/(|A|^2 x^{-\varepsilon})$. The second inequality uses $E \geq 0$, $x \geq 1$ and (Ct2).

Step 2: the one-sided bound. Apply Proposition 3.3 with $H = A$, the datum $\mathfrak{A}_{\mathfrak{h}}$ and $K' = K'_{\mathfrak{h}}$. The datum is of host type with respect to the prime factors of $Q_S q$ by Lemma 3.12. Condition (Ex1) is (Ct6); (Ex2) follows from the bounds on m, k above together with (Ct7); (Ex3) is item 4 of Lemma 3.17, with η compatible by (Ct1); and (Ex4) is Step 1. The proposition gives

$$G_{K'_{\mathfrak{h}}}(x) \leq x^{1/2+(e_D+e_L+6)\varepsilon} + \Sigma_{K'_{\mathfrak{h}}}(x).$$

Step 3: identification. By items 2 and 3 of Lemma 3.17, $G_{K'_{\mathfrak{h}}}(x) = G_{\mathfrak{L}}(x)$ and $\Sigma_{K'_{\mathfrak{h}}}(x) \leq 4o_{\mathfrak{L}} x^{1/2+\varepsilon} \leq 4o_{\mathfrak{L}} x^{1/2+(e_D+e_L+6)\varepsilon}$. Hence $G_{\mathfrak{L}}(x) \leq (1+4o_{\mathfrak{L}}) x^{1/2+(e_D+e_L+6)\varepsilon} \leq x^{1/2+(e_D+e_L+7)\varepsilon}$, by the choice of x_1. □

Lemma 6.6. *Let $\varepsilon > 0$, let $x, e_D, e_L, C_K, c^*, e_K$ be real numbers and let $\mathcal{F}_x = (\mathfrak{h}_{x,i}, T_{x,i}, S_{x,i})_{i \in I(x)}$ be a family as in Definition 6.1. If $\mathcal{F}_x$ is chain-tied at x with exponents $(e_D, e_L, C_K, c^*, e_K)$ and parameter ε, then there is $i \in I(x)$ such that $(\mathfrak{h}_{x,i}, T_{x,i})$ is core-tied at scale x with exponents $(e_D, e_L, C_K, c^*, e_K)$ and parameter ε.*

Proof. Take $i = i_0$ and the data of the chain tie, with translates $U = A$ and $\mathrm{tr} = \mathrm{id}_A$, whose image A contains A_0. By (Tc1), $T = T_{x,i_0}$ and η is compatible with $\mathfrak{h}_{x,i_0}$, which is (Ct1). Conditions (Ct2)–(Ct5) and (Ct7) are (Tc2). For (Ct6), the definition of weight data requires (Wd1)–(Wd5), with the weight clauses only for $u \in A_0$; these follow from (Tc3), which has them for every $u \in A$. The one clause not contained in (Tc3) is the inclusion $T_2 \subseteq A_0$ of (Wd1). It holds because every element of T_2 is a multiple $s\,a_0$ of the generator of A_0, by (Tc1). □

It remains to produce tied admissible families on $\frac{3}{2}$-windows for the cyclic level of every primitive character. This is Proposition 7.8, which Section 7 proves. We first assume it and deduce the two theorems. The proofs add only classical facts to the criterion: the Euler factors that separate a Dirichlet character from the primitive character inducing it, and, for the zeros of ζ to the left of the critical line, the functional equation [180].

Theorem 6.7. *Let $N \geq 1$ be an integer and let χ be a Dirichlet character modulo N. Then*

$$L(s,\chi) \neq 0 \qquad \text{for every } s \in \mathbb{C} \text{ with } \mathrm{Re}\, s > \tfrac{1}{2},$$

except at the pole $s = 1$ when $\chi = \chi_{0,N}$ is principal.

Proof. Primitive characters. Let $\psi \in \mathcal{P}$. By Proposition 7.8 there are $e_D, e_L \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is a host family with parameter ε that is admissible for $\mathfrak{L}_\psi$ (with the constant $C = 4$, so admissible in the sense of Definition 6.2), is tied with exponents (e_D, e_L), and whose scales contain $\frac{3}{2}$-windows. This is the hypothesis of Proposition 6.3 for the cyclic level $\mathfrak{L}_\psi$ and $\lambda = \frac{3}{2}$, and its last assertion gives

$$L(s, \psi) \neq 0 \qquad \text{for every } s \text{ with } \operatorname{Re} s > \tfrac{1}{2}.$$

Arbitrary characters. Let χ be a Dirichlet character modulo N, and let $\chi^* \in \mathcal{P}$ be the primitive character inducing it. Its modulus, the conductor of χ, divides N, and $\chi(n) = \chi^*(n)$ whenever $\gcd(n, N) = 1$. For $\operatorname{Re} s > 1$ the Euler products of $L(s, \chi)$ and $L(s, \chi^*)$ have the same factors at the primes $p \nmid N$, while at $p \mid N$ the factor of $L(s, \chi)$ is 1 and that of $L(s, \chi^*)$ is $(1 - \chi^*(p)p^{-s})^{-1}$. Hence

$$L(s, \chi) = L(s, \chi^*) \prod_{p \mid N} (1 - \chi^*(p)\, p^{-s}),$$

for $\operatorname{Re} s > 1$. Both sides are meromorphic on $\mathbb{C}$ with poles at most at $s = 1$ [180], so the identity extends to all s away from the pole at $s = 1$. This pole occurs only if $\chi^* = \mathbf{1}$, that is, if $\chi = \chi_{0,N}$ is principal. For $\operatorname{Re} s > 0$ and $p \mid N$ we have $|\chi^*(p)p^{-s}| \leq p^{-\operatorname{Re} s} < 1$, so no factor of the finite product vanishes. Applying the first part to χ^*, we obtain $L(s, \chi) \neq 0$ for every s with $\operatorname{Re} s > \frac{1}{2}$, except at the pole $s = 1$ when χ is principal. □

Theorem 6.8. *Every zero of the Riemann zeta function $\zeta(s)$ other than the trivial zeros $s = -2n$, $n \geq 1$, satisfies $\operatorname{Re} s = \frac{1}{2}$.*

Proof. Let $\zeta(s) = 0$, where s is not a trivial zero. By Theorem 6.7 with $N = 1$, for which $L(s, \chi_{0,1}) = \zeta(s)$, the zeta function has no zero with $\operatorname{Re} s > \frac{1}{2}$ (the point $s = 1$ is a pole, not a zero). It remains to exclude $\operatorname{Re} s < \frac{1}{2}$. Suppose, to the contrary, that $\operatorname{Re} s < \frac{1}{2}$.

Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, and let $\xi(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$ be the completed zeta function. By Riemann's functional equation [180], ξ is holomorphic on $\mathbb{C} \setminus \{0, 1\}$ and satisfies $\xi(1 - s) = \xi(s)$ there, and $\zeta(0) = -\frac{1}{2}$. The factor $\Gamma_{\mathbb{R}}$ has no zeros, and its poles are the points $0, -2, -4, \ldots$ (properties of the Gamma function). Since $\zeta(0) = -\frac{1}{2} \neq 0$, we have $s \neq 0$. As $s \neq 1$ and $s \neq 0$, the functional equation applies at s. Because s is not a trivial zero, $s \notin \{0, -2, -4, \ldots\}$, so $\Gamma_{\mathbb{R}}(s)$ is finite and non-zero, and $\xi(s) = 0$. Hence $\xi(1 - s) = 0$. Since $\operatorname{Re}(1 - s) > \frac{1}{2} > 0$, the factor $\Gamma_{\mathbb{R}}(1 - s)$ is finite and non-zero, so $\zeta(1 - s) = 0$. Here $\operatorname{Re}(1 - s) > \frac{1}{2}$ and $1 - s \neq 1$ (as $s \neq 0$), which contradicts the first paragraph. Hence $\operatorname{Re} s = \frac{1}{2}$. □

Corollary 6.9. *Let $N \geq 1$ and let χ be a Dirichlet character modulo N, primitive or not. Every zero s of $L(s, \chi)$ with $0 < \operatorname{Re} s < 1$ satisfies $\operatorname{Re} s = \frac{1}{2}$.*

Proof. Let $L(s, \chi) = 0$ with $0 < \operatorname{Re} s < 1$. Since $s \neq 1$, Theorem 6.7 gives $\operatorname{Re} s \leq \frac{1}{2}$; suppose $\operatorname{Re} s < \frac{1}{2}$.

Reduction to a primitive character. Let χ^* be the primitive character inducing χ, of modulus $f \mid N$. Comparing Euler products for $\operatorname{Re} s > 1$ and continuing meromorphically [180],

$$L(s, \chi) = L(s, \chi^*) \prod_{p \mid N} (1 - \chi^*(p)\, p^{-s}),$$

and for $\operatorname{Re} s > 0$ no factor of the finite product vanishes, since $|\chi^*(p)p^{-s}| \leq p^{-\operatorname{Re} s} < 1$. Hence $L(s, \chi^*) = 0$.

Reflection. Let $a \in \{0,1\}$ with $\chi^*(-1) = (-1)^a$, and put

$$\xi(s,\chi^*) = (f/\pi)^{(s+a)/2}\,\Gamma\big(\tfrac{s+a}{2}\big)L(s,\chi^*).$$

If $\chi^* \neq \mathbf{1}$, then $f \geq 3$ and the functional equation [180] gives $\xi(s,\chi^*) = W(\chi^*)\,\xi(1-s,\overline{\chi^*})$ for all s, with $|W(\chi^*)| = 1$. For $\chi^* = \mathbf{1}$ (so $f = 1$, $a = 0$) the functional equation is Riemann's $\xi(s) = \xi(1-s)$, valid for $s \notin \{0,1\}$, which holds in the strip. The factor $(f/\pi)^{(s+a)/2}\Gamma(\frac{s+a}{2})$ is finite and non-zero whenever $\operatorname{Re} s > 0$, because Γ has no zeros and its poles are at the non-positive integers. So $\xi(s,\chi^*) = 0$, hence $\xi(1-s,\overline{\chi^*}) = 0$, and since $\operatorname{Re}(1-s) > \frac{1}{2} > 0$ also $L(1-s,\overline{\chi^*}) = 0$. Now $\overline{\chi^*}$ is a primitive character modulo f, $\operatorname{Re}(1-s) > \frac{1}{2}$ and $1-s \neq 1$ because $s \neq 0$; this contradicts Theorem 6.7. Hence $\operatorname{Re} s = \frac{1}{2}$. □

7 The existence theorem from three inputs

This section proves the existence theorem Proposition 7.8: for every primitive character ψ there are exponents e_D, e_L such that for every small ε a host family with parameter ε is admissible for the cyclic level of ψ, is tied with exponents (e_D, e_L), and has scales containing $\frac{3}{2}$-windows. It is the statement that Section 6 left open. The proof uses three inputs: the supply of hosts on windows (Proposition 9.1, Part II), the window Sha bound (Theorem 36.4, Part VII), and a class of families with the link properties (Corollary 8.3, proved in Section 8 from Proposition 17.6 of Part III).

The key idea is to fix the exponents before ε is chosen, and for this the properties are asked of a class of families as a whole. A class with the link properties (Definition 7.7) contains, for every host requirement that the good-host requirement supplies on windows and for every small ε, a family with parameter ε whose members meet the requirement (design existence, (Lk1)). All its families have the linking property and well-chosen tops ((Lk2), (Lk3)). And all its families that satisfy the Sha condition carry slope data with one pair of exponents (e_D, e_L) for the whole class (Lk4). The shape theorem 7.10 turns such a class into tied families. The slope data of (Lk4) satisfy every slope condition except the loss condition (Sl8), and (Sl8) follows from (Lk2) and (Lk3) by the loss bound of Section 5, applied member by member (Corollary 7.11).

The existence theorem then follows in three steps, for a host requirement P_ψ that combines the host conditions for the cyclic level of ψ with the Sha bound for the host field. Hosts that meet P_ψ and the good-host requirement exist on $\frac{3}{2}$-windows, by the first two inputs and the supply of good hosts (Gh1). Every family whose members meet P_ψ satisfies the Sha condition, because the Sha bound is part of P_ψ. So the class contains tied families whose members meet P_ψ, and these are admissible because the host conditions are part of P_ψ (Proposition 7.9).

Several definitions precede the theorem. Three of them carry the argument: parametrised host families, in which every member has a rank and a loss datum (Definition 7.3); host requirements and the Sha condition (Definition 7.4); and the link properties (Definition 7.7). The notions of a normalised host, of the kernel form, of field rows and of a good window pair occur in Part I only inside the definition of the supply of good hosts; no proof of Part I uses them, and Part II works with them.

Definition 7.1. Let E be a number field, ℓ a prime, S_ℓ the set of places of E above ℓ, G_{E,S_ℓ} the Galois group of the maximal extension of E unramified outside S_ℓ, and $k \in \mathbb{Z}$. Put

$$\mathrm{Sha}^1_\ell(E,k) = \ker\Big(H^1(G_{E,S_\ell}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-k)) \to \prod_{v\in S_\ell} H^1(E_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-k))\Big),$$

with continuous cohomology, and let $\mathrm{len}_\ell(E,k) \in \mathbb{N} \cup \{\infty\}$ be its length as a $\mathbb{Z}_\ell$-module. For an integer $t \geq 1$ and $B \in \mathbb{R}$, the field E satisfies the **Sha bound** $\mathrm{V}(E;t,B)$ if $\mathrm{len}_\ell(E,k) < \infty$ for

all primes ℓ and all $1 \le k < t$, and

$$\sum_{\ell} \log \ell \sum_{k=1}^{t-1} \mathrm{len}_\ell(E,k) \le B.$$

Let $0 < \varepsilon < 1/52$. A **window pair at scale** Y is a pair of primes (q,d) with

$$q \equiv 3\ (8), \quad d \equiv 7\ (16), \quad \tfrac{1}{2}Y^{\varepsilon/5} < \log q \le Y^{\varepsilon/5}, \quad \tfrac{1}{2}Y^{\varepsilon/5}\log^2 Y < \log d \le Y^{\varepsilon/5}\log^2 Y,$$

$q \mid d+1$, and $p \mid d+1$ for every odd prime $p \le Y^{\varepsilon/5}$. A **page function** is a function $e_\varepsilon\colon \mathbb{R} \to \mathbb{N}$ with $x^{2+8\varepsilon} \le 2^{e_\varepsilon(x)} \le 2x^{2+8\varepsilon}$ for all $x \ge 1$.

Let $\psi \in \mathcal{P}$ have modulus N_ψ, let $e \ge 0$ be an integer and let q, d be primes. A number field E is a (ψ, e, q, d)**-field** if for some N divisible by 2^{e+2}, N_ψ, q and d, there is an embedding $E \hookrightarrow \mathbb{Q}(\zeta_N)$ whose image is the fixed field of the automorphisms $\zeta_N \mapsto \zeta_N^a$ with

$$a \equiv \pm 1\ (2^{e+2}), \qquad \psi(a) = 1, \qquad (a/q) = (a/d) = 1.$$

For $C_t > 0$, the character ψ satisfies the **window Sha bound with exponent** C_t if for every $\varepsilon \in (0, 1/52)$ and every page function e_ε there are $J \ge 1$ and X_0 such that for all $X \ge X_0$ and every window pair (q,d) at scale $X^{1/J}$ there is Y with $X^{1/J} \le Y$ and $Y^2 \le X$ such that every $(\psi, e_\varepsilon(x), q, d)$-field E with $x \in [Y, Y^2]$ satisfies $\mathrm{V}\big(E; \lfloor x^{C_t} \rfloor, \exp(x^\varepsilon/2)\big)$. The exponent used throughout is $C_t = 401$.

The group $\mathrm{Sha}^1_\ell(E,k)$ is the Tate–Shafarevich group of $\mathbb{Q}_\ell/\mathbb{Z}_\ell(-k)$ in the sense of Poitou–Tate duality [191]. The Sha bound $\mathrm{V}(E; t, B)$ bounds, with the weights $\log \ell$, the total length of these groups over all primes ℓ and all twists $1 \le k < t$. The bound $\exp(x^\varepsilon/2)$ of the window Sha bound is the square root of $b = \exp(x^\varepsilon)$, the size parameter of the scale x. The ranks of good hosts are at most x^{400} (Definition 7.7), and the exponent 401 covers the twists below them.

The fields to which the bound is applied are explicit. For a window pair (q,d), the Galois correspondence shows that a (ψ, e, q, d)-field is the compositum of the field cut out by ψ, of $\mathbb{Q}(\sqrt{-q}, \sqrt{-d})$ and of the real subfield of $\mathbb{Q}(\zeta_{2^{e+2}})$. In a window pair at scale Y one has $\log q \asymp Y^{\varepsilon/5}$ and $\log d \asymp Y^{\varepsilon/5}\log^2 Y$, so the conductors are enormous compared with the scale. Since $d \equiv -1$ modulo q and modulo every odd prime $p \le Y^{\varepsilon/5}$, all these primes split in $\mathbb{Q}(\sqrt{-d})$ (Lemma 9.4). A page function fixes the power of 2 in the conductor at the scale x, with 2^e between $x^{2+8\varepsilon}$ and $2x^{2+8\varepsilon}$.

The Sha bound is used in Parts III and IV. In Part III the divisor package (Sp4) bounds local defects by the lengths of these groups (Theorem 13.18, Theorem 13.2). In Part IV the packets are completed under the hypothesis that the Sha bound holds from some scale on (Proposition 19.3).

Definition 7.2. Let $\mathfrak{h} = (q, d, Q_S, \alpha)$ be a host for a frame (e, A, h, a_0) of index o, and let $N_\mathfrak{h} = Q_S q d$. The **field** $E_\mathfrak{h}$ of $\mathfrak{h}$ is the subfield of $\mathbb{Q}(\zeta_{N_\mathfrak{h}})$ fixed by the automorphisms $\zeta_{N_\mathfrak{h}} \mapsto \zeta_{N_\mathfrak{h}}^a$ with $\alpha(a) = 1$. The map α induces an isomorphism $\mathrm{Gal}(E_\mathfrak{h}/\mathbb{Q}) \cong \mathcal{G}$, and through it we write $\epsilon_1(\sigma)$ for $\sigma \in \mathrm{Gal}(E_\mathfrak{h}/\mathbb{Q})$.

Let $\psi \in \mathcal{P}$ have modulus N, and let $\langle\psi\rangle$ be the group of Dirichlet characters modulo N generated by ψ. The host $\mathfrak{h}$ has

- the **kernel form for** ψ if $N \mid Q_S$ and, for every a coprime to $N_\mathfrak{h}$, $\alpha(a) = 1$ if and only if $a \equiv \pm 1 \pmod{2^{e+2}}$, $\psi(a) = 1$ and $(a/q) = (a/d) = 1$;
- the **twist form for** ψ if for every character θ of $\mathcal{G}$ trivial on $\{(1,1)\} \times A_0$ there are $\xi \in \langle\psi\rangle$ and $i, j \in \{0,1\}$ with $\theta(\iota_d) = (-1)^j$ and $\chi_\theta(a) = \xi(a)(a/q)^i(a/d)^j$ for all a coprime to $N_\mathfrak{h}$.

For an integer $b \geq 0$, the host has **field rows up to** b if

(Fr1) $\epsilon_1(\alpha(u)) = (u/q)$ for every u coprime to $N_\mathfrak{h}$;

(Fr2) there is $s \in E_\mathfrak{h}$ with $s^2 = -q$ and $\sigma(s) = \epsilon_1(\sigma)\, s$ for every $\sigma \in \mathrm{Gal}(E_\mathfrak{h}/\mathbb{Q})$;

(Fr3) for every prime $l \mid Q_S$ there is u coprime to $N_\mathfrak{h}$ with $u \equiv 1 \pmod{N_\mathfrak{h}/l}$ and $\alpha(u) \neq 1$;

(Fr4) $Q_S \leq |A|^2$;

(Fr5) $o^{64} \leq 2^e$;

(Fr6) $q \neq 3$;

(Fr7) $(-d/p) = 1$ for every odd prime $p \leq b$;

(Fr8) $4l \leq b$ for every odd prime $l \mid Q_S$.

Definition 7.3. Let $\varepsilon > 0$. A **parametrised host family with parameter** ε is a host family $\mathcal{F} = (W, (\mathcal{F}_x)_{x \in \mathbb{R}})$ with parameter ε, whose members at x are $(\mathfrak{h}_{x,i}, T_{x,i}, S_{x,i})_{i \in I(x)}$ for the frame (e, A, h, a_0) at x, together with the following data:

- a real number $x_1 > 1$, the **threshold** of $\mathcal{F}$;
- for every real $x > x_1$ and every $i \in I(x)$, an integer $t_{x,i} \geq 1$, the **rank** of the member, and a loss datum $\mathfrak{M}_{x,i}$, the **loss datum** of the member;

subject to the following conditions.

(Pf1) Ties. For every $x > x_1$ and $i \in I(x)$, the loss datum $\mathfrak{M}_{x,i}$ is on the group $H = A$, its arithmetic datum is $\mathfrak{A}_{\mathfrak{h}_{x,i}}$, its set of tops is $T_{x,i}$, its orbit selection has scale x and parameter ε, and its size parameter t_R equals $t_{x,i}$.

(Pf2) Frame, heights, place record. For every $x > x_1$ and $i \in I(x)$, $\mathfrak{M}_{x,i}$ satisfies (Lm1), (Lm2) and (Lm6).

(Pf3) Sizes. There are real numbers $\varepsilon_z > 0$ and C_t such that for every $x > x_1$ and $i \in I(x)$, $\mathfrak{M}_{x,i}$ satisfies (Lm7) for (ε_z, C_t).

Put $b(x) = \exp(x^\varepsilon)$.

Properties of a parametrised host family. The parametrised host family $\mathcal{F}$ with parameter ε has

- the **linking property** if there are real numbers C, x_0 and a function $\delta\colon \mathbb{R} \to \mathbb{R}$ with $\delta(x) \to 0$ as $x \to \infty$ such that for every real $x \geq x_0$ with $x > x_1$ and every $i \in I(x)$ the loss datum $\mathfrak{M}_{x,i}$ satisfies (Lm3) for the loss constants $\big(C,\, b(x)^{\delta(x)}\big)$, that is, the served-prime inequality for $\big(C, b(x)^{\delta(x)}\big)$ and the surplus bound (thus the constant C and the function δ are the same for every x and every i);
- **few bad tops** if there is a real x_0 such that $\mathfrak{M}_{x,i}$ satisfies (Lm4) for every $x \geq x_0$ with $x > x_1$ and every $i \in I(x)$;
- **small shares** if there is a real x_0 such that $\mathfrak{M}_{x,i}$ satisfies (Lm5) for every $x \geq x_0$ with $x > x_1$ and every $i \in I(x)$;

- **non-empty top sets** if there is a real x_0 such that $T_{x,i} \neq \emptyset$ for every $x \geq x_0$ with $x > x_1$ and every $i \in I(x)$.

Definition 7.4. A **host requirement** is a property P of tuples $(x, \varepsilon, \mathfrak{h}, T, t)$ consisting of real numbers x and ε, a host $\mathfrak{h}$ for a frame (e, A, h, a_0), a finite set $T \subseteq A$ and an integer $t \geq 1$.

Let $\lambda > 1$ and let P be a host requirement. The class $\mathcal{C}_\lambda(\mathsf{P})$ consists of the parametrised host families $\mathcal{F}$, with any parameter ε, such that

(Rq1) the set of scales W of $\mathcal{F}$ contains λ-windows;

(Rq2) $\mathsf{P}(x, \varepsilon, \mathfrak{h}_{x,i}, T_{x,i}, t_{x,i})$ holds for every real $x > x_1$ and every $i \in I(x)$.

A parametrised host family $\mathcal{F}$ with parameter ε satisfies the **Sha condition** if there is a real x_0 such that $\mathrm{V}(E_{\mathfrak{h}_{x,i}};\, t_{x,i},\, \exp(x^\varepsilon/2))$ holds for every real $x \geq x_0$ with $x > x_1$ and every $i \in I(x)$.

Definition 7.5. Let $\mathfrak{h} = (q, d, Q_S, \alpha)$ be a host for a frame (e, A, h, a_0), and let $N_\mathfrak{h} = Q_S q d$. For $u \in (\mathbb{Z}/N_\mathfrak{h}\mathbb{Z})^\times$ let $\alpha_A(u) \in A$ be the **component of** $\alpha(u)$ **in** A, so that $\alpha(u) = ((s_1, s_2), \alpha_A(u))$ for some signs s_1, s_2. Since $2^{e+2} \mid Q_S$, every such u has a class $[u]_e \in H_e$, the image of u mod 2^{e+2}. The host $\mathfrak{h}$ is **normalised** if

(Hn1) $\alpha(-1) = ((-1,-1), 0)$;

(Hn2) $h(\alpha_A(u)) = [u]_e$ for every $u \in (\mathbb{Z}/N_\mathfrak{h}\mathbb{Z})^\times$;

(Hn3) $\alpha(u)^2 = 1$ for every $u \in (\mathbb{Z}/N_\mathfrak{h}\mathbb{Z})^\times$ with $u \equiv 1 \pmod{Q_S d}$.

When α is viewed as the Artin map of the field of the host, (Hn1) says that complex conjugation corresponds to $((-1,-1),0)$, (Hn2) that h reads off the action on the 2^{e+2}-th roots of unity modulo ± 1, and (Hn3) that the inertia group at q has exponent dividing 2.

Definition 7.6. Let $Q > 0$. A Dirichlet character χ has a **zero-free box of height** Q if $L(s, \chi) \neq 0$ for every s with $\operatorname{Re} s > 1 - \frac{1}{400}$ and $|\operatorname{Im} s| \leq Q$.

Let $M \geq 1$ be an integer, ξ a Dirichlet character modulo M, q, d odd primes not dividing M, and $(i, j) \in \{0,1\}^2$. The **twist** $\xi_{q,d}^{(i,j)}$ is the Dirichlet character modulo $2Mqd$

$$\xi_{q,d}^{(i,j)}(n) = \xi(n)\Big(\frac{n}{q}\Big)^i \Big(\frac{n}{d}\Big)^j \qquad (\gcd(n, 2Mqd) = 1),$$

and $(\xi_{q,d}^{(i,j)})^*$ is the primitive character inducing it. For $0 < \varepsilon < 1/52$, real $Y \geq 1$ and $j \in \{0,1\}$ put

$$h_j(M, \varepsilon, Y) = 8M \exp\big(Y^{\varepsilon/5}\big) \exp\big(Y^{\varepsilon/5} \log^2 Y\big)^j.$$

Let $0 < \varepsilon < 1/52$ and $Y \leq X$. A window pair (q, d) at scale Y is **good for** (M, X) if

(Wp1) $\gcd(M, qd) = 1$;

(Wp2) for every Dirichlet character ξ modulo M and every $(i, j) \neq (0, 0)$, the character $\chi = (\xi_{q,d}^{(i,j)})^*$ satisfies, for all $y \in [Y, X]$,

$$|g(y, \chi)| \leq y^{1/2+\varepsilon/2}, \qquad \Big|\frac{L'(1,\chi)}{L(1,\chi)}\Big| \leq 4 \log y;$$

(Wp3) for every real primitive character ϱ of modulus dividing $8M$ and every $(i,j) \neq (0,0)$, the character $(\varrho_{q,d}^{(i,j)})^*$ has a zero-free box of height $h_j(M,\varepsilon,Y)$, where the twist is formed with $8M$ in place of M.

Since $q, d \nmid M$ and $q \neq d$ (as $q \equiv 3$ and $d \equiv 7 \pmod 8$), a twist with $(i,j) \neq (0,0)$ has a non-trivial Jacobi-symbol factor modulo q or d that the other factors cannot cancel, so the characters in (Wp2) and (Wp3) are non-principal. Hence $L(1,\chi) \neq 0$ and $g(y,\chi)$ is given by (2.2).

Definition 7.7. A **good-host requirement** is a property $\mathfrak{G}$ of tuples $(x, \varepsilon, J, \kappa, \mathfrak{h}, T, t)$ consisting of real numbers x, ε, J, a finite list κ of real numbers, a host $\mathfrak{h}$ for a frame (e, A, h, a_0), a finite set $T \subseteq A$ and an integer $t \geq 1$. Let $\lambda > 1$. A host requirement P is **supplied by $\mathfrak{G}$ on λ-windows** if for every $\varepsilon \in (0, 1/52)$ there are a finite list κ of real numbers, a real number $J \geq 1$, a set $W \subseteq \mathbb{R}$ containing λ-windows and a real number x_0 such that for every $x \in W$ with $x \geq x_0$ there are a frame (e, A, h, a_0), a host $\mathfrak{h}$ for it, a finite set $T \subseteq A$ and an integer $t \geq 1$ with $\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}, T, t)$ and $\mathsf{P}(x, \varepsilon, \mathfrak{h}, T, t)$.

Supply of good hosts. Let $\psi \in \mathcal{P}$ have modulus N. A good-host requirement $\mathfrak{G}$ **supplies good hosts for** ψ if for every $\varepsilon \in (0, 1/52)$ and every $C_0 \geq 1$ there are a finite list κ of real numbers and a real number x_B with the following property. Let $x \geq x_B$ and $J \in [1, C_0]$, let (e, A, h, a_0) be a frame of index o_ψ, where $o_\psi \leq C_0$, with $x^{2+8\varepsilon} \leq |A| \leq 2x^{2+8\varepsilon}$, let M be an integer with $0 < M \leq C_0$, and let $\mathfrak{h} = (q, d, Q_S, \alpha)$ be a host for this frame such that

- $Q_S \leq C_0|A|$ and $Q_S \mid \operatorname{lcm}(2^{e+2}, M)$;
- $\mathfrak{h}$ is normalised and has the kernel form for ψ;
- for some scale $Y \in [x^{1/J}, x]$, the pair (q, d) is a window pair at scale Y that is good for (M, x), and $\mathfrak{h}$ has field rows up to $\lfloor Y^{\varepsilon/5} \rfloor$.

Then there are a finite set $T \subseteq A$ for which $(\mathfrak{h}, T)$ satisfies (Ec4) at scale x with parameter ε, and an integer t with $1 \leq t \leq x^{400}$, such that $\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}, T, t)$ holds. The requirement $\mathfrak{G}$ **supplies good hosts** if it supplies good hosts for every $\psi \in \mathcal{P}$.

Slope data at the members. Let $\mathcal{F}$ be a parametrised host family with parameter ε, threshold x_1, scales W and loss data $\mathfrak{M}_{x,i}$, and let e_D, e_L be real numbers. Then $\mathcal{F}$ has **slope data at its members with exponents** (e_D, e_L) if there is a real number c_0 such that for every real $c_L \geq c_0$ there are real numbers C_K, c^*, e_K, x_0 with the following property: for every $x \in W$ with $x \geq x_0$ and $x > x_1$ there are $i_0 \in I(x)$, data as in the definition of core-tied hosts for $(\mathfrak{h}_{x,i_0}, T_{x,i_0})$ at scale x with exponents $(e_D, e_L, C_K, c^*, e_K)$ and parameter ε, whose slope datum $\mathfrak{S}$ has the translates A, the map $\mathrm{tr} = \mathrm{id}_A$ and the constant vector c with $c_{\mathrm{Loss}} = c_L$ and $e_{\mathrm{Loss}} = 0$, and members $i_u \in I(x)$ $(u \in A)$, such that

- (Tc1), (Tc2) and (Tc3) hold, except that the loss condition (Sl8) is not required in (Ct4);
- the parameter b of $\mathfrak{S}$ is $\exp(x^\varepsilon)$, and for every $u \in A$ the loss datum $\mathfrak{M}_{x,i_u}$ has the integer m of $\mathfrak{S}$, a set of tops with $|\mathcal{T}|$ elements, and $S_u = S(\mathfrak{M}_{x,i_u})$ formed with the cut-off $\lceil 2x\exp(x^\varepsilon) \rceil$, $\Theta_u = \Theta(\mathfrak{M}_{x,i_u})$, $I_u = I(\mathfrak{M}_{x,i_u})$.

Link properties. Let $\mathfrak{G}$ be a good-host requirement. A class $\mathcal{A}$ of parametrised host families has the **link properties for $\mathfrak{G}$** if:

(Lk1) Design existence. For every $\lambda > 1$ and every host requirement P supplied by $\mathfrak{G}$ on λ-windows there is $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ some family in $\mathcal{A} \cap \mathcal{C}_\lambda(\mathsf{P})$ has parameter ε.

(Lk2) Linking property. Every family in $\mathcal{A}$ has the linking property.

(Lk3) Choice of tops. Every family in $\mathcal{A}$ has few bad tops, small shares and non-empty top sets.

(Lk4) Slope data. There are real numbers e_D, e_L such that every family in $\mathcal{A}$ that satisfies the Sha condition has slope data at its members with exponents (e_D, e_L).

Proposition 7.8. *Let $\psi \in \mathcal{P}$. There are $e_D, e_L \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is a host family with parameter ε with the following properties: it is admissible for $\mathfrak{L}_\psi$, with the constant $C = 4$ in the definition of admissibility; it is tied with exponents (e_D, e_L); and its scales contain $\frac{3}{2}$-windows.*

Proof. Fix $\psi \in \mathcal{P}$. Let P_ψ be the host requirement such that $\mathsf{P}_\psi(x, \varepsilon, \mathfrak{h}, T, t)$ holds if and only if $x > 1$, the pair $(\mathfrak{h}, T)$ satisfies the host conditions for $\mathfrak{L}_\psi$ at scale x with parameters $(\varepsilon, 4)$, and $\mathrm{V}(E_\mathfrak{h}; t, \exp(x^\varepsilon/2))$ holds. Let $\mathfrak{G}$ be a good-host requirement with (Gh1) and (Gh2) as in Proposition 7.9.

Hosts on windows. By Theorem 36.4, ψ satisfies the window Sha bound with exponent 401, and by (Gh1) $\mathfrak{G}$ supplies good hosts, in particular for ψ, which is hypothesis (Sb) of Proposition 9.1. So Proposition 9.1 gives, for every $\varepsilon \in (0, 1/52)$, a finite list κ, a real $J \geq 1$, a set W_ε containing $\frac{3}{2}$-windows and a real x_0 such that every $x \in W_\varepsilon$ with $x \geq x_0$ carries a frame, a host $\mathfrak{h}$ for it, a finite set $T \subseteq A$ and an integer $t \geq 1$ satisfying its items 1–3. Replace x_0 by $\max(x_0, 2)$. For $x \geq x_0$ items 1 and 2 give $\mathsf{P}_\psi(x, \varepsilon, \mathfrak{h}, T, t)$, since $x > 1$, and item 3 gives $\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}, T, t)$. This is hypothesis 1 of (Gh2) for $\lambda = \frac{3}{2}$ and $\mathsf{P} = \mathsf{P}_\psi$.

The Sha condition. Let $\mathcal{F} \in \mathcal{C}_{3/2}(\mathsf{P}_\psi)$ have parameter ε and threshold x_1. By (Rq2), $\mathsf{P}_\psi(x, \varepsilon, \mathfrak{h}_{x,i}, T_{x,i}, t_{x,i})$ holds for every $x > x_1$ and $i \in I(x)$; in particular, $\mathrm{V}(E_{\mathfrak{h}_{x,i}}; t_{x,i}, \exp(x^\varepsilon/2))$ holds. So $\mathcal{F}$ satisfies the Sha condition (with $x_0 = x_1 + 1$). This is hypothesis 2 of (Gh2).

Conclusion. By (Gh2) there are $e_D, e_L \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ some $\mathcal{F} \in \mathcal{C}_{3/2}(\mathsf{P}_\psi)$ with parameter ε is tied with exponents (e_D, e_L). It is a host family with parameter ε. Its scales contain $\frac{3}{2}$-windows by (Rq1). It is admissible for $\mathfrak{L}_\psi$ with $C = 4$: put $x_0' = x_1 + 1 > 1$; for every $x \geq x_0'$ and $i \in I(x)$ we have $x > x_1$, so (Rq2) gives P_ψ at (x, i), whose first clause is that $(\mathfrak{h}_{x,i}, T_{x,i})$ satisfies the host conditions for $\mathfrak{L}_\psi$ at scale x with parameters $(\varepsilon, 4)$. □

Proposition 7.9. *There is a good-host requirement $\mathfrak{G}$ with the following two properties.*

(Gh1) Supply of good hosts. $\mathfrak{G}$ *supplies good hosts.*

(Gh2) Families from hosts. *Let $\lambda > 1$ and let P be a host requirement. Suppose that*

1. *for every $\varepsilon \in (0, 1/52)$ there are a finite list κ of real numbers, a real number $J \geq 1$, a set $W \subseteq \mathbb{R}$ containing λ-windows and a real number x_0 such that for every $x \in W$ with $x \geq x_0$ there are a frame (e, A, h, a_0), a host $\mathfrak{h}$ for it, a finite set $T \subseteq A$ and an integer $t \geq 1$ with $\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}, T, t)$ and $\mathsf{P}(x, \varepsilon, \mathfrak{h}, T, t)$;*
2. *every parametrised host family in $\mathcal{C}_\lambda(\mathsf{P})$ satisfies the Sha condition.*

Then there are $e_D, e_L \in \mathbb{R}$ and $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1)$ some parametrised host family in $\mathcal{C}_\lambda(\mathsf{P})$ with parameter ε is tied with exponents (e_D, e_L).

Proof. By Corollary 8.3 there are a good-host requirement $\mathfrak{G}$ that supplies good hosts and a class $\mathcal{A}$ of parametrised host families with the link properties for $\mathfrak{G}$. So $\mathfrak{G}$ has (Gh1). For (Gh2), let $\lambda > 1$ and let P be a host requirement satisfying 1 and 2. Hypothesis 1 says exactly that P is supplied by $\mathfrak{G}$ on λ-windows. By hypothesis 2 every family in $\mathcal{C}_\lambda(\mathsf{P})$, in particular every family in $\mathcal{A} \cap \mathcal{C}_\lambda(\mathsf{P})$, satisfies the Sha condition. So Theorem 7.10 applies to $(\mathfrak{G}, \mathcal{A}, \lambda, \mathsf{P})$ and gives $e_D, e_L \in \mathbb{R}$ and $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1)$ some family in $\mathcal{A} \cap \mathcal{C}_\lambda(\mathsf{P}) \subseteq \mathcal{C}_\lambda(\mathsf{P})$ with parameter ε is tied with exponents (e_D, e_L). This is the conclusion of (Gh2). □

Theorem 7.10. *Let $\mathfrak{G}$ be a good-host requirement and let $\mathcal{A}$ be a class of parametrised host families with the link properties for $\mathfrak{G}$. Let $\lambda > 1$, and let P be a host requirement supplied by $\mathfrak{G}$ on λ-windows such that every family in $\mathcal{A} \cap \mathcal{C}_\lambda(\mathsf{P})$ satisfies the Sha condition. Then there are $e_D, e_L \in \mathbb{R}$ and $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1)$ some family in $\mathcal{A} \cap \mathcal{C}_\lambda(\mathsf{P})$ with parameter ε is tied with exponents (e_D, e_L).*

Proof. Choice of e_D, e_L, ε_1. Let e_D, e_L be the real numbers of (Lk4) for $\mathcal{A}$, and let $\varepsilon_0 > 0$ be the number that (Lk1) gives for λ and P (which is supplied by $\mathfrak{G}$ on λ-windows). Put $\varepsilon_1 = \min(\varepsilon_0, \frac{1}{52}) > 0$. Let $\varepsilon \in (0, \varepsilon_1)$. By (Lk1) there is a family $\mathcal{F} \in \mathcal{A} \cap \mathcal{C}_\lambda(\mathsf{P})$ with parameter ε; let W be its set of scales and x_1 its threshold. We show that $\mathcal{F}$ is tied with exponents (e_D, e_L).

The loss constant. By (Lk2) and (Lk3), $\mathcal{F}$ has the linking property, few bad tops, small shares and non-empty top sets, and $0 < \varepsilon < \frac{1}{52}$. Hence Corollary 7.11 applies to $\mathcal{F}$ and gives real numbers $c_0^{\mathrm{L}} \geq 1$ and x_0^{L} with its properties 1 and 2. By small shares there is x_W such that $\mathfrak{M}_{x,i}$ satisfies (Lm5) for every $x \geq x_W$ with $x > x_1$ and every $i \in I(x)$.

The slope data. By hypothesis $\mathcal{F}$ satisfies the Sha condition, so by (Lk4) it has slope data at its members with exponents (e_D, e_L): there is c_0^{S} such that for every $c_L \geq c_0^{\mathrm{S}}$ there are $C_K, c^*, e_K, x_0^{\mathrm{S}}$ with the property stated there. Take $c_L = \max(c_0^{\mathrm{L}}, c_0^{\mathrm{S}})$, let $C_K, c^*, e_K, x_0^{\mathrm{S}}$ be the corresponding numbers, and put

$$x_0 = \max(x_0^{\mathrm{L}},\, x_W,\, x_0^{\mathrm{S}},\, x_1 + 1).$$

These numbers depend on $\mathcal{F}$ only, not on the scale.

Chain ties at a scale. Let $x \in W$ with $x \geq x_0$; then $x > x_1$. By the choice of $C_K, c^*, e_K, x_0^{\mathrm{S}}$ there are an element $i_0 \in I(x)$, data as in the definition of core-tied hosts for $(\mathfrak{h}_{x,i_0}, T_{x,i_0})$ at scale x with exponents $(e_D, e_L, C_K, c^*, e_K)$ and parameter ε, and members $i_u \in I(x)$ $(u \in A)$, with the following properties.

- The slope datum $\mathfrak{S}$ of these data has the translates A, the map $\mathrm{tr} = \mathrm{id}_A$ and the constant vector c with $c_{\mathrm{Loss}} = c_L$ and $e_{\mathrm{Loss}} = 0$, and its parameter b is $b(x)$.
- (Tc1), (Tc2) and (Tc3) hold except for the loss condition (Sl8) in (Ct4).
- For every $u \in A$, the loss datum $\mathfrak{M}_{x,i_u}$ has m weights, where m is the integer of $\mathfrak{S}$, and a set of tops with $n_{\mathcal{T}}$ elements; moreover $S_u = S(\mathfrak{M}_{x,i_u})$, formed with the cut-off $\lceil 2x \exp(x^\varepsilon) \rceil \geq 1$, $\Theta_u = \Theta(\mathfrak{M}_{x,i_u})$ and $I_u = I(\mathfrak{M}_{x,i_u})$.

We check the hypotheses of Corollary 7.11, property 2, for $\mathfrak{S}$ and c:

- by (Ct2), which is part of (Tc2) and does not involve (Sl8), the parameters $x(\mathfrak{S})$ and $\varepsilon(\mathfrak{S})$ are x and ε; and $x \geq x_0^{\mathrm{L}}$, $x > x_1$, $b = b(x)$;
- $c_{\mathrm{Loss}} = c_L \geq c_0^{\mathrm{L}}$ and $e_{\mathrm{Loss}} = 0 \geq 0$;

- for every translate $u \in A$, the member $i_u \in I(x)$ has a loss datum satisfying (Lm5), because $x \geq x_W$ and $x > x_1$; its number of weights is m; its set of tops is T_{x,i_u} by (Pf1), so $|T_{x,i_u}| = n_{\mathcal{T}}$; and S_u, Θ_u, I_u are its prime sum with the cut-off $P_u = \lceil 2x \exp(x^{\varepsilon}) \rceil$, its lattice term and its companion sum.

Hence, by Corollary 7.11, property 2, $\mathfrak{S}$ satisfies (Sl8). So (Ct4) holds in full, and with it (Tc1), (Tc2) and (Tc3): the family $\mathcal{F}_x$ is chain-tied at x with exponents $(e_D, e_L, C_K, c^*, e_K)$ and parameter ε, with the element i_0 and the data above.

Conclusion. The numbers C_K, c^*, e_K, x_0 do not depend on x, and $\mathcal{F}_x$ is chain-tied at every $x \in W$ with $x \geq x_0$. Thus, for every $\varepsilon \in (0, \varepsilon_1)$, the family $\mathcal{F} \in \mathcal{A} \cap \mathcal{C}_\lambda(\mathsf{P})$ has parameter ε and is tied with exponents (e_D, e_L), where e_D, e_L, ε_1 do not depend on ε. □

Corollary 7.11. *Let $0 < \varepsilon < \frac{1}{52}$ and let $\mathcal{F}$ be a parametrised host family with parameter ε and threshold x_1 that has the linking property, few bad tops and non-empty top sets. Then there are real numbers $c_0 \geq 1$ and x_0 with the following properties.*

1. (Loss bound at the members.) *For every real $x \geq x_0$ with $x > x_1$, every $i \in I(x)$ whose loss datum $\mathfrak{M}_{x,i}$ satisfies (Lm5), and every cut-off $P \geq 1$,*

$$S(\mathfrak{M}_{x,i}) - \Theta(\mathfrak{M}_{x,i}) \leq c_0 \, |T_{x,i}| \, m \, b(x) \qquad \text{and} \qquad I(\mathfrak{M}_{x,i}) \leq |T_{x,i}| \, m \, b(x),$$

   where m is the number of weights of $\mathfrak{M}_{x,i}$. In particular, if $\mathcal{F}$ also has small shares, there is x_0' such that these bounds hold for every $x \geq x_0'$ with $x > x_1$ and every $i \in I(x)$.

2. (The loss condition.) *Let $\mathfrak{S}$ be a slope datum with parameters x, ε and b, where $x \geq x_0$ is a real number with $x > x_1$, ε is the parameter of $\mathcal{F}$ and $b = b(x)$, and let c be a constant vector with $c_{\mathrm{Loss}} \geq c_0$ and $e_{\mathrm{Loss}} \geq 0$. Suppose that to every translate u of $\mathfrak{S}$ there are attached a member $i_u \in I(x)$ whose loss datum $\mathfrak{M}_{x,i_u}$ satisfies (Lm5) and a cut-off $P_u \geq 1$, such that the number of weights of $\mathfrak{M}_{x,i_u}$ is the parameter m of $\mathfrak{S}$, $|T_{x,i_u}| = n_{\mathcal{T}}$, and*

$$S_u = S(\mathfrak{M}_{x,i_u}) \text{ (formed with the cut-off } P_u), \qquad \Theta_u = \Theta(\mathfrak{M}_{x,i_u}), \qquad I_u = I(\mathfrak{M}_{x,i_u}).$$

   Then $\mathfrak{S}$ satisfies the loss condition (Sl8): $S_u - \Theta_u \leq \mathrm{Lo}$ and $I_u \leq \mathrm{Lo}$ for every translate u.

The constant c_0 depends only on the constant C of the linking property: one may take $c_0 = 3\max(C, 0) + 13$.

Proof. The constants. By the linking property there are real numbers C, x_G and a function δ with $\delta(x) \to 0$ as $x \to \infty$ such that for every $x \geq x_G$ with $x > x_1$ and every $i \in I(x)$ the loss datum $\mathfrak{M}_{x,i}$ satisfies (Lm3) for $(C, b(x)^{\delta(x)})$. By few bad tops there is x_B such that $\mathfrak{M}_{x,i}$ satisfies (Lm4) for $x \geq x_B$, $x > x_1$, $i \in I(x)$. By non-empty top sets there is x_T with $T_{x,i} \neq \emptyset$ for $x \geq x_T$, $x > x_1$, $i \in I(x)$. By (Pf3) there are $\varepsilon_z > 0$ and C_t such that every $\mathfrak{M}_{x,i}$ $(x > x_1)$ satisfies (Lm7) for (ε_z, C_t). Let x^* be the number that Theorem 5.6 gives for $(\varepsilon, \varepsilon_z, C_t, C)$, and choose x_δ with $\delta(x) \leq \frac{1}{2}$ for $x \geq x_\delta$. Put

$$c_0 = 3\max(C, 0) + 13 \geq 1, \qquad x_0 = \max(x_G, x_B, x_T, x^*, x_\delta).$$

Item 1. Let $x \geq x_0$ with $x > x_1$, let $i \in I(x)$ be such that $\mathfrak{M}_{x,i}$ satisfies (Lm5), and let $P \geq 1$ be a cut-off. Put $b_o = b(x)^{\delta(x)} = \exp(\delta(x) x^{\varepsilon})$; then $0 < b_o \leq \exp(x^{\varepsilon}/2)$, since $\delta(x) \leq \frac{1}{2}$ and $x^{\varepsilon} > 0$. We check the hypotheses of Theorem 5.6 for $\mathfrak{M} = \mathfrak{M}_{x,i}$ and this b_o. By (Pf1) the orbit selection of $\mathfrak{M}_{x,i}$ has scale $x \geq x^*$ and parameter ε, and its set of tops is $T_{x,i}$, which

is non-empty as $x \geq x_T$. (Lm1), (Lm2) and (Lm6) hold by (Pf2); (Lm3) for (C, b_o) holds as $x \geq x_G$; (Lm4) holds as $x \geq x_B$; (Lm5) holds by assumption; and (Lm7) for (ε_z, C_t) holds by (Pf3). Hence, by Theorem 5.6, with $b = \exp(x^{\varepsilon}) = b(x)$,

$$S(\mathfrak{M}_{x,i}) - \Theta(\mathfrak{M}_{x,i}) \leq c_0\,|T_{x,i}|\,m\,b(x), \qquad I(\mathfrak{M}_{x,i}) \leq |T_{x,i}|\,m\,b(x).$$

If $\mathcal{F}$ has small shares, there is x_W such that $\mathfrak{M}_{x,i}$ satisfies (Lm5) for every $x \geq x_W$ with $x > x_1$ and every $i \in I(x)$; then $x_0' = \max(x_0, x_W)$ has the stated property.

Item 2. Let u be a translate. By item 1 for the member i_u and the cut-off P_u, and by the identifications,

$$S_u - \Theta_u \leq c_0\,n_{\mathcal{T}}\,m\,b, \qquad I_u \leq n_{\mathcal{T}}\,m\,b \leq c_0\,n_{\mathcal{T}}\,m\,b,$$

the last step because $c_0 \geq 1$ and $n_{\mathcal{T}} mb \geq 0$. Since $x > x_1 > 1$ and $e_{\mathrm{Loss}}\varepsilon \geq 0$, we have $x^{e_{\mathrm{Loss}}\varepsilon} \geq 1$; with $c_{\mathrm{Loss}} \geq c_0 \geq 0$ and $n_{\mathcal{T}} mb \geq 0$ this gives $c_0\,n_{\mathcal{T}}\,m\,b \leq c_{\mathrm{Loss}}\,n_{\mathcal{T}}\,m\,b\,x^{e_{\mathrm{Loss}}\varepsilon} = \mathrm{Lo}$. Hence $S_u - \Theta_u \leq \mathrm{Lo}$ and $I_u \leq \mathrm{Lo}$. □

The constant $c_0 = 3\max(C, 0) + 13$ of the corollary depends only on the constant C of the linking property, and the loss exponent is 0. So the loss condition costs no power of x^{ε}, and the second exponent of a tied family produced by the shape theorem is the one that (Lk4) provides.

With this the existence theorem is reduced to three statements: Proposition 9.1, Theorem 36.4 and Corollary 8.3. Parts II and VII prove the first two. The next section proves the third from Proposition 17.6.

8 Slope records: what a class must supply

This section proves the last link of the reduction and states what the arithmetic Parts must supply. The link property (Lk4) asks for slope data at the members of a family. A slope datum is an abstract record, and Parts III and IV do not produce it directly. They produce, for each member of a family, a slope record (Definition 8.1): the words of the member with their ranks, Gauss primes, slopes, heights, ambient slopes and exponents, the dual slopes, the frame term and two auxiliary primes. The properties (Sp1)–(Sp8) of slope records (Definition 8.2) are the slope conditions other than (Sl8), together with the size and weight conditions of a core-tied host, written for these records, with the constants of the host in place of abstract parameters: half the root discriminant $v(\mathfrak{h})$ of the host field, the number $\beta(\mathfrak{h}) = L'(1, \chi_d)/L(1, \chi_d)$, and $\log \Omega(\mathfrak{h}) = \gamma + \log(\pi/v(\mathfrak{h})) - \beta(\mathfrak{h})$.

The main result is Theorem 8.4: if every family of a class that satisfies the Sha condition has (Sp1)–(Sp8), with the exponent e in the divisor package (Sp4), then every such family has slope data at its members with exponents $(\max(e, 0), 0)$. The proof assembles one slope datum at each scale from the slope records and loss data of the members that carry the translates $u \in A$, and verifies the slope conditions other than (Sl8), the size conditions and the conditions on weight data one by one. It is long and contains no idea beyond the dictionary of Section 4; the table below records which property gives which condition.

Corollary 8.3 then repackages Proposition 17.6 of Part III. That proposition provides a good-host requirement and a class of families with slope records that has (Lk1), (Lk2), (Lk3) and, in place of (Lk4), the property (Lk5): the families of the class that satisfy the Sha condition have (Sp1)–(Sp8). The corollary passes to the underlying families and obtains (Lk4) from (Lk5) by Theorem 8.4. In Part III the divisor exponent is $e = 1$ (Theorem 17.7). The proofs therefore give the tied families of the existence theorem the exponents $(e_D, e_L) = (1, 0)$, and the bound of Proposition 6.4 reads $G_{\mathfrak{L}_\psi}(x) \leq x^{1/2+8\varepsilon}$ at the large scales of these families.

Definition 8.1. Let $m \geq 0$ be an integer. A **slope record** for m is a tuple $\mathfrak{V}$ consisting of

- a non-empty finite set $\mathcal{W}(\mathfrak{V})$, the **words** of $\mathfrak{V}$;
- for every word $\omega \in \mathcal{W}(\mathfrak{V})$: an integer $d(\mathfrak{V},\omega) \geq 0$, the **rank** of ω; a prime $T(\mathfrak{V},\omega)$, the **Gauss prime** of ω; and real numbers $\mu(\mathfrak{V},\omega)$, $\mu^{\max}(\mathfrak{V},\omega)$ (the **slope** and the **maximal slope** of ω), $h(\mathfrak{V},\omega)$ (the **height** of ω), $\mu'(\mathfrak{V},\omega)$, $\mu'^{\max}(\mathfrak{V},\omega)$ (the **ambient slope** and the **maximal ambient slope** of ω) and $E_j(\mathfrak{V},\omega)$ for $1 \leq j \leq m$ (the **exponents** of ω);
- real numbers $\mu^{\vee}(\mathfrak{V})$ and $\mu^{\vee}_{\min}(\mathfrak{V})$ (the **dual slope** and the **minimal dual slope**), $F(\mathfrak{V})$ (the **frame term**) and $\delta(\mathfrak{V})$ (the **irreducibility defect**);
- two primes $g_1(\mathfrak{V})$ and $g_2(\mathfrak{V})$, the **auxiliary primes**.

Attached to $\mathfrak{V}$ are the **total exponent** $E(\mathfrak{V},\omega) = \sum_{j=1}^{m} E_j(\mathfrak{V},\omega)$ of a word ω, the **mean slope** $s(\mathfrak{V})$, the **top slope** $s^{\max}(\mathfrak{V})$ and the **spread** $s^{\max}(\mathfrak{V}) - s(\mathfrak{V})$, where

$$s(\mathfrak{V}) = \frac{\sum_{\omega\in\mathcal{W}(\mathfrak{V})} d(\mathfrak{V},\omega)\big(\mu(\mathfrak{V},\omega) - h(\mathfrak{V},\omega)\big)}{\sum_{\omega\in\mathcal{W}(\mathfrak{V})} d(\mathfrak{V},\omega)}, \qquad s^{\max}(\mathfrak{V}) = \max_{\omega\in\mathcal{W}(\mathfrak{V})} \big(\mu^{\max}(\mathfrak{V},\omega) - h(\mathfrak{V},\omega)\big),$$

and $s(\mathfrak{V}) = 0$ if all ranks are 0.

Let $\mathcal{F}$ be a parametrised host family with parameter ε, threshold x_1 and frame (e, A, h, a_0) at x. **Slope records for** $\mathcal{F}$ are, for every real $x > x_1$ and every $i \in I(x)$, a slope record $\mathfrak{V}_{x,i}$ for the integer m of $\mathfrak{M}_{x,i}$ and an element $u_{x,i} \in A$, the **translate** of the member, such that the following hold.

(Sr1) Ties within a scale. For every $x > x_1$ and all $i, i' \in I(x)$: $\mathfrak{h}_{x,i} = \mathfrak{h}_{x,i'}$, $T_{x,i} = T_{x,i'}$ and $S_{x,i} = S_{x,i'}$; the loss data $\mathfrak{M}_{x,i}$ and $\mathfrak{M}_{x,i'}$ have the same parameters k, m, η and L_a; and $g_1(\mathfrak{V}_{x,i}) = g_1(\mathfrak{V}_{x,i'})$, $g_2(\mathfrak{V}_{x,i}) = g_2(\mathfrak{V}_{x,i'})$.

(Sr2) Carriers of translates. For every $x > x_1$, every $i \in I(x)$ and every $u \in A$ there is $i' \in I(x)$ with $u_{x,i'} = u$ such that $\mathfrak{M}_{x,i'}$ and $\mathfrak{M}_{x,i}$ have the same heights a_t ($t \in T_{x,i}$).

A **parametrised host family with slope records** is a parametrised host family together with slope records for it.

For a member at $x > x_1$ we write $T^{\mathrm{th}}_{x,i} = \{s\, a_0 : s \in S_{x,i}\} \subseteq A$ (the **thick set**). With $n = |A|$, $b(x) = \exp(x^{\varepsilon})$ and the parameters k, m of $\mathfrak{M}_{x,i}$, the **budgets** of the member are

$$B_0(x,i) = 2\,n\,m\,b(x)\,x^{\varepsilon}, \qquad N_c(x,i) = \frac{|T^{\mathrm{th}}_{x,i}|\,k\,b(x)\,x^{2\varepsilon}}{n^2}.$$

Definition 8.2. For a host $\mathfrak{h} = (q, d, Q_S, \alpha)$, put $v(\mathfrak{h}) = \frac{1}{2}\,|\mathrm{disc}(E_{\mathfrak{h}})|^{1/[E_{\mathfrak{h}}:\mathbb{Q}]}$, let χ_d be the Legendre symbol modulo the prime d, and put

$$\beta(\mathfrak{h}) = \frac{L'(1,\chi_d)}{L(1,\chi_d)}, \qquad \log\Omega(\mathfrak{h}) = \gamma + \log\frac{\pi}{v(\mathfrak{h})} - \beta(\mathfrak{h}).$$

Let $\mathcal{F}$ be a parametrised host family with slope records, with parameter ε, threshold x_1, scales W and frame (e, A, h, a_0) at x. In each item below, for a member (x, i) with $x > x_1$, we write $\mathfrak{V} = \mathfrak{V}_{x,i}$ and $\mathfrak{M} = \mathfrak{M}_{x,i}$, with its parameters k, m, η, heights a_t ($t \in T_{x,i}$) and weights w_j ($1 \leq j \leq m$), and

$$n = |A|, \quad f = 2n, \quad b = e^{x^{\varepsilon}}, \quad a = xb, \quad L_a = \log\big(2a\,v(\mathfrak{h}_{x,i})\big), \quad X' = e^{x^{\varepsilon/3}},$$
$$b' = \frac{a}{X'}, \quad X'' = xnX', \quad A_{\mathfrak{M}} = \sum_{t\in T_{x,i}} a_t, \quad B_0 = B_0(x,i), \quad N_c = N_c(x,i).$$

Here L_a is the **analytic loss** of the member, $L_a = \log\bigl(a\,|\mathrm{disc}(E_{\mathfrak{h}_{x,i}})|^{1/[E_{\mathfrak{h}_{x,i}}:\mathbb{Q}]}\bigr)$. It is not the size parameter of the loss datum $\mathfrak{M}$, which is not used in the items below. A statement holds "for all large members" if there is a real x_0 such that it holds for every real $x \ge x_0$ with $x > x_1$, every $i \in I(x)$ and all words $\omega, \omega' \in \mathcal{W}(\mathfrak{V})$. The family $\mathcal{F}$ has each of the following properties if the stated condition holds.

(Sp1) Counting bounds There is a real $C \ge 0$ such that, for all large members,

$$\frac{1}{f}\sum_{j=1}^{m}\sum_{\tau\in\Phi}\log\frac{w_j(\tau)\,v(\mathfrak{h}_{x,i})}{\pi} + m\log(g_1(\mathfrak{V})g_2(\mathfrak{V})) - C\Bigl(L_a + \frac{m^2 L_a}{k}\Bigr) \le \log T(\mathfrak{V},\omega) \le C\,m\,x^{\varepsilon}$$

and $|\log T(\mathfrak{V},\omega) - \log T(\mathfrak{V},\omega')| \le C\log k$.

(Sp2) Ambient slope bounds There is a real $C \ge 0$ such that, for all large members,

$$-C|T_{x,i}|\,a\,L_a(1+m/k) \le \mu'(\mathfrak{V},\omega) \quad\text{and}\quad \mu'^{\max}(\mathfrak{V},\omega) \le C|T_{x,i}|\,a\,L_a.$$

(Sp3) Transfer of slopes There is a real $C \ge 0$ such that, for all large members,

$$\begin{aligned}\mu^{\max}(\mathfrak{V},\omega) &\le \mu'^{\max}(\mathfrak{V},\omega) + \Theta(\mathfrak{M}) + F(\mathfrak{V}) - E(\mathfrak{V},\omega) + C\,m\,B_0,\\ \mu(\mathfrak{V},\omega) &\ge \mu'(\mathfrak{V},\omega) + \Theta(\mathfrak{M}) + F(\mathfrak{V}) - E(\mathfrak{V},\omega)\\ &\quad - \Bigl(C\,I(\mathfrak{M}) + C|T_{x,i}|\,m\Bigl(\frac{a}{k}+b'\Bigr)\log T(\mathfrak{V},\omega) + CX'' + 1\Bigr).\end{aligned}$$

(Sp4) Divisor package with exponent e Here e is a real number. There is a real C such that, for all large members, the following hold.

- $d(\mathfrak{V},\omega) > 0$ and $\mu(\mathfrak{V},\omega) \le \mu^{\max}(\mathfrak{V},\omega)$.
- We have

$$\mu^{\vee}(\mathfrak{V}) + h(\mathfrak{V},\omega) - F(\mathfrak{V}) \le \sum_{t\in T_{x,i}}\Bigl(m\bigl(a_t\log a_t - a_t - a_t\log\Omega(\mathfrak{h}_{x,i})\bigr) - a_t\log T(\mathfrak{V},\omega)\Bigr) + C|T_{x,i}|\,a\,L_a + C\,m\,(B_0+N_c)\,x^{e\varepsilon}.$$

- $|E(\mathfrak{V},\omega) - m\,A_{\mathfrak{M}}\log(g_1(\mathfrak{V})g_2(\mathfrak{V}))| \le C\,m\,B_0$.
- We have

$$\bigl|(h(\mathfrak{V},\omega) + A_{\mathfrak{M}}\log T(\mathfrak{V},\omega)) - (h(\mathfrak{V},\omega') + A_{\mathfrak{M}}\log T(\mathfrak{V},\omega'))\bigr| \le C(|T_{x,i}|\,a\,L_a + m\,B_0).$$

(Sp5) Dual-slope bound There are real numbers $c_\vee \ge 0$ and x_0 with the following property. For every real $x \ge x_0$ with $x > x_1$ there are real numbers $\bar\eta \ge 0$ and $\bar e \in [0,\frac14]$ with $(\bar\eta + 4\sqrt{\bar e})\,x^{c_\vee} \le 6$ such that, with $H = x^{c_\vee}b$, every $i \in I(x)$ satisfies the following.

- $0 \le \delta(\mathfrak{V}) \le \frac12$ and $s(\mathfrak{V}) - \mu^{\vee}_{\min}(\mathfrak{V}) \le H$.
- There are $\eta_{\mathfrak{V}} \in [0,\bar\eta]$ and $e_{\mathfrak{V}} \in [0,\bar e]$ with

$$s(\mathfrak{V}) - \log|\mathcal{W}(\mathfrak{V})| - \frac{1}{1-\delta(\mathfrak{V})}\Bigl(4\phi\log\Bigl(1+\frac{H}{\phi}\Bigr) + (\eta_{\mathfrak{V}} + 4\sqrt{e_{\mathfrak{V}}})H\Bigr) \le \mu^{\vee}(\mathfrak{V}),$$
$$\phi = s^{\max}(\mathfrak{V}) - s(\mathfrak{V}),$$

where $4\phi\log(1+H/\phi)$ is read as 0 when $\phi = 0$.

(Sp6) Weights at the nodes For all large members, every j and every $\tau \in \Phi$, we have $w_j(\tau) \geq 1$; moreover $b \leq w_j(\tau) \leq b + \eta + 1$ if τ is thick at $u_{x,i}$ for $T^{\mathrm{th}}_{x,i}$, and $w_j(\tau) \leq 2b'$ otherwise.

(Sp7) Thin scales There are real numbers $c > 0$ and x_0 such that, for every real $x \geq x_0$ with $x > x_1$ and every $i \in I(x)$, there is an $\eta \colon \widehat{\Gamma}_A \to \mathbb{C}$ compatible with $\mathfrak{h}_{x,i}$ satisfying $|G_{\mathfrak{A}_{\mathfrak{h}_{x,i}},\eta}(y,\phi)| \leq e^{-cx^{\varepsilon/8}}\, y$ for every $\phi \in \widehat{\Gamma}_A$ and every $y \geq X'/3$.

(Sp8) Sizes We have $\varepsilon \leq 1$, and for every real $c > 0$ there are real numbers $e_K \geq 0$ and x_0 such that, for every real $x \geq x_0$ with $x > x_1$, we have $I(x) \neq \emptyset$ if $x \in W$, and every $i \in I(x)$ satisfies

$$m \leq x^{1/2+\varepsilon/2}, \quad x^{1/2-\varepsilon} \leq m, \quad xm \leq k, \quad k \leq x^{e_K}, \quad k \leq a, \quad k \leq n^2 x^{1/2-\varepsilon},$$
$$0 \leq L_a \leq 2x^{\varepsilon}, \quad kb' \leq a, \quad X'' \leq a, \quad \log|\mathcal{W}(\mathfrak{V})| \leq (k-3)\log 2,$$
$$2f(x+1)e^{-cx^{\varepsilon/8}} \leq 1/x, \quad 6f(x+1)(\eta+1) \leq b/x, \quad x^{-\varepsilon}n \leq |T^{\mathrm{th}}_{x,i}|, \quad \tfrac{1}{2}x^{-\varepsilon/8}n \leq |T_{x,i}|,$$

and $|a_t/b - x| \leq 1$ for every $t \in T_{x,i}$.

Corollary 8.3. *There are a good-host requirement $\mathfrak{G}$ that supplies good hosts and a class $\mathcal{A}$ of parametrised host families that has the link properties for $\mathfrak{G}$.*

Proof. By Proposition 17.6 there are a good-host requirement $\mathfrak{G}$ with (Gh1), that is, a good-host requirement that supplies good hosts, and a class $\mathcal{A}_0$ of parametrised host families with slope records satisfying (Lk1), (Lk2), (Lk3) and (Lk4′) of that statement, the last with a real exponent e. In that statement a family with slope records is identified with its underlying parametrised host family. Let $\mathcal{A}$ be the class of the parametrised host families underlying the members of $\mathcal{A}_0$. We check the link properties of $\mathcal{A}$ for $\mathfrak{G}$.

(Lk1)–(Lk3). The clauses (Lk1), (Lk2), (Lk3) of Proposition 17.6 are conditions on the underlying parametrised host families: membership in $\mathcal{C}_\lambda(\mathsf{P})$, the parameter, the linking property, few bad tops, small shares and non-empty top sets all depend only on the underlying family. Hence they are the clauses (Lk1), (Lk2), (Lk3) of the link properties for $\mathcal{A}$ and $\mathfrak{G}$. For (Lk1), a member of $\mathcal{A}_0$ in $\mathcal{C}_\lambda(\mathsf{P})$ with parameter ε has its underlying family in $\mathcal{A} \cap \mathcal{C}_\lambda(\mathsf{P})$ with parameter ε; for (Lk2) and (Lk3), every family in $\mathcal{A}$ underlies a member of $\mathcal{A}_0$.

(Lk4). By (Lk4′) of Proposition 17.6, every family in $\mathcal{A}_0$ that satisfies the Sha condition has the properties (Sp1)–(Sp8), with (Sp4) for the exponent e. Hence Theorem 8.4 applies to the class $\mathcal{A}_0$ and the exponent e: every family in $\mathcal{A}_0$ that satisfies the Sha condition has slope data at its members with exponents $(\max(e,0),0)$. The Sha condition and slope data at the members depend only on the threshold, the scales, the members (hosts, top sets, index sets) and the loss data of the underlying parametrised host family. Let $\mathcal{F} \in \mathcal{A}$ satisfy the Sha condition, and let $\mathcal{F}_0 \in \mathcal{A}_0$ have underlying family $\mathcal{F}$. Then $\mathcal{F}_0$ satisfies the Sha condition, so $\mathcal{F}$ has slope data at its members with exponents $(\max(e,0),0)$. This is (Lk4) with $(e_D, e_L) = (\max(e,0),0)$. □

Theorem 8.4. *Let $\mathcal{A}$ be a class of parametrised host families with slope records and let e be a real number. Assume that every family in $\mathcal{A}$ that satisfies the Sha condition has the counting bounds (Sp1), the ambient slope bounds (Sp2), the transfer of slopes (Sp3), the divisor package with exponent e (Sp4), the dual-slope bound (Sp5), the weights at the nodes (Sp6), thin scales (Sp7) and the sizes (Sp8). Put $e_D = \max(e,0)$. Then every family in $\mathcal{A}$ that satisfies the Sha condition has slope data at its members with exponents $(e_D, 0)$.*

Proof. Let $\mathcal{F} \in \mathcal{A}$ satisfy the Sha condition; let ε be its parameter, x_1 its threshold and W its scales. By hypothesis, $\mathcal{F}$ has (Sp1)–(Sp8), with (Sp4) for the exponent e.

Constants. Let $C_1, C_2, C_3 \geq 0$ be the constants of (Sp1), (Sp2), (Sp3), let C_d be the constant of (Sp4) and $C_4 = \max(C_d, 0)$, let $c_\vee \geq 0$ be the constant of (Sp5), and $c > 0$ the constant of (Sp7); apply (Sp8) with this c and let $e_K \geq 0$ be the exponent it gives. Let X_0 be the largest of the thresholds x_0 of (Sp1)–(Sp8) (for (Sp8) the one belonging to c). Put $c_0 = 0$. Given a real $c_L \geq c_0$, put

$$C_K = C_4 + C_1 + C_2 + C_3 + c_L + 6, \qquad c^* = c_\vee, \qquad x_0 = X_0,$$

and let e_K be as above. We show that for every $x \in W$ with $x \geq x_0$ and $x > x_1$ there are i_0, core-tied data and members i_u with the properties required in the definition of slope data at the members with exponents $(e_D, 0)$; since $c_L \geq c_0$ was arbitrary, this proves the theorem.

The members at x. Fix such an x. By (Sp8), $I(x) \neq \emptyset$; fix $i_0 \in I(x)$. By (Sr2), for every $u \in A$ there is $i_u \in I(x)$ with translate $u_{x,i_u} = u$ such that $\mathfrak{M}_{x,i_u}$ has the heights a_t of $\mathfrak{M}_{x,i_0}$; fix such i_u. By (Sr1) all members at x have the host $\mathfrak{h} := \mathfrak{h}_{x,i_0}$, the top set $T := T_{x,i_0}$, the thick set $T_2 := T^{\mathrm{th}}_{x,i_0}$ and the same parameters k, m and $\nu := \eta$ of their loss data, and the same auxiliary primes g_1, g_2; hence also the same $B_0 := B_0(x,i)$, $N_c := N_c(x,i)$ and the same analytic loss $L_a := \log(2x\, e^{x^\varepsilon} v(\mathfrak{h}))$, which depends only on x and the host. By (Pf1), $\mathfrak{M}_u := \mathfrak{M}_{x,i_u}$ is on the group A with arithmetic datum $\mathfrak{A}_\mathfrak{h}$, set of tops T and orbit selection at scale x with parameter ε. Write $\mathfrak{V}_u = \mathfrak{V}_{x,i_u}$, $w_j^{(u)} \colon \Phi \to \mathbb{R}$ for the weights of $\mathfrak{M}_u$, $n = |A|$, $f = 2n$, $b = e^{x^\varepsilon}$, $a = xb$, $X' = e^{x^{\varepsilon/3}}$, $b' = a/X'$, $X'' = xnX'$, $A_T = \sum_{t\in T} a_t$ and $P = \lceil 2a \rceil$. All the properties (Sp1)–(Sp8) apply at x to every member, since $x \geq X_0$.

Elementary facts at x. (i) $x > x_1 > 1$, so $x \geq 1$, $x^\varepsilon \geq 1$, $b > 1$ and $a \geq b > 0$; by (Sp8), $0 < \varepsilon \leq 1$. (ii) By (Sp8), $m \geq x^{1/2-\varepsilon} > 0$, so the integer m is at least 1, and $k \geq xm \geq x^{3/2-\varepsilon} \geq 1$ because $3/2 - \varepsilon > 0$. (iii) By (Sp8), $|T| \geq \frac{1}{2} x^{-\varepsilon/8} n > 0$, so $|T| \geq 1$. (iv) By (Lm2) (through (Pf2)), $0 \leq a_t \leq a(1+\vartheta)$ with $\vartheta = 1/(kx^{13}) \leq 1$, so $0 \leq a_t \leq 2a \leq P$; by (Sp8), $a_t \geq (x-1)b > 0$. (v) $|\Phi| = f$: $\Phi = \tau(\ker \epsilon_2)$ and $|\ker \epsilon_2| = 2|A|$. (vi) $v(\mathfrak{h}) > 0$, since $|\mathrm{disc}(E_\mathfrak{h})| \geq 1$.

The character function. By (Sp7) at i_0 there is $\eta \colon \widehat{\Gamma}_A \to \mathbb{C}$ compatible with $\mathfrak{h}$ such that $|G_{\mathfrak{A}_\mathfrak{h},\eta}(y,\phi)| \leq \theta_0\, y$ for all ϕ and all $y \geq X'/3$, where $\theta_0 = e^{-cx^{\varepsilon/8}}$. We compute $\eta(\chi_C)$. The character $\theta = \chi_C \circ \tau$ of $\mathcal{G}$ is ϵ_2, and $\chi_\theta = \epsilon_2 \circ \alpha$ is $u \mapsto (u/d)$ by (Hr4); since d is prime, the primitive character inducing it is the Legendre symbol χ_d, which is not trivial, so $\chi_C \neq 1$. By Lemma 3.18, item 1, $(\chi_C)_{\mathfrak{A}_\mathfrak{h}}(n) = \chi_d(n)$ for all $n \geq 1$, and by compatibility $\eta(\chi_C) = L'(1,\chi_d)/L(1,\chi_d) = \beta(\mathfrak{h})$, a real number ($\chi_d$ is real and $L(1,\chi_d) \neq 0$). Hence $\mathrm{Re}\,\eta(\chi_C) = \beta(\mathfrak{h})$.

The core-tied data. Choose a bijection $\{1, \dots, f\} \to \Phi$, $\rho \mapsto g_\rho$ (possible by (v)). Put $w_{u,j}(g) = w_j^{(u)}(g)$ for $g \in \Phi$ and $w_{u,j}(g) = 1$ for $g \in \Gamma_A \setminus \Phi$ ($u \in A$, $1 \leq j \leq m$), and $a(t) = a_t$ for $t \in T$, $a(t) = 0$ for $t \in A \setminus T$. Let $\bar\eta \geq 0$, $\bar e \in [0, \frac{1}{4}]$ be the numbers of (Sp5) at x, and for $u \in A$ let $\eta_u = \eta_{\mathfrak{V}_u} \in [0, \bar\eta]$, $e_u = e_{\mathfrak{V}_u} \in [0, \bar e]$ be those of (Sp5) at the member i_u. The slope datum $\mathfrak{S}$ has the following entries.

- The translates $U = A$, $\mathcal{T} = T$, the words $\mathcal{W}(u) = \mathcal{W}(\mathfrak{V}_u)$, and the integers m and f.
- The parameters

$$x,\ \varepsilon,\ a,\ b,\ b',\ k,\ L = x^\varepsilon,\ L_a,\ B_0,\ N_c,\ \gamma_0 = \gamma,$$
$$\beta = \beta(\mathfrak{h}),\ v = v(\mathfrak{h}),\ l_g = \log(g_1 g_2), \qquad \bar\delta = \tfrac{1}{2},\ \bar\eta,\ \bar e.$$

- The numbers a_t ($t \in T$) and $w_{u,j,\rho} = w_{u,j}(g_\rho)$.

- For $\omega \in \mathcal{W}(u)$, the entries $d_{u,\omega}, T_{u,\omega}, \mu_{u,\omega}, \mu_{u,\omega}^{\max}, h_{u,\omega}, \mu'_{u,\omega}, \mu'^{\max}_{u,\omega}, E_{u,\omega,j}$, equal to the rank, Gauss prime, slope, maximal slope, height, ambient slope, maximal ambient slope and exponents of ω in $\mathfrak{V}_u$.
- For $u \in A$,

$$\mu_u^\vee = \mu^\vee(\mathfrak{V}_u),\ \mu_{\min,u}^\vee = \mu_{\min}^\vee(\mathfrak{V}_u),\ F_u = F(\mathfrak{V}_u),\ \Theta_u = \Theta(\mathfrak{M}_u),$$
$$S_u = S(\mathfrak{M}_u),\ I_u = I(\mathfrak{M}_u),\ \mathrm{HH}_u = x^{c_\vee} b,\ \delta_u = \delta(\mathfrak{V}_u),\ \eta_u,\ e_u,$$

  with $S(\mathfrak{M}_u)$ formed with the cut-off $P = \lceil 2x \exp(x^\varepsilon) \rceil$.

The constant vector is $c_{\mathrm{Div}} = c_E = c_H = C_4$, $e_{\mathrm{Div}} = e_D$, $c_{\mathrm{Count}} = C_1$, $c_{\mathrm{Slope}} = C_2$, $c_{\mathrm{Sl}} = C_3$, $c_{\mathrm{Loss}} = c_L$, $e_{\mathrm{Loss}} = 0$, $c_{\mathrm{HH}} = c_\vee$. The remaining core-tied data are $X'' = xnX'$, $\mathrm{tr} = \mathrm{id}_A$, the identity $\mathcal{T} \to T$, the bijection $\rho \mapsto g_\rho$, the function η, the integer $P \geq 1$, the set T_2, the function a, the weights $w_{u,j}$, and the reals ν, X', θ_0. In this slope datum $n_{\mathcal{T}} = |T|$, $A = A_T$ (the sum of the a_τ), $\log \Omega = \gamma + \log(\pi/v(\mathfrak{h})) - \beta(\mathfrak{h}) = \log \Omega(\mathfrak{h})$, $E_{u,\omega} = E(\mathfrak{V}_u, \omega)$; since all ranks are positive by (Sp4), $s_u = s(\mathfrak{V}_u)$ and $\phi_u = s^{\max}(\mathfrak{V}_u) - s(\mathfrak{V}_u)$.

(Tc1), (Ct2), (Ct3), (Ct5), (Ct7) and the identifications. (Tc1): $T = T_{x,i_0}$; every element of T_2 is $s\, a_0$ with $s \in S_{x,i_0}$; η is compatible with $\mathfrak{h}_{x,i_0}$. (Ct2): $x(\mathfrak{S}) = x$, $\varepsilon(\mathfrak{S}) = \varepsilon$, $e_{\mathrm{Div}} = e_D$, $e_{\mathrm{Loss}} = 0$, $c_{\mathrm{HH}} = c_\vee = c^*$. (Ct3) with $\mathrm{tr} = \mathrm{id}_A$: $a_\tau = a(\tau)$, $w_{u,j,\rho} = w_{u,j}(g_\rho)$ and $\gamma_0 = \gamma$ by construction, and $\beta = \mathrm{Re}\, \eta(\chi_C)$ by the computation above. For S_u: $S(\mathfrak{M}_u) = \sum_{t \in T} \sum_j A_{\mathrm{in}}(a_t, t, w_j^{(u)})$ formed with $(\mathfrak{A}_\mathfrak{h}, P)$, the weights extended off Φ in any way, since A_{in} depends on a weight only through its values on Φ; with the extension $w_{u,j}$ this is $\sum_{t \in T} \sum_j A_{\mathrm{in}}(a(t), t, w_{u,j})$, as (Ct3) requires. (Ct5) holds with equality: $B_0 = 2|A| m b x^\varepsilon$ and $N_c = |T_2| k b x^{2\varepsilon} / |A|^2 = |T_2| k b x^\varepsilon / (|A|^2 x^{-\varepsilon})$. (Ct7): $a = xb \leq 2xb$; $\frac{1}{2} x^{-\varepsilon/8} |A| \leq |T|$, $x^{-\varepsilon}|A| \leq |T_2|$, $x^{1/2-\varepsilon} \leq m$ and $k \leq |A|^2 x^{1/2-\varepsilon}$ by (Sp8). Finally $b = \exp(x^\varepsilon)$, and for every u the loss datum $\mathfrak{M}_u$ has the integer m, the set of tops T with $|\mathcal{T}|$ elements, and $S_u = S(\mathfrak{M}_u)$ with the cut-off $\lceil 2x \exp(x^\varepsilon) \rceil$, $\Theta_u = \Theta(\mathfrak{M}_u)$, $I_u = I(\mathfrak{M}_u)$.

The slope conditions other than (Sl8). Fix $u \in A$ and $\omega, \omega' \in \mathcal{W}(u)$; every property below is applied at the member i_u and its slope record $\mathfrak{V}_u$, whose loss datum has the common k, m, ν, tops T and heights a_t, and whose analytic loss is L_a.

- (Sl1): $x, b > 0$, $a, b' \geq 0$, $k \geq 1$, $v > 0$, $f > 0$ by (i), (ii), (vi); $a_\tau \geq 0$ by (iv); $w_{u,j,\rho} = w_j^{(u)}(g_\rho) \geq 1$ by (Sp6); $d_{u,\omega} > 0$ and $\mu_{u,\omega} \leq \mu_{u,\omega}^{\max}$ by (Sp4); $T_{u,\omega} > 0$ as a prime; $c_{\mathrm{Sl}} = C_3 \geq 0$; $\bar{\delta} = \frac{1}{2} < 1$, $\bar{\eta} \geq 0$, $0 \leq \bar{e} \leq \frac{1}{4}$.
- (Sl2): (Sp4) gives the bound with C_d and $x^{e\varepsilon}$; its left side and its sum over t are those of (Sl2), because $m(\varphi(a_t) - a_t \log \Omega) = m(a_t \log a_t - a_t - a_t \log \Omega(\mathfrak{h}))$ (here $a_t > 0$ by (iv)). The two error terms are non-negative after C_d is replaced by $C_4 \geq C_d$, since $L_a \geq 0$ by (Sp8) and $B_0, N_c \geq 0$, and $x^{e\varepsilon} \leq x^{e_D \varepsilon}$ since $x \geq 1$ and $e \leq e_D$; so (Sl2) holds with $c_{\mathrm{Div}} = C_4$ and $e_{\mathrm{Div}} = e_D$.
- (Sl3) and (Sl4): the second and third bounds of (Sp4), with $C_d \leq C_4$ and non-negative right sides, are (Sl3) with $c_E = C_4$ (as $E_{u,\omega} = E(\mathfrak{V}_u, \omega)$ and $A = A_T$) and (Sl4) with $c_H = C_4$.
- (Sl5): (Sp1) with C_1, since $\sum_{\rho=1}^f \log(w_{u,j,\rho} v/\pi) = \sum_{\tau \in \Phi} \log(w_j^{(u)}(\tau) v(\mathfrak{h})/\pi)$ by the bijection $\rho \mapsto g_\rho$, $l_g = \log(g_1 g_2)$ and $L = x^\varepsilon$.
- (Sl6): (Sp2) with C_2, as $n_{\mathcal{T}} = |T|$.
- (Sl7): (Sp3) with C_3; its subtracted term is exactly $\mathrm{Lo}''_{u,\omega} = c_{\mathrm{Sl}} I_u + c_{\mathrm{Sl}} n_{\mathcal{T}} m(a/k + b') \log T_{u,\omega} + c_{\mathrm{Sl}} X'' + 1$, with $\Theta_u + F_u - E_{u,\omega}$ as in (Sl7).

- (Sl9): (Sp5) at i_u gives the inequality of (Sl9) for $\mathrm{HH}_u = x^{c_\vee} b$, δ_u, η_u, e_u, since s_u, ϕ_u and $|\mathcal{W}(u)|$ are the quantities of (Sp5); so (Sl9) holds regardless of its hypotheses.
- (Sl10): $s_u - \mu^\vee_{\min,u} \le x^{c_\vee} b = \mathrm{HH}_u$ by (Sp5), and $0 < \mathrm{HH}_u \le x^{c_{\mathrm{HH}}} b$ with $c_{\mathrm{HH}} = c_\vee$.
- (Sl11): $0 \le \delta_u \le \frac{1}{2} = \bar\delta$ by (Sp5), $0 \le \eta_u \le \bar\eta$ and $0 \le e_u \le \bar e$ by the choice of η_u, e_u.

The size conditions for (c, X'', e_K, C_K). (Er1): $x \ge 1$ and $0 < \varepsilon \le 1$ by (i). (Er2): $b \ge 1$; $xb \le a$; $n_{\mathcal{T}} \ge 1$ by (iii); $m \le x^{1/2+\varepsilon/2}$, $xm \le k$, $k \le x^{e_K}$, $k \le a$, $0 \le L_a \le 2x^\varepsilon$, $kb' \le a$ and $X'' \le a$ by (Sp8); $e_K \ge 0$; $1 \le k$ by (ii); $0 \le L = x^\varepsilon \le 2x^\varepsilon$; $b' \ge 0$; $X'' \ge 0$; $0 \le a_\tau \le 2a$ by (iv); $B_0, N_c \ge 0$. (Er3): $\log|\mathcal{W}(u)| \le (k-3)\log 2$ by (Sp8) at i_u. (Er4): $\bar\delta = \frac{1}{2}$, $\bar\eta, \bar e \ge 0$, and by (Sp5), (iii) and (i) we have $(\bar\eta + 4\sqrt{\bar e}) x^{c_{\mathrm{HH}}} b \le C_K x^\varepsilon n_{\mathcal{T}} a$ (see the computation below). (Er5): $C_4, C_1, C_2, C_3, c_L \in [0, C_K]$ (each is non-negative, $c_L \ge c_0 = 0$, and C_K is their sum plus 6); $e_{\mathrm{Div}} = e_D \ge 0$, $e_{\mathrm{Loss}} = 0$, $c_{\mathrm{HH}} = c_\vee \ge 0$. Hence (Ct4) holds except for (Sl8), and with (Ct2), (Ct3), (Ct5), (Ct7) this is (Tc2).

(Tc3). We check (Wd1)–(Wd5) for $(A_0, T, T_2, a|_T, (w_{u,j}), P)$ with parameters $(b, b', \nu, X', \theta_0)$ for $(\mathfrak{A}_\mathfrak{h}, \eta)$, without the inclusion $T_2 \subseteq A_0$, and (Wd3) for every $u \in A$. (Wd1): $\nu > 0$ by (Lm7) (through (Pf3)); $X' > 0$; $b' > 0$; $b'X' = a = xb$; $\theta_0 \ge 0$. (Wd2): $|a_t/b - x| \le 1$ by (Sp8). (Wd3): let $u \in A$, $1 \le j \le m$, $\rho \in \Phi$; then $w_{u,j}(\rho) = w_j^{(u)}(\rho) \ge 1 > 0$ by (Sp6), so $a_t / w_{u,j}(\rho) \le a_t \le 2a \le P$ by (iv); the member i_u has translate u and thick set $T^{\mathrm{th}}_{x,i_u} = T_2$ (by (Sr1)), so (Sp6) at i_u gives $b \le w_{u,j}(\rho) \le b + \nu + 1$ if ρ is thick at u for T_2, and $w_{u,j}(\rho) \le 2b'$ otherwise. (Wd4) is the property of η chosen above. (Wd5): $2|\Phi|(x+1)\theta_0 \le 1/x$ and $6|\Phi|(x+1)(\nu+1) \le b/x$ by (Sp8) (for the constant c) and (v).

Thus i_0, the data above and the members i_u ($u \in A$) satisfy (Tc1), (Tc2) and (Tc3) except for (Sl8), with exponents $(e_D, 0, C_K, c^*, e_K)$ and parameter ε, the slope datum has translates A, $\mathrm{tr} = \mathrm{id}_A$, $c_{\mathrm{Loss}} = c_L$, $e_{\mathrm{Loss}} = 0$, and the identifications hold. As this holds for every $x \in W$ with $x \ge x_0$ and $x > x_1$, and for every $c_L \ge c_0$, the family $\mathcal{F}$ has slope data at its members with exponents $(e_D, 0)$. □

The reduction of Part I is now complete. The three tables below list the conditions it uses and the result that proves each. The first gives the slope conditions of the slope data at the members of a family, as they are verified in the proof of Theorem 8.4. The second gives the properties of slope records for the families of the class $\mathcal{A}_{\mathrm{prop}}$ of Definition 17.5 that satisfy the Sha condition, as in the proof of Theorem 17.7. The third gives the clauses of Proposition 17.6 and the link property (Lk4). In the second and third tables the words *condition of the class* mean that the condition is part of the definition of $\mathcal{A}_{\mathrm{prop}}$; the result named after these words shows that the families constructed in Part IV satisfy it.

Slope condition	follows from	by
(Sl1) positivity	(Sp4), (Sp6), (Sp8), (Lm2)	Theorem 8.4, Section 8
(Sl2) divisor bound	(Sp4), second item	the same
(Sl3) exponent comparison	(Sp4), third item	the same
(Sl4) height comparison	(Sp4), fourth item	the same
(Sl5) counting	(Sp1)	the same
(Sl6) ambient slopes	(Sp2)	the same
(Sl7) transfer of slopes	(Sp3)	the same
(Sl8) losses	(Lk2), (Lk3)	Corollary 7.11, Section 7, from the loss bound Theorem 5.6, Section 5
(Sl9) dual-slope inequality	(Sp5)	Theorem 8.4, Section 8
(Sl10) height bound	(Sp5)	the same
(Sl11) parameters	(Sp5)	the same

Property	proved by	Part
(Sp1) counting bounds	condition (Pc2) of the class; Proposition 18.1	IV
(Sp2) ambient slope bounds	condition (Pc2) of the class; Proposition 18.1	IV
(Sp3) transfer of slopes	Theorem 12.3, with Lemma 12.5	III
(Sp4) divisor package, $e = 1$	Theorem 13.2, from Theorem 13.4 and Theorem 13.18; the hypotheses of the latter come from the Ribet budget Theorem 22.1 of Part V	III
(Sp5) dual-slope bound	Theorem 14.2, with Theorem 14.8 and Theorem 15.3	III
(Sp6) weights at the nodes	condition (Pc2) of the class; Proposition 18.1	IV
(Sp7) thin scales	condition (Pc2) of the class; Proposition 18.1	IV
(Sp8) sizes	Lemma 12.1, from condition (Pc6)	III

Clause	proved by	Part
(Gh1) supply of good hosts	Proposition 11.5	II
(Lk1) design existence	Proposition 18.1, assuming (TCS)	IV
(Lk2) linking property	condition (Pc1) of the class (Theorem 17.7); Theorem 18.4	III, IV
(Lk3) choice of tops	condition (Pc1) of the class (Theorem 17.7); Proposition 18.1	III, IV
(Lk4) slope data at the members	Corollary 8.3, from (Lk5) by Theorem 8.4	I
(Lk5) slope records, (Sp1)–(Sp8)	Theorem 17.7, by the second table	III

What remains to be proved after Part I is therefore Proposition 9.1 (Part II), Proposition 17.6 (Part III, with inputs from Parts II, IV and V and, through Part IV, from Part VI and (TCS)) and Theorem 36.4 (Part VII, with Parts VIII and IX).

Part II
Hosts on windows

Part II constructs the auxiliary fields of the argument and shows that they exist at the scales where Part I needs them. It proves two statements. The first, Proposition 9.1, is the supply of hosts on windows, the first of the three inputs of the existence theorem Proposition 7.8. Let ψ be a primitive character that satisfies the window Sha bound with exponent 401, as every primitive character does by Theorem 36.4, and let $\mathfrak{G}$ be a good-host requirement that supplies good hosts for ψ. Then for every $\varepsilon \in (0, 1/52)$ there is a set of scales containing $\frac{3}{2}$-windows such that at every large scale x of the set some host satisfies the host conditions (Ec1)–(Ec5) for $\mathfrak{L}_\psi$, its field satisfies the Sha bound $\mathrm{V}\big(E_\mathfrak{h}; t, \exp(x^\varepsilon/2)\big)$, and $\mathfrak{G}$ holds. The second statement, Proposition 11.5, says that the concrete good-host requirement of Definition 11.4 supplies good hosts. It is clause (Gh1) of Proposition 17.6, and the concrete requirement is the one under which Parts III and IV work.

At a scale x the field $E_\mathfrak{h}$ of a host is an abelian CM field of degree $4|A|$, where $x^{2+8\varepsilon} \le |A| \le 2x^{2+8\varepsilon}$. Inside a cyclotomic field it is the fixed field of the automorphisms $\zeta \mapsto \zeta^a$ with $a \equiv \pm 1 \pmod{2^{e+2}}$, $\psi(a) = 1$ and $(a/q) = (a/d) = 1$ (Definition 7.1). By the Galois correspondence it is the compositum of the field cut out by ψ, of $\mathbb{Q}(\sqrt{-q}, \sqrt{-d})$, and of the real subfield of $\mathbb{Q}(\zeta_{2^{e+2}})$, which is the layer of degree 2^e of the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$; here $|A| = c\,2^e$ with c depending only on ψ (Lemma 9.2). The primes $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ have a prescribed size and form what we call a window pair. Alpöge speaks of an Iwasawa tower and of a precise choice of it made with Behrend's construction [4]. We read these as the cyclotomic 2-power layer and as the spacing sets of Proposition 11.12, which are built by Behrend's construction; this identification is our reading, and no statement of the paper depends on it.

The method is analytic number theory, with class field theory for the shape of the fields. Section 9 assembles Proposition 9.1 from three facts: a window pair gives a frame and a host whose field is of the shape just described; if the pair is good, the characters of the host whose twisted functions do not enter $G_{\mathfrak{L}_\psi}$ satisfy a square-root bound at the scale; and the ranges of scales on which pairs are good and on which the Sha bound holds overlap in $\frac{3}{2}$-windows. Section 10 proves that good window pairs exist at every large scale, by a mean square over quadratic twists in two stages; this is the analytic core of the Part. Section 11 constructs, for a host built on a good window pair, the further data that the concrete good-host requirement asks for: rank data, tops, a spacing set, level primes, the CM structure and two character estimates. The external inputs are published: Montgomery's zero-density estimate, the Landau–Page theorem, an effective bound for a real zero, the prime number theorem in progressions and the Chebotarev density theorem in the uniform form of Thorner and Zaman, Linnik's theorem on the least prime in a progression, and class field theory. No result of Part II depends on (TCS).

9 Supply of hosts on windows

This section proves Proposition 9.1, the supply of hosts on $\frac{3}{2}$-windows, which the proof of the existence theorem Proposition 7.8 uses in its first step. There its two hypotheses are met by Theorem 36.4, which gives the window Sha bound with exponent 401, and by clause (Gh1) of Proposition 7.9, which gives hypothesis (Sb). The proof assembles four lemmas. 9.2 attaches to ψ and to two primes q, d of the shape of a window pair a frame and a normalised host whose

field is a (ψ, e, q, d)-field, and checks (Ec1)–(Ec3) for every set of tops; it rests on the Galois theory of cyclotomic fields. Lemma 9.4 records the splitting of the small primes in $\mathbb{Q}(\sqrt{-d})$ that a window pair forces. Lemma 9.5 turns the goodness of the pair into (Ec5): the controlled characters, whose twisted functions do not enter $G_{\mathfrak{L}_\psi}$, satisfy the square-root bound at the scale x and the bound $4\log x$ for the logarithmic derivative at 1. Condition (Ec4), the distribution of the tops over the cosets of A_0, is part of the supply of good hosts and is proved in Section 11.

Lemma 9.3 explains the ratio $\frac{3}{2}$. Three ranges of scales must overlap. A window pair at scale Y can be chosen good up to Y^J (Proposition 10.2). The window Sha bound gives, for that pair, a range $[W', W'^2]$ of scales on which the fields satisfy the Sha bound. And a frame of order $c\,2^e$ must have order between $x^{2+8\varepsilon}$ and $2x^{2+8\varepsilon}$. The third condition is met by multiplying the scale by the constant $c^{1/(2+8\varepsilon)}$, and this shrinks the window of ratio 2 to one of ratio $\frac{3}{2}$. This is the origin of the ratio $\frac{3}{2}$ of the windows in Part I.

Proposition 9.1. *Let $\psi \in \mathcal{P}$ have modulus N and satisfy the window Sha bound with exponent 401. Let $\mathfrak{G}$ be a good-host requirement, and suppose:*

(Sb) *$\mathfrak{G}$ supplies good hosts for ψ.*

Then for every $\varepsilon \in (0, 1/52)$ there are a finite list κ of reals, a real $J \geq 1$, a set $W_\varepsilon \subseteq \mathbb{R}$ containing $\frac{3}{2}$-windows and a real x_0 such that for every $x \in W_\varepsilon$ with $x \geq x_0$ there are a frame (e, A, h, a_0) with $x^{2+8\varepsilon} \leq |A| \leq 2x^{2+8\varepsilon}$, a host $\mathfrak{h}$ for it, a finite set $T \subseteq A$ and an integer $t \geq 1$ with:

1. *$(\mathfrak{h}, T)$ satisfies the host conditions (Ec1)–(Ec5) for $\mathfrak{L}_\psi$ at scale x with parameters $(\varepsilon, 4)$;*
2. *$E_\mathfrak{h}$ is a (ψ, e, q, d)-field and satisfies $\mathrm{V}(E_\mathfrak{h}; t, \exp(x^\varepsilon/2))$;*
3. *$\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}, T, t)$ holds.*

Proof. Fix $\varepsilon \in (0, 1/52)$. Let c, e_0 be given by Lemma 9.2 for ψ, and let $J \geq 1$ and the set W_ε (containing $\frac{3}{2}$-windows) be given by Lemma 9.3 for (ψ, ε, c). Put $C_0 = J + o_\psi + 8N \geq 1$, and let κ, x_B be given by (Sb), that is, by the supply of good hosts for ψ, for (ε, C_0). These constants depend only on ψ and ε, not on the scale. Let $x_0 = \max\bigl(x_B,\ c\,2^{e_0} + 1,\ (4N+1)^{5J/\varepsilon}\bigr)$.

Let $x \in W_\varepsilon$ with $x \geq x_0$, and let Y, q, d, e be as in the definition of W_ε (Lemma 9.3): $x^{1/J} \leq Y \leq x$, (q, d) is a window pair at scale Y good for (N, x), $x^{2+8\varepsilon} \leq c2^e \leq 2x^{2+8\varepsilon}$, and every (ψ, e, q, d)-field satisfies $\mathrm{V}(\cdot\,; \lfloor x^{400} \rfloor, \sqrt{\exp(x^\varepsilon)})$. Put $b = \lfloor Y^{\varepsilon/5} \rfloor$. Then:

- $e \geq e_0$, since $c\,2^e \geq x^{2+8\varepsilon} \geq x > c\,2^{e_0}$;
- $Y \geq x^{1/J} \geq (4N+1)^{5/\varepsilon}$, so $b \geq 4N + 1$;
- by Lemma 9.4, $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$, $(-d/q) = 1$ and $(-d/p) = 1$ for every odd prime $p \leq b \leq Y^{\varepsilon/5}$; $q \neq 3$ because $\log q > \frac{1}{2} Y^{\varepsilon/5} \geq \frac{1}{2}(4N+1) > \log 3$; and $\gcd(N, qd) = 1$ by (Wp1).

So Lemma 9.2 gives a frame (e, A, h, a_0) with $|A| = c2^e \in [x^{2+8\varepsilon}, 2x^{2+8\varepsilon}]$ and index o_ψ, and a host $\mathfrak{h}$ with its items 1–6. The hypotheses of (Sb) hold with $M = N$: $J \leq C_0$, $o_\psi \leq C_0$, $0 < N \leq C_0$, $Q_S \leq 4N|A| \leq C_0|A|$ and $Q_S \mid \mathrm{lcm}(2^{e+2}, N)$ by item 1, the kernel form by item 2, field rows up to b by item 5, normalisation by item 6, and the pair is good for (N, x) at the scale $Y \in [x^{1/J}, x]$. So (Sb) gives T and t with (Ec4) and item 3.

Item 1: (Ec1)–(Ec3) are item 4 of Lemma 9.2 (valid for every T), (Ec4) is given by (Sb), and (Ec5) with $C = 4$ is Lemma 9.5, whose hypotheses are $N \mid Q_S$ (part of the kernel form, item 2

of Lemma 9.2), the prime factors of Q_S dividing $2N$ (item 1), the twist form (item 3) and the goodness of (q,d) for (N,x) at the scale $Y \le x$.

Item 2: $E_{\mathfrak{h}}$ is a (ψ,e,q,d)-field by item 2 of Lemma 9.2, so it satisfies $\mathrm{V}(E_{\mathfrak{h}};\lfloor x^{400}\rfloor,\sqrt{\exp(x^{\varepsilon})})$, and $\sqrt{\exp(x^{\varepsilon})}=\exp(x^{\varepsilon}/2)$. Since t is an integer with $t\le x^{400}$, $t\le\lfloor x^{400}\rfloor$, and $\mathrm{V}(E;t',B)$ follows from $\mathrm{V}(E;t,B)$ for $t'\le t$ (fewer non-negative lengths in the sum). □

Lemma 9.2. *Let $\psi\in\mathcal{P}$ have modulus N. There are integers $c\ge 1$ and $e_0\ge 2$ such that for every $e\ge e_0$, all primes q,d with $q\equiv 3 \pmod 8$, $q\ne 3$, $d\equiv 7 \pmod{16}$, $(-d/q)=1$ and $\gcd(N,qd)=1$, and every integer $b\ge 4N$ with $(-d/p)=1$ for all odd primes $p\le b$, there are a frame (e,A,h,a_0) and a host $\mathfrak{h}=(q,d,Q_S,\alpha)$ for it with the following properties.*

1. *$|A|=c\,2^e$; $Q_S\mid \mathrm{lcm}(2^{e+2},N)$, so every prime factor of Q_S divides $2N$; and $Q_S\le 4N|A|$.*
2. *$\mathfrak{h}$ has the kernel form for ψ; hence $E_{\mathfrak{h}}$ is a (ψ,e,q,d)-field.*
3. *$\mathfrak{h}$ has the twist form for ψ.*
4. *The frame has index o_ψ, and $(\mathfrak{h},T)$ satisfies (Ec1)–(Ec3) for $\mathfrak{L}_\psi$, for every $T\subseteq A$.*
5. *$\mathfrak{h}$ has field rows up to b.*
6. *$\mathfrak{h}$ is normalised.*

Proof. Construction. Let $\ker\psi\subseteq(\mathbb{Z}/N)^\times$ be the kernel of ψ (the common kernel of $\langle\psi\rangle$) and $B=(\mathbb{Z}/N)^\times/\ker\psi$, cyclic of order o_ψ; the characters of $(\mathbb{Z}/N)^\times$ trivial on $\ker\psi$ are exactly $\langle\psi\rangle$. Let Q_S be the least multiple of 2^{e+2} dividing $\mathrm{lcm}(2^{e+2},N)$ through which $u\mapsto[u]_B$ factors; since ψ is primitive of modulus N, $N\mid Q_S$, and $Q_S=2^{e+2}N'$ for $e+2\ge v$, where $N=2^vN'$ with N' odd. Let

$$\eta\colon(\mathbb{Z}/Q_S)^\times\to H_e\times B,\qquad u\mapsto(u \bmod 2^{e+2} \bmod\{\pm1\},\ [u]_B),$$

let A be its image (written additively), h the projection to H_e, which is surjective, and $A_0=A\cap(H_e\times\{0\})$, the kernel of the projection to B. The map h is injective on A_0, so A_0 embeds in the cyclic group H_e and is cyclic; let a_0 generate it. The projection $A\to B$ is surjective, so $[A:A_0]=|B|=o_\psi$. Define

$$\alpha\colon(\mathbb{Z}/Q_Sqd)^\times\to\mathcal{G},\qquad \alpha(u)=\Big(\Big(\big(\tfrac{u}{q}\big),\big(\tfrac{u}{d}\big)\Big),\ \eta\big(\big(\tfrac{u}{q}\big)u\big)\Big).$$

The sign (u/q) inside η is the key feature of the construction: since $q\equiv d\equiv 3 \pmod 4$, it gives $\alpha(-1)=((-1,-1),\eta(1))=((-1,-1),0)$, which is (Hn1) in item 6.

The constants. Since $h\colon A\to H_e$ is onto and $|H_e|=2^e$, $|A|=|\ker h|\,2^e$. For $e+2\ge v$ write $\psi=\psi_2\psi'$ with ψ_2 of 2-power modulus and ψ' of modulus N', let o' be the order of ψ' and r that of $\psi_2^{o'}$. Then $h(A_0)$ is the image in H_e of $\ker\psi_2^{o'}$, a subgroup of index r in $(\mathbb{Z}/2^{e+2})^\times$, so $|A|=o_\psi|A_0|=o_\psi 2^{e+1}/(rt)$ with $t=|\ker\psi_2^{o'}\cap\{\pm1\}|\in\{1,2\}$. Hence $c:=|\ker h|=2o_\psi/(rt)$ is an integer ≥ 1 independent of e, and $|A|=c\,2^e\ge 2^e$. Let e_0 be the least integer $\ge\max(v,2)$ with $2^{e_0}\ge 4N$ and $e_0\ge 64\,o_\psi$.

Host axioms. (Hr1) is the hypothesis on q,d. (Hr2): $2^{e+2}\mid Q_S$; $\gcd(Q_S,qd)=1$ as $Q_S\mid 2^{e+2}N$ and q,d are odd primes not dividing N; an odd prime $l\mid Q_S$ divides $N\le b$, so $(-d/l)=1$. (Hr3): let $((s_1,s_2),a)\in\mathcal{G}$. As η is onto A, there is $w\in(\mathbb{Z}/Q_S)^\times$ with $\eta(w)=a$. By the Chinese remainder theorem (Q_S, q, d pairwise coprime by (Hr2) and $q\ne d$, as $q\equiv 3$ and $d\equiv 7 \pmod 8$) there is u coprime to Q_Sqd with $(u/q)=s_1$, $(u/d)=s_2$ and $u\equiv s_1w \pmod{Q_S}$, since each

of ±1 is the Legendre symbol of some residue modulo an odd prime. Then $(u/q)u \equiv s_1^2 w = w \pmod{Q_S}$, so $\alpha(u) = ((s_1,s_2),\eta(w)) = ((s_1,s_2),a)$; hence α is onto. (Hr4): $\epsilon_2(\alpha(u)) = (u/d)$. (Hr5): if $u \equiv 1 \pmod{Q_S q}$ then $(u/q) = 1$ and $\eta(u) = 0$, so $\alpha(u) = ((1,(u/d)),0) \in \{1,\iota_d\}$.

Item 1. $|A| = c2^e$ was shown; $Q_S \mid 2^{e+2}N$, so its prime factors divide $2N$; and $Q_S \le 2^{e+2}N \le 4N|A|$.

Item 2. $N \mid Q_S$, and $\alpha(a) = 1$ if and only if $(a/q) = (a/d) = 1$ and $\eta(a) = 0$, i.e. $a \equiv \pm1 \pmod{2^{e+2}}$ and $\psi(a) = 1$. So $E_\mathfrak{h}$ is the fixed field of exactly the automorphisms in the definition of a (ψ,e,q,d)-field, with $N_\mathfrak{h} = Q_S qd$ divisible by $2^{e+2}, N, q, d$.

Item 3. A character θ trivial on $\{(1,1)\} \times A_0$ factors through $\{\pm1\}^2 \times A/A_0 \cong \{\pm1\}^2 \times B$, so $\theta = \epsilon_1^{i_0}\epsilon_2^j \cdot (\beta \circ \mathrm{pr}_B)$ with β a character of B, and $\xi = \beta([\cdot]_B) \in \langle\psi\rangle$. Then $\theta(\iota_d) = (-1)^j$ and $\chi_\theta(u) = (u/q)^{i_0}(u/d)^j\xi((u/q)u) = \xi(u)(u/q)^{i_0+k}(u/d)^j$, where $\xi(-1) = (-1)^k$; take $i \equiv i_0 + k \pmod 2$.

Item 4. (Ec1) is $[A : A_0] = o_\psi$. (Ec2): for $\psi' = (\psi^m)^* \in K_\psi$ let β_m be the character of B with $\beta_m([u]_B) = \psi^m(u)$ and put $\theta_m((s_1,s_2),a) = \beta_m(\mathrm{pr}_B(a) \cdot [s_1]_B)$. Then $\theta_m(\alpha(u)) = \beta_m([(u/q)^2u]_B) = \psi^m(u)$, θ_m is trivial on $\{(1,1)\} \times A_0$ and $\theta_m(\iota_d) = 1$, so $\theta_m \in X^+(\mathfrak{h})$ and $\chi^*_{\theta_m} = \psi'$. (Ec3): $\mathfrak{L}_\psi$ has no exceptional character, so we need $\chi^*_{\epsilon_1} \notin K_\psi$; but $\chi_{\epsilon_1}(u) = (u/q)$, so $\chi^*_{\epsilon_1}$ is the Legendre symbol modulo q, of modulus $q \nmid N$, while the characters of K_ψ have moduli dividing N.

Item 5. (Fr1) is the first coordinate of α. (Fr2): as $q \equiv 3 \pmod 4$, $\mathbb{Q}(\sqrt{-q})$ is the quadratic subfield of $\mathbb{Q}(\zeta_q) \subseteq \mathbb{Q}(\zeta_{N_\mathfrak{h}})$, and $\zeta \mapsto \zeta^a$ maps $\sqrt{-q}$ to $(a/q)\sqrt{-q}$; since $\alpha(a) = 1$ implies $(a/q) = 1$, $s = \sqrt{-q}$ lies in $E_\mathfrak{h}$ and $\sigma(s) = \epsilon_1(\sigma)s$ by (Fr1). (Fr3): for $l = 2$ take $u \equiv 1 + 2^{e+1} \pmod{2^{e+2}}$ and $u \equiv 1$ modulo the odd part of $N_\mathfrak{h}$; then $u \equiv 1 \pmod{N_\mathfrak{h}/2}$ and $u \not\equiv \pm1 \pmod{2^{e+2}}$ (as $e \ge 2$), so $\alpha(u) \ne 1$ by item 2. For odd $l \mid Q_S$ we have $l \mid N'$ and $v_l(Q_S) = v_l(N)$; by the minimality of Q_S (equivalently, as ψ has conductor N) there is $u \equiv 1 \pmod{Q_S/l}$ with $\psi(u) \ne 1$, and choosing $u \equiv 1 \pmod{qd}$ gives $u \equiv 1 \pmod{N_\mathfrak{h}/l}$ and $\alpha(u) \ne 1$. (Fr4): $Q_S \le 2^{e+2}N \le 2^{2e} \le |A|^2$ since $2^e \ge 4N$. (Fr5): $o_\psi^{64} \le 2^{64o_\psi} \le 2^e$. (Fr6) and (Fr7) are hypotheses. (Fr8): an odd prime $l \mid Q_S$ divides N, so $4l \le 4N \le b$.

Item 6. The component of $\alpha(u)$ in A is $\alpha_A(u) = \eta((u/q)u)$, where η is applied to the class of $(u/q)u$ modulo Q_S. (Hn1) was shown after the construction: $(-1/q) = (-1/d) = -1$ because $q \equiv d \equiv 3 \pmod 4$, so $\alpha(-1) = ((-1,-1),\eta(1)) = ((-1,-1),0)$. (Hn2): h is the projection $A \to H_e$, and the first coordinate of $\eta(v)$ is the class of $v \bmod 2^{e+2}$ modulo $\{\pm1\}$; for $v = (u/q)u = \pm u$ this class is $[u]_e$, so $h(\alpha_A(u)) = [u]_e$. (Hn3): let $u \equiv 1 \pmod{Q_S d}$. Then $(u/d) = 1$ and $(u/q)u \equiv (u/q) = \pm1 \pmod{Q_S}$, so $\alpha(u) = (((u/q),1),\eta(\pm1))$ and

$$\alpha(u)^2 = (((u/q)^2,1),\eta((\pm1)^2)) = ((1,1),\eta(1)) = 1.$$

□

Lemma 9.3. *Let $\psi \in \mathcal{P}$ have modulus N and satisfy the window Sha bound with exponent 401, let $0 < \varepsilon < 1/52$ and let $c \ge 1$. There is $J \ge 1$ such that the set W of real x for which there are a real Y, primes q,d and an integer $e \ge 0$ with*

1. *$x^{1/J} \le Y \le x$,*
2. *(q,d) is a window pair at scale Y that is good for (N,x),*
3. *$x^{2+8\varepsilon} \le c\,2^e \le 2x^{2+8\varepsilon}$,*
4. *every (ψ,e,q,d)-field E satisfies $\mathrm{V}(E;\lfloor x^{400}\rfloor,\sqrt{\exp(x^\varepsilon)})$,*

contains $\frac{3}{2}$-windows.

Proof. Choose a page function e_ε (for $x \geq 1$ let $e_\varepsilon(x)$ be the least integer with $2^{e_\varepsilon(x)} \geq x^{2+8\varepsilon}$). The window Sha bound gives $J \geq 1$ and X_0 such that for all $X \geq X_0$ and every window pair (q,d) at scale $X^{1/J}$ there is W' with $X^{1/J} \leq W'$, $W'^2 \leq X$ and $\mathrm{V}(E; \lfloor x'^{401} \rfloor, \sqrt{\exp(x'^\varepsilon)})$ for every $(\psi, e_\varepsilon(x'), q, d)$-field E and every $x' \in [W', W'^2]$. Let Y_1 be the threshold of Proposition 10.2 for $(N, \varepsilon, A = J)$, and let $k = c^{1/(2+8\varepsilon)} \geq 1$, so that $k^{2+8\varepsilon} = c$.

Let $y \in \mathbb{R}$. Choose $M \geq \max(1, y, Y_1, k^3, k^{400})$ and $X \geq \max(X_0, M^J)$; then $X^{1/J} \geq M \geq Y_1$. By Proposition 10.2 there is a window pair (q,d) at scale $Y := X^{1/J}$ good for $(N, Y^J) = (N, X)$; let W' be given by the window Sha bound for this pair. Put $Z = kW' \geq y$. We show $[Z, Z^{3/2}] \subseteq W$.

Let $x \in [Z, Z^{3/2}]$ and write $x = kx'$. Since $k^3 \leq M \leq W'$, $Z^{3/2} = (kW')^{3/2} \leq W'^2$, so $W' \leq x' \leq x \leq W'^2 \leq X$. Take $e = e_\varepsilon(x')$. Then:

- item 1: $x^{1/J} \leq X^{1/J} = Y \leq W' \leq x$;
- item 2: (q,d) is good for (N, X), hence for (N, x), since $x \leq X$ and the range in (Wp2) only shrinks;
- item 3: $x^{2+8\varepsilon} = k^{2+8\varepsilon} x'^{2+8\varepsilon} = c\, x'^{2+8\varepsilon}$, and $x'^{2+8\varepsilon} \leq 2^e \leq 2x'^{2+8\varepsilon}$;
- item 4: since $x' \in [W', W'^2]$, every (ψ, e, q, d)-field satisfies $\mathrm{V}(E; \lfloor x'^{401} \rfloor, \sqrt{\exp(x'^\varepsilon)})$; and $\mathrm{V}(E; t, B)$ is monotone (it implies $\mathrm{V}(E; t', B')$ for $t' \leq t$, $B' \geq B$), with $x^{400} = k^{400} x'^{400} \leq x' \cdot x'^{400}$ (as $k^{400} \leq M \leq W' \leq x'$) and $x'^\varepsilon \leq x^\varepsilon$.

So $x \in W$. The three ranges fit together as follows. The pair is at the scale $Y = X^{1/J} \leq x$ given by the window Sha bound, and it is good up to $X = Y^J \geq x$. The Sha window is $[W', W'^2]$ in the variable $x' = x/k$. Finally, the factor k converts the page $2^e \approx x'^{2+8\varepsilon}$ into frame orders $c\,2^e \approx x^{2+8\varepsilon}$, at the cost of shrinking the window of ratio 2 to one of ratio $\frac{3}{2}$. □

Lemma 9.4. *Let $0 < \varepsilon < 1/52$, $Y > 0$, and let (q,d) be a window pair at scale Y. Then:*

1. *q and d are primes with $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$ and $(-d/q) = 1$;*
2. *$(-d/p) = 1$ for every odd prime $p \leq Y^{\varepsilon/5}$;*
3. *$(-q/a) = (a/q)$ and $(-d/a) = (a/d)$ for every odd integer $a \geq 1$.*

In particular q, 2 and every odd prime $p \leq Y^{\varepsilon/5}$ split in $\mathbb{Q}(\sqrt{-d})$.

Proof. Items 1 and 2: the congruences are part of the definition. If p is an odd prime with $p \mid d+1$, then $-d \equiv 1 \pmod p$, so $(-d/p) = (1/p) = 1$. This applies to $p = q$ and to every odd prime $p \leq Y^{\varepsilon/5}$, which divide $d+1$ by definition.

Item 3: let $n \in \{q, d\}$; then $n \equiv 3 \pmod 4$. For odd $a \geq 1$, $(-n/a) = (-1/a)(n/a) = (-1)^{(a-1)/2}(n/a)$, and by quadratic reciprocity for the Jacobi symbol $(n/a)(a/n) = (-1)^{\frac{a-1}{2}\cdot\frac{n-1}{2}} = (-1)^{(a-1)/2}$, since $\frac{n-1}{2}$ is odd. Hence $(-n/a) = (a/n)$.

Splitting: $-d \equiv 1 \pmod 8$ because $d \equiv 7 \pmod 8$, so $\mathbb{Q}(\sqrt{-d})$ has discriminant $-d$, the prime 2 splits in it, and an odd prime $p \nmid d$ splits in it if and only if $(-d/p) = 1$; apply items 1 and 2. □

Lemma 9.5. *Let $\psi \in \mathcal{P}$ have modulus N, let $0 < \varepsilon < 1/52$ and $1 \leq Y \leq x$, and let $\mathfrak{h} = (q, d, Q_S, \alpha)$ be a host for a frame (e, A, h, a_0) such that $N \mid Q_S$, every prime factor of Q_S*

divides $2N$, $\mathfrak{h}$ *has the twist form for* ψ, *and* (q,d) *is a window pair at scale* Y *that is good for* (N,x). *Then every character* θ *that is controlled relative to* $\mathfrak{L}_\psi$ *satisfies*

$$|g(x,\chi_\theta^*)| \le x^{1/2+\varepsilon}, \qquad \Big|\frac{L'(1,\chi_\theta^*)}{L(1,\chi_\theta^*)}\Big| \le 4\log x.$$

In particular $(\mathfrak{h},T)$ *satisfies (Ec5) for* $\mathfrak{L}_\psi$ *at scale* x *with parameters* $(\varepsilon,4)$, *for every* $T \subseteq A$.

Proof. Let θ be controlled: $\theta \in X^-(\mathfrak{h})$, or $\theta \in X^+(\mathfrak{h})$ with $\chi_\theta^* \notin K_\psi$. In both cases θ is trivial on $\{(1,1)\} \times A_0$, so the twist form gives $\xi \in \langle\psi\rangle$ and $i,j \in \{0,1\}$ with $\theta(\iota_d) = (-1)^j$ and $\chi_\theta(a) = \xi(a)(a/q)^i(a/d)^j$ for a coprime to $Q_S qd$. We have $(i,j) \neq (0,0)$: if $\theta \in X^-$ then $j = 1$; if $\theta \in X^+$ and $(i,j) = (0,0)$, then χ_θ and ξ agree on the integers coprime to $Q_S qd$, so $\chi_\theta^* = \xi^* = (\psi^m)^* \in K_\psi$ for some m, which is excluded.

An integer a coprime to $2Nqd$ is coprime to $Q_S qd$, because the prime factors of Q_S divide $2N$; for such a, $\chi_\theta(a) = \xi_{q,d}^{(i,j)}(a)$, with ξ regarded as a character modulo N. Two Dirichlet characters that agree on the integers coprime to a common multiple of their moduli are induced by the same primitive character, so $\chi_\theta^* = (\xi_{q,d}^{(i,j)})^*$. Since $q, d \nmid N$ by (Wp1), property (Wp2) of the good pair at $y = x \in [Y,x]$ gives $|g(x,\chi_\theta^*)| \le x^{1/2+\varepsilon/2} \le x^{1/2+\varepsilon}$ (as $x \ge 1$) and $|L'(1,\chi_\theta^*)/L(1,\chi_\theta^*)| \le 4\log x$. □

10 Good window pairs: almost all quadratic twists

A host built on a window pair (q,d) has, besides the characters that make up $G_{\mathfrak{L}_\psi}$, the twists $\xi\cdot(\cdot/q)^i(\cdot/d)^j$ with $(i,j) \neq (0,0)$ and ξ of small modulus. The conductors of these twists exceed every power of the scale, so the square-root bound that (Ec5) asks of them does not follow from a zero-free region for any fixed L-function. This section proves that it holds nevertheless for suitable pairs, on a whole range of scales: Proposition 10.2 gives, for every $A \ge 1$ and every large Y, a window pair at scale Y that is good up to Y^A. Its consumer is Lemma 9.3.

The core is a mean square over a family of prime moduli, and the square root comes from it; no zero-free half-plane is used for any character. For a set P of primes in an arithmetic progression, Proposition 10.8 bounds the mean over $q \in P$ of the square of a smoothed prime sum twisted by $(\cdot/q)$ by $C\,y\log y$. It rests on the equidistribution of the Jacobi symbols (q/s) over $q \in P$ with at most one exceptional modulus s (Lemma 10.6), which comes from the prime number theorem in progressions with an exceptional pair (Proposition 10.7), proved from the theorem of Thorner and Zaman (Theorem B.7) and an effective bound for a real zero (Theorem B.5). By Lemma 10.9, few q have a large smoothed sum anywhere on the range of scales. Under a zero-free box to the right of $1 - \frac{1}{400}$ the smoothed sum controls $g(y,\chi)$ and the logarithmic derivative at 1 (Lemma 10.10, Lemma 10.11), and by Montgomery's zero-density estimate few characters lack such a box (Lemma 10.15). Together these give one stage, Proposition 10.5: almost all quadratic twists over primes in a progression are good.

Proposition 10.4 applies the stage twice. The first stage chooses $q \equiv 3 \pmod 8$ with $\log q$ of size $Y^{\varepsilon/5}$. The second chooses, for a good q, a prime $d \equiv 7 \pmod{16}$ in the progression $d \equiv -1$ modulo q and modulo every odd prime $p \le Y^{\varepsilon/5}$, with $\log d$ of size $Y^{\varepsilon/5}\log^2 Y$. The congruences make every such p split in $\mathbb{Q}(\sqrt{-d})$ (Lemma 9.4), and 2 splits as well since $d \equiv 7 \pmod{16}$; so every prime inert in $\mathbb{Q}(\sqrt{-d})$ exceeds $Y^{\varepsilon/5}$, which the proof of Proposition 11.11 uses. The modulus of the progression has logarithm of order $Y^{\varepsilon/5}$, while $\log d$ is larger by the factor $\log^2 Y$; this ratio is what places the count of the primes d in the range of the prime number theorem in progressions. We follow the exposition of [2] for the two stages.

Definition 10.1. For real numbers Q, τ, Y, X, a primitive Dirichlet character χ of modulus N is (Q,τ,Y,X)**-good** if $\chi \neq \mathbf{1}$, $N \le Q$, χ has a zero-free box of height Q,

$$\Big|\frac{L'(1,\chi)}{L(1,\chi)}\Big| \le 251 \log Y, \qquad \text{and} \qquad |g(y,\chi)| \le y^{1/2+\tau} \ \text{ for all } y \in [Y,X].$$

Proposition 10.2. *Let $M \ge 1$ be an integer, $0 < \varepsilon < 1/52$ and $A \ge 1$. There is Y_1 such that for every $Y \ge Y_1$ there is a window pair (q,d) at scale Y that is good for (M, Y^A).*

Proof. Apply Proposition 10.4 with $N_0 = 8M$, $\tau = \varepsilon/2$, $K = A$, $\eta = \frac12$, and $z' = Y^{\varepsilon/5}\log^2 Y$; let Y_1 be at least its threshold, at least the threshold of item 2 of Lemma 10.11 for $(\varepsilon, 3)$, and at least $e^{1+\log(8M)}$, and large enough that $e^{z/2} > 8M$. For $Y \ge Y_1$ the sets $\mathcal{Q}^{\mathrm{good}}$ and $\mathcal{D}_q^{\mathrm{good}}$ have at least half the elements of the non-empty sets $\mathcal{Q}$ and $\mathcal{D}_q$; choose $q \in \mathcal{Q}^{\mathrm{good}}$ and $d \in \mathcal{D}_q^{\mathrm{good}}$.

Window pair. With $z = Y^{\varepsilon/5}$: $q \equiv 3 \pmod 8$ and $\frac12 Y^{\varepsilon/5} < \log q \le Y^{\varepsilon/5}$; $d \equiv 7 \pmod{16}$ and $\frac12 Y^{\varepsilon/5}\log^2 Y < \log d \le Y^{\varepsilon/5}\log^2 Y$; $q \mid d+1$ and every odd prime $p \le Y^{\varepsilon/5}$ divides $d+1$. These are the conditions of a window pair at scale Y.

(Wp1) holds since $q, d > e^{z/2} > 8M$ are primes.

(Wp2): let ξ be a character modulo M and ξ' the character modulo $8M$ it induces. For n coprime to $2Mqd$, $\xi_{q,d}^{(i,j)}(n) = \xi'(n)(n/q)^i(n/d)^j$, so the two have the same inducing primitive character χ. If $j = 0$ (so $i = 1$), (Aq) for ξ' shows that χ is $(8Me^z, \varepsilon/2, Y, Y^A)$-good; if $j = 1$, (Ad) for ξ' and $a = i$ shows that it is $(8Me^{z+z'}, \varepsilon/2, Y, Y^A)$-good. In both cases $|g(y,\chi)| \le y^{1/2+\varepsilon/2}$ for $y \in [Y, Y^A]$, and χ is non-principal, of modulus at most $Q_j = h_j(M,\varepsilon,Y)$, with a zero-free box of height Q_j. Since

$$Y^{\varepsilon/6} \le \log Q_j = \log(8M) + Y^{\varepsilon/5} + jY^{\varepsilon/5}\log^2 Y \le 3Y^{\varepsilon/5}\log^2 Y$$

(using $\log Y \ge 1 + \log(8M)$), item 2 of Lemma 10.11 gives $|L'(1,\chi)/L(1,\chi)| \le 4\log Y \le 4\log y$ for $y \ge Y$.

(Wp3): a real primitive ϱ of modulus dividing $8M$ induces a character ϱ' modulo $8M$; applying (Aq), (Ad) to $\xi = \varrho'$ gives a zero-free box of height $h_j(M,\varepsilon,Y)$ for $(\varrho_{q,d}^{(i,j)})^*$. □

Definition 10.3. For integers $m_0 \ge 1$, a_0 and a real number $U \ge 1$, let

$$\mathcal{P}(m_0, a_0, U) = \{q \text{ prime} : \ U^{1/2} < q \le U, \ q \equiv a_0 \pmod{m_0}\}.$$

For $\varepsilon > 0$ and $Y > 1$, a triple (U, m_0, a_0) is **admissible at** (ε, Y) if $8 \mid m_0$, $\gcd(a_0, m_0) = 1$ and

$$\tfrac14 \log^2 Y \cdot \log m_0 \le \log U, \qquad Y^{\varepsilon/6} \le \log U \le Y^{\varepsilon/5}\log^2 Y.$$

Proposition 10.4. *Let $0 < \varepsilon < 1/52$, $\tau > 0$, $K \ge 1$, $\eta > 0$ and $N_0 \ge 1$. There is Y_0 such that for every $Y \ge Y_0$, with $z = Y^{\varepsilon/5}$, and every z' with $Y^{\varepsilon/5}\log^2 Y \le z' \le K^2 Y^{\varepsilon/5}\log^2 Y$, the following holds. Let $\mathcal{Q}$ be the set of primes $q \equiv 3 \pmod 8$ with $e^{z/2} < q \le e^z$, and for $q \in \mathcal{Q}$ let $\mathcal{D}_q$ be the set of primes $d \equiv 7 \pmod{16}$ with $e^{z'/2} < d \le e^{z'}$ and $q\prod_{3\le p\le z} p \mid d+1$ (product over primes). Then $\mathcal{Q} \neq \emptyset$, and there is $\mathcal{Q}^{\mathrm{good}} \subseteq \mathcal{Q}$ with $|\mathcal{Q}^{\mathrm{good}}| \ge (1-\eta)|\mathcal{Q}|$ such that for every $q \in \mathcal{Q}^{\mathrm{good}}$:*

(Aq) *for every Dirichlet character ξ modulo N_0, $(\xi\cdot(\cdot/q))^*$ is (N_0e^z, τ, Y, Y^K)-good;*

(Ad) *$\mathcal{D}_q \neq \emptyset$, and there is $\mathcal{D}_q^{\mathrm{good}} \subseteq \mathcal{D}_q$ with $|\mathcal{D}_q^{\mathrm{good}}| \geq (1-\eta)|\mathcal{D}_q|$ such that for every $d \in \mathcal{D}_q^{\mathrm{good}}$, every ξ modulo N_0 and $a \in \{0,1\}$, $(\xi \cdot (\cdot/q)^a \cdot (\cdot/d))^*$ is $(N_0 e^{z+z'}, \tau, Y, Y^K)$-good.*

Proof. Stage 1 (the prime q). Apply Proposition 10.5 with $(\varepsilon, \tau, K, \eta)$, R the number of characters modulo N_0, $U = e^z$, $m_0 = 8$, $a_0 = 3$, $F = N_0$ and $\Xi = \{\xi^* : \xi \bmod N_0\}$. Then $\mathcal{P}(8, 3, e^z) = \mathcal{Q}$. The triple is admissible at (ε, Y) for large Y: $\frac{1}{4}\log^2 Y \log 8 \leq Y^{\varepsilon/5} = \log U \leq Y^{\varepsilon/5}\log^2 Y$ and $Y^{\varepsilon/6} \leq \log U$; the only prime factor of m_0 is $2 \leq Y^{\varepsilon/5}$; and $F = N_0 \leq e^{z/2}$. For $q \in \mathcal{Q}$, $q > e^{z/2} > N_0$, so $(\xi \cdot (\cdot/q))^* = \xi^* \cdot (\cdot/q)$. The proposition gives $\mathcal{Q} \neq \emptyset$ and at most $\eta|\mathcal{Q}|$ primes q failing (Aq), with $FU = N_0 e^z$. Let $\mathcal{Q}^{\mathrm{good}}$ be the set of the remaining primes. We show (Ad) for every $q \in \mathcal{Q}$.

Stage 2 (the prime d, for fixed $q \in \mathcal{Q}$). Put $\varepsilon' = 6\varepsilon/5 \leq \frac{1}{4}$, $o = q\prod_{3 \leq p \leq z} p$, $m_0 = 16o$, and let a_0 be the class with $a_0 \equiv 7 \pmod{16}$ and $a_0 \equiv -1 \pmod{o}$ (Chinese remainder theorem; $\gcd(a_0, m_0) = 1$). Then $\mathcal{P}(m_0, a_0, e^{z'}) = \mathcal{D}_q$. Apply Proposition 10.5 with $(\varepsilon', \tau, K, \eta)$, R twice the number of characters modulo N_0, $U = e^{z'}$, $F = N_0 e^z$ and $\Xi = \{(\xi \cdot (\cdot/q)^a)^* : \xi \bmod N_0,\ a \in \{0,1\}\}$, whose moduli are at most $N_0 q \leq F$. The hypotheses hold for large Y:

- admissibility at (ε', Y): $\log m_0 \leq \log 16 + \log q + \sum_{p \leq z} \log p \leq 4z$, since $\log q \leq z$, $\sum_{p \leq z} \log p \leq z \log 4 \leq 2z$ by Chebyshev's bound [180] and $\log 16 \leq z$ for large Y, so $\frac{1}{4}\log^2 Y \log m_0 \leq z \log^2 Y \leq z'$; $Y^{\varepsilon'/6} = Y^{\varepsilon/5} \leq z'$; $z' \leq K^2 Y^{\varepsilon/5} \log^2 Y \leq Y^{\varepsilon'/5} \log^2 Y$;
- the prime factors of m_0 are 2, the odd primes $p \leq z \leq Y^{\varepsilon'/5}$, and $q > e^{z/2} > Y^{400K}$;
- $1 \leq F = N_0 e^z \leq e^{z'/2} = U^{1/2}$.

For $d \in \mathcal{D}_q$ we have $d > e^{z'/2} \geq F$, so $(\xi \cdot (\cdot/q)^a \cdot (\cdot/d))^* = (\xi \cdot (\cdot/q)^a)^* \cdot (\cdot/d)$. The proposition gives $\mathcal{D}_q \neq \emptyset$ and at most $\eta|\mathcal{D}_q|$ primes d failing (Ad), with $FU = N_0 e^{z+z'}$. □

Proposition 10.5. *Let $0 < \varepsilon \leq \frac{1}{4}$, $\tau > 0$, $K \geq 1$, $\eta > 0$ and $R \geq 1$. There is Y_0 with the following property. Let $Y \geq Y_0$, let (U, m_0, a_0) be a triple admissible at (ε, Y) such that every prime factor of m_0 is at most $Y^{\varepsilon/5}$ or larger than Y^{400K}, let F be real with $1 \leq F \leq U^{1/2}$, and let Ξ be a set of at most R primitive Dirichlet characters of moduli at most F. Then the set $P = \mathcal{P}(m_0, a_0, U)$ is non-empty, and at most $\eta|P|$ primes $q \in P$ have a $\xi \in \Xi$ for which $(\xi \cdot (\cdot/q))^*$ is not (FU, τ, Y, Y^K)-good.*

Proof. Let $q \in P$ and $\xi \in \Xi$, of modulus $f \leq F$. Since $q > U^{1/2} \geq F \geq f$ and $q > U^{1/2} > 2$, q is an odd prime with $q \nmid f$, so $\chi_{q,\xi} : n \mapsto \xi(n)(n/q)$ is a product of primitive characters of coprime moduli, hence primitive of modulus $fq \leq FU$, and $\chi_{q,\xi} \neq \mathbf{1}$. Thus $(\xi \cdot (\cdot/q))^* = \chi_{q,\xi}$ and $S_Z(y, \chi_{q,\xi}) = S_Z(y; q, \xi)$ in the notation of Lemma 10.10. Put $Q = FU$; then $\log U \leq \log Q \leq \frac{3}{2}\log U$, so $Y^{\varepsilon/6} \leq \log Q \leq \frac{3}{2}Y^{\varepsilon/5}\log^2 Y \leq Y^{\varepsilon/4}$ for Y large.

Non-emptiness and size. By item 1 of Lemma 10.6, $|P| \geq U^{9/10} > 0$.

Primes with a zero in the box. Let B_1 be the set of $q \in P$ such that some $\chi_{q,\xi}$ has no zero-free box of height Q. For each such q choose one such ξ. The map $q \mapsto \chi_{q,\xi}$ is injective on B_1, since the modulus fq with $f < q$ determines q. By Lemma 10.15, a consequence of Montgomery's zero-density estimate, there is an absolute constant C_z such that for $Q \geq 3$ at most $C_z Q^{1/4}$ primitive characters of modulus at most Q have a zero s with $\operatorname{Re} s > 1 - \frac{1}{400}$, $|\operatorname{Im} s| \leq Q$. It applies with $Q = FU \geq U \geq 3$, and the characters $\chi_{q,\xi}$, $q \in B_1$, are primitive of modulus at most FU, pairwise distinct for distinct q and each with such a zero. Hence $|B_1| \leq C_z(FU)^{1/4} \leq C_z U^{3/8} \leq \frac{\eta}{2}U^{9/10} \leq \frac{\eta}{2}|P|$ for Y large (recall $\log U \geq Y^{\varepsilon/6}$).

Primes with a large smoothed sum. For $\xi \in \Xi$ let $B_2(\xi)$ be the set of $q \in P$ with $|S_Z(y; q, \xi)| > \frac{1}{2}y^{1/2+\tau}$ for some $y \in [Y, Y^K]$. By Proposition 10.8 the mean-square hypothesis of Lemma 10.9

holds on $[Y, Y^K]$ with its absolute constant C, for every κ' with $|\kappa'| \le 1$; so Lemma 10.9 with $\eta/(2R)$ gives $|B_2(\xi)| \le \frac{\eta}{2R}|P|$, and $|\bigcup_\xi B_2(\xi)| \le \frac{\eta}{2}|P|$.

The other primes are good. Let $q \in P \setminus (B_1 \cup \bigcup_\xi B_2(\xi))$ and $\xi \in \Xi$. Then $\chi_{q,\xi}$ is non-principal of modulus at most Q with a zero-free box of height Q, and $|S_Z(y, \chi_{q,\xi})| \le \frac{1}{2} y^{1/2+\tau}$ on $[Y, Y^K]$. By Lemma 10.10, $|L'(1, \chi_{q,\xi})/L(1, \chi_{q,\xi})| \le 251 \log Y$ and $|g(y, \chi_{q,\xi})| \le y^{1/2+\tau}$ on $[Y, Y^K]$. So $\chi_{q,\xi}$ is (FU, τ, Y, Y^K)-good, and the bad primes lie in $B_1 \cup \bigcup_\xi B_2(\xi)$, of size at most $\eta|P|$. $\square$

Lemma 10.6. *Let $\varepsilon > 0$, $A > 0$, $B > 0$. There is Y_0 such that for every $Y \ge Y_0$ and every triple (U, m_0, a_0) admissible at (ε, Y):*

1. *$|\mathcal{P}(m_0, a_0, U)| \ge U^{9/10}$;*
2. *there is an integer $n^* \ge 0$ such that for every odd squarefree integer $s > 1$ with $\gcd(s, m_0) = 1$, $s \le Y^A$ and $s \ne n^*$,*

$$\Big| \sum_{q \in \mathcal{P}(m_0, a_0, U)} \Big(\frac{q}{s}\Big) \Big| \le Y^{-B} \, |\mathcal{P}(m_0, a_0, U)|.$$

Proof. We use two inputs. The first is the prime number theorem for progressions with relative error and an exceptional pair, Proposition 10.7 (a consequence of the theorem of Thorner and Zaman [259], of the type of Gallagher's theorem [100]): there are $c_3 > 0$, A_0, C_0 such that for every M with $8 \mid M$ there is a choice e_M, either empty or an *exceptional pair* (χ_1, β_1) modulo M (a real non-principal character χ_1 modulo M and a zero $\beta_1 \in (1 - 1/(8 \log M), 1)$ of $L(s, \chi_1)$), such that for $X \ge 2$ with $\log X \ge C_0 \log M \log(\log M + 2)$ and $\gcd(b, M) = 1$,

$$\Big|\pi(X; M, b) - \frac{\mathrm{li} X - \vartheta(b) \, \mathrm{li}(X^{\beta_1})}{\varphi(M)}\Big| \le A_0\Big(e^{-c_3 \log X / \log M} + e^{-\sqrt{c_3 \log X}}\Big) \frac{\mathrm{li} X - \vartheta(b) \, \mathrm{li}(X^{\beta_1})}{\varphi(M)},$$

with $\vartheta(b) = \chi_1(b)$ if $e_M = (\chi_1, \beta_1)$ and $\vartheta = 0$ if e_M is empty (no uniqueness of the pair is used). The second is the Landau–Page theorem [75]: there is $c_0 > 0$ such that for every $Z \ge 3$ at most one real primitive character of modulus at most Z has a real zero in $(1 - c_0/\log Z, 1)$.

Sizes. Put $L = \log Y$ and $u = \log U$. Admissibility gives $\log m_0 \le 4u/L^2$ and $u \ge Y^{\varepsilon/6}$; for $s \le Y^A$ the modulus $M = m_0 s$ satisfies $\log M \le 4u/L^2 + AL$. Hence, for $Y \ge Y_0(\varepsilon, A, B)$ and $X \in \{U^{1/2}, U\}$,

$$\frac{\log X}{\log M} \ge \min\Big(\frac{L^2}{16}, \, \frac{Y^{\varepsilon/6}}{4AL}\Big), \qquad \text{so} \qquad 2A_0\big(e^{-c_3 \log X/\log M} + e^{-\sqrt{c_3 \log X}}\big) \le \tfrac{1}{8} Y^{-B},$$

and $\log X \ge C_0 \log M \log(\log M + 2)$.

Item 1. Apply the prime number theorem with $M = m_0 \ge 8$ and $b = a_0$ at $X = U$; the relative error is at most $\frac{1}{8}$ by the size estimate (the case $s = 1$). We also use the effective bound $1 - \beta_1 \ge c/m_0$, with an absolute $c > 0$, for a real zero $\beta_1 \in (0, 1)$ of a real non-principal character χ_1 modulo m_0. Indeed, χ_1^* is real, primitive and $\ne \mathbf{1}$, of modulus $f \mid m_0$ with $f \ge 3$, and β_1 is a zero of $L(s, \chi_1^*)$, since the finitely many Euler factors $1 - \chi_1^*(p) p^{-\beta_1}$, $p \mid m_0$, by which $L(s, \chi_1)$ and $L(s, \chi_1^*)$ differ do not vanish. The bound for real zeros [75] gives $1 - \beta_1 \ge c_{\mathrm{P}} f^{-1/2} \log^{-2} f \ge c_{\mathrm{P}} m_0^{-1/2} \log^{-2} m_0$, as $x^{1/2} \log^2 x$ increases for $x \ge 1$, and this is at least c/m_0 with $c = c_{\mathrm{P}} \inf_{m \ge 3} m^{1/2} \log^{-2} m > 0$. Then

$$\mathrm{li} U - \vartheta \, \mathrm{li}(U^{\beta_1}) \ge \int_{U^{\beta_1}}^{U} \frac{dt}{\log t} \ge \frac{U(1 - e^{-(1-\beta_1)u})}{u} \ge \frac{U \min(1, cu/m_0)}{2u},$$

and the same lower bound $U/(2u)$ holds without exceptional pair. Hence $\pi(U;m_0,a_0) \ge \frac{7}{8}\,U\min(1,cu/m_0)/(2u\,m_0)$, and $|\mathcal{P}(m_0,a_0,U)| \ge \pi(U;m_0,a_0) - U^{1/2} \ge U^{9/10}$ for $Y \ge Y_0$, because $\log m_0 \le 4u/L^2 \le u/100$ and $u \ge Y^{\varepsilon/6}$.

Item 2. Let s be as in the statement and $M = m_0 s$. The Jacobi symbol $(\cdot/s)$ is a non-principal character modulo s, as $s>1$ is squarefree and odd. Since $\gcd(m_0,s)=1$, the classes b modulo M with $b \equiv a_0 \pmod{m_0}$ and $\gcd(b,s)=1$ map bijectively to $(\mathbb{Z}/s\mathbb{Z})^\times$, so

$$\sum_{q\in\mathcal{P}(m_0,a_0,U)} \Big(\frac{q}{s}\Big) = \sum_{b} \Big(\frac{b}{s}\Big)\big(\pi(U;M,b) - \pi(U^{1/2};M,b)\big).$$

Insert the prime number theorem. The terms $\mathrm{li}(X)/\varphi(M)$ cancel, as $\sum_b (b/s)=0$. If $e_M = (\chi_1,\beta_1)$, its terms are $\mathrm{li}(X^{\beta_1})\varphi(M)^{-1}\sum_b (b/s)\chi_1(b)$; writing $b = a_0' u$ with a fixed $a_0' \equiv a_0 \pmod{m_0}$, the sum over b is $(a_0'/s)\chi_1(a_0')\sum_u (u/s)\chi_1(u)$ over the subgroup of $u \equiv 1 \pmod{m_0}$, which is 0 unless $\chi_1(u) = (u/s)$ on that subgroup. Call s *exceptional* if this happens and $1-\beta_1 < c_0/\log(m_0 Y^A)$. By Landau–Page, applied with $Z = m_0 Y^A \ge 8$, the primitive characters inducing the χ_1 of two exceptional values s_1, s_2 coincide (for an exceptional s, χ_1 is real and non-principal modulo $M = m_0 s \le Z$, so χ_1^* is real, primitive, $\ne \mathbf{1}$, of modulus at most Z, and has the real zero $\beta_1 \in (1 - c_0/\log Z, 1)$, a zero of $L(s,\chi_1^*)$ as in item 1), and then $(u/s_1) = (u/s_2)$ for all $u \equiv 1 \pmod{m_0}$ coprime to $s_1 s_2$, which forces $s_1 = s_2$. Let n^* be the exceptional value if there is one, and $n^* = 0$ otherwise. Write $\mathrm{m}(X,b)$ for the main term of the prime number theorem and $\mathrm{err} \le \frac{1}{16}Y^{-B}$ for its relative error. *Error terms.* Since $\pi(X;M,b) \ge (1-\mathrm{err})\mathrm{m}(X,b)$, they total at most $2\,\mathrm{err}\sum_b(\pi(U;M,b)+\pi(U^{1/2};M,b)) \le 4\,\mathrm{err}\,\pi(U;m_0,a_0) \le 8\,\mathrm{err}\,|\mathcal{P}(m_0,a_0,U)|$, because $|\mathcal{P}(m_0,a_0,U)| \ge \pi(U;m_0,a_0) - U^{1/2} \ge \frac12\pi(U;m_0,a_0)$ by item 1. *Exceptional terms, $s \ne n^*$.* Either they vanish, or χ_1 is not trivial on the $u \equiv 1 \pmod{m_0}$. In the second case $\sum_b \chi_1(b) = 0$, $\sum_b \mathrm{m}(U,b) = \mathrm{li}(U)/\varphi(m_0)$ and $|\mathcal{P}(m_0,a_0,U)| \ge \frac14\mathrm{li}(U)/\varphi(m_0)$. Moreover $1-\beta_1 \ge c_0/\log(m_0 Y^A)$, so $\mathrm{li}(X^{\beta_1}) \le \frac{1}{16}Y^{-B}\mathrm{li}(U)$ for $X \le U$, and the exceptional terms total at most $\frac18 Y^{-B}\mathrm{li}(U)/\varphi(m_0) \le \frac12 Y^{-B}|\mathcal{P}(m_0,a_0,U)|$.

Adding up gives the bound $Y^{-B}|\mathcal{P}(m_0,a_0,U)|$. □

Proposition 10.7. *For real $x>1$ let $\mathrm{li}\,x = \int_2^x \frac{dt}{\log t}$ (an oriented integral, negative for $1<x<2$), and for an integer $M \ge 1$, an integer b and real X let $\pi(X;M,b)$ be the number of primes $p \le X$ with $p \equiv b \pmod M$. For an integer $M \ge 3$, an* exceptional pair modulo M *is a pair (χ_1,β_1) of a real non-principal Dirichlet character χ_1 modulo M and a real number β_1 with*

$$1 - \frac{1}{8\log M} < \beta_1 < 1, \qquad L(\beta_1,\chi_1) = 0.$$

There are real constants $c_3>0$, A_0 and C_0 such that for every integer M with $8 \mid M$ there is e_M, either empty or an exceptional pair modulo M, such that for every real $X \ge 2$ with $\log X \ge C_0 \log M \log(\log M + 2)$ and every integer $b \ge 0$ with $\gcd(b,M)=1$,

$$\Big|\pi(X;M,b) - m(X,b)\Big| \le A_0\Big(e^{-c_3\log X/\log M} + e^{-\sqrt{c_3\log X}}\Big)\, m(X,b),$$

where $m(X,b) = \mathrm{li}\,X/\varphi(M)$ if e_M is empty and $m(X,b) = (\mathrm{li}\,X - \chi_1(b)\,\mathrm{li}(X^{\beta_1}))/\varphi(M)$ if $e_M = (\chi_1,\beta_1)$.

Proof. Constants. We use the prime number theorem for progressions of Thorner and Zaman [259] (Corollary 1.4) with its constants c_4, A_4, and the effective bound $1-\beta \ge c_{\mathrm{P}} q^{-1/2}\log^{-2} q$

for a real zero $\beta \in (0,1)$ of $L(s,\chi)$, χ real primitive of modulus $q \geq 3$ [75]. Put

$$c_5 = \min\Big(1,\ c_{\mathrm{P}} \inf_{n\geq 3} \frac{n^{1/2}}{\log^2 n}\Big) > 0, \qquad C_0 = 32,$$
$$c_3 = \min\Big(\frac{c_4}{2}, \frac{c_4^2}{4}, \frac{3}{4}\Big), \qquad A_0 = A_4 + \frac{16(14A_4+4)}{c_5};$$

the infimum is positive because $n^{1/2}\log^{-2} n \to \infty$. By definition $c_{\mathrm{P}} n^{-1/2}\log^{-2} n \geq c_5/n$ for every integer $n \geq 3$.

Choice of e_M. Let $8 \mid M$. If there is an exceptional zero β_1 modulo M in the sense of item 1 of the theorem of Thorner and Zaman, with its character χ_1, put $e_M = (\chi_1, \beta_1)$; otherwise let e_M be empty. In the first case e_M is an exceptional pair modulo M: χ_1 is real and non-principal, $L(\beta_1,\chi_1) = 0$, $\beta_1 < 1$, and $\beta_1 \geq 1 - 1/(13\log(3M)) > 1 - 1/(8\log M)$ since $13\log(3M) > 8\log M$. In particular $\beta_1 > 1 - 1/(8\log 8) > \frac{9}{10}$.

Fix $X \geq 2$ with $\log X \geq C_0 \log M \log(\log M + 2)$ and $b \geq 0$ with $\gcd(b,M) = 1$. Let $\epsilon_b = \chi_1(b) \in \{1,-1\}$ if $e_M = (\chi_1,\beta_1)$ and $\epsilon_b = 0$ if e_M is empty, and for $t \geq 1$ put

$$f_b(t) = t - \epsilon_b \frac{t^{\beta_1}}{\beta_1}, \qquad f_b'(t) = 1 - \epsilon_b t^{\beta_1 - 1} \in [0,2]$$

(read $f_b(t) = t$ when $\epsilon_b = 0$). Then the main term of the theorem of Thorner and Zaman at (t,M,b) is $f_b(t)/\varphi(M)$, and the main term of the statement is $m(X,b)$.

Step 1. Sizes. As $M \geq 8$, $\log(\log M + 2) \geq \log(\log 8 + 2) > 1.4$, so $\log X \geq 44.8 \log M$. Put $t_0 = X^{1/2}$. Then $t_0 \geq M^{12} \geq 8^{12} > e^2$, $X^{\beta_1} \geq X^{9/10} \geq 2$, $M^2 \leq X^{1/20}$ and $\log X \leq 8X^{1/20}$.

For $t \geq e^2$ one has $f_b(t) \geq 0$ and $f_b(t) \leq 3t$: if $\epsilon_b = 1$, then $t^{\beta_1 - 1}/\beta_1 \leq 1$ because $-\log\beta_1 \leq (1-\beta_1)/\beta_1 \leq 2(1-\beta_1) \leq (1-\beta_1)\log t$; and $t^{\beta_1}/\beta_1 \leq 2t$ for $t \geq 1$.

Step 2. The main term as an integral, and its lower bound. The substitution $u = t^{\beta_1}$ maps $[2,X]$ onto $[2^{\beta_1}, X^{\beta_1}] \subset (1,\infty)$ and turns $t^{\beta_1-1}dt/\log t$ into $du/\log u$; hence

$$\varphi(M)\, m(X,b) = \int_2^X \frac{f_b'(t)}{\log t}\, dt - \epsilon_b \,\mathrm{li}(2^{\beta_1}), \qquad |\mathrm{li}(2^{\beta_1})| = \int_{2^{\beta_1}}^2 \frac{dt}{\log t} \leq \frac{2 - 2^{9/10}}{\log 2^{9/10}} < 1. \tag{10.1}$$

We show

$$\varphi(M)\, m(X,b) \geq \frac{c_5 X}{2M\log X}.$$

If $\epsilon_b \in \{0,-1\}$, then $\varphi(M)m(X,b) \geq \mathrm{li}X$, because $\mathrm{li}(X^{\beta_1}) \geq 0$ as $X^{\beta_1} \geq 2$; and $\mathrm{li}X \geq (X-2)/\log X \geq X/(2\log X) \geq c_5 X/(2M\log X)$. If $\epsilon_b = 1$, then

$$\varphi(M)m(X,b) = \int_{X^{\beta_1}}^X \frac{dt}{\log t} \geq \frac{X - X^{\beta_1}}{\log X} = \frac{X\big(1 - e^{-(1-\beta_1)\log X}\big)}{\log X} \geq \frac{X\min\big(1,(1-\beta_1)\log X\big)}{2\log X},$$

using $1 - e^{-u} \geq \min(1,u)/2$ for $u \geq 0$. Let χ_1^* be the primitive character inducing χ_1, of modulus $M^* \mid M$. It is real and non-principal, so $M^* \geq 3$; and β_1 is a zero of $L(s,\chi_1^*)$, since $L(s,\chi_1) = L(s,\chi_1^*)\prod_{p\mid M}(1 - \chi_1^*(p)p^{-s})$ and the factors do not vanish at $s = \beta_1 > 0$. The effective bound gives $1 - \beta_1 \geq c_{\mathrm{P}} M^{*-1/2}\log^{-2} M^* \geq c_5/M^* \geq c_5/M$. As $\log X \geq 1$ and $c_5/M \leq 1$, the minimum is at least c_5/M, and the claim follows. Since $\varphi(M) \leq M$, we get $m(X,b) \geq c_5 X/(2M^2\log X)$, and by Step 1

$$t_0 \leq \frac{2M^2\log X}{c_5 X^{1/2}}\, m(X,b) \leq \frac{16}{c_5} X^{-2/5} m(X,b) \leq \frac{16}{c_5} e^{-c_3 \log X/\log M}\, m(X,b),$$

the last step because $c_3 \le \frac{3}{4} \le \frac{2}{5}\log 8 \le \frac{2}{5}\log M$.

Step 3. Partial summation on $(t_0, X]$. Write $\vartheta(t) = \vartheta(t; M, b)$, $\pi(t) = \pi(t; M, b)$, $R(t) = \vartheta(t) - f_b(t)/\varphi(M)$ and $I = \int_{t_0}^{X} f_b'(t)\,dt/\log t$. Partial summation (Stieltjes integration of $1/\log t$ against ϑ, and integration by parts for the C^1 function f_b) gives

$$\pi(X) - \pi(t_0) = \frac{\vartheta(X)}{\log X} - \frac{\vartheta(t_0)}{\log t_0} + \int_{t_0}^{X} \frac{\vartheta(t)\,dt}{t\log^2 t}, \qquad I = \frac{f_b(X)}{\log X} - \frac{f_b(t_0)}{\log t_0} + \int_{t_0}^{X} \frac{f_b(t)\,dt}{t\log^2 t},$$

hence

$$\pi(X) - \pi(t_0) - \frac{I}{\varphi(M)} = \frac{R(X)}{\log X} - \frac{R(t_0)}{\log t_0} + \int_{t_0}^{X} \frac{R(t)\,dt}{t\log^2 t}.$$

For $t \in [t_0, X]$ we have $t \ge M^{12}$, so item 2 of the theorem of Thorner and Zaman gives $|R(t)| \le A_4 E(t) f_b(t)/\varphi(M)$ with $E(t) = e^{-c_4\log t/\log M} + e^{-c_4(\log t)^{3/5}(\log\log t)^{-1/5}}$. Since $\log t \ge \frac{1}{2}\log X$ and $(\log t)^{3/5}(\log\log t)^{-1/5} \ge (\log t)^{1/2}$ (equivalent to $\log\log t \le (\log t)^{1/2}$),

$$E(t) \le e^{-\frac{c_4}{2}\log X/\log M} + e^{-\frac{c_4}{\sqrt{2}}\sqrt{\log X}} \le E^* := e^{-c_3\log X/\log M} + e^{-\sqrt{c_3\log X}},$$

by $c_3 \le c_4/2$ and $\sqrt{c_3} \le c_4/2$. As $f_b \ge 0$ on $[t_0, X]$ (Step 1), the bound for R and the identity for I in the first display of this step give

$$\begin{aligned}\Big|\pi(X) - \pi(t_0) - \frac{I}{\varphi(M)}\Big| &\le \frac{A_4E^*}{\varphi(M)}\Big(\frac{f_b(X)}{\log X} + \frac{f_b(t_0)}{\log t_0} + \int_{t_0}^{X}\frac{f_b(t)\,dt}{t\log^2 t}\Big)\\ &= \frac{A_4E^*}{\varphi(M)}\Big(I + \frac{2f_b(t_0)}{\log t_0}\Big) \le \frac{A_4E^*}{\varphi(M)}(I + 6t_0).\end{aligned}$$

Step 4. Assembly. By the identity of Step 2, $\varphi(M)m(X,b) = \int_2^{t_0} f_b'(t)\,dt/\log t + I - \epsilon_b\mathrm{li}(2^{\beta_1})$, so

$$\pi(X) - m(X,b) = \Big(\pi(X) - \pi(t_0) - \frac{I}{\varphi(M)}\Big) + \Big(\pi(t_0) - \frac{1}{\varphi(M)}\int_2^{t_0}\frac{f_b'(t)\,dt}{\log t}\Big) + \frac{\epsilon_b\mathrm{li}(2^{\beta_1})}{\varphi(M)}.$$

The second bracket lies in $[-3t_0, t_0]$, as $0 \le \pi(t_0) \le t_0$ and $0 \le \int_2^{t_0} f_b'(t)\,dt/\log t \le 2t_0/\log 2 \le 3t_0$; the last term is at most $1 \le t_0$ in absolute value. Moreover $I \le \varphi(M)m(X,b) + 1$, since $f_b' \ge 0$ and $|\mathrm{li}(2^{\beta_1})| < 1$; and $E^* \le 2$, $\varphi(M) \ge 1$. With Step 3,

$$|\pi(X) - m(X,b)| \le A_4E^*m(X,b) + A_4E^*(1+6t_0) + 4t_0 \le A_4E^*m(X,b) + (14A_4+4)\,t_0.$$

By Step 2, $(14A_4+4)t_0 \le \frac{16(14A_4+4)}{c_5}e^{-c_3\log X/\log M}m(X,b) \le \frac{16(14A_4+4)}{c_5}E^*m(X,b)$. Hence $|\pi(X;M,b) - m(X,b)| \le A_0E^*m(X,b)$, which is the assertion. □

Proposition 10.8. *For real $y \ge 0$, $Z \ge 1$, a prime q and $\kappa\colon \mathbb{N} \to \mathbb{C}$ put*

$$S_Z(y; q, \kappa) = \sum_{n \le Z} \Lambda(n)\min\Big(1, \frac{y}{n}\Big)\kappa(n)\Big(\frac{n}{q}\Big).$$

There is an absolute constant $C > 0$ with the following property. Let $0 < \varepsilon \le 1$ and $K \ge 1$. There is Y_0 such that the following holds for every $Y \ge Y_0$, with $Z = Y^{400K}$: for every triple (U, m_0, a_0) admissible at (ε, Y) such that every prime factor p of m_0 satisfies $p \le Y^{\varepsilon/5}$ or $p > Y^{400K}$, every $y \in [Y, Y^K]$ and every κ with $|\kappa(n)| \le 1$ for all n,

$$\sum_{q \in \mathcal{P}(m_0, a_0, U)} \big|S_Z(y; q, \kappa)\big|^2 \le C\,y\log y\,|\mathcal{P}(m_0, a_0, U)|.$$

Proof. Write $P = \mathcal{P}(m_0, a_0, U)$, $N = \lfloor Z \rfloor$, $w = Y^{\varepsilon/5}$ and $c_n = \Lambda(n)\min(1, y/n) \geq 0$, which is supported on prime powers. Take $Y_0 \geq 3$ so large that $Y^{\varepsilon/6} \geq 1600K\log Y$, $(400K\log Y)^2 \leq Y$, and item 2 of Lemma 10.6 holds with $A = 800K$, $B = K+1$; let n^* be its exceptional value.

Step 1: expansion. Since $|\kappa| \leq 1$,

$$\sum_{q\in P} |S_Z(y; q, \kappa)|^2 \leq \sum_{n,n'\leq N} c_n c_{n'} \Big|\sum_{q\in P} \Big(\frac{nn'}{q}\Big)\Big|.$$

Step 2: reciprocity. Every $q \in P$ exceeds $U^{1/2} \geq e^{800K\log Y} = Z^2 \geq nn'$. For $n, n' \leq N$ let $s(nn')$ be the product of the primes $p \nmid m_0$ that divide nn' to an odd power; it is odd (as $2 \mid m_0$), squarefree, coprime to m_0 and at most $Z^2 = Y^{800K}$. By quadratic reciprocity, $(nn'/q) = \sigma\,(q/s(nn'))$ with a sign σ independent of $q \in P$. Indeed, for $p \mid m_0$ the symbol (p/q) depends only on q modulo $8p$, hence only on $q \bmod m_0$, which is fixed (for odd p use $(p/q) = \pm(q/p)$ with the sign given by $q \bmod 4$; for $p = 2$ use $q \bmod 8$). A prime to an even power contributes 1. For odd $p \nmid m_0$, $(p/q) = \pm(q/p)$ with the sign given by $q \bmod 4$. Hence $|\sum_{q\in P}(nn'/q)| = |\sum_{q\in P}(q/s(nn'))|$.

Step 3: three classes of pairs. If $s(nn') \notin \{1, n^*\}$, Lemma 10.6 bounds the inner sum by $Y^{-K-1}|P|$. Otherwise we bound it by $|P|$. Let $\mathrm{sP}(p) = \sum_{k\geq 1,\, p^k\leq N} c_{p^k}$ and let R be the sum of c_n over the prime powers $n = p^k \leq N$ with k even or $p \mid m_0$. For prime powers $n = p^a$, $n' = p'^b$: $s(nn') = 1$ exactly when $p = p'$ and ($a + b$ even or $p \mid m_0$), or $p \neq p'$ and both p^a and p'^b have $s = 1$. So these pairs contribute at most $\sum_p \mathrm{sP}(p)^2 + R^2$. The value n^*, if it occurs, is a product of at most two primes, and the pairs with $s(nn') = n^*$ contribute at most $2B_0^2 + 2B_0R$, where $B_0 = \max_p \mathrm{sP}(p)$. Hence

$$\sum_{q\in P} |S_Z|^2 \leq \Big(\sum_p \mathrm{sP}(p)^2 + R^2 + 2B_0^2 + 2B_0R + Y^{-K-1}\Big(\sum_{n\leq N} c_n\Big)^2\Big)|P|.$$

Step 4: the four estimates. We use Chebyshev's and Mertens' estimates in the form of items 2 and 3 of Lemma 3.5, namely $\sum_{n\leq x}\Lambda(n) \leq 3x$ for $x \geq 0$ and $\sum_{n\leq x}\Lambda(n)/n \leq \log x + 3 + \log 2$ for $x \geq 1$. The sums over primes below are bounded by these estimates, by partial summation where a weight $\log p/p^2$ occurs. Hence there are absolute constants $C_1, \dots, C_4$ such that for $y \geq 3$:

1. $\mathrm{sP}(p) \leq C_2 \log y$ for every p (the terms with $p^k \leq y$ give at most $\log y$, those with $p^k > y$ form a geometric series of total at most $2y\log p/\max(p, y) \leq 2\log y$);
2. $\sum_p \mathrm{sP}(p)^2 \leq C_3\, y\log y$ (split at $p \leq \sqrt{y}$, $\sqrt{y} < p \leq y$ and $p > y$; the middle range gives $\sum_{p\leq y} 9\log^2 p \ll y\log y$, the last $\sum_{p>y} 4y^2\log^2 p/p^2 \ll y\log y$);
3. $R \leq C_1(\sqrt{y} + w\log y)$ (prime powers with even exponent contribute $\ll \sqrt{y}$; the primes $p \mid m_0$ with $p \leq N$ satisfy $p \leq w$ by hypothesis and contribute $\leq \sum_{p\leq w} \mathrm{sP}(p)$);
4. $\sum_{n\leq Z} c_n \leq C_4\, y\log Z$.

Step 5: assembly. As $\varepsilon \leq 1$ and $y \geq Y$, $w \leq y^{1/5}$ and $w\log y \leq 4\sqrt{y}$, so $R \leq 5C_1\sqrt{y}$. Then $R^2 \leq 25C_1^2 y$, $2B_0^2 \leq 4C_2^2 y\log y$, $2B_0R \leq 20C_1C_2\, y$, and, with $\log Z = 400K\log Y$, $y \leq Y^K$ and $(400K\log Y)^2 \leq Y$,

$$Y^{-K-1}\Big(\sum_{n\leq N} c_n\Big)^2 \leq C_4^2\, y^2\, Y^{-K-1}(400K\log Y)^2 \leq C_4^2\, y.$$

Since $\log y \geq 1$, the bracket is at most $(C_3 + 25C_1^2 + 4C_2^2 + 20C_1C_2 + C_4^2)\, y\log y$. □

Lemma 10.9. *Let $C \ge 0$, $\tau > 0$ and $\eta > 0$. There is Y_0 such that the following holds for every $Y \ge Y_0$, all $X \ge Y$ and $Z \ge 1$, every finite set P of primes, every $B \subseteq P$ and every κ with $|\kappa(n)| \le 1$. Suppose that*

$$\sum_{q \in P} |S_Z(y; q, \kappa')|^2 \le C\, y \log y\, |P| \qquad \text{for every } y \in [Y, X] \text{ and every } \kappa' \text{ with } |\kappa'| \le 1,$$

and that every $q \in B$ has some $y \in [Y, X]$ with $|S_Z(y; q, \kappa)| > \frac{1}{2} y^{1/2+\tau}$. Then $|B| \le \eta\, |P|$.

Proof. Fix q and write $S(y) = S_Z(y; q, \kappa)$. Since $\frac{d}{dy} \min(1, y/n) = 1/n$ for $y < n$ and 0 for $y > n$, the function S is continuous and piecewise linear, with right derivative

$$D(v) = \sum_{v < n \le Z} \frac{\Lambda(n)\kappa(n)}{n} \Big(\frac{n}{q}\Big), \qquad v\, D(v) = S_Z(v; q, \kappa \mathbf{1}_{(v,\infty)}),$$

the second identity because $\min(1, v/n) = v/n$ for $n > v$. For $i \ge 0$ with $Y_i = 2^i Y \le X$ let $I_i = [Y_i, \min(2Y_i, X)]$. For $y \in I_i$, writing $S(y) = S(Y_i) + \int_{Y_i}^{y} D(v)\, dv$, the Cauchy–Schwarz inequality gives

$$|S(y)|^2 \le 2|S(Y_i)|^2 + 2Y_i \int_{I_i} |D(v)|^2\, dv.$$

Summing over $q \in P$ and applying the hypothesis at $y = Y_i$ with κ, and at each $v \in I_i$ with $\kappa \mathbf{1}_{(v,\infty)}$ (so $\sum_q |D(v)|^2 \le C \log v\, |P|/v$),

$$\sum_{q \in P} \sup_{y \in I_i} |S(y)|^2 \le 2C Y_i \log Y_i\, |P| + 2Y_i\, C \log(2Y_i) \log 2\, |P| \le 4C\, Y_i \log(2Y_i)\, |P|.$$

A prime $q \in B$ whose large value occurs in I_i has $\sup_{I_i} |S|^2 > \frac{1}{4} Y_i^{1+2\tau}$, so there are at most $16C \log(2Y_i)\, Y_i^{-2\tau} |P|$ of them. Summing over i,

$$|B| \le 16C\, |P| \sum_{i \ge 0} \log(2^{i+1}Y)\, (2^i Y)^{-2\tau} \le \frac{16C\, (\log(2Y) + \log 2/(1 - 2^{-2\tau}))}{(1 - 2^{-2\tau})\, Y^{2\tau}}\, |P|,$$

which is at most $\eta |P|$ for $Y \ge Y_0(C, \tau, \eta)$.

□

Lemma 10.10. *For real $y \ge 0$, $Z \ge 1$ and a Dirichlet character χ put*

$$S_Z(y, \chi) = \sum_{n \le Z} \Lambda(n) \min\Big(1, \frac{y}{n}\Big) \chi(n).$$

Let $0 < \varepsilon \le \frac{1}{4}$, $\tau > 0$ and $K \ge 1$. There is Y_0 such that the following holds for every $Y \ge Y_0$, with $Z = Y^{400K}$: for every Q with $Y^{\varepsilon/6} \le \log Q \le Y^{\varepsilon/4}$ and every non-principal Dirichlet character χ of modulus $N \le Q$ with a zero-free box of height Q:

1. *$|L'(1, \chi)/L(1, \chi)| \le 251 \log Y$;*
2. *$|g(y, \chi)| \le |S_Z(y, \chi)| + Y^{\varepsilon}$ for every $y \in [Y, Y^K]$; in particular, if $|S_Z(y, \chi)| \le \frac{1}{2} y^{1/2+\tau}$ for all $y \in [Y, Y^K]$, then $|g(y, \chi)| \le y^{1/2+\tau}$ for all $y \in [Y, Y^K]$.*

Proof. Item 1. Take Y_0 with $Y_0^{\varepsilon/6} \ge l_0$ and $C_0 \le \log Y_0$, where l_0, C_0 are the constants of item 1 of Lemma 10.11. Then $\log Q \ge l_0$, and $|L'(1,\chi)/L(1,\chi)| \le 1000 \log\log Q + C_0 \le 1000 \cdot \frac{\varepsilon}{4} \log Y + \log Y \le 251 \log Y$.

The splitting identity. Let $T_Z(\chi) = -L'(1,\chi)/L(1,\chi) - \sum_{n \le Z} \Lambda(n)\chi(n)/n$. For $0 < y \le Z$,

$$S_Z(y,\chi) = \sum_{n \le y} \Lambda(n)\chi(n) + y \sum_{y < n \le Z} \frac{\Lambda(n)\chi(n)}{n},$$

and comparing with (2.2) gives

$$g(y,\chi) = -S_Z(y,\chi) - y\,T_Z(\chi) \qquad (0 < y \le Z). \tag{$*$}$$

The tail. Let $x = Y^K$, so $Z = x^{400}$; put $\eta = x^{-2}$ and $h = Z\eta$. By (2.2),

$$g(Z+h,\chi) - g(Z,\chi) = -h\,T_Z(\chi) + \sum_{Z < n \le Z+h} \Lambda(n)\chi(n)\Big(\frac{Z+h}{n} - 1\Big).$$

The last sum has at most $h+1$ terms, each of modulus at most $h\log(Z+h)/Z$. By Lemma 10.12 with $\delta = \varepsilon/(1600K) \le 1/800$ (constant K_δ), the left side is at most $K_\delta \log Q(\eta \log Q\, Z^{1-1/400+\delta} + Z^2/Q)$. Hence

$$x\,|T_Z(\chi)| \le K_\delta \log^2 Q \cdot x\, Z^{-1/400+\delta} + K_\delta \log Q\, \frac{xZ}{\eta Q} + \frac{x(h+1)\log(Z+h)}{Z}.$$

Here $xZ^{-1/400+\delta} = x^{400\delta} = Y^{\varepsilon/4}$ and $\log^2 Q \le Y^{\varepsilon/2}$, so the first term is at most $K_\delta Y^{3\varepsilon/4}$. The second is at most $K_\delta Y^{\varepsilon/4} Y^{403K} \exp(-Y^{\varepsilon/6})$, and the third is at most $2x^{-1}(400\log x + 1)$. After enlarging Y_0, each is at most $\frac{1}{3}Y^\varepsilon$, so $x|T_Z(\chi)| \le Y^\varepsilon$.

Item 2. For $y \in [Y, Y^K]$ we have $y \le x \le Z$, so by $(*)$, $|g(y,\chi)| \le |S_Z(y,\chi)| + x|T_Z(\chi)| \le |S_Z(y,\chi)| + Y^\varepsilon$. If moreover $|S_Z(y,\chi)| \le \frac{1}{2} y^{1/2+\tau}$, then $|g(y,\chi)| \le y^{1/2+\tau}$ as soon as $Y^\varepsilon \le \frac{1}{2} Y^{1/2}$, which holds for $Y \ge 16$ since $\varepsilon \le \frac{1}{4}$. □

Lemma 10.11. *1. There are absolute constants C_0 and $l_0 \ge 1$ such that for every real Q with $\log Q \ge l_0$ and every non-principal Dirichlet character χ of modulus $N \le Q$ with a zero-free box of height Q,*

$$\Big|\frac{L'(1,\chi)}{L(1,\chi)}\Big| \le 1000 \log\log Q + C_0.$$

2. Let $0 < \varepsilon < 1/52$ and $A > 0$. There is Y_0 such that for every $Y \ge Y_0$, every Q with $Y^{\varepsilon/6} \le \log Q \le A Y^{\varepsilon/5} \log^2 Y$ and every χ as in item 1,

$$\Big|\frac{L'(1,\chi)}{L(1,\chi)}\Big| \le 4 \log Y.$$

Proof. We use (A) Mertens' estimate in the form $\sum_{n \le z} \Lambda(n)/n \le \log z + 3 + \log 2$ for $z \ge 1$, item 3 of Lemma 3.5, and (B) the increment bound Lemma 10.12.

Item 1. Let K be the constant of Lemma 10.12 for $\delta = 1/4000$, and choose $l_0 \geq 81$ with $l^{1006} \leq e^l$ for $l \geq l_0$. Put $l = \log Q \geq l_0$, $z = l^{1000}$, $\eta = l^{-5}$ and $h = z\eta$, and let $P = \sum_{n\leq z} \Lambda(n)\chi(n)/n$. By (2.2),

$$g(z+h,\chi) - g(z,\chi) = h\Big(P + \frac{L'(1,\chi)}{L(1,\chi)}\Big) + \sum_{z<n\leq z+h} \Lambda(n)\chi(n)\Big(\frac{z+h}{n} - 1\Big),$$

and the last sum has at most $h+1$ terms, each of modulus at most $\log(z+h)\, h/z$. Hence

$$\Big|P + \frac{L'(1,\chi)}{L(1,\chi)}\Big| \leq \frac{Kl(\eta l z^{1-1/400+1/4000} + z^2/Q)}{h} + \frac{(h+1)\log(z+h)}{z} \leq K + K + 2.$$

By (A), $|P| \leq \log z + 3 + \log 2 = 1000 \log\log Q + 3 + \log 2$. So item 1 holds with $C_0 = 2K + 5 + \log 2$.

Item 2. If $Y^{\varepsilon/6} \geq l_0$, item 1 applies, and $\log\log Q \leq \log A + \frac{\varepsilon}{5}\log Y + 2\log\log Y$. Thus

$$\Big|\frac{L'(1,\chi)}{L(1,\chi)}\Big| \leq 200\varepsilon \log Y + 2000 \log\log Y + 1000 \log A + C_0.$$

Since $\varepsilon < 1/52$, $200\varepsilon \log Y \leq \frac{50}{13}\log Y$. Choose Y_0 so that for $Y \geq Y_0$: $Y^{\varepsilon/6} \geq l_0$, $2000\log\log Y \leq \frac{1}{13}\log Y$ and $1000\log A + C_0 \leq \frac{1}{13}\log Y$. Then the right side is at most $(\frac{50}{13} + \frac{2}{13})\log Y = 4\log Y$. This is where the bound $\varepsilon < 1/52$ on the window parameter enters: $1000 \cdot \varepsilon/5 < 4$ with room $\frac{2}{13}\log Y$ for the lower-order terms. □

Lemma 10.12. *For every δ with $0 < \delta \leq 1/800$ there is $K \geq 0$ such that for every non-principal Dirichlet character χ of modulus N, every $Q \geq 2$ with $N \leq Q$ such that χ has a zero-free box of height Q, every $z \geq 1$ and every $\eta \in (0,1]$,*

$$|g(z(1+\eta),\chi) - g(z,\chi)| \leq K \log Q\Big(\eta \log Q \cdot z^{1-1/400+\delta} + \frac{z^2}{Q}\Big).$$

Proof. Write $\sigma_1 = 1 - \frac{1}{400} + \delta$, $T = Q/2$ and $F(y,w) = -\frac{L'}{L}(w,\chi)\,\frac{y^w}{w(w-1)}$. Three inputs are used. (P) Perron's formula for the Riesz kernel, Lemma 10.13, with $c = 2$ and $a_n = \Lambda(n)\chi(n)$: here $\sum_n |a_n| n^{-2} \leq \sum_n \Lambda(n) n^{-2} < \infty$ and $\sum_n a_n n^{-w} = -\frac{L'}{L}(w,\chi)$ for $\operatorname{Re} w > 1$, so for $y \geq 1$

$$\sum_{n\leq y} \Lambda(n)\chi(n)\Big(\frac{y}{n} - 1\Big) = \frac{1}{2\pi i}\int_{(2)} F(y,w)\, dw.$$

(L) The bound Lemma 10.14 for the logarithmic derivative in a zero-free box (its hypotheses $0 < \delta \leq 1/800$, $Q \geq 2$, $N \leq Q$, χ non-principal with a zero-free box of height Q are those of the statement): with $K_B = K_B(\delta) = 100/\delta^2$, $|L'/L(w,\chi)| \leq K_B \log Q$ for $\sigma_1 \leq \operatorname{Re} w \leq 2$, $|\operatorname{Im} w| \leq Q - 1$, for every χ and Q as in the statement. (R) $|L'/L(2+it,\chi)| \leq C_0 := -\zeta'(2)/\zeta(2) = \sum_n \Lambda(n) n^{-2}$, by the Dirichlet series of $-L'/L$.

Contour shift. $L(w,\chi)$ has no zero in the rectangle $\sigma_1 \leq \operatorname{Re} w \leq 2$, $|\operatorname{Im} w| \leq T$ (it lies in the box, as $T \leq Q$), and χ is non-principal, so that $L(w,\chi)$ is entire [180] and the only pole of $F(y,\cdot)$ there is $w = 1$, with residue $-y\,L'(1,\chi)/L(1,\chi)$. By Cauchy's theorem, for $y \geq 1$,

$$g(y,\chi) = \frac{1}{2\pi i}\Big(\int_{\substack{\operatorname{Re} w = 2 \\ |\operatorname{Im} w| > T}} + \int_{[2-iT,\sigma_1-iT]} + \int_{[\sigma_1+iT,2+iT]} + \int_{[\sigma_1-iT,\sigma_1+iT]}\Big) F(y,w)\, dw,$$

since $g(y,\chi)$ is the left side of (P) plus $y\,L'(1,\chi)/L(1,\chi)$ (by (2.2)).

The four pieces, for $y_1 = z(1+\eta)$ and $y_0 = z$, both in $[1,2z]$. Take the difference of the identity at y_1 and y_0.

- Vertical tails: by (R) and $|w(w-1)| \ge t^2$, each is at most $C_0 y^2 \int_{|t|>T} t^{-2}dt \le 4C_0 z^2 \cdot 2/T = 16C_0 z^2/Q$.
- Horizontal segments (length ≤ 2): by (L), as $T \le Q-1$, and $|w(w-1)| \ge T^2$, each is at most $2K_B \log Q \cdot 4z^2/T^2 \le 16K_B \log Q\, z^2/Q$ (using $Q \ge 2$).
- Left side: for $\operatorname{Re} w = \sigma_1 \le 1$ and $0 < \eta \le 1$, $|y_1^w - y_0^w| = z^{\sigma_1}|(1+\eta)^w - 1| \le 2\eta|w|z^{\sigma_1}$, so by (L) the difference of the two left integrals is at most $2\eta z^{\sigma_1} K_B \log Q \int_{-T}^{T} \frac{du}{|\sigma_1 - 1 + iu|} \le 2\eta z^{\sigma_1} K_B \log Q\,(2 + 2\log(1+800T))$, because $1-\sigma_1 \ge 1/800$; and $\log(1+800T) \le 10 \log Q$ for $Q \ge 2$.

Adding up, with $\log Q \ge \log 2$,

$$|g(y_1,\chi) - g(y_0,\chi)| \le K \log Q(\eta \log Q\, z^{\sigma_1} + z^2/Q)$$

with $K = K(\delta)$ depending only on $K_B(\delta)$ and C_0 (for instance $K = 60K_B + 40C_0$).

□

Lemma 10.13. *Let $c > 1$, let $(a_n)_{n\ge1}$ be complex numbers with $\sum_{n\ge1} |a_n| n^{-c} < \infty$, and put $F(w) = \sum_{n\ge1} a_n n^{-w}$ for $\operatorname{Re} w = c$. Then for every real $y > 0$ the integral below converges absolutely and*

$$\sum_{n \le y} a_n \Big(\frac{y}{n} - 1\Big) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(w)\, \frac{y^w}{w(w-1)}\, dw.$$

Proof. The kernel. For $a > 0$ let $I(a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{a^w}{w(w-1)}\, dw$. On $\operatorname{Re} w = c$ we have $|a^w| = a^c$ and $|w(w-1)| \ge \max(c(c-1), |\operatorname{Im} w|^2)$, so the integral converges absolutely, and $I(a) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT}$. We show $I(a) = \max(a-1, 0)$. Let $T > c+1$.

- If $a \ge 1$, apply Cauchy's residue theorem to the rectangle with vertices $c \pm iT$, $-T \pm iT$. The integrand has simple poles at $w = 0$ and $w = 1$, with residues -1 and a. On the left side $\operatorname{Re} w = -T$ we have $|a^w| = a^{-T} \le 1$ and $|w(w-1)| \ge T^2$, so its integral is at most $2T/T^2$; on the two horizontal sides $|a^w| \le a^c$ and $|w(w-1)| \ge T^2$, so each integral is at most $(T+c)a^c/T^2$. Letting $T \to \infty$ gives $I(a) = a - 1$.
- If $0 < a < 1$, use the rectangle with vertices $c \pm iT$, $T \pm iT$, which contains no pole. On the right side $|a^w| = a^T \le 1$ and $|w(w-1)| \ge T(T-1)$, so its integral is at most $2T/(T(T-1))$; on the horizontal sides $|a^w| \le a^c \le 1$ and $|w(w-1)| \ge T^2$, so each integral is at most T/T^2. Letting $T \to \infty$ gives $I(a) = 0$.

The series. By hypothesis

$$\sum_{n\ge1} \int_{c-i\infty}^{c+i\infty} \Big| a_n \Big(\frac{y}{n}\Big)^w \frac{1}{w(w-1)} \Big|\, |dw| = y^c \sum_{n\ge1} |a_n| n^{-c} \int_{-\infty}^{\infty} \frac{dt}{|(c+it)(c-1+it)|} < \infty,$$

so summation and integration may be interchanged (Fubini), and the integral in the statement converges absolutely and equals $\sum_{n\ge1} a_n I(y/n)$. By the first part, $I(y/n) = y/n - 1$ for $n \le y$ and $I(y/n) = 0$ for $n > y$. □

Lemma 10.14. *Let $0 < \delta \le 1/800$ and $Q \ge 2$, and let χ be a non-principal Dirichlet character of modulus $N \le Q$ with a zero-free box of height Q. Then for every w with $1 - \frac{1}{400} + \delta \le \operatorname{Re} w \le 2$ and $|\operatorname{Im} w| \le Q - 1$,*

$$\Big|\frac{L'(w,\chi)}{L(w,\chi)}\Big| \le \frac{100}{\delta^2}\log Q.$$

Proof. Growth. Let $A(u) = \sum_{n \le u} \chi(n)$; since χ is non-principal, the sum over any N consecutive integers vanishes, so $|A(u)| \le N$. For $\operatorname{Re} s > 1$, partial summation gives $L(s,\chi) = s\int_1^\infty A(u)u^{-s-1}\,du$; the integral converges absolutely and locally uniformly on $\operatorname{Re} s > 0$, so both sides are holomorphic there and, $L(s,\chi)$ being entire [180], they agree on $\operatorname{Re} s > 0$. Hence

$$|L(s,\chi)| \le \frac{N|s|}{\operatorname{Re} s} \qquad (\operatorname{Re} s > 0). \tag{10.2}$$

The centre. For $\operatorname{Re} s_0 = \frac{3}{2}$, $1/L(s_0,\chi) = \sum_{n\ge1}\mu(n)\chi(n)n^{-s_0}$, so $|L(s_0,\chi)| \ge 1/\zeta(\frac{3}{2}) \ge \frac{1}{3}$, because $\zeta(\frac{3}{2}) \le 1 + \int_1^\infty u^{-3/2}\,du = 3$.

The disc. Fix w as in the statement, let $t = \operatorname{Im} w$, $s_0 = \frac{3}{2} + it$, $R = \frac{1}{2} + \frac{1}{400} - \frac{\delta}{2}$ and $r = |w - s_0| = |\operatorname{Re} w - \frac{3}{2}|$. As $1 - \frac{1}{400} + \delta \le \operatorname{Re} w \le 2$ and $\delta < \frac{1}{400}$,

$$r \le \max\Big(\tfrac{1}{2} + \tfrac{1}{400} - \delta,\ \tfrac{1}{2}\Big) = \tfrac{1}{2} + \tfrac{1}{400} - \delta = R - \tfrac{\delta}{2}.$$

Every s with $|s - s_0| \le R$ has $\operatorname{Re} s \ge \frac{3}{2} - R = 1 - \frac{1}{400} + \frac{\delta}{2}$ and $|\operatorname{Im} s| \le Q - 1 + R < Q$. So the closed disc $\overline{D} = \{|s - s_0| \le R\}$ lies in the open set $U = \{\operatorname{Re} s > 1 - \frac{1}{400},\ |\operatorname{Im} s| < Q\}$, on which $L(\cdot,\chi)$ is holomorphic and, by the zero-free box, has no zero. Choose $R' > R$ with $\{|s - s_0| < R'\} \subseteq U$. On this disc, which is simply connected, there is a holomorphic h with $e^{h(s)} = L(s,\chi)/L(s_0,\chi)$ and $h(s_0) = 0$; then $h' = L'/L$ and $\operatorname{Re} h(s) = \log|L(s,\chi)| - \log|L(s_0,\chi)|$.

The bound for $\operatorname{Re} h$. For $s \in \overline{D}$ we have $\operatorname{Re} s \ge \frac{1}{2}$ and $|s| \le (\frac{3}{2} + R) + Q \le Q + 3$, so by (10.2) and $N \le Q$, $|L(s,\chi)| \le 2Q(Q+3) \le 5Q^2$ (as $Q \ge 2$). Hence, with the centre bound, $\operatorname{Re} h \le \log(15Q^2) \le M := 6\log Q$ on $\overline{D}$, because $15 \le Q^4$ for $Q \ge 2$.

Borel–Carathéodory. Let $g(z) = h(s_0 + z)$ for $|z| \le R$; then $g(0) = 0$ and $\operatorname{Re} g \le M$. By the Borel–Carathéodory lemma [180], $|g(z)| \le 2\rho M/(R - \rho)$ for $|z| \le \rho < R$. For $|z| \le r$ put $\rho = (R + r)/2$; the circle of radius $(R - r)/2$ about z lies in $\{|\zeta| \le \rho\}$, and Cauchy's estimate gives

$$|g'(z)| \le \frac{2}{R - r}\cdot\frac{2\rho M}{R - \rho} = \frac{8\rho M}{(R-r)^2} \le \frac{8RM}{(R-r)^2}.$$

At $z = w - s_0$, with $R - r \ge \delta/2$ and $R < 0.51$,

$$\Big|\frac{L'(w,\chi)}{L(w,\chi)}\Big| = |g'(w - s_0)| \le \frac{8\cdot 0.51\cdot 6\log Q}{\delta^2/4} \le \frac{100}{\delta^2}\log Q.$$

□

Lemma 10.15. *There is an absolute constant C_z such that for every real $Q \ge 3$ at most $C_z Q^{1/4}$ primitive Dirichlet characters of modulus at most Q have a zero s with $\operatorname{Re} s > 1 - \frac{1}{400}$ and $|\operatorname{Im} s| \le Q$.*

Proof. Put $\alpha = 1 - \frac{1}{400}$ and $T = Q \ge 3$. Let χ be primitive of modulus at most Q with a zero s_0 such that $\operatorname{Re} s_0 > \alpha$ and $|\operatorname{Im} s_0| \le Q$. For $\operatorname{Re} s > 1$ the Euler product of $L(s,\chi)$ converges absolutely and its factors are non-zero, so $\operatorname{Re} s_0 \le 1$; hence s_0 is counted by $N(\alpha, Q, \chi)$,

and $N(\alpha,Q,\chi)\ge 1$. By Montgomery's density theorem ([179, Theorem 12.2], quoted as Theorem B.8), as $3(1-\alpha)/(2-\alpha)=3/401$ and $Q^2T=Q^3$, the number of such χ is at most

$$\sum_{q\le Q}\sideset{}{^*}\sum_{\chi \bmod q} N(\alpha,Q,\chi)\le C_{\mathrm{M}}\,Q^{9/401}(\log Q^2)^9 = 2^9 C_{\mathrm{M}}\,Q^{9/401}(\log Q)^9.$$

Since $\frac14-\frac{9}{401}=\frac{365}{1604}>0$, the number $B=\sup_{Q\ge 3}(\log Q)^9Q^{-365/1604}$ is finite, and the count is at most $C_zQ^{1/4}$ with $C_z=2^9C_{\mathrm{M}}B$. □

11 Good hosts

A host satisfies the host conditions, but Parts III and IV need more of it: integers that fix the ranks of the lattices, a set of tops, a spacing set, auxiliary primes and analytic estimates beyond the scale x. The concrete good-host requirement (Definition 11.4) lists these data as existential witnesses with the conditions (Hg1)–(Hg8), and Proposition 11.5 proves that it supplies good hosts: for every host built on a good window pair, with the sizes of Section 9, the witnesses exist. This is clause (Gh1) of Proposition 17.6, the requirement for which the class of Part III is designed, and through Proposition 7.9 it is hypothesis (Sb) of Proposition 9.1. The proof of Proposition 11.5 takes the witnesses one condition at a time, each from its own result, and the definitions 11.1 and Definition 11.2 fix the sizes: $|A|$ of order $x^{2+8\varepsilon}$, $\sqrt{x\delta_1}\le m\le 2\sqrt{x\delta_1}$ with $\delta_1=x^{-\varepsilon/8}$, and k of order $x^{9/2+10\varepsilon}$.

Three results carry the weight. Proposition 11.11 chooses the tops at random, with density δ_1 on A_0 and $\frac34\delta_1$ off it. Chebyshev's inequality on the o cosets of A_0 gives the shares $4/(3o+1)$ and $3/(3o+1)$ of (Ec4), which produce the weight $(3o+1)^{-1}$ in $G_{\mathfrak{L}_\psi}$, and a first-moment bound with Markov's inequality gives few bad tops along the Frobenius orbits of the inert primes. Proposition 11.12 gives the spacing set: a set of about n_0/L_w small integers, up to a factor $\exp\bigl(-C\sqrt{\log n_0\log L_w}\bigr)$, without any relation $j(s_2-s_1)=i(s_3-s_1)$ with $0<|i|,|j|\le L_w$. Its proof is Behrend's construction of large sets without three-term progressions [20], with the base enlarged so that the relations with small coefficients are excluded as well. Proposition 11.9 gives the level primes: two primes g_1,g_2 of size about $\bigl(k\exp(x^{\varepsilon})\bigr)^8$ that are norms from $\mathbb{Q}(\sqrt{-d})$ of elements congruent to 1 modulo the host modulus, with disjoint exceptional sets (Lemma 11.10), and small primes p_1,p_2,q_0 congruent to 1 modulo it. It uses the Chebotarev density theorem of Thorner and Zaman (Theorem B.10) and ray class fields. In the proof of Proposition 11.5 the prime q_0 is taken instead from Linnik's theorem (Lemma 11.14). The primes g_1,g_2 are the auxiliary primes of the slope records of the packets in Part IV (Definition 19.1).

The other conditions are routine. Rank data with the rank sizes exist on every frame of the right order, with $t\le x^{400}$ (Lemma 11.6), and the primes of a window pair exceed $4tr+1$ (Lemma 11.7). The field of a normalised host is CM, contains a square root of $-d$, and its relative different over $\mathbb{Q}(\sqrt{-d})$ is the product of two conjugate coprime ideals (Lemma 11.13). The zero-free boxes of (Hg7) and the estimates at the scale in (Hg8) are those of a good window pair. The last estimate of (Hg8), a saving $\exp(-c\,x^{\varepsilon/8})$ for every character of the host at arguments $y\ge\frac13\exp(x^{\varepsilon/3})$, is Proposition 11.8, a Chebyshev-type bound that uses the absence of real zeros near 1 guaranteed by the boxes; it matches the thin-scale property (Sp7) of slope records.

Definition 11.1. A **rank datum** is a tuple $\mathfrak{R}=(e,A,h,a_0;k,m,t)$ consisting of a frame (e,A,h,a_0) and integers k,m,t such that

(R1) k and m are even and t is odd;

(R2) $2\le m$, $m+3\le k$, $k+m+4\le t$ and $3m<t$;

(R3) $m \le 2|A| \le k$;

(R4) $m^2 \le k + m(t+1)$.

Its **derived ranks** are

$$f = 2|A|, \qquad r = k + m - 1, \qquad n_V = r + t, \qquad r_S = \tfrac{m}{2} - 1, \qquad k_j = k + j(t+1) \quad (j \ge 0),$$

and its **signatures** are $\mathrm{sig}_Y = (k + r_S,\, r_S + 1)$ and $\mathrm{sig}_W(j) = (k + j - 1,\, m - j)$ for $1 \le j \le m$.

A **Harish-Chandra parameter** for $\mathfrak{R}$ is a tuple $(u, v, b, \tilde{b})$ with $u \in (\frac{1}{2} + \mathbb{Z})^k$, $v \in \mathbb{Z}^{k-1}$ and $b, \tilde{b} \in \mathbb{Z}^m$ such that

(Pa1) $u_1 > v_1 > u_2 > v_2 > \cdots > v_{k-1} > u_k > -b_1$;

(Pa2) $\tilde{b}_j - b_j \ge \frac{t+3}{2}$ for $1 \le j \le m$, and $b_{j+1} - \tilde{b}_j \ge \frac{t+3}{2}$ for $1 \le j < m$.

Its **shift exponents** are the integers $s_j^\sharp = \tilde{b}_j - \frac{1}{2}k_j$ and $s_j^\flat = \tilde{b}_j - \frac{1}{2}k_{j-1}$ for $1 \le j \le m$.

Definition 11.2. A **scale** is a triple $\mathfrak{s} = (x, \varepsilon, J)$ of real numbers such that

$$x > 1, \qquad 0 < \varepsilon < \frac{1}{52}, \qquad J \ge 1.$$

For this scale put

$$L(\mathfrak{s}) = x^{\varepsilon}, \qquad \delta_1(\mathfrak{s}) = x^{-\varepsilon/8}, \qquad \varepsilon_z(\mathfrak{s}) = \frac{\varepsilon}{J}.$$

For a rank datum $\mathfrak{R}$ with derived ranks f, r, define the integer

$$n_W(\mathfrak{R}, \mathfrak{s}) = 8 \left\lceil \frac{10r}{8} \left(f + \frac{f^2 (k-1)^3 L(\mathfrak{s})^2}{m} \right) \right\rceil.$$

The rank datum $\mathfrak{R}$ has the **rank sizes at** $\mathfrak{s}$ if

$$\begin{aligned} x^{2+8\varepsilon} &\le |A| \le 2x^{2+8\varepsilon}, \\ \sqrt{x\delta_1(\mathfrak{s})} &\le m \le 2\sqrt{x\delta_1(\mathfrak{s})}, \\ x^{9/2+10\varepsilon} &\le k \le 2x^{9/2+10\varepsilon}, \\ r + t &= 8fr + 4 + \lceil x^{377} \rceil\, n_W(\mathfrak{R}, \mathfrak{s}). \end{aligned}$$

Definition 11.3. Fix a host $\mathfrak{h}$ for a frame with group A, a set $T \subseteq A$, real numbers $x > 1$ and $0 < \varepsilon < 1/52$, an integer $k \ge 1$, and a total ordering of the finite set A. At each inert prime p, choose $\rho_p(t)$ to be the least element, in this ordering, of the coset containing t of $\langle h_p \rangle$ if p is not sign-changing, and of $\langle 2h_p \rangle$ if p is sign-changing. Thus ρ_p, R_p and $P_p(v)$ are the choices of Definition 5.1 determined by this ordering.

The following auxiliary constants are used only in this definition. Put

$$\begin{aligned} \delta_{\mathrm{orb}} &= x^{-\varepsilon/8}, & h_{\mathrm{orb}} &= 2 + 8\varepsilon, \\ K_{\mathrm{orb}} &= \left\lceil \frac{8(h_{\mathrm{orb}} + 2)}{\varepsilon} \right\rceil, & r_{\mathrm{orb}} &= K_{\mathrm{orb}} + 1, \\ u_{\mathrm{orb}} &= \lceil 1/\delta_{\mathrm{orb}} \rceil, & \ell_{\mathrm{orb}} &= (6r_{\mathrm{orb}} + 1)u_{\mathrm{orb}}, \\ M_{\mathrm{orb}} &= \left\lceil (h_{\mathrm{orb}} + 3) \log x\, 10^{6r_{\mathrm{orb}}+1} \right\rceil, & G_{\mathrm{orb}} &= (M_{\mathrm{orb}} + 5)\ell_{\mathrm{orb}}, \\ N_{\mathrm{orb}} &= \left\lceil \max\left(6\exp(1) G_{\mathrm{orb}} \delta_{\mathrm{orb}},\ (h_{\mathrm{orb}} + 3)\frac{\log x}{\log 2} \right) \right\rceil + 1. \end{aligned}$$

Define also the even nonnegative integer

$$L_{\mathrm{orb}} = 2\max\left(\{0\} \cup \left\{ \left\lceil \frac{\log((kx^{26})^4)}{\log p} \right\rceil : p \text{ is inert and not sign-changing} \right\}\right).$$

The maximum exists, since all its entries are nonnegative integers at most $\lceil \log((kx^{26})^4)/\log 2 \rceil$. The adjoined 0 gives the value when there is no such prime.

For a positive integer n and $P \subseteq \{0, \dots, n-1\}$ consider the following three finite conditions. Here and below the residue modulo n is taken in $\{0, \dots, n-1\}$, and all interval endpoints and quantified indices are integers.

(Go1) For every $0 \le i < n$,

$$|P \cap \{(i+j) \bmod n : 0 \le j < L_{\mathrm{orb}}\}| \le K_{\mathrm{orb}}.$$

(Go2) Put $B = \lfloor n/\ell_{\mathrm{orb}} \rfloor$. For every $0 \le s_0 < B$ there is an integer j with $0 \le j \le M_{\mathrm{orb}}$ such that, for $s = (s_0 + j) \bmod B$ and every $1 \le c \le 6r_{\mathrm{orb}} + 1$, the set

$$P \cap \{s\ell_{\mathrm{orb}} + (c-1)u_{\mathrm{orb}}, \dots, s\ell_{\mathrm{orb}} + cu_{\mathrm{orb}} - 1\}$$

is empty when c is odd and is a singleton whose element is even when c is even. This condition is vacuous if $B = 0$. These intervals do not wrap modulo n.

(Go3) For every $0 \le i < n$,

$$|P \cap \{(i+j) \bmod n : 0 \le j < G_{\mathrm{orb}}\}| \le N_{\mathrm{orb}}.$$

The **good representatives selected by the finite conditions** are

$$G_p = \{v \in R_p : \ P_p(v) \text{ satisfies (Go1)–(Go3) with } n = \mathrm{ord}(h_p)\}.$$

The parity in (Go2) refers to integer positions relative to the chosen representative. These choices depend on the ordering of A.

For a real number $C_b > 0$, these choices satisfy the **bad-top bound with constant** C_b if

$$\sum_{\substack{p \text{ inert, not sign-changing} \\ p < \exp(x^{\varepsilon/3}) \\ \mathrm{ord}(h_p) \ge x^{\varepsilon/4}}} \frac{\log p}{p} \left|\{t \in T : \rho_p(t) \notin G_p\}\right| \le 4C_b x^{-2} \delta_{\mathrm{orb}} |A| \sum_{\substack{p \text{ inert, not sign-changing} \\ p < \exp(x^{\varepsilon/3}) \\ \mathrm{ord}(h_p) \ge x^{\varepsilon/4}}} \frac{\log p}{p}.$$

Both sums are finite and have the same prime range.

Definition 11.4. The following defines a predicate $\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}, T, t)$, the **concrete good-host requirement**. It is false unless $x > 1$, $0 < \varepsilon < 1/52$, $J \ge 1$, $\mathfrak{h}$ is a host for a frame (e, A, h, a_0), $T \subseteq A$, t is an integer, and the finite real list κ has the form

$$\kappa = (C_{T_2}, C_8, C_5, c_6, C_b), \qquad C_8 \ge 1, \quad C_5 > 0, \quad c_6 > 0, \quad C_b > 0.$$

There is no sign restriction on C_{T_2}. For such inputs put $\mathfrak{s} = (x, \varepsilon, J)$, $E = E_{\mathfrak{h}}$ and

$$n_0 = |A_0|, \qquad Q_{\mathrm{odd}} = Q_S / 2^{v_2(Q_S)}, \qquad L_w = 10\lceil 405 \log x \rceil + 2\lceil 30/\varepsilon \rceil + 22.$$

Here $v_2(Q_S)$ is the exponent of 2 in the positive integer Q_S.

The predicate holds precisely when there exist a rank datum $\mathfrak{R} = (e, A, h, a_0; k, m, t)$, a total ordering of A, an integer $B \ge 0$, an element $s_d \in E$, a field embedding $\varphi_0 : E \hookrightarrow \mathbb{C}$, an integral ideal $\mathfrak{b}$ of E, five primes g_1, g_2, p_1, p_2, q_0, a finite set S of nonnegative integers, an element $u_0 \in A$ and a real number Y satisfying the following conditions. The integer L_w, the prime ℓ_0 and the set of level primes in (Hg5) are not among these data: they are given by explicit formulas.

(Hg1) Rank and host. The datum $\mathfrak{R}$ has the rank sizes at $\mathfrak{s}$, the host $\mathfrak{h}$ is normalised and has field rows up to B, and

$$4tr+1<q, \qquad 4tr+1<d, \qquad B\geq \left\lfloor x^{\varepsilon/(5J)}\right\rfloor, \qquad J\leq C_8, \quad o\leq C_8, \quad Q_S\leq C_8|A|.$$

(Hg2) Quadratic subfield and relative different. The field E is CM. Under the identification of Galois groups given by the host,

$$s_d^2=-d, \qquad \sigma(s_d)=\epsilon_2(\sigma)s_d \quad (\sigma\in \mathrm{Gal}(E/\mathbb{Q})), \qquad \varphi_0(s_d)=\sqrt{-1}\sqrt{d}.$$

Write $K=\mathbb{Q}(s_d)\subseteq E$ and use a bar for complex conjugation on E. The relative different satisfies

$$\mathcal{D}_{E/K}=\mathfrak{b}\bar{\mathfrak{b}}, \qquad \mathfrak{b}+\bar{\mathfrak{b}}=\mathcal{O}_E, \qquad \sigma(\mathfrak{b})=\mathfrak{b} \quad \big(\sigma\in\mathrm{Gal}(E/\mathbb{Q}),\ \epsilon_2(\sigma)=1\big).$$

The chosen CM type is the set of embeddings $\varphi: E\hookrightarrow\mathbb{C}$ with $\varphi(s_d)=\sqrt{-1}\sqrt{d}$. The auxiliary ideal in K is $\mathcal{O}_K$.

(Hg3) Tops and the spacing set. With $\delta_1=\delta_1(\mathfrak{s})$,

$$\tfrac{1}{2}\delta_1|A|\leq|T|\leq 2\delta_1|A|, \qquad x^{-\varepsilon}|A|\leq|S|, \qquad 4L_w|S|\leq n_0,$$

and

$$\frac{n_0}{L_w}\exp\Big(-C_{T_2}\sqrt{\log n_0\log L_w}\Big)\leq|S|.$$

Every $s\in S$ satisfies $4L_ws<n_0$. For pairwise distinct $s_1,s_2,s_3\in S$ and integers $i\neq j$ with $0<|i|,|j|\leq L_w$,

$$j(s_2-s_1)\neq i(s_3-s_1).$$

The associated thick places in $\ker\epsilon_2$ are $\{((-1,1),sa_0+u_0): s\in S\}$.

(Hg4) Good representatives. Choose ρ_p and G_p by Definition 11.3, using the given total ordering of A and the parameters (x,ε,k). They satisfy its bad-top bound with constant C_b.

(Hg5) Auxiliary and level primes. The primes g_1,g_2 are distinct. The primes p_1,p_2 are distinct and different from both g_1,g_2. We have $q_0\neq p_1,p_2$ and

$$(k\exp(x^\varepsilon))^8\leq g_i\leq 2(k\exp(x^\varepsilon))^8 \quad (i=1,2), \qquad p_1,p_2,q_0\leq N_{\mathfrak{h}}^{C_8},$$

$$g_i\equiv 1 \pmod{N_{\mathfrak{h}}} \quad (i=1,2), \qquad p_1,p_2\equiv 1 \pmod{\mathrm{lcm}(15,N_{\mathfrak{h}})},$$
$$q_0\equiv 1 \pmod{\ell_0N_{\mathfrak{h}}},$$

where ℓ_0 is the least prime not dividing $N_{\mathfrak{h}}(p_1-1)(p_2-1)$ (it exists since $p_1,p_2\geq 2$). For each $i=1,2$ there exists $\pi_i\in\mathcal{O}_E\cap K$ with

$$\pi_i\bar{\pi}_i=g_i, \qquad \pi_i-1\in N_{\mathfrak{h}}\mathcal{O}_E.$$

The primes p_1,p_2 split completely in E. The level primes are exactly $\{g_1,g_2,p_1,p_2,q_0\}$, and every one of them splits in K. The following two sets, for $i=1$ and $i=2$, are disjoint:

$$\{g_i\}\ \cup\ \{\ell:\ \ell \text{ is prime and there is an integer } 1\leq a\leq r \text{ with } g_i^a<\ell^{4v_\ell(g_i^a-1)}\}.$$

Here v_ℓ is the exponent of the prime ℓ in a positive integer.

(Hg6) A common scale for the pair and the boxes. One has $x^{1/J} \le Y \le x$. For use in (Hg6) and (Hg7), put

$$X_1 = \exp(Y^{\varepsilon/5}), \qquad X_2 = \exp(Y^{\varepsilon/5}(\log Y)^2).$$

Then

$$\sqrt{X_1} < q \le X_1, \qquad \sqrt{X_2} < d \le X_2, \qquad (-d/p) = 1 \quad (p \text{ prime},\ p \le Y^{\varepsilon/5}),$$

and $\log N_{\mathfrak{h}} \le 2Y^{\varepsilon/5}(\log Y)^2$. The congruences $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$ and $(-d/q) = 1$ are part of the host conditions.

(Hg7) Zero-free boxes. For every fundamental discriminant D below, let $\chi_D(n) = (D/n)$ be its primitive quadratic Dirichlet character. For $D = -q, -8q$, require $|D| \le 8X_1$ and a zero-free box of height $8X_1$ for χ_D. For $D = -d, -8d, qd, 8qd$, require $|D| \le 8X_1X_2$ and a zero-free box of height $8X_1X_2$ for χ_D. By the congruences in (Hg6), all these integers are fundamental discriminants.

For every character θ of $\mathcal{G}$ such that χ_θ^* is real, write $\mathrm{cond}(\chi_\theta^*)$ for its positive modulus. If $d \mid \mathrm{cond}(\chi_\theta^*)$, require

$$\mathrm{cond}(\chi_\theta^*) \le 8Q_{\mathrm{odd}}X_1X_2 \quad \text{and a zero-free box of height } 8Q_{\mathrm{odd}}X_1X_2 \text{ for } \chi_\theta^*.$$

If $q \mid \mathrm{cond}(\chi_\theta^*)$ and $d \nmid \mathrm{cond}(\chi_\theta^*)$, require

$$\mathrm{cond}(\chi_\theta^*) \le 8Q_{\mathrm{odd}}X_1 \quad \text{and a zero-free box of height } 8Q_{\mathrm{odd}}X_1 \text{ for } \chi_\theta^*.$$

(Hg8) Character estimates. For $D = -q, -d, qd$, with the primitive quadratic characters of (Hg7),

$$|g(x, \chi_D)| \le x^{1/2+\varepsilon}, \qquad \left|\frac{L'(1, \chi_D)}{L(1, \chi_D)}\right| \le C_5 \log x.$$

In addition,

$$\left|\mathrm{Re}\,\frac{L'(1, \chi_{-d})}{L(1, \chi_{-d})}\right| \le x^{\varepsilon}.$$

For every character θ of $\mathcal{G}$ and every real $y \ge \frac{1}{3}\exp(x^{\varepsilon/3})$,

$$|g(y, \chi_\theta^*)| \le y \exp(-c_6 x^{\varepsilon/8}).$$

Proposition 11.5. *Let $\mathfrak{G}$ be the good-host requirement obtained by restricting the concrete good-host requirement to integers $t \ge 1$. Then $\mathfrak{G}$ supplies good hosts.*

Proof. Let $\psi \in \mathcal{P}$ have modulus N; we show that $\mathfrak{G}$ supplies good hosts for ψ.

Choice of κ and x_B. Fix $\varepsilon \in (0, 1/52)$ and $C_0 \ge 1$. Take: $C \ge 1$ and x_0^{lv} from Proposition 11.9 for ε; $C_b > 0$ and x_0^{tp} from Proposition 11.11 for ε, C_0; C_T and x_0^{th} from Proposition 11.12 for ε, C_0; $c > 0$ and x_0^{ch} from Proposition 11.8 for $B = C_0$ and ε; x_B^{sz}, the number x_B of Lemma 11.7 for ε, C_0. Let $A \ge 1$ be the absolute exponent of Theorem B.9 (as in Lemma 11.14) and $X_* = \exp(1 + 2(3 + 2C)A/(4\varepsilon/5)^3)$, the cutoff of Lemma 11.14 for ε and its parameter $C_8 := C$. Let $x_6 \ge 1$ be such that for every $x \ge x_6$

$$4\log x \le x^{\varepsilon}, \qquad \log(2C_0) + 3\log x \le 15\, x^{\varepsilon/(5C_0)}, \qquad C_0 \le x^{\varepsilon/(5C_0)};$$

it exists because $\log x = o(x^\delta)$ for every $\delta > 0$. Put

$$\kappa = (C_{T_2}, C_8, C_5, c_6, C_b) = (C_T,\ \max(C_0, C, (3+2C)A),\ 4,\ c,\ C_b),$$

$$x_B = \max(16,\ 10708^{832/319},\ e^{4C_0},\ x_6,\ x_B^{\mathrm{sz}},\ x_0^{\mathrm{lv}},\ x_0^{\mathrm{tp}},\ x_0^{\mathrm{th}},\ x_0^{\mathrm{ch}},\ X_*).$$

Then $C_8 \geq 1$, $C_5 > 0$, $c_6 > 0$, $C_b > 0$, so κ has the form required in Definition 11.4. Fix $x \geq x_B$ and $J, (e, A, h, a_0), M, \mathfrak{h} = (q, d, Q_S, \alpha), Y$ as in the definition of the supply of good hosts for ψ; put $\mathfrak{s} = (x, \varepsilon, J)$, a scale since $x > 1$, $0 < \varepsilon < 1/52$, $J \geq 1$; and $N_{\mathfrak{h}} = Q_S q d$, $X_1 = \exp(Y^{\varepsilon/5})$, $X_2 = \exp(Y^{\varepsilon/5}(\log Y)^2)$, $Q_{\mathrm{odd}} = Q_S/2^{v_2(Q_S)}$. We construct the data required in Definition 11.4, condition by condition; t will be the last entry of the rank datum.

Elementary consequences of the hypotheses. (a) Since $Y \geq x^{1/J}$ and $J \leq C_0$, $Y^{\varepsilon/5} \geq x^{\varepsilon/(5C_0)}$ and $\log Y \geq (\log x)/C_0 \geq 4$ (as $x \geq e^{4C_0}$); also $Y \leq x$. (b) Q_{odd} divides the odd part of $\mathrm{lcm}(2^{e+2}, M)$, which divides M; so $Q_{\mathrm{odd}} \mid M$, $Q_{\mathrm{odd}} \leq M \leq C_0$, and every odd prime factor of Q_S divides M. (c) By (Wp1), $q, d \nmid M$. (d) For every prime $p \leq Y^{\varepsilon/5}$, $(-d/p) = 1$: for odd p, $p \mid d+1$ by the definition of window pairs, so $p \nmid d$ and $-d \equiv 1 \pmod p$; for $p = 2$, $d \equiv 7 \pmod{16}$ gives $-d \equiv 1 \pmod 8$. (e) From $\frac{1}{2}Y^{\varepsilon/5} < \log q \leq Y^{\varepsilon/5}$ and $\frac{1}{2}Y^{\varepsilon/5}(\log Y)^2 < \log d \leq Y^{\varepsilon/5}(\log Y)^2$: $\sqrt{X_1} < q \leq X_1$, $\sqrt{X_2} < d \leq X_2$, and $d > \exp(\frac{1}{2}\cdot 16) > 100$ by (a). (f) By (a), $|A| \leq 2x^3$ and $x \geq x_6$,

$$\log N_{\mathfrak{h}} \leq \log(C_0|A|) + Y^{\varepsilon/5} + Y^{\varepsilon/5}(\log Y)^2 \leq 15Y^{\varepsilon/5} + Y^{\varepsilon/5} + Y^{\varepsilon/5}(\log Y)^2 \leq 2Y^{\varepsilon/5}(\log Y)^2,$$

using $\log(C_0|A|) \leq \log(2C_0) + 3\log x \leq 15x^{\varepsilon/(5C_0)} \leq 15Y^{\varepsilon/5}$ and $(\log Y)^2 \geq 16$. As $1 < Y \leq x$, also $\log N_{\mathfrak{h}} \leq 2x^{\varepsilon/5}(\log x)^2$.

(Hg1). As $x \geq 16$, Lemma 11.6, item 1, gives integers k, m, t such that $\mathfrak{R} = (e, A, h, a_0; k, m, t)$ is a rank datum with the rank sizes at $\mathfrak{s}$; as $x \geq 10708^{832/319}$, item 2 gives $t \leq x^{400}$; and (R2) gives $t \geq k + m + 4 \geq 1$. Put $r = k + m - 1$. By Lemma 11.7 ($x \geq x_B^{\mathrm{sz}}$, $J \leq C_0$, $t \leq x^{400}$, $x^{1/J} \leq Y \leq x$), $4tr + 1 < q$ and $4tr + 1 < d$. Put $B = \lfloor Y^{\varepsilon/5} \rfloor$. By hypothesis, $\mathfrak{h}$ is normalised and has field rows up to B, and $B \geq \lfloor x^{\varepsilon/(5J)} \rfloor$ since $Y^{\varepsilon/5} \geq x^{\varepsilon/(5J)}$. Finally $J \leq C_0 \leq C_8$, $o_\psi \leq C_0 \leq C_8$ and $Q_S \leq C_0|A| \leq C_8|A|$.

(Hg2). The host is normalised, so $\alpha(-1) = ((-1,-1), 0)$ by (Hn1), and Lemma 11.13 applies to $E = E_{\mathfrak{h}}$: E is CM, with complex conjugation c, and there are $s_d \in E$, an embedding φ_0 and an integral ideal $\mathfrak{b}$ with $s_d^2 = -d$, $\sigma(s_d) = \epsilon_2(\sigma)s_d$, $\varphi_0(s_d) = \sqrt{-1}\sqrt{d}$ and, for $K = \mathbb{Q}(s_d)$, $\mathcal{D}_{E/K} = \mathfrak{b}\bar{\mathfrak{b}}$, $\mathfrak{b} + \bar{\mathfrak{b}} = \mathcal{O}_E$, $\sigma(\mathfrak{b}) = \mathfrak{b}$ whenever $\epsilon_2(\sigma) = 1$. This is (Hg2).

(Hg3) and (Hg4). We have $x \geq x_0^{\mathrm{tp}}$, and $\mathfrak{R}$, $\mathfrak{h}$, Y satisfy the hypotheses of Proposition 11.11 (rank sizes at $\mathfrak{s}$, $J \in [1, C_0]$, index $o_\psi \leq C_0$, a host whose pair (q, d) is a window pair at a scale $Y \geq x^{1/J}$). Fix an arbitrary total ordering of A. By that proposition there is $T \subseteq A$ with $\frac{1}{2}\delta_1(\mathfrak{s})|A| \leq |T| \leq 2\delta_1(\mathfrak{s})|A|$, with (Ec4) for $(\mathfrak{h}, T)$ at scale x with parameter ε, and such that the choices ρ_p, G_p determined by this ordering for (x, ε, k) satisfy the bad-top bound with constant C_b. This is (Hg4) and the first part of (Hg3). As $x \geq x_0^{\mathrm{th}}$, Proposition 11.12 gives a finite set S of non-negative integers with $4L_w s < n_0$ ($s \in S$), $x^{-\varepsilon}|A| \leq |S|$, $4L_w|S| \leq n_0$, $\frac{n_0}{L_w}\exp(-C_{T_2}\sqrt{\log n_0 \log L_w}) \leq |S|$ and no relation $j(s_2 - s_1) = i(s_3 - s_1)$ as in (Hg3), with $n_0 = |A_0|$ and L_w as in Definition 11.4. Take $u_0 = 0 \in A$. This is (Hg3).

(Hg5). The prime d satisfies $d \equiv 7 \pmod{16}$ by (Hr1), so $d \equiv 3 \pmod 4$. Moreover $d \geq 100$ by (e), $d \mid N_{\mathfrak{h}}$, and $\log N_{\mathfrak{h}} \leq 2x^{\varepsilon/5}(\log x)^2$ by (f). As $x \geq x_0^{\mathrm{lv}}$, Proposition 11.9 with $N_{\mathfrak{h}}$ and $\mathfrak{R}$ gives primes g_1, g_2, p_1, p_2 (and a fifth prime, which we do not use) and elements π_i of the ring of integers $\mathcal{O}$ of $\mathbb{Q}(\sqrt{-d})$ with its properties 1–3. As $x \geq X_*$, Lemma 11.14, applied with its parameter $C_8 := C \geq 1$, $N = N_{\mathfrak{h}} \geq d \geq 100$, these p_1, p_2 (primes $\leq N_{\mathfrak{h}}^C$ by property 3), the integer $k \geq 1$ of $\mathfrak{R}$ (rank sizes), $g_1, g_2 \geq (k\exp(x^\varepsilon))^8$ (property 1) and $\log N_{\mathfrak{h}} \leq 2x^{\varepsilon/5}(\log x)^2$ (by (f)), gives a prime q_0 (its q^*) with $q_0 \equiv 1 \pmod{\ell_0 N_{\mathfrak{h}}}$, where ℓ_0 is the least prime not

dividing $N_\mathfrak{h}(p_1-1)(p_2-1)$, $q_0 \le N_\mathfrak{h}^{(3+2C)A}$ and $q_0 \neq p_1, p_2$. Identify $\mathbb{Q}(\sqrt{-d})$ with $K = \mathbb{Q}(s_d)$ by $\sqrt{-d} \mapsto s_d$; then $\mathcal{O} = \mathcal{O}_K \subseteq \mathcal{O}_E$ and its non-trivial automorphism is the restriction of c. We check (Hg5).

- Distinctness: $g_1 \neq g_2$, $p_1 \neq p_2$, $p_1, p_2 \notin \{g_1, g_2\}$ (property 3, 2), and $q_0 \notin \{p_1, p_2\}$ (Lemma 11.14). Sizes: $(k\exp(x^\varepsilon))^8 \le g_i \le 2(k\exp(x^\varepsilon))^8$ (property 1), $p_1, p_2 \le N_\mathfrak{h}^C \le N_\mathfrak{h}^{C_8}$ and $q_0 \le N_\mathfrak{h}^{(3+2C)A} \le N_\mathfrak{h}^{C_8}$, as $N_\mathfrak{h} \ge 1$ and $C, (3+2C)A \le C_8$.
- $\pi_i \in \mathcal{O}_E \cap K$, $\pi_i\bar{\pi}_i = g_i$ and $\pi_i - 1 \in N_\mathfrak{h}\mathcal{O}_K \subseteq N_\mathfrak{h}\mathcal{O}_E$. Hence $g_i \equiv 1 \pmod{N_\mathfrak{h}}$: $\bar{\pi}_i - 1 \in N_\mathfrak{h}\mathcal{O}_K$, so $g_i - 1 = (\pi_i - 1)\bar{\pi}_i + (\bar{\pi}_i - 1) \in N_\mathfrak{h}\mathcal{O}_K$, and $(g_i-1)/N_\mathfrak{h} \in \mathcal{O}_K \cap \mathbb{Q} = \mathbb{Z}$.
- $p_1 \equiv p_2 \equiv 1 \pmod{\operatorname{lcm}(15, N_\mathfrak{h})}$ (property 3), and $q_0 \equiv 1 \pmod{\ell_0 N_\mathfrak{h}}$ as just shown; in particular $q_0 \equiv 1 \pmod{N_\mathfrak{h}}$.
- Splitting. Every $\ell \in \{g_1, g_2, p_1, p_2, q_0\}$ satisfies $\ell \equiv 1 \pmod{N_\mathfrak{h}}$, so $\ell \nmid N_\mathfrak{h}$ and the Frobenius of ℓ in $\mathrm{Gal}(\mathbb{Q}(\zeta_{N_\mathfrak{h}})/\mathbb{Q})$ is $\sigma_\ell = \mathrm{id}$; hence ℓ splits completely in $\mathbb{Q}(\zeta_{N_\mathfrak{h}})$ and in its subfields $E \supseteq K$. In particular p_1, p_2 split completely in E and every level prime splits in K.
- The sets $\{g_i\} \cup \{\ell \text{ prime} : g_i^a < \ell^{4v_\ell(g_i^a-1)} \text{ for some } 1 \le a \le r\}$, $i = 1, 2$, are disjoint (property 2, with $r = k + m - 1$ of $\mathfrak{R}$).

(Hg6). $x^{1/J} \le Y \le x$; $\sqrt{X_1} < q \le X_1$ and $\sqrt{X_2} < d \le X_2$ by (e); $(-d/p) = 1$ for primes $p \le Y^{\varepsilon/5}$ by (d); $\log N_\mathfrak{h} \le 2Y^{\varepsilon/5}(\log Y)^2$ by (f).

(Hg7), the six quadratic characters. For odd $n > 0$ coprime to qd, Jacobi reciprocity and $q \equiv d \equiv 3 \pmod 4$ give

$$\Big(\frac{-q}{n}\Big) = \Big(\frac{n}{q}\Big), \qquad \Big(\frac{-d}{n}\Big) = \Big(\frac{n}{d}\Big), \qquad \Big(\frac{qd}{n}\Big) = \Big(\frac{n}{q}\Big)\Big(\frac{n}{d}\Big), \qquad \Big(\frac{8D'}{n}\Big) = \Big(\frac{8}{n}\Big)\Big(\frac{D'}{n}\Big).$$

Let $\mathbf{1}$ be the trivial character and $\chi_8 = (8/\cdot)$, both real primitive of modulus dividing $8M$. Then χ_{-q} and $(\mathbf{1}_{q,d}^{(1,0)})^*$ are primitive characters agreeing on all n coprime to $16Mqd$, hence equal (a character is induced by a unique primitive character); likewise $\chi_{-8q} = (\chi 8_{q,d}^{(1,0)})^*$, $\chi_{-d} = (\mathbf{1}_{q,d}^{(0,1)})^*$, $\chi_{-8d} = (\chi 8_{q,d}^{(0,1)})^*$, $\chi_{qd} = (\mathbf{1}_{q,d}^{(1,1)})^*$, $\chi_{8qd} = (\chi 8_{q,d}^{(1,1)})^*$, the twists formed with $8M$. By (Wp3) these have zero-free boxes of heights $h_0(M, \varepsilon, Y) = 8MX_1 \ge 8X_1$ for $D = -q, -8q$ and $h_1(M, \varepsilon, Y) = 8MX_1X_2 \ge 8X_1X_2$ for $D = -d, -8d, qd, 8qd$; a zero-free box of some height is one of every smaller height. The size conditions $|D| \le 8X_1$, resp. $|D| \le 8X_1X_2$, hold by (e).

(Hg7), the real characters of the host. Let θ be a character of $\mathcal{G}$ with χ_θ^* real. As $\gcd(Q_S, qd) = 1$ and $q \neq d$, $\chi_\theta = \chi_S\chi_q\chi_d$ with characters modulo Q_S, q, d, and $\chi_\theta^* = \chi_S^*\chi_q^*\chi_d^*$, a product of primitive characters of coprime conductors, each real. A real character modulo an odd prime p is trivial or $(\cdot/p)$; so $\chi_q^* = (\cdot/q)^i$, $\chi_d^* = (\cdot/d)^j$ with $i, j \in \{0, 1\}$, and $\varrho = \chi_S^*$ is real primitive of conductor $f_S \mid Q_S$. The conductor of a real primitive character is $2^a m'$ with $a \in \{0, 2, 3\}$ and m' odd squarefree; here $m' \mid Q_{\mathrm{odd}}$, so $f_S \mid 8Q_{\mathrm{odd}} \mid 8M$ by (b), and $f_S \le 8Q_{\mathrm{odd}}$. For n coprime to $16Mqd$, $\chi_\theta^*(n) = \varrho(n)(n/q)^i(n/d)^j$, so $\chi_\theta^* = (\varrho_{q,d}^{(i,j)})^*$ (twist formed with $8M$) and $\mathrm{cond}(\chi_\theta^*) = f_S q^i d^j$. If $d \mid \mathrm{cond}(\chi_\theta^*)$, then $j = 1$, $\mathrm{cond}(\chi_\theta^*) \le 8Q_{\mathrm{odd}}X_1X_2$ by (e), and (Wp3) gives a zero-free box of height $8MX_1X_2 \ge 8Q_{\mathrm{odd}}X_1X_2$. If $q \mid \mathrm{cond}(\chi_\theta^*)$ and $d \nmid \mathrm{cond}(\chi_\theta^*)$, then $(i, j) = (1, 0)$, $\mathrm{cond}(\chi_\theta^*) \le 8Q_{\mathrm{odd}}X_1$, and (Wp3) gives a zero-free box of height $8MX_1 \ge 8Q_{\mathrm{odd}}X_1$. This is (Hg7).

(Hg8). By the identification above, $\chi_{-q}, \chi_{-d}, \chi_{qd}$ are $(\xi_{q,d}^{(i,j)})^*$ for the trivial character ξ modulo M and $(i, j) = (1, 0), (0, 1), (1, 1)$. As (q, d) is good for (M, x) and $Y \le x$, (Wp2) at $y = x$ gives $|g(x, \chi_D)| \le x^{1/2+\varepsilon/2} \le x^{1/2+\varepsilon}$ and $|L'(1, \chi_D)/L(1, \chi_D)| \le 4\log x = C_5 \log x$;

in particular $|\operatorname{Re} L'(1,\chi_{-d})/L(1,\chi_{-d})| \le 4\log x \le x^{\varepsilon}$ as $x \ge x_6$. For the last bound we apply Proposition 11.8 with $B = C_0$: $x \ge x_0^{\mathrm{ch}}$; $\mathfrak{h}$ is a host; the largest odd divisor Q_{odd} of Q_S is at most C_0 by (b); $\log(Q_S q d) \le 2x^{\varepsilon/5}\log^2 x$ by (f); and for every θ with χ_θ^* real and modulus divisible by q or d, the zero-free box just obtained contains every real $\beta > 1 - \frac{1}{400}$, so $L(\beta,\chi_\theta^*) \ne 0$. Hence $|g(y,\chi_\theta^*)| \le y\exp(-c_6 x^{\varepsilon/8})$ for every θ and every $y \ge \frac{1}{3}\exp(x^{\varepsilon/3})$. This is (Hg8).

Conclusion. All conditions (Hg1)–(Hg8) hold for the data $\mathfrak{R}$, the ordering, B, s_d, φ_0, $\mathfrak{b}$, g_1, g_2, p_1, p_2, q_0, S, u_0 and Y, and the preconditions of Definition 11.4 ($x > 1$, $0 < \varepsilon < 1/52$, $J \ge 1$, $T \subseteq A$, $t \in \mathbb{Z}$, the form of κ) hold. So the concrete good-host requirement holds at $(x,\varepsilon,J,\kappa,\mathfrak{h},T,t)$, with T finite satisfying (Ec4) and $1 \le t \le x^{400}$; as $t \ge 1$, $\mathfrak{G}(x,\varepsilon,J,\kappa,\mathfrak{h},T,t)$ holds. The list κ and x_B depend only on ψ, ε, C_0. Hence $\mathfrak{G}$ supplies good hosts for ψ. Since ψ was arbitrary, $\mathfrak{G}$ supplies good hosts. □

Lemma 11.6. *Let $\mathfrak{s} = (x,\varepsilon,J)$ be a scale.*

1. *If $x \ge 16$, then for every frame (e,A,h,a_0) with $x^{2+8\varepsilon} \le |A| \le 2x^{2+8\varepsilon}$ there are integers k,m,t such that $(e,A,h,a_0;k,m,t)$ is a rank datum with the rank sizes at $\mathfrak{s}$.*
2. *If $x \ge 10708^{832/319}$, then every rank datum $(e,A,h,a_0;k,m,t)$ with the rank sizes at $\mathfrak{s}$ has $t \le x^{400}$.*

Proof. Write $\delta_1 = \delta_1(\mathfrak{s}) = x^{-\varepsilon/8}$, so $x\delta_1 = x^{1-\varepsilon/8}$, and $T = \lceil x^{377}\rceil$.

Item 1: choice of m and k. Since $1-\varepsilon/8 \ge \frac{1}{2}$ and $x \ge 16$, $x\delta_1 \ge x^{1/2} \ge 4$, so $a := \sqrt{x\delta_1} \ge 2$; also $a \le x^{1/2}$. For real $a \ge 2$ the even integer $2\lceil a/2\rceil$ lies in $[a,2a]$ (as $\lceil a/2\rceil < a/2+1 \le a$). Let m be the even integer $2\lceil a/2\rceil \in [a,2a]$ and k the even integer $2\lceil b/2\rceil \in [b,2b]$ for $b = x^{9/2+10\varepsilon} \ge 2$. Then $r := k+m-1$ is odd, and with $f = 2|A|$ put

$$t = 8fr + 4 + T\,n - r, \qquad n = 8\Big\lceil \frac{10r}{8}\Big(f + \frac{f^2(k-1)^3 L(\mathfrak{s})^2}{m}\Big)\Big\rceil.$$

The quantity n depends only on f, r, k, m and $\mathfrak{s}$, so once $(e,A,h,a_0;k,m,t)$ is shown to be a rank datum, $n = n_W(\mathfrak{R},\mathfrak{s})$ and $r + t = 8fr + 4 + T\,n_W(\mathfrak{R},\mathfrak{s})$, which is the last of the rank sizes. The first three hold by the hypothesis on $|A|$ and the choice of m, k, since $\sqrt{x\delta_1(\mathfrak{s})} = a$.

Item 1: (R1)–(R4). (R1): k, m are even and $t = (8fr+4+Tn) - r$ is an even number minus an odd one, hence odd. (R2): $m \ge a \ge 2$. Since $m \le 2a \le 2x^{1/2}$ and $k \ge b \ge x^4 \ge 2x^{1/2} + 3$ for $x \ge 16$, $m+3 \le k$. As $f \ge 2$ and $Tn \ge 0$, $t \ge 16r + 4 - r \ge 7r + 4$, and $7r = 7(k+m-1) \ge k+m$ because $k + m \ge 7$; so $t \ge k + m + 4$, and $t \ge 7r + 4 > 3m$. (R3): $m \le 2x^{1/2} \le 2x^{2+8\varepsilon} \le 2|A|$, and $2|A| \le 4x^{2+8\varepsilon} \le 4x^3 \le x^4 \le b \le k$ (using $8\varepsilon < 1$ and $x \ge 4$). (R4): $t + 1 > 3m \ge m$, so $m^2 \le m(t+1) \le k + m(t+1)$.

Item 2. Let $\mathfrak{R} = (e,A,h,a_0;k,m,t)$ have the rank sizes at $\mathfrak{s}$, and put $E = \frac{797}{2} + \frac{929}{16}\varepsilon$. As $x > 1$, all powers below are ≥ 1. The rank sizes give $f = 2|A| \le 4x^{2+8\varepsilon}$, $k - 1 < k \le 2x^{9/2+10\varepsilon}$, $m \ge \sqrt{x\delta_1} = x^{1/2-\varepsilon/16}$ and $m \le 2x^{1/2}$, hence $r = k + m - 1 \le 4x^{9/2+10\varepsilon}$, and $L(\mathfrak{s})^2 = x^{2\varepsilon}$. Therefore

$$\frac{f^2(k-1)^3 L(\mathfrak{s})^2}{m} \le 16x^{4+16\varepsilon}\cdot 8x^{27/2+30\varepsilon}\cdot x^{2\varepsilon}\cdot x^{-1/2+\varepsilon/16} = 128\,x^{17+\frac{769}{16}\varepsilon}.$$

Moreover $f \le 4x^{2+8\varepsilon} \le 4x^{17+\frac{769}{16}\varepsilon}$, so the bracket in n_W is at most $132x^{17+\frac{769}{16}\varepsilon}$. With $8\lceil y\rceil \le 8y + 8$,

$$n_W(\mathfrak{R},\mathfrak{s}) \le 10r\cdot 132x^{17+\frac{769}{16}\varepsilon} + 8 \le 5280\,x^{\frac{43}{2}+\frac{929}{16}\varepsilon} + 8 \le 5288\,x^{\frac{43}{2}+\frac{929}{16}\varepsilon}.$$

Since $T \le x^{377}+1 \le 2x^{377}$, it follows that $T\,n_W(\mathfrak{R},\mathfrak{s}) \le 10576\,x^E$. Also $8fr \le 128x^{13/2+18\varepsilon} \le 128x^E$ and $4 \le 4x^E$. From the rank equation and $r \ge 0$,

$$t \le 8fr + 4 + T\,n_W(\mathfrak{R},\mathfrak{s}) \le 10708\,x^E.$$

Since $\varepsilon < \frac{1}{52}$, $400 - E = \frac{3}{2} - \frac{929}{16}\varepsilon > \frac{3}{2} - \frac{929}{832} = \frac{319}{832}$, so for $x \ge 10708^{832/319}$ we get $x^{400-E} \ge x^{319/832} \ge 10708$, i.e. $10708\,x^E \le x^{400}$, and $t \le x^{400}$. □

Lemma 11.7. *Let $0 < \varepsilon < 1/52$ and $C_0 \ge 1$, and put $\delta = \varepsilon/(5C_0)$ and*

$$x_B = \max\Bigl(e^{C_0},\ (1700/\delta + 6)^{2/\delta}\Bigr).$$

Let $\mathfrak{s} = (x,\varepsilon,J)$ be a scale with $x \ge x_B$ and $J \le C_0$, let $(e,A,h,a_0;k,m,t)$ be a rank datum with the rank sizes at $\mathfrak{s}$ and $t \le x^{400}$, and let (q,d) be a window pair at a scale Y with $x^{1/J} \le Y \le x$. Then

$$4tr+1 < q \qquad \text{and} \qquad 4tr+1 < d.$$

Proof. Upper bound for $4tr+1$. By (R2) of the rank datum, $m \ge 2$ and $k \ge m+3$, so $r = k+m-1 \ge 1$ and $t \ge k+m+4 \ge 1$. The rank sizes give $k \le 2x^{9/2+10\varepsilon}$ and $m \le 2\sqrt{x\delta_1(\mathfrak{s})} \le 2x^{1/2}$, since $\delta_1(\mathfrak{s}) = x^{-\varepsilon/8} \le 1$. As $x > 1$ and $9/2 + 10\varepsilon < 9/2 + 10/52 < 5$,

$$r < k+m \le 2x^{9/2+10\varepsilon} + 2x^{1/2} \le 4x^5, \qquad 4tr+1 \le 16x^{405}+1 \le 17x^{405},$$

so $\log(4tr+1) \le \log 17 + 405\log x < 3 + 405\log x$.

Lower bounds for $\log q$ and $\log d$. Since $J \le C_0$ and $x > 1$, $Y \ge x^{1/J} \ge x^{1/C_0}$, hence $Y^{\varepsilon/5} \ge x^{\varepsilon/(5C_0)} = x^{\delta}$; and $x \ge e^{C_0}$ gives $Y \ge e$, so $\log^2 Y \ge 1$. By the definition of a window pair at scale Y,

$$\log q > \tfrac{1}{2}Y^{\varepsilon/5} \ge \tfrac{1}{2}x^{\delta}, \qquad \log d > \tfrac{1}{2}Y^{\varepsilon/5}\log^2 Y \ge \tfrac{1}{2}x^{\delta}.$$

Comparison. Put $z = x^{\delta/2}$; then $z \ge 1700/\delta + 6 \ge 1$ because $x \ge (1700/\delta+6)^{2/\delta}$. From $\log w \le w$ for $w > 0$ applied to $w = z$ we get $\log x = \frac{2}{\delta}\log z \le \frac{2}{\delta}z$, so

$$3 + 405\log x \le 3 + \frac{810}{\delta}z < 3z + \frac{850}{\delta}z = \frac{z}{2}\Bigl(6 + \frac{1700}{\delta}\Bigr) \le \frac{z^2}{2} = \frac{1}{2}x^{\delta}.$$

Hence $\log(4tr+1) < \frac{1}{2}x^{\delta} < \log q$ and $\log(4tr+1) < \log d$, which gives both inequalities. □

Proposition 11.8. *There is a real number $c > 0$ such that for every real B and every $\varepsilon \in (0,1/52)$ there is a real number x_0 with the following property. Let $x \ge x_0$, let $\mathfrak{h} = (q,d,Q_S,\alpha)$ be a host for a frame, let Q' be the largest odd divisor of Q_S, and suppose that*

1. *$Q' \le B$ and $\log(Q_S q d) \le 2x^{\varepsilon/5}\log^2 x$;*
2. *for every character θ of $\mathcal{G}$ such that χ_θ^* is real and its modulus is divisible by q or by d, $L(\beta,\chi_\theta^*) \ne 0$ for every real $\beta > 1 - \frac{1}{400}$.*

Then for every character θ of $\mathcal{G}$ and every real $y \ge \frac{1}{3}\exp(x^{\varepsilon/3})$,

$$|g(y,\chi_\theta^*)| \le y\exp\bigl(-c\,x^{\varepsilon/8}\bigr).$$

Proof. We take $c = 1$. The proof applies the prime number theorem in progressions with the exceptional zero as secondary term (Theorem B.7, [259]) with modulus the conductor N of χ_θ^*; the modulus is so small against y ($\log N \ll x^{\varepsilon/5}\log^2 x$, $\log y \gg x^{\varepsilon/3}$) that the error saves $\exp(-x^{2\varepsilon/15+o(1)})$. Hypothesis 2 removes the exceptional zero when q or d divides N; otherwise a real χ_θ^* has modulus dividing $8Q' \le 8B$, one of finitely many characters, whose real zeros stay away from 1 by a margin depending on B only. The Riesz weight is then recovered from ψ by partial summation over $[y,\infty)$.

Constants. Fix B and $\varepsilon \in (0, 1/52)$. Let $\mathcal{X}_B$ be the finite set of real primitive characters $\chi \neq \mathbf{1}$ of modulus at most $8B$. For $\chi \in \mathcal{X}_B$, $L(s,\chi)$ is entire (Theorem B.1) and $L(1,\chi) \neq 0$ (Theorem B.2), so there is $c_\chi \in (0, \frac12)$ with $L(\beta,\chi) \neq 0$ for $\beta \in [1-c_\chi, 1]$. Put $c_B = \min(\{\frac{1}{400}\} \cup \{c_\chi : \chi \in \mathcal{X}_B\}) > 0$. For $x > 1$ write

$$v_0 = \tfrac12 x^{\varepsilon/3}, \qquad \mathcal{L} = 2x^{\varepsilon/5}\log^2 x,$$

$$T_1 = \frac{3A_4\mathcal{L}}{c_4}e^{-c_4 v_0/\mathcal{L}}, \quad T_2 = 6A_4 e^{-c_4\sqrt{v_0}}\Big(\frac{\sqrt{v_0}}{c_4} + \frac{1}{c_4^2}\Big),$$
$$T_3 = \frac{2}{c_B}e^{-c_B v_0}, \quad T_4 = (\log 2)e^{-v_0}, \quad T_5 = 4e^{-v_0/2}(2v_0+4).$$

Since $v_0/\mathcal{L} = x^{2\varepsilon/15}/(4\log^2 x)$, $\sqrt{v_0} \asymp x^{\varepsilon/6}$, and $\varepsilon/8 < 2\varepsilon/15 < \varepsilon/6 < \varepsilon/3$, each $T_i e^{x^{\varepsilon/8}}$ tends to 0; choose $x_0 \ge e$ such that every $x \ge x_0$ satisfies

(a) $x^{\varepsilon/3} \ge 2\log 3$ and $v_0 \ge 12\mathcal{L}$;

(b) $T_1 + T_2 + T_3 + T_4 + T_5 \le \exp(-x^{\varepsilon/8})$.

Fix $x \ge x_0$, a host $\mathfrak{h}$ with hypotheses 1 and 2, a character θ of $\mathcal{G}$ and $y \ge \frac13\exp(x^{\varepsilon/3})$. Let $\chi = \chi_\theta^*$, primitive of modulus N; as χ_θ is a character modulo $Q_S qd$, $N \mid Q_S qd$, so $\log N \le \mathcal{L}$ by hypothesis 1. Put $\delta_\chi = 1$ if $N = 1$ (then $\chi = \mathbf{1}$) and $\delta_\chi = 0$ otherwise, and $M = N$ if $N \ge 2$, $M = 2$ if $N = 1$; $\log M \le \mathcal{L}$, as $\mathcal{L} \ge 2$. By (a), $\log y \ge x^{\varepsilon/3} - \log 3 \ge v_0$.

Step 1. The exceptional zero is harmless. Suppose the exceptional zero β_1 modulo M exists and $\chi_1 = \chi$ as characters modulo M. By Theorem B.7, item 1, $\beta_1 < 1$ lies in the region $\operatorname{Re} s \ge 1 - 1/(13\log(3M))$, so $\beta_1 > \frac12$. As χ_1 is real and non-principal, $N = M \ge 2$, χ is real and $L(\beta_1,\chi) = 0$. If $q \mid N$ or $d \mid N$, hypothesis 2 gives $\beta_1 \le 1 - \frac{1}{400} \le 1 - c_B$. Otherwise $N \mid Q_S$; the modulus of a real primitive character is $2^a m'$ with $a \in \{0,2,3\}$ and m' odd and squarefree, here $m' \mid Q'$, so $N \mid 8Q'$ and $N \le 8B$; thus $\chi \in \mathcal{X}_B$ and $\beta_1 < 1 - c_\chi \le 1 - c_B$. In both cases $\beta_1 \le 1 - c_B$.

Step 2. A bound for ψ. We show that $|\psi(X,\chi) - \delta_\chi X| \le X\eta(X)$ for every $X \ge y$, where

$$\eta(X) = 3A_4\big(X^{-c_4/\mathcal{L}} + e^{-c_4\sqrt{\log X}}\big) + 2X^{-c_B} + \frac{\log 2}{X} + 4X^{-1/2}\log X.$$

Let $X \ge y$. Then $\log X \ge v_0 \ge 12\mathcal{L} \ge 12\log M$, so $X \ge M^{12}$ and Theorem B.7, item 2, applies to every b coprime to M: with $m_b = (X - \chi_1(b)X^{\beta_1}/\beta_1)/\varphi(M)$ (the second term omitted if there is no exceptional zero), $|m_b| \le 3X/\varphi(M)$ as $\beta_1 > \frac12$, and

$$|\vartheta(X;M,b) - m_b| \le A_4 E(X)\,|m_b|, \qquad E(X) = e^{-c_4\log X/\log M} + e^{-c_4(\log X)^{3/5}(\log\log X)^{-1/5}}.$$

Multiply by $\chi(b)$ (of modulus 1) and sum over the $\varphi(M)$ classes b: by orthogonality $\sum_b \chi(b) = \varphi(M)\delta_\chi$ (for $N \ge 2$, χ is non-principal modulo $M = N$) and $\sum_b \chi(b)\chi_1(b) = \varphi(M)$ if $\chi = \chi_1$

and 0 otherwise (χ_1 is real). So

$$\Big|\sum_{p\le X,\ p\nmid M}\chi(p)\log p-\delta_\chi X+[\chi=\chi_1]\frac{X^{\beta_1}}{\beta_1}\Big|\le 3A_4E(X)X.$$

The primes $p\mid M$ contribute 0 if $N\ge 2$ (then $\chi(p)=0$) and $\log 2$ if $N=1$. By Step 1, $[\chi=\chi_1]X^{\beta_1}/\beta_1\le 2X^{1-c_B}$. The prime powers contribute at most $\sum_{2\le k\le \log X/\log 2}\sum_{p\le X^{1/k}}\log p\le \log 4\,(X^{1/2}+X^{1/3}\log X/\log 2)\le 4X^{1/2}\log X$ by Lemma B.3, as $\log X\ge 1$. Finally $\log M\le \mathcal{L}$ gives $e^{-c_4\log X/\log M}\le X^{-c_4/\mathcal{L}}$, and for $v=\log X>e$ the inequality $\log v\le v^{1/2}$ gives $v^{3/5}(\log v)^{-1/5}\ge v^{1/2}$, so the second term of $E(X)$ is at most $e^{-c_4\sqrt{\log X}}$. This proves Step 2.

Step 3. An integral formula for g. We show that $g(y,\chi)=-y\int_y^\infty(\psi(u,\chi)-\delta_\chi u)\,u^{-2}\,du$, the integral converging absolutely. Put $R(u)=\psi(u,\chi)-\delta_\chi u$. By Step 2 and $\int_y^\infty\eta(u)u^{-1}du<\infty$ (Step 4), and as R is bounded on $[1,y]$, $\int_1^\infty|R(u)|u^{-2}du<\infty$. Partial summation ($\frac{y}{n}-1=\int_n^y y\,u^{-2}du$) gives $\sum_{n\le y}\Lambda(n)\chi(n)(\frac{y}{n}-1)=y\int_1^y\psi(u,\chi)u^{-2}du$. For real $s>1$, $-\frac{L'}{L}(s,\chi)=\sum_n\Lambda(n)\chi(n)n^{-s}=s\int_1^\infty\psi(u,\chi)u^{-s-1}du$ (Euler product; Fubini with $\sum_n\Lambda(n)n^{-s}<\infty$), and $s\int_1^\infty\delta_\chi u\cdot u^{-s-1}du=\delta_\chi\frac{s}{s-1}$. Let $s\to 1^+$; by dominated convergence (dominant $|R(u)|u^{-2}$), $s\int_1^\infty R(u)u^{-s-1}du\to\int_1^\infty R(u)u^{-2}du$. If $N\ge 2$, $L(s,\chi)$ is entire with $L(1,\chi)\ne 0$ (Theorem B.1, Theorem B.2), so $\int_1^\infty R(u)u^{-2}du=-\frac{L'}{L}(1,\chi)$ and $g(y,\chi)=y\int_1^y R(u)u^{-2}du+y\frac{L'}{L}(1,\chi)=-y\int_y^\infty R(u)u^{-2}du$ by (2.2). If $N=1$, $\zeta(s)=\frac{1}{s-1}+\gamma+O(|s-1|)$ (Theorem B.1) gives $-\frac{\zeta'}{\zeta}(s)=\frac{1}{s-1}-\gamma+O(|s-1|)$, so $s\int_1^\infty R(u)u^{-s-1}du=-\frac{\zeta'}{\zeta}(s)-\frac{1}{s-1}-1\to -1-\gamma$; then by (2.1), $G(y)=y\int_1^y\psi(u)u^{-2}du-y(\log y-1-\gamma)=y\int_1^y R(u)u^{-2}du+y(1+\gamma)=-y\int_y^\infty R(u)u^{-2}du$.

Step 4. The bound. By Steps 2 and 3, $|g(y,\chi)|\le y\int_y^\infty\eta(u)u^{-1}du$. Substituting $u=e^v$, with $v_y=\log y\ge v_0$, and using that $e^{-aw}(w/a+1/a^2)$ ($a>0$) and $e^{-v/2}(2v+4)$ decrease for $w,v\ge 0$:

$$\int_{v_y}^\infty 3A_4e^{-c_4v/\mathcal{L}}dv\le T_1,$$

$$\int_{v_y}^\infty 3A_4e^{-c_4\sqrt{v}}dv=6A_4e^{-c_4\sqrt{v_y}}\Big(\frac{\sqrt{v_y}}{c_4}+\frac{1}{c_4^2}\Big)\le T_2,\qquad \int_{v_y}^\infty 2e^{-c_Bv}dv\le T_3,$$

$$\int_y^\infty\frac{\log 2}{u^2}du=\frac{\log 2}{y}\le T_4,\qquad \int_{v_y}^\infty 4ve^{-v/2}dv=4e^{-v_y/2}(2v_y+4)\le T_5.$$

By (b), $|g(y,\chi_\theta^*)|\le y(T_1+\cdots+T_5)\le y\exp(-x^{\varepsilon/8})$, which is the claim with $c=1$. □

Proposition 11.9. *There is a real number $C\ge 1$ such that for every $\varepsilon\in(0,1/52)$ there is a real number x_0 with the following property. Let $x\ge x_0$ and $J\ge 1$, let $(e,A,h,a_0;k,m,t)$ be a rank datum with the rank sizes at (x,ε,J), and put $r=k+m-1$. Let $d\ge 100$ be a prime with $d\equiv 3\pmod 4$, let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\sqrt{-d})$, and let $N\ge 1$ be an integer with $d\mid N$ and*

$$\log N\le 2x^{\varepsilon/5}(\log x)^2.$$

Then there are primes g_1,g_2,p_1,p_2,q_0 with the following properties.

1. *For $i=1,2$,*

$$\big(k\exp(x^\varepsilon)\big)^8\le g_i\le 2\big(k\exp(x^\varepsilon)\big)^8,$$

and there is $\pi_i\in\mathcal{O}$ with $\pi_i\bar\pi_i=g_i$ and $\pi_i-1\in N\mathcal{O}$.

2. $g_1 \neq g_2$, *and the two sets*

$$B_i = \{g_i\} \cup \{\ell \text{ prime}: \ g_i^a < \ell^{4v_\ell(g_i^a-1)} \text{ for some integer } 1 \le a \le r\} \qquad (i = 1, 2)$$

are disjoint.

3. $p_1 \neq p_2$, $q_0 \notin \{p_1, p_2\}$ *and* $p_1, p_2 \notin \{g_1, g_2\}$; $p_1 \equiv p_2 \equiv 1 \pmod{\operatorname{lcm}(15, N)}$ *and* $q_0 \equiv 1 \pmod N$; *and* $p_1, p_2, q_0 \le N^C$.

Proof. The auxiliary primes g_1, g_2 come from the Chebotarev density theorem of Thorner and Zaman in the ray class field of $\mathbb{Q}(\sqrt{-d})$ modulo N, whose uniformity is in dN^2 only. By Stark's bound, the Landau–Siegel zero costs at most a factor Q^{-2}, which is negligible because $\log N \ll x^{\varepsilon/5}\log^2 x$ while $\log g_i \ge 8x^\varepsilon$. Among the many such primes in $[Z, 2Z]$, two with disjoint exceptional sets exist by Lemma 11.10. The primes p_1, p_2, q_0 come from Linnik's bound, derived here from Theorem B.7.

Step 1 (Linnik's bound). There is an integer $K_0 \ge 12$ such that every integer $M \ge 2$ has a prime $p \equiv 1 \pmod M$ with $p \le M^{K_0}$. Choose $K_0 \ge 12$ with $K_0 \log 2 \ge e$, $A_4 e^{-c_4 K_0} \le \frac14$ and $A_4 e^{-c_4\sqrt{K_0 \log 2}} \le \frac14$. Let $M \ge 2$ and $X = M^{K_0} \ge M^{12}$. By Theorem B.7, item 2, with $b = 1$: $|\vartheta(X; M, 1) - m| \le A_4 E\,|m|$, where $m = X(1 - X^{\beta_1 - 1}/\beta_1)/\varphi(M)$ (or $X/\varphi(M)$ if there is no exceptional zero) and $E = e^{-c_4 \log X/\log M} + e^{-c_4 (\log X)^{3/5}(\log\log X)^{-1/5}}$. Here $\log X/\log M = K_0$; and with $v = \log X \ge K_0 \log 2 \ge e$, $\log v \le v^{1/2}$ gives $v^{3/5}(\log v)^{-1/5} \ge v^{1/2} \ge \sqrt{K_0 \log 2}$; so $A_4 E \le \frac12$. Moreover $m > 0$: by item 1 of Theorem B.7, $\beta_1 \ge 1 - 1/(13\log(3M)) > \frac12 > 1/\log X$, and $f(\beta) = \beta X^{1-\beta}$ decreases on $[1/\log X, 1]$ (its derivative is $X^{1-\beta}(1 - \beta\log X)$) with $f(1) = 1$, so $X^{\beta_1 - 1}/\beta_1 < 1$. Hence $\vartheta(X; M, 1) \ge m/2 > 0$, and some prime $p \le X$ satisfies $p \equiv 1 \pmod M$.

Put $C = K_0(1 + 4K_0 + 2K_0^2) \ge 1$, an absolute constant.

Step 2 (choice of x_0). Fix $\varepsilon \in (0, 1/52)$ and write $\mathcal{L} = 2x^{\varepsilon/5}\log^2 x$. Let $c_6 = \frac12 \min(1, c_S)$. Choose $x_0 \ge e$ such that every $x \ge x_0$ satisfies

(a) $x^{1/2 - 10\varepsilon} \ge 4$, $8x^\varepsilon \ge 4\max(c_2, 1)\mathcal{L}$, $8x^\varepsilon > C\mathcal{L}$ and $20x^{15} < \exp(\frac65 x^\varepsilon)$;

(b) $10A_5(\exp(-2c_3 x^\varepsilon/\mathcal{L}) + \exp(-2\sqrt{c_3}\, x^{\varepsilon/2})) \le \frac{c_6}{2} e^{-8\mathcal{L}}$;

(c) $\dfrac{c_6 Z}{32 e^{10\mathcal{L}} \log(2Z)} \ge 3Z^{9/10} \ge \sqrt{2Z} + 2Z^{9/10}$ for every $Z \ge e^{8x^\varepsilon}$.

This is possible: $\mathcal{L} = x^{\varepsilon/5 + o(1)}$, so $x^\varepsilon/\mathcal{L} \ge x^{3\varepsilon/5}$ dominates $\mathcal{L}$ eventually, $x^{\varepsilon/2}$ dominates $\mathcal{L}$, and in (c) $e^{10\mathcal{L}} \log(2Z) \le Z^{1/20}$ once $\log Z \ge 8x^\varepsilon$ and x is large.

Fix $x \ge x_0$, $J \ge 1$, a rank datum $(e, A, h, a_0; k, m, t)$ with the rank sizes at (x, ε, J), $r = k + m - 1$, d, $\mathcal{O}$ and N as in the statement. By the rank sizes, $1 \le k \le 2x^{9/2 + 10\varepsilon}$ and $m \le 2\sqrt{x}$, so $r \le 4x^{9/2+10\varepsilon} \le x^5$ by (a). As $d \mid N$, $100 \le d \le N$ and $\log N \le \mathcal{L}$. Put

$$Z = \big(k \exp(x^\varepsilon)\big)^8, \qquad \text{so} \qquad \log Z \ge 8x^\varepsilon.$$

Step 3 (the ray class field). Let $K = \mathbb{Q}(\sqrt{-d})$, so $\mathcal{O} = \mathcal{O}_K$, $D_K = d$ (as $d \equiv 3 \pmod 4$), $n_K = 2$ and $\mathcal{O}^\times = \{\pm 1\}$ (as $d > 3$). Let L be the ray class field of K modulo $N\mathcal{O}$ and $G = \operatorname{Gal}(L/K)$. By Theorem B.13, item 2, L/K is finite abelian, unramified at the prime ideals not dividing N, and the Artin map induces an isomorphism of the ray class group $I(N)/P_1(N)$ onto G, where $I(N)$ is the group of fractional ideals prime to N and $P_1(N)$ the subgroup of principal ideals (α) with $\alpha \equiv 1 \pmod{}^\times N\mathcal{O}$. By item 3, every character of G corresponds

to a primitive Hecke character of conductor dividing $N\mathcal{O}$; so $\mathcal{Q} \le \mathrm{N}(N\mathcal{O}) = N^2$ and $Q = D_K \mathcal{Q}\, n_K^{n_K} \le 4dN^2 \le 4N^3$, whence $\log Q \le 4\mathcal{L}$. Also $|G| \le h_K\,|(\mathcal{O}/N\mathcal{O})^\times| \le h_K N^2$ by item 1, and $h_K \le d$: by Minkowski's bound every ideal class of K contains an integral ideal of norm at most $\frac{2}{\pi}\sqrt{d} < \sqrt{d}$, and, as the number of integral ideals of norm n in K is $\sum_{m|n}(\frac{-d}{m}) \le \tau(n)$ (Kronecker symbol), there are at most $\sum_{n<\sqrt{d}} \tau(n) \le \sum_{n<\sqrt{d}} n \le d$ integral ideals of norm $< \sqrt{d}$. So $|G| \le N^3$ and $|G|\,Q^2 \le 16N^9 \le N^{10} \le e^{10\mathcal{L}}$ (as $16 \le N$).

Apply Theorem B.10 with $F = K$, $H = G$, $C = \{1\}$, $g_C = 1$ (so the field fixed by H is K and $\theta_1 = \chi_1(1) = 1$ if the Landau–Siegel zero exists). If a prime ideal $\mathfrak{p}$ of K with $\mathrm{N}\mathfrak{p} = p$ prime is unramified in L, does not divide N, and has trivial Artin symbol, then $\mathfrak{p} \in P_1(N)$ by item 2 of Theorem B.13, i.e. $\mathfrak{p} = (\pi)$ with $\pi \equiv 1 \pmod{}^\times N\mathcal{O}$; as $\mathfrak{p}$ is integral and prime to N, $\pi \in \mathcal{O}$ and $\pi - 1 \in N\mathcal{O}$, and $\pi\bar{\pi} = \mathrm{N}\mathfrak{p} = p$. Let $\mathcal{S}$ be the set of primes $p \in (Z, 2Z]$ admitting such a π. Every prime ideal counted by $\pi_C(2Z) - \pi_C(Z)$ has norm in $(Z, 2Z]$, hence does not divide N (its norm exceeds N^2 by (a)); it has degree one with norm in $\mathcal{S}$, or degree two with norm p^2, $p \le \sqrt{2Z}$; and each p is the norm of at most two prime ideals. Hence

$$|\mathcal{S}| \ge \tfrac{1}{2}\big(\pi_C(2Z) - \pi_C(Z) - \sqrt{2Z}\big). \tag{11.1}$$

Step 4 (counting). By (a), $Z \ge Q^{\max(c_2,1)}$, so item 2 of Theorem B.10 applies at Z and $2Z$ with errors $E(Z)$, $E(2Z)$ of modulus at most

$$\rho = A_5\Big(\exp\Big[-\frac{c_3 \log Z}{\log Q}\Big] + \exp\Big[-\sqrt{c_3 \log Z/2}\Big]\Big) \le A_5\big(\exp(-2c_3 x^{\varepsilon}/\mathcal{L}) + \exp(-2\sqrt{c_3}\,x^{\varepsilon/2})\big).$$

Write $\mathcal{M}(y) = \mathrm{Li}(y) - \theta_1 \mathrm{Li}(y^{\beta_1})$. Then $0 \le \mathcal{M}(y) \le \mathrm{Li}(y) \le 5Z/\log(2Z)$ for $y \in \{Z, 2Z\}$. The first two inequalities: if β_1 exists, $\theta_1 = 1$, $\beta_1 > 1 - 1/(8\log Q) > \frac{1}{2}$ and $y^{\beta_1} \ge Z^{1/2} \ge e^4 > 2$, so $0 \le \mathrm{Li}(y^{\beta_1}) \le \mathrm{Li}(y)$. For the third, $\sqrt{y} \ge 2$ and

$$\mathrm{Li}(y) = \int_2^{\sqrt{y}} \frac{dt}{\log t} + \int_{\sqrt{y}}^{y} \frac{dt}{\log t} \le \frac{\sqrt{y}}{\log 2} + \frac{y}{\log\sqrt{y}} = \frac{\sqrt{y}}{\log 2} + \frac{2y}{\log y} \le \frac{\sqrt{2Z}}{\log 2} + \frac{4Z}{\log(2Z)} \le \frac{5Z}{\log(2Z)},$$

where the fourth step uses $y \le 2Z$ and that $t/\log t$ increases for $t \ge e$, and the last step is $\sqrt{2Z}/\log 2 \le Z/\log(2Z)$, valid as $\log Z \ge 8$.

And, since $\frac{d}{dt}\mathrm{Li}(t^\beta) = t^{\beta-1}/\log t$,

$$\mathcal{M}(2Z) - \mathcal{M}(Z) = \int_Z^{2Z} (1 - \theta_1 t^{\beta_1 - 1}) \frac{dt}{\log t} \ge (1 - \theta_1 Z^{\beta_1 - 1}) \frac{Z}{\log(2Z)}.$$

If β_1 exists, item 3 of Theorem B.10 gives $1 - \beta_1 \ge c_S Q^{-2}/\log Q$; with $w = (1-\beta_1)\log Z \ge c_S Q^{-2}$ (as $\log Z \ge \log Q$) and $1 - e^{-w} \ge \min(\frac{1}{2}, \frac{w}{2})$, we get $1 - Z^{\beta_1 - 1} \ge c_6 Q^{-2}$; this holds trivially if β_1 does not exist. Therefore

$$\begin{aligned}\pi_C(2Z) - \pi_C(Z) &= \frac{1}{|G|}\big(\mathcal{M}(2Z) - \mathcal{M}(Z) + E(2Z)\mathcal{M}(2Z) - E(Z)\mathcal{M}(Z)\big) \\ &\ge \frac{1}{|G|}\big(c_6 Q^{-2} - 10\rho\big)\frac{Z}{\log(2Z)} \ge \frac{c_6 Z}{2|G|Q^2 \log(2Z)},\end{aligned}$$

using $10\rho \le \frac{c_6}{2} e^{-8\mathcal{L}} \le \frac{c_6}{2} Q^{-2}$ by (b) and $Q \le e^{4\mathcal{L}}$. With $|G|Q^2 \le e^{10\mathcal{L}}$, (11.1) and (c), $|\mathcal{S}| \ge \frac{1}{2}(3Z^{9/10} - \sqrt{2Z}) \ge Z^{9/10}$.

Step 5 (the primes g_1, g_2). $Z \ge 2$, $r \ge 1$, and $20r^3 \le 20x^{15} < \exp(\frac{6}{5}x^{\varepsilon}) \le Z^{3/20}$ by (a). By Lemma 11.10 applied to $Y = Z$ and $\mathcal{S} \subseteq [Z, 2Z]$, there are $g_1 \ne g_2$ in $\mathcal{S}$ with $B_r(g_1) \cap B_r(g_2) = \varnothing$. Each g_i is a prime in $(Z, 2Z]$, i.e. $(k\exp(x^{\varepsilon}))^8 \le g_i \le 2(k\exp(x^{\varepsilon}))^8$, and has $\pi_i \in \mathcal{O}$ with

$\pi_i\bar\pi_i = g_i$ and $\pi_i - 1 \in N\mathcal{O}$. This is item 1 and the first half of item 2; $B_r(g_i)$ is the set B_i of item 2.

Step 6 (the primes p_1, p_2, q_0). By Step 1 there are primes $p_1 \equiv 1 \pmod{15N}$ with $p_1 \le (15N)^{K_0} \le N^{2K_0}$ (as $N \ge 15$), $p_2 \equiv 1 \pmod{15Np_1}$ with $p_2 \le (15Np_1)^{K_0} \le N^{K_0(2+2K_0)}$, and $q_0 \equiv 1 \pmod{Np_1p_2}$ with $q_0 \le (Np_1p_2)^{K_0} \le N^{K_0(1+4K_0+2K_0^2)} = N^C$. Then $p_1 \equiv p_2 \equiv 1 \pmod{\operatorname{lcm}(15, N)}$ and $q_0 \equiv 1 \pmod N$; $p_2 \ne p_1$ because $p_2 \equiv 1 \pmod{p_1}$; $q_0 \notin \{p_1, p_2\}$ because $q_0 \equiv 1 \pmod{p_1p_2}$; and $p_1, p_2, q_0 \le N^C$. Finally $p_j \le N^C \le e^{C\mathcal{L}} < e^{8x^\varepsilon} \le Z < g_i$ by (a), so $p_1, p_2 \notin \{g_1, g_2\}$. This is item 3. The constant C is absolute, and x_0 depends only on ε. □

Lemma 11.10. *For integers $r \ge 1$ and $g \ge 2$ put*

$$B_r(g) = \{g\} \cup \{\ell \text{ prime}: \ g^a < \ell^{4v_\ell(g^a-1)} \text{ for some integer } 1 \le a \le r\}.$$

Let $Y \ge 2$ be real and $r \ge 1$ an integer with $20r^3 < Y^{3/20}$, and let $\mathcal{S}$ be a set of integers in $[Y, 2Y]$ with $|\mathcal{S}| \ge Y^{9/10}$. Then there are $g \ne g'$ in $\mathcal{S}$ with $B_r(g) \cap B_r(g') = \varnothing$.

Proof. Size of $B_r(g)$. Fix $g \ge 2$ and $1 \le a \le r$. If $\ell_1, \dots, \ell_4$ were distinct primes with $\ell_j^{v_j} > g^{a/4}$, $v_j = v_{\ell_j}(g^a - 1)$, then $\prod_j \ell_j^{v_j}$ would divide $g^a - 1 \ge 1$ and exceed g^a, which is impossible. So each a contributes at most three primes, and $|B_r(g)| \le 3r + 1$.

How often a given integer lies in $B_r(g)$. Let $\ell \ge 2$ be an integer. We show

$$\#\{g \in \mathbb{Z} \cap [Y, 2Y]: \ \ell \in B_r(g)\} \le 1 + 2r^2(Y^{3/4} + 1).$$

If $\ell \in B_r(g)$ and $\ell \ne g$, then ℓ is prime and for some $1 \le a \le r$, $\ell^v > g^{a/4} \ge Y^{a/4}$ with $v = v_\ell(g^a - 1)$. Let $e_a \ge 1$ be the least integer with $\ell^{e_a} > Y^{a/4}$; then $v \ge e_a$, so $g^a \equiv 1 \pmod{\ell^{e_a}}$ and g is a unit modulo ℓ^{e_a}. The equation $u^a = 1$ has at most $2a$ solutions in $(\mathbb{Z}/\ell^{e_a}\mathbb{Z})^\times$: this group is cyclic for odd ℓ (at most a solutions), and for $\ell = 2$ it is cyclic or the product of a group of order 2 and a cyclic group (at most $2a$ solutions). Each residue class modulo ℓ^{e_a} contains at most $Y/\ell^{e_a} + 1 \le Y^{1-a/4} + 1 \le Y^{3/4} + 1$ integers of $[Y, 2Y]$. Summing over $a \le r$ gives at most $\sum_{a \le r} 2a(Y^{3/4} + 1) = r(r+1)(Y^{3/4} + 1) \le 2r^2(Y^{3/4} + 1)$ such g, plus at most one g with $g = \ell$.

Conclusion. Suppose that $B_r(g) \cap B_r(g') \ne \varnothing$ for all $g \ne g'$ in $\mathcal{S}$. As $|\mathcal{S}| \ge Y^{9/10} > 0$, fix $g_1 \in \mathcal{S}$; then every $g \in \mathcal{S}$ satisfies $B_r(g) \cap B_r(g_1) \ne \varnothing$ (for $g = g_1$ it contains g_1), so g lies in one of the at most $3r + 1$ sets $\{g : \ell \in B_r(g)\}$, $\ell \in B_r(g_1)$. Hence, as $Y^{3/4} \ge 1$ and $r \ge 1$,

$$|\mathcal{S}| \le (3r+1)(1 + 2r^2(Y^{3/4} + 1)) \le 4r \cdot 5r^2 Y^{3/4} = 20r^3 Y^{3/4} < Y^{3/20} Y^{3/4} = Y^{9/10},$$

a contradiction. □

Proposition 11.11. *For every $\varepsilon \in (0, 1/52)$ and every $C_0 \ge 1$ there are real numbers $C_b > 0$ and x_0 with the following property. Let $x \ge x_0$ and $J \in [1, C_0]$, put $\mathfrak{s} = (x, \varepsilon, J)$, let $(e, A, h, a_0; k, m, t)$ be a rank datum with the rank sizes at $\mathfrak{s}$ whose frame has index $o \le C_0$, and let $\mathfrak{h} = (q, d, Q_S, \alpha)$ be a host for the frame (e, A, h, a_0) such that (q, d) is a window pair at a scale $Y \ge x^{1/J}$. Then for every total ordering of A there is a set $T \subseteq A$ such that*

1. *$\frac{1}{2}\delta_1(\mathfrak{s})|A| \le |T| \le 2\delta_1(\mathfrak{s})|A|$;*
2. *$(\mathfrak{h}, T)$ satisfies (Ec4) at scale x with parameter ε, that is, $\big|\#\{u \in T : u - b \in A_0\} - \kappa(b)|T|\big| \le x^{-1+\varepsilon}|T|$ for every $b \in A$, with $\kappa(b) = 4/(3o+1)$ for $b \in A_0$ and $\kappa(b) = 3/(3o+1)$ otherwise;*
3. *the choices ρ_p and G_p prescribed by this ordering for $\mathfrak{h}$, T, the parameters (x, ε) and the integer k satisfy the bad-top bound with constant C_b.*

Proof. The top set is chosen at random, each element of A independently, with probability $\delta = \delta_1(\mathfrak{s})$ on A_0 and $\frac{3}{4}\delta$ off A_0; Chebyshev's inequality on the o cosets of A_0 gives (Ec4) and the size, and a first-moment bound with Markov's inequality gives the bad-top bound. The only arithmetic input is that inert primes are large, which bounds L_{orb}.

Constants. Fix $\varepsilon \in (0, 1/52)$ and $C_0 \geq 1$. Write h_{orb}, $K = K_{\mathrm{orb}}$ and $r = r_{\mathrm{orb}}$ as in Definition 11.3; K and r depend only on ε. Put

$$L_b = 2\Big(\frac{620C_0}{\varepsilon} + 1\Big), \qquad C_b = 2L_b^K + 4 > 0.$$

Choose $x_0 \geq 16$ such that every $x \geq x_0$ satisfies

(a) $x^{\varepsilon/8} \geq 10$ and $x^{\varepsilon} \geq 144C_0^2$;

(b) $2(6r+1)((h_{\mathrm{orb}}+3)10^{6r+1}\log x + 2)x^{\varepsilon/8} \leq x^{\varepsilon/4}$,

possible because $\log x = o(x^{\varepsilon/8})$. Fix $x \geq x_0$, J, $\mathfrak{s}$, the rank datum, the host, Y and a total ordering of A as in the statement, and write $\delta = \delta_1(\mathfrak{s}) = x^{-\varepsilon/8} = \delta_{\mathrm{orb}}$ and $n_0 = |A_0|$, so $|A| = o\, n_0$ and $x^{h_{\mathrm{orb}}} \leq |A| \leq 2x^{h_{\mathrm{orb}}}$. Let $\mathcal{P}$ be the finite set of primes p that are inert for $\mathfrak{A}_{\mathfrak{h}}$, not sign-changing, with $p < \exp(x^{\varepsilon/3})$ and $\mathrm{ord}(h_p) \geq x^{\varepsilon/4}$, and $\omega_p = \log p/p > 0$. The orbit labellings ρ_p are determined by the ordering and do not depend on T; the sets G_p do. Since $\rho_p(t) \in R_p$, a top $t \in T$ has $\rho_p(t) \notin G_p$ exactly when $P_p(\rho_p(t))$ violates one of (Go1)–(Go3) with $n = \mathrm{ord}(h_p)$.

Step 1. Inert primes exceed $Y^{\varepsilon/5}$, and $L_{\mathrm{orb}} \leq L_b$. Let p be inert. Then $\epsilon_2(\alpha(f_p)) = -1$, i.e. $(f_p/d) = -1$ by (Hr4). For an odd prime $l \neq d$, quadratic reciprocity with $d \equiv 3 \pmod 4$ gives $(l/d) = (-d/l)$. If $p \mid Q_S$, then p is odd and $f_p \equiv p \pmod d$ (as $d \mid N_p'$), so $(f_p/d) = (p/d) = (-d/p) = 1$ by (Hr2), a contradiction; hence $p \nmid Q_S q d$, f_p is the class of p, and $(-d/p) = (p/d) = -1$. For an odd prime $p \leq Y^{\varepsilon/5}$, $p \mid d+1$ gives $-d \equiv 1 \pmod p$ and $(-d/p) = 1$. Hence $p > Y^{\varepsilon/5} \geq x^{\varepsilon/(5J)} \geq x^{\varepsilon/(5C_0)}$. By the rank sizes $k \leq 2x^{9/2+10\varepsilon}$, so, using $4\log 2 \leq \log x$ and $40\varepsilon < 1$,

$$\log\big((kx^{26})^4\big) = 4\log k + 104\log x \leq 4\log 2 + (18+40\varepsilon)\log x + 104\log x \leq 124\log x,$$

and $\log\big((kx^{26})^4\big)/\log p < 124\log x/\big(\frac{\varepsilon}{5C_0}\log x\big) = 620C_0/\varepsilon$. So every entry in the definition of L_{orb} is at most $620C_0/\varepsilon + 1$, and $L_{\mathrm{orb}} \leq L_b$.

Step 2. The random set. Put $\vartheta(a) = \delta$ for $a \in A_0$ and $\vartheta(a) = \frac{3}{4}\delta$ for $a \in A \setminus A_0$, and give each $T \subseteq A$ the probability $\prod_{a\in T}\vartheta(a)\prod_{a\notin T}(1-\vartheta(a))$; thus the events $[a \in T]$, $a \in A$, are independent with probabilities $\vartheta(a) \in [\frac{3}{4}\delta, \delta]$, and $0 < \delta < 1$. For $b \in A$ let $Z_b(T) = \#\{u \in T : u - b \in A_0\} = |T \cap (b + A_0)|$, and let

$$W(T) = \sum_{p\in\mathcal{P}} \omega_p \,\big|\{t \in T : \rho_p(t) \notin G_p\}\big|,$$

the left side of the bad-top bound. Since $\delta = \delta_{\mathrm{orb}}$, the bad-top bound with constant C_b reads $W(T) \leq 4C_b x^{-2}\delta|A|\sum_{p\in\mathcal{P}}\omega_p$.

Step 3. One orbit fails with probability at most $C_b x^{-2}$. Let $p \in \mathcal{P}$, $n_p = \mathrm{ord}(h_p)$, so $x^{\varepsilon/4} \leq n_p \leq |A| \leq 2x^{h_{\mathrm{orb}}}$. Let $t \in A$ and $v = \rho_p(t)$; as $t \in v + \mathbb{N}h_p$, $t = v + i_0 h_p$ for a unique $0 \leq i_0 < n_p$. The map $i \mapsto v + ih_p$ is injective on $\{0, \dots, n_p - 1\}$, so the events $[i \in P_p(v)] = [v + ih_p \in T]$ are independent with probabilities $\vartheta_i \in [\frac{3}{4}\delta, \delta]$; conditionally on $t \in T$ (an event of probability $\vartheta(t) > 0$), $i_0 \in P_p(v)$ and the events for $i \neq i_0$ stay independent with the same probabilities. Write $\Pr_t$ for this conditional probability and $P = P_p(v)$. If $I \subseteq \{0, \dots, n_p - 1\} \setminus \{i_0\}$ and $m \geq 1$, then, taking the union over the m-subsets of I,

$$\Pr_t(|P \cap I| \geq m) \leq \binom{|I|}{m}\delta^m. \tag{11.2}$$

- (Go1) fails if $|P \cap W_i| \ge K+1$ for some window $W_i = \{(i+j) \bmod n_p : 0 \le j < L_{\mathrm{orb}}\}$, $0 \le i < n_p$. Then $|P \cap (W_i \setminus \{i_0\})| \ge K$, and $|W_i| \le L_{\mathrm{orb}} \le L_b$; by (11.2) and $\binom{L}{K} \le L^K$, the probability is at most $n_p L_b^K \delta^K$. As $K \ge 8(h_{\mathrm{orb}}+2)/\varepsilon$, $\delta^K = x^{-\varepsilon K/8} \le x^{-h_{\mathrm{orb}}-2}$; so this is at most $2L_b^K x^{-2}$.

- (Go3) fails if some window of G_{orb} consecutive residues contains $N = N_{\mathrm{orb}}+1$ elements of P, hence at least N_{orb} elements other than i_0. By (11.2) and $\binom{G}{N'} \le G^{N'}/N'! \le (eG/N')^{N'}$ (from $N'! \ge (N'/e)^{N'}$), with $G = G_{\mathrm{orb}}$, $N' = N_{\mathrm{orb}}$, the probability is at most $n_p(eG\delta/N')^{N'} \le n_p 6^{-N'} \le n_p 2^{-N'} \le 2x^{h_{\mathrm{orb}}} x^{-(h_{\mathrm{orb}}+3)} = 2x^{-3}$, since $N' \ge 6eG\delta$ and $N' \ge (h_{\mathrm{orb}}+3)\log x/\log 2$.

- (Go2). Put $u = u_{\mathrm{orb}} = \lceil 1/\delta \rceil$, $\ell = \ell_{\mathrm{orb}} = (6r+1)u$, $M = M_{\mathrm{orb}}$ and $B = \lfloor n_p/\ell \rfloor$. As $u \le 2x^{\varepsilon/8}$ and $M+1 \le (h_{\mathrm{orb}}+3)10^{6r+1}\log x + 2$, (b) gives $(M+1)\ell \le x^{\varepsilon/4} \le n_p$, so $B \ge M+1$. For $0 \le s < B$ the tile $\{s\ell, \dots, (s+1)\ell - 1\} \subseteq \{0, \dots, n_p - 1\}$ is the disjoint union of the cells $C_{s,c} = \{s\ell + (c-1)u, \dots, s\ell + cu - 1\}$, $1 \le c \le 6r+1$; call the tile *marked* if $P \cap C_{s,c} = \varnothing$ for odd c and $P \cap C_{s,c}$ is a singleton with even element for even c. (Go2) fails exactly when for some $s_0 < B$ none of the tiles $(s_0 + j) \bmod B$, $0 \le j \le M$, is marked. These are $M+1$ distinct, pairwise disjoint tiles, at least M of which do not contain i_0; whether a tile is marked depends only on P inside it, so for these M tiles the events are independent under $\Pr_t$. Each has probability at least $\mu_0 = 10^{-(6r+1)}$: the cell events are independent; $\delta \le 1/10$ by (a), $u \le 1/\delta + 1$ and $\log(1-\delta) \ge -\delta - \delta^2$ give $(1-\delta)^u \ge e^{-(1+\delta)^2} > \frac{1}{4}$; a cell has u consecutive integers, at least $\lfloor u/2 \rfloor \ge (1/\delta - 1)/2$ of them even, so

$$\Pr_t(P \cap C_{s,c} = \varnothing) \ge (1-\delta)^u > \tfrac{1}{4},$$
$$\Pr_t(P \cap C_{s,c} = \{y\},\ y \text{ even}) \ge \lfloor u/2 \rfloor \tfrac{3}{4}\delta(1-\delta)^{u-1} \ge \tfrac{3(1-\delta)}{8} \cdot \tfrac{1}{4} > \tfrac{1}{12};$$

  with $3r+1$ odd and $3r$ even cells, a tile is marked with probability at least $4^{-(3r+1)}12^{-3r} = \frac{1}{4}\,48^{-3r} \ge \frac{1}{4}\,10^{-6r} \ge \mu_0$. Hence for each s_0 the failure has probability at most $(1-\mu_0)^M \le e^{-\mu_0 M} \le e^{-(h_{\mathrm{orb}}+3)\log x} = x^{-(h_{\mathrm{orb}}+3)}$, as $\mu_0 M \ge (h_{\mathrm{orb}}+3)\log x$; summing over $s_0 < B \le n_p$ gives at most $2x^{-3}$.

Therefore $\Pr_t(\rho_p(t) \notin G_p) \le 2L_b^K x^{-2} + 4x^{-3} \le C_b x^{-2}$.

Step 4. The first moment. By Step 3,

$$\mathbb{E}\,W = \sum_{p \in \mathcal{P}} \omega_p \sum_{t \in A} \vartheta(t) \Pr_t(\rho_p(t) \notin G_p) \le C_b x^{-2} \Big(\sum_{t \in A} \vartheta(t)\Big) \sum_{p \in \mathcal{P}} \omega_p \le C_b x^{-2} \delta |A| \sum_{p \in \mathcal{P}} \omega_p =: \Omega.$$

If $\mathcal{P} = \varnothing$, $W = 0$ and the bad-top bound holds for every T. Otherwise $\Omega > 0$ and Markov's inequality gives $\Pr(W > 4\Omega) \le \frac{1}{4}$.

Step 5. The cosets. For $b \in A$, Z_b is a sum of n_0 independent indicators of probability $\vartheta(b)$ (constant on $b + A_0$); its mean is $n_0\vartheta(b)$ and its variance at most $n_0\delta$. Put $D = 2\sqrt{\delta|A|}$. By Chebyshev's inequality $\Pr(|Z_b - n_0\vartheta(b)| > D) \le n_0\delta/D^2$; Z_b depends only on the coset $b + A_0$, and there are o cosets, so the probability that $|Z_b - n_0\vartheta(b)| > D$ for some b is at most $o\, n_0\delta/D^2 = \frac{1}{4}$.

Step 6. Choice of T. With probability at least $\frac{1}{2}$ both $W \le 4\Omega$ (or $\mathcal{P} = \varnothing$) and $|Z_b - n_0\vartheta(b)| \le D$ for all $b \in A$; fix such a T. Item 3 holds: $W(T) \le 4C_b x^{-2}\delta_{\mathrm{orb}}|A| \sum_{p \in \mathcal{P}} \omega_p$. Summing over a set of representatives of the cosets, $\big||T| - \mu\big| \le oD$ with

$$\mu = n_0\delta + (o-1)n_0\tfrac{3}{4}\delta = \tfrac{3o+1}{4}\,\delta n_0 \in \big[\tfrac{3}{4}\delta|A|,\ \delta|A|\big].$$

Here $\delta|A| \ge x^{2+8\varepsilon-\varepsilon/8} \ge x^{2+7\varepsilon}$, so $\delta|A| \ge x^{\varepsilon} \ge 144C_0^2 \ge 64o^2$ by (a), which gives $oD = 2o\sqrt{\delta|A|} \le \frac14\delta|A|$; hence $\frac12\delta|A| \le |T| \le \frac54\delta|A|$, item 1. For item 2, we have $\kappa(b)\mu = n_0\vartheta(b)$ (for $b \in A_0$: $\frac{4}{3o+1}\cdot\frac{3o+1}{4}\delta n_0$; otherwise $\frac{3}{3o+1}\cdot\frac{3o+1}{4}\delta n_0$) and $\kappa(b)\,o \le \frac{4o}{3o+1} < \frac43$, so

$$\big|Z_b - \kappa(b)|T|\big| \le |Z_b - n_0\vartheta(b)| + \kappa(b)\big||T| - \mu\big| \le D + \kappa(b)oD < 3D = 6\sqrt{\delta|A|}.$$

Finally $6\sqrt{\delta|A|} \le \frac12 x^{-1+\varepsilon}\delta|A|$ is equivalent to $144x^{2-2\varepsilon} \le \delta|A|$, which holds as $\delta|A| \ge x^{2+7\varepsilon}$ and $x^{9\varepsilon} \ge 144$ by (a); with $|T| \ge \frac12\delta|A|$ this gives $|Z_b - \kappa(b)|T|| \le x^{-1+\varepsilon}|T|$ for every $b \in A$, which is (Ec4). □

Proposition 11.12. *There is a real number C_T such that for every $\varepsilon \in (0, 1/52)$ and every $C_0 \ge 1$ there is a real number x_0 with the following property. Let $x \ge x_0$, let (e, A, h, a_0) be a frame of index $o \le C_0$ with $x^{2+8\varepsilon} \le |A| \le 2x^{2+8\varepsilon}$, and put $n_0 = |A_0|$ and*

$$L_w = 10\lceil 405\log x\rceil + 2\lceil 30/\varepsilon\rceil + 22.$$

Then there is a finite set S of non-negative integers such that $4L_w s < n_0$ for every $s \in S$,

$$x^{-\varepsilon}|A| \le |S|, \qquad 4L_w|S| \le n_0, \qquad \frac{n_0}{L_w}\exp\Big(-C_T\sqrt{\log n_0\,\log L_w}\Big) \le |S|,$$

and $j(s_2 - s_1) \ne i(s_3 - s_1)$ for all pairwise distinct $s_1, s_2, s_3 \in S$ and all integers $i \ne j$ with $0 < |i| \le L_w$ and $0 < |j| \le L_w$.

Proof. We take $C_T = 8$. The proof has two steps: a construction for arbitrary integers $N \ge 1$, $L \ge 2$ with $\log N \ge 100\log L$ (Behrend's sphere construction, with the base enlarged so that the linear relations of the statement cannot carry), and the choice $N = n_0$, $L = L_w$ at large x.

Step 1. Let $N \ge 1$ and $L \ge 2$ be integers, $X = \log N$, $Y = \log L$, and suppose $X \ge 100Y$. Then there is a set S of positive integers such that $4Ls < N$ for every $s \in S$, $4L|S| \le N$, $j(s_2 - s_1) \ne i(s_3 - s_1)$ for all pairwise distinct $s_1, s_2, s_3 \in S$ and all integers $i \ne j$ with $0 < |i|, |j| \le L$, and

$$|S| \ge \frac{N}{L}\exp(-8\sqrt{XY}). \tag{11.3}$$

We have $Y \ge \log 2 > 0.69$, $Y \le X/100$ and $\sqrt{XY} \le X/10$.

Parameters. Let $n = \lceil\sqrt{X/Y}\,\rceil$. As $X/Y \ge 100$, $n \ge 10$ and $\sqrt{X/Y} \le n \le 1.1\sqrt{X/Y}$. Let $Z = (N/(4L))^{1/n}$ and $Q = \lfloor Z/(4L)\rfloor$. Since $\log 8 = 3\log 2 \le 3Y$ and $\log 4 \le 2Y$,

$$n\log(8L) \le 4nY \le 4.4\sqrt{XY} \le 0.44X, \qquad \log(4L) \le 3Y \le 0.03X,$$

so $n\log Z = X - \log(4L) \ge 0.97X \ge n\log(8L)$, that is, $Z \ge 8L$. Hence $Z/(4L) \ge 2$, so $Q \ge Z/(4L) - 1 \ge Z/(8L)$ and $Q \ge 2$; and $4LQ \le Z$.

A large sphere. Let $V = \{0, 1, \dots, Q-1\}^n \subseteq \mathbb{Z}^n$, with $\|v\|^2 = \sum_{\nu=1}^n v_\nu^2$ and $\langle v, w\rangle = \sum_\nu v_\nu w_\nu$. For $v \in V$, $\|v\|^2$ is an integer in $[0, n(Q-1)^2]$, which takes at most $n(Q-1)^2 + 1 \le nQ^2$ values. As $|V| = Q^n$, by the pigeonhole principle there is an integer ρ such that $\Sigma = \{v \in V : \|v\|^2 = \rho\}$ has

$$|\Sigma| \ge \frac{Q^n}{nQ^2} = \frac{Q^{n-2}}{n} \ge \frac1n\Big(\frac{Z}{8L}\Big)^{n-2} \ge \frac{Z^n\,Z^{-2}\,(8L)^{-n}}{n} \ge \frac{N}{L}\cdot\frac{1}{4n}\,N^{-2/n}(8L)^{-n},$$

using $n - 2 \ge 8$, $Q \ge Z/(8L)$, $8L > 1$, $Z^n = N/(4L)$ and $Z^2 \le N^{2/n}$. Here $\log(4n) \le n \le 1.1\sqrt{X/Y} = 1.1\sqrt{XY}/Y \le 1.6\sqrt{XY}$ (as $n \ge 10$ and $Y \ge 0.69$), $n\log(8L) \le 4.4\sqrt{XY}$ and $2X/n \le 2X/\sqrt{X/Y} = 2\sqrt{XY}$; so $|\Sigma| \ge \frac{N}{L}\exp(-8\sqrt{XY})$. In particular $|\Sigma| \ge \exp(X - Y -$

$0.8X) \geq \exp(0.19X) \geq \exp(13) > 1$, as $X \geq 100Y \geq 69$. Since $\{v \in V : \|v\|^2 = 0\} = \{0\}$, it follows that $\rho \neq 0$ and $0 \notin \Sigma$.

Digits. Put $B = 4LQ$ and $\phi(v) = \sum_{\nu=1}^{n} v_\nu B^{\nu-1}$, a $\mathbb{Z}$-linear map $\mathbb{Z}^n \to \mathbb{Z}$. If $c \in \mathbb{Z}^n$ satisfies $|c_\nu| < B$ for every ν and $\phi(c) = 0$, then $c = 0$: otherwise let ν_0 be the least index with $c_{\nu_0} \neq 0$; dividing $\phi(c) = 0$ by $B^{\nu_0 - 1}$ gives $c_{\nu_0} = -B \sum_{\nu > \nu_0} c_\nu B^{\nu - \nu_0 - 1}$, a multiple of B with $0 < |c_{\nu_0}| < B$, which is impossible. For $v, w \in V$, $|v_\nu - w_\nu| \leq Q - 1 < B$; so ϕ is injective on V.

The set. Put $S = \phi(\Sigma)$; then $|S| = |\Sigma|$, which gives (11.3). Every $v \in \Sigma$ is non-zero with non-negative entries, so $\phi(v) \geq 1$; and $\phi(v) \leq (Q-1)\sum_{\nu=1}^{n} B^{\nu-1} < B^n$, so $4L\phi(v) < 4LB^n \leq 4LZ^n = N$. As S consists of $|S|$ distinct positive integers, its largest element s^* satisfies $|S| \leq s^*$, hence $4L|S| \leq 4Ls^* < N$.

No relation. Let $s_k = \phi(v_k)$ with $v_k \in \Sigma$ $(k = 1, 2, 3)$ be pairwise distinct, and let $i \neq j$ be integers with $0 < |i|, |j| \leq L$ and $j(s_2 - s_1) = i(s_3 - s_1)$. Put $a = v_2 - v_1$, $b = v_3 - v_1 \in \mathbb{Z}^n$ and $c = ja - ib$. Then $|c_\nu| \leq (|i| + |j|)(Q-1) \leq 2L(Q-1) < B$ and $\phi(c) = j(s_2 - s_1) - i(s_3 - s_1) = 0$, so $c = 0$, that is, $ja = ib$. As v_1, v_2, v_3 lie on the sphere $\|v\|^2 = \rho$, $\|v_1 + a\|^2 = \|v_1\|^2$ and $\|v_1 + b\|^2 = \|v_1\|^2$, i.e. $2\langle v_1, a\rangle + \|a\|^2 = 0 = 2\langle v_1, b\rangle + \|b\|^2$. Hence

$$0 = j^2(2\langle v_1, a\rangle + \|a\|^2) - ij(2\langle v_1, b\rangle + \|b\|^2) = 2j\langle v_1, ja - ib\rangle + \|ja\|^2 - ij\|b\|^2 = i(i-j)\|b\|^2,$$

using $ja = ib$ twice ($\|ja\|^2 = \|ib\|^2 = i^2\|b\|^2$). As $i \neq 0$ and $i \neq j$, $b = 0$, so $v_3 = v_1$ and $s_3 = s_1$, a contradiction. This proves Step 1.

Step 2. The frame. Fix $\varepsilon \in (0, 1/52)$ and $C_0 \geq 1$, and write $u = \log x$. As $\log(4051u)/u \to 0$ and $\sqrt{u\log(4051u)}/u \to 0$ when $u \to \infty$, there is x_0 such that every $x \geq x_0$ satisfies

(a) $x \geq \max(2, C_0)$ and $u \geq 34 + 60/\varepsilon$;

(b) $u \geq 100\log(4051u)$;

(c) $\log C_0 + \log(4051u) + 16\sqrt{u\log(4051u)} \leq \varepsilon u$.

Let $x \geq x_0$ and let (e, A, h, a_0) be a frame of index $o \leq C_0$ with $x^{2+8\varepsilon} \leq |A| \leq 2x^{2+8\varepsilon}$. Put $N = n_0$ and $L = L_w$, an integer with $L \geq 22$; from $\lceil y \rceil \leq y + 1$, $L \leq 4050u + 34 + 60/\varepsilon \leq 4051u$ by (a), so $Y = \log L \leq \log(4051u)$. As $|A| = o\, n_0$ and $o \leq C_0 \leq x$, $N \geq |A|/C_0 \geq x^2/C_0 \geq x$; and $N \leq |A| \leq 2x^3 \leq x^4$ (as $8\varepsilon < 1$ and $x \geq 2$). So $u \leq X \leq 4u$, and $X \geq u \geq 100\log(4051u) \geq 100Y$ by (b). Step 1 gives a set S of positive (in particular non-negative) integers with $4L_w s < n_0$ for $s \in S$, $4L_w|S| \leq n_0$, the absence of the relations of the statement, and

$$|S| \geq \frac{n_0}{L_w}\exp\bigl(-8\sqrt{\log n_0\ \log L_w}\bigr),$$

which is the third displayed inequality with $C_T = 8$. Finally $\sqrt{XY} \leq \sqrt{4u\log(4051u)} = 2\sqrt{u\log(4051u)}$, so by (c)

$$\log L_w + \log C_0 + 8\sqrt{XY} \leq \log(4051u) + \log C_0 + 16\sqrt{u\log(4051u)} \leq \varepsilon \log x,$$

that is, $L_w C_0 \exp(8\sqrt{XY}) \leq x^\varepsilon$; hence $|S| \geq \frac{n_0}{L_w}\exp(-8\sqrt{XY}) \geq x^{-\varepsilon} C_0 n_0 \geq x^{-\varepsilon} o\, n_0 = x^{-\varepsilon}|A|$. All requirements hold, with $C_T = 8$ independent of ε and C_0. □

Lemma 11.13. *Let $\mathfrak{h} = (q, d, Q_S, \alpha)$ be a host for a frame such that $\alpha(-1) = ((-1,-1), 0)$, let $E = E_\mathfrak{h}$, and regard ϵ_2 as a character of $\mathrm{Gal}(E/\mathbb{Q})$ via the isomorphism induced by α. Then:*

1. *E is a CM field, and its complex conjugation c is the automorphism corresponding to $((-1,-1), 0)$; for every field embedding $\varphi\colon E \hookrightarrow \mathbb{C}$, $\overline{\varphi(y)} = \varphi(c(y))$ for all $y \in E$.*

2. *There are $s_d \in E$ and a field embedding $\varphi_0 \colon E \hookrightarrow \mathbb{C}$ with $s_d^2 = -d$, $\sigma(s_d) = \epsilon_2(\sigma) s_d$ for every $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$, and $\varphi_0(s_d) = \sqrt{-1}\sqrt{d}$.*

3. *For every s_d as in item 2, put $K = \mathbb{Q}(s_d) \subseteq E$. There is an integral ideal $\mathfrak{b}$ of $\mathcal{O}_E$ such that*

$$\mathcal{D}_{E/K} = \mathfrak{b}\, c(\mathfrak{b}), \qquad \mathfrak{b} + c(\mathfrak{b}) = \mathcal{O}_E, \qquad \sigma(\mathfrak{b}) = \mathfrak{b} \quad \text{for every } \sigma \in \mathrm{Gal}(E/\mathbb{Q}) \text{ with } \epsilon_2(\sigma) = 1.$$

Proof. Put $N = Q_S q d$, fix a primitive N-th root of unity ζ_N, and for a coprime to N let σ_a be the automorphism $\zeta_N \mapsto \zeta_N^a$ of $\mathbb{Q}(\zeta_N)$ and $\bar\sigma_a$ its restriction to E. By definition E is the fixed field of $\{\sigma_a : \alpha(a) = 1\}$, and the isomorphism $\mathrm{Gal}(E/\mathbb{Q}) \to \mathcal{G}$ sends $\bar\sigma_a$ to $\alpha(a)$; so $\epsilon_2(\bar\sigma_a) = \epsilon_2(\alpha(a)) = (a/d)$ by (Hr4).

E is CM with complex conjugation $c = \bar\sigma_{-1}$. Let $\varphi \colon E \hookrightarrow \mathbb{C}$ be an embedding and $\tilde\varphi$ an extension to $\mathbb{Q}(\zeta_N)$. Then $\omega = \tilde\varphi(\zeta_N)$ is a root of unity, $\overline{\omega} = \omega^{-1} = \tilde\varphi(\sigma_{-1}(\zeta_N))$, and hence $\overline{\tilde\varphi(y)} = \tilde\varphi(\sigma_{-1}(y))$ for all $y \in \mathbb{Q}(\zeta_N)$; restricting, $\overline{\varphi(y)} = \varphi(c(y))$ for $y \in E$, with $c = \bar\sigma_{-1}$. By hypothesis c corresponds to $\alpha(-1) = ((-1,-1),0) \neq 1$, so $c \neq \mathrm{id}$ and $c^2 = \mathrm{id}$. Let F be the fixed field of c, so $[E : F] = 2$. For $y \in F$, $\overline{\varphi(y)} = \varphi(y)$, so F is totally real. If $\varphi(E) \subseteq \mathbb{R}$ for some φ, then $\varphi \circ c = \varphi$ and, φ being injective, $c = \mathrm{id}$, a contradiction; so E is totally imaginary. Hence E is a totally imaginary quadratic extension of a totally real field, a CM field, and complex conjugation (under any embedding) is c. This is item 1.

The square root of $-d$. Let $\zeta_d = \zeta_N^{N/d}$ and $\gamma = \sum_{a=1}^{d-1} (a/d)\zeta_d^a \in \mathbb{Q}(\zeta_N)$. By the classical evaluation of the quadratic Gauss sum, $\gamma^2 = (-1/d)\, d = -d$, as $d \equiv 3 \pmod 4$ by (Hr1); and $\sigma_u(\gamma) = (u/d)\gamma$ for u coprime to N. If $\alpha(u) = 1$ then $(u/d) = \epsilon_2(\alpha(u)) = 1$, so $\sigma_u(\gamma) = \gamma$; thus $\gamma \in E$. Put $s_d = \gamma$. For $\sigma = \bar\sigma_u$, $\sigma(s_d) = (u/d) s_d = \epsilon_2(\sigma) s_d$. For any embedding φ, $\varphi(s_d)^2 = -d$, so $\varphi(s_d) = \pm\sqrt{-1}\sqrt{d}$. If the sign is $-$, the embedding $\varphi \circ c$ satisfies $\varphi(c(s_d)) = \overline{\varphi(s_d)} = \sqrt{-1}\sqrt{d}$ by item 1. This gives φ_0 and item 2.

Galois structure over K. Let s_d be as in item 2 and $K = \mathbb{Q}(s_d)$, an imaginary quadratic field, Galois over $\mathbb{Q}$. As $s_d \neq 0$, an automorphism σ of E fixes K if and only if $\epsilon_2(\sigma) = 1$; so $\mathrm{Gal}(E/K) = \ker \epsilon_2$, and c acts on K non-trivially since $\epsilon_2(c) = -1$. Every $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ satisfies $\sigma(\mathcal{D}_{E/K}) = \mathcal{D}_{E/K}$: indeed $\mathcal{D}_{E/K}^{-1} = \{y \in E : \mathrm{Tr}_{E/K}(y\mathcal{O}_E) \subseteq \mathcal{O}_K\}$, $\sigma(\mathcal{O}_E) = \mathcal{O}_E$, $\sigma(\mathcal{O}_K) = \mathcal{O}_K$, and $\mathrm{Tr}_{E/K}(\sigma(y)) = \sum_{\tau \in \ker\epsilon_2} \tau\sigma(y) = \sigma(\sum_\tau \sigma^{-1}\tau\sigma(y)) = \sigma(\mathrm{Tr}_{E/K}(y))$ because $\mathrm{Gal}(E/\mathbb{Q})$ is abelian. Writing $\mathcal{D}_{E/K} = \prod_{\mathfrak{P}} \mathfrak{P}^{n_{\mathfrak{P}}}$ over the primes of E, it follows that $n_{\sigma(\mathfrak{P})} = n_{\mathfrak{P}}$ for all σ and $\mathfrak{P}$; and $n_{\mathfrak{P}} > 0$ exactly when $\mathfrak{P}$ is ramified in E/K (Dedekind).

Every prime of E ramified over K lies above a rational prime that splits in K. Let $\mathfrak{P}$ be ramified in E/K and ℓ the rational prime below it. Then ℓ is ramified in $E/\mathbb{Q}$, hence in $\mathbb{Q}(\zeta_N)$, so $\ell \mid N$. If $\ell = 2$: $d \equiv 7 \pmod{16}$ gives $-d \equiv 1 \pmod 8$, so 2 splits in $K = \mathbb{Q}(\sqrt{-d})$. If ℓ is an odd prime dividing Q_S: $(-d/\ell) = 1$ by (Hr2) and $\ell \neq d$ since $\gcd(Q_S, qd) = 1$, so ℓ splits in K. If $\ell = q$: $(-d/q) = 1$ by (Hr1) and $q \neq d$, so q splits. The case $\ell = d$ does not occur. Indeed, since $d \nmid Q_S q$, the inertia group in $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ of a prime above d is $\{\sigma_u : u \equiv 1 \pmod{Q_S q}\}$. So the inertia group I of $\mathfrak{P}$ in $\mathrm{Gal}(E/\mathbb{Q})$ consists of the $\bar\sigma_u$ with $u \equiv 1 \pmod{Q_S q}$, which correspond to elements of $\{1, \iota_d\}$ by (Hr5). As $\epsilon_2(\iota_d) = -1$, the inertia group $I \cap \ker\epsilon_2$ of $\mathfrak{P}$ in E/K is trivial, contradicting the ramification of $\mathfrak{P}$ in E/K.

The ideal $\mathfrak{b}$. Let L be the finite set of rational primes below primes of E ramified in E/K. For $\ell \in L$ we have $\ell\mathcal{O}_K = \mathfrak{p}_\ell\, c(\mathfrak{p}_\ell)$ with $\mathfrak{p}_\ell \neq c(\mathfrak{p}_\ell)$; fix such $\mathfrak{p}_\ell$, and put

$$\mathfrak{b} = \prod_{\ell \in L} \prod_{\mathfrak{P} \mid \mathfrak{p}_\ell} \mathfrak{P}^{n_{\mathfrak{P}}} \qquad (\mathfrak{b} = \mathcal{O}_E \text{ if } L = \varnothing).$$

It is an integral ideal. The map $\mathfrak{P} \mapsto c(\mathfrak{P})$ is a bijection from the primes above $\mathfrak{p}_\ell$ to those above $c(\mathfrak{p}_\ell)$, and $n_{c(\mathfrak{P})} = n_{\mathfrak{P}}$; hence $c(\mathfrak{b}) = \prod_{\ell \in L} \prod_{\mathfrak{P} \mid c(\mathfrak{p}_\ell)} \mathfrak{P}^{n_{\mathfrak{P}}}$. Every $\mathfrak{P}$ with $n_{\mathfrak{P}} > 0$ lies above

$\mathfrak{p}_\ell$ or $c(\mathfrak{p}_\ell)$ for some $\ell \in L$, so $\mathfrak{b}\,c(\mathfrak{b}) = \mathcal{D}_{E/K}$. The ideals $\mathfrak{b}$ and $c(\mathfrak{b})$ are products of powers of two disjoint sets of primes (above the distinct primes $\mathfrak{p}_\ell$, resp. $c(\mathfrak{p}_\ell)$, of K), so they are coprime: $\mathfrak{b} + c(\mathfrak{b}) = \mathcal{O}_E$. Finally, if $\epsilon_2(\sigma) = 1$ then σ fixes K pointwise, hence fixes each $\mathfrak{p}_\ell$ and permutes the primes above it, preserving the exponents $n_{\mathfrak{P}}$; so $\sigma(\mathfrak{b}) = \mathfrak{b}$. This is item 3. □

Lemma 11.14. *Fix $\varepsilon > 0$ and $C_8 \geq 1$, and put*

$$C_8' = (3 + 2C_8)A, \qquad a = \frac{4\varepsilon}{5}, \qquad X_* = \exp\left(1 + \frac{2C_8'}{a^3}\right).$$

Let $x \geq X_*$ be real, let $N \geq 3$ and $p_1, p_2 \geq 2$ be integers, let $k \geq 1$ be real and let g_1, g_2 be positive real numbers such that

$$p_i \leq N^{C_8}, \qquad \log N \leq 2x^{\varepsilon/5}(\log x)^2, \qquad g_i \geq \left(k \exp(x^\varepsilon)\right)^8 \quad (i = 1, 2).$$

Then the following minima exist:

$$M = N(p_1 - 1)(p_2 - 1), \qquad r = \min\{\ell \geq 2 : \ell \text{ is prime and } \ell \nmid M\},$$

$$q^* = \min\{\ell \geq 2 : \ell \text{ is prime and } \ell \equiv 1 \pmod{rN}\}.$$

They satisfy

$$q^* \equiv 1 \pmod{rN}, \qquad q^* \leq N^{C_8'}, \qquad q^* \neq p_1, p_2, \qquad q^* < g_1, g_2, \qquad \gcd(q^*, N) = 1.$$

In particular, consider data satisfying the concrete good-host requirement at such a scale, and take $N = N_{\mathfrak{h}} = Q_S q d$ together with its primes p_1, p_2, g_1, g_2 and the integer k of its rank datum. Then the set of level primes may be replaced by

$$\{g_1, g_2, p_1, p_2, q^*\},$$

using the bound $q^* \leq N_{\mathfrak{h}}^{C_8'}$. This replacement leaves unchanged the list κ, the frame, the host (q, d, Q_S, α) and its arithmetic datum, the chosen field embeddings, the sets T, S, the element u_0, the rank datum, g_1, g_2, the ordering of A, the functions ρ_p and the sets G_p. All heights, weights, correction degrees, thresholds and correction errors that have been fixed keep their values.

Proof. The modulus in the second minimum is the product rN. We first prove that both minima exist and bound the second one. Since $M \geq 1$, choose a prime divisor s of $M + 1$. Then $s \nmid M$, so the first set is nonempty and

$$r \leq s \leq M + 1 \leq 2M.$$

In particular $rN \geq 6$. Linnik's theorem [163], with modulus rN and residue 1, gives a prime in the second set not exceeding $(rN)^A$. The least such prime therefore exists and satisfies

$$q^* \leq (rN)^A.$$

Because $N \geq 3$ and $p_i \geq 2$,

$$rN \leq 2N^2(p_1 - 1)(p_2 - 1) \leq N^3 p_1 p_2 \leq N^{3+2C_8}.$$

Raising this inequality to the positive exponent A proves $q^* \leq N^{C_8'}$.

The congruence implies $r \mid q^* - 1$. If $q^* = p_i$, then $r \mid p_i - 1$ and hence $r \mid M$, contradicting the choice of r. Thus $q^* \neq p_1, p_2$. Also $q^* \equiv 1 \pmod{N}$, which proves $\gcd(q^*, N) = 1$.

It remains to compare q^* with the numbers g_i. Write $y = \log x$. Then

$$y \geq 1 + \frac{2C_8'}{a^3} > 0.$$

The bound $\exp(z) \geq z^3/6$ for $z \geq 0$, from the exponential series, gives

$$8\exp(ay) \geq \frac{4}{3}a^3y^3 > 2C_8'y^2.$$

Multiplying by $x^{\varepsilon/5}$ and using $a = 4\varepsilon/5$, we obtain

$$\log q^* \leq C_8' \log N \leq 2C_8'x^{\varepsilon/5}(\log x)^2 < 8x^{\varepsilon} \leq 8\log k + 8x^{\varepsilon} \leq \log g_i.$$

Thus $q^* < g_i$ for $i = 1, 2$.

For the final assertion, data satisfying the concrete good-host requirement have $N_{\mathfrak{h}} \geq 3$, the required bounds on p_i and g_i, and $k \geq 1$. Their common scale Y satisfies $1 < Y \leq x$, so (Hg6) yields

$$\log N_{\mathfrak{h}} \leq 2Y^{\varepsilon/5}(\log Y)^2 \leq 2x^{\varepsilon/5}(\log x)^2.$$

Therefore the first part applies, with the same X_*, to all such data at every scale $x \geq X_*$. Take the new set of level primes to be the displayed set and keep all other listed data. The arithmetic datum is determined by the host, which is unchanged. The orbit choices are determined by that datum, the ordering of A and the parameters (x, ε, k), which are also unchanged. Hence the functions ρ_p and the sets G_p are the same as before, not merely of the same cardinalities. The listed numerical data are kept, so none of their values changes. Only the set of level primes is replaced. The exponent C_8' bounds the new prime q^*; it does not replace the parameter C_8 in κ. □

Part III
Slope records: the class defined by linking conditions

Part III proves the third input of Part I, Proposition 17.6: there are a good-host requirement and a class of parametrised host families with slope records that has the link properties (Gh1), (Lk1)–(Lk3) and (Lk5). Part I turns this into tied families through Corollary 8.3 and Proposition 7.9. The requirement is the concrete good-host requirement of Part II, and (Gh1) is Proposition 11.5. The class is $\mathcal{A}_{\mathrm{prop}}$ (Definition 17.5): a family belongs to it if it has the linking property and the properties of its tops (Pc1), and if, whenever it satisfies the Sha condition, it carries one compatible collection of records with the properties (Pc2)–(Pc6). Design existence (Lk1) for this class is Proposition 18.1, proved in Part IV. The main theorem of Part III is Theorem 17.7: every family of the class that satisfies the Sha condition has the properties (Sp1)–(Sp8) of its slope records, with the divisor exponent 1, and therefore slope data at its members with exponents $(e_D, e_L) = (1, 0)$. These exponents give the bound $G_{\mathfrak{L}_\psi}(x) \le x^{1/2+8\varepsilon}$ of Part I.

Four of the eight properties, (Sp1), (Sp2), (Sp6) and (Sp7), are required of the class directly. The other four are proved, each from records that the class requires, by theorems about hermitian lattices, Hecke lattices and Hilbert spaces: the sizes (Sp8) and the transfer of slopes (Sp3) in Section 12, the divisor package (Sp4) in Section 13, and the dual-slope bound (Sp5) in Sections 14–16. In the terms of Alpöge's account [4], (Sp4) carries the norms of the lifts and (Sp5) the slope inequality of Bost with approximate orthogonality; no result of Part III is about theta lifts, and the Part sees them only through the records. The Sha condition enters in Section 13, where the defects of the lattices are bounded by lengths of Tate–Shafarevich groups, and the local losses there are bounded by the Ribet budget Theorem 22.1 of Part V. Section 17 defines the class, which quotes the hypothesis lists of the theorems of Sections 12–16 and of the Ribet budget, and proves Theorem 17.7 and Proposition 17.6. Each of Sections 12–16 proves statements about families that carry given records, and can be read without the definition of the class. Nothing in Part III depends on (TCS) except Proposition 17.6, through Proposition 18.1.

12 Sizes and transfer of slopes: (Sp8), (Sp3)

This section proves two of the four properties of slope records that Theorem 17.7 derives, the sizes (Sp8) and the transfer of slopes (Sp3). The sizes are routine. Lemma 12.1 reads the size inequalities of a member off its loss datum and the rank sizes of Part II, and it gives (Sp8), with $k \le x^5$, once the few residual bounds collected in (Pc6) hold.

The transfer of slopes is the step in which divisibility becomes slope. The loss bound of Part I compares the prime sum with the lattice term Θ of the loss datum, which is built from the gains at the inert primes (Definition 5.5); (Sp3) moves Θ into the slope and the maximal slope of the lattices of the slope record, up to the terms F and E and an explicit error. The lattices are described by modification data (Definition 12.2): each is obtained from a level lattice by local twists at a finite set of places, and the modification formulas (Mf3) give its slope as the slope of the level lattice plus a weighted sum of local terms at these places. Theorem 12.3 computes the change place by place, with a rounding bound for the twists (Lemma 12.4). Of its hypotheses (Mf1)–(Mf8), the class requires the first seven, and the correction budget (Mf8) is derived: Lemma 12.5 obtains it from the linking property of the family, with Chebyshev's bound and Lemma 3.5; so (Sp3) uses the same linking property as the loss bound.

Lemma 12.1. *Let $\mathcal{F}$ be a parametrised host family with parameter ε, threshold x_1 and frame (e, A, h, a_0) at x. For a member (x, i) with $x > x_1$ let k, m, η, the size parameter L_c and a_t ($t \in T_{x,i}$) be the data of $\mathfrak{M}_{x,i}$, and put $n = |A|$, $f = 2n$, $b = e^{x^\varepsilon}$, $a = xb$, $X' = e^{x^{\varepsilon/3}}$, $b' = a/X'$, $X'' = xnX'$ and $L_a = \log\bigl(2a\, v(\mathfrak{h}_{x,i})\bigr)$, the analytic loss of the member.*

1. *If $I(x) \neq \emptyset$ for some $x > x_1$, then $\varepsilon < \frac{1}{52}$.*
2. *For every real $c > 0$ there is a real x_0 such that for every real $x \geq x_0$ with $x > x_1$, every $i \in I(x)$ and every $t \in T_{x,i}$:*

$$m \leq x^{1/2+\varepsilon/2}, \quad xm \leq k, \quad k \leq x^5, \quad k \leq a, \quad k \leq n^2 x^{1/2-\varepsilon},$$
$$0 \leq L_c, \quad 0 \leq L_a, \quad kb' \leq a, \quad X'' \leq a,$$
$$2f(x+1)e^{-cx^{\varepsilon/8}} \leq \frac{1}{x}, \qquad 6f(x+1)(\eta+1) \leq \frac{b}{x}, \qquad \frac{a_t}{b} \leq x+1.$$

3. *Let $\mathcal{F}$ carry slope records, and suppose that $\varepsilon \leq 1$ and that there is a real x_0' such that for every real $x \geq x_0'$ with $x > x_1$: $I(x) \neq \emptyset$ if $x \in W$; and for every $i \in I(x)$, with $\mathfrak{V} = \mathfrak{V}_{x,i}$,*

$$x^{1/2-\varepsilon} \leq m, \quad L_a \leq 2x^\varepsilon, \quad \log|\mathcal{W}(\mathfrak{V})| \leq (k-3)\log 2,$$
$$x^{-\varepsilon} n \leq |T^{\mathrm{th}}_{x,i}|, \quad \tfrac{1}{2}x^{-\varepsilon/8} n \leq |T_{x,i}|, \quad a_t \geq (x-1)\, b \quad (t \in T_{x,i}).$$

Then $\mathcal{F}$ has the sizes (Sp8), with $e_K = 5$.

Proof. **Item 1.** If $i \in I(x)$ with $x > x_1$, then by (Pf1) the orbit selection of $\mathfrak{M}_{x,i}$ has parameter ε, and orbit selections have $0 < \varepsilon < \frac{1}{52}$.

Item 2: choice of x_0. Let $c > 0$. Since $\log y \leq y$ for $y > 0$, applied to $y = x^{\varepsilon/16}$, we have $\log x \leq \frac{16}{\varepsilon} x^{\varepsilon/16}$ for $x \geq 1$. Choose $x_0 \geq 1$ such that for $x \geq x_0$

$$x^{\varepsilon/3-\varepsilon/16} \geq \frac{80}{\varepsilon}, \qquad x^{\varepsilon/16} \geq \frac{1}{c}\Bigl(\log 8 + \frac{80}{\varepsilon}\Bigr), \qquad x^{2\varepsilon/3} \geq \log 48 + 2.$$

Then for $x \geq x_0$ (using $x^{\varepsilon/16} \geq 1$, $x^{\varepsilon/3} \geq 1$):

(L1) $5\log x \leq \frac{80}{\varepsilon} x^{\varepsilon/16} \leq x^{\varepsilon/3}$, i.e. $x^5 \leq X'$;

(L2) $\log 8 + 5\log x \leq (\log 8 + \frac{80}{\varepsilon}) x^{\varepsilon/16} \leq c\, x^{\varepsilon/8}$, i.e. $8x^5 \leq e^{cx^{\varepsilon/8}}$;

(L3) $\log 48 + 2x^{\varepsilon/3} \leq (\log 48 + 2) x^{\varepsilon/3} \leq x^\varepsilon$, i.e. $48\, e^{2x^{\varepsilon/3}} \leq b$; in particular $X'^2 \leq b$.

Item 2: the inequalities. Fix $x \geq x_0$ with $x > x_1$ and $i \in I(x)$. By item 1, $0 < \varepsilon < \frac{1}{52}$; by (Pf1) $\mathfrak{M}_{x,i}$ is on $H = A$, so $n = |H|$; by (Pf3) it satisfies (Lm7), and by (Pf2) it satisfies (Lm2). With $\theta_s = \varepsilon/100$ and $x \geq 1$:

- $m \leq x^{\theta_s}(x\delta_1)^{1/2} = x^{1/2-\varepsilon/16+\varepsilon/100} \leq x^{1/2} \leq x^{1/2+\varepsilon/2}$; also $m \leq x$.
- $k \geq x^{9/2+10\varepsilon-\theta_s} \geq x^{9/2} \geq x^{3/2} \geq xm$.
- $k \leq x^{9/2+10\varepsilon+\theta_s} \leq x^5$, since $10.01\,\varepsilon < \frac{1}{2}$.
- $n \geq x^{2+8\varepsilon-\theta_s}$, so $n^2 x^{1/2-\varepsilon} \geq x^{9/2+15\varepsilon-\varepsilon/50} \geq x^{9/2+10\varepsilon+\varepsilon/100} \geq k$; and $n \leq x^{2+8\varepsilon+\theta_s} \leq x^3$, since $8.01\,\varepsilon < 1$.

- $L_c \geq x^{\varepsilon} > 0$ by (Lm7). Further $L_a = \log a + \log\bigl(2v(\mathfrak{h}_{x,i})\bigr) \geq \log a = \log x + x^{\varepsilon} > 0$, because $2v(\mathfrak{h}_{x,i}) = |\mathrm{disc}(E_{\mathfrak{h}_{x,i}})|^{1/[E_{\mathfrak{h}_{x,i}}:\mathbb{Q}]} \geq 1$ (the discriminant of a number field is a non-zero integer) and $x \geq 1$.
- By (Lm2), $a_t \leq a(1+\vartheta)$ with $\vartheta = 1/(kx^{13}) \leq 1/x$ (as $k \geq 1$), so $a_t/b \leq x(1+\vartheta) \leq x+1$.
- By (L1), $k \leq x^5 \leq X'$; as $a > 0$ and $b' = a/X'$, this is $kb' \leq a$. Since $x^{\varepsilon/3} \leq x^{\varepsilon}$, $X' \leq b \leq xb = a$, so $k \leq a$.
- $n \leq x^3 \leq X'$ by (L1), so $nX' \leq X'^2 \leq b$ by (L3), and $X'' = xnX' \leq xb = a$.
- $f = 2n \leq 2x^3$ and $x+1 \leq 2x$, so $x \cdot 2f(x+1) \leq 8x^5 \leq e^{cx^{\varepsilon/8}}$ by (L2); this is $2f(x+1)e^{-cx^{\varepsilon/8}} \leq 1/x$.
- By (Lm7), $\eta > 0$ and $\log\eta \leq x^{\varepsilon/4}$, so $\eta + 1 \leq 2e^{x^{\varepsilon/4}} \leq 2X'$. Hence $x \cdot 6f(x+1)(\eta+1) \leq 6 \cdot 2x^3 \cdot 2x \cdot x \cdot 2e^{x^{\varepsilon/4}} = 48x^5 e^{x^{\varepsilon/4}} \leq 48X'e^{x^{\varepsilon/3}} = 48e^{2x^{\varepsilon/3}} \leq b$ by (L1) and (L3); this is $6f(x+1)(\eta+1) \leq b/x$.

The number x_0 depends only on ε and c.

Item 3. Let $c > 0$; take $e_K = 5$ and the maximum of x_0' and the x_0 of item 2 for c. For x beyond it with $x > x_1$, (Sp8) requires: $I(x) \neq \emptyset$ if $x \in W$ (hypothesis); and for every $i \in I(x)$ the fifteen inequalities of (Sp8). Of these, $m \leq x^{1/2+\varepsilon/2}$, $xm \leq k$, $k \leq x^{e_K}$, $k \leq a$, $k \leq n^2 x^{1/2-\varepsilon}$, $kb' \leq a$, $X'' \leq a$ and the two inequalities with $f(x+1)$ are in item 2; $x^{1/2-\varepsilon} \leq m$, $\log|\mathcal{W}(\mathfrak{V})| \leq (k-3)\log 2$, $x^{-\varepsilon}n \leq |T^{\mathrm{th}}_{x,i}|$ and $\frac{1}{2}x^{-\varepsilon/8}n \leq |T_{x,i}|$ are hypotheses; $0 \leq L_a \leq 2x^{\varepsilon}$ combines item 2 with the hypothesis; and $|a_t/b - x| \leq 1$ combines $a_t/b \leq x+1$ (item 2) with $a_t \geq (x-1)b$ (hypothesis). Together with $\varepsilon \leq 1$ this is (Sp8). □

Definition 12.2. Let $\mathcal{F}$ be a parametrised host family with slope records, with parameter ε, threshold x_1 and frame (e, A, h, a_0) at x. For a member (x,i) with $x > x_1$, write $\mathfrak{V} = \mathfrak{V}_{x,i}$ and $\mathfrak{M} = \mathfrak{M}_{x,i}$, and let m, $(K, S, \ell, (g_j)_{1\leq j\leq m}, \tau, \mathrm{io})$ and $p(w)$ be the integer, the place record and the residue characteristics of $\mathfrak{M}$. Let d be the prime of the host $\mathfrak{h}_{x,i}$, and put

$$n = |A|, \quad f = 2n, \quad b = e^{x^{\varepsilon}}, \quad a = xb, \quad X' = e^{x^{\varepsilon/3}}, \quad b' = \frac{a}{X'}, \quad X'' = xnX'.$$

Modification data for $\mathcal{F}$ are, for every real $x > x_1$ and every $i \in I(x)$,

- for every $w \in S$ and $1 \leq j \leq m$ a real number $F_j(\mathfrak{V}, w)$, the **local frame numbers**;
- for every word $\omega \in \mathcal{W}(\mathfrak{V})$: an integer $d^{\circ}(\mathfrak{V}, \omega) \geq 1$; real numbers $\mu^{\circ}(\mathfrak{V}, \omega)$ and $\mu^{\circ\max}(\mathfrak{V}, \omega)$, the **slope of the level lattice** and the **maximal slope of the level lattice** of ω; an integer $N_{\Gamma}(\mathfrak{V}, \omega) \geq 1$, the **exceptional modulus** of ω; and for every $w \in S$ and $1 \leq j \leq m$ real numbers $E_j(\mathfrak{V}, \omega, w)$ (the **local exponents**), $\mathrm{len}(\mathfrak{V}, \omega, w)$ (the **local length**) and $\mathrm{tw}^{\flat}(\mathfrak{V}, \omega, w)$ (the **rounded twist**),

such that for every word ω

$$F(\mathfrak{V}) = 2\sum_{j=1}^{m}\sum_{w\in S} \ell(w)\, F_j(\mathfrak{V}, w), \qquad E(\mathfrak{V}, \omega) = 2\sum_{j=1}^{m}\sum_{w\in S} \ell(w)\, E_j(\mathfrak{V}, \omega, w). \tag{12.1}$$

The **twist** of ω at $w \in S$ and the set of places at the Gauss prime are

$$\mathrm{tw}(\mathfrak{V}, \omega, w) = \sum_{j=1}^{m}\bigl(g_j(w) + F_j(\mathfrak{V}, w) - E_j(\mathfrak{V}, \omega, w)\bigr), \qquad S_T(\mathfrak{V}, \omega) = \{w \in S :\ p(w) = T(\mathfrak{V}, \omega)\}.$$

A **parametrised host family with slope records and modification data** is a parametrised host family with slope records together with modification data for it.

As in Definition 8.2, a statement holds "for all large members" if there is a real x_0 such that it holds for every real $x \geq x_0$ with $x > x_1$, every $i \in I(x)$ and every word $\omega \in \mathcal{W}(\mathfrak{V}_{x,i})$. We write $T = T(\mathfrak{V}, \omega)$. Such a family has each of the following properties if the stated condition holds.

(Mf1) Non-negative local weights For all large members, $\ell(w) \geq 0$ for every finite place w of K.

(Mf2) Level lattices For all large members, $T > 2$ and

$$d(\mathfrak{V}, \omega) = \Big(1 - \frac{1}{T-1}\Big)^f d^\circ(\mathfrak{V}, \omega), \qquad \mu^\circ(\mathfrak{V}, \omega) = \mu'(\mathfrak{V}, \omega) - \frac{f \log(T-1)}{T-2},$$
$$\mu^{\circ\max}(\mathfrak{V}, \omega) \leq \mu'^{\max}(\mathfrak{V}, \omega).$$

(Mf3) Modification formulas For all large members, $d(\mathfrak{V}, \omega) > 0$ and

$$\mu(\mathfrak{V}, \omega) = \mu^\circ(\mathfrak{V}, \omega) + \sum_{w \in S} 2\ell(w)\, \mathrm{tw}^\flat(\mathfrak{V}, \omega, w) - \frac{1}{d(\mathfrak{V}, \omega)} \sum_{w \in S} 2\ell(w)\, \mathrm{len}(\mathfrak{V}, \omega, w),$$
$$\mu^{\max}(\mathfrak{V}, \omega) \leq \mu^{\circ\max}(\mathfrak{V}, \omega) + \sum_{w \in S} 2\ell(w)\big(\mathrm{tw}^\flat(\mathfrak{V}, \omega, w) - \tau(w)\big).$$

(Mf4) Local lengths There is a real $C \geq 0$ such that, for all large members, $\mathrm{len}(\mathfrak{V}, \omega, w) \leq d^\circ(\mathfrak{V}, \omega)(\tau(w) + \mathrm{io}(w))$ for every $w \in S \setminus S_T(\mathfrak{V}, \omega)$, and

$$\sum_{w \in S_T(\mathfrak{V}, \omega)} 2\ell(w)\, \mathrm{len}(\mathfrak{V}, \omega, w) \leq C\, d(\mathfrak{V}, \omega)\, |T_{x,i}|\, m\Big(\frac{a}{k} + b'\Big) \log T,$$

where k is the size parameter of $\mathfrak{M}$.

(Mf5) Rounding For all large members and every $w \in S$, $\mathrm{tw}^\flat(\mathfrak{V}, \omega, w) \leq \mathrm{tw}(\mathfrak{V}, \omega, w) \leq \mathrm{tw}^\flat(\mathfrak{V}, \omega, w) + 1$, and $\mathrm{tw}^\flat(\mathfrak{V}, \omega, w) = 0$ when $\mathrm{tw}(\mathfrak{V}, \omega, w) = 0$.

(Mf6) Support of the twists For all large members, $\mathrm{tw}(\mathfrak{V}, \omega, w) = 0$ for every $w \in S$ with $p(w) > X''$, $p(w) \notin \{T,\, g_1(\mathfrak{V}),\, g_2(\mathfrak{V}),\, d\}$ and $p(w) \nmid N_\Gamma(\mathfrak{V}, \omega)$.

(Mf7) Sizes of the special primes There is a real $C \geq 0$ such that, for all large members, $T \geq b^3$ and

$$\log T + \log g_1(\mathfrak{V}) + \log g_2(\mathfrak{V}) + \log d + \log N_\Gamma(\mathfrak{V}, \omega) \leq C\, X''.$$

(Mf8) Correction budget For all large members, $\sum_{w \in S} 2\ell(w)\, \tau(w) \leq b^2$.

Theorem 12.3. *Let $\mathcal{F}$ be a parametrised host family with slope records and modification data that has the properties (Mf1)–(Mf8). Then $\mathcal{F}$ has the transfer of slopes (Sp3). More precisely, there are real numbers $C \geq 0$ and x_0 such that for every real $x \geq x_0$ with $x > x_1$, every $i \in I(x)$ and every word $\omega \in \mathcal{W}(\mathfrak{V})$, with $\mathfrak{V} = \mathfrak{V}_{x,i}$, $\mathfrak{M} = \mathfrak{M}_{x,i}$ and the notation of (Sp3),*

$$\mu^{\max}(\mathfrak{V}, \omega) \leq \mu'^{\max}(\mathfrak{V}, \omega) + \Theta(\mathfrak{M}) + F(\mathfrak{V}) - E(\mathfrak{V}, \omega),$$
$$\mu(\mathfrak{V}, \omega) \geq \mu'(\mathfrak{V}, \omega) + \Theta(\mathfrak{M}) + F(\mathfrak{V}) - E(\mathfrak{V}, \omega)$$
$$- \Big(C\, I(\mathfrak{M}) + C|T_{x,i}|\, m\Big(\frac{a}{k} + b'\Big) \log T(\mathfrak{V}, \omega) + CX'' + 1\Big).$$

Proof. Let ε be the parameter of $\mathcal{F}$. Let $C_4 \geq 0$ be the constant of (Mf4) and $C_R \geq 0$ the constant of Lemma 12.4 (which applies by (Mf1), (Mf5), (Mf6), (Mf7)); put $C = \max(4, C_4, C_R)$. Let $N = \lceil 4/\varepsilon \rceil$ and let x_0 be the largest of $32\,N!$, the thresholds of (Mf1)–(Mf8) and that of Lemma 12.4. Fix $x \geq x_0$ with $x > x_1$, $i \in I(x)$ and $\omega \in \mathcal{W}(\mathfrak{V})$, and write $T = T(\mathfrak{V}, \omega)$, $d_\omega = d(\mathfrak{V}, \omega)$, $d_\omega^\circ = d^\circ(\mathfrak{V}, \omega)$, and n, f, b, a, X', b', X'' as in Definition 12.2; let k be the size parameter of $\mathfrak{M}$.

Signs. (i) By (Pf1) the orbit selection of $\mathfrak{M}$ has parameter ε and integer m, so $0 < \varepsilon < \frac{1}{52}$ and $m \geq 1$; $x > x_1 > 1$. (ii) For $w \in S$: $\tau(w) \geq 0$ and $\mathrm{io}(w) \geq 0$. Indeed, by (Lm6) (through (Pf2)) either $p(w)$ is not inert and $\tau(w) = \mathrm{io}(w) = 0$, or $\tau(w) = \tau_{p(w)}$ and $\mathrm{io}(w) = \mathrm{io}_{p(w)}$; and τ_p is 0 or $\epsilon_p + m \sum_{v \in V_p} (\lceil t_p(v) \rceil^+ + D_p + 1) \geq 0$, while io_p is 0 or $2\Sigma_p \max_{v \in V_p} S_p(v) \geq 0$, as $\Sigma_p \geq 0$ and $S_p(v) \geq 0$. (iii) Hence, with $\ell \geq 0$ (Mf1), the numbers $A := \sum_{w \in S} 2\ell(w)\tau(w)$ and $I(\mathfrak{M})$ are ≥ 0, and $A \leq b^2$ by (Mf8). (iv) By (Lm7) (through (Pf3)) $k > 0$, so $\Lambda := |T_{x,i}|\, m\, (a/k + b') \log T \geq 0$ (T is a prime); also $X'' > 0$ and $B_0(x, i) = 2nmbx^\varepsilon \geq 0$.

Sizes. We show $b \geq 16f$ and $b \geq 4$. By (Lm7), $n \leq x^{2+8\varepsilon+\varepsilon/100} \leq x^3$, since $x \geq 1$ and $8.01\,\varepsilon < 1$; so $f = 2n \leq 2x^3$. As $e^y \geq y^N/N!$ for $y \geq 0$, and $\varepsilon N \geq 4$, $x \geq 1$,

$b \geq 16f$; and $f \geq 2$ gives $b \geq 32 \geq 4$. By (Mf7), $T \geq b^3$. Consequently:

- $T \geq 64$; $\dfrac{2f}{T} \leq \dfrac{b/8}{b^3} \leq \dfrac{1}{2}$; $\dfrac{4fb^2}{T} \leq \dfrac{4f}{b} \leq \dfrac{1}{4}$;
- $\dfrac{f \log(T-1)}{T-2} \leq \dfrac{1}{4}$: since $\log y < y$ for $y > 0$, $\log(T-1) < \log T = 2 \log \sqrt{T} < 2\sqrt{T}$; since $b^2 \leq b^3 \leq T$, $b \leq \sqrt{T}$, so $f \log(T-1) \leq 2f\sqrt{T} \leq \frac{b}{8}\sqrt{T} \leq \frac{T}{8}$, while $T - 2 \geq \frac{T}{2}$.

Hence

$$\frac{4fb^2}{T} + \frac{f \log(T-1)}{T-2} \leq 1. \tag{12.2}$$

The deficit of the level lattice. Put $\delta = 1 - d_\omega/d_\omega^\circ$. By (Mf2), $d_\omega = (1-y)^f d_\omega^\circ$ with $y = 1/(T-1) \in (0,1)$, so $\delta = 1 - (1-y)^f$. By Bernoulli's inequality $(1-y)^f \geq 1 - fy$ (as f is a non-negative integer), so $0 \leq \delta \leq fy = f/(T-1) \leq 2f/T \leq \frac{1}{2}$, the middle step because $2(T-1) \geq T$. Therefore $d_\omega^\circ/d_\omega = 1/(1-\delta) \leq 2$ and $d_\omega^\circ/d_\omega - 1 = \delta/(1-\delta) \leq 2\delta \leq 4f/T$.

The twisting sum. By the definition of the twist, (12.1) and the formula for $\Theta(\mathfrak{M})$,

$$\sum_{w \in S} 2\ell(w)\, \mathrm{tw}(\mathfrak{V}, \omega, w) = 2 \sum_{w \in S} \ell(w) \sum_{j=1}^{m} g_j(w) + F(\mathfrak{V}) - E(\mathfrak{V}, \omega) = \Theta(\mathfrak{M}) + A + F(\mathfrak{V}) - E(\mathfrak{V}, \omega). \tag{12.3}$$

By Lemma 12.4,

$$\sum_{w \in S} 2\ell(w)\, \mathrm{tw}(\mathfrak{V}, \omega, w) - C_R X'' \leq \sum_{w \in S} 2\ell(w)\, \mathrm{tw}^\flat(\mathfrak{V}, \omega, w) \leq \sum_{w \in S} 2\ell(w)\, \mathrm{tw}(\mathfrak{V}, \omega, w). \tag{12.4}$$

Upper bound. By (Mf3), the right half of (12.4), (Mf2) and (12.3),

$$\mu^{\max}(\mathfrak{V}, \omega) \leq \mu^{\circ\,\max}(\mathfrak{V}, \omega) + \sum_{w \in S} 2\ell(w) \mathrm{tw}^\flat(\mathfrak{V}, \omega, w) - A \leq \mu'^{\max}(\mathfrak{V}, \omega) + \Theta(\mathfrak{M}) + F(\mathfrak{V}) - E(\mathfrak{V}, \omega).$$

Since $C\, m\, B_0(x, i) \geq 0$ by (iv), this gives the first inequality of (Sp3).

The local lengths. Split S into $S_T = S_T(\mathfrak{V}, \omega)$ and $S \setminus S_T$. By the second part of (Mf4), $\frac{1}{d_\omega} \sum_{w \in S_T} 2\ell(w)\, \mathrm{len}(\mathfrak{V}, \omega, w) \leq C_4 \Lambda$. By the first part, $\ell \geq 0$, and the signs (ii),

$$\frac{1}{d_\omega} \sum_{w \in S \setminus S_T} 2\ell(w)\, \mathrm{len}(\mathfrak{V}, \omega, w) \leq \frac{d_\omega^\circ}{d_\omega} \sum_{w \in S \setminus S_T} 2\ell(w)\big(\tau(w) + \mathrm{io}(w)\big) \leq \frac{d_\omega^\circ}{d_\omega}\big(A + 2I(\mathfrak{M})\big).$$

Lower bound. By (Mf3), the left half of (12.4), (12.3), (Mf2) and the two bounds for the local lengths,

$$\mu(\mathfrak{V},\omega) \geq \mu'(\mathfrak{V},\omega) - \frac{f\log(T-1)}{T-2} + \Theta(\mathfrak{M}) + F(\mathfrak{V}) - E(\mathfrak{V},\omega) + A - C_R X'' - C_4\Lambda - \frac{d_\omega^\circ}{d_\omega}(A + 2I(\mathfrak{M})).$$

By the deficit bounds, $0 \leq A \leq b^2$ and $I(\mathfrak{M}) \geq 0$, we have $A - \frac{d_\omega^\circ}{d_\omega} A \geq -\frac{4f}{T} b^2$ and $\frac{d_\omega^\circ}{d_\omega}\, 2I(\mathfrak{M}) \leq 4I(\mathfrak{M})$. With (12.2),

$$\mu(\mathfrak{V},\omega) \geq \mu'(\mathfrak{V},\omega) + \Theta(\mathfrak{M}) + F(\mathfrak{V}) - E(\mathfrak{V},\omega) - \big(4I(\mathfrak{M}) + C_4\Lambda + C_R X'' + 1\big).$$

As $I(\mathfrak{M}), \Lambda, X'' \geq 0$ and $4, C_4, C_R \leq C$, the bracket is at most $C\, I(\mathfrak{M}) + C\Lambda + CX'' + 1$, which is the second inequality of (Sp3). The constants $C \geq 0$ and x_0 do not depend on x, i or ω, so $\mathcal{F}$ has (Sp3). □

Lemma 12.4. *Let $\mathcal{F}$ be a parametrised host family with slope records and modification data that has non-negative local weights (Mf1), rounding (Mf5), support of the twists (Mf6) and sizes of the special primes (Mf7). Then there is a real number $C_R \geq 0$ such that for all large members*

$$0 \leq \sum_{w \in S} 2\ell(w)\big(\mathrm{tw}(\mathfrak{V},\omega,w) - \mathrm{tw}^\flat(\mathfrak{V},\omega,w)\big) \leq C_R\, X''.$$

Proof. Let $C \geq 0$ be the constant of (Mf7), let x_0 be the largest of the thresholds of (Mf1), (Mf5), (Mf6), (Mf7), and put $C_R = 2(\log 4 + C)$. Fix $x \geq x_0$ with $x > x_1$, $i \in I(x)$ and a word $\omega \in \mathcal{W}(\mathfrak{V})$, $\mathfrak{V} = \mathfrak{V}_{x,i}$; write $\mathrm{tw}(w)$, $\mathrm{tw}^\flat(w)$ for $\mathrm{tw}(\mathfrak{V},\omega,w)$, $\mathrm{tw}^\flat(\mathfrak{V},\omega,w)$, and $\Sigma = \sum_{w\in S} 2\ell(w)(\mathrm{tw}(w) - \mathrm{tw}^\flat(w))$.

Each term lies in $[0, 2\ell(w)]$ and vanishes unless $\mathrm{tw}(w) \neq 0$. By (Mf5), $0 \leq \mathrm{tw}(w) - \mathrm{tw}^\flat(w) \leq 1$ for $w \in S$, and $\mathrm{tw}(w) - \mathrm{tw}^\flat(w) = 0$ when $\mathrm{tw}(w) = 0$; by (Mf1), $\ell(w) \geq 0$. Hence every term of Σ is non-negative, so $\Sigma \geq 0$, and

$$\Sigma \leq \sum_{w \in S_0} 2\ell(w), \qquad S_0 = \{w \in S :\ \mathrm{tw}(w) \neq 0\}.$$

The places of S_0 lie over a small set of primes. Let P be the set of primes p with $p \leq X''$, together with T, $g_1(\mathfrak{V})$, $g_2(\mathfrak{V})$, d and the prime divisors of $N_\Gamma(\mathfrak{V},\omega)$. If $w \in S$ and $p(w) \notin P$, then $p(w) > X''$, $p(w) \notin \{T, g_1(\mathfrak{V}), g_2(\mathfrak{V}), d\}$ and $p(w) \nmid N_\Gamma(\mathfrak{V},\omega)$, so $\mathrm{tw}(w) = 0$ by (Mf6). Thus $p(w) \in P$ for every $w \in S_0$. Grouping the places of S_0 by their residue characteristic and using $\ell \geq 0$ on all finite places of K (Mf1) and the normalisation of the place record,

$$\sum_{w \in S_0} 2\ell(w) \leq 2 \sum_{p \in P} \sum_{w \mid p} \ell(w) = 2 \sum_{p \in P} \log p.$$

The primes of P. The set P is the union of $\{p \leq X''\}$, of the four primes $T, g_1(\mathfrak{V}), g_2(\mathfrak{V}), d$ and of the prime divisors of $N_\Gamma(\mathfrak{V},\omega)$; as all $\log p \geq 0$, the sum over the union is at most the sum of the sums over the parts. By Chebyshev's bound $\sum_{p \leq y} \log p \leq y \log 4$ for $y \geq 0$ (here $X'' > 0$); and $\sum_{p \mid N} \log p \leq \log N$ for an integer $N \geq 1$. Hence, by (Mf7),

$$\sum_{p \in P} \log p \leq X'' \log 4 + \log T + \log g_1(\mathfrak{V}) + \log g_2(\mathfrak{V}) + \log d + \log N_\Gamma(\mathfrak{V},\omega) \leq (\log 4 + C)\, X''.$$

Combining the three displays, $0 \leq \Sigma \leq 2(\log 4 + C)X'' = C_R X''$. □

Lemma 12.5. *Let $\mathcal{F}$ be a parametrised host family with parameter ε and threshold x_1 that has the linking property. Then there are real numbers $K_\tau \geq 0$, $E_\tau \geq 0$ and x_0 such that for every real $x \geq x_0$ with $x > x_1$ and every $i \in I(x)$, with $(K, S, \ell, (g_j), \tau, \mathrm{io})$ the place record of $\mathfrak{M}_{x,i}$,*

$$0 \leq \sum_{w \in S} 2\ell(w)\,\tau(w) \leq K_\tau\, x^{E_\tau}\, b(x) \leq b(x)^2.$$

In particular every parametrised host family with slope records and modification data whose underlying parametrised host family has the linking property has the correction budget (Mf8).

Proof. Let C, x_{L} and δ be the constants and the function of the linking property, so that for $x \geq x_{\mathrm{L}}$, $x > x_1$, every $\mathfrak{M}_{x,i}$ satisfies (Lm3) for (C, b_o) with $b_o = b(x)^{\delta(x)}$; let ε_z, C_t be the numbers of (Pf3). Put $C^+ = \max(C, 0)$, $c_t = |C_t| + 1$,

$$E_\tau = 2c_t + 42, \qquad K_\tau = 64 + 192\,C^+ + 4\,C^+\Big(\frac{32}{\varepsilon} + 66\Big),$$

and let $x_0 \geq \max(x_{\mathrm{L}}, 1)$ be such that for every $x \geq x_0$: $|\delta(x)| \leq \frac{1}{2}$, $x^{2\varepsilon/3} \geq 2$ and $K_\tau x^{E_\tau} \leq \exp(x^\varepsilon)$. Such x_0 exists: $\delta(x) \to 0$; and since $\log y \leq y$ for $y > 0$, $\log x \leq \frac{2}{\varepsilon} x^{\varepsilon/2}$, so $\log K_\tau + E_\tau \log x \leq (|\log K_\tau| + \frac{2E_\tau}{\varepsilon}) x^{\varepsilon/2} \leq x^\varepsilon$ as soon as $x^{\varepsilon/2} \geq \max(1, |\log K_\tau| + \frac{2E_\tau}{\varepsilon})$ (here $K_\tau \geq 64$).

Fix $x \geq x_0$ with $x > x_1$ and $i \in I(x)$, and write $\mathfrak{M} = \mathfrak{M}_{x,i}$ with its data as in Definition 5.5, $T = T_{x,i}$ its set of tops, $b = b(x)$, $b_o = b^{\delta(x)}$ and $\Upsilon = \exp(x^{\varepsilon/4})$. By (Pf1) the orbit selection of $\mathfrak{M}$ has scale x and parameter ε, so $0 < \varepsilon < \frac{1}{52}$ and $m \geq 1$; $x > 1$.

Sizes. By (Lm7) (through (Pf3)), with $\theta_s = \varepsilon/100$ and $0 < \varepsilon < \frac{1}{52}$, $x \geq 1$:

- $n \leq x^{2+8\varepsilon+\theta_s} \leq x^3$, $|T| \leq n$ (as $T \subseteq H$), so $m|T| \leq x^4$ by the next item;
- $m \leq x^{\theta_s}(x\delta_1)^{1/2} = x^{1/2-\varepsilon/16+\varepsilon/100} \leq x$;
- $1 \leq x^{9/2+10\varepsilon-\theta_s} \leq k \leq x^{9/2+10\varepsilon+\theta_s} \leq x^5$, hence $p_0^* = k^2 x^{26} \leq x^{36}$;
- $0 \leq t_R \leq x^{C_t+\theta_s} \leq x^{c_t}$, hence $N = k + m(t_R + 1) \leq x^5 + 2x^{c_t+1} \leq 3x^{c_t+5}$;
- $0 < x^\varepsilon \leq L_a \leq x^{\varepsilon+\theta_s} \leq x$;
- $S_c = 4k\Delta$ with $0 < \Delta \leq e^{x^{\varepsilon/4}/2} \leq \Upsilon$, so $0 < S_c \leq 4x^5\Upsilon$ and $c_G = 4S_c > 0$; $0 < \eta \leq \Upsilon$; hence $S_c + k + \eta \leq 6x^5\Upsilon$;
- $r_G = \lceil (32 + 64\varepsilon)/\varepsilon \rceil + 1 \leq \frac{32}{\varepsilon} + 66$; $\vartheta = 1/(kx^{13}) \leq 1$.

Moreover $X' \geq 1$, $b \geq 1$, $b' = a/X' \leq a = xb$, $X'b' = a$, $w'' > 0$, $\lambda = \log x \leq x$, $1/\delta_1 = x^{\varepsilon/8} \leq x$, $\log a = \log x + x^\varepsilon \leq 2x$, $\log(2a) \leq 3x$, and, by the choice of x_0,

$$\Upsilon X' \leq e^{2x^{\varepsilon/3}} \leq b, \qquad X' b^{1/2} \leq b, \qquad b_o \leq b^{1/2}, \qquad K_\tau x^{E_\tau} \leq b, \tag{12.5}$$

the first two because $2x^{\varepsilon/3} \leq x^\varepsilon$ and $x^{\varepsilon/3} \leq x^\varepsilon/2$ follow from $x^{2\varepsilon/3} \geq 2$, the third because $b \geq 1$ and $\delta(x) \leq \frac{1}{2}$.

Reduction to the candidates with served orbits. Let $\mathbb{P}_V$ be the set of candidates p with $V_p \neq \emptyset$; every $p \in \mathbb{P}_V$ is a prime with $p < X'$. For $w \in S$, (Lm6) (through (Pf2)) gives $\tau(w) = \tau_{p(w)}$ if $p(w)$ is inert and $\tau(w) = 0$ otherwise; and $\tau_p = 0$ when $V_p = \emptyset$, while $V_p \neq \emptyset$ forces p to be a candidate. So $\tau(w) \neq 0$ only if $p(w) \in \mathbb{P}_V$. For $p \in \mathbb{P}_V$, $p < X' \leq X''$, so by

(Lm6) every finite place of K above p lies in S, and the normalisation of the place record gives $\sum_{w\in S,\,w|p}\ell(w)=\log p$. Grouping the places of S by residue characteristic,

$$\sum_{w\in S}2\ell(w)\tau(w)=2\sum_{p\in\mathbb{P}_V}\tau_p\log p. \tag{12.6}$$

Each τ_p is ≥ 0: it is 0 or $\epsilon_p+m\sum_{v\in V_p}(\lceil t_p(v)\rceil^+ + D_p+1)$ with $\epsilon_p, D_p\geq 0$. This gives the lower bound 0 of the statement.

The primes of $\mathbb{P}_V$ are served. Let $p\in\mathbb{P}_V$. It is not of kind (Dr1), as $p<X'\leq X''$. It is a candidate, so it is not sign-changing and h_p has order at least $x^{\varepsilon/4}$; by (Lm1) h_p then has even order, so p is not of kind (Dr2). It is not of kind (Dr3): the order of h_p is not below $x^{\varepsilon/4}$; if $p\geq p_0^*$ no power of p lies in a band (definition of a candidate; the threshold p_0 of the orbit selection is p_0^*); and $\Sigma_p\leq x^{-\varepsilon}$ because $V_p\neq\emptyset$. Hence p is served, and the served-prime inequality of (Lm3) for (C,b_o) holds at p.

The gains at $p\in\mathbb{P}_V$ are at most $6m|T|a/p$. Fix j. If $\mathrm{gain}_j(p)=0$ by the exceptional clause of its definition there is nothing to show. Otherwise

$$\mathrm{gain}_j(p)=\sum_{v\in V_p}\max\bigl(g_j^U(v)-\mathrm{Pr}_j(v),\,g_j^0(v),\,0\bigr)+\sum_{v\in R_p\setminus V_p}\max\bigl(g_j^0(v),0\bigr).$$

(a) For $v\in R_p$, $\max(g_j^0(v),0)\leq 2a\gamma_p N_p(v)$. Indeed $g_j^0(v)=\sum_{s\in T_p(v)}\sigma_s-\delta_p(v)$, where σ_s is the series over the odd e not excluded for s of $(\alpha_s/p^e-c_j(s,e))^+$, and $\delta_p(v)\geq 0$ since $2a\geq 1$. We show $0\leq\sigma_s\leq\alpha_s^+\gamma_p$. If the series does not converge, $\sigma_s=0$ by the convention of Definition 5.3. If it converges, its terms tend to 0. Since $z_p(s,e+2n_p)=z_p(s,e)+2n_pf_p=z_p(s,e)$, the capacity $c_j(s,e)$ depends only on e modulo $2n_p$; and only finitely many e are excluded for s (each top $s'\neq s$ has at most one first hit from s). So every odd e_0 has infinitely many non-excluded odd $e\equiv e_0\pmod{2n_p}$, along which the terms tend to $(-c_j(s,e_0))^+$; hence $c_j(s,e_0)\geq 0$ for every odd e_0, each term is at most $(\alpha_s/p^e)^+=\alpha_s^+/p^e$, and $\sigma_s\leq\alpha_s^+\sum_{e\text{ odd}}p^{-e}=\alpha_s^+\gamma_p$. Finally $\alpha_s=a_s-2c_G\leq a_s\leq a(1+\vartheta)\leq 2a$ by (Lm2) and $c_G>0$, so $\alpha_s^+\leq 2a$, and $\max(g_j^0(v),0)\leq\sum_{s\in T_p(v)}\alpha_s^+\gamma_p\leq 2a\gamma_pN_p(v)$.

(b) For $v\in V_p$, using $\max(A,B,0)=\max(B,0)+(A-\max(B,0))^+$ for reals A,B,

$$\sum_{j=1}^m\max\bigl(g_j^U(v)-\mathrm{Pr}_j(v),g_j^0(v),0\bigr)=\sum_{j=1}^m\max(g_j^0(v),0)$$
$$+\sum_{j=1}^m\bigl(g_j^U(v)-\mathrm{Pr}_j(v)-\max(g_j^0(v),0)\bigr)^+\leq\sum_{j=1}^m\max(g_j^0(v),0)+S_p(v),$$

and $S_p(v)\leq 2mN_p(v)a/p$ by the surplus bound of (Lm3).

The sets $T_p(v)$, $v\in R_p$, partition T, so $\sum_{v\in R_p}N_p(v)=|T|$; $V_p\subseteq G_p\subseteq R_p$; and $\gamma_p=p/(p^2-1)\leq 2/p$ as $p^2\geq 2$. Summing (a) over $v\in R_p$ and j, and (b) over $v\in V_p$ (every term of the gains is non-negative),

$$\sum_{j=1}^m\mathrm{gain}_j(p)\leq m\cdot 2a\cdot\frac{2}{p}\,|T|+\frac{2ma}{p}|T|=\frac{6m|T|a}{p}. \tag{12.7}$$

The correction at $p\in\mathbb{P}_V$. The series $A_p(a_t,t,w_j)$ has non-negative terms and, as $w_j\geq w''>0$ on Φ (Lm2), finitely many non-zero ones, so $A_p\geq 0$. Writing $y^+=\max(y,0)$, the served-prime inequality and $T_p^{\mathrm{s}}\subseteq T$ give

$$\tau_p\leq\sum_{j=1}^m\mathrm{gain}_j(p)+\sum_{j=1}^m\sum_{t\in T_p^{\mathrm{s}}}\mathrm{lo}_{C,b_o}(p)+m\,\mathrm{fl}_C(p)\leq\frac{6m|T|a}{p}+m|T|\,\mathrm{lo}_{C,b_o}(p)^+ +m\,\mathrm{fl}_C(p)^+. \tag{12.8}$$

Since $p \in \mathbb{P}_V$ is inert and not sign-changing, $p < X'$, and all brackets in the definitions of the losses are non-negative while $b_o \geq 0$,

$$\mathrm{fl}_C(p)^+ \leq C^+ f(N^2 \log_p a + (\log_p a)^2),$$
$$\mathrm{lo}_{C,b_o}(p)^+ \leq C^+(S_c + k + \eta)\log_p(2a) + \mathrm{pr}^+ + C^+\lambda\Big(\frac{L_a a}{pk} + \frac{a}{p^2}\Big),$$

where $\mathrm{pr}^+ = C^+ r_G b'\lambda$ if $p < p_0^*$ and $\mathrm{pr}^+ = C^+(b'/x + \eta + S_c + (w'' + t_R)\lambda/\delta_1) + b_o$ otherwise; in the last term $p^{-L_1(p)} \leq p^{-2}$ because $k^2 x^{78} > 1$ gives $L_1(p) \geq 2$.

Summation. Insert (12.8) into (12.6) and sum over $p \in \mathbb{P}_V$, a set of fewer than X' primes below X'. We use: $\sum_{p<y} \log p \leq y \log 4 \leq 2y$ (Chebyshev's bound, Lemma B.3); $\sum_{p<X'} \frac{\log p}{p} \leq \sum_{n \leq X'} \frac{\Lambda(n)}{n} \leq \log X' + 3 + \log 2 \leq x^{\varepsilon/3} + 4 \leq 5x$ (Lemma 3.5, item 3, since $X' \geq 1$); $\sum_{p<X'} \frac{\log p}{p^2} \leq 5x$ likewise; and the sizes above. The five contributions are:

Each of them is at most a constant times $x^{E_\tau} b$, and their sum is at most $K_\tau x^{E_\tau} b$. Hence

$$\sum_{w \in S} 2\ell(w)\tau(w) \leq K_\tau x^{E_\tau} b \leq b^2,$$

the last step by (12.5). The numbers K_τ, E_τ, x_0 depend only on $\mathcal{F}$ (through C, x_{L}, δ, C_t, ε), not on x or i.

The correction budget. For a parametrised host family with slope records and modification data, (Mf8) is the requirement that $\sum_{w \in S} 2\ell(w)\tau(w) \leq b^2$ for all large members, where the sum does not depend on the word; this has been shown. □

13 Norms of the lifts and the divisor package: (Sp4)

This section proves the divisor package (Sp4) for the class, with the exponent 1 (Theorem 13.2). Its central inequality, the divisor bound, bounds the dual slope of the target, corrected by a height, by a sum over the tops t in which each of the m lifts contributes $a_t \log a_t - a_t - a_t \log \Omega$, where $\log \Omega = \gamma + \log(\pi/v) - \beta$ involves $\beta = L'(1, \chi_d)/L(1, \chi_d)$ (Definition 8.2). In Alpöge's account these are the norms of the lifts, computed with the Chowla–Selberg formula for the CM period [4]; in this paper they are numbers, and that they are norms of theta lifts is our reading. The package also contains the exponent comparison, in which the terms $m\, A_{\mathfrak{M}} \log(g_1 g_2)$ of order m at the level primes appear; Part I cancels them in the accounting.

The section runs top down, and the point is the role of the Sha condition. Theorem 13.2 obtains (Sp4) from seven lower properties (Dv1)–(Dv7) and the Sha condition. Of these, (Dv5) says that the divisor bound holds as soon as the defects of the lattices are small, and it is proved with exponent 1 from the bridge records (Theorem 13.4); the record inequality (Bk3) among them comes from the numerical records, through which $\log \Omega$ enters as the main term (Nr2) (Lemma 13.6). Property (Dv6) bounds the defects by lengths of Tate–Shafarevich groups: by Theorem 13.18, at large members with finite Sha lengths, each defect is at most its rank times the sum of an explicit budget and the total Sha length $\mathrm{len}(E; t)$. Under the Sha condition $\mathrm{len}(E; t) \leq \exp(x^\varepsilon/2) \leq b = \exp(x^\varepsilon)$, which fits within the budget $B_0 = 2nm\, b\, x^\varepsilon$, so the defects are small in the sense of (Dv5). This is where the window Sha bound of Parts VII–IX is used, through the Sha condition; in our reading it is the "significant weakening of Vandiver" that the note asks for [4].

The bound for the defects is local. The defect sum of a member is a weighted sum over finite places of defects of Hecke lattices, which are controlled by congruence modules and congruence ideals (Definition 13.8, Definition 13.9). Theorem 13.18 derives (Dv6) from the local defect

data (Ld1)–(Ld9); Theorem 13.16 and Theorem 13.17 reduce (Ld7)–(Ld9) to the local bound records (Lq1)–(Lq8); and the key step is the lattice index bound (Lq1), Theorem 13.13, a quantitative form of Ribet's lemma [208]. A congruence ideal is bounded by the lengths of the homomorphisms from a finitely presented module to the constituents of a reducible determinant (Lemma 13.14), and the Ribet records bound these, at a place above ℓ, by a Selmer term, a local term, the Sha length $\mathrm{len}_\ell(E;t)$ and the ℓ-adic valuation of the class number (Lemma 13.15). The local terms are bounded by the Ribet budget Theorem 22.1 of Part V, which supplies (Rf1), (Rf2), (Rf4), (Rf5), (Lq3) and (Lq7) in the proof of Theorem 17.7. The cut lemma 13.19 passes from the cut congruence modules to the defects. The conventions on Galois cohomology, p-adic Hodge theory, Hecke characters, unitary dual pairs and automorphic Galois representations are stated in this section at their first use; Parts V–IX use them as well.

Definition 13.1. For a number field E and an integer $t \geq 1$ the **total Sha length** is

$$\mathrm{len}(E;t) = \sum_{\ell} \log \ell \sum_{k=1}^{t-1} \mathrm{len}_\ell(E,k) \in [0,\infty].$$

Here ℓ runs over all primes and the terms are non-negative. The sum is ∞ if some $\mathrm{len}_\ell(E,k)$ with $1 \leq k < t$ is infinite or the series diverges.

Let $\mathcal{F}$ be a parametrised host family with slope records, with parameter ε and threshold x_1, and write A for the group of the frame of $\mathcal{F}$ at x. **Defect data for $\mathcal{F}$** consist of the following data, given for every real $x > x_1$ and every $i \in I(x)$, where m is the integer of $\mathfrak{M}_{x,i}$:

- real numbers $\mathrm{Df}_j(x,i)$ for $1 \leq j \leq m$, the **defect sums** of the member;
- an integer $\mathrm{rk}_0(x,i) \geq 0$, the **level rank**;
- integers $\mathrm{rk}_j(x,i) \geq 0$ for $1 \leq j \leq m$, the **zero-space ranks**.

A **parametrised host family with slope records and defect data** is a parametrised host family with slope records together with defect data for it.

In each item below, for a member (x,i) with $x > x_1$ we write $\mathfrak{V} = \mathfrak{V}_{x,i}$, $\mathfrak{M} = \mathfrak{M}_{x,i}$ with its parameters k, m and heights a_t ($t \in T_{x,i}$), $E = E_{\mathfrak{h}_{x,i}}$, L_a for the analytic loss of the member, and

$$n = |A|, \quad f = 2n, \quad b = e^{x^\varepsilon}, \quad a = xb, \quad L_a = \log\bigl(2a\, v(\mathfrak{h}_{x,i})\bigr),$$
$$A_{\mathfrak{M}} = \sum_{t \in T_{x,i}} a_t, \quad B_0 = B_0(x,i), \quad N_c = N_c(x,i).$$

The phrase "for all large members" means that there is a real x_0 such that the statement holds for every real $x \geq x_0$ with $x > x_1$, every $i \in I(x)$, all words $\omega, \omega' \in \mathcal{W}(\mathfrak{V})$ and every integer j with $1 \leq j \leq m$. Let e be a real number. The family has

(Dv1) Ranks and slopes if for all large members $d(\mathfrak{V},\omega) > 0$ and $\mu(\mathfrak{V},\omega) \leq \mu^{\max}(\mathfrak{V},\omega)$;

(Dv2) Non-negative size if $L_a \geq 0$ for all large members;

(Dv3) Exponent comparison if there is a real C such that for all large members $\bigl|E(\mathfrak{V},\omega) - m\, A_{\mathfrak{M}} \log(g_1(\mathfrak{V})g_2(\mathfrak{V}))\bigr| \leq C\, m\, B_0$;

(Dv4) Height comparison if there is a real C such that for all large members

$$\bigl|\bigl(h(\mathfrak{V},\omega) + A_{\mathfrak{M}} \log T(\mathfrak{V},\omega)\bigr) - \bigl(h(\mathfrak{V},\omega') + A_{\mathfrak{M}} \log T(\mathfrak{V},\omega')\bigr)\bigr| \leq C\bigl(|T_{x,i}|\, a\, L_a + m\, B_0\bigr);$$

(Dv5) Divisor bound from small defects, with exponent e if the following holds. Whenever there are real numbers C' and x_0' such that $\mathrm{Df}_j(x,i) \le \mathrm{rk}_0(x,i)\, C'\, x^{\varepsilon}\,(B_0 + N_c)$ for every real $x \ge x_0'$ with $x > x_1$, every $i \in I(x)$ and every $1 \le j \le m$, there is a real C such that for all large members

$$\mu^{\vee}(\mathfrak{V}) + h(\mathfrak{V},\omega) - F(\mathfrak{V}) \le \sum_{t \in T_{x,i}} \Big(m\big(a_t \log a_t - a_t - a_t \log \Omega(\mathfrak{h}_{x,i})\big) - a_t \log T(\mathfrak{V},\omega)\Big) + C|T_{x,i}|\, a\, L_a + C\, m\, (B_0 + N_c)\, x^{e\varepsilon}, \tag{13.1}$$

where $0 \log 0 = 0$;

(Dv6) Defects bounded by Sha lengths if there is a real C such that for all large members at which $\mathrm{len}_\ell(E,k) < \infty$ for every prime ℓ and every $1 \le k < t_{x,i}$, and $\mathrm{len}(E; t_{x,i}) < \infty$,

$$\mathrm{Df}_j(x,i) \le \mathrm{rk}_j(x,i)\Big(C\big(f\, m\, b\, x^{2\varepsilon} + (\log x)^2 N_c\big) + \mathrm{len}(E; t_{x,i})\Big);$$

(Dv7) Level ranks if $\mathrm{rk}_j(x,i) = \mathrm{rk}_0(x,i)$ for all large members.

Theorem 13.2. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, and let e be a real number. Assume that $\mathcal{F}$ has ranks and slopes (Dv1), non-negative size (Dv2), exponent comparison (Dv3), height comparison (Dv4), the divisor bound from small defects with exponent e (Dv5), defects bounded by Sha lengths (Dv6) and level ranks (Dv7). If $\mathcal{F}$ satisfies the Sha condition, then $\mathcal{F}$ has the divisor package with exponent e (Sp4), the expression $a_t \log a_t$ in (Sp4) being read as 0 when $a_t = 0$.*

Proof. Let $\varepsilon > 0$ be the parameter and $x_1 > 1$ the threshold of $\mathcal{F}$. Let C_3, C_4, C_6 be the constants of (Dv3), (Dv4), (Dv6), let $x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}, x^{(6)}, x^{(7)}$ be the thresholds of (Dv1), (Dv2), (Dv3), (Dv4), (Dv6), (Dv7), and let x_S be a real number such that $\mathrm{V}(E_{\mathfrak{h}_{x,i}}; t_{x,i}, e^{x^{\varepsilon}/2})$ holds for every real $x \ge x_S$ with $x > x_1$ and every $i \in I(x)$ (the Sha condition).

A logarithmic bound. Since $\log x / x^{\varepsilon/2} \to 0$ as $x \to \infty$, there is $x_L \ge 1$ with $0 \le \log x \le x^{\varepsilon/2}$ for all $x \ge x_L$; squaring, $(\log x)^2 \le x^{\varepsilon}$ for all $x \ge x_L$.

The defects are small. Put $X = \max(x_S, x^{(6)}, x^{(7)}, x_L)$. Let $x \ge X$ with $x > x_1$, let $i \in I(x)$, let m be the integer of $\mathfrak{M}_{x,i}$ and $1 \le j \le m$; write $t = t_{x,i}$, $E = E_{\mathfrak{h}_{x,i}}$, $y = x^{\varepsilon}$, $B_0 = B_0(x,i)$, $N_c = N_c(x,i)$. Then $x > 1$ and $\varepsilon > 0$ give $y \ge 1$; $B_0 \ge 0$ and $N_c \ge 0$ since $k \ge 0$; and $m \ge j \ge 1$.

- By the Sha condition, $\mathrm{len}_\ell(E,k) < \infty$ for every prime ℓ and $1 \le k < t$, and the series of non-negative terms $\sum_\ell \log \ell \sum_{k=1}^{t-1} \mathrm{len}_\ell(E,k)$ is at most $e^{x^{\varepsilon}/2}$; hence $\mathrm{len}(E;t) \le e^{x^{\varepsilon}/2} < \infty$. So the member satisfies the hypotheses of (Dv6), and since $x \ge x^{(6)}$,

$$\mathrm{Df}_j(x,i) \le \mathrm{rk}_j(x,i)\Big(C_6\big(f\, m\, b\, x^{2\varepsilon} + (\log x)^2 N_c\big) + \mathrm{len}(E;t)\Big).$$

- By (Dv7), $\mathrm{rk}_j(x,i) = \mathrm{rk}_0(x,i)$, since $x \ge x^{(7)}$.

- Put $P = f\, m\, b\, x^{2\varepsilon} + (\log x)^2 N_c$. Since $f = 2n$, $f\, m\, b\, x^{2\varepsilon} = y\, B_0$; since $x \ge x_L$ and $N_c \ge 0$, $(\log x)^2 N_c \le y\, N_c$. Hence $0 \le P \le y(B_0 + N_c)$ and $C_6 P \le |C_6| P \le |C_6|\, y(B_0 + N_c)$.

- $\mathrm{len}(E;t) \le e^{x^{\varepsilon}/2} \le e^{x^{\varepsilon}} = b$; and $b \le B_0 = f\, m\, y\, b$ because $f\, m\, y \ge 2 \cdot 1 \cdot 1$; and $B_0 \le y(B_0 + N_c)$ because $y \ge 1$ and $B_0, N_c \ge 0$. Hence $\mathrm{len}(E;t) \le y(B_0 + N_c)$.

Since $\mathrm{rk}_0(x,i) \geq 0$, the four points give

$$\mathrm{Df}_j(x,i) \leq \mathrm{rk}_0(x,i)\,(|C_6|+1)\,x^{\varepsilon}\,(B_0(x,i)+N_c(x,i)) \tag{13.2}$$

for every real $x \geq X$ with $x > x_1$, every $i \in I(x)$ and every $1 \leq j \leq m$.

The divisor bound. By (13.2), the hypothesis of (Dv5) holds with $C' = |C_6|+1$ and $x_0' = X$. Hence (Dv5) gives a real C_5 and a threshold $x^{(5)}$ such that the divisor bound (13.1) with the constant C_5 holds for all members (x,i) with $x \geq x^{(5)}$, $x > x_1$, and all words ω.

Assembly. Put $x_0 = \max(x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}, x^{(5)})$ and $C = \max(C_3, C_4, C_5, 0)$. Let $x \geq x_0$ with $x > x_1$, $i \in I(x)$ and $\omega, \omega' \in \mathcal{W}(\mathfrak{V}_{x,i})$. The error terms $m\,B_0$ of (Dv3), $|T_{x,i}|\,a\,L_a + m\,B_0$ of (Dv4), and $|T_{x,i}|\,a\,L_a$, $m(B_0+N_c)x^{e\varepsilon}$ of (13.1) are non-negative: $a = xb > 0$, $L_a \geq 0$ by (Dv2), $m, B_0, N_c \geq 0$ and $x^{e\varepsilon} > 0$. Each bound therefore stays true when its constant is replaced by the larger constant C. So, at the member (x,i) and for the words ω, ω': $d(\mathfrak{V}_{x,i}, \omega) > 0$ and $\mu(\mathfrak{V}_{x,i}, \omega) \leq \mu^{\max}(\mathfrak{V}_{x,i}, \omega)$ by (Dv1); the divisor bound of (Sp4) with constant C and exponent e is (13.1) with C (the convention $0 \log 0 = 0$ being the stated reading of (Sp4)); the exponent comparison of (Sp4) is (Dv3) with C; and the height comparison of (Sp4) is (Dv4) with C. This is (Sp4) with exponent e, the constant C and the threshold x_0. □

Definition 13.3. Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with parameter ε and threshold x_1, and write A for the group of the frame of $\mathcal{F}$ at x. **Bridge records for** $\mathcal{F}$ consist of the following real numbers, given for every real $x > x_1$ and every $i \in I(x)$, where m is the integer of $\mathfrak{M}_{x,i}$:

- $\mu_{\mathrm{b}}^{\vee}(x,i)$, the **base dual slope**, and $\mathrm{sc}(x,i)$, the **scaling term**;
- $\mathrm{pa}(x,i)$, the **passage allowance**;
- for every word $\omega \in \mathcal{W}(\mathfrak{V}_{x,i})$, $\mathrm{lm}(x,i,\omega)$, the **local main term**, and $\mathrm{le}(x,i,\omega)$, the **local error**;
- $\mathrm{tc}_j(x,i)$ for $1 \leq j \leq m$, the **twisted-place corrections**.

A **parametrised host family with slope records, defect data and bridge records** is a parametrised host family with slope records and defect data together with bridge records for it.

In each item below, for a member (x,i) with $x > x_1$ we write $\mathfrak{V} = \mathfrak{V}_{x,i}$, $\mathfrak{M} = \mathfrak{M}_{x,i}$ with its parameters k, m and heights a_t ($t \in T_{x,i}$), L_a for the analytic loss of the member, and

$$n = |A|, \quad b = e^{x^{\varepsilon}}, \quad a = xb, \quad L_a = \log(2a\,v(\mathfrak{h}_{x,i})), \quad B_0 = B_0(x,i), \quad N_c = N_c(x,i).$$

The phrase "for all large members" means that there is a real x_0 such that the statement holds for every real $x \geq x_0$ with $x > x_1$, every $i \in I(x)$ and every word $\omega \in \mathcal{W}(\mathfrak{V})$. The family has

(Bk1) Base divisor inequality if for all large members $\mathrm{rk}_0(x,i) \geq 1$ and

$$\mu_{\mathrm{b}}^{\vee}(x,i) - \mathrm{sc}(x,i) \leq \mathrm{lm}(x,i,\omega) + F(\mathfrak{V}) + \mathrm{le}(x,i,\omega) + \sum_{j=1}^{m}\Big(\frac{\mathrm{Df}_j(x,i)}{\mathrm{rk}_0(x,i)} + \mathrm{tc}_j(x,i)\Big); \tag{13.3}$$

(Bk2) Passage if for all large members $\mu^{\vee}(\mathfrak{V}) \leq \mu_{\mathrm{b}}^{\vee}(x,i) + \mathrm{pa}(x,i)$;

(Bk3) Record inequality if there is a real number C_0 such that for all large members, every word ω, every real $c \geq C_0$ and all real numbers $z, z_{\mathrm{b}}, z_{\mathrm{e}}$ with

$$z \leq z_{\mathrm{b}} + \mathrm{pa}(x,i), \qquad z_{\mathrm{b}} \leq \mathrm{lm}(x,i,\omega) + F(\mathfrak{V}) + z_{\mathrm{e}}, \qquad z_{\mathrm{e}} \leq c\,m\,x^{\varepsilon}(B_0+N_c),$$

one has

$$z + h(\mathfrak{V},\omega) + \mathrm{sc}(x,i) - F(\mathfrak{V}) \le \sum_{t\in T_{x,i}} \Big(m\big(a_t \log a_t - a_t - a_t \log \Omega(\mathfrak{h}_{x,i})\big) - a_t \log T(\mathfrak{V},\omega)\Big) + (50C_0+10)|T_{x,i}|\, a\, L_a + 60\, c\, m\, x^{\varepsilon}(B_0+N_c), \tag{13.4}$$

where $0 \log 0 = 0$;

(Bk4) Error budget if there is a real number c_{G} such that for all large members $\mathrm{le}(x,i,\omega) + \sum_{j=1}^{m} \mathrm{tc}_j(x,i) \le c_{\mathrm{G}}\, m\, x^{\varepsilon}(B_0+N_c)$.

Theorem 13.4. *Let $\mathcal{F}$ be a parametrised host family with slope records, defect data and bridge records. Assume that $\mathcal{F}$ has non-negative size (Dv2), the base divisor inequality (Bk1), the passage (Bk2), the record inequality (Bk3) and the error budget (Bk4). Then $\mathcal{F}$ has the divisor bound from small defects (Dv5) with exponent e for every real $e \ge 1$; in particular with exponent 1.*

Proof. Let $\varepsilon > 0$ be the parameter and $x_1 > 1$ the threshold of $\mathcal{F}$, and let $e \ge 1$ be real. Let $x^{(2)}$ be the threshold of (Dv2), $x^{(\mathrm{b1})}, x^{(\mathrm{b2})}, x^{(\mathrm{b3})}, x^{(\mathrm{b4})}$ those of (Bk1)–(Bk4), C_0 the constant of (Bk3) and c_{G} that of (Bk4).

To verify (Dv5) with exponent e, let C' and x_0' be real numbers such that

$$\mathrm{Df}_j(x,i) \le \mathrm{rk}_0(x,i)\, C'\, x^{\varepsilon}\big(B_0(x,i) + N_c(x,i)\big) \tag{13.5}$$

for every real $x \ge x_0'$ with $x > x_1$, every $i \in I(x)$ and every $1 \le j \le m$. Put

$$c = \max\big(C_0,\ c_{\mathrm{G}} + |C'|\big), \qquad C = \max\big(50C_0+10,\ 60c,\ 0\big),$$
$$x_0 = \max\big(x_0', x^{(2)}, x^{(\mathrm{b1})}, x^{(\mathrm{b2})}, x^{(\mathrm{b3})}, x^{(\mathrm{b4})}\big).$$

Let $x \ge x_0$ with $x > x_1$, let $i \in I(x)$ and $\omega \in \mathcal{W}(\mathfrak{V}_{x,i})$; write $\mathfrak{V} = \mathfrak{V}_{x,i}$, m, k, a_t for the data of $\mathfrak{M}_{x,i}$, L_a for the analytic loss of the member, $a = xb$, $B_0 = B_0(x,i)$, $N_c = N_c(x,i)$, and

$$U = x^{\varepsilon}(B_0 + N_c).$$

The unit is non-negative. Since $x > 1$, $x^{\varepsilon} > 0$ and $b > 0$; with $n \ge 1$, $m \ge 0$ and $k \ge 0$ this gives $B_0 \ge 0$ and $N_c \ge 0$, hence $U \ge 0$ and $mU \ge 0$.

The defect term is at most $|C'|\, m\, U$. By (Bk1), $\mathrm{rk}_0(x,i) \ge 1$. For $1 \le j \le m$, dividing (13.5) by $\mathrm{rk}_0(x,i) > 0$ gives $\mathrm{Df}_j(x,i)/\mathrm{rk}_0(x,i) \le C'U \le |C'|U$, the last step because $U \ge 0$. Summing over the m values of j,

$$\sum_{j=1}^{m} \frac{\mathrm{Df}_j(x,i)}{\mathrm{rk}_0(x,i)} \le |C'|\, m\, U.$$

The error of the base divisor inequality is within the record's budget. Put

$$z_{\mathrm{e}} = \mathrm{le}(x,i,\omega) + \sum_{j=1}^{m}\Big(\frac{\mathrm{Df}_j(x,i)}{\mathrm{rk}_0(x,i)} + \mathrm{tc}_j(x,i)\Big) = \Big(\mathrm{le}(x,i,\omega) + \sum_{j=1}^{m} \mathrm{tc}_j(x,i)\Big) + \sum_{j=1}^{m} \frac{\mathrm{Df}_j(x,i)}{\mathrm{rk}_0(x,i)}.$$

By (Bk4) the first bracket is at most $c_{\mathrm{G}}\, mU$; with the previous display, $z_{\mathrm{e}} \le (c_{\mathrm{G}} + |C'|)\, mU \le c\, mU$, the last step because $mU \ge 0$ and $c_{\mathrm{G}} + |C'| \le c$.

The record inequality applies. Put $z_{\mathrm{b}} = \mu_{\mathrm{b}}^{\vee}(x,i) - \mathrm{sc}(x,i)$ and $z = \mu^{\vee}(\mathfrak{V}) - \mathrm{sc}(x,i)$. By the passage (Bk2), $z \le z_{\mathrm{b}} + \mathrm{pa}(x,i)$. By the base divisor inequality (13.3) and the definition of z_{e}, $z_{\mathrm{b}} \le \mathrm{lm}(x,i,\omega) + F(\mathfrak{V}) + z_{\mathrm{e}}$. With $z_{\mathrm{e}} \le c\,m\,x^{\varepsilon}(B_0+N_c)$ and $c \ge C_0$, the record inequality (13.4) holds for $(c, z, z_{\mathrm{b}}, z_{\mathrm{e}})$. Its left side is $z + h(\mathfrak{V},\omega) + \mathrm{sc}(x,i) - F(\mathfrak{V}) = \mu^{\vee}(\mathfrak{V}) + h(\mathfrak{V},\omega) - F(\mathfrak{V})$. Hence

$$\mu^{\vee}(\mathfrak{V}) + h(\mathfrak{V},\omega) - F(\mathfrak{V}) \le \mathrm{Main} + (50C_0+10)|T_{x,i}|\,a\,L_a + 60\,c\,m\,x^{\varepsilon}(B_0+N_c),$$

where $\mathrm{Main} = \sum_{t\in T_{x,i}} \big(m(a_t \log a_t - a_t - a_t \log \Omega(\mathfrak{h}_{x,i})) - a_t \log T(\mathfrak{V},\omega)\big)$ with $0\log 0 = 0$, which is the main term of (13.1).

Passing to the constant C and the exponent e. By (Dv2), $L_a \ge 0$; with $a > 0$, $|T_{x,i}|\,a\,L_a \ge 0$, so $(50C_0+10)|T_{x,i}|\,a\,L_a \le C|T_{x,i}|\,a\,L_a$. Since $m(B_0+N_c) \ge 0$ and $x^{\varepsilon} > 0$, $60\,c\,m\,x^{\varepsilon}(B_0+N_c) \le C\,m\,(B_0+N_c)\,x^{\varepsilon}$. Finally $x > 1$ and $e\varepsilon \ge \varepsilon$ give $x^{\varepsilon} \le x^{e\varepsilon}$, and $C\,m\,(B_0+N_c) \ge 0$, so $C\,m\,(B_0+N_c)\,x^{\varepsilon} \le C\,m\,(B_0+N_c)\,x^{e\varepsilon}$. Therefore

$$\mu^{\vee}(\mathfrak{V}) + h(\mathfrak{V},\omega) - F(\mathfrak{V}) \le \mathrm{Main} + C|T_{x,i}|\,a\,L_a + C\,m\,(B_0+N_c)\,x^{e\varepsilon},$$

which is (13.1) with the constant C. Since C and x_0 depend only on $\mathcal{F}$, C' and x_0', and the member (x,i) with $x \ge x_0$, $x > x_1$, and the word ω were arbitrary, $\mathcal{F}$ has (Dv5) with exponent e. □

Definition 13.5. Let $\mathcal{F}$ be a parametrised host family with slope records, defect data and bridge records, with parameter ε and threshold x_1. **Numerical records for $\mathcal{F}$** consist of the following data, given for every real $x > x_1$, every $i \in I(x)$ and every word $\omega \in \mathcal{W}(\mathfrak{V}_{x,i})$, where m and k are the integers of $\mathfrak{M}_{x,i}$:

- a finite set $\mathrm{Pl}(x,i,\omega)$ containing $T_{x,i}$, the **record places**;
- for $1 \le j \le m$ a real number $\mathrm{ed}_j(x,i,\omega)$, the **eigenvalue degree** of the j-th lift;
- for $1 \le j \le m$ and $\rho \in \mathrm{Pl}(x,i,\omega)$ real numbers $\mathrm{rw}_j(x,i,\omega,\rho)$, $\mathrm{ms}_j(x,i,\omega,\rho)$ and $\mathrm{en}_j(x,i,\omega,\rho)$, the **record weight**, the **weight mass** and the **entropy term**;
- a real number $\mathrm{eh}(x,i,\omega)$, the **height error**.

A **parametrised host family with slope records, defect data, bridge records and numerical records** is a parametrised host family with slope records, defect data and bridge records together with numerical records for it.

In the items below, for a member (x,i) with $x > x_1$ and a word ω, we write $\mathfrak{V} = \mathfrak{V}_{x,i}$, $\mathfrak{M} = \mathfrak{M}_{x,i}$ with k, m and heights a_t ($t \in T = T_{x,i}$), $b = e^{x^{\varepsilon}}$, $a = xb$, $L_a = \log(2a\,v(\mathfrak{h}_{x,i}))$ for the analytic loss of the member, $B_0 = B_0(x,i)$, $\log\Omega = \log\Omega(\mathfrak{h}_{x,i})$ and $\mathrm{Pl} = \mathrm{Pl}(x,i,\omega)$. We put $\varphi(y) = y\log y - y$ for $y \ge 0$ (with $0\log 0 = 0$). Within one item the arguments (x,i,ω) are suppressed, so that we write ed_j, $\mathrm{rw}_j(\rho)$, $\mathrm{ms}_j(\rho)$, $\mathrm{en}_j(\rho)$, eh. We let $[\rho \in T]$ be 1 if $\rho \in T$ and 0 otherwise; a_ρ and $\varphi(a_\rho)$ are used only for $\rho \in T$. The phrase "for all large members" means that there is a real x_0 such that the statement holds for every real $x \ge x_0$ with $x > x_1$, every $i \in I(x)$ and every word $\omega \in \mathcal{W}(\mathfrak{V})$. The family has

(Nr1) Height inequality of the record if for all large members

$$h(\mathfrak{V},\omega) + \mathrm{sc}(x,i) \le -\sum_{t\in T} a_t \log T(\mathfrak{V},\omega) + \sum_{j=1}^{m} \sum_{\rho\in\mathrm{Pl}} \big(\mathrm{en}_j(\rho) - \mathrm{ms}_j(\rho)\log\Omega\big) + \mathrm{eh}; \quad (13.6)$$

(Nr2) Main term of the realisation if for all large members

$$\mathrm{lm}(x,i,\omega)=\sum_{j=1}^{m}\sum_{\rho\in\mathrm{Pl}}\mathrm{ed}_j\,\mathrm{rw}_j(\rho)\log\Omega;$$

(Nr3) Record budget if there is a real number C_{N} such that for all large members

$$\mathrm{pa}(x,i)+\mathrm{eh}+\sum_{j=1}^{m}\sum_{\rho\in\mathrm{Pl}}\Big((\mathrm{en}_j(\rho)-[\rho\in T]\,\varphi(a_\rho))+|\mathrm{ms}_j(\rho)-[\rho\in T]\,a_\rho-k\,\mathrm{rw}_j(\rho)|\,|\log\Omega|$$
$$+\,|\mathrm{ed}_j-k|\,|\mathrm{rw}_j(\rho)|\,|\log\Omega|\Big)\le C_{\mathrm{N}}(|T|\,a\,L_a+m\,x^{\varepsilon}B_0).$$

Lemma 13.6. *Let $\mathcal{F}$ be a parametrised host family with slope records, defect data, bridge records and numerical records that has the height inequality of the record (Nr1), the main term of the realisation (Nr2) and the record budget (Nr3). Then $\mathcal{F}$ has the record inequality (Bk3), with $C_0=\max(C_{\mathrm{N}},0)$, where C_{N} is the constant of (Nr3). More precisely, for all large members, every word ω, every real $c\ge C_0$ and all real $z,z_{\mathrm{b}},z_{\mathrm{e}}$ with $z\le z_{\mathrm{b}}+\mathrm{pa}(x,i)$, $z_{\mathrm{b}}\le\mathrm{lm}(x,i,\omega)+F(\mathfrak{V})+z_{\mathrm{e}}$ and $z_{\mathrm{e}}\le c\,m\,x^{\varepsilon}(B_0+N_c)$,*

$$z+h(\mathfrak{V},\omega)+\mathrm{sc}(x,i)-F(\mathfrak{V})\le\sum_{t\in T_{x,i}}\Big(m(a_t\log a_t-a_t-a_t\log\Omega(\mathfrak{h}_{x,i}))-a_t\log T(\mathfrak{V},\omega)\Big)$$
$$+\,C_0|T_{x,i}|\,a\,L_a+2c\,m\,x^{\varepsilon}(B_0+N_c),$$

with $0\log 0=0$ and the notation of (Bk3).

Proof. Let x_0 be the largest of the thresholds of (Nr1), (Nr2), (Nr3), and $C_0=\max(C_{\mathrm{N}},0)$. Fix $x\ge x_0$ with $x>x_1$, $i\in I(x)$, a word ω, a real $c\ge C_0$ and reals $z,z_{\mathrm{b}},z_{\mathrm{e}}$ as in the statement. Write $T=T_{x,i}$, $\mathrm{Pl}=\mathrm{Pl}(x,i,\omega)$, $\log\Omega=\log\Omega(\mathfrak{h}_{x,i})$, $\log T_\omega=\log T(\mathfrak{V},\omega)$, $\sum_{j,\rho}$ for $\sum_{j=1}^{m}\sum_{\rho\in\mathrm{Pl}}$, and

$$\mathrm{wd}_j(\rho)=\mathrm{ms}_j(\rho)-[\rho\in T]\,a_\rho-k\,\mathrm{rw}_j(\rho)\qquad\text{(local symbol)}.$$

Signs. By (Pf3) and (Lm7), $k>0$; the analytic loss satisfies $L_a=\log a+\log(2v(\mathfrak{h}_{x,i}))\ge\log a>0$, since $2v(\mathfrak{h}_{x,i})=|\mathrm{disc}(E_{\mathfrak{h}_{x,i}})|^{1/[E_{\mathfrak{h}_{x,i}}:\mathbb{Q}]}\ge 1$ and $a=x\,e^{x^{\varepsilon}}>1$ (as $x>x_1>1$); by (Pf2) and (Lm2), $a_t\ge 0$, so $\varphi(a_t)$ is defined; $m\ge 0$, $a>0$, $x^{\varepsilon}>0$, $B_0\ge 0$ and $N_c\ge 0$. Hence $|T|\,a\,L_a\ge 0$ and $m\,x^{\varepsilon}(B_0+N_c)\ge m\,x^{\varepsilon}B_0\ge 0$; and $c\ge C_0\ge 0$.

Chaining the hypotheses. From $z\le z_{\mathrm{b}}+\mathrm{pa}(x,i)$ and $z_{\mathrm{b}}\le\mathrm{lm}(x,i,\omega)+F(\mathfrak{V})+z_{\mathrm{e}}$, and then (13.6),

$$z+h(\mathfrak{V},\omega)+\mathrm{sc}(x,i)-F(\mathfrak{V})\le\mathrm{lm}(x,i,\omega)+z_{\mathrm{e}}+\mathrm{pa}(x,i)-\sum_{t\in T}a_t\log T_\omega$$
$$+\sum_{j,\rho}(\mathrm{en}_j(\rho)-\mathrm{ms}_j(\rho)\log\Omega)+\mathrm{eh}.$$

Separating the main term. Since $T\subseteq\mathrm{Pl}$ and j takes m values, $\sum_{j,\rho}[\rho\in T]\varphi(a_\rho)=m\sum_{t\in T}\varphi(a_t)$ and $\sum_{j,\rho}[\rho\in T]a_\rho=m\sum_{t\in T}a_t$. Writing $\mathrm{en}_j(\rho)=[\rho\in T]\varphi(a_\rho)+(\mathrm{en}_j(\rho)-[\rho\in T]\varphi(a_\rho))$ and $\mathrm{ms}_j(\rho)=[\rho\in T]a_\rho+k\,\mathrm{rw}_j(\rho)+\mathrm{wd}_j(\rho)$,

$$\sum_{j,\rho}(\mathrm{en}_j(\rho)-\mathrm{ms}_j(\rho)\log\Omega)=m\sum_{t\in T}(\varphi(a_t)-a_t\log\Omega)-k\sum_{j,\rho}\mathrm{rw}_j(\rho)\log\Omega$$
$$+\sum_{j,\rho}\Big((\mathrm{en}_j(\rho)-[\rho\in T]\varphi(a_\rho))-\mathrm{wd}_j(\rho)\log\Omega\Big).$$

By (Nr2),

$$\mathrm{lm}(x,i,\omega) - k\sum_{j,\rho}\mathrm{rw}_j(\rho)\log\Omega = \sum_{j,\rho}(\mathrm{ed}_j - k)\,\mathrm{rw}_j(\rho)\log\Omega \le \sum_{j,\rho}|\mathrm{ed}_j - k|\,|\mathrm{rw}_j(\rho)|\,|\log\Omega|,$$

and $-\mathrm{wd}_j(\rho)\log\Omega \le |\mathrm{wd}_j(\rho)|\,|\log\Omega|$. With $\mathrm{Main} = \sum_{t\in T}\big(m(\varphi(a_t) - a_t\log\Omega) - a_t\log T_\omega\big)$ we obtain

$$z + h(\mathfrak{V},\omega) + \mathrm{sc}(x,i) - F(\mathfrak{V}) \le \mathrm{Main} + z_{\mathrm{e}} + \mathcal{E},$$

where $\mathcal{E}$ (local symbol) is the left side of the record budget (Nr3).

The errors. By (Nr3), $\mathcal{E} \le C_{\mathrm{N}}(|T|aL_a + mx^{\varepsilon}B_0) \le C_0|T|aL_a + C_0\,mx^{\varepsilon}B_0$, using the signs. Since $C_0 \le c$ and $mx^{\varepsilon}B_0 \le mx^{\varepsilon}(B_0+N_c)$, and $z_{\mathrm{e}} \le c\,mx^{\varepsilon}(B_0+N_c)$,

$$z + h(\mathfrak{V},\omega) + \mathrm{sc}(x,i) - F(\mathfrak{V}) \le \mathrm{Main} + C_0|T|\,a\,L_a + 2c\,m\,x^{\varepsilon}(B_0+N_c),$$

which is the displayed inequality. As $C_0 \le 50C_0+10$, $2c \le 60c$, and $|T|aL_a \ge 0$, $mx^{\varepsilon}(B_0+N_c) \ge 0$, its right side is at most the right side of (13.4); x_0 and C_0 do not depend on the member, the word, c or $z, z_{\mathrm{b}}, z_{\mathrm{e}}$. Hence $\mathcal{F}$ has (Bk3) with the constant C_0. □

Notation for number fields, local fields and adeles is that of [190] and [63]. Let F be a number field with ring of integers $\mathcal{O}_F$. Places of F are written v, and finite places are identified with the nonzero primes of $\mathcal{O}_F$. We write F_v for the completion and $|\cdot|_v$ for its normalised absolute value. For finite v, ϖ_v is a uniformiser of $\mathcal{O}_{F_v}$ and $q_v = \mathrm{N}v$ is the cardinality of the residue field. For a finite extension E/F, $w \mid v$ means that the place w of E lies above v, and $\mathrm{N}_{E/F}$ is the norm (on elements, ideles and idele classes). A **non-archimedean local field** is a field complete for a discrete valuation with finite residue field. The adele ring is $\mathbb{A}_F = F_\infty \times \mathbb{A}_{F,f}$, where $F_\infty = F\otimes_{\mathbb{Q}}\mathbb{R}$ and $\mathbb{A}_{F,f}$ is the ring of finite adeles. On the ideles $\mathbb{A}_F^\times$ we set $|x|_{\mathbb{A}} = \prod_v |x_v|_v$. The idele class group is $C_F = \mathbb{A}_F^\times/F^\times$; C_F^1 is its subgroup of norm one and C_F^0 the connected component of the identity. We write $G_F = \mathrm{Gal}(\overline{F}/F)$; $G_{F_v} \subset G_F$ is a decomposition group, I_v its inertia subgroup and Frob_v an arithmetic Frobenius. For a finite set S of places, $G_{F,S}$ is the Galois group of the maximal extension of F unramified outside S.

Galois cohomology is continuous cohomology of profinite groups, as in [222] and [172]. For a discrete module M (or, for $\mathbb{Z}_\ell$-coefficients, a continuous ℓ-adic module M), we write $H^i(F,M)$ for $H^i(G_F,M)$ and $H^i(F_v,M)$ for $H^i(G_{F_v},M)$. Tate cohomology of a finite group is denoted $\hat{H}^i$. For a prime ℓ, χ_{cyc} is the ℓ-adic cyclotomic character, $\mathbb{Z}_\ell(k)$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell(k)$ are the twists by χ_{cyc}^k, and μ_{ℓ^r} is the group of ℓ^r-th roots of unity. For a finite $\mathbb{Z}/\ell^r\mathbb{Z}$-module M, the Cartier dual is $M^\vee = \mathrm{Hom}(M,\mu_{\ell^r})$. The unramified classes are

$$H^1_{\mathrm{ur}}(F_v,M) = \ker\big(H^1(F_v,M) \to H^1(I_v,M)\big),$$

and inv_v is the invariant map of the local Brauer group. For a finite set S of places containing those above ℓ and ∞,

$$\mathrm{Sha}^i_S(F,M) = \ker\Big(H^i(G_{F,S},M) \to \prod_{v\in S} H^i(F_v,M)\Big).$$

Local Tate duality, the local and global Euler characteristic formulas and Poitou–Tate duality are used in the form of [172], Ch. I, and the Greenberg–Wiles formula for Selmer groups in the form of [72].

Notation for p-adic Hodge theory is that of [94] and [52]. Let K be a finite extension of $\mathbb{Q}_p$, G_K its absolute Galois group and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$. For a perfectoid field L, $L^\flat$ denotes its tilt. Fix a compatible system $\underline{\zeta} = (\zeta_{p^n})_{n\ge 0}$ of p-power roots of unity. The period rings are the following:

- $A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}_p^\flat})$, with $\theta\colon A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_p}$;
- B_{dR}^+, the $\ker\theta$-adic completion of $A_{\mathrm{inf}}[1/p]$, and $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[1/t]$, where $t = \log[\underline{\zeta}]$, with $\mathrm{Fil}^i B_{\mathrm{dR}} = t^i B_{\mathrm{dR}}^+$;
- $A_{\mathrm{cris}} \subset B_{\mathrm{cris}}^+$ and $B_{\mathrm{cris}} = B_{\mathrm{cris}}^+[1/t]$, with Frobenius φ;
- B_{st}.

A finite-dimensional $\mathbb{Q}_p$-representation V of G_K is **de Rham**, **semistable** or **crystalline** if $(B \otimes_{\mathbb{Q}_p} V)^{G_K}$ has dimension $\dim_{\mathbb{Q}_p} V$ over B^{G_K} for $B = B_{\mathrm{dR}}, B_{\mathrm{st}}, B_{\mathrm{cris}}$ respectively. Hodge–Tate weights are normalised so that the cyclotomic character χ_{cyc} has Hodge–Tate weight 1.

Notation follows Tate's thesis [250] and [190], Ch. VII. A **Hecke character** of a number field F is a continuous character $\chi = \prod_v \chi_v\colon C_F \to \mathbb{C}^\times$. Hecke characters are unitary unless stated otherwise. Fix a subgroup of C_F that $|\cdot|_{\mathbb{A}}$ maps isomorphically onto $\mathbb{R}_{>0}$. Every unitary Hecke character can be written $\chi = \chi_0 |\cdot|_{\mathbb{A}}^{i\tau}$ with a unique $\tau \in \mathbb{R}$ and χ_0 trivial on this subgroup. The conductor of χ is $\mathfrak{f}(\chi)$. We write $\mathcal{S}(\mathbb{A}_F)$ for the Schwartz–Bruhat space, $\psi_{\mathrm{add}} = \prod_v \psi_{\mathrm{add},v}$ for the standard additive character of $\mathbb{A}_F/F$ (trivial on F), and $\hat{f}$ for the Fourier transform with respect to the self-dual measure. The **Tate zeta integral** $Z(f,\chi,s)$ is $\int_{\mathbb{A}_F^\times} f(x)\chi(x)|x|_{\mathbb{A}}^s\, d^\times x$ for $\mathrm{Re}\, s > 1$, together with its meromorphic continuation; it satisfies $Z(f,\chi,s) = Z(\hat{f},\chi^{-1},1-s)$. The **Hecke L-function** is $L(s,\chi) = \prod_{v\nmid\infty,\ \chi_v \text{ unramified}} (1-\chi_v(\varpi_v)q_v^{-s})^{-1}$ for $\mathrm{Re}\, s > 1$. For a Dirichlet character, viewed as a Hecke character of $\mathbb{Q}$ through class field theory, this is the Dirichlet L-function of the inducing primitive character. The **completed L-function** is $\xi(s,\chi) = (|d_F|\,\mathrm{N}\mathfrak{f}(\chi))^{s/2} L_\infty(s,\chi)L(s,\chi)$, with the archimedean gamma factors L_∞. The **root number** $W(\chi)$ is defined by $\xi(s,\chi) = W(\chi)\,\xi(1-s,\overline{\chi})$; it satisfies $|W(\chi)| = 1$. The Dirichlet density of a set of primes is meant in the sense of [190], VII.13.

Let E/F be a CM extension of number fields with complex conjugation c, and write $x \mapsto \bar{x}$ for c. Let Y be a finite-dimensional E-vector space with a nondegenerate form $\langle\,,\,\rangle$ that is hermitian ($\langle y,x\rangle = \overline{\langle x,y\rangle}$) or skew-hermitian ($\langle y,x\rangle = -\overline{\langle x,y\rangle}$). Then $\mathrm{U}(Y)$ denotes its unitary group over F, and $[\mathrm{U}(Y)] = \mathrm{U}(Y)(F)\backslash\mathrm{U}(Y)(\mathbb{A}_F)$. The signature of Y at a real place w of F is that of the hermitian form $\langle\,,\,\rangle$, resp. $i\langle\,,\,\rangle$, at the complex place of E above w.

Let V be a hermitian space and W a skew-hermitian space. Then $V \otimes_E W$ with the form $\mathrm{tr}_{E/F}(\langle\,,\,\rangle_V \otimes \overline{\langle\,,\,\rangle_W})$ is a symplectic F-space, and $(\mathrm{U}(V),\mathrm{U}(W))$ is a dual pair in its symplectic group. Let η be the quadratic character of E/F. Fix a nontrivial additive character ψ_{add} of $\mathbb{A}_F/F$ and Hecke characters χ_V, χ_W of $\mathbb{A}_E^\times/E^\times$ whose restrictions to $\mathbb{A}_F^\times$ are $\eta^{\dim V}$, resp. $\eta^{\dim W}$. Then $\omega = \omega_{\psi_{\mathrm{add}},\chi_V,\chi_W}$ denotes the Weil representation of $\mathrm{U}(V)(\mathbb{A}_F) \times \mathrm{U}(W)(\mathbb{A}_F)$ on a model of the Schwartz space $\mathcal{S}(V^{\dim W}(\mathbb{A}_F))$ (Schrödinger or mixed model) [151], [120]. For φ in this space, set

$$\theta(g,h;\varphi) = \sum_{x \in V^{\dim W}(F)} (\omega(g,h)\varphi)(x), \qquad \Theta(\varphi)(h) = \int_{[\mathrm{U}(V)]} \theta(g,h;\varphi)\, dg,$$

when the integral converges. The function $\Theta(\varphi)$ on $[\mathrm{U}(W)]$ is the **theta integral** of φ (the theta lift of the trivial representation). Let P be a parabolic subgroup of $\mathrm{U}(W)$ stabilising an isotropic line $Ex \subset W$. Its Levi factor is $\mathrm{Res}_{E/F}\mathbb{G}_m \times \mathrm{U}(W_1)$, where $W_1 = \langle x,y\rangle^\perp$ for an isotropic y with $\langle x,y\rangle = 1$. The constant term of a function f on $[\mathrm{U}(W)]$ along P is $f_P(h) = \int_{N(F)\backslash N(\mathbb{A}_F)} f(nh)\, dn$, where N is the unipotent radical and the measure has total mass 1. This notation holds throughout the parts of this paper on theta lifting.

Class field theory is normalised as in [190] and [11]: uniformisers correspond to arithmetic Frobenius elements. For a non-archimedean local field F_v, $\mathrm{Art}_{F_v}\colon F_v^\times \to G_{F_v}^{ab}$ is the local Artin

map and F_v^{ur} the maximal unramified extension; for a number field F, $\mathrm{Art}_F\colon C_F \to G_F^{ab}$ is the global Artin map and F^{ab} the maximal abelian extension. The **norm group** of a finite abelian extension E/F (respectively E_w/F_v) is $\mathrm{N}_{E/F}C_E \subseteq C_F$ (respectively $\mathrm{N}_{E_w/F_v}E_w^\times \subseteq F_v^\times$).

Definition 13.7. Let p be a prime, K a finite extension of $\mathbb{Q}_p$, L a finite extension of $\mathbb{Q}_p$, and V a finite-dimensional L-vector space with a continuous L-linear action of G_K. Put

$$D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}, \qquad \mathrm{Fil}^j D_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V) \cap (\mathrm{Fil}^j B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \quad (j \in \mathbb{Z}).$$

This is a module over $K \otimes_{\mathbb{Q}_p} L$, where $K \subset B_{\mathrm{dR}}$ acts on the first factor and L on V, with a decreasing filtration by $K \otimes_{\mathbb{Q}_p} L$-submodules. For a $\mathbb{Q}_p$-embedding $\tau\colon K \to L$ let

$$D_{\mathrm{dR},\tau}(V) = L \otimes_{K\otimes_{\mathbb{Q}_p} L,\, \tau\otimes\mathrm{id}} D_{\mathrm{dR}}(V),$$

with the filtration $\mathrm{Fil}^j D_{\mathrm{dR},\tau}(V)$ given by the image of $L \otimes \mathrm{Fil}^j D_{\mathrm{dR}}(V)$. An integer a is a **Hodge jump of V at τ** if $\mathrm{Fil}^{a+1} D_{\mathrm{dR},\tau}(V) \neq \mathrm{Fil}^a D_{\mathrm{dR},\tau}(V)$.

If $F \subseteq K$ is a subfield and $\tau_0\colon F \to L$ is a $\mathbb{Q}_p$-embedding, the **embeddings of K over τ_0** are the $\mathbb{Q}_p$-embeddings $\tau\colon K \to L$ with $\tau|_F = \tau_0$.

We fix the notation for ℓ-adic coefficients, Galois representations and automorphic representations used in this paper, following [69] and [17].

Coefficients. Let ℓ be a prime, let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$ with absolute value $|\cdot|_\ell$ and valuation v_ℓ normalised by $v_\ell(\ell) = 1$, and let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism of fields. A **coefficient ring** is a $\mathbb{Z}_\ell$-subalgebra $\mathcal{O} \subset \overline{\mathbb{Q}}_\ell$ that is a discrete valuation ring finite over $\mathbb{Z}_\ell$. Its ramification index is $e_{\mathcal{O}} = \mathrm{length}_{\mathcal{O}}(\mathcal{O}/\ell\mathcal{O})$. For an $\mathcal{O}$-module M of finite length, the **normalised length** of M is $\mathrm{length}_{\mathcal{O}}(M)/e_{\mathcal{O}}$, so that $\ell^{[\mathcal{O}:\mathbb{Z}_\ell]\cdot \mathrm{length}_{\mathcal{O}}(M)/e_{\mathcal{O}}} = \#M$. For a character θ of a group with values in $\mathcal{O}^\times$, we write $\mathcal{O}^\vee(\theta)$ for $\mathcal{O} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ with the discrete topology and with the group acting through θ. Dirichlet characters with values in $\mathcal{O}^\times$ are identified with characters of $G_{\mathbb{Q}}$ by class field theory.

Galois representations. An ℓ-adic representation of G_E is a continuous homomorphism $r\colon G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. We write $r^\vee$ for its contragredient. For a CM field E with complex conjugation c, we set $r^c(g) = r(cgc)$. Two representations are **isomorphic** if they are conjugate in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$.

Automorphic side. For a number field E, the notion of a **regular algebraic** cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ is that of [68]. For E CM, π is **conjugate self-dual** if $\pi^\vee \cong \pi^c$. An ℓ-adic representation r of G_E is **attached to** π, written $r \leftrightarrow_\iota \pi$, if the following holds for all but finitely many finite places w of E at which π is unramified: r is unramified at w, and ι carries the characteristic polynomial of $r(\mathrm{Frob}_w^{-1})$ (a geometric Frobenius) to $\prod_\alpha (X - \mathrm{N}w^{(n-1)/2}\alpha)$, where α runs over the Satake parameters of π_w.

The **Satake parameters** of π_w are defined as follows. Let $K_w = \mathrm{GL}_n(\mathcal{O}_{E_w})$. For a uniformiser ϖ_w and $0 \le j \le n$, let $d_j = \mathrm{diag}(\varpi_w, \dots, \varpi_w, 1, \dots, 1)$ with j entries ϖ_w, and let T_j be the Hecke operator of the double coset $K_w d_j K_w$, acting on $\pi_w^{K_w}$ by $x \mapsto \sum_{gK_w \subseteq K_w d_j K_w} \pi_w(g)x$. The Satake parameters of π_w form the multiset $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{C}^\times$ such that, for every uniformiser ϖ_w and every such j, the operator T_j is the scalar

$$T_j = \mathrm{N}w^{j(n-j)/2}\, e_j(\alpha_1, \dots, \alpha_n),$$

where e_j is the j-th elementary symmetric polynomial. They are the values $\chi_i(\varpi_w)$ of unramified characters $\chi_1, \dots, \chi_n$ of $E_w^\times$ such that π_w is a subquotient of the normalised induction of $\chi_1 \otimes \cdots \otimes \chi_n$ [60], [115]. Thus the relation $r \leftrightarrow_\iota \pi$ is the normalisation $\mathrm{rec}(\pi_w \otimes |\det|^{(1-n)/2})$ of

[121]. For E CM, r is **automorphic of conjugate self-dual type** if $r \leftrightarrow_\iota \pi$ for some regular algebraic conjugate self-dual cuspidal π. A normalised newform f of weight k, level $\Gamma_1(M)$ and character χ has the Galois representation $\rho_{f,\iota}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ constructed in [76].

Algebraic Hecke characters. Let E be a number field and let $n = (n_\varphi)_\varphi$ be integers indexed by the field embeddings $\varphi\colon E \to \mathbb{C}$. A (not necessarily unitary) Hecke character χ of E is **algebraic of infinity type** n if

$$\chi_\infty(k) = \prod_{\varphi\colon E\to\mathbb{C}} \varphi(k)^{n_\varphi} \qquad \text{for every } k \in E^\times \text{ positive at every real place},$$

where $\chi_\infty = \prod_{v\mid\infty}\chi_v$ and k is embedded diagonally in $\prod_{v\mid\infty} E_v^\times$. For these characters the global Artin map is used in the **geometric normalisation** $\mathrm{Art}_E^{\mathrm{geo}}(x) = \mathrm{Art}_E(x)^{-1}$, which sends uniformisers to geometric Frobenius elements. The **ℓ-adic character attached to** (χ,ι) is the continuous character $\chi_\iota\colon G_E \to \overline{\mathbb{Q}}_\ell^\times$ such that

$$\chi_\iota(\sigma) = \iota^{-1}\Big(\chi(x)\prod_{v\mid\infty}\prod_{i} i(x_v)^{-n_i}\Big)\cdot\prod_{w\mid\ell}\prod_{j} j(x_w)^{n_{\iota\circ j}}$$

for every $x \in \mathbb{A}_E^\times$ and every $\sigma \in G_E$ whose image in G_E^{ab} is $\mathrm{Art}_E^{\mathrm{geo}}(x)$. Here i runs over the continuous embeddings $E_v \to \mathbb{C}$, j runs over the continuous embeddings $E_w \to \overline{\mathbb{Q}}_\ell$, and n_i, $n_{\iota\circ j}$ are the exponents of their restrictions to E. This character exists and is unique [220].

Sign of the Hodge jumps. With these normalisations, let $w \nmid \ell$ be a finite place at which χ is unramified. Then χ_ι is unramified at w and $\chi_\iota(\mathrm{Frob}_w^{-1}) = \iota^{-1}\chi_w(\varpi_w)$ for an arithmetic Frobenius Frob_w and a uniformiser ϖ_w. At a place $w \mid \ell$, the restriction $\chi_\iota|_{G_{E_w}}$ is de Rham and, for every embedding $\tau\colon E \to \overline{\mathbb{Q}}_\ell$ inducing w, its only Hodge jump at τ (in the sense of Definition 13.7) is $-n_{\iota\circ\tau}$. In other words, its Hodge–Tate weight at τ is $n_{\iota\circ\tau}$ when χ_{cyc} has weight 1 [220], [94]. For $E = \mathbb{Q}$ and $\chi = |\cdot|_{\mathbb{A}}$, of infinity type 1, we have $\chi_\iota = \chi_{\mathrm{cyc}}$, with Hodge jump -1. The character defined by the same formula with Art_E in place of $\mathrm{Art}_E^{\mathrm{geo}}$ is χ_ι^{-1}, with Hodge jump $n_{\iota\circ\tau}$ at τ.

Definition 13.8. Let ℓ be a prime. A **Hecke lattice over** ℓ is a tuple $(\mathcal{O}, \mathcal{Z}, \mathbb{T}, M)$ consisting of

- a coefficient ring $\mathcal{O}$ for ℓ, with fraction field $\mathcal{O}[\ell^{-1}]$;
- a non-empty finite set $\mathcal{Z}$, whose elements are the **eigensystems**;
- an $\mathcal{O}$-subalgebra $\mathbb{T}$ of the algebra $\mathcal{O}^{\mathcal{Z}}$ of functions $\mathcal{Z} \to \mathcal{O}$ (pointwise operations, containing the constants) which separates the points of $\mathcal{Z}$, the **Hecke algebra**;
- a finitely generated free $\mathcal{O}$-module M with a homomorphism of $\mathcal{O}$-algebras $\mathbb{T} \to \mathrm{End}_{\mathcal{O}}(M)$, $f \mapsto f_M$.

Put $V = \mathcal{O}[\ell^{-1}] \otimes_{\mathcal{O}} M$ and regard M as an $\mathcal{O}$-submodule of V (possible since M is free). An $\mathcal{O}$-linear endomorphism α of M extends uniquely to an $\mathcal{O}[\ell^{-1}]$-linear endomorphism α_V of V; we write $f_V = (f_M)_V$. For $\pi \in \mathcal{Z}$ the **eigenspace** and the **multiplicity** of π are

$$V_\pi = \{v \in V :\ f_V v = f(\pi)\, v \text{ for all } f \in \mathbb{T}\}, \qquad m(\pi) = \dim_{\mathcal{O}[\ell^{-1}]} V_\pi,$$

and for a subset $Y \subseteq \mathcal{Z}$ we put $V_Y = \sum_{\pi\in Y} V_\pi$ (so $V_\emptyset = 0$).

Let $Z \subseteq \widehat{Z} \subseteq \mathcal{Z}$; Z is called the **good set** and $\widehat{Z}$ the **kept set**. Let $q\colon V \to V/V_{\mathcal{Z}\setminus\widehat{Z}}$ be the quotient map.

- The **congruence module** of M for Z and the **local defect** are

$$\mathrm{Cong}_Z(M) = \frac{V_Z \cap (M + V_{\mathcal{Z}\setminus Z})}{M \cap V_Z}, \qquad \mathrm{df}_Z(M) \in [0,\infty],$$

  where $\mathrm{df}_Z(M)$ is the normalised length of the $\mathcal{O}$-module $\mathrm{Cong}_Z(M)$ if it has finite length, and ∞ otherwise.

- The **cut lattice** for $\widehat{Z}$ is $q(M)$, and the **cut congruence module** and the **cut local defect** are

$$\mathrm{Cong}_Z^{\widehat{Z}}(M) = \frac{q(V_Z) \cap (q(M) + q(V_{\widehat{Z}\setminus Z}))}{q(M) \cap q(V_Z)}, \qquad \mathrm{df}_Z^{\widehat{Z}}(M) \in [0,\infty],$$

  the latter being the normalised length of $\mathrm{Cong}_Z^{\widehat{Z}}(M)$ if it has finite length, and ∞ otherwise. For $\widehat{Z} = \mathcal{Z}$, q is the identity and $\mathrm{df}_Z^{\mathcal{Z}}(M) = \mathrm{df}_Z(M)$.

- For $\pi \in Z$ the **congruence ideal** of π for $(Z, \widehat{Z})$ and its normalised length are

$$\mathfrak{c}_Z^{\widehat{Z}}(\pi) = \{f(\pi) :\ f \in \mathbb{T},\ f(\rho) = 0 \text{ for all } \rho \in \widehat{Z} \setminus Z\} \subseteq \mathcal{O}, \qquad \mathrm{cg}_Z^{\widehat{Z}}(\pi) \in [0,\infty],$$

  where $\mathfrak{c}_Z^{\widehat{Z}}(\pi)$ is an ideal of $\mathcal{O}$ and $\mathrm{cg}_Z^{\widehat{Z}}(\pi)$ is the normalised length of $\mathcal{O}/\mathfrak{c}_Z^{\widehat{Z}}(\pi)$, equal to ∞ when $\mathfrak{c}_Z^{\widehat{Z}}(\pi) = 0$.

Definition 13.9. For a number field E, a prime ℓ and an integer $t \geq 1$, the **ℓ-part of the Sha length** is $\mathrm{len}_\ell(E;t) = \sum_{k=1}^{t-1} \mathrm{len}_\ell(E,k) \in \mathbb{N} \cup \{\infty\}$; thus $\mathrm{len}(E;t) = \sum_\ell \log \ell \cdot \mathrm{len}_\ell(E;t)$.

Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with parameter ε and threshold x_1. **Local defect data for $\mathcal{F}$** consist of the following data, given for every real $x > x_1$ and every $i \in I(x)$, where m is the integer of $\mathfrak{M}_{x,i}$:

- a finite set $W_{x,i}$, the **twisted places** of the member, and for every $w \in W_{x,i}$ a prime $\ell_{x,i}(w)$, the **residue characteristic** of w, and a real number $\mathrm{lw}_{x,i}(w) \geq 0$, the **weight** of w;
- for every integer $1 \leq j \leq m$ and every $w \in W_{x,i}$, a finite set $R_{x,i}(j,w)$ of **local components**, such that the sets for distinct pairs (j,w) are disjoint;
- for every $r \in R_{x,i}(j,w)$, a Hecke lattice $(\mathcal{O}_r, \mathcal{Z}_r, \mathbb{T}_r, M_r)$ over $\ell_{x,i}(w)$, with eigenspaces $V_{r,Y}$ ($Y \subseteq \mathcal{Z}_r$) and multiplicities $m_r(\pi)$, together with a good set and a kept set $Z_r \subseteq \widehat{Z}_r \subseteq \mathcal{Z}_r$;
- an integer $N_{\mathrm{lev}}(x,i) \geq 1$, the **order of the level group**;
- a function $\mathrm{ex}_{x,i}$ from the set of primes to $[0,\infty)$, the **error allowances**.

For such a member and $r \in R_{x,i}(j,w)$ we write $\mathrm{df}(r) = \mathrm{df}_{Z_r}(M_r)$, $\mathrm{df}^{\mathrm{cut}}(r) = \mathrm{df}_{Z_r}^{\widehat{Z}_r}(M_r)$, $\mu(r) = \sum_{\pi \in Z_r} m_r(\pi)$ and $\mathrm{cg}(r,\pi) = \mathrm{cg}_{Z_r}^{\widehat{Z}_r}(\pi)$ for $\pi \in Z_r$. For a prime ℓ the **cut defect sum at ℓ** is

$$\mathrm{Df}_j^{\mathrm{cut}}(x,i;\ell) = \sum_{w \in W_{x,i},\ \ell_{x,i}(w) = \ell} \mathrm{lw}_{x,i}(w) \sum_{r \in R_{x,i}(j,w)} \mathrm{df}^{\mathrm{cut}}(r) \in [0,\infty].$$

We also write $P_{x,i} = \{\ell_{x,i}(w) : w \in W_{x,i}\}$, $G_{x,i} = \{g_1(\mathfrak{V}_{x,i}), g_2(\mathfrak{V}_{x,i})\}$, $E = E_{\mathfrak{h}_{x,i}}$ and $t = t_{x,i}$; n, f, b are as in Definition 13.1. A member (x,i) has **finite Sha lengths** if $\mathrm{len}_\ell(E,k) < \infty$ for

every prime ℓ and every $1 \le k < t$. As in Definition 13.1, the phrase "for all large members" means that there is a real x_0 such that the statement holds for every real $x \ge x_0$ with $x > x_1$, every $i \in I(x)$ and every integer j with $1 \le j \le m$.

The local defect data have

(Ld1) Defect sums if for all large members every $\mathrm{df}(r)$ is finite and

$$\mathrm{Df}_j(x,i) = \sum_{w \in W_{x,i}} \mathrm{lw}_{x,i}(w) \sum_{r \in R_{x,i}(j,w)} \mathrm{df}(r);$$

(Ld2) Mean condition if for all large members $\sum_{w \in W_{x,i},\, \ell_{x,i}(w) = \ell} \mathrm{lw}_{x,i}(w) \le \log \ell$ for every prime ℓ;

(Ld3) Multiplicities if for all large members $\sum_{r \in R_{x,i}(j,w)} \mu(r) \le \mathrm{rk}_j(x,i)$ for every $w \in W_{x,i}$;

(Ld4) Cut operators if for all large members, every $w \in W_{x,i}$ and every $r \in R_{x,i}(j,w)$ there is an $\mathcal{O}_r$-linear endomorphism α of M_r with $\alpha_V v = N_{\mathrm{lev}}(x,i)\, v$ for $v \in V_{r,Z_r}$ and $\alpha_V v = 0$ for $v \in V_{r,\mathcal{Z}_r \setminus \widehat{\mathcal{Z}}_r}$;

(Ld5) Sizes if there is a real C such that for all large members $\log N_{\mathrm{lev}}(x,i) \le C\, x^C x^{\varepsilon}$ and $\log \#\mathrm{Cl}(E) \le C\, x^C$;

(Ld6) Cut congruence modules if for all large members, every $w \in W_{x,i}$ with $\ell = \ell_{x,i}(w) \notin G_{x,i}$ and every $r \in R_{x,i}(j,w)$,

$$\mathrm{df}^{\mathrm{cut}}(r) \le \sum_{\pi \in Z_r} m_r(\pi)\big(\mathrm{cg}(r,\pi) + v_\ell(N_{\mathrm{lev}}(x,i))\big);$$

(Ld7) Congruence ideals bounded by Sha lengths if for all large members with finite Sha lengths, every $w \in W_{x,i}$ with $\ell = \ell_{x,i}(w) \notin G_{x,i}$, every $r \in R_{x,i}(j,w)$ and every $\pi \in Z_r$,

$$\mathrm{cg}(r,\pi) \le \mathrm{len}_\ell(E;t) + v_\ell(\#\mathrm{Cl}(E)) + \mathrm{ex}_{x,i}(\ell);$$

(Ld8) Auxiliary primes if for all large members with finite Sha lengths and every $\ell \in G_{x,i}$,

$$\mathrm{Df}_j^{\mathrm{cut}}(x,i;\ell) \le \log \ell \cdot \mathrm{rk}_j(x,i)\big(\mathrm{len}_\ell(E;t) + v_\ell(\#\mathrm{Cl}(E)) + \mathrm{ex}_{x,i}(\ell)\big);$$

(Ld9) Error budget if there is a real C such that for all large members with finite Sha lengths $\sum_{\ell \in P_{x,i}} \log \ell \cdot \mathrm{ex}_{x,i}(\ell) \le C\, f\, m\, b\, x^{2\varepsilon}$.

Definition 13.10. Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with parameter ε and threshold x_1, together with local defect data. **Local bound records** for $\mathcal{F}$ consist of the following data, given for every real $x > x_1$ and every $i \in I(x)$, where m is the integer of $\mathfrak{M}_{x,i}$:

- for every integer $1 \le j \le m$, every $w \in W_{x,i}$ with $\ell_{x,i}(w) \notin G_{x,i}$, every $r \in R_{x,i}(j,w)$ and every $\pi \in Z_r$, a real number $\mathrm{sl}(r,\pi)$, the **Selmer length** of π at r;
- for every such r, a real number $\mathrm{lt}(r)$, the **local term** of r;
- functions $\mathrm{sa}_{x,i}$, $\mathrm{la}_{x,i}$, $\mathrm{ga}_{x,i}$ from the set of primes to $\mathbb{R}$, the **Selmer allowances**, the **local allowances** and the **auxiliary allowances**.

Write $E = E_{\mathfrak{h}_{x,i}}$ and $t = t_{x,i}$. The records have

(Lq1) Lattice index bounds if for all large members with finite Sha lengths, every $w \in W_{x,i}$ with $\ell = \ell_{x,i}(w) \notin G_{x,i}$, every $r \in R_{x,i}(j,w)$ and every $\pi \in Z_r$,

$$\mathrm{cg}(r,\pi) \le \mathrm{sl}(r,\pi) + \mathrm{lt}(r) + \mathrm{len}_\ell(E;t) + v_\ell(\#\mathrm{Cl}(E)) + \ell^{-2};$$

(Lq2) Selmer bounds if for all large members with finite Sha lengths and all w, r, π as in (Lq1), $\mathrm{sl}(r,\pi) \le \mathrm{sa}_{x,i}(\ell) + \ell^{-2}$ with $\ell = \ell_{x,i}(w)$;

(Lq3) Local bounds if for all large members with finite Sha lengths and all w, r as in (Lq1), $\mathrm{lt}(r) \le \mathrm{la}_{x,i}(\ell)$ with $\ell = \ell_{x,i}(w)$;

(Lq4) Auxiliary bounds if for all large members with finite Sha lengths and every $\ell \in G_{x,i}$,

$$\mathrm{Df}_j^{\mathrm{cut}}(x,i;\ell) \le \log \ell \cdot \mathrm{rk}_j(x,i)\big(\mathrm{ga}_{x,i}(\ell) + 2\ell^{-2} + \mathrm{len}_\ell(E;t) + v_\ell(\#\mathrm{Cl}(E))\big);$$

(Lq5) Composition of the allowances if for all large members with finite Sha lengths

$$\mathrm{ex}_{x,i}(\ell) = \begin{cases} \mathrm{sa}_{x,i}(\ell) + 2\ell^{-2} + \mathrm{la}_{x,i}(\ell) & \ell \in P_{x,i} \setminus G_{x,i}, \\ \mathrm{ga}_{x,i}(\ell) + 2\ell^{-2} & \ell \in G_{x,i}; \end{cases} \tag{13.7}$$

(Lq6) Selmer budget if there is a real C such that for all large members with finite Sha lengths $\sum_{\ell \in P_{x,i} \setminus G_{x,i}} \log \ell \cdot \mathrm{sa}_{x,i}(\ell) \le C\, f\, m\, b\, x^{2\varepsilon}$;

(Lq7) Local budget if there is a real C such that for all large members with finite Sha lengths $\sum_{\ell \in P_{x,i} \setminus G_{x,i}} \log \ell \cdot \mathrm{la}_{x,i}(\ell) \le C\, f\, m\, b\, x^{2\varepsilon}$;

(Lq8) Auxiliary budget if there is a real C such that for all large members with finite Sha lengths $\sum_{\ell \in P_{x,i} \cap G_{x,i}} \log \ell \cdot \mathrm{ga}_{x,i}(\ell) \le C\, f\, m\, b\, x^{2\varepsilon}$.

Definition 13.11. Let $(\mathcal{O}, \mathcal{Z}, \mathbb{T}, M)$ be a Hecke lattice over a prime ℓ, and let $\mathbb{F}_{\mathcal{O}}$ be the residue field of $\mathcal{O}$. For an integer $n \ge 1$ we identify $\mathrm{M}_g(\mathrm{M}_n(\mathcal{O}[\ell^{-1}]))$ with $\mathrm{M}_{gn}(\mathcal{O}[\ell^{-1}])$, and similarly for rectangular matrices.

Determinant algebras. A **determinant algebra** over $\mathbb{T}$ is a tuple $(\mathcal{Z}', \phi, (z_1, \dots, z_{n_z}), n_{\mathcal{R}}, \mathcal{R})$ consisting of

- a finite set $\mathcal{Z}'$, whose elements are the **points**, and a map $\phi\colon \mathcal{Z}' \to \mathcal{Z}$;
- functions $z_1, \dots, z_{n_z}\colon \mathcal{Z}' \to \mathcal{O}$, the **descent functions**;
- an integer $n_{\mathcal{R}} \ge 1$, the **rank**;
- an $\mathcal{O}$-subalgebra $\mathcal{R}$ of $\prod_{x' \in \mathcal{Z}'} \mathrm{M}_{n_{\mathcal{R}}}(\mathcal{O}[\ell^{-1}])$.

For $a \in \mathcal{R}$ and $x' \in \mathcal{Z}'$ we write $a(x') \in \mathrm{M}_{n_{\mathcal{R}}}(\mathcal{O}[\ell^{-1}])$ for its component. For a matrix $\mathbf{X}$ with entries in $\mathcal{R}$, $\mathbf{X}(x')$ is the matrix of the components. For $x' \in \mathcal{Z}'$, $\mathcal{O}[\ell^{-1}]_{x'}^{n_{\mathcal{R}}}$ is the left $\mathcal{R}$-module $\mathcal{O}[\ell^{-1}]^{n_{\mathcal{R}}}$ (column vectors) on which a acts by $a(x')$. For an integer $g \ge 1$ the determinant algebra satisfies the **determinant condition in degree** g if

$$\begin{gathered} \text{for every } \mathbf{X} \in \mathrm{M}_g(\mathcal{R}) \text{ there are } a_1, \dots, a_{n_z} \in \mathbb{T} \text{ with} \\ \det \mathbf{X}(x') = \sum_{k=1}^{n_z} a_k(\phi(x'))\, z_k(x') \text{ for all } x' \in \mathcal{Z}', \end{gathered} \tag{13.8}$$

where the determinant is that of the $gn_{\mathcal{R}} \times gn_{\mathcal{R}}$ matrix $\mathbf{X}(x')$. A subset $\mathcal{Y} \subseteq \mathcal{Z}$ is **detected by the descent functions** if for every $x \in \mathcal{Y}$ and every $(c_1, \dots, c_{n_z}) \in \mathcal{O}^{n_z}$ with $\sum_k c_k z_k(x') = 0$ for all $x' \in \phi^{-1}(x)$, all c_k vanish.

Block forms. Let $\pi' \in \mathcal{Z}'$. A **block form at** π' with $s+1$ blocks ($s \geq 0$) is a sequence of integers $n_{\mathcal{R},0}, \dots, n_{\mathcal{R},s} \geq 1$ with sum $n_{\mathcal{R}}$, defining the consecutive intervals $I_0 < I_1 < \cdots < I_s$ of $\{1, \dots, n_{\mathcal{R}}\}$ of these lengths, such that for every $a \in \mathcal{R}$:

(Bf1) $a(\pi') \in \mathrm{M}_{n_{\mathcal{R}}}(\mathcal{O})$, and $a(\pi')$ is block upper triangular: its entries in rows I_k and columns $I_{k'}$ vanish whenever $k > k'$;

(Bf2) for each k, the map $\rho_k \colon \mathcal{R} \to \mathrm{M}_{n_{\mathcal{R},k}}(\mathcal{O})$ sending a to the diagonal block of $a(\pi')$ in rows and columns I_k is surjective.

By (Bf1) each ρ_k is a homomorphism of $\mathcal{O}$-algebras, since the diagonal blocks of a product of block upper triangular matrices are the products of the diagonal blocks. The **constituent module** of the k-th block is the left $\mathcal{R}$-module

$$\mathrm{Cs}_k = \left(\mathcal{O}[\ell^{-1}]/\mathcal{O}\right)^{n_{\mathcal{R},k}} \qquad \text{(column vectors)}, \qquad a \cdot v = \rho_k(a)\, v.$$

Presented modules. A **presentation** of a left $\mathcal{R}$-module $\mathcal{N}$ with $n_{\mathrm{gen}} \geq 1$ generators and $n_{\mathrm{rel}} \geq 1$ relations is a matrix $\mathbf{P} \in \mathrm{M}_{n_{\mathrm{rel}} \times n_{\mathrm{gen}}}(\mathcal{R})$ together with an isomorphism of left $\mathcal{R}$-modules between $\mathcal{N}$ and the cokernel of $\mathcal{R}^{n_{\mathrm{rel}}} \to \mathcal{R}^{n_{\mathrm{gen}}}$, $u \mapsto u\mathbf{P}$ (row vectors). For a left $\mathcal{R}$-module V, $\mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, V)$ is an $\mathcal{O}$-module. Let e_j be the images of the standard basis vectors. The map $\varphi \mapsto (\varphi(e_1), \dots, \varphi(e_{n_{\mathrm{gen}}}))$ identifies $\mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, V)$ with $\{v \in V^{n_{\mathrm{gen}}} : \mathbf{P}v = 0\}$, where $(\mathbf{P}v)_i = \sum_j \mathbf{P}_{ij} v_j$.

Definition 13.12. Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with threshold x_1, local defect data and local bound records. **Ribet records** for $\mathcal{F}$ consist, for every real $x > x_1$, every $i \in I(x)$, every integer $1 \leq j \leq m$, every $w \in W_{x,i}$ with $\ell_{x,i}(w) \notin G_{x,i}$, every $r \in R_{x,i}(j, w)$ and every $\pi \in Z_r$, of

- a determinant algebra $(\mathcal{Z}'_{r,\pi}, \phi_{r,\pi}, (z_k), n_{\mathcal{R}}, \mathcal{R}_{r,\pi})$ over $\mathbb{T}_r$ and a point $\pi' \in \phi_{r,\pi}^{-1}(\pi)$ with a block form with $t+1$ blocks, numbered $0, 1, \dots, t$, with constituent modules $\mathrm{Cs}_0, \dots, \mathrm{Cs}_t$;
- a left $\mathcal{R}_{r,\pi}$-module $\mathcal{N}_{r,\pi}$, the **module of extensions**, with a presentation with n_{gen} generators;
- real numbers $\mathrm{lc}_0(r,\pi), \dots, \mathrm{lc}_t(r,\pi) \geq 0$, the **local contributions**.

Write $\ell = \ell_{x,i}(w)$, $E = E_{\mathfrak{h}_{x,i}}$ and $t = t_{x,i}$, and let $\mathrm{nl}(\cdot)$ denote the normalised length of an $\mathcal{O}_r$-module. Let $\mathrm{Hom}_{\mathrm{ur}}(G_E, \mathcal{O}_r[\ell^{-1}]/\mathcal{O}_r)$ be the $\mathcal{O}_r$-module of **unramified homomorphisms**, that is, of the continuous homomorphisms $G_E \to \mathcal{O}_r[\ell^{-1}]/\mathcal{O}_r$ (for the discrete topology on the target) that vanish on the inertia group I_w of every finite place w of E. The records have each of the following properties if the stated condition holds for all large members with finite Sha lengths and all w, r, π as above.

(Rf1) Determinants The determinant algebra satisfies the determinant condition in degree n_{gen}, and $\widehat{Z}_r \setminus Z_r$ is detected by its descent functions.

(Rf2) Extensions away from the good set For every point x' with $\phi_{r,\pi}(x') \in \widehat{Z}_r \setminus Z_r$,

$$\mathrm{Hom}_{\mathcal{R}_{r,\pi}}\left(\mathcal{N}_{r,\pi}, \mathcal{O}_r[\ell^{-1}]^{n_{\mathcal{R}}}_{x'}\right) \neq 0.$$

(Rf3) Sizes The residue field of $\mathcal{O}_r$ has more than $n_{\mathrm{gen}} n_{\mathcal{R}}$ elements, and E has no real place if $\ell = 2$.

(Rf4) Block bounds The modules $\mathrm{Hom}_{\mathcal{R}_{r,\pi}}(\mathcal{N}_{r,\pi}, \mathrm{Cs}_k)$ have finite length, and

$$\begin{aligned}
&\mathrm{nl}\,\mathrm{Hom}(\mathcal{N}_{r,\pi}, \mathrm{Cs}_0) \le \mathrm{sl}(r,\pi) + \mathrm{lc}_0(r,\pi),\\
&\mathrm{nl}\,\mathrm{Hom}(\mathcal{N}_{r,\pi}, \mathrm{Cs}_1) \le \mathrm{nl}\,\mathrm{Hom}_{\mathrm{ur}}(G_E, \mathcal{O}_r[\ell^{-1}]/\mathcal{O}_r) + \mathrm{lc}_1(r,\pi),\\
&\mathrm{nl}\,\mathrm{Hom}(\mathcal{N}_{r,\pi}, \mathrm{Cs}_k) \le \mathrm{len}_\ell(E, k-1) + \mathrm{lc}_k(r,\pi) \qquad (2 \le k \le t),
\end{aligned}$$

where the homomorphisms are $\mathcal{R}_{r,\pi}$-linear.

(Rf5) Local term $\sum_{k=0}^{t} \mathrm{lc}_k(r,\pi) \le \mathrm{lt}(r)$.

Theorem 13.13. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with local defect data and local bound records, and suppose that $\mathcal{F}$ has Ribet records with the properties (Rf1)–(Rf5). Then the local bound records have the lattice index bounds (Lq1), in the stronger form without the term ℓ^{-2}: for all large members (x,i) with finite Sha lengths, every $1 \le j \le m$, every $w \in W_{x,i}$ with $\ell = \ell_{x,i}(w) \notin G_{x,i}$, every $r \in R_{x,i}(j,w)$ and every $\pi \in Z_r$,*

$$\mathrm{cg}(r,\pi) \le \mathrm{sl}(r,\pi) + \mathrm{lt}(r) + \mathrm{len}_\ell(E;t) + v_\ell(\#\mathrm{Cl}(E)), \qquad E = E_{\mathfrak{h}_{x,i}},\ t = t_{x,i}.$$

Proof. Let $x^{(1)}, \dots, x^{(5)}$ be thresholds for (Rf1)–(Rf5) and $x_0 = \max(x^{(1)}, \dots, x^{(5)})$. Fix a real $x \ge x_0$ with $x > x_1$, a member $i \in I(x)$ with finite Sha lengths, an integer $1 \le j \le m$, a place $w \in W_{x,i}$ with $\ell = \ell_{x,i}(w) \notin G_{x,i}$, a local component $r \in R_{x,i}(j,w)$ and $\pi \in Z_r$; all five properties apply. Write $\mathcal{O} = \mathcal{O}_r$, $\mathcal{R} = \mathcal{R}_{r,\pi}$, $\mathcal{N} = \mathcal{N}_{r,\pi}$, $\mathrm{lc}_k = \mathrm{lc}_k(r,\pi)$.

The Fitting step. Put $\mathcal{Y} = \widehat{Z}_r \setminus Z_r$; it does not contain $\pi \in Z_r$, and the ideal $\mathfrak{c} = \{f(\pi) : f \in \mathbb{T}_r,\ f|_{\mathcal{Y}} = 0\}$ is the congruence ideal $\mathfrak{c}_{Z_r}^{\widehat{Z}_r}(\pi)$, so the normalised length of $\mathcal{O}/\mathfrak{c}$ is $\mathrm{cg}(r,\pi)$. The hypotheses of Lemma 13.14 for the Hecke lattice of r, the point π, the set $\mathcal{Y}$, the determinant algebra $\mathcal{R}$ with the block form at $\pi' \in \phi_{r,\pi}^{-1}(\pi)$ and the presented module $\mathcal{N}$ hold: the determinant condition in degree n_{gen} and the detection of $\mathcal{Y}$ by (Rf1), the non-vanishing at the points over $\mathcal{Y}$ by (Rf2), the size of the residue field by (Rf3). Hence

$$\mathrm{cg}(r,\pi) \le \sum_{k=0}^{t} \mathrm{nl}\,\mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathrm{Cs}_k).$$

The four kinds of terms. By (Rf4) the summand for $k = 0$ is at most $\mathrm{sl}(r,\pi) + \mathrm{lc}_0$, the summand for $k = 1$ at most the normalised length of $\mathrm{Hom}_{\mathrm{ur}}(G_E, \mathcal{O}[\ell^{-1}]/\mathcal{O})$ plus lc_1, and the summand for $2 \le k \le t$ at most $\mathrm{len}_\ell(E, k-1) + \mathrm{lc}_k$. By (Rf3), ℓ is odd or E has no real place, so Lemma 13.15 gives that the normalised length of $\mathrm{Hom}_{\mathrm{ur}}(G_E, \mathcal{O}[\ell^{-1}]/\mathcal{O})$ is $v_\ell(\#\mathrm{Cl}(E))$. The substitution $k' = k-1$ gives $\sum_{k=2}^{t} \mathrm{len}_\ell(E, k-1) = \sum_{k'=1}^{t-1} \mathrm{len}_\ell(E, k') = \mathrm{len}_\ell(E;t)$, a finite number because the member has finite Sha lengths (the sum is empty, and $\mathrm{len}_\ell(E;1) = 0$, when $t = 1$). Adding,

$$\mathrm{cg}(r,\pi) \le \mathrm{sl}(r,\pi) + v_\ell(\#\mathrm{Cl}(E)) + \mathrm{len}_\ell(E;t) + \sum_{k=0}^{t} \mathrm{lc}_k \le \mathrm{sl}(r,\pi) + \mathrm{lt}(r) + \mathrm{len}_\ell(E;t) + v_\ell(\#\mathrm{Cl}(E)),$$

the last step by (Rf5).

This holds for every real $x \ge x_0$ with $x > x_1$, every member $i \in I(x)$ with finite Sha lengths and all j, w, r, π as in the statement. Since $\ell^{-2} > 0$, it implies (Lq1) with the threshold x_0. □

Lemma 13.14. *Let $(\mathcal{O}, \mathcal{Z}, \mathbb{T}, M)$ be a Hecke lattice over a prime ℓ, let $\pi \in \mathcal{Z}$, let $\mathcal{Y} \subseteq \mathcal{Z} \setminus \{\pi\}$, and let*

$$\mathfrak{c} = \{f(\pi) : \ f \in \mathbb{T}, \ f(y) = 0 \text{ for all } y \in \mathcal{Y}\} \subseteq \mathcal{O}.$$

Let $(\mathcal{Z}', \phi, (z_k)_{k \le n_z}, n_{\mathcal{R}}, \mathcal{R})$ be a determinant algebra over $\mathbb{T}$, let $\pi' \in \phi^{-1}(\pi)$ carry a block form with $s+1$ blocks and constituent modules $\mathrm{Cs}_0, \dots, \mathrm{Cs}_s$, and let $\mathcal{N}$ be a left $\mathcal{R}$-module with a presentation with n_{gen} generators and n_{rel} relations. Assume:

- *the determinant algebra satisfies the determinant condition in degree n_{gen}, and $\mathcal{Y}$ is detected by the descent functions;*
- *$\mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathcal{O}[\ell^{-1}]^{n_{\mathcal{R}}}_{x'}) \neq 0$ for every $x' \in \phi^{-1}(\mathcal{Y})$;*
- *the residue field $\mathbb{F}_{\mathcal{O}}$ has more than $n_{\mathrm{gen}}\, n_{\mathcal{R}}$ elements.*

Then $\mathfrak{c}$ is an ideal of $\mathcal{O}$ and

$$\mathrm{length}_{\mathcal{O}}(\mathcal{O}/\mathfrak{c}) \le \sum_{k=0}^{s} \mathrm{length}_{\mathcal{O}} \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathrm{Cs}_k) \qquad \text{in } \mathbb{N} \cup \{\infty\};$$

equivalently, the same inequality holds for normalised lengths.

Proof. Write $K = \mathcal{O}[\ell^{-1}]$, fix a uniformiser $\varpi_{\mathcal{O}}$, let v be the valuation of K with $v(\varpi_{\mathcal{O}}) = 1$ and $v(0) = \infty$, and let $\mathbf{P} \in \mathrm{M}_{n_{\mathrm{rel}} \times n_{\mathrm{gen}}}(\mathcal{R})$ be the presentation matrix. Since $\mathbb{T}$ is an $\mathcal{O}$-algebra of functions, $c\,f \in \mathbb{T}$ for $c \in \mathcal{O}$ and $f \in \mathbb{T}$, so $\mathfrak{c}$ is an $\mathcal{O}$-submodule of $\mathcal{O}$, i.e. an ideal; for $y \in \mathfrak{c}$ we have $\mathrm{length}_{\mathcal{O}}(\mathcal{O}/\mathfrak{c}) \le \mathrm{length}_{\mathcal{O}}(\mathcal{O}/y\mathcal{O}) = v(y)$.

Step 1: the determinants of the combinations of the relations lie in $\mathfrak{c}$. Let $\mathbf{Q} \in \mathrm{M}_{n_{\mathrm{gen}} \times n_{\mathrm{rel}}}(\mathcal{R})$ and $\mathbf{X} = \mathbf{Q}\mathbf{P} \in \mathrm{M}_{n_{\mathrm{gen}}}(\mathcal{R})$. By the determinant condition in degree n_{gen} there are $a_1, \dots, a_{n_z} \in \mathbb{T}$ with $\det \mathbf{X}(x') = \sum_k a_k(\phi(x')) z_k(x')$ for all $x' \in \mathcal{Z}'$. Let $x \in \mathcal{Y}$ and $x' \in \phi^{-1}(x)$. On $V = K^{n_{\mathcal{R}}}_{x'}$ the element a acts by $a(x')$, so $V^{n_{\mathrm{gen}}} = K^{n_{\mathrm{gen}} n_{\mathcal{R}}}$ and $\mathbf{P}$ acts on it by the matrix $\mathbf{P}(x')$; by the second hypothesis and the identification of $\mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, V)$ there is $w \neq 0$ with $\mathbf{P}(x')w = 0$. Evaluation at x' is a ring homomorphism $\mathcal{R} \to \mathrm{M}_{n_{\mathcal{R}}}(K)$, so $\mathbf{X}(x') = \mathbf{Q}(x')\mathbf{P}(x')$ and $\mathbf{X}(x')w = 0$; hence $\det \mathbf{X}(x') = 0$. Thus $\sum_k a_k(x) z_k(x') = 0$ for every $x' \in \phi^{-1}(x)$, and since $\mathcal{Y}$ is detected by the descent functions, $a_k(x) = 0$ for all k. So every a_k vanishes on $\mathcal{Y}$, $a_k(\pi) \in \mathfrak{c}$, and as $z_k(\pi') \in \mathcal{O}$ and $\phi(\pi') = \pi$,

$$\det \mathbf{X}(\pi') = \sum_k a_k(\pi) z_k(\pi') \in \mathfrak{c}, \qquad \text{hence} \qquad \mathrm{length}_{\mathcal{O}}(\mathcal{O}/\mathfrak{c}) \le v(\det \mathbf{X}(\pi')). \tag{13.9}$$

Step 2: the determinant at π' factors through the blocks. By (Bf1) every entry of $\mathbf{Q}(\pi')$ and $\mathbf{P}(\pi')$, and hence of $\mathbf{X}(\pi')$, is a block upper triangular matrix in $\mathrm{M}_{n_{\mathcal{R}}}(\mathcal{O})$. Index the basis of $\mathcal{O}^{n_{\mathrm{gen}} n_{\mathcal{R}}}$ by pairs (j, a), $1 \le j \le n_{\mathrm{gen}}$, $1 \le a \le n_{\mathcal{R}}$, and let $\beta(a)$ be the block containing a. The entry of $\mathbf{X}(\pi')$ at $((j,a),(j',a'))$ is the (a,a') entry of $\mathbf{X}_{jj'}(\pi')$, which vanishes when $\beta(a) > \beta(a')$. Ordering the basis first by $\beta(a)$ therefore conjugates $\mathbf{X}(\pi')$ by a permutation matrix into a block upper triangular matrix whose k-th diagonal block is $\rho_k(\mathbf{X}) = (\rho_k(\mathbf{X}_{jj'}))_{j,j'}$. Since ρ_k is a ring homomorphism, $\rho_k(\mathbf{X}) = \rho_k(\mathbf{Q})\rho_k(\mathbf{P})$ (entrywise application). Conjugation by a permutation matrix preserves the determinant, so

$$\det \mathbf{X}(\pi') = \prod_{k=0}^{s} \det\big(\rho_k(\mathbf{Q})\, \rho_k(\mathbf{P})\big). \tag{13.10}$$

Step 3: one block. For $B \in \mathrm{M}_{n' \times n}(\mathcal{O})$ let $\kappa(B) \in \mathbb{N} \cup \{\infty\}$ be the length of the kernel of $B \colon (K/\mathcal{O})^n \to (K/\mathcal{O})^{n'}$. By the Smith normal form over the discrete valuation ring $\mathcal{O}$, $B =$

UDW with $U \in \mathrm{GL}_{n'}(\mathcal{O})$, $W \in \mathrm{GL}_n(\mathcal{O})$ and D zero except for $D_{ii} = \varpi_{\mathcal{O}}^{\alpha_i}$, $1 \le i \le \min(n, n')$, with $\alpha_i \in \mathbb{N} \cup \{\infty\}$ and $\varpi_{\mathcal{O}}^{\infty} = 0$. The kernel of B is W^{-1} of the kernel of D. If $n > n'$ or some $\alpha_i = \infty$, the kernel of D contains a copy of $K/\mathcal{O}$, which has infinite length (the submodules $\varpi_{\mathcal{O}}^{-r}\mathcal{O}/\mathcal{O}$ increase strictly), so $\kappa(B) = \infty$. Otherwise the kernel of D is $\bigoplus_{i \le n} \varpi_{\mathcal{O}}^{-\alpha_i}\mathcal{O}/\mathcal{O} \cong \bigoplus_i \mathcal{O}/\varpi_{\mathcal{O}}^{\alpha_i}\mathcal{O}$, and $\kappa(B) = \sum_{i \le n} \alpha_i$. In this case let U_1 be the $n' \times n$ matrix of the first n columns of U and $\Delta = \mathrm{diag}(\varpi_{\mathcal{O}}^{\alpha_1}, \dots, \varpi_{\mathcal{O}}^{\alpha_n})$; then $UD = U_1\Delta$, and for every $C \in \mathrm{M}_{n \times n'}(\mathcal{O})$

$$\det(CB) = \det(CU_1)\,\det(\Delta)\,\det(W), \qquad v\big(\det(CB)\big) = v\big(\det(CU_1)\big) + \kappa(B).$$

So $v(\det(CB)) = \kappa(B)$ as soon as the reduction $\det(\overline{C}\,\overline{U}_1) \in \mathbb{F}_{\mathcal{O}}$ is non-zero. The first n rows C_B of U^{-1} satisfy $C_B U_1 = I_n$.

Step 4: a common choice of Q. We may assume that the right side of the assertion is finite. For each k, $\mathrm{Cs}_k^{n_{\mathrm{gen}}} = (K/\mathcal{O})^{n_{\mathrm{gen}} n_{\mathcal{R},k}}$ and $\mathbf{P}$ acts on it by the matrix $B_k = \rho_k(\mathbf{P}) \in \mathrm{M}_{n_{\mathrm{rel}} n_{\mathcal{R},k} \times n_{\mathrm{gen}} n_{\mathcal{R},k}}(\mathcal{O})$, so by the identification of homomorphisms $\mathrm{length}_{\mathcal{O}} \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathrm{Cs}_k) = \kappa(B_k) < \infty$. Let $U_{k,1}$ and $C_k = C_{B_k}$ be as in Step 3 for B_k. By (Bf2), applied entrywise, there is $\mathbf{Q}^{(k)} \in \mathrm{M}_{n_{\mathrm{gen}} \times n_{\mathrm{rel}}}(\mathcal{R})$ with $\rho_k(\mathbf{Q}^{(k)}) = C_k$. For $c = (c_0, \dots, c_s) \in \mathcal{O}^{s+1}$ put $\mathbf{Q}(c) = \sum_{k'} c_{k'} \mathbf{Q}^{(k')}$. The entries of the matrix $\overline{\rho_k(\mathbf{Q}(c))}\,\overline{U}_{k,1} = \sum_{k'} \bar{c}_{k'}\, \overline{\rho_k(\mathbf{Q}^{(k')})}\,\overline{U}_{k,1}$ of size $n_{\mathrm{gen}} n_{\mathcal{R},k}$ are linear forms in $\bar{c} \in \mathbb{F}_{\mathcal{O}}^{s+1}$, so $F_k(\bar{c}) = \det\big(\overline{\rho_k(\mathbf{Q}(c))}\,\overline{U}_{k,1}\big)$ is a polynomial of total degree at most $n_{\mathrm{gen}} n_{\mathcal{R},k}$, and $F_k(e_k) = \det(\overline{C}_k \overline{U}_{k,1}) = 1$ at the k-th unit vector. The product $F = \prod_k F_k$ is a non-zero polynomial of total degree at most $n_{\mathrm{gen}} \sum_k n_{\mathcal{R},k} = n_{\mathrm{gen}} n_{\mathcal{R}} < \#\mathbb{F}_{\mathcal{O}}$. A non-zero polynomial $F \in \mathbb{F}[X_0, \dots, X_s]$ of total degree $D < \#\mathbb{F}$ over a field $\mathbb{F}$ takes a non-zero value on $\mathbb{F}^{s+1}$: for $s = 0$ it has at most D roots; for $s \ge 1$ write $F = \sum_{i \le D} F_i(X_1, \dots, X_s) X_0^i$, choose j with $F_j \ne 0$ and, by induction, a point $(c_1, \dots, c_s)$ with $F_j(c_1, \dots, c_s) \ne 0$; then $F(X_0, c_1, \dots, c_s)$ is a non-zero polynomial of degree at most D in X_0 and has a non-root. Choose $\bar{c}$ with $F(\bar{c}) \ne 0$, lift it to $c \in \mathcal{O}^{s+1}$ and put $\mathbf{Q} = \mathbf{Q}(c)$. Then $F_k(\bar{c}) \ne 0$ for every k, and Step 3 with $C = \rho_k(\mathbf{Q})$ gives $v\big(\det(\rho_k(\mathbf{Q}) B_k)\big) = \kappa(B_k)$.

Step 5: conclusion. With this $\mathbf{Q}$, (13.9) and (13.10) give

$$\mathrm{length}_{\mathcal{O}}(\mathcal{O}/\mathfrak{c}) \le \sum_{k=0}^{s} v\big(\det(\rho_k(\mathbf{Q})\rho_k(\mathbf{P}))\big) = \sum_{k=0}^{s} \kappa(B_k) = \sum_{k=0}^{s} \mathrm{length}_{\mathcal{O}} \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathrm{Cs}_k).$$

Dividing by $e_{\mathcal{O}} \ge 1$ gives the form with normalised lengths. □

Lemma 13.15. *Let E be a number field, ℓ a prime and $\mathcal{O}$ a coefficient ring for ℓ. Let*

$$\mathrm{Hom}_{\mathrm{ur}}(G_E, \mathcal{O}[\ell^{-1}]/\mathcal{O})$$

be the $\mathcal{O}$-module of continuous homomorphisms $G_E \to \mathcal{O}[\ell^{-1}]/\mathcal{O}$ (discrete topology) that vanish on the inertia group I_w of every finite place w of E. If ℓ is odd or E has no real place, this module has finite length and its normalised length is $v_\ell(\#\mathrm{Cl}(E))$.

Proof. Write $K = \mathcal{O}[\ell^{-1}]$, $U = \mathrm{Hom}_{\mathrm{ur}}(G_E, K/\mathcal{O})$, and let H be the Hilbert class field of E, the maximal abelian extension of E unramified at all places, finite and infinite; by class field theory the Artin map induces $\mathrm{Gal}(H/E) \cong \mathrm{Cl}(E)$ [190] (Ch. VI, the Hilbert class field).

Every $\varphi \in U$ factors through $\mathrm{Gal}(H/E)$. The kernel of φ is open (continuity into a discrete group) and contains the commutator subgroup (the target is abelian), so its fixed field L is a finite abelian extension of E and φ induces an injection $\mathrm{Gal}(L/E) \hookrightarrow K/\mathcal{O}$. For a finite place w the inertia group of L/E at w is the image of I_w, on which φ vanishes, so L/E is unramified at w. For a real place, the decomposition group of L/E is generated by the image σ of a complex conjugation and has order at most 2, so $2\varphi(\sigma) = 0$; if ℓ is odd, multiplication

by 2 is injective on $K/\mathcal{O}$ and $\varphi(\sigma) = 0$, and if E has no real place there is nothing to check. Hence L/E is unramified at all places, $L \subseteq H$, and φ factors through $\mathrm{Gal}(H/E)$. Conversely, every homomorphism $\mathrm{Gal}(H/E) \to K/\mathcal{O}$ composed with the restriction $G_E \to \mathrm{Gal}(H/E)$ is continuous and vanishes on every I_w, because H/E is unramified at w. So restriction induces an isomorphism of $\mathcal{O}$-modules $U \cong \mathrm{Hom}(\mathrm{Gal}(H/E), K/\mathcal{O}) \cong \mathrm{Hom}(\mathrm{Cl}(E), K/\mathcal{O})$.

The length. Write $\mathrm{Cl}(E) \cong \bigoplus_i \mathbb{Z}/n_i\mathbb{Z}$, a finite sum, and $n_i = \ell^{a_i}u_i$ with $\ell \nmid u_i$. Then $\mathrm{Hom}(\mathbb{Z}/n_i\mathbb{Z}, K/\mathcal{O})$ is the n_i-torsion of $K/\mathcal{O}$; as u_i is a unit of $\mathcal{O}$ this is the ℓ^{a_i}-torsion $\ell^{-a_i}\mathcal{O}/\mathcal{O} \cong \mathcal{O}/\ell^{a_i}\mathcal{O}$, of length $a_i e_{\mathcal{O}}$ (the quotients $\ell^j\mathcal{O}/\ell^{j+1}\mathcal{O} \cong \mathcal{O}/\ell\mathcal{O}$ have length $e_{\mathcal{O}}$). Hence U has length $e_{\mathcal{O}} \sum_i a_i = e_{\mathcal{O}}\, v_\ell(\#\mathrm{Cl}(E))$, and its normalised length is $v_\ell(\#\mathrm{Cl}(E))$. □

Theorem 13.16. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with local defect data, and suppose that $\mathcal{F}$ has local bound records with the properties (Lq1)–(Lq5). Then the local defect data have the properties (Ld7), congruence ideals bounded by Sha lengths, and (Ld8), auxiliary primes.*

Proof. Let $x^{(1)}, \dots, x^{(5)}$ be thresholds for (Lq1)–(Lq5) and $x_0 = \max(x^{(1)}, \dots, x^{(5)})$. Fix a real $x \geq x_0$ with $x > x_1$, a member $i \in I(x)$ with finite Sha lengths, and an integer $1 \leq j \leq m$, m the integer of $\mathfrak{M}_{x,i}$. All five properties apply at (x, i, j). Write $U(\ell) = \mathrm{len}_\ell(E; t) + v_\ell(\#\mathrm{Cl}(E))$ for a prime ℓ; it is finite, since the member has finite Sha lengths.

(Ld7). Let $w \in W_{x,i}$ with $\ell = \ell_{x,i}(w) \notin G_{x,i}$, let $r \in R_{x,i}(j, w)$ and $\pi \in Z_r$. Then $\ell \in P_{x,i} \setminus G_{x,i}$, because ℓ is the residue characteristic of the twisted place w. By (Lq1), then (Lq2) for $\mathrm{sl}(r, \pi)$ and (Lq3) for $\mathrm{lt}(r)$,

$$\mathrm{cg}(r, \pi) \leq \mathrm{sl}(r, \pi) + \mathrm{lt}(r) + U(\ell) + \ell^{-2} \leq \big(\mathrm{sa}_{x,i}(\ell) + \ell^{-2}\big) + \mathrm{la}_{x,i}(\ell) + U(\ell) + \ell^{-2}.$$

By the first case of the composition of the allowances (Lq5), $\mathrm{sa}_{x,i}(\ell) + 2\ell^{-2} + \mathrm{la}_{x,i}(\ell) = \mathrm{ex}_{x,i}(\ell)$, so

$$\mathrm{cg}(r, \pi) \leq \mathrm{len}_\ell(E; t) + v_\ell(\#\mathrm{Cl}(E)) + \mathrm{ex}_{x,i}(\ell).$$

(Ld8). Let $\ell \in G_{x,i}$. By (Lq4) and the second case of (Lq5), $\mathrm{ga}_{x,i}(\ell) + 2\ell^{-2} = \mathrm{ex}_{x,i}(\ell)$ and

$$\mathrm{Df}_j^{\mathrm{cut}}(x, i; \ell) \leq \log \ell \cdot \mathrm{rk}_j(x, i)\big(\mathrm{len}_\ell(E; t) + v_\ell(\#\mathrm{Cl}(E)) + \mathrm{ex}_{x,i}(\ell)\big).$$

Both bounds hold for every real $x \geq x_0$ with $x > x_1$, every member $i \in I(x)$ with finite Sha lengths and every $1 \leq j \leq m$, which is (Ld7) and (Ld8) with the threshold x_0. □

Theorem 13.17. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with local defect data, and suppose that $\mathcal{F}$ has local bound records with the properties (Lq5)–(Lq8). Then the local defect data have the error budget (Ld9).*

Proof. **The prime sum $\sum_\ell \log \ell/\ell^2$.** For real $y \geq 1$ we have $\log y \leq \sqrt{y}$: the function $g(y) = \sqrt{y} - \log y$ has $g'(y) = (\sqrt{y} - 2)/(2y)$, so its minimum on $[1, \infty)$ is $g(4) = 2 - \log 4 > 0$, because $4 < e^2$. Hence $\log n/n^2 \leq n^{-3/2}$ for every integer $n \geq 2$, and $n^{-3/2} \leq \int_{n-1}^{n} y^{-3/2}\, dy$ since $y^{-3/2}$ decreases. Therefore, for every finite set S of primes,

$$\sum_{\ell \in S} \frac{\log \ell}{\ell^2} \leq \sum_{n \geq 2} n^{-3/2} \leq \int_1^\infty y^{-3/2}\, dy = 2. \tag{13.11}$$

The budget. Let C_6, C_7, C_8 be constants and $x^{(5)}, \dots, x^{(8)}$ thresholds for (Lq5)–(Lq8), $x_0 = \max(x^{(5)}, \dots, x^{(8)})$, and $C = |C_6| + |C_7| + |C_8| + 2$. Fix a real $x \geq x_0$ with $x > x_1$ and a member $i \in I(x)$ with finite Sha lengths; write $P = P_{x,i}$, $G = G_{x,i}$ and $Y = f\, m\, b\, x^{2\varepsilon}$. Since

$f \ge 2$, $m \ge 1$, $b \ge 1$ and $x^{2\varepsilon} \ge 1$ (as $x > 1$, $\varepsilon > 0$), $Y \ge 2$. Split P into the disjoint sets $P \setminus G$ and $P \cap G$ and insert the composition (Lq5) of the allowances:

$$\sum_{\ell \in P} \log \ell \cdot \mathrm{ex}_{x,i}(\ell) = \sum_{\ell \in P \setminus G} \log \ell \cdot \mathrm{sa}_{x,i}(\ell) + \sum_{\ell \in P \setminus G} \log \ell \cdot \mathrm{la}_{x,i}(\ell) + \sum_{\ell \in P \cap G} \log \ell \cdot \mathrm{ga}_{x,i}(\ell) + 2 \sum_{\ell \in P} \frac{\log \ell}{\ell^2}.$$

By (Lq6), (Lq7), (Lq8) the first three sums are at most $C_6 Y$, $C_7 Y$, $C_8 Y$, hence at most $|C_6| Y$, $|C_7| Y$, $|C_8| Y$ as $Y \ge 0$; by (13.11) the last term is at most $4 \le 2Y$. So

$$\sum_{\ell \in P_{x,i}} \log \ell \cdot \mathrm{ex}_{x,i}(\ell) \le \bigl(|C_6| + |C_7| + |C_8| + 2\bigr)\, f\, m\, b\, x^{2\varepsilon} = C\, f\, m\, b\, x^{2\varepsilon}$$

for every real $x \ge x_0$ with $x > x_1$ and every member $i \in I(x)$ with finite Sha lengths, which is (Ld9) with the constant C and the threshold x_0. □

Theorem 13.18. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with parameter ε, and suppose that $\mathcal{F}$ has local defect data with the properties (Ld1)–(Ld9). Then there is a real number $C \ge 0$ such that for all large members (x, i) with finite Sha lengths and $\mathrm{len}(E; t_{x,i}) < \infty$, and every $1 \le j \le m$,*

$$\mathrm{Df}_j(x,i) \le \mathrm{rk}_j(x,i)\Bigl(C\, f\, m\, b\, x^{2\varepsilon} + \mathrm{len}(E; t_{x,i})\Bigr).$$

In particular $\mathcal{F}$ has defects bounded by Sha lengths (Dv6).

Proof. Let C_5 and C_9 be the constants of (Ld5) and (Ld9), and $x^{(1)}, \dots, x^{(9)}$ the thresholds of (Ld1)–(Ld9). Since $\log x / x^{\varepsilon} \to 0$ as $x \to \infty$, there is $x_L \ge 1$ with $|C_5| \log x \le x^{\varepsilon}$, hence $x^{|C_5|} \le b$, for all $x \ge x_L$. Put $X = \max(x^{(1)}, \dots, x^{(9)}, x_L)$ and $C = 3|C_5| + |C_9|$.

Fix a real $x \ge X$ with $x > x_1$, a member $i \in I(x)$ with finite Sha lengths and $\mathrm{len}(E; t) < \infty$ (where $E = E_{\mathfrak{h}_{x,i}}$, $t = t_{x,i}$), and an integer $1 \le j \le m$. Write $W = W_{x,i}$, $\ell(w) = \ell_{x,i}(w)$, $\mathrm{lw} = \mathrm{lw}_{x,i}$, $R(w) = R_{x,i}(j, w)$, $P = P_{x,i}$, $G = G_{x,i}$, $N = N_{\mathrm{lev}}(x, i)$, $\mathrm{rk} = \mathrm{rk}_j(x, i) \ge 0$, $y = x^{\varepsilon} \ge 1$. For a prime ℓ put

$$U(\ell) = \mathrm{len}_\ell(E; t) + v_\ell(\#\mathrm{Cl}(E)) + \mathrm{ex}_{x,i}(\ell) + v_\ell(N),$$

a finite non-negative real number, because the member has finite Sha lengths and $\mathrm{ex}_{x,i}(\ell) \ge 0$.

Cut reduction at one component. Let $w \in W$ and $r \in R(w)$. By (Ld4) there is an $\mathcal{O}_r$-linear endomorphism of M_r acting as N on V_{r,Z_r} and as 0 on $V_{r,\mathcal{Z}_r \setminus \widehat{Z}_r}$. By Lemma 13.19, applied to the Hecke lattice $(\mathcal{O}_r, \mathcal{Z}_r, \mathbb{T}_r, M_r)$ over $\ell(w)$, the sets $Z_r \subseteq \widehat{Z}_r$ and the integer $N \ge 1$, $\mathrm{df}^{\mathrm{cut}}(r)$ is finite and

$$\mathrm{df}(r) \le \mathrm{df}^{\mathrm{cut}}(r) + \mu(r)\, v_{\ell(w)}(N). \tag{13.12}$$

Cut reduction of the defect sum. Multiply (13.12) by $\mathrm{lw}(w) \ge 0$ and sum over $r \in R(w)$ and $w \in W$. By (Ld1) the left side becomes $\mathrm{Df}_j(x, i)$. Grouping the places by their residue characteristic, the first term on the right becomes $\sum_{\ell \in P} \mathrm{Df}_j^{\mathrm{cut}}(x, i; \ell)$. In the second, (Ld3) gives $\sum_{r \in R(w)} \mu(r) \le \mathrm{rk}$, and $\mathrm{lw}(w) v_{\ell(w)}(N) \ge 0$, so it is at most

$$\mathrm{rk} \sum_{w \in W} \mathrm{lw}(w)\, v_{\ell(w)}(N) = \mathrm{rk} \sum_{\ell \in P} v_\ell(N) \sum_{w \in W,\, \ell(w) = \ell} \mathrm{lw}(w) \le \mathrm{rk} \sum_{\ell \in P} v_\ell(N) \log \ell \le \mathrm{rk}\, \log N,$$

by the mean condition (Ld2), and because $\sum_\ell v_\ell(N) \log \ell = \log N$ over all primes with non-negative terms. Hence

$$\mathrm{Df}_j(x,i) \le \sum_{\ell \in P} \mathrm{Df}_j^{\mathrm{cut}}(x, i; \ell) + \mathrm{rk}\, \log N. \tag{13.13}$$

The cut defect sum at a prime outside G. Let $\ell \in P \setminus G$, $w \in W$ with $\ell(w) = \ell$, and $r \in R(w)$. By (Ld6), then (Ld7) for each $\pi \in Z_r$ together with $m_r(\pi) \geq 0$,

$$\mathrm{df}^{\mathrm{cut}}(r) \leq \sum_{\pi \in Z_r} m_r(\pi)(\mathrm{cg}(r,\pi) + v_\ell(N)) \leq \sum_{\pi \in Z_r} m_r(\pi)\, U(\ell) = \mu(r)\, U(\ell).$$

Summing over $r \in R(w)$ and using (Ld3) with $U(\ell) \geq 0$ gives $\sum_{r \in R(w)} \mathrm{df}^{\mathrm{cut}}(r) \leq \mathrm{rk}\, U(\ell)$. Multiplying by $\mathrm{lw}(w) \geq 0$, summing over the w with $\ell(w) = \ell$ and using (Ld2) with $\mathrm{rk}\, U(\ell) \geq 0$,

$$\mathrm{Df}_j^{\mathrm{cut}}(x, i; \ell) \leq \mathrm{rk}\, U(\ell) \sum_{w \in W,\, \ell(w) = \ell} \mathrm{lw}(w) \leq \log \ell \cdot \mathrm{rk}\, U(\ell).$$

The cut defect sum at a prime of G. For $\ell \in P \cap G$, (Ld8) gives $\mathrm{Df}_j^{\mathrm{cut}}(x, i; \ell) \leq \log \ell \cdot \mathrm{rk}(U(\ell) - v_\ell(N)) \leq \log \ell \cdot \mathrm{rk}\, U(\ell)$.

Summation over the primes: the Sha lengths enter. By the last two paragraphs and (13.13),

$$\mathrm{Df}_j(x,i) \leq \mathrm{rk}\Big(\sum_{\ell \in P} \log \ell\ \mathrm{len}_\ell(E;t) + \sum_{\ell \in P} \log \ell\, v_\ell(\#\mathrm{Cl}(E)) + \sum_{\ell \in P} \log \ell\, \mathrm{ex}_{x,i}(\ell) + \sum_{\ell \in P} \log \ell\, v_\ell(N)\Big) + \mathrm{rk}\, \log N.$$

The four sums are bounded as follows. The first is a finite partial sum of the series

$$\mathrm{len}(E;t) = \sum_{\ell} \log \ell\ \mathrm{len}_\ell(E;t)$$

of non-negative terms, hence at most $\mathrm{len}(E;t)$. The second is at most $\sum_\ell \log \ell\, v_\ell(\#\mathrm{Cl}(E)) = \log \#\mathrm{Cl}(E)$, and the fourth at most $\log N$, for the same reason. The third is at most $C_9\, f\, m\, b\, x^{2\varepsilon} \leq |C_9|\, f\, m\, b\, x^{2\varepsilon}$ by (Ld9). Since $\mathrm{rk} \geq 0$,

$$\mathrm{Df}_j(x,i) \leq \mathrm{rk}\Big(\mathrm{len}(E;t) + \log \#\mathrm{Cl}(E) + 2 \log N + |C_9|\, f\, m\, b\, x^{2\varepsilon}\Big). \tag{13.14}$$

The sizes. For $x \geq 1$, $C_5 x^{C_5} \leq |C_5| x^{|C_5|}$ (both sides agree if $C_5 \geq 0$, and the left side is ≤ 0 otherwise); and $x^{|C_5|} \leq b$ as $x \geq x_L$. So (Ld5) gives $\log \#\mathrm{Cl}(E) \leq |C_5|\, b$ and $\log N \leq |C_5|\, b\, y$. As $f = 2n \geq 2$, $m \geq j \geq 1$ and $y \geq 1$, we have $b \leq b\, y \leq f\, m\, b\, y^2 = f\, m\, b\, x^{2\varepsilon}$. Hence $\log \#\mathrm{Cl}(E) + 2 \log N \leq 3|C_5|\, f\, m\, b\, x^{2\varepsilon}$, and (13.14) becomes

$$\mathrm{Df}_j(x,i) \leq \mathrm{rk}_j(x,i)\Big(C\, f\, m\, b\, x^{2\varepsilon} + \mathrm{len}(E;t)\Big), \qquad C = 3|C_5| + |C_9| \geq 0.$$

This holds for every real $x \geq X$ with $x > x_1$, every member $i \in I(x)$ with finite Sha lengths and $\mathrm{len}(E;t) < \infty$, and every $1 \leq j \leq m$: the first assertion, with threshold X.

(Dv6). Since $C \geq 0$ and $(\log x)^2 N_c(x,i) \geq 0$, the bound gives

$$\mathrm{Df}_j(x,i) \leq \mathrm{rk}_j(x,i)(C(f\, m\, b\, x^{2\varepsilon} + (\log x)^2 N_c(x,i)) + \mathrm{len}(E;t))$$

at the same members, which is (Dv6) with the constant C and the threshold X. □

Lemma 13.19. *Let $(\mathcal{O}, \mathcal{Z}, \mathbb{T}, M)$ be a Hecke lattice over a prime ℓ, let $Z \subseteq \widehat{Z} \subseteq \mathcal{Z}$, let $N \geq 1$ be an integer, and let α be an $\mathcal{O}$-linear endomorphism of M such that $\alpha_V v = Nv$ for every $v \in V_Z$ and $\alpha_V v = 0$ for every $v \in V_{\mathcal{Z} \setminus \widehat{Z}}$. Then $\mathrm{df}_Z(M)$ and $\mathrm{df}_Z^{\widehat{Z}}(M)$ are finite, and*

$$\mathrm{df}_Z(M) \leq \mathrm{df}_Z^{\widehat{Z}}(M) + \Big(\sum_{\pi \in Z} m(\pi)\Big)\, v_\ell(N).$$

Proof. Write $K = \mathcal{O}[\ell^{-1}]$. The map $f \mapsto f_V$ is a homomorphism of rings $\mathbb{T} \to \mathrm{End}_K(V)$, since $f \mapsto f_M$ is one and extension of scalars respects sums and composition.

The eigenspace decomposition. For $\pi \neq \rho$ in $\mathcal{Z}$ choose $f_{\pi,\rho} \in \mathbb{T}$ with $f_{\pi,\rho}(\pi) \neq f_{\pi,\rho}(\rho)$ ($\mathbb{T}$ separates points). Since $\mathbb{T}$ contains the constants,

$$g_\pi = \prod_{\rho \neq \pi} \big(f_{\pi,\rho} - f_{\pi,\rho}(\rho)\big) \in \mathbb{T}, \qquad g_\pi(\rho) = 0 \ (\rho \neq \pi), \qquad c_\pi := g_\pi(\pi) = \prod_{\rho \neq \pi} \big(f_{\pi,\rho}(\pi) - f_{\pi,\rho}(\rho)\big) \neq 0,$$

the last because $\mathcal{O}$ is a domain. Put $d = \prod_{\pi \in \mathcal{Z}} c_\pi \in \mathcal{O} \setminus \{0\}$ and $h_\pi = (d/c_\pi)\, g_\pi \in \mathbb{T}$, so that $h_\pi(\rho) = d$ if $\rho = \pi$ and 0 otherwise. As functions on $\mathcal{Z}$, $\sum_\pi h_\pi = d$, $h_\pi h_\rho = 0$ for $\pi \neq \rho$, $h_\pi^2 = d\, h_\pi$, and $f h_\pi = f(\pi) h_\pi$ for every $f \in \mathbb{T}$. Put $E_\pi = d^{-1}(h_\pi)_V \in \mathrm{End}_K(V)$. Applying the ring homomorphism $f \mapsto f_V$:

$$\sum_\pi E_\pi = \mathrm{id}_V, \qquad E_\pi E_\rho = \begin{cases} E_\pi & \rho = \pi, \\ 0 & \rho \neq \pi, \end{cases} \qquad f_V E_\pi = f(\pi) E_\pi \quad (f \in \mathbb{T}).$$

The last identity gives $E_\pi V \subseteq V_\pi$. Conversely, if $v \in V_\pi$ then $(h_\rho)_V v = h_\rho(\pi) v$, so $E_\rho v = v$ for $\rho = \pi$ and $E_\rho v = 0$ for $\rho \neq \pi$. Hence $V = \bigoplus_{\pi \in \mathcal{Z}} V_\pi$ and E_π is the projection onto V_π along $\bigoplus_{\rho \neq \pi} V_\rho$. For $Y \subseteq \mathcal{Z}$ the map $E_Y = \sum_{\pi \in Y} E_\pi$ is the projection onto V_Y along $V_{\mathcal{Z} \setminus Y}$, and $dE_Y = \big(\sum_{\pi \in Y} h_\pi\big)_V$ maps M into M. In particular $\dim_K V_Y = \sum_{\pi \in Y} m(\pi)$.

The congruence module of M. Put $U = V_Z \cap (M + V_{\mathcal{Z} \setminus Z})$ and $L = M \cap V_Z$, so $\mathrm{Cong}_Z(M) = U/L$ and $L \subseteq U$. We have $U = E_Z(M)$: if $u = x + y$ with $u \in V_Z$, $x \in M$, $y \in V_{\mathcal{Z} \setminus Z}$, then $u = E_Z u = E_Z x$; conversely, for $x \in M$, $E_Z x \in V_Z$ and $E_Z x = x - E_{\mathcal{Z} \setminus Z} x \in M + V_{\mathcal{Z} \setminus Z}$. So $U = d^{-1}((dE_Z)(M))$ is a finitely generated $\mathcal{O}$-module, and $dU = (dE_Z)(M) \subseteq M \cap V_Z = L$. Thus U/L is a finitely generated $\mathcal{O}$-module killed by $d \neq 0$; over the discrete valuation ring $\mathcal{O}$ it is a finite direct sum of modules $\mathcal{O}/\mathfrak{m}^k$, so it has finite length, and $\mathrm{df}_Z(M) < \infty$.

The cut congruence module. Each E_π maps $V_{\mathcal{Z} \setminus \widehat{Z}}$ into itself, so it induces E'_π on $V' = V/V_{\mathcal{Z} \setminus \widehat{Z}}$ with $qE_\pi = E'_\pi q$. Since $V = V_{\widehat{Z}} \oplus V_{\mathcal{Z} \setminus \widehat{Z}}$, the restriction of q to $V_{\widehat{Z}}$ is an isomorphism onto V', and it carries $V_{\widehat{Z}} = V_Z \oplus V_{\widehat{Z} \setminus Z}$ to $V' = q(V_Z) \oplus q(V_{\widehat{Z} \setminus Z})$; the map $E'_Z = \sum_{\pi \in Z} E'_\pi$ is the projection of V' onto $q(V_Z)$ along $q(V_{\widehat{Z} \setminus Z})$. Put $U' = q(V_Z) \cap \big(q(M) + q(V_{\widehat{Z} \setminus Z})\big)$ and $L' = q(M) \cap q(V_Z)$, so $\mathrm{Cong}_Z^{\widehat{Z}}(M) = U'/L'$. The argument of the previous paragraph, applied to the projection E'_Z, gives $U' = E'_Z(q(M)) = q(E_Z(M)) = q(U)$; and $q(L) \subseteq L'$. As q is injective on V_Z, it restricts to an isomorphism of $\mathcal{O}$-modules $U \to U'$ carrying L onto $q(L)$, so $U/L \cong U'/q(L)$. Hence U'/L', a quotient of $U'/q(L)$, has finite length, $\mathrm{df}_Z^{\widehat{Z}}(M) < \infty$, and the exact sequence $0 \to L'/q(L) \to U'/q(L) \to U'/L' \to 0$ gives

$$\mathrm{length}_{\mathcal{O}}(U/L) = \mathrm{length}_{\mathcal{O}}(U'/L') + \mathrm{length}_{\mathcal{O}}(L'/q(L)). \tag{13.15}$$

The operator kills the difference. Let $y \in L'$. Then $y = q(x)$ with $x \in M$, and since $y \in q(V_Z)$ there is $z \in V_Z$ with $x - z \in V_{\mathcal{Z} \setminus \widehat{Z}}$. By the hypotheses on α, $\alpha_V(x) = \alpha_V(z) + \alpha_V(x - z) = Nz$; as $\alpha(x) \in M$, we get $Nz \in M \cap V_Z = L$. Therefore $Ny = q(Nx) = q(Nz) + q\big(N(x - z)\big) = q(Nz) \in q(L)$. So $N\,L' \subseteq q(L)$.

Bounding the difference. L' is a submodule of the finitely generated module U' over the noetherian ring $\mathcal{O}$, hence finitely generated, and it is torsion-free, hence free of some rank r'. An $\mathcal{O}$-basis of $L' \subseteq q(V_Z)$ is linearly independent over K (clear denominators), so $r' \leq \dim_K q(V_Z) = \dim_K V_Z = \sum_{\pi \in Z} m(\pi)$. By the previous paragraph $L'/q(L)$ is a quotient of $L'/NL' \cong (\mathcal{O}/N\mathcal{O})^{r'}$, so

$$\mathrm{length}_{\mathcal{O}}(L'/q(L)) \leq r' \,\mathrm{length}_{\mathcal{O}}(\mathcal{O}/N\mathcal{O}) \leq \Big(\sum_{\pi \in Z} m(\pi)\Big) \,\mathrm{length}_{\mathcal{O}}(\mathcal{O}/N\mathcal{O}).$$

Write $N = \ell^v u$ with $v = v_\ell(N)$ and u an integer prime to ℓ; u is a unit of $\mathbb{Z}_\ell$, hence of $\mathcal{O}$, so $N\mathcal{O} = \ell^v\mathcal{O}$. Multiplication by ℓ^i induces isomorphisms $\mathcal{O}/\ell\mathcal{O} \cong \ell^i\mathcal{O}/\ell^{i+1}\mathcal{O}$, so $\operatorname{length}_{\mathcal{O}}(\mathcal{O}/N\mathcal{O}) = v\, e_{\mathcal{O}}$.

Conclusion. Dividing (13.15) by $e_{\mathcal{O}} \geq 1$ and inserting the bound just proved,

$$\operatorname{df}_Z(M) = \frac{\operatorname{length}_{\mathcal{O}}(U/L)}{e_{\mathcal{O}}} \leq \frac{\operatorname{length}_{\mathcal{O}}(U'/L')}{e_{\mathcal{O}}} + \Big(\sum_{\pi\in Z} m(\pi)\Big) v_\ell(N) = \operatorname{df}_Z^{\widehat{Z}}(M) + \Big(\sum_{\pi\in Z} m(\pi)\Big) v_\ell(N).$$

□

14 Approximate orthogonality and the slope inequality: (Sp5)

This section proves the dual-slope bound (Sp5) for the class (Theorem 14.2). In our reading it is the count of zeros against poles of Stepanov's method, in the form of a slope inequality: the mean of the slopes of the source lattices of a slope record, each corrected by the height of its map, is bounded by the dual slope of the common target, up to the loss that (Sp5) makes explicit. For one map this is Bost's slope inequality (Lemma B.127), which bounds the slope of the source, corrected by the height, by the maximal slope of the target. For the family of maps it is not enough, because the maximal slope of the target can exceed its mean slope by far. Theorem 14.4 replaces the maximal slope by the mean slope. It cuts each source lattice along its Harder–Narasimhan filtration, maps each piece into one step of the filtration of the target by Bost's inequality, and bounds the rank of that step from below by counting dimensions at one complex place, where the maps form a nearly tight frame (Lemma 14.5). We follow the exposition of [2] for this argument. What Theorem 14.2 needs is therefore a frame at one complex place whose Gram blocks are nearly those of random subspaces, which is the approximate orthogonality of Alpöge's account [4].

The rest of the section reduces near-orthogonality (Nf4) to estimates for the Gram blocks $\mathrm{Gr}_{\omega\omega'}$, and the reductions are linear algebra in a finite-dimensional Hilbert space. Theorem 14.8 derives (Nf4) from six lower properties (No1)–(No6). The off-diagonal condition (No3), that $\|\mathrm{Gr}_{\omega\omega'}\|_{\mathrm{HS}}^2$ exceeds the value $D_\omega D_{\omega'}/D^*$ for random subspaces by at most a small factor, follows from pair expansions of the Gram blocks (Theorem 14.9), and these follow from row estimates (Theorem 14.13), whose main row is bounded by that same value (Theorem 14.14). The diagonal near-isometry (No4) follows from diagonal expansions (Theorem 14.16), and the dimension bound (No5) from a splitting of the kernel diagonal into a thick part and a cusp part by reduction theory (Corollary 14.18). The row estimates themselves are the subject of Sections 15 and 16. In the construction beneath (TCS), outlined in Appendix A, the frames are frames of Poincaré series; the present section uses only the Hilbert space, the maps and the estimates.

Notation for Arakelov geometry follows [110], [48] and [183]. An **arithmetic variety** is an integral scheme $\mathcal{X}$, flat and projective over $\operatorname{Spec}\mathbb{Z}$ (or over $\operatorname{Spec}\mathcal{O}_F$), of absolute dimension $D+1$, with generic fibre X. A **Hermitian line bundle** $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ is a line bundle on $\mathcal{X}$ together with a continuous metric on $\mathcal{L}_{\mathbb{C}}$ invariant under complex conjugation. The properties smooth, semipositive, nef and big of Hermitian line bundles are meant in the usual sense. Unless stated otherwise, $\overline{\mathcal{O}}(1)$ on a product of projective spaces carries the Fubini–Study metric. A **Hermitian lattice** $\overline{M} = (M, \|\cdot\|)$ is a projective $\mathcal{O}_F$-module of finite rank with Hermitian norms at the infinite places. It has an Arakelov degree $\widehat{\deg}\,\overline{M}$, a normalised slope $\widehat{\mu}(\overline{M}) = \widehat{\deg}\,\overline{M}/([F:\mathbb{Q}]\operatorname{rk} M)$ and successive minima. We write $\widehat{c}_1(\overline{\mathcal{L}})$ for the arithmetic first Chern class and $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{D+1})$ for the arithmetic self-intersection. For a cycle $\mathcal{Z}$ the **height** is $h_{\overline{\mathcal{L}}}(\mathcal{Z}) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{\dim\mathcal{Z}} \mid \mathcal{Z})$. The height of a point is normalised by $[F:\mathbb{Q}]$ only where this is stated. We set $\widehat{h}^0(\mathcal{X}, \overline{\mathcal{L}}) = \log\#\{s \in H^0(\mathcal{X}, \mathcal{L}) : \|s\|_{\sup} \leq 1\}$, and the **arithmetic volume**

is $\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) = \limsup_n \widehat{h}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})/(n^{D+1}/(D+1)!)$. Sections s with $\|s\|_{\sup} \le 1$ are called **small**. Plurisubharmonic functions and currents are understood in the sense of [79].

Definition 14.1. For a hermitian lattice $\overline{M}$ over $\mathcal{O}_K$ of positive rank, $\widehat{\mu}_{\max}(\overline{M})$ is the largest slope of a non-zero submodule (with the induced norms) and $\widehat{\mu}_{\min}(\overline{M})$ the smallest slope of a non-zero torsion-free quotient (with the quotient norms) [248], [113]. For hermitian lattices $\overline{E}, \overline{F}$ and a K-linear map $\varphi\colon E_K \to F_K$, the **height** of φ is $h(\varphi) = [K:\mathbb{Q}]^{-1} \sum_{\sigma\colon K\to\mathbb{C}} \log \|\varphi_\sigma\|$, where $\|\varphi_\sigma\|$ is the operator norm for the hermitian norms at σ.

Let $\mathcal{F}$ be a parametrised host family with slope records, with parameter ε and threshold x_1. **Frame realisations for** $\mathcal{F}$ consist of the following data, given for every real $x > x_1$ and every $i \in I(x)$, where $\mathfrak{V} = \mathfrak{V}_{x,i}$ and $d_\omega = d(\mathfrak{V}, \omega)$. When the member is fixed, the index (x, i) of these data is suppressed, in this definition and in the results that cite it.

- A number field K, an embedding $v\colon K \to \mathbb{C}$, and a hermitian lattice $\overline{R}$ over $\mathcal{O}_K$, the **realisation lattice** of the member, with dual $\overline{R}^\vee$. For every word $\omega \in \mathcal{W}(\mathfrak{V})$, a hermitian lattice $\overline{N}_\omega$ over $\mathcal{O}_K$ of rank d_ω, the **divisor lattice** of ω, and a K-linear map $B_\omega\colon N_{\omega,K} \to R_K^\vee$. These satisfy

$$\mu(\mathfrak{V},\omega) = 2\widehat{\mu}(\overline{N}_\omega), \quad \mu^{\max}(\mathfrak{V},\omega) = 2\widehat{\mu}_{\max}(\overline{N}_\omega),$$
$$\mu^\vee(\mathfrak{V}) = 2\widehat{\mu}(\overline{R}^\vee), \quad \mu^\vee_{\min}(\mathfrak{V}) = 2\widehat{\mu}_{\min}(\overline{R}^\vee).$$

- The **frame data at** v. These are a finite-dimensional complex Hilbert space $\mathcal{H}$ with a surjective $\mathbb{C}$-linear map $\Pi\colon \mathcal{H} \to R^\vee_{K,v}$, and a real number $D^* > 0$, the **reference dimension**. Moreover, for every word ω, they comprise an integer $D_\omega \ge d_\omega$, the **frame size**; a real number $\kappa_\omega > 0$, the **weight**; $\mathbb{C}$-linear maps $A_\omega\colon \mathbb{C}^{D_\omega} \to \mathcal{H}$ (the **frame map**) and $U_\omega\colon \mathbb{C}^{d_\omega} \to \mathbb{C}^{D_\omega}$ with $U_\omega^* U_\omega = 1$; and a $\mathbb{C}$-linear isomorphism $Z_\omega\colon \mathbb{C}^{d_\omega} \to N_{\omega,K,v}$. These satisfy $\Pi \circ A_\omega \circ U_\omega = B_{\omega,v} \circ Z_\omega$. The **Gram blocks** are $\mathrm{Gr}_{\omega\omega'} = (\kappa_\omega\kappa_{\omega'})^{1/2} A_\omega^* A_{\omega'}$.

A **parametrised host family with slope records and frame realisations** is a parametrised host family with slope records together with frame realisations for it. Let $c > 0$ be a real number. The phrase "for all large members" means that there is a real x_0 such that the statement holds for every real $x \ge x_0$ with $x > x_1$, every $i \in I(x)$ and every word ω of $\mathfrak{V}_{x,i}$. The family has

(Nf1) Small irreducibility defect if $0 \le \delta(\mathfrak{V}_{x,i}) \le \frac{1}{2}$ for all large members;

(Nf2) Height bound with exponent c if $s(\mathfrak{V}_{x,i}) - \mu^\vee_{\min}(\mathfrak{V}_{x,i}) \le x^c\, b(x)$ for all large members;

(Nf3) Integral maps of bounded height if for all large members we have $\mathrm{rk}\, R \ge 1$, $d_\omega \ge 1$, $B_\omega(N_\omega) \subseteq R^\vee$, $2h(B_\omega) \le h(\mathfrak{V}_{x,i}, \omega)$ if $B_\omega \ne 0$, and

$$(1 - \delta(\mathfrak{V}_{x,i}))\dim \mathcal{H} \le \mathrm{rk}\, R;$$

(Nf4) Near-orthogonal frames with exponent c if there is a real x_0 with the following property. For every real $x \ge x_0$ with $x > x_1$ there are real numbers $\bar{\varrho} \ge 0$ and $\bar{e} \ge 0$ with $(3\sqrt{\bar{\varrho}} + 4\sqrt{\bar{e}})\, x^c \le 1$ such that, with $\sigma = x^{-c}$, the frame data of every member $i \in I(x)$ satisfy the following for some real numbers $\eta_{\mathrm{off}}, \eta_{\dim}, e_{\mathrm{dg}}$ with $-1 \le \eta_{\mathrm{off}} \le \sigma/2$, $\eta_{\dim} \le \sigma/2$, $0 \le e_{\mathrm{dg}}$, $\sqrt{e_{\mathrm{dg}}} \le \sigma/2$ and $2e_{\mathrm{dg}} \le \bar{e}$:

(i) $\|\mathrm{Gr}_{\omega\omega'}\|^2_{\mathrm{HS}} \le (1 + \eta_{\mathrm{off}})\, D_\omega D_{\omega'}/D^*$ for all words $\omega \ne \omega'$ (*near-orthogonality*);

(ii) $\|\mathrm{Gr}_{\omega\omega} - 1\|^2_{\mathrm{HS}} \le e_{\mathrm{dg}}\, D_\omega$ for every word ω;

(iii) $\dim \mathcal{H} \le (1+\eta_{\mathrm{dim}})\, D^*$;
(iv) $\sum_\omega D_\omega \ge x^c D^*$;
(v) $\sum_\omega (D_\omega - d_\omega) \le \bar\varrho \sum_\omega D_\omega$.

Theorem 14.2. *Let $c > 0$ be a real number and let $\mathcal{F}$ be a parametrised host family with slope records and frame realisations that has a small irreducibility defect (Nf1), the height bound with exponent c (Nf2), integral maps of bounded height (Nf3) and near-orthogonal frames with exponent c (Nf4). Then $\mathcal{F}$ has the dual-slope bound (Sp5) with $c_\vee = c$.*

Proof. Let ε be the parameter and x_1 the threshold of $\mathcal{F}$, and let x_0 be the largest of 1, $20^{1/c}$ and the thresholds of (Nf1)–(Nf4). Fix a real $x \ge x_0$ with $x > x_1$ and put $\sigma = x^{-c}$, $b = b(x)$ and $H = x^c b$. Since $x^c \ge 20$, $0 < \sigma \le \frac{1}{20}$.

The numbers at x. Let $\bar\varrho, \bar e \ge 0$ be the numbers of (Nf4) at x, so $(3\sqrt{\bar\varrho} + 4\sqrt{\bar e})x^c \le 1$; hence $\sqrt{\bar\varrho} \le \frac{1}{3}\sigma$ and $\sqrt{\bar e} \le \frac{1}{4}\sigma$, so $\bar\varrho \le \frac{1}{4}$ and $\bar e \le \frac{1}{4}$. Put $\bar\eta = 5\sigma + 3\sqrt{\bar\varrho} \ge 0$. Then

$$(\bar\eta + 4\sqrt{\bar e})\, x^c = 5 + (3\sqrt{\bar\varrho} + 4\sqrt{\bar e})\, x^c \le 6.$$

A member. Fix $i \in I(x)$, write $\mathfrak{V} = \mathfrak{V}_{x,i}$, $\delta = \delta(\mathfrak{V})$, $s = s(\mathfrak{V})$, $\phi = s^{\max}(\mathfrak{V}) - s$, and use the frame realisation of the member. By (Nf1), $0 \le \delta \le \frac{1}{2}$, and by (Nf2), $s - \mu^\vee_{\min}(\mathfrak{V}) \le H$. It remains to find $\eta_{\mathfrak{V}} \in [0, \bar\eta]$ and $e_{\mathfrak{V}} \in [0, \bar e]$ with the inequality of (Sp5).

Compressed frames. Let $\eta_{\mathrm{off}}, \eta_{\mathrm{dim}}, e_{\mathrm{dg}}$ be the numbers of (Nf4) at the member. Lemma 14.3, applied to $I = \mathcal{W}(\mathfrak{V})$, the integers $D_\omega \ge d_\omega$, the maps A_ω, U_ω, the weights κ_ω, D^*, σ, η_{off}, η_{dim}, $e = e_{\mathrm{dg}}$ and $\delta_{\mathrm{rk}} = \bar\varrho \in [0, \frac{1}{4}]$ (its hypotheses (i)–(v) are (i)–(v) of (Nf4), (iv) because $x^c D^* = D^*/\sigma$), shows for $A'_\omega = A_\omega U_\omega$ and $Q' = \sum_\omega \kappa_\omega A'_\omega A'^*_\omega$:

$$\sum_\omega \|\kappa_\omega A'^*_\omega A'_\omega - 1\|^2_{\mathrm{HS}} \le 2e_{\mathrm{dg}} \sum_\omega d_\omega, \qquad \dim \mathcal{H} \cdot \operatorname{tr}(Q'^2) \le (1+\bar\eta)(\operatorname{tr} Q')^2.$$

Put $e_{\mathfrak{V}} = 2e_{\mathrm{dg}} \in [0, \bar e]$ and $\eta_{\mathfrak{V}} = \bar\eta$; then $e_{\mathfrak{V}} \le \frac{1}{4}$.

The slope bound. Apply Theorem 14.4 with $\overline{E} = \overline{R}^\vee$, of rank $\operatorname{rk} R \ge 1$ by (Nf3); the lattices $\overline{N}_\omega$ of rank $d_\omega \ge 1$ (Nf3) and the maps B_ω, with $B_\omega(N_\omega) \subseteq R^\vee$ and $h(B_\omega) \le h_\omega := h(\mathfrak{V}, \omega)/2$ if $B_\omega \ne 0$ (Nf3); the numbers $s/2$, $u = \phi/2$, $\widetilde{H} = H/2$; the frame data $\mathcal{H}$, Π, A'_ω, Z_ω (with $\Pi \circ A'_\omega = B_{\omega,v} \circ Z_\omega$), κ_ω; and $\delta < 1$, $e = e_{\mathfrak{V}}$, $\eta = \eta_{\mathfrak{V}}$, with $(1-\delta)\dim \mathcal{H} \le \operatorname{rk} R$ by (Nf3). Its hypotheses hold: since all ranks are positive, the identities of the frame realisation give

$$\sum_\omega d_\omega(\widehat{\mu}(\overline{N}_\omega) - h_\omega) = \frac{1}{2}\sum_\omega d_\omega(\mu(\mathfrak{V}, \omega) - h(\mathfrak{V}, \omega)) = \frac{s}{2}\sum_\omega d_\omega,$$

which is (a); (b) is $\widehat{\mu}_{\max}(\overline{N}_\omega) - h_\omega = \frac{1}{2}(\mu^{\max}(\mathfrak{V}, \omega) - h(\mathfrak{V}, \omega)) \le \frac{1}{2}s^{\max}(\mathfrak{V}) = \frac{s}{2} + u$; (c) is $\frac{s}{2} - \widehat{\mu}_{\min}(\overline{R}^\vee) = \frac{1}{2}(s - \mu^\vee_{\min}(\mathfrak{V})) \le \widetilde{H}$; $\widetilde{H} \ge 0$; $u \ge 0$, because $\mu(\mathfrak{V}, \omega) = 2\widehat{\mu}(\overline{N}_\omega) \le 2\widehat{\mu}_{\max}(\overline{N}_\omega) = \mu^{\max}(\mathfrak{V}, \omega)$ for every word, so the weighted mean s of the numbers $\mu(\mathfrak{V}, \omega) - h(\mathfrak{V}, \omega)$ is at most $s^{\max}(\mathfrak{V})$; (d) and (e) are the two displays above. Hence, with $4u\widetilde{H}/(u + \widetilde{H}) = \phi H/(\phi + H)$ (here $H > 0$),

$$\frac{s}{2} - \frac{1}{1-\delta}\Big(\frac{\phi H}{\phi + H} + (\eta_{\mathfrak{V}} + 4\sqrt{e_{\mathfrak{V}}})\frac{H}{2}\Big) \le \widehat{\mu}(\overline{R}^\vee) = \frac{1}{2}\mu^\vee(\mathfrak{V}).$$

Since $\phi H/(\phi + H) \le \phi \log(1 + H/\phi) \le 2\phi \log(1 + H/\phi)$ for $\phi > 0$ (by $v/(1+v) \le \log(1+v)$ for $v = H/\phi \ge 0$), and both sides vanish for $\phi = 0$ under the convention below,

$$\frac{s}{2} - \frac{1}{1-\delta}\Big(2\phi \log\Big(1 + \frac{H}{\phi}\Big) + (\eta_{\mathfrak{V}} + 4\sqrt{e_{\mathfrak{V}}})\frac{H}{2}\Big) \le \frac{1}{2}\mu^\vee(\mathfrak{V}),$$

the logarithmic term read as 0 when $\phi = 0$. Multiplying by 2 and using $\log|\mathcal{W}(\mathfrak{V})| \geq 0$ ($\mathcal{W}(\mathfrak{V})$ is non-empty),

$$s - \log|\mathcal{W}(\mathfrak{V})| - \frac{1}{1-\delta}\Big(4\phi\log\Big(1+\frac{H}{\phi}\Big) + (\eta_{\mathfrak{V}} + 4\sqrt{e_{\mathfrak{V}}})H\Big) \leq \mu^{\vee}(\mathfrak{V}).$$

As $x \geq x_0$ with $x > x_1$ and $i \in I(x)$ were arbitrary, $\mathcal{F}$ has (Sp5) with $c_{\vee} = c$, the threshold x_0 and the numbers $\bar{\eta}, \bar{e}$ at each x. □

Lemma 14.3. *Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space and I a finite set; for $i \in I$ let $D_i \geq d_i \geq 0$ be integers, $A_i\colon \mathbb{C}^{D_i} \to \mathcal{H}$ a linear map, $U_i\colon \mathbb{C}^{d_i} \to \mathbb{C}^{D_i}$ a linear isometric embedding ($U_i^* U_i = 1$) and $\kappa_i > 0$ a real number; the spaces $\mathbb{C}^k$ carry the standard inner product. Put $\mathrm{Gr}_{ij} = (\kappa_i\kappa_j)^{1/2} A_i^* A_j\colon \mathbb{C}^{D_j} \to \mathbb{C}^{D_i}$. Let $D^* > 0$, $0 < \sigma \leq \frac{1}{20}$, and let $\eta_{\mathrm{off}}, \eta_{\dim}, e, \delta_{\mathrm{rk}}$ be real numbers with $-1 \leq \eta_{\mathrm{off}} \leq \sigma/2$, $\eta_{\dim} \leq \sigma/2$, $0 \leq e$, $\sqrt{e} \leq \sigma/2$ and $0 \leq \delta_{\mathrm{rk}} \leq \frac{1}{4}$. Assume*

(i) $\|\mathrm{Gr}_{ij}\|_{\mathrm{HS}}^2 \leq (1+\eta_{\mathrm{off}})\, D_i D_j / D^*$ *for all $i \neq j$;*

(ii) $\|\mathrm{Gr}_{ii} - 1\|_{\mathrm{HS}}^2 \leq e\, D_i$ *for all i;*

(iii) $\dim\mathcal{H} \leq (1+\eta_{\dim})\, D^*$;

(iv) $\sum_i D_i \geq D^*/\sigma$;

(v) $\sum_i (D_i - d_i) \leq \delta_{\mathrm{rk}} \sum_i D_i$.

Let $A_i' = A_i U_i\colon \mathbb{C}^{d_i} \to \mathcal{H}$ and $Q' = \sum_i \kappa_i A_i' A_i'^*$. Then $\dim\mathcal{H} \geq 1$, $\sum_i d_i \geq \frac{3}{4}\sum_i D_i > 0$,

$$\sum_{i\in I} \|\kappa_i A_i'^* A_i' - 1\|_{\mathrm{HS}}^2 \leq 2e\sum_{i\in I} d_i \qquad \text{and} \qquad \dim\mathcal{H}\cdot\mathrm{tr}(Q'^2) \leq (1+5\sigma+3\sqrt{\delta_{\mathrm{rk}}})(\mathrm{tr}\, Q')^2.$$

Proof. Write $R = \sum_i D_i$, $n = \dim\mathcal{H}$, $Q = \sum_i \kappa_i A_i A_i^*$ and $a = \sigma/2 \leq \frac{1}{40}$; then $\eta_{\mathrm{off}}, \eta_{\dim}, \sqrt{e} \leq a$. By (iv), $R \geq D^*/\sigma > 0$, and by (v), $\sum_i d_i \geq (1-\delta_{\mathrm{rk}})R \geq \frac{3}{4}R > 0$.

Traces of Q. $\mathrm{tr}\, Q = \sum_i \kappa_i \mathrm{tr}(A_i^* A_i) = \sum_i \mathrm{tr}\,\mathrm{Gr}_{ii}$ and

$$\mathrm{tr}(Q^2) = \sum_{i,j} \kappa_i\kappa_j \mathrm{tr}(A_i A_i^* A_j A_j^*) = \sum_{i,j} \kappa_i\kappa_j \mathrm{tr}\big((A_i^* A_j)^*(A_i^* A_j)\big) = \sum_{i,j} \|\mathrm{Gr}_{ij}\|_{\mathrm{HS}}^2.$$

For the identity 1 of $\mathbb{C}^{D_i}$, $|\mathrm{tr}(\mathrm{Gr}_{ii} - 1)| = |\langle 1, \mathrm{Gr}_{ii} - 1\rangle_{\mathrm{HS}}| \leq \sqrt{D_i}\,\|\mathrm{Gr}_{ii} - 1\|_{\mathrm{HS}}$, so by (ii) $\mathrm{tr}\,\mathrm{Gr}_{ii} \geq D_i(1-\sqrt{e})$ and

$$\mathrm{tr}\, Q \geq (1-\sqrt{e})R > 0. \tag{14.1}$$

In particular $Q \neq 0$, so $\mathcal{H} \neq 0$ and $n \geq 1$; by (iii) then $1 + \eta_{\dim} > 0$. Further $\|\mathrm{Gr}_{ii}\|_{\mathrm{HS}} \leq \|1\|_{\mathrm{HS}} + \|\mathrm{Gr}_{ii} - 1\|_{\mathrm{HS}} \leq (1+\sqrt{e})\sqrt{D_i}$, and by (i) and $1 + \eta_{\mathrm{off}} \geq 0$,

$$\mathrm{tr}(Q^2) \leq (1+\eta_{\mathrm{off}})\frac{1}{D^*}\sum_{i\neq j} D_i D_j + (1+\sqrt{e})^2 R \leq \big((1+\eta_{\mathrm{off}}) + (1+\sqrt{e})^2\sigma\big)\frac{R^2}{D^*},$$

using $R \leq \sigma R^2/D^*$ (from (iv)). By (iii), $1/D^* \leq (1+\eta_{\dim})/n$, and by (14.1), $R^2 \leq (\mathrm{tr}\, Q)^2/(1-\sqrt{e})^2$. Hence

$$n\,\mathrm{tr}(Q^2) \leq \frac{\big((1+\eta_{\mathrm{off}}) + (1+\sqrt{e})^2\sigma\big)(1+\eta_{\dim})}{(1-\sqrt{e})^2}(\mathrm{tr}\, Q)^2 \leq (1+5\sigma)(\mathrm{tr}\, Q)^2. \tag{14.2}$$

Compression. Let $E_i = 1 - U_i U_i^*$, the orthogonal projection of $\mathbb{C}^{D_i}$ onto $U_i(\mathbb{C}^{d_i})^\perp$, of rank $D_i - d_i$. Then $Q - Q' = \sum_i \kappa_i A_i E_i A_i^*$ is positive semidefinite, and Q' is positive semidefinite, so $0 \le Q' \le Q$ and therefore $\operatorname{tr}(Q'^2) \le \operatorname{tr}(Q'^{1/2} Q Q'^{1/2}) = \operatorname{tr}(Q^{1/2} Q' Q^{1/2}) \le \operatorname{tr}(Q^2)$. For the traces, $\operatorname{tr} Q - \operatorname{tr} Q' = \sum_i \operatorname{tr}(\mathrm{Gr}_{ii} E_i)$ and, as above, $\operatorname{tr}(\mathrm{Gr}_{ii} E_i) = \operatorname{tr} E_i + \langle E_i, \mathrm{Gr}_{ii} - 1\rangle_{\mathrm{HS}} \le (D_i - d_i) + \sqrt{D_i - d_i}\,\sqrt{e\,D_i}$. Summing, with the Cauchy–Schwarz inequality and (v),

$$\operatorname{tr} Q - \operatorname{tr} Q' \le \delta_{\mathrm{rk}} R + \sqrt{\delta_{\mathrm{rk}} R}\sqrt{eR} = \big(\delta_{\mathrm{rk}} + \sqrt{\delta_{\mathrm{rk}} e}\big) R \le \beta \operatorname{tr} Q, \qquad \beta = \frac{\delta_{\mathrm{rk}} + \sqrt{\delta_{\mathrm{rk}} e}}{1 - \sqrt{e}},$$

by (14.1). With $\rho = \sqrt{\delta_{\mathrm{rk}}} \le \frac{1}{2}$ and $\sqrt{e} \le \frac{1}{40}$, $\beta \le \frac{40}{39}\rho(\rho + \frac{1}{40}) < 1$, so $\operatorname{tr} Q' \ge (1 - \beta) \operatorname{tr} Q > 0$, and by (14.2)

$$n \operatorname{tr}(Q'^2) \le n \operatorname{tr}(Q^2) \le \frac{1 + 5\sigma}{(1 - \beta)^2} (\operatorname{tr} Q')^2 \le (1 + 5\sigma + 3\rho)(\operatorname{tr} Q')^2.$$

Diagonal defects after compression. $\kappa_i A_i'^* A_i' - 1 = U_i^* (\mathrm{Gr}_{ii} - 1) U_i$ since $U_i^* U_i = 1$, and $\|U_i^* X U_i\|_{\mathrm{HS}} \le \|X\|_{\mathrm{HS}}$ for the isometry U_i. By (ii), $\sum_i \|\kappa_i A_i'^* A_i' - 1\|_{\mathrm{HS}}^2 \le eR \le \frac{4}{3} e \sum_i d_i \le 2e \sum_i d_i$. □

Theorem 14.4. *Let K be a number field and $v\colon K \to \mathbb{C}$ an embedding. Let $\overline{E}$ be a hermitian lattice over $\mathcal{O}_K$ of rank $r_E \ge 1$ and I a finite non-empty set; for $i \in I$ let $\overline{N}_i$ be a hermitian lattice over $\mathcal{O}_K$ of rank $d_i \ge 1$, let $B_i\colon N_{i,K} \to E_K$ be a K-linear map with $B_i(N_i) \subseteq E$, and let h_i be a real number with $h(B_i) \le h_i$ if $B_i \ne 0$. Let $s, u, \widetilde{H}$ be real numbers with $u \ge 0$, $\widetilde{H} \ge 0$ and*

(a) $\sum_i d_i(\widehat{\mu}(\overline{N}_i) - h_i) \ge s \sum_i d_i$;

(b) $\widehat{\mu}_{\max}(\overline{N}_i) - h_i \le s + u$ *for every* $i \in I$;

(c) $s - \widehat{\mu}_{\min}(\overline{E}) \le \widetilde{H}$.

Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space and $\Pi\colon \mathcal{H} \to E_{K,v}$ a surjective $\mathbb{C}$-linear map; for $i \in I$ let $A_i\colon \mathbb{C}^{d_i} \to \mathcal{H}$ be $\mathbb{C}$-linear and $Z_i\colon \mathbb{C}^{d_i} \to N_{i,K,v}$ a $\mathbb{C}$-linear isomorphism with $\Pi \circ A_i = B_{i,v} \circ Z_i$, and let $\kappa_i > 0$. Let δ, e, η be real numbers with $0 \le \delta < 1$, $(1 - \delta)\dim \mathcal{H} \le r_E$, $0 \le e \le \frac{1}{4}$, $\eta \ge 0$, and with $Q = \sum_i \kappa_i A_i A_i^*$ ($\mathbb{C}^{d_i}$ with the standard inner product)

(d) $\sum_i \|\kappa_i A_i^* A_i - 1\|_{\mathrm{HS}}^2 \le e \sum_i d_i$;

(e) $\dim \mathcal{H} \cdot \operatorname{tr}(Q^2) \le (1 + \eta)(\operatorname{tr} Q)^2$.

Then

$$s - \frac{1}{1 - \delta}\Big(\frac{4u\widetilde{H}}{u + \widetilde{H}} + (\eta + 4\sqrt{e})\widetilde{H}\Big) \le \widehat{\mu}(\overline{E}),$$

where $u\widetilde{H}/(u + \widetilde{H})$ is read as 0 when $u = \widetilde{H} = 0$.

Proof. We use the following standard facts of this theory: (HN1) $\sum_j \lambda_j(\overline{N}) = d\,\widehat{\mu}(\overline{N})$, $\lambda_1 = \widehat{\mu}_{\max}(\overline{N})$, $\lambda_d = \widehat{\mu}_{\min}(\overline{N})$; (HN2) $\operatorname{rk} N^{\ge\vartheta} = \#\{j : \lambda_j(\overline{N}) \ge \vartheta\}$, $\widehat{\mu}_{\min}(\overline{N^{\ge\vartheta}}) \ge \vartheta$ if $N^{\ge\vartheta} \ne 0$, and $\widehat{\mu}_{\max}(\overline{N/N^{\ge\vartheta}}) < \vartheta$ if $N^{\ge\vartheta} \ne N$; (HN3) the last subquotient $\overline{Q}$ of the filtration of $\overline{N}$ is semistable with $\widehat{\mu}(\overline{Q}) = \widehat{\mu}_{\max}(\overline{Q}) = \widehat{\mu}_{\min}(\overline{N})$. Put $d = \sum_i d_i \ge 1$.

Step 1: the integrated count of low slopes in the sources. Consider the d numbers $x_{ij} = \lambda_j(\overline{N}_i) - h_i$ ($i \in I$, $1 \le j \le d_i$) and put $z_{ij} = s - x_{ij}$. By (HN1) and (a), $\sum_{i,j} z_{ij} \le 0$; by

(HN1) and (b), $z_{ij} \geq -u$. For $\tau \geq 0$ let $c(\tau) = \#\{(i,j) : x_{ij} < s - \tau\}$, a step function of τ with finitely many steps. We show

$$\int_0^{\widetilde{H}} c(\tau)\, d\tau \leq \frac{u\widetilde{H}}{u+\widetilde{H}}\, d,$$

the right side read as 0 when $u = \widetilde{H} = 0$. For each (i,j) the set of $\tau \in [0,\widetilde{H}]$ with $x_{ij} < s - \tau$ is $[0,\widetilde{H}] \cap [0, z_{ij})$, of length $\min\{\widetilde{H}, (z_{ij})_+\}$, so $\int_0^{\widetilde{H}} c = \sum_{i,j} \min\{\widetilde{H}, (z_{ij})_+\}$. If $u = \widetilde{H} = 0$ this is 0. Let $u + \widetilde{H} > 0$. For every real $z \geq -u$,

$$\min\{\widetilde{H}, z_+\} \leq \frac{\widetilde{H}\,(z+u)}{u+\widetilde{H}}\,:$$

for $-u \leq z \leq 0$ the left side is 0 and the right side is ≥ 0; for $0 \leq z \leq \widetilde{H}$ the inequality $z(u+\widetilde{H}) \leq \widetilde{H}(z+u)$ reads $zu \leq \widetilde{H}u$; for $z \geq \widetilde{H}$ it reads $u + \widetilde{H} \leq z + u$. (The right side is the line through $(-u, 0)$ and $(\widetilde{H}, \widetilde{H})$.) Applying this to $z = z_{ij}$ and summing,

$$\int_0^{\widetilde{H}} c(\tau)\, d\tau \leq \frac{\widetilde{H}}{u+\widetilde{H}}\Big(\sum_{i,j} z_{ij} + du\Big) \leq \frac{u\widetilde{H}}{u+\widetilde{H}}\, d.$$

The estimate in this form, which integrates the exact count $c(\tau)$ rather than its Markov bound $c(\tau)(u+\tau) \leq du$ (whose integral gives $u\log(1+\widetilde{H}/u)$), is due to [261], Proposition 2.1.

Step 2: the high parts of the sources. For $\tau \geq 0$ let $F_i(\tau) = N_i^{\geq s-\tau+h_i}$, with the induced norms. By (HN2), $d_i - \operatorname{rk} F_i(\tau) = \#\{j : x_{ij} < s - \tau\}$, so $\sum_i (d_i - \operatorname{rk} F_i(\tau)) = c(\tau)$.

Step 3: the images lie in the high part of the target. Let $S(\tau) = E^{\geq s-\tau}$. We show $B_i(F_i(\tau)) \subseteq S(\tau)$ for every i. Assume $B_i(F_i(\tau)) \neq 0$; then $B_i \neq 0$ and $F_i(\tau) \neq 0$. Let $F' = E \cap B_i(F_i(\tau))_K$, a non-zero saturated submodule of E, with the induced norms; it contains $B_i(F_i(\tau))$. First, $\widehat{\mu}_{\min}(\overline{F'}) \geq s - \tau$. Let $\overline{Q}$ be the last subquotient of the filtration of $\overline{F'}$ and $\varphi\colon F_i(\tau)_K \to Q_K$ the composite of B_i with the projection $F'_K \to Q_K$. Then $\varphi(F_i(\tau)) \subseteq Q$, $\varphi \neq 0$ (the image of B_i on $F_i(\tau)_K$ spans F'_K, which maps onto $Q_K \neq 0$), and $\|\varphi_\sigma\| \leq \|B_{i,\sigma}\|$ for every embedding σ, since restriction to a subspace with the induced norm, the isometric inclusion $F' \subseteq E$ and the quotient map (of norm ≤ 1) do not increase operator norms; hence $h(\varphi) \leq h(B_i) \leq h_i$. By Lemma B.127, (HN2) and (HN3),

$$s - \tau + h_i \leq \widehat{\mu}_{\min}(\overline{F_i(\tau)}) \leq \widehat{\mu}_{\max}(\overline{Q}) + h(\varphi) \leq \widehat{\mu}_{\min}(\overline{F'}) + h_i.$$

Second, $F' \subseteq S(\tau)$. Otherwise $S(\tau) \neq E$ and the composite $\psi\colon F'_K \to (E/S(\tau))_K$ of the inclusion and the projection is non-zero, maps F' into the torsion-free quotient $E/S(\tau)$ (quotient norms) and has $\|\psi_\sigma\| \leq 1$, so $h(\psi) \leq 0$; Lemma B.127 and (HN2) give $s - \tau \leq \widehat{\mu}_{\min}(\overline{F'}) \leq \widehat{\mu}_{\max}(\overline{E/S(\tau)}) < s - \tau$, a contradiction. Hence $B_i(F_i(\tau)) \subseteq F' \subseteq S(\tau)$.

Step 4: the place v. Let $T(\tau) = \sum_i B_i(F_i(\tau)_K) \subseteq S(\tau)_K$; then $\dim_K T(\tau) \leq \operatorname{rk} S(\tau)$ and $\dim_{\mathbb{C}} T(\tau)_v = \dim_K T(\tau)$. Let $U_i' = Z_i^{-1}(F_i(\tau)_{K,v}) \subseteq \mathbb{C}^{d_i}$, of dimension $\operatorname{rk} F_i(\tau)$, and $\mathcal{S} = \sum_i A_i(U_i') \subseteq \mathcal{H}$. Since $\Pi(A_i(U_i')) = B_{i,v}(F_i(\tau)_{K,v}) \subseteq T(\tau)_v$, we have $\Pi(\mathcal{S}) \subseteq T(\tau)_v$. As Π is surjective onto $E_{K,v}$, of dimension r_E, $\dim\ker\Pi = \dim\mathcal{H} - r_E$, so $\dim\mathcal{S} \leq \dim\Pi(\mathcal{S}) + \dim\mathcal{H} - r_E$ and

$$r_E - \operatorname{rk} S(\tau) \leq r_E - \dim\Pi(\mathcal{S}) \leq \dim\mathcal{H} - \dim\mathcal{S} = \dim\mathcal{S}^\perp.$$

Lemma 14.5, applied with $U_i = \mathbb{C}^{d_i}$, the subspaces U_i' and $\theta = c(\tau)/d$ (its (a), (b), (c) are (d), (e) and Step 2, and $r = d > 0$), gives $\dim\mathcal{S}^\perp \leq (\eta + 4c(\tau)/d + 4\sqrt{e})\dim\mathcal{H}$. With $\dim\mathcal{H} \leq r_E/(1-\delta)$,

$$r_E - \operatorname{rk} S(\tau) \leq w(\tau)\, r_E, \qquad w(\tau) = \frac{\eta + 4c(\tau)/d + 4\sqrt{e}}{1-\delta} \qquad (\tau \geq 0).$$

Step 5: layer cake for the target. Let $\lambda_j = \lambda_j(\overline{E})$. By (HN2), $\#\{j : \lambda_j < s - \tau\} = r_E - \operatorname{rk} S(\tau) \le w(\tau) r_E$. By (HN1) and (c), $\lambda_j \ge \widehat{\mu}_{\min}(\overline{E}) \ge s - \widetilde{H}$, so for every j, $s - \lambda_j \le \int_0^{\widetilde{H}} \mathbf{1}[\lambda_j < s - \tau]\, d\tau$ (if $\lambda_j < s$ the integral is the length of $[0, s - \lambda_j) \subseteq [0, \widetilde{H}]$; otherwise the left side is ≤ 0). Summing over j and using (HN1),

$$\widehat{\mu}(\overline{E}) = \frac{1}{r_E} \sum_j \lambda_j \ge s - \int_0^{\widetilde{H}} \frac{\#\{j : \lambda_j < s - \tau\}}{r_E}\, d\tau \ge s - \int_0^{\widetilde{H}} w(\tau)\, d\tau$$

(the integrands are step functions of τ). Finally, by Step 1,

$$\int_0^{\widetilde{H}} w(\tau)\, d\tau = \frac{1}{1-\delta}\Big(\frac{4}{d}\int_0^{\widetilde{H}} c(\tau)\, d\tau + (\eta + 4\sqrt{e})\widetilde{H}\Big) \le \frac{1}{1-\delta}\Big(\frac{4u\widetilde{H}}{u + \widetilde{H}} + (\eta + 4\sqrt{e})\widetilde{H}\Big),$$

which is the assertion. □

Lemma 14.5. *Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space and I a finite set; for $i \in I$ let U_i be a finite-dimensional complex Hilbert space, $A_i \colon U_i \to \mathcal{H}$ a $\mathbb{C}$-linear map, $\kappa_i > 0$ a real number and $U_i' \subseteq U_i$ a linear subspace. Put $r = \sum_{i\in I} \dim U_i$ and $Q = \sum_{i\in I} \kappa_i A_i A_i^*$, and let e, η, θ be real numbers with $0 \le e \le \frac{1}{4}$ and $\eta \ge 0$. Assume $r > 0$ and*

(a) $\sum_{i\in I} \|\kappa_i A_i^* A_i - 1\|_{\mathrm{HS}}^2 \le e\, r$;

(b) $\dim \mathcal{H} \cdot \operatorname{tr}(Q^2) \le (1+\eta)\,(\operatorname{tr} Q)^2$;

(c) $\sum_{i\in I} (\dim U_i - \dim U_i') \le \theta\, r$.

Then

$$\dim\Big(\sum_{i\in I} A_i(U_i')\Big)^{\perp} \le (\eta + 4\theta + 4\sqrt{e})\dim \mathcal{H}.$$

Proof. Write $n = \dim \mathcal{H}$, $M_i = \kappa_i A_i^* A_i - 1$ (a self-adjoint operator on U_i), $t = \operatorname{tr} Q$, $W = \big(\sum_i A_i(U_i')\big)^{\perp}$ and $w = \dim W$. If $n = 0$ then $w = 0$ and there is nothing to prove; let $n \ge 1$. By (c), $\theta r \ge 0$, so $\theta \ge 0$. If $\theta \ge \frac{1}{4}$, then $w \le n \le 4\theta n$ and the claim holds since $\eta, \sqrt{e} \ge 0$; let $\theta < \frac{1}{4}$.

Compressed traces. For a self-adjoint operator M on a finite-dimensional Hilbert space U and an orthonormal family $f_1, \dots, f_k$ in U with orthogonal projection P onto its span,

$$\Big|\sum_{m=1}^{k} \langle f_m, M f_m\rangle\Big| = |\operatorname{tr}(PM)| \le \|P\|_{\mathrm{HS}} \|M\|_{\mathrm{HS}} = \sqrt{k}\, \|M\|_{\mathrm{HS}} \tag{14.3}$$

by the Cauchy–Schwarz inequality for the Hilbert–Schmidt inner product. Applied to M_i, with $\langle f_m, M_i f_m\rangle = \kappa_i \|A_i f_m\|^2 - 1$:

$$\Big|\kappa_i \sum_{m=1}^{k} \|A_i f_m\|^2 - k\Big| \le \sqrt{k}\, \|M_i\|_{\mathrm{HS}}. \tag{14.4}$$

Lower bound for t. Let $(f_{i,m})_m$ be an orthonormal basis of U_i. Then $t = \sum_i \kappa_i \operatorname{tr}(A_i^* A_i) = \sum_i \kappa_i \sum_m \|A_i f_{i,m}\|^2$, and (14.4), the Cauchy–Schwarz inequality for sums and (a) give

$$t \ge \sum_i \Big(\dim U_i - \sqrt{\dim U_i}\, \|M_i\|_{\mathrm{HS}}\Big) \ge r - \sqrt{r}\Big(\sum_i \|M_i\|_{\mathrm{HS}}^2\Big)^{1/2} \ge r - \sqrt{r}\sqrt{e\, r} = r(1 - \sqrt{e}).$$

Since $\sqrt{e} \le \frac12$, $t \ge r/2 > 0$.

Upper bound for the trace of Q on W. Let $g_1, \dots, g_w$ be an orthonormal basis of W and $\tau = \sum_l \langle g_l, Q g_l\rangle = \sum_i \kappa_i \sum_l \|A_i^* g_l\|^2$. Fix i; let P_i be the orthogonal projection of U_i onto $U_i'^{\perp}$, of rank $k_i = \dim U_i - \dim U_i'$, and P_W that of $\mathcal{H}$ onto W. For $u \in U_i'$, $\langle A_i^* g_l, u\rangle = \langle g_l, A_i u\rangle = 0$, so $A_i^* g_l \in U_i'^{\perp}$ and $A_i^* P_W = P_i A_i^* P_W$. Hence

$$\sum_l \|A_i^* g_l\|^2 = \|P_i A_i^* P_W\|_{\mathrm{HS}}^2 \le \|P_i A_i^*\|_{\mathrm{HS}}^2 = \|A_i P_i\|_{\mathrm{HS}}^2 = \sum_{m=1}^{k_i} \|A_i f_{i,m}'\|^2$$

for an orthonormal basis $(f_{i,m}')_m$ of $U_i'^{\perp}$, and by (14.4) $\kappa_i \sum_l \|A_i^* g_l\|^2 \le k_i + \sqrt{k_i}\|M_i\|_{\mathrm{HS}}$. Summing over i and using the Cauchy–Schwarz inequality, (c) and (a),

$$\tau \le \sum_i k_i + \Big(\sum_i k_i\Big)^{1/2}\Big(\sum_i \|M_i\|_{\mathrm{HS}}^2\Big)^{1/2} \le \theta r + \sqrt{\theta r}\sqrt{er} = r\big(\theta + \sqrt{\theta e}\big).$$

Lower bound for the trace of Q on W. Put $\bar s = t/n$ and $B = Q - \bar s \cdot 1$, self-adjoint. By (14.3) for B and (g_l), $|\tau - \bar s\, w| = \big|\sum_l \langle g_l, B g_l\rangle\big| \le \sqrt{w}\,\|B\|_{\mathrm{HS}}$, so $\tau \ge \bar s\, w - \sqrt{w}\,\|B\|_{\mathrm{HS}}$. Moreover $\|B\|_{\mathrm{HS}}^2 = \operatorname{tr}(Q^2) - t^2/n \le \eta\, t^2/n$ by (b).

Conclusion. Combining the two bounds for τ and multiplying by $n/t > 0$: with $y = (n/t)\|B\|_{\mathrm{HS}}$, so that $y^2 \le \eta n$,

$$w - \sqrt{w}\, y \le \frac{n\, r}{t}\big(\theta + \sqrt{\theta e}\big) \le 2n\big(\theta + \sqrt{\theta e}\big),$$

using $r \le 2t$. Since $\sqrt{w}\, y \le \frac12(w + y^2)$, this gives $\frac12 w \le \frac12 y^2 + 2n(\theta + \sqrt{\theta e})$, i.e.

$$w \le y^2 + 4n\theta + 4n\sqrt{\theta e} \le (\eta + 4\theta + 4\sqrt{e})n,$$

because $\sqrt{\theta e} \le \sqrt{e}$ for $\theta < \frac14$. □

Definition 14.6. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces, $K \in \mathcal{H}_1$, and let $\vartheta \ge 0$, $C_{\mathrm{pr}} \ge 0$ and $C_{\mathrm{mn}}, \beta_{\mathrm{lp}}, \beta_{\mathrm{ng}}$ be real numbers. A **pair expansion** of K at scale ϑ with **price** C_{pr}, **main constant** C_{mn}, **loop share** β_{lp} and **non-generic share** β_{ng} consists of

- a countable set $\mathcal{M}$, the **terms**, with a partition $\mathcal{M} = \mathcal{M}_{\mathrm{g}} \sqcup \mathcal{M}_{\mathrm{n}}$ into **generic** and **non-generic** terms;
- for every term t, vectors $y_t \in \mathcal{H}_1$ and $z_t \in \mathcal{H}_2$;
- an equivalence relation $\sim$ on $\mathcal{M}$ (terms in the same **row**), a finite set I and a map $\mathrm{lb}\colon \mathcal{M} \to I$, the **label**, constant on the rows;
- a real number $c_{\mathrm{rw}} > 0$, the **row normalisation**, and a matrix $\Upsilon \in \mathbb{C}^{I\times I}$, the **label matrix**;
- a function $\mathrm{cr}\colon \mathcal{M}\times\mathcal{M} \to \mathbb{Z}_{\ge 0}$, the **corank**, and a set $\mathrm{Sc} \subseteq \mathcal{M}\times\mathcal{M}$ of **scalar pairs**.

These data are subject to the conditions below. In them we use the **row pairs** $\mathrm{Pr} = \{(t,t') \in \mathcal{M}\times\mathcal{M} : t \sim t'\}$, their parts $\mathrm{Pr}_{\ge 2} = \{(t,t') \in \mathrm{Pr} : \mathrm{cr}(t,t') \ge 2\}$ and $\mathrm{Pr}_1 = \{(t,t') \in \mathrm{Pr} : \mathrm{cr}(t,t') = 1\}$, and, for $S \subseteq \mathcal{M}$ and $i \in I$, the **label sums**

$$z_{S,i} = \sum_{t\in S,\ \mathrm{lb}(t)=i} z_t \in \mathcal{H}_2.$$

(Pe1) Expansion. $\sum_{t\in\mathcal{M}}\|y_t\|<\infty$, $\sum_{t\in\mathcal{M}}\|z_t\|<\infty$ and $K=\sum_{t\in\mathcal{M}}y_t$.

(Pe2) Norm expansion. For $S=\mathcal{M}_{\mathrm{g}}$ and for $S=\mathcal{M}_{\mathrm{n}}$,

$$\Big\|\sum_{t\in S}y_t\Big\|^2=\operatorname{Re}\sum_{i,j\in I}\Upsilon_{ij}\,\langle z_{S,i},z_{S,j}\rangle.$$

(Pe3) Price. The matrix $C_{\mathrm{pr}}\,c_{\mathrm{rw}}\,1_I-\Upsilon$ is positive semidefinite.

(Pe4) Row orthogonality. $\langle z_t,z_{t'}\rangle=0$ whenever $\mathrm{lb}(t)=\mathrm{lb}(t')$ and $t\not\sim t'$.

(Pe5) Positive corank. $\mathrm{cr}(t,t')\geq 1$ for all $(t,t')\in\mathrm{Pr}$.

(Pe6) Loops. $c_{\mathrm{rw}}\sum_{(t,t')\in\mathrm{Pr}_{\geq 2}}|\langle z_t,z_{t'}\rangle|\leq\beta_{\mathrm{lp}}\vartheta$ and $c_{\mathrm{rw}}\sum_{(t,t')\in\mathrm{Pr}_1\cap(\mathcal{M}_{\mathrm{g}}\times\mathcal{M}_{\mathrm{g}})\setminus\mathrm{Sc}}|\langle z_t,z_{t'}\rangle|\leq\beta_{\mathrm{lp}}\vartheta$.

(Pe7) Main row. $c_{\mathrm{rw}}\sum_{(t,t')\in\mathrm{Pr}_1\cap(\mathcal{M}_{\mathrm{g}}\times\mathcal{M}_{\mathrm{g}})\cap\mathrm{Sc}}\operatorname{Re}\langle z_t,z_{t'}\rangle\leq C_{\mathrm{mn}}\vartheta$.

(Pe8) Non-generic row. $c_{\mathrm{rw}}\sum_{(t,t')\in\mathrm{Pr}_1\cap(\mathcal{M}_{\mathrm{n}}\times\mathcal{M}_{\mathrm{n}})}\operatorname{Re}\langle z_t,z_{t'}\rangle\leq\beta_{\mathrm{ng}}\vartheta$.

Definition 14.7. Let $\mathcal{F}$ be a parametrised host family with slope records and frame realisations, with parameter ε and threshold x_1. **Word scales for** $\mathcal{F}$ consist of a real number $\Gamma_{\mathrm{w}}=\Gamma_{\mathrm{w}}(x,i)\geq 1$ for every real $x>x_1$ and every $i\in I(x)$, the **word scale** of the member. The index is suppressed when the member is fixed. For a real number $C_0\geq 1$, a member **has word sizes for** C_0 if

$$C_0^{-1}\leq\frac{D_\omega\,\Gamma_{\mathrm{w}}^2}{D^*}\leq C_0\qquad\text{for every word }\omega.$$

The phrase "for all large members" means that there is a real x_0 such that the statement holds for every real $x\geq x_0$ with $x>x_1$, every $i\in I(x)$ and every word ω (and ω') of $\mathfrak{V}_{x,i}$. Let $C_0\geq 1$ and $c>0$ be real numbers. The family with word scales has

(No1) Word sizes for C_0 if all large members have word sizes for C_0;

(No2) Word count for C_0 **with exponent** c if $|\mathcal{W}(\mathfrak{V}_{x,i})|\geq C_0\,\Gamma_{\mathrm{w}}^2\,x^c$ for all large members;

(No3) Off-diagonal near-orthogonality for C_0 **with exponent** c if for all large members that have word sizes for C_0 and all words $\omega\neq\omega'$

$$\|\mathrm{Gr}_{\omega\omega'}\|_{\mathrm{HS}}^2\leq\Big(1+\frac{x^{-c}}{12}\Big)\frac{D_\omega D_{\omega'}}{D^*};$$

(No4) Diagonal near-isometry for C_0 **with exponent** c if for all large members that have word sizes for C_0 and every word ω, $\|\mathrm{Gr}_{\omega\omega}-1\|_{\mathrm{HS}}^2\leq x^{-2c}D_\omega/144$;

(No5) Dimension bound for C_0 **with exponent** c if $\dim\mathcal{H}\leq(1+x^{-c}/12)\,D^*$ for all large members that have word sizes for C_0;

(No6) Rank deficit with exponent c if $\sum_\omega(D_\omega-d_\omega)\leq\frac{x^{-2c}}{36}\sum_\omega D_\omega$ for all large members;

(No7) Pair expansions of the Gram blocks for C_0 **with exponent** c if for all large members that have word sizes for C_0 and all words $\omega\neq\omega'$, the Gram block $\mathrm{Gr}_{\omega\omega'}$, as an element of the space of linear maps $\mathbb{C}^{D_{\omega'}}\to\mathbb{C}^{D_\omega}$ with the Hilbert–Schmidt inner product, has a pair expansion at scale $D_\omega D_{\omega'}/D^*$ with price at most $1+x^{-c}/100$, main constant at most $1+x^{-c}/100$, and loop share and non-generic share at most $x^{-2c}/10^4$.

Theorem 14.8. *Let $c > 0$ and $C_0 \geq 1$ be real numbers and let $\mathcal{F}$ be a parametrised host family with slope records, frame realisations and word scales that has word sizes for C_0 (No1), the word count for C_0 with exponent c (No2), off-diagonal near-orthogonality (No3), diagonal near-isometry (No4) and the dimension bound (No5) for C_0 with exponent c, and the rank deficit with exponent c (No6). Then $\mathcal{F}$ has near-orthogonal frames with exponent c (Nf4).*

Proof. Let x_0 be the largest of the thresholds of (No1)–(No6). Fix a real $x \geq x_0$ with $x > x_1$, put $\sigma = x^{-c}$ and

$$\bar{\varrho} = \frac{x^{-2c}}{36}, \qquad \bar{e} = \frac{x^{-2c}}{72}.$$

Then $\bar{\varrho}, \bar{e} \geq 0$ and $(3\sqrt{\bar{\varrho}} + 4\sqrt{\bar{e}})\, x^c = \frac{1}{2} + \frac{4}{\sqrt{72}} = \frac{1}{2} + \frac{\sqrt{2}}{3} < 1$.

Fix $i \in I(x)$ and take $\eta_{\mathrm{off}} = \eta_{\mathrm{dim}} = \sigma/12$ and $e_{\mathrm{dg}} = \sigma^2/144$. Then $-1 \leq \eta_{\mathrm{off}} \leq \sigma/2$, $\eta_{\mathrm{dim}} \leq \sigma/2$, $e_{\mathrm{dg}} \geq 0$, $\sqrt{e_{\mathrm{dg}}} = \sigma/12 \leq \sigma/2$ and $2e_{\mathrm{dg}} = \bar{e}$. By (No1) the member has word sizes for C_0, so (No3), (No4) and (No5) apply to it and give (i), (ii) and (iii) of (Nf4) with these numbers. (v) is (No6) with $\bar{\varrho}$. For (iv): by word sizes $D_\omega \geq D^*/(C_0\Gamma_{\mathrm{w}}^2)$ for every word, and by (No2) there are at least $C_0\Gamma_{\mathrm{w}}^2 x^c$ words, so

$$\sum_{\omega} D_\omega \geq |\mathcal{W}(\mathfrak{V}_{x,i})|\, \frac{D^*}{C_0\Gamma_{\mathrm{w}}^2} \geq x^c D^*.$$

As $x \geq x_0$ with $x > x_1$ and $i \in I(x)$ were arbitrary, and $\bar{\varrho}, \bar{e}$ depend on x only, $\mathcal{F}$ has (Nf4) with exponent c and threshold x_0. □

Theorem 14.9. *Let $c > 0$ and $C_0 \geq 1$ be real numbers and let $\mathcal{F}$ be a parametrised host family with slope records, frame realisations and word scales that has pair expansions of the Gram blocks for C_0 with exponent c (No7). Then $\mathcal{F}$ has off-diagonal near-orthogonality for C_0 with exponent c (No3).*

Proof. Let x_0 be the threshold of (No7) and put $x_0' = \max(x_0, 1)$. Fix a real $x \geq x_0'$ with $x > x_1$, a member $i \in I(x)$ that has word sizes for C_0, and words $\omega \neq \omega'$ of $\mathfrak{V}_{x,i}$; put $a = x^{-c}/100$, so $0 < a \leq \frac{1}{100}$ because $x \geq 1$. Word sizes give $D_\omega, D_{\omega'} \geq D^*/(C_0\Gamma_{\mathrm{w}}^2) > 0$, so $\vartheta = D_\omega D_{\omega'}/D^* > 0$.

By (No7), $\mathrm{Gr}_{\omega\omega'}$, in the space of linear maps $\mathbb{C}^{D_{\omega'}} \to \mathbb{C}^{D_\omega}$ with the Hilbert–Schmidt inner product, has a pair expansion at scale ϑ with price $C_{\mathrm{pr}} \in [0, 1+a]$, main constant $C_{\mathrm{mn}} \leq 1+a$, and loop share β_{lp} and non-generic share β_{ng} at most a^2. By Lemma 14.10,

$$\|\mathrm{Gr}_{\omega\omega'}\|_{\mathrm{HS}} \leq \sqrt{C_{\mathrm{pr}}(C_{\mathrm{mn}} + 2\beta_{\mathrm{lp}})\vartheta} + \sqrt{C_{\mathrm{pr}}(\beta_{\mathrm{lp}} + \beta_{\mathrm{ng}})\vartheta}.$$

Since $C_{\mathrm{pr}} \geq 0$, $C_{\mathrm{mn}} + 2\beta_{\mathrm{lp}} \leq 1 + a + 2a^2$ and $\beta_{\mathrm{lp}} + \beta_{\mathrm{ng}} \leq 2a^2$, the radicands are at most $X\vartheta$ and $Y\vartheta$ with $X = (1+a)(1+a+2a^2)$ and $Y = 2a^2(1+a)$, and the square root is increasing; hence

$$\|\mathrm{Gr}_{\omega\omega'}\|_{\mathrm{HS}}^2 \leq \big(X + Y + 2\sqrt{XY}\big)\vartheta \leq (1+5a)\,\vartheta \leq \Big(1 + \frac{x^{-c}}{12}\Big)\frac{D_\omega D_{\omega'}}{D^*}.$$

As $x \geq x_0'$ with $x > x_1$, the member with word sizes for C_0 and the words $\omega \neq \omega'$ were arbitrary, (No3) holds for C_0 with exponent c and threshold x_0'. □

Lemma 14.10. *Let $\mathcal{H}_1, \mathcal{H}_2$ be complex Hilbert spaces and $K \in \mathcal{H}_1$. Let $\vartheta \geq 0$ and $C_{\mathrm{pr}} \geq 0$, let $C_{\mathrm{mn}}, \beta_{\mathrm{lp}}, \beta_{\mathrm{ng}}$ be real numbers, and suppose that K has a pair expansion at scale ϑ with price C_{pr}, main constant C_{mn}, loop share β_{lp} and non-generic share β_{ng}. Then $C_{\mathrm{pr}}(C_{\mathrm{mn}} + 2\beta_{\mathrm{lp}})\vartheta \geq 0$, $C_{\mathrm{pr}}(\beta_{\mathrm{lp}} + \beta_{\mathrm{ng}})\vartheta \geq 0$ and*

$$\|K\| \leq \sqrt{C_{\mathrm{pr}}\,(C_{\mathrm{mn}} + 2\beta_{\mathrm{lp}})\,\vartheta} + \sqrt{C_{\mathrm{pr}}\,(\beta_{\mathrm{lp}} + \beta_{\mathrm{ng}})\,\vartheta}\,.$$

Proof. Fix the data of the pair expansion. For $S \subseteq \mathcal{M}$ put $Y_S = \sum_{t\in S} y_t$.

Convergence. By (Pe1) the families (y_t) and (z_t) are absolutely summable, so every Y_S and every $z_{S,i}$ is an absolutely convergent sum, and $K = Y_{\mathcal{M}_\mathrm{g}} + Y_{\mathcal{M}_\mathrm{n}}$ because $\mathcal{M} = \mathcal{M}_\mathrm{g} \sqcup \mathcal{M}_\mathrm{n}$. By the Cauchy–Schwarz inequality $\sum_{(t,t')\in\mathcal{M}\times\mathcal{M}} |\langle z_t, z_{t'}\rangle| \le \big(\sum_t \|z_t\|\big)^2 < \infty$, so every sum over a set of pairs below converges absolutely and may be split and rearranged.

The label matrix. Let $M \in \mathbb{C}^{I\times I}$ be positive semidefinite and $a_i \in \mathcal{H}_2$ $(i \in I)$. Choose an orthonormal basis $e_1, \dots, e_n$ of the finite-dimensional span of the a_i and put $w_{k,i} = \langle e_k, a_i\rangle$; then $a_i = \sum_k w_{k,i} e_k$ and $\langle a_i, a_j\rangle = \sum_k \overline{w_{k,i}}\, w_{k,j}$, so

$$\sum_{i,j} M_{ij}\langle a_i, a_j\rangle = \sum_{k=1}^{n} \sum_{i,j} \overline{w_{k,i}}\, M_{ij}\, w_{k,j} \ge 0.$$

With $M = C_\mathrm{pr}c_\mathrm{rw}1_I - \Upsilon$, positive semidefinite by (Pe3), this gives

$$\mathrm{Re}\sum_{i,j} \Upsilon_{ij}\langle a_i, a_j\rangle = C_\mathrm{pr}c_\mathrm{rw}\sum_i \|a_i\|^2 - \sum_{i,j} M_{ij}\langle a_i, a_j\rangle \le C_\mathrm{pr}c_\mathrm{rw}\sum_i \|a_i\|^2.$$

Applied to $a_i = z_{S,i}$, (Pe2) yields for $S \in \{\mathcal{M}_\mathrm{g}, \mathcal{M}_\mathrm{n}\}$

$$\|Y_S\|^2 \le C_\mathrm{pr}\, c_\mathrm{rw} \sum_{i\in I} \|z_{S,i}\|^2. \tag{14.5}$$

Pair form of the label sums. For $S \subseteq \mathcal{M}$ and $i \in I$, $\|z_{S,i}\|^2 = \sum\langle z_t, z_{t'}\rangle$ over the pairs $t, t' \in S$ with $\mathrm{lb}(t) = \mathrm{lb}(t') = i$. By (Pe4) the pairs with $t \not\sim t'$ contribute 0; since lb is constant on rows, the remaining pairs are the $(t, t') \in \mathrm{Pr} \cap (S \times S)$ with $\mathrm{lb}(t) = i$. Summing over $i \in I$, every row pair is counted once, and since the left side is real,

$$\sum_{i\in I} \|z_{S,i}\|^2 = \sum_{(t,t')\in \mathrm{Pr}\cap(S\times S)} \mathrm{Re}\langle z_t, z_{t'}\rangle. \tag{14.6}$$

Generic terms. By (Pe5), $\mathrm{Pr} \cap (\mathcal{M}_\mathrm{g} \times \mathcal{M}_\mathrm{g})$ is the disjoint union of its parts in $\mathrm{Pr}_{\ge 2}$, in $\mathrm{Pr}_1 \setminus \mathrm{Sc}$ and in $\mathrm{Pr}_1 \cap \mathrm{Sc}$. Bounding $\mathrm{Re}\langle z_t, z_{t'}\rangle \le |\langle z_t, z_{t'}\rangle|$ on the first two parts and enlarging the first part to $\mathrm{Pr}_{\ge 2}$, (14.6), (Pe6) and (Pe7) give

$$c_\mathrm{rw} \sum_i \|z_{\mathcal{M}_\mathrm{g},i}\|^2 \le \beta_\mathrm{lp}\vartheta + \beta_\mathrm{lp}\vartheta + C_\mathrm{mn}\vartheta.$$

Non-generic terms. In the same way, by (Pe5) $\mathrm{Pr}\cap(\mathcal{M}_\mathrm{n}\times\mathcal{M}_\mathrm{n})$ is the disjoint union of its parts in $\mathrm{Pr}_{\ge 2}$ and in Pr_1, and (14.6), the first inequality of (Pe6) and (Pe8) give $c_\mathrm{rw}\sum_i \|z_{\mathcal{M}_\mathrm{n},i}\|^2 \le (\beta_\mathrm{lp} + \beta_\mathrm{ng})\vartheta$.

Conclusion. Multiplying the last two bounds by $C_\mathrm{pr} \ge 0$ and using (14.5),

$$0 \le \|Y_{\mathcal{M}_\mathrm{g}}\|^2 \le C_\mathrm{pr}(C_\mathrm{mn} + 2\beta_\mathrm{lp})\vartheta, \qquad 0 \le \|Y_{\mathcal{M}_\mathrm{n}}\|^2 \le C_\mathrm{pr}(\beta_\mathrm{lp} + \beta_\mathrm{ng})\vartheta,$$

which proves the two non-negativity claims. Taking square roots and using $\|K\| \le \|Y_{\mathcal{M}_\mathrm{g}}\| + \|Y_{\mathcal{M}_\mathrm{n}}\|$ proves the bound. □

Definition 14.11. Let S_mn be a real number and let $d, d', D^* > 0$ and $c_\mathrm{rw} > 0$ be real numbers. **Main-row mass data** for $(S_\mathrm{mn}; c_\mathrm{rw}, d, d', D^*)$ with **density factor** e_1 and **Poisson error** δ_P consist of real numbers

$$V,\ J,\ J_\infty,\ e_1,\ \delta_\mathrm{P}$$

and positive real numbers

$$\rho,\ m_0,\ c,\ c',\ C_\infty,\ \deg_\pi,\ U_{\mathrm{loc}},\ M_{\mathrm{loc}},\ v_Z,\ v_B,\ v_G,\ v_H,\ v_{H'},$$
$$v_{H,\infty},\ v_{H',\infty},\ n_\pi,\ n_{\pi'},\ \mu_K,\ \mu_K^\natural,\ \mu_K^\circ,\ \mu_S,\ \mu_{S'},\ \mu_H,\ \mu_{H'},\ \mu_H^\circ,\ \mu_{H'}^\circ$$

such that the following hold.

(Ma1) Form of the main row. $S_{\mathrm{mn}} = \rho\, V$.

(Ma2) Main-row inequality. $e_1 \geq 0$ and $c\, c'\, V \leq (1+\delta_{\mathrm{P}})\, e_1\, m_0\, J$.

(Ma3) Mass of the main orbit. $m_0\, \rho = v_B\, C_\infty\, M_{\mathrm{loc}}/(v_Z\, \mu_K)$.

(Ma4) Poisson error. $\delta_{\mathrm{P}} \geq 0$.

(Ma5) Archimedean integral. $J = v_Z\, J_\infty/(C_\infty\, U_{\mathrm{loc}})$ and $J_\infty \leq 1/\deg_\pi$.

(Ma6) Local product formula. $M_{\mathrm{loc}}/U_{\mathrm{loc}} = \mu_K\, \mu_K^\natural/(\mu_S\, \mu_{S'})$ and $\mu_K^\natural = c_{\mathrm{rw}}\, \mu_K^\circ$.

(Ma7) Unfolding constants. $c\, \mu_S = \mu_H\, v_{H,\infty}/n_\pi$ and $c'\, \mu_{S'} = \mu_{H'}\, v_{H',\infty}/n_{\pi'}$.

(Ma8) Word dimensions. $d = n_\pi\, v_H/(v_{H,\infty}\, \mu_H^\circ)$, $d' = n_{\pi'}\, v_{H'}/(v_{H',\infty}\, \mu_{H'}^\circ)$, $\mu_H = c_{\mathrm{rw}}\, \mu_H^\circ$ and $\mu_{H'} = c_{\mathrm{rw}}\, \mu_{H'}^\circ$.

(Ma9) Volume identity and reference dimension. $v_B\, v_G = v_H\, v_{H'}$ and $D^* = \deg_\pi\, v_G/\mu_K^\circ$.

Definition 14.12. Let $\mathcal{F}$ be a parametrised host family with slope records, frame realisations and word scales, with parameter ε and threshold x_1. **Row parameters for $\mathcal{F}$** are, for every real $x > x_1$ and every $i \in I(x)$, integers $k_{\mathrm{d}} \geq 1$ (the **definite rank**), $r_{\mathrm{f}} \geq 0$ (the **flag rank**) and $f_\infty \geq 1$, integers $g_1^{\mathrm{aux}}, g_2^{\mathrm{aux}} \geq 3$ (the **auxiliary moduli**) and a real number $T_{\mathrm{amp}} > 0$ (the **amplitude bound**).

Fix a member and words $\omega \neq \omega'$, and put $\vartheta = D_\omega D_{\omega'}/D^*$. Regard $\mathrm{Gr}_{\omega\omega'}$ as a vector of the space of linear maps $\mathbb{C}^{D_{\omega'}} \to \mathbb{C}^{D_\omega}$ with the Hilbert–Schmidt inner product. A **main-row expansion** of $\mathrm{Gr}_{\omega\omega'}$ with price C_{pr}, loop share β_{lp} and non-generic share β_{ng} consists of data as in Definition 14.6 for this vector at scale ϑ, satisfying (Pe1)–(Pe6) and (Pe8), together with main-row mass data for $(S_{\mathrm{mn}}; c_{\mathrm{rw}}, D_\omega, D_{\omega'}, D^*)$, where

$$S_{\mathrm{mn}} = \sum_{(t,t') \in \mathrm{Pr}_1 \cap (\mathcal{M}_{\mathrm{g}} \times \mathcal{M}_{\mathrm{g}}) \cap \mathrm{Sc}} \mathrm{Re}\langle z_t, z_{t'} \rangle$$

is the **main-row sum**.

Let $C_0 \geq 1$, $C_{\mathrm{A}} \geq 1$, $c_{\mathrm{lp}} > 0$, $C_{\mathrm{ng}} \geq 0$ and $c > 0$ be real numbers. The family with row parameters has each of the following properties if the stated condition holds.

(Rw1) Row estimates for C_0 with constants $(C_{\mathrm{A}}, c_{\mathrm{lp}}, C_{\mathrm{ng}})$ For all large members that have word sizes for C_0 and all words $\omega \neq \omega'$, the Gram block $\mathrm{Gr}_{\omega\omega'}$ has a main-row expansion with

$$C_{\mathrm{pr}} = \Big(1 + \frac{3}{g_1^{\mathrm{aux}}}\Big)^{f_\infty} \Big(1 + \frac{3}{g_2^{\mathrm{aux}}}\Big)^{f_\infty}, \qquad \beta_{\mathrm{lp}} = \exp(-c_{\mathrm{lp}}\, k_{\mathrm{d}} f_\infty \log k_{\mathrm{d}}),$$
$$\beta_{\mathrm{ng}} = 2^{2r_{\mathrm{f}}+5-k_{\mathrm{d}}} \exp\big(C_{\mathrm{ng}}(2r_{\mathrm{f}}+2)^5 f_\infty \log b(x)\big),$$

whose mass data have density factor $e_1 \leq \exp\big(12 f_\infty\, 2^{2r_{\mathrm{f}}+2-k_{\mathrm{d}}}\big)$ and Poisson error $\delta_{\mathrm{P}} \leq C_{\mathrm{A}}(r_{\mathrm{f}}+1)(C_{\mathrm{A}}/T_{\mathrm{amp}})^{f_\infty}$.

(Rw2) Row sizes with exponent c for $(C_{\mathrm{A}}, c_{\mathrm{lp}}, C_{\mathrm{ng}})$ For all large members,

$$g_1^{\mathrm{aux}},\, g_2^{\mathrm{aux}} \ge 1300\, f_\infty\, x^c, \qquad C_{\mathrm{A}}(r_{\mathrm{f}}+1)\Big(\frac{C_{\mathrm{A}}}{T_{\mathrm{amp}}}\Big)^{f_\infty} \le \frac{x^{-c}}{400}, \qquad 12 f_\infty\, 2^{2r_{\mathrm{f}}+2-k_{\mathrm{d}}} \le \frac{x^{-c}}{400},$$

$$\exp(-c_{\mathrm{lp}}\, k_{\mathrm{d}} f_\infty \log k_{\mathrm{d}}) \le \frac{x^{-2c}}{10^4}, \qquad 2^{2r_{\mathrm{f}}+5-k_{\mathrm{d}}} \exp\big(C_{\mathrm{ng}}(2r_{\mathrm{f}}+2)^5 f_\infty \log b(x)\big) \le \frac{x^{-2c}}{10^4}.$$

Theorem 14.13. *Let $c > 0$, $C_0 \ge 1$, $C_{\mathrm{A}} \ge 1$, $c_{\mathrm{lp}} > 0$ and $C_{\mathrm{ng}} \ge 0$ be real numbers, and let $\mathcal{F}$ be a parametrised host family with slope records, frame realisations, word scales and row parameters that has row estimates for C_0 with constants $(C_{\mathrm{A}}, c_{\mathrm{lp}}, C_{\mathrm{ng}})$ (Rw1) and row sizes with exponent c for $(C_{\mathrm{A}}, c_{\mathrm{lp}}, C_{\mathrm{ng}})$ (Rw2). Then $\mathcal{F}$ has pair expansions of the Gram blocks for C_0 with exponent c (No7).*

Proof. Let x_0 be a real number above which both (Rw1) and (Rw2) hold, and put $x_0' = \max(x_0, 1)$. Fix a real $x \ge x_0'$ with $x > x_1$, a member $i \in I(x)$ that has word sizes for C_0, and words $\omega \ne \omega'$; write $\sigma = x^{-c}$, so $0 < \sigma \le 1$. Word sizes give $D_\omega, D_{\omega'} \ge D^*/(C_0\Gamma_{\mathrm{w}}^2) > 0$, so $\vartheta = D_\omega D_{\omega'}/D^* > 0$.

A pair expansion. By (Rw1), $\mathrm{Gr}_{\omega\omega'}$ has a main-row expansion at scale ϑ with price $C_{\mathrm{pr}} = (1+3/g_1^{\mathrm{aux}})^{f_\infty}(1+3/g_2^{\mathrm{aux}})^{f_\infty}$, loop share β_{lp} and non-generic share β_{ng} as in (Rw1), and with main-row mass data for $(S_{\mathrm{mn}}; c_{\mathrm{rw}}, D_\omega, D_{\omega'}, D^*)$ whose density factor e_1 and Poisson error δ_{P} obey the bounds of (Rw1). By Theorem 14.14,

$$c_{\mathrm{rw}} \sum_{(t,t')\in \mathrm{Pr}_1 \cap (\mathcal{M}_{\mathrm{g}}\times\mathcal{M}_{\mathrm{g}})\cap \mathrm{Sc}} \mathrm{Re}\langle z_t, z_{t'}\rangle = c_{\mathrm{rw}} S_{\mathrm{mn}} \le (1+\delta_{\mathrm{P}})\, e_1\, \vartheta,$$

which is (Pe7) with $C_{\mathrm{mn}} = (1+\delta_{\mathrm{P}})e_1$. Together with (Pe1)–(Pe6) and (Pe8), and since $\vartheta \ge 0$ and $C_{\mathrm{pr}} \ge 0$, the data are a pair expansion of $\mathrm{Gr}_{\omega\omega'}$ at scale ϑ with price C_{pr}, main constant C_{mn}, loop share β_{lp} and non-generic share β_{ng}.

The price. Since $1 + t \le e^t$ for real t, and by the first size condition of (Rw2),

$$C_{\mathrm{pr}} \le \exp\Big(\frac{3f_\infty}{g_1^{\mathrm{aux}}} + \frac{3f_\infty}{g_2^{\mathrm{aux}}}\Big) \le \exp(a), \qquad a = \frac{6f_\infty}{1300 f_\infty x^c} = \frac{3\sigma}{650} \in (0, 1].$$

By convexity $e^a \le 1 + (e-1)a \le 1 + 2a$ for $0 \le a \le 1$, so $C_{\mathrm{pr}} \le 1 + 6\sigma/650 \le 1 + \sigma/100$.

The main constant. By (Ma4) and (Rw1), (Rw2), $0 \le \delta_{\mathrm{P}} \le C_{\mathrm{A}}(r_{\mathrm{f}}+1)(C_{\mathrm{A}}/T_{\mathrm{amp}})^{f_\infty} \le \sigma/400$; by (Ma2) $e_1 \ge 0$, and $e_1 \le \exp(12 f_\infty 2^{2r_{\mathrm{f}}+2-k_{\mathrm{d}}}) \le \exp(\sigma/400)$. Hence, as $1 + \delta_{\mathrm{P}} \ge 0$,

$$C_{\mathrm{mn}} = (1+\delta_{\mathrm{P}})e_1 \le e^{\delta_{\mathrm{P}}} e^{\sigma/400} \le e^{\sigma/200} \le 1 + \frac{\sigma}{100},$$

the last step by $e^a \le 1 + 2a$ with $a = \sigma/200 \in (0, 1]$.

The shares. By the last two size conditions of (Rw2), $\beta_{\mathrm{lp}} \le \sigma^2/10^4 = x^{-2c}/10^4$ and $\beta_{\mathrm{ng}} \le x^{-2c}/10^4$.

As the member with word sizes for C_0 and the words $\omega \ne \omega'$ were arbitrary, (No7) holds for C_0 with exponent c and threshold x_0'. □

Theorem 14.14. *Let S_{mn} be a real number and let $c_{\mathrm{rw}}, d, d', D^* > 0$. Suppose that $(S_{\mathrm{mn}}; c_{\mathrm{rw}}, d, d', D^*)$ has main-row mass data with density factor e_1 and Poisson error δ_{P}. Then*

$$c_{\mathrm{rw}}\, S_{\mathrm{mn}} \le (1+\delta_{\mathrm{P}})\, e_1\, \frac{d\, d'}{D^*}.$$

Proof. **The kernel value is at most the Poisson and density factors times the orbit mass over the formal degree.** By (Ma5) and since $v_Z/(C_\infty U_{\mathrm{loc}}) > 0$, $J = \frac{v_Z}{C_\infty U_{\mathrm{loc}}} J_\infty \le \frac{v_Z}{C_\infty U_{\mathrm{loc}} \deg_\pi}$. By (Ma2) and (Ma4), $e_1 \ge 0$ and $1 + \delta_{\mathrm{P}} \ge 1$, and $m_0 > 0$, $cc' > 0$; so $(1 + \delta_{\mathrm{P}}) e_1 m_0/(cc') \ge 0$ and the inequality of (Ma2) gives

$$V \le \frac{(1+\delta_{\mathrm{P}})\, e_1\, m_0\, J}{c\, c'} \le \frac{(1+\delta_{\mathrm{P}})\, e_1\, m_0\, v_Z}{c\, c'\, C_\infty\, U_{\mathrm{loc}}\, \deg_\pi}. \tag{14.7}$$

The main row. Since $\rho > 0$, (Ma1) and (14.7) give, with $A = m_0\rho = v_B C_\infty M_{\mathrm{loc}}/(v_Z \mu_K)$ by (Ma3),

$$S_{\mathrm{mn}} = \rho V \le (1+\delta_{\mathrm{P}})\, e_1\, A\, \frac{v_Z}{c\, c'\, C_\infty\, U_{\mathrm{loc}}\, \deg_\pi}, \tag{14.8}$$

that is

$$S_{\mathrm{mn}} \le (1+\delta_{\mathrm{P}})\, e_1\, \frac{v_B}{\mu_K \deg_\pi\, c\, c'} \cdot \frac{M_{\mathrm{loc}}}{U_{\mathrm{loc}}}. \tag{14.9}$$

The masses combine to the random-subspace value. By (Ma6), $\frac{M_{\mathrm{loc}}}{U_{\mathrm{loc}}} = \frac{\mu_K\, c_{\mathrm{rw}}\, \mu_K^\circ}{\mu_S\, \mu_{S'}}$. By (Ma7) and (Ma8),

$$c\, \mu_S = \frac{\mu_H v_{H,\infty}}{n_\pi} = \frac{c_{\mathrm{rw}}\, \mu_H^\circ v_{H,\infty}}{n_\pi} = \frac{c_{\mathrm{rw}}\, v_H}{d}, \qquad c'\, \mu_{S'} = \frac{c_{\mathrm{rw}}\, v_{H'}}{d'}.$$

Hence, using (Ma9) in the last two steps,

$$\frac{v_B}{\mu_K \deg_\pi\, c\, c'} \cdot \frac{M_{\mathrm{loc}}}{U_{\mathrm{loc}}} = \frac{v_B\, c_{\mathrm{rw}}\, \mu_K^\circ}{\deg_\pi\, (c\mu_S)(c'\mu_{S'})} = \frac{v_B\, \mu_K^\circ\, d\, d'}{c_{\mathrm{rw}} \deg_\pi\, v_H v_{H'}} = \frac{\mu_K^\circ\, d\, d'}{c_{\mathrm{rw}} \deg_\pi\, v_G} = \frac{d\, d'}{c_{\mathrm{rw}} D^*}.$$

Inserting this into (14.9) and multiplying by $c_{\mathrm{rw}} > 0$ proves the claim. □

Definition 14.15. Let $\mathcal{F}$ be a parametrised host family with slope records, frame realisations and word scales, and let $C_0 \ge 1$ and $c > 0$ be real numbers. For a member and a word ω, a **diagonal expansion** of $\mathrm{Gr}_{\omega\omega}$ with **degenerate share** $\epsilon \ge 0$ is a decomposition

$$\mathrm{Gr}_{\omega\omega} - 1 = K_{\mathrm{dg}} + K_{\mathrm{dm}}$$

in the space of linear maps $\mathbb{C}^{D_\omega} \to \mathbb{C}^{D_\omega}$ with the Hilbert–Schmidt inner product, together with a pair expansion of K_{dm} at scale D_ω^2/D^* with price C_{pr}, main constant C_{mn}, loop share β_{lp} and non-generic share β_{ng}, such that $\|K_{\mathrm{dg}}\|_{\mathrm{HS}}^2 \le \epsilon D_\omega$, $C_{\mathrm{mn}} + 2\beta_{\mathrm{lp}} \ge 0$, $\beta_{\mathrm{lp}} + \beta_{\mathrm{ng}} \ge 0$ and

$$C_{\mathrm{pr}}\Big(\sqrt{C_{\mathrm{mn}} + 2\beta_{\mathrm{lp}}} + \sqrt{\beta_{\mathrm{lp}} + \beta_{\mathrm{ng}}}\Big)^2 \le \Gamma_{\mathrm{w}}.$$

The family has

(Dx1) Diagonal expansions for C_0 with exponent c if for all large members that have word sizes for C_0 and every word ω, $\mathrm{Gr}_{\omega\omega}$ has a diagonal expansion with a degenerate share ϵ satisfying

$$2\epsilon + \frac{2C_0}{\Gamma_{\mathrm{w}}} \le \frac{x^{-2c}}{144}.$$

Theorem 14.16. *Let $C_0 \ge 1$ and $c > 0$ be real numbers and let $\mathcal{F}$ be a parametrised host family with slope records, frame realisations and word scales that has diagonal expansions for C_0 with exponent c (Dx1). Then $\mathcal{F}$ has the diagonal near-isometry for C_0 with exponent c (No4).*

Proof. Let x_0 be a threshold for (Dx1). Fix a real $x \geq x_0$ with $x > x_1$, a member $i \in I(x)$ that has word sizes for C_0, and a word ω, and take a diagonal expansion $\mathrm{Gr}_{\omega\omega} - 1 = K_{\mathrm{dg}} + K_{\mathrm{dm}}$ with degenerate share ϵ as in (Dx1); let $C_{\mathrm{pr}}, C_{\mathrm{mn}}, \beta_{\mathrm{lp}}, \beta_{\mathrm{ng}}$ be the price, main constant and shares of the pair expansion of K_{dm}, at scale $\vartheta = D_\omega^2/D^* \geq 0$.

The generic part. By Lemma 14.10, and since $C_{\mathrm{pr}} \geq 0$, $\vartheta \geq 0$, $C_{\mathrm{mn}} + 2\beta_{\mathrm{lp}} \geq 0$ and $\beta_{\mathrm{lp}} + \beta_{\mathrm{ng}} \geq 0$,

$$\|K_{\mathrm{dm}}\|_{\mathrm{HS}} \leq \sqrt{C_{\mathrm{pr}}(C_{\mathrm{mn}} + 2\beta_{\mathrm{lp}})\vartheta} + \sqrt{C_{\mathrm{pr}}(\beta_{\mathrm{lp}} + \beta_{\mathrm{ng}})\vartheta} = \sqrt{C_{\mathrm{pr}}\vartheta}\Big(\sqrt{C_{\mathrm{mn}} + 2\beta_{\mathrm{lp}}} + \sqrt{\beta_{\mathrm{lp}} + \beta_{\mathrm{ng}}}\Big),$$

so $\|K_{\mathrm{dm}}\|_{\mathrm{HS}}^2 \leq \Gamma_{\mathrm{w}}\vartheta$ by the defining inequality of a diagonal expansion. Word sizes give $D_\omega\Gamma_{\mathrm{w}}^2/D^* \leq C_0$, hence $\vartheta = D_\omega \cdot D_\omega/D^* \leq C_0 D_\omega/\Gamma_{\mathrm{w}}^2$ and

$$\|K_{\mathrm{dm}}\|_{\mathrm{HS}}^2 \leq \frac{C_0 D_\omega}{\Gamma_{\mathrm{w}}}.$$

Conclusion. By the triangle inequality and $(a+b)^2 \leq 2a^2 + 2b^2$,

$$\|\mathrm{Gr}_{\omega\omega} - 1\|_{\mathrm{HS}}^2 \leq 2\|K_{\mathrm{dg}}\|_{\mathrm{HS}}^2 + 2\|K_{\mathrm{dm}}\|_{\mathrm{HS}}^2 \leq \Big(2\epsilon + \frac{2C_0}{\Gamma_{\mathrm{w}}}\Big)D_\omega \leq \frac{x^{-2c}}{144}D_\omega.$$

As x, the member with word sizes and ω were arbitrary, (No4) holds for C_0 with exponent c and threshold x_0. □

Definition 14.17. The symbols $X, \mu, X_{\mathrm{t}}, X_{\mathrm{c}}, \mathrm{kd}, a, c_{\mathrm{id}}, \epsilon_{\mathrm{o}}, \epsilon_{\mathrm{c}}$ are local to this definition and to the results that apply it. Let $\mathcal{F}$ be a parametrised host family with slope records, frame realisations and word scales. For a member, with frame data $\mathcal{H}$ and D^*, and real numbers $\epsilon_{\mathrm{o}}, \epsilon_{\mathrm{c}} \geq 0$, a **kernel splitting** of the member with **off-identity error** ϵ_{o} and **cusp error** ϵ_{c} consists of a measure space (X, μ), disjoint measurable sets $X_{\mathrm{t}}, X_{\mathrm{c}} \subseteq X$ with $X = X_{\mathrm{t}} \cup X_{\mathrm{c}}$ (the **thick part** and the **cusp part**), a measurable function $\mathrm{kd}\colon X \to [0, \infty)$ (the **kernel diagonal**), a measurable function $a\colon X_{\mathrm{t}} \to [0, \infty]$ (the **off-identity majorant**) and a real number $c_{\mathrm{id}} \geq 0$ (the **identity density**), such that

$$\int_X \mathrm{kd}\, d\mu = \dim \mathcal{H}, \qquad c_{\mathrm{id}}\, \mu(X_{\mathrm{t}}) \leq D^*, \tag{14.10}$$

$$\mathrm{kd}(x) \leq c_{\mathrm{id}} + a(x) \quad \text{for every } x \in X_{\mathrm{t}}, \tag{14.11}$$

$$\int_{X_{\mathrm{t}}} a\, d\mu \leq \epsilon_{\mathrm{o}}\, D^*, \qquad \int_{X_{\mathrm{c}}} \mathrm{kd}\, d\mu \leq \epsilon_{\mathrm{c}} \dim \mathcal{H}. \tag{14.12}$$

Let $C_0 \geq 1$ and $c > 0$ be real numbers. The family has

(Dk1) Kernel splittings for C_0 with exponent c if for all large members that have word sizes for C_0 there are real numbers $\epsilon_{\mathrm{o}} \geq 0$ and $\epsilon_{\mathrm{c}} \in [0, 1)$ with

$$1 + \epsilon_{\mathrm{o}} \leq (1 - \epsilon_{\mathrm{c}})\Big(1 + \frac{x^{-c}}{12}\Big),$$

where x is the scale of the member, and a kernel splitting of the member with off-identity error ϵ_{o} and cusp error ϵ_{c}.

Corollary 14.18. *Let $C_0 \geq 1$ and $c > 0$ be real numbers and let $\mathcal{F}$ be a parametrised host family with slope records, frame realisations and word scales. Assume that for all large members that have word sizes for C_0, with frame data $\mathcal{H}$ and D^*, there are a measure space (X, μ) with*

$\mu(X) < \infty$, *a measurable function* $\mathrm{ls}\colon X \to (0,\infty)$, *a real number* $\tau_* > 0$, *measurable functions* $\mathrm{kd}\colon X \to [0,\infty)$ *and* $\mathrm{na}, \mathrm{fa}\colon X \to [0,\infty]$ *and real numbers* $c_{\mathrm{id}}, \epsilon_{\mathrm{N}}, \epsilon_{\mathrm{F}}, \kappa \geq 0$ *with the properties (Kc1)–(Kc4) of Lemma 14.19 (for* $X_{\mathrm{t}} = \{\mathrm{ls} \geq \tau_*\}$, $X_{\mathrm{c}} = X \setminus X_{\mathrm{t}}$*), such that* $\kappa\,\mu(X) < 1$ *and*

$$1 + \epsilon_{\mathrm{N}} + \epsilon_{\mathrm{F}} \leq (1 - \kappa\,\mu(X))\Big(1 + \frac{x^{-c}}{12}\Big),$$

x *the scale of the member. Then* $\mathcal{F}$ *has the dimension bound for* C_0 *with exponent* c *(No5).*

Proof. Let x_0 be a threshold beyond which the assumption holds, and let a member at a scale $x \geq x_0$, $x > x_1$ (x_1 the threshold of the parametrised host family), with word sizes for C_0 be given, with the data of the assumption. By Lemma 14.19 the member has a kernel splitting with off-identity error $\epsilon_{\mathrm{o}} = \epsilon_{\mathrm{N}} + \epsilon_{\mathrm{F}} \geq 0$ and cusp error $\epsilon_{\mathrm{c}} = \kappa\,\mu(X)$. Since $\kappa \geq 0$ and $0 \leq \mu(X) < \infty$, $\epsilon_{\mathrm{c}} \geq 0$, and $\epsilon_{\mathrm{c}} < 1$ by assumption, so $\epsilon_{\mathrm{c}} \in [0,1)$; the displayed inequality is $1 + \epsilon_{\mathrm{o}} \leq (1 - \epsilon_{\mathrm{c}})(1 + x^{-c}/12)$. As the member was arbitrary, $\mathcal{F}$ has kernel splittings for C_0 with exponent c (Dk1), with threshold x_0. By Theorem 14.20, $\mathcal{F}$ has the dimension bound (No5) for C_0 with exponent c. □

Lemma 14.19. *The notation introduced here is local to this result and to the results that apply it. Fix a member of a parametrised host family with slope records, frame realisations and word scales, with frame data* $\mathcal{H}$ *and* D^**. Let* (X,μ) *be a measure space with* $\mu(X) < \infty$, $\mathrm{ls}\colon X \to (0,\infty)$ *a measurable function (the* ***least size****) and* $\tau_* > 0$*; put* $X_{\mathrm{t}} = \{x : \mathrm{ls}(x) \geq \tau_*\}$ *and* $X_{\mathrm{c}} = X \setminus X_{\mathrm{t}}$*. Let* $\mathrm{kd}\colon X \to [0,\infty)$ *and* $\mathrm{na}, \mathrm{fa}\colon X \to [0,\infty]$ *be measurable, and let* $c_{\mathrm{id}}, \epsilon_{\mathrm{N}}, \epsilon_{\mathrm{F}}, \kappa \geq 0$ *be real numbers such that*

(Kc1) Trace and identity term $\int_X \mathrm{kd}\,d\mu = \dim \mathcal{H}$ *and* $c_{\mathrm{id}}\,\mu(X) = D^*$*;*

(Kc2) Thick split $\mathrm{kd}(x) \leq c_{\mathrm{id}} + \mathrm{na}(x) + \mathrm{fa}(x)$ *for* $x \in X_{\mathrm{t}}$*;*

(Kc3) Near and far terms $\int_{X_{\mathrm{t}}} \mathrm{na}\,d\mu \leq \epsilon_{\mathrm{N}} D^*$ *and* $\int_{X_{\mathrm{t}}} \mathrm{fa}\,d\mu \leq \epsilon_{\mathrm{F}} D^*$*;*

(Kc4) Cusp: pointwise bound $\mathrm{kd}(x) \leq \kappa \dim \mathcal{H}$ *for* $x \in X_{\mathrm{c}}$*.*

Then the member has a kernel splitting with off-identity error $\epsilon_{\mathrm{N}} + \epsilon_{\mathrm{F}}$ *and cusp error* $\kappa\,\mu(X)$*.*

Proof. Take the measure space (X,μ), the thick part X_{t} and the cusp part X_{c}, which are measurable (as ls is), disjoint and cover X; the kernel diagonal kd; the off-identity majorant $a = (\mathrm{na} + \mathrm{fa})|_{X_{\mathrm{t}}}$, measurable with values in $[0,\infty]$; and the identity density c_{id}. Then (14.10) holds: the trace identity is (Kc1), and $c_{\mathrm{id}}\mu(X_{\mathrm{t}}) \leq c_{\mathrm{id}}\mu(X) = D^*$. (14.11) is (Kc2). For (14.12): $\int_{X_{\mathrm{t}}} a\,d\mu = \int_{X_{\mathrm{t}}} \mathrm{na}\,d\mu + \int_{X_{\mathrm{t}}} \mathrm{fa}\,d\mu \leq (\epsilon_{\mathrm{N}} + \epsilon_{\mathrm{F}})D^*$ by (Kc3) (additivity of the integral of nonnegative functions), and by (Kc4)

$$\int_{X_{\mathrm{c}}} \mathrm{kd}\,d\mu \leq \kappa \dim \mathcal{H}\,\mu(X_{\mathrm{c}}) \leq \kappa\,\mu(X) \dim \mathcal{H}.$$

□

Theorem 14.20. *Let* $C_0 \geq 1$ *and* $c > 0$ *be real numbers and let* $\mathcal{F}$ *be a parametrised host family with slope records, frame realisations and word scales that has kernel splittings for* C_0 *with exponent* c *(Dk1). Then* $\mathcal{F}$ *has the dimension bound for* C_0 *with exponent* c *(No5).*

Proof. Let x_0 be a threshold for (Dk1). Fix a real $x \geq x_0$ with $x > x_1$ and a member $i \in I(x)$ that has word sizes for C_0; take $\epsilon_{\mathrm{o}} \geq 0$, $\epsilon_{\mathrm{c}} \in [0,1)$ with $1+\epsilon_{\mathrm{o}} \leq (1-\epsilon_{\mathrm{c}})(1+x^{-c}/12)$ and a kernel splitting $(X, \mu, X_{\mathrm{t}}, X_{\mathrm{c}}, \mathrm{kd}, a, c_{\mathrm{id}})$ of the member with these errors, as (Dk1) provides. Put $N_{\mathcal{H}} = \dim \mathcal{H}$, a non-negative integer since $\mathcal{H}$ is finite-dimensional.

Splitting the trace. The sets $X_{\mathrm{t}}, X_{\mathrm{c}}$ are measurable, disjoint and cover X, and $\mathrm{kd} \geq 0$ is measurable, so by additivity of the integral of non-negative functions and (14.10)

$$N_{\mathcal{H}} = \int_X \mathrm{kd}\, d\mu = \int_{X_{\mathrm{t}}} \mathrm{kd}\, d\mu + \int_{X_{\mathrm{c}}} \mathrm{kd}\, d\mu;$$

both summands are finite because their sum is $N_{\mathcal{H}}$.

The thick part. By (14.11), $\mathrm{kd} \leq c_{\mathrm{id}} + a$ on X_{t}, both sides measurable with values in $[0,\infty]$; monotonicity and additivity of the integral, (14.10) and (14.12) give

$$\int_{X_{\mathrm{t}}} \mathrm{kd}\, d\mu \leq c_{\mathrm{id}}\, \mu(X_{\mathrm{t}}) + \int_{X_{\mathrm{t}}} a\, d\mu \leq D^* + \epsilon_{\mathrm{o}} D^*.$$

(If $\mu(X_{\mathrm{t}}) = \infty$, then $c_{\mathrm{id}} = 0$ because $c_{\mathrm{id}}\, \mu(X_{\mathrm{t}}) \leq D^* < \infty$, and the first term is 0.)

The cusp part. By (14.12), $\int_{X_{\mathrm{c}}} \mathrm{kd}\, d\mu \leq \epsilon_{\mathrm{c}} N_{\mathcal{H}}$.

Conclusion. Adding the two bounds, $N_{\mathcal{H}} \leq (1+\epsilon_{\mathrm{o}}) D^* + \epsilon_{\mathrm{c}} N_{\mathcal{H}}$, i.e.

$$(1-\epsilon_{\mathrm{c}})\, N_{\mathcal{H}} \leq (1+\epsilon_{\mathrm{o}})\, D^* \leq (1-\epsilon_{\mathrm{c}})\Big(1+\frac{x^{-c}}{12}\Big) D^*,$$

using $D^* > 0$. Since $1-\epsilon_{\mathrm{c}} > 0$, $\dim \mathcal{H} = N_{\mathcal{H}} \leq (1+x^{-c}/12) D^*$. As x and the member with word sizes were arbitrary, (No5) holds for C_0 with exponent c and threshold x_0. □

15 Row estimates for the Gram blocks

This section proves the row estimates (Rw1) from the row records (Ro1)–(Ro4) that the class requires (Theorem 15.3); Theorem 14.13 of Section 14 turns them into pair expansions of the Gram blocks. A Gram block $\mathrm{Gr}_{\omega\omega'}$ is written as a sum of terms indexed by a countable set (Definition 14.6), and the pairs of terms are grouped into three parts: the main row, the loop pairs and the non-generic row. They are bounded separately and to different standards. The main row is bounded by the value $d\, d'/D^*$ for random subspaces times the factor $(1+\delta_{\mathrm{P}})e_1$ (Theorem 14.14); the loop pairs and the non-generic row need only be small against it. This makes the main row the heart of the section.

The main row is controlled by its mass data (Lemma 15.8), which combine four inputs. A Poisson recursion along a tower of points bounds the energy of coefficients dominated by a positive kernel (Theorem 15.10, Corollary 15.9); its archimedean clauses come from a product structure and Poisson summation (Theorem 15.14). The orbit energy is bounded by an energy bound for orbital integrals of Bessel coefficients of a square-integrable representation, which rests on the Schur orthogonality relations with the formal degree (Proposition 15.18). The mass and the class number of a group compact at infinity are expressed by its covolume (Lemma 15.19), and the stabilisers of nondegenerate tuples have the covolume of the unitary group, by Siegel's formula for unitary groups (Proposition 15.20, Theorem B.158). The loop pairs are bounded through configuration masses, with the share $\exp(-\frac{1}{3} k f \log k)$ (Theorem 15.4). The non-generic row is bounded through point data with a local factor (Theorem 15.5), and the local factor $2^{-(k-2r-5)}$ comes from a gain $q^{-(k-2r-5)}$, with $q \geq 2$, at one non-generic place; Section 16 proves it (Lemma 16.4).

Definition 15.1. Let $C \ge 1$ and $K_0 > 0$ be real numbers, let k, r, f be integers with $k \ge 1$, $r \ge 0$, $f \ge 1$, put $m = 2r + 2$, and let $b \ge 2$, $g > 0$, $v > 0$, $N_T > 0$, $\Gamma \ge 1$ be real numbers and $\Xi = (\Xi(s))_{2 \le s \le k}$ real numbers, the **configuration masses**. Put $\lambda = \log b$ and $h = b^{Cm^2 f}$ (the **height scale**). The **loop-row numbers** are, for $2 \le s \le k$,

$$\begin{aligned} \mathrm{nd}_s &= \Big(\frac{2}{g}\Big)^{(s-1)kf} (kv)^{kf}\, \Xi(s)\, C^{kf}\, h^{C(s+m)m^2} (Csf)^{6sf}\, \Gamma, \\ \mathrm{dg}_s &= \Big(\frac{2}{g}\Big)^{(s-2)kf} (kv)^{kf}\, \Xi(s)\, C^{kf}\, h^{C(s+m)m^2} (Csf)^{6sf}\, \Gamma\, e^{Cm^6 f\lambda} \Big(\frac{C}{k}\Big)^{kf}, \\ \mathrm{on} &= C^f\, h^{Cm^3}\, \Gamma\, e^{Cm^6 f\lambda} \big(C\, g^{-(2r+1)} k^{-r}\big)^{kf/2}. \end{aligned}$$

The data $(C, K_0; k, r, f, b, g, v, N_T, \Gamma, \Xi)$ satisfy

(Lb1) Loop parameter conditions if $m \ge 4$, $m \le f \le k$, $g \ge (kb)^8$, $K_0 m^5 f\lambda \le k$, $v \le b$, $N_T \le b^{Cmf}$ and $v \ge 1$, $N_T \ge 1$, $\Gamma \le \exp(Cm^5 f\lambda)$;

(Lb2) Configuration mass bounds if $0 \le \Xi(s)$ and $(kv)^{kf}\, \Xi(s) \le h^C C^{skf}$ for $2 \le s \le k-2$, and $0 \le \Xi(s) \le (Ck)^{kf} (h^{Cm} N_T)^{Ck}$ for $s \in \{k-1, k\}$.

Let moreover $A \ge 1$ and $\vartheta \ge 0$ be real numbers and let a pair expansion at scale ϑ (with some price, main constant and shares) be given. Its **loop pairs** are the elements of

$$\mathrm{Lp} = \mathrm{Pr}_{\ge 2} \cup \big((\mathrm{Pr}_1 \cap (\mathcal{M}_\mathrm{g} \times \mathcal{M}_\mathrm{g})) \setminus \mathrm{Sc}\big).$$

The pair expansion has

(Lb3) The loop-row bound with constant A for the data above if

$$c_\mathrm{rw} \sum_{(t,t') \in \mathrm{Lp}} |\langle z_t, z_{t'} \rangle| \le A\Big(\sum_{s=2}^{k} \mathrm{nd}_s + \Gamma^2 \sum_{s=2}^{k} \mathrm{dg}_s + \mathrm{on}\Big)\vartheta.$$

Definition 15.2. Let a pair expansion be given and let $r \ge 0$, $k \ge 2r + 5$ be integers. **Non-generic row data** for it consist of sets Cl (the **classes**), P (the **points**) and N (the **normals**), maps $\mathrm{cl}\colon \mathcal{M}_\mathrm{n} \to \mathrm{Cl}$, $\mathrm{pt}\colon \mathcal{M}_\mathrm{n} \to P$, $\mathrm{nm}\colon \mathcal{M}_\mathrm{n} \to N$, functions $w, P_\mathrm{n}\colon P \to (0, \infty)$, $J, n_\mathrm{g}\colon P \to [0, \infty)$ and $a_2\colon \mathrm{Cl} \times P \times N \to [0, \infty)$, and real numbers $S_w, T_\mathrm{pt} \ge 0$. For $(c, \gamma, \nu) \in \mathrm{Cl} \times P \times N$ put

$$u(c, \gamma, \nu) = \sum_{t \in \mathcal{M}_\mathrm{n},\ (\mathrm{cl}(t), \mathrm{pt}(t), \mathrm{nm}(t)) = (c, \gamma, \nu)} z_t \in \mathcal{H}_2$$

(absolutely convergent by (Pe1)); (c, γ, ν) is **occupied** if $z_t \ne 0$ for some term t in this sum, and $P_\mathrm{oc} \subseteq P$ is the set of points γ such that (c, γ, ν) is occupied for some c, ν. The data have

(Ng1) Classes if for all $t, t' \in \mathcal{M}_\mathrm{n}$: $(t, t') \in \mathrm{Pr}_1$ if and only if $\mathrm{cl}(t) = \mathrm{cl}(t')$;

(Ng2) Normals if for all $c \in \mathrm{Cl}$ and $\gamma \in P$ the set of ν with (c, γ, ν) occupied is finite with at most $P_\mathrm{n}(\gamma)$ elements;

(Ng3) Point weights if $\sum_{\gamma \in F} w(\gamma)^{-1} \le S_w$ for every finite $F \subseteq P_\mathrm{oc}$;

(Ng4) Factorisation if $\|u(c, \gamma, \nu)\|^2 \le a_2(c, \gamma, \nu)\, J(\gamma)$ for all (c, γ, ν);

(Ng5) Local gain if $\sum_{(c,\nu) \in F} a_2(c, \gamma, \nu) \le 2^{-(k-2r-5)} n_\mathrm{g}(\gamma)$ for every $\gamma \in P_\mathrm{oc}$ and every finite $F \subseteq \mathrm{Cl} \times N$;

(Ng6) Point sum if $\sum_{\gamma\in F} w(\gamma)P_{\mathrm{n}}(\gamma)n_{\mathrm{g}}(\gamma)J(\gamma)\le T_{\mathrm{pt}}$ for every finite $F\subseteq P_{\mathrm{oc}}$.

Theorem 15.3. *Let $C_0\ge 1$, $C_{\mathrm{A}}\ge 1$, $0<c_{\mathrm{lp}}\le\frac{1}{3}$, $C_{\mathrm{ng}}\ge 0$, $C_{\mathrm{lb}}\ge 1$ and $A_{\mathrm{lb}}\ge 1$ be real numbers, let $C_0^{\mathrm{ng}}\ge 0$ be an integer, and put $K_0=32C_{\mathrm{lb}}^{32}$. Let $\mathcal{F}$ be a parametrised host family with slope records, frame realisations, word scales and row parameters, with parameter ε and threshold x_1. Suppose that for all large members that have word sizes for C_0 and all words $\omega\ne\omega'$, with $\vartheta=D_\omega D_{\omega'}/D^*$ and the row parameters of the member, there are data with the following properties.*

(Ro1) Expansion. *Data as in Definition 14.6 for the vector $\mathrm{Gr}_{\omega\omega'}$ of the space of linear maps $\mathbb{C}^{D_{\omega'}}\to\mathbb{C}^{D_\omega}$ with the Hilbert–Schmidt inner product, at scale ϑ, satisfying (Pe1)–(Pe5) with price*

$$C_{\mathrm{pr}}=\Big(1+\frac{3}{g_1^{\mathrm{aux}}}\Big)^{f_\infty}\Big(1+\frac{3}{g_2^{\mathrm{aux}}}\Big)^{f_\infty}.$$

(Ro2) Loops. *Real numbers $v,N_T>0$ and $\Xi=(\Xi(s))_{2\le s\le k_{\mathrm{d}}}$ such that*

$$(C_{\mathrm{lb}},K_0;k_{\mathrm{d}},r_{\mathrm{f}},f_\infty,b(x),g_1^{\mathrm{aux}}g_2^{\mathrm{aux}},v,N_T,\Gamma_{\mathrm{w}},\Xi)$$

satisfy the loop parameter conditions (Lb1) and the configuration mass bounds (Lb2), the data of (Ro1) have the loop-row bound (Lb3) with constant A_{lb} for these data, and

$$6\log(3A_{\mathrm{lb}})\le k_{\mathrm{d}}f_\infty\log k_{\mathrm{d}}.$$

(Ro3) Non-generic row. *Non-generic row data for the data of (Ro1) with $r=r_{\mathrm{f}}$ and $k=k_{\mathrm{d}}$ having the properties (Ng1)–(Ng4) and (Ng6), with*

$$c_{\mathrm{rw}}\,S_w\,T_{\mathrm{pt}}\le\exp\big(C_{\mathrm{ng}}(2r_{\mathrm{f}}+2)^5f_\infty\log b(x)\big)\,\vartheta,$$

and, for every $\gamma\in P_{\mathrm{oc}}$, data as in Lemma 16.4 with $r=r_{\mathrm{f}}$, $k=k_{\mathrm{d}}$ and $C_0=C_0^{\mathrm{ng}}$ satisfying its items 1 and 2 and (Nv1), (Nv2).

(Ro4) Main row. *With c_{rw} the row normalisation of the data of (Ro1) and*

$$S_{\mathrm{mn}}=\sum_{(t,t')\in\mathrm{Pr}_1\cap(\mathcal{M}_{\mathrm{g}}\times\mathcal{M}_{\mathrm{g}})\cap\mathrm{Sc}}\mathrm{Re}\langle z_t,z_{t'}\rangle$$

its main-row sum, numbers and data satisfying (Mx1)–(Mx5) of Lemma 15.8 for $(S_{\mathrm{mn}};c_{\mathrm{rw}},D_\omega,D_{\omega'},D^*)$, with Poisson recursion data with parameters (r,f,n_{d}), lattice-sum bound β_{L} and block ratio β_{B} in (Mx1), such that

$$e_1\le\exp\big(12f_\infty\,2^{2r_{\mathrm{f}}+2-k_{\mathrm{d}}}\big),$$
$$15\,(r+1)\big(2\cdot 3^f\beta_{\mathrm{L}}+4\beta_{\mathrm{B}}\big)\le C_{\mathrm{A}}(r_{\mathrm{f}}+1)\Big(\frac{C_{\mathrm{A}}}{T_{\mathrm{amp}}}\Big)^{f_\infty}.$$

Then $\mathcal{F}$ has row estimates for C_0 with constants $(C_{\mathrm{A}},c_{\mathrm{lp}},C_{\mathrm{ng}})$ (Rw1).

Proof. Fix a large member that has word sizes for C_0, words $\omega\ne\omega'$ and the data of (Ro1)–(Ro4); write $k=k_{\mathrm{d}}$, r_{f}, $f=f_\infty$, $b=b(x)$. Word sizes give $D_\omega,D_{\omega'}\ge D^*/(C_0\Gamma_{\mathrm{w}}^2)>0$, so $\vartheta>0$. We show that the data of (Ro1), with the mass data of (Ro4), form a main-row expansion of $\mathrm{Gr}_{\omega\omega'}$ with the price, shares and bounds of (Rw1).

Rank. By (Lb1) of (Ro2), $m \ge 4$, $f \ge m$, and $k \ge K_0 m^5 f\lambda$ with $K_0 = 32C_{\mathrm{lb}}^{32} \ge 32$ and $\lambda \ge \log 2 > \frac{1}{2}$ (as $b \ge 2$). Hence $k \ge 16m^6 \ge 2m + 3 = 4r_{\mathrm{f}} + 7$; in particular $k \ge 2r_{\mathrm{f}} + 5$ and $k \ge 1$, so $X := kf \log k \ge 0$.

The loop condition (Pe6). The data of (Ro2) satisfy (Lb1) and (Lb2) with $C = C_{\mathrm{lb}} \ge 1$ and $K_0 = 32C^{32}$, and $A = A_{\mathrm{lb}} \ge 1$ satisfies $6\log(3A) \le X$. The data of (Ro1) are a pair expansion at the scale $\vartheta \ge 0$ with the loop-row bound (Lb3) with constant A. By Theorem 15.4, both sums of (Pe6), multiplied by c_{rw}, are at most $e^{-X/3}\vartheta$. Since $X \ge 0$ and $c_{\mathrm{lp}} \le \frac{1}{3}$, $e^{-X/3} \le e^{-c_{\mathrm{lp}}X}$; so (Pe6) holds with $\beta_{\mathrm{lp}} = \exp(-c_{\mathrm{lp}}k_{\mathrm{d}}f_\infty \log k_{\mathrm{d}})$.

The non-generic condition (Pe8). Since $k \ge 4r_{\mathrm{f}} + 7$, Lemma 16.4 applies to the non-generic row data of (Ro3) with $r = r_{\mathrm{f}}$, k, $C_0 = C_0^{\mathrm{ng}}$ and, at every $\gamma \in P_{\mathrm{oc}}$, the data of (Ro3), so the data have the local gain (Ng5). With (Ng1)–(Ng4) and (Ng6) of (Ro3) they have (Ng1)–(Ng6), and $k \ge 2r_{\mathrm{f}} + 5$. Put $Q = \exp(C_{\mathrm{ng}}m^5 f \log b) \ge 0$; by (Ro3), $c_{\mathrm{rw}}S_w T_{\mathrm{pt}} \le Q\vartheta$. By (ii) of Theorem 15.5, the pair expansion satisfies (Pe8) with non-generic share $2^{-(k-2r_{\mathrm{f}}-5)}Q = 2^{2r_{\mathrm{f}}+5-k_{\mathrm{d}}}\exp\bigl(C_{\mathrm{ng}}(2r_{\mathrm{f}}+2)^5 f_\infty \log b(x)\bigr)$, the β_{ng} of (Rw1).

The main-row mass data. By (Ro4) and Lemma 15.8, $(S_{\mathrm{mn}}; c_{\mathrm{rw}}, D_\omega, D_{\omega'}, D^*)$, with $c_{\mathrm{rw}} > 0$ the row normalisation and $D_\omega, D_{\omega'}, D^* > 0$, has main-row mass data with density factor e_1 and Poisson error $\delta_{\mathrm{P}} = 15(r+1)(2\cdot 3^f\beta_{\mathrm{L}} + 4\beta_{\mathrm{B}})$, and by the two inequalities of (Ro4), $e_1 \le \exp(12f_\infty 2^{2r_{\mathrm{f}}+2-k_{\mathrm{d}}})$ and $\delta_{\mathrm{P}} \le C_{\mathrm{A}}(r_{\mathrm{f}}+1)(C_{\mathrm{A}}/T_{\mathrm{amp}})^{f_\infty}$.

Conclusion. The data of (Ro1) satisfy (Pe1)–(Pe5) with the price C_{pr} of (Rw1), (Pe6) with the loop share and (Pe8) with the non-generic share of (Rw1), and come with main-row mass data for their main-row sum whose density factor and Poisson error obey the bounds of (Rw1). This is a main-row expansion of $\mathrm{Gr}_{\omega\omega'}$ as required in (Rw1). As the large member with word sizes for C_0 and the words $\omega \ne \omega'$ were arbitrary (with the same threshold as in the hypothesis), $\mathcal{F}$ has row estimates for C_0 with constants $(C_{\mathrm{A}}, c_{\mathrm{lp}}, C_{\mathrm{ng}})$. □

Theorem 15.4. *Let $C \ge 1$ and put $K_0 = 32C^{32}$. Let $(C, K_0; k, r, f, b, g, v, N_T, \Gamma, \Xi)$ satisfy the loop parameter conditions (Lb1) and the configuration mass bounds (Lb2), and let $A \ge 1$ with $6\log(3A) \le kf\log k$. Then*

$$\sum_{s=2}^{k} \mathrm{nd}_s \le g^{-kf/2} \le e^{-4kf\log k}, \qquad \Gamma^2 \sum_{s=2}^{k} \mathrm{dg}_s \le e^{-\frac{1}{2}kf\log k}, \qquad \mathrm{on} \le e^{-kf\log k},$$

and every pair expansion at a scale $\vartheta \ge 0$ that has the loop-row bound (Lb3) with constant A for these data satisfies the loop condition (Pe6) with loop share $\beta_{\mathrm{lp}} = \exp(-\frac{1}{3}kf\log k)$:

$$c_{\mathrm{rw}} \sum_{(t,t')\in \mathrm{Pr}_{\ge 2}} |\langle z_t, z_{t'}\rangle| \le e^{-\frac{1}{3}kf\log k}\vartheta, \qquad c_{\mathrm{rw}} \sum_{(t,t')\in(\mathrm{Pr}_1\cap(\mathcal{M}_{\mathrm{g}}\times\mathcal{M}_{\mathrm{g}}))\setminus \mathrm{Sc}} |\langle z_t, z_{t'}\rangle| \le e^{-\frac{1}{3}kf\log k}\vartheta.$$

Proof. Write $\kappa = \log k$ and $q = g^{-kf/2}$.

Consequences of (Lb1). Since $f \ge m \ge 4$ and $\lambda \ge \log 2 > 1/2$, $m^5 f\lambda \ge 4^6/2 \ge 2048$, so $k \ge K_0 m^5 f\lambda \ge 2048K_0$. Hence:

(a) $k \ge K_0 \ge C^{32}$, so $C \le k$ and $0 \le \log C \le \kappa/32$; also $k \ge 2048$, so $\kappa \ge 7$;

(b) $k \ge K_0 m^6\lambda \ge 32C^2m^6\lambda$ (as $f \ge m$) and $k \ge K_0 m^5 \ge 8C^3m^5$ (as $f\lambda \ge 2$);

(c) $\log g \ge 8\kappa + 8\lambda$ (as $g \ge (kb)^8$), so $g \ge 2^8$ and $q \le 2^{-4}$, and $q \le (kb)^{-4kf} \le e^{-4kf\kappa}$;

(d) for $2 \le s \le k$, $Csf \le k^3$, so $\log(Csf) \le 3\kappa$; $\log(kv) \le \kappa + \lambda$; $\log N_T \le Cmf\lambda$; $\log\Gamma \le Cm^5 f\lambda$; $\log h = Cm^2 f\lambda$; and $kf\lambda \ge \log 2$.

By (Lb2), $\Xi(s) \geq 0$ for all s; if $\Xi(s) = 0$ then $\mathrm{nd}_s = \mathrm{dg}_s = 0$ and the bounds of Steps 1 and 3 below hold trivially, so we assume $\Xi(s) > 0$ there and take logarithms.

Step 1: $\mathrm{nd}_s \leq \frac{1}{2} q^{s-1}$ for $2 \leq s \leq k$. Put $j = s-1 \geq 1$; we show $\log \mathrm{nd}_s \leq -\frac{jkf}{2} \log g - \log 2$.

Case $2 \leq s \leq k-2$. By (Lb2), $\log\big((kv)^{kf}\Xi(s)\big) \leq C \log h + skf \log C = C^2 m^2 f\lambda + skf \log C$. Hence

$$\log \mathrm{nd}_s \leq -jkf \log g + jkf \log 2 + (s+1)kf \log C + C^2 m^2 f\lambda \\ + C^2 m^4 (s+m) f\lambda + 6sf \log(Csf) + Cm^5 f\lambda.$$

With $s+1 \leq 3j$, $s \leq 2j$, $s+m \leq (m+2)j$ and $1 \leq j$, the right side is at most

$$-jkf \log g + jkf\lambda + 3jkf\kappa + C^2 m^4 (m+3) jf\lambda + 36jf\kappa + Cm^5 f\lambda.$$

By (b), $C^2 m^4 (m+3) \leq 2C^2 m^5 \leq k$ and $Cm^5 \leq k$, and $36 \leq k$; so this is at most $-jkf \log g + 3jkf\lambda + 4jkf\kappa = -\frac{jkf}{2} \log g - jkf\big(\frac{1}{2} \log g - 3\lambda - 4\kappa\big) \leq -\frac{jkf}{2} \log g - jkf\lambda$ by (c), and $jkf\lambda \geq \log 2$.

Case $s \in \{k-1, k\}$. By (Lb2) and (d), $\log \Xi(s) \leq kf \log(Ck) + Ck(C^2 m^3 f\lambda + Cmf\lambda) \leq 2kf\kappa + 2C^3 m^3 kf\lambda$. With $s+m \leq 2k$, $6sf \log(Csf) \leq 18kf\kappa$ and $Cm^5 f\lambda \leq kf\lambda$,

$$\begin{aligned} \log \mathrm{nd}_s &\leq -jkf \log g + jkf\lambda + 22kf\kappa + kf\lambda\,(2 + 2C^3 m^3 + 2C^2 m^4) \\ &\leq -jkf \log g + jkf\lambda + 22kf\kappa + 6C^3 m^4 kf\lambda. \end{aligned}$$

Here $j \geq k-2 \geq k/2$, so by (c) $\frac{jkf}{2} \log g - jkf\lambda \geq jkf(4\kappa + 3\lambda) \geq 2k \cdot kf\kappa + \frac{3}{2}k \cdot kf\lambda$, which exceeds $22kf\kappa + 6C^3 m^4 kf\lambda + \log 2$ because $k \geq 12$ and $\frac{3}{2}k \geq 7C^3 m^4$ by (b). This gives the claim.

Step 2: the nondegenerate rows. Distinct s give distinct $j = s-1 \geq 1$, so by Step 1 and $q \leq \frac{1}{2}$,

$$\sum_{s=2}^{k} \mathrm{nd}_s \leq \frac{1}{2} \sum_{j \geq 1} q^j = \frac{q}{2(1-q)} \leq q = g^{-kf/2} \leq e^{-4kf \log k}.$$

Step 3: $\Gamma^2 \mathrm{dg}_s \leq \frac{1}{2} e^{-kf\kappa/2} q^{s-2}$ for $2 \leq s \leq k$. Put $j' = s-2 \geq 0$, and recall that $\log \Gamma^2 \leq 2Cm^5 f\lambda$ and $\log(C/k)^{kf} = kf \log C - kf\kappa$.

Case $2 \leq s \leq k-2$. As in Step 1,

$$\log(\Gamma^2 \mathrm{dg}_s) \leq -j'kf \log g + j'kf \log 2 + (s+2)kf \log C - kf\kappa \\ + \Big[C^2 m^2 + C^2 m^4 (s+m) + 3Cm^5 + Cm^6\Big] f\lambda + 6sf \log(Csf).$$

Since $s = j' + 2$: $(s+2)kf \log C \leq j'kf\kappa + 4kf \log C \leq j'kf\kappa + \frac{1}{8}kf\kappa$ by (a); the bracket is at most $4C^2 m^6 + C^2 m^4 j'$ (each of $C^2 m^2$, $C^2 m^4 (m+2)$, $3Cm^5$, Cm^6 is at most $C^2 m^6$), and $4C^2 m^6 f\lambda \leq \frac{1}{8}kf \leq \frac{1}{8}kf\kappa$ by (b), while $C^2 m^4 j' f\lambda \leq j'kf\lambda$; and $6sf \log(Csf) \leq 36f\kappa + 18j'f\kappa \leq \frac{1}{8}kf\kappa + j'kf\kappa$ as $k \geq 288$. Hence

$$\log(\Gamma^2 \mathrm{dg}_s) \leq -j'kf \log g + j'kf(\lambda + 2\kappa + \lambda) - \frac{5}{8}kf\kappa \leq -\frac{j'kf}{2} \log g - \frac{1}{2}kf\kappa - \frac{1}{8}kf\kappa,$$

using $\frac{1}{2} \log g \geq 2\kappa + 2\lambda$ and $\log 2 \leq \lambda$; and $\frac{1}{8}kf\kappa \geq \log 2$.

Case $s \in \{k-1, k\}$. Now $j' \geq k-3 \geq k/2$. As in the second case of Step 1, with the additional factors,

$$\begin{aligned} \log(\Gamma^2 \mathrm{dg}_s) &\leq -j'kf \log g + j'kf\lambda + 22kf\kappa + kf\lambda\Big(3 + 2C^3 m^3 + 2C^2 m^4\Big) \\ &\leq -j'kf \log g + j'kf\lambda + 22kf\kappa + 7C^3 m^4 kf\lambda, \end{aligned}$$

where $3Cm^5f\lambda + Cm^6f\lambda \le 2kf\lambda$ by (b) was used and $kf\log C - kf\kappa \le 0$ absorbed the factor $(C/k)^{kf}$ together with one $kf\log C$. As before $\frac{j'kf}{2}\log g - j'kf\lambda \ge 2k\cdot kf\kappa + \frac{3}{2}k\cdot kf\lambda \ge 22kf\kappa + \frac{1}{2}kf\kappa + 7C^3m^4kf\lambda + \log 2$, since $k \ge 12$ and $\frac{3}{2}k \ge 8C^3m^4$.

Step 4: the degenerate rows. Distinct s give distinct $j' \ge 0$, so by Step 3

$$\Gamma^2 \sum_{s=2}^{k} \mathrm{dg}_s \le \frac{1}{2} e^{-kf\kappa/2} \sum_{j'\ge 0} q^{j'} \le \frac{e^{-kf\kappa/2}}{2(1-q)} \le e^{-\frac{1}{2}kf\log k}.$$

Step 5: the non-scalar pairs of corank one. As $m \ge 4$, $r \ge 1$, so $g^{-(2r+1)}k^{-r} \le g^{-3}$ (as $g \ge 1$, $k \ge 1$). Hence, with (d),

$$\begin{aligned}\log \mathrm{on} &\le f\log C + C^2m^5f\lambda + Cm^5f\lambda + Cm^6f\lambda + \frac{kf}{2}\log C - \frac{3kf}{2}\log g \\ &\le kf\kappa + 3C^2m^6f\lambda - 12kf\kappa - 12kf\lambda \le -11kf\kappa,\end{aligned}$$

using $\log C \le \kappa$, (c), and $3C^2m^6 \le k$ from (b). So $\mathrm{on} \le e^{-kf\log k}$.

Step 6: the loop share. Put $X = kf\log k$. By Steps 2, 4, 5 and the hypothesis $\log(3A) \le X/6$,

$$\begin{aligned}A\Big(\sum_s \mathrm{nd}_s + \Gamma^2\sum_s \mathrm{dg}_s + \mathrm{on}\Big) &\le A(e^{-4X} + e^{-X/2} + e^{-X}) \le 3A\,e^{-X/2} \\ &= e^{-X/3}\exp\Big(\log(3A) - \frac{X}{6}\Big) \le e^{-X/3}.\end{aligned}$$

Now let a pair expansion at scale $\vartheta \ge 0$ have (Lb3) with constant A. The sets $\mathrm{Pr}_{\ge 2}$ and $(\mathrm{Pr}_1 \cap (\mathcal{M}_\mathrm{g}\times\mathcal{M}_\mathrm{g})) \setminus \mathrm{Sc}$ are contained in Lp, the summands $|\langle z_t, z_{t'}\rangle|$ are nonnegative and $c_\mathrm{rw} > 0$, so each of the two sums of (Pe6), multiplied by c_rw, is at most $c_\mathrm{rw}\sum_\mathrm{Lp}|\langle z_t, z_{t'}\rangle| \le A(\dots)\vartheta \le e^{-X/3}\vartheta$. □

Theorem 15.5. *(i) Let $\mathcal{H}_2$ be a complex Hilbert space, S a countable set with vectors $z_t \in \mathcal{H}_2$ ($t \in S$) such that $\sum_{t\in S}\|z_t\| < \infty$, let $R \subseteq S\times S$ and let $\lambda \ge 0$ be real. Let sets Cl, P, N, maps $\mathrm{cl}, \mathrm{pt}, \mathrm{nm}$ on S, functions $w, P_\mathrm{n}, J, n_\mathrm{g}, a_2$ and reals $S_w, T_\mathrm{pt} \ge 0$ be given as in Definition 15.2 with S in place of $\mathcal{M}_\mathrm{n}$, having (Ng1)–(Ng4) and (Ng6) with S and R in place of $\mathcal{M}_\mathrm{n}$ and $\mathrm{Pr}_1 \cap (\mathcal{M}_\mathrm{n}\times\mathcal{M}_\mathrm{n})$, and (Ng5) with λ in place of $2^{-(k-2r-5)}$. Then*

$$\sum_{(t,t')\in R} \mathrm{Re}\langle z_t, z_{t'}\rangle \le \lambda\, S_w\, T_\mathrm{pt},$$

the series converging absolutely.

(ii) Let a pair expansion at scale $\vartheta \ge 0$ be given, let $r \ge 0$ and $k \ge 2r+5$ be integers, and let non-generic row data for it have the properties (Ng1)–(Ng6). Then

$$c_\mathrm{rw} \sum_{(t,t')\in \mathrm{Pr}_1\cap(\mathcal{M}_\mathrm{n}\times\mathcal{M}_\mathrm{n})} \mathrm{Re}\langle z_t, z_{t'}\rangle \le 2^{-(k-2r-5)}\, c_\mathrm{rw}\, S_w\, T_\mathrm{pt}.$$

In particular, if $Q \ge 0$ and $c_\mathrm{rw}S_wT_\mathrm{pt} \le Q\vartheta$, the pair expansion satisfies the non-generic row condition (Pe8) with non-generic share $\beta_\mathrm{ng} = 2^{-(k-2r-5)}Q$.

Proof. **(i): the sum over R is a sum of squares over classes.** By (Ng1), $R = \{(t,t') \in S^2 : \mathrm{cl}(t) = \mathrm{cl}(t')\}$. As $\sum_{t\in S}\|z_t\| < \infty$, $\sum_{(t,t')\in S^2}|\langle z_t, z_{t'}\rangle| \le (\sum_t \|z_t\|)^2 < \infty$, and absolutely

convergent series may be regrouped. For a class c put $U_c = \sum_{t \in S,\, \mathrm{cl}(t)=c} z_t$. Grouping the pairs by the common class and using bilinearity of the inner product on absolutely convergent sums,

$$\sum_{(t,t')\in R} \mathrm{Re}\langle z_t, z_{t'}\rangle = \sum_{c\in \mathrm{Cl}} \mathrm{Re}\big\langle U_c, U_c\big\rangle = \sum_{c\in \mathrm{Cl}} \|U_c\|^2,$$

only countably many classes contributing (S is countable).

Cauchy–Schwarz over points and normals. Fix c and let O_c be the set of (γ,ν) with (c,γ,ν) occupied. If (c,γ,ν) is not occupied, every z_t in the sum defining $u(c,\gamma,\nu)$ vanishes and $u(c,\gamma,\nu)=0$; so, regrouping, $U_c = \sum_{(\gamma,\nu)\in O_c} u(c,\gamma,\nu)$, absolutely convergent, O_c countable. For a finite $F \subseteq O_c$, the Cauchy–Schwarz inequality gives

$$\Big\| \sum_{(\gamma,\nu)\in F} u(c,\gamma,\nu)\Big\|^2 \le \Big(\sum_{(\gamma,\nu)\in F} \frac{1}{w(\gamma)P_{\mathrm{n}}(\gamma)}\Big)\Big(\sum_{(\gamma,\nu)\in F} w(\gamma)P_{\mathrm{n}}(\gamma)\|u(c,\gamma,\nu)\|^2\Big).$$

Let $F_P \subseteq P_{\mathrm{oc}}$ be the finite set of points occurring in F. By (Ng2), each $\gamma \in F_P$ occurs with at most $P_{\mathrm{n}}(\gamma)$ normals, so by (Ng3) the first factor is at most $\sum_{\gamma\in F_P} w(\gamma)^{-1} \le S_w$. Letting F exhaust O_c and using continuity of the norm,

$$\|U_c\|^2 \le S_w \sum_{(\gamma,\nu)\in O_c} w(\gamma)P_{\mathrm{n}}(\gamma)\|u(c,\gamma,\nu)\|^2.$$

Local factor and point sum. Summing over c (all terms nonnegative), then using (Ng4), and for each $\gamma \in P_{\mathrm{oc}}$ the bound (Ng5) with λ on every finite set of pairs (c,ν) (hence on their supremum),

$$\begin{aligned}\sum_c \|U_c\|^2 &\le S_w \sum_{\gamma\in P_{\mathrm{oc}}} w(\gamma)P_{\mathrm{n}}(\gamma)J(\gamma) \sum_{(c,\nu):\,(c,\gamma,\nu)\ \text{occupied}} a_2(c,\gamma,\nu)\\ &\le \lambda\, S_w \sum_{\gamma\in P_{\mathrm{oc}}} w(\gamma)P_{\mathrm{n}}(\gamma)n_{\mathrm{g}}(\gamma)J(\gamma),\end{aligned}$$

and the last sum is at most T_{pt} by (Ng6), applied to its finite partial sums. This is (i).

(ii). Take $\mathcal{H}_2$ the second space of the pair expansion, $S = \mathcal{M}_{\mathrm{n}}$ with its vectors z_t, $R = \mathrm{Pr}_1 \cap (\mathcal{M}_{\mathrm{n}} \times \mathcal{M}_{\mathrm{n}})$ and $\lambda = 2^{-(k-2r-5)}$. Then $\sum_{t\in S}\|z_t\| < \infty$ by (Pe1), and (Ng1)–(Ng6) of the non-generic row data are the hypotheses of (i) for these choices. So (i) gives the sum bound with $2^{-(k-2r-5)}S_wT_{\mathrm{pt}}$; multiplying by $c_{\mathrm{rw}} > 0$ gives the inequality. If $c_{\mathrm{rw}}S_wT_{\mathrm{pt}} \le Q\vartheta$, the right side is at most $2^{-(k-2r-5)}Q\vartheta$, which is (Pe8) with $\beta_{\mathrm{ng}} = 2^{-(k-2r-5)}Q$.

□

Definition 15.6. Points and towers. Let K be a field. A **point** is a finite sequence $u = ((m_1,t_1),\dots,(m_n,t_n))$, $n \ge 0$, of **blocks** (m_a,t_a) with $m_a \in \mathbb{Z}_{\ge 1}$ (the **size**) and $t_a \in K^\times$ (the **coordinate**); its **level** is $|u| = m_1+\cdots+m_n$, the block (m_a,t_a) **starts at** $1+m_1+\cdots+m_{a-1}$, $\varnothing$ is the empty point and $u\cdot(m,t)$ is u followed by the block (m,t). A **tower** is a family $\mathcal{S} = (\mathcal{S}_{j,m})_{j,m\ge 1}$ of finite subsets of $K^\times$; for $i \ge 0$, $\mathcal{X}_i(\mathcal{S})$ is the (finite) set of points u of level i such that every block (m_a,t_a) of u has $t_a \in \mathcal{S}_{j,m_a}$, j the place where it starts. A function κ defined on $P\times P$ for a finite set P of points (or on a larger set) is **positive semidefinite on** P if $\sum_{u,u'\in P} \overline{w_u}\,\kappa(u,u')\,w_{u'} \ge 0$ for all $w \in \mathbb{C}^P$. For κ defined on pairs of points and $t,t' \in K^\times$, $m \ge 1$, put $\kappa_m^{t,t'}(h,h') = \kappa(h\cdot(m,t), h'\cdot(m,t'))$.

Data. Let $r \ge 0$, $f \ge 1$ and $n_{\mathrm{d}} \ge 1$ be integers, put $N_{\mathrm{top}} = 4n_{\mathrm{d}}(r+1)$, and let $\beta_{\mathrm{L}}, \beta_{\mathrm{B}} \ge 0$ be real numbers. **Poisson recursion data** over K with parameters (r,f,n_{d}), **lattice-sum bound** β_{L} and **block ratio** β_{B} consist of

- a finite set $\mathcal{A}$ of **derivative words** with a **length** $|\alpha| \in \mathbb{Z}_{\geq 0}$ and a **weight** $\mathrm{wt}(\alpha) > 0$ for each $\alpha \in \mathcal{A}$, and an **empty word** $\alpha_\varnothing \in \mathcal{A}$ with $|\alpha_\varnothing| = 0$, $\mathrm{wt}(\alpha_\varnothing) = 1$; $\mathcal{A}_n = \{\alpha \in \mathcal{A} : |\alpha| \leq n\}$;
- for every real $\eta > 0$, $0 \leq i \leq r+1$ and $\alpha \in \mathcal{A}$ a **wave kernel** $g_{\eta,i,\alpha}$ and for every $0 \leq i \leq r+1$ a **coefficient kernel** k_i, complex functions on pairs of points;
- real numbers $\mathrm{ex}_1, \dots, \mathrm{ex}_{r+1}$ (the **excesses**), additive subgroups $L_{j,m}$, $L'_{j,m}$ $(j, m \geq 1)$ and $L_i^\vee$ $(1 \leq i \leq r+1)$ of K, and real functions $\mathrm{mj}_i^\vee$, $\mathrm{mj}'_{j,m}$ on K (the **majorants**);
- for all $\eta > 0$, $1 \leq i \leq r+1$, $\alpha \in \mathcal{A}$, finite sets P of points of level $i-1$ and functions κ on $P \times P$ a complex function $z \mapsto \widehat{\mathrm{En}}_{\eta,i,\alpha}(P, \kappa; z)$ on K (the **Fourier transform**);
- a real number J, a complex Hilbert space $\mathcal{H}$ and a map W from points to $\mathcal{H}$,

with the following properties, in which, for a finite set P of points, a function κ on $P \times P$, $t, t' \in K^\times$, $m \geq 1$ and $0 \leq i \leq r+1$, the **energies** are

$$\mathrm{En}_{\eta,i,\alpha}(P, \kappa) = \frac{1}{\mathrm{wt}(\alpha)} \operatorname{Re} \sum_{u,u' \in P} \kappa(u, u')\, g_{\eta,i,\alpha}(u, u'),$$

$$\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(P, \kappa; t, t') = \frac{1}{\mathrm{wt}(\alpha)} \sum_{h,h' \in P} \kappa(h, h')\, g_{\eta,i,\alpha}(h \cdot (m, t), h' \cdot (m, t')),$$

and $j = i - m + 1$ whenever i and m are given.

(Pt1) Positivity. For all $\eta > 0$, i, α and finite sets P of points, $g_{\eta,i,\alpha}$ and k_i are positive semidefinite on P; $k_0(\varnothing, \varnothing) = 1$.

(Pt2) Support of the coefficient kernels. If $1 \leq m \leq i \leq r+1$, h, h' are points of level $i-m$, $t \neq t'$ in $K^\times$ and $k_i(h \cdot (m, t), h' \cdot (m, t')) \neq 0$, then $(i, m) \neq (r+1, 1)$ and $t^{-1} - t'^{-1} \in L'_{j,m}$.

(Pt3) Theta majorants. $0 \leq \mathrm{ex}_i \leq 1$ for $1 \leq i \leq r+1$, and for every tower $\mathcal{S}$ there are complex functions $\Theta_{i,m,t}$ on pairs of points $(1 \leq m \leq i \leq r+1,\ t \in \mathcal{S}_{j,m})$ such that, with $\mathcal{X} = \mathcal{X}_{i-m}(\mathcal{S})$: (a) $\Theta_{i,m,t} - (k_i)^{t,t}_m$ is positive semidefinite on $\mathcal{X}$; (b) $\Theta_{i,m,t} = \Theta_{i,m,t'}$ on $\mathcal{X} \times \mathcal{X}$ if $t - t' \in L_{j,m}$; (c) $(1 + \mathrm{ex}_i) k_{i-m} - \sum_{t \in R} \Theta_{i,m,t}$ is positive semidefinite on $\mathcal{X}$ for every $R \subseteq \mathcal{S}_{j,m}$ whose elements are pairwise incongruent modulo $L_{j,m}$.

(Pt4) Poisson summation. Let $\eta > 0$, $1 \leq i \leq r+1$, $\alpha \in \mathcal{A}$, P a finite set of points of level $i-1$, κ positive semidefinite on P, $t_0 \in K$ and $B \in \mathbb{R}$ with $\sum_{z \in Z} |\widehat{\mathrm{En}}_{\eta,i,\alpha}(P, \kappa; z)| \leq B$ for every finite $Z \subseteq L_i^\vee \setminus \{0\}$. Then $\sum_{t \in S} \operatorname{Re} \mathrm{En}^{\uparrow}_{\eta,i,1,\alpha}(P, \kappa; t, t) \leq |\widehat{\mathrm{En}}_{\eta,i,\alpha}(P, \kappa; 0)| + B$ for every finite $S \subseteq (t_0 + L_{i,1}) \setminus \{0\}$.

(Pt5) Fourier decay of the wave packets. For all $\eta > 0$, $1 \leq i \leq r+1$ and $\alpha \in \mathcal{A}$ with $|\alpha| + n_{\mathrm{d}} \leq N_{\mathrm{top}}$ there is $\theta \colon \mathcal{A}_{|\alpha|+n_{\mathrm{d}}} \to [0, \infty)$ with $\sum_{\alpha'} \theta(\alpha') \leq 2^f$ such that for every finite set P of points of level $i-1$ and every κ positive semidefinite on P: $\widehat{\mathrm{En}}_{\eta,i,\alpha}(P, \kappa; 0) = \mathrm{En}_{\eta,i-1,\alpha}(P, \kappa)$ and $|\widehat{\mathrm{En}}_{\eta,i,\alpha}(P, \kappa; z)| \leq \mathrm{mj}_i^\vee(z) \sum_{\alpha'} \theta(\alpha') \mathrm{En}_{\eta,i-1,\alpha'}(P, \kappa)$ for $z \in L_i^\vee \setminus \{0\}$.

(Pt6) Off-diagonal decay. For all $\eta > 0$, $1 \leq m \leq i \leq r+1$ with $(i, m) \neq (r+1, 1)$ and $\alpha \in \mathcal{A}$ with $|\alpha| + n_{\mathrm{d}} \leq N_{\mathrm{top}}$ there is $\theta \colon \mathcal{A}_{|\alpha|+n_{\mathrm{d}}} \to [0, \infty)$ with $\sum_{\alpha'} \theta(\alpha') \leq 2 \cdot 3^f$ such that for every finite set P of points of level $i-m$, finite $S \subseteq K^\times$, κ positive semidefinite on

$\{h\cdot(m,t): h\in P,\ t\in S\}$ and $t\neq t'$ in S,

$$|\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(P,\kappa_m^{t,t'};t,t')| \le \mathrm{mj}'_{j,m}(t^{-1}-t'^{-1})\cdot\frac{1}{2}\sum_{\alpha'}\theta(\alpha')\Big(\mathrm{Re}\,\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha'}(P,\kappa_m^{t,t};t,t) + \mathrm{Re}\,\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha'}(P,\kappa_m^{t',t'};t',t')\Big).$$

(Pt7) Degenerate blocks. For all $\eta>0$, $2\le m\le i\le r+1$ and $\alpha\in\mathcal{A}$ with $|\alpha|+(m+1)n_{\mathrm{d}}\le N_{\mathrm{top}}$ there is $\theta\colon \mathcal{A}_{|\alpha|+(m+1)n_{\mathrm{d}}}\to[0,\infty)$ with $\sum_{\alpha'}\theta(\alpha')\le 1$ such that for every finite set P of points of level $i-m$, κ positive semidefinite on P, $t_0\in K$ and finite $S\subseteq(t_0+L_{j,m})\setminus\{0\}$: $\sum_{t\in S}\mathrm{Re}\,\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(P,\kappa;t,t)\le\beta_{\mathrm{B}}^{2(m-1)}\sum_{\alpha'}\theta(\alpha')\,\mathrm{En}_{\eta,i-m,\alpha'}(P,\kappa)$.

(Pt8) Lattice sums. For $1\le i\le r+1$ and every finite $Z\subseteq L_i^{\vee}\setminus\{0\}$ we have $\sum_{z\in Z}\mathrm{mj}_i^{\vee}(z)\le\beta_{\mathrm{L}}$; for $1\le m\le i\le r+1$ with $(i,m)\neq(r+1,1)$ and every finite $Z\subseteq L'_{j,m}\setminus\{0\}$ we have $\sum_{z\in Z}\mathrm{mj}'_{j,m}(z)\le\beta_{\mathrm{L}}$.

(Pt9) Limit and cutoff. For every $\alpha\in\mathcal{A}_{N_{\mathrm{top}}}$, $\mathrm{Re}\,g_{\eta,0,\alpha}(\varnothing,\varnothing)/\mathrm{wt}(\alpha)\to J$ as $\eta\to0^+$; for every finite set P of points of level $r+1$ there is $\eta_0>0$ with $g_{\eta,r+1,\alpha_\varnothing}(u,u')=\langle W(u'),W(u)\rangle$ for all $0<\eta<\eta_0$ and $u,u'\in P$ with $k_{r+1}(u,u')\neq0$.

Definition 15.7. Let Poisson recursion data over a field K with parameters (r,f,n_{d}) and block ratio β_{B} be given. A **local point** is a finite sequence of blocks (m,c) with $m\in\mathbb{Z}_{\ge1}$ and $c\in\mathbb{C}^{\times}$; level, $\varnothing$, $p\cdot(m,c)$ and positive semidefiniteness are as for points. For a field embedding $\sigma_v\colon K\to\mathbb{C}$ and a point $u=((m_a,t_a))_a$ put $\sigma_v(u)=((m_a,\sigma_v(t_a)))_a$. An **archimedean product structure** on the data consists of field embeddings $\sigma_1,\dots,\sigma_f\colon K\to\mathbb{C}$, with $\sigma=(\sigma_v)_v\colon K\to\mathbb{C}^f$; real numbers $g>0$, $T\ge1$, $C_4\ge1$ and $\mathrm{vol}(j,m)\ge0$ $(j,m\ge1)$; for each v, **local wave kernels** $g^v_{\eta,i,\alpha}$ ($\eta>0$, $0\le i\le r+1$, $\alpha\in\mathcal{A}$), complex functions on pairs of local points, **local weights** $\mathrm{wt}_v(\alpha)>0$, a real number J_v, a complex Hilbert space $\mathcal{H}_v$ and a map W_v from local points to $\mathcal{H}_v$; such that, with $\langle\xi,c\rangle=\frac{2}{g}\sum_v\mathrm{Re}(\xi_vc_v)$ on $\mathbb{C}^f\cong\mathbb{R}^{2f}$:

(Aw1) Product form. $g_{\eta,i,\alpha}(u,u')=\prod_v g^v_{\eta,i,\alpha}(\sigma_v(u),\sigma_v(u'))$, $\mathrm{wt}(\alpha)=\prod_v\mathrm{wt}_v(\alpha)$, $J=\prod_vJ_v$ and $\langle W(u'),W(u)\rangle=\prod_v\langle W_v(\sigma_v(u')),W_v(\sigma_v(u))\rangle$ for all points u,u', η, i, α.

(Aw2) Positivity. Every $g^v_{\eta,i,\alpha}$ is positive semidefinite on every finite set of local points; every k_i is positive semidefinite on every finite set of points, and $k_0(\varnothing,\varnothing)=1$.

(Aw3) Integration over the new coordinate. For $1\le i\le r+1$: $\Lambda_i=\sigma(L_{i,1})$ is a lattice in $\mathbb{C}^f$; $\sigma(L_i^{\vee})=\{\xi:\langle\xi,\lambda\rangle\in\mathbb{Z}\ \forall\lambda\in\Lambda_i\}$; there is C_i with $\#\{\lambda\in\Lambda_i:|\lambda_v|\le R_v\ \forall v\}\le C_i\prod_v(1+R_v)^2$ for all $R_v\ge0$; there are real numbers $d_v(i)>0$ $(1\le v\le f)$ with $\prod_v d_v(i)=\mathrm{covol}(\Lambda_i)$; for every v, $\eta>0$, $\alpha\in\mathcal{A}$ and local points p,q of level $i-1$ there are a continuous function $c\mapsto G^v_{\eta,i,\alpha}(p,q;c)$ on $\mathbb{C}$ and $C\ge0$ with $G^v_{\eta,i,\alpha}(p,q;c)=g^v_{\eta,i,\alpha}(p\cdot(1,c),q\cdot(1,c))$ for $c\neq0$, $|G^v_{\eta,i,\alpha}(p,q;c)|\le C(1+|c|)^{-3}$ for all $c\in\mathbb{C}$, and $\int_{\mathbb{C}}G^v_{\eta,i,\alpha}(p,q;c)\,dc=d_v(i)\,g^v_{\eta,i-1,\alpha}(p,q)$; and for all $\eta>0$, $\alpha\in\mathcal{A}$, finite sets P of points of level $i-1$, κ positive semidefinite on P and $z\in L_i^{\vee}$, $\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;z)=\mathrm{covol}(\Lambda_i)^{-1}\int_{\mathbb{C}^f}\varphi_{\eta,i,\alpha}(P,\kappa;c)\,e^{2\pi i\langle\sigma(z),c\rangle}dc$, where $\varphi_{\eta,i,\alpha}(P,\kappa;c)=\frac{1}{\mathrm{wt}(\alpha)}\sum_{h,h'\in P}\kappa(h,h')\prod_v G^v_{\eta,i,\alpha}(\sigma_v(h),\sigma_v(h');c_v)$ (Lebesgue measure on $\mathbb{C}=\mathbb{R}^2$ and on $\mathbb{C}^f=\mathbb{R}^{2f}$; the covolume is taken for the latter).

(Aw4) Wave decay at every set of places. There are $0\le\varrho'_{i,v}\le\varrho_{i,v}$ $(1\le i\le r+1,\ 1\le v\le f)$ with $\mathrm{mj}_i^{\vee}(z)=\prod_v\min(1,(\varrho_{i,v}/|\sigma_v(z)|)^{n_{\mathrm{d}}})$ for $z\in L_i^{\vee}\setminus\{0\}$, such that for all $\eta>0$, $1\le i\le r+1$, α with $|\alpha|+n_{\mathrm{d}}\le N_{\mathrm{top}}$ and $A\subseteq\{1,\dots,f\}$ there is $\theta_A\colon\mathcal{A}_{|\alpha|+n_{\mathrm{d}}}\to[0,\infty)$ with $\sum\theta_A\le1$ and, for all P, κ as in (Aw3) and $z\in L_i^{\vee}\setminus\{0\}$, $\prod_{v\in A}|\sigma_v(z)|^{n_{\mathrm{d}}}\,|\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;z)|\le\prod_{v\in A}\varrho'^{\,n_{\mathrm{d}}}_{i,v}\sum_{\alpha'}\theta_A(\alpha')\mathrm{En}_{\eta,i-1,\alpha'}(P,\kappa)$.

(Aw5) Lift bounds. There are $\varrho_{j,m,v} \geq 0$ with $\mathrm{mj}'_{j,m}(z) = \prod_v \min(1, (\varrho_{j,m,v}/|\sigma_v(z)|)^{n_\mathrm{d}})$ for $z \in K^\times$ $(1 \leq m \leq i \leq r+1,\ (i,m) \neq (r+1,1))$, and $0 \leq \varrho'_{j,m,v} \leq \varrho_{j,m,v}$, $\lambda_{j,m,v} \geq 0$ with $2\lambda_{j,m,v} \leq \varrho_{j,m,v}$, such that for all $\eta > 0$ and α with $|\alpha| + n_\mathrm{d} \leq N_\mathrm{top}$, writing $d(P,\kappa;t;\alpha') = \mathrm{Re}\,\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha'}(P,\kappa;t,t)$: (a) for $1 \leq m \leq i \leq r$ and every $A \subseteq \{1,\dots,f\}$ there is $\theta_A\colon \mathcal{A}_{|\alpha|+n_\mathrm{d}} \to [0,\infty)$ with $\sum \theta_A \leq 2$ such that for every finite set P of points of level $i-m$, finite $S \subseteq K^\times$, κ positive semidefinite on $\{h\cdot(m,t) : h \in P, t \in S\}$ and $t \neq t'$ in S, $\prod_{v\in A}|\sigma_v(t^{-1}-t'^{-1})|^{n_\mathrm{d}}|\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(P,\kappa^{t,t'}_m;t,t')| \leq \prod_{v\in A}\varrho'^{\,n_\mathrm{d}}_{j,m,v}\cdot \frac{1}{2}\sum_{\alpha'}\theta_A(\alpha')(d(P,\kappa^{t,t}_m;t;\alpha') + d(P,\kappa^{t',t'}_m;t';\alpha'))$; (b) for $i = r+1$, $2 \leq m \leq r+1$ and every $B \subseteq \{1,\dots,f\}$ there is $\theta_B\colon \mathcal{A}_{|\alpha|+n_\mathrm{d}} \to [0,\infty)$ with $\sum\theta_B \leq 1$ such that for every finite set P of points of level $r+1-m$, κ positive semidefinite on P and $t \in K^\times$, $d(P,\kappa;t;\alpha) \leq \prod_{v\in B}(|\sigma_v(t)|\lambda_{j,m,v})^{2n_\mathrm{d}}\sum_{\alpha'}\theta_B(\alpha')d(P,\kappa;t;\alpha')$.

(Aw6) Degenerate blocks. $\beta_\mathrm{B} = (C_4/T)^f$ and $\mathrm{vol}(j,m) \leq T^{-2f}$ for all j,m; for all $\eta > 0$, $2 \leq m \leq i \leq r+1$ and α with $|\alpha|+(m+1)n_\mathrm{d} \leq N_\mathrm{top}$ there are $0 \leq M \leq C_4^{fm}\prod_{1\leq m'<m}\mathrm{vol}(j,m')$ and $\theta\colon \mathcal{A}_{|\alpha|+mn_\mathrm{d}} \to [0,\infty)$ with $\sum\theta \leq 1$ such that for every finite set P of points of level $i-m$, κ positive semidefinite on P, $t_0 \in K$ and finite $S \subseteq (t_0 + L_{j,m}) \setminus \{0\}$, $\sum_{t\in S}\mathrm{Re}\,\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(P,\kappa;t,t) \leq M\sum_{\alpha'}\theta(\alpha')\mathrm{En}_{\eta,i-m,\alpha'}(P,\kappa)$.

(Aw7) Limit and top. For every v and $\alpha \in \mathcal{A}_{N_\mathrm{top}}$, $g^v_{\eta,0,\alpha}(\varnothing,\varnothing)/\mathrm{wt}_v(\alpha) \to J_v$ as $\eta \to 0^+$; for every v and every finite set Q of local points of level $r+1$ there is $\eta_0 > 0$ with $g^v_{\eta,r+1,\alpha\varnothing}(p,q) = \langle W_v(q), W_v(p)\rangle$ for $0 < \eta < \eta_0$ and $p,q \in Q$.

Lemma 15.8. *The notation introduced here is local to this result and to the results that apply it. Let S_mn, V, e_1 be real numbers, let $c_\mathrm{rw}, d, d', D^* > 0$, and let $\rho, m_0, c, c', C_\infty, U_\mathrm{loc}, M_\mathrm{loc}, v_Z$, $\mu_K, \mu^\natural_K, \mu^\circ_K, \mu_S, \mu_{S'}, n_\pi, n_{\pi'}, \deg_\pi$ be positive real numbers. Assume the following.*

(Mx1) Poisson part. *Poisson recursion data over a field K with parameters (r,f,n_d), lattice-sum bound β_L and block ratio β_B satisfy (Pt2), (Pt3) and (Pt8), carry an archimedean product structure, and satisfy $(r+1)(2\cdot 3^f\beta_\mathrm{L} + 4\beta_\mathrm{B}) \leq 1$. There are a set I, complex functions a_ι on points $(\iota \in I)$ with $a_\iota(u) \neq 0$ only if u has level $r+1$ and $\sum_u |a_\iota(u)|\,\|W(u)\| < \infty$, and a real number $s \geq 0$ with $\max(s,1)\prod_{i=1}^{r+1}(1+\mathrm{ex}_i) \leq e_1$, such that (Mi1) and (Mi2) hold for these and for c, c', V, m_0.*

(Mx2) Archimedean integral. *There are data satisfying all hypotheses of Proposition 15.18 with $d_\sigma = \deg_\pi$; let $J_\infty \in [0,\infty]$ be the lower integral on the left side of its conclusion. The number J of the Poisson recursion data of (Mx1) satisfies $J = v_Z\,J_\infty/(C_\infty U_\mathrm{loc})$.*

(Mx3) Stabilisers. *There are a CM extension E/F, a nondegenerate hermitian space V over E of dimension n, a Haar measure μ_G on $\mathrm{U}(\mathsf{V})(\mathbb{A}_F)$, and tuples X_1, X_2, X_3 of sizes a_1, a_2, a_3 as in item 1 of Proposition 15.20 (so $a_j \geq 1$, $2a_j < n$, nondegenerate Gram matrices), with stabilisers H_1, H_2, H_3 and Haar measures μ'_1, μ'_2, μ'_3 on their adelic points satisfying (15.12). Put $v_H = \mu'_1([H_1])$, $v_{H'} = \mu'_2([H_2])$, $v_B = \mu'_3([H_3])$, $v_G = \mu_G([\mathrm{U}(\mathsf{V})])$, and assume $D^* = \deg_\pi\, v_G/\mu^\circ_K$.*

(Mx4) Word levels. *H_1 is a connected reductive group over F with $H_1(F_\infty)$ compact, $\mu'_1 = \mu_\infty \otimes \mu_f$ for Haar measures μ_∞ on $H_1(F_\infty)$ and μ_f on $H_1(\mathbb{A}_{F,f})$, and there are compact open subgroups $K'' \subseteq K_1 \subseteq H_1(\mathbb{A}_{F,f})$ with $H_1(F) \cap cK''c^{-1} = \{1\}$ for every $c \in H_1(\mathbb{A}_{F,f})$, $[K_1 : K''] = c_\mathrm{rw}$ and*

$$d = n_\pi\,\big|H_1(F)\backslash H_1(\mathbb{A}_{F,f})/K''\big|.$$

Put $v_{H,\infty} = \mu_\infty(H_1(F_\infty))$, $\mu^\circ_H = \mu_f(K'')$, $\mu_H = \mu_f(K_1)$. The same holds for H_2, d', $n_{\pi'}$, with numbers $v_{H',\infty}$, $\mu^\circ_{H'}$, $\mu_{H'}$.

(Mx5) Relations. *With these numbers, (Ma1), (Ma3), (Ma6) and (Ma7) hold.*

Then $(S_{\mathrm{mn}}; c_{\mathrm{rw}}, d, d', D^*)$ *has main-row mass data, with the numbers* V, J, J_∞, e_1 *and* ρ, m_0, $c, c', C_\infty, \deg_\pi, U_{\mathrm{loc}}, M_{\mathrm{loc}}, v_Z, v_B, v_G, v_H, v_{H'}, v_{H,\infty}, v_{H',\infty}, n_\pi, n_{\pi'}, \mu_K, \mu_K^\natural, \mu_K^\circ, \mu_S, \mu_{S'}, \mu_H, \mu_{H'}$, $\mu_H^\circ, \mu_{H'}^\circ$ *above, density factor* e_1 *and Poisson error*

$$\delta_{\mathrm{P}} = 15\,(r+1)(2\cdot 3^f\beta_{\mathrm{L}} + 4\beta_{\mathrm{B}}).$$

Proof. We check that the numbers listed are real, that those required positive in Definition 14.11 are positive, and (Ma1)–(Ma9).

(Ma2) and (Ma4). By Theorem 15.14, the Poisson recursion data of (Mx1), which carry an archimedean product structure, satisfy (Pt1), (Pt4), (Pt5), (Pt6), (Pt7) and (Pt9); with (Pt2), (Pt3), (Pt8) from (Mx1) they satisfy (Pt1)–(Pt9). Together with $(r+1)(2\cdot 3^f\beta_{\mathrm{L}}+4\beta_{\mathrm{B}}) \le 1$, the functions a_ι with their support and summability conditions, $s \ge 0$, $m_0 > 0$, $\max(s,1)\prod_i(1+\mathrm{ex}_i) \le e_1$ and (Mi1), (Mi2), these are all hypotheses of Corollary 15.9. It gives $\delta_{\mathrm{P}} \ge 0$, $e_1 \ge 1$, and $c\,c'\,V \le (1+\delta_{\mathrm{P}})\,e_1\,m_0\,J$; so $e_1 \ge 0$ and (Ma2), (Ma4) hold.

(Ma5). By Proposition 15.18, applied to the data of (Mx2) with $d_\sigma = \deg_\pi$, $J_\infty \le 1/\deg_\pi$. As a lower integral of a nonnegative function, $J_\infty \ge 0$; hence J_∞ is a real number in $[0, 1/\deg_\pi]$, and $J = v_Z J_\infty/(C_\infty U_{\mathrm{loc}})$ by (Mx2). This is (Ma5).

(Ma8). The group H_1, the measure $\mu_1' = \mu_\infty \otimes \mu_f$ and the subgroups $K'' \subseteq K_1$ satisfy the hypotheses of Lemma 15.19 (with $H = H_1$, $K = K_1$). $H_1(F_\infty)$ is compact and nonempty, so $0 < v_{H,\infty} < \infty$; K_1 and K'' are compact open, so $0 < \mu_H, \mu_H^\circ < \infty$. By item 1 of Lemma 15.19, $v_H = \mu_1'([H_1]) = v_{H,\infty}\,\mu_H \sum_{c\in R(K_1)} |H_1(F) \cap cK_1c^{-1}|^{-1}$ is a finite sum of positive numbers over a nonempty finite set, so $0 < v_H < \infty$. By item 2, $\mu_H = [K_1 : K'']\,\mu_H^\circ = c_{\mathrm{rw}}\,\mu_H^\circ$ and $|H_1(F)\backslash H_1(\mathbb{A}_{F,f})/K''| = v_H/(v_{H,\infty}\,\mu_H^\circ)$; by (Mx4), $d = n_\pi v_H/(v_{H,\infty}\mu_H^\circ)$. The same argument for H_2 gives $0 < v_{H'}, v_{H',\infty}, \mu_{H'}, \mu_{H'}^\circ < \infty$, $d' = n_{\pi'}v_{H'}/(v_{H',\infty}\mu_{H'}^\circ)$ and $\mu_{H'} = c_{\mathrm{rw}}\mu_{H'}^\circ$. This is (Ma8).

(Ma9). By item 1 of Proposition 15.20, applied to X_1 and to X_3, $\mu_1'([H_1]) = \mu_G([\mathrm{U}(\mathsf{V})]) = \mu_3'([H_3]) < \infty$; so $v_G = v_B = v_H \in (0,\infty)$. By item 2, $\mu_3'([H_3])\,\mu_G([\mathrm{U}(\mathsf{V})]) = \mu_1'([H_1])\,\mu_2'([H_2])$, that is $v_B v_G = v_H v_{H'}$; and $D^* = \deg_\pi v_G/\mu_K^\circ$ by (Mx3). This is (Ma9).

The rest. (Ma1), (Ma3), (Ma6), (Ma7) hold by (Mx5). The numbers $\rho, m_0, c, c', C_\infty$, $\deg_\pi, U_{\mathrm{loc}}, M_{\mathrm{loc}}, v_Z, n_\pi, n_{\pi'}, \mu_K, \mu_K^\natural, \mu_K^\circ, \mu_S, \mu_{S'}$ are positive by hypothesis, and $v_B, v_G, v_H, v_{H'}$, $v_{H,\infty}, v_{H',\infty}, \mu_H, \mu_{H'}, \mu_H^\circ, \mu_{H'}^\circ$ were shown positive; $V, J, J_\infty, e_1, \delta_{\mathrm{P}}$ are real. Hence all clauses of Definition 14.11 hold, with density factor e_1 and Poisson error δ_{P}. □

Corollary 15.9. *Let Poisson recursion data over a field* K *with parameters* (r, f, n_{d})*, lattice-sum bound* β_{L} *and block ratio* β_{B} *satisfy (Pt1)–(Pt9) and* $(r+1)(2\cdot 3^f\beta_{\mathrm{L}} + 4\beta_{\mathrm{B}}) \le 1$*, and put* $\delta_{\mathrm{P}} = 15(r+1)(2\cdot 3^f\beta_{\mathrm{L}} + 4\beta_{\mathrm{B}})$*. Let* I *be a set and* a_ι *(*$\iota \in I$*) complex functions on points with* $a_\iota(u) \ne 0$ *only if* u *has level* $r+1$ *and* $\sum_u |a_\iota(u)|\,\|W(u)\| < \infty$*. Let* $s \ge 0$*,* c, c', V, m_0, e_1 *be real numbers with* $m_0 \ge 0$ *and* $\max(s,1)\prod_{i=1}^{r+1}(1+\mathrm{ex}_i) \le e_1$*, such that*

(Mi1) Density domination. $s\,k_{r+1}(u,u') - \sum_{\iota\in R} a_\iota(u)\overline{a_\iota(u')}$ *is positive semidefinite on* P *for every finite set* P *of points of level* $r+1$ *and every finite* $R \subseteq I$*;*

(Mi2) Unfolding link. $c\,c'\,V \le m_0\,M$ *for every real* M *such that* $\sum_{\iota\in R} \|\sum_u a_\iota(u)W(u)\|^2 \le M$ *for all finite* $R \subseteq I$*.*

Then $\delta_{\mathrm{P}} \ge 0$*,* $e_1 \ge 1$*,* $J \ge 0$ *and*

$$c\,c'\,V \le (1+\delta_{\mathrm{P}})\,e_1\,m_0\,J;$$

in particular the numbers $V, J, e_1, \delta_{\mathrm{P}}, c, c', m_0$ satisfy (Ma2) and (Ma4).

Proof. **Signs.** We have $\delta_{\mathrm{P}} \geq 0$ since $\beta_{\mathrm{L}}, \beta_{\mathrm{B}} \geq 0$. By (Pt3), $\mathrm{ex}_i \geq 0$, so $e_1 \geq \max(s,1) \geq 1$. By (Pt1) the kernel $g_{\eta,0,\alpha}$ is positive semidefinite on $\{\varnothing\}$, so $g_{\eta,0,\alpha}(\varnothing,\varnothing) \geq 0$, and by (Pt9) J is a limit of the numbers $\mathrm{Re}\, g_{\eta,0,\alpha_\varnothing}(\varnothing,\varnothing)/\mathrm{wt}(\alpha_\varnothing) \geq 0$; hence $J \geq 0$.

Domination with a positive factor. Put $\beta_{\mathrm{d}} = \max(s,1) > 0$. For P and R as in (Mi1),

$$\beta_{\mathrm{d}}\, k_{r+1}(u,u') - \sum_{\iota\in R} a_\iota(u)\overline{a_\iota(u')} = (\beta_{\mathrm{d}} - s)\, k_{r+1}(u,u') + \Big(s\, k_{r+1}(u,u') - \sum_{\iota\in R} a_\iota(u)\overline{a_\iota(u')}\Big)$$

is positive semidefinite on P, as a nonnegative multiple of k_{r+1} (positive semidefinite by (Pt1)) plus a kernel positive semidefinite by (Mi1).

Conclusion. By Theorem 15.10 with this β_{d}, every finite $R \subseteq I$ has $\sum_{\iota\in R} \| \sum_u a_\iota(u) W(u)\|^2 \leq M$ with $M = \max(s,1)(1+\delta_{\mathrm{P}}) \prod_i (1+\mathrm{ex}_i)\, J$. By (Mi2), $c\,c'\,V \leq m_0 M$, and as $m_0 \geq 0$, $1+\delta_{\mathrm{P}} \geq 0$, $J \geq 0$ and $\max(s,1)\prod_i(1+\mathrm{ex}_i) \leq e_1$, $m_0 M \leq (1+\delta_{\mathrm{P}}) e_1 m_0 J$. The final assertion of the statement is, by definition, (Ma2) and (Ma4) for these numbers. □

Theorem 15.10. *Let Poisson recursion data over a field K with parameters (r, f, n_{d}), lattice-sum bound β_{L} and block ratio β_{B} satisfy (Pt1)–(Pt9), and suppose*

$$(r+1)\big(2\cdot 3^f \beta_{\mathrm{L}} + 4\beta_{\mathrm{B}}\big) \leq 1; \qquad \textit{put} \qquad \delta_{\mathrm{P}} = 15\,(r+1)\big(2\cdot 3^f \beta_{\mathrm{L}} + 4\beta_{\mathrm{B}}\big).$$

Let I be a set and, for $\iota \in I$, a_ι a complex function on points such that

1. *$a_\iota(u) \neq 0$ only if u has level $r+1$;*
2. *$\sum_u |a_\iota(u)|\, \|W(u)\| < \infty$, the sum over all points;*
3. *there is a real $\beta_{\mathrm{d}} > 0$ such that $\beta_{\mathrm{d}}\, k_{r+1}(u,u') - \sum_{\iota\in R} a_\iota(u)\overline{a_\iota(u')}$ is positive semidefinite on P for every finite set P of points of level $r+1$ and every finite $R \subseteq I$.*

Then for every finite $R \subseteq I$ the sums $\sum_u a_\iota(u) W(u)$ converge absolutely in $\mathcal{H}$, and

$$\sum_{\iota\in R} \Big\| \sum_u a_\iota(u)\, W(u) \Big\|^2 \leq \beta_{\mathrm{d}}\,(1+\delta_{\mathrm{P}}) \prod_{i=1}^{r+1} (1+\mathrm{ex}_i)\; J.$$

Proof. **The recursion on one tower.** Fix a tower $\mathcal{S}$ and $\eta > 0$, and let $\sigma_0, \dots, \sigma_{r+1} \geq 0$ be as in Lemma 15.11. Put $\delta_1 = 2\cdot 3^f \beta_{\mathrm{L}}$; then $(r+1)(\delta_1 + 4\beta_{\mathrm{B}}) \leq 1$, in particular $\delta_1 \leq 1$. By Lemma 15.11 the hypothesis of Lemma 15.13 holds with $\beta = \beta_{\mathrm{B}}$ (and $\mathrm{ex}_i \geq 0$ by (Pt3)), so

$$\sigma_{r+1}^2 \leq (1+\delta_{\mathrm{P}}) \prod_{i=1}^{r+1} (1+\mathrm{ex}_i)\, \sigma_0^2.$$

The only point of level 0 is $\varnothing$, so $\mathcal{X}_0 = \{\varnothing\}$, and $k_0(\varnothing,\varnothing) = 1$ by (Pt1); hence $\sigma_0^2 = s_0(\eta)$ with $s_0(\eta) = \max_{\alpha\in\mathcal{A}_{N_{\mathrm{top}}}} \mathrm{Re}\, g_{\eta,0,\alpha}(\varnothing,\varnothing)/\mathrm{wt}(\alpha)$. As $\alpha_\varnothing \in \mathcal{A}_0$ and $\mathrm{wt}(\alpha_\varnothing) = 1$,

$$\begin{aligned} \mathrm{Re} \sum_{u,u'\in\mathcal{X}_{r+1}} k_{r+1}(u,u')\, g_{\eta,r+1,\alpha_\varnothing}(u,u') = \mathrm{En}_{\eta,r+1,\alpha_\varnothing}(\mathcal{X}_{r+1}, k_{r+1}) &\leq \sigma_{r+1}^2 \\ &\leq (1+\delta_{\mathrm{P}}) \prod_{i=1}^{r+1} (1+\mathrm{ex}_i)\, s_0(\eta). \end{aligned} \tag{15.1}$$

The coefficient energy on one tower. Fix a tower $\mathcal{S}$ and a finite $R \subseteq I$; $\mathcal{X} = \mathcal{X}_{r+1}(\mathcal{S})$ is a finite set of points of level $r+1$. Put $\Gamma(u,u') = \langle W(u'), W(u)\rangle$; it is positive semidefinite on $\mathcal{X}$, since $\sum_{u,u'} \overline{w_u}\, w_{u'} \langle W(u'), W(u)\rangle = \|\sum_u \overline{w_u} W(u)\|^2$. Put $C(u,u') = \sum_{\iota \in R} a_\iota(u)\overline{a_\iota(u')}$. Expanding the norms,

$$E_{\mathcal{X}} = \sum_{\iota \in R} \Big\| \sum_{u \in \mathcal{X}} a_\iota(u) W(u) \Big\|^2 = \sum_{u,u' \in \mathcal{X}} C(u,u')\, \Gamma(u,u').$$

By hypothesis 3 and the Schur product theorem, $(\beta_{\mathrm{d}} k_{r+1} - C)\Gamma$ is positive semidefinite on $\mathcal{X}$, so the sum of its entries is real and ≥ 0; hence $E_{\mathcal{X}} \leq \beta_{\mathrm{d}} \operatorname{Re} \sum_{u,u' \in \mathcal{X}} k_{r+1}(u,u')\Gamma(u,u')$. By (Pt9) there is $\eta_0 > 0$ such that for $0 < \eta < \eta_0$, $k_{r+1}(u,u')\Gamma(u,u') = k_{r+1}(u,u') g_{\eta,r+1,\alpha\varnothing}(u,u')$ for all $u, u' \in \mathcal{X}$ (both sides vanish where $k_{r+1}(u,u') = 0$). By (15.1) and $\beta_{\mathrm{d}} > 0$, $E_{\mathcal{X}} \leq \beta_{\mathrm{d}}(1+\delta_{\mathrm{P}}) \prod_i (1 + \mathrm{ex}_i)\, s_0(\eta)$ for $0 < \eta < \eta_0$. By (Pt9), each of the finitely many functions $\eta \mapsto \operatorname{Re} g_{\eta,0,\alpha}(\varnothing,\varnothing)/\mathrm{wt}(\alpha)$, $\alpha \in \mathcal{A}_{N_{\mathrm{top}}}$, tends to J, hence so does their maximum $s_0(\eta)$; letting $\eta \to 0^+$ gives

$$E_{\mathcal{X}} \leq B_{\mathrm{A}} \qquad \text{for every tower } \mathcal{S}; \tag{15.2}$$

in particular $B_{\mathrm{A}} \geq 0$.

Exhaustion by towers. By hypothesis 2 and completeness of $\mathcal{H}$, $v_\iota = \sum_u a_\iota(u) W(u)$ converges absolutely, hence unconditionally: for $\tau > 0$ there is a finite set P_0 of points with $\|v_\iota - \sum_{u \in P} a_\iota(u) W(u)\| \leq \tau$ for every finite $P \supseteq P_0$ and every $\iota \in R$ (R is finite). Let $\mathcal{S}_{j,m}$ be the set of coordinates t of the blocks (m,t) starting at j of the points of P_0 of level $r+1$; this is a tower, and every point of P_0 of level $r+1$ lies in $\mathcal{X} = \mathcal{X}_{r+1}(\mathcal{S})$. For $P = \mathcal{X} \cup P_0$, the points of $P \setminus \mathcal{X}$ have level $\neq r+1$, so a_ι vanishes on them by hypothesis 1 and $\sum_{u \in P} a_\iota(u) W(u) = \sum_{u \in \mathcal{X}} a_\iota(u) W(u)$. Hence, by the triangle inequality in $\mathcal{H}^R$ and (15.2),

$$\Big(\sum_{\iota \in R} \|v_\iota\|^2 \Big)^{1/2} \leq E_{\mathcal{X}}^{1/2} + \tau |R|^{1/2} \leq B_{\mathrm{A}}^{1/2} + \tau |R|^{1/2}.$$

Letting $\tau \to 0$ proves the theorem. □

Lemma 15.11. *Let Poisson recursion data over a field K with parameters (r, f, n_{d}), lattice-sum bound β_{L} and block ratio β_{B} satisfy (Pt1)–(Pt8), and suppose $\delta_1 = 2 \cdot 3^f \beta_{\mathrm{L}} \leq 1$. Fix $\eta > 0$ and a tower $\mathcal{S}$, and let*

$$\sigma_k = \Big(\max_{\alpha \in \mathcal{A}_{4n_{\mathrm{d}}(r+1-k)}} \mathrm{En}_{\eta,k,\alpha}(\mathcal{X}_k(\mathcal{S}), k_k) \Big)^{1/2} \qquad (0 \leq k \leq r+1).$$

Then for $1 \leq i \leq r+1$

$$\sigma_i \leq (1+\delta_1)\sqrt{1+\mathrm{ex}_i}\, \sigma_{i-1} + 2 \sum_{m=2}^{i} \beta_{\mathrm{B}}^{m-1}\, \sigma_{i-m}.$$

Proof. Fix $1 \leq i \leq r+1$ and $\alpha \in \mathcal{A}_{4n_{\mathrm{d}}(r+1-i)}$; it suffices to bound $\mathrm{En}_{\eta,i,\alpha}(\mathcal{X}_i, k_i)^{1/2}$ by the right side.

Splitting by the last block. For $1 \leq m \leq i$ let $\mathcal{X}^{(m)} = \{h \cdot (m,t) : h \in \mathcal{X}_{i-m},\ t \in \mathcal{S}_{j,m}\}$. Every $u \in \mathcal{X}_i$ has level $i \geq 1$, hence a last block (m,t) with $1 \leq m \leq i$, and $u = h \cdot (m,t)$ with $|h| = i - m$; the blocks of h start where they start in u, and the last block starts at $|h| + 1 = j$; so $u \in \mathcal{X}_i$ if and only if $h \in \mathcal{X}_{i-m}$ and $t \in \mathcal{S}_{j,m}$. Hence $\mathcal{X}_i = \bigsqcup_{m=1}^{i} \mathcal{X}^{(m)}$, and $(h,t) \mapsto h \cdot (m,t)$ is a bijection $\mathcal{X}_{i-m} \times \mathcal{S}_{j,m} \to \mathcal{X}^{(m)}$ (the last block and the rest determine the point). The kernel

$M(u,u') = k_i(u,u')g_{\eta,i,\alpha}(u,u')$ is positive semidefinite on $\mathcal{X}_i$ by (Pt1) and the Schur product theorem, so $x \mapsto (\sum_{u,u'} \overline{x_u} M(u,u') x_{u'})^{1/2}$ is a seminorm on $\mathbb{C}^{\mathcal{X}_i}$. Applying it to the indicator vectors $\mathbf{1}_{\mathcal{X}^{(m)}}$, whose sum is $\mathbf{1}_{\mathcal{X}_i}$, and dividing by $\mathrm{wt}(\alpha) > 0$,

$$\mathrm{En}_{\eta,i,\alpha}(\mathcal{X}_i, k_i)^{1/2} \le \sum_{m=1}^{i} \mathrm{En}_{\eta,i,\alpha}(\mathcal{X}^{(m)}, k_i)^{1/2}, \tag{15.3}$$

all energies being ≥ 0 (positive semidefinite kernels have a real nonnegative sum of entries).

Off-diagonal terms of one block size. Fix m. By the bijection,

$$\begin{aligned}\mathrm{En}_{\eta,i,\alpha}(\mathcal{X}^{(m)}, k_i) &= \mathrm{Re} \sum_{t,t' \in \mathcal{S}_{j,m}} \mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(\mathcal{X}_{i-m}, (k_i)_m^{t,t'}; t, t') \\ &= D_{i,m}(\alpha) + \mathrm{Re} \sum_{t \ne t'} \mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(\mathcal{X}_{i-m}, (k_i)_m^{t,t'}; t, t').\end{aligned}$$

If $(i,m) = (r+1,1)$, every $(k_i)_m^{t,t'}$ with $t \ne t'$ vanishes on points of level $i-m$ by (Pt2), so the off-diagonal sum is 0. Otherwise a pair $t \ne t'$ with $t^{-1} - t'^{-1} \notin L'_{j,m}$ contributes 0, by (Pt2). Let θ be the weight of (Pt6) for (η, i, m, α); $|\alpha| + n_{\mathrm{d}} \le N_{\mathrm{top}}$ as $i \ge 1$. Apply (Pt6) with $P = \mathcal{X}_{i-m}$, $S = \mathcal{S}_{j,m}$, $\kappa = k_i$ (positive semidefinite by (Pt1)), and write $d_{\alpha'}(t) = \mathrm{Re}\,\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha'}(\mathcal{X}_{i-m}, (k_i)_m^{t,t}; t, t) \ge 0$ (the kernels $(k_i)_m^{t,t}$ and $(h,h') \mapsto g_{\eta,i,\alpha'}(h \cdot (m,t), h' \cdot (m,t))$ are positive semidefinite on $\mathcal{X}_{i-m}$ by (Pt1), as $h \mapsto h \cdot (m,t)$ is injective, and the Schur product theorem applies), so that $\sum_t d_{\alpha'}(t) = D_{i,m}(\alpha')$. Let Π be the set of ordered pairs $(t,t') \in \mathcal{S}_{j,m}^2$ with $t \ne t'$ and $t^{-1} - t'^{-1} \in L'_{j,m}$. For fixed t the elements $t^{-1} - t'^{-1}$, $(t,t') \in \Pi$, are pairwise distinct, nonzero and in $L'_{j,m}$, so their majorants sum to at most β_{L} by (Pt8); likewise for fixed t'. Since $\theta \ge 0$ and $d \ge 0$,

$$\begin{aligned}&\Big| \sum_{(t,t') \in \Pi} \mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(\mathcal{X}_{i-m}, (k_i)_m^{t,t'}; t, t') \Big| \\ &\qquad \le \frac{1}{2} \sum_{\alpha'} \theta(\alpha') \sum_{(t,t') \in \Pi} \mathrm{mj}'_{j,m}(t^{-1} - t'^{-1})(d_{\alpha'}(t) + d_{\alpha'}(t')) \le \beta_{\mathrm{L}} \sum_{\alpha'} \theta(\alpha')\, D_{i,m}(\alpha').\end{aligned}$$

With $\sum \theta \le 2 \cdot 3^f$, in all cases

$$\mathrm{En}_{\eta,i,\alpha}(\mathcal{X}^{(m)}, k_i) \le D_{i,m}(\alpha) + \delta_1 \max_{\alpha' \in \mathcal{A}_{|\alpha|+n_{\mathrm{d}}}} D_{i,m}(\alpha'). \tag{15.4}$$

The diagonal blocks. α and every $\alpha' \in \mathcal{A}_{|\alpha|+n_{\mathrm{d}}}$ lie in $\mathcal{A}_{4n_{\mathrm{d}}(r+1-i)+n_{\mathrm{d}}}$, so Lemma 15.12 applies to them. For $m = 1$, as $2^f \beta_{\mathrm{L}} \le \delta_1$, (15.4) gives $\mathrm{En}_{\eta,i,\alpha}(\mathcal{X}^{(1)}, k_i) \le (1+\delta_1)^2(1+\mathrm{ex}_i)\sigma_{i-1}^2$. For $m \ge 2$, as $\delta_1 \le 1$ and $\mathrm{ex}_i \le 1$ by (Pt3), $\mathrm{En}_{\eta,i,\alpha}(\mathcal{X}^{(m)}, k_i) \le (1+\delta_1)(1+\mathrm{ex}_i)\beta_{\mathrm{B}}^{2(m-1)}\sigma_{i-m}^2 \le 4\beta_{\mathrm{B}}^{2(m-1)}\sigma_{i-m}^2$. Taking square roots ($\sigma \ge 0$, $\beta_{\mathrm{B}} \ge 0$, $1 + \mathrm{ex}_i \ge 0$) and inserting into (15.3) gives the bound for α; maximising over α proves the lemma. □

Lemma 15.12. *Let Poisson recursion data over a field K with parameters (r, f, n_{d}), lattice-sum bound β_{L} and block ratio β_{B} satisfy (Pt1)–(Pt8). Fix $\eta > 0$ and a tower $\mathcal{S}$, write $\mathcal{X}_i = \mathcal{X}_i(\mathcal{S})$, and for $1 \le m \le i \le r+1$ and $\alpha \in \mathcal{A}$ put*

$$D_{i,m}(\alpha) = \sum_{t \in \mathcal{S}_{i-m+1,m}} \mathrm{Re}\,\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(\mathcal{X}_{i-m}, (k_i)_m^{t,t}; t, t).$$

Then $\mathrm{En}_{\eta,k,\alpha}(\mathcal{X}_k, k_k) \ge 0$ for all k, α, so that

$$\sigma_k = \Big(\max_{\alpha \in \mathcal{A}_{4n_{\mathrm{d}}(r+1-k)}} \mathrm{En}_{\eta,k,\alpha}(\mathcal{X}_k, k_k) \Big)^{1/2} \qquad (0 \le k \le r+1)$$

is a well-defined nonnegative number; $D_{i,m}(\alpha) \ge 0$*; and for* $1 \le m \le i \le r+1$ *and* $\alpha \in \mathcal{A}_{4n_{\mathrm{d}}(r+1-i)+n_{\mathrm{d}}}$

$$D_{i,1}(\alpha) \le (1+\mathrm{ex}_i)(1+2^f\beta_{\mathrm{L}})\,\sigma_{i-1}^2, \qquad D_{i,m}(\alpha) \le (1+\mathrm{ex}_i)\,\beta_{\mathrm{B}}^{2(m-1)}\,\sigma_{i-m}^2 \quad (m \ge 2).$$

Proof. **Positivity facts.** We use the Schur product theorem: the entrywise product of two kernels positive semidefinite on a finite set P is positive semidefinite on P; and the sum of all entries of a kernel positive semidefinite on P is ≥ 0 (take $w = (1, \dots, 1)$).

(F1) If κ is positive semidefinite on a finite set P of points, then $\mathrm{En}_{\eta,k,\alpha}(P,\kappa) \ge 0$, by (Pt1), the Schur product theorem and $\mathrm{wt}(\alpha) > 0$. With $\kappa = k_k$ and $P = \mathcal{X}_k$ this is the first claim; $\mathcal{A}_n \ni \alpha_\varnothing$ is finite and nonempty, so σ_k is well defined and ≥ 0.

(F2) If P is a finite set of points of level $i-m$ and κ is positive semidefinite on P, then $\mathrm{Re}\,\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(P,\kappa;t,t) \ge 0$: the map $h \mapsto h\cdot(m,t)$ is injective, so $(h,h') \mapsto g_{\eta,i,\alpha}(h\cdot(m,t), h'\cdot(m,t))$ is positive semidefinite on P by (Pt1), and the Schur product theorem applies. In particular $D_{i,m}(\alpha) \ge 0$, since $(k_i)^{t,t}_m$ is positive semidefinite on $\mathcal{X}$ by (Pt1) and the same injectivity.

(F3) Both energies are linear in κ (the first over $\mathbb{R}$), and depend only on the values of κ on $P \times P$.

Reduction to the theta majorants. Let $\Theta_{i,m,t}$ be the functions of (Pt3) for the tower $\mathcal{S}$. For $t \in \mathcal{S}_{j,m}$, $\Theta_{i,m,t} = (\Theta_{i,m,t} - (k_i)^{t,t}_m) + (k_i)^{t,t}_m$ is positive semidefinite on $\mathcal{X}$ by (Pt3)(a) and (F2). Congruence modulo the subgroup $L_{j,m}$ is an equivalence relation on the finite set $\mathcal{S}_{j,m}$; let $R \subseteq \mathcal{S}_{j,m}$ contain exactly one element of each class, and for $t_0 \in R$ let $S(t_0) = \{t \in \mathcal{S}_{j,m} : t - t_0 \in L_{j,m}\}$, so that $\mathcal{S}_{j,m} = \bigsqcup_{t_0\in R} S(t_0)$ and $S(t_0) \subseteq (t_0 + L_{j,m}) \setminus \{0\}$. By (F2) and (F3) applied to $\Theta_{i,m,t} - (k_i)^{t,t}_m$, and by (Pt3)(b) with (F3),

$$D_{i,m}(\alpha) \le \sum_{t\in\mathcal{S}_{j,m}} \mathrm{Re}\,\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(\mathcal{X},\Theta_{i,m,t};t,t) = \sum_{t_0\in R}\sum_{t\in S(t_0)} \mathrm{Re}\,\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(\mathcal{X},\Theta_{i,m,t_0};t,t). \quad (15.5)$$

Moreover, as the elements of R are pairwise incongruent, (Pt3)(c) with (F1) and (F3) gives, for every k and α',

$$\sum_{t_0\in R} \mathrm{En}_{\eta,i-m,\alpha'}(\mathcal{X},\Theta_{i,m,t_0}) = \mathrm{En}_{\eta,i-m,\alpha'}\Big(\mathcal{X}, \sum_{t_0\in R}\Theta_{i,m,t_0}\Big) \le (1+\mathrm{ex}_i)\,\mathrm{En}_{\eta,i-m,\alpha'}(\mathcal{X},k_{i-m}). \quad (15.6)$$

Word lengths. Let $\alpha \in \mathcal{A}_n$, $n = 4n_{\mathrm{d}}(r+1-i)+n_{\mathrm{d}}$. For $m=1$: $|\alpha|+n_{\mathrm{d}} \le N_{\mathrm{top}} - 4n_{\mathrm{d}}i + 2n_{\mathrm{d}} \le N_{\mathrm{top}}$, and $\mathcal{A}_{|\alpha|+n_{\mathrm{d}}} \subseteq \mathcal{A}_{4n_{\mathrm{d}}(r+2-i)}$, which also contains α. For $m \ge 2$: $|\alpha| + (m+1)n_{\mathrm{d}} \le N_{\mathrm{top}} - n_{\mathrm{d}}(4i-m-2) \le N_{\mathrm{top}}$ as $m \le i$ and $i \ge 1$, and $\mathcal{A}_{|\alpha|+(m+1)n_{\mathrm{d}}} \subseteq \mathcal{A}_{4n_{\mathrm{d}}(r+1-i+m)}$ as $m+2 \le 4m$. Hence, by the definition of σ, $\mathrm{En}_{\eta,i-1,\alpha'}(\mathcal{X}_{i-1},k_{i-1}) \le \sigma_{i-1}^2$ for $\alpha' \in \mathcal{A}_{|\alpha|+n_{\mathrm{d}}} \cup \{\alpha\}$ when $m=1$, and $\mathrm{En}_{\eta,i-m,\alpha'}(\mathcal{X}_{i-m},k_{i-m}) \le \sigma_{i-m}^2$ for $\alpha' \in \mathcal{A}_{|\alpha|+(m+1)n_{\mathrm{d}}}$ when $m \ge 2$.

The case $m = 1$: Poisson summation. Here $j = i$. Let θ be the weight of (Pt5) for (η,i,α); it does not depend on P or κ. Fix $t_0 \in R$ and put $\kappa = \Theta_{i,1,t_0}$, positive semidefinite on $\mathcal{X} = \mathcal{X}_{i-1}$, and $B = \beta_{\mathrm{L}}\sum_{\alpha'}\theta(\alpha')\mathrm{En}_{\eta,i-1,\alpha'}(\mathcal{X},\kappa)$. For finite $Z \subseteq L_i^\vee \setminus \{0\}$, (Pt5) and (Pt8) give, as $\sum_{\alpha'}\theta(\alpha')\mathrm{En}_{\eta,i-1,\alpha'}(\mathcal{X},\kappa) \ge 0$ by (F1),

$$\sum_{z\in Z} |\widehat{\mathrm{En}}_{\eta,i,\alpha}(\mathcal{X},\kappa;z)| \le \Big(\sum_{z\in Z}\mathrm{mj}_i^\vee(z)\Big)\sum_{\alpha'}\theta(\alpha')\mathrm{En}_{\eta,i-1,\alpha'}(\mathcal{X},\kappa) \le B.$$

By (Pt4) with $S = S(t_0)$, and since $\widehat{\mathrm{En}}_{\eta,i,\alpha}(\mathcal{X},\kappa;0) = \mathrm{En}_{\eta,i-1,\alpha}(\mathcal{X},\kappa) \geq 0$ by (Pt5) and (F1),

$$\sum_{t\in S(t_0)} \mathrm{Re}\,\mathrm{En}^{\uparrow}_{\eta,i,1,\alpha}(\mathcal{X},\Theta_{i,1,t_0};t,t) \leq \mathrm{En}_{\eta,i-1,\alpha}(\mathcal{X},\Theta_{i,1,t_0}) + \beta_{\mathrm{L}} \sum_{\alpha'} \theta(\alpha')\,\mathrm{En}_{\eta,i-1,\alpha'}(\mathcal{X},\Theta_{i,1,t_0}).$$

Summing over $t_0 \in R$ and using (15.5), (15.6) (with $\theta \geq 0$, $\beta_{\mathrm{L}} \geq 0$), then the word lengths, $\sum\theta \leq 2^f$ and $1+\mathrm{ex}_i \geq 0$:

$$D_{i,1}(\alpha) \leq (1+\mathrm{ex}_i)\Big(\mathrm{En}_{\eta,i-1,\alpha}(\mathcal{X},k_{i-1}) + \beta_{\mathrm{L}} \sum_{\alpha'} \theta(\alpha')\,\mathrm{En}_{\eta,i-1,\alpha'}(\mathcal{X},k_{i-1})\Big) \leq (1+\mathrm{ex}_i)(1+2^f\beta_{\mathrm{L}})\,\sigma_{i-1}^2.$$

The case $m \geq 2$: degenerate blocks. Let θ be the weight of (Pt7) for (η,i,m,α), independent of P,κ,t_0,S. For $t_0 \in R$, (Pt7) with $P = \mathcal{X}$, $\kappa = \Theta_{i,m,t_0}$ and $S = S(t_0)$ bounds the inner sum of (15.5) by $\beta_{\mathrm{B}}^{2(m-1)} \sum_{\alpha'} \theta(\alpha')\mathrm{En}_{\eta,i-m,\alpha'}(\mathcal{X},\Theta_{i,m,t_0})$. Summing over t_0 and using (15.6), the word lengths and $\sum\theta \leq 1$, $\theta \geq 0$:

$$D_{i,m}(\alpha) \leq (1+\mathrm{ex}_i)\,\beta_{\mathrm{B}}^{2(m-1)} \sum_{\alpha'} \theta(\alpha')\,\mathrm{En}_{\eta,i-m,\alpha'}(\mathcal{X}_{i-m},k_{i-m}) \leq (1+\mathrm{ex}_i)\,\beta_{\mathrm{B}}^{2(m-1)}\,\sigma_{i-m}^2.$$

□

Lemma 15.13. *Let $r \geq 0$ be an integer and $\delta_1,\beta \geq 0$ real numbers with $(r+1)(\delta_1+4\beta) \leq 1$. Let $\mathrm{ex}_1,\dots,\mathrm{ex}_{r+1} \geq 0$ and $\sigma_0,\dots,\sigma_{r+1} \geq 0$ be real numbers such that*

$$\sigma_i \leq (1+\delta_1)\sqrt{1+\mathrm{ex}_i}\;\sigma_{i-1} + 2\sum_{m=2}^{i} \beta^{m-1}\sigma_{i-m} \qquad (1 \leq i \leq r+1).$$

Then

$$\sigma_{r+1}^2 \leq \big(1+15(r+1)(\delta_1+4\beta)\big)\prod_{i=1}^{r+1}(1+\mathrm{ex}_i)\;\sigma_0^2.$$

Proof. **Normalisation.** Put $Q_0 = 1$, $Q_i = \prod_{a=1}^{i}\sqrt{1+\mathrm{ex}_a}$, so $1 \leq Q_{i-m} \leq Q_i$ and $Q_i = \sqrt{1+\mathrm{ex}_i}\,Q_{i-1}$, and $\tau_i = \sigma_i/Q_i \geq 0$. Dividing the hypothesis by Q_i and using $\sigma_{i-m}/Q_i \leq \sigma_{i-m}/Q_{i-m}$,

$$\tau_i \leq (1+\delta_1)\tau_{i-1} + 2\sum_{m=2}^{i}\beta^{m-1}\tau_{i-m} \qquad (1 \leq i \leq r+1).$$

Running maximum. Let $M_i = \max_{0\leq a\leq i}\tau_a$. Since $4\beta \leq 1$, $\beta \leq \frac14$ and $2\sum_{m=2}^{i}\beta^{m-1} \leq 2\beta/(1-\beta) \leq 4\beta$, so $\tau_i \leq (1+\delta_1+4\beta)M_{i-1}$; as also $M_{i-1} \leq (1+\delta_1+4\beta)M_{i-1}$, we get $M_i \leq (1+\delta_1+4\beta)M_{i-1}$ and by induction $\tau_{r+1} \leq M_{r+1} \leq (1+\delta_1+4\beta)^{r+1}\sigma_0$.

The constant. Hence $\sigma_{r+1}^2 = Q_{r+1}^2\tau_{r+1}^2 \leq \prod_i(1+\mathrm{ex}_i)\,(1+\delta_1+4\beta)^{2(r+1)}\sigma_0^2$. With $z = (r+1)(\delta_1+4\beta) \in [0,1]$, $1+w \leq e^w$ and $e^w \leq 1+we^w$ ($w \geq 0$, mean value theorem) give $(1+\delta_1+4\beta)^{2(r+1)} \leq e^{2z} \leq 1+2ze^{2z} \leq 1+2e^2z \leq 1+15z$. □

Theorem 15.14. *Let Poisson recursion data over a field K with parameters (r,f,n_{d}) carry an archimedean product structure. Then the data satisfy (Pt1), (Pt4), (Pt5), (Pt6), (Pt7) and (Pt9).*

Proof. **(Pt1).** The coefficient part is (Aw2). Let P be a finite set of points and fix η,i,α. For each v the kernel $(u,u') \mapsto g^v_{\eta,i,\alpha}(\sigma_v(u),\sigma_v(u'))$ is positive semidefinite on P: for $w \in \mathbb{C}^P$, $\sum_{u,u'}\overline{w_u}g^v(\sigma_v u,\sigma_v u')w_{u'} = \sum_{p,q\in\sigma_v(P)}\overline{W_p}\,g^v(p,q)\,W_q \geq 0$ with $W_p = \sum_{\sigma_v(u)=p} w_u$, by (Aw2) on

the finite set $\sigma_v(P)$ of local points. By (Aw1) and the Schur product theorem, $g_{\eta,i,\alpha}$ is positive semidefinite on P.

Consequences of (Aw3). Fix $\eta>0$, $1\le i\le r+1$, α, a finite set P of points of level $i-1$ and κ positive semidefinite on P; write $\varphi(c)=\varphi_{\eta,i,\alpha}(P,\kappa;c)$ and $G_v(h,h';c)=G^v_{\eta,i,\alpha}(\sigma_v(h),\sigma_v(h');c)$ for $h,h'\in P$ (the local points $\sigma_v(h)$ have level $i-1$), and let $C_v(h,h')$ be the constants of (Aw3) for them. (a) φ is continuous on $\mathbb{C}^f$, being a finite sum of products of continuous functions of single coordinates, and $|\varphi(c)|\le C\prod_v(1+|c_v|)^{-3}$ with $C=\mathrm{wt}(\alpha)^{-1}\sum_{h,h'}|\kappa(h,h')|\prod_v C_v(h,h')$. (b) For $t\in K^\times$ and a point h, $\sigma_v(t)\ne0$ and $\sigma_v(h\cdot(1,t))=\sigma_v(h)\cdot(1,\sigma_v(t))$; so by (Aw3) and (Aw1), $\varphi(\sigma(t))=\mathrm{wt}(\alpha)^{-1}\sum_{h,h'}\kappa(h,h')g_{\eta,i,\alpha}(h\cdot(1,t),h'\cdot(1,t))=\mathrm{En}^{\uparrow}_{\eta,i,1,\alpha}(P,\kappa;t,t)$. (c) Let $c\in(\mathbb{C}^\times)^f$. Then $G_v(h,h';c_v)=g^v_{\eta,i,\alpha}(\sigma_v(h)\cdot(1,c_v),\sigma_v(h')\cdot(1,c_v))$; the map $h\mapsto\sigma_v(h)\cdot(1,c_v)$ sends P to a finite set of local points, so $(h,h')\mapsto G_v(h,h';c_v)$ is positive semidefinite on P by (Aw2) and the pull-back argument of the proof of (Pt1). By the Schur product theorem $\kappa\prod_v G_v(\cdot,\cdot;c_v)$ is positive semidefinite on P, so the sum of its entries (its quadratic form at the vector with all entries 1) is real and ≥0; as $\mathrm{wt}(\alpha)>0$, $\varphi(c)\ge0$. As $(\mathbb{C}^\times)^f$ is dense in $\mathbb{C}^f$ and $\mathrm{Re}\,\varphi$ is continuous, $\mathrm{Re}\,\varphi\ge0$ on $\mathbb{C}^f$. (d) Zero frequency. Since $\int_{\mathbb{C}}(1+|c|)^{-3}dc=2\pi\int_0^\infty\rho(1+\rho)^{-3}d\rho=\pi$, each function $c\mapsto\prod_v G_v(h,h';c_v)$ is integrable on $\mathbb{C}^f$, and by Fubini's theorem and (Aw3) its integral is $\prod_v\int_{\mathbb{C}}G_v(h,h';c_v)\,dc_v=\prod_v d_v(i)\,g^v_{\eta,i-1,\alpha}(\sigma_v(h),\sigma_v(h'))=\mathrm{covol}(\Lambda_i)\,g_{\eta,i-1,\alpha}(h,h')$, the last step by (Aw1). Hence, by the Fourier formula of (Aw3) at $z=0$, $\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;0)=\mathrm{covol}(\Lambda_i)^{-1}\int\varphi=\mathrm{wt}(\alpha)^{-1}\sum_{h,h'}\kappa(h,h')g_{\eta,i-1,\alpha}(h,h')$. By (Pt1), proved above, and the Schur product theorem this sum of entries of a positive semidefinite kernel is real, so $\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;0)=\mathrm{En}_{\eta,i-1,\alpha}(P,\kappa)$.

(Pt4). By (a)–(c), the Fourier formula and the lattice conditions of (Aw3), the hypotheses of Lemma 15.15 hold for σ and g with the functions $\varphi_{\eta,i,\alpha}(P,\kappa;\cdot)$ (σ is injective, being a field embedding in each coordinate); it gives (Pt4).

(Pt5). Put $|z|_v=|\sigma_v(z)|$; it is positive for $z\ne0$. By (Pt1), the zero-frequency identity (d) and (Aw4), the hypotheses of Lemma 15.16 hold with these $|\cdot|_v$, $\varrho_{i,v}$, $\varrho'_{i,v}$ and θ_A; it gives (Pt5).

(Pt6). $z\mapsto|\sigma_v(z)|$ is an absolute value on K. By (Pt1) and (Aw5), the hypotheses of Lemma 15.17 hold with these absolute values, $\varrho_{j,m,v}$, $\varrho'_{j,m,v}$, $\lambda_{j,m,v}$ and the weights of (Aw5); it gives (Pt6).

(Pt7). Let $\eta>0$, $2\le m\le i\le r+1$, α with $|\alpha|+(m+1)n_{\mathrm{d}}\le N_{\mathrm{top}}$, and M, θ as in (Aw6); extend θ by 0 to $\mathcal{A}_{|\alpha|+(m+1)n_{\mathrm{d}}}\supseteq\mathcal{A}_{|\alpha|+mn_{\mathrm{d}}}$. As $\mathrm{vol}(j,m')\le T^{-2f}$, $M\le C_4^{fm}T^{-2f(m-1)}$, and as $C_4\ge1$ and $fm\le2f(m-1)$ for $m\ge2$, $C_4^{fm}\le C_4^{2f(m-1)}$; hence $M\le(C_4/T)^{2f(m-1)}=\beta_{\mathrm{B}}^{2(m-1)}$. The energies $\mathrm{En}_{\eta,i-m,\alpha'}(P,\kappa)$ are ≥0 (the kernel $\kappa\,g_{\eta,i-m,\alpha'}$ is positive semidefinite on P by (Pt1) and the Schur product theorem, and $\mathrm{wt}(\alpha')>0$), so the bound of (Aw6) gives that of (Pt7).

(Pt9). Since $\sigma_v(\varnothing)=\varnothing$, (Aw1) gives $g_{\eta,0,\alpha}(\varnothing,\varnothing)/\mathrm{wt}(\alpha)=\prod_v g^v_{\eta,0,\alpha}(\varnothing,\varnothing)/\mathrm{wt}_v(\alpha)$, which tends to $\prod_v J_v=J$ by (Aw7); as J is real, the real parts tend to J. Let P be a finite set of points of level $r+1$; $\sigma_v(P)$ is a finite set of local points of level $r+1$, and let η_0 be the least of the η_0 of (Aw7) for $Q=\sigma_v(P)$, $v=1,\dots,f$. For $0<\eta<\eta_0$ and $u,u'\in P$, by (Aw1) and (Aw7), $g_{\eta,r+1,\alpha\varnothing}(u,u')=\prod_v\langle W_v(\sigma_v u'),W_v(\sigma_v u)\rangle=\langle W(u'),W(u)\rangle$, for all pairs and in particular for those with $k_{r+1}(u,u')\ne0$. □

Lemma 15.15. *Fix Poisson recursion data over a field K with parameters (r,f,n_{d}), an injective additive map $\sigma=(\sigma_1,\dots,\sigma_f)\colon K\to\mathbb{C}^f$ and a real number $g>0$. Put $\langle\xi,c\rangle=\frac{2}{g}\sum_{v=1}^f\mathrm{Re}(\xi_v c_v)$ for $\xi,c\in\mathbb{C}^f$, and identify $\mathbb{C}^f$ with $\mathbb{R}^{2f}$ (Lebesgue measure dc). Suppose that for every $1\le i\le r+1$:*

1. *$\Lambda_i=\sigma(L_{i,1})$ is a lattice in $\mathbb{C}^f$, $\sigma(L_i^\vee)=\{\xi:\langle\xi,\lambda\rangle\in\mathbb{Z}$ for all $\lambda\in\Lambda_i\}$, and there is C_i*

with $\#\{\lambda\in\Lambda_i : |\lambda_v|\le R_v \text{ for all } v\}\le C_i\prod_v(1+R_v)^2$ *for all* $R_1,\dots,R_f\ge 0$*;*

2. *for all* $\eta>0$*,* $\alpha\in\mathcal{A}$*, finite sets* P *of points of level* $i-1$ *and* κ *positive semidefinite on* P *there are a continuous* $\varphi\colon \mathbb{C}^f\to\mathbb{C}$ *and* $C\ge 0$ *with* $\operatorname{Re}\varphi\ge 0$*,* $|\varphi(c)|\le C\prod_v(1+|c_v|)^{-3}$*,* $\varphi(\sigma(t))=\mathrm{En}^{\uparrow}_{\eta,i,1,\alpha}(P,\kappa;t,t)$ *for* $t\in K^\times$*, and*

$$\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;z)=\operatorname{covol}(\Lambda_i)^{-1}\int_{\mathbb{C}^f}\varphi(c)\,e^{2\pi i\langle\sigma(z),c\rangle}\,dc\qquad(z\in L_i^\vee).$$

Then the data satisfy (Pt4).

Proof. Fix η,i,α,P,κ as in (Pt4), $t_0\in K$, B with $\sum_{z\in Z}|\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;z)|\le B$ for every finite $Z\subseteq L_i^\vee\setminus\{0\}$, and a finite $S\subseteq(t_0+L_{i,1})\setminus\{0\}$. Let φ, C be as in item 2, $\Lambda=\Lambda_i$, $\Lambda'=\sigma(L_i^\vee)$, $x=\sigma(t_0)$.

The periodisation converges. For $\lambda\in\Lambda$ let $k_v(\lambda)=\lfloor\log_2(1+|\lambda_v|)\rfloor\ge 0$. For $k\in\mathbb{Z}^f_{\ge 0}$ the λ with $k(\lambda)=k$ satisfy $|\lambda_v|<2^{k_v+1}$, so there are at most $C_i\prod_v(1+2^{k_v+1})^2\le C_i 9^f\prod_v 2^{2k_v}$ of them by item 1, and each has $\prod_v(1+|\lambda_v|)^{-3}\le\prod_v 2^{-3k_v}$. Hence $\sum_{\lambda\in\Lambda}\prod_v(1+|\lambda_v|)^{-3}\le C_i9^f\sum_k\prod_v 2^{-k_v}=C_i18^f$. If $|y_v|\le R$ for all v, then $1+|y_v+\lambda_v|\ge(1+|\lambda_v|)/(1+R)$, so $|\varphi(y+\lambda)|\le C(1+R)^{3f}\prod_v(1+|\lambda_v|)^{-3}$. Therefore $F(y)=\sum_{\lambda\in\Lambda}\varphi(y+\lambda)$ converges absolutely and uniformly on compact sets; F is continuous and Λ-periodic.

Its Fourier series. The pairing $\langle\cdot,\cdot\rangle$ is a nondegenerate real bilinear form on $\mathbb{R}^{2f}$, so the characters of $\mathbb{R}^{2f}/\Lambda$ are $y\mapsto e^{2\pi i\langle\xi,y\rangle}$, $\xi\in\Lambda'$ (item 1). Let D be a fundamental domain of Λ. By absolute convergence and Fubini, and as $e^{-2\pi i\langle\xi,\lambda\rangle}=1$,

$$\begin{aligned}c_\xi:=\frac{1}{\operatorname{covol}(\Lambda)}\int_D F(y)e^{-2\pi i\langle\xi,y\rangle}dy&=\frac{1}{\operatorname{covol}(\Lambda)}\int_{\mathbb{C}^f}\varphi(y)e^{-2\pi i\langle\xi,y\rangle}dy\\&=\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;-z)\qquad(\xi=\sigma(z)).\end{aligned}$$

As $L_i^\vee=-L_i^\vee$ and finite sums of $|\widehat{\mathrm{En}}(z)|$ over $z\ne 0$ are at most B, $\sum_{\xi\in\Lambda'}|c_\xi|\le|\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;0)|+B<\infty$. The series $\sum_\xi c_\xi e^{2\pi i\langle\xi,y\rangle}$ then converges uniformly to a continuous periodic function with the same Fourier coefficients as F, so it equals F (a continuous function on a torus all of whose Fourier coefficients vanish is zero, by the density of trigonometric polynomials). In particular $|F(x)|\le|\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;0)|+B$.

Comparison with the sum over S**.** For $t\in S$, $t\ne 0$ and $\sigma(t)=x+\sigma(t-t_0)$ with $t-t_0\in L_{i,1}$; distinct t give distinct $\lambda=\sigma(t-t_0)\in\Lambda$ (σ is injective). By item 2 and $\operatorname{Re}\varphi\ge 0$,

$$\sum_{t\in S}\operatorname{Re}\mathrm{En}^{\uparrow}_{\eta,i,1,\alpha}(P,\kappa;t,t)=\sum_{t\in S}\operatorname{Re}\varphi(\sigma(t))\le\sum_{\lambda\in\Lambda}\operatorname{Re}\varphi(x+\lambda)=\operatorname{Re}F(x)\le|\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;0)|+B,$$

which is (Pt4). □

Lemma 15.16. *Let Poisson recursion data over a field* K *with parameters* (r,f,n_{d}) *satisfy (Pt1). Let* $z\mapsto|z|_v$ *(*$1\le v\le f$*) be functions* $K\to[0,\infty)$ *with* $|z|_v>0$ *for* $z\ne 0$*, and let* $0\le\varrho'_{i,v}\le\varrho_{i,v}$ *(*$1\le i\le r+1$*,* $1\le v\le f$*) be real numbers such that*

$$\mathrm{mj}_i^\vee(z)=\prod_{v=1}^f\min\Bigl(1,\Bigl(\frac{\varrho_{i,v}}{|z|_v}\Bigr)^{n_{\mathrm{d}}}\Bigr)\qquad(1\le i\le r+1,\ z\in L_i^\vee\setminus\{0\}).$$

Suppose that for all $\eta>0$*,* $1\le i\le r+1$ *and* $\alpha\in\mathcal{A}$ *with* $|\alpha|+n_{\mathrm{d}}\le N_{\mathrm{top}}$*:*

1. $\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;0)=\mathrm{En}_{\eta,i-1,\alpha}(P,\kappa)$ *for every finite set* P *of points of level* $i-1$ *and every* κ *positive semidefinite on* P*;*

2. *for every subset* $A\subseteq\{1,\dots,f\}$ *there is* $\theta_A\colon \mathcal{A}_{|\alpha|+n_{\mathrm{d}}}\to[0,\infty)$ *with* $\sum_{\alpha'}\theta_A(\alpha')\le 1$ *such that for every finite set* P *of points of level* $i-1$*, every* κ *positive semidefinite on* P *and every* $z\in L_i^\vee\setminus\{0\}$

$$\Big(\prod_{v\in A}|z|_v^{n_{\mathrm{d}}}\Big)\big|\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;z)\big|\le\Big(\prod_{v\in A}\varrho_{i,v}'^{\,n_{\mathrm{d}}}\Big)\sum_{\alpha'\in\mathcal{A}_{|\alpha|+n_{\mathrm{d}}}}\theta_A(\alpha')\,\mathrm{En}_{\eta,i-1,\alpha'}(P,\kappa).$$

Then the data satisfy (Pt5), with the weight $\theta=\sum_{A\subseteq\{1,\dots,f\}}\theta_A$*.*

Proof. Fix $\eta>0$, $1\le i\le r+1$ and $\alpha\in\mathcal{A}$ with $|\alpha|+n_{\mathrm{d}}\le N_{\mathrm{top}}$, and let $(\theta_A)_A$ be as in item 2. Then $\theta=\sum_A\theta_A$ maps $\mathcal{A}_{|\alpha|+n_{\mathrm{d}}}$ to $[0,\infty)$, does not depend on P, κ or z, satisfies $\theta_A\le\theta$ pointwise for every A, and has $\sum_{\alpha'}\theta(\alpha')\le 2^f$, the number of subsets of $\{1,\dots,f\}$. The zero-frequency identity of (Pt5) is item 1.

Let P be a finite set of points of level $i-1$, κ positive semidefinite on P and $z\in L_i^\vee\setminus\{0\}$. Every energy $\mathrm{En}_{\eta,i-1,\alpha'}(P,\kappa)$ is ≥ 0: the kernel $\kappa\,g_{\eta,i-1,\alpha'}$ is positive semidefinite on P by (Pt1) and the Schur product theorem, so the sum of its entries is real and ≥ 0, and $\mathrm{wt}(\alpha')>0$. Let $A=\{v:\varrho_{i,v}<|z|_v\}$. For $v\in A$, $0\le\varrho_{i,v}/|z|_v<1$, so the v-th factor of $\mathrm{mj}_i^\vee(z)$ is $(\varrho_{i,v}/|z|_v)^{n_{\mathrm{d}}}$; for $v\notin A$, $\varrho_{i,v}/|z|_v\ge 1$ and the factor is 1. Hence

$$\mathrm{mj}_i^\vee(z)=\prod_{v\in A}\Big(\frac{\varrho_{i,v}}{|z|_v}\Big)^{n_{\mathrm{d}}}.$$

Dividing item 2 for this A by $\prod_{v\in A}|z|_v^{n_{\mathrm{d}}}>0$, and using $0\le\varrho_{i,v}'\le\varrho_{i,v}$, the nonnegativity of the energies and $0\le\theta_A\le\theta$,

$$\big|\widehat{\mathrm{En}}_{\eta,i,\alpha}(P,\kappa;z)\big|\le\prod_{v\in A}\Big(\frac{\varrho_{i,v}'}{|z|_v}\Big)^{n_{\mathrm{d}}}\sum_{\alpha'}\theta_A(\alpha')\,\mathrm{En}_{\eta,i-1,\alpha'}(P,\kappa)\le\mathrm{mj}_i^\vee(z)\sum_{\alpha'}\theta(\alpha')\,\mathrm{En}_{\eta,i-1,\alpha'}(P,\kappa),$$

which is the decay bound of (Pt5). □

Lemma 15.17. *Let Poisson recursion data over a field* K *with parameters* (r,f,n_{d}) *satisfy (Pt1). Let* $|\cdot|_1,\dots,|\cdot|_f$ *be absolute values on* K*, and for* $1\le m\le i\le r+1$ *with* $(i,m)\ne(r+1,1)$ *and* $1\le v\le f$ *let* $\varrho_{j,m,v}\ge 0$ *be real numbers with*

$$\mathrm{mj}_{j,m}'(z)=\prod_{v=1}^{f}\min\Big(1,\Big(\frac{\varrho_{j,m,v}}{|z|_v}\Big)^{n_{\mathrm{d}}}\Big)\qquad(z\in K^\times).$$

For a finite set P *of points of level* $i-m$*, a kernel* κ *positive semidefinite on* P*,* $t\in K^\times$ *and* $\alpha\in\mathcal{A}$ *write* $d_{\eta,i,m,\alpha}(P,\kappa;t)=\mathrm{Re}\,\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(P,\kappa;t,t)$*. Suppose that for all* $\eta>0$ *and all* $\alpha\in\mathcal{A}$ *with* $|\alpha|+n_{\mathrm{d}}\le N_{\mathrm{top}}$*:*

1. *(below the top) there are* $0\le\varrho_{j,m,v}'\le\varrho_{j,m,v}$*, not depending on* η *or* α*, such that for* $1\le m\le i\le r$ *and every* $A\subseteq\{1,\dots,f\}$ *there is* $\theta_A\colon\mathcal{A}_{|\alpha|+n_{\mathrm{d}}}\to[0,\infty)$ *with* $\sum_{\alpha'}\theta_A(\alpha')\le 2$ *such that for every finite set* P *of points of level* $i-m$*, finite* $S\subseteq K^\times$*,* κ *positive semidefinite on* $\{h\cdot(m,t):h\in P,\ t\in S\}$ *and* $t\ne t'$ *in* S

$$\begin{aligned}&\Big(\prod_{v\in A}|t^{-1}-t'^{-1}|_v^{n_{\mathrm{d}}}\Big)\big|\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(P,\kappa_m^{t,t'};t,t')\big|\\&\qquad\le\Big(\prod_{v\in A}\varrho_{j,m,v}'^{\,n_{\mathrm{d}}}\Big)\frac{1}{2}\sum_{\alpha'}\theta_A(\alpha')\big(d_{\eta,i,m,\alpha'}(P,\kappa_m^{t,t};t)+d_{\eta,i,m,\alpha'}(P,\kappa_m^{t',t'};t')\big);\end{aligned}$$

2. *(the top diagonal) there are $\lambda_{j,m,v} \ge 0$ with $2\lambda_{j,m,v} \le \varrho_{j,m,v}$, not depending on η or α, such that for $i = r+1$, $2 \le m \le r+1$ and every $B \subseteq \{1,\dots,f\}$ there is $\theta_B \colon \mathcal{A}_{|\alpha|+n_{\mathrm{d}}} \to [0,\infty)$ with $\sum_{\alpha'} \theta_B(\alpha') \le 1$ such that for every finite set P of points of level $r+1-m$, κ positive semidefinite on P and $t \in K^\times$*

$$d_{\eta,r+1,m,\alpha}(P,\kappa;t) \le \Big(\prod_{v\in B} \big(|t|_v \lambda_{j,m,v}\big)^{2n_{\mathrm{d}}}\Big) \sum_{\alpha'} \theta_B(\alpha')\, d_{\eta,r+1,m,\alpha'}(P,\kappa;t).$$

Then the data satisfy (Pt6).

Proof. Fix $\eta > 0$, $1 \le m \le i \le r+1$ with $(i,m) \ne (r+1,1)$ and α with $|\alpha| + n_{\mathrm{d}} \le N_{\mathrm{top}}$; then either $i \le r$ (item 1 applies) or $i = r+1$ and $m \ge 2$ (item 2 applies). Let P, S, κ and $t \ne t'$ in S be as in (Pt6). As $t \ne t'$, $t^{-1} - t'^{-1} \ne 0$, so $o_v > 0$ for every v.

Positivity and the Cauchy–Schwarz inequality. Put $Q = \{h \cdot (m,s) : h \in P,\ s \in \{t,t'\}\}$. The maps $h \mapsto h\cdot(m,t)$ and $h \mapsto h\cdot(m,t')$ are injective with disjoint images. The kernel $M = \kappa\, g_{\eta,i,\alpha}$ is positive semidefinite on Q by (Pt1) and the Schur product theorem, hence hermitian on Q. For $a,b \in \mathbb{C}$ take the vector equal to a on $P\cdot(m,t)$ and to b on $P\cdot(m,t')$: with $X = \sum_{h,h'\in P} M(h\cdot(m,t), h'\cdot(m,t))$, $Y = \sum_{h,h'} M(h\cdot(m,t), h'\cdot(m,t'))$ and $Z = \sum_{h,h'} M(h\cdot(m,t'), h'\cdot(m,t'))$ the quadratic form is $|a|^2X + \bar a b Y + a\bar b \bar Y + |b|^2 Z \ge 0$, so $X, Z \ge 0$ and $|Y|^2 \le XZ$. Dividing by $\mathrm{wt}(\alpha)^2$, with $X/\mathrm{wt}(\alpha) = d_{\eta,i,m,\alpha}(P,\kappa_m^{t,t};t)$ etc.,

$$\big|\mathrm{En}^{\uparrow}_{\eta,i,m,\alpha}(P,\kappa_m^{t,t'};t,t')\big| \le \big(d_{\eta,i,m,\alpha}(P,\kappa_m^{t,t};t)\, d_{\eta,i,m,\alpha}(P,\kappa_m^{t',t'};t')\big)^{1/2}. \tag{15.7}$$

In the same way every $d_{\eta,i,m,\alpha'}(P,\kappa_m^{s,s};s)$, $s \in S$, is ≥ 0; write $d_s(\alpha')$ for it.

Below the top, $i \le r$. Let $\theta = \sum_A \theta_A$ over all $A \subseteq \{1,\dots,f\}$; it depends only on (η,i,m,α), and $\sum_{\alpha'} \theta(\alpha') \le 2\cdot 2^f \le 2\cdot 3^f$. Let $A = \{v : \varrho_{j,m,v} < o_v\}$. As in the factorisation of the majorant, $\mathrm{mj}'_{j,m}(t^{-1}-t'^{-1}) = \prod_{v\in A} (\varrho_{j,m,v}/o_v)^{n_{\mathrm{d}}}$ (the factors with $v \notin A$ equal 1). Item 1 for this A, divided by $\prod_{v\in A} o_v^{n_{\mathrm{d}}} > 0$, with $\varrho' \le \varrho$, $d \ge 0$ and $\theta_A \le \theta$, gives the bound of (Pt6).

The top, $i = r+1$, $m \ge 2$. Let $\theta = \sum_B \theta_B$ over all B; $\sum_{\alpha'} \theta(\alpha') \le 2^f \le 2\cdot 3^f$. Write $\lambda_v = \lambda_{j,m,v}$, $\varrho_v = \varrho_{j,m,v}$, and let $A = \{v : 2\lambda_v < o_v\}$, $A_1 = \{v \in A : |t|_v \le |t'|_v\}$, $A_2 = A \setminus A_1$. Item 2 with $B = A_1$ for the kernel $\kappa_m^{t,t}$ and the point t, and with $B = A_2$ for $\kappa_m^{t',t'}$ and t' (both positive semidefinite on P, as $h \mapsto h\cdot(m,s)$ is injective), gives

$$d_t(\alpha) \le \prod_{v\in A_1} (|t|_v\lambda_v)^{2n_{\mathrm{d}}}\, S_1, \qquad d_{t'}(\alpha) \le \prod_{v\in A_2} (|t'|_v\lambda_v)^{2n_{\mathrm{d}}}\, S_2,$$

where $S_1 = \sum_{\alpha'} \theta_{A_1}(\alpha') d_t(\alpha')$ and $S_2 = \sum_{\alpha'} \theta_{A_2}(\alpha') d_{t'}(\alpha')$. For $v \in A_1$, $o_v \le |t|_v^{-1} + |t'|_v^{-1} \le 2|t|_v^{-1}$, so $|t|_v\lambda_v \le 2\lambda_v/o_v < 1$; likewise $|t'|_v\lambda_v \le 2\lambda_v/o_v < 1$ for $v \in A_2$. For $v \in A$, $2\lambda_v/o_v \le \min(1, \varrho_v/o_v)$ as $2\lambda_v \le \varrho_v$; for $v \notin A$, $\varrho_v \ge 2\lambda_v \ge o_v$ and the factor of $\mathrm{mj}'_{j,m}$ at v is 1. Hence, by (15.7) and $\sqrt{S_1S_2} \le (S_1+S_2)/2$,

$$\begin{aligned}\big|\mathrm{En}^{\uparrow}_{\eta,r+1,m,\alpha}(P,\kappa_m^{t,t'};t,t')\big| &\le \prod_{v\in A} \Big(\frac{2\lambda_v}{o_v}\Big)^{n_{\mathrm{d}}} \sqrt{S_1S_2} \le \mathrm{mj}'_{j,m}(t^{-1}-t'^{-1})\,\frac{S_1+S_2}{2}\\ &\le \mathrm{mj}'_{j,m}(t^{-1}-t'^{-1})\,\frac{1}{2}\sum_{\alpha'} \theta(\alpha')\big(d_t(\alpha')+d_{t'}(\alpha')\big),\end{aligned}$$

using $\theta_{A_1}, \theta_{A_2} \le \theta$ and $d \ge 0$. This is the bound of (Pt6). □

Proposition 15.18. *Let G be a second countable, locally compact, unimodular group with Haar measure μ. Let $(\sigma,\mathcal{H})$ be a strongly continuous unitary representation of G and $d_\sigma > 0$ such*

that for all $a, b, a', b' \in \mathcal{H}$ the function $g \mapsto \langle b, \sigma(g)a\rangle\overline{\langle b', \sigma(g)a'\rangle}$ is μ-integrable with

$$\int_G \langle b, \sigma(g)a\rangle\overline{\langle b', \sigma(g)a'\rangle}\, d\mu(g) = \frac{\langle a', a\rangle\langle b, b'\rangle}{d_\sigma},$$

and let $w \in \mathcal{H}$ with $\|w\| = 1$. Let $S \subseteq G$ be a closed unimodular subgroup with Haar measure μ_S that is amenable in the following sense: for every compact $K \subseteq S$ and every $\epsilon > 0$ there is a measurable $\Omega \subseteq S$ with $0 < \mu_S(\Omega) < \infty$ and $\mu_S(\Omega) \le (1+\epsilon)\,\mu_S^*(\{t \in \Omega : Kt \subseteq \Omega\})$, where μ_S^* is the outer measure of μ_S. Let ν be a σ-finite measure on G with $\int_G f\, d\mu = \int_G \int_S f(sx)\, d\mu_S(s)\, d\nu(x)$ for all measurable $f \ge 0$. Let (τ, V) be a unitary representation of S on a finite-dimensional Hilbert space and $W\colon G \to V$ a continuous map such that

1. *$W(sx) = \tau(s)W(x)$ for $s \in S$, $x \in G$;*
2. *for every $x \in G$ the function $y \mapsto d_\sigma\, \overline{\langle w, \sigma(y)w\rangle}\, W(xy)$ is μ-integrable with integral $W(x)$;*
3. *$\int_G \|W\|\, d\nu < \infty$ and $\int_G \|W\|^2\, d\nu = 1$.*

Let $(S', \mu_{S'}, \nu', \tau', V', W')$ be data with the same properties. Let $(\mathcal{C}, m_{\mathcal{C}})$ be a measure space and $\sec\colon \mathcal{C} \to G$ a measurable map with

$$\int_G \varphi\, d\mu = \int_{\mathcal{C}} \int_S \int_{S'} \varphi\big(s \sec(c)\, s'^{-1}\big)\, d\mu_{S'}(s')\, d\mu_S(s)\, dm_{\mathcal{C}}(c)$$

for all measurable $\varphi \ge 0$ on G. Assume that for every $c \in \mathcal{C}$ the map $x \mapsto W(x) \otimes W'(\sec(c)^{-1}x)^*$ is μ-integrable, and that the integrals

$$\iiiint |\langle \sigma(z)w, \sigma(s\sec(c)\, s'^{-1})\sigma(z')w\rangle|\, \|W(z)\|\, \|W'(z')\|\, d\mu_{S'}(s')\, d\mu_S(s)\, d\nu'(z')\, d\nu(z) \quad (c \in \mathcal{C}),$$

$$\iiint |\langle \sigma(z)w, \sigma(s)\sigma(z')w\rangle|\, \|W(z)\|\, \|W(z')\|\, d\mu_S(s)\, d\nu(z')\, d\nu(z)$$

and the analogue of the last one for (S', ν', W') are finite. For $\gamma \in G$ put $\mathrm{Or}(\gamma) = \int_G W(x) \otimes W'(\gamma^{-1}x)^*\, d\mu(x) \in \mathrm{Hom}(V', V)$. Then

$$\int_{\mathcal{C}} \big\|\mathrm{Or}(\sec(c))\big\|_{\mathrm{HS}}^2\, dm_{\mathcal{C}}(c) \le \frac{1}{d_\sigma},$$

the integral being the lower Lebesgue integral (the supremum of the integrals of the measurable simple functions below the integrand) if the integrand is not known to be measurable.

Proof. **Standing facts.** (a) G, S, S' are second countable and locally compact, hence σ-compact; $\mu, \mu_S, \mu_{S'}$ are σ-finite, Borel sets of products are products of Borel sets, continuous functions on products are measurable, and Tonelli's and Fubini's theorems apply to $\mu, \mu_S, \mu_{S'}, \nu, \nu'$. (b) By unimodularity μ and μ_S are invariant under left and right translations and under inversion; these maps preserve measurable sets, hence also outer measures. (c) The quotient identity for ν extends to every μ-integrable complex function f (apply it to the positive and negative parts of the real and imaginary parts): $\int_G f\, d\mu = \int_G \int_S f(sx)\, d\mu_S(s)\, d\nu(x) = \int_G \int_S f(s^{-1}x)\, d\mu_S(s)\, d\nu(x)$, the inner integrals converging absolutely for ν-almost every x; likewise for ν'. (d) $\|W(sx)\| = \|W(x)\|$ for $s \in S$, by item 1 and the unitarity of τ. (e) Pairing item 2 with $e \in V$ (the inner product is conjugate-linear in the first argument) gives, for $x \in G$,

$$\langle W(x), e\rangle = d \int_G \langle w, \sigma(y)w\rangle\, \langle W(xy), e\rangle\, d\mu(y), \tag{15.8}$$

and the same for W'.

Truncated Bessel vectors. For a measurable $\Omega \subseteq S$ with $\mu_S(\Omega) < \infty$ and $e \in V$ put

$$y_\Omega(e) = \int_G \int_\Omega \langle W(s^{-1}z), e\rangle\, \sigma(s^{-1}z) w \, d\mu_S(s)\, d\nu(z) \in \mathcal{H}.$$

The integrand is continuous on $S \times G$ (strong continuity of σ, continuity of W), its norm is at most $\|W(z)\|\,\|e\|$ by (d), and $\int_G \int_\Omega \|W(z)\|\, d\mu_S\, d\nu = \mu_S(\Omega) \int_G \|W\|\, d\nu < \infty$ by item 3; so the Bochner integral exists, and it is linear in e. Put $E(\Omega) = \sum_i \|y_\Omega(e_i)\|^2$. Define $y'_{\Omega'}(e')$ and $E'(\Omega')$ in the same way for measurable $\Omega' \subseteq S'$ of finite measure, and let $X_{\Omega,\Omega'}(g) \in \mathrm{Hom}(V', V)$ be the operator with

$$\langle e, X_{\Omega,\Omega'}(g) e' \rangle = d\, \langle y_\Omega(e), \sigma(g)\, y'_{\Omega'}(e') \rangle \qquad (e \in V,\ e' \in V',\ g \in G);$$

it exists since the right side is conjugate-linear in e and linear in e', and $g \mapsto X_{\Omega,\Omega'}(g)$ is continuous.

Schur orthogonality for the truncations. By the orthogonality hypothesis with $a = a' = y'_{\Omega'}(e'_j)$ and $b = b' = y_\Omega(e_i)$,

$$\int_G \|X_{\Omega,\Omega'}(g)\|_{\mathrm{HS}}^2\, d\mu(g) = d^2 \sum_{i,j} \int_G \big|\langle y_\Omega(e_i), \sigma(g) y'_{\Omega'}(e'_j)\rangle\big|^2 d\mu(g) = d\, E(\Omega)\, E'(\Omega'). \tag{15.9}$$

Translation rule. For $s \in S$, $s' \in S'$ and $\gamma \in G$,

$$X_{\Omega,\Omega'}(s\gamma s'^{-1}) = \tau(s)\, X_{\Omega s, \Omega' s'}(\gamma)\, \tau'(s')^{-1}, \qquad \text{so} \qquad \|X_{\Omega,\Omega'}(s\gamma s'^{-1})\|_{\mathrm{HS}} = \|X_{\Omega s,\Omega' s'}(\gamma)\|_{\mathrm{HS}}. \tag{15.10}$$

Indeed, substituting $t = t_1 s^{-1}$ (right invariance of μ_S; $t \in \Omega$ if and only if $t_1 \in \Omega s$) and using $W(t^{-1}z) = W(st_1^{-1}z) = \tau(s) W(t_1^{-1}z)$ and $\langle \tau(s)v, e\rangle = \langle v, \tau(s)^{-1} e\rangle$, one gets $\sigma(s)^{-1} y_\Omega(e) = y_{\Omega s}(\tau(s)^{-1} e)$, and likewise for S'; hence

$$\langle e, X_{\Omega,\Omega'}(s\gamma s'^{-1}) e' \rangle = d \langle \sigma(s)^{-1} y_\Omega(e), \sigma(\gamma)\sigma(s')^{-1} y'_{\Omega'}(e') \rangle = \langle \tau(s)^{-1} e, X_{\Omega s, \Omega' s'}(\gamma) \tau'(s')^{-1} e' \rangle.$$

The Hilbert–Schmidt norm is invariant under multiplication by unitaries.

Følner sequences. By (a) there are compact sets $K_1 \subseteq K_2 \subseteq \cdots$ in S with $1 \in K_1$ and $\bigcup_N K_N = S$. By the amenability hypothesis there are measurable $\Omega_n \subseteq S$ with $0 < \mu_S(\Omega_n) < \infty$ and $\mu_S(\Omega_n) \le (1 + \frac{1}{n}) \mu_S^*(A_n)$, where $A_n = \{t \in \Omega_n : K_n t \subseteq \Omega_n\}$. Choose K'_N, Ω'_n, A'_n in S' in the same way. If $u \in K_N$ and $n \ge N$, then $A_n \subseteq \{t \in \Omega_n : ut \in \Omega_n\} = \Omega_n \cap u^{-1}\Omega_n$.

The energy of the truncations is $\mu_S(\Omega_n)/d$ asymptotically:

$$\lim_{n\to\infty} \frac{E(\Omega_n)}{\mu_S(\Omega_n)} = \frac{1}{d}, \qquad \lim_{n\to\infty} \frac{E'(\Omega'_n)}{\mu_{S'}(\Omega'_n)} = \frac{1}{d}. \tag{15.11}$$

We prove the first. Write $\|y_\Omega(e_i)\|^2 = \langle y_\Omega(e_i), y_\Omega(e_i)\rangle$ as an integral over $(z, s, z', t) \in (G \times \Omega)^2$; the integrand is bounded by $\|W(z)\|\|W(z')\|$, which is integrable, so we may sum over i inside. By Parseval, item 1 and unitarity, $\sum_i \overline{\langle W(s^{-1}z), e_i\rangle} \langle W(t^{-1}z'), e_i\rangle = \langle W(t^{-1}z'), W(s^{-1}z)\rangle = \langle W(st^{-1}z'), W(z)\rangle$, and $\langle \sigma(s^{-1}z)w, \sigma(t^{-1}z')w\rangle = \langle \sigma(z)w, \sigma(st^{-1})\sigma(z')w\rangle$. So, with

$$k(z, z', u) = \langle W(uz'), W(z)\rangle\, \langle \sigma(z)w, \sigma(u)\sigma(z')w\rangle,$$

$E(\Omega) = \int\int_\Omega\int_\Omega k(z, z', st^{-1})\, d\mu_S(t)\, d\mu_S(s)\, d\nu(z')\, d\nu(z)$. For fixed z, z', s the map $t \mapsto u = st^{-1}$ preserves μ_S (unimodularity), and $t = u^{-1}s \in \Omega$ if and only if $s \in u\Omega$; by Fubini on $\Omega \times S$ (k is bounded for fixed z, z') the inner double integral is $\int_S k(z, z', u)\, \mu_S(\Omega \cap u\Omega)\, d\mu_S(u)$. Hence

$$\frac{E(\Omega_n)}{\mu_S(\Omega_n)} = \int_G \int_G \int_S k(z, z', u)\, a_n(u)\, d\mu_S(u)\, d\nu(z')\, d\nu(z), \qquad a_n(u) = \frac{\mu_S(\Omega_n \cap u\Omega_n)}{\mu_S(\Omega_n)} \in [0, 1],$$

the interchange being justified by $|k(z,z',u)| \le \|W(z)\|\|W(z')\||\langle\sigma(z)w,\sigma(u)\sigma(z')w\rangle|$ (by (d)), whose integral is the second finite integral of the hypotheses. For fixed u, take N with $u^{-1}\in K_N$; for $n\ge N$, $A_n\subseteq\Omega_n\cap u\Omega_n$, so $a_n(u)\ge \mu_S^*(A_n)/\mu_S(\Omega_n)\ge n/(n+1)$; thus $a_n\to 1$ pointwise. By dominated convergence the limit in (15.11) exists and equals $\iint\int k$. For ν-almost every z the function $g_z(x)=\langle W(x),W(z)\rangle\langle\sigma(z)w,\sigma(x)w\rangle$ is μ-integrable with

$$\int_G g_z\,d\mu=\int_G\int_S k(z,z',u)\,d\mu_S(u)\,d\nu(z')$$

by (c), as $k(z,z',u)=g_z(uz')$. Since $\langle\sigma(z)w,\sigma(x)w\rangle=\langle w,\sigma(z^{-1}x)w\rangle$, the substitution $x=zy$ and item 2 give

$$\int_G g_z\,d\mu=\Big\langle\int_G\overline{\langle w,\sigma(y)w\rangle}\,W(zy)\,d\mu(y),\,W(z)\Big\rangle=\frac{1}{d}\,\|W(z)\|^2,$$

so the limit is $\frac{1}{d}\int_G\|W\|^2d\nu=\frac{1}{d}$ by item 3.

Each double coset is bounded below by the averaged truncations. Fix $c\in\mathcal{C}$, put $\gamma=\sec(c)$, and

$$F_n(c)=\frac{1}{\mu_S(\Omega_n)\mu_{S'}(\Omega_n')}\int_S\int_{S'}\big\|X_{\Omega_n,\Omega_n'}(s\gamma s'^{-1})\big\|_{\mathrm{HS}}^2\,d\mu_{S'}(s')\,d\mu_S(s).$$

We show $\|\mathrm{Or}(\gamma)\|_{\mathrm{HS}}^2\le\liminf_n F_n(c)$. Let $M(x,x')=|\langle\sigma(x)w,\sigma(\gamma)\sigma(x')w\rangle|\,\|W(x)\|\,\|W'(x')\|$. Since $\langle\sigma(t^{-1}z)w,\sigma(\gamma)\sigma(t'^{-1}z')w\rangle=\langle\sigma(z)w,\sigma(t\gamma t'^{-1})\sigma(z')w\rangle$ and by (d), the first finite integral of the hypotheses is

$$\Theta=\int_G\int_G\int_S\int_{S'}M(t^{-1}z,t'^{-1}z')\,d\mu_{S'}(t')\,d\mu_S(t)\,d\nu'(z')\,d\nu(z)<\infty,$$

and by (c) applied in each variable $\int_{G\times G}M\,d(\mu\otimes\mu)=\Theta$. Fix unit vectors $e\in V$, $e'\in V'$ and put $\kappa(x,x')=\overline{\langle W(x),e\rangle}\langle W'(x'),e'\rangle\langle\sigma(x)w,\sigma(\gamma)\sigma(x')w\rangle$, so $|\kappa|\le M$.

- By the integrability of $x\mapsto W(x)\otimes W'(\gamma^{-1}x)^*$,

$$\langle e,\mathrm{Or}(\gamma)e'\rangle=\int_G\overline{\langle W(x),e\rangle}\langle W'(\gamma^{-1}x),e'\rangle\,d\mu(x).$$

  By (15.8) for W' at $\gamma^{-1}x$ and the substitution $x'=\gamma^{-1}xy$ (left invariance),

$$\langle W'(\gamma^{-1}x),e'\rangle=d\int_G\langle w,\sigma(x^{-1}\gamma x')w\rangle\langle W'(x'),e'\rangle\,d\mu(x'),$$

  and $\langle w,\sigma(x^{-1}\gamma x')w\rangle=\langle\sigma(x)w,\sigma(\gamma)\sigma(x')w\rangle$. So $\langle e,\mathrm{Or}(\gamma)e'\rangle=d\int_G\int_G\kappa\,d\mu(x')\,d\mu(x)$, and since $M\in L^1(\mu\otimes\mu)$, by Fubini and (c),

$$\langle e,\mathrm{Or}(\gamma)e'\rangle=d\int_G\int_S\int_G\int_{S'}\kappa(t^{-1}z,t'^{-1}z')\,d\mu_{S'}(t')\,d\nu'(z')\,d\mu_S(t)\,d\nu(z).$$

- For measurable $B\subseteq S$, $B'\subseteq S'$ of finite measure, expanding the definitions (Fubini; the integrand is bounded by $\|W(z)\|\|W'(z')\|$) gives the same expression for $\langle e,X_{B,B'}(\gamma)e'\rangle$ with t restricted to B and t' to B'. Hence

$$\big|\langle e,X_{B,B'}(\gamma)e'\rangle-\langle e,\mathrm{Or}(\gamma)e'\rangle\big|\le d\iiint(1-\mathbf{1}_B(t)\mathbf{1}_{B'}(t'))M(t^{-1}z,t'^{-1}z').$$

- Let T_N be the same integral with $\mathbf{1}_{K_N}(t)\mathbf{1}_{K'_N}(t')$ in place of $\mathbf{1}_B(t)\mathbf{1}_{B'}(t')$. Since $K_N \uparrow S$, $K'_N \uparrow S'$ and $\Theta < \infty$, $T_N \to 0$ by dominated convergence. If $K_N \subseteq B$ and $K'_N \subseteq B'$, every matrix entry of $X_{B,B'}(\gamma) - \mathrm{Or}(\gamma)$ is at most dT_N in modulus, so $\|X_{B,B'}(\gamma)\|_{\mathrm{HS}} \geq \|\mathrm{Or}(\gamma)\|_{\mathrm{HS}} - dT_N\sqrt{nn'}$.

Fix N and let $n \geq N$. By (15.10) the integrand of $F_n(c)$ is $f_n(s,s') = \|X_{\Omega_n s,\Omega'_n s'}(\gamma)\|^2_{\mathrm{HS}}$, a continuous function of (s,s'). For $t \in A_n$, $K_N t \subseteq K_n t \subseteq \Omega_n$, i.e. $K_N \subseteq \Omega_n t^{-1}$; likewise for $t' \in A'_n$. With $\beta_N = \max\bigl(0, \|\mathrm{Or}(\gamma)\|_{\mathrm{HS}} - dT_N\sqrt{nn'}\bigr)^2$ we get $f_n \geq \beta_N$ on $A_n^{-1} \times A_n'^{-1}$. Let $P = \{f_n \geq \beta_N\}$, a closed set. For $s \in A_n^{-1}$ the section P_s contains $A_n'^{-1}$, so $\mu_{S'}(P_s) \geq \mu^*_{S'}(A'_n)$ by (b); the measurable set $\{s : \mu_{S'}(P_s) \geq \mu^*_{S'}(A'_n)\}$ contains A_n^{-1}, so its measure is at least $\mu^*_S(A_n)$. By Tonelli,

$$\int_S \int_{S'} f_n \geq \beta_N \int_S \mu_{S'}(P_s)\, d\mu_S(s) \geq \beta_N\, \mu^*_S(A_n)\, \mu^*_{S'}(A'_n), \qquad \text{so} \qquad F_n(c) \geq \beta_N \Bigl(\frac{n}{n+1}\Bigr)^2.$$

Hence $\liminf_n F_n(c) \geq \beta_N$ for every N, and $\beta_N \to \|\mathrm{Or}(\gamma)\|^2_{\mathrm{HS}}$ as $T_N \to 0$.

Fatou's lemma concludes. The map $(c,s,s') \mapsto \|X_{\Omega_n,\Omega'_n}(s\sec(c)\, s'^{-1})\|^2_{\mathrm{HS}}$ is measurable (a continuous function of a measurable map), so F_n is measurable by Tonelli. By the disintegration hypothesis for $\varphi = \|X_{\Omega_n,\Omega'_n}\|^2_{\mathrm{HS}}$ and (15.9),

$$\int_{\mathcal{C}} F_n\, dm_{\mathcal{C}} = \frac{1}{\mu_S(\Omega_n)\mu_{S'}(\Omega'_n)} \int_G \|X_{\Omega_n,\Omega'_n}\|^2_{\mathrm{HS}}\, d\mu = d \cdot \frac{E(\Omega_n)}{\mu_S(\Omega_n)} \cdot \frac{E'(\Omega'_n)}{\mu_{S'}(\Omega'_n)} \longrightarrow d \cdot \frac{1}{d} \cdot \frac{1}{d} = \frac{1}{d}$$

by (15.11). By the pointwise bound, the monotonicity of the lower integral and Fatou's lemma,

$$\int_{\mathcal{C}} \|\mathrm{Or}(\sec(c))\|^2_{\mathrm{HS}}\, dm_{\mathcal{C}}(c) \leq \int_{\mathcal{C}} \liminf_n F_n\, dm_{\mathcal{C}} \leq \liminf_n \int_{\mathcal{C}} F_n\, dm_{\mathcal{C}} = \frac{1}{d}.$$

□

Lemma 15.19. *Let F be a number field and H a connected reductive group over F with $H(F_\infty)$ compact. Let μ_∞ and μ_f be Haar measures on $H(F_\infty)$ and $H(\mathbb{A}_{F,f})$, put $\mu = \mu_\infty \otimes \mu_f$ on $H(\mathbb{A}_F) = H(F_\infty) \times H(\mathbb{A}_{F,f})$, and let $K \subseteq H(\mathbb{A}_{F,f})$ be a compact open subgroup. For $c \in H(\mathbb{A}_{F,f})$ put $\Gamma_c(K) = H(F) \cap cKc^{-1}$.*

1. *The set $H(F)\backslash H(\mathbb{A}_{F,f})/K$ is finite, every $\Gamma_c(K)$ is finite, and for any set $R(K)$ of representatives c of the double cosets*

$$\mu([H]) = \mu_\infty\bigl(H(F_\infty)\bigr)\, \mu_f(K) \sum_{c \in R(K)} \frac{1}{|\Gamma_c(K)|}.$$

2. *If $K'' \subseteq K$ is an open subgroup with $\Gamma_c(K'') = \{1\}$ for every $c \in H(\mathbb{A}_{F,f})$, then $\mu_f(K) = [K : K'']\, \mu_f(K'')$ and*

$$\bigl|H(F)\backslash H(\mathbb{A}_{F,f})/K''\bigr| = \frac{\mu([H])}{\mu_\infty\bigl(H(F_\infty)\bigr)\, \mu_f(K'')}.$$

Proof. **Finiteness.** The double coset set $H(F)\backslash H(\mathbb{A}_{F,f})/K$ is finite by Borel's finiteness of class numbers [38]. Each $\Gamma_c(K)$ is the intersection of a discrete subgroup with the compact group cKc^{-1}, hence finite.

Decomposition of the quotient. Let $R(K)$ be a set of representatives. Then $H(\mathbb{A}_F) = \bigsqcup_{c \in R(K)} H(F)\bigl(H(F_\infty) \times cK\bigr)$, a disjoint union of open sets invariant under left multiplication

by $H(F)$. For fixed c, $\gamma \in H(F)$ and $(h, ck), (h', ck') \in H(F_\infty) \times cK$ with $\gamma(h, ck) = (h', ck')$, one has $\gamma = ck'k^{-1}c^{-1} \in \Gamma_c(K)$. Hence the natural map

$$\Gamma_c(K)\backslash\big(H(F_\infty) \times cK\big) \longrightarrow H(F)\backslash H(F)\big(H(F_\infty) \times cK\big)$$

is a bijection, and the action of the finite group $\Gamma_c(K)$ on $H(F_\infty) \times cK$ by left multiplication is free (already on the first factor). A fundamental domain for this action has measure $\mu(H(F_\infty) \times cK)/|\Gamma_c(K)| = \mu_\infty(H(F_\infty))\mu_f(K)/|\Gamma_c(K)|$, using the left invariance of μ_f. The union over $c \in R(K)$ of these fundamental domains is a fundamental domain of $H(F)$ in $H(\mathbb{A}_F)$, which proves item 1.

Item 2. K is the disjoint union of $[K : K'']$ left cosets kK'', each of measure $\mu_f(K'')$, and $[K : K''] < \infty$ since K is compact and K'' open. Item 1 for K'', with all $|\Gamma_c(K'')| = 1$, gives $\mu([H]) = \mu_\infty(H(F_\infty))\,\mu_f(K'')\,|R(K'')|$. □

Proposition 15.20. *Let E/F be a CM extension, V a nondegenerate hermitian space over E of dimension n, $G = \mathrm{U}(V)$, and μ a Haar measure on $G(\mathbb{A}_F)$.*

1. *Let $X_0 \in V^a$ with $a \geq 1$, $2a < n$ and $T = \mathrm{gram}(X_0)$ nondegenerate, let $H \subseteq G$ be the stabiliser of X_0 (the elements fixing each entry of X_0), and let μ' be a Haar measure on $H(\mathbb{A}_F)$ such that*

$$\int_{G(\mathbb{A}_F)/H(\mathbb{A}_F)} \Phi(gX_0)\, d\dot{\mu}(gH(\mathbb{A}_F)) = g_\infty(\Phi_\infty, T) \prod_{w<\infty} g_w(\Phi_w, T) \tag{15.12}$$

   for every admissible test function Φ on $V^a_{\mathbb{A}_F}$. Then $\mu'([H]) = \mu([G]) < \infty$.

2. *Let X_1, X_2, X_3 be tuples as in item 1, of sizes a_1, a_2, a_3, with stabilisers H_1, H_2, H_3 and Haar measures μ'_1, μ'_2, μ'_3 satisfying* (15.12) *for X_1, X_2, X_3 respectively. Then*

$$\mu'_3([H_3])\,\mu([G]) = \mu'_1([H_1])\,\mu'_2([H_2]).$$

Proof. **Item 1. A positive test function.** Let $L \subset V$ be an $\mathcal{O}_E$-lattice containing the entries of X_0 (enlarge any lattice by the $\mathcal{O}_E$-span of these entries). Put $\Phi_w = \mathbf{1}_{(L\otimes\mathcal{O}_{F_w})^a}$ for every finite w and let $\Phi_\infty \geq 0$ be smooth and compactly supported with $\Phi_\infty(X_0) > 0$. Then $\Phi = \Phi_\infty \otimes \bigotimes_w \Phi_w$ is admissible, $\Phi \geq 0$, Φ is continuous on $V^a_{\mathbb{A}}$ and $\Phi(X_0) = \Phi_\infty(X_0) > 0$. Write I for the right side of (15.12) for this Φ, and $f(g) = \Phi(g^{-1}X_0)$ for $g \in G(\mathbb{A})$. Since $H(\mathbb{A})$ fixes X_0, $f(hg) = f(g)$ for $h \in H(\mathbb{A})$; f is continuous, $f \geq 0$ and $f(1) > 0$.

The rational points of the orbit form one orbit (Witt). Let $X \in V^a$ with $\mathrm{gram}(X) = T$. As T is nondegenerate, the entries of X_0 and of X are linearly independent over E, and the E-linear map from the span of X_0 to the span of X sending the i-th entry of X_0 to that of X preserves $\langle\,,\rangle$; both spans are nondegenerate. By Witt's extension theorem for hermitian forms [214] the map extends to an isometry $\gamma \in G(F)$ of V, so $X = \gamma X_0$. Hence $\gamma \mapsto \gamma X_0$ induces a bijection $G(F)/H(F) \to \{X \in V^a : \mathrm{gram}(X) = T\}$, $H(F) = G(F) \cap H(\mathbb{A})$, and

$$\vartheta_T(g) = \sum_{\gamma \in G(F)/H(F)} \Phi(g^{-1}\gamma X_0) = \sum_{\gamma \in H(F)\backslash G(F)} f(\gamma g) \qquad (g \in G(\mathbb{A})).$$

Unfolding. By the unfolding identity for the discrete subgroups $H(F) \subseteq G(F)$ and the quotient integral formula for $H(F)\backslash H(\mathbb{A})$, with the left $H(\mathbb{A})$-invariance of f,

$$\int_{[G]} \vartheta_T\, d\mu = \int_{H(F)\backslash G(\mathbb{A})} f = \int_{H(\mathbb{A})\backslash G(\mathbb{A})} \int_{H(F)\backslash H(\mathbb{A})} f(hg)\, d\mu'(h)\, d\dot{\mu} = \mu'([H])\, I. \tag{15.13}$$

Siegel's formula. By Theorem B.158 (hypotheses: E/F CM, V nondegenerate, $1 \le a$, $2a < n$, T nondegenerate, Φ admissible), $\mu([G]) < \infty$, ϑ_T is integrable on $[G]$ and $\int_{[G]} \vartheta_T \, d\mu = \mu([G])\, I$. Moreover $\mu([G]) > 0$, since $G(\mathbb{A})$ is the countable union of the $G(F)$-translates of a fundamental domain. In particular $I < \infty$.

The integral is positive. Choose an open neighbourhood U of 1 in $G(\mathbb{A})$ with $f > 0$ on U and $(UU^{-1}) \cap H(F) = \{1\}$ (discreteness of $H(F)$). For $\varphi = \mathbf{1}_U f$ and every g, at most one $h \in H(F)$ has $hg \in U$, so $\sum_{h \in H(F)} \varphi(hg) \le f(g)$ by the invariance of f. Hence $0 < \int_{G(\mathbb{A})} \varphi \, d\mu = \int_{H(F)\backslash G(\mathbb{A})} \sum_{h \in H(F)} \varphi(hg) \le \int_{H(F)\backslash G(\mathbb{A})} f$, the first inequality because U is a nonempty open set and $\varphi > 0$ on it. With (15.13) and Siegel's formula, $\mu([G])\, I > 0$, so $0 < I < \infty$.

Conclusion of item 1. Comparing (15.13) with Siegel's formula gives $\mu'([H])\, I = \mu([G])\, I$, and dividing by $I \in (0, \infty)$ gives $\mu'([H]) = \mu([G]) < \infty$.

Item 2. By item 1 applied to X_1, X_2, X_3, $\mu_i'([H_i]) = \mu([G])$ for $i = 1, 2, 3$, so both sides equal $\mu([G])^2$. □

16 The local gain of the non-generic row

This section proves the local gain (Ng5) of the non-generic row (Lemma 16.4), which the row estimates of Section 15 use (Theorem 15.3). It is the longest section of Part III and it is entirely local: its objects are hermitian lattices over a local quadratic algebra at one finite place, with their unitary groups, masses and local densities. Apart from Lemma 16.4, the only result of the section used elsewhere is Lemma 16.27, the conjugacy of self-dual hermitian lattices over an unramified or split local quadratic algebra, which Appendix D uses as well; a reader who accepts Lemma 16.4 may skip the rest.

The statement to be proved is that the amplitudes of the non-generic terms gain a factor $q^{-(k-2r-5)}$ against the total mass, up to a factor $H^{39(r+1)^2}$, when k is large compared with r (Theorem 16.38); one non-generic place suffices for the global gain (Lemma 16.40). A term is generic when its defect order is 1; otherwise its defect order is q^i with $i \ge 1$ (Proposition 16.8). Three facts produce the gain. Terms of large defect order are rare: a defect of order β costs $\beta^{-(k-1)}$ in mass (Proposition 16.20). Non-generic terms of the first type have fewer defect classes than their defect order by a factor q (Proposition 16.10). Terms of the second type have phases that cancel on average over Eichler moves (Proposition 16.30). Subsection 16.1 reduces (Ng5) to the local non-generic bound at a local frame pair. Subsection 16.2 shows that local frame pairs carry local non-generic records with the properties (Lg1)–(Lg5), Subsection 16.3 proves the mass bound for a given defect order, and Subsection 16.4 proves the cancellation and the local non-generic bound.

16.1 The local gain from frame realisations

This subsection reduces the local gain (Ng5) to a statement at one place. Lemma 16.4 derives (Ng5) from frame realisations of the local amplitudes at the places of a finite set, through Lemma 16.40: if the non-generic amplitudes gain $q_p^{-(k-2r-5)}$ at the places p of a set Ω and the other local amplitudes are bounded, the gain at a single place of Ω passes to the global amplitude. Corollary 16.5 supplies the hypothesis of that lemma for realised local data, and it rests on Corollary 16.6, the local non-generic bound at a local frame pair. The rest of the section proves Corollary 16.6, which follows from Theorem 16.7 and Theorem 16.38.

Definition 16.1. The notation introduced here is local to this definition and to the results that apply it. Let $r \ge 0$ and $k \ge 1$ be integers. **Local non-generic records** (with parameters r, k)

consist of

- a group G acting on a set X and subgroups $K_0 \subseteq \Sigma \subseteq G$; for $x \in X$, $\Sigma_x = \{g \in \Sigma : gx = x\}$; a subgroup $A \subseteq G$ is **commensurable with** K_0 if $[A : A \cap K_0]$ and $[K_0 : A \cap K_0]$ are finite, and then $\mathrm{vol}(A) = [A : A \cap K_0]/[K_0 : A \cap K_0]$;
- a countable set D (the **data**) with weights $\nu\colon D \to [0, \infty)$;
- for every $d \in D$ a Σ-stable subset $X_d \subseteq X$ (the **terms over** d) that is the union of finitely many Σ-orbits, such that for $x \in X_d$ the group Σ_x is commensurable with K_0 and $\mathrm{vol}(\Sigma_{gx}) = \mathrm{vol}(\Sigma_x)$ for $g \in \Sigma$; the **mass** of x is $\mu(x) = 1/\mathrm{vol}(\Sigma_x)$;
- for every d, Σ-invariant functions on X_d: the **phase** $\omega_d\colon X_d \to \mathbb{C}$ with $|\omega_d| \le 1$, the **defect order** $b_d\colon X_d \to \mathbb{Z}_{\ge 1}$ and the **class count** $c_d\colon X_d \to \mathbb{Z}_{\ge 1}$; a Σ-stable subset $X_d^{\mathrm{II}} \subseteq X_d$ (the terms of **second type**) with a Σ-invariant **label** $\lambda_d\colon X_d^{\mathrm{II}} \to \mathbb{Z}^2$;
- integers $q \ge 2$, $a \ge 0$ and a real number $H \ge 1$ (the **height**).

For a Σ-stable subset $Y \subseteq X_d$ put

$$\mathrm{amp}_d(Y) = \sum_{O \in \Sigma\backslash Y} \mu(O)\,\omega_d(O), \qquad \mathrm{mass}_d(Y) = \sum_{O \in \Sigma\backslash Y} \mu(O)$$

(finite sums; μ and ω_d are constant on orbits). A term $x \in X_d$ is **generic** if $b_d(x) = 1$; X_d^{g} and $X_d^{\mathrm{n}} = X_d \setminus X_d^{\mathrm{g}}$ are the sets of generic and non-generic terms, and the **total mass** is

$$\mathrm{nn} = \sum_{d \in D} \nu(d)\,\mathrm{mass}_d(X_d) \in [0, \infty].$$

The records have

(Lg1) Defect orders if $q^a \le H$ and $b_d(x) \in \{q^i : 0 \le i \le a\}$ for all d and $x \in X_d$;

(Lg2) Class counts if $c_d(x) \in \{q^j : j \ge 0\}$ and $c_d(x) \le b_d(x)$ for all d, x, and $q\,c_d(x) \le b_d(x)$ for $x \in X_d^{\mathrm{n}} \setminus X_d^{\mathrm{II}}$;

(Lg3) Mass of a class count if for every d and every integer $c \ge 1$

$$\mathrm{mass}_d(\{x \in X_d : c_d(x) = c\}) \le H^{9(r+1)^2+12(r+1)}\,c^{k-1};$$

(Lg4) Share of a defect order if for every integer $\beta \ge 1$

$$\sum_{d \in D} \nu(d)\,\mathrm{mass}_d(\{x \in X_d : b_d(x) = \beta\}) \le 4H^{4(r+1)}\,\beta^{-(k-1)}\,\mathrm{nn};$$

(Lg5) Averaging kernels on the second type if $X_d^{\mathrm{II}} \subseteq X_d^{\mathrm{n}}$, $\lambda_d(x) \in \{1, \dots, r\} \times \{0, \dots, i-1\}$ whenever $x \in X_d^{\mathrm{II}}$ and $b_d(x) = q^i$, and for all d, $i \ge 1$ and $\lambda \in \mathbb{Z}^2$ the set $\mathcal{P} = \{x \in X_d^{\mathrm{II}} : b_d(x) = q^i,\ \lambda_d(x) = \lambda\}$ carries a function $m\colon \mathcal{P} \times \mathcal{P} \to [0, \infty)$ with $m(gx, gy) = m(x, y)$ for $g \in \Sigma$, $m(x, y) = m(y, x)$, $\{y : m(x, y) \ne 0\}$ finite for every x, and

$$\Big|\sum_{y \in \mathcal{P}} m(x, y) - 1\Big| \le \tfrac{24}{7} q^{-(k-2r-4)}, \qquad \Big|\sum_{y \in \mathcal{P}} m(x, y)\,\omega_d(y)\Big| \le \tfrac{24}{7} q^{-(k-2r-4)} \qquad (x \in \mathcal{P}).$$

Definition 16.2. The notation introduced here is local to this definition and to the results that apply it.

Local setting. Let F be a nonarchimedean local field of characteristic 0, with ring of integers $\mathcal{O}_F$, uniformiser ϖ, normalised valuation v ($v(\varpi)=1$) and $q=|\mathcal{O}_F/\varpi\mathcal{O}_F|$. Let E be either the unramified quadratic field extension of F or the algebra $F\times F$; $x\mapsto\bar{x}$ is its nontrivial F-automorphism ($(x,y)\mapsto(y,x)$ on $F\times F$), $\mathrm{N}(x)=x\bar{x}\in F$, and $\mathcal{O}_E$ is the integral closure of $\mathcal{O}_F$ in E. For $x\in E$ and $m\in\mathbb{Z}$ write $\varpi^m\mid x$ if $x\in\varpi^m\mathcal{O}_E$. Let $\psi_0\colon E\to\mathbb{C}^\times$ be a continuous unitary character of the additive group with $\psi_0(\bar{x})=\psi_0(x)$ for $x\in E$ and with conductor $\mathcal{O}_E$: for $x\in E$, $\psi_0(xy)=1$ for all $y\in\mathcal{O}_E$ if and only if $x\in\mathcal{O}_E$. Let $r\geq 0$ and $k\geq 1$ be integers, and put $n=k+2r+1$.

Local frame pairs. A **local frame pair** with parameters r,k consists of $V=E^n$ with a hermitian form $h\colon V\times V\to E$ (E-linear in the first variable, $h(y,x)=\overline{h(x,y)}$, with invertible Gram matrix), elements $T,T'\in F^\times$ and $\theta\in\{1,\varpi\}$, where $\theta=\varpi$ is allowed only if $E=F\times F$ and $T,T'\in\mathcal{O}_F^\times$, and families $f=(f_0,\dots,f_r)$, $f'=(f'_0,\dots,f'_r)\in V^{r+1}$ and $e^*=(e^*_0,\dots,e^*_{r-1})$, $e'^*\in V^r$, such that

(Fp1) Frames $h(f_a,f_b)=0$ unless $a=b=r$, $h(f_r,f_r)=-T$, $h(f_a,e^*_j)=\delta_{aj}$ ($0\leq a\leq r$, $0\leq j<r$) and $h(e^*_i,e^*_j)=0$; and the same for f',e'^*,T';

(Fp2) Point form for the matrix $Z=(h(f'_b,f_a))_{0\leq a,b\leq r}$, the matrix $((-1)^{r-b}Z_{ab})_{a,b}$ is block diagonal with diagonal blocks $t_1J_{s_1},\dots,t_mJ_{s_m}$, where $t_i\in E^\times$, $s_i\geq 1$, $\sum_i s_i=r+1$ and J_s is the $s\times s$ matrix with ones on the antidiagonal and zeros elsewhere;

(Fp3) Gram quantity with D_l the leading principal $l\times l$ minor of Z ($D_0=1$),

$$G=D_{r+1}\bar{D}_{r+1}-TT'\,D_r\bar{D}_r\neq 0.$$

Flags, groups, characters. For $g\in V^{r+1}$ and $0\leq j\leq r+1$ let $F_j(g)$ be the E-span of $g_0,\dots,g_{j-1}$; put $X=F_r(f)$, $X'=F_r(f')$, $M_E=F_{r+1}(f)+F_{r+1}(f')$ and $W_0=M_E^\perp$. Let $\mathrm{U}(V)$ be the unitary group of h. For $g\in\{f,f'\}$ let $N(g)$ be the group of $u\in\mathrm{U}(V)$ such that for $0\leq j<r$, $uF_{j+1}(g)=F_{j+1}(g)$ and $uw-w\in F_j(g)$ for $w\in F_{j+1}(g)$, and $u(F_r(g)^\perp)=F_r(g)^\perp$ and $uw-w\in F_r(g)$ for $w\in F_r(g)^\perp$. Put $N=N(f)$, $N'=N(f')$, $\Sigma=N\cap N'$, and

$$\psi_N(u)=\psi_0\Big(\theta^{-1}\sum_{j<r}h(uf_{j+1},e^*_j)\Big),\qquad \psi_{N'}(u)=\psi_0\Big(\theta^{-1}\sum_{j<r}h(uf'_{j+1},e'^*_j)\Big).$$

Let K_0 be the group of $u\in\Sigma$ stabilising the $\mathcal{O}_E$-spans of $f_0,\dots,f_r$ and of $f'_0,\dots,f'_r$. For a subgroup $A\subseteq\mathrm{U}(V)$ and a reference subgroup K put $\mathrm{vol}_K(A)^{-1}=[K:A\cap K]/[A:A\cap K]$ if both indices are finite, and $\mathrm{vol}_K(A)^{-1}=0$ otherwise. Let H_0 be the group of $u\in\mathrm{U}(V)$ with $uw=w$ for $w\in M_E$, and K_{ref} its subgroup stabilising $\mathcal{O}_E^n$.

Data and terms. A **datum** is a triple $d=(A,A',\Gamma)$ of $\mathcal{O}_E$-submodules $A,A'\subseteq V$ and a subset $\Gamma\subseteq V\times V$; H_0 acts on data, and the **weight** of d is $\nu(d)=\mathrm{vol}_{K_{\mathrm{ref}}}(\mathrm{Stab}_{H_0}(d))^{-1}$. An $\mathcal{O}_E$-submodule A is **inner** for (f,e^*) if $F_r(f)\subseteq A\subseteq X^\perp$, $A\cap(\mathrm{span}_E e^*)^\perp$ is finitely generated, a vector $y\in X^\perp$ lies in A if and only if $h(A,y)\subseteq\mathcal{O}_E$, and $f_r\in A$ with $h(f_r,y)=1$ for some $y\in A$; then

$$\Lambda_{\mathrm{std}}(A)=\textstyle\sum_{j<r}\mathcal{O}_Ef_j+(A\cap(\mathrm{span}_E e^*)^\perp)+\sum_{j<r}\mathcal{O}_Ee^*_j,$$

and likewise for (f',e'^*). An $\mathcal{O}_E$-submodule $\Lambda\subseteq V$ is a **term over** d if A is inner for (f,e^*), A' is inner for (f',e'^*), and there are $u\in N$, $u'\in N'$ (a **witness**) with

$$\Lambda=u\Lambda_{\mathrm{std}}(A)=u'\Lambda'_{\mathrm{std}}(A'),\qquad u'f'_b-uf_b\in\theta\Lambda\ (0\leq b\leq r),$$

and Γ is the set of pairs $(p,p')\in V\times V$ for which there are $y,y'\in\Lambda$ with $y\perp X$, $y'\perp X'$, $y'-p\in X'$, $y-p'\in X$ and $y'-y\in\theta\Lambda$. For every term we fix one witness (u,u') and put $\mathrm{fr}_b=uf_b$ and $\mathrm{fr}'_b=u'f'_b$. The **mass** of a term is $\mu(\Lambda)=\mathrm{vol}_{K_0}(\mathrm{Stab}_\Sigma(\Lambda))^{-1}$ and its **phase** is $\omega_d(\Lambda)=\psi_N(u)\overline{\psi_{N'}(u')}$.

Joint frame lattice and defects. For a term Λ over d, the **joint frame lattice** is the $\mathcal{O}_E$-span $M(\Lambda)$ of the $2r+2$ vectors fr_b and $\theta^{-1}(\mathrm{fr}'_b-\mathrm{fr}_b)$, the **saturation** is $\mathrm{sat}(\Lambda)=\Lambda\cap M_E$, and the **defect order** is the group index $b_d(\Lambda)=[\mathrm{sat}(\Lambda):M(\Lambda)]$ (set to 0 if $M(\Lambda)\not\subseteq\mathrm{sat}(\Lambda)$ or the index is infinite); Λ is **generic** if $M(\Lambda)=\mathrm{sat}(\Lambda)$. The **level** of $y\in M_E$ is the least $j\in\{0,\dots,r+1\}$ such that $\theta y=\sum_b\alpha_b\mathrm{fr}_b+\sum_b\alpha'_b\mathrm{fr}'_b$ with $\alpha'_b\in\mathcal{O}_E$ and $\alpha_b+\alpha'_b\in\theta\mathcal{O}_E$ for all $b\ge j$ (and 0 if there is none); write $\ell(y)$. A term is of **second type** if it is not generic, $\ell(y)\le r$ for all $y\in\mathrm{sat}(\Lambda)$, and $h(x_1,x'_2)\in\theta\mathcal{O}_E$ whenever $x_1,x_2\in X$ and $x'_1,x'_2\in X'$ satisfy $x_1+x'_1,\ x_2+x'_2\in\theta\Lambda$. Its **label** is $\lambda_d(\Lambda)=(J,e)$, where $J=\max\{\ell(y):y\in\mathrm{sat}(\Lambda)\}$ and e is the integer with $q^e=[\mathrm{sat}(\Lambda)\cap(M(\Lambda)+F_{J-1}(f)+F_{J-1}(f')):M(\Lambda)]$ if this index is a power of q, $e=0$ otherwise.

Class count. Let $\hat I(\Lambda)$ be the sum of $M(\Lambda)$ and the $\mathcal{O}_E$-span of the vectors $y\in V$ for which there are $w\in W_0\cap A\cap A'$ with $(w,w)\in\Gamma$, $\lambda\in X$ and $\lambda'\in X'$ with $w+\lambda\in\Lambda$, $w+\lambda'\in\Lambda$, $\lambda'-\lambda\in\theta\Lambda$ and $y-\theta^{-1}(\lambda'-\lambda)\in M(\Lambda)$. The **class count** is $c_d(\Lambda)=[\hat I(\Lambda):M(\Lambda)]$ (set to 0 if infinite).

Frame classes. A **frame pair** is an element (p,p') of $V^{r+1}\times V^{r+1}$; for a matrix $B\in\mathrm{M}_{r+1}(E)$ put $(pB)_j=\sum_i B_{ij}p_i$. The **class** of a frame pair is

$$\mathrm{cl}(p,p')=\{(pB,p'B'):\ B,B'\in\mathrm{M}_{r+1}(\mathcal{O}_E)\text{ upper unitriangular},\ B'-B\in\theta\,\mathrm{M}_{r+1}(\mathcal{O}_E)\}.$$

A **frame class** is a set $\mathrm{cl}(uf,u'f')$ with $u\in N$, $u'\in N'$; it is **integral** if $h(p'_b,p_a)\in\mathcal{O}_E$ for all (p,p') in it and all a,b. $\mathrm{U}(V)$ acts on frame pairs by $g(p,p')=(gp,gp')$ and hence on sets of frame pairs. The **weight** of a frame class C is $w(C)=\mathrm{vol}_{K_0}(\mathrm{Stab}_\Sigma(C))^{-1}$, and the **class of a term** Λ is $\mathrm{Fr}(\Lambda)=\mathrm{cl}(\mathrm{fr},\mathrm{fr}')$. For a frame pair, $M(p,p')$ is the $\mathcal{O}_E$-span of the p_b and $\theta^{-1}(p'_b-p_b)$ $(0\le b\le r)$; so $M(\Lambda)=M(\mathrm{fr},\mathrm{fr}')$.

Saturated masses. An $\mathcal{O}_E$-submodule $\Lambda\subseteq V$ is **self-dual** if it is finitely generated and $\Lambda=\{y\in V:h(\Lambda,y)\subseteq\mathcal{O}_E\}$. For an $\mathcal{O}_E$-submodule $\hat M\subseteq M_E$, the **saturated mass** is

$$\sigma(\hat M)=\sum_{[\Lambda]}\mathrm{vol}_{K_{\mathrm{ref}}}(\mathrm{Stab}_{H_0}(\Lambda))^{-1}\in[0,\infty],$$

the sum over the H_0-orbits of self-dual Λ with $\Lambda\cap M_E=\hat M$ (one fixed representative per orbit).

Numbers.

$$a=v(G)-2(r+1)v(\theta),\qquad v_D=\sum_{1\le l\le r+1,\ D_l\neq0}v(D_l\bar D_l),\qquad H=q^{\,v(\theta)+a+v_D}.$$

Definition 16.3. The notation introduced here is local to this definition and to the results that apply it. Let $r\ge0$ and $k\ge1$ be integers, Π a finite set, $\Omega\subseteq\Pi$, and for every $p\in\Pi$ let D_p be a set (not assumed finite or countable), $\mathrm{wt}_p\colon D_p\to[0,\infty)$, $a_p^{\mathrm{N}},a_p^{\mathrm{Z}}\colon D_p\to\mathbb{C}$, $n_p\ge0$ and $H_p\ge1$ real numbers and $q_p\ge2$ an integer (**local non-generic data**). Sums over D_p of nonnegative terms, here and in the results citing this definition, are unordered sums in $[0,\infty]$. A **frame realisation** of these data is a family $(S_p,c_p)_{p\in\Pi}$ with

(Fb1) Local frame pairs S_p is a local frame pair with parameters r,k (over a local field with residue field of size q_p) for which some datum has a term, D_p is a set of data of S_p containing exactly one datum from each H_0-orbit of data having a term, each element of D_p has a term, and H_p is the height H of S_p;

(Fb2) Amplitudes $c_p > 0$ is a real number and, for $\delta \in D_p$ and a set R_δ of representatives of the Σ-orbits of terms over δ,

$$\mathrm{wt}_p(\delta) = \nu(\delta),$$
$$a_p^{\mathrm{N}}(\delta) = c_p \sum_{\Lambda \in R_\delta \text{ not generic}} \mu(\Lambda)\,\omega_\delta(\Lambda), \qquad a_p^{\mathrm{Z}}(\delta) = c_p \sum_{\Lambda \in R_\delta \text{ generic}} \mu(\Lambda)\,\omega_\delta(\Lambda),$$
$$n_p = c_p^2 \sum_{\delta \in D_p} \nu(\delta) \sum_{\Lambda \in R_\delta} \mu(\Lambda);$$

(Fb3) Non-generic places $q_p \le H_p$ for every $p \in \Omega$.

Lemma 16.4. *The notation introduced here is local to this result and to the results that apply it. Let $r \ge 0$, $k \ge 4r{+}7$ and $C_0 \ge 0$ be integers, and put $e = (39{+}C_0)(r{+}1)^2{+}2(2r{+}2){+}5$. Fix a pair expansion and non-generic row data for it, with sets* Cl*, P, N. Suppose that for every $\gamma \in P_{\mathrm{oc}}$ there are a finite set Π, a subset $\Omega \subseteq \Pi$, local non-generic data $(D_p, \mathrm{wt}_p, a_p^{\mathrm{N}}, a_p^{\mathrm{Z}}, n_p, H_p, q_p)_{p\in\Pi}$ with a frame realisation with parameters r, k, real numbers $n_p^\circ \ge 0$ ($p \in \Pi$) and $V \ge 0$, and a set X with maps $\rho\colon X \to [0,\infty)$ and $\delta\colon X \to \prod_{p\in\Pi} D_p$ (all depending on γ; X need not be finite, and sums over subsets of X are unordered sums in $[0,\infty]$), such that*

1. *$n_p = n_p^\circ$ for $p \notin \Omega$ and $n_p \le 8H_p^{C_0(r+1)^2+2(2r+2)} n_p^\circ$ for $p \in \Omega$;*
2. *$\sum_{x\in X,\ \delta(x)=\delta_0} \rho(x) \le V \prod_{p\in\Pi} \mathrm{wt}_p(\delta_{0,p})$ for every $\delta_0 \in \prod_p D_p$;*

and, with $\mathrm{amp}(\delta_0) = \sum_{\varnothing\neq B\subseteq\Omega} \prod_{p\in B} a_p^{\mathrm{N}}(\delta_{0,p}) \prod_{p\in\Pi\setminus B} a_p^{\mathrm{Z}}(\delta_{0,p})$,

(Nv1) Amplitude domination *$\sum_{(c,\nu)\in F} a_2(c,\gamma,\nu) \le \sum_{x\in X} \rho(x)\,|\mathrm{amp}(\delta(x))|^2$ for every finite $F \subseteq \mathrm{Cl} \times N$;*

(Nv2) Generic mass $\bigl(\prod_{p\in\Pi} H_p\bigr)^e V \prod_{p\in\Pi} n_p^\circ \le n_{\mathrm{g}}(\gamma)$.

Then the non-generic row data have the local gain (Ng5) with these r and k.

Proof. Fix $\gamma \in P_{\mathrm{oc}}$ and the data attached to it. By Definition 16.3, $\mathrm{wt}_p \ge 0$, $H_p \ge 1$, $n_p \ge 0$ and $q_p \ge 2$ is an integer; D_p is an arbitrary set. Let $X' \subseteq X$ be finite; we show $\sum_{x\in X'} \rho(x)|\mathrm{amp}(\delta(x))|^2 \le 2^{-(k-2r-5)} n_{\mathrm{g}}(\gamma)$. If $X' = \emptyset$, the left side is 0; let $X' \neq \emptyset$. Item 2 holds with X' in place of X, since restricting the sum to X' only decreases its left side. Put $F_p = \{\delta(x)_p : x \in X'\}$, a finite nonempty subset of D_p, so δ maps X' into $\prod_p F_p$. Since $k \ge 4r+7$ and the data have a frame realisation, Corollary 16.5 gives $q_p \le H_p$ for $p \in \Omega$ and hypothesis (1) of Lemma 16.40 with $C_1 = 39$ for the restrictions of $\mathrm{wt}_p, a_p^{\mathrm{N}}, a_p^{\mathrm{Z}}$ to the finite sets F_p: for $p \in \Omega$, $\sum_{\delta_0\in F_p} \mathrm{wt}_p(\delta_0)|a_p^{\mathrm{N}}(\delta_0)|^2 \le H_p^{39(r+1)^2} q_p^{-(k-2r-5)} n_p$, and for every $p \in \Pi$, $\sum_{\delta_0\in F_p} \mathrm{wt}_p(\delta_0)|a_p^{\mathrm{Z}}(\delta_0)|^2 \le H_p^{39(r+1)^2} n_p$. We apply Lemma 16.40 with the finite sets F_p in place of D_p and the finite set X' with $\rho|_{X'}$, $\delta|_{X'}$: hypothesis (2) there is item 1 above, and hypothesis (3) there is item 2 above for X' and $\delta_0 \in \prod_p F_p \subseteq \prod_p D_p$; and $k \ge 4r{+}7 \ge 2r{+}5$. With $C_1 = 39$ the exponent e_{ng} of Lemma 16.40 equals e, and the function amp above is the one defined there. Hence

$$\sum_{x\in X'} \rho(x)\,|\mathrm{amp}(\delta(x))|^2 \le 2^{-(k-2r-5)} \Bigl(\prod_{p\in\Pi} H_p\Bigr)^e V \prod_{p\in\Pi} n_p^\circ \le 2^{-(k-2r-5)}\, n_{\mathrm{g}}(\gamma),$$

the second step by (Nv2). The unordered sum over X of the nonnegative numbers $\rho(x)|\mathrm{amp}(\delta(x))|^2$ is the supremum of its finite partial sums, so $\sum_{x\in X} \rho(x)|\mathrm{amp}(\delta(x))|^2 \le 2^{-(k-2r-5)} n_{\mathrm{g}}(\gamma)$. With (Nv1), $\sum_{(c,\nu)\in F} a_2(c,\gamma,\nu) \le 2^{-(k-2r-5)} n_{\mathrm{g}}(\gamma)$ for every finite $F \subseteq \mathrm{Cl} \times N$. As $\gamma \in P_{\mathrm{oc}}$ was arbitrary, this is (Ng5). □

Corollary 16.5. *Let $r \geq 0$ and $k \geq 4r+7$ be integers and let local non-generic data $(D_p, \mathrm{wt}_p, a_p^{\mathrm{N}}, a_p^{\mathrm{Z}}, n_p, H_p, q_p)_{p\in\Pi}$, $\Omega \subseteq \Pi$, have a frame realisation. Then for every $p \in \Omega$*

$$\sum_{\delta\in D_p} \mathrm{wt}_p(\delta)\,|a_p^{\mathrm{N}}(\delta)|^2 \leq H_p^{39(r+1)^2} q_p^{-(k-2r-5)} n_p \qquad \text{and} \qquad q_p \leq H_p,$$

and for every $p \in \Pi$, $\sum_{\delta\in D_p} \mathrm{wt}_p(\delta)|a_p^{\mathrm{Z}}(\delta)|^2 \leq H_p^{39(r+1)^2} n_p$. In particular, for every choice of finite sets $F_p \subseteq D_p$ ($p \in \Pi$), the restrictions of wt_p, a_p^{N}, a_p^{Z} to F_p satisfy hypothesis (1) of Lemma 16.40 with $C_1 = 39$.

Proof. Let (S_p, c_p) be the realisation and $p \in \Pi$. By (Fb1), S_p is a local frame pair with parameters r and $k \geq 4r+7$ and D_p is a set of data as in Corollary 16.6; its q is q_p and its H is H_p. By (Fb2) the sums of Corollary 16.6 for S_p and D_p are $\mathrm{amp}_\delta^{\mathrm{n}} = a_p^{\mathrm{N}}(\delta)/c_p$, $\mathrm{amp}_\delta^{\mathrm{g}} = a_p^{\mathrm{Z}}(\delta)/c_p$, $\mathrm{nn} = n_p/c_p^2$ and $\nu = \mathrm{wt}_p$. Multiplying the two inequalities of Corollary 16.6 by c_p^2 gives the two amplitude bounds. For $p \in \Omega$, $q_p \leq H_p$ by (Fb3). All sums over D_p here are unordered sums of nonnegative terms (Corollary 16.6 allows an arbitrary set of representatives D); for finite $F_p \subseteq D_p$ the sums over F_p are at most those over D_p, which gives hypothesis (1) of Lemma 16.40 for the restrictions to F_p (with $q_p \leq H_p$ for $p \in \Omega$). □

Corollary 16.6. *Let a local frame pair with parameters r and $k \geq 4r+7$ be given. Let D be a set of representatives of the H_0-orbits of data that contain a datum having a term, every element of D having a term; for $d \in D$ let R_d be a set of representatives of the Σ-orbits of terms over d, and put*

$$\mathrm{amp}_d^{\mathrm{n}} = \sum_{\Lambda\in R_d \ \text{not generic}} \mu(\Lambda)\,\omega_d(\Lambda), \qquad \mathrm{amp}_d^{\mathrm{g}} = \sum_{\Lambda\in R_d \ \text{generic}} \mu(\Lambda)\,\omega_d(\Lambda),$$
$$\mathrm{nn} = \sum_{d\in D} \nu(d) \sum_{\Lambda\in R_d} \mu(\Lambda).$$

Then

$$\sum_{d\in D} \nu(d)\,|\mathrm{amp}_d^{\mathrm{n}}|^2 \leq H^{39(r+1)^2} q^{-(k-2r-5)}\,\mathrm{nn}, \qquad \sum_{d\in D} \nu(d)\,|\mathrm{amp}_d^{\mathrm{g}}|^2 \leq H^{39(r+1)^2}\,\mathrm{nn}.$$

Proof. If no datum has a term, D is empty and both sides vanish. Otherwise Theorem 16.7 shows that these data form local non-generic records with parameters r, k and the properties (Lg1)–(Lg5), with total mass nn; their sets of generic and non-generic terms over d are those of Definition 16.2 (generic means $b_d = 1$, by Proposition 16.8(3)), and since μ, ω_d and genericity are constant on Σ-orbits (Proposition 16.12), $\mathrm{amp}_d^{\mathrm{n}}$ and $\mathrm{amp}_d^{\mathrm{g}}$ are the amplitudes $\mathrm{amp}_d(X_d^{\mathrm{n}})$ and $\mathrm{amp}_d(X_d^{\mathrm{g}})$ of the records. As $k \geq 4r+7 \geq 2r+3$, Theorem 16.38 gives both bounds. □

16.2 Local non-generic records of a frame pair: defect orders and class counts

This subsection proves Theorem 16.7: a local frame pair, with its terms, phases, defect orders, class counts and terms of second type, forms local non-generic records with the properties (Lg1)–(Lg5), which are the hypotheses of the local non-generic bound Theorem 16.38. Most of the properties are structural. The terms are self-dual lattices that meet the flags in the translated frames (Lemma 16.18); their defect order is a power q^i with $0 \leq i \leq a$, and it is 1 exactly for the generic terms (Proposition 16.8); a common unipotent group acts on them and preserves phases,

defects, types and labels (Proposition 16.12). The estimates are three: the class count of a non-generic term of first type is smaller than its defect order by a factor q (Proposition 16.10), proved here; the share of the terms of a given defect order (Proposition 16.20), proved in the next subsection; and the averaging kernel of Eichler moves on the terms of second type (Proposition 16.30), proved in the last one.

Theorem 16.7. *Let a local frame pair with parameters r and $k \geq 4r+7$ be given such that some datum has a term. Let D be a set of representatives of the H_0-orbits of data that contain a datum having a term, every element of D having a term. Then the group $\mathrm{U}(V)$ acting on the set of $\mathcal{O}_E$-submodules of V, the subgroups $K_0 \subseteq \Sigma$, the data D with the weights ν, for $d \in D$ the set X_d of terms over d with the phase ω_d, the defect order b_d, the class count c_d, the set X_d^{II} of terms of second type over d with the label λ_d, and the numbers q, a, H form local non-generic records with parameters r, k; their masses are the masses μ of Definition 16.2, and they have the properties (Lg1)–(Lg5).*

Proof. **Step 1: the data are countable.** Let $d = (A, A', \Gamma)$ have a term Λ. Put $Y = X^\perp \cap (\mathrm{span}_E e^*)^\perp = (X + \mathrm{span}_E e^*)^\perp$; by (Fp1) $X + \mathrm{span}_E e^*$ is a nondegenerate (hyperbolic) subspace, so Y is a nondegenerate complement. For $w \in A$, $w' = w - \sum_j h(w, e_j^*) f_j$ lies in A (as $F_r(f) \subseteq A$), in $X^\perp$ and in $(\mathrm{span}_E e^*)^\perp$; hence $A = F_r(f) + A_0$ with $A_0 = A \cap Y$, a finitely generated $\mathcal{O}_E$-module. If the E-span of A_0 were not Y, nondegeneracy of Y would give $0 \neq y \in Y$ orthogonal to A_0, and the inner condition would put Ey into A_0, which is impossible for a finitely generated $\mathcal{O}_E$-module. So A_0 is a lattice in Y, and likewise for A'; $\Lambda = u\Lambda_{\mathrm{std}}(A)$ is a lattice in V, and Γ is determined by Λ. A finite-dimensional F-vector space has countably many lattices (each lies between $\varpi^s L_0$ and $\varpi^{-s} L_0$ for a fixed lattice L_0 and some s, and is determined by its image in the finite group $\varpi^{-s}L_0/\varpi^s L_0$). So the data having a term form a countable set, and D is countable.

Step 2: the records. $K_0 \subseteq \Sigma \subseteq \mathrm{U}(V)$ by definition, and $\nu \geq 0$. For $d \in D$, X_d is Σ-stable and, for $x \in X_d$, $\Sigma_x = \mathrm{Stab}_\Sigma(x)$ is commensurable with K_0 with $\mathrm{vol}(\Sigma_{gx}) = \mathrm{vol}(\Sigma_x)$ ($g \in \Sigma$), by Proposition 16.12; so the mass $1/\mathrm{vol}(\Sigma_x)$ of the records is $\mu(x)$. X_d is the union of finitely many Σ-orbits by Proposition 16.11. $|\omega_d| \leq 1$ since ψ_0 is unitary. By Proposition 16.8(3),(5), b_d and c_d take values in $\{q^j : j \geq 0\} \subseteq \mathbb{Z}_{\geq 1}$. By Proposition 16.12, ω_d, b_d, c_d and λ_d are Σ-invariant and X_d^{II} is Σ-stable; λ_d takes values in $\mathbb{Z}^2$. Finally $q \geq 2$ is an integer, a is an integer with $a \geq 0$ and $H \geq 1$, by Proposition 16.8(2) applied to a term of some datum. The total mass of the records is $\sum_{d \in D} \nu(d) \sum_{O \in \Sigma \backslash X_d} \mu(O)$.

(Lg1) and (Lg2). By Proposition 16.8(2),(3), $q^a \leq H$ and $b_d(x) = q^i$ with $0 \leq i \leq a$; by (5), $c_d(x) = q^{i'}$ and $c_d(x) \leq b_d(x)$. A term in $X_d^{\mathrm{n}} \setminus X_d^{\mathrm{II}}$ has $b_d \neq 1$, so it is not generic by (3), and it is not of second type; Proposition 16.10 gives $q\, c_d(x) \leq b_d(x)$.

(Lg3). For $c \geq 1$ the set $\{x \in X_d : c_d(x) = c\}$ is Σ-stable, and its mass is $\sum \mu(\Lambda)$ over the representatives Λ with $c_d(\Lambda) = c$ of any set of representatives of the Σ-orbits of X_d; Proposition 16.11 bounds it by $H^{9(r+1)^2+12(r+1)} c^{k-1}$.

(Lg4). Enlarge D to a set D' of representatives of all H_0-orbits of data, adding for each orbit not met by D an arbitrary datum. Such an orbit contains no datum having a term, so the added data contribute 0 to both sides of Proposition 16.20; for D' and representatives R_d of the Σ-orbits of terms it gives, for every $\beta \geq 1$,

$$\sum_{d \in D} \nu(d)\, \mathrm{mass}_d(\{b_d = \beta\}) \leq 4H^{4(r+1)} \beta^{-(k-1)} \sum_{d \in D} \nu(d)\, \mathrm{mass}_d(X_d),$$

the right side being $4H^{4(r+1)}\beta^{-(k-1)}\mathrm{nn}$.

(Lg5): the labels. A term of second type is not generic, so $X_d^{\mathrm{II}} \subseteq X_d^{\mathrm{n}}$. Let $x = \Lambda \in X_d^{\mathrm{II}}$ with $b_d(\Lambda) = q^i$ and $\lambda_d(\Lambda) = (J, e)$. By Proposition 16.8(4) with $j = 0$, $\ell(y) = 0$ exactly for $y \in M(\Lambda)$; as Λ is not generic some $y \in \mathrm{sat}(\Lambda) \setminus M(\Lambda)$ has $\ell(y) \geq 1$, so $J \geq 1$, and $J \leq r$ as Λ is of second type. Put $S = \mathrm{sat}(\Lambda) \cap (M(\Lambda) + F_{J-1}(f) + F_{J-1}(f'))$, a submodule with $M(\Lambda) \subseteq S \subseteq \mathrm{sat}(\Lambda)$; by (4) it is the set of $y \in \mathrm{sat}(\Lambda)$ with $\ell(y) \leq J - 1$, so it does not contain a vector of level J and $S \neq \mathrm{sat}(\Lambda)$. By Lemma 16.9(1) applied to $S/M(\Lambda)$ and to the nonzero module $\mathrm{sat}(\Lambda)/S$, $[S : M(\Lambda)] = q^e$ (so e is the exponent in the definition of the label) and $q^e \leq q^i/q$; so $0 \leq e \leq i - 1$. (By (4) the level is constant on cosets of $M(\Lambda)$, so exactly $q^i - q^e$ cosets have level J.)

(Lg5): the kernels. For $d \in D$, $i \geq 1$ and $\lambda \in \mathbb{Z}^2$, the set $\mathcal{P}$ of (Lg5) is the set of Proposition 16.30, which provides the kernel m with the required invariance, symmetry, finiteness and the two averaging bounds with $\frac{24}{7} q^{-(k-2r-4)}$. □

Proposition 16.8. *Let a local frame pair with parameters r, k be given, d a datum and Λ a term over d. Then:*

1. *$h(\Lambda, \Lambda) \subseteq \mathcal{O}_E$; the $2r+2$ vectors fr_b, $\theta^{-1}(\mathrm{fr}'_b - \mathrm{fr}_b)$ $(0 \leq b \leq r)$ lie in Λ, are E-linearly independent, span M_E over E, and the determinant of their Gram matrix is $(-1)^{r+1}\theta^{-2(r+1)}G$;*
2. *$a \geq 0$, $D_l \in \mathcal{O}_E$ for all l, $v_D \geq 0$, and $1 \leq q^a \leq H$;*
3. *$\varpi^a \mathrm{sat}(\Lambda) \subseteq M(\Lambda) \subseteq \mathrm{sat}(\Lambda)$ and $b_d(\Lambda) = q^i$ for an integer $0 \leq i \leq a$; Λ is generic if and only if $b_d(\Lambda) = 1$; if Λ is not generic, then $q \leq H$;*
4. *for $y \in \mathrm{sat}(\Lambda)$ and $0 \leq j \leq r + 1$: $\ell(y) \leq j$ if and only if $y \in M(\Lambda) + F_j(f) + F_j(f')$;*
5. *$M(\Lambda) \subseteq \hat{I}(\Lambda) \subseteq \mathrm{sat}(\Lambda)$, and $c_d(\Lambda) = q^{i'}$ for an integer $i' \geq 0$ with $c_d(\Lambda) \leq b_d(\Lambda)$.*

Proof. **Step 1: the translated frames are unitriangular.** For $j < r$, $f_j \in F_{j+1}(f)$, so $uf_j - f_j \in F_j(f)$ by the definition of N; and $f_r \in F_r(f)^\perp$ by (Fp1), so $uf_r - f_r \in F_r(f)$. Hence $\mathrm{fr}_b = \sum_{i \leq b} U_{ib} f_i$ with U upper triangular with ones on the diagonal, and $F_j(\mathrm{fr}) = F_j(f)$ for all j; likewise $\mathrm{fr}' = f'U'$ and $F_j(\mathrm{fr}') = F_j(f')$.

Step 2: integrality and membership. The generators of $\Lambda_{\mathrm{std}}(A)$ are the f_j $(j < r)$, the $y \in A \cap (\mathrm{span}_E e^*)^\perp$ and the e_j^*. Their pairings lie in $\mathcal{O}_E$: $h(f_i, f_j) = 0$ and $h(f_i, e_j^*) = \delta_{ij}$, $h(e_i^*, e_j^*) = 0$ by (Fp1); $h(f_i, y) = 0$ as $y \in A \subseteq X^\perp$; $h(y, e_j^*) = 0$; and for $y, y' \in A$, $y' \in X^\perp$ lies in A, so the inner condition gives $h(A, y') \subseteq \mathcal{O}_E$. By sesquilinearity and hermitian symmetry $h(\Lambda_{\mathrm{std}}(A), \Lambda_{\mathrm{std}}(A)) \subseteq \mathcal{O}_E$, and $\Lambda = u\Lambda_{\mathrm{std}}(A)$ with u unitary is integral. For $j < r$, $\mathrm{fr}_j = uf_j \in \Lambda$; $f_r \in A$ (inner) and $h(f_r, e_j^*) = 0$ by (Fp1), so $f_r \in \Lambda_{\mathrm{std}}(A)$ and $\mathrm{fr}_r \in \Lambda$; finally $\theta^{-1}(\mathrm{fr}'_b - \mathrm{fr}_b) \in \Lambda$ by the witness condition.

Step 3: the Gram determinant. As u, u' are isometries, $(h(\mathrm{fr}_a, \mathrm{fr}_b)) = (h(f_a, f_b)) = -T\,E_{rr}$ and $(h(\mathrm{fr}'_a, \mathrm{fr}'_b)) = -T'E_{rr}$ (E_{rr} the matrix unit), by (Fp1). Put $\tilde{Z}_{ab} = h(\mathrm{fr}'_b, \mathrm{fr}_a)$; by Step 1, $\tilde{Z} = \bar{U}^{\mathsf{T}} Z U'$ with $\bar{U}^{\mathsf{T}}$ lower and U' upper unitriangular, so the leading $l \times l$ block of $\tilde{Z}$ is the product of the leading blocks and $\tilde{Z}$ has the same leading principal minors D_l as Z. The Gram matrix of $(\mathrm{fr}, \mathrm{fr}')$ is

$$P = \begin{pmatrix} -TE_{rr} & \overline{\tilde{Z}} \\ \tilde{Z}^{\mathsf{T}} & -T'E_{rr} \end{pmatrix}.$$

Expand $\det P$ by multilinearity in the two rows of index r in the two blocks, each the sum of its part in $-TE_{rr}$ (resp. $-T'E_{rr}$) and its part in the off-diagonal block. With both off-diagonal parts the matrix is $\begin{pmatrix} 0 & \overline{\tilde{Z}} \\ \tilde{Z}^{\mathsf{T}} & 0 \end{pmatrix}$, of determinant $(-1)^{r+1} \det \overline{\tilde{Z}} \det \tilde{Z}^{\mathsf{T}} = (-1)^{r+1} D_{r+1} \bar{D}_{r+1}$. With

exactly one diagonal part, say the first, the expansion along that row leaves the matrix without the first row and column of index r, in which the $r+1$ rows of the second block are supported on the r remaining columns of the first block, so its determinant vanishes; likewise for the second. With both diagonal parts the expansion gives $(-T)(-T')$ times (sign $+1$, both positions diagonal) the determinant of $\begin{pmatrix} 0 & \overline{\tilde{Z}}_{[r]} \\ \tilde{Z}_{[r]}^{\mathsf{T}} & 0 \end{pmatrix}$, $\tilde{Z}_{[r]}$ the leading $r \times r$ block, that is $(-1)^r TT' D_r \bar{D}_r$. So $\det P = (-1)^{r+1} G$. The family $g = (\mathrm{fr}_b, \theta^{-1}(\mathrm{fr}'_b - \mathrm{fr}_b))_b$ is $(\mathrm{fr}, \mathrm{fr}')C$ with $C = \begin{pmatrix} 1 & -\theta^{-1} \\ 0 & \theta^{-1} \end{pmatrix}$ (blocks of scalar matrices), so its Gram matrix is $C^{\mathsf{T}} P \bar{C}$, of determinant $\det P \cdot \mathrm{N}(\theta^{-(r+1)}) = (-1)^{r+1}\theta^{-2(r+1)}G$, as $\theta \in F$. This is a nonzero element of F, hence a unit of E; if $\sum_i c_i g_i = 0$, then c^{T} times the Gram matrix vanishes and so does c (multiply by the adjugate). So g is independent, and as C is invertible it spans the span of $(\mathrm{fr}, \mathrm{fr}')$, which is $F_{r+1}(f) + F_{r+1}(f') = M_E$ by Step 1. This proves (1).

Step 4: the numbers. The entries of the Gram matrix of g and of $\tilde{Z}$ lie in $\mathcal{O}_E$ by Step 2. By Lemma 16.9(2) (applied below), $v((-1)^{r+1}\theta^{-2(r+1)}G) = v(G) - 2(r+1)v(\theta) = a \geq 0$. The minors D_l of $\tilde{Z}$ lie in $\mathcal{O}_E$, so $D_l \bar{D}_l \in \mathcal{O}_E \cap F = \mathcal{O}_F$ and each summand of v_D is ≥ 0. With $v(\theta) \geq 0$ and $q \geq 2$, $H = q^{v(\theta)+a+v_D} \geq q^a \geq 1$. This proves (2).

Step 5: the defect order. Apply Lemma 16.9(2) to $b = g$ (so $U = M_E$, $M = M(\Lambda)$, $\delta = (-1)^{r+1}\theta^{-2(r+1)}G$) and $N = \mathrm{sat}(\Lambda) = \Lambda \cap M_E$: $M(\Lambda) \subseteq \mathrm{sat}(\Lambda)$ since $g \subseteq \Lambda \cap M_E$, and $h(\mathrm{sat}, \mathrm{sat}) \subseteq \mathcal{O}_E$ by Step 2. It gives $\varpi^a \mathrm{sat}(\Lambda) \subseteq M(\Lambda)$ and $b_d(\Lambda) = q^i$, $0 \leq i \leq a$. Generic means $M(\Lambda) = \mathrm{sat}(\Lambda)$, i.e. index 1. If Λ is not generic, $i \geq 1$, so $a \geq 1$ and $q \leq q^a \leq H$. This proves (3).

Step 6: levels. By Step 3, $(\mathrm{fr}, \mathrm{fr}')$ is an E-basis of M_E, so every $w \in M_E$ has unique coordinates. The set of j for which the condition in the definition of $\ell(y)$ holds is closed upwards and contains $r+1$; so $\ell(y) \leq j$ if and only if it holds at j. Now

$$\theta M(\Lambda) = \Big\{ \sum_b \alpha_b \mathrm{fr}_b + \sum_b \alpha'_b \mathrm{fr}'_b : \ \alpha'_b \in \mathcal{O}_E, \ \alpha_b + \alpha'_b \in \theta\mathcal{O}_E \text{ for all } b \Big\},$$

since $\theta(\beta_b \mathrm{fr}_b + \gamma_b \theta^{-1}(\mathrm{fr}'_b - \mathrm{fr}_b)) = (\theta\beta_b - \gamma_b)\mathrm{fr}_b + \gamma_b \mathrm{fr}'_b$, and conversely $\gamma_b = \alpha'_b$, $\beta_b = \theta^{-1}(\alpha_b + \alpha'_b)$. If the condition holds at j, the part of θy with indices $b \geq j$ lies in $\theta M(\Lambda)$ and the rest in $F_j(\mathrm{fr}) + F_j(\mathrm{fr}')$; if conversely $\theta y = \theta m + w$ with $m \in M(\Lambda)$ and $w \in F_j(\mathrm{fr}) + F_j(\mathrm{fr}')$, the coordinates of θy with $b \geq j$ are those of θm, which satisfy the condition. So $\ell(y) \leq j$ if and only if $\theta y \in \theta M(\Lambda) + F_j(\mathrm{fr}) + F_j(\mathrm{fr}')$, i.e. (dividing by θ and using Step 1) $y \in M(\Lambda) + F_j(f) + F_j(f')$. This proves (4).

Step 7: the class count. Let y be one of the generating vectors of $\hat{I}(\Lambda)$ beyond $M(\Lambda)$, with w, λ, λ' as in the definition. Then $y_1 = \theta^{-1}(\lambda' - \lambda) \in \Lambda$ as $\lambda' - \lambda \in \theta\Lambda$, and $y_1 \in M_E$ as $\lambda \in X$, $\lambda' \in X'$; so $y_1 \in \mathrm{sat}(\Lambda)$ and $y \in M(\Lambda) + y_1 \subseteq \mathrm{sat}(\Lambda)$. Hence $M(\Lambda) \subseteq \hat{I}(\Lambda) \subseteq \mathrm{sat}(\Lambda)$. The quotient $\hat{I}(\Lambda)/M(\Lambda)$ is an $\mathcal{O}_E$-submodule of the finite module $\mathrm{sat}(\Lambda)/M(\Lambda)$ of order q^i; by Lemma 16.9(1) its order is $q^{i'}$, and $q^{i'} \leq q^i$. This proves (5). □

Lemma 16.9. *The notation introduced here is local to this result and to the results that apply it.*

1. *Every $\mathcal{O}_E$-module of finite cardinality has cardinality q^j for an integer $j \geq 0$, and $j \geq 1$ if it is nonzero.*

2. *Let V be a free E-module of finite rank with a hermitian form h (E-linear in the first variable, $h(y,x) = \overline{h(x,y)}$). Let $b_1, \ldots, b_m \in V$ be E-linearly independent with $h(b_i, b_j) \in$*

$\mathcal{O}_E$ for all i, j and $\delta = \det(h(b_i, b_j))_{i,j} \neq 0$, let U and M be their E-span and their $\mathcal{O}_E$-span, and let N be an $\mathcal{O}_E$-submodule with $M \subseteq N \subseteq U$ and $h(N, N) \subseteq \mathcal{O}_E$. Then $\delta \in \mathcal{O}_F$, $a := v(\delta) \geq 0$, $\varpi^a N \subseteq M$, and $[N : M] = q^i$ for an integer $0 \leq i \leq a$.

Proof. **(i).** $\mathcal{O}_E$ is a principal ideal ring: a discrete valuation ring with residue field of q^2 elements and uniformiser ϖ if E is the unramified extension, and $\mathcal{O}_F \times \mathcal{O}_F$ if $E = F \times F$. A finite $\mathcal{O}_E$-module P is finitely generated and torsion. In the first case the structure theorem gives $P \cong \bigoplus_i \mathcal{O}_E/\varpi^{e_i}\mathcal{O}_E$ with $|\mathcal{O}_E/\varpi^e\mathcal{O}_E| = q^{2e}$. In the second, the idempotents $(1,0), (0,1)$ split $P = P_1 \times P_2$ with finite $\mathcal{O}_F$-modules $P_i \cong \bigoplus \mathcal{O}_F/\varpi^e\mathcal{O}_F$, of orders powers of q. So $|P| = q^j$, and $j \geq 1$ if $P \neq 0$.

(ii): the determinant. The Gram matrix $\mathrm{Gm} = (h(b_i, b_j))$ satisfies $\overline{\mathrm{Gm}}^{\mathsf{T}} = \mathrm{Gm}$, so $\bar{\delta} = \det \overline{\mathrm{Gm}}^{\mathsf{T}} = \delta$ and $\delta \in F$; as its entries lie in $\mathcal{O}_E$, $\delta \in \mathcal{O}_E \cap F = \mathcal{O}_F$, and $a = v(\delta) \geq 0$ since $\delta \neq 0$.

(ii): killing. Let $y \in N$; as $y \in U$, $y = \sum_i c_i b_i$ with $c \in E^m$. Since $y, b_j \in N$, the row vector $(h(y, b_j))_j = c^{\mathsf{T}}\mathrm{Gm}$ lies in $\mathcal{O}_E^m$. Multiplying on the right by the adjugate $\mathrm{adj}(\mathrm{Gm}) \in M_m(\mathcal{O}_E)$ gives $\delta\, c^{\mathsf{T}} \in \mathcal{O}_E^m$. Writing $\delta = \varpi^a\varepsilon$ with $\varepsilon \in \mathcal{O}_F^\times$, $\varpi^a c_i \in \mathcal{O}_E$ for all i, so $\varpi^a y \in M$.

(ii): the index. By the previous step $M \subseteq N \subseteq \varpi^{-a}M$, and $\varpi^{-a}M$ is free of rank m with basis $\varpi^{-a}b_i$ (the b_i are independent). Over the principal ideal ring $\mathcal{O}_E$ (componentwise in the split case), the submodule N of a free module of rank m containing M is free of rank m; let $b'_1, \ldots, b'_m$ be a basis and write $b_j = \sum_i A_{ij} b'_i$ with $A \in M_m(\mathcal{O}_E)$. Then $\mathrm{Gm} = A^{\mathsf{T}}\mathrm{Gm}'\bar{A}$ with $\mathrm{Gm}' = (h(b'_k, b'_l))$, hence

$$\delta = \det A \cdot \overline{\det A} \cdot \delta' = \mathrm{N}(\det A)\, \delta', \qquad \delta' = \det \mathrm{Gm}'.$$

As $h(N, N) \subseteq \mathcal{O}_E$, Gm' is an integral hermitian matrix, so $\delta' \in \mathcal{O}_F$ as in the first step, and $\delta' \neq 0$. Thus $v(\mathrm{N}(\det A)) = a - v(\delta') \leq a$, and $v(\mathrm{N}(\det A)) \geq 0$ as $\det A \in \mathcal{O}_E$. By elementary divisors, $[N : M] = |\mathcal{O}_E^m/A\mathcal{O}_E^m| = |\mathcal{O}_E/\det A\, \mathcal{O}_E|$. If E is a field, $\det A = \varpi^e\eta$ with $\eta \in \mathcal{O}_E^\times$, $|\mathcal{O}_E/\varpi^e\mathcal{O}_E| = q^{2e}$ and $v(\mathrm{N}(\det A)) = 2e$. If $E = F \times F$, $\det A = (x, y)$ with $x, y \neq 0$, $|\mathcal{O}_E/\det A\, \mathcal{O}_E| = q^{v(x)+v(y)}$ and $\mathrm{N}(\det A) = xy$. In both cases $[N : M] = q^i$ with $i = v(\mathrm{N}(\det A)) \in [0, a]$. □

Proposition 16.10. *Let a local frame pair with parameters r, k be given, d a datum and Λ a term over d that is neither generic nor of second type. Then $q\, c_d(\Lambda) \leq b_d(\Lambda)$.*

Proof. **Step 1: the frames f, f' are independent.** For $j < r$, $f_j \in F_{j+1}(f)$, so $uf_j - f_j \in F_j(f)$ by the definition of N; and $f_r \in F_r(f)^\perp$ by (Fp1), so $uf_r - f_r \in F_r(f)$. Hence $(\mathrm{fr}_0, \ldots, \mathrm{fr}_r) = (f_0, \ldots, f_r)U$ with U upper unitriangular, and likewise $\mathrm{fr}' = f'U'$. By Proposition 16.8(1) the vectors $g_b = \mathrm{fr}_b$, $g_{r+1+b} = \theta^{-1}(\mathrm{fr}'_b - \mathrm{fr}_b)$ form an E-basis of M_E; as (f, f') is obtained from g by the invertible matrix $\begin{pmatrix} 1 & -\theta^{-1} \\ 0 & \theta^{-1} \end{pmatrix}^{-1} \mathrm{diag}(U^{-1}, U'^{-1})$ (blocks of size $r+1$), the $2r+2$ vectors f_b, f'_b form an E-basis of M_E. In particular every vector of M_E is uniquely a sum of a vector of $F_{r+1}(f)$ and one of $F_{r+1}(f')$, and $F_{r+1}(f) \cap F_{r+1}(f') = 0$. Also X and X' are totally isotropic ($h(f_a, f_b) = 0$ for $a, b < r$ by (Fp1)), $W_0 \perp M_E$ by definition, and $h(\Lambda, \Lambda) \subseteq \mathcal{O}_E$ by Proposition 16.8(1). Since $\theta \in F$, $h(v, \theta w) = \theta h(v, w)$.

Step 2: the module R of glued vectors. Let R be the set of vectors $\theta^{-1}(\lambda' - \lambda)$ for which there are $w \in W_0$, $\lambda \in X$, $\lambda' \in X'$ with $w + \lambda \in \Lambda$ and $\lambda' - \lambda \in \theta\Lambda$. Componentwise sums and $\mathcal{O}_E$-multiples of such triples (w, λ, λ') are again such triples, so R is an $\mathcal{O}_E$-submodule; $R \subseteq \Lambda$,

and $R \subseteq X + X'$. Every generator y of $\hat{I}(\Lambda)$ beyond M comes with $w \in W_0$, $\lambda \in X$, $\lambda' \in X'$, $w + \lambda \in \Lambda$, $\lambda' - \lambda \in \theta\Lambda$ and $y - \theta^{-1}(\lambda' - \lambda) \in M$, so $y \in M + R$; hence

$$\hat{I}(\Lambda) \subseteq M + R.$$

Step 3: the pairing condition on R. Let $y_1, y_2 \in R$ with triples $(w_i, \lambda_i, \lambda_i')$, and let $x_i \in X$, $x_i' \in X'$ with $x_i + x_i' = \theta y_i$ $(i = 1, 2)$. Then $x_i + \lambda_i = \lambda_i' - x_i' \in F_{r+1}(f) \cap F_{r+1}(f') = 0$, so $x_i = -\lambda_i$ and $x_i' = \lambda_i'$. As $w_1 \perp M_E$ and $h(\lambda_1, \lambda_2) = 0$,

$$h(x_1, x_2') = -h(\lambda_1, \lambda_2') = -h(w_1 + \lambda_1, \lambda_2' - \lambda_2) = -\theta\, h(w_1 + \lambda_1, y_2) \in \theta\mathcal{O}_E,$$

since $w_1 + \lambda_1$ and y_2 lie in Λ.

Step 4: if $\hat{I}(\Lambda) = \mathrm{sat}(\Lambda)$, then Λ is of second type. Assume $\hat{I}(\Lambda) = \mathrm{sat}(\Lambda)$; so $\mathrm{sat}(\Lambda) \subseteq M + R$ by Step 2. Λ is not generic by hypothesis.

Levels. For $y \in \mathrm{sat}(\Lambda)$, $y \in M + R \subseteq M + F_r(f) + F_r(f')$, so $\ell(y) \leq r$ by Proposition 16.8(4).

Pairing. Let $x_i \in X$, $x_i' \in X'$ with $x_i + x_i' = \theta z_i$, $z_i \in \Lambda$ $(i = 1, 2)$. As $X, X' \subseteq M_E$, $z_i \in \mathrm{sat}(\Lambda)$, so $z_i = m_i + y_i$ with $m_i \in M$, $y_i \in R$; write $y_i = u_i + u_i'$ with $u_i = -\theta^{-1}\lambda_i \in X$, $u_i' = \theta^{-1}\lambda_i' \in X'$ for a triple of y_i. From $m = \sum_b \beta_b \mathrm{fr}_b + \sum_b \gamma_b \theta^{-1}(\mathrm{fr}_b' - \mathrm{fr}_b)$ $(\beta_b, \gamma_b \in \mathcal{O}_E)$ one gets $\theta m = \sum_b (\theta\beta_b - \gamma_b)\mathrm{fr}_b + \sum_b \gamma_b \mathrm{fr}_b'$, so $\theta m_i = c_i + c_i'$ with $c_i \in F_{r+1}(f)$ and $c_i' \in F_{r+1}(f') \cap \Lambda$ (by Proposition 16.8(1), $\mathrm{fr}_b' = \mathrm{fr}_b + \theta \cdot \theta^{-1}(\mathrm{fr}_b' - \mathrm{fr}_b) \in \Lambda$). Put $a_i = x_i - \theta u_i \in X$ and $a_i' = x_i' - \theta u_i' \in X'$; then $a_i + a_i' = \theta m_i = c_i + c_i'$, and the uniqueness of Step 1 gives $a_i' = c_i' \in \Lambda$. Now

$$h(x_1, x_2') = h(\theta u_1, \theta u_2') + \theta h(u_1, a_2') + \theta h(a_1, u_2') + h(a_1, a_2'),$$

and each term lies in $\theta\mathcal{O}_E$:

- $\theta u_i \in X$, $\theta u_i' \in X'$ and $\theta u_i + \theta u_i' = \theta y_i$ with $y_i \in R$, so Step 3 gives $h(\theta u_1, \theta u_2') \in \theta\mathcal{O}_E$;
- $h(u_1, a_2') = h(y_1, a_2')$ since $h(u_1', a_2') = 0$, and $y_1, a_2' \in \Lambda$;
- $h(a_1, u_2') = h(a_1, y_2)$ since $h(a_1, u_2) = 0$, and $h(a_1, y_2) = \theta h(m_1, y_2) - h(a_1', y_2) \in \mathcal{O}_E$ as $a_1 = \theta m_1 - a_1'$ and $m_1, a_1', y_2 \in \Lambda$;
- $h(a_1, a_2') = \theta h(m_1, a_2') - h(a_1', a_2') = \theta h(m_1, a_2') \in \theta\mathcal{O}_E$.

So $h(x_1, x_2') \in \theta\mathcal{O}_E$, and Λ is of second type.

Step 5: the index. By hypothesis Λ is not of second type, so $\hat{I}(\Lambda) \neq \mathrm{sat}(\Lambda)$ by Step 4. By Proposition 16.8(3),(5), $M \subseteq \hat{I}(\Lambda) \subseteq \mathrm{sat}(\Lambda)$ and $b_d(\Lambda) = [\mathrm{sat}(\Lambda) : M]$ is finite, so

$$b_d(\Lambda) = [\mathrm{sat}(\Lambda) : \hat{I}(\Lambda)] \cdot [\hat{I}(\Lambda) : M] = [\mathrm{sat}(\Lambda) : \hat{I}(\Lambda)] \cdot c_d(\Lambda).$$

The quotient $\mathrm{sat}(\Lambda)/\hat{I}(\Lambda)$ is a nonzero finite $\mathcal{O}_E$-module, of order at least q by Lemma 16.9(1). Hence $q\, c_d(\Lambda) \leq b_d(\Lambda)$. □

Proposition 16.11. *Let a local frame pair with parameters r, k be given and d a datum. Then the terms over d form finitely many Σ-orbits, and for every set R of representatives of these orbits and every integer $c \geq 1$,*

$$\sum_{\Lambda \in R,\ c_d(\Lambda) = c} \mu(\Lambda) \leq H^{9(r+1)^2 + 12(r+1)}\, c^{\,k-1}.$$

Proof. If d has no term, both claims hold trivially. Let Λ_0 be a term over d. By Proposition 16.8(2), $a \ge 0$, $v_D \ge 0$ and $H = q^{v(\theta)+a+v_D} \ge 1$; hence $q^a \le H$ and $q^{v(\theta)+v_D} \le H$. Let $\mathcal{T}$ be the set of terms over d and, for $c \ge 1$, $\mathcal{T}_c = \{\Lambda \in \mathcal{T} : c_d(\Lambda) = c\}$; by Proposition 16.12 both are Σ-stable, μ is constant on Σ-orbits, and $\mathrm{Fr}\colon \mathcal{T} \to \{\text{integral frame classes}\}$ is Σ-equivariant. By Proposition 16.13 the integral frame classes form finitely many Σ-orbits; let R_{cl} be a set of representatives, and for $C \in R_{\mathrm{cl}}$ let $\mathcal{F}_C = \{\Lambda \in \mathcal{T}_c : \mathrm{Fr}(\Lambda) = C\}$.

Step 1: the fibre formula. By Lemma 16.19 applied to $\mathcal{T}' = \mathcal{T}_c$,

$$\sum_{O \in \Sigma\backslash\mathcal{T}_c} \mu(O) = \sum_{C \in R_{\mathrm{cl}}} w(C)\,|\mathcal{F}_C| \qquad \text{in } [0,\infty].$$

Step 2: the bound. By Proposition 16.16, $|\mathcal{F}_C| \le q^{10a(r+1)} c^{k-1}$ for every C (the bound is trivial if $\mathcal{F}_C$ is empty). With Proposition 16.13 and Step 1, for any set R of representatives of $\Sigma\backslash\mathcal{T}$,

$$\begin{aligned}\sum_{\Lambda \in R,\ c_d(\Lambda)=c} \mu(\Lambda) = \sum_{O \in \Sigma\backslash\mathcal{T}_c} \mu(O) &\le q^{r(r+1)(v(\theta)+v_D)}\, q^{10a(r+1)}\, c^{k-1} \\ &\le H^{r(r+1)+10(r+1)} c^{k-1} \le H^{9(r+1)^2+12(r+1)} c^{k-1},\end{aligned}$$

using $q^{v(\theta)+v_D} \le H$, $q^a \le H$ and $H \ge 1$.

Step 3: finitely many orbits. By Proposition 16.8(3),(5), $c_d(\Lambda) \in \{q^i : 0 \le i \le a\}$ for every term, so $\mathcal{T}$ is the union of finitely many $\mathcal{T}_c$. Every Σ-orbit of $\mathcal{T}_c$ meets $\mathcal{F}_C$ for the representative $C \in R_{\mathrm{cl}}$ of the orbit of its classes (Fr is equivariant), and the finitely many C and the finite sets $\mathcal{F}_C$ (Proposition 16.16) leave finitely many such orbits; so $\mathcal{T}$ is the union of finitely many Σ-orbits. □

Proposition 16.12. *Let a local frame pair with parameters r, k be given, let d be a datum, let Λ be a term over d and let $g \in \Sigma$. Then $g\Lambda$ is a term over d. For every choice of the fixed witnesses of Λ and of $g\Lambda$, we have $\omega_d(g\Lambda) = \omega_d(\Lambda)$, $b_d(g\Lambda) = b_d(\Lambda)$, $c_d(g\Lambda) = c_d(\Lambda)$, $g\Lambda$ is of second type if and only if Λ is, and then $\lambda_d(g\Lambda) = \lambda_d(\Lambda)$. The class $\mathrm{Fr}(\Lambda)$ does not depend on the choice of the witness and is an integral frame class; $M(p,p') = M(\Lambda)$ for every $(p,p') \in \mathrm{Fr}(\Lambda)$, and $\mathrm{Fr}(g\Lambda) = g\,\mathrm{Fr}(\Lambda)$. For every frame class C, the group $\mathrm{Stab}_\Sigma(C)$ is commensurable with K_0 and $w(gC) = w(C)$. Finally, $\mathrm{Stab}_\Sigma(\Lambda)$ is commensurable with K_0, that is, both indices $[\mathrm{Stab}_\Sigma(\Lambda) : \mathrm{Stab}_\Sigma(\Lambda) \cap K_0]$ and $[K_0 : \mathrm{Stab}_\Sigma(\Lambda) \cap K_0]$ are finite, and $\mathrm{vol}_{K_0}(\mathrm{Stab}_\Sigma(g\Lambda)) = \mathrm{vol}_{K_0}(\mathrm{Stab}_\Sigma(\Lambda))$.*

Proof. **Step 0: the groups N, N', Σ.** (a) For $u \in N$ and $j < r$, $uf_j - f_j \in F_j(f)$, and $uf_r - f_r \in F_r(f)$ as $f_r \in F_r(f)^\perp$ by (Fp1); so $uf_b = \sum_{i \le b} s(u)_{ib} f_i$ with $s(u)$ upper unitriangular. From $uvf = (uf)s(v) = fs(u)s(v)$, $s(uv) = s(u)s(v)$, whose superdiagonal is $s(u)_{j,j+1} + s(v)_{j,j+1}$. By (Fp1), $h(f_i, e_j^*) = \delta_{ij}$, so $h(uf_{j+1}, e_j^*) = s(u)_{j,j+1}$ and $\psi_N(u) = \psi_0(\theta^{-1} \sum_{j<r} s(u)_{j,j+1})$; hence ψ_N is a unitary character of N, and likewise $\psi_{N'}(u') = \psi_0(\theta^{-1} \sum_j s'(u')_{j,j+1})$ with $u'f' = f's'(u')$. (b) As Λ is a term, Proposition 16.8(1) gives an E-basis of M_E of $2r+2$ vectors with nonzero Gram determinant; so M_E is nondegenerate, $V = M_E \oplus W_0$, and, the translated frames being f, f' times unitriangular matrices by (a), $f_0, \dots, f_r, f'_0, \dots, f'_r$ is an E-basis of M_E; in particular $F_{r+1}(f) \cap F_{r+1}(f') = 0$. (c) Every $g \in \Sigma$ preserves each $F_j(f)$ and $F_j(f')$ $(j \le r+1)$, hence M_E, and is an isometry, hence preserves $X^\perp$, $X'^\perp$ and W_0. For $w \in W_0 \subseteq X^\perp \cap X'^\perp$, $gw - w \in X \cap X' = 0$: g fixes W_0 pointwise. In the basis (f, f') of M_E, g acts by $\mathrm{diag}(s(g), s'(g))$; so $(g-1)^{r+1} = 0$ on V, and g is determined by $(s(g), s'(g))$.

Step 1: $g\Lambda$ is a term over d. Let (u, u') be a witness of Λ. Then $gu \in N$, $gu' \in N'$, $g\Lambda = gu\Lambda_{\mathrm{std}}(A) = gu'\Lambda'_{\mathrm{std}}(A')$ and $gu'f'_b - guf_b = g(u'f'_b - uf_b) \in \theta g\Lambda$. The set of pairs (p, p')

attached to $g\Lambda$ in the definition of a term is Γ: if $y, y' \in \Lambda$ satisfy the conditions for Λ, then $gy, gy' \in g\Lambda$, $gy \perp X$, $gy' \perp X'$ (Step 0(c)), $gy' - p = (gy' - y') + (y' - p) \in X'$ since $gy' - y' \in X'$ for $y' \in X'^{\perp}$ (definition of N'), likewise $gy - p' \in X$, and $gy' - gy \in \theta g\Lambda$; the converse follows with g^{-1} and $g\Lambda$. So (gu, gu') witnesses that $g\Lambda$ is a term over d.

Step 2: the invariants do not depend on the witness. First, $\Lambda_{\mathrm{std}}(A) \cap F_{r+1}(f) = \sum_b \mathcal{O}_E f_b$. Indeed $f_j \in \Lambda_{\mathrm{std}}(A)$ for $j < r$, and $f_r \in A$ (inner) with $h(f_r, e_j^*) = 0$, so $f_r \in \Lambda_{\mathrm{std}}(A)$. Conversely let $x = \sum_b c_b f_b \in \Lambda_{\mathrm{std}}(A)$. The generators of $\Lambda_{\mathrm{std}}(A)$ pair integrally with each other ($h(f_i, f_j) = 0$, $h(f_i, e_j^*) = \delta_{ij}$, $h(e_i^*, e_j^*) = 0$ by (Fp1), $A \perp X$, $A \cap (X^*)^{\perp} \perp e^*$, and $h(A, y) \subseteq \mathcal{O}_E$ for $y \in A \subseteq X^{\perp}$ by the inner condition); so $c_j = h(x, e_j^*) \in \mathcal{O}_E$ for $j < r$. Take $y \in A$ with $h(f_r, y) = 1$; $y_0 = y - \sum_j h(y, e_j^*) f_j \in A \cap (X^*)^{\perp} \subseteq \Lambda_{\mathrm{std}}(A)$ and $h(f_r, y_0) = 1$, so $c_r = h(c_r f_r, y_0) = h(x - \sum_{j<r} c_j f_j, y_0) \in \mathcal{O}_E$. Now let (u, u') and (v, v') be witnesses of Λ, and $\mathsf{A} = s(u^{-1}v)$, $\mathsf{A}' = s'(u'^{-1}v')$, so $vf_j = \sum_i \mathsf{A}_{ij} u f_i$ and $v' f'_j = \sum_i \mathsf{A}'_{ij} u' f'_i$. As $vf_j \in v\Lambda_{\mathrm{std}}(A) = \Lambda = u\Lambda_{\mathrm{std}}(A)$, $\sum_i \mathsf{A}_{ij} f_i \in \Lambda_{\mathrm{std}}(A) \cap F_{r+1}(f)$, so A is integral; likewise A'. With $y_i = \theta^{-1}(u' f'_i - u f_i)$ and $z_j = \theta^{-1}(v' f'_j - v f_j)$, both in Λ,

$$z_j - \sum_i \mathsf{A}'_{ij} y_i = \sum_i \theta^{-1}(\mathsf{A}'_{ij} - \mathsf{A}_{ij})\, u f_i \in \Lambda,$$

so $\theta^{-1}(\mathsf{A}' - \mathsf{A})$ is integral. Hence $vf_j \in M_u$ and $z_j = \sum_i \mathsf{A}'_{ij} y_i + \sum_i \theta^{-1}(\mathsf{A}'_{ij} - \mathsf{A}_{ij}) u f_i \in M_u$, where M_u, M_v are the joint frame lattices of the two witnesses; so $M_v \subseteq M_u$, and by symmetry $M_u = M_v$. By Step 0(a),

$$\frac{\psi_N(v)}{\psi_{N'}(v')} = \frac{\psi_N(u)}{\psi_{N'}(u')}\, \psi_0\Big(\theta^{-1} \sum_{j<r} (\mathsf{A}_{j,j+1} - \mathsf{A}'_{j,j+1})\Big) = \frac{\psi_N(u)}{\psi_{N'}(u')},$$

since the argument lies in $\mathcal{O}_E$ and ψ_0 has conductor $\mathcal{O}_E$. So $\omega_d(\Lambda)$ and $M(\Lambda)$ do not depend on the witness. By Proposition 16.8(4) (for either witness), for $y \in \mathrm{sat}(\Lambda)$ the level is determined by $\ell(y) \le j \iff y \in M(\Lambda) + F_j(f) + F_j(f')$; and $\mathrm{sat}(\Lambda)$, $\hat{I}(\Lambda)$ (built from $M(\Lambda)$ and witness-free data), b_d, c_d, genericity, the second type and the label are determined by Λ, $M(\Lambda)$ and the levels on $\mathrm{sat}(\Lambda)$. So all of them are independent of the witness, and below the witness (gu, gu') of Step 1 may be used for $g\Lambda$.

Step 3: $\psi_N = \psi_{N'}$ on Σ. Let $g \in \Sigma$, $s = s(g)$, $s' = s'(g)$. As g is an isometry, $h(gf'_b, gf_a) = Z_{ab}$, i.e. $\bar{s}^{\mathsf{T}} Z s' = Z$; with $\mathsf{L} = \bar{s}^{\mathsf{T}}$ (lower unitriangular) and $\mathsf{C} = s'^{-1}$ (upper unitriangular), $\mathsf{L}Z = Z\mathsf{C}$. By (Fp2), $Z_{a\sigma(a)} = z_a \neq 0$ and $Z_{ab} = 0$ for $b \neq \sigma(a)$, where σ is the involution reversing each block; within a block $\sigma(c+1) = \sigma(c) - 1$ and $z_{c+1} = -z_c$ (the sign $(-1)^{r-b}$ changes, t_i does not), and for c, $c+1$ in consecutive blocks $\sigma(c) < \sigma(c+1)$. Comparing the entries $(a, \sigma(c))$ of $\mathsf{L}Z = Z\mathsf{C}$ gives

$$\mathsf{L}_{ac}\, z_c = z_a\, \mathsf{C}_{\sigma(a)\sigma(c)} \qquad (0 \le a, c \le r).$$

For $c, c+1$ in one block, $a = c+1$ gives $\mathsf{L}_{c+1,c} = -\mathsf{C}_{\sigma(c)-1,\sigma(c)}$; for $c, c+1$ in different blocks, $\mathsf{C}_{\sigma(c+1)\sigma(c)} = 0$ (C upper) gives $\mathsf{L}_{c+1,c} = 0$; and for $b, b+1$ in different blocks, $a = \sigma(b)$, $c = \sigma(b+1)$ give $\mathsf{C}_{b,b+1} = 0$ (L lower, $\sigma(b) < \sigma(b+1)$). As $c \mapsto \sigma(c) - 1$ is a bijection between the within-block positions, $\sum_c \mathsf{L}_{c+1,c} = -\sum_b \mathsf{C}_{b,b+1} = \sum_b s'_{b,b+1}$ (the superdiagonal of the inverse of a unitriangular matrix is the negative of its superdiagonal). So $\sum_j s'_{j,j+1} = \overline{\sum_j s_{j,j+1}}$, and since $\theta \in F$ and $\psi_0(\bar{x}) = \psi_0(x)$, $\psi_{N'}(g) = \psi_N(g)$.

Step 4: the invariants of $g\Lambda$. With the witness (gu, gu'): $\mathrm{fr}_b(g\Lambda) = g\,\mathrm{fr}_b(\Lambda)$, so $M(g\Lambda) = gM(\Lambda)$; $\mathrm{sat}(g\Lambda) = g\Lambda \cap M_E = g\,\mathrm{sat}(\Lambda)$ (Step 0(c)). If y is a generator of $\hat{I}(\Lambda)$ with (w, λ, λ'), then gy is one of $\hat{I}(g\Lambda)$ with $(w, g\lambda, g\lambda')$: $gw = w$, $w + g\lambda = g(w + \lambda) \in g\Lambda$, $w + g\lambda' \in g\Lambda$, $g\lambda' - g\lambda \in \theta g\Lambda$, $g\lambda \in X$, $g\lambda' \in X'$ and $gy - \theta^{-1}(g\lambda' - g\lambda) \in gM(\Lambda)$; with g^{-1}, $\hat{I}(g\Lambda) = g\hat{I}(\Lambda)$.

As g is an additive bijection, $b_d(g\Lambda) = b_d(\Lambda)$ and $c_d(g\Lambda) = c_d(\Lambda)$, and $g\Lambda$ is generic if and only if Λ is. By Proposition 16.8(4) and Step 0(c), for $y \in \mathrm{sat}(\Lambda)$, $\ell(gy) \le j \iff gy \in g(M(\Lambda) + F_j(f) + F_j(f')) \iff \ell(y) \le j$; so the maximal level J is the same, and g maps $\mathrm{sat}(\Lambda) \cap (M(\Lambda) + F_{J-1}(f) + F_{J-1}(f'))$ onto the corresponding module of $g\Lambda$, so the labels agree. For the pairing condition, if $x_i \in X$, $x_i' \in X'$ and $x_i + x_i' \in \theta g\Lambda$, then $g^{-1}x_i \in X$, $g^{-1}x_i' \in X'$, $g^{-1}x_i + g^{-1}x_i' \in \theta\Lambda$ and $h(x_1, x_2') = h(g^{-1}x_1, g^{-1}x_2')$; so $g\Lambda$ is of second type if and only if Λ is. Finally, by Steps 0(a) and 3,

$$\omega_d(g\Lambda) = \frac{\psi_N(g)\psi_N(u)}{\psi_{N'}(g)\psi_{N'}(u')} = \omega_d(\Lambda).$$

Step 5: commensurability. Λ *is a lattice.* Put $Y = (X + X^*)^\perp$; $X + X^*$ is nondegenerate by (Fp1), so $V = X \oplus X^* \oplus Y$ with Y nondegenerate. $A_0 = A \cap (X^*)^\perp \subseteq X^\perp \cap (X^*)^\perp = Y$ is finitely generated (inner), and for $w \in A$, $w - \sum_j h(w, e_j^*) f_j \in A_0$, so $A = X + A_0$ (as $X = F_r(f) \subseteq A$). If A_0 did not span Y, some $0 \ne y \in Y$ would be orthogonal to A_0 and to X, hence $h(A, Ey) = 0$, and the inner condition would give $Ey \subseteq A \cap (X^*)^\perp = A_0$, impossible for a finitely generated module. So $\Lambda_{\mathrm{std}}(A)$, and $\Lambda = u\Lambda_{\mathrm{std}}(A)$, are finitely generated $\mathcal{O}_E$-modules spanning V (lattices). *Stabilisers of two lattices are commensurable.* For lattices Λ_1, Λ_2 choose $t \ge 0$ with $\varpi^t\Lambda_1 \subseteq \Lambda_2 \subseteq \varpi^{-t}\Lambda_1$. The kernel Γ_1 of $\mathrm{Stab}(\Lambda_1) \to \mathrm{Aut}(\Lambda_1/\varpi^{2t}\Lambda_1)$ has finite index (the quotient is finite), and for $x \in \Lambda_2$, $(g-1)x \in (g-1)\varpi^{-t}\Lambda_1 \subseteq \varpi^t\Lambda_1 \subseteq \Lambda_2$ ($g \in \Gamma_1$); with $g^{-1} \in \Gamma_1$, $\Gamma_1 \subseteq \mathrm{Stab}(\Lambda_2)$. Hence $[\Sigma \cap \mathrm{Stab}(\Lambda_1) : \Sigma \cap \mathrm{Stab}(\Lambda_1) \cap \mathrm{Stab}(\Lambda_2)] \le [\mathrm{Stab}(\Lambda_1) : \Gamma_1] < \infty$, and symmetrically. K_0 *is a lattice stabiliser.* Let L_W be a lattice in W_0 and $\Lambda_1 = \sum_b \mathcal{O}_E f_b + \sum_b \mathcal{O}_E f_b' + L_W$, a lattice by Step 0(b). For $g \in \Sigma$, by Step 0(c) $g\Lambda_1 = \Lambda_1$ if and only if $s(g)$ and $s'(g)$ are integral (then so are their inverses), if and only if g stabilises $\sum \mathcal{O}_E f_b$ and $\sum \mathcal{O}_E f_b'$. So $K_0 = \Sigma \cap \mathrm{Stab}(\Lambda_1)$, and $\mathrm{Stab}_\Sigma(\Lambda)$ is commensurable with K_0. *Volumes.* For pairwise commensurable subgroups put $[B_1 : B_2] = [B_1 : B_1 \cap B_2]/[B_2 : B_1 \cap B_2]$; computing both through $B_1 \cap B_2 \cap B_3$, which has finite index in each, gives $[B_1 : B_3] = [B_1 : B_2][B_2 : B_3]$, and $[gB_1g^{-1} : gB_2g^{-1}] = [B_1 : B_2]$; $\mathrm{vol}_{K_0}(B) = [B : K_0]$. Let $g \in \Sigma$ and $B = \mathrm{Stab}_\Sigma(\Lambda)$, so $\mathrm{Stab}_\Sigma(g\Lambda) = gBg^{-1}$. By Step 0(c), $\mathsf{n} = g - 1$ is nilpotent, so for any lattice Λ_0 the lattice $\Lambda_g = \sum_{i \le r} \mathsf{n}^i \Lambda_0$ is stable under $g = 1 + \mathsf{n}$ and $g^{-1} = \sum_{i \le r} (-\mathsf{n})^i$; then $B_g = \mathrm{Stab}_\Sigma(\Lambda_g)$ is commensurable with K_0 and $gB_gg^{-1} = B_g$. Hence

$$\mathrm{vol}_{K_0}(gBg^{-1}) = [gBg^{-1} : gB_gg^{-1}]\,[B_g : K_0] = [B : B_g]\,[B_g : K_0] = \mathrm{vol}_{K_0}(B).$$

The same computation applies to every subgroup $B \subseteq \Sigma$ commensurable with K_0.

Step 6: frame classes. The pairs (B, B') of integral upper unitriangular matrices with $B' - B \in \theta \mathrm{M}_{r+1}(\mathcal{O}_E)$ form a group Γ_θ under componentwise multiplication ($B'C' - BC = (B' - B)C' + B(C' - C)$, and $B'^{-1} - B^{-1} = B'^{-1}(B - B')B^{-1}$, inverses of integral unitriangular matrices being integral), acting on frame pairs on the right; the classes are its orbits. By Step 2, two witnesses give frame pairs in one orbit, so $\mathrm{Fr}(\Lambda)$ does not depend on the witness; with the witness (gu, gu'), $\mathrm{Fr}(g\Lambda) = \mathrm{cl}(guf, gu'f') = g\,\mathrm{Fr}(\Lambda)$ (g is linear). $\mathrm{Fr}(\Lambda)$ is the frame class of (u, u'), and it is integral: its pairs $(\mathrm{fr}B, \mathrm{fr}'B')$ consist of vectors of the integral lattice Λ (Proposition 16.8(1)). For $(p, p') = (\mathrm{fr}B, \mathrm{fr}'B') \in \mathrm{Fr}(\Lambda)$, $p_j = \sum_i B_{ij}\mathrm{fr}_i$ and $\theta^{-1}(p_j' - p_j) = \sum_i B_{ij}'\theta^{-1}(\mathrm{fr}_i' - \mathrm{fr}_i) + \sum_i \theta^{-1}(B_{ij}' - B_{ij})\mathrm{fr}_i$ lie in $M(\Lambda)$, so $M(p, p') \subseteq M(\Lambda)$; as $(\mathrm{fr}, \mathrm{fr}') = (p, p')(B, B')^{-1}$ with $(B, B')^{-1} \in \Gamma_\theta$, also $M(\Lambda) \subseteq M(p, p')$. Now let $C = \mathrm{cl}(p, p')$ with $p = uf$, $p' = u'f'$, $u \in N$, $u' \in N'$. By Step 0, $(p, p') = (fs(u), f's'(u'))$ is an E-basis of M_E, and $g \in \Sigma$ acts by $gp = p\tilde{s}(g)$, $gp' = p'\tilde{s}'(g)$ with $\tilde{s}(g) = s(u)^{-1}s(g)s(u)$, $\tilde{s}'(g) = s'(u')^{-1}s'(g)s'(u')$ unitriangular. As p, p' are bases, $gC = \mathrm{cl}(p\tilde{s}, p'\tilde{s}') = C$ if and only if $(\tilde{s}(g), \tilde{s}'(g)) \in \Gamma_\theta$. Put $\Lambda_C = \sum_b \mathcal{O}_E p_b + \sum_b \mathcal{O}_E p_b' + L_W$; as in Step 5, $g\Lambda_C = \Lambda_C$ if and only if $\tilde{s}(g)$, $\tilde{s}'(g)$ are integral. Hence $\mathrm{Stab}_\Sigma(C) \subseteq \mathrm{Stab}_\Sigma(\Lambda_C)$, and it contains the elements of $\mathrm{Stab}_\Sigma(\Lambda_C)$ acting trivially

on the finite module $\Lambda_C/\theta\Lambda_C$ (then $\tilde{s} \equiv 1 \equiv \tilde{s}'$ modulo θ), a subgroup of finite index. So $\mathrm{Stab}_\Sigma(C)$ is commensurable with the lattice stabiliser $\mathrm{Stab}_\Sigma(\Lambda_C)$, hence with K_0 (Step 5); and as $\mathrm{Stab}_\Sigma(gC) = g\,\mathrm{Stab}_\Sigma(C)g^{-1}$, the volume computation of Step 5 gives $w(gC) = w(C)$. □

Proposition 16.13. *Let a local frame pair with parameters r, k be given such that some datum has a term. Then $w(C)$ is a positive integer for every integral frame class C, the integral frame classes form finitely many Σ-orbits, and for every set R of representatives of these orbits*

$$\sum_{C\in R} w(C) \le q^{\,r(r+1)(v(\theta)+v_D)}.$$

Proof. **Step 1: stabilisers of classes stabilise lattices.** If $g \in \Sigma$ and $g\,\mathrm{cl}(p,p') = \mathrm{cl}(p,p')$, then $(gp, gp') \in \mathrm{cl}(p,p')$, so $gp = pB$ with $B \in \mathrm{U}(\mathcal{O})$ and $g\Lambda_p \subseteq \Lambda_p$; the same for g^{-1} gives $g\Lambda_p = \Lambda_p$.

Step 2: a representative in each orbit. Let $C = \mathrm{cl}(nf, n'f')$ be an integral frame class, $n \in N$, $n' \in N'$. Take h, u as in Lemma 16.14(3) and put $u' = s'(h^{-1}n')$ (upper unitriangular, $h^{-1}n' \in N'$). Then $C^\natural := h^{-1}C = \mathrm{cl}(fu, f'u')$ (h^{-1} is linear), and $C^\natural$ is integral because h^{-1} is an isometry. If $g \in \mathrm{Stab}_\Sigma(C^\natural)$, then $hgh^{-1} \in \mathrm{Stab}_\Sigma(C)$ stabilises Λ_{nf} by Step 1, so $g = h^{-1}(hgh^{-1})h \in K_0$ by Lemma 16.14(3). Thus $\mathrm{Stab}_\Sigma(C^\natural) \subseteq K_0$, and by Proposition 16.12 this group is commensurable with K_0 and $w(C) = w(C^\natural)$. Hence

$$w(C) = \frac{[K_0 : \mathrm{Stab}_\Sigma(C^\natural)]}{[\mathrm{Stab}_\Sigma(C^\natural) : \mathrm{Stab}_\Sigma(C^\natural)]} = [K_0 : \mathrm{Stab}_{K_0}(C^\natural)] = |K_0 C^\natural|.$$

In particular $w(C)$ is a positive integer: the index is finite by the commensurability, and $K_0C^\natural \ni C^\natural$ is nonempty.

Step 3: the K_0-orbit of $C^\natural$ lies in a set of normal forms. Let T be the set of integral frame classes of the form $\mathrm{cl}(fx, f'x')$ with x, x' upper unitriangular and $x_{ij} \in \mathcal{O}_E$ whenever i, j lie in one block. For $g \in K_0$, $gC^\natural = \mathrm{cl}(fs(g)u, f's'(g)u')$ is integral; $s(g)$ is block diagonal (Lemma 16.14(1)) and integral (as $g\Lambda_f = \Lambda_f$ and f is linearly independent), so on a block B_β, $(s(g)u)_{B_\beta B_\beta} = s(g)_{B_\beta B_\beta} u_{B_\beta B_\beta} = s(g)_{B_\beta B_\beta}$ is integral. So $K_0C^\natural \subseteq T$.

Step 4: summation. If C_1, C_2 lie in different Σ-orbits, then $K_0C_1^\natural$ and $K_0C_2^\natural$ are disjoint, since $K_0 \subseteq \Sigma$ and $C_i^\natural \in \Sigma C_i$. So for every set R of representatives of the Σ-orbits of integral frame classes, in $[0,\infty]$,

$$\sum_{C\in R} w(C) = \sum_{C\in R} |K_0 C^\natural| \le |T|,$$

and the number of these orbits is at most $|T|$, each $K_0C^\natural$ being nonempty. It remains to show $|T| \le q^{\,r(r+1)(v(\theta)+v_D)}$; this is clear if $T = \emptyset$.

Step 5: counting the normal forms. Let $\mathrm{cl}(fx, f'x') \in T$. Since $(fx, f'x')$ lies in this integral class, the matrix $(h((f'x')_b, (fx)_a))_{ab} = \bar{x}^{\mathsf{T}} Z x'$ is integral, and Lemma 16.15 applies: Δ (which depends only on the frame pair) is an invertible element of $\mathcal{O}_E$ with $v(\Delta\bar{\Delta}) = v_D$, and $\bar{\Delta}x$, $\Delta x'$ are integral. *Cosets.* Let $\delta \in \mathcal{O}_E$ be invertible in E and R_δ a set of representatives of $\delta^{-1}\mathcal{O}_E/\mathcal{O}_E$, which has $|\mathcal{O}_E/\delta\mathcal{O}_E| = q^{v(\delta\bar{\delta})}$ elements (if E is a field, $\delta = \varepsilon\varpi^e$ with ε a unit and $|\mathcal{O}_E/\varpi^e\mathcal{O}_E| = q^{2e}$; if $E = F \times F$, $\delta = (\delta_1, \delta_2)$ and the order is $q^{v(\delta_1)+v(\delta_2)}$). If x is upper unitriangular with δx integral, then $x\,\mathrm{U}(\mathcal{O})$ contains a matrix with all entries above the diagonal in R_δ: column by column, and in column j for $i = j-1, j-2, \dots, 0$, the entry $(xA)_{ij} = x_{ij} + A_{ij} + \sum_{i<c<j} x_{ic}A_{cj}$ lies in $\delta^{-1}\mathcal{O}_E$ and is moved into R_δ by the choice of $A_{ij} \in \mathcal{O}_E$, which changes only the entries $(xA)_{i'j}$ with $i' \le i$. So there are at most $q^{v(\delta\bar{\delta})r(r+1)/2}$ such cosets;

we fix a representative $\hat{x}$ of each, $x = \hat{x}A_x$ with $A_x \in \mathrm{U}(\mathcal{O})$. *Congruences.* Let $R_\theta \subseteq \mathrm{U}(\mathcal{O})$ contain one element of each class of $\mathrm{U}(\mathcal{O})$ modulo $\theta \mathrm{M}_{r+1}(\mathcal{O}_E)$; $|R_\theta| \le |\mathcal{O}_E/\theta\mathcal{O}_E|^{r(r+1)/2} = q^{v(\theta)r(r+1)}$. *The map.* For $\mathrm{cl}(fx, f'x') \in T$ write $x = \hat{x}A$, $x' = \hat{x}'A'$ and choose $B_0 \in R_\theta$ with $A'A^{-1} \equiv B_0$ modulo θ. Then $(A, B_0^{-1}A') \in \Gamma_\theta$, because $B_0^{-1}A' - A = B_0^{-1}(A'A^{-1} - B_0)A$, and $(fx, f'x') = (f\hat{x}, f'\hat{x}'B_0)(A, B_0^{-1}A')$; hence $\mathrm{cl}(fx, f'x') = \mathrm{cl}(f\hat{x}, f'\hat{x}'B_0)$ is determined by the coset of x (with $\delta = \bar{\Delta}$), the coset of x' (with $\delta = \Delta$) and B_0. Since $v(\Delta\bar{\Delta}) = v(\bar{\Delta}\Delta) = v_D$,

$$|T| \le q^{v_D r(r+1)/2} \cdot q^{v_D r(r+1)/2} \cdot q^{v(\theta)r(r+1)} = q^{r(r+1)(v(\theta)+v_D)}.$$

□

Lemma 16.14. *Let a local frame pair with parameters r, k be given such that some datum has a term. Then $f_0, \dots, f_r, f'_0, \dots, f'_r$ is an E-basis of M_E, and:*

1. *for $g \in N$ (resp. $g \in N'$) there is a unique upper unitriangular $s(g)$ (resp. $s'(g)$) in $\mathrm{M}_{r+1}(E)$ with $gf_b = \sum_i s(g)_{ib} f_i$ (resp. $gf'_b = \sum_i s'(g)_{ib} f'_i$) for $0 \le b \le r$; $s(g_1g_2) = s(g_1)s(g_2)$; and for $g \in \Sigma$, $s(g)$ and $s'(g)$ are block diagonal and $\overline{s(g)}^{\mathsf{T}} Z\, s'(g) = Z$;*
2. *for every block-diagonal upper unitriangular $s \in \mathrm{M}_{r+1}(E)$ there is $g \in \Sigma$ with $s(g) = s$;*
3. *for every $n \in N$ there are $h \in \Sigma$ and an upper unitriangular $u \in \mathrm{M}_{r+1}(E)$ whose diagonal blocks are identity matrices such that $h^{-1}nf_b = \sum_i u_{ib} f_i$ for $0 \le b \le r$, and $h^{-1}gh \in K_0$ for every $g \in \Sigma$ with $g\Lambda_n = \Lambda_n$, where $\Lambda_n = \sum_b \mathcal{O}_E\, nf_b$.*

Proof. **Step 1: matrices and the basis.** Let $g \in N$. For $j < r$, $f_j \in F_{j+1}(f)$ gives $gf_j - f_j \in F_j(f)$; and $f_r \perp F_r(f)$ by (Fp1) gives $gf_r - f_r \in F_r(f)$. So $gf = fs$ with s upper unitriangular; likewise for N'. Let Λ be a term over some datum with witness (u_0, u'_0). By Proposition 16.8(1), the $2r+2$ vectors $u_0 f_b$, $\theta^{-1}(u'_0 f'_b - u_0 f_b)$ are E-linearly independent with nonzero Gram determinant and span M_E; so M_E is a nondegenerate free E-module of rank $2r+2$ and $V = M_E \oplus W_0$ orthogonally. As $u_0 f = f s_0$, $u'_0 f' = f' s'_0$ with unitriangular s_0, s'_0, the $2r+2$ vectors f_b, f'_b span M_E, hence form a basis (a surjective endomorphism of a finitely generated free module over a commutative ring is bijective). So $s(g)$, $s'(g)$ are unique, and $g_1 g_2 f = g_1(f s(g_2)) = f s(g_1)s(g_2)$ by linearity.

Step 2: proof of (1). Let $g \in \Sigma$, $s = s(g)$, $s' = s'(g)$. As g is an isometry, $Z_{ab} = h(gf'_b, gf_a) = \sum_{i,j} s'_{ib} \bar{s}_{ja} Z_{ji}$, i.e. $\bar{s}^{\mathsf{T}} Z s' = Z$. Put $\mathsf{M} = (\bar{s}^{\mathsf{T}})^{-1}$, lower unitriangular; then $Zs' = \mathsf{M}Z$, whose (a,b) entries read $z_a s'_{\sigma(a)b} = \mathsf{M}_{a\sigma(b)} z_{\sigma(b)}$. Let $c < b$ lie in different blocks and take $a = \sigma(c)$: since σ maps each block to itself and the blocks are consecutive intervals, $\sigma(c) < \sigma(b)$, so $\mathsf{M}_{\sigma(c)\sigma(b)} = 0$ and $s'_{cb} = 0$. So s' is block diagonal, and then so is $\bar{s}^{\mathsf{T}} = Z s'^{-1} Z^{-1}$, hence s.

Step 3: proof of (2). Let s be block diagonal upper unitriangular and $s' = Z^{-1}(\bar{s}^{\mathsf{T}})^{-1} Z$. On the block B_β, $s'_\beta = \mathcal{E}_\beta J_{s_\beta} (\bar{s}_\beta^{\mathsf{T}})^{-1} J_{s_\beta} \mathcal{E}_\beta$ (t_β cancels); conjugation by J_s maps lower unitriangular matrices to upper unitriangular ones ($(JAJ)_{ab} = A_{s-1-a,s-1-b}$) and conjugation by $\mathcal{E}_\beta$ preserves them, so s' is block diagonal upper unitriangular, and $\bar{s}^{\mathsf{T}} Z s' = Z$. By Step 1 there is a unique E-linear $g\colon V \to V$ with $gf = fs$, $gf' = f's'$ and $g|_{W_0} = \mathrm{id}$. It is an isometry: by (Fp1), $h((fs)_a, (fs)_b) = \sum_{i,j} s_{ia} \bar{s}_{jb} h(f_i, f_j) = -T s_{ra} \bar{s}_{rb} = h(f_a, f_b)$, as the row r of s is that of the identity; likewise for f'; $h((f's')_b, (fs)_a) = (\bar{s}^{\mathsf{T}} Z s')_{ab} = Z_{ab}$, and the reversed pairings follow by hermitian symmetry; g preserves M_E and W_0, which are orthogonal. So $g \in \mathrm{U}(V)$. $g \in N$: for $j < r$, $gF_{j+1}(f) = F_{j+1}(f)$ and $(g-1)F_{j+1}(f) \subseteq F_j(f)$ as s is unitriangular; $gX = X$ and g is an isometry, so $g(X^\perp) = X^\perp$. Let $w \in X^\perp$, $w = m + w_0$ with $w_0 \in W_0$ and $m = \sum_b \alpha_b f_b + \sum_b \beta_b f'_b$. As $W_0 \perp M_E \supseteq X$, $m \perp X$, and for $a < r$, using $h(f_b, f_a) = 0$,

$$0 = h(m, f_a) = \sum_b \beta_b Z_{ab} = \beta_{\sigma(a)} z_a.$$

So $\beta_c = 0$ unless $c = \sigma(r) = l_{m-1}$, the first index of the last block. Hence $(g-1)w = \sum_b \alpha_b(g-1)f_b + \beta_{l_{m-1}}(g-1)f'_{l_{m-1}}$; the first sum lies in $F_r(f) = X$, and $(g-1)f'_{l_{m-1}} = \sum_{i<l_{m-1}} s'_{i,l_{m-1}} f'_i = 0$ because s' is block diagonal and l_{m-1} is the first index of its block. So $(g-1)w \in X$ and $g \in N$. Symmetrically, for $w \in X'^{\perp}$ and $a < r$, $0 = h(m, f'_a) = \sum_b \alpha_b \overline{Z_{ba}} = \alpha_{\sigma(a)}\overline{z_{\sigma(a)}}$, so only $\alpha_{l_{m-1}}$ may be nonzero, $(g-1)f_{l_{m-1}} = 0$ since s is block diagonal, and $(g-1)f'_b \in X'$; so $g \in N'$. Thus $g \in \Sigma$ and $s(g) = s$.

Step 4: proof of (3). Let $n \in N$, $s = s(n)$, and s_0 the block-diagonal part of s (equal to s on $\bigcup_\beta B_\beta \times B_\beta$, zero elsewhere), block diagonal upper unitriangular. By (2) there is $h \in \Sigma$ with $s(h) = s_0$; then $h^{-1}n \in N$ and $u := s(h^{-1}n) = s_0^{-1}s$ by (1). As s_0^{-1} is block diagonal, $u_{B_\beta B_\beta} = (s_{B_\beta B_\beta})^{-1} s_{B_\beta B_\beta} = 1$; and $h^{-1}nf = fu$. Let $g \in \Sigma$ with $g\Lambda_n = \Lambda_n$ and $g' = h^{-1}gh \in \Sigma$, $\sigma_g = s(g')$. Then g' stabilises $h^{-1}\Lambda_n = \sum_b \mathcal{O}_E(fu)_b$, whose generators fu are E-linearly independent; since $g'(fu) = fs(g')u = (fu)(u^{-1}\sigma_g u)$, the matrix $Q = u^{-1}\sigma_g u$ is integral. The matrices u, u^{-1} are upper unitriangular, hence block upper triangular, and σ_g is block diagonal by (1); so the diagonal blocks are $Q_{B_\beta B_\beta} = (u^{-1})_{B_\beta B_\beta}(\sigma_g)_{B_\beta B_\beta} u_{B_\beta B_\beta} = (\sigma_g)_{B_\beta B_\beta}$, and σ_g is integral. By (1), $s'(g') = Z^{-1}(\bar\sigma_g^{\mathsf T})^{-1}Z$, whose block on B_β is $\mathcal{E}_\beta J(\bar\sigma_{g,\beta}^{\mathsf T})^{-1} J\mathcal{E}_\beta$, integral since the inverse of an integral unitriangular matrix is integral. So g' and g'^{-1} map $\sum_b \mathcal{O}_E f_b$ and $\sum_b \mathcal{O}_E f'_b$ into themselves, i.e. $g' \in K_0$. □

Lemma 16.15. *Let a local frame pair with parameters r, k be given, and let $u, u' \in \mathrm{M}_{r+1}(E)$ be upper unitriangular, with $u_{ij} \in \mathcal{O}_E$ whenever i and j lie in one block, such that $P = \bar u^{\mathsf T} Z u'$ has all entries in $\mathcal{O}_E$. Then $D_l \in \mathcal{O}_E$ for $0 \le l \le r+1$, and $D_l \ne 0$ if and only if $l \in \{l_0, \dots, l_m\}$. The element*

$$\Delta = \prod_{1 \le l \le r+1,\ D_l \ne 0} D_l$$

lies in $\mathcal{O}_E$, is invertible in E, satisfies $v(\Delta\bar\Delta) = v_D$, and $\bar\Delta u_{ij} \in \mathcal{O}_E$ and $\Delta u'_{ij} \in \mathcal{O}_E$ for all i, j.

Proof. **Step 1: the shape of Z.** By (Fp2), $Z_{ab} = (-1)^{r-b}(t_\beta J_{s_\beta})_{ab}$ on the block $B_\beta \times B_\beta$ and $Z_{ab} = 0$ off the diagonal blocks. So $Z = \mathrm{diag}(Z_1, \dots, Z_m)$ with $Z_\beta = t_\beta J_{s_\beta}\mathcal{E}_\beta$, $\mathcal{E}_\beta$ diagonal with entries ± 1; Z_β is invertible, $t_\beta Z_\beta^{-1} = \mathcal{E}_\beta J_{s_\beta}$ has entries in $\{0, \pm 1\}$, and $\det Z_\beta = \pm t_\beta^{s_\beta}$. For $l = l_\beta$, $Z_{[l]} = \mathrm{diag}(Z_1, \dots, Z_\beta)$ and $D_{l_\beta} = \prod_{\gamma \le \beta} \det Z_\gamma$ is invertible. For $l_{\beta-1} < l < l_\beta$ the row $l_{\beta-1}$ of $Z_{[l]}$ is zero, because the only nonzero entry of that row of Z is in the column $l_\beta - 1 \ge l$; so $D_l = 0$.

Step 2: the minors. Put $L = \bar u^{\mathsf T}$ (lower unitriangular, $L_{ij} \in \mathcal{O}_E$ for i, j in one block) and $U = u'$, so $P = LZU$. As L is lower and U upper triangular, $P_{[l]} = L_{[l]}Z_{[l]}U_{[l]}$ and $\det P_{[l]} = D_l$ for every l; as P is integral, $D_l \in \mathcal{O}_E$. Hence $\Delta = \prod_{\beta=1}^m D_{l_\beta} \in \mathcal{O}_E$ is invertible, and $v(\Delta\bar\Delta) = \sum_{l \ge 1,\, D_l \ne 0} v(D_l \bar D_l) = v_D$.

Step 3: the columns of L. Fix β, $l = l_\beta$, and $i \ge l$. For $c < l$, $P_{ic} = \sum_{a,b} L_{ia} Z_{ab} U_{bc}$ with $b \le c < l$, and $Z_{ab} \ne 0$ forces a into the block of b, so $a < l$ (l is a block end). Hence $P_{\{i\},[l]} = L_{\{i\},[l]} Z_{[l]} U_{[l]}$, and with $P_{[l]} = L_{[l]}Z_{[l]}U_{[l]}$,

$$L_{\{i\},[l]} = y\, L_{[l]}, \qquad y = P_{\{i\},[l]} P_{[l]}^{-1} = D_l^{-1} P_{\{i\},[l]} \operatorname{adj}(P_{[l]}),$$

so $D_l y$ is integral. For $c \in B_\beta$, $L_{ic} = \sum_{a<l} y_a L_{ac}$, and $L_{ac} \ne 0$ forces $c \le a < l$, i.e. $a \in B_\beta$, where $L_{ac} \in \mathcal{O}_E$. So $D_{l_\beta} L_{ic} \in \mathcal{O}_E$ for $i \ge l_\beta$, $c \in B_\beta$; for $i \in B_\beta$ already $L_{ic} \in \mathcal{O}_E$, and for $i < l_{\beta-1}$, $L_{ic} = 0$.

Step 4: the rows of U. Fix β and $l' = l_{\beta-1}$; write 1 for the indices $< l'$ and 2 for those $\ge l'$. If $\beta = 1$ put $S = P$. If $\beta \ge 2$, block multiplication ($L_{12} = 0$, $U_{21} = 0$, $Z = \mathrm{diag}(Z_{11}, Z_{22})$

since l' is a block end) gives $P_{11} = L_{11}Z_{11}U_{11}$, $P_{12} = L_{11}Z_{11}U_{12}$, $P_{21} = L_{21}Z_{11}U_{11}$, $P_{22} = L_{21}Z_{11}U_{12} + L_{22}Z_{22}U_{22}$, and P_{11} is invertible with determinant $D_{l'}$; hence

$$S := P_{22} - P_{21}P_{11}^{-1}P_{12} = L_{22}Z_{22}U_{22}.$$

By Schur's determinant formula $\det \begin{pmatrix} A & b \\ c & d \end{pmatrix} = \det A\,(d - cA^{-1}b)$, applied to $P_{[l']\cup\{i\},[l']\cup\{j\}}$, $D_{l'}S_{ij} = \det P_{[l']\cup\{i\},[l']\cup\{j\}} \in \mathcal{O}_E$. So in both cases $D_{l'}S$ is integral ($D_0 = 1$). The rows of L_{22} indexed by B_β (the first block of 2) are supported on the columns B_β and Z_{22} is block diagonal, so the rows B_β of $S = L_{22}Z_{22}U_{22}$ are

$$S_{B_\beta,2} = L_{B_\beta B_\beta}\, Z_\beta\, U_{B_\beta,2}, \qquad U_{B_\beta,2} = Z_\beta^{-1} L_{B_\beta B_\beta}^{-1} S_{B_\beta,2}.$$

$L_{B_\beta B_\beta}$ is integral unitriangular, so its inverse is integral; with Step 1, $t_\beta D_{l'} U_{ij} \in \mathcal{O}_E$ for $i \in B_\beta$ and $j \geq l'$, while $U_{ij} = 0$ for $j < l' \leq i$.

Step 5: valuations. Let $w \in \mathcal{W}$ and $i \in B_\beta$. All D_l are integral, so $w(D_{l_\gamma}) \geq 0$ and $w(D_{l_\gamma}) \leq w(\Delta)$ for every γ, and $w(D_{l_\beta}) = s_\beta w(t_\beta) + w(D_{l'})$ by Step 1. If $w(t_\beta) \geq 0$, then $w(t_\beta) + w(D_{l'}) \leq w(D_{l_\beta}) \leq w(\Delta)$; if $w(t_\beta) < 0$, then $w(t_\beta) + w(D_{l'}) < w(D_{l'}) \leq w(\Delta)$ (for $\beta = 1$, $w(D_0) = 0 \leq w(\Delta)$). With Step 4, $w(\Delta U_{ij}) \geq w(\Delta) - w(t_\beta) - w(D_{l'}) \geq 0$ for all j; with Step 3, $w(\Delta L_{ic}) \geq w(\Delta) - w(D_{l_\beta}) \geq 0$ for $c \in B_\beta$ and all i. As β and w are arbitrary, $\Delta u' = \Delta U$ and ΔL are integral, and $\bar{\Delta} u_{ji} = \overline{\Delta L_{ij}} \in \mathcal{O}_E$, since $\mathcal{O}_E$ is stable under conjugation. □

Proposition 16.16. *Let a local frame pair with parameters r, k be given, d a datum, C a set of frame pairs and $c \geq 1$ an integer. If some term over d has class C, then $a \geq 0$ and the number of terms Λ over d with $\mathrm{Fr}(\Lambda) = C$ and $c_d(\Lambda) = c$ is at most $q^{10a(r+1)}c^{k-1}$.*

Proof. **Step 0: set-up.** Let Λ_1 be a term over d with $\mathrm{Fr}(\Lambda_1) = C$. By Proposition 16.8(1),(2), $a \geq 0$, M_E is nondegenerate, $V = M_E \oplus W_0$ orthogonally, and $M := M(\Lambda_1)$ has the basis fr_b, $\theta^{-1}(\mathrm{fr}'_b - \mathrm{fr}_b)$ with Gram determinant $g_M = (-1)^{r+1}\theta^{-2(r+1)}G \in F^\times$, $v(g_M) = a$; hence $[M^\# : M] = |\mathcal{O}_E/g_M\mathcal{O}_E| = q^{2a}$. Let $\mathcal{T}_c$ be the set of terms Λ over d with $\mathrm{Fr}(\Lambda) = C$ and $c_d(\Lambda) = c$. For $\Lambda \in \mathcal{T}_c$, choosing $(p, p') \in C = \mathrm{Fr}(\Lambda) = \mathrm{Fr}(\Lambda_1)$, Proposition 16.12 gives $M(\Lambda) = M(p, p') = M$. By Proposition 16.8(1),(3),(5), $\mathrm{sat}(\Lambda)$ is an integral module with $M \subseteq \hat{I}(\Lambda) \subseteq \mathrm{sat}(\Lambda) \subseteq M_E$ and $[\mathrm{sat}(\Lambda) : M] = b_d(\Lambda) \leq q^a$. We bound $|\mathcal{T}_c|$ through the map $\Lambda \mapsto (\mathrm{sat}(\Lambda), \Lambda \cap W_0)$.

Step 1: saturations. By Lemma 16.17 for Λ_1, $\mathrm{sat}(\Lambda)$ is one of at most $q^{2a(2r+2)}$ modules $\hat{M}$.

Step 2: the gluing homomorphism. Let $L_d = \{w \in W_0 \cap A \cap A' : (w, w) \in \Gamma\}$, and for a term Λ over d write $w \sim_\Lambda y$ if there are $\lambda \in X$, $\lambda' \in X'$ with $w + \lambda \in \Lambda$, $w + \lambda' \in \Lambda$, $\lambda' - \lambda \in \theta\Lambda$ and $y - \theta^{-1}(\lambda' - \lambda) \in M(\Lambda)$. Let $\mathrm{fr}, \mathrm{fr}'$ be the frames of the fixed witness of Λ. (a) *Existence.* For $w \in L_d$, as Γ is the set of pairs attached to Λ, there are $y_1, y'_1 \in \Lambda$ with $y_1 - w \in X$, $y'_1 - w \in X'$, $y'_1 - y_1 \in \theta\Lambda$; with $\lambda = y_1 - w$, $\lambda' = y'_1 - w$, the vector $y = \theta^{-1}(y'_1 - y_1) \in \Lambda \cap M_E = \mathrm{sat}(\Lambda)$ satisfies $w \sim_\Lambda y$. (b) *Uniqueness modulo $M(\Lambda)$.* If $w \sim_\Lambda y$ and $y' - y \in M(\Lambda)$, then $w \sim_\Lambda y'$ (same λ, λ'). Conversely let $w \sim_\Lambda y$ by (λ_1, λ'_1) and $w \sim_\Lambda y'$ by (λ_2, λ'_2). Then $\mu = \lambda_1 - \lambda_2 \in \Lambda \cap X$ and $\mu' = \lambda'_1 - \lambda'_2 \in \Lambda \cap X'$, so by Lemma 16.18(2) $\mu = \sum_{j<r} \alpha_j \mathrm{fr}_j$, $\mu' = \sum_{j<r} \alpha'_j \mathrm{fr}'_j$ with integral coefficients. The vector $\theta^{-1}(\mu' - \mu) = \theta^{-1}(\lambda'_1 - \lambda_1) - \theta^{-1}(\lambda'_2 - \lambda_2)$ lies in Λ and equals $\sum_j \alpha'_j \theta^{-1}(\mathrm{fr}'_j - \mathrm{fr}_j) + z$ with $z = \sum_{j<r} \theta^{-1}(\alpha'_j - \alpha_j)\mathrm{fr}_j$; the first sum lies in $M(\Lambda) \subseteq \Lambda$, so $z \in \Lambda \cap X$, and by Lemma 16.18(2) and linear independence $\theta^{-1}(\alpha'_j - \alpha_j) \in \mathcal{O}_E$, so $z \in M(\Lambda)$. Hence $\theta^{-1}(\mu' - \mu) \in M(\Lambda)$ and $y - y' \in M(\Lambda)$. (c) *Linearity.* If $w_i \sim_\Lambda y_i$ by (λ_i, λ'_i) $(i = 1, 2)$ and $\gamma \in \mathcal{O}_E$, then $\gamma w_1 + w_2 \sim_\Lambda \gamma y_1 + y_2$ by $(\gamma\lambda_1 + \lambda_2, \gamma\lambda'_1 + \lambda'_2)$. (d) *Kernel.* For $w \in W_0$: $w \in \Lambda$ if and only if $w \in L_d$ and $w \sim_\Lambda 0$. If $w \in \Lambda$, then $w \in A \cap A'$ by Lemma 16.18(3),

$(w,w) \in \Gamma$ (take $y = y' = w$; $W_0 \perp X, X'$) and $w \sim_\Lambda 0$ with $\lambda = \lambda' = 0$. Conversely, if $w \sim_\Lambda 0$ by (λ, λ'), then $m = \theta^{-1}(\lambda' - \lambda) \in M(\Lambda)$, $m = \sum_b \mu_b \mathrm{fr}_b + \sum_b \mu'_b \theta^{-1}(\mathrm{fr}'_b - \mathrm{fr}_b)$ with integral coefficients, and $\lambda' - \lambda = \sum_b (\theta\mu_b - \mu'_b)\mathrm{fr}_b + \sum_b \mu'_b \mathrm{fr}'_b$. As $\mathrm{fr}, \mathrm{fr}'$ is an E-basis of M_E (Proposition 16.8(1)), $\lambda \in F_{r+1}(f) = \operatorname{span} \mathrm{fr}$ and $\lambda' \in X' = \operatorname{span}(\mathrm{fr}'_0, \dots, \mathrm{fr}'_{r-1})$, comparing coordinates gives $\lambda' = \sum_{b<r} \mu'_b \mathrm{fr}'_b \in \Lambda$, so $w = (w + \lambda') - \lambda' \in \Lambda$. By (a)–(c) there is an $\mathcal{O}_E$-linear map $\tau_\Lambda \colon L_d \to \mathrm{sat}(\Lambda)/M$, $w \mapsto y + M$ for any y with $w \sim_\Lambda y$; by (d) and (b) its kernel is $\Lambda \cap W_0$, and by the definition of $\hat{I}(\Lambda)$ its image is $\hat{I}(\Lambda)/M$, of order $c_d(\Lambda)$. *L_d is a lattice of rank $k-1$ in W_0.* By (c) for Λ_1 (applied to the pairs (w,w)), L_d is an $\mathcal{O}_E$-module; by (a), every $w \in L_d$ is the orthogonal projection to W_0 of a vector of Λ_1, so L_d is finitely generated; and $L_d \supseteq \Lambda_1 \cap W_0$ by (d), which spans W_0 since the finitely generated self-dual module Λ_1 (Lemma 16.18(1)) spans V (otherwise a nonzero subspace orthogonal to it would lie in $\Lambda_1^\# = \Lambda_1$). So L_d is free of rank $\dim W_0 = n - (2r+2) = k-1$; fix a basis $b_1, \dots, b_{k-1}$.

Step 3: the complements. Fix $\hat{M}$ and let $\Lambda \in \mathcal{T}_c$ with $\mathrm{sat}(\Lambda) = \hat{M}$; put $D = \hat{M}/M$, of order $b_d(\Lambda) \le q^a$, and $x_i = \tau_\Lambda(b_i)$. Then $I = \hat{I}(\Lambda)/M$ is a submodule of D of order c, $x \in I^{k-1}$, and

$$\Lambda \cap W_0 = \Big\{ \sum_i \gamma_i b_i : \ \gamma \in \mathcal{O}_E^{k-1}, \ \sum_i \gamma_i x_i = 0 \Big\}$$

is determined by x. Since $M \subseteq \hat{M} \subseteq M^\#$ and $M^\#$ is free of rank $2r+2$, every submodule of D is generated by $2r+2$ elements, so D has at most $|D|^{2r+2} \le q^{a(2r+2)}$ submodules. Hence, for fixed $\hat{M}$, $\Lambda \cap W_0$ takes at most $q^{a(2r+2)} c^{k-1}$ values.

Step 4: the gluings. Let Λ be self-dual (Lemma 16.18(1)) with $\Lambda \cap M_E = \hat{M}$ and $\Lambda \cap W_0 = B$, and let π_M, π_W be the orthogonal projections. For $y \in \Lambda$ and $x \in \hat{M}$, $h(\pi_M y, x) = h(y,x) \in \mathcal{O}_E$, so $\pi_M \Lambda \subseteq \hat{M}^\#$; and $\{x \in M_E : h(x, \pi_M \Lambda) \subseteq \mathcal{O}_E\} = M_E \cap \Lambda^\# = \hat{M}$, so the lattice $\pi_M \Lambda$ equals $\hat{M}^\#$; likewise $\pi_W \Lambda = B^\#$. If $\pi_M y \in \hat{M} \subseteq \Lambda$, then $\pi_W y = y - \pi_M y \in \Lambda \cap W_0 = B$; so $y \mapsto \pi_M y + \hat{M}$ and $y \mapsto \pi_W y + B$ are surjections $\Lambda \to \hat{M}^\#/\hat{M}$, $\Lambda \to B^\#/B$, both with kernel $\hat{M} \oplus B$. Hence there is an isomorphism $\varphi \colon \hat{M}^\#/\hat{M} \to B^\#/B$ with

$$\Lambda = \{ x + z : \ x \in \hat{M}^\#, \ z \in B^\#, \ \varphi(x + \hat{M}) = z + B \},$$

and Λ is determined by $(\hat{M}, B, \varphi)$. As $|B^\#/B| = |\hat{M}^\#/\hat{M}| \le [M^\# : M] = q^{2a}$ and $\hat{M}^\#/\hat{M}$ is generated by $2r+2$ elements, there are at most $q^{2a(2r+2)}$ homomorphisms φ.

Step 5: conclusion. By Steps 1, 3 and 4,

$$|\mathcal{T}_c| \le q^{2a(2r+2)} \cdot q^{a(2r+2)} c^{k-1} \cdot q^{2a(2r+2)} = q^{10a(r+1)} c^{k-1}.$$

□

Lemma 16.17. *Fix a local frame pair with parameters r, k and a term Λ over some datum. Then the number of $\mathcal{O}_E$-submodules $\hat{M}$ with $M(\Lambda) \subseteq \hat{M} \subseteq M_E$ and $h(\hat{M}, \hat{M}) \subseteq \mathcal{O}_E$ is at most $q^{2a(2r+2)}$, and each of them satisfies $\varpi^a \hat{M} \subseteq M(\Lambda)$.*

Proof. By Proposition 16.8(1), $M = M(\Lambda)$ has an $\mathcal{O}_E$-basis $g_1, \dots, g_m$ ($m = 2r+2$), which is an E-basis of M_E, with integral Gram matrix $P = (h(g_i, g_j))$ of determinant $\delta = (-1)^{r+1} \theta^{-2(r+1)} G$, and $v(\delta) = a \ge 0$ (Proposition 16.8(2) and Lemma 16.9(2)). Let $\hat{M}$ be as in the statement. By Lemma 16.9(2) (with $N = \hat{M}$), $\varpi^a \hat{M} \subseteq M$. Moreover $\hat{M} \subseteq M^\# = \{x \in M_E : h(x, M) \subseteq \mathcal{O}_E\}$, since $M \subseteq \hat{M}$ and $\hat{M}$ is integral. Writing $x = \sum_i c_i g_i$, $h(x, g_j) = (c^\mathsf{T} P)_j$, so $M^\# = \{\sum c_i g_i : c^\mathsf{T} \in \mathcal{O}_E^m P^{-1}\}$ and $M^\#/M \cong \mathcal{O}_E^m P^{-1} / \mathcal{O}_E^m$. By elementary divisors over $\mathcal{O}_E$ (a principal ideal ring: $\mathcal{O}_E$ is a discrete valuation ring if E is a field and $\mathcal{O}_F \times \mathcal{O}_F$ if $E = F \times F$), this module has order $|\mathcal{O}_E/\delta\mathcal{O}_E| = |\mathcal{O}_E/\varpi^a \mathcal{O}_E| = q^{2a}$ ($\delta \in F$ has valuation a; $\mathcal{O}_E/\varpi\mathcal{O}_E$ has q^2 elements in both cases), and it is generated by the m images of the dual basis. The modules $\hat{M}$ correspond

injectively to the submodules $\hat{M}/M$ of $M^{\#}/M$. A submodule of an $\mathcal{O}_E$-module generated by m elements is generated by m elements (its preimage in $\mathcal{O}_E^m$ is a submodule of a free module of rank m over a principal ideal ring, hence generated by m elements), so it is the image of a map $\mathcal{O}_E^m \to M^{\#}/M$, of which there are at most $|M^{\#}/M|^m = q^{2am}$. Hence there are at most $q^{2a(2r+2)}$ modules $\hat{M}$. □

Lemma 16.18. *Fix a local frame pair with parameters r, k, a datum $d = (A, A', \Gamma)$ and a term Λ over d with witness (u, u'). Then:*

1. *Λ is self-dual;*
2. *$\Lambda \cap F_{r+1}(f) = \sum_{b \le r} \mathcal{O}_E u f_b$ and $\Lambda \cap X = \sum_{j<r} \mathcal{O}_E u f_j$; likewise $\Lambda \cap F_{r+1}(f') = \sum_{b \le r} \mathcal{O}_E u' f'_b$ and $\Lambda \cap X' = \sum_{j<r} \mathcal{O}_E u' f'_j$;*
3. *$\Lambda \cap W_0 \subseteq A \cap A'$.*

Proof. **Step 1: the decomposition.** By (Fp1), $h(f_i, f_j) = 0$, $h(f_i, e_j^*) = \delta_{ij}$, $h(e_i^*, e_j^*) = 0$ for $i, j < r$; so $X + X^*$ is nondegenerate with E-basis $f_0, \dots, f_{r-1}, e_0^*, \dots, e_{r-1}^*$, and $V = (X \oplus X^*) \oplus Y$ orthogonally. For $w \in A$, $w_0 = w - \sum_{j<r} h(w, e_j^*) f_j$ lies in A (as $X \subseteq A$) and satisfies $h(w_0, e_i^*) = 0$; so $A = X + A_0$, and $A_0 \subseteq X^\perp \cap (X^*)^\perp = Y$ since $A \subseteq X^\perp$. By the inner condition, $h(A, y) \subseteq \mathcal{O}_E$ for $y \in A$, so A_0 is integral; and if $y \in Y$ satisfies $h(A_0, y) \subseteq \mathcal{O}_E$, then $h(A, y) = h(X + A_0, y) \subseteq \mathcal{O}_E$ and $y \in X^\perp$, so $y \in A$ by the inner condition, and $y \perp X^*$, so $y \in A_0$. Thus $A_0 = \{y \in Y : h(A_0, y) \subseteq \mathcal{O}_E\}$, and A_0 is finitely generated (inner condition).

Step 2: proof of (1). H is its own dual in $X \oplus X^*$ (dual bases), and A_0 is its own dual in Y (Step 1). For $z = x + y$ with $x \in X \oplus X^*$, $y \in Y$, $h(H + A_0, z) \subseteq \mathcal{O}_E$ holds if and only if $h(H, x) \subseteq \mathcal{O}_E$ and $h(A_0, y) \subseteq \mathcal{O}_E$, i.e. $x \in H$ and $y \in A_0$. So $\Lambda_{\mathrm{std}}(A) = H \oplus A_0$ is finitely generated and self-dual, and so is $\Lambda = u\Lambda_{\mathrm{std}}(A)$, u being an isometry.

Step 3: proof of (2). First $\Lambda_{\mathrm{std}}(A) \cap F_{r+1}(f) = \sum_b \mathcal{O}_E f_b$: the vectors f_j ($j < r$) lie in H, and $f_r \in A$ (inner) with $h(f_r, e_j^*) = 0$, so $f_r \in A_0$. Conversely let $x = \sum_b c_b f_b \in \Lambda_{\mathrm{std}}(A)$. For $j < r$, $c_j = h(x, e_j^*) \in \mathcal{O}_E$ by (1). The inner condition gives $y \in A$ with $h(f_r, y) = 1$; then $y_0 = y - \sum_j h(y, e_j^*) f_j \in A_0 \subseteq \Lambda_{\mathrm{std}}(A)$ and $h(f_r, y_0) = 1$ ($f_r \perp X$), so $c_r = h(x - \sum_{j<r} c_j f_j, y_0) \in \mathcal{O}_E$ by (1). Now $u \in N$ preserves $F_r(f) = X$ and maps f_r into $f_r + X$, so $uF_{r+1}(f) = F_{r+1}(f)$ and $\Lambda \cap F_{r+1}(f) = u(\Lambda_{\mathrm{std}}(A) \cap F_{r+1}(f)) = \sum_b \mathcal{O}_E u f_b$. An element $\sum_b c_b u f_b = u(\sum_b c_b f_b)$ lies in $X = uX$ if and only if $c_r = 0$, as $f_0, \dots, f_r$ are linearly independent ($f_r \notin X$ because $h(f_r, f_r) = -T \neq 0$ while $X \perp f_r$). The statements for f' follow in the same way from $\Lambda = u'\Lambda'_{\mathrm{std}}(A')$.

Step 4: proof of (3). Let $w \in \Lambda \cap W_0$. Since $u^{-1} \in N$ and $w \in W_0 \subseteq X^\perp$, $u^{-1}w - w \in X$, so $u^{-1}w \in X^\perp \cap \Lambda_{\mathrm{std}}(A)$. Write $u^{-1}w = x_0 + a_0 + \sum_j \beta_j e_j^*$ with $x_0 \in \sum_{j<r} \mathcal{O}_E f_j$, $a_0 \in A_0$; then $0 = h(u^{-1}w, f_i) = \beta_i$ for $i < r$ ($a_0 \in Y \perp X$, X isotropic). So $u^{-1}w \in X + A_0 = A$ and $w \in A + X = A$. Likewise $w \in A'$. □

Lemma 16.19. *Let a local frame pair with parameters r, k be given, d a datum, $\mathcal{T}'$ a Σ-stable set of terms over d, and R_{cl} a set of representatives of the Σ-orbits of integral frame classes. Then μ is constant on Σ-orbits and, in $[0, \infty]$,*

$$\sum_{O \in \Sigma \backslash \mathcal{T}'} \mu(O) = \sum_{C \in R_{\mathrm{cl}}} w(C) \, \big|\{\Lambda \in \mathcal{T}' : \mathrm{Fr}(\Lambda) = C\}\big|.$$

Proof. By Proposition 16.12, μ is constant on Σ-orbits, Fr maps $\mathcal{T}'$ to integral frame classes and $\mathrm{Fr}(g\Lambda) = g\,\mathrm{Fr}(\Lambda)$ for $g \in \Sigma$. For $C \in R_{\mathrm{cl}}$ put $\mathcal{F}_C = \{\Lambda \in \mathcal{T}' : \mathrm{Fr}(\Lambda) = C\}$ and $\Sigma_C = \mathrm{Stab}_\Sigma(C)$. Every Σ-orbit $O \subseteq \mathcal{T}'$ has its classes in one Σ-orbit of integral classes, with representative $C \in R_{\mathrm{cl}}$, and meets $\mathcal{F}_C$; if $\Lambda, g\Lambda \in \mathcal{F}_C$ ($g \in \Sigma$), then $gC = \mathrm{Fr}(g\Lambda) = C$, so $g \in \Sigma_C$. Hence $O \mapsto O \cap \mathcal{F}_C$ is a bijection from the orbits over the orbit of C onto the Σ_C-orbits in $\mathcal{F}_C$. For $\Lambda \in \mathcal{F}_C$, $\Sigma_\Lambda = \mathrm{Stab}_\Sigma(\Lambda) \subseteq \Sigma_C$, both commensurable with K_0 (Proposition 16.12), so $[\Sigma_C : \Sigma_\Lambda]$ is finite and equals $|\Sigma_C\Lambda|$, and

$$\mu(\Lambda) = [\Sigma_\Lambda : K_0]^{-1} = \big([\Sigma_\Lambda : \Sigma_C]\,[\Sigma_C : K_0]\big)^{-1} = [\Sigma_C : \Sigma_\Lambda]\,w(C) = |\Sigma_C\Lambda|\,w(C).$$

Summing over the Σ_C-orbits in $\mathcal{F}_C$ gives $w(C)|\mathcal{F}_C|$, and summing over $C \in R_{\mathrm{cl}}$ (all terms nonnegative) gives the formula. □

16.3 The share of the terms of a given defect order

This subsection proves that a defect of order β costs a factor $\beta^{-(k-1)}$ in mass, up to the factor $4H^{4(r+1)}$ (Proposition 16.20). The terms are counted with masses over frame classes (Lemma 16.19, Proposition 16.21). The key estimate is Proposition 16.23: for an integral overlattice of index β of the module of a term, the saturated mass is $\kappa\,\beta^{-(k-1)}$ up to a factor 2, for a constant $\kappa > 0$. Its proof identifies glue masses with primitive local densities (Proposition 16.25), and these with counts of free tuples with a prescribed Gram matrix in a hermitian space over the residue ring (Lemma 16.28). The other lemmas of the subsection describe the self-dual lattices with a given trace on the frame span (Lemma 16.22), the fibre measures of the Gram map (Lemma 16.26), the conjugacy of self-dual lattices (Lemma 16.27) and the exact count of free tuples (Lemma 16.29).

Proposition 16.20. *Let a local frame pair with parameters r and $k \geq 4r+7$ be given, let D be a set of representatives of the H_0-orbits of data and, for $d \in D$, R_d a set of representatives of the Σ-orbits of terms over d. Then for every integer $\beta \geq 1$, in $[0,\infty]$,*

$$\sum_{d\in D} \nu(d) \sum_{\Lambda\in R_d,\ b_d(\Lambda)=\beta} \mu(\Lambda) \leq 4H^{4(r+1)}\beta^{-(k-1)} \sum_{d\in D} \nu(d) \sum_{\Lambda\in R_d} \mu(\Lambda).$$

Proof. If no datum has a term, both sides vanish. Otherwise let $\kappa > 0$ be given by Proposition 16.23 ($k \geq 4r+7 \geq 2r+6$), and let R_{cl} be a set of representatives of the Σ-orbits of integral frame classes. For $d \in D$, a frame class C and $\beta \geq 1$ put

$$n_d(C,\beta) = \big|\{\Lambda \text{ term over } d : \mathrm{Fr}(\Lambda) = C,\ b_d(\Lambda) = \beta\}\big|,$$
$$n_d(C) = \big|\{\Lambda \text{ term over } d : \mathrm{Fr}(\Lambda) = C\}\big|.$$

Step 1: by classes. For $d \in D$ the set of terms over d with $b_d = \beta$ is Σ-stable (Proposition 16.12), so Lemma 16.19 gives

$$\sum_{\Lambda\in R_d,\, b_d(\Lambda)=\beta} \mu(\Lambda) = \sum_{C\in R_{\mathrm{cl}}} w(C) n_d(C,\beta),$$

and likewise $\sum_{\Lambda\in R_d} \mu(\Lambda) = \sum_C w(C) n_d(C)$. Exchanging the sums of nonnegative terms,

$$\sum_{d\in D} \nu(d) \sum_{\Lambda\in R_d,\, b_d(\Lambda)=\beta} \mu(\Lambda) = \sum_{C\in R_{\mathrm{cl}}} w(C)\,S_\beta(C), \qquad \sum_{d\in D} \nu(d) \sum_{\Lambda\in R_d} \mu(\Lambda) = \sum_{C\in R_{\mathrm{cl}}} w(C)\,S(C),$$

with $S_\beta(C) = \sum_d \nu(d) n_d(C,\beta)$ and $S(C) = \sum_d \nu(d) n_d(C)$.

Step 2: one class. We show $S_\beta(C) \le 4H^{4(r+1)}\beta^{-(k-1)} S(C)$ for every $C \in R_{\mathrm{cl}}$. If $S_\beta(C) = 0$ there is nothing to show. Otherwise some term Λ_0 (over some datum) has $\mathrm{Fr}(\Lambda_0) = C$; put $M = M(\Lambda_0)$. By Proposition 16.21 with $Q = \{\beta\}$, then Proposition 16.23 and Lemma 16.17,

$$S_\beta(C) \le \sum_{\hat M:\,[\hat M:M]=\beta} \sigma(\hat M) \le q^{2a(2r+2)} \cdot 2\kappa\, \beta^{-(k-1)},$$

the sum over the integral $\hat M$ with $M \subseteq \hat M \subseteq M_E$. By Proposition 16.23 with $\hat M = M$ (index 1) and the equality of Proposition 16.21,

$$\frac{\kappa}{2} \le \sigma(M) = \sum_d \nu(d)\,\big|\{\Lambda \text{ generic term over } d : \mathrm{Fr}(\Lambda) = C\}\big| \le S(C).$$

Hence $S_\beta(C) \le 4q^{4a(r+1)}\beta^{-(k-1)} S(C) \le 4H^{4(r+1)}\beta^{-(k-1)} S(C)$, as $q^a \le H$ by Proposition 16.8(2) applied to Λ_0.

Step 3: conclusion. Multiplying Step 2 by $w(C)$ and summing over R_{cl}, Step 1 gives the claim. □

Proposition 16.21. *Let a local frame pair with parameters r, k be given, D a set of representatives of the H_0-orbits of data, Λ_0 a term over some datum, $C = \mathrm{Fr}(\Lambda_0)$ and $M = M(\Lambda_0)$. Then for every set Q of positive integers, in $[0,\infty]$,*

$$\sum_{d\in D} \nu(d)\,\big|\{\Lambda \text{ term over } d : \mathrm{Fr}(\Lambda) = C,\ b_d(\Lambda) \in Q\}\big| \le \sum_{\hat M} \sigma(\hat M),$$

the sum over the $\mathcal{O}_E$-submodules $\hat M$ with $M \subseteq \hat M \subseteq M_E$, $h(\hat M, \hat M) \subseteq \mathcal{O}_E$ and $[\hat M : M] \in Q$; and

$$\sum_{d\in D} \nu(d)\,\big|\{\Lambda \text{ generic term over } d : \mathrm{Fr}(\Lambda) = C\}\big| = \sigma(M).$$

Proof. **Step 1: a term determines its datum.** If Λ is a term over $d = (A, A', \Gamma)$ with witness (u, u'), then $A = X + (\Lambda \cap X^\perp)$ and $A' = X' + (\Lambda \cap X'^\perp)$ by Lemma 16.22(1), and Γ is the set of pairs attached to Λ by definition. So Λ is a term over exactly one datum, $d(\Lambda)$.

Step 2: H_0 acts on terms. Let Λ be a term over d with witness (u, u') and $g \in H_0$. As g fixes M_E pointwise, it fixes X, X', every $F_j(f)$, $F_j(f')$ and every vector uf_b, $u'f'_b$, and preserves $X^\perp$, $X'^\perp$. By Lemma 16.18(1),(2), $g\Lambda$ is self-dual and $g\Lambda \cap F_{r+1}(f) = g(\sum_b \mathcal{O}_E u f_b) = \sum_b \mathcal{O}_E u f_b$; by Lemma 16.22(2),(1), $g\Lambda = m\Lambda_{\mathrm{std}}(\tilde A)$ with $m \in N$, $mf = uf$ and $\tilde A = X + (g\Lambda \cap X^\perp) = g(X + (\Lambda \cap X^\perp)) = gA$. Likewise $g\Lambda = m'\Lambda'_{\mathrm{std}}(gA')$ with $m' \in N'$, $m'f' = u'f'$. Then $m'f'_b - mf_b = u'f'_b - uf_b \in \theta\Lambda \cap M_E$ is fixed by g, so it lies in $\theta g\Lambda$; and the set of pairs attached to $g\Lambda$ is the image under g of that of Λ (g maps the vectors y, y' of the definition for Λ to vectors for $g\Lambda$, and g^{-1} conversely). So $g\Lambda$ is a term over gd with witness (m, m'), and by Proposition 16.12 (independence of the witness) $\mathrm{Fr}(g\Lambda) = \mathrm{cl}(uf, u'f') = \mathrm{Fr}(\Lambda)$ and $M(g\Lambda) = M(\Lambda)$; also $\mathrm{sat}(g\Lambda) = g\,\mathrm{sat}(\Lambda) = \mathrm{sat}(\Lambda)$. Hence $b_{gd}(g\Lambda) = b_d(\Lambda)$, and $g\Lambda$ is generic if and only if Λ is.

Step 3: stabilisers of data with a term. Let Λ be a term over d. By Step 1, $\mathrm{Stab}_{H_0}(\Lambda) \subseteq \mathrm{Stab}_{H_0}(d)$. The module $A \cap W_0$ is finitely generated: with $Y = (X + \mathrm{span}_E e^*)^\perp$ and $A_0 = A \cap (\mathrm{span}_E e^*)^\perp$ (finitely generated, inner condition), $A = X + A_0$ with $A_0 \subseteq Y$ (subtract $\sum_j h(w, e_j^*) f_j$ from $w \in A$), and the projection $X \oplus Y \to Y$ along X is injective on W_0 ($W_0 \cap X = 0$) and maps $A \cap W_0$ into A_0. It contains $\Lambda \cap W_0$ (Lemma 16.18(3)), which spans W_0 because the

self-dual Λ spans V. So $L_0 = (A \cap W_0) \oplus M$ is a lattice of V, and $\mathrm{Stab}_{H_0}(d) \subseteq \mathrm{Stab}_{H_0}(L_0)$, as H_0 fixes M and preserves W_0. Being between two lattice stabilisers, $\mathrm{Stab}_{H_0}(d)$ is commensurable with K_{ref}.

Step 4: data-weighted counts are lattice masses. Let $\mathcal{P}$ be a condition on terms preserved by the action of Step 2, and $\mathcal{L}$ the set of lattices that are terms over some datum and satisfy $\mathcal{P}$; $\mathcal{L}$ is H_0-stable. Let $O \subseteq \mathcal{L}$ be an H_0-orbit. By Steps 1 and 2, the data $d(\Lambda)$, $\Lambda \in O$, form one H_0-orbit, which contains exactly one $d_O \in D$, and the elements of O that are terms over a datum in D are those with $d(\Lambda) = d_O$; they form one $\mathrm{Stab}_{H_0}(d_O)$-orbit. Fixing one of them, Λ_O, there are $[\mathrm{Stab}_{H_0}(d_O) : \mathrm{Stab}_{H_0}(\Lambda_O)]$ of them, so the contribution of O to the left side below is, by Step 3,

$$\nu(d_O)\,[\mathrm{Stab}_{H_0}(d_O) : \mathrm{Stab}_{H_0}(\Lambda_O)] = [\mathrm{Stab}_{H_0}(\Lambda_O)].$$

Summing over the orbits, in $[0, \infty]$,

$$\sum_{d \in D} \nu(d)\,\big|\{\Lambda \text{ term over } d \text{ with } \mathcal{P}\}\big| = \sum_{O \subseteq \mathcal{L}} \mathrm{vol}_{K_{\mathrm{ref}}}(\mathrm{Stab}_{H_0}(\Lambda_O))^{-1}.$$

Step 5: the inequality. Take $\mathcal{P}$: $\mathrm{Fr}(\Lambda) = C$ and $b_d(\Lambda) \in Q$ (preserved by Step 2). For $\Lambda \in \mathcal{L}$, choosing $(p, p') \in C = \mathrm{Fr}(\Lambda)$, Proposition 16.12 gives $M(\Lambda) = M(p, p') = M(\Lambda_0) = M$; Λ is self-dual (Lemma 16.18(1)); and $\hat{M} = \mathrm{sat}(\Lambda) = \Lambda \cap M_E$ is integral with $M \subseteq \hat{M} \subseteq M_E$ and $[\hat{M} : M] = b_d(\Lambda) \in Q$ (Proposition 16.8(1),(3)). As $\Lambda \cap M_E$ is constant on H_0-orbits, each orbit $O \subseteq \mathcal{L}$ is one of the orbits summed in $\sigma(\hat{M})$ for exactly one such $\hat{M}$, with the same summand. With Step 4 this gives the inequality.

Step 6: generic terms of class C are the self-dual lattices with saturation M. If Λ is a generic term with $\mathrm{Fr}(\Lambda) = C$, it is self-dual with $\Lambda \cap M_E = M(\Lambda) = M$. Conversely let Λ be self-dual with $\Lambda \cap M_E = M$. By Proposition 16.8(1), $u_0 f$, $u_0' f'$ form an E-basis of M_E, and M is spanned by $u_0 f_b$, $\theta^{-1}(u_0' f_b' - u_0 f_b)$, and also by $u_0' f_b'$, $\theta^{-1}(u_0 f_b - u_0' f_b')$ ($\theta \in \mathcal{O}_F$). An element $\sum_b \alpha_b u_0 f_b + \sum_b \beta_b \theta^{-1}(u_0' f_b' - u_0 f_b)$ of M lies in $F_{r+1}(f)$ only if $\beta = 0$, as $F_{r+1}(f) \cap F_{r+1}(f') = 0$; so $\Lambda \cap F_{r+1}(f) = M \cap F_{r+1}(f) = \sum_b \mathcal{O}_E u_0 f_b$, and likewise $\Lambda \cap F_{r+1}(f') = \sum_b \mathcal{O}_E u_0' f_b'$. By Lemma 16.22(2), $\Lambda = m\Lambda_{\mathrm{std}}(A) = m'\Lambda'_{\mathrm{std}}(A')$ with inner A, A', $m \in N$, $m' \in N'$, $mf = u_0 f$, $m'f' = u_0' f'$; and $m' f_b' - m f_b = \theta \cdot \theta^{-1}(u_0' f_b' - u_0 f_b) \in \theta M \subseteq \theta\Lambda$. With Γ the set of pairs attached to Λ, Λ is a term over (A, A', Γ) with witness (m, m'), $\mathrm{Fr}(\Lambda) = \mathrm{cl}(u_0 f, u_0' f') = C$ (Proposition 16.12) and $M(\Lambda) = M = \mathrm{sat}(\Lambda)$, i.e. generic. Taking $\mathcal{P}$: $\mathrm{Fr}(\Lambda) = C$ and generic, $\mathcal{L}$ is the set of self-dual Λ with $\Lambda \cap M_E = M$, and Step 4 gives the equality with $\sigma(M)$. □

Lemma 16.22. *Fix a local frame pair with parameters r, k.*

1. *If A is inner for (f, e^*) and $m \in N$, then $mA = A$ and $A = X + (m\Lambda_{\mathrm{std}}(A) \cap X^\perp)$.*
2. *Let $\Lambda \subseteq V$ be self-dual and $n \in N$ with $\Lambda \cap F_{r+1}(f) = \sum_{b \le r} \mathcal{O}_E\, n f_b$. Then there are an inner module A for (f, e^*) and $m \in N$ with $m f_b = n f_b$ for $0 \le b \le r$ and $\Lambda = m\Lambda_{\mathrm{std}}(A)$.*

The same holds with $(f', e'^*, X', N', \Lambda'_{\mathrm{std}})$ in place of $(f, e^*, X, N, \Lambda_{\mathrm{std}})$.

Proof. **Step 1: proof of (1).** As $A \subseteq X^\perp$ and $m \in N$, $(m-1)A \subseteq X \subseteq A$, so $mA \subseteq A$, and likewise $m^{-1}A \subseteq A$. Put $A_0 = A \cap (X^*)^\perp$. For $w \in A$, $w - \sum_j h(w, e_j^*) f_j \in A_0$, so $A = X + A_0$, and $A_0 \subseteq X^\perp \cap (X^*)^\perp = Y$. An element $\sum_j \alpha_j f_j + a_0 + \sum_j \beta_j e_j^*$ of $\Lambda_{\mathrm{std}}(A)$ ($a_0 \in A_0$) lies in $X^\perp$ if and only if all $\beta_i = h(\cdot, f_i)$ vanish; so $\Lambda_{\mathrm{std}}(A) \cap X^\perp = \sum_j \mathcal{O}_E f_j + A_0$. Since m preserves $X^\perp$, $m\Lambda_{\mathrm{std}}(A) \cap X^\perp = m(\sum_j \mathcal{O}_E f_j + A_0) \subseteq mA = A$, and $X + m(\sum_j \mathcal{O}_E f_j + A_0) \supseteq X + mA_0 \supseteq X + A_0 = A$, because $mA_0 \subseteq A_0 + X$ and $m^{-1}A_0 \subseteq A_0 + X$. Hence $A = X + (m\Lambda_{\mathrm{std}}(A) \cap X^\perp)$.

Step 2: reduction to $n = 1$. $n^{-1}\Lambda$ is self-dual (n is an isometry) and, as n preserves $F_{r+1}(f)$ (it preserves X and maps f_r into $f_r + X$), $n^{-1}\Lambda \cap F_{r+1}(f) = \sum_b \mathcal{O}_E f_b$. If $n^{-1}\Lambda = m_1\Lambda_{\mathrm{std}}(A)$ with $m_1 \in N$, $m_1 f = f$, then $m = nm_1$ works. So assume $\Lambda \cap F_{r+1}(f) = \sum_b \mathcal{O}_E f_b$.

Step 3: dual vectors. Λ is finitely generated and self-dual, hence free with a basis $(\lambda_i)_i$ whose dual vectors λ_i^* ($h(\lambda_i, \lambda_j^*) = \delta_{ij}$) lie in $\Lambda^\# = \Lambda$; so every $\mathcal{O}_E$-linear map $\phi\colon \Lambda \to \mathcal{O}_E$ is $x \mapsto h(x, c)$ for $c = \sum_j \overline{\phi(\lambda_j)}\lambda_j^* \in \Lambda$. The module $P_0 = \Lambda \cap F_{r+1}(f)$ is saturated in Λ, so Λ/P_0 is finitely generated and torsion-free, hence free ($\mathcal{O}_E$ is a principal ideal ring), and P_0 is a direct summand with basis $f_0, \dots, f_r$. Extending the coordinate functionals by zero on a complement gives $c_0, \dots, c_r \in \Lambda$ with $h(f_b, c_j) = \delta_{bj}$ for $0 \le b, j \le r$.

Step 4: isotropic dual vectors. As E/F is unramified or split, the trace $\mathcal{O}_E \to \mathcal{O}_F$ is surjective; fix $\sigma \in \mathcal{O}_E$ with $\sigma + \bar\sigma = 1$. For $j < r$ put

$$y_j = c_j - \sum_{i<r} \sigma\, h(c_j, c_i)\, f_i \in \Lambda.$$

Then $h(f_a, y_j) = \delta_{aj}$ for $0 \le a \le r$ (as $h(f_a, f_i) = 0$ for $i < r$), and, using $h(c_i, f_l) = \delta_{il}$ for $l < r$,

$$h(y_i, y_j) = h(c_i, c_j) - \overline{\sigma\, h(c_j, c_i)} - \sigma\, h(c_i, c_j) = (1 - \bar\sigma - \sigma)\, h(c_i, c_j) = 0.$$

Step 5: a unipotent element carrying e^* to y. Write $y_j - e_j^* = \xi_j + \zeta_j + \eta_j$ with $\xi_j \in X$, $\zeta_j \in X^*$, $\eta_j \in Y$. For $a < r$, $0 = h(f_a, y_j - e_j^*) = h(f_a, \zeta_j)$, so $\zeta_j = 0$; and $0 = h(f_r, y_j - e_j^*) = h(f_r, \eta_j)$. Define E-linear $u\colon V \to V$ by $u|_X = \mathrm{id}$, $ue_j^* = e_j^* + \xi_j + \eta_j = y_j$, and $uy = y - \sum_{j<r} h(y, \eta_j) f_j$ for $y \in Y$. It is an isometry: $h(ux, ue_j^*) = h(x, e_j^*)$ and $h(ux, uy) = 0$ for $x \in X$, $y \in Y$; $h(ue_i^*, ue_j^*) = h(y_i, y_j) = 0$; $h(uy, ue_j^*) = h(y, \eta_j) - h(y, \eta_j)h(f_j, e_j^*) = 0$; $h(uy, uy') = h(y, y')$, as the f_j are isotropic and orthogonal to Y. It lies in N: it fixes X pointwise, so $uF_{j+1}(f) = F_{j+1}(f)$ and $(u-1)F_{j+1}(f) = 0$ for $j < r$; it preserves $X^\perp = X \oplus Y$ with $(u-1)X^\perp \subseteq X$. And $uf_r = f_r$ since $f_r \in Y$ and $h(f_r, \eta_j) = 0$. So $uf_b = f_b$ for all b.

Step 6: the standard form. $\Lambda_1 = u^{-1}\Lambda$ is self-dual and contains $f_b = u^{-1}f_b$, $e_j^* = u^{-1}y_j$ and $c' = u^{-1}c_r$, with $h(f_b, c') = h(uf_b, c_r) = \delta_{br}$. The module $H = \sum_{j<r}(\mathcal{O}_E f_j + \mathcal{O}_E e_j^*) \subseteq \Lambda_1$ is self-dual in $X \oplus X^*$, and for $v \in \Lambda_1$, $v - \sum_j (h(v, e_j^*)f_j + h(v, f_j)e_j^*) \in \Lambda_1 \cap Y$ (integral coefficients); so $\Lambda_1 = H \oplus A_0$ with $A_0 = \Lambda_1 \cap Y$, and A_0 is self-dual in Y (the dual of an orthogonal sum is the sum of the duals). Put $A = X + A_0$. Then $A \cap (X^*)^\perp = A_0$ (an element $x + a$, $x \in X$, $a \in A_0$, is orthogonal to X^* only if $x = 0$), so $\Lambda_{\mathrm{std}}(A) = H + A_0 = \Lambda_1$. A is inner: $X \subseteq A \subseteq X^\perp$; $A \cap (X^*)^\perp = A_0$ is finitely generated; for $y = x + y_Y \in X^\perp$ ($x \in X$, $y_Y \in Y$), $h(A, y) = h(A_0, y_Y)$, which lies in $\mathcal{O}_E$ if and only if $y_Y \in A_0$, i.e. $y \in A$; $f_r \in \Lambda_1 \cap Y = A_0 \subseteq A$; and $c'_Y = c' - \sum_j (h(c', e_j^*)f_j + h(c', f_j)e_j^*) \in A_0$ has $h(f_r, c'_Y) = h(f_r, c') = 1$. Hence $\Lambda = u\Lambda_{\mathrm{std}}(A)$ with $u \in N$, $uf = f$. □

Proposition 16.23. *Let a local frame pair with parameters r and $k \ge 2r + 6$ be given such that some datum has a term. Then there is a real number $\kappa > 0$ such that for every term Λ_0 over any datum and every $\mathcal{O}_E$-submodule $\hat M$ with $M(\Lambda_0) \subseteq \hat M \subseteq M_E$ and $h(\hat M, \hat M) \subseteq \mathcal{O}_E$,*

$$\frac{\kappa}{2}\,[\hat M : M(\Lambda_0)]^{-(k-1)} \le \sigma(\hat M) \le 2\kappa\,[\hat M : M(\Lambda_0)]^{-(k-1)}.$$

Proof. **Step 1: the reference lattice.** Let Λ_s be a term over some datum; it is self-dual (Lemma 16.18(1)), hence a free $\mathcal{O}_E$-module of rank n whose dual basis lies in Λ_s; so $\Lambda_s/\varpi\Lambda_s$ is a free κ_E-module of rank n with a nondegenerate hermitian form induced by h. By Lemma 16.28, as $n \ge 2m + 3$, for every hermitian $G \in \mathrm{M}_m(\mathcal{O}_E)$,

$$\tfrac{1}{2} \le q^{-(2nm - m^2)} N_{\Lambda_s}(G) \le 2.$$

Let κ_0 be the constant of Proposition 16.25(2) for Λ_s, and put $\kappa = \kappa_0\, q^{(k-1)a} > 0$.

Step 2: the Gram determinant of $\hat M$. Let Λ_0 be a term, $M = M(\Lambda_0)$, and $\hat M$ as in the statement. By Proposition 16.8(1), M has the basis fr_b, $\theta^{-1}(\mathrm{fr}'_b - \mathrm{fr}_b)$ with Gram determinant $(-1)^{r+1}\theta^{-2(r+1)}G$, of valuation $v(G) - 2(r+1)v(\theta) = a$. As $M \subseteq \hat M \subseteq M^{\#} = \{x \in M_E : h(M,x) \subseteq \mathcal{O}_E\}$, $\hat M$ is finitely generated and spans M_E, hence free of rank m; if the basis of M is the basis of $\hat M$ times $C \in \mathrm{M}_m(\mathcal{O}_E)$, then $G_M = \bar C^{\mathsf T} G_{\hat M} C$, so $v(\det G_M) = v(\mathrm{N}(\det C)) + v(\det G_{\hat M})$, and $[\hat M : M] = |\mathcal{O}_E/\det C\,\mathcal{O}_E| = q^{v(\mathrm{N}(\det C))}$ (elementary divisors; $|\mathcal{O}_E/\varpi^e\mathcal{O}_E| = q^{2e}$ if E is a field, and componentwise if $E = F \times F$). Hence $q^{v(\det G_{\hat M})} = q^a[\hat M : M]^{-1}$.

Step 3: conclusion. By Proposition 16.25(2) and Steps 1–2,

$$\sigma(\hat M) = \kappa_0\, q^{(k-1)a}\,[\hat M : M]^{-(k-1)}\, q^{-(2nm-m^2)} N_{\Lambda_s}(G_{\hat M}) \in \left[\tfrac{\kappa}{2}, 2\kappa\right][\hat M : M]^{-(k-1)},$$

with κ independent of Λ_0 and $\hat M$, as a depends only on the frame pair. □

Definition 16.24. The notation introduced here is local to this definition and to the results that apply it. Put $\kappa_E = \mathcal{O}_E/\varpi\mathcal{O}_E$ and $\kappa_F = \mathcal{O}_F/\varpi\mathcal{O}_F$.

Spaces. A **local hermitian space** of rank n is a free E-module V of rank n with a form $h\colon V \times V \to E$, E-linear in the first variable, with $h(y,x) = \overline{h(x,y)}$ and invertible Gram matrix. Its unitary group $\mathrm{U}(V)$ is the group of E-linear $g\colon V \to V$ with $h(gx,gy) = h(x,y)$, with the topology induced from $\mathrm{GL}_E(V)$. An $\mathcal{O}_E$-submodule $\Lambda \subseteq V$ is **self-dual** if it is finitely generated and $\Lambda = \{y \in V : h(\Lambda, y) \subseteq \mathcal{O}_E\}$. An E-submodule $M \subseteq V$ is **nondegenerate** if it is free and the Gram matrix of h on an E-basis of M is invertible; its **rank** is its rank as a free module, $M^{\perp} = \{x \in V : h(M,x) = 0\}$, and

$$H_M = \{u \in \mathrm{U}(V) : uw = w \text{ for all } w \in M\}.$$

Lattices. Let U be a local hermitian space (for instance a nondegenerate M or $M^{\perp}$ with h restricted). A **lattice** in U is a finitely generated $\mathcal{O}_E$-submodule $P \subseteq U$ spanning U over E; it is free over $\mathcal{O}_E$ of rank $\operatorname{rank} U$. Its **dual** is $P^{\#} = \{x \in U : h(P,x) \subseteq \mathcal{O}_E\}$; P is **integral** if $h(P,P) \subseteq \mathcal{O}_E$ (equivalently $P \subseteq P^{\#}$), **maximal integral** if it is integral and no integral lattice in U contains it properly, and then its **discriminant module** is $D_P = P^{\#}/P$. For an $\mathcal{O}_E$-basis $l = (l_1, \dots, l_g)$ of an integral lattice P let $G_l = (h(l_a, l_b))_{a,b}$; put

$$v_{\det}(P) = v(\det G_l) \in \mathbb{Z}_{\geq 0}, \qquad \#\ker(\bar G_l\colon \kappa_E^g \to \kappa_E^g) = q^{2c(P)},$$

$\bar G_l$ the reduction of G_l modulo ϖ; $c(P) \in \{0, \dots, g\}$ is the **corank** of P.

Standard factors. Let $\epsilon = -1$ if E is a field and $\epsilon = 1$ if $E = F \times F$. For integers $j \geq 0$ put

$$\alpha^{\mathrm{st}}(j) = \prod_{i=1}^{j}(1 - \epsilon^i q^{-i}) \qquad (\alpha^{\mathrm{st}}(0) = 1).$$

Glue masses. Let V be a local hermitian space, $M \subseteq V$ a nondegenerate E-submodule and $K \subseteq H_M$ a subgroup. For an $\mathcal{O}_E$-submodule $L \subseteq M$ the **glue mass of L relative to K** is

$$\mathrm{mass}^{\mathrm{gl}}_K(L) = \sum_{[\Lambda]} \mathrm{vol}_K\big(\mathrm{Stab}_{H_M}(\Lambda)\big)^{-1} \in [0, \infty],$$

the sum over the H_M-orbits of self-dual $\Lambda \subseteq V$ with $\Lambda \cap M = L$ (one fixed representative per orbit). For a lattice T in $M^{\perp}$ put $K_T = \{u \in H_M : uT = T\}$ and $\mathrm{mass}^{\mathrm{gl}}_T(L) = \mathrm{mass}^{\mathrm{gl}}_{K_T}(L)$.

Proposition 16.25. *For a local hermitian space (V,h), a self-dual $\Lambda_s \subseteq V$, an integer $m \geq 1$ and a hermitian $G \in \mathrm{M}_m(\mathcal{O}_E)$ let $N_{\Lambda_s}(G)$ be the number of free m-tuples in $\Lambda_s/\varpi\Lambda_s$ whose Gram matrix for the form induced by h is G modulo ϖ.*

1. *Let (V,h) be a local hermitian space of rank n, $\Lambda_s \subseteq V$ self-dual, $M \subseteq V$ a nondegenerate E-submodule of rank $m \geq 1$ and $K \subseteq H_M$ a compact open subgroup. Then there is a real number $\kappa_0 > 0$ such that for every finitely generated $\mathcal{O}_E$-submodule $L \subseteq M$ spanning M with $h(L,L) \subseteq \mathcal{O}_E$, and the Gram matrix G_L of any $\mathcal{O}_E$-basis of L,*

$$\mathrm{mass}_K^{\mathrm{gl}}(L) = \kappa_0\, q^{(n-m)\,v(\det G_L)}\, q^{-(2nm-m^2)}\, N_{\Lambda_s}(G_L).$$

   Moreover, for every such L the self-dual $\Lambda \subseteq V$ with $\Lambda \cap M = L$ form finitely many H_M-orbits, and their stabilisers in H_M are compact open.

2. *Let a local frame pair with parameters r,k be given such that some datum has a term, put $m = 2r+2$, and let $\Lambda_s \subseteq V$ be self-dual. Then M_E is a nondegenerate submodule of rank m of the local hermitian space (V,h), $H_0 = H_{M_E}$, K_{ref} is a compact open subgroup of H_0, and $\sigma(\hat{M}) = \mathrm{mass}_{K_{\mathrm{ref}}}^{\mathrm{gl}}(\hat{M})$ for every $\mathcal{O}_E$-submodule $\hat{M} \subseteq M_E$. Hence there is a real number $\kappa_0 > 0$ such that for every finitely generated $\mathcal{O}_E$-submodule $L \subseteq M_E$ spanning M_E with $h(L,L) \subseteq \mathcal{O}_E$, and the Gram matrix G_L of any $\mathcal{O}_E$-basis of L,*

$$\sigma(L) = \kappa_0\, q^{(n-m)\,v(\det G_L)}\, q^{-(2nm-m^2)}\, N_{\Lambda_s}(G_L).$$

Proof. **Proof of (1). Step 0: the groups and measures.** As M is nondegenerate, $V = M \oplus W_0$ and restriction identifies H_M with $\mathrm{U}(W_0)$ (Definition 16.24). Fix an E-basis y of M; $\det \Phi(y) \neq 0$. G_{U} and H_M are unimodular (unitary groups of nondegenerate hermitian spaces, GL over F in the split case). Fix the Haar measure μ_{U} on G_{U} with $\mu_{\mathrm{U}}(K_s) = 1$, the Haar measure μ_0 on H_M with $\mu_0(K) = 1$, and let ν be the G_{U}-invariant measure on G_{U}/H_M with $\int_{G_{\mathrm{U}}} f = \int_{G_{\mathrm{U}}/H_M} \int_{H_M} f(xh)\, d\mu_0(h)\, d\nu(xH_M)$ (Weil's quotient measure). For $x \in G_{\mathrm{U}}$, applying this to $f = 1_{K_s x}$ (and $\mu_{\mathrm{U}}(K_s x) = \mu_{\mathrm{U}}(K_s)$) gives

$$\nu(K_s x H_M/H_M)\, \mu_0(H_M \cap x^{-1}K_s x) = 1, \tag{16.1}$$

since for $z = kxh_0 \in K_s x H_M$, $\{h \in H_M : zh \in K_s x\} = h_0^{-1}(H_M \cap x^{-1}K_s x)$. For a compact open subgroup $A \subseteq H_M$, $\mathrm{vol}_K(A)^{-1} = [K : A\cap K]/[A : A\cap K] = \mu_0(K)/\mu_0(A) = \mu_0(A)^{-1}$.

Step 1: orbits of tuples. Let L be as in the statement; it is free of rank m (finitely generated, torsion free, spanning M); let l be a basis, $G_L = \Phi(l)$, $\det G_L \neq 0$, and $O_L = \Phi^{-1}(G_L) \subseteq V^m$. If $t \in O_L$, the t_i are independent and $l_i \mapsto t_i$ is an isometry $M \to \mathrm{span}_E(t)$; by Witt's extension theorem for hermitian forms (Scharlau, *Quadratic and Hermitian forms*, Ch. 7; for $E = F \times F$, where V is a pair of F-spaces V_1, V_2 in perfect duality and $\mathrm{U}(V) \cong \mathrm{GL}(V_1)$, extend ι_1 on M_1 by any isomorphism of the annihilators $M_2^{\perp} \subseteq V_1$ and $\iota(M_2)^{\perp}$, and ι_2 by the contragredient) it extends to an element of G_{U}. So $O_L = G_{\mathrm{U}} l$, $\mathrm{Stab}_{G_{\mathrm{U}}}(l) = H_M$, and $\varphi_L : G_{\mathrm{U}}/H_M \to O_L$, $xH_M \mapsto xl$, is a G_{U}-equivariant continuous bijection, a homeomorphism by the open mapping theorem for transitive actions of σ-compact locally compact groups (Arens). O_L is closed in V^m.

Step 2: the glue mass as an orbit volume. Let $T_L = O_L \cap \mathrm{Prim}$, compact open in O_L and K_s-stable. For $t \in T_L$ choose $g \in G_{\mathrm{U}}$ with $t = g^{-1}l$ and put $\Lambda_t = g\Lambda_s$; as $\mathrm{span}_E(t) = g^{-1}M$ and t is primitive, $\Lambda_s \cap \mathrm{span}_E(t) = \mathrm{span}_{\mathcal{O}_E}(t)$, so $\Lambda_t \cap M = g\,\mathrm{span}_{\mathcal{O}_E}(t) = L$. The map $K_s t \mapsto H_M \Lambda_t$ is a bijection from K_s-orbits in T_L onto H_M-orbits of self-dual Λ with $\Lambda \cap M = L$: well defined (another g' with $t = g'^{-1}l$ has $g'g^{-1} \in H_M$; and $kt = (gk^{-1})^{-1}l$ gives $gk^{-1}\Lambda_s = \Lambda_t$); surjective,

because all self-dual lattices in V form one G_{U}-orbit (Lemma 16.27, applied to (V,h); self-dual modules are lattices, Definition 16.24), so $\Lambda = g\Lambda_s$ and $t = g^{-1}l$ lies in Λ_s^m, and $\Lambda_s \cap \mathrm{span}_E(t) = g^{-1}L = \mathrm{span}_{\mathcal{O}_E}(t)$ is a direct summand, so $t \in T_L$; injective, because $g'\Lambda_s = hg\Lambda_s$ with $h \in H_M$ gives $k = g'^{-1}hg \in K_s$ and $g'^{-1}l = kg^{-1}h^{-1}l = kt$. Moreover $\mathrm{Stab}_{H_M}(\Lambda_t) = H_M \cap gK_sg^{-1}$, compact open in H_M, and $K_st = \varphi_L(K_sg^{-1}H_M/H_M)$ is open in O_L; so T_L is a finite disjoint union of K_s-orbits, and by Step 0 and (16.1) with $x = g^{-1}$,

$$\mathrm{mass}_K^{\mathrm{gl}}(L) = \sum_{K_st \subseteq T_L} \mu_0(H_M \cap gK_sg^{-1})^{-1} = \sum_{K_st \subseteq T_L} \nu(K_sg^{-1}H_M/H_M) = \nu(\varphi_L^{-1}(T_L)).$$

Step 3: comparison with the fibre measure. By Lemma 16.26(1), $B \cap O_L \mapsto \lambda_{G_L}(B)$ ($B \subseteq V^m$ compact open; every compact open subset of the closed set O_L is of this form) is a finitely additive, nonnegative, nonzero, G_{U}-invariant function on the compact open subsets of O_L, i.e. a nonzero G_{U}-invariant positive functional on the locally constant compactly supported functions on $O_L \cong G_{\mathrm{U}}/H_M$. By the uniqueness of invariant measures on homogeneous spaces (Weil; Bourbaki, *Intégration*, Ch. VII, §2), $\lambda_{G_L} \circ \varphi_L = c_L\,\nu$ for a constant $c_L > 0$. Write $l = yA$ with $A \in \mathrm{GL}_m(E)$ ($l_j = \sum_i A_{ij}y_i$). Then $G_L = A^{\mathsf{T}}\Phi(y)\bar{A}$, $\varphi_L(Z) = \varphi_y(Z)A$, and Lemma 16.26(2) gives $c_L = |\mathrm{N}(\det A)|_F^{n-m}c_y$. As $\det G_L = \mathrm{N}(\det A)\det\Phi(y)$, $|\mathrm{N}(\det A)|_F^{n-m} = q^{-(n-m)(v(\det G_L)-v(\det\Phi(y)))}$.

Step 4: conclusion of (1). By Steps 2–3 and Lemma 16.26(3) ($\lambda_{G_L}(T_L) = \lambda_{G_L}(\mathrm{Prim})$ as $T_L = \mathrm{Prim} \cap O_L$),

$$\mathrm{mass}_K^{\mathrm{gl}}(L) = \frac{\lambda_{G_L}(T_L)}{c_L} = \kappa_0\, q^{(n-m)v(\det G_L)}\, q^{-(2nm-m^2)} N_{\Lambda_s}(G_L), \qquad \kappa_0 = \frac{q^{-(n-m)v(\det\Phi(y))}}{c_y} > 0,$$

and κ_0 does not depend on L. The finiteness of the set of H_M-orbits and the compactness and openness of the stabilisers were shown in Step 2 (T_L is a finite union of K_s-orbits, in bijection with these orbits). (Explicitly: $\kappa_0^{-1} = q^{(n-m)v(\det\Phi(y))}\,\mu^y(K_s)$, where μ^y is the Haar measure on $\mathrm{U}(V)$ whose quotient by μ_0 is the fibre measure $\lambda_{\Phi(y)}$ transported to $\mathrm{U}(V)/H_M$ by φ_y; the identity holds with y replaced by any basis of any such L.)

Proof of (2). $V = E^n$ with h is a local hermitian space of rank n (its Gram matrix is invertible). Let Λ_0 be a term over some datum. By Proposition 16.8(1) the m vectors $\mathrm{fr}_b, \theta^{-1}(\mathrm{fr}'_b - \mathrm{fr}_b)$ form an E-basis y of M_E with $\det\Phi(y) = \pm\theta^{-2(r+1)}G$, and $G \in F^\times$ by (Fp3) ($D_l\bar{D}_l \in F$); so M_E is free of rank m with invertible Gram matrix, i.e. nondegenerate. $H_0 = H_{M_E}$ by the two definitions. $K_{\mathrm{ref}} = H_0 \cap \mathrm{Stab}_{\mathrm{U}(V)}(\mathcal{O}_E^n)$, and the stabiliser of a lattice is a compact open subgroup of $\mathrm{U}(V)$ (it is closed in the compact open subgroup $\mathrm{GL}_n(\mathcal{O}_E)$ of $\mathrm{GL}_n(E)$, intersected with $\mathrm{U}(V)$); so K_{ref} is compact open in H_0. Self-dual modules and $\mathrm{vol}_{K_{\mathrm{ref}}}$ have the same meaning in Definition 16.2 and Definition 16.24, so the formula for $\sigma(\hat{M})$ is that for $\mathrm{mass}_{K_{\mathrm{ref}}}^{\mathrm{gl}}(\hat{M})$. Now apply (1) with $M = M_E$ and $K = K_{\mathrm{ref}}$. □

Lemma 16.26. *Let (V,h) be a local hermitian space of rank n, let $\Lambda_s \subseteq V$ be self-dual and let $m \geq 1$. Put*

$$\mathrm{Herm}_m(E) = \{X \in \mathrm{M}_m(E) : \bar{X}^{\mathsf{T}} = X\},$$
$$\Phi\colon V^m \to \mathrm{Herm}_m(E), \qquad \Phi(t) = (h(t_i,t_j))_{i,j}.$$

Give V^m and $\mathrm{Herm}_m(E)$ the Haar measures (of F-vector spaces) with

$$\mathrm{vol}(\Lambda_s^m) = 1, \qquad \mathrm{vol}(\mathrm{Herm}_m(\mathcal{O}_E)) = 1.$$

Let $S \in \mathrm{Herm}_m(E)$ with $\det S \neq 0$ and $O_S = \Phi^{-1}(S)$.

1. *For every compact open $B \subseteq V^m$ there is a lattice $U_0 \subseteq \mathrm{Herm}_m(E)$ such that* $\mathrm{vol}(B \cap \Phi^{-1}(S+U))/\mathrm{vol}(U)$ *has the same value $\lambda_S(B)$ for all $\mathcal{O}_F$-lattices $U \subseteq U_0$. The number $\lambda_S(B)$ depends only on $B \cap O_S$, is finitely additive in B, is > 0 if $B \cap O_S \neq \emptyset$, and $\lambda_S(gB) = \lambda_S(B)$ for $g \in \mathrm{U}(V)$ (acting diagonally).*

2. *Let $A \in \mathrm{GL}_m(E)$ and $BA = \{tA : t \in B\}$, where $(tA)_j = \sum_i A_{ij} t_i$. Then $\lambda_{A^\intercal S \bar{A}}(BA) = |\mathrm{N}(\det A)|_F^{n-m} \lambda_S(B)$.*

3. *Let $\mathrm{Prim} \subseteq \Lambda_s^m$ be the set of tuples whose reduction modulo ϖ is free, $G \in \mathrm{Herm}_m(\mathcal{O}_E)$ with $\det G \neq 0$, and $N_{\Lambda_s}(G)$ the number of free m-tuples in $\Lambda_s/\varpi\Lambda_s$ with Gram matrix G modulo ϖ. Then $\lambda_G(\mathrm{Prim}) = q^{-(2nm-m^2)} N_{\Lambda_s}(G)$.*

Proof. **Step 0: the differential.** For $t, \eta \in V^m$, $\Phi(t+\eta) = \Phi(t) + D_t(\eta) + \Phi(\eta)$ with $D_t(\eta)_{ij} = h(t_i, \eta_j) + h(\eta_i, t_j)$, F-linear in η. If $t_1, \dots, t_m$ are E-linearly independent, $D_t \colon V^m \to \mathrm{Herm}_m(E)$ is surjective: given X, let Y be upper triangular with $Y_{ij} = X_{ij}$ $(i < j)$, $Y_{ii} = \sigma X_{ii}$, so $X = Y + \bar{Y}^\intercal$; as h is nondegenerate there are η_i with $h(\eta_i, t_j) = Y_{ij}$ for all j, and then $D_t(\eta) = \bar{Y}^\intercal + Y = X$. The same argument over κ_E (trace $\kappa_E \to \kappa_F$ surjective, $\bar{h}$ nondegenerate) shows: if $t \in \Lambda_s^m$ has free reduction $\bar{t}$, then $\bar{D}_t \colon (\Lambda_s/\varpi\Lambda_s)^m \to \mathrm{Herm}_m(\kappa_E)$ is surjective ($\bar{t}$ extends to a basis, which has an $\bar{h}$-dual basis), with kernel of order q^{2nm-m^2}.

Step 1: a fibre count. Let P be an $\mathcal{O}_F$-lattice in $\mathrm{Herm}_m(E)$ and $\psi \colon \Lambda_s^m \to P$ a map with $\psi(\delta + \varpi^j \eta) \equiv \psi(\delta) + \varpi^j D(\eta) \pmod{\varpi^{j+1} P}$ for all $j \geq 0$, $\delta, \eta \in \Lambda_s^m$, where $D \colon \Lambda_s^m \to P$ is $\mathcal{O}_F$-linear and surjective. Then for every lattice $U' \subseteq P$ and $c \in P$, $\mathrm{vol}\{\delta : \psi(\delta) \in c + U'\} = [P : U']^{-1}$. Indeed ψ induces maps $\psi_j \colon \Lambda_s^m/\varpi^j\Lambda_s^m \to P/\varpi^j P$; the q^{2nm} lifts $\delta + \varpi^j\eta$ of a class δ map to $\psi(\delta) + \varpi^j \bar{D}(\bar{\eta})$ modulo $\varpi^{j+1}P$, and $\bar{D} \colon \Lambda_s^m/\varpi \to P/\varpi P$ is surjective, so each point of $P/\varpi^{j+1}P$ above $\psi_j(\delta)$ has exactly q^{2nm-m^2} preimages among them. By induction every fibre of ψ_j has $q^{j(2nm-m^2)}$ elements. For $\varpi^j P \subseteq U'$ the set in question is the union of $[U' : \varpi^j P]$ fibres of ψ_j, each a union of classes of volume q^{-2nmj}; its volume is $[U' : \varpi^j P] q^{-jm^2} = [P : U']^{-1}$.

Step 2: good balls. Let $t_0 \in O_S$; t_0 is linearly independent ($\det S \neq 0$), so $P_0 = D_{t_0}(\Lambda_s^m)$ is an $\mathcal{O}_F$-lattice in $\mathrm{Herm}_m(E)$ (Step 0). Choose $M_0 \geq 1$ with $\varpi^{M_0 - 1}\mathrm{Herm}_m(\mathcal{O}_E) \subseteq P_0$. Let $M \geq M_0$ and $t_1 \in O_S$ with $t_1 - t_0 \in \varpi^{M_0}\Lambda_s^m$. Then $D_{t_1} - D_{t_0}$ maps Λ_s^m into $\varpi^{M_0}\mathrm{Herm}_m(\mathcal{O}_E) \subseteq \varpi P_0$, so $D_{t_1}(\Lambda_s^m) = P_0$ (Nakayama). Put $\psi(\delta) = D_{t_1}(\delta) + \varpi^M \Phi(\delta) \in P_0$, so that $\Phi(t_1 + \varpi^M \delta) = S + \varpi^M \psi(\delta)$; ψ satisfies the hypothesis of Step 1 with $D = D_{t_1}$, $P = P_0$, since $\varpi^M(\Phi(\delta + \varpi^j\eta) - \Phi(\delta)) \in \varpi^{M+j}\mathrm{Herm}_m(\mathcal{O}_E) \subseteq \varpi^{j+1}P_0$. Hence, for the ball $\beta = t_1 + \varpi^M\Lambda_s^m$ and every lattice $U \subseteq \varpi^M P_0$,

$$\frac{\mathrm{vol}(\beta \cap \Phi^{-1}(S+U))}{\mathrm{vol}(U)} = \frac{q^{-2nmM}[P_0 : \varpi^{-M}U]^{-1}}{q^{-Mm^2}\mathrm{vol}(P_0)[P_0 : \varpi^{-M}U]^{-1}} = \frac{q^{-M(2nm-m^2)}}{\mathrm{vol}(P_0)} > 0,$$

independent of U. Call such β *good* (for t_0).

Step 3: proof of (1). O_S is closed, so $B \cap O_S$ is compact; cover it by finitely many good balls $t^{(\nu)} + \varpi^{M_\nu}\Lambda_s^m$ (Step 2 with $t_1 = t_0$) and let $M^* = \max_\nu M_\nu$. Partition B into finitely many balls of radius ϖ^{M^*} (translates of $\varpi^{M^*}\Lambda_s^m$). A ball β meeting O_S in a point t_1 is $t_1 + \varpi^{M^*}\Lambda_s^m$, contained (ultrametric) in some $t^{(\nu)} + \varpi^{M_\nu}\Lambda_s^m$, hence good by Step 2. A ball β not meeting O_S has compact image $\Phi(\beta) \not\ni S$, so $\beta \cap \Phi^{-1}(S+U) = \emptyset$ for small U. Summing over the partition gives U_0 and the constant value $\lambda_S(B)$. If B, B' are compact open with $B \cap O_S = B' \cap O_S$, apply this to $B \setminus B'$ and $B' \setminus B$ (which miss O_S) and to $B \cap B'$: $\lambda_S(B) = \lambda_S(B')$. Finite additivity is that of volume; positivity follows from Step 2. For $g \in \mathrm{U}(V)$, $\bar{g}^\intercal H g = H$ for the Gram matrix H of h, so $|\mathrm{N}(\det g)|_F = 1$; the diagonal action on $V^m = E^{n \times m}$ has E-determinant $(\det g)^m$, hence preserves volume ($|\det_F T|_F = |\mathrm{N}(\det_E T)|_F$ for E-linear T), and $\Phi(gt) = \Phi(t)$. So $\lambda_S(gB) = \lambda_S(B)$.

Step 4: proof of (2). $\Phi(tA) = A^{\mathsf T}\Phi(t)\bar A$, so $BA \cap \Phi^{-1}(A^{\mathsf T}S\bar A + U) = (B \cap \Phi^{-1}(S + U'))A$ with $U' = A^{-\mathsf T}U\bar A^{-1}$. Right multiplication by A on $E^{n\times m}$ has E-determinant $(\det A)^n$, so it multiplies volumes by $|\mathrm N(\det A)|_F^n$. The F-linear map $c_A\colon X \mapsto A^{\mathsf T}X\bar A$ of $\mathrm{Herm}_m(E)$ satisfies $c_{AA'} = c_{A'}c_A$, so $A \mapsto |\det_F c_A|_F$ is a character of $\mathrm{GL}_m(E)$; it is 1 on elementary matrices (then c_A is unipotent) and for $A = \mathrm{diag}(a_1,\dots,a_m)$ it is $\prod_i |\mathrm N a_i|_F \prod_{i<j}|\mathrm N(a_ia_j)|_F = |\mathrm N(\det A)|_F^m$ (the entry X_{ij} is multiplied by $a_i\bar a_j$); as these generate $\mathrm{GL}_m(E)$, $\mathrm{vol}(U) = |\mathrm N(\det A)|_F^m \mathrm{vol}(U')$. When U runs through small lattices, so does U'; dividing, $\lambda_{A^{\mathsf T}S\bar A}(BA) = |\mathrm N(\det A)|_F^{n-m}\lambda_S(B)$. (For $E = F \times F$ all of this holds componentwise.)

Step 5: proof of (3). $\Phi(t + \varpi^N\eta) - \Phi(t) \in \varpi^N\mathrm{Herm}_m(\mathcal O_E)$ for $t,\eta \in \Lambda_s^m$, and Prim is a union of classes modulo $\varpi\Lambda_s^m$. Let R_N be the set of classes modulo $\varpi^N\Lambda_s^m$ in Prim with $\Phi \equiv G \pmod{\varpi^N}$; $|R_1| = N_{\Lambda_s}(G)$. For $j \ge 1$ and a class $t \in R_j$, the lifts $t + \varpi^j\eta$ (η modulo ϖ) satisfy $\Phi(t + \varpi^j\eta) \equiv \Phi(t) + \varpi^j D_t(\eta) \pmod{\varpi^{j+1}}$ ($2j \ge j+1$), so the condition $\Phi \equiv G \pmod{\varpi^{j+1}}$ is the affine equation $\bar D_t(\bar\eta) = \varpi^{-j}(G - \Phi(t)) \bmod \varpi$, with exactly q^{2nm-m^2} solutions (Step 0). Hence $|R_N| = q^{(N-1)(2nm-m^2)}N_{\Lambda_s}(G)$ and

$$\frac{\mathrm{vol}(\mathrm{Prim} \cap \Phi^{-1}(G + \varpi^N\mathrm{Herm}_m(\mathcal O_E)))}{\mathrm{vol}(\varpi^N\mathrm{Herm}_m(\mathcal O_E))} = \frac{q^{-2nmN}|R_N|}{q^{-Nm^2}} = q^{-(2nm-m^2)}N_{\Lambda_s}(G)$$

for every $N \ge 1$. By (1) this is $\lambda_G(\mathrm{Prim})$. □

Lemma 16.27. *Let F be a non-archimedean local field with valuation ring $\mathcal O$, and let K be either the unramified quadratic field extension of F or $F \times F$, with its non-trivial F-involution c and maximal order R (R is the valuation ring of K, resp. $\mathcal O \times \mathcal O$). Let W be a free K-module of finite rank with a non-degenerate form $k\colon W \times W \to K$, K-linear in the first variable, with $k(y,x) = c(k(x,y))$. Call an R-lattice $L \subseteq W$ (a finitely generated R-submodule spanning W over K) **self-dual** if $L = \{x \in W :\ k(x,L) \subseteq R\}$. Then for any two self-dual R-lattices L, L' of W there is a K-linear $g\colon W \to W$ with $k(gx,gy) = k(x,y)$ for all x,y and $gL = L'$.*

Proof. ***K a field: a self-dual lattice has an orthonormal basis.*** Here R is a discrete valuation ring with uniformiser ϖ (the extension is unramified), every finitely generated torsion-free R-module is free, and $\mathrm{Tr}\colon R \to \mathcal O$ and $N\colon R^\times \to \mathcal O^\times$ are surjective (unramified extensions of local fields; Serre, *Local fields*, Ch. V, §2). Self-duality of L means $k(L,L) \subseteq R$ and that $x \mapsto k(\cdot,x)$ maps L onto $\mathrm{Hom}_R(L,R)$ (every R-linear $L \to R$ extends to a K-linear form on W, which is $k(\cdot,x)$ for a unique $x \in W$, and then $k(L,x) \subseteq R$ gives $x \in L$). We show by induction on $\mathrm{rk}_R L$ that L has a basis $e_1,\dots,e_r$ with $k(e_i,e_j) = \delta_{ij}$. Let $x \in L \smallsetminus \varpi L$. Then $k(x,\cdot)\colon L \to R$ is onto: otherwise its image lies in ϖR, so the R-linear map $k(\cdot,x) = c \circ k(x,\cdot)$ is ϖ times an element of $\mathrm{Hom}_R(L,R)$, which by self-duality is $k(\cdot,x')$ with $x' \in L$; as $k(\cdot,\varpi x') = \varpi\, k(\cdot,x')$, non-degeneracy gives $x = \varpi x' \in \varpi L$. Choose $y \in L$ with $k(x,y) = 1$. We find $z \in L$ with $u = k(z,z) \in \mathcal O^\times$ ($k(z,z)$ is c-invariant, so lies in $R \cap F = \mathcal O$): take $z = x$ or $z = y$ if $k(x,x)$ or $k(y,y)$ is a unit; otherwise choose $a \in R$ with $\mathrm{Tr}(a) = 1$ and $z = x + ay$, for which

$$k(z,z) = k(x,x) + \mathrm{Tr}\big(\bar a\, k(x,y)\big) + N(a)\, k(y,y) \equiv \mathrm{Tr}(\bar a) = 1 \pmod{\varpi}.$$

Then $L = Rz \oplus L_1$ with $L_1 = \{w \in L :\ k(w,z) = 0\}$, via $w = k(w,z)u^{-1}z + (w - k(w,z)u^{-1}z)$; the Gram matrix of L in a basis adapted to this sum is block diagonal with the unit u in one block, so its other block is invertible over R and L_1 is self-dual in $z^\perp$. Choose $t \in R^\times$ with $N(t) = u$ and put $e_1 = t^{-1}z$, so $k(e_1,e_1) = 1$, and apply the induction hypothesis to L_1. Given orthonormal bases (e_i) of L and (e_i') of L', the K-linear g with $ge_i = e_i'$ preserves k and maps L onto L'.

$K = F \times F$. Let ϵ_1, ϵ_2 be the idempotents of K (interchanged by c) and $W_i = \epsilon_i W$, so $W = W_1 \oplus W_2$. Writing $k = \epsilon_1 k_1 + \epsilon_2 k_2$ with F-valued k_i, linearity in the first variable

and semilinearity in the second give $k_1(x,y) = b(x_1, y_2)$ for an F-bilinear $b\colon W_1 \times W_2 \to F$ ($x = x_1 + x_2$, $x_i \in W_i$), and hermitian symmetry gives $k_2(x,y) = b(y_1, x_2)$; non-degeneracy of k makes b perfect. A K-linear $g = (g_1, g_2)$, $g_i \in \mathrm{GL}(W_i)$, preserves k if and only if $g_2 = (g_1^*)^{-1}$, with g_1^* the adjoint for b. Every R-lattice is $L = L_1 \oplus L_2$ with $L_i = \epsilon_i L$ an $\mathcal{O}$-lattice of W_i, and $\{x :\ k(x,L) \subseteq R\} = \{x_1 :\ b(x_1, L_2) \subseteq \mathcal{O}\} \oplus \{x_2 :\ b(L_1, x_2) \subseteq \mathcal{O}\}$; hence L is self-dual if and only if $L_2 = L_1^\perp := \{x_2 :\ b(L_1, x_2) \subseteq \mathcal{O}\}$ (the condition on L_1 then follows from perfectness of b). For self-dual $L = L_1 \oplus L_1^\perp$ and $L' = L_1' \oplus L_1'^\perp$ choose $g_1 \in \mathrm{GL}(W_1)$ with $g_1 L_1 = L_1'$ (both are free $\mathcal{O}$-modules of rank $\dim W_1$); then $(g_1^*)^{-1} L_1^\perp = (g_1 L_1)^\perp = L_1'^\perp$, and $g = (g_1, (g_1^*)^{-1})$ preserves k and maps L onto L'. □

Lemma 16.28. *Let $\bar V$ be a free κ_E-module of rank n with a nondegenerate hermitian form $\bar h$, and $m \geq 0$ an integer with $n \geq 2m+3$. For every hermitian matrix $\bar G \in \mathrm{M}_m(\kappa_E)$ the number $N(\bar G)$ of free m-tuples $(z_1, \dots, z_m) \in \bar V^m$ with $\bar h(z_a, z_b) = \bar G_{ab}$ for all a, b satisfies*

$$\tfrac{1}{2}\, q^{2nm-m^2} \leq N(\bar G) \leq 2\, q^{2nm-m^2}.$$

Proof. Let c be the integer with $\#\ker \bar G = q^{2c}$; $c \leq m$, since in the field case $\ker \bar G$ is a subspace of κ_E^m, and in the split case $\bar G = (B, B^{\mathsf T})$ with $\ker \bar G = \ker B \times \ker B^{\mathsf T}$, of order $q^{2(m - \mathrm{rk}\, B)}$. As $n \geq 2m+3$, $n - m \geq m + 3 > c$, so Lemma 16.29(2) gives

$$q^{-(2nm-m^2)} N(\bar G) = \frac{\alpha(n)}{\alpha(n-m-c)} = \prod_{i=n-m-c+1}^{n} (1 - \epsilon^i q^{-i}),$$

a product over indices $i \geq n - 2m + 1 \geq 4$. With $q \geq 2$, $\sum_{i \geq 4} q^{-i} \leq \sum_{i \geq 4} 2^{-i} = \frac{1}{8}$, so the product lies between $1 - \frac{1}{8} \geq \frac{1}{2}$ and $\prod_{i \geq 4}(1 + 2^{-i}) \leq e^{1/8} < 2$ (for $m = 0$ it is the empty product 1). □

Lemma 16.29. *The notation introduced here is local to this result and to the results that apply it. Let $\epsilon = -1$ if κ_E is a field and $\epsilon = 1$ otherwise, and $\alpha(j) = \prod_{i=1}^{j}(1 - \epsilon^i q^{-i})$ for integers $j \geq 0$ ($\alpha(0) = 1$). Let $\bar V$ be a free κ_E-module of rank $n \geq 0$ with a form $\bar h$, κ_E-linear in the first variable, with $\bar h(y,x) = \overline{\bar h(x,y)}$ and invertible Gram matrix.*

1. *For $g \in \kappa_F$ the number of $x \in \bar V$ with $\bar h(x,x) = g$ is $q^{2n-1} + \epsilon^n(q^n - q^{n-1})$ if $g = 0$ and $q^{2n-1} - \epsilon^n q^{n-1}$ if $g \neq 0$; the number of free $x \in \bar V$ with $\bar h(x,x) = 0$ is $(q^{n-1} - \epsilon^{n-1})(q^n - \epsilon^n)$.*
2. *Let $m \geq 0$, let $\bar G \in \mathrm{M}_m(\kappa_E)$ be hermitian ($\bar G_{ba} = \overline{\bar G_{ab}}$), and let $c \geq 0$ be the integer with $\#\ker(\bar G\colon \kappa_E^m \to \kappa_E^m) = q^{2c}$. Then the number $N(\bar G)$ of free m-tuples $(z_1, \dots, z_m) \in \bar V^m$ with $\bar h(z_a, z_b) = \bar G_{ab}$ for all a, b is*

$$N(\bar G) = \begin{cases} q^{2nm-m^2}\, \dfrac{\alpha(n)}{\alpha(n-m-c)} & \text{if } c \leq n-m, \\ 0 & \text{if } c > n-m. \end{cases}$$

Proof. **Step 0: orthogonal splittings and orthonormal bases.** (a) Let $x_1, \dots, x_s \in \bar V$ have invertible Gram matrix $C = (\bar h(x_a, x_b))$. If $\sum_i a_i x_i = 0$, pairing with x_b gives $\sum_i a_i C_{ib} = 0$ for all b, so $a = 0$: the family is free. For $v \in \bar V$ there is a unique $c \in \kappa_E^s$ with $\bar h(v - \sum_i c_i x_i, x_b) = 0$ for all b (C invertible), so $\bar V = X \oplus X^\perp$ with $X = \mathrm{span}(x)$, $X^\perp = \{v : \bar h(v, x_b) = 0\ \forall b\}$; $X^\perp$ is free of rank $n - s$ (componentwise dimensions) and its Gram matrix is invertible (an element of $X^\perp$ orthogonal to $X^\perp$ is orthogonal to $\bar V$). (b) $\bar V$ has a basis e with $\bar h(e_a, e_b) = \delta_{ab}$. Field case,

induction on $n \geq 1$: some z has $\bar{h}(z,z) \neq 0$, since otherwise $0 = \bar{h}(x+ay, x+ay) = \mathrm{Tr}(\bar{a}\,\bar{h}(x,y))$ for all x, y, a, so $\bar{h}(x,y) = 0$ (Tr is onto), contradicting invertibility; as $\bar{h}(z,z) \in \kappa_F^\times = \mathrm{N}(\kappa_E^\times)$ we may scale z to norm 1, and apply (a) and the induction hypothesis to $z^\perp$. Split case: let $\epsilon_1 = (1,0)$, $\epsilon_2 = (0,1)$ and $V_i = \epsilon_i \bar{V}$, so $\bar{V} = V_1 \times V_2$, $x = (x_1, x_2)$. Write $\bar{h} = (h_1, h_2)$. Linearity in the first and semilinearity in the second variable give $h_1(x,y) = h_1(\epsilon_1 x, \epsilon_2 y)$, so $h_1(x,y) = b(x_1, y_2)$ for an $\mathbb{F}_q$-bilinear $b\colon V_1 \times V_2 \to \mathbb{F}_q$, and $\bar{h}(y,x) = \overline{\bar{h}(x,y)}$ gives $h_2(x,y) = b(y_1, x_2)$; b is perfect as the Gram matrix is invertible; dual bases (e'_a), (e''_a) for b give $e_a = (e'_a, e''_a)$.

Step 1: proof of (1). Field case: write $\bar{V} = \kappa_E u \oplus u^\perp$ with $\bar{h}(u,u) = 1$ (Step 0). For $x = au + x'$, $\bar{h}(x,x) = \mathrm{N}(a) + \bar{h}(x',x')$, so by induction on n (the count for $g \neq 0$ not depending on g)

$$s_n = s_{n-1} + (q^2-1)t_{n-1}, \qquad t_n = (q+1)s_{n-1} + (q^2-q-1)t_{n-1}, \qquad s_0 = 1,\ t_0 = 0,$$

(for t_n: $a = 0$ contributes t_{n-1}; the $q+1$ elements a with $\mathrm{N}(a) = g$ contribute s_{n-1} each; the $(q+1)(q-2)$ others t_{n-1} each).

Every nonzero vector is free, so $J(n) = s_n - 1 = (q^{n-1} + (-1)^n)(q^n - (-1)^n)$ (expand). Split case: by Step 0(b), $\bar{V} = \mathbb{F}_q^n \times \mathbb{F}_q^n$ with $\bar{h}(x,x) = (x_1 \cdot x_2, x_1 \cdot x_2)$. For $g \neq 0$: $x_1 \neq 0$ ($q^n - 1$ choices) and x_2 in an affine hyperplane (q^{n-1}): $t_n = q^{2n-1} - q^{n-1}$. For $g = 0$: $x_1 = 0$ gives q^n, $x_1 \neq 0$ gives $(q^n-1)q^{n-1}$: $s_n = q^{2n-1} + q^n - q^{n-1}$. Free means $x_1 \neq 0 \neq x_2$: $J(n) = (q^n-1)(q^{n-1}-1)$. In both cases these are the formulas of (1).

Step 2: invariance. For $A \in \mathrm{GL}_m(\kappa_E)$ and $z \in \bar{V}^m$ put $(zA)_b = \sum_a A_{ab} z_a$. Then $\bar{h}((zA)_a, (zA)_b) = (A^\mathsf{T} \bar{G} \bar{A})_{ab}$ if z has Gram matrix $\bar{G}$, $z \mapsto zA$ is a bijection of free m-tuples (inverse $z \mapsto zA^{-1}$), and $\ker(A^\mathsf{T}\bar{G}\bar{A}) = \bar{A}^{-1} \ker \bar{G}$. So $N(\bar{G}) = N(A^\mathsf{T}\bar{G}\bar{A})$ and c is the same for both.

Step 3: normal form. Put $s = m - c$. There is $A \in \mathrm{GL}_m(\kappa_E)$ with $A^\mathsf{T}\bar{G}\bar{A} = \mathrm{diag}(I_s, 0_c)$. Field case: $\beta(x,y) = x^\mathsf{T}\bar{G}\bar{y}$ is a hermitian form on κ_E^m (column vectors) with radical $R = \{y : \bar{G}\bar{y} = 0\} = \overline{\ker \bar{G}}$ of dimension c; on a complement C of R, β is nondegenerate (if $y \in C$ and $\beta(C,y) = 0$ then $\beta(\kappa_E^m, y) = 0$, so $y \in R \cap C = 0$), so C has a β-orthonormal basis $a_1, \ldots, a_s$ (Step 0(b)); with a basis $a_{s+1}, \ldots, a_m$ of R, the matrix A with columns a_j has $A^\mathsf{T}\bar{G}\bar{A} = (\beta(a_i, a_j)) = \mathrm{diag}(I_s, 0_c)$. Split case: $\bar{G} = (B, B^\mathsf{T})$ with $B \in \mathrm{M}_m(\mathbb{F}_q)$, $\ker \bar{G} = \ker B \times \ker B^\mathsf{T}$, so $c = m - \mathrm{rk}\, B$; for $A = (A_1, A_2)$, $\bar{A} = (A_2, A_1)$ and $A^\mathsf{T}\bar{G}\bar{A} = (A_1^\mathsf{T} B A_2, (A_1^\mathsf{T} B A_2)^\mathsf{T})$, and invertible A_1, A_2 with $A_1^\mathsf{T} B A_2 = \mathrm{diag}(I_s, 0_c)$ exist. By Step 2 we may assume $\bar{G} = \mathrm{diag}(I_s, 0_c)$.

Step 4: the nondegenerate part. If $s > n$ there is no orthonormal s-tuple (it would be free, Step 0(a)), so $N(\bar{G}) = 0$; and then $n - m < -c \leq c$, as claimed. Let $s \leq n$. A tuple $(x_1, \ldots, x_s, y_1, \ldots, y_c)$ has Gram matrix $\mathrm{diag}(I_s, 0_c)$ if and only if x is orthonormal, every y_b lies in $X^\perp$ ($X = \mathrm{span}(x)$) and $\bar{h}(y_a, y_b) = 0$ for all a, b; it is free if and only if y is free, as $\bar{V} = X \oplus X^\perp$ with basis x of X (Step 0(a)). Choosing $x_1, \ldots, x_s$ in turn, x_{i+1} is a vector of norm 1 in $(\mathrm{span}(x_1, \ldots, x_i))^\perp$, which has rank $n - i$ and invertible Gram matrix; so by (1) the number of orthonormal s-tuples is

$$\prod_{i=0}^{s-1} t_{n-i} = \prod_{i=0}^{s-1} q^{2(n-i)-1} \frac{\alpha(n-i)}{\alpha(n-i-1)} = q^{2ns-s^2} \frac{\alpha(n)}{\alpha(n-s)}.$$

For each of them $X^\perp$ has rank $d = n - s$ and invertible Gram matrix, so

$$N(\bar{G}) = q^{2ns-s^2} \frac{\alpha(n)}{\alpha(n-s)}\, I(d,c),$$

where $I(d,c)$ is the number of free c-tuples with zero Gram matrix in a space of rank d with invertible Gram matrix (it depends only on d, c by Step 5).

Step 5: the isotropic part. $I(d,0)=1$; for $c\geq 1$, $I(d,c)=0$ if $d\leq 1$ (then $J(d)=0$), and $I(d,c)=J(d)\,q^{2(c-1)}\,I(d-2,c-1)$ if $d\geq 2$. Indeed, choose y_1 free of norm 0 ($J(d)$ ways). As y_1 is free and the Gram matrix is invertible, $v\mapsto \bar{h}(v,y_1)$ is onto κ_E; take w with $\bar{h}(w,y_1)=1$, $\sigma\in\kappa_E$ with $\mathrm{Tr}(\sigma)=1$, and $y^*=w-\sigma\bar{h}(w,w)y_1$. Then $\bar{h}(y^*,y_1)=1$ and $\bar{h}(y^*,y^*)=\bar{h}(w,w)(1-\bar{\sigma}-\sigma)=0$ ($\bar{h}(w,w)\in\kappa_F$). The pair (y_1,y^*) has Gram matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, so by Step 0(a) $\bar{V}=P\oplus Y'$ with $P=\mathrm{span}(y_1,y^*)$ and $Y'=P^{\perp}$ of rank $d-2$ with invertible Gram matrix. A vector $v=ay_1+by^*+w'$ ($w'\in Y'$) has $\bar{h}(v,y_1)=b$; so the vectors orthogonal to y_1 form $\kappa_E y_1\oplus Y'$. Hence the tuples $(y_2,\dots,y_c)$ completing y_1 are $y_b=a_by_1+w_b$ with $a_b\in\kappa_E$ arbitrary (q^2 choices each) and $w_b\in Y'$; their Gram matrix is that of $(w_2,\dots,w_c)$ (y_1 is isotropic and orthogonal to Y'), and $(y_1,\dots,y_c)$ is free if and only if $(y_1,w_2,\dots,w_c)$ is (a unitriangular change), if and only if $(w_2,\dots,w_c)$ is free in Y' (as $\kappa_E y_1\oplus Y'$ is direct and y_1 is free). This gives the recursion. Solving it with Step 1: if $2c\leq d$,

$$I(d,c)=q^{c(c-1)}\prod_{i=0}^{c-1}J(d-2i)=q^{c(c-1)}\prod_{i=0}^{c-1}q^{2(d-2i)-1}\frac{\alpha(d-2i)}{\alpha(d-2i-2)}=q^{2dc-c^2}\frac{\alpha(d)}{\alpha(d-2c)},$$

and if $2c>d$ the recursion reaches $I(d',c')$ with $c'\geq 1$, $d'\leq 1$, so $I(d,c)=0$.

Step 6: conclusion. With $d=n-s=n-m+c$: $2c\leq d$ if and only if $c\leq n-m$; then $d-2c=n-m-c$ and

$$N(\bar{G})=q^{2ns-s^2}\frac{\alpha(n)}{\alpha(n-s)}\cdot q^{2(n-s)c-c^2}\frac{\alpha(n-s)}{\alpha(n-m-c)}=q^{2nm-m^2}\frac{\alpha(n)}{\alpha(n-m-c)},$$

since $2ns-s^2+2(n-s)c-c^2=2n(s+c)-(s+c)^2$ and $s+c=m$; if $c>n-m$, $N(\bar{G})=0$. □

16.4 Averaging over Eichler moves and the local non-generic bound

This subsection proves the cancellation for the terms of second type and then the local non-generic bound Theorem 16.38. For the terms of second type with a given defect order and label, Proposition 16.30 constructs from Eichler moves an invariant symmetric averaging kernel whose total weight differs from 1, and whose average of the phases differs from 0, by at most $\frac{24}{7}q^{-(k-2r-4)}$. The phases are controlled by Gauss-sum bounds for the quadric condition (Lemma 16.37), and the geometry of the moves reduces the kernel to the orthogonal complement of the frame span (Proposition 16.31, Proposition 16.32). Lemma 16.39 turns such a kernel into cancellation of the orbit sums of the phases. Theorem 16.38 then derives from (Lg1)–(Lg5), which encode the mass bound for the defect orders, the class count of the terms of first type and this cancellation, the bound $H^{39(r+1)^2}q^{-(k-2r-5)}$ for the non-generic amplitudes against the total mass.

Proposition 16.30. *Let a local frame pair with parameters r and $k\geq 4r+7$ be given, d a datum, $i\geq 1$ an integer and $\lambda\in\mathbb{Z}^2$, and let $\mathcal{P}$ be the set of terms Λ over d of second type with $b_d(\Lambda)=q^i$ and $\lambda_d(\Lambda)=\lambda$. Then there is a function $m\colon\mathcal{P}\times\mathcal{P}\to[0,\infty)$ with $m(g\Lambda,g\Lambda_1)=m(\Lambda,\Lambda_1)$ for $g\in\Sigma$, $m(\Lambda,\Lambda_1)=m(\Lambda_1,\Lambda)$, $\{\Lambda_1: m(\Lambda,\Lambda_1)\neq 0\}$ finite for every Λ, and for every $\Lambda\in\mathcal{P}$*

$$\Big|\sum_{\Lambda_1\in\mathcal{P}}m(\Lambda,\Lambda_1)-1\Big|\leq\tfrac{24}{7}q^{-(k-2r-4)},\qquad\Big|\sum_{\Lambda_1\in\mathcal{P}}m(\Lambda,\Lambda_1)\,\omega_d(\Lambda_1)\Big|\leq\tfrac{24}{7}q^{-(k-2r-4)}.$$

Proof. Put $\rho=k-2r-3\geq 2r+4\geq 4$, so $q^{1-\rho}=q^{-(k-2r-4)}$. Take m from Proposition 16.31(Mr1); it has the required invariance, symmetry and finiteness. Fix $\Lambda\in\mathcal{P}$ and write $\lambda=(J,e)$; J is a level, so $0\leq J\leq r+1$ and $F_J(f)+F_J(f')\subseteq M_E$.

Step 1: the complement U_Λ splits off ρ orthonormal vectors. By (Mr0) there are $e_1,\dots,e_\rho \in \Lambda\cap W_0$ with $h(e_a,e_b)=\delta_{ab}$. As $F_J(f)+F_J(f')\subseteq M_E$ and $W_0=M_E^\perp$, $e_a\in U_\Lambda$. For $u\in U_\Lambda$, $h(u,e_a)\in\mathcal{O}_E$ (Λ is integral, Proposition 16.8(1)), so $u'=u-\sum_a h(u,e_a)e_a\in U_\Lambda$ and $u'\perp e_a$; hence $U_\Lambda=\mathcal{O}_Ee_1\oplus\cdots\oplus\mathcal{O}_Ee_\rho\oplus U'$ with $U'=U_\Lambda\cap\{e_1,\dots,e_\rho\}^\perp$, an orthogonal decomposition, $\mathcal{O}_Ee_a\cong\mathcal{O}_E$ (as $h(ce_a,e_a)=c$), and U_Λ is finitely generated ($\Lambda=u\Lambda_{\mathrm{std}}(A)$ is finitely generated by the inner condition, and $\mathcal{O}_E$ is noetherian). So Lemma 16.37 applies to U_Λ with these e_a and a ($a\ge0$ by Proposition 16.8(2)); its set Z is Z_Λ, and $L'_\Lambda\subseteq L$: if $h(u,e_1),\dots,h(u,e_\rho)$ generate $\mathcal{O}_E$, say $\sum_a\gamma_a h(u,e_a)=1$, then $y=\sum_a\bar\gamma_a e_a\in\Lambda\cap W_0$ and $h(u,y)=\sum_a\gamma_a h(u,e_a)=1$.

Step 2: the row sum. By (Mr1) and Lemma 16.37(1),(2),

$$\Big|\sum_{\Lambda_1} m(\Lambda,\Lambda_1)-1\Big|\le|q^a\mathrm{vol}(Z_\Lambda)-1|+q^a\mathrm{vol}(Z_\Lambda\cap L)\le\tfrac{8}{7}q^{1-\rho}+\tfrac{16}{7}q^{1-\rho}=\tfrac{24}{7}q^{-(k-2r-4)}.$$

Step 3: the phase average. By (Mr2), $|\omega_d(\Lambda)|\le1$, $\sum_b p_b=1$ and Lemma 16.37(3) (each $u\mapsto\psi_0(h(y_b,u))$ is a nontrivial continuous character of U_Λ) and (2),

$$\begin{aligned}\Big|\sum_{\Lambda_1} m(\Lambda,\Lambda_1)\,\omega_d(\Lambda_1)\Big|&\le\max_b q^a\Big|\int_{Z_\Lambda}\psi_0(h(y_b,u))\,du\Big|+q^a\mathrm{vol}(Z_\Lambda\cap L)\\&\le\tfrac{8}{7}q^{1-\rho}+\tfrac{16}{7}q^{1-\rho}=\tfrac{24}{7}q^{-(k-2r-4)}.\end{aligned}$$

□

Proposition 16.31. *Let a local frame pair with parameters r and $k\ge4r+7$ be given, d a datum, $i\ge1$ an integer, $\lambda\in\mathbb{Z}^2$, and $\mathcal{P}$ the set of terms over d of second type with $b_d=q^i$ and $\lambda_d=\lambda$. For $\Lambda\in\mathcal{P}$ with $\lambda=(J,e)$ put $U_\Lambda=\{u\in\Lambda: h(u,F_J(f)+F_J(f'))=0\}$ with its Haar probability measure du, $Z_\Lambda=\{u\in U_\Lambda: h(u,u)\in\varpi^a\mathcal{O}_F\}$ and $L'_\Lambda=\{u\in U_\Lambda: h(u,y)\ne1$ for all $y\in\Lambda\cap W_0\}$. Then:*

(Mr0) Orthonormal complement *for every term Λ over d, $\Lambda\cap W_0$ contains $k-2r-3$ vectors $e_1,\dots,e_{k-2r-3}$ with $h(e_a,e_b)=\delta_{ab}$;*

(Mr1) Kernel *there is a function $m\colon\mathcal{P}\times\mathcal{P}\to[0,\infty)$ with $m(g\Lambda,g\Lambda_1)=m(\Lambda,\Lambda_1)$ for $g\in\Sigma$, $m(\Lambda,\Lambda_1)=m(\Lambda_1,\Lambda)$, $\{\Lambda_1: m(\Lambda,\Lambda_1)\ne0\}$ finite for every Λ, and, for every $\Lambda\in\mathcal{P}$,*

$$q^a\mathrm{vol}(Z_\Lambda)-q^a\mathrm{vol}(Z_\Lambda\cap L'_\Lambda)\le\sum_{\Lambda_1\in\mathcal{P}}m(\Lambda,\Lambda_1)\le q^a\mathrm{vol}(Z_\Lambda);$$

(Mr2) Phases *for this m and every $\Lambda\in\mathcal{P}$ there are a nonempty finite set B, weights $p_b\ge0$ with $\sum_b p_b=1$ and vectors $y_b\in V$ ($b\in B$) such that $u\mapsto\psi_0(h(y_b,u))$ is a nontrivial character of U_Λ for every b and*

$$\Big|\sum_{\Lambda_1\in\mathcal{P}}m(\Lambda,\Lambda_1)\,\omega_d(\Lambda_1)-\omega_d(\Lambda)\sum_{b\in B}p_b\,q^a\int_{Z_\Lambda}\psi_0(h(y_b,u))\,du\Big|\le q^a\mathrm{vol}(Z_\Lambda\cap L'_\Lambda).$$

Proof. **Step 0: proof of (Mr0).** Let Λ be a term, $S=\mathrm{sat}(\Lambda)$, $B=\Lambda\cap W_0$. Λ is self-dual (Lemma 16.18(1)) and $V=M_E\oplus W_0$ (Proposition 16.8(1)). With the orthogonal projections, $\pi_M\Lambda\subseteq S^\#$ (as $h(\pi_My,x)=h(y,x)$ for $x\in S$) and $\{x\in M_E: h(x,\pi_M\Lambda)\subseteq\mathcal{O}_E\}=M_E\cap\Lambda^\#=S$, so $\pi_M\Lambda=S^\#$; likewise $\pi_W\Lambda=B^\#$; and $y\mapsto\pi_My+S$, $y\mapsto\pi_Wy+B$ are surjections $\Lambda\to S^\#/S$, $\Lambda\to B^\#/B$ with the common kernel $S\oplus B$ (if $\pi_My\in S\subseteq\Lambda$ then $\pi_Wy\in\Lambda\cap W_0$). Hence

$B^\#/B \cong S^\#/S$, which is generated by $2r+2$ elements ($S^\#$ is a lattice in M_E). Reduce modulo ϖ: $\bar B = B/\varpi B$ is free of rank $k-1$ over κ_E with the hermitian form $\bar h$ induced by h ($h(B,B) \subseteq \mathcal{O}_E$). Its radical is $(B \cap \varpi B^\#)/\varpi B \cong (B^\# \cap \varpi^{-1}B)/B$, the ϖ-torsion of $B^\#/B$, hence generated by $2r+2$ elements; so $\bar B$ has a nondegenerate quotient of rank at least $\rho = k-1-(2r+2) = k-2r-3$. A nondegenerate hermitian κ_E-module $\neq 0$ contains a vector $\bar v$ with $\bar h(\bar v,\bar v) \in \kappa_F^\times$ (if all $\bar h(\bar v,\bar v)$ vanished, polarisation would give $\mathrm{tr}(\lambda \bar h(\bar v,\bar w)) = 0$ for all λ, hence $\bar h = 0$). Lift $\bar v$ to $v \in B$: $h(v,v) \in \mathcal{O}_F^\times$ is a norm $c\bar c$, and $e_1 = c^{-1}v$ has $h(e_1,e_1) = 1$. Then $B = \mathcal{O}_E e_1 \oplus B_1$ with $B_1 = B \cap e_1^\perp$ ($v \mapsto v - h(v,e_1)e_1$), $B_1^\#/B_1 \cong B^\#/B$, and the nondegenerate quotient of $\bar B_1$ has rank one less. After ρ steps we have $e_1,\dots,e_\rho \in \Lambda \cap W_0$ with $h(e_a,e_b) = \delta_{ab}$.

Step 1: set-up. If $\mathcal{P} = \emptyset$ there is nothing to prove; let $\lambda = (J,e)$ with $\mathcal{P} \neq \emptyset$. For $\Lambda \in \mathcal{P}$ the term is not generic, so some $y \in \mathrm{sat}(\Lambda) \setminus M(\Lambda)$ has level ≥ 1 (Proposition 16.8(4)), whence $1 \leq J \leq r$. $\mathrm{sat}(\Lambda)/M(\Lambda)$ is an $\mathcal{O}_E$-module of order $\beta = q^i$; its classes of level $\leq J-1$ form a submodule of order q^e (finite $\mathcal{O}_E$-modules have order a power of q), so $\beta_J = q^i - q^e \geq 1$ classes have level exactly J, the same for all $\Lambda \in \mathcal{P}$; levels are constant on $M(\Lambda)$-classes (Proposition 16.8(4)). Let $\mathfrak{G} = F_J(f) \times F_J(f') \times Y_J^\perp \times E_0$ with $E_0 = \{t \in E : t+\bar t = 0\}$, a finite-dimensional F-vector space, $T = \varpi^a\mathcal{O}_E \cap E_0$, and for $w = (x,x',u,t) \in \mathfrak{G}$ put $s_w = h(u,u)/2 + t$ (then $s_w + \bar s_w = h(u,u)$, and $t \mapsto s_w$ is a bijection onto the solutions of this equation) and $E_w = E(x,u,s_w)$. For $\Lambda \in \mathcal{P}$ let $A_\Lambda = B_\Lambda \times U_\Lambda \times T$ (a compact open subgroup: $F_J(f) \cap F_J(f') = 0$, so B_Λ is the preimage of the lattice $\theta\Lambda \cap Y_J$), $\mathrm{Mv}_\Lambda = \{w : (x,x',u,s_w) \text{ a move of } \Lambda\}$, and $S(\Lambda,\Lambda_1) = \{w \in \mathrm{Mv}_\Lambda : E_w^{-1}\Lambda = \Lambda_1\}$. Fix a Haar measure μ on $\mathfrak{G}$ and define

$$m(\Lambda,\Lambda_1) = q^a \frac{\beta}{\beta_J} \frac{\mu(S(\Lambda,\Lambda_1))}{\mu(A_\Lambda)} \qquad (\Lambda,\Lambda_1 \in \mathcal{P}).$$

Step 2: finiteness. Mv_Λ is compact and open: $(x,x') \mapsto \theta^{-1}(x+x') + M(\Lambda) \in \mathrm{sat}(\Lambda)/M(\Lambda)$ is continuous into a finite discrete set on the open B_Λ, U_Λ is compact open, and $\{(u,t) : s_w \in \varpi^a\mathcal{O}_E\}$ is open, closed and bounded over the compact U_Λ. The map $w \mapsto E_w$ is continuous into $\mathrm{GL}(V)$, and $E_w^{-1}\Lambda = E_{w_0}^{-1}\Lambda$ as soon as $E_{w_0}E_w^{-1}$ lies in the open subgroup $\mathrm{Stab}(\Lambda)$; so $w \mapsto E_w^{-1}\Lambda$ is locally constant on Mv_Λ, takes finitely many values, and the sets $S(\Lambda,\Lambda_1)$ are open and closed. Hence $\{\Lambda_1 : m(\Lambda,\Lambda_1) \neq 0\}$ is finite, and $\sum_{\Lambda_1 \in \mathcal{P}} m(\Lambda,\Lambda_1)\phi(\Lambda_1) = q^a \frac{\beta}{\beta_J} \mu(A_\Lambda)^{-1} \int_{\mathrm{Mv}_\Lambda} 1[E_w^{-1}\Lambda \in \mathcal{P}]\, \phi(E_w^{-1}\Lambda)\, d\mu(w)$ for every bounded ϕ.

Step 3: Σ-invariance. Let $g \in \Sigma$ and $\Lambda \in \mathcal{P}$; then $g\Lambda \in \mathcal{P}$ (Proposition 16.12). $\phi_g(x,x',u,t) = (gx,gx',gu,t)$ is an F-linear automorphism of $\mathfrak{G}$ (g preserves $F_J(f)$, $F_J(f')$, hence $Y_J^\perp$), so $\mu \circ \phi_g = c_g\mu$ for a constant $c_g > 0$. It maps A_Λ onto $A_{g\Lambda}$ and Mv_Λ onto $\mathrm{Mv}_{g\Lambda}$: $x+x' \in \theta\Lambda \iff gx+gx' \in \theta g\Lambda$, $h(gu,gu) = h(u,u)$, and the level of $\theta^{-1}(gx+gx')$ for $g\Lambda$ equals that of $\theta^{-1}(x+x')$ for Λ, by the characterisation $\ell \leq j \iff y \in M + F_j(f) + F_j(f')$ of Proposition 16.8(4) and $M(g\Lambda) = gM(\Lambda)$ (Proposition 16.12). As $gE(x,u,s)g^{-1} = E(gx,gu,s)$ (g is an isometry), $E_{\phi_g w}^{-1} g\Lambda = gE_w^{-1}\Lambda$, so $\phi_g(S(\Lambda,\Lambda_1)) = S(g\Lambda,g\Lambda_1)$, and $m(g\Lambda,g\Lambda_1) = m(\Lambda,\Lambda_1)$.

Step 4: symmetry. Let $\Lambda,\Lambda_1 \in \mathcal{P}$ and $\iota(x,x',u,t) = (x,x',-u,-t)$, an involutive automorphism of $\mathfrak{G}$, so $\mu \circ \iota = \mu$. For $w \in S(\Lambda,\Lambda_1)$, $s_{\iota w} = h(u,u)/2 - t = \bar s_w$ ($h(u,u) \in F$, $\bar t = -t$), so by (Mg4) of Proposition 16.32 $\iota w \in \mathrm{Mv}_{\Lambda_1}$ and $E_{\iota w}^{-1}\Lambda_1 = E_w\Lambda_1 = \Lambda$; thus $\iota S(\Lambda,\Lambda_1) \subseteq S(\Lambda_1,\Lambda)$, and symmetrically, so $\mu(S(\Lambda,\Lambda_1)) = \mu(S(\Lambda_1,\Lambda))$. By (Mg5), $\mu(A_\Lambda) = \mu(A_{\Lambda_1})$ (for commensurable compact open subgroups $\mu(H) = [H:C]\mu(C)$). Hence $m(\Lambda,\Lambda_1) = m(\Lambda_1,\Lambda)$.

Step 5: the measure of the moves. Fix $\Lambda \in \mathcal{P}$ and write $\mu = \mu_1 \otimes \mu_2 \otimes \mu_3$ with Haar measures on $F_J(f) \times F_J(f')$, $Y_J^\perp$, E_0 normalised by $\mu_1(B_\Lambda) = \mu_2(U_\Lambda) = \mu_3(T) = 1$, so $\mu(A_\Lambda) = 1$ and $\mu_2|_{U_\Lambda}$ is the Haar probability du. The indicator of Mv_Λ is the product of $1[(x,x') \in B^J]$, $B^J = \{(x,x') \in B_\Lambda : \ell(\theta^{-1}(x+x')) = J\}$, of $1[u \in U_\Lambda]$ and of $1[s_w \in \varpi^a\mathcal{O}_E]$. (a) The homomorphism $\pi\colon B_\Lambda \to \mathrm{sat}(\Lambda)/M(\Lambda)$, $(x,x') \mapsto \theta^{-1}(x+x') + M(\Lambda)$, is surjective: every $y \in \mathrm{sat}(\Lambda)$ has level $\leq J$, so $y = m_0 + z_1 + z_2$ with $m_0 \in M(\Lambda)$, $z_1 \in F_J(f)$, $z_2 \in F_J(f')$

(Proposition 16.8(4)), $z_1+z_2\in\Lambda$, and $(\theta z_1,\theta z_2)\in B_\Lambda$. B^J is the preimage of the β_J classes of level J, each a coset of $\ker\pi$, of measure $1/\beta$; so $\mu_1(B^J)=\beta_J/\beta$. (b) For $u\in U_\Lambda$, the set $\{t\in E_0: h(u,u)/2+t\in\varpi^a\mathcal{O}_E\}$ is empty unless $h(u,u)\in\varpi^a\mathcal{O}_F$ ($s+\bar s\in\varpi^a\mathcal{O}_E\cap F$), and then contains $\sigma h(u,u)-h(u,u)/2$ and is a coset of T, of measure 1. So the t-fibre has measure $1[u\in Z_\Lambda]$. By Fubini, $\mu(\mathrm{Mv}_\Lambda)=\frac{\beta_J}{\beta}\mathrm{vol}(Z_\Lambda)$ and $\mu(\mathrm{Mv}_\Lambda\cap\{u\in L'_\Lambda\})=\frac{\beta_J}{\beta}\mathrm{vol}(Z_\Lambda\cap L'_\Lambda)$.

Step 6: proof of (Mr1). By Step 2 with $\phi=1$, $\sum_{\Lambda_1\in\mathcal{P}}m(\Lambda,\Lambda_1)=q^a\frac{\beta}{\beta_J}\mu\{w\in\mathrm{Mv}_\Lambda: E_w^{-1}\Lambda\in\mathcal{P}\}$. By (Mg3), every $w\in\mathrm{Mv}_\Lambda$ with $u\notin L'_\Lambda$ (i.e. $h(u,y)=1$ for some $y\in\Lambda\cap W_0$) has $E_w^{-1}\Lambda\in\mathcal{P}$. With Step 5, $q^a\mathrm{vol}(Z_\Lambda)-q^a\mathrm{vol}(Z_\Lambda\cap L'_\Lambda)\le\sum_{\Lambda_1}m(\Lambda,\Lambda_1)\le q^a\mathrm{vol}(Z_\Lambda)$. Steps 2–4 give the other properties of m.

Step 7: proof of (Mr2). For $b=(x,x')\in B^J$ put $y_b=-\theta^{-1}(h(x,e^*_{J-1})\mathrm{fr}_J+h(x',e'^*_{J-1})\mathrm{fr}'_J)$. By (Mg1) and the independence of ω_d of the witness (Proposition 16.12),

$$\omega_d(E_w^{-1}\Lambda)=\omega_d(\Lambda)\psi_0(h(y_b,u))\qquad(w\in\mathrm{Mv}_\Lambda).$$

By Step 2 with $\phi=\omega_d$,

$$\sum_{\Lambda_1}m(\Lambda,\Lambda_1)\omega_d(\Lambda_1)=q^a\frac{\beta}{\beta_J}\,\omega_d(\Lambda)\int_{\mathrm{Mv}_\Lambda}1[E_w^{-1}\Lambda\in\mathcal{P}]\,\psi_0(h(y_b,u))\,d\mu(w),$$

and removing the indicator changes the right side by at most $q^a\frac{\beta}{\beta_J}\mu(\mathrm{Mv}_\Lambda\cap\{u\in L'_\Lambda\})=q^a\mathrm{vol}(Z_\Lambda\cap L'_\Lambda)$ (Step 6, all values of modulus 1). By Fubini and Step 5(b) the full integral is $\int_{B^J}\int_{Z_\Lambda}\psi_0(h(y_b,u))\,du\,d\mu_1(b)$. Since $\mathrm{fr}_J,\mathrm{fr}'_J\in\Lambda$ and $h(\Lambda,\Lambda)\subseteq\mathcal{O}_E$, the open subgroup $B'=\ker\pi\cap\{(x,x')\in B_\Lambda: h(x,e^*_{J-1}),h(x',e'^*_{J-1})\in\theta\mathcal{O}_E\}$ of B_Λ satisfies $h(y_{b+b'}-y_b,u)\in\mathcal{O}_E$ for $b'\in B'$, $u\in U_\Lambda$; so $\psi_0(h(y_b,u))$ depends only on $b+B'$, and B^J is a finite union of cosets $b+B'$, $b\in\mathfrak{B}$. With $p_b=\mu_1(B')/\mu_1(B^J)$ ($\sum_b p_b=1$) the main term is $\omega_d(\Lambda)\sum_{b\in\mathfrak{B}}p_b\,q^a\int_{Z_\Lambda}\psi_0(h(y_b,u))\,du$. Each $u\mapsto\psi_0(h(y_b,u))$ is a nontrivial character of U_Λ by (Mg2), applied to the move $(x,x',0,0)$. This is (Mr2). □

Proposition 16.32. *Let a local frame pair with parameters r,k be given, d a datum, $i\ge1$, $J\ge1$, $e\ge0$ integers, and $\mathcal{P}$ the set of terms over d of second type with $b_d=q^i$ and $\lambda_d=(J,e)$. For $\Lambda\in\mathcal{P}$ put $Y_J=F_J(f)+F_J(f')$, $U_\Lambda=\Lambda\cap Y_J^\perp$, $B_\Lambda=\{(x,x')\in F_J(f)\times F_J(f'): x+x'\in\theta\Lambda\}$, and call (x,x',u,s) a* move *of Λ if $(x,x')\in B_\Lambda$, $\ell(\theta^{-1}(x+x'))=J$, $u\in U_\Lambda$, $s\in\varpi^a\mathcal{O}_E$ and $s+\bar s=h(u,u)$; put $E(x,u,s)v=v+h(v,u)x-h(v,x)(u+sx)$ ($v\in V$). For a move of Λ and $\Lambda_1=E(x,u,s)^{-1}\Lambda$:*

(Mg1) Terms and phases *$E(x,u,s)\in\mathrm{U}(V)$, Λ_1 is a term over d, and*

$$\omega_d(\Lambda_1)=\omega_d(\Lambda)\,\psi_0(h(y,u))\quad\text{with}$$
$$y=-\theta^{-1}(h(x,e^*_{J-1})\mathrm{fr}_J+h(x',e'^*_{J-1})\mathrm{fr}'_J)\qquad(\text{frames of }\Lambda);$$

(Mg2) Nontrivial character *$u\mapsto\psi_0(h(y,u))$ is a nontrivial character of U_Λ;*

(Mg3) Non-leaking moves stay *if $h(u,w)=1$ for some $w\in\Lambda\cap W_0$, then $\Lambda_1\in\mathcal{P}$;*

(Mg4) Reversal *if $\Lambda_1\in\mathcal{P}$, then $(x,x',-u,\bar s)$ is a move of Λ_1 (with the frames of Λ_1), and $E(x,-u,\bar s)=E(x,u,s)^{-1}$;*

(Mg5) Constant reference volume *for $\Lambda,\Lambda_1\in\mathcal{P}$, the subgroups $B_\Lambda\times U_\Lambda$ and $B_{\Lambda_1}\times U_{\Lambda_1}$ of $F_J(f)\times F_J(f')\times Y_J^\perp$ are commensurable and $[B_\Lambda\times U_\Lambda:C]=[B_{\Lambda_1}\times U_{\Lambda_1}:C]$ for C their intersection.*

Proof. **Step 1: facts on $\Lambda \in \mathcal{P}$.** Let $\Lambda \in \mathcal{P}$ with fixed witness (n, n'). $J \le r$, as the second type gives $\ell \le r$ on $\mathrm{sat}(\Lambda)$. By Proposition 16.8(1), $\mathrm{fr}_0, \dots, \mathrm{fr}_r, \mathrm{fr}'_0, \dots, \mathrm{fr}'_r$ are linearly independent, so $F_{r+1}(f) \cap F_{r+1}(f') = 0$ (these spaces are the spans of the fr_b, resp. fr'_b, as $n \in N$ preserves the flag of f and $n' \in N'$ that of f'). For $b < J \le r$, $nf_b - f_b \in F_b(f)$ and $n'f'_b - f'_b \in F_b(f')$ (definition of N, N'), so $F_J(f)$ is spanned by fr_b and $F_J(f')$ by fr'_b ($b < J$), and Y_J by both. (a) For linearly independent (p, p'), $M(p,p') = \{\sum_b \alpha_b p_b + \sum_b \alpha'_b p'_b : \theta\alpha'_b \in \mathcal{O}_E,\ \alpha_b + \alpha'_b \in \mathcal{O}_E\ \forall b\}$: the generators have this form, the set is a module, and $\sum_b \alpha_b p_b + \alpha'_b p'_b = \sum_b (\alpha_b + \alpha'_b) p_b + (\theta\alpha'_b)\theta^{-1}(p'_b - p_b)$. By Proposition 16.12, $M(\Lambda) = M(p,p')$ for the frames (p,p') of every witness of Λ (they lie in $\mathrm{Fr}(\Lambda)$). Consequently $M(\Lambda) \cap Y_J = M_J(\Lambda)$, the $\mathcal{O}_E$-span of fr_b, $\theta^{-1}(\mathrm{fr}'_b - \mathrm{fr}_b)$ ($b < J$): the coefficients of $y \in Y_J$ vanish at $b \ge J$. (b) By Proposition 16.8(4), for $y \in \mathrm{sat}(\Lambda)$ and $0 \le j \le r+1$, $\ell(y) \le j$ if and only if $y \in M(\Lambda) + F_j(f) + F_j(f')$; so ℓ is constant on the cosets of $M(\Lambda)$ in $\mathrm{sat}(\Lambda)$ and does not depend on the witness. (c) $\mathrm{sat}(\Lambda) = (\Lambda \cap Y_J) + M(\Lambda)$: every $y \in \mathrm{sat}(\Lambda)$ has $\ell(y) \le J$, so $y = m + z$ with $m \in M(\Lambda)$, $z \in Y_J$, and $z = y - m \in \Lambda$ since $M(\Lambda) \subseteq \mathrm{sat}(\Lambda) \subseteq \Lambda$ (Proposition 16.8(3)). As $(\Lambda \cap Y_J) \cap M(\Lambda) = M_J(\Lambda)$, $[\Lambda \cap Y_J : M_J(\Lambda)] = [\mathrm{sat}(\Lambda) : M(\Lambda)] = b_d(\Lambda) = q^i$. A move (x, x', u, s) of Λ satisfies the hypotheses of Lemma 16.33 ($x + x' \in \theta\Lambda$ as $(x,x') \in B_\Lambda$; $u \in \Lambda$, $h(u, Y_J) = 0$ as $u \in U_\Lambda$).

Step 2: proof of (Mg1). This is Lemma 16.33(4) and (5).

Step 3: proof of (Mg2). Write $x = \sum_{b<J} \alpha_b \mathrm{fr}_b$, $x' = \sum_{b<J} \alpha'_b \mathrm{fr}'_b$. Then $\alpha_{J-1} = h(x, e^*_{J-1})$: $z_0 = n^{-1}x = \sum_{b<J} \alpha_b f_b \in F_J(f)$, $x - z_0 = nz_0 - z_0 \in F_{J-1}(f)$, and $h(f_b, e^*_{J-1}) = \delta_{b,J-1}$; likewise $\alpha'_{J-1} = h(x', e'^*_{J-1})$. So $y = -\theta^{-1}z$ with $z = \alpha_{J-1}\mathrm{fr}_J + \alpha'_{J-1}\mathrm{fr}'_J$. Since the frame vectors are independent, $\theta \cdot \theta^{-1}(x + x')$ has the unique coefficients (α_b, α'_b), zero for $b \ge J$; so $\ell(\theta^{-1}(x+x')) = J$ means that the condition $[\alpha'_{J-1} \in \mathcal{O}_E$ and $\alpha_{J-1} + \alpha'_{J-1} \in \theta\mathcal{O}_E]$ fails. Suppose $\psi_0(h(y,u)) = 1$ for all $u \in U_\Lambda$. For $c \in \mathcal{O}_E$, $\bar{c}u \in U_\Lambda$ and $h(y, \bar{c}u) = ch(y,u)$; by the conductor of ψ_0, $h(y,u) \in \mathcal{O}_E$, i.e. $h(U_\Lambda, y) \subseteq \mathcal{O}_E$. As $U_\Lambda = \Lambda \cap Y_J^\perp$ and $(Y_J^\perp)^\perp = Y_J$, Lemma 16.36(1) gives $\theta^{-1}z = -y = \lambda + w$ with $\lambda \in \Lambda$, $w \in Y_J$. Then $\lambda \in M_E$ ($z \in M_E$ as $J \le r$), so $\lambda \in \mathrm{sat}(\Lambda)$ and $\ell(\lambda) \le J$. But $\theta\lambda = \alpha_{J-1}\mathrm{fr}_J + \alpha'_{J-1}\mathrm{fr}'_J - \theta w$ with θw a combination of $\mathrm{fr}_b, \mathrm{fr}'_b$, $b < J$; its unique coefficients at index J are $(\alpha_{J-1}, \alpha'_{J-1})$, and $\ell(\lambda) \le J$ forces $\alpha'_{J-1} \in \mathcal{O}_E$, $\alpha_{J-1} + \alpha'_{J-1} \in \theta\mathcal{O}_E$: a contradiction. So the character is nontrivial.

Step 4: traces on Y_J under a move. Let $\Lambda \in \mathcal{P}$, (x, x', u, s) a move, $E = E(x,u,s)$, $\Lambda_1 = E^{-1}\Lambda$ (a term over d by Step 2), $B_0 = \Lambda \cap Y_J$. (t1) $B_0 \subseteq \Lambda_1$. Let $y = v + v' \in B_0$, $v \in F_J(f)$, $v' \in F_J(f')$. Then $h(y,u) = 0$ and $h(y,x) = h(v',x)$ (X is isotropic). The pairing condition of the second type for $x_1 = x$, $x'_1 = x'$, $x_2 = \theta v$, $x'_2 = \theta v'$ ($\theta v + \theta v' = \theta y \in \theta\Lambda$) gives $\theta h(x, v') \in \theta\mathcal{O}_E$, so $h(v', x) \in \mathcal{O}_E$, and $Ey = y - h(v',x)(u + sx) \in \Lambda$ (Lemma 16.33(1)). So $y \in \Lambda_1$. (t2) $M(\Lambda_1) \cap Y_J = M(\Lambda) \cap Y_J$. By Step 1(a), $M(\Lambda_1) = M(p,p')$ for the frames (p,p') of the witness of Lemma 16.33(4), which are linearly independent (they span $F_{r+1}(f)$ and $F_{r+1}(f')$) and agree with $(\mathrm{fr}, \mathrm{fr}')$ at $b < J$. An element of Y_J has the same coefficients in both systems (zero at $b \ge J$), so the membership conditions of Step 1(a) coincide. (t3) For $y \in B_0$ (in $\mathrm{sat}(\Lambda) \cap \mathrm{sat}(\Lambda_1)$ by (t1), $Y_J \subseteq M_E$), $\ell_{\Lambda_1}(y) = \ell_\Lambda(y)$. For $j \ge J$ both levels are $\le j$ ($y \in Y_J$). For $j < J$, if $y = m + w$ with $w \in F_j(f) + F_j(f') \subseteq Y_J$, then $m = y - w \in M(\Lambda) \cap Y_J = M(\Lambda_1) \cap Y_J$ by (t2), and conversely; conclude with Step 1(b) for Λ and Λ_1. (t4) If $h(u,w) = 1$ for some $w \in \Lambda \cap W_0$, every $y \in \mathrm{sat}(\Lambda_1)$ is congruent modulo $M(\Lambda_1)$ to an element of B_0. Indeed $Ey \in \Lambda$ and, as $y, x \in M_E \perp w$, $h(Ey, w) = h(y,w) + h(y,u)h(x,w) - h(y,x)(h(u,w) + sh(x,w)) = -h(y,x)$, so $h(y,x) \in \mathcal{O}_E$ ($w \in \Lambda$). Hence $b := y + h(y,u)x = Ey + h(y,x)(u + sx) \in \Lambda \cap M_E = \mathrm{sat}(\Lambda)$, and by Step 1(c) $b = m + y_0$ with $m \in M(\Lambda)$, $y_0 \in B_0$. Let $\tau(v) = v - h(v,u)x$, an E-linear map. For each generator of $M(\Lambda)$ there is $y_1 \in B_0$ with $\tau(\cdot) + y_1 \in M(p,p') = M(\Lambda_1)$: $\tau(\mathrm{fr}_b) = p_b$ ($y_1 = 0$), and $\tau(\theta^{-1}(\mathrm{fr}'_b - \mathrm{fr}_b)) + h(\mathrm{fr}'_b, u)\theta^{-1}(x + x') = \theta^{-1}(p'_b - p_b)$, where $h(\mathrm{fr}'_b, u) \in \mathcal{O}_E$ and $\theta^{-1}(x + x') \in B_0$; by linearity there is $y_1 \in B_0$ with $\tau(m) + y_1 \in M(\Lambda_1)$. Since $h(b,u) = h(y,u)$ ($h(x,u) = 0$) and $h(y_0, u) = 0$, $y = \tau(b) = \tau(m) + y_0$, so $y - (y_0 - y_1) = \tau(m) + y_1 \in M(\Lambda_1)$

with $y_0 - y_1 \in B_0$.

Step 5: a test for the second type. Let Λ' be a non-generic term over d and $B \subseteq \Lambda' \cap (X + X')$ a set such that every $y \in \mathrm{sat}(\Lambda')$ is congruent modulo $M(\Lambda')$ to an element of B, and $h(x_1, x_2') \in \theta\mathcal{O}_E$ whenever $x_1, x_2 \in X$, $x_1', x_2' \in X'$ and $\theta^{-1}(x_1 + x_1'), \theta^{-1}(x_2 + x_2') \in B$. Then Λ' is of second type. Levels: $y \equiv y_0 \in B \subseteq F_r(f) + F_r(f')$, so $\ell(y) \le r$ by Step 1(b). Pairing: let $x_\nu \in X$, $x_\nu' \in X'$ with $x_\nu + x_\nu' = \theta y_\nu$, $y_\nu \in \Lambda'$ ($\nu = 1, 2$). Then $y_\nu \in \mathrm{sat}(\Lambda')$, so $y_\nu = (u_\nu + u_\nu') + m_\nu$ with $u_\nu + u_\nu' \in B$, $u_\nu \in X$, $u_\nu' \in X'$, $m_\nu \in M(\Lambda') \subseteq \Lambda'$. Put $a_\nu = x_\nu - \theta u_\nu \in X$, $a_\nu' = x_\nu' - \theta u_\nu' \in X'$; then $a_\nu + a_\nu' = \theta m_\nu = \xi + \xi'$ with $\xi \in \sum_b \mathcal{O}_E \mathrm{fr}_b$, $\xi' \in \sum_b \mathcal{O}_E \mathrm{fr}_b' \subseteq \Lambda'$ (frames of Λ'), and $a_\nu - \xi = \xi' - a_\nu' \in F_{r+1}(f) \cap F_{r+1}(f') = 0$; so $a_\nu' \in \Lambda'$. Using $h(X, X) = h(X', X') = 0$, $h(\Lambda', \Lambda') \subseteq \mathcal{O}_E$ and $\bar\theta = \theta$:

$$h(x_1, x_2') = h(\theta u_1, \theta u_2') + \theta h(u_1 + u_1', a_2') + \theta(\theta h(m_1, u_2 + u_2') - h(a_1', u_2 + u_2')) + \theta h(m_1, a_2'),$$

and the first term is in $\theta\mathcal{O}_E$ by hypothesis (applied to $\theta u_\nu, \theta u_\nu'$), the others since $u_\nu + u_\nu', a_\nu', m_\nu \in \Lambda'$. So Λ' is of second type.

Step 6: proof of (Mg3). By (t1), $B_0 \subseteq \mathrm{sat}(\Lambda_1)$; by Step 1(c) and (t4), $\mathrm{sat}(\Lambda) = B_0 + M(\Lambda)$ and $\mathrm{sat}(\Lambda_1) = B_0 + M(\Lambda_1)$, and $B_0 \cap M(\Lambda) = B_0 \cap M(\Lambda_1)$ by (t2). So $B_0/(B_0 \cap M(\Lambda))$ maps isomorphically onto $\mathrm{sat}(\Lambda)/M(\Lambda)$ and onto $\mathrm{sat}(\Lambda_1)/M(\Lambda_1)$, and by (t3) and Step 1(b) both isomorphisms preserve levels. Hence $b_d(\Lambda_1) = b_d(\Lambda) = q^i \ne 1$, so Λ_1 is not generic; the maximal level on $\mathrm{sat}(\Lambda_1)$ is J; and by Step 1(b) the classes of level $\le J-1$ form the submodule $\mathrm{sat} \cap (M + F_{J-1}(f) + F_{J-1}(f'))$ modulo M, of the same order for Λ and Λ_1, so $\lambda_d(\Lambda_1) = (J, e)$. Step 5 with $\Lambda' = \Lambda_1$ and $B = B_0$ ($B_0 \subseteq \Lambda_1 \cap Y_J \subseteq \Lambda_1 \cap (X + X')$; representatives by (t4); if $\theta^{-1}(x_\nu + x_\nu') \in B_0 \subseteq \Lambda$, then $h(x_1, x_2') \in \theta\mathcal{O}_E$ by the second type of Λ) shows that Λ_1 is of second type. So $\Lambda_1 \in \mathcal{P}$.

Step 7: proof of (Mg4). Let $\Lambda_1 \in \mathcal{P}$. By Lemma 16.33(3), $x + x' \in \theta\Lambda_1$, so $(x, x') \in B_{\Lambda_1}$; $\theta^{-1}(x + x') \in B_0$, so by (t3) its level for Λ_1 is J; $-u \in \Lambda_1$ and $h(-u, Y_J) = 0$, so $-u \in U_{\Lambda_1}$; $\bar s \in \varpi^a\mathcal{O}_E$ and $\bar s + s = h(-u, -u)$. So $(x, x', -u, \bar s)$ is a move of Λ_1, and $E(x, -u, \bar s) = E(x, u, s)^{-1}$ by Lemma 16.34(2).

Step 8: proof of (Mg5). Let $\Lambda, \Lambda' \in \mathcal{P}$, $Q = \Lambda \cap \Lambda'$, μ a Haar measure on Y_J. For Λ with witness (n, n') let T_Λ be the E-linear automorphism of Y_J with $T_\Lambda f_b = \mathrm{fr}_b$, $T_\Lambda f_b' = \mathrm{fr}_b'$ ($b < J$). As $nf_b - f_b \in F_b(f)$ and $n'f_b' - f_b' \in F_b(f')$, its matrix in the basis $(f_b, f_b')_{b<J}$ is unitriangular in each block, so $\det_E T_\Lambda = 1$; and T_Λ maps $M_J^0 = \sum_{b<J} \mathcal{O}_E f_b + \mathcal{O}_E\theta^{-1}(f_b' - f_b)$ onto $M_J(\Lambda)$. For an E-linear automorphism S of Y_J, $\mu(SP) = |\mathrm{N}_{E/F}(\det_E S)|_F\, \mu(P)$; hence $\mu(M_J(\Lambda)) = \mu(M_J^0) = \mu(M_J(\Lambda'))$, and by Step 1(c) $\mu(\Lambda \cap Y_J) = q^i\mu(M_J(\Lambda)) = \mu(\Lambda' \cap Y_J)$. So $[\Lambda \cap Y_J : Q \cap Y_J] = [\Lambda' \cap Y_J : Q \cap Y_J]$, and Lemma 16.36(2) with $U = Y_J$ (Λ, Λ' self-dual by Lemma 16.18(1)) gives $[U_\Lambda : U_\Lambda \cap U_{\Lambda'}] = [U_{\Lambda'} : U_\Lambda \cap U_{\Lambda'}]$, all finite. The map $\varphi(x, x') = \theta^{-1}(x + x')$ is an isomorphism $F_J(f) \times F_J(f') \to Y_J$ ($F_J(f) \cap F_J(f') = 0$) with $B_\Lambda = \varphi^{-1}(\Lambda \cap Y_J)$, so $[B_\Lambda : B_\Lambda \cap B_{\Lambda'}] = [B_{\Lambda'} : B_\Lambda \cap B_{\Lambda'}]$. With $C = (B_\Lambda \cap B_{\Lambda'}) \times (U_\Lambda \cap U_{\Lambda'})$, the intersection of $B_\Lambda \times U_\Lambda$ and $B_{\Lambda'} \times U_{\Lambda'}$, the index $[B_\Lambda \times U_\Lambda : C]$ is the product of the two indices for Λ, and likewise for Λ'; so the two indices are finite and equal. □

Lemma 16.33. *Fix a local frame pair with parameters r, k, a datum d, an integer $J \ge 1$ and a term Λ over d of second type with $\lambda_d(\Lambda) = (J, e)$ for some e; let (n, n') be the fixed witness of Λ. Put $Y_J = F_J(f) + F_J(f')$. Let $x \in F_J(f)$, $x' \in F_J(f')$, $u \in \Lambda$, $s \in \varpi^a\mathcal{O}_E$ with $x + x' \in \theta\Lambda$, $h(u, Y_J) = 0$ and $s + \bar s = h(u, u)$; put $E = E(x, u, s)$, $E' = E(x', u, s)$ and $\Lambda_1 = E^{-1}\Lambda$. Then:*

1. *$J \le r$, $x \in X$, $x' \in X'$, $h(x', x) \in \theta\mathcal{O}_E$, and $sx, \bar s x, sx', \bar s x' \in \Lambda$;*
2. *$EE'v - v \in \theta\Lambda$ for every $v \in \Lambda$ with $h(v, x') = 0$;*
3. *$x + x' \in \theta\Lambda_1$, $-u \in \Lambda_1$, $\bar s x \in \Lambda_1$;*

4. *$E \in \mathrm{U}(V)$, and Λ_1 is a term over d with a witness $(E^{-1}n, gE'n')$, where $g \in \mathrm{U}(V)$ fixes every vector of $X'^{\perp}$; the frames of this witness are $E^{-1}\mathrm{fr}_b = \mathrm{fr}_b - h(\mathrm{fr}_b, u)x$ and $gE'\mathrm{fr}'_b = \mathrm{fr}'_b + h(\mathrm{fr}'_b, u)x'$ $(0 \le b \le r)$, equal to fr_b, fr'_b for $b < J$;*

5. *$\omega_d(\Lambda_1) = \omega_d(\Lambda)\,\psi_0(h(y,u))$ with $y = -\theta^{-1}\big(h(x, e^*_{J-1})\mathrm{fr}_J + h(x', e'^*_{J-1})\mathrm{fr}'_J\big)$.*

Proof. **Step 0: facts on Λ.** Λ is self-dual (Lemma 16.18(1)), so $h(\Lambda,\Lambda) \subseteq \mathcal{O}_E$; the vectors $\mathrm{fr}_b, \mathrm{fr}'_b$ lie in Λ, and $\mathrm{fr}_0, \dots, \mathrm{fr}_r, \mathrm{fr}'_0, \dots, \mathrm{fr}'_r$ are linearly independent over E (Proposition 16.8(1)); so $f_0, \dots, f_r$ and $f'_0, \dots, f'_r$ are linearly independent and $F_{r+1}(f) \cap F_{r+1}(f') = 0$. As $n \in N$, $nF_j(f) = F_j(f)$ for $j \le r+1$ ($nF_{j+1}(f) = F_{j+1}(f)$ for $j < r$ by definition; $nf_r \in f_r + X$), so $F_j(f)$ is the E-span of $\mathrm{fr}_0, \dots, \mathrm{fr}_{j-1}$; likewise for f'. Also $n(X^{\perp}) = X^{\perp}$ and $f_b \in X^{\perp}$ by (Fp1), so $\mathrm{fr}_b \in X^{\perp}$; likewise $\mathrm{fr}'_b \in X'^{\perp}$.

Step 1: proof of (1). The second type gives $\ell \le r$ on $\mathrm{sat}(\Lambda)$, so $J \le r$ and $x \in F_J(f) \subseteq X$, $x' \in X'$. Applying the pairing condition of the second type to $x_1 = x_2 = x$, $x'_1 = x'_2 = x'$ (with $x + x' \in \theta\Lambda$) gives $h(x,x') \in \theta\mathcal{O}_E$, hence $h(x',x) = \overline{h(x,x')} \in \theta\mathcal{O}_E$. Let $y_0 = \theta^{-1}(x + x') \in \Lambda \cap M_E = \mathrm{sat}(\Lambda)$. By Proposition 16.8(2),(3), $a \ge 0$ and $\varpi^a y_0 \in M(\Lambda)$; as $\theta M(\Lambda)$ is spanned by the $\theta\mathrm{fr}_b$ and the $\mathrm{fr}'_b - \mathrm{fr}_b$, $\varpi^a x + \varpi^a x' = \theta\varpi^a y_0 = \xi + \xi'$ with $\xi \in \sum_b \mathcal{O}_E\mathrm{fr}_b$, $\xi' \in \sum_b \mathcal{O}_E\mathrm{fr}'_b$. Then $\varpi^a x - \xi = \xi' - \varpi^a x'$ lies in $F_{r+1}(f) \cap F_{r+1}(f') = 0$ (Step 0), so $\varpi^a x = \xi \in \Lambda$ and $\varpi^a x' = \xi' \in \Lambda$. Writing $s = \varpi^a c$ with $c \in \mathcal{O}_E$, $sx = c\varpi^a x$ and $\bar{s}x = \bar{c}\varpi^a x$ lie in Λ, and likewise for x'.

Step 2: a congruence. Let L be an $\mathcal{O}_E$-module with $h(L,L) \subseteq \mathcal{O}_E$, $x + x' \in \theta L$, $u_0 \in L$ with $h(x', u_0) = 0$, $s_0 \in E$ with $s_0 x \in L$, and $t \in E$. Write $x + x' = \theta y_1$ ($y_1 \in L$) and $h(x', x) = \theta c_0$ ($c_0 \in \mathcal{O}_E$, by (1)). Then for $v \in L$ with $h(v, x') = 0$:

$$E(x, u_0, s_0)E(x', u_0, t)v - v = \theta\big(\alpha y_1 - (h(v, y_1) + \alpha c_0)(u_0 + s_0 x)\big) \in \theta L, \qquad \alpha = h(v, u_0). \tag{16.2}$$

Indeed $E(x', u_0, t)v = v + \alpha x'$; then $h(v + \alpha x', u_0) = \alpha$ and $h(v + \alpha x', x) = h(v,x) + \alpha\theta c_0$ with $h(v,x) = h(v, x + x') = \theta h(v, y_1)$; so $E(x, u_0, s_0)(v + \alpha x') - v = \alpha(x + x') - \theta(h(v, y_1) + \alpha c_0)(u_0 + s_0 x)$, and $\alpha, h(v, y_1) \in \mathcal{O}_E$. With $L = \Lambda$, $u_0 = u$ ($h(x', u) = 0$ as $x' \in Y_J$), $s_0 = s$, $t = s$, this is (2).

Step 3: proof of (3) and of $E \in \mathrm{U}(V)$. By Lemma 16.34(2) for $g = f$ ($x \in F_J(f)$, $h(F_J(f), u) = 0$, $1 \le J \le r$): $E \in N \subseteq \mathrm{U}(V)$, $E^{-1} = E(x, -u, \bar{s})$, $Ex = x$, $E(\bar{s}x) = \bar{s}x$. Likewise $E' \in N'$, $E'^{-1} = E(x', -u, \bar{s})$. Write $h(x', x) = \theta c_0$. Then $Ex' = x' - \theta c_0(u + sx)$ (as $h(x', u) = 0$), so $E(x + x') = (x + x') - \theta c_0(u + sx) \in \theta\Lambda$, i.e. $x + x' \in \theta\Lambda_1$. $E(-u) = -u - h(u,u)x = -(u + sx + \bar{s}x) \in \Lambda$ ($h(u,x) = 0$), so $-u \in \Lambda_1$; and $\bar{s}x = E(\bar{s}x) \in \Lambda$, so $\bar{s}x \in \Lambda_1$. Λ_1 is self-dual (E is an isometry). So Step 2 applies to $L = \Lambda_1$, $u_0 = -u$, $s_0 = \bar{s}$, $t = \bar{s}$: for $v \in \Lambda_1$ with $h(v, x') = 0$,

$$E^{-1}E'^{-1}v - v \in \theta\Lambda_1. \tag{16.3}$$

Step 4: the trace on $X'^{\perp}$. We show $E'\Lambda \cap X'^{\perp} = \Lambda_1 \cap X'^{\perp}$. As $E' \in N'$, $E'(X'^{\perp}) = X'^{\perp}$. If $v \in E'\Lambda \cap X'^{\perp}$, then $w = E'^{-1}v \in \Lambda \cap X'^{\perp}$, and by (2) $Ev = EE'w \in w + \theta\Lambda \subseteq \Lambda$, so $v \in \Lambda_1$. If $v \in \Lambda_1 \cap X'^{\perp}$, then by (16.3) $E^{-1}E'^{-1}v \in \Lambda_1 = E^{-1}\Lambda$, so $E'^{-1}v \in \Lambda$ and $v \in E'\Lambda$. Now X' is spanned by $f'_0, \dots, f'_{r-1}$, linearly independent (Step 0) with $h(f'_i, f'_j) = 0$ (Fp1), and $E'\Lambda$, Λ_1 are self-dual. By Lemma 16.35 there is $g \in \mathrm{U}(V)$ fixing $X'^{\perp}$ pointwise with $gE'\Lambda = \Lambda_1$.

Step 5: proof of (4). $g \in N'$: g fixes $X'^{\perp} \supseteq F_{r+1}(f')$ (Fp1) pointwise, so all conditions defining N' hold with $gw - w = 0$. Hence $E^{-1}n \in N$ and $gE'n' \in N'$, and $\Lambda_1 = E^{-1}n\Lambda_{\mathrm{std}}(A) = gE'n'\Lambda'_{\mathrm{std}}(A')$; A, A' are inner as before. Frames: as $\mathrm{fr}_b \in X^{\perp}$ and $x \in X$, Lemma 16.34(2) gives $E^{-1}\mathrm{fr}_b = \mathrm{fr}_b - h(\mathrm{fr}_b, u)x$; as $\mathrm{fr}'_b \in X'^{\perp}$, $E'\mathrm{fr}'_b = \mathrm{fr}'_b + h(\mathrm{fr}'_b, u)x' \in X'^{\perp}$, fixed by g. For $b < J$, $\mathrm{fr}_b, \mathrm{fr}'_b \in Y_J$ (Step 0) are orthogonal to u, so the frames are unchanged. Congruence: $gE'\mathrm{fr}'_b - E^{-1}\mathrm{fr}_b = E^{-1}\big((EE'\mathrm{fr}'_b - \mathrm{fr}'_b) + (\mathrm{fr}'_b - \mathrm{fr}_b)\big) \in E^{-1}\theta\Lambda = \theta\Lambda_1$, by (2) ($\mathrm{fr}'_b \in \Lambda \cap X'^{\perp}$)

and the congruence of the witness (n, n'). The congruence graph: let $(p, p') \in \Gamma$, with $y, y' \in \Lambda$, $y \perp X$, $y' \perp X'$, $y' - p \in X'$, $y - p' \in X$, $y' - y \in \theta\Lambda$. Put $y'_1 = E'y'$, $y_1 = E^{-1}y$. Then $Ey'_1 = EE'y' \in y' + \theta\Lambda \subseteq \Lambda$ by (2), so $y'_1 \in \Lambda_1$; $y_1 \in \Lambda_1$; $y'_1 \perp X'$ and $y_1 \perp X$ ($E' \in N'$, $E^{-1} \in N$ preserve $X'^{\perp}$, $X^{\perp}$); $y'_1 - p = (E'y' - y') + (y' - p) \in X'$ and $y_1 - p' \in X$ (elements of N', N move $X'^{\perp}$, $X^{\perp}$ by vectors of X', X); and $E(y'_1 - y_1) = (EE'y' - y') + (y' - y) \in \theta\Lambda$, so $y'_1 - y_1 \in \theta\Lambda_1$. Thus Γ is contained in the congruence graph of Λ_1. The same argument with $(\Lambda_1, E^{-1}, E'^{-1}, (16.3))$ in place of $(\Lambda, E, E', (2))$ (note $\Lambda = E\Lambda_1$, $E^{-1} \in N$, $E'^{-1} \in N'$) gives the converse inclusion. So Λ_1 is a term over d with witness $(E^{-1}n, gE'n')$.

Step 6: proof of (5). $\omega_d(\Lambda_1)$ does not depend on the choice of the fixed witness (Proposition 16.12 with the identity of Σ), so we compute it with $(E^{-1}n, gE'n')$. By Lemma 16.34(1), $\psi_N(E^{-1}n) = \psi_N(E)^{-1}\psi_N(n)$ and $\psi_{N'}(gE'n') = \psi_{N'}(g)\psi_{N'}(E')\psi_{N'}(n') = \psi_{N'}(E')\psi_{N'}(n')$ (g fixes $X'^{\perp} = F_r(f')^{\perp}$ pointwise). All values have modulus one, so

$$\omega_d(\Lambda_1) = \psi_N(E^{-1}n)\overline{\psi_{N'}(gE'n')} = \omega_d(\Lambda)\,\overline{\psi_N(E)}\,\overline{\psi_{N'}(E')}.$$

By Lemma 16.34(2) with $m = n$, resp. $m = n'$:

$$\psi_N(E) = \psi_0(\theta^{-1}h(x, e^*_{J-1})h(\mathrm{fr}_J, u)) \quad \text{and} \quad \psi_{N'}(E') = \psi_0(\theta^{-1}h(x', e'^*_{J-1})h(\mathrm{fr}'_J, u)).$$

As $\overline{\psi_0(z)} = \psi_0(-z)$ and

$$h(y, u) = -\theta^{-1}(h(x, e^*_{J-1})h(\mathrm{fr}_J, u) + h(x', e'^*_{J-1})h(\mathrm{fr}'_J, u)),$$

(5) follows. □

Lemma 16.34. *Let a local frame pair with parameters r, k be given, and let (g, g^*, ψ_g) be (f, e^*, ψ_N) or $(f', e'^*, \psi_{N'})$. For $x, u \in V$, $s \in E$ let $E(x, u, s)v = v + h(v, u)x - h(v, x)(u + sx)$ $(v \in V)$.*

1. *For $w \in N(g)$ and $0 \le j < r$ put $c_j(w) = h(wg_{j+1}, g^*_j)$. Then $c_j(ww_1) = c_j(w) + c_j(w_1)$ for $w, w_1 \in N(g)$; hence ψ_g is a homomorphism $N(g) \to \{z \in \mathbb{C} : |z| = 1\}$, and $\psi_g(w) = 1$ if w fixes every vector of $F_r(g)^{\perp}$.*

2. *Let $1 \le J \le r$, $x \in F_J(g)$, $u \in V$ with $h(F_J(g), u) = 0$, and $s \in E$ with $s + \bar{s} = h(u, u)$. Then $E(x, u, s) \in N(g)$, $E(x, u, s)^{-1} = E(x, -u, \bar{s})$, $E(x, u, s)w = w$ for $w \in F_J(g)$, $E(x, u, s)v - v = h(v, u)x$ for $v \in F_r(g)^{\perp}$, and*

$$\psi_g(E(x, u, s)) = \psi_0(\theta^{-1}h(x, g^*_{J-1})\, h(mg_J, u)) \qquad \text{for every } m \in N(g).$$

Proof. **Step 0: the flag.** By (Fp1), $h(g_a, g_b) = 0$ unless $a = b = r$, so $F_r(g)$ is totally isotropic and $g_b \in F_r(g)^{\perp}$ for $0 \le b \le r$; and $h(g_a, g^*_j) = \delta_{aj}$, so $h(F_j(g), g^*_j) = 0$. Let $w \in N(g)$ and $0 \le j < r$. For $z \in F_{j+1}(g)$, $wz - z \in F_j(g)$ (definition of $N(g)$). Moreover $wg_{j+1} - g_{j+1} \in F_{j+1}(g)$: if $j + 1 < r$ this is the condition for $j + 1$ applied to $g_{j+1} \in F_{j+2}(g)$; if $j + 1 = r$ it is the condition $wv - v \in F_r(g)$ for $v = g_r \in F_r(g)^{\perp}$.

Step 1: proof of (1). Let $w, w_1 \in N(g)$ and $z = w_1g_{j+1} - g_{j+1} \in F_{j+1}(g)$. Then $ww_1g_{j+1} = wg_{j+1} + z + (wz - z)$ with $wz - z \in F_j(g) \perp g^*_j$, so $c_j(ww_1) = c_j(w) + h(z, g^*_j) = c_j(w) + c_j(w_1) - h(g_{j+1}, g^*_j) = c_j(w) + c_j(w_1)$. As $\psi_g(w) = \psi_0(\theta^{-1}\sum_j c_j(w))$ and ψ_0 is a unitary character of $(E, +)$, ψ_g is a homomorphism into the unit circle. If w fixes $F_r(g)^{\perp}$ pointwise, $wg_{j+1} = g_{j+1}$ (Step 0), so $c_j(w) = 0$ for all j and $\psi_g(w) = 1$.

Step 2: isometry and inverse. Write $E = E(x,u,s)$. As $J \le r$, $x \in F_r(g)$, so $h(x,x) = 0$; and $h(x,u) = h(u,x) = 0$. For $v, w \in V$ put $\alpha = h(v,u)$, $\beta = h(v,x)$, $\alpha' = h(w,u)$, $\beta' = h(w,x)$. Then

$$h(Ev, Ew) = h(v,w) + \beta\bar{\beta}'\big(h(u,u) - s - \bar{s}\big) = h(v,w).$$

For $w = Ev$: $h(w,x) = \beta$ and $h(w,u) = \alpha - \beta h(u,u)$, so $E(x,-u,\bar{s})w = w - (\alpha - \beta h(u,u))x - \beta(-u + \bar{s}x) = v + \beta(h(u,u) - s - \bar{s})x = v$. The pair $(-u, \bar{s})$ satisfies the same hypotheses as (u,s), so also $E\,E(x,-u,\bar{s}) = 1$. Hence $E \in \mathrm{U}(V)$ and $E^{-1} = E(x,-u,\bar{s})$.

Step 3: $E \in N(g)$. For $v \in F_r(g)^\perp$, $h(v,x) = 0$, so $Ev - v = h(v,u)x \in F_J(g) \subseteq F_r(g)$; in particular $E(F_r(g)^\perp) \subseteq F_r(g)^\perp$ ($F_r(g) \subseteq F_r(g)^\perp$), and the same for E^{-1}, so $E(F_r(g)^\perp) = F_r(g)^\perp$. Let $0 \le j < r$ and $w \in F_{j+1}(g) \subseteq F_r(g) \subseteq F_r(g)^\perp$, so $Ew - w = h(w,u)x$. If $j+1 \le J$, then $w \in F_J(g)$ and $h(w,u) = 0$, so $Ew = w$; if $j \ge J$, then $x \in F_J(g) \subseteq F_j(g)$. In both cases $Ew - w \in F_j(g)$; hence $EF_{j+1}(g) \subseteq F_{j+1}(g)$, and equality holds by the same for E^{-1}. So $E \in N(g)$, and E fixes $F_J(g)$ pointwise.

Step 4: the character. By Step 0, $g_{j+1} \in F_r(g)^\perp$, so $Eg_{j+1} = g_{j+1} + h(g_{j+1},u)x$ and $c_j(E) = h(g_{j+1},u)\,h(x,g_j^*)$. If $j < J-1$, then $g_{j+1} \in F_J(g)$ and $h(g_{j+1},u) = 0$; if $j \ge J$, then $x \in F_J(g) \subseteq F_j(g)$ and $h(x,g_j^*) = 0$. So $\psi_g(E) = \psi_0(\theta^{-1}h(x,g_{J-1}^*)h(g_J,u))$. Finally, for $m \in N(g)$, Step 0 with $j = J-1$ gives $mg_J - g_J \in F_J(g)$, which is orthogonal to u; so $h(mg_J,u) = h(g_J,u)$. □

Lemma 16.35. *Let (V,h) be the hermitian space of a local frame pair. Let $e_1,\dots,e_m \in V$ be linearly independent over E with $h(e_i,e_j) = 0$ for all i,j, let X be their E-span and $X^\perp = \{w \in V : h(X,w) = 0\}$, and let $\Lambda_a, \Lambda_b \subseteq V$ be self-dual $\mathcal{O}_E$-modules with $\Lambda_a \cap X^\perp = \Lambda_b \cap X^\perp$. Then there is $g \in \mathrm{U}(V)$ with $gw = w$ for all $w \in X^\perp$ and $g\Lambda_a = \Lambda_b$.*

Proof. **Step 1: dual bases.** Since X is totally isotropic, $X \subseteq X^\perp$. $P = \Lambda_a \cap X$ is a finitely generated $\mathcal{O}_E$-module spanning X (Lemma 16.36(1)); X is free of rank m over E, so P (finitely generated, ϖ-torsion free) is free of rank m: let $b_1,\dots,b_m$ be a basis; it is an E-basis of X. Applying Lemma 16.36(1) to the coordinate functionals of this basis gives $c_1,\dots,c_m \in \Lambda_a$ with $h(b_k,c_j) = \delta_{kj}$, hence $h(c_j,b_k) = \delta_{jk}$. Also $b_k \in \Lambda_a \cap X^\perp = \Lambda_b \cap X^\perp$, so $b_k \in \Lambda_b$. For every $v \in V$,

$$v - \sum_j h(v,b_j)c_j \in X^\perp, \tag{16.4}$$

since its pairing with b_k is $h(v,b_k) - h(v,b_k) = 0$ and the b_k span X.

Step 2: translates of the c_j in Λ_b. $h(\Lambda_b \cap X^\perp, c_j) = h(\Lambda_a \cap X^\perp, c_j) \subseteq h(\Lambda_a,\Lambda_a) \subseteq \mathcal{O}_E$. Since h is nondegenerate, $(X^\perp)^\perp = X$, so Lemma 16.36(1) for Λ_b and $U = X^\perp$ gives $c_j \in \Lambda_b + X$: choose $c_j' \in \Lambda_b$ with $d_j^0 = c_j' - c_j \in X$. Put $H_{jk} = h(c_j,d_k^0) + h(d_j^0,c_k)$. As $h(d_j^0,d_k^0) = 0$, $H_{jk} = h(c_j',c_k') - h(c_j,c_k) \in \mathcal{O}_E$, and $\overline{H_{kj}} = H_{jk}$. Put

$$d_j = d_j^0 - \sum_l \tau H_{jl}\, b_l \in X.$$

Then $c_j + d_j = c_j' - \sum_l \tau H_{jl} b_l \in \Lambda_b$, $h(d_j,b_k) = h(d_j,d_k) = 0$, and

$$h(c_j,d_k) + h(d_j,c_k) = \big(h(c_j,d_k^0) - \bar{\tau}\overline{H_{kj}}\big) + \big(h(d_j^0,c_k) - \tau H_{jk}\big) = H_{jk} - (\tau + \bar{\tau})H_{jk} = 0.$$

Step 3: the isometry. Define $g(v) = v + \sum_j h(v,b_j)d_j$. It is E-linear, and $v \mapsto v - \sum_j h(v,b_j)d_j$ is its inverse because $h(d_j,b_k) = 0$. For $v \in V$ and k, (16.4) and $d_k \in X$

give $h(v,d_k)=\sum_j h(v,b_j)h(c_j,d_k)$, and likewise $h(d_j,w)=\sum_k \overline{h(w,b_k)}\,h(d_j,c_k)$. Hence, using $h(d_j,d_k)=0$,

$$\begin{aligned} h(gv,gw)-h(v,w) &= \sum_k \overline{h(w,b_k)}\,h(v,d_k)+\sum_j h(v,b_j)\,h(d_j,w)\\ &= \sum_{j,k} h(v,b_j)\overline{h(w,b_k)}\big(h(c_j,d_k)+h(d_j,c_k)\big)=0, \end{aligned}$$

so $g\in \mathrm{U}(V)$. If $w\in X^\perp$ then $h(w,b_j)=0$ and $gw=w$.

Step 4: $g\Lambda_a=\Lambda_b$. Let $v\in\Lambda_a$ and $\beta_j=h(v,b_j)\in\mathcal{O}_E$ ($b_j\in\Lambda_a$). Then $gv=(v-\sum_j\beta_j c_j)+\sum_j\beta_j(c_j+d_j)$; the first summand lies in $\Lambda_a\cap X^\perp=\Lambda_b\cap X^\perp$ by (16.4), the second in Λ_b. So $g\Lambda_a\subseteq\Lambda_b$. Conversely let $w\in\Lambda_b$ and $\beta_j=h(w,b_j)\in\mathcal{O}_E$ ($b_j\in\Lambda_b$). Then $g^{-1}w=(w-\sum_j\beta_j(c_j+d_j))+\sum_j\beta_j c_j$; the first summand lies in Λ_b and is orthogonal to every b_k (as $h(c_j+d_j,b_k)=\delta_{jk}$), so it lies in $\Lambda_b\cap X^\perp=\Lambda_a\cap X^\perp$, and the second lies in Λ_a. So $g^{-1}\Lambda_b\subseteq\Lambda_a$, and $g\Lambda_a=\Lambda_b$. □

Lemma 16.36. *Let (V,h) be the hermitian space of a local frame pair. Let $\Lambda,\Lambda'\subseteq V$ be self-dual $\mathcal{O}_E$-modules and $Q=\Lambda\cap\Lambda'$, let $U\subseteq V$ be an E-submodule, and put $U^\perp=\{w\in V : h(a,w)=0$ for all $a\in U\}$. Then:*

1. *$\Lambda\cap U$ is a finitely generated $\mathcal{O}_E$-module spanning U over E; every $\mathcal{O}_E$-linear map $\varphi\colon \Lambda\cap U\to\mathcal{O}_E$ is of the form $\varphi(a)=h(a,\lambda)$ for some $\lambda\in\Lambda$; and for $w\in V$, $h(\Lambda\cap U,w)\subseteq\mathcal{O}_E$ if and only if $w\in\Lambda+U^\perp$;*
2. *all indices below are finite and*

$$[\Lambda\cap U^\perp : Q\cap U^\perp]\cdot[\Lambda'\cap U : Q\cap U]=[\Lambda'\cap U^\perp : Q\cap U^\perp]\cdot[\Lambda\cap U : Q\cap U].$$

Proof. **Step 1: lattices.** Λ is finitely generated. If $W=E\Lambda$ were different from V, then $W^\perp\neq 0$ (as h has invertible Gram matrix), and $W^\perp\subseteq\{y : h(\Lambda,y)\subseteq\mathcal{O}_E\}=\Lambda$; but $W^\perp$ contains $\varpi^{-N}w$ for all N and some $w\neq 0$, so it is unbounded, while a finitely generated module is bounded. So Λ is a lattice in V, and $h(\Lambda,\Lambda)\subseteq\mathcal{O}_E$ since $\Lambda\subseteq\Lambda^\#=\Lambda$. Λ contains an E-basis $v_1,\dots,v_n$ of V; for $y=\sum_i c_i v_i\in U$ and N large, $\varpi^N c_i\in\mathcal{O}_E$, so $\varpi^N y\in\Lambda\cap U$. Hence $\Lambda\cap U$ (finitely generated, $\mathcal{O}_E$ being noetherian) spans U: it is a lattice in U; the same holds for Λ' and for $U^\perp$ in place of U.

Step 2: proof of (1). $\Lambda/(\Lambda\cap U)$ embeds into V/U, on which ϖ acts invertibly; so it is finitely generated and ϖ-torsion free, hence projective, and $\Lambda=(\Lambda\cap U)\oplus C$ for a submodule C. Given φ, let $\psi_0\colon\Lambda\to\mathcal{O}_E$ be φ on $\Lambda\cap U$ and 0 on C, and $\psi\colon V\to E$ its unique E-linear extension ($V=E\Lambda$, $\psi(v)=\varpi^{-N}\psi_0(\varpi^N v)$ for $\varpi^N v\in\Lambda$). As the Gram matrix of h is invertible, $\lambda\mapsto h(\cdot,\lambda)$ is a bijection of V onto the E-linear forms on V; so $\psi=h(\cdot,\lambda)$ for some $\lambda\in V$. Then $h(\Lambda,\lambda)=\psi_0(\Lambda)\subseteq\mathcal{O}_E$, so $\lambda\in\Lambda$ by self-duality, and $\varphi(a)=h(a,\lambda)$ on $\Lambda\cap U$. If $w=\lambda+z$ with $\lambda\in\Lambda$, $z\in U^\perp$, then $h(a,w)=h(a,\lambda)\in\mathcal{O}_E$ for $a\in\Lambda\cap U$. Conversely, if $h(\Lambda\cap U,w)\subseteq\mathcal{O}_E$, the map $a\mapsto h(a,w)$ is $\mathcal{O}_E$-linear into $\mathcal{O}_E$, so it equals $h(\cdot,\lambda)$ for some $\lambda\in\Lambda$; then $h(a,w-\lambda)=0$ for $a\in\Lambda\cap U$, hence for $a\in U$ (Step 1 and E-linearity in a), i.e. $w-\lambda\in U^\perp$.

Step 3: dual lattices. For a lattice P in W put $P^\vee=\{\phi\in\mathrm{Hom}_E(W,E) : \phi(P)\subseteq\mathcal{O}_E\}$, a lattice in $\mathrm{Hom}_E(W,E)$. For lattices $P\subseteq R$ in W, $R^\vee\subseteq P^\vee$ and $[P^\vee:R^\vee]=[R:P]$ (elementary divisors: a basis e_i of R with $\varpi^{a_i}e_i$ a basis of P gives the bases e_i^* of $R^\vee$ and $\varpi^{-a_i}e_i^*$ of $P^\vee$). Since $(P+P')^\vee=P^\vee\cap P'^\vee$, for any lattices P,P' in W:

$$[P^\vee : P^\vee\cap P'^\vee]=[P^\vee:(P+P')^\vee]=[P+P':P]=[P':P\cap P']. \tag{16.5}$$

Take $W = V$: the bijection $\lambda \mapsto h(\cdot, \lambda)$ is additive and maps $P^{\#} = \{y : h(P, y) \subseteq \mathcal{O}_E\}$ onto $P^{\vee}$; with $P = \Lambda$, $P' = \Lambda'$ (so $P^{\#} = \Lambda$, $P'^{\#} = \Lambda'$), (16.5) gives $[\Lambda : Q] = [\Lambda' : Q]$.

Step 4: the projection to U^*. Let $U^* = \mathrm{Hom}_E(U, E)$ and $\rho\colon V \to U^*$, $\rho(w)(a) = h(a, w)$. It is F-linear with kernel $U^{\perp}$, and surjective (a form on U extends to V and is then $h(\cdot, w)$). By (1), $\rho(\Lambda) = (\Lambda \cap U)^{\vee}$: $\rho(\Lambda) \subseteq (\Lambda \cap U)^{\vee}$ as $h(\Lambda, \Lambda) \subseteq \mathcal{O}_E$; and for $\phi \in (\Lambda \cap U)^{\vee}$, (1) gives $\lambda \in \Lambda$ with $\phi = \rho(\lambda)$ on $\Lambda \cap U$, hence on U. Likewise $\rho(\Lambda') = (\Lambda' \cap U)^{\vee}$. Write $Z = U^{\perp}$, fix Haar measures μ_Z on Z and μ^* on U^*, an F-linear section σ of ρ, and the Haar measure μ on $V = Z \oplus \sigma(U^*)$ with $\int_V f\,d\mu = \int_{U^*}\int_Z f(\sigma(\phi) + z)\,d\mu_Z(z)\,d\mu^*(\phi)$. For a compact open subgroup $P \subseteq V$ and $\phi \in U^*$, $\{z : \sigma(\phi) + z \in P\}$ is a coset of $P \cap Z$ if $\phi \in \rho(P)$ and empty otherwise, so

$$\mu(P) = \mu_Z(P \cap Z)\,\mu^*(\rho(P)).$$

For commensurable compact open subgroups P, P' of a vector space with Haar measure ν, $\nu(P) = [P : P \cap P']\,\nu(P \cap P')$, so $\nu(P)/\nu(P') = [P : P \cap P']/[P' : P \cap P']$.

Step 5: proof of (2). The indices are finite by Step 1. By Step 3, $\mu(\Lambda) = \mu(\Lambda')$; by Step 4,

$$\frac{\mu_Z(\Lambda \cap Z)}{\mu_Z(\Lambda' \cap Z)} = \frac{\mu^*((\Lambda' \cap U)^{\vee})}{\mu^*((\Lambda \cap U)^{\vee})} = \frac{[(\Lambda' \cap U)^{\vee} : (\Lambda \cap U)^{\vee} \cap (\Lambda' \cap U)^{\vee}]}{[(\Lambda \cap U)^{\vee} : (\Lambda \cap U)^{\vee} \cap (\Lambda' \cap U)^{\vee}]} = \frac{[\Lambda \cap U : Q \cap U]}{[\Lambda' \cap U : Q \cap U]},$$

the last equality by (16.5) in $W = U$ with $P = \Lambda \cap U$, $P' = \Lambda' \cap U$, $P \cap P' = Q \cap U$. The left side is $[\Lambda \cap Z : Q \cap Z]/[\Lambda' \cap Z : Q \cap Z]$. Cross-multiplying gives (2). □

Lemma 16.37. *The notation introduced here is local to this result and to the results that apply it. Let U be a finitely generated $\mathcal{O}_E$-module with a hermitian form h (E-linear in the first variable, $h(y, x) = \overline{h(x, y)}$) such that $h(U, U) \subseteq \mathcal{O}_E$ and*

$$U = \mathcal{O}_E e_1 \oplus \cdots \oplus \mathcal{O}_E e_\rho \oplus U', \qquad h(e_a, e_b) = 0 \ (a \neq b), \quad h(e_a, U') = 0, \quad h(e_a, e_a) \in \mathcal{O}_F^{\times},$$

with $\mathcal{O}_E e_a \cong \mathcal{O}_E$ and $\rho \geq 4$. Let du be the Haar probability measure on U, $a \geq 0$ an integer, $Z = \{u \in U : h(u, u) \in \varpi^a \mathcal{O}_F\}$, and L the set of $u \in U$ for which $h(u, e_1), \ldots, h(u, e_\rho)$ generate a proper ideal of $\mathcal{O}_E$. Then

1. *$|q^a \mathrm{vol}(Z) - 1| \leq \frac{8}{7} q^{1-\rho}$;*
2. *$q^a \mathrm{vol}(Z \cap L) \leq \frac{16}{7} q^{1-\rho}$;*
3. *for every nontrivial continuous character ϕ of the additive group U, $q^a \left|\int_Z \phi(u)\,du\right| \leq \frac{8}{7} q^{1-\rho}$.*

Proof. Write $v_E(x) = \max\{m : x \in \varpi^m \mathcal{O}_E\}$ for $0 \neq x \in \mathcal{O}_E$. As E/F is unramified or split, $|\mathcal{O}_E/\varpi\mathcal{O}_E| = q^2$, the maximal ideals of $\mathcal{O}_E$ are $\varpi\mathcal{O}_E$ (field case) or $\mathfrak{p}_1 = \varpi\mathcal{O}_F \times \mathcal{O}_F$ and $\mathfrak{p}_2 = \mathcal{O}_F \times \varpi\mathcal{O}_F$ (split case), and $\{\mathrm{Tr}(cx) : c \in \mathcal{O}_E\} = \varpi^{v_E(x)}\mathcal{O}_F$ (the trace maps $\mathcal{O}_E$ onto $\mathcal{O}_F$, and $x\varpi^{-v_E(x)}$ is a unit, resp. has a unit component). All characters below are continuous, and dc is the Haar probability measure of $\mathcal{O}_E$.

Step 1: expansion of the condition. For $x \in \mathcal{O}_F$, $\mathbf{1}[x \in \varpi^a \mathcal{O}_F] = q^{-a} \sum_\chi \chi(x)$, the sum over the q^a characters χ of $\mathcal{O}_F/\varpi^a\mathcal{O}_F$. The *conductor* of χ is the least $i \geq 0$ with $\chi(\varpi^i \mathcal{O}_F) = 1$; there are $q^i - q^{i-1}$ characters of conductor exactly $i \geq 1$ (those trivial on $\varpi^i\mathcal{O}_F$ are the q^i characters of $\mathcal{O}_F/\varpi^i\mathcal{O}_F$), and only $\chi = 1$ has conductor 0.

Step 2: one coordinate. Let χ have conductor $i \geq 1$, $\varepsilon \in \mathcal{O}_F^{\times}$, ϕ_1 a character of $\mathcal{O}_E$, and D either $\mathcal{O}_E$ or a maximal ideal of $\mathcal{O}_E$. We show $|G| \leq q^{-i}$ for $G = \int_D \chi(\varepsilon \mathrm{N}(c))\phi_1(c)\,dc$. Since $\mathrm{N}(c' + \delta) = \mathrm{N}(c') + \mathrm{N}(\delta) + \mathrm{Tr}(c'\bar{\delta})$ and D is an additive group, substituting $c = c' + \delta$,

$$|G|^2 = \int_D \chi(\varepsilon \mathrm{N}(\delta))\phi_1(\delta)\Big(\int_D \chi(\varepsilon\,\mathrm{Tr}(c'\bar{\delta}))\,dc'\Big)d\delta \leq \mathrm{vol}(D)\,\mathrm{vol}(D_\chi),$$

because the inner integral of a character over the group D is $\mathrm{vol}(D)$ if it is trivial and 0 otherwise; D_χ is the set of $\delta \in D$ with $\chi(\varepsilon \mathrm{Tr}(D\bar{\delta})) = 1$. If $\mathrm{Tr}(D\bar{\delta}) = \varpi^m \mathcal{O}_F$ with $m \le i-1$, then χ is nontrivial on it (ε is a unit); so $\delta \in D_\chi$ forces $m \ge i$. For $D = \mathcal{O}_E$: $m = v_E(\delta)$, $D_\chi \subseteq \varpi^i \mathcal{O}_E$, and $|G|^2 \le q^{-2i}$. For $D = \varpi \mathcal{O}_E$: $m = 1 + v_E(\delta)$, $D_\chi \subseteq \varpi^{\max(1,i-1)}\mathcal{O}_E$, and $|G|^2 \le q^{-2}q^{-2\max(1,i-1)} \le q^{-2i}$. For $D = \mathfrak{p}_1$ (and symmetrically $\mathfrak{p}_2$): with $c' = (c_1, c_2)$, $\delta = (\delta_1, \delta_2)$, $\bar{\delta} = (\delta_2, \delta_1)$, $\mathrm{Tr}(c'\bar{\delta}) = c_1\delta_2 + c_2\delta_1$ ranges over $\varpi^{\min(1+v(\delta_2), v(\delta_1))}\mathcal{O}_F$; so $D_\chi \subseteq \varpi^i \mathcal{O}_F \times \varpi^{i-1}\mathcal{O}_F$, of volume $q^{-i}q^{-(i-1)}$, and $|G|^2 \le q^{-1}q^{-(2i-1)} = q^{-2i}$.

Step 3: factorisation. Write $u = \sum_a c_a e_a + u'$; the Haar probability measure of U is the product of dc on each $\mathcal{O}_E e_a$ and the Haar probability measure du' of U', and $h(u,u) = \sum_a \varepsilon_a \mathrm{N}(c_a) + h(u',u')$ with $\varepsilon_a = h(e_a, e_a) \in \mathcal{O}_F^\times$ and $h(u',u') \in \mathcal{O}_E \cap F = \mathcal{O}_F$. A character ϕ of U is $\phi(u) = \prod_a \phi(c_a e_a) \cdot \phi(u')$. Hence, for $D_1, \dots, D_\rho \in \{\mathcal{O}_E\} \cup \{\text{maximal ideals}\}$ and χ of conductor $i \ge 1$,

$$\Big| \int_{D_1 e_1 \oplus \cdots \oplus D_\rho e_\rho \oplus U'} \chi(h(u,u))\phi(u)\, du \Big| = \prod_a \Big| \int_{D_a} \chi(\varepsilon_a \mathrm{N}(c))\phi(ce_a)\, dc \Big| \cdot \Big| \int_{U'} \chi(h(u',u'))\phi(u')\, du' \Big| \le q^{-i\rho}$$

by Step 2 (the last factor has modulus at most 1). Since $q \ge 2$ and $\rho \ge 4$, $q^{1-\rho} \le \frac{1}{8}$ and

$$\sum_{i \ge 1} (q^i - q^{i-1}) q^{-i\rho} = \Big(1 - \frac{1}{q}\Big) \frac{q^{1-\rho}}{1 - q^{1-\rho}} \le \frac{8}{7}\Big(1 - \frac{1}{q}\Big) q^{1-\rho}.$$

Step 4: the three bounds. By Step 1, $q^a \int_Y \mathbf{1}_Z \phi\, du = \sum_\chi \int_Y \chi(h(u,u))\phi(u)\, du$ for every product set Y as in Step 3 and every character ϕ (possibly trivial). (1): $Y = U$, $\phi = 1$; the term $\chi = 1$ is 1, the others contribute at most $\frac{8}{7}q^{1-\rho}$ by Step 3. (3): $Y = U$; the term $\chi = 1$ is $\int_U \phi = 0$ as ϕ is nontrivial, and the others contribute at most $\frac{8}{7}q^{1-\rho}$. (2): if the $h(u, e_a) = c_a \varepsilon_a$ generate a proper ideal, it lies in a maximal ideal $\mathfrak{p}$, and then all $c_a \in \mathfrak{p}$ (ε_a is a unit); so $L \subseteq \bigcup_{\mathfrak{p}} Y_{\mathfrak{p}}$ with $Y_{\mathfrak{p}} = \mathfrak{p}e_1 \oplus \cdots \oplus \mathfrak{p}e_\rho \oplus U'$, over the at most two maximal ideals. For each, with $\phi = 1$, the term $\chi = 1$ is $\mathrm{vol}(Y_{\mathfrak{p}}) \le q^{-\rho}$ and the others contribute at most $\frac{8}{7}(1 - \frac{1}{q})q^{1-\rho}$, so

$$q^a \mathrm{vol}(Z \cap Y_{\mathfrak{p}}) \le q^{1-\rho}\Big(\frac{1}{q} + \frac{8}{7} - \frac{8}{7q}\Big) \le \frac{8}{7} q^{1-\rho},$$

and summing over at most two maximal ideals gives (2). □

Theorem 16.38. *Let $r \ge 0$ and $k \ge 2r+3$ be integers and let local non-generic records with parameters r, k have the properties (Lg1)–(Lg5). Then*

$$\sum_{d \in D} \nu(d) \left|\mathrm{amp}_d(X_d^{\mathrm{n}})\right|^2 \le H^{39(r+1)^2} q^{-(k-2r-5)}\, \mathrm{nn}, \qquad \sum_{d \in D} \nu(d) \left|\mathrm{amp}_d(X_d^{\mathrm{g}})\right|^2 \le H^{39(r+1)^2}\, \mathrm{nn}.$$

Proof. Put $A_0 = H^{9(r+1)^2 + 12(r+1)}$, $A_1 = 4H^{4(r+1)}$ and, for a family $Y = (Y_d)_{d \in D}$ of Σ-stable subsets $Y_d \subseteq X_d$, $\mathrm{S}(Y) = \sum_d \nu(d) |\mathrm{amp}_d(Y_d)|^2$. We have $k - 1 \ge 2$, so $x = q^{k-1} \ge 2$ and $\sum_{j=0}^n x^j \le 2x^n$ for $n \ge 0$.

Step 1: the basic inequality. If $B, \varepsilon \ge 0$, $\mathrm{mass}_d(Y_d) \le B$ and $|\mathrm{amp}_d(Y_d)| \le \varepsilon\, \mathrm{mass}_d(Y_d)$ for every d, then

$$\mathrm{S}(Y) \le \sum_d \nu(d) \varepsilon^2 \mathrm{mass}_d(Y_d)^2 \le \varepsilon^2 B \sum_d \nu(d)\, \mathrm{mass}_d(Y_d).$$

Since $|\omega_d| \le 1$, the hypothesis $|\mathrm{amp}_d(Y_d)| \le \mathrm{mass}_d(Y_d)$ holds always ($\varepsilon = 1$).

Step 2: mass bounds per datum. Let $i \ge 0$ and $Y_d \subseteq \{x \in X_d : b_d(x) = q^i\}$ be Σ-stable. By (Lg2) every $x \in Y_d$ has $c_d(x) = q^j$ with $0 \le j \le i$, so by (Lg3) and mass_d being additive over disjoint stable sets,

$$\mathrm{mass}_d(Y_d) \le \sum_{j=0}^{i} A_0 q^{j(k-1)} \le 2A_0 q^{i(k-1)}.$$

If moreover $i \ge 1$ and $Y_d \cap X_d^{\mathrm{II}} = \varnothing$, then (Lg2) gives $q^j \le q^{i-1}$, i.e. $j \le i-1$, and $\mathrm{mass}_d(Y_d) \le 2A_0 q^{(i-1)(k-1)}$. In all cases, by (Lg4) with $\beta = q^i$,

$$\sum_d \nu(d)\,\mathrm{mass}_d(Y_d) \le A_1 q^{-i(k-1)}\,\mathrm{nn}.$$

Step 3: the generic terms. On X_d^{g} we have $c_d \le b_d = 1$, hence $c_d = 1$, so $\mathrm{mass}_d(X_d^{\mathrm{g}}) \le A_0$ by (Lg3), and $\sum_d \nu(d)\mathrm{mass}_d(X_d^{\mathrm{g}}) \le \mathrm{nn}$; Step 1 with $\varepsilon = 1$, $B = A_0$ gives $\mathrm{S}(X^{\mathrm{g}}) \le A_0\,\mathrm{nn} \le H^{39(r+1)^2}\mathrm{nn}$, as $H \ge 1$ and $9(r+1)^2 + 12(r+1) \le 39(r+1)^2$.

Step 4: the pieces of the non-generic terms. If $a = 0$, (Lg1) gives $b_d \equiv 1$, every X_d^{n} is empty and the first bound is trivial. Let $a \ge 1$; then $H \ge q^a \ge 2$. For $1 \le i \le a$ let $Y_d^{\mathrm{I},i} = \{x \in X_d^{\mathrm{n}} \setminus X_d^{\mathrm{II}} : b_d(x) = q^i\}$, and for $\lambda \in \{1,\dots,r\} \times \{0,\dots,i-1\}$ let $Y_d^{i,\lambda} = \{x \in X_d^{\mathrm{II}} : b_d(x) = q^i,\ \lambda_d(x) = \lambda\}$. These sets are Σ-stable (b_d, λ_d, X_d^{II} are), pairwise disjoint, and by (Lg1) (a non-generic term has $b_d = q^i$ with $1 \le i \le a$) and (Lg5) ($X_d^{\mathrm{II}} \subseteq X_d^{\mathrm{n}}$ and the range of λ_d) their union is X_d^{n}. Their number is

$$N \le a + a \cdot ra = a(1+ra) \le 2^a \cdot 2^{ra} \le q^{a(r+1)} \le H^{r+1}.$$

First type. By Steps 1 and 2 ($\varepsilon = 1$, $B = 2A_0 q^{(i-1)(k-1)}$), $\mathrm{S}(Y^{\mathrm{I},i}) \le 2A_0 q^{(i-1)(k-1)} A_1 q^{-i(k-1)}\mathrm{nn} = 2A_0A_1 q^{-(k-1)}\mathrm{nn}$.

Second type. Fix d, i, λ and $\mathcal{P} = Y_d^{i,\lambda}$. By (Lg5) there is a kernel m on $\mathcal{P}$, and Lemma 16.39 applies to G, X, Σ, K_0, $\mathcal{P}$, $\omega_d|_{\mathcal{P}}$, m and $e = \frac{24}{7} q^{-(k-2r-4)}$: $\mathcal{P}$ is Σ-stable and a union of finitely many orbits (as X_d is), (Vk1) is the volume condition of the records, (Vk2) holds as ω_d and m are Σ-invariant with $|\omega_d| \le 1$, and (Vk3), (Vk4) are (Lg5). Its conclusion is $|\mathrm{amp}_d(\mathcal{P})| \le 2e\,\mathrm{mass}_d(\mathcal{P}) \le 7q^{-(k-2r-4)}\mathrm{mass}_d(\mathcal{P})$. Steps 1 and 2 with $\varepsilon = 7q^{-(k-2r-4)}$ and $B = 2A_0 q^{i(k-1)}$ give

$$\mathrm{S}(Y^{i,\lambda}) \le 49q^{-2(k-2r-4)} \cdot 2A_0 q^{i(k-1)} \cdot A_1 q^{-i(k-1)}\mathrm{nn} = 98A_0A_1 q^{-2(k-2r-4)}\mathrm{nn}.$$

Comparison. Since $k-1 \ge k-2r-5$ and $2(k-2r-4) \ge k-2r-5$ (this is $k \ge 2r+3$), and $q \ge 2$, every piece has $\mathrm{S}(Y) \le 98A_0A_1 q^{-(k-2r-5)}\mathrm{nn}$.

Step 5: Cauchy–Schwarz over the pieces. For each d, $\mathrm{amp}_d(X_d^{\mathrm{n}})$ is the sum of the N amplitudes of the pieces, so $|\mathrm{amp}_d(X_d^{\mathrm{n}})|^2 \le N \sum_Y |\mathrm{amp}_d(Y_d)|^2$; multiplying by $\nu(d)$ and summing over d,

$$\begin{aligned}\mathrm{S}(X^{\mathrm{n}}) \le N \sum_Y \mathrm{S}(Y) &\le N^2 \cdot 98A_0A_1 q^{-(k-2r-5)}\mathrm{nn} \\ &\le 392\, H^{2(r+1)+9(r+1)^2+12(r+1)+4(r+1)} q^{-(k-2r-5)}\mathrm{nn}.\end{aligned}$$

As $H \ge 2$, $392 \le 2^9 \le H^9$, and with $x = r+1 \ge 1$ the exponent is $9x^2 + 18x + 9 = 9(x+1)^2 \le 36x^2 \le 39x^2$. This is the first bound. □

Lemma 16.39. *The notation introduced here is local to this result and to the results that apply it. Let G be a group acting on a set X, $\Sigma \subseteq G$ a subgroup and $K_0 \subseteq \Sigma$ a subgroup. A subgroup $A \subseteq G$ is **commensurable with** K_0 if $[A : A \cap K_0]$ and $[K_0 : A \cap K_0]$ are finite; then put*

$$\operatorname{vol}(A) = \frac{[A : A \cap K_0]}{[K_0 : A \cap K_0]} \in (0, \infty).$$

For $x \in X$ let $\Sigma_x = \{g \in \Sigma : gx = x\}$. Let $\mathcal{P} \subseteq X$ be a Σ-stable subset that is the union of finitely many Σ-orbits, $\omega \colon \mathcal{P} \to \mathbb{C}$, $m \colon \mathcal{P} \times \mathcal{P} \to [0, \infty)$ and $e \geq 0$ such that

(Vk1) Volumes *for every $x \in \mathcal{P}$, Σ_x is commensurable with K_0, and $\operatorname{vol}(\Sigma_{gx}) = \operatorname{vol}(\Sigma_x)$ for $g \in \Sigma$;*

(Vk2) Invariance *$\omega(gx) = \omega(x)$ and $m(gx, gy) = m(x, y)$ for all $g \in \Sigma$ and $x, y \in \mathcal{P}$, and $|\omega(x)| \leq 1$;*

(Vk3) Symmetric finite kernel *$m(x, y) = m(y, x)$, and for every x the set $\{y \in \mathcal{P} : m(x, y) \neq 0\}$ is finite;*

(Vk4) Averaging *for every $x \in \mathcal{P}$,*

$$\Big|\sum_{y \in \mathcal{P}} m(x, y) - 1\Big| \leq e, \qquad \Big|\sum_{y \in \mathcal{P}} m(x, y)\, \omega(y)\Big| \leq e.$$

Put $\mu(x) = 1/\operatorname{vol}(\Sigma_x)$ for $x \in \mathcal{P}$. Then μ and ω are constant on Σ-orbits, and

$$\Big|\sum_{O \in \Sigma \backslash \mathcal{P}} \mu(O)\, \omega(O)\Big| \leq 2e \sum_{O \in \Sigma \backslash \mathcal{P}} \mu(O).$$

Proof. **Step 1: indices as volume ratios.** Let $B \subseteq A$ be subgroups of G commensurable with K_0. Then $[A : B]$ is finite and $[A : B] = \operatorname{vol}(A)/\operatorname{vol}(B)$. Indeed, put $A_0 = A \cap K_0 \supseteq B_0 = B \cap K_0$; $[K_0 : B_0]$ is finite and $[K_0 : B_0] = [K_0 : A_0][A_0 : B_0]$, so $[A_0 : B_0]$ is finite, and $[A : B_0] = [A : A_0][A_0 : B_0]$ is finite, hence so is $[A : B]$, with $[A : B][B : B_0] = [A : B_0]$. Therefore

$$\frac{\operatorname{vol}(A)}{\operatorname{vol}(B)} = \frac{[A : A_0]\,[K_0 : B_0]}{[K_0 : A_0]\,[B : B_0]} = \frac{[A : A_0][A_0 : B_0]}{[B : B_0]} = \frac{[A : B_0]}{[B : B_0]} = [A : B].$$

If A, B are commensurable with K_0, so is $A \cap B$: $[K_0 : K_0 \cap A \cap B] \leq [K_0 : K_0 \cap A][K_0 : K_0 \cap B]$, and $[A \cap B : A \cap B \cap K_0] \leq [A : A \cap K_0]$ since $A \cap B \cap K_0 = (A \cap B) \cap (A \cap K_0)$.

Step 2: the pair weight. For $g \in \Sigma$, $\Sigma_{gx} = g \Sigma_x g^{-1}$; with (Vk1) and (Vk2), μ and ω are constant on Σ-orbits. For $x, y \in \mathcal{P}$ put

$$\nu(x, y) = \mu(x)\,[\Sigma_x : \Sigma_x \cap \Sigma_y] = \frac{1}{\operatorname{vol}(\Sigma_x \cap \Sigma_y)},$$

the equality by Step 1 applied to $\Sigma_x \cap \Sigma_y \subseteq \Sigma_x$. So $\nu(x, y) = \nu(y, x)$. For $g \in \Sigma$, conjugation by g carries $\Sigma_x \cap \Sigma_y$ onto $\Sigma_{gx} \cap \Sigma_{gy}$ and Σ_x onto Σ_{gx}, so the index is unchanged, and $\mu(gx) = \mu(x)$; hence $\nu(gx, gy) = \nu(x, y)$. The Σ_x-orbit of y has $[\Sigma_x : \Sigma_x \cap \Sigma_y]$ elements, the stabiliser of y in Σ_x being $\Sigma_x \cap \Sigma_y$.

Step 3: an exchange identity. Fix representatives $x_1, \dots, x_n$ of the Σ-orbits in $\mathcal{P}$. Let $\Phi\colon \mathcal{P}\times\mathcal{P}\to\mathbb{C}$ satisfy $\Phi(gx,gy)=\Phi(x,y)$ for $g\in\Sigma$ and vanish outside $\{(x,y): m(x,y)\neq 0\}$. We claim

$$\sum_{i=1}^{n}\mu(x_i)\sum_{y\in\mathcal{P}}\Phi(x_i,y)=\sum_{i=1}^{n}\mu(x_i)\sum_{y\in\mathcal{P}}\Phi(y,x_i), \tag{16.6}$$

all sums being finite by (Vk3) (the support of $y\mapsto\Phi(y,x_i)$ lies in $\{y: m(y,x_i)\neq 0\}=\{y: m(x_i,y)\neq 0\}$). Let $\mathcal{O}$ be the set of Σ-orbits on $\mathcal{P}\times\mathcal{P}$ (diagonal action). Each $Q\in\mathcal{O}$ contains pairs with first entry x_i for exactly one i, and those pairs form one Σ_{x_i}-orbit $\{(x_i,sy): s\in\Sigma_{x_i}\}$: if $(x_i,y),(x_i,y')\in Q$ then $(x_i,y')=g(x_i,y)$ with $g\in\Sigma$, $gx_i=x_i$. Grouping the inner sum on the left of (16.6) by Σ_{x_i}-orbits of y and using Step 2,

$$\sum_{i}\mu(x_i)\sum_{y}\Phi(x_i,y)=\sum_{Q\in\mathcal{O}}\nu(Q)\,\Phi(Q),$$

where $\nu(Q)$ and $\Phi(Q)$ are the values at any pair of Q (both are Σ-invariant), and only the finitely many Q with $\Phi(Q)\neq 0$ contribute. The same identity for $\Phi^{\top}(x,y)=\Phi(y,x)$ (which is Σ-invariant and vanishes outside $\{m\neq 0\}$, m being symmetric) gives the right side of (16.6) as $\sum_Q\nu(Q)\Phi(Q^{\top})$, $Q^{\top}$ the image of Q under $(x,y)\mapsto(y,x)$. As $Q\mapsto Q^{\top}$ is a bijection of $\mathcal{O}$ and $\nu(Q^{\top})=\nu(Q)$ by the symmetry of ν, the two sums agree.

Step 4: the bound. Put $R(x)=\sum_y m(x,y)$. Then

$$\sum_{i}\mu(x_i)\omega(x_i)=\sum_{i}\mu(x_i)\omega(x_i)\bigl(1-R(x_i)\bigr)+\sum_{i}\mu(x_i)\sum_{y}m(x_i,y)\,\omega(x_i).$$

The first sum has modulus at most $e\sum_i\mu(x_i)$ by $|\omega|\le 1$ and (Vk4). In the second apply (16.6) to $\Phi(x,y)=m(x,y)\omega(x)$, which is Σ-invariant by (Vk2): it equals $\sum_i\mu(x_i)\sum_y m(y,x_i)\omega(y)=\sum_i\mu(x_i)\sum_y m(x_i,y)\omega(y)$, of modulus at most $e\sum_i\mu(x_i)$ by (Vk3) and (Vk4). Adding, and noting that the sums over i are the sums over $\Sigma\backslash\mathcal{P}$, gives the claim. □

Lemma 16.40. *The notation introduced here is local to this result and to the results that apply it. Let $r\ge 0$, $k\ge 2r+5$, $C_1,C_0\ge 0$ be integers and put $e_{\mathrm{ng}}=(C_1+C_0)(r+1)^2+2(2r+2)+5$. Let Π be a finite set (the places), $\Omega\subseteq\Pi$, and for $p\in\Pi$ let D_p be a finite set, $\mathrm{wt}_p\colon D_p\to[0,\infty)$, $a_p^{\mathrm{N}},a_p^{\mathrm{Z}}\colon D_p\to\mathbb{C}$, and $H_p\ge 1$, $n_p\ge 0$, $n_p^{\circ}\ge 0$ real numbers and $q_p\ge 2$ integers with $q_p\le H_p$ for $p\in\Omega$. Let $V\ge 0$, let X be a finite set with $\rho\colon X\to[0,\infty)$ and $d\colon X\to\prod_{p\in\Pi}D_p$. Assume:*

1. *for $p\in\Omega$,*
$$\sum_{\delta\in D_p}\mathrm{wt}_p(\delta)|a_p^{\mathrm{N}}(\delta)|^2\le H_p^{C_1(r+1)^2}q_p^{-(k-2r-5)}n_p,$$
and for every $p\in\Pi$,
$$\sum_{\delta\in D_p}\mathrm{wt}_p(\delta)|a_p^{\mathrm{Z}}(\delta)|^2\le H_p^{C_1(r+1)^2}n_p;$$
2. *$n_p=n_p^{\circ}$ for $p\notin\Omega$, and $n_p\le 8H_p^{C_0(r+1)^2+2(2r+2)}n_p^{\circ}$ for $p\in\Omega$;*
3. *for every $\delta\in\prod_p D_p$, $\sum_{x\in X,\ d(x)=\delta}\rho(x)\le V\prod_{p\in\Pi}\mathrm{wt}_p(\delta_p)$.*

For $\delta\in\prod_p D_p$ put

$$\mathrm{amp}(\delta)=\sum_{\emptyset\neq B\subseteq\Omega}\prod_{p\in B}a_p^{\mathrm{N}}(\delta_p)\prod_{p\in\Pi\backslash B}a_p^{\mathrm{Z}}(\delta_p).$$

Then

$$\sum_{x\in X}\rho(x)\,|\mathrm{amp}(d(x))|^2\le 2^{-(k-2r-5)}\Bigl(\prod_{p\in\Pi}H_p\Bigr)^{e_{\mathrm{ng}}}V\prod_{p\in\Pi}n_p^{\circ}.$$

Proof. If $\Omega = \emptyset$ the sum defining amp is empty, the left side is 0 and the right side is nonnegative. Let $\Omega \neq \emptyset$, let $\mathcal{B}$ be the set of nonempty subsets of Ω, so $|\mathcal{B}| \le 2^{|\Omega|}$, and put $H = \prod_p H_p \ge 1$. For $B \in \mathcal{B}$ and $p \in \Pi$ let $b_p^B = a_p^{\mathrm{N}}$ if $p \in B$ and $b_p^B = a_p^{\mathrm{Z}}$ otherwise, and $\Phi_B(\delta) = \prod_p |b_p^B(\delta_p)|^2 \ge 0$.

Cauchy–Schwarz over the subsets. $|\mathrm{amp}(\delta)|^2 \le |\mathcal{B}| \sum_{B \in \mathcal{B}} \Phi_B(\delta)$.

Unfolding to the local sums. Grouping $x \in X$ by $d(x)$ and using (3) and $\Phi_B \ge 0$,

$$\sum_{x \in X} \rho(x) \Phi_B(d(x)) = \sum_{\delta \in d(X)} \Phi_B(\delta) \sum_{d(x) = \delta} \rho(x) \le V \sum_{\delta \in \prod_p D_p} \prod_p \mathrm{wt}_p(\delta_p) |b_p^B(\delta_p)|^2$$
$$= V \prod_{p \in \Pi} \Big(\sum_{\delta \in D_p} \mathrm{wt}_p(\delta) |b_p^B(\delta)|^2 \Big).$$

The gain. By (1) the product is at most $H^{C_1(r+1)^2} \prod_p n_p \prod_{p \in B} q_p^{-(k-2r-5)}$. As $k - 2r - 5 \ge 0$ and $q_p \ge 2$, each factor $q_p^{-(k-2r-5)}$ is at most 1, and since $B \neq \emptyset$ one of them is at most $2^{-(k-2r-5)}$. By (2) and $H_p \ge 1$, $\prod_p n_p \le 8^{|\Omega|} H^{C_0(r+1)^2 + 2(2r+2)} \prod_p n_p^\circ$. Hence

$$\sum_{x \in X} \rho(x) |\mathrm{amp}(d(x))|^2 \le |\mathcal{B}|^2 \, 8^{|\Omega|} \, 2^{-(k-2r-5)} H^{(C_1 + C_0)(r+1)^2 + 2(2r+2)} \, V \prod_p n_p^\circ.$$

The combinatorial factor. $|\mathcal{B}|^2 8^{|\Omega|} \le 32^{|\Omega|} \le \prod_{p \in \Omega} q_p^5 \le \prod_{p \in \Omega} H_p^5 \le H^5$, since $q_p \ge 2$, $q_p \le H_p$ on Ω and $H_p \ge 1$. This gives the claim with exponent e_{ng}.

□

17 The class and its universal properties

This section defines the class $\mathcal{A}_{\mathrm{prop}}$ and proves the two results for which Part III exists: Theorem 17.7, the universal properties of the class, and Proposition 17.6, the third input of Part I. The class is defined by lower conditions only. A family belongs to it if it has the linking property and the properties of its tops (Pc1), and if, whenever it satisfies the Sha condition, it carries one compatible collection of modification data, defect data, bridge, numerical, local defect, local bound and Ribet records, and frame, expansion, kernel, row and budget data, with the properties (Pc2)–(Pc6). Each of these is a hypothesis of a theorem of Sections 12–16 or of the Ribet budget Theorem 22.1; none of the properties (Sp3), (Sp4), (Sp5) and (Sp8) is required. The class is defined in this way because Part IV must show that it contains a family at every small ε: it asks for data and their lower properties, which a construction can be asked to supply, and the inequalities that follow from them are proved, here and in Part IV, rather than assumed. The four definitions before the class introduce the budget data for the local allowances, the Galois Ribet data, the inert budget data and the coefficient extensions of Hecke lattices, which (Pc5) asks for together with the hypotheses of the Ribet budget.

The proof of Theorem 17.7 is an assembly of the earlier sections, in the order in which their outputs are needed: (Sp3) from Section 12, (Sp4) from Section 13 with the Ribet budget, (Sp5) from Sections 14–16, (Sp8) from Lemma 12.1, while (Sp1), (Sp2), (Sp6) and (Sp7) are (Pc2). With Theorem 8.4 of Part I this gives slope data at the members with exponents $(1, 0)$. The only number in the argument that is the same for every family of the class is the divisor exponent 1, and hence the pair $(1, 0)$; all other constants and thresholds may depend on the family. Proposition 17.6 then takes the concrete good-host requirement for $\mathfrak{G}$, so that (Gh1) is Proposition 11.5; (Lk1) is design existence, Proposition 18.1 of Part IV, which is where (TCS) enters; and (Lk2), (Lk3) and (Lk5) are Theorem 17.7.

Definition 17.1. Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with parameter ε and threshold x_1, with local defect data and local bound records. **Budget data** for $\mathcal{F}$ consist of the following data, given for every real $x > x_1$ and every $i \in I(x)$:

- an integer $\varrho_{x,i} \geq 1$, the **base rank**, and real numbers $H_{x,i} \geq 1$, the **weight bound**, and $\ell^*_{x,i} \geq 3$, the **cut-off**;
- finite sets of primes $\mathcal{P}^{\mathrm{sm}}_{x,i}$, $\mathcal{P}^{\mathrm{rm}}_{x,i}$, $\mathcal{P}^{\mathrm{in}}_{x,i}$ and $\mathcal{P}^{\mathrm{ex}}_{x,i}$, the **small**, **ramified**, **inert** and **exceptional primes**.

For a member (x,i) write $t = t_{x,i}$, $\varrho = \varrho_{x,i}$, $H = H_{x,i}$, $\ell^* = \ell^*_{x,i}$ and $Y = f\,m\,b\,x^{2\varepsilon}$. A prime $\ell \in P_{x,i} \setminus G_{x,i}$ is of **small type** if $\ell \in \mathcal{P}^{\mathrm{sm}}_{x,i} \cup \mathcal{P}^{\mathrm{rm}}_{x,i}$; otherwise it is of **inert type** if $\ell \in \mathcal{P}^{\mathrm{in}}_{x,i}$ and ($\ell \leq \ell^*$ or $\ell \in \mathcal{P}^{\mathrm{ex}}_{x,i}$), of **split type** if $\ell \notin \mathcal{P}^{\mathrm{in}}_{x,i}$ and ($\ell \leq \ell^*$ or $\ell \in \mathcal{P}^{\mathrm{ex}}_{x,i}$), and **large** if $\ell > \ell^*$ and $\ell \notin \mathcal{P}^{\mathrm{ex}}_{x,i}$. The four types partition $P_{x,i} \setminus G_{x,i}$; the set of large primes is written $\mathcal{P}^{\mathrm{lg}}_{x,i}$. The budget data have

(Lz1) Sizes if there is a real C such that for all large members with finite Sha lengths: $1 \leq m \leq t$, $\varrho \leq t$, $t \leq b$, $H \leq C\,b$,

$$\ell^*\,n\,t^2(\varrho+t)\,x^{2\varepsilon} \leq C\,m\,b, \qquad |\mathcal{P}^{\mathrm{ex}}_{x,i}|\Big(1+\frac{b}{\ell^*}\Big)n\,t^2(\varrho+t) \leq C\,m\,b,$$

and $\ell \leq 3b$ for every $\ell \in \mathcal{P}^{\mathrm{ex}}_{x,i}$;

(Lz2) Small type if there is a real C such that for all large members with finite Sha lengths the following hold:

- $\mathrm{la}_{x,i}(\ell) \leq C\big(f\,m\,b\log x + x^C x^{2\varepsilon}\big)$ for $\ell \in \mathcal{P}^{\mathrm{sm}}_{x,i} \setminus G_{x,i}$, and $\sum_{\ell \in \mathcal{P}^{\mathrm{sm}}_{x,i}} \log \ell \leq C\,x^{\varepsilon}$;
- $\mathrm{la}_{x,i}(\ell) \leq C\,f\,m\,b$ and $\log \ell \leq x^{\varepsilon}$ for $\ell \in \mathcal{P}^{\mathrm{rm}}_{x,i}$;
- $|\mathcal{P}^{\mathrm{rm}}_{x,i}| \leq 2$;

(Lz3) Inert type if there is a real C such that for all large members with finite Sha lengths $\sum \log \ell \cdot \mathrm{la}_{x,i}(\ell) \leq C\,Y$, where the sum runs over the primes of inert type;

(Lz4) Split type if there is a real C such that for all large members with finite Sha lengths and every prime ℓ of split type

$$\mathrm{la}_{x,i}(\ell) \leq C\,f\Big(n\,t^2(\varrho+t)(1+\log_\ell(H+t)) + \frac{m\,H}{\ell}\Big);$$

(Lz5) Large primes if for all large members with finite Sha lengths $\mathrm{la}_{x,i}(\ell) \leq \ell^{-2}$ for every $\ell \in \mathcal{P}^{\mathrm{lg}}_{x,i}$.

Definition 17.2. For a square matrix A of size N over a commutative ring and $0 \leq j \leq N$, $\sigma_j(A)$ is the sum of the principal $j \times j$ minors of A, so that $\det(1 - XA) = \sum_{j=0}^{N} (-1)^j \sigma_j(A) X^j$.

Galois Ribet data at a point. Let E be a number field, ℓ a prime, S_ℓ the set of places of E above ℓ, and fix for every finite place v of E a decomposition group $G_{E_v} \subseteq G_E$ with its inertia group I_v. Let $(\mathcal{O}, \mathcal{Z}, \mathbb{T}, M)$ be a Hecke lattice over ℓ, put $K = \mathcal{O}[\ell^{-1}]$, and let $t \geq 1$ be an integer. Let $(\mathcal{Z}', \phi, (z_k), n_{\mathcal{R}}, \mathcal{R})$ be a determinant algebra over $\mathbb{T}$, and let $\pi' \in \mathcal{Z}'$ be a point with a block form with $t+1$ blocks numbered $0, 1, \ldots, t$, block sizes $n_{\mathcal{R},0}, \ldots, n_{\mathcal{R},t}$, maps ρ_k and constituent modules Cs_k. Let $\mathcal{N}$ be a left $\mathcal{R}$-module. **Galois Ribet data** on $(\mathcal{R}, \pi', \mathcal{N})$ over E consist of:

- an $\mathcal{O}$-subalgebra $\mathbb{T}'$ of the ring of functions $\mathcal{Z}' \to \mathcal{O}$ (pointwise operations), the **coefficient algebra**; a function $f \in \mathbb{T}'$ is identified with the central element $(f(x') \cdot 1)_{x' \in \mathcal{Z}'}$ of $\prod_{x' \in \mathcal{Z}'} \mathrm{M}_{n_{\mathcal{R}}}(K)$;
- a group homomorphism $\mathbf{r} \colon G_E \to \mathcal{R}^\times$, the **representation**, such that $g \mapsto \mathbf{r}(g)(x') \in \mathrm{GL}_{n_{\mathcal{R}}}(K)$ is continuous for every $x' \in \mathcal{Z}'$ and $\mathcal{R}$ is the $\mathbb{T}'$-submodule spanned by $\mathbf{r}(G_E)$, that is, $\mathcal{R} = \{\sum_{h \in S} f_h \, \mathbf{r}(h) : \ S \subseteq G_E \text{ finite}, \ f_h \in \mathbb{T}'\}$;
- a continuous character $\chi_1 \colon G_E \to \mathcal{O}^\times$; we put $\chi_c = \chi_1 \, \chi_{\mathrm{cyc}}^{1-c}$ for $1 \le c \le t$ and $u(g) = \chi_1(g)^{-1} \mathbf{r}(g) \in \mathcal{R}^\times$;
- an element $\mathcal{E} = \sum_{h \in S_{\mathcal{E}}} t_h \, \mathbf{r}(h) \in \mathcal{R}$, with $S_{\mathcal{E}} \subseteq G_E$ finite and $t_h \in \mathbb{T}'$, the **projector**, and a function $\tilde{\delta} \in \mathbb{T}'$ with $\tilde{\delta}(x') \neq 0$ for every $x' \in \mathcal{Z}'$, its **denominator function**; we put $\delta = \tilde{\delta}(\pi') \in \mathcal{O} \setminus \{0\}$, the **denominator**;
- an $\mathcal{R}$-linear map $\iota \colon \mathcal{N}_{\mathcal{E}} \to \mathcal{N}$, where $\mathcal{N}_{\mathcal{E}} = \sum_{g \in G_E} \mathcal{R} \, y(g) \subseteq \mathcal{R}$ is the left ideal generated by the **Ribet cocycle** $y(g) = (u(g) - 1)\mathcal{E}$;
- disjoint finite sets Σ_{T} and Σ_{P} of finite places of E not above ℓ, the **tame places** and the **polynomial places**, an element $\gamma_v \in I_v$ for each $v \in \Sigma_{\mathrm{T}}$ (we put $\zeta_v = \chi_1(\gamma_v)$), and an element $p_v \in \mathcal{R}$ for each $v \in \Sigma_{\mathrm{P}}$;
- for each $v \in S_\ell$ an $\mathcal{O}$-submodule $F_0(v) \subseteq H^1(G_{E_v}, \mathrm{Cs}_0(u))$, the **local condition** at v;
- real numbers $\kappa_k \ge 0$ and $\lambda_k(v) \ge 0$ for $0 \le k \le t$ and $v \in S_\ell$, the **column losses** and the **local losses**.

For $f \in \mathbb{T}'$ the matrix $f(\pi') \cdot 1$ is block diagonal, so $\rho_k(f) = f(\pi') \cdot 1$ for every k; in particular $\tilde{\delta}$ acts on every Cs_k as the scalar δ. Here $\mathrm{Cs}_k(u)$ is the discrete $\mathcal{O}[G_E]$-module Cs_k on which g acts by $\rho_k(u(g))$. For $\varphi \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathrm{Cs}_k)$ put $c_\varphi(g) = \varphi(\iota(y(g)))$, and let $X_k \subseteq H^1(G_E, \mathrm{Cs}_k(u))$ be the set of classes of those c_φ that are continuous 1-cocycles, the **Ribet classes** of block k. Put $F_1(v) = H^1_{\mathrm{ur}}(E_v, \mathrm{Cs}_1(u))$ and $F_k(v) = 0$ for $2 \le k \le t$ and $v \in S_\ell$. The **Selmer group** of block 0 is

$$\mathrm{Sel}_0 = \{x \in H^1(G_E, \mathrm{Cs}_0(u)) : x|_{I_w} = 0 \text{ for every finite } w \notin S_\ell \cup \Sigma_{\mathrm{T}} \cup \Sigma_{\mathrm{P}}, \\ x|_{G_{E_v}} \in F_0(v) \text{ for every } v \in S_\ell\}.$$

We consider the following properties of the data.

(Gd1) Block characters for $1 \le k \le t$, $n_{\mathcal{R},k} = 1$ and $\rho_k(\mathbf{r}(g)) = \chi_k(g)$ for all $g \in G_E$;

(Gd2) Projector $\mathcal{E}^2 = \tilde{\delta} \, \mathcal{E}$, $\rho_1(\mathcal{E}) = \delta$, and $\rho_k(\mathcal{E}) = 0$ for $k \neq 1$;

(Gd3) Unramified places $\mathbf{r}(\gamma) = 1$ for every finite place $w \notin S_\ell \cup \Sigma_{\mathrm{T}} \cup \Sigma_{\mathrm{P}}$ and every $\gamma \in I_w$;

(Gd4) Tame places for $v \in \Sigma_{\mathrm{T}}$: the image of γ_v generates topologically the maximal pro-ℓ quotient of I_v, $\zeta_v \neq 1$, and $(\mathbf{r}(\gamma) - 1)(\mathbf{r}(\gamma) - \chi_1(\gamma)) = 0$ in $\mathcal{R}$ for every $\gamma \in I_v$;

(Gd5) Polynomial places for $v \in \Sigma_{\mathrm{P}}$: $p_v \, (\mathbf{r}(\gamma) - \chi_1(\gamma)) = 0$ in $\mathcal{R}$ for every $\gamma \in I_v$, and $\rho_k(p_v) \in \mathcal{O}^\times$ for $1 \le k \le t$;

(Gd6) Column losses for $0 \le k \le t$ the kernel of $\mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathrm{Cs}_k) \to \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}_{\mathcal{E}}, \mathrm{Cs}_k)$, $\varphi \mapsto \varphi \circ \iota$, has normalised length at most κ_k;

(Gd7) Local losses for $0 \le k \le t$ and $v \in S_\ell$ the image of X_k in $H^1(G_{E_v}, \mathrm{Cs}_k(u))/F_k(v)$ under restriction has normalised length at most $\lambda_k(v)$.

For $a \in \mathcal{O} \setminus \{0\}$ put $\nu(a) = \mathrm{length}_{\mathcal{O}}(\mathcal{O}/a\mathcal{O})/e_{\mathcal{O}}$. The **loss sums** of the data are

$$\mathrm{ls}_0 = \kappa_0 + 2\, n_{\mathcal{R},0}\, \nu(\delta) + \sum_{v \in S_\ell} \lambda_0(v), \qquad \mathrm{ls}_1 = \kappa_1 + \sum_{v \in S_\ell} \lambda_1(v) + \sum_{v \in \Sigma_{\mathrm{T}}} \nu(\zeta_v - 1),$$
$$\mathrm{ls}_k = \kappa_k + 2\, \nu(\delta) + \sum_{v \in S_\ell} \lambda_k(v) + \sum_{v \in \Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) \qquad (2 \le k \le t).$$

Galois Ribet records of a family. Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with local defect data, local bound records and Ribet records. **Galois Ribet records** for $\mathcal{F}$ consist, for every (x, i, j, w, r, π) for which the Ribet records carry data, of Galois Ribet data on $(\mathcal{R}_{r,\pi}, \pi', \mathcal{N}_{r,\pi})$ over $E = E_{\mathfrak{h}_{x,i}}$, with $\ell = \ell_{x,i}(w)$, $t = t_{x,i}$ and the Hecke lattice $(\mathcal{O}_r, \mathcal{Z}_r, \mathbb{T}_r, M_r)$ of r. Their loss sums are written $\mathrm{ls}_k(r, \pi)$, and their Selmer group $\mathrm{Sel}_0(r, \pi)$. The records have

(Gd8) Records if for all large members with finite Sha lengths and all w, r, π with $\ell_{x,i}(w) \notin G_{x,i}$, $r \in R_{x,i}(j, w)$, $\pi \in Z_r$: the data have (Gd1)–(Gd7), E has no real place, $\mathrm{Sel}_0(r, \pi)$ has normalised length at most $\mathrm{sl}(r, \pi)$, and $\mathrm{lc}_k(r, \pi) \ge \mathrm{ls}_k(r, \pi)$ for $0 \le k \le t$;

(Gd9) Hecke-valued characteristic polynomials if for all large members with finite Sha lengths and all w, r, π as in (Gd8): the $\mathcal{O}$-module $\mathcal{D}_{r,\pi} = \{\sum_k (a_k \circ \phi_{r,\pi})\, z_k : \ a_k \in \mathbb{T}_r\}$ of functions $\mathcal{Z}'_{r,\pi} \to \mathcal{O}$ contains the constant function 1 and the coefficient algebra $\mathbb{T}'$ of the data and is closed under products; $\widehat{Z}_r \setminus Z_r$ is detected by the descent functions; and there is a finite set B of places of E such that for every finite place $w \notin B$ whose residue field has prime order, every $x' \in \mathcal{Z}'_{r,\pi}$ has $\mathbf{r}(\gamma)(x') = 1$ for $\gamma \in I_w$, and the functions $x' \mapsto \sigma_j(\mathbf{r}(\mathrm{Frob}_w)(x'))$, $1 \le j \le n_{\mathcal{R}}$, lie in $\mathcal{D}_{r,\pi}$;

(Gd10) Lossless at the large primes, relative to budget data for $\mathcal{F}$ (Definition 17.1) if for all large members with finite Sha lengths and all w, r, π as in (Gd8) with $\ell_{x,i}(w) \in \mathcal{P}^{\mathrm{lg}}_{x,i}$: $\delta \in \mathcal{O}^\times$, $t \sum_{v \in \Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) \le \ell^{-2}$, $\kappa_k = 0$ for $0 \le k \le t$, and $\lambda_k(v) = 0$ for $0 \le k \le t$ and $v \in S_\ell$.

Definition 17.3. Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with parameter ε and threshold x_1, with local defect data, local bound records and budget data. **Inert budget data** for $\mathcal{F}$ consist of the following data, given for every real $x > x_1$ and every $i \in I(x)$:

- a real number $\ell_{\mathrm{A}} \ge 2$, the **orbit cut-off**, and a real number $b^\circ \ge 1$, the **thin scale**;
- real numbers $w^\circ, \eta^\circ, S^\circ \ge 0$, the **thin bound**, the **width** and the **spread**, respectively;
- a real number $c_{\mathrm{bd}} \ge 0$, the **band threshold**;
- sets of primes $\mathcal{P}^{\mathrm{lo}}_{x,i}$ and $\mathcal{P}^{\mathrm{so}}_{x,i}$, the **long-orbit primes** and the **small-order primes**;
- a function $\beta_{x,i}$ from the set of primes to $\{0, 1\}$, the **band indicator**.

For a member (x, i), write $t = t_{x,i}$, write ϱ, H, ℓ^* for the corresponding entries of its budget data, and put $L = x^\varepsilon$. The **row excess** of a prime $\ell \ge 3$ is

$$A_{x,i}(\ell) = \begin{cases} \dfrac{H}{\ell_{\mathrm{A}}(\ell - 1)} + (w^\circ + S^\circ + \eta^\circ + 1)(3 + \log \ell_{\mathrm{A}}) + L, & \ell \in \mathcal{P}^{\mathrm{lo}}_{x,i}, \\ \dfrac{H}{\ell^2} + \dfrac{w^\circ + \eta^\circ}{\ell} + S^\circ + L + \eta^\circ\, \beta_{x,i}(\ell), & \ell \notin \mathcal{P}^{\mathrm{lo}}_{x,i}. \end{cases}$$

The inert budget data have

(Iq1) Sizes if there is a real $C \geq 1$ such that for all large members with finite Sha lengths:

1. $H \leq 2b$, $\ell_{\mathrm{A}} \leq \ell^*$, $(3+\log \ell_{\mathrm{A}})^2 \leq L$, $\varrho\, L \leq 2m\, \ell_{\mathrm{A}}$, $\varrho\, b^\circ \ell_{\mathrm{A}} \leq m\, b$, $L \leq b^\circ$, $w^\circ + S^\circ + \eta^\circ + 1 \leq 11\, b^\circ$, $\varrho\,(S^\circ + L)\,\ell^* \leq 2m\, b$, $\varrho\, \eta^\circ \leq 4m\, b$ and $c_{\mathrm{bd}} \leq C$;
2. every prime ℓ of inert type satisfies $\ell \geq 11$; if moreover $\ell \in \mathcal{P}^{\mathrm{lo}}_{x,i}$ then $\ell < \ell_{\mathrm{A}}$, and if $\ell \notin \mathcal{P}^{\mathrm{lo}}_{x,i} \cup \mathcal{P}^{\mathrm{so}}_{x,i}$ and $\ell \leq \ell^*$ then $\ell \geq \ell_{\mathrm{A}}$;
3. $\varrho \sum_{\ell \in \mathcal{P}^{\mathrm{so}}_{x,i}} \log \ell/\ell^2 \leq 4m\, L^2$ (a finite set);
4. for every prime ℓ with $\beta_{x,i}(\ell) = 1$ and $\ell \geq c_{\mathrm{bd}}$: $\varrho\, \eta^\circ \ell \leq 16\, m\, b$;
5. $\big|\mathcal{P}^{\mathrm{ex}}_{x,i}\big|\, \varrho\,(1 + H/\ell^* + w^\circ + \eta^\circ + S^\circ + L) \leq C\, m\, b$;

(Iq2) Inert allowance if there is a real $C \geq 1$ such that for all large members with finite Sha lengths and every prime ℓ of inert type

$$\mathrm{la}_{x,i}(\ell) \leq C\, f\Big(n\, t^2(\varrho + t)(1 + \log_\ell(H + t)) + \frac{m\, H}{\ell} + \varrho\, A_{x,i}(\ell)\Big).$$

Definition 17.4. Let ℓ be a prime and $\mathcal{O} \subseteq \mathcal{O}'$ coefficient rings for ℓ; put $K = \mathcal{O}[\ell^{-1}]$ and $K' = \mathcal{O}'[\ell^{-1}]$. For an $\mathcal{O}$-submodule X of a K-vector space W, $X_{\mathcal{O}'}$ denotes the $\mathcal{O}'$-span of X in $K' \otimes_K W$, where X is identified with $1 \otimes X$. For $W = K^{\mathcal{Z}}$ or $W = \prod_{x'} \mathrm{M}_n(K)$ we identify $K' \otimes_K W$ with $K'^{\mathcal{Z}}$, respectively $\prod_{x'} \mathrm{M}_n(K')$.

Hecke lattices. The **coefficient extension** of a Hecke lattice $(\mathcal{O}, \mathcal{Z}, \mathbb{T}, M)$ over ℓ to $\mathcal{O}'$ is $(\mathcal{O}', \mathcal{Z}, \mathbb{T}_{\mathcal{O}'}, M_{\mathcal{O}'})$. Here $M_{\mathcal{O}'} = \mathcal{O}' \otimes_{\mathcal{O}} M$, and $\sum_i a_i f_i \in \mathbb{T}_{\mathcal{O}'}$ ($a_i \in \mathcal{O}'$, $f_i \in \mathbb{T}$) acts on $M_{\mathcal{O}'}$ by $\sum_i a_i \otimes f_{i,M}$. This is well defined: $\mathcal{O}'$ is a free $\mathcal{O}$-module, so $\mathcal{O}' \otimes_{\mathcal{O}} \mathbb{T} \to \mathbb{T}_{\mathcal{O}'}$ is an isomorphism. The good and kept sets are unchanged.

Determinant algebras and presented modules. The coefficient extension of a determinant algebra $(\mathcal{Z}', \phi, (z_k), n_{\mathcal{R}}, \mathcal{R})$ over $\mathbb{T}$ is $(\mathcal{Z}', \phi, (z_k), n_{\mathcal{R}}, \mathcal{R}_{\mathcal{O}'})$ over $\mathbb{T}_{\mathcal{O}'}$, with the same points, block sizes at a point π' and descent functions. For a left $\mathcal{R}$-module $\mathcal{N}$ with a presentation $\mathbf{P}$, its coefficient extension is $\mathcal{N}_{\mathcal{O}'} = \mathcal{R}_{\mathcal{O}'} \otimes_{\mathcal{R}} \mathcal{N}$, presented by the same matrix $\mathbf{P}$ over $\mathcal{R}_{\mathcal{O}'}$.

Galois Ribet items. Galois Ribet items are the entries of Galois Ribet data other than the losses $\kappa_k, \lambda_k(v)$. The coefficient extension of Galois Ribet items on $(\mathcal{R}, \pi', \mathcal{N})$ over E with the Hecke lattice $(\mathcal{O}, \mathcal{Z}, \mathbb{T}, M)$ consists of the following:

- the coefficient algebra $\mathbb{T}'_{\mathcal{O}'}$;
- the representation $\mathbf{r}$ followed by $\mathcal{R} \subseteq \mathcal{R}_{\mathcal{O}'}$, and χ_1 followed by $\mathcal{O}^\times \subseteq \mathcal{O}'^\times$;
- the same projector $\mathcal{E}$, denominator function $\tilde{\delta}$, tame and polynomial places, and elements γ_v and p_v;
- the local conditions $F_0(v)_{\mathcal{O}'}$, the $\mathcal{O}'$-span of the image of $F_0(v)$ in $H^1(G_{E_v}, \mathcal{O}' \otimes_{\mathcal{O}} \mathrm{Cs}_0(u))$;
- the map $\iota_{\mathcal{O}'} \colon (\mathcal{N}_{\mathcal{E}})_{\mathcal{O}'} \to \mathcal{N}_{\mathcal{O}'}$, $\sum_j a_j n_j \mapsto \sum_j a_j \otimes \iota(n_j)$ ($a_j \in \mathcal{O}'$, $n_j \in \mathcal{N}_{\mathcal{E}}$), on the left ideal $(\mathcal{N}_{\mathcal{E}})_{\mathcal{O}'}$ of $\mathcal{R}_{\mathcal{O}'}$ generated by the Ribet cocycle. This map is well defined: since $\mathcal{O}'$ is free over $\mathcal{O}$, the map $\mathcal{O}' \otimes_{\mathcal{O}} \mathcal{N}_{\mathcal{E}} \to (\mathcal{N}_{\mathcal{E}})_{\mathcal{O}'}$ is an isomorphism.

These are Galois Ribet items on $(\mathcal{R}_{\mathcal{O}'}, \pi', \mathcal{N}_{\mathcal{O}'})$ over E with the extended Hecke lattice.

Families. Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with local defect data, local bound records, Ribet records, budget data, inert budget data and Galois

Ribet items at points (j, w, r, π) of members. For every local component r, let a coefficient ring $\mathcal{O}'_r \supseteq \mathcal{O}_r$ be given. The **coefficient extension** $\mathcal{F}'$ of $\mathcal{F}$ along $(\mathcal{O}_r \subseteq \mathcal{O}'_r)_r$ is $\mathcal{F}$ with the same data, except that the following are replaced by their coefficient extensions to $\mathcal{O}'_r$: the Hecke lattice of every local component r, the determinant algebras and modules of extensions of the Ribet records at r, and the Galois Ribet items at the points with this r. The local contributions $\mathrm{lc}_k(r, \pi)$, the local bound records, the budget data and the inert budget data are unchanged.

The root-of-unity extension. For a coefficient ring $\mathcal{O}$ for ℓ and an integer $N \geq 1$, $\mathcal{O}(N)$ denotes the integral closure of $\mathcal{O}$ in $K(\zeta) \subseteq \overline{\mathbb{Q}}_\ell$, where ζ is a primitive $\mathrm{lcm}(\ell, N)$-th root of unity.

Definition 17.5. The **class defined by linking and lower slope conditions**, denoted $\mathcal{A}_{\mathrm{prop}}$, consists of the parametrised host families $\mathcal{F}$ with slope records that satisfy (Pc1) below and the following conditional requirement: if $\mathcal{F}$ satisfies the Sha condition, then it admits a compatible collection of the records appearing in (Pc3)–(Pc5), and it satisfies (Pc2)–(Pc6).

Here a compatible collection consists of the given slope records, a choice of modification data, defect data, bridge records, numerical records, local defect data, local bound records and Ribet records, and all the frame, expansion, kernel, row, Galois Ribet, column, weight-word, position and budget data specified in (Pc5). All of them are indexed over the same family, scales and members. Write L_c for the size parameter of a loss datum. The collection is required to satisfy the following identifications; these are equalities of the indicated data, not identifications up to isomorphism.

- The underlying family $\mathcal{F}$, its parameter ε, threshold x_1, index sets $I(x)$, hosts $\mathfrak{h}_{x,i}$, frames, top and spacing sets, ranks $t_{x,i}$, translates $u_{x,i}$, loss data $\mathfrak{M}_{x,i}$ and slope records $\mathfrak{V}_{x,i}$ are the given ones in every record. Thus all records share every entry of $\mathfrak{M}_{x,i}$ and $\mathfrak{V}_{x,i}$, including k, m, η, L_c, a_t, the loss place record, the words and their ranks, slopes, heights, Gauss primes and exponents, the dual slopes, and F, δ, g_1, g_2. Consequently they also share the derived budgets B_0, N_c, the thick set and the host field.

- The modification data use these F and total exponents E in their two defining local sum identities. Their local frame numbers, local exponents, local lengths, rounded twists, level slopes, level ranks and exceptional moduli are the same throughout the collection.

- The defect sums Df_j and ranks $\mathrm{rk}_0, \mathrm{rk}_j$ in the bridge and local defect conditions are those of the defect data of the collection. The base dual slope $\mu^\vee_{\mathrm{b}}$, scaling term sc, passage allowance pa, local main and error terms lm, le and corrections tc_j are those of the bridge records of the collection. In particular the numerical conditions use these $\mathrm{sc}, \mathrm{pa}, \mathrm{lm}$ and the given word heights and Gauss primes.

- The twisted-place sets $W_{x,i}$, residue characteristics $\ell_{x,i}(w)$, weights $\mathrm{lw}_{x,i}(w)$, local component sets $R_{x,i}(j, w)$, Hecke lattices $(\mathcal{O}_r, \mathcal{Z}_r, \mathbb{T}_r, M_r)$, good and kept sets $Z_r, \widehat{Z}_r$, level order N_{lev} and allowances $\mathrm{ex}_{x,i}$ used by the local bound and Ribet conditions are those of the local defect data of the collection. Their eigenspaces, multiplicities, congruence lengths, cut defect sums and sets $P_{x,i}, G_{x,i}$ are the objects derived from these data. The Selmer lengths $\mathrm{sl}(r, \pi)$ and local terms $\mathrm{lt}(r)$ in the Ribet conditions are those of the local bound records of the collection, and so are the functions $\mathrm{sa}_{x,i}, \mathrm{la}_{x,i}, \mathrm{ga}_{x,i}$. The determinant algebra of each Ribet record is over the same $\mathbb{T}_r$.

- At each member there is a single frame realisation, and its data

$$K, v, \overline{R}, \overline{N}_\omega, B_\omega, \mathcal{H}, \Pi, D^*, D_\omega, \kappa_\omega, A_\omega, U_\omega, Z_\omega$$

are used throughout the frame, word-scale and row conditions. Their metric, rank, slope and map identities are the defining identities of Definition 14.1, also where they enter the diagonal, kernel and row data of (Pc5). In particular, every Gram block is the one computed from these A_ω, κ_ω. The word scale Γ_{w} of the member is the same throughout, and every word-size condition uses it together with these D_ω, D^*.

- The row parameters $k_{\mathrm{d}}, r_{\mathrm{f}}, f_\infty, g_1^{\mathrm{aux}}, g_2^{\mathrm{aux}}, T_{\mathrm{amp}}$ are the same in (Rw2) and (Ro1)–(Ro4). Each expansion required there is an expansion of the Gram block just specified; its main-row mass data use the same $D_\omega, D_{\omega'}, D^*$ and its own row normalisation.

The diagonal and row expansions use the Gram blocks just specified. In their main-row data $D^* = \deg_\pi v_G/\mu_K^\circ$, with the same measures and normalisation in every occurrence. The Galois Ribet, column, weight-word and budget data use the same host field, Hecke lattices, good and kept sets, local modules and inclusions specified by (Ub2), and the same Selmer and local-bound entries; the local loss sums and allowances are those required by (Ub9). Compatibility of the collection means these identifications, together with the defining types and identities of the cited data definitions. The numbered conditions imposed on the collection are exactly (Pc2)–(Pc6) below.

(Pc1) Linking and tops. The underlying parametrised host family has the linking property, few bad tops, small shares and non-empty top sets.

(Pc2) Direct slope conditions. The family has counting bounds (Sp1), ambient slope bounds (Sp2), weights at the nodes (Sp6) and thin scales (Sp7).

(Pc3) Modification conditions. The family admits modification data with properties (Mf1)–(Mf7).

(Pc4) Divisor conditions. The family admits defect data, bridge records, numerical records, local defect data, local bound records and Ribet records, with the same defect data and bridge records wherever they are used, having the following properties:

$$\begin{gathered}\text{(Dv1), (Dv2), (Dv3), (Dv4), (Dv7),}\\ \text{(Bk1), (Bk2), (Bk4),}\qquad\text{(Nr1), (Nr2), (Nr3),}\\ \text{(Ld1)–(Ld6),}\qquad\text{(Lq2), (Lq4), (Lq5), (Lq6), (Lq8),}\qquad\text{(Rf3).}\end{gathered}$$

(Pc5) Construction enrichment. The family admits frame realisations, word scales, row parameters, diagonal expansions, kernel and row data, Galois Ribet items, column data, weight words, large-prime position data, budget data and inert budget data, compatible with (Pc3), (Pc4). There are constants

$$\begin{gathered}c > 0,\quad C_0 \geq 1,\quad C_{\mathrm{A}} \geq 1,\quad 0 < c_{\mathrm{lp}} \leq \tfrac{1}{3},\quad C_{\mathrm{ng}} \geq 0,\\ C_{\mathrm{lb}} \geq 1,\quad A_{\mathrm{lb}} \geq 1,\quad C_0^{\mathrm{ng}} \in \mathbb{Z}_{\geq 0},\qquad c_{\mathrm{sp}}, c_{\mathrm{in}}, c_{\mathrm{sm}}, C_{\mathrm{bud}} \geq 1,\end{gathered}$$

depending only on the family, and in particular not on the scale, the member or any subsequent local index. Put $K_0 = 32C_{\mathrm{lb}}^{32}$. Each property named below is required in full as defined, with its hypotheses, its inner quantifiers and its uniform threshold for large members. The following hold.

- The frame conditions (Nf1), (Nf2) with exponent c, and (Nf3); word sizes (No1) for C_0, (No2) for C_0 with exponent c, and (No6) with exponent c; and (Rw2) with exponent c for $(C_{\mathrm{A}}, c_{\mathrm{lp}}, C_{\mathrm{ng}})$.

- The diagonal expansions have (Dx1) of Definition 14.15 for these C_0, c and the same Gram blocks.
- For all large members with word sizes for C_0, there are a measure space (X, μ) with $\mu(X) < \infty$, measurable functions $\mathrm{ls} : X \to (0, \infty)$, $\mathrm{kd} : X \to [0, \infty)$ and $\mathrm{na}, \mathrm{fa} : X \to [0, \infty]$, a real $\tau_* > 0$, and $c_{\mathrm{id}}, \epsilon_{\mathrm{N}}, \epsilon_{\mathrm{F}}, \kappa \geq 0$, satisfying (Kc1)–(Kc4) of Lemma 14.19 for $X_{\mathrm{t}} = \{\mathrm{ls} \geq \tau_*\}$, $X_{\mathrm{c}} = X \setminus X_{\mathrm{t}}$ and the same $\mathcal{H}, D^*$, such that

$$\kappa\mu(X) < 1, \qquad 1 + \epsilon_{\mathrm{N}} + \epsilon_{\mathrm{F}} \leq (1 - \kappa\mu(X))(1 + x^{-c}/12).$$

  These are the kernel hypotheses of Corollary 14.18.
- For all large members with word sizes for C_0 and every $\omega \neq \omega'$, there are data with (Ro1)–(Ro4) of Theorem 15.3, with the constants above, the same row parameters, Gram block and Hilbert–Schmidt metric, and scale $D_\omega D_{\omega'}/D^*$. In particular the loop modulus is $g_1^{\mathrm{aux}} g_2^{\mathrm{aux}}$; all non-generic data refer to this expansion, its a_2 and its n_{g}; and the main-row data are those of its main-row sum and its row normalisation. All hypotheses of (Mx1)–(Mx5) are required, including both word-group data and the complete archimedean product structure on the same Poisson data.
- The Galois Ribet items are given at every point of every member as in the setting of Theorem 22.4. There is a common x_0 such that for every $x \geq x_0$, $x > x_1$, and every member with finite Sha lengths there are an integer $N_{x,i} \geq 1$ with $4 \mid N_{x,i}$ and integers $S_{x,i}^{\mathrm{rm}}, \eta_{x,i}^{\mathrm{rm}} \geq 0$, independent of all subsequent point and local indices, for which the Galois Ribet items of the coefficient extension along $\mathcal{O}_r \subseteq \mathcal{O}_r(N_{x,i})$ satisfy (Ub1), (Ub2), (Ub3°), (Ub4)–(Ub10), including (Ub7'), of Theorem 22.1. The extension is that of Definition 17.4 at the local components of these members, and the identity extension at all other local components. The constants $c_{\mathrm{sp}}, c_{\mathrm{in}}, c_{\mathrm{sm}}, C_{\mathrm{bud}}$ are those above, and these properties are required with all their quantifiers over points, lifts, words, primes, places, components and eigensystems. The Galois Ribet items before extension, the local defect data, Ribet records, local bound records, budget data and inert budget data are those of the compatible collection. In particular (Ub9) uses the minimal losses, which are unchanged by Lemma 22.17, and the same allowance defined there, not separately chosen loss bounds.

(Pc6) Residual size conditions. The parameter of the family satisfies $\varepsilon \leq 1$. Moreover, there is a real x_0 such that for every $x \geq x_0$ with $x > x_1$ we have $I(x) \neq \varnothing$ when x lies in the set W of scales of $\mathcal{F}$, and every $i \in I(x)$ satisfies

$$x^{1/2-\varepsilon} \leq m, \qquad \log(2a\, v(\mathfrak{h}_{x,i})) \leq 2x^{\varepsilon}, \qquad \log|\mathcal{W}(\mathfrak{V})| \leq (k-3)\log 2,$$
$$x^{-\varepsilon} n \leq |T_{x,i}^{\mathrm{th}}|, \qquad \tfrac{1}{2}x^{-\varepsilon/8} n \leq |T_{x,i}|, \qquad a_t \geq (x-1)b \quad (t \in T_{x,i}),$$

where, in the usual member notation, $n = |A|$, $b = \exp(x^{\varepsilon})$, $a = xb$ and $\mathfrak{V} = \mathfrak{V}_{x,i}$.

The class $\mathcal{A}_{\mathrm{prop}}$ is a single class: it does not depend on a window ratio, on a supplied host requirement or on a requested parameter. The constants and the records whose existence is required may depend on the family.

Proposition 17.6. *There are a good-host requirement $\mathfrak{G}$ and a class $\mathcal{A}$ of parametrised host families with slope records such that the following hold. In (Lk1), (Lk2), (Lk3), in the intersection $\mathcal{A} \cap \mathcal{C}_\lambda(\mathsf{P})$ and in the Sha condition, a family with slope records is identified with its underlying parametrised host family.*

(Gh1) Supply of good hosts. $\mathfrak{G}$ *supplies good hosts.*

(Lk1) Design existence for $\mathfrak{G}$. *For every $\lambda > 1$ and every host requirement P supplied by $\mathfrak{G}$ on λ-windows there is $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ some family in $\mathcal{A} \cap \mathcal{C}_\lambda(\mathsf{P})$ has parameter ε.*

(Lk2) *Every family in $\mathcal{A}$ has the linking property.*

(Lk3) *Every family in $\mathcal{A}$ has few bad tops, small shares and non-empty top sets.*

(Lk4′) Slope records. *There is a real number e such that every family in $\mathcal{A}$ that satisfies the Sha condition has the properties (Sp1)–(Sp8), with (Sp4) for the exponent e.*

Proof. The choice. Let $\mathfrak{G}$ be the concrete good-host requirement of Definition 11.4. It is a predicate on tuples $(x, \varepsilon, J, \kappa, \mathfrak{h}, T, t)$ of real numbers x, ε, J, a finite list κ of real numbers, a host $\mathfrak{h}$ for a frame (e, A, h, a_0), a finite set $T \subseteq A$ and an integer t; restricted to $t \geq 1$ it is a good-host requirement. Let $\mathcal{A} = \mathcal{A}_{\mathrm{prop}}$ be the class defined by linking and lower slope conditions; it is a class of parametrised host families with slope records. Neither $\mathfrak{G}$ nor $\mathcal{A}$ depends on a window ratio, a host requirement or a parameter. We verify the clauses of the statement for this $\mathfrak{G}$ and this $\mathcal{A}$, where, as in the statement, a family with slope records is identified with its underlying parametrised host family in (Lk1)–(Lk3), in $\mathcal{A} \cap \mathcal{C}_\lambda(\mathsf{P})$ and in the Sha condition.

(Gh1). By Proposition 11.5, the concrete good-host requirement, restricted to $t \geq 1$, supplies good hosts.

(Lk1). Let $\lambda > 1$ and let P be a host requirement supplied by $\mathfrak{G}$ on λ-windows. We apply Proposition 18.1 to the requirement $\mathfrak{G}$ and the class $\mathcal{A}_{\mathrm{prop}}$, which do not depend on λ and P. It gives $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is a parametrised host family with slope records of parameter ε in $\mathcal{A}_{\mathrm{prop}} \cap \mathcal{C}_\lambda(\mathsf{P})$, where membership in $\mathcal{C}_\lambda(\mathsf{P})$ refers to its underlying parametrised host family. This is (Lk1).

(Lk2) and (Lk3). By Theorem 17.7, every family in $\mathcal{A}_{\mathrm{prop}}$ has the linking property, few bad tops, small shares and non-empty top sets. These are properties of the underlying parametrised host family, as in the identification.

(Lk4′). By Theorem 17.7, every family in $\mathcal{A}_{\mathrm{prop}}$ that satisfies the Sha condition (a condition on its underlying parametrised host family) has the properties (Sp1)–(Sp8), with (Sp4) for the exponent 1, the same for every family of the class. So (Lk4′) holds with $e = 1$. □

Theorem 17.7. *Every family in $\mathcal{A}_{\mathrm{prop}}$ has the linking property, few bad tops, small shares and non-empty top sets. Every family in $\mathcal{A}_{\mathrm{prop}}$ that satisfies the Sha condition has (Sp1)–(Sp8), with (Sp4) for the exponent $e = 1$, which is the same for every family of the class. Consequently every such family has slope data at its members with the same exponents $(e_D, e_L) = (1, 0)$. The coefficients, thresholds and auxiliary exponents in the named family properties may depend on the family; the divisor exponent 1 and the slope-data exponents $(1, 0)$ do not.*

Proof. Fix $\mathcal{F} \in \mathcal{A}_{\mathrm{prop}}$. Its linking property and its three tops properties are (Pc1); they hold without the Sha condition.

Now suppose that $\mathcal{F}$ satisfies the Sha condition. By the definition of $\mathcal{A}_{\mathrm{prop}}$ there is a compatible collection of the records in (Pc3)–(Pc5) with all their stated family-wide properties; fix one. This collection is chosen for the family $\mathcal{F}$ and is the same for all scales and members.

We begin with the transfer of slopes. The linking property and Lemma 12.5 give the correction budget (Mf8) for the modification data of (Pc3). Together with (Mf1)–(Mf7), this verifies the hypotheses of Theorem 12.3, which gives (Sp3).

For the divisor conditions, (Nr1)–(Nr3) of (Pc4) give (Bk3) by Lemma 13.6. Since (Pc4) uses the same numerical and bridge records throughout, (Bk1), (Bk2), (Bk3), (Bk4) and (Dv2) hold for one choice of records. Applying Theorem 13.4 at $e = 1$ gives (Dv5) with exponent 1.

Apply Theorem 22.1 to the Galois Ribet items and the data of (Pc5), whose hypotheses concern the coefficient extension specified there, with the family constants $c_{\mathrm{sp}}, c_{\mathrm{in}}, c_{\mathrm{sm}}, C_{\mathrm{bud}}$. By compatibility, the host, the original Hecke lattices, the point modules, the inclusions, the minimal-loss choices and the allowance records agree; the conclusions of that theorem concern the original family, with the same minimal-loss numbers and allowances. On a common tail of scales, on which moreover $x^{\varepsilon} \geq 3$ and the thresholds of (Lz1), (Iq1) are met, that theorem gives (Rf1), (Rf2), (Rf4), (Rf5), (Lq3), and (Lq7) with its derived coefficient $19C_3^3$. These conclusions keep their finite-Sha antecedents and their local indices. Together with (Rf3) and (Lq2), (Lq4)–(Lq6), (Lq8) of (Pc4), the local bound records therefore have (Lq2)–(Lq8) and the Ribet records have (Rf1)–(Rf5). Hence Theorem 13.13 gives (Lq1) for these same local components and congruence lengths. Thus Theorem 13.16, applied to (Lq1)–(Lq5), gives (Ld7) and (Ld8), and Theorem 13.17, applied to (Lq5)–(Lq8), gives (Ld9). Properties (Ld1)–(Ld6) were required in (Pc4). All nine hypotheses of Theorem 13.18 therefore hold, and that theorem gives (Dv6). Each of these local assertions remains conditional on the member having finite Sha lengths.

We have now obtained (Dv5) with exponent 1 and (Dv6), while (Dv1)–(Dv4) and (Dv7) are part of (Pc4). The family satisfies the Sha condition by assumption. Hence Theorem 13.2 applies with exponent 1 and proves (Sp4) for that exponent.

For the frame bounds, use the family constants of (Pc5). Its diagonal expansions give (No4) by Theorem 14.16. Its kernel hypotheses, including the finite measure, positivity and both scalar inequalities, give (No5) by Corollary 14.18. Its (Ro1)–(Ro4) data give (Rw1) by Theorem 15.3. All three use the same C_0, c, frame Hilbert space, Gram blocks, word scale and row parameters, and keep their word-size antecedents. Thus (Rw1) and (Rw2) hold with the same $(C_{\mathrm{A}}, c_{\mathrm{lp}}, C_{\mathrm{ng}})$ and exponent c, so the hypotheses of Theorem 14.13 are satisfied, giving (No7) for C_0 with exponent c. By Theorem 14.9 this implies (No3) for the same C_0, c. Properties (No1), (No2), (No6) are in (Pc5), and (No4), (No5) have just been derived. Applying Theorem 14.8 therefore gives (Nf4) with exponent c. The height condition (Nf2) holds with the same exponent, and (Nf1), (Nf3) hold as well. Thus the four hypotheses of Theorem 14.2 hold, and that theorem gives (Sp5). In particular, at each scale its envelopes are common to all members, as (Sp5) requires.

Condition (Pc6) is the residual size hypothesis in item 3 of Lemma 12.1. The family is parametrised as that lemma requires, and $\varepsilon \leq 1$ is part of (Pc6). The lemma gives (Sp8) with $e_K = 5$, with a threshold depending on the requested decay constant as stated there.

Finally (Sp1), (Sp2), (Sp6), (Sp7) are (Pc2). Together with the four consequences just proved, all of (Sp1)–(Sp8) hold, and the divisor exponent in (Sp4) is 1. Applying Theorem 8.4 gives slope data at the members with exponents $(\max(1,0),0) = (1,0)$.

The only number in this argument that is the same for all families of the class is the divisor exponent 1, and hence the pair $(1,0)$; every other constant and threshold depends on the family as in the cited property or theorem. Since $\mathcal{F}$ was arbitrary, all conclusions hold for the class $\mathcal{A}_{\mathrm{prop}}$. □

Part IV
The chain of theta lifts: design existence, assuming (TCS)

In Alpöge's account the auxiliary object of the proof is a chain of iterated theta lifts $\mathrm{U}(X_0) \to \mathrm{U}(X_1) \to \cdots \to \mathrm{U}(X_m)$ between totally definite unitary groups, which wins multiplicity m [4]. This paper sees the chain only through its records: the slope data of Part I, the slope records of the members of a family, and the translate packets of this Part, each of which consists, at one scale, of one loss datum and one slope record for every translate $u \in A$ (Definition 19.1). That these records come from a chain of theta lifts is our reading; Appendix A outlines the construction. Part IV is where the chain would be constructed, and the one standing assumption of the paper, the existence of a theta chain system (TCS), is what is assumed of it. (TCS) is stated in Section 21 as Hypothesis 21.1.

The Part proves design existence, Proposition 18.1: the class $\mathcal{A}_{\mathrm{prop}}$ of Part III contains, for every small ε, a family with parameter ε whose scales contain λ-windows and whose members satisfy a given host requirement supplied by the concrete good-host requirement. This is clause (Lk1) of Proposition 17.6. The family is assembled from packets, one at each scale, and the work of the Part is to prove every property the class asks for that (TCS) does not state. The linking property follows from the numerical data of the packets alone (Theorem 18.4), and it is proved without (TCS). The other properties are derived from the data that (TCS) supplies: the bounds (Ms1) and (Ms3) for model slopes by the tower theorems 29.14 and Proposition 31.14 of Part VI, the realisation estimate (Ri1) by the reference budget Theorem 20.2, and the remaining model-slope and height conditions and eight lower conditions of the class by the results of Section 19. Apart from these two tower theorems, Part IV uses no result of another Part.

(TCS) is used directly only once, in the proof of Proposition 21.2. Nine results depend on it through this use: Proposition 19.9, Proposition 19.3, Proposition 18.1, Proposition 17.6, Corollary 8.3, Proposition 7.9, Proposition 7.8, Theorem 6.7 and Theorem 6.8. They form a chain in this order, each used in the proof of the next, and no other result of the paper depends on (TCS). The sections run from the conclusion to the assumption. Section 18 proves design existence from a joint supply of packets and the linking property. Section 19 derives that joint supply from a supply with realisations, and this from model-level packet supply. Section 20 proves the reference budget, and Section 21 states (TCS) and proves model-level packet supply from it.

18 Packets and design existence

This section proves design existence, Proposition 18.1, which is (Lk1) of Proposition 17.6. The proof takes the joint supply of packets of Proposition 19.3 and builds from it, for every small ε, a family whose members at a scale x are indexed by the translates $u \in A_x$ and carry the host, tops, loss datum and slope record of the packet. Two points need an argument. The first is the linking property (Pc1), which Proposition 19.3 does not provide; it comes from Theorem 18.4, whose four hypotheses are exactly the numerical conditions (Js2) of the packets. The second is the Sha condition. The class requires the records (Pc2)–(Pc6) only of families that satisfy it, and the packets carry their lower completions only on a tail of scales at which the Sha bound holds for every member. The proof therefore splits into two cases: if the family does not satisfy the Sha condition, the conditional requirement is vacuous; if it does, the lower completions supplied

in (Js3) give (Pc2)–(Pc6).

Theorem 18.4 is the main theorem proved in this section, and it does not depend on (TCS). It derives the linking property (Lm3), the main hypothesis of the loss bound Theorem 5.6, from the integer construction alone: heights, depths and weights in prescribed integer boxes (Definition 18.2), the finite good-orbit selections of the good host (Definition 11.3), and numerical orbit corrections (Definition 18.3). The constants of the linking property are uniform over all families with these data. The core of the proof is a counting argument along the Frobenius orbits of the inert primes. The marked tiles that condition (Go2) of the good-orbit selection provides give every good orbit a lower density of tops (Lemma 18.7), and the floor loss of the served-prime inequality pays for the orbit losses and the correction error (Lemma 18.12). Lemma 18.9 combines the orbit counts with the scalar bounds of the integer weights (Lemma 18.6) into the linking property.

Proposition 18.1. *For every real $\lambda > 1$ and every host requirement P supplied by the concrete good-host requirement $\mathfrak{G}$ on λ-windows there is a real $\varepsilon_0 > 0$ with the following property. For every $\varepsilon \in (0, \varepsilon_0)$ there is a parametrised host family with slope records $\mathcal{F}$ of parameter ε such that*

$$\mathcal{F} \in \mathcal{A}_{\mathrm{prop}} \cap \mathcal{C}_\lambda(\mathsf{P}).$$

Here $\mathfrak{G}$ and $\mathcal{A}_{\mathrm{prop}}$ are fixed; they do not depend on λ, P or ε.

Proof. Fix $\lambda > 1$ and a host requirement P supplied by $\mathfrak{G}$ on λ-windows. Let $\varepsilon_* = \varepsilon_*(\lambda, \mathsf{P}) > 0$ be given by Proposition 19.3 for this λ and P, and set

$$\varepsilon_0 = \min(\varepsilon_*, 1/52) > 0.$$

Fix $0 < \varepsilon < \varepsilon_0$. The supply condition gives a fixed $J \geq 1$, a finite list κ, a set W containing λ-windows, and a cutoff above which each $x \in W$ admits a host, tops and rank satisfying both $\mathfrak{G}$ and P. Increase that cutoff to $X_H > 1$. At every $x \in W$ with $x \geq X_H$ choose one such tuple, together with all the data required by $\mathfrak{G}$. Write the tuple as $\mathfrak{h}_x, T_x, t_x$, with frame group A_x and spacing set S_x.

Apply Proposition 19.3, for the same λ and P, to this collection. Its hypotheses hold, since $\varepsilon < \min(\varepsilon_*, 1/52)$ and every chosen tuple satisfies both $\mathfrak{G}$ and P. It gives the common constants, a cutoff $X \geq X_H$, and packets P_x for all $x \in W$ with $x \geq X$. For any real $Q \geq X$ define a family $\mathcal{F}^Q$ with scales W, threshold $x_1 = Q$ and index sets

$$I_Q(x) = \begin{cases} A_x, & x \in W,\ x > Q, \\ \varnothing, & \text{otherwise.} \end{cases}$$

At a nonempty scale use the host, tops, spacing set, rank, loss datum, slope record and translate of the packet, indexed by $u \in A_x$. At an empty scale choose any fixed frame. Such a frame exists because W is unbounded, so that there is a packet at some $x \geq X$. Indeed, a λ-window with left endpoint larger than $\max(X, 1)$ contains such a scale.

We first check that $\mathcal{F}^Q$ is a parametrised host family with slope records. The packet ties give (Pf1), including the group, arithmetic datum, tops, scale, parameter and rank. Conditions (Lm1), (Lm2), (Lm6) of (Js1) give (Pf2), and its common pair (ε_z, C_t) gives (Pf3). The common host, tops, spacing set, scalar parameters and auxiliary primes give (Sr1). For a member and any $u \in A_x$, the member indexed by u has translate u and the same heights, which gives (Sr2). Conditions at empty index sets are vacuous. Moreover $\mathcal{F}^Q \in \mathcal{C}_\lambda(\mathsf{P})$, since its scales are W and every member carries the tuple chosen above, which satisfies P.

The family $\mathcal{F}^Q$ satisfies (Pc1) for every $Q \geq X$. The four hypotheses of Theorem 18.4 are the conditions of (Js2), with the same fixed $\varepsilon, c_D, c_h, \mathfrak{k}_{\mathrm{orb}}, C_e$ at every member. That theorem therefore gives the linking property, with a coefficient, a cutoff and a function δ that do not depend on the member. By (Js1), (Lm4) and (Lm5) hold at every member above its threshold, which gives few bad tops and small shares. Finally (Hg3) gives

$$|T_x| \geq \tfrac{1}{2} x^{-\varepsilon/8} |A_x| > 0$$

because A_x is a finite group. Thus the top set of every member is nonempty.

If $\mathcal{F}^X$ does not satisfy the Sha condition, the conditional requirement defining $\mathcal{A}_{\mathrm{prop}}$ is vacuous. Since $\mathcal{F}^X$ satisfies (Pc1), it is the required family.

Suppose instead that $\mathcal{F}^X$ satisfies the Sha condition. There is a cutoff Z such that every member at every $x \geq Z$ above its threshold satisfies the stated Sha bound. At each $x \in W$ with $x > \max(X, Z)$ the index set is the nonempty group A_x, and all members have host $\mathfrak{h}_x$ and rank t_x. Hence, after increasing the cutoff by 1, there is a single cutoff such that

$$\mathrm{V}(E_{\mathfrak{h}_x}; t_x, \exp(x^{\varepsilon}/2))$$

holds at every scale $x \in W$ above it. Condition (Js3) now supplies uniform lower completions of these packets at all sufficiently large scales. Let $Y \geq X$, $Y > 1$, and the common scalar parameters be as in (Pl3).

Consider $\mathcal{F}^Y$. Every member has $x > Y$, so its completion is defined. Attach all the record and input-data types of (Pl1) to that member. Their defining identities and compatibility give one compatible collection of family records. The pointwise interpretation (Pl2), with common parameters that do not depend on the scale, gives the corresponding family properties. Here Y serves as their eventual cutoff, all inner quantifiers over words, places, primes and components are kept, and so is each antecedent (finiteness of a Tate–Shafarevich group, a word-size condition, or nonvanishing of a map). The displayed lists in (Pl3), together with its separate kernel requirement, give (Pc2), (Pc3), (Pc4) and (Pc5). The twelve common constants of (Pc5), including the nonnegative integer, are the same in all the properties in which they occur. In its Ribet conditions, the integers attached to a member are chosen after that member's finiteness antecedent for the Tate–Shafarevich group and before all its local indices. Thus the compatible data give the quantified hypotheses required there.

The last inequalities of (Pl3) give the residual bounds in (Pc6), and $\varepsilon \leq 1$ is also part of that condition. For every $x \in W$ with $x > Y$ the index set A_x is nonempty, so the required eventual nonemptiness holds. Thus (Pc6) holds. The family $\mathcal{F}^Y$ still satisfies (Pc1) and lies in $\mathcal{C}_\lambda(\mathsf{P})$ by the checks above. It therefore lies in $\mathcal{A}_{\mathrm{prop}} \cap \mathcal{C}_\lambda(\mathsf{P})$, which completes both cases. Neither $\mathfrak{G}$ nor $\mathcal{A}_{\mathrm{prop}}$ depends on λ, P or ε.

□

Definition 18.2. Fix a loss datum with $x > 1$, $0 < \varepsilon < 1/52$, $k \geq 1$ and $m \geq 1$, a number field E, a subset $\Phi_{\mathrm{th}} \subseteq \Phi \setminus (\{0\} \times T)$ and real constants $c_D \geq 0$, $c_h \geq 402$. In this definition and in the results that refer to it, the following quantities are formed from the scale and these constants:

$$\Delta = \lceil x^{c_D} \rceil, \qquad h_0 = 2\lceil x^{c_h} \rceil, \qquad b = \exp(x^{\varepsilon}), \qquad n = |H|.$$

The places in Φ_{th} are called thick; the places outside $\Phi_{\mathrm{th}} \cup (\{0\} \times T)$ are called thin.

The loss datum has the **integer weights and boxes for** $(E, \Phi_{\mathrm{th}}, c_D, c_h)$ if the following conditions hold.

(Iw1) Its cloud size and total gap are

$$S_c = 4k\Delta, \qquad \eta = mh_0.$$

(Iw2) At every top $t \in T$ its integer heights and depths satisfy

$$\lceil a \rceil \le a_t \le \left\lfloor a + \frac{a}{kx^{13}} \right\rfloor, \qquad \lceil b' \rceil \le b_t.$$

(Iw3) Every $w_1(z)$ is an integer, and

$$\begin{aligned} w_1(0,t) &= b_t + \lfloor b'/x \rfloor + h_0/2 && (t \in T),\\ w_1(z) &= \lceil w'' \rceil + h_0/2 && (z \text{ thin}),\\ \lceil b \rceil + h_0/2 \le w_1(z) &< \lceil b \rceil + |\mathcal{O}_{E\mathrm{tors}}^{\times}| + h_0/2 && (z \text{ thick}). \end{aligned}$$

For $1 \le j \le m$ and $z \in \Phi$, the other weights are

$$w_j(z) = w_1(z) + (j-1)h_0.$$

(Iw4) For every $1 \le j \le m$,

$$\varsigma_j = \frac{k + (2j-1)(t_R+1)}{2}.$$

Definition 18.3. A **tuple of numerical orbit constants** is a tuple of eight nonnegative real numbers

$$\mathfrak{k}_{\mathrm{orb}} = (C_{\mathrm{La}}, C_{\mathrm{band}}, C_{\mathrm{Ds}}, C_{\mathrm{Dl}}, C_{\mathrm{ts}}, C_{\mathrm{t1}}, C_{\mathrm{t2}}, C_{\mathrm{orb}}).$$

The entries C_{La}, C_{band} and C_{orb} do not enter the conditions below. Fix such a tuple, a real number $C_e \ge 0$ and a loss datum with $x > 1$, $0 < \varepsilon < 1/52$, $k, m \ge 1$ and $S_c \ge 0$. In this definition and in the statements that refer to it, we use the following quantities, which depend on the tuple and on the fixed loss datum:

$$\begin{aligned} u_{\mathrm{orb}} &= \lceil \delta_1^{-1} \rceil, & \ell_{\mathrm{orb}} &= (6r_G+1)u_{\mathrm{orb}},\\ M_{\mathrm{orb}} &= \left\lceil (5+8\varepsilon)(\log x) 10^{6r_G+1} \right\rceil, & G_{\mathrm{orb}} &= (M_{\mathrm{orb}}+5)\ell_{\mathrm{orb}},\\ N_{\mathrm{orb}} &= \left\lceil \max\left(6\exp(1) G_{\mathrm{orb}} \delta_1,\ (5+8\varepsilon)\frac{\log x}{\log 2} \right) \right\rceil + 1. \end{aligned}$$

For a prime p and a nonnegative integer d_{cyc} put

$$Q_{\mathrm{cor}}(p) = p^{G_{\mathrm{orb}}}, \qquad \ell_{\mathrm{cor}}(p) = \left\lceil \log_p((kx^{26})^4) \right\rceil, \qquad u_{\mathrm{cor}}(d_{\mathrm{cyc}}) = 40 d_{\mathrm{cyc}} \delta_1.$$

Define nonnegative integers

$$\begin{aligned} D^{\mathrm{s}}_{\mathrm{cor}}(p, d_{\mathrm{cyc}}) = \Big\lceil & C_{\mathrm{Ds}} u_{\mathrm{cor}}(d_{\mathrm{cyc}})(N_{\mathrm{orb}}^3 + r_G^5 + k^2) \log_p a \\ & + 2r_G d_{\mathrm{cyc}} c_G + r_G u_{\mathrm{cor}}(d_{\mathrm{cyc}})(2b' + 2c_G) \Big\rceil, \\ D^{\mathrm{l}}_{\mathrm{cor}}(p, d_{\mathrm{cyc}}) = \Big\lceil & C_{\mathrm{Dl}} u_{\mathrm{cor}}(d_{\mathrm{cyc}}) x^{26} (N_{\mathrm{orb}}^5 + k^3) \log_p a \Big\rceil \end{aligned}$$

and the real number

$$\begin{aligned} c_{\mathrm{cor}}(p, d_{\mathrm{cyc}}) = {} & 8(D^{\mathrm{l}}_{\mathrm{cor}}(p, d_{\mathrm{cyc}}) + 1) + C_{\mathrm{t2}} u_{\mathrm{cor}}(d_{\mathrm{cyc}}) k \log_p a \\ & + 4u_{\mathrm{cor}}(d_{\mathrm{cyc}}) c_G + 2 d_{\mathrm{cyc}} c_G. \end{aligned}$$

The loss datum has the **numerical orbit corrections for** $(\mathfrak{k}_{\mathrm{orb}}, C_e)$ if the following identities hold.

(Nc2) Correction degree. For every inert prime p,

$$D_p = \begin{cases} D^{\mathrm{s}}_{\mathrm{cor}}(p, n_p), & p < k^2 x^{26}, \\ D^{\mathrm{l}}_{\mathrm{cor}}(p, n_p), & p \geq k^2 x^{26}. \end{cases}$$

(Nc3) Orbit thresholds. Let p be a candidate prime and $v \in V_p$. Candidate primes are not sign-changing, so $n_p = \mathrm{ord}(h_p)$. Residues modulo n_p are taken in $\{0, \dots, n_p - 1\}$. For an odd $i \in P_p(v)$ let $d^-_{p,v}(i)$ be the least integer in

$$\{e : 1 \leq e \leq n_p,\ (i-e) \bmod n_p \in P_p(v),\ (i-e) \bmod n_p \text{ is even}\},$$

and put $d^-_{p,v}(i) = 0$ if this set is empty. Set

$$\begin{aligned} t^{\mathrm{s}}_{\mathrm{cor}}(p, v) &= C_{\mathrm{ts}} r_G N_p(v) \left(\frac{L_a a}{pk} + a p^{-\ell_{\mathrm{cor}}(p)} + 2b' + 2c_G + kQ_{\mathrm{cor}}(p) \right) + 2r_G n_p c_G, \\ t^{\mathrm{l}}_{\mathrm{cor}}(p, v) &= \sum_{\substack{i \in P_p(v),\ i \text{ odd} \\ d^-_{p,v}(i) > 0}} \min \left(b_{v+ih_p}, \frac{a_{v+((i-d^-_{p,v}(i)) \bmod n_p)h_p}}{p^{d^-_{p,v}(i)}} \right) \\ &\quad + C_{\mathrm{t1}} N_p(v) N_{\mathrm{orb}} \left(\frac{a}{p^2} + \frac{L_a a}{pk} \right) \\ &\quad + C_{\mathrm{t2}} N_p(v) \left(Q_{\mathrm{cor}}(p) c_{\mathrm{cor}}(p, n_p) + kQ_{\mathrm{cor}}(p) + k \log_p a + c_G \right) + 2n_p c_G. \end{aligned}$$

The condition is

$$t_p(v) = \begin{cases} t^{\mathrm{s}}_{\mathrm{cor}}(p, v), & p < k^2 x^{26}, \\ t^{\mathrm{l}}_{\mathrm{cor}}(p, v), & p \geq k^2 x^{26}. \end{cases}$$

(Nc4) Correction error. For every inert prime p, with $f = 2|H|$,

$$\epsilon_p = \left\lceil C_e f k^2 \log_p a \right\rceil.$$

Theorem 18.4. *Fix*

$$0 < \varepsilon < 1/52, \qquad c_D > 397, \qquad c_h \geq 402,$$

a tuple $\mathfrak{k}_{\mathrm{orb}}$ of eight nonnegative real constants and a real constant $C_e \geq 0$. There are constants $C_0 \geq 0$ and $x_0 > 1$, depending only on these fixed data, with the following property.

Let $\mathcal{F}$ be any parametrised host family with parameter ε and threshold x_1. Suppose that for every $x > x_1$ and every $i \in I(x)$ the following hold.

1. *Its loss datum has the group $H = A$ and parameters k, m, t_R of a rank datum with the rank sizes at (x, ε, J) for some $J \geq 1$, with $t_R = t$.*
2. *Its heights, depths, weights and shifts have the integer weights and boxes for the fixed exponents c_D, c_h. The arithmetic field and thick subset in that condition may depend on the member.*
3. *Its top set, Frobenius translations, orbit representatives and good representatives form a finite good-orbit selection, and these are the choices used in its loss datum.*
4. *Its correction degrees, orbit thresholds and correction errors satisfy (Nc2), (Nc3) and (Nc4) for the fixed tuple $\mathfrak{k}_{\mathrm{orb}}$ and C_e.*

Then $\mathcal{F}$ has the linking property. More precisely, put

$$C = \max(C_0, 11 + C_e, 16)$$

and

$$\delta(x) = \begin{cases} x^{-\varepsilon/2}, & x > 1, \\ 0, & x \le 1. \end{cases}$$

For every such family, every $x \ge x_0$ with $x > x_1$ and every member at x, the loss datum of the member satisfies both inequalities of (Lm3) for

$$(C, b(x)^{\delta(x)}) = (C, \exp(x^{\varepsilon/2})).$$

In particular, C, x_0 and δ are the same for all such families and all their members.

Proof. All thresholds in this proof depend only on the fixed data of the statement. They are increased finitely many times. None depends on a family, a member, an orbit, a prime, a thick subset or an arithmetic field.

Take C_{den} to be the explicit positive constant of Lemma 18.7, and put

$$C_{\mathrm{Norb}} = \max\left(6\exp(1)C_{\mathrm{den}}, \frac{5 + 8\varepsilon}{\log 2}\right) + 2.$$

This constant is local to this proof. For $x \ge \exp(1)$ the estimates proved in that lemma and the definition of N_{orb} give

$$G_{\mathrm{orb}} \le C_{\mathrm{den}} x^{\varepsilon/8} \log x, \qquad N_{\mathrm{orb}} \le C_{\mathrm{Norb}} \log x. \tag{1}$$

Indeed, multiply the first inequality by $\delta_1 = x^{-\varepsilon/8}$, apply the maximum in the definition of N_{orb}, and use $\lceil u \rceil + 1 \le u + 2$ and $\log x \ge 1$ for the second inequality. Choose

$$C_0 = \max\left(12,\ C_{\mathrm{ts}} r_G,\ 13 + 4C_{\mathrm{ts}} + 160C_{\mathrm{den}},\ C_{\mathrm{den}},\ C_{\mathrm{t1}} C_{\mathrm{Norb}}\right). \tag{2}$$

Here $r_G = \lceil (32 + 64\varepsilon)/\varepsilon \rceil + 1$ depends only on ε. Thus C_0 is nonnegative and has the required dependence.

Fix a family satisfying the four hypotheses, a scale $x > x_1$, and a member. By Lemma 18.6, after a uniform increase of the threshold, its scalar inequalities satisfy

$$k \ge 5, \quad S_c \ge 100, \quad N \le S_c, \quad kS_c \le w'', \quad 0 \le \varsigma_j \le S_c,$$

and all the height, depth and weight bounds in item 1 of Lemma 18.9. The top weights also satisfy the upper offset bound needed by Lemma 18.5, since their defining formula gives

$$w_j(0,t) - b_t = \lfloor b'/x \rfloor + (j - \tfrac{1}{2})h_0 \le b'/x + mh_0 = b'/x + \eta. \tag{3}$$

The lower offset bound is supplied by Lemma 18.6.

Every candidate prime is inert and not sign-changing, has $p < X'$ and $n_p = \mathrm{ord}(h_p) \ge x^{\varepsilon/4}$. Its representatives $v \in V_p$ belong to G_p. The second part of Lemma 18.7, applied to these top sets and representatives, therefore gives, uniformly,

$$N_p(v) \ge 1, \qquad n_p \delta_1 \le N_p(v) C_{\mathrm{den}} \log x. \tag{4}$$

Condition (Lm1), which is part of a parametrised host family, gives even n_p at every such prime. This checks the density and parity hypotheses of the finite estimate.

We check its remaining numerical hypotheses. The rank sizes imply, after another uniform increase of the threshold,

$$x^{2+8\varepsilon} \le n = |H| \le x^3, \qquad k \le x^6. \tag{5}$$

The identities for the integer weights give

$$S_c = 4k\lceil x^{c_D}\rceil, \qquad \eta = 2m\lceil x^{c_h}\rceil, \qquad c_G = 4S_c,$$

with $m \le 2x^{1/2}$. Together with $N \le S_c$, (1) and

$$0 < \log_p a \le \frac{\log a}{\log 2} = \frac{\log x + x^{\varepsilon}}{\log 2} \quad (p \ge 2),$$

these show that there is a fixed real exponent $c_{\mathrm{pol}} \ge 26$ such that, for all sufficiently large x, every one of

$$n,\ k,\ N,\ S_c,\ \eta,\ c_G,\ r_G,\ N_{\mathrm{orb}},\ \frac{C_{\mathrm{den}}\log x}{\delta_1},\ C_{\mathrm{den}}\log x,\ x^{26},\ \log_p a \tag{6}$$

is at most $x^{c_{\mathrm{pol}}}$, uniformly for $p \ge 2$. The same bound holds for each fixed nonnegative entry of $\mathfrak{k}_{\mathrm{orb}}$ and for 1. To see the choice explicitly, take c_{pol} strictly greater than

$$26, \quad 7 + c_D, \quad 1 + c_h, \quad 3$$

and increase it if necessary. The estimates $S_c \le 8x^{6+c_D}$, $\eta \le 8x^{1/2+c_h}$, $N \le S_c$, (1), and $\log x = o(x^{\alpha})$ for every fixed $\alpha > 0$ then prove (6), after enlarging the threshold to absorb the finitely many fixed coefficients. We may enlarge c_{pol} once more; it remains fixed.

First consider the scalar inequality containing w''. Since $w'' = b'/(xn)$, (5) gives

$$\frac{1}{b'}\frac{C_{\mathrm{den}}\log x}{\delta_1} w'' \le C_{\mathrm{den}}(\log x)x^{-3-63\varepsilon/8} \longrightarrow 0. \tag{7}$$

Also

$$b' = x\exp(x^{\varepsilon} - x^{\varepsilon/3}),$$

so $x^a/b' \to 0$ for every fixed real exponent a. This follows by taking logarithms and dividing by x^{ε}. From (6),

$$\frac{1}{b'}\frac{C_{\mathrm{den}}\log x}{\delta_1}(\eta + c_G + N) \le \frac{3x^{2c_{\mathrm{pol}}}}{b'} \longrightarrow 0. \tag{8}$$

Choose the threshold so that each expression in (7) and (8) is at most $1/2$. Their sum proves the first common scalar bound in Lemma 18.5. The same exponential comparison gives $2c_G \le b'$, proving $2b' + 2c_G \le 3b'$. It also gives, for every prime p,

$$C_{\mathrm{Ds}}(N_{\mathrm{orb}}^3 + r_G^5 + k^2)\log_p a \le 3x^{7c_{\mathrm{pol}}} \le b' \tag{9}$$

eventually. All replacements in the first inequality use (6).

For a prime $p < k^2x^{26}$, (5) implies $\log p \le 38\log x$. Equation (1) then yields

$$\log Q_{\mathrm{cor}}(p) \le 38C_{\mathrm{den}}x^{\varepsilon/8}(\log x)^2 \le x^{7\varepsilon/48} \tag{10}$$

eventually, since $\varepsilon/8 = 6\varepsilon/48$ and $(\log x)^2/x^{\varepsilon/48} \to 0$. Thus

$$\log(kQ_{\mathrm{cor}}(p)) \le 6\log x + x^{7\varepsilon/48} \le \log x + x^{\varepsilon} - x^{\varepsilon/3} = \log b'$$

eventually. The penultimate inequality follows by dividing its difference by x^{ε}, whose limit is 1. This proves the required small-prime bound uniformly in p.

For every prime $p < X'$, the same argument uses $\log p < x^{\varepsilon/3}$ instead and gives

$$\log Q_{\mathrm{cor}}(p) \le C_{\mathrm{den}}(\log x)x^{11\varepsilon/24} \le x^{23\varepsilon/48} \tag{11}$$

eventually, because $11/24 = 22/48$. We use this to check the last, large-prime budget of the finite estimate.

For every integer $0 \le d_{\mathrm{cyc}} \le n$, (6) and $\delta_1 \le 1$ imply

$$u_{\mathrm{cor}}(d_{\mathrm{cyc}}) = 40d_{\mathrm{cyc}}\delta_1 \le x^{2c_{\mathrm{pol}}}$$

after increasing the threshold. The defining formula and $\lceil u \rceil \le u + 1$ give

$$D^{\mathrm{l}}_{\mathrm{cor}}(p, d_{\mathrm{cyc}}) \le 2x^{10c_{\mathrm{pol}}} + 1 \le x^{11c_{\mathrm{pol}}}. \tag{12}$$

Indeed its product has exponents at most $c_{\mathrm{pol}} + 2c_{\mathrm{pol}} + c_{\mathrm{pol}} + 5c_{\mathrm{pol}} + c_{\mathrm{pol}}$; the factor 2 bounds $N^5_{\mathrm{orb}} + k^3$. Substituting (12) into the four terms of c_{cor} gives

$$\begin{aligned} c_{\mathrm{cor}}(p, d_{\mathrm{cyc}}) &\le 16x^{11c_{\mathrm{pol}}} + x^{5c_{\mathrm{pol}}} + 4x^{3c_{\mathrm{pol}}} + 2x^{2c_{\mathrm{pol}}} \\ &\le x^{12c_{\mathrm{pol}}} \end{aligned} \tag{13}$$

eventually. The bounds are uniform in $p \ge 2$ and $d_{\mathrm{cyc}} \le n$.

In the last budget of Lemma 18.5, take $d_{\mathrm{cyc}} = n_p \le n$. Its first term is at most $3x^{2c_{\mathrm{pol}}}$. The four terms multiplied by C_{t2} are, respectively, at most

$$x^{13c_{\mathrm{pol}}} \exp(x^{23\varepsilon/48}), \quad x^{2c_{\mathrm{pol}}} \exp(x^{23\varepsilon/48}), \quad x^{3c_{\mathrm{pol}}}, \quad x^{2c_{\mathrm{pol}}},$$

by (6), (11) and (13). Its next term is at most $2x^{2c_{\mathrm{pol}}}$. Its term multiplied by 40 is at most $80x^{9c_{\mathrm{pol}}}$, using (6). The final constant is 3. For sufficiently large x, their sum is consequently at most

$$x^{14c_{\mathrm{pol}}} \exp(x^{23\varepsilon/48}).$$

Indeed, first bound $80x^{9c_{\mathrm{pol}}} \le x^{10c_{\mathrm{pol}}}$ and then bound the remaining sum by

$$12x^{13c_{\mathrm{pol}}} \exp(x^{23\varepsilon/48}) \le x^{14c_{\mathrm{pol}}} \exp(x^{23\varepsilon/48}).$$

Finally

$$14c_{\mathrm{pol}} \log x + x^{23\varepsilon/48} \le x^{\varepsilon/2} \tag{14}$$

eventually: the ratio of the left side to $x^{\varepsilon/2}$ tends to 0, because $23/48 < 1/2$ and $\varepsilon > 0$. This proves that last budget with $b_o = \exp(x^{\varepsilon/2})$.

Equations (1) and (2) check all the coefficient inequalities of the finite estimate. Every other hypothesis has been checked in (3)–(14) or by Lemma 18.6. Also $\eta > 0$ and $L_a \ge x^{\varepsilon} > 0$ follow from (Pf3) and (Lm7). Thus Lemma 18.5, with C_0 as coefficient and $b_o = \exp(x^{\varepsilon/2})$, gives item 3 of Lemma 18.9 at every candidate prime, representative and lift.

For its item 2, a served prime satisfies $p \le X''$. Condition (Pf3) supplies (Lm7), and (Nc4) supplies the canonical ceiling. Applying Lemma 18.8 therefore gives

$$\epsilon_p \le (C_e + 1)fk^2 \log_p a.$$

Its item 1 was already checked by Lemma 18.6. Apply Lemma 18.9 with correction coefficient $C_e + 1$. It gives the linking property with

$$\max(C_0, 10 + (C_e + 1), 16) = \max(C_0, 11 + C_e, 16)$$

and the stated function δ. Take x_0 greater than all the finitely many thresholds chosen in this proof. Since each of them depends only on the fixed data, x_0, C and δ are the same for every family and member, as claimed. □

Lemma 18.5. *Consider a loss datum whose correction degrees and orbit thresholds satisfy (Nc2) and (Nc3) of Definition 18.3 for a tuple* $\mathfrak{k}_{\mathrm{orb}}$ *of eight nonnegative real constants, and let* $C_{\mathrm{den}} > 0$, C *and* $b_o > 0$ *be real numbers. Suppose that*

$$x \geq \exp(1), \quad 0 < \varepsilon < 1/52, \quad m \geq 1, \quad k \geq 5, \quad S_c \geq 100,$$
$$N \leq S_c, \quad kS_c \leq w'', \quad \eta > 0, \quad L_a \geq 0,$$

and that for every lift $1 \leq j \leq m$, *place* $z \in \Phi$ *and top* $t \in T$,

$$0 \leq \varsigma_j \leq S_c, \qquad w_j(z) \geq 1, \qquad a \leq a_t \leq a\left(1 + \frac{1}{kx^{13}}\right), \qquad b_t \geq b',$$
$$\frac{b'}{x} \leq w_j(0,t) - b_t \leq \frac{b'}{x} + \eta.$$

Assume the following further inequalities.

1. *For every candidate prime* p *and* $v \in V_p$,

$$n_p \delta_1 \leq N_p(v) C_{\mathrm{den}} \log x.$$

For every candidate $p \geq p_0^* = k^2 x^{26}$, *the integer* n_p *is even.*

2. *The following common scalar bounds hold:*

$$\frac{C_{\mathrm{den}} \log x}{\delta_1}(w'' + \eta + c_G + N) \leq b', \qquad 2b' + 2c_G \leq 3b'.$$

For every candidate prime $p < p_0^*$,

$$kQ_{\mathrm{cor}}(p) \leq b'.$$

For every candidate prime p,

$$C_{\mathrm{Ds}}(N_{\mathrm{orb}}^3 + r_G^5 + k^2) \log_p a \leq b'.$$

3. *The loss coefficient satisfies*

$$C \geq 12, \qquad C_{\mathrm{ts}} r_G \leq C, \qquad 13 + 4C_{\mathrm{ts}} + 160 C_{\mathrm{den}} \leq C, \qquad C_{\mathrm{den}} \leq C,$$
$$C_{\mathrm{t1}} N_{\mathrm{orb}} \leq C \log x.$$

4. *For every candidate prime* $p \geq p_0^*$,

$$\begin{aligned} &\frac{C_{\mathrm{den}} \log x}{\delta_1}(\eta + c_G + N) \\ &\quad + C_{\mathrm{t2}}\left(Q_{\mathrm{cor}}(p) c_{\mathrm{cor}}(p, n_p) + kQ_{\mathrm{cor}}(p) + k \log_p a + c_G\right) \\ &\quad + 2\frac{C_{\mathrm{den}} \log x}{\delta_1} c_G \\ &\quad + 40 C_{\mathrm{den}} (\log x)\, C_{\mathrm{Dl}} x^{26} (N_{\mathrm{orb}}^5 + k^3) \log_p a + 3 \leq b_o. \end{aligned}$$

Then, for every candidate prime p, representative $v \in V_p$ and lift $1 \le j \le m$,

$$\sum_{t \in T_p(v)} (A_p(a_t, t, w_j) - \mathrm{lo}_{C,b_o}(p)) \le g_j^U(v) - \mathrm{Pr}_j(v) + \delta_p(v) - \left(\lceil t_p(v) \rceil^+ + D_p + 1 \right).$$

Proof. Fix a candidate prime p, a representative $v \in V_p$ and a lift j. Thus p is not sign-changing, $p < X'$ and $n_p = \mathrm{ord}(h_p) \ge 1$. The auxiliary notation below depends on these choices and on the fixed loss datum. Put

$$g_{\mathrm{den}} = \frac{C_{\mathrm{den}} \log x}{\delta_1}, \qquad u_{\mathrm{gain}} = w'' + \eta + c_G + N, \qquad z_{\mathrm{orb}} = \log_p(2a).$$

The density hypothesis gives

$$n_p \le N_p(v) g_{\mathrm{den}}, \qquad n_p \delta_1 \le N_p(v) C_{\mathrm{den}} \log x. \tag{1}$$

Since $\delta_1 > 0$, $n_p \ge 1$ and $C_{\mathrm{den}} \log x > 0$, we also have $N_p(v) \ge 1$.

We first check the signs used below. The identities of a loss datum and the scalar hypotheses give

$$b' \ge w'' \ge k S_c \ge 500, \qquad \log x \ge 1, \qquad r_G \ge 1, \qquad g_{\mathrm{den}} u_{\mathrm{gain}} \le b', \qquad 2c_G \le b'. \tag{2}$$

Here $w'' = b'/(x|H|) \le b'$ because $x > 1$ and $|H| \ge 1$. All terms in u_{gain} are nonnegative. Consequently

$$n_p c_G \le n_p u_{\mathrm{gain}} \le N_p(v) b'. \tag{3}$$

Moreover $a = b' X' > b'$, and the height bounds imply

$$0 < a_t - 2c_G \le a_t \le 2a \qquad (t \in T). \tag{4}$$

Every capacity is positive:

$$w_j(z_p(t,e)) - \varsigma_j + c_G \ge 1 - S_c + 4S_c > 0. \tag{5}$$

Since $p < X' < a$, we have $z_{\mathrm{orb}} \ge 1$.

We next compare the positive-part sums with the served gain. For a top $t \in T_p(v)$ put

$$m_{\mathrm{top}}(t) = \sum_{\substack{e \ge 1 \\ e \text{ odd}}} \min\left(\frac{a_t}{p^e}, w_j(z_p(t,e)) \right),$$

and put

$$m_{\mathrm{retained}} = \sum_{\substack{t \in T_p(v) \\ t \text{ at an even position}}} m_{\mathrm{top}}(t).$$

The first series converges by comparison with $a_t \sum_{e \text{ odd}} p^{-e}$. The positive-part series defining $A_p(a_t, t, w_j)$ is eventually zero because $w_j \ge 1$. The identity $y = (y - w)^+ + \min(y, w)$ for $y, w \ge 0$ therefore gives

$$A_p(a_t, t, w_j) + m_{\mathrm{top}}(t) = \gamma_p a_t. \tag{6}$$

For a top at an even position, its contribution on the left of (6) differs from $\gamma_p(a_t - 2c_G)$ by $2c_G\gamma_p \le 3c_G z_{\mathrm{orb}}$. For a top at an odd position, write $\alpha_t = a_t - 2c_G$ and use the capacity $c_j(t,e)$ of the gain datum. The inequality

$$\left(\frac{a_t}{p^e} - w_j(z_p(t,e)) \right)^+ - \left(\frac{\alpha_t}{p^e} - c_j(t,e) \right)^+ \le \frac{2c_G}{p^e} + c_G$$

follows from $c_j(t,e) - w_j(z_p(t,e)) = c_G - \varsigma_j \le c_G$ and the fact that the positive-part function is nondecreasing and has Lipschitz constant 1. If the first positive part is zero, its difference from the second is nonpositive. Otherwise $p^e < a_t \le 2a$, so there are at most z_{orb} such positive integer exponents. The total difference is thus at most

$$2c_G\gamma_p + c_G z_{\mathrm{orb}} \le 3c_G z_{\mathrm{orb}}.$$

Both α_t and the capacities are nonnegative by (4) and (5). Hence their minimum series converges, and

$$\sum_{e \text{ odd}} \left(\frac{\alpha_t}{p^e} - c_j(t,e)\right)^+ = \gamma_p\alpha_t - \sum_{e \text{ odd}} \min(\alpha_t/p^e, c_j(t,e)).$$

Summing these estimates over even and odd top positions, and using the definition of $g_j^U(v)$, proves

$$\sum_{t\in T_p(v)} A_p(a_t,t,w_j) + m_{\mathrm{retained}} \le g_j^U(v) + \delta_p(v) + 3N_p(v)c_G z_{\mathrm{orb}}. \tag{7}$$

The coefficient of this last error is covered by the first part of the local loss, since

$$3c_G z_{\mathrm{orb}} = 12S_c z_{\mathrm{orb}} \le C(S_c + k + \eta)z_{\mathrm{orb}}. \tag{8}$$

It remains to prove the following price and correction estimate:

$$\mathrm{Pr}_j(v) + \lceil t_p(v)\rceil^+ + D_p + 1 \le m_{\mathrm{retained}} + N_p(v)\big(\mathrm{pr}_{C,b_o}(p) + \mathrm{or}_C(p)\big). \tag{9}$$

We prove it separately in the two ranges determined by $p_0^* = k^2x^{26}$, the price threshold of this loss datum.

Suppose first that $p < p_0^*$. Put

$$s_{\mathrm{small}} = \frac{L_a a}{pk} + ap^{-\ell_{\mathrm{cor}}(p)}, \qquad e_{\mathrm{small}} = C_{\mathrm{Ds}}(N_{\mathrm{orb}}^3 + r_G^5 + k^2)\log_p a.$$

Both are nonnegative, using $L_a \ge 0$, and $e_{\mathrm{small}} \le b'$. The correction formulas and the ceiling bound give

$$\begin{aligned} t_p(v) &= C_{\mathrm{ts}} r_G N_p(v)\big(s_{\mathrm{small}} + 2b' + 2c_G + kQ_{\mathrm{cor}}(p)\big) + 2r_G n_p c_G \ge 0,\\ D_p &\le 40n_p\delta_1 e_{\mathrm{small}} + 2r_G n_p c_G + 40r_G n_p\delta_1(2b' + 2c_G) + 1. \end{aligned}$$

Also $2b' + 2c_G \le 3b'$ and

$$\begin{aligned} \mathrm{Pr}_j(v) &= r_G\big(N_p(v)(2b' + c_G) + n_p u_{\mathrm{gain}}\big)\\ &\le 3r_G\big(N_p(v)b' + n_p u_{\mathrm{gain}}\big) \le 6r_G N_p(v)b'. \end{aligned}$$

Since $t_p(v) \ge 0$, its positive ceiling is at most $t_p(v) + 1$. Using (1)–(3), $kQ_{\mathrm{cor}}(p) \le b'$ and the preceding bounds, we obtain

$$\begin{aligned} &\mathrm{Pr}_j(v) + \lceil t_p(v)\rceil^+ + D_p + 1\\ &\quad\le C_{\mathrm{ts}} r_G N_p(v) s_{\mathrm{small}} + (10 + 4C_{\mathrm{ts}})r_G N_p(v)b'\\ &\qquad + 40C_{\mathrm{den}}(\log x)N_p(v)b' + 120r_G C_{\mathrm{den}}(\log x)N_p(v)b' + 3\\ &\quad\le C(\log x)N_p(v)s_{\mathrm{small}}\\ &\qquad + (13 + 4C_{\mathrm{ts}} + 160C_{\mathrm{den}})r_G(\log x)N_p(v)b'\\ &\quad\le N_p(v)\big(C(\log x)s_{\mathrm{small}} + Cr_G b'\log x\big). \end{aligned} \tag{10}$$

For the first inequality, the two terms $2r_G n_p c_G$ contribute at most $4r_G N_p(v)b'$, and the two terms involving $n_p\delta_1$ contribute the displayed 40 and 120 terms. For the second, use $C_{\mathrm{ts}}r_G \le C$, $\log x, r_G \ge 1$ and $r_G(\log x)N_p(v)b' \ge 1$; this last bound absorbs the constant 3.

The cutoff in s_{small} is at least the cutoff in the local loss. Indeed

$$\log_p((kx^{26})^4) = \log_p(k^2x^{78}) + \log_p(k^2x^{26}),$$

and the second summand is greater than 1 because $p < k^2x^{26}$. Taking ceilings therefore gives

$$\ell_{\mathrm{cor}}(p) \ge \lceil \log_p(k^2x^{78}) \rceil + 1 = L_1(p).$$

Thus

$$s_{\mathrm{small}} \le \frac{L_a a}{pk} + ap^{-L_1(p)}.$$

The last expression in (10) is consequently at most $N_p(v)(\mathrm{or}_C(p)+\mathrm{pr}_{C,b_o}(p))$. Since $m_{\mathrm{retained}} \ge 0$, this proves (9) in the small-prime range.

Now suppose that $p \ge p_0^*$. The integer n_p is even by hypothesis. Let k_{back} be the first, finite-sum term in $t^{\mathrm{l}}_{\mathrm{cor}}(p,v)$ of (Nc3), and put

$$h_{\mathrm{top}} = b'/x + \eta.$$

We claim

$$\sum_{\substack{t\in T_p(v)\\ t \text{ at an odd position}}} (w_j(0,t) - \varsigma_j - b_t) + k_{\mathrm{back}} \le m_{\mathrm{retained}} + N_p(v)h_{\mathrm{top}}. \tag{11}$$

To prove this, label the orbit by its residues $0,\dots,n_p-1$ relative to v. For each odd $i \in P_p(v)$ with $d^-_{p,v}(i) > 0$, associate the pair

$$\left((i - d^-_{p,v}(i)) \bmod n_p,\ d^-_{p,v}(i)\right).$$

The first entry is an even top position by definition. The second entry is odd, since subtracting it from odd i gives an even residue modulo the even integer n_p. The association is injective: the pair determines i modulo n_p, hence determines its representative in $\{0,\dots,n_p-1\}$. The bound $b_{v+ih_p} \le w_j(0, v+ih_p)$ shows that the corresponding summand of k_{back} is at most

$$\min\left(w_j(0,v+ih_p), \frac{a_{v+((i-d^-_{p,v}(i)) \bmod n_p)h_p}}{p^{d^-_{p,v}(i)}}\right).$$

This is the term with that odd exponent in m_{top} of the associated even top. Distinct pairs give distinct terms, and all remaining terms are nonnegative. Therefore $k_{\mathrm{back}} \le m_{\mathrm{retained}}$. For each odd-position top the remaining summand in (11) is at most h_{top}, by $\varsigma_j \ge 0$ and the upper top-offset bound. There are at most $N_p(v)$ such tops, proving (11).

Put

$$\begin{aligned} s_{\mathrm{large}} &= \frac{a}{p^2} + \frac{L_a a}{pk},\\ e_{\mathrm{large}} &= C_{\mathrm{Dl}}x^{26}(N_{\mathrm{orb}}^5 + k^3)\log_p a,\\ b_{\mathrm{large}} &= Q_{\mathrm{cor}}(p)c_{\mathrm{cor}}(p,n_p) + kQ_{\mathrm{cor}}(p) + k\log_p a + c_G. \end{aligned}$$

These numbers and k_{back} are nonnegative. Conditions (Nc2) and (Nc3) give

$$\begin{aligned} t_p(v) &= k_{\mathrm{back}} + C_{\mathrm{t1}}N_p(v)N_{\mathrm{orb}}s_{\mathrm{large}} + C_{\mathrm{t2}}N_p(v)b_{\mathrm{large}} + 2n_pc_G \ge 0,\\ D_p &\le 40n_p\delta_1 e_{\mathrm{large}} + 1. \end{aligned}$$

Using the large-prime price, (11), (1), and $C_{\mathrm{t1}}N_{\mathrm{orb}} \le C\log x$, we obtain

$$\begin{aligned}
&\mathrm{Pr}_j(v) + \lceil t_p(v)\rceil^+ + D_p + 1\\
&\quad\le m_{\mathrm{retained}} + N_p(v)h_{\mathrm{top}} + n_p u_{\mathrm{gain}} + C(\log x)N_p(v)s_{\mathrm{large}}\\
&\qquad + C_{\mathrm{t2}}N_p(v)b_{\mathrm{large}} + 2n_p c_G + 40n_p\delta_1 e_{\mathrm{large}} + 3\\
&\quad\le m_{\mathrm{retained}} + N_p(v)\,[h_{\mathrm{top}} + g_{\mathrm{den}}w'' + C(\log x)s_{\mathrm{large}}\\
&\qquad\qquad + g_{\mathrm{den}}(\eta + c_G + N) + C_{\mathrm{t2}}b_{\mathrm{large}} + 2g_{\mathrm{den}}c_G + 40C_{\mathrm{den}}(\log x)e_{\mathrm{large}} + 3]\\
&\quad\le m_{\mathrm{retained}} + N_p(v)\big(h_{\mathrm{top}} + g_{\mathrm{den}}w'' + C(\log x)s_{\mathrm{large}} + b_o\big).
\end{aligned} \tag{12}$$

The factor $N_p(v) \ge 1$ absorbs the constant 3. The last inequality is hypothesis 4.

Finally $C_{\mathrm{den}} \le C$ implies $g_{\mathrm{den}}w'' \le Cw''(\log x)/\delta_1$. Since $C \ge 1$ and $S_c, t_R \ge 0$,

$$\begin{aligned}
h_{\mathrm{top}} + g_{\mathrm{den}}w'' + b_o &\le C\left(\frac{b'}{x} + \eta + S_c + \frac{(w''+t_R)\log x}{\delta_1}\right) + b_o\\
&= \mathrm{pr}_{C,b_o}(p).
\end{aligned}$$

Also $C(\log x)s_{\mathrm{large}} = \mathrm{or}_C(p)$. Thus (12) proves (9) in the large-prime range.

Subtract $N_p(v)\mathrm{lo}_{C,b_o}(p)$ from (7). Using (8), the definition of the local loss and then (9), we find

$$\begin{aligned}
&\sum_{t\in T_p(v)} \big(A_p(a_t,t,w_j) - \mathrm{lo}_{C,b_o}(p)\big)\\
&\quad\le g_j^U(v) + \delta_p(v) - m_{\mathrm{retained}} - N_p(v)\big(\mathrm{pr}_{C,b_o}(p) + \mathrm{or}_C(p)\big)\\
&\quad\le g_j^U(v) - \mathrm{Pr}_j(v) + \delta_p(v) - \big(\lceil t_p(v)\rceil^+ + D_p + 1\big).
\end{aligned}$$

This is the asserted inequality, and p, v and j were arbitrary. □

Lemma 18.6. *Suppose a loss datum has the integer weights and boxes and satisfies*

$$k \ge 5, \qquad S_c \ge 100, \qquad kS_c \le w'', \qquad N = k + m(t_R+1) \le S_c.$$

Then, for every $1 \le j \le m$, every $z \in \Phi$ and every top t,

$$0 \le \varsigma_j \le S_c, \qquad w_j(z) \ge 1, \qquad a \le a_t \le a\left(1 + \frac{1}{kx^{13}}\right), \qquad b_t \ge b', \qquad w_j(0,t) \ge b_t + b'/x.$$

Moreover, fix $0 < \varepsilon < 1/52$, $c_D > 397$ and $c_h \ge 402$. There is a real number $x_0 > 1$, depending only on these constants, with the following property. If $x \ge x_0$, a loss datum at (x,ε) has the integer weights and boxes for these exponents, and its group H and parameters k, m, t_R are the group A and parameters k, m, t of a rank datum with the rank sizes at (x,ε,J) for some $J \ge 1$, then all four hypotheses in the first display, and hence all its conclusions, hold.

Proof. First assume the four displayed scalar hypotheses. Since $x > 1$ and $c_D \ge 0$, $\Delta \ge 1$; also $h_0/2 = \lceil x^{c_h}\rceil \ge 1$. The shifts increase with j, and

$$0 < \varsigma_j \le \varsigma_m = \frac{k + (2m-1)(t_R+1)}{2} < k + m(t_R+1) = N \le S_c.$$

The strict inequality follows because its right side minus its middle expression is $(k+t_R+1)/2 > 0$. The height and depth conclusions follow directly from the integer boxes.

Put $u = b'/x > 0$ for this paragraph. At a top,

$$w_j(0,t) - b_t = \lfloor u\rfloor + (j - \tfrac{1}{2})h_0 \ge \lfloor u\rfloor + 1 \ge u.$$

In particular b_t is a positive integer and all top weights are at least 1. At a thin place, $\lceil w'' \rceil \geq 1$ because $w'' > 0$. At a thick place, $\lceil b \rceil \geq 1$. The positive half-gap and the nonnegative later gaps show that all those weights are at least 1 as well. This proves every conclusion of the first part.

We now find a threshold independent of the rank datum and the weights. Write $n = |A|$ and $f = 2n$. The rank conditions give $m \leq k$ and thus $r = k + m - 1 \leq 2k$. The rank sizes imply

$$\begin{gathered} n \leq 2x^{2+8\varepsilon}, \qquad x^{1/2-\varepsilon/16} \leq m \leq 2x^{1/2-\varepsilon/16}, \\ x^{9/2+10\varepsilon} \leq k \leq 2x^{9/2+10\varepsilon}, \qquad r \leq 4x^{9/2+10\varepsilon}, \qquad f \leq 4x^{2+8\varepsilon}. \end{gathered}$$

Since $\lceil v \rceil \leq v + 1$ for $v \geq 0$, the defining formula for n_W yields

$$\begin{aligned} n_W &\leq 10r \left(f + \frac{f^2 (k-1)^3 x^{2\varepsilon}}{m} \right) + 8 \\ &\leq 160 x^{13/2+18\varepsilon} + 5120 x^{43/2+(929/16)\varepsilon} + 8 \\ &\leq 5288 x^{43/2+(929/16)\varepsilon}. \end{aligned}$$

Every replacement here is an upper bound by the preceding display and $k - 1 \leq k$; the last exponent exceeds the other exponents for $x > 1$. Also $\lceil x^{377} \rceil \leq 2x^{377}$. Using $t_R = t \leq r + t = 8fr + 4 + \lceil x^{377} \rceil n_W$ gives

$$\begin{aligned} t_R &\leq 128 x^{13/2+18\varepsilon} + 4 + 10576 x^{797/2+(929/16)\varepsilon} \\ &\leq 10708 x^{797/2+(929/16)\varepsilon}. \end{aligned}$$

The exponent $797/2 + (929/16)\varepsilon$ is strictly less than 401 for $\varepsilon < 1/52$. We may therefore choose a threshold such that $t_R \leq x^{401}$ for all the data under consideration.

For x above this threshold, the inequalities $9/2 + 10\varepsilon < 5$ and $m \leq 2x^{1/2}$ imply

$$N = k + m(t_R + 1) \leq 2x^5 + 4x^{401+1/2} \leq 6x^{401+1/2}.$$

On the other hand

$$S_c = 4k \lceil x^{c_D} \rceil \geq 4x^{9/2+10\varepsilon+c_D}.$$

Because $c_D - 397 + 10\varepsilon > 0$, increasing the threshold until $x^{c_D - 397 + 10\varepsilon} \geq 3/2$ proves $N \leq S_c$ uniformly. Increasing it also until $x^{9/2+10\varepsilon} \geq 25$ gives $k \geq 25$ and $S_c \geq 4k \geq 100$.

Finally $\Delta \leq 2x^{c_D}$ and the rank sizes give

$$nkS_c = 4nk^2 \Delta \leq 64 x^{11+28\varepsilon+c_D}.$$

The loss identities give

$$nw'' = \exp(x^{\varepsilon} - x^{\varepsilon/3}).$$

For every fixed positive ε and fixed real exponent, this last exponential eventually exceeds $64x^{11+28\varepsilon+c_D}$: taking logarithms reduces this to

$$x^{\varepsilon} - x^{\varepsilon/3} - (11 + 28\varepsilon + c_D) \log x - \log 64 \geq 0,$$

whose left side divided by x^{ε} tends to 1. Thus $kS_c \leq w''$ after one further uniform increase of the threshold. All thresholds used depend only on the fixed exponents and ε. The first part now applies, completing the uniform assertion. $\square$

Lemma 18.7. *Fix $0 < \varepsilon < 1/52$. There is a constant $C_{\mathrm{den}} > 0$, depending only on ε, such that for every $x \geq \exp(1)$, every positive integer $d_{\mathrm{cyc}} \geq 3G_{\mathrm{orb}}$ and every $P_{\mathrm{cyc}} \subseteq \{0, \dots, d_{\mathrm{cyc}} - 1\}$ satisfying (Go2) with $n = d_{\mathrm{cyc}}$,*

$$|P_{\mathrm{cyc}}| \geq 1, \qquad d_{\mathrm{cyc}}\delta_{\mathrm{orb}} \leq |P_{\mathrm{cyc}}| C_{\mathrm{den}} \log x.$$

One may take

$$C_{\mathrm{den}} = 2(6r_{\mathrm{orb}} + 1)\left((5 + 8\varepsilon)10^{6r_{\mathrm{orb}}+1} + 6\right).$$

The integer r_{orb} depends only on ε.

There is also a threshold $x_0 > 1$, depending only on ε, such that for every $x \geq x_0$, every finite good-orbit selection, every inert prime p that is not sign-changing and has $\mathrm{ord}(h_p) \geq x^{\varepsilon/4}$, and every $v \in G_p$,

$$N_p(v) \geq 1, \qquad \mathrm{ord}(h_p)\, x^{-\varepsilon/8} \leq N_p(v) C_{\mathrm{den}} \log x.$$

Proof. Throughout the proof, all numerical orbit quantities have the fixed parameters (x, ε) of the statement. Put $B = \lfloor d_{\mathrm{cyc}}/\ell_{\mathrm{orb}} \rfloor$ and $d_{\mathrm{rem}} = d_{\mathrm{cyc}} - B\ell_{\mathrm{orb}}$, so $0 \leq d_{\mathrm{rem}} < \ell_{\mathrm{orb}}$. The hypothesis on the length gives

$$B \geq 3(M_{\mathrm{orb}} + 5) > M_{\mathrm{orb}}.$$

Every block selected by (Go2) contains a point of P_{cyc} in its second cell. Write this point as $s\ell_{\mathrm{orb}} + a_{\mathrm{cell}}$, where s is the block index and $u_{\mathrm{orb}} \leq a_{\mathrm{cell}} < 2u_{\mathrm{orb}} \leq \ell_{\mathrm{orb}}$.

We show that every integer $0 \leq i < d_{\mathrm{cyc}}$ has a point of P_{cyc} within forward cyclic distance G_{orb}. If $i < B\ell_{\mathrm{orb}}$, write $i = j_{\mathrm{blk}}\ell_{\mathrm{orb}} + i_{\mathrm{cell}}$ with $0 \leq j_{\mathrm{blk}} < B$ and $0 \leq i_{\mathrm{cell}} < \ell_{\mathrm{orb}}$. Apply (Go2) starting from $(j_{\mathrm{blk}} + 1) \bmod B$ and let $0 \leq j \leq M_{\mathrm{orb}}$ be the resulting displacement. Since $M_{\mathrm{orb}} < B$, the selected block index is either $j_{\mathrm{blk}} + 1 + j$ or $j_{\mathrm{blk}} + 1 + j - B$.

In the first case the selected point is strictly after i, with distance

$$(j + 1)\ell_{\mathrm{orb}} + a_{\mathrm{cell}} - i_{\mathrm{cell}} < (M_{\mathrm{orb}} + 2)\ell_{\mathrm{orb}}.$$

In the second case travel once through the end of the cycle; the positive distance is

$$(j + 1)\ell_{\mathrm{orb}} + d_{\mathrm{rem}} + a_{\mathrm{cell}} - i_{\mathrm{cell}} < (M_{\mathrm{orb}} + 3)\ell_{\mathrm{orb}}.$$

If instead $i \geq B\ell_{\mathrm{orb}}$, start (Go2) at block 0. Its selected block has index $j \leq M_{\mathrm{orb}} < B$, and the positive distance through the end of the cycle to its selected point is

$$d_{\mathrm{cyc}} - i + j\ell_{\mathrm{orb}} + a_{\mathrm{cell}} < (M_{\mathrm{orb}} + 2)\ell_{\mathrm{orb}}.$$

In every case the distance is a positive integer at most $G_{\mathrm{orb}} = (M_{\mathrm{orb}} + 5)\ell_{\mathrm{orb}}$.

For each i choose one such pair consisting of a point of P_{cyc} and its distance. The pair determines i by subtracting the distance modulo d_{cyc}, so this is an injection into $P_{\mathrm{cyc}} \times \{1, \dots, G_{\mathrm{orb}}\}$. Hence

$$d_{\mathrm{cyc}} \leq |P_{\mathrm{cyc}}| G_{\mathrm{orb}}, \qquad |P_{\mathrm{cyc}}| \geq 1.$$

To bound the last factor, put $A_{\mathrm{tile}} = (5 + 8\varepsilon)10^{6r_{\mathrm{orb}}+1}$. Since $0 < \delta_{\mathrm{orb}} < 1$ and $\log x \geq 1$,

$$u_{\mathrm{orb}}\delta_{\mathrm{orb}} \leq 2, \qquad M_{\mathrm{orb}} + 5 \leq A_{\mathrm{tile}} \log x + 6 \leq (A_{\mathrm{tile}} + 6) \log x.$$

It follows that

$$G_{\mathrm{orb}}\delta_{\mathrm{orb}} \leq 2(6r_{\mathrm{orb}} + 1)(A_{\mathrm{tile}} + 6) \log x = C_{\mathrm{den}} \log x.$$

Multiplying the finite covering inequality by δ_{orb} proves the first assertion.

For the uniform threshold, the same estimates give

$$G_{\mathrm{orb}} \le C_{\mathrm{den}} x^{\varepsilon/8} \log x \quad (x \ge \exp(1)).$$

As $\log x / x^{\varepsilon/8} \to 0$, one threshold depending only on ε gives $3G_{\mathrm{orb}} \le x^{\varepsilon/4}$ for every larger x. Now take a prime and representative in the second assertion. Its position set satisfies (Go2) by the definition of G_p, and its cycle length is at least $3G_{\mathrm{orb}}$. Since p is not sign-changing, the map $i \mapsto v + ih_p$, for $0 \le i < \mathrm{ord}(h_p)$, is a bijection onto its orbit; restricting to the top positions gives $|P_p(v)| = N_p(v)$. The first assertion applied to this position set gives both required conclusions. □

Lemma 18.8. *Let $\mathfrak{M}$ be a loss datum satisfying (Lm7) for some $\varepsilon_z > 0$ and real C_t. Let $C_e \ge 0$ be real, and let $p \le X''$ be an inert prime. Then*

$$fk^2 \log_p a \ge 1.$$

Consequently, if

$$\epsilon_p = \left\lceil C_e f k^2 \log_p a \right\rceil,$$

then

$$\epsilon_p \le (C_e + 1) f k^2 \log_p a.$$

Proof. Put $\theta_s = \varepsilon/100$, as in (Lm7). A loss datum has $x > 1$ and $0 < \varepsilon < 1/52$. Therefore

$$2 + 8\varepsilon + \theta_s < 3, \qquad \frac{9}{2} + 10\varepsilon - \theta_s > 0.$$

The upper bound for n and lower bound for k in (Lm7) imply

$$1 \le n \le x^3, \qquad k \ge x^{9/2 + 10\varepsilon - \theta_s} > 1.$$

Since k is an integer, $k \ge 2$, and hence $fk^2 = 2nk^2 \ge 8$.

The prime p is at least 2, so $\log p > 0$. Using $p \le X'' = xn \exp(x^{\varepsilon/3})$ and $n \le x^3$ gives

$$\begin{aligned} \log p &\le \log X'' = \log x + \log n + x^{\varepsilon/3} \\ &\le 4 \log x + x^{\varepsilon/3} \le 4 \log x + 4x^{\varepsilon} = 4 \log a. \end{aligned}$$

Here $x^{\varepsilon/3} \le x^{\varepsilon}$ because $x > 1$ and $\varepsilon > 0$, and $a = x \exp(x^{\varepsilon})$. In particular $\log_p a \ge 1/4$. Combining this with $fk^2 \ge 8$ yields $fk^2 \log_p a \ge 2 \ge 1$.

The argument of the ceiling in the assumed formula for ϵ_p is nonnegative. The ceiling of a real number is strictly less than that number plus 1. It follows that

$$\epsilon_p < C_e f k^2 \log_p a + 1 \le (C_e + 1) f k^2 \log_p a,$$

where the last inequality uses the lower bound just proved. This implies the required non-strict bound. □

Lemma 18.9. *Let $\mathcal{F}$ be a parametrised host family with parameter ε. Suppose there are real constants $C_e \ge 0$, C_0 and x_2 such that, for every $x \ge x_2$ with $x > x_1$ and every $i \in I(x)$, the following hold.*

1. *The member satisfies*

$$k \ge 5, \qquad S_c \ge 100, \qquad kS_c \le w'', \qquad 0 \le \varsigma_j \le S_c \quad (1 \le j \le m),$$

and, for all tops t and lifts j,

$$a \le a_t \le a\left(1 + \frac{1}{kx^{13}}\right), \quad b_t \ge b', \quad w_j(z) \ge 1 \ (z \in \Phi), \quad w_j(0,t) \ge b_t + \frac{b'}{x}.$$

2. *At every served inert prime p,*

$$\epsilon_p \le C_e f k^2 \log_p a, \qquad f = 2|A|.$$

3. *At every candidate prime p, every $v \in V_p$ and every $1 \le j \le m$,*

$$\sum_{t \in T_p(v)} (A_p(a_t, t, w_j) - \mathrm{lo}_{C_0, \exp(x^{\varepsilon/2})}(p)) \le g_j^U(v) - \mathrm{Pr}_j(v) + \delta_p(v) - (\lceil t_p(v) \rceil^+ + D_p + 1).$$

Then $\mathcal{F}$ has the linking property. It holds with

$$C = \max(C_0, 10 + C_e, 16), \qquad \delta(x) = \begin{cases} x^{-\varepsilon/2} & x > 1, \\ 0 & x \le 1, \end{cases}$$

and threshold x_2. In particular, the same constant C and the same function δ serve every member.

Proof. Fix $x \ge x_2$ with $x > x_1$ and a member $i \in I(x)$. We prove both inequalities of (Lm3) for its loss datum and for $(C, \exp(x^{\varepsilon/2}))$.

The identities of a loss datum give $c_G = 4S_c$, $N = k + m(t_R + 1) \ge 0$, $b' = a/X' > 0$, $w'' = b'/(x|A|) > 0$, and $X' = \exp(x^{\varepsilon/3}) > 2$. Also $a = x\exp(x^{\varepsilon}) > 2$. Its size conditions give $\eta > 0$ and $L_a \ge x^{\varepsilon} > 0$. Every coefficient of C in $\mathrm{lo}_{C,b_o}(p)$ is therefore nonnegative. This follows term by term from its formula, since $\log x > 0$, $\delta_1 > 0$, $t_R \ge 0$, $r_G \ge 1$ and every positive power of p is positive. Thus increasing C_0 to C can only increase the local loss, with $b_o = \exp(x^{\varepsilon/2})$ fixed. The candidate estimate in item 3 consequently remains valid with C in place of C_0.

Since $C \ge 16$, all terms of the local loss are nonnegative and its first term alone gives

$$\mathrm{lo}_{C,\exp(x^{\varepsilon/2})}(p) \ge CS_c \log_p(2a) \ge 12S_c \log_p(2a) = 3c_G \log_p(2a)$$

at every inert prime. Here the middle-prime summand b_o is positive, and all other summands have the signs just checked. Item 2 and Lemma 18.12, with $C \ge 10 + C_e$ and $C \ge 16$, give

$$\sum_{v \in R_p} \delta_p(v) + \epsilon_p \le \mathrm{fl}_C(p)$$

at every served inert prime. Moreover $k \ge 1$, $c_G \ge 0$, $0 \le \varsigma_j \le S_c \le c_G$, and the assumed top-weight bound implies $w_j(0,t) \ge b' + b'/x$. All scalar hypotheses of Lemma 18.10 are now verified, and that lemma gives the served-prime inequality for these loss constants.

It remains to prove the surplus bound. The identity $b'/x = |A|w''$ and $|A| \ge 1$ imply

$$\frac{b'}{x} \ge kS_c \ge 500, \qquad b' > 500, \qquad a = X'b' > 2b' > 10S_c.$$

The last inequality uses $b' \geq xkS_c > kS_c \geq 5S_c$ and $X' > 2$. Hence, for each top,

$$0 < a_t - 8S_c = \alpha_t \leq a_t \leq a\left(1 + \frac{1}{kx^{13}}\right) \leq \frac{6}{5}a.$$

The capacities satisfy $c_j(t,e) \geq 1 - S_c + 4S_c > 0$, and every orbit loss is nonnegative by its formula. We check both price regimes. If $p \geq p_0^*$, then each summand at an odd-position top obeys

$$w_j(0,t) - \varsigma_j - b_t \geq b'/x - S_c \geq 0,$$

and $n_p(w'' + \eta + c_G + N) \geq 0$. Thus $\mathrm{Pr}_j(v) \geq 0$. If $p < p_0^*$, all factors in

$$r_G\big(N_p(v)(2b' + c_G) + n_p(w'' + \eta + c_G + N)\big)$$

are nonnegative, so its price is nonnegative too.

For $v \in V_p$ the prime p is a candidate, so $p < X'$. If $N_p(v) > 0$, then $mN_p(v) \geq 1$ and

$$\frac{mN_p(v)a}{p} \geq \frac{a}{p} > \frac{a}{X'} = b' > 500 > \frac{5}{2}.$$

If $N_p(v) = 0$ the other alternative of Lemma 18.13 applies. That lemma therefore gives $S_p(v) \leq 2mN_p(v)a/p$ for all such p, v. This proves the surplus bound and hence (Lm3).

Finally $\varepsilon > 0$, so the stated function δ tends to 0 as $x \to \infty$. For $x > 1$, the positive real power identity gives

$$b(x)^{\delta(x)} = \exp(x^{\varepsilon}x^{-\varepsilon/2}) = \exp(x^{\varepsilon/2}).$$

Since C and the threshold do not depend on x or i, and δ is the same function for all of them, the two inequalities proved above are the linking property for $\mathcal{F}$. □

Lemma 18.10. *Let $\mathfrak{M}$ be a loss datum and let C, b_o be real numbers. Suppose that $k \geq 1$, $c_G \geq 0$, and for every top t and $1 \leq j \leq m$,*

$$0 \leq a_t \leq a\left(1 + \frac{1}{kx^{13}}\right), \qquad w_j(z) \geq 1 \quad (z \in \Phi), \qquad w_j(0,t) \geq b' + \frac{b'}{x}, \qquad 0 \leq \varsigma_j \leq c_G.$$

Assume that for every served inert prime p,

$$\mathrm{lo}_{C,b_o}(p) \geq 3c_G \log_p(2a), \qquad \sum_{v \in R_p} \delta_p(v) + \epsilon_p \leq \mathrm{fl}_C(p).$$

Suppose also that for every candidate prime p, every $v \in V_p$ and $1 \leq j \leq m$,

$$\sum_{t \in T_p(v)} \big(A_p(a_t, t, w_j) - \mathrm{lo}_{C,b_o}(p)\big) \leq g_j^U(v) - \mathrm{Pr}_j(v) + \delta_p(v) - \big(\lceil t_p(v) \rceil^+ + D_p + 1\big).$$

Then the served-prime inequality holds: for every served inert prime p,

$$\sum_{j=1}^{m} \sum_{t \in T_p^{\mathrm{s}}} \big(A_p(a_t, t, w_j) - \mathrm{lo}_{C,b_o}(p)\big) - m\,\mathrm{fl}_C(p) \leq \sum_{j=1}^{m} \mathrm{gain}_j(p) - \tau_p.$$

Proof. We begin with signs and convergence. A loss datum has $x > 1$, $\varepsilon > 0$ and $a = x\exp(x^{\varepsilon}) > 1$. Thus its orbit losses are nonnegative by their formula, and the floor-budget hypothesis implies $\mathrm{fl}_C(p) \geq 0$ at a served prime. Also $\mathrm{lo}_{C,b_o}(p) \geq 0$ there. The weight bound makes each original odd-exponent sum finite. Moreover

$$c_j(t,e) = w_j(z_p(t,e)) - \varsigma_j + c_G \geq 1.$$

The reduced positive-part sums are therefore finite too. The series of minima defining the served gains converge by comparison with the geometric series, since their first arguments are α_t/p^e and their second arguments are at least 1.

Fix a served inert prime p. If it is a candidate, all the hypotheses of Lemma 18.11 hold: the orbit losses are nonnegative, the floor budget is assumed, and the stated orbitwise estimate applies at each $v \in V_p$. That lemma gives the conclusion. Hence assume from now on that p is not a candidate. Then $V_p = \emptyset$ and $\tau_p = 0$. Since a served prime is not dropped, $p \leq X''$. All gains are nonnegative, including when their exceptional zero rule applies.

If p is not sign-changing and $p < X'$, its served-top set is empty. The left side of the required inequality is then $-m\,\mathrm{fl}_C(p) \leq 0$, whereas its right side is a sum of nonnegative gains. It remains to treat a sign-changing prime or a prime $p \geq X'$. All tops are served in these cases.

We show first that every excluded exponent contributes zero to the original sum. For a sign-changing prime an odd exponent has partner with first coordinate 1, whereas a top place has first coordinate 0. Thus no odd exponent is excluded. If $p \geq X'$ and $e \geq 1$, then $p^e \geq X'$ and, using $kx^{13} \geq x$ and $b' = a/X'$, we have

$$\frac{a_t}{p^e} \leq b'\left(1 + \frac{1}{kx^{13}}\right) \leq b' + \frac{b'}{x}.$$

At an excluded exponent the partner is a top place, whose weight is at least this last quantity. The corresponding positive part is zero. This includes $p = X'$.

For $v \in R_p$ and a fixed j we next prove

$$\sum_{t \in T_p(v)} \left(A_p(a_t, t, w_j) - 3c_G \log_p(2a)\right) \leq g_j^0(v) + \delta_p(v).$$

At a nonexcluded odd exponent, the original argument of the positive part minus its reduced argument equals

$$\left(\frac{a_t}{p^e} - w_j(z_p(t,e))\right) - \left(\frac{a_t - 2c_G}{p^e} - w_j(z_p(t,e)) + \varsigma_j - c_G\right) = \frac{2c_G}{p^e} + c_G - \varsigma_j \in [0, 3c_G].$$

The positive-part function is increasing and has Lipschitz constant 1. Thus the original positive part is at most the reduced one plus $3c_G$. This extra term is needed only when the original positive part is nonzero, since at every other exponent the original term is zero and the reduced term is nonnegative. A nonzero original term implies $p^e < a_t \leq 2a$, because the weight is at least 1 and $1/(kx^{13}) \leq 1$. There are at most $\log_p(2a)$ positive integer exponents with $p^e < 2a$, and therefore at most that many odd exponents. Summing the termwise comparison, and using the vanishing at excluded exponents already proved, gives the displayed orbit estimate by the definition of $g_j^0(v)$.

Suppose now that the exceptional zero rule for the gain does not apply. Since $V_p = \emptyset$, its formula is

$$\mathrm{gain}_j(p) = \sum_{v \in R_p} \max(g_j^0(v), 0).$$

Sum the orbit estimate over the partition $(T_p(v))_{v\in R_p}$ of T. Replace each plain gain by its nonnegative part, use $\mathrm{lo}_{C,b_o}(p) \geq 3c_G \log_p(2a)$ and then use the floor budget. This gives, for each j,

$$\begin{aligned}\sum_{t\in T}(A_p(a_t,t,w_j) - \mathrm{lo}_{C,b_o}(p)) &\leq \sum_{v\in R_p} g_j^0(v) + \sum_{v\in R_p}\delta_p(v)\\ &\leq \mathrm{gain}_j(p) + \mathrm{fl}_C(p).\end{aligned}$$

Sum over j and subtract $m\,\mathrm{fl}_C(p)$. Since all tops are served and $\tau_p = 0$, the desired inequality follows.

Finally suppose that the exceptional zero rule applies. The alternative $p > X''$ is impossible for a served prime. Thus p is not sign-changing and $h_p = 0$. In the present case this forces $p \geq X'$. Every partner of a top is now that same top place. The preceding height comparison and the top-weight bound show that every term of $A_p(a_t,t,w_j)$ is zero. The left side of the required inequality is therefore a sum of nonpositive local-loss terms and a nonpositive floor-loss term. It is at most 0, which is the value of every gain and of τ_p. This proves the last case. □

Lemma 18.11. *Let $\mathfrak{M}$ be a loss datum, let p be a candidate prime, and let C, b_o be real numbers. Assume that $\delta_p(v) \geq 0$ for every $v \in R_p$ and*

$$\sum_{v\in R_p}\delta_p(v) + \epsilon_p \leq \mathrm{fl}_C(p).$$

Suppose also that for every $v \in V_p$ and $1 \leq j \leq m$,

$$\sum_{t\in T_p(v)}(A_p(a_t,t,w_j) - \mathrm{lo}_{C,b_o}(p)) \leq g_j^U(v) - \mathrm{Pr}_j(v) + \delta_p(v) - (\lceil t_p(v)\rceil^+ + D_p + 1).$$

Then

$$\sum_{j=1}^{m}\sum_{t\in T_p^{\mathrm{s}}}(A_p(a_t,t,w_j) - \mathrm{lo}_{C,b_o}(p)) - m\,\mathrm{fl}_C(p) \leq \sum_{j=1}^{m}\mathrm{gain}_j(p) - \tau_p.$$

Proof. Because p is a candidate, it is not sign-changing and $p < X'$. Its Frobenius element h_p has order at least $x^{\varepsilon/4} > 1$, so $h_p \neq 0$. Moreover $X'' = x|H|X' > X'$, since $x > 1$ and a finite group has at least one element. Thus neither condition suppressing $\mathrm{gain}_j(p)$ applies. Its defining expression is a sum of nonnegative maxima, and in particular $\mathrm{gain}_j(p) \geq 0$.

The orbit selection has $m \geq 1$, and $\epsilon_p \geq 0$ by definition. The nonnegative orbit losses and the floor-budget hypothesis imply $\mathrm{fl}_C(p) \geq 0$. If $V_p = \emptyset$, then $T_p^{\mathrm{s}} = \emptyset$ and $\tau_p = 0$. The required inequality reduces to $-m\,\mathrm{fl}_C(p) \leq \sum_j \mathrm{gain}_j(p)$, which follows from these signs.

Assume now that $V_p \neq \emptyset$. The sets $T_p(v)$ for $v \in V_p$ are pairwise disjoint and have union T_p^{s}, because they are the fibres of the orbit labelling on T and p is neither sign-changing nor at least X'. For each j, summing the assumed orbitwise estimates therefore gives

$$\begin{aligned}&\sum_{t\in T_p^{\mathrm{s}}}(A_p(a_t,t,w_j) - \mathrm{lo}_{C,b_o}(p))\\ &\quad\leq \sum_{v\in V_p}(g_j^U(v) - \mathrm{Pr}_j(v)) + \sum_{v\in V_p}\delta_p(v) - \sum_{v\in V_p}(\lceil t_p(v)\rceil^+ + D_p + 1).\end{aligned}$$

The first sum on the right is at most $\mathrm{gain}_j(p)$: each summand is at most the corresponding maximum in that gain, and every contribution from $R_p \setminus V_p$ is nonnegative. Summing over j

and subtracting $m\,\mathrm{fl}_C(p)$ bounds the desired left-hand side by

$$\sum_{j=1}^{m}\mathrm{gain}_j(p)+m\left(\sum_{v\in V_p}\delta_p(v)-\mathrm{fl}_C(p)\right)\\ -m\sum_{v\in V_p}\bigl(\lceil t_p(v)\rceil^{+}+D_p+1\bigr).$$

Here $V_p\subseteq R_p$. The nonnegative orbit losses and the floor budget give

$$m\left(\sum_{v\in V_p}\delta_p(v)-\mathrm{fl}_C(p)\right)\le -m\epsilon_p\le-\epsilon_p,$$

using $m\ge 1$. Since $V_p\neq\emptyset$, its correction is

$$\tau_p=\epsilon_p+m\sum_{v\in V_p}\bigl(\lceil t_p(v)\rceil^{+}+D_p+1\bigr).$$

Substituting these two relations gives exactly the required upper bound. □

Lemma 18.12. *Let $\mathfrak{M}$ be a loss datum with $a\ge 2$, and let p be an inert prime. Let $C_e\ge 0$ and $C\ge\max(10+C_e,16)$ be real numbers. If*

$$\epsilon_p\le C_e f k^2\log_p a,$$

then

$$\delta_p(v)\ge 0\quad(v\in R_p),\qquad\sum_{v\in R_p}\delta_p(v)+\epsilon_p\le\mathrm{fl}_C(p).$$

Proof. Every orbit on H has $\mathrm{ord}(h_p)$ elements when p is not sign-changing, and $\mathrm{ord}(2h_p)$ elements when p is sign-changing. These orbits partition H. In the former case $n_p=\mathrm{ord}(h_p)$; in the latter case $n_p=2\,\mathrm{ord}(2h_p)$. It follows that

$$|R_p|n_p\le 2|H|=f.$$

The sets $T_p(v)$, $v\in R_p$, partition T, because they are the fibres of the orbit labelling on T. Consequently

$$\sum_{v\in R_p}N_p(v)=|T|\le|H|\le f.$$

Since $p\ge 2$ and $a\ge 2$, we have $\log_p a\ge 0$ and

$$0\le\log_p(2a)\le 2\log_p a.$$

The formula for $\delta_p(v)$ has only nonnegative factors, so each orbit loss is nonnegative. Summing that formula and using the preceding bounds gives

$$\sum_{v\in R_p}\delta_p(v)=5|R_p|n_pN^2\log_p(2a)+4|T|\bigl(\log_p(2a)\bigr)^2\\ \le 10fN^2\log_p a+16f(\log_p a)^2.$$

By the parameter identities of a loss datum, $N=k+m(t_R+1)\ge k\ge 0$. Thus the assumed error bound implies

$$\epsilon_p\le C_e f N^2\log_p a.$$

Adding the last two estimates and using $C \geq 10 + C_e$ and $C \geq 16$ yields

$$\sum_{v \in R_p} \delta_p(v) + \epsilon_p \leq CfN^2 \log_p a + Cf(\log_p a)^2 = \mathrm{fl}_C(p),$$

as required. □

Lemma 18.13. *Let a gain datum be given, let p be an inert prime and let $v \in V_p$. Suppose that*

$$0 \leq \alpha_t \leq \frac{6}{5}a \quad (t \in T_p(v)), \qquad c_j(t,e) \geq 0 \quad (1 \leq j \leq m,\ t \in T_p(v),\ e \geq 1 \text{ odd}),$$

and that $\delta_p(v) \geq 0$ and $\mathrm{Pr}_j(v) \geq 0$ for $1 \leq j \leq m$. Assume also that

$$N_p(v) = 0 \quad \text{or} \quad \frac{mN_p(v)a}{p} \geq \frac{5}{2}.$$

Then

$$S_p(v) \leq \frac{2mN_p(v)a}{p}.$$

Proof. We first bound the served gain. Both entries of each minimum in its definition are nonnegative, and

$$0 \leq \min\left(\frac{\alpha_t}{p^e}, c_j(t,e)\right) \leq \frac{\alpha_t}{p^e}.$$

Consequently the series of these minima converges and is nonnegative, by comparison with the geometric series over the odd positive exponents. Since the orbit loss is nonnegative,

$$g_j^U(v) \leq \gamma_p \sum_{t \in T_p(v)} \alpha_t.$$

The prime p is at least 2, so $p^2 \geq 4$ and

$$0 < \gamma_p = \frac{p}{p^2 - 1} \leq \frac{4}{3p}.$$

Using the upper bound on each reduced height and $N_p(v) = |T_p(v)|$ gives

$$g_j^U(v) \leq \frac{6}{5}\gamma_p N_p(v)a \leq \frac{8}{5}\,\frac{N_p(v)a}{p}.$$

The price and $\max(g_j^0(v), 0)$ are nonnegative. The last upper bound is also nonnegative, because $a > 0$. Subtracting the two nonnegative terms and then taking the positive part therefore yields

$$0 \leq \left(g_j^U(v) - \mathrm{Pr}_j(v) - \max(g_j^0(v), 0)\right)^+ \leq \frac{8}{5}\,\frac{N_p(v)a}{p}.$$

Summing over the m indices shows that the sum inside the defining ceiling for $S_p(v)$ lies between 0 and $8mN_p(v)a/(5p)$.

If $N_p(v) = 0$, that sum is zero, hence $S_p(v) = 0$. Otherwise the second alternative in the hypothesis applies. The elementary inequality $\lceil y \rceil < y + 1$ and the preceding bound give

$$S_p(v) < \frac{8}{5}\,\frac{mN_p(v)a}{p} + 1 \leq \frac{2mN_p(v)a}{p},$$

where the final inequality is equivalent to $mN_p(v)a/p \geq 5/2$. This proves the stated bound. □

19 Joint packet supply from realisations

This section proves the joint supply of packets, Proposition 19.3, which Proposition 18.1 uses. It is the second of three steps that lead from (TCS) to the packets that design existence needs, and each step adds properties that the previous one does not state. Proposition 21.2 of Section 21 supplies packets with model-level data. Proposition 19.9 adds the remaining model-slope and height properties, and Proposition 19.3 adds the conditions on the loss data and the eight lower conditions of the class that the earlier statements leave out. All three statements have the same shape: packets with conditions on their loss data and numerical data at every large scale, and, under the hypothesis that the Sha bound holds eventually, compatible completions on a tail of scales.

The step from Proposition 19.9 to Proposition 19.3 is mostly bookkeeping. Normalised hosts give the frame condition (Lm1) (Lemma 19.21), and the bad-top bound of the good host gives few bad tops (Lm4) (Lemma 19.22). Five lower conditions, (Mf1), (Mf5), (Dv1), (Dv2) and (Nf1), follow from the common data of the members, with the analytic loss of the host bounded by Lemma 19.28. The other three, (Bk1), (Bk2) and (Nf3), are the substance of the step: Theorem 19.23 derives them from the realisation estimates. Its proof passes through the covolume bound from the lattice chain (Theorem 19.26) and the core lower bound from descent along deep covers (Theorem 19.24).

The step from Proposition 21.2 to Proposition 19.9 supplies the model-slope properties (Ms2) and (Ms4), with the constant 83400, by the model-budget attachment Theorem 19.12, whose inputs are explicit bounds for the host field: its discriminant and degree (Lemma 19.16), its roots of unity (Lemma 19.15) and the depths of the tops (Lemma 19.14). From (Ms1)–(Ms4), Theorem 19.18 gives the mean-slope bound (Ht3) at the divided weight, with the Hodge–Tate cost bounded by Lemma 19.19, and Theorem 19.20 then gives the uniform lower bound (Dp6) for the twisted slope. The bounds (Ms1) and (Ms3) themselves come from Part VI through Proposition 21.2. In our reading the model slopes, which through (Dp6) and the descent along deep covers feed the covolume bound, are the "computation of covolumes" of Alpöge's account [4].

Definition 19.1. Fix real numbers $x > 1$ and $\varepsilon > 0$, a frame (e, A, h, a_0), a host $\mathfrak{h}$ for that frame, finite sets $T \subseteq A$ and $S \subseteq \mathbb{Z}_{\geq 0}$, an integer $t \geq 1$, integers $k, m \geq 0$, real numbers η, L_a, integers a_s for $s \in T$, and primes g_1, g_2.

A **base translate packet** over these data is a finite tuple

$$(\mathfrak{h}, T, S, t, \mathfrak{M}_u, \mathfrak{V}_u, u)_{u \in A}$$

satisfying the following conditions.

(Bt1) Loss data. For every $u \in A$, $\mathfrak{M}_u$ is a loss datum on the group A with arithmetic datum $\mathfrak{A}_{\mathfrak{h}}$ and top set T. Its orbit selection has scale x and parameter ε, its number of lifts is m, and its size parameters k, t_R are k, t. Each $\mathfrak{M}_u$ carries its own place record as part of the loss datum.

(Bt2) Common parameters and heights. For every $u \in A$, the parameters η, L_a of $\mathfrak{M}_u$ are the given η, L_a, and the height of $\mathfrak{M}_u$ at each $s \in T$ is a_s.

(Bt3) Slope records and translates. For every $u \in A$, $\mathfrak{V}_u$ is a slope record for m with auxiliary primes g_1, g_2, and the translate of the member indexed by u is u.

Every member of the packet has host $\mathfrak{h}$, top set T, spacing set S and rank t.

Definition 19.2. Fix $\varepsilon > 0$, a set $W \subseteq \mathbb{R}$, a real number $X > 1$ and, for every $x \in W$ with $x \geq X$, a base translate packet P_x over a host $\mathfrak{h}_x$, frame group A_x, top set T_x, spacing set S_x and rank t_x, with loss data $\mathfrak{M}_{x,u}$ and slope records $\mathfrak{V}_{x,u}$ for $u \in A_x$.

(Pl1) Completion data. A **lower completion of the packet at** x consists, for every $u \in A_x$, of the member data of all record and input-data types in (Pc3)–(Pc5) of Definition 17.5, attached to $\mathfrak{M}_{x,u}$ and $\mathfrak{V}_{x,u}$. Precisely, these are the data of the corresponding definitions, with the outer indexing over every real $x > x_1$ and every $i \in I(x)$ replaced by the fixed scale x and an index $u \in A_x$: the objects of the stated types, subject to all their defining identities and all their defining ties. All further indices are kept, among them all words, lifts, places, local components and eigensystems. In particular, a lower completion includes the local sum identities of the modification data, the Hecke lattices and the good and kept sets of the local defect data, the determinant algebras and extension modules of the Ribet records, and every metric, slope and map identity of the frame realisations. The additional expansion, kernel, row, Galois Ribet, column, weight-word, position and budget data of (Pc5) are included as well, with their defining types and identities and for the same frame and arithmetic data. Condition (Pl1) imposes none of the numbered properties and inequalities.

These data form one compatible collection, with the identifications of shared records and shared fields listed in Definition 17.5 and no others, read at the fixed scale x with member set A_x. The loss data, slope records, hosts, heights and all other data of P_x are unchanged. A **lower completion on a tail** is a choice of such completion data at every $x \in W$ with $x \geq Y$, for some real $Y \geq X$.

(Pl2) Pointwise interpretation of the numbered formulas. At a fixed scale x, a numbered property listed in (Pl3) is interpreted as follows. The family data at (x, i) are replaced by the packet and completion data at (x, u), with $I(x) = A_x$. The outer eventual-scale quantifier of the property, namely the assertion that there is a cutoff such that the formula holds at all scales above it, is removed, and the remaining formula is required for every $u \in A_x$. All universal quantifiers over words, pairs of words, lifts, places, primes, components, eigensystems and any other local indices remain in force, including those contained in the property's convention for all large members. All inner existential quantifiers keep their positions.

An existentially quantified scalar bound constant that precedes the eventual-scale quantifier of the property is not quantified there; it is taken from the common list of (Pl3). No other quantifier is moved, and every antecedent is kept. In particular, the following antecedents are kept: the finite-Sha-length antecedents in the remaining (Lq) and (Rf) conditions and in (Ub1), (Ub2), (Ub3°), (Ub4)–(Ub10), including (Ub7'), for the coefficient extension along $\mathcal{O}_r \subseteq \mathcal{O}_r(N_{x,u})$; the word-size antecedents in (Dx1), the kernel hypotheses and (Ro1)–(Ro4); and the nonzero-map antecedent in (Nf3). Finite Sha lengths are as defined in Definition 13.9. The Galois Ribet items that are required at every point before the finite-Sha antecedent remain original completion data at every point. The coefficient-extended items, with the extended data accompanying them, are obtained from these original completion data by Definition 17.4; they do not replace the original completion records.

For these coefficient-extension hypotheses, again only the eventual-scale quantifier is removed: the integer $N_{x,u} \geq 1$ with $4 \mid N_{x,u}$ and the integers $S^{\mathrm{rm}}_{x,u}, \eta^{\mathrm{rm}}_{x,u} \geq 0$ are chosen together, after the member and the finite-Sha antecedent and before all point and local indices. In this pointwise interpretation all the hypotheses of Theorem 22.1, including

(Ub7'), are taken for that coefficient extension. No family property is evaluated on a family supported at a single scale.

(Pl3) Uniform lower conditions. The collection of packets **admits uniform lower completions on a tail** if $\varepsilon \le 1$ and there exist a finite list of twenty-four scalar constants, a real number $Y \ge X$ with $Y > 1$ and a lower completion at every $x \in W$ with $x \ge Y$ such that the requirements below hold.

The list contains a separate entry for each scalar constant that is existentially quantified in

(Sp1), (Sp2), (Sp7), (Mf4), (Mf7), (Dv3), (Dv4), (Bk4), (Nr3), (Ld5), (Lq6), (Lq8),

subject to the sign restrictions stated in those properties; constants belonging to different properties need not be equal. The remaining twelve entries of the list are

$$c > 0, \quad C_0 \ge 1, \quad C_{\mathrm{A}} \ge 1, \quad 0 < c_{\mathrm{lp}} \le \tfrac{1}{3}, \quad C_{\mathrm{ng}} \ge 0,$$
$$C_{\mathrm{lb}} \ge 1, \quad A_{\mathrm{lb}} \ge 1, \quad C_0^{\mathrm{ng}} \in \mathbb{Z}_{\ge 0}, \qquad c_{\mathrm{sp}}, c_{\mathrm{in}}, c_{\mathrm{sm}}, C_{\mathrm{bud}} \ge 1.$$

All entries are real numbers, except C_0^{ng}, which is a nonnegative integer. The same c is used in (Nf2), (No2), (No6), (Rw2), (Dx1) and the kernel scalar bound of (Pc5). The same C_0 is used in (No1), (No2), (Dx1), the kernel hypotheses and (Ro1)–(Ro4). The same $(C_{\mathrm{A}}, c_{\mathrm{lp}}, C_{\mathrm{ng}})$ is used in the row data and (Rw2). The four constants $c_{\mathrm{sp}}, c_{\mathrm{in}}, c_{\mathrm{sm}}, C_{\mathrm{bud}}$ are used in (Ub1), (Ub2), (Ub3°), (Ub4)–(Ub10), including (Ub7'), for the coefficient extension specified in (Pl2); C_{bud} is distinct from the frame constant C_0. The bound of (Lq7), which is removed, is supplied by the original minimal losses and the derived allowance coefficient of Theorem 22.1; these are not additional entries of the list.

For every $x \in W$ with $x \ge Y$, the following numbered formulas hold in the pointwise interpretation (Pl2), with the constants of this list:

(Sp1), (Sp2), (Sp6), (Sp7),
(Mf1)–(Mf7), (Dv1), (Dv2), (Dv3), (Dv4), (Dv7),
(Bk1), (Bk2), (Bk4), (Nr1), (Nr2), (Nr3),
(Ld1)–(Ld6), (Lq2), (Lq4), (Lq5), (Lq6), (Lq8), (Rf3),
(Nf1), (Nf2), (Nf3), (No1), (No2), (No6), (Rw2),
(Dx1), (Ro1)–(Ro4), (Ub1), (Ub2), (Ub3°), (Ub4)–(Ub10), including (Ub7').

The kernel hypotheses of (Pc5), including (Kc1)–(Kc4) and both displayed scalar inequalities, hold in the same pointwise interpretation. All these formulas are taken with the common constants and the compatible data of (Pc5), and with every hypothesis of the statements they cite. The list of constants and Y do not depend on the scale, the member, the words or any other local index. The completion objects, and the objects whose existence is asserted by the inner existential quantifiers that are kept, may depend on their permitted indices.

Finally, at every such x the packet index set A_x is nonempty and every $u \in A_x$ satisfies, with $n = |A_x|$, $b = \exp(x^{\varepsilon})$, $a = xb$ and $\mathfrak{V} = \mathfrak{V}_{x,u}$,

$$x^{1/2-\varepsilon} \le m, \qquad \log(2a\, v(\mathfrak{h}_x)) \le 2x^{\varepsilon}, \qquad \log|\mathcal{W}(\mathfrak{V})| \le (k-3)\log 2,$$
$$x^{-\varepsilon} n \le |T_x^{\mathrm{th}}|, \qquad \tfrac{1}{2}x^{-\varepsilon/8} n \le |T_x|, \qquad a_s \ge (x-1)b \quad (s \in T_x),$$

where $T_x^{\mathrm{th}} = \{sa_0 : s \in S_x\} \subseteq A_x$, and k, m, a_s are the common packet parameters at x.

Proposition 19.3. *Let $\lambda > 1$ and let P be a host requirement supplied by the concrete good-host requirement $\mathfrak{G}$ on λ-windows. There is a real number $\varepsilon_* = \varepsilon_*(\lambda, \mathsf{P}) > 0$ with the following property. Fix*

$$0 < \varepsilon < \min(\varepsilon_*, 1/52), \qquad J \geq 1,$$

a finite list κ, a set $W \subseteq \mathbb{R}$ containing λ-windows, and a real $X_H > 1$. For every $x \in W$ with $x \geq X_H$, fix a frame, a host $\mathfrak{h}_x$, a top set T_x, an integer $t_x \geq 1$ with $\mathsf{P}(x, \varepsilon, \mathfrak{h}_x, T_x, t_x)$, and a choice of all the data whose existence is asserted by

$$\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}_x, T_x, t_x).$$

In particular this fixes its rank datum, ordering, good-orbit selections, spacing set S_x, spacing translate and auxiliary primes. These choices are made once, simultaneously for all such x.

There are real numbers

$$c_D > 397, \quad c_h \geq 402, \quad C_e \geq 0, \quad \varepsilon_z > 0, \quad C_t, \quad X \geq X_H,$$

a tuple $\mathfrak{k}_{\mathrm{orb}}$ of eight nonnegative real numbers, and, for every $x \in W$ with $x \geq X$, a base translate packet over the chosen frame, host $\mathfrak{h}_x$, top set T_x, spacing set S_x, rank t_x and auxiliary primes, with the following properties.

(Js1) Member conditions. *For every such x and every translate $u \in A_x$, the loss datum of the packet satisfies (Lm1), (Lm2), (Lm4), (Lm5), (Lm6), and (Lm7) for the same (ε_z, C_t).*

(Js2) Numerical conditions. *For every such x and u, the rank datum of the loss datum is the rank datum fixed above, and it has its rank sizes at (x, ε, J), with $t_R = t_x$. Its heights, depths, weights and shifts have the integer weights and boxes for (c_D, c_h), for some number field and thick subset. Its top set, Frobenius translations, orbit representatives and good representatives are those of the finite good-orbit selections fixed above. Its correction degrees, orbit thresholds and correction errors satisfy (Nc2), (Nc3), (Nc4) for the same $(\mathfrak{k}_{\mathrm{orb}}, C_e)$.*

(Js3) Conditional compatible completion. *Suppose that there is a real number X_V such that*

$$\mathrm{V}(E_{\mathfrak{h}_x}; t_x, \exp(x^{\varepsilon}/2))$$

holds for every $x \in W$ with $x \geq \max(X, X_V)$. Then this system of packets admits a uniform lower completion on a tail, in the sense of Definition 19.2, whose loss data, place records, slope records, heights, depths, weights, orbit selections and orbit masses are those of the packets above.

The constants, the cutoff and the packet system may depend on the whole collection of choices fixed above. The constants do not depend on an individual scale, translate, prime, orbit or word.

Proof. Let $\varepsilon_\dagger = \varepsilon_\dagger(\lambda, \mathsf{P}) > 0$ be given by Proposition 19.9 for the same λ and P, and put $\varepsilon_* = \varepsilon_\dagger$. Fix all the data of the statement. Since W contains λ-windows, there is a scale $x \in W$ above X_H. The good-host requirement at that scale implies that the list κ has the form

$$\kappa = (C_{T_2}, C_8, C_5, c_6, C_b), \qquad C_8 \geq 1, \quad C_5, c_6, C_b > 0.$$

Apply Proposition 19.9 to the data fixed in the statement. Denote its cutoff by X_{R} and its packets by P_x. Let X_{env} be the threshold of Lemma 19.28 with $C_{\mathrm{La}} = 1$, for its analytic conclusion.

We first verify the remaining frame condition at every packet member. By (Hg1), the host is normalised and the frame index is at most C_8. By (Ra1), $e_2 = e$, and $x \geq X_{\mathrm{R}} \geq \max\{2, (2C_8)^{4/\varepsilon}\}$. Thus Lemma 19.21 gives (Lm1). By (Hg4), the good-orbit selections satisfy their bad-top bound with the same constant C_b, and (Hg3) gives $|T_x| \geq \frac{1}{2}x^{-\varepsilon/8}|A_x|$. By Lemma 19.22, (Lm4) therefore holds for every packet above

$$X_{\mathrm{B}} = \max\{2, (8C_b)^{3/\varepsilon}\}.$$

Take $X' = \max(X_{\mathrm{R}}, X_{\mathrm{B}}, X_{\mathrm{env}})$ and keep the same packets at scales $x \in W$, $x \geq X'$. Condition (Ra1), the two deductions just made, and the unchanged common pair (ε_z, C_t) give all of (Js1). Condition (Ra2) is (Js2).

Suppose that the Sha premise of (Js3) holds for these packets. Then the premise of (Ra3) holds as well, after replacing its cutoff X_V by $\max(X', X_V)$ if necessary. This gives a completion on a tail starting at $Y \geq X_{\mathrm{R}}$, its twenty-four common parameters, and the family $\mathcal{F}$ specified there, whose members are the members of the same packets at $x \in W$, $x > Y$.

The rank datum has $m \geq 2$, so there is at least one lift at every member. The chain and deep-cover hypotheses and the coefficient supplied by (Ra3) give every hypothesis of Theorem 19.23. Hence $\mathcal{F}$ has (Bk1), (Bk2) and (Nf3), each with a threshold that is uniform over its members and words. The realisation estimates also have a uniform threshold, say Z_{R}. Choose Z at least X', $Y+1$, Z_{R} and the thresholds for those three conclusions. At every scale $x \geq Z$, all the members indexed by A_x are members of $\mathcal{F}$, and all subsequent deductions concern these members.

Five further lower conditions follow from the common data of these members.

- By (Ml1), at every finite place w the weight is $\ell(w) = e_w f_w \log p(w)/[K : \mathbb{Q}] \geq 0$, since e_w, f_w and $[K : \mathbb{Q}]$ are positive and $p(w) \geq 2$. This gives (Mf1) at all finite places.

- Put $y = e_w \mathrm{tw}(\mathfrak{V}, \omega, w)$. The floor inequalities and $e_w \geq 1$ give

  $$\frac{\lfloor y \rfloor}{e_w} \leq \mathrm{tw}(\mathfrak{V}, \omega, w) < \frac{\lfloor y \rfloor}{e_w} + \frac{1}{e_w} \leq \frac{\lfloor y \rfloor}{e_w} + 1.$$

  By (Ml3), the fraction is the specified rounded twist. If the twist is zero, its floor is zero. This proves (Mf5), for either sign of the twist.

- The divisor lattice N_ω has positive rank by (Ml2). It is one of its own nonzero submodules, so $\widehat{\mu}(N_\omega) \leq \widehat{\mu}_{\max}(N_\omega)$. The frame slope identities therefore give $d(\mathfrak{V}, \omega) > 0$ and $\mu(\mathfrak{V}, \omega) \leq \mu^{\max}(\mathfrak{V}, \omega)$, which is (Dv1).

- The host and Lemma 19.28 give the analytic loss

  $$L_a = \log(a \ \mathrm{rd}_{E_{\mathfrak{h}_x}}) = \log(2a\, v(\mathfrak{h}_x)) \geq x^{\varepsilon} > 0,$$

  which gives (Dv2).

- By (Ri2), $0 \leq \delta(\mathfrak{V}) \leq 1/3 \leq 1/2$, which gives (Nf1).

Besides the constants already supplied, these deductions use only the common host threshold. The other numbered properties of (Pl3) are supplied by (Ra3), with the same twenty-four common parameters. They remain true after restriction to $x \geq Z$, with all their inner quantifiers and antecedents unchanged.

It remains to check the residual requirements. The bound $\varepsilon \leq 1$ follows from $\varepsilon < 1/52$, and A_x is nonempty because it is a group. Rank sizes give

$$m \geq \sqrt{x\, x^{-\varepsilon/8}} = x^{1/2-\varepsilon/16} \geq x^{1/2-\varepsilon}.$$

The top cardinality bound is (Hg3). For the thick cardinality bound put $n_0 = |\langle a_0\rangle|$ and let L_w be the positive integer in the good-host requirement. Its displayed formula gives $L_w \geq 1$, and every $s \in S_x$ satisfies $0 \leq s < n_0/(4L_w) \leq n_0$. If $sa_0 = s'a_0$ for two such integers, then n_0 divides $s - s'$ while $|s - s'| < n_0$, so $s = s'$. Thus

$$|T_x^{\mathrm{th}}| = |\{sa_0 : s \in S_x\}| = |S_x| \geq x^{-\varepsilon}|A_x|$$

by (Hg3). The integer height box gives, for every top s,

$$a_s \geq \lceil a \rceil \geq a = xb \geq (x-1)b.$$

The word-count bound is supplied explicitly in (Ra3). The analytic conclusion of Lemma 19.28, at the same host and scale, gives

$$\log(2a\, v(\mathfrak{h}_x)) \leq 2x^{\varepsilon}.$$

Therefore all the residual inequalities of (Pl3) hold.

Restrict the completion and the packet system to the tail $x \in W$, $x \geq Z$. All numbered formulas, residual bounds and common scalar parameters required by (Pl3) hold there. Its separate kernel requirement is supplied by (Ra3): the data with (Kc1)–(Kc4), the finite measure and both scalar inequalities remain valid on this tail, with all inner quantifiers and antecedents and with the same common constants. No further constant is needed. This is the uniform lower completion required by (Js3), and its loss data, place records, slope records, numerical choices and orbit masses are those of the packets above. Taking X' as the cutoff in the statement proves all its assertions.

□

Definition 19.4. Let $\mathcal{F}$ be a parametrised host family with slope records, modification data and frame realisations. **Simultaneous lattice realisations for** $\mathcal{F}$ consist of the following data and identities for every $x > x_1$ and every $i \in I(x)$. In this definition and in the results that refer to it, the member (x, i) is fixed and its index is suppressed from these data; we write $\mathfrak{V} = \mathfrak{V}_{x,i}$.

(Ml1) One coefficient field and its places. The field K of the frame realisation is the field K of the place record, and every finite place in the records is a finite place of this field. For such a place w, let $\mathfrak{p}_w$ be its prime ideal and let e_w and f_w be its ramification index and residue degree over $p(w)$. Then

$$\ell(w) = \frac{e_w f_w \log p(w)}{[K : \mathbb{Q}]}.$$

(Ml2) The ambient and level lattices. For every word $\omega \in \mathcal{W}(\mathfrak{V})$ there is a hermitian $\mathcal{O}_K$-lattice $\overline{M}_\omega$ of rank $d^\circ(\mathfrak{V}, \omega)$ and an injective K-linear map

$$\iota_{\mathrm{lev},\omega} \colon N_{\omega,K} \longrightarrow M_{\omega,K}.$$

The divisor lattice has positive rank, and its archimedean norms are the pullbacks under this map:

$$d(\mathfrak{V}, \omega) \geq 1, \qquad \|z\|_{\overline{N}_\omega,\sigma} = \|\iota_{\mathrm{lev},\omega,\sigma}(z)\|_{\overline{M}_\omega,\sigma} \quad (\sigma \colon K \hookrightarrow \mathbb{C},\ z \in N_{\omega,K} \otimes_{K,\sigma} \mathbb{C}).$$

Define the level lattice, in the space $N_{\omega,K}$, by

$$M_\omega^{\mathrm{lev}} = \iota_{\mathrm{lev},\omega}^{-1}(M_\omega)$$

with the same pullback norms. The following projection identities hold:

$$\mu'(\mathfrak{V},\omega)=2\widehat{\mu}(\overline{M}_\omega),\qquad \mu'^{\max}(\mathfrak{V},\omega)=2\widehat{\mu}_{\max}(\overline{M}_\omega),$$
$$\mu^\circ(\mathfrak{V},\omega)=2\widehat{\mu}(\overline{M}^{\mathrm{lev}}_\omega),\quad \mu^{\circ\max}(\mathfrak{V},\omega)=2\widehat{\mu}_{\max}(\overline{M}^{\mathrm{lev}}_\omega).$$

The divisor lattices, their norms, the maps B_ω, the dual slopes and the complex frame data are those of the given frame realisation, together with its slope identities.

(Ml3) The intermediate lattice and ideal multiple. For every word ω there is a finitely generated full-rank $\mathcal{O}_K$-submodule N'_ω of $N_{\omega,K}$, with the same archimedean norms. Put

$$\mathfrak{a}_\omega=\prod_{w\in S}\mathfrak{p}_w^{-\lfloor e_w\mathrm{tw}(\mathfrak{V},\omega,w)\rfloor}.$$

Then

$$N_\omega=\mathfrak{a}_\omega N'_\omega,\qquad \mathrm{tw}^\flat(\mathfrak{V},\omega,w)=\frac{\lfloor e_w\mathrm{tw}(\mathfrak{V},\omega,w)\rfloor}{e_w}\quad(w\in S).$$

Here $\mathfrak{a}_\omega N'_\omega$ means the $\mathcal{O}_K$-span of all products of an element of $\mathfrak{a}_\omega$ with an element of N'_ω; the equality is in the common space $N_{\omega,K}$.

(Ml4) The local quotients. Let $\mathcal{O}_{K,w}$ denote the localisation of $\mathcal{O}_K$ at $\mathfrak{p}_w$, and write a subscript w for localisation of a lattice. For every finite place w with $p(w)\neq T(\mathfrak{V},\omega)$,

$$N'_{\omega,w}\subseteq M^{\mathrm{lev}}_{\omega,w},$$

with equality if moreover $p(w)$ is dropped or is not inert. At every $w\in S$, including the places above $T(\mathfrak{V},\omega)$,

$$\mathrm{len}(\mathfrak{V},\omega,w)=\frac{1}{e_w}\,\mathrm{length}_{\mathcal{O}_{K,w}}\left(\frac{M^{\mathrm{lev}}_{\omega,w}}{N'_{\omega,w}\cap M^{\mathrm{lev}}_{\omega,w}}\right).$$

The intersection and quotient are taken in $N_{\omega,K}$. Both lattices have full rank there, so this quotient is a finite-length module.

Definition 19.5. Frames and normalised log-covolumes. Let K be a totally complex number field and fix, for every infinite place v of K, one of its two complex embeddings, also written $v\colon K\to\mathbb{C}$. Let $d\geq 0$ be an integer. A **frame** on K^d is a family $X=(X(v))_v$ of matrices $X(v)\in\mathrm{GL}_d(\mathbb{C})$, one for each infinite place v; it defines the norms $\|y\|_{X,v}=|X(v)\,v(y)|$ on K^d, where $v(y)$ is the coordinatewise image of y and $|\cdot|$ the standard hermitian norm of $\mathbb{C}^d$. Thus the frame matrix is applied directly: if hermitian norms on K^d are given as $y\mapsto|Y(v)^{-1}v(y)|$ by matrices $Y(v)\in\mathrm{GL}_d(\mathbb{C})$, they are the norms of the frame $X=(Y(v)^{-1})_v$; only the induced norms enter ht and lcv below. Let $M\subseteq K^d$ be a finitely generated $\mathcal{O}_K$-submodule and $r=\dim_K(KM)$ its rank. The map $y\mapsto(X(v)\,v(y))_v$ sends M onto a lattice of rank $r[K:\mathbb{Q}]$ in its real span inside $\bigoplus_v\mathbb{C}^d$, which carries the euclidean structure $\mathrm{Re}\sum_v\langle\cdot,\cdot\rangle_{\mathbb{C}^d}$; let $\mathrm{covol}_X(M)$ be its covolume ($=1$ if $M=0$). The **height** and the **normalised log-covolume** of (M,X) are

$$\mathrm{ht}(M,X)=2^{r[K:\mathbb{Q}]/2}\,|\mathrm{disc}(K)|^{-r/2}\,\mathrm{covol}_X(M),\qquad \mathrm{lcv}(M,X)=\frac{2}{[K:\mathbb{Q}]}\log\mathrm{ht}(M,X).\quad(19.1)$$

Covolume records. Let $\mathcal{F}$ be a parametrised host family with parameter ε and threshold x_1. A **parametrised host family with covolume records** is such a family $\mathcal{F}$ together with the following data. For every real $x>x_1$ and every $i\in I(x)$ there is a set $\mathrm{Pt}_{x,i}$, the **realised points** of the member (x,i), and for every $\mathfrak{z}\in\mathrm{Pt}_{x,i}$ there is a **covolume record**, consisting of

- a totally complex number field $K_{\mathfrak{z}}$, with a fixed embedding for each of its infinite places, and an integer $D_{\mathfrak{z}} \geq 1$, the **realisation dimension**;
- for every $j \in \{1, \dots, m\}$, where m is the parameter of $\mathfrak{M}_{x,i}$ (the **lifts**): an integer $d_{\mathfrak{z},j} \geq 0$, a frame $X_{\mathfrak{z},j}$ on $K_{\mathfrak{z}}^{d_{\mathfrak{z},j}}$ and a finitely generated $\mathcal{O}_{K_{\mathfrak{z}}}$-submodule $M_{\mathfrak{z},j} \subseteq K_{\mathfrak{z}}^{d_{\mathfrak{z},j}}$, the **realisation lattice of the lift** j;
- a finite set $\mathrm{RP}_{\mathfrak{z}}$, the **realisation places**, and real numbers $w^{\mathrm{re}}_{\mathfrak{z},j}(\rho)$ for $\rho \in \mathrm{RP}_{\mathfrak{z}}$ and $1 \leq j \leq m$, the **realisation weights**.

The realised points are the points of the construction of the member that satisfy its point conditions and its arithmetic ties. The results based on this definition use only the data listed here.

Main term and unit error. For $x > x_1$, $i \in I(x)$, $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and $1 \leq j \leq m$, with k, m the parameters of $\mathfrak{M}_{x,i}$, put

$$\mathrm{mt}_j(\mathfrak{z}) = \sum_{\rho \in \mathrm{RP}_{\mathfrak{z}}} (k - m + 2j - 1)\, w^{\mathrm{re}}_{\mathfrak{z},j}(\rho)\, \log \Omega(\mathfrak{h}_{x,i}), \qquad \mathrm{ue}(x,i) = B_0(x,i)\, \log\log x, \tag{19.2}$$

the **main term** of the lift j at $\mathfrak{z}$ and the **unit error** of the member.

The covolume bound. The family has the **covolume bound** if there are real numbers $C_0 \geq 0$ and x_0 such that for every real $x \geq x_0$ with $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$, every real $C \geq C_0$ and every $j \in \{1, \dots, m\}$

$$\frac{\mathrm{lcv}(M_{\mathfrak{z},j}, X_{\mathfrak{z},j})}{D_{\mathfrak{z}}} \leq \mathrm{mt}_j(\mathfrak{z}) + C\, \mathrm{ue}(x,i). \tag{19.3}$$

Definition 19.6. Let $\mathcal{F}$ be a parametrised host family with slope records, modification data, defect data, bridge records, frame realisations, simultaneous lattice realisations and covolume records, all built on the same underlying objects wherever they overlap. **Base covolume realisations for** $\mathcal{F}$ consist, for every real $x > x_1$ and every $i \in I(x)$, of the following data. In these clauses (x,i) is fixed, and K, $\overline{R}$, Φ, m and w_j denote the coefficient field, level realisation lattice, place set, number of lifts and loss weights of the member (x,i).

1. A nonzero hermitian $\mathcal{O}_K$-lattice $\overline{R}_{\mathrm{b}}$, the **base realisation lattice**, and a point $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ with coefficient field K, the **selected point**, such that

   $$K_{\mathfrak{z}} = K, \qquad D_{\mathfrak{z}} = \mathrm{rk}_{\mathcal{O}_K} R_{\mathrm{b}} = \mathrm{rk}_0(x,i) \geq 1, \qquad \mu^{\vee}_{\mathrm{b}}(x,i) = 2\widehat{\mu}(\overline{R}^{\vee}_{\mathrm{b}}).$$

   The reference lattices attached to $\mathfrak{z}$, and their archimedean frames, are those of the covolume record of $\mathfrak{z}$.

2. A bijection $\beta_{\mathfrak{z}} \colon \mathrm{RP}_{\mathfrak{z}} \to \Phi$ such that

   $$w^{\mathrm{re}}_{\mathfrak{z},j}(\rho) = w_j(\beta_{\mathfrak{z}}(\rho)) \qquad (1 \leq j \leq m,\ \rho \in \mathrm{RP}_{\mathfrak{z}}).$$

In these data the level realisation is the lattice $\overline{R}$, and the divisor lattices, intermediate lattices, fractional ideals and maps B_ω are those of the given simultaneous and frame realisations of the member. A parametrised host family **with base covolume realisations** is such a family together with such data.

Realisation estimates. Let $C \geq 0$ be a real number. The family **has realisation estimates for** C if there is a real number $X > 1$ such that the following conditions hold for every real $x \geq X$ with $x > x_1$ and every $i \in I(x)$. In these conditions the index (x,i) is suppressed where convenient: $\mathfrak{V} = \mathfrak{V}_{x,i}$, $\mathfrak{z}$ is the selected point, $D = D_{\mathfrak{z}} = \mathrm{rk}_0(x,i)$ and $\delta = \delta(\mathfrak{V})$.

(Ri1) Reference comparison and numerical budget. For every word ω there is a real number E_ω such that

$$\mu_{\mathrm{b}}^{\vee} - \mathrm{sc} \le \sum_{j=1}^{m} \frac{\mathrm{lcv}(M_{\mathfrak{z},j}, X_{\mathfrak{z},j})}{D} + F(\mathfrak{V}) + E_\omega,$$

$$\mathrm{lm}(x,i,\omega) = \sum_{j=1}^{m} \sum_{\tau \in \Phi} (k - m + 2j - 1) w_j(\tau) \log \Omega(\mathfrak{h}_{x,i}),$$

and

$$E_\omega + mC\,\mathrm{ue}(x,i) \le \mathrm{le}(x,i,\omega) + \sum_{j=1}^{m} \left(\frac{\mathrm{Df}_j(x,i)}{D} + \mathrm{tc}_j(x,i) \right).$$

(Ri2) Passage. One has $0 \le \delta \le 1/3$, and there are real numbers H_0, L_{grp} such that

$$\mu^{\vee}(\mathfrak{V}) \le \mu_{\mathrm{b}}^{\vee} + 3\delta(\mu_{\mathrm{b}}^{\vee} + H_0 + 28L_{\mathrm{grp}}) + 28L_{\mathrm{grp}}, \qquad 3\delta(\mu_{\mathrm{b}}^{\vee} + H_0 + 28L_{\mathrm{grp}}) + 28L_{\mathrm{grp}} \le \mathrm{pa}(x,i).$$

(Ri3) Valuations before the ideal twist. The level realisation lattice R is nonzero. For a word ω and a finite place w of K, put

$$a_{\omega,w} = \begin{cases} \lfloor e_w \,\mathrm{tw}(\mathfrak{V}, \omega, w) \rfloor, & w \in S, \\ 0, & w \notin S, \end{cases}$$

and let $v_w \colon K^{\times} \to \mathbb{Z}$ be the discrete valuation at w that takes the value 1 on a local uniformizer. Then for every word ω, every finite place w, every $n \in N'_{\omega,w}$ and every $r \in R_w$,

$$B_\omega(n)(r) = 0 \quad \text{or} \quad v_w(B_\omega(n)(r)) \ge a_{\omega,w}.$$

(Ri4) Archimedean norms and the frame quotient. For every word ω with $B_\omega \neq 0$ there are real numbers $q_{\omega,\sigma}$, one for each embedding $\sigma \colon K \hookrightarrow \mathbb{C}$, such that

$$\log \|B_{\omega,\sigma}\|^2 \le q_{\omega,\sigma}, \qquad \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma} q_{\omega,\sigma} \le h(\mathfrak{V}, \omega),$$

where the norms are operator norms for the given divisor and dual-realisation metrics. Moreover, the frame quotient satisfies

$$\dim_{\mathbb{C}} \ker \Pi \le \delta \, \dim_{\mathbb{C}} \mathcal{H}.$$

Definition 19.7. Hermitian spaces. Let K be a totally complex number field with a fixed embedding $v \colon K \to \mathbb{C}$ for each infinite place. A **hermitian space** over K is a finite-dimensional K-vector space V with a hermitian norm $\|\cdot\|_v$ on $V \otimes_{K,v} \mathbb{C}$ for every infinite place v. A finitely generated $\mathcal{O}_K$-submodule $L \subseteq V$ is torsion free, hence projective, and with the restricted norms it is a hermitian lattice; its rank is $\mathrm{rk}\, L = \dim_K(KL)$. For $L \neq 0$ we write $\widehat{\mu}(L)$ for its normalised slope and $\widehat{\mu}_{\max}(L)$ for the maximum of $\widehat{\mu}(L')$ over the non-zero $\mathcal{O}_K$-submodules $L' \subseteq L$. An **orthogonal projection** of V is a K-linear map $\pi \colon V \to V$ with $\pi \circ \pi = \pi$ such that $\pi(V)$ and $\ker \pi$ are orthogonal for every $\|\cdot\|_v$, that is $\|\pi y + (y' - \pi y')\|_v^2 = \|\pi y\|_v^2 + \|y' - \pi y'\|_v^2$ for all $y, y' \in V \otimes_{K,v} \mathbb{C}$.

Chain data. A parametrised host family with covolume records **carries chain data** if for every real $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and every lift $j \in \{1, \dots, m\}$ the following data are given, where $K = K_{\mathfrak{z}}$:

- a hermitian space $V_{\mathfrak{z},j}$ over K and a finitely generated $\mathcal{O}_K$-submodule $L_{\mathfrak{z},j} \subseteq V_{\mathfrak{z},j}$, the **type lattice**;
- an orthogonal projection $\pi_{\mathfrak{z},j}$ of $V_{\mathfrak{z},j}$ and an integer $g_{\mathfrak{z},j} \geq 1$, the **denominator**, with $g_{\mathfrak{z},j}\,\pi_{\mathfrak{z},j}(L_{\mathfrak{z},j}) \subseteq L_{\mathfrak{z},j}$; the **good part** and the **bad part** of the type lattice are
$$L^{\mathrm{g}}_{\mathfrak{z},j} = L_{\mathfrak{z},j} \cap \pi_{\mathfrak{z},j}(V_{\mathfrak{z},j}), \qquad L^{\mathrm{b}}_{\mathfrak{z},j} = L_{\mathfrak{z},j} \cap \ker \pi_{\mathfrak{z},j};$$
- a non-zero hermitian lattice $L^{\mathrm{c}}_{\mathfrak{z},j}$ over K, the **core lattice**.

Properties. "For all large points" means: there is a real x_0 such that the statement holds for every real $x \geq x_0$ with $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and every $j \in \{1, \dots, m\}$. A family carrying chain data has

(Lc1) Recursion if for all large points $L^{\mathrm{g}}_{\mathfrak{z},j} \neq 0$ and
$$\frac{\mathrm{lcv}(M_{\mathfrak{z},j}, X_{\mathfrak{z},j})}{D_{\mathfrak{z}}} \leq -2\,\widehat{\mu}(L^{\mathrm{g}}_{\mathfrak{z},j});$$

(Lc2) Small denominators if for all large points $2 \log g_{\mathfrak{z},j} \leq b(x)$;

(Lc3) Small bad parts if there is a real $c > 0$ such that for all large points
$$\mathrm{rk}\, L^{\mathrm{b}}_{\mathfrak{z},j} \leq 3\, e^{-x^c}\, \mathrm{rk}\, L_{\mathfrak{z},j};$$

(Lc4) Maximal slopes if there is a real A such that for all large points $2\,\widehat{\mu}_{\max}(L_{\mathfrak{z},j}) \leq x^A\, b(x)$;

(Lc5) Size of the main terms if there is a real A such that for all large points $|\mathrm{mt}_j(\mathfrak{z})| \leq x^A\, b(x)$;

(Lc6) Core comparison if there is a real $C \geq 0$ such that for all large points $2\,\widehat{\mu}(L_{\mathfrak{z},j}) \geq 2\,\widehat{\mu}(L^{\mathrm{c}}_{\mathfrak{z},j}) - C\,\mathrm{ue}(x,i)$;

(Lc7) Core lower bound if there is a real $C \geq 0$ such that for all large points $2\,\widehat{\mu}(L^{\mathrm{c}}_{\mathfrak{z},j}) \geq -\mathrm{mt}_j(\mathfrak{z}) - C\,\mathrm{ue}(x,i)$.

In (Lc4) and (Lc6) the slopes of $L_{\mathfrak{z},j}$ are taken where $L_{\mathfrak{z},j} \neq 0$; by (Lc1) this holds at all large points.

Definition 19.8. A parametrised host family with covolume records carrying chain data **carries deep-cover data** if for every real $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and every lift j with $1 \leq j < m$ the following data are given:

- a totally complex number field $K' = K'_{\mathfrak{z},j}$, with a fixed embedding for each infinite place, and two hermitian spaces $V^-_{\mathfrak{z},j}$, $V^+_{\mathfrak{z},j}$ over K', with finitely generated $\mathcal{O}_{K'}$-submodules $A_{\mathfrak{z},j} \subseteq V^-_{\mathfrak{z},j}$, the **lowered lattice**, and $B_{\mathfrak{z},j} \subseteq V^+_{\mathfrak{z},j}$, the **deep lattice**;
- an integer $n_{\mathfrak{z},j} \geq 0$ and K'-linear maps $\phi_{\mathfrak{z},j,h} \colon V^-_{\mathfrak{z},j} \to V^+_{\mathfrak{z},j}$ for $1 \leq h \leq n_{\mathfrak{z},j}$, the **descent maps**;
- a real number $\Lambda_{\mathfrak{z},j} > 0$, the **generator size**, and a real number $\sigma_{\mathfrak{z},j}$, the **twisted slope**.

A family carrying deep-cover data has the following properties if the stated condition holds for all large points, where for each of (Dp0), (Dp5) and (Dp6) there is a real constant $C \geq 0$ of its own, chosen before the large points, at the lift $j = m$ for (Dp0) and at every lift $j < m$ for (Dp1)–(Dp6). We write $A = A_{\mathfrak{z},j}$, $B = B_{\mathfrak{z},j}$, $\phi_h = \phi_{\mathfrak{z},j,h}$, $n = n_{\mathfrak{z},j}$, $\Lambda = \Lambda_{\mathfrak{z},j}$, $\sigma = \sigma_{\mathfrak{z},j}$ and $b = b(x)$.

(Dp0) Top lift $2\,\widehat{\mu}(L^{\mathrm{c}}_{\mathfrak{z},m}) \geq -\mathrm{mt}_m(\mathfrak{z}) - C\,\mathrm{ue}(x,i)$.

(Dp1) Sizes $b \leq \Lambda$ and $|\mathrm{mt}_j(\mathfrak{z})| \leq \Lambda$.

(Dp2) Descent maps $A \neq 0$, and $\phi_h(A) \subseteq B$ for every h. If $y \in V^-_{\mathfrak{z},j}$ and $\phi_h(y) = 0$ for all h, then $y = 0$. For every infinite place v of K' and every $y \in V^-_{\mathfrak{z},j} \otimes_{K',v} \mathbb{C}$,

$$\sum_{h=1}^{n} \|\phi_h(y)\|_v^2 \leq e^{2b}\|y\|_v^2.$$

Moreover, $n \operatorname{rk} B \leq (1 + b/\Lambda) \operatorname{rk} A$.

(Dp3) Short generators There is a set $s \subseteq B$ that spans $K'B$ over K' and satisfies $\|y\|_v \leq e^{\Lambda}$ for every $y \in s$ and every infinite place v.

(Dp4) Ring comparison $2\,\widehat{\mu}(B) - 2b \leq 2\,\widehat{\mu}(L^{\mathrm{c}}_{\mathfrak{z},j})$.

(Dp5) Flat bound $2\,\widehat{\mu}(A) \geq \sigma - \mathrm{mt}_j(\mathfrak{z}) - C\,\mathrm{ue}(x,i)$.

(Dp6) Uniform lower bound $\sigma \geq -C\,\mathrm{ue}(x,i)$.

Condition (Dp4) includes the requirement $B \neq 0$, which follows from (Dp2) when $n \geq 1$.

Proposition 19.9. *Let $\lambda > 1$ and let P be a host requirement supplied by the concrete good-host requirement $\mathfrak{G}$ on λ-windows. There is a real $\varepsilon_\dagger = \varepsilon_\dagger(\lambda, \mathsf{P}) > 0$ with the following property. Fix*

$$0 < \varepsilon < \min(\varepsilon_\dagger, 1/52), \quad J \geq 1, \quad \kappa = (C_{T_2}, C_8, C_5, c_6, C_b),$$

where $C_8 \geq 1$ and $C_5, c_6, C_b > 0$, a set W containing λ-windows, and $X_H > 1$. For every $x \in W$ with $x \geq X_H$, fix a frame, a host $\mathfrak{h}_x$, tops T_x, a rank t_x with $\mathsf{P}(x, \varepsilon, \mathfrak{h}_x, T_x, t_x)$, and a choice of all the data whose existence is asserted by the concrete good-host requirement $\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}_x, T_x, t_x)$. These choices, including the rank data, orderings, good-orbit choices, spacing sets and auxiliary primes, are fixed once and for all.

Then there are constants

$$c_D > 397, \quad c_h \geq 402, \quad C_e \geq 0, \quad \varepsilon_z > 0, \quad C_t, \quad X \geq \max\{X_H, 2, (2C_8)^{4/\varepsilon}\},$$

a tuple $\mathfrak{k}_{\mathrm{orb}}$ of eight nonnegative real numbers, and, for every $x \in W$ with $x \geq X$, a base translate packet P_x over the chosen frame, host, tops, spacing set, rank and auxiliary primes, with the following properties.

(Ra1) Loss data. *Every packet member satisfies (Lm2), (Lm5), (Lm6), and (Lm7) for the common pair (ε_z, C_t). Its loss parameter e_2 equals the exponent e of its frame.*

(Ra2) Numerical data. *At every packet member, the loss datum has the rank datum fixed above, with its rank sizes at (x, ε, J) and with $t_R = t_x$. Its heights, depths, weights and shifts have the integer weights and boxes for (c_D, c_h), for some number field and thick subset. Its tops, Frobenius translations, orbit representatives and good representatives are given by the finite good-orbit choices fixed above, and these are the choices used by the loss datum. Its correction degrees, orbit thresholds and correction errors satisfy (Nc2)–(Nc4) for the common pair $(\mathfrak{k}_{\mathrm{orb}}, C_e)$.*

(Ra3) Joint realisation under an eventual Sha bound. *Suppose that there is a real number X_V such that the Sha bound*

$$\mathrm{V}(E_{\mathfrak{h}_x}; t_x, \exp(x^{\varepsilon}/2))$$

holds for all $x \in W$ with $x \geq \max(X, X_V)$. Then there are the twenty-four common scalar constants specified in (Pl3), a real $Y \geq X$, and compatible lower completion data (Pl1) for these packets at every $x \in W$ with $x \geq Y$, with the following properties.

- *All the numbered formulas of (Pl3) hold at those scales with those constants, in the pointwise interpretation (Pl2), except (Mf1), (Mf5), (Dv1), (Dv2), (Bk1), (Bk2), (Nf1) and (Nf3).*
- *The kernel hypotheses of (Pc5), namely its data with (Kc1)–(Kc4), finite total measure and both displayed scalar inequalities, hold there in the pointwise interpretation (Pl2), with the same constants and data.*
- *Every packet member there satisfies*

$$\log |\mathcal{W}(\mathfrak{V})| \leq (k-3)\log 2.$$

Let $\mathcal{F}$ be the family with scales W, threshold Y, and index sets

$$I(x) = \begin{cases} A_x, & x \in W,\ x > Y, \\ \varnothing, & \text{otherwise}, \end{cases}$$

whose members at the nonempty index sets are the packet members with their lower completion data. At empty index sets, take any fixed frame. This family admits base covolume realisations, chain data with (Lc1)–(Lc6), and deep-cover data with (Dp0)–(Dp6), all compatible with the packets and completions above. There are constants $C_6, C_{\mathrm{top}}, C_{\mathrm{F}}, C_{\mathrm{U}} \geq 0$ with which (Lc6), (Dp0), (Dp5), (Dp6) hold, respectively, and a constant

$$C \geq C_6 + \max\{C_{\mathrm{top}}, 2(C_{\mathrm{F}} + C_{\mathrm{U}}) + 9\} + 1$$

such that $\mathcal{F}$ has realisation estimates for C.

A field, lattice, metric, map or scalar that occurs in several of these records is the same in all of them. The twenty-four common constants and the additional covolume constants do not depend on the scale, the translate, the word or any other local index. The loss data, place records, slope records, numerical choices and orbit masses are those of the packets above.

All the constants, cutoffs and packets asserted to exist may depend on the whole collection of choices fixed at the start.

Proof. Apply Proposition 21.2 for the same λ and P, and take its $\varepsilon_\dagger(\lambda, \mathsf{P})$ as $\varepsilon_\dagger$. Apply it to the data fixed in the statement, which satisfy its hypotheses, and take its constants and packets. Conditions (Ma1),(Ma2) are (Ra1),(Ra2), and the packets are chosen before the eventual Sha hypothesis is assumed.

Now assume that hypothesis, and take the common constants, the twenty-four common parameters and the compatible data supplied there. We show that these packets and data satisfy (Ra3) after one restriction to a common tail.

Fix a realised point and a lift $1 \le j < m$. Put

$$d = d_{\mathrm{flag}} = f\binom{r}{2}, \quad p = k + j - 1, \quad q = m - j, \quad Z = f^2 r^4.$$

The rank data give $f \ge 1$, $k \ge m + 3$, $m \ge 2$, $r = k + m - 1$, hence $r \ge 1$, $d \ge 1$ and $p + q = r$. In particular $pq \le r^2$, so

$$\binom{fpq+1}{2} \le \frac{fr^2(fr^2+1)}{2} \le f^2 r^4 = Z.$$

Also $r \le Z$, $fr \le Z$ and $Z \ge 1$. Thus

$$r + \binom{fpq+1}{2} + 2 \le 4Z, \qquad 8fr + 4 \le 12Z.$$

The maximum defining c_0 is an integer. Rounding it upwards to a multiple of 8 adds at most 7, and therefore

$$0 \le c_0 \le 12Z + 7 \le 19Z \le 22 f^2 r^4.$$

The length identity, $P \ge 40$ and $k/m \ge 1$ imply

$$T^{\mathrm{d}} = \left\lceil P^2 d^2 k/m \right\rceil \ge 1600 d^2 \ge 4(d+1).$$

For $x \ge 2$, rank sizes and $\varepsilon < 1/52$ give $k \le 2x^{9/2+10\varepsilon} \le 2x^5$. The common bounds $P, d \le x^{C_{\mathrm{prec}}}$ and $m \ge 1$ yield

$$T^{\mathrm{d}} \le 2x^{4C_{\mathrm{prec}}+5} + 1 \le x^{4C_{\mathrm{prec}}+7}.$$

The last inequality follows since $x^2 \ge 4$ and $x^{4C_{\mathrm{prec}}+5} \ge 1$. Set $C_T = 4C_{\mathrm{prec}} + 7$. This constant does not depend on the point or the lift.

Condition (Lm7), for the size parameter L_c, gives directly

$$0 < x^{\varepsilon} \le L_c \le x^{\varepsilon + \varepsilon/100}.$$

These are the size bounds used in Theorem 19.12. The same-host, CM-type, final-depth, numerator, anchored division, norm, spread, weight-bound, derivative and level identities in Theorem 19.12 are those supplied in (Ma3), with its $d = d_{\mathrm{flag}}$. Apply that theorem to the tail family. Outside W the index sets are empty, so its universal hypotheses there are vacuous. It gives (Ms2),(Ms4) with the constant 83400, beyond a cutoff that depends only on the constants already fixed.

On a further tail $x \ge \exp(\exp(1))$, the unit error is nonnegative. Since $B \ge 128$, $\mathrm{rd}_E \ge 1$ and $\mathrm{wn}, \mathrm{wsp} \ge 0$, we also have

$$\mathrm{lz}\,\mathrm{wn} + \mathrm{wsp} \ge 0.$$

Increasing the constants in (Ms3),(Ms4) to $C_{\mathrm{mod}} = \max\{C_{\mathrm{ms}}, 83400\}$ therefore preserves both inequalities. Conditions (Ms1),(Ms2) have no constant. Apply Theorem 19.18 to these component and model data, using the identity for e^{ms} supplied above. It gives (Ht3) with the constant $C_{\mathrm{mod}}(C_{\mathrm{mod}} + 120)$.

Increase the remaining component constant to

$$C_{\mathrm{H}} = \max\{C_{\mathrm{ht}}, C_{\mathrm{mod}}(C_{\mathrm{mod}} + 120)\}.$$

The budget inequalities are preserved since the unit error is nonnegative, and (Ht1),(Ht5) do not involve the constant. All of (Ht1)–(Ht5) now hold with C_{H}. By Theorem 19.20, the twisted slope satisfies (Dp6) with

$$C_{\mathrm{U}} = 8C_{\mathrm{H}} + 4.$$

Let Y' be the maximum of Y, the attachment cutoff, $\exp(\exp(1))$ and the cutoffs of the model and component consequences above. Restrict the family and its data to $x > Y'$. Each packet member, inner datum, shared object and common constant restricts without change. At empty index sets one may use the fixed frame allowed in the statement. The numbered lower formulas also hold at every $x \in W$, $x \geq Y'$, since they already hold at all scales at least Y. The kernel data with (Kc1)–(Kc4), the finite measure and both scalar inequalities, supplied in (Ma3), remain valid after this restriction, with all their inner quantifiers and antecedents and the same common constants. No additional constant is needed.

Conditions (Lc1)–(Lc6),(Dp0)–(Dp5) remain true, and (Dp6) has just been proved. The constant C supplied above already satisfies

$$C \geq C_6 + \max\{C_{\mathrm{top}}, 2(C_{\mathrm{F}} + C_{\mathrm{U}}) + 9\} + 1$$

for this C_{U}, and the family has realisation estimates for it, so C is not enlarged. Hence (Ra3) holds on the new tail, including its compatibility requirements and the requirement that the packet data are those of the packets above. Together with (Ra1),(Ra2), which come from Proposition 21.2, this proves the proposition.

□

Definition 19.10. A family carrying deep-cover data **carries component data** if for every real $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and every lift $j < m$ the following data are given; the index $(\mathfrak{z}, j)$ is suppressed below.

- A finite non-empty set $\mathcal{Z}$, the **components**.
- For every $Z \in \mathcal{Z}$, real numbers $\mathrm{rc}(Z)$ and $\mathrm{gc}(Z)$, the rank and the degree of the lattice of sections of the twisted weight on Z, as well as $\mathrm{dg}(Z) > 0$, the **covering degree**, $\mathrm{hc}(Z)$, the **height** of Z, and $\mathrm{vs}(Z)$, the **volume slope** of Z at the divided weight.
- An integer $T^{\mathrm{d}} \geq 1$, the **division length**, and an integer $D^{\mathrm{fl}} \geq 0$, the **flag dimension**.
- Real numbers $N^{\mathrm{b}} > 0$, $\rho^{\mathrm{d}} \in [0, 1)$, $c^{\mathrm{s}}, c^{\mathrm{c}}, e^{\mathrm{ms}} \geq 0$, $a^{\mathrm{hs}} > 0$ and $r^{\mathrm{hs}} \geq 0$.

Put $\mathrm{dg}_{\min} = \min_Z \mathrm{dg}(Z)$.

The family has the following properties if there is a real number $C \geq 0$ such that the conditions below hold for all large points with $j < m$, where $E = \mathrm{ue}(x, i)$.

(Ht1) Degrees and shares $T^{\mathrm{d}} \geq 3D^{\mathrm{fl}} + 5$, and for every Z one has $\mathrm{dg}(Z) \leq \mathrm{dg}_{\min}/(1 - \rho^{\mathrm{d}})$ and $N^{\mathrm{b}} \mathrm{dg}(Z) \leq \mathrm{rc}(Z)$.

(Ht2) Heights from volume slopes For every Z one has $\mathrm{hc}(Z) + c^{\mathrm{s}} \geq 0$ and

$$(1 - \rho^{\mathrm{d}}) \max\bigl(0, \mathrm{vs}(Z) + c^{\mathrm{s}}\bigr) \leq \mathrm{hc}(Z) + c^{\mathrm{s}} + c^{\mathrm{c}};$$

moreover $T^{\mathrm{d}} c^{\mathrm{c}} \leq C\,E$ and $c^{\mathrm{s}}\bigl(6D^{\mathrm{fl}} + 10 + 2\rho^{\mathrm{d}} T^{\mathrm{d}}\bigr) \leq C\,E$.

(Ht3) Mean-slope bound at the divided weight $\frac{1}{|\mathcal{Z}|} \sum_Z \mathrm{vs}(Z) \geq -e^{\mathrm{ms}}$ and $T^{\mathrm{d}} e^{\mathrm{ms}} \leq C\,E$.

(Ht4) Effective Hilbert–Samuel inequality For every Z,

$$N^{\mathrm{b}}\mathrm{dg}(Z)(T^{\mathrm{d}}\mathrm{hc}(Z) - 3(D^{\mathrm{fl}}+1)|\mathrm{hc}(Z)| - a^{\mathrm{hs}}\log(2+|\mathrm{hc}(Z)|) - r^{\mathrm{hs}}) \le \mathrm{gc}(Z),$$

and $a^{\mathrm{hs}}\log\max(a^{\mathrm{hs}},1) + r^{\mathrm{hs}} \le C\,E$.

(Ht5) Sum bound $2\sum_Z \mathrm{gc}(Z)/\sum_Z \mathrm{rc}(Z) \le \sigma_{\mathfrak{z},j}$.

Definition 19.11. The Hodge–Tate cost. Let E be a number field, $M \ge 1$ an integer and $B > 0$ a real number. Let $P_{E,M}(B)$ be the set of primes p with $p \le B$ or $p \mid 2\,\mathrm{disc}(E)\,M$; it is finite. The **Hodge–Tate cost** is

$$c^{\mathrm{HT}}_{E,M}(B) = \sum_{p\in P_{E,M}(B)} \Big(\frac{1}{p-1} + \frac{40\,v_p(\mathrm{disc}(E))}{[E:\mathbb{Q}]} + \delta_{p,2}\Big)\log p, \tag{19.4}$$

where v_p is the p-adic valuation on $\mathbb{Z}$ and $\delta_{p,2}$ is 1 for $p = 2$ and 0 otherwise. The **root discriminant** of E is $\mathrm{rd}_E = |\mathrm{disc}(E)|^{1/[E:\mathbb{Q}]} \ge 1$.

Model-slope data. A family carrying deep-cover data and component data **carries model-slope data** if for every real $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and every lift $j < m$ the following data are given; the index $(\mathfrak{z}, j)$ is suppressed.

- A number field E, the **host field**, and an integer $M \ge 1$, the **level**.
- Real numbers $B \ge 128$, the **weight bound**, L, the **logarithmic derivative**, $\mathrm{wn} \ge 0$, the **weight norm**, and $\mathrm{wsp} \ge 0$, the **spread term**.
- For every component $Z \in \mathcal{Z}$, a real number $\mathrm{ns}(Z)$, the **model slope** of Z.

Put

$$\mathrm{lz} = \log(B\,\mathrm{rd}_E) + |L|, \tag{19.5}$$

the **logarithmic size**.

The family has the following properties if there is a real number $C \ge 0$ such that the conditions below hold for all large points with $j < m$.

(Ms1) Volume above model slope $\mathrm{vs}(Z) \ge \mathrm{ns}(Z) - 2\,c^{\mathrm{HT}}_{E,M}(B)\,\mathrm{wn}$ for every $Z \in \mathcal{Z}$.

(Ms2) Few bad primes The number of primes dividing $2\,\mathrm{disc}(E)\,M$ is at most $\lfloor B \rfloor$.

(Ms3) Mean model slope $\dfrac{1}{|\mathcal{Z}|}\displaystyle\sum_{Z\in\mathcal{Z}} \mathrm{ns}(Z) \ge -C\,(\mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp})$.

(Ms4) Budget at the divided weight $T^{\mathrm{d}}(\mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp}) \le C\,\mathrm{ue}(x,i)$.

Theorem 19.12. *Fix real constants*

$$0 < \epsilon < 1/52, \quad c_D \ge 0, \quad c_h \ge 402, \quad C_T \ge 0, \quad J \ge 1, \quad \kappa = (C_{T_2}, C_8, C_5, c_6, C_b),$$

where $C_8 \ge 1$ and $C_5, c_6, C_b > 0$. Let $\mathcal{F}$ be a parametrised host family of parameter ϵ carrying model-slope data. Suppose that there is $X_{\mathrm{att}} > 1$ such that, for every real $x \ge X_{\mathrm{att}}$ with $x > x_1$ and every $i \in I(x)$, the following hypotheses hold.

The member carries a choice of all the data whose existence is asserted by the concrete good-host requirement $\mathfrak{G}(x,\epsilon,J,\kappa,\mathfrak{h},T,t)$. Its loss datum has the rank datum of these data, with its rank sizes. Write

$$n=|A|,\quad f=2n,\quad r=k+m-1,\quad \theta=\epsilon/100,\quad \delta_1=x^{-\epsilon/8},$$

$$b=\exp(x^{\epsilon}),\quad b'=x\exp(x^{\epsilon}-x^{\epsilon/3}),\quad \Delta=\lceil x^{c_D}\rceil,\quad h=2\lceil x^{c_h}\rceil.$$

The loss datum has integer weights and boxes for the host field $E_{\mathfrak{h}}$, the constants (c_D,c_h), and a thick subset of Φ. Its final integer depths b_t and its parameter L_a satisfy

$$0\le L_a\le x^{\epsilon+\theta},\qquad b_t\le b'+\frac{2n\delta_1 x^{\epsilon}L_a b'}{k}\quad (t\in T).$$

For every realised point $\mathfrak{z}\in \mathrm{Pt}_{x,i}$ and every integer $1\le j<m$, the following ties hold. The model host field E equals $E_{\mathfrak{h}}$. There is a CM type $\mathcal{I}$ of E, choosing one complex embedding from each conjugate pair, and a bijection $\beta:\mathcal{I}\to\Phi$. Put $B_\tau=w_1(\beta(\tau))-h/2$ and $p=k-1+j$. Define

$$U_{\tau,a}=\begin{cases}(4\Delta-1)(k-2-a), & 0\le a<k-1,\\ -B_\tau-(l-1)h+l-r\mathbf{1}_{l>j}, & a=k+l-2,\quad 1\le l\le m.\end{cases}$$

There are a real $d\ge 0$ and an integer c_0 such that

$$0\le c_0\le 22f^2r^4,\qquad 4(d+1)\le T^{\mathrm{d}}\le x^{C_T}.$$

Put $D=T^{\mathrm{d}}+d+1$, let v be the unique integer array on $\mathcal{I}\times\{0,\dots,r-1\}$ determined by

$$v_{\tau,k-2}=0,\qquad v_{\tau,a}-v_{\tau,a+1}=\left\lfloor\frac{U_{\tau,a}-U_{\tau,a+1}+2-(c_0+2r)\mathbf{1}_{a+1=p}}{D}\right\rfloor\quad (0\le a<r-1),$$

and put $V_{\tau,a}=v_{\tau,a}+3r\mathbf{1}_{a\ge p}$. The numerical entries of the model satisfy

$$\begin{aligned}\mathrm{wn}&=\sum_{\tau\in\mathcal{I}}\sum_{a=0}^{r-1}|V_{\tau,a}|,\\ \mathrm{wsp}&=(m-j)\sum_{\tau\in\mathcal{I}}\Big(\max_a V_{\tau,a}-\min_a V_{\tau,a}\Big),\\ B&=4fr^2\left(1+\frac{\max_{\tau,a}|V_{\tau,a}|}{h/(4T^{\mathrm{d}})}\right),\\ L&=\mathrm{Re}\,\frac{L'(1,\chi_{-d_{\mathfrak{h}}})}{L(1,\chi_{-d_{\mathfrak{h}}})},\\ M&=g_1g_2\,15^{a_{\mathrm{lev}}},\end{aligned}$$

where a_{lev} is a nonnegative integer and g_1,g_2 are the auxiliary primes of the good-host data above. The choices $\mathcal{I},\beta,d,c_0,a_{\mathrm{lev}}$ may vary with the point and lift.

Then there exists X_0, depending only on $X_{\mathrm{att}},\epsilon,c_D,c_h,C_T,J,\kappa$, such that for every $x\ge X_0$ with $x>x_1$, every $i\in I(x)$, every $\mathfrak{z}\in\mathrm{Pt}_{x,i}$, and every integer $1\le j<m$, properties (Ms2),(Ms4) hold with the common constant $C=83400$. The theorem assumes the data and identities above; it does not assert that such model data exist.

Proof. Let X_{dw} be the threshold in Lemma 19.13 for $(\epsilon,c_D,c_h,C_T,6)$. Put

$$X_{\mathrm{host}}=\exp\!\left(1+\frac{6}{(4\epsilon/5)^3}\right),\quad X_{\mathrm{tor}}=128^{1/(2-16\epsilon)},\quad X_{\mathrm{top}}=2^{1/(5/2+19\epsilon/200)},\quad X_{\mathrm{slack}}=2^{100/\epsilon}.$$

Choose

$$X_0 = \max\left\{X_{\mathrm{att}}, X_{\mathrm{dw}}, X_{\mathrm{host}}, X_{\mathrm{tor}}, X_{\mathrm{top}}, X_{\mathrm{slack}}, \exp(\exp(1)), 10^{1/\epsilon}\right\}.$$

Fix a member, a point and a lift in the range of the statement, beyond this threshold. Rank sizes give

$$n \le 2x^{2+8\epsilon}, \qquad m \le 2\sqrt{x\delta_1} \le 2\sqrt{x}, \qquad k \le 2x^{9/2+10\epsilon}.$$

Condition (R2) of the rank datum gives $m \ge 2$ and $k \ge m+3$; hence also $r \ge 4$ and $m \le r$. Since $x^\theta \ge 2$, we have $n \le x^{2+8\epsilon+\theta}$. The lower rank-size bound also gives $k \ge x^{9/2+10\epsilon-\theta}$. The assumed bounds on the depths and Lemma 19.14 therefore imply

$$\lceil b' \rceil \le b_t \le 2b' \quad (t \in T)$$

for the final depths.

By Lemma 19.15, the torsion-unit group of the host field has order at most x^6. Pulling (Iw3) back along the given bijection β places every B_τ in the appropriate one of the three permitted intervals in Lemma 19.13: the top interval using the preceding depth bound, the thin singleton, or the thick interval with $C_\mu = 6$. By Lemma 19.16, E is CM of degree $4n$; therefore $|\mathcal{I}| = 2n = f$. All hypotheses of Lemma 19.13 now hold for the displayed arrays, with its $T = T^{\mathrm{d}}$. Its conclusions give the required raw and divided-weight bounds, spread inequality, and scalar and rank-size bounds, including $h \ge fr^2$.

Again Lemma 19.16 gives $Q_S \le n^2$, the discriminant-prime support containment in the primes dividing Q_S together with $q, d_{\mathfrak{h}}$, and

$$\log \mathrm{rd}_E + |L| \le 9x^\epsilon$$

for the real logarithmic derivative specified in the statement. The level identity implies that every prime divisor of M belongs to $\{g_1, g_2, 3, 5\}$; hence M has at most $4 \le 15$ prime divisors. We also have $x^\epsilon \ge 10$ and $\log\log x \ge 1$.

Apply Lemma 19.17, reindexing $a \in \{0, \dots, r-1\}$ as $a+1$, with

$$Q = Q_S, \quad q_{\mathrm{H}} = q, \quad d_{\mathrm{H}} = d_{\mathfrak{h}}, \quad S = 4k\Delta, \quad T = T^{\mathrm{d}}, \quad q_{\mathrm{s}} = m - j.$$

By the identities assumed in the statement, the quantities $\mathrm{wn}, \mathrm{wsp}, B, T$ in that lemma are those of the model. We obtain

$$\#\{p : p \text{ prime},\ p \mid 2\,\mathrm{disc}(E)\,M\} \le \lfloor B \rfloor$$

and

$$T^{\mathrm{d}}\left[(\log(B\mathrm{rd}_E) + |L|)\mathrm{wn} + \mathrm{wsp}\right] \le 83400\,2nmbx^\epsilon \log\log x = 83400\,\mathrm{ue}(x,i).$$

These are (Ms2) and (Ms4). No threshold depends on the member, the point, the lift or the integer a_{lev}, and the common constant is 83400. □

Lemma 19.13. *Fix real constants $0 < \epsilon < 1/52$, $c_D \ge 0$, $c_h > 12$, and $C_T, C_\mu \ge 0$. There exists $X > 1$, depending only on these constants, with the following property for every real $x \ge X$. Put*

$$b = \exp(x^\epsilon), \qquad b' = x\exp(x^\epsilon - x^{\epsilon/3}), \qquad \Delta = \lceil x^{c_D} \rceil, \qquad h = 2\lceil x^{c_h} \rceil.$$

Let n, k, m be integers satisfying

$$1 \le n \le 2x^{2+8\epsilon}, \qquad 2 \le m \le 2x^{1/2}, \qquad m+3 \le k \le 2x^{9/2+10\epsilon},$$

and put $f = 2n$, $r = k + m - 1$, $S_c = 4k\Delta$, and $\eta = mh$. Let I be a finite set of cardinality f. For each $\tau \in I$, let B_τ be a real number belonging to at least one of the following sets:

$$\{b_t + \lfloor b'/x \rfloor : \lceil b' \rceil \le b_t \le 2b'\}, \qquad \left\{ \left\lceil \frac{b'}{xn} \right\rceil \right\}, \qquad [\,\lceil b \rceil, \lceil b \rceil + x^{C_\mu}\,).$$

Let j be an integer with $1 \le j < m$, put $p = k - 1 + j$, and define

$$U_{\tau,a} = \begin{cases} (4\Delta - 1)(k - 2 - a), & 0 \le a < k - 1, \\ -B_\tau - (l-1)h + l - r\mathbf{1}_{l>j}, & a = k + l - 2, \quad 1 \le l \le m. \end{cases}$$

Let real numbers $d \ge 0$, $T > 0$ and an integer c_0 satisfy

$$4(d+1) \le T \le x^{C_T}, \qquad 0 \le c_0 \le 22 f^2 r^4.$$

Set $D = T + d + 1$. Let v be the unique integer array on $I \times \{0, \dots, r-1\}$ determined by

$$v_{\tau,k-2} = 0, \qquad v_{\tau,a} - v_{\tau,a+1} = \left\lfloor \frac{U_{\tau,a} - U_{\tau,a+1} + 2 - (c_0 + 2r)\mathbf{1}_{a+1=p}}{D} \right\rfloor \quad (0 \le a < r-1),$$

and put $V_{\tau,a} = v_{\tau,a} + 3r\mathbf{1}_{a \ge p}$. Then

1. *For $a < r - m$, one has $0 \le U_{\tau,a} \le S_c$; for $r - m \le a < r$, one has $|U_{\tau,a}| \le 8b$.*
2. *For all τ, a,*
$$T|V_{\tau,a}| \le |U_{\tau,a}| + c_0 + 4r, \qquad |V_{\tau,a}| \le \frac{8b}{T}.$$
3. *The rank-size and scalar bounds*
$$rS_c \le 800b, \qquad r(c_0 + 4r) \le 26b, \qquad fr^2 \le b, \qquad h \ge fr^2$$
hold.
4. *One has*
$$(m-j) \sum_{\tau \in I} \left(\max_{0 \le a < r} V_{\tau,a} - \min_{0 \le a < r} V_{\tau,a} \right) \le 6 \sum_{\tau \in I} \sum_{a=0}^{r-1} |V_{\tau,a}|.$$

Proof. We first choose a tail uniform in all the data of the statement. Write $L = x^\epsilon$. For $x \ge 1$, the hypotheses imply

$$f \le 4x^{2+8\epsilon}, \qquad r \le 4x^{9/2+10\epsilon}, \qquad \Delta \le 2x^{c_D}, \qquad h \le 4x^{c_h},$$
$$S_c \le 16x^{c_D+9/2+10\epsilon}, \qquad \eta \le 8x^{c_h+1/2}, \qquad fr^2 \le 64x^{11+28\epsilon}, \qquad c_0 \le 90112x^{22+56\epsilon}.$$

Consequently

$$rS_c \le 64x^{c_D+9+20\epsilon}, \qquad r(c_0 + 4r) \le 360512x^{28}, \qquad rT \le 4x^{9/2+10\epsilon+C_T}.$$

For the middle bound, the two terms are bounded by $360448x^{53/2+66\epsilon}$ and $64x^{9+20\epsilon}$; both exponents are less than 28. Since $11 + 28\epsilon < 12 < c_h$, the bounds $h \ge 2x^{c_h}$ and $fr^2 \le 64x^{11+28\epsilon}$ give $h \ge fr^2$ on a uniform tail.

Put $s = x^{\epsilon/3}$ and $w = b'/(xn) = \exp(L - s)/n$. On a uniform tail $s \le L/3$, since $s/L = x^{-2\epsilon/3} \longrightarrow 0$. Therefore

$$w \ge \tfrac{1}{2} x^{-2-8\epsilon} \exp(2L/3).$$

Thus w dominates each fixed polynomial displayed above. The three permitted sets for B_τ give, uniformly in the choice of B_τ,

$$w \le B_\tau \le 2b$$

on a tail. Indeed, for the first set the upper bound $B_\tau \le 3b'$ suffices, and $b'/b = x\exp(-x^{\epsilon/3}) \to 0$; its lower bound follows from $b' \ge w$. For the second set, $w \le B_\tau \le w+1$. For the third, $B_\tau < b + 1 + x^{C_\mu}$, and $b \ge w$ eventually.

Choose $X > 1$, depending only on the five fixed constants, such that all the preceding bounds hold for every $x \ge X$ and, for every permitted choice of B_τ,

$$\begin{gathered} B_\tau \ge 2(S_c + \eta + 5r), \qquad B_\tau \ge 2c_0 + 6r + 10rT, \\ S_c + c_0 + 4r \le 8b, \qquad B_\tau + \eta + c_0 + 5r \le 8b, \\ rS_c \le 800b, \qquad r(c_0+4r) \le 26b, \qquad fr^2 \le b, \qquad h \ge fr^2. \end{gathered}$$

This simultaneous choice is justified by the explicit polynomial bounds: for every fixed $A, R, a > 0$, $\exp(ax^\epsilon)/(Ax^R) \to \infty$. In particular, the two lower bounds for B_τ follow from the lower bound for w. Its upper bound $2b$, together with the polynomial bounds for the remaining summands, gives the fourth displayed inequality. No threshold depends on $n, k, m, I, j, B_\tau, d, T$, or c_0.

We next estimate the recurrence. The length assumption gives

$$T \le D \le 5T/4, \qquad T \ge 4.$$

All the differences specified for v are integers, so the anchor determines a unique integer array by recursion in both directions. For $a \le k-2$, the steps from a to the anchor have numerator $4\Delta + 1 > 0$ and do not contain the exceptional step. Summing the upper floor inequality gives the stronger bound

$$0 \le v_{\tau,a} \le \frac{U_{\tau,a} + 2r}{D}.$$

Also $0 \le U_{\tau,a} \le S_c$, since $\Delta \ge 1$.

Fix now a node $a = k + l - 2$, $1 \le l \le m$, and write $\delta = \mathbf{1}_{l>j}$. Summing the l recurrence numerators from the anchor to this node telescopes to

$$B_\tau + (l-1)h + l - (c_0 + r)\delta.$$

The sum of the corresponding floors equals $-v_{\tau,a}$. The inequalities $z - 1 \le \lfloor z \rfloor \le z$, and the shift $V_{\tau,a} = v_{\tau,a} + 3r\delta$, imply

$$-V_{\tau,a} \ge \frac{B_\tau - c_0 - 3r}{D} - 4r \ge \frac{B_\tau}{2D} \ge \frac{B_\tau}{4T} > 0.$$

For the middle inequality, use $B_\tau/2 \ge c_0 + 3r + 5rT$ and $D \le 5T/4$. The upper floor inequality also gives

$$-V_{\tau,a} \le \frac{B_\tau + \eta + r}{D}.$$

The node numerator is negative, and $|U_{\tau,a}| \le B_\tau + \eta + r$. When $\delta = 0$, the upper floor inequality gives $T|V_{\tau,a}| \le |U_{\tau,a}| + 2r$. When $\delta = 1$, it gives

$$|V_{\tau,a}| \le \frac{|U_{\tau,a}| + 2r}{D} - 3r \le \frac{|U_{\tau,a}|}{D},$$

since $D \ge 1$. Combining these bounds with the cloud estimate proves $T|V_{\tau,a}| \le |U_{\tau,a}| + c_0 + 4r$ at every index. The chosen tail bounds prove both the claimed node bound for U and $|V_{\tau,a}| \le 8b/T$.

It remains to estimate the spread. The anchor is zero, the cloud values are nonnegative, and all node values are negative. The cloud and node estimates above yield

$$\max_a V_{\tau,a} - \min_a V_{\tau,a} \le \frac{S_c + 2r + B_\tau + \eta + r}{D} \le \frac{3B_\tau}{2T},$$

using $B_\tau \ge 2(S_c + \eta + 5r)$ and $D \ge T$. Each of the m node entries has absolute value at least $B_\tau/(4T)$, whence

$$\sum_a |V_{\tau,a}| \ge \frac{mB_\tau}{4T}.$$

As $m - j \le m$, these two inequalities imply

$$(m-j)\left(\max_a V_{\tau,a} - \min_a V_{\tau,a}\right) \le 6\sum_a |V_{\tau,a}|.$$

Summation over τ proves the last assertion. The remaining scalar assertions are among the uniform tail bounds already established. □

Lemma 19.14. *Fix $\epsilon > 0$, and put*

$$\theta = \epsilon/100, \qquad X_\epsilon = 2^{1/(5/2+19\epsilon/200)}.$$

Let $x \ge \max(1, X_\epsilon)$ be real, and let real numbers n, k, L_a satisfy

$$n > 0, \quad k > 0, \quad L_a \ge 0, \qquad n \le x^{2+8\epsilon+\theta}, \quad L_a \le x^{\epsilon+\theta}, \quad k \ge x^{9/2+10\epsilon-\theta}.$$

Put $\delta_1 = x^{-\epsilon/8}$, and let $b' > 0$ be any real number. Let $\mathcal{T}$ be a set, and let $b_t \in \mathbb{Z}$ for each $t \in \mathcal{T}$. If

$$b' \le b_t \le b' + \frac{2n\delta_1 x^\epsilon L_a b'}{k} \qquad (t \in \mathcal{T}),$$

then

$$\lceil b' \rceil \le b_t \le 2b' \qquad (t \in \mathcal{T}).$$

Proof. All factors in $2n\delta_1 x^\epsilon L_a/k$ are nonnegative, and $k > 0$. Substituting the bounds for n, L_a, k gives

$$\begin{aligned}\frac{2n\delta_1 x^\epsilon L_a}{k} &\le 2x^{(2+8\epsilon+\theta)-\epsilon/8+\epsilon+(\epsilon+\theta)-(9/2+10\epsilon-\theta)} \\ &= 2x^{-5/2-\epsilon/8+3\theta} = 2x^{-5/2-19\epsilon/200} \le 1.\end{aligned}$$

The last inequality follows from $x \ge X_\epsilon$ and $5/2 + 19\epsilon/200 > 0$. Since $b' > 0$, the assumed upper bound for b_t therefore gives $b_t \le (1+1)b' = 2b'$. The lower bound $b' \le b_t$, together with $b_t \in \mathbb{Z}$, gives $\lceil b' \rceil \le b_t$. □

Lemma 19.15. *Let E be a number field of degree $D = [E : \mathbb{Q}]$, and let $\mu(E)$ be its group of roots of unity. Then*

$$|\mathcal{O}_{E\mathrm{tors}}^\times| = |\mu(E)| \le 2D^2.$$

In particular, let E be the field of a host on a finite group A, and put $n = |A|$. If

$$0 < \epsilon < 1/52, \qquad x \ge 128^{1/(2-16\epsilon)}, \qquad n \le 2x^{2+8\epsilon},$$

then

$$|\mu(E)| \le 32n^2 \le 128x^{4+16\epsilon} \le x^6.$$

Consequently, for the integer weights and boxes of Definition 18.2 with this same field E, every thick place z satisfies

$$\lceil b \rceil \le w_1(z) - h_0/2 < \lceil b \rceil + x^6.$$

Thus $w_1(z) - h_0/2$ belongs to the third of the sets listed in Lemma 19.13, with $C_\mu = 6$.

Proof. The finite cyclic assertion in Dirichlet's unit theorem gives $\mu(E) = \langle \zeta \rangle$, where ζ has order $w = |\mu(E)|$; see [190, Chapter I]. A primitive w-th root of unity has degree

$$[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(w),$$

where φ is Euler's totient function. Since $\mathbb{Q}(\zeta) \subseteq E$, one has $\varphi(w) \le D$.

For $w = 1$, the claimed bound is immediate. For $w > 1$, Euler's product formula gives

$$\frac{\varphi(w)^2}{w} = \prod_{p^a \| w} p^{a-2}(p-1)^2.$$

If $a \ge 2$, the corresponding factor is at least 1. If $a = 1$ and p is odd, then $(p-1)^2 - p = p(p-3) + 1 \ge 1$, so that factor is again at least 1. The only possible factor less than 1 occurs for $p = 2, a = 1$, and equals $1/2$. Therefore

$$\frac{\varphi(w)^2}{w} \ge \frac{1}{2}, \qquad w \le 2\varphi(w)^2 \le 2D^2.$$

Every root of unity is an algebraic-integer unit, and every torsion unit is a root of unity. This proves the group identification and the first assertion.

For a host field, Lemma 19.16 gives $D = 4n$. Hence

$$|\mu(E)| \le 2(4n)^2 = 32n^2 \le 128x^{4+16\epsilon}.$$

The exponent $2 - 16\epsilon$ is positive. The stated cutoff therefore implies $128 \le x^{2-16\epsilon}$, proving $|\mu(E)| \le x^6$. Finally, (Iw3) of the integer weights and boxes gives at every thick place

$$\lceil b \rceil \le w_1(z) - h_0/2 < \lceil b \rceil + |\mathcal{O}_{E^{\mathrm{tors}}}^\times|.$$

Substitute the proved bound. This is the third permitted set, with $C_\mu = 6$. □

Lemma 19.16. *Let $\mathfrak{h}$ be a host on a finite group A, and put*

$$n = |A|, \qquad E = E_{\mathfrak{h}}, \qquad N = N_{\mathfrak{h}}, \qquad \mathrm{rd}_E = |\operatorname{disc}(E)|^{1/[E:\mathbb{Q}]}.$$

Then

$$[E : \mathbb{Q}] = 4n, \qquad 1 \le \mathrm{rd}_E \le N, \qquad p \mid |\operatorname{disc}(E)| \implies p \mid N \quad (p \text{ prime}).$$

Since $2 \mid Q_S$,

$$\{p : p \text{ prime},\ p \mid 2|\operatorname{disc}(E)|\} \subseteq \{p : p \text{ prime},\ p \mid Q_S\} \cup \{q, d\}.$$

Suppose that data are given, as in the concrete good-host requirement, for which this requirement holds for $\mathfrak{h}$. Then E is CM, the chosen CM type has cardinality $2n$, and $Q_S \le n^2$.

Fix $\varepsilon > 0$, and define

$$a = \frac{4\varepsilon}{5}, \qquad X_\varepsilon = \exp\Big(1 + \frac{6}{a^3}\Big).$$

For all such data at a scale $x \geq X_\varepsilon$,

$$\log \mathrm{rd}_E \leq 2x^\varepsilon.$$

Consequently every real number L_{der} with $|L_{\mathrm{der}}| \leq x^\varepsilon$ satisfies

$$\log \mathrm{rd}_E + |L_{\mathrm{der}}| \leq 3x^\varepsilon \leq 9x^\varepsilon.$$

In particular, this applies to the real part of $L'(1, \chi_{-d})/L(1, \chi_{-d})$ in (Hg8). The threshold X_ε depends only on ε; it does not depend on the host, the translates or the words.

Proof. The surjection α identifies $\mathrm{Gal}(E/\mathbb{Q})$ with $\{\pm 1\}^2 \times A$. Galois theory therefore gives $[E : \mathbb{Q}] = 4n$. Every character of this Galois group, composed with α, is a Dirichlet character modulo N. Its inducing primitive character has positive conductor dividing N. Under the cyclotomic Artin map these are precisely the characters of the abelian extension $E/\mathbb{Q}$. Applying Theorem B.15 over $\mathbb{Q}$ gives

$$|\operatorname{disc}(E)| = \prod_\chi \operatorname{cond}(\chi),$$

with $4n$ factors, each a positive divisor of N. Hence every prime divisor of the discriminant divides N, and

$$1 \leq |\operatorname{disc}(E)| \leq N^{4n}.$$

Taking $(4n)$-th roots proves the root-discriminant bounds. The support assertion follows from $N = Q_S q d$ and $2 \mid Q_S$. The CM assertion is (Hg2); its type chooses exactly one member from each pair of conjugate embeddings, so its cardinality is $[E : \mathbb{Q}]/2 = 2n$. Finally (Hg1) includes field row (Fr4), which is $Q_S \leq n^2$.

For the scale estimate write $y = \log x$. The chosen cutoff gives $y \geq 1 + 6/a^3 > 0$. The exponential series, whose terms are nonnegative at $ay > 0$, gives

$$e^{ay} \geq \frac{(ay)^3}{6} \geq y^2.$$

Thus $(\log x)^2 \leq x^{4\varepsilon/5}$. The good-host scale Y satisfies $1 < Y \leq x$, and (Hg6) gives

$$\log N \leq 2Y^{\varepsilon/5}(\log Y)^2 \leq 2x^{\varepsilon/5}(\log x)^2 \leq 2x^\varepsilon.$$

Combine this with $\mathrm{rd}_E \leq N$. The assertion for L_{der} follows by addition, and its stated example is the explicit real-part bound in (Hg8). Every cutoff and estimate above is independent of the remaining choices. □

Lemma 19.17. *Fix $\varepsilon > 0$ and $x > 1$ with*

$$x^\varepsilon \geq 10, \qquad \log\log x \geq 1, \qquad b = \exp(x^\varepsilon).$$

Let n, m, r be integers with $n \geq 1$, $1 \leq m \leq r$ and $r \geq 4$. Let E be a CM number field with $[E : \mathbb{Q}] = 4n$, let I be a choice of one embedding $E \hookrightarrow \mathbb{C}$ from each complex-conjugate pair, and put $f = |I| = 2n$. Write

$$\mathrm{rd}_E = |\mathrm{disc}(E)|^{1/[E:\mathbb{Q}]}.$$

Let $Q, M \geq 1$ and $q_{\mathrm{H}}, d_{\mathrm{H}} \geq 2$ be integers satisfying

$$Q \leq n^2, \qquad |\{p : p \text{ prime}, \ p \mid M\}| \leq 15,$$
$$\{p : p \text{ prime}, \ p \mid 2|\mathrm{disc}(E)|\} \subseteq \{p : p \text{ prime}, \ p \mid Q\} \cup \{q_{\mathrm{H}}, d_{\mathrm{H}}\}.$$

Let

$$U, V : I \times \{1, \dots, r\} \longrightarrow \mathbb{R}, \qquad T > 0, \quad h \geq fr^2, \quad S \geq 0, \quad c_0 \in \mathbb{R}, \quad q_{\mathrm{s}} \geq 0$$

be data, where T, h, S, q_{s} are real. Suppose, for every $\tau \in I$, that

$$0 \leq U_{\tau,i} \leq S \quad (1 \leq i \leq r - m), \qquad |U_{\tau,i}| \leq 8b \quad (r - m < i \leq r),$$

$$T|V_{\tau,i}| \leq |U_{\tau,i}| + c_0 + 4r, \qquad |V_{\tau,i}| \leq \frac{8b}{T} \quad (1 \leq i \leq r),$$

and suppose also that

$$rS \leq 800b, \qquad r(c_0 + 4r) \leq 26b, \qquad fr^2 \leq b.$$

Put

$$g = \frac{h}{4T}, \qquad H = \max_{\tau \in I, \ 1 \leq i \leq r} |V_{\tau,i}|,$$
$$\mathrm{wn} = \sum_{\tau \in I} \sum_{i=1}^{r} |V_{\tau,i}|, \qquad \mathrm{wsp} = q_{\mathrm{s}} \sum_{\tau \in I} \left(\max_{1 \leq i \leq r} V_{\tau,i} - \min_{1 \leq i \leq r} V_{\tau,i} \right),$$
$$B = 4fr^2 \left(1 + \frac{H}{g}\right).$$

Assume $\mathrm{wsp} \leq 6\,\mathrm{wn}$*, and let $L_{\mathrm{der}} \in \mathbb{R}$ satisfy*

$$\log \mathrm{rd}_E + |L_{\mathrm{der}}| \leq 9x^{\varepsilon}.$$

Then

$$128 \leq B \leq 132b, \qquad |\{p : p \text{ prime}, \ p \mid 2|\mathrm{disc}(E)|M\}| \leq \lfloor B \rfloor,$$

and

$$T\left[(\log(B\,\mathrm{rd}_E) + |L_{\mathrm{der}}|)\mathrm{wn} + \mathrm{wsp}\right] \leq 83400\left[2nmb\,x^{\varepsilon} \log\log x\right].$$

The constant 83400 *does not depend on any of the data above. Consequently, if the displayed hypotheses hold beyond a common threshold at every selected point and for every lift $1 \leq j < m$, then the few-bad-primes inequality and the budget inequality at the divided weight hold at all these points and lifts, with this same constant.*

Proof. The CM field has $4n$ embeddings, in conjugate pairs, so $f = 2n$. There is at most one prime divisor of Q exceeding n: otherwise two distinct such primes would have product greater than n^2, whereas their product divides $Q \leq n^2$. By the discriminant-support and level-prime assumptions, among the primes dividing $2|\mathrm{disc}(E)|M$ at most $1 + 2 + 15 = 18$ exceed n. At most $n + 1$ of them are at most n. Their total number is therefore at most

$$n + 19 \leq 128n = 4f \cdot 16 \leq 4fr^2.$$

Since $h \geq fr^2 > 0$ and $T > 0$, we have $g > 0$; also $H \geq 0$. Hence

$$B \geq 4fr^2 \geq 128.$$

The integer cardinality is at most B, and consequently is at most $\lfloor B \rfloor$.

Summing the pointwise division inequality and then using the bounds on U gives

$$\begin{aligned} T\,\mathrm{wn} &\le \sum_{\tau,i} |U_{\tau,i}| + fr(c_0+4r) \\ &\le f\big(rS + 8mb + r(c_0+4r)\big) \\ &\le f(826b + 8mb) \le 834 fmb. \end{aligned}$$

Here the cloud contribution is at most $f(r-m)S \le frS$, and the last inequality uses $m \ge 1$.

The entry bound gives $H \le 8b/T$. Therefore

$$\frac{H}{g} \le \frac{32b}{h}, \qquad B \le 4fr^2 + \frac{128fr^2 b}{h} \le 4b + 128b = 132b.$$

In particular

$$\log B \le x^{\varepsilon} + \log 132 \le x^{\varepsilon} + 22,$$

because $132 \le 12^2$ and $\log 12 \le 11$. Since $\mathrm{rd}_E \ge 1$, all logarithms in the statement are defined. The norm wn is nonnegative, and the spread assumption and logarithmic-size assumption yield

$$\begin{aligned} \big(\log(B\,\mathrm{rd}_E) + |L_{\mathrm{der}}|\big)\mathrm{wn} + \mathrm{wsp} &\le \big(\log B + \log \mathrm{rd}_E + |L_{\mathrm{der}}| + 6\big)\mathrm{wn} \\ &\le (10x^{\varepsilon} + 28)\mathrm{wn} \le 100 x^{\varepsilon}\,\mathrm{wn}. \end{aligned}$$

Multiplying by T and using the preceding norm bound, $f = 2n$ and $\log\log x \ge 1$, we obtain

$$T\left[\big(\log(B\,\mathrm{rd}_E) + |L_{\mathrm{der}}|\big)\mathrm{wn} + \mathrm{wsp}\right] \le 83400 fmbx^{\varepsilon} \le 83400\left[2nmbx^{\varepsilon}\log\log x\right].$$

The numerical constants in these estimates do not depend on the data, so applying the estimates beyond the common threshold gives the asserted uniform form. □

Theorem 19.18. *Let $\mathcal{F}$ be a parametrised host family with covolume records carrying chain data, deep-cover data, component data and model-slope data with the properties (Ms1)–(Ms4) for the constant $C \ge 0$. At every large point with $j < m$ put*

$$e_* = C\big(\mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp}\big) + 2\,c^{\mathrm{HT}}_{E,M}(B)\,\mathrm{wn}.$$

Then for all large points with $j < m$

$$e_* \ge 0, \qquad \frac{1}{|\mathcal{Z}|}\sum_{Z\in\mathcal{Z}} \mathrm{vs}(Z) \ge -e_*, \qquad T^{\mathrm{d}} e_* \le C(C+120)\,\mathrm{ue}(x,i).$$

Consequently, if the component data is taken with $e^{\mathrm{ms}} = e_*$, it has the mean-slope bound (Ht3) for the constant $C(C+120)$; and if moreover (Ht1), (Ht2), (Ht4), (Ht5) hold for a constant C', then (Ht1)–(Ht5) hold for the constant $\max\big(C', C(C+120)\big)$.

Proof. Fix a large point with $j < m$ (beyond the threshold of (Ms1)–(Ms4)) and write $c = c^{\mathrm{HT}}_{E,M}(B)$, $N = |\mathcal{Z}| \ge 1$, $U = \mathrm{ue}(x,i)$.

The cost and the logarithmic size. By (Ms2) and $B \ge 128$, Lemma 19.19 gives $0 \le c \le 60\log(B\,\mathrm{rd}_E)$. Since $\mathrm{rd}_E \ge 1$ and $|L| \ge 0$,

$$\log(B\,\mathrm{rd}_E) \le \mathrm{lz}, \qquad \mathrm{lz} \ge \log 128 > 0. \tag{19.6}$$

With $C, \mathrm{wn}, \mathrm{wsp} \ge 0$ every term of e_* is non-negative, so $e_* \ge 0$.

The mean. Summing (Ms1) over $Z \in \mathcal{Z}$, dividing by N and using (Ms3),

$$\frac{1}{N}\sum_{Z} \mathrm{vs}(Z) \geq \frac{1}{N}\sum_{Z} \mathrm{ns}(Z) - 2c\,\mathrm{wn} \geq -C\big(\mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp}\big) - 2c\,\mathrm{wn} = -e_*.$$

The budget. Since $T^{\mathrm{d}} \geq 1$, $\mathrm{wn} \geq 0$ and $\mathrm{wsp} \geq 0$, the cost bound, (19.6) and (Ms4) give

$$\begin{aligned} T^{\mathrm{d}} e_* &= C\,T^{\mathrm{d}}\big(\mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp}\big) + 2c\,T^{\mathrm{d}}\mathrm{wn} \\ &\leq C\cdot CU + 120\,T^{\mathrm{d}}\,\mathrm{lz}\cdot\mathrm{wn} \\ &\leq C^2U + 120\,T^{\mathrm{d}}\big(\mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp}\big) \leq C(C+120)\,U. \end{aligned}$$

The consequences. With $e^{\mathrm{ms}} = e_* \geq 0$ the two displayed inequalities are (Ht3) for the constant $C(C+120)$, for all large points. Each of (Ht2), (Ht3), (Ht4) bounds a quantity by its constant times U. Moreover $U \geq 0$: by Definition 19.5, $U = 2|A|\,m\,b(x)\,x^{\varepsilon}\log\log x$ with $b(x) > 0$, which is ≥ 0 for $x \geq e$. Hence each of these bounds stays true when its constant is replaced by a larger one, while (Ht1) and (Ht5) contain no constant. Taking as threshold the largest of e and the two thresholds, and the constant $\max(C', C(C+120))$, the family has (Ht1)–(Ht5). □

Lemma 19.19. *Let E be a number field, $M \geq 1$ an integer and $B \geq 128$ a real number such that the number of primes dividing $2\,\mathrm{disc}(E)\,M$ is at most $\lfloor B \rfloor$. Then*

$$0 \leq c^{\mathrm{HT}}_{E,M}(B) \leq 60\,\log\big(B\,\mathrm{rd}_E\big).$$

Proof. Every summand of (19.4) is non-negative, since $p \geq 2$ and $v_p(\mathrm{disc}(E)) \geq 0$; this gives the lower bound. For the upper bound split the sum into three parts.

The discriminant part. Every prime dividing $\mathrm{disc}(E)$ lies in $P_{E,M}(B)$, and $\mathrm{disc}(E)$ is a non-zero integer, so

$$\sum_{p\in P_{E,M}(B)} \frac{40\,v_p(\mathrm{disc}(E))}{[E:\mathbb{Q}]}\log p = \frac{40}{[E:\mathbb{Q}]}\log|\mathrm{disc}(E)| = 40\log\mathrm{rd}_E.$$

The term $\delta_{p,2}\log p$ contributes $\log 2$, as $2 \in P_{E,M}(B)$.

The primes $p \leq B$. We use Chebyshev's bound $\theta(y) = \sum_{p\leq y}\log p < y\log 4$ for $y \geq 1$. Let $n = \lfloor B \rfloor \geq 128$. By Legendre's formula, $\log n! = \sum_{p\leq n}\log p\sum_{k\geq 1}\lfloor n/p^k\rfloor \geq \sum_{p\leq n}\log p\,(n/p-1)$, and $\log n! \leq n\log n$. Hence

$$\sum_{p\leq B}\frac{\log p}{p} = \sum_{p\leq n}\frac{\log p}{p} \leq \log n + \frac{\theta(n)}{n} < \log B + \log 4.$$

Since $p - 1 \geq p/2$, the primes $p \leq B$ contribute $\sum_{p\leq B}\frac{\log p}{p-1} \leq 2\sum_{p\leq B}\frac{\log p}{p} < 2\log B + 2\log 4$.

The primes $p > B$ in $P_{E,M}(B)$. They divide $2\,\mathrm{disc}(E)\,M$, so there are at most $\lfloor B\rfloor \leq B$ of them. The function $t \mapsto \log t/(t-1)$ is decreasing on $(1,\infty)$: its derivative is $\big((t-1)/t - \log t\big)/(t-1)^2 < 0$ because $\log t > 1 - 1/t$ for $t > 1$. So each such prime contributes $\log p/(p-1) < \log B/(B-1)$, and together they contribute less than $B\log B/(B-1) \leq \frac{128}{127}\log B < 2\log B$.

Conclusion. Adding the parts,

$$\begin{aligned} c^{\mathrm{HT}}_{E,M}(B) &< 4\log B + 5\log 2 + 40\log\mathrm{rd}_E \leq 5\log B + 40\log\mathrm{rd}_E \\ &\leq 60\big(\log B + \log\mathrm{rd}_E\big) = 60\log\big(B\,\mathrm{rd}_E\big), \end{aligned}$$

using $5\log 2 < 7\log 2 = \log 128 \leq \log B$ and $\log\mathrm{rd}_E \geq 0$. □

Theorem 19.20. *Let $\mathcal{F}$ be a parametrised host family with covolume records carrying chain data, deep-cover data and component data with the properties (Ht1)–(Ht5) for the constant $C \geq 0$. Then $\mathcal{F}$ has the uniform lower bound (Dp6), with the constant $8C + 4$.*

Proof. Let x_0 be the maximum of e^e and the threshold of (Ht1)–(Ht5). Fix $x \geq x_0$ with $x > x_1$, $i \in I(x)$, $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and a lift $j < m$; use the notation of Definition 19.10 and write $E = \mathrm{ue}(x,i)$, $T = T^{\mathrm{d}}$, $D = D^{\mathrm{fl}}$, $\rho = \rho^{\mathrm{d}}$, $N = |\mathcal{Z}|$, $\delta' = \mathrm{dg}_{\min}$, $S = \sum_Z \mathrm{dg}(Z) > 0$. Since $|A|, m \geq 1$ (A the group of the frame at x), $x^{\varepsilon} \geq 1$ and $\log\log x \geq 1$, we have $E = 2|A|\, m\, b(x)\, x^{\varepsilon} \log\log x \geq b(x) \geq 1$.

The weighted mean of the heights. For every Z, (Ht2) gives

$$\mathrm{hc}(Z) + c^{\mathrm{s}} + c^{\mathrm{c}} \geq (1-\rho)\max(0, \mathrm{vs}(Z) + c^{\mathrm{s}}) \geq 0,$$

hence, as $\mathrm{dg}(Z) \geq \delta'$,

$$\mathrm{dg}(Z)\big(\mathrm{hc}(Z) + c^{\mathrm{s}} + c^{\mathrm{c}}\big) \geq \delta'\big(\mathrm{hc}(Z) + c^{\mathrm{s}} + c^{\mathrm{c}}\big) \geq \delta'(1-\rho)\big(\mathrm{vs}(Z) + c^{\mathrm{s}}\big).$$

Summing over Z and using (Ht3), $\sum_Z \mathrm{dg}(Z)(\mathrm{hc}(Z) + c^{\mathrm{s}} + c^{\mathrm{c}}) \geq \delta'(1-\rho)N(c^{\mathrm{s}} - e^{\mathrm{ms}})$. By (Ht1), $N\delta' \leq S \leq N\delta'/(1-\rho)$, so $\kappa = N\delta'/S$ satisfies $(1-\rho)^2 \leq \kappa(1-\rho) \leq 1$. Dividing by S, the weighted mean $\bar{H} = \sum_Z \mathrm{dg}(Z)\mathrm{hc}(Z)/S$ satisfies

$$\bar{H} \geq \kappa(1-\rho)(c^{\mathrm{s}} - e^{\mathrm{ms}}) - c^{\mathrm{s}} - c^{\mathrm{c}} \geq (1-\rho)^2 c^{\mathrm{s}} - c^{\mathrm{s}} - e^{\mathrm{ms}} - c^{\mathrm{c}} \geq -\big(2\rho c^{\mathrm{s}} + e^{\mathrm{ms}} + c^{\mathrm{c}}\big). \tag{19.7}$$

A linear lower bound for the bracket of (Ht4). Let $h = \mathrm{hc}(Z)$ and $f(Z) = Th - 3(D+1)|h| - a^{\mathrm{hs}} \log(2 + |h|) - r^{\mathrm{hs}}$. For $a > 0$ and $y > 0$ one has $a \log y \leq y + a \log \max(a, 1)$ (the function $a \log y - y$ is maximal at $y = a$, with value $a \log a - a$). With $y = 2 + |h|$, and $|h| \leq h + 2c^{\mathrm{s}}$ (since $h \geq -c^{\mathrm{s}}$ by (Ht2)),

$$\begin{aligned} f(Z) &\geq Th - (3D+4)|h| - 2 - a^{\mathrm{hs}} \log \max(a^{\mathrm{hs}}, 1) - r^{\mathrm{hs}} \\ &\geq (T - 3D - 4)h - (6D+8)c^{\mathrm{s}} - 2 - a^{\mathrm{hs}} \log \max(a^{\mathrm{hs}}, 1) - r^{\mathrm{hs}}. \end{aligned}$$

The weighted mean of the bracket. Put $\bar{F} = \sum_Z \mathrm{dg}(Z) f(Z)/S$. By (Ht1), $1 \leq T - 3D - 4 \leq T$, and the right side of (19.7) is ≤ 0, so $(T - 3D - 4)\bar{H} \geq \min(0, T\bar{H}) \geq -T(2\rho c^{\mathrm{s}} + e^{\mathrm{ms}} + c^{\mathrm{c}})$. Hence, with the budgets of (Ht2), (Ht3), (Ht4) and $E \geq 1$,

$$\bar{F} \geq -Te^{\mathrm{ms}} - Tc^{\mathrm{c}} - c^{\mathrm{s}}(6D + 8 + 2\rho T) - 2 - \big(a^{\mathrm{hs}} \log \max(a^{\mathrm{hs}}, 1) + r^{\mathrm{hs}}\big) \geq -(4C+2)E.$$

The twisted slope. By (Ht4), $\sum_Z \mathrm{gc}(Z) \geq N^{\mathrm{b}} S \bar{F}$, and by (Ht1), $\sum_Z \mathrm{rc}(Z) \geq N^{\mathrm{b}} S > 0$. If $\bar{F} \geq 0$ the quotient $\sum \mathrm{gc} / \sum \mathrm{rc}$ is ≥ 0; if $\bar{F} < 0$ it is $\geq N^{\mathrm{b}} S \bar{F} / \sum \mathrm{rc} \geq \bar{F}$. In both cases it is $\geq -(4C+2)E$, and (Ht5) gives

$$\sigma_{\mathfrak{z},j} \geq 2\,\frac{\sum_Z \mathrm{gc}(Z)}{\sum_Z \mathrm{rc}(Z)} \geq -(8C+4)E.$$

This is (Dp6) for all large points with the constant $8C + 4$. □

Lemma 19.21. *Let $x > 1$ and $\varepsilon > 0$. Let $\mathfrak{h} = (q, d, Q_S, \alpha)$ be a normalised host for a frame (e, A, h, a_0), and let $\mathfrak{M}$ be a loss datum at scale x with parameter ε, group $H = A$ and arithmetic datum $\mathfrak{A}_{\mathfrak{h}}$. Assume*

$$e_2 = e, \qquad o < x^{\varepsilon/4}.$$

Then $\mathfrak{M}$ satisfies (Lm1).

In particular, fix $\varepsilon > 0$ and $C_8 \geq 1$, and put

$$x_0 = \max\{2, (2C_8)^{4/\varepsilon}\}.$$

For every such normalised host and loss datum with $x \geq x_0$, $e_2 = e$ and frame index $o \leq C_8$, condition (Lm1) holds. This cutoff is independent of the host and loss datum.

Proof. First consider a prime p inert for $\mathfrak{A}_\mathfrak{h}$. By the definition of inertness, $p \neq 2$, so p is odd. Put $N = Q_S qd$, and write $N = N_p N_p'$ with N_p a power of p and $p \nmid N_p'$. The host condition gives $2^{e+2} \mid Q_S$, hence $2^{e+2} \mid N$. Since N_p is odd, $2^{e+2} \mid N_p'$. Let u_p denote the unit used in the definition of the host's Frobenius: it satisfies

$$u_p \equiv p \pmod{N_p'}, \qquad u_p \equiv 1 \pmod{N_p}.$$

It follows that $u_p \equiv p \pmod{2^{e+2}}$, so $[u_p]_e = [p]_e$. Write α_A for the A-component of α. Normalisation (Hn2) gives

$$h(\alpha_A(u_p)) = [u_p]_e = [p]_e.$$

The identification τ defining the arithmetic datum leaves the A-component unchanged. The element c_∞ has zero A-component. Consequently the A-component h_p of $c_\infty + \mathrm{Fr}_p$ equals $\alpha_A(u_p)$, and therefore

$$h(h_p) = [p]_e. \tag{1}$$

This argument also covers primes dividing Q_S: it uses only that p is odd and the CRT congruence modulo N_p'.

There are 2^{e+1} units modulo 2^{e+2}, namely its odd residue classes. Multiplication by -1 pairs them without fixed points: a fixed unit u would give $2u = 0$ modulo 2^{e+2} and hence $2 = 0$ there, contradicting $e \geq 2$. Thus the quotient group H_e has 2^e elements. By Lagrange's theorem, every element of H_e has order dividing 2^e.

For any element g of a finite group and any homomorphism, the order of its image divides the order of g: applying the homomorphism to $g^{\mathrm{ord}(g)} = 1$ proves this assertion (or use additive multiples for A). By (1), $\mathrm{ord}([p]_e)$ divides $\mathrm{ord}(h_p)$. If the latter order is odd, the former divides both an odd integer and 2^e, so it is 1 and $[p]_e = 1$. Since $e_2 = e$, these are the first two assertions of (Lm1), even without its restriction to primes that are not sign-changing.

It remains to prove the evenness assertion. Put $A_0 = \langle a_0 \rangle$ and $n_0 = |A_0|$. Injectivity of h on A_0 identifies it with a subgroup of H_e, so $n_0 \mid 2^e$. In particular n_0 is a power of 2. Counting the cosets of A_0 gives $|A| = on_0$. If $g \in A$ has odd order d_g, then Lagrange's theorem gives $d_g \mid on_0$. Since $\gcd(d_g, n_0) = 1$, Euclid's divisibility lemma gives $d_g \mid o$. Both integers are positive, whence

$$d_g \leq o < x^{\varepsilon/4}.$$

Thus every element of A with order at least $x^{\varepsilon/4}$ has even order. Applying this to h_p proves the final assertion of (Lm1).

For the uniform assertion, $x \geq x_0$ implies $x^{\varepsilon/4} \geq 2C_8 > C_8 \geq o$, because $\varepsilon > 0$ and $C_8 \geq 1$. The first assertion therefore applies with the same cutoff for every host and loss datum under consideration. □

Lemma 19.22. *Fix $0 < \varepsilon < 1/52$ and $C_b > 0$, and put*

$$x_0 = \max\{2, (8C_b)^{3/\varepsilon}\}.$$

Let $\mathfrak{M}$ be a loss datum with parameter ε and scale $x \geq x_0$, on the group $H = A$ of a host. Suppose that its top set, Frobenius translations, orbit representatives and good representatives are those of a finite good-orbit selection satisfying the bad-top bound with constant C_b, and that

$$|T| \geq \tfrac{1}{2} x^{-\varepsilon/8} |A|.$$

Then $T \neq \varnothing$ and $\mathfrak{M}$ satisfies (Lm4). The cutoff depends only on ε, C_b.

Proof. Put $n = |A|$, $\delta_1 = x^{-\varepsilon/8}$ and $U = \exp(x^{\varepsilon/3})$. The group A contains its identity, so $n \geq 1$. Since $\delta_1 > 0$, the assumed lower bound gives $|T| > 0$, proving nonemptiness.

Let s_B be the prime range in (Lm4), and put $M = \lfloor U \rfloor$. All its primes are less than U. If $M < 2$, there are no such primes and their sum is zero. If $M \geq 2$, nonnegativity, $\log p \leq \log U$, and inclusion of the primes among the integers give

$$\sum_{p \in s_B} \frac{\log p}{p} \leq \sum_{\substack{p \text{ prime} \\ p < U}} \frac{\log p}{p} \leq (\log U) \sum_{j=2}^{M} \frac{1}{j}.$$

For every integer $2 \leq j \leq M$, one has $1/t \geq 1/j$ throughout $[j-1, j]$, hence

$$\frac{1}{j} \leq \int_{j-1}^{j} \frac{dt}{t}.$$

Summing these inequalities yields

$$\sum_{j=2}^{M} \frac{1}{j} \leq \int_{1}^{M} \frac{dt}{t} = \log M \leq \log U.$$

Consequently, in both cases,

$$\sum_{p \in s_B} \frac{\log p}{p} \leq (\log U)^2 = x^{2\varepsilon/3}.$$

The finite good-orbit bad-top bound has the same prime range s_B. Its left-hand side, therefore, is at most

$$4C_b x^{-2} \delta_1 n \sum_{p \in s_B} \frac{\log p}{p} \leq 4C_b \delta_1 n x^{-2+2\varepsilon/3} \leq 8C_b |T| x^{-2+2\varepsilon/3}.$$

Finally $x \geq (8C_b)^{3/\varepsilon}$ implies $x^{\varepsilon/3} \geq 8C_b$. The last expression is at most $|T| x^{-2+\varepsilon}$, which is the bound required by (Lm4). All sums here are finite; if s_B is empty the same inequalities hold with zero left-hand side. □

Theorem 19.23. *Let $\mathcal{F}$ be a parametrised host family with base covolume realisations, with at least one lift at every member. Suppose it carries chain data with (Lc1)–(Lc6) and deep-cover data with (Dp0)–(Dp6). Let $C_6, C_{\mathrm{top}}, C_{\mathrm{F}}, C_{\mathrm{U}} \geq 0$ be constants with which (Lc6), (Dp0), (Dp5), (Dp6) hold, respectively, and put*

$$C_{\mathrm{cv}} = C_6 + \max\{C_{\mathrm{top}},\, 2(C_{\mathrm{F}} + C_{\mathrm{U}}) + 9\} + 1.$$

If $\mathcal{F}$ has realisation estimates for some real $C \geq C_{\mathrm{cv}}$, then it has (Bk1), (Bk2) and (Nf3).

Proof. Choose a cutoff $X > 1$ beyond which the realisation estimates for C hold.

By Theorem 19.24, the family has (Lc7) with

$$C_7 = \max\{C_{\mathrm{top}}, 2(C_{\mathrm{F}} + C_{\mathrm{U}}) + 9\}.$$

Apply Theorem 19.26. There is one cutoff X_{cv}, uniform over all members, realised points and lifts, such that the covolume inequality holds for every coefficient at least $C_6 + C_7 + 1 = C_{\mathrm{cv}}$. In particular, it holds for the constant C of the hypothesis. Increase the cutoff to $\max(X, X_{\mathrm{cv}})$. All the following assertions concern members above this common cutoff and all their words.

The base divisor inequality follows by summing the covolume inequalities. The weight transport in the base covolume realisation is bijective, so for each lift

$$\mathrm{mt}_j(\mathfrak{z}) = \sum_{\tau \in \Phi} (k - m + 2j - 1) w_j(\tau) \log \Omega(\mathfrak{h}_{x,i}).$$

Consequently (Ri1) and the summed covolume bound give

$$\begin{aligned} \mu_{\mathrm{b}}^{\vee} - \mathrm{sc} &\le \sum_j \mathrm{mt}_j(\mathfrak{z}) + mC\,\mathrm{ue} + F + E_\omega \\ &\le \mathrm{lm}(x, i, \omega) + F + \mathrm{le}(x, i, \omega) + \sum_j \left(\frac{\mathrm{Df}_j(x, i)}{\mathrm{rk}_0(x, i)} + \mathrm{tc}_j(x, i) \right). \end{aligned}$$

The base rank is at least 1 by the base covolume realisation, so this proves every part of (Bk1). The two inequalities in (Ri2) give $\mu^{\vee}(\mathfrak{V}) \le \mu_{\mathrm{b}}^{\vee} + \mathrm{pa}(x, i)$, which is (Bk2).

We next prove integrality of the divisor maps. Fix a word and write $\mathfrak{a}_\omega$ for its fractional ideal in the simultaneous lattice realisation. At every finite place w,

$$\mathfrak{a}_\omega \mathcal{O}_{K,w} = \mathfrak{p}_w^{-a_{\omega,w}} \mathcal{O}_{K,w}, \qquad N_{\omega,w} = \mathfrak{p}_w^{-a_{\omega,w}} N'_{\omega,w}.$$

The valuation condition (Ri3), and linearity in both arguments, therefore imply

$$B_\omega(N_{\omega,w})(R_w) \subseteq \mathcal{O}_{K,w}.$$

Explicitly, multiplication by an element of valuation at least $-a_{\omega,w}$ offsets the lower bound $a_{\omega,w}$ in (Ri3), and finite sums preserve nonnegative valuation. Thus for every $n \in N_\omega$ and $r \in R$, the scalar $B_\omega(n)(r) \in K$ is integral at every finite place. The intersection of these local rings inside K is $\mathcal{O}_K$: a nonzero scalar with nonnegative valuation at every prime has an integral principal fractional ideal, hence belongs to $\mathcal{O}_K$. It follows that $B_\omega(N_\omega) \subseteq \mathrm{Hom}_{\mathcal{O}_K}(R, \mathcal{O}_K) = R^{\vee}$.

For a nonzero map, average (Ri4) over the embeddings. By the definition of its height,

$$2h(B_\omega) = \frac{1}{[K : \mathbb{Q}]} \sum_\sigma \log \|B_{\omega,\sigma}\|^2 \le \frac{1}{[K : \mathbb{Q}]} \sum_\sigma q_{\omega,\sigma} \le h(\mathfrak{V}, \omega).$$

The level lattice is nonzero by (Ri3), hence has rank at least 1. Each divisor lattice has positive rank by the simultaneous realisation. Finally the frame map is surjective, so rank–nullity and (Ri4) give

$$\mathrm{rk}\, R = \dim_{\mathbb{C}} R_{K,v}^{\vee} = \dim_{\mathbb{C}} \mathcal{H} - \dim_{\mathbb{C}} \ker \Pi \ge (1 - \delta) \dim_{\mathbb{C}} \mathcal{H}.$$

These are the positive-rank, integral-image, height and frame-dimension clauses of (Nf3), all with the same common cutoff. □

Theorem 19.24. *Let $\mathcal{F}$ be a parametrised host family with covolume records carrying chain data and deep-cover data with the properties (Dp0)–(Dp6), (Dp0) for the constant C_{top}, (Dp5) for C_{F} and (Dp6) for C_{U}. Then $\mathcal{F}$ has the core lower bound (Lc7), with the constant* $\max\bigl(C_{\mathrm{top}}, 2(C_{\mathrm{F}} + C_{\mathrm{U}}) + 9\bigr)$.

Proof. Let x_0 be the maximum of e^e and the thresholds of (Dp0)–(Dp6), and put

$$C_7 = \max\bigl(C_{\mathrm{top}}, 2(C_{\mathrm{F}} + C_{\mathrm{U}}) + 9\bigr).$$

Fix $x \geq x_0$ with $x > x_1$, $i \in I(x)$, $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and a lift j; write $E = \mathrm{ue}(x,i)$, $b = b(x)$ and $M = \mathrm{mt}_j(\mathfrak{z})$. As $|A|, m \geq 1$, $x^{\varepsilon} \geq 1$ and $\log\log x \geq 1$, we have $E \geq b > 0$.

The top lift. If $j = m$, (Dp0) gives $2\widehat{\mu}(L^{\mathrm{c}}_{\mathfrak{z},m}) \geq -M - C_{\mathrm{top}}E \geq -M - C_7E$.

A lift below the top: the lowered lattice. Let $j < m$ and use the notation of Definition 19.8. By (Dp5) and (Dp6),

$$2\,\widehat{\mu}(A) \geq \sigma - M - C_{\mathrm{F}}E \geq -M - (C_{\mathrm{F}} + C_{\mathrm{U}})E. \tag{19.8}$$

Descent to the deep lattice. Apply Lemma 19.25 to $A \subseteq V^-_{\mathfrak{z},j}$, $B \subseteq V^+_{\mathfrak{z},j}$, the descent maps, $P = b$, Λ and $\epsilon = b/\Lambda$: its hypotheses 1, 2 and 4 are (Dp2) and hypothesis 3 is (Dp3); $\Lambda > 0$ and $\epsilon \geq 0$. By (Dp1), $\epsilon \leq 1$, and then $s \mapsto s - \epsilon|s|$ is non-decreasing on $\mathbb{R}$. Since $\tau = \widehat{\mu}(A) - b \geq \tau_0 := -\frac{1}{2}M - \frac{1}{2}(C_{\mathrm{F}} + C_{\mathrm{U}})E - b$ by (19.8), the lemma gives

$$\widehat{\mu}(B) \geq \tau - \epsilon(|\tau| + \Lambda) \geq \tau_0 - \epsilon|\tau_0| - \epsilon\Lambda.$$

Here $\epsilon\Lambda = b$, and $\epsilon|\tau_0| \leq \frac{1}{2}\epsilon|M| + \frac{1}{2}(C_{\mathrm{F}} + C_{\mathrm{U}})E + b \leq \frac{1}{2}b + \frac{1}{2}(C_{\mathrm{F}} + C_{\mathrm{U}})E + b$, using $\epsilon \leq 1$ and $\epsilon|M| \leq (b/\Lambda)\Lambda = b$ by (Dp1). Hence

$$2\,\widehat{\mu}(B) \geq -M - 2(C_{\mathrm{F}} + C_{\mathrm{U}})E - 7b.$$

Back to the core lattice. By (Dp4) and $b \leq E$,

$$2\,\widehat{\mu}(L^{\mathrm{c}}_{\mathfrak{z},j}) \geq 2\,\widehat{\mu}(B) - 2b \geq -M - 2(C_{\mathrm{F}} + C_{\mathrm{U}})E - 9b \geq -M - \bigl(2(C_{\mathrm{F}} + C_{\mathrm{U}}) + 9\bigr)E \geq -M - C_7E.$$

So (Lc7) holds for all large points with the constant C_7. □

Lemma 19.25. *Let K be a totally complex number field, V and W hermitian spaces over K, $A \subseteq V$ and $B \subseteq W$ finitely generated $\mathcal{O}_K$-submodules with $A \neq 0$, $n \geq 0$ an integer and $\phi_1, \dots, \phi_n \colon V \to W$ K-linear maps. Let P, $\Lambda \geq 0$ and $\epsilon \geq 0$ be real numbers. Assume*

1. *$\phi_h(A) \subseteq B$ for every h, and $y = 0$ whenever $y \in V$ and $\phi_h(y) = 0$ for all h;*
2. *$\sum_{h=1}^{n} \|\phi_h(y)\|_v^2 \leq e^{2P}\|y\|_v^2$ for every infinite place v and every $y \in V \otimes_{K,v} \mathbb{C}$;*
3. *some subset of B spans KB over K and consists of vectors y with $\|y\|_v \leq e^{\Lambda}$ for every infinite place v;*
4. *$n \operatorname{rk} B \leq (1+\epsilon)\operatorname{rk} A$.*

Then $B \neq 0$ and, with $\tau = \widehat{\mu}(A) - P$,

$$\widehat{\mu}(B) \geq \tau - \epsilon(|\tau| + \Lambda).$$

Proof. Write $d_K = [K : \mathbb{Q}]$, $a = \operatorname{rk} A \geq 1$ and $r = \operatorname{rk} B$. In this proof the Arakelov degree uses, for each complex embedding σ, the norm of the place of σ; all standard facts used (additivity in short exact sequences with sub and quotient norms, the degree of a sublattice of full rank, the degree of a free lattice by its Gram determinants) are those of Section 14.

$n \geq 1$ *and* $B \neq 0$. Take $y \in A$, $y \neq 0$. By hypothesis 1 some $\phi_h(y) \neq 0$; it lies in B.

The product map. Give W^n the orthogonal sum norms; $B^n \subseteq W^n$ is a hermitian lattice of rank nr and $\widehat{\deg}(B^n) = n\,\widehat{\deg}(B)$. Let $\Phi = (\phi_1, \dots, \phi_n)\colon V \to W^n$. By hypothesis 1, Φ is injective and $\Phi(A) \subseteq B^n$; by hypothesis 2, $\|\Phi(y)\|_v \leq e^P \|y\|_v$ for every v and $y \in V \otimes_{K,v} \mathbb{C}$. In particular $a \leq nr$.

Degree of the image. Φ restricts to an isomorphism $A \to \Phi(A)$ of $\mathcal{O}_K$-modules, hence induces an isomorphism of the determinant lattices $\det A = \wedge^a A \to \det \Phi(A)$, whose norm at every place is at most $\|\Phi_v\|^a \leq e^{aP}$ (the operator norm of $\wedge^a \Phi_v$ is the product of the a largest singular values of Φ_v restricted to $KA \otimes \mathbb{C}$). For an isomorphism of rank-one hermitian lattices that multiplies norms by at most c at every place, the degree drops by at most $d_K \log c$. Since the degree of a hermitian lattice is that of its determinant,

$$\widehat{\deg}\,\Phi(A) \geq \widehat{\deg}\,A - d_K\, a\, P. \tag{19.9}$$

Quotients of B^n *have slope at least* $-\Lambda$. Let $s \subseteq B$ be the set of hypothesis 3 and $s^{(n)} \subseteq B^n$ the set of its images under the n coordinate inclusions; it spans KB^n and its vectors have norm $\leq e^\Lambda$ everywhere. Let $Q \neq 0$ be a torsion-free quotient of B^n, of rank q, with the quotient norms. The image of $s^{(n)}$ spans KQ, so it contains a K-basis $\bar{y}_1, \dots, \bar{y}_q$ of KQ, with $\|\bar{y}_t\|_v \leq e^\Lambda$ (quotient norms do not exceed the norms). The $\mathcal{O}_K$-span N of the $\bar{y}_t$ is free of rank q and of finite index in Q, so by Hadamard's inequality for the Gram determinants

$$\widehat{\deg}\,Q \geq \widehat{\deg}\,N = -\tfrac{1}{2}\sum_\sigma \log\det\bigl(\langle \bar{y}_t, \bar{y}_{t'}\rangle_\sigma\bigr) \geq -\sum_\sigma \sum_{t=1}^{q} \log\|\bar{y}_t\|_\sigma \geq -q\, d_K\, \Lambda.$$

The slope inequality. Let $S = B^n \cap K\Phi(A)$, saturated in B^n, of rank a; it contains $\Phi(A)$ with finite index, so $\widehat{\deg}\,S \geq \widehat{\deg}\,\Phi(A)$. If $a < nr$, the quotient B^n/S is torsion free of rank $nr - a$, and by additivity, the previous step and (19.9),

$$n\,\widehat{\deg}\,B = \widehat{\deg}\,S + \widehat{\deg}(B^n/S) \geq \widehat{\deg}\,A - d_K aP - (nr - a)\, d_K\Lambda;$$

if $a = nr$ the same inequality holds with $S = B^n$. Dividing by $d_K\, nr$ and writing $\rho = a/(nr) \in (0, 1]$,

$$\widehat{\mu}(B) \geq \rho\,\tau - (1-\rho)\Lambda = \tau - (1-\rho)(\tau + \Lambda) \geq \tau - (1-\rho)\bigl(|\tau| + \Lambda\bigr).$$

By hypothesis 4, $1/\rho = nr/a \leq 1 + \epsilon$, so $1 - \rho \leq \epsilon/(1+\epsilon) \leq \epsilon$; since $|\tau| + \Lambda \geq 0$ the claim follows. □

Theorem 19.26. *Let* $\mathcal{F}$ *be a parametrised host family with covolume records that carries chain data with the properties (Lc1)–(Lc7), (Lc6) for the constant* C_6 *and (Lc7) for the constant* C_7*. Then* $\mathcal{F}$ *has the covolume bound, with* $C_0 = C_6 + C_7 + 1$.

Proof. Choice of the threshold. Let $c > 0$ be the constant of (Lc3), A_4 that of (Lc4) and A_5 that of (Lc5), and put $S(x) = x^{A_4} + x^{A_5} + C_6 + C_7 + 1$. Since $e^{-x^c} x^A \to 0$ as $x \to \infty$ for every real A, there is x_* with $12\, e^{-x^c} S(x) \leq 1$ for $x \geq x_*$. Let x_0 be the maximum of x_*, e^e and the thresholds of (Lc1)–(Lc7). Fix $x \geq x_0$ with $x > x_1$, $i \in I(x)$, $\mathfrak{z} \in \mathrm{Pt}_{x,i}$, a lift j (so $m \geq 1$) and $C \geq C_0$; write $L = L_{\mathfrak{z},j}$, L^{g}, L^{b}, L^{c}, $g = g_{\mathfrak{z},j}$, $b = b(x)$, $E = \mathrm{ue}(x,i)$ and $\mathrm{mt} = \mathrm{mt}_j(\mathfrak{z})$.

The unit error dominates b. Since $n, m \geq 1$, $x^\varepsilon \geq 1$ and $\log\log x \geq 1$ for $x \geq e^e$, we have $E \geq 2b \geq b \geq 1$; in particular $E \geq 0$.

The hypotheses of Lemma 19.27, item 2, hold for $\delta = e^{-x^c}$ *and* $Q = S(x)\,E$. We have $Q \geq 0$, and $S(x) \geq 1$ gives $12\delta \leq 12\delta S(x) \leq 1$, so $0 \leq \delta \leq \frac{1}{12}$. By (Lc1) $L^{\mathrm{g}} \neq 0$, hence $L \neq 0$. By (Lc3)

$\operatorname{rk} L^{\mathrm{b}} \le 3\delta \operatorname{rk} L$. By (Lc4) $2\widehat{\mu}_{\max}(L) \le x^{A_4} b \le x^{A_4} E \le Q$. By (Lc2) $2\log g \le b \le E \le Q$. By (Lc6), (Lc7) and (Lc5),

$$2\,\widehat{\mu}(L) \ge 2\,\widehat{\mu}(L^{\mathrm{c}}) - C_6 E \ge -\mathrm{mt} - (C_6 + C_7)E \ge -x^{A_5} b - (C_6 + C_7)E \ge -Q. \tag{19.10}$$

The orthogonal projection $\pi_{\mathfrak{z},j}$ and the denominator g satisfy $g\,\pi_{\mathfrak{z},j}(L) \subseteq L$ by Definition 19.7.

Conclusion. By Lemma 19.27, item 2, and the second inequality of (19.10),

$$2\,\widehat{\mu}(L^{\mathrm{g}}) \ge 2\,\widehat{\mu}(L) - 12\,\delta\,Q \ge -\mathrm{mt} - (C_6 + C_7)E - 12\,\delta\,S(x)\,E \ge -\mathrm{mt} - (C_6 + C_7 + 1)E,$$

since $12\delta S(x) \le 1$ and $E \ge 0$. With (Lc1),

$$\frac{\operatorname{lcv}(M_{\mathfrak{z},j}, X_{\mathfrak{z},j})}{D_{\mathfrak{z}}} \le -2\,\widehat{\mu}(L^{\mathrm{g}}) \le \mathrm{mt} + C_0 E \le \mathrm{mt} + C\,E,$$

the last step because $C \ge C_0$ and $E \ge 0$. This is (19.3) for every $x \ge x_0$ with $x > x_1$, every member, realised point, lift and $C \ge C_0$. □

Lemma 19.27. *Let K be a totally complex number field, V a hermitian space over K, $L \subseteq V$ a non-zero finitely generated $\mathcal{O}_K$-submodule, π an orthogonal projection of V and $g \ge 1$ an integer with $g\,\pi(L) \subseteq L$. Put $L^{\mathrm{g}} = L \cap \pi(V)$ and $L^{\mathrm{b}} = L \cap \ker \pi$, of ranks r_{g} and r_{b}.*

1. *$\operatorname{rk} L = r_{\mathrm{g}} + r_{\mathrm{b}}$, and*

$$\operatorname{rk}(L)\,\widehat{\mu}(L) \le r_{\mathrm{g}}\,\widehat{\mu}(L^{\mathrm{g}}) + r_{\mathrm{b}}(\widehat{\mu}_{\max}(L) + \log g),$$

   where a product with a rank 0 is read as 0.

2. *Let $0 \le \delta \le \frac{1}{12}$ and $Q \ge 0$ be real numbers with $r_{\mathrm{b}} \le 3\delta \operatorname{rk} L$, $2\,\widehat{\mu}_{\max}(L) \le Q$, $2\log g \le Q$ and $-Q \le 2\,\widehat{\mu}(L)$. Then $L^{\mathrm{g}} \ne 0$ and*

$$2\,\widehat{\mu}(L^{\mathrm{g}}) \ge 2\,\widehat{\mu}(L) - 12\,\delta\,Q.$$

Proof. Write $W = KL$, $W_v = W \otimes_{K,v} \mathbb{C}$, and $d_K = [K : \mathbb{Q}]$.

The two parts span complementary subspaces of W. Since $g\,\pi(L) \subseteq L$ and L spans W, $\pi(W) \subseteq W$. Hence every $y \in W$ is $y = \pi y + (y - \pi y)$ with $\pi y \in W \cap \pi(V)$ and $y - \pi y \in W \cap \ker \pi$, so $W = (W \cap \pi(V)) \oplus (W \cap \ker \pi)$. Moreover $KL^{\mathrm{g}} = W \cap \pi(V)$: the inclusion $\subseteq$ is clear, and for $y \in W \cap \pi(V)$ there is an integer $N \ge 1$ with $Ny \in L$, so $Ny \in L^{\mathrm{g}}$. In the same way $KL^{\mathrm{b}} = W \cap \ker \pi$. Taking dimensions gives $\operatorname{rk} L = r_{\mathrm{g}} + r_{\mathrm{b}}$, and $(1-\pi)(W) = W \cap \ker \pi = KL^{\mathrm{b}}$.

The quotient by the good part is isometric to $(1-\pi)(L)$. $L \cap KL^{\mathrm{g}} = L \cap \pi(V) = L^{\mathrm{g}}$, so L^{g} is saturated in L and L/L^{g} is torsion free of rank r_{b}; give it the quotient norms. The restriction of $1 - \pi$ to W has kernel $W \cap \pi(V) = KL^{\mathrm{g}}$, so it induces an isomorphism of L/L^{g} onto $(1-\pi)(L) \subseteq V$. It is an isometry: for $y \in W_v$ the quotient norm of its class is $\inf \|y - z\|_v$ over $z \in (KL^{\mathrm{g}})_v = (W \cap \pi(V))_v$; since πy lies in that space and $y - z = (1-\pi)y + (\pi y - z)$ with $(1-\pi)y \perp (\pi y - z)$, we get $\|y - z\|_v^2 = \|(1-\pi)y\|_v^2 + \|\pi y - z\|_v^2$, minimal at $z = \pi y$. By additivity of the Arakelov degree in the short exact sequence $0 \to L^{\mathrm{g}} \to L \to L/L^{\mathrm{g}} \to 0$ with the induced sub and quotient norms,

$$\widehat{\deg}\, L = \widehat{\deg}\, L^{\mathrm{g}} + \widehat{\deg}\,(1-\pi)(L). \tag{19.11}$$

Degree of $(1-\pi)(L)$. If $r_{\mathrm{b}} = 0$ then $(1-\pi)(L) = 0$ and its degree is 0. Let $r_{\mathrm{b}} > 0$. For $y \in L$, $g(1-\pi)y = gy - g\pi y \in L$ and $\pi\big(g(1-\pi)y\big) = 0$, so $g\,(1-\pi)(L) \subseteq L^{\mathrm{b}}$, and

both have rank r_{b} (they span $(1-\pi)(W) = KL^{\mathrm{b}}$). A sublattice $N' \subseteq N$ of the same rank satisfies $\widehat{\deg} N' = \widehat{\deg} N - \log \#(N/N') \le \widehat{\deg} N$, and multiplication by g is an isomorphism $N \to gN$ multiplying every norm by g, so $\widehat{\deg}(gN) = \widehat{\deg} N - d_K \operatorname{rk}(N) \log g$. Hence, using $\widehat{\mu}(L^{\mathrm{b}}) \le \widehat{\mu}_{\max}(L)$ (L^{b} is a non-zero submodule of L),

$$\widehat{\deg}\,(1-\pi)(L) = \widehat{\deg}\, g(1-\pi)(L) + d_K r_{\mathrm{b}} \log g \le \widehat{\deg}\, L^{\mathrm{b}} + d_K r_{\mathrm{b}} \log g \le d_K\, r_{\mathrm{b}} (\widehat{\mu}_{\max}(L) + \log g).$$

Inserting this into (19.11) and dividing by d_K gives item 1, since $\widehat{\deg} N = d_K \operatorname{rk}(N)\, \widehat{\mu}(N)$ for $N \neq 0$ and $\widehat{\deg} 0 = 0$.

Item 2. Put $r = \operatorname{rk} L > 0$, $\mu = 2\widehat{\mu}(L)$, $\mu_{\mathrm{g}} = 2\widehat{\mu}(L^{\mathrm{g}})$. From $r_{\mathrm{b}} \le 3\delta(r_{\mathrm{g}} + r_{\mathrm{b}})$ we get $(1-3\delta) r_{\mathrm{b}} \le 3\delta r_{\mathrm{g}}$, and $1 - 3\delta \ge \frac{3}{4}$ gives $r_{\mathrm{b}} \le 4\delta r_{\mathrm{g}}$; also $r_{\mathrm{g}} = r - r_{\mathrm{b}} \ge (1-3\delta) r > 0$, so $L^{\mathrm{g}} \neq 0$. Twice item 1, with $2\widehat{\mu}_{\max}(L) + 2\log g \le 2Q$ when $r_{\mathrm{b}} > 0$, gives $(r_{\mathrm{g}} + r_{\mathrm{b}})\mu \le r_{\mathrm{g}} \mu_{\mathrm{g}} + 2Q r_{\mathrm{b}}$. With $\mu \ge -Q$ this yields

$$r_{\mathrm{g}} \mu_{\mathrm{g}} \ge r_{\mathrm{g}} \mu + r_{\mathrm{b}} \mu - 2Q r_{\mathrm{b}} \ge r_{\mathrm{g}} \mu - 3Q\, r_{\mathrm{b}} \ge r_{\mathrm{g}} \mu - 12\, \delta\, Q\, r_{\mathrm{g}},$$

and dividing by $r_{\mathrm{g}} > 0$ proves item 2. □

Lemma 19.28. *Fix $0 < \varepsilon < 1/52$ and $C_{\mathrm{La}} > 0$. Put*

$$\begin{aligned}
r_G &= \left\lceil \frac{32 + 64\varepsilon}{\varepsilon} \right\rceil + 1, \qquad A_\varepsilon = (5 + 8\varepsilon) 10^{6 r_G + 1}, \qquad B_\varepsilon = \frac{5 + 8\varepsilon}{\log 2}, \\
K_\varepsilon &= 12 e (6 r_G + 1)(A_\varepsilon + 6) + B_\varepsilon + 2, \qquad \alpha_0 = 4\varepsilon/5, \qquad \theta = \varepsilon/100, \\
y_{\mathrm{env}} &= \max\left\{ 1, \frac{24}{\alpha_0^3}, \frac{4}{\varepsilon^2}, \frac{1}{B_\varepsilon \sqrt{C_{\mathrm{La}}}}, \frac{12 C_{\mathrm{La}} K_\varepsilon^2}{\theta^3} \right\}, \qquad X_{\mathrm{env}} = \exp(y_{\mathrm{env}}).
\end{aligned}$$

For $x > 1$ define

$$\begin{aligned}
L &= x^{\varepsilon}, \qquad a = x \exp(L), \qquad \delta_1 = x^{-\varepsilon/8}, \qquad u_{\mathrm{orb}} = \lceil \delta_1^{-1} \rceil, \qquad M_{\mathrm{orb}} = \lceil A_\varepsilon \log x \rceil, \\
G_{\mathrm{orb}} &= (M_{\mathrm{orb}} + 5)(6 r_G + 1) u_{\mathrm{orb}}, \qquad N_{\mathrm{orb}} = \lceil \max\{ 6 e G_{\mathrm{orb}} \delta_1, B_\varepsilon \log x \} \rceil + 1.
\end{aligned}$$

Then for every $x \ge e$,

$$B_\varepsilon \log x \le N_{\mathrm{orb}} \le K_\varepsilon \log x.$$

Now let $x \ge X_{\mathrm{env}}$ and $J \ge 1$, let κ be admissible, and fix data, as in the concrete good-host requirement, for which $\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}, T, t)$ holds. Put

$$E = E_{\mathfrak{h}}, \qquad L_a = \log(a \operatorname{rd}_E), \qquad L_c = C_{\mathrm{La}} N_{\mathrm{orb}}^2 \log a.$$

Then

$$x^{\varepsilon} \le L_a \le 2x^{\varepsilon}, \qquad x^{\varepsilon} \le L_c \le x^{\varepsilon + \varepsilon/100}.$$

The threshold X_{env} depends only on ε and C_{La}; it does not depend on the host, on the remaining data or on any integer tables.

Proof. First let $x \ge e$ and write $y = \log x \ge 1$. Since $0 < \delta_1 \le 1$,

$$\delta_1 u_{\mathrm{orb}} \le 1 + \delta_1 \le 2, \qquad M_{\mathrm{orb}} + 5 \le A_\varepsilon y + 6 \le (A_\varepsilon + 6) y.$$

It follows that

$$G_{\mathrm{orb}} \delta_1 \le 2(6 r_G + 1)(A_\varepsilon + 6) y.$$

For nonnegative real numbers u, v, $\max(u,v) \le u+v$, and $\lceil u \rceil + 1 \le u+2$. Consequently

$$\begin{aligned} N_{\mathrm{orb}} &\le 6eG_{\mathrm{orb}}\delta_1 + B_\varepsilon y + 2 \\ &\le \bigl(12e(6r_G+1)(A_\varepsilon+6) + B_\varepsilon + 2\bigr)y = K_\varepsilon y. \end{aligned}$$

Its defining maximum also gives $N_{\mathrm{orb}} \ge B_\varepsilon y$.

Now let $x \ge X_{\mathrm{env}}$ and fix data as in the second part of the statement. By Lemma 19.16, $1 \le \mathrm{rd}_E \le N_{\mathfrak{h}}$. Condition (Hg6) gives $1 < Y \le x$, and hence

$$0 \le \log \mathrm{rd}_E \le \log N_{\mathfrak{h}} \le 2Y^{\varepsilon/5}(\log Y)^2 \le 2x^{\varepsilon/5}y^2.$$

For every $z \ge 0$, the exponential series gives $e^z \ge z^j/j!$ for each nonnegative integer j. Since $y \ge 24/\alpha_0^3$,

$$e^{\alpha_0 y} \ge \frac{\alpha_0^3 y^3}{6} \ge 4y^2,$$

so $2x^{\varepsilon/5}y^2 \le x^\varepsilon/2 = L/2$. Since $y \ge 4/\varepsilon^2$,

$$e^{\varepsilon y} \ge \frac{\varepsilon^2 y^2}{2} \ge 2y,$$

so $y \le L/2$. Therefore

$$L \le L_a = L + y + \log \mathrm{rd}_E \le 2L, \qquad L \le \log a = L + y \le \tfrac{3}{2}L \le 2L.$$

The lower orbit-order bound and $y \ge 1/(B_\varepsilon\sqrt{C_{\mathrm{La}}})$ give $C_{\mathrm{La}}N_{\mathrm{orb}}^2 \ge 1$. Together with $\log a \ge L$, this proves $L_c \ge L$. For the upper bound, the orbit-order estimate and $\log a \le 2L$ give

$$L_c \le 2C_{\mathrm{La}}K_\varepsilon^2 y^2 L.$$

Finally $y \ge 12C_{\mathrm{La}}K_\varepsilon^2/\theta^3$, so

$$e^{\theta y} \ge \frac{\theta^3 y^3}{6} \ge 2C_{\mathrm{La}}K_\varepsilon^2 y^2.$$

Multiplying by L yields $L_c \le Lx^\theta = x^{\varepsilon+\varepsilon/100}$. All these inequalities hold for $x \ge X_{\mathrm{env}}$, and this common threshold depends neither on the host nor on any tables chosen later. □

20 The reference budget

This section proves the reference budget Theorem 20.2, from which Proposition 21.2 obtains the realisation estimate (Ri1) of the base covolume realisations. It is placed before (TCS) because the statement of (TCS) asks for reference comparison data that satisfy every hypothesis of this theorem, and does not ask for (Ri1) itself; the reader can thus see, before reading (TCS), exactly what is assumed and what is proved. The method is a comparison of covolumes. Theorem 20.4 bounds the normalised log-covolume of the base lattice by D times the mean of the logarithms of the archimedean scalings, plus the log-covolumes of the reference lattices, plus a sum over finite places of local index terms. Lemma 20.3 bounds the local terms by local estimates whose denominators are integers: a group index, a level and one denominator for each lift. Theorem 20.2 then sums the local bounds, and its hypotheses bound the logarithms of these integers by the budgets of the record: those of the group index and the level by a multiple of m times a bound for the unit error, and those of the denominators by the terms tc_j. The passage between log-covolumes and slopes is Lemma 20.5. No result of this section depends on (TCS).

Definition 20.1. Reference comparison data consist of the following objects and conditions.

(Rc1) Fields, types and dimensions. A totally complex number field K, an involutive field automorphism $a \mapsto \bar{a}$, a subfield $K_0 \subseteq K$, and an embedding $\sigma_0 : K \hookrightarrow \mathbb{C}$ satisfy

$$\sigma(\bar{a}) = \overline{\sigma(a)} \quad (a \in K_0,\ \sigma : K \hookrightarrow \mathbb{C}), \qquad \sigma_0(\bar{a}) = \overline{\sigma_0(a)} \quad (a \in K).$$

Write $n_K = [K : \mathbb{Q}]$. A set of types $\mathcal{T}$ has a distinguished element t_0 and a map μ from the finite places of K to $\mathcal{T}$, with

$$\mu(\bar{w}) = \mu(w), \qquad \{w : \mu(w) \neq t_0\} \text{ finite.}$$

Here $\bar{w}$ is the conjugate prime under the involution. Finally, the data include an integer $D \geq 1$, a finite set J and integers $d_j \geq 0$ for $j \in J$.

(Rc2) Based spaces, metrics and lattices. The spaces

$$E = K^D, \qquad Q_t = K^D, \qquad W_j = K^{d_j}, \qquad T_{j,t} = K^{d_j}$$

carry their standard bases and positive definite Hermitian metrics at every embedding of K into $\mathbb{C}$. The metrics at a pair of complex-conjugate embeddings are conjugate: their Gram matrices in these bases are complex conjugates. The data include full $\mathcal{O}_K$-lattices

$$R_{\mathrm{b}} \subset E, \qquad L_t \subset Q_t, \qquad M_j \subset W_j, \qquad A_{j,t} \subset T_{j,t}$$

and K-linear isomorphisms

$$\iota_t : Q_t \longrightarrow E, \qquad \theta_{j,t} : T_{j,t} \longrightarrow W_j.$$

Here a lattice is full if it is finitely generated over $\mathcal{O}_K$ and spans the indicated K-space.

For any one of these metrised lattices put

$$\mathrm{lcv}(\overline{L}) = -\frac{2}{n_K}\widehat{\deg}(\overline{L}).$$

In dimension zero the lattice and space are zero, determinants of empty matrices are 1, and $\mathrm{lcv}(0) = 0$. Every occurrence of lcv refers to these metrics.

(Rc3) Algebraic metric matrices. For all j and t the data include a nondegenerate matrix $H_{j,t} \in M_{d_j}(K)$ such that

$$\bar{H}_{j,t}^{T} = H_{j,t}, \qquad \mathrm{Gram}_{T_{j,t},\sigma_0} = \sigma_0(H_{j,t}).$$

At the distinguished type, in addition,

$$H_{j,t_0} \in M_{d_j}(K_0), \qquad \mathrm{Gram}_{T_{j,t_0},\sigma} = \sigma(H_{j,t_0}) \quad \text{for every } \sigma : K \hookrightarrow \mathbb{C}.$$

These are matrix identities in the fixed bases.

(Rc4) Determinant metric ratios. For an isomorphism $f : U \to V$ between two of the metrised spaces of the same dimension, write

$$r_\sigma(f) = \frac{\det\big(\langle fe_a, fe_b\rangle_{V,\sigma}\big)}{\det\big(\langle e_a, e_b\rangle_{U,\sigma}\big)},$$

where (e_a) is the specified basis of U. The data include positive real numbers κ_σ, equal at conjugate embeddings, and elements $\rho_t \in K^\times$ such that

$$\rho_{t_0} = 1, \qquad \sigma_0(\rho_t) \in \mathbb{R}_{>0}, \qquad v_w(\rho_{\mu(w)}) = 0 \quad \text{for every finite } w.$$

For a nonzero element $a \in K$, $v_w(a)$ denotes the exponent of w in the principal fractional ideal $a\mathcal{O}_K$, so v_w takes the value 1 on a local uniformizer. These data satisfy

$$r_\sigma(\iota_{t_0}) = \kappa_\sigma^D \prod_{j\in J} r_\sigma(\theta_{j,t_0}) \quad \text{for every } \sigma,$$

$$r_{\sigma_0}(\iota_t) = \bigl(\kappa_{\sigma_0}\sigma_0(\rho_t)\bigr)^D \prod_{j\in J} r_{\sigma_0}(\theta_{j,t}) \quad \text{for every } t.$$

(Rc5) Local bases, Gram determinants and gluing. For every type t the data include an element $g_t \in K^\times$ with

$$\sigma_0(g_t) = \det \mathrm{Gram}_{Q_t,\sigma_0}\,.$$

For every t, w the data include a K-basis $(e_{t,w,a})_{a=1}^D$ of Q_t; let $E_{t,w}$ be its column matrix in the fixed basis. This basis generates L_t over the semilocal ring

$$\mathcal{O}_{K,\{w,\bar w\}} = S^{-1}\mathcal{O}_K, \qquad S = \mathcal{O}_K \setminus (w \cup \bar w);$$

that is, the spans of this basis and of L_t over that ring are equal. Put

$$G_{t,w} = g_t\, \overline{\det E_{t,w}}\, \det E_{t,w} \in K^\times.$$

The single-prime local gluing identities hold:

$$(R_{\mathrm{b}})_w = \iota_{\mu(w)}(L_{\mu(w)})_w \quad \text{for every finite } w.$$

(Rc6) Signed indices and the base Gram bound. For full lattices A, B in the same K-space put

$$I_w(A,B) = \mathrm{length}_{\mathcal{O}_{K,w}}(A_w/(A\cap B)_w) - \mathrm{length}_{\mathcal{O}_{K,w}}(B_w/(A\cap B)_w).$$

For a nondegenerate conjugate-symmetric matrix H in a fixed basis put

$$A^{\#H} = \{z \in K^d : \bar a^{\,T} H z \in \mathcal{O}_K \text{ for every } a \in A\}, \qquad J_w(H,A) = I_w(A, A^{\#H}).$$

The finite-place weight in these data is

$$\lambda_w = \frac{\log|\mathcal{O}_K/w|}{n_K}.$$

The function $w \mapsto \lambda_w v_w(G_{t_0,w})$ has finite support, and the **base Gram bound** holds:

$$\mathrm{lcv}(\overline{L}_{t_0}) \le \sum_{w \text{ finite}} \lambda_w v_w(G_{t_0,w}).$$

The sum runs over this finite support.

(Rc7) Attachment to a selected base realisation. When the data are attached to a member with a base covolume realisation, K is the field of the member, $\overline{R}_{\mathrm{b}}$ its base lattice with its metric, D its base dimension and J its lift index set, and each $\overline{M}_j$ is the reference lattice of its selected point, with its metric. Objects shared by the data and the realisation are equal. No equality between d_j and D is imposed. Equality of a metric specified by a frame means equality of the induced norms in the fixed coordinates.

Theorem 20.2. *Fix reference comparison data attached to a member with base covolume realisations, a word ω of that member, a real $C \geq 0$, and a finite set $\mathcal{T}_{\mathrm{w}} \subseteq \mathcal{S}$ of finite places. Suppose $C_w = 0$ outside $\mathcal{S}$ and $U_{wj} = 0$ for $w \in \mathcal{S} \setminus \mathcal{T}_{\mathrm{w}}$. Let $a_{\mathrm{grp}}, a_{\mathrm{lev}}, a_{\mathrm{den},j}$ be positive integers, and let $f_{wj}, e_{wj}, a_{wj}, \eta_{wj}, q_w$ be real numbers on the indicated finite sets. Assume*

$$v_w(G_{\mu(w),w}) \leq D v_w(a_{\mathrm{grp}} a_{\mathrm{lev}}) \quad (w \in \mathcal{S}),$$

$$\sum_{j=1}^{m} (I_{wj} + I'_{wj}) \leq D \left(\sum_{j=1}^{m} (2f_{wj} + e_{wj}) + q_w \right) \quad (w \in \mathcal{T}_{\mathrm{w}}),$$

$$J_{wj} \leq D a_{wj} + \eta_{wj} + D\big(v_w(a_{\mathrm{den},j}) + v_w(a_{\mathrm{lev}})\big) \quad (w \in \mathcal{T}_{\mathrm{w}},\ 1 \leq j \leq m).$$

Let $u_\star$ be real and $c_1, c_{\mathrm{G}}, c_{\mathrm{e}} \geq 0$ satisfy

$$0 \leq \mathrm{ue}(x,i) \leq u_\star, \quad 1 \leq m u_\star, \quad c_{\mathrm{e}} \geq C + c_{\mathrm{G}} + c_1 + 1,$$

and suppose

$$\sum_{w \in \mathcal{T}_{\mathrm{w}}} \lambda_w (e_{wj} + a_{wj}) \leq c_1 u_\star, \qquad \sum_{w \in \mathcal{T}_{\mathrm{w}}} \lambda_w \eta_{wj} \leq \mathrm{Df}_j(x,i), \qquad \sum_{w \in \mathcal{T}_{\mathrm{w}}} \lambda_w q_w \leq 1,$$

$$\log a_{\mathrm{grp}} + \log a_{\mathrm{lev}} \leq c_{\mathrm{G}} m u_\star, \qquad \mathrm{tc}_j(x,i) \geq \log a_{\mathrm{den},j} + \log a_{\mathrm{lev}}, \qquad \mathrm{le}(x,i,\omega) = c_{\mathrm{e}} m u_\star.$$

Finally suppose

$$\mathrm{sc}(x,i) = \frac{1}{n_K} \sum_{\sigma} \log \kappa_\sigma, \qquad 2 \sum_{j=1}^{m} \sum_{w \in \mathcal{T}_{\mathrm{w}}} \lambda_w f_{wj} \leq F(\mathfrak{V}),$$

and

$$\mathrm{lm}(x,i,\omega) = \sum_{j=1}^{m} \sum_{\tau \in \Phi} (k - m + 2j - 1) w_j(\tau) \log \Omega(\mathfrak{h}_{x,i}).$$

Then all three inequalities and identities of (Ri1) hold at this member and word, for this C and a suitable real E_ω. In particular, if these hypotheses hold on a common tail of a family, for all its members and words and with the same constants $C, c_1, c_{\mathrm{G}}, c_{\mathrm{e}}$, then the family satisfies (Ri1) there. The theorem does not assert that reference data or such local inputs exist.

Proof. For every positive rational integer a, unique factorisation of its principal ideal and its ideal norm give

$$\sum_{w \text{ finite}} \lambda_w v_w(a) = \frac{\log N_{K/\mathbb{Q}}(a\mathcal{O}_K)}{n_K} = \log a.$$

Indeed $a\mathcal{O}_K$ has index a^{n_K} in the free abelian group $\mathcal{O}_K$, of rank n_K. Each summand is nonnegative. Its sum over any subset of finite places is therefore at most $\log a$.

Apply Lemma 20.3. Use $g_w = v_w(a_{\mathrm{grp}} a_{\mathrm{lev}})$ and

$$L_G = \log a_{\mathrm{grp}} + \log a_{\mathrm{lev}}, \qquad E_{1j} = c_1 u_\star, \qquad E_3 = 1.$$

In the local bound for J_{wj}, use the error

$$\widetilde{\eta}_{wj} = \eta_{wj} + D\big(v_w(a_{\mathrm{den},j}) + v_w(a_{\mathrm{lev}})\big).$$

As $D \geq 1$, the defect assumption and the positive-integer place-sum bound prove

$$\sum_{w \in \mathcal{T}_{\mathrm{w}}} \frac{\lambda_w \widetilde{\eta}_{wj}}{D} \leq E_{2j} := \frac{\mathrm{Df}_j(x,i)}{D} + \log a_{\mathrm{den},j} + \log a_{\mathrm{lev}}.$$

The same place-sum bound gives the required L_G estimate over $\mathcal{S}$. All other local and summed hypotheses of Lemma 20.3 are among those assumed above. Its attachment conclusion, using the specified scaling and frame ties, proves the first inequality of (Ri1) with

$$E_\omega = L_G + \sum_{j=1}^{m} (E_{1j} + E_{2j}) + 1.$$

The second identity is a stated tie. For the remaining budget, the hypotheses give

$$\begin{aligned} E_\omega + mC\,\mathrm{ue}(x,i) &\le (c_{\mathrm{G}} + c_1 + C) m u_\star + 1 + \sum_{j=1}^{m} \left(\frac{\mathrm{Df}_j(x,i)}{D} + \mathrm{tc}_j(x,i) \right) \\ &\le c_{\mathrm{e}} m u_\star + \sum_{j=1}^{m} \left(\frac{\mathrm{Df}_j(x,i)}{D} + \mathrm{tc}_j(x,i) \right). \end{aligned}$$

Here multiplication of $\mathrm{ue} \le u_\star$ by mC is valid since $m, C \ge 0$, and the last step uses $1 \le m u_\star$ and the bound on c_{e}. Substitute the specified value of le. This is the last inequality in (Ri1). The assertion about families follows from the same pointwise argument, with the same common cutoff and constants. □

Lemma 20.3. *Fix reference comparison data and a finite set $\mathcal{T}_{\mathrm{w}} \subseteq \mathcal{S}$ of finite places of K. For a place w and $j \in J$ put*

$$I_{wj} = I_w(M_j, \theta_{j,\mu(w)} A_{j,\mu(w)}), \quad I'_{wj} = I_{\bar{w}}(M_j, \theta_{j,\mu(w)} A_{j,\mu(w)}), \quad J_{wj} = J_w(H_{j,\mu(w)}, A_{j,\mu(w)}),$$

$$U_{wj} = I_{wj} + I'_{wj} + J_{wj}, \qquad C_w = v_w(G_{\mu(w),w}) + \sum_{j \in J} U_{wj}.$$

Suppose $C_w = 0$ for $w \notin \mathcal{S}$, and $U_{wj} = 0$ for every $w \in \mathcal{S} \setminus \mathcal{T}_{\mathrm{w}}$ and $j \in J$. Let g_w, q_w $(w \in \mathcal{S})$, and $f_{wj}, e_{wj}, a_{wj}, \eta_{wj}$ $(w \in \mathcal{T}_{\mathrm{w}},\ j \in J)$ be real numbers. Assume

$$v_w(G_{\mu(w),w}) \le D g_w \quad (w \in \mathcal{S}),$$

$$\sum_{j \in J} (I_{wj} + I'_{wj}) \le D \left(\sum_{j \in J} (2 f_{wj} + e_{wj}) + q_w \right) \quad (w \in \mathcal{T}_{\mathrm{w}}), \qquad J_{wj} \le D a_{wj} + \eta_{wj}.$$

Finally suppose that real numbers L_G, E_3, E_{1j}, E_{2j} satisfy

$$\sum_{w \in \mathcal{S}} \lambda_w g_w \le L_G, \qquad \sum_{w \in \mathcal{T}_{\mathrm{w}}} \lambda_w q_w \le E_3,$$

$$\sum_{w \in \mathcal{T}_{\mathrm{w}}} \lambda_w (e_{wj} + a_{wj}) \le E_{1j}, \qquad \sum_{w \in \mathcal{T}_{\mathrm{w}}} \frac{\lambda_w \eta_{wj}}{D} \le E_{2j} \quad (j \in J).$$

Then

$$\frac{\mathrm{lcv}(R_{\mathrm{b}})}{D} \le \frac{1}{n_K} \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log \kappa_\sigma + \sum_{j \in J} \frac{\mathrm{lcv}(M_j)}{D} + 2 \sum_{j \in J} \sum_{w \in \mathcal{T}_{\mathrm{w}}} \lambda_w f_{wj} + L_G + \sum_{j \in J} (E_{1j} + E_{2j}) + E_3.$$

No sign condition is imposed on the local numbers or on the error bounds above. The notation introduced here is local to this lemma.

Suppose moreover that these data are attached, as in (Rc7), to a member with base covolume realisations as in Definition 19.6. Fix a word ω of that member and suppose

$$\mathrm{sc} = \frac{1}{n_K}\sum_{\sigma}\log\kappa_\sigma, \qquad 2\sum_{j\in J}\sum_{w\in\mathcal{T}_{\mathrm{w}}}\lambda_w f_{wj} \le F(\mathfrak{V}).$$

Set $E_\omega = L_G + \sum_{j\in J}(E_{1j}+E_{2j}) + E_3$. Then the reference comparison in (Ri1) holds:

$$\mu_{\mathrm{b}}^{\vee} - \mathrm{sc} \le \sum_{j=1}^{m}\frac{\mathrm{lcv}(M_{\mathfrak{z},j}, X_{\mathfrak{z},j})}{D} + F(\mathfrak{V}) + E_\omega.$$

Proof. For a finite place, $\lambda_w = \log|\mathcal{O}_K/w|/n_K > 0$, and $D \ge 1$. By Theorem 20.4 and the support assumption,

$$\frac{\mathrm{lcv}(R_{\mathrm{b}})}{D} \le \frac{1}{n_K}\sum_{\sigma}\log\kappa_\sigma + \sum_{j\in J}\frac{\mathrm{lcv}(M_j)}{D} + \frac{1}{D}\sum_{w\in\mathcal{S}}\lambda_w C_w.$$

The combined signed local term U_{wj} vanishes off $\mathcal{T}_{\mathrm{w}}$. Thus

$$\frac{1}{D}\sum_{w\in\mathcal{S}}\lambda_w C_w = \frac{1}{D}\sum_{w\in\mathcal{S}}\lambda_w v_w(G_{\mu(w),w}) + \frac{1}{D}\sum_{w\in\mathcal{T}_{\mathrm{w}}}\lambda_w\sum_{j\in J}(I_{wj}+I'_{wj}+J_{wj}).$$

The first term is at most L_G. For each place in $\mathcal{T}_{\mathrm{w}}$ the index hypotheses give

$$\frac{1}{D}\sum_{j\in J}(I_{wj}+I'_{wj}+J_{wj}) \le 2\sum_{j\in J}f_{wj} + \sum_{j\in J}(e_{wj}+a_{wj}) + q_w + \frac{1}{D}\sum_{j\in J}\eta_{wj}.$$

Multiply by the nonnegative weight λ_w and sum. Interchange the two finite sums and use the three remaining budget bounds. This gives

$$\frac{1}{D}\sum_{w\in\mathcal{S}}\lambda_w C_w \le L_G + 2\sum_{j\in J}\sum_{w\in\mathcal{T}_{\mathrm{w}}}\lambda_w f_{wj} + \sum_{j\in J}(E_{1j}+E_{2j}) + E_3.$$

Substituting this bound proves the first assertion. The signed index terms are combined before any upper bound is taken, so their individual signs are not used.

For the attachment assertion, the determinant of a hermitian dual is the dual determinant line with the dual norm, so its Arakelov degree is the negative of the original degree. Using the rank and slope identities in the base covolume realisation gives

$$\mu_{\mathrm{b}}^{\vee} = 2\widehat{\mu}(\overline{R}_{\mathrm{b}}^{\vee}) = -\frac{2\widehat{\deg}(\overline{R}_{\mathrm{b}})}{n_K D} = \frac{\mathrm{lcv}(R_{\mathrm{b}})}{D}.$$

By (Rc7), the reference lattices and their metrics in the preceding inequality are the covolume lattices and metrics of the selected point. Their intrinsic log-covolumes agree with the frame log-covolumes by Lemma 20.5, with the stated zero-rank convention if needed. Subtract the specified scaling term from the preceding bound and use the assumed upper bound by $F(\mathfrak{V})$. The result is the reference comparison displayed in the statement. □

Theorem 20.4. *For reference comparison data put*

$$I_{wjt} = I_w(M_j, \theta_{j,t}A_{j,t}), \qquad J_{wjt} = J_w(H_{j,t}, A_{j,t}), \qquad C_{wt} = v_w(G_{t,w}) + \sum_{j\in J}(I_{wjt} + I_{\bar{w}jt} + J_{wjt}).$$

Then $w \mapsto C_{w,\mu(w)}$ has finite support and

$$\operatorname{lcv}(R_{\mathrm{b}}) \le D\,\frac{1}{n_K}\sum_{\sigma:K\hookrightarrow\mathbb{C}}\log\kappa_\sigma + \sum_{j\in J}\operatorname{lcv}(M_j) + \sum_{w\ \text{finite}}\lambda_w C_{w,\mu(w)}.$$

The finite-place sum is taken over its finite support. Here the reference ranks d_j may differ from one another and from D.

Proof. Write $N = [K:\mathbb{Q}]$, and, for a full lattice L in one of the fixed coordinate spaces, let $\mathfrak{a}_L$ be its determinant fractional ideal and $a_L(w) = v_w(\mathfrak{a}_L)$. The determinant-line definition of degree and Lemma 20.5 give

$$\operatorname{lcv}(L) = \frac{2\log N\mathfrak{a}_L + \sum_{\sigma:K\hookrightarrow\mathbb{C}}\log\det H_{L,\sigma}}{N}, \tag{1}$$

where $H_{L,\sigma}$ is the metric matrix in the fixed coordinate basis. This formula is also valid in dimension zero, with determinant 1 and log-covolume 0. Consequently two lattices in the same metrised space satisfy

$$\operatorname{lcv}(L') - \operatorname{lcv}(L) = 2\sum_w \lambda_w(a_{L'}(w) - a_L(w)). \tag{2}$$

For a linear isomorphism f between two of the metrised spaces, the determinant metric ratio in the definition gives

$$\operatorname{lcv}(fL) - \operatorname{lcv}(L) = \frac{1}{N}\sum_\sigma \log r_\sigma(f). \tag{3}$$

Indeed the determinant ideal changes by $\det f$, the principal-ideal norm formula changes its finite term into $2N^{-1}\sum_\sigma \log|\sigma(\det f)|$, and the remaining metric terms give exactly the logarithm of $r_\sigma(f)$.

The signed local index has the formula

$$I_w(A,B) = a_B(w) - a_A(w). \tag{4}$$

To see this, put $C = A_w \cap B_w$. For either $L = A_w$ or $L = B_w$, the length of L/C over the discrete valuation ring is $a_C(w) - a_L(w)$, by diagonalising an inclusion of free modules over that ring. Subtraction proves (4).

Put $h_{jt} = \det H_{j,t}$. These determinants are nonzero: their images at σ_0 are determinants of positive definite metric matrices. Moreover $\bar{H}_{j,t}^T = H_{j,t}$, since the two sides agree at σ_0 and that embedding is injective. The dual for $\bar{x}^T H_{j,t} y$ has determinant ideal valuation

$$a_{(A_{j,t})^{\#H_{j,t}}}(w) = -v_w(h_{jt}) - a_{A_{j,t}}(\bar{w}).$$

This follows by writing its defining pairing in bases over the two conjugate local rings: the matrix $H_{j,t}$ identifies the dual with the ordinary linear dual of the conjugate lattice. Thus (4) gives

$$J_w(H_{j,t}, A_{j,t}) = -v_w(h_{jt}) - a_{A_{j,t}}(w) - a_{A_{j,t}}(\bar{w}). \tag{5}$$

Write $I_{wjt} = I_w(M_j, \theta_{j,t}A_{j,t})$ and $J_{wjt} = J_w(H_{j,t}, A_{j,t})$, and set

$$C_{wt} = v_w(G_{t,w}) + \sum_{j\in J}(I_{wjt} + I_{\bar{w}jt} + J_{wjt}).$$

The same semilocal basis generates L_t at both w and $\bar{w}$. Hence

$$v_w(G_{t,w}) = v_w(g_t) + a_{L_t}(w) + a_{L_t}(\bar{w}).$$

Equations (4),(5) cancel every determinant-ideal valuation of $A_{j,t}$ and give

$$\begin{aligned} C_{wt} = {} & v_w(g_t) + a_{L_t}(w) + a_{L_t}(\bar{w}) \\ & + \sum_{j\in J} (v_w(\det \theta_{j,t}) + v_{\bar{w}}(\det \theta_{j,t}) - v_w(h_{jt}) - a_{M_j}(w) - a_{M_j}(\bar{w})). \end{aligned} \tag{6}$$

The determinant-ratio identities at σ_0, with the fixed metric determinants of E and W_j cancelled, imply the following equality in $K^\times$:

$$\frac{\overline{\det \iota_t}\, \det \iota_t / g_t}{\overline{\det \iota_{t_0}}\, \det \iota_{t_0} / g_{t_0}} = \rho_t^D \prod_{j\in J} \frac{\overline{\det \theta_{j,t}}\, \det \theta_{j,t}/h_{jt}}{\overline{\det \theta_{j,t_0}}\, \det \theta_{j,t_0}/h_{jt_0}}. \tag{7}$$

In fact its two sides have the same image under σ_0, and σ_0 commutes with conjugation on K. Its injectivity proves (7). For $t = \mu(w)$ the factor ρ_t has valuation zero at w. Taking v_w of (7) and using (6) therefore yields, with $\widetilde{L}_t = \iota_t L_t$,

$$C_{w,\mu(w)} - C_{w,t_0} = a_{\widetilde{L}_{\mu(w)}}(w) + a_{\widetilde{L}_{\mu(w)}}(\bar{w}) - a_{\widetilde{L}_{t_0}}(w) - a_{\widetilde{L}_{t_0}}(\bar{w}). \tag{8}$$

The local gluing identities and $\mu(\bar{w}) = \mu(w)$ identify the two selected terms with $a_{R_\mathrm{b}}(w)$ and $a_{R_\mathrm{b}}(\bar{w})$. Since $\lambda_{\bar{w}} = \lambda_w$, summing (8) and applying (2) gives

$$\sum_w \lambda_w (C_{w,\mu(w)} - C_{w,t_0}) = 2 \sum_w \lambda_w (a_{R_\mathrm{b}}(w) - a_{\widetilde{L}_{t_0}}(w)) = \mathrm{lcv}(R_\mathrm{b}) - \mathrm{lcv}(\widetilde{L}_{t_0}). \tag{9}$$

All these sums are finite. At the base type this follows from (6), since each fixed nonzero scalar and each fixed fractional ideal has only finitely many nonzero valuations. The selected type differs from the base at only finitely many places, so selection changes this support by a finite set.

It remains to estimate the base expression. At the base type, H_{j,t_0} is the metric matrix of T_{j,t_0} at every embedding. Equations (1),(5) and the principal-ideal norm formula give

$$\mathrm{lcv}(A_{j,t_0}) = -\sum_w \lambda_w J_{wj,t_0}. \tag{10}$$

Also (2),(4) give

$$\mathrm{lcv}(\theta_{j,t_0} A_{j,t_0}) = \mathrm{lcv}(M_j) + 2 \sum_w \lambda_w I_{wj,t_0}.$$

By (3),(10), and conjugation of the finite-place sum,

$$\frac{1}{N} \sum_\sigma \log r_\sigma(\theta_{j,t_0}) = \mathrm{lcv}(M_j) + \sum_w \lambda_w (I_{wj,t_0} + I_{\bar{w}j,t_0} + J_{wj,t_0}). \tag{11}$$

The base determinant-ratio identity and (3) now yield

$$\mathrm{lcv}(\widetilde{L}_{t_0}) = \mathrm{lcv}(L_{t_0}) + D\, \frac{1}{N} \sum_\sigma \log \kappa_\sigma + \sum_{j\in J} \frac{1}{N} \sum_\sigma \log r_\sigma(\theta_{j,t_0}).$$

The base Gram bound and (11) imply

$$\mathrm{lcv}(\widetilde{L}_{t_0}) \le D\, \frac{1}{N} \sum_\sigma \log \kappa_\sigma + \sum_{j\in J} \mathrm{lcv}(M_j) + \sum_w \lambda_w C_{w,t_0}.$$

Substitute this into (9). The terms with C_{w,t_0} cancel, giving the claimed inequality. No equality between a reference rank d_j and D has been used. □

Lemma 20.5. *Let K be a totally complex number field, let $d \geq 1$, and let X be a frame on K^d. Let $M \subseteq K^d$ be a finitely generated $\mathcal{O}_K$-submodule of rank r with $1 \leq r \leq d$, and let $\overline{M}_X$ be M with the induced hermitian norms $\|y\|_{X,v} = |X(v)v(y)|$ on its K-span. Then*

$$\mathrm{ht}(M, X) = \exp\bigl(-\widehat{\deg}(\overline{M}_X)\bigr), \qquad \mathrm{lcv}(M, X) = -2r\,\widehat{\mu}(\overline{M}_X).$$

Proof. Put $N = [K : \mathbb{Q}] = 2s$. The intrinsic Minkowski space of $\overline{M}_X$ is isometric to the real span of the image of M used to define $\mathrm{covol}_X(M)$. The degree–Minkowski-covolume formula, with one embedding per complex pair and the unweighted real Euclidean metric, therefore gives

$$\mathrm{covol}_X(M) = 2^{-rs}\,|\operatorname{disc}(K)|^{r/2} \exp\bigl(-\widehat{\deg}(\overline{M}_X)\bigr).$$

Substituting this into the definition of the height and using $N/2 = s$ yields

$$\mathrm{ht}(M, X) = 2^{rN/2}\,|\operatorname{disc}(K)|^{-r/2}\,\mathrm{covol}_X(M) = \exp\bigl(-\widehat{\deg}(\overline{M}_X)\bigr).$$

The definitions of normalised log-covolume and slope now give

$$\mathrm{lcv}(M, X) = \frac{2}{N}\log \mathrm{ht}(M, X) = -\frac{2}{N}\widehat{\deg}(\overline{M}_X) = -2r\,\widehat{\mu}(\overline{M}_X).$$

Throughout, r is the rank of M, and the covolume is taken in its real span. □

21 The theta chain system (TCS) and model-level packet supply

This section states the one standing assumption of the paper, the existence of a theta chain system (TCS), as Hypothesis 21.1, and proves from it model-level packet supply, Proposition 21.2. The statement describes a theta chain system only through the records it must carry, its packets; Appendix A explains how chains of theta lifts are expected to produce these records. The proof of Proposition 21.2 is the only place where (TCS) is used directly, and Section 19 carries its conclusion to design existence. The proof keeps the packets of (TCS) and adds three properties. It checks a gap condition in the Chern-root calculus (Definition 21.3) for the integer arrays of the models (Lemma 21.4), using the bounds for the host field and the depths proved in Section 19; with the model identities this verifies the remaining hypothesis (Aj2) of Theorem 29.14, and that theorem gives the bounds (Ms1) and (Ms3) for model slopes. It then applies the reference budget Theorem 20.2 of Section 20, which gives the realisation estimate (Ri1). The definition and the lemma of the gap condition close the section. The existence of a theta chain system (TCS) is not proved in this paper. Appendix A outlines the construction beneath it, the chain of theta lifts of Alpöge's account, and gives the status of each of its steps.

The statement of (TCS) is long because it names every record that the proofs above it use, so we describe it first in words. The data are those of design existence: a window ratio $\lambda > 1$, a host requirement supplied on λ-windows by the concrete good-host requirement, a parameter ε, and, at every large scale x of a set containing λ-windows, a host $\mathfrak{h}_x$ with tops T_x, a rank t_x and a fixed choice of all the witnesses of the good-host requirement, among them the rank datum, the good-orbit choices, the spacing set and the auxiliary primes. (TCS) asserts constants, common to all scales, and at every large scale of the set a base translate packet over exactly these data: one loss datum and one slope record for every translate $u \in A_x$ (Definition 19.1). The packets are subject to three clauses.

The first clause, (Rr1), concerns the loss data. They satisfy the conditions on heights and weights, small shares, the place record and sizes of Definition 5.5, with common constants, and their loss parameter is the exponent of the frame. The second, (Rr2), makes the numerical data explicit. The loss data carry the rank datum fixed by the good host, with its rank sizes. Their heights, depths, weights and shifts are integers in the boxes of Definition 18.2, with exponents $c_D > 397$ and $c_h \geq 402$. Their tops and orbit representatives are the good-orbit choices of the good host, and their orbit corrections satisfy the bounds of Definition 18.3 with common constants. From these numerical data alone Section 18 derives the linking property (Theorem 18.4).

The third clause, (Rr3), asks for models and local comparisons, and only under a condition: that the Sha bound $\mathrm{V}(E_{\mathfrak{h}_x}; t_x, \exp(x^{\varepsilon}/2))$ holds at all large scales of the set. Under this condition the same packets carry, on a tail of scales, the following data, compatible with one another and with common constants. First, lower completions: the records that the class $\mathcal{A}_{\mathrm{prop}}$ of Part III asks of its members (Definition 19.2), with all the numbered conditions listed there except eight, which Proposition 19.3 derives. Second, the realisation data of the family formed by the packet members, over common realised points: base covolume realisations, chain data, deep-cover data, and component and model-slope data, with the height conditions (Ht1), (Ht2), (Ht4) and (Ht5). Third, the model data: the model field is the host field, and the entries of the models are given by explicit identities in terms of integer arrays built from the weights and the ranks, of a level formed from the auxiliary primes g_1, g_2, and of $\mathrm{Re}\, L'(1, \chi_{-d})/L(1, \chi_{-d})$. At each realised point and each lift $1 \leq j < m$ there is a unitary Shimura datum over the host field, of signature $(k+j-1,\, m-j)$, that satisfies the hypothesis (Aj1) of Theorem 29.14 and carries the data of its hypothesis (Aj3), with a constant from Proposition 31.14. Fourth, the local comparison: the passage, valuation and archimedean conditions (Ri2)–(Ri4) of Definition 19.6, and, for every member and every word, reference comparison data that satisfy every hypothesis of Theorem 20.2.

The clauses do not contain the bounds (Ms1) and (Ms3), nor the realisation estimate (Ri1); these are the three properties that the proof of Proposition 21.2 adds. Besides definitions, the statement of (TCS) refers only to three results, Theorem 29.14, Proposition 31.14 and Theorem 20.2, and all three are proved without it. In one sentence, (TCS) says that the construction of the lattices produces packets whose numerical data are explicit and whose models and local comparisons, once the bound for Tate–Shafarevich groups holds, satisfy explicit identities and local estimates with uniform constants. In our reading the packets are the records of the chain of theta lifts, and (TCS) is what the paper assumes of the chain.

The construction that proves the following statement is outlined in Appendix A. Since this proof is not complete in the form presented here, we state the statement as a hypothesis, for logical rigour.

Hypothesis 21.1. *Let $\lambda > 1$ and let P be a host requirement supplied by the concrete good-host requirement $\mathfrak{G}$ on λ-windows. There is a real $\varepsilon_\dagger = \varepsilon_\dagger(\lambda, \mathsf{P}) > 0$ with the following property. Fix*

$$0 < \varepsilon < \min(\varepsilon_\dagger, 1/52), \quad J \geq 1, \quad \kappa = (C_{T_2}, C_8, C_5, c_6, C_b),$$

where $C_8 \geq 1$ and $C_5, c_6, C_b > 0$, a set W containing λ-windows, and $X_H > 1$. For every $x \in W$ with $x \geq X_H$, fix a frame, a host $\mathfrak{h}_x$, tops T_x, a rank t_x with $\mathsf{P}(x, \varepsilon, \mathfrak{h}_x, T_x, t_x)$, and a choice of all the data whose existence is asserted by the concrete good-host requirement $\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}_x, T_x, t_x)$. These choices, including the rank data, orderings, good-orbit choices, spacing sets and auxiliary primes, are fixed once and for all.

Then there are constants

$$c_D > 397, \quad c_h \geq 402, \quad C_e \geq 0, \quad \varepsilon_z > 0, \quad C_t, \quad X \geq \max\{X_H, 2, (2C_8)^{4/\varepsilon}\},$$

a tuple $\mathfrak{k}_{\mathrm{orb}}$ of eight nonnegative real numbers, and, for every $x \in W$ with $x \geq X$, a base translate packet P_x over the chosen frame, host, tops, spacing set, rank and auxiliary primes, with the following properties.

(Rr1) Loss data. *Every packet member satisfies (Lm2), (Lm5), (Lm6), and (Lm7) for a common pair (ε_z, C_t). Its loss parameter e_2 equals the exponent e of its frame.*

(Rr2) Numerical data. *At every packet member, the loss datum has the rank datum fixed above, with its rank sizes at (x, ε, J) and with $t_R = t_x$. Its heights, depths, weights and shifts have integer weights and boxes for (c_D, c_h), for some number field and thick subset. Its tops, Frobenius translations, orbit representatives and good representatives are given by the finite good-orbit choices fixed above, and these are the choices used by the loss datum. Its correction degrees, orbit thresholds and correction errors satisfy (Nc2)–(Nc4) for a common pair $(\mathfrak{k}_{\mathrm{orb}}, C_e)$. The first component C_{La} of that common tuple satisfies $C_{\mathrm{La}} \geq 1$, and the size parameter is*

$$L_c = C_{\mathrm{La}} N_{\mathrm{orb}}^2 \log a,$$

with the numerical orbit expressions of Definition 18.3.

(Rr3) Models and local comparison under an eventual Sha bound. *Suppose that there is a real number X_V such that the Sha bound*

$$\mathrm{V}(E_{\mathfrak{h}_x}; t_x, \exp(x^{\varepsilon}/2))$$

holds for all $x \in W$ with $x \geq \max(X, X_V)$. Then the following hold.

Lower completions. *There are the twenty-four common scalar constants specified in (Pl3), common constants $C_{\mathrm{prec}}, C_{\mathrm{ms}}, C_{\mathrm{ht}} \geq 0$, a real $Y \geq X$ with $Y > 1$, and compatible lower completion data (Pl1) for these packets at every $x \in W$ with $x \geq Y$, with the following properties.*

- *All numbered formulas of (Pl3) hold there with those constants, in the pointwise interpretation (Pl2), except (Mf1), (Mf5), (Dv1), (Dv2), (Bk1), (Bk2), (Nf1) and (Nf3).*
- *The kernel hypotheses of (Pc5), namely its data with (Kc1)–(Kc4), finite total measure and both displayed scalar inequalities, hold there in the pointwise interpretation (Pl2), with the same constants and data.*
- *Every packet member there satisfies*

$$\log |\mathcal{W}(\mathfrak{V})| \leq (k-3) \log 2.$$

The family. *Let $\mathcal{F}$ be the family with scales W, threshold Y, and index sets*

$$I(x) = \begin{cases} A_x, & x \in W, \quad x > Y, \\ \varnothing, & \text{otherwise}, \end{cases}$$

whose members are the packet members with their lower completions. At scales with empty index set, take any fixed frame. This family admits compatible base covolume realisations, chain data with (Lc1)–(Lc6), deep-cover data with (Dp0)–(Dp5), and component and model-slope data. All of them are over the same realised points and are compatible with the packets and completions above.

Heights and the component identity. *Set*

$$C_{\mathrm{mod}} = \max\{83400, C_{\mathrm{ms}}\}, \quad C_{\mathrm{Hs}} = \max\{C_{\mathrm{ht}}, C_{\mathrm{mod}}(C_{\mathrm{mod}} + 120)\}, \quad C_{\mathrm{U}} = 8C_{\mathrm{Hs}} + 4.$$

Here C_{ms} is the absolute constant of Theorem 29.14. For every $x \in W$ with $x > Y$, every member $u \in A_x$, every realised point, and every $1 \le j < m$, conditions (Ht1), (Ht2), (Ht4), (Ht5) hold with C_{ht}, with all their remaining quantifiers as stated there. The model slopes $\mathrm{ns}(Z)$ may be taken from the tower conclusion below. Moreover, the following component identity holds:

$$e^{\mathrm{ms}} = C_{\mathrm{mod}}(\mathrm{lz}\,\mathrm{wn} + \mathrm{wsp}) + 2c^{\mathrm{HT}}_{E,M}(B)\,\mathrm{wn}.$$

Model data. *At these scales, the number field of the integer weights and boxes of (Rr2) is $E_{\mathfrak{h}_x}$, with the thick subset already chosen and the same final depths. Put*

$$n = |A_x|, \quad f = 2n, \quad r = k + m - 1, \quad \delta_1 = x^{-\varepsilon/8},$$
$$b' = x\exp(x^{\varepsilon} - x^{\varepsilon/3}), \quad \Delta = \lceil x^{c_D}\rceil, \quad h = 2\lceil x^{c_h}\rceil.$$

At every packet member, these depths satisfy

$$b_t \le b' + \frac{2n\delta_1 x^{\varepsilon} L_c b'}{k} \quad (t \in T_x).$$

For every realised point and every $1 \le j < m$, the model field E is $E_{\mathfrak{h}_x}$. There are a CM type $\mathcal{I}$ of E, choosing one embedding from each conjugate pair, and a bijection $\beta : \mathcal{I} \to \Phi$. Put $p_j = k + j - 1$, $q_j = m - j$, and $d_{\mathrm{flag}} = f\binom{r}{2}$. The component flag dimension is $D^{\mathrm{fl}} = d_{\mathrm{flag}}$, and there is a real P such that

$$40 \le P \le x^{C_{\mathrm{prec}}}, \qquad d_{\mathrm{flag}} \le x^{C_{\mathrm{prec}}}, \qquad T^{\mathrm{d}} = \left\lceil P^2 d_{\mathrm{flag}}^2 k/m \right\rceil.$$

Put

$$c_0 = 8\left\lceil \frac{\max\{r + \binom{fp_jq_j+1}{2} + 2,\, 8fr + 4\}}{8} \right\rceil, \qquad B_\tau = w_1(\beta(\tau)) - h/2.$$

Moreover, the following joint inequalities hold:

$$T^{\mathrm{d}} \ge 4(d_{\mathrm{flag}} + 1), \qquad T^{\mathrm{d}} + d_{\mathrm{flag}} + 1 \le 4\Delta + 1, \qquad h \ge 2T^{\mathrm{d}}(3r + 1) + 2(r + c_0).$$

Define

$$U_{\tau,a} = \begin{cases} (4\Delta - 1)(k - 2 - a), & 0 \le a < k - 1, \\ -B_\tau - (l - 1)h + l - r\mathbf{1}_{l>j}, & a = k + l - 2, \quad 1 \le l \le m. \end{cases}$$

Let v be the integer array with $v_{\tau,k-2} = 0$ and

$$v_{\tau,a} - v_{\tau,a+1} = \left\lfloor \frac{U_{\tau,a} - U_{\tau,a+1} + 2 - (c_0 + 2r)\mathbf{1}_{a+1=p_j}}{T^{\mathrm{d}} + d_{\mathrm{flag}} + 1} \right\rfloor \quad (0 \le a < r - 1),$$

and put $V_{\tau,a} = v_{\tau,a} + 3r\mathbf{1}_{a\ge p_j}$. *All array indices* a *run over* $\{0,\dots,r-1\}$. *The model identities*

$$\begin{aligned}
\mathrm{wn} &= \sum_{\tau\in\mathcal{I}}\sum_{a=0}^{r-1}|V_{\tau,a}|,\\
\mathrm{wsp} &= (m-j)\sum_{\tau\in\mathcal{I}}(\max_a V_{\tau,a} - \min_a V_{\tau,a}),\\
B &= 4fr^2\left(1+\frac{\max_{\tau,a}|V_{\tau,a}|}{h/(4T^{\mathrm{d}})}\right),\\
L &= \mathrm{Re}\,\frac{L'(1,\chi_{-d_{\mathfrak{h}_x}})}{L(1,\chi_{-d_{\mathfrak{h}_x}})},\\
M &= g_1g_2\,15^{a_{\mathrm{lev}}}
\end{aligned}$$

hold, where a_{lev} *is a nonnegative integer and* g_1, g_2 *are the auxiliary primes fixed above. The choices* $P, \mathcal{I}, \beta, a_{\mathrm{lev}}$ *may depend on the point and the lift, but* C_{prec} *is common to all of them.*

Towers. *At each of these points and lifts there are a unitary Shimura datum and a component bijection satisfying (Aj1) of Theorem 29.14, with host field* $E = E_{\mathfrak{h}_x}$, *host prime* $d_{\mathfrak{h}_x}$, *signature* (p_j, q_j), *CM type* $\Phi_{\mathsf{H}} = \mathcal{I}$, *and level* $(M, \mathfrak{n}, 1)$. *The component set and the model entries are those specified above. For*

$$u_{\tau,i} = V_{\tau,i-1} \quad (1\le i\le r), \qquad g = \frac{h}{4T^{\mathrm{d}}},$$

all the data and hypotheses of (Aj3) of that theorem are available, with its absolute constants (the constant of Proposition 31.14 and $C' = 4$*), for every attached tower as quantified there. In particular, the counting, transfer and two volume identities there concern these components and the given* $\mathrm{vs}(Z)$. *The integer* M *of the model identities is the one in the level of the datum.*

Local comparison. *There are constants* $C_6, C_{\mathrm{top}}, C_{\mathrm{F}} \ge 0$ *with which (Lc6), (Dp0) and (Dp5) hold, respectively, and a constant*

$$C \ge C_6 + \max\{C_{\mathrm{top}},\, 2(C_{\mathrm{F}} + C_{\mathrm{U}}) + 9\} + 1$$

such that the following local-comparison requirements hold. There are a real $Z_{\mathrm{ref}} \ge Y$ *and nonnegative real constants* $c_1, c_{\mathrm{G}}, c_{\mathrm{e}}$, *independent of the scale, the member, the word and all local indices. Every member of* $\mathcal{F}$ *with* $x \ge Z_{\mathrm{ref}}$ *satisfies conditions (Ri2), (Ri3), (Ri4) of Definition 19.6, with all their quantifiers over words, places and vectors as stated there.*

For every $x \ge Z_{\mathrm{ref}}$ *with* $x > Y$, *every* $i \in I(x)$, *and every* $\omega \in \mathcal{W}(\mathfrak{V}_{x,i})$, *there are reference comparison data satisfying (Rc1)–(Rc7) of Definition 20.1. They are attached to the given base point, with the same field* K, *base lattice* $\overline{R}_{\mathrm{b}}$, *dimension* D, *reference lattices and induced metrics, and their local lift index set is* $\{1,\dots,m\}$. *For these data there are finite sets* $\mathcal{T}_{\mathrm{w}} \subseteq \mathcal{S}$, *positive integers* $a_{\mathrm{grp}}, a_{\mathrm{lev}}, a_{\mathrm{den},j}$, *real local arrays* $f_{wj}, e_{wj}, a_{wj}, \eta_{wj}, q_w$, *and a real* $u_\star$, *satisfying all hypotheses of Theorem 20.2. These include* $m \ge 1$ *and all its displayed support conditions, local estimates, summed bounds and scalar record formulas, and its ties to the scaling, frame and main terms. The coefficient of that theorem is the constant* C *above, and its* $c_1, c_{\mathrm{G}}, c_{\mathrm{e}}$ *are the common constants above. In particular, its quantities* $\mathrm{Df}_j, \mathrm{tc}_j, \mathrm{le}, \mathrm{sc}, F(\mathfrak{V}_{x,i}), \mathrm{lm}, \mathrm{ue}$

are the records already specified. The reference data, denominators and local arrays may depend on the member and the word, but the common constants and the cutoff do not. The conclusion (Ri1) is not assumed.

A field, lattice, metric, map or scalar that occurs in several of the requirements above is the same in all of them. The twenty-four common constants and all the other constants above do not depend on the scale, the member, the point, the word, the lift or any other local index. The loss data, place records, slope records, numerical choices, orbit masses and final profiles are those of the packets above.

All the constants, cutoffs and packets asserted to exist may depend on the whole collection of choices fixed at the start.

Proposition 21.2. *Let $\lambda > 1$ and let P be a host requirement supplied by the concrete good-host requirement $\mathfrak{G}$ on λ-windows. There is a real $\varepsilon_\dagger = \varepsilon_\dagger(\lambda, \mathsf{P}) > 0$ with the following property. Fix*

$$0 < \varepsilon < \min(\varepsilon_\dagger, 1/52), \quad J \geq 1, \quad \kappa = (C_{T_2}, C_8, C_5, c_6, C_b),$$

where $C_8 \geq 1$ and $C_5, c_6, C_b > 0$, a set W containing λ-windows, and $X_H > 1$. For every $x \in W$ with $x \geq X_H$, fix a frame, a host $\mathfrak{h}_x$, tops T_x, a rank t_x with $\mathsf{P}(x, \varepsilon, \mathfrak{h}_x, T_x, t_x)$, and a choice of all the data whose existence is asserted by the concrete good-host requirement $\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}_x, T_x, t_x)$. These choices, including the rank data, orderings, good-orbit choices, spacing sets and auxiliary primes, are fixed once and for all.

Then there are constants

$$c_D > 397, \quad c_h \geq 402, \quad C_e \geq 0, \quad \varepsilon_z > 0, \quad C_t, \quad X \geq \max\{X_H, 2, (2C_8)^{4/\varepsilon}\},$$

a tuple $\mathfrak{k}_{\mathrm{orb}}$ of eight nonnegative real numbers, and, for every $x \in W$ with $x \geq X$, a base translate packet P_x over the chosen frame, host, tops, spacing set, rank and auxiliary primes, with the following properties.

(Ma1) Loss data. *Every packet member satisfies (Lm2), (Lm5), (Lm6), and (Lm7) for a common pair (ε_z, C_t). Its loss parameter e_2 equals the exponent e of its frame.*

(Ma2) Numerical data. *At every packet member, the loss datum has the rank datum fixed above, with its rank sizes at (x, ε, J) and with $t_R = t_x$. Its heights, depths, weights and shifts have integer weights and boxes for (c_D, c_h), for some number field and thick subset. Its tops, Frobenius translations, orbit representatives and good representatives are given by the finite good-orbit choices fixed above, and these are the choices used by the loss datum. Its correction degrees, orbit thresholds and correction errors satisfy (Nc2)–(Nc4) for a common pair $(\mathfrak{k}_{\mathrm{orb}}, C_e)$.*

(Ma3) Primitive models under an eventual Sha bound. *Suppose that there is a real number X_V such that the Sha bound*

$$\mathrm{V}(E_{\mathfrak{h}_x}; t_x, \exp(x^\varepsilon/2))$$

holds for all $x \in W$ with $x \geq \max(X, X_V)$. Then the following hold.

Lower completions. *There are the twenty-four common scalar constants specified in (Pl3), common constants $C_{\mathrm{prec}}, C_{\mathrm{ms}}, C_{\mathrm{ht}} \geq 0$, a real $Y \geq X$ with $Y > 1$, and compatible lower completion data (Pl1) for these packets at every $x \in W$ with $x \geq Y$, with the following properties.*

- *All numbered formulas of (Pl3) hold there with those constants, in the pointwise interpretation (Pl2), except (Mf1), (Mf5), (Dv1), (Dv2), (Bk1), (Bk2), (Nf1) and (Nf3).*
- *The kernel hypotheses of (Pc5), namely its data with (Kc1)–(Kc4), finite total measure and both displayed scalar inequalities, hold there in the pointwise interpretation (Pl2), with the same constants and data.*
- *Every packet member there satisfies*

$$\log|\mathcal{W}(\mathfrak{V})| \le (k-3)\log 2.$$

The family. *Let $\mathcal{F}$ be the family with scales W, threshold Y, and index sets*

$$I(x) = \begin{cases} A_x, & x \in W, \quad x > Y, \\ \varnothing, & \text{otherwise}, \end{cases}$$

whose members are the packet members with their lower completions. At scales with empty index set, take any fixed frame. This family admits compatible base covolume realisations, chain data with (Lc1)–(Lc6), deep-cover data with (Dp0)–(Dp5), and component and model-slope data. All of them are over the same realised points and are compatible with the packets and completions above.

Heights, model slopes and the component identity. *Set*

$$C_{\mathrm{mod}} = \max\{83400, C_{\mathrm{ms}}\}, \quad C_{\mathrm{H}} = \max\{C_{\mathrm{ht}}, C_{\mathrm{mod}}(C_{\mathrm{mod}}+120)\}, \quad C_{\mathrm{U}} = 8C_{\mathrm{H}} + 4.$$

For every $x \in W$ with $x > Y$, every member $u \in A_x$, every realised point, and every $1 \le j < m$, conditions (Ht1), (Ht2), (Ht4), (Ht5) hold with C_{ht}, and (Ms1), (Ms3) hold with C_{ms}, with all their remaining quantifiers as stated there. Moreover, the following component identity holds:

$$e^{\mathrm{ms}} = C_{\mathrm{mod}}(\mathrm{lz}\,\mathrm{wn} + \mathrm{wsp}) + 2c^{\mathrm{HT}}_{E,M}(B)\,\mathrm{wn}.$$

Model data. *At these scales, the number field of the integer weights and boxes of (Ma2) is $E_{\mathfrak{h}_x}$, with the thick subset already chosen and the same final depths. Put*

$$n = |A_x|, \quad f = 2n, \quad r = k+m-1, \quad \delta_1 = x^{-\varepsilon/8},$$
$$b' = x\exp(x^{\varepsilon} - x^{\varepsilon/3}), \quad \Delta = \lceil x^{c_D}\rceil, \quad h = 2\lceil x^{c_h}\rceil.$$

At every packet member, these depths satisfy

$$b_t \le b' + \frac{2n\delta_1 x^{\varepsilon} L_c b'}{k} \quad (t \in T_x).$$

For every realised point and every $1 \le j < m$, the model field E is $E_{\mathfrak{h}_x}$. There are a CM type $\mathcal{I}$ of E, choosing one embedding from each conjugate pair, and a bijection $\beta : \mathcal{I} \to \Phi$. Put $p_j = k + j - 1$, $q_j = m - j$, and $d_{\mathrm{flag}} = f\binom{r}{2}$. The component flag dimension is $D^{\mathrm{fl}} = d_{\mathrm{flag}}$, and there is a real P such that

$$40 \le P \le x^{C_{\mathrm{prec}}}, \qquad d_{\mathrm{flag}} \le x^{C_{\mathrm{prec}}}, \qquad T^{\mathrm{d}} = \left\lceil P^2 d_{\mathrm{flag}}^2 k/m \right\rceil.$$

Put

$$c_0 = 8\left\lceil \frac{\max\{r + \binom{fp_jq_j+1}{2} + 2,\ 8fr+4\}}{8} \right\rceil, \qquad B_\tau = w_1(\beta(\tau)) - h/2,$$

and define

$$U_{\tau,a} = \begin{cases} (4\Delta-1)(k-2-a), & 0 \le a < k-1, \\ -B_\tau - (l-1)h + l - r\mathbf{1}_{l>j}, & a = k+l-2, \quad 1 \le l \le m. \end{cases}$$

Let v be the integer array with $v_{\tau,k-2} = 0$ and

$$v_{\tau,a} - v_{\tau,a+1} = \left\lfloor \frac{U_{\tau,a} - U_{\tau,a+1} + 2 - (c_0+2r)\mathbf{1}_{a+1=p_j}}{T^{\mathrm{d}} + d_{\mathrm{flag}} + 1} \right\rfloor \quad (0 \le a < r-1),$$

and put $V_{\tau,a} = v_{\tau,a} + 3r\mathbf{1}_{a \ge p_j}$. All array indices a run over $\{0,\dots,r-1\}$. The model identities

$$\begin{aligned} \mathrm{wn} &= \sum_{\tau\in\mathcal{I}} \sum_{a=0}^{r-1} |V_{\tau,a}|, \\ \mathrm{wsp} &= (m-j) \sum_{\tau\in\mathcal{I}} (\max_a V_{\tau,a} - \min_a V_{\tau,a}), \\ B &= 4fr^2 \left(1 + \frac{\max_{\tau,a} |V_{\tau,a}|}{h/(4T^{\mathrm{d}})}\right), \\ L &= \mathrm{Re}\, \frac{L'(1,\chi_{-d_{\mathfrak{h}x}})}{L(1,\chi_{-d_{\mathfrak{h}x}})}, \\ M &= g_1 g_2\, 15^{a_{\mathrm{lev}}} \end{aligned}$$

hold, where a_{lev} is a nonnegative integer and g_1, g_2 are the auxiliary primes fixed above. The choices $P, \mathcal{I}, \beta, a_{\mathrm{lev}}$ may depend on the point and the lift, but C_{prec} is common to all of them.

Realisation estimates. *There are constants $C_6, C_{\mathrm{top}}, C_{\mathrm{F}} \ge 0$ with which (Lc6), (Dp0) and (Dp5) hold, respectively, and a constant*

$$C \ge C_6 + \max\{C_{\mathrm{top}},\, 2(C_{\mathrm{F}} + C_{\mathrm{U}}) + 9\} + 1$$

such that $\mathcal{F}$ has realisation estimates for C.

A field, lattice, metric, map or scalar that occurs in several of the requirements above is the same in all of them. The twenty-four common constants and all the other constants above do not depend on the scale, the member, the point, the word, the lift or any other local index. The loss data, place records, slope records, numerical choices, orbit masses and final profiles are those of the packets above.

All the constants, cutoffs and packets asserted to exist may depend on the whole collection of choices fixed at the start.

Proof. Apply Hypothesis 21.1 for the same λ and P, and take its $\varepsilon_\dagger(\lambda, \mathsf{P})$ as $\varepsilon_\dagger$. Apply it to the data fixed in the statement, which satisfy its hypotheses, and use its outer constants and its packets. Conditions (Rr1),(Rr2) imply (Ma1),(Ma2), respectively, and the packets are chosen before the eventual Sha premise is assumed.

Now assume that premise, and keep the twenty-four common constants, the other constants, the lower completions, the base covolume realisations, the chain and deep-cover data and the component and model data supplied there. Write Y_0 for the threshold of the tail supplied there.

We prove the two model-slope properties and the realisation estimates on a common further tail, with the same packets and the same scalar coefficient C.

We first prepare the divided weight at a point and a lift $1 \le j < m$. Put $d = d_{\mathrm{flag}} = f\binom{r}{2}$ and $Z = f^2 r^4$. Rank sizes give

$$n \le 2x^{2+8\varepsilon}, \quad 2 \le m \le 2\sqrt{x}, \quad m+3 \le k \le 2x^{9/2+10\varepsilon}, \quad r = k+m-1.$$

In particular $d \ge 1$, $p_j + q_j = r$ and $p_j q_j \le r^2$. Hence $\binom{fp_jq_j+1}{2} \le Z$, $r \le Z$, $fr \le Z$, and the explicit upward rounding in c_0 gives

$$0 \le c_0 \le 12Z + 7 \le 22 f^2 r^4.$$

The length identity and $P \ge 40$ give $T^{\mathrm{d}} \ge 1600 d^2 \ge 4(d+1)$. For $x \ge 2$, the bounds $k \le 2x^5$ and $P, d \le x^{C_{\mathrm{prec}}}$ give

$$T^{\mathrm{d}} \le 2x^{4C_{\mathrm{prec}}+5} + 1 \le x^{4C_{\mathrm{prec}}+7}.$$

Put $C_T = 4C_{\mathrm{prec}} + 7$; it does not depend on the point or the lift.

Condition (Lm7) gives $0 < L_c \le x^{\varepsilon+\varepsilon/100}$. On a uniform tail, rank sizes also give $n \le x^{2+8\varepsilon+\varepsilon/100}$ and $k \ge x^{9/2+10\varepsilon-\varepsilon/100}$. The depth bound supplied above and Lemma 19.14 therefore give $\lceil b' \rceil \le b_t \le 2b'$ for the final depths. By Lemma 19.15, the order of the torsion units of the host field is at most x^6 on a uniform tail. Consequently (Iw3), pulled back through β, puts every B_τ in one of the three permitted ranges of Lemma 19.13, with $C_\mu = 6$. By Lemma 19.16, $[E : \mathbb{Q}] = 4n$, so $|\mathcal{I}| = 2n = f$. All the data hypotheses of Lemma 21.4 are now verified for these arrays, with $T = T^{\mathrm{d}}$. Its additional joint inequalities are explicit requirements of (Rr3). Thus, after one uniform restriction of the tail,

$$u_{\tau,i} = V_{\tau,i-1}, \qquad g = h/(4T^{\mathrm{d}})$$

satisfy all the gap hypotheses.

We check the remaining conditions in (Aj2) of Theorem 29.14. By the model identities,

$$b_0(u,g) = 4fr^2(1 + \|u\|_\infty/g) = B < \lfloor B \rfloor + 1, \qquad D^{\mathrm{fl}} = f\binom{r}{2}.$$

The signature has $q_j = m - j < p_j = k + j - 1$, so $\min(p_j, q_j)\mathrm{spr}(u) = \mathrm{wsp}$, and $\|u\|_1 = \mathrm{wn}$. The prime $d = d_{\mathfrak{h}_x}$ of the CM host satisfies $d \equiv 7 \pmod{16}$ by (Ud1). Thus the primitive quadratic character χ_{-d} equals the Legendre character $(\cdot/d)$, namely the character χ_d in Proposition 31.14. Since the model field, the host prime and the logarithmic derivative are the same, $L = L_{\mathsf{H}}$. Hence every numerical and weight condition of (Aj2) holds for the datum of (Rr3). Conditions (Aj1),(Aj3) are supplied for this datum, this component set and every attached tower.

Apply Theorem 29.14 to the tail family. That theorem chooses the tower and its compatible model data jointly, and gives (Ms1),(Ms3) with the absolute constant C_{ms} specified there. Choose $\mathrm{ns}(Z)$ as the theorem prescribes. No other entry is changed, since the other required lower, reference, component, chain and deep-cover conditions do not involve the model slopes. Their objects, the component function vs, the fields, the metrics and the scalars stay as before. The hypotheses and the conclusion hold with thresholds uniform over the family, so there is one common tail on which both required model-slope properties hold.

For the realisation estimates, fix a member and a word on the reference tail. The reference comparison data and the local data satisfy every hypothesis of Theorem 20.2, with the given point, its field, base lattice, reference metrics and scalar records, and with the same coefficient C. That theorem gives (Ri1) for this member and word, with an error E_ω that may depend

on the member and the word. The constants $C, c_1, c_{\mathrm{G}}, c_{\mathrm{e}}$ do not depend on the member or the word. Conditions (Ri2),(Ri3),(Ri4) are supplied as stated, with all their quantifiers over places, words and vectors. Together these are the realisation estimates for C.

Choose Y at least Y_0, the reference cutoff and all the uniform cutoffs just obtained, and restrict the lower completion and the family records to this tail. The index sets are then A_x for $x \in W$, $x > Y$, and empty otherwise, as in (Ma3). All numbered lower conditions and residual word bounds persist. The kernel data with (Kc1)–(Kc4), the finite measure and both scalar inequalities are supplied by (Rr3), and they remain valid on this tail with all inner quantifiers, antecedents and common constants. They do not involve the model slopes and need no further constant. The definitions of $C_{\mathrm{mod}}, C_{\mathrm{H}}, C_{\mathrm{U}}$ and the component identity for e^{ms} remain in force, as does the lower bound on C. Thus (Ht1),(Ht2),(Ht4),(Ht5), the properties (Ms1),(Ms3) derived above, and the realisation estimates hold with the data and constants required in (Ma3). The remaining compatibility requirements and the requirement on the final profiles restrict to the same tail. This proves the proposition, assuming Hypothesis 21.1.

□

Definition 21.3. Throughout, Φ is a non-empty finite set (the **embeddings**), $n \geq 2$ an integer (the **rank**) and κ an integer with $1 \leq \kappa < n$ (the **cut**); all objects below depend on (Φ, n, κ), which is suppressed from the notation.

Polynomials and the flag degree. Let $\mathcal{R} = \mathbb{R}[x_{a,i} : a \in \Phi,\ 1 \leq i \leq n]$, the **Chern-root ring**, and $D = |\Phi|\, n(n-1)/2$, the **flag dimension**. For $m \in \mathbb{N}^{\Phi \times \{1,\dots,n\}}$ write $x^m = \prod x_{a,i}^{m_{a,i}}$ and $[x^m]F$ for the coefficient of x^m in $F \in \mathcal{R}$. Let $\delta_{a,i} = n - i$. For $w = (w_a)_{a \in \Phi} \in S_n^{\Phi}$ put $\operatorname{sgn}(w) = \prod_a \operatorname{sgn}(w_a)$ and $(\delta \circ w)_{a,i} = \delta_{a,w_a(i)}$. The **flag degree** is the linear form

$$\deg^{\mathrm{fl}}(F) = (-1)^D \sum_{w \in S_n^{\Phi}} \operatorname{sgn}(w)\, [x^{\delta \circ w}]F \qquad (F \in \mathcal{R}). \tag{21.1}$$

Weights. A **weight** is an element $u = (u_{a,i})$ of $\mathbb{R}^{\Phi \times \{1,\dots,n\}}$; its **linear root form** is $\mathrm{lr}(u) = -\sum_{a,i} u_{a,i} x_{a,i} \in \mathcal{R}$. It is **dominant** if $u_{a,1} \geq u_{a,2} \geq \cdots \geq u_{a,n}$ for every a, and **regular dominant** if these inequalities are strict. Put $\|u\|_1 = \sum_{a,i} |u_{a,i}|$, $\|u\|_\infty = \max_{a,i} |u_{a,i}|$ and $\operatorname{spr}(u) = \sum_a (\max_i u_{a,i} - \min_i u_{a,i})$, the **spread**. The **small block** is $\{1, \dots, \kappa\}$ if $\kappa \leq n - \kappa$ and $\{\kappa + 1, \dots, n\}$ otherwise. A weight u satisfies the **gap hypothesis with gap** $g > 0$ if it is regular dominant, $u_{a,\kappa} - u_{a,\kappa+1} \geq g$ for every a, and $|u_{a,i} - u_{a,i'}| \geq g$ for every a, every i in the small block and every $i' \neq i$; its **threshold** is then $b_0(u, g) = 4|\Phi| n^2 (1 + \|u\|_\infty / g)$.

Power sums and central derivatives. For $a \in \Phi$ and $k \geq 0$ let $\mathrm{ps}_{a,k} = \sum_{i=1}^n x_{a,i}^k$ (the **power sums**). A **label** is a pair $\sigma = (a, \pm) \in \Phi \times \{+, -\}$; its **partial power sums** are

$$\mathrm{ps}_k^{(a,+)} = \sum_{i=1}^{\kappa} x_{a,i}^k, \qquad \mathrm{ps}_k^{(a,-)} = \sum_{i=\kappa+1}^{n} (-x_{a,i})^k \qquad (k \geq 0).$$

The **central derivatives** at a are the derivations $\partial_a^{<} = \sum_{i=1}^{\kappa} \partial/\partial x_{a,i}$ and $\partial_a = \sum_{i=1}^{n} \partial/\partial x_{a,i}$ of $\mathcal{R}$. A **power-sum decomposition** of $F \in \mathcal{R}$ is a family $(Q_{a,k})_{a \in \Phi,\, 1 \leq k \leq n}$ in $\mathcal{R}$ with $Q_{a,k}$ homogeneous of degree $D + 1 - k$ (the zero polynomial allowed) and $F = \sum_{a \in \Phi} \sum_{k=1}^n \mathrm{ps}_{a,k} Q_{a,k}$.

Frobenius terms. A **Frobenius labelling** is a map fr assigning to every prime p, every $a \in \Phi$ and every $m \in \mathbb{N}$ a label $\mathrm{fr}(p, a, m) \in \Phi \times \{+, -\}$ whose sign does not depend on a. For such fr, a prime p, $a \in \Phi$, $1 \leq k \leq n$ and $Q \in \mathcal{R}$ put

$$\Sigma_p(a, k, Q) = \sum_{m=0}^{\infty} p^{-k(m+1)} \deg^{\mathrm{fl}}(\mathrm{ps}_{k-1}^{\mathrm{fr}(p,a,m)} Q)$$

(an absolutely convergent series: the labels form a finite set and $p^{-k} \le 1/2$), the **Frobenius term** $\rho_p(a,k,Q) = k\log p\cdot\Sigma_p(a,k,Q)$ if $k \ge 2$ and $\rho_p(a,1,Q) = 0$, and the **stage-change term**

$$\Delta_p(a,k,Q) = k\log p\Big(\Sigma_p(a,k,Q) - \frac{\deg^{\mathrm{fl}}(\mathrm{ps}_{k-1}^{(a,+)}Q)}{p-1}\Big). \tag{21.2}$$

For a finite set S of primes and an integer B_0, the **Frobenius tail** $\rho_{>B_0}(a,k,Q)$ is the limit, as the integer $B\to\infty$, of $\sum\rho_p(a,k,Q)$ over the primes $p\notin S$ with $B_0 < p \le B$, whenever this limit exists (the dependence on S and fr is suppressed from the notation).

The test class. Let $D^{\mathrm{gr}} = |\Phi|\kappa(n-\kappa)$. The **Grassmannian class** $\mathrm{pl}\in\mathcal{R}$ is $\sum_a\sum_{i=\kappa+1}^{n} x_{a,i}$ if $n-\kappa\le\kappa$ and $-\sum_a\sum_{i=1}^{\kappa}x_{a,i}$ otherwise; the **fibre class** is

$$\mathrm{fb} = \prod_{a\in\Phi}\prod_{\substack{1\le i<i'\le n\\ i,i'\le\kappa \text{ or } i,i'>\kappa}}(x_{a,i'}-x_{a,i}),$$

and the **test class** is $\mathrm{pl}^{D^{\mathrm{gr}}+1}\mathrm{fb}$, with **test degree** $\deg^{\mathrm{fl}}(\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb})$.

Lemma 21.4. *Fix the constants of Lemma 19.13. There exists $X>1$, depending only on these constants, with the following property. Let $x\ge X$ be real, and let the remaining data satisfy the hypotheses of that lemma, with its permitted choices B_τ, its arrays U,v,V and $D = T+d+1$. Then the conclusions of that lemma hold. Suppose moreover that*

$$D\le 4\Delta+1, \qquad h\ge 2T(3r+1)+2(r+c_0),$$

and put

$$u_{\tau,i} = V_{\tau,i-1}\quad(\tau\in I,\ 1\le i\le r), \qquad p = k-1+j, \qquad g = \frac{h}{4T}.$$

Then u satisfies the full gap hypothesis of Definition 21.3 for embeddings I, rank r, cut p, and gap g.

Proof. Take the threshold of Lemma 19.13 and enlarge it uniformly so that $B_\tau+1\ge D$ for every permitted choice. The three allowed ranges for B_τ give

$$B_\tau \ge \frac{b'}{xn} \ge \tfrac{1}{2}x^{-2-8\epsilon}\exp(x^{\epsilon}-x^{\epsilon/3})$$

on a common tail, whereas

$$D = T+d+1 \le \tfrac{5}{4}T \le \tfrac{5}{4}x^{C_T}.$$

The exponential lower bound dominates this fixed polynomial, so the enlargement depends only on the stated constants. No further restriction on C_T is needed. The numerical conclusions of Lemma 19.13 persist.

Fix τ. The defining recurrence and final shift give

$$V_{\tau,a}-V_{\tau,a+1} = \begin{cases} \lfloor(4\Delta+1)/D\rfloor, & 0\le a<k-2,\\ \lfloor(B_\tau+1)/D\rfloor, & a=k-2,\\ \lfloor(h+1)/D\rfloor, & a=k+l-2,\ 1\le l<m,\ l\ne j,\\ \lfloor(h+1-r-c_0)/D\rfloor-3r, & a=p-1. \end{cases}$$

The first two expressions are at least 1. For the last, write $R = r + c_0$ and $Q = 3r + 1$. The additional height inequality gives $TQ \le h/2 - R$, so $h - R > 0$. Since $\lfloor z \rfloor \ge z - 1$ and $D \le 5T/4$,

$$\begin{aligned}\left\lfloor \frac{h+1-R}{D} \right\rfloor - 3r &\ge \frac{h-R}{D} - Q \\ &\ge \frac{4(h-R)}{5T} - Q \\ &\ge \frac{3h/10 + R/5}{T} \ge \frac{h}{4T} = g.\end{aligned}$$

Every nonexceptional node difference is also at least g, because

$$\left\lfloor \frac{h+1}{D} \right\rfloor \ge \left\lfloor \frac{h+1-R}{D} \right\rfloor \ge g + 3r.$$

Thus all adjacent differences are positive and the cut difference is at least g.

Finally $r - p = m - j < p$, since $1 \le j < m$ and $k \ge m + 3$. The small block is therefore $\{p+1, \ldots, r\}$. The difference between a weight in this block and any distinct weight is a sum, up to sign, of positive adjacent differences containing either the cut difference or a difference internal to this small block. At least one summand is at least g. Hence

$$|u_{\tau,i} - u_{\tau,i'}| \ge g \quad (i \in \{p+1, \ldots, r\},\ i' \ne i).$$

This is the remaining part of the gap hypothesis. □

Part V
The Ribet budget

Part V proves one theorem, the Ribet budget Theorem 22.1, and it has one consumer: the proof of Theorem 17.7 in Part III. There it gives the properties (Rf1), (Rf2), (Rf4) and (Rf5) of the Ribet records and the properties (Lq3) and (Lq7) of the local bound records of a family of the class $\mathcal{A}_{\mathrm{prop}}$. With them Theorem 13.13 bounds, at every local component r of a large member and every good eigensystem π, the congruence length $\mathrm{cg}(r,\pi)$ by the Selmer length, the local term $\mathrm{lt}(r)$, the Sha length $\mathrm{len}_\ell(E;t)$ and $v_\ell(\#\mathrm{Cl}(E))$, and this is how the defects of the divisor package (Sp4) come to be bounded by Sha lengths (Section 13).

The mechanism is Ribet's [208]. A homomorphism from a module of extensions to one of the constituent modules of a determinant algebra gives a cocycle (Lemma 22.12), and its class lies in a Selmer group, in a group of unramified homomorphisms or in a Tate–Shafarevich group of a Tate twist, up to two kinds of loss: a global loss κ_k, and local losses $\lambda_k(v)$ at the places v above the residue characteristic ℓ, which measure how far the class is from the local Bloch–Kato condition (Theorem 22.11). The budget bounds the smallest admissible losses at each ℓ by an explicit allowance $a_{x,i}(\ell)$, and the sum over ℓ of $\log\ell\cdot a_{x,i}(\ell)$ by a constant times $f\,m\,b\,x^{2\varepsilon}$.

Everything else in the Part is local theory, done place by place. The primes ℓ are of small type (including at most two ramified primes, such as the primes q and d of the host), of split type, of inert type, or large, and each type has its own allowance. The common tool is a lifting exponent for integral crystalline classes along a surjection of crystalline lattices over an unramified field (Proposition 24.1): the exponent is at most an absolute constant times $2+\mathrm{wt}/\ell+\log_\ell(\mathrm{wt}+1)$, where wt is the width of the Hodge window. It is proved with Wach modules [265], [26], which are obtained here from prismatic F-crystals [31], and with Herr's complex [126]; and at the large primes the local conditions are computed exactly with residual (φ,Γ)-modules [93].

Section 22 states the theorem and assembles it from the local results. Sections 23 and 24 build the theory of Wach modules that is needed and prove the lifting exponents, over unramified fields and over subfields of the first cyclotomic layer. Section 25 turns lifting exponents into losses at the places above ℓ. Sections 26, 27 and 28 bound the total loss at a prime of each type: inert, then ramified, split and small, then large. The bounds at the primes of inert type rest on three colength bounds for weight words along Frobenius orbits; their proofs are combinatorial and long, and they form Appendix C. No result of Part V depends on (TCS), and apart from one lemma of Section 13 (Lemma 13.15) the Part cites no other Part.

22 The Ribet budget theorem

This section states the Ribet budget and assembles it from the local results of Sections 23–28. The statement used in Part III is Theorem 22.1. Its hypotheses are asked of one coefficient extension of the family, along $\mathcal{O}_r\subseteq\mathcal{O}_r(N_{x,i})$, because the local results need coefficient rings that contain certain roots of unity; its conclusions are for the original family. The passage between the two is formal: the extended coefficient ring contains the cyclotomic fields that the budget reads (Lemma 22.16), and a coefficient extension leaves normalised lengths, minimal losses and the Ribet properties unchanged (Lemma 22.17).

The budget itself is Theorem 22.4. Its minimal losses are the normalised lengths of the kernel in (Gd6) and of the local images in (Gd7), and its allowance $a_{x,i}(\ell)$ is an explicit expression in the sizes of the member, one for each type of prime. The proof treats one point (j,w,r,π) at a time: according to the type of $\ell=\ell_{x,i}(w)$ it exhibits losses with (Gd6) and (Gd7) whose

total is at most $a_{x,i}(\ell)$, by Theorem 25.1 and Theorem 27.15 at the split primes, Theorem 26.1 at the inert primes, Proposition 27.16 and Theorem 27.1 at the small and ramified primes, and Proposition 28.1 at the large primes, where the minimal losses vanish. Two summations close the proof: Theorem 22.10 turns the Galois Ribet records into Ribet records with (Rf1), (Rf4) and (Rf5), and Theorem 22.9, with Theorem 22.8 for the primes of inert type, sums the allowances into the local budget (Lq7). Property (Rf2), non-vanishing at the points outside the good set, comes from Proposition 22.7.

The idea of the section is in Lemma 22.12 and Theorem 22.11, Ribet's construction [208] in the form needed here: a homomorphism φ from the module of extensions to the constituent module Cs_k gives a cocycle c_φ, and the normalised length of $\mathrm{Hom}(\mathcal{N}, \mathrm{Cs}_k)$ is bounded by the loss sum ls_k plus the length of a Selmer group ($k = 0$), of a group of unramified homomorphisms ($k = 1$), or of $\mathrm{Sha}^1_\ell(E, k-1)$ ($k \geq 2$). The other results of the section are auxiliary; among them are local facts at the primes q, d of the host, the determinant condition from Hecke-valued characteristic polynomials, and Tate–Shafarevich groups of Tate twists with coefficients in a coefficient ring.

Theorem 22.1. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with local defect data, local bound records, Ribet records, budget data and inert budget data, and Galois Ribet items at every point of every member (x, i), $x > x_1$, as in Theorem 22.4; let $c_{\mathrm{sp}}, c_{\mathrm{in}}, c_{\mathrm{sm}}, C_{\mathrm{bud}} \geq 1$. Write (Ub3°) for (Ub3) without its two clauses on K (that K contains $\sigma(E_v(\mu_\ell))$ at primes of split or inert type and large primes, and that K contains a primitive $\mathrm{lcm}(\ell, N_{x,i})$-th root of unity at primes of small type). Suppose there is a real x_0 such that for every $x \geq x_0$ with $x > x_1$ and every member (x, i) with finite Sha lengths there are an integer $N_{x,i} \geq 1$ with $4 \mid N_{x,i}$ and integers $S^{\mathrm{rm}}_{x,i}, \eta^{\mathrm{rm}}_{x,i} \geq 0$ with the following property, where $\mathcal{F}'$ is the coefficient extension of $\mathcal{F}$ along $\mathcal{O}_r \subseteq \mathcal{O}_r(N_{x,i})$ for the local components r of these members and along $\mathcal{O}_r \subseteq \mathcal{O}_r$ for all other local components: at every point (j, w, r, π) of (x, i) the Galois Ribet items of $\mathcal{F}'$ satisfy (Ub1), (Ub2), (Ub3°), (Ub4)–(Ub10), with $K = \mathcal{O}_r(N_{x,i})[\ell^{-1}]$. Then, for all $x \geq x_0$ with $x > x_1$, $x^\varepsilon \geq 3$ and x beyond the thresholds of (Lz1) and (Iq1), and all members with finite Sha lengths:*

1. *at every (j, w, r, π) the minimal losses of the Galois Ribet items of $\mathcal{F}$ are finite and*

$$\sum_{k=0}^{t} \mathrm{ls}_k^{\min}(r, \pi) \leq a_{x,i}(\ell);$$

   the Galois Ribet items of $\mathcal{F}$ with the minimal losses where these are finite (and the losses 0 elsewhere) are Galois Ribet records having (Gd8), (Gd9) and (Gd10);

2. *the Ribet records of $\mathcal{F}$ have (Rf1), (Rf2), (Rf4) and (Rf5);*

3. *the local bound records of $\mathcal{F}$ have (Lq3), and (Lq7) with the constant $19C_3^3$.*

Proof. **Step 1: $\mathcal{F}'$ satisfies the hypotheses of Theorem 22.4.** By item 3 of Lemma 22.17, $\mathcal{F}'$ is a parametrised host family with slope records and defect data, with local defect data, local bound records, Ribet records, budget data and inert budget data, and it carries Galois Ribet items at every point of every member (x, i), $x > x_1$ (the coefficient extensions of those of $\mathcal{F}$, item 2 there), with the same ℓ, t and the extended Hecke lattice of r. Let x_0, a member (x, i) with $x \geq x_0$, $x > x_1$ and finite Sha lengths, and $N = N_{x,i}$, $S^{\mathrm{rm}}_{x,i}$, $\eta^{\mathrm{rm}}_{x,i}$ be as in the hypothesis, and let (j, w, r, π) be a point of (x, i), $\ell = \ell_{x,i}(w)$, $K' = \mathcal{O}_r(N)[\ell^{-1}]$. It remains to check the two clauses on K' in (Ub3). By (Ub3°), E is isomorphic to a subfield of $\mathbb{Q}(\mu_N)$; for $v \mid \ell$ the

completion E_v is then $\mathbb{Q}_\ell$-isomorphic to a subfield of a completion of $\mathbb{Q}(\mu_N)$ at a place above ℓ, that is, of $\mathbb{Q}_\ell(\mu_N)$. Item 3 of Lemma 22.16 with $F = E_v$ and $F_1 = E_v(\mu_\ell)$ gives $\sigma(E_v(\mu_\ell)) \subseteq K'$ for every $\mathbb{Q}_\ell$-embedding σ of $E_v(\mu_\ell)$, at every prime (in particular at the primes of split and inert type and the large primes); item 1 there gives a primitive $\operatorname{lcm}(\ell, N)$-th root of unity in K'. So (Ub1)–(Ub10) hold for $\mathcal{F}'$ at every point of every member with $x \geq x_0$, $x > x_1$ and finite Sha lengths, with the same x_0, $N_{x,i}$, $S^{\mathrm{rm}}_{x,i}$, $\eta^{\mathrm{rm}}_{x,i}$ and constants.

Step 2: the conclusions for $\mathcal{F}'$. By Theorem 22.4, for all x and members as in the conclusion: at every point the minimal losses of the items of $\mathcal{F}'$ are finite with $\sum_k \mathrm{ls}^{\min}_k \leq a_{x,i}(\ell)$, and these items with the minimal losses (losses 0 where they are infinite) are Galois Ribet records with (Gd8), (Gd9), (Gd10); the Ribet records of $\mathcal{F}'$ have (Rf1), (Rf2), (Rf4), (Rf5); the local bound records of $\mathcal{F}'$ have (Lq3) and (Lq7) with the constant $19C_3^3$.

Step 3: descent to $\mathcal{F}$. By item 2 of Lemma 22.17, at every point the minimal losses of the items of $\mathcal{F}$ equal those of $\mathcal{F}'$, and the loss sums formed with them are equal (same $n_{\mathcal{R},0}$, $\nu(\delta)$, $\nu(\zeta_v - 1)$); so they are finite exactly where those of $\mathcal{F}'$ are, and $\sum_k \mathrm{ls}^{\min}_k \leq a_{x,i}(\ell)$, the allowance being the same number. The losses chosen for $\mathcal{F}$ (minimal where finite, 0 elsewhere) are the losses chosen for $\mathcal{F}'$; by item 3 of Lemma 22.17 the items of $\mathcal{F}$ with them are Galois Ribet records with (Gd8), (Gd9), (Gd10). This is item 1. By item 3 of Lemma 22.17, (Rf2), (Rf4), (Rf5) for $\mathcal{F}'$ give them for $\mathcal{F}$, and (Rf1) for $\mathcal{F}'$ gives (Rf1) for $\mathcal{F}$: item 2. The local bound records of $\mathcal{F}$ are those of $\mathcal{F}'$: item 3. □

Definition 22.2. Let $F \subset \overline{\mathbb{Q}}$ be a number field and ℓ a prime. Let L be a field with $\mathbb{Q}_\ell \subseteq L \subseteq \overline{\mathbb{Q}}_\ell$ which is finite over $\mathbb{Q}_\ell$ or equal to $\overline{\mathbb{Q}}_\ell$, with the topology of $|\cdot|_\ell$, and let V be a finite-dimensional L-vector space with a continuous L-linear action of G_F. For a finite place v of F, a **decomposition group at** v is the image $D_j \subseteq G_F$ of $\mathrm{Gal}(\overline{F_v}/F_v)$ under restriction along an embedding $j\colon \overline{\mathbb{Q}} \to \overline{F_v}$ over F; $I_j \subseteq D_j$ is the image of the inertia group. For a closed subgroup $D \subseteq G_F$, $H^1(D, V)$ is computed with continuous cocycles.

Local conditions. Let v be a place of F and D a decomposition group at v (for v archimedean: the image of $\mathrm{Gal}(\mathbb{C}/F_v)$ under an embedding $\overline{\mathbb{Q}} \to \mathbb{C}$ over F). The subspace $H^1_f(D, V) \subseteq H^1(D, V)$ consists of the classes $[c]$ such that:

- if v is finite and $v \nmid \ell$, $D = D_j$: there is $x \in V$ with $c(g) = gx - x$ for all $g \in I_j$;
- if $v \mid \ell$, $D = D_j$: there is $y \in V \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $c(g) \otimes 1 = gy - y$ for all $g \in D_j$, where $D_j \cong \mathrm{Gal}(\overline{F_v}/F_v)$ acts on V and on B_{cris} (formed for $\overline{F_v} = \overline{\mathbb{Q}}_\ell$) diagonally;
- if v is archimedean: $H^1_f(D, V) = 0$.

Rational coefficients. A continuous 1-cocycle $c\colon G_F \to V$ is a **Bloch–Kato cocycle** if for every place v of F and every decomposition group D at v the class of $c|_D$ lies in $H^1_f(D, V)$. The **Bloch–Kato Selmer group**

$$H^1_f(F, V) \subseteq H^1(F, V)$$

is the space of classes of Bloch–Kato cocycles.

Divisible coefficients [141], 14.1. Let L be finite over $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_L$, and let $T \subset V$ be a G_F-stable $\mathcal{O}_L$-lattice; $V/T \cong T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is a discrete G_F-module. The **Selmer group** of T over F is

$$\mathrm{Sel}(F, T) = \{x \in H^1(F, V/T) \ :\ x|_D \in \mathrm{Im}(H^1_f(D, V) \to H^1(D, V/T)) \text{ for every place } v \text{ and every decomposition group } D \text{ at } v\}.$$

Definition 22.3. Let E be a number field, ℓ a prime, $(\mathcal{O}, \mathcal{Z}, \mathbb{T}, M)$ a Hecke lattice over ℓ with $K = \mathcal{O}[\ell^{-1}]$, $\mathcal{R}$ a determinant algebra over $\mathbb{T}$, π' a point with a block form with $t+1$ blocks numbered $0, \dots, t$, and $\mathbf{r}$, χ_1, $\mathcal{E}$, δ items of Galois Ribet data over E on $(\mathcal{R}, \pi', \mathcal{N})$ for some $\mathcal{N}$, taken together with the coefficient algebra $\mathbb{T}'$ and the denominator function $\tilde{\delta}$ of these data; these items, and u, y and $\mathcal{N}_{\mathcal{E}}$, do not depend on $\mathcal{N}$. Write $W_0 = K^{n_{\mathcal{R},0}}$ and $\Lambda_0 = \mathcal{O}^{n_{\mathcal{R},0}}$, on which $g \in G_E$ acts by $\rho_0(u(g))$, so that $\mathrm{Cs}_0(u) = W_0/\Lambda_0$; for $\lambda \in \Lambda_0$ and $x \in K/\mathcal{O}$ write $x\lambda = \tilde{x}\lambda + \Lambda_0 \in \mathrm{Cs}_0$ for any lift $\tilde{x} \in K$ of x.

Column data for these items consist of a set $C \subseteq S_\ell$, the **column places**, an idempotent $e_1 \in \mathcal{R}$, and for every $v \in C$ an integer $j_v \geq 0$, maps $\alpha_{v,a} \colon G_{E_v} \to \mathcal{R}$ for $1 \leq a \leq j_v$, and a map $\alpha_v^0 \colon G_{E_v} \to \mathcal{R}\mathcal{E}$, the **column coefficients**. Let $\mathcal{F} = \bigoplus_{v \in C} (\mathcal{R}\epsilon_v \oplus \bigoplus_{a=1}^{j_v} \mathcal{R}\epsilon_{v,a} \oplus \mathcal{R}\epsilon_v^0)$ be the free left $\mathcal{R}$-module on symbols $\epsilon_v, \epsilon_{v,a}, \epsilon_v^0$, and $\mathrm{Rel} \subseteq \mathcal{N}_{\mathcal{E}} \oplus \mathcal{F}$ the left $\mathcal{R}$-submodule generated by the elements, for $v \in C$, $1 \leq a \leq j_v$ and $g \in G_{E_v}$,

$$\mathcal{E}\epsilon_v, \qquad (1-e_1)\epsilon_{v,a}, \qquad (\tilde{\delta} - \mathcal{E})\epsilon_v^0, \qquad \Big(y(g),\ (u(g)-1)\epsilon_v - \delta \sum_{a=1}^{j_v} \alpha_{v,a}(g)\epsilon_{v,a} - \delta\, \alpha_v^0(g)\epsilon_v^0\Big).$$

The **column-enlarged module of extensions** is the left $\mathcal{R}$-module

$$\mathcal{N}^{\mathrm{col}} = (\mathcal{N}_{\mathcal{E}} \oplus \mathcal{F})/\mathrm{Rel}, \qquad \iota^{\mathrm{col}} \colon \mathcal{N}_{\mathcal{E}} \to \mathcal{N}^{\mathrm{col}}, \quad n \mapsto (n, 0) + \mathrm{Rel}. \tag{22.1}$$

At $v \in C$ the column data have the **column properties** if there are a vector $z_1 \in \Lambda_0$, continuous maps $\beta_{v,a} \colon G_{E_v} \to \Lambda_0$ $(1 \leq a \leq j_v)$, integers $e_v' \geq 0$ and $b_v \geq 0$ and an element $g_v \in G_{E_v}$ such that:

(Cm1) The column line $\rho_k(e_1) = 0$ for $1 \leq k \leq t$, and every $m \in \mathrm{Cs}_0$ with $\rho_0(e_1)m = m$ is of the form $m = xz_1$ with $x \in K/\mathcal{O}$;

(Cm2) Realisation each $\beta_{v,a}$ is a 1-cocycle,

$$\beta_{v,a}(gh) = \beta_{v,a}(g) + \rho_0(u(g))\beta_{v,a}(h),$$

and $\rho_0(\alpha_{v,a}(g))\, z_1 = \ell^{e_v'}\, \beta_{v,a}(g)$ for $g \in G_{E_v}$;

(Cm3) Crystalline classes the class of each $\beta_{v,a}$ in $H^1(G_{E_v}, W_0)$ lies in $H^1_f(G_{E_v}, W_0)$;

(Cm4) Independence if $x_1, \dots, x_{j_v} \in K/\mathcal{O}$ and $g \mapsto \sum_a x_a \beta_{v,a}(g)$ is a coboundary of G_{E_v} with values in $\mathrm{Cs}_0(u)$, then $x_1 = \cdots = x_{j_v} = 0$;

(Cm5) The character column $\rho_1(\alpha_v^0(g)) = 0$ for $g \in I_v$, and

$$\rho_1(\alpha_v^0(g_v)) \neq 0 \quad \text{with} \quad \nu(\rho_1(\alpha_v^0(g_v))) \leq b_v.$$

Theorem 22.4. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with local defect data, local bound records, Ribet records, budget data and inert budget data. For Galois Ribet data, the **minimal losses** $\kappa_k^{\min}, \lambda_k^{\min}(v) \in [0, \infty]$ are the normalised lengths of the kernel in (Gd6) and of the image in (Gd7); they depend only on the entries other than the losses, the **Galois Ribet items**. When they are finite, $\mathrm{ls}_k^{\min}$ denotes the loss sums formed with them. For a member (x, i) write t, ϱ, H, ℓ^* for its budget data, $L = x^{\varepsilon}$, and for an integer $N \geq 1$ and a prime ℓ put $s_\ell(N) = \max(1, v_\ell(N))$. Let $c_{\mathrm{sp}}, c_{\mathrm{in}}, c_{\mathrm{sm}}, C_{\mathrm{bud}} \geq 1$ be real numbers, $c_1 = \max(c_{\mathrm{sp}}, 3c_{\mathrm{ur}})$,*

$c_2 = \max(c_{\rm in}, 3c_{\rm ur})$, *and define the* ***allowance*** $a_{x,i}(\ell)$ *of* $\ell \in P_{x,i} \setminus G_{x,i}$, *given an integer* $N_{x,i} \geq 1$ *and integers* $S^{\rm rm}_{x,i}, \eta^{\rm rm}_{x,i} \geq 0$, *as*

$$
\begin{aligned}
&c_{\rm cy}\, c_{\rm sm}\, f\Big(m\lceil H\rceil s_\ell(N_{x,i}) \\
&\qquad + f\, t^2(\varrho+t)\big(s_\ell(N_{x,i}) + \log_\ell(2\lceil H\rceil) + \log_\ell N_{x,i} + 1\big)\Big) && (\ell \text{ of small type, } \ell \notin \mathcal{P}^{\rm rm}_{x,i}),\\
&160\, c_{\rm Q}\, c_{\rm sm}\, f\Big(\varrho\big(\tfrac{H}{\ell} + S^{\rm rm}_{x,i} + \eta^{\rm rm}_{x,i} + \log_\ell(2H) + 1\big) + m\,H \\
&\qquad + n\, t^2(\varrho+t)\big(1 + \log_\ell(H+t)\big)\Big) && (\ell \in \mathcal{P}^{\rm rm}_{x,i}),\\
&190\, c_1\, f\Big(n\, t^2(\varrho+t)(1+\log_\ell(H+t)) + \tfrac{m\,H}{\ell}\Big) && (\ell \text{ of split type}),\\
&300\, c_{\rm R}(c_2+1)\, f\Big(n\, t^2(\varrho+t)(1+\log_\ell(H+t)) + \tfrac{m\,H}{\ell} + \varrho\, A_{x,i}(\ell)\Big) && (\ell \text{ of inert type}),\\
&\ell^{-2} && (\ell \text{ large}).
\end{aligned}
$$

Suppose that at every point (j, w, r, π) *(as below) of every member* (x, i), $x > x_1$, *Galois Ribet items on* $(\mathcal{R}_{r,\pi}, \pi', \mathcal{N}_{r,\pi})$ *over* E *are given, with* ℓ, t *and the Hecke lattice of* r, *and that there is a real* x_0 *such that for every* $x \geq x_0$ *with* $x > x_1$ *and every member* (x, i) *with finite Sha lengths there are an integer* $N_{x,i} \geq 1$ *with* $4 \mid N_{x,i}$ *and integers* $S^{\rm rm}_{x,i}, \eta^{\rm rm}_{x,i} \geq 0$ *such that the following hold, where* (j, w, r, π) *runs over* $1 \leq j \leq m$, $w \in W_{x,i}$ *with* $\ell = \ell_{x,i}(w) \notin G_{x,i}$, $r \in R_{x,i}(j, w)$, $\pi \in Z_r$, *and* $E = E_{\mathfrak{h}_{x,i}}$, $K = \mathcal{O}_r[\ell^{-1}]$, $D_v = G_{E_v}$, $n_v = [E_v : \mathbb{Q}_\ell]$:

(Ub1) Items *at every* (j, w, r, π) *the Galois Ribet items have (Gd1)–(Gd5); they have (Gd9); E has no real place; and* $\mathrm{Sel}_0(r, \pi)$ *has normalised length at most* $\mathrm{sl}(r, \pi)$;

(Ub2) Modules *if* ℓ *is of split type, or of small type with* $\ell \notin \mathcal{P}^{\rm rm}_{x,i}$, *there are column data with column places* $C \subseteq S_\ell$ *for the items* $\mathbf{r}, \chi_1, \mathcal{E}, \delta$, *and* $\mathcal{N}_{r,\pi} = \mathcal{N}^{\rm col}$, $\iota = \iota^{\rm col}$; *if* ℓ *is of inert type, large or in* $\mathcal{P}^{\rm rm}_{x,i}$, $\mathcal{N}_{r,\pi} = \mathcal{N}_{\mathcal{E}}$ *and* ι *is the identity;*

(Ub3) Host and coefficients E *is isomorphic to a subfield of* $\mathbb{Q}(\mu_{N_{x,i}})$; *every prime of split type, of inert type and every large prime is prime to* $N_{x,i}$; *every prime of split type is odd; if* ℓ *is of split or inert type or large,* K *contains* $\sigma(E_v(\mu_\ell))$ *for every* $v \mid \ell$ *and every* $\mathbb{Q}_\ell$*-embedding* σ *of* $E_v(\mu_\ell)$; *if* ℓ *is of small type,* K *contains a primitive* $\mathrm{lcm}(\ell, N_{x,i})$*-th root of unity; and (Ll1), (Ll2) hold at every* $v \mid \ell$;

(Ub4) Split type *if* ℓ *is of split type: there are integers* $M_0, h_0 \geq 0$ *with* $M_0 + h_0 \leq H$ *such that at every* $v \mid \ell$ *all Hodge jumps of* $V_{\mathcal{N}}$, *of* $W_0, \ldots, W_t$ *and of* W_0' *lie in* $[-M_0, h_0]$; *at every* $v \in C$ *the column data have the column properties with* $e'_v \leq c_{\rm sp}(1 + H/\ell + \log_\ell H)$ *and* $b_v \leq c_{\rm sp} t$, $H^0(D_v, W_0) = 0$, W_0 *has at most* m *Hodge jumps* $a < 0$ *at each embedding, and* 0 *is not a Hodge jump of* W_0 *at some embedding; at every* $v \mid \ell$, $v \notin C$: $H^0(D_v, W_0') = 0$, W_0 *has at most* m *Hodge jumps* $a \geq 0$ *at each embedding,* 0 *is not a Hodge jump of* W_0' *at some embedding, and* $|J(V_{\mathcal{N}})| \leq c_{\rm sp} n_v(\varrho + t)$; *and* $\sum_{v|\ell} n_v \leq 2f$, $n_{\mathcal{R},0} \leq \varrho$, $\nu(\delta) \leq 2t$, $\sum_{v \in \Sigma_{\rm T}} \nu(\zeta_v - 1) \leq c_{\rm sp}$;

(Ub5) Inert type *if* ℓ *is of inert type: at every* $v \mid \ell$, $H^0(D_v, W_0') = 0$; *there are integers* $M_0, h_0 \geq 0$ *with* $M_0 + h_0 \leq H$ *such that all Hodge jumps of* $V_{\mathcal{N}}$ *and of* $W_0, \ldots, W_t$ *lie in* $[-M_0, h_0]$; $|J(V_{\mathcal{N}})| \leq c_{\rm in} n_v(\varrho + t)$; *and there is a weight word at* v *with the integer* m *above whose integers* b, w, S, η, M, h' *satisfy* $b + 2\eta + S \leq H$, $w + S + \eta + 1 \leq H$, $M + h' \leq H$, $w \leq w^\circ$, $S \leq S^\circ$, $\eta \leq \eta^\circ$, $\beta_v \leq \beta_{x,i}(\ell)$, *and the hypotheses of item 3 of Proposition 26.2 with* $j = \max(1, \lceil \log_{\ell-1} \ell_{\rm A} \rceil)$ *if* $\ell \in \mathcal{P}^{\rm lo}_{x,i}$, *those of item 1 or of item 2 if* $\ell \notin \mathcal{P}^{\rm lo}_{x,i}$; *and* $\sum_{v|\ell} n_v \leq 2f$, $n_{\mathcal{R},0} \leq \varrho$, $\nu(\delta) \leq 2t$, $\sum_{v \in \Sigma_{\rm T}} \nu(\zeta_v - 1) \leq c_{\rm in}$;

(Ub6) Large primes *if ℓ is large: $\delta \in \mathcal{O}_r^\times$, $t\sum_{v\in\Sigma_{\mathrm{T}}}\nu(\zeta_v-1)\le \ell^{-2}$, and at every $v\mid\ell$ there are a Frobenius numbering and data $h,w,\eta,\mathrm{typ},(U_a)$ with items 1–3 and (Ph1)–(Ph5) of Proposition 28.1;*

(Ub7) Small type *if ℓ is of small type and $\ell\notin\mathcal{P}^{\mathrm{rm}}_{x,i}$: at every $v\mid\ell$ all Hodge jumps of $V_{\mathcal{N}}$, of $W_0,\dots,W_t$ and of W_0' lie in $[-\lceil H\rceil,\lceil H\rceil]$; at every $v\in C$ the column data have the column properties with $e_v'\le c_{\mathrm{sm}}(\lceil H\rceil s_\ell(N_{x,i})+1)$ and $b_v\le c_{\mathrm{sm}}t$, $H^0(D_v,W_0)=0$, W_0 has at most m Hodge jumps $a<0$ at each embedding, and 0 is not a Hodge jump of W_0 at some embedding; at every $v\mid\ell$, $v\notin C$: $H^0(D_v,W_0')=0$, W_0 has at most m Hodge jumps $a\ge 0$ at each embedding, 0 is not a Hodge jump of W_0' at some embedding, and $|J(V_{\mathcal{N}})|\le c_{\mathrm{sm}}n_v(\varrho+t)$; and $\sum_{v\mid\ell}n_v\le 2f$, $n_{\mathcal{R},0}\le\varrho$, $\nu(\delta)\le 2t$, $\sum_{v\in\Sigma_{\mathrm{T}}}\nu(\zeta_v-1)\le c_{\mathrm{sm}}$;*

(Ub7') Ramified primes *if $\ell\in\mathcal{P}^{\mathrm{rm}}_{x,i}$: at every $v\mid\ell$, with $E_{v,0}$ the maximal unramified subextension of $E_v/\mathbb{Q}_\ell$, either $E_v\subseteq E_{v,0}(\mu_\ell)$ and $[E_v:E_{v,0}]=2$, or $\mathfrak{h}_{x,i}=(q,d,Q_S,\alpha)$ admits a complete good-host witness and $\ell\in\{q,d\}$; $J(V_{\mathcal{N}})\subseteq[-M_0,h_0]$ with integers $M_0,h_0\ge 0$, $M_0+h_0\le H$, and $|J(V_{\mathcal{N}})|\le c_{\mathrm{sm}}n_v(\varrho+t)$; at every $\mathbb{Q}_\ell$-embedding $\tau\colon E_v\to K$ there are a sign $\epsilon_\tau\in\{+1,-1\}$ and an integer $b_\tau>\eta^{\mathrm{rm}}_{x,i}$ such that every Hodge jump a of every $U'_{x'}$, $x'\in\mathcal{Z}'_{r,\pi}$, at τ satisfies $|a|\le 2H$ and either $|a|\le\eta^{\mathrm{rm}}_{x,i}$ or $\epsilon_\tau a\in[b_\tau,b_\tau+S^{\mathrm{rm}}_{x,i}]$, and W_0' has at most m Hodge jumps in $[-\eta^{\mathrm{rm}}_{x,i},\eta^{\mathrm{rm}}_{x,i}]$ at τ, counted with multiplicity; and $\sum_{v\mid\ell}n_v\le 2f$, $n_{\mathcal{R},0}\le\varrho$, $\nu(\delta)\le 2t$, $\sum_{v\in\Sigma_{\mathrm{T}}}\nu(\zeta_v-1)\le c_{\mathrm{sm}}$;*

(Ub8) Sizes *(Lz1) holds with the constant C_{bud}; $\sum_{\ell\in\mathcal{P}^{\mathrm{sm}}_{x,i}}\log\ell\le C_{\mathrm{bud}}L$, $|\mathcal{P}^{\mathrm{rm}}_{x,i}|\le 2$ and $\log\ell\le L$ for $\ell\in\mathcal{P}^{\mathrm{rm}}_{x,i}$; $a_{x,i}(\ell)\le C_{\mathrm{bud}}(f\,m\,b\log x+x^{C_{\mathrm{bud}}}L^2)$ for $\ell\in\mathcal{P}^{\mathrm{sm}}_{x,i}\setminus G_{x,i}$ and $a_{x,i}(\ell)\le C_{\mathrm{bud}}f\,m\,b$ for $\ell\in\mathcal{P}^{\mathrm{rm}}_{x,i}\setminus G_{x,i}$; and (Iq1) holds with the constant C_{bud};*

(Ub9) Record choices $\mathrm{lc}_k(r,\pi)=\mathrm{ls}^{\min}_k(r,\pi)$ *for $0\le k\le t$ whenever the minimal losses at (r,π) are finite;* $\mathrm{lt}(r)=a_{x,i}(\ell)$; *and* $\mathrm{la}_{x,i}(\ell)=a_{x,i}(\ell)$ *for $\ell\in P_{x,i}\setminus G_{x,i}$,* $\mathrm{la}_{x,i}(\ell)=0$ *for every other prime;*

(Ub10) Points without the character line *for every $x'\in\mathcal{Z}'_{r,\pi}$ with $\phi_{r,\pi}(x')\in\widehat{Z}_r\setminus Z_r$, the hypotheses 1–3 of Proposition 22.7 hold at x' for the column data of (Ub2), or, if ℓ is of inert type, large or in $\mathcal{P}^{\mathrm{rm}}_{x,i}$, for the column data with no column places.*

Then, for all $x\ge x_0$ with $x>x_1$, $x^\varepsilon\ge 3$ and x beyond the thresholds of (Lz1) and (Iq1), and all members with finite Sha lengths:

1. *at every (j,w,r,π) the minimal losses are finite and $\sum_{k=0}^t\mathrm{ls}^{\min}_k(r,\pi)\le a_{x,i}(\ell)$; the Galois Ribet items with the minimal losses where these are finite (and the losses 0 elsewhere) are Galois Ribet records having (Gd8), (Gd9) and (Gd10);*
2. *the Ribet records have (Rf1), (Rf2), (Rf4) and (Rf5);*
3. *the local bound records have (Lq3), and (Lq7) with the constant $19C_3^3$, where*

$$C_2=\max(C_{\mathrm{bud}},300c_{\mathrm{R}}(c_2+1)),\qquad C_3=\max(190c_1,120C_2^3).$$

Proof. **Step 0: minimal losses.** Fix Galois Ribet items at a point. (i) If real numbers $\kappa_k,\lambda_k(v)\ge 0$ make (Gd6) and (Gd7) true, then by definition $\kappa_k^{\min}\le\kappa_k$ and $\lambda_k^{\min}(v)\le\lambda_k(v)$; so the minimal losses are finite, (Gd6) and (Gd7) hold with the minimal losses (the modules there

have normalised length equal to them), and, every loss sum being a non-decreasing function of the losses, $\mathrm{ls}_k^{\min} \le \mathrm{ls}_k$. (ii) For any finite losses,

$$\sum_{k=0}^{t} \mathrm{ls}_k = T(\kappa,\lambda) := \sum_{k=0}^{t} \kappa_k + \sum_{v\in S_\ell} \sum_{k=0}^{t} \lambda_k(v) + t \sum_{v\in\Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) + 2(n_{\mathcal{R},0} + t - 1)\,\nu(\delta),$$

so by (i) it suffices, at each point, to exhibit losses with (Gd6), (Gd7) and $T(\kappa,\lambda) \le a_{x,i}(\ell)$: then $\sum_k \mathrm{ls}_k^{\min} \le a_{x,i}(\ell)$.

Step 1: thresholds and the host. As $\varepsilon > 0$, choose $x_0' \ge x_0$ beyond the thresholds of (Lz1) and (Iq1) with $L = x^\varepsilon \ge 3$ for $x \ge x_0'$. Fix such $x > x_1$, a member (x,i) with finite Sha lengths, $N = N_{x,i}$, a point (j,w,r,π) and $\ell = \ell_{x,i}(w) \in P_{x,i} \setminus G_{x,i}$. By (Lz1), $1 \le m \le t$, $\varrho \le t$ and $\ell^* \ge 3$. If $\ell \nmid N$ then every E_v, $v \mid \ell$, is unramified over $\mathbb{Q}_\ell$: by (Ub3) E_v is isomorphic to a subfield of $\mathbb{Q}_\ell(\mu_N)$, which is unramified over $\mathbb{Q}_\ell$ as $\ell \nmid N$. Two elementary bounds are used: for $0 \le a \le H$ and $\ell \ge 3$,

$$2 + \frac{a}{\ell} + \log_\ell(a+1) \le 3\Big(1 + \frac{H}{\ell} + \log_\ell H\Big), \qquad \text{(B1)}$$

and, for $\ell \ge 11$ and $1 \le H \le 2b$ (the elementary bound of the inert case),

$$1 + \log_\ell(2H) \le L. \qquad \text{(B2)}$$

In (B2) the scale is $L = x^\varepsilon = \log b$, the parameter scale of the family.

Step 2: split type. Here ℓ is odd and prime to N (Ub3), so E_v is unramified, $E_v = E_{v,0} \subseteq E_{v,0}(\mu_\ell)$, K contains every $\sigma(E_{v,0}(\mu_\ell))$, and (Ll1), (Ll2) hold. At $v \mid \ell$, $v \notin C$, apply item 1 of Corollary 25.39 with $a_\tau^- = M_0$, $a_\tau^+ = h_0$ for all τ (admissible by (Ub4), all jumps of $V_{\mathcal{N}}$ and of the W_k being in $[-M_0, h_0]$): (Ll3) holds with $e_0 = \lfloor c_{\mathrm{ur}}(2 + \mathrm{wt}/\ell + \log_\ell(\mathrm{wt}+1))\rfloor$, $\mathrm{wt} = M_0 + h_0 \le H$. Its case $k = 0$ is the inclusion required of $e_0(v)$ in (Lr2) of Theorem 25.1; put $e_0(v) = e_0$, so $e_0(v) \le 3c_{\mathrm{ur}}(1 + H/\ell + \log_\ell H) \le c_1(1 + H/\ell + \log_\ell H)$ by (B1). With $\mu_\tau^{\mp}(v) = m$ and the embeddings τ_v of (Ub4), (Lr1) and (Lr2) hold, the remaining hypotheses of Theorem 25.1 hold by (Ub2)–(Ub4) and (Gd1), (Gd2) (Ub1), so Theorem 25.1 gives losses with (Gd6) and (Gd7). Items 1–4 of Theorem 27.15 hold with $c = c_1$: item 1 by (Ub4) and (Lz1); item 2 by (Ub4); item 3 by $e_v' \le c_{\mathrm{sp}}(1 + H/\ell + \log_\ell H)$ and the bound on $e_0(v)$, as $c_{\mathrm{sp}} \le c_1$; item 4 by (Ub4). So $T(\kappa,\lambda) \le 190c_1 f(n\,t^2(\varrho+t)(1 + \log_\ell(H+t)) + mH/\ell) = a_{x,i}(\ell)$.

Step 3: inert type. By (Iq1) item 2, $\ell \ge 11$; $\ell \nmid N$ (Ub3), so every E_v is unramified, K contains every $\sigma(E_v(\mu_\ell))$, and (Ll1), (Ll2) hold; $\mathcal{N}_{r,\pi} = \mathcal{N}_{\mathcal{E}}$ with ι the identity (Ub2). As in Step 2, item 1 of Corollary 25.39 with $a_\tau^- = M_0$, $a_\tau^+ = h_0$ (Ub5) gives at every $v \mid \ell$ an integer $e_0(v)$ with the inclusion of item 2 of Theorem 26.1. The lifting exponent e_v of the weight word of (Ub5) is the integer e_{A} of Proposition 24.1 for the window $[-M, h']$ and the duals $U'_{x'}$, which are crystalline by (Ll1); by the final assertion of Proposition 24.1 (ℓ odd, E_v unramified, $K \supseteq \sigma(E_v(\mu_\ell))$) and (B1) with $a = M + h' \le H$, $e_v \le c_{\mathrm{ur}}(2 + (M+h')/\ell + \log_\ell(M+h'+1)) \le c_2(1 + H/\ell + \log_\ell H)$. Take $A = A_{x,i}(\ell)$: for $\ell \in \mathcal{P}_{x,i}^{\mathrm{lo}}$ this is the number A_{lo} of Theorem 26.1 formed with $\ell_{\mathrm{A}}, w^\circ, S^\circ, \eta^\circ$ and L, otherwise A_{md} with $\beta = \beta_{x,i}(\ell) \in \{0,1\}$. The hypotheses of Theorem 26.1 hold with $c = c_2$: $n, \varrho, m \ge 1$, $H, c_2, L \ge 1$, $\ell_{\mathrm{A}} \ge 2$, $w^\circ, \eta^\circ, S^\circ \ge 0$; item 1 by (Ub5), (Lz1) and (B2), using $H \le 2b$ from (Iq1) item 1; item 2 by (Ub3), (Ub5), $e_0(v)$ and $c_{\mathrm{in}} \le c_2$; item 3 by (Ub5), with item 3 of Proposition 26.2 for $j = \max(1, \lceil \log_{\ell-1} \ell_{\mathrm{A}} \rceil)$ when $A = A_{\mathrm{lo}}$ and item 1 or 2 when $A = A_{\mathrm{md}}$, and the bound on e_v; item 4 by (Ub5). It gives $\kappa_k = 0$ and losses $\lambda_k(v)$ with (Gd6), (Gd7) and $T(\kappa,\lambda) \le 300c_{\mathrm{R}}(c_2+1)f(n\,t^2(\varrho+t)(1+\log_\ell(H+t)) + mH/\ell + \varrho A) = a_{x,i}(\ell)$.

Step 4: large primes. Here $\ell > \ell^* \ge 3$, so $\ell \ge 5$, and $\ell \nmid N$ (Ub3), so every E_v is unramified and K contains every $\sigma(E_v)$. As $\mathcal{N}_{r,\pi} = \mathcal{N}_{\mathcal{E}}$ and ι is the identity (Ub2), the map

$\varphi \mapsto \varphi \circ \iota$ of (Gd6) is the identity and $\kappa_k^{\min} = 0$. At every $v \mid \ell$, Proposition 28.1 applies: (Gd1), (Gd2) by (Ub1), δ a unit, (Ll1), (Ll2), E_v unramified and $K \supseteq \sigma(E_v(\mu_\ell))$ for every embedding σ by (Ub3), and the data with items 1–3 and (Ph1)–(Ph5) by (Ub6); it gives $\lambda_k^{\min}(v) = 0$. So the minimal losses are finite and, as $\nu(\delta) = 0$, $T(\kappa^{\min}, \lambda^{\min}) = t \sum_{v \in \Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) \leq \ell^{-2} = a_{x,i}(\ell)$ by (Ub6).

Step 5: small type, $\ell \notin \mathcal{P}_{x,i}^{\mathrm{rm}}$. Apply Proposition 27.16 with $N = N_{x,i}$ (so $4 \mid N$), the integer $\lceil H \rceil \geq 1$ in place of H, $c = c_{\mathrm{sm}}$, the integers n, ϱ, m and the column data of (Ub2): its item 1 holds because E_v is isomorphic to a subfield of $\mathbb{Q}_\ell(\mu_N) \subseteq \mathbb{Q}_\ell(\mu_{\mathrm{lcm}(\ell,N)})$ (Ub3), K contains a primitive $\mathrm{lcm}(\ell, N)$-th root of unity (Ub3), (Ll1), (Ll2) hold, and the jumps lie in $[-\lceil H \rceil, \lceil H \rceil]$ (Ub7); items 2–4 hold by (Ub7) and $\varrho \leq t$, $m \leq t$ from (Lz1); there $s = s_\ell(N)$. It gives losses with (Gd6), (Gd7) and $T(\kappa, \lambda) \leq a_{x,i}(\ell)$.

Step 5′: the ramified primes. Let $\ell \in \mathcal{P}_{x,i}^{\mathrm{rm}}$, so $\mathcal{N}_{r,\pi} = \mathcal{N}_{\mathcal{E}}$ and ι is the identity (Ub2). *The local fields.* Let $v \mid \ell$. If the second alternative of the first clause of (Ub7') holds, then $E = E_{\mathfrak{h}_{x,i}}$ for a host with a complete good-host witness and $\ell \in \{q, d\}$, and item 5 of Lemma 22.5 gives $E_v = E_{v,0}(\sqrt{-\ell}) \subseteq E_{v,0}(\mu_\ell)$ and $[E_v : E_{v,0}] = 2$; so in either case $E_v \subseteq E_{v,0}(\mu_\ell)$ and $[E_v : E_{v,0}] = 2$. In particular $[E_{v,0}(\mu_\ell) : E_{v,0}] \geq 2$, so $\mu_\ell \not\subseteq \mathbb{Q}_\ell$ and ℓ is odd. By (Ub3), E_v is isomorphic to a subfield of $\mathbb{Q}_\ell(\mu_N)$, $N = N_{x,i}$; write $N = \ell^a N'$ with $\ell \nmid N'$. The maximal unramified subextension of $\mathbb{Q}_\ell(\mu_N)/\mathbb{Q}_\ell$ is $\mathbb{Q}_\ell(\mu_{N'})$, so $E_{v,0}$ is isomorphic to a subfield of $\mathbb{Q}_\ell(\mu_{N'})$ and $E_{v,0}(\mu_\ell)$ to a subfield of $\mathbb{Q}_\ell(\mu_{N'\ell})$. This field is normal over $\mathbb{Q}_\ell$, so $\sigma(E_{v,0}(\mu_\ell)) \subseteq \mathbb{Q}_\ell(\mu_{N'\ell})$ for every $\mathbb{Q}_\ell$-embedding σ of $E_{v,0}(\mu_\ell)$; and $\mathbb{Q}_\ell(\mu_{N'\ell}) \subseteq K$, since K contains a primitive $\mathrm{lcm}(\ell, N)$-th root of unity (Ub3) and $N'\ell \mid \mathrm{lcm}(\ell, N)$. (Ll1), (Ll2) hold by (Ub3). *The losses.* Apply Theorem 27.1 with the integers n, ϱ, m, $S = S_{x,i}^{\mathrm{rm}}$, $\eta = \eta_{x,i}^{\mathrm{rm}}$, the real number H and $c = c_{\mathrm{sm}}$: its hypothesis 1 holds by (Ub7') and (Lz1) ($\varrho \leq t$, $m \leq t$), its hypothesis 2 by the local fields just discussed and (Ub7'), its hypotheses 3 and 4 by (Ub7'). Its items 1 and 2 give losses with (Gd6), (Gd7) and

$$T(\kappa, \lambda) \leq 160\, c_{\mathrm{Q}}\, c_{\mathrm{sm}}\, f\Big(\varrho\big(\tfrac{H}{\ell} + S_{x,i}^{\mathrm{rm}} + \eta_{x,i}^{\mathrm{rm}} + \log_\ell(2H) + 1\big) + mH + nt^2(\varrho + t)\big(1 + \log_\ell(H + t)\big)\Big) = a_{x,i}(\ell).$$

Step 6: the Galois Ribet records; (Rf1), (Rf2), (Rf4), (Rf5). Since $\ell \in P_{x,i} \setminus G_{x,i}$ is of exactly one of the four types, and a prime of small type either lies in $\mathcal{P}_{x,i}^{\mathrm{rm}}$ or not, Steps 0, 2–5 and 5′ show that at every point the minimal losses are finite and $\sum_k \mathrm{ls}_k^{\min}(r, \pi) \leq a_{x,i}(\ell)$. With them (and with the losses 0 at the points of members below the threshold where minimal losses are infinite; such members are not read by (Gd8)–(Gd10)) the items are Galois Ribet records, and at the points of large members they are Galois Ribet data with (Gd1)–(Gd7) ((Ub1), Step 0); with (Ub1) and $\mathrm{lc}_k = \mathrm{ls}_k^{\min}$ (Ub9) they form Galois Ribet records with (Gd8) and (Gd9), and with (Gd10) by Step 4 and (Ub6). By Theorem 22.10, items 1 and 2, the Ribet records have (Rf4) and (Rf1). (Rf5): $\sum_k \mathrm{lc}_k(r, \pi) = \sum_k \mathrm{ls}_k^{\min}(r, \pi) \leq a_{x,i}(\ell) = \mathrm{lt}(r)$ by (Ub9). (Rf2): let $\phi_{r,\pi}(x') \in \widehat{Z}_r \setminus Z_r$. At a prime of split type or of small type not in $\mathcal{P}_{x,i}^{\mathrm{rm}}$, Proposition 22.7 with the column data of (Ub2) (its hypotheses by (Ub10), and $\mathcal{E}^2 = \tilde{\delta}\mathcal{E}$ for the denominator function $\tilde{\delta}$ of the items by (Gd2)) gives $\mathrm{Hom}_{\mathcal{R}_{r,\pi}}(\mathcal{N}^{\mathrm{col}}, \mathcal{O}_r[\ell^{-1}]_{x'}^{n_{\mathcal{R}}}) \neq 0$, and $\mathcal{N}^{\mathrm{col}} = \mathcal{N}_{r,\pi}$. At a prime of inert type, a large prime or a prime in $\mathcal{P}_{x,i}^{\mathrm{rm}}$, take the column data with no column places and $e_1 = 1$: the free module $\mathcal{F}$ and the relations are then 0, so $\mathcal{N}^{\mathrm{col}} = \mathcal{N}_{\mathcal{E}} = \mathcal{N}_{r,\pi}$ and ι^{col} is the identity, and Proposition 22.7 (hypotheses by (Ub10)) gives the same non-vanishing. This is (Rf2).

Step 7: (Lq3) and (Lq7). (Lq3): $\mathrm{lt}(r) = a_{x,i}(\ell) = \mathrm{la}_{x,i}(\ell)$ by (Ub9). For (Lq7) we check the clauses of the budget data for the local allowances $\mathrm{la}_{x,i}$ of (Ub9). (Lz1) holds with C_{bud} (Ub8). (Lz2) holds with C_{bud}: for $\ell \in \mathcal{P}_{x,i}^{\mathrm{sm}} \setminus G_{x,i}$ and $\ell \in \mathcal{P}_{x,i}^{\mathrm{rm}} \setminus G_{x,i}$ by $\mathrm{la} = a$ and (Ub8), for $\ell \in \mathcal{P}_{x,i}^{\mathrm{rm}} \cap G_{x,i}$ because $\mathrm{la}_{x,i}(\ell) = 0$, and the size parts by (Ub8). (Lz4) holds with

$190c_1$ since $\mathrm{la} = a$ at split type. (Lz5) holds since $\mathrm{la}_{x,i}(\ell) = \ell^{-2}$ at large primes. The inert budget data have (Iq1) with C_{bud} (Ub8) and (Iq2) with $300c_{\mathrm{R}}(c_2 + 1)$, since $\mathrm{la} = a$ at inert type. Each of these clauses remains true when its constant is enlarged, so C_2 is a constant for (Lz1), (Iq1) and (Iq2), and Theorem 22.8 gives (Lz3) with the constant $120C_2^3$. Then $C_3 = \max(190c_1, 120C_2^3) \geq \max(C_{\mathrm{bud}}, 190c_1, 120C_2^3)$ is a constant for each of (Lz1)–(Lz4), and Theorem 22.9 gives (Lq7) with the constant $19C_3^3$. □

Lemma 22.5. *Fix data for which the concrete good-host requirement holds, together with objects satisfying its conditions. Put*

$$E = E_{\mathfrak{h}}, \qquad K = \mathbb{Q}(s_d), \qquad \Phi_{\mathsf{H}} = \{\tau : E \hookrightarrow \mathbb{C} : \tau(s_d) = \varphi_0(s_d)\}.$$

Write a bar for CM conjugation. For a finite prime v of E, let D_v, I_v be its decomposition and inertia subgroups in $\mathrm{Gal}(E/\mathbb{Q})$. *Then:*

1. $q \equiv d \equiv 3 \pmod 4$.
2. *For $\ell \in \{q, d\}$ and every $v \mid \ell$,*
$$e(v/\ell) = 2, \qquad \bar{v} \neq v.$$
3. *For $v \mid q$, $\sigma \in D_v$ and every embedding $\tau : E \hookrightarrow \mathbb{C}$,*
$$\tau \circ \sigma \in \Phi_{\mathsf{H}} \iff \tau \in \Phi_{\mathsf{H}}.$$
4. *For $v \mid d$, $1 \neq \sigma \in I_v$ and every such embedding,*
$$\tau \circ \sigma \in \Phi_{\mathsf{H}} \iff \tau \notin \Phi_{\mathsf{H}}.$$
5. *For $\ell \in \{q, d\}$ and every $v \mid \ell$, fix an algebraic closure of $\mathbb{Q}_\ell$ and a realisation of E_v in it. Let $E_{v,0}$ be the maximal unramified subextension of $E_v/\mathbb{Q}_\ell$. Then*
$$E_v = E_{v,0}(\sqrt{-\ell}) \subseteq E_{v,0}(\mu_\ell), \qquad [E_v : E_{v,0}] = 2.$$

Moreover, fix $0 < \varepsilon < 1/52$, $J \geq 1$ and $R \geq 0$, and put

$$\alpha_0 = \frac{\varepsilon}{5J}, \qquad X_{\varepsilon,J,R} = \exp\!\left(\max\left\{J,\, 1 + \frac{4R}{\alpha_0^2}\right\}\right).$$

For all data for which the concrete good-host requirement holds with these ε, J and with a scale $x \geq X_{\varepsilon,J,R}$, and all objects satisfying its conditions, one has

$$q > x^R, \qquad d > x^R.$$

The threshold $X_{\varepsilon,J,R}$ is the same for every host and for all member, lift and local indices.

Proof. Condition (Hr1) gives $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$, proving the first assertion. In particular $q \neq d$. Write $N = Q_S qd$.

By Theorem B.16, inertia at q is the image under α of

$$U_q = \{u \in (\mathbb{Z}/N\mathbb{Z})^\times : u \equiv 1 \pmod{Q_S d}\}.$$

Condition (Hr2) and the Chinese remainder theorem identify U_q with $\mathbb{F}_q^\times$, so its image is cyclic. Condition (Hn3) makes that image have exponent dividing 2. Condition (Fr1) gives $\epsilon_1(\alpha(u)) =$

(u/q). Choosing a nonsquare modulo q and 1 modulo $Q_S d$ shows that the image is nontrivial. Thus it has order 2.

At d, the same cyclotomic formula and (Hr5) give

$$I_v = \alpha(U_d) \subseteq \{1, \iota_d\}, \qquad U_d = \{u : u \equiv 1 \pmod{Q_S q}\}.$$

A nonsquare modulo d, extended by 1 modulo $Q_S q$, has second sign -1 by (Hr4). Hence $I_v = \{1, \iota_d\}$. The Galois ramification identity $e(v/\ell) = |I_v|$ proves both ramification assertions. By item 1 of Lemma 22.6, every prime of E^+ above a divisor of N splits in E. Consequently $\bar{v} \neq v$ at both distinguished primes.

Since $d \equiv 3 \pmod 4$, the quadratic field K has discriminant $-d$. The relation $(-d/q) = 1$ and $q \neq d$ say that q splits in K. Thus every D_v at q restricts trivially to K, and in particular fixes s_d. This proves assertion 3. At d, the unique nonidentity inertia element is labelled ι_d. Condition (Hg2) gives

$$\iota_d(s_d) = \epsilon_2(\iota_d) s_d = -s_d.$$

Every embedding takes s_d to one of $i\sqrt{d}, -i\sqrt{d}$. Composing with ι_d exchanges these values, proving assertion 4.

We turn to assertion 5. The field E contains an element of square $-q$ by (Fr2), and the chosen s_d has square $-d$. Thus E_v contains $\sqrt{-\ell}$. The unramified field $E_{v,0}$ has uniformiser ℓ, so $X^2 + \ell$ is Eisenstein over it. Since the already proved ramification index is 2, both E_v and $E_{v,0}(\sqrt{-\ell})$ have degree 2 over $E_{v,0}$, and their containment is equality.

It remains to prove the inclusion in assertion 5, which follows from a finite sum. For an odd prime ℓ, write χ for the quadratic character on $\mathbb{F}_\ell^\times$ and choose a primitive ℓ-th root ζ. Put

$$t = \sum_{a \in \mathbb{F}_\ell^\times} \chi(a)\zeta^a, \qquad t^- = \sum_{a \in \mathbb{F}_\ell^\times} \chi(a)\zeta^{-a}.$$

Changing a to $-a$ gives $t^- = \chi(-1)t$. Changing a to ub in the product gives

$$tt^- = \sum_{u \in \mathbb{F}_\ell^\times} \chi(u) \sum_{b \in \mathbb{F}_\ell^\times} \zeta^{b(u-1)} = (\ell - 1) - \sum_{u \neq 0,1} \chi(u) = \ell.$$

Here the inner sum is $\ell - 1$ for $u = 1$ and -1 otherwise, and the nontrivial quadratic character has sum zero. Consequently $t^2 = \chi(-1)\ell = -\ell$ when $\ell \equiv 3 \pmod 4$. This calculation is valid in characteristic zero, including the fixed ℓ-adic closure. Therefore $\sqrt{-\ell} \in \mathbb{Q}_\ell(\mu_\ell)$, proving the inclusion and assertion 5.

For the uniform bounds, set $y = \log x$. The definition of $X_{\varepsilon,J,R}$ gives $y \geq J \geq 1$ and $y > 4R/\alpha_0^2$. The exponential series implies

$$e^{\alpha_0 y} \geq \frac{\alpha_0^2 y^2}{2} > 2Ry,$$

where the strict last inequality also holds for $R = 0$. The real number Y of (Hg6) satisfies $Y \geq x^{1/J} \geq e$. Therefore

$$\log q > \tfrac{1}{2} Y^{\varepsilon/5} \geq \tfrac{1}{2} x^{\alpha_0} > R \log x$$

and

$$\log d > \tfrac{1}{2} Y^{\varepsilon/5} (\log Y)^2 \geq \tfrac{1}{2} x^{\alpha_0} > R \log x.$$

Exponentiating proves both bounds. Every estimate uses only ε, J, R, so the threshold has the asserted uniformity. □

Lemma 22.6. *Fix data for which the concrete good-host requirement of Definition 11.4 holds, together with objects satisfying its conditions. Put $E = E_{\mathfrak{h}}$, $F = E^+$, $K = \mathbb{Q}(s_d)$ and $N = N_{\mathfrak{h}}$. Let φ_0 be the given embedding and Φ the given CM type,*

$$\Phi = \{\tau : E \hookrightarrow \mathbb{C} : \tau(s_d) = i\sqrt{d}\}.$$

Let c, ι and s denote the automorphisms labelled, respectively, by

$$((-1,-1),0), \qquad ((1,-1),0), \qquad ((-1,1),0).$$

Then:

1. *E/F is unramified at every finite place, and every prime of F above a rational prime dividing N splits in E.*

2. *Every prime divisor of the order of a root of unity in E divides N. Moreover ι fixes every root of unity, $s(\zeta) = \zeta^{-1}$, and*

$$N_{E/K}(\zeta) = \prod_{\tau \in \Phi} \tau(\zeta) = 1 \qquad (\zeta \in \mu(E)).$$

   In the norm equality, K is embedded in $\mathbb{C}$ by $\varphi_0|_K$.

3. *For $\zeta \in \mu(E)$, put*

$$d_\zeta = \begin{cases} 1+\zeta, & \zeta \neq -1, \\ s_d, & \zeta = -1. \end{cases}$$

   This is a nonzero element of E, and

$$d_\zeta = \zeta \bar{d}_\zeta, \qquad \mathrm{ord}_v(d_\zeta) = 0 \quad (v \nmid N), \qquad \prod_{\tau \in \Phi} \frac{\tau(d_\zeta)}{|\tau(d_\zeta)|} = 1.$$

4. *For every $u \in \mathcal{O}_E^\times$,*

$$\prod_{\tau \in \Phi} \frac{\tau(u)}{|\tau(u)|} = 1.$$

Proof. The normalization identifies c with complex conjugation. Its restriction to K is nontrivial. We use the cyclotomic inertia and decomposition descriptions of Theorem B.16, including their images in the specified subfield E.

For an odd prime $p \mid Q_S$, (Hr2) says that p splits in K. The prime q splits in K by (Hr1), and 2 splits because $d \equiv 7 \pmod 8$. Therefore c belongs to no decomposition group above any prime dividing $Q_S q$.

Consider a prime above d. Each element of its decomposition group in $E/\mathbb{Q}$ is represented by a unit u modulo N with

$$u \equiv d^a \pmod{Q_S q}$$

for some $a \geq 0$. If its label were c, (Fr1) would give

$$-1 = (u/q) = (d/q)^a = (-1)^a.$$

Here $(d/q) = -1$, since $(-d/q) = 1$ and $q \equiv 3 \pmod 4$. Thus a is odd. On the other hand, the A-component of c is zero, so (Hn2), with $16 \mid 2^{e+2} \mid Q_S$, gives $u \equiv \pm 1 \pmod{16}$. This

contradicts $u \equiv d^a \equiv 7 \pmod{16}$. Consequently c belongs to no decomposition group above any rational divisor of N. Every corresponding prime of F therefore splits in the quadratic extension E/F.

Outside N, $E/\mathbb{Q}$ is unramified, being a subextension of the cyclotomic field of level N. Hence E/F is unramified there too. At primes above N it is split, which proves item 1.

By (Hr5), inertia at d has image contained in $\{1, \iota\}$. By (Hr4), its second sign is the nontrivial quadratic character modulo d, so its image equals this group. In particular its ramification index is 2. Since $d \geq 7$, E cannot contain a primitive d-th root of unity: the subfield generated by such a root would have ramification index $d - 1 > 2$ at d. Thus the order of every root of unity in E is prime to d. The field generated by that root is unramified at d; therefore ι fixes it. Since $s = c\iota$, it follows that $s(\zeta) = \zeta^{-1}$.

If an odd prime $\ell \nmid N$ divided the order of a root of unity in E, then E would contain a primitive ℓ-th root. Its cyclotomic subfield ramifies at ℓ, contrary to the unramifiedness of $E/\mathbb{Q}$ there. The prime 2 already divides N. This proves the prime-support assertion.

The automorphism s is a nonidentity involution fixing K. The embeddings in Φ are exactly $\varphi_0 \circ g$ for $g \in \mathrm{Gal}(E/K)$: the Galois labels with second sign 1 fix s_d, and these embeddings have the specified value on s_d. Composition by s pairs them without fixed points. Their values on a root of unity are inverse pairs, so their product is 1. This product is $\varphi_0(N_{E/K}(\zeta))$, and injectivity of $\varphi_0|_K$ proves the norm assertion of item 2.

Suppose $\zeta \neq -1$. Then $1 + \zeta \neq 0$, and

$$\zeta\overline{(1+\zeta)} = \zeta(1 + \zeta^{-1}) = 1 + \zeta.$$

Let $v \nmid N$, and let p be its residue characteristic. Put $M = \mathrm{lcm}(2, \mathrm{ord}(\zeta))$. The prime-support assertion gives $p \nmid M$. Both ζ and -1 have order dividing M. Reduction is injective on these roots: if distinct M-th roots x, y had equal reductions, the geometric-sum identity

$$(x - y) \sum_{a=0}^{M-1} x^{M-1-a} y^a = x^M - y^M = 0$$

would force the sum to be zero in E; its reduction is $M\bar{y}^{M-1} \neq 0$, a contradiction. Thus ζ does not reduce to -1, and the integral element $1+\zeta$ has valuation zero at v. Composition by s pairs its phase factors into

$$\frac{1+z}{|1+z|}\frac{1+\bar{z}}{|1+\bar{z}|} = 1.$$

This proves item 3 for $\zeta \neq -1$.

For $\zeta = -1$, use $d_\zeta = s_d$. Its square is $-d$, so its valuation vanishes away from d. Its phase under every member of Φ is i. The surjection $A \to H_e$, with $|H_e| = 2^e$ and $e \geq 2$, implies $4 \mid |A|$, hence $4 \mid |\Phi| = 2|A|$. The product of these phases is therefore 1.

Finally take $u \in \mathcal{O}_E^\times$. The quotient $\zeta = u/\bar{u}$ is an algebraic integer unit, and all its conjugates have modulus one. It is a root of unity: the monic characteristic polynomials over $\mathbb{Q}$ of its positive powers have integral coefficients bounded by the binomial coefficients of degree $[E : \mathbb{Q}]$; only finitely many such polynomials, and hence finitely many roots, occur. Two positive powers therefore coincide.

Put $t = u/d_\zeta$. The equation in item 3 gives $t = \bar{t}$, so $t \in F^\times$. At an inert finite place of F, item 1 excludes residue characteristics dividing N, and item 3 gives $\mathrm{ord}(d_\zeta) = 0$. Since u is a unit and E/F is unramified, $\mathrm{ord}(t) = 0$ at that place of F. By Theorem B.17, the quadratic Artin character of E/F is trivial on finite local units; it is also trivial at split places. Its value

on every finite component of t is consequently 1. It is trivial on the principal idele t, and its infinite components are signs. Therefore

$$\prod_{\sigma: F \hookrightarrow \mathbb{R}} \operatorname{sgn} \sigma(t) = 1.$$

Multiplying this identity by the phase identity for d_ζ proves item 4. □

Proposition 22.7. *Let column data be given for items $\mathbf{r}, \chi_1, \mathcal{E}, \delta$ of Galois Ribet data over a number field E on $(\mathcal{R}, \pi', \mathcal{N})$ for some $\mathcal{N}$, with $\mathcal{E}^2 = \tilde{\delta}\mathcal{E}$, and let $K = \mathcal{O}[\ell^{-1}]$, $x' \in \mathcal{Z}'$ and $V = K_{x'}^{n_{\mathcal{R}}}$, on which $a \in \mathcal{R}$ acts by $a(x')$. Suppose:*

1. *the image $\mathcal{E}(x')V$ is a K-line, and, if $C \neq \emptyset$, $e_1^2 = e_1$ and the image $e_1(x')V$ is a K-line;*
2. *for every $0 \neq \eta \in \mathcal{E}(x')V$ there is $g \in G_E$ with $u(g)(x')\eta \neq \eta$;*
3. *for every $v \in C$, the K-subspace L_v of the space of functions $G_{E_v} \to V$ spanned by the coboundaries $g \mapsto (u(g)(x') - 1)m$ ($m \in V$), the functions $g \mapsto \alpha_{v,a}(g)(x')s$ ($1 \leq a \leq j_v$, $s \in e_1(x')V$) and the functions $g \mapsto \alpha_v^0(g)(x')s$ ($s \in \mathcal{E}(x')V$) satisfies*

$$\dim_K L_v \leq \dim_K B_v + j_v + \dim_K V^{G_{E_v}},$$

where B_v is the space of the coboundaries and $V^{G_{E_v}} = \{m \in V : u(g)(x')m = m \text{ for all } g \in G_{E_v}\}$.

Then $\operatorname{Hom}_{\mathcal{R}}(\mathcal{N}^{\mathrm{col}}, V) \neq 0$.

Proof. **Homomorphisms from data.** Let $\psi \colon \mathcal{N}_{\mathcal{E}} \to V$ be $\mathcal{R}$-linear and let $m_v, s_{v,a}, s_v^0 \in V$ ($v \in C$, $1 \leq a \leq j_v$) satisfy, for all v, a and $g \in G_{E_v}$,

$$\begin{gathered} \mathcal{E}m_v = 0, \quad e_1 s_{v,a} = s_{v,a}, \quad \mathcal{E}s_v^0 = \tilde{\delta}(x')s_v^0, \\ \psi(y(g)) = -(u(g) - 1)m_v + \delta \sum_a \alpha_{v,a}(g)s_{v,a} + \delta\, \alpha_v^0(g)s_v^0. \end{gathered} \tag{$*$}$$

The $\mathcal{R}$-linear map $\mathcal{N}_{\mathcal{E}} \oplus \mathcal{F} \to V$ given by ψ on $\mathcal{N}_{\mathcal{E}}$ and by $\epsilon_v \mapsto m_v$, $\epsilon_{v,a} \mapsto s_{v,a}$, $\epsilon_v^0 \mapsto s_v^0$ on the free module $\mathcal{F}$ kills the generators $\mathcal{E}\epsilon_v$, $(1 - e_1)\epsilon_{v,a}$, $(\tilde{\delta} - \mathcal{E})\epsilon_v^0$ and $(y(g), (u(g) - 1)\epsilon_v - \delta \sum_a \alpha_{v,a}(g)\epsilon_{v,a} - \delta\alpha_v^0(g)\epsilon_v^0)$ of Rel, by $(*)$; so it induces $\varphi \in \operatorname{Hom}_{\mathcal{R}}(\mathcal{N}^{\mathrm{col}}, V)$ with $\varphi \circ \iota^{\mathrm{col}} = \psi$ and the values $m_v, s_{v,a}, s_v^0$ on the classes of the symbols.

A kernel vector at each column place. Fix $0 \neq \eta \in \mathcal{E}V$ and, if $C \neq \emptyset$, $0 \neq s_1 \in e_1 V$ (hypothesis 1); if $C = \emptyset$ this paragraph and Case 1 are empty. For $v \in C$ consider the K-linear map

$$\Phi_v \colon V \oplus K^{j_v} \oplus K \to L_v, \qquad (m, (c_a), c_0) \mapsto \Big(g \mapsto \sum_a c_a\, \alpha_{v,a}(g)s_1 + c_0\, \alpha_v^0(g)\eta - (u(g) - 1)m\Big),$$

which takes values in L_v by definition of L_v. As $m \mapsto (g \mapsto (u(g) - 1)m)$ maps V onto B_v with kernel $V^{G_{E_v}}$, $\dim B_v = \dim V - \dim V^{G_{E_v}}$, and hypothesis 3 gives $\dim L_v \leq \dim V + j_v < \dim(V \oplus K^{j_v} \oplus K)$. So there is $p_v = (m_v^\circ, (c_{v,a}), c_{v,0}) \neq 0$ with $\Phi_v(p_v) = 0$, i.e.

$$\sum_a c_{v,a}\, \alpha_{v,a}(g)s_1 + c_{v,0}\, \alpha_v^0(g)\eta = (u(g) - 1)m_v^\circ \qquad (g \in G_{E_v}). \tag{$**$}$$

Since $\mathcal{E}^2 = \tilde{\delta}\mathcal{E}$ and $\eta \in \mathcal{E}V$, $\mathcal{E}\eta = \tilde{\delta}(x')\eta$; since e_1 is idempotent when $C \neq \emptyset$, $e_1 s_1 = s_1$.

Case 1: $\mathcal{E}m^\circ_{v_0} = 0$ for some $v_0 \in C$. Take $\psi = 0$; at v_0 put $m_{v_0} = \delta m^\circ_{v_0}$, $s_{v_0,a} = c_{v_0,a}s_1$, $s^0_{v_0} = c_{v_0,0}\eta$, and all other values 0. Then $\mathcal{E}m_{v_0} = 0$, $e_1 s_{v_0,a} = s_{v_0,a}$, $\mathcal{E}s^0_{v_0} = \tilde\delta(x')s^0_{v_0}$, and the last relation of $(*)$ at v_0 is δ times $(**)$, while at the other places it reads $0 = 0$. The resulting φ is non-zero: if $m^\circ_{v_0} \neq 0$ it sends ϵ_{v_0} to $\delta m^\circ_{v_0} \neq 0$; otherwise either some $c_{v_0,a} \neq 0$, and then φ sends $\epsilon_{v_0,a}$ to $c_{v_0,a}s_1 \neq 0$, or $c_{v_0,0} \neq 0$, and then φ sends $\epsilon^0_{v_0}$ to $c_{v_0,0}\eta \neq 0$.

Case 2: $\mathcal{E}m^\circ_v \neq 0$ for every $v \in C$. As $\mathcal{E}V = K\eta$, $\mathcal{E}m^\circ_v = t_v\eta$ with $t_v \in K^\times$; replacing p_v by $t_v^{-1}p_v$ we may assume $\mathcal{E}m^\circ_v = \eta$. Put $\theta = \delta/\tilde\delta(x') \in K^\times$. Choose $w \in V$ with $\mathcal{E}w = \theta\eta$ (possible as $\mathcal{E}V = K\eta$) and let $\psi(n) = n(x')w$ for $n \in \mathcal{N}_\mathcal{E} \subseteq \mathcal{R}$; it is $\mathcal{R}$-linear, and $\psi(y(g)) = (u(g)-1)\mathcal{E}w = \theta(u(g)-1)\eta$. Put $m_v = \delta m^\circ_v - \theta\eta$, $s_{v,a} = c_{v,a}s_1$, $s^0_v = c_{v,0}\eta$. Then $\mathcal{E}m_v = \delta\eta - \theta\tilde\delta(x')\eta = 0$, $e_1 s_{v,a} = s_{v,a}$, $\mathcal{E}s^0_v = \tilde\delta(x')s^0_v$, and by $(**)$

$$\begin{aligned}&-(u(g)-1)m_v + \delta\sum_a c_{v,a}\alpha_{v,a}(g)s_1 + \delta c_{v,0}\alpha^0_v(g)\eta\\ &\quad= -(u(g)-1)(\delta m^\circ_v - \theta\eta) + \delta(u(g)-1)m^\circ_v = \theta(u(g)-1)\eta = \psi(y(g)),\end{aligned}$$

which is $(*)$. By hypothesis 2 there is $g \in G_E$ with $(u(g)-1)\eta \neq 0$, so $\varphi(\iota^{\mathrm{col}}(y(g))) = \psi(y(g)) = \theta(u(g)-1)\eta \neq 0$ and $\varphi \neq 0$. (If $C = \emptyset$ we are in this case.)

The two cases are exhaustive, which proves the claim. □

Theorem 22.8. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with local defect data, local bound records, budget data having (Lz1), and inert budget data having (Iq1) and (Iq2). Then the budget data have (Lz3). More precisely, if $C \geq 1$ is a constant for each of (Lz1), (Iq1) and (Iq2), then for all large members with finite Sha lengths*

$$\sum_{\ell \text{ of inert type}} \log\ell \cdot \mathrm{la}_{x,i}(\ell) \leq 120\, C^3\, f\, m\, b\, x^{2\varepsilon}.$$

Proof. **Thresholds.** Let $C \geq 1$ be a constant for (Lz1), (Iq1), (Iq2) and x_0' a common threshold. As $\varepsilon > 0$ there is $x_0 \geq \max(x_0', 1)$ such that every real $x \geq x_0$ has, with $L = x^\varepsilon$, $L \geq \log(3(C+1)) + 1$. Fix such an x with $x > x_1$ and a member $i \in I(x)$ with finite Sha lengths; (Lz1), (Iq1), (Iq2) apply. Then $L \geq 1$, $\log 3, \log C, \log(C+1) \leq L$, and we put $Y = f\,m\,b\,L^2$, $Q = n\,t^2(\varrho + t)$, $\lambda_\ell = 1 + \log_\ell(H+t)$.

Consequences of (Lz1) and (Iq1). As $1 \leq m \leq t \leq Q\,L^2$, the first display of (Lz1) gives $\ell^* \leq C\,m\,b/(Q\,L^2) \leq C\,b$, so $\log\ell^* \leq 2L$; as $H \leq C\,b$ and $t \leq b$, $\log(H+t) \leq \log(C+1)+L \leq 2L$. From (Iq1)(1): $\log\ell_\mathrm{A} + 2 \leq (3+\log\ell_\mathrm{A})^2 \leq L$, $\varrho/\ell_\mathrm{A} \leq 2m/L$, $\varrho\, b^\circ \leq \varrho\, b^\circ \ell_\mathrm{A}/2 \leq m\,b/2$ and $w^\circ + \eta^\circ \leq 11 b^\circ$. Two elementary sums are used:

$$\sum_{p \leq y} \frac{\log p}{p} \leq 1 + \frac{(\log y)^2}{2} \quad (y \geq 2), \qquad \sum_{k \geq N} \frac{\log k}{k^2} \leq \frac{2(\log N + 2)}{N} \quad (N \geq 2 \text{ an integer}),$$

and $u \mapsto (\log u + 2)/u$ is decreasing on $[1,\infty)$ (its derivative is $-(1+\log u)/u^2$). In particular $\sum_{p \leq \ell^*} \log p/p \leq 1 + 2L^2 \leq 3L^2$.

The split. By (Iq2), for every prime ℓ of inert type

$$\log\ell \cdot \mathrm{la}_{x,i}(\ell) \leq C\, f \log\ell\Big(Q\lambda_\ell + \frac{m\,H}{\ell}\Big) + C\, f\, \varrho \log\ell \cdot A(\ell),$$

and both summands are non-negative. We bound the sums of the two summands over the primes of inert type, first those with $\ell \leq \ell^*$, then the exceptional ones with $\ell > \ell^*$.

First summand, $\ell \le \ell^*$. Since $\log\ell \cdot \log_\ell(H+t) = \log(H+t)$, the first summand is $C\,f(Q(\log\ell + \log(H+t)) + m\,H\log\ell/\ell)$. Summed over all primes $p \le \ell^*$ (at most ℓ^* of them): $C\,f\,Q\,\ell^* \cdot 4L \le C\,f\,Q \cdot \frac{C\,m\,b}{Q\,L^2} \cdot 4L \le 4C^2Y$, and $C\,f\,m \cdot C\,b \cdot 3L^2 = 3C^2Y$. Total at most $7C^2Y$.

First summand, $\ell > \ell^*$. Such ℓ lies in $\mathcal{P}^{\mathrm{ex}}_{x,i}$, so $\ell \le 3b$ by (Lz1) and $\log\ell \le \log 3 + L \le 2L$; as $\ell > \ell^* \ge 3$, $\log\ell > 1$ and $\log_\ell(H+t) \le 2L$; $H/\ell \le C\,b/\ell^*$. With $m \le Q$ and $C, L \ge 1$: $\log\ell \cdot C\,f(Q(1+2L) + C\,m\,b/\ell^*) \le 2L \cdot C\,f \cdot 3C\,L\,Q(1+b/\ell^*)$. There are at most $|\mathcal{P}^{\mathrm{ex}}_{x,i}|$ such primes, and the second display of (Lz1) gives a total of at most $6C^2L^2f \cdot C\,m\,b = 6C^3Y$.

Second summand: the long-orbit primes. Let $\ell \in \mathcal{P}^{\mathrm{lo}}_{x,i}$ be of inert type; by (Iq1)(2), $5 \le \ell < \ell_{\mathrm{A}} \le \ell^*$. Here $\varrho\log\ell \cdot A(\ell)$ is the sum of $\varrho H\log\ell/(\ell_{\mathrm{A}}(\ell-1))$, $\varrho(w^\circ + S^\circ + \eta^\circ + 1)(3 + \log\ell_{\mathrm{A}})\log\ell$ and $\varrho L\log\ell$. For $\ell \ge 5$, $\frac{1}{\ell-1} \le \frac{5}{4\ell}$, so $\sum_{5\le p<\ell_{\mathrm{A}}} \frac{\log p}{p-1} \le \frac{5}{4}(1 + \frac{(\log\ell_{\mathrm{A}})^2}{2}) \le (3+\log\ell_{\mathrm{A}})^2 \le L$, and $\sum_{p<\ell_{\mathrm{A}}} \log p \le \ell_{\mathrm{A}}\log\ell_{\mathrm{A}}$. Hence the three parts sum over these primes to at most

$$\varrho\frac{H}{\ell_{\mathrm{A}}}L \le 2b \cdot 2m = 4m\,b, \qquad 11\,\varrho\,b^\circ\ell_{\mathrm{A}}(3+\log\ell_{\mathrm{A}})^2 \le 11\,m\,b\,L,$$
$$\varrho\,L\,\ell_{\mathrm{A}}\log\ell_{\mathrm{A}} \le \varrho\,b^\circ\ell_{\mathrm{A}}\log\ell_{\mathrm{A}} \le m\,b\,L,$$

using $H \le 2b$, $\varrho L \le 2m\ell_{\mathrm{A}}$, $w^\circ + S^\circ + \eta^\circ + 1 \le 11b^\circ$, $\varrho b^\circ\ell_{\mathrm{A}} \le mb$, $L \le b^\circ$ and $\log\ell_{\mathrm{A}} \le L$. Total at most $16\,m\,b\,L^2$.

Second summand: the other primes up to ℓ^*. Let $\ell \le \ell^*$ be of inert type, $\ell \notin \mathcal{P}^{\mathrm{lo}}_{x,i}$; then $\varrho\log\ell\cdot A(\ell)$ is the sum of the four non-negative terms $\varrho H\log\ell/\ell^2$, $\varrho(w^\circ+\eta^\circ)\log\ell/\ell$, $\varrho(S^\circ+L)\log\ell$ and $\varrho\eta^\circ\beta(\ell)\log\ell$. (i) The H-terms with $\ell \in \mathcal{P}^{\mathrm{so}}_{x,i}$: by (Iq1)(3), at most $H \cdot 4mL^2 \le 8\,m\,b\,L^2$. (ii) The H-terms with $\ell \notin \mathcal{P}^{\mathrm{so}}_{x,i}$: by (Iq1)(2) these ℓ satisfy $\ell \ge \ell_{\mathrm{A}}$, so they are at least $N = \lceil\ell_{\mathrm{A}}\rceil \ge 2$, and the second elementary sum with the monotonicity of $(\log u + 2)/u$ gives at most $\varrho H \cdot \frac{2(\log\ell_{\mathrm{A}}+2)}{\ell_{\mathrm{A}}} \le 2b \cdot 2L \cdot \frac{2m}{L} = 8\,m\,b$. (iii) The thin terms: at most $\varrho \cdot 11b^\circ \cdot 3L^2 \le \frac{33}{2}m\,b\,L^2$. (iv) The spread terms: at most $\varrho(S^\circ + L)\ell^*\log\ell^* \le 2mb \cdot 2L = 4\,m\,b\,L$. (v) The band terms: if $\beta(\ell) = 1$ and $\ell \ge c_{\mathrm{bd}}$, (Iq1)(4) gives $\varrho\eta^\circ\log\ell \le 16\,m\,b\log\ell/\ell$, which sums to at most $48\,m\,b\,L^2$; if $\ell < c_{\mathrm{bd}}$, then $\varrho\eta^\circ \le 4mb$ and $\sum_{p<c_{\mathrm{bd}}}\log p \le c_{\mathrm{bd}}\log\max(c_{\mathrm{bd}},1) \le C\log C \le C\,L$ (no prime lies below c_{bd} if $c_{\mathrm{bd}} \le 2$), so these sum to at most $4C\,m\,b\,L$. Total at most $(85+4C)\,m\,b\,L^2$.

Second summand: the exceptional primes above ℓ^*. Such ℓ is not long-orbit ($\mathcal{P}^{\mathrm{lo}}$-primes of inert type lie below $\ell_{\mathrm{A}} \le \ell^*$), so with $\ell > \ell^* \ge 3$ and $\beta \le 1$, $A(\ell) \le 1 + H/\ell^* + w^\circ + \eta^\circ + S^\circ + L$, and $\log\ell \le 2L$. By (Iq1)(5) these terms sum to at most $2L \cdot C\,m\,b$.

Total. The second summands sum to at most $C\,f\,(16 + 85 + 4C + 2C)\,m\,b\,L^2 \le 107\,C^2\,Y$. Adding the first summands, $\sum\log\ell \cdot \mathrm{la}_{x,i}(\ell) \le (7C^2 + 6C^3 + 107C^2)\,Y \le 120\,C^3\,f\,m\,b\,x^{2\varepsilon}$ for every real $x \ge x_0$ with $x > x_1$ and every member with finite Sha lengths. This is (Lz3). □

Theorem 22.9. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with local defect data and local bound records, and suppose that $\mathcal{F}$ has budget data with the properties (Lz1)–(Lz5). Then the local bound records have the local budget (Lq7). More precisely, if $C \ge 1$ is a constant for each of (Lz1)–(Lz4), then for all large members with finite Sha lengths*

$$\sum_{\ell \in P_{x,i}\setminus G_{x,i}} \log\ell \cdot \mathrm{la}_{x,i}(\ell) \le 19\,C^3\,f\,m\,b\,x^{2\varepsilon}.$$

Proof. **Thresholds.** Let $C \ge 1$ be a constant for each of (Lz1)–(Lz4) and x_0' a common threshold for (Lz1)–(Lz5). As $\varepsilon > 0$, there is $x_0 \ge \max(x_0', 1)$ such that every real $x \ge x_0$ satisfies, with $L = x^\varepsilon$: $L \ge \log(3(C+1)) + 1$, $\log x \le L$ and $x^{C+\varepsilon} \le e^L$ (indeed $L \to \infty$, and $(C+\varepsilon)\log x/x^\varepsilon \to 0$). Fix $x \ge x_0$ with $x > x_1$ and a member $i \in I(x)$ with finite Sha lengths; then (Lz1)–(Lz5) apply. We have $L \ge 1$, $b = e^L$, $Y = f\,m\,b\,L^2$, and $\log 3,\ \log C,\ \log(C+1) \le L$.

Two consequences of (Lz1). Since $1 \le m \le t \le n\,t^2(\varrho+t)L^2$, the first display of (Lz1) gives $\ell^* \le C\,m\,b/(n\,t^2(\varrho+t)L^2) \le C\,b$, hence $\log \ell^* \le \log C + L \le 2L$. Since $H \le C\,b$ and $t \le b$, $\log(H+t) \le \log(C+1) + L \le 2L$.

The split. The types partition $P_{x,i} \setminus G_{x,i}$, so the sum is the sum of the four partial sums $\Sigma_{\mathrm{sm}}, \Sigma_{\mathrm{in}}, \Sigma_{\mathrm{sp}}, \Sigma_{\mathrm{lg}}$ over the primes of small, inert, split type and the large primes. By (Lz5), $\Sigma_{\mathrm{lg}} \le \sum_{k\ge 2} \log k/k^2 = -\zeta'(2) < 1 \le Y$ (the series is the classical value $0.9375\ldots$; $Y \ge 1$ as $f, m, b, L \ge 1$); by (Lz3), $\Sigma_{\mathrm{in}} \le C\,Y$.

Small type. For $\ell \in \mathcal{P}^{\mathrm{sm}}_{x,i} \setminus G_{x,i}$, (Lz2) bounds $\log \ell \cdot \mathrm{la}_{x,i}(\ell)$ by the non-negative number $\log \ell \cdot C(f\,m\,b \log x + x^C L^2)$; summing over all of $\mathcal{P}^{\mathrm{sm}}_{x,i}$ and using $\sum_{\ell \in \mathcal{P}^{\mathrm{sm}}_{x,i}} \log \ell \le C\,L$, $\log x \le L$ and $x^C L^3 = x^{C+\varepsilon} L^2 \le b\,L^2 \le f\,m\,b\,L^2$,

$$\sum_{\ell \in \mathcal{P}^{\mathrm{sm}}_{x,i} \setminus G_{x,i}} \log \ell \cdot \mathrm{la}_{x,i}(\ell) \le C^2\big(f\,m\,b\,L^2 + x^C L^3\big) \le 2C^2\,Y.$$

The remaining primes of small type lie in $\mathcal{P}^{\mathrm{rm}}_{x,i}$, which has at most two elements, each with $\log \ell \le L$ and $\mathrm{la}_{x,i}(\ell) \le C\,f\,m\,b$; their contribution is at most $2C\,f\,m\,b\,L \le 2C\,Y$. So $\Sigma_{\mathrm{sm}} \le 4C^2 Y$.

Split type up to the cut-off. For ℓ of split type with $\ell \le \ell^*$, (Lz4) and $\log \ell \cdot \log_\ell(H+t) = \log(H+t)$ give

$$\log \ell \cdot \mathrm{la}_{x,i}(\ell) \le C\,f\Big(n\,t^2(\varrho+t)\big(\log \ell + \log(H+t)\big) + m\,H\,\frac{\log \ell}{\ell}\Big),$$

a non-negative bound, which we sum over all primes $p \le \ell^*$. There are at most ℓ^* of them, so the first part sums to at most $C\,f\,n\,t^2(\varrho+t)\,\ell^*(\log \ell^* + \log(H+t)) \le C\,f\,n\,t^2(\varrho+t) \cdot \frac{C\,m\,b}{n\,t^2(\varrho+t)L^2} \cdot 4L \le 4C^2\,Y$. For the second part,

$$\sum_{p \le \ell^*} \frac{\log p}{p} \le \sum_{k=2}^{\lfloor \ell^* \rfloor} \frac{\log k}{k} \le 1 + \frac{(\log \ell^*)^2}{2} \le 1 + 2L^2 \le 3L^2,$$

so the second part is at most $C\,f\,m \cdot C\,b \cdot 3L^2 = 3C^2\,Y$. These primes contribute at most $7C^2 Y$.

Exceptional primes of split type. Let ℓ be of split type with $\ell > \ell^*$; then $\ell \in \mathcal{P}^{\mathrm{ex}}_{x,i}$, so $\ell \le 3b$ and $\log \ell \le \log 3 + L \le 2L$; as $\ell > \ell^* \ge 3$, $\log \ell > 1$ and $\log_\ell(H+t) \le \log(H+t) \le 2L$; and $H/\ell \le C\,b/\ell^*$. With $Q = n\,t^2(\varrho+t) \ge m$, (Lz4) gives

$$\log \ell \cdot \mathrm{la}_{x,i}(\ell) \le 2L \cdot C\,f\Big(Q(1+2L) + \frac{C\,m\,b}{\ell^*}\Big) \le 2L \cdot C\,f \cdot 3C\,L\,Q\Big(1 + \frac{b}{\ell^*}\Big) = 6C^2L^2 f\,Q\Big(1 + \frac{b}{\ell^*}\Big).$$

There are at most $|\mathcal{P}^{\mathrm{ex}}_{x,i}|$ such primes, and by the second display of (Lz1) their total is at most $6C^2L^2 f \cdot C\,m\,b = 6C^3\,Y$.

Total. $\sum \log \ell \cdot \mathrm{la}_{x,i}(\ell) \le (1 + 4C^2 + C + 7C^2 + 6C^3)\,Y \le 19C^3\,Y$, for every real $x \ge x_0$ with $x > x_1$ and every member $i \in I(x)$ with finite Sha lengths. This is (Lq7) with the constant $19C^3$ and the threshold x_0. □

Theorem 22.10. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with local defect data, local bound records and Ribet records, and let Galois Ribet records for $\mathcal{F}$ have (Gd8). Then:*

1. *the Ribet records have the block bounds (Rf4);*
2. *if the Galois Ribet records also have (Gd9), the Ribet records have the determinant property (Rf1);*

3. for all (x,i,j,w,r,π) as in (Gd8), with $t=t_{x,i}$, $\ell=\ell_{x,i}(w)$ and $n_{\mathcal{R},0}$ the size of block 0,

$$\sum_{k=0}^{t}\mathrm{ls}_k(r,\pi)=\sum_{k=0}^{t}\kappa_k+\sum_{v\in S_\ell}\sum_{k=0}^{t}\lambda_k(v)+t\sum_{v\in\Sigma_{\mathrm{T}}}\nu(\zeta_v-1)+2\,(n_{\mathcal{R},0}+t-1)\,\nu(\delta);$$

consequently, if the local contributions are the loss sums, $\mathrm{lc}_k(r,\pi)=\mathrm{ls}_k(r,\pi)$, and the right side is at most $\mathrm{lt}(r)$ for all large members with finite Sha lengths and all such w,r,π, then the Ribet records have (Rf4) and (Rf5).

Proof. Let $x^{(8)}$ be a threshold for (Gd8) and, in item 2, $x^{(9)}$ one for (Gd9); put $x_0=x^{(8)}$, respectively $x_0=\max(x^{(8)},x^{(9)})$. Fix a real $x\geq x_0$ with $x>x_1$, a member $i\in I(x)$ with finite Sha lengths, $1\leq j\leq m$, $w\in W_{x,i}$ with $\ell=\ell_{x,i}(w)\notin G_{x,i}$, $r\in R_{x,i}(j,w)$ and $\pi\in Z_r$; write $E=E_{\mathfrak{h}_{x,i}}$, $t=t_{x,i}$, $\mathcal{O}=\mathcal{O}_r$, $\mathcal{R}=\mathcal{R}_{r,\pi}$, $\mathcal{N}=\mathcal{N}_{r,\pi}$.

Item 1. By (Gd8) the Galois Ribet data on $(\mathcal{R},\pi',\mathcal{N})$ have (Gd1)–(Gd7) and E has no real place; $\mathcal{N}$ has a presentation, so it is finitely generated. Theorem 22.11 gives, for $k=0$, $\mathrm{nl}\,\mathrm{Hom}_{\mathcal{R}}(\mathcal{N},\mathrm{Cs}_0)\leq\mathrm{ls}_0(r,\pi)+\mathrm{nl}\,\mathrm{Sel}_0(r,\pi)\leq\mathrm{lc}_0(r,\pi)+\mathrm{sl}(r,\pi)$, using $\mathrm{ls}_0\leq\mathrm{lc}_0$ and $\mathrm{nl}\,\mathrm{Sel}_0\leq\mathrm{sl}$ from (Gd8); for $k=1$, $\mathrm{nl}\,\mathrm{Hom}_{\mathcal{R}}(\mathcal{N},\mathrm{Cs}_1)\leq\mathrm{nl}\,\mathrm{Hom}_{\mathrm{ur}}(G_E,\mathcal{O}[\ell^{-1}]/\mathcal{O})+\mathrm{lc}_1(r,\pi)$; for $2\leq k\leq t$, $\mathrm{nl}\,\mathrm{Hom}_{\mathcal{R}}(\mathcal{N},\mathrm{Cs}_k)\leq\mathrm{len}_\ell(E,k-1)+\mathrm{lc}_k(r,\pi)$. These are the three inequalities of (Rf4). The right sides are finite: $\mathrm{sl}(r,\pi)$ and $\mathrm{lc}_k(r,\pi)$ are real numbers; $\mathrm{Hom}_{\mathrm{ur}}$ has finite length by Lemma 13.15, as E has no real place; $\mathrm{len}_\ell(E,k-1)<\infty$ for $1\leq k-1<t$ because the member has finite Sha lengths. So the modules $\mathrm{Hom}_{\mathcal{R}}(\mathcal{N},\mathrm{Cs}_k)$ have finite length, and (Rf4) holds with the threshold $x^{(8)}$.

Item 2. By (Gd9), the $\mathcal{O}$-module $\mathcal{D}_{r,\pi}$ contains 1 and is closed under products, and hypothesis (3) of Lemma 22.14 holds for the representation $\mathbf{r}$ of the data (in the role of the homomorphism of the lemma); hypothesis (2) of that lemma holds with the coefficient algebra $\mathbb{T}'$ of the data: by the definition of Galois Ribet data $\mathcal{R}$ is the $\mathbb{T}'$-span of $\mathbf{r}(G_E)$ and each $g\mapsto\mathbf{r}(g)(x')$ is continuous, and $\mathbb{T}'\subseteq\mathcal{D}_{r,\pi}$ by (Gd9). So the determinant algebra satisfies the determinant condition in every degree, in particular in degree n_{gen}, and by (Gd9) $\widehat{Z}_r\setminus Z_r$ is detected by the descent functions. This is (Rf1), with the threshold x_0.

Item 3. By the definition of the loss sums,

$$\sum_{k=0}^{t}\mathrm{ls}_k=\sum_{k=0}^{t}\kappa_k+\sum_{k=0}^{t}\sum_{v\in S_\ell}\lambda_k(v)+\sum_{k=1}^{t}\sum_{v\in\Sigma_{\mathrm{T}}}\nu(\zeta_v-1)+2n_{\mathcal{R},0}\nu(\delta)+\sum_{k=2}^{t}2\nu(\delta),$$

and the last three sums are $t\sum_{v\in\Sigma_{\mathrm{T}}}\nu(\zeta_v-1)$ and $2(n_{\mathcal{R},0}+t-1)\nu(\delta)$. If $\mathrm{lc}_k=\mathrm{ls}_k$, then (Gd8) holds in its clause $\mathrm{lc}_k\geq\mathrm{ls}_k$, so item 1 gives (Rf4), and $\sum_k\mathrm{lc}_k(r,\pi)=\sum_k\mathrm{ls}_k(r,\pi)\leq\mathrm{lt}(r)$ is (Rf5). □

Theorem 22.11. *Let E be a number field, ℓ a prime, and let Galois Ribet data over E on $(\mathcal{R},\pi',\mathcal{N})$, with $t+1$ blocks, have the properties (Gd1)–(Gd7), where $\mathcal{N}$ is finitely generated. Then for every $0\leq k\leq t$ and $\varphi\in\mathrm{Hom}_{\mathcal{R}}(\mathcal{N},\mathrm{Cs}_k)$ the function c_φ is a continuous 1-cocycle of G_E with values in $\mathrm{Cs}_k(u)$, and, with* nl *the normalised length,*

$$\mathrm{nl}\,\mathrm{Hom}_{\mathcal{R}}(\mathcal{N},\mathrm{Cs}_0)\leq\mathrm{ls}_0+\mathrm{nl}\,\mathrm{Sel}_0,$$
$$\mathrm{nl}\,\mathrm{Hom}_{\mathcal{R}}(\mathcal{N},\mathrm{Cs}_1)\leq\mathrm{ls}_1+\mathrm{nl}\,\mathrm{Hom}_{\mathrm{ur}}(G_E,\mathcal{O}[\ell^{-1}]/\mathcal{O}),$$

and, if E has no real place,

$$\mathrm{nl}\,\mathrm{Hom}_{\mathcal{R}}(\mathcal{N},\mathrm{Cs}_k)\leq\mathrm{ls}_k+\mathrm{len}_\ell(E,k-1)\qquad(2\leq k\leq t),$$

all inequalities in $[0,\infty]$.

Proof. Write $n_k = n_{\mathcal{R},k}$. Each Cs_k is a left $\mathcal{R}$-module through ρ_k, the scalars $\mathcal{O}$ are central in $\mathcal{R}$, and $\mathrm{Cs}_k(u)$ is Cs_k with g acting by $u(g)$ in the sense of Lemma 22.12.

Step 0: the Ribet cocycles are continuous cocycles. For each $x' \in \mathcal{Z}'$ the set $\mathbf{r}(G_E)(x')$ is compact in $\mathrm{GL}_{n_\mathcal{R}}(K)$, hence contained in $\varpi^{-a}\mathrm{M}_{n_\mathcal{R}}(\mathcal{O})$ for some a; as $\mathcal{Z}'$ is finite and the elements of $\mathbb{T}'$ are $\mathcal{O}$-valued, the component at each x' of an element $\sum_h f_h\mathbf{r}(h)$ ($f_h \in \mathbb{T}'$) of $\mathcal{R}$, the $\mathbb{T}'$-span of $\mathbf{r}(G_E)$, lies in the $\mathcal{O}$-span of $\mathbf{r}(G_E)(x')$; so $\mathcal{R}$ is a submodule of a finitely generated $\mathcal{O}$-module, hence finitely generated and free, and its ϖ-adic topology is the topology induced from $\prod_{x'} \mathrm{M}_{n_\mathcal{R}}(K)$ (a lattice in a finite-dimensional K-space). So $g \mapsto \mathbf{r}(g)$, and with χ_1 also $g \mapsto u(g)$ and $g \mapsto y(g)$, are continuous into $\mathcal{R}$. The left ideal $\mathcal{N}_\mathcal{E}$ is a finitely generated $\mathcal{O}$-module and, by the Artin–Rees lemma, its ϖ-adic topology is induced from $\mathcal{R}$; so $y\colon G_E \to \mathcal{N}_\mathcal{E}$ is continuous for the ϖ-adic topology. Let $\varphi \in \mathrm{Hom}_\mathcal{R}(\mathcal{N}, \mathrm{Cs}_k)$ and $\psi = \varphi \circ \iota \in \mathrm{Hom}_\mathcal{R}(\mathcal{N}_\mathcal{E}, \mathrm{Cs}_k)$. Since $\mathcal{N}$ is finitely generated, $\varphi(\mathcal{N})$ is a finitely generated submodule of $(K/\mathcal{O})^{n_k}$, killed by some ϖ^b; so ψ kills $\varpi^b\mathcal{N}_\mathcal{E}$, and $c_\varphi = \psi \circ y$ is locally constant. By Lemma 22.12(1), applied with $G = G_E$, the ring $\mathcal{R}$, $\mathcal{B} = \mathbb{T}'$ (central in $\mathcal{R}$; $f \in \mathbb{T}'$ acts on Cs_k as the scalar $f(\pi')$ by Definition 17.2), $r = \mathbf{r}$, $\chi = \chi_1$, the element $\mathcal{E}$ and $V = \mathrm{Cs}_k$, the function $c_\varphi = c_\psi$ is a 1-cocycle for $\mathrm{Cs}_k(u)$, and $\psi \mapsto c_\psi$ is injective and $\mathcal{O}$-linear on $\mathrm{Hom}_\mathcal{R}(\mathcal{N}_\mathcal{E}, \mathrm{Cs}_k)$. In particular X_k is the image of the $\mathcal{O}$-linear map $\Theta_k\colon \psi \mapsto [c_\psi]$ on the image J_k of $\iota^*\colon \varphi \mapsto \varphi \circ \iota$.

Step 1: the column kernel. The exact sequence $0 \to \ker \iota^* \to \mathrm{Hom}_\mathcal{R}(\mathcal{N}, \mathrm{Cs}_k) \to J_k \to 0$, the sequence $0 \to \ker\Theta_k \to J_k \to X_k \to 0$, additivity of length and (Gd6) give

$$\mathrm{nl}\,\mathrm{Hom}_\mathcal{R}(\mathcal{N}, \mathrm{Cs}_k) \le \kappa_k + \mathrm{nl}\ker\Theta_k + \mathrm{nl}\,X_k. \tag{22.2}$$

Step 2: coboundaries. For $k = 1$, (Gd1) gives $\rho_1(u(g)) = \chi_1(g)^{-1}\chi_1(g) = 1$: the action on $\mathrm{Cs}_1(u)$ is trivial, there are no non-zero coboundaries, and $\ker\Theta_1 = 0$ by the injectivity of Step 0. Let $k \ne 1$. By (Gd2), $\mathcal{E}^2 = \tilde\delta\mathcal{E}$ with $\tilde\delta \in \mathbb{T}'$, and $\mathcal{E}$ acts on Cs_k by $\rho_k(\mathcal{E}) = 0$. Put $\tau = \sum_h t_h\chi_1(h) \in \mathbb{T}'$. As $f \in \mathbb{T}'$ acts on Cs_k as the scalar $f(\pi')$, $\tilde\delta$ acts as δ and τ as $\tau(\pi') = \sum_h t_h(\pi')\chi_1(h) = \rho_1(\sum_h t_h\mathbf{r}(h)) = \rho_1(\mathcal{E}) = \delta$, by (Gd1), (Gd2) and because ρ_1 is a homomorphism with $\rho_1(t_h) = t_h(\pi')$. If $\psi \in \ker\Theta_k$, then $c_\psi(g) = u(g)v - v$ for some $v \in \mathrm{Cs}_k$, and Lemma 22.12(2), with $\mathcal{B} = \mathbb{T}'$ and $\tilde\delta$ in the role of its δ, gives $\tilde\delta\tau v = \delta^2 v = 0$. The element v is unique modulo the invariants $\mathrm{Cs}_k(u)^{G_E}$, so $\psi \mapsto v$ is a well-defined $\mathcal{O}$-linear map $\ker\Theta_k \to \mathrm{Cs}_k[\delta^2]/(\mathrm{Cs}_k[\delta^2] \cap \mathrm{Cs}_k(u)^{G_E})$; it is injective, because v determines c_ψ, which determines ψ. As $(K/\mathcal{O})[\delta^2] = \delta^{-2}\mathcal{O}/\mathcal{O} \cong \mathcal{O}/\delta^2\mathcal{O}$,

$$\mathrm{nl}\ker\Theta_k \le \mathrm{nl}\,\mathrm{Cs}_k[\delta^2] = n_k\,\nu(\delta^2) = 2\,n_k\,\nu(\delta) \qquad (k \ne 1),$$

with $n_k = 1$ for $k \ge 2$ by (Gd1).

Step 3: the Ribet classes away from ℓ. Let $x = [c] \in X_k$ with $c = c_\psi$. Then $u(\gamma) - 1 = \chi_1(\gamma)^{-1}(\mathbf{r}(\gamma) - \chi_1(\gamma))$.

(a) Let $w \notin S_\ell \cup \Sigma_\mathrm{T} \cup \Sigma_\mathrm{P}$ be finite and $\gamma \in I_w$. By (Gd3) $\mathbf{r}(\gamma) = 1$, so $\chi_1(\gamma) = \rho_1(\mathbf{r}(\gamma)) = 1$ by (Gd1), $u(\gamma) = 1$, $y(\gamma) = 0$ and $c(\gamma) = 0$. Hence $x|_{I_w} = 0$, for every k.

(b) Let $k \ge 1$ and v finite, $v \notin S_\ell$. For $\gamma \in I_v$, $\rho_k(u(\gamma)) = \chi_k(\gamma)\chi_1(\gamma)^{-1} = \chi_\mathrm{cyc}(\gamma)^{1-k} = 1$, since χ_cyc is unramified at v. So I_v acts trivially on $\mathrm{Cs}_k(u)$, hence $H^1(I_v, \mathrm{Cs}_k(u)) = \mathrm{Hom}_\mathrm{cont}(I_v, \mathrm{Cs}_k)$, and $x|_{I_v} = c|_{I_v}$ is a continuous homomorphism; also $\chi_k(\gamma) = \chi_1(\gamma)$.

(c) Let $k \ge 1$, $v \in \Sigma_\mathrm{P}$, $\gamma \in I_v$. By (Gd5), $p_v y(\gamma) = \chi_1(\gamma)^{-1}p_v(\mathbf{r}(\gamma) - \chi_1(\gamma))\mathcal{E} = 0$, and $p_v y(\gamma) \in \mathcal{N}_\mathcal{E}$, so $0 = \psi(p_v y(\gamma)) = \rho_k(p_v)\,c(\gamma)$; as $\rho_k(p_v)$ is a unit, $c(\gamma) = 0$. Hence $x|_{I_v} = 0$.

(d) Let $k \ge 1$, $v \in \Sigma_\mathrm{T}$, $\gamma \in I_v$. By (Gd4), $(\mathbf{r}(\gamma)-1)y(\gamma) = \chi_1(\gamma)^{-1}(\mathbf{r}(\gamma)-1)(\mathbf{r}(\gamma)-\chi_1(\gamma))\mathcal{E} = 0$, so $0 = \psi((\mathbf{r}(\gamma) - 1)y(\gamma)) = (\chi_k(\gamma) - 1)c(\gamma) = (\chi_1(\gamma) - 1)c(\gamma)$ by (b); in particular

$(\zeta_v-1)c(\gamma_v)=0$. By (b), $c|_{I_v}$ is a continuous homomorphism into the discrete ℓ-primary group Cs_k; its image is compact, hence a finite ℓ-group, so $c|_{I_v}$ factors through the maximal pro-ℓ quotient of I_v, which is topologically generated by the image of γ_v (Gd4); a continuous homomorphism from it into a discrete group is determined by its value at that generator. Hence $x|_{I_v}\mapsto c(\gamma_v)$ is an injective $\mathcal{O}$-linear map from the image of X_k in $H^1(I_v,\mathrm{Cs}_k(u))$ into $\mathrm{Cs}_k[\zeta_v-1]\cong\mathcal{O}/(\zeta_v-1)\mathcal{O}$ ($n_k=1$, and $\zeta_v\neq 1$ by (Gd4)). So that image has normalised length at most $\nu(\zeta_v-1)$.

Step 4: the local conditions. Let $\Pi_0=S_\ell$ and $\Pi_k=S_\ell\cup\Sigma_{\mathrm{T}}$ for $k\geq 1$; for $v\in S_\ell$ let $Q_v=H^1(G_{E_v},\mathrm{Cs}_k(u))/F_k(v)$ and for $v\in\Sigma_{\mathrm{T}}$ let $Q_v=H^1(I_v,\mathrm{Cs}_k(u))$, and let $\mathrm{res}\colon X_k\to\bigoplus_{v\in\Pi_k}Q_v$ be the map of restrictions, with components res_v. Then $X_k/\ker\mathrm{res}$ embeds into $\bigoplus_v\mathrm{res}_v(X_k)$, so by (Gd7) and Step 3(d)

$$\mathrm{nl}\,X_k\leq\mathrm{nl}\ker\mathrm{res}+\sum_{v\in S_\ell}\lambda_k(v)+[k\geq 1]\sum_{v\in\Sigma_{\mathrm{T}}}\nu(\zeta_v-1). \tag{22.3}$$

We identify $\ker\mathrm{res}$.

- $k=0$: an element of $\ker\mathrm{res}$ vanishes on I_w for every finite $w\notin S_\ell\cup\Sigma_{\mathrm{T}}\cup\Sigma_{\mathrm{P}}$ by Step 3(a) and lies in $F_0(v)$ at $v\in S_\ell$; so $\ker\mathrm{res}\subseteq\mathrm{Sel}_0$.
- $k=1$: $\mathrm{Cs}_1(u)=K/\mathcal{O}$ with trivial action, so $H^1(G_E,K/\mathcal{O})=\mathrm{Hom}_{\mathrm{cont}}(G_E,K/\mathcal{O})$. An element of $\ker\mathrm{res}$ vanishes on I_w for $w\notin S_\ell\cup\Sigma_{\mathrm{T}}\cup\Sigma_{\mathrm{P}}$ (Step 3(a)), for $w\in\Sigma_{\mathrm{P}}$ (Step 3(c)), for $w\in\Sigma_{\mathrm{T}}$ (definition of res), and for $w\in S_\ell$, because its restriction to G_{E_w} lies in $F_1(w)=H^1_{\mathrm{ur}}(E_w,K/\mathcal{O})$. So $\ker\mathrm{res}\subseteq\mathrm{Hom}_{\mathrm{ur}}(G_E,K/\mathcal{O})$.
- $k\geq 2$: by (Gd1), g acts on $\mathrm{Cs}_k(u)=K/\mathcal{O}=\mathcal{O}^\vee$ by $\chi_{\mathrm{cyc}}(g)^{1-k}$, so $\mathrm{Cs}_k(u)=\mathcal{O}^\vee(\chi_{\mathrm{cyc}}^{-(k-1)})$. An element of $\ker\mathrm{res}$ vanishes on I_w for every finite $w\notin S_\ell$ (Step 3(a), 3(c), and the definition of res at Σ_{T}) and restricts to 0 on G_{E_v} for $v\in S_\ell$ ($F_k(v)=0$). So $\ker\mathrm{res}\subseteq\mathrm{Sh}_{\mathcal{O}}(E,k-1)$ in the notation of Lemma 22.13, whose normalised length is $\mathrm{len}_\ell(E,k-1)$ when E has no real place.

Step 5: conclusion. Combining (22.2), Step 2, (22.3) and Step 4, and comparing with the definition of the loss sums ls_k: for $k=0$ the right side is $\kappa_0+2n_0\nu(\delta)+\sum_{v\in S_\ell}\lambda_0(v)+\mathrm{nl}\,\mathrm{Sel}_0=\mathrm{ls}_0+\mathrm{nl}\,\mathrm{Sel}_0$; for $k=1$ it is $\kappa_1+\sum_{v\in S_\ell}\lambda_1(v)+\sum_{v\in\Sigma_{\mathrm{T}}}\nu(\zeta_v-1)+\mathrm{nl}\,\mathrm{Hom}_{\mathrm{ur}}=\mathrm{ls}_1+\mathrm{nl}\,\mathrm{Hom}_{\mathrm{ur}}$; for $k\geq 2$ it is $\kappa_k+2\nu(\delta)+\sum_{v\in S_\ell}\lambda_k(v)+\sum_{v\in\Sigma_{\mathrm{T}}}\nu(\zeta_v-1)+\mathrm{len}_\ell(E,k-1)=\mathrm{ls}_k+\mathrm{len}_\ell(E,k-1)$. A submodule has at most the length of the module, so the lengths of $\ker\mathrm{res}$ are bounded by those of Sel_0, $\mathrm{Hom}_{\mathrm{ur}}$ and $\mathrm{Sh}_{\mathcal{O}}(E,k-1)$; all steps hold in $[0,\infty]$. □

Lemma 22.12. *Let G be a group, $\mathcal{R}$ a ring, $\mathcal{B}$ a commutative ring with a ring homomorphism into the centre of $\mathcal{R}$ (through which $\mathcal{B}$ acts on every $\mathcal{R}$-module), $r\colon G\to\mathcal{R}^\times$ and $\chi\colon G\to\mathcal{B}^\times$ homomorphisms, and $u(g)=\chi(g)^{-1}r(g)$. Let $\mathcal{E}=\sum_{h\in S}t_h\,r(h)$ for a finite set $S\subseteq G$ and $t_h\in\mathcal{B}$, put $\tau=\sum_{h\in S}t_h\chi(h)\in\mathcal{B}$, and let*

$$y(g)=(u(g)-1)\mathcal{E}\quad(g\in G),\qquad\mathcal{N}_{\mathcal{E}}=\sum_{g\in G}\mathcal{R}\,y(g)\subseteq\mathcal{R}$$

(the Ribet cocycle *and the left ideal it generates). For a left $\mathcal{R}$-module V let $V(u)$ be V with $g\in G$ acting by $u(g)$.*

1. *The map $\varphi\mapsto c_\varphi$, $c_\varphi(g)=\varphi(y(g))$, is an injective $\mathcal{B}$-linear map*

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{N}_{\mathcal{E}},V)\to Z^1(G,V(u))$$

into the inhomogeneous 1-cocycles.

2. *Suppose $\mathcal{E}^2 = \delta\mathcal{E}$ for some $\delta \in \mathcal{B}$ and $\mathcal{E}$ acts as 0 on V. If c_φ is a coboundary, $c_\varphi(g) = (u(g)-1)v$ for some $v \in V$ and all g, then $\delta\tau\, v = 0$.*

Proof. **(1).** Since u is a homomorphism, $y(gh) = (u(g)u(h)-1)\mathcal{E} = (u(g)-1)\mathcal{E} + u(g)(u(h)-1)\mathcal{E} = y(g) + u(g)\,y(h)$. Applying the $\mathcal{R}$-linear map φ gives $c_\varphi(gh) = c_\varphi(g) + u(g)c_\varphi(h)$, the cocycle identity for $V(u)$. The map is $\mathcal{B}$-linear because $\mathcal{B}$ acts through the centre. It is injective because the $y(g)$ generate $\mathcal{N}_\mathcal{E}$ as a left $\mathcal{R}$-module, so φ is determined by its values on them.

(2). As $\sum_h t_h\chi(h)u(h) = \sum_h t_h r(h) = \mathcal{E}$,

$$(\mathcal{E}-\tau)\mathcal{E} = \sum_{h\in S} t_h\chi(h)(u(h)-1)\mathcal{E} = \sum_{h\in S} t_h\chi(h)\,y(h) \in \mathcal{N}_\mathcal{E},$$

and by $\mathcal{R}$-linearity and the hypothesis on c_φ,

$$\varphi\big((\mathcal{E}-\tau)\mathcal{E}\big) = \sum_h t_h\chi(h)(u(h)-1)v = (\mathcal{E}-\tau)v = -\tau v,$$

because $\mathcal{E}v = 0$. Since τ and δ are central and $\mathcal{E}^2 = \delta\mathcal{E}$, both $\mathcal{E}\cdot(\mathcal{E}-\tau)\mathcal{E}$ and $\delta\cdot(\mathcal{E}-\tau)\mathcal{E}$ equal $\delta(\delta-\tau)\mathcal{E}$. Applying φ and using $\mathcal{R}$-linearity twice,

$$\mathcal{E}\cdot\varphi\big((\mathcal{E}-\tau)\mathcal{E}\big) = \delta\cdot\varphi\big((\mathcal{E}-\tau)\mathcal{E}\big), \qquad \text{i.e.} \qquad -\tau\,\mathcal{E}v = -\delta\tau\, v.$$

The left side is 0 because $\mathcal{E}$ acts as 0 on V; hence $\delta\tau v = 0$. □

Lemma 22.13. *Let E be a number field without real place, ℓ a prime, $\mathcal{O}$ a coefficient ring for ℓ and $k \in \mathbb{Z}$. Let $\mathrm{Sh}_\mathcal{O}(E,k) \subseteq H^1\big(G_E, \mathcal{O}^\vee(\chi_{\mathrm{cyc}}^{-k})\big)$ be the $\mathcal{O}$-submodule of the classes x with $x|_{I_w} = 0$ for every finite place w of E not above ℓ and $x|_{G_{E_v}} = 0$ for every place v above ℓ. Then*

$$\mathrm{length}_\mathcal{O}\,\mathrm{Sh}_\mathcal{O}(E,k) = e_\mathcal{O}\,\mathrm{len}_\ell(E,k) \qquad \text{in } \mathbb{N}\cup\{\infty\};$$

in particular, when $\mathrm{len}_\ell(E,k)$ is finite, the normalised length of $\mathrm{Sh}_\mathcal{O}(E,k)$ is $\mathrm{len}_\ell(E,k)$.

Proof. Write $M = \mathcal{O}^\vee(\chi_{\mathrm{cyc}}^{-k})$, a discrete ℓ-primary $\mathcal{O}[G_E]$-module, and S_ℓ for the set of places above ℓ.

Step 1: inflation from G_{E,S_ℓ}. Let $H \subseteq G_E$ be the closed normal subgroup generated by the inertia groups of all finite places $w \notin S_\ell$ (all conjugates of the I_w). Since E has no real place, the fixed field of H is the maximal extension of E unramified outside S_ℓ, so $G_{E,S_\ell} = G_E/H$. The character χ_{cyc} is unramified at every finite $w \notin S_\ell$, so H acts trivially on M and M is a G_{E,S_ℓ}-module. The inflation–restriction sequence for continuous cohomology with discrete coefficients,

$$0 \to H^1(G_{E,S_\ell}, M) \xrightarrow{\ \mathrm{inf}\ } H^1(G_E, M) \to H^1(H, M) = \mathrm{Hom}_{\mathrm{cont}}(H, M),$$

is exact. Let c be a continuous cocycle of G_E with class x. For a finite $w \notin S_\ell$, I_w acts trivially on M, so $H^1(I_w, M) = \mathrm{Hom}_{\mathrm{cont}}(I_w, M)$ and $x|_{I_w} = 0$ means $c|_{I_w} = 0$. If γ acts trivially on M, then for every g

$$c(g\gamma g^{-1}) = c(g) + g\,c(\gamma) + g\gamma\,c(g^{-1}) = g\,c(\gamma) + \big(c(g) + g\,c(g^{-1})\big) = g\,c(\gamma),$$

because $c(g) + g\,c(g^{-1}) = c(1) = 0$. Hence if $c|_{I_w} = 0$ for all finite $w \notin S_\ell$, then c vanishes on all conjugates of these groups; as $c|_H$ is a continuous homomorphism (the cocycle identity with trivial action), its kernel is a closed subgroup containing them, hence contains H, and x

restricts to 0 on H. Conversely an inflated class vanishes on every $I_w \subseteq H$. So inflation identifies $H^1(G_{E,S_\ell}, M)$ with the classes of $H^1(G_E, M)$ that vanish on I_w for every finite $w \notin S_\ell$. For $v \in S_\ell$ the restriction of an inflated class to G_{E_v} is the restriction of the original class along $G_{E_v} \to G_E \to G_{E,S_\ell}$, which is the map $H^1(G_{E,S_\ell}, M) \to H^1(E_v, M)$ of Definition 7.1 (another choice of the decomposition group is conjugate and has the same kernel). Therefore inflation is an isomorphism of $\mathcal{O}$-modules

$$\mathrm{Sha}_{\mathcal{O}} := \ker\Big(H^1(G_{E,S_\ell}, M) \to \prod_{v \in S_\ell} H^1(E_v, M)\Big) \xrightarrow{\sim} \mathrm{Sh}_{\mathcal{O}}(E, k).$$

Step 2: change of coefficients. Let $d = [\mathcal{O} : \mathbb{Z}_\ell]$ and $b_1, \dots, b_d$ a $\mathbb{Z}_\ell$-basis of $\mathcal{O}$. Since χ_{cyc}^{-k} takes values in $\mathbb{Z}_\ell^\times$, the map $(a_i) \mapsto \sum_i b_i \otimes a_i$ is an isomorphism of discrete $\mathbb{Z}_\ell[G_{E,S_\ell}]$-modules $(\mathbb{Q}_\ell/\mathbb{Z}_\ell(-k))^d \cong M$. Continuous cohomology commutes with finite direct sums, compatibly with the restrictions to the G_{E_v}, so $\mathrm{Sha}_{\mathcal{O}} \cong \mathrm{Sha}^1_\ell(E,k)^d$ as $\mathbb{Z}_\ell$-modules and $\mathrm{length}_{\mathbb{Z}_\ell} \mathrm{Sha}_{\mathcal{O}} = d\,\mathrm{len}_\ell(E,k)$.

Step 3: lengths over $\mathcal{O}$ and over $\mathbb{Z}_\ell$. Let $f_{\mathcal{O}}$ be the degree of the residue field $\mathbb{F}_{\mathcal{O}}$ over $\mathbb{F}_\ell$, so $d = e_{\mathcal{O}} f_{\mathcal{O}}$. For an $\mathcal{O}$-module Y, $\mathrm{length}_{\mathbb{Z}_\ell} Y = f_{\mathcal{O}} \mathrm{length}_{\mathcal{O}} Y$ in $\mathbb{N} \cup \{\infty\}$: if $\mathrm{length}_{\mathcal{O}} Y$ is finite, a composition series over $\mathcal{O}$ has factors $\mathbb{F}_{\mathcal{O}}$, each of length $f_{\mathcal{O}}$ over $\mathbb{Z}_\ell$ (its $\mathbb{Z}_\ell$-submodules are its $\mathbb{F}_\ell$-subspaces), and length is additive; if it is infinite, there are arbitrarily long chains of $\mathcal{O}$-submodules, which are chains of $\mathbb{Z}_\ell$-submodules. With Steps 1 and 2,

$$f_{\mathcal{O}} \mathrm{length}_{\mathcal{O}} \mathrm{Sh}_{\mathcal{O}}(E,k) = d\,\mathrm{len}_\ell(E,k) = e_{\mathcal{O}} f_{\mathcal{O}} \mathrm{len}_\ell(E,k),$$

which gives the assertion after division by $f_{\mathcal{O}}$, and after a further division by $e_{\mathcal{O}}$ the statement on normalised lengths. □

Lemma 22.14. *Let $(\mathcal{O}, \mathcal{Z}, \mathbb{T}, M)$ be a Hecke lattice over a prime ℓ, $K = \mathcal{O}[\ell^{-1}]$, and $(\mathcal{Z}', \phi, (z_1, \dots, z_{n_z}), n_{\mathcal{R}}, \mathcal{R})$ a determinant algebra over $\mathbb{T}$. Let $\mathcal{D} = \{\sum_k (a_k \circ \phi)\, z_k : \ a_k \in \mathbb{T}\}$, an $\mathcal{O}$-module of functions $\mathcal{Z}' \to \mathcal{O}$. Let E be a number field and $r \colon G_E \to \mathcal{R}^\times$ a group homomorphism. Assume:*

1. *$\mathcal{D}$ contains the constant function 1 and is closed under products;*
2. *there is an $\mathcal{O}$-subalgebra $\mathbb{T}' \subseteq \mathcal{D}$ of the ring of functions $\mathcal{Z}' \to \mathcal{O}$ such that $\mathcal{R}$ is the $\mathbb{T}'$-span of $r(G_E)$, that is, $\mathcal{R} = \{\sum_{h \in S} f_h\, r(h) : \ S \subseteq G_E \text{ finite}, \ f_h \in \mathbb{T}'\}$ with $f \in \mathbb{T}'$ acting by $f(x')$ at x'; and $g \mapsto r(g)(x') \in \mathrm{GL}_{n_{\mathcal{R}}}(K)$ is continuous for every $x' \in \mathcal{Z}'$;*
3. *there is a finite set B of places of E such that for every finite place $w \notin B$ of E whose residue field has prime order, $r(\gamma)(x') = 1$ for all $\gamma \in I_w$ and $x' \in \mathcal{Z}'$, and the functions $x' \mapsto \sigma_j(r(\mathrm{Frob}_w)(x'))$, $1 \le j \le n_{\mathcal{R}}$, lie in $\mathcal{D}$.*

Then for every $g \in G_E$ and $1 \le j \le n_{\mathcal{R}}$ the function $x' \mapsto \sigma_j(r(g)(x'))$ lies in $\mathcal{D}$, and the determinant algebra satisfies the determinant condition in every degree $g \ge 1$.

Proof. **Step 1: $\mathcal{D}$ is closed in $K^{\mathcal{Z}'}$.** $\mathbb{T}$ is an $\mathcal{O}$-submodule of the finitely generated free module $\mathcal{O}^{\mathcal{Z}}$, hence finitely generated ($\mathcal{O}$ is noetherian), so $\mathcal{D}$, the image of $\mathbb{T}^{n_z}$ under $(a_k) \mapsto \sum_k (a_k \circ \phi) z_k$, is a finitely generated $\mathcal{O}$-module; it is the image of some $\mathcal{O}^d$ under a continuous map, hence compact, hence closed in the Hausdorff space $K^{\mathcal{Z}'}$.

Step 2: all g. Let $A \subseteq G_E$ be the set of g with $x' \mapsto \sigma_j(r(g)(x'))$ in $\mathcal{D}$ for all j. The maps $g \mapsto \sigma_j(r(g)(x'))$ are continuous (polynomials in the entries), so A is closed by Step 1. Since $r(hgh^{-1})(x')$ is conjugate to $r(g)(x')$ for every x', A is stable under conjugation. For w as in (3),

$r(\cdot)(x')$ is trivial on I_w, so every element of $\mathrm{Frob}_w I_w$ and every conjugate lies in A. Let $U \subseteq G_E$ be an open normal subgroup with fixed field L and $s \in G_E$. By the Chebotarev density theorem (Theorem B.18) the finite places of E unramified in L whose Frobenius lies in the conjugacy class of sU in $\mathrm{Gal}(L/E)$ have positive Dirichlet density; the places of residue degree at least 2 over $\mathbb{Q}$ have Dirichlet density 0 (the series $\sum_w \mathrm{N}w^{-\sigma}$ over them converges at $\sigma = 1$, as there are at most $[E:\mathbb{Q}]$ places above each rational prime p and each has norm at least p^2), and B is finite. So there is w as in (3), unramified in L, with a Frobenius element at w mapping to a conjugate of sU; by the stability of A under conjugation, $A \cap sU \neq \emptyset$. Thus A meets every coset of every open normal subgroup, so A is dense, and closed, so $A = G_E$.

Step 3: matrices over $\mathcal{R}$. By (1), $\mathcal{D}$ is a subring of the ring $K^{\mathcal{Z}'}$ of functions $\mathcal{Z}' \to K$: an $\mathcal{O}$-module containing 1 and closed under products. By Step 2, the functions $x' \mapsto \sigma_j(r(h)(x'))$ lie in $\mathcal{D}$ for every $h \in G_E$ and $1 \le j \le n_{\mathcal{R}}$. Let $g \ge 1$ and $\mathbf{X} \in \mathrm{M}_g(\mathcal{R})$. By (2) each entry is $\mathbf{X}_{pq} = \sum_{h \in H_0} f_{pqh}\, r(h)$ with a finite $H_0 \subseteq G_E$ and $f_{pqh} \in \mathbb{T}' \subseteq \mathcal{D}$, so $x' \mapsto \mathbf{X}_{pq}(x')$ is a $\mathcal{D}$-combination in the sense of Lemma 22.15, applied with $F = K$, $Y = \mathcal{Z}'$, the group G_E, the homomorphisms $\varrho_{x'} = r(\cdot)(x') \colon G_E \to \mathrm{GL}_{n_{\mathcal{R}}}(K) \subseteq \mathrm{M}_{n_{\mathcal{R}}}(K)$ and $A = \mathcal{D}$. That lemma gives that $x' \mapsto \det \mathbf{X}(x')$ lies in $\mathcal{D}$: there are $a_1, \dots, a_{n_z} \in \mathbb{T}$ with $\det \mathbf{X}(x') = \sum_k a_k(\phi(x'))z_k(x')$ for all $x' \in \mathcal{Z}'$, which is (13.8) in degree g. □

Lemma 22.15. *Let F be a commutative ring, Y a set, $n \ge 1$, G a monoid, and for every $y \in Y$ let $\varrho_y \colon G \to \mathrm{M}_n(F)$ be a homomorphism of monoids. Let A be a subring of the ring F^Y of functions $Y \to F$ (pointwise operations) such that the functions $y \mapsto \sigma_j(\varrho_y(h))$ lie in A for every $h \in G$ and $1 \le j \le n$. Call a function $f \colon Y \to \mathrm{M}_n(F)$ an* A-combination *if $f(y) = \sum_{h \in H} a_h(y)\, \varrho_y(h)$ for a finite $H \subseteq G$ and $a_h \in A$. Then for every $g \ge 1$ and every $g \times g$ matrix $\mathbf{X} = (\mathbf{X}_{pq})$ of A-combinations, the function $y \mapsto \det \mathbf{X}(y)$ (the determinant of the $gn \times gn$ matrix $(\mathbf{X}_{pq}(y))_{p,q}$) lies in A.*

Proof. **Step 1: words.** Choose a finite $H \subseteq G$ and $a_{pqh} \in A$ with $\mathbf{X}_{pq}(y) = \sum_{h \in H} a_{pqh}(y)\varrho_y(h)$ for all p, q (take the union of the finite sets of the entries and coefficients 0 elsewhere). For $Z \in \mathrm{M}_n(F)$ let $e_{pq} \otimes Z \in \mathrm{M}_{gn}(F)$ be the matrix with Z in the block (p,q) and 0 elsewhere; $(e_{pq} \otimes Z)(e_{p'q'} \otimes Z')$ is $e_{pq'} \otimes ZZ'$ if $q = p'$ and 0 otherwise. Enumerate the triples $i = (p, q, h)$ $(1 \le p, q \le g,\ h \in H)$ as $1, \dots, k$, and for $y \in Y$ put $A_i(y) = a_{pqh}(y)\, e_{pq} \otimes \varrho_y(h)$, so that $\mathbf{X}(y) = \sum_i A_i(y)$, of size $N = gn$. For a word $w = i_1 \cdots i_m$, $i_\nu = (p_\nu, q_\nu, h_\nu)$, put $a_w = \prod_\nu a_{p_\nu q_\nu h_\nu} \in A$ and $h_w = h_1 \cdots h_m \in G$. As ϱ_y is multiplicative, $A_w(y) = A_{i_1}(y) \cdots A_{i_m}(y)$ equals $a_w(y)\, e_{p_1 q_m} \otimes \varrho_y(h_w)$ if $q_\nu = p_{\nu+1}$ for all $\nu < m$, and 0 otherwise. For $j \ge 1$, $\sigma_j(e_{pq} \otimes Z) = 0$ if $p \neq q$ (a principal minor of positive size of this matrix has a zero row or column) and $\sigma_j(e_{pp} \otimes Z) = \sigma_j(Z)$ (the principal minors that do not lie in the block (p,p) vanish). With $\sigma_j(bZ) = b^j \sigma_j(Z)$ for $b \in F$, for every word w and $j \ge 1$

$$\sigma_j\big(A_w(y)\big) = \epsilon_w\, a_w(y)^j\, \sigma_j\big(\varrho_y(h_w)\big), \qquad \epsilon_w \in \{0, 1\},$$

where $\epsilon_w = 1$ exactly when $q_\nu = p_{\nu+1}$ for all $\nu < m$ and $p_1 = q_m$; ϵ_w, a_w, h_w do not depend on y.

Step 2: Amitsur's formula. Fix y. By Theorem B.169 over $C = F$ with the matrices $A_i(y)$,

$$\det\Big(1 - \sum_i t_i A_i(y)\Big) = \prod_{w \text{ Lyndon}} \det(1 - t_w A_w(y)) = \prod_w \sum_{j=0}^{N} (-1)^j \sigma_j\big(A_w(y)\big)\, t_w^j$$

in $F[[t_1, \dots, t_k]]$ (for an $N \times N$ matrix Z, $\det(1 - XZ) = \sum_j (-1)^j \sigma_j(Z) X^j$). The left side is $\sum_{j=0}^{N} (-1)^j \sigma_j(\sum_i t_i A_i(y))$, and $\sigma_j(\sum_i t_i A_i(y))$ is homogeneous of degree j in t; so its homogeneous part of degree N is $(-1)^N \det(\sum_i t_i A_i(y))$. On the right, t_w has degree the length $|w|$ of

w, so only the finitely many Lyndon words of length at most N contribute in degree N, and the homogeneous part of degree N is the finite sum, over the families (j_w) of non-negative integers indexed by these words with $\sum_w j_w|w| = N$, of $\prod_w (-1)^{j_w}\sigma_{j_w}(A_w(y))\,t_w^{j_w}$. Equating the two homogeneous polynomials of degree N and evaluating at $t_1 = \cdots = t_k = 1$, with Step 1,

$$\det \mathbf{X}(y) = (-1)^N \sum_{(j_w)} \prod_w (-1)^{j_w} \left(\epsilon_w a_w(y)^{j_w}\right)^{[j_w \geq 1]} \sigma_{j_w}\big(\varrho_y(h_w)\big),$$

where the index set, the signs, ϵ_w, a_w and h_w do not depend on y, and $\sigma_0 = 1$.

Step 3: conclusion. Each a_w is in A, each function $y \mapsto \sigma_j(\varrho_y(h_w))$ with $j \geq 1$ is in A by hypothesis, and the constants $0, \pm 1$ are in A; A is closed under sums and products. So the right side of the last display, as a function of y, lies in A. □

Lemma 22.16. *Let ℓ be a prime, $N \geq 1$ an integer, $N_\ell = \operatorname{lcm}(\ell, N)$, $s = \max(1, v_\ell(N))$, $\mathcal{O}$ a coefficient ring for ℓ, $K = \mathcal{O}[\ell^{-1}]$, and $K' = \mathcal{O}(N)[\ell^{-1}]$. Then:*

1. *$\mathcal{O}(N)$ is a coefficient ring for ℓ containing $\mathcal{O}$, and $K' = K(\zeta)$ for a primitive N_ℓ-th root of unity ζ; in particular K' contains a primitive N_ℓ-th root of unity;*
2. *the ramification index of K'/K divides $(\ell - 1)\ell^{s-1}$;*
3. *if F is a finite extension of $\mathbb{Q}_\ell$ which is $\mathbb{Q}_\ell$-isomorphic to a subfield of $\mathbb{Q}_\ell(\mu_N)$ and $F_1 = F(\xi)$ with ξ a primitive ℓ-th root of unity in an algebraic closure of F, then $\sigma(F_1) \subseteq K'$ for every $\mathbb{Q}_\ell$-embedding $\sigma\colon F_1 \to \overline{\mathbb{Q}}_\ell$.*

Proof. **Item 1.** Let $\zeta \in \overline{\mathbb{Q}}_\ell$ be a primitive N_ℓ-th root of unity; $K(\zeta)$ is a finite extension of $\mathbb{Q}_\ell$. Since $\mathcal{O}$ is integral over $\mathbb{Z}_\ell$, the integral closure $\mathcal{O}(N)$ of $\mathcal{O}$ in $K(\zeta)$ equals the integral closure of $\mathbb{Z}_\ell$ in $K(\zeta)$, its valuation ring: a discrete valuation ring, finite over $\mathbb{Z}_\ell$, containing $\mathcal{O}$. Every element of $K(\zeta)$ becomes integral after multiplication by a power of ℓ, so $\mathcal{O}(N)[\ell^{-1}] = K(\zeta)$.

Item 2. Write $N_\ell = \ell^s N'$ with $\ell \nmid N'$; indeed $v_\ell(N_\ell) = \max(1, v_\ell(N)) = s$. Then $\zeta_1 = \zeta^{\ell^s}$ has order N', $\zeta_2 = \zeta^{N'}$ has order ℓ^s, and $\zeta \in \mathbb{Q}(\zeta_1, \zeta_2)$ since $\gcd(\ell^s, N') = 1$; so $K' = K_1(\zeta_2)$ with $K_1 = K(\zeta_1)$. The extension K_1/K is unramified, being generated by a root of unity of order prime to ℓ (the polynomial $X^{N'} - 1$ is separable modulo the maximal ideal). The extension $\mathbb{Q}_\ell(\zeta_2)/\mathbb{Q}_\ell$ is Galois with group $(\mathbb{Z}/\ell^s\mathbb{Z})^\times$ of order $(\ell - 1)\ell^{s-1}$, and $K' = K_1 \cdot \mathbb{Q}_\ell(\zeta_2)$, so restriction embeds $\operatorname{Gal}(K'/K_1)$ into it and $[K' : K_1]$ divides $(\ell - 1)\ell^{s-1}$. Hence $e(K'/K) = e(K'/K_1)\,e(K_1/K) = e(K'/K_1)$ divides $e(K'/K_1)f(K'/K_1) = [K' : K_1]$, which divides $(\ell - 1)\ell^{s-1}$.

Item 3. Let $C = \mathbb{Q}_\ell(\zeta)$; it is the splitting field of $X^{N_\ell} - 1$ over $\mathbb{Q}_\ell$, hence normal over $\mathbb{Q}_\ell$, and $C \subseteq K'$ by item 1. Since $N \mid N_\ell$, $\mathbb{Q}_\ell(\mu_N) \subseteq C$, so there is a $\mathbb{Q}_\ell$-embedding $\tau\colon F \to C$. Extend τ to $\tau_1\colon F_1 \to \overline{\mathbb{Q}}_\ell$; $\tau_1(\xi)$ is a primitive ℓ-th root of unity, which lies in C as $\ell \mid N_\ell$, so $\tau_1(F_1) \subseteq C$. For a $\mathbb{Q}_\ell$-embedding $\sigma\colon F_1 \to \overline{\mathbb{Q}}_\ell$, the embedding $\sigma \circ \tau_1^{-1}$ of $\tau_1(F_1)$ extends to a $\mathbb{Q}_\ell$-automorphism of $\overline{\mathbb{Q}}_\ell$, which maps the normal extension C onto itself. So $\sigma(F_1) \subseteq C \subseteq K'$. □

Lemma 22.17. *Let ℓ be a prime and $\mathcal{O} \subseteq \mathcal{O}'$ coefficient rings for ℓ.*

1. *For every $\mathcal{O}$-module X, $\mathcal{O}' \otimes_{\mathcal{O}} X$ has the same normalised length in $[0, \infty]$ (over $\mathcal{O}'$) as X (over $\mathcal{O}$); in particular ν of an element $a \in \mathcal{O} \setminus \{0\}$ is the same over $\mathcal{O}$ and over $\mathcal{O}'$.*
2. *Let Galois Ribet items on $(\mathcal{R}, \pi', \mathcal{N})$ over E with a Hecke lattice over ℓ with coefficient ring $\mathcal{O}$ be given, and consider their coefficient extension to $\mathcal{O}'$. Then the coefficient extension of the block form at π' is a block form with the same block sizes; the items have (Gd1)–(Gd5)*

if and only if the extension has them; at every $v \in S_\ell$ they have (Ll1), respectively (Ll2), if and only if the extension has it; and the minimal losses $\kappa_k^{\min}$, $\lambda_k^{\min}(v)$, the normalised length of Sel_0*, the denominator δ, $\nu(\delta)$ and $\nu(\zeta_v - 1)$ ($v \in \Sigma_{\mathrm{T}}$) are the same for the items and for the extension.*

3. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with local defect data, local bound records, Ribet records, budget data, inert budget data and Galois Ribet items at points, and let $\mathcal{F}'$ be its coefficient extension along coefficient rings $\mathcal{O}'_r \supseteq \mathcal{O}_r$. Then $\mathcal{F}'$ is such a family with such data. For every choice of losses, the items of $\mathcal{F}$ with these losses are Galois Ribet records having (Gd8), (Gd9), respectively (Gd10), if and only if the items of $\mathcal{F}'$ with the same losses are. The Ribet records of $\mathcal{F}$ have (Rf2), (Rf4), respectively (Rf5), if and only if those of $\mathcal{F}'$ have it, and they have (Rf1) if those of $\mathcal{F}'$ have it. The local bound records of $\mathcal{F}$ and $\mathcal{F}'$ coincide. At every member and every integer $N_{x,i}$, (Ub1), (Ub8) and (Ub9) hold for $\mathcal{F}$ if and only if they hold for $\mathcal{F}'$.*

Proof. **Step 0: a basis.** $\mathcal{O}'$ is a finitely generated torsion-free module over the discrete valuation ring $\mathcal{O}$, hence free, of rank $d = [K' : K]$. The quotient $\mathcal{O}'/\mathcal{O}$ is torsion-free: if $x \in \mathcal{O}'$ and $ax \in \mathcal{O}$ with $a \in \mathcal{O} \setminus \{0\}$, then $x \in K$ is integral over $\mathcal{O}$, so $x \in \mathcal{O}$. Hence $\mathcal{O}'$ has an $\mathcal{O}$-basis $\varepsilon_0 = 1, \varepsilon_1, \dots, \varepsilon_{d-1}$, which is also a K-basis of K'. Consequently, for every $\mathcal{O}$-module X, $\mathcal{O}' \otimes_{\mathcal{O}} X = \bigoplus_j \varepsilon_j \otimes X$, and every $\mathcal{O}$-linear map, group action or module structure over an $\mathcal{O}$-algebra extends componentwise; and for an $\mathcal{O}$-submodule X of a K-vector space W, $K' \otimes_K W = \bigoplus_j \varepsilon_j W$ and $X_{\mathcal{O}'} = \bigoplus_j \varepsilon_j X$, so $\mathcal{O}' \otimes_{\mathcal{O}} X \to X_{\mathcal{O}'}$ is an isomorphism, and $X_{\mathcal{O}'} \cap (\varepsilon_0 W) = X$.

Step 1: item 1. Let $\mathfrak{m}, \mathfrak{m}'$ be the maximal ideals and $e = e(K'/K)$, so $\mathfrak{m}\mathcal{O}' = \mathfrak{m}'^e$. Tensoring with the free module $\mathcal{O}'$ is exact, and $\mathcal{O}' \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{m} = \mathcal{O}'/\mathfrak{m}'^e$ has $\mathcal{O}'$-length e; by induction along a composition series, $\mathrm{length}_{\mathcal{O}'}(\mathcal{O}' \otimes X) = e\,\mathrm{length}_{\mathcal{O}}(X)$ for X of finite length. Applied to $X = \mathcal{O}/\ell\mathcal{O}$, this gives $e_{\mathcal{O}'} = e\,e_{\mathcal{O}}$; dividing, the normalised lengths agree. If X has infinite length, then $\mathcal{O}' \otimes X \supseteq \varepsilon_0 \otimes X \cong X$ has infinite $\mathcal{O}$-length, and since $\mathcal{O}'/\mathfrak{m}'$ is finite over $\mathcal{O}/\mathfrak{m}$, an $\mathcal{O}'$-module of finite length has finite $\mathcal{O}$-length; so $\mathcal{O}' \otimes X$ has infinite length. For $a \in \mathcal{O} \setminus \{0\}$ apply this to $X = \mathcal{O}/a\mathcal{O}$, with $\mathcal{O}' \otimes X = \mathcal{O}'/a\mathcal{O}'$.

Step 2: the extended objects decompose. By Step 0, $\mathcal{R}_{\mathcal{O}'} = \bigoplus_j \varepsilon_j \mathcal{R}$ and $\mathbb{T}'_{\mathcal{O}'} = \bigoplus_j \varepsilon_j \mathbb{T}'$; for $a = \sum_j \varepsilon_j a_j$ ($a_j \in \mathcal{R}$) and $x' \in \mathcal{Z}'$, $a(x') = \sum_j \varepsilon_j a_j(x')$. *Block form.* By (Bf1) for the a_j, $a(\pi') \in \mathrm{M}_{n_{\mathcal{R}}}(\mathcal{O}')$ is block upper triangular, and its k-th diagonal block is $\rho'_k(a) = \sum_j \varepsilon_j \rho_k(a_j)$; as ρ_k is onto $\mathrm{M}_{n_{\mathcal{R},k}}(\mathcal{O})$, ρ'_k is onto $\bigoplus_j \varepsilon_j \mathrm{M}_{n_{\mathcal{R},k}}(\mathcal{O}) = \mathrm{M}_{n_{\mathcal{R},k}}(\mathcal{O}')$. The constituent module of the extension is $(K'/\mathcal{O}')^{n_{\mathcal{R},k}} = \bigoplus_j \varepsilon_j \mathrm{Cs}_k$, on which $a \in \mathcal{R}$ acts by $\rho_k(a)$ in each summand; we write it Cs'_k. *Modules.* As $\mathcal{R}_{\mathcal{O}'} = \mathcal{O}' \otimes_{\mathcal{O}} \mathcal{R}$, $\mathcal{N}_{\mathcal{O}'} = \mathcal{R}_{\mathcal{O}'} \otimes_{\mathcal{R}} \mathcal{N} = \mathcal{O}' \otimes_{\mathcal{O}} \mathcal{N}$. The Ribet ideal of the extension is $\sum_g \mathcal{R}_{\mathcal{O}'} y(g) = \bigoplus_j \varepsilon_j \mathcal{N}_{\mathcal{E}} = (\mathcal{N}_{\mathcal{E}})_{\mathcal{O}'}$, and $\iota_{\mathcal{O}'} = \mathrm{id} \otimes \iota$. For a left $\mathcal{R}$-module Y and any left $\mathcal{R}$-module X, adjunction and $\mathcal{O}' \otimes Y = \bigoplus_j \varepsilon_j Y$ give

$$\mathrm{Hom}_{\mathcal{R}_{\mathcal{O}'}}(\mathcal{R}_{\mathcal{O}'} \otimes_{\mathcal{R}} X,\ \mathcal{O}' \otimes_{\mathcal{O}} Y) = \mathrm{Hom}_{\mathcal{R}}(X, \textstyle\bigoplus_j \varepsilon_j Y) = \bigoplus_j \varepsilon_j \,\mathrm{Hom}_{\mathcal{R}}(X, Y), \qquad (*)$$

naturally in X; we use it for $X = \mathcal{N}, \mathcal{N}_{\mathcal{E}}$ and $Y = \mathrm{Cs}_k$ or $Y = K_{x'}^{n_{\mathcal{R}}}$ (with $\mathcal{O}' \otimes_{\mathcal{O}} K_{x'}^{n_{\mathcal{R}}} = K_{x'}^{\prime n_{\mathcal{R}}}$). *Cohomology.* Continuous cochains with values in $\bigoplus_j \varepsilon_j \mathrm{Cs}_k(u)$ are the sums of their components, a cochain being continuous if and only if its components are; so $H^1(H, \mathrm{Cs}'_k(u)) = \bigoplus_j \varepsilon_j H^1(H, \mathrm{Cs}_k(u))$ for every closed subgroup $H \subseteq G_E$, compatibly with restriction.

Step 3: item 2. *(Gd1)–(Gd5).* The elements $\mathbf{r}(g)$, $\mathcal{E}$, $\tilde{\delta}$, γ_v, p_v and the characters χ_c are the same, ρ'_k restricts to ρ_k on $\mathcal{R}$, and identities in $\mathcal{R}$ hold in $\mathcal{R}_{\mathcal{O}'} \supseteq \mathcal{R}$ and conversely; this gives (Gd1)–(Gd4) in both directions. For (Gd5), $\rho_k(p_v) \in \mathcal{O}$ is a unit of $\mathcal{O}'$ if and only if it is a unit of $\mathcal{O}$, since $\mathcal{O}' \cap K = \mathcal{O}$. The entries of the extension satisfy the requirements of

Galois Ribet data (the coefficient algebra is an $\mathcal{O}'$-subalgebra, $\mathcal{R}_{\mathcal{O}'}$ is the $\mathbb{T}'_{\mathcal{O}'}$-span of $\mathbf{r}(G_E)$, $\tilde{\delta}(x') \neq 0$, $\delta = \tilde{\delta}(\pi') \in \mathcal{O} \setminus \{0\}$ is the same denominator). *Column losses.* By $(*)$, the map $\varphi \mapsto \varphi \circ \iota_{\mathcal{O}'}$ of (Gd6) for the extension is $\bigoplus_j \varepsilon_j$ of the map for the items, so its kernel is $\mathcal{O}'\otimes$ the kernel for the items; by item 1, $\kappa_k^{\min}$ is the same. *Local losses.* For $\varphi = \sum_j \varepsilon_j \varphi_j$, $c_\varphi = \sum_j \varepsilon_j c_{\varphi_j}$, which is a continuous cocycle if and only if every c_{φ_j} is; so the Ribet classes of the extension are $X'_k = \bigoplus_j \varepsilon_j X_k$ (Step 2). The local conditions are $F'_0(v) = \bigoplus_j \varepsilon_j F_0(v)$ by definition, $F'_1(v) = H^1_{\mathrm{ur}}(E_v, \mathrm{Cs}'_1(u)) = \bigoplus_j \varepsilon_j F_1(v)$ (the classes vanishing on inertia, componentwise), and $F'_k(v) = 0$ for $k \geq 2$. Hence the image of X'_k in $H^1(G_{E_v}, \mathrm{Cs}'_k(u))/F'_k(v)$ is $\mathcal{O}'\otimes$ the image of X_k, and $\lambda_k^{\min}(v)$ is the same by item 1. *Selmer group.* Its defining conditions (vanishing on inertia groups, local conditions $F_0(v)$) are componentwise, so the Selmer group of the extension is $\bigoplus_j \varepsilon_j \mathrm{Sel}_0$, of the same normalised length. *ν.* $\delta, \zeta_v - 1 \in \mathcal{O} \setminus \{0\}$, and item 1 applies. *(Ll1), (Ll2).* As a $\mathbb{Q}_\ell$-linear representation of $D = G_{E_v}$, $K'^{n_\mathcal{R}}$ with $g \mapsto \mathbf{r}(g)(x')$ is the direct sum of the d subrepresentations $\varepsilon_j K^{n_\mathcal{R}}$, each isomorphic to $K^{n_\mathcal{R}}$; a finite direct sum of representations is crystalline if and only if each summand is (the functor D_{cris} commutes with finite direct sums and $\dim_{\mathbb{Q}_\ell}$ is additive), so (Ll1) holds for the extension if and only if it holds for the items. For (Ll2), $\mathrm{Cs}'_0(u) = \bigoplus_j \varepsilon_j \mathrm{Cs}_0(u)$ is the quotient of $\bigoplus_j \varepsilon_j V_0$ by $\bigoplus_j \varepsilon_j T_0$ ($V_0 = K^{n_{\mathcal{R},0}}$, $T_0 = \mathcal{O}^{n_{\mathcal{R},0}}$), and the Bloch–Kato condition of Definition 22.2 is compatible with finite direct sums, so $H^1_f(D, \mathrm{Cs}'_0(u)) = \bigoplus_j \varepsilon_j H^1_f(D, \mathrm{Cs}_0(u))$; as $F'_0(v) = \bigoplus_j \varepsilon_j F_0(v)$, the equality (Ll2) holds for the extension if and only if it holds for the items.

Step 4: item 3, the structure. The coefficient extension of a Hecke lattice is a Hecke lattice: $\mathbb{T}_{\mathcal{O}'}$ is an $\mathcal{O}'$-subalgebra of $\mathcal{O}'^{\mathcal{Z}}$ containing the constants and $\mathbb{T}$, so it separates the points; $M_{\mathcal{O}'}$ is free; the action is an algebra homomorphism by Step 0. The extended determinant algebras have, by Step 3, block forms with $t+1$ blocks at the same points $\pi' \in \phi^{-1}(\pi)$, and $\mathcal{N}_{\mathcal{O}'}$ is presented by $\mathbf{P}$ with n_{gen} generators (right exactness of $\mathcal{R}_{\mathcal{O}'} \otimes_\mathcal{R} -$). All other data are unchanged. So $\mathcal{F}'$ is a family with the same kinds of data.

Step 5: item 3, (Gd8)–(Gd10). (Gd8): by item 2 and item 1, for fixed losses the conditions (Gd1)–(Gd7) hold for the items if and only if they hold for the extension (the modules in (Gd6) and (Gd7) have the same normalised lengths); E, $\mathrm{sl}(r,\pi)$, $\mathrm{lc}_k(r,\pi)$ are unchanged, and the loss sums are the same as $n_{\mathcal{R},0}$, $\nu(\delta)$, $\nu(\zeta_v - 1)$ are. (Gd10): $\delta \in \mathcal{O}$ is a unit of $\mathcal{O}'$ if and only if of $\mathcal{O}$; the other conditions are on unchanged numbers. (Gd9): the module of the extension is $\mathcal{D}'_{r,\pi} = \{\sum_k (a_k \circ \phi) z_k : \ a_k \in \mathbb{T}_{r,\mathcal{O}'}\} = \bigoplus_j \varepsilon_j \mathcal{D}_{r,\pi}$ inside $\bigoplus_j \varepsilon_j K^{\mathcal{Z}'}$, and by Step 0 $\mathcal{D}' \cap K^{\mathcal{Z}'} = \mathcal{D}$. If $\mathcal{D}$ contains 1 and $\mathbb{T}'$ and is closed under products, then $\mathcal{D}'$ contains 1 and $\mathbb{T}'_{\mathcal{O}'}$ and is closed under products (products of ε_i-multiples land in $\bigoplus_j \varepsilon_j \mathcal{D}$, as $\varepsilon_i \varepsilon_{i'} \in \mathcal{O}'$); conversely, if $\mathcal{D}'$ has these properties, then $1 \in \mathcal{D}' \cap K^{\mathcal{Z}'} = \mathcal{D}$, $\mathbb{T}' \subseteq \mathcal{D}' \cap K^{\mathcal{Z}'} = \mathcal{D}$ and $\mathcal{D}\mathcal{D} \subseteq \mathcal{D}' \cap K^{\mathcal{Z}'} = \mathcal{D}$. Detection: if $\sum_k c_k z_k(x') = 0$ on $\phi^{-1}(x)$ with $c_k = \sum_j \varepsilon_j c_{kj} \in \mathcal{O}'$, then, the z_k being $\mathcal{O}$-valued, $\sum_k c_{kj} z_k(x') = 0$ for every j; so detection over $\mathcal{O}$ implies detection over $\mathcal{O}'$, and the converse holds as $\mathcal{O} \subseteq \mathcal{O}'$. The functions $x' \mapsto \sigma_j(\mathbf{r}(\mathrm{Frob}_w)(x'))$ are K-valued and $\mathbf{r}(\gamma)(x')$ is unchanged, so they lie in $\mathcal{D}'$ if and only if they lie in $\mathcal{D}' \cap K^{\mathcal{Z}'} = \mathcal{D}$, with the same set B.

Step 6: item 3, (Rf1)–(Rf5), records, (Ub1), (Ub8), (Ub9). (Rf2): by $(*)$ with $Y = K^{n_\mathcal{R}}_{x'}$, $\mathrm{Hom}_{\mathcal{R}_{\mathcal{O}'}}(\mathcal{N}_{\mathcal{O}'}, K'^{n_\mathcal{R}}_{x'}) = \bigoplus_j \varepsilon_j \mathrm{Hom}_\mathcal{R}(\mathcal{N}, K^{n_\mathcal{R}}_{x'})$, nonzero if and only if $\mathrm{Hom}_\mathcal{R}(\mathcal{N}, K^{n_\mathcal{R}}_{x'}) \neq 0$. (Rf4): by $(*)$ and item 1, $\mathrm{Hom}(\mathcal{N}_{\mathcal{O}'}, \mathrm{Cs}'_k)$ has finite length and the same normalised length as $\mathrm{Hom}(\mathcal{N}, \mathrm{Cs}_k)$; likewise $\mathrm{Hom}_{\mathrm{ur}}(G_E, K'/\mathcal{O}') = \bigoplus_j \varepsilon_j \mathrm{Hom}_{\mathrm{ur}}(G_E, K/\mathcal{O})$; $\mathrm{len}_\ell(E, k-1)$, sl and lc_k do not involve the coefficient ring. (Rf5): unchanged numbers. (Rf1): detection is equivalent as in Step 5. Suppose the determinant condition in degree $g = n_{\mathrm{gen}}$ holds for the extension, and let $\mathbf{X} \in \mathrm{M}_g(\mathcal{R}) \subseteq \mathrm{M}_g(\mathcal{R}_{\mathcal{O}'})$. There are $a'_k \in \mathbb{T}_{\mathcal{O}'}$, $a'_k = \sum_j \varepsilon_j a_{kj}$ with $a_{kj} \in \mathbb{T}$, such that $\det \mathbf{X}(x') = \sum_k a'_k(\phi(x')) z_k(x') = \sum_j \varepsilon_j (\sum_k a_{kj}(\phi(x')) z_k(x'))$ for all x'. The left side lies in $K = \varepsilon_0 K$ and the inner sums lie in $\mathcal{O} \subseteq K$; comparing components in $K' = \bigoplus_j \varepsilon_j K$ gives $\det \mathbf{X}(x') = \sum_k a_{k0}(\phi(x')) z_k(x')$ with $a_{k0} \in \mathbb{T}$, the determinant condition for $\mathcal{F}$. *Records.* The

local bound records, budget data and inert budget data are unchanged by definition. (Ub1): (Gd1)–(Gd5) and (Gd9) by item 2 and Step 5, E unchanged, Sel_0 of the same normalised length (item 2). (Ub8): it involves only the budget data, inert budget data and the allowances $a_{x,i}(\ell)$, which are numbers formed from them and from $N_{x,i}$, $S^{\mathrm{rm}}_{x,i}$, $\eta^{\mathrm{rm}}_{x,i}$. (Ub9): the minimal losses, hence $\mathrm{ls}^{\min}_k$, are the same (item 2), as are lc_k, lt, $\mathrm{la}_{x,i}$ and $a_{x,i}(\ell)$. □

23 Wach modules: sites, levels and the crystalline criterion

This section develops the Wach modules on which all local bounds of the Part rest. Let ℓ be odd, F unramified over $\mathbb{Q}_\ell$ of degree n, and K a field with $F \subseteq K \subseteq F(\mu_\ell)$. A Wach module of a crystalline lattice over K with coefficients L is a module over $(\mathcal{O}_F \otimes \mathcal{O}_L)[[X]]$ with a Frobenius, sending X to $(1+X)^\ell - 1$, and an action of Γ_K; it splits into n *sites*, one for each embedding of F in a Frobenius numbering, and Frobenius carries the site a to the site $a+1$. The Hodge jumps enter through a *window* (M_a, h_a) at each site. Most results also have a version of level s, over subfields of $F(\mu_{\ell^s})$ and for every prime ℓ; it serves the small primes.

Four results are used in the rest of the Part. Every crystalline lattice over K has a Wach module with the window of its Hodge jumps (Proposition 23.7); we obtain it from the prismatic F-crystal of the lattice [31], evaluated on the cyclotomic prism (Lemma 23.8). Membership of an element in a site is detected by its *levels*, its images under the maps into $B_{\mathrm{dR}} \otimes T$ attached to the Frobenius twists along an arc of sites (Lemma 23.5). The cokernel of the Wach map of a surjection of lattices is killed at every site by $X^{H_{\max}}$, where $H_{\max}$ is the largest width of the window (Proposition 23.18). And the Herr class [126] of a Wach vector with poles only at the Frobenius twists Q_s of $Q = \varphi(X)/X$, of precision at least $1 + \max_a M_a$, lies in the Bloch–Kato subgroup H^1_f [34] (Proposition 23.27). Their consumers are the lifting exponents of Section 24, the local conditions at large primes in Section 28, the colength bounds of Appendix C, and the ramified places of Section 27.

The section is long because everything is integral and every constant is explicit; the ideas are those of the theory of Wach modules [265], [26] and of (φ, Γ)-modules [93]. The subsections follow the four results: levels and membership (23.1), existence with level maps and horizontal frames (23.2), the length lemma (23.3), and the crystalline criterion (23.4).

23.1 Levels of a Wach module and membership in a site

This subsection fixes the language of the local theory. Definition 23.1 defines onto maps of crystalline lattices over an unramified field, their colength (the normalised length of the cokernel of the induced map on integral Bloch–Kato groups), Frobenius numberings and weight words; the colength is the quantity bounded at the inert primes (Section 26 and Appendix C). The next three definitions introduce Wach modules with sites and windows, the arc coordinates of an element along an arc of consecutive sites, and the level data of an arc. The one result, Lemma 23.5, says that an element of the arc domain lies in the site $\mathbf{N}_c$ exactly when all its level expansions are de Rham integral; it is how membership in a site is verified in Sections 24 and 28 and in Appendix C.

Definition 23.1. Let ℓ be an odd prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of degree n, $\varphi \in \mathrm{Gal}(F/\mathbb{Q}_\ell)$ its arithmetic Frobenius (inducing $x \mapsto x^\ell$ on the residue field), $D = G_F$, and L a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_L$ such that L contains $\sigma(F(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $F(\mu_\ell)$ into $\overline{\mathbb{Q}}_\ell$.

Frobenius numberings. A **Frobenius numbering** is a bijection $\mathbb{Z}/n \to \mathrm{Hom}_{\mathbb{Q}_\ell}(F, L)$, $a \mapsto \tau_a$, with $\tau_{a+1} \circ \varphi = \tau_a$ for all a. (It exists: φ generates the cyclic group $\mathrm{Gal}(F/\mathbb{Q}_\ell)$ of

order n, which acts simply transitively on the n embeddings by $\tau \mapsto \tau \circ \varphi^{-1}$; fix τ_0 and put $\tau_a = \tau_0 \circ \varphi^{-a}$.) With this numbering the crystalline Frobenius, which is φ-semilinear, maps the τ_a-component of $D_{\mathrm{cris}}(V) \otimes_F L$ to the τ_{a+1}-component.

Onto maps of crystalline lattices and their colength. An **onto map of crystalline lattices** over F with coefficients L is a tuple $(V_1, T_1, V_2, T_2, \Phi)$ of finite-dimensional L-vector spaces V_1, V_2 with continuous L-linear actions of D that are crystalline as $\mathbb{Q}_\ell$-representations, D-stable $\mathcal{O}_L$-lattices $T_\nu \subseteq V_\nu$, and an L-linear D-equivariant map $\Phi\colon V_1 \to V_2$ with $\Phi(T_1) = T_2$. Since Φ carries $H^1_f(D, V_1)$ into $H^1_f(D, V_2)$, it carries $H^1_f(D, T_1)$ into $H^1_f(D, T_2)$; the **colength** of Φ is

$$\mathrm{nl}\Big(H^1_f(D, T_2)/\Phi_* H^1_f(D, T_1)\Big) \in [0, \infty].$$

Weight words. Fix a Frobenius numbering. A **weight word** for an onto map Φ of crystalline lattices consists of integers $b \geq 1$ and $w, S, \eta, m \geq 0$, **sizes** $s_a > \eta$ and **signs** $\epsilon_a \in \{1, -1\}$ for $a \in \mathbb{Z}/n$, such that:

(Ow1) Windows for every a, every Hodge jump of V_1 at τ_a lies in $[s_a, s_a + S] \cup [-\eta, \eta]$ if $\epsilon_a = 1$, and in $[-s_a - S, -s_a] \cup [-\eta, \eta]$ if $\epsilon_a = -1$;

(Ow2) Low jumps for every a, $\sum_{|j| \leq \eta} \dim_L\big(\mathrm{Fil}^j D_{\mathrm{dR},\tau_a}(V_2)/\mathrm{Fil}^{j+1} D_{\mathrm{dR},\tau_a}(V_2)\big) \leq m$;

(Ow3) Thick and thin sites every a is **thick** ($b \leq s_a \leq b + \eta$) or **thin** ($s_a \leq w$).

Attached quantities: $h = b + \eta + S$ if some site is thick and $h = w + S + \eta + 1$ otherwise; $G = w + S + \eta$; $H' = \max_a s_a + S + \eta$; $d_2 = \dim_L V_2$. For integers $M_a, h_a \geq 0$ ($a \in \mathbb{Z}/n$) such that every Hodge jump of V_1 at τ_a lies in $[-M_a, h_a]$, the **lifting exponent** $e(M_\bullet, h_\bullet)$ is the integer e_{A} of Proposition 24.1 formed for V_1 with $a^-_{\tau_a} = M_a$ and $a^+_{\tau_a} = h_a$. For a real $c \geq 0$ the **band indicator** $\beta_c \in \{0, 1\}$ is 1 if some thin site a has a thick neighbour $a + 1$ or $a - 1$ and

$$|\ell s_a - b| \leq c(\ell\eta + \ell S + b/\ell + s_a + \ell \log_\ell(2h)),$$

and 0 otherwise.

Definition 23.2. Let ℓ, F, n, φ, L, $\mathcal{O}_L$ be as in the definition of Frobenius numberings and fix a Frobenius numbering $(\tau_a)_{a \in \mathbb{Z}/n}$. Let $\mathcal{O}_F = W(k)$, k the residue field of F, and write φ also for the Frobenius of $\mathcal{O}_F$.

The ring. Let $\mathbf{A}_F = \mathcal{O}_F[[X]]$ with the ring endomorphism φ that is the Frobenius on $\mathcal{O}_F$ and sends X to $(1 + X)^\ell - 1$, and, for $u \in \mathbb{Z}_\ell^\times$, the $\mathcal{O}_F$-linear ring automorphism γ_u with $\gamma_u(X) = (1 + X)^u - 1 = \sum_{k \geq 1} \binom{u}{k} X^k$. Put

$$Q = \frac{\varphi(X)}{X} = \sum_{i=0}^{\ell-1} (1 + X)^i, \qquad Q_j = \varphi^j(Q) \quad (j \geq 0), \qquad \lambda_e = \prod_{j=0}^{e-1} Q_j \quad (e \geq 0),$$

elements of $\mathbb{Z}_\ell[[X]]$.

Sites. The **site ring** is $\mathbf{A}_{F,L} = (\mathcal{O}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L)[[X]]$, with φ and γ_u extended $\mathcal{O}_L$-linearly (on coefficients $\varphi \otimes \mathrm{id}$, resp. the identity). The **site idempotents** $e_a \in \mathcal{O}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$, $a \in \mathbb{Z}/n$, are the idempotents of the ring isomorphism

$$\mathcal{O}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L \to \prod_{a \in \mathbb{Z}/n} \mathcal{O}_L, \qquad x \otimes \lambda \mapsto (\tau_a(x)\lambda)_a,$$

(an isomorphism because $\mathcal{O}_F$ is finite étale over $\mathbb{Z}_\ell$ of degree n and the τ_a are its n embeddings into $\mathcal{O}_L$). So $\mathbf{A}_{F,L} = \prod_a e_a \mathbf{A}_{F,L}$ with $e_a \mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$, and $\varphi(e_a) = e_{a+1}$, $\gamma_u(e_a) = e_a$; the elements of $\mathbb{Z}_\ell[[X]]$, such as X, Q_j, λ_e, act on every site.

The base field. Let K be a field with $F \subseteq K \subseteq F(\mu_\ell)$ inside a fixed algebraic closure $\overline{\mathbb{Q}}_\ell$, χ_{cyc} the cyclotomic character, $\Gamma_K = \chi_{\mathrm{cyc}}(G_K) \subseteq \mathbb{Z}_\ell^\times$ and Δ_K the torsion subgroup of Γ_K. For $K = F$ one has $\Gamma_F = \mathbb{Z}_\ell^\times$ and $\Delta_F = \mu_{\ell-1}$: $F/\mathbb{Q}_\ell$ is unramified and $\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell$ totally ramified, so the two are linearly disjoint and χ_{cyc} maps G_F onto $\mathrm{Gal}(\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell) \cong \mathbb{Z}_\ell^\times$.

Wach modules. Let $M_a, h_a \geq 0$ be integers ($a \in \mathbb{Z}/n$). A **Wach module** over K with coefficients L and **window** $(M_a, h_a)_{a\in\mathbb{Z}/n}$ is a finitely generated $\mathbf{A}_{F,L}$-module $\mathbf{N}$ with a φ-semilinear map $\varphi_{\mathbf{N}}\colon \mathbf{N} \to \mathbf{N}[1/Q] = \mathbf{N} \otimes_{\mathbf{A}_{F,L}} \mathbf{A}_{F,L}[1/Q]$ and γ_u-semilinear maps $\gamma_{u,\mathbf{N}}\colon \mathbf{N} \to \mathbf{N}$ ($u \in \Gamma_K$) such that, with the **site components** $\mathbf{N}_a = e_a\mathbf{N}$:

(Wm1) Sites there is $r \geq 0$ such that every $\mathbf{N}_a$ is a free $e_a\mathbf{A}_{F,L}$-module of rank r;

(Wm2) Heights for every $a \in \mathbb{Z}/n$,

$$Q^{M_a}\varphi_{\mathbf{N}}(\mathbf{N}_{a-1}) \subseteq \mathbf{N}_a \qquad \text{and} \qquad Q^{h_a}\mathbf{N}_a \subseteq \mathbf{A}_{F,L}\,\varphi_{\mathbf{N}}(\mathbf{N}_{a-1}),$$

where $\mathbf{A}_{F,L}\,S$ denotes the $\mathbf{A}_{F,L}$-submodule generated by a set S;

(Wm3) Γ-action $\gamma_{1,\mathbf{N}} = \mathrm{id}$, $\gamma_{uv,\mathbf{N}} = \gamma_{u,\mathbf{N}} \circ \gamma_{v,\mathbf{N}}$, $\gamma_{u,\mathbf{N}} \circ \varphi_{\mathbf{N}} = \varphi_{\mathbf{N}} \circ \gamma_{u,\mathbf{N}}$ (on $\mathbf{N}[1/Q]$, $\gamma_{u,\mathbf{N}}$ extended semilinearly), and $(\gamma_{u,\mathbf{N}} - 1)(\mathbf{N}) \subseteq X\mathbf{N}$ for all $u, v \in \Gamma_K$.

A **Wach module over** F is a Wach module over $K = F$; its maps $\gamma_{u,\mathbf{N}}$ are given for every $u \in \Gamma_F = \mathbb{Z}_\ell^\times$. A statement that does not name K is about Wach modules over F. Write φ and γ_u for $\varphi_{\mathbf{N}}$ and $\gamma_{u,\mathbf{N}}$ when no confusion arises.

Level s. Let $s \geq 1$ be an integer. The ring $\mathbf{A}_F$, φ, γ_u, Q, Q_j, λ_e, the site ring, the site idempotents and Frobenius numberings are defined by the same formulas for every prime ℓ, including $\ell = 2$ (none of the formulas, and neither the existence of a Frobenius numbering nor the product decomposition of $\mathcal{O}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$, uses that ℓ is odd). Let $s \geq 2$ if $\ell = 2$, let L contain $\sigma(F(\mu_{\ell^s}))$ for every $\mathbb{Q}_\ell$-embedding σ of $F(\mu_{\ell^s})$, and let K be a field with $F \subseteq K \subseteq F(\mu_{\ell^s})$, $\Gamma_K = \chi_{\mathrm{cyc}}(G_K)$ and Δ_K its torsion subgroup. A **Wach module of level s** over K with coefficients L and window $(M_a, h_a)_{a\in\mathbb{Z}/n}$ is defined as a Wach module above with Q replaced by Q_{s-1} throughout: $\varphi_{\mathbf{N}}$ maps $\mathbf{N}$ to $\mathbf{N}[1/Q_{s-1}]$, (Wm2) reads $Q_{s-1}^{M_a}\varphi_{\mathbf{N}}(\mathbf{N}_{a-1}) \subseteq \mathbf{N}_a$ and $Q_{s-1}^{h_a}\mathbf{N}_a \subseteq \mathbf{A}_{F,L}\,\varphi_{\mathbf{N}}(\mathbf{N}_{a-1})$, the commutation in (Wm3) holds on $\mathbf{N}[1/Q_{s-1}]$, and (Wm1) and the congruence $(\gamma_{u,\mathbf{N}} - 1)(\mathbf{N}) \subseteq X\mathbf{N}$ for $u \in \Gamma_K$ are unchanged; in addition, with the **level variable**

$$X_{s-1} = \varphi^{s-1}(X) = (1+X)^{\ell^{s-1}} - 1 = X\lambda_{s-1} \in \mathbb{Z}_\ell[[X]],$$

(Wm3') Level congruence $(\gamma_{u,\mathbf{N}} - 1)(\mathbf{N}) \subseteq X_{s-1}\mathbf{N}$ for every $u \in \Gamma_K$ with $u \equiv 1 \bmod \ell^{s-1}$.

For $s = 1$ one has $X_0 = X$ and every $u \in \Gamma_K$ satisfies $u \equiv 1 \bmod \ell^0$, so (Wm3') is the congruence of (Wm3). For $K = F(\mu_{\ell^s})$ one has $\Gamma_K = 1 + \ell^s\mathbb{Z}_\ell$ ($F/\mathbb{Q}_\ell$ is unramified, so F and $\mathbb{Q}_\ell(\mu_{\ell^\infty})$ are linearly disjoint), and (Wm3') holds for every $u \in \Gamma_K$. For ℓ odd a Wach module of level 1 is a Wach module as defined above ($Q_0 = Q$); a Wach module without a named level has level 1.

Frobenius pull-backs and arc factors. Let $\mathbf{N}[1/\lambda_\infty] = \mathbf{N} \otimes_{\mathbf{A}_{F,L}} \mathbf{A}_{F,L}[1/Q_j : j \geq 0]$, which contains $\mathbf{N}$ by (Wm1); $\varphi_{\mathbf{N}}$ extends to the φ-semilinear map $x/f \mapsto \varphi_{\mathbf{N}}(x)/\varphi(f)$ of $\mathbf{N}[1/\lambda_\infty]$ (f a product of the Q_j; $\varphi(Q_j) = Q_{j+1}$). For $c \in \mathbb{Z}/n$ and $e \geq 0$ the **Frobenius pull-back** $P_e(\mathbf{N}_c)$ is the $\mathbf{A}_{F,L}$-submodule of $\mathbf{N}[1/\lambda_\infty]$ generated by $\varphi_{\mathbf{N}}^e(\mathbf{N}_c)$; thus $P_0(\mathbf{N}_c) = \mathbf{N}_c$ and $P_e(\mathbf{N}_c) \subseteq e_{c+e}\mathbf{N}[1/\lambda_\infty]$. For $a \in \mathbb{Z}/n$, $e \geq 0$ and a function $H\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$ the **arc factor** is

$$\Lambda_{a,e}(H) = \prod_{j=0}^{e-1} Q_j^{H(a-j)} \in \mathbb{Z}_\ell[[X]].$$

Definition 23.3. Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a pair (V,T) be as in the existence of Wach modules over unramified fields, and $\mathbf{N}=\mathbf{N}(T)$, of rank $r=\dim_L V$. Put $E=(\mathcal{O}_F\otimes_{\mathbb{Z}_\ell}\mathcal{O}_L)[1/\ell]$. Fix $c\in\mathbb{Z}/n$, an integer $e\geq 0$, $a=c+e$, a function $H\colon \mathbb{Z}/n\to\mathbb{Z}_{\geq 0}$, $\Pi=\Lambda_{a,e}(H)$, and an integer $m\geq 1$. Let $\mathcal{K}_a$ be the fraction field of $e_a\mathbf{A}_{F,L}\cong\mathcal{O}_L[[X]]$; an element $\lambda\in\mathcal{O}_L$ acts on $e_a\mathbf{A}_{F,L}$ and on $\mathcal{K}_a$ as $e_a(1\otimes\lambda)$.

Frame data. A **frame** at c is an $e_c\mathbf{A}_{F,L}$-basis $\mathbf{n}=(n_k)_{k\leq r}$ of $\mathbf{N}_c$. Let $H_k\in\mathbf{N}_E$ be the horizontal section through n_k, and write $H_k=\sum_j h_{kj}n_j$ with $h_{kj}=\sum_{i\geq 0}h_{kj,i}X^i\in e_cE[[X]]$. An integer $\kappa\geq 0$ is **admissible** (for $\mathbf{n}$ and m) if $\ell^\kappa h_{kj,i}\in e_c(\mathcal{O}_F\otimes\mathcal{O}_L)$ for all k,j and $i<m$; then the **κ-truncation** is

$$T_\kappa(H_k)=\sum_j\sum_{i<m}\ell^\kappa h_{kj,i}X^i\,n_j\ \in\ \mathbf{N}_c.$$

The **crystalline frame** is $d_k=\mathrm{Ev}(H_k)\in D_{\mathrm{cris}}(V)=(B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V)^{G_F}$. Let $\varphi=\varphi\otimes 1$ on $B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V$ and

$$\mathrm{Fil}^u D_{\mathrm{cris}}(V)=D_{\mathrm{cris}}(V)\cap\big(\mathrm{Fil}^u B_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell}V\big)\quad(u\in\mathbb{Z}),\qquad \delta_e(\mathbf{v})=\sum_k v_k\,\varphi^e(d_k)\quad(\mathbf{v}\in L^r).$$

Coordinate vectors. Let $y\in\mathbf{N}[1/\lambda_\infty]$. A vector $f=(f_k)_{k\leq r}\in\mathcal{K}_a^r$ is an **arc coordinate vector** of y (for $\mathbf{n},e,H,m$) if $\Pi f_k\in e_a\mathbf{A}_{F,L}$ for every k and there are an admissible κ and $\kappa'\geq 0$ with

$$\ell^{\kappa'}\Big(\ell^\kappa\,\Pi\,y-\sum_k\Pi f_k\,\varphi^e\big(T_\kappa(H_k)\big)\Big)\in X\Pi^2\,P_e(\mathbf{N}_c).\tag{23.1}$$

Graded sections. For $\alpha\colon\{0,\dots,e-1\}\to\mathbb{Z}$ put $Q^\alpha=\prod_{s<e}Q_s^{\alpha(s)}\in\mathcal{K}_a^\times$. A **section decomposition** of $f\in\mathcal{K}_a^r$ is a finite family $(g_i,\mathbf{v}_i,\alpha_i)_{i\in I}$ with $g_i\in Q^{\alpha_i}e_a\mathbf{A}_{F,L}$, $\mathbf{v}_i\in\mathcal{O}_L^r$ and $f_k=\sum_i g_i\,v_{i,k}$ for all k. For $B=(B_s)_{s<e}$ with $B_s\in\mathbb{Z}_{\geq 0}\cup\{\infty\}$, the vector f is

(Gs1) Hodge-graded if it has a section decomposition with $\delta_e(\mathbf{v}_i)\in\varphi^{s+4}\big(\mathrm{Fil}^{-\alpha_i(s)}D_{\mathrm{cris}}(V)\big)$ for all i and all $s<e$;

(Gs2) pole-bounded by B if it has a section decomposition such that for all i and $s<e$: $\alpha_i(s)\geq 0$, or $\alpha_i(s)\geq -B_s$ and $\delta_e(\mathbf{v}_i)\in\varphi^{s+4}\big(\mathrm{Fil}^{-\alpha_i(s)}D_{\mathrm{cris}}(V)\big)$.

Since $\delta_e(\mathbf{v})\in\varphi^e(\epsilon_{c+3}D_{\mathrm{cris}}(V))=\epsilon_{a+3}D_{\mathrm{cris}}(V)$ (Lemma 23.10), the vector $\varphi^{-(s+4)}(\delta_e(\mathbf{v}))$ lies in $\epsilon_{a-s-1}D_{\mathrm{cris}}(V)$, the component of the embedding τ_{a-s} ($\epsilon_b=\iota_\Lambda(e_b)$ is the idempotent of τ_{b+1} for the canonical inclusion $F\subset B_{\mathrm{dR}}$, because $\iota=\varphi\circ w$ on $\mathcal{O}_F$): the level-s condition reads the Hodge filtration at τ_{a-s}, the embedding whose exponent $H(a-s)$ the arc factor Π carries at Q_s. f is **polar** if it is pole-bounded by $B\equiv\infty$.

Arc coordinate properties. $\mathbf{N}$ has the **arc coordinate properties** at (c,e,H,m) for the frame $\mathbf{n}$ if:

(Ac1) for every $y\in\mathbf{N}[1/\lambda_\infty]$ with $\Pi y\in P_e(\mathbf{N}_c)$ there are $t\geq 0$ and an arc coordinate vector of $\ell^t y$;

(Ac2) for every $y\in P_e(\mathbf{N}_c)$ there are $t\geq 0$ and an arc coordinate vector f of $\ell^t y$ with all $f_k\in e_a\mathbf{A}_{F,L}$;

(Ac3) for every $y\in\mathbf{N}_a$ with $\Pi y\in P_e(\mathbf{N}_c)$ there are $t,\kappa\geq 0$ and an arc coordinate vector f of $\ell^t y$ such that $\ell^\kappa f$ is Hodge-graded;

(Ac4) if f is an arc coordinate vector of 0, there is $t\geq 0$ with $\ell^t f_k\in X\Pi\,e_a\mathbf{A}_{F,L}$ for every k.

Definition 23.4. Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ be as in Proposition 23.7, and fix $u_0 \in 1+\ell\mathbb{Z}_\ell$ with $v_\ell(u_0-1)=1$.

The arc. Let (V,T) be a pair, $\mathbf{N}=\mathbf{N}(T)$ of rank r, $D=D_{\mathrm{cris}}(V)$. Fix a site $c\in\mathbb{Z}/n$ (the *cut*), an integer $1\le \mathrm{len}\le n$, $c_0=c-\mathrm{len}$, a function $H\colon \mathbb{Z}/n\to\mathbb{Z}_{\ge 0}$, $\Pi=\Lambda_{c,\mathrm{len}}(H)=\prod_{s<\mathrm{len}} Q_s^{H(c-s)}$, and a frame $\mathbf{n}=(n_k)_{k\le r}$ at c_0 with horizontal sections H_k and crystalline frame d_k. The **arc domain** is

$$W=\{w\in\mathbf{N}[1/\lambda_\infty]:\ \Pi w\in P_{\mathrm{len}}(\mathbf{N}_{c_0})\}\ \subseteq\ e_c\mathbf{N}[1/\lambda_{\mathrm{len}}].$$

Levels. For $0\le s<\mathrm{len}$ let $\Lambda_s=\varphi^{-(s+1)}\circ\iota_T$ on $\mathbf{N}[1/\lambda_{\mathrm{len}}]$ and $\lambda_s=\varphi^{-(s+1)}\circ(\iota\otimes\mathrm{id})$ on $\mathbf{A}_{F,L}[1/\lambda_{\mathrm{len}}]$, with values in $B_{\mathrm{dR}}\otimes_{\mathbb{Z}_\ell}T$ and $B_{\mathrm{dR}}\otimes_{\mathbb{Z}_\ell}\mathcal{O}_L$ (the level maps of Lemma 23.13; $\Lambda_s(gx)=\lambda_s(g)\Lambda_s(x)$). Put $\epsilon_s=\epsilon_{c-s-1}=\lambda_s(e_c)$, the idempotent of the embedding τ_{c-s}, $R_s^+=\epsilon_s(B_{\mathrm{dR}}^+\otimes_{\mathbb{Q}_\ell}L)$, $R_s=R_s^+[1/t]$, and the **level frame**

$$D_k^{(s)}=\varphi^{\mathrm{len}-s-4}(d_k)\in\epsilon_sD\qquad(k\le r),$$

so that $\Lambda_s(\varphi^{\mathrm{len}}(H_k))=D_k^{(s)}$ by Lemma 23.13 (1). The $D_k^{(s)}$ are an L-basis of ϵ_sD and an R_s-basis of $\epsilon_s(B_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell}V)$ (by Lemma 23.10 (2), as φ maps ϵ_bD bijectively onto $\epsilon_{b+1}D$, and by Lemma 23.16 with $b=c-s-1$). For $v\in L^r$ the level-s condition of Definition 23.3 reads $\sum_k v_kD_k^{(s)}=\varphi^{-(s+4)}(\delta_{\mathrm{len}}(v))$.

The algebraic level model. Let $\overline{\mathbb{Q}}_\ell$ be the algebraic closure of $\mathbb{Q}_\ell$ in B_{dR}^+; θ maps it isomorphically onto the algebraic closure of $\mathbb{Q}_\ell$ in $\mathbb{C}_\ell$ (Hensel's lemma in the complete discrete valuation ring B_{dR}^+, whose residue field $\mathbb{C}_\ell$ has characteristic 0), and it contains $W(k)[1/\ell]=F$. Let $\zeta_{\ell^k}\in\overline{\mathbb{Q}}_\ell$ be the element with $\theta(\zeta_{\ell^k})=\theta([\varepsilon^{1/\ell^k}])$. The **level coefficient ring** is

$$K_{(s)}=\epsilon_s\big(F(\zeta_{\ell^{s+1}})\otimes_{\mathbb{Q}_\ell}L\big)\subseteq R_s^+.$$

θ is injective on $K_{(s)}$ (L is flat over $\mathbb{Q}_\ell$) and R_s^+ is t-torsion free and t-adically complete, so $\sum_{i\ge i_0}a_it^i\mapsto$ its sum is an injective ring homomorphism $K_{(s)}((t))\to R_s$ (apply θ to t^{-i_0} times a vanishing sum); we identify $K_{(s)}((t))$ with its image. An element $g\in G_F$ preserves $\overline{\mathbb{Q}}_\ell$, fixes F and ϵ_s, sends t to $\chi(g)t$ (χ the cyclotomic character) and $\zeta_{\ell^{s+1}}$ to $\zeta_{\ell^{s+1}}^{\chi(g)}$; for $u\in\mathbb{Z}_\ell^\times$ let σ_u be the automorphism of $K_{(s)}$ induced by any g with $\chi(g)=u$, and on $K_{(s)}((t))^r$ put, coordinatewise,

$$\gamma_u^{(s)}\Big(\sum_i a_it^i\Big)=\sum_i u^i\sigma_u(a_i)\,t^i.$$

Level expansions. An element $w\in e_c\mathbf{N}[1/\lambda_{\mathrm{len}}]$ is **algebraic at level** s if $\Lambda_s(w)=\sum_k f_kD_k^{(s)}$ with all $f_k\in K_{(s)}((t))$; its **level-s expansion** is then $\mathrm{lv}_s(w)=(f_k)_k\in K_{(s)}((t))^r$ (unique, the $D_k^{(s)}$ being an R_s-basis). For $a\in\mathcal{O}_L^r$ (constants $e_c(1\otimes a_k)$ of $e_c\mathbf{A}_{F,L}$), λ_s maps a to $\epsilon_sa\in L^r\subseteq K_{(s)}^r$.

Filtrations and lattices at level s. Fix integers $G,S,\eta,g,M\ge 0$, $1\le j\le\mathrm{len}$, and for each $s<\mathrm{len}$ a *kind* $\kappa_s\in\{\mathrm{thin},\mathrm{up},\mathrm{down}\}$ and a *size* $b_s\ge 0$. Put

$$\mathrm{Fil}_s(u)=\Big\{v\in L^r:\ \sum_k v_kD_k^{(s)}\in\mathrm{Fil}^uD\Big\}=\{v:\ \delta_{\mathrm{len}}(v)\in\varphi^{s+4}(\mathrm{Fil}^uD)\}\qquad(u\in\mathbb{Z}),$$

the *kind space* $U_s^L=L^r$, $\mathrm{Fil}_s(b_s)$, $\mathrm{Fil}_s(-\eta)$ for $\kappa_s=$ thin, up, down, $U_s=U_s^L\cap\mathcal{O}_L^r$, and the *kind family* $(\mathrm{Nb}_s(i))_{i\in\mathbb{Z}}$: for thin, 0 if $i<0$ and L^r if $i\ge 0$; for up, 0 if $i<-b_s$, U_s^L if $-b_s\le i<0$, L^r if $i\ge 0$; for down, 0 if $i<0$, U_s^L if $0\le i<b_s$, L^r if $i\ge b_s$. For a family $\mathcal{X}=(\mathcal{X}(i))_{i\in\mathbb{Z}}$ of L-subspaces of L^r let $\Xi_s(\mathcal{X})$ be the set of $f\in K_{(s)}((t))^r$ whose t^i-coefficient lies in $K_{(s)}\otimes_L\mathcal{X}(i)$ for every i. A vector $f\in K_{(s)}((t))^r$ is

- **de Rham integral** at s if $f \in \Xi_s((\mathrm{Fil}_s(-i))_i)$;
- **kind integral** at s if $f \in \Xi_s((\mathrm{Nb}_s(i))_i)$;
- **truncation small** at s, written $f \in \mathcal{T}_s$, if

$$\lambda_s(\Pi) f \in \lambda_s(f')\, K_{(s)}[[t]]^r,$$

where $f' = \varphi^{\mathrm{len}-j}(X^{(\ell-1)M} \prod_{s'<j} Q_{s'}^{3g})$.

A family $(f_s)_{s<\mathrm{len}}$, $f_s \in K_{(s)}((t))^r$, is **constant** if there is $v \in L^r$ with $f_s = v$ for every s.

Arc level data of a map. Let $\Phi\colon T_1 \to T_2$ be a G_F-equivariant $\mathcal{O}_L$-linear map of the lattices of pairs (V_1, T_1), (V_2, T_2), of ranks r_1, r_2, and $\mathbf{N}(\Phi)\colon \mathbf{N}(T_1) \to \mathbf{N}(T_2)$ its map of Wach modules. Fix $(c, \mathrm{len}, H, G, S, \eta, g, M, j, (\kappa_s, b_s)_s)$ as above and frames $\mathbf{n}^\nu$ at c_0 of $\mathbf{N}(T_\nu)$. The **frame matrix** $o \in M_{r_2 \times r_1}(\mathcal{O}_L)$ is defined by $\mathbf{N}(\Phi)(n_k^1) \equiv \sum_l o_{lk} n_l^2 \bmod X\mathbf{N}(T_2)_{c_0}$ (via $e_{c_0}\mathbf{A}_{F,L}/X \cong \mathcal{O}_L$). The **arc level data** of Φ are the objects above for T_1 and for T_2 (indices 1, 2), together with $\mathbf{N}(\Phi)\colon W_1 \to W_2$ and o ($\mathbf{N}(\Phi)$ commutes with φ and preserves sites, hence maps W_1 into W_2).

Arc level properties. The arc level data of Φ have the **arc level properties** if, for $\nu = 1, 2$ and every $s < \mathrm{len}$:

(Lp1) Algebraic levels $\lambda_s(e_c\mathbf{A}_{F,L}) \subseteq K_{(s)}[[t]]$, $\lambda_s(Q_s) \in t\,K_{(s)}[[t]]^\times$, $\lambda_s(X), \lambda_s(Q_{s'}) \in K_{(s)}[[t]]^\times$ for $s' \neq s$, and every element of W_ν is algebraic at level s;

(Lp2) Equivariance $\gamma_u(W_\nu) \subseteq W_\nu$ and $\mathrm{lv}_s(\gamma_u w) = \gamma_u^{(s)}(\mathrm{lv}_s(w))$ for $w \in W_\nu$, $u \in \mathbb{Z}_\ell^\times$;

(Lp3) Membership for $w \in W_\nu$: $w \in \mathbf{N}(T_\nu)_c$ if and only if $\mathrm{lv}_{s'}(w)$ is de Rham integral at s' for every $s' < \mathrm{len}$;

(Lp4) Regularity $\lambda_s(\Pi)\,\mathrm{lv}_s(w) \in K_{(s)}[[t]]^{r_\nu}$ for $w \in W_\nu$;

(Lp5) Completeness if $a \in e_c\mathbf{A}_{F,L}$ and $\lambda_s(a) \in t^k K_{(s)}[[t]]$, then $\ell^N a \in Q_s^k e_c \mathbf{A}_{F,L}$ for some $N \geq 0$;

(Lp6) Constants if $x \in \mathbf{N}(T_\nu)_{c_0}$, $(\gamma_{u_0} - 1)x \in X^{(\ell-1)M}\mathbf{N}(T_\nu)_{c_0}$, $x - \sum_k a_k n_k^\nu \in X\mathbf{N}(T_\nu)_{c_0}$ with $a \in \mathcal{O}_L^{r_\nu}$, and $w \in W_\nu$ with $\Pi w = \varphi^{\mathrm{len}}(x)$, then $\mathrm{lv}_s(w) - \lambda_s(\Pi)^{-1} a \in \mathcal{T}_s$;

(Lp7) Windows Fil_s^ν is decreasing, equal to L^{r_ν} for $u \ll 0$ and to 0 for $u \gg 0$, and each jump u ($\mathrm{Fil}_s^\nu(u) \neq \mathrm{Fil}_s^\nu(u+1)$) satisfies $|u| \leq G$ if κ_s is thin, $u \in [b_s, b_s + S] \cup [-\eta, \eta]$ if up, $u \in [-b_s - S, -b_s] \cup [-\eta, \eta]$ if down;

(Lp8) Galois structure $\sigma_u^{\ell^s} = \mathrm{id}$ on $K_{(s)}$ for $u \in 1 + \ell\mathbb{Z}_\ell$, and the elements of $K_{(s)}$ fixed by every σ_u ($u \in \mathbb{Z}_\ell^\times$) are those of $\epsilon_s L = L$;

and, for every $s < \mathrm{len}$:

(Lh1) Level map $\mathrm{lv}_s(\mathbf{N}(\Phi)w) = o\,\mathrm{lv}_s(w)$ for $w \in W_1$;

(Lh2) Kind spaces if κ_s is up, then $o(U_s^1) \subseteq U_s^2$.

Along a weight word. For an onto map Φ of crystalline lattices with a weight word $(b, w, S, \eta, m, (s_a), (\mathrm{sg}_a))$ and a cut c of length len, with $G = w + S + \eta$, $g = \max(G, 1)$ and h_* as in Proposition C.83, the **arc level data along the word** use: κ_s thin if $c - s$ is thin, up (resp. down) if $c - s$ is thick with $\mathrm{sg}_{c-s} = 1$ (resp. -1); $b_s = s_{c-s}$; $H(c-s) = g + G$, $g + s_{c-s} + S$, $g + \eta$ for κ_s thin, up, down; $M = \lfloor (h_* + 2g)/(\ell - 1) \rfloor + 1$; and the j of the cut configuration.

Lemma 23.5. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ be as in Proposition 23.7, (V,T) a pair, $\mathbf{N}=\mathbf{N}(T)$, and fix arc level data for (V,T) (cut c, length $1\le \mathrm{len}\le n$, H, frame at $c-\mathrm{len}$). Then:*

1. *for $s<\mathrm{len}$ and $f\in K_{(s)}((t))^r$: f is de Rham integral at s if and only if $\sum_k f_k D_k^{(s)}\in \epsilon_s(B^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell} V)$;*
2. *(Lp3) holds: for $w\in W$, $w\in\mathbf{N}_c$ if and only if $\mathrm{lv}_s(w)$ is de Rham integral at s for every $s<\mathrm{len}$.*

Proof. Write $B=B^+_{\mathrm{dR}}$, $K=K_{(s)}$, $D=D_{\mathrm{cris}}(V)$, $D_k=D_k^{(s)}$.

Step 1: $K((t))\cap t^mR_s^+=t^mK[[t]]$. If $f=\sum_{i\ge i_0}a_it^i\in t^mR_s^+$ with $a_{i_0}\neq 0$ and $i_0<m$, then $t^{-i_0}f\in tR_s^+$ and $\theta(t^{-i_0}f)=a_{i_0}$ (identified with its image) vanishes; θ is injective on K, a contradiction.

Step 2: proof of 1. By Lemma 23.16 (with $b=c-s-1$, so that its ϵ, R^+ are ϵ_s, R_s^+), ϵ_sD has an L-basis $(D_i^{\mathrm{ad}})_i$ adapted to the filtration ($D_i^{\mathrm{ad}}\in\mathrm{Fil}^{u_i}D$, and $\epsilon_s\mathrm{Fil}^jD$ spanned by the D_i^{ad} with $u_i\ge j$), and

$$\epsilon_s(B\otimes_{\mathbb{Q}_\ell}V)=\bigoplus_i t^{-u_i}R_s^+D_i^{\mathrm{ad}}.$$

Let $D_k=\sum_i W_{ki}D_i^{\mathrm{ad}}$, $W\in GL_r(L)$. For $f\in K((t))^r$ put $\phi_i=\sum_k f_kW_{ki}\in K((t))$, so $\sum_k f_kD_k=\sum_i\phi_iD_i^{\mathrm{ad}}$. By Step 1, this lies in $\epsilon_s(B\otimes V)$ iff $\phi_i\in t^{-u_i}K[[t]]$ for all i, i.e. iff for every m the t^m-coefficients satisfy $\phi_i^{(m)}=0$ whenever $u_i<-m$, i.e. iff $\sum_i\phi_i^{(m)}D_i^{\mathrm{ad}}\in K\otimes_L\epsilon_s\mathrm{Fil}^{-m}D$ for every m. Since $\sum_i\phi_i^{(m)}D_i^{\mathrm{ad}}=\sum_k f_k^{(m)}D_k$ ($f^{(m)}$ the t^m-coefficient of f) and the D_k form an L-basis of ϵ_sD, the last condition says $f^{(m)}\in K\otimes_L\mathrm{Fil}_s(-m)$ for every m: f is de Rham integral at s.

Step 3: a basis of the integral level. Let $\mathbf{n}'=(n'_j)_{j\le r}$ be an $e_c\mathbf{A}_{F,L}$-basis of $\mathbf{N}_c$ (Wm1). Then $(\Lambda_s(n'_j))_j$ is an R_s^+-basis of $M=\epsilon_s(B\otimes_{\mathbb{Z}_\ell}T)=\epsilon_s(B\otimes_{\mathbb{Q}_\ell}V)$. This is (Lm2) of Lemma 23.6, applied with the site $c-\mathrm{len}$ in place of its c, $e=\mathrm{len}$ and $a=c$: its λ_s, Λ_s, $\epsilon=\epsilon_{a-s-1}$ and R^+ are our λ_s, Λ_s, ϵ_s and R_s^+.

Step 4: proof of 2. Let $w\in W\subseteq e_c\mathbf{N}[1/\lambda_{\mathrm{len}}]$ and write $w=\sum_j g_jn'_j$ with $g_j\in e_c\mathbf{A}_{F,L}[1/\lambda_{\mathrm{len}}]$. Let $\mathcal{O}=e_c\mathbf{A}_{F,L}[1/\ell]\cong\mathcal{O}_L[[X]][1/\ell]$ (factorial; the $Q_{s'}$ squarefree and pairwise coprime) and, for $g\in\mathcal{O}[1/\lambda_{\mathrm{len}}]$, $\mathrm{ord}_s(g)$ the minimum of $v_\pi(g)$ over the prime factors π of Q_s. (a) $g\in e_c\mathbf{A}_{F,L}$ iff $\mathrm{ord}_s(g)\ge 0$ for all $s<\mathrm{len}$: if so, $g=g'/\lambda_{\mathrm{len}}^{K'}$ with $g'\in e_c\mathbf{A}_{F,L}$ and $\lambda_{\mathrm{len}}^{K'}$ divides g' in $\mathcal{O}$, so $g=\varpi^{-N}h_0$ with $h_0\in\mathcal{O}_L[[X]]$, ϖ a uniformiser of $\mathcal{O}_L$, and $\lambda_{\mathrm{len}}^{K'}h_0=\varpi^Ng'$; if $N>0$, reduction modulo ϖ gives $X^{K'\deg\lambda_{\mathrm{len}}}\bar{h}_0=0$ in the domain $k_L[[X]]$ (λ_{len} is distinguished), so $\varpi\mid h_0$, and induction on N gives $g\in\mathcal{O}_L[[X]]$; the converse is clear. (b) By (Lm3) of Lemma 23.6 (in the setting of Step 3, with $\alpha=0$; its $\mathcal{O}$ and ord_s are ours), $\lambda_s(g)\in R_s^+$ iff $\mathrm{ord}_s(g)\ge 0$. (c) $\Lambda_s(w)=\sum_j\lambda_s(g_j)\Lambda_s(n'_j)$ with $\lambda_s(g_j)\in R_s$, so by Step 3 (a basis of M is a basis of $M[1/t]$), $\Lambda_s(w)\in M$ iff $\lambda_s(g_j)\in R_s^+$ for all j. Combining (a)–(c): $w\in\mathbf{N}_c$ iff $\Lambda_s(w)\in\epsilon_s(B\otimes V)$ for every $s<\mathrm{len}$. Finally w is algebraic at every level with $\Lambda_s(w)=\sum_k\mathrm{lv}_s(w)_kD_k$ (Lemma 23.17 (Lp1)), and 1 turns the condition into de Rham integrality of $\mathrm{lv}_s(w)$. □

23.2 Existence of Wach modules, level maps and horizontal frames

The main result is Proposition 23.7: every crystalline lattice over a field between F and $F(\mu_\ell)$ has a Wach module whose window at each site is the window of its Hodge jumps at the corresponding embeddings, together with a realisation map into $A_{\mathrm{inf}}[1/\mu_\varepsilon]\otimes T$. The proof evaluates the prismatic F-crystal of the lattice [31] on the cyclotomic prism $(\mathbf{A}_F, Q\,\mathbf{A}_F)$, whose maps to Fontaine's ring are set up in Lemma 23.8. The lemmas that follow supply the tools of the later

lifting arguments: level maps, which evaluate a Wach module at the Frobenius twists of μ_ε and land in $B_{\mathrm{dR}} \otimes T$; horizontal sections and frames, which evaluate to bases of D_{cris} site by site; and the algebraicity of the levels (Lemma 23.17), which places the level expansions in $K_{(s)}((t))$ with an explicit Galois structure. These lemmas are technical; their point is integral control, with explicit denominators, at every site.

Lemma 23.6. *Let ℓ, F, L, $(\tau_b)_{b\in\mathbb{Z}/n}$ and a pair (V,T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$, $r = \dim_L V$, $D = D_{\mathrm{cris}}(V)$ and $B = B^+_{\mathrm{dR}}$. Fix $c \in \mathbb{Z}/n$, an integer $e \geq 1$, $a = c + e$, an integer s with $0 \leq s < e$, and put $q = e - s - 1$, $\epsilon = \epsilon_{a-s-1}$ and $R^+ = \epsilon(B \otimes_{\mathbb{Q}_\ell} L)$. Let $\mathcal{O} = e_a \mathbf{A}_{F,L}[1/\ell]$ ($\cong \mathcal{O}_L[[X]][1/\ell]$, a factorial domain), and for $g \in \mathcal{O}[1/\lambda_e]$ let $\mathrm{ord}_s(g) \in \mathbb{Z} \cup \{\infty\}$ be the minimum of $v_\pi(g)$ over the prime elements π of $\mathcal{O}$ dividing Q_s.*

(Lm1) Scalars *There is a unique ring homomorphism $\lambda_s \colon \mathbf{A}_{F,L}[1/\lambda_e] \to B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} L$ equal to $(\varphi^{-(s+1)} \otimes 1) \circ \iota_L$ on $\mathbf{A}_{F,L}$. It maps $\mathbf{A}_{F,L}$ into $B \otimes_{\mathbb{Q}_\ell} L$, $\lambda_s(e_a) = \epsilon$, $\lambda_s(Q_s) = \xi_0$, and $\lambda_s(X)$ and $\lambda_s(Q_{s'})$ ($s' < e$, $s' \neq s$) are units of B; $\lambda_s(\varphi^e(g)) = (\varphi^q \otimes 1)(\iota_L(g))$ for $g \in \mathbf{A}_{F,L}$.*

(Lm2) Vectors *There is a unique λ_s-semilinear map $\Lambda_s \colon \mathbf{N}[1/\lambda_e] \to B_{\mathrm{dR}} \otimes_{\mathbb{Z}_\ell} T$ with $\Lambda_s(x) = (\varphi^{-(s+1)} \otimes 1)(\iota_T(x))$ for $x \in \mathbf{N}$. It satisfies $\Lambda_s(\mathbf{N}) \subseteq B \otimes_{\mathbb{Z}_\ell} T$, $\Lambda_s(\varphi^e_{\mathbf{N}}(z)) = (\varphi^q \otimes 1)(\iota_T(z))$ for $z \in \mathbf{N}$, and for every $e_a\mathbf{A}_{F,L}$-basis $(n'_j)_{j\leq r}$ of $\mathbf{N}_a$ the vectors $\Lambda_s(n'_j)$ form a basis of the R^+-module $\epsilon(B \otimes_{\mathbb{Q}_\ell} V)$.*

(Lm3) Orders *For $g \in \mathcal{O}[1/\lambda_e]$ and $\alpha \in \mathbb{Z}$: $\lambda_s(g) \in \xi_0^\alpha R^+$ if and only if $\mathrm{ord}_s(g) \geq \alpha$.*

(Lm4) Lattice criterion *An element $y \in \mathbf{N}_a[1/\lambda_e] = e_a\mathbf{N}[1/\lambda_e]$ lies in $\mathbf{N}_a$ if and only if, for every $s' < e$, the map $\Lambda_{s'}$ (formed with s' in place of s) sends y into $\epsilon_{a-s'-1}(B \otimes_{\mathbb{Q}_\ell} V)$.*

(Lm5) Frobenius pull-backs *Let $\mathbf{n} = (n_k)_{k\leq r}$ be a frame at c, (H_k) the horizontal sections through it, (d_k) its crystalline frame, $D_k = \varphi^{e-s-4}(d_k) \in D$ (φ is bijective on D) and $\Lambda_D = \sum_k R^+ D_k$. Then $D_k \in \epsilon D$, $\Lambda_s(P_e(\mathbf{N}_c)) \subseteq \Lambda_D$, and for every integer $m \geq 1$, every κ admissible for $\mathbf{n}$ and m, and every k,*

$$\Lambda_s(\varphi^e_{\mathbf{N}}(T_\kappa(H_k))) - \ell^\kappa D_k \in \xi_0^m \Lambda_D.$$

Proof. Throughout, B is a complete discrete valuation ring with uniformiser ξ_0 and residue map θ, ℓ is invertible in B, $B_{\mathrm{dR}} = B[1/\xi_0]$ and $t \in \xi_0 B^\times$ (Theorem B.40 (Ai3)); $A_{\mathrm{inf}} \subseteq B$ compatibly with θ, and φ is a ring automorphism of A_{inf} with $\varphi([x]) = [x^\ell]$. Write $\zeta_{\ell^k} = (\varepsilon^{1/\ell^k})^\sharp$, a primitive ℓ^k-th root of unity, and $B_L = B \otimes_{\mathbb{Q}_\ell} L$, a free B-module of rank $[L : \mathbb{Q}_\ell]$, hence ξ_0-torsion free and ξ_0-adically separated; the same holds for its direct summand $R^+ = \epsilon B_L$, and $\epsilon(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} L) = R^+[1/\xi_0]$.

Step 1: the elements $\varphi^j(\xi_0)$ and $\varphi^j(\mu)$. We have $\varphi^j(\xi_0) = \sum_{k<\ell} [\varepsilon^{\ell^{j-1}}]^k$ and $\varphi^j(\mu) = [\varepsilon^{\ell^j}] - 1$ for $j \in \mathbb{Z}$. By (Ai1), $\theta(\varphi^j(\xi_0)) = \ell$ for $j \geq 1$ and $\theta(\varphi^j(\xi_0)) = \sum_{k<\ell} \zeta_{\ell^{1-j}}^k \neq 0$ for $j \leq -1$ (the sum of the powers of a primitive ℓ^{1-j}-th root of unity ζ, $1-j \geq 2$, equals $(\zeta^\ell - 1)/(\zeta - 1)$ and $\zeta^\ell \neq 1$); so $\varphi^j(\xi_0) \in B^\times$ for $j \neq 0$, as B is local with residue map θ. Likewise $\theta(\varphi^{-k}(\mu)) = \zeta_{\ell^k} - 1 \neq 0$ for $k \geq 1$, so $\varphi^{-k}(\mu) \in B^\times$, while $\theta(\varphi^j(\mu)) = 1 - 1 = 0$ for $j \geq 0$, so $\varphi^j(\mu) \in \xi_0 A_{\mathrm{inf}} \subseteq \xi_0 B$ by (Ai1).

Step 2: (Lm1). By Lemma 23.8 (4), $\iota(X) = \mu$, $\iota(Q) = \tilde{\xi} = \sum_{k<\ell} [\varepsilon]^k = \varphi(\xi_0)$ and $\iota \circ \varphi = \varphi \circ \iota$ (ι is a morphism of prisms), so $\iota(Q_{s'}) = \varphi^{s'+1}(\xi_0)$. Hence $(\varphi^{-(s+1)} \otimes 1)\iota_L$ sends X to $\varphi^{-(s+1)}(\mu) \in B^\times$, Q_s to ξ_0 and $Q_{s'}$ to $\varphi^{s'-s}(\xi_0) \in B^\times$ for $s' \neq s$ (Step 1). As $\lambda_e = \prod_{s'<e} Q_{s'}$ goes to ξ_0 times a unit, which is invertible in B_{dR}, the homomorphism $(\varphi^{-(s+1)} \otimes 1) \circ \iota_L \colon \mathbf{A}_{F,L} \to A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L \subseteq B_L$ extends uniquely to $\mathbf{A}_{F,L}[1/\lambda_e]$. Further $\lambda_s(e_a) = \varphi^{-(s+1)}(\iota_\Lambda(e_a)) = \iota_\Lambda(e_{a-s-1}) = \epsilon_{a-s-1}$, since

ι_Λ commutes with φ on $\mathcal{O}_F \otimes \mathcal{O}_L$ and $\varphi(e_b) = e_{b+1}$ (Definition 23.2). Finally $\lambda_s(\varphi^e(g)) = (\varphi^{-(s+1)}\varphi^e \otimes 1)(\iota_L(g)) = (\varphi^q \otimes 1)(\iota_L(g))$.

Step 3: (Lm2). For $x \in \mathbf{N}$, $\iota_T(x) \in A_{\mathrm{inf}}[1/\mu] \otimes T$ and $\varphi^{-(s+1)}$ maps $A_{\mathrm{inf}}[1/\mu]$ into $A_{\mathrm{inf}}[1/\varphi^{-(s+1)}(\mu)] \subseteq B$ (Step 1); so $x \mapsto (\varphi^{-(s+1)} \otimes 1)(\iota_T(x))$ maps $\mathbf{N}$ into $B \otimes_{\mathbb{Z}_\ell} T$, and it is λ_s-semilinear on $\mathbf{N}$ by (Ex2) ($\iota_T(fx) = \iota(f)\iota_T(x)$ for $f \in \mathbf{A}_F$, $\mathcal{O}_L$-linearity, and $\mathbf{A}_{F,L} = \mathbf{A}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$ as $\mathcal{O}_L$ is finite free over $\mathbb{Z}_\ell$). Since $\mathbf{N}[1/\lambda_e] = \mathbf{N} \otimes_{\mathbf{A}_{F,L}} \mathbf{A}_{F,L}[1/\lambda_e]$ and $\lambda_s(\lambda_e)$ is invertible, it extends uniquely to a λ_s-semilinear Λ_s on $\mathbf{N}[1/\lambda_e]$. *Frobenius.* The elements μ and $\iota(\lambda_e) = \prod_{s'<e} \varphi^{s'+1}(\xi_0)$ are units of $W(\mathbb{C}_\ell^\flat)$ (the image of μ modulo ℓ is $\varepsilon - 1 \neq 0$ in the field $\mathbb{C}_\ell^\flat$, and $\tilde{\xi}$ is a unit there by (Ex2), hence so are its Frobenius twists), so $A^\circ = A_{\mathrm{inf}}[1/\mu, 1/\iota(\lambda_e)] \subseteq W(\mathbb{C}_\ell^\flat)$, and ι_T extends ι_L-semilinearly to $\mathbf{N}[1/\lambda_e] \to A^\circ \otimes T$. Iterating the Frobenius clause of (Ex2) e times gives $\iota_T(\varphi^e_{\mathbf{N}}(z)) = (\varphi^e \otimes 1)(\iota_T(z))$ in $A^\circ \otimes T$ for $z \in \mathbf{N}$ (each $\varphi^j_{\mathbf{N}}(z)$, $j \leq e$, lies in $\mathbf{N}[1/\lambda_j]$, as $\varphi(Q_{s'}) = Q_{s'+1}$). By Steps 1 and 2 the localisation $\varphi^{-(s+1)}\colon A^\circ \to B_{\mathrm{dR}}$ is defined and restricts to λ_s on $\iota_L(\mathbf{A}_{F,L}[1/\lambda_e])$; thus $\Lambda_s = (\varphi^{-(s+1)} \otimes 1) \circ \iota_T$ on $\mathbf{N}[1/\lambda_e]$ (both are λ_s-semilinear and agree on $\mathbf{N}$), and $\Lambda_s(\varphi^e_{\mathbf{N}}(z)) = (\varphi^{e-s-1} \otimes 1)(\iota_T(z))$. *Basis.* By (Ex2), $A_{\mathrm{inf}}[1/\mu] \cdot \iota_T(\mathbf{N}) = A_{\mathrm{inf}}[1/\mu] \otimes T$; applying the ring automorphism $\varphi^{-(s+1)} \otimes 1$ and extending scalars to B, $1 \otimes T \subseteq B \cdot \Lambda_s(\mathbf{N})$, so $B \cdot \Lambda_s(\mathbf{N}) = B \otimes_{\mathbb{Z}_\ell} T = B \otimes_{\mathbb{Q}_\ell} V$. Since $\Lambda_s(e_b x) = \epsilon_{b-s-1}\Lambda_s(x)$ (Step 2 for b in place of a) and the ϵ_b are orthogonal idempotents with sum 1, multiplying by ϵ gives $\epsilon(B \otimes V) = B \cdot \Lambda_s(\mathbf{N}_a)$. Now $\Lambda_s(\mathbf{N}_a) = \sum_j \lambda_s(e_a \mathbf{A}_{F,L})\Lambda_s(n'_j) \subseteq \sum_j R^+ \Lambda_s(n'_j)$, and R^+ contains the image of B acting on $\epsilon(B \otimes V)$; so the r vectors $\Lambda_s(n'_j)$ generate the R^+-module $\epsilon(B \otimes_{\mathbb{Q}_\ell} V) = R^+ \otimes_L V$, which is free of rank r. A surjective endomorphism of a finitely generated module over a commutative ring is bijective, so the map $(R^+)^r \to R^+ \otimes_L V$, $(x_j) \mapsto \sum_j x_j \Lambda_s(n'_j)$, is an isomorphism, i.e. the $\Lambda_s(n'_j)$ form a basis.

Step 4: (Lm3). The $Q_{s'}$ ($s' < e$) are squarefree and pairwise coprime in $\mathcal{O}$: they are monic polynomials (Lemma 23.8 (3)), products of distinct irreducible distinguished polynomials, and distinct cyclotomic polynomials have no common root. Let $g \in \mathcal{O}[1/\lambda_e]$, $g \neq 0$, and $\beta = \mathrm{ord}_s(g)$. Since Q_s is squarefree, $g = Q_s^\beta g_0$ with $g_0 \in \mathcal{O}[1/Q_{s'} : s' \neq s,\ s' < e]$ and $g_0 \notin Q_s\mathcal{O}[1/Q_{s'} : s' \neq s]$. By (Lm1), $\lambda_s(\mathcal{O}) \subseteq \epsilon B_L = R^+$ (note $\lambda_s(1/\ell) = 1/\ell$) and $\lambda_s(Q_{s'}) \in B^\times$ for $s' \neq s$, so $\lambda_s(g_0) \in R^+$ and $\lambda_s(g) = \xi_0^\beta \lambda_s(g_0)$. If $\beta \geq \alpha$ this lies in $\xi_0^\alpha R^+$. Conversely it suffices to show $\lambda_s(g_0) \notin \xi_0 R^+$, for then $\lambda_s(g) \notin \xi_0^{\beta+1} R^+ \supseteq \xi_0^\alpha R^+$ when $\beta < \alpha$ (R^+ is ξ_0-torsion free). Consider $\bar{\lambda} = \theta \circ \lambda_s\colon \mathcal{O} \to R^+/\xi_0 R^+ = \epsilon(\mathbb{C}_\ell \otimes_{\mathbb{Q}_\ell} L)$. On $e_a(\mathcal{O}_F \otimes \mathcal{O}_L)[1/\ell] = e_a(1 \otimes L) \cong L$ it is the L-linear map $\lambda \mapsto \epsilon(1 \otimes \lambda)$, and it sends X to $\epsilon(\zeta_{\ell^{s+1}} - 1)$ (Step 1); it is continuous for the X-adic topology on $\mathcal{O}_L[[X]]$ since $\zeta_{\ell^{s+1}} - 1$ has positive valuation. By Weierstrass division by the distinguished polynomial Q_s, $\mathcal{O}/Q_s\mathcal{O} \cong L[X]/(Q_s) \cong \mathbb{Q}_\ell(\zeta_{\ell^{s+1}}) \otimes_{\mathbb{Q}_\ell} L$, and $\bar{\lambda}$ vanishes on Q_s and induces the map $\mathbb{Q}_\ell(\zeta_{\ell^{s+1}}) \otimes_{\mathbb{Q}_\ell} L \to \epsilon(\mathbb{C}_\ell \otimes_{\mathbb{Q}_\ell} L)$, $x \otimes \lambda \mapsto \epsilon(x \otimes \lambda)$. Write $\epsilon(\mathbb{C}_\ell \otimes_{\mathbb{Q}_\ell} L) = \mathbb{C}_\ell \otimes_F L$, where $F \subset \mathbb{C}_\ell$ canonically and F acts on L through the embedding attached to the idempotent ϵ of $F \otimes_{\mathbb{Q}_\ell} L$. The induced map is $(\mathbb{Q}_\ell(\zeta_{\ell^{s+1}}) \otimes_{\mathbb{Q}_\ell} F) \otimes_F L \to \mathbb{C}_\ell \otimes_F L$; here $\mathbb{Q}_\ell(\zeta_{\ell^{s+1}}) \otimes_{\mathbb{Q}_\ell} F \cong F(\zeta_{\ell^{s+1}}) \subset \mathbb{C}_\ell$ is a field ($F/\mathbb{Q}_\ell$ unramified, $\mathbb{Q}_\ell(\zeta_{\ell^{s+1}})/\mathbb{Q}_\ell$ totally ramified) and L is flat over F, so the map is injective. Hence $\ker \bar{\lambda} = Q_s\mathcal{O}$, and $\bar{\lambda}(Q_{s'}) \neq 0$ is a unit in each factor for $s' \neq s$; as $g_0 = g_1 / \prod_{s' \neq s} Q_{s'}^k$ with $g_1 \in \mathcal{O} \setminus Q_s\mathcal{O}$, $\theta(\lambda_s(g_0)) \neq 0$, i.e. $\lambda_s(g_0) \notin \xi_0 R^+$.

Step 5: (Lm4). If $y \in \mathbf{N}_a$ then $\Lambda_{s'}(y) \in \Lambda_{s'}(e_a\mathbf{N}) \subseteq \epsilon_{a-s'-1}(B \otimes V)$ by Step 3. Conversely let (n'_j) be a basis of $\mathbf{N}_a$ and $y = \sum_j g_j n'_j$, $g_j \in e_a\mathbf{A}_{F,L}[1/\lambda_e]$. For each $s' < e$, $\Lambda_{s'}(y) = \sum_j \lambda_{s'}(g_j)\Lambda_{s'}(n'_j)$ with $\lambda_{s'}(g_j) \in \epsilon_{a-s'-1}(B_{\mathrm{dR}} \otimes L)$; by the basis property (Lm2) the hypothesis forces $\lambda_{s'}(g_j) \in \epsilon_{a-s'-1}B_L$, so $\mathrm{ord}_{s'}(g_j) \geq 0$ by (Lm3). Write $g_j = h_j/\lambda_e^k$ with $h_j \in e_a\mathbf{A}_{F,L}$. Every prime of $\mathcal{O}$ dividing λ_e divides exactly one $Q_{s'}$, to the first power, so $\mathrm{ord}_{s'}(g_j) \geq 0$ for all $s' < e$ says that λ_e^k divides h_j in $\mathcal{O}$: $h_j = \lambda_e^k u_j$ with $u_j \in \mathcal{O}$. Let ϖ be a uniformiser of $\mathcal{O}_L$ and $i \geq 0$ minimal with $\varpi^i u_j \in e_a\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$. If $i > 0$, reduce $\varpi^i h_j = \lambda_e^k(\varpi^i u_j)$ modulo ϖ: $\lambda_e^k \equiv X^d$ ($d = k \deg \lambda_e$, Lemma 23.8 (3)) and $k_L[[X]]$ is a domain, so $\varpi^i u_j \equiv 0$, contradicting minimality. So $u_j = g_j \in e_a\mathbf{A}_{F,L}$ and $y \in \mathbf{N}_a$.

Step 6: (Lm5). Let $\mathbf{h} = (h_{kj}) \in M_r(e_cE[[X]])$ with $H_k = \sum_j h_{kj} n_j$; by (Hf2) of Lemma 23.9 the H_k are an $e_cE[[X]]$-basis of $e_c\mathbf{N}_E$, so $\mathbf{h}$ is invertible and $n_j = \sum_k (\mathbf{h}^{-1})_{jk} H_k$. The map $\mathrm{ev}^{(q)}$ of Lemma 23.13 is a ring homomorphism (a ξ_0-adically convergent substitution, item 1 there) with $\mathrm{ev}^{(q)}(X) = \varphi^q(\mu)$, and $\mathrm{ev}^{(q)}(e_cE) \subseteq \varphi^q(\iota_\Lambda(e_c))B_L = \epsilon_{c+q}B_L = R^+$ since $c+q = a-s-1$; so $\mathrm{ev}^{(q)}(e_cE[[X]]) \subseteq R^+$. By its definition $G^{(q)}$ is additive with $G^{(q)}(gW) = \mathrm{ev}^{(q)}(g)G^{(q)}(W)$ for $g \in e_cE[[X]]$ and $G^{(q)}(n_j) = (\varphi^q \otimes 1)(\iota_T(n_j))$, and by Lemma 23.13 (1) $G^{(q)}(H_k) = \varphi^{q-3}(d_k) = D_k$. Hence $D_k = \sum_j \mathrm{ev}^{(q)}(h_{kj})(\varphi^q \otimes 1)(\iota_T(n_j)) \in \epsilon(B_{\mathrm{dR}} \otimes V)$, so $D_k \in \epsilon D$ (the idempotent ϵ of $F \otimes L$ acts on $D \subseteq B_{\mathrm{cris}} \otimes V \subseteq B_{\mathrm{dR}} \otimes V$ compatibly), and

$$(\varphi^q \otimes 1)(\iota_T(n_j)) = G^{(q)}(n_j) = \sum_k \mathrm{ev}^{(q)}((\mathbf{h}^{-1})_{jk}) D_k \in \Lambda_D. \tag{1}$$

Pull-backs. $P_e(\mathbf{N}_c)$ is generated over $\mathbf{A}_{F,L}$ by the $\varphi^e_{\mathbf{N}}(n_j)$ (as $\varphi^e_{\mathbf{N}}(\sum f_j n_j) = \sum \varphi^e(f_j)\varphi^e_{\mathbf{N}}(n_j)$) and lies in $e_a\mathbf{N}[1/\lambda_\infty]$, so each $p \in P_e(\mathbf{N}_c)$ is $\sum_j g_j \varphi^e_{\mathbf{N}}(n_j)$ with $g_j \in e_a\mathbf{A}_{F,L}$; then $\Lambda_s(p) = \sum_j \lambda_s(g_j)G^{(q)}(n_j) \in \Lambda_D$ by (Lm2), (1) and $\lambda_s(g_j) \in \epsilon B_L = R^+$. *Truncations.* Write $T_\kappa(H_k) = \sum_j p_{kj} n_j$ with the polynomials $p_{kj} = \sum_{i<m} \ell^\kappa h_{kj,i} X^i \in e_c\mathbf{A}_{F,L}$, and $\ell^\kappa h_{kj} = p_{kj} + X^m \sigma_{kj}$ with $\sigma_{kj} \in e_cE[[X]]$. By (Lm2) and (Lm1), $\Lambda_s(\varphi^e_{\mathbf{N}}(T_\kappa H_k)) = \sum_j (\varphi^q \otimes 1)(\iota_L(p_{kj}))\, G^{(q)}(n_j)$, and $(\varphi^q \otimes 1)(\iota_L(p_{kj})) = \sum_{i<m} \varphi^q(\iota_\Lambda(\ell^\kappa h_{kj,i}))\varphi^q(\mu)^i = \mathrm{ev}^{(q)}(p_{kj})$ (a finite sum; $\iota_L = \iota_\Lambda$ on coefficients and $\iota_L(X) = \mu$). Hence

$$\Lambda_s(\varphi^e_{\mathbf{N}}(T_\kappa H_k)) = G^{(q)}\Big(\sum_j p_{kj} n_j\Big) = \ell^\kappa G^{(q)}(H_k) - \varphi^q(\mu)^m \sum_j \mathrm{ev}^{(q)}(\sigma_{kj}) G^{(q)}(n_j) \in \ell^\kappa D_k + \xi_0^m \Lambda_D,$$

by (1), $\mathrm{ev}^{(q)}(\sigma_{kj}) \in R^+$ and $\varphi^q(\mu) \in \xi_0 B$ (Step 1, $q \geq 0$). □

Proposition 23.7. *Let ℓ be an odd prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of degree n, L a finite extension of $\mathbb{Q}_\ell$ containing $\sigma(F(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $F(\mu_\ell)$, $(\tau_a)_{a\in\mathbb{Z}/n}$ a Frobenius numbering, and K a field with $F \subseteq K \subseteq F(\mu_\ell)$. Call a* pair over K *a pair (V,T) of a finite-dimensional L-vector space V with a continuous L-linear action of G_K, crystalline as a $\mathbb{Q}_\ell$-representation, and a G_K-stable $\mathcal{O}_L$-lattice $T \subseteq V$; a* pair *is a pair over F. To every pair (V,T) over K one can attach a finitely generated $\mathbf{A}_{F,L}$-module $\mathbf{N}(T)$ with maps $\varphi_{\mathbf{N}}$, $\gamma_{u,\mathbf{N}}$ and an injective additive map $\iota_T\colon \mathbf{N}(T) \to A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes_{\mathbb{Z}_\ell} T$ such that:*

(Ex1) Window *if $M_a, h_a \geq 0$ are integers such that every Hodge jump of V at every embedding of K over τ_a lies in $[-M_a, h_a]$ ($a \in \mathbb{Z}/n$), then $\mathbf{N}(T)$ is a Wach module over K with coefficients L and window $(M_a,h_a)_a$, with $r = \dim_L V$ in (Wm1); in particular $(\gamma_{\delta,\mathbf{N}} - 1)(\mathbf{N}(T)) \subseteq X\mathbf{N}(T)$ for every $\delta \in \Delta_K$;*

(Ex2) Realisation *$\iota_T(fx) = \iota(f)\,\iota_T(x)$ for $f \in \mathbf{A}_F$, $\iota_T(\lambda x) = (1 \otimes \lambda)\iota_T(x)$ for $\lambda \in \mathcal{O}_L$, $\iota_T(\varphi_{\mathbf{N}}(x)) = (\varphi \otimes 1)(\iota_T(x))$ (in $W(\mathbb{C}^\flat_\ell) \otimes_{\mathbb{Z}_\ell} T$, where $\tilde{\xi} = \iota(Q)$ is a unit, ι_T being extended to $\mathbf{N}(T)[1/Q]$), $\sigma(\iota_T(x)) = \iota_T(\gamma_{\chi_{\mathrm{cyc}}(\sigma),\mathbf{N}}(x))$ for $\sigma \in G_K$ acting diagonally (so $\iota_T(\mathbf{N}(T))$ is fixed by $H_F \subseteq G_K$), and $A_{\mathrm{inf}}[1/\mu_\varepsilon] \cdot \iota_T(\mathbf{N}(T)) = A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes_{\mathbb{Z}_\ell} T$;*

(Ex3) Functoriality *for another pair (V',T') over K and every G_K-equivariant $\mathcal{O}_L$-linear map $f\colon T \to T'$ there is an $\mathbf{A}_{F,L}$-linear map $\mathbf{N}(f)\colon \mathbf{N}(T) \to \mathbf{N}(T')$ commuting with $\varphi_{\mathbf{N}}$ and all $\gamma_{u,\mathbf{N}}$, with $\iota_{T'} \circ \mathbf{N}(f) = (1 \otimes f) \circ \iota_T$, $\mathbf{N}(\mathrm{id}) = \mathrm{id}$ and $\mathbf{N}(f' \circ f) = \mathbf{N}(f') \circ \mathbf{N}(f)$;*

(Ex4) Level *s more generally, let ℓ be any prime, $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, L a finite extension of $\mathbb{Q}_\ell$ containing $\sigma(F(\mu_{\ell^s}))$ for every $\mathbb{Q}_\ell$-embedding σ of $F(\mu_{\ell^s})$, and K a field with $F \subseteq K \subseteq F(\mu_{\ell^s})$, pairs over K being defined as above. Then to every pair (V,T) over K one can attach a finitely generated $\mathbf{A}_{F,L}$-module $\mathbf{N}_s(T)$ with maps $\varphi_{\mathbf{N}}$, $\gamma_{u,\mathbf{N}}$*

($u \in \Gamma_K$) and an injective additive map $\iota_{s,T}\colon \mathbf{N}_s(T) \to A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes_{\mathbb{Z}_\ell} T$ satisfying (Ex1)–(Ex3) with Wach module replaced by Wach module of level s (Definition 23.2, paragraph Level s), $\mathbf{N}(T)$ by $\mathbf{N}_s(T)$, Q by Q_{s-1}, ι by the map ι_s of Lemma 23.8(6) and ι_T by $\iota_{s,T}$. For $s = 1$ and ℓ odd one may take $\mathbf{N}_1(T) = \mathbf{N}(T)$.

(Ex5) Origin *the modules and maps of (Ex1)–(Ex4) can be chosen to arise from prismatic data, as follows. For ℓ, s, L, K as in (Ex4) and every pair (V,T) over K there are a datum $\mathcal{E}_T$ given by Theorem B.41 (for the field K and $p = \ell$) for the $\mathbb{Z}_\ell$-module T with its G_K-action, and an isomorphism c_T with the properties of (Bs3) for $\mathcal{E}_T$, such that, for the object $(\mathbf{A}_F, Q_{s-1}\mathbf{A}_F)$ over $\mathcal{O}_K$ of Lemma 23.8(6): $\mathbf{N}_s(T) = \mathcal{E}_T(\mathbf{A}_F)$; $\varphi_{\mathbf{N}}(x) = \varphi_{\mathcal{E}_T,\mathbf{A}_F}(1 \otimes x)$; $\gamma_{u,\mathbf{N}} = \mathcal{E}_T(\gamma_u)$ for $u \in \Gamma_K$; $\iota_{s,T} = c_T \circ \mathcal{E}_T(\iota_s)$; each $\lambda \in \mathcal{O}_L$ acts on $\mathbf{N}_s(T)$ by $\mathcal{E}(\lambda)_{\mathbf{A}_F}$, the map of (Bs5) for multiplication by λ on T; and $\mathbf{N}_s(f) = \mathcal{E}(f)_{\mathbf{A}_F}$, the map of (Bs5), for f as in (Ex3). For $s = 1$ and ℓ odd this holds with $\mathbf{N}(T) = \mathbf{N}_1(T)$, $\iota_T = \iota_{1,T}$, $\iota_1 = \iota$ and $Q_0 = Q$.*

Proof. We prove (Ex4) for every admissible (ℓ, s); (Ex1)–(Ex3) are its case $s = 1$, ℓ odd, treated in the last paragraph. Fix a prime ℓ, an integer $s \geq 1$ with $s \geq 2$ if $\ell = 2$, a finite extension L of $\mathbb{Q}_\ell$ containing $\sigma(F(\mu_{\ell^s}))$ for every $\mathbb{Q}_\ell$-embedding σ of $F(\mu_{\ell^s})$, and a field K with $F \subseteq K \subseteq F(\mu_{\ell^s})$. Write $P = Q_{s-1} = \Phi_{\ell^s}(1+X)$ (Lemma 23.8(3)). We use the following facts, all from Lemma 23.8: (i) $(\mathbf{A}_F, P\mathbf{A}_F)$ and $(\mathcal{O}_F, \ell\mathcal{O}_F)$ are objects over $\mathcal{O}_K$, and red, ι_s and γ_u ($u \in \Gamma_K$) are morphisms over $\mathcal{O}_K$ (item 6, which gives item 5 for level s); (ii) ι_s is injective, $\iota_s(P) = \tilde{\xi}$ and $\iota_s^{-1}(\tilde{\xi}^k A_{\mathrm{inf}}) = P^k\mathbf{A}_F$ for $k \geq 0$ (item 6); (iii) $\iota_s = \varphi \circ w$ on $\mathcal{O}_F$ (item 6); (iv) $\sigma \circ \iota_s = \iota_s \circ \gamma_{\chi_{\mathrm{cyc}}(\sigma)}$ and $\chi_{\mathrm{cyc}}(\sigma) \in \Gamma_K$ for $\sigma \in G_K$ (item 6); (v) $\gamma_u(P) \in P\,\mathbb{Z}_\ell[[X]]^\times$, $\gamma_u\gamma_v = \gamma_{uv}$, $\gamma_1 = \mathrm{id}$, $\gamma_u \circ \varphi = \varphi \circ \gamma_u$ and $\mathrm{red} \circ \gamma_u = \mathrm{red}$ for $u, v \in \mathbb{Z}_\ell^\times$ (item 2 and its proof, which hold for every prime by the paragraph 'No oddness' there). Moreover (vi) L contains $\sigma(K)$ for every $\mathbb{Q}_\ell$-embedding σ of K, since $K \subseteq F(\mu_{\ell^s})$ and every such σ extends to $F(\mu_{\ell^s})$.

The construction. Apply Theorem B.41 to the field K and $p = \ell$ (it holds for every prime). For each pair (V,T) over K fix a datum $\mathcal{E}_T$ attached to T (a free $\mathbb{Z}_\ell$-module of rank $d = [L : \mathbb{Q}_\ell] \dim_L V$), and let $c_T\colon \mathcal{E}_T(A_{\mathrm{inf}}) \otimes W(\mathbb{C}_\ell^\flat) \to T \otimes W(\mathbb{C}_\ell^\flat)$ be the isomorphism of (Bs3). Put $\mathbf{N} = \mathbf{N}_s(T) = \mathcal{E}_T(\mathbf{A}_F)$ for the object $(\mathbf{A}_F, P\mathbf{A}_F)$ of (i), $M = \mathcal{E}_T(A_{\mathrm{inf}})$, $\varphi_{\mathbf{N}}(x) = \varphi_{\mathcal{E},\mathbf{A}_F}(1 \otimes x) \in \mathbf{N}[1/P]$, $\gamma_{u,\mathbf{N}} = \mathcal{E}_T(\gamma_u)$ ($u \in \Gamma_K$), $\iota_{s,T} = c_T \circ \mathcal{E}_T(\iota_s)$, and $\mathbf{N}_s(f) = \mathcal{E}(f)_{\mathbf{A}_F}$ with $\mathcal{E}(f)$ from (Bs5). Since $\mathbf{A}_F$ is local, the finite projective module $\mathbf{N}$ is free of rank d; let (x_j) be a basis. By (Bs2) for ι_s, M is free on $m_j = \mathcal{E}_T(\iota_s)(x_j)$; since ι_s is injective (ii), $\mathcal{E}_T(\iota_s)$ is injective, and c_T is injective on M ($A_{\mathrm{inf}} \subset W(\mathbb{C}_\ell^\flat)$), so $\iota_{s,T}$ is injective; by (Bs3) its image lies in $A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes T$ and $A_{\mathrm{inf}}[1/\mu_\varepsilon]\iota_{s,T}(\mathbf{N}) = c_T(M[1/\mu_\varepsilon]) = A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes T$.

The $\mathcal{O}_L$-action and the sites. For $\lambda \in \mathcal{O}_L$ multiplication by λ is a G_K-equivariant endomorphism of T; let λ act on $\mathcal{E}_T(A)$ by $\mathcal{E}(\lambda)_A$. This is a ring action commuting with A: by (Bs5), $c_T \circ (\mathcal{E}(g)_{A_{\mathrm{inf}}} \otimes 1) = (g \otimes 1) \circ c_T$ for every endomorphism g, so on M (which embeds into $M \otimes W(\mathbb{C}_\ell^\flat)$) the maps $\mathcal{E}(\lambda+\lambda')$ and $\mathcal{E}(\lambda)+\mathcal{E}(\lambda')$ agree, as do $\mathcal{E}(\lambda\lambda')$ and $\mathcal{E}(\lambda)\mathcal{E}(\lambda')$, and $\mathcal{E}(\lambda)$ for $\lambda \in \mathbb{Z}_\ell$ and multiplication by λ; as $\mathcal{E}_T(\iota_s)$ is injective and intertwines $\mathcal{E}(g)_{\mathbf{A}_F}$ with $\mathcal{E}(g)_{A_{\mathrm{inf}}}$ (Bs5), the same holds on $\mathbf{N}$. So $\mathbf{N}$ is a module over $\mathbf{A}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$; being finite over $\mathbf{A}_F$ it is X-adically complete and separated, so $\mathbf{A}_{F,L} = (\mathcal{O}_F \otimes \mathcal{O}_L)[[X]]$ acts. By (Bs5) the $\mathcal{O}_L$-action commutes with $\varphi_{\mathbf{N}}$ and with every $\gamma_{u,\mathbf{N}}$, so $\varphi_{\mathbf{N}}$ is semilinear for $\varphi \otimes \mathrm{id}$ and $\gamma_{u,\mathbf{N}}$ for γ_u on $\mathbf{A}_{F,L}$, and the clause of (Ex2) for $\lambda \in \mathcal{O}_L$ holds. Each $\mathbf{N}_a = e_a\mathbf{N}$ is a direct summand of the free $\mathbf{A}_F$-module $\mathbf{N}$, hence free over the local ring $\mathbf{A}_F$; it is a finitely generated module over $R = e_a\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ (with $\mathcal{O}_F$ acting through τ_a), a regular local ring of dimension 2, finite and free over $\mathbf{A}_F$. Since (ℓ, X) is a regular sequence of $\mathbf{A}_F$, it is regular on $\mathbf{N}_a$, and it generates an ideal primary to the maximal ideal of R; so $\mathrm{depth}_R\mathbf{N}_a = 2$ and, by the Auslander–Buchsbaum formula, $\mathbf{N}_a$ is a free R-module, of rank r_a say.

Frobenius moves the sites. For $x \in \mathcal{O}_F$, $\lambda \in \mathcal{O}_L$ and $y \in \mathbf{N}$, $\varphi_{\mathbf{N}}((x \otimes \lambda)y) = (\varphi(x) \otimes \lambda)\varphi_{\mathbf{N}}(y)$, and $(\varphi \otimes \mathrm{id})(e_{a-1}) = e_a$ because $\tau_a(\varphi(x)) = \tau_{a-1}(x)$; hence $\varphi_{\mathbf{N}}(\mathbf{N}_{a-1}) \subseteq e_a\mathbf{N}[1/P]$. On M let $\mathcal{O}_F \otimes \mathcal{O}_L$ act with $\mathcal{O}_F$ through ι_s; its idempotents e_a^M satisfy $\mathcal{E}_T(\iota_s)(\mathbf{N}_a) \subseteq e_a^M M =: M_a = A_{\mathrm{inf}}\mathcal{E}_T(\iota_s)(\mathbf{N}_a)$, and, since $\varphi \circ \iota_s = \iota_s \circ \varphi$ and φ_M commutes with $\mathcal{O}_L$, $\varphi_M(1 \otimes e_{a-1}^M m) = e_a^M \varphi_M(1 \otimes m)$.

Heights at A_{inf}, site by site. Let V have its Hodge jumps at every embedding of K over τ_a in $[-M_a, h_a]$. The Witt vector Frobenius φ is a G_K-equivariant automorphism of A_{inf} with $\varphi(\xi_0) = \tilde{\xi}$; it induces isomorphisms $A_{\mathrm{inf}}[1/\ell]/\xi_0^b \to A_{\mathrm{inf}}[1/\ell]/\tilde{\xi}^b$, hence, by (Ai3), a G_K-equivariant ring isomorphism $\Phi\colon B_{\mathrm{dR}}^+ \to B'^+$ extending φ, with $\Phi(tB_{\mathrm{dR}}^+) = \tilde{\xi}B'^+$; so B'^+ is a discrete valuation ring with uniformiser $\tilde{\xi}$ and $B'^+/\tilde{\xi} = A_{\mathrm{inf}}[1/\ell]/\tilde{\xi}$. The field $F \subset B_{\mathrm{dR}}^+$ of (Ai3) is $w(\mathcal{O}_F)[1/\ell]$ (both lift the inclusion $F \subset \mathbb{C}_\ell$ through θ, and such a lift is unique as $F/\mathbb{Q}_\ell$ is étale and B_{dR}^+ is complete), and $\Phi(w(x)) = \varphi(w(x)) = \iota_s(x)$ by (iii). Let $D = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V)^{G_K}$ with the filtration of (Ai4) (for the finite extension K), a $K \otimes_{\mathbb{Q}_\ell} L$-module with filtration by submodules, and $D_a = e_a D$, where $e_a \in F \otimes_{\mathbb{Q}_\ell} L \subseteq K \otimes_{\mathbb{Q}_\ell} L$. Since $K \otimes_{\mathbb{Q}_\ell} L = K \otimes_F (F \otimes_{\mathbb{Q}_\ell} L)$, $e_a(K \otimes_{\mathbb{Q}_\ell} L) = K \otimes_{F,\tau_a} L$; writing $K = F(y)$ with minimal polynomial R_y over F, this is $L[Y]/(\tau_a R_y)$, and by (vi) $\tau_a R_y$ splits into distinct linear factors $Y - \tau(y)$ over L, one for each embedding τ of K over τ_a. So $e_a(K \otimes_{\mathbb{Q}_\ell} L) \cong \prod_\tau L$, the product over the embeddings τ of K over τ_a, and, as the filtration is by $K \otimes_{\mathbb{Q}_\ell} L$-submodules, $D_a = \bigoplus_\tau D_{\mathrm{dR},\tau}(V)$ as filtered modules. Applying $1 \otimes \Phi$ and e_a to (Ai4) (with $k = 0$) gives in $V \otimes B'_{\mathrm{dR}}$, where $x \otimes \lambda$ acts as $\iota_s(x) \otimes \lambda$,

$$\sum_{j \in \mathbb{Z}} \tilde{\xi}^{-j} B'^+ \Phi(\mathrm{Fil}^j D_a) = e_a(V \otimes B'^+), \qquad \Phi(D) = (V \otimes B'_{\mathrm{dR}})^{G_K}, \qquad e_a\Phi(D) = \Phi(D_a).$$

Every $D_{\mathrm{dR},\tau}(V)$, τ over τ_a, is finite-dimensional with an exhaustive and separated filtration whose jumps lie in $[-M_a, h_a]$; so $\mathrm{Fil}^{-M_a} D_a = D_a$ and $\mathrm{Fil}^{h_a+1} D_a = 0$, and the term $j = -M_a$ gives $\tilde{\xi}^{M_a} B'^+ \Phi(D_a) \subseteq e_a(V \otimes B'^+)$, and only $j \le h_a$ contribute, so $e_a(V \otimes B'^+) \subseteq \tilde{\xi}^{-h_a} B'^+ \Phi(D_a)$. The isomorphism $M \otimes B'_{\mathrm{dR}} \cong T \otimes B'_{\mathrm{dR}}$ of (Bs4) is A_{inf}-linear and $\mathcal{O}_L$-linear (Bs5), so it carries e_a^M to e_a; it carries the image Λ_0 of $M \otimes B'^+$ onto $T \otimes B'^+$ and the B'^+-span Λ_1 of the $\varphi_M(1 \otimes m)$ onto $B'^+\Phi(D)$. With $e_a\Lambda_1 = B'^+\varphi_M(1 \otimes M_{a-1})$ (previous paragraph) we get

$$\tilde{\xi}^{M_a}\, B'^+ \varphi_M(1 \otimes M_{a-1}) \subseteq e_a\Lambda_0, \qquad \tilde{\xi}^{h_a}\, e_a\Lambda_0 \subseteq B'^+\varphi_M(1 \otimes M_{a-1}). \qquad (*)$$

Descent from B'^+: $A_{\mathrm{inf}} \cap \tilde{\xi}^k B'^+ = \tilde{\xi}^k A_{\mathrm{inf}}$. For $k = 1$: if $a \in A_{\mathrm{inf}}$ maps to 0 in $B'^+/\tilde{\xi} = A_{\mathrm{inf}}[1/\ell]/\tilde{\xi}$, then $\ell^m a \in \tilde{\xi}A_{\mathrm{inf}} = \ker\tilde{\theta}$ for some m, so $\tilde{\theta}(a) = 0$ ($\mathcal{O}_{\mathbb{C}_\ell}$ has no ℓ-torsion) and $a \in \tilde{\xi}A_{\mathrm{inf}}$; for $k \ge 2$ write $a = \tilde{\xi}a_1$ and use that B'^+ is a domain. Hence, for a free A_{inf}-module P', an element of $P'[1/\tilde{\xi}]$ whose image in $P' \otimes B'_{\mathrm{dR}}$ lies in the image of $P' \otimes B'^+$ lies in P' (compare coordinates). Apply this to $P' = M$ and to $P' = A_{\mathrm{inf}}\varphi_M(1 \otimes M)$, the image of the free module φ^*M, which maps injectively (φ_M is an isomorphism after inverting $\tilde{\xi}$, and $\tilde{\xi}$ is a non-zero-divisor) and has $P' \otimes B'_{\mathrm{dR}} = M \otimes B'_{\mathrm{dR}}$ with $P' \otimes B'^+$ mapping onto Λ_1. By $(*)$, $\tilde{\xi}^{M_a}\varphi_M(1 \otimes m) \in M$ for $m \in M_{a-1}$, and $\tilde{\xi}^{h_a} m \in P'$ for $m \in M_a$; taking e_a^M-components,

$$\tilde{\xi}^{M_a}\varphi_M(1 \otimes M_{a-1}) \subseteq M_a, \qquad \tilde{\xi}^{h_a} M_a \subseteq A_{\mathrm{inf}}\varphi_M(1 \otimes M_{a-1}).$$

Heights at $\mathbf{A}_F$: (Wm2) of level s. The map $\mathcal{E}_T(\iota_s)$ extends to $\mathbf{N}[1/P] \to M[1/\tilde{\xi}]$ (as $\iota_s(P) = \tilde{\xi}$) and intertwines $\varphi_{\mathbf{N}}$ with $m \mapsto \varphi_M(1 \otimes m)$ (Bs2). Let $x \in \mathbf{N}_{a-1}$ and $z = P^{M_a}\varphi_{\mathbf{N}}(x) = \sum_j (b_j/P^k)x_j$ with $b_j \in \mathbf{A}_F$. Then $\mathcal{E}_T(\iota_s)(z) = \sum_j (\iota_s(b_j)/\tilde{\xi}^k)m_j \in M$, so $\iota_s(b_j) \in \tilde{\xi}^k A_{\mathrm{inf}}$ and, by (ii), $b_j \in P^k\mathbf{A}_F$: $z \in \mathbf{N}$, and $z = e_a z \in \mathbf{N}_a$. Likewise $P_{\mathbf{N}} = \mathbf{A}_F\varphi_{\mathbf{N}}(\mathbf{N})$ is free on the $\varphi_{\mathbf{N}}(x_j)$ (the image of the free module $\mathbf{A}_F \otimes_\varphi \mathbf{N}$, injective as before), $\mathcal{E}_T(\iota_s)$ maps this basis to the basis $\varphi_M(1 \otimes m_j)$ of $A_{\mathrm{inf}}\varphi_M(1 \otimes M)$, and the same comparison of coordinates, with (ii), shows $P^{h_a}\mathbf{N}_a \subseteq P_{\mathbf{N}}$. As $P_{\mathbf{N}}$ is stable under $\mathcal{O}_L$, $e_a P_{\mathbf{N}} = \mathbf{A}_{F,L}\varphi_{\mathbf{N}}(\mathbf{N}_{a-1})$, which gives $P^{h_a}\mathbf{N}_a \subseteq \mathbf{A}_{F,L}\varphi_{\mathbf{N}}(\mathbf{N}_{a-1})$. These are the two inclusions of (Wm2) with $Q_{s-1} = P$ in place of Q.

Ranks: (Wm1). By (Wm2),

$$P^{h_a+M_a}\mathbf{N}_a \subseteq P^{M_a}\mathbf{A}_{F,L}\varphi_{\mathbf{N}}(\mathbf{N}_{a-1}) \subseteq \mathbf{N}_a,$$

and $\mathbf{A}_{F,L}\varphi_{\mathbf{N}}(\mathbf{N}_{a-1}) = e_aP_{\mathbf{N}}$ is the image of $R \otimes_\varphi \mathbf{N}_{a-1}$, free of rank r_{a-1} over $R = e_a\mathbf{A}_{F,L}$, injectively; as P is a non-zero-divisor on the free R-module $\mathbf{N}_a$, comparing ranks after inverting P gives $r_a = r_{a-1}$ for all a, i.e. $r_a = r$. Over $\mathbf{A}_F$, R is free of rank $[L:\mathbb{Q}_\ell]/n$, so $d = n\,r\,[L:\mathbb{Q}_\ell]/n$, i.e. $r = \dim_L V$.

(Wm3). Let $u, v \in \Gamma_K$. By (Bs2), (i) and (v), $\gamma_{uv,\mathbf{N}} = \mathcal{E}_T(\gamma_u\gamma_v) = \gamma_{u,\mathbf{N}}\gamma_{v,\mathbf{N}}$, $\gamma_{1,\mathbf{N}} = \mathrm{id}$, and $\gamma_{u,\mathbf{N}}$ commutes with $\varphi_{\mathbf{N}}$ (compatibility of (Bs2) with the Frobenius isomorphisms, as $\gamma_u\varphi = \varphi\gamma_u$; $\gamma_u(P) \in P\mathbf{A}_F^\times$ by (v), so $\gamma_{u,\mathbf{N}}$ extends to $\mathbf{N}[1/P]$). Since $\mathrm{red}\circ\gamma_u = \mathrm{red}$, $\mathcal{E}_T(\mathrm{red})\circ\gamma_{u,\mathbf{N}} = \mathcal{E}_T(\mathrm{red})$; by (Bs2) the $\mathcal{O}_F$-linear extension $\mathcal{O}_F \otimes_{\mathrm{red},\mathbf{A}_F} \mathbf{N} = \mathbf{N}/X\mathbf{N} \to \mathcal{E}_T(\mathcal{O}_F)$ is an isomorphism ($\ker\mathrm{red} = X\mathbf{A}_F$), so $\ker\mathcal{E}_T(\mathrm{red}) = X\mathbf{N}$ and $(\gamma_{u,\mathbf{N}} - 1)\mathbf{N} \subseteq X\mathbf{N}$; this holds for every $u \in \Gamma_K$, in particular for $u = \delta \in \Delta_K \subseteq \Gamma_K$.

(Wm3'). Put $Y = X_{s-1} = (1+X)^{\ell^{s-1}} - 1$, $B = \mathbf{A}_F/Y\mathbf{A}_F$, $\pi\colon \mathbf{A}_F \to B$ the quotient map and $\eta = \pi(1+X)$. The δ-structure of the ℓ-torsion-free ring $\mathbf{A}_F$ is $\delta(f) = (\varphi(f) - f^\ell)/\ell$. (a) *The ring B.* Y is a monic polynomial in X of degree ℓ^{s-1} with $Y \equiv X^{\ell^{s-1}} \bmod \ell$, so by Weierstrass division $B = \mathcal{O}_F[X]/(Y)$ is free over $\mathcal{O}_F$ with basis $1, \eta, \dots, \eta^{\ell^{s-1}-1}$, and $\eta^{\ell^{s-1}} = 1$; hence B is ℓ-torsion-free and ℓ-adically complete, and $\pi(X)^{\ell^{s-1}} \in \ell B$. (b) *$Y\mathbf{A}_F$ is a δ-ideal.* As $\varphi(1+X) = (1+X)^\ell$, $\varphi(Y) = (1+Y)^\ell - 1$, so $\delta(Y) = \sum_{0<i<\ell} \ell^{-1}\binom{\ell}{i}Y^i \in Y\mathbf{A}_F$. For $f \in \mathbf{A}_F$, $\delta(fY) = f^\ell\delta(Y) + Y^\ell\delta(f) + \ell\,\delta(f)\delta(Y) \in Y\mathbf{A}_F$, and for $y, y' \in Y\mathbf{A}_F$, $\delta(y+y') = \delta(y)+\delta(y')-\sum_{0<i<\ell}\ell^{-1}\binom{\ell}{i}y^iy'^{\ell-i} \in Y\mathbf{A}_F$. So δ induces a δ-structure on B (with Frobenius $\mathcal{O}_F$-Frobenius-semilinear and $\eta \mapsto \eta^\ell$) for which π is a homomorphism of δ-rings. (c) *$(B, \ell B)$ is an object over $\mathcal{O}_K$ and π a morphism over $\mathcal{O}_K$.* $(B, \ell B)$ is a bounded prism: ℓ is a non-zero-divisor of B, B is ℓ-adically complete, $\ell \in \ell B$, and $B/\ell B$ is killed by ℓ. Since $Q_{s-1} = \Phi_{\ell^s}(1+X) = \Phi_\ell((1+X)^{\ell^{s-1}}) = \Phi_\ell(1+Y)$ (Lemma 23.8(3)), $\pi(Q_{s-1}) = \Phi_\ell(1) = \ell$; so $\pi(P\mathbf{A}_F) \subseteq \ell B$, π induces $\mathbf{A}_F/P\mathbf{A}_F \to B/\ell B$, and with the composite $\mathcal{O}_K \to \mathbf{A}_F/P\mathbf{A}_F \to B/\ell B$ (fact (i)) $(B, \ell B)$ is an object over $\mathcal{O}_K$ and π is a morphism over $\mathcal{O}_K$. (d) *$\pi\circ\gamma_u = \pi$ for $u \in \Gamma_K$ with $u \equiv 1 \bmod \ell^{s-1}$.* Write $u = 1 + \ell^{s-1}w$ with $w \in \mathbb{Z}_\ell$. In $\mathbf{A}_F$, $(1+X)^u = (1+X)(1+Y)^w$, where $(1+Y)^w = \sum_{k\ge0}\binom{w}{k}Y^k$: both sides are continuous in $w \in \mathbb{Z}_\ell$ for the (ℓ, X)-adic topology and agree for $w \in \mathbb{Z}_{\ge0}$. Hence $\pi(\gamma_u(1+X)) = \eta\,\pi(1+Y)^w = \eta$. The maps $\pi\circ\gamma_u$ and π are $\mathcal{O}_F$-linear ring homomorphisms, continuous from the (ℓ, X)-adic topology to the ℓ-adic topology of B (by (a), $\pi(X)$ is topologically nilpotent), and agree on X; as $\mathcal{O}_F[X]$ is dense in $\mathbf{A}_F$, they are equal. (e) *Conclusion.* By (Bs2) for π, the B-linear extension $B \otimes_{\pi,\mathbf{A}_F} \mathbf{N} = \mathbf{N}/Y\mathbf{N} \to \mathcal{E}_T(B)$ of $\mathcal{E}_T(\pi)$ is an isomorphism, so $\ker\mathcal{E}_T(\pi) = Y\mathbf{N}$. For u as in (d), $u \in \Gamma_K$, γ_u is a morphism over $\mathcal{O}_K$ (fact (i)), and by (Bs2) $\mathcal{E}_T(\pi)\circ\gamma_{u,\mathbf{N}} = \mathcal{E}_T(\pi\circ\gamma_u) = \mathcal{E}_T(\pi)$. Hence $(\gamma_{u,\mathbf{N}} - 1)(\mathbf{N}) \subseteq Y\mathbf{N} = X_{s-1}\mathbf{N}$, which is (Wm3'). For $s = 1$, $Y = X$, $B = \mathcal{O}_F$ and $\pi = \mathrm{red}$, and this is the argument for the congruence in (Wm3).

So $\mathbf{N}_s(T)$ is a Wach module of level s over K with coefficients L and window $(M_a, h_a)_a$, with $r = \dim_L V$: this is (Ex1) of level s.

(Ex2), (Ex3) of level s. $\iota_{s,T}(fx) = \iota_s(f)\iota_{s,T}(x)$ because $\mathcal{E}_T(\iota_s)$ is ι_s-semilinear and c_T is A_{inf}-linear; $\iota_{s,T}(\lambda x) = (1\otimes\lambda)\iota_{s,T}(x)$ by the $\mathcal{O}_L$-paragraph and (Bs5). By (Bs2), $\mathcal{E}_T(\iota_s)(\varphi_{\mathbf{N}}(x)) = \varphi_M(1\otimes\mathcal{E}_T(\iota_s)x)$, and by (Bs3) $c_T(\varphi_M(1\otimes m)) = (\varphi\otimes1)(c_T(m))$; so $\iota_{s,T}\circ\varphi_{\mathbf{N}} = (\varphi\otimes1)\circ\iota_{s,T}$ on $\mathbf{N}[1/P]$, where $\tilde\xi = \iota_s(P)$ is a unit of $W(\mathbb{C}_\ell^\flat)$. For $\sigma \in G_K$, c_T is G_K-equivariant, σ acts on M by $\mathcal{E}_T(\sigma)$, and $\mathcal{E}_T(\sigma)\circ\mathcal{E}_T(\iota_s) = \mathcal{E}_T(\sigma\circ\iota_s) = \mathcal{E}_T(\iota_s\circ\gamma_{\chi_{\mathrm{cyc}}(\sigma)}) = \mathcal{E}_T(\iota_s)\circ\gamma_{\chi_{\mathrm{cyc}}(\sigma),\mathbf{N}}$ by (iv), (i) and (Bs2); so $\sigma\circ\iota_{s,T} = \iota_{s,T}\circ\gamma_{\chi_{\mathrm{cyc}}(\sigma),\mathbf{N}}$, and for $h \in H_F$, $\chi_{\mathrm{cyc}}(h) = 1$. The spanning clause was shown in the construction. For (Ex3), $\mathbf{N}_s(f)$ is $\mathbf{A}_F$-linear and commutes with $\varphi_{\mathbf{N}}$ and with $\mathcal{E}(\gamma_u)$ by (Bs5); it commutes with the $\mathcal{O}_L$-action because $\mathcal{E}(f)\mathcal{E}(\lambda) = \mathcal{E}(f\lambda) = \mathcal{E}(\lambda f) = \mathcal{E}(\lambda)\mathcal{E}(f)$, hence is $\mathbf{A}_{F,L}$-linear; and $\iota_{s,T'}\circ\mathbf{N}_s(f) = c_{T'}\circ\mathcal{E}(f)_{A_{\mathrm{inf}}}\circ\mathcal{E}_T(\iota_s) = (f\otimes1)\circ c_T\circ\mathcal{E}_T(\iota_s)$ by (Bs5) twice. The identities for id and composites are those of (Bs5). This proves (Ex4).

(Ex1)–(Ex3). Let ℓ be odd and $s = 1$. Then the hypotheses on ℓ, L, K in (Ex4) are those of the statement, $P = Q_0 = Q$, $\iota_1 = \iota$ (Lemma 23.8(6)), and a Wach module of level 1 is a Wach module (Definition 23.2, paragraph Level s). So $\mathbf{N}(T) = \mathbf{N}_1(T)$, $\iota_T = \iota_{1,T}$, $\mathbf{N}(f) = \mathbf{N}_1(f)$ satisfy (Ex1)–(Ex3).

(Ex5). The construction above defines $\mathbf{N}_s(T)$, $\varphi_{\mathbf{N}}$, $\gamma_{u,\mathbf{N}}$, $\iota_{s,T}$, the $\mathcal{O}_L$-action and $\mathbf{N}_s(f)$ in the terms of (Ex5), with the datum $\mathcal{E}_T$ and the isomorphism c_T of (Bs3) fixed there; for $s = 1$ and ℓ odd, $\mathbf{N}(T) = \mathbf{N}_1(T)$ by the preceding paragraph. □

Lemma 23.8. *Let ℓ be a prime, F a finite unramified extension of $\mathbb{Q}_\ell$ with residue field k, $F_1 = F(\mu_\ell)$, $H_F = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/F(\mu_{\ell^\infty}))$, and $w\colon \mathcal{O}_F = W(k) \to A_{\mathrm{inf}}$ the map of Witt vectors induced by the unique ring homomorphism $k \to \mathcal{O}_{\mathbb{C}_\ell^\flat}$ lifting the inclusion of k into the residue field. Then:*

1. *$(\mathbf{A}_F, Q\mathbf{A}_F)$ is a bounded prism; $X \mapsto \zeta_\ell - 1$ induces $\mathbf{A}_F/Q\mathbf{A}_F \cong \mathcal{O}_{F_1}$, which makes it an object over $\mathcal{O}_F$. With $\mathcal{O}_F \to k$ the reduction, $(\mathcal{O}_F, \ell\mathcal{O}_F)$ is an object over $\mathcal{O}_F$, and* $\mathrm{red}\colon \mathbf{A}_F \to \mathcal{O}_F$, $X \mapsto 0$, *is a morphism over $\mathcal{O}_F$.*

2. *For every $u \in \mathbb{Z}_\ell^\times$, γ_u is an automorphism of $(\mathbf{A}_F, Q\mathbf{A}_F)$ over $\mathcal{O}_F$, $\gamma_u\gamma_v = \gamma_{uv}$,* $\mathrm{red} \circ \gamma_u = \mathrm{red}$, *$\gamma_u(X) \in X\mathbf{A}_F^\times$, and $\gamma_u(Q_s) \in Q_s\,\mathbb{Z}_\ell[[X]]^\times$ for every $s \geq 0$.*

3. *For $s \geq 0$, $Q_s = \Phi_{\ell^{s+1}}(1 + X)$ is a monic polynomial of degree $\ell^s(\ell - 1)$ with $Q_s \equiv X^{\ell^s(\ell-1)} \bmod \ell$, its roots are the $\zeta - 1$, ζ running over the primitive ℓ^{s+1}-th roots of unity, and an element of $\mathcal{O}_L[[X]]$ (L any finite extension of $\mathbb{Q}_\ell$) vanishing at all of them is divisible by Q_s in $\mathcal{O}_L[[X]]$.*

4. *The map $\iota\colon \mathbf{A}_F \to A_{\mathrm{inf}}$ equal to $\varphi \circ w$ on $\mathcal{O}_F$ with $\iota(X) = \mu_\varepsilon$ is a well-defined injective ring homomorphism, a morphism $(\mathbf{A}_F, Q\mathbf{A}_F) \to (A_{\mathrm{inf}}, \tilde{\xi}A_{\mathrm{inf}})$ over $\mathcal{O}_F$, with $\iota(Q) = \tilde{\xi}$, $\tilde{\theta} \circ \iota|_{\mathcal{O}_F}$ the inclusion $\mathcal{O}_F \subset \mathcal{O}_{\mathbb{C}_\ell}$, $\sigma \circ \iota = \iota \circ \gamma_{\chi_{\mathrm{cyc}}(\sigma)}$ for every $\sigma \in G_F$ (so $h \circ \iota = \iota$ for $h \in H_F$), and*
$$\iota^{-1}(\tilde{\xi}^k A_{\mathrm{inf}}) = Q^k\mathbf{A}_F \qquad (k \geq 0).$$

5. *Let K be a field with $F \subseteq K \subseteq F_1$ and $\Gamma_K = \chi_{\mathrm{cyc}}(G_K)$. Then $(\mathbf{A}_F, Q\mathbf{A}_F)$, through $\mathcal{O}_K \subseteq \mathcal{O}_{F_1} \cong \mathbf{A}_F/Q\mathbf{A}_F$, and $(\mathcal{O}_F, \ell\mathcal{O}_F)$, through the reduction $\mathcal{O}_K \to k$, are objects over $\mathcal{O}_K$;* red *is a morphism over $\mathcal{O}_K$, and ι is a morphism over $\mathcal{O}_K$ to the object $(A_{\mathrm{inf}}, \tilde{\xi}A_{\mathrm{inf}})$ over $\mathcal{O}_K$ of Theorem B.41; for every $u \in \Gamma_K$, γ_u is an automorphism of $(\mathbf{A}_F, Q\mathbf{A}_F)$ over $\mathcal{O}_K$; and $\sigma \circ \iota = \iota \circ \gamma_{\chi_{\mathrm{cyc}}(\sigma)}$ with $\chi_{\mathrm{cyc}}(\sigma) \in \Gamma_K$ for every $\sigma \in G_K$.*

6. *(Level s.) Let $s \geq 1$ be an integer, $F_s = F(\mu_{\ell^s})$, and $\mu_{\varepsilon,s} = [\varepsilon^{1/\ell^{s-1}}] - 1 \in A_{\mathrm{inf}}$, where $\varepsilon^{1/\ell^{s-1}}$ is the unique ℓ^{s-1}-th root of ε in $\mathcal{O}_{\mathbb{C}_\ell^\flat}$. Then $(\mathbf{A}_F, Q_{s-1}\mathbf{A}_F)$ is a bounded prism and $X \mapsto \zeta_{\ell^s} - 1$ induces $\mathbf{A}_F/Q_{s-1}\mathbf{A}_F \cong \mathcal{O}_{F_s}$. The map $\iota_s\colon \mathbf{A}_F \to A_{\mathrm{inf}}$ equal to $\varphi \circ w$ on $\mathcal{O}_F$ with $\iota_s(X) = \mu_{\varepsilon,s}$ is a well-defined injective ring homomorphism commuting with the Frobenius lifts, with $\iota_s(Q_{s-1}) = \tilde{\xi}$, $\tilde{\theta} \circ \iota_s|_{\mathcal{O}_F}$ the inclusion $\mathcal{O}_F \subset \mathcal{O}_{\mathbb{C}_\ell}$, $\sigma \circ \iota_s = \iota_s \circ \gamma_{\chi_{\mathrm{cyc}}(\sigma)}$ for every $\sigma \in G_F$, and $\iota_s^{-1}(\tilde{\xi}^k A_{\mathrm{inf}}) = Q_{s-1}^k\mathbf{A}_F$ ($k \geq 0$). For every field K with $F \subseteq K \subseteq F_s$ and $\Gamma_K = \chi_{\mathrm{cyc}}(G_K)$, the assertions of item 5 hold with Q, F_1, ι replaced by Q_{s-1}, F_s, ι_s. For $s = 1$ ($Q_0 = Q$, $\iota_1 = \iota$) this is items 1, 4 and 5.*

Proof. **Item 3.** Since $\varphi^s(X) = (1 + X)^{\ell^s} - 1$, $Q_s = \varphi^s(Q) = \sum_{i<\ell}(1 + X)^{i\ell^s} = ((1 + X)^{\ell^{s+1}} - 1)/((1 + X)^{\ell^s} - 1) = \Phi_{\ell^{s+1}}(1 + X)$, a monic polynomial of degree $\ell^s(\ell - 1)$ whose roots are the $\zeta - 1$ with ζ primitive of order ℓ^{s+1}; they are pairwise distinct and lie in the maximal ideal of the ring of integers of $\mathbb{Q}_\ell(\zeta)$. Modulo ℓ, $((1+X)^{\ell^s} - 1)Q_s \equiv X^{\ell^{s+1}}$ and $(1+X)^{\ell^s} - 1 \equiv X^{\ell^s}$, and

X^{ℓ^s} is a non-zero-divisor of $\mathbb{F}_\ell[[X]]$, so $Q_s \equiv X^{\ell^s(\ell-1)} \bmod \ell$: Q_s is a distinguished polynomial. Let $f \in \mathcal{O}_L[[X]]$ vanish at its roots. By Weierstrass division $f = Q_s g + R$ with $g \in \mathcal{O}_L[[X]]$ and $R \in \mathcal{O}_L[X]$ of degree $< \deg Q_s$; evaluating at the roots (the series converge there) shows that R has $\deg Q_s$ distinct roots, so $R = 0$.

Item 1. The map φ lifts the Frobenius modulo ℓ: on $\mathcal{O}_F$ it is the Witt vector Frobenius, $\varphi(X) \equiv X^\ell \bmod \ell$, hence $\varphi(f) \equiv f^\ell \bmod \ell$ for all $f \in \mathbf{A}_F$; as $\mathbf{A}_F$ has no ℓ-torsion, $\delta(f) = (\varphi(f) - f^\ell)/\ell$ makes it a δ-ring. By item 3 ($s = 0$) $Q \equiv X^{\ell-1} \bmod \ell$, so (ℓ, Q) and (ℓ, X) have the same radical and $\mathbf{A}_F$ is (ℓ, Q)-adically complete. The map red commutes with the Frobenius lifts ($\mathrm{red}(\varphi(X)) = 0$) and $\mathrm{red}(Q) = \ell$; as $\delta(\ell) = 1 - \ell^{\ell-1}$ is a unit, $\delta(Q) \notin (\ell, X)$, the maximal ideal of the local ring $\mathbf{A}_F$, so $\delta(Q)$ is a unit and Q is distinguished. Since $Q = \Phi_\ell(1 + X)$ is an Eisenstein polynomial over $\mathcal{O}_F$ in the variable X (monic of degree $\ell - 1$, the other coefficients in $\ell\mathcal{O}_F$, constant term ℓ), Weierstrass division gives $\mathbf{A}_F/Q = \mathcal{O}_F[X]/(\Phi_\ell(1+X)) \cong \mathcal{O}_F[\zeta_\ell] = \mathcal{O}_{F_1}$, which has no ℓ-torsion; so the prism is bounded, and $\mathcal{O}_F \subseteq \mathcal{O}_{F_1}$ makes it an object over $\mathcal{O}_F$. The pair $(\mathcal{O}_F, \ell\mathcal{O}_F)$ is a bounded prism ($\delta(\ell)$ is a unit) and an object over $\mathcal{O}_F$ through $\mathcal{O}_F \to k$. Finally $\mathrm{red}(Q\mathbf{A}_F) \subseteq \ell\mathcal{O}_F$ and the induced map $\mathcal{O}_{F_1} \to k$ is the reduction, which restricts to the reduction on $\mathcal{O}_F$; so red is a morphism over $\mathcal{O}_F$.

Item 2. γ_u is well defined since $\gamma_u(X) = uX + \cdots \in X\mathbf{A}_F^\times$ (u a unit), and $\gamma_u\gamma_v = \gamma_{uv}$ because $(1 + \gamma_v(X))^u = (1 + X)^{uv}$; so γ_u is an automorphism with inverse $\gamma_{u^{-1}}$. It commutes with φ: on $\mathcal{O}_F$ the map φ is the Frobenius and γ_u is the identity, and both composites send X to $(1 + X)^{u\ell} - 1$. The element $\gamma_u(Q_s) = \Phi_{\ell^{s+1}}((1 + X)^u)$ vanishes at every root $\zeta - 1$ of Q_s, since ζ^u is again primitive; by item 3, $\gamma_u(Q_s) = Q_s v$ with $v \in \mathbb{Z}_\ell[[X]]$, and symmetrically $\gamma_{u^{-1}}(Q_s) = Q_s v'$; applying γ_u to the second identity gives $Q_s = Q_s v \gamma_u(v')$, so v is a unit. In particular $\gamma_u(Q\mathbf{A}_F) = Q\mathbf{A}_F$, and the induced automorphism of $\mathcal{O}_{F_1}$ sends ζ_ℓ to ζ_ℓ^u and is the identity on $\mathcal{O}_F$: γ_u is an automorphism over $\mathcal{O}_F$. Since $\gamma_u(X) \in X\mathbf{A}_F$, $\mathrm{red} \circ \gamma_u = \mathrm{red}$.

Item 4. The element μ_ε is topologically nilpotent for the topology of (Ai2) (its image modulo ℓ is $\varepsilon - 1$, of positive valuation), so ι is a well-defined continuous ring homomorphism. It commutes with the Frobenius lifts: w is Frobenius-equivariant, and $\varphi(\mu_\varepsilon) = [\varepsilon]^\ell - 1 = (1 + \mu_\varepsilon)^\ell - 1 = \iota(\varphi(X))$; so ι is a map of δ-rings, and $\iota(Q) = \sum_{i<\ell}[\varepsilon]^i = \tilde{\xi}$. On $\mathcal{O}_F$, $\tilde{\theta} \circ \iota = \theta \circ \varphi^{-1} \circ \varphi \circ w = \theta \circ w$, the inclusion $\mathcal{O}_F \subset \mathcal{O}_{\mathbb{C}_\ell}$, and $\tilde{\theta}(\mu_\varepsilon) = \theta([\varepsilon^{1/\ell}]) - 1 = \zeta_\ell - 1$; hence the induced map $\mathcal{O}_{F_1} = \mathbf{A}_F/Q \to A_{\mathrm{inf}}/\tilde{\xi} = \mathcal{O}_{\mathbb{C}_\ell}$ is the inclusion, and ι is a morphism over $\mathcal{O}_F$. For $\sigma \in G_F$: σ fixes the image of k in $\mathcal{O}_{\mathbb{C}_\ell^\flat}$, hence fixes $w(\mathcal{O}_F)$ and $\varphi(w(\mathcal{O}_F))$, and $\sigma(\mu_\varepsilon) = [\varepsilon^{\chi_{\mathrm{cyc}}(\sigma)}] - 1 = (1 + \mu_\varepsilon)^{\chi_{\mathrm{cyc}}(\sigma)} - 1 = \iota(\gamma_{\chi_{\mathrm{cyc}}(\sigma)}(X))$ by multiplicativity and continuity of the Teichmüller map and continuity of the binomial series; both sides being continuous ring homomorphisms, $\sigma \circ \iota = \iota \circ \gamma_{\chi_{\mathrm{cyc}}(\sigma)}$. For $h \in H_F$, $\chi_{\mathrm{cyc}}(h) = 1$ and $\gamma_1 = \mathrm{id}$.

Injectivity: modulo ℓ, ι induces $k[[X]] \to \mathcal{O}_{\mathbb{C}_\ell^\flat}$, $f \mapsto f(\varepsilon - 1)$ (coefficients through $k \subset \mathcal{O}_{\mathbb{C}_\ell^\flat}$); if $f \neq 0$ has first non-zero coefficient at X^i, that coefficient is a unit and $|f(\varepsilon - 1)| = |\varepsilon - 1|^i \neq 0$. So $\iota \bmod \ell$ is injective. If $\iota(a) = 0$ with $a \neq 0$, write $a = \ell^m a'$ with $a' \notin \ell\mathbf{A}_F$ ($\mathbf{A}_F$ is ℓ-adically separated); since A_{inf} has no ℓ-torsion, $\iota(a') = 0$, contradicting injectivity modulo ℓ.

We determine the inverse image of $\tilde{\xi}^k A_{\mathrm{inf}}$ by induction on k, the case $k = 0$ being trivial. Let $k \geq 1$ and $\iota(a) \in \tilde{\xi}^k A_{\mathrm{inf}} \subseteq \ker\tilde{\theta}$. By the above, $\tilde{\theta} \circ \iota$ factors as $\mathbf{A}_F \to \mathbf{A}_F/Q = \mathcal{O}_{F_1} \hookrightarrow \mathcal{O}_{\mathbb{C}_\ell}$, so $a \in Q\mathbf{A}_F$, $a = Qa_1$. Then $\tilde{\xi}\,\iota(a_1) \in \tilde{\xi}^k A_{\mathrm{inf}}$, and as A_{inf} is a domain (Ai1), $\iota(a_1) \in \tilde{\xi}^{k-1} A_{\mathrm{inf}}$; by induction $a_1 \in Q^{k-1}\mathbf{A}_F$, so $a \in Q^k\mathbf{A}_F$. The reverse inclusion is $\iota(Q) = \tilde{\xi}$.

The case $\ell = 2$. Items 1 to 5 hold for $\ell = 2$ with the same proofs: $\delta(2) = (2 - 4)/2 = -1$ is a unit, $Q = \Phi_2(1 + X) = 2 + X$ is Eisenstein of degree 1, $F_1 = F$, and $\iota(Q) = 1 + [\varepsilon] = \tilde{\xi}$.

Item 5. The field K lies between F and the totally ramified extension F_1, so $\mathcal{O}_K \subseteq \mathcal{O}_{F_1}$ is a local subring with residue field k. Objects: by item 1, $(\mathbf{A}_F, Q\mathbf{A}_F)$ and $(\mathcal{O}_F, \ell\mathcal{O}_F)$ are bounded prisms, and $\mathcal{O}_K \subseteq \mathcal{O}_{F_1} = \mathbf{A}_F/Q$, $\mathcal{O}_K \to k = \mathcal{O}_F/\ell$ are ring homomorphisms. red: it is a map

of δ-rings with $\mathrm{red}(Q\mathbf{A}_F) \subseteq \ell\mathcal{O}_F$ (item 1), and the induced map $\mathcal{O}_{F_1} \to k$ is the reduction of the local ring $\mathcal{O}_{F_1}$ (proof of item 1), whose restriction to $\mathcal{O}_K$ is the reduction of $\mathcal{O}_K$. ι: it is a map of δ-rings with $\iota(Q) = \tilde{\xi}$, and the induced map $\mathcal{O}_{F_1} \to A_{\mathrm{inf}}/\tilde{\xi} = \mathcal{O}_{\mathbb{C}_\ell}$ (via $\tilde{\theta}$) is the inclusion (proof of item 4), whose restriction to $\mathcal{O}_K$ is the inclusion $\mathcal{O}_K \subseteq \mathcal{O}_{\mathbb{C}_\ell}$ that makes A_{inf} an object over $\mathcal{O}_K$ in Theorem B.41. γ_u: let $u = \chi_{\mathrm{cyc}}(\sigma)$ with $\sigma \in G_K$. By item 2, γ_u is an automorphism of the prism, and the induced automorphism of $\mathcal{O}_{F_1} = \mathcal{O}_F[\zeta_\ell]$ is the identity on $\mathcal{O}_F$ and sends ζ_ℓ to ζ_ℓ^u. The restriction of σ to F_1 has the same two properties ($\sigma \in G_K \subseteq G_F$ and $\sigma(\zeta_\ell) = \zeta_\ell^{\chi_{\mathrm{cyc}}(\sigma)}$), so the two automorphisms of $\mathcal{O}_{F_1}$ coincide; as σ fixes K, the induced automorphism is the identity on $\mathcal{O}_K$, i.e. γ_u is an automorphism over $\mathcal{O}_K$. Finally $G_K \subseteq G_F$, so $\sigma \circ \iota = \iota \circ \gamma_{\chi_{\mathrm{cyc}}(\sigma)}$ is item 4, and $\chi_{\mathrm{cyc}}(\sigma) \in \Gamma_K$ by definition.

Item 6. *The prism.* By item 3, $Q_{s-1} = \Phi_{\ell^s}(1+X)$ is monic of degree $\ell^{s-1}(\ell-1)$ and $\equiv X^{\ell^{s-1}(\ell-1)} \bmod \ell$, so (ℓ, Q_{s-1}) and (ℓ, X) have the same radical and $\mathbf{A}_F$ is (ℓ, Q_{s-1})-adically complete. As red is a map of δ-rings with $\mathrm{red}(Q_{s-1}) = \Phi_{\ell^s}(1) = \ell$ and $\delta(\ell)$ is a unit, $\delta(Q_{s-1}) \notin (\ell, X)$ is a unit: Q_{s-1} is distinguished. Its constant term is ℓ and its other non-leading coefficients lie in $\ell\mathbb{Z}_\ell$, so it is an Eisenstein polynomial over $\mathcal{O}_F$; by Weierstrass division $\mathbf{A}_F/Q_{s-1} = \mathcal{O}_F[X]/(\Phi_{\ell^s}(1+X)) \cong \mathcal{O}_F[\zeta_{\ell^s}] = \mathcal{O}_{F_s}$, with $X \mapsto \zeta_{\ell^s} - 1$, which has no ℓ-torsion. So the prism is bounded. *The map ι_s.* Its image of X is topologically nilpotent for the topology of (Ai2) ($\mu_{\varepsilon,s} \bmod \ell = \varepsilon^{1/\ell^{s-1}} - 1$ has positive valuation), so ι_s is a well-defined continuous ring homomorphism; $\varphi(\mu_{\varepsilon,s}) = [\varepsilon^{1/\ell^{s-1}}]^\ell - 1 = (1+\mu_{\varepsilon,s})^\ell - 1 = \iota_s(\varphi(X))$ and w is Frobenius-equivariant, so ι_s commutes with the Frobenius lifts; $\iota_s(Q_{s-1}) = \sum_{i<\ell}[\varepsilon^{1/\ell^{s-1}}]^{i\ell^{s-1}} = \sum_{i<\ell}[\varepsilon]^i = \tilde{\xi}$. On $\mathcal{O}_F$, $\tilde{\theta} \circ \iota_s = \theta \circ w$ as in item 4, and $\tilde{\theta}(\mu_{\varepsilon,s}) = \theta([\varepsilon^{1/\ell^s}]) - 1 = \zeta_{\ell^s} - 1$; so the induced map $\mathcal{O}_{F_s} = \mathbf{A}_F/Q_{s-1} \to A_{\mathrm{inf}}/\tilde{\xi} = \mathcal{O}_{\mathbb{C}_\ell}$ is the inclusion. For $\sigma \in G_F$, σ fixes $\varphi(w(\mathcal{O}_F))$, and since σ raises every ζ_{ℓ^n} to the power $\chi_{\mathrm{cyc}}(\sigma)$ it maps $\varepsilon^{1/\ell^{s-1}}$ to $(\varepsilon^{1/\ell^{s-1}})^{\chi_{\mathrm{cyc}}(\sigma)}$; so $\sigma(\mu_{\varepsilon,s}) = (1+\mu_{\varepsilon,s})^{\chi_{\mathrm{cyc}}(\sigma)} - 1 = \iota_s(\gamma_{\chi_{\mathrm{cyc}}(\sigma)}(X))$ by multiplicativity and continuity of the Teichmüller map and of the binomial series, and $\sigma \circ \iota_s = \iota_s \circ \gamma_{\chi_{\mathrm{cyc}}(\sigma)}$. Injectivity: modulo ℓ, ι_s induces $f \mapsto f(\varepsilon^{1/\ell^{s-1}} - 1)$ on $k[[X]]$, injective since $|\varepsilon^{1/\ell^{s-1}} - 1| \neq 0$; then as in item 4. The inverse image of $\tilde{\xi}^k A_{\mathrm{inf}}$: as $\tilde{\theta} \circ \iota_s$ factors through the injection $\mathbf{A}_F/Q_{s-1} = \mathcal{O}_{F_s} \hookrightarrow \mathcal{O}_{\mathbb{C}_\ell}$, the induction of item 4 applies with Q_{s-1} in place of Q. *Over $\mathcal{O}_K$.* The extension F_s/F is totally ramified, so $\mathcal{O}_K \subseteq \mathcal{O}_{F_s}$ is a local subring with residue field k, and the argument of item 5 applies with F_s, Q_{s-1}, ι_s: $\mathrm{red}(Q_{s-1}) = \ell$ and the induced map $\mathcal{O}_{F_s} \to k$ is the reduction; the map induced by ι_s is the inclusion; for $u = \chi_{\mathrm{cyc}}(\sigma)$, $\sigma \in G_K$, item 2 (with Q_{s-1}) gives $\gamma_u(Q_{s-1}\mathbf{A}_F) = Q_{s-1}\mathbf{A}_F$, and the induced automorphism of $\mathcal{O}_{F_s} = \mathcal{O}_F[\zeta_{\ell^s}]$ is the identity on $\mathcal{O}_F$ and sends ζ_{ℓ^s} to $\zeta_{\ell^s}^u$, i.e. it is the restriction of σ, which fixes $\mathcal{O}_K$. □

Lemma 23.9. *Let ℓ be an odd prime, F, L as in Definition 23.2, and $\mathbf{N}$ a Wach module over F with coefficients L (any window), of rank r in (Wm1). Put $\Lambda = \mathcal{O}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$, $E = \Lambda[1/\ell]$, $h_u = (1+X)^u - 1 \in X\mathbb{Z}_\ell[[X]]$ for $u \in \mathbb{Z}_\ell^\times$, and $\mathbf{N}_E = \mathbf{N} \otimes_{\mathbf{A}_{F,L}} E[[X]]$, with the semilinear operators $\gamma_u(f \otimes x) = (f \circ h_u) \otimes \gamma_u(x)$ ($f \circ h_u$ the formal composition) and the submodules $X^k\mathbf{N}_E = \mathbf{N} \otimes X^k E[[X]]$. For $k \geq 0$ let*

$$s_\ell(k) = J + v_\ell(J!), \qquad J = \Big\lfloor \frac{k}{\ell - 1} \Big\rfloor; \qquad \text{so } s_\ell(k) \leq \frac{k\,\ell}{(\ell-1)^2}.$$

(Hf1) Trivialisation of cocycles *Let $m \geq 1$ and let $(z_u)_{u \in \mathbb{Z}_\ell^\times}$ be elements of $X^m\mathbf{N}$ with $z_{uv} = z_u + \gamma_u(z_v)$ for all u, v. Then there is a unique $W \in X^m\mathbf{N}_E$ with $(\gamma_u - 1)(W) = z_u$ for every $u \in \mathbb{Z}_\ell^\times$; uniqueness already holds for one u that is not a root of unity.*

(Hf2) Horizontal sections *For every $x_0 \in \mathbf{N}$ there is a unique $H \in x_0 + X\mathbf{N}_E$ with $\gamma_u(H) = H$ for every $u \in \mathbb{Z}_\ell^\times$. If $x_1, \ldots, x_r$ is an $\mathbf{A}_{F,L}$-basis of $\mathbf{N}$, the corresponding $H_1, \ldots, H_r$ form an $E[[X]]$-basis of $\mathbf{N}_E$.*

(Hf3) Denominators *Let (n_i) be an $\mathbf{A}_{F,L}$-basis of $\mathbf{N}$. The coordinates $\sum_k a_k X^k \in E[[X]]$ of W in (Hf1) and of H in (Hf2) satisfy $\ell^{s_\ell(k)} a_k \in \Lambda$ for every $k \geq 0$. Consequently, for every $R \geq 1$ there is $x_R \in \mathbf{N}$ with $\ell^{s_\ell(R-1)} W - x_R \in X^R \mathbf{N}_E$ (resp. $\ell^{s_\ell(R-1)} H - x_R \in X^R \mathbf{N}_E$), and for every such x_R and every u,*

$$(\gamma_u - 1)(x_R) - \ell^{s_\ell(R-1)} z_u \in X^R \mathbf{N} \qquad (\text{resp. } (\gamma_u - 1)(x_R) \in X^R \mathbf{N}).$$

In (Hf1) one can take $x_R \in X^m \mathbf{N}$.

Over a base field. *Let K be a field with $F \subseteq K \subseteq F(\mu_\ell)$, Γ_K, Δ_K as in Definition 23.2, $d = |\Delta_K|$, and let $\mathbf{N}$ be a Wach module over K with coefficients L. Then (Hf1)–(Hf3) hold with $u, v \in \Gamma_K$ in place of $u, v \in \mathbb{Z}_\ell^\times$ (the operators γ_u of $\mathbf{N}_E$ being defined for $u \in \Gamma_K$) and with $s_\ell(k)$ replaced by*

$$s_{\ell,d}(k) = J + v_\ell(J!), \qquad J = \Big\lfloor \frac{k}{d} \Big\rfloor; \qquad \text{so } s_{\ell,d}(k) \leq \frac{k\,\ell}{d(\ell-1)} \leq \frac{3k}{2}.$$

For $K = F$ one has $\Gamma_F = \mathbb{Z}_\ell^\times$, $d = \ell - 1$ and $s_{\ell,\ell-1} = s_\ell$: this is the statement above.

Level s. *Let ℓ be any prime, $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, L and K ($F \subseteq K \subseteq F(\mu_{\ell^s})$) as in the paragraph Level s of Definition 23.2, and $\mathbf{N}$ a Wach module of level s over K with coefficients L. With Λ, E, h_u, $\mathbf{N}_E$ as above, (Hf1)–(Hf3) hold with $u, v \in \Gamma_K$ in place of $u, v \in \mathbb{Z}_\ell^\times$ and with $s_\ell(k)$ replaced by*

$$\nu_{\ell,s}(k) = sk + v_\ell(k!) \leq (s+1)k.$$

(For $s = 1$ and ℓ odd the paragraph over a base field gives the smaller $s_{\ell,d}$.)

Proof. **Preliminaries.** By (Wm1), each $\mathbf{N}_a$ is free of rank r over $e_a \mathbf{A}_{F,L}$, so $\mathbf{N} = \bigoplus_a \mathbf{N}_a$ is free of rank r over $\mathbf{A}_{F,L} = \Lambda[[X]]$; fix a basis (n_i) (any basis, for (Hf3)). Then $\mathbf{N}_E = \bigoplus_i E[[X]]\, n_i \supseteq \mathbf{N}$, $X^k \mathbf{N}_E$ consists of the elements whose coordinates lie in $X^k E[[X]]$, and $\mathbf{N} \cap X^k \mathbf{N}_E = X^k \mathbf{N}$ (as $\Lambda[[X]] \cap X^k E[[X]] = X^k \Lambda[[X]]$). The Λ-linear ring endomorphism γ_u of $\Lambda[[X]]$ with $\gamma_u(X) = h_u$ is $f \mapsto f \circ h_u$ (for $f = \sum_{k<N} c_k X^k + X^N g$, $\gamma_u(f) - \sum_{k<N} c_k h_u^k = h_u^N \gamma_u(g) \in X^N \Lambda[[X]]$); formal composition with $h \in XE[[X]]$ is a ring endomorphism of $E[[X]]$ preserving each $X^k E[[X]]$, composition is associative, and $h_v \circ h_u = (1+X)^{uv} - 1 = h_{uv}$. Hence γ_u on $\mathbf{N}_E$ is well defined (the map is $\Lambda[[X]]$-balanced because $\gamma_{u,\mathbf{N}}$ is γ_u-semilinear), $\gamma_u \gamma_v = \gamma_{uv}$ (by (Wm3) on $\mathbf{N}$), all γ_u commute, and they preserve each $X^k \mathbf{N}_E$. Let $G_u \in M_r(\Lambda[[X]])$ be the matrix with $\gamma_u(n_i) = \sum_j (G_u)_{ji} n_j$; by (Wm3), $G_u \equiv I$ modulo X, and $\gamma_u(\sum_i w_i n_i) = \sum_j (G_u\,(w \circ h_u))_j n_j$ for $w \in E[[X]]^r$.

The generator and its eigenvalues. Fix a topological generator g of $\mathbb{Z}_\ell^\times$ (procyclic, ℓ odd). For $i \geq 1$ let $\delta_i = v_\ell(g^i - 1)$. If $(\ell-1) \nmid i$, then $g^i \not\equiv 1 \bmod \ell$ (the image of g generates $\mathbb{F}_\ell^\times$), so $\delta_i = 0$. If $i = (\ell-1)j$, then $g^{\ell-1} = 1 + \ell t$ with $t \in \mathbb{Z}_\ell^\times$ ($g^{\ell-1}$ generates $1 + \ell\mathbb{Z}_\ell$), and in $(1+\ell t)^j - 1 = j\ell t + \sum_{q \geq 2} \binom{j}{q} (\ell t)^q$ the first term has valuation $1 + v_\ell(j)$ while $v_\ell(\binom{j}{q} \ell^q) \geq v_\ell(j) - v_\ell(q) + q \geq v_\ell(j) + 2$ for $q \geq 2$ ($\binom{j}{q} = \frac{j}{q}\binom{j-1}{q-1}$, and $q - v_\ell(q) \geq 2$ for $q \geq 2$, $\ell \geq 3$); so $\delta_i = 1 + v_\ell(j)$. Hence, for $k \geq 1$,

$$\sum_{i=1}^{k} \delta_i = \sum_{j=1}^{J} (1 + v_\ell(j)) = J + v_\ell(J!) = s_\ell(k), \qquad J = \lfloor k/(\ell-1) \rfloor,$$

and $s_\ell(k) \leq J + J/(\ell-1) = J\ell/(\ell-1) \leq k\ell/(\ell-1)^2$ by Legendre's formula $v_\ell(J!) \leq J/(\ell-1)$.

(Hf1) for $u = g$: existence, uniqueness, denominators. Write $G = G_g = \sum_{j \geq 0} G_j X^j$ ($G_j \in M_r(\Lambda)$, $G_0 = I$), $h = h_g$ and $h^i = \sum_{j \geq i} h_j^{(i)} X^j$ with $h_j^{(i)} \in \mathbb{Z}_\ell$, $h_i^{(i)} = g^i$. Let $\zeta =$

$\sum_{k\ge m}\zeta_k X^k$ ($\zeta_k\in\Lambda^r$) be the coordinates of z_g. For $w=\sum_{k\ge m}w_kX^k\in(X^mE[[X]])^r$, the coefficient of X^k in $G(w\circ h)-w$ is

$$(g^k-1)w_k+\sum_{m\le i<k}M_{k,i}w_i,\qquad M_{k,i}=\sum_{j=i}^{k}G_{k-j}h_j^{(i)}\in M_r(\Lambda),$$

since w_k enters only through $G_0h_k^{(k)}X^k$. As $g^k\ne1$ for $k\ge1$, the equation $(\gamma_g-1)(\sum_i w_in_i)=z_g$ is equivalent to the recursion $w_k=(g^k-1)^{-1}(\zeta_k-\sum_{m\le i<k}M_{k,i}w_i)$, $k\ge m$, which has exactly one solution. By induction, $\ell^{t_k}w_k\in\Lambda^r$ with $t_k=\sum_{i=m}^{k}\delta_i\le s_\ell(k)$: indeed $\ell^{t_{k-1}}(\zeta_k-\sum_{i<k}M_{k,i}w_i)\in\Lambda^r$ (t is non-decreasing, $t_{m-1}=0$) and $(g^k-1)^{-1}\in\ell^{-\delta_k}\mathbb{Z}_\ell$. So there is a unique $W\in X^m\mathbf{N}_E$ with $(\gamma_g-1)W=z_g$, and its coordinates satisfy the bound of (Hf3). The same lowest-degree argument shows: for any u not a root of unity, $(\gamma_u-1)D=0$ and $D\in X\mathbf{N}_E$ imply $D=0$ (if $D\ne0$ has lowest degree $k\ge1$, its degree-k coefficient is multiplied by $u^k-1\ne0$); this is the uniqueness statement.

(Hf1) for all u. The cocycle relation and $uv=vu$ give $z_u+\gamma_u(z_g)=z_{ug}=z_{gu}=z_g+\gamma_g(z_u)$, i.e. $(\gamma_g-1)(z_u)=(\gamma_u-1)(z_g)$. Let $W'=\gamma_u(W)-z_u\in X^m\mathbf{N}_E$. Then $(\gamma_g-1)W'=\gamma_u(z_g)-(\gamma_u-1)(z_g)=z_g$, so $W'=W$ by uniqueness, i.e. $(\gamma_u-1)W=z_u$.

(Hf2). The elements $z_u=(1-\gamma_u)(x_0)$ lie in $X\mathbf{N}$ by (Wm3) and satisfy $z_{uv}=z_u+\gamma_u(z_v)$. By (Hf1) with $m=1$ there is a unique $W\in X\mathbf{N}_E$ with $(\gamma_u-1)W=(1-\gamma_u)x_0$ for all u, i.e. $H=x_0+W$ is γ_u-invariant; and H is unique in $x_0+X\mathbf{N}_E$ since the difference of two such lies in $X\mathbf{N}_E$ and is γ_g-invariant. The coordinates of H are those of x_0 (in $\Lambda[[X]]$) plus those of W, so they satisfy the bound of (Hf3). For a basis $x_1,\dots,x_r$, write $H_i=\sum_j P_{ji}x_j$ with $P\in M_r(E[[X]])$ ((x_j) is also an $E[[X]]$-basis of $\mathbf{N}_E$); $H_i-x_i\in X\mathbf{N}_E$ gives $P\equiv I$ modulo X, so $\det P\in1+XE[[X]]$ is a unit and (H_i) is a basis.

(Hf3): truncations. Let W be as in (Hf1) with coordinates $w_i=\sum_k w_{i,k}X^k$, and $R\ge1$. Since s_ℓ is non-decreasing, $x_R=\sum_i(\sum_{k<R}\ell^{s_\ell(R-1)}w_{i,k}X^k)n_i$ lies in $X^m\mathbf{N}$ (coefficients in Λ, vanishing below degree m) and $\ell^{s_\ell(R-1)}W-x_R\in X^R\mathbf{N}_E$. For any $x_R\in\mathbf{N}$ with this property, $(\gamma_u-1)(x_R)-\ell^{s_\ell(R-1)}z_u=(\gamma_u-1)(x_R-\ell^{s_\ell(R-1)}W)\in X^R\mathbf{N}_E$, and it lies in $\mathbf{N}$; so it lies in $\mathbf{N}\cap X^R\mathbf{N}_E=X^R\mathbf{N}$. The case of H is identical, with z_u replaced by 0.

Over a base field. Let K, Γ_K, Δ_K, d and $\mathbf{N}$ be as there. The group Γ_K contains $1+\ell\mathbb{Z}_\ell$: by Definition 23.2, $\chi_{\mathrm{cyc}}(G_F)=\mathbb{Z}_\ell^\times$, and $\sigma\in G_F$ fixes μ_ℓ if and only if $\chi_{\mathrm{cyc}}(\sigma)\equiv1\bmod\ell$, so $\chi_{\mathrm{cyc}}(G_{F(\mu_\ell)})=1+\ell\mathbb{Z}_\ell$, and $G_{F(\mu_\ell)}\subseteq G_K$. As ℓ is odd, $\mathbb{Z}_\ell^\times=\mu_{\ell-1}\times(1+\ell\mathbb{Z}_\ell)$, hence $\Gamma_K=\Delta_K\times(1+\ell\mathbb{Z}_\ell)$ with $\Delta_K=\Gamma_K\cap\mu_{\ell-1}$ cyclic of order d, $d\mid\ell-1$; and Γ_K is closed (image of a compact group). In the proof above replace $\mathbb{Z}_\ell^\times$ by Γ_K throughout; the preliminaries are unchanged ((Wm1), (Wm3) for $u\in\Gamma_K$; h_vh_u-composition and commutativity hold in Γ_K). Only the paragraph on the generator changes: let δ_0 generate Δ_K and $g=\delta_0(1+\ell)\in\Gamma_K$. The closure of the group generated by g contains $g^d=(1+\ell)^d$, which generates $1+\ell\mathbb{Z}_\ell$ topologically (d is prime to ℓ), hence contains $(1+\ell)$ and $\delta_0=g(1+\ell)^{-1}$; so g is a topological generator of Γ_K, of infinite order. Reduction modulo ℓ maps $\mu_{\ell-1}$ isomorphically onto $\mathbb{F}_\ell^\times$, so $g\bmod\ell=\delta_0\bmod\ell$ has order d and $\delta_i=v_\ell(g^i-1)=0$ for $d\nmid i$; for $i=dj$, $g^i=(1+\ell)^{dj}$ and $\delta_i=1+v_\ell(dj)=1+v_\ell(j)$ (the identity $v_\ell((1+\ell)^N-1)=1+v_\ell(N)$ for ℓ odd, proved in the paragraph on the generator with $t=1$). Hence $\sum_{i\le k}\delta_i=J+v_\ell(J!)=s_{\ell,d}(k)$ with $J=\lfloor k/d\rfloor$, and by Legendre's formula $s_{\ell,d}(k)\le J\ell/(\ell-1)\le k\ell/(d(\ell-1))\le k\ell/(\ell-1)\le3k/2$. The recursion, the passage to all $u\in\Gamma_K$ (which uses only $uv=vu$ in Γ_K and uniqueness for g), (Hf2) and the truncations then go through unchanged, with $s_{\ell,d}$ in place of s_ℓ. For $K=F$, $\Delta_F=\mu_{\ell-1}$, $d=\ell-1$, and this is the proof above.

Level s. Let ℓ, s, K and $\mathbf{N}$ be as there. The preliminaries are unchanged: they use (Wm1), the congruence $(\gamma_u-1)(\mathbf{N})\subseteq X\mathbf{N}$ and $\gamma_u\gamma_v=\gamma_{uv}$, which a Wach module of level s

has for $u, v \in \Gamma_K$ (Definition 23.2, paragraph Level s), and $h_v \circ h_u = h_{uv}$, which holds for every prime. Only the generator changes. Put $g = 1 + \ell^s$. It lies in Γ_K: $\chi_{\mathrm{cyc}}(G_{F(\mu_{\ell^s})}) = 1 + \ell^s\mathbb{Z}_\ell$ (Definition 23.2, paragraph Level s) and $G_{F(\mu_{\ell^s})} \subseteq G_K$; and it is not a root of unity. For $i \geq 1$ we claim $\delta_i = v_\ell(g^i - 1) = s + v_\ell(i)$. Write $i = \ell^a b$ with $\ell \nmid b$. First $(1+\ell^s)^b - 1 = b\ell^s + \sum_{q \geq 2} \binom{b}{q} \ell^{sq}$ has valuation exactly s. Next, if $v_\ell(y) \geq 1$ for ℓ odd, resp. $v_2(y) \geq 2$ for $\ell = 2$, then $(1+y)^\ell - 1 = \ell y + \sum_{1<q<\ell} \binom{\ell}{q} y^q + y^\ell$ has valuation exactly $v_\ell(y) + 1$: the middle terms have valuation at least $1 + 2v_\ell(y)$, and y^ℓ has valuation $\ell v_\ell(y) \geq v_\ell(y) + 2$ because $(\ell - 1)v_\ell(y) \geq 2$; the hypothesis is kept (valuations only grow). Since $s \geq 1$, resp. $s \geq 2$ for $\ell = 2$, induction on a gives $\delta_i = s + a$. Hence, in the recursion of (Hf1) for $u = g$ (which uses only that the matrix G_g is $\equiv I$ modulo X and that $g^k \neq 1$ for $k \geq 1$, and therefore goes through unchanged), $\ell^{t_k} w_k \in \Lambda^r$ with

$$t_k = \sum_{i=m}^{k} \delta_i \leq \sum_{i=1}^{k} (s + v_\ell(i)) = sk + v_\ell(k!) = \nu_{\ell,s}(k),$$

and $v_\ell(k!) \leq (k-1)/(\ell - 1) \leq k$ by Legendre's formula, so $\nu_{\ell,s}(k) \leq (s+1)k$. Uniqueness for g is the lowest-degree argument; the passage to all $u \in \Gamma_K$ uses only that Γ_K is commutative and uniqueness for g; (Hf2) uses the congruence of (Wm3) of level s; the truncations use only that $\nu_{\ell,s}$ is non-decreasing. All go through unchanged with $\nu_{\ell,s}$ in place of s_ℓ. □

Lemma 23.10. *Let ℓ, F, L, (τ_a) and a pair (V, T) be as in Proposition 23.7, $d = \dim_L V$, $\mathbf{N} = \mathbf{N}(T)$, and $D_{\mathrm{cris}}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V)^{G_F}$, a module over $F \otimes_{\mathbb{Q}_\ell} L$ with the orthogonal idempotents $\epsilon_b = \iota_\Lambda(e_b)$, $b \in \mathbb{Z}/n$. For each $a \in \mathbb{Z}/n$ let $(n_{a,i})_{i \leq d}$ be an $e_a\mathbf{A}_{F,L}$-basis of $\mathbf{N}_a$, and $H_{a,i} \in \mathbf{N}_E$ the horizontal section through $n_{a,i}$ ((Hf2)). Then $H_{a,i} \in e_a\mathbf{N}_R$, the elements $d_{a,i} = \mathrm{Ev}(H_{a,i})$ lie in $\epsilon_{a+3}D_{\mathrm{cris}}(V)$, and:*

1. *for each a, $(d_{a,i})_{i \leq d}$ is a basis of $\epsilon_{a+3}(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V)$ over $\epsilon_{a+3}(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} L)$;*
2. *$(d_{a,i})_{a,i}$ is a basis of $D_{\mathrm{cris}}(V)$ over $F \otimes_{\mathbb{Q}_\ell} L$, and for each a, $(d_{a,i})_{i \leq d}$ is a basis of $\epsilon_{a+3}D_{\mathrm{cris}}(V)$ over $\epsilon_{a+3}(F \otimes_{\mathbb{Q}_\ell} L) \cong L$.*

Over a base field. *Let K be a field with $F \subseteq K \subseteq F(\mu_\ell)$ as in Definition 23.2, (V, T) a pair over K as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$, $D_{\mathrm{cris}}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V)^{G_K}$, and $H_{a,i}$ the horizontal sections of (Hf2) over K (Lemma 23.9, over a base field). Then the same holds: $H_{a,i} \in e_a\mathbf{N}_R$, $d_{a,i} = \mathrm{Ev}(H_{a,i}) \in \epsilon_{a+3}D_{\mathrm{cris}}(V)$, and items 1 and 2 hold as stated (still over $F \otimes_{\mathbb{Q}_\ell} L$). For $K = F$ this is the statement above.*

Proof. By (Ex1) and (Wm1), $(n_{a,i})_{a,i}$ is an $\mathbf{A}_{F,L}$-basis of $\mathbf{N} = \bigoplus_a \mathbf{N}_a$, of size nd. Write $H = H_{a,i}$, $x = n_{a,i}$ for the moment.

Sites and growth. Since $\gamma_u(e_a) = e_a$, e_aH is γ_u-invariant for every u and $e_aH - x = e_a(H - x) \in X\mathbf{N}_E$; by the uniqueness in (Hf2), $e_aH = H$. By (Hf3) the coordinates $\sum_k c_k X^k$ of H in the basis $(n_{b,j})$ satisfy $\ell^{s_\ell(k)} c_k \in \Lambda$ with $s_\ell(k) \leq k\ell/(\ell-1)^2 \leq 2k$, so they lie in R and $H \in e_a\mathbf{N}_R$.

Invariance and the site of $d_{a,i}$. For $\sigma \in G_F$, (Ev3) and (Hf2) give

$$\sigma(\mathrm{Ev}(H)) = \mathrm{Ev}(\gamma_{\chi_{\mathrm{cyc}}(\sigma)}(H)) = \mathrm{Ev}(H),$$

so $d_{a,i} \in D_{\mathrm{cris}}(V)$. From the defining formula of Ev, $\mathrm{Ev}(fW) = e(f)\mathrm{Ev}(W)$ for $f \in R$, $W \in \mathbf{N}_R$; so $d_{a,i} = e(e_a)d_{a,i}$ with $e(e_a) = \varphi^3(\iota_\Lambda(e_a))$ (Ev2). As ι is φ-equivariant (Lemma 23.8(4)) and $\varphi(e_b) = e_{b+1}$, $\varphi^3(\iota_\Lambda(e_a)) = \iota_\Lambda(e_{a+3}) = \epsilon_{a+3}$. The elements ϵ_b lie in $F \otimes_{\mathbb{Q}_\ell} L \subseteq B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} L$ ($\iota|_{\mathcal{O}_F} = \varphi \circ w$ has the same image as the canonical $w\colon W(k) \to A_{\mathrm{inf}}$, φ being an automorphism of $W(k)$), and they are orthogonal idempotents with sum 1, as the e_b are.

Item 1. Write $H_{a,i} = \sum_{b,j} P_{(b,j),(a,i)} n_{b,j}$ with $P \in M_{nd}(R)$; as $H_{a,i} - n_{a,i} \in X\mathbf{N}_E$, $P - I$ has entries in XR. Put $v_{b,j} = \mathrm{Ev}(n_{b,j}) = \varphi^3((\rho \otimes 1)(\iota_T(n_{b,j})))$. Then $d_{a,i} = \sum_{b,j} e(P_{(b,j),(a,i)}) v_{b,j}$, and $e(P) \in M_{nd}(A)$ with $e(P) - I \in \ell M_{nd}(A)$ by (Ev2) ($e(XR) \subseteq \ell A$, as $e(1+f) \in 1 + \ell A$ for $f \in XR$); since A is ℓ-adically complete, $\det e(P) \in 1 + \ell A$ is a unit and $e(P) \in \mathrm{GL}_{nd}(A)$. So it suffices to prove the basis property of item 1 for $(v_{b,j})$ in place of $(d_{b,j})$ (the matrix $e(P)$ respects the components: $P_{(b,j),(a,i)} = 0$ for $b \neq a$, as $H_{a,i} \in e_a \mathbf{N}_E$). By (Ex2), $\iota_T(gx) = \iota_L(g)\iota_T(x)$ for $g \in \mathbf{A}_{F,L}$ (write $g = \sum_j f_j \otimes \lambda_j$ with $f_j \in \mathbf{A}_F$, $\lambda_j \in \mathcal{O}_L$), and $A_{\mathrm{inf}}[1/\mu] \cdot \iota_T(\mathbf{N}) = A_{\mathrm{inf}}[1/\mu] \otimes T$; so the nd elements $\iota_T(n_{b,j})$ generate $A_{\mathrm{inf}}[1/\mu] \otimes_{\mathbb{Z}_\ell} T$ over $A_{\mathrm{inf}}[1/\mu] \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$. This module is free of rank d over $A_{\mathrm{inf}}[1/\mu] \otimes \mathcal{O}_L$; decomposing along the idempotents $\iota_\Lambda(e_b)$, each component $\iota_\Lambda(e_b)(A_{\mathrm{inf}}[1/\mu] \otimes T)$ is free of rank d over $\iota_\Lambda(e_b)(A_{\mathrm{inf}}[1/\mu] \otimes \mathcal{O}_L)$ and generated by the d elements $\iota_T(n_{b,j}) = \iota_\Lambda(e_b)\iota_T(n_{b,j})$, $j \leq d$; a surjective endomorphism of a finitely generated module over a commutative ring is bijective, so these form a basis, and $(\iota_T(n_{b,j}))_{b,j}$ is a basis of $A_{\mathrm{inf}}[1/\mu] \otimes T$ over $A_{\mathrm{inf}}[1/\mu] \otimes \mathcal{O}_L$ in the sense that every element is uniquely $\sum_{b,j} c_{b,j} \iota_T(n_{b,j})$ with $c_{b,j} \in \iota_\Lambda(e_b)(A_{\mathrm{inf}}[1/\mu] \otimes \mathcal{O}_L)$. Applying the ring homomorphism $\rho \otimes 1$ preserves this property over $B_{\mathrm{cris}} \otimes \mathcal{O}_L$. Finally $\varphi^3 = \varphi^3 \otimes 1$ fixes $1 \otimes T$; in a fixed $\mathcal{O}_L$-basis (t_k) of T, the matrix of $(\varphi^3 \rho \iota_T(n_{b,j}))$ is φ^3 applied to that of $(\rho\iota_T(n_{b,j}))$, and a ring endomorphism maps invertible matrices to invertible matrices (and $\varphi^3(\iota_\Lambda(e_b)) = \epsilon_{b+3}$); so $(v_{b,j})$ has the same basis property, with $v_{b,j} \in \epsilon_{b+3}(B_{\mathrm{cris}} \otimes T)$. So for each b, $(v_{b,j})_{j \leq d}$ is a basis of $\epsilon_{b+3}(B_{\mathrm{cris}} \otimes T)$ over $\epsilon_{b+3}(B_{\mathrm{cris}} \otimes \mathcal{O}_L)$; with the invertible block-diagonal $e(P)$ the same holds for $(d_{a,i})_{i \leq d}$, and inverting ℓ ($B_{\mathrm{cris}} \otimes_{\mathbb{Z}_\ell} T = B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V$, $B_{\mathrm{cris}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L = B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} L$) gives item 1. Since $\sum_b \epsilon_b = 1$, every element of $B_{\mathrm{cris}} \otimes V$ is then uniquely $\sum_{a,i} c_{a,i} d_{a,i}$ with $c_{a,i} \in \epsilon_{a+3}(B_{\mathrm{cris}} \otimes L)$.

Item 2. Let $v \in D_{\mathrm{cris}}(V)$ and write $v = \sum_{a,i} c_{a,i} d_{a,i}$ with $c_{a,i} \in \epsilon_{a+3}(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} L)$, uniquely by the last sentence of the proof of item 1. For $\sigma \in G_F$, $v = \sigma(v) = \sum \sigma(c_{a,i}) d_{a,i}$ (the $d_{a,i}$ are invariant, $\sigma(\epsilon_b) = \epsilon_b$), so $\sigma(c_{a,i}) = c_{a,i}$ by uniqueness. Now $(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} L)^{G_F} = B_{\mathrm{cris}}^{G_F} \otimes_{\mathbb{Q}_\ell} L = F \otimes_{\mathbb{Q}_\ell} L$, because L is finite free over $\mathbb{Q}_\ell$ with trivial action and $B_{\mathrm{cris}}^{G_F} = F$ for F unramified [94]. So $c_{a,i} \in \epsilon_{a+3}(F \otimes L)$; conversely every such combination is invariant. Hence $(d_{a,i})$ is an $F \otimes L$-basis of $D_{\mathrm{cris}}(V)$, and multiplying by ϵ_{a+3} (which kills $d_{b,j}$ for $b \neq a$) shows that $(d_{a,i})_i$ is a basis of $\epsilon_{a+3} D_{\mathrm{cris}}(V)$ over $\epsilon_{a+3}(F \otimes L)$, which is L via the isomorphism $\mathcal{O}_F \otimes \mathcal{O}_L \cong \prod_b \mathcal{O}_L$ of Definition 23.2 (transported by ι_Λ).

Over a base field. For a pair over K the proof is the same, with the following three substitutions. (i) Sites and growth: H is the horizontal section of (Hf2) over K (γ_u-invariant for $u \in \Gamma_K$); $e_a H$ is γ_u-invariant for $u \in \Gamma_K$, so $e_a H = H$ by the uniqueness there; its coordinates satisfy $\ell^{s_{\ell,|\Delta_K|}(k)} c_k \in \Lambda$ with $s_{\ell,|\Delta_K|}(k) \leq 3k/2 \leq 2k$ (here d is $\dim_L V$, so the index of s is written $|\Delta_K|$), so $H \in e_a \mathbf{N}_R$. (ii) Invariance: for $\sigma \in G_K$, (Ev3) over K (Lemma 23.11, over a base field) and the invariance of H under $\gamma_{\chi_{\mathrm{cyc}}(\sigma)}$, $\chi_{\mathrm{cyc}}(\sigma) \in \Gamma_K$, give $\sigma(d_{a,i}) = d_{a,i}$, so $d_{a,i} \in D_{\mathrm{cris}}(V)$; the computation of the site is unchanged. Item 1 involves no Galois group: it uses (Ex2) and $A_{\mathrm{inf}}[1/\mu] \cdot \iota_T(\mathbf{N}) = A_{\mathrm{inf}}[1/\mu] \otimes T$, which Proposition 23.7 gives for pairs over K. (iii) Item 2: the uniqueness argument runs with $\sigma \in G_K$, and $(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} L)^{G_K} = B_{\mathrm{cris}}^{G_K} \otimes_{\mathbb{Q}_\ell} L = F \otimes_{\mathbb{Q}_\ell} L$, because $B_{\mathrm{cris}}^{G_K}$ is the maximal unramified subextension K_0 of $K/\mathbb{Q}_\ell$ [94], and $K_0 = F$: $F \subseteq K_0$, and $K \subseteq F(\mu_\ell)$ with $F(\mu_\ell)/F$ totally ramified, so K/F is totally ramified. This is the only place where the base field enters beyond the substitution $G_F \to G_K$, $\mathbb{Z}_\ell^\times \to \Gamma_K$. □

Lemma 23.11. *Let ℓ, F, L, (τ_a) and a pair (V,T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$, and ρ: $A_{\mathrm{inf}}[1/\mu] \to B_{\mathrm{cris}}$ the G_F- and φ-equivariant ring homomorphism extending $A_{\mathrm{inf}} \subset A_{\mathrm{cris}} \subset B_{\mathrm{cris}}$. Let $R \subseteq E[[X]]$ be the set of series $\sum_k a_k X^k$ with $\ell^{2k} a_k \in \Lambda$ for all k, and $\mathbf{N}_R = \mathbf{N} \otimes_{\mathbf{A}_{F,L}} R \subseteq \mathbf{N}_E$. Let $\iota_\Lambda \colon E \to (A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L)[1/\ell]$ be $\iota|_{\mathcal{O}_F} \otimes \mathrm{id}$, extended to E, and $A = A_{\mathrm{cris}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$, with φ and G_F acting on the first factor. Then:*

(Ev1) *R is a subring of $E[[X]]$ containing $\mathbf{A}_{F,L}$, stable under $f \mapsto f \circ h_u$ for every $u \in \mathbb{Z}_\ell^\times$,*

and $\mathbf{N}_R$ is stable under the operators γ_u of $\mathbf{N}_E$.

(Ev2) *For $f = \sum a_k X^k \in R$ the series $e(f) = \sum_{k\ge 0} \varphi^3(\iota_\Lambda(a_k))\varphi^3(\mu)^k$ converges in A, its k-th term lying in $\ell^k A$. The map $e\colon R \to A$ is a ring homomorphism; $e(f) = \varphi^3(\iota_L(f))$ for $f \in \mathbf{A}_{F,L}$, where $\iota_L = \iota \otimes \mathrm{id}\colon \mathbf{A}_{F,L} = \mathbf{A}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L \to A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$; $e(f) \in 1 + \ell A$ if $f \in 1 + XR$; and $\sigma(e(f)) = e(f \circ h_{\chi_{\mathrm{cyc}}(\sigma)})$ for $\sigma \in G_F$.*

(Ev3) *There is a unique additive map $\mathrm{Ev}\colon \mathbf{N}_R \to B_{\mathrm{cris}} \otimes_{\mathbb{Z}_\ell} T$ with $\mathrm{Ev}(f \otimes x) = e(f) \cdot \varphi^3((\rho \otimes 1)(\iota_T(x)))$ for $f \in R$, $x \in \mathbf{N}$ (with $\varphi = \varphi \otimes 1$ and $B_{\mathrm{cris}} \otimes_{\mathbb{Z}_\ell} T$ a module over $B_{\mathrm{cris}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L \supseteq A$). It satisfies $\mathrm{Ev}(x) = \varphi^3((\rho \otimes 1)(\iota_T(x)))$ for $x \in \mathbf{N}$, and $\sigma(\mathrm{Ev}(W)) = \mathrm{Ev}(\gamma_{\chi_{\mathrm{cyc}}(\sigma)}(W))$ for $W \in \mathbf{N}_R$ and $\sigma \in G_F$ acting diagonally.*

Over a base field. *Let K be a field with $F \subseteq K \subseteq F(\mu_\ell)$, Γ_K as in Definition 23.2, (V,T) a pair over K as in Proposition 23.7 and $\mathbf{N} = \mathbf{N}(T)$. Then (Ev1)–(Ev3) hold with $u \in \Gamma_K$ in the last clause of (Ev1) and with $\sigma \in G_K$ in (Ev3); (Ev2) is unchanged. For $K = F$ this is the statement above.*

Level s. *Let ℓ be any prime, $s \ge 1$ an integer with $s \ge 2$ if $\ell = 2$, and L, K ($F \subseteq K \subseteq F(\mu_{\ell^s})$) and a pair (V,T) over K as in (Ex4) of Proposition 23.7; let $\mathbf{N} = \mathbf{N}_s(T)$ with $\iota_{s,T}$, ι_s as in Lemma 23.8(6), $\iota_{s,L} = \iota_s \otimes \mathrm{id}$, and ρ, Λ, E, ι_Λ, A as above ($\iota_s = \iota$ on $\mathcal{O}_F$, so ι_Λ is unchanged). Let $R_s \subseteq E[[X]]$ be the set of series $\sum_k a_k X^k$ with $\ell^{(s+1)k} a_k \in \Lambda$ for all k, $\mathbf{N}_{R_s} = \mathbf{N} \otimes_{\mathbf{A}_{F,L}} R_s$, and for $f = \sum a_k X^k \in R_s$, $x \in \mathbf{N}$,*

$$e_s(f) = \sum_{k\ge 0} \varphi^{2s+1}(\iota_\Lambda(a_k))\, \varphi^{s+2}(\mu)^k, \qquad \mathrm{Ev}_s(f \otimes x) = e_s(f) \cdot \varphi^{2s+1}((\rho \otimes 1)(\iota_{s,T}(x))).$$

Then (Ev1)–(Ev3), in the form over a base field ($u \in \Gamma_K$, $\sigma \in G_K$), hold with R, $\mathbf{N}_R$, e, Ev, ι_L, ι_T, φ^3 replaced by R_s, $\mathbf{N}_{R_s}$, e_s, Ev_s, $\iota_{s,L}$, $\iota_{s,T}$, φ^{2s+1}; in particular $e_s(f) = \varphi^{2s+1}(\iota_{s,L}(f))$ for $f \in \mathbf{A}_{F,L}$, $\mathrm{Ev}_s(x) = \varphi^{2s+1}((\rho \otimes 1)(\iota_{s,T}(x)))$ for $x \in \mathbf{N}$, and $\sigma(\mathrm{Ev}_s(W)) = \mathrm{Ev}_s(\gamma_{\chi_{\mathrm{cyc}}(\sigma)}(W))$ for $W \in \mathbf{N}_{R_s}$, $\sigma \in G_K$. For $s = 1$ and ℓ odd ($R_1 = R$, $e_1 = e$, $\mathrm{Ev}_1 = \mathrm{Ev}$) this is the paragraph over a base field.

Proof. **(Ev1).** The coefficient of X^k in a product is $\sum_{i+j=k} a_i b_j$, and $\ell^{2k} a_i b_j = (\ell^{2i} a_i)(\ell^{2j} b_j) \in \Lambda$; so R is a ring, and it contains $\Lambda[[X]] = \mathbf{A}_{F,L}$. For $h \in X\mathbb{Z}_\ell[[X]]$ write $h^k = \sum_{j\ge k} h_j^{(k)} X^j$ with $h_j^{(k)} \in \mathbb{Z}_\ell$; the coefficient of X^j in $f \circ h$ is $\sum_{k\le j} a_k h_j^{(k)}$, and $\ell^{2j} a_k h_j^{(k)} \in \Lambda$ for $k \le j$; so $f \circ h \in R$. By (Ex1) and (Wm1) of Definition 23.2, every $\mathbf{N}_a$ is free of rank $r = \dim_L V$ over $e_a \mathbf{A}_{F,L}$, so $\mathbf{N} = \bigoplus_a \mathbf{N}_a$ is free over $\mathbf{A}_{F,L}$ and $\mathbf{N}_R = \bigoplus_i R\, n_i$ for a basis (n_i), and $\gamma_u(\sum_i w_i n_i) = \sum_j (G_u(w \circ h_u))_j n_j$ with $G_u \in M_r(\mathbf{A}_{F,L}) \subseteq M_r(R)$ the matrix of $\gamma_{u,\mathbf{N}}$ in this basis; hence $\gamma_u(\mathbf{N}_R) \subseteq \mathbf{N}_R$.

(Ev2). The module A is a finite free A_{cris}-module, hence ℓ-torsion-free and ℓ-adically complete and separated. By (Ac1), $\varphi^3(\mu)^k \in \ell^{3k} A_{\mathrm{cris}}$, and $\ell^{2k} a_k \in \Lambda$ gives $\varphi^3(\iota_\Lambda(a_k)) \in \ell^{-2k}(A_{\mathrm{inf}} \otimes \mathcal{O}_L)$; so the k-th term lies in $\ell^k A$ and the series converges. The map e is additive, and multiplicative because the product of two series in A whose terms tend to 0 is the sum of their Cauchy product, while $\varphi^3 \circ \iota_\Lambda$ is a ring homomorphism on E. If $f \in 1 + XR$, all terms of index $k \ge 1$ lie in ℓA and the term of index 0 is 1. For $f = \sum c_k X^k \in \mathbf{A}_{F,L}$ and $N \ge 0$, $f = \sum_{k<N} c_k X^k + X^N g_N$ with $g_N \in \mathbf{A}_{F,L}$, and since ι_L and φ^3 are ring homomorphisms with $\iota_L(X) = \mu$,

$$\varphi^3(\iota_L(f)) - \sum_{k<N} \varphi^3(\iota_\Lambda(c_k))\varphi^3(\mu)^k = \varphi^3(\mu)^N \varphi^3(\iota_L(g_N)) \in \ell^{3N} A;$$

so $e(f) = \varphi^3(\iota_L(f))$. Let $\sigma \in G_F$, $u = \chi_{\mathrm{cyc}}(\sigma)$. By Lemma 23.8(4), $\sigma \circ \iota = \iota \circ \gamma_u$; as γ_u fixes $\mathcal{O}_F$, σ fixes $\iota_\Lambda(E)$, and, commuting with φ, every $\varphi^3(\iota_\Lambda(a))$; and $\sigma(\varphi^3(\mu)) = \varphi^3(\sigma(\iota(X))) =$

$\varphi^3(\iota_L(h_u)) = e(h_u)$ by the previous identity. The ring automorphism σ of A preserves every $\ell^N A$, hence is continuous, so $\sigma(e(f)) = \sum_k \varphi^3(\iota_\Lambda(a_k))\, e(h_u)^k$. On the other hand, with $h = h_u$,

$$e(f \circ h_u) = \sum_j \sum_{k \le j} \varphi^3(\iota_\Lambda(a_k))\, h_j^{(k)}\, \varphi^3(\mu)^j,$$

whose term of index (j,k) lies in $\ell^{3j-2k}A \subseteq \ell^j A$; so the double family is summable in any order in A, and summing first over $j \ge k$ gives $\sum_k \varphi^3(\iota_\Lambda(a_k))\, e(h_u^k) = \sum_k \varphi^3(\iota_\Lambda(a_k))\, e(h_u)^k$, e being multiplicative.

(Ev3). The map $R \times \mathbf{N} \to B_{\mathrm{cris}} \otimes T$, $(f,x) \mapsto e(f)\,\varphi^3\rho(\iota_T(x))$ (writing ρ for $\rho \otimes 1$; $\iota_T(x) \in A_{\mathrm{inf}}[1/\mu] \otimes T$ by (Ex2)) is biadditive, and $\mathbf{A}_{F,L}$-balanced: for $g \in \mathbf{A}_{F,L}$, (Ex2) gives $\iota_T(gx) = \iota_L(g)\iota_T(x)$ (write $g = \sum_j f_j \otimes \lambda_j$ with $f_j \in \mathbf{A}_F$, $\lambda_j \in \mathcal{O}_L$), and ρ is a ring homomorphism, $A_{\mathrm{inf}} \otimes \mathcal{O}_L$-linear on $A_{\mathrm{inf}}[1/\mu] \otimes T$ and φ-equivariant, so $e(fg)\varphi^3\rho\iota_T(x) = e(f)\varphi^3(\iota_L(g))\varphi^3\rho\iota_T(x) = e(f)\varphi^3\rho\iota_T(gx)$. Hence Ev exists and is unique; $\mathrm{Ev}(1 \otimes x) = \varphi^3\rho\iota_T(x)$ as $e(1) = 1$. For $\sigma \in G_F$, $u = \chi_{\mathrm{cyc}}(\sigma)$: G_F acts $\mathcal{O}_L$-linearly on T, so $\sigma(b \cdot w) = \sigma(b)\sigma(w)$ for $b \in B_{\mathrm{cris}} \otimes \mathcal{O}_L$, $w \in B_{\mathrm{cris}} \otimes T$; σ commutes with φ and ρ, and $\sigma(\iota_T(x)) = \iota_T(\gamma_u(x))$ (Ex2); with (Ev2), $\sigma(e(f)\varphi^3\rho\iota_T(x)) = e(f \circ h_u)\varphi^3\rho\iota_T(\gamma_u(x)) = \mathrm{Ev}(\gamma_u(f \otimes x))$, and both sides of the identity are additive in W.

Over a base field. For a pair over K: the ring R, the map e and (Ev2) do not involve the pair; (Ev2) for $\sigma \in G_F$ contains the case $\sigma \in G_K \subseteq G_F$. In (Ev1), $\mathbf{N}$ is free over $\mathbf{A}_{F,L}$ by (Ex1) and (Wm1) over K, and $\gamma_{u,\mathbf{N}}$ is given for $u \in \Gamma_K$, so the argument gives $\gamma_u(\mathbf{N}_R) \subseteq \mathbf{N}_R$ for $u \in \Gamma_K$. In (Ev3) the pair enters only through (Ex2), which Proposition 23.7 gives for pairs over K with $\sigma \in G_K$ and $u = \chi_{\mathrm{cyc}}(\sigma) \in \Gamma_K$; ρ is G_K-equivariant as $G_K \subseteq G_F$. The rest of the argument is unchanged. No step uses that K is unramified.

Level s. (Ev1): the argument is the same, with $2k$ replaced by $(s+1)k$: $\ell^{(s+1)k}a_i b_j = (\ell^{(s+1)i}a_i)(\ell^{(s+1)j}b_j)$ for $i+j=k$, and $\ell^{(s+1)j}a_k h_j^{(k)} \in \Lambda$ for $k \le j$; $\mathbf{N}$ is free over $\mathbf{A}_{F,L}$ by (Wm1) of level s, and $\gamma_{u,\mathbf{N}}$ is given for $u \in \Gamma_K$. (Ev2): by (Ac1) of Lemma 23.12 (paragraph Every prime), $\varphi^{s+2}(\mu)^k \in \ell^{(s+2)k}A_{\mathrm{cris}}$, and $\ell^{(s+1)k}a_k \in \Lambda$ gives $\varphi^{2s+1}(\iota_\Lambda(a_k)) \in \ell^{-(s+1)k}(A_{\mathrm{inf}} \otimes \mathcal{O}_L)$; so the k-th term lies in $\ell^k A$, the series converges, e_s is a ring homomorphism, and $e_s(f) \in 1 + \ell A$ for $f \in 1 + XR_s$, as before. Since $\varphi[x] = [x^\ell]$ and $(\varepsilon^{1/\ell^{s-1}})^{\ell^{s-1}} = \varepsilon$, $\varphi^{s-1}(\mu_{\varepsilon,s}) = \mu$, so $\varphi^{2s+1}(\iota_{s,L}(X)) = \varphi^{2s+1}(\mu_{\varepsilon,s}) = \varphi^{s+2}(\mu)$; the identity $e_s(f) = \varphi^{2s+1}(\iota_{s,L}(f))$ for $f \in \mathbf{A}_{F,L}$ follows as before, the remainder after N terms lying in $\varphi^{s+2}(\mu)^N A \subseteq \ell^{(s+2)N}A$. For $\sigma \in G_K$ and $u = \chi_{\mathrm{cyc}}(\sigma)$, Lemma 23.8(6) gives $\sigma \circ \iota_s = \iota_s \circ \gamma_u$; so σ fixes every $\varphi^{2s+1}(\iota_\Lambda(a))$ and $\sigma(\varphi^{s+2}(\mu)) = \varphi^{2s+1}(\iota_{s,L}(h_u)) = e_s(h_u)$; in $e_s(f \circ h_u)$ the term of index (j,k), $k \le j$, lies in $\ell^{(s+2)j-(s+1)k}A \subseteq \ell^j A$, and the rearrangement is as before. (Ev3): as before, with φ^{2s+1} in place of φ^3 and (Ex2) of level s (Proposition 23.7, (Ex4)): $\iota_{s,T}(x) \in A_{\mathrm{inf}}[1/\mu] \otimes T$, $\iota_{s,T}(gx) = \iota_{s,L}(g)\iota_{s,T}(x)$ for $g \in \mathbf{A}_{F,L}$, and $\sigma(\iota_{s,T}(x)) = \iota_{s,T}(\gamma_u(x))$ for $\sigma \in G_K$; ρ exists and is G_K- and φ-equivariant for every prime because μ is a unit of B_{cris} ((Ac2), Every prime). □

Lemma 23.12. *Let ℓ be an odd prime, use the notation of Theorem B.40 with $p = \ell$, let F be a finite unramified extension of $\mathbb{Q}_\ell$ and write $\mu = \mu_\varepsilon$. Then:*

(Ac1) Divisibility *$\tilde{\xi} \in \ell A_{\mathrm{cris}}$ and $\mu^{\ell-1} \in \ell A_{\mathrm{cris}}$. Hence $\varphi^n(\mu) \in \ell^n A_{\mathrm{cris}}$ for every $n \ge 0$, and $\mu^{(\ell-1)N} A_{\mathrm{inf}} \subseteq \ell^N A_{\mathrm{cris}}$ for every $N \ge 0$.*

(Ac2) Units *For every $j \ge 0$, $\varphi^j(\mu)$ is a unit of $W(\mathbb{C}_\ell^\flat)$ and of B_{cris}. Let S be the subring of $W(\mathbb{C}_\ell^\flat)$ generated by A_{inf} and the elements $\varphi^j(\mu)^{-1}$, $j \ge 0$. Then S is stable under φ and under G_F, and there is a unique ring homomorphism $\rho_S \colon S \to B_{\mathrm{cris}}$ extending the inclusion $A_{\mathrm{inf}} \subset A_{\mathrm{cris}} \subset B_{\mathrm{cris}}$; it satisfies $\rho_S \circ \varphi = \varphi \circ \rho_S$ and $\rho_S \circ \sigma = \sigma \circ \rho_S$ for every $\sigma \in G_F$.*

Every prime. *(Ac1) and (Ac2) hold for every prime ℓ, including $\ell = 2$, with F a finite unramified extension of $\mathbb{Q}_\ell$; for $\ell = 2$, (Ac1) reads $\tilde{\xi} \in 2A_{\mathrm{cris}}$ and $\mu \in 2A_{\mathrm{cris}}$. For ℓ odd this is the statement above.*

Proof. We use the classical facts on A_{cris} [94], [52]: A_{cris} is the ℓ-adic completion of the divided power envelope of A_{inf} with respect to $\ker\theta = \xi_0 A_{\mathrm{inf}}$ (Ai1); it is ℓ-torsion-free and ℓ-adically complete and separated, $A_{\mathrm{inf}} \subset A_{\mathrm{cris}} \subset A_{\mathrm{cris}}[1/\ell] = B^+_{\mathrm{cris}}$, the Frobenius φ and the action of $G_{\mathbb{Q}_\ell}$ on A_{inf} extend to A_{cris} and commute; and $t = \sum_{k\geq 1}(-1)^{k-1}\mu^k/k$, the series converging in A_{cris}.

(Ac1). By definition of ξ_0 in Theorem B.40, $\xi_0 = \sum_{i=0}^{\ell-1}[\varepsilon^{1/\ell}]^i$, so $\varphi(\xi_0) = \sum_{i=0}^{\ell-1}[\varepsilon]^i = \tilde{\xi}$. The Frobenius of $A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}^\flat_\ell})$ lifts the ℓ-th power map, so $\tilde{\xi} = \varphi(\xi_0) \in \xi_0^\ell + \ell A_{\mathrm{inf}}$; since $\xi_0 \in \ker\theta$ has divided powers in A_{cris}, $\xi_0^\ell = \ell!\,\xi_0^{[\ell]} \in \ell A_{\mathrm{cris}}$. Hence $\tilde{\xi} \in \ell A_{\mathrm{cris}}$. Next, $[\varepsilon] = 1 + \mu$ gives

$$\tilde{\xi} = \frac{(1+\mu)^\ell - 1}{\mu} = \sum_{i=1}^{\ell}\binom{\ell}{i}\mu^{i-1} \in \mu^{\ell-1} + \ell A_{\mathrm{inf}},$$

since ℓ divides $\binom{\ell}{i}$ for $0 < i < \ell$; so $\mu^{\ell-1} \in \ell A_{\mathrm{cris}}$. As $\varphi(\mu) = \mu\tilde{\xi}$, induction gives $\varphi^n(\mu) = \mu\prod_{j=0}^{n-1}\varphi^j(\tilde{\xi})$, and each $\varphi^j(\tilde{\xi})$ lies in $\varphi^j(\ell A_{\mathrm{cris}}) \subseteq \ell A_{\mathrm{cris}}$; so $\varphi^n(\mu) \in \ell^n A_{\mathrm{cris}}$. Finally $\mu^{(\ell-1)N} = (\mu^{\ell-1})^N \in \ell^N A_{\mathrm{cris}}$.

(Ac2): units. The reduction of $\varphi^j(\mu) = [\varepsilon^{\ell^j}] - 1$ modulo ℓ is $\varepsilon^{\ell^j} - 1 \neq 0$ in the field $\mathbb{C}^\flat_\ell$ ($\varepsilon \neq 1$), so $\varphi^j(\mu)$ is a unit of the complete discrete valuation ring $W(\mathbb{C}^\flat_\ell)$. In A_{cris} we have $t = \mu v$ with $v = \sum_{k\geq 1}(-1)^{k-1}\mu^{k-1}/k$: by (Ac1), $\mu^{k-1}/k \in \ell^{\lfloor (k-1)/(\ell-1)\rfloor - v_\ell(k)}A_{\mathrm{cris}}$, and $\lfloor (k-1)/(\ell-1)\rfloor \geq \lfloor (\ell^{v_\ell(k)}-1)/(\ell-1)\rfloor = \sum_{i<v_\ell(k)}\ell^i \geq v_\ell(k)$ with $\lfloor (k-1)/(\ell-1)\rfloor - v_\ell(k) \to \infty$, so the series converges in A_{cris}. Since t is a unit of $B_{\mathrm{cris}} = B^+_{\mathrm{cris}}[1/t]$, so is μ (with inverse vt^{-1}). Applying φ^j and using $\varphi(t) = \ell t$ [94], $\varphi^j(\mu)\varphi^j(v) = \ell^j t$, a unit of B_{cris}; so $\varphi^j(\mu)$ is a unit of B_{cris}.

(Ac2): the ring S and ρ_S. Let $\Sigma \subseteq A_{\mathrm{inf}}$ be the multiplicative set generated by the $\varphi^j(\mu)$, $j \geq 0$. As A_{inf} is a domain (Ai1) contained in $W(\mathbb{C}^\flat_\ell)$, in which the elements of Σ are units, the map $\Sigma^{-1}A_{\mathrm{inf}} \to W(\mathbb{C}^\flat_\ell)$, $a/s \mapsto as^{-1}$, is injective with image S; so $S \cong \Sigma^{-1}A_{\mathrm{inf}}$. By the universal property of localisation, a ring homomorphism $A_{\mathrm{inf}} \to B$ that maps Σ to units of B extends uniquely to S. Applied to $A_{\mathrm{inf}} \subset B_{\mathrm{cris}}$ (units by the previous paragraph), this gives existence and uniqueness of ρ_S. Stability: $\varphi(\varphi^j(\mu)) = \varphi^{j+1}(\mu) \in \Sigma$, so $\varphi(\Sigma) \subseteq \Sigma$ and $\varphi(S) \subseteq S$. For $\sigma \in G_F$ let $u = \chi_{\mathrm{cyc}}(\sigma)$. By Lemma 23.8(4), $\sigma(\mu) = \sigma(\iota(X)) = \iota(\gamma_u(X))$, and by Lemma 23.8(2), $\gamma_u(X) = Xw$ with $w \in \mathbf{A}_F^\times$; so $\sigma(\mu) = \mu\beta_\sigma$ with $\beta_\sigma = \iota(w) \in A_{\mathrm{inf}}^\times$ (ι is a ring homomorphism, so it maps units to units). Since σ commutes with φ, $\sigma(\varphi^j(\mu)) = \varphi^j(\mu)\varphi^j(\beta_\sigma)$ with $\varphi^j(\beta_\sigma) \in A_{\mathrm{inf}}^\times$; so $\sigma(\varphi^j(\mu))^{-1} = \varphi^j(\mu)^{-1}\varphi^j(\beta_\sigma)^{-1} \in S$ and $\sigma(S) \subseteq S$. Equivariance: $\rho_S \circ \varphi$ and $\varphi \circ \rho_S$ are ring homomorphisms $S \to B_{\mathrm{cris}}$ which agree on A_{inf}, because φ on B_{cris} extends φ on A_{inf}; by the uniqueness in the universal property of $S \cong \Sigma^{-1}A_{\mathrm{inf}}$ they are equal. The same argument applies to $\rho_S \circ \sigma$ and $\sigma \circ \rho_S$.

Every prime. No step above uses that ℓ is odd. The facts on A_{cris} recalled at the start hold for every prime [94]. In (Ac1), $\varphi(\xi_0) \in \xi_0^\ell + \ell A_{\mathrm{inf}}$, $\xi_0^\ell = \ell!\,\xi_0^{[\ell]}$ and $\ell \mid \binom{\ell}{i}$ for $0 < i < \ell$ hold for every prime (for $\ell = 2$: $\tilde{\xi} = 2 + \mu$, and $\mu^{\ell-1} = \mu$). In (Ac2), the bound $\lfloor (k-1)/(\ell-1)\rfloor \geq \lfloor (\ell^{v_\ell(k)}-1)/(\ell-1)\rfloor \geq v_\ell(k)$ uses only $k \geq \ell^{v_\ell(k)}$, and $\lfloor (k-1)/(\ell-1)\rfloor - v_\ell(k) \geq (k-1)/(\ell-1) - 1 - \log_\ell k \to \infty$; the stability of S under G_F uses items (2) and (4) of Lemma 23.8, which are stated for every prime. □

Lemma 23.13. *Let ℓ, F, L, $(\tau_b)_{b\in\mathbb{Z}/n}$ and a pair (V,T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$*

of rank r, $D = D_{\mathrm{cris}}(V)$, and $c \in \mathbb{Z}/n$. For an integer $q \geq 0$ and $f = \sum_{i\geq 0} a_i X^i \in E[[X]]$ put

$$\mathrm{ev}^{(q)}(f) = \sum_{i\geq 0} \varphi^q(\iota_\Lambda(a_i))\,\varphi^q(\mu)^i \;\in\; B^+_{\mathrm{dR}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L,$$

and for a frame $\mathbf{n} = (n_j)_{j\leq r}$ at c and $W = \sum_j c_j n_j \in e_c\mathbf{N}_E$ ($c_j \in e_cE[[X]]$) put $G^{(q)}(W) = \sum_j \mathrm{ev}^{(q)}(c_j)\,\varphi^q(\iota_T(n_j)) \in B_{\mathrm{dR}} \otimes_{\mathbb{Z}_\ell} T$.

1. *The series $\mathrm{ev}^{(q)}(f)$ converges for the topology of the discrete valuation ring B^+_{dR}, $G^{(q)}$ does not depend on the frame, and for the horizontal sections H_k through a frame (n_k) at c, with $d_k = \mathrm{Ev}(H_k)$,*
$$G^{(q)}(H_k) = \varphi^{q-3}(d_k) \qquad (q \geq 0),$$
where φ^{q-3} is the Frobenius of D (bijective on D) and $D \subset B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V \subset B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V$.
2. *Let $e \geq 0$, $a = c + e$, $H\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$, $\Pi = \Lambda_{a,e}(H)$, and $m \geq 1$ an integer with $m \geq 2H(a-s)$ for $0 \leq s < e$. Then for every frame $\mathbf{n}$ of $\mathbf{N}$ at c, $\mathbf{N}$ has the property (Ac3) at (c, e, H, m): for every $y \in \mathbf{N}_a$ with $\Pi y \in P_e(\mathbf{N}_c)$ there are $t, \kappa \geq 0$ and an arc coordinate vector f of $\ell^t y$ such that $\ell^\kappa f$ is Hodge-graded, i.e. has a section decomposition $(g_i, \mathbf{v}_i, \alpha_i)$ with $\delta_e(\mathbf{v}_i) \in \varphi^{s+4}\big(\mathrm{Fil}^{-\alpha_i(s)}D\big)$ for all i and $s < e$.*

Proof. Throughout, $B = B^+_{\mathrm{dR}}$, a complete discrete valuation ring with residue map θ (Theorem B.40 (Ai3)); ℓ is invertible in B, and $B_{\mathrm{dR}} = B[1/t]$ is its fraction field.

Step 0: the elements ξ, $\varphi^j(\xi)$, $\varphi^j(\mu)$ in B. By Theorem B.41, $\tilde\theta = \theta \circ \varphi^{-1}$ is surjective with kernel $\tilde\xi A_{\mathrm{inf}}$; hence $\ker\theta = \xi A_{\mathrm{inf}}$ with $\xi = \varphi^{-1}(\tilde\xi) = \sum_{i<\ell}[\varepsilon^{1/\ell}]^i$, and ξ is a uniformiser of B; as $t \in \mathrm{Fil}^1 \setminus \mathrm{Fil}^2$, $t \in \xi B^\times$. With $\zeta_{\ell^k} = \theta([\varepsilon^{1/\ell^k}])$ a primitive ℓ^k-th root of unity: $\theta(\varphi^j(\xi)) = \ell$ for $j \geq 1$ and $\theta(\varphi^j(\xi)) = \sum_{i<\ell}\zeta_{\ell^{1-j}}^i = (\zeta_{\ell^{-j}} - 1)/(\zeta_{\ell^{1-j}} - 1) \neq 0$ for $j \leq -1$; so $\varphi^j(\xi) \in B^\times$ for $j \neq 0$. Further $\theta(\varphi^q(\mu)) = 0$ for $q \geq 0$, so $\varphi^q(\mu) \in \xi A_{\mathrm{inf}}$, and $\theta(\varphi^{-k}(\mu)) = \zeta_{\ell^k} - 1 \neq 0$ for $k \geq 1$. Since $\iota(Q) = \tilde\xi$ (Lemma 23.8 (4)) and $\iota\varphi = \varphi\iota$, $\iota(Q_{s'}) = \varphi^{s'+1}(\xi)$.

Step 1: $\mathrm{ev}^{(q)}$. The coefficients $\iota_\Lambda(a_i)$ lie in $(W(k)\otimes\mathcal{O}_L)[1/\ell] \subset B \otimes_{\mathbb{Z}_\ell}\mathcal{O}_L$ and $\varphi^q(\mu)^i \in \xi^i B$, so the series converges ξ-adically and $\mathrm{ev}^{(q)}\colon E[[X]] \to B \otimes \mathcal{O}_L$ is a ring homomorphism. (a) For $f \in \mathbf{A}_{F,L}$, $\mathrm{ev}^{(q)}(f) = \varphi^q(\iota_L(f))$ ($\iota_L = \iota \otimes \mathrm{id}$): the tail $\sum_{i\geq b}$ of the series $\varphi^q(\iota_L(f))$, convergent in $A_{\mathrm{inf}} \otimes \mathcal{O}_L$, lies in $\varphi^q(\mu)^b A_{\mathrm{inf}} \otimes \mathcal{O}_L \subseteq \xi^b B \otimes \mathcal{O}_L$, so both sides agree modulo ξ^b for every b. Since $\mathrm{ev}^{(q)}(Q) = \varphi^{q+1}(\xi)$ is a unit, $\mathrm{ev}^{(q)}$ extends to $\mathbf{A}_{F,L}[1/Q]$ with the same formula. (b) $\mathrm{ev}^{(q)}(\varphi(f)) = \mathrm{ev}^{(q+1)}(f)$: $\varphi(f) = \sum\varphi(a_i)\varphi(X)^i$, $\iota_\Lambda\varphi = \varphi\iota_\Lambda$, $\mathrm{ev}^{(q)}(\varphi(X)) = \varphi^q(\iota(\varphi X)) = \varphi^{q+1}(\mu)$, and $\mathrm{ev}^{(q)}$ is continuous for the X-adic and ξ-adic topologies.

Step 2: $G^{(q)}$. If $\mathbf{n}'$ is another frame at c and $n_j = \sum_l g_{jl}n'_l$ ($g_{jl} \in e_c\mathbf{A}_{F,L}$), then $\iota_T(n_j) = \sum_l \iota_L(g_{jl})\iota_T(n'_l)$ by (Ex2), so $\varphi^q(\iota_T(n_j)) = \sum_l \mathrm{ev}^{(q)}(g_{jl})\varphi^q(\iota_T(n'_l))$ by Step 1 (a), and $G^{(q)}$ computed in either frame agrees. $G^{(q)}$ is additive with $G^{(q)}(gW) = \mathrm{ev}^{(q)}(g)G^{(q)}(W)$ and $G^{(q)}(x) = \varphi^q(\iota_T(x))$ for $x \in \mathbf{N}_c$. *Frobenius*: for $W \in e_c\mathbf{N}_E$, $\varphi(W) \in e_{c+1}\mathbf{N}_E$ (Q is a unit of $E[[X]]$, $Q(0) = \ell$) and $G^{(q)}(\varphi(W)) = G^{(q+1)}(W)$: with a frame $\mathbf{n}'$ at $c+1$ and $\varphi(n_j) = \sum_l A_{jl}n'_l$, $A_{jl} \in \mathbf{A}_{F,L}[1/Q]$, (Ex2) gives $\varphi(\iota_T(n_j)) = \sum_l \iota_L(A_{jl})\iota_T(n'_l)$ in $W(\mathbb{C}^\flat_\ell)\otimes T$, hence in $A_{\mathrm{inf}}[1/\mu, 1/\tilde\xi]\otimes T$, and by Step 1

$$\begin{aligned} G^{(q)}(\varphi W) &= \sum_l \mathrm{ev}^{(q)}\Big(\sum_j \varphi(c_j)A_{jl}\Big)\varphi^q(\iota_T n'_l) \\ &= \sum_j \mathrm{ev}^{(q+1)}(c_j)\,\varphi^q\Big(\sum_l \iota_L(A_{jl})\iota_T(n'_l)\Big) = G^{(q+1)}(W). \end{aligned}$$

Step 3: the constant Frobenius matrix of horizontal frames. Let (H_k), (H'_l) be the horizontal sections through frames at c and $c+1$. The operators γ_u of $\mathbf{N}_E$ commute with φ

((Wm3), and $\varphi(f)\circ h_u=\varphi(f\circ h_u)$ since $h_u(\varphi(X))=(1+X)^{\ell u}-1=\varphi(X)\circ h_u$), so $\varphi(H_k)$ is fixed by all γ_u. Write $\varphi(H_k)=\sum_l P_{kl}H'_l$ with $P_{kl}\in e_{c+1}E[[X]]$ ((Hf2): the H'_l are a basis); then $P_{kl}\circ h_u=P_{kl}$ for all u, and comparing the lowest non-constant terms ($h_u(X)=uX+\dots$, u not a root of unity) shows $P_{kl}\in e_{c+1}E$. The matrix P is invertible: by (Wm2) $\varphi(\mathbf{N}_c)$ generates $\mathbf{N}_{c+1}[1/Q]$, so $(\varphi(H_k))$ is an $E[[X]]$-basis of $e_{c+1}\mathbf{N}_E$.

Step 4: comparison with Ev **at** $q=3,4$**.** Let $h\in e_cE[[X]]$ be a coordinate of some H_k, so $\ell^{2i}a_i\in\Lambda$ (Lemma 23.11, Lemma 23.10). By Lemma 23.11 (Ev2) the series $e(h)=\sum_i\varphi^3(\iota_\Lambda(a_i))\varphi^3(\mu)^i$ converges ℓ-adically in $A_{\mathrm{cris}}\otimes\mathcal{O}_L$ with i-th term in $\ell^iA_{\mathrm{cris}}\otimes\mathcal{O}_L$, and so does $\varphi(e(h))=\sum_i\varphi^4(\iota_\Lambda(a_i))\varphi^4(\mu)^i$ (φ is ℓ-adically continuous on A_{cris}). For $b\geq 1$, the image of A_{cris} (the ℓ-adic completion of $A_{\mathrm{inf}}[\xi^j/j!:j\geq 1]$) in $B/\xi^bB=A_{\mathrm{inf}}[1/\ell]/\xi^b$ ((Ai3)) lies in $((b-1)!)^{-1}A_{\mathrm{inf}}/\xi^bA_{\mathrm{inf}}$, which is ℓ-adically separated, and the map is ℓ-adically continuous; the terms with $i\geq b$ vanish modulo ξ^b. Hence the images in $B\otimes\mathcal{O}_L$ of $e(h)$ and $\varphi(e(h))$ agree modulo every ξ^b with $\mathrm{ev}^{(3)}(h)$ and $\mathrm{ev}^{(4)}(h)$, so are equal to them. With $\mathrm{Ev}(H_k)=\sum_j e(h_{kj})\varphi^3(\rho\iota_T(n_j))$ (Ev3) this gives, in $B_{\mathrm{dR}}\otimes T$: $d_k=G^{(3)}(H_k)$ and $\varphi(d_k)=G^{(4)}(H_k)$. By Steps 2 and 3, $G^{(4)}(H_k)=G^{(3)}(\varphi H_k)=\sum_l\varphi^3(\iota_\Lambda(P_{kl}))\,G^{(3)}(H'_l)$, so, $B_{\mathrm{cris}}\to B_{\mathrm{dR}}$ being injective,

$$\varphi(d_k)=\sum_l\varphi^3(\iota_\Lambda(P_{kl}))\,d'_l\qquad\text{in } D.\qquad(\mathrm{R})$$

Step 5: proof of item 1. Convergence was shown in Step 1 and independence of the frame in Step 2. We show $G^{(q)}(H)=\varphi^{q-3}(d)$ for all horizontal frames at all sites, by induction upwards and downwards from $q=3$ (Step 4). If it holds for q (all sites), then by Steps 2, 3 and (R) (with φ^{q-3} semilinear: $\varphi^{q-3}(\varphi^3(\iota_\Lambda(P))d')=\varphi^q(\iota_\Lambda(P))\varphi^{q-3}(d')$)

$$G^{(q+1)}(H_k)=\sum_l\varphi^q(\iota_\Lambda(P_{kl}))\varphi^{q-3}(d'_l)=\varphi^{q-3}(\varphi(d_k))=\varphi^{q-2}(d_k).$$

If it holds for $q+1\geq 1$ (all sites), the same identity read backwards gives $\sum_l\varphi^q(\iota_\Lambda(P_{kl}))(G^{(q)}(H'_l)-\varphi^{q-3}(d'_l))=0$; the matrix $\varphi^q(\iota_\Lambda(P))$ is invertible, so $G^{(q)}(H'_l)=\varphi^{q-3}(d'_l)$ for the horizontal frame at $c+1$; as c and the frames are arbitrary, this is the claim for q.

Step 6: the level map. Fix e, a, H, Π, m, the frame $\mathbf{n}$ as in item 2, and $s<e$; put $q=e-s-1\geq 0$, $h=H(a-s)$. Extend ι_T ι_L-semilinearly to $\mathbf{N}[1/\lambda_e]\to A_{\mathrm{inf}}[1/\mu,1/\iota(\lambda_e)]\otimes T$; then $\iota_T(\varphi^e(z))=\varphi^e(\iota_T(z))$ for $z\in\mathbf{N}$ (iterate (Ex2); $\varphi^e(\mu)=\mu\,\iota(\lambda_e)$). By Step 0, $\varphi^{-(s+1)}$ maps $A_{\mathrm{inf}}[1/\mu,1/\iota(\lambda_e)]$ into B_{dR}, and $A_{\mathrm{inf}}[1/\mu]$ into B. Put $\Lambda_s=\varphi^{-(s+1)}\circ\iota_T\colon\mathbf{N}[1/\lambda_e]\to B_{\mathrm{dR}}\otimes T$ and $\lambda_s=\varphi^{-(s+1)}\circ\iota_L$ on $\mathbf{A}_{F,L}[1/\lambda_e]$. Then: $\Lambda_s(gx)=\lambda_s(g)\Lambda_s(x)$; $\Lambda_s(\mathbf{N})\subseteq B\otimes T$; $\Lambda_s(\varphi^e(z))=\varphi^q(\iota_T(z))=G^{(q)}(z)$ for $z\in\mathbf{N}_c$; $\lambda_s(Q_s)=\xi$, $\lambda_s(Q_{s'})=\varphi^{s'-s}(\xi)\in B^\times$ for $s'\neq s$, $\lambda_s(X)=\varphi^{-(s+1)}(\mu)\in B^\times$; and $\lambda_s(e_a)=\iota_\Lambda(\varphi^{-(s+1)}(e_a))=\varepsilon_{a-s-1}=:\epsilon$. Let $R^+=\epsilon(B\otimes_{\mathbb{Q}_\ell}L)$ (a free B-module, so ξ-torsion free and ξ-adically separated) and $R=R^+[1/\xi]$. Put $D_k=G^{(q)}(H_k)$, which equals $\varphi^{e-s-4}(d_k)$ by item 1; the D_k lie in ϵD and form an L-basis of it (Lemma 23.10 (2); φ maps ε_bD bijectively onto $\varepsilon_{b+1}D$). Put $\Lambda_D=\sum_k R^+D_k$. Then $\Lambda_s(P_e(\mathbf{N}_c))\subseteq\Lambda_D$: for $p=\sum_j g_j\varphi^e(n_j)$, $g_j\in e_a\mathbf{A}_{F,L}$, $\Lambda_s(p)=\sum_j\lambda_s(g_j)G^{(q)}(n_j)$, and $n_j=\sum_k(\mathbf{h}^{-1})_{jk}H_k$ in $\mathbf{N}_E$ with $\mathbf{h}=(h_{kj})\in GL_r(e_cE[[X]])$ ($\mathbf{h}\equiv 1 \bmod X$), so $G^{(q)}(n_j)=\sum_k\mathrm{ev}^{(q)}((\mathbf{h}^{-1})_{jk})D_k$.

Step 7: the de Rham lattice at level s**.** V is crystalline, so $D\subseteq D_{\mathrm{dR}}(V)$ have the same F-dimension and $D=D_{\mathrm{dR}}(V)$; the filtration of Definition 23.3 is that of Theorem B.40 (Ai4). The map of (Ai4) is $F\otimes L$-linear and Fil^jD is an $F\otimes L$-submodule, so applying ϵ to (Ai4) with $k=0$: $\epsilon(B\otimes_{\mathbb{Q}_\ell}V)=\sum_j t^{-j}R^+\otimes_L\epsilon\mathrm{Fil}^jD$ inside $\epsilon(B_{\mathrm{dR}}\otimes V)=R\otimes_L\epsilon D$. Choose an L-basis (D_i^{ad}) of ϵD adapted to the filtration, $D_i^{\mathrm{ad}}\in\mathrm{Fil}^{u_i}$ with $\epsilon\mathrm{Fil}^jD$ spanned by the D_i^{ad} with $u_i\geq j$. Then

$$\epsilon(B\otimes V)=\bigoplus_i\xi^{-u_i}R^+D_i^{\mathrm{ad}},\qquad\Lambda_D=\bigoplus_iR^+D_i^{\mathrm{ad}}.$$

Let $W \in GL_r(L)$ with $D_k = \sum_i W_{ki} D_i^{\mathrm{ad}}$ and $\mathbf{v}_i^{(s)} \in L^r$ the i-th row of W^{-1}; then $\sum_k v_{i,k}^{(s)} D_k = D_i^{\mathrm{ad}}$, i.e. $\varphi^{-(s+4)}(\delta_e(\mathbf{v}_i^{(s)})) = D_i^{\mathrm{ad}} \in \mathrm{Fil}^{u_i} D$.

Step 8: orders at Q_s. $\mathcal{O} = e_a \mathbf{A}_{F,L}[1/\ell] \cong \mathcal{O}_L[[X]][1/\ell]$ is a factorial domain; the $Q_{s'}$ are squarefree and pairwise coprime in it (they are products of distinct irreducible distinguished polynomials, Lemma 23.8 (3), and distinct cyclotomic polynomials are coprime in $L[X]$). For g in $\mathcal{O}[1/\lambda_e]$ let $\mathrm{ord}_s(g)$ be the minimum of $v_\pi(g)$ over the prime factors π of Q_s. We claim that $\lambda_s(g) \in \xi^\alpha R^+$ if and only if $\mathrm{ord}_s(g) \geq \alpha$. Write $g = Q_s^\beta g_0$, $\beta = \mathrm{ord}_s(g)$, $g_0 \in \mathcal{O}[1/Q_{s'} : s' \neq s]$, $Q_s \nmid g_0$; then $\lambda_s(g) = \xi^\beta \lambda_s(g_0)$ with $\lambda_s(g_0) \in R^+$, and it suffices that $\lambda_s(g_0) \notin \xi R^+$. The map $\theta \circ \lambda_s \colon \mathcal{O} \to R^+/\xi R^+ = \epsilon(\mathbb{C}_\ell \otimes_{\mathbb{Q}_\ell} L) = \mathbb{C}_\ell \otimes_F L$ (F acting on L through the embedding of ϵ) sends X to $\zeta_{\ell^{s+1}} - 1$ and is L-linear; it factors through $\mathcal{O}/Q_s\mathcal{O} \cong L[X]/(Q_s) \cong \mathbb{Q}_\ell(\zeta_{\ell^{s+1}}) \otimes_{\mathbb{Q}_\ell} L = (\mathbb{Q}_\ell(\zeta_{\ell^{s+1}}) \otimes_{\mathbb{Q}_\ell} F) \otimes_F L \cong F(\zeta_{\ell^{s+1}}) \otimes_F L \hookrightarrow \mathbb{C}_\ell \otimes_F L$ (Weierstrass division by the distinguished polynomial Q_s; $F/\mathbb{Q}_\ell$ unramified and $\mathbb{Q}_\ell(\zeta_{\ell^{s+1}})$ totally ramified, so the middle map is an isomorphism; L is flat over F). So $\ker(\theta \circ \lambda_s) = Q_s\mathcal{O}$, the elements $Q_{s'}$ ($s' \neq s$) map to units, and $\theta(\lambda_s(g_0)) \neq 0$.

Step 9: the level-s condition on arc coordinates. Let $y \in \mathbf{N}_a$ with $\Pi y \in P_e(\mathbf{N}_c)$. By Lemma 23.14 (Ac1) there are t and an arc coordinate vector f of $\ell^t y$ for some (κ, κ'): $\ell^{\kappa'}(\ell^{\kappa+t}\Pi y - \sum_k \Pi f_k \varphi^e(T_\kappa(H_k))) = X\Pi^2 p$ with $p \in P_e(\mathbf{N}_c)$. Here $T_\kappa(H_k) = \ell^\kappa H_k - S_k$ with $S_k \in X^m e_c \mathbf{N}_E$, so $\Lambda_s(\varphi^e(T_\kappa H_k)) = G^{(q)}(T_\kappa H_k) = \ell^\kappa D_k - G^{(q)}(S_k)$ and $G^{(q)}(S_k) \in \varphi^q(\mu)^m \Lambda_D \subseteq \xi^m \Lambda_D$ (Step 6). Apply Λ_s and divide by $\ell^{\kappa+\kappa'}\lambda_s(\Pi)$, where $\lambda_s(\Pi) \in \xi^h B^\times$ and $\lambda_s(f_k) \in \xi^{-h} R^+$ ($\Pi f_k \in e_a \mathbf{A}_{F,L}$):

$$\sum_k \lambda_s(f_k) D_k \in \ell^t \Lambda_s(y) + \xi^{m-h}\Lambda_D + \xi^h \Lambda_D \subseteq \epsilon(B \otimes V) + \xi^h \Lambda_D,$$

using $\Lambda_s(y) = \epsilon\Lambda_s(y) \in \epsilon(B \otimes T)$ and $m \geq 2h$. With $\varphi_i^{(s)} = \sum_k f_k W_{ki}$ (so $f = \sum_i \varphi_i^{(s)} \mathbf{v}_i^{(s)}$), Step 7 gives $\lambda_s(\varphi_i^{(s)}) \in \xi^{\min(-u_i, h)} R^+$, and Step 8:

$$\mathrm{ord}_s(\varphi_i^{(s)}) \geq \min(-u_i^{(s)},\, H(a-s)) \qquad (s < e,\ i \leq r). \qquad \text{(L)}$$

Step 10: removing the uncontrolled orders. Let $J \geq 0$ bound the absolute values of all Hodge jumps of V and $N = J + 2\max_{s<e} H(a-s)$. For $s < e$ let $\mathcal{O}_{(s)} = \mathcal{O}[1/X, 1/Q_{s'} : s' \neq s]$; then $\mathcal{O}_{(s)}/Q_s^N \cong L[X]/(Q_s^N)$. Let $z_s \in L[X]^r$ with $z_s \equiv \sum_{i:\, u_i^{(s)} < -H(a-s)} (\varphi_i^{(s)}/(X\Pi)) \mathbf{v}_i^{(s)} \bmod Q_s^N \mathcal{O}_{(s)}^r$ (these quotients lie in $\mathcal{O}_{(s)}$ by (L)), and by the Chinese remainder theorem $z \in L[X]^r$ with $z \equiv z_s \bmod Q_s^N$ for all $s < e$. Choose t_z with $\ell^{t_z} z \in \mathcal{O}_L[X]^r$ and put $f' = \ell^{t_z}(f - X\Pi z)$. Then f' is an arc coordinate vector of $\ell^{t+t_z} y$: $\Pi f_k' \in e_a \mathbf{A}_{F,L}$, and the defining relation for f' is ℓ^{t_z} times that for f minus $\ell^{\kappa'} \sum_k X\Pi^2(\ell^{t_z} z_k)\varphi^e(T_\kappa H_k) \in X\Pi^2 P_e(\mathbf{N}_c)$. Its s-coordinates $\varphi_i'^{(s)} = \ell^{t_z}(\varphi_i^{(s)} - X\Pi z_i^{(s)})$ satisfy: if $u_i^{(s)} < -H(a-s)$, $\varphi_i'^{(s)} \in X\Pi Q_s^N \mathcal{O}_{(s)}$, so $\mathrm{ord}_s \geq N \geq -u_i^{(s)}$; otherwise $z_i^{(s)} \in Q_s^N \mathcal{O}_{(s)}$ and (L) gives $\mathrm{ord}_s(\varphi_i'^{(s)}) \geq \min(-u_i^{(s)}, H(a-s)) = -u_i^{(s)}$. So

$$\mathrm{ord}_s(\varphi_i'^{(s)}) \geq -u_i^{(s)} \qquad (s < e,\ i \leq r), \qquad \text{and } \mathrm{ord}_s(f_k') \geq -H(a-s). \qquad \text{(L')}$$

Step 11: the section decomposition. Let $E_s \in L[X]$ with $E_s \equiv 1 \bmod Q_s^N$ and $E_s \equiv 0 \bmod Q_{s'}^N$ for $s' \neq s$ (Chinese remainder theorem). Then $f' = \sum_{s<e} \sum_i E_s \varphi_i'^{(s)} \mathbf{v}_i^{(s)} + \sum_k ((1 - \sum_s E_s) f_k') \mathbf{e}_k$. Consider the terms $(E_s \varphi_i'^{(s)}, \mathbf{v}_i^{(s)}, \alpha_{s,i})$ with $\alpha_{s,i}(s) = -u_i^{(s)}$ and $\alpha_{s,i}(s') = N - H(a-s')$ for $s' \neq s$, and $((1 - \sum_s E_s) f_k', \mathbf{e}_k, \alpha_0)$ with $\alpha_0(s') = N - H(a-s')$. Each first entry g lies in $\mathcal{O}[1/\lambda_e]$ and has $\mathrm{ord}_{s'}(g) \geq \alpha(s')$ for every $s' < e$ by (L'), $E_s \equiv 1 \bmod Q_s$ and the congruences of the E_s; hence $gQ^{-\alpha}$ has no prime of any $Q_{s'}$ in its denominator and lies in $\mathcal{O}$, i.e. $g \in Q^\alpha e_a \mathbf{A}_{F,L}[1/\ell]$. Conditions (Gs1): for (s,i) at level s, $\delta_e(\mathbf{v}_i^{(s)}) = \varphi^{s+4}(D_i^{\mathrm{ad}}) \in \varphi^{s+4}(\mathrm{Fil}^{u_i} D)$

by Step 7; at every other level s', and for the last terms at every level, $-\alpha(s') = H(a-s') - N \le -J$, so $\mathrm{Fil}^{-\alpha(s')}D = D$ and the condition holds because φ is bijective on D. Finally choose κ with $\ell^{\kappa_1} g \in Q^{\alpha} e_a \mathbf{A}_{F,L}$ and $\ell^{\kappa_2}\mathbf{v} \in \mathcal{O}_L^r$ for all terms, $\kappa = \kappa_1 + \kappa_2$: the terms $(\ell^{\kappa_1} g, \ell^{\kappa_2}\mathbf{v}, \alpha)$ form a section decomposition of $\ell^{\kappa} f'$ which is Hodge-graded. This proves (Ac3) at (c, e, H, m). □

Lemma 23.14. *Let ℓ, F, L, (τ_a) and a pair (V,T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$, $c \in \mathbb{Z}/n$, $e \ge 0$, $a = c + e$, $H\colon \mathbb{Z}/n \to \mathbb{Z}_{\ge 0}$, and $m \ge 1$ an integer with $m \ge 2H(a-s)$ for $0 \le s < e$. Then for every frame $\mathbf{n}$ of $\mathbf{N}$ at c, $\mathbf{N}$ has the properties (Ac1), (Ac2) and (Ac4) at (c, e, H, m); in (Ac1) and (Ac2) the exponent t can be chosen depending only on $\mathbf{n}$ and m, and in (Ac4) one can take $t = \kappa'$ for any (κ, κ') for which f satisfies the defining relation of arc coordinate vectors.*

Proof. Let $\Pi = \Lambda_{a,e}(H)$, $\mathbf{n} = (n_k)_{k \le r}$, $H_k = \sum_j h_{kj} n_j$ the horizontal sections and $h = (h_{kj}) \in M_r(e_c E[[X]])$. Since $H_k \in n_k + X\mathbf{N}_E$ (Lemma 23.9 (Hf2)), $h \equiv I \bmod X$, so h is invertible; let $g = h^{-1}$, so $n_j = \sum_k g_{jk} H_k$. Under $e_c\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]] \cong e_a\mathbf{A}_{F,L}$ the map $\varphi^e\colon e_c\mathbf{A}_{F,L} \to e_a\mathbf{A}_{F,L}$ is $u(X) \mapsto u((1+X)^{\ell^e} - 1)$, because $\tau_{b+e} \circ \varphi^e = \tau_b$; and $\varphi^e(X) = X\lambda_e$. We use twice:

(T) If $z \in \mathbf{N}_c \cap X^m \mathbf{N}_E$ then $\varphi^e(z) \in X\Pi^2 P_e(\mathbf{N}_c)$. Indeed, in the basis $\mathbf{n}$ the coordinates of z lie in $e_c\mathbf{A}_{F,L} \cap X^m e_c E[[X]] = X^m e_c \mathbf{A}_{F,L}$, so $\varphi^e(z) \in X^m \lambda_e^m P_e(\mathbf{N}_c)$, and $X\Pi^2$ divides $X^m\lambda_e^m = X^m \prod_{s<e} Q_s^m$ since $m \ge 1$ and $m \ge 2H(a-s)$.

(Ac1) and (Ac2). Fix κ admissible (it exists: finitely many coefficients $h_{kj,i}$, $i < m$, are involved), and $\kappa_0 \ge 0$ with $\ell^{\kappa_0} g_{jk,i} \in e_c(\mathcal{O}_F \otimes \mathcal{O}_L)$ for all j, k and $i < m$, where $g_{jk} = \sum_i g_{jk,i} X^i$. Let $\hat{g}_{jk} = \sum_{i<m} \ell^{\kappa_0} g_{jk,i} X^i \in e_c\mathbf{A}_{F,L}$. Then $\ell^{\kappa_0} g_{jk} - \hat{g}_{jk} \in X^m e_c E[[X]]$ and $\ell^{\kappa} H_k - T_\kappa(H_k) \in X^m \mathbf{N}_E$, so

$$z_j := \ell^{\kappa_0+\kappa} n_j - \sum_k \hat{g}_{jk}\, T_\kappa(H_k) = \sum_k (\ell^{\kappa_0} g_{jk} - \hat{g}_{jk}) \ell^{\kappa} H_k + \sum_k \hat{g}_{jk} (\ell^{\kappa} H_k - T_\kappa(H_k)) \in \mathbf{N}_c \cap X^m \mathbf{N}_E,$$

and by (T), $\ell^{\kappa_0+\kappa}\varphi^e(n_j) - \sum_k \varphi^e(\hat{g}_{jk})\varphi^e(T_\kappa(H_k)) \in X\Pi^2 P_e(\mathbf{N}_c)$. Let $y \in \mathbf{N}[1/\lambda_\infty]$ with $\Pi y \in P_e(\mathbf{N}_c)$ (resp. $y \in P_e(\mathbf{N}_c)$), and write $\Pi y = \sum_j \beta_j \varphi^e(n_j)$ (resp. $y = \sum_j \beta_j \varphi^e(n_j)$) with $\beta_j \in e_a\mathbf{A}_{F,L}$, possible since $P_e(\mathbf{N}_c) \subseteq e_a\mathbf{N}[1/\lambda_\infty]$ is generated by $\varphi^e(\mathbf{N}_c)$. Put $f_k = \Pi^{-1} \sum_j \beta_j \varphi^e(\hat{g}_{jk})$ (resp. $f_k = \sum_j \beta_j \varphi^e(\hat{g}_{jk})$). Multiplying the last display by β_j (resp. by $\Pi\beta_j$) and summing over j,

$$\ell^{\kappa}\, \Pi\, (\ell^{\kappa_0} y) - \sum_k \Pi f_k\, \varphi^e(T_\kappa(H_k)) \in X\Pi^2 P_e(\mathbf{N}_c),$$

and $\Pi f_k \in e_a\mathbf{A}_{F,L}$ (resp. $f_k \in e_a\mathbf{A}_{F,L}$). So f is an arc coordinate vector of $\ell^{\kappa_0} y$ (with $\kappa' = 0$), integral in the second case: this is (Ac1) and (Ac2) with $t = \kappa_0$.

(Ac4). Let f be an arc coordinate vector of 0 for (κ, κ') and $F_k = \Pi f_k \in e_a\mathbf{A}_{F,L}$. Write $T_\kappa(H_k) = \sum_j \tau_{kj} n_j$ with $\tau_{kj} = \sum_{i<m} \ell^{\kappa} h_{kj,i} X^i$; since $h \equiv I \bmod X$, $\tau \equiv \ell^{\kappa} I \bmod X$ and $\det\tau \in \ell^{r\kappa} + Xe_c\mathbf{A}_{F,L}$. By Lemma 23.15 (1) the $\varphi^e(n_j)$ form an $e_a\mathbf{A}_{F,L}$-basis of $P_e(\mathbf{N}_c)$, so the defining relation $\ell^{\kappa'} \sum_k F_k \varphi^e(T_\kappa(H_k)) \in X\Pi^2 P_e(\mathbf{N}_c)$ says that the row vector $\ell^{\kappa'} F\vartheta$, $\vartheta = (\varphi^e(\tau_{kj}))$, has entries in $X\Pi^2 e_a\mathbf{A}_{F,L}$. Multiplying on the right by the adjugate of ϑ gives $\ell^{\kappa'} F_k\, \varphi^e(\det\tau) \in X\Pi^2 e_a\mathbf{A}_{F,L}$ for every k.

Identify $e_a\mathbf{A}_{F,L} = \mathcal{O}_L[[X]]$. The element $P = X\Pi^2 = X\prod_{s<e} Q_s^{2H(a-s)}$ is a distinguished polynomial ($Q_s = \Phi_{\ell^{s+1}}(1+X)$ is monic and $\equiv X^{\deg Q_s} \bmod \ell$). Its roots in $\mathbb{C}_\ell$ are 0 and the $\zeta - 1$, ζ a primitive ℓ^{s+1}-th root of unity with $s < e$; at each of them $\delta := \varphi^e(\det\tau)$ takes the value $(\det\tau)(\zeta^{\ell^e} - 1) = (\det\tau)(0) = \ell^{r\kappa} \ne 0$. By Weierstrass division write $\ell^{\kappa'} F_k = PA + R$ with $A \in \mathcal{O}_L[[X]]$ and $R \in \mathcal{O}_L[X]$ of degree $< \deg P$. Then $R\delta \in P\,\mathcal{O}_L[[X]]$, so the analytic function $R\delta$ on the open unit disc vanishes at every root z of P to at least its multiplicity; as $\delta(z) \ne 0$, so does R. Hence P divides R in $\mathbb{C}_\ell[X]$, and $\deg R < \deg P$ forces $R = 0$. Thus $\ell^{\kappa'}\Pi f_k = \ell^{\kappa'} F_k \in X\Pi^2 e_a\mathbf{A}_{F,L}$, i.e. $\ell^{\kappa'} f_k \in X\Pi e_a\mathbf{A}_{F,L}$, which is (Ac4) with $t = \kappa'$. □

Lemma 23.15. *Let* $\mathbf{N}$ *be a Wach module over* F *with coefficients* L *and window* $(M_a, h_a)_{a\in\mathbb{Z}/n}$*, of rank* r*; regard* M *and* h *as functions* $a\mapsto M_a$*,* $a\mapsto h_a$*. Let* $c\in\mathbb{Z}/n$*,* $e\geq 0$*,* $a=c+e$*, and* $R=e_a\mathbf{A}_{F,L}\cong\mathcal{O}_L[[X]]$*.*

1. *$P_e(\mathbf{N}_c)$ is a free R-module of rank r, and the R-linear map $R\otimes_{\varphi^e,\,e_c\mathbf{A}_{F,L}}\mathbf{N}_c\to P_e(\mathbf{N}_c)$, $g\otimes x\mapsto g\varphi^e_{\mathbf{N}}(x)$, is an isomorphism.*
2. *For every function $H\colon\mathbb{Z}/n\to\mathbb{Z}_{\geq 0}$ with $H\geq M$ pointwise, $\Lambda_{a,e}(H)\,P_e(\mathbf{N}_c)\subseteq\mathbf{N}_a$; for every $H\geq h$ pointwise, $\Lambda_{a,e}(H)\,\mathbf{N}_a\subseteq P_e(\mathbf{N}_c)$.*
3. *(Saturation.) Let $S\subset\mathbb{Z}_\ell[[X]]$ be the multiplicative monoid generated by X and the Q_s $(s\geq 0)$, let U be $\mathbf{N}_a$ or $P_e(\mathbf{N}_c)$, $f,f'\in S$, $k\geq 0$, and $x\in U\otimes_R\operatorname{Frac}(R)$. If $f'x\in U$ and $\ell^k x\in fU$, then $x\in fU$.*

Proof. Write $\Lambda(H)=\Lambda_{a,e}(H)$ and, for $e\geq 1$, $a'=a-1$, so that $P_{e-1}(\mathbf{N}_c)\subseteq e_{a'}\mathbf{N}[1/\lambda_\infty]$. Since $\varphi_{\mathbf{N}}$ is φ-semilinear, the $\mathbf{A}_{F,L}$-span of $\varphi_{\mathbf{N}}$ of a set Y equals the $\mathbf{A}_{F,L}$-span of $\varphi_{\mathbf{N}}$ of the $\mathbf{A}_{F,L}$-span of Y; hence

$$P_e(\mathbf{N}_c)=\mathbf{A}_{F,L}\,\varphi_{\mathbf{N}}\big(P_{e-1}(\mathbf{N}_c)\big)\qquad(e\geq 1).\qquad(1)$$

Also $\varphi(\Lambda_{a',e-1}(H))=\prod_{s=0}^{e-2}Q_{s+1}^{H(a'-s)}=\prod_{s=1}^{e-1}Q_s^{H(a-s)}$, so

$$\Lambda_{a,e}(H)=Q^{H(a)}\,\varphi(\Lambda_{a',e-1}(H)).\qquad(2)$$

Item 2, by induction on e; for $e=0$, $\Lambda=1$ and $P_0(\mathbf{N}_c)=\mathbf{N}_c=\mathbf{N}_a$. Let $e\geq 1$, $H\geq M$. By induction $\Lambda_{a',e-1}(H)P_{e-1}(\mathbf{N}_c)\subseteq\mathbf{N}_{a'}$; applying $\varphi_{\mathbf{N}}$ and taking spans, $\varphi(\Lambda_{a',e-1}(H))\,P_e(\mathbf{N}_c)\subseteq\mathbf{A}_{F,L}\varphi_{\mathbf{N}}(\mathbf{N}_{a'})$ by (1). By (Wm2), $Q^{M_a}\mathbf{A}_{F,L}\varphi_{\mathbf{N}}(\mathbf{N}_{a'})\subseteq\mathbf{N}_a$, and $Q^{H(a)}=Q^{H(a)-M_a}Q^{M_a}$; with (2), $\Lambda(H)P_e(\mathbf{N}_c)\subseteq\mathbf{N}_a$. For $H\geq h$: by induction $\Lambda_{a',e-1}(H)\mathbf{N}_{a'}\subseteq P_{e-1}(\mathbf{N}_c)$, so $\varphi(\Lambda_{a',e-1}(H))\,\mathbf{A}_{F,L}\varphi_{\mathbf{N}}(\mathbf{N}_{a'})\subseteq P_e(\mathbf{N}_c)$ by (1); by (Wm2), $Q^{H(a)}\mathbf{N}_a\subseteq\mathbf{A}_{F,L}\varphi_{\mathbf{N}}(\mathbf{N}_{a'})$; with (2), $\Lambda(H)\mathbf{N}_a\subseteq P_e(\mathbf{N}_c)$.

Item 1, by induction on e, trivial for $e=0$. Let $e\geq 1$; by induction $P_{e-1}(\mathbf{N}_c)$ is free of rank r over $R'=e_{a'}\mathbf{A}_{F,L}$, with basis $y_1,\dots,y_r$ the images of $1\otimes x_i$ for a basis (x_i) of $\mathbf{N}_c$. By (1), $P_e(\mathbf{N}_c)$ is generated over R by $\varphi_{\mathbf{N}}(y_1),\dots,\varphi_{\mathbf{N}}(y_r)$, so $\rho\colon R^r\to P_e(\mathbf{N}_c)$, $(g_i)\mapsto\sum g_i\varphi_{\mathbf{N}}(y_i)$, is onto, and the map of the statement is ρ composed with the isomorphism $R\otimes_{\varphi^e}\mathbf{N}_c\cong R\otimes_\varphi P_{e-1}(\mathbf{N}_c)\cong R^r$. By item 2 with $H=h$, $P_e(\mathbf{N}_c)$ contains $\Lambda(h)\mathbf{N}_a$, a free R-module of rank r ($\Lambda(h)\neq 0$ in the domain R); and $P_e(\mathbf{N}_c)\subseteq\mathbf{N}_a\otimes_R\operatorname{Frac}(R)$, a vector space of dimension r. So $P_e(\mathbf{N}_c)\otimes\operatorname{Frac}(R)$ has dimension r, $\rho\otimes\operatorname{Frac}(R)$ is an isomorphism, and $\ker\rho$, a submodule of the torsion-free R^r that vanishes after $\otimes\operatorname{Frac}(R)$, is 0.

Item 3. First, every $g\in S$ is a non-zero-divisor on R/ϖ^jR for every $j\geq 1$, ϖ a uniformiser of $\mathcal{O}_L$: modulo ϖ, X is X and $Q_s\equiv X^{\ell^s(\ell-1)}$ (Lemma 23.8(3); $\ell\in\varpi\mathcal{O}_L$), so $g\equiv X^N\not\equiv 0$ in the domain $R/\varpi R=k_L[[X]]$; if $gy\in\varpi^jR$ with $y\notin\varpi^jR$, write $y=\varpi^iy'$ with $i<j$ and $y'\notin\varpi R$ (R is ϖ-adically separated, R/ϖ is a domain); then $gy'\in\varpi^{j-i}R\subseteq\varpi R$, contradicting $g,y'\not\equiv 0 \bmod\varpi$. Now let $\ell^kx=fy$ with $y\in U$. Then $ff'y=\ell^k(f'x)\in\ell^kU$. Since U is free over R (item 1, (Wm1)) and $\ell^kR=\varpi^jR$ for $j=k\,v_\varpi(\ell)$, the non-zero-divisor ff' on R/ϖ^jR gives $y\in\ell^kU$, $y=\ell^ky'$ with $y'\in U$. Then $\ell^k(x-fy')=0$ in the torsion-free $U\otimes\operatorname{Frac}(R)$, so $x=fy'\in fU$. □

Lemma 23.16. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a pair (V,T) be as in Proposition 23.7, $D=D_{\mathrm{cris}}(V)=(B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V)^{G_F}$ with $\operatorname{Fil}^jD=D\cap(t^jB^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell}V)$, $b\in\mathbb{Z}/n$, $\epsilon=\epsilon_b$, $R^+=\epsilon(B^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell}L)$ and $R=R^+[1/t]$. Then ϵD has an L-basis $(D_i^{\mathrm{ad}})_i$* adapted to the filtration*: there are integers*

u_i with $D_i^{\mathrm{ad}} \in \mathrm{Fil}^{u_i} D$ such that, for every $j \in \mathbb{Z}$, the D_i^{ad} with $u_i \geq j$ form an L-basis of $\epsilon\,\mathrm{Fil}^j D$. For every such basis

$$\epsilon(B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_\ell} V) = \bigoplus_i t^{-u_i} R^+ D_i^{\mathrm{ad}},$$

and every L-basis of ϵD is an R-basis of $\epsilon(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V)$.

Proof. D **is the de Rham module.** As $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ G_F-equivariantly, $D \subseteq D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V)^{G_F}$; V is crystalline, hence de Rham, and both spaces have F-dimension $\dim_{\mathbb{Q}_\ell} V$, so $D = D_{\mathrm{dR}}(V)$, and $\mathrm{Fil}^j D$ is the filtration of Theorem B.40 (Ai4). The ring $F \otimes_{\mathbb{Q}_\ell} L$ acts on $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V$ ($F \subset B_{\mathrm{dR}}^+$ on the left factor, L on V); this action preserves D and every $t^j B_{\mathrm{dR}}^+ \otimes V$, hence every $\mathrm{Fil}^j D$.

Adapted bases. The space ϵD is a finite-dimensional L-vector space ($\epsilon(F \otimes_{\mathbb{Q}_\ell} L) \cong L$), and $(\epsilon \mathrm{Fil}^j D)_j$ is a decreasing family of subspaces. It equals ϵD for $j \ll 0$, since each of finitely many generators of D lies in some $t^j B_{\mathrm{dR}}^+ \otimes V$ ($B_{\mathrm{dR}} = \bigcup_j t^{-j} B_{\mathrm{dR}}^+$), and it is 0 for $j \gg 0$, since a decreasing chain of subspaces stabilises and $\bigcap_j t^j B_{\mathrm{dR}}^+ = 0$. Choosing bases successively from the largest j with $\epsilon \mathrm{Fil}^j D \neq 0$ downwards gives an adapted basis.

The lattice. The isomorphism $B_{\mathrm{dR}} \otimes_F D \to B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V$ of (Ai4) is linear for $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} L$ (acting on the source through B_{dR} and L on D), so it maps $\epsilon(B_{\mathrm{dR}} \otimes_F D) = R \otimes_L \epsilon D$ isomorphically onto $\epsilon(B_{\mathrm{dR}} \otimes V)$; as ϵD is free over L, every L-basis of ϵD is an R-basis of $\epsilon(B_{\mathrm{dR}} \otimes V)$. With $k = 0$, (Ai4) says that $\sum_j t^{-j} B_{\mathrm{dR}}^+ \otimes_F \mathrm{Fil}^j D$ maps onto $B_{\mathrm{dR}}^+ \otimes V$; applying ϵ, and since $\mathrm{Fil}^j D$ is an $F \otimes L$-submodule,

$$\epsilon(B_{\mathrm{dR}}^+ \otimes V) = \sum_j t^{-j} R^+ \,\epsilon \mathrm{Fil}^j D = \sum_j \sum_{i:\, u_i \geq j} t^{-j} R^+ D_i^{\mathrm{ad}} = \sum_i t^{-u_i} R^+ D_i^{\mathrm{ad}},$$

because $t^{-j} R^+ \subseteq t^{-u_i} R^+$ for $j \leq u_i$, with equality at $j = u_i$. The sum is direct, the D_i^{ad} being an R-basis of $\epsilon(B_{\mathrm{dR}} \otimes V)$. □

Lemma 23.17. *Let ℓ, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ be as in Proposition 23.7, $\Phi \colon T_1 \to T_2$ a G_F-equivariant $\mathcal{O}_L$-linear map of the lattices of pairs (V_1, T_1), (V_2, T_2), and fix arc level data of Φ (any cut c, length $1 \leq \mathrm{len} \leq n$, H, frames, and parameters). Then (Lp1), (Lp2), (Lp4), (Lp5), (Lp8), (Lh1) and (Lh2) hold, for $\nu = 1, 2$ and every $s < \mathrm{len}$. Moreover $\lambda_s(\Pi) \in t^{H(c-s)} K_{(s)}[[t]]^\times$, so $\mathrm{lv}_s(w) \in t^{-H(c-s)} K_{(s)}[[t]]^{r_\nu}$ for $w \in W_\nu$, and the elements σ_u ($u \in \mathbb{Z}_\ell^\times$) commute.*

Proof. Fix ν and $s < \mathrm{len}$ and drop ν from the notation. Write $B = B_{\mathrm{dR}}^+$, $K = K_{(s)}$, $\zeta = \zeta_{\ell^{s+1}}$, $\mu = [\varepsilon] - 1$, $\mu_s = \varphi^{-(s+1)}(\mu) = [\varepsilon^{1/\ell^{s+1}}] - 1$, $q = \mathrm{len} - s - 1 \geq 0$, $D_k = D_k^{(s)}$. Let $\xi = \sum_{i<\ell} [\varepsilon^{1/\ell}]^i$, the element ξ_0 of Theorem B.40; by (Ai1) and (Ai3) there, $\ker\theta = \xi A_{\mathrm{inf}}$ and $\xi B = tB$, so $t \in \xi B^\times$. By (Lm1) and (Lm2) of Lemma 23.6, applied with the site $c - \mathrm{len}$ in place of its c, $e = \mathrm{len}$ and $a = c$ (its λ_s, Λ_s and ϵ are our λ_s, Λ_s and ϵ_s): $\Lambda_s(gx) = \lambda_s(g)\Lambda_s(x)$, $\lambda_s(e_c Q_s) = \epsilon_s \xi$, and $\lambda_s(e_c Q_{s'}) = \epsilon_s v_{s'}$ with $v_{s'} \in B^\times$, so $\theta(v_{s'}) \neq 0$, for $s' < \mathrm{len}$, $s' \neq s$. Finally, as $\iota(X) = \mu$ (Lemma 23.8), $\lambda_s(e_c X) = \epsilon_s \mu_s$, and $\theta(\mu_s) = \theta([\varepsilon^{1/\ell^{s+1}}]) - 1 = \theta(\zeta - 1) \neq 0$ by (Ai1).

Step 1: $[\varepsilon^{1/\ell^{s+1}}] = \zeta \exp(t/\ell^{s+1})$ **in** B**.** Put $N = \ell^{s+1}$. Since θ is injective on $\overline{\mathbb{Q}}_\ell$ and $\theta(\zeta^N) = \theta([\varepsilon]) = 1$, $\zeta^N = 1$. The elements $y = \zeta^{-1}[\varepsilon^{1/N}]$ and $z = \exp(t/N)$ (t-adically convergent, $t \in \mathrm{Fil}^1 B$) satisfy $\theta(y) = \theta(z) = 1$ and $y^N = [\varepsilon] = \exp(t) = z^N$ ($t = \log[\varepsilon]$). If $y/z = 1 + x$ with $0 \neq x \in \mathrm{Fil}^k B \setminus \mathrm{Fil}^{k+1} B$, $k \geq 1$, then $0 = (1+x)^N - 1 \in Nx + \mathrm{Fil}^{2k} B$, so $x \in \mathrm{Fil}^{k+1} B$ (N is invertible in B), a contradiction. So $y = z$.

Step 2: $\lambda_s(e_c \mathbf{A}_{F,L}) \subseteq K[[t]]$**.** Let $a = e_c \sum_i a_i X^i$ with $a_i \in \mathcal{O}_L$ (acting as $e_c(1 \otimes a_i)$) and $P_m = \epsilon_s \sum_{i<m} a_i \mu_s^i$. Since $\iota(\sum_{i \geq m} a_i X^i) = \mu^m \iota(\sum_i a_{i+m} X^i)$, we have $\lambda_s(a) - P_m \in \mu_s^m (A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L)$. As $\zeta - 1$ is a uniformiser of $\mathbb{Q}_\ell(\zeta)$, totally ramified of degree $d = \ell^s(\ell - 1)$, $\theta(\mu_s^d) \in \ell \mathcal{O}_{\mathbb{C}_\ell}$, so

$\mu_s^d \in \ell A_{\inf} + \xi A_{\inf}$ and $\mu_s^{d(b+k)} \in \ell^{k+1} A_{\inf} + \xi^b A_{\inf}$ for $b, k \geq 0$. Let $\mathcal{V}_b = (B/\xi^b) \otimes_{\mathbb{Q}_\ell} L$ with the ℓ-adic topology whose unit ball is the image of $(A_{\inf}/\xi^b) \otimes \mathcal{O}_L$ (separated: $B/\xi^b = A_{\inf}[1/\ell]/\xi^b$ by Theorem B.40 (Ai3), and $A_{\inf}/\xi^b$ is ℓ-torsion free and ℓ-adically separated). The last claim holds by induction on b. For $b = 1$, $A_{\inf}/\xi \cong \mathcal{O}_{\mathbb{C}_\ell}$ via θ (Ai1). For $b \geq 2$, multiplication by ξ gives an exact sequence $0 \to A_{\inf}/\xi^{b-1} \to A_{\inf}/\xi^b \to A_{\inf}/\xi \to 0$ ($A_{\inf}$ is a domain), and an extension of ℓ-torsion free modules is ℓ-torsion free. If $x \in \bigcap_k \ell^k (A_{\inf}/\xi^b)$, its image in $\mathcal{O}_{\mathbb{C}_\ell}$ is 0, so $x = \xi y$ with $y \in A_{\inf}/\xi^{b-1}$; writing $x = \ell^k z_k$, the image of z_k in $\mathcal{O}_{\mathbb{C}_\ell}$ is killed by ℓ^k, hence 0, so $z_k = \xi y_k$, and $\xi(y - \ell^k y_k) = 0$ gives $y = \ell^k y_k$ for every k; by induction $y = 0$. Hence $P_m \to \lambda_s(a)$ in $\mathcal{V}_b$. On the other hand, by Step 1, $\mu_s = \pi + \tau$ with $\pi = \zeta - 1$ and $\tau = \zeta(\exp(t/\ell^{s+1}) - 1) \in t\,\mathbb{Q}_\ell(\zeta)[[t]]$, so in $K[[t]]$ the t^j-coefficient of P_m is $\sum_{i<m} a_i c_{i,j}$ with $c_{i,j} = \epsilon_s \sum_{k \leq j} \binom{i}{k} \pi^{i-k} [t^j](\tau^k)$, and $c_{i,j} \to 0$ in K as $i \to \infty$ ($|\pi| < 1$). So $F(a) = \sum_j (\sum_i a_i c_{i,j}) t^j \in K[[t]]$ is defined, and $F(a) - P_m \equiv \sum_{j<b} \delta_{m,j} t^j \bmod t^b$ with $\delta_{m,j} \to 0$ in K; the $\mathbb{Q}_\ell$-linear map $K^b \to \mathcal{V}_b$, $(x_j) \mapsto \sum_j x_j t^j$, has finite-dimensional source and is continuous, so $P_m \to F(a)$ in $\mathcal{V}_b$ too. Therefore $\lambda_s(a) \equiv F(a) \bmod \xi^b$ for all b, and $\lambda_s(a) = F(a) \in K[[t]]$ (R_s^+ is ξ-adically separated).

Step 3: units, and $\lambda_s(\Pi)$. For $f \in K[[t]]$ with constant term f_0, $\theta(f) = f_0$ in $\epsilon_s(\mathbb{C}_\ell \otimes_{\mathbb{Q}_\ell} L)$. If $\theta(f)$ is a unit there, then $f_0 \in K^\times$: K is a finite product of fields, and a primitive idempotent e of K with $ef_0 = 0$ would give $e\theta(f) = 0$ with $e \neq 0$ (θ is injective on K). Hence $f \in K[[t]]^\times$. By Step 2 this applies to $\lambda_s(X)$ ($\theta = \epsilon_s(\zeta - 1)$) and to $\lambda_s(Q_{s'})$, $s' \neq s$. For $f = \lambda_s(Q_s) = \epsilon_s \xi$: $f_0 = \theta(f) = 0$, so $f = tf'$ with $f' \in K[[t]]$; and $\xi = tv$ with $v \in B^\times$, so $f' = \epsilon_s v$ (R_s^+ is t-torsion free), $\theta(f')$ is a unit, and $f' \in K[[t]]^\times$. Thus $\lambda_s(Q_s) \in tK[[t]]^\times$, $\lambda_s(\Pi) = \prod_{s'<\mathrm{len}} \lambda_s(Q_{s'})^{H(c-s')} \in t^{H(c-s)} K[[t]]^\times$, and λ_s maps $e_c \mathbf{A}_{F,L}[1/\lambda_{\mathrm{len}}]$ into $K((t))$.

Step 4: algebraicity and regularity. Let $w \in W$, so $\Pi w = \sum_j g_j \varphi^{\mathrm{len}}(n_j)$ with $g_j \in e_c \mathbf{A}_{F,L}$. By (Ex2), $\Lambda_s(\varphi^{\mathrm{len}}(n_j)) = \varphi^q(\iota_T(n_j)) = G^{(q)}(n_j)$ with $G^{(q)}$ as in Lemma 23.13; writing $n_j = \sum_k (\mathbf{h}^{-1})_{jk} H_k$ in $\mathbf{N}_E$ ($\mathbf{h} = (h_{kj}) \in GL_r(e_{c_0} E[[X]])$, Lemma 23.9 (Hf2)), semilinearity of $G^{(q)}$ and Lemma 23.13 (1) give $G^{(q)}(n_j) = \sum_k \mathrm{ev}^{(q)}((\mathbf{h}^{-1})_{jk}) D_k$. For $f = \sum_i b_i X^i \in e_{c_0} E[[X]]$ ($b_i \in e_{c_0}(1 \otimes L)$), $\mathrm{ev}^{(q)}(f) = \sum_i \varphi^q(\iota_\Lambda(b_i))(\exp(\ell^q t) - 1)^i$ with $\varphi^q(\iota_\Lambda(b_i)) \in \epsilon_{c_0+q} L = \epsilon_s L \subseteq K$ and $\varphi^q(\mu) = [\varepsilon]^{\ell^q} - 1 = \exp(\ell^q t) - 1 \in t\mathbb{Q}_\ell[[t]]$; modulo t^b only the terms $i < b$ contribute, both in B and in $K[[t]]$, so $\mathrm{ev}^{(q)}(f) \in K[[t]]$. With Step 2, $\Lambda_s(\Pi w) = \sum_k c_k D_k$ with $c_k = \sum_j \lambda_s(g_j) \mathrm{ev}^{(q)}((\mathbf{h}^{-1})_{jk}) \in K[[t]]$. As $\Lambda_s(w) = \lambda_s(\Pi)^{-1} \Lambda_s(\Pi w)$, w is algebraic at level s with $\mathrm{lv}_s(w) = \lambda_s(\Pi)^{-1}(c_k)_k \in t^{-H(c-s)} K[[t]]^r$ by Step 3, and $\lambda_s(\Pi)\mathrm{lv}_s(w) = (c_k)_k \in K[[t]]^r$. This proves (Lp1) and (Lp4).

Step 5: (Lp5). Let $\mathcal{O} = e_c \mathbf{A}_{F,L}[1/\ell] \cong \mathcal{O}_L[[X]][1/\ell]$, factorial, with Q_s squarefree. By (Lm3) of Lemma 23.6 with $\alpha = 1$ (setting as above; its $\mathcal{O}$ is ours), $g \in \mathcal{O}$ has $\lambda_s(g) \in \xi R_s^+$ if and only if every prime factor of Q_s divides g, i.e. (Q_s being squarefree) if and only if $Q_s \mid g$; as $\ker\theta \cap R_s^+ = \xi R_s^+$, $\theta \circ \lambda_s \colon \mathcal{O} \to \epsilon_s(\mathbb{C}_\ell \otimes L)$ has kernel $Q_s \mathcal{O}$. Let $a \in e_c \mathbf{A}_{F,L}$, $a \neq 0$, with $\lambda_s(a) \in t^k K[[t]] \subseteq \xi^k R_s^+$, and write $a = Q_s^\beta a_0$ in $\mathcal{O}$ with $\beta \geq 0$ maximal, so $Q_s \nmid a_0$. Then $\theta(\lambda_s(a_0)) \neq 0$, i.e. $\lambda_s(a_0) \notin \xi R_s^+$; as $\lambda_s(a) = \xi^\beta \epsilon_s \lambda_s(a_0)$ and R_s^+ is ξ-torsion free, $\beta \geq k$. So $a = Q_s^k a'$ with $a' \in \mathcal{O}$, and $\ell^N a' \in e_c \mathbf{A}_{F,L}$ for some N.

Step 6: (Lp2). For $u \in \mathbb{Z}_\ell^\times$, $\gamma_u(Q_{s'}) \in Q_{s'} \mathbb{Z}_\ell[[X]]^\times$: by Weierstrass preparation $\gamma_u(Q_{s'}) = Pv$ with v a unit and P distinguished of degree $\deg Q_{s'}$ (modulo ℓ, $Q_{s'} \equiv X^{\deg Q_{s'}}$ and $\gamma_u(X) \in uX + X^2 \mathbb{Z}_\ell[[X]]$), and the zeros of $\gamma_u(Q_{s'})$ in the maximal ideal of $\mathcal{O}_{\mathbb{C}_\ell}$ are the x with $(1+x)^u$ a primitive $\ell^{s'+1}$-th root of unity, i.e. the $\deg Q_{s'}$ distinct zeros of $Q_{s'}$; so $P = Q_{s'}$. Hence $\gamma_u(\Pi) = v_u \Pi$ with v_u a unit. Since γ_u commutes with φ (Wm3) and preserves $\mathbf{N}_{c_0}$, it preserves $P_{\mathrm{len}}(\mathbf{N}_{c_0})$, and $\Pi \gamma_u(w) = v_u^{-1} \gamma_u(\Pi w) \in P_{\mathrm{len}}(\mathbf{N}_{c_0})$ for $w \in W$: $\gamma_u(W) \subseteq W$. Let $g \in G_F$ with $\chi(g) = u$. On $\mathbf{A}_F$, $g \circ \iota = \iota \circ \gamma_u$ (g fixes $W(k)$ and $g(\mu) = [\varepsilon]^u - 1 = \iota((1+X)^u - 1)$), so (Ex2) extends to $\iota_T(\gamma_u w) = g(\iota_T(w))$ on $\mathbf{N}[1/\lambda_{\mathrm{len}}]$, and, g commuting with φ, $\Lambda_s(\gamma_u w) = g(\Lambda_s(w)) = \sum_k g(f_k) D_k$ ($D_k \in D_{\mathrm{cris}}(V)$ is G_F-fixed). The automorphism g of B_{dR} preserves every Fil^b, so $g(\sum_i a_i t^i) = \sum_i g(a_i) u^i t^i = \sum_i \sigma_u(a_i) u^i t^i$ for $\sum_i a_i t^i \in K((t))$. So $\mathrm{lv}_s(\gamma_u w) = \gamma_u^{(s)}(\mathrm{lv}_s(w))$.

Step 7: (Lp8). $F/\mathbb{Q}_\ell$ is unramified and $\mathbb{Q}_\ell(\zeta)/\mathbb{Q}_\ell$ totally ramified, so $F(\zeta)/F$ is Galois with group $(\mathbb{Z}/\ell^{s+1})^\times$, $\bar{u}$ acting by $\zeta \mapsto \zeta^u$; $\sigma_u = \epsilon_s(\rho_{\bar{u}} \otimes \mathrm{id}_L)$ with $\rho_{\bar{u}}$ this automorphism. So the σ_u commute ($\sigma_u\sigma_v = \sigma_{uv}$). For $u \in 1 + \ell\mathbb{Z}_\ell$, $u^{\ell^s} \equiv 1 \bmod \ell^{s+1}$, so $\sigma_u^{\ell^s} = \sigma_{u^{\ell^s}} = \mathrm{id}$. Taking invariants under the finite group commutes with $- \otimes_{\mathbb{Q}_\ell} L$ (L is flat), so the common fixed points of the σ_u in K are $\epsilon_s(F \otimes_{\mathbb{Q}_\ell} L) = \epsilon_s L$.

Step 8: (Lh1), (Lh2). $\mathbf{N}(\Phi)$ is $\mathbf{A}_{F,L}$-linear and commutes with φ and the γ_u (Ex3), so it maps W_1 into W_2; extend it $E[[X]]$-linearly to $\mathbf{N}(T_1)_E \to \mathbf{N}(T_2)_E$. Then $\mathbf{N}(\Phi)(H_k^1)$ is γ_u-fixed and lies in $\mathbf{N}(\Phi)(n_k^1) + X\mathbf{N}(T_2)_E = \sum_l o_{lk} n_l^2 + X\mathbf{N}(T_2)_E$; by the uniqueness in (Hf2), $\mathbf{N}(\Phi)(H_k^1) = \sum_l o_{lk} H_l^2$. From $\iota_{T_2} \circ \mathbf{N}(\Phi) = (1 \otimes \Phi) \circ \iota_{T_1}$ (Ex3) and the definition of $G^{(q)}$, $G_2^{(q)}(\mathbf{N}(\Phi)Y) = (1 \otimes \Phi)G_1^{(q)}(Y)$ for $Y \in e_{c_0}\mathbf{N}(T_1)_E$; with Lemma 23.13 (1), $(1 \otimes \Phi)(D_k^1) = \sum_l o_{lk} D_l^2$. Also $\Lambda_s^2 \circ \mathbf{N}(\Phi) = (1 \otimes \Phi) \circ \Lambda_s^1$. For $w \in W_1$ with $\Lambda_s^1(w) = \sum_k f_k D_k^1$, the B_{dR}-linearity of $1 \otimes \Phi$ gives $\Lambda_s^2(\mathbf{N}(\Phi)w) = \sum_l (\sum_k o_{lk} f_k) D_l^2$: (Lh1). If κ_s is up and $v \in U_s^1$, then $\sum_l (ov)_l D_l^2 = (1 \otimes \Phi)(\sum_k v_k D_k^1) \in (1 \otimes \Phi)(\mathrm{Fil}^{b_s} D_1) \subseteq \mathrm{Fil}^{b_s} D_2$ ($1 \otimes \Phi$ maps $D_{\mathrm{cris}}(V_1)$ to $D_{\mathrm{cris}}(V_2)$ and $t^b B \otimes V_1$ to $t^b B \otimes V_2$), and $ov \in \mathcal{O}_L^{r_2}$: (Lh2). □

23.3 The Wach length lemma

The Wach length lemma (Proposition 23.18) is the first half of the lifting exponent over unramified fields (Proposition 24.1): for a surjection $T \to T''$ of crystalline lattices, the cokernel of the induced map of Wach modules is killed at every site by $X^{H_{\max}}$, where $H_{\max}$ is the largest width $M_a + h_a$ of the window. The proof first shows that some power of X kills the cokernel (Proposition 23.19), and then reaches the exponent $H_{\max}$ by an induction around the Frobenius cycle of sites, using the Frobenius recursion of Lemma 23.20. The second half is Γ-descent (Lemma 23.21), which turns an X-power annihilator of a Γ-stable quotient of Wach type into an ℓ-power annihilator; the valuations of the operators that occur there are computed in Lemma 23.22. The subsection also introduces the Δ-invariant Wach ring at ramification e (Definition 23.23), the ring of the Δ-invariant lattices of Sections 24, 27 and 28.

Proposition 23.18. *Let ℓ be an odd prime, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a field K with $F \subseteq K \subseteq F(\mu_\ell)$ as in Proposition 23.7, and $f\colon T \to T''$ a G_K-equivariant $\mathcal{O}_L$-linear surjection of the lattices of pairs (V,T), (V'',T'') over K (for $K = F$: pairs, G_F, and the embedding τ_a itself below). Let $M_a, h_a \geq 0$ be integers such that every Hodge jump of V at every embedding of K over τ_a lies in $[-M_a, h_a]$ ($a \in \mathbb{Z}/n$), and $H_{\max} = \max_a(M_a + h_a)$. Then for every $c \in \mathbb{Z}/n$*

$$X^{H_{\max}}\,\mathbf{N}(T'')_c \subseteq \mathbf{N}(f)(\mathbf{N}(T)_c).$$

***Level** s. Let ℓ be any prime, $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, L, K ($F \subseteq K \subseteq F(\mu_{\ell^s})$) and pairs over K as in (Ex4) of Proposition 23.7, $X_{s-1} = \varphi^{s-1}(X)$ the level variable of Definition 23.2, $f\colon T \to T''$ a G_K-equivariant $\mathcal{O}_L$-linear surjection of the lattices of pairs (V,T), (V'',T'') over K, and M_a, h_a, $H_{\max}$ as above. Then for every $c \in \mathbb{Z}/n$*

$$X_{s-1}^{H_{\max}}\,\mathbf{N}_s(T'')_c \subseteq \mathbf{N}_s(f)(\mathbf{N}_s(T)_c).$$

For $s = 1$ and ℓ odd this is the statement above.

Proof. **Setting.** Put $\mathbf{N} = \mathbf{N}(T'')$, $\mathbf{N}' = \mathbf{N}(f)(\mathbf{N}(T))$ and $H = H_{\max}$; then $\mathbf{N}'_a = e_a\mathbf{N}' = \mathbf{N}(f)(\mathbf{N}(T)_a)$. By (Ex1), $\mathbf{N}(T)$ is a Wach module with window $(M_a, h_a)_a$; by Lemma 23.25 (V is crystalline, hence de Rham) every Hodge jump of V'' at an embedding of K over τ_a is one of V at that embedding, so by (Ex1) $\mathbf{N}$ is a Wach module with the same window. By (Ex3), $\mathbf{N}(f)$ is $\mathbf{A}_{F,L}$-linear and commutes with φ (extended to $\mathbf{N}(T)[1/Q]$) and with every γ_u.

Hence $(\gamma_u - 1)\mathbf{N}' = \mathbf{N}(f)((\gamma_u - 1)\mathbf{N}(T)) \subseteq X\mathbf{N}'$ by (Wm3) for $\mathbf{N}(T)$, and applying $\mathbf{N}(f)$ to (Wm2) for $\mathbf{N}(T)$ gives $Q^{h_a}\mathbf{N}'_a \subseteq \mathbf{A}_{F,L}\varphi(\mathbf{N}'_{a-1})$; so Lemma 23.20 applies to $\mathbf{N}$, $\mathbf{N}'$. Let $R_a = e_a\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ and

$$I_a = \{g \in R_a : \ g\mathbf{N}_a \subseteq \mathbf{N}'_a\},$$

an ideal of R_a. By Proposition 23.19 there is $c_0 \geq 0$ with $X^{c_0}e_a \in I_a$ for all a. We must show $X^H e_a \in I_a$ for all a.

Step 1: if $g \in I_a$, then $Q^H g \in \varphi(I_{a-1})R_a$. By item 1 of Lemma 23.20, $Q^H g = \sum_{k<\ell} X^k\varphi(z_k)$ with $z_k \in R_{a-1}$; we show $z_k \in I_{a-1}$. Let $y \in \mathbf{N}_{a-1}$. By (Wm2) for $\mathbf{N}$, $\rho = Q^{M_a}\varphi(y) \in \mathbf{N}_a$, so $g\rho = \mathbf{N}(f)(x)$ with $x \in \mathbf{N}(T)_a$; by (Wm2) for $\mathbf{N}(T)$, $Q^{h_a}x \in \mathbf{A}_{F,L}\varphi(\mathbf{N}(T)_{a-1})$, and applying $\mathbf{N}(f)$ and multiplying by $Q^{H-M_a-h_a}$,

$$\sum_{k<\ell} X^k\varphi(z_k y) = Q^H g\,\varphi(y) \in \mathbf{A}_{F,L}\varphi(\mathbf{N}'_{a-1}).$$

By item 3 of Lemma 23.20 the right-hand element equals $\sum_k X^k\varphi(u_k)$ with $u_k \in \mathbf{N}'_{a-1}$, and item 2 applied to $z_k y - u_k \in \mathbf{N}_{a-1}$ gives $z_k y = u_k \in \mathbf{N}'_{a-1}$. Hence $z_k \in I_{a-1}$.

Step 2: an induction along the Frobenius cycle. Fix a and suppose $X^H e_a \notin I_a$. Let ϖ be a uniformiser of $\mathcal{O}_L$ and, for every b, identify R_b with $\mathcal{O}_L[[X]]$; then $\varphi\colon R_b \to R_{b+1}$ is $\mathcal{O}_L$-linear and sends X to $(1+X)^\ell - 1 \in (\varpi, X^\ell)$. We show for all $m \geq 0$: if $y \in R_{a+m}$ and $X^H y \in I_{a+m}$, then $y \in (\varpi, X^{m+1})$.

For $m = 0$: if $y \notin (\varpi, X)$, its constant term is a unit, so y is a unit of R_a and $X^H e_a \in I_a$, a contradiction.

For $m \to m+1$: let $y \in R_{a+m+1}$ with $X^H y \in I_{a+m+1}$, and write $y = \sum_{k<\ell} X^k\varphi(z_k)$ with $z_k \in R_{a+m}$ (item 1 of Lemma 23.20). Since $\varphi(X) = XQ$,

$$Q^H X^H y = \sum_{k<\ell} X^k\varphi(X^H z_k),$$

and by Step 1 this lies in $\varphi(I_{a+m})R_{a+m+1}$. Every element $\sum_i r_i\varphi(w'_i)$ of that ideal ($r_i \in R_{a+m+1}$, $w'_i \in I_{a+m}$) can be written, expanding $r_i = \sum_k X^k\varphi(g_{ik})$ by item 1, as $\sum_k X^k\varphi(w_k)$ with $w_k = \sum_i g_{ik}w'_i \in I_{a+m}$. Uniqueness in item 1 gives $X^H z_k = w_k \in I_{a+m}$, so $z_k \in (\varpi, X^{m+1})$ by the induction hypothesis, and $\varphi(z_k) \in (\varpi, \varphi(X)^{m+1}) \subseteq (\varpi, X^{\ell(m+1)}) \subseteq (\varpi, X^{m+2})$. Hence $y \in (\varpi, X^{m+2})$.

Conclusion. Take $m = c_0$ and $y = X^{c_0} \in R_{a+c_0}$: then $X^H y \in I_{a+c_0}$ because $X^{c_0} \in I_{a+c_0}$, so $X^{c_0} \in (\varpi, X^{c_0+1})$. But every element $\varpi\alpha + X^{c_0+1}\beta$ has its coefficient of X^{c_0} in $\varpi\mathcal{O}_L$, while that of X^{c_0} is 1. This contradiction shows $X^H e_a \in I_a$, i.e. $X^H\mathbf{N}_a \subseteq \mathbf{N}'_a$.

Level s. Put $\mathbf{N} = \mathbf{N}_s(T'')$, $\mathbf{N}' = \mathbf{N}_s(f)(\mathbf{N}_s(T))$, $H = H_{\max}$, and R_a, I_a as in the Setting. By (Ex1)–(Ex3) of level s and Lemma 23.25, $\mathbf{N}_s(T)$ and $\mathbf{N}$ are Wach modules of level s with window $(M_a, h_a)_a$, $\mathbf{N}_s(f)$ is $\mathbf{A}_{F,L}$-linear and commutes with φ, and $Q_{s-1}^{h_a}\mathbf{N}'_a \subseteq \mathbf{A}_{F,L}\varphi(\mathbf{N}'_{a-1})$; items 1–3 of Lemma 23.20 apply (paragraph Level s there), and Proposition 23.19 (paragraph Level s) gives $c_0 \geq 0$ with $X_{s-1}^{c_0}e_a \in I_a$ for all a. Step 1 goes through unchanged with Q_{s-1} in place of Q (it uses only (Wm2) of level s for $\mathbf{N}$ and $\mathbf{N}_s(T)$ and items 1–3): if $g \in I_a$, then $Q_{s-1}^H g \in \varphi(I_{a-1})R_a$. Step 2: suppose $X_{s-1}^H e_a \notin I_a$; we show for all $m \geq 0$: if $y \in R_{a+m}$ and $X_{s-1}^H y \in I_{a+m}$, then $y \in (\varpi, X^{m+1})$. For $m = 0$ this is as before. For $m \to m+1$ write $y = \sum_{k<\ell} X^k\varphi(z_k)$; since $\varphi(X_{s-1}) = X_{s-1}Q_{s-1}$ (indeed $\varphi(X_{s-1}) = (1+X)^{\ell^s} - 1 = X_{s-1}\Phi_\ell(1+X_{s-1})$, and $\Phi_\ell(1+X_{s-1}) = \Phi_{\ell^s}(1+X) = Q_{s-1}$ by Lemma 23.8(3)), $Q_{s-1}^H X_{s-1}^H y = \sum_k X^k\varphi(X_{s-1}^H z_k)$, and the rest of the step is unchanged. Conclusion: take $m = \ell^{s-1}c_0$ and $y = X_{s-1}^{c_0} \in R_{a+m}$; then $X_{s-1}^H y \in I_{a+m}$, so $X_{s-1}^{c_0} \in (\varpi, X^{m+1})$. But $X_{s-1} = (1+X)^{\ell^{s-1}} - 1 \equiv X^{\ell^{s-1}}$ modulo ℓ, so the coefficient of X^m in $X_{s-1}^{c_0}$ is $\equiv 1$ modulo ϖ, while every element of (ϖ, X^{m+1}) has that coefficient in $\varpi\mathcal{O}_L$. This contradiction shows $X_{s-1}^H\mathbf{N}_a \subseteq \mathbf{N}'_a$. □

Proposition 23.19. *Let ℓ be an odd prime, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a field K with $F\subseteq K\subseteq F(\mu_\ell)$ as in Proposition 23.7, and $f\colon T\to T''$ a G_K-equivariant $\mathcal{O}_L$-linear surjection of the lattices of pairs (V,T), (V'',T'') over K (for $K=F$: pairs and G_F). Then there is an integer $c\geq 0$ with*

$$X^c\,\mathbf{N}(T'')\subseteq \mathbf{N}(f)(\mathbf{N}(T)).$$

***Level** s. Let ℓ be any prime, $s\geq 1$ an integer with $s\geq 2$ if $\ell=2$, L, K ($F\subseteq K\subseteq F(\mu_{\ell^s})$) and pairs over K as in (Ex4) of Proposition 23.7, $X_{s-1}=\varphi^{s-1}(X)$ the level variable of Definition 23.2, and $f\colon T\to T''$ a G_K-equivariant $\mathcal{O}_L$-linear surjection of the lattices of pairs (V,T), (V'',T'') over K. Then there is an integer $c\geq 0$ with $X_{s-1}^c\,\mathbf{N}_s(T'')\subseteq \mathbf{N}_s(f)(\mathbf{N}_s(T))$. For $s=1$ and ℓ odd this is the statement above.*

Proof. **Setting.** Put $\mathbf{N}=\mathbf{N}(T'')$, $\mathbf{N}'=\mathbf{N}(f)(\mathbf{N}(T))$, $R_a=e_a\mathbf{A}_{F,L}\cong\mathcal{O}_L[[X]]$, $C_a=\mathbf{N}_a/\mathbf{N}'_a$ and $I_a=\{g\in R_a:\ g\mathbf{N}_a\subseteq\mathbf{N}'_a\}$, the annihilator of the finitely generated R_a-module C_a. Choose $M_a,h_a\geq 0$ with all Hodge jumps of V at every embedding of K over τ_a in $[-M_a,h_a]$ and put $H=\max_a(M_a+h_a)$; by (Ex1), $\mathbf{N}(T)$ is a Wach module with window $(M_a,h_a)_a$; by Lemma 23.25 (V is crystalline, hence de Rham) every Hodge jump of V'' at an embedding of K over τ_a is one of V at that embedding, so by (Ex1) $\mathbf{N}$ is a Wach module with the same window. By (Ex3), $\mathbf{N}(f)$ is $\mathbf{A}_{F,L}$-linear and commutes with φ (extended to $\mathbf{N}(T)[1/Q]$) and with every γ_u; hence $(\gamma_u-1)\mathbf{N}'=\mathbf{N}(f)((\gamma_u-1)\mathbf{N}(T))\subseteq X\mathbf{N}'$ for all $u\in\Gamma_K$ by (Wm3) for $\mathbf{N}(T)$, and applying $\mathbf{N}(f)$ to (Wm2) for $\mathbf{N}(T)$ gives $Q^{h_a}\mathbf{N}'_a\subseteq\mathbf{A}_{F,L}\varphi(\mathbf{N}'_{a-1})$ for all a; so Lemma 23.20 applies to $\mathbf{N},\mathbf{N}'$. Let ϖ be a uniformiser of $\mathcal{O}_L$ and $\kappa_L=\mathcal{O}_L/\varpi$. Ring-theoretic facts: R_a is a two-dimensional regular local ring; its height-one primes are (ϖ) and the (P) with P an irreducible distinguished polynomial, and each (P) is the set of g vanishing at a root z of P in the open unit disc $\mathfrak{m}_{\mathbb{C}_\ell}$ (Weierstrass preparation: every $0\neq g$ is $\varpi^k P_g u$ with u a unit and P_g distinguished, so g has finitely many zeros in $\mathfrak{m}_{\mathbb{C}_\ell}$, and we write $\mathrm{ord}_z(g)$ for the order of vanishing at z).

Step 1: $I_a\not\subseteq(\varpi)$; in particular $I_a\neq 0$. Reduce the realisations modulo ϖ. Since $\ell\in\varpi\mathcal{O}_L$, for a pair (V°,T°) over K we have $(A_{\mathrm{inf}}[1/\mu_\varepsilon]\otimes_{\mathbb{Z}_\ell}T^\circ)/\varpi=\mathbb{C}_\ell^\flat\otimes_{\mathbb{F}_\ell}(T^\circ/\varpi T^\circ)$, because $A_{\mathrm{inf}}[1/\mu_\varepsilon]/\ell=\mathcal{O}_{\mathbb{C}_\ell^\flat}[1/(\varepsilon-1)]=\mathbb{C}_\ell^\flat$. Let $B=\mathbb{C}_\ell^\flat\otimes_{\mathbb{F}_\ell}\kappa_L$, a finite product of copies of $\mathbb{C}_\ell^\flat$ (κ_L is finite separable over $\mathbb{F}_\ell$ and $\mathbb{C}_\ell^\flat$ is algebraically closed), and $B_a=\bar\varepsilon_a B$ with $\bar\varepsilon_a$ the image of $(\iota\otimes\mathrm{id})(e_a)$; so $B=\prod_a B_a$ and each B_a is a non-zero product of fields. Reducing $\iota\otimes\mathrm{id}\colon R_a\to(A_{\mathrm{inf}}\otimes\mathcal{O}_L)\bar\varepsilon_a$ gives a ring map $\kappa_L[[X]]=R_a/\varpi\to B_a$ sending X to the unit $(\varepsilon-1)\bar\varepsilon_a$, hence $k_a:=\kappa_L((X))\to B_a$. By (Ex2), ι_{T° induces a k_a-semilinear map from $\bar{\mathbf{N}}^\circ_a:=(\mathbf{N}(T^\circ)_a/\varpi)[1/X]$ to $Y^\circ_a:=\bar\varepsilon_a(\mathbb{C}_\ell^\flat\otimes_{\mathbb{F}_\ell}T^\circ/\varpi)$, and its B_a-linear extension $\Psi^\circ_a\colon B_a\otimes_{k_a}\bar{\mathbf{N}}^\circ_a\to Y^\circ_a$ is onto, since $A_{\mathrm{inf}}[1/\mu_\varepsilon]\iota_{T^\circ}(\mathbf{N}(T^\circ))=A_{\mathrm{inf}}[1/\mu_\varepsilon]\otimes T^\circ$. Both sides are free B_a-modules of rank $\dim_L V^\circ$: the source by (Wm1) and (Ex1) ($\mathbf{N}(T^\circ)_a$ free of rank $\dim_L V^\circ$), the target because $\mathbb{C}_\ell^\flat\otimes_{\mathbb{F}_\ell}T^\circ/\varpi=B\otimes_{\kappa_L}(T^\circ/\varpi)$ is free of that rank over B. A surjection between free modules of the same finite rank over a finite product of fields is bijective, so Ψ°_a is an isomorphism. Apply this to T'' and T. As f is onto, $(1\otimes f)(Y_a(T))=Y_a(T'')$, and $(1\otimes f)\circ\Psi_a(T)=\Psi_a(T'')\circ(1\otimes\overline{\mathbf{N}(f)})$ by (Ex3); hence $B_a\otimes U=B_a\otimes\bar{\mathbf{N}}_a$, where $U\subseteq\bar{\mathbf{N}}_a$ is the k_a-span of the image of $\mathbf{N}'_a$. Since $B_a\neq 0$ is faithfully flat over the field k_a, $U=\bar{\mathbf{N}}_a$. Now k_a is the residue field of the discrete valuation ring $R_{a,(\varpi)}$, so $\mathbf{N}'_{a,(\varpi)}+\varpi\mathbf{N}_{a,(\varpi)}=\mathbf{N}_{a,(\varpi)}$, and by Nakayama $(C_a)_{(\varpi)}=0$. As C_a is finitely generated, some $s\in R_a\setminus(\varpi)$ kills C_a, i.e. $I_a\not\subseteq(\varpi)$.

Step 2: the annihilators along Frobenius. If $g\in I_a$, then $Q^Hg=\sum_{k<\ell}X^k\varphi(w_k)$ with $w_k\in I_{a-1}$. Indeed, by item 1 of Lemma 23.20, $Q^Hg=\sum_k X^k\varphi(w_k)$ with $w_k\in R_{a-1}$; for $y\in\mathbf{N}_{a-1}$, (Wm2) for $\mathbf{N}$ gives $g\,Q^{M_a}\varphi(y)=\mathbf{N}(f)(x)$ with $x\in\mathbf{N}(T)_a$, (Wm2) for $\mathbf{N}(T)$ and $\mathbf{N}(f)$ give $Q^Hg\varphi(y)=\sum_k X^k\varphi(w_ky)\in\mathbf{A}_{F,L}\varphi(\mathbf{N}'_{a-1})$, and items 3 and 2 of Lemma 23.20 give $w_ky\in\mathbf{N}'_{a-1}$.

Step 3: no zeros off 0. In the coordinates $R_b\cong\mathcal{O}_L[[X]]$, $\varphi(g)(X)=g((1+X)^\ell-1)$ (under the identifications $R_b\cong\mathcal{O}_L[[X]]$, $\varphi\colon R_{b-1}\to R_b$ is the $\mathcal{O}_L$-linear ring map sending X

to $(1+X)^\ell - 1$, Definition 23.2). For $z \in \mathfrak{m}_{\mathbb{C}_\ell}$ put $z^{[\ell]} = (1+z)^\ell - 1$; as $(1+X)^\ell - 1 - z^{[\ell]}$ has a simple zero at z (its derivative $\ell(1+z)^{\ell-1}$ is non-zero), $\mathrm{ord}_z(\varphi(g)) = \mathrm{ord}_{z^{[\ell]}}(g)$. For $b \in \mathbb{Z}/n$ let Z_b be the set of $z \in \mathfrak{m}_{\mathbb{C}_\ell} \setminus \{0\}$ at which every element of I_b vanishes; Z_b is finite, as $I_b \neq 0$. Let $w \in Z_{a-1}$ and $z \in \mathfrak{m}_{\mathbb{C}_\ell}$ with $z^{[\ell]} = w$; there are exactly ℓ such z (the roots of $(1+z)^\ell = 1+w$ are distinct as $1+w \neq 0$, and each lies in the open disc because every ℓ-th root of $1+w$ is $\equiv 1$ modulo the maximal ideal), and $z \neq 0$, $Q(z) = w/z \neq 0$. For $g \in I_a$, Step 2 gives $\mathrm{ord}_z(g) = \mathrm{ord}_z(Q^H g) \geq \min_k \mathrm{ord}_z(\varphi(w_k)) = \min_k \mathrm{ord}_w(w_k) \geq 1$. Hence all ℓ such z lie in Z_a. By induction, if $Z_{a-1} \neq \emptyset$ then Z_{a-1+k} contains the ℓ^k distinct solutions of $(1+z)^{\ell^k} = 1+w$ for every $k \geq 1$; taking $k \equiv 0 \pmod{n}$ large contradicts the finiteness of Z_{a-1}. So $Z_b = \emptyset$ for every b.

Step 4: conclusion. Let $\mathfrak{p}$ be a minimal prime of R_a over I_a. Then $\mathfrak{p} \neq 0$ (Step 1), and $\mathfrak{p}$ is either maximal, or of height one; in the latter case $\mathfrak{p} \neq (\varpi)$ by Step 1, so $\mathfrak{p} = (P)$ with P irreducible distinguished, and every element of I_a vanishes at the roots of P; by Step 3 these roots are all 0, so $P = X$. Hence X lies in every minimal prime over I_a, i.e. in $\sqrt{I_a}$, and since R_a is Noetherian, $X^{c_a} \in I_a$ for some c_a. With $c = \max_a c_a$, $X^c \mathbf{N}(T'') = \sum_a X^c \mathbf{N}_a \subseteq \mathbf{N}'$.

Level s. Put $\mathbf{N} = \mathbf{N}_s(T'')$, $\mathbf{N}' = \mathbf{N}_s(f)(\mathbf{N}_s(T))$ and R_a, C_a, I_a, H as in the Setting, with (Ex1)–(Ex3) of level s (Proposition 23.7, (Ex4)) and Wach modules of level s; items 1–3 of Lemma 23.20 hold for them (paragraph Level s there). Step 1 goes through unchanged with ι_s, ι_{s,T° in place of ι, ι_{T°: by Lemma 23.8(6), $\iota_s(X) = \mu_{\varepsilon,s} = [\varepsilon^{1/\ell^{s-1}}] - 1$, whose reduction $\varepsilon^{1/\ell^{s-1}} - 1$ is a non-zero element, hence a unit, of $\mathbb{C}_\ell^\flat$; the spanning clause is (Ex2) of level s; and $\ell \in \varpi\mathcal{O}_L$, κ_L separable over $\mathbb{F}_\ell$, hold for every prime. Step 2 goes through unchanged with Q_{s-1} in place of Q ((Wm2) of level s): if $g \in I_a$, then $Q_{s-1}^H g = \sum_{k<\ell} X^k \varphi(w_k)$ with $w_k \in I_{a-1}$. Step 3 becomes: *every $w \in Z_b$ satisfies* $(1+w)^{\ell^{s-1}} = 1$. Suppose $w \in Z_{a-1}$ with $(1+w)^{\ell^{s-1}} \neq 1$, and let z be one of the ℓ points of the open disc with $z^{[\ell]} = w$ (they are distinct and non-zero, as before). Then $(1+z)^{\ell^s} = (1+w)^{\ell^{s-1}} \neq 1$, so $1+z$ is not a primitive ℓ^s-th root of unity and $Q_{s-1}(z) \neq 0$ (Lemma 23.8(3)); hence for $g \in I_a$, $\mathrm{ord}_z(g) = \mathrm{ord}_z(Q_{s-1}^H g) \geq \min_k \mathrm{ord}_w(w_k) \geq 1$, and $z \in Z_a$; moreover $(1+z)^{\ell^{s-1}} \neq 1$ (its ℓ-th power is $(1+w)^{\ell^{s-1}} \neq 1$). By induction Z_{a-1+k} contains the ℓ^k distinct solutions of $(1+z)^{\ell^k} = 1+w$, and $k \equiv 0 \pmod{n}$ large contradicts the finiteness of Z_{a-1}. Step 4: let $\mathfrak{p}$ be a minimal prime of R_a over I_a; as before $\mathfrak{p}$ is maximal or $\mathfrak{p} = (P)$ with P irreducible distinguished, all of whose roots are zeros of every element of I_a, hence lie in $Z_a \cup \{0\}$, hence are roots of $X_{s-1} = (1+X)^{\ell^{s-1}} - 1$. Weierstrass division $X_{s-1} = qP + \rho$ with $\deg \rho < \deg P$ gives a polynomial ρ vanishing at the $\deg P$ distinct roots of the separable P, so $\rho = 0$ and $X_{s-1} \in (P)$. The maximal ideal (ϖ, X) contains X_{s-1} too. So $X_{s-1} \in \sqrt{I_a}$, $X_{s-1}^{c_a} \in I_a$ for some c_a, and $c = \max_a c_a$ works. □

Lemma 23.20. *Let ℓ be an odd prime, F, n, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ as in the definition of sites, K a base field as there, $d = |\Delta_K|$, $\mathbf{N}$ a Wach module over K with coefficients L and window $(M_a, h_a)_{a\in\mathbb{Z}/n}$, and $\mathbf{N}' \subseteq \mathbf{N}$ an $\mathbf{A}_{F,L}$-submodule with $\mathbf{N}'_a = e_a\mathbf{N}'$, such that for all $u \in \Gamma_K$ and $a \in \mathbb{Z}/n$*

$$(\gamma_u - 1)(\mathbf{N}') \subseteq X\mathbf{N}', \qquad Q^{h_a}\mathbf{N}'_a \subseteq \mathbf{A}_{F,L}\,\varphi(\mathbf{N}'_{a-1}).$$

Let $a \in \mathbb{Z}/n$. Then:

1. *every $f \in e_a\mathbf{A}_{F,L}$ can be written uniquely as $f = \sum_{k=0}^{\ell-1} X^k \varphi(g_k)$ with $g_k \in e_{a-1}\mathbf{A}_{F,L}$;*
2. *if $x_0, \ldots, x_{\ell-1} \in \mathbf{N}_{a-1}$ and $\sum_k X^k\varphi(x_k) = 0$, then all $x_k = 0$;*
3. *every element of $\mathbf{A}_{F,L}\varphi(\mathbf{N}'_{a-1})$ lying in $e_a\mathbf{N}[1/Q]$ is of the form $\sum_{k<\ell} X^k\varphi(y_k)$ with $y_k \in \mathbf{N}'_{a-1}$; and if $x \in \mathbf{N}_{a-1}$ and $\varphi(x) \in \mathbf{A}_{F,L}\varphi(\mathbf{N}'_{a-1})$, then $x \in \mathbf{N}'_{a-1}$;*

4. if $s \geq 1$ and b are integers with $X^s \mathbf{N}_a \subseteq \mathbf{N}'_a$ and $b \geq 2 + \log_\ell s$, then

$$X^{M_a + h_a + \lceil s/d \rceil + b} \mathbf{N}_{a-1} \subseteq \mathbf{N}'_{a-1}.$$

For $K = F$ ($\Gamma_F = \mathbb{Z}_\ell^\times$, $d = \ell - 1$) the exponent is $M_a + h_a + \lceil s/(\ell-1) \rceil + b$.

Items 1–3 use neither the hypothesis on γ_u nor the one on $Q^{h_a}\mathbf{N}'_a$.

Level s. *Let ℓ be any prime, $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, K a field with $F \subseteq K \subseteq F(\mu_{\ell^s})$, $\mathbf{N}$ a Wach module of level s over K with coefficients L and window $(M_a, h_a)_a$ (Definition 23.2, paragraph Level s), and $\mathbf{N}' \subseteq \mathbf{N}$ an $\mathbf{A}_{F,L}$-submodule. Then items 1–3 hold, with $\mathbf{N}[1/Q]$ replaced by $\mathbf{N}[1/Q_{s-1}]$ in item 3.*

Proof. Write $R_a = e_a \mathbf{A}_{F,L}$, a ring isomorphic to $\mathcal{O}_L[[X]]$ by the site decomposition; $\varphi(e_{a-1}) = e_a$, so φ maps R_{a-1} into R_a and $\mathbf{N}_{a-1}$ into $e_a\mathbf{N}[1/Q]$.

Step 1 (item 1): R_a is free over $\varphi(R_{a-1})$ with basis $1, X, \dots, X^{\ell-1}$. On coefficients, $\varphi \otimes \mathrm{id}$ maps $e_{a-1}(\mathcal{O}_F \otimes \mathcal{O}_L)$ isomorphically onto $e_a(\mathcal{O}_F \otimes \mathcal{O}_L)$, and both are identified with $\mathcal{O}_L$ compatibly ($\tau_a \circ \varphi = \tau_{a-1}$). So in the coordinates $R_{a-1} \cong \mathcal{O}_L[[X]] \cong R_a$ the map φ is $g(X) \mapsto g(P)$ with $P = (1+X)^\ell - 1 = X^\ell + \ell(\cdots)$, a distinguished polynomial of degree ℓ with $P(0) = 0$; in particular φ is injective on R_{a-1}. Existence of $f = \sum_{k<\ell} X^k g_k(P)$ for $f \in \mathcal{O}_L[[X]]$: by the Weierstrass division theorem write $f = r_0 + q_1 P$, $q_1 = r_1 + q_2 P$, …, with polynomials r_i of degree $< \ell$; then $f = \sum_{i \geq 0} r_i P^i$ (convergent for the (ℓ, X)-adic topology, as $P^i \in (\ell, X)^i$ and $\mathcal{O}_L[[X]]$ is (ℓ, X)-adically complete), and collecting the coefficient of X^k in the r_i gives the g_k. Uniqueness: if $\sum_{k<\ell} X^k g_k(P) = 0$ with not all $g_k = 0$, divide by the largest power of a uniformiser of $\mathcal{O}_L$ dividing all g_k and reduce modulo it; since $P \equiv X^\ell$, this gives $\sum_k X^k \bar{g}_k(X^\ell) = 0$ with some $\bar{g}_k \neq 0$, impossible because the exponents occurring in $X^k \bar{g}_k(X^\ell)$ lie in $k + \ell\mathbb{Z}$.

Step 2 (item 2): if $x_0, \dots, x_{\ell-1} \in \mathbf{N}_{a-1}$ and $\sum_k X^k \varphi(x_k) = 0$, then all $x_k = 0$. Let $b_1, \dots, b_r$ be an R_{a-1}-basis of $\mathbf{N}_{a-1}$ (Wm1). By (Wm2), $Q^{h_a}\mathbf{N}_a \subseteq \mathbf{A}_{F,L}\varphi(\mathbf{N}_{a-1}) = \sum_t R_a \varphi(b_t)$; as $\mathbf{N}_a$ is free of rank r over the domain R_a, the r elements $\varphi(b_t)$ span an r-dimensional space over the fraction field of R_a, hence are linearly independent over R_a. Writing $x_k = \sum_t g_{kt} b_t$, the relation becomes $\sum_t (\sum_k X^k \varphi(g_{kt}))\varphi(b_t) = 0$, so $\sum_k X^k \varphi(g_{kt}) = 0$ for every t, and Step 1 gives $g_{kt} = 0$.

Step 3 (item 3): if $x \in \mathbf{N}_{a-1}$ and $\varphi(x) \in \mathbf{A}_{F,L}\varphi(\mathbf{N}'_{a-1})$, then $x \in \mathbf{N}'_{a-1}$. Write $\varphi(x) = \sum_m f_m \varphi(n'_m)$ with $n'_m \in \mathbf{N}'_{a-1}$ and (multiplying by e_a) $f_m \in R_a$, and by Step 1 $f_m = \sum_{k<\ell} X^k \varphi(g_{mk})$ with $g_{mk} \in R_{a-1}$. Then $\varphi(x) = \sum_k X^k \varphi(y_k)$ with $y_k = \sum_m g_{mk} n'_m \in \mathbf{N}'_{a-1}$; this computation applies to every element $\sum_m f_m \varphi(n'_m)$ of $\mathbf{A}_{F,L}\varphi(\mathbf{N}'_{a-1}) \cap e_a\mathbf{N}[1/Q]$ and proves the first assertion of item 3. Step 2 applied to $x - y_0, -y_1, \dots, -y_{\ell-1}$ gives $x = y_0 \in \mathbf{N}'_{a-1}$.

Step 4 (item 4): the exponent. Put $k = \lceil s/d \rceil + b$ and $E = M_a + h_a + k$; let $n \in \mathbf{N}_{a-1}$. Since $\varphi(X) = XQ$,

$$\varphi(X^E n) = X^E Q^E \varphi(n) = Q^{h_a} \cdot X^E Q^k \rho, \qquad \rho = Q^{M_a}\varphi(n) \in \mathbf{N}_a$$

by (Wm2). We show $X^E Q^k \rho \in \mathbf{N}'_a$. Then $\varphi(X^E n) \in Q^{h_a}\mathbf{N}'_a \subseteq \mathbf{A}_{F,L}\varphi(\mathbf{N}'_{a-1})$ by hypothesis, and Step 3 gives $X^E n \in \mathbf{N}'_{a-1}$, which is the assertion.

Now $Q = \sum_{m=1}^{\ell} \binom{\ell}{m} X^{m-1} = X^{\ell-1} + \ell w$ with $w \in \mathbb{Z}_\ell[X]$, so

$$X^E Q^k \rho = \sum_{i=0}^{k} \binom{k}{i} w^i \cdot \ell^i X^{j_i} \rho, \qquad j_i = E + (\ell-1)(k-i).$$

Apply Lemma 23.21 with $\Gamma = \Gamma_K$, a closed subgroup of $\mathbb{Z}_\ell^\times$ containing $1 + \ell\mathbb{Z}_\ell$ with $[\Gamma_K : 1 + \ell\mathbb{Z}_\ell] = |\Delta_K| = d$ (Lemma 23.24(Rr7): $\Gamma_K = \Delta_K \cdot (1 + \ell\mathbb{Z}_\ell)$), and $D = N = \mathbf{N}_a$ (free,

so X is injective), $N' = \mathbf{N}'_a$: both are stable under γ_u with $(\gamma_u - 1)\mathbf{N}_a \subseteq X\mathbf{N}_a$ by (Wm3) and $(\gamma_u - 1)\mathbf{N}'_a = e_a(\gamma_u - 1)\mathbf{N}' \subseteq X\mathbf{N}'_a$ ($\gamma_u(e_a) = e_a$), and X^s kills $\mathbf{N}_a/\mathbf{N}'_a$. Fix i and put $t = k - i \geq 0$. If $j_i \geq s$, then $\ell^i X^{j_i}\rho \in X^s\mathbf{N}_a \subseteq \mathbf{N}'_a$. If $j_i < s$, by items 1 and 3 of Lemma 23.21 it suffices that $i \geq (s - j_i)\ell/(d(\ell-1)) + 2 + \log_\ell s$, and for this it suffices, as $b \geq 2 + \log_\ell s$, that $(s - j_i)\ell/(d(\ell-1)) \leq k - t - b$. Since $E \geq k$, $j_i \geq k + (\ell-1)t$, and, as $\ell/d \geq 1$,

$$\frac{(s-j_i)\ell}{d(\ell-1)} \leq \frac{(s-k)\ell}{d(\ell-1)} - \frac{t\ell}{d} \leq \frac{(s-k)\ell}{d(\ell-1)} - t,$$

so it suffices that $s\ell/(d(\ell-1)) + b \leq k(1 + \ell/(d(\ell-1)))$. This holds: $k \geq s/d + b$ and

$$\Big(\frac{s}{d} + b\Big)\Big(1 + \frac{\ell}{d(\ell-1)}\Big) \geq \frac{s}{d} + \frac{s\ell}{d^2(\ell-1)} + b \geq \frac{s}{d} + \frac{s}{d(\ell-1)} + b = \frac{s\ell}{d(\ell-1)} + b,$$

using $\ell/d \geq 1$ in the middle step. Hence every term lies in $\mathbf{N}'_a$, and so does $X^E Q^k \rho$.

Level s. Step 1 concerns only $\mathbf{A}_{F,L}$ and φ, which are defined by the same formulas for every prime (Definition 23.2, paragraph Level s), and it does not use that ℓ is odd: $(1+X)^\ell - 1$ is a distinguished polynomial of degree ℓ with constant term 0 for every prime. Step 2 uses (Wm1) and the second inclusion of (Wm2); for a Wach module of level s the latter reads $Q_{s-1}^{h_a}\mathbf{N}_a \subseteq \mathbf{A}_{F,L}\varphi(\mathbf{N}_{a-1})$, and the rank argument is unchanged ($Q_{s-1}^{h_a}\mathbf{N}_a$ is free of rank r over the domain R_a). Step 3 uses only Steps 1 and 2, with $\varphi(\mathbf{N}_{a-1}) \subseteq e_a\mathbf{N}[1/Q_{s-1}]$. □

Lemma 23.21. *Let ℓ be an odd prime and $\mathbf{A}_{F,L}$, X, γ_u ($u \in \mathbb{Z}_\ell^\times$) as in the definition of sites. Let $\Gamma \subseteq \mathbb{Z}_\ell^\times$ be a closed subgroup containing $1+\ell\mathbb{Z}_\ell$ and $d = [\Gamma : 1+\ell\mathbb{Z}_\ell]$, a divisor of $\ell-1$ (for a field K as in Definition 23.2: $\Gamma = \Gamma_K$ and $d = |\Delta_K|$; for $K = F$: $\Gamma = \mathbb{Z}_\ell^\times$ and $d = \ell-1$). Let D be an $\mathbf{A}_{F,L}$-module on which multiplication by X is injective, with additive maps $\gamma_{u,D}\colon D \to D$ ($u \in \Gamma$) satisfying $\gamma_{u,D}(fx) = \gamma_u(f)\gamma_{u,D}(x)$ ($f \in \mathbf{A}_{F,L}$, $x \in D$), and let $N' \subseteq N \subseteq D$ be $\mathbf{A}_{F,L}$-submodules with*

$$(\gamma_{u,D} - 1)(N) \subseteq XN, \qquad (\gamma_{u,D} - 1)(N') \subseteq XN' \qquad (u \in \Gamma).$$

Say that $f \in \mathbf{A}_{F,L}$ kills N/N' if $fN \subseteq N'$. For integers $0 \leq j \leq s$ put

$$\mathrm{dsc}_{\ell,d}(j,s) = \sum_{\substack{j<k\leq s \\ d|k}} (1 + v_\ell(k)), \qquad \mathrm{dsc}_\ell(j,s) = \mathrm{dsc}_{\ell,\ell-1}(j,s).$$

1. *If X^s kills N/N', then $\ell^a X^j$ kills N/N' for all integers $0 \leq j \leq s$ and $a \geq \mathrm{dsc}_{\ell,d}(j,s)$.*
2. *$\mathrm{dsc}_{\ell,d}(0,s) = A + v_\ell(A!)$ with $A = \lfloor s/d \rfloor$, and $\mathrm{dsc}_{\ell,d}(0,s) \leq \lfloor s\ell/(d(\ell-1)) \rfloor$.*
3. *If $s \geq 1$, then $\mathrm{dsc}_{\ell,d}(j,s) < (s-j)\ell/(d(\ell-1)) + 2 + \log_\ell s$ for $0 \leq j \leq s$.*

For $\Gamma = \mathbb{Z}_\ell^\times$ ($d = \ell - 1$) the three items read with dsc_ℓ and the bounds $\lfloor s\ell/(\ell-1)^2 \rfloor$ and $(s-j)\ell/(\ell-1)^2 + 2 + \log_\ell s$.

Level s. *In this paragraph $s \geq 1$ is a cyclotomic level and $S \geq 0$ a killing exponent. Let ℓ be any prime, $s \geq 2$ if $\ell = 2$, $X_{s-1} = \varphi^{s-1}(X) = (1+X)^{\ell^{s-1}} - 1$ (the level variable of Definition 23.2) and $u_s = 1 + \ell^s$. Let D be an $\mathbf{A}_{F,L}$-module on which multiplication by X_{s-1} is injective, with an additive map $\gamma_D\colon D \to D$ such that $\gamma_D(fx) = \gamma_{u_s}(f)\gamma_D(x)$, and let $N' \subseteq N \subseteq D$ be $\mathbf{A}_{F,L}$-submodules with $(\gamma_D - 1)(N) \subseteq X_{s-1}N$ and $(\gamma_D - 1)(N') \subseteq X_{s-1}N'$. For integers $0 \leq j \leq S$ and $M \geq 1$ put*

$$\mathrm{dsc}_\ell^{(s)}(j,S) = \sum_{j<k\leq S} (s + v_\ell(k)), \qquad \sigma_M^{(s)} = \sum_{i=1}^{M-1} (s + v_\ell(i)) = (M-1)s + v_\ell((M-1)!).$$

4. *(Descent.) If X_{s-1}^S kills N/N', then $\ell^a X_{s-1}^j$ kills N/N' for all $0 \le j \le S$ and $a \ge \mathrm{dsc}_\ell^{(s)}(j,S)$; and $\mathrm{dsc}_\ell^{(s)}(0,S) = sS + v_\ell(S!) \le (s+1)S$.*

5. *(Precision raise.) For every $M \ge 1$ and $z \in X_{s-1}N$ there is $k \in X_{s-1}N$ with $\ell^{\sigma_M^{(s)}} z - (\gamma_D - 1)(k) \in X_{s-1}^M N$.*

6. *(Rigidity.) Suppose moreover that $N/X_{s-1}N$ has no ℓ-torsion (for instance N a finite free $\mathbf{A}_{F,L}$-module, or a site component e_a of one). If $M \ge 1$, $y \in X_{s-1}N$ and $(\gamma_D - 1)(y) \in X_{s-1}^M N$, then $y \in X_{s-1}^M N$.*

For $s = 1$ and ℓ odd, item 4 with $d = 1$ is item 1 for $\Gamma \supseteq 1 + \ell\mathbb{Z}_\ell$ restricted to the single element $u_1 = 1 + \ell$.

Proof. **The descent step.** Let $c \in \mathbb{Z}_\ell$, $j \ge 0$ and $u \in \Gamma$, and suppose cX^{j+1} kills N/N'; we show that $c(u^{j+1}-1)X^j$ kills N/N'. Write $\gamma = \gamma_{u,D}$. Since $\gamma_u(X) = (1+X)^u - 1 = \sum_{k\ge1} \binom{u}{k} X^k$, we have $\gamma_u(X) = X(u + Xt)$ with $t \in \mathbb{Z}_\ell[[X]]$. Let $n \in N$ and $m = cX^j n \in N$; by hypothesis $Xm \in N'$. Write $\gamma(n) = n + Xn_1$ with $n_1 \in N$. Then

$$\gamma(m) = cX^j(u + Xt)^j(n + Xn_1) = u^j m + cX^{j+1} n_2 \qquad (n_2 \in N),$$

so $\gamma(m) - u^j m \in N'$. Next, $Xm \in N'$ gives $\gamma(Xm) - Xm = Xn'$ with $n' \in N'$, and $\gamma(Xm) = \gamma_u(X)\gamma(m) = X(u + Xt)\gamma(m)$; since X is injective on D, $(u + Xt)\gamma(m) - m = n' \in N'$. Now $X\gamma(m) = u^j Xm + X(\gamma(m) - u^j m) \in N'$, hence $u\gamma(m) - m \in N'$, and with $u(\gamma(m) - u^j m) \in N'$ we get $(u^{j+1} - 1)m \in N'$, which is the claim.

The element u. As ℓ is odd, $\mathbb{Z}_\ell^\times = \mu_{\ell-1} \times (1 + \ell\mathbb{Z}_\ell)$, and since $\Gamma \supseteq 1 + \ell\mathbb{Z}_\ell$, $\Gamma = \Delta \times (1 + \ell\mathbb{Z}_\ell)$ with $\Delta = \Gamma \cap \mu_{\ell-1}$ cyclic of order d. Let δ_0 generate Δ and $u = \delta_0(1 + \ell) \in \Gamma$. Reduction modulo ℓ maps $\mu_{\ell-1}$ isomorphically onto $\mathbb{F}_\ell^\times$, so $u \bmod \ell = \delta_0 \bmod \ell$ has order d, and $u^k - 1 \in \mathbb{Z}_\ell^\times$ when $d \nmid k$. For $k = dm$, $u^k = (1 + \ell)^{dm}$, and the classical identity $v_\ell((1 + \ell)^n - 1) = 1 + v_\ell(n)$ (ℓ odd) with $v_\ell(d) = 0$ gives $v_\ell(u^k - 1) = 1 + v_\ell(m) = 1 + v_\ell(k)$.

Item 1. Applying the descent step with $c = \prod_{k=i+1}^{s}(u^k - 1)$ for $i = s-1, s-2, \ldots, j$ (induction downwards from $c = 1$, $i + 1 = s$) shows that $\prod_{k=j+1}^{s}(u^k - 1)\, X^j$ kills N/N'. By the previous paragraph $\prod_{k=j+1}^{s}(u^k - 1) = \epsilon\, \ell^{\mathrm{dsc}_{\ell,d}(j,s)}$ with $\epsilon \in \mathbb{Z}_\ell^\times$; as N' is an $\mathbf{A}_{F,L}$-module, $\ell^{\mathrm{dsc}_{\ell,d}(j,s)} X^j = \epsilon^{-1} \cdot \epsilon \ell^{\mathrm{dsc}_{\ell,d}(j,s)} X^j$ kills N/N', and so does every $\ell^a X^j$ with $a \ge \mathrm{dsc}_{\ell,d}(j,s)$.

Item 2. The $k \in (0, s]$ divisible by d are $k = dm$, $1 \le m \le A$, and $v_\ell(k) = v_\ell(m)$, so $\mathrm{dsc}_{\ell,d}(0,s) = A + \sum_{m \le A} v_\ell(m) = A + v_\ell(A!)$. For $A = 0$ the bound is trivial. For $A \ge 1$, Legendre's formula $v_\ell(A!) = (A - S_\ell(A))/(\ell - 1)$, with $S_\ell(A) \ge 1$ the sum of the base-ℓ digits, gives $\mathrm{dsc}_{\ell,d}(0,s) \le A + (A-1)/(\ell-1) < A\ell/(\ell-1) \le s\ell/(d(\ell-1))$; an integer below a real number x is at most $\lfloor x \rfloor$.

Item 3. Put $m_1 = \lfloor s/d \rfloor$ and $m_0 = \lfloor j/d \rfloor$; as in item 2, $\mathrm{dsc}_{\ell,d}(j,s) = (m_1 - m_0) + v_\ell(m_1!) - v_\ell(m_0!)$. If $m_1 = m_0$ this is 0 and the bound holds. Otherwise $m_1 \ge 1$ and, by Legendre's formula,

$$\mathrm{dsc}_{\ell,d}(j,s) = (m_1 - m_0)\frac{\ell}{\ell-1} - \frac{S_\ell(m_1) - S_\ell(m_0)}{\ell - 1} \le (m_1 - m_0)\frac{\ell}{\ell-1} + 1 + \log_\ell s,$$

because $S_\ell(m_1) \ge 1$, and $S_\ell(m_0) \le (\ell-1)(1 + \log_\ell m_0)$ for $m_0 \ge 1$ (at most $1 + \log_\ell m_0$ digits, each at most $\ell - 1$) while $S_\ell(0) = 0$, and $m_0 \le s$. Finally $m_1 \le s/d$ and $m_0 \ge (j - d + 1)/d$ give $m_1 - m_0 \le (s - j + d - 1)/d$, so

$$\mathrm{dsc}_{\ell,d}(j,s) \le \frac{(s-j)\ell}{d(\ell-1)} + \frac{(d-1)\ell}{d(\ell-1)} + 1 + \log_\ell s < \frac{(s-j)\ell}{d(\ell-1)} + 2 + \log_\ell s,$$

since $(d-1)\ell < d(\ell-1)$, i.e. $d<\ell$. For $d=\ell-1$ these are the bounds stated for dsc_ℓ.

Level s. Write $\gamma=\gamma_D$, $Y=X_{s-1}$, $u=u_s$. (i) *Graded action.* $\gamma_u(1+Y)=\gamma_u((1+X)^{\ell^{s-1}})=(1+X)^{u\ell^{s-1}}=(1+Y)^u$, so $\gamma_u(Y)=(1+Y)^u-1=\sum_{k\geq 1}\binom{u}{k}Y^k=Y(u+Yt)$ with $t\in\mathbb{Z}_\ell[[X]]$. For $j\geq 0$ and $n\in N$, writing $\gamma(n)=n+Yn_1$ with $n_1\in N$,

$$\gamma(Y^j n)=Y^j(u+Yt)^j(n+Yn_1)\in u^jY^jn+Y^{j+1}N. \qquad (*)$$

(ii) *Eigenvalue gaps.* By item 1 of Lemma 23.22 ($p=\ell$, $m=s$, which is allowed as $s\geq 2$ if $\ell=2$), $u^k-1=\epsilon_k\ell^{s+v_\ell(k)}$ with $\epsilon_k\in\mathbb{Z}_\ell^\times$ for $k\geq 1$. (iii) *Item 4.* The descent step above holds with X replaced by Y: it uses only $\gamma_u(Y)=Y(u+Yt)$, $(\gamma-1)(N)\subseteq YN$, $(\gamma-1)(N')\subseteq YN'$ and the injectivity of Y on D; so if cY^{j+1} kills N/N' then $c(u^{j+1}-1)Y^j$ does. Induction downwards from $j+1=S$ shows that $\prod_{k=j+1}^{S}(u^k-1)\,Y^j=\epsilon\,\ell^{\mathrm{dsc}_\ell^{(s)}(j,S)}Y^j$ (ϵ a unit, by (ii)) kills N/N', hence so does every ℓ^aY^j with $a\geq \mathrm{dsc}_\ell^{(s)}(j,S)$. Finally $\mathrm{dsc}_\ell^{(s)}(0,S)=sS+\sum_{k\leq S}v_\ell(k)=sS+v_\ell(S!)$, and $v_\ell(S!)\leq S$ (Legendre). (iv) *Item 5.* We show by induction on $j=1,\dots,M$: there is $k_j\in YN$ with $\ell^{\sigma_j^{(s)}}z-(\gamma-1)(k_j)\in Y^jN$. For $j=1$ take $k_1=0$ ($\sigma_1^{(s)}=0$). Given k_j with $z_j=\ell^{\sigma_j^{(s)}}z-(\gamma-1)(k_j)\in Y^jN$, $j<M$, put $k'=\epsilon_j^{-1}z_j\in Y^jN\subseteq YN$; by $(*)$, $(\gamma-1)(k')\in\epsilon_j^{-1}(u^j-1)z_j+Y^{j+1}N=\ell^{s+v_\ell(j)}z_j+Y^{j+1}N$. So $k_{j+1}=\ell^{s+v_\ell(j)}k_j+k'$ gives $\ell^{\sigma_{j+1}^{(s)}}z-(\gamma-1)(k_{j+1})=\ell^{s+v_\ell(j)}z_j-(\gamma-1)(k')\in Y^{j+1}N$, as $\sigma_{j+1}^{(s)}=\sigma_j^{(s)}+s+v_\ell(j)$. Take $k=k_M$. (v) *Item 6.* Let $y\in Y^jN$ with $1\leq j<M$, say $y=Y^jn$. By $(*)$, $(\gamma-1)(y)\in(u^j-1)Y^jn+Y^{j+1}N$; as $(\gamma-1)(y)\in Y^MN\subseteq Y^{j+1}N$, we get $Y^j(u^j-1)n\in Y^{j+1}N$, hence $(u^j-1)n\in YN$ (Y is injective on D), hence $\ell^{s+v_\ell(j)}n\in YN$ by (ii), hence $n\in YN$ because N/YN has no ℓ-torsion; so $y\in Y^{j+1}N$. Induction from $j=1$ gives $y\in Y^MN$. For the example in the statement: if N is finite free over $\mathbf{A}_{F,L}=\prod_a e_a\mathbf{A}_{F,L}$, $e_a\mathbf{A}_{F,L}\cong\mathcal{O}_L[[X]]$, then N/YN is a finite direct sum of copies of $\mathcal{O}_L[[X]]/(Y)=\mathcal{O}_L[X]/(Y)$ (Weierstrass division by the distinguished polynomial Y), which is free over $\mathcal{O}_L$. □

Lemma 23.22. *Let p be a prime, $m\geq 1$ an integer with $m\geq 2$ if $p=2$, and $u\in\mathbb{Z}_p$ with $v_p(u-1)=m$.*

1. *For every non-zero integer n, $v_p(u^n-1)=m+v_p(n)$.*
2. *For an integer $e\geq 0$ and a finite set $J\subset\mathbb{Z}$, the polynomial*

$$P_{J,e}(X)=\prod_{i\in J\setminus\{0\}}(X^{p^e}-u^{ip^e})\cdot\sum_{k=0}^{p^e-1}X^k\in\mathbb{Z}_p[X]$$

   satisfies $v_p\big(P_{J,e}(1)\big)=e+\sum_{i\in J\setminus\{0\}}\big(e+m+v_p(i)\big)$.
3. *Let M be a $\mathbb{Z}_p$-module, A a $\mathbb{Z}_p$-linear endomorphism of M, $M'\subseteq M$ an A-stable submodule, $f\in\mathbb{Z}_p$ and $Q\in\mathbb{Z}_p[X]$. There is $Q_1\in\mathbb{Z}_p[X]$ with $Q(A)-Q(f)=Q_1(A)(A-f)$; in particular, if $w\in M$ and $(A-f)w\in M'$, then $Q(A)w-Q(f)w\in M'$.*

Proof. Item 1. As u is a unit and $u^{-n}-1=-u^{-n}(u^n-1)$, we may assume $n>0$; write $n=p^kn'$ with $p\nmid n'$. First, $u^{n'}-1=(u-1)(1+u+\cdots+u^{n'-1})$ and the second factor is $\equiv n'\not\equiv 0 \pmod p$, so $v_p(u^{n'}-1)=m$. Next, if $x\in\mathbb{Z}_p$ has $v_p(x)=t\geq 1$, then

$$(1+x)^p-1=px+\sum_{k=2}^{p-1}\binom{p}{k}x^k+x^p,$$

where $v_p(px) = t+1$, $v_p(\binom{p}{k}x^k) \geq 1+kt > t+1$ for $2 \leq k \leq p-1$, and $v_p(x^p) = pt \geq t+2$ because $(p-1)t \geq 2$ (if $p \geq 3$ since $t \geq 1$; if $p = 2$ since $t \geq m \geq 2$, the valuations met in the induction below being $m, m+1, \ldots$; for $p = 2$ the middle sum is empty); hence $v_p((1+x)^p - 1) = t+1$. Applying this k times, starting from $1 + x = u^{n'}$, gives $v_p(u^n - 1) = m + k = m + v_p(n)$.

Item 2. $P_{J,e}(1) = p^e \prod_{i \in J \setminus \{0\}} (1 - u^{ip^e})$, and by item 1 with $n = ip^e \neq 0$, $v_p(1 - u^{ip^e}) = m + e + v_p(i)$.

Item 3. $X - f$ divides $Q(X) - Q(f)$ in $\mathbb{Z}_p[X]$; write $Q(X) - Q(f) = Q_1(X)(X - f)$ and substitute A (polynomials in A commute). If $(A-f)w \in M'$, then $Q(A)w - Q(f)w = Q_1(A)((A-f)w) \in M'$, since M' is an A-stable $\mathbb{Z}_p$-submodule. □

Definition 23.23. Let ℓ be an odd prime and $\mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$ the group of roots of unity of order dividing $\ell - 1$. For $a \in \mathbb{Z}_\ell$ write $(1+X)^a = \sum_{k \geq 0} \binom{a}{k} X^k \in \mathbb{Z}_\ell[[X]]$. Let $\Delta \subseteq \mu_{\ell-1}$ be a subgroup and call $e = (\ell-1)/|\Delta|$ its **ramification index**.

(Dr1) The group $\Gamma_\Delta = \{u \in \mathbb{Z}_\ell^\times : \ u \equiv \delta \pmod{\ell} \text{ for some } \delta \in \Delta\} = \Delta \cdot (1 + \ell\mathbb{Z}_\ell)$.

(Dr2) The invariant uniformiser

$$\pi_\Delta = \sum_{\delta \in \Delta} \big((1+X)^\delta - 1\big) \in \mathbb{Z}_\ell[[X]].$$

For $\Delta = \mu_{\ell-1}$ ($e = 1$) it is written

$$\pi_0 = \pi_{\mu_{\ell-1}} = \sum_{\delta \in \mu_{\ell-1}} (1+X)^\delta - (\ell - 1).$$

(Dr3) The Eisenstein polynomial Fix an algebraic closure $\overline{\mathbb{Q}}_\ell$. Let $S_\Delta \subset \overline{\mathbb{Q}}_\ell$ be the set of values $\pi_\Delta(\zeta - 1)$, ζ running over the primitive ℓ-th roots of unity in $\overline{\mathbb{Q}}_\ell$ (the series converges at $\zeta - 1$, which has positive valuation), and

$$E_\Delta(Y) = \prod_{y \in S_\Delta} (Y - y), \qquad q_\Delta = E_\Delta(\pi_\Delta).$$

(Dr4) The ring For a ring R on which the γ_u act (in particular $R = \mathcal{O}_L[[X]]$ with γ_u the $\mathcal{O}_L$-algebra automorphism with $\gamma_u(X) = (1+X)^u - 1$, or a site ring $e_a \mathbf{A}_{F,L}$), the **Δ-invariant Wach ring** is

$$R^\Delta = \{f \in R : \ \gamma_\delta(f) = f \text{ for every } \delta \in \Delta\}.$$

Lemma 23.24. *Let ℓ be an odd prime, L a finite extension of $\mathbb{Q}_\ell$, $R = \mathcal{O}_L[[X]]$, $\Delta \subseteq \mu_{\ell-1}$ a subgroup and $e = (\ell-1)/|\Delta|$.*

(Rr1) Uniformiser *$\gamma_\delta(\pi_\Delta) = \pi_\Delta$ for $\delta \in \Delta$, and $\pi_\Delta = X^{|\Delta|} v_\Delta$ with $v_\Delta \in \mathbb{Z}_\ell[[X]]^\times$, $v_\Delta(0) = 1/(|\Delta| - 1)!$.*

(Rr2) Variable *The $\mathcal{O}_L$-algebra homomorphism $\mathcal{O}_L[[T]] \to R$, $T \mapsto \pi_\Delta$, is injective with image R^Δ; that is, every $f \in R^\Delta$ is $g(\pi_\Delta)$ for a unique $g \in \mathcal{O}_L[[T]]$. Moreover $R^\Delta \cap XR = \pi_\Delta R^\Delta$.*

(Rr3) Eisenstein polynomial *$|S_\Delta| = e$, and $E_\Delta \in \mathbb{Z}_\ell[Y]$ is an Eisenstein polynomial of degree e. Every $y \in S_\Delta$ lies in the subfield $\mathbb{Q}_\ell(\mu_\ell)^\Delta$ of $\mathbb{Q}_\ell(\mu_\ell)$ fixed by the automorphisms $\zeta \mapsto \zeta^\delta$ ($\delta \in \Delta$, read modulo ℓ), which is totally ramified of degree e over $\mathbb{Q}_\ell$, and y is a uniformiser of it. For $\Delta = \mu_{\ell-1}$: $S_\Delta = \{-\ell\}$ and $E_\Delta(Y) = Y + \ell$, $q_\Delta = \pi_0 + \ell$.*

(Rr4) The ideal of Q $q_\Delta = Q\, w_\Delta$ *with* $w_\Delta \in \mathbb{Z}_\ell[[X]]^\times$.

(Rr5) Frobenius $\varphi(\pi_\Delta) = w'_\Delta\, \pi_\Delta\, q_\Delta^{|\Delta|}$ *with* $w'_\Delta \in \mathbb{Z}_\ell[[X]]^\times$.

(Rr6) Gamma *Let $u \in \Gamma_\Delta$. Then $\gamma_u(\pi_\Delta)$ is Δ-invariant and*

$$\gamma_u(\pi_\Delta) - \pi_\Delta = t_u\, \pi_\Delta\, q_\Delta, \qquad \gamma_u(q_\Delta) = h_u\, q_\Delta,$$

with $t_u \in \mathbb{Z}_\ell[[X]]$ and $h_u \in 1 + X\mathbb{Z}_\ell[[X]]$, both Δ-invariant. If moreover $v_\ell(u-1) = 1$, then $t_u \in \mathbb{Z}_\ell[[X]]^\times$.

(Rr7) The field *Let F be a finite unramified extension of $\mathbb{Q}_\ell$ inside $\overline{\mathbb{Q}}_\ell$ and $F \subseteq K \subseteq F(\mu_\ell)$ a field with $[K:F] = e$. Then $\chi(G_K) = \Gamma_\Delta$ and Δ is the torsion subgroup of $\chi(G_K)$; every $y \in S_\Delta$ lies in K and is a uniformiser of K, and E_Δ is its minimal polynomial over F.*

Proof. **Two standard facts.** (a) For $a \in \mathbb{Z}_\ell$, $\gamma_u((1+X)^a) = (1+X)^{ua}$ and $\gamma_u\gamma_{u'} = \gamma_{uu'}$: both hold for $a \in \mathbb{Z}_{\geq 0}$, and $a \mapsto (1+X)^a$ is continuous for the (ℓ, X)-adic topology, as are the ring endomorphisms γ_u ($\gamma_u(X) \in X\mathbb{Z}_\ell[[X]]$). Likewise $(1+x)^a = \zeta^n$ for every integer $n \equiv a \pmod{\ell}$, since $(1+x)^a$ is continuous in a, equals ζ^a for $a \in \mathbb{Z}_{\geq 0}$, and ζ^n depends only on n modulo ℓ. We write ζ^a for this value. (b) *Divisibility criterion.* If $h \in \mathbb{Z}_\ell[[X]]$, $\bar{h}$ has X-adic order exactly $\ell - 1$, and $h(\zeta - 1) = 0$ for all $\ell - 1$ primitive ζ, then $h \in Q\,\mathbb{Z}_\ell[[X]]^\times$: by Weierstrass preparation $h = P\,w$ with w a unit and P distinguished of degree $\ell - 1$; $w(\zeta - 1)$ is a unit, so P vanishes at the $\ell - 1$ distinct roots of the monic Q of degree $\ell - 1$, and $P = Q$. Similarly, if $g \in \mathbb{Z}_\ell[[X]]$ vanishes at all $\zeta - 1$, then $g \in Q\,\mathbb{Z}_\ell[[X]]$: Weierstrass division $g = Qt + r$ by the distinguished polynomial Q leaves $r \in \mathbb{Z}_\ell[X]$ of degree $< \ell - 1$ with $\ell - 1$ distinct roots, so $r = 0$.

(Rr1). By (a), $\gamma_{\delta'}(\pi_\Delta) = \sum_{\delta \in \Delta}((1+X)^{\delta'\delta} - 1) = \pi_\Delta$ for $\delta' \in \Delta$, as $\delta \mapsto \delta'\delta$ permutes Δ. The constant term of π_Δ is 0; for $k \geq 1$ its coefficient of X^k is

$$\sum_{\delta \in \Delta} \binom{\delta}{k} = \frac{1}{k!} \sum_{i=1}^{k} s(k,i) \sum_{\delta \in \Delta} \delta^i,$$

with $s(k,i)$ the Stirling numbers of the first kind. Since Δ is cyclic of order $|\Delta|$, $\sum_{\delta \in \Delta} \delta^i$ is $|\Delta|$ if $|\Delta| \mid i$ and 0 otherwise. For $1 \leq k \leq |\Delta|$ (so $k! \in \mathbb{Z}_\ell^\times$, as $|\Delta| < \ell$) only $k = i = |\Delta|$ contributes, giving the coefficient 0 for $k < |\Delta|$ and $|\Delta|/|\Delta|! = 1/(|\Delta| - 1)! \in \mathbb{Z}_\ell^\times$ for $k = |\Delta|$. So $\pi_\Delta = X^{|\Delta|} v_\Delta$ with $v_\Delta(0) = 1/(|\Delta| - 1)!$ a unit.

(Rr2). $R^\Delta \cap XR = \pi_\Delta R^\Delta$. The inclusion $\supseteq$ holds by (Rr1). Let $f \in R^\Delta \cap XR$, $f \neq 0$, with term of least degree $a_i X^i$, $i \geq 1$. As $\gamma_\delta(X) = \delta X + X^2(\dots)$, the term of least degree of $\gamma_\delta(f) = f$ is $\delta^i a_i X^i$, so $\delta^i = 1$ for all $\delta \in \Delta$ ($\mathcal{O}_L$ is a domain), and $|\Delta|$ divides i. Hence $f \in X^{|\Delta|}R = \pi_\Delta R$; as π_Δ is Δ-invariant and R is a domain, $f/\pi_\Delta \in R^\Delta$. *Image.* The homomorphism $T \mapsto \pi_\Delta$ is well defined as $\pi_\Delta \in XR$, and its image lies in R^Δ by continuity of the γ_δ. Conversely, for $f \in R^\Delta$ put $f_0 = f$, $c_j = f_j(0) \in \mathcal{O}_L$, $f_{j+1} = (f_j - c_j)/\pi_\Delta \in R^\Delta$ (previous paragraph). Then $f - \sum_{j<m} c_j \pi_\Delta^j = \pi_\Delta^m f_m \in X^{|\Delta|m}R$ for every m, so $f = \sum_j c_j \pi_\Delta^j$. *Injectivity.* If $\sum_j c_j \pi_\Delta^j = 0$ with some $c_j \neq 0$ and j_0 is the least such j, then $c_{j_0}\pi_\Delta^{j_0}$ has X-adic order exactly $|\Delta| j_0$ (leading coefficient $c_{j_0} v_\Delta(0)^{j_0} \neq 0$) while the other terms lie in $X^{|\Delta|(j_0+1)}R$; a contradiction.

(Rr3). Let ζ be primitive, $x = \zeta - 1$, so $v(x) = 1/(\ell - 1)$, and put $y_\zeta = \pi_\Delta(x)$. By (Rr1), $y_\zeta = x^{|\Delta|} v_\Delta(x)$ with $v_\Delta(x) \equiv v_\Delta(0)$ modulo the maximal ideal, a unit; so $v(y_\zeta) = |\Delta|/(\ell - 1) = 1/e$. By (a), $y_\zeta = \sum_{\delta \in \Delta}(\zeta^\delta - 1)$. For $a \in (\mathbb{Z}/\ell)^\times$ let $\sigma_a \in \mathrm{Gal}(\mathbb{Q}_\ell(\zeta)/\mathbb{Q}_\ell)$ be $\zeta \mapsto \zeta^a$ (an isomorphism $(\mathbb{Z}/\ell)^\times \to \mathrm{Gal}$, Φ_ℓ being irreducible over $\mathbb{Q}_\ell$). Since π_Δ has coefficients in $\mathbb{Z}_\ell$ and σ_a is continuous, $\sigma_a(y_\zeta) = y_{\zeta^a} = \sum_\delta(\zeta^{a\delta} - 1)$, which equals y_ζ when $a \equiv \delta' \pmod{\ell}$ with $\delta' \in \Delta$

(permutation of Δ). Reduction modulo ℓ is an isomorphism $\mu_{\ell-1} \to (\mathbb{Z}/\ell)^\times$, so the fixed field $F_\Delta = \mathbb{Q}_\ell(\zeta)^\Delta$ of the image of Δ has degree $(\ell-1)/|\Delta| = e$ over $\mathbb{Q}_\ell$; it is totally ramified because $\mathbb{Q}_\ell(\zeta)$ is. So $y_\zeta \in F_\Delta$ with $v(y_\zeta) = 1/e$: y_ζ is a uniformiser of F_Δ, hence $\mathbb{Q}_\ell(y_\zeta)$ has ramification index e and equals F_Δ. Its e conjugates $\sigma_a(y_\zeta) = y_{\zeta^a}$ (a modulo the image of Δ) are therefore distinct, and as ζ^a runs over all primitive roots they exhaust S_Δ (which does not depend on ζ). So $|S_\Delta| = e$ and E_Δ is the minimal polynomial of the uniformiser y_ζ of the totally ramified extension $F_\Delta/\mathbb{Q}_\ell$, which is an Eisenstein polynomial in $\mathbb{Z}_\ell[Y]$. For $\Delta = \mu_{\ell-1}$, $y_\zeta = \sum_{a=1}^{\ell-1} \zeta^a - (\ell-1) = -1-(\ell-1) = -\ell$.

(Rr4). The series $q_\Delta = E_\Delta(\pi_\Delta) \in \mathbb{Z}_\ell[[X]]$ vanishes at every $\zeta - 1$, since $\pi_\Delta(\zeta-1) \in S_\Delta$. Modulo ℓ, $E_\Delta(Y) \equiv Y^e$ (Eisenstein), so $\bar{q}_\Delta = \bar{\pi}_\Delta^e = X^{e|\Delta|}\bar{v}_\Delta^e = X^{\ell-1}\bar{v}_\Delta^e$ has X-adic order exactly $\ell-1$. By (b), $q_\Delta = Q\,w_\Delta$ with w_Δ a unit.

(Rr5). We have $\varphi(X) = (1+X)^\ell - 1 = XQ$, and $\varphi(v_\Delta)$ is a unit (its constant term is $v_\Delta(0)$, as $\varphi(X)(0) = 0$). So, by (Rr1) and (Rr4),

$$\varphi(\pi_\Delta) = X^{|\Delta|}Q^{|\Delta|}\varphi(v_\Delta) = \pi_\Delta\, q_\Delta^{|\Delta|} \cdot v_\Delta^{-1} w_\Delta^{-|\Delta|}\varphi(v_\Delta).$$

(Rr6). By (a), $\gamma_\delta\gamma_u = \gamma_u\gamma_\delta$, so γ_u maps Δ-invariant elements to Δ-invariant elements; in particular $\gamma_u(\pi_\Delta)$ and $\gamma_u(q_\Delta)$ are Δ-invariant. Write $\gamma_u(X) = X\,c_u$ with $c_u = u + X(\dots) \in \mathbb{Z}_\ell[[X]]^\times$; then $\gamma_u(\pi_\Delta) = \pi_\Delta m_u$ with $m_u = c_u^{|\Delta|}\gamma_u(v_\Delta)v_\Delta^{-1} \in \mathbb{Z}_\ell[[X]]^\times$, Δ-invariant (quotient of Δ-invariants in the domain $\mathbb{Z}_\ell[[X]]$). Let $u \equiv \delta_u \pmod{\ell}$ with $\delta_u \in \Delta$. By (a), $\gamma_u(\pi_\Delta)(\zeta-1) = \pi_\Delta(\zeta^u - 1) = \sum_\delta(\zeta^{\delta_u\delta} - 1) = \pi_\Delta(\zeta-1)$, which is non-zero by (Rr3); so $m_u - 1$ vanishes at every $\zeta - 1$. By (b) and (Rr4), $m_u - 1 = t_u\, q_\Delta$ with $t_u \in \mathbb{Z}_\ell[[X]]$, Δ-invariant as m_u and q_Δ are. This is the first identity. Next, $\gamma_u(q_\Delta) = E_\Delta(\gamma_u(\pi_\Delta))$ vanishes at every $\zeta - 1$ (its value there is $E_\Delta(\pi_\Delta(\zeta-1)) = 0$), and $\overline{\gamma_u(q_\Delta)} = \gamma_u(\bar{q}_\Delta) = (Xc_u)^{\ell-1}\gamma_u(\bar{v}_\Delta)^e$ has X-adic order $\ell-1$; by (b) and (Rr4), $\gamma_u(q_\Delta) = q_\Delta h_u$ with h_u a unit, Δ-invariant. Since $\gamma_u(f) \equiv f \pmod{X}$ for $f \in \mathbb{Z}_\ell[[X]]$, $q_\Delta(h_u - 1) \in X\mathbb{Z}_\ell[[X]]$; as $q_\Delta(0) = E_\Delta(0) \neq 0$ and X is prime, $h_u \in 1 + X\mathbb{Z}_\ell[[X]]$. Finally let $v_\ell(u-1) = 1$. Evaluating $m_u - 1 = t_u q_\Delta$ at $X = 0$: $m_u(0) = u^{|\Delta|}$ and $q_\Delta(0) = E_\Delta(0)$ has valuation 1 (Eisenstein). Now $u^{|\Delta|} - 1 = (u-1)(1 + u + \dots + u^{|\Delta|-1})$ and the second factor is $\equiv |\Delta| \not\equiv 0 \pmod{\ell}$, so $v_\ell(u^{|\Delta|} - 1) = 1$ and $t_u(0)$ is a unit; hence t_u is a unit.

(Rr7). As $F/\mathbb{Q}_\ell$ is unramified and $\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell$ is totally ramified, $F \cap \mathbb{Q}_\ell(\mu_{\ell^\infty}) = \mathbb{Q}_\ell$, so restriction identifies $\mathrm{Gal}(F(\mu_{\ell^\infty})/F)$ with $\mathrm{Gal}(\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell)$ and $\chi(G_F) = \mathbb{Z}_\ell^\times$. Since $K \subseteq F(\mu_\ell)$, $\chi(G_K)$ is a subgroup of index $[K:F] = e$ of $\mathbb{Z}_\ell^\times$ containing $\chi(G_{F(\mu_\ell)}) = 1 + \ell\mathbb{Z}_\ell$. Since $\mathbb{Z}_\ell^\times/(1+\ell\mathbb{Z}_\ell) \cong (\mathbb{Z}/\ell)^\times$ is cyclic, it has exactly one subgroup of index e, the image of Δ; so $\chi(G_K) = \Delta \cdot (1+\ell\mathbb{Z}_\ell) = \Gamma_\Delta$, whose torsion subgroup is Δ ($1+\ell\mathbb{Z}_\ell$ is torsion-free for odd ℓ). For $\sigma \in G_K$ and $y = \pi_\Delta(\zeta-1) \in S_\Delta$, $\sigma(y) = \pi_\Delta(\zeta^{\chi(\sigma)} - 1) = y$ as in (Rr6), so $y \in K$. The extension K/F is totally ramified of degree e (inside the totally ramified $F(\mu_\ell)/F$) and $v(y) = 1/e$ by (Rr3), so y is a uniformiser of K. The polynomial E_Δ is Eisenstein over $\mathcal{O}_F$ as well (ℓ is a uniformiser of F), hence irreducible over F, and it is the minimal polynomial of y over F. □

Lemma 23.25. *Let p be a prime, K and L finite extensions of $\mathbb{Q}_p$, and $\Phi\colon V \to V''$ an L-linear G_K-equivariant surjection of finite-dimensional L-vector spaces with continuous L-linear actions of G_K, with V de Rham as a $\mathbb{Q}_p$-representation. Then V'' is de Rham, and for every $\mathbb{Q}_p$-embedding $\tau\colon K \to L$ every Hodge jump of V'' at τ is a Hodge jump of V at τ.*

Proof. By Fontaine's theory, de Rham representations are stable under quotients, and on de Rham representations the functor D_{dR} is exact and strictly compatible with the filtrations (Fontaine, Représentations p-adiques semi-stables, 1994; Brinon–Conrad, CMI notes on p-adic Hodge theory, §6.3). So V'' is de Rham, and for every j the map $\mathrm{Fil}^j D_{\mathrm{dR}}(V) \to \mathrm{Fil}^j D_{\mathrm{dR}}(V'')$ induced by Φ is onto; hence so is $\mathrm{gr}^j D_{\mathrm{dR}}(V) \to \mathrm{gr}^j D_{\mathrm{dR}}(V'')$. These maps are $K \otimes_{\mathbb{Q}_p} L$-linear. As

$K/\mathbb{Q}_p$ is separable, $K\otimes_{\mathbb{Q}_p}L$ is a finite product of fields, and $\tau\otimes\mathrm{id}\colon K\otimes_{\mathbb{Q}_p}L\to L$ is the projection onto one of them; so $L\otimes_{K\otimes L,\tau\otimes\mathrm{id}}(-)$ is multiplication by an idempotent ϵ_τ, which is exact and carries Fil^j and gr^j of D_{dR} to those of $D_{\mathrm{dR},\tau}$. Hence $\mathrm{gr}^jD_{\mathrm{dR},\tau}(V)\to\mathrm{gr}^jD_{\mathrm{dR},\tau}(V'')$ is onto for every j. An integer a is a Hodge jump of V'' at τ exactly when $\mathrm{gr}^aD_{\mathrm{dR},\tau}(V'')=\mathrm{Fil}^a/\mathrm{Fil}^{a+1}\neq 0$; then $\mathrm{gr}^aD_{\mathrm{dR},\tau}(V)\neq 0$, i.e. a is a Hodge jump of V at τ. □

23.4 Bloch–Kato classes as crystalline extensions

This subsection connects Wach modules with Galois cohomology, and it is what lets the later sections work with Wach vectors in place of classes. A class in H^1_f is the class of a crystalline extension by the trivial lattice, with the Hodge jumps of the lattice together with 0 (Lemma 23.26). Such a class is the Herr class of the Frobenius defect of a lift (Lemma 23.32), where Herr classes of Wach vectors are made explicit in Lemma 23.28. Conversely, the integral crystalline criterion Proposition 23.27 says that a Herr vector of precision at least $1+\max_a M_a$ whose poles lie only at finitely many Q_s has a Bloch–Kato class. The proof trivialises the Γ-part of such a vector crystallinely (Proposition 23.31), which rests on the contraction of Frobenius on $X^m\mathbf{N}$ (Lemma 23.30). The criterion is used by both lifting exponents of Section 24 and by the relay classes there, and again in Section 27 and Appendix C.

Lemma 23.26. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a field K with $F\subseteq K\subseteq F(\mu_\ell)$ be as in Proposition 23.7, (V,T) a pair over K, $H^1_f(K,T)$ the inverse image in $H^1(K,T)$ of $H^1_f(K,V)=\ker\bigl(H^1(K,V)\to H^1(K,B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V)\bigr)$, and $x\in H^1_f(K,T)$. Then there are a pair (V_Y,Y) over K and an exact sequence $0\to T\xrightarrow{\iota'}Y\xrightarrow{\pi}\mathcal{O}_L\to 0$ of G_K-equivariant $\mathcal{O}_L$-linear maps ($\mathcal{O}_L$ the trivial representation) with $[Y]=x$, where $[Y]\in H^1(K,T)$ is the class of the cocycle $g\mapsto\iota'^{-1}(g\tilde e-\tilde e)$ for any $\tilde e\in Y$ with $\pi(\tilde e)=1$, such that for every $a\in\mathbb{Z}/n$ and every embedding τ of K over τ_a, every Hodge jump of V_Y at τ is 0 or a Hodge jump of V at τ. In particular, if $M_a,h_a\geq 0$ are integers such that every Hodge jump of V at every embedding of K over τ_a lies in $[-M_a,h_a]$, the same holds for V_Y. For $K=F$ this is the statement with pairs, G_F, $H^1_f(F,T)$ and the embeddings τ_a. **Level s.** The same holds for any prime ℓ, an integer $s\geq 1$ with $s\geq 2$ if $\ell=2$, L and K ($F\subseteq K\subseteq F(\mu_{\ell^s})$) and pairs over K as in (Ex4) of Proposition 23.7; for $s=1$ and ℓ odd this is the statement above.*

Proof. **The extension.** Let $\xi\colon G_K\to T$ be a continuous cocycle representing x. Put $Y=T\oplus\mathcal{O}_L$ with $g(t,a)=(gt+a\,\xi(g),a)$, $\iota'(t)=(t,0)$, $\pi(t,a)=a$; the cocycle relation makes this a continuous $\mathcal{O}_L$-linear action, Y is a G_K-stable lattice in $V_Y=Y\otimes_{\mathcal{O}_L}L$, and ι', π are equivariant with $0\to T\to Y\to\mathcal{O}_L\to 0$ exact. With $\tilde e=(0,1)$ we get $g\tilde e-\tilde e=\iota'(\xi(g))$, so $[Y]=x$.

V_Y is crystalline. Put $D_{\mathrm{cris}}(W)=(B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}W)^{G_K}$, an $F\otimes_{\mathbb{Q}_\ell}L$-module ($B_{\mathrm{cris}}^{G_K}$ is the maximal unramified subextension of K, which is F, as $K\subseteq F(\mu_\ell)$ is totally ramified over F). Since B_{cris} is flat over $\mathbb{Q}_\ell$, $0\to B_{\mathrm{cris}}\otimes V\to B_{\mathrm{cris}}\otimes V_Y\to B_{\mathrm{cris}}\otimes L\to 0$ is exact, and taking invariants gives the exact sequence of $F\otimes L$-modules

$$0\to D_{\mathrm{cris}}(V)\to D_{\mathrm{cris}}(V_Y)\to D_{\mathrm{cris}}(L)\xrightarrow{\delta}H^1(K,B_{\mathrm{cris}}\otimes V),$$

with δ $F\otimes L$-linear. $D_{\mathrm{cris}}(L)=F\otimes L$ is generated by $1\otimes 1$, which lifts to $1\otimes\tilde e$; $g(1\otimes\tilde e)-1\otimes\tilde e=1\otimes\xi(g)$, so $\delta(1\otimes 1)$ is the image of x in $H^1(K,B_{\mathrm{cris}}\otimes V)$, which vanishes because the image of x in $H^1(K,V)$ lies in $H^1_f(K,V)$. Hence $\delta=0$ and $\dim_F D_{\mathrm{cris}}(V_Y)=\dim_F D_{\mathrm{cris}}(V)+\dim_F(F\otimes L)=\dim_{\mathbb{Q}_\ell}V+\dim_{\mathbb{Q}_\ell}L=\dim_{\mathbb{Q}_\ell}V_Y$, as V is crystalline. So V_Y is crystalline, and (V_Y,Y) is a pair over K.

Hodge jumps. V, V_Y, L are crystalline, hence de Rham. On de Rham representations the functor D_{dR} is exact and strictly compatible with the filtrations (Fontaine), so for every j

the sequences $0 \to \mathrm{gr}^j D_{\mathrm{dR}}(V) \to \mathrm{gr}^j D_{\mathrm{dR}}(V_Y) \to \mathrm{gr}^j D_{\mathrm{dR}}(L) \to 0$ are exact; they are sequences of $K \otimes_{\mathbb{Q}_\ell} L$-modules, and $K \otimes_{\mathbb{Q}_\ell} L \cong \prod_\tau L$ over the $\mathbb{Q}_\ell$-embeddings $\tau\colon K \to L$ (L contains $\sigma(F(\mu_\ell)) \supseteq \sigma(K)$ for every embedding σ), so they stay exact after applying the idempotent of an embedding τ, which gives $\mathrm{gr}^j D_{\mathrm{dR},\tau}$. As $\mathrm{gr}^j D_{\mathrm{dR}}(L) = 0$ for $j \neq 0$, a $j \neq 0$ with $\mathrm{gr}^j D_{\mathrm{dR},\tau}(V_Y) \neq 0$ has $\mathrm{gr}^j D_{\mathrm{dR},\tau}(V) \neq 0$. The last assertion follows because $0 \in [-M_a, h_a]$.

Level s. The proof uses two properties of K and L and nothing about ℓ: $B_{\mathrm{cris}}^{G_K} = F$, the maximal unramified subextension of K, which holds because $K \subseteq F(\mu_{\ell^s})$ is totally ramified over F ($\mathbb{Q}_\ell(\mu_{\ell^s})/\mathbb{Q}_\ell$ is totally ramified and $F/\mathbb{Q}_\ell$ unramified); and $L \supseteq \sigma(K)$ for every embedding σ of K, which holds because $L \supseteq \sigma(F(\mu_{\ell^s}))$ for every embedding of $F(\mu_{\ell^s})$ and every embedding of K extends to one of $F(\mu_{\ell^s})$. So the proof applies without change. □

Proposition 23.27. *Let ℓ, F, L, (τ_a) and a pair (V,T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$, $(M_a, h_a)_a$ a window of V as in (Ex1) and $m \geq 1 + \max_a M_a$. If $y \in \mathbf{N}[1/\lambda_\infty]$ is a Herr vector of precision m and $fy \in \mathbf{N}$ for some finite product f of the Q_s, $s \geq 0$, then $\mathrm{cl}(y) \in H^1_f(F,T)$.*
Over a base field. *Let K be a field with $F \subseteq K \subseteq F(\mu_\ell)$ and Γ_K as in Definition 23.2, (V,T) a pair over K as in Proposition 23.7, $(M_a, h_a)_a$ a window as in (Ex1) over K, $m \geq 1 + \max_a M_a$, and y a Herr vector of precision m over K (Lemma 23.28, over a base field) with $fy \in \mathbf{N}$ for a finite product f of the Q_s. Then $\mathrm{cl}(y) \in H^1_f(K,T)$, the inverse image in $H^1(K,T)$ of $H^1_f(K,V) = \ker(H^1(K,V) \to H^1(K, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V))$. For $K = F$ this is the statement above.*
Level s. *Let ℓ be any prime, $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, L, K ($F \subseteq K \subseteq F(\mu_{\ell^s})$) and a pair (V,T) over K as in (Ex4) of Proposition 23.7, $\mathbf{N} = \mathbf{N}_s(T)$, $(M_a, h_a)_a$ a window of V as in (Ex1) of level s, $m \geq 1 + \max_a M_a$, and $y \in \mathbf{N}[1/\lambda_\infty]$ a Herr vector of precision m at level s (Lemma 23.28, paragraph Level s) with $fy \in \mathbf{N}$ for a finite product f of the Q_j with $j \geq s-1$. Then its Herr class satisfies $\mathrm{cl}(y) \in H^1_f(K,T)$. For $s = 1$ and ℓ odd this is the paragraph over a base field.*

Proof. Write $\mu = \mu_\varepsilon$ and φ for $\varphi \otimes 1$; fix a $\mathbb{Z}_\ell$-basis of T, so that all equations below in $A_{\mathrm{inf}} \otimes T$ or $W(\mathbb{C}_\ell^\flat) \otimes T$ are coordinatewise.

Step 1: the poles of $\iota_T(y)$. Since $\iota(\varphi(X)) = \varphi(\mu)$ and $\varphi(X) = XQ$, we have $\varphi(\mu) = \mu\tilde{\xi}$ with $\tilde{\xi} = \iota(Q)$, hence $\iota(Q_s) = \varphi^s(\tilde{\xi}) = \varphi^{s+1}(\mu)/\varphi^s(\mu)$ in the domain $W(\mathbb{C}_\ell^\flat)$, where all $\varphi^j(\mu)$ are units (their reductions $(\varepsilon - 1)^{\ell^j}$ are non-zero). Write $f = \prod_s Q_s^{k_s}$. By (Gc0) and (Gc1), $\iota_T(y) = \iota(f)^{-1}\iota_T(fy)$ and $\mu^\kappa \iota_T(fy) \in A_{\mathrm{inf}} \otimes T$, so

$$P_0\, \iota_T(y) \in A_{\mathrm{inf}} \otimes T, \qquad P_0 = \mu^\kappa \prod_s \varphi^{s+1}(\mu)^{k_s}.$$

Step 2: an Artin–Schreier solution with poles. We have $\varphi^i(\mu) \mid \varphi^j(\mu)$ in A_{inf} for $i \leq j$ (apply φ^i to $\varphi^{j-i}(\mu) = \mu \prod_{r<j-i} \varphi^r(\tilde{\xi})$). Let $J = 1 + \max\{s : k_s \neq 0\}$ (or $J = 1$), $c = \kappa + \sum_s k_s$ and $P' = \big(\prod_{j=0}^{J} \varphi^j(\mu)\big)^c$. Then $\varphi(P') = \big(\prod_{j=1}^{J+1} \varphi^j(\mu)\big)^c = P'R$ with $R = (\varphi^{J+1}(\mu)/\mu)^c \in A_{\mathrm{inf}}$, and $P_0 \mid \varphi(P')$ in A_{inf} (each factor μ^κ, $\varphi^{s+1}(\mu)^{k_s}$ divides a corresponding power of some $\varphi^j(\mu)$, $1 \leq j \leq J+1$, and c exceeds the total exponent). Put $a' = (\varphi(P')/P_0)\, P_0 \iota_T(y) \in A_{\mathrm{inf}} \otimes T$. We solve $\varphi(b_0) - Rb_0 = a'$ with $b_0 \in A_{\mathrm{inf}} \otimes T$, coordinatewise: for $a \in A_{\mathrm{inf}}$ with reduction $\bar{a} \in \mathcal{O}_{\mathbb{C}_\ell^\flat}$, the monic polynomial $Y^\ell - \bar{R}Y - \bar{a}$ has a root t_0 in the algebraically closed field $\mathbb{C}_\ell^\flat$, integral since $\mathcal{O}_{\mathbb{C}_\ell^\flat}$ is integrally closed; as reduction modulo ℓ is a ring homomorphism $A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_\ell^\flat}$ carrying φ to the ℓ-th power map, $\varphi([t_0]) - R[t_0] - a \in \ell A_{\mathrm{inf}}$, say $= -\ell a_1$; repeating with a_1 and summing, $b_0 = \sum_i \ell^i [t_i]$ converges (A_{inf} is ℓ-adically complete) and solves the equation. Put $b = P'^{-1} b_0 \in W(\mathbb{C}_\ell^\flat) \otimes T$. Then

$$(\varphi - 1)(b) = \varphi(P')^{-1}\big(\varphi(b_0) - Rb_0\big) = \varphi(P')^{-1} a' = \iota_T(y).$$

Step 3: the ring S. Let $S \subseteq W(\mathbb{C}_\ell^\flat)$ be the subring generated by A_{inf} and the $\varphi^j(\mu)^{-1}$, $j \geq 0$. By Lemma 23.12(Ac2) it is stable under φ and G_F, and the inclusion $A_{\mathrm{inf}} \subset B_{\mathrm{cris}}$ extends uniquely to a ring homomorphism $\rho_S \colon S \to B_{\mathrm{cris}}$, equivariant for φ and G_F; it extends ρ of Proposition 23.31, since both restrict to the inclusion on A_{inf} and $A_{\mathrm{inf}}[1/\mu] \subseteq S$ is a localisation of A_{inf}.

Step 4: the class is crystalline. Let $u = \chi_{\mathrm{cyc}}(\sigma)$ and $z_u = z_u(y)$. By (Gc1), $\iota_T(z_u) \in \mu^{-\kappa} A_{\mathrm{inf}} \otimes T \subseteq S \otimes T$, and $b \in S \otimes T$, so $c_{y,b}(\sigma) = \iota_T(z_u) - (\sigma - 1)(b)$, which lies in T (Lemma 23.28(Hc1)), is an identity in $S \otimes T$. Applying the G_F-equivariant map $\rho_S \otimes 1 \colon S \otimes_{\mathbb{Z}_\ell} T \to B_{\mathrm{cris}} \otimes_{\mathbb{Z}_\ell} T = B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V$, which is the identity on T, and Proposition 23.31,

$$c_{y,b}(\sigma) = (\sigma - 1)(\beta) - (\sigma - 1)(\rho_S(b)) = (\sigma - 1)\big(\beta - \rho_S(b)\big) \qquad (\sigma \in G_F).$$

So the image of the cocycle $c_{y,b}$ in $H^1(F, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V)$ is 0, i.e. the image of $\mathrm{cl}(y)$ in $H^1(F, V)$ lies in $H^1_f(F, V)$, and $\mathrm{cl}(y) \in H^1_f(F, T)$.

Over a base field. For a pair over K: Steps 1 and 2 are statements in $A_{\mathrm{inf}} \otimes T$ and $W(\mathbb{C}_\ell^\flat) \otimes T$ that use only (Gc0), (Gc1) (which Lemma 23.29 gives over K), the ring identities $\iota(Q_s) = \varphi^{s+1}(\mu)/\varphi^s(\mu)$ and the Artin–Schreier solution in A_{inf}; no Galois group enters. Step 3: S and ρ_S are G_F-equivariant, hence G_K-equivariant. Step 4 runs with $\sigma \in G_K$, $u = \chi_{\mathrm{cyc}}(\sigma) \in \Gamma_K$: (Gc1) over K, (Hc1) of Lemma 23.28 over K ($c_{y,b}(\sigma) \in T$ for $\sigma \in G_K$), and Proposition 23.31 over K give $c_{y,b}(\sigma) = (\sigma - 1)(\beta - \rho_S(b))$ for $\sigma \in G_K$; so the image of $\mathrm{cl}(y)$ in $H^1(K, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V)$ is 0 and $\mathrm{cl}(y) \in H^1_f(K, T)$. No step uses that the base is unramified.

Level s. Let the data be as in the paragraph Level s of the statement, $\sigma \in G_K$, $u = \chi_{\mathrm{cyc}}(\sigma) \in \Gamma_K$. Step 1: write $f = \prod_{k \geq 0} Q_{s-1+k}^{c_k}$. By Lemma 23.8(6), ι_s commutes with the Frobenius lifts and $\iota_s(Q_{s-1}) = \tilde{\xi} = \varphi(\mu)/\mu$, so $\iota_s(Q_{s-1+k}) = \varphi^k(\tilde{\xi}) = \varphi^{k+1}(\mu)/\varphi^k(\mu)$; with (Gc0), (Gc1) of level s (Lemma 23.29, paragraph Level s; pole factor μ^κ) this gives $P_0\, \iota_{s,T}(y) \in A_{\mathrm{inf}} \otimes T$ with $P_0 = \mu^\kappa \prod_k \varphi^{k+1}(\mu)^{c_k}$. Step 2 goes through unchanged with $J = 1 + \max\{k : c_k \neq 0\}$ (or $J = 1$), $c = \kappa + \sum_k c_k$ and $\iota_{s,T}$ in place of ι_T; the Artin–Schreier argument holds for every prime (the polynomial $Y^\ell - \bar{R}Y - \bar{a}$ has a root in $\mathbb{C}_\ell^\flat$). Step 3: S and ρ_S exist for every prime and are G_F-, hence G_K-equivariant (Lemma 23.12, paragraph Every prime), and ρ_S extends ρ. Step 4: by (Gc1) of level s, $\iota_{s,T}(z_u) \in \mu^{-\kappa} A_{\mathrm{inf}} \otimes T \subseteq S \otimes T$; by (Hc1) of Lemma 23.28, paragraph Level s, $c_{y,b}(\sigma) = \iota_{s,T}(z_u) - (\sigma - 1)(b) \in T$; and Proposition 23.31, paragraph Level s (which applies: y and f satisfy its hypotheses), gives β with $(\sigma - 1)(\beta) = (\rho \otimes 1)(\iota_{s,T}(z_u))$. Applying $\rho_S \otimes 1$, $c_{y,b}(\sigma) = (\sigma - 1)(\beta - \rho_S(b))$ for every $\sigma \in G_K$, so the image of $\mathrm{cl}(y)$ in $H^1(K, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V)$ is 0 and $\mathrm{cl}(y) \in H^1_f(K, T)$. □

Lemma 23.28. *Let ℓ, F, L, (τ_a) and a pair (V, T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$, let $(M_a, h_a)_a$ be a window of V as in (Ex1), and let $m \geq 1 + \max_a M_a$ be an integer. Call $y \in \mathbf{N}[1/\lambda_\infty]$ a* Herr vector of precision m *if $(\gamma_u - 1)(y) \in X^m \mathbf{N}$ for every $u \in \mathbb{Z}_\ell^\times$; for such y and u let $z_u(y)$ be the unique element of $X^m \mathbf{N}$ with $(\varphi - 1)(z_u(y)) = (\gamma_u - 1)(y)$ (it exists and is unique by Lemma 23.30(2)). Then:*

(Hc1) The class *For a Herr vector y of precision m there is $b \in W(\mathbb{C}_\ell^\flat) \otimes_{\mathbb{Z}_\ell} T$ with $(\varphi \otimes 1 - 1)(b) = \iota_T(y)$. For every such b, the map*

$$c_{y,b} \colon G_F \to W(\mathbb{C}_\ell^\flat) \otimes_{\mathbb{Z}_\ell} T, \qquad c_{y,b}(\sigma) = \iota_T\big(z_{\chi_{\mathrm{cyc}}(\sigma)}(y)\big) - (\sigma - 1)(b),$$

takes values in T and is a continuous 1-cocycle; its class $\mathrm{cl}(y) \in H^1(F, T)$ (the Herr class *of y) does not depend on b, nor on m (if y has precision $m' \geq m$ too).*

(Hc2) One generator suffices *If $y \in \mathbf{N}[1/\lambda_\infty]$ and $(\gamma_u - 1)(y) \in X^m \mathbf{N}$ for all u in a subset of $\mathbb{Z}_\ell^\times$ generating a dense subgroup (for instance a topological generator of $\mathbb{Z}_\ell^\times$, or $\mu_{\ell-1}$ together with a topological generator of $1 + \ell\mathbb{Z}_\ell$), then y is a Herr vector of precision m.*

(Hc3) Additivity *Herr vectors of precision m form an $\mathcal{O}_L$-submodule of $\mathbf{N}[1/\lambda_\infty]$, and* cl *is $\mathcal{O}_L$-linear on it.*

(Hc4) Coboundaries *If $w \in \mathbf{N}[1/\lambda_\infty]$ and $(\gamma_u - 1)(w) \in X^m\mathbf{N}$ for all u, then $(\varphi - 1)(w)$ is a Herr vector of precision m, $z_u((\varphi-1)(w)) = (\gamma_u - 1)(w)$, and $\mathrm{cl}((\varphi-1)(w)) = 0$. In particular every $y \in X^m\mathbf{N}$ is a Herr vector of precision m with $\mathrm{cl}(y) = 0$.*

(Hc5) Naturality *Let (V', T') be another pair with window (M'_a, h'_a), $m \geq 1 + \max_a M'_a$, and $f\colon T \to T'$ a G_F-equivariant $\mathcal{O}_L$-linear map. Then $\mathbf{N}(f)$, extended $\mathbf{A}_{F,L}$-linearly to $\mathbf{N}[1/\lambda_\infty]$, carries Herr vectors of precision m to Herr vectors of precision m, with $z_u(\mathbf{N}(f)(y)) = \mathbf{N}(f)(z_u(y))$ and $\mathrm{cl}(\mathbf{N}(f)(y)) = f_*\,\mathrm{cl}(y)$.*

Over a base field. *Let K be a field with $F \subseteq K \subseteq F(\mu_\ell)$ and Γ_K as in Definition 23.2. For pairs over K as in Proposition 23.7, (Hc1)–(Hc5) hold with u ranging over Γ_K instead of $\mathbb{Z}_\ell^\times$ (in the definition of Herr vectors and of $z_u(y)$, and in (Hc4)), G_F replaced by G_K, $H^1(F,-)$ by $H^1(K,-)$ and G_F-equivariant maps by G_K-equivariant ones; in (Hc2) the subset is one of Γ_K generating a dense subgroup of Γ_K. For $K = F$ this is the statement above.*

Level s. *Let ℓ be any prime, $s \geq 1$ with $s \geq 2$ if $\ell = 2$, and L, K ($F \subseteq K \subseteq F(\mu_{\ell^s})$) and pairs over K as in (Ex4) of Proposition 23.7; let $\mathbf{N} = \mathbf{N}_s(T)$, $(M_a, h_a)_a$ a window of V as in (Ex1) of level s, $m \geq 1 + \max_a M_a$, and $X_{s-1} = \varphi^{s-1}(X)$ the level variable of Definition 23.2. Call $y \in \mathbf{N}[1/\lambda_\infty]$ a* Herr vector of precision m at level s *if $(\gamma_u - 1)(y) \in X_{s-1}^m\mathbf{N}$ for every $u \in \Gamma_K$, and let $z_u(y)$ be the unique element of $X_{s-1}^m\mathbf{N}$ with $(\varphi-1)(z_u(y)) = (\gamma_u - 1)(y)$ (Lemma 23.30, Level s). Then (Hc1)–(Hc5), in the form over a base field, hold for Herr vectors of precision m at level s, with $\mathbf{N}(T)$, ι_T, $\mathbf{N}(f)$ replaced by $\mathbf{N}_s(T)$, $\iota_{s,T}$, $\mathbf{N}_s(f)$ and $X^m\mathbf{N}$ by $X_{s-1}^m\mathbf{N}$ throughout (also in (Hc2), (Hc4), (Hc5)); in (Hc5) the second pair is a pair over K with $\mathbf{N}_s(T')$. For $s = 1$ and ℓ odd this is the paragraph over a base field.*

Proof. Write φ for $\varphi \otimes 1$ on $W(\mathbb{C}_\ell^\flat) \otimes_{\mathbb{Z}_\ell} T$; it commutes with the diagonal action of G_F. T is free of finite rank over $\mathbb{Z}_\ell$; fix a $\mathbb{Z}_\ell$-basis, so $W(\mathbb{C}_\ell^\flat) \otimes T$ is a finite power of $W(\mathbb{C}_\ell^\flat)$ with φ acting coordinatewise. For $x \in W(\mathbb{C}_\ell^\flat)$ write $x = \sum_{i \geq 0} \ell^i [x_i]$ (unique, $\mathbb{C}_\ell^\flat$ being perfect); then $\varphi(x) = \sum \ell^i [x_i^\ell]$.

Artin–Schreier. (a) $W(\mathbb{C}_\ell^\flat)^{\varphi=1} = \mathbb{Z}_\ell$: $\varphi(x) = x$ forces $x_i^\ell = x_i$, so $x_i \in \mathbb{F}_\ell$ and $x \in W(\mathbb{F}_\ell) = \mathbb{Z}_\ell$. Hence $(W(\mathbb{C}_\ell^\flat) \otimes T)^{\varphi=1} = T$. (b) $\varphi - 1$ is onto $W(\mathbb{C}_\ell^\flat)$: given a, choose $t_0 \in \mathbb{C}_\ell^\flat$ with $t_0^\ell - t_0 = a_0$ ($\mathbb{C}_\ell^\flat$ is algebraically closed); then $a - (\varphi - 1)[t_0] \in \ell W(\mathbb{C}_\ell^\flat)$ because $[t_0^\ell] - [t_0] \equiv [t_0^\ell - t_0]$ modulo ℓ; iterating and using that $W(\mathbb{C}_\ell^\flat)$ is ℓ-adically complete and separated gives t with $(\varphi - 1)t = a$. Hence $\varphi - 1$ is onto $W(\mathbb{C}_\ell^\flat) \otimes T$.

(Hc1): values in T and the cocycle identity. Let y be a Herr vector of precision m and $z_u = z_u(y)$. A b exists by (b). For $u, v \in \mathbb{Z}_\ell^\times$, $z_u + \gamma_u(z_v) \in X^m\mathbf{N}$ ($\gamma_u(X) \in X\mathbf{A}_F$) and, as γ_u commutes with φ (Wm3) and $\gamma_u\gamma_v = \gamma_{uv}$,

$$(\varphi - 1)(z_u + \gamma_u(z_v)) = (\gamma_u - 1)(y) + \gamma_u(\gamma_v - 1)(y) = (\gamma_{uv} - 1)(y),$$

so $z_{uv} = z_u + \gamma_u(z_v)$ by uniqueness. Let $\sigma, \tau \in G_F$, $u = \chi_{\mathrm{cyc}}(\sigma)$, $v = \chi_{\mathrm{cyc}}(\tau)$. By (Gc0), $\iota_T \circ \varphi = \varphi \circ \iota_T$ and $\sigma \circ \iota_T = \iota_T \circ \gamma_u$, so

$$(\varphi - 1)(c_{y,b}(\sigma)) = \iota_T((\gamma_u - 1)(y)) - (\sigma - 1)(\varphi - 1)(b) = (\sigma - 1)\iota_T(y) - (\sigma - 1)\iota_T(y) = 0,$$

and $c_{y,b}(\sigma) \in T$ by (a). Also $c_{y,b}(\sigma\tau) = \iota_T(z_u) + \sigma\iota_T(z_v) - (\sigma - 1)(b) - \sigma(\tau - 1)(b) = c_{y,b}(\sigma) + \sigma c_{y,b}(\tau)$.

(Hc1): continuity. By the cocycle identity, $c_{y,b}(\sigma\tau) - c_{y,b}(\sigma) = \sigma c_{y,b}(\tau)$, and G_F acts continuously on T, so it suffices to show: for every $n \geq 1$ there is an open subgroup $U \subseteq G_F$

with $c_{y,b}(U) \subseteq \ell^n T$. Give $W_n(\mathbb{C}_\ell^\flat) = W(\mathbb{C}_\ell^\flat)/\ell^n$ the product topology of the valuation topology on its n Witt coordinates. The Witt addition and multiplication are given by integral polynomials in the coordinates, so $W_n(\mathbb{C}_\ell^\flat)$ is a topological ring; G_F acts coordinatewise, continuously since it acts continuously on $\mathbb{C}_\ell^\flat$ [94]; for an ideal I of $\mathcal{O}_{\mathbb{C}_\ell^\flat}$, $W_n(I)$ (coordinates in I) is an ideal of $W_n(\mathcal{O}_{\mathbb{C}_\ell^\flat})$ and $W_n(I)W_n(J) \subseteq W_n(IJ)$ (each coordinate of a product is isobaric of positive degree in both arguments); and $W_n(\mathbb{F}_\ell) = \mathbb{Z}/\ell^n$ is discrete, since a non-zero element has a coordinate in $\mathbb{F}_\ell^\times$. Equip $W_n(\mathbb{C}_\ell^\flat) \otimes T$ with the product topology via the fixed basis; then T/ℓ^n is discrete in it. Now: (i) $\sigma \mapsto (\sigma-1)(b)$ modulo ℓ^n is continuous, with value 0 at $\sigma = 1$. (ii) Let $k \geq 0$. By (Gc3) of Lemma 23.29 (applicable since $(\gamma_u - 1)(y) \in X^m\mathbf{N} \subseteq \mathbf{N}$) there is s with $(\gamma_u - 1)(y) \in (\ell, X)^{k+m}\mathbf{N} \cap X^m\mathbf{N}$ for $u \in 1 + \ell^s\mathbb{Z}_\ell$; in coordinates of $\mathcal{O}_L[[X]]$-bases of the $\mathbf{N}_a$, additivity of the (ϖ, X)-order (ϖ a uniformiser of $\mathcal{O}_L$) shows that this intersection lies in $(\varpi, X)^k X^m\mathbf{N}$. By Lemma 23.30(1), φ maps $(\varpi, X)^k X^m\mathbf{N}$ into itself, and $z_u = -\sum_{j\geq 0} \varphi^j((\gamma_u - 1)(y))$ (Lemma 23.30(2)) lies in this closed submodule. Let K be as in (Gc1), so that $\mu_\varepsilon^K \iota_T(\mathbf{N}) \subseteq A_{\mathrm{inf}} \otimes T$, and let $\iota_L = \iota \otimes \mathrm{id}\colon \mathbf{A}_{F,L} = \mathbf{A}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L \to A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$ ($\mathcal{O}_L$ is finite free over $\mathbb{Z}_\ell$), a ring homomorphism with $\iota_L(\varpi) = 1 \otimes \varpi$ and $\iota_L(X) = \iota(X) = \mu_\varepsilon$ (Lemma 23.8); by (Ex2), $\iota_T(gx) = \iota_L(g)\iota_T(x)$ for $g \in \mathbf{A}_{F,L}$ and $x \in \mathbf{N}$. Every element of $(\varpi, X)^k X^m\mathbf{N}$ is a finite sum of products gx with $g \in (\varpi, X)^k$ and $x \in \mathbf{N}$, so $\mu_\varepsilon^K \iota_T(z_u) \in (\varpi, \mu_\varepsilon)^k(A_{\mathrm{inf}} \otimes T)$; and for $k \geq en$, e the ramification index of L, $(\varpi, \mu_\varepsilon)^k \subseteq \ell^n(A_{\mathrm{inf}} \otimes \mathcal{O}_L) + \mu_\varepsilon^{k-en}(A_{\mathrm{inf}} \otimes \mathcal{O}_L)$ (a monomial $\varpi^i\mu_\varepsilon^{k-i}$ with $i \geq en$ lies in $\ell^n(\dots)$). Modulo ℓ^n, $\mu_\varepsilon = [\varepsilon] - 1$ has all its coordinates in the maximal ideal (its image in $W(\bar{\mathbb{F}}_\ell)$ is 0), hence of valuation $\geq c$ for some $c > 0$ (finitely many coordinates), so $\mu_\varepsilon^r W_n(\mathcal{O}_{\mathbb{C}_\ell^\flat}) \subseteq W_n(\{v \geq rc\})$, and μ_ε^{-K} is a fixed unit of $W_n(\mathbb{C}_\ell^\flat)$. Hence $\iota_T(z_{\chi_{\mathrm{cyc}}(\sigma)}) \to 0$ in $W_n(\mathbb{C}_\ell^\flat) \otimes T$ as $\chi_{\mathrm{cyc}}(\sigma) \to 1$, i.e. along an open subgroup of G_F. By (i) and (ii), $c_{y,b}(\sigma)$ modulo ℓ^n tends to 0 as $\sigma \to 1$; it lies in the discrete subset T/ℓ^n, so it is 0 on an open subgroup.

(Hc1): independence. Two solutions b, b' differ by an element of $(W(\mathbb{C}_\ell^\flat) \otimes T)^{\varphi=1} = T$, and $c_{y,b'} = c_{y,b} - (\sigma \mapsto (\sigma-1)(b'-b))$ differs by a coboundary. If y has precision $m' \geq m$, the element z_u defined in $X^{m'}\mathbf{N} \subseteq X^m\mathbf{N}$ is the unique solution in $X^m\mathbf{N}$, so the cocycle is the same.

(Hc2). Let S be the set of $u \in \mathbb{Z}_\ell^\times$ with $(\gamma_u - 1)(y) \in X^m\mathbf{N}$. It is a subgroup: $(\gamma_{uv} - 1)(y) = \gamma_u(\gamma_v - 1)(y) + (\gamma_u - 1)(y)$, $(\gamma_{u^{-1}} - 1)(y) = -\gamma_{u^{-1}}(\gamma_u - 1)(y)$, and γ_u preserves $X^m\mathbf{N}$. It is closed: write $y = x/f$ with $x \in \mathbf{N}$, f a product of Q_s's, $v_w = f/\gamma_w(f)$. For $u \in \bar{S}$ choose $u_i \in S$ with $w_i = u_i^{-1}u \to 1$. Then $f(\gamma_u - 1)(y) - f(\gamma_{u_i} - 1)(y) = f\gamma_{u_i}(\gamma_{w_i} - 1)(y)$, and $f(\gamma_w - 1)(y) = v_w(\gamma_w - 1)(x) + (v_w - 1)x \to 0$ in $\mathbf{N}$ as $w \to 1$: the first term by (Gc3) applied to $x \in \mathbf{N}$ (v_w is a unit), the second because $v_w - 1 \in (\ell, X)^{s+1-\deg f}$ for $w \in 1 + \ell^s\mathbb{Z}_\ell$, where $\deg f$ is the number of factors Q_t of f. Indeed, for such w, $\gamma_w(X) - X = (1+X)((1+X)^{w-1} - 1) \in (\ell, X)^{s+1}\mathbb{Z}_\ell[[X]]$, hence $\gamma_w(g) - g \in (\ell, X)^{s+1}$ for $g \in \mathbb{Z}_\ell[[X]]$; in the regular local ring $\mathbb{Z}_\ell[[X]]$ the (ℓ, X)-adic order is additive, each Q_t has order 1 ($Q_t(0) = \ell$), and $\gamma_w(f) \in f\,\mathbb{Z}_\ell[[X]]^\times$ (Lemma 23.8(2)), so f and $\gamma_w(f)$ have order $\deg f$ and $v_w - 1 = (f - \gamma_w(f))/\gamma_w(f)$ has order at least $s + 1 - \deg f$; $\gamma_{u_i}(f)/f$ is a unit, so $f\gamma_{u_i}(\gamma_{w_i} - 1)(y) \to 0$ as well. Hence $f(\gamma_u - 1)(y)$ is a limit of elements of $fX^m\mathbf{N}$, a finitely generated submodule of the finitely generated module $\mathbf{N}$ over the noetherian (ℓ, X)-adically complete ring $\mathbf{A}_{F,L}$, hence closed. So $f(\gamma_u - 1)(y) \in fX^m\mathbf{N}$, and $(\gamma_u - 1)(y) \in X^m\mathbf{N}$ since f is a non-zero-divisor on $\mathbf{N}[1/\lambda_\infty]$. A closed subgroup containing a set that generates a dense subgroup is $\mathbb{Z}_\ell^\times$. ($\mathbb{Z}_\ell^\times = \mu_{\ell-1} \times (1 + \ell\mathbb{Z}_\ell)$ is procyclic for ℓ odd.)

(Hc3). If y_1, y_2 are Herr vectors and $\lambda \in \mathcal{O}_L$, then $y_1 + \lambda y_2$ is one (γ_u is $\mathcal{O}_L$-linear), $z_u(y_1 + \lambda y_2) = z_u(y_1) + \lambda z_u(y_2)$ by uniqueness, $b_1 + (1 \otimes \lambda)b_2$ solves the equation for $y_1 + \lambda y_2$ (ι_T is $\mathcal{O}_L$-linear by (Ex2)), and $c_{y_1+\lambda y_2, b_1+\lambda b_2} = c_{y_1,b_1} + \lambda c_{y_2,b_2}$ since G_F acts $\mathcal{O}_L$-linearly on T.

(Hc4). Put $y = (\varphi - 1)(w)$. Then $(\gamma_u - 1)(y) = (\varphi - 1)((\gamma_u - 1)(w)) \in X^m\mathbf{N}$, because φ maps $X^m\mathbf{N}$ into $X^m\mathbf{N}$ (Lemma 23.30(1)); so y is a Herr vector, and $z_u(y) = (\gamma_u - 1)(w)$ by uniqueness. Take $b = \iota_T(w)$, which satisfies $(\varphi - 1)(b) = \iota_T(y)$ by (Gc0); then $c_{y,b}(\sigma) = \iota_T((\gamma_u - 1)(w)) - (\sigma - 1)\iota_T(w) = 0$ by (Gc0). If $y \in X^m\mathbf{N}$, apply this to $w = -\sum_{j\geq 0} \varphi^j(y) \in X^m\mathbf{N}$,

which satisfies $(\varphi-1)(w)=y$ (Lemma 23.30(2)); $(\gamma_u-1)(w)\in X^m\mathbf{N}$ since γ_u preserves $X^m\mathbf{N}$.

(Hc5). $\mathbf{N}(f)$ is $\mathbf{A}_{F,L}$-linear and commutes with φ and all γ_u (Ex3), so its extension to $\mathbf{N}[1/\lambda_\infty]$ commutes with them too and maps $X^m\mathbf{N}$ into $X^m\mathbf{N}'$; hence it maps Herr vectors to Herr vectors and, by uniqueness in $X^m\mathbf{N}'$, $z_u(\mathbf{N}(f)(y))=\mathbf{N}(f)(z_u(y))$. As $\iota_{T'}\circ\mathbf{N}(f)=(1\otimes f)\circ\iota_T$ on $\mathbf{N}$ (Ex3), and both sides are ι-semilinear, the identity extends to $\mathbf{N}[1/\lambda_\infty]$. If b solves the equation for y, then $b'=(1\otimes f)(b)$ solves it for $\mathbf{N}(f)(y)$, and $c_{\mathbf{N}(f)(y),b'}(\sigma)=(1\otimes f)(c_{y,b}(\sigma))=f(c_{y,b}(\sigma))$, since $1\otimes f$ commutes with φ and G_F. So $\mathrm{cl}(\mathbf{N}(f)(y))=f_*\mathrm{cl}(y)$.

Over a base field. For pairs over K the proof is unchanged, with $\sigma,\tau\in G_K$, $u=\chi_{\mathrm{cyc}}(\sigma)\in\Gamma_K$, (Gc0) and (Gc3) of Lemma 23.29 over K (its closing paragraph), and the continuity of the action of $G_K\subseteq G_F$ on $\mathbb{C}_\ell^\flat$ and on T; in the continuity step, $\chi_{\mathrm{cyc}}(\sigma)\to1$ along an open subgroup of G_K. In (Hc2), S is a closed subgroup of Γ_K (same argument, with (Gc3) over K) containing a set that generates a dense subgroup of Γ_K, so $S=\Gamma_K$.

Level s. The proof over a base field applies with the replacements of the statement; we list what it uses and where the level-s form comes from. The Artin–Schreier facts (a), (b) do not involve $\mathbf{N}$. The existence, uniqueness and formula of z_u, that φ maps $X^m_{s-1}\mathbf{N}$ into itself and that γ_u preserves it, are Lemma 23.30, Level s; $\gamma_u\gamma_v=\gamma_{uv}$ and the commutation of γ_u with φ are (Wm3) of level s; (Gc0), (Gc1), (Gc3) are those of Lemma 23.29, Level s, with $\iota_{s,T}$ and the ideal $(\ell,\mu_{\varepsilon,s})$. The one step that changes is in the continuity of (Hc1): given k, (Gc3) gives $(\gamma_u-1)(y)\in(\ell,X)^{k'}\mathbf{N}\cap X^m_{s-1}\mathbf{N}$ with $k'=k+\ell^{s-1}m$ for u near 1; in coordinates of $\mathcal{O}_L[[X]]$-bases of the $\mathbf{N}_a$, an element $X^m_{s-1}c$ of $(\varpi,X)^{k'}$ has $c\in(\varpi,X)^{k'-m\ell^{s-1}}=(\varpi,X)^k$, because the (ϖ,X)-order is additive and that of X_{s-1} is at most ℓ^{s-1} (its monomial $X^{\ell^{s-1}}$ has coefficient 1). So $(\gamma_u-1)(y)\in(\varpi,X)^kX^m_{s-1}\mathbf{N}$, a closed submodule mapped into itself by φ, which therefore contains z_u; then $\mu_\varepsilon^\kappa\iota_{s,T}(z_u)\in(\varpi,\mu_{\varepsilon,s})^k(A_{\mathrm{inf}}\otimes T)$ by (Gc1) of level s and the semilinearity argument above, with $\iota_L=\iota_s\otimes\mathrm{id}$, $\iota_s(X)=\mu_{\varepsilon,s}$ (Lemma 23.8(6)) and (Ex2) of level s, and modulo ℓ^n the element $\mu_{\varepsilon,s}$, whose image in $W(\bar{\mathbb{F}}_\ell)$ is 0, has all its Witt coordinates of positive valuation; the rest of the step is unchanged. In (Hc2) the closedness argument uses (Gc3) of level s and that $fX^m_{s-1}\mathbf{N}$ is a finitely generated, hence closed, submodule of $\mathbf{N}$; (Hc4) uses $\varphi(X^m_{s-1}\mathbf{N})\subseteq X^m_{s-1}\mathbf{N}$; (Hc5) uses (Ex3) of level s. The clause (Wm3') is not used. □

Lemma 23.29. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a pair (V,T) be as in Proposition 23.7, $d=\dim_L V$, and $\mathbf{N}=\mathbf{N}(T)$ with $\varphi=\varphi_{\mathbf{N}}$, $\gamma_u=\gamma_{u,\mathbf{N}}$ and ι_T. Write (ℓ,X) for the ideal of $\mathbf{A}_{F,L}$ generated by ℓ and X, and (ℓ,μ_ε) for the ideal of A_{inf} generated by ℓ and μ_ε.*

(Gc0) Extension to poles *For f a finite product of the Q_s ($s\geq0$) and $x\in\mathbf{N}$, the formulas $\gamma_u(x/f)=\gamma_u(x)/\gamma_u(f)$ and $\iota_T(x/f)=\iota(f)^{-1}\iota_T(x)$ give a well-defined γ_u-semilinear action on $\mathbf{N}[1/\lambda_\infty]$ and a well-defined additive injective map $\iota_T\colon\mathbf{N}[1/\lambda_\infty]\to W(\mathbb{C}_\ell^\flat)\otimes_{\mathbb{Z}_\ell}T$, and for $y\in\mathbf{N}[1/\lambda_\infty]$, $\sigma\in G_F$:*

$$\sigma(\iota_T(y))=\iota_T(\gamma_{\chi_{\mathrm{cyc}}(\sigma)}(y)),\qquad(\varphi\otimes1)(\iota_T(y))=\iota_T(\varphi(y)).$$

(Gc1) Bounded poles *There is an integer $\kappa\geq0$ with $\mu_\varepsilon^\kappa\iota_T(\mathbf{N})\subseteq A_{\mathrm{inf}}\otimes_{\mathbb{Z}_\ell}T$; for such κ, $\mu_\varepsilon^\kappa\iota_T((\ell,X)^k\mathbf{N})\subseteq(\ell,\mu_\varepsilon)^k(A_{\mathrm{inf}}\otimes_{\mathbb{Z}_\ell}T)$ for every $k\geq0$.*

(Gc2) Strictness *For κ as in (Gc1) and every $k\geq0$ there is $j\geq0$ such that every $x\in\mathbf{N}$ with $\mu_\varepsilon^\kappa\iota_T(x)\in(\ell,\mu_\varepsilon)^j(A_{\mathrm{inf}}\otimes_{\mathbb{Z}_\ell}T)$ lies in $(\ell,X)^k\mathbf{N}$.*

(Gc3) Continuity *If $y\in\mathbf{N}[1/\lambda_\infty]$ and $(\gamma_u-1)(y)\in\mathbf{N}$ for every $u\in\mathbb{Z}_\ell^\times$, then for every $k\geq0$ there is $s\geq1$ such that $(\gamma_u-1)(y)\in(\ell,X)^k\mathbf{N}$ for all $u\in1+\ell^s\mathbb{Z}_\ell$.*

Over a base field. *Let K be a field with $F\subseteq K\subseteq F(\mu_\ell)$ and Γ_K as in Definition 23.2. For a pair (V,T) over K as in Proposition 23.7, (Gc0)–(Gc3) hold with G_F replaced by G_K and $\mathbb{Z}_\ell^\times$*

by Γ_K (in (Gc0) for $\sigma \in G_K$; in (Gc3) the hypothesis is for every $u \in \Gamma_K$, and the conclusion for $u \in 1+\ell^s\mathbb{Z}_\ell$, s large, which lies in Γ_K). For $K = F$ this is the statement above.

Level s. *Let ℓ be any prime, $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, and L, K ($F \subseteq K \subseteq F(\mu_{\ell^s})$) and pairs (V,T) over K as in (Ex4) of Proposition 23.7; let $\mathbf{N} = \mathbf{N}_s(T)$ with its maps and $\iota_{s,T}$, and $\mu_{\varepsilon,s} = [\varepsilon^{1/\ell^{s-1}}] - 1$ and ι_s as in Lemma 23.8(6). Then (Gc0)–(Gc3) hold, in the form over a base field, with $\mathbf{N}(T)$, ι_T, ι replaced by $\mathbf{N}_s(T)$, $\iota_{s,T}$, ι_s, and with the ideal (ℓ, μ_ε) replaced by $(\ell, \mu_{\varepsilon,s})$ in (Gc1) and (Gc2); the pole factor μ_ε^κ in (Gc1), (Gc2) is unchanged. For $s = 1$ and ℓ odd ($\mu_{\varepsilon,1} = \mu_\varepsilon$) this is the paragraph over a base field.*

Proof. Throughout, $R = A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$, a free A_{inf}-module of rank $[L : \mathbb{Q}_\ell]$, and ϖ is a uniformiser of $\mathcal{O}_L$, e the ramification index of $L/\mathbb{Q}_\ell$; T is free over $\mathcal{O}_L$ of rank d, so $A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} T = R \otimes_{\mathcal{O}_L} T \cong R^d$ after the choice of an $\mathcal{O}_L$-basis of T, which we fix. If $d = 0$ everything is trivial; let $d \geq 1$.

Topologies. The (ℓ, μ_ε)-adic topology of A_{inf} is the topology of (Ai2): modulo ℓ, μ_ε reduces to $\varepsilon - 1$, of valuation $\ell/(\ell-1) \in (1,2]$, and $p^\flat$ has valuation 1, so $(\ell, \mu_\varepsilon) \subseteq (\ell, [p^\flat])$ and $[p^\flat]^2 \in (\ell, \mu_\varepsilon)$. Hence A_{inf} and R are (ℓ, μ_ε)-adically complete and separated, and G_F acts continuously on A_{inf} for this topology (Ai2). Let $\iota_L \colon \mathbf{A}_{F,L} = (\mathcal{O}_F \otimes \mathcal{O}_L)[[X]] \to R$ be $\sum_k c_k X^k \mapsto \sum_k (\iota \otimes \mathrm{id})(c_k)\mu_\varepsilon^k$ (convergent). It is a ring homomorphism extending ι and $\lambda \mapsto 1 \otimes \lambda$.

(Gc0). By Lemma 23.8(2), $\gamma_u(Q_s) \in Q_s \mathbb{Z}_\ell[[X]]^\times$, so $\gamma_u(f) \in f\,\mathbb{Z}_\ell[[X]]^\times$ and $\gamma_u(x)/\gamma_u(f) \in \mathbf{N}[1/\lambda_\infty]$; if $x/f = x'/f'$ then $f'x = fx'$ ($\mathbf{N}$ is torsion-free, being free over the product of the domains $\mathcal{O}_L[[X]]$ by (Wm1)), so the formula is well defined, and it is semilinear. By Lemma 23.8(4), $\iota(Q_s) = \varphi^s(\iota(Q)) = \varphi^s(\tilde{\xi})$, whose reduction modulo ℓ is $(\varepsilon - 1)^{\ell^s(\ell-1)} \neq 0$ in the field $\mathbb{C}_\ell^\flat$, so it is a unit of $W(\mathbb{C}_\ell^\flat)$; the extension of ι_T is well defined as before and injective because ι_T is (Ex2). For $\sigma \in G_F$ with $u = \chi_{\mathrm{cyc}}(\sigma)$, (Ex2) and $\sigma \circ \iota = \iota \circ \gamma_u$ (Lemma 23.8(4)) give $\sigma(\iota(f)^{-1}\iota_T(x)) = \iota(\gamma_u(f))^{-1}\iota_T(\gamma_u(x)) = \iota_T(\gamma_u(x/f))$; similarly $\varphi \circ \iota = \iota \circ \varphi$ (ι is a morphism of prisms, hence of δ-rings) and (Ex2) give the second identity.

ι_T is ι_L-semilinear. Since $\mathcal{O}_L$ is finite free over $\mathbb{Z}_\ell$, $\mathbf{A}_{F,L} = \mathbf{A}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$, so every $g \in \mathbf{A}_{F,L}$ is a finite sum $\sum_j f_j \otimes \lambda_j$ with $f_j \in \mathbf{A}_F = \mathcal{O}_F[[X]]$, $\lambda_j \in \mathcal{O}_L$, and $\iota_L(g) = \sum_j \iota(f_j) \otimes \lambda_j$. By (Ex2), $\iota_T(fx) = \iota(f)\iota_T(x)$ for $f \in \mathbf{A}_F$ and $\iota_T(\lambda x) = (1 \otimes \lambda)\iota_T(x)$; hence $\iota_T(gx) = \iota_L(g)\iota_T(x)$ for all $g \in \mathbf{A}_{F,L}$, $x \in \mathbf{N}$. Let κ be such that $\mu_\varepsilon^\kappa \iota_T(x_i) \in A_{\mathrm{inf}} \otimes T$ for a finite generating set (x_i) of $\mathbf{N}$ (it exists since $\iota_T(\mathbf{N}) \subseteq A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes T$ by (Ex2)).

(Gc1). With κ as above, semilinearity gives $\mu_\varepsilon^\kappa \iota_T(\mathbf{N}) = \sum_i \iota_L(\mathbf{A}_{F,L})\mu_\varepsilon^\kappa \iota_T(x_i) \subseteq A_{\mathrm{inf}} \otimes T$, and $\iota_L((\ell, X)^k) \subseteq (\ell, \mu_\varepsilon)^k R$ gives the second claim.

(Gc2): the sites. Put $\epsilon_a = \iota_L(e_a)$, orthogonal idempotents of R with sum 1, and

$$\iota_a \colon \mathcal{O}_L[[X]] \cong e_a \mathbf{A}_{F,L} \to \epsilon_a R$$

the restriction of ι_L. Let $n_{a,1}, \ldots, n_{a,d}$ be an $e_a\mathbf{A}_{F,L}$-basis of $\mathbf{N}_a$ ((Wm1) with $r = d$ by (Ex1)). Multiplying $A_{\mathrm{inf}}[1/\mu_\varepsilon] \cdot \iota_T(\mathbf{N}) = A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes T$ (Ex2) by ϵ_a and using $\epsilon_a \iota_T(\mathbf{N}) = \iota_T(\mathbf{N}_a)$ gives $\epsilon_a R[1/\mu_\varepsilon]^d = \sum_i \epsilon_a R[1/\mu_\varepsilon]\,\iota_T(n_{a,i})$. A surjective endomorphism of a finitely generated module over a commutative ring is bijective, so the matrix A_a with columns $\iota_T(n_{a,i})$ lies in $\mathrm{GL}_d(\epsilon_a R[1/\mu_\varepsilon])$; fix κ' with $\mu_\varepsilon^{\kappa'} A_a^{-1} \in M_d(\epsilon_a R)$ for all a. If $x = \sum_{a,i} c_{a,i} n_{a,i}$ and $\mu_\varepsilon^\kappa \iota_T(x) \in (\ell, \mu_\varepsilon)^j R^d$, then $\epsilon_a \mu_\varepsilon^\kappa \iota_T(x) = A_a\,(\iota_a(X^\kappa c_{a,i}))_i$, hence

$$\iota_a\big(X^{\kappa+\kappa'} c_{a,i}\big) \in \epsilon_a(\ell, \mu_\varepsilon)^j R \subseteq \epsilon_a(\varpi, \mu_\varepsilon)^j R \qquad \text{for all } a, i.$$

(Gc2): strictness of ι_a. Claim: if $g \in \mathcal{O}_L[[X]]$ and $\iota_a(g) \in (\varpi, \mu_\varepsilon)^j R$, then $g \in (\varpi, X)^j$. Modulo ϖ, $R/\varpi R = A_{\mathrm{inf}}/\ell \otimes_{\mathbb{F}_\ell} k_L = \mathcal{O}_{\mathbb{C}_\ell^\flat} \otimes_{\mathbb{F}_\ell} k_L \cong \prod_\rho \mathcal{O}_{\mathbb{C}_\ell^\flat}$, the product over the embeddings $\rho \colon k_L \to \mathcal{O}_{\mathbb{C}_\ell^\flat}$ ($k_L/\mathbb{F}_\ell$ is separable and the residue field of $\mathcal{O}_{\mathbb{C}_\ell^\flat}$ is algebraically closed). The idempotent ϵ_a reduces to the indicator of a set S_a of factors, non-empty ($\epsilon_a \neq 0$ since $\iota_T(n_{a,1}) \neq 0$,

and an idempotent in ϖR is 0 because R is ϖ-adically separated). In a factor $\rho \in S_a$, $\iota_a(\bar{g})$ for $\bar{g} = \sum_{k \geq m} a_k X^k$, $a_m \neq 0$, is $(\varepsilon-1)^m(\rho(a_m)+(\varepsilon-1)(\cdots))$, of valuation exactly $m\ell/(\ell-1)$, while the reduction of $(\varpi, \mu_\varepsilon)^j R$ is $(\varepsilon-1)^j(R/\varpi R)$, of valuation $\geq j\ell/(\ell-1)$ in every factor. So $m \geq j$, i.e. $g = X^j g_0 + \varpi g_1$ with $g_0, g_1 \in \mathcal{O}_L[[X]]$, and $\iota_a(\varpi g_1) \in (\varpi, \mu_\varepsilon)^j R$. If $\varpi w = \sum_{i=0}^{j} \varpi^i \mu_\varepsilon^{j-i} r_i$ in R, then $\mu_\varepsilon^j r_0 \in \varpi R$; as $\varepsilon - 1$ is a non-zero-divisor of $R/\varpi R$, $r_0 \in \varpi R$, and since R is ϖ-torsion-free, $w \in (\varpi, \mu_\varepsilon)^{j-1} R$. So $\iota_a(g_1) \in (\varpi, \mu_\varepsilon)^{j-1} R$ and the claim follows by induction on j.

(Gc2): conclusion. The ring $\mathcal{O}_L[[X]]$ is regular local with maximal ideal (ϖ, X), so its associated graded ring is a polynomial ring over k_L in two variables, a domain, and the (ϖ, X)-adic order is additive. From the claim, $X^{\kappa+\kappa'} c_{a,i} \in (\varpi, X)^j$, hence $c_{a,i} \in (\varpi, X)^{j-\kappa-\kappa'}$. Finally $(\varpi, X)^{ek} \subseteq (\ell, X)^k$: a monomial $\varpi^i X^{ek-i}$ with $i = eq + r$, $0 \leq r < e$, equals a unit times $\ell^q \varpi^r X^{ek-i}$, and $q + (ek - i) \geq k$ since $(e-1)(k-q) \geq r$ when $q < k$ (and $r = 0$ when $q = k$). So $j = ek + \kappa + \kappa'$ works.

(Gc3). Write $y = x/f$ with $x \in \mathbf{N}$ and f a product of Q_s's. For $u \in 1 + \ell^s \mathbb{Z}_\ell$, $\gamma_u(X) - X = (1+X)((1+X)^{u-1}-1) \in (\ell, X)^{s+1}\mathbb{Z}_\ell[[X]]$, hence $\gamma_u(g) - g \in (\ell, X)^{s+1}$ for $g \in \mathbb{Z}_\ell[[X]]$; since each Q_t has (ℓ, X)-order 1 ($Q_t(0) = \ell$) and the order is additive on $\mathbb{Z}_\ell[[X]]$, the unit $v_u = f/\gamma_u(f)$ satisfies $v_u - 1 = (f - \gamma_u(f))/\gamma_u(f) \in (\ell, X)^{s+1-\deg}$ with deg the number of factors of f (in the regular local ring $\mathbb{Z}_\ell[[X]]$ membership in the r-th power of the maximal ideal is order $\geq r$). Now $f\,(\gamma_u - 1)(y) = v_u \gamma_u(x) - x = v_u(\gamma_u - 1)(x) + (v_u - 1)x$. Let $\sigma \in G_F$ with $\chi_{\mathrm{cyc}}(\sigma) = u$ ($\chi_{\mathrm{cyc}} \colon G_F \to \mathbb{Z}_\ell^\times$ is onto, F being unramified). With $w = \mu_\varepsilon^\kappa \iota_T(x) \in A_{\mathrm{inf}} \otimes T$ and $\sigma(\mu_\varepsilon) = \mu_\varepsilon \beta_\sigma$, $\beta_\sigma = \sum_{k \geq 1} \binom{u}{k} \mu_\varepsilon^{k-1} \in A_{\mathrm{inf}}^\times$,

$$\mu_\varepsilon^\kappa \iota_T((\gamma_u - 1)(x)) = \mu_\varepsilon^\kappa (\sigma - 1)\iota_T(x) = \beta_\sigma^{-\kappa}(\sigma(w) - w) + (\beta_\sigma^{-\kappa} - 1)w.$$

Given J, continuity of G_F on A_{inf} (Ai2) and on T gives an open subgroup $U \subseteq G_F$ with $\sigma(w) - w \in (\ell, \mu_\varepsilon)^J (A_{\mathrm{inf}} \otimes T)$ for $\sigma \in U$, and $\beta_\sigma - 1 = (u-1) + \sum_{k \geq 2} \binom{u}{k} \mu_\varepsilon^{k-1} \in (\ell, \mu_\varepsilon)^J$ once u is close enough to 1 ($\binom{u}{k} \to \binom{1}{k} = 0$ for each $k \geq 2$). Since $\chi_{\mathrm{cyc}}(U)$ is open, there is s such that every $u \in 1 + \ell^s \mathbb{Z}_\ell$ is $\chi_{\mathrm{cyc}}(\sigma)$ for such a σ; then (Gc2) gives $(\gamma_u - 1)(x) \in (\ell, X)^{k'} \mathbf{N}$, with k' as large as we like for s large. Together with $v_u - 1 \in (\ell, X)^{s+1-\deg}$, $f\,(\gamma_u - 1)(y) \in (\ell, X)^{k''} \mathbf{N}$ with $k'' \to \infty$ as $s \to \infty$. As $(\gamma_u - 1)(y) \in \mathbf{N}$ by hypothesis, and in coordinates of an $\mathcal{O}_L[[X]]$-basis of each $\mathbf{N}_a$ the (ϖ, X)-order is additive, $(\gamma_u - 1)(y) \in (\varpi, X)^{k''-\deg} \mathbf{N}$; with $(\varpi, X)^{ek} \subseteq (\ell, X)^k$ this gives (Gc3).

Over a base field. For a pair over K the proof is the same, with $\sigma \in G_K$, $u = \chi_{\mathrm{cyc}}(\sigma) \in \Gamma_K$, (Ex1)–(Ex2) of Proposition 23.7 over K, and Lemma 23.8(4), which holds for all $\sigma \in G_F \supseteq G_K$. The one property of F used is in (Gc3): that every $u \in 1 + \ell^s \mathbb{Z}_\ell$, s large, is $\chi_{\mathrm{cyc}}(\sigma)$ for some σ in a given open subgroup U. Over K: $\chi_{\mathrm{cyc}} \colon G_K \to \Gamma_K$ is a continuous surjection of profinite groups, hence open, so $\chi_{\mathrm{cyc}}(U)$ is open in Γ_K; and Γ_K is open in $\mathbb{Z}_\ell^\times$, being closed of finite index ($[\Gamma_F : \Gamma_K] \leq [G_F : G_K] = [K : F]$). So $\chi_{\mathrm{cyc}}(U) \supseteq 1 + \ell^s \mathbb{Z}_\ell$ for s large, and the argument is unchanged.

Level s. Fix the level s (written s_0 in this paragraph, to keep s for the exponent in (Gc3)). The proof above, over K, applies with the following replacements; we list every property of ι and μ_ε it uses. (1) Topology: modulo ℓ, μ_{ε,s_0} reduces to $\varepsilon^{1/\ell^{s_0-1}} - 1$, of valuation $c = \ell/((\ell-1)\ell^{s_0-1}) > 0$, so $(\ell, \mu_{\varepsilon,s_0})^{\lceil 1/c \rceil} \subseteq (\ell, [p^\flat])$ and $[p^\flat]^{\lceil c \rceil} \in (\ell, \mu_{\varepsilon,s_0})$: the $(\ell, \mu_{\varepsilon,s_0})$-adic topology is again that of (Ai2). Put $\iota_L(\sum c_k X^k) = \sum (\iota_{s_0} \otimes \mathrm{id})(c_k) \mu_{\varepsilon,s_0}^k$. (2) (Gc0): $\iota_{s_0}(Q_j) = \Phi_{\ell^{j+1}}([\varepsilon^{1/\ell^{s_0-1}}])$ reduces modulo ℓ to $(\varepsilon^{1/\ell^{s_0-1}} - 1)^{\ell^j(\ell-1)} \neq 0$, so it is a unit of $W(\mathbb{C}_\ell^\flat)$ for every $j \geq 0$; $\gamma_u(Q_j) \in Q_j \mathbb{Z}_\ell[[X]]^\times$ (Lemma 23.8(2)); $\sigma \circ \iota_{s_0} = \iota_{s_0} \circ \gamma_{\chi_{\mathrm{cyc}}(\sigma)}$ and $\varphi \circ \iota_{s_0} = \iota_{s_0} \circ \varphi$ (Lemma 23.8(6)); and (Ex1), (Ex2) of level s_0 (Proposition 23.7(Ex4)). (3) κ exists because $\iota_{s_0,T}(\mathbf{N}) \subseteq A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes T$ ((Ex2) of level s_0); (Gc1) follows from $\iota_L((\ell, X)^k) \subseteq (\ell, \mu_{\varepsilon,s_0})^k R$. (4) (Gc2): the matrices A_a lie in $\mathrm{GL}_d(\epsilon_a R[1/\mu_\varepsilon])$ by the spanning clause of (Ex2), and, since $\mu_\varepsilon = \iota_{s_0}(X_{s_0-1})$ with $X_{s_0-1} =$

$(1+X)^{\ell^{s_0-1}}-1$, the displayed relation reads $\iota_a(X_{s_0-1}^{\kappa+\kappa'}c_{a,i}) \in \epsilon_a(\varpi,\mu_{\varepsilon,s_0})^j R$. The strictness claim holds with μ_{ε,s_0} in place of μ_ε and the valuation c in place of $\ell/(\ell-1)$: its proof uses only that the reduction $\varepsilon^{1/\ell^{s_0-1}}-1$ has positive valuation and is a non-zero-divisor of $R/\varpi R$. Hence $X_{s_0-1}^{\kappa+\kappa'}c_{a,i} \in (\varpi,X)^j$; as X_{s_0-1} has the monomial $X^{\ell^{s_0-1}}$ with coefficient 1, its (ϖ,X)-order is at most ℓ^{s_0-1}, so $c_{a,i} \in (\varpi,X)^{j-\ell^{s_0-1}(\kappa+\kappa')}$, and $j = ek+\ell^{s_0-1}(\kappa+\kappa')$ works. (5) (Gc3): unchanged (it uses $w = \mu_\varepsilon^\kappa \iota_{s_0,T}(x)$, $\sigma(\mu_\varepsilon) = \mu_\varepsilon\beta_\sigma$, continuity, (Gc2) of level s_0, and the openness of $\chi_{\mathrm{cyc}}(U)$ in Γ_K, which holds for every finite K). □

Lemma 23.30. *Let $\mathbf{N}$ be a Wach module over F with coefficients L and window $(M_a,h_a)_{a\in\mathbb{Z}/n}$, and $m \geq 1+\max_a M_a$ an integer. Then:*

1. *$\varphi_{\mathbf{N}}(X^m\mathbf{N}) \subseteq X^m Q\,\mathbf{N}$, and $\varphi_{\mathbf{N}}^k(X^m\mathbf{N}) \subseteq X^m\lambda_k\mathbf{N}$ for every $k \geq 0$;*
2. *$1-\varphi_{\mathbf{N}}$ is a bijection of $X^m\mathbf{N}$ onto itself, with inverse $w \mapsto \sum_{k\geq 0}\varphi_{\mathbf{N}}^k(w)$ (convergent for the (ℓ,X)-adic topology);*
3. *let $u \in \mathbb{Z}_\ell^\times$ and $\gamma = \gamma_{u,\mathbf{N}}$. For every $y \in \mathbf{N}$ with $(\gamma-1)y \in X^m\mathbf{N}$ there is a unique $z \in X^m\mathbf{N}$ with $(\varphi_{\mathbf{N}}-1)z = (\gamma-1)y$, namely $z_y = -\sum_{k\geq 0}\varphi_{\mathbf{N}}^k((\gamma-1)y)$; $y \mapsto z_y$ is additive; if $y \in X^m\mathbf{N}$, then $w = -\sum_{k\geq 0}\varphi_{\mathbf{N}}^k(y) \in X^m\mathbf{N}$ satisfies $(\varphi_{\mathbf{N}}-1)w = y$ and $(\gamma-1)w = z_y$;*
4. *if $\mathbf{N}'$ is another Wach module over F with coefficients L whose window (M'_a,h'_a) also satisfies $m \geq 1+\max_a M'_a$, and $g\colon \mathbf{N} \to \mathbf{N}'$ is $\mathbf{A}_{F,L}$-linear and commutes with φ and γ_u, then $g(z_y) = z_{g(y)}$ for every y as in item 3.*

Over a base field. *Items 1–4 hold for Wach modules over a base field K as in Definition 23.2, with $u \in \Gamma_K$ in items 3 and 4; for $K = F$ this is the statement above.*

Level s. *Let ℓ be any prime, $s \geq 1$ with $s \geq 2$ if $\ell = 2$, K with $F \subseteq K \subseteq F(\mu_{\ell^s})$, $\mathbf{N}$ (and $\mathbf{N}'$ in item 4) a Wach module of level s over K with coefficients L (Definition 23.2, paragraph Level s, with the level variable X_{s-1} there), and $m \geq 1+\max_a M_a$. Then items 1–4 hold with X, Q, λ_k replaced by X_{s-1}, Q_{s-1}, $\varphi^{s-1}(\lambda_k) = \prod_{j=0}^{k-1} Q_{s-1+j}$ and $u \in \Gamma_K$: in particular $\varphi_{\mathbf{N}}(X_{s-1}^m\mathbf{N}) \subseteq X_{s-1}^m Q_{s-1}\mathbf{N}$, $1-\varphi_{\mathbf{N}}$ is bijective on $X_{s-1}^m\mathbf{N}$ with inverse $\sum_{k\geq 0}\varphi_{\mathbf{N}}^k$, and for $y \in \mathbf{N}$ with $(\gamma_{u,\mathbf{N}}-1)y \in X_{s-1}^m\mathbf{N}$ there is a unique $z_y \in X_{s-1}^m\mathbf{N}$ with $(\varphi_{\mathbf{N}}-1)z_y = (\gamma_{u,\mathbf{N}}-1)y$. For $s = 1$ and ℓ odd this is the statement over a base field.*

Proof. **Item 1.** Let $x \in \mathbf{N}_{a-1}$. Since $\varphi(X) = XQ$,

$$\varphi_{\mathbf{N}}(X^m x) = X^m Q^m \varphi_{\mathbf{N}}(x) = X^m Q^{m-M_a}\big(Q^{M_a}\varphi_{\mathbf{N}}(x)\big) \in X^m Q^{m-M_a}\mathbf{N}_a \subseteq X^m Q\mathbf{N}$$

by (Wm2) and $m - M_a \geq 1$; by additivity and $\mathbf{N} = \bigoplus_a \mathbf{N}_a$ the first claim follows. By induction on k, using $\varphi_{\mathbf{N}}(fy) = \varphi(f)\varphi_{\mathbf{N}}(y)$ and $\varphi(\lambda_k)Q = \lambda_{k+1}$: if $\varphi_{\mathbf{N}}^k(X^m\mathbf{N}) \subseteq \lambda_k X^m\mathbf{N}$ then $\varphi_{\mathbf{N}}^{k+1}(X^m\mathbf{N}) \subseteq \varphi(\lambda_k)\varphi_{\mathbf{N}}(X^m\mathbf{N}) \subseteq \varphi(\lambda_k)X^m Q\mathbf{N} = \lambda_{k+1}X^m\mathbf{N}$.

Item 2. Let $\mathfrak{m} = (\ell,X) \subseteq \mathbf{A}_{F,L}$. Each Q_s lies in $\mathfrak{m}$ ($Q_s(0) = \ell$), so $\lambda_k \in \mathfrak{m}^k$. The ring $\mathbf{A}_{F,L} = \prod_a \mathcal{O}_L[[X]]$ is noetherian and $\mathfrak{m}$-adically complete, and $\mathfrak{m}$ lies in its Jacobson radical; so the finitely generated module $X^m\mathbf{N}$ is $\mathfrak{m}$-adically complete and separated. By item 1, $\varphi_{\mathbf{N}}$ maps $X^m\mathbf{N}$ into itself and $\mathfrak{m}^j X^m\mathbf{N}$ into $\mathfrak{m}^j X^m\mathbf{N}$ ($\varphi(\mathfrak{m}) \subseteq \mathfrak{m}$), so it is continuous there, and $\varphi_{\mathbf{N}}^k(w) \in \mathfrak{m}^k X^m\mathbf{N}$ for $w \in X^m\mathbf{N}$. Hence $s(w) = \sum_{k\geq 0}\varphi_{\mathbf{N}}^k(w)$ converges in $X^m\mathbf{N}$, and $(1-\varphi_{\mathbf{N}})s(w) = w$ by continuity and telescoping. If $w \in X^m\mathbf{N}$ and $\varphi_{\mathbf{N}}(w) = w$, then $w = \varphi_{\mathbf{N}}^k(w) \in \mathfrak{m}^k X^m\mathbf{N}$ for all k, so $w = 0$. Thus $1-\varphi_{\mathbf{N}}$ is bijective on $X^m\mathbf{N}$ with inverse s.

Item 3. By item 2 the equation $(\varphi_{\mathbf{N}}-1)z = (\gamma-1)y$ has the unique solution $z_y = -s((\gamma-1)y)$ in $X^m\mathbf{N}$, and $y \mapsto z_y$ is additive since γ and s are. Let $y \in X^m\mathbf{N}$ and $w = -s(y)$,

so $(\varphi_{\mathbf{N}}-1)w=y$. Since $\gamma_u(X)\in X\mathbf{A}_F$, γ preserves $X^m\mathbf{N}$, so $(\gamma-1)w\in X^m\mathbf{N}$; and γ commutes with $\varphi_{\mathbf{N}}$ (Wm3), so $(\varphi_{\mathbf{N}}-1)(\gamma-1)w=(\gamma-1)(\varphi_{\mathbf{N}}-1)w=(\gamma-1)y$. By uniqueness, $(\gamma-1)w=z_y$.

Item 4. g maps $X^m\mathbf{N}$ into $X^m\mathbf{N}'$, and $(\varphi-1)g(z_y)=g((\varphi-1)z_y)=g((\gamma-1)y)=(\gamma-1)g(y)$, which lies in $X^m\mathbf{N}'$; by uniqueness in $\mathbf{N}'$, $g(z_y)=z_{g(y)}$.

Over a base field. For Wach modules over K the proof is unchanged: items 1 and 2 use only (Wm1), (Wm2), $\varphi(X)=XQ$ and the ring $\mathbf{A}_{F,L}$, none of which involves the base field; items 3 and 4 use, for the given $u\in\Gamma_K$, only that $\gamma_u(X)\in X\mathbf{A}_F$ (true for every $u\in\mathbb{Z}_\ell^\times$) and that $\gamma_{u,\mathbf{N}}$ commutes with $\varphi_{\mathbf{N}}$, which (Wm3) over K gives for $u\in\Gamma_K$.

Level s. The proof above applies with X, Q, λ_k replaced by X_{s-1}, Q_{s-1}, $\varphi^{s-1}(\lambda_k)$; we check the four facts it uses. (i) Applying the ring endomorphism φ^{s-1} to $\varphi(X)=XQ$ gives $\varphi(X_{s-1})=X_{s-1}Q_{s-1}$; with (Wm2) of level s, $\varphi_{\mathbf{N}}(X_{s-1}^m x)=X_{s-1}^m Q_{s-1}^{m-M_a}(Q_{s-1}^{M_a}\varphi_{\mathbf{N}}(x))\in X_{s-1}^m Q_{s-1}\mathbf{N}$ for $x\in\mathbf{N}_{a-1}$, and $\varphi(\varphi^{s-1}(\lambda_k))Q_{s-1}=\varphi^{s-1}(\lambda_{k+1})$ gives the induction of item 1. (ii) Each Q_{s-1+j} lies in $\mathfrak{m}=(\ell,X)$ ($\varphi(f)(0)=\varphi(f(0))$ since $\varphi(X)(0)=0$, and $Q(0)=\ell$, so $Q_{s-1+j}(0)=\ell$), so $\varphi^{s-1}(\lambda_k)\in\mathfrak{m}^k$ and item 2 follows as before. (iii) $\gamma_u(X_{s-1})=(1+X_{s-1})^u-1=\sum_{k\ge1}\binom{u}{k}X_{s-1}^k\in X_{s-1}\mathbf{A}_F$ for every $u\in\mathbb{Z}_\ell^\times$, so $\gamma_{u,\mathbf{N}}$ preserves $X_{s-1}^m\mathbf{N}$. (iv) $\gamma_{u,\mathbf{N}}$ commutes with $\varphi_{\mathbf{N}}$ on $\mathbf{N}[1/Q_{s-1}]$ for $u\in\Gamma_K$ ((Wm3) of level s). Items 3 and 4 then follow as before. (Clause (Wm3') is not used.) □

Proposition 23.31. *Let ℓ, F, L, (τ_a) and a pair (V,T) be as in Proposition 23.7, $\mathbf{N}=\mathbf{N}(T)$, $(M_a,h_a)_a$ a window of V as in (Ex1), $m\ge1+\max_a M_a$, and let $\rho\colon A_{\mathrm{inf}}[1/\mu_\varepsilon]\to B_{\mathrm{cris}}$ be the G_F- and φ-equivariant ring homomorphism extending $A_{\mathrm{inf}}\subset A_{\mathrm{cris}}\subset B_{\mathrm{cris}}$ (μ_ε divides t in A_{cris}, so it is a unit of B_{cris}). Let $y\in\mathbf{N}[1/\lambda_\infty]$ be a Herr vector of precision m such that $fy\in\mathbf{N}$ for some finite product f of the Q_s, $s\ge0$. Then there is $\beta\in B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V$ such that*

$$(\sigma-1)(\beta)=(\rho\otimes1)\big(\iota_T(z_{\chi_{\mathrm{cyc}}(\sigma)}(y))\big)\qquad\text{for every }\sigma\in G_F.$$

Over a base field. *Let K be a field with $F\subseteq K\subseteq F(\mu_\ell)$ and Γ_K as in Definition 23.2. For a pair (V,T) over K as in Proposition 23.7, a window as in (Ex1) over K, $m\ge1+\max_a M_a$, and a Herr vector y of precision m over K (Lemma 23.28, over a base field: $u\in\Gamma_K$) with $fy\in\mathbf{N}$ as above, there is $\beta\in B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V$ with $(\sigma-1)(\beta)=(\rho\otimes1)(\iota_T(z_{\chi_{\mathrm{cyc}}(\sigma)}(y)))$ for every $\sigma\in G_K$. For $K=F$ this is the statement above.* ***Level s.*** *Let ℓ be any prime, $s\ge1$ an integer with $s\ge2$ if $\ell=2$, L, K ($F\subseteq K\subseteq F(\mu_{\ell^s})$) and a pair (V,T) over K as in (Ex4) of Proposition 23.7, $\mathbf{N}=\mathbf{N}_s(T)$ with $\iota_{s,T}$, $(M_a,h_a)_a$ a window of V as in (Ex1) of level s, $m\ge1+\max_a M_a$, and ρ as above (it exists for every prime: μ_ε is a unit of B_{cris} by Lemma 23.12, paragraph Every prime). Let $y\in\mathbf{N}[1/\lambda_\infty]$ be a Herr vector of precision m at level s (Lemma 23.28, paragraph Level s) such that $fy\in\mathbf{N}$ for some finite product f of the Q_j with $j\ge s-1$. Then there is $\beta\in B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V$ with $(\sigma-1)(\beta)=(\rho\otimes1)(\iota_{s,T}(z_{\chi_{\mathrm{cyc}}(\sigma)}(y)))$ for every $\sigma\in G_K$. For $s=1$ and ℓ odd this is the paragraph over a base field.*

Proof. Write $\mu=\mu_\varepsilon$, $z_u=z_u(y)$, and φ for $\varphi\otimes1$ on $W(\mathbb{C}_\ell^\flat)\otimes_{\mathbb{Z}_\ell}T$ and on $B_{\mathrm{cris}}\otimes_{\mathbb{Z}_\ell}T=B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V$; G_F acts diagonally on both and commutes with φ. The plan is to trivialise the Γ-cocycle $u\mapsto z_u$ in power series of geometric growth, evaluate at $\varphi^3(\mu)\in\ell^3A_{\mathrm{cris}}$, where such series converge, and correct the Frobenius twist so introduced by $\sum_{j<3}\varphi^j(\iota_T(y))$.

Step 1: the cocycle relation. For $u,v\in\mathbb{Z}_\ell^\times$, $z_u+\gamma_u(z_v)\in X^m\mathbf{N}$ ($\gamma_u(X)\in X\mathbf{A}_F$) and, as γ_u commutes with φ and $\gamma_u\gamma_v=\gamma_{uv}$ ((Wm3), extended to $\mathbf{N}[1/\lambda_\infty]$ by (Gc0)),

$$(\varphi-1)\big(z_u+\gamma_u(z_v)\big)=(\gamma_u-1)(y)+\gamma_u(\gamma_v-1)(y)=(\gamma_{uv}-1)(y);$$

by the uniqueness in Lemma 23.30(2), $z_{uv}=z_u+\gamma_u(z_v)$.

Step 2: the trivialisation. By (Ex1), $\mathbf{N}$ is a Wach module; fix an $\mathbf{A}_{F,L}$-basis (n_i) of it. By Step 1 and Lemma 23.9(Hf1) there is $W \in X^m(\mathbf{N}\otimes_{\mathbf{A}_{F,L}} E[[X]])$ with $(\gamma_u-1)(W)=z_u$ for every $u\in\mathbb{Z}_\ell^\times$, and by (Hf3) its coordinates $\sum_k a_kX^k$ satisfy $\ell^{s_\ell(k)}a_k\in\Lambda$ with $s_\ell(k)\le k\ell/(\ell-1)^2\le 2k$. So the coordinates lie in R, and

$$W\in\mathbf{N}_R,\qquad (\gamma_u-1)(W)=z_u\quad\text{for every } u\in\mathbb{Z}_\ell^\times. \tag{23.2}$$

Step 3: the poles of $\iota_T(y)$. Write $f=\prod_s Q_s^{k_s}$. By (Gc0), $\iota_T(y)=\iota(f)^{-1}\iota_T(fy)$, where $\iota(f)=\prod_s\varphi^s(\tilde\xi)^{k_s}=\prod_s(\varphi^{s+1}(\mu)/\varphi^s(\mu))^{k_s}$ (Lemma 23.8(4) and $\varphi(\mu)=\mu\tilde\xi$) and $\iota_T(fy)\in\mu^{-\kappa}(A_{\inf}\otimes T)$ (Gc1). Hence $\iota_T(y)\in S\otimes_{\mathbb{Z}_\ell}T$ with S as in (Ac2), and so is each $\varphi^j(\iota_T(y))$ (S is φ-stable). The map $\rho_S\otimes 1\colon S\otimes T\to B_{\mathrm{cris}}\otimes T$ commutes with φ and G_F (Ac2) and restricts to $\rho\otimes1$ on $A_{\inf}[1/\mu]\otimes T$: $\rho_S|_{A_{\inf}[1/\mu]}$ and ρ both extend the inclusion of $A_{\inf}$, and $A_{\inf}[1/\mu]$ is a localisation of $A_{\inf}$.

Step 4: the element β. Define

$$\beta=\mathrm{Ev}(W)-\sum_{j=0}^{2}\varphi^j\big((\rho_S\otimes1)(\iota_T(y))\big)\in B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V,$$

and let $\sigma\in G_F$, $u=\chi_{\mathrm{cyc}}(\sigma)$. By (Ev3) and (23.2), $(\sigma-1)\mathrm{Ev}(W)=\mathrm{Ev}((\gamma_u-1)W)=\mathrm{Ev}(z_u)=\varphi^3(\rho\otimes1)(\iota_T(z_u))$, as $z_u\in\mathbf{N}$. By (Gc0) and the defining equation of z_u, $(\sigma-1)\iota_T(y)=\iota_T((\gamma_u-1)(y))=\iota_T((\varphi-1)(z_u))=(\varphi-1)\iota_T(z_u)$, and $(\rho_S\otimes1)(\iota_T(z_u))=(\rho\otimes1)(\iota_T(z_u))$ since $z_u\in\mathbf{N}$; so, as σ commutes with φ and with ρ_S,

$$(\sigma-1)\sum_{j=0}^{2}\varphi^j(\rho_S\otimes1)(\iota_T(y))=\sum_{j=0}^{2}\varphi^j(\varphi-1)(\rho\otimes1)(\iota_T(z_u))=\varphi^3(\rho\otimes1)(\iota_T(z_u))-(\rho\otimes1)(\iota_T(z_u)).$$

Subtracting, $(\sigma-1)(\beta)=(\rho\otimes1)\big(\iota_T(z_{\chi_{\mathrm{cyc}}(\sigma)}(y))\big)$ for every $\sigma\in G_F$.

Over a base field. For a pair over K the four steps run with $u,v\in\Gamma_K$ and $\sigma\in G_K$, each input taken in its form over K. Step 1: (Wm3) for $u,v\in\Gamma_K$, (Gc0) over K (Lemma 23.29) and the uniqueness of Lemma 23.30(2) over K give $z_{uv}=z_u+\gamma_u(z_v)$. Step 2: (Hf1), (Hf3) over K (Lemma 23.9, over a base field) give $W\in X^m\mathbf{N}_E$ with $(\gamma_u-1)(W)=z_u$ for every $u\in\Gamma_K$ and coordinates with $\ell^{s_{\ell,d}(k)}a_k\in\Lambda$, $s_{\ell,d}(k)\le 3k/2\le 2k$ ($d=|\Delta_K|$), so $W\in\mathbf{N}_R$. Step 3: (Gc0), (Gc1) over K; S and ρ_S are G_F-, hence G_K-equivariant. Step 4: (Ev3) over K (Lemma 23.11, over a base field) for $\sigma\in G_K$, $u=\chi_{\mathrm{cyc}}(\sigma)\in\Gamma_K$; the computation is the same. No step uses that the base is unramified beyond the denominators of Step 2, which stay below $2k$.

Level s. Let the data be as in the paragraph Level s of the statement; $u,v\in\Gamma_K$, $\sigma\in G_K$, $z_u=z_u(y)\in X_{s-1}^m\mathbf{N}$, $X_{s-1}=\varphi^{s-1}(X)$. Step 1: $\gamma_u(X_{s-1})=(1+X_{s-1})^u-1\in X_{s-1}\mathbf{A}_F$, so $z_u+\gamma_u(z_v)\in X_{s-1}^m\mathbf{N}$; by (Wm3) of level s and (Gc0) of level s (Lemma 23.29, paragraph Level s) the displayed computation of Step 1 holds, and the uniqueness of Lemma 23.30, paragraph Level s, gives $z_{uv}=z_u+\gamma_u(z_v)$. Step 2: $\mathbf{N}$ is a Wach module of level s over K ((Ex1) of level s), and $z_u\in X_{s-1}^m\mathbf{N}\subseteq X^m\mathbf{N}$ since $X_{s-1}=X\lambda_{s-1}$. By (Hf1), (Hf3) of Lemma 23.9, paragraph Level s, there is $W\in X^m\mathbf{N}_E$ with $(\gamma_u-1)(W)=z_u$ for every $u\in\Gamma_K$, whose coordinates satisfy $\ell^{\nu_{\ell,s}(k)}a_k\in\Lambda$ with $\nu_{\ell,s}(k)\le(s+1)k$; so $W\in\mathbf{N}_{R_s}$ (Lemma 23.11, paragraph Level s). Step 3: write $f=\prod_{k\ge0}Q_{s-1+k}^{c_k}$. By Lemma 23.8(6), ι_s commutes with the Frobenius lifts and $\iota_s(Q_{s-1})=\tilde\xi=\varphi(\mu)/\mu$, so $\iota_s(Q_{s-1+k})=\varphi^k(\tilde\xi)=\varphi^{k+1}(\mu)/\varphi^k(\mu)$ and $\iota_s(f)^{-1}\in S$, S as in (Ac2) of Lemma 23.12 (paragraph Every prime). (This is where $j\ge s-1$ is used: for $j<s-1$, $\iota_s(Q_j)=\varphi^{j-s+1}(\tilde\xi)$ is a negative Frobenius twist of $\tilde\xi$, which (Ac2) does not invert.) By (Gc0) and (Gc1) of level s, $\iota_{s,T}(y)=\iota_s(f)^{-1}\iota_{s,T}(fy)$ with $\iota_{s,T}(fy)\in\mu^{-\kappa}(A_{\inf}\otimes T)$ (the pole factor

of (Gc1) is μ_ε^κ at every level); so $\iota_{s,T}(y)$ and all $\varphi^j(\iota_{s,T}(y))$ lie in $S\otimes T$, and $\rho_S\otimes 1$ restricts to $\rho\otimes 1$ on $A_{\mathrm{inf}}[1/\mu]\otimes T$ as before. Step 4: put

$$\beta=\mathrm{Ev}_s(W)-\sum_{j=0}^{2s}\varphi^j\big((\rho_S\otimes 1)(\iota_{s,T}(y))\big)\in B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V.$$

Let $\sigma\in G_K$, $u=\chi_{\mathrm{cyc}}(\sigma)$. By (Ev3) of level s and Step 2, $(\sigma-1)\mathrm{Ev}_s(W)=\mathrm{Ev}_s(z_u)=\varphi^{2s+1}(\rho\otimes 1)(\iota_{s,T}(z_u))$, as $z_u\in\mathbf{N}$. By (Gc0) of level s and the defining equation of z_u, $(\sigma-1)\iota_{s,T}(y)=\iota_{s,T}((\gamma_u-1)(y))=(\varphi-1)\iota_{s,T}(z_u)$, so

$$\begin{aligned}(\sigma-1)\sum_{j=0}^{2s}\varphi^j(\rho_S\otimes 1)(\iota_{s,T}(y))&=\sum_{j=0}^{2s}\varphi^j(\varphi-1)(\rho\otimes 1)(\iota_{s,T}(z_u))\\&=\varphi^{2s+1}(\rho\otimes 1)(\iota_{s,T}(z_u))-(\rho\otimes 1)(\iota_{s,T}(z_u)).\end{aligned}$$

Subtracting, $(\sigma-1)(\beta)=(\rho\otimes 1)(\iota_{s,T}(z_u))$ for every $\sigma\in G_K$. For $s=1$, ℓ odd, $2s+1=3$ and this is the proof over a base field with the weaker denominators $(s+1)k=2k$. □

Lemma 23.32. *Let ℓ, F, L, (τ_a) be as in Proposition 23.7, let (V',T') and (V,T) be pairs as there, $\mathcal{O}_L$ the lattice of the trivial one-dimensional representation, and*

$$0\to T'\xrightarrow{\iota'}T\xrightarrow{\pi}\mathcal{O}_L\to 0$$

an exact sequence of G_F-equivariant $\mathcal{O}_L$-linear maps. Fix $\tilde e\in T$ with $\pi(\tilde e)=1$ and let $[T]\in H^1(F,T')$ be the class of the cocycle $\sigma\mapsto\iota'^{-1}(\sigma\tilde e-\tilde e)$ (independent of $\tilde e$). Let $(M_a,h_a)_a$ be a window of V' as in (Ex1) and $m\ge 1+\max_a M_a$. Let $\tilde x\in\mathbf{N}(T)[1/\lambda_\infty]$ and $c\in\mathcal{O}_L$ satisfy

$$(1\otimes\pi)(\iota_T(\tilde x))=1\otimes c\quad\text{in } W(\mathbb{C}_\ell^\flat)\otimes_{\mathbb{Z}_\ell}\mathcal{O}_L,\qquad(\gamma_u-1)(\tilde x)\in\mathbf{N}(\iota')(X^m\mathbf{N}(T'))\quad\text{for all } u\in\mathbb{Z}_\ell^\times.$$

Then there is a unique $y\in\mathbf{N}(T')[1/\lambda_\infty]$ with $\mathbf{N}(\iota')(y)=(\varphi-1)(\tilde x)$; it is a Herr vector of precision m for $\mathbf{N}(T')$, with $\mathbf{N}(\iota')(z_u(y))=(\gamma_u-1)(\tilde x)$ for every u, and

$$\mathrm{cl}(y)=c\,[T]\qquad\text{in } H^1(F,T').$$

Over a base field. *Let K be a field with $F\subseteq K\subseteq F(\mu_\ell)$ and Γ_K as in Definition 23.2. The same holds for pairs over K (Proposition 23.7), G_K-equivariant maps, the hypothesis on $\tilde x$ for all $u\in\Gamma_K$, the class $[T]\in H^1(K,T')$ of the cocycle on G_K, and Herr vectors and classes over K (closing paragraph of Lemma 23.28). For $K=F$ this is the statement above.* ***Level s.*** *Let ℓ be any prime, $s\ge 1$ with $s\ge 2$ if $\ell=2$, and L, K ($F\subseteq K\subseteq F(\mu_{\ell^s})$) and pairs over K as in (Ex4) of Proposition 23.7. The same holds, in the form over a base field, with $\mathbf{N}$, ι_T, $\mathbf{N}(\iota')$ replaced by $\mathbf{N}_s$, $\iota_{s,T}$, $\mathbf{N}_s(\iota')$, the window of V' as in (Ex1) of level s, the hypothesis $(\gamma_u-1)(\tilde x)\in\mathbf{N}_s(\iota')(X_{s-1}^m\mathbf{N}_s(T'))$ for all $u\in\Gamma_K$ (X_{s-1} the level variable of Definition 23.2), and Herr vectors of precision m at level s and their classes (Lemma 23.28, Level s). For $s=1$ and ℓ odd this is the paragraph over a base field.*

Proof. **The element y.** Let $\pi_1=\iota_{\mathcal{O}_L}$ be the realisation of the trivial pair and $w=(\varphi-1)(\tilde x)\in\mathbf{N}(T)[1/\lambda_\infty]$ (by (Wm2), $Q^M\varphi(\mathbf{N}(T))\subseteq\mathbf{N}(T)$ for some M, and $\varphi(Q_s)=Q_{s+1}$). By (Ex3) and (Gc0),

$$\pi_1(\mathbf{N}(\pi)(w))=(1\otimes\pi)\iota_T(w)=(\varphi\otimes 1-1)\big((1\otimes\pi)\iota_T(\tilde x)\big)=(\varphi\otimes 1-1)(1\otimes c)=0,$$

and π_1 is injective on $\mathbf{N}(\mathcal{O}_L)[1/\lambda_\infty]$ by (Gc0); so $\mathbf{N}(\pi)(w)=0$. Choose a finite product f of the Q_s with $fw\in\mathbf{N}(T)$; then $\mathbf{N}(\pi)(fw)=0$, so $fw=\mathbf{N}(\iota')(x)$ with $x\in\mathbf{N}(T')$ by Lemma 23.33,

and $y = x/f$ satisfies $\mathbf{N}(\iota')(y) = w$. It is unique because $\mathbf{N}(\iota')$ is injective (Lemma 23.33) and remains so on $\mathbf{N}(T')[1/\lambda_\infty]$ (localisation is exact).

Precision. By hypothesis, $(\gamma_u - 1)(\tilde{x}) = \mathbf{N}(\iota')(z_u)$ with $z_u \in X^m\mathbf{N}(T')$, unique by injectivity. Since γ_u and φ commute (Wm3) and $\mathbf{N}(\iota')$ commutes with both (Ex3),

$$\mathbf{N}(\iota')\big((\gamma_u - 1)y\big) = (\gamma_u - 1)(\varphi - 1)\tilde{x} = (\varphi - 1)(\gamma_u - 1)\tilde{x} = \mathbf{N}(\iota')\big((\varphi - 1)z_u\big),$$

so $(\gamma_u - 1)y = (\varphi - 1)z_u$, which lies in $X^m\mathbf{N}(T')$ by Lemma 23.30(1). Hence y is a Herr vector of precision m, and by the uniqueness in Lemma 23.30(2), $z_u(y) = z_u$.

A solution b of the Frobenius equation. The sequence splits over $\mathcal{O}_L$ ($\mathcal{O}_L$ is free), so for $B = A_{\mathrm{inf}}[1/\mu_\varepsilon]$ and for $B = W(\mathbb{C}_\ell^\flat)$ the map $1 \otimes \iota' \colon B \otimes T' \to B \otimes T$ is injective with image $\ker(1 \otimes \pi)$, and it commutes with $\varphi \otimes 1$ and with the diagonal action of G_F. The element $\iota_T(\tilde{x}) - 1 \otimes c\tilde{e}$ lies in $\ker(1 \otimes \pi)$, since $(1 \otimes \pi)(1 \otimes c\tilde{e}) = 1 \otimes c$; let $b \in W(\mathbb{C}_\ell^\flat) \otimes T'$ be its preimage (here $\iota_T(\tilde{x}) \in W(\mathbb{C}_\ell^\flat) \otimes T$ by (Gc0)). As $1 \otimes c\tilde{e}$ is fixed by $\varphi \otimes 1$,

$$(1 \otimes \iota')\big((\varphi \otimes 1 - 1)b\big) = (\varphi \otimes 1 - 1)\iota_T(\tilde{x}) = \iota_T(w) = \iota_T\big(\mathbf{N}(\iota')(y)\big) = (1 \otimes \iota')\big(\iota_{T'}(y)\big),$$

using (Gc0) and (Ex3) (extended to poles by the formula of (Gc0)). Hence $(\varphi \otimes 1 - 1)b = \iota_{T'}(y)$.

The cocycle. By (Hc1) of Lemma 23.28, $\mathrm{cl}(y)$ is the class of $c_{y,b}(\sigma) = \iota_{T'}(z_{\chi_{\mathrm{cyc}}(\sigma)}(y)) - (\sigma - 1)b$. Applying $1 \otimes \iota'$ and using $\sigma(\iota_T(\tilde{x})) = \iota_T(\gamma_{\chi_{\mathrm{cyc}}(\sigma)}\tilde{x})$ (Gc0):

$$(1 \otimes \iota')\big(c_{y,b}(\sigma)\big) = \iota_T\big((\gamma_{\chi_{\mathrm{cyc}}(\sigma)} - 1)\tilde{x}\big) - (\sigma - 1)\iota_T(\tilde{x}) + (\sigma - 1)(1 \otimes c\tilde{e}) = 1 \otimes c(\sigma\tilde{e} - \tilde{e}).$$

Since $\sigma\tilde{e} - \tilde{e} \in \ker \pi = \iota'(T')$, this says $c_{y,b}(\sigma) = c\,\iota'^{-1}(\sigma\tilde{e} - \tilde{e})$ for every $\sigma \in G_F$. Therefore $\mathrm{cl}(y) = c\,[T]$. (Independence of $\tilde{e}$: replacing $\tilde{e}$ by $\tilde{e} + \iota'(t)$ changes the cocycle by the coboundary $\sigma \mapsto \sigma t - t$.)

Over a base field. For pairs over K the proof is unchanged, with $\sigma \in G_K$, $u \in \Gamma_K$, and Lemma 23.33, (Gc0) of Lemma 23.29 and (Hc1) of Lemma 23.28 in their versions over K (closing paragraphs).

Level s. The proof over K applies with the replacements of the statement: the element y uses (Wm2) of level s ($Q_{s-1}^M\varphi(\mathbf{N}_s(T)) \subseteq \mathbf{N}_s(T)$, and $\varphi(Q_j) = Q_{j+1}$), (Ex3) of level s, the injectivity of $\iota_{s,\mathcal{O}_L}$ on $\mathbf{N}_s(\mathcal{O}_L)[1/\lambda_\infty]$ ((Gc0) of Lemma 23.29, Level s) and Lemma 23.33, Level s; the precision step uses (Wm3) of level s and Lemma 23.30, Level s (items 1, 2 with X_{s-1}^m); the solution b and the cocycle use (Gc0) of level s and (Hc1) of Lemma 23.28, Level s. □

Lemma 23.33. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ be as in Proposition 23.7, and let (V', T'), (V, T), (V'', T'') be three pairs as there, with G_F-equivariant $\mathcal{O}_L$-linear maps forming an exact sequence*

$$0 \to T' \xrightarrow{\iota'} T \xrightarrow{\pi} T'' \to 0.$$

Then $\mathbf{N}(\iota') \colon \mathbf{N}(T') \to \mathbf{N}(T)$ is injective, $\mathbf{N}(\pi) \circ \mathbf{N}(\iota') = 0$, and

$$\ker\big(\mathbf{N}(\pi) \colon \mathbf{N}(T) \to \mathbf{N}(T'')\big) = \mathbf{N}(\iota')\big(\mathbf{N}(T')\big).$$

Equivalently: if $y \in \mathbf{N}(T)$ and $\mathbf{N}(\pi)(y) = 0$, then $y = \mathbf{N}(\iota')(x)$ for a unique $x \in \mathbf{N}(T')$. ***Over a base field.*** *The same holds for three pairs over a field K with $F \subseteq K \subseteq F(\mu_\ell)$ (Proposition 23.7) and G_K-equivariant maps; for $K = F$ this is the statement above.* ***Level s.*** *Let ℓ be any prime, $s \geq 1$ with $s \geq 2$ if $\ell = 2$, and L, K ($F \subseteq K \subseteq F(\mu_{\ell^s})$) and pairs over K as in (Ex4) of Proposition 23.7. The same holds for three pairs over K, G_K-equivariant maps, and $\mathbf{N}_s$, $\mathbf{N}_s(\iota')$, $\mathbf{N}_s(\pi)$ in place of $\mathbf{N}$, $\mathbf{N}(\iota')$, $\mathbf{N}(\pi)$. For $s = 1$ and ℓ odd this is the paragraph over a base field.*

Proof. Throughout, $R = \mathcal{O}_L[[X]]$, ϖ is a uniformiser of $\mathcal{O}_L$, $k_L = \mathcal{O}_L/\varpi$, and $r' = \operatorname{rank}_{\mathcal{O}_L} T' = \dim_L V'$, $r = \operatorname{rank}_{\mathcal{O}_L} T$. By definition of the sites, $e_a\mathbf{A}_{F,L} \cong R$ via $x \otimes \lambda \mapsto \tau_a(x)\lambda$ on coefficients; under these identifications $\gamma_u \colon e_a\mathbf{A}_{F,L} \to e_a\mathbf{A}_{F,L}$ is $f(X) \mapsto f((1+X)^u - 1)$, and $\varphi \colon e_{a-1}\mathbf{A}_{F,L} \to e_a\mathbf{A}_{F,L}$ is $f(X) \mapsto f((1+X)^\ell - 1)$, because $\tau_a(\varphi(x))\lambda = \tau_{a-1}(x)\lambda$. For $f \in R$ and z in the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{\mathbb{C}_\ell}$ the series $f(z)$ converges, and $f \mapsto f(z)$ is a ring homomorphism $R \to \mathcal{O}_{\mathbb{C}_\ell}$; thus $\varphi(f)(z) = f((1+z)^\ell - 1)$. Put $K = \ker \mathbf{N}(\pi)$ and $N' = \mathbf{N}(\iota')(\mathbf{N}(T'))$; both are $\mathbf{A}_{F,L}$-submodules of $\mathbf{N}(T)$, so $K = \bigoplus_a K_a$, $N' = \bigoplus_a N'_a$ with $K_a = e_aK$, $N'_a = e_aN'$, and it suffices to prove $K_a = N'_a$ for every a.

The realisation ring. Let $A' = A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$ and $\iota_L = \iota \otimes \mathrm{id} \colon \mathbf{A}_{F,L} = \mathbf{A}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L \to A'$ (the equality holds because $\mathcal{O}_L$ is finite free over $\mathbb{Z}_\ell$). Since ι is injective (Lemma 23.8(4)), A_{inf} is a domain and $\mathcal{O}_L$ is free over $\mathbb{Z}_\ell$, ι_L is injective. Put $\varepsilon_a = \iota_L(e_a)$, an idempotent of A'. By (Ex2), $\iota_T(fx) = \iota_L(f)\iota_T(x)$ for $f \in \mathbf{A}_{F,L}$, $x \in \mathbf{N}(T)$ (write $f = \sum g_i \otimes \lambda_i$), and likewise for T', T''. The module $W = A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes_{\mathbb{Z}_\ell} T = A' \otimes_{\mathcal{O}_L} T$ is free of rank r over A'. The ring ε_aA' is non-zero and ϖ is not a unit in it: indeed $A'/\varpi A' = (A_{\mathrm{inf}}/\ell)[1/\bar{\mu}] \otimes_{\mathbb{F}_\ell} k_L = \mathbb{C}_\ell^\flat \otimes_{\mathbb{F}_\ell} k_L$ ($\bar{\mu} = \varepsilon - 1$ has positive valuation in $\mathcal{O}_{\mathbb{C}_\ell^\flat}$), and the image of e_a there is the image of the non-zero idempotent $\bar{e}_a$ of $(\mathcal{O}_F \otimes \mathcal{O}_L)/\varpi = k \otimes_{\mathbb{F}_\ell} k_L \cong \prod_a k_L$ under the map $k \otimes k_L \to \mathbb{C}_\ell^\flat \otimes k_L$ induced by the field embedding $k \to \mathbb{C}_\ell^\flat$ (the reduction of $\varphi \circ w$), which is injective since k_L is flat over $\mathbb{F}_\ell$.

Step 0: $\mathbf{N}(\iota')$ is injective and $\mathbf{N}(\pi)\mathbf{N}(\iota') = 0$. As T'' is free over $\mathcal{O}_L$, the sequence splits over $\mathcal{O}_L$, so $1 \otimes \iota' \colon A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes T' \to W$ is injective and its image W' is the kernel of $1 \otimes \pi$ and a direct summand of W, free of rank r' over A'. By (Ex3), $\iota_T \circ \mathbf{N}(\iota') = (1 \otimes \iota') \circ \iota_{T'}$, which is injective, so $\mathbf{N}(\iota')$ is injective; and $\iota_{T''} \circ \mathbf{N}(\pi) \circ \mathbf{N}(\iota') = (1 \otimes \pi\iota') \circ \iota_{T'} = 0$, so $\mathbf{N}(\pi)\mathbf{N}(\iota') = 0$ by injectivity of $\iota_{T''}$. Hence $N' \subseteq K$, and $N'_a \cong \mathbf{N}(T')_a$ is free of rank r' over R by (Wm1) and (Ex1). Moreover $\iota_T(K) \subseteq W'$, since $(1 \otimes \pi)\iota_T(y) = \iota_{T''}(\mathbf{N}(\pi)y) = 0$ for $y \in K$.

Step 1: K_a is free of rank r', and the index of N'_a in K_a is prime to ϖ. K_a is the kernel of the R-linear map $\mathbf{N}(T)_a \to \mathbf{N}(T'')_a$ between free R-modules (Wm1). Its image I is torsion-free, so $\operatorname{depth}_R I \geq 1$, and the depth lemma for $0 \to K_a \to \mathbf{N}(T)_a \to I \to 0$ gives $\operatorname{depth}_R K_a \geq 2$; since R is regular local of dimension 2, the Auslander–Buchsbaum formula shows that K_a is free, of rank s say. As $N'_a \subseteq K_a$ is free of rank r', $s \geq r'$. Choose R-bases $b = (b_1, \ldots, b_s)$ of K_a, $c = (c_1, \ldots, c_{r'})$ of N'_a and $d = (d_1, \ldots, d_r)$ of $\mathbf{N}(T)_a$, and write $b = dP$, $c = bB$ with matrices $P \in M_{r \times s}(R)$, $B \in M_{s \times r'}(R)$.

By (Ex2), $A_{\mathrm{inf}}[1/\mu_\varepsilon] \cdot \iota_T(\mathbf{N}(T)) = W$; this span is an A'-module ($\iota_T(\lambda x) = (1 \otimes \lambda)\iota_T(x)$), so $\varepsilon_aW = A'\iota_T(\mathbf{N}(T)_a)$ is generated by the r elements $\iota_T(d)$. A generating family of r elements of a free module of rank r over a commutative ring is a basis, so $\iota_T(d)$ is an ε_aA'-basis of ε_aW. In the same way (for the pair (V', T'), transported by $1 \otimes \iota'$) $\iota_T(c)$ is a basis of ε_aW', a free ε_aA'-module of rank r' and a direct summand of ε_aW. Now $\iota_T(b) = \iota_T(d)\,\iota_L(P)$ has entries in ε_aW'. If $s > r'$, the s-fold exterior product of these s elements vanishes ($\wedge^s$ of a free module of rank $r' < s$ is 0, and $\wedge^s\varepsilon_aW' \to \wedge^s\varepsilon_aW$); its coordinates in the basis of $\wedge^s\varepsilon_aW$ induced by $\iota_T(d)$ are the images under ι_L of the $s \times s$ minors of P. Some such minor is non-zero, as b is linearly independent over R; this contradicts the injectivity of ι_L. So $s = r'$ and B is square. Since $c = bB$, the family $\iota_T(b)$ generates $\varepsilon_aW' \supseteq A'\iota_T(c) = \varepsilon_aW'$, so it is a basis of ε_aW', and $\iota_L(B)$ is the matrix of a change of basis. Hence

$$\delta_a := \det B \in R \qquad \text{has} \qquad \iota_L(\delta_a) \in (\varepsilon_aA')^\times.$$

In particular $\delta_a \neq 0$, and $\delta_a \notin \varpi R$, since ϖ is not a unit of ε_aA'.

Step 2: Frobenius forces $\delta_a \in X^{j_a}R^\times$, hence $X^{j_a}K_a \subseteq N'_a$. Let $(M_a, h_a)_a$ be a window of V and $(M'_a, h'_a)_a$ a window of V' as in (Ex1). Write $b^{(a)}$, $c^{(a)}$, B_a, δ_a for the data of Step 1 at the site a. For $x \in K_{a-1}$, $Q^{M_a}\varphi(x) \in \mathbf{N}(T)_a$ by (Wm2), and $\mathbf{N}(\pi)(Q^{M_a}\varphi(x)) = Q^{M_a}\varphi(\mathbf{N}(\pi)x) = 0$

by (Ex3) (all maps extended to the localisations at Q, which are injective on the free modules), so $Q^{M_a}\varphi(K_{a-1}) \subseteq K_a$. Hence $\varphi(b^{(a-1)}) = b^{(a)}Q^{-M_a}G_a$ with $G_a \in M_{r'}(R)$. Since $\mathbf{N}(\iota')$ commutes with φ, (Wm2) for $\mathbf{N}(T')$ gives $\varphi(c^{(a-1)}) = c^{(a)}Q^{-M'_a}H_a$ with $H_a \in M_{r'}(R)$, and $Q^{h'_a}c^{(a)} = \varphi(c^{(a-1)})C_a$ with $C_a \in M_{r'}(R)$ (the module generated by $\varphi(N'_{a-1})$ is generated by $\varphi(c^{(a-1)})$, by semilinearity). As $c^{(a)}$ is linearly independent over $R[1/Q]$, $H_aC_a = Q^{h'_a+M'_a}\cdot 1$, so $\det H_a$ divides a power of Q in R. Applying φ to $c^{(a-1)} = b^{(a-1)}B_{a-1}$ and comparing in the basis $b^{(a)}$ of $K_a[1/Q]$:

$$Q^{M'_a}G_a\,\varphi(B_{a-1}) = Q^{M_a}B_aH_a, \qquad \text{so} \qquad Q^{r'M'_a}\det(G_a)\,\varphi(\delta_{a-1}) = Q^{r'M_a}\,\delta_a\,\det(H_a). \qquad (*)$$

By Step 1 and the Weierstrass preparation theorem, $\delta_a = P_au_a$ with $P_a \in \mathcal{O}_L[X]$ a distinguished polynomial and $u_a \in R^\times$; so the set Z_a of zeros of δ_a in $\mathfrak{m}$ is finite. Let $\alpha \in Z_{a-1}$ with $\alpha \neq 0$, and $\beta \in \mathfrak{m}$ with $(1+\beta)^\ell = 1+\alpha$. Then $\varphi(\delta_{a-1})(\beta) = \delta_{a-1}(\alpha) = 0$; and $Q(\beta) = \Phi_\ell(1+\beta) \neq 0$ because $(1+\beta)^\ell \neq 1$, so also $\det(H_a)(\beta) \neq 0$. Evaluating $(*)$ at β gives $\delta_a(\beta) = 0$, and $\beta \neq 0$. By induction, for every $k \geq 1$ all $\beta \in \mathfrak{m}$ with $(1+\beta)^{\ell^k} = 1+\alpha$ lie in Z_{a-1+k}. There are ℓ^k of them: if $w \in \mathcal{O}_{\mathbb{C}_\ell}$ has $w^{\ell^k} = 1+\alpha$, its residue is an ℓ^k-th root of 1 in characteristic ℓ, so $w \in 1+\mathfrak{m}$, and the $\zeta w - 1$, $\zeta \in \mu_{\ell^k}$, are ℓ^k distinct elements of $\mathfrak{m}$. Taking k a multiple of n gives $|Z_{a-1}| \geq \ell^k$ for arbitrarily large k, a contradiction. Hence $Z_a \subseteq \{0\}$ for all a, so $P_a = X^{j_a}$ with $j_a = \deg P_a$, and $\delta_a = X^{j_a}u_a$. From $B_a\,\mathrm{adj}(B_a) = \delta_a\cdot 1$ we get $\delta_a b^{(a)} = c^{(a)}\mathrm{adj}(B_a)$, so $X^{j_a}K_a \subseteq N'_a$.

Step 3: the Γ-action removes the power of X. Both K and N' are stable under all γ_u (Ex3), and $(\gamma_u - 1)N' \subseteq XN'$ by (Wm3) for $\mathbf{N}(T')$. Also $(\gamma_u-1)K \subseteq XK$: for $y \in K$, (Wm3) gives $(\gamma_u-1)y = Xz$ with $z \in \mathbf{N}(T)$, and $X\,\mathbf{N}(\pi)(z) = (\gamma_u-1)\mathbf{N}(\pi)(y) = 0$, so $\mathbf{N}(\pi)(z) = 0$ as X is a non-zero-divisor on the free modules $\mathbf{N}(T'')_a$. Let $j \geq 0$ be minimal with $X^jK_a \subseteq N'_a$ (it exists by Step 2) and suppose $j \geq 1$. Write $X^jb = cD$ with $D \in M_{r'}(R)$, $\gamma_u(b) = b(1+XE)$, $\gamma_u(c) = c(1+XE')$ with $E, E' \in M_{r'}(R)$, and $\gamma_u(X) = Xw_u$ with $w_u = \sum_{k\geq 1}\binom{u}{k}X^{k-1} \in \mathbb{Z}_\ell[[X]]$, $w_u(0) = u$. Applying γ_u to $X^jb = cD$:

$$w_u^j\,c\,D(1+XE) = X^jw_u^j\,b(1+XE) = \gamma_u(X^jb) = c(1+XE')\gamma_u(D),$$
$$\text{so} \qquad w_u^jD(1+XE) = (1+XE')\gamma_u(D).$$

Reduce modulo X (the reduction $R \to \mathcal{O}_L$, $X \mapsto 0$, satisfies $\mathrm{red}\circ\gamma_u = \mathrm{red}$): $u^j\,\mathrm{red}(D) = \mathrm{red}(D)$. Take $u = 1+\ell$; then $u^j \neq 1$, and $\mathcal{O}_L$ is torsion-free, so $\mathrm{red}(D) = 0$, i.e. $D = XD'$ with $D' \in M_{r'}(R)$. Then $X(X^{j-1}b - cD') = 0$ in the free module $\mathbf{N}(T)_a$, so $X^{j-1}b = cD'$ and $X^{j-1}K_a \subseteq N'_a$, contradicting the minimality of j. Hence $j = 0$, $K_a \subseteq N'_a$, and with Step 0, $K_a = N'_a$. Uniqueness of x is the injectivity of $\mathbf{N}(\iota')$.

Over a base field. For pairs over K the proof is unchanged: (Wm1)–(Wm3), (Ex2), (Ex3) hold over K by Proposition 23.7, the maps γ_u are given for $u \in \Gamma_K$, and the one element used in Step 3, $u = 1+\ell$, lies in Γ_K, since $\Gamma_K = \Delta(1+\ell\mathbb{Z}_\ell)$ for a subgroup Δ of $\mu_{\ell-1}$ by (Rr7) of Lemma 23.24 (applied to the subgroup of order $(\ell-1)/[K:F]$).

Level s. Write $\mathbf{N} = \mathbf{N}_s$, $P = Q_{s-1}$ and $Y = X_{s-1} = (1+X)^{\ell^{s-1}} - 1$. Steps 0 and 1 are unchanged, with ι_s, $\iota_{s,T}$ in place of ι, ι_T: they use (Wm1), (Ex1)–(Ex3) of level s (Proposition 23.7(Ex4)), the injectivity of ι_s and $\iota_s = \varphi\circ w$ on $\mathcal{O}_F$ (Lemma 23.8(6)). Steps 2 and 3 change as follows. *Step 2 at level s: δ_a divides a power of Y.* The identity $(*)$ holds with Q replaced by P ((Wm2) of level s), and $\det H_a$ divides a power of P. Let $\alpha \in Z_{a-1}$ be such that $1+\alpha$ is not a root of unity of order dividing ℓ^{s-1}, and $\beta \in \mathfrak{m}$ with $(1+\beta)^\ell = 1+\alpha$. Then $1+\beta$ is not a primitive ℓ^s-th root of unity (otherwise $1+\alpha$ would have order ℓ^{s-1}), so $P(\beta) = \Phi_{\ell^s}(1+\beta) \neq 0$ and $\det(H_a)(\beta) \neq 0$; evaluating $(*)$ at β gives $\delta_a(\beta) = 0$; and $1+\beta$ is again not a root of unity of order dividing ℓ^{s-1} (its ℓ-th power $1+\alpha$ is not). By induction all ℓ^k elements β with $(1+\beta)^{\ell^k} = 1+\alpha$

lie in Z_{a-1+k}, and taking k a multiple of n contradicts the finiteness of Z_{a-1} as before. Hence every zero of δ_a in $\mathfrak{m}$ is of the form $\zeta - 1$ with $\zeta^{\ell^{s-1}} = 1$, i.e. a root of Y. As $L \supseteq \mu_{\ell^s}$ (the hypothesis on L in (Ex4)), the distinguished polynomial P_a of $\delta_a = P_a u_a$ is $\prod_\zeta (X - (\zeta - 1))^{m_\zeta}$ over these roots, and $Y = \prod_{\zeta^{\ell^{s-1}}=1}(X - (\zeta - 1))$; so with $j_a = \deg P_a$, $Y^{j_a} = P_a G_a$ for a monic $G_a \in \mathcal{O}_L[X]$, and $Y^{j_a} b^{(a)} = c^{(a)} \mathrm{adj}(B_a) u_a^{-1} G_a$ gives $Y^{j_a} K_a \subseteq N'_a$. *Step 3 at level s: the Γ-action removes the power of Y.* Let $u = 1 + \ell^s \in \Gamma_K$ ($\Gamma_K \supseteq \Gamma_{F(\mu_{\ell^s})} = 1 + \ell^s \mathbb{Z}_\ell$, Definition 23.2, paragraph Level s). By (Wm3') of level s, $(\gamma_u - 1)N' \subseteq YN'$, and $(\gamma_u - 1)K \subseteq YK$: for $y \in K$, (Wm3') gives $(\gamma_u - 1)y = Yz$ with $z \in \mathbf{N}(T)$, and $Y\,\mathbf{N}(\pi)(z) = 0$, so $\mathbf{N}(\pi)(z) = 0$ (Y is a non-zero-divisor on the free modules $\mathbf{N}(T'')_a$). Let $j \geq 0$ be minimal with $Y^j K_a \subseteq N'_a$ and suppose $j \geq 1$. Write $Y^j b = cD$, $\gamma_u(b) = b(1 + YE)$, $\gamma_u(c) = c(1 + YE')$ and $\gamma_u(Y) = Y w_u$ with $w_u = \sum_{k \geq 1} \binom{u}{k} Y^{k-1}$. As in Step 3, $w_u^j D(1 + YE) = (1 + YE')\gamma_u(D)$. Let $\rho\colon R \to R/YR$ be the quotient map; $R/YR = \mathcal{O}_L[X]/(Y)$ is free over $\mathcal{O}_L$ (Weierstrass division), $\rho \circ \gamma_u = \rho$ (both are continuous $\mathcal{O}_L$-algebra maps, $\rho(Y) = 0$, and $\gamma_u(1+X) = (1+X)^{1+\ell^s} = (1+X)(1+Y)^\ell$, so they agree on X) and $\rho(w_u) = u$. Hence $u^j \rho(D) = \rho(D)$; as $u^j \neq 1$ and R/YR is $\mathcal{O}_L$-torsion-free, $\rho(D) = 0$, $D = YD'$, and $Y^{j-1} b = cD'$ since Y is a non-zero-divisor on $\mathbf{N}(T)_a$, contradicting minimality. So $K_a = N'_a$. □

24 Relay classes, class splitting and lifting exponents

This section proves the two lifting exponents of the Part. For a surjection $F\colon T_1 \to T_2$ of crystalline lattices whose Hodge jumps lie in a window of width wt, a lifting exponent is an integer e with $\ell^e H^1_f(K, T_2) \subseteq F_* H^1_f(K, T_1)$. Over an unramified field K one can take $e = \lfloor c_{\mathrm{ur}}(2 + \mathrm{wt}/\ell + \log_\ell(\mathrm{wt} + 1)) \rfloor$ with an absolute constant c_{ur} (Proposition 24.1); over a field between K_0 unramified and $K_0(\mu_\ell)$ one can take $e = \lfloor c_{\mathrm{cy}}(\mathrm{wt} + 1) \rfloor$ (Proposition 24.10). The first enters the budget at the primes of split and inert type, through the explicit local losses of Section 25 and the row bound of Section 26; the second serves the small and ramified primes in Section 27.

The core of the unramified case is a conversion of exponents. The length lemma gives the annihilator X^{wt} for the cokernel of a surjection of Wach modules, and Γ-descent turns it into the power ℓ^{r_0} with $r_0 = \lfloor \mathrm{wt}\,\ell/(\ell-1)^2 \rfloor$; this is where the denominator ℓ in wt/ℓ comes from. A class in $H^1_f(K, T_2)$ is then realised as a crystalline extension by the trivial lattice, whose splitting at each site lifts up to ℓ^{r_0}, and the class is split into *relay classes*, one for each cut of a set of sites (Proposition 24.4). A relay class is the Herr class of a Frobenius twist of a Wach vector at the cut, corrected by a truncated horizontal part (Lemma 24.5), and it lifts as soon as the Wach vector lifts modulo X (Lemma 24.9). The one-site lifts of Lemma 24.7 cost the ℓ-adic valuations of the pole indices, which Legendre's formula bounds by a multiple of wt/ℓ; the proof gives in fact an exponent at most $\frac{63}{4}\,\mathrm{wt}/\ell + \frac{11}{4}$. Class splitting and relay classes are used again, with longer arcs between the cuts, in Appendix C.

24.1 Lifting exponents in the unramified case

The subsection opens with the statement Proposition 24.1 and then proves the lemmas on which its proof rests, in the order in which the proof uses them: level filtrations lie in their windows and the Hodge jumps of a quotient are Hodge jumps of the source (Lemma 24.2); the Wach module of the trivial lattice (Lemma 24.3); class splitting (Proposition 24.4) with relay classes and relay truncation (Lemma 24.5, Lemma 24.6); the one-site lift of Γ-almost-invariant Wach vectors, with the killing polynomial of the normalised Γ-operator (Lemma 24.7, Lemma 24.8); and relay lifting (Lemma 24.9). Class splitting and the relay lemmas are stated for general cuts

and arcs, more generally than the proof of Proposition 24.1 needs, because Appendix C applies them to arcs of length greater than one.

Proposition 24.1. *Let p be an odd prime, K a finite unramified extension of $\mathbb{Q}_p$, L a finite extension of $\mathbb{Q}_p$ containing $\sigma(K(\mu_p))$ for every $\mathbb{Q}_p$-embedding σ of $K(\mu_p)$ into $\overline{\mathbb{Q}}_p$, and $\mathcal{O}_L$ its ring of integers. For a continuous L-linear representation V of G_K on a finite-dimensional L-vector space and a G_K-stable $\mathcal{O}_L$-lattice $T \subseteq V$ let $H^1_f(K,T)$ be the inverse image in $H^1(K,T)$ of $H^1_f(K,V) = \ker\bigl(H^1(K,V) \to H^1(K, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)\bigr)$.*

There is an absolute constant $c_{\mathrm{ur}} \geq 1$ with the following property. Let V_1, V_2 be such representations, crystalline as representations over $\mathbb{Q}_p$, let $T_\nu \subseteq V_\nu$ be G_K-stable $\mathcal{O}_L$-lattices, and let $F\colon V_1 \to V_2$ be an L-linear G_K-equivariant map with $F(T_1) = T_2$. Let $a^-_\tau, a^+_\tau \geq 0$ be integers, for the $\mathbb{Q}_p$-embeddings $\tau\colon K \to L$, such that every Hodge jump of V_1 at τ lies in $[-a^-_\tau, a^+_\tau]$, put $\mathrm{wt} = \max_\tau(a^-_\tau + a^+_\tau)$ and

$$e_{\mathrm{A}} = \Bigl\lfloor c_{\mathrm{ur}}\Bigl(2 + \frac{\mathrm{wt}}{p} + \log_p(\mathrm{wt}+1)\Bigr)\Bigr\rfloor.$$

Then

$$p^{e_{\mathrm{A}}}\, H^1_f(K,T_2) \subseteq F_*\bigl(H^1_f(K,T_1)\bigr).$$

The integer e_{A} depends only on p and wt and is non-decreasing in wt, hence in each $a^\pm_\tau$; and $e_{\mathrm{A}} \leq c_{\mathrm{ur}}(2 + \mathrm{wt}/p + \log_p(\mathrm{wt}+1))$.

Proof. We show that $c_{\mathrm{ur}} = 18$ has the property. Write $\ell = p$, F for the field K (the map of the statement is written Φ in this proof), $n = [F : \mathbb{Q}_\ell]$, and fix a Frobenius numbering $(\tau_a)_{a\in\mathbb{Z}/n}$; the τ_a are the n embeddings of F into L. Put $M_a = a^-_{\tau_a}$, $h_a = a^+_{\tau_a}$, $A = \max_a M_a$, $B = \max_a h_a$, $w = \mathrm{wt} = \max_a(M_a + h_a)$; so $A, B \leq w$. The hypotheses on ℓ, F, L are those of Proposition 23.7, so (V_1,T_1) and (V_2,T_2) are pairs. Since $\Phi(T_1) = T_2$ and T_2 spans V_2, $\Phi\colon V_1 \to V_2$ is onto, and by item 1 of Lemma 24.2 every Hodge jump of V_2 at τ_a is one of V_1; so every Hodge jump of V_1 and of V_2 at τ_a lies in $[-M_a, h_a]$.

Step 1: lifting along onto maps up to ℓ^{r_0}, $r_0 = \lfloor w\ell/(\ell-1)^2 \rfloor$. Let $f\colon T \to T''$ be a G_F-equivariant $\mathcal{O}_L$-linear surjection of the lattices of pairs (V,T), (V'',T'') such that every Hodge jump of V at τ_a lies in $[-M_a, h_a]$, and let $c \in \mathbb{Z}/n$. We claim: for every $y \in \mathbf{N}(T'')_c$ there is $x \in \mathbf{N}(T)_c$ with $\mathbf{N}(f)(x) = \ell^{r_0} y$. By Proposition 23.18, $X^w \mathbf{N}(T'')_c \subseteq \mathbf{N}(f)(\mathbf{N}(T)_c)$ (its $H_{\max}$ is w). Apply item 1 of Lemma 23.21 with $D = N = \mathbf{N}(T'')_c$, $N' = \mathbf{N}(f)(\mathbf{N}(T)_c)$, $\gamma_{u,D} = \gamma_{u,\mathbf{N}}$, $s = w$, $j = 0$: multiplication by X is injective on D, which is free over $e_c\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ by (Wm1) for the window of (Ex1); $\gamma_{u,\mathbf{N}}$ is γ_u-semilinear, fixes e_c and has $(\gamma_u - 1)(N) \subseteq e_c X \mathbf{N}(T'') = XN$ by (Wm3); and N' is an $\mathbf{A}_{F,L}$-submodule with $(\gamma_u - 1)(N') = \mathbf{N}(f)\bigl((\gamma_u - 1)\mathbf{N}(T)_c\bigr) \subseteq XN'$, since $\mathbf{N}(f)$ is $\mathbf{A}_{F,L}$-linear and commutes with γ_u (Ex3). So $\ell^a N \subseteq N'$ for $a = \mathrm{dsc}_\ell(0,w)$, and $\mathrm{dsc}_\ell(0,w) \leq r_0$ by item 2 there; hence $\ell^{r_0} N \subseteq N'$.

Step 2: the extension. Let $x \in H^1_f(F,T_2)$. By Lemma 23.26 there are a pair (V_Y, Y) and an exact sequence $0 \to T_2 \xrightarrow{\iota'} Y \xrightarrow{\pi} \mathcal{O}_L \to 0$ of G_F-equivariant $\mathcal{O}_L$-linear maps with $[Y] = x$, every Hodge jump of V_Y at τ_a being 0 or one of V_2, hence in $[-M_a, h_a]$ (as $M_a, h_a \geq 0$). So $(M_a, h_a)_a$ is a window, as in (Ex1), of V_1, V_2 and V_Y. For $c \in \mathbb{Z}/n$, Step 1 for π and $y = e^\circ_c \in \mathbf{N}(\mathcal{O}_L)_c$ (Lemma 24.3) gives $x_c \in \mathbf{N}(Y)_c$ with $\mathbf{N}(\pi)(x_c) = \ell^{r_0} e^\circ_c$, so by (Ex3) $(1 \otimes \pi)(\iota_Y(x_c)) = \iota_{\mathcal{O}_L}(\ell^{r_0} e^\circ_c) = \ell^{r_0}\varepsilon_c$.

Step 3: splitting. Apply Proposition 24.4 to $(V',T') = (V_2,T_2)$, $(V,T) = (V_Y,Y)$, ι', π, the window (M_a,h_a), $C = \mathbb{Z}/n$ with $\mathrm{len}_c = 1$, $j = 1$, $G = g = A$, $m = R = A+1$, $r = r_0$ and the x_c of Step 2. Its hypotheses hold: $c - 1 \in C$ and there is no d with $1 \leq d < 1$; $j \leq \mathrm{len}_c$;

$M_{c+1} \le A = G$; $g \ge G$; $m = \max(g, 1 + \max_a M_a)$; $R \ge m$. With $\sigma = \sigma_\ell(A, A+1, 1)$ it gives $\tau'_c \in P_1(\mathbf{N}(T_2)_c)$ and $w_c \in I_c$ (formed for $\mathbf{N}(T_2)$) with

$$\mathbf{N}(\iota')(w_c) = X_{c-1} - \ell^\sigma x_c, \qquad X_{c-1} = \ell^\sigma \varphi(x_{c-1}) - \mathbf{N}(\iota')(\tau'_{c-1}), \qquad \ell^{r_0+2\sigma} x = \sum_{c \in \mathbb{Z}/n} \rho_{c,1}(w_c).$$

Step 4: the poles of w_c. By (Ry0) of Lemma 24.5, formed at the site $c-1$ with $j = 1$ and $G = A \ge M_c$ for $\mathbf{N}(Y)$ and for $\mathbf{N}(T_2)$, $\varphi(x_{c-1}) \in P_1(\mathbf{N}(Y)_{c-1}) \subseteq Q^{-A}\mathbf{N}(Y)_c$ and $\tau'_{c-1} \in Q^{-A}\mathbf{N}(T_2)_c$; also $x_c \in \mathbf{N}(Y)_c$. Hence $y := \mathbf{N}(\iota')(Q^A w_c) = Q^A \mathbf{N}(\iota')(w_c) \in \mathbf{N}(Y)$, and $\mathbf{N}(\pi)(y) = Q^A\, \mathbf{N}(\pi \circ \iota')(w_c) = 0$. By Lemma 23.33 (with $T'' = \mathcal{O}_L$), $y = \mathbf{N}(\iota')(z)$ with $z \in \mathbf{N}(T_2)$. The map $\mathbf{N}(\iota')$ is injective on $\mathbf{N}(T_2)$ (Lemma 23.33), hence on $\mathbf{N}(T_2)[1/\lambda_\infty]$, its base change along the flat localisation $\mathbf{A}_{F,L} \to \mathbf{A}_{F,L}[1/Q_s :\ s \ge 0]$. So $Q^A w_c = z \in \mathbf{N}(T_2)$.

Step 5: lifting each relay vector. Let J be the set of Hodge jumps of V_1 at the τ_a and $\kappa = \kappa_\ell(J, A)$. By Step 1 for Φ, every $y \in \mathbf{N}(T_2)_c$ has $x \in \mathbf{N}(T_1)_c$ with $\mathbf{N}(\Phi)(x) = \ell^{r_0} y$. So Lemma 24.7, with M_a, J, $G = A \ge \max_a M_a$, $r = r_0$ and $w = w_c \in I_c$ (Step 3) with $Q^A w_c \in \mathbf{N}(T_2)$ (Step 4), gives $u_c \in e_c \mathbf{N}(T_1)[1/\lambda_\infty]$ with $(\gamma_u - 1)(u_c) \in X\mathbf{N}(T_1)$ for all u, $Q^A u_c \in \mathbf{N}(T_1)$ and $\ell^{r_0+\kappa} w_c - \mathbf{N}(\Phi)(u_c) \in X\mathbf{N}(T_2)_c$.

Step 6: the relay classes lift. Apply Lemma 24.9 to (V_1, T_1), (V_2, T_2) with the window (M_a, h_a) for both, Φ, c, $j = 1$, $G = g = A$, $m = R = A + 1$ (so $m \ge 1 + \max_{a,\nu} M_a^\nu$, $R \ge \max(m, g)$, $M_{c+1}^\nu \le G$), $w = w_c$, $u = u_c$, $k = r_0 + \kappa$ and $\lambda' = Q^A = Q_0^A$; the sets and classes I_c^2, $\rho_{c,1}^2$ are those of Step 3. It gives $\ell^{r_0+\kappa}\rho_{c,1}(w_c) \in \Phi_* H_f^1(F, T_1)$, where $H_f^1(F, T_1)$, the group of Proposition 23.27, is the inverse image of $H_f^1(F, V_1)$, as here. Summing over c with Step 3,

$$\ell^e x \in \Phi_* H_f^1(F, T_1), \qquad e = 2r_0 + 2\sigma + \kappa.$$

Step 7: the exponent. For $\ell \ge 3$, $\ell/(\ell-1) \le 3/2$ and $\ell/(\ell-1)^2 \le \frac{9}{4} \cdot \frac{1}{\ell}$. (a) $r_0 \le \frac{9}{4}\, w/\ell$. (b) $\sigma = \lfloor (A + \frac{A+1}{\ell-1}) \frac{\ell}{(\ell-1)^2} \rfloor + 1 \le \frac{9}{4} \cdot \frac{3A+1}{2\ell} + 1 \le \frac{27}{8}\frac{w}{\ell} + \frac{11}{8}$. (c) Every $x \in J'_A$ is non-zero with $-(A-1) \le x \le A + B$, since $J \subseteq [-A, B]$ and $1 \le k \le A$; so by Legendre's formula ($v_\ell(N!) \le N/(\ell-1)$)

$$\sum_{x \in J'_A} v_\ell(x) \le v_\ell((A-1)!) + v_\ell((A+B)!) \le \frac{2A+B}{\ell-1} \le \frac{3w}{\ell-1} \le \frac{9}{2}\frac{w}{\ell}$$

(if $A = 0$ then $J'_A = \emptyset$); this sum is κ. Hence

$$e \le \Big(\frac{9}{2} + \frac{27}{4} + \frac{9}{2}\Big)\frac{w}{\ell} + \frac{11}{4} = \frac{63}{4}\,\frac{w}{\ell} + \frac{11}{4} < 18\Big(2 + \frac{w}{\ell} + \log_\ell(w+1)\Big).$$

As e is an integer, $e \le e_{\mathrm{A}}$ for $c_{\mathrm{ur}} = 18$, and $\ell^{e_{\mathrm{A}}} x = \ell^{e_{\mathrm{A}} - e}(\ell^e x) \in \Phi_* H_f^1(F, T_1)$, a subgroup. As $x \in H_f^1(F, T_2)$ was arbitrary, the inclusion follows. Finally $t \mapsto c_{\mathrm{ur}}(2 + t/\ell + \log_\ell(t+1))$ is increasing on $[0, \infty)$, so e_{A} is non-decreasing in wt, and $e_{\mathrm{A}} \le c_{\mathrm{ur}}(2 + \mathrm{wt}/\ell + \log_\ell(\mathrm{wt}+1))$ by definition of the floor. □

Lemma 24.2. *Let ℓ, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ be as in Proposition 23.7 and $\Phi\colon V_1 \to V_2$ an onto map of crystalline lattices $T_1 \to T_2$ over F with coefficients L.*

1. *For every $a \in \mathbb{Z}/n$, every Hodge jump of V_2 at τ_a is a Hodge jump of V_1 at τ_a.*
2. *For a weight word $(b, w, S, \eta, m, (s_a), (\mathrm{sg}_a))$ for Φ, $G = w + S + \eta$, a cut c, a length $1 \le \mathrm{len} \le n$ and frames at $c - \mathrm{len}$, the arc level data of Φ along the word have (Lp7) for $\nu = 1, 2$ and every $s < \mathrm{len}$: the jumps of Fil_s^ν are the Hodge jumps of V_ν at τ_{c-s}.*

Proof. Write $B = B^+_{\mathrm{dR}}$, $D_\nu = D_{\mathrm{cris}}(V_\nu) = D_{\mathrm{dR}}(V_\nu)$ (crystalline, so de Rham), with the filtration of Theorem B.40 (Ai4).

Step 1: components. For $b \in \mathbb{Z}/n$, ϵ_b is the idempotent of $F \otimes_{\mathbb{Q}_\ell} L$ belonging to τ_{b+1} (for the inclusion $F \subset B_{\mathrm{dR}}$ of Theorem B.40 (Ai3); see the remark in Definition 23.3), and $\mathrm{Fil}^j D_\nu$ is an $F \otimes L$-submodule; so $\epsilon_b \mathrm{Fil}^j D_\nu = \mathrm{Fil}^j D_\nu \cap \epsilon_b D_\nu$ and $\epsilon_b D_\nu = D_{\mathrm{dR},\tau_{b+1}}(V_\nu)$ as filtered L-spaces. Fix a, put $\epsilon = \epsilon_{a-1}$, $R^+ = \epsilon(B \otimes_{\mathbb{Q}_\ell} L)$, and choose L-bases $(D^\nu_i)_i$ of ϵD_ν adapted to the filtration, $D^\nu_i \in \mathrm{Fil}^{u_{\nu,i}} D_\nu$ with $\epsilon \mathrm{Fil}^j D_\nu$ spanned by the D^ν_i with $u_{\nu,i} \geq j$. The Hodge jumps of V_ν at τ_a are the $u_{\nu,i}$. Applying ϵ to (Ai4) for $k = 0$ and $k = 1$ (the comparison is $F \otimes L$-linear):

$$\epsilon(B \otimes_{\mathbb{Q}_\ell} V_\nu) = \bigoplus_i t^{-u_{\nu,i}} R^+ D^\nu_i, \qquad \epsilon(tB \otimes_{\mathbb{Q}_\ell} V_\nu) = \bigoplus_i t^{1-u_{\nu,i}} R^+ D^\nu_i.$$

Step 2: proof of 1. Let $M_\nu = \epsilon(B \otimes V_\nu)/\epsilon(tB \otimes V_\nu)$ and, for $u \in \mathbb{Z}$, $M^{(u)}_\nu$ the image of $\bigoplus_{i:\, u_{\nu,i}=u} t^{-u} R^+ D^\nu_i$; by Step 1, $M_\nu = \bigoplus_u M^{(u)}_\nu$, each $M^{(u)}_\nu$ is G_F-stable, and $M^{(u)}_\nu \neq 0$ iff u is a Hodge jump of V_ν at τ_a. The map $1 \otimes \Phi$ is $B \otimes L$-linear and G_F-equivariant and maps $\epsilon(B \otimes V_1)$ onto $\epsilon(B \otimes V_2)$ (Φ is onto), hence induces a surjection $f\colon M_1 \to M_2$. Let $u \neq u'$, i with $u_{1,i} = u$, and write the $M^{(u')}_2$-component of $f(t^{-u} D^1_i)$ as $\sum_{j:\, u_{2,j}=u'} t^{-u'} c_j D^2_j$ with $c_j \in R^+/tR^+ = \epsilon(\mathbb{C}_\ell \otimes_{\mathbb{Q}_\ell} L)$. For $g \in G_F$, $g(t^{-u} D^1_i) = \chi(g)^{-u} t^{-u} D^1_i$ (D_1 is G_F-fixed), and the projections to the $M^{(u')}_2$ commute with g; so $g(c_j) = \chi(g)^{u'-u} c_j$. Writing $c_j = \sum_m c_{jm} \otimes \lambda_m$ for a $\mathbb{Q}_\ell$-basis (λ_m) of L (G_F acting on $\mathbb{C}_\ell$ only), $\chi(g)^{u-u'} g(c_{jm}) = c_{jm}$ for all g, so $c_{jm} = 0$ by Theorem B.42 (2). Hence $f(M^{(u)}_1) \subseteq M^{(u)}_2$ for every u, and as f is onto, $f(M^{(u)}_1) = M^{(u)}_2$. If u is a Hodge jump of V_2 at τ_a, then $M^{(u)}_2 \neq 0$, so $M^{(u)}_1 \neq 0$ and u is a Hodge jump of V_1 at τ_a.

Step 3: proof of 2. Let $s < \mathrm{len}$ and $a = c - s$, so $\epsilon_s = \epsilon_{c-s-1}$ and $\epsilon_s D_\nu = D_{\mathrm{dR},\tau_{c-s}}(V_\nu)$ by Step 1. The level frame $(D^{(s)}_k)_k$ is an L-basis of $\epsilon_s D_\nu$, and $\sum_k v_k D^{(s)}_k \in \epsilon_s D_\nu$, so $\mathrm{Fil}^\nu_s(u)$ is the coordinate space of $\epsilon_s \mathrm{Fil}^u D_\nu$: Fil^ν_s is decreasing, equal to L^{r_ν} for $u \ll 0$ and to 0 for $u \gg 0$ (the filtration of D_{dR} is exhaustive and separated), and its jumps are the Hodge jumps of V_ν at τ_{c-s}. By 1 these lie among those of V_1, and by (Ow1): if $c - s$ is thin (κ_s thin; $s_{c-s} \leq w$ by (Ow3)), every jump u has $|u| \leq \max(s_{c-s} + S, \eta) \leq w + S + \eta = G$; if $c - s$ is thick with $\mathrm{sg}_{c-s} = 1$ (κ_s up, $b_s = s_{c-s}$), $u \in [b_s, b_s + S] \cup [-\eta, \eta]$; if $\mathrm{sg}_{c-s} = -1$ (κ_s down), $u \in [-b_s - S, -b_s] \cup [-\eta, \eta]$. This is (Lp7). □

Lemma 24.3. *Let ℓ, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ be as in Proposition 23.7 and let $\mathcal{O}_L$ be the lattice of the trivial pair $(L, \mathcal{O}_L)$ (G_F acting trivially). Then there is an $\mathbf{A}_{F,L}$-linear isomorphism $\psi\colon \mathbf{N}(\mathcal{O}_L) \to \mathbf{A}_{F,L}$ with*

$$\iota_{\mathcal{O}_L} = (\iota \otimes \mathrm{id}_{\mathcal{O}_L}) \circ \psi.$$

In particular, for every $c \in \mathbb{Z}/n$ the element $e^\circ_c = \psi^{-1}(e_c)$ lies in $\mathbf{N}(\mathcal{O}_L)_c$ and $\iota_{\mathcal{O}_L}(e^\circ_c) = \varepsilon_c := (\iota \otimes \mathrm{id})(e_c)$.

Over a base field. *The same holds for a field K with $F \subseteq K \subseteq F(\mu_\ell)$ as in Definition 23.2 and the trivial pair $(L, \mathcal{O}_L)$ over K (G_K acting trivially), $\mathbf{N}(\mathcal{O}_L)$ and $\iota_{\mathcal{O}_L}$ being those of Proposition 23.7 for pairs over K. For $K = F$ this is the statement above.*

Level s. *Let ℓ be any prime, $s \geq 1$ with $s \geq 2$ if $\ell = 2$, and L, K ($F \subseteq K \subseteq F(\mu_{\ell^s})$) as in (Ex4) of Proposition 23.7. For the trivial pair $(L, \mathcal{O}_L)$ over K there is an $\mathbf{A}_{F,L}$-linear isomorphism $\psi\colon \mathbf{N}_s(\mathcal{O}_L) \to \mathbf{A}_{F,L}$ with $\iota_{s,\mathcal{O}_L} = (\iota_s \otimes \mathrm{id}_{\mathcal{O}_L}) \circ \psi$ (ι_s as in Lemma 23.8(6)); in particular $e^\circ_c = \psi^{-1}(e_c)$ satisfies $\iota_{s,\mathcal{O}_L}(e^\circ_c) = (\iota_s \otimes \mathrm{id})(e_c)$. For $s = 1$ and ℓ odd this is the paragraph over a base field.*

Proof. **The unit datum.** Put $T_0 = \mathcal{O}_L$, a free $\mathbb{Z}_\ell$-module of rank $[L : \mathbb{Q}_\ell]$ with trivial action, so $T_0[1/\ell]$ is crystalline. For an object (A, I) over $\mathcal{O}_F$ let $\mathcal{U}(A) = A \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$, with $\varphi_{\mathcal{U},A}(a \otimes (b \otimes \lambda)) =$

$ab' \otimes \lambda$ where $b' = \varphi_A(b)$, i.e. the linearisation of $\varphi_A \otimes \mathrm{id}$, an isomorphism already before inverting I; and for a morphism f let $\mathcal{U}(f) = f \otimes \mathrm{id}$. Then (Bs1) and (Bs2) of Theorem B.41 hold. For (Bs3) take the identity of $W(\mathbb{C}_\ell^\flat) \otimes \mathcal{O}_L$: it is G_F-equivariant, carries $\varphi_M \otimes \varphi = \varphi \otimes \mathrm{id}$ to $1 \otimes \varphi$, and restricts to the identity of $A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes \mathcal{O}_L$. For (Bs4): the elements $\varphi_M(1 \otimes m)$ generate $M = A_{\mathrm{inf}} \otimes \mathcal{O}_L$, so their B'^+-span is $B'^+ \otimes \mathcal{O}_L$; on the other side $(\mathcal{O}_L \otimes B'_{\mathrm{dR}})^{G_F} = \mathcal{O}_L \otimes (B'_{\mathrm{dR}})^{G_F}$ contains $\mathcal{O}_L \otimes 1$, so its span contains $B'^+ \otimes \mathcal{O}_L$, and equality holds because $(B'_{\mathrm{dR}})^{G_F} \subseteq B'^+$. For the last inclusion: φ is a G_F-equivariant automorphism of A_{inf} with $\tilde{\theta} \circ \varphi = \theta$, hence induces G_F-equivariant isomorphisms $B_{\mathrm{dR}}^+ \to B'^+$ and $B_{\mathrm{dR}} \to B'_{\mathrm{dR}}$; and if $x \in B_{\mathrm{dR}}^{G_F}$ with $x \in t^{-k}B_{\mathrm{dR}}^+ \setminus t^{-k+1}B_{\mathrm{dR}}^+$ for some $k \geq 1$, the image of $t^k x$ in $B_{\mathrm{dR}}^+/tB_{\mathrm{dR}}^+ = \mathbb{C}_\ell$ (Theorem B.40) is a non-zero c with $\chi_{\mathrm{cyc}}(g)^k gc = c$ for all $g \in G_F$, contradicting item 2 of Theorem B.42. So $\mathcal{U}$ is a datum attached to T_0 as in Theorem B.41, with comparison isomorphism $c_{\mathcal{U}} = \mathrm{id}$.

Comparison with the chosen datum. By (Ex5) of Proposition 23.7 (with $K = F$, $s = 1$) there are a datum $\mathcal{E} = \mathcal{E}_{T_0}$ given by Theorem B.41 for T_0 and a comparison c_{T_0} with the properties of (Bs3) such that $\mathbf{N}(\mathcal{O}_L) = \mathcal{E}(\mathbf{A}_F)$, $\iota_{\mathcal{O}_L} = c_{T_0} \circ \mathcal{E}(\iota)$ and $\lambda \in \mathcal{O}_L$ acts on $\mathbf{N}(\mathcal{O}_L)$ by $\mathcal{E}(\lambda)_{\mathbf{A}_F}$. Apply (Bs5) to $\mathrm{id}\colon T_0 \to T_0$ with the data $\mathcal{E}$, $\mathcal{U}$ and with $\mathcal{U}$, $\mathcal{E}$: we get A-linear maps $\psi_A = \mathcal{E}(\mathrm{id})_A\colon \mathcal{E}(A) \to \mathcal{U}(A)$ and $\psi'_A\colon \mathcal{U}(A) \to \mathcal{E}(A)$ with $\psi'_A \circ \psi_A = \mathcal{E}(\mathrm{id} \circ \mathrm{id})_A = \mathrm{id}$ and $\psi_A \circ \psi'_A = \mathrm{id}$, commuting with $\mathcal{E}(g), \mathcal{U}(g)$ for every morphism g, and with $c_{\mathcal{U}} \circ (\psi_{A_{\mathrm{inf}}} \otimes 1) = c_{T_0}$. Let $\psi = \psi_{\mathbf{A}_F}\colon \mathbf{N}(\mathcal{O}_L) \to \mathbf{A}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L = \mathbf{A}_{F,L}$, an $\mathbf{A}_F$-linear bijection. It is $\mathcal{O}_L$-linear: $\lambda \in \mathcal{O}_L$ acts on $\mathcal{E}(\mathbf{A}_F) = \mathbf{N}(\mathcal{O}_L)$ by $\mathcal{E}(\lambda)_{\mathbf{A}_F}$ ((Ex5) of Proposition 23.7) and on $\mathcal{U}(A)$ by $\mathcal{U}(\lambda)_A = \mathrm{id} \otimes \lambda$, and (Bs5) gives $\psi_A \circ \mathcal{E}(\lambda)_A = \mathcal{U}(\lambda)_A \circ \psi_A$ (both equal the map attached to $\lambda \circ \mathrm{id} = \mathrm{id} \circ \lambda$); hence ψ is $\mathbf{A}_{F,L}$-linear. Finally, for $x \in \mathbf{N}(\mathcal{O}_L)$,

$$\iota_{\mathcal{O}_L}(x) = c_{T_0}(\mathcal{E}(\iota)x) = c_{\mathcal{U}}(\psi_{A_{\mathrm{inf}}}(\mathcal{E}(\iota)x)) = c_{\mathcal{U}}(\mathcal{U}(\iota)(\psi x)) = (\iota \otimes \mathrm{id})(\psi x).$$

As ψ is $\mathbf{A}_{F,L}$-linear it maps $\mathbf{N}(\mathcal{O}_L)_c = e_c\mathbf{N}(\mathcal{O}_L)$ onto $e_c\mathbf{A}_{F,L}$, which gives the last assertion.

Over a base field. For the trivial pair over K the proof is the same with G_F replaced by G_K and objects and morphisms over $\mathcal{O}_F$ by those over $\mathcal{O}_K$, with (Ex5) of Proposition 23.7 for pairs over K (Theorem B.41 is stated for every finite extension of $\mathbb{Q}_\ell$). The places that involve the field: in (Bs3) the identity is G_K-equivariant; in (Bs4) one needs $(B'_{\mathrm{dR}})^{G_K} \subseteq B'^+$, and the argument given uses only that φ induces G_K-equivariant isomorphisms $B_{\mathrm{dR}}^+ \to B'^+$, $B_{\mathrm{dR}} \to B'_{\mathrm{dR}}$ and item 2 of Theorem B.42, which is stated for every finite extension of $\mathbb{Q}_\ell$, in particular for K; the comparison with the chosen datum uses (Bs5) and (Ex5) of Proposition 23.7 over K in the same way. No step uses that the base is unramified.

Level s. The proof over K applies without change to $K \subseteq F(\mu_{\ell^s})$ and any prime ℓ: the unit datum $\mathcal{U}$ and its properties (Bs1)–(Bs4) do not involve the prism $(\mathbf{A}_F, Q_{s-1}\mathbf{A}_F)$ or ℓ odd (Theorem B.41 and item 2 of Theorem B.42 hold for every prime and every finite extension); the comparison with the chosen datum uses (Bs5) and (Ex5) of Proposition 23.7 at level s, by which $\mathbf{N}_s(\mathcal{O}_L) = \mathcal{E}(\mathbf{A}_F)$ for the object $(\mathbf{A}_F, Q_{s-1}\mathbf{A}_F)$, $\iota_{s,\mathcal{O}_L} = c_{T_0} \circ \mathcal{E}(\iota_s)$ and $\lambda \in \mathcal{O}_L$ acts by $\mathcal{E}(\lambda)_{\mathbf{A}_F}$; the displayed computation then reads $\iota_{s,\mathcal{O}_L}(x) = c_{\mathcal{U}}(\mathcal{U}(\iota_s)(\psi x)) = (\iota_s \otimes \mathrm{id})(\psi x)$. □

Proposition 24.4. *Let ℓ, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ be as in Proposition 23.7; let (V', T'), (V, T) be pairs and $0 \to T' \xrightarrow{\iota'} T \xrightarrow{\pi} \mathcal{O}_L \to 0$ an exact sequence of G_F-equivariant $\mathcal{O}_L$-linear maps ($\mathcal{O}_L$ the trivial representation), with extension class $[T] \in H^1(F, T')$. Let $(M_a, h_a)_a$ be a window both of V' and of V as in (Ex1). Let $C \subseteq \mathbb{Z}/n$ be non-empty and $\mathrm{len}\colon C \to \mathbb{Z}_{\geq 1}$ such that for every $c \in C$: $c - \mathrm{len}_c \in C$ and $c - d \notin C$ for $1 \leq d < \mathrm{len}_c$. Let $j \geq 1$, $G \geq 0$, $g \geq G$, $m \geq \max(g, 1 + \max_a M_a)$, $R \geq m$ be integers with $j \leq \mathrm{len}_c$ and $M_{c+s} \leq G$ for all $c \in C$ and $1 \leq s \leq j$; put $\sigma = \sigma_\ell(g, m, j)$, and let I_c, $\rho_{c,j}\colon I_c \to H^1(F, T')$ be formed for $\mathbf{N}(T')$ with these c, j, G, g, m, R. Write $\varepsilon_a = (\iota \otimes \mathrm{id})(e_a) \in A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$. Let $r \geq 0$ and suppose that for every $c \in C$ there is $x_c \in \mathbf{N}(T)_c$ with*

$$(1 \otimes \pi)(\iota_T(x_c)) = \ell^r \varepsilon_c.$$

Then there are $\tau'_c \in P_j(\mathbf{N}(T')_c)$ *and* $w_c \in I_c$ *(*$c \in C$*) such that, with* $X_c = \ell^\sigma \varphi^j(x_c) - \mathbf{N}(\iota')(\tau'_c)$,

$$\mathbf{N}(\iota')(w_c) = \varphi^{\mathrm{len}_c - j}(X_{c-\mathrm{len}_c}) - \ell^\sigma x_c, \qquad (\gamma_u - 1)(X_c) \in \mathbf{N}(\iota')(X^m \mathbf{N}(T'))\ (u \in \mathbb{Z}_\ell^\times),$$

$-\ell^\sigma \tau'_c$ *is a relay correction of* w_c*, and*

$$\ell^{r+2\sigma}\,[T] = \sum_{c \in C} \rho_{c,j}(w_c) \qquad \text{in } H^1(F, T').$$

Proof. Write $\iota_L = \iota \otimes \mathrm{id}$, $p(c) = c - \mathrm{len}_c$, $N' = \mathbf{N}(T')$, $N = \mathbf{N}(T)$, and π_1 for the realisation of the trivial pair.

Step 0: two facts. (F1) The map ι is a morphism of δ-rings (Lemma 23.8(4)), so $(\varphi \otimes 1) \circ \iota_L = \iota_L \circ \varphi$; and $\sigma \circ \iota_L = \iota_L \circ \gamma_{\chi_{\mathrm{cyc}}(\sigma)}$ for $\sigma \in G_F$. As $\gamma_u(e_a) = e_a$ and $\varphi(e_a) = e_{a+1}$, the ε_a are G_F-invariant, $(\varphi \otimes 1)(\varepsilon_a) = \varepsilon_{a+1}$, and $\sum_a \varepsilon_a = 1$. (F2) If $u \in N[1/\lambda_\infty]$ and $(1 \otimes \pi)\iota_T(u) = 0$, then $u = \mathbf{N}(\iota')(t)$ for a unique $t \in N'[1/\lambda_\infty]$, and $t \in X^k N'$ if $u \in X^k N$. Indeed $\pi_1(\mathbf{N}(\pi)u) = (1 \otimes \pi)\iota_T(u) = 0$ and π_1 is injective on $\mathbf{N}(\mathcal{O}_L)[1/\lambda_\infty]$ (Lemma 23.29(Gc0)), so $\mathbf{N}(\pi)u = 0$; clearing poles by a product f of the Q_s and applying Lemma 23.33 gives t; if $u = X^k u'$ with $u' \in N$, then $X^k \mathbf{N}(\pi)(u') = 0$, so $\mathbf{N}(\pi)(u') = 0$, $u' = \mathbf{N}(\iota')(t')$ with $t' \in N'$, and $t = X^k t'$ by injectivity of $\mathbf{N}(\iota')$. Moreover $e_a t = t$ when $e_a u = u$.

Step 1: the arcs. For $c \in C$, len_c is the least $e \geq 1$ with $p(c) + e \in C$ (as $p(c) + e' = c - (\mathrm{len}_c - e') \notin C$ for $1 \leq e' < \mathrm{len}_c$), so c is determined by $p(c)$ and $p\colon C \to C$ is injective, hence bijective. Every $i \in \mathbb{Z}/n$ is $p(c) + k$ for exactly one pair (c, k) with $c \in C$, $0 \leq k < \mathrm{len}_c$: let $d \geq 0$ be least with $c_0 := i - d \in C$, $e \geq 1$ least with $c_0 + e \in C$, and $c = c_0 + e$; then $p(c) = c_0$, and $d < e = \mathrm{len}_c$ (otherwise $i - (d - e) = c_0 + e \in C$ contradicts the choice of d); conversely if $i = p(c) + k$ with $0 \leq k < \mathrm{len}_c$, then $p(c) + k - d' \notin C$ for $0 \leq d' < k$ (it equals $c - (\mathrm{len}_c - k + d')$), so $k = d$ and $p(c) = c_0$, which determines c. Hence $\sum_{i \in \mathbb{Z}/n} f(i) = \sum_{c \in C} \sum_{k < \mathrm{len}_c} f(p(c) + k)$ for every function f.

Step 2: the pieces at a cut. Fix $c \in C$. For $u \in \mathbb{Z}_\ell^\times$ and $\sigma_u \in G_F$ with $\chi_{\mathrm{cyc}}(\sigma_u) = u$, $(1 \otimes \pi)\iota_T(\gamma_u x_c) = \sigma_u(\ell^r \varepsilon_c) = \ell^r \varepsilon_c$ by (Ex2) and (F1); with (Wm3) and (F2), $(\gamma_u - 1)(x_c) = \mathbf{N}(\iota')(t_{c,u})$ with $t_{c,u} \in X N'_c$, and $u \mapsto t_{c,u}$ satisfies the cocycle relation of (Hf1) (it holds for $(\gamma_u - 1)(x_c)$, and $\mathbf{N}(\iota')$ is injective and commutes with γ_u). Let $W_c \in X N'_E$ be the solution of (Hf1) of Lemma 23.9 for $(t_{c,u})_u$. By uniqueness in (Hf1), $e_c W_c = W_c$. Let $(n_i)_i$ be an $e_c \mathbf{A}_{F,L}$-basis of N'_c (Wm1), completed by bases of the other sites to an $\mathbf{A}_{F,L}$-basis of N', and write $W_c = \sum_i W_i n_i$ with $W_i = \sum_{k \geq 1} \omega_{ik} X^k$, $\omega_{ik} \in e_c E$; by (Hf3), $\ell^{s_\ell(k)} \omega_{ik} \in \Lambda = \mathcal{O}_F \otimes \mathcal{O}_L$. Put $V_c = \sum_i \sum_{1 \leq k < R} \omega_{ik} X^k n_i \in N'_c[1/\ell]$, so that $V_c - W_c \in X^R N'_E$. As $\varphi^j(X) = X \lambda_j$ and $\ell^{s_\ell(k)} \varphi^j(\omega_{ik}) \in \Lambda$, (Rt1) of Lemma 24.6 writes each of the finitely many terms $\ell^\sigma \varphi^j(\omega_{ik})(X\lambda_j)^k$ as $\alpha_{ik} + X^m \lambda_j^g \beta_{ik}$ with $\alpha_{ik} \in \Lambda[[X]]$ and $\beta_{ik} \in \Lambda[[X]][1/\ell]$. Then $\tau'_c = \sum_{i,k} \alpha_{ik} \varphi^j(n_i) \in P_j(N'_c)$ and $\tau'_c - \ell^\sigma \varphi^j(V_c) = -X^m \lambda_j^g \sum_{i,k} \beta_{ik} \varphi^j(n_i) \in X^m \lambda_j^g P_j(N'_c)[1/\ell]$. Now $x_c \in I_c(N)$ (the set I_c formed for N, legitimate as (M_a, h_a) is a window of V), and by uniqueness in (Hf1), $\mathbf{N}(\iota')(W_c)$ is the element W_{x_c} for N. Since $\mathbf{N}(\iota')$ maps $P_j(N'_c)$ into $P_j(N_c)$ and $X^R N'_E$ into $X^R N_E$, $\mathbf{N}(\iota')(\tau'_c)$ is a relay correction of x_c for N, with $\mathbf{N}(\iota')(V_c)$ in the role of V in the definition of a relay correction. By (Ry2) for N, $(\gamma_u - 1)(X_c) \in X^m N$. Also, by (Ex2), (F1) and $\pi \circ \iota' = 0$,

$$(1 \otimes \pi)\iota_T(X_c) = \ell^\sigma (\varphi \otimes 1)^j (\ell^r \varepsilon_c) = \ell^{\sigma + r} \varepsilon_{c+j}, \qquad (1)$$

which is G_F-invariant; so $(1 \otimes \pi)\iota_T((\gamma_u - 1)X_c) = 0$, and (F2) gives $(\gamma_u - 1)(X_c) = \mathbf{N}(\iota')(X^m \mu_{c,u})$ with $\mu_{c,u} \in N'$.

Step 3: the global lift and its Herr vector. Put $\tilde{x} = \sum_{c \in C} \sum_{0 \leq k < \mathrm{len}_c} \varphi^k(X_{p(c)}) \in N[1/\lambda_\infty]$. By (1), (F1) and Step 1,

$$(1 \otimes \pi)\iota_T(\tilde{x}) = \sum_{c \in C} \sum_{k < \mathrm{len}_c} \ell^{\sigma + r} \varepsilon_{p(c)+j+k} = \ell^{\sigma + r} \sum_{i \in \mathbb{Z}/n} \varepsilon_{i+j} = 1 \otimes \ell^{\sigma + r}.$$

Further $(\gamma_u-1)(\tilde{x}) = \mathbf{N}(\iota')(\sum_{c,k}\varphi^k(X^m\mu_{p(c),u})) \in \mathbf{N}(\iota')(X^mN')$ by Lemma 23.30(1) for N'. By Lemma 23.32 (with $c=\ell^{\sigma+r}$), there is $y\in N'[1/\lambda_\infty]$ with $\mathbf{N}(\iota')(y)=(\varphi-1)(\tilde{x})$; it is a Herr vector of precision m and $\mathrm{cl}(y)=\ell^{\sigma+r}[T]$.

Step 4: the arc pieces. For $c\in C$ put $u_c=\varphi^{\mathrm{len}_c-j}(X_{p(c)})-\ell^\sigma x_c\in e_cN[1/\lambda_\infty]$. By (1) and $p(c)+\mathrm{len}_c=c$, $(1\otimes\pi)\iota_T(u_c)=\ell^{\sigma+r}\varepsilon_c-\ell^{\sigma+r}\varepsilon_c=0$, so $u_c=\mathbf{N}(\iota')(w_c)$ with $w_c\in e_cN'[1/\lambda_\infty]$ by (F2). Applying γ_u-1 and injectivity of $\mathbf{N}(\iota')$,

$$(\gamma_u-1)(w_c)=\varphi^{\mathrm{len}_c-j}(X^m\mu_{p(c),u})-\ell^\sigma t_{c,u}\in X^mN'+XN'\subseteq XN',$$

using Lemma 23.30(1); so $w_c\in I_c$. The family $z''_u=\varphi^{\mathrm{len}_c-j}(X^m\mu_{p(c),u})=(\gamma_u-1)(w_c)+\ell^\sigma t_{c,u}$ lies in X^mN' and satisfies the cocycle relation (a sum of two cocycles); let $W''\in X^mN'_E$ be its solution in (Hf1). Then $W''-\ell^\sigma W_c\in XN'_E$ solves the equation of w_c, so $W_{w_c}=W''-\ell^\sigma W_c$. Truncating the coordinates of W'' below degree R gives $V''\in X^mN'_c[1/\ell]$ with $V''-W''\in X^RN'_E$ ((Hf3); the coordinates have no terms of degree $<m$). With $V=V''-\ell^\sigma V_c$ we have $V-W_{w_c}\in X^RN'_E$ and

$$-\ell^\sigma\tau'_c-\ell^\sigma\varphi^j(V)=-\ell^\sigma(\tau'_c-\ell^\sigma\varphi^j(V_c))-\ell^\sigma\varphi^j(V'')\in X^m\lambda_j^gP_j(N'_c)[1/\ell],$$

since $\varphi^j(X^mN'_c[1/\ell])=X^m\lambda_j^mP_j(N'_c)[1/\ell]$ and $m\ge g$. As $-\ell^\sigma\tau'_c\in P_j(N'_c)$, it is a relay correction of w_c, and $\rho_{c,j}(w_c)=\mathrm{cl}(v_c)$ with $v_c=\ell^\sigma\varphi^j(w_c)+\ell^\sigma\tau'_c$, a Herr vector of precision m by (Ry2), (Ry3).

Step 5: telescoping. By definition of w_c and X_c,

$$\mathbf{N}(\iota')(v_c)=\ell^\sigma\varphi^{\mathrm{len}_c}(X_{p(c)})-\ell^\sigma(\ell^\sigma\varphi^j(x_c)-\mathbf{N}(\iota')(\tau'_c))=\ell^\sigma(\varphi^{\mathrm{len}_c}(X_{p(c)})-X_c).$$

On the other hand $(\varphi-1)(\tilde{x})=\sum_{c\in C}(\varphi^{\mathrm{len}_c}(X_{p(c)})-X_{p(c)})=\sum_{c\in C}(\varphi^{\mathrm{len}_c}(X_{p(c)})-X_c)$, the second equality because p is a bijection of C (Step 1). Hence $\mathbf{N}(\iota')(\sum_c v_c)=\mathbf{N}(\iota')(\ell^\sigma y)$ and, by injectivity, $\sum_{c\in C}v_c=\ell^\sigma y$.

Step 6: classes. By (Hc3) of Lemma 23.28 (Herr vectors of precision m form an $\mathcal{O}_L$-module on which cl is linear),

$$\sum_{c\in C}\rho_{c,j}(w_c)=\sum_{c\in C}\mathrm{cl}(v_c)=\mathrm{cl}(\ell^\sigma y)=\ell^\sigma\,\mathrm{cl}(y)=\ell^{\,r+2\sigma}[T].$$

□

Lemma 24.5. *Let ℓ, F, L, (τ_a) and a pair (V,T) be as in Proposition 23.7, $\mathbf{N}=\mathbf{N}(T)$ with a window $(M_a,h_a)_a$ as in (Ex1). Let $c\in\mathbb{Z}/n$ and integers $j\ge1$, $G\ge0$, $g\ge G$, $m\ge1+\max_a M_a$ and $R\ge\max(m,g)$, with $M_{c+s}\le G$ for $1\le s\le j$, and put $\sigma=\sigma_\ell(g,m,j)$. Let I_c be the set of $w\in e_c\mathbf{N}[1/\lambda_\infty]$ with $(\gamma_u-1)(w)\in X\mathbf{N}$ for all $u\in\mathbb{Z}_\ell^\times$; for $w\in I_c$ let $W_w\in X\mathbf{N}_E$ be the unique element with $(\gamma_u-1)(W_w)=(\gamma_u-1)(w)$ for all u ((Hf1) with $z_u=(\gamma_u-1)(w)$). An element $\tau\in P_j(\mathbf{N}_c)$ is a* relay correction *of $w\in I_c$ if there is $V\in\mathbf{N}_c[1/\ell]$ with*

$$V-W_w\in X^R\mathbf{N}_E\qquad\text{and}\qquad\tau-\ell^\sigma\varphi^j(V)\in X^m\lambda_j^g\,P_j(\mathbf{N}_c)[1/\ell].$$

(Ry0) Poles $P_j(\mathbf{N}_c)\subseteq\lambda_j^{-G}\mathbf{N}_{c+j}$.

(Ry1) Existence *Every $w\in I_c$ has a relay correction.*

(Ry2) Precision *If τ is a relay correction of w, then $\ell^\sigma\varphi^j(w)-\tau$ is a Herr vector of precision m.*

(Ry3) The relay class $\rho_{c,j}(w) := \mathrm{cl}(\ell^\sigma\varphi^j(w)-\tau) \in H^1(F,T)$ *does not depend on the relay correction* τ.

(Ry4) Linearity I_c *is an* $\mathcal{O}_L$*-submodule of* $e_c\mathbf{N}[1/\lambda_\infty]$ *and* $\rho_{c,j}\colon I_c \to H^1(F,T)$ *is* $\mathcal{O}_L$*-linear.*

(Ry5) Vanishing *If* $w \in X\mathbf{N}_c$*, then* $w \in I_c$ *and* $\rho_{c,j}(w)=0$.

(Ry6) Naturality *Let* (V',T') *be another pair with a window* (M'_a,h'_a) *such that* $m \geq 1+\max_a M'_a$ *and* $M'_{c+s} \leq G$ *for* $1\leq s\leq j$*, and* $f\colon T\to T'$ *a* G_F*-equivariant* $\mathcal{O}_L$*-linear map. Then* $\mathbf{N}(f)(I_c)\subseteq I'_c$ *and* $\rho'_{c,j}(\mathbf{N}(f)(w)) = f_*\rho_{c,j}(w)$*, where* I'_c*,* $\rho'_{c,j}$ *are formed for* $\mathbf{N}(T')$ *with the same* c,j,G,g,m,R.

(Ry7) Crystalline classes *If* $w\in I_c$ *and* $fw\in\mathbf{N}$ *for some finite product* f *of the* Q_s *(*$s\geq 0$*), then* $\rho_{c,j}(w)\in H^1_f(F,T)$*, the group of Proposition 23.27.*

Proof. Fix a site a and write $R_a = e_a\mathbf{A}_{F,L}\cong\mathcal{O}_L[[X]]$, ϖ a uniformiser of $\mathcal{O}_L$.

Two intersection facts. (C1) For $k,G'\geq 0$: $X^kR_a[1/\ell]\cap\lambda_j^{-G'}R_a = X^kR_a$ inside the fraction field. Indeed, if $f=g_1/\lambda_j^{G'} = X^kh/\ell^b$ with $g_1,h\in R_a$, then $\ell^b g_1 = X^k\lambda_j^{G'}h$; R_a is a unique factorisation domain in which ϖ is prime and does not divide $X^k\lambda_j^{G'}$ (its reduction is $X^{k+G'\deg\lambda_j}$, since each Q_s is monic and $\equiv X^{\deg Q_s} \bmod \ell$ by Lemma 23.8(3)); so $\ell^b \mid h$ and $f\in X^kR_a$. (C2) $R_a[1/\ell]\cap X^kL[[X]] = X^kR_a[1/\ell]$, by comparing coefficients. Since $\mathbf{N}_a$ is free over R_a (Wm1), (C1) and (C2) hold for $\mathbf{N}_a$ in place of R_a, coordinatewise in a basis: $X^k\mathbf{N}_a[1/\ell]\cap\lambda_j^{-G'}\mathbf{N}_a = X^k\mathbf{N}_a$ and $\mathbf{N}_a[1/\ell]\cap X^k\mathbf{N}_E = X^k\mathbf{N}_a[1/\ell]$ (here $\mathbf{N}_a[1/\ell]\subseteq e_a\mathbf{N}_E$, the elements Q_s, λ_j being units of $E[[X]]$ as $Q_s(0)=\ell$).

(Ry0). By induction on $i\leq j$: $\varphi^i(\mathbf{N}_c)\subseteq\prod_{s=0}^{i-1}Q_s^{-M_{c+i-s}}\mathbf{N}_{c+i}$. For $i=1$ this is (Wm2). If it holds for $i<j$, then, as φ is semilinear with $\varphi(Q_s)=Q_{s+1}$, $\varphi^{i+1}(\mathbf{N}_c)\subseteq\prod_{s=0}^{i-1}Q_{s+1}^{-M_{c+i-s}}\varphi(\mathbf{N}_{c+i})\subseteq\prod_{s=0}^{i-1}Q_{s+1}^{-M_{c+i-s}}\cdot Q_0^{-M_{c+i+1}}\mathbf{N}_{c+i+1}$, which is the claim for $i+1$. For $i=j$ all exponents are $M_{c+j-s}\leq G$, so $\varphi^j(\mathbf{N}_c)\subseteq\lambda_j^{-G}\mathbf{N}_{c+j}$, and the same holds for the module $P_j(\mathbf{N}_c)$ it generates.

(Ry1). Let $n_1,\dots,n_r$ be an R_c-basis of $\mathbf{N}_c$ and $W_w=\sum_i W_in_i$, $W_i=\sum_{k\geq 1}a_{ik}X^k$ with $a_{ik}\in e_cE$. By (Hf3) of Lemma 23.9, $\ell^{s_\ell(k)}a_{ik}\in\Lambda$. Put $V=\sum_i\sum_{1\leq k<R}a_{ik}X^kn_i\in\mathbf{N}_c[1/\ell]$; then $V-W_w\in X^R\mathbf{N}_E$. Now $\varphi^j(V)=\sum_i\sum_{k<R}\varphi^j(a_{ik})(X\lambda_j)^k\varphi^j(n_i)$, and $\ell^{s_\ell(k)}\varphi^j(a_{ik})\in\Lambda$. By (Rt1) of Lemma 24.6, $\ell^\sigma\varphi^j(a_{ik})(X\lambda_j)^k = f_{ik}+X^m\lambda_j^gh_{ik}$ with $f_{ik}\in\Lambda[[X]]$ and $h_{ik}\in\Lambda[[X]][1/\ell]$. Then $\tau=\sum_{i,k}f_{ik}\varphi^j(n_i)\in P_j(\mathbf{N}_c)$ and $\tau-\ell^\sigma\varphi^j(V) = -X^m\lambda_j^g\sum_{i,k}h_{ik}\varphi^j(n_i)\in X^m\lambda_j^gP_j(\mathbf{N}_c)[1/\ell]$.

(Ry2). Let τ, V be as in the definition, $d=\tau-\ell^\sigma\varphi^j(V)$, and $v=\ell^\sigma\varphi^j(w)-\tau = \ell^\sigma\varphi^j(w-V)-d$. Since γ_u commutes with φ (Wm3),

$$(\gamma_u-1)(v) = \ell^\sigma\varphi^j((\gamma_u-1)(w-V)) - (\gamma_u-1)(d).$$

First, $(\gamma_u-1)(w-V)=(\gamma_u-1)(W_w-V)$ lies in $X^R\mathbf{N}_E$ ($\gamma_u(X)\in X\mathbb{Z}_\ell[[X]]^\times$; $e_c\mathbf{N}[1/\lambda_\infty]\subseteq e_c\mathbf{N}_E$ as the Q_s are units of $E[[X]]$) and in $\mathbf{N}_c[1/\ell]$ (as $(\gamma_u-1)(w)\in X\mathbf{N}_c$ and $V\in\mathbf{N}_c[1/\ell]$), hence in $X^R\mathbf{N}_c[1/\ell]$ by (C2); so, using $\varphi^j(X)=X\lambda_j$ and (Ry0), $\ell^\sigma\varphi^j((\gamma_u-1)(w-V))\in X^R\lambda_j^{R-G}\mathbf{N}_{c+j}[1/\ell]\subseteq X^m\mathbf{N}_{c+j}[1/\ell]$, as $R\geq m$ and $R\geq g\geq G$. Second, γ_u preserves $P_j(\mathbf{N}_c)$ (it preserves $\mathbf{N}_c$ and commutes with φ) and $\gamma_u(X^m\lambda_j^g)\in X^m\lambda_j^g\mathbb{Z}_\ell[[X]]^\times$ (Lemma 23.8(2)), so $(\gamma_u-1)(d)\in X^m\lambda_j^gP_j(\mathbf{N}_c)[1/\ell]\subseteq X^m\lambda_j^{g-G}\mathbf{N}_{c+j}[1/\ell]\subseteq X^m\mathbf{N}_{c+j}[1/\ell]$. Hence $(\gamma_u-1)(v)\in X^m\mathbf{N}_{c+j}[1/\ell]$. On the other hand $(\gamma_u-1)(v)=\ell^\sigma\varphi^j((\gamma_u-1)(w))-(\gamma_u-1)(\tau)$ with $(\gamma_u-1)(w)\in X\mathbf{N}_c$ and $(\gamma_u-1)(\tau)\in P_j(\mathbf{N}_c)$, so $(\gamma_u-1)(v)\in P_j(\mathbf{N}_c)\subseteq\lambda_j^{-G}\mathbf{N}_{c+j}$ by (Ry0). By (C1), $(\gamma_u-1)(v)\in X^m\mathbf{N}$ for every u, and $v\in\mathbf{N}[1/\lambda_\infty]$: v is a Herr vector of precision m.

(Ry3). Let τ and τ' be two relay corrections of w, with elements V and V' as in the definition. Then $V'-V \in \mathbf{N}_c[1/\ell] \cap X^R\mathbf{N}_E = X^R\mathbf{N}_c[1/\ell]$ (C2), so $\ell^\sigma\varphi^j(V'-V) \in X^R\lambda_j^R P_j(\mathbf{N}_c)[1/\ell] \subseteq X^m\lambda_j^g P_j(\mathbf{N}_c)[1/\ell]$. Hence $\tau'-\tau \in P_j(\mathbf{N}_c) \cap X^m\lambda_j^g P_j(\mathbf{N}_c)[1/\ell] \subseteq \lambda_j^{-G}\mathbf{N}_{c+j} \cap X^m\mathbf{N}_{c+j}[1/\ell] = X^m\mathbf{N}_{c+j}$, by (Ry0), $g \geq G$ and (C1). By (Hc4) of Lemma 23.28, $\tau'-\tau$ is a Herr vector of precision m with class 0, and by (Hc3), $\mathrm{cl}(\ell^\sigma\varphi^j(w)-\tau) = \mathrm{cl}(\ell^\sigma\varphi^j(w)-\tau') + \mathrm{cl}(\tau'-\tau) = \mathrm{cl}(\ell^\sigma\varphi^j(w)-\tau')$.

(Ry4). I_c is an $\mathcal{O}_L$-submodule since the γ_u are additive and $\mathcal{O}_L$-linear. By the uniqueness in (Hf1), $W_{w_1+\lambda w_2} = W_{w_1} + \lambda W_{w_2}$ for $\lambda \in \mathcal{O}_L$; so if (τ_i, V_i) are relay corrections of w_i, then $(\tau_1+\lambda\tau_2, V_1+\lambda V_2)$ is one of $w_1+\lambda w_2$, and (Hc3) gives $\rho_{c,j}(w_1+\lambda w_2) = \rho_{c,j}(w_1) + \lambda\rho_{c,j}(w_2)$.

(Ry5). If $w \in X\mathbf{N}_c$, then $(\gamma_u - 1)(w) \in X\mathbf{N}$ by (Wm3) and $\gamma_u(X) \in X\mathbb{Z}_\ell[[X]]$, so $w \in I_c$, and $W_w = w$ by uniqueness. With $V = w$, $\tau = \ell^\sigma\varphi^j(w) \in P_j(\mathbf{N}_c)$ is a relay correction, and $\rho_{c,j}(w) = \mathrm{cl}(0) = 0$.

(Ry6). $\mathbf{N}(f)$ is $\mathbf{A}_{F,L}$-linear and commutes with φ and γ_u (Ex3); extend it $E[[X]]$-linearly to $\mathbf{N}_E \to \mathbf{N}(T')_E$, where it commutes with γ_u and preserves $X^k\mathbf{N}_E$. It maps $e_c\mathbf{N}[1/\lambda_\infty]$ into $e_c\mathbf{N}(T')[1/\lambda_\infty]$, $\mathbf{N}_c$ into $\mathbf{N}(T')_c$ and I_c into I'_c, and $W_{\mathbf{N}(f)w} = \mathbf{N}(f)(W_w)$ by uniqueness in (Hf1). If (τ, V) is a relay correction of w, then $(\mathbf{N}(f)\tau, \mathbf{N}(f)V)$ is one of $\mathbf{N}(f)w$, as $\mathbf{N}(f)(P_j(\mathbf{N}_c)) \subseteq P_j(\mathbf{N}(T')_c)$. Hence $\rho'_{c,j}(\mathbf{N}(f)w) = \mathrm{cl}(\mathbf{N}(f)(\ell^\sigma\varphi^j(w)-\tau)) = f_*\rho_{c,j}(w)$ by (Hc5) of Lemma 23.28.

(Ry7). Let τ be a relay correction of w (Ry1) and $v = \ell^\sigma\varphi^j(w)-\tau$, a Herr vector of precision m by (Ry2), with $\rho_{c,j}(w) = \mathrm{cl}(v)$. As $w \in e_c\mathbf{N}[1/\lambda_\infty]$, $fw \in e_c\mathbf{N} = \mathbf{N}_c$, so $\varphi^j(f)\varphi^j(w) = \varphi^j(fw) \in P_j(\mathbf{N}_c) \subseteq \lambda_j^{-G}\mathbf{N}$ by (Ry0), and $\tau \in P_j(\mathbf{N}_c) \subseteq \lambda_j^{-G}\mathbf{N}$. Hence, with the finite product $f' = \lambda_j^G\varphi^j(f)$ of the Q_s ($\varphi^j(Q_s) = Q_{s+j}$),

$$f'v = \ell^\sigma\lambda_j^G\varphi^j(fw) - \varphi^j(f)\,\lambda_j^G\tau \in \mathbf{N}.$$

Since $m \geq 1 + \max_a M_a$, Proposition 23.27 gives $\rho_{c,j}(w) = \mathrm{cl}(v) \in H^1_f(F,T)$. □

Lemma 24.6. *Let ℓ be an odd prime, F, L as in Definition 23.2, $E = \Lambda[1/\ell]$, and let φ also denote the endomorphism of $E[[X]]$ that is $\varphi \otimes \mathrm{id}$ on E and sends X to $(1+X)^\ell - 1$; thus $\varphi^j(X) = X\lambda_j$. For integers $j \geq 1$, $g \geq 0$, $m \geq 0$ put*

$$N_j = (\ell-1)\ell^{j-1}, \qquad \sigma_\ell(g,m,j) = \Big\lfloor \Big(g + \frac{m}{N_j}\Big)\frac{\ell}{(\ell-1)^2} \Big\rfloor + 1.$$

Write $\sigma = \sigma_\ell(g,m,j)$.

(Rt1) One term *For every $k \geq 0$ and every $c \in E$ with $\ell^{s_\ell(k)}c \in \Lambda$,*

$$\ell^\sigma c\,(X\lambda_j)^k \in \Lambda[[X]] + X^m\lambda_j^g\,\Lambda[[X]][1/\ell].$$

(Rt2) Series *For every $W = \sum_{k\geq 0} a_k X^k \in E[[X]]$ with $\ell^{s_\ell(k)}a_k \in \Lambda$ for all k,*

$$\ell^\sigma\varphi^j(W) \in \Lambda[[X]] + X^m\lambda_j^g\,E[[X]].$$

Proof. **An ideal-power inclusion.** Let A be a commutative ring, $a, Y, p \in A$, and $N \geq 1$, $M, z, t \geq 0$ integers with $p \in aA + Y^NA$ and $t + \lceil M/N \rceil \leq z+1$. Then $p^z \in a^tA + Y^MA$. Indeed, write $p = aa' + Y^Nb'$; in $p^z = \sum_{i=0}^z \binom{z}{i} a^i a'^i Y^{N(z-i)} b'^{z-i}$ the terms with $i \geq t$ lie in a^tA, and for $i < t$ we have $z - i \geq z - t + 1 \geq \lceil M/N \rceil$, so $N(z-i) \geq M$ and the term lies in Y^MA.

Proof of (Rt1). Put $\kappa = g + m/N_j$. Since $\ell \geq 3$, $\ell < (\ell-1)^2$, so the function $\rho(x) = x\ell/(\ell-1)^2 - x + \kappa$ is strictly decreasing, with $\rho(\kappa) = \kappa\ell/(\ell-1)^2 < \sigma$ (definition of σ). Recall $s_\ell(k) \leq k\ell/(\ell-1)^2$ (Lemma 23.9).

Case $k \le \kappa$. Then $s_\ell(k) \le \kappa\ell/(\ell-1)^2 < \sigma$, so $\ell^\sigma c = \ell^{\sigma - s_\ell(k)} \cdot \ell^{s_\ell(k)} c \in \Lambda$ and $\ell^\sigma c (X\lambda_j)^k \in \Lambda[[X]]$.

Case $k > \kappa$. Then $z := k - g > m/N_j$, so $z \ge \lceil m/N_j \rceil$ and $t := z + 1 - \lceil m/N_j \rceil \ge 1$. By Lemma 23.8(3), Q_{j-1} is a monic polynomial congruent to X^{N_j} modulo ℓ, so $Q_{j-1} \in \ell\,\mathbb{Z}_\ell[[X]] + X^{N_j}\mathbb{Z}_\ell[[X]]$. The inclusion above (with $A = \mathbb{Z}_\ell[[X]]$, $a = \ell$, $Y = X$, $p = Q_{j-1}$, $N = N_j$, $M = m$) gives $Q_{j-1}^z \in \ell^t \mathbb{Z}_\ell[[X]] + X^m \mathbb{Z}_\ell[[X]]$. As $k = g + z$ and $\lambda_j = \lambda_{j-1} Q_{j-1}$,

$$(X\lambda_j)^k = \lambda_j^g \cdot X^k \lambda_{j-1}^z \cdot Q_{j-1}^z \in \ell^t \mathbb{Z}_\ell[[X]] + X^m \lambda_j^g \mathbb{Z}_\ell[[X]],$$

say $(X\lambda_j)^k = \ell^t f + X^m \lambda_j^g h$. Moreover

$$s_\ell(k) - t \le \frac{k\ell}{(\ell-1)^2} - k + g - 1 + \Big\lceil \frac{m}{N_j} \Big\rceil < \rho(k) < \rho(\kappa) < \sigma,$$

using $\lceil x \rceil < x+1$. Hence $\ell^{\sigma+t} c = \ell^{\sigma+t-s_\ell(k)} \cdot \ell^{s_\ell(k)} c \in \Lambda$, and $\ell^\sigma c (X\lambda_j)^k = (\ell^{\sigma+t} c) f + X^m \lambda_j^g (\ell^\sigma c h)$ is of the required form, since $\ell^\sigma c h \in E \cdot \mathbb{Z}_\ell[[X]] \subseteq \Lambda[[X]][1/\ell]$.

Proof of (Rt2). $\varphi^j(W) = \sum_k \varphi^j(a_k)(X\lambda_j)^k$, the series converging X-adically since $(X\lambda_j)^k \in X^k \mathbb{Z}_\ell[[X]]$; and $\ell^{s_\ell(k)} \varphi^j(a_k) = \varphi^j(\ell^{s_\ell(k)} a_k) \in \Lambda$, as $\varphi \otimes \mathrm{id}$ preserves Λ. Let $k_0 = \max(m, g)$. For $k \ge k_0$, $(X\lambda_j)^k = X^m \lambda_j^g \cdot X^{k-m} \lambda_j^{k-g}$, so

$$\sum_{k \ge k_0} \ell^\sigma \varphi^j(a_k)(X\lambda_j)^k = X^m \lambda_j^g \sum_{k \ge k_0} \ell^\sigma \varphi^j(a_k) X^{k-m} \lambda_j^{k-g} \in X^m \lambda_j^g E[[X]],$$

the inner series converging X-adically. The finitely many terms with $k < k_0$ are of the required form by (Rt1) applied to $c = \varphi^j(a_k)$, as $\Lambda[[X]][1/\ell] \subseteq E[[X]]$. □

Lemma 24.7. *Let ℓ, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ be as in Proposition 23.7, and let $\Phi\colon T_1 \to T_2$ be a G_F-equivariant $\mathcal{O}_L$-linear surjection of the lattices of pairs (V_1, T_1), (V_2, T_2). Let $M_a, h_a \ge 0$ be integers such that every Hodge jump of V_1 at τ_a lies in $[-M_a, h_a]$ $(a \in \mathbb{Z}/n)$, let J be the set of integers that are Hodge jumps of V_1 at some τ_a, let $G \ge \max_a M_a$ be an integer, and put*

$$J'_G = \{j + k :\ j \in J,\ 1 \le k \le G,\ j + k \ne 0\}, \qquad \kappa_\ell(J, G) = \sum_{x \in J'_G} v_\ell(x).$$

For $\nu = 1, 2$ and $c \in \mathbb{Z}/n$ let I_c^ν be the set of $w \in e_c \mathbf{N}(T_\nu)[1/\lambda_\infty]$ with $(\gamma_s - 1)(w) \in X\mathbf{N}(T_\nu)$ for all $s \in \mathbb{Z}_\ell^\times$. Let $c \in \mathbb{Z}/n$ and $r \ge 0$ be such that for every $y \in \mathbf{N}(T_2)_c$ there is $x \in \mathbf{N}(T_1)_c$ with $\mathbf{N}(\Phi)(x) = \ell^r y$. Then for every $w \in I_c^2$ with $Q^G w \in \mathbf{N}(T_2)$ there is $u \in I_c^1$ with $Q^G u \in \mathbf{N}(T_1)$ and

$$\ell^{\,r + \kappa_\ell(J,G)}\, w - \mathbf{N}(\Phi)(u) \in X\, \mathbf{N}(T_2)_c.$$

Proof. By (Ex1), $\mathbf{N}_1$ and $\mathbf{N}_2$ are Wach modules (for $\mathbf{N}_1$ with the window (M_a, h_a); for $\mathbf{N}_2$ only (Wm1) and (Wm3) are used, valid for any window of V_2). So each $\mathbf{N}_{\nu,c}$ is free of finite rank over the domain A_c, and $\mathbf{N}_\nu[1/\lambda_\infty]$ is torsion-free over $\mathbf{A}_{F,L}[1/Q_s :\ s \ge 0]$.

Step 0: the invariant uniformiser. (i) For $\delta \in \Delta$, $\gamma_\delta((1+X)^a) = (1+X)^{\delta a}$ $(a \in \mathbb{Z}_\ell)$, so γ_δ permutes the summands of π_0 and $\gamma_\delta(\pi_0) = \pi_0$; since the γ_s commute, $\gamma_s(\pi_0)$ is also Δ-fixed. Items (ii)–(iv) are Lemma 23.24 for $\Delta = \mu_{\ell-1}$, where $e = 1$, $\pi_\Delta = \pi_0$ and $\Gamma_\Delta = \mu_{\ell-1}(1 + \ell\mathbb{Z}_\ell) = \mathbb{Z}_\ell^\times$. (ii) By (Rr1), $\pi_0 = X^{\ell-1} v_0$ with $v_0 \in \mathbb{Z}_\ell[[X]]^\times$. (iii) Put $q_0 = \pi_0 + \ell$; by (Rr3) $q_0 = q_\Delta$, so by (Rr4) $q_0 = Q\, v_1$ with $v_1 \in \mathbb{Z}_\ell[[X]]^\times$. In particular $Q^G z \in \mathbf{N}_\nu$ if and only if $q_0^G z \in \mathbf{N}_\nu$. (iv) For $s \in \mathbb{Z}_\ell^\times = \Gamma_\Delta$, by (Rr6) $\gamma_s(q_0) = q_0 h_s$ with $h_s \in 1 + X\mathbb{Z}_\ell[[X]]$ Δ-fixed; h_s is a unit, its constant term being 1.

Step 1: $X\mathbf{N}_{\nu,c} \cap \mathbf{N}_{\nu,c}^{\Delta} = \pi_0 \mathbf{N}_{\nu,c}^{\Delta}$**.** The inclusion $\supseteq$ holds by (i), (ii). Let δ_0 be a generator of Δ; then $\delta_0^i - 1 \in \mathbb{Z}_\ell^\times$ for $1 \le i \le \ell - 2$. Let $y \in \mathbf{N}_{\nu,c}^{\Delta}$ with $y = X^i y_i$, $y_i \in \mathbf{N}_{\nu,c}$, for some $1 \le i \le \ell - 2$. Write $\gamma_{\delta_0}(X) = (1+X)^{\delta_0} - 1 = X(\delta_0 + Xb)$ with $b \in \mathbb{Z}_\ell[[X]]$. Then $X^i y_i = \gamma_{\delta_0}(y) = X^i(\delta_0 + Xb)^i \gamma_{\delta_0}(y_i)$; cancelling X^i (no X-torsion) and reducing modulo $X\mathbf{N}_{\nu,c}$, with $\gamma_{\delta_0}(y_i) \equiv y_i$ by (Wm3), gives $(\delta_0^i - 1)y_i \in X\mathbf{N}_{\nu,c}$, so $y_i \in X\mathbf{N}_{\nu,c}$. By induction from $i = 1$, $y \in X^{\ell-1}\mathbf{N}_{\nu,c} = \pi_0 \mathbf{N}_{\nu,c}$ (by (ii)); if $y = \pi_0 y'$, then $\pi_0 \gamma_\delta(y') = \gamma_\delta(y) = \pi_0 y'$, so y' is Δ-fixed.

Step 2: the normalised operator. Fix $g = 1 + \ell$, so $v_\ell(g-1) = 1$, and let $\beta(x) \in \mathbb{Z}_\ell$ be the numbers with $\ell\beta(x) = 1 - g^{-x}$ (the $c(x)$ of Lemma 24.8 for $u = g$). For $H \ge 0$ put $D_\nu(H) = \{z \in (e_c \mathbf{N}_\nu[1/\lambda_\infty])^{\Delta} : \ q_0^H z \in \mathbf{N}_\nu\}$; thus $D_\nu(0) = \mathbf{N}_{\nu,c}^{\Delta}$. Claim: for $z \in D_\nu(H)$ and $s \in \mathbb{Z}_\ell^\times$, $(\gamma_s - 1)(z) = X\, z'$ with $z' \in q_0^{-H}\mathbf{N}_{\nu,c}$, and $(\gamma_g - 1)(z) = \pi_0\, \delta z$ for a unique $\delta z \in D_\nu(H)$. Indeed, with $n = q_0^H z \in \mathbf{N}_{\nu,c}^{\Delta}$ (by (i)),

$$q_0^H(\gamma_s - 1)(z) = h_s^{-H}\gamma_s(n) - n = (h_s^{-H} - 1)\gamma_s(n) + (\gamma_s - 1)(n) \in X\mathbf{N}_{\nu,c}$$

by (iv) and (Wm3); for $s = g$ this element is Δ-fixed (the γ's commute and q_0 is Δ-fixed), so by Step 1 it equals $\pi_0 n'$ with $n' \in \mathbf{N}_{\nu,c}^{\Delta}$, and $\delta z = q_0^{-H} n'$; uniqueness holds as π_0 is a non-zero-divisor. So δ is an $\mathcal{O}_L$-linear endomorphism of each $D_\nu(H)$, in particular of $\mathbf{N}_{\nu,c}^{\Delta}$, and every polynomial $P \in \mathbb{Z}_\ell[Y]$ gives $P(\delta)$ on these sets. Since $\mathbf{N}(\Phi)$, extended to $\mathbf{N}_1[1/\lambda_\infty]$, is $\mathbf{A}_{F,L}[1/Q_s]$-linear and commutes with all γ_s (Ex3), it maps $D_1(H)$ into $D_2(H)$ and $\mathbf{N}(\Phi) \circ \delta = \delta \circ \mathbf{N}(\Phi)$ there.

Step 3: reduction to Δ-fixed w. Let $w \in I_c^2$ with $Q^G w \in \mathbf{N}_2$ and put $\bar{w} = (\ell - 1)^{-1} \sum_{\delta \in \Delta} \gamma_\delta(w)$ ($\ell - 1$ is a unit). Then $w - \bar{w} = (\ell-1)^{-1} \sum_\delta (1 - \gamma_\delta)(w) \in X\mathbf{N}_{2,c}$ (definition of I_c^2; γ_δ fixes e_c), $\bar{w}$ is Δ-fixed, $\bar{w} \in I_c^2$ (as $(\gamma_s - 1)\gamma_\delta(w) = \gamma_\delta((\gamma_s - 1)w) \in \gamma_\delta(X\mathbf{N}_2) = X\mathbf{N}_2$), and $q_0^G \bar{w} \in \mathbf{N}_2$ since $q_0^G \gamma_\delta(w) = \gamma_\delta(q_0^G w)$ and $q_0^G w \in \mathbf{N}_2$ by (iii). If u satisfies the conclusion for $\bar{w}$, then $\ell^{r+\kappa_\ell(J,G)} w - \mathbf{N}(\Phi)(u) = \ell^{r+\kappa_\ell(J,G)}(w - \bar{w}) + \big(\ell^{r+\kappa_\ell(J,G)}\bar{w} - \mathbf{N}(\Phi)(u)\big) \in X\mathbf{N}_{2,c}$. So we assume w Δ-fixed, that is $w \in D_2(G)$. Then $(\gamma_g - 1)(w) \in X\mathbf{N}_2$ is Δ-fixed, so $\delta w \in \mathbf{N}_{2,c}^{\Delta}$ by Step 1.

Averaging also gives Δ-fixed lifts: if $y \in \mathbf{N}_{2,c}^{\Delta}$ and $x \in \mathbf{N}_{1,c}$ has $\mathbf{N}(\Phi)(x) = \ell^r y$, then $\bar{x} = (\ell-1)^{-1} \sum_\delta \gamma_\delta(x) \in \mathbf{N}_{1,c}^{\Delta}$ has $\mathbf{N}(\Phi)(\bar{x}) = \ell^r y$ (Ex3). $(*)$

Step 4: the construction. Let $Z = \{j + k : \ j \in J,\ 1 \le k \le G\}$, $\epsilon = 1$ if $0 \in Z$ and $\epsilon = 0$ otherwise, and

$$P(Y) = \prod_{x \in Z} (Y - \beta(x)), \qquad R(Y) = \prod_{x \in J'_G} (Y - \beta(x)).$$

Since $\beta(0) = 0$ and $J'_G = Z \setminus \{0\}$, we have $Y\, R(Y) = Y^{1-\epsilon} P(Y)$. Write $R(Y) = R(0) + Y\, S(Y)$ with $S \in \mathbb{Z}_\ell[Y]$. (a) By $(*)$ for $y = q_0^G w \in \mathbf{N}_{2,c}^{\Delta}$ there is $x \in \mathbf{N}_{1,c}^{\Delta}$ with $\mathbf{N}(\Phi)(x) = \ell^r q_0^G w$; put $u_0 = q_0^{-G} x \in D_1(G)$, so $\mathbf{N}(\Phi)(u_0) = \ell^r w$. (b) Put $y_0 = u_0$ and $y_i = (\delta - \beta(x_i))(y_{i-1}) \in D_1(G)$ for an enumeration $x_1, \dots, x_m$ of Z; then $\pi_0(y_i + \beta(x_i) y_{i-1}) = \pi_0 \delta y_{i-1} = (\gamma_g - 1)(y_{i-1})$, y_0 is Δ-fixed and $Q^G y_0 \in \mathbf{N}_1$ by (iii). By Lemma 24.8 for the pair (V_1, T_1), $u = g$, $Y = G$ and the set J of the statement (a finite set containing every Hodge jump of V_1 at τ_c), $y_m = P(\delta)u_0 \in \mathbf{N}_1$, hence $P(\delta)u_0 \in \mathbf{N}_{1,c}^{\Delta}$, and since δ preserves $\mathbf{N}_{1,c}^{\Delta}$ (Step 2),

$$\delta R(\delta) u_0 = \delta^{1-\epsilon} P(\delta) u_0 \in \mathbf{N}_{1,c}^{\Delta}.$$

(c) $n := R(\delta)w - R(0)w = S(\delta)(\delta w) \in \mathbf{N}_{2,c}^{\Delta}$ by Step 3 and Step 2. By $(*)$ there is $x' \in \mathbf{N}_{1,c}^{\Delta}$ with $\mathbf{N}(\Phi)(x') = \ell^r n$. (d) Put $u_2 = R(\delta)u_0 - x' \in D_1(G)$. By Step 2, $\mathbf{N}(\Phi)(u_2) = \ell^r R(\delta) w - \ell^r n = \ell^r R(0)\, w$, and $\delta u_2 = \delta R(\delta) u_0 - \delta x' \in \mathbf{N}_{1,c}^{\Delta}$ by (b) and Step 2.

Step 5: the exponent. For $x \in J'_G$ we have $x \ne 0$, and by item 1 of Lemma 23.22 ($p = \ell$ odd, $m = 1$, $u = g$), $v_\ell(g^{-x} - 1) = 1 + v_\ell(x)$, so $v_\ell(\beta(x)) = v_\ell(x)$ and $v_\ell(R(0)) = \sum_{x \in J'_G} v_\ell(x) =$

$\kappa_\ell(J,G)$. Hence $t = \ell^{\kappa_\ell(J,G)}/R(0) \in \mathbb{Z}_\ell$. Put $u = t\,u_2 \in D_1(G)$. Then $Q^G u \in \mathbf{N}_1$ by (iii), $\mathbf{N}(\Phi)(u) = \ell^{r+\kappa_\ell(J,G)} w$ exactly, and $(\gamma_g - 1)(u) = \pi_0\, t\, \delta u_2 \in \mathbf{N}_1$.

Step 6: $u \in I_c^1$. Let $\bar{M} = D_1(G)/\mathbf{N}_{1,c}^\Delta$. It is a free $\mathcal{O}_L$-module of finite rank: multiplication by q_0^G identifies $q_0^{-G}\mathbf{N}_{1,c}/\mathbf{N}_{1,c}$ with $\mathbf{N}_{1,c}/q_0^G\mathbf{N}_{1,c} \cong (A_c/Q^G A_c)^d$ (d the rank, (Wm1)), free over $\mathcal{O}_L$ of rank $dG(\ell-1)$ by Weierstrass division by the distinguished polynomial Q^G; taking Δ-invariants is exact (averaging), so $\bar{M}$ is the image of the averaging projector on this module, a direct summand, hence free. Each γ_s preserves $D_1(G)$ and $\mathbf{N}_{1,c}^\Delta$ (Step 2 and (Wm3)) and acts $\mathcal{O}_L$-linearly on $\bar{M}$, so $\rho(a) = \gamma_{g^a}$ defines a homomorphism $\rho\colon \mathbb{Z}_\ell \to \mathrm{GL}(\bar{M}) \cong \mathrm{GL}_N(\mathcal{O}_L)$ (by (Wm3) and $g^{a+b} = g^a g^b$). It is continuous: let ℓ^e be the exact power of ℓ dividing $g_0 = |\mathrm{GL}_N(\mathcal{O}_L/\ell\mathcal{O}_L)|$ and $K_i = 1 + \ell^i M_N(\mathcal{O}_L)$; for $a \in \ell^{i+e}\mathbb{Z}_\ell$ write $a = g_0\ell^i b$ with $b \in \mathbb{Z}_\ell$; then $\rho(b)^{g_0} \in K_1$ and $\rho(a) = (\rho(b)^{g_0})^{\ell^i} \in K_{1+i}$, since $(1+\ell^k A)^\ell \in K_{k+1}$ for $k \geq 1$ (binomial expansion). The class $[u] \in \bar{M}$ satisfies $\rho(1)[u] = [u]$ by Step 5, so its stabiliser in $\mathbb{Z}_\ell$, a closed subgroup (preimage of a closed subgroup under a continuous map), contains $\mathbb{Z}$ and therefore $\mathbb{Z}_\ell$. As $\mathbb{Z}_\ell^\times = \Delta \times g^{\mathbb{Z}_\ell}$ and u is Δ-fixed, $(\gamma_s - 1)(u) \in \mathbf{N}_{1,c}$ for every $s \in \mathbb{Z}_\ell^\times$. By Step 2, also $(\gamma_s - 1)(u) = Xz'$ with $q_0^G z' \in \mathbf{N}_{1,c}$. In coordinates for an A_c-basis of $\mathbf{N}_{1,c}$, each coordinate f of z' satisfies $q_0^G f \in A_c$ and $Xf \in A_c$; as $A_c \cong \mathcal{O}_L[[X]]$ is a unique factorisation domain and the prime X does not divide q_0, $f \in A_c$. So $(\gamma_s - 1)(u) \in X\mathbf{N}_{1,c}$ for all s, and $u \in e_c\mathbf{N}_1[1/\lambda_\infty]$: $u \in I_c^1$.

Steps 5 and 6 give $u \in I_c^1$ with $Q^G u \in \mathbf{N}_1$ and $\ell^{r+\kappa_\ell(J,G)} w - \mathbf{N}(\Phi)(u) = 0 \in X\mathbf{N}_{2,c}$ for Δ-fixed w, and Step 3 gives the general case.

Remark. For Δ-fixed w the lift is exact. □

Lemma 24.8. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a pair (V,T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$, let $c \in \mathbb{Z}/n$, and let $J \subset \mathbb{Z}$ be a finite set containing every Hodge jump of V at τ_c (for instance the set of all Hodge jumps of V at all τ_a). Let $u \in 1 + \ell\mathbb{Z}_\ell$, and for $x \in \mathbb{Z}$ let $c(x) \in \mathbb{Z}_\ell$ be defined by $\ell\, c(x) = 1 - u^{-x}$. Let $Y \geq 0$ be an integer and $x_1, \ldots, x_m$ an enumeration, without repetition, of the finite set $\{j + k :\ j \in J,\ 1 \leq k \leq Y\}$. Let $y_0, y_1, \ldots, y_m \in e_c\mathbf{N}[1/\lambda_\infty]$ be such that $Q^Y y_0 \in \mathbf{N}$ and*

$$\pi_0(y_i + c(x_i)\, y_{i-1}) = (\gamma_u - 1)(y_{i-1}) \qquad (1 \leq i \leq m).$$

Then $y_m \in \mathbf{N}$.

Proof. **Step 0: level data at the cut c.** Choose integers $M_a, h_a \geq 0$ such that every Hodge jump of V at τ_a lies in $[-M_a, h_a]$ (there are finitely many). By (Ex1), $\mathbf{N}$ is a Wach module with window $(M_a, h_a)_a$ and rank r. Fix arc level data for id_T (Definition 23.4) with cut c, length $\mathrm{len} = 1$ (so $c_0 = c - 1$), $H(c) = Y + h_c$ (other values of H arbitrary), a frame at $c-1$ (an A_{c-1}-basis of $\mathbf{N}_{c-1}$, which exists by (Wm1)) with its horizontal sections (Hf2) and crystalline frame, and parameters $G = S = \eta = g = M = 0$, $j = 1$, κ_0 thin, $b_0 = 0$. Then $\Pi = Q^{Y+h_c}$ and $W = \{w \in \mathbf{N}[1/\lambda_\infty] :\ Q^{Y+h_c} w \in P_1(\mathbf{N}_{c-1})\}$. By Lemma 23.17 (for $\Phi = \mathrm{id}_T$) the properties (Lp1), (Lp2), (Lp8) hold at $s = 0$, and by Lemma 23.5 (2) so does (Lp3). Explicitly:

(a) $\lambda_0(A_c) \subseteq K[[t]]$ and $\lambda_0(e_c Q) = t\,\eta$ with $\eta \in K[[t]]^\times$ (Lp1);

(b) every $w \in W$ is algebraic at level 0: $\Lambda_0(w) = \sum_k \mathrm{lv}_0(w)_k D_k$ with $\mathrm{lv}_0(w) \in K((t))^r$ (Lp1). Since Λ_0 is additive, $\Lambda_0(gw) = \lambda_0(g)\Lambda_0(w)$, $\epsilon_0 = \lambda_0(e_c)$, $D_k \in \epsilon_0 D$, and the D_k are an R_0-basis, lv_0 is additive and $\mathrm{lv}_0(gw) = \lambda_0(e_c g)\,\mathrm{lv}_0(w)$ for $g \in \mathbf{A}_{F,L}$ and $w, gw \in W$;

(c) $\gamma_u(W) \subseteq W$ and $\mathrm{lv}_0(\gamma_u w) = \gamma_u^{(0)}(\mathrm{lv}_0(w))$ (Lp2);

(d) $\sigma_u = \mathrm{id}$ on K, because $u \in 1 + \ell\mathbb{Z}_\ell$ (Lp8 at $s = 0$); so $\gamma_u^{(0)}(\sum_i a_i t^i) = \sum_i u^i a_i t^i$ coordinatewise, and $\gamma_u^{(0)}(g\,v) = g(ut)\,v$ for $g = \sum_i a_i t^i \in K((t))$, $g(ut) = \sum_i u^i a_i t^i$, and $v \in L^r$;

(e) for $w \in W$: $w \in \mathbf{N}_c$ if and only if $\mathrm{lv}_0(w)$ is de Rham integral at 0 (Lp3).

Step 1: the de Rham integral vectors. Recall $\mathrm{Fil}_0(u) = \{v \in L^r : \sum_k v_k D_k \in \mathrm{Fil}^u D\}$. As V is crystalline and F is unramified, $D \subseteq D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V)^{G_F}$, both of F-dimension $\dim_{\mathbb{Q}_\ell} V$, so $D = D_{\mathrm{dR}}(V)$ and $\mathrm{Fil}^u D$ is the Hodge filtration: decreasing, equal to D for $u \ll 0$ and to 0 for $u \gg 0$, by $F \otimes_{\mathbb{Q}_\ell} L$-submodules ($\mathrm{Fil}^u B_{\mathrm{dR}} = t^u B_{\mathrm{dR}}^+$ is stable under $F \subset B_{\mathrm{dR}}^+$, and L acts on V). Hence $\mathrm{Fil}^u D \cap \epsilon_0 D = \epsilon_0 \mathrm{Fil}^u D$. As ϵ_0 is the idempotent of $F \otimes_{\mathbb{Q}_\ell} L$ belonging to τ_c (Definition 23.4), $\epsilon_0 M = L \otimes_{F \otimes L, \tau_c \otimes \mathrm{id}} M$ for every $F \otimes_{\mathbb{Q}_\ell} L$-module M, so $v \mapsto \sum_k v_k D_k$ is an L-linear isomorphism $L^r \to \epsilon_0 D = D_{\mathrm{dR},\tau_c}(V)$ carrying $\mathrm{Fil}_0(u)$ onto $\mathrm{Fil}^u D_{\mathrm{dR},\tau_c}(V)$ for every u. Choose an L-basis $v_1, \dots, v_r$ of L^r adapted to this finite filtration: integers $j_1, \dots, j_r$ with $\mathrm{Fil}_0(u) = \bigoplus_{b:\ j_b \geq u} L v_b$ for all u (choose, for each u, a complement of $\mathrm{Fil}_0(u+1)$ in $\mathrm{Fil}_0(u)$ and put $j_b = u$ on a basis of it). Each j_b satisfies $\mathrm{Fil}_0(j_b + 1) \neq \mathrm{Fil}_0(j_b)$, so it is a Hodge jump of V at τ_c, and $j_b \in J$ by the hypothesis on J. The v_b are also a K-basis of K^r; write $f \in K((t))^r$ as $f = \sum_b g_b v_b$ with $g_b \in K((t))$. The t^i-coefficient of f lies in $K \otimes_L \mathrm{Fil}_0(-i) = \bigoplus_{j_b \geq -i} K v_b$ for every i if and only if $g_b \in t^{-j_b} K[[t]]$ for every b. So the set of de Rham integral vectors at 0 is

$$\Lambda = \bigoplus_{b=1}^{r} t^{-j_b} K[[t]]\, v_b,$$

a $K[[t]]$-submodule of $K((t))^r$.

Step 2: $Q^Y y_i \in \mathbf{N}_c$ and $y_i \in W$ for every i. (i) If $z \in e_c \mathbf{N}[1/\lambda_\infty]$ and $\pi_0 z \in \mathbf{N}_c$, then $z \in \mathbf{N}_c$. Indeed, in an A_c-basis of $\mathbf{N}_c$ (Wm1) the coordinates of z are a/λ_e with $a \in A_c$ and some $e \geq 0$, and $\pi_0 a \in \lambda_e A_c$. Let $\zeta' \neq 1$ be a root of unity of order dividing ℓ^e; the evaluation at $\zeta' - 1$ is a ring homomorphism on $\mathcal{O}_L[[X]]$, and it sends $(1+X)^\delta$ to ζ'^δ (the power by any integer congruent to δ modulo the order of ζ'). In an embedding $\mathbb{Q}(\zeta') \subset \mathbb{C}$, $\left|\sum_\delta \zeta'^\delta\right| \leq \ell - 1$ with equality and value $\ell - 1$ only if all $\zeta'^\delta = 1$, which fails since δ is a unit; so $\pi_0(\zeta' - 1) \neq 0$, hence $a(\zeta' - 1) = 0$. So a vanishes at all roots of every Q_s, $s < e$, and by Lemma 23.8 (3), applied to $Q_0, \dots, Q_{e-1}$ in turn (the roots of different Q_s are distinct), λ_e divides a. (ii) Induction on i. For $i = 0$, $Q^Y y_0 \in \mathbf{N} \cap e_c \mathbf{N}[1/\lambda_\infty] = \mathbf{N}_c$. Let $i \geq 1$ and $n = Q^Y y_{i-1} \in \mathbf{N}_c$. By Lemma 23.8 (2), $\gamma_u(Q) = Qh$ with $h \in \mathbb{Z}_\ell[[X]]^\times$, so $Q^Y \gamma_u(y_{i-1}) = h^{-Y} \gamma_u(n) \in \mathbf{N}_c$ by (Wm3) (γ_u fixes e_c). Hence $\pi_0 Q^Y y_i = Q^Y(\gamma_u - 1)(y_{i-1}) - c(x_i)\pi_0 n \in \mathbf{N}_c$, and (i) for $z = Q^Y y_i$ gives $Q^Y y_i \in \mathbf{N}_c$. (iii) By (Wm2) at $a = c$, $Q^{h_c} \mathbf{N}_c \subseteq \mathbf{A}_{F,L} \varphi_{\mathbf{N}}(\mathbf{N}_{c-1}) = P_1(\mathbf{N}_{c-1})$, so $Q^{Y+h_c} y_i \in P_1(\mathbf{N}_{c-1})$: $y_i \in W$. Likewise $Q^Y y_0 \in W$.

Step 3: the constant term of $\lambda_0(\pi_0)$. Let $p = \lambda_0(e_c \pi_0) \in K[[t]]$ (Step 0 (a)). Its constant term $p_0 \in K$ is determined by $\theta(p) = \theta(p_0)$ (θ extended L-linearly), since $\theta(t) = 0$ and θ is injective on K (Definition 23.4). On $\mathbb{Z}_\ell[[X]]$, $\lambda_0 = \varphi^{-1} \circ \iota$ (followed by $\otimes 1$), and $\psi = \theta \circ \varphi^{-1} \circ \iota \colon \mathbb{Z}_\ell[[X]] \to \mathcal{O}_{\mathbb{C}_\ell}$ is a $\mathbb{Z}_\ell$-algebra homomorphism with $\psi(X) = \theta([\varepsilon^{1/\ell}]) - 1 = \zeta - 1$, ζ a primitive ℓ-th root of unity (ε is a compatible system of primitive ℓ^k-th roots of unity). Such a homomorphism is the evaluation at $\zeta - 1$: for $g \in \mathbb{Z}_\ell[[X]]$ and $k \geq 1$ write $g = g_{<k} + X^k g_k$ with $g_{<k}$ a polynomial; then $\psi(g) - g(\zeta - 1) = (\zeta - 1)^k(\psi(g_k) - g_k(\zeta - 1))$ has absolute value at most $|\zeta - 1|^k \to 0$. Hence, as in (i) and since $\delta \mapsto \delta \bmod \ell$ is a bijection $\mu_{\ell-1} \to \mathbb{F}_\ell^\times$,

$$\psi(\pi_0) = \sum_{\delta \in \mu_{\ell-1}} \zeta^\delta - (\ell - 1) = \sum_{a=1}^{\ell-1} \zeta^a - (\ell - 1) = -1 - (\ell - 1) = -\ell.$$

So $p_0 = -\ell$ (that is, $-\ell\epsilon_0$, the unit of K being ϵ_0); as ℓ is invertible in the $\mathbb{Q}_\ell$-algebra K, $p \in K[[t]]^\times$ and $p^{-1} \equiv -\ell^{-1} \bmod tK[[t]]$.

Step 4: the operator at level 0. Put $f_i = \mathrm{lv}_0(y_i)$ (Step 2). Applying lv_0 to the relation of the statement, by Step 0 (b), (c) (all terms lie in W: W is an $\mathbf{A}_{F,L}$-submodule of $\mathbf{N}[1/\lambda_\infty]$,

as $P_1(\mathbf{N}_{c-1})$ is one, and $\gamma_u(y_{i-1}) \in W$ by (c)),

$$p\,(f_i + c(x_i) f_{i-1}) = \gamma_u^{(0)}(f_{i-1}) - f_{i-1} \qquad (1 \le i \le m).$$

Let $\mathcal{E}$ be the K-linear operator $\mathcal{E}(f) = p^{-1}(\gamma_u^{(0)}(f) - f)$ on $K((t))^r$. Then $f_i = (\mathcal{E} - c(x_i))(f_{i-1})$, and so $f_m = \Omega(\mathcal{E})(f_0)$ with $\Omega(T) = \prod_{x \in Z}(T - c(x))$ (the factors commute). By Step 0 (d), $\mathcal{E}(g\,v) = \mathcal{E}_1(g)\,v$ for $g \in K((t))$ and $v \in L^r$, where $\mathcal{E}_1(g) = p^{-1}(g(ut) - g(t))$. For $q \in \mathbb{Z}$, $\mathcal{E}_1$ maps $t^q K[[t]]$ into itself, and

$$\mathcal{E}_1(g) - c(-q)\,g \in t^{q+1} K[[t]] \qquad (g \in t^q K[[t]]).$$

So for $x = j + k$ (with $q = -x$): $\mathcal{E}_1 - c(j+k)$ maps $t^{-j-k}K[[t]]$ into $t^{-j-k+1}K[[t]]$.

Step 5: conclusion. By Step 2, $Q^Y y_0 \in \mathbf{N}_c \cap W$, so by Step 0 (e), (b), (a), $\lambda_0(e_c Q)^Y f_0 = t^Y \eta^Y f_0 \in \Lambda$, and $f_0 \in t^{-Y}\Lambda$ (Λ is a $K[[t]]$-module). Write $f_0 = \sum_b g_b v_b$ with $g_b \in t^{-j_b - Y} K[[t]]$ (Step 1). Fix b. The set $Z_b = \{j_b + k :\ 1 \le k \le Y\}$ has Y elements and is contained in Z (as $j_b \in J$), so $\Omega(T) = \Omega_b(T) \prod_{k=1}^{Y}(T - c(j_b + k))$ with $\Omega_b(T) = \prod_{x \in Z \setminus Z_b}(T - c(x))$. Applying Step 4 successively for $k = Y, Y-1, \dots, 1$,

$$\prod_{k=1}^{Y} (\mathcal{E}_1 - c(j_b + k))(g_b) \in t^{-j_b} K[[t]],$$

and $\Omega_b(\mathcal{E}_1)$ preserves $t^{-j_b}K[[t]]$. Hence $\Omega(\mathcal{E}_1)(g_b) \in t^{-j_b}K[[t]]$ for every b, and

$$f_m = \Omega(\mathcal{E})(f_0) = \sum_b \Omega(\mathcal{E}_1)(g_b)\,v_b \in \Lambda.$$

So $\mathrm{lv}_0(y_m)$ is de Rham integral at 0; as $y_m \in W$ (Step 2), Step 0 (e) gives $y_m \in \mathbf{N}_c \subseteq \mathbf{N}$.

Remark. In words: at the point $X = \zeta - 1$ the Wach module, expanded in the uniformiser t, is $\bigoplus_b t^{-j_b}K[[t]]v_b$ with γ_u acting by $t \mapsto ut$, so on the k-th pole layer of the line of v_b the operator $\pi_0^{-1}(\gamma_u - 1)$ acts by $c(j_b + k)$ modulo the next layer, and each line is a single Jordan-free chain with distinct eigenvalues; this is why the product over the set Z, each value once, suffices. The proof uses neither invariance under $\mu_{\ell-1}$ nor $v_\ell(u-1) = 1$, and of J only that it contains the Hodge jumps at τ_c. For $Y = 0$ and $\mathbf{N}/Q\mathbf{N}$ after inverting ℓ, the eigenvalue statement is the proposition on the action of Γ_n on $N_V/\varphi^{n-1}(q)$ in [24], §III (case $n = 1$). □

Lemma 24.9. *Let ℓ, F, L, (τ_a) be as in Proposition 23.7, let (V_1, T_1), (V_2, T_2) be pairs with windows $(M_a^1, h_a^1)_a$, $(M_a^2, h_a^2)_a$ as in (Ex1), and let $\Phi\colon T_1 \to T_2$ be a G_F-equivariant $\mathcal{O}_L$-linear map. Let $c \in \mathbb{Z}/n$ and integers $j \ge 1$, $G \ge 0$, $g \ge G$, $m \ge 1 + \max_{a,\nu} M_a^\nu$ and $R \ge \max(m, g)$ with $M_{c+s}^\nu \le G$ for $1 \le s \le j$ and $\nu = 1, 2$; let I_c^ν and $\rho_{c,j}^\nu$ be formed for $\mathbf{N}(T_\nu)$ with these c, j, G, g, m, R. Let $w \in I_c^2$, $u \in I_c^1$ and $k \ge 0$ with*

$$\ell^k w - \mathbf{N}(\Phi)(u) \in X\,\mathbf{N}(T_2)_c.$$

Then $\ell^k \rho_{c,j}^2(w) = \Phi_* \rho_{c,j}^1(u)$ in $H^1(F, T_2)$. If moreover $\lambda' u \in \mathbf{N}(T_1)$ for some finite product λ' of the Q_s ($s \ge 0$), then $\ell^k \rho_{c,j}^2(w) \in \Phi_* H_f^1(F, T_1)$.

Proof. By (Ry6) of Lemma 24.5, applied to $f = \Phi\colon T_1 \to T_2$ (its hypotheses on the window of T_2 are among ours), $\mathbf{N}(\Phi)(u) \in I_c^2$ and $\rho_{c,j}^2(\mathbf{N}(\Phi)(u)) = \Phi_* \rho_{c,j}^1(u)$. By (Ry4), I_c^2 is an $\mathcal{O}_L$-submodule and $\rho_{c,j}^2$ is $\mathcal{O}_L$-linear; hence $y := \ell^k w - \mathbf{N}(\Phi)(u) \in I_c^2$ and

$$\ell^k \rho_{c,j}^2(w) = \rho_{c,j}^2(\ell^k w) = \rho_{c,j}^2(\mathbf{N}(\Phi)(u)) + \rho_{c,j}^2(y).$$

Since $y \in X\mathbf{N}(T_2)_c$, (Ry5) gives $\rho_{c,j}^2(y) = 0$, which proves the identity. If $\lambda' u \in \mathbf{N}(T_1)$, then (Ry7) for $\mathbf{N}(T_1)$ gives $\rho_{c,j}^1(u) \in H_f^1(F, T_1)$, and the identity gives $\ell^k \rho_{c,j}^2(w) \in \Phi_* H_f^1(F, T_1)$. □

24.2 Lifting exponents over the first cyclotomic layer

Over a field K with $K_0 \subseteq K \subseteq K_0(\mu_\ell)$ and K_0 unramified, the Wach modules are Δ-invariant: they live over the ring of Definition 23.23, with the uniformiser π_Δ invariant under the torsion Δ_K of Γ_K (Lemma 24.14). The subsection proves Proposition 24.10 along the lines of the unramified case. The two inputs are a Δ-invariant lift along a surjection of lattices up to a power of ℓ of size about $2(\mathrm{wt}+1)/|\Delta_K|$ (Lemma 24.11), and a letter split, which writes a crystalline class, up to a bounded power of ℓ, as a sum of Herr classes of Δ-invariant vectors, one for each site (Lemma 24.12, from Proposition 24.13). The resulting exponent is linear in wt, not in wt/ℓ; this suffices at the small and ramified primes, whose number and size are restricted by (Ub8) of Theorem 22.4, and whose losses are bounded in Section 27. The statement has a version of level s, over $K_0(\mu_{\ell^s})$ and for every prime ℓ, which is the one used at the small primes.

Proposition 24.10. *There is an absolute constant $c_{\mathrm{cy}} \geq 1$ with the following property. Let p be an odd prime, K_0 a finite unramified extension of $\mathbb{Q}_p$ and K a field with $K_0 \subseteq K \subseteq K_0(\mu_p)$ inside $\overline{\mathbb{Q}}_p$. Let L be a finite extension of $\mathbb{Q}_p$ containing $\sigma(K_0(\mu_p))$ for every $\mathbb{Q}_p$-embedding σ of $K_0(\mu_p)$ into $\overline{\mathbb{Q}}_p$, and $\mathcal{O}_L$ its ring of integers. Let V_1, V_2 be continuous L-linear representations of G_K on finite-dimensional L-vector spaces, crystalline as representations over $\mathbb{Q}_p$, G_K-stable $\mathcal{O}_L$-lattices $T_\nu \subseteq V_\nu$, and $F\colon V_1 \to V_2$ an L-linear G_K-equivariant map with $F(T_1) = T_2$. Let $a_\tau^-, a_\tau^+ \geq 0$ be integers, for the $\mathbb{Q}_p$-embeddings $\tau\colon K \to L$, such that every Hodge jump of V_1 at τ lies in $[-a_\tau^-, a_\tau^+]$, and put $\mathrm{wt} = \max_\tau(a_\tau^- + a_\tau^+)$ and*

$$e_{\mathrm{B}} = \lfloor c_{\mathrm{cy}}(\mathrm{wt}+1) \rfloor.$$

Then

$$p^{e_{\mathrm{B}}}\, H^1_f(K, T_2) \subseteq F_*(H^1_f(K, T_1)).$$

Level s. *Moreover the same constant c_{cy} has the following property. Let p be any prime, $s \geq 1$ an integer with $s \geq 2$ if $p = 2$, K_0 a finite unramified extension of $\mathbb{Q}_p$, $K = K_0(\mu_{p^s})$, L a finite extension of $\mathbb{Q}_p$ containing $\sigma(K)$ for every $\mathbb{Q}_p$-embedding σ of K, and V_ν, T_ν, F, $a_\tau^\pm$ (for the $\mathbb{Q}_p$-embeddings $\tau\colon K \to L$) and* wt *as above. Put $e_{\mathrm{B}}^{(s)} = \lfloor c_{\mathrm{cy}}(\mathrm{wt}+1)(s+1) \rfloor$. Then*

$$p^{e_{\mathrm{B}}^{(s)}}\, H^1_f(K, T_2) \subseteq F_*(H^1_f(K, T_1)).$$

Proof. We show that $c_{\mathrm{cy}} = 24$ has the property. Write $\ell = p$, $F = K_0$ (the map of the statement is written Φ in this proof), $n = [F : \mathbb{Q}_\ell]$, and fix a Frobenius numbering $(\tau_a)_{a \in \mathbb{Z}/n}$; the $\mathbb{Q}_\ell$-embeddings $\tau\colon K \to L$ are the embeddings of K over the τ_a. The hypotheses on ℓ, F, K, L are those of Proposition 23.7, so (V_1, T_1), (V_2, T_2) are pairs over K. Put $d = |\Delta_K|$, $w = \mathrm{wt}$ and $h' = w + 1 \geq 1$. Since $\Phi(T_1) = T_2$ spans V_2, $\Phi\colon V_1 \to V_2$ is onto, and by Lemma 23.25 every Hodge jump of V_2 at τ is one of V_1 at τ. As $a_\tau^\pm \geq 0$, $[-a_\tau^-, a_\tau^+] \subseteq [-w, w]$; so every Hodge jump of V_1 and of V_2 at every τ lies in $[-w, w] \subseteq [-h', h']$, and $M_a = h_a = h'$ ($a \in \mathbb{Z}/n$) is a window as in (Ex1) for both.

Step 1: lifting along onto maps up to ℓ^{r_0}, $r_0 = \mathrm{dsc}_{\ell,d}(0, 2h')$. By Lemma 24.11 for $\Phi\colon T_1 \to T_2$ (onto, and every Hodge jump of V_1 at every embedding of K lies in $[-h', h']$), for every $c \in \mathbb{Z}/n$ every $y \in \mathbf{N}(T_2)_c$ has $x \in \mathbf{N}(T_1)_c$ with $\mathbf{N}(\Phi)(x) = \ell^{r_0} y$, and $r_0 \leq \lfloor 2h'\ell/(d(\ell-1)) \rfloor \leq 3h'$.

Steps 2–3: the letter split. Let $x \in H^1_f(K, T_2)$. By Lemma 24.12 for (V_2, T_2) and h' (every Hodge jump of V_2 lies in $[-h', h']$), with $M = \lceil 4h'/d \rceil$, $m = Md$ and $\sigma = M - 1 +$

$v_\ell((M-1)!)$, there are $w_c \in e_c\mathbf{N}(T_2)[1/\lambda_\infty]$ ($c \in \mathbb{Z}/n$), fixed by every γ_δ, $\delta \in \Delta_K$, with $Q^{h'}w_c \in \mathbf{A}_{F,L}\varphi(\mathbf{N}(T_2)_{c-1})$, $(\gamma_u - 1)(w_c) \in \pi_\Delta^M \mathbf{N}(T_2)_c^\Delta$ for every $u \in \Gamma_K$, and

$$\ell^{\,r_0+\sigma}x = \sum_{c\in\mathbb{Z}/n} \mathrm{cl}(w_c) \qquad \text{in } H^1(K,T_2),$$

each w_c a Herr vector of precision m over K. Moreover $m \geq 4h' \geq 1 + h' = 1 + \max_a M_a$.

Step 4: the poles of w_c. By Steps 2–3, $Q^{h'}w_c \in \mathbf{A}_{F,L}\varphi(\mathbf{N}(T_2)_{c-1})$, and by (Wm2) for the window (h',h') of $\mathbf{N}(T_2)$ (Ex1), $Q^{h'}\varphi(\mathbf{N}(T_2)_{c-1}) \subseteq \mathbf{N}(T_2)_c$. So, with $G = 2h'$, $Q^G w_c \in \mathbf{N}(T_2)$. By Steps 2–3, w_c is Δ_K-fixed; by Steps 2–3 and (Dm2), $(\gamma_s - 1)(w_c) \in \pi_\Delta^M \mathbf{N}(T_2)_c^\Delta \subseteq \pi_\Delta \mathbf{N}(T_2)_c^\Delta \subseteq X\mathbf{N}(T_2)$ for every $s \in \Gamma_K$. So $w_c \in I_c^2$ in the notation of Lemma 24.18.

Step 5: the one-site lift. Let J be the set of Hodge jumps of V_1 at all embeddings of K (a subset of $[-w,w]$), $J' = \{j+k :\ j \in J,\ 1 \leq k \leq G,\ j+k \neq 0\}$ and $\kappa = \sum_{x\in J'}(1+v_\ell(x))$. By Step 1 for Φ, every $y \in \mathbf{N}(T_2)_c$ has $x \in \mathbf{N}(T_1)_c$ with $\mathbf{N}(\Phi)(x) = \ell^{r_0}y$. So Lemma 24.18 (with c, J, G, $r = r_0$, $w = w_c$) gives $u_c \in I_c^1$ with $Q^G u_c \in \mathbf{N}(T_1)$ and $\mathbf{N}(\Phi)(u_c) = \ell^{\,r_0+\kappa}w_c$.

Step 6: precision. Fix $u = 1 + \ell \in 1 + \ell\mathbb{Z}_\ell \subseteq \Gamma_K$, so $v_\ell(u-1) = 1$. Let $z = (\gamma_u - 1)(u_c)$. It lies in $X\mathbf{N}(T_1)$ ($u_c \in I_c^1$), in $e_c\mathbf{N}(T_1)$ (γ_u fixes e_c), and it is Δ_K-fixed (γ_δ commutes with γ_u); so $z \in X\mathbf{N}(T_1)_c \cap \mathbf{N}(T_1)_c^\Delta = \pi_\Delta \mathbf{N}(T_1)_c^\Delta$ by (Dm2). By Lemma 24.16 for $\mathbf{N}(T_1)$ (a Wach module over K by (Ex1)), with M as in Step 3 and $\sigma_M = \sigma$, there is $k \in \pi_\Delta \mathbf{N}(T_1)_c^\Delta$ with $\ell^\sigma z - (\gamma_u - 1)(k) \in \pi_\Delta^M \mathbf{N}(T_1)_c^\Delta$. Put $\bar{u}_c = \ell^\sigma u_c - k$. Then $(\gamma_u - 1)(\bar{u}_c) \in \pi_\Delta^M \mathbf{N}(T_1)_c^\Delta \subseteq X^m\mathbf{N}(T_1)$ (sidecalc above), and $(\gamma_\delta - 1)(\bar{u}_c) = 0$ for $\delta \in \Delta_K$. As $\Delta_K \cup \{u\}$ generates a dense subgroup of $\Gamma_K = \Delta_K \times (1 + \ell\mathbb{Z}_\ell)$, (Hc2) over K shows that $\bar{u}_c$ is a Herr vector of precision m for $\mathbf{N}(T_1)$ (window (h',h'), $m \geq 1 + h'$). Moreover $Q^G\bar{u}_c = \ell^\sigma Q^G u_c - Q^G k \in \mathbf{N}(T_1)$. By Proposition 23.27 over K, $\mathrm{cl}(\bar{u}_c) \in H_f^1(K,T_1)$.

Step 7: the correction term vanishes. Let $k' = \mathbf{N}(\Phi)(k)$. As $\mathbf{N}(\Phi)$ is $\mathbf{A}_{F,L}$-linear and commutes with every γ_s, $k' \in \pi_\Delta \mathbf{N}(T_2)_c^\Delta$, and

$$(\gamma_u - 1)(k') = \ell^\sigma \mathbf{N}(\Phi)(z) - \mathbf{N}(\Phi)\big(\ell^\sigma z - (\gamma_u - 1)(k)\big),$$
$$\mathbf{N}(\Phi)(z) = (\gamma_u - 1)\mathbf{N}(\Phi)(u_c) = \ell^{\,r_0+\kappa}(\gamma_u - 1)(w_c).$$

The first term lies in $\pi_\Delta^M \mathbf{N}(T_2)_c^\Delta$ by Steps 2–3, the second by the choice of k. By Lemma 24.21 for $\mathbf{N}(T_2)$, $k' \in \pi_\Delta^M \mathbf{N}(T_2)_c^\Delta \subseteq X^m\mathbf{N}(T_2)$, so by (Hc4) over K it is a Herr vector of precision m with $\mathrm{cl}(k') = 0$.

Step 8: the classes lift. By Steps 5 and 7, $\mathbf{N}(\Phi)(\bar{u}_c) = \ell^{\,\sigma+r_0+\kappa}w_c - k'$. By (Hc5) over K (for Φ, both windows (h',h'), $m \geq 1 + h'$) and (Hc3), (Hc4) over K,

$$\Phi_*\mathrm{cl}(\bar{u}_c) = \mathrm{cl}\big(\mathbf{N}(\Phi)(\bar{u}_c)\big) = \ell^{\,\sigma+r_0+\kappa}\mathrm{cl}(w_c) - \mathrm{cl}(k') = \ell^{\,\sigma+r_0+\kappa}\mathrm{cl}(w_c).$$

Multiplying Step 3 by $\ell^{\sigma+r_0+\kappa}$ and summing over c,

$$\ell^{\,e}x = \sum_{c\in\mathbb{Z}/n} \Phi_*\mathrm{cl}(\bar{u}_c) \in \Phi_*\big(H_f^1(K,T_1)\big), \qquad e = 2r_0 + 2\sigma + \kappa.$$

Step 9: the exponent. (a) $2r_0 \leq 6h'$ (Step 1). (b) $M - 1 < 4h'/d \leq 4h'$, and by Legendre's formula $\sigma \leq (M-1)\ell/(\ell-1) \leq \frac{3}{2}(M-1) < 6h'$; so $2\sigma < 12h'$. (c) $J' \subseteq [-w+1, w+G] \setminus \{0\} = [-w+1, 3w+2] \setminus \{0\}$, so $|J'| \leq 4w+2$, and, the elements of J' being distinct non-zero integers of absolute value at most $3w+2$ with at most $w-1$ of them negative, $\sum_{x\in J'} v_\ell(x) \leq v_\ell((w-1)!) + v_\ell((3w+2)!) \leq (4w+1)/(\ell-1) \leq (4w+1)/2$ (for $w = 0$ the first factorial is absent). So $\kappa \leq 6w + 3 < 6h'$. Hence $e < 24h' = 24(\mathrm{wt}+1) = e_{\mathrm{B}}$ for $c_{\mathrm{cy}} = 24$, and $\ell^{e_{\mathrm{B}}}x = \ell^{e_{\mathrm{B}}-e}(\ell^e x) \in \Phi_* H_f^1(K,T_1)$, a subgroup. As $x \in H_f^1(K,T_2)$ was arbitrary, the inclusion follows.

Level s. Let the data be as in the paragraph Level s of the statement; write $\ell = p$, $F = K_0$, $\Phi = F$ (the map), $n = [F : \mathbb{Q}_\ell]$, fix a Frobenius numbering, and put $w = \mathrm{wt}$, $h' = w + 1$, $Y = X_{s-1}$ (the level variable of Definition 23.2), $u_s = 1 + \ell^s$, $M = 4h'$. The hypotheses on ℓ, s, F, $K = F(\mu_{\ell^s})$, L are those of (Ex4) of Proposition 23.7, so (V_ν, T_ν) are pairs over K; $\Gamma_K = 1 + \ell^s\mathbb{Z}_\ell$, procyclic with generator u_s ($\ell^s \neq 2$), and Δ_K is trivial. As in the first paragraph, every Hodge jump of V_1 and V_2 at every embedding lies in $[-h', h']$, and $(h', h')_a$ is a window of level s for both. All cited clauses are taken in their paragraphs Level s. (1) By Lemma 24.11 for Φ, every $y \in \mathbf{N}_s(T_2)_c$ has $x \in \mathbf{N}_s(T_1)_c$ with $\mathbf{N}_s(\Phi)(x) = \ell^{r_0} y$, $r_0 = 2h's + v_\ell((2h')!)$. (2) Let $x \in H^1_f(K, T_2)$. By Lemma 24.12 for (V_2, T_2) there are $w_c \in e_c\mathbf{N}_s(T_2)[1/\lambda_\infty]$ with $Q^{h'}_{s-1} w_c \in \mathbf{A}_{F,L}\varphi(\mathbf{N}_s(T_2)_{c-1})$, $(\gamma_v - 1)(w_c) \in Y^M\mathbf{N}_s(T_2)_c$ for every $v \in \Gamma_K$, and $\ell^{r_0+\sigma}x = \sum_c \mathrm{cl}(w_c)$, $\sigma = (M-1)s + v_\ell((M-1)!)$. (3) By (Wm2) of level s, $Q^{h'}_{s-1}\varphi(\mathbf{N}_s(T_2)_{c-1}) \subseteq \mathbf{N}_s(T_2)_c$, so $Q^G_{s-1} w_c \in \mathbf{N}_s(T_2)$ with $G = 2h'$, and $w_c \in I^{2,s}_c$ in the notation of Lemma 24.18. (4) Let J be the set of Hodge jumps of V_1 at all embeddings of K ($J \subseteq [-w, w]$), J' as there and $\kappa = \sum_{x \in J'}(s + v_\ell(x))$. By (1), Lemma 24.18 (with $r = r_0$) gives $u_c \in I^{1,s}_c$ with $Q^G_{s-1}u_c \in \mathbf{N}_s(T_1)$ and $\mathbf{N}_s(\Phi)(u_c) = \ell^{r_0+\kappa}w_c$. (5) $z = (\gamma_{u_s} - 1)(u_c) \in Y\mathbf{N}_s(T_1) \cap e_c\mathbf{N}_s(T_1) = Y\mathbf{N}_s(T_1)_c$. Item 5 of Lemma 23.21 ($D = N = \mathbf{N}_s(T_1)_c$, free over $e_c\mathbf{A}_{F,L}$, $\gamma_D = \gamma_{u_s}$, (Wm3')) gives $k \in Y\mathbf{N}_s(T_1)_c$ with $\ell^\sigma z - (\gamma_{u_s} - 1)(k) \in Y^M\mathbf{N}_s(T_1)_c$. Put $\bar{u}_c = \ell^\sigma u_c - k$: $(\gamma_{u_s} - 1)(\bar{u}_c) \in Y^M\mathbf{N}_s(T_1)$, so by (Hc2) of Lemma 23.28 (the subset $\{u_s\}$) $\bar{u}_c$ is a Herr vector of precision $M \geq 1 + h'$ at level s; and $Q^G_{s-1}\bar{u}_c \in \mathbf{N}_s(T_1)$. By Proposition 23.27 ($f = Q^G_{s-1}$), $\mathrm{cl}(\bar{u}_c) \in H^1_f(K, T_1)$. (6) $k' = \mathbf{N}_s(\Phi)(k) \in Y\mathbf{N}_s(T_2)_c$ and $(\gamma_{u_s} - 1)(k') = \ell^\sigma\mathbf{N}_s(\Phi)(z) - \mathbf{N}_s(\Phi)(\ell^\sigma z - (\gamma_{u_s} - 1)(k))$ with $\mathbf{N}_s(\Phi)(z) = \ell^{r_0+\kappa}(\gamma_{u_s} - 1)(w_c)$; both terms lie in $Y^M\mathbf{N}_s(T_2)_c$ by (2) and (5). Item 6 of Lemma 23.21 ($N = \mathbf{N}_s(T_2)_c$, free over $e_c\mathbf{A}_{F,L}$) gives $k' \in Y^M\mathbf{N}_s(T_2)$, so by (Hc4) k' is a Herr vector of precision M at level s with $\mathrm{cl}(k') = 0$. (7) $\mathbf{N}_s(\Phi)(\bar{u}_c) = \ell^{\sigma+r_0+\kappa}w_c - k'$, and (Hc5), (Hc3), (Hc4) give $\Phi_*\mathrm{cl}(\bar{u}_c) = \ell^{\sigma+r_0+\kappa}\mathrm{cl}(w_c)$. Multiplying (2) by $\ell^{\sigma+r_0+\kappa}$ and summing over c, $\ell^e x \in \Phi_* H^1_f(K, T_1)$ with $e = 2r_0 + 2\sigma + \kappa$. (8) The exponent. By Legendre's formula $v_\ell(N!) \leq N - 1$ for $N \geq 1$. So $r_0 \leq 2h's + 2h' - 1$ and $\sigma \leq (4h' - 1)s + 4h' - 2$. As in Step 9(c), $J' \subseteq [-w+1, 3w+2] \setminus \{0\}$ has at most $4w+2 = 4h'-2$ elements, at most $w-1$ of them negative, so $\sum_{x \in J'} v_\ell(x) \leq v_\ell((w-1)!) + v_\ell((3w+2)!) \leq 4w + 1 = 4h' - 3$ and $\kappa \leq (4h' - 2)s + 4h' - 3$. Hence

$$e \leq (16h' - 4)s + 16h' - 9 < 16h'(s+1) \leq 24(\mathrm{wt} + 1)(s+1) = e^{(s)}_{\mathrm{B}} \quad \text{for } c_{\mathrm{cy}} = 24,$$

and $\ell^{e^{(s)}_{\mathrm{B}}} x \in \Phi_* H^1_f(K, T_1)$. □

Lemma 24.11. *Let ℓ be an odd prime, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ and a field K with $F \subseteq K \subseteq F(\mu_\ell)$ as in Proposition 23.7, $d = |\Delta_K|$, $h' \geq 1$ an integer, and $f\colon T \to T''$ a G_K-equivariant $\mathcal{O}_L$-linear surjection of the lattices of pairs (V, T), (V'', T'') over K such that every Hodge jump of V at every embedding of K lies in $[-h', h']$. Put $r_0 = \mathrm{dsc}_{\ell,d}(0, 2h')$; then $r_0 \leq \lfloor 2h'\ell/(d(\ell-1))\rfloor$, and for every $c \in \mathbb{Z}/n$ and every $y \in \mathbf{N}(T'')_c$ there is $x \in \mathbf{N}(T)_c$ with $\mathbf{N}(f)(x) = \ell^{r_0}y$; if y is Δ_K-fixed, x can be taken Δ_K-fixed.* ***Level*** *s. Let ℓ be any prime, $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, L, K ($F \subseteq K \subseteq F(\mu_{\ell^s})$) and pairs over K as in (Ex4) of Proposition 23.7, $h' \geq 1$, and $f\colon T \to T''$ a G_K-equivariant $\mathcal{O}_L$-linear surjection of the lattices of pairs (V, T), (V'', T'') over K with every Hodge jump of V at every embedding of K in $[-h', h']$. Put $r^{(s)}_0 = 2h's + v_\ell((2h')!) \leq 2h'(s+1)$. Then for every $c \in \mathbb{Z}/n$ and every $y \in \mathbf{N}_s(T'')_c$ there is $x \in \mathbf{N}_s(T)_c$ with $\mathbf{N}_s(f)(x) = \ell^{r^{(s)}_0}y$.*

Proof. By (Ex1), $\mathbf{N}(T)$ and $\mathbf{N}(T'')$ are Wach modules over K with window $(h', h')_a$ (the map $V \to V''$ induced by f is onto, so by Lemma 23.25 every Hodge jump of V'' at an embedding is one of V there). By Proposition 23.18 with $M_a = h_a = h'$, $X^{2h'}\mathbf{N}(T'')_c \subseteq \mathbf{N}(f)(\mathbf{N}(T)_c)$. Apply item 1 of Lemma 23.21 with $\Gamma = \Gamma_K$ (index d), $D = N = \mathbf{N}(T'')_c$, $N' = \mathbf{N}(f)(\mathbf{N}(T)_c)$, $\gamma_{u,D} = \gamma_{u,\mathbf{N}}$

$(u \in \Gamma_K)$, $s = 2h'$, $j = 0$. Its hypotheses hold: D is free over $e_c\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ (Wm1), so multiplication by X is injective; $\gamma_{u,\mathbf{N}}$ is γ_u-semilinear and fixes e_c; $(\gamma_u - 1)(N) \subseteq XN$ by (Wm3); and N' is an $\mathbf{A}_{F,L}$-submodule with $(\gamma_u - 1)(N') = \mathbf{N}(f)((\gamma_u - 1)\mathbf{N}(T)_c) \subseteq XN'$, as $\mathbf{N}(f)$ is $\mathbf{A}_{F,L}$-linear and commutes with every γ_u (Ex3). So $\ell^{r_0}N \subseteq N'$, which is the lifting assertion; item 2 there gives the bound on r_0. If y is Δ_K-fixed and $\mathbf{N}(f)(x) = \ell^{r_0}y$, put $x' = d^{-1}\sum_{\delta\in\Delta_K}\gamma_\delta(x)$; this is defined since $d \mid \ell - 1$ is a unit of $\mathbb{Z}_\ell$, lies in $\mathbf{N}(T)_c$ (γ_δ preserves $\mathbf{N}(T)$ and fixes e_c), is Δ_K-fixed (Δ_K is a group acting through γ, (Wm3)), and $\mathbf{N}(f)(x') = d^{-1}\sum_\delta \gamma_\delta(\ell^{r_0}y) = \ell^{r_0}y$.

Level s. By (Ex1) of level s (Proposition 23.7, (Ex4)) and Lemma 23.25, $\mathbf{N}_s(T)$ and $\mathbf{N}_s(T'')$ are Wach modules of level s over K with window $(h', h')_a$. By Proposition 23.18 (paragraph Level s) with $M_a = h_a = h'$, $X_{s-1}^{2h'}\mathbf{N}_s(T'')_c \subseteq \mathbf{N}_s(f)(\mathbf{N}_s(T)_c)$. Apply item 4 of Lemma 23.21 (paragraph Level s) with $D = N = \mathbf{N}_s(T'')_c$, $N' = \mathbf{N}_s(f)(\mathbf{N}_s(T)_c)$, $\gamma_D = \gamma_{u_s,\mathbf{N}}$, $S = 2h'$, $j = 0$. Its hypotheses hold: $u_s = 1 + \ell^s$ lies in Γ_K (it lies in $\chi_{\mathrm{cyc}}(G_{F(\mu_{\ell^s})}) = 1 + \ell^s\mathbb{Z}_\ell$, Definition 23.2) and $u_s \equiv 1 \bmod \ell^{s-1}$, so (Wm3') gives $(\gamma_{u_s} - 1)(N) \subseteq X_{s-1}N$; N' is an $\mathbf{A}_{F,L}$-submodule with $(\gamma_{u_s} - 1)(N') = \mathbf{N}_s(f)((\gamma_{u_s} - 1)\mathbf{N}_s(T)_c) \subseteq X_{s-1}N'$ by (Wm3') for $\mathbf{N}_s(T)$ and (Ex3) of level s; D is free over the domain $e_c\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ (Wm1), so $X_{s-1} \neq 0$ acts injectively; and $\gamma_{u_s,\mathbf{N}}$ is γ_{u_s}-semilinear and fixes e_c. So $\ell^{r_0^{(s)}}N \subseteq N'$ with $r_0^{(s)} = \mathrm{dsc}_\ell^{(s)}(0, 2h') = 2h's + v_\ell((2h')!)$. □

Lemma 24.12. *Let ℓ be an odd prime, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a field K with $F \subseteq K \subseteq F(\mu_\ell)$ as in Proposition 23.7, $d = |\Delta_K|$, (V, T) a pair over K and $h' \geq 1$ an integer such that every Hodge jump of V at every embedding of K lies in $[-h', h']$. Put $M = \lceil 4h'/d \rceil$, $m = Md$, $\sigma = M - 1 + v_\ell((M-1)!)$ and $r_0 = \mathrm{dsc}_{\ell,d}(0, 2h')$. Then for every $x \in H^1_f(K, T)$ there are $w_c \in e_c\mathbf{N}(T)[1/\lambda_\infty]$ $(c \in \mathbb{Z}/n)$ such that $\gamma_\delta(w_c) = w_c$ for $\delta \in \Delta_K$, $Q^{h'}w_c \in \mathbf{A}_{F,L}\,\varphi(\mathbf{N}(T)_{c-1})$, $(\gamma_u - 1)(w_c) \in \pi_\Delta^M\mathbf{N}(T)_c^\Delta$ for every $u \in \Gamma_K$ (so w_c is a Herr vector of precision m over K), and*

$$\ell^{r_0+\sigma}\,x = \sum_{c\in\mathbb{Z}/n} \mathrm{cl}(w_c) \qquad \text{in } H^1(K, T).$$

Level s. *Let ℓ be any prime, $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, L as in (Ex4) of Proposition 23.7, $K = F(\mu_{\ell^s})$, X_{s-1} the level variable of Definition 23.2, (V, T) a pair over K and $h' \geq 1$ with every Hodge jump of V at every embedding of K in $[-h', h']$. Put $M = 4h'$, $\sigma^{(s)} = (M-1)s + v_\ell((M-1)!)$ and $r_0^{(s)} = 2h's + v_\ell((2h')!)$. Then for every $x \in H^1_f(K, T)$ there are $w_c \in e_c\mathbf{N}_s(T)[1/\lambda_\infty]$ $(c \in \mathbb{Z}/n)$ with $Q_{s-1}^{h'}w_c \in \mathbf{A}_{F,L}\,\varphi(\mathbf{N}_s(T)_{c-1})$, $(\gamma_u - 1)(w_c) \in X_{s-1}^M\mathbf{N}_s(T)_c$ for every $u \in \Gamma_K$ (so w_c is a Herr vector of precision M at level s), and $\ell^{r_0^{(s)}+\sigma^{(s)}}x = \sum_c \mathrm{cl}(w_c)$ in $H^1(K, T)$.*

Proof. By Lemma 23.26 (over K) there are a pair (V_Y, Y) over K and an exact sequence $0 \to T \xrightarrow{\iota'} Y \xrightarrow{p} \mathcal{O}_L \to 0$ of G_K-equivariant $\mathcal{O}_L$-linear maps with $[Y] = x$, every Hodge jump of V_Y at every embedding being 0 or one of V, hence in $[-h', h']$. For $c \in \mathbb{Z}/n$, Lemma 24.11 for p and $y = e_c^\circ \in \mathbf{N}(\mathcal{O}_L)_c$ (Lemma 24.3, over a base field) gives $x_c \in \mathbf{N}(Y)_c$ with $\mathbf{N}(p)(x_c) = \ell^{r_0}e_c^\circ$, so by (Ex3) $(1 \otimes p)(\iota_Y(x_c)) = \iota_{\mathcal{O}_L}(\ell^{r_0}e_c^\circ) = \ell^{r_0}\varepsilon_c$. Now Proposition 24.13 applies to $(V', T') = (V, T)$, (V_Y, Y), ι', p, h', $r = r_0$ and the x_c (the Hodge jumps of V and V_Y lie in $[-h', h']$), with $s' = d$, so its M, m, σ are those defined above; it gives the w_c with (Rs1)–(Rs3) and $\ell^{r_0+\sigma}[Y] = \sum_c \mathrm{cl}(w_c)$.

Level s. By Lemma 23.26 (paragraph Level s) there are a pair (V_Y, Y) over K and an exact sequence $0 \to T \xrightarrow{\iota'} Y \xrightarrow{p} \mathcal{O}_L \to 0$ with $[Y] = x$ and all Hodge jumps of V_Y in $[-h', h']$. For $c \in \mathbb{Z}/n$, Lemma 24.11 (paragraph Level s) for p and $y = e_c^\circ \in \mathbf{N}_s(\mathcal{O}_L)_c$ (Lemma 24.3, paragraph Level s) gives $x_c \in \mathbf{N}_s(Y)_c$ with $\mathbf{N}_s(p)(x_c) = \ell^{r_0^{(s)}}e_c^\circ$, so by (Ex3) of level s, $(1 \otimes p)(\iota_{s,Y}(x_c)) = \iota_{s,\mathcal{O}_L}(\ell^{r_0^{(s)}}e_c^\circ) = \ell^{r_0^{(s)}}(\iota_s \otimes \mathrm{id})(e_c) = \ell^{r_0^{(s)}}\varepsilon_c$. Finally, Proposition 24.13 (paragraph Level s), applied to (V, T), (V_Y, Y), ι', p, h', $r = r_0^{(s)}$ and the x_c, gives the w_c and $\ell^{r_0^{(s)}+\sigma^{(s)}}[Y] = \sum_c \mathrm{cl}(w_c)$. □

Proposition 24.13. *Let ℓ be an odd prime, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ as in Proposition 23.7, K a field with $F \subseteq K \subseteq F(\mu_\ell)$, $\Delta = \Delta_K$ and $s' = |\Delta|$. Let (V',T'), (V,T) be pairs over K and $0 \to T' \xrightarrow{\iota'} T \xrightarrow{\pi} \mathcal{O}_L \to 0$ an exact sequence of G_K-equivariant $\mathcal{O}_L$-linear maps ($\mathcal{O}_L$ the trivial representation), with extension class $[T] \in H^1(K,T')$. Let $h' \geq 1$ be an integer such that every Hodge jump of V' and of V at every $\mathbb{Q}_\ell$-embedding $K \to L$ lies in $[-h',h']$, and put*

$$M = \lceil 4h'/s' \rceil, \qquad m = Ms', \qquad \sigma = M - 1 + v_\ell((M-1)!).$$

Write $\varepsilon_a = (\iota \otimes \mathrm{id})(e_a) \in A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$. Let $r \geq 0$ and suppose that for every $c \in \mathbb{Z}/n$ there is $x_c \in \mathbf{N}(T)_c$ with $(1\otimes\pi)(\iota_T(x_c)) = \ell^r \varepsilon_c$. Then there are $w_c \in e_c\mathbf{N}(T')[1/\lambda_\infty]$ ($c \in \mathbb{Z}/n$) such that

(Rs1) Invariance *$\gamma_\delta(w_c) = w_c$ for every $\delta \in \Delta$;*

(Rs2) Poles *$Q^{h'} w_c \in \mathbf{A}_{F,L}\,\varphi(\mathbf{N}(T')_{c-1})$;*

(Rs3) Precision *$(\gamma_u - 1)(w_c) \in \pi_\Delta^M\, \mathbf{N}(T')_c^\Delta$ for every $u \in \Gamma_K$; in particular w_c is a Herr vector of precision m over K;*

and

$$\ell^{r+\sigma}\,[T] = \sum_{c\in\mathbb{Z}/n} \mathrm{cl}(w_c) \qquad \text{in } H^1(K,T').$$

Level s. *Let ℓ be any prime, $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, L as in (Ex4) of Proposition 23.7, $K = F(\mu_{\ell^s})$, X_{s-1} the level variable of Definition 23.2, and (V',T'), (V,T) pairs over K with an exact sequence $0 \to T' \xrightarrow{\iota'} T \xrightarrow{\pi} \mathcal{O}_L \to 0$ of G_K-equivariant $\mathcal{O}_L$-linear maps and extension class $[T] \in H^1(K,T')$; let $h' \geq 1$ be such that every Hodge jump of V' and of V at every embedding $K \to L$ lies in $[-h',h']$, and put*

$$M = 4h', \qquad \sigma^{(s)} = (M-1)s + v_\ell((M-1)!).$$

Let $r \geq 0$ and suppose that for every $c \in \mathbb{Z}/n$ there is $x_c \in \mathbf{N}_s(T)_c$ with $(1\otimes\pi)(\iota_{s,T}(x_c)) = \ell^r\varepsilon_c$ (here $\varepsilon_c = (\iota_s \otimes \mathrm{id})(e_c)$, as $\iota_s = \iota$ on $\mathcal{O}_F$). Then there are $w_c \in e_c\mathbf{N}_s(T')[1/\lambda_\infty]$ ($c \in \mathbb{Z}/n$) with $Q_{s-1}^{h'} w_c \in \mathbf{A}_{F,L}\,\varphi(\mathbf{N}_s(T')_{c-1})$ and $(\gamma_u - 1)(w_c) \in X_{s-1}^M \mathbf{N}_s(T')_c$ for every $u \in \Gamma_K$ (so w_c is a Herr vector of precision M at level s, Lemma 23.28), and $\ell^{r+\sigma^{(s)}}[T] = \sum_c \mathrm{cl}(w_c)$ in $H^1(K,T')$.

Proof. **Step 0: preliminaries.** (a) By (Ex1), N' and N are Wach modules over K with window $(h',h')_a$, and $m = Ms' \geq 4h' \geq 1 + h'$, so the Herr calculus of Lemma 23.28 applies to both with precision m. (b) By Lemma 24.14, $(\ell-1)/s' = [K:F]$, and by (Rr7) $\Gamma_K = \Delta \cdot (1+\ell\mathbb{Z}_\ell) = \Delta \times (1+\ell\mathbb{Z}_\ell)$; $u_0 \in \Gamma_K$, $v_\ell(u_0 - 1) = 1$, and $\Delta \cup \{u_0\}$ generates a dense subgroup of Γ_K ($u_0^{\mathbb{Z}}$ is dense in $1+\ell\mathbb{Z}_\ell$, ℓ odd). (c) By Lemma 23.8(4) the ε_a are G_K-invariant, $(\varphi\otimes 1)(\varepsilon_a) = \varepsilon_{a+1}$ and $\sum_a \varepsilon_a = 1$. (d) If $y \in N[1/\lambda_\infty]$ and $(1\otimes\pi)\iota_T(y) = 0$, then $y = \iota_*(t)$ for a unique $t \in N'[1/\lambda_\infty]$; $t \in X^k N'$ if $y \in X^k N$; $e_c t = t$ if $e_c y = y$; t is Δ-fixed if y is. Indeed, by (Ex3) $\iota_{\mathcal{O}_L}(\pi_* y) = (1\otimes\pi)\iota_T(y) = 0$, and $\iota_{\mathcal{O}_L}$ is injective on $\mathbf{N}(\mathcal{O}_L)[1/\lambda_\infty]$ (Gc0), so $\pi_* y = 0$; clearing poles by a product f of the Q_s, Lemma 23.33 gives $fy = \iota_*(t_0)$, $t = t_0/f$, unique as ι_* is injective (also after localisation). If $y = X^k y'$ with $y' \in N$, then $X^k \pi_*(y') = 0$, so $\pi_*(y') = 0$ ($N(\mathcal{O}_L)$ is torsion-free) and $y' = \iota_*(t')$, $t = X^k t'$. The last two claims follow from injectivity, as ι_* is $\mathbf{A}_{F,L}$-linear and commutes with γ_δ. (e) For $k \geq 0$, $X^k N_c' \cap N_c'^\Delta = \pi_\Delta^{\lceil k/s' \rceil} N_c'^\Delta$, and in particular $X^m N_c' \cap N_c'^\Delta = \pi_\Delta^M N_c'^\Delta$. Indeed, in a Δ-fixed basis (Dm1) the coordinates f of such an element lie in $R_c^\Delta \cap X^k R_c$; by (Rr2) $f = \sum_j c_j \pi_\Delta^j$, and by (Rr1) the X-adic order of f is $s'j_0$

for the least j_0 with $c_{j_0} \neq 0$; so $j_0 \geq k/s'$ and $f \in \pi_\Delta^{\lceil k/s' \rceil} R_c^\Delta$. The converse inclusion holds as $\pi_\Delta^j \in X^{s'j} R_c$.

Step 1: Δ-fixed lifts of the units. Replace x_c by $|\Delta|^{-1} \sum_{\delta \in \Delta} \gamma_\delta(x_c)$ ($|\Delta|$ divides $\ell - 1$): for $\delta \in \Delta$ and $\sigma_\delta \in G_K$ with $\chi_{\mathrm{cyc}}(\sigma_\delta) = \delta$, (Gc0) and (c) give $(1 \otimes \pi)\iota_T(\gamma_\delta x_c) = \sigma_\delta(\ell^r \varepsilon_c) = \ell^r \varepsilon_c$. So we may assume $x_c \in N_c^\Delta$.

Step 2: the Γ-defect. By (Dm3), $(\gamma_{u_0} - 1)(x_c) \in \pi_\Delta N_c^\Delta \subseteq X N_c$, and its image under $(1 \otimes \pi)\iota_T$ is $\sigma_0(\ell^r \varepsilon_c) - \ell^r \varepsilon_c = 0$ ($\chi_{\mathrm{cyc}}(\sigma_0) = u_0$). By (d), $(\gamma_{u_0} - 1)(x_c) = \iota_*(z_c)$ with $z_c \in X N_c'$ Δ-fixed, so $z_c \in \pi_\Delta N_c'^\Delta$ by (Dm2). By Lemma 24.16 (for N', u_0, M) there is $k_c \in \pi_\Delta N_c'^\Delta$ with

$$\kappa_c := \ell^\sigma z_c - (\gamma_{u_0} - 1)(k_c) \in \pi_\Delta^M N_c'^\Delta.$$

Put $b_c = \ell^\sigma x_c - \iota_*(k_c) \in N_c^\Delta$. Then $(\gamma_{u_0} - 1)(b_c) = \iota_*(\kappa_c)$ (ι_* commutes with γ_{u_0}, (Ex3)) and $(1 \otimes \pi)\iota_T(b_c) = \ell^{r+\sigma} \varepsilon_c$ ($\pi \circ \iota' = 0$).

Step 3: Frobenius preserves $\pi_\Delta^M N'^\Delta$. Let $y = \pi_\Delta^M n$ with $n \in N_{c-1}'^\Delta$. By (Rr5) and (Rr4), $\varphi(y) = w'^M \pi_\Delta^M q_\Delta^{Ms'} \varphi(n) = w'^M w_\Delta^{Ms'} \pi_\Delta^M Q^{Ms'-h'} \cdot Q^{h'} \varphi(n)$ with units $w' = w'_\Delta$, w_Δ, and $Q^{h'}\varphi(n) \in N_c'$ by (Wm2); as $Ms' \geq h'$, $\varphi(y) \in \pi_\Delta^M N_c'$. It is Δ-fixed because φ commutes with every γ_δ (Wm3). Writing $\varphi(y) = \pi_\Delta^M n'$, n' is Δ-fixed (π_Δ is, and N_c' is torsion-free), so $\varphi(y) \in \pi_\Delta^M N_c'^\Delta$.

Step 4: the vectors w_c. Put $s_c = \varphi(b_{c-1}) \in e_c N[1/Q]$ (by (Wm2) and $\varphi(e_{c-1}) = e_c$); it is Δ-fixed, and $(1 \otimes \pi)\iota_T(s_c) = (\varphi \otimes 1)(\ell^{r+\sigma} \varepsilon_{c-1}) = \ell^{r+\sigma} \varepsilon_c$ by (Gc0) and (c). So $s_c - b_c$ has image 0, and by (d) $s_c - b_c = \iota_*(w_c)$ with $w_c \in e_c N'[1/\lambda_\infty]$ Δ-fixed: this is (Rs1).

(Rs2). Let $P = \mathbf{A}_{F,L}\varphi(N_{c-1})$ and $P' = \mathbf{A}_{F,L}\varphi(N_{c-1}')$. For a Wach module $\mathbf{N}$ over K with an R_{c-1}-basis $(d_j)_{j \leq d}$ of $\mathbf{N}_{c-1}$, the $\varphi(d_j)$ form an R_c-basis of $\mathbf{A}_{F,L}\varphi(\mathbf{N}_{c-1})$: they generate it by semilinearity, and they are independent because by (Wm2) their span contains $Q^{h_c}\mathbf{N}_c$, of rank d over the domain R_c. Hence $R_c \otimes_{\varphi, R_{c-1}} \mathbf{N}_{c-1} \to \mathbf{A}_{F,L}\varphi(\mathbf{N}_{c-1})$, $a \otimes x \mapsto a\varphi(x)$, is an isomorphism, natural in $\mathbf{N}$ for maps commuting with φ. Now $\varphi\colon R_{c-1} \to R_c$ is $f(X) \mapsto f((1+X)^\ell - 1)$, and R_c is free over its image with basis $(1+X)^i$, $0 \leq i < \ell$ (Weierstrass division by the distinguished polynomial $(1+X)^\ell - 1$); so $R_c \otimes_{\varphi, R_{c-1}} -$ is exact. Applied to the exact sequence $0 \to N_{c-1}' \xrightarrow{\iota_*} N_{c-1} \xrightarrow{\pi_*} \mathbf{N}(\mathcal{O}_L)_{c-1}$ (Lemma 23.33), it shows: an element of P killed by π_* lies in $\iota_*(P')$. Now $Q^{h'} s_c \in P$, and $Q^{h'} b_c \in P$ by (Wm2) for N ($h_c \leq h'$), so $Q^{h'}(s_c - b_c) \in P$, and $\pi_*(Q^{h'}(s_c - b_c)) = 0$ (Step 0(d)). Hence $Q^{h'}\iota_*(w_c) = \iota_*(p')$ with $p' \in P'$, and $Q^{h'} w_c = p'$ by injectivity.

(Rs3). Since ι_* and φ commute with γ_{u_0} (Ex3), (Wm3), Step 2 gives

$$\iota_*\big((\gamma_{u_0} - 1)(w_c)\big) = \varphi\big(\iota_*(\kappa_{c-1})\big) - \iota_*(\kappa_c) = \iota_*\big(\varphi(\kappa_{c-1}) - \kappa_c\big),$$

so $(\gamma_{u_0} - 1)(w_c) = \varphi(\kappa_{c-1}) - \kappa_c \in \pi_\Delta^M N_c'^\Delta \subseteq X^m N'$ by Step 3 and (Rr1). Also $(\gamma_\delta - 1)(w_c) = 0$ for $\delta \in \Delta$. By (b) and (Hc2), w_c is a Herr vector of precision m: $(\gamma_u - 1)(w_c) \in X^m N'$ for all $u \in \Gamma_K$. This element lies in $e_c N'$ and is Δ-fixed (γ_u and γ_δ commute, Γ_K being commutative), so by (e) it lies in $\pi_\Delta^M N_c'^\Delta$.

Step 5: the global vector and its class. Put $\tilde{x} = \sum_c s_c \in N[1/\lambda_\infty]$. By Step 4 and (c), $(1 \otimes \pi)\iota_T(\tilde{x}) = \ell^{r+\sigma} \sum_c \varepsilon_c = 1 \otimes \ell^{r+\sigma}$. Further $(\gamma_\delta - 1)(\tilde{x}) = 0$ for $\delta \in \Delta$, and $(\gamma_{u_0} - 1)(\tilde{x}) = \sum_c \varphi(\iota_*(\kappa_{c-1})) = \iota_*(\sum_c \varphi(\kappa_{c-1})) \in \iota_*(X^m N') \subseteq X^m N$ by Step 3. By (b) and (Hc2) for N, $(\gamma_u - 1)(\tilde{x}) \in X^m N$ for all $u \in \Gamma_K$; its image under $(1 \otimes \pi)\iota_T$ is $(\sigma_u - 1)(1 \otimes \ell^{r+\sigma}) = 0$ ((Gc0), $\chi_{\mathrm{cyc}}(\sigma_u) = u$), so by (d) $(\gamma_u - 1)(\tilde{x}) \in \iota_*(X^m N')$. By Lemma 23.32 (over K, $c = \ell^{r+\sigma}$, window (h', h') of V'), the unique $y \in N'[1/\lambda_\infty]$ with $\iota_*(y) = (\varphi - 1)(\tilde{x})$ is a Herr vector of precision m with $\mathrm{cl}(y) = \ell^{r+\sigma}[T]$.

Step 6: telescoping. Let $W = \sum_c w_c$. Then

$$\iota_*(\varphi(W)) = \sum_c \varphi(s_c - b_c) = \sum_c \varphi(s_c) - \sum_c s_{c+1} = \varphi(\tilde{x}) - \tilde{x},$$

so $y = \varphi(W)$ by injectivity of ι_*. By Step 4 and (Hc3), W is a Herr vector of precision m with $(\gamma_u - 1)(W) \in X^m N'$ for all $u \in \Gamma_K$; by (Hc4), $(\varphi - 1)(W)$ is a Herr vector with $\mathrm{cl}((\varphi - 1)(W)) = 0$. By (Hc3),

$$\ell^{r+\sigma}[T] = \mathrm{cl}(\varphi(W)) = \mathrm{cl}(W) + \mathrm{cl}((\varphi - 1)(W)) = \sum_c \mathrm{cl}(w_c).$$

Level s. Let the data be as in the paragraph Level s of the statement; now $N' = \mathbf{N}_s(T')$, $N = \mathbf{N}_s(T)$, $\iota_* = \mathbf{N}_s(\iota')$, $\pi_* = \mathbf{N}_s(\pi)$, $Y = X_{s-1}$, $u_s = 1 + \ell^s$, $\sigma = \sigma^{(s)}$, and every cited clause is taken in its level-s form: (Ex1)–(Ex3) of (Ex4) of Proposition 23.7, (Hc1)–(Hc4) of Lemma 23.28, Lemma 23.32, Lemma 23.33, (Gc0) of Lemma 23.29, items 1–3 of Lemma 23.30, each in its paragraph Level s. *Step 0.* (a) N', N are Wach modules of level s with window $(h', h')_a$, and $M = 4h' \geq 1 + h'$. (b) $\Gamma_K = 1 + \ell^s \mathbb{Z}_\ell$ (Definition 23.2, paragraph Level s), which is procyclic with topological generator u_s ($\ell^s \neq 2$); so $\{u_s\}$ generates a dense subgroup of Γ_K, (Hc2) applies with this subset, and (Wm3') holds for every $u \in \Gamma_K$. (c) By Lemma 23.8(6) the ε_a are G_K-invariant, $(\varphi \otimes 1)(\varepsilon_a) = \varepsilon_{a+1}$ and $\sum_a \varepsilon_a = 1$. (d) If $y \in N[1/\lambda_\infty]$ and $(1 \otimes \pi)\iota_{s,T}(y) = 0$, then $y = \iota_*(t)$ for a unique $t \in N'[1/\lambda_\infty]$, with $t \in Y^k N'$ if $y \in Y^k N$ and $e_c t = t$ if $e_c y = y$: the proof is that of Step 0(d) with Y in place of X ($Y^k \pi_*(y') = 0$ forces $\pi_*(y') = 0$ as $\mathbf{N}_s(\mathcal{O}_L)$ is torsion-free). *Step 1* is void (Δ_K is trivial). *Step 2.* By (Wm3'), $(\gamma_{u_s} - 1)(x_c) \in Y N_c$, and its image under $(1 \otimes \pi)\iota_{s,T}$ is $\sigma_0(\ell^r \varepsilon_c) - \ell^r \varepsilon_c = 0$ ($\chi_{\mathrm{cyc}}(\sigma_0) = u_s$, (Gc0), (c)); by (d), $(\gamma_{u_s} - 1)(x_c) = \iota_*(z_c)$ with $z_c \in Y N'_c$. Item 5 of Lemma 23.21 (paragraph Level s; $D = N = N'_c$, free over $e_c \mathbf{A}_{F,L}$, $\gamma_D = \gamma_{u_s}$, (Wm3')) gives $k_c \in Y N'_c$ with $\kappa_c = \ell^\sigma z_c - (\gamma_{u_s} - 1)(k_c) \in Y^M N'_c$. Put $b_c = \ell^\sigma x_c - \iota_*(k_c) \in N_c$; then $(\gamma_{u_s} - 1)(b_c) = \iota_*(\kappa_c)$ and $(1 \otimes \pi)\iota_{s,T}(b_c) = \ell^{r+\sigma}\varepsilon_c$. *Step 3.* By item 1 of Lemma 23.30 (Level s, $m = M \geq 1 + h'$), $\varphi(Y^M N'_{c-1}) \subseteq Y^M Q_{s-1} N' \cap e_c N'[1/Q_{s-1}]$, so $\varphi(Y^M N'_{c-1}) \subseteq Y^M N'_c$. *Step 4.* Put $s_c = \varphi(b_{c-1}) \in e_c N[1/Q_{s-1}]$; by (Gc0) and (c), $(1 \otimes \pi)\iota_{s,T}(s_c) = \ell^{r+\sigma}\varepsilon_c$, so $s_c - b_c = \iota_*(w_c)$ with $w_c \in e_c N'[1/\lambda_\infty]$ by (d). The pole clause is proved as (Rs2) above with Q_{s-1} in place of Q: the φ-images of an R_{c-1}-basis of $\mathbf{N}_{c-1}$ form an R_c-basis of $\mathbf{A}_{F,L}\varphi(\mathbf{N}_{c-1})$ by (Wm2) of level s, $R_c \otimes_{\varphi, R_{c-1}} -$ is exact, and Lemma 23.33 (Level s) gives $Q_{s-1}^{h'} w_c \in \mathbf{A}_{F,L}\varphi(N'_{c-1})$. As ι_*, φ commute with γ_{u_s}, $(\gamma_{u_s} - 1)(w_c) = \varphi(\kappa_{c-1}) - \kappa_c \in Y^M N'_c$ by Step 3. By (b) and (Hc2), $(\gamma_u - 1)(w_c) \in Y^M N'$ for all $u \in \Gamma_K$, and it lies in $e_c N'$, hence in $Y^M N'_c$. *Step 5.* $\tilde{x} = \sum_c s_c$ has $(1 \otimes \pi)\iota_{s,T}(\tilde{x}) = 1 \otimes \ell^{r+\sigma}$ and $(\gamma_{u_s} - 1)(\tilde{x}) = \iota_*(\sum_c \varphi(\kappa_{c-1})) \in \iota_*(Y^M N') \subseteq Y^M N$; by (Hc2) for N, $(\gamma_u - 1)(\tilde{x}) \in Y^M N$ for all $u \in \Gamma_K$, with image 0 under $(1 \otimes \pi)\iota_{s,T}$, so by (d) it lies in $\iota_*(Y^M N')$. Lemma 23.32 (Level s, $c = \ell^{r+\sigma}$, window (h', h'), $m = M$) gives the Herr vector y of precision M at level s with $\iota_*(y) = (\varphi - 1)(\tilde{x})$ and $\mathrm{cl}(y) = \ell^{r+\sigma}[T]$. *Step 6* is unchanged: $W = \sum_c w_c$ has $\iota_*(\varphi(W)) = \varphi(\tilde{x}) - \tilde{x}$, so $y = \varphi(W)$, and (Hc3), (Hc4) give $\ell^{r+\sigma}[T] = \mathrm{cl}(W) + \mathrm{cl}((\varphi - 1)(W)) = \sum_c \mathrm{cl}(w_c)$. □

Lemma 24.14. *Let ℓ be an odd prime, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ as in Definition 23.2, K a field with $F \subseteq K \subseteq F(\mu_\ell)$, $\Delta = \Delta_K$, and $\mathbf{N}$ a Wach module over K with coefficients L (any window), of rank r in (Wm1). Then Δ is a subgroup of $\mu_{\ell-1}$ with $(\ell - 1)/|\Delta| = [K : F]$, and for every $a \in \mathbb{Z}/n$, with $R = e_a \mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$:*

(Dm1) Invariant basis *$\mathbf{N}_a$ has an R-basis $(n_k)_{k \leq r}$ of Δ-fixed vectors, and for every such basis $\mathbf{N}_a^\Delta = \bigoplus_k R^\Delta n_k$; so $\mathbf{N}_a^\Delta$ is a free module of rank r over $R^\Delta = \mathcal{O}_L[[\pi_\Delta]]$, and an element of $\mathbf{N}_a$ is Δ-fixed if and only if its coordinates in (n_k) lie in R^Δ.*

(Dm2) Divisibility by X *$X\mathbf{N}_a \cap \mathbf{N}_a^\Delta = \pi_\Delta \mathbf{N}_a^\Delta$.*

(Dm3) Gamma *For every $u \in \Gamma_K$ and $y \in \mathbf{N}_a^\Delta$, $\gamma_u(y) \in \mathbf{N}_a^\Delta$ and $(\gamma_u - 1)(y) = \pi_\Delta y'$ for a unique $y' \in \mathbf{N}_a^\Delta$.*

Proof. **The group.** $\Delta = \Delta_K$ is a finite subgroup of $\mathbb{Z}_\ell^\times$, hence of its torsion subgroup $\mu_{\ell-1}$ (ℓ odd). Let $e = [K : F]$; it divides $\ell - 1 = [F(\mu_\ell) : F]$, and $\mu_{\ell-1}$, being cyclic, has exactly one subgroup Δ' of order $(\ell-1)/e$. By (Rr7) for Δ', Δ' is the torsion subgroup of $\chi_{\mathrm{cyc}}(G_K) = \Gamma_K$, i.e. $\Delta' = \Delta$; so $(\ell-1)/|\Delta| = e$.

The operators on R. The γ_u ($u \in \Gamma_K$) fix $\mathcal{O}_F \otimes \mathcal{O}_L$, hence e_a, so they preserve $\mathbf{N}_a$ and act on $R \cong \mathcal{O}_L[[X]]$ as the $\mathcal{O}_L$-algebra automorphisms with $X \mapsto (1+X)^u - 1$; this is the action of (Dr4) of Definition 23.23, so R^Δ is the ring of Lemma 23.24, and by (Rr2) $R^\Delta = \mathcal{O}_L[[\pi_\Delta]]$ (unique expansion) and $R^\Delta \cap XR = \pi_\Delta R^\Delta$.

(Dm1). By Lemma 24.15, $\mathbf{N}_a$ has an R-basis (n_k) with $\gamma_\delta(n_k) = n_k$ for all $\delta \in \Delta$. Let (n_k) be any such basis and $x = \sum_k f_k n_k \in \mathbf{N}_a$, $f_k \in R$. Since γ_δ is γ_δ-semilinear, $\gamma_\delta(x) = \sum_k \gamma_\delta(f_k) n_k$, and by uniqueness of coordinates $\gamma_\delta(x) = x$ if and only if $\gamma_\delta(f_k) = f_k$ for all k. So $x \in \mathbf{N}_a^\Delta$ if and only if every $f_k \in R^\Delta$, i.e. $\mathbf{N}_a^\Delta = \sum_k R^\Delta n_k$, and the sum is direct because the n_k are R-linearly independent. So $\mathbf{N}_a^\Delta$ is free over R^Δ on (n_k), of rank r.

(Dm2). By (Rr1), $\pi_\Delta \in XR$, and π_Δ is Δ-fixed, so $\pi_\Delta \mathbf{N}_a^\Delta \subseteq X\mathbf{N}_a \cap \mathbf{N}_a^\Delta$. Conversely let $x = \sum_k f_k n_k \in X\mathbf{N}_a \cap \mathbf{N}_a^\Delta$. Writing $x = X\sum_k g_k n_k$ with $g_k \in R$ and comparing coordinates, $f_k = Xg_k \in XR$; by (Dm1), $f_k \in R^\Delta$. Hence $f_k \in R^\Delta \cap XR = \pi_\Delta R^\Delta$, say $f_k = \pi_\Delta f_k'$ with $f_k' \in R^\Delta$, and $x = \pi_\Delta \sum_k f_k' n_k \in \pi_\Delta \mathbf{N}_a^\Delta$ by (Dm1).

(Dm3). Let $u \in \Gamma_K$, $\delta \in \Delta \subseteq \Gamma_K$ and $y \in \mathbf{N}_a^\Delta$. As $\Gamma_K \subseteq \mathbb{Z}_\ell^\times$ is commutative, (Wm3) gives $\gamma_\delta\gamma_u = \gamma_{\delta u} = \gamma_{u\delta} = \gamma_u\gamma_\delta$, so $\gamma_\delta(\gamma_u(y)) = \gamma_u(y)$: $\gamma_u(y) \in \mathbf{N}_a^\Delta$, and $(\gamma_u - 1)(y) \in \mathbf{N}_a^\Delta$. By (Wm3), $(\gamma_u - 1)(y) \in X\mathbf{N}$, and as it lies in $\mathbf{N}_a$, $(\gamma_u - 1)(y) = e_a(\gamma_u - 1)(y) \in e_a X\mathbf{N} = X\mathbf{N}_a$. By (Dm2), $(\gamma_u - 1)(y) = \pi_\Delta y'$ with $y' \in \mathbf{N}_a^\Delta$; y' is unique because $\pi_\Delta \neq 0$ and $\mathbf{N}_a$ is free over the domain R. □

Lemma 24.15. *Let ℓ be an odd prime, F, L as in Definition 23.2, K a base field as there, $\mathbf{N}$ a Wach module over K with coefficients L (any window) and $a \in \mathbb{Z}/n$. Then $\mathbf{N}_a$ has an $e_a\mathbf{A}_{F,L}$-basis $(n_k)_{k\le r}$ with $\gamma_u(n_k) = n_k$ for every $u \in \Delta_K$ and every k. For $K = F$, $\Delta_F = \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$.*

Proof. Write $R_0 = e_a\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ and $\Delta = \Delta_K$, a finite subgroup of $\mathbb{Z}_\ell^\times$, hence of $\mu_{\ell-1}$, so $|\Delta|$ divides $\ell - 1$ and is a unit of $\mathbb{Z}_\ell$ ($\Delta = \mu_{\ell-1}$ for $K = F$). The ring R_0 is local with maximal ideal $\mathfrak{m} = (\pi_L, X)$ (π_L a uniformiser of $\mathcal{O}_L$), and $\mathbf{N}_a$ is free of rank r over it by (Wm1). The operators γ_u ($u \in \Gamma_K \supseteq \Delta$) preserve $\mathbf{N}_a$ ($\gamma_u(e_a) = e_a$), act as the identity on $\mathcal{O}_F \otimes \mathcal{O}_L$, and satisfy $(\gamma_u - 1)\mathbf{N} \subseteq X\mathbf{N}$ and $\gamma_{uv} = \gamma_u\gamma_v$ by (Wm3). As $|\Delta| \in \mathbb{Z}_\ell^\times$, the map $\pi(x) = |\Delta|^{-1}\sum_{\delta\in\Delta}\gamma_\delta(x)$ sends $\mathbf{N}_a$ into its Δ-invariants ($\gamma_u\pi = \pi$ for $u \in \Delta$, since Δ is a group), and $\pi(x) - x = |\Delta|^{-1}\sum_\delta(\gamma_\delta - 1)(x) \in X\mathbf{N}_a \subseteq \mathfrak{m}\mathbf{N}_a$. Lift a basis of $\mathbf{N}_a/\mathfrak{m}\mathbf{N}_a$ to $x_1, \dots, x_r \in \mathbf{N}_a$ and put $n_k = \pi(x_k)$. The n_k reduce to the same basis, so they generate $\mathbf{N}_a$ by Nakayama's lemma; r generators of a free module of rank r over a commutative ring form a basis. So (n_k) is a Δ-invariant basis. □

Lemma 24.16. *Let ℓ be an odd prime, F, L, K with $F \subseteq K \subseteq F(\mu_\ell)$ as in Lemma 24.14, $\Delta = \Delta_K$, $\mathbf{N}$ a Wach module over K with coefficients L (any window), $a \in \mathbb{Z}/n$, $u \in \Gamma_K$ with $v_\ell(u-1) = 1$, and $M \ge 1$ an integer. Put*

$$\sigma_M = \sum_{i=1}^{M-1}(1 + v_\ell(i)) = M - 1 + v_\ell((M-1)!).$$

Then for every $z \in \pi_\Delta\mathbf{N}_a^\Delta$ there is $k \in \pi_\Delta\mathbf{N}_a^\Delta$ with

$$\ell^{\sigma_M} z - (\gamma_u - 1)(k) \in \pi_\Delta^M \mathbf{N}_a^\Delta.$$

Proof. By (Dm1), $\mathbf{N}_a^\Delta$ has an R^Δ-basis $(n_k)_{k\le r}$, so Lemma 24.17 applies to $\mathbf{N}$, a, u, M and this basis, with the same σ_M. Its item (Gf0) for $\mathbf{N}^\natural = \mathbf{N}_a^\Delta$ is the assertion. □

Lemma 24.17. *Let ℓ be an odd prime, F, L, K with $F \subseteq K \subseteq F(\mu_\ell)$ as in Lemma 24.14, $\Delta = \Delta_K$, $\mathbf{N}$ a Wach module over K with coefficients L (any window), $a \in \mathbb{Z}/n$, $R = e_a\mathbf{A}_{F,L}$, $u \in \Gamma_K$ with $v_\ell(u-1) = 1$, and $M \ge 1$ an integer. Put*

$$\sigma_M = \sum_{i=1}^{M-1} \big(1 + v_\ell(i)\big) = M - 1 + v_\ell\big((M-1)!\big).$$

Let $\mathbf{N}_a^{\Delta\vee} = \mathrm{Hom}_{R^\Delta}(\mathbf{N}_a^\Delta, R^\Delta)$, on which γ_u acts by $\gamma_u(\lambda) = \gamma_u \circ \lambda \circ \gamma_u^{-1}$. Let $(n_k)_{k\le r}$ be an R^Δ-basis of $\mathbf{N}_a^\Delta$ and $(n_k^\vee)_{k\le r}$ the dual basis of $\mathbf{N}_a^{\Delta\vee}$. Then:

(Gf0) Approximate solutions *for $\mathbf{N}^\natural$ either of $\mathbf{N}_a^\Delta$, $\mathbf{N}_a^{\Delta\vee}$ and every $z \in \pi_\Delta\mathbf{N}^\natural$ there is $k \in \pi_\Delta\mathbf{N}^\natural$ with $\ell^{\sigma_M} z - (\gamma_u - 1)(k) \in \pi_\Delta^M\mathbf{N}^\natural$;*

(Gf1) Frames *there are $h_1, \dots, h_r \in \mathbf{N}_a^\Delta$ with $h_k - \ell^{\sigma_M} n_k \in \pi_\Delta\mathbf{N}_a^\Delta$ and $(\gamma_u - 1)(h_k) \in \pi_\Delta^M\mathbf{N}_a^\Delta$ for every k;*

(Gf2) Dual frames *there are $\lambda_1, \dots, \lambda_r \in \mathbf{N}_a^{\Delta\vee}$ with $\lambda_k - \ell^{\sigma_M} n_k^\vee \in \pi_\Delta\mathbf{N}_a^{\Delta\vee}$ and $(\gamma_u - 1)(\lambda_k) \in \pi_\Delta^M\mathbf{N}_a^{\Delta\vee}$ for every k; equivalently, $\lambda_k(\gamma_u x) - \gamma_u(\lambda_k(x)) \in \pi_\Delta^M R^\Delta$ for every $x \in \mathbf{N}_a^\Delta$;*

(Gf3) Rigidity *if $g \in R^\Delta$ satisfies $\gamma_u(g) - g \in \pi_\Delta^M R^\Delta$ and $g = \sum_{i\ge 0} g_i \pi_\Delta^i$ with $g_i \in \mathcal{O}_L$, then $g - g_0 \in \pi_\Delta^M R^\Delta$;*

(Gf4) Products *for any h_k as in (Gf1) and λ_j as in (Gf2), $\lambda_j(h_k) - \ell^{2\sigma_M}\delta_{jk} \in \pi_\Delta^M R^\Delta$ for all j, k (δ_{jk} the Kronecker symbol);*

(Gf5) Reconstruction *for such h_k, λ_k and every $x \in \mathbf{N}_a^\Delta$: $\sum_k \lambda_k(x)\, h_k - \ell^{2\sigma_M} x \in \pi_\Delta^M\mathbf{N}_a^\Delta$;*

(Gf6) Uniqueness *if $d \in \pi_\Delta\mathbf{N}_a^\Delta$ and $(\gamma_u - 1)(d) \in \pi_\Delta^M\mathbf{N}_a^\Delta$, then $d \in \pi_\Delta^M\mathbf{N}_a^\Delta$. In particular the class modulo $\pi_\Delta^M\mathbf{N}_a^\Delta$ of a vector $h \in \mathbf{N}_a^\Delta$ with $h - \ell^{\sigma_M} x \in \pi_\Delta\mathbf{N}_a^\Delta$ and $(\gamma_u - 1)(h) \in \pi_\Delta^M\mathbf{N}_a^\Delta$ depends only on x modulo $\pi_\Delta\mathbf{N}_a^\Delta$, and such an h exists for every $x \in \mathbf{N}_a^\Delta$ ($h = \sum_k o_k h_k$ for $x \equiv \sum_k o_k n_k$, $o_k \in \mathcal{O}_L$).*

Proof. **Step 0: the two modules.** By (Dm1), $\mathbf{N}_a^\Delta$ is free over R^Δ with basis (n_k); hence $\mathbf{N}_a^{\Delta\vee}$ is free with the dual basis $(n_k^\vee)$, and since π_Δ is a non-zero-divisor of the domain R^Δ, a form with values in $\pi_\Delta^j R^\Delta$ lies in $\pi_\Delta^j\mathbf{N}_a^{\Delta\vee}$ (divide its values on the basis). By (Dm3), γ_u and $\gamma_{u^{-1}} = \gamma_u^{-1}$ map $\mathbf{N}_a^\Delta$ into itself, so γ_u is a bijection of $\mathbf{N}_a^\Delta$; it is semilinear, $\gamma_u(fx) = \gamma_u(f)\gamma_u(x)$ for $f \in R^\Delta$, and γ_u is a continuous ring automorphism of R^Δ (it fixes $\mathcal{O}_L$, and $\gamma_u(\pi_\Delta) \in R^\Delta$ by (Rr6)). On $\mathbf{N}_a^{\Delta\vee}$: $\gamma_u(\lambda)(fx) = \gamma_u(\lambda(\gamma_u^{-1}(f)\gamma_u^{-1}(x))) = f\,\gamma_u(\lambda)(x)$, so $\gamma_u(\lambda)$ is R^Δ-linear, and $\gamma_u(g\lambda) = \gamma_u(g)\gamma_u(\lambda)$; so γ_u is a semilinear bijection of $\mathbf{N}_a^{\Delta\vee}$ with inverse $\lambda \mapsto \gamma_u^{-1} \circ \lambda \circ \gamma_u$. The equivalence in (Gf2): for $x \in \mathbf{N}_a^\Delta$ and $y = \gamma_u x$, $(\gamma_u(\lambda) - \lambda)(y) = \gamma_u(\lambda(x)) - \lambda(\gamma_u x)$, and y runs over $\mathbf{N}_a^\Delta$ when x does.

Let $\mathbf{N}^\natural$ denote either of the two modules $\mathbf{N}_a^\Delta$, $\mathbf{N}_a^{\Delta\vee}$. We show

$$(\gamma_u - 1)(\mathbf{N}^\natural) \subseteq \pi_\Delta\mathbf{N}^\natural. \tag{H}$$

For $\mathbf{N}_a^\Delta$ this is (Dm3). For $g \in R^\Delta$, $\gamma_u(g) - g \in \pi_\Delta R^\Delta$: by (Rr2) g is a convergent series $\sum_i g_i\pi_\Delta^i$, $g_i \in \mathcal{O}_L$, and $\gamma_u(\pi_\Delta)^i - \pi_\Delta^i \in \pi_\Delta R^\Delta$ by (Rr6). For $\lambda \in \mathbf{N}_a^{\Delta\vee}$ and $x \in \mathbf{N}_a^\Delta$, (Dm3) for $u^{-1} \in \Gamma_K$ gives $\gamma_u^{-1}(x) = x + \pi_\Delta x'$ with $x' \in \mathbf{N}_a^\Delta$, so

$$(\gamma_u(\lambda) - \lambda)(x) = \big(\gamma_u(\lambda(x)) - \lambda(x)\big) + \gamma_u(\pi_\Delta)\,\gamma_u(\lambda(x')) \in \pi_\Delta R^\Delta,$$

and $\gamma_u(\lambda)-\lambda\in\pi_\Delta\mathbf{N}_a^{\Delta\vee}$ by Step 0.

Step 1: γ_u on π_Δ. By (Rr6), $\gamma_u(\pi_\Delta)=\pi_\Delta m_u$ with $m_u=1+t_uq_\Delta\in R^\Delta$. Write $\gamma_u(X)=(1+X)^u-1=X(u+Xb)$ with $b\in\mathbb{Z}_\ell[[X]]$ and $\pi_\Delta=X^{s'}v_\Delta$ with v_Δ a unit (Rr1). Then $\gamma_u(\pi_\Delta)=X^{s'}(u+Xb)^{s'}\gamma_u(v_\Delta)$ and $\gamma_u(v_\Delta)\equiv v_\Delta(0)\equiv v_\Delta \pmod X$; cancelling $X^{s'}$ in the domain $\mathbb{Z}_\ell[[X]]$ gives $m_uv_\Delta=(u+Xb)^{s'}\gamma_u(v_\Delta)$, so $m_u\equiv u^{s'}\pmod X$, hence $m_u-u^{s'}\in R^\Delta\cap XR=\pi_\Delta R^\Delta$ (Rr2). For $i\ge1$, $m_u^i-u^{is'}=(m_u-u^{s'})\sum_{j<i}m_u^ju^{(i-1-j)s'}\in\pi_\Delta R^\Delta$. Since ℓ is odd and $v_\ell(u-1)=1$, $v_\ell(u^k-1)=1+v_\ell(k)$ for every integer $k\ge1$ (classical), and $v_\ell(is')=v_\ell(i)$; so $u^{is'}-1=\ell^{1+v_\ell(i)}\varepsilon_i$ with $\varepsilon_i\in\mathbb{Z}_\ell^\times$.

Step 2: graded pieces. Let $i\ge1$ and $y\in\mathbf{N}^\natural$. By (H), $\gamma_u(y)=y+\pi_\Delta y'$ with $y'\in\mathbf{N}^\natural$, so

$$(\gamma_u-1)(\pi_\Delta^iy)=(u^{is'}-1)\pi_\Delta^iy+\pi_\Delta^i(m_u^i-u^{is'})y+\pi_\Delta^{i+1}m_u^iy',$$

and the last two terms lie in $\pi_\Delta^{i+1}\mathbf{N}^\natural$ by Step 1. Hence

$$(\gamma_u-1)(\varepsilon_i^{-1}\pi_\Delta^iy)\equiv\ell^{1+v_\ell(i)}\pi_\Delta^iy\pmod{\pi_\Delta^{i+1}\mathbf{N}^\natural}.\tag{1}$$

Step 3: approximate solutions. For $1\le j\le M$ put $\sigma_j=\sum_{i=1}^{j-1}(1+v_\ell(i))$. We show: for every $z\in\pi_\Delta\mathbf{N}^\natural$ there is $k_j\in\pi_\Delta\mathbf{N}^\natural$ with $r_j:=\ell^{\sigma_j}z-(\gamma_u-1)(k_j)\in\pi_\Delta^j\mathbf{N}^\natural$. For $j=1$ take $k_1=0$. Let $1\le j<M$ and k_j be given; $r_j\in\mathbf{N}^\natural$ by (H) and $r_j=\pi_\Delta^jy_j$ with $y_j\in\mathbf{N}^\natural$ (Step 0: π_Δ is a non-zero-divisor on the free module). Put $k'=\varepsilon_j^{-1}\pi_\Delta^jy_j\in\pi_\Delta\mathbf{N}^\natural$ and $k_{j+1}=\ell^{1+v_\ell(j)}k_j+k'$. Then $\sigma_{j+1}=\sigma_j+1+v_\ell(j)$ and, by (1),

$$\ell^{\sigma_{j+1}}z-(\gamma_u-1)(k_{j+1})=\ell^{1+v_\ell(j)}\pi_\Delta^jy_j-(\gamma_u-1)(\varepsilon_j^{-1}\pi_\Delta^jy_j)\in\pi_\Delta^{j+1}\mathbf{N}^\natural.$$

At $j=M$: σ_M is as in the statement, and $k=k_M$ proves (Gf0).

Step 4: (Gf1) and (Gf2). Let y be n_k (in $\mathbf{N}_a^\Delta$) or $n_k^\vee$ (in $\mathbf{N}_a^{\Delta\vee}$). By (H), $z=(\gamma_u-1)(y)\in\pi_\Delta\mathbf{N}^\natural$; let k_M be as in Step 3 and put $h=\ell^{\sigma_M}y-k_M$. Then $h-\ell^{\sigma_M}y=-k_M\in\pi_\Delta\mathbf{N}^\natural$ and $(\gamma_u-1)(h)=\ell^{\sigma_M}z-(\gamma_u-1)(k_M)\in\pi_\Delta^M\mathbf{N}^\natural$. This gives h_k, respectively λ_k.

Step 5: (Gf3). Suppose $g-g_0\notin\pi_\Delta^MR^\Delta$; then there is a least j with $1\le j<M$ and $g_j\ne0$, and $g-g_0=\pi_\Delta^j(g_j+\pi_\Delta w)$ with $w\in R^\Delta$. As $\gamma_u(g_0)=g_0$,

$$\gamma_u(g)-g=\pi_\Delta^jm_u^j\gamma_u(g_j+\pi_\Delta w)-\pi_\Delta^j(g_j+\pi_\Delta w)\equiv(u^{js'}-1)\,g_j\,\pi_\Delta^j\pmod{\pi_\Delta^{j+1}R^\Delta}$$

by Step 1 ($\gamma_u(g_j)=g_j$, $\gamma_u(\pi_\Delta w)\in\pi_\Delta R^\Delta$). Here $(u^{js'}-1)g_j\ne0$ in the domain $\mathcal{O}_L$, so by the uniqueness of the expansion in (Rr2) $\gamma_u(g)-g\notin\pi_\Delta^{j+1}R^\Delta\supseteq\pi_\Delta^MR^\Delta$, a contradiction.

Step 6: (Gf4). Put $g=\lambda_j(h_k)\in R^\Delta$ and write $\gamma_u(\lambda_j)=\lambda_j+\pi_\Delta^M\mu$, $\gamma_u(h_k)=h_k+\pi_\Delta^My$ with $\mu\in\mathbf{N}_a^{\Delta\vee}$, $y\in\mathbf{N}_a^\Delta$. By the definition of the action on forms, $\gamma_u(g)=\gamma_u(\lambda_j)(\gamma_u(h_k))=g+\pi_\Delta^M(\lambda_j(y)+\mu(h_k)+\pi_\Delta^M\mu(y))$, so $\gamma_u(g)-g\in\pi_\Delta^MR^\Delta$. Writing $\lambda_j=\ell^{\sigma_M}n_j^\vee+\pi_\Delta\mu'$ and $h_k=\ell^{\sigma_M}n_k+\pi_\Delta y''$ gives $g\equiv\ell^{2\sigma_M}n_j^\vee(n_k)=\ell^{2\sigma_M}\delta_{jk}\pmod{\pi_\Delta R^\Delta}$, so $g_0=\ell^{2\sigma_M}\delta_{jk}$, and (Gf3) gives $g-\ell^{2\sigma_M}\delta_{jk}\in\pi_\Delta^MR^\Delta$.

Step 7: (Gf5). Write $h_k=\sum_iH_{ki}n_i$ and $\lambda_j=\sum_i\Lambda_{ji}n_i^\vee$ with matrices H,Λ over R^Δ, and put $c=\ell^{2\sigma_M}$. Then $\lambda_j(h_k)=(\Lambda H^{\mathsf T})_{jk}$, so by (Gf4) $\Lambda H^{\mathsf T}\equiv c\,I$ modulo π_Δ^M. Let $\bar A=R^\Delta/\pi_\Delta^MR^\Delta$; by (Rr2) it is isomorphic to $\mathcal{O}_L[[T]]/(T^M)$, a free $\mathcal{O}_L$-module, so ℓ is a non-zero-divisor on $\bar A$ and $\bar A\to\bar A[1/\ell]$ is injective. In the commutative ring $\bar A[1/\ell]$, c is a unit and $\bar\Lambda\,(c^{-1}\bar H^{\mathsf T})=I$, so $c^{-1}\bar H^{\mathsf T}$ is a two-sided inverse of $\bar\Lambda$ (square matrices over a commutative ring) and $\bar H^{\mathsf T}\bar\Lambda=c\,I$ in $\bar A[1/\ell]$, hence in $\bar A$. For $x=n_j$: $\sum_k\lambda_k(n_j)h_k=\sum_i(\sum_k\Lambda_{kj}H_{ki})n_i=\sum_i(H^{\mathsf T}\Lambda)_{ij}\,n_i\equiv c\,n_j$ modulo $\pi_\Delta^M\mathbf{N}_a^\Delta$. Both sides of (Gf5) are R^Δ-linear in x and (n_j) is a basis, so (Gf5) holds for every x.

Step 8: (Gf6). Suppose $d \notin \pi_\Delta^M \mathbf{N}_a^\Delta$. Since $\mathbf{N}_a^\Delta \cong (R^\Delta)^r$ and $\bigcap_j \pi_\Delta^j R^\Delta = 0$ (Rr2), there is a largest j with $d \in \pi_\Delta^j \mathbf{N}_a^\Delta$, and $1 \le j < M$; write $d = \pi_\Delta^j y$ with $y \notin \pi_\Delta \mathbf{N}_a^\Delta$. By Step 2 (for $\mathbf{N}^\natural = \mathbf{N}_a^\Delta$), $(\gamma_u - 1)(d) \equiv (u^{js'} - 1)\pi_\Delta^j y$ modulo $\pi_\Delta^{j+1}\mathbf{N}_a^\Delta$, and $(\gamma_u - 1)(d) \in \pi_\Delta^M \mathbf{N}_a^\Delta \subseteq \pi_\Delta^{j+1}\mathbf{N}_a^\Delta$; cancelling π_Δ^j (a non-zero-divisor on the free module), $(u^{js'}-1)y \in \pi_\Delta \mathbf{N}_a^\Delta$. But $\mathbf{N}_a^\Delta/\pi_\Delta \mathbf{N}_a^\Delta \cong \mathcal{O}_L^r$ (as $R^\Delta/\pi_\Delta R^\Delta = \mathcal{O}_L$ by (Rr2)) is torsion-free and $u^{js'} - 1 \neq 0$ in $\mathcal{O}_L$ (Step 1), so $y \in \pi_\Delta \mathbf{N}_a^\Delta$, a contradiction. For the second sentence: if h, h' are two such vectors for $x \equiv x'$ modulo $\pi_\Delta \mathbf{N}_a^\Delta$, then $d = h - h' \in \pi_\Delta \mathbf{N}_a^\Delta$ and $(\gamma_u - 1)(d) \in \pi_\Delta^M \mathbf{N}_a^\Delta$, so $d \in \pi_\Delta^M \mathbf{N}_a^\Delta$; and $\sum_k o_k h_k$ has both properties by (Gf1), since γ_u fixes $o_k \in \mathcal{O}_L$. □

Lemma 24.18. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a field K with $F \subseteq K \subseteq F(\mu_\ell)$ be as in Proposition 23.7, $\Delta = \Delta_K$, and $\Phi\colon T_1 \to T_2$ a G_K-equivariant $\mathcal{O}_L$-linear map of the lattices of pairs (V_1, T_1), (V_2, T_2) over K. Let $c \in \mathbb{Z}/n$, let $J \subset \mathbb{Z}$ be a finite set containing every Hodge jump of V_1 at every embedding of K over τ_c, $G \ge 0$ an integer, and*

$$J' = \{j + k :\ j \in J,\ 1 \le k \le G,\ j + k \neq 0\}, \qquad \kappa = \sum_{x \in J'} \big(1 + v_\ell(x)\big).$$

For $\nu = 1, 2$ let I_c^ν be the set of $w \in e_c \mathbf{N}(T_\nu)[1/\lambda_\infty]$ with $\gamma_\delta(w) = w$ for every $\delta \in \Delta$ and $(\gamma_s - 1)(w) \in X\mathbf{N}(T_\nu)$ for every $s \in \Gamma_K$. Let $r \ge 0$ be such that for every $y \in \mathbf{N}(T_2)_c$ there is $x \in \mathbf{N}(T_1)_c$ with $\mathbf{N}(\Phi)(x) = \ell^r y$. Then for every $w \in I_c^2$ with $Q^G w \in \mathbf{N}(T_2)$ there is $u \in I_c^1$ with $Q^G u \in \mathbf{N}(T_1)$ and

$$\mathbf{N}(\Phi)(u) = \ell^{\,r+\kappa}\, w.$$

Level s. *Let ℓ be any prime, $s \ge 1$ an integer with $s \ge 2$ if $\ell = 2$, L as in (Ex4) of Proposition 23.7, $K = F(\mu_{\ell^s})$, X_{s-1} the level variable of Definition 23.2, $\Phi\colon T_1 \to T_2$ a G_K-equivariant $\mathcal{O}_L$-linear map of the lattices of pairs (V_1, T_1), (V_2, T_2) over K, and c, J, G, J' as above, with*

$$\kappa^{(s)} = \sum_{x \in J'} \big(s + v_\ell(x)\big).$$

For $\nu = 1, 2$ let $I_c^{\nu,s}$ be the set of $w \in e_c \mathbf{N}_s(T_\nu)[1/\lambda_\infty]$ with $(\gamma_v - 1)(w) \in X_{s-1}\mathbf{N}_s(T_\nu)$ for every $v \in \Gamma_K$. Let $r \ge 0$ be such that every $y \in \mathbf{N}_s(T_2)_c$ has $x \in \mathbf{N}_s(T_1)_c$ with $\mathbf{N}_s(\Phi)(x) = \ell^r y$. Then for every $w \in I_c^{2,s}$ with $Q_{s-1}^G w \in \mathbf{N}_s(T_2)$ there is $u \in I_c^{1,s}$ with $Q_{s-1}^G u \in \mathbf{N}_s(T_1)$ and $\mathbf{N}_s(\Phi)(u) = \ell^{\,r+\kappa^{(s)}} w$.

Proof. By (Ex1) (for any window of V_ν) each $\mathbf{N}_{\nu,c}$ is free of finite rank over the domain A_c (Wm1), and $(\gamma_s - 1)(\mathbf{N}_\nu) \subseteq X\mathbf{N}_\nu$, $\gamma_s(e_c) = e_c$ (Wm3), so γ_s preserves $\mathbf{N}_{\nu,c}$ for $s \in \Gamma_K$.

Step 0: the element q. By (Rr7), $\Gamma_K = \Gamma_\Delta = \Delta \cdot (1 + \ell\mathbb{Z}_\ell)$. By (Rr4), $q = Q\, w_\Delta$ with $w_\Delta \in \mathbb{Z}_\ell[[X]]^\times$, so $Q^G z \in \mathbf{N}_\nu$ if and only if $q^G z \in \mathbf{N}_\nu$. By (Rr2), (Rr3), $q = E_\Delta(\pi_\Delta)$ is Δ-fixed and $q(0) = E_\Delta(0) \neq 0$, so the prime X of A_c does not divide q. By (Rr6), $\gamma_s(q) = h_s q$ with $h_s \in 1 + X\mathbb{Z}_\ell[[X]]$, a unit, for $s \in \Gamma_K$.

Step 1: vectors with poles along q. For $H \ge 0$ let $D_\nu(H) = \{z \in (e_c \mathbf{N}_\nu[1/\lambda_\infty])^\Delta : q^H z \in \mathbf{N}_\nu\}$; so $D_\nu(0) = \mathbf{N}_{\nu,c}^\Delta$. For $z \in D_\nu(H)$, $n = q^H z \in \mathbf{N}_{\nu,c}$ and $s \in \Gamma_K$,

$$q^H(\gamma_s - 1)(z) = h_s^{-H}\gamma_s(n) - n = (h_s^{-H} - 1)\gamma_s(n) + (\gamma_s - 1)(n) \in X\mathbf{N}_{\nu,c},$$

so $(\gamma_s - 1)(z) = Xz'$ with $q^H z' \in \mathbf{N}_{\nu,c}$; and $\gamma_s(z) \in D_\nu(H)$ (the γ's commute, $q^H\gamma_s(z) = h_s^{-H}\gamma_s(n)$). So every polynomial in γ_s with coefficients in $\mathbb{Z}_\ell$ preserves $D_\nu(H)$ and $\mathbf{N}_{\nu,c}^\Delta$. $\mathbf{N}(\Phi)$ commutes with every γ_s (Ex3) and maps $D_1(H)$ into $D_2(H)$.

Step 2: invariant lifts. $|\Delta|$ divides $\ell - 1$, so it is a unit of $\mathbb{Z}_\ell$. If $y \in \mathbf{N}_{2,c}^\Delta$ and $x \in \mathbf{N}_{1,c}$ has $\mathbf{N}(\Phi)(x) = \ell^r y$, then $\bar{x} = |\Delta|^{-1} \sum_{\delta\in\Delta} \gamma_\delta(x) \in \mathbf{N}_{1,c}^\Delta$ and $\mathbf{N}(\Phi)(\bar{x}) = \ell^r y$ (Ex3). $(*)$

Step 3: the construction. Let $g = 1+\ell \in \Gamma_K$, $Z = \{j+k:\ j\in J,\ 1\le k\le G\}$, $\epsilon = 1$ if $0\in Z$ and $\epsilon = 0$ otherwise, and

$$P(Y) = \prod_{x\in Z}(Y - g^{-x}), \qquad R(Y) = \prod_{x\in J'}(Y-g^{-x}) \in \mathbb{Z}_\ell[Y].$$

As $J' = Z\setminus\{0\}$ and $g^0 = 1$, $(Y-1)R(Y) = (Y-1)^{1-\epsilon}P(Y)$. Write $R(Y) = R(1)+(Y-1)S(Y)$ with $S\in\mathbb{Z}_\ell[Y]$. Let $w\in I_c^2$ with $Q^Gw\in\mathbf{N}_2$; then $w\in D_2(G)$ by Step 0. (a) $q^Gw\in\mathbf{N}_{2,c}^\Delta$; by $(*)$ there is $x\in\mathbf{N}_{1,c}^\Delta$ with $\mathbf{N}(\Phi)(x) = \ell^r q^G w$. Put $u_0 = q^{-G}x\in D_1(G)$; then $\mathbf{N}(\Phi)(u_0) = \ell^r w$ and $Q^Gu_0\in\mathbf{N}_1$. (b) By Lemma 24.19 for the pair (V_1,T_1), c, J, $u = g\in 1+\ell\mathbb{Z}_\ell$, $Y = G$, an enumeration $x_1,\dots,x_m$ of Z and $y_0 = u_0$, the vector $y_m = P(\gamma_g)(u_0)$ lies in $\mathbf{N}_{1,c}$ (the factors $\gamma_g - g^{-x_i}$ commute). Hence

$$(\gamma_g-1)R(\gamma_g)(u_0) = (\gamma_g-1)^{1-\epsilon}P(\gamma_g)(u_0)\in\mathbf{N}_{1,c}.$$

(c) $(\gamma_g-1)(w)\in X\mathbf{N}_2\cap e_c\mathbf{N}_2[1/\lambda_\infty] = X\mathbf{N}_{2,c}$ is Δ-fixed, so $n := R(\gamma_g)(w) - R(1)w = S(\gamma_g)((\gamma_g-1)(w))\in\mathbf{N}_{2,c}^\Delta$ (Step 1). By $(*)$ there is $x'\in\mathbf{N}_{1,c}^\Delta$ with $\mathbf{N}(\Phi)(x') = \ell^r n$. (d) Put $u_2 = R(\gamma_g)(u_0) - x'\in D_1(G)$ (Step 1). Then $\mathbf{N}(\Phi)(u_2) = R(\gamma_g)(\ell^r w) - \ell^r n = \ell^r R(1)\,w$, and $(\gamma_g-1)(u_2)\in\mathbf{N}_{1,c}$ by (b) and since γ_g preserves $\mathbf{N}_{1,c}$.

Step 4: the exponent. Let $x\in J'$. Then $x\neq 0$, and by item 1 of Lemma 23.22 ($p=\ell$ odd, $m=1$, $u=g$), $v_\ell(1-g^{-x}) = 1+v_\ell(x)$. So $v_\ell(R(1)) = \kappa$ and $t = \ell^\kappa/R(1)\in\mathbb{Z}_\ell^\times$. Put $u = t\,u_2\in D_1(G)$: u is Δ-fixed, $Q^Gu\in\mathbf{N}_1$ (Step 0), $\mathbf{N}(\Phi)(u) = \ell^{r+\kappa}w$, and $(\gamma_g-1)(u)\in\mathbf{N}_{1,c}$.

Step 5: $u\in I_c^1$. Let $\bar M = D_1(G)/\mathbf{N}_{1,c}^\Delta$. Multiplication by q^G identifies $q^{-G}\mathbf{N}_{1,c}/\mathbf{N}_{1,c}$ with $\mathbf{N}_{1,c}/q^G\mathbf{N}_{1,c}\cong (A_c/Q^GA_c)^d$ (d the rank; $q^G = Q^Gw_\Delta^G$), a free $\mathcal{O}_L$-module of finite rank by Weierstrass division by the distinguished polynomial Q^G; the γ_δ act on it, taking Δ-invariants is exact (averaging by $|\Delta|^{-1}\sum_\delta\gamma_\delta$), and $\bar M = (q^{-G}\mathbf{N}_{1,c}/\mathbf{N}_{1,c})^\Delta$ is the image of this projector, a direct summand, hence free of finite rank N over $\mathcal{O}_L$. Each γ_s ($s\in\Gamma_K$) preserves $D_1(G)$ and $\mathbf{N}_{1,c}^\Delta$ (Step 1) and acts $\mathcal{O}_L$-linearly on $\bar M$; $\rho(a) = \gamma_{g^a}$ defines a homomorphism $\rho\colon\mathbb{Z}_\ell\to\mathrm{GL}(\bar M)\cong\mathrm{GL}_N(\mathcal{O}_L)$ (Wm3). It is continuous: let ℓ^e be the exact power of ℓ dividing $g_0 = |\mathrm{GL}_N(\mathcal{O}_L/\ell\mathcal{O}_L)|$ and $K_i = 1+\ell^iM_N(\mathcal{O}_L)$; for $a\in\ell^{i+e}\mathbb{Z}_\ell$ write $a = g_0\ell^ib$ with $b\in\mathbb{Z}_\ell$; then $\rho(b)^{g_0}\in K_1$ and $\rho(a) = (\rho(b)^{g_0})^{\ell^i}\in K_{1+i}$, since $(1+\ell^kA)^\ell\in K_{k+1}$ for $k\ge 1$. The class $[u]\in\bar M$ is fixed by $\rho(1)$ (Step 4), so its stabiliser, a closed subgroup of $\mathbb{Z}_\ell$ containing $\mathbb{Z}$, is $\mathbb{Z}_\ell$. As $1+\ell\mathbb{Z}_\ell = g^{\mathbb{Z}_\ell}$ (ℓ odd) and $\Gamma_K = \Delta\cdot g^{\mathbb{Z}_\ell}$ (Step 0), every $s\in\Gamma_K$ is δg^a, and $(\gamma_s-1)(u) = \gamma_\delta((\gamma_{g^a}-1)(u)) + (\gamma_\delta-1)(u) = \gamma_\delta((\gamma_{g^a}-1)(u))\in\mathbf{N}_{1,c}$. By Step 1, also $(\gamma_s-1)(u) = Xz'$ with $q^Gz'\in\mathbf{N}_{1,c}$. In coordinates for an A_c-basis of $\mathbf{N}_{1,c}$, each coordinate f of z' has $Xf\in A_c$ and $q^Gf\in A_c$; writing $f = a/X$ with $a\in A_c$, X divides q^Ga in the unique factorisation domain $A_c\cong\mathcal{O}_L[[X]]$ and not q (Step 0), so X divides a and $f\in A_c$. Hence $(\gamma_s-1)(u)\in X\mathbf{N}_1$ for every $s\in\Gamma_K$, and $u\in I_c^1$.

Remark. Compared with Lemma 24.7 (the case $K=F$), the operator $\pi_0^{-1}(\gamma_g-1)$ is replaced by γ_g and the pole killer by Lemma 24.19; the price is 1 per index of J' on top of $v_\ell(x)$. With $J\subseteq[-A,B]$ and $G=A$, $J'\subseteq[-(A-1),A+B]\setminus\{0\}$, so $|J'|\le 2A+B$ and $\kappa\le(2A+B)+v_\ell((A-1)!)+v_\ell((A+B)!)$.

Level s. Let the data be as in the paragraph Level s of the statement; $\mathbf{N}_\nu = \mathbf{N}_s(T_\nu)$, $A_c = e_c\mathbf{A}_{F,L}\cong\mathcal{O}_L[[X]]$, $q = Q_{s-1}$, $Y = X_{s-1}$, and (Ex1), (Ex3), (Wm1), (Wm3') are taken at level s. Here Δ_K is trivial and $\Gamma_K = 1+\ell^s\mathbb{Z}_\ell$ (Definition 23.2, paragraph Level s), so (Wm3') gives $(\gamma_v-1)(\mathbf{N}_\nu)\subseteq Y\mathbf{N}_\nu$ for every $v\in\Gamma_K$. *Step 0.* Y and q are coprime in the unique factorisation domains $\mathbb{Z}_\ell[[X]]$ and A_c: neither is divisible by a uniformiser ($Y\equiv X^{\ell^{s-1}}$, $q\equiv X^{\ell^{s-1}(\ell-1)}$ modulo ℓ), and a common prime factor would be a distinguished polynomial with a common root, while the roots of Y are the $\zeta-1$ with $\zeta^{\ell^{s-1}}=1$ and those of q the $\zeta-1$ with ζ of exact order ℓ^s (Lemma 23.8(3)). For $v\in\Gamma_K$, $\gamma_v(q) = h_vq$ with $h_v\in\mathbb{Z}_\ell[[X]]^\times$ (Lemma 23.8(2)); since $q = \Phi_\ell(1+Y)$ and $\gamma_v(1+Y) = (1+Y)^v\in(1+Y)+Y\mathbb{Z}_\ell[[X]]$,

$\gamma_v(q) - q \in Y\mathbb{Z}_\ell[[X]] \cap q\mathbb{Z}_\ell[[X]] = Yq\,\mathbb{Z}_\ell[[X]]$, so $h_v \in 1 + Y\mathbb{Z}_\ell[[X]]$. *Step 1.* With $D_\nu(H) = \{z \in e_c\mathbf{N}_\nu[1/\lambda_\infty] :\ q^H z \in \mathbf{N}_\nu\}$, the computation of Step 1 gives, for $z \in D_\nu(H)$ and $n = q^H z$, $q^H(\gamma_v - 1)(z) = (h_v^{-H} - 1)\gamma_v(n) + (\gamma_v - 1)(n) \in Y\mathbf{N}_{\nu,c}$; so $(\gamma_v - 1)(z) = Yz'$ with $q^H z' \in \mathbf{N}_{\nu,c}$, γ_v preserves $D_\nu(H)$ and $\mathbf{N}_{\nu,c}$, and $\mathbf{N}_s(\Phi)$ maps $D_1(H)$ into $D_2(H)$. *Step 2* is void (no averaging is needed). *Step 3* carries over with $g = u_s = 1 + \ell^s$, Q replaced by q, invariance under Δ dropped, and Lemma 24.19 (paragraph Level s; $u = g \in 1 + \ell^s\mathbb{Z}_\ell$, $Y = G$) in (b); in (c), $(\gamma_g - 1)(w) \in Y\mathbf{N}_2 \cap e_c\mathbf{N}_2[1/\lambda_\infty] \subseteq \mathbf{N}_{2,c}$. It gives $u_2 \in D_1(G)$ with $\mathbf{N}_s(\Phi)(u_2) = \ell^r R(1)w$ and $(\gamma_g - 1)(u_2) \in \mathbf{N}_{1,c}$. *Step 4.* By item 1 of Lemma 23.22 ($p = \ell$, $m = s$, allowed as $s \geq 2$ if $\ell = 2$), $v_\ell(1 - g^{-x}) = s + v_\ell(x)$ for $x \in J'$, so $v_\ell(R(1)) = \kappa^{(s)}$, and $u = (\ell^{\kappa^{(s)}}/R(1))\,u_2 \in D_1(G)$ has $Q_{s-1}^G u \in \mathbf{N}_1$, $\mathbf{N}_s(\Phi)(u) = \ell^{r+\kappa^{(s)}}w$ and $(\gamma_g - 1)(u) \in \mathbf{N}_{1,c}$. *Step 5.* $\bar{M} = D_1(G)/\mathbf{N}_{1,c} \cong \mathbf{N}_{1,c}/q^G\mathbf{N}_{1,c}$ is free of finite rank over $\mathcal{O}_L$ (Weierstrass division by the distinguished polynomial q^G); the continuity argument for $\rho(a) = \gamma_{g^a}$ carries over unchanged, so $[u]$ is fixed by $\rho(\mathbb{Z}_\ell)$, and $\Gamma_K = g^{\mathbb{Z}_\ell}$ ($1+\ell^s\mathbb{Z}_\ell$ is procyclic with generator $1+\ell^s$, $\ell^s \neq 2$). Hence $(\gamma_v - 1)(u) \in \mathbf{N}_{1,c}$ for every $v \in \Gamma_K$; by Step 1 also $(\gamma_v - 1)(u) = Yz'$ with $q^G z' \in \mathbf{N}_{1,c}$. In coordinates, each coordinate f of z' has $Yf \in A_c$ and $q^G f \in A_c$; as Y and q are coprime in the unique factorisation domain A_c (Step 0), $f \in A_c$. So $(\gamma_v - 1)(u) \in Y\mathbf{N}_1$ for every $v \in \Gamma_K$, and $u \in I_c^{1,s}$. □

Lemma 24.19. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a field K with $F \subseteq K \subseteq F(\mu_\ell)$ be as in Proposition 23.7, (V,T) a pair over K, $\mathbf{N} = \mathbf{N}(T)$, $c \in \mathbb{Z}/n$, and $J \subset \mathbb{Z}$ a finite set containing every Hodge jump of V at every embedding of K over τ_c. Let $u \in 1 + \ell\mathbb{Z}_\ell$, $Y \geq 0$ an integer, and $x_1, \ldots, x_m$ an enumeration, without repetition, of the finite set $Z = \{j + k :\ j \in J,\ 1 \leq k \leq Y\}$. Let $y_0 \in e_c\mathbf{N}[1/\lambda_\infty]$ with $Q^Y y_0 \in \mathbf{N}$, and put*

$$y_i = \gamma_u(y_{i-1}) - u^{-x_i}\,y_{i-1} \qquad (1 \leq i \leq m).$$

Then $y_m \in \mathbf{N}_c$.

Level s. *Let ℓ be any prime, $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, L, K ($F \subseteq K \subseteq F(\mu_{\ell^s})$) and a pair (V,T) over K as in (Ex4) of Proposition 23.7, $\mathbf{N} = \mathbf{N}_s(T)$, and c, J, Y, Z, $x_1, \ldots, x_m$ as above. Let $u \in 1 + \ell^s\mathbb{Z}_\ell$ and $y_0 \in e_c\mathbf{N}[1/\lambda_\infty]$ with $Q_{s-1}^Y y_0 \in \mathbf{N}$, and define y_i as above. Then $y_m \in \mathbf{N}_c$. For $s = 1$ and ℓ odd this is the statement above with $u \in 1 + \ell\mathbb{Z}_\ell$.*

Proof. **Step 1: the level-0 recursion.** $y_0 \in W$, and W is an $\mathcal{O}_L$-submodule of $e_c\mathbf{N}[1/\lambda_\infty]$ (if $Q^Y w, Q^{Y'}w' \in \mathbf{N}$ then $Q^{\max(Y,Y')}(w + w') \in \mathbf{N}$) stable under γ_u (Lz2); as $u^{-x_i} \in \mathbb{Z}_\ell$, all y_i lie in W. Put $f_i = \mathrm{lv}(y_i) \in \mathcal{F}((t))^r$. By (Lz2) and the $\mathcal{O}_L$-linearity of lv,

$$f_i = (\gamma_u^{(0)} - u^{-x_i})(f_{i-1}) \qquad (1 \leq i \leq m),$$

so $f_m = \Omega(f_0)$ with $\Omega = \prod_{x\in Z}(\gamma_u^{(0)} - u^{-x})$, the factors being commuting $\mathbb{Z}_\ell$-linear operators. Since $u \in 1 + \ell\mathbb{Z}_\ell$, σ_u is the identity, so $\gamma_u^{(0)}(g\,v) = g(ut)\,v$ for $g \in \mathcal{F}((t))$ and $v \in \mathcal{F}^r$, where $g(ut) = \sum_i u^i a_i t^i$ for $g = \sum_i a_i t^i$.

Step 2: the integral vectors. By (Lz4), for each embedding τ of K over τ_c the filtration $\epsilon_\tau\mathrm{Fil}$ of $\epsilon_\tau\mathcal{F}_K^r = L^r$ is decreasing, equal to L^r for $j \ll 0$ and to 0 for $j \gg 0$; choose an L-basis $(v_{\tau,b})_{b\leq r}$ of L^r adapted to it: integers $j_{\tau,b}$ with $\epsilon_\tau\mathrm{Fil}(j) = \bigoplus_{b:\ j_{\tau,b}\geq j} Lv_{\tau,b}$ for every j (for each j choose a complement of $\epsilon_\tau\mathrm{Fil}(j+1)$ in $\epsilon_\tau\mathrm{Fil}(j)$, and put $j_{\tau,b} = j$ on a basis of it). Each $j_{\tau,b}$ satisfies $\epsilon_\tau\mathrm{Fil}(j_{\tau,b}+1) \neq \epsilon_\tau\mathrm{Fil}(j_{\tau,b})$, so it is a Hodge jump of V at τ (Lz4), and $j_{\tau,b} \in J$. As $\mathrm{Fil}(j)$ is an $\mathcal{F}_K$-submodule and $\mathcal{F}_K = \prod_\tau L\epsilon_\tau$, $\mathrm{Fil}(j) = \bigoplus_\tau \epsilon_\tau\mathrm{Fil}(j)$; and $\mathcal{F}$ is flat over the product of fields $\mathcal{F}_K$, so $\mathcal{F} \otimes_{\mathcal{F}_K} \mathrm{Fil}(j)$ is the $\mathcal{F}$-span of $\mathrm{Fil}(j)$ in $\mathcal{F}^r$, namely $\bigoplus_\tau \bigoplus_{b:\ j_{\tau,b}\geq j} \epsilon_\tau\mathcal{F}\,v_{\tau,b}$ (here $v_{\tau,b}$ is viewed in $\epsilon_\tau\mathcal{F}_K^r$). Every $f \in \mathcal{F}((t))^r$ is uniquely $f = \sum_{\tau,b} g_{\tau,b}v_{\tau,b}$ with $g_{\tau,b} \in \epsilon_\tau\mathcal{F}((t))$, and the t^i-coefficient of f lies in $\mathcal{F} \otimes_{\mathcal{F}_K} \mathrm{Fil}(-i)$ for every i if and only if $g_{\tau,b} \in t^{-j_{\tau,b}}\epsilon_\tau\mathcal{F}[[t]]$ for all τ, b.

So the set of f satisfying the condition of (Lz3) is

$$\Lambda = \bigoplus_{\tau,b} t^{-j_{\tau,b}}\,\epsilon_\tau \mathcal{F}[[t]]\,v_{\tau,b},$$

an $\mathcal{F}[[t]]$-submodule of $\mathcal{F}((t))^r$.

Step 3: the start. $Q^Y y_0 \in \mathbf{N} \cap e_c\mathbf{N}[1/\lambda_\infty] = \mathbf{N}_c$ and $Q^Y y_0 \in W$, so $\mathrm{lv}(Q^Y y_0) \in \Lambda$ by (Lz3). By (Lz1), $\mathrm{lv}(Q^Y y_0) = \lambda(e_cQ)^Y f_0 = t^Y \eta^Y f_0$ with $\eta \in \mathcal{F}[[t]]^\times$; hence $f_0 \in t^{-Y}\Lambda$, that is $f_0 = \sum_{\tau,b} g_{\tau,b} v_{\tau,b}$ with $g_{\tau,b} \in t^{-j_{\tau,b}-Y}\epsilon_\tau\mathcal{F}[[t]]$.

Step 4: one factor. Let $x \in \mathbb{Z}$ and $q \in \mathbb{Z}$. The operator $g \mapsto g(ut) - u^{-x}g$ maps $t^q\epsilon_\tau\mathcal{F}[[t]]$ into itself, and maps $t^{-x}\epsilon_\tau\mathcal{F}[[t]]$ into $t^{-x+1}\epsilon_\tau\mathcal{F}[[t]]$.

Step 5: conclusion. Fix (τ, b) and $j = j_{\tau,b} \in J$. The Y integers $j+1, \dots, j+Y$ lie in Z, so Ω is the product of $\prod_{k=1}^{Y}(\gamma_u^{(0)} - u^{-(j+k)})$ and of the factors with $x \in Z\setminus\{j+1, \dots, j+Y\}$. By Step 1, $\Omega(g_{\tau,b}v_{\tau,b}) = \Omega_1(g_{\tau,b})\,v_{\tau,b}$, where Ω_1 is the same product of the operators $g \mapsto g(ut) - u^{-x}g$. Applying Step 4 for $x = j+Y, j+Y-1, \dots, j+1$ in turn,

$$\prod_{k=1}^{Y} \left(g \mapsto g(ut) - u^{-(j+k)}g\right)(g_{\tau,b}) \in t^{-j}\epsilon_\tau\mathcal{F}[[t]],$$

and the remaining factors preserve $t^{-j}\epsilon_\tau\mathcal{F}[[t]]$ (Step 4). Hence $\Omega_1(g_{\tau,b}) \in t^{-j_{\tau,b}}\epsilon_\tau\mathcal{F}[[t]]$ for all (τ, b), and $f_m = \Omega(f_0) \in \Lambda$. As $y_m \in W$ (Step 1), (Lz3) gives $y_m \in \mathbf{N}_c$. (For $Y = 0$, $Z = \emptyset$, $m = 0$ and $y_0 \in \mathbf{N} \cap e_c\mathbf{N}[1/\lambda_\infty] = \mathbf{N}_c$ directly.)

Remark. Each factor costs $v_\ell(1-u^{-x}) = v_\ell(u-1) + v_\ell(x)$ when it is turned into a lift: with $v_\ell(u-1) = 1$ that is $1 + v_\ell(x)$ per index. The eigenvalue u^{-x} is the same at every embedding τ over τ_c, which is why one product serves all of them; this is where the plain operator $\gamma_u - u^{-x}$ is used instead of $\pi_0^{-1}(\gamma_u - 1)$ of Lemma 24.8, whose eigenvalues at the points over different τ differ when $K \neq F$.

Level s. Use Lemma 24.20, paragraph Level s (with $\zeta = \zeta_{\ell^s}$, W defined by Q_{s-1}-poles), in place of Lemma 24.20. Step 1 applies without change: W is an $\mathcal{O}_L$-module stable under γ_u ((Lz2); $u \in 1 + \ell^s\mathbb{Z}_\ell \subseteq \Gamma_K$), and σ_u is the identity because $\zeta_{\ell^s}^u = \zeta_{\ell^s}$, so $\gamma_u^{(0)}(g\,v) = g(ut)\,v$. Step 2 applies without change ((Lz3), (Lz4)). Step 3: $\mathrm{lv}(Q_{s-1}^Y y_0) = \lambda(e_cQ_{s-1})^Y f_0 = t^Y\eta^Y f_0$ with $\eta \in \mathcal{F}[[t]]^\times$ by (Lz1) of level s, so $f_0 \in t^{-Y}\Lambda$. Steps 4 and 5 apply without change. □

Lemma 24.20. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a field K with $F \subseteq K \subseteq F(\mu_\ell)$ be as in Proposition 23.7, (V,T) a pair over K, $\mathbf{N} = \mathbf{N}(T)$, $r = \dim_L V$, and $c \in \mathbb{Z}/n$. Let $\zeta = \zeta_\ell \in \overline{\mathbb{Q}}_\ell$ be the primitive ℓ-th root of unity of the compatible system $\underline{\zeta}$ fixed in Section 13 (the one defining t), so that $F(\zeta) = F(\mu_\ell) \supseteq K$. Let $\epsilon \in F \otimes_{\mathbb{Q}_\ell} L$ be the idempotent of τ_c (the one with $(x \otimes b)\epsilon = \tau_c(x)b\,\epsilon$), and put*

$$\mathcal{F} = \epsilon\big(F(\zeta) \otimes_{\mathbb{Q}_\ell} L\big) \;\supseteq\; \mathcal{F}_K = \epsilon\big(K \otimes_{\mathbb{Q}_\ell} L\big) = \prod_\tau L\,\epsilon_\tau,$$

the product over the embeddings τ of K over τ_c, ϵ_τ the idempotent of τ. For $u \in \Gamma_K$ let σ_u be the automorphism of $\mathcal{F}$ that is the identity on F and L and sends ζ to ζ^u (it fixes $\mathcal{F}_K$, and is the identity if $u \in 1 + \ell\mathbb{Z}_\ell$), and on $\mathcal{F}((t))^r$ let $\gamma_u^{(0)}(\sum_i a_i t^i) = \sum_i u^i\sigma_u(a_i)\,t^i$, coordinatewise. Let

$$W = \{w \in e_c\mathbf{N}[1/\lambda_\infty] :\; Q^Y w \in \mathbf{N} \text{ for some integer } Y \geq 0\}.$$

Then there are a ring homomorphism $\lambda\colon e_c\mathbf{A}_{F,L} \to \mathcal{F}[[t]]$, an $\mathcal{O}_L$-linear map $\mathrm{lv}\colon W \to \mathcal{F}((t))^r$ and a decreasing filtration $(\mathrm{Fil}(j))_{j\in\mathbb{Z}}$ of $\mathcal{F}_K^r$ by $\mathcal{F}_K$-submodules such that:

(Lz1) Ring $\lambda(e_c(x\otimes b))=\epsilon(x\otimes b)$ *for* $x\in\mathcal{O}_F$, $b\in\mathcal{O}_L$, $\lambda(e_cX)=\epsilon\,(\zeta e^{t/\ell}-1)$, $\lambda(e_cQ)\in t\,\mathcal{F}[[t]]^\times$, *and* $\mathrm{lv}(gw)=\lambda(g)\,\mathrm{lv}(w)$ *for* $g\in e_c\mathbf{A}_{F,L}$ *and* $w\in W$; *moreover* λ *is the substitution* $\lambda(e_c\sum_i b_iX^i)=\sum_i\epsilon\,b_i\,(\zeta e^{t/\ell}-1)^i$ *(*$b_i\in\mathcal{O}_L$, *under* $e_c\mathbf{A}_{F,L}\cong\mathcal{O}_L[[X]]$*), the sum converging coefficientwise in* $\mathcal{F}[[t]]$; *in particular* λ *does not depend on* (V,T);

(Lz2) Gamma *for* $u\in\Gamma_K$: $\gamma_u(W)\subseteq W$ *and* $\mathrm{lv}(\gamma_u w)=\gamma_u^{(0)}(\mathrm{lv}(w))$ *for* $w\in W$;

(Lz3) Membership *for* $w\in W$: $w\in\mathbf{N}_c$ *if and only if, for every* $i\in\mathbb{Z}$, *the* t^i*-coefficient of* $\mathrm{lv}(w)$ *lies in* $\mathcal{F}\otimes_{\mathcal{F}_K}\mathrm{Fil}(-i)$;

(Lz4) Hodge filtrations *for every embedding* τ *of* K *over* τ_c *there is an* L*-linear isomorphism* $\epsilon_\tau\mathcal{F}_K^r=L^r\to D_{\mathrm{dR},\tau}(V)$ *carrying* $\epsilon_\tau\mathrm{Fil}(j)$ *onto* $\mathrm{Fil}^jD_{\mathrm{dR},\tau}(V)$ *for every* j; *in particular* $\epsilon_\tau\mathrm{Fil}(j)=L^r$ *for* $j\ll0$, $=0$ *for* $j\gg0$, *and the integers* j *with* $\epsilon_\tau\mathrm{Fil}(j+1)\neq\epsilon_\tau\mathrm{Fil}(j)$ *are the Hodge jumps of* V *at* τ;

(Lz5) Frobenius images *for every* $x\in\mathbf{N}_{c-1}$: $\varphi(x)\in W$, $\mathrm{lv}(\varphi(x))\in\mathcal{F}[[t]]^r$, *and there is an* $\mathcal{O}_L$*-linear surjection* $\mathrm{val}\colon\mathbf{N}_{c-1}\to\mathcal{O}_L^r$ *with kernel* $X\mathbf{N}_{c-1}$ *such that* $\mathrm{lv}(\varphi(x))-\epsilon\,\mathrm{val}(x)\in t\,\mathcal{F}[[t]]^r$ *for every* $x\in\mathbf{N}_{c-1}$ *(here* $\epsilon\,\mathrm{val}(x)$ *is the vector with entries* $\epsilon\,(1\otimes\mathrm{val}(x)_k)\in\mathcal{F}_K$*).*

Level s. *Let* ℓ *be any prime,* $s\geq1$ *an integer with* $s\geq2$ *if* $\ell=2$, L, K *(*$F\subseteq K\subseteq F(\mu_{\ell^s})$*) and a pair* (V,T) *over* K *as in (Ex4) of Proposition 23.7,* $\mathbf{N}=\mathbf{N}_s(T)$, $r=\dim_L V$, $c\in\mathbb{Z}/n$. *Let* $\zeta=\zeta_{\ell^s}$ *be the primitive* ℓ^s*-th root of unity of the compatible system* $\underline{\zeta}$, *and define* ϵ, $\mathcal{F}=\epsilon(F(\zeta)\otimes_{\mathbb{Q}_\ell}L)\supseteq\mathcal{F}_K=\prod_\tau L\epsilon_\tau$, σ_u *(*$u\in\Gamma_K$; $\zeta\mapsto\zeta^u$*),* $\gamma_u^{(0)}$ *as above with this* ζ, *and* $W=\{w\in e_c\mathbf{N}[1/\lambda_\infty]:\ Q_{s-1}^Yw\in\mathbf{N}$ *for some* $Y\geq0\}$. *Then there are* λ, lv *and* Fil *with (Lz1)–(Lz4), where in (Lz1)* λ *is the substitution* $X\mapsto\zeta e^{t/\ell^s}-1$, *i.e.* $\lambda(e_cX)=\epsilon(\zeta e^{t/\ell^s}-1)$, *and* $\lambda(e_cQ_{s-1})\in t\,\mathcal{F}[[t]]^\times$ *in place of* $\lambda(e_cQ)\in t\,\mathcal{F}[[t]]^\times$. *(*$\sigma_u$ *is the identity for* $u\in1+\ell^s\mathbb{Z}_\ell$, *in particular for every* $u\in\Gamma_K$ *when* $K=F(\mu_{\ell^s})$*.) For* $s=1$ *and* ℓ *odd these are (Lz1)–(Lz4) above.*

Proof. Write $B=B_{\mathrm{dR}}^+$, $\mu=\mu_\varepsilon=q^\ell-1$, $\mu'=q-1$, and φ also for $\varphi\otimes1$ on $W(\mathbb{C}_\ell^\flat)\otimes_{\mathbb{Z}_\ell}T$. Put $R^+=\epsilon(B\otimes_{\mathbb{Q}_\ell}L)$ and $R=R^+[1/t]=\epsilon(B_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell}L)$; here $F\subset\overline{\mathbb{Q}}_\ell\subset B$ by (Ai3), so $\epsilon\in F\otimes L\subset B\otimes L$.

Step 0: identities in B**.** By (Ai3), $tB=\xi_0B=\ker(\theta)$, B is a complete discrete valuation ring and $\overline{\mathbb{Q}}_\ell\subset B$ compatibly with θ; $t=\log(q^\ell)$, so $q^\ell=\exp(t)$. The elements $x=\zeta^{-1}q$ and $y=\exp(t/\ell)$ lie in $1+tB$ ($\theta(q)=\zeta_\ell=\theta(\zeta)$) and $x^\ell=q^\ell=y^\ell$; so $x/y\in1+tB$ is an ℓ-th root of unity of the domain B. The ℓ elements of $\mu_\ell\subset\overline{\mathbb{Q}}_\ell\subset B$ are all its ℓ-th roots of unity, and only 1 among them has θ-value 1; so $x=y$:

$$q=\zeta e^{t/\ell},\qquad\mu'=\zeta e^{t/\ell}-1,\qquad\mu=e^t-1\in tB^\times,\qquad\xi_0=\frac{\mu}{\mu'}=\frac{e^t-1}{\zeta e^{t/\ell}-1}\in tB^\times,$$

μ' being a unit ($\theta(\mu')=\zeta-1\neq0$). Also $\varphi^{-1}(\tilde{\xi})=\xi_0$ ($\tilde{\xi}=\varphi(\mu)/\mu$, Theorem B.41).

Step 1: the algebraic model. θ is injective on $\mathcal{F}$: $F(\zeta)\to\mathbb{C}_\ell$ is injective and L is flat over $\mathbb{Q}_\ell$. R^+ is a direct summand of the free B-module $B\otimes_{\mathbb{Q}_\ell}L$, hence t-torsion free and t-adically complete. So $\sum_{i\geq i_0}a_it^i\mapsto$ its sum is a ring homomorphism $\mathcal{F}((t))\to R$, injective (if a sum with $a_{i_0}\neq0$ vanished, θ of t^{-i_0} times it would give $a_{i_0}=0$); we identify $\mathcal{F}((t))$ with its image, and the same argument gives $\mathcal{F}((t))\cap t^mR^+=t^m\mathcal{F}[[t]]$ for $m\in\mathbb{Z}$.

Step 2: the ring map λ**.** Let $\lambda=\varphi^{-1}\circ(\iota\otimes\mathrm{id})$ on $e_c\mathbf{A}_{F,L}$, with values in $A_{\mathrm{inf}}\otimes_{\mathbb{Z}_\ell}\mathcal{O}_L\subset B\otimes\mathcal{O}_L$. By Lemma 23.8(4), $\iota|_{\mathcal{O}_F}=\varphi\circ w$ and $\iota(X)=\mu$, so $\lambda(x\otimes b)=w(x)\otimes b$ and $\lambda(X)=\mu'$. The element $w(x)\in B$ is a root of the minimal polynomial of x over $\mathbb{Q}_\ell$ with $\theta(w(x))=x$ (Lemma 23.8(4): $\tilde{\theta}\circ\iota|_{\mathcal{O}_F}=\theta\circ w$ is the inclusion), so it is $x\in\overline{\mathbb{Q}}_\ell\subset B$ (the roots of that separable

polynomial in the domain B are its roots in $\overline{\mathbb{Q}}_\ell$, with distinct θ-values). Hence $\lambda(e_c)=\epsilon$ and $\lambda(e_c(x\otimes b))=\epsilon(x\otimes b)$. Now let $g=\sum_i b_iX^i\in e_c\mathbf{A}_{F,L}\cong\mathcal{O}_L[[X]]$ and put $g^{\mathrm{al}}=\sum_i \epsilon b_i(\zeta e^{t/\ell}-1)^i$. This converges coefficientwise in $\mathcal{F}[[t]]$: modulo t^m, $(\zeta e^{t/\ell}-1)^i=\sum_{j<m}\binom{i}{j}(\zeta-1)^{i-j}\zeta^j(e^{t/\ell}-1)^j$ lies in $(\zeta-1)^{i-m}\ell^{-C_m}\mathcal{O}[t]$ with $\mathcal{O}=\epsilon(\mathcal{O}_{F(\zeta)}\otimes\mathcal{O}_L)$ and C_m independent of i, and $v_\ell(\zeta-1)>0$. We claim $\lambda(g)=g^{\mathrm{al}}$. Fix $m\geq 1$. By (Ai3) and (Ai1), $B/t^mB=A_{\mathrm{inf}}[1/\ell]/\xi_0^m$ contains $\bar{A}=A_{\mathrm{inf}}/\xi_0^mA_{\mathrm{inf}}$ ($\bar{A}$ is ℓ-torsion free, an iterated extension of copies of $A_{\mathrm{inf}}/\xi_0\cong\mathcal{O}_{\mathbb{C}_\ell}$), and $\bar{A}\otimes\mathcal{O}_L$ is ℓ-adically separated (an extension of ℓ-torsion free ℓ-adically separated modules is such). Since $\theta(\mu')^{\ell-1}=(\zeta-1)^{\ell-1}\in\ell\mathcal{O}_{\mathbb{C}_\ell}$, $\mu'^{\ell-1}\in\ell A_{\mathrm{inf}}+\xi_0A_{\mathrm{inf}}$, hence $\mu'^N\in\ell^kA_{\mathrm{inf}}+\xi_0^mA_{\mathrm{inf}}$ for $N\geq(\ell-1)(m+k)$. With $S_N=\sum_{i<N}\epsilon b_i\mu'^i$: $\lambda(g)-S_N=\mu'^N\lambda(g_N)$ with $g_N\in e_c\mathbf{A}_{F,L}$ (μ' is topologically nilpotent in A_{inf}, its reduction lying in the maximal ideal of $\mathcal{O}_{\mathbb{C}_\ell^\flat}$), so its image modulo t^m lies in $\ell^k(\bar{A}\otimes\mathcal{O}_L)$ for N large. On the other side S_N is also the partial sum of g^{al} ($\mu'=\zeta e^{t/\ell}-1$ by Step 0), and $g^{\mathrm{al}}-S_N$ modulo t^m lies in $(\zeta-1)^{N-m}\ell^{-C_m}\mathcal{O}[t]$; the images in B/t^m of $\mathcal{O}_{F(\zeta)}=\mathcal{O}_F[\zeta]$ and of t lie in $\ell^{-C'_m}\bar{A}$ ($\zeta=q\,e^{-t/\ell}$, and $t=\log(1+\mu)\equiv\sum_{n<m}(-1)^{n+1}\mu^n/n$ modulo ξ_0^m, as $\mu=\xi_0\mu'$). So $\lambda(g)-g^{\mathrm{al}}$ modulo t^m lies in $\bigcap_k\ell^{k-C}(\bar{A}\otimes\mathcal{O}_L)=0$ for a constant C; as m is arbitrary and R^+ is t-adically separated, $\lambda(g)=g^{\mathrm{al}}\in\mathcal{F}[[t]]$. Finally $\lambda(e_cQ)=\epsilon\varphi^{-1}(\tilde{\xi})=\epsilon\xi_0=\epsilon(e^t-1)/(\zeta e^{t/\ell}-1)\in t\mathcal{F}[[t]]^\times$ ($e^t-1\in t\mathbb{Q}[[t]]^\times$, and $\zeta e^{t/\ell}-1$ has constant term $\zeta-1\neq 0$). So λ is a ring homomorphism $e_c\mathbf{A}_{F,L}\to\mathcal{F}[[t]]$ with the properties of (Lz1) on λ, and extends to $e_c\mathbf{A}_{F,L}[1/Q]\to\mathcal{F}((t))$.

Step 3: the horizontal level frame. By (Ex1) and (Wm1), $\mathbf{N}_{c-1}$ has an $e_{c-1}\mathbf{A}_{F,L}$-basis $(n_j)_{j\leq r}$; let $H_k=\sum_j h_{kj}n_j$ be the horizontal sections of (Hf2) over K through n_k, $\mathbf{h}=(h_{kj})\in M_r(e_{c-1}E[[X]])$, $\mathbf{h}\equiv 1 \bmod X$, and $\gamma_u(H_k)=H_k$ for $u\in\Gamma_K$. For $f=\sum_i a_iX^i\in e_{c-1}E[[X]]$ put $\mathrm{ev}(f)=\sum_i\iota_\Lambda(a_i)\mu^i$, $\iota_\Lambda=(\iota|_{\mathcal{O}_F}\otimes\mathrm{id})[1/\ell]$; $\iota_\Lambda(e_{c-1})=(w\otimes\mathrm{id})(\varphi(e_{c-1}))=\epsilon$ and $\mu=e^t-1\in t\mathbb{Q}_\ell[[t]]$, so the series converges t-adically in $\epsilon(F\otimes L)[[t]]=L[[t]]\epsilon\subseteq\mathcal{F}[[t]]$ and ev is a ring homomorphism. For $f\in e_{c-1}\mathbf{A}_{F,L}$, $\mathrm{ev}(f)=(\iota\otimes\mathrm{id})(f)$ (the tails of the second series lie in $\mu^bA_{\mathrm{inf}}\otimes\mathcal{O}_L\subseteq t^bB\otimes\mathcal{O}_L$). For $\sigma\in G_K$ and $u=\chi_{\mathrm{cyc}}(\sigma)\in\Gamma_K$: σ fixes $F\supseteq$ the coefficients, $\sigma(\mu)=(1+\mu)^u-1=h_u(\mu)$, and σ is t-adically continuous ($\sigma(t)=ut$), so $\sigma(\mathrm{ev}(f))=\mathrm{ev}(f\circ h_u)$. Put

$$D_k=\sum_j\mathrm{ev}(h_{kj})\,\iota_T(n_j)\in B_{\mathrm{dR}}\otimes_{\mathbb{Z}_\ell}T\qquad(k\leq r),$$

($\iota_T(n_j)\in A_{\mathrm{inf}}[1/\mu]\otimes T$, and $\mu\in tB^\times$ is a unit of B_{dR}). (a) *Basis.* By (Ex2), $A_{\mathrm{inf}}[1/\mu]\cdot\iota_T(\mathbf{N})=A_{\mathrm{inf}}[1/\mu]\otimes T$, and $\iota_T(e_bx)=(\iota\otimes\mathrm{id})(e_b)\iota_T(x)$ with $(\iota\otimes\mathrm{id})(e_{c-1})=\epsilon$; so $\epsilon(A_{\mathrm{inf}}[1/\mu]\otimes T)$ is generated over $A_{\mathrm{inf}}[1/\mu]\otimes\mathcal{O}_L$ by $\iota_T(\mathbf{N}_{c-1})$, hence by the r elements $\iota_T(n_j)$; it is free of rank r over $\epsilon(A_{\mathrm{inf}}[1/\mu]\otimes\mathcal{O}_L)$ (T is free of rank r over $\mathcal{O}_L$), so they form a basis, and after base change $(\iota_T(n_j))_j$ is an R-basis of $\epsilon(B_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell}V)$. As $\mathrm{ev}(\mathbf{h})\equiv 1 \bmod t$ is invertible over $L[[t]]$, $(D_k)_k$ is an R-basis of $\epsilon(B_{\mathrm{dR}}\otimes V)$, and $\iota_T(n_j)=\sum_k\mathrm{ev}((\mathbf{h}^{-1})_{jk})D_k$. (b) *Invariance.* Let $\sigma\in G_K$, $u=\chi_{\mathrm{cyc}}(\sigma)$, and $\gamma_u(n_j)=\sum_l(G_u)_{lj}n_l$ with $G_u\in M_r(e_{c-1}\mathbf{A}_{F,L})$. By (Ex2), $\sigma(\iota_T(n_j))=\iota_T(\gamma_un_j)=\sum_l\mathrm{ev}((G_u)_{lj})\iota_T(n_l)$, so $\sigma(D_k)=\sum_l\mathrm{ev}(\sum_j(h_{kj}\circ h_u)(G_u)_{lj})\iota_T(n_l)$; the inner sum is the n_l-coordinate of $\gamma_u(H_k)=H_k$, i.e. h_{kl}. So $\sigma(D_k)=D_k$, and $D_k\in\epsilon D_{\mathrm{dR}}(V)$, $D_{\mathrm{dR}}(V)=(B_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell}V)^{G_K}$. (c) *De Rham basis.* If $v\in\epsilon D_{\mathrm{dR}}(V)$, $v=\sum_k c_kD_k$ with unique $c_k\in R$ by (a), and σ-invariance and uniqueness give $c_k\in R^{G_K}=\epsilon(B_{\mathrm{dR}}^{G_K}\otimes_{\mathbb{Q}_\ell}L)=\epsilon(K\otimes_{\mathbb{Q}_\ell}L)=\mathcal{F}_K$, by $B_{\mathrm{dR}}^{G_K}=K$ [94]. So (D_k) is an $\mathcal{F}_K$-basis of $\epsilon D_{\mathrm{dR}}(V)$, and $(\epsilon_\tau D_k)_k$ an L-basis of $\epsilon_\tau D_{\mathrm{dR}}(V)=D_{\mathrm{dR},\tau}(V)$ for each τ over τ_c (Definition 13.7: $D_{\mathrm{dR},\tau}(V)=L\otimes_{K\otimes L,\tau\otimes\mathrm{id}}D_{\mathrm{dR}}(V)$ is the ϵ_τ-component).

Step 4: the level map. Extend ι_T and the γ_u to $\mathbf{N}[1/\lambda_\infty]$ as in (Gc0) over K, and put $\Lambda=\varphi^{-1}\circ\iota_T$ on $e_c\mathbf{N}[1/Q]$. Let $w\in W$, $Q^Yw\in\mathbf{N}$; then $Q^Yw\in\mathbf{N}_c$. Fix integers $M_a,h_a\geq 0$ bounding the Hodge jumps as in (Ex1); by (Wm2), $Q^{h_c}\mathbf{N}_c\subseteq\mathbf{A}_{F,L}\varphi(\mathbf{N}_{c-1})$, so

$Q^{Y+h_c}w = \sum_j g_j\varphi(n_j)$ with $g_j \in e_c\mathbf{A}_{F,L}$. By (Ex2) and Steps 2–3,

$$\Lambda(w) = \lambda(e_cQ)^{-(Y+h_c)}\sum_j \lambda(g_j)\,\iota_T(n_j) = \sum_k \Big(\lambda(e_cQ)^{-(Y+h_c)}\sum_j \lambda(g_j)\,\mathrm{ev}((\mathbf{h}^{-1})_{jk})\Big)D_k,$$

with coefficients in $\mathcal{F}((t))$. Define $\mathrm{lv}(w) \in \mathcal{F}((t))^r$ as the coordinate vector of $\Lambda(w)$ in the R-basis (D_k); it depends only on w, and lv is $\mathcal{O}_L$-linear. For $g \in e_c\mathbf{A}_{F,L}$, $gw \in W$ and $\Lambda(gw) = \lambda(g)\Lambda(w)$ ((Ex2)), so $\mathrm{lv}(gw) = \lambda(g)\mathrm{lv}(w)$. This completes (Lz1).

Step 5: (Lz2). Let $u \in \Gamma_K$, $w \in W$. By Lemma 23.8(2), $\gamma_u(Q) \in Q\,\mathbb{Z}_\ell[[X]]^\times$, so $Q^Y\gamma_u(w) = (Q/\gamma_u(Q))^Y\gamma_u(Q^Yw) \in \mathbf{N}$, and $\gamma_u(e_c) = e_c$: $\gamma_u(w) \in W$. Choose $\sigma \in G_K$ with $\chi_{\mathrm{cyc}}(\sigma) = u$; by (Gc0), $\iota_T(\gamma_u w) = \sigma(\iota_T(w))$, and σ commutes with φ, so $\Lambda(\gamma_u w) = \sigma(\Lambda(w)) = \sum_k \sigma(\mathrm{lv}(w)_k)D_k$ by Step 3(b). On $\mathcal{F}((t))$, σ is the identity on F and L, sends ζ to ζ^u and t to ut (Ai3), and is t-adically continuous; so $\sigma(\sum_i a_it^i) = \sum_i u^i\sigma_u(a_i)t^i$, and $\mathrm{lv}(\gamma_u w) = \gamma_u^{(0)}(\mathrm{lv}(w))$ by uniqueness of coordinates. (σ_u fixes $\mathcal{F}_K$, as σ fixes K; for $u \in 1+\ell\mathbb{Z}_\ell$, $\zeta^u = \zeta$.)

Step 6: the filtration and (Lz4). V is crystalline, hence de Rham [94]. Put $\mathrm{Fil}(j) = \{v \in \mathcal{F}_K^r : \sum_k v_kD_k \in \mathrm{Fil}^j D_{\mathrm{dR}}(V)\}$, an $\mathcal{F}_K$-submodule, decreasing in j, since $\mathrm{Fil}^j D_{\mathrm{dR}}(V)$ is a decreasing family of $K\otimes L$-submodules (Definition 13.7). For τ over τ_c, $v \mapsto \sum_k v_k\epsilon_\tau D_k$ is an L-isomorphism $\epsilon_\tau\mathcal{F}_K^r = L^r \to D_{\mathrm{dR},\tau}(V)$ (Step 3(c)) carrying $\epsilon_\tau\mathrm{Fil}(j)$ onto $\epsilon_\tau\mathrm{Fil}^j D_{\mathrm{dR}}(V) = \mathrm{Fil}^j D_{\mathrm{dR},\tau}(V)$. The filtration of the finite-dimensional space $D_{\mathrm{dR},\tau}(V)$ is exhaustive and separated ($\bigcup_j t^jB = B_{\mathrm{dR}}$, $\bigcap_j t^jB = 0$), which gives the remaining assertions of (Lz4).

Step 7: de Rham integrality in coordinates. Claim: for $f \in \mathcal{F}((t))^r$, $\sum_k f_kD_k \in \epsilon(B\otimes_{\mathbb{Q}_\ell} V)$ if and only if the t^i-coefficient $f^{(i)}$ lies in $\mathcal{F}\otimes_{\mathcal{F}_K}\mathrm{Fil}(-i)$ for every i. Both sides decompose along the idempotents ϵ_τ ($\epsilon = \sum_\tau\epsilon_\tau$ in $K\otimes L$; $\mathcal{F} = \prod_\tau \epsilon_\tau\mathcal{F}$, $\epsilon_\tau\mathcal{F} = F(\zeta)\otimes_{K,\tau} L$). Fix τ, put $R_\tau^+ = \epsilon_\tau(B\otimes_{\mathbb{Q}_\ell} L)$ and choose an L-basis (D_i^{ad}) of $D_{\mathrm{dR},\tau}(V)$ with $D_i^{\mathrm{ad}} \in \mathrm{Fil}^{u_i}$ such that each $\mathrm{Fil}^j D_{\mathrm{dR},\tau}(V)$ is spanned by the D_i^{ad} with $u_i \ge j$. Applying ϵ_τ to (Ai4) for K and V with $k=0$ (the comparison map is $K\otimes L$-linear),

$$\epsilon_\tau(B\otimes_{\mathbb{Q}_\ell} V) = \sum_j t^{-j}R_\tau^+ \otimes_L \mathrm{Fil}^j D_{\mathrm{dR},\tau}(V) = \bigoplus_i t^{-u_i}R_\tau^+ D_i^{\mathrm{ad}},$$

a direct sum because (D_i^{ad}) is a basis of $\epsilon_\tau(B_{\mathrm{dR}}\otimes V)$ over $R_\tau^+[1/t]$ ((Ai4) again). Write $\epsilon_\tau D_k = \sum_i W_{ki}D_i^{\mathrm{ad}}$ ($W \in GL_r(L)$) and $\phi_i = \sum_k \epsilon_\tau f_k W_{ki} \in \epsilon_\tau\mathcal{F}((t))$. The ϵ_τ-component of $\sum_k f_kD_k$ lies in $\epsilon_\tau(B\otimes V)$ iff $\phi_i \in t^{-u_i}R_\tau^+$ for all i, iff (Step 1, with $\epsilon_\tau\mathcal{F}$) $\phi_i \in t^{-u_i}\epsilon_\tau\mathcal{F}[[t]]$, iff for every m the t^m-coefficients vanish when $u_i < -m$, iff $\sum_i \phi_i^{(m)}D_i^{\mathrm{ad}} \in \epsilon_\tau\mathcal{F}\otimes_L \mathrm{Fil}^{-m}D_{\mathrm{dR},\tau}(V)$ for every m; as $\sum_i \phi_i^{(m)}D_i^{\mathrm{ad}} = \sum_k \epsilon_\tau f_k^{(m)}\epsilon_\tau D_k$, Step 6 turns this into $\epsilon_\tau f^{(m)} \in \epsilon_\tau\mathcal{F}\otimes_L \epsilon_\tau\mathrm{Fil}(-m)$. Taking all τ gives the claim.

Step 8: (Lz3). Let $(n'_j)_{j\le r}$ be an $e_c\mathbf{A}_{F,L}$-basis of $\mathbf{N}_c$. (a) $(\Lambda(n'_j))_j$ is an R^+-basis of $\epsilon(B\otimes_{\mathbb{Z}_\ell} T) = \epsilon(B\otimes_{\mathbb{Q}_\ell} V)$: φ^{-1} maps $A_{\mathrm{inf}}[1/\mu]$ into $A_{\mathrm{inf}}[1/\mu'] \subseteq B$, so $\Lambda(\mathbf{N}) \subseteq B\otimes T$ and, by (Ex2), $B\cdot\Lambda(\mathbf{N}) \supseteq \varphi^{-1}(A_{\mathrm{inf}}[1/\mu]\otimes T) \supseteq 1\otimes T$, i.e. $B\cdot\Lambda(\mathbf{N}) = B\otimes T$; multiplying by $\epsilon = \lambda(e_c)$ and using $\Lambda(e_bx) = \lambda(e_b)\Lambda(x)$, $\epsilon(B\otimes T) = \sum_j R^+\Lambda(n'_j)$, a free R^+-module of rank r generated by r elements. (b) For $g \in Q^{-Y}e_c\mathbf{A}_{F,L}$: $g \in e_c\mathbf{A}_{F,L}$ iff $\lambda(g) \in R^+$. Indeed, if $Y \ge 1$, $g = Q^{-Y}g'$ and $\lambda(g) \in R^+$, then $\lambda(g') \in \xi_0^Y R^+ \subseteq tR^+$, so $\theta(\lambda(g')) = 0$; under $e_c\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$, $\theta\circ\lambda$ is $g' \mapsto \epsilon\,g'(\zeta-1)$ (Step 2), and $\epsilon(F(\zeta)\otimes L) \cong L[X]/(Q)$ via $X \mapsto \zeta - 1$ ($Q = \Phi_\ell(1+X)$ is irreducible over the unramified F); so g' vanishes at every root of Q and $Q \mid g'$ in $\mathcal{O}_L[[X]]$ (Lemma 23.8(3)); induction on Y. The converse is clear. (c) Let $w \in W$ and write $w = \sum_j g_jn'_j$, $g_j \in Q^{-Y}e_c\mathbf{A}_{F,L}$. Then $\Lambda(w) = \sum_j \lambda(g_j)\Lambda(n'_j)$, and by (a) (a basis of $\epsilon(B\otimes V)$ is a basis of $\epsilon(B_{\mathrm{dR}}\otimes V)$) and (b): $w \in \mathbf{N}_c$ iff all $g_j \in e_c\mathbf{A}_{F,L}$ iff $\Lambda(w) \in \epsilon(B\otimes V)$. Since $\Lambda(w) = \sum_k \mathrm{lv}(w)_kD_k$, Step 7 turns this into the condition of (Lz3).

Step 9: (Lz5). Let (n_j), $\mathbf{h}$, ev and (D_k) be as in Step 3, and $x = \sum_j a_jn_j \in \mathbf{N}_{c-1}$ with $a_j \in e_{c-1}\mathbf{A}_{F,L}$. By (Wm2) for the window of Step 4, $Q^{M_c}\varphi(x) \in \mathbf{N}$, and $\varphi(x) = e_c\varphi(x)$ as

$\varphi(e_{c-1}) = e_c$; so $\varphi(x) \in W$. By (Ex2), $\iota_T(\varphi(x)) = \varphi(\iota_T(x))$, hence $\Lambda(\varphi(x)) = \iota_T(x)$. Since $e_{c-1}\mathbf{A}_{F,L} = \mathcal{O}_F[[X]] \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$ ($\mathcal{O}_L$ is finite free over $\mathbb{Z}_\ell$), each a_j is a finite sum $\sum f \otimes b$ with $f \in \mathbf{A}_F$, $b \in \mathcal{O}_L$, and (Ex2) gives $\iota_T(a_j n_j) = (\iota \otimes \mathrm{id})(a_j)\, \iota_T(n_j) = \mathrm{ev}(a_j)\, \iota_T(n_j)$ (Step 3). With Step 3(a),

$$\Lambda(\varphi(x)) = \sum_k \Big(\sum_j \mathrm{ev}(a_j)\, \mathrm{ev}((\mathbf{h}^{-1})_{jk}) \Big) D_k,$$

so $\mathrm{lv}(\varphi(x))_k = \sum_j \mathrm{ev}(a_j)\mathrm{ev}((\mathbf{h}^{-1})_{jk}) \in L[[t]]\epsilon \subseteq \mathcal{F}[[t]]$. Under $e_{c-1}\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ write $a_j(0) \in \mathcal{O}_L$ for the constant term; as $\mu = e^t - 1 \in t\mathbb{Q}_\ell[[t]]$ and $\iota_\Lambda(e_{c-1}) = \epsilon$, $\mathrm{ev}(a_j) \equiv \epsilon\, a_j(0)$ modulo t, and $\mathrm{ev}(\mathbf{h}^{-1}) \equiv \epsilon\, I$ modulo t ($\mathbf{h} \equiv 1$ modulo X). Hence $\mathrm{lv}(\varphi(x))_k \equiv \epsilon\, a_k(0)$ modulo t. Put $\mathrm{val}(x) = (a_k(0))_k$: it is $\mathcal{O}_L$-linear, surjective (take $x = bn_k$), and its kernel is $\{x :\ a_k(0) = 0 \text{ for all } k\} = X\mathbf{N}_{c-1}$.

Level s. Let the data be as in the paragraph Level s of the statement; every cited clause is taken in its level-s form ((Ex1), (Ex2) of (Ex4) of Proposition 23.7, (Wm1), (Wm2) of level s, (Gc0) of Lemma 23.29 and (Hf2) of Lemma 23.9, paragraphs Level s), and ι_s, $\mu_{\varepsilon,s} = [\varepsilon^{1/\ell^{s-1}}] - 1$ are as in Lemma 23.8(6). *Step 0.* Put $q_s = [\varepsilon^{1/\ell^s}]$, so $\theta(q_s) = \zeta$ and $q_s^{\ell^s} = [\varepsilon] = e^t$. The argument of Step 0 with $x = \zeta^{-1}q_s$, $y = e^{t/\ell^s}$ ($x^{\ell^s} = y^{\ell^s}$, $x/y \in 1 + tB$ an ℓ^s-th root of unity, and only $1 \in \mu_{\ell^s}$ has θ-value 1) gives $q_s = \zeta e^{t/\ell^s}$, and $\mu_s' := q_s - 1$ is a unit of B ($\theta(\mu_s') = \zeta - 1 \neq 0$); $\xi_0 = \varphi^{-1}(\tilde{\xi}) \in tB^\times$ as there. *Step 1* carries over with the new $\mathcal{F}$. *Step 2.* Let $\lambda = \varphi^{-1} \circ (\iota_s \otimes \mathrm{id})$ on $e_c\mathbf{A}_{F,L}$. As $\iota_s = \varphi \circ w$ on $\mathcal{O}_F$ and $\varphi^{-1}(\mu_{\varepsilon,s}) = q_s - 1 = \mu_s'$, $\lambda(x \otimes b) = w(x) \otimes b = x \otimes b$ and $\lambda(X) = \mu_s'$; the proof that $\lambda(g) = \sum_i \epsilon b_i (\zeta e^{t/\ell^s} - 1)^i$ carries over with μ_s' for μ' and $(\zeta - 1)^{\varphi(\ell^s)} \in \ell\mathcal{O}_{\mathbb{C}_\ell}$ for $(\zeta - 1)^{\ell - 1} \in \ell\mathcal{O}_{\mathbb{C}_\ell}$ (μ_s' is topologically nilpotent in A_{inf}, its reduction $\varepsilon^{1/\ell^s} - 1$ lying in the maximal ideal; $\zeta = q_s e^{-t/\ell^s}$). Finally $\lambda(e_c Q_{s-1}) = \epsilon\varphi^{-1}(\iota_s(Q_{s-1})) = \epsilon\varphi^{-1}(\tilde{\xi}) = \epsilon\xi_0 \in t\mathcal{F}[[t]]^\times$, and for $i \geq 1$, $\lambda(e_c Q_{s-1+i}) = \epsilon\, \Phi_{\ell^{s+i}}(\zeta e^{t/\ell^s})$ has constant term $\epsilon\Phi_{\ell^{s+i}}(\zeta) \neq 0$, so is a unit of $\mathcal{F}[[t]]$. *Step 3 (the frame, moved $s-1$ sites back).* This is the only substantial change: $\iota_s(X) = \mu_{\varepsilon,s}$ has $\theta(\mu_{\varepsilon,s}) = \zeta_{\ell^{s-1}} - 1$, which is non-zero for $s \geq 2$, so the series ev of Step 3 would not converge; we apply φ^{s-1}, which sends $\mu_{\varepsilon,s}$ to $\mu = e^t - 1 \in tB$. Let $(n_j)_{j \leq r}$ be an $e_{c-s}\mathbf{A}_{F,L}$-basis of $\mathbf{N}_{c-s}$ and $H_k = \sum_j h_{kj} n_j$ the horizontal sections of (Hf2) through n_k, $\mathbf{h} \equiv 1 \bmod X$. For $f = \sum_i a_i X^i \in e_{c-s}E[[X]]$ put $\mathrm{ev}_s(f) = \sum_i \varphi^{s-1}(\iota_\Lambda(a_i))\mu^i$, t-adically convergent; $\varphi^{s-1}(\iota_\Lambda(e_{c-s})) = (w \otimes \mathrm{id})(e_c) = \epsilon$ (as $(\varphi \otimes \mathrm{id})(e_b) = e_{b+1}$ and $w(x) = x$ in B), and $\mathrm{ev}_s(f) = \varphi^{s-1}((\iota_s \otimes \mathrm{id})(f))$ for $f \in e_{c-s}\mathbf{A}_{F,L}$ ($\varphi^{s-1}(\mu_{\varepsilon,s}) = \mu$). Put $D_k = \sum_j \mathrm{ev}_s(h_{kj})\, \varphi^{s-1}(\iota_{s,T}(n_j)) \in B_{\mathrm{dR}} \otimes T$. (a) By (Ex2), $A_{\mathrm{inf}}[1/\mu]\iota_{s,T}(\mathbf{N}) = A_{\mathrm{inf}}[1/\mu] \otimes T$; applying the automorphism φ^{s-1} of $W(\mathbb{C}_\ell^\flat) \otimes T$, $A_{\mathrm{inf}}[1/\varphi^{s-1}(\mu)] \cdot \varphi^{s-1}\iota_{s,T}(\mathbf{N}) = A_{\mathrm{inf}}[1/\varphi^{s-1}(\mu)] \otimes T$, and $\varphi^{s-1}(\mu) = \mu \prod_{i<s-1} \varphi^i(\tilde{\xi})$ is a unit of B_{dR} ($\mu \in tB^\times$, and $\theta(\varphi^i(\tilde{\xi})) = \ell \neq 0$ since $\theta([\varepsilon^{\ell^i}]) = 1$). With $\iota_{s,T}(e_b x) = (\iota_s \otimes \mathrm{id})(e_b)\iota_{s,T}(x)$ and $\varphi^{s-1}((\iota_s \otimes \mathrm{id})(e_{c-s})) = \epsilon$, the $\varphi^{s-1}\iota_{s,T}(n_j)$ form an R-basis of $\epsilon(B_{\mathrm{dR}} \otimes V)$, hence so do the D_k, and $\varphi^{s-1}\iota_{s,T}(n_j) = \sum_k \mathrm{ev}_s((\mathbf{h}^{-1})_{jk})D_k$. (b) For $\sigma \in G_K$, $u = \chi_{\mathrm{cyc}}(\sigma)$: $\sigma(\varphi^{s-1}\iota_{s,T}(n_j)) = \varphi^{s-1}\iota_{s,T}(\gamma_u n_j)$ by (Ex2), $\sigma(\mathrm{ev}_s(f)) = \mathrm{ev}_s(f \circ h_u)$ (σ fixes the coefficients, $\sigma(\mu) = (1+\mu)^u - 1$), and horizontality gives $\sigma(D_k) = D_k$ exactly as in Step 3(b). (c) carries over unchanged. *Step 4.* For $w \in W$ with $Q_{s-1}^Y w \in \mathbf{N}_c$, iterating (Wm2) of level s along $c, c-1, \ldots, c-s+1$ (and $\varphi(Q_{s-1+i}) = Q_{s+i}$) gives $P\, Q_{s-1}^Y w = \sum_j g_j \varphi^s(n_j)$ with $g_j \in e_c\mathbf{A}_{F,L}$ and $P = \prod_{i=0}^{s-1} Q_{s-1+i}^{h_{c-i}}$. With $\Lambda = \varphi^{-1} \circ \iota_{s,T}$ and $\Lambda(\varphi^s(n_j)) = \varphi^{s-1}(\iota_{s,T}(n_j))$ ((Ex2)),

$$\Lambda(w) = \sum_k \Big(\lambda(e_c P Q_{s-1}^Y)^{-1} \sum_j \lambda(g_j)\, \mathrm{ev}_s((\mathbf{h}^{-1})_{jk}) \Big) D_k,$$

with coefficients in $\mathcal{F}((t))$ by Step 2; $\mathrm{lv}(w)$ is the coordinate vector, and (Lz1) follows as before. *Step 5* carries over with Q_{s-1} ($\gamma_u(Q_{s-1}) \in Q_{s-1}\mathbb{Z}_\ell[[X]]^\times$, Lemma 23.8(2)) and $\sigma(\zeta) = \zeta^u$ for $\zeta = \zeta_{\ell^s}$. *Steps 6 and 7* carry over unchanged. *Step 8.* (a) carries over unchanged ($\varphi^{-1}(\mu) = [\varepsilon^{1/\ell}] - 1$ is a unit of B). (b) For $g \in Q_{s-1}^{-Y} e_c \mathbf{A}_{F,L}$: if $\lambda(g) \in R^+$ and $g = Q_{s-1}^{-Y} g'$, $Y \geq 1$, then $\lambda(g') \in \xi_0^Y R^+ \subseteq tR^+$, so $\epsilon\, g'(\zeta - 1) = \theta(\lambda(g')) = 0$; as $\epsilon(F(\zeta) \otimes L) \cong L[X]/(Q_{s-1})$ via $X \mapsto \zeta - 1$

($Q_{s-1} = \Phi_{\ell^s}(1+X)$ is irreducible over the unramified F), g' vanishes at every root of Q_{s-1} and $Q_{s-1} \mid g'$ (Lemma 23.8(3)); induction on Y. (c) carries over unchanged. This proves (Lz1)–(Lz4) at level s. □

Lemma 24.21. *Let ℓ be an odd prime, F, L, K with $F \subseteq K \subseteq F(\mu_\ell)$ as in Lemma 24.14, $\Delta = \Delta_K$, $\mathbf{N}$ a Wach module over K with coefficients L (any window), $a \in \mathbb{Z}/n$, $u \in \Gamma_K$ with $v_\ell(u-1) = 1$, and $M \geq 1$ an integer. If $y \in \pi_\Delta \mathbf{N}_a^\Delta$ and $(\gamma_u - 1)(y) \in \pi_\Delta^M \mathbf{N}_a^\Delta$, then $y \in \pi_\Delta^M \mathbf{N}_a^\Delta$.*

Proof. Write $\pi = \pi_\Delta$, $q = q_\Delta$ and $R = e_a \mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$. By (Dm1), $\mathbf{N}_a^\Delta$ is free over $R^\Delta = \mathcal{O}_L[[\pi]]$, with a basis $(n_k)_k$ of Δ-fixed vectors.

Step 1: the constant. By (Rr7), $\Gamma_K = \Gamma_\Delta$, so (Rr6) applies to u: $\gamma_u(\pi) = \pi(1 + t_u q)$ with $t_u \in \mathbb{Z}_\ell[[X]]^\times$ Δ-invariant (as $v_\ell(u-1) = 1$). The element $t_u q \in \mathbb{Z}_\ell[[X]]$ is Δ-invariant, so by (Rr2) (for the coefficient field $\mathbb{Q}_\ell$) $t_u q = g(\pi)$ with $g \in \mathbb{Z}_\ell[[T]]$, and $g(0)$ is the value of $t_u q$ at $X = 0$ (as π has no constant term, (Rr2) and (Rr1) of Lemma 23.24). Put $b = g(0) = t_u(0)\, E_\Delta(0)$: here $q(0) = E_\Delta(\pi(0)) = E_\Delta(0)$, $t_u(0) \in \mathbb{Z}_\ell^\times$ since t_u is a unit, and $v_\ell(E_\Delta(0)) = 1$ since E_Δ is Eisenstein (Rr3). So $v_\ell(b) = 1$, $c = 1 + b \in 1 + \ell\mathbb{Z}_\ell$ and $c \neq 1$; as $1 + \ell\mathbb{Z}_\ell$ is torsion free (ℓ odd), $c^j \neq 1$ for every $j \geq 1$. Moreover $(1 + t_u q)^j = (1 + g(\pi))^j \equiv c^j \pmod{\pi \mathbb{Z}_\ell[[\pi]]}$.

Step 2: one step of the induction. Let $1 \leq j < M$ and $y \in \pi^j \mathbf{N}_a^\Delta$ with $(\gamma_u - 1)(y) \in \pi^M \mathbf{N}_a^\Delta$; write $y = \pi^j y'$ with $y' \in \mathbf{N}_a^\Delta$. Since γ_u is γ_u-semilinear,

$$(\gamma_u - 1)(y) = \gamma_u(\pi)^j \gamma_u(y') - \pi^j y' = \pi^j \Big((1+t_u q)^j (\gamma_u(y') - y') + ((1+t_u q)^j - c^j) y' + (c^j - 1)\, y' \Big).$$

By (Dm3), $\gamma_u(y') - y' \in \pi \mathbf{N}_a^\Delta$, and $(1+t_u q)^j \in \mathbb{Z}_\ell[[\pi]] \subseteq R^\Delta$; by Step 1, $(1+t_u q)^j - c^j \in \pi \mathbb{Z}_\ell[[\pi]]$. Hence the first two terms in the bracket lie in $\pi \mathbf{N}_a^\Delta$, and as $(\gamma_u - 1)(y) \in \pi^M \mathbf{N}_a^\Delta \subseteq \pi^{j+1} \mathbf{N}_a^\Delta$,

$$\pi^j (c^j - 1)\, y' \in \pi^{j+1} \mathbf{N}_a^\Delta.$$

In the basis (n_k), with $y' = \sum_k f_k n_k$, $f_k \in \mathcal{O}_L[[\pi]]$, this says $\pi^j (c^j - 1) f_k \in \pi^{j+1} \mathcal{O}_L[[\pi]]$ for every k; as $\mathcal{O}_L[[\pi]]$ is a domain, $(c^j - 1) f_k \in \pi \mathcal{O}_L[[\pi]]$, that is $(c^j - 1) f_k(0) = 0$ in $\mathcal{O}_L$, and since $c^j - 1 \neq 0$, $f_k(0) = 0$. So $y' \in \pi \mathbf{N}_a^\Delta$ and $y \in \pi^{j+1} \mathbf{N}_a^\Delta$.

Step 3: conclusion. Starting from $y \in \pi \mathbf{N}_a^\Delta$ and applying Step 2 for $j = 1, 2, \ldots, M-1$ (the hypothesis on $(\gamma_u - 1)(y)$ does not change), $y \in \pi^M \mathbf{N}_a^\Delta$. For $M = 1$ there is nothing to prove. □

25 Losses at the places above ℓ

This section converts lifting exponents into the local losses $\lambda_k(v)$ of Galois Ribet data at the places v above ℓ, and into the global losses κ_k where the module of extensions is enlarged. Its outputs are Theorem 25.1, used at the primes of split type and of small type, Corollary 25.39, the explicit losses at unramified places that the budget uses at the split and inert primes, and Corollary 25.6, used at the inert and ramified primes (Sections 26 and 27). The local condition is the Bloch–Kato condition (Ll2), and the points of the determinant algebra are crystalline at v (Ll1).

There are two ways to bound the loss at v, and Theorem 25.1 uses both. At a *row place* the loss is counted: a lifting exponent bounds the exponent of the Bloch–Kato defect, and the dual count Lemma 25.5, an exact formula from local Tate duality and the Bloch–Kato dimension formula [34], turns an exponent into a length (Theorem 25.3, Theorem 25.7). At a *column place* the module of extensions is replaced by its column-enlargement (Definition 22.3), which

makes every Ribet class satisfy the local condition at v; the price is paid in the global losses κ_k (Theorem 25.2). The constituent block 0 is treated with the lifting exponents of Section 24. For the character blocks $k \geq 1$, where the classes are pushed into negative Tate twists, the section proves a separate exponent, built from the ℓ-adic valuations of the differences between the Hodge jumps and the twist (Proposition 25.8); Subsections 25.2 and 25.3 contain its proof.

25.1 Column places and row places above ℓ

The subsection states the main result of the section, Theorem 25.1, and proves it from four ingredients: the column-enlarged module at the column places (Theorem 25.2), local losses from lifting exponents (Theorem 25.3), the push of integral Bloch–Kato classes into a torsion quotient (Lemma 25.4), and the dual count (Lemma 25.5), which gives the exact length of a Bloch–Kato defect of bounded exponent. Corollary 25.6 combines the lifting exponent for block 0 with the weight-logarithmic exponent of Proposition 25.8 for the character blocks, and Theorem 25.7 is the general form of the dual count for Galois Ribet data.

Theorem 25.1. *Let E be a number field, ℓ an odd prime, $K = \mathcal{O}[\ell^{-1}]$, column data with column places $C \subseteq S_\ell$ for items $\mathbf{r}, \chi_1, \mathcal{E}, \delta$, and Galois Ribet data over E on $(\mathcal{R}, \pi', \mathcal{N}^{\mathrm{col}})$ with these items, $\iota = \iota^{\mathrm{col}}$ and $t+1$ blocks, having (Gd1) and (Gd2); write $n_0 = n_{\mathcal{R},0}$, $D_v = G_{E_v}$ and $n_v = [E_v : \mathbb{Q}_\ell]$. Let $M_0, h_0 \geq 0$ be integers. Suppose that at every $v \in S_\ell$: $E_v \subseteq E_{v,0}(\mu_\ell)$, $E_{v,0}$ the maximal unramified subextension of $E_v/\mathbb{Q}_\ell$; K contains $\sigma(E_{v,0}(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $E_{v,0}(\mu_\ell)$ into $\overline{\mathbb{Q}}_\ell$; (Ll1) and (Ll2) hold; and all Hodge jumps of $V_{\mathcal{N}}$, W_0 and W_0' lie in $[-M_0, h_0]$. Suppose moreover:*

(Lr1) Column places *at every $v \in C$ the column data have the column properties; $H^0(D_v, W_0) = 0$; W_0 has at most $\mu_\tau^-(v)$ Hodge jumps $a < 0$ at each τ, counted with multiplicity; and 0 is not a Hodge jump of W_0 at some embedding τ_v;*

(Lr2) Row places *at every $v \in S_\ell \setminus C$: $e_0(v) \geq 0$ is an integer with $\ell^{e_0(v)} H^1_f(D_v, \mathcal{N}_\mathcal{E}) \subseteq \mathrm{pr}_{1*} H^1_f(D_v, \widetilde{T}_\psi)$ for every $\mathcal{O}$-linear D_v-equivariant $\psi\colon \mathcal{N}_\mathcal{E} \to \mathrm{Cs}_0(u)$; $H^0(D_v, W_0') = 0$; W_0 has at most $\mu_\tau^+(v)$ Hodge jumps $a \geq 0$ at each τ, counted with multiplicity; and 0 is not a Hodge jump of W_0' at some embedding τ_v.*

Then (Gd7) holds at every $v \in S_\ell$ and (Gd6) holds with

$$\lambda_k(v) = 0 \quad (v \in C,\ 0 \leq k \leq t),$$
$$\lambda_0(v) = e_0(v) \sum_\tau \mu_\tau^+(v) + n_0\, c_{\mathrm{inv}}(W_0', \tau_v), \quad \lambda_k(v) = c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)\, n_v + w_k(v) \quad (v \notin C,\ k \geq 1),$$
$$\kappa_0 = \sum_{v \in C} \Big((e_v' + \nu(\delta)) \sum_\tau \mu_\tau^-(v) + n_0 \big(c_{\mathrm{inv}}(W_0, \tau_v) + \nu(\delta) \big) \Big),$$
$$\kappa_1 = \sum_{v \in C} \big(2\nu(\delta) + b_v \big), \qquad \kappa_k = \sum_{v \in C} \big(\nu(\delta) + w_{k-1}(v) \big)\ (k \geq 2).$$

In particular $\sum_{k=0}^t \kappa_k + \sum_{v \in S_\ell} \sum_{k=0}^t \lambda_k(v) = \Lambda_\ell$ with

$$\Lambda_\ell = \sum_{v \in C} \Big((e_v' + \nu(\delta)) \sum_\tau \mu_\tau^-(v) + n_0\, c_{\mathrm{inv}}(W_0, \tau_v) + (n_0 + t + 1)\nu(\delta) + b_v + \sum_{i=1}^{t-1} w_i(v) \Big)$$
$$+ \sum_{v \in S_\ell \setminus C} \Big(e_0(v) \sum_\tau \mu_\tau^+(v) + n_0\, c_{\mathrm{inv}}(W_0', \tau_v) + \sum_{k=1}^{t} \big(c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)\, n_v + w_k(v) \big) \Big).$$

Level s. *More generally, let ℓ be a prime and $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, and replace the two field conditions above by: at every $v \in S_\ell$, $E_v \subseteq E_{v,0}(\mu_{\ell^s})$, and K contains $\sigma(E_{v,0}(\mu_{\ell^s}))$ for every $\mathbb{Q}_\ell$-embedding σ of $E_{v,0}(\mu_{\ell^s})$ into $\overline{\mathbb{Q}}_\ell$; keep the other hypotheses. Then all assertions above hold with $c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)$ replaced by $c_{\mathrm{B},s}(V_{\mathcal{N}}, k-1)$ of Proposition 25.8 (paragraph Level s, $p = \ell$) and $c_{\mathrm{inv}}(\cdot, \tau_v)$ by $c_{\mathrm{inv},s}(\cdot, \tau_v)$ of Proposition 25.35 (formed with $p = \ell$ and M_0, h_0), in $\lambda_0(v)$, $\lambda_k(v)$, κ_0 and Λ_ℓ. For $s = 1$ and ℓ odd this is the statement above.* ***Descent.*** *Let ℓ be a prime, $N \geq 1$ an integer with $4 \mid N$ if $\ell = 2$, $s = \max(1, v_\ell(N))$ and $N_\ell = \mathrm{lcm}(\ell, N)$, and replace the two field conditions by: at every $v \in S_\ell$, E_v is $\mathbb{Q}_\ell$-isomorphic to a subfield of $\mathbb{Q}_\ell(\mu_{N_\ell})$, and K contains a primitive N_ℓ-th root of unity; keep the other hypotheses. Then all assertions of the paragraph Level s hold (with this s) with $c_{\mathrm{B},s}(V_{\mathcal{N}}, k-1)$ replaced by $c_{\mathrm{B},s}(V_{\mathcal{N}}, k-1) + s - 1$.*

Proof. We prove the first assertion and the assertion of the paragraph Level s together. In the first case put $s = 1$, $c_k^{\mathrm{B}} = c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)$ and $c^{\mathrm{inv}}(\cdot, \tau) = c_{\mathrm{inv}}(\cdot, \tau) = c_{\mathrm{inv},1}(\cdot, \tau)$; in the second put $c_k^{\mathrm{B}} = c_{\mathrm{B},s}(V_{\mathcal{N}}, k-1)$ and $c^{\mathrm{inv}}(\cdot, \tau) = c_{\mathrm{inv},s}(\cdot, \tau)$. In both cases $s \geq 2$ if $\ell = 2$, and at every $v \in S_\ell$, $E_v \subseteq E_{v,0}(\mu_{\ell^s})$ and K contains every $\sigma(E_{v,0}(\mu_{\ell^s}))$; in particular every embedding τ of E_v extends to an embedding σ of $E_{v,0}(\mu_{\ell^s})$, so $\tau(E_v) \subseteq \sigma(E_{v,0}(\mu_{\ell^s})) \subseteq K$.

Step 1: the column places. Let $v \in C$. By Theorem 25.3 ((Ll1), (Gd1), (Gd2)) W_0 is crystalline, and Λ_0 is a D_v-stable lattice with $W_0/\Lambda_0 = \mathrm{Cs}_0(u)$, $\dim_K W_0 = n_0$. By Lemma 25.34(1) (at our level s, with $W = W_0$, $\Lambda = \Lambda_0$, $\tau = \tau_v$ and the window $[-M_0, h_0]$; its field conditions are ours), $h^{\mathrm{c}}(v) = \mathrm{nl}\, H^0(D_v, \mathrm{Cs}_0(u)) \leq n_0\, c^{\mathrm{inv}}(W_0, \tau_v)$. By Theorem 25.2(4), with $K \supseteq \tau(E_v)$ for all τ and $H^0(D_v, W_0) = 0$, $j_v \leq \sum_\tau \mu_\tau^-(v)$. By Theorem 25.2(2) ((Ll2) at v), (Gd7) holds at v with $\lambda_k(v) = 0$; by Theorem 25.2(3) and the two bounds just proved (and $e'_v + \nu(\delta) \geq 0$), (Gd6) holds with the stated κ_k (with c^{inv} in κ_0).

Step 2: the row places. Let $v \in S_\ell \setminus C$. All hypotheses of Corollary 25.6 (in the second case of its paragraph Level s, with this s) hold at v: the prime ℓ (odd in the first case), the two field conditions, (Ll1), (Ll2), $H^0(D_v, W_0') = 0$, the integer $e_0 = e_0(v)$ of (Lr2), and $J(V_{\mathcal{N}}) \subseteq [-M_0, h_0]$. It gives (Gd7) at v with $\lambda_0(v) = e_0(v)\, d^0(W_0) + h_0(v)$, $h_0(v) = \mathrm{nl}\, H^0(D_v, W_0'/\Lambda_0')$, and $\lambda_k(v) = c_k^{\mathrm{B}} n_v + w_k(v)$ for $k \geq 1$. By Theorem 25.7(4), $d^0(W_0) \leq \sum_\tau \mu_\tau^+(v)$. W_0 is crystalline (Step 1), so by Lemma 25.34(2) W_0' is crystalline of K-dimension n_0 with the D_v-stable lattice Λ_0', and by Lemma 25.34(1) (at our level s, with $\tau = \tau_v$ and the window $[-M_0, h_0]$) $h_0(v) \leq n_0\, c^{\mathrm{inv}}(W_0', \tau_v)$. So (Gd7) holds at v with the stated $\lambda_0(v)$.

Step 3: the sum. Adding the stated κ_k and $\lambda_k(v)$: for $v \in C$ the contributions of $\kappa_1, \ldots, \kappa_t$ are $2\nu(\delta) + b_v + \sum_{k=2}^t (\nu(\delta) + w_{k-1}(v)) = (t+1)\nu(\delta) + b_v + \sum_{i=1}^{t-1} w_i(v)$, and that of κ_0 is $(e'_v + \nu(\delta)) \sum_\tau \mu_\tau^-(v) + n_0 c^{\mathrm{inv}}(W_0, \tau_v) + n_0 \nu(\delta)$; for $v \notin C$ the contribution is $\sum_{k=0}^t \lambda_k(v)$. This is Λ_ℓ (with c_k^{B} and c^{inv} in place of $c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)$ and c_{inv}).

Descent. Put $c_k^{\mathrm{B}} = c_{\mathrm{B},s}(V_{\mathcal{N}}, k-1) + s - 1$ and $c^{\mathrm{inv}} = c_{\mathrm{inv},s}$. As $\mathbb{Q}_\ell(\mu_{N_\ell})$ is Galois over $\mathbb{Q}_\ell$, every E_v ($v \in S_\ell$) lies in $\mathbb{Q}_\ell(\mu_{N_\ell})$, and every $\mathbb{Q}_\ell$-embedding τ of E_v has $\tau(E_v) \subseteq \mathbb{Q}_\ell(\mu_{N_\ell}) \subseteq K$. Steps 1–3 apply unchanged, with Lemma 25.34 and Corollary 25.6 in their paragraphs Descent (whose field conditions are ours) in place of their level-s forms; Theorem 25.2(4) needs only $K \supseteq \tau(E_v)$ for all τ. □

Theorem 25.2. *Let E be a number field, ℓ a prime and $K = \mathcal{O}[\ell^{-1}]$. Let column data $(C, e_1, (j_v, \alpha_{v,a}, \alpha_v^0)_{v \in C})$ be given for items $\mathbf{r}, \chi_1, \mathcal{E}, \delta$, and let Galois Ribet data over E on $(\mathcal{R}, \pi', \mathcal{N}^{\mathrm{col}})$ with these items, with $\iota = \iota^{\mathrm{col}}$ and $t+1$ blocks, have (Gd1) and (Gd2). Suppose that the column data have the column properties at every $v \in C$. For $v \in C$ put $h^{\mathrm{c}}(v) = \mathrm{nl}\, H^0(G_{E_v}, \mathrm{Cs}_0(u)) \in [0, \infty]$, and for an integer $i \geq 1$ let $w_i(v)$ be the largest integer $w \geq 0$ with $\chi_{\mathrm{cyc}}(g)^i \equiv 1 \bmod \ell^w$ for all $g \in G_{E_v}$. Then:*

1. *if $\mathcal{N}_{\mathcal{E}}$ is a finitely generated $\mathcal{R}$-module, so is $\mathcal{N}^{\mathrm{col}}$;*

2. *for every $v \in C$ at which (Ll2) holds and every $0 \le k \le t$, the image of X_k in*

$$H^1(G_{E_v}, \mathrm{Cs}_k(u))/F_k(v)$$

is zero; so (Gd7) holds at v with $\lambda_k(v) = 0$ for $0 \le k \le t$;

3. *(Gd6) holds whenever*

$$\begin{aligned}
\kappa_0 &\ge \sum_{v\in C} \Big(j_v\big(e'_v + \nu(\delta)\big) + h^{\mathrm{c}}(v) + n_{\mathcal{R},0}\,\nu(\delta) \Big), \\
\kappa_1 &\ge \sum_{v\in C} \big(2\nu(\delta) + b_v\big), \\
\kappa_k &\ge \sum_{v\in C} \big(\nu(\delta) + w_{k-1}(v)\big) \quad (2 \le k \le t);
\end{aligned}$$

4. *if $v \in C$ is a place at which (Ll1) holds, K contains $\tau(E_v)$ for every $\mathbb{Q}_\ell$-embedding τ of E_v into $\overline{\mathbb{Q}}_\ell$, and W_0 has at each such τ at most μ_τ^- Hodge jumps $a < 0$ counted with multiplicity, then*

$$j_v \le \dim_K H^0(G_{E_v}, W_0) + \sum_\tau \mu_\tau^-.$$

Proof. **Step 0: homomorphisms out of $\mathcal{N}^{\mathrm{col}}$.** Let $0 \le k \le t$. An $\mathcal{R}$-linear map $\varphi\colon \mathcal{N}^{\mathrm{col}} \to \mathrm{Cs}_k$ is the same as an $\mathcal{R}$-linear map $\tilde\varphi\colon \mathcal{N}_{\mathcal{E}} \oplus \mathcal{F} \to \mathrm{Cs}_k$ vanishing on Rel, i.e. on its generators. As $\mathcal{F}$ is free on the symbols, $\tilde\varphi$ is determined by $\psi = \tilde\varphi|_{\mathcal{N}_{\mathcal{E}}} = \varphi \circ \iota^{\mathrm{col}}$ and the values $m_v = \tilde\varphi(\epsilon_v)$, $s_{v,a} = \tilde\varphi(\epsilon_{v,a})$, $s_v^0 = \tilde\varphi(\epsilon_v^0)$, which may be any elements of Cs_k subject to: for all $v \in C$, a and $g \in G_{E_v}$,

$$\begin{aligned}
&\text{(i) } \rho_k(\mathcal{E})m_v = 0, \qquad \text{(ii) } \rho_k(e_1)s_{v,a} = s_{v,a}, \qquad \text{(iii) } (\delta - \rho_k(\mathcal{E}))s_v^0 = 0, \\
&\text{(iv) } \psi(y(g)) = -\big(\rho_k(u(g)) - 1\big)m_v + \delta \sum_a \rho_k(\alpha_{v,a}(g))\, s_{v,a} + \delta\, \rho_k(\alpha_v^0(g))\, s_v^0.
\end{aligned}$$

Since $c_\varphi(g) = \varphi(\iota^{\mathrm{col}}(y(g))) = \psi(y(g))$, (iv) says that on G_{E_v} the cocycle c_φ is the coboundary of $-m_v$ plus

$$c'_{\varphi,v}(g) = \delta \sum_a \rho_k(\alpha_{v,a}(g))\, s_{v,a} + \delta\, \rho_k(\alpha_v^0(g))\, s_v^0. \tag{25.1}$$

Three facts on the blocks are used repeatedly. (a) For $k \ge 1$, (Cm1) and (ii) give $s_{v,a} = \rho_k(e_1)s_{v,a} = 0$. (b) For $k \ne 1$, $\rho_k(\mathcal{E}) = 0$ by (Gd2), and as $\alpha_v^0(g) = r\mathcal{E}$ for some $r \in \mathcal{R}$, $\rho_k(\alpha_v^0(g)) = \rho_k(r)\rho_k(\mathcal{E}) = 0$. (c) For $k = 0$, by (ii) and (Cm1) $s_{v,a} = x_a z_1$ with $x_a \in K/\mathcal{O}$; for a lift $\tilde x_a$ and $g \in G_{E_v}$, as $\rho_0(\alpha_{v,a}(g))$ has entries in $\mathcal{O}$ and preserves Λ_0, (Cm2) gives

$$\delta\, \rho_0(\alpha_{v,a}(g))\, s_{v,a} = \delta \tilde x_a\, \rho_0(\alpha_{v,a}(g)) z_1 + \Lambda_0 = \delta \ell^{e'_v} \tilde x_a\, \beta_{v,a}(g) + \Lambda_0.$$

Item 1. $C \subseteq S_\ell$ is finite, so $\mathcal{N}^{\mathrm{col}}$ is generated by the image of a finite generating set of $\mathcal{N}_{\mathcal{E}}$ and the images of the finitely many symbols.

Item 2. Let $v \in C$ satisfy (Ll2), $0 \le k \le t$, and $x \in X_k$ the class of a continuous cocycle c_φ. By Step 0, $x|_{G_{E_v}}$ is the class of the continuous cocycle $c'_{\varphi,v}$ of (25.1) (a cocycle minus a coboundary). $k = 0$. By (b) and (c), $c'_{\varphi,v}(g) = b(g) + \Lambda_0$ with $b(g) = \sum_a \delta \ell^{e'_v} \tilde x_a \beta_{v,a}(g) \in W_0$. By (Cm2) b is a continuous W_0-valued 1-cocycle, and its class $\sum_a \delta \ell^{e'_v} \tilde x_a [\beta_{v,a}]$ lies in $H^1_f(G_{E_v}, W_0)$ by (Cm3), because $H^1_f(G_{E_v}, W_0)$ is a K-subspace of $H^1(G_{E_v}, W_0)$ (the kernel of the map to $H^1(G_{E_v}, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} W_0)$, which is K-linear for K acting through W_0). So $x|_{G_{E_v}}$ is the image of

a class of $H^1_f(G_{E_v}, W_0)$ in $H^1(G_{E_v}, W_0/\Lambda_0)$, i.e. it lies in $H^1_f(G_{E_v}, \mathrm{Cs}_0(u))$, which is $F_0(v)$ by (Ll2). $k = 1$. By (Gd1) $n_{\mathcal{R},1} = 1$ and $\rho_1(u(g)) = \chi_1(g)^{-1}\chi_1(g) = 1$, so $\mathrm{Cs}_1(u) = K/\mathcal{O}$ with trivial action, and by (a) $c'_{\varphi,v}(g) = \delta\rho_1(\alpha^0_v(g))s^0_v$. A cocycle with values in a trivial module is a homomorphism; this one vanishes on I_v by (Cm5). So $x|_{G_{E_v}}$ restricts to 0 on I_v: $x|_{G_{E_v}} \in H^1_{\mathrm{ur}}(E_v, \mathrm{Cs}_1(u)) = F_1(v)$. $k \geq 2$. By (a) and (b), $c'_{\varphi,v} = 0$, so $x|_{G_{E_v}} = 0 \in F_k(v)$. In all cases the image of X_k in $H^1(G_{E_v}, \mathrm{Cs}_k(u))/F_k(v)$ is 0, which is (Gd7) at v with $\lambda_k(v) = 0$.

Item 3: the kernel splits over the column places. By Step 0, φ lies in the kernel $\mathcal{K}_k$ of $\varphi \mapsto \varphi \circ \iota^{\mathrm{col}}$ if and only if $\psi = 0$, and then φ is determined by the tuple $(m_v, (s_{v,a})_a, s^0_v)_{v\in C}$, which satisfies (i)–(iv) with $\psi = 0$; conversely every such tuple defines an element of $\mathcal{K}_k$. The correspondence is $\mathcal{O}$-linear and the conditions involve each v separately, so $\mathcal{K}_k \cong \prod_{v\in C} \mathcal{K}_{k,v}$ with

$$\mathcal{K}_{k,v} = \Big\{(m, (s_a)_a, s^0) \in \mathrm{Cs}_k^{j_v+2} : \text{ (i)–(iii) hold and } (\rho_k(u(g)) - 1)m = \delta\sum_a \rho_k(\alpha_{v,a}(g))s_a + \delta\rho_k(\alpha^0_v(g))s^0 \ \ \forall g \in G_{E_v}\Big\},$$

and $\mathrm{nl}\,\mathcal{K}_k = \sum_v \mathrm{nl}\,\mathcal{K}_{k,v}$. For a submodule P of a product, $\mathrm{nl}P \leq \mathrm{nl}(\ker p|_P) + \mathrm{nl}(p(P))$ for every projection p; we use this with $p(m, s, s^0) = (s, s^0)$.

$k = 0$. The kernel of p on $\mathcal{K}_{0,v}$ consists of the m with $(\rho_0(u(g)) - 1)m = 0$ for all $g \in G_{E_v}$ ((i) holds as $\rho_0(\mathcal{E}) = 0$), i.e. of $H^0(G_{E_v}, \mathrm{Cs}_0(u))$, of normalised length $h^{\mathrm{c}}(v)$. In the image, $s^0 \in \mathrm{Cs}_0[\delta]$ by (iii) and $\rho_0(\mathcal{E}) = 0$, of normalised length $n_{\mathcal{R},0}\nu(\delta)$. For the s_a: by (b), (c) the last condition says that $g \mapsto \sum_a x'_a\beta_{v,a}(g)$, with $x'_a = \delta\ell^{e'_v}x_a \in K/\mathcal{O}$, is the coboundary of m; by (Cm4), $\delta\ell^{e'_v}x_a = 0$. So each s_a lies in the image of $(K/\mathcal{O})[\delta\ell^{e'_v}]$ under the $\mathcal{O}$-linear map $x \mapsto xz_1$, of normalised length at most $\nu(\delta\ell^{e'_v}) = \nu(\delta)+e'_v$. Hence $\mathrm{nl}\,\mathcal{K}_{0,v} \leq h^{\mathrm{c}}(v)+j_v(e'_v+\nu(\delta))+n_{\mathcal{R},0}\nu(\delta)$.

$k = 1$. Here $\mathrm{Cs}_1 = K/\mathcal{O}$, $\rho_1(\mathcal{E}) = \delta$ (Gd2), $\rho_1(u(g)) = 1$ and $s_a = 0$ by (a). So $\mathcal{K}_{1,v}$ consists of the $(m, 0, s^0)$ with $\delta m = 0$ (by (i)) and $\delta\rho_1(\alpha^0_v(g))s^0 = 0$ for all $g \in G_{E_v}$ ((iii) is automatic). Taking $g = g_v$ and using (Cm5), $s^0 \in (K/\mathcal{O})[\delta\rho_1(\alpha^0_v(g_v))]$, of normalised length $\nu(\delta) + \nu(\rho_1(\alpha^0_v(g_v))) \leq \nu(\delta) + b_v$ ($\delta \neq 0$ and $\rho_1(\alpha^0_v(g_v)) \neq 0$ in $\mathcal{O}$). So $\mathrm{nl}\,\mathcal{K}_{1,v} \leq 2\nu(\delta) + b_v$.

$2 \leq k \leq t$. Here $\mathrm{Cs}_k = K/\mathcal{O}$, $s_a = 0$ by (a), $\rho_k(\mathcal{E}) = 0$ and $\rho_k(\alpha^0_v(g)) = 0$ by (b), and $\rho_k(u(g)) = \chi_1(g)^{-1}\chi_k(g) = \chi_{\mathrm{cyc}}(g)^{1-k}$ by (Gd1). So $\mathcal{K}_{k,v}$ consists of the $(m, 0, s^0)$ with $\delta s^0 = 0$ and $(\chi_{\mathrm{cyc}}(g)^{1-k} - 1)m = 0$ for $g \in G_{E_v}$. As $\chi_{\mathrm{cyc}}(g)^{1-k} - 1 = -\chi_{\mathrm{cyc}}(g)^{1-k}(\chi_{\mathrm{cyc}}(g)^{k-1} - 1)$ with $\chi_{\mathrm{cyc}}(g)^{1-k} \in \mathbb{Z}_\ell^\times$, these elements generate the ideal $\ell^w\mathbb{Z}_\ell$ of $\mathbb{Z}_\ell$, $w = w_{k-1}(v)$; w is finite because $\chi_{\mathrm{cyc}}(G_{E_v})$ is an open subgroup of $\mathbb{Z}_\ell^\times$ and so contains an element of infinite order. So $m \in (K/\mathcal{O})[\ell^w]$, and $\mathrm{nl}\,\mathcal{K}_{k,v} \leq w_{k-1}(v) + \nu(\delta)$.

Summing over $v \in C$, $\mathrm{nl}\,\mathcal{K}_k$ is at most the stated lower bound for κ_k; this is (Gd6).

Item 4. *The classes $[\beta_{v,a}] \in H^1(G_{E_v}, W_0)$ are K-linearly independent.* Suppose $\sum_a c_a[\beta_{v,a}] = 0$ with $c \in K^{j_v}$ not zero; multiplying by a power of ϖ we may take all $c_a \in \mathcal{O}$ and some $c_{a_0} \in \mathcal{O}^\times$. Then $\sum_a c_a\beta_{v,a}(g) = (\rho_0(u(g)) - 1)w$ for some $w \in W_0$ and all $g \in G_{E_v}$, so with $x_a = \varpi^{-1}c_a + \mathcal{O}$ the function $g \mapsto \sum_a x_a\beta_{v,a}(g)$ is the coboundary of $\varpi^{-1}w + \Lambda_0$. By (Cm4) all $x_a = 0$, but $x_{a_0} \neq 0$ since c_{a_0} is a unit: a contradiction. By (Cm3) these classes lie in $H^1_f(G_{E_v}, W_0)$, so $j_v \leq \dim_K H^1_f(G_{E_v}, W_0)$. *The dimension.* By Theorem 25.3 ((Ll1), (Gd1), (Gd2)), W_0 is crystalline, hence de Rham [52]. By Theorem B.43, $\dim_{\mathbb{Q}_\ell} H^1_f = \dim_{\mathbb{Q}_\ell} H^0 + \dim_{\mathbb{Q}_\ell} D_{\mathrm{dR}}(W_0) - \dim_{\mathbb{Q}_\ell} \mathrm{Fil}^0 D_{\mathrm{dR}}(W_0)$; all four are K-vector spaces (K acts through W_0), and $\dim_{E_v} D_{\mathrm{dR}}(W_0) = \dim_{\mathbb{Q}_\ell} W_0$, so $\dim_K D_{\mathrm{dR}}(W_0) = n_v n_{\mathcal{R},0}$, $n_v = [E_v : \mathbb{Q}_\ell]$. Hence

$$\dim_K H^1_f(G_{E_v}, W_0) = \dim_K H^0(G_{E_v}, W_0) + n_v n_{\mathcal{R},0} - d^0(W_0), \qquad d^0(W_0) = \dim_K \mathrm{Fil}^0 D_{\mathrm{dR}}(W_0).$$

The jump count. As K contains every $\tau(E_v)$, $E_v \otimes_{\mathbb{Q}_\ell} K \cong \prod_\tau K$, and with the idempotents e_τ, $D_{\mathrm{dR}}(W_0) = \bigoplus_\tau D_{\mathrm{dR},\tau}(W_0)$ compatibly with the filtrations, so $\sum_\tau \dim_K D_{\mathrm{dR},\tau}(W_0) =$

$n_v n_{\mathcal{R},0}$. On each $D_{\mathrm{dR},\tau}(W_0)$ the filtration is decreasing, equal to the whole space for $a \ll 0$ (every element lies in $t^a B_{\mathrm{dR}}^+ \otimes W_0$ for some a, and the space is finite-dimensional) and 0 for $a \gg 0$; so $\dim_K D_{\mathrm{dR},\tau}(W_0) = \sum_{a\in\mathbb{Z}} \dim_K \mathrm{gr}^a$. By Lemma 25.5(3), $d^0(W_0) = \sum_\tau \sum_{a\geq 0} \dim_K \mathrm{gr}^a D_{\mathrm{dR},\tau}(W_0)$. Therefore $n_v n_{\mathcal{R},0} - d^0(W_0) = \sum_\tau \sum_{a<0} \dim_K \mathrm{gr}^a D_{\mathrm{dR},\tau}(W_0) \leq \sum_\tau \mu_\tau^-$, which with the two previous displays gives item 4. □

Theorem 25.3. *Let E be a number field, ℓ a prime, and let Galois Ribet data over E on $(\mathcal{R}, \pi', \mathcal{N})$, with $t+1$ blocks of sizes $n_{\mathcal{R},0}, \dots, n_{\mathcal{R},t}$ over the Hecke lattice $(\mathcal{O}, \mathcal{Z}, \mathbb{T}, M)$, have (Gd1) and (Gd2); put $K = \mathcal{O}[\ell^{-1}]$. Let $V_{\mathcal{N}} = K\mathcal{N}_{\mathcal{E}} \subseteq \prod_{x'\in\mathcal{Z}'} \mathrm{M}_{n_{\mathcal{R}}}(K)$, on which $g \in G_E$ acts by left multiplication by $u(g)$, and for $0 \leq k \leq t$ let $W_k = K^{n_{\mathcal{R},k}}$, on which g acts by $\rho_k(u(g))$, with the lattice $\Lambda_k = \mathcal{O}^{n_{\mathcal{R},k}}$, so that $\mathrm{Cs}_k(u) = W_k/\Lambda_k$. Let v be a place of E above ℓ, $D = G_{E_v}$, and suppose:*

(Ll1) Crystalline points *for every $x' \in \mathcal{Z}'$ the representation $g \mapsto \mathbf{r}(g)(x')$ of D on $K^{n_{\mathcal{R}}}$ is crystalline;*

(Ll1') Potentially crystalline points *there is a finite Galois extension K' of E_v inside $\overline{E_v}$ such that, with $D_{K'} \subseteq D$ the image of $\mathrm{Gal}(\overline{E_v}/K')$ (an open normal subgroup of index $[K' : E_v]$, identified with $G_{K'}$), for every $x' \in \mathcal{Z}'$ the restriction of $g \mapsto \mathbf{r}(g)(x')$ to $D_{K'}$ is a crystalline representation of $G_{K'}$ (for $K' = E_v$ this is (Ll1));*

(Ll2) Bloch–Kato condition *$F_0(v) = H^1_f(D, \mathrm{Cs}_0(u))$;*

(Ll3) Lifting exponents *$e_0, \dots, e_t \geq 0$ are integers such that for every $0 \leq k \leq t$ and every $\mathcal{O}$-linear D-equivariant $\psi\colon \mathcal{N}_{\mathcal{E}} \to \mathrm{Cs}_k(u)$,*

$$\ell^{e_k}\, H^1_f(D, \mathcal{N}_{\mathcal{E}}) \subseteq \mathrm{pr}_{1*}\, H^1_f(D, \widetilde{T}_\psi), \qquad \widetilde{T}_\psi \subseteq V_{\mathcal{N}} \oplus W_k.$$

Then, under (Ll1) alone, $V_{\mathcal{N}}$ and every W_k are crystalline representations of D; under (Ll1') alone, their restrictions to $D_{K'}$ are crystalline representations of $G_{K'}$; under either of (Ll1), (Ll1'), $\mathcal{N}_{\mathcal{E}}$ is a D-stable lattice in $V_{\mathcal{N}}$ and the restriction of y to D has class in $H^1_f(D, \mathcal{N}_{\mathcal{E}})$; and, under (Ll1) or (Ll1') together with (Ll2) and (Ll3), for $0 \leq k \leq t$ the image of X_k in $H^1(G_{E_v}, \mathrm{Cs}_k(u))/F_k(v)$ has normalised length at most

$$\lambda_k(v) = \left([E_v : \mathbb{Q}_\ell] + 2\right) n_{\mathcal{R},k}\, e_k;$$

that is, (Gd7) holds at v with these local losses. If moreover $\mathcal{N} = \mathcal{N}_{\mathcal{E}}$ and ι is the identity, (Gd6) holds with $\kappa_k = 0$ for all k.

Proof. **Step 0: crystalline objects.** Fix $x' \in \mathcal{Z}'$ and write $r_{x'}$ for $g \mapsto \mathbf{r}(g)(x')$ on $K^{n_{\mathcal{R}}}$; $r_{\pi'}$ is the case $x' = \pi'$. By (Bf1) of Definition 13.11 every $r_{\pi'}(g)$ is block upper triangular, so the spans of the coordinates in the first $k+1$ blocks form a D-stable flag of $K^{n_{\mathcal{R}}}$, and the k-th graded piece is K^{n_k} with g acting by the diagonal block $\rho_k(\mathbf{r}(g))$. By (Ll1) and Theorem B.45 (subquotients), these pieces are crystalline; for $k = 1$, $n_1 = 1$ and the piece is $K(\chi_1)$ by (Gd1). Hence $K(\chi_1^{-1})$ is crystalline (dual), and so are $K(\chi_1^{-1}) \otimes_K r_{x'}$ for every x' (a quotient of the tensor product over $\mathbb{Q}_\ell$) and $W_k = K(\chi_1^{-1}) \otimes_K$ (k-th piece), on which g acts by $\chi_1(g)^{-1}\rho_k(\mathbf{r}(g)) = \rho_k(u(g))$ (ρ_k is $\mathcal{O}$-linear). The representation A with g acting by left multiplication by $u(g)$ is the direct sum over x' and over the $n_{\mathcal{R}}$ columns of copies of $K(\chi_1^{-1}) \otimes_K r_{x'}$, so it is crystalline, and so is every D-stable subspace of it. Under (Ll1') the same argument, applied to the restrictions to $D_{K'} \cong G_{K'}$ with Theorem B.45 for the finite extension K' of $\mathbb{Q}_\ell$, shows that the restrictions to $D_{K'}$ of the graded pieces, of $K(\chi_1^{\pm 1})$, of every W_k, of A and of every D-stable subspace of A are crystalline representations of $G_{K'}$. The algebra $\mathcal{R}$ is a finitely generated free $\mathcal{O}$-module: for each

x' the set $\mathbf{r}(G_E)(x')$ is compact, hence contained in $\varpi^{-a}\mathrm{M}_{n_\mathcal{R}}(\mathcal{O})$ for some a (ϖ a uniformiser of $\mathcal{O}$); $\mathcal{Z}'$ is finite, and the component at x' of an element $\sum_h f_h\mathbf{r}(h)$ of $\mathcal{R}$ (f_h in the coefficient algebra of Definition 17.2, whose elements are $\mathcal{O}$-valued) is an $\mathcal{O}$-combination of the $\mathbf{r}(h)(x')$; so $\mathcal{R}$ is a submodule of the finitely generated free module $\varpi^{-a}\prod_{x'}\mathrm{M}_{n_\mathcal{R}}(\mathcal{O})$ over the discrete valuation ring $\mathcal{O}$, hence finitely generated and free, and so is its submodule $\mathcal{N}_\mathcal{E}$; as $u(g)\in\mathcal{R}$ and $\mathcal{N}_\mathcal{E}$ is a left ideal, $\mathcal{N}_\mathcal{E}$ is $u(D)$-stable, and it spans $V_\mathcal{N}$. So $V_\mathcal{N}$ is crystalline and $\mathcal{N}_\mathcal{E}$ is a D-stable lattice in it; the action is continuous since $g\mapsto u(g)$ is continuous into $\mathcal{R}$ (each $g\mapsto\mathbf{r}(g)(x')$ and χ_1 is continuous, and the ϖ-adic topology of the lattice $\mathcal{R}$ is induced from the finite-dimensional K-space A).

Step 1: the Ribet cocycle is crystalline at v. The map $\rho_1\colon\mathcal{R}\to\mathrm{M}_1(\mathcal{O})=\mathcal{O}$ is a homomorphism of $\mathcal{O}$-algebras (Definition 13.11); it extends K-linearly to $K\mathcal{R}=K\otimes_\mathcal{O}\mathcal{R}\subseteq A$. By (Gd1) and (Gd2), $\rho_1(y(g))=(\rho_1(u(g))-1)\rho_1(\mathcal{E})=(\chi_1(g)^{-1}\chi_1(g)-1)\delta=0$, so $\rho_1(ay(g))=\rho_1(a)\cdot 0=0$ for $a\in\mathcal{R}$, i.e. $\rho_1(V_\mathcal{N})=0$; and $\rho_1(\mathcal{E})=\delta\neq 0$. Let $V_P=V_\mathcal{N}+K\mathcal{E}\subseteq A$. Then $\mathcal{E}\notin V_\mathcal{N}$, $V_P=V_\mathcal{N}\oplus K\mathcal{E}$, and V_P is D-stable because $u(g)\mathcal{E}=\mathcal{E}+y(g)$ with $y(g)\in\mathcal{N}_\mathcal{E}$. The map $\beta=\delta^{-1}\rho_1\colon V_P\to K$ is K-linear, surjective, with kernel $V_\mathcal{N}$ (compare dimensions), and D-equivariant for the trivial action on K: $\beta(u(g)z)=\delta^{-1}\rho_1(u(g))\rho_1(z)=\beta(z)$. So $0\to V_\mathcal{N}\to V_P\xrightarrow{\beta}K\to 0$ is an exact sequence of crystalline representations of D (Step 0; K trivial is crystalline). Write $D_\mathrm{cris}(X)=(X\otimes_{\mathbb{Q}_\ell}B_\mathrm{cris})^D$; it is left exact in X ($\otimes_{\mathbb{Q}_\ell}B_\mathrm{cris}$ is exact, invariants are left exact) and $\dim_{E_{v,0}}D_\mathrm{cris}(X)=\dim_{\mathbb{Q}_\ell}X$ for crystalline X. Hence the image of $D_\mathrm{cris}(V_P)$ in $D_\mathrm{cris}(K)$ has dimension $\dim_{\mathbb{Q}_\ell}V_P-\dim_{\mathbb{Q}_\ell}V_\mathcal{N}=\dim_{\mathbb{Q}_\ell}K=\dim D_\mathrm{cris}(K)$, and $D_\mathrm{cris}(V_P)\to D_\mathrm{cris}(K)$ is onto. Choose $z\in D_\mathrm{cris}(V_P)$ mapping to the invariant element $1\otimes 1$. Then $\mathcal{E}\otimes 1-z$ lies in the kernel of $V_P\otimes B_\mathrm{cris}\to K\otimes B_\mathrm{cris}$, which is $V_\mathcal{N}\otimes B_\mathrm{cris}$, and for $g\in D$

$$g(\mathcal{E}\otimes 1-z)-(\mathcal{E}\otimes 1-z)=(u(g)\mathcal{E}-\mathcal{E})\otimes 1=y(g)\otimes 1.$$

So $y|_D$ is a cocycle with values in $V_\mathcal{N}$ whose class lies in $H^1_f(D,V_\mathcal{N})$ (Definition 22.2); it takes values in $\mathcal{N}_\mathcal{E}$ and is continuous (each component of $y(g)=(u(g)-1)\mathcal{E}$ is continuous in g, as $g\mapsto\mathbf{r}(g)(x')$ and χ_1 are, and the ϖ-adic topology of the lattice $\mathcal{N}_\mathcal{E}$ is the topology induced from the finite-dimensional K-space $V_\mathcal{N}$). Hence $[y|_D]\in H^1_f(D,\mathcal{N}_\mathcal{E})$.

Under (Ll1'). The subspaces $V_\mathcal{N}\subset V_P$, the map β and the identity $u(g)\mathcal{E}=\mathcal{E}+y(g)$ do not depend on (Ll1), and by Step 0 the restriction of $0\to V_\mathcal{N}\to V_P\to K\to 0$ to $D_{K'}$ is an exact sequence of crystalline representations of $G_{K'}$. Put $D_{\mathrm{cris},K'}(X)=(X\otimes_{\mathbb{Q}_\ell}B_\mathrm{cris})^{D_{K'}}$, a vector space over $B_\mathrm{cris}^{D_{K'}}=K'_0$, the maximal unramified subextension of $K'/\mathbb{Q}_\ell$; it is left exact in X and of dimension $\dim_{\mathbb{Q}_\ell}X$ for X crystalline over K'. The dimension count above, with $D_{K'}$ in place of D, gives $z'\in D_{\mathrm{cris},K'}(V_P)$ mapping to $1\otimes 1$, and $m_0=\mathcal{E}\otimes 1-z'$ lies in $M=V_\mathcal{N}\otimes_{\mathbb{Q}_\ell}B_\mathrm{cris}$ with $y(h)\otimes 1=hm_0-m_0$ for $h\in D_{K'}$. We descend to D. The map $c(g)=y(g)\otimes 1-(gm_0-m_0)$ is a 1-cocycle $D\to M$ vanishing on $D_{K'}$. For $g\in D$ and $h\in D_{K'}$, $c(gh)=c(g)+gc(h)=c(g)$, and $c(hg)=c(h)+hc(g)=hc(g)$ while $hg=g(g^{-1}hg)$ with $g^{-1}hg\in D_{K'}$ (normality), so $c(hg)=c(g)$. Hence c takes values in $M^{D_{K'}}$ and is constant on the cosets of $D_{K'}$; write $\Gamma=D/D_{K'}$, of order $m=[K':E_v]$, and $c(\gamma)$ for its value on a coset. Summing $c(g\gamma)=c(g)+gc(\gamma)$ over $\gamma\in\Gamma$ and reindexing gives $\sum_\gamma c(\gamma)=m\,c(g)+g\sum_\gamma c(\gamma)$. Since M is a $\mathbb{Q}_\ell$-vector space we may put $m_1=-m^{-1}\sum_{\gamma\in\Gamma}c(\gamma)\in M$, and then $c(g)=gm_1-m_1$ for every $g\in D$. So $y(g)\otimes 1=g(m_0+m_1)-(m_0+m_1)$ for all $g\in D$ with $m_0+m_1\in M$, that is, the class of $y|_D$ in $H^1(D,V_\mathcal{N})$ lies in $H^1_f(D,V_\mathcal{N})$ (Definition 22.2), and as before $[y|_D]\in H^1_f(D,\mathcal{N}_\mathcal{E})$. The division by $m=[K':E_v]$ happens in the rational space M: membership in $H^1_f(D,\mathcal{N}_\mathcal{E})$ is the condition on the rational class, so no power of ℓ is lost and the exponents e_k and the losses $\lambda_k(v)$ below are the same under (Ll1) and (Ll1'). Steps 0 and 1 use only (Ll1) or (Ll1'), together with (Gd1) and (Gd2); Steps 2–5 do not use (Ll1).

Step 2: the local conditions contain the Bloch–Kato images and are divisible.

Put $F'_k = H^1_f(D, \mathrm{Cs}_k(u))$, the image of $H^1_f(D, W_k)$. We show $F'_k \subseteq F_k(v)$ and that $F_k(v)$ is divisible.

- $k = 0$: $F_0(v) = F'_0$ by (Ll2), divisible by Lemma 25.4(3).
- $k = 1$: $\rho_1(u(g)) = 1$, so $W_1 = K$ with trivial action and $H^1(D, K) = \mathrm{Hom}_{\mathrm{cont}}(D, K)$. Choosing a $\mathbb{Q}_\ell$-basis of K writes K as a sum of copies of $\mathbb{Q}_\ell$, and the condition of Definition 22.2 is componentwise; by Theorem B.44, $H^1_f(D, K)$ consists of the homomorphisms vanishing on I_v. Their images in $\mathrm{Hom}_{\mathrm{cont}}(D, K/\mathcal{O}) = H^1(D, K/\mathcal{O})$ vanish on I_v, so $F'_1 \subseteq H^1_{\mathrm{ur}}(E_v, K/\mathcal{O}) = F_1(v)$. The latter is the group of continuous homomorphisms $D/I_v \to K/\mathcal{O}$; $D/I_v \cong \widehat{\mathbb{Z}}$ is topologically generated by a Frobenius element, so evaluation at it is injective, and it is onto because every element of $K/\mathcal{O}$ has finite ℓ-power order ℓ^s and $\widehat{\mathbb{Z}} \to \mathbb{Z}/\ell^s$ is continuous. So $F_1(v) \cong K/\mathcal{O}$ is divisible.
- $k \geq 2$: by (Gd1), $\rho_k(u(g)) = \chi_1(g)^{-1}\chi_k(g) = \chi_{\mathrm{cyc}}(g)^{1-k}$, so as a $\mathbb{Q}_\ell$-representation W_k is a sum of copies of $\mathbb{Q}_\ell(s)$, $s = 1-k \leq -1$, the line on which g acts by $\chi_{\mathrm{cyc}}(g)^s$. For $V = \mathbb{Q}_\ell(s)$ with basis ϵ: $H^0(D, V) = 0$, since $\chi_{\mathrm{cyc}}(D)$ is infinite; an element $b \otimes \epsilon$ of $B_{\mathrm{dR}} \otimes V$ is D-invariant if and only if $g(b) = \chi_{\mathrm{cyc}}(g)^{-s} b$, i.e. if and only if $bt^s \in B_{\mathrm{dR}}^D = E_v$ (t is invertible in the field B_{dR} and $g(t) = \chi_{\mathrm{cyc}}(g)t$); so $D_{\mathrm{dR}}(V) = E_v\, t^{-s} \otimes \epsilon$, of dimension 1 over E_v, and V is de Rham; as $t^{-s} \in \mathrm{Fil}^{-s} B_{\mathrm{dR}} \subseteq \mathrm{Fil}^0 B_{\mathrm{dR}}$, $\mathrm{Fil}^0 D_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V)$. By Theorem B.43, $H^1_f(D, \mathbb{Q}_\ell(s)) = 0$, hence $H^1_f(D, W_k) = 0$ (componentwise), and $F'_k = 0 = F_k(v)$, which is divisible.

Step 3: the Ribet classes at v. Let $\varphi \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathrm{Cs}_k)$ with c_φ a continuous cocycle, and $\psi = \varphi \circ \iota \colon \mathcal{N}_{\mathcal{E}} \to \mathrm{Cs}_k$. Since ψ is $\mathcal{R}$-linear and $\mathcal{R}$ acts on Cs_k through ρ_k, $\psi(u(g)n) = \rho_k(u(g))\psi(n)$: ψ is $\mathcal{O}$-linear and D-equivariant into $\mathrm{Cs}_k(u) = W_k/\Lambda_k$. As $c_\varphi = \psi \circ y$, the restriction of $[c_\varphi]$ to D is $\psi_*[y|_D]$ with $[y|_D] \in H^1_f(D, \mathcal{N}_{\mathcal{E}})$ (Step 1). By (Ll3) the hypothesis of Lemma 25.4(2) holds for $F = E$, v, D, the coefficient field K, $V = V_{\mathcal{N}}$, $T = \mathcal{N}_{\mathcal{E}}$, $W = W_k$, $\Lambda = \Lambda_k$, ψ and $e = e_k$; it gives $\ell^{e_k}\,[c_\varphi|_D] \in F'_k \subseteq F_k(v)$ (Step 2). As $\ell^{e_k} = \varpi^{e_k e_{\mathcal{O}}} w$ with $w \in \mathcal{O}^\times$ and $F_k(v)$ is an $\mathcal{O}$-module, $\varpi^{e_k e_{\mathcal{O}}}[c_\varphi|_D] \in F_k(v)$.

Step 4: the length. By Step 3 the image of X_k under restriction lies in $Y_k = \{z \in H^1(D, \mathrm{Cs}_k(u)) : \varpi^{e_k e_{\mathcal{O}}} z \in F_k(v)\}$, so its image in $H^1(D, \mathrm{Cs}_k(u))/F_k(v)$ is a submodule of $Y_k/F_k(v)$. By Lemma 25.4(3), with the divisible submodule $F' = F_k(v)$ (Step 2), $a = e_k e_{\mathcal{O}}$ and $\dim_K W_k = n_k$, $\mathrm{nl}(Y_k/F_k(v)) \leq ([E_v : \mathbb{Q}_\ell] + 2)\, n_k\, e_k = \lambda_k(v)$. A submodule has at most the length of the module; this is (Gd7) at v.

Step 5: no column loss. If $\mathcal{N} = \mathcal{N}_{\mathcal{E}}$ and $\iota = \mathrm{id}$, the map $\varphi \mapsto \varphi \circ \iota$ of (Gd6) is the identity, with kernel 0; so (Gd6) holds with $\kappa_k = 0$. □

Lemma 25.4. *Let F be a number field, ℓ a prime, v a place of F above ℓ and $D \subseteq G_F$ a decomposition group at v. Let K be a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$, uniformiser ϖ and $e_{\mathcal{O}} = v_\varpi(\ell)$. Let V and W be finite-dimensional K-vector spaces with continuous K-linear actions of D, and $T \subset V$, $\Lambda \subset W$ D-stable $\mathcal{O}$-lattices (finitely generated $\mathcal{O}$-submodules spanning V, W over K). For a D-stable $\mathcal{O}$-lattice T' in such a representation V' let $H^1_f(D, T')$ be the inverse image of $H^1_f(D, V')$ under $H^1(D, T') \to H^1(D, V')$, and put*

$$H^1_f(D, W/\Lambda) = \mathrm{Im}\big(H^1_f(D, W) \to H^1(D, W/\Lambda)\big),$$

W/Λ carrying the discrete topology. Let $\psi \colon T \to W/\Lambda$ be an $\mathcal{O}$-linear D-equivariant map. Then:

1. *$\widetilde{T}_\psi = \{(t, w) \in T \oplus W : \psi(t) = w + \Lambda\}$ is a D-stable $\mathcal{O}$-lattice in $V \oplus W$, and the first projection pr_1 maps it onto T;*

2. *ψ is continuous, and if $e \geq 0$ is an integer with $\ell^e H^1_f(D,T) \subseteq \mathrm{pr}_{1*} H^1_f(D,\widetilde{T}_\psi)$, then $\ell^e \psi_*(x) \in H^1_f(D,W/\Lambda)$ for every $x \in H^1_f(D,T)$;*

3. *let $F' \subseteq H^1(D,W/\Lambda)$ be a divisible $\mathcal{O}$-submodule and $a \geq 0$ an integer; then*

$$\mathrm{nl}\Big(\{z \in H^1(D,W/\Lambda) :\ \varpi^a z \in F'\}/F'\Big) \leq \big([F_v:\mathbb{Q}_\ell]+2\big)\,\dim_K W\,\frac{a}{e_{\mathcal{O}}};$$

and $H^1_f(D,W/\Lambda)$ is divisible.

Proof. **Item 1.** $\widetilde{T}_\psi$ is an $\mathcal{O}$-submodule because ψ is $\mathcal{O}$-linear, and it is D-stable because $\psi(gt) = g\psi(t) = gw+\Lambda$ when $\psi(t) = w+\Lambda$ (Λ is D-stable). The image $\psi(T)$ is a finitely generated $\mathcal{O}$-submodule of the torsion module W/Λ, so it is killed by ϖ^b for some $b \geq 0$, i.e. $\psi(T) \subseteq \varpi^{-b}\Lambda/\Lambda$. Hence

$$\varpi^b T \oplus \Lambda \ \subseteq\ \widetilde{T}_\psi \ \subseteq\ T \oplus \varpi^{-b}\Lambda :$$

for the left inclusion, $\psi(\varpi^b t) = \varpi^b\psi(t) = 0 = 0+\Lambda$ and $\psi(0) = 0 = \lambda+\Lambda$ for $\lambda \in \Lambda$; for the right one, $w+\Lambda = \psi(t) \in \varpi^{-b}\Lambda/\Lambda$ gives $w \in \varpi^{-b}\Lambda$. The right side is finitely generated over the noetherian ring $\mathcal{O}$, so $\widetilde{T}_\psi$ is finitely generated (and free, being torsion-free over a discrete valuation ring), and the left side spans $V \oplus W$ over K. So $\widetilde{T}_\psi$ is a D-stable lattice. For $t \in T$ choose $w \in W$ with $w+\Lambda = \psi(t)$; then $(t,w) \in \widetilde{T}_\psi$, so $\mathrm{pr}_1(\widetilde{T}_\psi) = T$.

Item 2. ψ vanishes on the open subgroup $\varpi^b T$ of T, so it is continuous for the ϖ-adic topology of T and the discrete topology of W/Λ; hence $\psi \circ c$ is a continuous cocycle for every continuous cocycle $c\colon D \to T$, and ψ_* is defined on $H^1(D,T)$. Let $x = [c] \in H^1_f(D,T)$. By hypothesis there is a continuous cocycle $\tilde{c} = (c_1,c_2)\colon D \to \widetilde{T}_\psi$ whose class in $H^1(D,V\oplus W)$ lies in $H^1_f(D,V\oplus W)$ and with $\ell^e x = \mathrm{pr}_{1*}[\tilde{c}] = [c_1]$; so there is $t_0 \in T$ with $\ell^e c(g) = c_1(g) + gt_0 - t_0$ for all $g \in D$. Since $\tilde{c}(g) \in \widetilde{T}_\psi$, $\psi(c_1(g)) = c_2(g)+\Lambda$. The map $c_2\colon D \to W$ is a continuous cocycle (the second component of one), and its class lies in $H^1_f(D,W)$: there is $y \in (V\oplus W)\otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}} = (V\otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}) \oplus (W \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}})$ with $\tilde{c}(g)\otimes 1 = gy - y$, and the second component y_2 of y satisfies $c_2(g)\otimes 1 = gy_2 - y_2$, the action being componentwise. Therefore, for $g \in D$,

$$\psi(\ell^e c(g)) = \psi(c_1(g)) + \psi(gt_0 - t_0) = \big(c_2(g)+\Lambda\big) + \big(g\psi(t_0) - \psi(t_0)\big),$$

so $\ell^e\psi_*(x)$ is the image of $[c_2] \in H^1_f(D,W)$ in $H^1(D,W/\Lambda)$, the second term being a coboundary of the discrete module W/Λ. This is the claim.

Item 3. Write $\mathrm{Cs} = W/\Lambda$, $d = \dim_K W$, $H = H^1(D,\mathrm{Cs})$ and $Y = \{z \in H : \varpi^a z \in F'\}$. $Y = H[\varpi^a] + F'$. If $z \in Y$, then $\varpi^a z \in F' = \varpi^a F'$ (F' is divisible), say $\varpi^a z = \varpi^a f'$ with $f' \in F'$, and $z - f' \in H[\varpi^a]$; the other inclusion is clear. So Y/F' is a quotient of $H[\varpi^a]$. *$H[\varpi^a]$ is a quotient of $H^1(D,M)$, $M = \mathrm{Cs}[\varpi^a]$.* Multiplication by ϖ^a on Cs is surjective ($w+\Lambda = \varpi^a(\varpi^{-a}w+\Lambda)$) with kernel $M = \varpi^{-a}\Lambda/\Lambda$; the long exact cohomology sequence of $0 \to M \to \mathrm{Cs} \xrightarrow{\varpi^a} \mathrm{Cs} \to 0$ (discrete D-modules) contains $H^1(D,M) \to H \xrightarrow{\varpi^a} H$, so the image of the first map is $H[\varpi^a]$. *The size of $H^1(D,M)$.* $M \cong \Lambda/\varpi^a\Lambda \cong (\mathcal{O}/\varpi^a)^d$ is a finite $\mathcal{O}$-module of length ad, killed by ℓ^r for $r = \lceil a/e_{\mathcal{O}}\rceil$, with a continuous action of $D = G_{F_v}$. By Theorem B.29(4) ($v \mid \ell$), $\#H^1(D,M) = \#H^0(D,M)\,\#H^2(D,M)\,(\#M)^{[F_v:\mathbb{Q}_\ell]}$. Here $\#H^0(D,M) \leq \#M$, and by Theorem B.29(2) with $i=2$ the pairing $H^2(D,M)\times H^0(D,M^\vee) \to \mathbb{Z}/\ell^r$ is perfect, so $\#H^2(D,M) = \#H^0(D,M^\vee) \leq \#M^\vee = \#M$. Hence $\#H^1(D,M) \leq (\#M)^{[F_v:\mathbb{Q}_\ell]+2}$. All these groups are finite $\mathcal{O}$-modules, and a finite $\mathcal{O}$-module X has $\#X = q^{\mathrm{length}_{\mathcal{O}}X}$; so $\mathrm{length}_{\mathcal{O}} H^1(D,M) \leq ([F_v:\mathbb{Q}_\ell]+2)\,a\,d$. Lengths do not increase under quotients, and dividing by $e_{\mathcal{O}}$ gives the bound. Finally $H^1_f(D,W)$ is a K-subspace of $H^1(D,W)$, hence divisible, and its image under the $\mathcal{O}$-linear map $H^1(D,W) \to H^1(D,W/\Lambda)$ is divisible. □

Lemma 25.5. *Let F be a number field, ℓ a prime, v a place of F above ℓ, $D \subseteq G_F$ a decomposition group at v, and $n_v = [F_v : \mathbb{Q}_\ell]$. Let K be a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$ and uniformiser ϖ, W a K-vector space of finite dimension d with a continuous K-linear action of D which is de Rham as a $\mathbb{Q}_\ell$-representation, $\Lambda \subset W$ a D-stable $\mathcal{O}$-lattice, $\mathrm{Cs} = W/\Lambda$, $H = H^1(D, \mathrm{Cs})$ and $F' = H^1_f(D, \mathrm{Cs})$. Put*

$$d^0(W) = \dim_K \mathrm{Fil}^0 D_{\mathrm{dR}}(W), \qquad D_{\mathrm{dR}}(W) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} W)^D,$$

K acting through W. Let $W' = \mathrm{Hom}_{\mathbb{Q}_\ell}(W, \mathbb{Q}_\ell(1))$ and $\Lambda' = \mathrm{Hom}_{\mathbb{Z}_\ell}(\Lambda, \mathbb{Z}_\ell(1))$, with D acting by $(gf)(w) = g\, f(g^{-1}w)$ and $\mathcal{O}$ by $(\alpha f)(w) = f(\alpha w)$, and $A' = W'/\Lambda'$. For an integer $a \geq 0$ let $Y_a = \{z \in H : \varpi^a z \in F'\}$. Then:

1. $\mathrm{nl}(Y_a/F') = \dfrac{a}{e_{\mathcal{O}}}\, d^0(W) + \mathrm{nl}\big(H^2(D, \Lambda/\varpi^a\Lambda)\big)$;
2. $\mathrm{nl}\big(H^2(D, \Lambda/\varpi^a\Lambda)\big) = \mathrm{nl}\big(H^0(D, A')[\varpi^a]\big) \leq \mathrm{nl}\big(H^0(D, A')\big)$, *and $H^0(D, A')$ is finite if $H^0(D, W') = 0$;*
3. $d^0(W) \leq n_v\, d$; *and if K contains $\tau(F_v)$ for every $\mathbb{Q}_\ell$-embedding τ of F_v into $\overline{\mathbb{Q}}_\ell$, then $d^0(W) = \sum_\tau \sum_{a \geq 0} \dim_K\big(\mathrm{Fil}^a D_{\mathrm{dR},\tau}(W)/\mathrm{Fil}^{a+1} D_{\mathrm{dR},\tau}(W)\big)$, the number of Hodge jumps $a \geq 0$ of W counted with multiplicity over all τ.*

Proof. **Step 0: two facts on torsion modules.** (i) A torsion $\mathcal{O}$-module X with $X[\varpi]$ finite is isomorphic to $(K/\mathcal{O})^c \oplus X_0$ with X_0 finite (dual form of the structure theorem for finitely generated $\mathcal{O}$-modules); c is the *corank* of X, and it is the corank of any divisible submodule X_1 with X/X_1 finite (the image of X_1 in $X/(K/\mathcal{O})^c \cong X_0$ is divisible and finite, hence 0, so $X_1 \subseteq (K/\mathcal{O})^c$ with finite divisible quotient, hence $X_1 = (K/\mathcal{O})^c$). Every $\mathcal{O}$-submodule of $\mathrm{Cs} \cong (K/\mathcal{O})^d$ satisfies the hypothesis. (ii) For such X and $a \geq 0$,

$$\mathrm{length}\, X[\varpi^a] - \mathrm{length}\, X/\varpi^a X = a\, c,$$

because on $(K/\mathcal{O})^c$ multiplication by ϖ^a is onto with kernel of length ac, and on the finite X_0 its kernel and cokernel have equal length.

Step 1: $F' \cong (K/\mathcal{O})^{c_f}$ with $c_f = \dim_K H^1_f(D, W)$. The long exact sequence of $0 \to \Lambda \to W \to \mathrm{Cs} \to 0$ (a set-theoretic section of $W \to \mathrm{Cs}$ is continuous, Cs being discrete) shows that the kernel of $H^1_f(D, W) \to H$ is $L = \iota(H^1(D, \Lambda)) \cap H^1_f(D, W)$, $\iota\colon H^1(D, \Lambda) \to H^1(D, W)$. By Theorem B.30, $H^1(D, \Lambda)$ is finitely generated, so L is finitely generated, and torsion-free as a submodule of a K-vector space, hence free. L spans $H^1_f(D, W)$: if $x = [c] \in H^1_f(D, W)$, the image $c(D)$ of the compact group D is compact, hence contained in $\varpi^{-s}\Lambda$ for some s (the open sets $\varpi^{-s}\Lambda$ increase and cover W), so $\varpi^s c$ is a continuous cocycle with values in Λ and $\varpi^s x \in L$ ($H^1_f(D, W)$ is a K-subspace). Thus L is a lattice in $H^1_f(D, W)$ and $F' \cong H^1_f(D, W)/L \cong (K/\mathcal{O})^{c_f}$. In particular F' is divisible and $\mathrm{length}\, F'[\varpi^a] = a\, c_f$.

Step 2: $X = H^0(D, \mathrm{Cs})$ has corank $h^0 = \dim_K H^0(D, W)$. In $0 \to H^0(D, \Lambda) \to H^0(D, W) \to X \xrightarrow{\partial} H^1(D, \Lambda)$, $H^0(D, \Lambda) = \Lambda \cap H^0(D, W)$ is a lattice in $H^0(D, W)$ (finitely generated as a submodule of Λ; it spans, since $\varpi^s w \in \Lambda$ for $w \in W$ and large s), so the image of $H^0(D, W)$ in X is $\cong (K/\mathcal{O})^{h^0}$. The image of ∂ consists of torsion elements of the finitely generated module $H^1(D, \Lambda)$ (Theorem B.30), so it is finite. By Step 0(i), X has corank h^0.

Step 3: the length of Y_a/F'. Since F' is divisible, $Y_a = H[\varpi^a] + F'$: if $\varpi^a z = \varpi^a f'$ with $f' \in F'$, then $z - f' \in H[\varpi^a]$. Hence $Y_a/F' \cong H[\varpi^a]/(H[\varpi^a] \cap F') = H[\varpi^a]/F'[\varpi^a]$, and by Step 1

$$\mathrm{length}(Y_a/F') = \mathrm{length}\, H[\varpi^a] - a\, c_f.$$

Let $M = \mathrm{Cs}[\varpi^a] = \varpi^{-a}\Lambda/\Lambda \cong \Lambda/\varpi^a\Lambda$, a finite $\mathcal{O}$-module of length ad killed by ℓ^r, $r = \lceil a/e_{\mathcal{O}} \rceil$. The sequence $0 \to M \to \mathrm{Cs} \xrightarrow{\varpi^a} \mathrm{Cs} \to 0$ is exact (Cs is divisible) and gives the exact sequence

$$0 \to X/\varpi^a X \to H^1(D, M) \to H[\varpi^a] \to 0,$$

and $H^0(D, M) = X[\varpi^a]$. By Theorem B.29(4),

$$\operatorname{length} H^1(D, M) = \operatorname{length} H^0(D, M) + \operatorname{length} H^2(D, M) + n_v\, a\, d.$$

With Step 0(ii) and Step 2,

$$\begin{aligned} \operatorname{length} H[\varpi^a] &= \big(\operatorname{length} X[\varpi^a] - \operatorname{length} X/\varpi^a X\big) + \operatorname{length} H^2(D, M) + n_v ad \\ &= a\, h^0 + \operatorname{length} H^2(D, M) + n_v ad, \end{aligned}$$

so $\operatorname{length}(Y_a/F') = a\,(h^0 + n_v d - c_f) + \operatorname{length} H^2(D, M)$.

Step 4: the dimension count. W is de Rham as a $\mathbb{Q}_\ell$-representation, so by Theorem B.43

$$\dim_{\mathbb{Q}_\ell} H^1_f(D, W) = \dim_{\mathbb{Q}_\ell} H^0(D, W) + \dim_{\mathbb{Q}_\ell} D_{\mathrm{dR}}(W) - \dim_{\mathbb{Q}_\ell} \mathrm{Fil}^0 D_{\mathrm{dR}}(W).$$

All four spaces are K-vector spaces (K acts through W, commuting with D and preserving $\mathrm{Fil}^0 B_{\mathrm{dR}} \otimes W$, and Definition 22.2 defines H^1_f by a K-linear condition), so we may divide by $[K : \mathbb{Q}_\ell]$. Being de Rham, $\dim_{F_v} D_{\mathrm{dR}}(W) = \dim_{\mathbb{Q}_\ell} W = d[K : \mathbb{Q}_\ell]$ ($B^D_{\mathrm{dR}} = F_v$), so $\dim_K D_{\mathrm{dR}}(W) = n_v d$. Hence $c_f = h^0 + n_v d - d^0(W)$, and Step 3 gives $\operatorname{length}(Y_a/F') = a\, d^0(W) + \operatorname{length} H^2(D, M)$. Dividing by $e_{\mathcal{O}}$ gives item 1.

Item 2. By Theorem B.29(2) with $i = 2$, $\#H^2(D, M) = \#H^0(D, M^\vee)$. Since Λ is free over $\mathbb{Z}_\ell$, $\mathrm{Hom}_{\mathbb{Z}_\ell}(\Lambda, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) = W'/\Lambda' = A'$, and $M^\vee = \mathrm{Hom}(\Lambda/\varpi^a\Lambda, \mu_{\ell^r})$ is the submodule of homomorphisms vanishing on $\varpi^a\Lambda$, i.e. $A'[\varpi^a]$ for the $\mathcal{O}$-structure of the statement; the identification is D-equivariant. So $H^0(D, M^\vee) = H^0(D, A')[\varpi^a]$; both $H^2(D, M)$ and this group are finite $\mathcal{O}$-modules ($M^\vee$ is an $\mathcal{O}$-module through M), and equal orders give equal lengths. If $H^0(D, W') = 0$, the sequence $0 = H^0(D, W') \to H^0(D, A') \xrightarrow{\partial} H^1(D, \Lambda')$ of $0 \to \Lambda' \to W' \to A' \to 0$ embeds the torsion module $H^0(D, A')$ into the torsion of $H^1(D, \Lambda')$, which is finite by Theorem B.30 (Λ' is a finitely generated $\mathcal{O}$-module with continuous action).

Item 3. $d^0(W) \le \dim_K D_{\mathrm{dR}}(W) = n_v d$ (Step 4). If K contains every $\tau(F_v)$, then $F_v \otimes_{\mathbb{Q}_\ell} K \cong \prod_\tau K$ ($F_v/\mathbb{Q}_\ell$ is separable), with idempotents e_τ; $D_{\mathrm{dR}}(W)$ and each $\mathrm{Fil}^a D_{\mathrm{dR}}(W)$ are $F_v \otimes K$-modules, so $\mathrm{Fil}^a D_{\mathrm{dR}}(W) = \bigoplus_\tau e_\tau \mathrm{Fil}^a D_{\mathrm{dR}}(W)$ and $e_\tau \mathrm{Fil}^a D_{\mathrm{dR}}(W) = \mathrm{Fil}^a D_{\mathrm{dR},\tau}(W)$. The filtration of the finite-dimensional space $D_{\mathrm{dR},\tau}(W)$ is decreasing, with $\mathrm{Fil}^a = 0$ for large a (as $\bigcap_a t^a B^+_{\mathrm{dR}} = 0$), so $\dim_K \mathrm{Fil}^0 D_{\mathrm{dR},\tau}(W) = \sum_{a \ge 0} \dim_K(\mathrm{Fil}^a/\mathrm{Fil}^{a+1})$, and summing over τ gives the formula. □

Corollary 25.6. *Let E be a number field, ℓ an odd prime, Galois Ribet data over E with $t+1$ blocks having (Gd1) and (Gd2), $K = \mathcal{O}[\ell^{-1}]$, and v a place of E above ℓ, $D = G_{E_v}$, $n_v = [E_v : \mathbb{Q}_\ell]$, such that: $E_v \subseteq E_{v,0}(\mu_\ell)$, $E_{v,0}$ the maximal unramified subextension of $E_v/\mathbb{Q}_\ell$; K contains $\sigma(E_{v,0}(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $E_{v,0}(\mu_\ell)$ into $\overline{\mathbb{Q}}_\ell$; (Ll1) and (Ll2) hold; and $M_0, h_0 \ge 0$ are integers with $J(V_{\mathcal{N}}) \subseteq [-M_0, h_0]$. Then for $1 \le k \le t$ and every $x \in X_k$,*

$$\ell^{\,c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)}\, x|_D \in H^1_f(D, \mathrm{Cs}_k(u)),$$

and the image of X_k in $H^1(D, \mathrm{Cs}_k(u))/F_k(v)$ has normalised length at most $c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)\, n_v + w_k(v)$. If moreover $H^0(D, W_0') = 0$ and $e_0 \ge 0$ is an integer with $\ell^{e_0} H^1_f(D, \mathcal{N}_{\mathcal{E}}) \subseteq \mathrm{pr}_{1*} H^1_f(D, \widetilde{T}_\psi)$ for every $\mathcal{O}$-linear D-equivariant $\psi\colon \mathcal{N}_{\mathcal{E}} \to \mathrm{Cs}_0(u)$, then (Gd7) holds at v with

$$\lambda_0(v) = e_0\, d^0(W_0) + h_0(v), \qquad \lambda_k(v) = c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)\, n_v + w_k(v) \quad (1 \le k \le t).$$

Level s. *More generally, let ℓ be a prime and $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, and replace the two field conditions above by: $E_v \subseteq E_{v,0}(\mu_{\ell^s})$, and K contains $\sigma(E_{v,0}(\mu_{\ell^s}))$ for every $\mathbb{Q}_\ell$-embedding σ of $E_{v,0}(\mu_{\ell^s})$ into $\overline{\mathbb{Q}}_\ell$; keep the other hypotheses. Then all assertions above hold with $c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)$ replaced by $c_{\mathrm{B},s}(V_{\mathcal{N}}, k-1)$ of Proposition 25.8 (paragraph Level s, with $p = \ell$). For $s = 1$ and ℓ odd this is the statement above.* ***Descent.*** *Let ℓ be a prime, $N \geq 1$ an integer with $4 \mid N$ if $\ell = 2$, $s = \max(1, v_\ell(N))$ and $N_\ell = \mathrm{lcm}(\ell, N)$, and replace the two field conditions by: E_v is $\mathbb{Q}_\ell$-isomorphic to a subfield of $\mathbb{Q}_\ell(\mu_{N_\ell})$, and K contains a primitive N_ℓ-th root of unity; keep the other hypotheses. Then all assertions above hold with $c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)$ replaced by $c_{\mathrm{B},s}(V_{\mathcal{N}}, k-1) + s - 1$.*

Proof. We prove the first assertion and the assertion of the paragraph Level s together. In the first case put $s = 1$ and $c_k = c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)$; in the second put $c_k = c_{\mathrm{B},s}(V_{\mathcal{N}}, k-1)$ $(1 \leq k \leq t)$. In both cases $s \geq 2$ if $\ell = 2$, $E_v \subseteq E_{v,0}(\mu_{\ell^s})$ and K contains $\sigma(E_{v,0}(\mu_{\ell^s}))$ for every $\mathbb{Q}_\ell$-embedding σ; only Step 2 uses these conditions, and no step uses that ℓ is odd otherwise.

Step 1: the crystalline extension carrying the Ribet cocycle. By Theorem 25.3 ((Ll1), (Gd1), (Gd2)), $V_{\mathcal{N}}$ is crystalline, $\mathcal{N}_{\mathcal{E}}$ is a D-stable lattice in it, and $[y|_D] \in H^1_f(D, \mathcal{N}_{\mathcal{E}})$, i.e. the class of $y|_D$ in $H^1(D, V_{\mathcal{N}})$ lies in $H^1_f(D, V_{\mathcal{N}})$: there is $y_B \in V_{\mathcal{N}} \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $y(g) \otimes 1 = g y_B - y_B$ for $g \in D$. Let $Y = V_{\mathcal{N}} \oplus K$ with $g(n, a) = (u(g)n + a\, y(g), a)$; as y is a cocycle for the action $n \mapsto u(g)n$ $(y(gh) = (u(gh) - 1)\mathcal{E} = u(g)y(h) + y(g))$, this is a continuous K-linear action, $T_Y = \mathcal{N}_{\mathcal{E}} \oplus \mathcal{O}$ is a D-stable lattice $(y(g) \in \mathcal{N}_{\mathcal{E}})$, $\beta(n, a) = a$ is an invariant form with $\beta(T_Y) = \mathcal{O}$, $\ker \beta = V_{\mathcal{N}} \oplus 0$ with $T_Y \cap \ker \beta = \mathcal{N}_{\mathcal{E}} \oplus 0$, and $y_0 = (0, 1)$ has $\beta(y_0) = 1$ and $g y_0 - y_0 = (y(g), 0)$. *Y is crystalline.* For $\alpha \in K$, $z_\alpha = (-\alpha y_B, \alpha \otimes 1) \in Y \otimes B_{\mathrm{cris}}$ satisfies $g z_\alpha = (-\alpha\, g y_B + \alpha\, y(g) \otimes 1, \alpha \otimes 1) = z_\alpha$. Write $D_{\mathrm{cris}}(X) = (X \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}})^D$, left exact in X. The second projection $D_{\mathrm{cris}}(Y) \to D_{\mathrm{cris}}(K) = K \otimes_{\mathbb{Q}_\ell} E_{v,0}$ has kernel $D_{\mathrm{cris}}(V_{\mathcal{N}})$ and its image contains every $\alpha \otimes 1$, hence is onto (an $E_{v,0}$-linear map). So $\dim_{E_{v,0}} D_{\mathrm{cris}}(Y) = \dim_{\mathbb{Q}_\ell} V_{\mathcal{N}} + [K : \mathbb{Q}_\ell] = \dim_{\mathbb{Q}_\ell} Y$.

Step 2: the character blocks. Let $1 \leq k \leq t$, $x \in X_k$ the class of c_φ, and $\psi = \varphi \circ \iota$. As in Definition 17.2, ψ is $\mathcal{O}$-linear with $\psi(u(g)n) = \rho_k(u(g))\psi(n) = \chi_{\mathrm{cyc}}(g)^{1-k}\psi(n)$ by (Gd1) (φ, ι are $\mathcal{R}$-linear and $\mathcal{R}$ acts on Cs_k through ρ_k), and $\mathrm{Cs}_k(u) = K/\mathcal{O}$ with this action is $L(-e)/\mathcal{O}_L(-e)$ for $L = K$, $e = k - 1$. Apply Proposition 25.8 (in the second case its paragraph Level s, with this s) with $p = \ell$, its K equal to E_v (so its K_0 is $E_{v,0}$; the inclusion into $E_{v,0}(\mu_{\ell^s})$ and the condition on $L = K$ are hypotheses here), Y, T_Y, β, y_0 of Step 1, $V = V_{\mathcal{N}}$ (de Rham, being crystalline), $T = \mathcal{N}_{\mathcal{E}}$, M_0, h_0, $e = k - 1$ and φ replaced by ψ (on T, gx is $u(g)x$). Since $\psi(g y_0 - y_0) = \psi(y(g)) = c_\varphi(g)$ for $g \in D$, it gives $\ell^{c_k} x|_D \in H^1_f(D, \mathrm{Cs}_k(u))$.

Step 3: block 0, under the additional hypotheses on $H^0(D, W_0')$ and e_0. For $x \in X_0$ the class of c_φ and $\psi = \varphi \circ \iota$, ψ is $\mathcal{O}$-linear and D-equivariant into $\mathrm{Cs}_0(u) = W_0/\Lambda_0$ and $x|_D = \psi_*[y|_D]$ with $[y|_D] \in H^1_f(D, \mathcal{N}_{\mathcal{E}})$ (Step 1); the hypothesis on e_0 is that of Lemma 25.4(2) for $T = \mathcal{N}_{\mathcal{E}}$, $W = W_0$, $\Lambda = \Lambda_0$, ψ, $e = e_0$, which gives $\ell^{e_0} x|_D \in H^1_f(D, \mathrm{Cs}_0(u))$.

Step 4: the losses. For $1 \leq k \leq t$, Step 2 is the hypothesis of Theorem 25.7(1) for the block k with this c_k, and by Theorem 25.7(3), $d^0(W_k) = n_v$ and $h_k(v) = w_k(v) < \infty$; so Theorem 25.7(1) bounds the normalised length of the image of X_k by $c_k n_v + w_k(v)$. Under the additional hypotheses, Step 3 is the hypothesis of Theorem 25.7(1) for the block 0 with $c_0 = e_0$, and $h_0(v) < \infty$ by Theorem 25.7(4) as $H^0(D, W_0') = 0$; so Theorem 25.7(1) bounds the image of X_0 by $e_0 d^0(W_0) + h_0(v)$, and (Gd7) holds at v with the stated $\lambda_k(v)$, $0 \leq k \leq t$.

Descent. Put $c_k = c_{\mathrm{B},s}(V_{\mathcal{N}}, k-1) + s - 1$. As $\mathbb{Q}_\ell(\mu_{N_\ell})$ is Galois over $\mathbb{Q}_\ell$, $E_v \subseteq \mathbb{Q}_\ell(\mu_{N_\ell})$ inside $\overline{\mathbb{Q}}_\ell$. Steps 1, 3 and 4 do not use the field conditions. In Step 2 apply the paragraph Descent of Proposition 25.8 (with $p = \ell$, this N, its K equal to E_v and its L equal to our K, which contains a primitive N_ℓ-th root of unity) instead of its paragraph Level s; with the same

data it gives $\ell^{c_k} x|_D \in H^1_f(D, \mathrm{Cs}_k(u))$, and Step 4 then gives the bound $c_k n_v + w_k(v)$ and (Gd7) with $\lambda_k(v) = c_k n_v + w_k(v)$ for $k \geq 1$. □

Theorem 25.7. *Let E be a number field, ℓ a prime, and Galois Ribet data over E with $t+1$ blocks having (Gd1) and (Gd2); put $K = \mathcal{O}[\ell^{-1}]$. Let v be a place of E above ℓ, $D = G_{E_v}$, $n_v = [E_v : \mathbb{Q}_\ell]$, and suppose (Ll1) and (Ll2). For $0 \leq k \leq t$ put*

$$h_k(v) = \mathrm{nl}\big(H^0(D, W_k'/\Lambda_k')\big) \in [0, \infty].$$

1. *Let $0 \leq k \leq t$ and let $c_k \geq 0$ be an integer such that $\ell^{c_k} x|_D \in H^1_f(D, \mathrm{Cs}_k(u))$ for every $x \in X_k$. If $h_k(v) < \infty$, then the image of X_k in $H^1(G_{E_v}, \mathrm{Cs}_k(u))/F_k(v)$ has normalised length at most*
$$\lambda_k(v) = c_k\, d^0(W_k) + h_k(v);$$
so (Gd7) holds at v with these local losses when such integers $c_0, \dots, c_t$ are given and all $h_k(v)$ are finite.

2. *If (Ll3) holds with integers $e_0, \dots, e_t$, the hypothesis of item 1 holds for every k with $c_k = e_k$.*

3. *For $1 \leq k \leq t$: $d^0(W_k) = n_v$, and $h_k(v) = w_k(v)$, where $w_k(v) \geq 0$ is the largest integer w with $\chi_{\mathrm{cyc}}(g)^k \equiv 1 \bmod \ell^w$ for all $g \in D$ (finite).*

4. *For $k = 0$: $d^0(W_0) \leq n_v\, n_{\mathcal{R},0}$; if K contains $\tau(E_v)$ for every $\mathbb{Q}_\ell$-embedding τ of E_v into $\overline{\mathbb{Q}}_\ell$ and, for each τ, at most μ_τ Hodge jumps $a \geq 0$ of W_0 at τ counted with multiplicity occur, then $d^0(W_0) \leq \sum_\tau \mu_\tau$; and $h_0(v) < \infty$ if $H^0(D, W_0') = 0$.*

In particular, under (Ll1)–(Ll3), $H^0(D, W_0') = 0$ and the jump condition of item 4, (Gd7) holds at v with $\lambda_0(v) = e_0 \sum_\tau \mu_\tau + h_0(v)$ and $\lambda_k(v) = e_k\, n_v + w_k(v)$ for $1 \leq k \leq t$.

Proof. **Step 1: the blocks are de Rham, and $F_k' = H^1_f(D, \mathrm{Cs}_k(u)) \subseteq F_k(v)$.** By Theorem 25.3, under (Ll1) and (Gd1), (Gd2) every W_k is a crystalline representation of D, hence de Rham, and Λ_k is a D-stable lattice with $W_k/\Lambda_k = \mathrm{Cs}_k(u)$. For $k = 0$, $F_0' = F_0(v)$ by (Ll2). For $k = 1$, $\rho_1(u(g)) = \chi_1(g)^{-1}\chi_1(g) = 1$ by (Gd1), so $W_1 = K$ with trivial action; in a $\mathbb{Q}_\ell$-basis of K the condition of Definition 22.2 is componentwise, and by Theorem B.44 $H^1_f(D, K)$ consists of the homomorphisms $D \to K$ vanishing on the inertia group; their images in $H^1(D, K/\mathcal{O})$ vanish on inertia, so $F_1' \subseteq H^1_{\mathrm{ur}}(E_v, \mathrm{Cs}_1(u)) = F_1(v)$. For $k \geq 2$, $\rho_k(u(g)) = \chi_{\mathrm{cyc}}(g)^{1-k}$ by (Gd1), so W_k is a sum of copies of $\mathbb{Q}_\ell(s)$, $s = 1-k \leq -1$; for this line $H^0(D, \mathbb{Q}_\ell(s)) = 0$ ($\chi_{\mathrm{cyc}}(D)$ is infinite) and $D_{\mathrm{dR}}(\mathbb{Q}_\ell(s)) = E_v\, t^{-s} \otimes \epsilon$ (ϵ a basis; $b \otimes \epsilon$ is invariant if and only if $bt^s \in B_{\mathrm{dR}}^D = E_v$), which lies in $\mathrm{Fil}^{-s} \subseteq \mathrm{Fil}^0$; so Theorem B.43 gives $H^1_f(D, \mathbb{Q}_\ell(s)) = 0$, hence $F_k' = 0 = F_k(v)$.

Step 2: item 1. Fix k with $h_k(v) < \infty$ and put $a = c_k e_{\mathcal{O}}$. Since $\ell^{c_k} = \varpi^a w$ with $w \in \mathcal{O}^\times$ and F_k' is an $\mathcal{O}$-module, the hypothesis gives $\varpi^a x|_D \in F_k'$, i.e. $x|_D \in Y_a$ for every $x \in X_k$, with Y_a formed for $W = W_k$, $\Lambda = \Lambda_k$. The image of X_k in $H^1(D, \mathrm{Cs}_k(u))/F_k'$ is therefore a submodule of Y_a/F_k', and its image in $H^1(D, \mathrm{Cs}_k(u))/F_k(v)$ is a quotient of it, because $F_k' \subseteq F_k(v)$ (Step 1). By Lemma 25.5(1),(2), applied with $F = E$, v, D, the coefficient field K, $W = W_k$ (de Rham, Step 1) and $\Lambda = \Lambda_k$,

$$\mathrm{nl}(Y_a/F_k') \leq \frac{a}{e_{\mathcal{O}}}\, d^0(W_k) + \mathrm{nl}\big(H^0(D, W_k'/\Lambda_k')\big) = c_k\, d^0(W_k) + h_k(v).$$

Lengths do not increase under passage to submodules and quotients; this is the bound.

Step 3: item 2. Let $x \in X_k$ be the class of c_φ, $\varphi \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathrm{Cs}_k)$, and $\psi = \varphi \circ \iota \colon \mathcal{N}_{\mathcal{E}} \to \mathrm{Cs}_k$. Both maps are $\mathcal{R}$-linear and $\mathcal{R}$ acts on Cs_k through ρ_k, so $\psi(u(g)n) = \rho_k(u(g))\psi(n)$: ψ

is $\mathcal{O}$-linear and D-equivariant into $\mathrm{Cs}_k(u) = W_k/\Lambda_k$, and $c_\varphi = \psi \circ y$, so $x|_D = \psi_*[y|_D]$. By Theorem 25.3 (which uses (Ll1), (Gd1), (Gd2)), $V_{\mathcal{N}}$ is crystalline, $\mathcal{N}_{\mathcal{E}}$ is a D-stable lattice in it and $[y|_D] \in H^1_f(D, \mathcal{N}_{\mathcal{E}})$. (Ll3) is the hypothesis of Lemma 25.4(2) for $T = \mathcal{N}_{\mathcal{E}} \subset V = V_{\mathcal{N}}$, $\Lambda = \Lambda_k \subset W = W_k$, ψ and $e = e_k$; it gives $\ell^{e_k}\psi_*[y|_D] \in H^1_f(D, \mathrm{Cs}_k(u))$.

Step 4: item 3. Let $1 \le k \le t$. $W_k = K$ with g acting by $\chi_{\mathrm{cyc}}(g)^{1-k}$ (Step 1). $D_{\mathrm{dR}}(W_k) = D_{\mathrm{dR}}(\mathbb{Q}_\ell(1-k)) \otimes_{\mathbb{Q}_\ell} K$ has K-dimension n_v and lies in Fil^0 (Step 1, also for $k = 1$: $D_{\mathrm{dR}}(K) = E_v \otimes K \subseteq \mathrm{Fil}^0$), so $d^0(W_k) = n_v$. For $f \in \Lambda'_k = \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathcal{O}, \mathbb{Z}_\ell(1))$ and $g \in D$, $(gf)(x) = \chi_{\mathrm{cyc}}(g)\, f(\chi_{\mathrm{cyc}}(g)^{k-1}x) = \chi_{\mathrm{cyc}}(g)^k f(x)$, as f is $\mathbb{Z}_\ell$-linear; so $W'_k/\Lambda'_k \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{[K:\mathbb{Q}_\ell]}$ as a group, with g acting by the scalar $\chi_{\mathrm{cyc}}(g)^k$. An element is invariant if and only if each coordinate is killed by $\chi_{\mathrm{cyc}}(g)^k - 1$ for all $g \in D$, i.e. by ℓ^w, $w = w_k(v)$; $w_k(v)$ is finite since $\chi_{\mathrm{cyc}}(D)$ is an open subgroup of $\mathbb{Z}_\ell^\times$, which contains elements χ with $\chi^k \ne 1$. Hence $\#H^0(D, W'_k/\Lambda'_k) = \ell^{w[K:\mathbb{Q}_\ell]} = q^{w\, e_{\mathcal{O}}}$ with $q = \ell^{[K:\mathbb{Q}_\ell]/e_{\mathcal{O}}}$ the order of the residue field, so its length is $w\, e_{\mathcal{O}}$ and $h_k(v) = w_k(v)$.

Step 5: item 4. W_0 has dimension $n_{\mathcal{R},0}$; Lemma 25.5(3) gives $d^0(W_0) \le n_v n_{\mathcal{R},0}$ and, under the condition on K, $d^0(W_0) = \sum_\tau$(number of Hodge jumps $a \ge 0$ at τ with multiplicity) $\le \sum_\tau \mu_\tau$; Lemma 25.5(2) gives finiteness of $h_0(v)$ when $H^0(D, W'_0) = 0$. The final sentence combines items 1–4. □

25.2 Torsion classes with coefficients in truncations of A_{inf}

The first step towards Proposition 25.8 is Proposition 25.10: if a cocycle of $G_{K_0(\mu_\ell)}$ with values in $T \otimes \xi^a A_{\mathrm{inf}}/\xi^b A_{\mathrm{inf}}$, ξ a generator of $\ker\theta$, becomes a coboundary after multiplication by some power of ℓ, then it does so after multiplication by $\ell^{c_{\mathrm{LL}}(V;a,b)}$, an explicit power determined by the Hodge jumps. The proof decompletes: a crystalline lattice has an integral Sen frame at the level fixed by its Hodge window (Proposition 25.11, built from the Wach lattice of Proposition 25.12), and the twisted cohomology of a Sen frame has bounded torsion (Proposition 25.17), an integral form of the Tate–Sen method. The graded pieces of the truncations are Tate twists of $\mathcal{O}_{\mathbb{C}_\ell}$ (Lemma 25.21). The remaining lemmas are integral versions of standard facts: Frobenius integrality in $W(\mathbb{C}_\ell^\flat)$, Frobenius heights of the Breuil–Kisin–Fargues module, the normalised trace on the cyclotomic tower, and the almost vanishing of its first cohomology.

Proposition 25.8. *Let p be an odd prime, K a finite extension of $\mathbb{Q}_p$ with maximal unramified subextension K_0 such that $K \subseteq K_0(\mu_p)$, and L a finite extension of $\mathbb{Q}_p$ containing $\sigma(K_0(\mu_p))$ for every $\mathbb{Q}_p$-embedding σ of $K_0(\mu_p)$ into $\overline{\mathbb{Q}}_p$, with ring of integers $\mathcal{O}_L$. For a finite-dimensional continuous L-linear representation V of G_K, de Rham over $\mathbb{Q}_p$, let $J(V)$ be the set of integers a with $\mathrm{Fil}^{a+1}D_{\mathrm{dR}}(V) \ne \mathrm{Fil}^a D_{\mathrm{dR}}(V)$, $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$.*

Let Y be a crystalline L-linear representation of G_K with a G_K-stable $\mathcal{O}_L$-lattice T_Y, let $\beta\colon Y \to L$ be a G_K-invariant L-linear form with $\beta(T_Y) \subseteq \mathcal{O}_L$, $V = \ker\beta$ and $T = T_Y \cap V$, and let $y_0 \in T_Y$ with $\beta(y_0) = 1$. Let $M_0, h_0 \ge 0$ be integers with $J(V) \subseteq [-M_0, h_0]$, put $H = M_0 + h_0$, $n_0 = \min\{n \ge 0 :\ H \le p^n(p-1)\}$ and $m_N = n_0 + 3$. Let $e \ge 0$ be an integer, $\mathcal{O}_L(-e)$ the lattice $\mathcal{O}_L$ in $L(-e) = L$ with g acting by $\chi_{\mathrm{cyc}}(g)^{-e}$, and $\varphi\colon T \to L/\mathcal{O}_L$ an $\mathcal{O}_L$-linear map with $\varphi(gx) = \chi_{\mathrm{cyc}}(g)^{-e}\varphi(x)$ for $g \in G_K$, $x \in T$. Put

$$r_i = \sum_{u \in J(V),\, u \ne i} (m_N + v_p(u-i)) \quad (0 \le i \le e),$$

$$c_{\mathrm{B}}(V,e) = (m_N - 1) + \sum_{i=0}^{e}(2 + \max(1, r_i)) + (e+2) + v_p(e) + \Big\lfloor \frac{e}{p-1} \Big\rfloor + 2 + v_p(e+1).$$

Then $g \mapsto \varphi(gy_0 - y_0)$ is a continuous 1-cocycle of G_K with values in $L(-e)/\mathcal{O}_L(-e)$, and

$$p^{\,c_{\mathrm{B}}(V,e)}\,[g \mapsto \varphi(gy_0 - y_0)] \in H^1_f(G_K, L(-e)/\mathcal{O}_L(-e)).$$

Level s. *More generally, let p be a prime, $s \geq 1$ an integer with $s \geq 2$ if $p = 2$, K a finite extension of $\mathbb{Q}_p$ with maximal unramified subextension K_0 such that $K \subseteq K_s := K_0(\mu_{p^s})$, and L a finite extension of $\mathbb{Q}_p$ containing $\sigma(K_s)$ for every $\mathbb{Q}_p$-embedding σ of K_s, with ring of integers $\mathcal{O}_L$; let Y, T_Y, β, V, T, y_0, M_0, h_0, H, n_0, e and φ be as above, and put $m_{N,s} = 2s + n_0 + 1$,*

$$r_i^{(s)} = \sum_{u \in J(V),\, u \neq i} (m_{N,s} + v_p(u-i)) \quad (0 \leq i \leq e),$$

$$c_{\mathrm{B},s}(V,e) = (m_{N,s} - s) + \sum_{i=0}^{e} (2 + \max(1, r_i^{(s)})) + s(e+4) + v_p(e) + \Big\lfloor \frac{e}{p-1} \Big\rfloor + v_p(e+1).$$

Then $g \mapsto \varphi(gy_0 - y_0)$ is a continuous 1-cocycle of G_K with values in $L(-e)/\mathcal{O}_L(-e)$, and $p^{c_{\mathrm{B},s}(V,e)}[g \mapsto \varphi(gy_0 - y_0)] \in H^1_f(G_K, L(-e)/\mathcal{O}_L(-e))$. For $s = 1$ and p odd, $m_{N,1} = m_N$, $r_i^{(1)} = r_i$, $c_{\mathrm{B},1} = c_{\mathrm{B}}$, and this is the statement above.

Descent. *Let p be a prime, $N \geq 1$ an integer with $4 \mid N$ if $p = 2$, $s = \max(1, v_p(N))$, $N_p = \mathrm{lcm}(p, N)$, K a field with $\mathbb{Q}_p \subseteq K \subseteq \mathbb{Q}_p(\mu_{N_p})$ inside $\overline{\mathbb{Q}}_p$, and L a finite extension of $\mathbb{Q}_p$ containing a primitive N_p-th root of unity, with ring of integers $\mathcal{O}_L$; let Y, T_Y, β, V, T, y_0, M_0, h_0, e and φ be as above, for G_K. Then $p^{c_{\mathrm{B},s}(V,e)+s-1}[g \mapsto \varphi(gy_0 - y_0)] \in H^1_f(G_K, L(-e)/\mathcal{O}_L(-e))$.*

Proof. We prove the assertion of the paragraph Level s; the first assertion is its case $s = 1$, p odd (the hypotheses agree with $K_1 = K_0(\mu_p)$, and $c_{\mathrm{B},1}(V,e) = (m_N - 1) + \sum_{i=0}^{e}(2 + \max(1, r_i)) + (e+4) + v_p(e) + \lfloor e/(p-1) \rfloor + v_p(e+1) = c_{\mathrm{B}}(V,e)$). So let p, s, $K \subseteq K_s$, L and the data be as there.

The cocycle. As β is G_K-invariant, $\beta(gy_0 - y_0) = 0$, so $f(g) := gy_0 - y_0$ lies in $T_Y \cap V = T$; $f(gh) = g(hy_0 - y_0) + (gy_0 - y_0) = f(g) + gf(h)$, and f is continuous because G_K acts continuously on the finite free $\mathbb{Z}_p$-module T_Y. The image of φ is a finitely generated submodule of $L/\mathcal{O}_L$, hence killed by some p^n, so φ factors through T/p^nT and is continuous; with $\varphi(gx) = \chi_{\mathrm{cyc}}(g)^{-e}\varphi(x)$ this shows that $g \mapsto \varphi(f(g))$ is a continuous 1-cocycle with values in $L(-e)/\mathcal{O}_L(-e)$. Also $T = T_Y \cap V$ is a G_K-stable $\mathcal{O}_L$-lattice in V, and V is crystalline by Theorem B.45 (a G_K-stable subspace of the crystalline Y).

The class is torsion modulo ξ^{e+1}. Put $b = e + 1$ and $Q = A_{\mathrm{inf}}/\xi^b A_{\mathrm{inf}}$. The maximal unramified subfield of K_s is K_0 (K_s/K_0 is totally ramified), so $\dim_{K_0}(B_{\mathrm{cris}} \otimes Y)^{G_{K_s}} \geq \dim_{K_0}(B_{\mathrm{cris}} \otimes Y)^{G_K} = \dim_{\mathbb{Q}_p} Y$, and Y is crystalline, hence de Rham [52], as a representation of G_{K_s}; so is the trivial representation L. On de Rham representations $\mathrm{Fil}^0 D_{\mathrm{dR}}$ is exact [52], so the map $(Y \otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}})^{G_{K_s}} \to (L \otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}})^{G_{K_s}}$ induced by β is onto; choose y^+ mapping to $1 \otimes 1$ and let $\bar{y}^+$ be its image modulo ξ^b. Since $B^+_{\mathrm{dR}}/\xi^b B^+_{\mathrm{dR}} = Q[1/p]$, the element $m = y_0 \otimes 1 - \bar{y}^+$ of $Y \otimes_{\mathbb{Q}_p} Q[1/p]$ lies in the kernel $V \otimes_{\mathbb{Q}_p} Q[1/p]$ of $\beta \otimes 1$, and $(g-1)m = f(g) \otimes 1$ for $g \in G_{K_s}$, as $\bar{y}^+$ is invariant. Now Q has no p-torsion (it is filtered by the $\xi^i A_{\mathrm{inf}}/\xi^{i+1} A_{\mathrm{inf}} \cong \mathcal{O}_{\mathbb{C}_p}$, ξ being a non-zero-divisor and θ inducing $A_{\mathrm{inf}}/\xi \cong \mathcal{O}_{\mathbb{C}_p}$ [52]), and T is free over $\mathbb{Z}_p$, so $T \otimes_{\mathbb{Z}_p} Q$ embeds into $V \otimes_{\mathbb{Q}_p} Q[1/p]$; m is a finite sum of tensors, so $p^k m \in T \otimes_{\mathbb{Z}_p} Q$ for some $k \geq 0$, and $p^k(f(g) \otimes 1) = (g-1)(p^k m)$ in $T \otimes_{\mathbb{Z}_p} Q$. Thus $z = f|_{G_{K_s}} \otimes 1$, a continuous 1-cocycle of G_{K_s} with values in $T \otimes_{\mathbb{Z}_p} Q$, has $p^k z$ a coboundary.

The two exponents. By Proposition 25.10, paragraph Level s (with V, T, M_0, h_0, $a = 0$, $b = e + 1$; its hypotheses are ours), $p^N z$ is a coboundary for $N = c_{\mathrm{LL},s}(V; 0, e+1) = (m_{N,s} - s) + \sum_{i=0}^{e}(2 + \max(1, r_i^{(s)}))$. By Proposition 25.23, paragraph Level s (with $K \subseteq K_s$, L, T, e, φ, f and this N),

$$p^{N + s(e+4) + v_p(e) + \lfloor e/(p-1) \rfloor + v_p(e+1)}[g \mapsto \varphi(f(g))] \in H^1_f(G_K, L(-e)/\mathcal{O}_L(-e)),$$

and the exponent is $c_{\mathrm{B},s}(V,e)$.

Descent. Let the data be as in the paragraph Descent of the statement, and $K' = K(\mu_{p^s})$. By Lemma 25.32, $K' = K'_0(\mu_{p^s})$ with K'_0 the maximal unramified subextension of $K'/\mathbb{Q}_p$, $v_p([K' : K]) \le s-1$, and $s \ge 2$ if $p = 2$; and L contains $\sigma(K')$ for every $\mathbb{Q}_p$-embedding σ of K', because $\sigma(K') \subseteq \mathbb{Q}_p(\mu_{N_p}) \subseteq L$ ($\mathbb{Q}_p(\mu_{N_p})$ is Galois over $\mathbb{Q}_p$). Restrict all data to $G_{K'}$: Y is crystalline over K', β is $G_{K'}$-invariant, $\varphi(gx) = \chi_{\mathrm{cyc}}(g)^{-e}\varphi(x)$ for $g \in G_{K'}$, and $J(V|_{G_{K'}}) = J(V)$: for the de Rham representation V the natural map $K' \otimes_K D_{\mathrm{dR},K}(V) \to D_{\mathrm{dR},K'}(V)$ is an isomorphism of filtered K'-vector spaces [94], and K' is faithfully flat over K, so $\mathrm{Fil}^{a+1} \ne \mathrm{Fil}^a$ over K if and only if over K'. So M_0, h_0, H, n_0 and $c_{\mathrm{B},s}(V,e)$ are the same over K', and the paragraph Level s over K' (with K', K'_0 in the roles of K, K_0) gives $p^c\,\mathrm{res}(x) \in H^1_f(G_{K'}, L(-e)/\mathcal{O}_L(-e))$, where x is the class of $g \mapsto \varphi(gy_0 - y_0)$ on G_K, $\mathrm{res}(x)$ the class of its restriction, and $c = c_{\mathrm{B},s}(V,e)$. By item 1 of Lemma 25.33 (for $E = K$, K', $V = L(-e)$, $T = \mathcal{O}_L(-e)$; the group $H^1_f(G_K, L(-e)/\mathcal{O}_L(-e))$ is the image of $H^1_f(G_K, L(-e))$, as in Lemma 25.4), $[K' : K]\,p^c x = \mathrm{cor}(p^c \mathrm{res}(x)) \in H^1_f(G_K, L(-e)/\mathcal{O}_L(-e))$. Write $[K' : K] = p^v u$ with $v \le s-1$ and u prime to p; $H^1_f(G_K, L(-e)/\mathcal{O}_L(-e))$ is an $\mathcal{O}_L$-submodule, so $p^{c+v}x$, and hence $p^{c+s-1}x$, lies in it. □

Definition 25.9. Let p be a prime and $K_0 \subset \overline{\mathbb{Q}}_p$ a finite unramified extension of $\mathbb{Q}_p$. Write $\mathcal{O}_{\mathbb{C}_p}$ for the valuation ring of $\mathbb{C}_p$, $|\cdot|$ for its absolute value with $|p| = p^{-1}$, $\mathcal{O}_F$ for the valuation ring of a subfield $F \subseteq \mathbb{C}_p$, $K_0(\mu_{p^\infty}) = \bigcup_{n\ge 0} K_0(\mu_{p^n})$ and $H_{K_0} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K_0(\mu_{p^\infty}))$. A Galois group acts on $\mathbb{C}_p^r$ and on matrices over $\mathbb{C}_p$ entrywise.

Let $m \ge 1$ and $r \ge 0$ be integers, with $m \ge 2$ if $p = 2$, and $J \subset \mathbb{Z}$ a finite set. A **Sen frame** of level m and rank r over K_0 with weights in J is a pair (U, g) of a map $U\colon G_{K_0(\mu_{p^m})} \to M_r(\mathcal{O}_{K_0(\mu_{p^m})})$ and an element $g \in G_{K_0(\mu_{p^m})}$ such that:

(Sf1) U is continuous, $U(\sigma\tau) = U(\sigma)U(\tau)$ for $\sigma, \tau \in G_{K_0(\mu_{p^m})}$, and $U(h) = 1$ for $h \in H_{K_0}$;

(Sf2) $v_p(\chi_{\mathrm{cyc}}(g) - 1) = m$, and $|U(g)_{ab} - \delta_{ab}| < p^{-1/(p-1)}$ for all entries;

(Sf3) there are a $\mathbb{C}_p$-basis $e_1, \dots, e_r$ of $\mathbb{C}_p^r$ and integers $u_1, \dots, u_r \in J$ with

$$\chi_{\mathrm{cyc}}(\sigma)^{u_k}\, U(\sigma)\, \sigma(e_k) = e_k \qquad (\sigma \in G_{K_0(\mu_{p^m})},\ 1 \le k \le r).$$

For $i \in \mathbb{Z}$, the **i-twisted action** of $G_{K_0(\mu_{p^m})}$ on $\mathbb{C}_p^r$ attached to U is

$$\sigma *_i y = \chi_{\mathrm{cyc}}(\sigma)^i\, U(\sigma)\, \sigma(y) \qquad (\sigma \in G_{K_0(\mu_{p^m})},\ y \in \mathbb{C}_p^r). \tag{25.2}$$

Proposition 25.10. *Let p be an odd prime, K a finite extension of $\mathbb{Q}_p$ with maximal unramified subextension K_0 such that $K \subseteq K_1 := K_0(\mu_p)$, and L a finite extension of $\mathbb{Q}_p$ containing $\sigma(K_1)$ for every $\mathbb{Q}_p$-embedding σ of K_1, with ring of integers $\mathcal{O}_L$. Let V be a crystalline L-linear representation of G_K with a G_K-stable $\mathcal{O}_L$-lattice T, and $M_0, h_0 \ge 0$ integers with $J(V) \subseteq [-M_0, h_0]$; let m_N and r_i ($i \in \mathbb{Z}$) be formed as in Proposition 25.8. Let ξ be a generator of $\ker\theta$, $0 \le a < b$ integers, and $Q_{a,b} = \xi^a A_{\mathrm{inf}}/\xi^b A_{\mathrm{inf}}$, with the action of G_K induced from A_{inf}; give $T \otimes_{\mathbb{Z}_p} Q_{a,b}$ the diagonal action of G_{K_1} and the p-adic topology. Put*

$$c_{\mathrm{LL}}(V; a, b) = (m_N - 1) + \sum_{i=a}^{b-1} \bigl(2 + \max(1, r_i)\bigr).$$

If z is a continuous 1-cocycle of G_{K_1} with values in $T \otimes_{\mathbb{Z}_p} Q_{a,b}$ and $p^k z$ is a coboundary for some $k \ge 0$, then $p^{c_{\mathrm{LL}}(V;a,b)} z$ is a coboundary. ***Level s.*** *More generally, let p be a prime,*

$s \geq 1$ an integer with $s \geq 2$ if $p = 2$, K a finite extension of $\mathbb{Q}_p$ with maximal unramified subextension K_0 such that $K \subseteq K_s := K_0(\mu_{p^s})$, L a finite extension of $\mathbb{Q}_p$ containing $\sigma(K_s)$ for every $\mathbb{Q}_p$-embedding σ of K_s, and V, T, M_0, h_0, ξ, $0 \leq a < b$, $Q_{a,b}$ as above. Let $m_{N,s}$ be as in Proposition 25.36 and put

$$r_i^{(s)} = \sum_{u \in J(V),\, u \neq i} (m_{N,s} + v_p(u-i)) \quad (i \in \mathbb{Z}), \qquad c_{\mathrm{LL},s}(V; a, b) = (m_{N,s} - s) + \sum_{i=a}^{b-1} (2 + \max(1, r_i^{(s)})).$$

If z is a continuous 1-cocycle of G_{K_s} with values in $T \otimes_{\mathbb{Z}_p} Q_{a,b}$ (diagonal action, p-adic topology) and $p^k z$ is a coboundary for some $k \geq 0$, then $p^{c_{\mathrm{LL},s}(V;a,b)} z$ is a coboundary. For $s = 1$ and p odd, $m_{N,1} = m_N$, $r_i^{(1)} = r_i$, $c_{\mathrm{LL},1} = c_{\mathrm{LL}}$, and this is the statement above.

Proof. We prove the assertion of the paragraph Level s; the first assertion is its case $s = 1$, p odd (the hypotheses agree, and $m_{N,1} = n_0 + 3 = m_N$, so $r_i^{(1)} = r_i$ and $c_{\mathrm{LL},1} = c_{\mathrm{LL}}$). So let p, s, $K \subseteq K_s$, L, V, T, $a < b$ be as there. Write $m = m_{N,s} = 2s + n_0 + 1$, $K_n = K_0(\mu_{p^n})$ and $G_n = G_{K_n}$; thus $m \geq s$ and $G_m \subseteq G_s \subseteq G_K$, and $m \geq 3$ ($m \geq 5$ if $p = 2$). T is free of finite rank over $\mathbb{Z}_p$ (free over $\mathcal{O}_L$, which is free over $\mathbb{Z}_p$); fix a $\mathbb{Z}_p$-basis $b_1, \ldots, b_d$ and let $M(\sigma) \in \mathrm{GL}_d(\mathbb{Z}_p)$ be the matrix of $\sigma \in G_K$. Put $X_{a,b} = T \otimes_{\mathbb{Z}_p} Q_{a,b}$; in coordinates $X_{a,b} = Q_{a,b}^d$ with σ acting by $y \mapsto M(\sigma)\sigma(y)$, and the p-adic topology is the product topology. For a closed subgroup $G' \subseteq G_K$ and $c \geq 0$ we say that $X_{a,b}$ *has exponent c over G'* if for every continuous 1-cocycle $f\colon G' \to X_{a,b}$ such that $p^k f$ is a coboundary for some k, $p^c f$ is a coboundary; δx denotes the coboundary $\sigma \mapsto \sigma x - x$. By Lemma 25.21(2) applied coordinatewise and continuity of M, $\sigma \mapsto \sigma x$ is continuous for every $x \in X_{a,b}$, so coboundaries are continuous. Moreover V is crystalline, hence de Rham [52], as a representation of G_m: the maximal unramified subfield of K_m is K_0 and $\dim_{K_0}(B_{\mathrm{cris}} \otimes V)^{G_m} \geq \dim_{K_0}(B_{\mathrm{cris}} \otimes V)^{G_K} = \dim_{\mathbb{Q}_p} V$. We show that $X_{a,b}$ has exponent $c_{\mathrm{LL},s}(V; a, b)$ over G_s, which is the claim.

Step 1: the graded pieces have exponent $c_i = 2 + \max(1, r_i^{(s)})$ over G_m. Let $i \geq 0$. The map $\Theta = \vartheta_i^{\oplus d}\colon X_{i,i+1} \to \mathcal{O}_{\mathbb{C}_p}^d$ is an additive bijection with $\Theta(p^n x) = p^n \Theta(x)$, hence a homeomorphism for the p-adic topologies, and since $G_m \subseteq G_{\mathbb{Q}_p(\mu_p)}$ and $M(\sigma)$ has entries in $\mathbb{Z}_p$, Lemma 25.21(3) gives $\Theta(\sigma x) = \chi_{\mathrm{cyc}}(\sigma)^i M(\sigma)\sigma(\Theta(x))$ for $\sigma \in G_m$. So it suffices to treat continuous cocycles $f\colon G_m \to \mathcal{O}_{\mathbb{C}_p}^d$ for the action $\sigma \cdot y = \chi_{\mathrm{cyc}}(\sigma)^i M(\sigma)\sigma(y)$ with $p^k f = \delta x$, $x \in \mathcal{O}_{\mathbb{C}_p}^d$. Take D and the Sen frame (U, g) of level $m = m_{N,s}$ and rank d with weights in $J(V)$ from Proposition 25.11 (its hypotheses are ours: p, s, $K \subseteq K_s$, L, V, T, M_0, h_0). For $y \in \mathbb{C}_p^d$ and $\sigma \in G_m$,

$$\chi_{\mathrm{cyc}}(\sigma)^i M(\sigma)\sigma(Dy) = \chi_{\mathrm{cyc}}(\sigma)^i D\, U(\sigma)\sigma(y) = D\,(\sigma *_i y).$$

Hence $f' = pD^{-1}f$ is a continuous map into $\mathcal{O}_{\mathbb{C}_p}^d$ (pD^{-1} is integral) with $f'(\sigma\tau) = f'(\sigma) + pD^{-1}D(\sigma *_i (D^{-1}f(\tau))) = f'(\sigma) + \sigma *_i f'(\tau)$, a cocycle for $*_i$, and $p^k f'$ is the coboundary for $*_i$ of $pD^{-1}x \in \mathcal{O}_{\mathbb{C}_p}^d$. By Proposition 25.17(1) (level m; $m \geq 2$ if $p = 2$), $p^{1+N} f'$ is the $*_i$-coboundary of some $z \in \mathcal{O}_{\mathbb{C}_p}^d$, where $N = N_{J(V),m}(i) = \max(1, r_i^{(s)})$. Applying D: $p^{2+N} f(\sigma) = D(\sigma *_i z - z) = \sigma \cdot (Dz) - Dz$ with $Dz \in \mathcal{O}_{\mathbb{C}_p}^d$ (D is integral).

Step 2: $X_{a,b}$ has exponent $\sum_{i=a}^{b-1} c_i$ over G_m. By induction on $b - a$; $b = a + 1$ is Step 1. Assume it for $X_{a,b}$ and put $A = X_{b,b+1}$, $B = X_{a,b+1}$, $C = X_{a,b}$, $c_C = \sum_{i=a}^{b-1} c_i$. By Lemma 25.21(1) and freeness of T, $0 \to A \to B \xrightarrow{\pi} C \to 0$ is exact and equivariant, and $p^n B \cap A = p^n A$, so A carries the subspace topology. Let $f\colon G_m \to B$ be a continuous cocycle with $p^k f = \delta x$. Then πf is a continuous cocycle with $p^k \pi f = \delta(\pi x)$, so $p^{c_C} \pi f = \delta y$ for some $y \in C$; lift y to $\tilde{y} \in B$. The cocycle $f_1 = p^{c_C} f - \delta\tilde{y}$ is continuous and has $\pi f_1 = 0$, so it is a continuous cocycle with values in A. Next, $p^k f_1 = \delta w$ with $w = p^{c_C} x - p^k \tilde{y} \in B$, and $z = \pi(w)$

satisfies $\sigma z - z = \pi(p^k f_1(\sigma)) = 0$; by Lemma 25.22 (for $F = K_m$, the de Rham representation V of G_m, the lattice T and $a \le b \le b+1$) there are $k' \ge 0$ and $x' \in B$ fixed by G_m with $\pi(x') = p^{k'} z$. Then $a_0 = p^{k'} w - x' \in \ker \pi = A$ and $\delta a_0 = p^{k'} \delta w = p^{k+k'} f_1$. By Step 1 for A, $p^{c_b} f_1 = \delta a_1$ with $a_1 \in A$, and

$$p^{c_b + c_C} f = p^{c_b}(f_1 + \delta \tilde{y}) = \delta(a_1 + p^{c_b} \tilde{y}).$$

Step 3: descent from G_m to G_s. Since K_n/K_0 has degree $p^{n-1}(p-1)$ for $n \ge 1$ ($\Phi_{p^n}(X+1)$ is Eisenstein over $\mathcal{O}_{K_0}$, for every prime p), G_m is a normal subgroup of G_s of index $n_1 = [K_m : K_s] = p^{m-s}$; let S be a set of representatives of G_s/G_m. Let $f\colon G_s \to X_{a,b}$ be a continuous cocycle with $p^k f = \delta x$, and $c = \sum_{i=a}^{b-1} c_i$. By Step 2 applied to $f|_{G_m}$, $F = p^c f$ satisfies $F(h) = hy - y$ for $h \in G_m$, with $y \in X_{a,b}$. For $\sigma \in G_s$ and $s' \in S$ write $\sigma s' = s'' h$ with $s'' \in S$, $h \in G_m$; as s' runs through S so does s''. From $F(\sigma s') = F(\sigma) + \sigma F(s')$ and $F(s''h) = F(s'') + s''hy - s''y = F(s'') + \sigma s' y - s'' y$, summing over $s' \in S$,

$$n_1 F(\sigma) = \sum_{s''} F(s'') - \sigma \sum_{s'} F(s') + \sigma \sum_{s'} s'y - \sum_{s''} s''y = \sigma w - w, \qquad w = \sum_{s' \in S} \bigl(s'y - F(s')\bigr) \in X_{a,b}.$$

So $p^{(m-s)+c} f = n_1 F$ is a coboundary, and $(m-s) + c = (m_{N,s} - s) + \sum_{i=a}^{b-1} (2 + \max(1, r_i^{(s)})) = c_{\mathrm{LL},s}(V; a, b)$. □

Proposition 25.11. *Let p, s, $K \subseteq K_0(\mu_{p^s})$, L, V, T, M_0, h_0 and $m_{N,s}$ be as in Proposition 25.36, let $b_1, \dots, b_d$ be a $\mathbb{Z}_p$-basis of T and $M(\sigma) \in \mathrm{GL}_d(\mathbb{Z}_p)$ the matrix of $\sigma \in G_K$ in it. Then $J(V)$ is finite, and there are a matrix $D \in \mathrm{GL}_d(\mathbb{C}_p)$ such that D and pD^{-1} have all entries in $\mathcal{O}_{\mathbb{C}_p}$, and a Sen frame (U, g) of level $m_{N,s}$ and rank d over K_0 with weights in $J(V)$, such that*

$$D\,U(\sigma) = M(\sigma)\,\sigma(D) \qquad \text{for all } \sigma \in G_{K_0(\mu_{p^{m_{N,s}}})}.$$

Proof. Put $H = M_0 + h_0$, $n = n_0 + 1$, $n' = n + s - 1 = n_0 + s$ and $m = m_{N,s} = 2s + n_0 + 1 = n' + s + 1$. Since $J(V) \subseteq [-M_0, h_0]$, $J(V)$ is finite. Elements of $\mathcal{O} \otimes_{\mathbb{Z}_p} T$ and of $\mathbb{C}_p \otimes_{\mathbb{Z}_p} T$ are written in the coordinates of $b_\bullet$, so that $\sigma \in G_K$ acts by $y \mapsto M(\sigma)\sigma(y)$. Let $n_1, \dots, n_d$ be as in Proposition 25.12 at level s (whose hypotheses are ours: $K \subseteq K_s$ and L contains $\sigma(K_s)$ for every embedding σ).

Step 1: the specialisation θ_n. The ring $\mathcal{O}_{\mathbb{C}_p^\flat}$ is perfect of characteristic p, so the Witt vector Frobenius φ of A_{inf} is a ring automorphism with $\varphi([x]) = [x^p]$, commuting with $G_{\mathbb{Q}_p}$ (functoriality of Witt vectors). Put $\theta_n = \theta \circ \varphi^{-n}$, a surjective $G_{\mathbb{Q}_p}$-equivariant ring homomorphism $A_{\mathrm{inf}} \to \mathcal{O}$ by (Ai1), and $c = \theta_n(\mu_\varepsilon) = \theta([\varepsilon^{1/p^n}]) - 1 = \zeta_{p^n} - 1$. As $\varphi^{-n}(I) \subseteq (p, [(p^\flat)^{1/p^n}])$ and θ maps this ideal into $p^{1/p^n}\mathcal{O}$, θ_n is continuous for the I-adic and p-adic topologies. On the image of $W(k)$, θ_n maps $\sum_i p^i [s(a_i)]$ (s the lift $k \to \mathcal{O}_{\mathbb{C}_p^\flat}$, $a_i \in k$) to $\sum_i p^i\, \omega(a_i^{1/p^n}) \in \mathcal{O}_{K_0}$, ω the Teichmüller representatives of k in $\mathcal{O}_{K_0}$, since $s(a)^\sharp = \omega(a)$ [52]. Hence, by continuity, $\theta_n(\sum_k a_k \mu_\varepsilon^k) = \sum_k \theta_n(a_k) c^k$ and

$$\theta_n(\mathbf{A}^+_{K_0}) \subseteq \mathcal{O}_{K_n},$$

because $\mathcal{O}_{K_n} = \mathcal{O}_{K_0}[c]$ is complete: $\Phi_{p^n}(X+1)$ is Eisenstein over $\mathcal{O}_{K_0}$, so K_n/K_0 is totally ramified of degree $e = p^{n-1}(p-1)$ with uniformiser c, and $p \in c^e \mathcal{O}^\times_{K_n}$. In the same way $c' = \theta_n(\mu_{\varepsilon,s}) = \theta([\varepsilon^{1/p^{n'}}]) - 1 = \zeta_{p^{n'}} - 1$ is a uniformiser of $K_{n'}$, $p \in c'^{e'} \mathcal{O}^\times_{K_{n'}}$ with $e' = p^{n'-1}(p-1)$, $\theta_n(\mathbf{A}^+_{K_0,s}) \subseteq \mathcal{O}_{K_{n'}} = \mathcal{O}_{K_0}[c']$, and $c = (1+c')^{p^{s-1}} - 1 \in c' \mathcal{O}_{K_{n'}}$; for $s = 1$, $c' = c$. Finally, for

$\sigma \in G_{K_s}$, $\sigma(\mu_\varepsilon) = \mu_\varepsilon w_\sigma$ with

$$w_\sigma = \sum_{k\geq 1} \binom{\chi_{\mathrm{cyc}}(\sigma)}{k} \mu_\varepsilon^{k-1} \in \mathbf{A}_{K_0}^+ :$$

the binomial series $[\varepsilon]^\beta = \sum_k \binom{\beta}{k} \mu_\varepsilon^k$ converges for $\beta \in \mathbb{Z}_p$ by (Ai2) ($\mu_\varepsilon \equiv [\varepsilon - 1]$ modulo p and $\varepsilon - 1$ has positive valuation, so a power of μ_ε lies in I), is continuous in β and is the usual power for $\beta \in \mathbb{Z}_{\geq 0}$; if integers $\alpha_j \geq 0$ tend to $\chi_{\mathrm{cyc}}(\sigma)$, then $\varepsilon^{\alpha_j} \to \sigma(\varepsilon)$ componentwise, so $\sigma([\varepsilon]) = [\sigma(\varepsilon)] = \lim_j [\varepsilon]^{\alpha_j} = [\varepsilon]^{\chi_{\mathrm{cyc}}(\sigma)}$ by continuity of $[\cdot]$ (Ai2). As $\chi_{\mathrm{cyc}}(\sigma) \in 1 + p\mathbb{Z}_p$ and $p \in c\mathcal{O}_{K_n}$, $\theta_n(w_\sigma) \equiv \chi_{\mathrm{cyc}}(\sigma) \equiv 1$ modulo $c\mathcal{O}_{K_n}$.

Step 2: the matrix D and the sandwich. By (Wl1), $d_j = (\theta_n \otimes 1)(\mu_\varepsilon^{M_0} n_j) \in \mathcal{O}^d$; let $D \in M_d(\mathcal{O})$ have columns $d_1, \ldots, d_d$. The map $\theta_n \otimes 1 \colon A_{\mathrm{inf}} \otimes T \to \mathcal{O} \otimes T$ is additive, θ_n-semilinear and G_K-equivariant. Multiplying (Wl2) by $\mu_\varepsilon^{M_0}$ and applying $\theta_n \otimes 1$ gives, for $t = b_a$, $c^H f_a \in \sum_j \mathcal{O}\, d_j$ (f_a the standard basis), i.e. $DX = c^H \cdot 1$ for some $X \in M_d(\mathcal{O})$. As $c \neq 0$, D is invertible with $D^{-1} = c^{-H} X$. By the definition of n_0, $H \leq p^{n_0}(p-1) = e$, so $v_p(c^H) = H/e \leq 1$, $p/c^H \in \mathcal{O}$, and $pD^{-1} = (p/c^H)X$ has entries in $\mathcal{O}$.

Step 3: the cocycle U. For $\sigma \in G_{K_s}$ put $U(\sigma) = D^{-1} M(\sigma) \sigma(D)$. By (Wl4), $\sigma(\mu_\varepsilon^{M_0} n_j) = w_\sigma^{M_0} \sum_i a(\sigma)_{ij}\, \mu_\varepsilon^{M_0} n_i$, and applying $\theta_n \otimes 1$,

$$M(\sigma)\, \sigma(D) = D\, U(\sigma), \qquad U(\sigma) = \theta_n(w_\sigma)^{M_0}\, \theta_n(a(\sigma)) \in M_d(\mathcal{O}_{K_{n'}}), \qquad U(\sigma) \equiv 1 \bmod c'\, \mathcal{O}_{K_{n'}},$$

using Step 1 ($\theta_n(w_\sigma) \equiv 1$ modulo $c\mathcal{O}_{K_n} \subseteq c'\mathcal{O}_{K_{n'}}$) and $a(\sigma) - 1 \in M_d(\mu_{\varepsilon,s} \mathbf{A}_{K_0,s}^+)$, $\theta_n(\mu_{\varepsilon,s}) = c'$. Since $M(\tau)$ has rational entries and M is multiplicative,

$$U(\sigma\tau) = D^{-1} M(\sigma) \sigma(D)\, \sigma(D^{-1} M(\tau) \tau(D)) = U(\sigma)\, \sigma(U(\tau)).$$

For $h \in H_{K_0} \subseteq G_{K_s}$, $h([\varepsilon]) = [\varepsilon]$ (as $\chi_{\mathrm{cyc}}(h) = 1$) and $h(n_j) = n_j$ by (Wl3), so $M(h) h(d_j) = d_j$ and $U(h) = 1$. The cocycle U is continuous, as M and $\sigma \mapsto \sigma(D)$ are.

Step 4: (Sf1) at level m. As $m \geq n' \geq s$, $G_m \subseteq G_{K_{n'}} \subseteq G_{K_s}$ and G_m fixes $\mathcal{O}_{K_{n'}}$, so on G_m the cocycle relation reads $U(\sigma\tau) = U(\sigma)U(\tau)$; the entries of $U(\sigma)$ lie in $\mathcal{O}_{K_{n'}} \subseteq \mathcal{O}_{K_m}$; with Step 3 this is (Sf1) for the restriction of U to G_m.

Step 5: the element g and (Sf2). As $K_0/\mathbb{Q}_p$ is unramified and $\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p$ is totally ramified, χ_{cyc} maps G_{K_0} onto $\mathbb{Z}_p^\times$, hence G_{K_s} onto $1 + p^s\mathbb{Z}_p$; choose $g_1 \in G_{K_s}$ with $v_p(\chi_{\mathrm{cyc}}(g_1) - 1) = s$ and put $g = g_1^{p^{m-s}}$. Since $s \geq 1$, and $s \geq 2$ if $p = 2$, $v_p(x^p - 1) = v_p(x-1) + 1$ for $x \in 1 + p^s\mathbb{Z}_p$, so $v_p(\chi_{\mathrm{cyc}}(g) - 1) = s + (m - s) = m$; so g fixes μ_{p^m} and K_0, i.e. $g \in G_m$. Apply Lemma 25.15 to $R = \mathcal{O}_{K_{n'}}$, $\varpi = c'$, $e' = p^{n'-1}(p-1)$ in place of e, $\gamma = g_1|_{K_{n'}}$ and $Y(x) = U(g_1)\, g_1(x)$ on R^d. Its hypotheses: γ fixes $\mathcal{O}_{K_0}$ and $\gamma(c') - c' = \zeta_{p^{n'}}(\zeta_{p^{n'}}^{\chi_{\mathrm{cyc}}(g_1)-1} - 1)$, where $\zeta_{p^{n'}}^{\chi_{\mathrm{cyc}}(g_1)-1}$ is a $p^{n'-s}$-th root of unity (as p^s divides $\chi_{\mathrm{cyc}}(g_1) - 1$), so $\gamma(c') - c'$ is 0 (if $n' = s$) or has valuation at least $v_p(\zeta_{p^{n'-s}} - 1) = p^s v_p(c') \geq 2 v_p(c')$; hence $\gamma(c') - c' \in c'^2 R$, $\gamma(y) - y \in c'R$ for $y \in R = \mathcal{O}_{K_0}[c']$, and for $x = c'^j y$, $\gamma(x) - x = \gamma(c')^j(\gamma(y) - y) + (\gamma(c')^j - c'^j)y \in c'^{j+1}R$; Y is additive and γ-semilinear, and $Y(x) - x = (U(g_1) - 1)g_1(x) + (g_1(x) - x) \in c'R^d$ by Step 3. By the cocycle relation $Y^k(x) = U(g_1^k)\, g_1^k(x)$. Since $m - s = n' + 1 \geq n'$, the lemma (its last clause with n' in place of n) gives $(U(g) - 1)f_a = Y^{p^{m-s}}(f_a) - f_a \in c'^{p^{n'}} R^d$, and $v_p(c'^{p^{n'}}) = p^{n'}/e' = p/(p-1) > 1/(p-1)$; so $|U(g)_{ab} - \delta_{ab}| < \rho$. With $v_p(\chi_{\mathrm{cyc}}(g) - 1) = m$ this is (Sf2).

Step 6: (Sf3) with weights in $J(V)$. The representation V is crystalline, hence de Rham [52], and L contains the image of every embedding of $K \subseteq K_s$; let $e_1, \ldots, e_d$ and $u_k \in J(V)$ be as in Lemma 25.16(2). The vectors $D^{-1}e_k$ form a $\mathbb{C}_p$-basis, and for $\sigma \in G_m$,

$$\chi_{\mathrm{cyc}}(\sigma)^{u_k}\, U(\sigma)\, \sigma(D^{-1}e_k) = D^{-1}\big(\chi_{\mathrm{cyc}}(\sigma)^{u_k} M(\sigma)\sigma(e_k)\big) = D^{-1}e_k.$$

So $(U|_{G_m}, g)$ is a Sen frame of level $m = m_{N,s}$ and rank d over K_0 with weights in $J(V)$, D and pD^{-1} are integral (Step 2), and $DU(\sigma) = M(\sigma)\sigma(D)$ for $\sigma \in G_m$ (Step 3). □

Proposition 25.12. *Let p be a prime, $s \geq 1$ an integer, K_0 a finite unramified extension of $\mathbb{Q}_p$ with residue field k, K a field with $K_0 \subseteq K \subseteq K_s = K_0(\mu_{p^s})$, and L a finite extension of $\mathbb{Q}_p$ containing $\sigma(K_s)$ for every $\mathbb{Q}_p$-embedding σ of K_s. Let V be a crystalline L-linear representation of G_K with a G_K-stable $\mathcal{O}_L$-lattice T, $d = \operatorname{rank}_{\mathbb{Z}_p} T$, and $M_0, h_0 \geq 0$ integers with $J(V) \subseteq [-M_0, h_0]$. Put $\mu_\varepsilon = [\varepsilon] - 1$ and $\mu_{\varepsilon,s} = [\varepsilon^{1/p^{s-1}}] - 1$ in A_{inf}, where $\varepsilon^{1/p^{s-1}}$ is the unique p^{s-1}-th root of ε in the perfect ring $\mathcal{O}_{\mathbb{C}_p^\flat}$ (so $\mu_{\varepsilon,1} = \mu_\varepsilon$); let $W(k) \to A_{\mathrm{inf}}$ be the map of Witt vectors induced by the unique ring homomorphism $k \to \mathcal{O}_{\mathbb{C}_p^\flat}$ lifting the inclusion of k into the residue field, and $\mathbf{A}^+_{K_0,s} \subset A_{\mathrm{inf}}$ the image of $W(k)[[X]]$ under $X \mapsto \mu_{\varepsilon,s}$; write $\mathbf{A}^+_{K_0} = \mathbf{A}^+_{K_0,1}$. Let G_K act on $A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes_{\mathbb{Z}_p} T$ diagonally. Then there are $n_1, \dots, n_d \in A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes_{\mathbb{Z}_p} T$ such that:*

(Wl1) $\mu_\varepsilon^{M_0} n_j \in A_{\mathrm{inf}} \otimes_{\mathbb{Z}_p} T$ *for every j;*

(Wl2) $\mu_\varepsilon^{h_0}(1 \otimes t) \in \sum_{j=1}^d A_{\mathrm{inf}}\, n_j$ *for every $t \in T$;*

(Wl3) $h(n_j) = n_j$ *for every $h \in H_{K_0}$ and every j;*

(Wl4) *for every $\sigma \in G_{K_s}$ there is a matrix $a(\sigma) \in M_d(\mathbf{A}^+_{K_0,s})$ with $\sigma(n_j) = \sum_i a(\sigma)_{ij}\, n_i$ for all j and $a(\sigma) - 1 \in M_d(\mu_{\varepsilon,s}\mathbf{A}^+_{K_0,s})$.*

Proof. Forgetting the $\mathcal{O}_L$-structure, T is a free $\mathbb{Z}_p$-module of rank d with a continuous action of G_K, and $T[1/p] = V$ is crystalline. Let $\mathcal{E}$ be a datum attached to T by Theorem B.41, $M = \mathcal{E}(A_{\mathrm{inf}})$, and $c\colon M \otimes_{A_{\mathrm{inf}}} W(\mathbb{C}_p^\flat) \to T \otimes_{\mathbb{Z}_p} W(\mathbb{C}_p^\flat)$ the isomorphism of (Bs3), which maps M into $A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes_{\mathbb{Z}_p} T$. Write σ_0 for the Frobenius of $W(k)$ and $w\colon W(k) \to A_{\mathrm{inf}}$ for the map of the statement.

The cyclotomic prism is an object over $\mathcal{O}_K$. Let $A = W(k)[[X]]$ with the ring endomorphism φ_A equal to σ_0 on $W(k)$ and $X \mapsto (1+X)^p - 1$. It lifts the Frobenius modulo p and A has no p-torsion, so A is a δ-ring with $\delta(a) = (\varphi_A(a) - a^p)/p$. Put $d = \Phi_{p^s}(1+X) = \sum_{k=0}^{p-1}(1+X)^{kp^{s-1}}$, with Φ_{p^s} the p^s-th cyclotomic polynomial (for $s = 1$, $d = ((1+X)^p - 1)/X$). Since $(1+X)^{p^{s-1}} \equiv 1 + X^{p^{s-1}} \bmod p$, we have $d \equiv X^{p^{s-1}(p-1)} \bmod p$, so the ideals (p, d) and (p, X) have the same radical, so A is (p,d)-adically complete. The map $\mathrm{red}\colon A \to W(k)$, $X \mapsto 0$, is a homomorphism of δ-rings (it commutes with the Frobenius lifts, as $\varphi_A(X) \mapsto 0$) and sends d to p; as $\delta(p) = 1 - p^{p-1}$ is a unit of $W(k)$, $\delta(d)$ is not in the maximal ideal (p, X) of the local ring A, hence is a unit: d is distinguished. The polynomial $\Phi_{p^s}(X+1)$ is Eisenstein over $W(k)$ (it is congruent to $X^{p^{s-1}(p-1)}$ modulo p and its constant term is $\Phi_{p^s}(1) = p$), so $X \mapsto \zeta_{p^s} - 1$ induces $A/dA \cong W(k)[\zeta_{p^s}] = \mathcal{O}_{K_s}$, which has no p-torsion. So (A, dA) is a bounded prism, and $\mathcal{O}_K \subseteq \mathcal{O}_{K_s} = A/dA$ makes it an object over $\mathcal{O}_K$; this is where $K \subseteq K_s$ is used. Likewise $(W(k), pW(k))$ with $\mathcal{O}_K \to \mathcal{O}_{K_s} \to \mathcal{O}_{K_s}/(\zeta_{p^s} - 1) = k$ is an object over $\mathcal{O}_K$, and red is a morphism over $\mathcal{O}_K$ (it maps d to p and induces the reduction $\mathcal{O}_{K_s} \to k$).

The map to Fontaine's ring. Let $\iota\colon A \to A_{\mathrm{inf}}$ be $a \mapsto \varphi(w(a))$ on $W(k)$ and $X \mapsto \mu_{\varepsilon,s}$; it is well defined because $\mu_{\varepsilon,s}$ is topologically nilpotent for the topology of (Ai2) ($\mu_{\varepsilon,s} \bmod p = \varepsilon^{1/p^{s-1}} - 1$ has positive valuation). It commutes with the Frobenius lifts (w is Frobenius-equivariant and $\varphi(\mu_{\varepsilon,s}) = [\varepsilon^{1/p^{s-1}}]^p - 1 = (1+\mu_{\varepsilon,s})^p - 1$, as $\varphi([x]) = [x^p]$), and $\iota(d) = \sum_{k<p}[\varepsilon^{1/p^{s-1}}]^{kp^{s-1}} = \sum_{k<p}[\varepsilon]^k = \tilde{\xi}$. On the quotients it induces $\mathcal{O}_{K_s} \to \mathcal{O}_{\mathbb{C}_p}$, $\zeta_{p^s} - 1 \mapsto \tilde{\theta}(\mu_{\varepsilon,s}) = \theta([\varepsilon^{1/p^s}]) - 1 = \zeta_{p^s} - 1$, and on $W(k)$ the map $\tilde{\theta} \circ \varphi \circ w = \theta \circ w$, the inclusion $W(k) = \mathcal{O}_{K_0} \subset \mathcal{O}_{\mathbb{C}_p}$; so ι is a morphism $(A, dA) \to (A_{\mathrm{inf}}, \tilde{\xi} A_{\mathrm{inf}})$ over $\mathcal{O}_K$. Its image is $w(W(k))[[\mu_{\varepsilon,s}]] = \mathbf{A}^+_{K_0,s}$, because $\varphi \circ w = w \circ \sigma_0$ and σ_0 is bijective.

The elements n_j. As A is local, the finite projective module $\mathcal{E}(A)$ is free; let $e_1,\dots,e_d$ be a basis. By (Bs2) the elements $m_j=\mathcal{E}(\iota)(e_j)$ form an $A_{\inf}$-basis of M. Put $n_j=c(m_j)\in A_{\inf}[1/\mu_\varepsilon]\otimes_{\mathbb{Z}_p}T$.

(Wl3). Let $h\in H_{K_0}$. Then $h\in G_{K_s}\subseteq G_K$, h fixes every ζ_{p^n}, hence ε, its p-power roots and $\mu_{\varepsilon,s}$, and h fixes $w(W(k))$ (it acts trivially on k). So $h\circ\iota=\iota$ as morphisms over $\mathcal{O}_K$, and by (Bs2) $\mathcal{E}(h)\circ\mathcal{E}(\iota)=\mathcal{E}(\iota)$. By the G_K-equivariance of c, $h(n_j)=c(\mathcal{E}(h)(m_j))=c(m_j)=n_j$.

(Wl4), for every $\sigma\in G_K$. Let $\gamma_\sigma\colon A\to A$ be the identity on $W(k)$ and $X\mapsto(1+X)^{\chi_{\mathrm{cyc}}(\sigma)}-1=\sum_{k\geq1}\binom{\chi_{\mathrm{cyc}}(\sigma)}{k}X^k$. It commutes with φ_A (both composites send X to $(1+X)^{p\chi_{\mathrm{cyc}}(\sigma)}-1$). In $A/dA=\mathcal{O}_{K_s}$ the image of $\gamma_\sigma(d)=\sum_{k<p}(1+X)^{kp^{s-1}\chi_{\mathrm{cyc}}(\sigma)}$ is $\sum_{k<p}\zeta_{p^s}^{kp^{s-1}\chi_{\mathrm{cyc}}(\sigma)}=\sum_{k<p}\zeta_p^{k\chi_{\mathrm{cyc}}(\sigma)}=0$, since $\zeta_{p^s}^{p^{s-1}}=\zeta_p$ and $\zeta_p^{\chi_{\mathrm{cyc}}(\sigma)}$ is a primitive p-th root of unity; so $\gamma_\sigma(dA)\subseteq dA$, and the induced automorphism of $\mathcal{O}_{K_s}$ is $\zeta_{p^s}\mapsto\zeta_{p^s}^{\chi_{\mathrm{cyc}}(\sigma)}$, trivial on $W(k)$, i.e. the restriction of σ, which fixes $\mathcal{O}_K$. Hence γ_σ is a morphism over $\mathcal{O}_K$. Moreover $\sigma\circ\iota=\iota\circ\gamma_\sigma$: both are trivial on $W(k)$-parts ($\sigma\in G_{K_0}$), and $\sigma(\mu_{\varepsilon,s})=[(\varepsilon^{1/p^{s-1}})^{\chi_{\mathrm{cyc}}(\sigma)}]-1=(1+\mu_{\varepsilon,s})^{\chi_{\mathrm{cyc}}(\sigma)}-1$ (as σ raises every ζ_{p^n} to the power $\chi_{\mathrm{cyc}}(\sigma)$, it maps $\varepsilon^{1/p^{s-1}}$ to its $\chi_{\mathrm{cyc}}(\sigma)$-th power) by multiplicativity and continuity of the Teichmüller map. Write $\mathcal{E}(\gamma_\sigma)(e_j)=\sum_i\alpha_{ij}e_i$ with $\alpha=\alpha(\sigma)\in M_d(A)$. By (Bs2) and the equivariance of c,

$$\sigma(n_j)=c(\mathcal{E}(\sigma)\mathcal{E}(\iota)e_j)=c(\mathcal{E}(\iota)\mathcal{E}(\gamma_\sigma)e_j)=\sum_i\iota(\alpha_{ij})\,n_i,$$

so $a(\sigma)=\iota(\alpha)\in M_d(\mathbf{A}^+_{K_0,s})$. Finally $\mathrm{red}\circ\gamma_\sigma=\mathrm{red}$ (as $\gamma_\sigma(X)\in XA$), so $\mathcal{E}(\mathrm{red})\circ\mathcal{E}(\gamma_\sigma)=\mathcal{E}(\mathrm{red})$, i.e. $\sum_i\mathrm{red}(\alpha_{ij})\,\mathcal{E}(\mathrm{red})(e_i)=\mathcal{E}(\mathrm{red})(e_j)$. The $\mathcal{E}(\mathrm{red})(e_i)$ form a $W(k)$-basis of $\mathcal{E}(W(k))$ by (Bs2), so $\mathrm{red}(\alpha)=1$, i.e. $\alpha-1\in M_d(XA)$ and $a(\sigma)-1\in M_d(\mu_{\varepsilon,s}\mathbf{A}^+_{K_0,s})$. In particular (Wl4) holds for $\sigma\in G_{K_s}$.

(Wl1) and (Wl2). Fix a $\mathbb{Z}_p$-basis $t_1,\dots,t_d$ of T and write $n_j=\sum_k Z_{jk}\,t_k$ with $Z\in M_d(W(\mathbb{C}_p^\flat))$; since the m_j form a basis of M and c is an isomorphism after $\otimes W(\mathbb{C}_p^\flat)$, Z is invertible; let $U=Z^{-1}$, so $t_k=\sum_j U_{kj}n_j$. The element $\tilde\xi$ is a unit of $W(\mathbb{C}_p^\flat)$ (its reduction $(\varepsilon-1)^{p-1}$ is non-zero), so c extends to $M[1/\tilde\xi]$, and since c carries $\varphi_M\otimes\varphi$ to $1\otimes\varphi$, $c(\varphi_M(1\otimes m_j))=\sum_k\varphi(Z_{jk})t_k$. By Proposition 25.14 there are $P,R\in M_d(A_{\inf})$ with $\tilde\xi^{M_0}\varphi_M(1\otimes m_j)=\sum_i P_{ji}m_i$ and $\tilde\xi^{h_0}m_j=\sum_i R_{ji}\varphi_M(1\otimes m_i)$; applying c,

$$\tilde\xi^{M_0}\varphi(Z)=PZ,\qquad \tilde\xi^{h_0}Z=R\,\varphi(Z).$$

Recall $\varphi(\mu_\varepsilon)=\mu_\varepsilon\tilde\xi$. For (Wl1) put $Y=\mu_\varepsilon^{M_0}Z$: then $\varphi(Y)=\mu_\varepsilon^{M_0}\tilde\xi^{M_0}\varphi(Z)=PY$, so Y has entries in $A_{\inf}$ by Lemma 25.13 (with $S=P$, $S'=1$ and zero constant term), i.e. $\mu_\varepsilon^{M_0}n_j=\sum_k Y_{jk}t_k\in A_{\inf}\otimes_{\mathbb{Z}_p}T$. For (Wl2), multiplying $\tilde\xi^{h_0}Z=R\varphi(Z)$ by U on the left and by $\varphi(U)=\varphi(Z)^{-1}$ on the right gives $\tilde\xi^{h_0}\varphi(U)=UR$; with $Y'=\mu_\varepsilon^{h_0}U$ this is $\varphi(Y')=Y'R$, so Y' has entries in $A_{\inf}$ by Lemma 25.13 (with $S=1$, $S'=R$), and $\mu_\varepsilon^{h_0}(1\otimes t_k)=\sum_j Y'_{kj}n_j$. Since every $t\in T$ is a $\mathbb{Z}_p$-combination of the t_k, (Wl2) follows. □

Lemma 25.13. *Let p be a prime, $W(\mathbb{C}_p^\flat)$ the ring of Witt vectors of the perfect field $\mathbb{C}_p^\flat$, containing $A_{\inf}$ by functoriality of Witt vectors, and φ its Frobenius (the functorial extension of $x\mapsto x^p$, which restricts to the Frobenius of $A_{\inf}$), applied to matrices entrywise. Let $r,s\geq1$, $S\in M_r(A_{\inf})$, $S'\in M_s(A_{\inf})$, $Y\in M_{r\times s}(A_{\inf})$ and $Z\in M_{r\times s}(W(\mathbb{C}_p^\flat))$ with*

$$\varphi(Z)=S\,Z\,S'+Y.$$

Then $Z\in M_{r\times s}(A_{\inf})$.

Proof. We use the classical facts on Witt vectors of a perfect ring R of characteristic p: every element of $W(R)$ has a unique expansion $\sum_{i\geq 0} p^i[x_i]$ with $x_i \in R$; $W(R)/pW(R) = R$ via $x \mapsto x_0$; the Teichmüller map is multiplicative and $\varphi(\sum p^i[x_i]) = \sum p^i[x_i^p]$. Applied to $R = \mathcal{O}_{\mathbb{C}_p^\flat} \subset \mathbb{C}_p^\flat$, an element of $W(\mathbb{C}_p^\flat)$ lies in A_{inf} if and only if all its coordinates x_i lie in $\mathcal{O}_{\mathbb{C}_p^\flat}$; in particular $A_{\mathrm{inf}} \cap pW(\mathbb{C}_p^\flat) = pA_{\mathrm{inf}}$ (an element of A_{inf} with $x_0 = 0$ is $p\sum_{i\geq 1} p^{i-1}[x_i]$). Write v for the valuation of $\mathbb{C}_p^\flat$ and $\bar{X}$ for the entrywise reduction modulo p of a matrix X over $W(\mathbb{C}_p^\flat)$.

The first coordinates are integral. Reducing the equation modulo p gives $\bar{Z}^{(p)} = \bar{S}\,\bar{Z}\,\bar{S}' + \bar{Y}$ in $M_{r\times s}(\mathbb{C}_p^\flat)$, where $\bar{Z}^{(p)}$ is the entrywise p-th power and $\bar{S}, \bar{S}', \bar{Y}$ have entries in $\mathcal{O}_{\mathbb{C}_p^\flat}$. Suppose $\bar{Z} \neq 0$ and let m be the minimum of the valuations of its entries, attained at the entry (a, b). The (a, b)-entry of the left side has valuation pm; every entry of $\bar{S}\bar{Z}\bar{S}'$ is a sum of products of an entry of $\bar{Z}$ with two integral elements, so it has valuation at least m, and the entries of $\bar{Y}$ have valuation at least 0. Hence $pm \geq \min(m, 0)$. If $m < 0$ this reads $pm \geq m$, i.e. $(p-1)m \geq 0$, a contradiction. So $m \geq 0$: the entries of $\bar{Z}$ lie in $\mathcal{O}_{\mathbb{C}_p^\flat}$.

Induction on the coordinates. Let $z = \bar{Z}$ (entries in $\mathcal{O}_{\mathbb{C}_p^\flat}$) and $[z]$ its entrywise Teichmüller lift, a matrix over A_{inf}. Then $Z - [z] \in pM_{r\times s}(W(\mathbb{C}_p^\flat))$; write $Z = [z] + pZ_1$, where the coordinates of each entry of Z_1 are those of Z shifted by one. Since $\varphi([z]) = [z^{(p)}]$ is over A_{inf},

$$pY_1 := S[z]S' + Y - \varphi([z]) = p(\varphi(Z_1) - SZ_1S')$$

lies in $M_{r\times s}(A_{\mathrm{inf}}) \cap pM_{r\times s}(W(\mathbb{C}_p^\flat)) = pM_{r\times s}(A_{\mathrm{inf}})$, and as $W(\mathbb{C}_p^\flat)$ has no p-torsion, $\varphi(Z_1) = SZ_1S' + Y_1$ with Y_1 over A_{inf}. This is an equation of the same form, so by the first step the coordinate x_1 of every entry of Z is integral. Repeating, the coordinate x_k of every entry of Z is the coordinate x_0 of the corresponding entry of a matrix Z_k satisfying an equation $\varphi(Z_k) = SZ_kS' + Y_k$ with Y_k over A_{inf}, hence lies in $\mathcal{O}_{\mathbb{C}_p^\flat}$. All coordinates being integral, Z has entries in A_{inf}. □

Proposition 25.14. *Let p be a prime, K a finite extension of $\mathbb{Q}_p$, T a finite free $\mathbb{Z}_p$-module with a continuous action of G_K such that $V = T[1/p]$ is crystalline, $\mathcal{E}$ a datum attached to T as in Theorem B.41, $M = \mathcal{E}(A_{\mathrm{inf}})$, and $\varphi(M) \subseteq M[1/\tilde{\xi}]$ the A_{inf}-submodule generated by the elements $\varphi_M(1 \otimes m)$, $m \in M$. Let $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ with the filtration of (Ai4), and let $M_0, h_0 \geq 0$ be integers such that every integer a with $\mathrm{Fil}^{a+1} D_{\mathrm{dR}}(V) \neq \mathrm{Fil}^a D_{\mathrm{dR}}(V)$ satisfies $-M_0 \leq a \leq h_0$. Then*

$$\tilde{\xi}^{M_0}\,\varphi(M) \subseteq M \qquad \text{and} \qquad \tilde{\xi}^{h_0}\,M \subseteq \varphi(M).$$

Proof. If $V = 0$ there is nothing to prove; let $V \neq 0$.

The Frobenius identifies Fontaine's B_{dR}^+ with B'^+. The Witt vector Frobenius φ is a ring automorphism of A_{inf} commuting with G_K, and $\varphi(q) = [\varepsilon]$, so $\varphi(\xi_0) = \sum_{k=0}^{p-1}[\varepsilon]^k = \tilde{\xi}$. Hence φ induces isomorphisms $A_{\mathrm{inf}}[1/p]/\xi_0^b \to A_{\mathrm{inf}}[1/p]/\tilde{\xi}^b$ compatible in b. By (Ai3) the source is $B_{\mathrm{dR}}^+/\xi_0^b B_{\mathrm{dR}}^+$, and B_{dR}^+, a complete discrete valuation ring with uniformiser ξ_0, is the inverse limit of these quotients. Passing to the limit gives a G_K-equivariant ring isomorphism $\Phi\colon B_{\mathrm{dR}}^+ \to B'^+$ with $\Phi(\xi_0) = \tilde{\xi}$, extending φ on the image of $A_{\mathrm{inf}}[1/p]$. Consequently B'^+ is a discrete valuation ring with uniformiser $\tilde{\xi}$, $B'^+/\tilde{\xi}B'^+ = A_{\mathrm{inf}}[1/p]/\tilde{\xi}A_{\mathrm{inf}}[1/p]$, and Φ extends to an isomorphism $B_{\mathrm{dR}} \to B'_{\mathrm{dR}}$ with $\Phi(t^j B_{\mathrm{dR}}^+) = \tilde{\xi}^j B'^+$ for every $j \in \mathbb{Z}$, because $tB_{\mathrm{dR}}^+ = \xi_0 B_{\mathrm{dR}}^+$ by (Ai3).

The Hodge window in $V \otimes_{\mathbb{Q}_p} B'_{\mathrm{dR}}$. Let $D = D_{\mathrm{dR}}(V)$ with $\mathrm{Fil}^j D$ as in (Ai4), $F = K$. The filtration is decreasing, exhaustive ($B_{\mathrm{dR}} \otimes V = \bigcup_N t^{-N} B_{\mathrm{dR}}^+ \otimes V$ and D is finite-dimensional) and

separated ($\bigcap_j t^j B^+_{\mathrm{dR}} \otimes V = 0$). So $a_0 = \max\{a : \mathrm{Fil}^a D = D\}$ and $a_1 = \max\{a : \mathrm{Fil}^a D \neq 0\}$ exist, and both are jumps ($\mathrm{Fil}^{a_0+1} D \neq \mathrm{Fil}^{a_0} D$, $\mathrm{Fil}^{a_1+1} D = 0 \neq \mathrm{Fil}^{a_1} D$). By hypothesis $a_0 \geq -M_0$ and $a_1 \leq h_0$, hence

$$\mathrm{Fil}^{-M_0} D = D, \qquad \mathrm{Fil}^{h_0+1} D = 0.$$

The map $1 \otimes \Phi \colon V \otimes B_{\mathrm{dR}} \to V \otimes B'_{\mathrm{dR}}$ is a G_K-equivariant bijection, Φ-semilinear; so $D' = (1 \otimes \Phi)(D)$ equals $(V \otimes B'_{\mathrm{dR}})^{G_K}$, and with $\mathrm{Fil}'^j = (1 \otimes \Phi)(\mathrm{Fil}^j D)$ the case $k = 0$ of (Ai4) becomes

$$\sum_{j \in \mathbb{Z}} \tilde{\xi}^{-j} B'^{+} \, \mathrm{Fil}'^j = V \otimes_{\mathbb{Q}_p} B'^{+} = T \otimes_{\mathbb{Z}_p} B'^{+},$$

where $\tilde{\xi}^{-j} B'^{+} \mathrm{Fil}'^j$ is the B'^{+}-submodule generated by $\tilde{\xi}^{-j} \mathrm{Fil}'^j$ (the image under $1 \otimes \Phi$ of the B^+_{dR}-submodule generated by $t^{-j} \mathrm{Fil}^j D$). Let $\Lambda_{\mathrm{dR}} = B'^{+} D'$. The term $j = -M_0$ gives $\tilde{\xi}^{M_0} \Lambda_{\mathrm{dR}} \subseteq T \otimes B'^{+}$; and since only $j \leq h_0$ contribute and $\tilde{\xi}^{-j} \in \tilde{\xi}^{-h_0} B'^{+}$ for $j \leq h_0$, also $T \otimes B'^{+} \subseteq \tilde{\xi}^{-h_0} \Lambda_{\mathrm{dR}}$.

The two lattices of M. Inside $M \otimes_{A_{\mathrm{inf}}} B'_{\mathrm{dR}} \cong T \otimes B'_{\mathrm{dR}}$ let Λ_0 be the image of $M \otimes B'^{+}$ and Λ_1 the B'^{+}-submodule generated by the images of the $\varphi_M(1 \otimes m)$. By (Bs4), $\Lambda_0 = T \otimes B'^{+}$ and $\Lambda_1 = \Lambda_{\mathrm{dR}}$. By the previous paragraph,

$$\tilde{\xi}^{M_0} \Lambda_1 \subseteq \Lambda_0, \qquad \tilde{\xi}^{h_0} \Lambda_0 \subseteq \Lambda_1. \tag{$*$}$$

Descent from B'^{+} to A_{inf}. First, $\iota^{-1}(\tilde{\xi}^k B'^{+}) = \tilde{\xi}^k A_{\mathrm{inf}}$ for $k \geq 0$. For $k = 1$: if $\iota(a) \in \tilde{\xi} B'^{+}$, then the image of a in $B'^{+}/\tilde{\xi} = A_{\mathrm{inf}}[1/p]/\tilde{\xi} A_{\mathrm{inf}}[1/p]$ vanishes, so $p^n a \in \tilde{\xi} A_{\mathrm{inf}} = \ker \tilde{\theta}$ for some n, hence $p^n \tilde{\theta}(a) = 0$ in $\mathcal{O}_{\mathbb{C}_p}$ and $a \in \ker \tilde{\theta} = \tilde{\xi} A_{\mathrm{inf}}$. For $k \geq 2$, write $a = \tilde{\xi} a_1$; then $\tilde{\xi}\, \iota(a_1) \in \tilde{\xi}^k B'^{+}$ and, B'^{+} being a domain, $\iota(a_1) \in \tilde{\xi}^{k-1} B'^{+}$; induct. Second, let N be a finite projective A_{inf}-module and $x \in N[1/\tilde{\xi}]$ an element whose image in $N \otimes B'_{\mathrm{dR}}$ lies in the image of $N \otimes B'^{+}$; then $x \in N$. Indeed, for $N = A_{\mathrm{inf}}^r$ write $x = \tilde{\xi}^{-k}(a_1, \ldots, a_r)$; the hypothesis says $\iota(a_i) \in \tilde{\xi}^k B'^{+}$, so $a_i \in \tilde{\xi}^k A_{\mathrm{inf}}$ and $x \in A_{\mathrm{inf}}^r$. In general $N \oplus P = A_{\mathrm{inf}}^r$ for some P; all the base changes respect the decomposition, so $x \in A_{\mathrm{inf}}^r$, and its component in P is 0 because $\tilde{\xi}^k x \in N$ for some k and P is $\tilde{\xi}$-torsion free (A_{inf} is a domain by (Ai1)); so $x \in N$.

Conclusion. For $m \in M$ the element $x = \tilde{\xi}^{M_0} \varphi_M(1 \otimes m)$ lies in $M[1/\tilde{\xi}]$ and, by $(*)$, its image lies in Λ_0; by the descent with $N = M$, $x \in M$. These elements generate $\tilde{\xi}^{M_0} \varphi(M)$, so $\tilde{\xi}^{M_0} \varphi(M) \subseteq M$. For the second inclusion, $\varphi^* M = A_{\mathrm{inf}} \otimes_{\varphi, A_{\mathrm{inf}}} M$ is finite projective, hence $\tilde{\xi}$-torsion free, and φ_M is an isomorphism after inverting $\tilde{\xi}$; so $\varphi(M)$, the image of $\varphi^* M$ in $M[1/\tilde{\xi}]$, is isomorphic to $\varphi^* M$ (finite projective) and $\varphi(M)[1/\tilde{\xi}] = M[1/\tilde{\xi}]$. Thus $\varphi(M) \otimes B'_{\mathrm{dR}} = M \otimes B'_{\mathrm{dR}}$, and the image of $\varphi(M) \otimes B'^{+}$ there is Λ_1. For $m \in M$ the element $\tilde{\xi}^{h_0} m \in \varphi(M)[1/\tilde{\xi}]$ has image in $\tilde{\xi}^{h_0} \Lambda_0 \subseteq \Lambda_1$ by $(*)$, so $\tilde{\xi}^{h_0} m \in \varphi(M)$ by the descent with $N = \varphi(M)$. Hence $\tilde{\xi}^{h_0} M \subseteq \varphi(M)$. $\square$

Lemma 25.15. *Let p be a prime, R a discrete valuation ring with uniformiser ϖ and $p \in \varpi^e R^\times$ for an integer $e \geq 1$, and $d \geq 1$. Let γ be a ring automorphism of R with $\gamma(x) - x \in \varpi^{j+1} R$ for all $j \geq 0$ and $x \in \varpi^j R$, and $Y \colon R^d \to R^d$ an additive map with $Y(\lambda x) = \gamma(\lambda) Y(x)$ ($\lambda \in R$, $x \in R^d$) and $Y(x) - x \in \varpi R^d$ for all x. Put $c_0 = 1$ and $c_{b+1} = \min(c_b + e, p c_b)$. Then*

$$Y^{p^b}(x) - x \in \varpi^{c_b} R^d \qquad (x \in R^d,\ b \geq 0).$$

If $e = p^{n-1}(p-1)$ for an integer $n \geq 1$, then $c_b \geq p^n$ for all $b \geq n$.

Proof. Say that an additive map Z of $R^{d'}$ ($d' \geq 1$) has *level* c if $Z(\varpi^j R^{d'}) \subseteq \varpi^{j+c} R^{d'}$ for all $j \geq 0$.

Semilinear maps congruent to 1. Let γ' be a ring automorphism of R such that $\gamma' - 1$ has level $c \geq 1$ on R, and Z an additive γ'-semilinear map of $R^{d'}$ with $(Z - 1) R^{d'} \subseteq \varpi^c R^{d'}$. Then $Z - 1$ has level c: for $x = \varpi^j y$,

$$Z(x) - x = \gamma'(\varpi^j)(Z(y) - y) + (\gamma'(\varpi^j) - \varpi^j) y,$$

where $\gamma'(\varpi^j) \in \varpi^j R$ (an automorphism of R preserves its maximal ideal), $Z(y) - y \in \varpi^c R^{d'}$ and $\gamma'(\varpi^j) - \varpi^j \in \varpi^{j+c} R$.

p-th powers. If Z is additive and $\delta = Z - 1$ has level $c \geq 1$, then $Z^p - 1$ has level $\min(c+e, pc)$: indeed $Z^p - 1 = \sum_{k=1}^{p-1} \binom{p}{k} \delta^k + \delta^p$, where δ^k has level $kc \geq c$, $p \mid \binom{p}{k}$ for $1 \leq k \leq p-1$ and multiplication by p maps $\varpi^i R^{d'}$ into $\varpi^{i+e} R^{d'}$, while δ^p has level pc.

Induction. We show that $\gamma^{p^b} - 1$ has level c_b on R and $Y^{p^b} - 1$ has level c_b on R^d. For $b = 0$ the first is the hypothesis on γ, and the second follows from the first paragraph with $\gamma' = \gamma$, $Z = Y$, $c = 1$. If both hold for b, the second paragraph applied to $Z = \gamma^{p^b}$ on R ($d' = 1$) and to $Z = Y^{p^b}$ on R^d gives both for $b+1$, since $c_{b+1} = \min(c_b + e, pc_b)$. Level c_b for $j = 0$ is the claim.

The case $e = p^{n-1}(p-1)$. The sequence (c_b) is non-decreasing, as $c_b + e > c_b$ and $pc_b > c_b$. If $c_b \leq p^{n-1}$ then $(p-1)c_b \leq e$, i.e. $pc_b \leq c_b + e$, so $c_{b+1} = pc_b$; by induction $c_b = p^b$ for $0 \leq b \leq n$. Hence $c_b \geq c_n = p^n$ for $b \geq n$. □

Lemma 25.16. *Let p be a prime, $K \subset \overline{\mathbb{Q}}_p$ a finite extension of $\mathbb{Q}_p$, and $L \subset \overline{\mathbb{Q}}_p$ a finite extension of $\mathbb{Q}_p$ containing $\sigma(K)$ for every $\mathbb{Q}_p$-embedding $\sigma\colon K \to \overline{\mathbb{Q}}_p$. Let V be a finite-dimensional L-vector space with a continuous L-linear action of G_K, de Rham as a $\mathbb{Q}_p$-representation, $b_1, \dots, b_d$ a $\mathbb{Q}_p$-basis of V, $M(\sigma) \in \mathrm{GL}_d(\mathbb{Q}_p)$ the matrix of $\sigma \in G_K$ and $B(y) \in M_d(\mathbb{Q}_p)$ the matrix of multiplication by $y \in L$ in it.*

1. *The map $K \otimes_{\mathbb{Q}_p} L \to \prod_\tau L$, $x \otimes y \mapsto (\tau(x)y)_\tau$, the product over the $\mathbb{Q}_p$-embeddings $\tau\colon K \to L$, is a ring isomorphism. Let $\epsilon_\tau \in K \otimes_{\mathbb{Q}_p} L$ be the element with components 1 at τ and 0 elsewhere, write $\epsilon_\tau = \sum_j x_j \otimes y_j$, and put $E_\tau = \sum_j x_j B(y_j) \in M_d(K)$. Then E_τ does not depend on the choice of the x_j, y_j, $E_\tau^2 = E_\tau$, $E_\tau E_{\tau'} = 0$ for $\tau \neq \tau'$, $\sum_\tau E_\tau = 1$, $\sigma(E_\tau) = E_\tau$ and $E_\tau M(\sigma) = M(\sigma) E_\tau$ for $\sigma \in G_K$.*

2. *There are a $\mathbb{C}_p$-basis $e_1, \dots, e_d$ of $\mathbb{C}_p^d$, integers $u_1, \dots, u_d$ and embeddings $\tau_1, \dots, \tau_d\colon K \to L$ such that, for all k: $\chi_{\mathrm{cyc}}(\sigma)^{u_k} M(\sigma) \sigma(e_k) = e_k$ for $\sigma \in G_K$; u_k is a Hodge jump of V at τ_k, and $u_k \in J(V)$; and $E_\tau e_k = e_k$ if $\tau = \tau_k$, $E_\tau e_k = 0$ if $\tau \neq \tau_k$.*

Proof. **Item 1.** Let $K = \mathbb{Q}_p(\alpha)$ with minimal polynomial f of degree $n = [K : \mathbb{Q}_p]$. The n embeddings of K into $\overline{\mathbb{Q}}_p$ take values in L by hypothesis, so $f = \prod_\tau (X - \tau(\alpha))$ in $L[X]$ with distinct roots ($K/\mathbb{Q}_p$ is separable). By the Chinese remainder theorem $K \otimes_{\mathbb{Q}_p} L = L[X]/(f) \to \prod_\tau L[X]/(X - \tau(\alpha)) = \prod_\tau L$ is an isomorphism; it is the stated map, since $\alpha \otimes 1 \mapsto (\tau(\alpha))_\tau$. The ring $K \otimes_{\mathbb{Q}_p} L$ acts $\mathbb{C}_p$-linearly on $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ by $(x \otimes y)(c \otimes v) = xc \otimes yv$ ($K \subset \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$); in the coordinates of the basis $b_\bullet$, $x \otimes y$ acts by the matrix $x\,B(y)$, so E_τ is the matrix of the action of ϵ_τ and does not depend on the chosen expression. The relations $\epsilon_\tau^2 = \epsilon_\tau$, $\epsilon_\tau \epsilon_{\tau'} = 0$ ($\tau \neq \tau'$), $\sum_\tau \epsilon_\tau = 1$ give the corresponding ones for E_τ. The entries of E_τ lie in K, hence are fixed by G_K; and $B(y)M(\sigma) = M(\sigma)B(y)$ because G_K acts L-linearly.

Item 2: the adapted basis of D_{dR}. Let $D = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ with $\mathrm{Fil}^j D = D \cap (t^j B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V)$; $\dim_K D = d$ as V is de Rham. The ring $K \otimes_{\mathbb{Q}_p} L$ acts on $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ by $K \subset \overline{\mathbb{Q}}_p \subset B_{\mathrm{dR}}^+$ (Ai3) on the first factor and L on V; both commute with G_K (K is fixed by G_K, G_K acts L-linearly) and preserve $t^j B_{\mathrm{dR}}^+ \otimes V$, so they preserve D and every $\mathrm{Fil}^j D$. Hence $D = \bigoplus_\tau \epsilon_\tau D$ and $\mathrm{Fil}^j D = \bigoplus_\tau \epsilon_\tau \mathrm{Fil}^j D$ with $\epsilon_\tau \mathrm{Fil}^j D = \mathrm{Fil}^j D \cap \epsilon_\tau D$. The quotient map $D \to D_{\mathrm{dR},\tau}(V) = L \otimes_{K \otimes L, \tau \otimes \mathrm{id}} D$ is the projection onto $\epsilon_\tau D$, compatibly with the induced filtrations; so an integer a is a Hodge jump of V at τ exactly when $\epsilon_\tau \mathrm{Fil}^{a+1} D \neq \epsilon_\tau \mathrm{Fil}^a D$. On the finite-dimensional K-space $\epsilon_\tau D$ the filtration $(\epsilon_\tau \mathrm{Fil}^j D)_j$ is decreasing, equal to $\epsilon_\tau D$ for j small (a basis of D lies in $t^{-N} B_{\mathrm{dR}}^+ \otimes V$ for some N) and 0 for j large ($\bigcap_j t^j B_{\mathrm{dR}}^+ = 0$); so it has an adapted K-basis: elements d with integers u such that $\epsilon_\tau \mathrm{Fil}^j D$ is spanned by those d with $u \geq j$. Each such d lies in $\epsilon_\tau \mathrm{Fil}^u D \setminus \epsilon_\tau \mathrm{Fil}^{u+1} D$, so u is a Hodge jump of V at τ, and $\mathrm{Fil}^u D \neq \mathrm{Fil}^{u+1} D$,

so $u \in J(V)$. Taking the union over τ we get a K-basis $d_1, \dots, d_d$ of D, embeddings τ_k with $d_k \in \epsilon_{\tau_k} D$ and integers u_k with $\mathrm{Fil}^j D = \bigoplus_{u_k \ge j} K d_k$ for all j.

Item 2: the Hodge–Tate basis. Put $w_k = t^{-u_k} d_k \in B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$. Then $\sum_j t^{-j} B_{\mathrm{dR}}^+ \otimes_K \mathrm{Fil}^j D = \bigoplus_k t^{-u_k} B_{\mathrm{dR}}^+ d_k$, so by (Ai4) (with $k=0$) the w_k generate $B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V$ over B_{dR}^+, and they are linearly independent over B_{dR} because $B_{\mathrm{dR}} \otimes_K D \to B_{\mathrm{dR}} \otimes V$ is an isomorphism; so they form a B_{dR}^+-basis. As $\theta\colon B_{\mathrm{dR}}^+ \to \mathbb{C}_p$ is onto with kernel tB_{dR}^+ (Ai3), the images $e_k = (\theta \otimes 1)(w_k)$ form a $\mathbb{C}_p$-basis of $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$; we identify $\mathbb{C}_p \otimes V$ with $\mathbb{C}_p^d$ by the basis $b_\bullet$, so that σ acts by $y \mapsto M(\sigma)\sigma(y)$. Since $\sigma(t) = \chi_{\mathrm{cyc}}(\sigma) t$ (Ai3) and $\sigma(d_k) = d_k$, $\sigma(w_k) = \chi_{\mathrm{cyc}}(\sigma)^{-u_k} w_k$, and $\theta \otimes 1$ is G_K-equivariant; so $M(\sigma)\sigma(e_k) = \chi_{\mathrm{cyc}}(\sigma)^{-u_k} e_k$. Finally $\theta \otimes 1$ commutes with the action of $K \otimes_{\mathbb{Q}_p} L$ (θ restricts to the inclusion on $K \subset \overline{\mathbb{Q}}_p$ by (Ai3), and is $\mathbb{Q}_p$-linear), and multiplication by t^{-u_k} commutes with it too; as $\epsilon_\tau d_k = \epsilon_\tau \epsilon_{\tau_k} d_k$ is d_k for $\tau = \tau_k$ and 0 otherwise, $E_\tau e_k = (\theta \otimes 1)(t^{-u_k} \epsilon_\tau d_k)$ is e_k for $\tau = \tau_k$ and 0 otherwise. □

Proposition 25.17. *Let p be a prime, K_0 a finite unramified extension of $\mathbb{Q}_p$, $m \ge 1$ with $m \ge 2$ if $p = 2$, $J \subset \mathbb{Z}$ finite, (U, g) a Sen frame of level m and rank r over K_0 with weights in J, and $G_m = G_{K_0(\mu_{p^m})}$. For $i \in \mathbb{Z}$ put*

$$N_{J,m}(i) = \max\Big(1, \sum_{u \in J,\ u \ne i} \big(m + v_p(u-i)\big)\Big).$$

A cocycle for $*_i$ *is a continuous map $c\colon G_m \to \mathbb{C}_p^r$ with $c(\sigma\tau) = c(\sigma) + \sigma *_i c(\tau)$; the* coboundary *of $x \in \mathbb{C}_p^r$ is $\sigma \mapsto \sigma *_i x - x$.*

1. *If c is a cocycle for $*_i$ with values in $\mathcal{O}_{\mathbb{C}_p}^r$ and $p^k c$ is the coboundary of an element of $\mathcal{O}_{\mathbb{C}_p}^r$ for some $k \ge 0$, then $p^{1+N_{J,m}(i)} c$ is the coboundary of an element of $\mathcal{O}_{\mathbb{C}_p}^r$.*
2. *If $i \notin J$, $M \subseteq \mathbb{C}_p^r$ is an additive subgroup with $\mathcal{O}_{\mathbb{C}_p}^r \subseteq M$ and $pM \subseteq \mathcal{O}_{\mathbb{C}_p}^r$, and $y \in \mathbb{C}_p^r$ satisfies $\sigma *_i y - y \in M$ for all $\sigma \in G_m$, then $p^{2+N_{J,m}(i)} y \in M$.*

Proof. For $r = 0$ both items are trivial; let $r \ge 1$, write $N = N_{J,m}(i)$, $H = H_{K_0}$, δx for the coboundary of x, and ζ_p for a primitive p-th root of unity.

The twisted action. For $\sigma, \tau \in G_m$ and $y \in \mathbb{C}_p^r$, $\sigma *_i (\tau *_i y) = \chi_{\mathrm{cyc}}(\sigma\tau)^i U(\sigma) U(\tau)\, \sigma\tau(y) = (\sigma\tau) *_i y$, since σ fixes $\chi_{\mathrm{cyc}}(\tau)^i \in \mathbb{Q}_p$ and the entries of $U(\tau)$, which lie in K_m, and U is multiplicative (Sf1). So $*_i$ is an action, additive, with $\sigma *_i (\lambda y) = \sigma(\lambda)\, \sigma *_i y$; $\sigma \mapsto \sigma *_i x$ is continuous (χ_{cyc}, U and the Galois action on $\mathbb{C}_p$ are continuous), so coboundaries are cocycles; $\sigma *_i$ preserves $\mathcal{O}_{\mathbb{C}_p}^r$ because $U(\sigma) \in M_r(\mathcal{O}_{K_m})$; and for $h \in H$, $h *_i y = h(y)$, as $U(h) = 1$ and $\chi_{\mathrm{cyc}}(h) = 1$. The group H is normal in G_m.

Step 1: killing the cocycle on H. Let c be as in item 1 and $p^k c = \delta x_0$ with $x_0 \in \mathcal{O}_{\mathbb{C}_p}^r$. On H, c is a continuous cocycle for the Galois action with values in $\mathcal{O}_{\mathbb{C}_p}^r$; by Lemma 25.19 with $F = K_0$ and $a = p \in K_\infty$ there is $x_1 \in \mathcal{O}_{\mathbb{C}_p}^r$ with $p\, c(h) = h(x_1) - x_1$ for $h \in H$. Put $c_1 = p\, c - \delta x_1$, a cocycle for $*_i$ with values in $\mathcal{O}_{\mathbb{C}_p}^r$ and $c_1|_H = 0$.

Step 2: its values are fixed by H. For $h \in H$ and $\sigma \in G_m$, $c_1(h\sigma) = c_1(h) + h(c_1(\sigma)) = h(c_1(\sigma))$ and $c_1(h\sigma) = c_1(\sigma \cdot \sigma^{-1} h \sigma) = c_1(\sigma) + \sigma *_i c_1(\sigma^{-1} h \sigma) = c_1(\sigma)$. So the entries of $c_1(\sigma)$ are fixed by H and lie in $\widehat{K}_\infty \cap \mathcal{O}_{\mathbb{C}_p} = \mathcal{O}_{\widehat{K}_\infty}$ by Theorem B.42(1). Moreover $p^k c_1 = p^{k+1} c - p^k \delta x_1 = \delta x_2$ with $x_2 = p\, x_0 - p^k x_1 \in \mathcal{O}_{\mathbb{C}_p}^r$, and for $h \in H$, $0 = p^k c_1(h) = h(x_2) - x_2$; so $x_2 \in \mathcal{O}_{\widehat{K}_\infty}^r$ by Theorem B.42(1).

Step 3: the coinvariant equation at g. Put $w = c_1(g) \in \mathcal{O}_{\widehat{K}_\infty}^r$, so $g *_i x_2 - x_2 = p^k w$. *There is $y \in \mathcal{O}_{\widehat{K}_\infty}^r$ with $g *_i y - y = p^N w$.* Decompose coordinatewise (Lemma 25.18(1)):

$w = w_0 + w_X$, $x_2 = v_0 + v_X$ with $w_0, v_0 \in \mathcal{O}_{K_m}^r$ and $w_X, v_X \in X_m^r$. The map $g*_i$ sends $\mathcal{O}_{K_m}^r$ into itself (there $g *_i v = \chi_{\mathrm{cyc}}(g)^i U(g)v$, as g fixes K_m) and X_m^r into itself (X_m is a G_{K_0}-stable $\mathcal{O}_{K_m}$-module). By uniqueness of the decomposition, $\chi_{\mathrm{cyc}}(g)^i U(g)v_0 - v_0 = p^k w_0$. Put $f_u = \chi_{\mathrm{cyc}}(g)^{-u} \in \mathbb{Z}_p^\times$ for $u \in \mathbb{Z}$; then

$$(U(g) - f_i)v_0 = f_i\, p^k w_0.$$

The part over K_m. Let $Q(X) = \prod_{u\in J,\, u\neq i}(X - f_u) \in \mathbb{Z}_p[X]$ and $Q(X) - Q(f_i) = (X - f_i)Q_1(X)$ with $Q_1 \in \mathbb{Z}_p[X]$ (Lemma 23.22(3)). The polynomial $Q(X)(X - f_i)$ is $\prod_{u\in J}(X - f_u)$ if $i \in J$ and a multiple of it if $i \notin J$, so $Q(U(g))(U(g) - f_i) = 0$ by Lemma 25.20(1). Applied to v_0: $f_i p^k Q(U(g))w_0 = 0$, so $Q(U(g))w_0 = 0$ and

$$\beta\, w_0 = -(U(g) - f_i)Q_1(U(g))\, w_0, \qquad \beta = Q(f_i) = \prod_{u\in J,\, u\neq i}(f_i - f_u).$$

Since $v_p(\chi_{\mathrm{cyc}}(g) - 1) = m$ (Sf2), with $m \geq 2$ if $p = 2$, $v_p(\chi_{\mathrm{cyc}}(g)^n - 1) = m + v_p(n)$ for $n \neq 0$ by Lemma 23.22(1); as $f_i - f_u = f_u(\chi_{\mathrm{cyc}}(g)^{u-i} - 1)$, $v_p(\beta) = \sum_{u\in J, u\neq i}(m + v_p(u - i)) \leq N$, so $t = p^N/\beta \in \mathbb{Z}_p$. The vector $z_0 = -t\, f_i\, Q_1(U(g))w_0 \in \mathcal{O}_{K_m}^r$ satisfies

$$g *_i z_0 - z_0 = \chi_{\mathrm{cyc}}(g)^i(U(g) - f_i)z_0 = -t(U(g) - f_i)Q_1(U(g))w_0 = t\beta\, w_0 = p^N w_0.$$

The part in X_m. As $N \geq 1 \geq 1/(p-1)$, $v_p(p^N/(\zeta_p - 1)) = N - 1/(p-1) \geq 0$ and $p^N/(\zeta_p - 1) \in \mathcal{O}_{K_1} \subseteq \mathcal{O}_{K_m}$, so $x' = p^N(\zeta_p - 1)^{-1}w_X \in X_m^r$. The matrix $A = \chi_{\mathrm{cyc}}(g)^i U(g) \in M_r(\mathcal{O}_{K_m})$ has $|A_{ab} - \delta_{ab}| < p^{-1/(p-1)}$ (write $A - 1 = \chi_{\mathrm{cyc}}(g)^i(U(g) - 1) + (\chi_{\mathrm{cyc}}(g)^i - 1)$ with $|\chi_{\mathrm{cyc}}(g)^i| = 1$: the entries of $U(g) - 1$ have absolute value $< p^{-1/(p-1)}$ by (Sf2), and $|\chi_{\mathrm{cyc}}(g)^i - 1| \leq p^{-m} < p^{-1/(p-1)}$, as $m \geq 1$, and $m \geq 2$ if $p = 2$), and $g \in G_{K_m}$ has $v_p(\chi_{\mathrm{cyc}}(g) - 1) = m$; by Lemma 25.18(2) there is $y_X \in X_m^r$ with $g *_i y_X - y_X = A\, g(y_X) - y_X = (\zeta_p - 1)x' = p^N w_X$. So $y = z_0 + y_X \in \mathcal{O}_{\widehat{K}_\infty}^r$ satisfies $g *_i y - y = p^N w$.

Step 4: the cocycle $p^N c_1 - \delta y$ vanishes. Let $d = p^N c_1 - \delta y$, a cocycle for $*_i$. For $h \in H$, $d(h) = 0 - (h(y) - y) = 0$ by Step 2 and $y \in \mathcal{O}_{\widehat{K}_\infty}^r$ (fixed by H); and $d(g) = p^N w - (g *_i y - y) = 0$. For $\sigma \in G_m$, $h \in H$: $d(\sigma h) = d(\sigma) + \sigma *_i d(h) = d(\sigma)$, so d factors through a continuous map $\bar{d}$ on $\Gamma_m = G_m/H = \mathrm{Gal}(K_\infty/K_m)$. Since K_n/K_0 has degree $p^{n-1}(p-1)$ ($\Phi_{p^n}(X+1)$ is Eisenstein over $\mathcal{O}_{K_0}$), χ_{cyc} identifies Γ_m with $1 + p^m\mathbb{Z}_p$, and $\chi_{\mathrm{cyc}}(g)$ is a topological generator of $1 + p^m\mathbb{Z}_p$; as $n \mapsto \chi_{\mathrm{cyc}}(g)^n$ extends to a homeomorphism $\mathbb{Z}_p \to 1 + p^m\mathbb{Z}_p$ and the positive integers are dense in $\mathbb{Z}_p$, the images of g^n, $n \geq 1$, are dense in Γ_m. By the cocycle relation $d(g^n) = \sum_{k=0}^{n-1} g^k *_i d(g) = 0$ for $n \geq 1$. So $\bar{d}$ vanishes on a dense subset, and $d = 0$.

Item 1. By Step 4, $p^N c_1 = \delta y$, hence $p^{N+1}c = p^N c_1 + p^N \delta x_1 = \delta(y + p^N x_1)$ with $y + p^N x_1 \in \mathcal{O}_{\mathbb{C}_p}^r$.

Item 2. Let $c(\sigma) = p\,(\sigma *_i y - y)$. It is the coboundary of py, hence a cocycle for $*_i$, and takes values in $pM \subseteq \mathcal{O}_{\mathbb{C}_p}^r$. Choose $k \geq 0$ with $p^{k+1}y \in \mathcal{O}_{\mathbb{C}_p}^r$; then $p^k c = \delta(p^{k+1}y)$. By item 1, $p^{1+N}c = \delta z$ for some $z \in \mathcal{O}_{\mathbb{C}_p}^r$. Since $p^{1+N}c = \delta(p^{2+N}y)$, the vector $p^{2+N}y - z$ is fixed by $\sigma*_i$ for every $\sigma \in G_m$, and as $i \notin J$, Lemma 25.20(2) gives $p^{2+N}y = z \in \mathcal{O}_{\mathbb{C}_p}^r \subseteq M$. □

Lemma 25.18. *Let p be a prime, $K_0 \subset \overline{\mathbb{Q}}_p$ a finite unramified extension of $\mathbb{Q}_p$, $K_n = K_0(\mu_{p^n})$ for $n \geq 0$, $K_\infty = \bigcup_n K_n$, $\widehat{K}_\infty$ the closure of K_∞ in $\mathbb{C}_p$, ζ_p a primitive p-th root of unity, and $m \geq 1$ an integer, with $m \geq 2$ if $p = 2$.*

1. *There is a unique continuous K_m-linear map $R_m \colon \widehat{K}_\infty \to K_m$ with*

$$R_m(x) = p^{m-n}\,\mathrm{Tr}_{K_n/K_m}(x) \qquad (x \in K_n,\ n \geq m).$$

It satisfies $R_m(\mathcal{O}_{\widehat{K}_\infty}) \subseteq \mathcal{O}_{K_m}$ *and* $R_m(\sigma x) = \sigma R_m(x)$ *for* $\sigma \in G_{K_0}$, $x \in \widehat{K}_\infty$. *Put* $X_m = \{x \in \mathcal{O}_{\widehat{K}_\infty} : R_m(x) = 0\}$. *Every* $z \in \mathcal{O}_{\widehat{K}_\infty}$ *can be written uniquely as* $z = z_0 + z_X$ *with* $z_0 \in \mathcal{O}_{K_m}$ *and* $z_X \in X_m$. *The set* X_m *is closed in* $\mathbb{C}_p$, $\mathcal{O}_{K_m} X_m \subseteq X_m$, *and* $\sigma(X_m) \subseteq X_m$ *for* $\sigma \in G_{K_0}$.

2. *Let* $g \in G_{K_m}$ *with* $v_p(\chi_{\mathrm{cyc}}(g) - 1) = m$, *let* $r \geq 1$, *and let* $A \in M_r(\mathcal{O}_{K_m})$ *with* $|A_{ab} - \delta_{ab}| < p^{-1/(p-1)}$ *for all entries. For every* $x \in X_m^r$ *there is exactly one* $y \in X_m^r$ *with*

$$A\, g(y) - y = (\zeta_p - 1)\, x.$$

Proof. Fix a compatible system $(\zeta_{p^n})_{n\geq 1}$ of primitive p^n-th roots of unity with $\zeta_{p^1} = \zeta_p$ and $\zeta_{p^n}^p = \zeta_{p^{n-1}}$; v is the valuation of $\mathbb{C}_p$ with $v(p) = 1$, and ϖ a uniformiser of K_m.

The layers. For $n \geq 1$ the polynomial $\Phi_{p^n}(X+1)$ is Eisenstein over $\mathcal{O}_{K_0}$, because p is a uniformiser of the unramified field K_0; so K_n/K_0 is totally ramified of degree $p^{n-1}(p-1)$ with uniformiser $\zeta_{p^n} - 1$. For $n \geq 2$, $[K_n : K_{n-1}] = p$, so $X^p - \zeta_{p^{n-1}}$ is the minimal polynomial of ζ_{p^n} over K_{n-1}, with roots $\zeta_{p^n}\zeta_p^a$ $(0 \leq a < p)$; and as K_n/K_{n-1} is totally ramified with uniformiser $\zeta_{p^n} - 1$, $\mathcal{O}_{K_n} = \mathcal{O}_{K_{n-1}}[\zeta_{p^n} - 1] = \mathcal{O}_{K_{n-1}}[\zeta_{p^n}]$ is free over $\mathcal{O}_{K_{n-1}}$ with basis $1, \zeta_{p^n}, \ldots, \zeta_{p^n}^{p-1}$ (Serre, *Local fields*, I, §6, Prop. 18).

One step of the trace. For $n \geq 2$ put $\rho_n = p^{-1}\mathrm{Tr}_{K_n/K_{n-1}}$. For $1 \leq j \leq p-1$, $\mathrm{Tr}_{K_n/K_{n-1}}(\zeta_{p^n}^j) = \zeta_{p^n}^j \sum_{a=0}^{p-1} \zeta_p^{aj} = 0$, so

$$\rho_n\Big(\sum_{j=0}^{p-1} y_j \zeta_{p^n}^j\Big) = y_0 \qquad (y_j \in K_{n-1}).$$

Hence ρ_n is the identity on K_{n-1}, $\rho_n(\mathcal{O}_{K_n}) \subseteq \mathcal{O}_{K_{n-1}}$, $\ker \rho_n = X^{(n)} := \bigoplus_{j=1}^{p-1} K_{n-1}\zeta_{p^n}^j$ and $X^{(n)} \cap \mathcal{O}_{K_n} = \bigoplus_{j=1}^{p-1} \mathcal{O}_{K_{n-1}}\zeta_{p^n}^j$. Every $\sigma \in G_{K_0}$ preserves K_n and K_{n-1} and normalises $\mathrm{Gal}(K_n/K_{n-1})$ (the group $\mathrm{Gal}(K_n/K_0)$ is abelian), so $\rho_n \circ \sigma = \sigma \circ \rho_n$ on K_n, and $\sigma(X^{(n)}) = X^{(n)}$.

Item 1. For $n \geq m$ let $R_{m,n} = \rho_{m+1} \circ \cdots \circ \rho_n \colon K_n \to K_m$ (the identity for $n = m$); by transitivity of traces $R_{m,n} = p^{m-n}\mathrm{Tr}_{K_n/K_m}$, and $R_{m,n+1}|_{K_n} = R_{m,n}$ as ρ_{n+1} is the identity on K_n. So the $R_{m,n}$ define a K_m-linear map R_m on K_∞, the identity on K_m, commuting with G_{K_0} and with $R_m(\mathcal{O}_{K_\infty}) \subseteq \mathcal{O}_{K_m}$. If $x \in K_\infty$ and $v(x) \geq k \in \mathbb{Z}_{\geq 0}$, then $p^{-k}x \in \mathcal{O}_{K_\infty}$, so $v(R_m(x)) \geq k$: R_m is uniformly continuous, and since K_m is complete and K_∞ is dense in $\widehat{K}_\infty$ it extends uniquely to a continuous map R_m on $\widehat{K}_\infty$; by continuity it is K_m-linear, maps the closure $\mathcal{O}_{\widehat{K}_\infty}$ of $\mathcal{O}_{K_\infty}$ into the closed set $\mathcal{O}_{K_m}$, and commutes with each $\sigma \in G_{K_0}$ (an isometry of $\mathbb{C}_p$ preserving K_∞). Uniqueness of R_m follows from continuity and density. As R_m is the identity on K_m, $R_m \circ R_m = R_m$; for $z \in \mathcal{O}_{\widehat{K}_\infty}$ put $z_0 = R_m(z) \in \mathcal{O}_{K_m}$ and $z_X = z - z_0$, which lies in $\mathcal{O}_{\widehat{K}_\infty}$ and has $R_m(z_X) = 0$. If $z_0 + z_X = z_0' + z_X'$ are two such decompositions, then $z_0 - z_0' \in K_m \cap \ker R_m = 0$. X_m is closed (a continuous preimage intersected with the closed $\mathcal{O}_{\widehat{K}_\infty}$), an $\mathcal{O}_{K_m}$-module by linearity, and G_{K_0}-stable by equivariance.

The kernel layer by layer. For $n > m$ and $x \in K_n$, $x = \rho_n(x) + (x - \rho_n(x))$ with $x - \rho_n(x) \in X^{(n)}$, $\rho_n(x) \in K_{n-1}$, and R_m vanishes on $X^{(n)}$ (on K_n it is $R_{m,n-1} \circ \rho_n$). By induction on n,

$$K_n = K_m \oplus X^{(m+1)} \oplus \cdots \oplus X^{(n)}, \qquad X_m \cap \mathcal{O}_{K_n} = \bigoplus_{k=m+1}^{n} (X^{(k)} \cap \mathcal{O}_{K_k}),$$

with R_m the projection to K_m; the integral form holds because each ρ_k maps $\mathcal{O}_{K_k}$ into $\mathcal{O}_{K_{k-1}}$, so the components of an integral element are integral. Moreover X_m is the closure of $X_m \cap \mathcal{O}_{K_\infty}$: if $z \in X_m$ and $z_j \in \mathcal{O}_{K_\infty}$ tend to z, then $z_j - R_m(z_j) \in X_m \cap \mathcal{O}_{K_\infty}$ tend to $z - R_m(z) = z$.

Inverting $g-1$ on one layer. Let g be as in item 2 and $n \geq m+1$, and put $h_n = g^{p^{n-1-m}}$. We have $v_p(a^p-1) = v_p(a-1)+1$ for $a \in 1+p\mathbb{Z}_p$ if p is odd, and for $a \in 1+4\mathbb{Z}_2$ if $p=2$ (then $a^2-1=(a-1)(a+1)$ with $v_2(a+1)=1$); as $v_p(\chi_{\mathrm{cyc}}(g)-1)=m$, with $m \geq 2$ if $p=2$, and the valuations only increase along the powers, it follows that $v_p(\chi_{\mathrm{cyc}}(h_n)-1) = n-1$: h_n fixes $\mu_{p^{n-1}}$ and K_0, hence K_{n-1}, and $h_n(\zeta_{p^n}) = \zeta_{p^n}\zeta_p^w$ for an integer w prime to p. So $h_n(y\zeta_{p^n}^j) = \zeta_p^{jw} y \zeta_{p^n}^j$ for $y \in K_{n-1}$, and $h_n - 1$ acts on the summand $K_{n-1}\zeta_{p^n}^j$ of $X^{(n)}$ as multiplication by $\zeta_p^{jw}-1 \neq 0$. The numbers $\varepsilon_j = (\zeta_p-1)/(\zeta_p^{jw}-1)$, $1 \leq j \leq p-1$, are units of $\mathbb{Z}_p[\zeta_p] \subseteq \mathcal{O}_{K_{n-1}}$ (both ζ_p-1 and $\zeta_p^{jw}-1$ are uniformisers of $\mathbb{Q}_p(\zeta_p)$). So $S_n = (\zeta_p-1)(h_n-1)^{-1}$, $S_n(\sum_j y_j\zeta_{p^n}^j) = \sum_j \varepsilon_j y_j \zeta_{p^n}^j$, is a K_{n-1}-linear bijection of $X^{(n)}$ preserving $X^{(n)} \cap \mathcal{O}_{K_n}$. The element g preserves K_n, $X^{(n)}$ and $\mathcal{O}_{K_n}$, is K_m-linear on K_n and commutes with h_n, hence with S_n. With $N_n = \sum_{k=0}^{p^{n-1-m}-1} g^k$ we have $(g-1)N_n = N_n(g-1) = h_n - 1$; so $T_n = N_n S_n$ is a K_m-linear endomorphism of $X^{(n)}$ commuting with g, preserving $X^{(n)} \cap \mathcal{O}_{K_n}$, with

$$(g-1)\,T_n = T_n\,(g-1) = (h_n-1)S_n = \zeta_p - 1 \qquad \text{on } X^{(n)}.$$

The operator T on X_m. On $X_m \cap K_\infty = \bigoplus_{n>m} X^{(n)}$ let $T = \bigoplus_n T_n$. It is K_m-linear, commutes with g, satisfies $(g-1)T = T(g-1) = \zeta_p - 1$, and maps $X_m \cap \mathcal{O}_{K_\infty}$ into itself by the integral decomposition above. If $x, x' \in X_m \cap \mathcal{O}_{K_\infty}$ and $v(x-x') \geq k \geq 0$, then $p^{-k}(x-x') \in X_m \cap \mathcal{O}_{K_\infty}$ and $v(T(x)-T(x')) \geq k$; so T extends by continuity to X_m (the closure of $X_m \cap \mathcal{O}_{K_\infty}$, and closed), with values in X_m, and by continuity it is $\mathcal{O}_{K_m}$-linear, commutes with g, and $(g-1)T = T(g-1) = (\zeta_p-1)\,\mathrm{id}$ on X_m. Let g and T act on X_m^r coordinatewise; then $T(Cw) = C\,T(w)$ for $C \in M_r(\mathcal{O}_{K_m})$, $w \in X_m^r$, and $T(\varpi^k X_m^r) \subseteq \varpi^k X_m^r$.

Item 2. Put $B = (\zeta_p-1)^{-1}(A-1) \in M_r(K_m)$. Its entries have absolute value <1 (as $|\zeta_p - 1| = p^{-1/(p-1)}$), so they lie in $\varpi\mathcal{O}_{K_m}$. The map $\Phi(z) = B\,g(T(z))$ satisfies $\Phi(\varpi^k X_m^r) \subseteq \varpi^{k+1} X_m^r$ for $k \geq 0$. *Existence.* The series $z = \sum_{j\geq 0}(-\Phi)^j(x)$ has its j-th term in $\varpi^j X_m^r \subseteq \varpi^j \mathcal{O}_{\mathbb{C}_p}^r$, so it converges, and its sum lies in the closed set X_m^r and satisfies $z + \Phi(z) = x$ (Φ is continuous, being additive with $\Phi(\varpi^k X_m^r) \subseteq \varpi^k X_m^r$). Put $y = T(z) \in X_m^r$. Then

$$A\,g(y) - y = (g-1)T(z) + (A-1)\,g(T(z)) = (\zeta_p-1)\big(z + \Phi(z)\big) = (\zeta_p-1)x.$$

Uniqueness. If y, y' are two solutions, $d = y - y' \in X_m^r$ satisfies $A\,g(d) = d$, so $(g-1)d = -(A-1)g(d)$; applying T gives $(\zeta_p-1)d = -(A-1)T(g(d))$, i.e. $d = -B\,T(g(d))$. If $d \in \varpi^k X_m^r$ then $-B\,T(g(d)) \in \varpi^{k+1} X_m^r$; so $d \in \bigcap_k \varpi^k \mathcal{O}_{\mathbb{C}_p}^r = 0$. □

Lemma 25.19. *Let p be a prime, F a finite extension of $\mathbb{Q}_p$, $r \geq 1$, and $c\colon H_F \to \mathcal{O}_{\mathbb{C}_p}^r$ a continuous map with $c(\sigma\tau) = c(\sigma) + \sigma\big(c(\tau)\big)$ for $\sigma, \tau \in H_F$. Then for every $a \in F_\infty$ with $|a| < 1$ there is $x \in \mathcal{O}_{\mathbb{C}_p}^r$ with $a\,c(\sigma) = \sigma(x) - x$ for all $\sigma \in H_F$.*

Proof. The action is coordinatewise, so it suffices to treat $r=1$; for $a=0$ take $x=0$. Let $a \neq 0$, and fix an integer $n \geq 1$ with $|p|^n < |a|$; put $b = p^n/a \in F_\infty$, so $|b|<1$, and every $\sigma \in H_F$ fixes a and b. Write δx for the map $\sigma \mapsto \sigma(x) - x$.

One approximation. *For every continuous $c'\colon H_F \to \mathcal{O}_{\mathbb{C}_p}$ with $c'(\sigma\tau) = c'(\sigma) + \sigma c'(\tau)$ there is $x \in \mathcal{O}_{\mathbb{C}_p}$ with $a\,c'(\sigma) - (\sigma x - x) \in p^n\mathcal{O}_{\mathbb{C}_p}$ for all $\sigma \in H_F$.* The set $V = \{\sigma \in H_F :\ c'(\sigma) \in p^n\mathcal{O}_{\mathbb{C}_p}\}$ is open (continuity) and a subgroup: $c'(\sigma\tau) = c'(\sigma) + \sigma c'(\tau)$ and $c'(\sigma^{-1}) = -\sigma^{-1}c'(\sigma)$, and H_F preserves $p^n\mathcal{O}_{\mathbb{C}_p}$. So V contains an open normal subgroup N of the profinite group H_F, and $N = \mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ for a finite Galois extension L of F_∞. By Theorem B.42(3) there is $\alpha \in \mathcal{O}_L$ with $\mathrm{Tr}_{L/F_\infty}(\alpha) = a$. Let $S \subset H_F$ be a set of representatives of H_F/N and $x = -\sum_{s\in S} s(\alpha)\,c'(s) \in \mathcal{O}_{\mathbb{C}_p}$. For $\sigma \in H_F$,

$$\sigma x = -\sum_{s\in S} \sigma s(\alpha)\big(c'(\sigma s) - c'(\sigma)\big) = -\sum_{s\in S} \sigma s(\alpha)\,c'(\sigma s) + c'(\sigma)\sum_{s\in S}\sigma s(\alpha).$$

As s runs through S, σs runs through a set of representatives of $H_F/N = \mathrm{Gal}(L/F_\infty)$; so $\sum_s \sigma s(\alpha) = \mathrm{Tr}_{L/F_\infty}(\alpha) = a$. Writing $\sigma s = s'\nu$ with $s' \in S$, $\nu \in N$, we have $\sigma s(\alpha) = s'(\alpha)$ and $c'(\sigma s) = c'(s') + s'c'(\nu) \in c'(s') + p^n\mathcal{O}_{\mathbb{C}_p}$; since $|s'(\alpha)| \le 1$, $\sum_s \sigma s(\alpha) c'(\sigma s) \equiv \sum_{s'} s'(\alpha) c'(s') = -x$ modulo $p^n\mathcal{O}_{\mathbb{C}_p}$. Hence $\sigma x - x \equiv a\, c'(\sigma)$ modulo $p^n\mathcal{O}_{\mathbb{C}_p}$.

Iteration. Put $c_0 = c$. Given a continuous cocycle $c_j\colon H_F \to \mathcal{O}_{\mathbb{C}_p}$, choose x_j by the approximation and put $c_{j+1} = p^{-n}(a\, c_j - \delta x_j)$, again a continuous cocycle with values in $\mathcal{O}_{\mathbb{C}_p}$. By induction on $J \ge 0$,

$$a\, c = \delta\Big(\sum_{j<J} b^j x_j\Big) + b^J a\, c_J,$$

since $b^J a\, c_J = b^J(\delta x_J + p^n c_{J+1}) = \delta(b^J x_J) + b^{J+1} a\, c_{J+1}$ (b is fixed by H_F). As $|b| < 1$ and $|c_J(\sigma)| \le 1$, letting $J \to \infty$ gives $a\, c = \delta x$ with $x = \sum_{j\ge 0} b^j x_j \in \mathcal{O}_{\mathbb{C}_p}$. □

Lemma 25.20. *Let p be a prime, K_0 a finite unramified extension of $\mathbb{Q}_p$, $m \ge 1$ with $m \ge 2$ if $p = 2$, $J \subset \mathbb{Z}$ finite, and (U, g) a Sen frame of level m and rank r over K_0 with weights in J.*

1. *$\prod_{u\in J}(U(g) - \chi_{\mathrm{cyc}}(g)^{-u}\cdot 1) = 0$ in $M_r(K_0(\mu_{p^m}))$.*
2. *If $i \in \mathbb{Z}\setminus J$ and $w \in \mathbb{C}_p^r$ satisfies $\sigma *_i w = w$ for all $\sigma \in G_{K_0(\mu_{p^m})}$, then $w = 0$.*

Proof. For $r = 0$ there is nothing to prove; let $r \ge 1$. Let $E \in \mathrm{GL}_r(\mathbb{C}_p)$ be the matrix with columns $e_1, \dots, e_r$ of (Sf3) and $D(\sigma) = \mathrm{diag}(\chi_{\mathrm{cyc}}(\sigma)^{-u_1}, \dots, \chi_{\mathrm{cyc}}(\sigma)^{-u_r})$. Then (Sf3) reads

$$U(\sigma)\,\sigma(E) = E\, D(\sigma) \qquad (\sigma \in G_{K_m}).$$

Item 2. Write $w = Ea$ with $a \in \mathbb{C}_p^r$. Then

$$\sigma *_i w = \chi_{\mathrm{cyc}}(\sigma)^i U(\sigma)\sigma(E)\sigma(a) = E\big(\chi_{\mathrm{cyc}}(\sigma)^i D(\sigma)\sigma(a)\big).$$

Since E is invertible, $\sigma *_i w = w$ for all σ means $\chi_{\mathrm{cyc}}(\sigma)^{i-u_k}\sigma(a_k) = a_k$ for all $\sigma \in G_{K_m}$ and all k. Since $u_k \in J$ and $i \notin J$, $i - u_k \ne 0$, and Theorem B.42(2) for the finite extension K_m of $\mathbb{Q}_p$ gives $a_k = 0$. So $w = 0$.

E has entries in $\widehat{K}_\infty$. For $h \in H_{K_0} \subseteq G_{K_m}$, $U(h) = 1$ by (Sf1) and $\chi_{\mathrm{cyc}}(h) = 1$ (h fixes μ_{p^∞}), so $h(E) = E$, and by Theorem B.42(1) for $F = K_0$ the entries of E lie in $\widehat{K}_\infty$. Choose $s \ge 0$ with $p^s E \in M_r(\mathcal{O}_{\widehat{K}_\infty})$ and decompose entrywise by Lemma 25.18(1): $p^s E = E_0 + E_X$ with $E_0 \in M_r(\mathcal{O}_{K_m})$ and $E_X \in M_r(X_m)$.

The kernel part vanishes. Multiply the displayed relation at $\sigma = g$ by p^s: $U(g)\, g(E_0) + U(g)\, g(E_X) = E_0 D(g) + E_X D(g)$. Here $g(E_0) = E_0$ (g fixes K_m), so $U(g)g(E_0)$ and $E_0 D(g)$ have entries in $\mathcal{O}_{K_m}$, while $U(g)g(E_X)$ and $E_X D(g)$ have entries in X_m (X_m is a G_{K_0}-stable $\mathcal{O}_{K_m}$-module, $U(g) \in M_r(\mathcal{O}_{K_m})$, $D(g) \in M_r(\mathbb{Z}_p)$). By uniqueness of the decomposition, $U(g)E_0 = E_0 D(g)$ and $U(g)g(E_X) = E_X D(g)$. So the k-th column $x_k \in X_m^r$ of E_X satisfies $A_k g(x_k) - x_k = 0$ with $A_k = \chi_{\mathrm{cyc}}(g)^{u_k} U(g) \in M_r(\mathcal{O}_{K_m})$. Now $A_k - 1 = \chi_{\mathrm{cyc}}(g)^{u_k}(U(g) - 1) + (\chi_{\mathrm{cyc}}(g)^{u_k} - 1)\cdot 1$, where $|\chi_{\mathrm{cyc}}(g)^{u_k}| = 1$, the entries of $U(g) - 1$ have absolute value $< p^{-1/(p-1)}$ by (Sf2), and $|\chi_{\mathrm{cyc}}(g)^{u_k} - 1| \le p^{-m} < p^{-1/(p-1)}$ because $\chi_{\mathrm{cyc}}(g) \in 1 + p^m\mathbb{Z}_p$ and $m > 1/(p-1)$ ($m \ge 1$ and $p \ge 3$, or $m \ge 2$ and $p = 2$). So A_k satisfies the hypothesis of Lemma 25.18(2), whose uniqueness (with $x = 0$, of which $y = 0$ is a solution) gives $x_k = 0$. Hence $E_X = 0$ and $p^s E = E_0$.

Item 1. So $E \in M_r(K_m) \cap \mathrm{GL}_r(\mathbb{C}_p) = \mathrm{GL}_r(K_m)$, and the relation at $\sigma = g$ becomes $U(g) = E\, D(g)\, E^{-1}$. Therefore

$$\prod_{u\in J}(U(g) - \chi_{\mathrm{cyc}}(g)^{-u}) = E\Big(\prod_{u\in J}(D(g) - \chi_{\mathrm{cyc}}(g)^{-u})\Big)E^{-1} = 0,$$

since the product in the middle is diagonal with k-th entry $\prod_{u\in J}(\chi_{\mathrm{cyc}}(g)^{-u_k}-\chi_{\mathrm{cyc}}(g)^{-u})$, which contains the factor $u=u_k\in J$. □

Lemma 25.21. *Let p be a prime and ξ a generator of $\ker\theta$. For integers $0\le a\le b$ let $Q_{a,b}=\xi^aA_{\mathrm{inf}}/\xi^bA_{\mathrm{inf}}$, with the action of $G_{\mathbb{Q}_p}$ induced from A_{inf} and the p-adic topology.*

1. *For $a\le e\le b$ the sequence $0\to Q_{e,b}\to Q_{a,b}\to Q_{a,e}\to 0$ of inclusion and projection is exact and $G_{\mathbb{Q}_p}$-equivariant; $Q_{a,b}$ has no p-torsion; and $p^nQ_{a,b}\cap Q_{e,b}=p^nQ_{e,b}$ for $n\ge 0$.*
2. *For every $x\in Q_{a,b}$ the map $G_{\mathbb{Q}_p}\to Q_{a,b}$, $\sigma\mapsto\sigma x$, is continuous.*
3. *For every $i\ge 0$ there is a bijective additive map $\vartheta_i\colon Q_{i,i+1}\to\mathcal{O}_{\mathbb{C}_p}$ with $\vartheta_i(p^nx)=p^n\vartheta_i(x)$ for $n\ge 0$ and $\vartheta_i(\sigma x)=\chi_{\mathrm{cyc}}(\sigma)^i\,\sigma(\vartheta_i(x))$ for $\sigma\in G_{\mathbb{Q}_p(\mu_p)}$.*

Proof. Write $A=A_{\mathrm{inf}}$ and $I=(p,\xi_0)=(p,[p^\flat])$ (Ai1). As ξ and ξ_0 generate the same ideal of the domain A (Ai1), $\xi=v\xi_0$ with $v\in A^\times$, so $\xi^jA=\xi_0^jA$ for all j and we may take $\xi=\xi_0$. Since θ is $G_{\mathbb{Q}_p}$-equivariant, $\sigma(\xi_0)\in\ker\theta=\xi_0A$, so every σ preserves the ideals ξ_0^jA and acts on the $Q_{a,b}$.

Item 3 without the Galois action, and item 1. Multiplication by ξ_0^i is injective on the domain A and induces an isomorphism $A/\xi_0A\to\xi_0^iA/\xi_0^{i+1}A$; with $A/\xi_0A\cong\mathcal{O}_{\mathbb{C}_p}$ via θ (Ai1), $\vartheta_i(\xi_0^i\lambda \bmod \xi_0^{i+1})=\theta(\lambda)$ is a well-defined additive bijection $Q_{i,i+1}\to\mathcal{O}_{\mathbb{C}_p}$, and $\vartheta_i(p^nx)=p^n\vartheta_i(x)$. The sequence of item 1 is exact by the isomorphism $(\xi_0^aA/\xi_0^bA)/(\xi_0^eA/\xi_0^bA)\cong\xi_0^aA/\xi_0^eA$, and equivariant. By induction on $b-a$, using the sequences $0\to Q_{b-1,b}\to Q_{a,b}\to Q_{a,b-1}\to 0$ and $Q_{b-1,b}\cong\mathcal{O}_{\mathbb{C}_p}$, every $Q_{a,b}$ has no p-torsion. If $x\in Q_{a,b}$ and $p^nx\in Q_{e,b}$, the image of x in $Q_{a,e}$ is killed by p^n, hence zero, so $x\in Q_{e,b}$: this gives $p^nQ_{a,b}\cap Q_{e,b}=p^nQ_{e,b}$.

Item 2. Let $x\in Q_{a,b}$ with a lift $\tilde x\in\xi_0^aA$, let $n\ge 0$ and $N=n+b-1$. Every generator $p^j\xi_0^{N-j}$ of I^N has $j\ge n$ or $N-j\ge b$, so $I^N\subseteq p^nA+\xi_0^bA$. By (Ai2) the set $U=\{\sigma:\ \sigma\tilde x-\tilde x\in I^N\}$ is open. For $\sigma\in U$, $y=\sigma\tilde x-\tilde x\in\xi_0^aA$ is $p^ns+\xi_0^bs'$ with $s,s'\in A$; then $p^ns\in\xi_0^aA$, and since $A/\xi_0^aA=Q_{0,a}$ has no p-torsion, $s\in\xi_0^aA$. So $\sigma x-x\in p^nQ_{a,b}$ for $\sigma\in U$, and for any τ, $\sigma x-\tau x=\tau(\tau^{-1}\sigma x-x)\in p^nQ_{a,b}$ for $\sigma\in\tau U$. Hence $\sigma\mapsto\sigma x$ is continuous.

The cocycle of ξ_0. Let $\sigma\in G_{\mathbb{Q}_p(\mu_p)}$ and $\alpha=\chi_{\mathrm{cyc}}(\sigma)\in 1+p\mathbb{Z}_p$. Modulo p, $q-1\equiv[\varepsilon^{1/p}-1]$ and $\varepsilon^{1/p}-1$ has positive valuation, so $(q-1)^M\in I$ for some $M\ge 1$ and $(q-1)^k\to 0$ for the I-adic topology. By (Ai2) the series $q^\beta=\sum_{k\ge 0}\binom{\beta}{k}(q-1)^k$ converges for $\beta\in\mathbb{Z}_p$, depends continuously on β, and is the usual power for $\beta\in\mathbb{Z}_{\ge 0}$. The element $\varepsilon^{1/p}$ has p-power-compatible components $\zeta_{p^{k+1}}$ $(k\ge 0)$, and σ raises them to the power α; so if integers $\alpha_j\ge 0$ tend to α, the components of $(\varepsilon^{1/p})^{\alpha_j}$ converge to those of $\sigma(\varepsilon^{1/p})$, and by continuity of the Teichmüller map (Ai2) $\sigma(q)=[\sigma(\varepsilon^{1/p})]=\lim_j q^{\alpha_j}=q^\alpha$; hence $\sigma(q^p)=q^{p\alpha}=(q^p)^\alpha$, the last power taken by the binomial series in q^p-1 (both sides are continuous in α and agree on $\mathbb{Z}_{\ge 0}$). Put $S=\sum_{k\ge 1}\binom{\alpha}{k}(q-1)^{k-1}$ and $S'=\sum_{k\ge 1}\binom{\alpha}{k}(q^p-1)^{k-1}$, so $\sigma(q)-1=(q-1)S$ and $\sigma(q^p)-1=(q^p-1)S'$. Since $\theta(I)\subseteq p\mathcal{O}_{\mathbb{C}_p}$, θ is continuous; with $\theta(q)=\zeta_p$ and $\theta(q^p)=\theta([\varepsilon])=1$ (Ai1),

$$\theta(S)=\sum_{k\ge 1}\binom{\alpha}{k}(\zeta_p-1)^{k-1}=\frac{\zeta_p^\alpha-1}{\zeta_p-1}=1,\qquad \theta(S')=\alpha,$$

because $\alpha\equiv 1 \bmod p$. Now $\xi_0(q-1)=q^p-1$; applying σ, $\sigma(\xi_0)(q-1)S=(q^p-1)S'=\xi_0(q-1)S'$, and as A is a domain and $q-1\ne 0$ $(\theta(q-1)=\zeta_p-1\ne 0)$, $\sigma(\xi_0)S=\xi_0S'$. Writing $\sigma(\xi_0)=\xi_0c_\sigma$ with $c_\sigma\in A$ and cancelling ξ_0, $c_\sigma S=S'$, so $\theta(c_\sigma)=\alpha$.

Item 3. For $x=\xi_0^i\lambda \bmod \xi_0^{i+1}$ and $\sigma\in G_{\mathbb{Q}_p(\mu_p)}$, $\sigma x=\xi_0^ic_\sigma^i\sigma(\lambda) \bmod \xi_0^{i+1}$, so $\vartheta_i(\sigma x)=\theta(c_\sigma)^i\theta(\sigma\lambda)=\chi_{\mathrm{cyc}}(\sigma)^i\sigma(\vartheta_i(x))$ by equivariance of θ. □

Lemma 25.22. *Let p be a prime, F a finite extension of $\mathbb{Q}_p$, V a de Rham representation of G_F on a finite-dimensional $\mathbb{Q}_p$-vector space, $T \subset V$ a G_F-stable $\mathbb{Z}_p$-lattice, ξ a generator of $\ker\theta$ and $0 \le a \le e \le b$ integers; give $T \otimes_{\mathbb{Z}_p} Q_{a,b}$ the diagonal action of G_F. If $z \in T \otimes_{\mathbb{Z}_p} Q_{a,e}$ is fixed by G_F, there are $k \ge 0$ and $x \in T \otimes_{\mathbb{Z}_p} Q_{a,b}$ fixed by G_F whose image in $T \otimes_{\mathbb{Z}_p} Q_{a,e}$ is $p^k z$.*

Proof. Write $A = A_{\mathrm{inf}}$, $B = B_{\mathrm{dR}}^+$; as $\xi A = \xi_0 A$ we may take $\xi = \xi_0$, and $\xi_0 B = tB$ by (Ai3).

Rationalisation. By (Ai3) the natural map $A[1/p]/\xi_0^b A[1/p] \to B/t^b B$ is a $G_{\mathbb{Q}_p}$-equivariant isomorphism; it maps $\xi_0^a A[1/p]/\xi_0^b A[1/p] = Q_{a,b}[1/p]$ onto $\xi_0^a(B/t^bB) = t^aB/t^bB$. Put $W_{a,b} = V \otimes_{\mathbb{Q}_p} t^aB/t^bB$. As T is free over $\mathbb{Z}_p$ and $Q_{a,b}$ has no p-torsion (Lemma 25.21(1)), $T \otimes Q_{a,b} \to W_{a,b}$ is injective and G_F-equivariant, every element of $W_{a,b}$ has a p-power multiple in its image, and these maps commute with the projections to $W_{a,e}$, $T \otimes Q_{a,e}$. So it suffices to show that $W_{a,b}^{G_F} \to W_{a,e}^{G_F}$ is onto: if $x' \in W_{a,b}^{G_F}$ maps to the image of z, choose k with $p^k x'$ in the image of $T \otimes Q_{a,b}$; its preimage x is fixed by G_F (injectivity and equivariance) and maps to $p^k z$ (injectivity of $T \otimes Q_{a,e} \to W_{a,e}$).

An adapted basis. Let $D = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_F}$ with $\mathrm{Fil}^j D = D \cap (t^j B \otimes V)$; V being de Rham, $\dim_F D = \dim_{\mathbb{Q}_p} V$. The $\mathrm{Fil}^j D$ form a decreasing filtration by F-subspaces, equal to D for j small (a basis of D lies in $t^{-N}B \otimes V$ for some N) and to 0 for j large ($\bigcap_j t^j B = 0$). Choose an F-basis $d_1, \dots, d_n$ of D and integers $h_1, \dots, h_n$ with $\mathrm{Fil}^j D = \bigoplus_{h_k \ge j} F d_k$. Then $\sum_j t^{a-j}B \otimes_F \mathrm{Fil}^j D = \bigoplus_k t^{a-h_k} B\, d_k$, and by (Ai4) for $k = a$ and $k = b$,

$$W_{a,b} \cong \bigoplus_{k=1}^{n} (t^{a-h_k}B/t^{b-h_k}B)\, d_k,$$

G_F-equivariantly (G_F fixes the d_k) and compatibly with the projection to $W_{a,e}$.

Invariants of $t^cB/t^{c'}B$. Let $c \le c'$ be integers. B is a discrete valuation ring with uniformiser t and $\theta\colon B/tB \cong \mathbb{C}_p$ (Ai3), so $y \mapsto \theta(t^{-j}y)$ identifies $t^jB/t^{j+1}B$ with $\mathbb{C}_p$ on which σ acts by $\chi_{\mathrm{cyc}}(\sigma)^j\sigma$ (as $\sigma t = \chi_{\mathrm{cyc}}(\sigma)t$). Its G_F-invariants are 0 for $j \ne 0$ by Theorem B.42(2) and F for $j = 0$ by Theorem B.42(1). By left exactness of invariants and induction on $c' - c$, $(t^cB/t^{c'}B)^{G_F}$ is an F-space of dimension at most 1, and 0 unless $c \le 0 < c'$. If $c \le 0 < c'$, the image of $F \subset \overline{\mathbb{Q}}_p \subset B$ (Ai3) lies in it and is non-zero (θ is injective on F), so the invariants are exactly the image of F.

Conclusion. Hence $W_{a,b}^{G_F} = \bigoplus_{k:\, a \le h_k < b} F\,\bar{d}_k$ and $W_{a,e}^{G_F} = \bigoplus_{k:\, a \le h_k < e} F\,\bar{d}_k$, where $\bar{d}_k$ is the image of $1 \cdot d_k$; the projection maps $\bar{d}_k$ to $\bar{d}_k$, so $W_{a,b}^{G_F} \to W_{a,e}^{G_F}$ is onto. □

25.3 The Bloch–Kato condition in negative Tate twists; lengths of invariants

The second step towards Proposition 25.8 is Proposition 25.23: if a cocycle becomes, after multiplication by ℓ^N, a coboundary with values in $T \otimes A_{\mathrm{inf}}/\xi^{e+1}A_{\mathrm{inf}}$, then its push into the negative Tate twist $L(-e)/\mathcal{O}_L(-e)$ lies in H^1_f after multiplication by $\ell^{N+e+4+\lfloor e/(\ell-1)\rfloor + v_\ell(e) + v_\ell(e+1)}$. The proof uses an integral form of the dual exponential at a finite cyclotomic level (Definition 25.25, Proposition 25.26), whose image is controlled by values of ψ-invariant series and twisted Frobenius power sums. Two descent lemmas move lifting exponents between a subfield of a cyclotomic field and a cyclotomic layer over its unramified part (Lemma 25.32, Lemma 25.33), and Lemma 25.34 bounds the length of the invariants of a crystalline lattice quotient, a term in Theorem 25.1. The subsection ends with Corollary 25.39, the explicit local losses at unramified places that the budget reads at the split and inert primes.

Proposition 25.23. *Let p, K, $K_1 = K_0(\mu_p) \supseteq K$ and $L \supseteq \mathcal{O}_L$ be as in Proposition 25.10, ξ a generator of $\ker\theta$. Let T be a G_K-stable $\mathcal{O}_L$-lattice in a finite-dimensional continuous L-linear representation of G_K, $e \geq 0$ an integer, $\varphi\colon T \to L/\mathcal{O}_L$ an $\mathcal{O}_L$-linear map with $\varphi(gx) = \chi_{\mathrm{cyc}}(g)^{-e}\varphi(x)$ ($g \in G_K$, $x \in T$), and $f\colon G_K \to T$ a continuous 1-cocycle. Let $N \geq 0$ be an integer such that $g \mapsto p^N(f(g)\otimes 1)$ is a coboundary of G_{K_1} with values in $T \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}/\xi^{e+1}A_{\mathrm{inf}}$ (diagonal action, p-adic topology). Then*

$$p^{\,N+(e+2)+v_p(e)+\lfloor e/(p-1)\rfloor+2+v_p(e+1)}\,[g \mapsto \varphi(f(g))] \in H^1_f(G_K, L(-e)/\mathcal{O}_L(-e)).$$

Level s. *More generally, let p be a prime, $s \geq 1$ an integer with $s \geq 2$ if $p = 2$, K a finite extension of $\mathbb{Q}_p$ with maximal unramified subextension K_0 such that $K \subseteq K_s := K_0(\mu_{p^s})$, and L a finite extension of $\mathbb{Q}_p$ containing $\sigma(K_s)$ for every $\mathbb{Q}_p$-embedding σ of K_s, with ring of integers $\mathcal{O}_L$. Let ξ, T, e, φ and f be as above, and let $N \geq 0$ be an integer such that $g \mapsto p^N(f(g)\otimes 1)$ is a coboundary of G_{K_s} with values in $T \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}/\xi^{e+1}A_{\mathrm{inf}}$. Then*

$$p^{\,N+s(e+4)+v_p(e)+\lfloor e/(p-1)\rfloor+v_p(e+1)}\,[g \mapsto \varphi(f(g))] \in H^1_f(G_K, L(-e)/\mathcal{O}_L(-e)).$$

For $s = 1$ and p odd the exponent is $N + (e+2) + v_p(e) + \lfloor e/(p-1)\rfloor + 2 + v_p(e+1)$ and this is the statement above.

Proof. We prove the assertion of the paragraph Level s; the first assertion is its case $s = 1$, p odd (the hypotheses there, with $K_1 = K_0(\mu_p)$, are those of the case $s = 1$, and the exponents agree: $s(e+4) = e+4 = (e+2)+2$). So let p, s, $K \subseteq K_s$, L, T, e, φ, f and N be as in Level s. Write $W = L(-e)$, $\Lambda = \mathcal{O}_L(-e)$, $\chi = \chi_{\mathrm{cyc}}$, $Q = A_{\mathrm{inf}}/\xi^{e+1}A_{\mathrm{inf}} = Q_{0,e+1}$, $\kappa = s + v_p(e+1)$ and $M' = N + a_{e,s} + \kappa$. Since $\xi A_{\mathrm{inf}} = \xi_0 A_{\mathrm{inf}}$ and $\xi_0 B^+_{\mathrm{dR}} = tB^+_{\mathrm{dR}}$, (Ai3) gives a $G_{\mathbb{Q}_p}$-equivariant isomorphism $Q[1/p] \cong B^+_{\mathrm{dR}}/t^{e+1}B^+_{\mathrm{dR}}$ compatible with the maps from A_{inf}, and $Q \subset Q[1/p]$ as Q has no p-torsion (Lemma 25.21(1)). As $K_0/\mathbb{Q}_p$ is unramified, $[K_0(\mu_{p^j}) : K_0] = (p-1)p^{j-1}$ for $j \geq 1$, so χ maps G_{K_0} onto $\mathbb{Z}_p^\times$ and G_{K_s} onto $1 + p^s\mathbb{Z}_p$; so some $g \in G_{K_s}$ has $v_p(\chi(g) - 1) = s$.

Step 1: coordinates. $\varphi(T)$ is a finitely generated submodule of $L/\mathcal{O}_L$, hence killed by some p^r, so φ factors through T/p^rT and $g \mapsto \varphi(f(g))$ is a continuous cocycle. Let $w_1, \ldots, w_n$ be a $\mathbb{Z}_p$-basis of $\mathcal{O}_L$. As T is free over $\mathbb{Z}_p$, φ has a $\mathbb{Z}_p$-linear lift $\Phi\colon T \to p^{-r}\mathcal{O}_L$; write $\Phi(y) = \sum_i \psi_i(y)w_i$ with $\mathbb{Z}_p$-linear $\psi_i\colon T \to p^{-r}\mathbb{Z}_p$. Since $\Phi(gy)$ and $\chi(g)^{-e}\Phi(y)$ both lift $\varphi(gy)$, $\varepsilon_{g,i}(y) = \psi_i(gy) - \chi(g)^{-e}\psi_i(y) \in \mathbb{Z}_p$ for $g \in G_K$, $y \in T$. Put $c_i(\sigma) = \psi_i(f(\sigma))$. Then $c_i(\sigma\tau) - c_i(\sigma) - \chi(\sigma)^{-e}c_i(\tau) = \varepsilon_{\sigma,i}(f(\tau)) \in \mathbb{Z}_p$; and as f is continuous, every σ has a neighbourhood U with $f(\tau) - f(\sigma) \in p^rT$, hence $c_i(\tau) - c_i(\sigma) \in \mathbb{Z}_p$, for $\tau \in U$.

Step 2: genuine cocycles over K_s. By Lemma 25.24 (for $F = K_s$, this s and g, the restriction of c_i to G_{K_s}; for $p = 2$ its last sentence, as $s \geq 2$) there are continuous $z_i\colon G_{K_s} \to \mathbb{Q}_p$ with $z_i(\sigma\tau) = z_i(\sigma) + \chi(\sigma)^{-e}z_i(\tau)$ and $z_i - p^\kappa c_i$ with values in $\mathbb{Z}_p$.

Step 3: the trivialisation modulo t^{e+1} pushed to the coordinates. By hypothesis there is $X \in T \otimes_{\mathbb{Z}_p} Q$ with $p^N f(\sigma)\otimes 1 = \sigma X - X$ for $\sigma \in G_{K_s}$. The additive map $\Psi_i = \psi_i \otimes 1\colon T \otimes_{\mathbb{Z}_p} Q \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} Q = Q[1/p]$ satisfies, for $y \in T$, $a \in Q$, $\sigma \in G_K$,

$$\Psi_i(\sigma(y\otimes a)) = \psi_i(\sigma y)\,\sigma(a) = \chi(\sigma)^{-e}\sigma\big(\Psi_i(y\otimes a)\big) + \varepsilon_{\sigma,i}(y)\,\sigma(a),$$

so $\Psi_i(\sigma x) - \chi(\sigma)^{-e}\sigma(\Psi_i(x)) \in Q$ for all $x \in T \otimes Q$. With $\beta_i = \Psi_i(X)$ this gives $p^N c_i(\sigma) \equiv \chi(\sigma)^{-e}\sigma(\beta_i) - \beta_i$ modulo Q, and hence, by Step 2,

$$p^N z_i(\sigma) = p^{N+\kappa}c_i(\sigma) + p^N\big(z_i(\sigma) - p^\kappa c_i(\sigma)\big) \equiv \chi(\sigma)^{-e}\sigma(p^\kappa\beta_i) - p^\kappa\beta_i \pmod{Q} \qquad (\sigma \in G_{K_s}).$$

Transporting by $Q[1/p] \cong B^+_{\mathrm{dR}}/t^{e+1}B^+_{\mathrm{dR}}$, choose $b_i \in B^+_{\mathrm{dR}}$ lifting the image of $p^\kappa\beta_i$ and, for each σ, $a_{\sigma,i} \in A_{\mathrm{inf}}$ lifting the element of Q above; then $p^N z_i(\sigma) - (\chi(\sigma)^{-e}\sigma(b_i) - b_i) - a_{\sigma,i} \in t^{e+1}B^+_{\mathrm{dR}}$.

Step 4: the integral dual exponential. By Proposition 25.26 (paragraph Level s) applied to $p^N z_i$ there are continuous cocycles $z_{1,i}\colon G_{K_s} \to \mathbb{Z}_p$ (for χ^{-e}) and $b_{1,i} \in B_{\mathrm{dR}}^+$ such that $y_i = p^{N+a_{e,s}} z_i - z_{1,i}$ satisfies $\omega_i(\sigma) = y_i(\sigma) - (\chi(\sigma)^{-e}\sigma(b_{1,i}) - b_{1,i}) \in t^{e+1}B_{\mathrm{dR}}^+$ for $\sigma \in G_{K_s}$.

Step 5: the classes y_i are crystalline. Let $V_e = \mathbb{Q}_p(-e)$, the representation of G_{K_s} on $\mathbb{Q}_p$ by χ^{-e}, so that $B_{\mathrm{dR}}^+ \otimes V_e$ is B_{dR}^+ with $\sigma * x = \chi(\sigma)^{-e}\sigma(x)$. Then y_i is a continuous cocycle with values in V_e, and ω_i is a continuous cocycle for $*$ with values in $t^{e+1}B_{\mathrm{dR}}^+$ (the difference of y_i and a coboundary; $\sigma \mapsto \sigma(b_{1,i})$ is continuous). Multiplication by t^{-e} is an isomorphism of $(t^{e+1}B_{\mathrm{dR}}^+, *)$ onto tB_{dR}^+ with its Galois action, since $\sigma(t) = \chi(\sigma)t$. By Theorem B.46(1) for $F = K_s$, $\omega_i(\sigma) = \sigma * x_i - x_i$ for some $x_i \in t^{e+1}B_{\mathrm{dR}}^+$, so y_i is the coboundary of $b_{1,i} + x_i$ in $B_{\mathrm{dR}}^+ \otimes V_e \subset B_{\mathrm{dR}} \otimes V_e$, and $[y_i] \in H^1_g(K_s, V_e)$. Now $V_e^*(1) = \mathbb{Q}_p(1+e)$, the maximal unramified subfield of K_s is K_0 (K_s/K_0 is totally ramified), and $D_{\mathrm{cris}}(\mathbb{Q}_p(1+e))$ is the K_0-line spanned by t^{-1-e} times a basis vector, on which φ acts by $a\,t^{-1-e} \mapsto \varphi(a)p^{-1-e}t^{-1-e}$ ($\varphi(t) = pt$) [52]; as φ is an isometry of K_0, $\varphi(a)p^{-1-e} = a$ forces $a = 0$. By Theorem B.46(2), $H^1_g(K_s, V_e) = H^1_f(K_s, V_e)$, so $[y_i] \in H^1_f(K_s, V_e)$.

Step 6: the class over K_s. As a representation of G_{K_s} over $\mathbb{Q}_p$, $W = \bigoplus_i \mathbb{Q}_p w_i \cong V_e^{\oplus n}$, so $Y(\sigma) = \sum_i y_i(\sigma)w_i$ is a continuous cocycle with $[Y] \in H^1_f(K_s, W)$ (a coboundary in $B_{\mathrm{cris}} \otimes V_e$ in each coordinate). For $\sigma \in G_{K_s}$,

$$Y(\sigma) - p^{M'}\Phi(f(\sigma)) = \sum_i \Big(p^{N+a_{e,s}}\big(z_i(\sigma) - p^{\kappa}c_i(\sigma)\big) - z_{1,i}(\sigma)\Big)w_i \in \mathcal{O}_L,$$

so the image of $[Y]$ in $H^1(G_{K_s}, W/\Lambda)$ is the restriction of $p^{M'}[\varphi \circ f]$.

Step 7: descent to K. The degree $m = [K_s : K]$ divides $[K_s : K_0] = (p-1)p^{s-1}$, so $m = p^v u$ with $v \le s-1$ and $p \nmid u$. Corestriction $\mathrm{cor}\colon H^1(G_{K_s}, -) \to H^1(G_K, -)$ is natural in the coefficient module and $\mathrm{cor} \circ \mathrm{res}$ is multiplication by m (continuous cochains; Serre, *Cohomologie galoisienne*, I.2.4). Naturality for $W \to B_{\mathrm{cris}} \otimes W$ gives $\mathrm{cor}(H^1_f(K_s, W)) \subseteq H^1_f(K, W)$, and naturality for $W \to W/\Lambda$ gives

$$m\,p^{M'}[\varphi \circ f] = \mathrm{cor}\big(\mathrm{res}\,p^{M'}[\varphi \circ f]\big) = \text{image of } \mathrm{cor}[Y] \in H^1_f(G_K, W/\Lambda).$$

Every continuous cocycle of G_K with values in the discrete torsion module W/Λ has finite image, hence is killed by a power of p; so $H^1(G_K, W/\Lambda)$ is a $\mathbb{Z}_p$-module on which u acts invertibly, and $H^1_f(G_K, W/\Lambda)$, the image of an L-subspace under a $\mathbb{Z}_p$-linear map, is a $\mathbb{Z}_p$-submodule. Hence $p^{M'+v}[\varphi \circ f]$ and $p^{M'+s-1}[\varphi \circ f]$ lie in $H^1_f(G_K, L(-e)/\mathcal{O}_L(-e))$, and

$$\begin{aligned} M' + s - 1 &= N + s(e+2) + v_p(e) + \Big\lfloor \frac{e}{p-1} \Big\rfloor + 1 + s + v_p(e+1) + s - 1 \\ &= N + s(e+4) + v_p(e) + \Big\lfloor \frac{e}{p-1} \Big\rfloor + v_p(e+1). \end{aligned}$$

□

Lemma 25.24. *Let p be an odd prime, F a finite extension of $\mathbb{Q}_p$, and $s \ge 1$ an integer such that some $g \in G_F$ has $v_p(\chi_{\mathrm{cyc}}(g) - 1) = s$. Let $e, r \ge 0$ be integers, $\kappa = s + v_p(e+1)$, and $c\colon G_F \to p^{-r}\mathbb{Z}_p$ a map such that $c(\sigma\tau) - c(\sigma) - \chi_{\mathrm{cyc}}(\sigma)^{-e}c(\tau) \in \mathbb{Z}_p$ for all $\sigma, \tau \in G_F$, and every σ has a neighbourhood U with $c(\tau) - c(\sigma) \in \mathbb{Z}_p$ for $\tau \in U$. Then there is a continuous map $z\colon G_F \to \mathbb{Q}_p$ with*

$$z(\sigma\tau) = z(\sigma) + \chi_{\mathrm{cyc}}(\sigma)^{-e}z(\tau) \quad \textit{and} \quad z(\sigma) - p^{\kappa}c(\sigma) \in \mathbb{Z}_p \qquad (\sigma, \tau \in G_F).$$

The same holds for $p = 2$ when $s \ge 2$.

Proof. For a $\mathbb{Z}_p$-module A write $A(-e)$ for A with σ acting by multiplication by $\chi_{\mathrm{cyc}}(\sigma)^{-e}$, and d for the differential of continuous cochains, $(du)(\sigma,\tau) = u(\sigma) + \chi_{\mathrm{cyc}}(\sigma)^{-e}u(\tau) - u(\sigma\tau)$ on 1-cochains. The field F is the completion of a number field at a place above p (Krasner's lemma), so Theorem B.29 applies to G_F.

A continuous lift. Let $M = p^{-r}\mathbb{Z}_p/\mathbb{Z}_p$, finite and discrete, and $\bar{c}$ the image of c in M; by hypothesis $\bar{c}$ is locally constant. Choose representatives $\iota\colon M \to p^{-r}\mathbb{Z}_p$ and put $c' = \iota\circ\bar{c}$: it is locally constant, hence continuous, and $c' - c$ takes values in $\mathbb{Z}_p$. Then $w = dc'$ is a continuous 2-cochain with values in $\mathbb{Z}_p$ (the almost-cocycle relation holds for c' as for c), and $dw = ddc' = 0$, so w is a continuous 2-cocycle of G_F with values in $\mathbb{Z}_p(-e)$.

$H^2(G_F,\mathbb{Z}_p(-e))$ is killed by p^κ. Let $n \geq 1$ and $M_n = \mathbb{Z}/p^n(-e)$. Its Cartier dual $M_n^\vee = \mathrm{Hom}(M_n,\mu_{p^n})$ is $\mathbb{Z}/p^n$ with σ acting by $\chi_{\mathrm{cyc}}(\sigma)^{1+e}$. As $v_p(\chi_{\mathrm{cyc}}(g)-1) = s$, with p odd, or $p=2$ and $s\geq 2$, $v_p(\chi_{\mathrm{cyc}}(g)^{1+e}-1) = s + v_p(1+e) = \kappa$ (in both cases the p-adic logarithm is an isomorphism of $1+p^s\mathbb{Z}_p$ onto $p^s\mathbb{Z}_p$ preserving valuations of $x-1$ and $\log x$, and $\log(x^{1+e}) = (1+e)\log x$), so every $x \in H^0(G_F,M_n^\vee)$ satisfies $(\chi_{\mathrm{cyc}}(g)^{1+e}-1)x = 0$, hence $p^\kappa x = 0$. By Theorem B.29(2), $H^2(G_F,M_n)$ is the dual of the finite group $H^0(G_F,M_n^\vee)$, so it is killed by p^κ; and $H^1(G_F,M_n)$ is finite by Theorem B.29(1). By Theorem B.31 (with $T = \mathbb{Z}_p(-e)$, $i=2$), $H^2(G_F,\mathbb{Z}_p(-e)) \cong \varprojlim_n H^2(G_F,M_n)$, which is killed by p^κ.

The cocycle. Hence $p^\kappa w = du$ for a continuous $u\colon G_F \to \mathbb{Z}_p$. Put $z = p^\kappa c' - u$. It is continuous, $dz = p^\kappa w - du = 0$, i.e. $z(\sigma\tau) = z(\sigma) + \chi_{\mathrm{cyc}}(\sigma)^{-e}z(\tau)$, and $z - p^\kappa c = p^\kappa(c'-c) - u$ takes values in $\mathbb{Z}_p$. □

Definition 25.25. Let p be a prime, K_0 a finite unramified extension of $\mathbb{Q}_p$, $s\geq 1$ an integer, $K_s = K_0(\mu_{p^s})$, and $e\geq 0$ an integer. Write $\log\chi_{\mathrm{cyc}}(\sigma)\in\mathbb{Z}_p$ for the p-adic logarithm of $\chi_{\mathrm{cyc}}(\sigma)$. Let $z\colon G_{K_s}\to\mathbb{Q}_p$ be a continuous map with

$$z(\sigma\tau) = z(\sigma) + \chi_{\mathrm{cyc}}(\sigma)^{-e}z(\tau) \qquad (\sigma,\tau\in G_{K_s}), \tag{25.3}$$

a **cocycle for χ_{cyc}^{-e}**. An element $\alpha\in K_s$ is a **dual exponential of z at level s in weight e** if there is $b\in B_{\mathrm{dR}}^+$ with

$$z(\sigma) - \log\chi_{\mathrm{cyc}}(\sigma)\,\alpha\,t^e - \bigl(\chi_{\mathrm{cyc}}(\sigma)^{-e}\sigma(b) - b\bigr) \in t^{e+1}B_{\mathrm{dR}}^+ \qquad (\sigma\in G_{K_s}). \tag{25.4}$$

Proposition 25.26. *Let p be an odd prime, K_0 a finite unramified extension of $\mathbb{Q}_p$, $K_1 = K_0(\mu_p)$, $e\geq 0$ an integer, and*

$$a_e = (e+2) + v_p(e) + \Bigl\lfloor\frac{e}{p-1}\Bigr\rfloor + 1.$$

Let $z\colon G_{K_1}\to\mathbb{Q}_p$ be continuous with $z(\sigma\tau) = z(\sigma) + \chi_{\mathrm{cyc}}(\sigma)^{-e}z(\tau)$, and suppose that there are $b\in B_{\mathrm{dR}}^+$ and elements $a_\sigma\in A_{\mathrm{inf}}$ $(\sigma\in G_{K_1})$ with

$$z(\sigma) - \bigl(\chi_{\mathrm{cyc}}(\sigma)^{-e}\sigma(b) - b\bigr) - a_\sigma \in t^{e+1}B_{\mathrm{dR}}^+ \qquad (\sigma\in G_{K_1}).$$

Then there are a continuous $z_1\colon G_{K_1}\to\mathbb{Z}_p$ with $z_1(\sigma\tau) = z_1(\sigma) + \chi_{\mathrm{cyc}}(\sigma)^{-e}z_1(\tau)$ and $b_1\in B_{\mathrm{dR}}^+$ with

$$p^{a_e}z(\sigma) - z_1(\sigma) - \bigl(\chi_{\mathrm{cyc}}(\sigma)^{-e}\sigma(b_1) - b_1\bigr) \in t^{e+1}B_{\mathrm{dR}}^+ \qquad (\sigma\in G_{K_1}).$$

Level s. *More generally, let p be a prime, $s\geq 1$ an integer with $s\geq 2$ if $p=2$, K_0 a finite unramified extension of $\mathbb{Q}_p$, $K_s = K_0(\mu_{p^s})$, $e\geq 0$ an integer, and*

$$a_{e,s} = s(e+2) + v_p(e) + \Bigl\lfloor\frac{e}{p-1}\Bigr\rfloor + 1.$$

If $z\colon G_{K_s}\to\mathbb{Q}_p$ is continuous with $z(\sigma\tau)=z(\sigma)+\chi_{\mathrm{cyc}}(\sigma)^{-e}z(\tau)$ and there are $b\in B_{\mathrm{dR}}^+$ and $a_\sigma\in A_{\mathrm{inf}}$ ($\sigma\in G_{K_s}$) with the displayed hypothesis above for all $\sigma\in G_{K_s}$, then there are a continuous $z_1\colon G_{K_s}\to\mathbb{Z}_p$ with $z_1(\sigma\tau)=z_1(\sigma)+\chi_{\mathrm{cyc}}(\sigma)^{-e}z_1(\tau)$ and $b_1\in B_{\mathrm{dR}}^+$ with

$$p^{a_{e,s}}z(\sigma)-z_1(\sigma)-\bigl(\chi_{\mathrm{cyc}}(\sigma)^{-e}\sigma(b_1)-b_1\bigr)\in t^{e+1}B_{\mathrm{dR}}^+\qquad(\sigma\in G_{K_s}).$$

For $s=1$ and p odd, $a_{e,1}=a_e$ and this is the statement above.

Proof. We prove the assertion of the paragraph Level s; the first assertion is its case $s=1$, p odd. So let p, s, K_s, e, z, b and $(a_\sigma)_{\sigma\in G_{K_s}}$ be as there. Write $\chi=\chi_{\mathrm{cyc}}$, $B=B_{\mathrm{dR}}^+$, $\delta y(\sigma)=\chi(\sigma)^{-e}\sigma(y)-y$ for $y\in B$, and $C_{e,s}=(s+1)e+v_p(e!)+s+1+e/(p-1)$.

The dual exponential of z and its size. By Lemma 25.27 (at level s) the continuous cocycle z has a dual exponential $\alpha\in K_s$ at level s in weight e: there is $b'\in B$ with $z(\sigma)-\log\chi(\sigma)\alpha t^e-\delta b'(\sigma)\in t^{e+1}B$ for $\sigma\in G_{K_s}$. Subtracting this from the hypothesis $z(\sigma)-\delta b(\sigma)-a_\sigma\in t^{e+1}B$ gives, with $x=b-b'$,

$$\chi(\sigma)^{-e}\sigma(x)-x+a_\sigma-\log\chi(\sigma)\,\alpha\,t^e\in t^{e+1}B\qquad(\sigma\in G_{K_s}),$$

so $v_p(\alpha)\ge -C_{e,s}$ by Lemma 25.28 (paragraph Level s; its hypotheses on p, s are ours).

Comparison of exponents. Let $\beta_{s,e}$ be the exponent of Proposition 25.29, so $\beta_{s,0}=-1$ and $\beta_{s,e}=s-1-e-v_p((e-1)!)$ for $e\ge1$. For $e\ge1$, using $v_p(e!)=v_p(e)+v_p((e-1)!)$,

$$a_{e,s}-C_{e,s}-\beta_{s,e}=1+\Bigl\lfloor\frac{e}{p-1}\Bigr\rfloor-\frac{e}{p-1}>0,$$

and for $e=0$, $a_{0,s}-C_{0,s}-\beta_{s,0}=(2s+1)-(s+1)+1=s+1>0$. Hence $\alpha'=p^{a_{e,s}-\beta_{s,e}}\alpha\in K_s$ (the exponent is an integer) has $v_p(\alpha')\ge a_{e,s}-\beta_{s,e}-C_{e,s}>0$, so $\alpha'\in\mathcal{O}_{K_s}$.

An integral cocycle with the same dual exponential. By Proposition 25.29 (at level s; it allows every prime p with $s\ge2$ if $p=2$), applied to $\alpha'\in\mathcal{O}_{K_s}$, there are a continuous cocycle $z_1\colon G_{K_s}\to\mathbb{Z}_p$ for χ^{-e} and $b''\in B$ with $z_1(\sigma)-\log\chi(\sigma)\,p^{\beta_{s,e}}\alpha'\,t^e-\delta b''(\sigma)\in t^{e+1}B$ for $\sigma\in G_{K_s}$, and $p^{\beta_{s,e}}\alpha'=p^{a_{e,s}}\alpha$. Put $b_1=p^{a_{e,s}}b'-b''$. Since δ is additive,

$$\begin{aligned}p^{a_{e,s}}z(\sigma)-z_1(\sigma)-\delta b_1(\sigma)&=p^{a_{e,s}}\bigl(z(\sigma)-\log\chi(\sigma)\alpha t^e-\delta b'(\sigma)\bigr)\\&\quad-\bigl(z_1(\sigma)-\log\chi(\sigma)p^{a_{e,s}}\alpha t^e-\delta b''(\sigma)\bigr)\in t^{e+1}B\end{aligned}$$

for all $\sigma\in G_{K_s}$, which is the assertion. For $s=1$, p odd: $a_{e,1}=(e+2)+v_p(e)+\lfloor e/(p-1)\rfloor+1=a_e$, $C_{e,1}=2e+v_p(e!)+2+e/(p-1)$ and $\beta_{1,e}$ are the level-1 quantities, and the argument is the proof of the first assertion. □

Lemma 25.27. *Let p be a prime, K_0 a finite unramified extension of $\mathbb{Q}_p$, $s\ge1$, $K_s=K_0(\mu_{p^s})$ and $e\ge0$. Every continuous cocycle $z\colon G_{K_s}\to\mathbb{Q}_p$ for χ_{cyc}^{-e} has a dual exponential at level s in weight e.*

Proof. Write $A=A_{\mathrm{inf}}$, $B=B_{\mathrm{dR}}^+$, $G=G_{K_s}$, $\chi=\chi_{\mathrm{cyc}}$, $n=e+1$, and $\sigma*x=\chi(\sigma)^{-e}\sigma(x)$ for $x\in B$ (and on quotients of B); for an element v we write $(\delta v)(\sigma)=\sigma*v-v$.

The quotient $V=B/t^nB$ and its topology. By (Ai3) ($\xi_0B=tB$) the natural map $A[1/p]/\xi_0^nA[1/p]\to V$ is a $G_{\mathbb{Q}_p}$-equivariant isomorphism. Let $Q\subseteq V$ be the image of A and $V^i=t^iB/t^nB$ for $0\le i\le n$. If $a\in A$ lies in ξ_0^iB ($i\le n$), then $a\in\xi_0^iA[1/p]$ by (Ai3) for $b=i$, so $p^ka\in\xi_0^iA$ for some k, and $a\in\xi_0^iA$ because $A/\xi_0^iA=Q_{0,i}$ has no p-torsion (Lemma 25.21(1)). Hence $Q\cong A/\xi_0^nA=Q_{0,n}$, $Q\cap V^i$ is the image of ξ_0^iA, and $V=\bigcup_k p^{-k}Q$. Give V the topology for which the sets $v+p^kQ$ ($k\in\mathbb{Z}$) form a basis. For $v\in p^{-k}Q$, $\sigma\mapsto\sigma(p^kv)$

is continuous into Q with the p-adic topology (Lemma 25.21(2)); since χ is continuous and $\chi(\sigma)^{-e}\sigma(v)-\chi(\tau)^{-e}\tau(v)=\chi(\sigma)^{-e}(\sigma v-\tau v)+(\chi(\sigma)^{-e}-\chi(\tau)^{-e})\tau v$ with $\tau v\in p^{-k}Q$, the map $\sigma\mapsto\sigma*v$ is continuous $G\to V$.

The graded maps. For $0\le i\le e$ define $\rho_i\colon V^i\to\mathbb{C}_p$ by $\rho_i(x \bmod t^nB)=\theta(t^{-i}x)$ for $x\in t^iB$; it is well defined since $t^{-i}t^nB\subseteq tB=\ker\theta$. It is additive, $\mathbb{Q}_p$-linear and surjective ($\theta(A)=\mathcal{O}_{\mathbb{C}_p}$), its kernel is V^{i+1}, and since $\theta\colon B\to\mathbb{C}_p$ factors through $B/tB\cong A[1/p]/\xi_0A[1/p]$, on which it is equivariant (Ai1), and $\sigma(t)=\chi(\sigma)t$,

$$\rho_i(\sigma*v)=\chi(\sigma)^{i-e}\,\sigma\big(\rho_i(v)\big)\qquad(\sigma\in G,\ v\in V^i).$$

As $\xi_0/t\in B^\times$, $\rho_i(\xi_0^i\lambda)=\theta(\xi_0/t)^i\theta(\lambda)$ for $\lambda\in A$, so $\rho_i(p^kQ\cap V^i)=p^k\theta(\xi_0/t)^i\mathcal{O}_{\mathbb{C}_p}$ and ρ_i is continuous.

Induction. We show: for $0\le i\le e$ there is $b_i\in V$ such that $c_i(\sigma)=z(\sigma)-(\sigma*b_i-b_i)\in V^i$ for all $\sigma\in G$. For $i=0$ take $b_0=0$. Let $i<e$ and b_i be given. The map $c_i\colon G\to V^i$ is continuous (previous paragraphs; z is continuous into $\mathbb{Q}_p\subset V$) and satisfies $c_i(\sigma\tau)=c_i(\sigma)+\sigma*c_i(\tau)$, as z does by (25.3) (G fixes $\mathbb{Q}_p$) and every δv does. So $\rho_i\circ c_i\colon G\to\mathbb{C}_p$ is continuous and satisfies $(\rho_i\circ c_i)(\sigma\tau)=(\rho_i\circ c_i)(\sigma)+\chi(\sigma)^{i-e}\sigma\big((\rho_i\circ c_i)(\tau)\big)$. Since $i-e\neq0$, Theorem B.47(1) for $F=K_s$ gives $y\in\mathbb{C}_p$ with $\rho_i(c_i(\sigma))=\chi(\sigma)^{i-e}\sigma(y)-y$. Choose $v\in V^i$ with $\rho_i(v)=y$; then $\rho_i(c_i(\sigma)-(\sigma*v-v))=0$, so $b_{i+1}=b_i+v$ works.

The last step. For $i=e$ the same argument shows that $\rho_e\circ c_e$ is a continuous map with $(\rho_e\circ c_e)(\sigma\tau)=(\rho_e\circ c_e)(\sigma)+\sigma\big((\rho_e\circ c_e)(\tau)\big)$. By Theorem B.47(2) there are $\alpha\in K_s$ and $y\in\mathbb{C}_p$ with $\rho_e(c_e(\sigma))=\alpha\log\chi(\sigma)+\sigma(y)-y$. Choose $v\in V^e$ with $\rho_e(v)=y$, and let $w\in V^e$ be the class of αt^e ($\alpha\in\overline{\mathbb{Q}}_p\subset B$); $\rho_e(w)=\theta(\alpha)=\alpha$ by (Ai3). Then

$$\rho_e\big(c_e(\sigma)-\log\chi(\sigma)\,w-(\sigma*v-v)\big)=\alpha\log\chi(\sigma)+\sigma(y)-y-\alpha\log\chi(\sigma)-(\sigma(y)-y)=0,$$

and $\ker\rho_e=V^{e+1}=0$. Let $b\in B$ lift b_e+v. The vanishing in $V=B/t^{e+1}B$ reads $z(\sigma)-\log\chi(\sigma)\alpha t^e-(\chi(\sigma)^{-e}\sigma(b)-b)\in t^{e+1}B$ for all $\sigma\in G$, which is (25.4): α is a dual exponential of z at level s in weight e. □

Lemma 25.28. *Let p be an odd prime, K_0 a finite unramified extension of $\mathbb{Q}_p$, $K_1=K_0(\mu_p)$ and $e\ge0$. Let $x\in B_{\mathrm{dR}}^+$, $\alpha\in K_1$ and $a_\sigma\in A_{\mathrm{inf}}$ ($\sigma\in G_{K_1}$) satisfy*

$$\chi_{\mathrm{cyc}}(\sigma)^{-e}\sigma(x)-x+a_\sigma-\log\chi_{\mathrm{cyc}}(\sigma)\,\alpha\,t^e\in t^{e+1}B_{\mathrm{dR}}^+\qquad(\sigma\in G_{K_1}).$$

Then

$$v_p(\alpha)\ \ge\ -\Big(2e+v_p(e!)+2+\frac{e}{p-1}\Big).$$

Level s. *More generally, let p be a prime, $s\ge1$ an integer with $s\ge2$ if $p=2$, K_0 a finite unramified extension of $\mathbb{Q}_p$, $K_s=K_0(\mu_{p^s})$ and $e\ge0$. If $x\in B_{\mathrm{dR}}^+$, $\alpha\in K_s$ and $a_\sigma\in A_{\mathrm{inf}}$ ($\sigma\in G_{K_s}$) satisfy the displayed relation above for all $\sigma\in G_{K_s}$, then*

$$v_p(\alpha)\ \ge\ -\Big((s+1)e+v_p(e!)+s+1+\frac{e}{p-1}\Big).$$

For $s=1$ and p odd this is the statement above.

Proof. We prove the level-s assertion; the first assertion is its case $s=1$, p odd (then $(s+1)e+v_p(e!)+s+1=2e+v_p(e!)+2$). So let p, s, K_s, e, x, $\alpha\in K_s$ and $(a_\sigma)_{\sigma\in G_{K_s}}$ be as in the level-s assertion. Write $A=A_{\mathrm{inf}}$, $B=B_{\mathrm{dR}}^+$, $\mathcal{A}\subseteq B$ for the image of A, $\mathrm{Fil}^i=t^iB$, $G=G_{K_s}$, $\chi=\chi_{\mathrm{cyc}}$, $\sigma*y=\chi(\sigma)^{-e}\sigma(y)$ for $y\in B$, $\pi=\zeta_p-1\in K_1\subseteq K_s$ (so $v_p(\pi)=1/(p-1)$; for $p=2$, $\pi=-2$), and $w_\sigma=\sigma*x-x+a_\sigma-\log\chi(\sigma)\alpha t^e\in\mathrm{Fil}^{e+1}$. Since $K_0/\mathbb{Q}_p$ is unramified,

$[K_0(\mu_{p^j}) : K_0] = (p-1)p^{j-1}$ for $j \geq 1$, so χ maps G_{K_0} onto $\mathbb{Z}_p^\times$ and G onto $1 + p^s\mathbb{Z}_p$; fix $g \in G$ with $v_p(\chi(g) - 1) = s$. As p is odd, or $p = 2$ and $s \geq 2$, the logarithm maps $1 + p^s\mathbb{Z}_p$ isomorphically onto $p^s\mathbb{Z}_p$ preserving valuations, so $v_p(\log \chi(g)) = s$.

Three facts on B. (a) $\mathcal{A} \cap \mathrm{Fil}^i$ is the image of $\xi_0^i A$: if $a \in A$ lies in $\xi_0^i B = \mathrm{Fil}^i$ (Ai3), then $a \in \xi_0^i A[1/p]$ by (Ai3), so $p^k a \in \xi_0^i A$ for some k, and $a \in \xi_0^i A$ as $Q_{0,i}$ has no p-torsion (Lemma 25.21(1)). (b) $t = \xi_0 \cdot (q-1)\, u$ with $u \in B$, $\theta(u) = 1$: indeed $q^p = [\varepsilon]$, $\theta([\varepsilon]) = 1$, so $t = \log(1 + (q^p - 1)) = (q^p - 1)u$ with $u = \sum_{k \geq 1} (-1)^{k-1}(q^p - 1)^{k-1}/k$, convergent in the complete discrete valuation ring B with $u - 1 \in \ker \theta$, and $q^p - 1 = \xi_0(q-1)$. Hence $\theta(t/\xi_0) = \theta(q-1) = \zeta_p - 1 = \pi$. (c) For $0 \leq i \leq e$ let $\rho_i \colon \mathrm{Fil}^i \to \mathbb{C}_p$, $\rho_i(y) = \theta(t^{-i}y)$. It is additive, has kernel Fil^{i+1}, satisfies $\rho_i(\sigma * y) = \chi(\sigma)^{i-e}\sigma(\rho_i(y))$ (as $\sigma(t) = \chi(\sigma)t$ and θ is equivariant), $\rho_e(\beta t^e) = \beta$ for $\beta \in \overline{\mathbb{Q}}_p$, and by (a), (b), $\rho_i(\mathcal{A} \cap \mathrm{Fil}^i) = \theta(\xi_0/t)^i \theta(A) = \pi^{-i}\mathcal{O}_{\mathbb{C}_p}$.

Induction. Put $c_0 = 0$ and $c_{i+1} = c_i + 1 + s + v_p(e-i)$ for $0 \leq i < e$, so that $c_e = \sum_{k=1}^{e}(1 + s + v_p(k)) = (s+1)e + v_p(e!)$. We show for $0 \leq i \leq e$: $p^{c_i}x \in \mathcal{A} + \mathrm{Fil}^i$. For $i = 0$ this is trivial. Let $i < e$ and $p^{c_i}x = A_i + y$ with $A_i \in \mathcal{A}$, $y \in \mathrm{Fil}^i$. Then

$$\sigma * y - y = p^{c_i}\big(\log \chi(\sigma)\alpha t^e - a_\sigma + w_\sigma\big) - (\sigma * A_i - A_i).$$

Here $p^{c_i} \log \chi(\sigma)\alpha t^e$ and $p^{c_i} w_\sigma$ lie in $\mathrm{Fil}^e \subseteq \mathrm{Fil}^{i+1}$, and $E_\sigma = p^{c_i}a_\sigma + \sigma * A_i - A_i \in \mathcal{A}$ ($\chi(\sigma)^{-e} \in \mathbb{Z}_p$, and A is Galois stable). As $\sigma * y - y \in \mathrm{Fil}^i$, $E_\sigma \in \mathcal{A} \cap \mathrm{Fil}^i$, and $\rho_i(\sigma * y - y) = -\rho_i(E_\sigma) \in \pi^{-i}\mathcal{O}_{\mathbb{C}_p}$ by (c). Let $u_i = \pi^i \rho_i(y) \in \mathbb{C}_p$ and $c(\sigma) = \chi(\sigma)^{i-e}\sigma(u_i) - u_i = \pi^i \rho_i(\sigma * y - y)$ ($\pi \in K_s$ is fixed by G). Then $c \colon G \to \mathcal{O}_{\mathbb{C}_p}$ is continuous. The pair (U, g) with $U \equiv 1$ on $G = G_{K_0(\mu_{p^s})}$ is a Sen frame of level s and rank 1 over K_0 with weights in $J = \{0\}$: (Sf1) is clear, (Sf2) holds as $v_p(\chi(g) - 1) = s$ and $U(g) = 1$, (Sf3) holds with $e_1 = 1$, $u_1 = 0$, and the level s is admissible as $s \geq 2$ if $p = 2$. Its twisted action is $\sigma *_{i-e} y' = \chi(\sigma)^{i-e}\sigma(y')$, and c is a cocycle for $*_{i-e}$, the coboundary of u_i. Choose $k \geq 0$ with $p^k u_i \in \mathcal{O}_{\mathbb{C}_p}$; then $p^k c$ is the coboundary of an integral element, and Proposition 25.17(1) (level $m = s$) gives $u' \in \mathcal{O}_{\mathbb{C}_p}$ with $p^{1+N}c(\sigma) = \chi(\sigma)^{i-e}\sigma(u') - u'$, where $N = N_{\{0\},s}(i-e) = \max(1, s + v_p(e-i)) = s + v_p(e-i)$. The element $p^{1+N}u_i - u'$ satisfies $\chi(\sigma)^{i-e}\sigma(\cdot) = \cdot$ for all $\sigma \in G$, and $i - e \neq 0$, so it is 0 by Theorem B.42(2) for $F = K_s$. Hence $p^{1+s+v_p(e-i)}\rho_i(y) \in \pi^{-i}\mathcal{O}_{\mathbb{C}_p} = \rho_i(\mathcal{A} \cap \mathrm{Fil}^i)$: there is $A' \in \mathcal{A} \cap \mathrm{Fil}^i$ with $p^{1+s+v_p(e-i)}y - A' \in \ker \rho_i = \mathrm{Fil}^{i+1}$, and $p^{c_{i+1}}x = p^{1+s+v_p(e-i)}A_i + A' + (p^{1+s+v_p(e-i)}y - A') \in \mathcal{A} + \mathrm{Fil}^{i+1}$.

The top piece. Write $p^{c_e}x = A_e + y$ with $A_e \in \mathcal{A}$, $y \in \mathrm{Fil}^e$. As above, $E_\sigma = p^{c_e}a_\sigma + \sigma * A_e - A_e \in \mathcal{A}$ equals $p^{c_e}\log \chi(\sigma)\alpha t^e + p^{c_e}w_\sigma - (\sigma * y - y) \in \mathrm{Fil}^e$, so $\rho_e(E_\sigma) \in \pi^{-e}\mathcal{O}_{\mathbb{C}_p}$, and applying ρ_e (which kills Fil^{e+1} and satisfies $\rho_e(\sigma * y) = \sigma(\rho_e(y))$),

$$\sigma(\bar{y}) - \bar{y} = p^{c_e} \log \chi(\sigma)\, \alpha - \rho_e(E_\sigma), \qquad \bar{y} = \rho_e(y).$$

Put $u = \pi^e \bar{y}$ and $\beta = p^{c_e}\pi^e \alpha \in K_s$. Then $c(\sigma) = \beta \log \chi(\sigma) - (\sigma(u) - u) = \pi^e \rho_e(E_\sigma) \in \mathcal{O}_{\mathbb{C}_p}$ for all $\sigma \in G$, and c is continuous with $c(\sigma\tau) = c(\sigma) + \sigma(c(\tau))$ ($\log \chi$ is a homomorphism and β is fixed by G). On $H_{K_0} = \mathrm{Gal}(\overline{\mathbb{Q}}_p / K_0(\mu_{p^\infty})) = H_{K_s}$, $\log \chi = 0$, so c restricted to H_{K_s} is an $\mathcal{O}_{\mathbb{C}_p}$-valued continuous cocycle; by Lemma 25.19 (for $F = K_s$, $r = 1$, $a = p$) there is $x_1 \in \mathcal{O}_{\mathbb{C}_p}$ with $p\, c(h) = h(x_1) - x_1$ for $h \in H_{K_s}$. Then $u_2 = pu + x_1$ satisfies $h(u_2) - u_2 = p(h(u) - u) + p\, c(h) = 0$, so $u_2 \in \widehat{K}_\infty$ by Theorem B.42(1). For $\sigma \in G$,

$$c'(\sigma) := p\, c(\sigma) - (\sigma(x_1) - x_1) = p\beta \log \chi(\sigma) - (\sigma(u_2) - u_2)$$

lies in $\mathcal{O}_{\mathbb{C}_p}$ (left side) and in $\widehat{K}_\infty$ (right side; $\widehat{K}_\infty$ is stable under G_{K_0}), hence in $\mathcal{O}_{\widehat{K}_\infty}$. Apply R_s of Lemma 25.18(1) (level $m = s$, admissible as $s \geq 2$ if $p = 2$) at $\sigma = g$: R_s is K_s-linear with $R_s(1) = 1$, commutes with $g \in G_{K_0}$, and $R_s(u_2) \in K_s$ is fixed by $g \in G_{K_s}$, so $R_s(g(u_2) - u_2) = 0$ and

$$p\beta \log \chi(g) = R_s(c'(g)) \in R_s(\mathcal{O}_{\widehat{K}_\infty}) \subseteq \mathcal{O}_{K_s}.$$

Thus $1+v_p(\beta)+s\ge 0$, i.e. $v_p(\beta)\ge -(s+1)$, and $v_p(\alpha)=v_p(\beta)-c_e-e/(p-1)\ge -\big((s+1)e+v_p(e!)+s+1+e/(p-1)\big)$. □

Proposition 25.29. *Let p be a prime, $s\ge 1$ an integer with $s\ge 2$ if $p=2$, K_0 a finite unramified extension of $\mathbb{Q}_p$, $K_s=K_0(\mu_{p^s})$, $e\ge 0$, and*

$$\beta_{s,e}=-1\ \ (e=0),\qquad \beta_{s,e}=s-1-e-v_p\big((e-1)!\big)\ \ (e\ge 1).$$

For every $\alpha\in\mathcal{O}_{K_s}$ there is a continuous cocycle $z\colon G_{K_s}\to\mathbb{Z}_p$ for χ_{cyc}^{-e} such that $p^{\beta_{s,e}}\alpha$ is a dual exponential of z at level s in weight e.

Proof. Let $\alpha\in\mathcal{O}_{K_s}$, $u=e!/p^{v_p(e!)}\in\mathbb{Z}_p^\times$ and $x=u\alpha\in\mathcal{O}_{K_s}$. By Lemma 25.30 there is a ψ-invariant $f\in\mathcal{O}_{K_0}[[X]]$ with

$$(\partial^e f)^{\varphi^{-s}}(\zeta_{p^s}-1)=p^{N_{s,e}}x. \tag{25.5}$$

Step 1 (the class). Let $T=\mathbb{Z}_p(-e)$ with basis e_{-e} ($\sigma e_{-e}=\chi_{\mathrm{cyc}}(\sigma)^{-e}e_{-e}$) and $V=T\otimes\mathbb{Q}_p$. The ψ-invariance of f is the criterion of Theorem B.48 for $\psi(f(\pi))=f(\pi)$, so $y=f(\pi)\otimes e_{-e}\in\mathbf{D}(T)^{\psi=1}$, and Theorem B.48 applies with $n=s\ge 1$. By its item 1 there is a continuous cocycle $c\colon G_{K_s}\to T$ whose class maps to $h^1_{K_s,V}(y)$. Write $c(\sigma)=z(\sigma)e_{-e}$; then $z\colon G_{K_s}\to\mathbb{Z}_p$ is continuous, and the cocycle rule $c(\sigma\tau)=c(\sigma)+\sigma c(\tau)$ is (25.3).

Step 2 (its dual exponential). Since $\sigma(t)=\chi_{\mathrm{cyc}}(\sigma)t$, the element $d_e=t^e\otimes e_{-e}$ is G_{K_0}-invariant, $D_{\mathrm{cris}}(V)=K_0d_e$ and $\varphi(d_e)=p^ed_e$. With $t=\log(1+\pi)\in B^+_{\mathrm{rig},K_0}$ we have $y=f(\pi)t^{-e}\otimes d_e$, and since $\varphi^{-s}(t)=p^{-s}t$,

$$\varphi^{-s}(y)=f^{\varphi^{-s}}(\zeta_{p^s}e^{t/p^s}-1)\,(p^{-s}t)^{-e}\otimes p^{-se}d_e=F(t)\,t^{-e}\otimes d_e,\qquad F(t)=h(\zeta_{p^s}e^{t/p^s}-1),\ h=f^{\varphi^{-s}}.$$

By the chain rule $\frac{d}{dt}\,k(\zeta_{p^s}e^{t/p^s}-1)=p^{-s}(\partial k)(\zeta_{p^s}e^{t/p^s}-1)$ for every $k\in\mathcal{O}_{K_0}[[X]]$, so $F^{(e)}(0)=p^{-se}(\partial^e h)(\zeta_{p^s}-1)$, and $\partial^e h=(\partial^e f)^{\varphi^{-s}}$ since ∂ has integer coefficients. Hence the coefficient of t^0 is

$$\partial_V(\varphi^{-s}(y))=\frac{F^{(e)}(0)}{e!}\,d_e=\frac{p^{-se}}{e!}(\partial^e f)^{\varphi^{-s}}(\zeta_{p^s}-1)\,d_e=\frac{p^{N_{s,e}-se}\,u}{e!}\,\alpha\,d_e=p^{N_{s,e}-se-v_p(e!)}\alpha\,d_e$$

by (25.5). By item 2 of Theorem B.48 and the side calculation,

$$\exp^*_{K_s,V^*(1)}\big(h^1_{K_s,V}(y)\big)=p^{-s}\partial_V(\varphi^{-s}(y))=p^{\beta_{s,e}}\alpha\,d_e.$$

(The convention for φ^{-s} on the coefficients plays no role: applying the Frobenius to the coefficients preserves ψ-invariance, so the set of values in Lemma 25.30 is the same under either convention.)

Step 3 (Kato's formula). By item 3 of Theorem B.48, the image of the class of c in $H^1(K_s,B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}V)$ is the class of $\sigma\mapsto\log\chi_{\mathrm{cyc}}(\sigma)\,p^{\beta_{s,e}}\alpha\,d_e$. So there is $B\in B_{\mathrm{dR}}\otimes V$ with $c(\sigma)-\log\chi_{\mathrm{cyc}}(\sigma)p^{\beta_{s,e}}\alpha\,t^e\otimes e_{-e}=(\sigma-1)B$ for all $\sigma\in G_{K_s}$. Writing $B=b\otimes e_{-e}$ with $b\in B_{\mathrm{dR}}$, so that $(\sigma-1)B=(\chi_{\mathrm{cyc}}(\sigma)^{-e}\sigma(b)-b)\otimes e_{-e}$, this reads

$$z(\sigma)-\log\chi_{\mathrm{cyc}}(\sigma)\,p^{\beta_{s,e}}\alpha\,t^e=\chi_{\mathrm{cyc}}(\sigma)^{-e}\sigma(b)-b\qquad(\sigma\in G_{K_s}). \tag{25.6}$$

Step 4 ($b\in B^+_{\mathrm{dR}}$). The left side of (25.6) lies in B^+_{dR} ($e\ge 0$), so $c_\sigma:=\chi_{\mathrm{cyc}}(\sigma)^{-e}\sigma(b)-b\in B^+_{\mathrm{dR}}$. Suppose $b\notin B^+_{\mathrm{dR}}$, let $m\ge 1$ be least with $t^mb\in B^+_{\mathrm{dR}}$ ($B_{\mathrm{dR}}=B^+_{\mathrm{dR}}[1/t]$) and $w=\theta(t^mb)\in\mathbb{C}_p$; $w\ne 0$ by minimality, as $\ker\theta=tB^+_{\mathrm{dR}}$. For $\sigma\in G_{K_s}$, $\sigma(t^mb)=\chi_{\mathrm{cyc}}(\sigma)^mt^m\sigma(b)=\chi_{\mathrm{cyc}}(\sigma)^{m+e}(t^mb+t^mc_\sigma)$ and $\theta(t^mc_\sigma)=0$, so $\sigma(w)=\chi_{\mathrm{cyc}}(\sigma)^{m+e}w$ with $m+e\ge 1$. This contradicts Tate's theorem $\mathbb{C}_p(j)^{G_{K_s}}=0$ for $j\ne 0$. Hence $b\in B^+_{\mathrm{dR}}$, and (25.6) gives (25.4) for z and $p^{\beta_{s,e}}\alpha$ (with difference $0\in t^{e+1}B^+_{\mathrm{dR}}$). Thus $p^{\beta_{s,e}}\alpha$ is a dual exponential of z at level s in weight e (Definition 25.25). □

Lemma 25.30. *Let p be a prime, $s \geq 1$ an integer with $s \geq 2$ if $p = 2$, K_0 a finite unramified extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and Frobenius φ, $K_s = K_0(\mu_{p^s})$, and $e \geq 0$ an integer. For $h \in \mathcal{O}[[X]]$ and $j \in \mathbb{Z}$ let h^{φ^j} be h with φ^j applied to its coefficients, let $\partial = (1+X)\frac{d}{dX}$, and call h ψ-**invariant** if*

$$h^{\varphi}((1+X)^p - 1) = \frac{1}{p}\sum_{\eta^p=1} h(\eta(1+X) - 1).$$

Put $N_{s,0} = s - 1$ and $N_{s,e} = (e+1)(s-1) + s + v_p(e)$ for $e \geq 1$. Then for every $x \in \mathcal{O}_{K_s}$ there is a ψ-invariant $f \in \mathcal{O}[[X]]$ with

$$(\partial^e f)^{\varphi^{-s}}(\zeta_{p^s} - 1) = p^{N_{s,e}}\, x.$$

Proof. Write $\zeta_j = \zeta_{p^j}$ ($\zeta_0 = 1$), $\mathrm{Tr} = \mathrm{Tr}_{K_0/\mathbb{Q}_p}$, $\mathfrak{m} = (p, X) \subset \mathcal{O}[[X]]$; $\mathcal{O}[[X]]$ is $\mathfrak{m}$-adically complete. Let $\varphi(h) = h^{\varphi}((1+X)^p - 1)$, a continuous ring endomorphism, and for $0 \leq j \leq s$ let $\mathrm{ev}_j(h) = h^{\varphi^{-j}}(\zeta_j - 1) \in \mathcal{O}_{K_j}$; ev_j is additive, continuous ($v_p(\zeta_j - 1) > 0$) and $\mathrm{ev}_j(ch) = \varphi^{-j}(c)\mathrm{ev}_j(h)$ for $c \in \mathcal{O}$. We use the following classical facts, $K_0/\mathbb{Q}_p$ being unramified:

(a) $\mathrm{Tr}\colon \mathcal{O} \to \mathbb{Z}_p$ is onto (the residue trace is onto, the residue extension being separable) and $\mathrm{Tr} \circ \varphi = \mathrm{Tr}$; $(1-\varphi)\mathcal{O} = \{u \in \mathcal{O} : \mathrm{Tr}\, u = 0\}$ (additive Hilbert 90 on the residue field, lifted by successive approximation in the p-adically complete $\mathcal{O}$); for $j \geq 1$ the map $1 - p^j\varphi$ is a bijection of $\mathcal{O}$ with inverse $u \mapsto \sum_{i\geq 0} p^{ji}\varphi^i(u)$.

Moreover:

(F1) $\varphi^k(h) = h^{\varphi^k}((1+X)^{p^k} - 1)$, hence $\mathrm{ev}_s(\varphi^k h) = \mathrm{ev}_{s-k}(h)$ for $0 \leq k \leq s$ (as $\zeta_s^{p^k} = \zeta_{s-k}$) and $\mathrm{ev}_s(\varphi^k h) = \varphi^{k-s}(h(0))$ for $k \geq s$.

(F2) $\partial\varphi = p\,\varphi\partial$ (chain rule: $\partial(h^{\varphi}((1+X)^p - 1)) = p(1+X)^p(h^{\varphi})'((1+X)^p - 1)$), so $\partial^e\varphi^k = p^{ek}\varphi^k\partial^e$; $\partial^e(1+X)^a = a^e(1+X)^a$; ∂ is continuous and ∂^e kills constants for $e \geq 1$.

(F3) For $b = 1 + p^s$ and $e \geq 1$, $v_p(b^e - 1) = s + v_p(e)$, because p is odd or $s \geq 2$ (lifting the exponent).

Step 1 (a construction of ψ-invariant series). Let A be a finite set of positive integers prime to p, $g = \sum_{a\in A}\lambda_a(1+X)^a$ with $\lambda_a \in \mathcal{O}$ and $\mathrm{Tr}\, g(0) = 0$, and by (a) choose $d_f \in \mathcal{O}$ with $d_f - \varphi(d_f) = g(0)$. Put

$$f = d_f + \sum_{k\geq 0}\varphi^k(g - g(0)).$$

The series converges: with $u_k = (1+X)^{p^k} - 1$ one has $u_0 = X$ and $u_{k+1} = (1+u_k)^p - 1 \in u_k\mathfrak{m}$, so $u_k \in \mathfrak{m}^{k+1}$, and $g - g(0) \in X\mathcal{O}[[X]]$ gives $\varphi^k(g - g(0)) \in u_k\mathcal{O}[[X]] \subseteq \mathfrak{m}^{k+1}$. We check that f is ψ-invariant. Let $S(h) = \sum_{\eta^p=1} h(\eta(1+X) - 1)$, computed in $\mathcal{O}[\zeta_1][[X]]$; S is additive and continuous, since $\eta(1+X) - 1 = (\eta - 1) + \eta X$ lies in the ideal $(\zeta_1 - 1, X)$. We have $S(c) = pc$ for constants, $S((1+X)^a) = \big(\sum_\eta \eta^a\big)(1+X)^a = 0$ for $p \nmid a$, and $S(\varphi(h)) = p\varphi(h)$ because $\varphi(h)(\eta(1+X) - 1) = h^{\varphi}(\eta^p(1+X)^p - 1) = \varphi(h)$. Hence

$$S(f) = pd_f + S(g) - pg(0) + p\sum_{k\geq 1}\varphi^k(g - g(0)) = p\Big(\varphi(d_f) + \sum_{k\geq 0}\varphi^{k+1}(g - g(0))\Big) = p\,\varphi(f),$$

using $S(g) = 0$, $d_f - g(0) = \varphi(d_f)$ and the continuity of φ. This is the ψ-invariance $f^{\varphi}((1+X)^p - 1) = p^{-1}S(f)$.

Step 2 (the case $e=0$). Let (μ,d) be admissible at level s for $e=0$, i.e. $d-\varphi(d)=\varphi^s(M)$ with $M=\sum_{a\in P_s}\mu_a$; then $\operatorname{Tr} M=\operatorname{Tr}(d-\varphi(d))=0$. Put $A=P_s$, $\lambda_a=\varphi^s(\mu_a)$, so $g(0)=\varphi^s(M)$ has trace 0, and take $d_f=\varphi^s\big(d+\sum_{k=0}^{s-1}\varphi^k(M)\big)$; indeed $d_f-\varphi(d_f)=\varphi^s\big(d-\varphi(d)+M-\varphi^s(M)\big)=\varphi^s(M)=g(0)$. Let f be the series of Step 1. By (F1) and continuity, the terms $k\ge s$ contribute $\varphi^{k-s}(g(0)-g(0))=0$, and

$$\operatorname{ev}_s(f)=\varphi^{-s}(d_f)+\sum_{k=0}^{s-1}\big(\operatorname{ev}_{s-k}(g)-\varphi^{k-s}(g(0))\big).$$

Here $\operatorname{ev}_{s-k}(g)=\sum_a\varphi^{k-s}(\lambda_a)\zeta_{s-k}^{a}=\sum_a\varphi^k(\mu_a)\zeta_s^{ap^k}$ and $\varphi^{k-s}(g(0))=\varphi^k(M)$, so $\operatorname{ev}_s(f)=d+\sum_{k=0}^{s-1}\sum_a\varphi^k(\mu_a)\zeta_s^{ap^k}=C_s(\mu,d)$.

Step 3 (the case $e\ge 1$). Let (μ,d) be admissible at level s, put $\nu=s+v_p(e)$, $\mu'=p^\nu\mu$, $d'=p^\nu d$ (again admissible), $\lambda_a=a^{-e}\varphi^s(\mu'_a)$ for $a\in P_s$ (note $a^{-e}\in\mathbb{Z}_p^\times$), and $\tau=\operatorname{Tr}\sum_{a\in P_s}\lambda_a\in p^\nu\mathbb{Z}_p$. Let $b=1+p^s$; by (F3), $\tau/(1-b^e)\in\mathbb{Z}_p$, and by (a) there is $\kappa\in\mathcal{O}$ with $\operatorname{Tr}\kappa=-\tau/(1-b^e)$. Put

$$g=\sum_{a\in P_s}\lambda_a(1+X)^a-b^e\kappa(1+X)+\kappa(1+X)^b,$$

a series of the form of Step 1 with $A=P_s\cup\{b\}$ ($b\notin P_s$, $1\in P_s$). Then $\operatorname{Tr} g(0)=\tau+(1-b^e)\operatorname{Tr}\kappa=0$ ($b^e\in\mathbb{Z}_p$), so Step 1 gives a ψ-invariant f. By (F2) and continuity, $\partial^e f=\sum_{k\ge0}p^{ek}\varphi^k(G)$ with

$$G=\partial^e g=G_0+b^e\kappa\big((1+X)^b-(1+X)\big),\qquad G_0=\sum_{a\in P_s}a^e\lambda_a(1+X)^a.$$

For $0\le j\le s$, $\operatorname{ev}_j\big(b^e\kappa((1+X)^b-(1+X))\big)=b^e\varphi^{-j}(\kappa)(\zeta_j^b-\zeta_j)=0$ since $p^j\mid b-1$, and the constant term of this correction is 0. Hence by (F1)

$$\operatorname{ev}_s(\partial^e f)=\sum_{k=0}^{s-1}p^{ek}\operatorname{ev}_{s-k}(G_0)+p^{es}\delta,\qquad \delta=\sum_{i\ge0}p^{ei}\varphi^i(G_0(0)).$$

Since $a^e\varphi^{k-s}(\lambda_a)=\varphi^{k-s}(a^e\lambda_a)=\varphi^k(\mu'_a)$, we get $\operatorname{ev}_{s-k}(G_0)=\sum_a\varphi^k(\mu'_a)\zeta_s^{ap^k}$; and $\delta-p^e\varphi(\delta)=G_0(0)=\varphi^s(\sum_a\mu'_a)=d'-p^e\varphi(d')$, so $\delta=d'$ by the injectivity of $1-p^e\varphi$ (facts (a)). Therefore $\operatorname{ev}_s(\partial^e f)=C_s(\mu',d')=p^\nu C_s(\mu,d)$.

Step 4 (conclusion). Let $x\in\mathcal{O}_{K_s}$. By Lemma 25.31 (level $m=s$) there is an admissible (μ,d) with $C_s(\mu,d)=p^{(e+1)(s-1)}x$. Step 2 ($e=0$) gives a ψ-invariant f with $\operatorname{ev}_s(f)=p^{s-1}x=p^{N_{s,0}}x$, and Step 3 ($e\ge1$) one with $\operatorname{ev}_s(\partial^e f)=p^{(e+1)(s-1)+s+v_p(e)}x=p^{N_{s,e}}x$. Since $\operatorname{ev}_s(\partial^e f)=(\partial^e f)^{\varphi^{-s}}(\zeta_s-1)$, this is the claim. □

Lemma 25.31. *Let p be a prime, K_0 a finite unramified extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and Frobenius φ, $e\ge0$ an integer, $m\ge1$ an integer, $K_m=K_0(\mu_{p^m})$, and $P_m=\{a\in\mathbb{Z}:0<a<p^m,\ p\nmid a\}$. For $\mu=(\mu_a)_{a\in P_m}\in\mathcal{O}^{P_m}$ and $d\in\mathcal{O}$ put*

$$C_m(\mu,d)=\sum_{k=0}^{m-1}p^{ek}\sum_{a\in P_m}\varphi^k(\mu_a)\,\zeta_{p^m}^{ap^k}+p^{em}d\ \in\mathcal{O}_{K_m},$$

*and call (μ,d) **admissible** if $d-p^e\varphi(d)=\varphi^m\big(\sum_{a\in P_m}\mu_a\big)$. Then for every $x\in\mathcal{O}_{K_m}$ there is an admissible pair (μ,d) with $C_m(\mu,d)=p^{(e+1)(m-1)}x$.*

Proof. Write $\zeta_j=\zeta_{p^j}$, $\operatorname{Tr}=\operatorname{Tr}_{K_0/\mathbb{Q}_p}$, and let $J_m\subset\mathcal{O}_{K_m}$ be the set of values $C_m(\mu,d)$ of admissible pairs. Both admissibility and C_m are $\mathbb{Z}_p$-linear in (μ,d) (φ is $\mathbb{Z}_p$-linear), so J_m is a $\mathbb{Z}_p$-submodule. We use the following classical facts.

(a) Since $K_0/\mathbb{Q}_p$ is unramified: $\mathrm{Tr}\colon \mathcal{O} \to \mathbb{Z}_p$ is onto; $(1-\varphi)\mathcal{O} = \{u \in \mathcal{O} : \mathrm{Tr}\, u = 0\}$ and $\ker(1-\varphi) = \mathbb{Z}_p$; $\mathrm{Tr} \circ \varphi = \mathrm{Tr}$; for $j \geq 1$ the map $1 - p^j\varphi$ is a bijection of $\mathcal{O}$ with inverse $u \mapsto \sum_{i\geq 0} p^{ji}\varphi^i(u)$. In particular, for $e \geq 1$ every μ has exactly one $d = d(\mu)$ with (μ, d) admissible, and $d(\mu)$ depends continuously on μ.

(b) $\mathcal{O}_{K_j} = \mathcal{O}[\zeta_j]$ for $j \geq 0$ (K_0 unramified, $\mathbb{Q}_p(\zeta_j)$ totally ramified). For $j = 1$ the $p-1$ elements ζ_1^a, $a \in P_1$, form an $\mathcal{O}$-basis of $\mathcal{O}_{K_1}$ (they span, since $1 = -\sum_{a\in P_1} \zeta_1^a$, and the rank is $p-1$). For $j \geq 2$, ζ_j is a root of $X^p - \zeta_{j-1}$ and $\mathcal{O}_{K_j} = \bigoplus_{i=0}^{p-1} \mathcal{O}_{K_{j-1}}\zeta_j^i$; since $\zeta_{j-1}^l\zeta_j^i = \zeta_j^{pl+i}$, this gives $\mathcal{O}_{K_j} = \mathcal{O}_{K_{j-1}} + W_j$ with $W_j = \sum_{a\in P_j} \mathcal{O}\zeta_j^a$. Finally $\sum_{l=0}^{p-1} \zeta_j^{b+lp^{j-1}} = \zeta_j^b \sum_{l=0}^{p-1} \zeta_1^l = 0$ for every integer b.

We prove $J_m \supseteq p^{(e+1)(m-1)}\mathcal{O}_{K_m}$ by induction on m.

Step 1: $J_1 = \mathcal{O}_{K_1}$. Here $C_1(\mu, d) = \sum_{a\in P_1} \mu_a\zeta_1^a + p^e d$.

Case $e \geq 1$. Let $x \in \mathcal{O}_{K_1}$ and write $x = \sum_a c_a\zeta_1^a$ by (b). Then $C_1(c, d(c)) = x + p^e d(c)$, so $\mathcal{O}_{K_1} \subseteq J_1 + p^e\mathcal{O}_{K_1}$. The set J_1 is the image of the compact set $\mathcal{O}^{P_1}$ under the continuous map $\mu \mapsto C_1(\mu, d(\mu))$, hence closed. Iterating, $x = j_0 + p^e x_1$, $x_1 = j_1 + p^e x_2, \ldots$ with $j_i \in J_1$, so $x = \sum_{i\geq 0} p^{ei} j_i$ is a limit of elements of the $\mathbb{Z}_p$-module J_1 and lies in J_1.

Case $e = 0$. First, $\mathcal{O} \subseteq J_1$: for $v \in \mathcal{O}$ put $w = v - \varphi(v)$, $\mu_a = w$ for all $a \in P_1$, $d = (p-1)\varphi(v)$. Then $d - \varphi(d) = (p-1)\varphi(w) = \varphi(\sum_a \mu_a)$, so (μ, d) is admissible, and $C_1(\mu, d) = w\sum_a \zeta_1^a + d = -w + (p-1)\varphi(v) = -(v - p\varphi(v))$. As $v \mapsto v - p\varphi(v)$ is a bijection of $\mathcal{O}$ by (a), $\mathcal{O} \subseteq J_1$. If $p = 2$ then $K_1 = K_0$ and we are done. If p is odd, let $x = \sum_a c_a\zeta_1^a \in \mathcal{O}_{K_1}$, $\tau = \mathrm{Tr}\sum_a c_a \in \mathbb{Z}_p$, choose $u \in \mathcal{O}$ with $\mathrm{Tr}\, u = -\tau/(p-1)$ (by (a); $p-1 \in \mathbb{Z}_p^\times$) and put $\mu_a = c_a + u$. Then $\mathrm{Tr}\,\varphi(\sum_a \mu_a) = \tau + (p-1)\mathrm{Tr}\, u = 0$, so by (a) there is $d \in \mathcal{O}$ with $d - \varphi(d) = \varphi(\sum_a \mu_a)$, and $C_1(\mu, d) = x + u\sum_a \zeta_1^a + d = x - u + d$. Hence $x \in J_1 + \mathcal{O} = J_1$.

Step 2: $J_m \supseteq p^{e+1}J_{m-1}$ for $m \geq 2$ (with $\mathcal{O}_{K_{m-1}} \subset \mathcal{O}_{K_m}$). Let (μ', d') be admissible at level $m-1$. For $a \in P_m$ let $\bar{a} \in P_{m-1}$ be its residue modulo p^{m-1}; the $a \in P_m$ with $\bar{a} = b$ are exactly $b + lp^{m-1}$, $0 \leq l \leq p-1$. Put $\mu_a = \varphi^{-1}(\mu'_{\bar{a}})$ and $d = pd'$. For $k = 0$, by (b), $\sum_{a\in P_m} \mu_a\zeta_m^a = \sum_{b\in P_{m-1}} \varphi^{-1}(\mu'_b)\sum_{l=0}^{p-1} \zeta_m^{b+lp^{m-1}} = 0$. For $k \geq 1$, $\zeta_m^{(b+lp^{m-1})p^k} = \zeta_m^{bp^k} = \zeta_{m-1}^{bp^{k-1}}$, so $\sum_{a\in P_m} \varphi^k(\mu_a)\zeta_m^{ap^k} = p\sum_{b\in P_{m-1}} \varphi^{k-1}(\mu'_b)\zeta_{m-1}^{bp^{k-1}}$. Hence, with $k' = k-1$,

$$C_m(\mu, d) = p^{e+1}\Big(\sum_{k'=0}^{m-2} p^{ek'} \sum_{b\in P_{m-1}} \varphi^{k'}(\mu'_b)\zeta_{m-1}^{bp^{k'}} + p^{e(m-1)}d'\Big) = p^{e+1}C_{m-1}(\mu', d').$$

The pair (μ, d) is admissible: $\varphi^m(\sum_a \mu_a) = p\,\varphi^{m-1}(\sum_b \mu'_b) = p(d' - p^e\varphi(d')) = d - p^e\varphi(d)$.

Step 3: induction. Let $m \geq 2$ and assume $J_{m-1} \supseteq p^{(e+1)(m-2)}\mathcal{O}_{K_{m-1}}$. By Step 2,

$$J_m \supseteq p^{(e+1)(m-1)}\mathcal{O}_{K_{m-1}}. \qquad (*)$$

By (b) it remains to show $p^{(e+1)(m-1)}x \in J_m$ for $x = \sum_{a\in P_m} c_a\zeta_m^a \in W_m$. Moreover, $\zeta_m^{ap^k} \in \mathcal{O}_{K_{m-1}}$ for $k \geq 1$.

Case $e \geq 1$. Take $\mu = p^{(e+1)(m-1)}c$ and $d = d(\mu) = \sum_{i\geq 0} p^{ei}\varphi^{i+m}(\sum_a \mu_a) \in p^{(e+1)(m-1)}\mathcal{O}$. Then $C_m(\mu, d) = p^{(e+1)(m-1)}x + y$ with $y = \sum_{k=1}^{m-1} p^{ek}\sum_a \varphi^k(\mu_a)\zeta_m^{ap^k} + p^{em}d \in p^{(e+1)(m-1)}\mathcal{O}_{K_{m-1}}$, so $y \in J_m$ by $(*)$ and $p^{(e+1)(m-1)}x = C_m(\mu, d) - y \in J_m$.

Case $e = 0$. Let $\tau = \mathrm{Tr}\sum_a c_a \in \mathbb{Z}_p$, choose $t_0 \in \mathcal{O}$ with $\mathrm{Tr}\, t_0 = -\tau$ and put $t = p^{m-2}t_0$. Let $\mu_a = p^{m-1}c_a + t$ if $a \equiv 1 \pmod{p^{m-1}}$ and $\mu_a = p^{m-1}c_a$ otherwise. Then $\sum_a \mu_a = p^{m-1}(\sum_a c_a + t_0) =: p^{m-1}M_0$ with $\mathrm{Tr}\, M_0 = 0$, so by (a) there is $d_0 \in \mathcal{O}$ with $d_0 - \varphi(d_0) = \varphi^m(M_0)$, and $(\mu, p^{m-1}d_0)$ is admissible. By (b) the term $k = 0$ is $p^{m-1}x + t\sum_{l=0}^{p-1} \zeta_m^{1+lp^{m-1}} = p^{m-1}x$. The

remaining part $y = \sum_{k=1}^{m-1}\sum_a \varphi^k(\mu_a)\zeta_m^{ap^k} + p^{m-1}d_0$ lies in $p^{m-1}\mathcal{O}_{K_{m-1}}$: the contribution of the $p^{m-1}c_a$ is divisible by p^{m-1}, and that of t at level $k \geq 1$ is $\sum_{l=0}^{p-1}\varphi^k(t)\zeta_m^{(1+lp^{m-1})p^k} = p\,\varphi^k(t)\zeta_m^{p^k} \in p^{m-1}\mathcal{O}_{K_{m-1}}$. By $(*)$ (with $e=0$), $y \in J_m$, hence $p^{m-1}x \in J_m$.

In both cases $J_m \supseteq p^{(e+1)(m-1)}(\mathcal{O}_{K_{m-1}} + W_m) = p^{(e+1)(m-1)}\mathcal{O}_{K_m}$, which completes the induction. □

Lemma 25.32. *Let p be a prime, $N \geq 1$ an integer with $4 \mid N$ if $p = 2$, $s = \max(1, v_p(N))$ and $N_p = \operatorname{lcm}(p, N)$. Let E be a field with $\mathbb{Q}_p \subseteq E \subseteq \mathbb{Q}_p(\mu_{N_p})$ inside $\overline{\mathbb{Q}}_p$, put $K' = E(\mu_{p^s})$, and let K'_0 be the maximal unramified subextension of $K'/\mathbb{Q}_p$. Then*

$$K' = K'_0(\mu_{p^s}), \qquad [K' : K'_0] = \varphi(p^s), \qquad [K' : E] \text{ divides } (p-1)p^{s-1};$$

in particular $v_p([K' : E]) \leq s - 1$, and $s \geq 2$ if $p = 2$.

Proof. Write $N_p = p^s N_0$ with $p \nmid N_0$ (indeed $v_p(N_p) = \max(1, v_p(N)) = s$), and $e(\cdot)$ for the ramification index over $\mathbb{Q}_p$. The field $\mathbb{Q}_p(\mu_{N_0})$ is unramified and $\mathbb{Q}_p(\mu_{p^s})$ is totally ramified of degree $\varphi(p^s)$ over $\mathbb{Q}_p$; so their compositum $\mathbb{Q}_p(\mu_{N_p})$ has $e(\mathbb{Q}_p(\mu_{N_p})) = \varphi(p^s)$ (its ramification index is at least that of $\mathbb{Q}_p(\mu_{p^s})$, and it is the extension of the unramified $\mathbb{Q}_p(\mu_{N_0})$ by a root of Φ_{p^s}, of degree at most $\varphi(p^s)$). Since $E \subseteq \mathbb{Q}_p(\mu_{N_p})$ and $\mu_{p^s} \subseteq \mathbb{Q}_p(\mu_{N_p})$, $\mathbb{Q}_p(\mu_{p^s}) \subseteq K' \subseteq \mathbb{Q}_p(\mu_{N_p})$; as ramification indices are multiplicative in towers, $\varphi(p^s) \leq e(K') \leq \varphi(p^s)$, so $[K' : K'_0] = e(K') = \varphi(p^s)$. Next, $K'_0(\mu_{p^s}) \subseteq K'$, and $e(K'_0(\mu_{p^s})) \geq e(\mathbb{Q}_p(\mu_{p^s})) = \varphi(p^s)$ while K'_0 is unramified, so $[K'_0(\mu_{p^s}) : K'_0] \geq \varphi(p^s) = [K' : K'_0]$; hence $K' = K'_0(\mu_{p^s})$. Finally $K' = E(\mu_{p^s})$ is Galois over E with group mapping injectively into $(\mathbb{Z}/p^s\mathbb{Z})^\times$ (an automorphism is determined by its action on a primitive p^s-th root of unity), so $[K' : E]$ divides $\varphi(p^s) = (p-1)p^{s-1}$ and $v_p([K' : E]) \leq s - 1$. If $p = 2$, then $4 \mid N$ gives $s = v_2(N) \geq 2$.

For example, let p be odd, $s = 1$, $u \in \mathbb{Z}_p^\times$ a non-square and $p^* = (-1)^{(p-1)/2}p$. If $\mathbb{Q}_p(\mu_N)$ contains the unramified quadratic extension, then $E = \mathbb{Q}_p(\sqrt{up^*}) \subseteq \mathbb{Q}_p(\mu_{N_p})$, E is not contained in $\mathbb{Q}_p(\mu_p)$ (whose only quadratic subfield is $\mathbb{Q}_p(\sqrt{p^*})$), while $K' = E(\mu_p) = \mathbb{Q}_p(\sqrt{u}, \mu_p) = K'_0(\mu_p)$ with $K'_0 = \mathbb{Q}_p(\sqrt{u})$ and $[K' : E] = p - 1$, prime to p. □

Lemma 25.33. *Let p be a prime, $E \subseteq K'$ finite extensions of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$, L a finite extension of $\mathbb{Q}_p$, V_1, V_2 finite-dimensional L-vector spaces with continuous L-linear actions of G_E, $T_\nu \subseteq V_\nu$ G_E-stable $\mathcal{O}_L$-lattices, and $F\colon V_1 \to V_2$ an L-linear G_E-equivariant map with $F(T_1) \subseteq T_2$.*

1. *For $\nu = 1, 2$, restriction $H^1(E, -) \to H^1(K', -)$ and corestriction $H^1(K', -) \to H^1(E, -)$ map $H^1_f(\cdot, V_\nu)$, $H^1_f(\cdot, T_\nu)$ and $H^1_f(\cdot, V_\nu/T_\nu) = \operatorname{Im}\big(H^1_f(\cdot, V_\nu) \to H^1(\cdot, V_\nu/T_\nu)\big)$ into the corresponding groups;* $\mathrm{cor} \circ \mathrm{res}$ *is multiplication by $[K' : E]$; and both commute with F_*.*

2. *Let $e \geq 0$ be an integer with*

$$p^e\, H^1_f(K', T_2) \subseteq F_*(H^1_f(K', T_1)),$$

the groups being formed for the restrictions of the representations to $G_{K'}$. Then

$$p^{\,e+v_p([K':E])}\, H^1_f(E, T_2) \subseteq F_*\big(H^1_f(E, T_1)\big).$$

Proof. Write $G = G_E$ and $H = G_{K'}$, an open subgroup of index $d = [K' : E]$. For a topological G-module M (below M is T_ν, V_ν or $B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_\nu$) we use restriction $\mathrm{res}\colon H^1(G, M) \to H^1(H, M)$ and corestriction $\mathrm{cor}\colon H^1(H, M) \to H^1(G, M)$ on continuous cohomology. Both are natural in M (for G-equivariant continuous maps of coefficient modules), and $\mathrm{cor} \circ \mathrm{res}$ is multiplication

by d [222], I.2.4; for H open of finite index both maps are given on continuous 1-cochains by the explicit formulas there (the transfer formula for cor involves a fixed finite set of coset representatives and the action of G), which are continuous and natural in the coefficient module, so the statement holds for continuous cohomology with these topological coefficients.

Step 1: res and cor preserve the Bloch–Kato groups. Let $\beta\colon V_\nu \to B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_\nu$, $x \mapsto 1\otimes x$, a continuous G-equivariant map, and $j\colon T_\nu \to V_\nu$ the inclusion. By definition $H^1_f(K,V_\nu) = \ker(\beta_*\colon H^1(K,V_\nu) \to H^1(K, B_{\mathrm{cris}} \otimes V_\nu))$ and $H^1_f(K,T_\nu) = j_*^{-1} H^1_f(K,V_\nu)$, for $K = E$ and $K = K'$. By naturality, $\beta_* \circ \mathrm{res} = \mathrm{res} \circ \beta_*$ and $\beta_* \circ \mathrm{cor} = \mathrm{cor} \circ \beta_*$, so $\mathrm{res}(H^1_f(E,V_\nu)) \subseteq H^1_f(K',V_\nu)$ and $\mathrm{cor}(H^1_f(K',V_\nu)) \subseteq H^1_f(E,V_\nu)$; and j_* commutes with res and cor, so $\mathrm{res}(H^1_f(E,T_\nu)) \subseteq H^1_f(K',T_\nu)$ and $\mathrm{cor}(H^1_f(K',T_\nu)) \subseteq H^1_f(E,T_\nu)$. Likewise F_* (for $F|_{T_1}\colon T_1 \to T_2$) commutes with res and cor, and so does the projection $\pi\colon V_\nu \to V_\nu/T_\nu$ (discrete quotient; continuous and G_E-equivariant), so res and cor map $\pi_* H^1_f(\cdot, V_\nu)$ into $\pi_* H^1_f(\cdot, V_\nu)$. This is item 1.

Step 2: the descent (item 2). Let $x \in H^1_f(E,T_2)$. By Step 1, $\mathrm{res}(x) \in H^1_f(K',T_2)$, so by hypothesis $p^e \mathrm{res}(x) = F_*(y)$ with $y \in H^1_f(K',T_1)$. Applying cor,

$$d\, p^e x = \mathrm{cor}(\mathrm{res}(p^e x)) = \mathrm{cor}(F_* y) = F_*(\mathrm{cor}\, y), \qquad \mathrm{cor}\, y \in H^1_f(E,T_1)$$

by Step 1. Write $d = p^{v_p(d)} u$ with u prime to p, a unit of $\mathbb{Z}_p$. The group $F_*(H^1_f(E,T_1))$ is a $\mathbb{Z}_p$-submodule of $H^1(E,T_2)$ ($H^1_f(E,T_1)$ is the inverse image of a $\mathbb{Q}_p$-subspace under a $\mathbb{Z}_p$-linear map, and F_* is $\mathbb{Z}_p$-linear). Hence $p^{e+v_p(d)} x = u^{-1}\,(d\, p^e x) \in F_*(H^1_f(E,T_1))$. □

Lemma 25.34. *Let E be a number field, ℓ a prime, $s \geq 1$ an integer with $s \geq 2$ if $\ell = 2$, v a place of E above ℓ, $D = G_{E_v}$, $E_{v,0}$ the maximal unramified subextension of $E_v/\mathbb{Q}_\ell$ with $E_v \subseteq E_{v,0}(\mu_{\ell^s})$, and K a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$ containing $\sigma(E_{v,0}(\mu_{\ell^s}))$ for every $\mathbb{Q}_\ell$-embedding σ of $E_{v,0}(\mu_{\ell^s})$ into $\overline{\mathbb{Q}}_\ell$. Let W be a K-vector space of dimension d with a continuous K-linear action of D, crystalline as a $\mathbb{Q}_\ell$-representation, and $\Lambda \subset W$ a D-stable $\mathcal{O}$-lattice. Then:*

1. *if $M_0, h_0 \geq 0$ are integers with $J(W) \subseteq [-M_0, h_0]$ and $\tau\colon E_v \to K$ is a $\mathbb{Q}_\ell$-embedding at which 0 is not a Hodge jump of W, then*

$$\mathrm{nl}\, H^0(D, W/\Lambda) \leq d\; c_{\mathrm{inv},s}(W,\tau);$$

2. *W' is a K-vector space of dimension d with a continuous K-linear action of D, crystalline as a $\mathbb{Q}_\ell$-representation, and Λ' is a D-stable $\mathcal{O}$-lattice in W'; so item 1 applies to (W',Λ').*

Descent. *Items 1 and 2 also hold when the two field conditions are replaced by: $N \geq 1$ is an integer with $4 \mid N$ if $\ell = 2$, $s = \max(1, v_\ell(N))$, $N_\ell = \mathrm{lcm}(\ell, N)$, E_v is $\mathbb{Q}_\ell$-isomorphic to a subfield of $\mathbb{Q}_\ell(\mu_{N_\ell})$, and K contains a primitive N_ℓ-th root of unity.*

Proof. **Item 1.** Apply Proposition 25.35 at our level s with $p = \ell$ ($s \geq 2$ if $\ell = 2$), its field K equal to E_v (so its K_0 is $E_{v,0}$, and $E_v \subseteq E_{v,0}(\mu_{\ell^s})$ is a hypothesis), its L equal to our K (the condition on the conjugates of $E_{v,0}(\mu_{\ell^s})$ is a hypothesis), $V = W$, $T = \Lambda$, the integers M_0, h_0 and τ. It gives $\ell^c x = 0$ for every $x \in H^0(D, W/\Lambda)$, $c = c_{\mathrm{inv},s}(W,\tau)$, so $H^0(D,W/\Lambda) \subseteq (W/\Lambda)[\ell^c]$. As Λ is a finitely generated torsion-free module over the discrete valuation ring $\mathcal{O}$ spanning W, it is free of rank d, $W/\Lambda \cong (K/\mathcal{O})^d$, and $(K/\mathcal{O})^d[\ell^c] \cong (\mathcal{O}/\ell^c\mathcal{O})^d$ has length $d\, c\, e_{\mathcal{O}}$, i.e. normalised length $d\, c$.

Item 2. For $\alpha \in K$, $f \in W'$ and $g \in D$, $(g(\alpha f))(w) = g\, f(\alpha g^{-1} w) = (\alpha(gf))(w)$, since α commutes with g^{-1} on W; so W' is a K-linear representation, of $\mathbb{Q}_\ell$-dimension $\dim_{\mathbb{Q}_\ell} W$, hence

of K-dimension d; the action is continuous, being the contragredient twisted by χ_{cyc}. As a $\mathbb{Q}_\ell$-representation $W' \cong W^* \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(1)$; W^* is crystalline by Theorem B.45, $\mathbb{Q}_\ell(1)$ is crystalline [52], and so is their tensor product by Theorem B.45. Λ' is D-stable and stable under $\mathcal{O}$ (as $\alpha\Lambda \subseteq \Lambda$ for $\alpha \in \mathcal{O}$), finitely generated over $\mathbb{Z}_\ell$ (Λ is free of finite rank over $\mathbb{Z}_\ell$), hence over $\mathcal{O}$, and spans W' over $\mathbb{Q}_\ell$, hence over K: it is a D-stable $\mathcal{O}$-lattice.

Descent. As $\mathbb{Q}_\ell(\mu_{N_\ell})$ is Galois over $\mathbb{Q}_\ell$, the subfield E_v of $\overline{\mathbb{Q}}_\ell$ (via the embedding defining D), being isomorphic to a subfield of it, is contained in it. Item 1 is then the argument above with the descent assertion of Proposition 25.35 (its K equal to E_v, its L equal to our K) in place of its level-s assertion. Item 2 does not use the field conditions. □

Proposition 25.35. *Let p be a prime, $s \geq 1$ an integer with $s \geq 2$ if $p = 2$, K a finite extension of $\mathbb{Q}_p$ with maximal unramified subextension K_0 such that $K \subseteq K_0(\mu_{p^s})$, and L a finite extension of $\mathbb{Q}_p$ containing $\sigma(K_0(\mu_{p^s}))$ for every $\mathbb{Q}_p$-embedding σ of $K_0(\mu_{p^s})$ into $\overline{\mathbb{Q}}_p$, with ring of integers $\mathcal{O}_L$. Let V be a crystalline L-linear representation of G_K with a G_K-stable $\mathcal{O}_L$-lattice T, and $M_0, h_0 \geq 0$ integers with $J(V) \subseteq [-M_0, h_0]$; put $H = M_0 + h_0$, $n_0 = \min\{n \geq 0 :\ H \leq p^n(p-1)\}$, $m_{N,s} = 2s + n_0 + 1$ and $m_N = m_{N,1} = n_0 + 3$. Let $\tau\colon K \to L$ be a $\mathbb{Q}_p$-embedding such that 0 is not a Hodge jump of V at τ, let $J_\tau(V)$ be the set of Hodge jumps of V at τ, and put*

$$c_{\mathrm{inv},s}(V,\tau) = 2 + \sum_{a \in J_\tau(V)} \big(m_{N,s} + v_p(a)\big), \qquad c_{\mathrm{inv}}(V,\tau) = c_{\mathrm{inv},1}(V,\tau).$$

Then $p^{\,c_{\mathrm{inv},s}(V,\tau)} x = 0$ for every $x \in V/T$ with $gx = x$ for all $g \in G_K$.

Descent. *Let p be a prime, $N \geq 1$ an integer with $4 \mid N$ if $p = 2$, $s = \max(1, v_p(N))$, $N_p = \mathrm{lcm}(p, N)$, K a field with $\mathbb{Q}_p \subseteq K \subseteq \mathbb{Q}_p(\mu_{N_p})$ inside $\overline{\mathbb{Q}}_p$, L a finite extension of $\mathbb{Q}_p$ containing a primitive N_p-th root of unity, and V, T, M_0, h_0, $\tau\colon K \to L$ as above (for G_K). Then $p^{\,c_{\mathrm{inv},s}(V,\tau)} x = 0$ for every $x \in V/T$ fixed by G_K.*

Proof. Let $x \in V/T$ be fixed by G_K and $v \in V$ a lift of x. Put $m = m_{N,s} = 2s + n_0 + 1 \geq 3$ and $G_m = G_{K_0(\mu_{p^m})}$; as $m \geq s$, $K \subseteq K_0(\mu_{p^s}) \subseteq K_0(\mu_{p^m})$ and $G_m \subseteq G_K$.

Coordinates. The lattice T is finitely generated and torsion-free over the discrete valuation ring $\mathcal{O}_L$, hence free, and $\mathcal{O}_L$ is free of finite rank over $\mathbb{Z}_p$; so T has a $\mathbb{Z}_p$-basis $b_1, \dots, b_d$. Let $M(\sigma) \in \mathrm{GL}_d(\mathbb{Z}_p)$ be the matrix of $\sigma \in G_K$ in it. As T spans V over L, $V = T[1/p] = \bigoplus_j \mathbb{Q}_p b_j$; let $y \in \mathbb{Q}_p^d$ be the coordinate vector of v. Since $gx = x$ for $g \in G_K$, $\sigma v - v \in T$, i.e.

$$M(\sigma)y - y \in \mathbb{Z}_p^d \qquad (\sigma \in G_K).$$

By Proposition 25.36 (whose hypotheses are ours) there are $r \geq 0$, $P \in M_{r\times d}(\mathbb{C}_p)$, a finite set $J \subseteq J_\tau(V)$ and a Sen frame (U, g) of level m and rank r over K_0 with weights in J satisfying (Wt1)–(Wt3).

The case $J = \emptyset$. Then $r = 0$, since by (Sf3) every basis vector of $\mathbb{C}_p^r$ carries a weight in J. So P is the empty matrix, $Py = 0 = P \cdot 0 \in P(\mathcal{O}_{\mathbb{C}_p}^d)$, and (Wt3) gives $y \in \mathbb{Z}_p^d$: $v \in T$ and $x = 0$.

The case $J \neq \emptyset$. For $\sigma \in G_m$, σ fixes the entries of y, so by (Wt1)

$$\sigma *_0 (Py) - Py = U(\sigma)\sigma(P)\, y - Py = P\big(M(\sigma)y - y\big) \in P(\mathbb{Z}_p^d) \subseteq M_P := P(\mathcal{O}_{\mathbb{C}_p}^d).$$

The set M_P is an additive subgroup of $\mathbb{C}_p^r$ with $\mathcal{O}_{\mathbb{C}_p}^r \subseteq M_P$ and $pM_P \subseteq \mathcal{O}_{\mathbb{C}_p}^r$ by (Wt2), and $0 \notin J$ because $J \subseteq J_\tau(V)$ and 0 is not a Hodge jump of V at τ. So Proposition 25.17(2) with $i = 0$ (applicable at $p = 2$ as $m \geq 2$) gives $p^{\,2+N} Py \in M_P$ for $N = N_{J,m}(0) = \max\big(1, \sum_{u\in J}(m + v_p(u))\big)$. As $J \neq \emptyset$ and every term is at least $m \geq 1$, $N = \sum_{u\in J}(m + v_p(u)) \leq \sum_{a\in J_\tau(V)}(m +$

$v_p(a)$), so $2+N \le c_{\mathrm{inv},s}(V,\tau)$. Now $P(p^{2+N}y) \in P(\mathcal{O}_{\mathbb{C}_p}^d)$ with $p^{2+N}y \in \mathbb{Q}_p^d$, and (Wt3) gives $p^{2+N}y \in \mathbb{Z}_p^d$, i.e. $p^{2+N}v \in T$. Hence $p^{c_{\mathrm{inv},s}(V,\tau)}v \in T$ and $p^{c_{\mathrm{inv},s}(V,\tau)}x = 0$.

Descent. Let $K' = K(\mu_{p^s})$. By Lemma 25.32, $K' = K_0'(\mu_{p^s})$ with K_0' its maximal unramified subextension, and $s \ge 2$ if $p = 2$; every $\mathbb{Q}_p$-embedding of K' has image in $\mathbb{Q}_p(\mu_{N_p}) \subseteq L$. Extend τ to a $\mathbb{Q}_p$-embedding $\tau'\colon K' \to L$. The restriction of V to $G_{K'}$ is crystalline, and the natural map $K' \otimes_K D_{\mathrm{dR},K}(V) \to D_{\mathrm{dR},K'}(V)$ is an isomorphism of filtered $K' \otimes_{\mathbb{Q}_p} L$-modules [94]; hence $J(V|_{G_{K'}}) = J(V)$ (K' is faithfully flat over K), and $D_{\mathrm{dR},\tau'}(V|_{G_{K'}}) = L \otimes_{K'\otimes L,\tau'\otimes\mathrm{id}} (K' \otimes_K D_{\mathrm{dR},K}(V)) = L \otimes_{K\otimes L,\tau\otimes\mathrm{id}} D_{\mathrm{dR},K}(V) = D_{\mathrm{dR},\tau}(V)$ as filtered L-spaces, so $V|_{G_{K'}}$ has at τ' the Hodge jumps of V at τ (not 0) and $c_{\mathrm{inv},s}(V|_{G_{K'}},\tau') = c_{\mathrm{inv},s}(V,\tau)$, with the same M_0, h_0, n_0. A G_K-fixed $x \in V/T$ is $G_{K'}$-fixed, and the statement over K' (with K', K_0' in the roles of K, K_0) gives $p^{c_{\mathrm{inv},s}(V,\tau)}x = 0$. □

Proposition 25.36. *Let p be a prime, $s \ge 1$ an integer with $s \ge 2$ if $p = 2$, K a finite extension of $\mathbb{Q}_p$ with maximal unramified subextension K_0 such that $K \subseteq K_0(\mu_{p^s})$, and L a finite extension of $\mathbb{Q}_p$ containing $\sigma(K_0(\mu_{p^s}))$ for every $\mathbb{Q}_p$-embedding σ of $K_0(\mu_{p^s})$ into $\overline{\mathbb{Q}}_p$. Let V be a crystalline L-linear representation of G_K with a G_K-stable $\mathcal{O}_L$-lattice T, and $M_0, h_0 \ge 0$ integers with $J(V) \subseteq [-M_0, h_0]$; put $H = M_0 + h_0$, $n_0 = \min\{n \ge 0 :\ H \le p^n(p-1)\}$, $m_{N,s} = 2s + n_0 + 1$ and $m_N = m_{N,1} = n_0 + 3$. Let $\tau\colon K \to L$ be a $\mathbb{Q}_p$-embedding and $J_\tau(V)$ the set of Hodge jumps of V at τ. Let $b_1, \dots, b_d$ be a $\mathbb{Z}_p$-basis of T and $M(\sigma) \in \mathrm{GL}_d(\mathbb{Z}_p)$ the matrix of $\sigma \in G_K$ in it ($\sigma b_j = \sum_a M(\sigma)_{aj} b_a$). Then there are an integer $r \ge 0$, a matrix $P \in M_{r\times d}(\mathbb{C}_p)$, a finite set $J \subseteq J_\tau(V)$ and a Sen frame (U, g) of level $m_{N,s}$ and rank r over K_0 with weights in J such that, with $P(\mathcal{O}_{\mathbb{C}_p}^d) = \{Px :\ x \in \mathcal{O}_{\mathbb{C}_p}^d\}$:*

(Wt1) *$P\,M(\sigma) = U(\sigma)\,\sigma(P)$ for all $\sigma \in G_{K_0(\mu_{p^{m_{N,s}}})}$;*

(Wt2) *$\mathcal{O}_{\mathbb{C}_p}^r \subseteq P(\mathcal{O}_{\mathbb{C}_p}^d)$ and $p\,P(\mathcal{O}_{\mathbb{C}_p}^d) \subseteq \mathcal{O}_{\mathbb{C}_p}^r$;*

(Wt3) *if $y \in \mathbb{Q}_p^d$ and $Py \in P(\mathcal{O}_{\mathbb{C}_p}^d)$, then $y \in \mathbb{Z}_p^d$.*

Proof. Put $m = m_{N,s}$ and $G_m = G_{K_m}$; since $m = 2s+n_0+1 \ge s$, $K \subseteq K_s \subseteq K_m$ and $G_m \subseteq G_K$; also $m \ge 3$. As V is crystalline, it is de Rham [52]. Every $\mathbb{Q}_p$-embedding of K into $\overline{\mathbb{Q}}_p$ extends to K_s, so L contains its image, and Lemma 25.16 applies to K, L, V and the $\mathbb{Q}_p$-basis $b_\bullet$ of V. Let $E = E_\tau \in M_d(K)$ be the matrix of its item 1 and $B(y)$ the matrix of multiplication by $y \in L$. Take D and the Sen frame (U, g) of level m and rank d of Proposition 25.11 (whose hypotheses are ours), so that $D\,U(\sigma) = M(\sigma)\,\sigma(D)$ for $\sigma \in G_m$. As D, $\sigma(D)$ and $M(\sigma)$ are invertible, so is $U(\sigma)$, and

$$D^{-1}M(\sigma) = U(\sigma)\,\sigma(D)^{-1} \qquad (\sigma \in G_m). \tag{25.7}$$

Step 1: $e' = D^{-1}ED$ is an idempotent in $M_d(K_m)$ commuting with U. For $\sigma \in G_m$, $\sigma(E) = E$ and $EM(\sigma) = M(\sigma)E$ by Lemma 25.16(1), so by (25.7)

$$\sigma(e') = \sigma(D)^{-1}E\,\sigma(D) = U(\sigma)^{-1}D^{-1}M(\sigma)\,E\,M(\sigma)^{-1}D\,U(\sigma) = U(\sigma)^{-1}e'\,U(\sigma).$$

Thus $U(\sigma)\sigma(e') = e'U(\sigma)$ for $\sigma \in G_m$, and Lemma 25.37 (with $U' = U$, $X = e'$; (U, g) satisfies (Sf1), (Sf2), and $m \ge 3$, as that lemma requires $m \ge 2$ when $p = 2$) gives $e' \in M_d(K_m)$. Hence $\sigma(e') = e'$ and $e'U(\sigma) = U(\sigma)e'$ for $\sigma \in G_m$; and $e'^2 = D^{-1}E^2D = e'$.

Step 2: the cut. By Lemma 25.38 for $F = K_m$ with the absolute value of $\mathbb{C}_p$ (valuation ring $\mathcal{O}_{K_m}$), with $r = \operatorname{rank} e'$, there are $R \in M_{r\times d}(\mathcal{O}_{K_m})$ and $S \in M_{d\times r}(\mathcal{O}_{K_m})$ with $RS = 1$, $Re' = R$, $e'SR = e'$. For $\sigma \in G_m$, $RU(\sigma) = Re'U(\sigma) = RU(\sigma)e'$, hence

$$RU(\sigma)SR = RU(\sigma)e'SR = RU(\sigma)e' = RU(\sigma). \tag{25.8}$$

Put $U'(\sigma) = RU(\sigma)S \in M_r(\mathcal{O}_{K_m})$ and $P = RD^{-1} \in M_{r\times d}(\mathbb{C}_p)$. Then $PE = Re'D^{-1} = RD^{-1} = P$ and $PDS = RS = 1$.

Step 3: (Sf1), (Sf2) for (U', g). U' is continuous, as U is. By (25.8), $U'(\sigma)U'(\tau) = RU(\sigma)SRU(\tau)S = RU(\sigma)U(\tau)S = U'(\sigma\tau)$. For $h \in H_{K_0}$, $U'(h) = RS = 1$. The element g has $v_p(\chi_{\mathrm{cyc}}(g)-1) = m$, and every entry of $U'(g)-1 = R\,(U(g)-1)\,S$ is a finite sum of products of an entry of $U(g)-1$ (absolute value $< \rho$) with entries of R and S (absolute value ≤ 1), hence has absolute value $< \rho$.

Step 4: (Wt1) and (Wt2). For $\sigma \in G_m$, by (25.7), (25.8) and $\sigma(R) = R$,

$$P\,M(\sigma) = RU(\sigma)\sigma(D^{-1}) = RU(\sigma)SR\,\sigma(D^{-1}) = U'(\sigma)\,\sigma(RD^{-1}) = U'(\sigma)\,\sigma(P).$$

If $y \in \mathcal{O}^r$ then $y = RSy = P(DSy)$ with $DSy \in \mathcal{O}^d$, as D is integral; and if $x \in \mathcal{O}^d$ then $pPx = R(pD^{-1}x) \in \mathcal{O}^r$, as pD^{-1} is integral. This is (Wt2).

Step 5: (Sf3) with weights in $J_\tau(V)$. Let $e_1, \dots, e_d$, u_k, τ_k be as in Lemma 25.16(2), and $J = \{u_k : \tau_k = \tau\}$, a finite subset of $J_\tau(V)$. If $\tau_k \ne \tau$ then $Pe_k = PEe_k = 0$. Since P has the right inverse DS, it maps $\mathbb{C}_p^d$ onto $\mathbb{C}_p^r$, so the Pe_k span $\mathbb{C}_p^r$ and some r of them, $Pe_{k_1}, \dots, Pe_{k_r}$, form a basis; being non-zero, they have $\tau_{k_i} = \tau$, so $u_{k_i} \in J$. For $\sigma \in G_m$, by (Wt1),

$$\chi_{\mathrm{cyc}}(\sigma)^{u_k}\,U'(\sigma)\,\sigma(Pe_k) = P\big(\chi_{\mathrm{cyc}}(\sigma)^{u_k}M(\sigma)\sigma(e_k)\big) = Pe_k.$$

With Step 3, (U', g) is a Sen frame of level m and rank r over K_0 with weights in J.

Step 6: (Wt3). Let $y \in \mathbb{Q}_p^d$ and $x \in \mathcal{O}^d$ with $Py = Px$. For $Q = De'S$ we have $QP = De'SRD^{-1} = De'D^{-1} = E$, so $Ey = Ex$. Let $\iota\colon L \to \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$ be a $\mathbb{Q}_p$-embedding extending the inverse $\tau(K) \to K$ of τ (it exists as $L/\tau(K)$ is algebraic), so $\iota(\tau(x')) = x'$ for $x' \in K$. The lattice T is finitely generated and torsion-free over the discrete valuation ring $\mathcal{O}_L$, hence has an $\mathcal{O}_L$-basis $t_1, \dots, t_s$, which is an L-basis of V; write $b_a = \sum_i c_{ai}t_i$ with $c_{ai} \in \mathcal{O}_L$. The $\mathbb{C}_p$-linear map $\pi\colon \mathbb{C}_p^d \to \mathbb{C}_p^s$, $\pi(z)_i = \sum_a \iota(c_{ai})z_a$, is the map $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \to \mathbb{C}_p \otimes_{L,\iota} V$, $c \otimes v \mapsto c \otimes v$, in the coordinates $b_\bullet$ and $t_\bullet$; hence $\pi(B(y')z) = \iota(y')\pi(z)$ for $y' \in L$, and $\pi(x'z) = x'\pi(z) = \iota(\tau(x'))\pi(z)$ for $x' \in K$. With $\epsilon_\tau = \sum_j x_j \otimes y_j$ as in Lemma 25.16(1),

$$\pi(Ez) = \sum_j x_j\,\iota(y_j)\,\pi(z) = \iota\Big(\sum_j \tau(x_j)y_j\Big)\pi(z) = \pi(z),$$

since $\sum_j \tau(x_j)y_j$ is the τ-component of ϵ_τ, namely 1. So $\pi(y) = \pi(Ey) = \pi(Ex) = \pi(x) \in \mathcal{O}^s$, as $|\iota(c_{ai})| = |c_{ai}| \le 1$. Writing $v = \sum_a y_a b_a = \sum_i \gamma_i t_i$ with $\gamma_i \in L$, we have $\pi(y)_i = \iota(\gamma_i)$, so $|\gamma_i| \le 1$, $\gamma_i \in \mathcal{O}_L$, $v \in T$, and $y \in \mathbb{Z}_p^d$ because $b_\bullet$ is a $\mathbb{Z}_p$-basis of T. □

Lemma 25.37. *Let p be a prime, K_0 a finite unramified extension of $\mathbb{Q}_p$, $m \ge 1$ an integer with $m \ge 2$ if $p = 2$, $K_m = K_0(\mu_{p^m})$, $r, r' \ge 0$ integers, and $g \in G_{K_m}$. Let $U\colon G_{K_m} \to M_r(\mathcal{O}_{K_m})$ and $U'\colon G_{K_m} \to M_{r'}(\mathcal{O}_{K_m})$ be maps such that (U, g) and (U', g) satisfy (Sf1) and (Sf2). If $X \in M_{r\times r'}(\mathbb{C}_p)$ satisfies*

$$U(\sigma)\,\sigma(X) = X\,U'(\sigma) \qquad (\sigma \in G_{K_m}),$$

then $X \in M_{r\times r'}(K_m)$.

Proof. If $r = 0$ or $r' = 0$ there is nothing to prove; let $r, r' \ge 1$ and write $\rho = p^{-1/(p-1)} < 1$.

The entries of X lie in $\widehat{K}_\infty$. For $h \in H_{K_0} \subseteq G_{K_m}$, (Sf1) gives $U(h) = 1$ and $U'(h) = 1$, so $h(X) = X$; by Theorem B.42(1) for $F = K_0$ the entries of X lie in $\widehat{K}_\infty$. Choose $s \ge 0$ with $p^sX \in M_{r\times r'}(\mathcal{O}_{\widehat{K}_\infty})$ and decompose entrywise by Lemma 25.18(1): $p^sX = X_0 + X_X$ with $X_0 \in M_{r\times r'}(\mathcal{O}_{K_m})$ and $X_X \in M_{r\times r'}(X_m)$.

The kernel part satisfies a twisted fixed-point equation. The relation at $\sigma = g$, multiplied by p^s, reads $U(g)g(X_0) + U(g)g(X_X) = X_0U'(g) + X_XU'(g)$. Since g fixes K_m, $g(X_0) = X_0$, and $U(g)X_0$, $X_0U'(g)$ have entries in $\mathcal{O}_{K_m}$; since X_m is a G_{K_0}-stable $\mathcal{O}_{K_m}$-module and $U(g)$, $U'(g)$ have entries in $\mathcal{O}_{K_m}$, the matrices $U(g)g(X_X)$ and $X_XU'(g)$ have entries in X_m. By uniqueness of the decomposition, $U(g)\,g(X_X) = X_X\,U'(g)$. Write $U'(g) = 1 + N'$; by (Sf2) the entries of N' have absolute value $< \rho \le 1$, so $W = \sum_{k\ge0}(-N')^k$ converges in $M_{r'}(\mathcal{O}_{K_m})$ (K_m is complete), $W\,U'(g) = U'(g)\,W = 1$, and $W - 1 = \sum_{k\ge1}(-N')^k$ has entries of absolute value at most $\max_{a,b}|N'_{ab}| < \rho$. Hence

$$U(g)\,g(X_X)\,W = X_X.$$

Vectorisation. Identify $M_{r\times r'}(\mathbb{C}_p)$ with $\mathbb{C}_p^{rr'}$ by stacking columns, $Y \mapsto \mathrm{vec}(Y)$; then $\mathrm{vec}(U(g)YW) = A\,\mathrm{vec}(Y)$ with the Kronecker product $A = W^{\mathsf{T}} \otimes U(g) \in M_{rr'}(\mathcal{O}_{K_m})$, and $\mathrm{vec}(g(Y)) = g(\mathrm{vec}(Y))$. Since $A - 1 = (W^{\mathsf{T}} - 1) \otimes U(g) + 1 \otimes (U(g) - 1)$, every entry of $A - 1$ is a sum of two terms, each a product of an entry of absolute value $< \rho$ (from $W - 1$ or, by (Sf2), from $U(g) - 1$) with an entry of absolute value at most 1; so $|A_{ab} - \delta_{ab}| < \rho$. The vector $x = \mathrm{vec}(X_X) \in X_m^{rr'}$ satisfies $A\,g(x) - x = 0 = (\zeta_p - 1)\cdot 0$, and so does $x = 0$. By (Sf2), $v_p(\chi_{\mathrm{cyc}}(g) - 1) = m$, and $m \ge 2$ if $p = 2$, so the uniqueness in Lemma 25.18(2) (rank rr') gives $\mathrm{vec}(X_X) = 0$.

Hence $p^sX = X_0$ has entries in $\mathcal{O}_{K_m}$ and $X \in M_{r\times r'}(K_m)$. □

Lemma 25.38. *Let F be a field with a non-archimedean absolute value $|\cdot|$ and valuation ring $\mathcal{O} = \{x \in F : |x| \le 1\}$, let $d \ge 0$, and let $e \in M_d(F)$ satisfy $e^2 = e$, of rank r. Then there are $R \in M_{r\times d}(\mathcal{O})$ and $S \in M_{d\times r}(\mathcal{O})$ with*

$$RS = 1_r, \qquad Re = R, \qquad eSR = e.$$

Proof. If $r = 0$ then $e = 0$, and the empty matrices satisfy the three identities. Let $r \ge 1$. For sequences $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_r)$ in $\{1, \dots, d\}$ write $e_{I,J}$ for the $r \times r$ matrix with entries $e_{i_a j_b}$ ($1 \le a, b \le r$), and similarly $e_{I,*}$ (rows I, all columns) and $e_{*,J}$. Since e has rank r, some $\det e_{I,J}$ is non-zero; among the finitely many pairs choose (I, J) with $|\det e_{I,J}|$ maximal, and put $A = e_{I,J} \in \mathrm{GL}_r(F)$,

$$R = A^{-1}e_{I,*}, \qquad S_{ik} = \begin{cases} 1, & i = j_k, \\ 0, & \text{otherwise}, \end{cases}$$

so that $YS = Y_{*,J}$ for every matrix Y with d columns.

The identities $RS = 1$ and $Re = R$. $RS = A^{-1}e_{I,J} = 1_r$. As $e^2 = e$, $e_{I,*}\,e = (e^2)_{I,*} = e_{I,*}$, so $Re = R$.

Integrality of R. By Cramer's rule the (k, j) entry of $A^{-1}e_{I,*}$ is $\det A_{(k,j)}/\det A$, where $A_{(k,j)}$ is A with its k-th column replaced by the column of $e_{I,*}$ with index j, i.e. $A_{(k,j)} = e_{I,J'}$ for $J' = (j_1, \dots, j_{k-1}, j, j_{k+1}, \dots, j_r)$. By the choice of (I, J), $|\det e_{I,J'}| \le |\det A|$, so $|R_{kj}| \le 1$. S has entries $0, 1$.

The identity $eSR = e$. $eSR = e_{*,J}A^{-1}e_{I,*}$. Fix $i, j \in \{1, \dots, d\}$ and let B be the $(r + 1) \times (r + 1)$ matrix with rows $(i_1, \dots, i_r, i)$ and columns $(j_1, \dots, j_r, j)$ of e, i.e.

$$B = \begin{pmatrix} A & e_{I,j} \\ e_{i,J} & e_{ij} \end{pmatrix}.$$

Its rows are rows of e, and its columns are obtained by selecting columns, so $\operatorname{rank} B \le \operatorname{rank} e = r$ and $\det B = 0$. By the Schur complement formula $\det B = \det A\,(e_{ij} - e_{i,J}A^{-1}e_{I,j})$, and $\det A \neq 0$; so $e_{ij} = (e_{*,J}A^{-1}e_{I,*})_{ij}$. □

Corollary 25.39. *Let E be a number field, ℓ a prime, Galois Ribet data over E with $t+1$ blocks having (Gd1) and (Gd2), $K = \mathcal{O}[\ell^{-1}]$, and v a place of E above ℓ at which (Ll1) and (Ll2) hold. Let $a_\tau^-, a_\tau^+ \ge 0$ be integers, indexed by the $\mathbb{Q}_\ell$-embeddings $\tau\colon E_v \to K$, such that every Hodge jump at τ of $V_{\mathcal{N}}$ and of each W_k ($0 \le k \le t$) lies in $[-a_\tau^-, a_\tau^+]$, and put $\mathrm{wt} = \max_\tau(a_\tau^- + a_\tau^+)$.*

1. *If ℓ is odd, $E_v/\mathbb{Q}_\ell$ is unramified and K contains $\sigma(E_v(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $E_v(\mu_\ell)$ into $\overline{\mathbb{Q}}_\ell$, then (Ll3) holds with $e_k = \lfloor c_{\mathrm{ur}}(2 + \mathrm{wt}/\ell + \log_\ell(\mathrm{wt}+1))\rfloor$ for all k, and (Gd7) holds at v with*

$$\lambda_k(v) = \big([E_v : \mathbb{Q}_\ell] + 2\big)\, n_{\mathcal{R},k}\, \lfloor c_{\mathrm{ur}}(2 + \mathrm{wt}/\ell + \log_\ell(\mathrm{wt}+1))\rfloor.$$

Proof. Fix $0 \le k \le t$ and an $\mathcal{O}$-linear D-equivariant $\psi\colon \mathcal{N}_{\mathcal{E}} \to \mathrm{Cs}_k(u)$. By the first conclusions of Theorem 25.3, which need only (Ll1), (Gd1) and (Gd2), $V_{\mathcal{N}}$ and W_k are crystalline, so $V_1 = V_{\mathcal{N}} \oplus W_k$ is crystalline (Theorem B.45), and $\mathcal{N}_{\mathcal{E}}$ is a D-stable lattice in $V_2 = V_{\mathcal{N}}$; by Lemma 25.4(1), $T_1 = \widetilde{T}_\psi$ is a D-stable lattice in V_1 and the projection $F = \mathrm{pr}_1\colon V_1 \to V_2$, which is K-linear and D-equivariant, satisfies $F(T_1) = T_2$ with $T_2 = \mathcal{N}_{\mathcal{E}}$. For each τ, $D_{\mathrm{dR},\tau}(V_1) = D_{\mathrm{dR},\tau}(V_{\mathcal{N}}) \oplus D_{\mathrm{dR},\tau}(W_k)$ with the direct-sum filtration ($B_{\mathrm{dR}} \otimes -$, invariants, the base change along τ and Fil^a all commute with finite direct sums), so an integer a with $\mathrm{Fil}^{a+1} \neq \mathrm{Fil}^a$ on V_1 has the same property on a summand: every Hodge jump of V_1 at τ is one of $V_{\mathcal{N}}$ or W_k, hence lies in $[-a_\tau^-, a_\tau^+]$.

Item 1. All hypotheses of Proposition 24.1 hold with $p = \ell$, its field K equal to E_v, its coefficient field L equal to our K, V_1, V_2, T_1, T_2, F as above and the integers $a_\tau^\pm$. It gives an integer $e_{\mathrm{A}} \le c_{\mathrm{ur}}(2 + \mathrm{wt}/\ell + \log_\ell(\mathrm{wt}+1))$, hence $e_{\mathrm{A}} \le e_k$ for the integer e_k of the statement, with $\ell^{e_{\mathrm{A}}} H^1_f(D, \mathcal{N}_{\mathcal{E}}) \subseteq \mathrm{pr}_{1*} H^1_f(D, \widetilde{T}_\psi)$. The right side is an $\mathcal{O}$-module, so $\ell^{e_k} H^1_f(D, \mathcal{N}_{\mathcal{E}}) = \ell^{e_k - e_{\mathrm{A}}} \ell^{e_{\mathrm{A}}} H^1_f(D, \mathcal{N}_{\mathcal{E}})$ lies in it. This holds for every k and ψ: (Ll3). Theorem 25.3 gives (Gd7) with the stated $\lambda_k(v)$. □

26 Primes of inert type: the row bound

At a prime ℓ of inert type every place above ℓ is unramified and there are no column places, so the global losses vanish and the whole loss is local. This section bounds it (Theorem 26.1): the total is at most $300\, c_{\mathrm{R}}(c+1)\, f\big(n\, t^2(\varrho + t)(1 + \log_\ell(H + t)) + mH/\ell + \varrho A\big)$, where A is the row excess, one of the two numbers A_{lo}, A_{md} of the statement. This is the allowance of the inert primes in Theorem 22.4. The character blocks are handled by Corollary 25.6; the work is in block 0, the row loss Λ_v^{row}.

The row loss is a colength (Lemma 26.3): it is at most the normalised length of the cokernel of the map on integral Bloch–Kato groups induced by a surjection of crystalline lattices, from a lattice in a sum of duals of the points to the dual of the block-0 lattice. With the generator bound of Lemma 27.3, the lifting exponent of Section 24 alone bounds this colength by about $(n+1)\, n_0\, \mathrm{wt}/\ell$. Proposition 26.2 does better by reading the Hodge jumps along the Frobenius orbit of embeddings: a *weight word* records, at each site, whether the jumps there are large (thick) or small (thin), and with which sign. In three configurations, alternating words with a thin site, alternating words that are thick everywhere, and long orbits with isolated thick clusters, the term proportional to the rank n_0 carries h/ℓ^2, respectively H'/ℓ^2 and $h/(\ell-1)^{j+1}$,

in place of h/ℓ, and the terms of size h/ℓ, where they remain, are multiplied by $n\,m$ and not by the rank. The three colength bounds behind this are Proposition C.1, Proposition C.52 and Proposition C.77; their proofs are long and combinatorial and form Appendix C.

Theorem 26.1. *Let E be a number field, $\ell \geq 11$ a prime, $K = \mathcal{O}[\ell^{-1}]$, and Galois Ribet data over E on $(\mathcal{R}, \pi', \mathcal{N}_{\mathcal{E}})$ with ι the identity and $t+1$ blocks, having (Gd1) and (Gd2); $n_0 = n_{\mathcal{R},0}$, $n_v = [E_v : \mathbb{Q}_\ell]$. Let $n, \varrho, m \geq 1$ be integers, $f = 2n$, $H, c, L \geq 1$ and $\ell_{\mathrm{A}} \geq 2$, $w^\circ, \eta^\circ, S^\circ \geq 0$ real numbers, $\beta \in \{0,1\}$, and let A be one of the two numbers*

$$A_{\mathrm{lo}} = \frac{H}{\ell_{\mathrm{A}}(\ell-1)} + (w^\circ + S^\circ + \eta^\circ + 1)(3 + \log \ell_{\mathrm{A}}) + L, \qquad A_{\mathrm{md}} = \frac{H}{\ell^2} + \frac{w^\circ + \eta^\circ}{\ell} + S^\circ + L + \eta^\circ \beta.$$

Suppose:

1. *$\sum_{v \in S_\ell} n_v \leq 2f$, $n_0 \leq \varrho \leq t$, $m \leq t$ and $1 + \log_\ell(2H) \leq L$;*
2. *at every $v \in S_\ell$: $E_v/\mathbb{Q}_\ell$ is unramified; K contains $\sigma(E_v(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $E_v(\mu_\ell)$; (Ll1) and (Ll2) hold; $H^0(G_{E_v}, W_0') = 0$; there is an integer $e_0(v) \geq 0$ with $\ell^{e_0(v)} H^1_f(G_{E_v}, \mathcal{N}_{\mathcal{E}}) \subseteq \mathrm{pr}_{1*} H^1_f(G_{E_v}, \widetilde{T}_\psi)$ for every $\mathcal{O}$-linear G_{E_v}-equivariant $\psi \colon \mathcal{N}_{\mathcal{E}} \to \mathrm{Cs}_0(u)$; and $J(V_{\mathcal{N}}) \subseteq [-M_0, h_0]$ with integers $M_0, h_0 \geq 0$, $M_0 + h_0 \leq H$, and $|J(V_{\mathcal{N}})| \leq c\, n_v(\varrho + t)$;*
3. *at every $v \in S_\ell$ there is a weight word with the integer m above, $b + 2\eta + S \leq H$, $w + S + \eta + 1 \leq H$, $w \leq w^\circ$, $S \leq S^\circ$, $\eta \leq \eta^\circ$, $\beta_v \leq \beta$ and $e_v \leq c(1 + H/\ell + \log_\ell H)$, satisfying the hypotheses of item 3 of Proposition 26.2 with $j = j_{\mathrm{A}} := \max(1, \lceil \log_{\ell-1} \ell_{\mathrm{A}} \rceil)$ if $A = A_{\mathrm{lo}}$, and those of item 1 or of item 2 if $A = A_{\mathrm{md}}$;*
4. *$\nu(\delta) \leq 2t$ and $\sum_{v \in \Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) \leq c$.*

Then (Gd6) holds with $\kappa_k = 0$ and (Gd7) holds with $\lambda_0(v) = 34\, c_{\mathrm{R}}\big((n_v+1)\varrho A + (c+1)\, n_v\, m\,(1 + H/\ell + \log_\ell H)\big)$ and $\lambda_k(v) = c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)\, n_v + w_k(v)$ for $1 \leq k \leq t$, and these losses satisfy

$$\sum_{k=0}^{t} \kappa_k + \sum_{v \in S_\ell} \sum_{k=0}^{t} \lambda_k(v) + t \sum_{v \in \Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) + 2(n_0 + t - 1)\, \nu(\delta) \\ \leq 300\, c_{\mathrm{R}}(c+1)\, f\Big(n\, t^2(\varrho + t)(1 + \log_\ell(H + t)) + \frac{m\, H}{\ell} + \varrho\, A\Big).$$

Proof. **Step 1: (Gd6).** As ι is the identity, the map $\varphi \mapsto \varphi \circ \iota$ of (Gd6) is the identity, so its kernel is 0 and (Gd6) holds with $\kappa_k = 0$.

Step 2: the character blocks. Let $v \in S_\ell$. Since E_v is unramified, $E_{v,0} = E_v$, and all hypotheses of Corollary 25.6 at v are among hypothesis 2 (ℓ is odd). It gives (Gd7) at v for $1 \leq k \leq t$ with $\lambda_k(v) = c_{\mathrm{B}}(V_{\mathcal{N}}, k-1) n_v + w_k(v)$. We bound these. With $n_{\min}$ as in Proposition 25.8 for $M_0 + h_0 \leq H$: putting $n_1 = \lceil \log_\ell \max(M_0 + h_0, 1) \rceil$, $M_0 + h_0 \leq \ell^{n_1}(\ell - 1)$, so $m_N = n_{\min} + 3 \leq n_1 + 3 \leq \log_\ell H + 4 \leq 4\lambda$. For $0 \leq i \leq e \leq t-1$ and $u \in J = J(V_{\mathcal{N}})$ with $u \neq i$, $0 < |u - i| < H + t$, so $v_\ell(u - i) \leq \lambda$ and $r_i \leq 5|J|\lambda$. In $c_{\mathrm{B}}(V_{\mathcal{N}}, e)$: $m_N - 1 \leq 4\lambda$; $\sum_{i=0}^{e}(2 + \max(1, r_i)) \leq t(3 + 5|J|\lambda)$; $e + 2 \leq t + 1$; $v_\ell(e), v_\ell(e+1) \leq \log_\ell t \leq \lambda$ (with $v_\ell(0) = 0$); $\lfloor e/(\ell-1) \rfloor \leq t$. Hence $c_{\mathrm{B}}(V_{\mathcal{N}}, e) \leq 6\lambda + 5t + 3 + 5t|J|\lambda \leq 19\, t \max(1, |J|)\lambda \leq 19\, c\, t\, n_v(\varrho + t)\lambda$, and $\sum_{k=1}^{t} c_{\mathrm{B}}(V_{\mathcal{N}}, k-1) n_v \leq 19\, c\, t^2 n_v^2 (\varrho + t)\lambda$. As $E_v/\mathbb{Q}_\ell$ is unramified, $E_v \cap \mathbb{Q}_\ell(\mu_{\ell^\infty}) = \mathbb{Q}_\ell$ (an unramified and a totally ramified extension), so $\chi_{\mathrm{cyc}}(G_{E_v}) = \mathbb{Z}_\ell^\times$; for g with $\chi_{\mathrm{cyc}}(g) = 1 + \ell$ and ℓ odd, $v_\ell((1+\ell)^k - 1) = 1 + v_\ell(k)$, so $w_k(v) \leq 1 + \log_\ell t \leq \lambda$ and $\sum_{k=1}^{t} w_k(v) \leq t\lambda$.

Step 3: block 0. Let $v \in S_\ell$, with the weight word of hypothesis 3; its hypotheses (unramified E_v of degree n_v, $K \supseteq \tau(E_v)$ and μ_ℓ, (Ll1), (Ll2), $\ell \geq 11$, ι the identity) are those of Proposition 26.2. Put $\Lambda = \Lambda_v^{\mathrm{row}}$; (Gd7) for $k = 0$ at v holds with any $\lambda_0(v) \geq \Lambda$. Moreover $\eta \leq \eta^\circ$, $w \leq w^\circ$, $S \leq S^\circ$, $h \leq H$ (both $b + \eta + S$ and $w + S + \eta + 1$ are at most H), $\log_\ell h \leq \log_\ell(2h) \leq \log_\ell(2H) \leq L - 1$, and $e_v \leq c\, e^\sharp$. *Item 1.* $\Lambda \leq c_{\mathrm{R}}\big((n_v+1)n_0\, A_{\mathrm{md}} + n_v m\, e_v\big)$, since the bracket of item 1 is at most $H/\ell^2 + (w^\circ + \eta^\circ)/\ell + S^\circ + L + \eta^\circ\beta = A_{\mathrm{md}}$. *Item 2.* Every $s_a \leq b + \eta$, so $H' \leq b + 2\eta + S \leq H$. Hence $H'/\ell^2 + S + \eta/\ell + 1 + \log_\ell(2H') \leq H/\ell^2 + (w^\circ + \eta^\circ)/\ell + S^\circ + L + \eta^\circ\beta = A_{\mathrm{md}}$, and, as $\log_\ell 2 < 1$, $1 + H'/\ell + \log_\ell(2H') \leq 2 + H/\ell + \log_\ell H \leq 2e^\sharp$. Thus $\Lambda \leq c_{\mathrm{R}}\big((n_v+1)n_0 A_{\mathrm{md}} + 2n_v m\, e^\sharp\big)$. *Item 3.* Here $A = A_{\mathrm{lo}}$ and $j = j_{\mathrm{A}}$, so $(\ell-1)^j \geq \ell_{\mathrm{A}}$ and $h/(\ell-1)^{j+1} \leq H/(\ell_{\mathrm{A}}(\ell-1))$; as $\log(\ell-1) \geq 1$, $j + 1 \leq \max(2, 2 + \log \ell_{\mathrm{A}}) \leq 3 + \log \ell_{\mathrm{A}}$, so $(G+1)(j+1) \leq (w^\circ + S^\circ + \eta^\circ + 1)(3 + \log \ell_{\mathrm{A}})$; and $\log_\ell h \leq L$. Thus $\Lambda \leq c_{\mathrm{R}}(n_v+1)n_0 A_{\mathrm{lo}}$. In all cases, with $n_0 \leq \varrho$, $e_v \leq ce^\sharp$, $c_{\mathrm{R}} \geq 1$: $\Lambda \leq 34\, c_{\mathrm{R}}\big((n_v+1)\varrho A + (c+1)n_v m\, e^\sharp\big) = \lambda_0(v)$. So (Gd7) holds with the stated losses.

Step 4: the sum. By hypothesis 1, $n_v \geq 1$, $\sum_v (n_v + 1) \leq 2\sum_v n_v \leq 4f$, $|S_\ell| \leq 2f$ and $\sum_v n_v^2 \leq (\sum_v n_v)^2 \leq 4f^2 = 8fn$. Hence

$$\sum_v \lambda_0(v) \leq 136\, c_{\mathrm{R}} f\varrho A + 68\, c_{\mathrm{R}}(c+1) f\, m\, e^\sharp, \qquad \sum_v \sum_{k \geq 1} \lambda_k(v) \leq 152\, c\, f\, Q\lambda + 2f\, t\lambda.$$

By hypothesis 4, $t\sum_{\Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) \leq c\, t$ and $2(n_0 + t - 1)\nu(\delta) \leq 4t(\varrho + t)$. As $Q \geq t(\varrho + t) \geq t$, $f \geq 2$ and $c, \lambda \geq 1$: $2ft\lambda \leq 2fQ\lambda$, $ct \leq c\, fQ\lambda$, $4t(\varrho + t) \leq 2fQ\lambda$. As $e^\sharp \leq \lambda + H/\ell$ and $m \leq Q$, $m\, e^\sharp \leq Q\lambda + mH/\ell$. With $\kappa_k = 0$ the left side of the statement is at most

$$136\, c_{\mathrm{R}} f\varrho A + 68\, c_{\mathrm{R}}(c+1) f\big(Q\lambda + \tfrac{mH}{\ell}\big) + (153c + 4) fQ\lambda \leq 300\, c_{\mathrm{R}}(c+1)\, f\Big(Q\lambda + \frac{mH}{\ell} + \varrho A\Big).$$

□

Proposition 26.2. *There is an absolute constant $c_{\mathrm{R}} \geq 1$ with the following property. Let E be a number field, $\ell \geq 11$ a prime, $K = \mathcal{O}[\ell^{-1}]$, and Galois Ribet data over E on $(\mathcal{R}, \pi', \mathcal{N}_{\mathcal{E}})$ with ι the identity and $t+1$ blocks, having (Gd1) and (Gd2); $n_0 = n_{\mathcal{R},0}$. Let v be a place of E above ℓ such that $E_v/\mathbb{Q}_\ell$ is unramified of degree n, K contains $\tau(E_v)$ for every $\mathbb{Q}_\ell$-embedding τ of E_v and a primitive ℓ-th root of unity, and (Ll1), (Ll2) hold at v; $D = G_{E_v}$. Number the embeddings τ_a, $a \in \mathbb{Z}/n$, along the Frobenius orbit: $\tau_{a+1} \circ \varphi = \tau_a$, φ the arithmetic Frobenius of E_v (a Frobenius numbering: the crystalline Frobenius carries the τ_a-component to the τ_{a+1}-component). For a point $x' \in \mathcal{Z}'$ let $U_{x'}$ be $K^{n_{\mathcal{R}}}$ with $g \in D$ acting by $\chi_1(g)^{-1}\mathbf{r}(g)(x')$, and $U'_{x'}$ its dual.*

*A **weight word** at v consists of integers $b \geq 1$, $w, S, \eta, m \geq 0$, $M, h' \geq 0$, sizes $s_a > \eta$ and signs $\epsilon_a \in \{\pm 1\}$ ($a \in \mathbb{Z}/n$) such that:*

(Ww1) *for every $x' \in \mathcal{Z}'$ and a, every Hodge jump of $U'_{x'}$ at τ_a lies in $[s_a, s_a + S] \cup [-\eta, \eta]$ if $\epsilon_a = 1$ and in $[-s_a - S, -s_a] \cup [-\eta, \eta]$ if $\epsilon_a = -1$, and in $[-M, h']$;*

(Ww2) *at every τ_a, W'_0 has at most m Hodge jumps in $[-\eta, \eta]$, counted with multiplicity;*

(Ww3) *every a is* thick *($b \leq s_a \leq b + \eta$) or* thin *($s_a \leq w$).*

Put $h = b + \eta + S$ if some a is thick and $h = w + S + \eta + 1$ otherwise; $G = w + S + \eta$; $e_v = e_{\mathrm{A}}$ of Proposition 24.1 for the window $[-M, h']$ and the jumps of the $U'_{x'}$; $\beta_v = 1$ if some thin a has a thick neighbour $a \pm 1$ and $|\ell s_a - b| \leq c_{\mathrm{R}}(\ell\eta + \ell S + b/\ell + s_a + \ell \log_\ell(2h))$, and $\beta_v = 0$ otherwise. Let Λ_v^{row} be the normalised length of the image of X_0 in $H^1(D, \mathrm{Cs}_0(u))/F_0(v)$. Then:

1. *(alternating words with a thin site) if some a is thin, $\epsilon_{a+1} \neq \epsilon_a$ for all a, no three consecutive $a, a+1, a+2$ are thick, and $20w, 20\eta, 20S \leq b$, then*

$$\Lambda_v^{\mathrm{row}} \leq c_{\mathrm{R}}\Big((n+1)\,n_0\Big(\frac{h}{\ell^2} + \frac{w+\eta}{\ell} + S + 1 + \log_\ell(2h) + \eta\,\beta_v\Big) + n\,m\,e_v\Big);$$

2. *(alternating thick words) if n is even, every a is thick, there is $d \in \{0,1\}$ with $\epsilon_a = -1$ if and only if $a+d$ is even (for representatives $0 \leq a < n$), and, with $H' = \max_a s_a + S + \eta$ and $a_{\mathrm{R}} = H'(2\ell-1)/(\ell-1)^2 + 4 + \log_\ell(2H')$, every $s_a \geq 2a_{\mathrm{R}} + \eta + 1$ and $s_a \geq (H'-\eta)/2$, then*

$$\Lambda_v^{\mathrm{row}} \leq c_{\mathrm{R}}\Big((n+1)\,n_0\Big(\frac{H'}{\ell^2} + S + \frac{\eta}{\ell} + 1 + \log_\ell(2H')\Big) + n\,m\Big(1 + \frac{H'}{\ell} + \log_\ell(2H')\Big)\Big);$$

3. *(long orbits) if $j \geq 1$ is an integer with $j+1 \leq n$, $b, w, S, \eta \geq 1$, $c_{\mathrm{R}}(j+1)(G+1) \leq b$, and every thick a has either no thick site among $a-1, \ldots, a-j$, or exactly one thick site $a-d$ with $1 \leq d \leq j$ and no other thick site among $a-1, \ldots, a-d-j$, then*

$$\Lambda_v^{\mathrm{row}} \leq c_{\mathrm{R}}\,(n+1)\,n_0\Big(\frac{h}{(\ell-1)^{j+1}} + (G+1)(j+1) + \log_\ell h\Big).$$

Proof. We take $c_{\mathrm{R}} = \max(68\,c_{\mathrm{ur}}, 320, c_{\mathrm{A}}, c_{\mathrm{C}})$, with c_{ur} the absolute constant of Proposition 24.1.

Step 1: reduction to a colength. Put $L = K$, $\mathcal{O}_L = \mathcal{O}$. The field E_v is unramified over $\mathbb{Q}_\ell$ of degree n and $\ell \geq 11$ is odd; K contains $\sigma(E_v(\mu_\ell)) = \sigma(E_v)(\sigma(\zeta_\ell))$ for every embedding σ, since it contains $\sigma(E_v)$ and a primitive ℓ-th root of unity, of which $\sigma(\zeta_\ell)$ is a power. The numbering (τ_a) is a Frobenius numbering. The Galois Ribet data have ι the identity, (Gd1), (Gd2), and (Ll1), (Ll2) hold at v, so Lemma 26.3 applies: by its items 1 and 2, $F\colon V_1 \to V_2$ is an onto map of crystalline lattices $T_1 \to T_2$ over E_v with coefficients K, with $d_2 = \dim_K V_2 = n_0$; by its item 3,

$$\Lambda_v^{\mathrm{row}} \leq \text{the colength of } F.$$

It remains to bound the colength in the three cases.

Step 2: the weight word of F. Let the weight word at v be given. (Ow1): by Lemma 26.3(2) every Hodge jump of V_1 at τ_a is a Hodge jump of some $U'_{x'}$ at τ_a, so by (Ww1) it lies in the window prescribed by (s_a, ϵ_a) and in $[-M, h']$. (Ow2): V_2 and W_0' have the same Hodge jumps with multiplicities (Lemma 26.3(2)), so (Ww2) gives (Ow2). (Ow3) is (Ww3); $b \geq 1$ and $s_a > \eta$ are part of the word. So $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ is a weight word for F, and its attached h, G, H' are those of the statement. Two comparisons:

- *Lifting exponents.* If $M_a \leq M$ and $h_a \leq h'$ for all a and every jump of V_1 at τ_a lies in $[-M_a, h_a]$, then $e(M_\bullet, h_\bullet) \leq e_v$. Indeed both are the integer e_{A} of Proposition 24.1, which depends only on ℓ and wt and is non-decreasing in wt: for $e(M_\bullet, h_\bullet)$ formed with $a^{\pm}_{\tau_a} = M_a, h_a$, so wt $= \max_a(M_a + h_a)$, and for e_v with $a^{\pm} = M, h'$, so wt $= M + h' \geq \max_a(M_a + h_a)$.
- *Band indicators.* $\beta_{320} \leq \beta_v$: the conditions defining them differ only in the constant of the inequality, $320 \leq c_{\mathrm{R}}$, and $\beta_v = \beta_{c_{\mathrm{R}}}$.

Item 1. Its hypotheses are those of Proposition C.1 for F and this word (a thin site, $\epsilon_{a+1} \neq \epsilon_a$, no three consecutive thick sites, $20w, 20\eta, 20S \leq b$, $\ell \geq 11$). With the windows $M_a = M$, $h_a = h'$ it gives the colength bound $68\,c_{\mathrm{ur}}((n+1)n_0(h/\ell^2 + (w+\eta)/\ell + S + 1 +$

$\log_\ell(2h)+\eta\beta_{320})+n\,m\,e(M_\bullet,h_\bullet))$, which by Step 2 and $68\,c_{\mathrm{ur}}\le c_{\mathrm{R}}$ is at most the bound of item 1.

Item 2. Its hypotheses are those of Proposition C.52 for F and this word ($\ell\ge 11$, n even, the parity condition on the signs, the two lower bounds for the sizes with the same H' and a_{R}). It bounds the colength of F by $c_{\mathrm{C}}\big((n+1)d_2(H'/\ell^2+S+\eta/\ell+1+\log_\ell(2H'))+n\,m(1+H'/\ell+\log_\ell(2H'))\big)$ with $d_2=n_0$; as $c_{\mathrm{C}}\le c_{\mathrm{R}}$, this is item 2.

Item 3. Its hypotheses give those of Proposition C.77: $j\ge 1$, $j+1\le n$, $c_{\mathrm{A}}(j+1)(G+1)\le c_{\mathrm{R}}(j+1)(G+1)\le b$, and the condition on the thick sites is the same (it looks at the sites $a-1,a-2,\dots$ preceding a in the Frobenius numbering). So the colength is at most $c_{\mathrm{A}}(n+1)n_0\big(h/(\ell-1)^{j+1}+(G+1)(j+1)+\log_\ell h\big)$; the bracket is non-negative ($h\ge 1$), and $c_{\mathrm{A}}\le c_{\mathrm{R}}$ gives item 3.

With Step 1 the three bounds hold for Λ_v^{row}. □

Lemma 26.3. *Let E be a number field, ℓ a prime, $K=\mathcal{O}[\ell^{-1}]$, and Galois Ribet data over E on $(\mathcal{R},\pi',\mathcal{N}_{\mathcal{E}})$ with ι the identity and $t+1$ blocks, having (Gd1) and (Gd2); $n_0=n_{\mathcal{R},0}$. Let v be a place of E above ℓ at which (Ll1) and (Ll2) hold, and $D=G_{E_v}$. For an $\mathcal{O}$-module X write $X(1)=X\otimes_{\mathbb{Z}_\ell}\mathbb{Z}_\ell(1)$, fix a generator ζ of $\mathbb{Z}_\ell(1)$, and let $\mathbf{v}=(1,0,\dots,0)$ be the first coordinate row vector of length n_0. Put:*

- $T_1=\mathcal{R}(1)\subset V_1=(K\mathcal{R})(1)$, $K\mathcal{R}\subseteq\prod_{x'\in\mathcal{Z}'}\mathrm{M}_{n_{\mathcal{R}}}(K)$, *with $g\in D$ acting by $a\otimes\zeta\mapsto a\,u(g)^{-1}\otimes g\zeta$;*
- $T_2=\mathrm{Hom}_{\mathcal{O}}(\Lambda_0,\mathcal{O}(1))\subset V_2=\mathrm{Hom}_K(W_0,K(1))$, *with $(gf)(m)=g\,f(\rho_0(u(g))^{-1}m)$;*
- $F\colon V_1\to V_2$, $F(a\otimes\zeta)(m)=(\mathbf{v}\,\rho_0(a)\,m)\otimes\zeta$, *with ρ_0 extended K-linearly to $K\mathcal{R}$.*

For $x'\in\mathcal{Z}'$ let $U_{x'}$ be $K^{n_{\mathcal{R}}}$ with $g\in D$ acting by $u(g)(x')=\chi_1(g)^{-1}\mathbf{r}(g)(x')$, and $U'_{x'}$, W'_0 the duals of $U_{x'}$, W_0. Then:

1. *T_1, T_2 are D-stable $\mathcal{O}$-lattices in V_1, V_2, and F is K-linear and D-equivariant with $F(T_1)=T_2$;*
2. *V_1 is isomorphic to a D-stable subspace of $\bigoplus_{x'\in\mathcal{Z}'}(U'_{x'})^{n_{\mathcal{R}}}$ and V_2 is isomorphic to W'_0; V_1 and V_2 are crystalline; every Hodge jump of V_1 at an embedding τ is a Hodge jump of $U'_{x'}$ at τ for some x', and V_2 and W'_0 have the same Hodge jumps with the same multiplicities;*
3. *the image of X_0 in $H^1(D,\mathrm{Cs}_0(u))/F_0(v)$ under restriction has normalised length at most*

$$\mathrm{nl}\Big(H^1_f(D,T_2)/F_*H^1_f(D,T_1)\Big)\in[0,\infty].$$

Proof. **Step 1: the lattices and F.** For each x' the map $g\mapsto\mathbf{r}(g)(x')$ is continuous on the compact group G_E, so its image is bounded in $\mathrm{M}_{n_{\mathcal{R}}}(K)$; as $\mathcal{Z}'$ is finite there is $s\ge 0$ with $\varpi^s\mathbf{r}(g)(x')\in\mathrm{M}_{n_{\mathcal{R}}}(\mathcal{O})$ for all g and x'. Since $\mathcal{R}$ is the span of $\mathbf{r}(G_E)$ over the coefficient algebra $\mathbb{T}'$ of Definition 17.2, whose elements are $\mathcal{O}$-valued functions, the component at x' of every element of $\mathcal{R}$ is an $\mathcal{O}$-combination of the $\mathbf{r}(g)(x')$, and $\mathcal{R}\subseteq\varpi^{-s}\prod_{x'}\mathrm{M}_{n_{\mathcal{R}}}(\mathcal{O})$; over the discrete valuation ring $\mathcal{O}$ it is therefore finitely generated and torsion-free, hence free, and it spans $K\mathcal{R}$. So T_1 is a lattice in V_1. It is D-stable: $u(g)^{-1}=\chi_1(g)\,\mathbf{r}(g^{-1})\in\mathcal{R}$ because $\chi_1(g)\in\mathcal{O}^\times$, and $\mathcal{R}$ is a ring. The action is continuous (each component $a(x')u(g)(x')^{-1}$ is continuous in g). T_2 is free of rank n_0 and spans V_2; it is D-stable because $\rho_0(u(g))=\chi_1(g)^{-1}\rho_0(\mathbf{r}(g))$ lies in $\mathrm{GL}_{n_0}(\mathcal{O})$,

with inverse $\chi_1(g)\rho_0(\mathbf{r}(g^{-1}))$, and it varies continuously with g (a block of $\mathbf{r}(g)(\pi')$, by (Bf1)). F is K-linear, and since ρ_0 is multiplicative,

$$F(a\,u(g)^{-1}\otimes g\zeta)(m) = \big(\mathbf{v}\rho_0(a)\rho_0(u(g))^{-1}m\big)\otimes g\zeta = g\Big(F(a\otimes\zeta)\big(\rho_0(u(g))^{-1}m\big)\Big) = \big(gF(a\otimes\zeta)\big)(m),$$

so F is D-equivariant. For $a\in\mathcal{R}$, $\rho_0(a)\in \mathrm{M}_{n_0}(\mathcal{O})$ by (Bf1), so $F(T_1)\subseteq T_2$. Conversely, an element of T_2 is $m\mapsto(\lambda m)\otimes\zeta$ for a row vector $\lambda\in\mathcal{O}^{n_0}$; the matrix $A\in\mathrm{M}_{n_0}(\mathcal{O})$ with first row λ and other rows 0 is $\rho_0(a)$ for some $a\in\mathcal{R}$ by (Bf2), and $F(a\otimes\zeta)$ is that element. This is item 1.

Step 2: comparison with the duals. Let $\mathrm{Row}_{x'}$ be the row vectors $K^{n_\mathcal{R}}$, tensored with $\mathbb{Z}_\ell(1)$, with g acting by $w\otimes\zeta\mapsto w\,u(g)(x')^{-1}\otimes g\zeta$. The map $V_1\to\bigoplus_{x'}\mathrm{Row}_{x'}^{n_\mathcal{R}}$ sending $a\otimes\zeta$ to the rows of the components $a(x')$, each tensored with ζ, is K-linear, injective (an element of $\prod_{x'}\mathrm{M}_{n_\mathcal{R}}(K)$ is determined by its rows) and D-equivariant (the i-th row of $a(x')u(g)(x')^{-1}$ is the i-th row of $a(x')$ times $u(g)(x')^{-1}$). Next, $w\otimes\zeta\mapsto(z\mapsto(wz)\otimes\zeta)$ is a D-equivariant K-linear isomorphism $\mathrm{Row}_{x'}\to\mathrm{Hom}_K(U_{x'},K(1))$ for the action $(gf)(z)=g\,f(u(g)(x')^{-1}z)$. Finally, for a finite-dimensional K-representation U of D, $f\mapsto(\mathrm{Tr}_{K/\mathbb{Q}_\ell}\otimes 1)\circ f$ is an isomorphism $\mathrm{Hom}_K(U,K(1))\to U'=\mathrm{Hom}_{\mathbb{Q}_\ell}(U,\mathbb{Q}_\ell(1))$, K-linear for $(\alpha f)(z)=f(\alpha z)$ and D-equivariant: if $\mathrm{Tr}\circ f=0$ then $\mathrm{Tr}(\alpha f(z))=\mathrm{Tr}(f(\alpha z))=0$ for all $\alpha\in K$, so $f(z)=0$ by the non-degeneracy of the trace form, and both sides have $\mathbb{Q}_\ell$-dimension $[K:\mathbb{Q}_\ell]\dim_K U$. With $U=U_{x'}$ and $U=W_0$ this gives the two isomorphisms of item 2. *Crystalline.* By (Gd1) and (Bf1), $g\mapsto\chi_1(g)=\rho_1(\mathbf{r}(g))$ is the action on the subquotient of $\mathbf{r}(\cdot)(\pi')$ spanned by the basis vectors with indices in I_1 (the spans of the indices in I_0 and in $I_0\cup I_1$ are stable, $\mathbf{r}(g)(\pi')$ being block upper triangular); by (Ll1) and Theorem B.45 the character $\chi_1|_D$ (on K) is crystalline, hence so is its dual χ_1^{-1} (the K-dual and the $\mathbb{Q}_\ell$-dual agree by the trace isomorphism). $U_{x'}$ is the image of $K_{\mathbf{r}(x')}^{n_\mathcal{R}}\otimes_{\mathbb{Q}_\ell}K_{\chi_1^{-1}}$ under the equivariant surjection $z\otimes\alpha\mapsto\alpha z$, so it is crystalline by (Ll1) and Theorem B.45; so are $U'_{x'}=\mathrm{Hom}_{\mathbb{Q}_\ell}(U_{x'},\mathbb{Q}_\ell)\otimes\mathbb{Q}_\ell(1)$, their sum, and its subspace V_1. W_0 is crystalline by Theorem 25.3, hence so is $W_0'\cong V_2$. *Jumps.* Let $X\subseteq Y$ be a D-stable subspace of a de Rham representation. As X is a $\mathbb{Q}_\ell$-direct summand of Y, $\mathrm{Fil}^aB_{\mathrm{dR}}\otimes X=(\mathrm{Fil}^aB_{\mathrm{dR}}\otimes Y)\cap(B_{\mathrm{dR}}\otimes X)$, so $\mathrm{Fil}^aD_{\mathrm{dR}}(X)=D_{\mathrm{dR}}(X)\cap\mathrm{Fil}^aD_{\mathrm{dR}}(Y)$, and the same holds for the τ-components (the idempotents of $E_v\otimes_{\mathbb{Q}_\ell}K$ act compatibly). If a is a jump of X at τ, take $z\in\mathrm{Fil}^aD_{\mathrm{dR},\tau}(X)\setminus\mathrm{Fil}^{a+1}D_{\mathrm{dR},\tau}(X)$; then $z\in\mathrm{Fil}^aD_{\mathrm{dR},\tau}(Y)$ and $z\notin\mathrm{Fil}^{a+1}D_{\mathrm{dR},\tau}(Y)$, so a is a jump of Y at τ. For a direct sum the filtered spaces $D_{\mathrm{dR},\tau}$ add, so its jumps are the union of the jumps of the summands; isomorphic representations have the same jumps with multiplicities. This proves item 2.

Step 3: the pairing identity. The evaluation $\Lambda_0\to\mathrm{Hom}_\mathcal{O}(T_2,\mathcal{O}(1))$, $m\mapsto(f\mapsto f(m))$, is a D-equivariant isomorphism (as Λ_0 is free; equivariance: $f\mapsto g((g^{-1}f)(m))=f(\rho_0(u(g))m)$). Hence, in the notation of Theorem B.50 for the lattice $T=T_2$ and $L=K$: $V_2^*\cong W_0$, $W^*\cong W_0/\Lambda_0=\mathrm{Cs}_0(u)$ with the pairing $P\colon T_2\times\mathrm{Cs}_0(u)\to(K/\mathcal{O})(1)$, $P(f,c)=f(\tilde c)$ mod $\mathcal{O}(1)$ for a lift $\tilde c\in W_0$, and $H^1_f(D,W^*)$ is the image of $H^1_f(D,W_0)$, i.e. $H^1_f(D,\mathrm{Cs}_0(u))$ of Lemma 25.4, which is $F_0(v)$ by (Ll2). Write $\langle z,x\rangle\in K/\mathcal{O}$ for the second pairing of Theorem B.50(1), $z\in H^1(D,T_2)$, $x\in H^1(D,\mathrm{Cs}_0(u))$. Let $\varphi\in\mathrm{Hom}_\mathcal{R}(\mathcal{N}_\mathcal{E},\mathrm{Cs}_0)$ with $c_\varphi=\varphi\circ y$ continuous; its class x restricts to $\varphi_*[y|_D]$. Let M be the $\mathcal{O}$-module $\mathcal{N}_\mathcal{E}$ with the trivial action of D, so $M(1)$ has g acting only on $\mathbb{Z}_\ell(1)$, and define

$$\beta\colon T_1\times\mathcal{N}_\mathcal{E}\to M(1),\quad \beta(a\otimes\zeta,n)=(an)\otimes\zeta,$$
$$\Psi_\varphi\colon M(1)\to(K/\mathcal{O})(1),\quad \Psi_\varphi(n\otimes\zeta)=\big(\mathbf{v}\,\varphi(n)\big)\otimes\zeta.$$

$an\in\mathcal{N}_\mathcal{E}$ since $\mathcal{N}_\mathcal{E}$ is a left ideal; β is $\mathcal{O}$-bilinear and D-equivariant ($\beta(au(g)^{-1}\otimes g\zeta,u(g)n)=(an)\otimes g\zeta$); Ψ_φ is $\mathcal{O}$-linear, D-equivariant and continuous (it vanishes on $\varpi^kM(1)$ for k with $\varpi^k\varphi(n_j)=0$ on a finite generating set). Since φ is $\mathcal{R}$-linear, $\rho_0(a)\varphi(n)=\varphi(an)$, so

$$P(F(a\otimes\zeta),\varphi(n))=\big(\mathbf{v}\rho_0(a)\varphi(n)\big)\otimes\zeta=\Psi_\varphi(\beta(a\otimes\zeta,n)).$$

For continuous cocycles $z\colon D \to T_1$ and $c\colon D \to \mathcal{N}_{\mathcal{E}}$, the cup products

$$(g,h) \mapsto P(F(z(g)), g\,\varphi(c(h))) \qquad \text{and} \qquad (g,h) \mapsto \Psi_{\varphi}(\beta(z(g), g\,c(h)))$$

therefore coincide, that is, $F_*[z] \cup_P \varphi_*[c] = (\Psi_{\varphi})_*([z] \cup_{\beta} [c])$.

Step 4: $[z] \cup_{\beta} [y|_D] = 0$ for $[z] \in H^1_f(D, T_1)$. By Theorem 25.3 (under (Ll1), (Gd1), (Gd2)) $V_{\mathcal{N}} = K\mathcal{N}_{\mathcal{E}}$ is crystalline with the lattice $\mathcal{N}_{\mathcal{E}}$ and $[y|_D] \in H^1_f(D, \mathcal{N}_{\mathcal{E}})$. M is a submodule of the free module $\mathcal{R}$, hence free with a basis $m_1, \dots, m_N$ and coordinates κ_j; then $M(1) \cong \mathcal{O}(1)^N$ and $[z] \cup_{\beta} [y] = \sum_j m_j \otimes ([z] \cup_{\beta_j} [y])$ with $\beta_j = (\kappa_j \otimes 1) \circ \beta$. Let $\mathcal{N}^*_{\mathcal{E}} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{N}_{\mathcal{E}}, \mathcal{O}(1))$ (the dual lattice of Theorem B.50 for $T = \mathcal{N}_{\mathcal{E}}$) and $\beta'_j\colon T_1 \to \mathcal{N}^*_{\mathcal{E}}$, $\beta'_j(t)(n) = \beta_j(t,n)$; it is D-equivariant because β_j is, and $z \cup_{\beta_j} y = (\beta'_j \circ z) \cup_{\mathrm{ev}} y$ for the evaluation $\mathrm{ev}\colon \mathcal{N}^*_{\mathcal{E}} \times \mathcal{N}_{\mathcal{E}} \to \mathcal{O}(1)$. The K-linear extension $\beta'_j\colon V_1 \to V^*_{\mathcal{N}}$ carries $H^1_f(D, V_1)$ into $H^1_f(D, V^*_{\mathcal{N}})$: if $z(g) \otimes 1 = gY - Y$ in $V_1 \otimes B_{\mathrm{cris}}$, then $\beta'_j(z(g)) \otimes 1 = g(\beta'_j \otimes 1)(Y) - (\beta'_j \otimes 1)(Y)$ (Definition 22.2). So the rational classes of $\beta'_j \circ z$ and $y|_D$ lie in $H^1_f(D, V^*_{\mathcal{N}})$ and $H^1_f(D, V_{\mathcal{N}})$. Let $\gamma \in H^2(D, K(1))$ be their cup product under the evaluation $V^*_{\mathcal{N}} \times V_{\mathcal{N}} \to K(1)$. As $K(1) \cong \mathbb{Q}_{\ell}(1)^{[K:\mathbb{Q}_{\ell}]}$ (a basis of K), $H^2(D, K(1)) = K \otimes_{\mathbb{Q}_{\ell}} H^2(D, \mathbb{Q}_{\ell}(1))$ and $\mathrm{Tr}_{K/\mathbb{Q}_{\ell}} \otimes 1\colon K(1) \to \mathbb{Q}_{\ell}(1)$ induces $\mathrm{Tr} \otimes 1$ on H^2. For $\alpha \in K$, $(\mathrm{Tr} \otimes 1)_*(\alpha\gamma)$ is the cup product, under the $\mathbb{Q}_{\ell}$-evaluation pairing, of the class of $(\mathrm{Tr} \otimes 1) \circ \alpha\beta'_j z$ in $H^1(D, \mathrm{Hom}_{\mathbb{Q}_{\ell}}(V_{\mathcal{N}}, \mathbb{Q}_{\ell}(1)))$ with $y|_D$; the first lies in H^1_f (the trace isomorphism of Step 2 is linear and equivariant, so it preserves H^1_f as above) and the second in $H^1_f(D, V_{\mathcal{N}})$, and $V_{\mathcal{N}}$ is de Rham, so this cup product vanishes by Theorem B.49 (the order of the factors changes it only by a sign). Hence $\mathrm{Tr}_{K/\mathbb{Q}_{\ell}}(\alpha\gamma) = 0$ in $H^2(D, \mathbb{Q}_{\ell}(1))$ for every $\alpha \in K$, and $\gamma = 0$ by the non-degeneracy of the trace form. By Theorem B.50(1), $H^2(D, \mathcal{O}(1)) \cong \mathcal{O} \to H^2(D, K(1)) \cong K$ is the inclusion, hence injective, and it carries the integral cup product $[\beta'_j z] \cup_{\mathrm{ev}} [y|_D]$ to the rational one. So $[z] \cup_{\beta_j} [y|_D] = 0$ for every j and $[z] \cup_{\beta} [y|_D] = 0$. With Step 3, $\langle F_*[z], x|_D \rangle = 0$ for every $x \in X_0$ and $[z] \in H^1_f(D, T_1)$.

Step 5: the count. Put $H = H^1_f(D, T_2)$ and $Z = F_* H^1_f(D, T_1)$; $Z \subseteq H$ by the argument of Step 4 applied to F. Let $\Theta\colon H^1(D, \mathrm{Cs}_0(u)) \to \mathrm{Hom}_{\mathcal{O}}(H, K/\mathcal{O})$, $\Theta(x)(z) = \langle z, x \rangle$, which is $\mathcal{O}$-linear as the pairing is $\mathcal{O}$-bilinear. V_2 is de Rham (Step 2), so by Theorem B.50(2) the kernel of Θ is the annihilator of H, which is $H^1_f(D, W^*) = F_0(v)$ (Step 3). X_0 is an $\mathcal{O}$-submodule (the classes c_{φ} are $\mathcal{O}$-linear in φ), and by Step 4 $\Theta(x|_D)$ vanishes on Z for $x \in X_0$. Hence Θ induces an injective $\mathcal{O}$-linear map from the image Y of X_0 in $H^1(D, \mathrm{Cs}_0(u))/F_0(v)$ into $\mathrm{Hom}_{\mathcal{O}}(H/Z, K/\mathcal{O})$. If H/Z has infinite length there is nothing to prove. Otherwise $H/Z \cong \bigoplus_i \mathcal{O}/\varpi^{k_i}$ with $\sum_i k_i = \mathrm{length}(H/Z)$, and $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\varpi^k, K/\mathcal{O}) = \varpi^{-k}\mathcal{O}/\mathcal{O}$ has length k, so $\mathrm{length}\, Y \le \mathrm{length}(H/Z)$. Dividing by $e_{\mathcal{O}}$ gives item 3. □

27 Ramified, split and small primes

This section bounds the total loss at the remaining primes of bounded size, each by a result of the form used in Theorem 22.4. At a prime of split type, Theorem 27.15 sums the losses of Theorem 25.1 over the column and row places and bounds the total by $190\,c\,f(n\,t^2(\varrho + t)(1 + \log_{\ell}(H + t)) + mH/\ell)$; this is a summation. At a small prime, where the places above ℓ may be ramified in a cyclotomic field $\mathbb{Q}_{\ell}(\mu_{N_{\ell}})$, Proposition 27.16 combines Theorem 25.1 with the lifting exponent over cyclotomic layers (Proposition 24.10) and two descent lemmas of Section 25.

The substantial case is that of the ramified primes, at which each place is ramified quadratic in the first cyclotomic layer, $E_v \subseteq E_{v,0}(\mu_{\ell})$ with $[E_v : E_{v,0}] = 2$; for a host with a complete good-host witness the primes q and d are of this kind (Lemma 22.5). There, as at the inert primes, the block-0 loss is a colength, and Theorem 27.1 rests on the colength bound Proposition 27.2 over a ramified quadratic subfield of the first cyclotomic layer. Its proof works letter by letter: at

each site the Wach module carries Frobenius letter data (Definition 27.4), a universal frame lifts the Herr classes at a letter up to ℓ^{Y_c} modulo at most $6m$ crystalline classes (Proposition 27.6), and the exceptional classes are absorbed by a generator count (Proposition 27.5, Lemma 27.3, Lemma 27.14). The case analysis is by slot: a letter whose embeddings carry jumps of one sign (Proposition 27.11), or of opposite signs (Proposition 27.12). The two counting lemmas 27.3 and Lemma 27.14 are used again in Appendix C.

Theorem 27.1. *Let E be a number field, ℓ an odd prime, $K = \mathcal{O}[\ell^{-1}]$, and Galois Ribet data over E on $(\mathcal{R}, \pi', \mathcal{N}_{\mathcal{E}})$ with ι the identity and $t+1$ blocks, having (Gd1) and (Gd2); write $n_0 = n_{\mathcal{R},0}$, $n_v = [E_v : \mathbb{Q}_\ell]$ and $E_{v,0}$ for the maximal unramified subextension of $E_v/\mathbb{Q}_\ell$. Let $n, \varrho, m \geq 1$ and $S, \eta \geq 0$ be integers, $f = 2n$, and $H, c \geq 1$ real numbers. Suppose:*

1. *$\sum_{v \in S_\ell} n_v \leq 2f$, $n_0 \leq \varrho \leq t$ and $m \leq t$;*
2. *at every $v \in S_\ell$: $E_v \subseteq E_{v,0}(\mu_\ell)$ and $[E_v : E_{v,0}] = 2$; K contains $\sigma(E_{v,0}(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $E_{v,0}(\mu_\ell)$ into $\overline{\mathbb{Q}}_\ell$; (Ll1) and (Ll2) hold; and $J(V_{\mathcal{N}}) \subseteq [-M_0, h_0]$ with integers $M_0, h_0 \geq 0$, $M_0 + h_0 \leq H$, and $|J(V_{\mathcal{N}})| \leq c\, n_v(\varrho + t)$;*
3. *at every $v \in S_\ell$ and every $\mathbb{Q}_\ell$-embedding $\tau\colon E_v \to K$ there are a sign $\epsilon_\tau \in \{+1, -1\}$ and an integer $b_\tau > \eta$ such that every Hodge jump a of every $U'_{x'}$, $x' \in \mathcal{Z}'$, at τ satisfies $|a| \leq 2H$ and either $|a| \leq \eta$ or $\epsilon_\tau a \in [b_\tau, b_\tau + S]$; and W'_0 has at most m Hodge jumps in $[-\eta, \eta]$ at τ, counted with multiplicity;*
4. *$\nu(\delta) \leq 2t$ and $\sum_{v \in \Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) \leq c$.*

Put $h = \lfloor 2H \rfloor$, $A = H/\ell + S + \eta + \log_\ell(2H) + 1$ and $Q = n\, t^2(\varrho + t)$. Then:

1. *(Gd6) holds with $\kappa_k = 0$, and (Gd7) holds with*

$$\lambda_0(v) = c_{\mathrm{Q}}\Big((n_v+1)\, n_0\Big(\frac{2h}{\ell} + 2S + 2\log_\ell h + 1 + \eta\Big) + 3\, n_v\, m\,(h+1)\Big),$$
$$\lambda_k(v) = c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)\, n_v + w_k(v) \quad (1 \leq k \leq t);$$

2. *these losses satisfy*

$$\sum_{k=0}^{t} \kappa_k + \sum_{v \in S_\ell} \sum_{k=0}^{t} \lambda_k(v) + t \sum_{v \in \Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) + 2(n_0 + t - 1)\, \nu(\delta)$$
$$\leq 160\, c_{\mathrm{Q}}\, c\, f\Big(\varrho\, A + m\, H + Q\big(1 + \log_\ell(H + t)\big)\Big);$$

3. *if moreover $b, C_1 \geq 1$ are real numbers with $2\varrho \leq \ell$, $H \leq C_1 b$,*

$$\varrho\,(S + \eta + 1 + \log_\ell(2H)) \leq C_1\, m\, b \quad \text{and} \quad Q\,(1 + \log_\ell(H + t)) \leq C_1\, m\, b,$$

then the right side of item 2 is at most $560\, c_{\mathrm{Q}}\, c\, C_1\, f\, m\, b$.

Proof. We have $t \geq m \geq 1$, and $h = \lfloor 2H \rfloor \geq 2$ is an integer; as Hodge jumps are integers, $|a| \leq 2H$ is equivalent to $|a| \leq h$.

Step 1: (Gd6). As ι is the identity, the map $\varphi \mapsto \varphi \circ \iota$ of (Gd6) is the identity, so its kernel is 0 and (Gd6) holds with $\kappa_k = 0$.

Step 2: block 0, by local duality and the colength. Let $v \in S_\ell$, $D = G_{E_v}$. The Galois Ribet data have ι the identity, (Gd1), (Gd2), and (Ll1), (Ll2) hold at v, so Lemma 26.3 applies

at v; let $T_1 \subset V_1$, $T_2 \subset V_2$ and F be as there. By its item 1, T_1, T_2 are D-stable $\mathcal{O}$-lattices and F is K-linear, D-equivariant, with $F(T_1) = T_2$; by its item 2, V_1 and V_2 are crystalline, every Hodge jump of V_1 at τ is a Hodge jump of some $U'_{x'}$ at τ, and V_2 and W'_0 have the same Hodge jumps with the same multiplicities; by its item 3,

$$\Lambda_v^{\mathrm{row}} \le \mathrm{nl}\big(H^1_f(D,T_2)/F_*H^1_f(D,T_1)\big).$$

Apply Proposition 27.2 with its prime ℓ, its field K equal to E_v (so its K_0 is $E_{v,0}$ and its n is n'_v), its coefficient field L equal to K, the representations V_1, V_2, the lattices T_1, T_2, $\Phi = F$, and the integers S, η, m, h and the data ϵ_τ, b_τ of hypothesis 3. Its hypotheses hold: ℓ is odd; $E_v \subseteq E_{v,0}(\mu_\ell)$, $[E_v : E_{v,0}] = 2$ and the condition on K are hypothesis 2; V_1, V_2 are crystalline and $F(T_1) = T_2$ (above); $h \ge 1$ and $b_\tau > \eta$. (Qa1): a Hodge jump a of V_1 at τ is a jump of some $U'_{x'}$ at τ, so by hypothesis 3 $|a| \le 2H$, i.e. $|a| \le h$, and $|a| \le \eta$ or $\epsilon_\tau a \in [b_\tau, b_\tau + S]$. (Qa2): $\sum_{|j|\le\eta} m_{V_2}(\tau, j)$ is the number of Hodge jumps of V_2 at τ in $[-\eta, \eta]$ counted with multiplicity, which equals that of W'_0, at most m by hypothesis 3. Here $r = \dim_K V_2 = n_0$, since $V_2 = \mathrm{Hom}_K(W_0, K(1))$ and $\dim_K W_0 = n_0$. With $2n'_v + 1 = n_v + 1$ and $6n'_v = 3n_v$ it gives

$$\mathrm{nl}\big(H^1_f(D,T_2)/F_*H^1_f(D,T_1)\big) \le c_{\mathrm{Q}}\Big((n_v{+}1)\,n_0\Big(\frac{2h}{\ell}{+}2S{+}2\log_\ell h{+}1{+}\eta\Big){+}3\,n_v\,m\,(h{+}1)\Big) = \lambda_0(v).$$

So $\Lambda_v^{\mathrm{row}} \le \lambda_0(v)$: this is (Gd7) for $k = 0$ at v.

Step 3: the character blocks. Let $v \in S_\ell$. The hypotheses of the first part of Corollary 25.6 at v are: ℓ odd, (Gd1), (Gd2), $E_v \subseteq E_{v,0}(\mu_\ell)$, the condition on K, (Ll1), (Ll2), and $J(V_\mathcal{N}) \subseteq [-M_0, h_0]$; all hold by hypothesis 2. It gives, for $1 \le k \le t$, that the image of X_k in $H^1(G_{E_v}, \mathrm{Cs}_k(u))/F_k(v)$ has normalised length at most $c_{\mathrm{B}}(V_\mathcal{N}, k-1)n_v + w_k(v) = \lambda_k(v)$. With Step 2 this is (Gd7) at v for all k, and item 1 is proved.

Step 4: the character exponents. Let $v \in S_\ell$ and $0 \le e \le t-1$; write $J = J(V_\mathcal{N})$. Put $n_1 = \lceil \log_\ell \max(M_0 + h_0, 1) \rceil$; then $\ell^{n_1}(\ell-1) \ge \ell^{n_1} \ge M_0 + h_0$, so $n_{\min} \le n_1 \le \log_\ell H + 1$ (as $\max(M_0 + h_0, 1) \le H$), and $m_N = n_{\min} + 3 \le \log_\ell H + 4 \le \lambda + 3 \le 4\lambda$. For $0 \le i \le e$ and $u \in J$ with $u \ne i$: $|u| \le \max(M_0, h_0) \le H$, so $0 < |u - i| \le H + t - 1 < H + t$ and $v_\ell(u - i) \le \log_\ell |u - i| < \lambda$; hence $r_i \le |J|(m_N + \lambda) \le 5|J|\lambda$. In $c_{\mathrm{B}}(V_\mathcal{N}, e)$: $m_N - 1 \le 4\lambda$; $\sum_{i=0}^{e}(2 + \max(1, r_i)) \le t(3 + 5|J|\lambda)$; $e + 2 \le t + 1$; $v_\ell(e) \le \log_\ell t \le \lambda$ (with $v_\ell(0) = 0$) and $v_\ell(e+1) \le \log_\ell t \le \lambda$ as $1 \le e + 1 \le t$; $\lfloor e/(\ell-1) \rfloor \le t$. So

$$c_{\mathrm{B}}(V_\mathcal{N}, e) \le 6\lambda + 5t + 3 + 5t|J|\lambda \le 14\,t\lambda + 5\,t|J|\lambda \le 19\,t\max(1, |J|)\,\lambda \le 19\,c\,t\,n_v(\varrho + t)\,\lambda,$$

and $\sum_{k=1}^{t} c_{\mathrm{B}}(V_\mathcal{N}, k-1)\,n_v \le 19\,c\,t^2 n_v^2(\varrho + t)\lambda$.

Step 5: the terms $w_k(v)$. Since $E_{v,0}$ is unramified over $\mathbb{Q}_\ell$ and every subextension of $\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell$ other than $\mathbb{Q}_\ell$ is ramified, $E_{v,0} \cap \mathbb{Q}_\ell(\mu_{\ell^\infty}) = \mathbb{Q}_\ell$ and $\chi_{\mathrm{cyc}}(G_{E_{v,0}}) = \mathbb{Z}_\ell^\times$. Choose $g \in G_{E_{v,0}}$ with $\chi_{\mathrm{cyc}}(g) = 1 + \ell$. Then g acts trivially on μ_ℓ (as $1 + \ell \equiv 1 \bmod \ell$) and on $E_{v,0}$, hence on $E_{v,0}(\mu_\ell) \supseteq E_v$ (hypothesis 2), so $g \in G_{E_v}$. As ℓ is odd, $v_\ell((1+\ell)^k - 1) = 1 + v_\ell(k)$ for $k \ge 1$ (classical), so $w_k(v) \le 1 + v_\ell(k) \le 1 + \log_\ell t \le \lambda$ for $1 \le k \le t$, and $\sum_{k=1}^{t} w_k(v) \le t\lambda$.

Step 6: the sum (item 2). *Block* 0. As $h \le 2H$, $h + 1 \le 2H + 1 \le 3H$, $\log_\ell h \le \log_\ell(2H)$, $n_v + 1 \le 2n_v$ and $n_0 \le \varrho$,

$$\lambda_0(v) \le c_{\mathrm{Q}}\Big(2n_v\varrho\Big(\frac{4H}{\ell} + 2S + 2\log_\ell(2H) + 1 + \eta\Big) + 9\,n_v mH\Big) \le c_{\mathrm{Q}}\,n_v\,(8\varrho A + 9mH),$$

so by hypothesis 1, $\sum_v \lambda_0(v) \le c_{\mathrm{Q}} f(16\varrho A + 18mH)$. *Character blocks.* By hypothesis 1, $n_v \ge 1$, $|S_\ell| \le \sum_v n_v \le 2f$ and $\sum_v n_v^2 \le (\sum_v n_v)^2 \le 4f^2 = 8fn$; by Steps 4 and 5, $\sum_v \sum_{k\ge1} \lambda_k(v) \le 152\,c\,fQ\lambda + 2ft\lambda$. *Tame and δ terms.* By hypothesis 4, $t\sum_{\Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) \le ct$ and $2(n_0 + t - 1)\nu(\delta) \le$

$4t(\varrho{+}t)$. As $Q \geq t(\varrho{+}t) \geq t$, $f \geq 2$ and $c, \lambda \geq 1$: $2ft\lambda \leq 2fQ\lambda$, $ct \leq cfQ\lambda$ and $4t(\varrho{+}t) \leq 2fQ\lambda$. With $\kappa_k = 0$ the left side of item 2 is at most

$$c_{\mathrm{Q}} f(16\varrho A + 18mH) + (153c + 4)fQ\lambda \leq 160\, c_{\mathrm{Q}}\, c\, f(\varrho A + mH + Q\lambda),$$

since $c_{\mathrm{Q}}, c \geq 1$.

Step 7: item 3. From $2\varrho \leq \ell$ and $H \leq C_1 b \leq C_1 mb$: $\varrho H/\ell \leq H/2 \leq C_1 mb/2$; so $\varrho A \leq C_1 mb/2 + \varrho(S + \eta + 1 + \log_\ell(2H)) \leq \frac{3}{2} C_1 mb$. Also $mH \leq C_1 mb$ and $Q\lambda \leq C_1 mb$. Hence $\varrho A + mH + Q\lambda \leq \frac{7}{2} C_1 mb$, and the right side of item 2 is at most $160 \cdot \frac{7}{2}\, c_{\mathrm{Q}} c C_1 fmb = 560\, c_{\mathrm{Q}} c C_1 fmb$. □

Proposition 27.2. *There is an absolute constant $c_{\mathrm{Q}} \geq 1$ with the following property. Let ℓ be an odd prime, K a finite extension of $\mathbb{Q}_\ell$ with maximal unramified subfield K_0, $n = [K_0 : \mathbb{Q}_\ell]$, such that $K \subseteq K_0(\mu_\ell)$ and $[K : K_0] = 2$; let L be a finite extension of $\mathbb{Q}_\ell$ containing $\sigma(K_0(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $K_0(\mu_\ell)$, and $\mathcal{O} = \mathcal{O}_L$. Let V_1, V_2 be continuous L-linear representations of G_K, crystalline as representations over $\mathbb{Q}_\ell$, G_K-stable $\mathcal{O}$-lattices $T_\nu \subseteq V_\nu$, and $\Phi\colon V_1 \to V_2$ an L-linear G_K-equivariant map with $\Phi(T_1) = T_2$; put $r = \dim_L V_2$. Let $S, \eta, m \geq 0$ and $h \geq 1$ be integers and, for every $\mathbb{Q}_\ell$-embedding $\tau\colon K \to L$, let $\epsilon_\tau \in \{+1, -1\}$ and $b_\tau > \eta$ be an integer, such that for every τ:*

(Qa1) *every Hodge jump a of V_1 at τ satisfies $|a| \leq h$, and either $|a| \leq \eta$ or $\epsilon_\tau a \in [b_\tau, b_\tau + S]$;*

(Qa2) $\sum_{|j| \leq \eta} m_{V_2}(\tau, j) \leq m$.

Then

$$\mathrm{nl}\Big(H^1_f(K, T_2)/\Phi_* H^1_f(K, T_1)\Big) \leq c_{\mathrm{Q}}\Big((2n+1)\, r\Big(\frac{2h}{\ell} + 2S + 2\log_\ell h + 1 + \eta\Big) + 6\, n\, m\, (h+1)\Big).$$

Proof. Let c_{cy} be the absolute constant of Proposition 24.10; we show that $c_{\mathrm{Q}} = \max(80,\, 3c_{\mathrm{cy}})$ has the property. Let the data be as in the statement, $H = H^1_f(K, T_2)$ and $H_1 = \Phi_* H^1_f(K, T_1) \subseteq H$, and put $\Sigma = 2h/\ell + 2S + 2\log_\ell h + 1 + \eta \geq 0$.

(i) Generators. V_2 is crystalline, hence de Rham, and $[K : \mathbb{Q}_\ell] = 2n$; by Lemma 27.3, H is generated as an $\mathcal{O}$-module by at most $g = (2n+1)r$ elements.

(ii) The lifting exponent. The field K_0 is finite unramified over $\mathbb{Q}_\ell$ and $K_0 \subseteq K \subseteq K_0(\mu_\ell)$; L contains $\sigma(K_0(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $K_0(\mu_\ell)$; by (Qa1) every Hodge jump of V_1 at every $\tau\colon K \to L$ lies in $[-h, h]$. So Proposition 24.10 applies with $p = \ell$ (odd), its fields $K_0 \subseteq K$ equal to ours, the lattices T_1, T_2, the map Φ, and $a^-_\tau = a^+_\tau = h$ for every τ; then $\mathrm{wt} = 2h$, and it gives the integer $e_{\mathrm{B}} = \lfloor c_{\mathrm{cy}}(2h+1) \rfloor$, with $0 \leq e_{\mathrm{B}} \leq 3c_{\mathrm{cy}} h$ (as $h \geq 1$), and

$$\ell^{e_{\mathrm{B}}} H \subseteq H_1.$$

(iii) The killing exponent. By Proposition 27.5, whose hypotheses are those of the statement, there are an integer e with $0 \leq e \leq 80\,\Sigma$ and a finite $Z \subseteq H$ with $|Z| \leq 6nm$ and $\ell^e H \subseteq H_1 + \sum_{z \in Z} \mathcal{O} z$.

(iv) The colength. By (ii), $\ell^{e_{\mathrm{B}}} z \in H_1$ for every $z \in Z$. So Lemma 27.14 applies with g, $H' = H_1$, Z, e and $e' = e_{\mathrm{B}}$: H/H_1 has finite length and

$$\mathrm{nl}(H/H_1) \leq g\, e + |Z|\, e_{\mathrm{B}} \leq (2n+1)\, r \cdot 80\,\Sigma + 6nm \cdot 3c_{\mathrm{cy}} h \leq c_{\mathrm{Q}}\Big((2n+1)\, r\, \Sigma + 6\, n\, m\, (h+1)\Big),$$

which is the assertion. □

Lemma 27.3. *Let ℓ be a prime, F a finite extension of $\mathbb{Q}_\ell$ of degree n, L a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$, V an L-vector space of dimension d with a continuous L-linear action of G_F which is de Rham as a $\mathbb{Q}_\ell$-representation, and $T \subseteq V$ a G_F-stable $\mathcal{O}$-lattice. Let $H^1_f(G_F, T)$ be the inverse image of $H^1_f(G_F, V)$ in $H^1(G_F, T)$. Then $H^1_f(G_F, T)$ is generated as an $\mathcal{O}$-module by at most $(n+1)\,d$ elements.*

Proof. Let ϖ be a uniformiser of $\mathcal{O}$ and k its residue field. For a finitely generated $\mathcal{O}$-module $X \cong \mathcal{O}^r \oplus \Theta$ with Θ finite, the minimal number of generators is $\dim_k X/\varpi X = r + \dim_k \Theta[\varpi]$ (structure theorem and Nakayama).

Finite generation and torsion. $H^i(G_F, M)$ is finite for finite ℓ-primary M (Tate), so $H^1(G_F, T)$ is finitely generated by Theorem B.30, and so is its submodule $X = H^1_f(G_F, T)$. Since G_F is compact and $V = \bigcup_k \varpi^{-k} T$ with each $\varpi^{-k}T$ open, every continuous cochain with values in V takes values in some $\varpi^{-k}T$; hence $H^1(G_F, V) = H^1(G_F, T) \otimes_{\mathcal{O}} L$ and the kernel of $H^1(G_F, T) \to H^1(G_F, V)$ is the torsion submodule Θ of $H^1(G_F, T)$. As $0 \in H^1_f(G_F, V)$, $\Theta \subseteq X$, so Θ is also the torsion of X, and X/Θ embeds in $H^1_f(G_F, V)$.

The torsion needs at most $d - a$ generators, $a = \dim_L H^0(G_F, V)$. The sequence $0 \to T \to V \to V/T \to 0$ gives the exact sequence $H^0(G_F, V) \to N \xrightarrow{\delta} H^1(G_F, T) \to H^1(G_F, V)$ with $N = H^0(G_F, V/T)$, so $\Theta = \delta(N) \cong N/I$ with I the image of $H^0(G_F, V)$. Now $N \subseteq V/T \cong (L/\mathcal{O})^d$ and $N[\varpi] \subseteq (V/T)[\varpi] \cong k^d$, so $N \cong (L/\mathcal{O})^{a'} \oplus F'$ with F' finite and $a' + \dim_k F'[\varpi] = \dim_k N[\varpi] \le d$. The image $I \cong H^0(G_F, V)/(H^0(G_F, V) \cap T) \cong (L/\mathcal{O})^a$ is divisible, hence contained in the maximal divisible submodule $(L/\mathcal{O})^{a'}$ of N; since $N/I \cong \Theta$ is finite, $a' = a$ and I is that submodule. Thus $\Theta \cong F'$ and $\dim_k \Theta[\varpi] \le d - a$.

The free part has rank at most $a + nd$. L acts on V by G_F-equivariant endomorphisms, so $H^1_f(G_F, V)$ is an L-subspace of $H^1(G_F, V)$, and X/Θ, a finitely generated torsion-free $\mathcal{O}$-submodule of it, is free of rank at most $\dim_L H^1_f(G_F, V)$. By the Bloch–Kato dimension formula Theorem B.43,

$$\dim_{\mathbb{Q}_\ell} H^1_f(G_F, V) = \dim_{\mathbb{Q}_\ell} H^0(G_F, V) + \dim_{\mathbb{Q}_\ell}\big(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)\big) \le [L:\mathbb{Q}_\ell]\,a + n\,[L:\mathbb{Q}_\ell]\,d,$$

because $D_{\mathrm{dR}}(V)$ has dimension $\dim_{\mathbb{Q}_\ell} V = [L:\mathbb{Q}_\ell]d$ over F (V de Rham). Dividing by $[L:\mathbb{Q}_\ell]$, $\dim_L H^1_f(G_F, V) \le a + nd$.

Hence X is generated by at most $(a + nd) + (d - a) = (n+1)d$ elements. □

Definition 27.4. Let ℓ be an odd prime, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ and a field K with $F \subseteq K \subseteq F(\mu_\ell)$ as in the existence of Wach modules over K, $\Delta = \Delta_K$, $s' = |\Delta|$, (V, T) a pair over K, and $h' \ge 1$ an integer such that every Hodge jump of V at every $\mathbb{Q}_\ell$-embedding $K \to L$ lies in $[-h', h']$; let $\mathbf{N} = \mathbf{N}(T)$, a Wach module over K with window $(h', h')_{a \in \mathbb{Z}/n}$, of rank $r = \dim_L V$. Put

$$M = \lceil 4h'/s' \rceil, \qquad \sigma = M - 1 + v_\ell\big((M-1)!\big),$$

fix $u \in \Gamma_K$ with $v_\ell(u-1) = 1$, write $\pi = \pi_\Delta$, $q = q_\Delta$, $R^\Delta_a = (e_a \mathbf{A}_{F,L})^\Delta$, and fix a **letter** $c \in \mathbb{Z}/n$.

(Fd1) Pole lattice $U_c = \mathbf{A}_{F,L}\,\varphi(\mathbf{N}_{c-1}) \subseteq e_c \mathbf{N}[1/\lambda_\infty]$, and

$$\Lambda_c = \{y \in (e_c \mathbf{N}[1/\lambda_\infty])^\Delta : \ Q^{h'} y \in U_c\}, \qquad J_c = \pi^M q^{3h'} U_c^\Delta.$$

(Fd2) Letter module $\bar{L}_c = \mathbf{N}^\Delta_{c-1}/\pi \mathbf{N}^\Delta_{c-1}$, an $\mathcal{O}_L$-module; for $x \in \mathbf{N}^\Delta_{c-1}$ write $\bar{x}$ for its class. Let $q^{-h'} R^\Delta_c$ be the R^Δ_c-submodule of the fraction field of $e_c \mathbf{A}_{F,L}$ generated by $q^{-h'}$, and

$$\Lambda^{\mathrm{L}}_c = q^{-h'} R^\Delta_c \otimes_{\mathcal{O}_L} \bar{L}_c, \qquad J^{\mathrm{L}}_c = \pi^M q^{3h'} R^\Delta_c \otimes_{\mathcal{O}_L} \bar{L}_c \subseteq \Lambda^{\mathrm{L}}_c,$$

an R^Δ_c-module through the first factor, on which γ_u acts by $\gamma_u(f \otimes \bar{x}) = \gamma_u(f) \otimes \bar{x}$. Its elements are **letter vectors**.

(Fd3) Frame lifts A **frame lift** of $x \in \mathbf{N}^{\Delta}_{c-1}$ is an element $s \in \mathbf{N}^{\Delta}_{c-1}$ with

$$s - \ell^{\sigma} x \in \pi \mathbf{N}^{\Delta}_{c-1} \qquad \text{and} \qquad (\gamma_u - 1)(s) \in \pi^{M} \mathbf{N}^{\Delta}_{c-1}.$$

(Fd4) Frobenius relation A letter vector x and $y \in \Lambda_c$ are **Frobenius related**, written $x \rightsquigarrow y$, if there are $f_1, \dots, f_k \in q^{-h'} R^{\Delta}_c$, $x_1, \dots, x_k \in \mathbf{N}^{\Delta}_{c-1}$ and frame lifts s_j of x_j with

$$x = \sum_j f_j \otimes \bar{x}_j \qquad \text{and} \qquad y - \sum_j f_j \, \varphi(s_j) \in J_c.$$

(Fd5) Hodge-integral letter vectors $\mathbf{N}^{\mathrm{L}}_c = \{x \in \Lambda^{\mathrm{L}}_c : \ x \rightsquigarrow y \text{ for some } y \in \mathbf{N}_c\}$.

Proposition 27.5. *Let ℓ be an odd prime, K a finite extension of $\mathbb{Q}_\ell$ with maximal unramified subfield K_0, $n = [K_0 : \mathbb{Q}_\ell]$, such that $K \subseteq K_0(\mu_\ell)$ and $[K : K_0] = 2$; let L be a finite extension of $\mathbb{Q}_\ell$ containing $\sigma(K_0(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $K_0(\mu_\ell)$, and $\mathcal{O} = \mathcal{O}_L$. Let V_1, V_2 be continuous L-linear representations of G_K, crystalline as representations over $\mathbb{Q}_\ell$, G_K-stable $\mathcal{O}$-lattices $T_\nu \subseteq V_\nu$, and $\Phi\colon V_1 \to V_2$ an L-linear G_K-equivariant map with $\Phi(T_1) = T_2$. Let $S, \eta, m \geq 0$ and $h \geq 1$ be integers and, for every $\mathbb{Q}_\ell$-embedding $\tau\colon K \to L$, let $\epsilon_\tau \in \{+1, -1\}$ and $b_\tau > \eta$ be an integer, such that for every τ: every Hodge jump a of V_1 at τ satisfies $|a| \leq h$, and either $|a| \leq \eta$ or $\epsilon_\tau a \in [b_\tau, b_\tau + S]$; and $\sum_{|j| \leq \eta} m_{V_2}(\tau, j) \leq m$. Then there are an integer e with*

$$0 \leq e \leq 80\Big(\frac{2h}{\ell} + 2S + 2\log_\ell h + 1 + \eta\Big)$$

and a finite subset $Z \subseteq H^1_f(K, T_2)$ with $|Z| \leq 6nm$ such that

$$\ell^e \, H^1_f(K, T_2) \subseteq \Phi_* H^1_f(K, T_1) + \sum_{z \in Z} \mathcal{O} z.$$

Proof. Write $F = K_0$, fix a Frobenius numbering $(\tau_a)_{a \in \mathbb{Z}/n}$ of the embeddings $F \to L$; every $\mathbb{Q}_\ell$-embedding $\tau\colon K \to L$ lies over exactly one τ_a. The hypotheses on ℓ, F, K, L are those of Proposition 23.7, so (V_ν, T_ν) are pairs over K. Since $\Phi(T_1) = T_2$ spans V_2, Φ is onto, and by Lemma 23.25 every Hodge jump of V_2 at τ is one of V_1 at τ. By Lemma 24.14, $d := |\Delta_K| = (\ell - 1)/[K : F] = (\ell - 1)/2$.

Step 1: normalising the data. Put $\eta_1 = \min(\eta, h)$, $h' = h + S + 1$ and $b'_\tau = \min(b_\tau, h + 1)$. Then $\eta_1 < b'_\tau$ and $b'_\tau + S \leq h'$. Let a be a Hodge jump of V_ν ($\nu = 1, 2$) at τ; then $|a| \leq h$. If $|a| \leq \eta$, then $|a| \leq \eta_1$. Otherwise $\epsilon_\tau a \in [b_\tau, b_\tau + S]$ and $b_\tau \leq |a| \leq h$, so $b'_\tau = b_\tau$ and $\epsilon_\tau a \in [b'_\tau, b'_\tau + S]$. Also $\sum_{|j| \leq \eta_1} m_{V_2}(\tau, j) \leq m$, and every Hodge jump lies in $[-h, h] \subseteq [-h', h']$, $h' \geq 1$. So for every letter $c \in \mathbb{Z}/n$ the hypotheses of Proposition 27.6 hold with h', S, η_1, (ϵ_τ, b'_τ), m and $u = 1 + \ell \in 1 + \ell\mathbb{Z}_\ell \subseteq \Gamma_K$ (so $v_\ell(u - 1) = 1$); here $M = \lceil 4h'/d \rceil$ and $\sigma = M - 1 + v_\ell((M-1)!)$, the Ω_c, κ_c and Y_c are formed with b'_τ and η_1, and $y_c = \lceil \eta_1 + \frac{1}{2} + \log_\ell h' \rceil$.

Step 2: the exceptional set and the exponent. For each c let $s_c \subseteq H^1_f(K, T_2)$ be the set of Proposition 27.6, and $Z = \bigcup_c s_c$; then $|Z| \leq 6nm$. Put $Y = \max_c Y_c$ and $e = r_0 + \sigma + Y$, with $r_0 = \mathrm{dsc}_{\ell,d}(0, 2h')$.

Step 3: the inclusion. Let $x \in H^1_f(K, T_2)$. By Lemma 24.12 for (V_2, T_2) and h' (its M, σ, r_0 are ours) there are w_c with $\ell^{r_0 + \sigma} x = \sum_c \mathrm{cl}(w_c)$, each w_c satisfying the hypotheses on w in Proposition 27.6. Hence

$$\ell^e x = \sum_c \ell^{Y - Y_c} \, \ell^{Y_c} \mathrm{cl}(w_c) \in \Phi_* H^1_f(K, T_1) + \sum_{z \in Z} \mathcal{O} z,$$

both summands being $\mathcal{O}$-modules.

Step 4: the bound $e \le 80\Sigma$, $\Sigma = \frac{2h}{\ell} + 2S + 2\log_\ell h + 1 + \eta$. (a) By Lemma 24.11, $r_0 \le 2h'\ell/(d(\ell-1)) = 4h'\ell/(\ell-1)^2$. (b) $M - 1 < 4h'/d = 8h'/(\ell-1)$ and, by Legendre's formula, $\sigma \le (M-1)\ell/(\ell-1) < 8h'\ell/(\ell-1)^2$. (c) Ω_c is contained in the union over the two τ over τ_c of a set of S integers: for a down τ the integers $b'_\tau, \dots, b'_\tau + S - 1$, for an up τ the integers $-(b'_\tau + S), \dots, -(b'_\tau + 1)$, whose absolute values are $b'_\tau + 1, \dots, b'_\tau + S$; so, if $S \ge 1$,

$$\kappa_c \le \sum_{\tau \text{ down}} \Big(S + v_\ell(S!) + v_\ell\binom{b'_\tau + S - 1}{S}\Big) + \sum_{\tau \text{ up}} \Big(S + v_\ell(S!) + v_\ell\binom{b'_\tau + S}{S}\Big) \le 2\Big(S + \frac{S}{2} + \log_\ell h'\Big),$$

by Legendre ($v_\ell(S!) \le S/(\ell-1) \le S/2$) and Kummer ($v_\ell\binom{N}{k} \le \log_\ell N$, here $N \le b'_\tau + S \le h'$); if $S = 0$, $\kappa_c = 0$. So $\kappa_c \le 3S + 2\log_\ell h'$. (d) $\max(2\sigma, y_c) \le 2\sigma + y_c \le 2\sigma + \eta + \frac{3}{2} + \log_\ell h'$. Hence

$$e = 2r_0 + 4\sigma + \kappa_c + \max(2\sigma, y_c) < \frac{56\, h'\ell}{(\ell-1)^2} + 3S + 3\log_\ell h' + \eta + \frac{3}{2}$$

for a c realising the maximum. As $\ell \ge 3$, $\ell/(\ell-1)^2 \le \frac{9}{4}\cdot\frac{1}{\ell}$, so $56h'\ell/(\ell-1)^2 \le 126(h+S+1)/\ell \le 63\cdot\frac{2h}{\ell} + 42S + 42$. As $h \ge 1$, $h' = h+S+1 \le h(S+2)$ and $\ell^{S+1} \ge S+2$, so $\log_\ell h' \le \log_\ell h + S + 1$. Altogether

$$e < 63\cdot\frac{2h}{\ell} + 48S + 3\log_\ell h + \eta + 46.5 \le 80\,\Sigma.$$

Also $e \ge 0$. This proves the proposition with this e and Z. □

Proposition 27.6. *Let ℓ, F, L, K, h', u, the letter c, $\Phi\colon T_1 \to T_2$, S, η and (ϵ_τ, b_τ) for the embeddings τ of K over τ_c be as in Proposition 27.11, with $[K:F] = 2$, and let $m \ge 0$ be an integer with $\sum_{|j|\le\eta} m_{V_2}(\tau, j) \le m$ for both τ over τ_c. Put*

$$\kappa_c = \sum_{i\in\Omega_c} (1 + v_\ell(|i|)), \qquad Y_c = 3\sigma + \kappa_c + r_0 + \max(2\sigma, y_c).$$

Then there is a finite subset $s_c \subseteq H^1_f(K, T_2)$ with $|s_c| \le 6m$ such that for every $w \in e_c\mathbf{N}(T_2)[1/\lambda_\infty]$ with $\gamma_\delta(w) = w$ ($\delta \in \Delta_K$), $Q^{h'}w \in \mathbf{A}_{F,L}\varphi(\mathbf{N}(T_2)_{c-1})$ and $(\gamma_{u'} - 1)(w) \in \pi^M\mathbf{N}(T_2)^\Delta_c$ for every $u' \in \Gamma_K$,

$$\ell^{Y_c}\,\mathrm{cl}(w) \in \Phi_* H^1_f(K, T_1) + \sum_{z\in s_c} \mathcal{O}_L z.$$

Proof. Write $\mathbf{N}^{(\nu)} = \mathbf{N}(T_\nu)$, $g = \mathbf{N}(\Phi)$ and $d = |\Delta_K|$. By Lemma 23.25 and (Ex1), both are Wach modules over K with window $(h', h')_a$, and $m_0 := Md \ge 4h' \ge 1 + h'$, so Herr vectors of precision m_0 and their classes are defined for both. Since $\pi = X^d v_\Delta$ (Rr1), $\pi^M\mathbf{N}^{(\nu)} \subseteq X^{m_0}\mathbf{N}^{(\nu)}$. As $v_\ell(u-1) = 1$, u topologically generates $1 + \ell\mathbb{Z}_\ell$, and $\Delta_K \cup \{u\}$ generates a dense subgroup of $\Gamma_K = \Delta_K \times (1 + \ell\mathbb{Z}_\ell)$.

Step 0: lifting precision. Claim: if $v \in \Lambda^{(\nu)}_c$ and $(\gamma_u - 1)(v) \in \pi(\mathbf{N}^{(\nu)}_c)^\Delta$, there is $k \in \pi(\mathbf{N}^{(\nu)}_c)^\Delta$ such that $\bar v = \ell^\sigma v - k$ satisfies $(\gamma_u - 1)(\bar v) \in \pi^M(\mathbf{N}^{(\nu)}_c)^\Delta$; then $\bar v$ is a Herr vector of precision m_0 with $Q^{2h'}\bar v \in \mathbf{N}^{(\nu)}$ and $\mathrm{cl}(\bar v) \in H^1_f(K, T_\nu)$. Indeed Lemma 24.16 (with $z = (\gamma_u - 1)(v)$) gives k; $\bar v$ is Δ_K-fixed and $(\gamma_u - 1)(\bar v) \in X^{m_0}\mathbf{N}^{(\nu)}$, so (Hc2) applies; $Q^{h'}v \in U^{(\nu)}_c$ and $Q^{h'}U^{(\nu)}_c \subseteq \mathbf{N}^{(\nu)}_c$ (Wm2), $k \in \mathbf{N}^{(\nu)}$; and Proposition 23.27 over K gives $\mathrm{cl}(\bar v) \in H^1_f(K, T_\nu)$.

Step 1: the exceptional classes. Apply Proposition 27.9 to (V_2, T_2) with the given data (its hypotheses are among ours; $e = 2$): it gives L^{r}_c of rank $\le 2m$ and letters $\ell_1, \dots, \ell_k$. By

Proposition 27.10 with $S = L_c^{\mathrm{r}}$, $\mathrm{Inv}_c(L_c^{\mathrm{r}})/(\mathrm{Inv}_c(L_c^{\mathrm{r}}) \cap \pi\mathbf{N}_c^{\mathrm{L},(2)})$ is free of rank $R \le 3 \cdot 2m = 6m$; choose $\xi_1, \dots, \xi_R \in \mathrm{Inv}_c(L_c^{\mathrm{r}})$ whose classes form a basis, and $Y_i \in \Lambda_c^{(2)}$ with $\xi_i \rightsquigarrow_2 Y_i$ (Fq0). By (Mo3) (with $a = 0$), $(\gamma_u - 1)(Y_i) \in \pi(\mathbf{N}_c^{(2)})^\Delta$; let $\bar{Y}_i = \ell^\sigma Y_i - k_i$ be as in Step 0 and $s_c = \{\mathrm{cl}(\bar{Y}_i)\}_{i \le R} \subseteq H_f^1(K, T_2)$.

Step 2: decomposing w. Let w be as in the statement; then $w \in \Lambda_c^{(2)}$ ($Q^{h'}w \in U_c^{(2)}$, Δ-fixed) and $(\gamma_u - 1)(w) \in \pi(\mathbf{N}_c^{(2)})^\Delta$. (Fp1)–(Fp3) give $f_1, \dots, f_k$, x, x_{r}; as $x_{\mathrm{r}} \in \mathrm{Inv}_c(L_c^{\mathrm{r}})$ by (Fp2), there are $o_i \in \mathcal{O}_L$ with $x_{\mathrm{r}} - \sum_i o_i\xi_i \in \pi\mathbf{N}_c^{\mathrm{L},(2)}$. Choose Y_j' with $f_j \otimes \bar{\ell}_j \rightsquigarrow_2 Y_j'$. By (Fq0), $x - \sum_j f_j \otimes \bar{\ell}_j - \sum_i o_i\xi_i \rightsquigarrow_2 D_0 := \ell^{2\sigma}\varpi_c w - \sum_j Y_j' - \sum_i o_i Y_i$, and this letter vector lies in $\pi\mathbf{N}_c^{\mathrm{L},(2)}$; by (Mo2) (with $j = 1$, $a = 0$), $D_0 \in \pi(\mathbf{N}_c^{(2)})^\Delta$.

Step 3: the slots. Put $y_1 = r_0 + \max(2\sigma, y_c)$. For each j, f_j satisfies (Fp3), $\ell_j \in (\mathbf{N}_{c-1}^{(2)})^\Delta$ has $\mathrm{val}_2(\ell_j) \in \epsilon_\tau \mathrm{Fil}_2(b_\tau)$ at the up τ, and $f_j \otimes \bar{\ell}_j \rightsquigarrow_2 Y_j'$. The two embeddings over τ_c are both down, both up, or of opposite signs; by Proposition 27.11 in the first two cases (then multiply its w_1 by $\ell^{y_1 - r_0}$) and Proposition 27.12 in the third, there is $w_{1,j} \in \Lambda_c^{(1)}$ with $(\gamma_u - 1)(w_{1,j}) \in \pi(\mathbf{N}_c^{(1)})^\Delta$ and $\ell^{y_1}Y_j' - g(w_{1,j}) \in \pi(\mathbf{N}_c^{(2)})^\Delta$. Put $v_1 = \sum_j w_{1,j}$.

Step 4: the scalar ϖ_c. We have $0 \notin \Omega_c$ (as $b_\tau > \eta \ge 0$), and for $i \ne 0$, $v_\ell(1 - u^i) = v_\ell(1 - u^{|i|}) = 1 + v_\ell(|i|)$ (ℓ odd, $v_\ell(u - 1) = 1$). So $\varpi_c = \varepsilon\ell^{\kappa_c}$ with $\varepsilon \in \mathbb{Z}_\ell^\times$. Multiplying Step 2 by $\ell^{y_1}\varepsilon^{-1}$ and using Step 3,

$$\ell^{y_1 + 2\sigma + \kappa_c} w - \varepsilon^{-1} g(v_1) - \sum_i \varepsilon^{-1}\ell^{y_1} o_i Y_i \in \pi(\mathbf{N}_c^{(2)})^\Delta.$$

Step 5: the classes. Let $\bar{v}_1 = \ell^\sigma v_1 - k$ be as in Step 0 (for $\nu = 1$), and

$$D = \ell^{\sigma + y_1 + 2\sigma + \kappa_c} w - \varepsilon^{-1} g(\bar{v}_1) - \sum_i \varepsilon^{-1}\ell^{y_1} o_i \bar{Y}_i.$$

By Step 4, $D \in \ell^\sigma\pi(\mathbf{N}_c^{(2)})^\Delta + \varepsilon^{-1}g(k) + \sum_i \varepsilon^{-1}\ell^{y_1}o_i k_i \subseteq \pi(\mathbf{N}_c^{(2)})^\Delta$ (g is $\mathbf{A}_{F,L}$-linear and commutes with the γ_δ, (Ex3)). Each of w, $g(\bar{v}_1)$, $\bar{Y}_i$ has $(\gamma_u - 1)(\cdot) \in \pi^M(\mathbf{N}_c^{(2)})^\Delta$ (for $g(\bar{v}_1)$ because g commutes with γ_u), so $(\gamma_u - 1)(D) \in \pi^M(\mathbf{N}_c^{(2)})^\Delta$, and Lemma 24.21 gives $D \in \pi^M(\mathbf{N}_c^{(2)})^\Delta \subseteq X^{m_0}\mathbf{N}^{(2)}$; by (Hc4), D is a Herr vector of precision m_0 with $\mathrm{cl}(D) = 0$. The vectors w, $\bar{Y}_i$, $\bar{v}_1$ are Herr vectors of precision m_0 ((Hc2), Step 0), and by (Hc5) so is $g(\bar{v}_1)$ with $\mathrm{cl}(g(\bar{v}_1)) = \Phi_*\mathrm{cl}(\bar{v}_1)$. By (Hc3),

$$\ell^{Y_c}\mathrm{cl}(w) = \varepsilon^{-1}\Phi_*\mathrm{cl}(\bar{v}_1) + \sum_i \varepsilon^{-1}\ell^{y_1}o_i\,\mathrm{cl}(\bar{Y}_i),$$

with $\mathrm{cl}(\bar{v}_1) \in H_f^1(K, T_1)$ (Step 0), which is the assertion. □

Proposition 27.7. *Let ℓ, F, L, K, (V, T), h', $\mathbf{N} = \mathbf{N}(T)$ and the letter c be as in Definition 27.4. Then:*

(Fq0) Well defined *For every letter vector x there is $y \in \Lambda_c$ with $x \rightsquigarrow y$; if $x \rightsquigarrow y$, then $x \rightsquigarrow y'$ holds exactly for $y' \in y + J_c$. If $x \rightsquigarrow y$, $x' \rightsquigarrow y'$ and $g \in R_c^\Delta$, then $x + x' \rightsquigarrow y + y'$ and $gx \rightsquigarrow gy$; and $x \rightsquigarrow 0$ for every $x \in J_c^{\mathrm{L}}$.*

(Fq1) Surjective up to $\ell^{2\sigma}$ *For every $y \in \Lambda_c$ there is a letter vector x with $x \rightsquigarrow \ell^{2\sigma}y$.*

(Fq2) Injective up to $\ell^{2\sigma}$ *If $x \rightsquigarrow 0$, then $\ell^{2\sigma}x \in J_c^{\mathrm{L}}$.*

(Fq3) Gamma *$\gamma_u(\Lambda_c) \subseteq \Lambda_c$, $\gamma_u(J_c) \subseteq J_c$, $\gamma_u(\Lambda_c^{\mathrm{L}}) \subseteq \Lambda_c^{\mathrm{L}}$, and $x \rightsquigarrow y$ implies $\gamma_u(x) \rightsquigarrow \gamma_u(y)$.*

(Fq4) Level values $\Lambda_c \subseteq W$. *Extend* λ *to* $q^{-h'}R_c^\Delta$ *by* $\lambda(f) = \lambda(q)^{-h'}\lambda(q^{h'}f) \in \mathcal{F}((t))$. *If* $x = \sum_j f_j \otimes \bar{x}_j$ *with* $f_j \in q^{-h'}R_c^\Delta$, $x_j \in \mathbf{N}_{c-1}^\Delta$, *and* $x \rightsquigarrow y$, *then*

$$\mathrm{lv}(y) - \ell^\sigma \sum_j \lambda(f_j)\,\epsilon\,\mathrm{val}(x_j) \;\in\; t^{3h'}\mathcal{F}[[t]]^r.$$

(Fq5) Naturality *Let* (V', T') *be a pair over* K *whose Hodge jumps at every embedding lie in* $[-h', h']$, *form the objects of Definition 27.4 for it with the same* u *and* c *(marked with a prime), and let* $f\colon T \to T'$ *be* G_K*-equivariant and* $\mathcal{O}_L$*-linear,* $g = \mathbf{N}(f)$, *extended* $\mathbf{A}_{F,L}$*-linearly to* $\mathbf{N}[1/\lambda_\infty]$. *Then* $g(\mathbf{N}_{c-1}^\Delta) \subseteq \mathbf{N}_{c-1}'^\Delta$, $g(\Lambda_c) \subseteq \Lambda_c'$, $g(J_c) \subseteq J_c'$, g *induces* $\bar{g}\colon \bar{L}_c \to \bar{L}_c'$, *and* $x \rightsquigarrow y$ *implies* $(\mathrm{id} \otimes \bar{g})(x) \rightsquigarrow g(y)$.

(Fq6) Expansions of coefficients *With* λ *extended as in (Fq4):* $\lambda(\pi)$ *is a unit of* $\mathcal{F}[[t]]$, *and* $\lambda(\gamma_u(f))_i = u^i\lambda(f)_i$ *for every* $f \in q^{-h'}R_c^\Delta$ *and* $i \in \mathbb{Z}$, *where* g_i *denotes the* t^i*-coefficient of* $g \in \mathcal{F}((t))$.

Proof. By (Ex1), $\mathbf{N}$ is a Wach module over K with window $(h', h')_a$; in particular, by (Wm2), $Q^{h'}\varphi(\mathbf{N}_{c-1}) \subseteq \mathbf{N}_c$ and $Q^{h'}\mathbf{N}_c \subseteq U_c$.

Step 0: coordinates. By (Dm1) choose a Δ-fixed $e_{c-1}\mathbf{A}_{F,L}$-basis $(n_k)_{k\le r}$ of $\mathbf{N}_{c-1}$; it is an R_{c-1}^Δ-basis of $\mathbf{N}_{c-1}^\Delta$, and since $R_{c-1}^\Delta/\pi R_{c-1}^\Delta = \mathcal{O}_L$ (Rr2), $(\bar{n}_k)$ is an $\mathcal{O}_L$-basis of $\bar{L}_c$; so every letter vector is $\sum_k F_k \otimes \bar{n}_k$ with unique $F_k \in q^{-h'}R_c^\Delta$. (a) U_c *is free over* $e_c\mathbf{A}_{F,L}$ *with basis* $(\varphi(n_k))$. The $\varphi(n_k)$ generate U_c, since $\varphi(\sum_k a_k n_k) = \sum_k \varphi(a_k)\varphi(n_k)$ and $\varphi(e_{c-1}\mathbf{A}_{F,L}) \subseteq e_c\mathbf{A}_{F,L}$. The module $e_c\mathbf{N}[1/\lambda_\infty]$ is free over the domain $e_c\mathbf{A}_{F,L}[1/\lambda_\infty]$ (Wm1), so U_c is torsion free; as $Q^{h'}\mathbf{N}_c \subseteq U_c \subseteq Q^{-h'}\mathbf{N}_c$, U_c has rank r. Being generated by r elements, these are linearly independent over the domain $e_c\mathbf{A}_{F,L}$. (b) *Coordinates of* Λ_c. By (Rr4), $q = Qw_\Delta$ with $w_\Delta \in \mathbb{Z}_\ell[[X]]^\times$, so $Q^{h'}y \in U_c$ if and only if $q^{h'}y \in U_c$. By (Wm3), $\gamma_\delta(\varphi(n_k)) = \varphi(\gamma_\delta n_k) = \varphi(n_k)$ for $\delta \in \Delta$, and $q = E_\Delta(\pi)$ is Δ-invariant (Rr1). Writing $q^{h'}y = \sum_k b_k\varphi(n_k)$, y is Δ-fixed if and only if every b_k is, by (a). Hence

$$\Lambda_c = \bigoplus_k q^{-h'}R_c^\Delta\,\varphi(n_k), \qquad U_c^\Delta = \bigoplus_k R_c^\Delta\,\varphi(n_k), \qquad J_c = \bigoplus_k \pi^M q^{3h'}R_c^\Delta\,\varphi(n_k),$$

with unique coordinates. (c) *The Frobenius of* π. The map φ commutes with the γ_δ, so $\varphi(R_{c-1}^\Delta) \subseteq R_c^\Delta$; by (Rr5), $\varphi(\pi) = w'\pi q^{s'}$ with $w' \in \mathbb{Z}_\ell[[X]]^\times$, and w' is Δ-invariant because $\varphi(\pi)$, π, q are. As $s'M \ge 4h'$, for $f \in q^{-h'}R_c^\Delta$ and $z \in \mathbf{N}_{c-1}^\Delta$:

$$f\,\varphi(\pi^M z) = f\,w'^M\pi^M q^{s'M}\varphi(z) \in \pi^M q^{3h'}U_c^\Delta = J_c. \qquad (*)$$

Step 1: (Fq0). Let $x = \sum_j f_j \otimes \bar{x}_j$ be any representation. Frame lifts s_j of x_j exist by (Gf6), and $y = \sum_j f_j\varphi(s_j)$ lies in Λ_c: it is Δ-fixed, and $q^{h'}y = \sum_j (q^{h'}f_j)\varphi(s_j) \in U_c$. So $x \rightsquigarrow y$. Fix frame lifts h_k of the n_k ((Gf1)) and write $x = \sum_k F_k \otimes \bar{n}_k$; put $y_0 = \sum_k F_k\varphi(h_k)$. For any representation and frame lifts as above, write $x_j \equiv \sum_k o_{jk}n_k$ modulo $\pi\mathbf{N}_{c-1}^\Delta$ with $o_{jk} \in \mathcal{O}_L$; then $F_k = \sum_j o_{jk}f_j$ (uniqueness of the F_k), and by (Gf6) $s_j - \sum_k o_{jk}h_k \in \pi^M\mathbf{N}_{c-1}^\Delta$ ($\sum_k o_{jk}h_k$ is a frame lift of $\sum_k o_{jk}n_k$, as γ_u fixes $\mathcal{O}_L$). By $(*)$,

$$\sum_j f_j\varphi(s_j) - y_0 = \sum_j f_j\varphi\Big(s_j - \sum_k o_{jk}h_k\Big) \in J_c.$$

Hence $x \rightsquigarrow y$ if and only if $y \in y_0 + J_c$, which is the first assertion. Concatenating representations gives $x + x' \rightsquigarrow y + y'$; multiplying the f_j by g (and J_c being an R_c^Δ-module) gives $gx \rightsquigarrow gy$. For $x = \sum_k \pi^M q^{3h'}g_k \otimes \bar{n}_k \in J_c^{\mathrm{L}}$, $y_0 = \pi^M q^{3h'}\sum_k g_k\varphi(h_k) \in J_c$, so $x \rightsquigarrow 0$.

Step 2: the frame matrices. Let λ_i be dual frames as in (Gf2) for the basis (n_k), and put $\Lambda_{ij} = \lambda_i(n_j) \in R_{c-1}^\Delta$ and $h_k = \sum_j H_{kj} n_j$, $H_{kj} \in R_{c-1}^\Delta$. By (Gf4), $\sum_j H_{kj}\Lambda_{ij} = \lambda_i(h_k) = \ell^{2\sigma}\delta_{ik} + \pi^M \rho_{ik}$ with $\rho_{ik} \in R_{c-1}^\Delta$; by (Gf5) for $x = n_j$, $\sum_k \Lambda_{kj} h_k = \ell^{2\sigma} n_j + \pi^M z_j$ with $z_j \in \mathbf{N}_{c-1}^\Delta$.

Step 3: (Fq1) and (Fq2). Let $y = \sum_j y_j \varphi(n_j) \in \Lambda_c$, $y_j \in q^{-h'} R_c^\Delta$ (Step 0(b)). Put $f_k = \sum_j y_j \varphi(\Lambda_{kj}) \in q^{-h'} R_c^\Delta$ and $x = \sum_k f_k \otimes \bar{n}_k$. Then $x \rightsquigarrow \sum_k f_k \varphi(h_k)$, and

$$\sum_k f_k \varphi(h_k) = \sum_j y_j \, \varphi\Big(\sum_k \Lambda_{kj} h_k\Big) = \ell^{2\sigma} y + \sum_j y_j \, \varphi(\pi^M z_j) \in \ell^{2\sigma} y + J_c$$

by $(*)$; by (Fq0), $x \rightsquigarrow \ell^{2\sigma} y$. For (Fq2), let $x = \sum_k f_k \otimes \bar{n}_k \rightsquigarrow 0$. By (Fq0), $\sum_k f_k \varphi(h_k) \in J_c$, i.e. $g_j := \sum_k f_k \varphi(H_{kj}) \in \pi^M q^{3h'} R_c^\Delta$ for every j (Step 0(b)). For each i,

$$\sum_j g_j \, \varphi(\Lambda_{ij}) = \sum_k f_k \, \varphi\Big(\sum_j H_{kj}\Lambda_{ij}\Big) = \ell^{2\sigma} f_i + \sum_k f_k \, w'^M \pi^M q^{s'M} \varphi(\rho_{ik}),$$

and the left side and the last sum lie in $\pi^M q^{3h'} R_c^\Delta$ ($f_k q^{s'M} \in q^{3h'} R_c^\Delta$). So $\ell^{2\sigma} f_i \in \pi^M q^{3h'} R_c^\Delta$ for all i, that is $\ell^{2\sigma} x \in J_c^{\mathrm{L}}$.

Step 4: (Fq3). By (Rr7), $u \in \Gamma_K = \Gamma_\Delta$, so by (Rr6) $\gamma_u(\pi) = \pi(1 + t_u q)$ and $\gamma_u(q) = h_u q$ with $t_u \in \mathbb{Z}_\ell[[X]]$, $h_u \in 1 + X\mathbb{Z}_\ell[[X]]$, both Δ-invariant; h_u is a unit, and so is $1 + t_u q$, whose constant term $1 + t_u(0)E_\Delta(0)$ lies in $1 + \ell\mathbb{Z}_\ell$ (E_Δ is Eisenstein, (Rr3)). Hence γ_u preserves R_c^Δ, $q^{-h'} R_c^\Delta$ and the ideal $\pi^M q^{3h'} R_c^\Delta$; as γ_u fixes $\mathcal{O}_L$, the map $\gamma_u(f \otimes \bar{x}) = \gamma_u(f) \otimes \bar{x}$ of (Fd2) is well defined on $\Lambda_c^{\mathrm{L}} = q^{-h'} R_c^\Delta \otimes_{\mathcal{O}_L} \bar{L}_c$ and $\gamma_u(\Lambda_c^{\mathrm{L}}) \subseteq \Lambda_c^{\mathrm{L}}$. As Γ_K is commutative, γ_u preserves Δ-fixed vectors, and $\gamma_u(a\varphi(z)) = \gamma_u(a)\varphi(\gamma_u z)$ (Wm3) shows $\gamma_u(U_c) \subseteq U_c$; with $\gamma_u(q^{h'}) = h_u^{h'} q^{h'}$ this gives $\gamma_u(\Lambda_c) \subseteq \Lambda_c$ (Step 0(b)) and $\gamma_u(J_c) \subseteq J_c$. Let $x = \sum_j f_j \otimes \bar{x}_j \rightsquigarrow y$ with frame lifts s_j, $y - \sum_j f_j \varphi(s_j) \in J_c$. Then

$$\gamma_u(y) - \sum_j \gamma_u(f_j)\varphi(s_j) \in \sum_j \gamma_u(f_j) \, \varphi\big((\gamma_u - 1)(s_j)\big) + J_c \subseteq J_c$$

by $(*)$, since $(\gamma_u - 1)(s_j) \in \pi^M \mathbf{N}_{c-1}^\Delta$. So $\gamma_u(x) = \sum_j \gamma_u(f_j) \otimes \bar{x}_j \rightsquigarrow \gamma_u(y)$.

Step 5: (Fq4). If $y \in \Lambda_c$ then $Q^{2h'} y \in Q^{h'} U_c \subseteq \mathbf{N}_c$, so $y \in W$. By (Rr4) and (Lz1), $\lambda(q) = \lambda(e_c Q)\lambda(e_c w_\Delta) \in t\mathcal{F}[[t]]^\times$, so the extension of λ is defined, $\lambda(q^{-h'} R_c^\Delta) \subseteq t^{-h'}\mathcal{F}[[t]]$, and by (Lz1) $\lambda(q)^{h'} \mathrm{lv}(fw) = \mathrm{lv}(q^{h'} fw) = \lambda(q^{h'} f)\mathrm{lv}(w)$, i.e. $\mathrm{lv}(fw) = \lambda(f)\mathrm{lv}(w)$ for $f \in q^{-h'} R_c^\Delta$ and $w \in W$ with $fw \in W$. (a) $\mathrm{lv}(J_c) \subseteq t^{3h'}\mathcal{F}[[t]]^r$. For $z = \sum_k b_k \varphi(n_k) \in U_c^\Delta$, $\mathrm{lv}(z) = \sum_k \lambda(b_k)\mathrm{lv}(\varphi(n_k)) \in \mathcal{F}[[t]]^r$ by (Lz1) and (Lz5); and $\lambda(\pi^M q^{3h'}) \in t^{3h'}\mathcal{F}[[t]]$. (b) *Frame lifts.* Let s be a frame lift of $x \in \mathbf{N}_{c-1}^\Delta$ and $P = \varphi(s)$. By (Lz5), $\mathrm{lv}(P) = \sum_{i \geq 0} a_i t^i \in \mathcal{F}[[t]]^r$ with $a_0 = \epsilon\,\mathrm{val}(s) = \ell^\sigma \epsilon\,\mathrm{val}(x)$, because $s - \ell^\sigma x \in \pi\mathbf{N}_{c-1} \subseteq X\mathbf{N}_{c-1} = \ker(\mathrm{val})$ ((Rr1): $\pi = X^{s'} v_\Delta$). Write $(\gamma_u - 1)(s) = \pi^M z$, $z \in \mathbf{N}_{c-1}^\Delta$; then $(\gamma_u - 1)(P) = \varphi(\pi)^M \varphi(z)$ (Wm3) and, as $\lambda(\varphi(\pi)) = \lambda(w')\lambda(\pi)\lambda(q)^{s'} \in t^{s'}\mathcal{F}[[t]]$, $\mathrm{lv}((\gamma_u - 1)P) \in t^{s'M}\mathcal{F}[[t]]^r$ by (Lz1), (Lz5). By (Lz2), $\mathrm{lv}(\gamma_u P) = \gamma_u^{(0)}(\mathrm{lv}(P))$, and since $u \in 1 + \ell\mathbb{Z}_\ell$ the automorphism σ_u is the identity, so $\gamma_u^{(0)}(\sum_i a_i t^i) = \sum_i u^i a_i t^i$. Hence $(u^i - 1)a_i = 0$ for $1 \leq i < s'M$; as $v_\ell(u^i - 1) = 1 + v_\ell(i)$ is finite, $u^i \neq 1$ and $a_i = 0$. So $\mathrm{lv}(\varphi(s)) - \ell^\sigma \epsilon\,\mathrm{val}(x) \in t^{s'M}\mathcal{F}[[t]]^r$. (c) Let $x = \sum_j f_j \otimes \bar{x}_j \rightsquigarrow y$. By (Fq0) and the definition, $y = \sum_j f_j \varphi(s_j) + y_1$ with frame lifts s_j of x_j and $y_1 \in J_c$ (a different representation of x changes y only within J_c, and $\mathrm{val}(x_j)$ depends only on $\bar{x}_j$, as $\pi\mathbf{N}_{c-1} \subseteq \ker \mathrm{val}$; so we may use the given representation). By (b), (a) and the multiplicativity above,

$$\mathrm{lv}(y) - \ell^\sigma \sum_j \lambda(f_j)\epsilon\,\mathrm{val}(x_j) \in \sum_j \lambda(f_j)\, t^{s'M}\mathcal{F}[[t]]^r + t^{3h'}\mathcal{F}[[t]]^r \subseteq t^{3h'}\mathcal{F}[[t]]^r,$$

since $s'M - h' \geq 3h'$.

Step 6: (Fq5). By (Ex3), g is $\mathbf{A}_{F,L}$-linear and commutes with φ and every γ_v; so it commutes with e_{c-1}, e_c, maps Δ-fixed vectors to Δ-fixed vectors, $\mathbf{N}_{c-1}^{\Delta}$ into $\mathbf{N}_{c-1}'^{\Delta}$, $\pi\mathbf{N}_{c-1}^{\Delta}$ into $\pi\mathbf{N}_{c-1}'^{\Delta}$ (hence induces $\bar{g}$), U_c into U_c', Λ_c into Λ_c' and J_c into J_c'. If s is a frame lift of x, then $g(s) - \ell^{\sigma} g(x) = g(s - \ell^{\sigma} x) \in \pi\mathbf{N}_{c-1}'^{\Delta}$ and $(\gamma_u - 1)(g(s)) = g((\gamma_u - 1)(s)) \in \pi^M \mathbf{N}_{c-1}'^{\Delta}$, so $g(s)$ is a frame lift of $g(x)$ (the numbers M, σ depend only on h' and Δ). Applying g to $y - \sum_j f_j \varphi(s_j) \in J_c$ gives $g(y) - \sum_j f_j \varphi(g(s_j)) \in J_c'$, i.e. $\sum_j f_j \otimes \overline{g(x_j)} \rightsquigarrow g(y)$.

Step 7: (Fq6). We have $\pi = X^{s'} v_{\Delta}$ with v_{Δ} a unit (Rr1), and $\lambda(X) = \epsilon(\zeta e^{t/\ell} - 1)$ has the unit constant term $\epsilon(\zeta - 1)$ of $\mathcal{F}$ (Lz1); so $\lambda(\pi)$ is a unit. For $g \in e_c \mathbf{A}_{F,L}$ and $w' \in W$, (Lz1) and (Lz2) give $\lambda(\gamma_u g)\, \gamma_u^{(0)}(\mathrm{lv}(w')) = \mathrm{lv}(\gamma_u(gw')) = \gamma_u^{(0)}(\lambda(g))\, \gamma_u^{(0)}(\mathrm{lv}(w'))$ ($\gamma_u^{(0)}$ is a ring automorphism of $\mathcal{F}((t))$, applied coordinatewise). Take $w' = \varphi(n)$ with $n \in \mathbf{N}_{c-1}$ and $\mathrm{val}(n)$ the k-th standard basis vector ((Lz5)); the k-th coordinate of $\gamma_u^{(0)}(\mathrm{lv}(w'))$ is ϵ plus a multiple of t, a unit of $\mathcal{F}[[t]]$. Hence $\lambda \circ \gamma_u = \gamma_u^{(0)} \circ \lambda$ on $e_c \mathbf{A}_{F,L}$. For $f \in q^{-h'} R_c^{\Delta}$, using $\gamma_u(q) = h_u q$ (Step 4), $\lambda(\gamma_u f) = \lambda(\gamma_u q)^{-h'} \lambda(\gamma_u(q^{h'} f)) = \gamma_u^{(0)}(\lambda(q))^{-h'} \gamma_u^{(0)}(\lambda(q^{h'} f)) = \gamma_u^{(0)}(\lambda(f))$. As $u \in 1 + \ell\mathbb{Z}_{\ell}$, σ_u is the identity and $\gamma_u^{(0)}(\sum_i a_i t^i) = \sum_i u^i a_i t^i$. □

Lemma 27.8. *Let ℓ, F, L, K, (V,T), h', $\mathbf{N} = \mathbf{N}(T)$ and the letter c be as in Definition 27.4. Let x be a letter vector, $y \in \Lambda_c$ and $a \geq 0$ an integer with $x \rightsquigarrow \ell^a y$. Then:*

(Mo1) Integrality *$J_c \subseteq \pi^M \mathbf{N}_c^{\Delta}$, and $y \in \mathbf{N}_c$ if and only if $x \in \mathbf{N}_c^{\mathrm{L}}$.*

(Mo2) Divisibility *For $1 \leq j \leq M$: $y \in \pi^j \mathbf{N}_c^{\Delta}$ if and only if $x \in \pi^j \mathbf{N}_c^{\mathrm{L}}$.*

(Mo3) Precision *$(\gamma_u - 1)(y) \in \pi\mathbf{N}_c^{\Delta}$ if and only if $(\gamma_u - 1)(x) \in \pi\mathbf{N}_c^{\mathrm{L}}$.*

(Mo4) Structure *$\mathbf{N}_c^{\mathrm{L}}$ is an R_c^{Δ}-submodule of Λ_c^{L} with $\gamma_u(\mathbf{N}_c^{\mathrm{L}}) \subseteq \mathbf{N}_c^{\mathrm{L}}$ and $q^{h'} R_c^{\Delta} \otimes_{\mathcal{O}_L} \bar{L}_c \subseteq \mathbf{N}_c^{\mathrm{L}}$, hence $J_c^{\mathrm{L}} \subseteq \pi^M \mathbf{N}_c^{\mathrm{L}}$; and for integers $b, j \geq 0$ and $x' \in \Lambda_c^{\mathrm{L}}$, $\ell^b x' \in \pi^j \mathbf{N}_c^{\mathrm{L}}$ implies $x' \in \pi^j \mathbf{N}_c^{\mathrm{L}}$.*

(Mo5) Hodge description *Write $x = \sum_k f_k \otimes \bar{x}_k$ with $f_k \in q^{-h'} R_c^{\Delta}$ and $x_k \in \mathbf{N}_{c-1}^{\Delta}$, and*

$$\sum_k \lambda(f_k)\, \epsilon\, \mathrm{val}(x_k) = \sum_{i \geq -h'} e_i t^i \qquad \text{with } e_i \in \mathcal{F}^r.$$

Then $x \in \mathbf{N}_c^{\mathrm{L}}$ if and only if $e_i \in \mathcal{F} \otimes_{\mathcal{F}_K} \mathrm{Fil}(-i)$ for every $i < h'$.

(Mo6) Sen condition *If $(\gamma_u - 1)(x) \in \mathbf{N}_c^{\mathrm{L}}$, then for every $i \neq 0$ the t^i-coefficient of $\mathrm{lv}(y)$ lies in $\mathcal{F} \otimes_{\mathcal{F}_K} \mathrm{Fil}(-i)$; in the notation of (Mo5), $e_i \in \mathcal{F} \otimes_{\mathcal{F}_K} \mathrm{Fil}(-i)$ for every $i \neq 0$ with $i < 3h'$.*

Proof. **Step 0: three facts.** (i) *$\mathbf{N}_c$ is ℓ-saturated in W*: if $w \in W$ and $\ell^b w \in \mathbf{N}_c$ then $w \in \mathbf{N}_c$, by (Lz3), since $\mathrm{lv}(\ell^b w) = \ell^b \mathrm{lv}(w)$ and each $\mathcal{F} \otimes_{\mathcal{F}_K} \mathrm{Fil}(-i)$ is an $\mathcal{F}$-module. Also $\Lambda_c \subseteq W$ by (Fq4). (ii) $J_c \subseteq \pi^M \mathbf{N}_c^{\Delta}$: by (Wm2) and (Rr4) ($q = Q w_{\Delta}$, w_{Δ} a unit), $q^{h'} U_c \subseteq \mathbf{N}_c$, so $q^{3h'} U_c^{\Delta}$ consists of Δ-fixed elements of $\mathbf{N}_c$. (iii) *Coordinates*: as $R_{c-1}^{\Delta}/\pi R_{c-1}^{\Delta} = \mathcal{O}_L$ (Rr2), a Δ-fixed basis (n_k) of $\mathbf{N}_{c-1}$ ((Dm1)) gives an $\mathcal{O}_L$-basis $(\bar{n}_k)$ of $\bar{L}_c$, so $\Lambda_c^{\mathrm{L}} = \bigoplus_k q^{-h'} R_c^{\Delta} \otimes \bar{n}_k$; and $q^{-h'} R_c^{\Delta} \cong R_c^{\Delta} \cong \mathcal{O}_L[[T]]$ (multiplication by $q^{h'}$; (Rr2)), on which π is a non-zero-divisor and modulo π^j of which ℓ is a non-zero-divisor. Likewise $\mathbf{N}_c^{\Delta}/\pi^j \mathbf{N}_c^{\Delta} \cong (R_c^{\Delta}/\pi^j)^r$ has no ℓ-torsion ((Dm1)).

(Mo1). The first assertion is Step 0(ii). If $y \in \mathbf{N}_c$, then $\ell^a y \in \mathbf{N}_c$ and $x \in \mathbf{N}_c^{\mathrm{L}}$ by definition. Conversely, if $x \rightsquigarrow y'$ with $y' \in \mathbf{N}_c$, then $\ell^a y - y' \in J_c \subseteq \mathbf{N}_c$ by (Fq0), so $\ell^a y \in \mathbf{N}_c$ and $y \in \mathbf{N}_c$ by Step 0(i).

(Mo4). Let $x_1 \rightsquigarrow y_1$, $x_2 \rightsquigarrow y_2$ with $y_1, y_2 \in \mathbf{N}_c$ and $g \in R_c^\Delta$. By (Fq0), $x_1 + x_2 \rightsquigarrow y_1 + y_2$ and $gx_1 \rightsquigarrow gy_1$, and by (Fq3) $\gamma_u(x_1) \rightsquigarrow \gamma_u(y_1)$; these targets lie in $\mathbf{N}_c$. For $f \in R_c^\Delta$ and $z \in \mathbf{N}_{c-1}^\Delta$ with frame lift s, $q^{h'} f \otimes \bar{z} \rightsquigarrow q^{h'} f\varphi(s) \in q^{h'} U_c \subseteq \mathbf{N}_c$; so $q^{h'} R_c^\Delta \otimes \bar{L}_c \subseteq \mathbf{N}_c^{\mathrm{L}}$, and $J_c^{\mathrm{L}} = \pi^M q^{2h'} \cdot q^{h'} R_c^\Delta \otimes \bar{L}_c \subseteq \pi^M \mathbf{N}_c^{\mathrm{L}}$. Saturation for $j = 0$: if $\ell^b x' \in \mathbf{N}_c^{\mathrm{L}}$, take $x' \rightsquigarrow y'$ ((Fq0)); then $\ell^b x' \rightsquigarrow \ell^b y'$, so $\ell^b y' \in \mathbf{N}_c$ by (Mo1) (applied with $a = 0$), hence $y' \in \mathbf{N}_c$ (Step 0(i)) and $x' \in \mathbf{N}_c^{\mathrm{L}}$. For $j \geq 1$: let $\ell^b x' = \pi^j n$ with $n \in \mathbf{N}_c^{\mathrm{L}}$. In the coordinates of Step 0(iii), $\ell^b F_k = \pi^j G_k$; since ℓ is a non-zero-divisor modulo π^j, $F_k = \pi^j F_k'$ with $F_k' \in q^{-h'} R_c^\Delta$, and cancelling π^j gives $\ell^b F_k' = G_k$. So $x' = \pi^j x''$ with $\ell^b x'' = n \in \mathbf{N}_c^{\mathrm{L}}$, and $x'' \in \mathbf{N}_c^{\mathrm{L}}$ by the case $j = 0$.

(Mo2). Suppose $x = \pi^j x'$ with $x' \in \mathbf{N}_c^{\mathrm{L}}$, and let $x' \rightsquigarrow y'$ with $y' \in \mathbf{N}_c$; then $y' \in \Lambda_c \cap \mathbf{N}_c = \mathbf{N}_c^\Delta \cap \Lambda_c$. By (Fq0), $x \rightsquigarrow \pi^j y'$, so $\ell^a y - \pi^j y' \in J_c \subseteq \pi^M \mathbf{N}_c^\Delta$ and $\ell^a y \in \pi^j \mathbf{N}_c^\Delta$. By Step 0(i) $y \in \mathbf{N}_c$, so $y \in \mathbf{N}_c^\Delta$, and as $\mathbf{N}_c^\Delta / \pi^j \mathbf{N}_c^\Delta$ has no ℓ-torsion, $y \in \pi^j \mathbf{N}_c^\Delta$. Conversely let $y = \pi^j z$ with $z \in \mathbf{N}_c^\Delta$. Then $z \in \Lambda_c$ ($Q^{h'} z \in Q^{h'} \mathbf{N}_c \subseteq U_c$ by (Wm2)). By (Fq1) there is $x'' \rightsquigarrow \ell^{2\sigma} z$, and $x'' \in \mathbf{N}_c^{\mathrm{L}}$ by (Mo1). By (Fq0), $\ell^a \pi^j x'' \rightsquigarrow \ell^{2\sigma+a} y$ and $\ell^{2\sigma} x \rightsquigarrow \ell^{2\sigma+a} y$, so $\ell^a \pi^j x'' - \ell^{2\sigma} x \rightsquigarrow 0$, and by (Fq2) and (Mo4)

$$\ell^{2\sigma+a} \pi^j x'' - \ell^{4\sigma} x \in J_c^{\mathrm{L}} \subseteq \pi^M \mathbf{N}_c^{\mathrm{L}} \subseteq \pi^j \mathbf{N}_c^{\mathrm{L}}.$$

Hence $\ell^{4\sigma} x \in \pi^j \mathbf{N}_c^{\mathrm{L}}$, and $x \in \pi^j \mathbf{N}_c^{\mathrm{L}}$ by (Mo4).

(Mo3). By (Fq3), $(\gamma_u - 1)(y) \in \Lambda_c$ and $\gamma_u(x) \rightsquigarrow \ell^a \gamma_u(y)$, so $(\gamma_u - 1)(x) \rightsquigarrow \ell^a (\gamma_u - 1)(y)$ by (Fq0). Apply (Mo2) with $j = 1$ to this pair.

(Mo5). Take y_0 with $x \rightsquigarrow y_0$. By (Mo1) (with $a = 0$), $x \in \mathbf{N}_c^{\mathrm{L}}$ if and only if $y_0 \in \mathbf{N}_c$, which by (Lz3) means that the t^i-coefficient of $\mathrm{lv}(y_0)$ lies in $\mathcal{F} \otimes \mathrm{Fil}(-i)$ for every i. For $i \geq h'$ this holds automatically: every Hodge jump of V is $\geq -h'$, so by (Lz4) $\epsilon_\tau \mathrm{Fil}(-i) = L^r$ for every τ, i.e. $\mathrm{Fil}(-i) = \mathcal{F}_K^r$. For $i < h' < 3h'$, by (Fq4) the t^i-coefficient of $\mathrm{lv}(y_0)$ is $\ell^\sigma e_i$, and $\ell^\sigma e_i \in \mathcal{F} \otimes \mathrm{Fil}(-i)$ if and only if e_i is.

(Mo6). By (Fq3) and (Fq0), $(\gamma_u - 1)(x) \rightsquigarrow \ell^a (\gamma_u - 1)(y)$, so $(\gamma_u - 1)(y) \in \mathbf{N}_c$ by (Mo1). Write $\mathrm{lv}(y) = \sum_i a_i t^i$. By (Lz2), with $u \in 1 + \ell\mathbb{Z}_\ell$ so that σ_u is the identity, $\mathrm{lv}((\gamma_u - 1)y) = \sum_i (u^i - 1) a_i t^i$; by (Lz3), $(u^i - 1) a_i \in \mathcal{F} \otimes \mathrm{Fil}(-i)$ for every i. For $i \neq 0$, $u^i - 1 \in \mathbb{Q}_\ell^\times$ ($v_\ell(u^{|i|} - 1) = 1 + v_\ell(|i|)$), so $a_i \in \mathcal{F} \otimes \mathrm{Fil}(-i)$. For $i < 3h'$, $\ell^a a_i = \ell^\sigma e_i$ by (Fq4) applied to $x \rightsquigarrow \ell^a y$ (the expansion there is of $\mathrm{lv}(\ell^a y) = \ell^a \mathrm{lv}(y)$); hence $e_i \in \mathcal{F} \otimes \mathrm{Fil}(-i)$ for $i \neq 0$, $i < 3h'$. □

Proposition 27.9. *Let ℓ, F, L, K, (V, T), h', $\mathbf{N} = \mathbf{N}(T)$ and the letter c be as in Definition 27.4, and $e = [K : F]$. Let $S, \eta, m \geq 0$ be integers and, for every embedding τ of K over τ_c, let $\epsilon_\tau \in \{+1, -1\}$ and b_τ be an integer with $\eta < b_\tau$ and $b_\tau + S \leq h'$, such that every Hodge jump a of V at τ satisfies $|a| \leq \eta$ or $\epsilon_\tau a \in [b_\tau, b_\tau + S]$, and $\sum_{|j| \leq \eta} m_V(\tau, j) \leq m$. Call τ* up *if $\epsilon_\tau = 1$ and* down *otherwise. Let Ω_c be the set of integers i such that for some τ over τ_c either τ is up and $-(b_\tau + S) \leq i < -b_\tau$, or τ is down and $b_\tau \leq i < b_\tau + S$, and put*

$$\varpi_c = \prod_{i \in \Omega_c} (1 - u^i) \in \mathbb{Z}_\ell.$$

Then there are an $\mathcal{O}_L$-submodule $L_c^{\mathrm{r}} \subseteq \bar{L}_c$ of rank at most $e\,m$ and elements $\ell_1, \ldots, \ell_k \in \mathbf{N}_{c-1}^\Delta$ with $\mathrm{val}(\ell_j) \in \epsilon_\tau \mathrm{Fil}(b_\tau)$ for every up τ over τ_c and every j (identifying $\epsilon_\tau \mathcal{F}_K^r = L^r$), such that for every $w \in \Lambda_c$ with $(\gamma_u - 1)(w) \in \pi \mathbf{N}_c^\Delta$ there are $f_1, \ldots, f_k \in q^{-h'} R_c^\Delta$, a letter vector x and an element x_{r} of the image of $q^{-h'} R_c^\Delta \otimes_{\mathcal{O}_L} L_c^{\mathrm{r}}$ in Λ_c^{L} with:

(Fp1) *$x \rightsquigarrow \ell^{2\sigma} \varpi_c w$, and $x - \sum_j f_j \otimes \bar{\ell}_j - x_{\mathrm{r}} \in \pi \mathbf{N}_c^{\mathrm{L}}$;*

(Fp2) *$(\gamma_u - 1)(x_{\mathrm{r}}) \in \pi \mathbf{N}_c^{\mathrm{L}}$ and $(\gamma_u - 1)(f_j \otimes \bar{\ell}_j) \in \pi \mathbf{N}_c^{\mathrm{L}}$ for every j;*

(Fp3) Coordinates *for every j: $(\gamma_u - 1)(f_j) \in \pi q^{-h'} R_c^\Delta$; $(\epsilon_\tau \lambda(f_j))_i = 0$ for every up τ over τ_c and $i < -b_\tau$; and $(\epsilon_\tau \lambda(f_j))_i = 0$ for every down τ over τ_c and $0 \neq i < b_\tau + S$.*

Proof. **Step 0: preliminaries.** (a) By (Fq6), $\lambda(\gamma_u f)_i = u^i \lambda(f)_i$ for $f \in q^{-h'} R_c^\Delta$ and all i. (b) By (Fq6), $\lambda(\pi)$ is a unit of $\mathcal{F}[[t]]$. (c) *The value map on* $\bar{L}_c$. By (Dm1) and (Dm2), $\bar{L}_c \to \mathbf{N}_{c-1}/X\mathbf{N}_{c-1}$ is an isomorphism, so by (Lz5) val induces an isomorphism $\bar{L}_c \to \mathcal{O}_L^r$. For τ over τ_c and $j \in \mathbb{Z}$ put $A_\tau(j) = \{\bar{x} \in \bar{L}_c : \ \mathrm{val}(x) \in \epsilon_\tau \mathrm{Fil}(j)\}$, a saturated submodule of rank $\dim_L \mathrm{Fil}^j D_{\mathrm{dR},\tau}(V)$ (Lz4). If τ is up, the jumps below b_τ are those with $|a| \le \eta$, so $\operatorname{rank} A_\tau(b_\tau) \ge r - m$; if τ is down, the jumps $\ge -b_\tau + 1$ are those with $|a| \le \eta$ (as $\eta < b_\tau$), so $\operatorname{rank} A_\tau(1 - b_\tau) \le m$. Moreover $\epsilon_\tau \mathrm{Fil}(j) = 0$ for $j > b_\tau + S$ if τ is up, and $\epsilon_\tau \mathrm{Fil}(j) = L^r$ for $j \le -(b_\tau + S)$ if τ is down (all jumps lie in $[-(b_\tau + S), b_\tau + S]$).

Step 1: the letters and the rest. Let $A = \bigcap_{\tau \text{ up}} A_\tau(b_\tau)$, a saturated submodule with $r - \operatorname{rank} A \le (\#\text{up})\, m$, and let $\tilde{B}$ be the saturation of $\sum_{\tau \text{ down}} A_\tau(1 - b_\tau)$, of rank $\le (\#\text{down})\, m$. Over the discrete valuation ring $\mathcal{O}_L$ choose a complement A^c of A in $\bar{L}_c$ and a basis $\bar{\ell}_1, \dots, \bar{\ell}_k$ of A with lifts $\ell_j \in \mathbf{N}_{c-1}^\Delta$, and put $L_c^{\mathrm{r}} = \tilde{B} + A^c$, of rank $\le e\, m$ ($\#\text{up} + \#\text{down} = e$). Since $\bar{L}_c = A + A^c$, the map $A \to \bar{L}_c / L_c^{\mathrm{r}}$ is onto; $\bar{L}_c / \tilde{B}$ is free, so the projection $\bar{L}_c / \tilde{B} \to \bar{L}_c / L_c^{\mathrm{r}}$ lifts to $\bar{p}\colon \bar{L}_c / \tilde{B} \to A$. Let $p = (p_1, \dots, p_k)\colon \bar{L}_c \to \mathcal{O}_L^k$ be the coordinates of $\bar{p}$ composed with the projection, in the basis $(\bar{\ell}_j)$. Then $p(\tilde{B}) = 0$ and $\bar{y} - \sum_j p_j(\bar{y}) \bar{\ell}_j \in L_c^{\mathrm{r}}$ for every $\bar{y} \in \bar{L}_c$. By construction $\mathrm{val}(\ell_j) \in \epsilon_\tau \mathrm{Fil}(b_\tau)$ for every up τ.

Step 2: the Herr vector. Let w be as in the statement. By (Fq1) there is b_0 with $b_0 \rightsquigarrow \ell^{2\sigma} w$, and by (Mo3) (with $a = 2\sigma$) $(\gamma_u - 1)(b_0) \in \pi \mathbf{N}_c^{\mathrm{L}}$. Choose a Δ-fixed basis (n_i) of $\mathbf{N}_{c-1}$ ((Dm1)); $(\bar{n}_i)$ is a basis of $\bar{L}_c$. Write $b_0 = \sum_i X_i \otimes \bar{n}_i$ and $(\gamma_u - 1)(b_0) = \pi \sum_i Y_i \otimes \bar{n}_i$ with $X_i, Y_i \in q^{-h'} R_c^\Delta$, so $(\gamma_u - 1)(X_i) = \pi Y_i$. By (Mo6), for every integer i with $0 \ne i < 3h'$,

$$\sum_{i'} \lambda(X_{i'})_i \, \epsilon \, \mathrm{val}(n_{i'}) \in \mathcal{F} \otimes_{\mathcal{F}_K} \mathrm{Fil}(-i). \qquad (1)$$

For τ over τ_c and an integer i let $d_\tau(i) = \sum_{i'} (\epsilon_\tau \lambda(X_{i'})_i) \otimes \bar{n}_{i'} \in \epsilon_\tau \mathcal{F} \otimes_{\mathcal{O}_L} \bar{L}_c$. Since $\epsilon_\tau \mathcal{F}$ is flat over L and val is an isomorphism, (1) says: $d_\tau(i) \in \epsilon_\tau \mathcal{F} \otimes A_\tau(-i)$ for $0 \ne i < 3h'$.

Step 3: projecting. Put $F_j = \sum_i p_j(\bar{n}_i) X_i \in q^{-h'} R_c^\Delta$ and $\rho_i = \bar{n}_i - \sum_j p_j(\bar{n}_i) \bar{\ell}_j \in L_c^{\mathrm{r}}$. Then

$$b_0 = \sum_j F_j \otimes \bar{\ell}_j + \sum_i X_i \otimes \rho_i,$$

and $\epsilon_\tau \lambda(F_j)_i = (\mathrm{id} \otimes p_j)(d_\tau(i))$. If τ is up and $i < -(b_\tau + S)$, then $-i > b_\tau + S$, $A_\tau(-i) = 0$ (Step 0(c)), so $d_\tau(i) = 0$ and $\epsilon_\tau \lambda(F_j)_i = 0$. If τ is down and $0 \ne i < b_\tau$ ($\le h' < 3h'$), then $-i \ge 1 - b_\tau$, so $A_\tau(-i) \subseteq A_\tau(1 - b_\tau) \subseteq \tilde{B}$ and p_j kills $d_\tau(i)$: $\epsilon_\tau \lambda(F_j)_i = 0$.

Step 4: the killing polynomial. Let $P(Z) = \prod_{i \in \Omega_c} (Z - u^i) \in \mathbb{Z}_\ell[Z]$, so $P(1) = \varpi_c$, and put

$$f_j = P(\gamma_u)(F_j), \qquad x_{\mathrm{r}} = \sum_i P(\gamma_u)(X_i) \otimes \rho_i, \qquad x = \varpi_c b_0.$$

As γ_u preserves $q^{-h'} R_c^\Delta$ (Fq3), $f_j \in q^{-h'} R_c^\Delta$ and x_{r} lies in the image of $q^{-h'} R_c^\Delta \otimes L_c^{\mathrm{r}}$. By Step 0(a), $\lambda(f_j)_i = P(u^i) \lambda(F_j)_i$. If τ is up and $i < -b_\tau$: either $i < -(b_\tau + S)$ and $\epsilon_\tau \lambda(F_j)_i = 0$, or $i \in \Omega_c$ and $P(u^i) = 0$. If τ is down and $0 \ne i < b_\tau + S$: either $i < b_\tau$ and $\epsilon_\tau \lambda(F_j)_i = 0$, or $i \in \Omega_c$. This proves the last two clauses of (Fp3). As $(\gamma_u - 1)(F_j) = \pi \sum_i p_j(\bar{n}_i) Y_i$, $\gamma_u(\pi) \in \pi R_c^\Delta$ ((Rr6), $u \in \Gamma_\Delta$ by (Rr7)) and γ_u preserves $q^{-h'} R_c^\Delta$, also $(\gamma_u - 1)(f_j) = P(\gamma_u)((\gamma_u - 1) F_j) \in \pi q^{-h'} R_c^\Delta$.

Step 5: (Fp1). We have $x = \varpi_c b_0 \rightsquigarrow \ell^{2\sigma} \varpi_c w$ by (Fq0) ($\varpi_c \in \mathbb{Z}_\ell \subseteq R_c^\Delta$). Since γ_u acts on letter vectors through the first factor, $P(\gamma_u)(b_0) = \sum_j f_j \otimes \bar{\ell}_j + x_{\mathrm{r}}$ by Step 3. Write $P(Z) - P(1) = R(Z)(Z - 1)$ with $R \in \mathbb{Z}_\ell[Z]$. The module $\pi \mathbf{N}_c^{\mathrm{L}}$ is stable under γ_u and $\mathbb{Z}_\ell$ ((Mo4), $\gamma_u(\pi) \in \pi R_c^\Delta$), so

$$x - \sum_j f_j \otimes \bar{\ell}_j - x_{\mathrm{r}} = (P(1) - P(\gamma_u))(b_0) = -R(\gamma_u)((\gamma_u - 1)(b_0)) \in \pi \mathbf{N}_c^{\mathrm{L}}.$$

Step 6: (Fp2). By Step 4, $(\gamma_u-1)(f_j)=\pi z_j$ with $z_j\in q^{-h'}R_c^\Delta$, so $(\gamma_u-1)(f_j\otimes\bar\ell_j)=\pi(z_j\otimes\bar\ell_j)$; we show $z_j\otimes\bar\ell_j\in\mathbf{N}_c^{\mathrm{L}}$ by (Mo5). By Step 0(a), $\lambda(\pi)\lambda(z_j)=\sum_i(u^i-1)\lambda(f_j)_i t^i$, and $\lambda(\pi)$ is a unit (Step 0(b)). If τ is up, $\epsilon_\tau\lambda(f_j)\in t^{-b_\tau}\epsilon_\tau\mathcal{F}[[t]]$, hence so is $\epsilon_\tau\lambda(z_j)$; its coefficients with $i\ge -b_\tau$ multiply $\mathrm{val}(\ell_j)\in\epsilon_\tau\mathrm{Fil}(b_\tau)\subseteq\epsilon_\tau\mathrm{Fil}(-i)$. If τ is down, $(u^i-1)\epsilon_\tau\lambda(f_j)_i=0$ for $i<b_\tau+S$ (for $i=0$ because $u^0=1$), so $\epsilon_\tau\lambda(z_j)\in t^{b_\tau+S}\epsilon_\tau\mathcal{F}[[t]]$, and for $i\ge b_\tau+S$ we have $\epsilon_\tau\mathrm{Fil}(-i)=L^r$ (Step 0(c)). Since $\epsilon=\sum_\tau\epsilon_\tau$ over the τ over τ_c, every coefficient of $\lambda(z_j)\epsilon\,\mathrm{val}(\ell_j)$ at t^i lies in $\mathcal{F}\otimes\mathrm{Fil}(-i)$, and (Mo5) gives $z_j\otimes\bar\ell_j\in\mathbf{N}_c^{\mathrm{L}}$. Finally $(\gamma_u-1)(x_{\mathrm{r}})=P(\gamma_u)((\gamma_u-1)b_0)-\sum_j(\gamma_u-1)(f_j\otimes\bar\ell_j)\in\pi\mathbf{N}_c^{\mathrm{L}}$, using the stability of $\pi\mathbf{N}_c^{\mathrm{L}}$ under γ_u once more. □

Proposition 27.10. *Let ℓ, F, L, K, (V,T), h', $\mathbf{N}=\mathbf{N}(T)$ and the letter c be as in Definition 27.4, and $e=[K:F]$. Let $S\subseteq\bar L_c$ be an $\mathcal{O}_L$-submodule, $\Lambda_c^{\mathrm{L}}(S)$ the image of $q^{-h'}R_c^\Delta\otimes_{\mathcal{O}_L}S$ in Λ_c^{L}, and*

$$\mathrm{Inv}_c(S)=\{x\in\Lambda_c^{\mathrm{L}}(S):\ (\gamma_u-1)(x)\in\pi\mathbf{N}_c^{\mathrm{L}}\}.$$

Then $\mathrm{Inv}_c(S)/(\mathrm{Inv}_c(S)\cap\pi\mathbf{N}_c^{\mathrm{L}})$ is a finitely generated free $\mathcal{O}_L$-module of rank at most $(e+1)\,\mathrm{rank}_{\mathcal{O}_L}S$.

Proof. Write $R=e_c\mathbf{A}_{F,L}\cong\mathcal{O}_L[[X]]$, $R^\Delta=R_c^\Delta=\mathcal{O}_L[[\pi]]$ (Rr2), $K'=\pi\mathbf{N}_c^{\mathrm{L}}$ and $\gamma=\gamma_u$. The module S is free ($\bar L_c$ is a finitely generated free $\mathcal{O}_L$-module) and $q^{-h'}R^\Delta\cong R^\Delta$ is flat over $\mathcal{O}_L$, so $q^{-h'}R^\Delta\otimes S\to\Lambda_c^{\mathrm{L}}$ is injective. By (Fq3) and (Mo4), K' is a γ-stable $\mathcal{O}_L$-module ($\gamma(\pi)\in\pi R^\Delta$ by (Rr6), $u\in\Gamma_\Delta$ by (Rr7)).

Step 0: the roots of q and evaluations. By (Rr7), $E_\Delta\in\mathbb{Z}_\ell[Y]$ is the minimal polynomial over F of a uniformiser $y_0\in S_\Delta\subseteq K$, $y_0=\pi_\Delta(\zeta-1)$; for each of the e embeddings τ of K over τ_c choose an extension $\tilde\tau$ to $F(\mu_\ell)$ and put $\zeta_\tau=\tilde\tau(\zeta)\in L$, $x_\tau=\zeta_\tau-1$, $y_\tau=\tau(y_0)=\pi_\Delta(x_\tau)$. The y_τ are the e distinct roots of E_Δ, of positive valuation, so $q=\prod_\tau(\pi-y_\tau)$ in R^Δ. Let $\mathrm{ev}_\tau\colon R\to\mathcal{O}_L$, $g\mapsto g(x_\tau)$, a continuous ring homomorphism; on R^Δ it is $g\mapsto g(y_\tau)$ (as a series in π), with kernel $(\pi-y_\tau)R^\Delta$. Let $\mathrm{ev}_0\colon R^\Delta\to\mathcal{O}_L$ be the constant term in π, with kernel πR^Δ. Since $\gamma(X)=(1+X)^u-1$ and $\zeta_\tau^u=\zeta_\tau$ ($u\equiv1$ modulo ℓ), $\mathrm{ev}_\tau\circ\gamma=\mathrm{ev}_\tau$; and $\mathrm{ev}_0\circ\gamma=\mathrm{ev}_0$ on R^Δ as $\gamma(\pi)\in\pi R^\Delta$. Also $\mathrm{ev}_\tau(q)=0$, since $q=Qw_\Delta$ (Rr4) and $Q(x_\tau)=\Phi_\ell(\zeta_\tau)=0$.

Step 1: the multipliers. By (Rr6), $\gamma(\pi)=\pi+t_u\pi q$ with $t_u\in\mathbb{Z}_\ell[[\pi]]$ and $\gamma(q)=h_uq$ with $h_u\in1+X\mathbb{Z}_\ell[[X]]$ Δ-invariant. Put $m=\gamma(\pi)/\pi=1+t_uq$ and, for each τ, $D_\tau=(\gamma(\pi)-y_\tau)/(\pi-y_\tau)=1+t_u\pi\,q/(\pi-y_\tau)$; these lie in R^Δ, and $h_u=\gamma(q)/q=\prod_\tau D_\tau$. We claim: $\mathrm{ev}_0(h_u)=1$; $\mathrm{ev}_\tau(m)=1$; $\mathrm{ev}_\tau(D_{\tau'})=1$ for $\tau'\ne\tau$; $\mathrm{ev}_\tau(D_\tau)=u$; hence $\mathrm{ev}_\tau(h_u)=u$. The first holds as $h_u\equiv1$ modulo X; the second and third as $\mathrm{ev}_\tau(q)=0$ and $\mathrm{ev}_\tau(q/(\pi-y_{\tau'}))=\prod_{\tau''\ne\tau'}(y_\tau-y_{\tau''})=0$. For the fourth: Q has the simple root x_τ and $q=Qw_\Delta=\prod_{\tau'}(\pi-y_{\tau'})$ with $\pi-y_{\tau'}$ not vanishing at x_τ for $\tau'\ne\tau$; so $\pi-y_\tau=(X-x_\tau)A$ with $A\in R$, $A(x_\tau)\ne0$ (division by $X-x_\tau$ in $\mathcal{O}_L[[X]]$). Further $\gamma(X)-x_\tau=(1+X)^u-(1+x_\tau)^u=(X-x_\tau)B$ with $B(x_\tau)=u(1+x_\tau)^{u-1}=u$. Hence $D_\tau A=B\,\gamma(A)$, and evaluating at x_τ (where $\gamma(A)(x_\tau)=A(x_\tau)$ by Step 0) gives $\mathrm{ev}_\tau(D_\tau)A(x_\tau)=uA(x_\tau)$, so $\mathrm{ev}_\tau(D_\tau)=u$.

Step 2: a filtration. Number the embeddings over τ_c as $\tau_1,\dots,\tau_e$ and let $(f_k)_{0\le k<N}$, $N=1+2h'e$, be the sequence π, then $2h'$ copies of $\pi-y_{\tau_1}$, …, then $2h'$ copies of $\pi-y_{\tau_e}$; so $\prod_{k<N}f_k=\pi q^{2h'}$. Put $G_k=q^{-h'}\prod_{j<k}f_j$ and $F_k=G_kR^\Delta\otimes S\subseteq\Lambda_c^{\mathrm{L}}(S)$, so $F_0=\Lambda_c^{\mathrm{L}}(S)$, $F_{k+1}\subseteq F_k$, and $F_N=\pi q^{h'}R^\Delta\otimes S\subseteq K'$ by (Mo4). Multiplication by G_k gives $F_k\cong R^\Delta\otimes S$ and $F_k/F_{k+1}\cong(R^\Delta/f_kR^\Delta)\otimes S\cong S$, of rank $\mathrm{rank}\,S$. Now $\gamma(G_k)=\rho_kG_k$ with $\rho_k=h_u^{-h'}\prod_{j<k}\gamma(f_j)/f_j\in R^\Delta$, a product of $h_u^{-h'}$, m and the D_τ. Let $c_k=\mathrm{ev}(\rho_k)$, where ev is ev_0 if $k=0$ and ev_{τ_i} if $f_k=\pi-y_{\tau_i}$. For $x=G_kg\otimes s\in F_k$, $\gamma(x)-c_kx=G_k(\rho_k\gamma(g)-c_kg)\otimes s$ and $\mathrm{ev}(\rho_k\gamma(g)-c_kg)=c_k\mathrm{ev}(g)-c_k\mathrm{ev}(g)=0$, so

$$(\gamma-c_k)(F_k)\subseteq F_{k+1}.$$

By Step 1, $c_0 = \mathrm{ev}_0(h_u)^{-h'} = 1$, and if f_k is the $(a+1)$-st copy of $\pi - y_{\tau_i}$ $(0 \le a < 2h')$, then $c_k = u^{-h'} \cdot 1 \cdot u^a = u^{a-h'}$. As $u \in 1 + \ell\mathbb{Z}_\ell$, $u \neq 1$, is not a root of unity, $c_k = 1$ for exactly $1 + e$ indices k ($k = 0$ and $a = h'$ in each block).

Step 3: the rank count. For a submodule $P \subseteq F_0$ put $\mathrm{Inv}(P) = \{x \in P : (\gamma - 1)(x) \in K'\}$ and $\mathcal{Q}(P) = \mathrm{Inv}(P)/(\mathrm{Inv}(P) \cap K')$; then $\mathcal{Q}(F_0)$ is the module of the statement and $\mathcal{Q}(F_N) = 0$. Fix $k < N$, let $\iota\colon \mathcal{Q}(F_{k+1}) \to \mathcal{Q}(F_k)$ be induced by inclusion and C its cokernel, so $\operatorname{rank} \mathcal{Q}(F_k) \le \operatorname{rank} \mathcal{Q}(F_{k+1}) + \operatorname{rank} C$. If $c_k \neq 1$: for $x \in \mathrm{Inv}(F_k)$, $y = (\gamma - c_k)(x) \in F_{k+1}$ and $(\gamma - 1)(y) = (\gamma - c_k)((\gamma - 1)x) \in K'$, so $y \in \mathrm{Inv}(F_{k+1})$, and $(1 - c_k)x = y - (\gamma - 1)(x) \in \mathrm{Inv}(F_{k+1}) + K'$; thus C is killed by $1 - c_k \neq 0$ and has rank 0. If $c_k = 1$: C is a quotient of $\mathrm{Inv}(F_k)/\mathrm{Inv}(F_{k+1})$, which embeds in F_k/F_{k+1} (as $\mathrm{Inv}(F_k) \cap F_{k+1} = \mathrm{Inv}(F_{k+1})$), so $\operatorname{rank} C \le \operatorname{rank} S$. Summing over k with Step 2, $\operatorname{rank} \mathcal{Q}(F_0) \le (e+1) \operatorname{rank} S$.

Step 4: finite and free. The element $\pi q^{2h'}$ is a monic polynomial in π of degree N whose other coefficients lie in $\ell\mathcal{O}_L$ (E_Δ is Eisenstein, (Rr3)), so by Weierstrass division $R^\Delta/\pi q^{2h'} R^\Delta$ is free of rank N over $\mathcal{O}_L$, and F_0/F_N is finitely generated. The module $\mathcal{Q}(F_0)$ is a quotient of $\mathrm{Inv}(F_0)/(\mathrm{Inv}(F_0) \cap F_N)$ ($F_N \subseteq K'$), a submodule of F_0/F_N; so it is finitely generated. It is torsion free: if $x \in \mathrm{Inv}(F_0)$ and $ax \in K'$ with $0 \neq a \in \mathcal{O}_L$, since $\mathcal{O}_L$ is a discrete valuation ring whose maximal ideal contains ℓ, some power ℓ^b lies in $a\mathcal{O}_L$, say $\ell^b = ac$ with $c \in \mathcal{O}_L$; then $\ell^b x = c\,(ax) \in K' = \pi \mathbf{N}_c^{\mathrm{L}}$ and $x \in K'$ by (Mo4). Hence $\mathcal{Q}(F_0)$ is free. □

Proposition 27.11. *Let ℓ, F, L, K, h', u and the letter c be as in Definition 27.4, and $\Phi\colon T_1 \to T_2$ a G_K-equivariant $\mathcal{O}_L$-linear surjection of the lattices of pairs (V_1, T_1), (V_2, T_2) over K, such that every Hodge jump of V_1 at every $\mathbb{Q}_\ell$-embedding $K \to L$ lies in $[-h', h']$. Let $S, \eta \ge 0$ be integers and, for every embedding τ of K over τ_c, $\epsilon_\tau \in \{+1, -1\}$ and an integer b_τ with $\eta < b_\tau$ and $b_\tau + S \le h'$, such that for $\nu = 1, 2$ every Hodge jump a of V_ν at τ satisfies $|a| \le \eta$ or $\epsilon_\tau a \in [b_\tau, b_\tau + S]$. Let $f \in q^{-h'} R_c^\Delta$ satisfy the three conditions of (Fp3), $x \in (\mathbf{N}_{c-1}^{(2)})^\Delta$ with $\mathrm{val}_2(x) \in \epsilon_\tau \mathrm{Fil}_2(b_\tau)$ for every up τ over τ_c, and $y \in \Lambda_c^{(2)}$ with $f \otimes \bar{x} \rightsquigarrow_2 y$. If all embeddings of K over τ_c are down, or all are up, then there is $w_1 \in \Lambda_c^{(1)}$ with*

$$(\gamma_u - 1)(w_1) \in \pi(\mathbf{N}_c^{(1)})^\Delta \qquad \text{and} \qquad \ell^{r_0} y - \mathbf{N}(\Phi)(w_1) \in \pi(\mathbf{N}_c^{(2)})^\Delta.$$

Proof. By Lemma 23.25 every Hodge jump of V_2 is one of V_1, so Definition 27.4 applies to both pairs with the same h', M, σ. Write $g = \mathbf{N}(\Phi)$. Every Hodge jump of V_ν at τ over τ_c lies in $[-(b_\tau + S), b_\tau + S]$, so by (Lz4): $\epsilon_\tau \mathrm{Fil}_\nu(j) = L^{r_\nu}$ for $j \le -(b_\tau + S)$, and $\epsilon_\tau \mathrm{Fil}_\nu(b_\tau) \subseteq \epsilon_\tau \mathrm{Fil}_\nu(j)$ for $j \le b_\tau$.

Down letters. Suppose every τ over τ_c is down. By Lemma 24.11 (for Φ, site $c-1$) there is $x_1 \in (\mathbf{N}_{c-1}^{(1)})^\Delta$ with $g(x_1) = \ell^{r_0} x$. By (Fq0) choose $w_1 \in \Lambda_c^{(1)}$ with $f \otimes \bar{x}_1 \rightsquigarrow_1 w_1$. By (Fq5), $f \otimes \overline{g(x_1)} = \ell^{r_0}(f \otimes \bar{x}) \rightsquigarrow_2 g(w_1)$, and by (Fq0) also $\ell^{r_0}(f \otimes \bar{x}) \rightsquigarrow_2 \ell^{r_0} y$; so $\ell^{r_0} y - g(w_1) \in J_c^{(2)} \subseteq \pi^M (\mathbf{N}_c^{(2)})^\Delta$ by (Fq0) and (Mo1). For the precision of w_1: by (Fp3), $(\gamma_u - 1)(f) = \pi z$ with $z \in q^{-h'} R_c^\Delta$, so $(\gamma_u - 1)(f \otimes \bar{x}_1) = \pi(z \otimes \bar{x}_1)$. By (Fq6), $\lambda(\pi)\lambda(z) = \sum_i (u^i - 1)\lambda(f)_i t^i$ with $\lambda(\pi)$ a unit; for down τ and $i < b_\tau + S$ the coefficient $(u^i - 1)\epsilon_\tau \lambda(f)_i$ vanishes (for $i = 0$ as $u^0 = 1$, otherwise by (Fp3)), so $\epsilon_\tau \lambda(z) \in t^{b_\tau + S} \epsilon_\tau \mathcal{F}[[t]]$. Hence every t^i-coefficient of $\lambda(z) \epsilon \operatorname{val}_1(x_1)$ with nonzero ϵ_τ-component has $i \ge b_\tau + S$, where $\epsilon_\tau \mathrm{Fil}_1(-i) = L^{r_1}$; by (Mo5) for (V_1, T_1), $z \otimes \bar{x}_1 \in \mathbf{N}_c^{\mathrm{L},(1)}$. So $(\gamma_u - 1)(f \otimes \bar{x}_1) \in \pi \mathbf{N}_c^{\mathrm{L},(1)}$, and by (Mo3) (with $a = 0$) $(\gamma_u - 1)(w_1) \in \pi(\mathbf{N}_c^{(1)})^\Delta$.

Up letters. Suppose every τ over τ_c is up. By (Fp3), $\epsilon_\tau \lambda(f) \in t^{-b_\tau} \epsilon_\tau \mathcal{F}[[t]]$ for every τ over τ_c, so the t^i-coefficient of $\epsilon_\tau \lambda(f) \operatorname{val}_2(x)$ vanishes for $i < -b_\tau$ and lies in $\epsilon_\tau \mathcal{F} \otimes \epsilon_\tau \mathrm{Fil}_2(b_\tau) \subseteq$

$\epsilon_\tau\mathcal{F}\otimes\epsilon_\tau\mathrm{Fil}_2(-i)$ for $i\geq -b_\tau$. By (Mo5), $f\otimes\bar{x}\in\mathbf{N}_c^{\mathrm{L},(2)}$, so $y\in\mathbf{N}_c^{(2)}$ by (Mo1); y is Δ-fixed, being in $\Lambda_c^{(2)}$. By Lemma 24.11 (site c) there is $w_1\in(\mathbf{N}_c^{(1)})^\Delta$ with $g(w_1)=\ell^{r_0}y$. Then $Q^{h'}w_1\in Q^{h'}\mathbf{N}_c^{(1)}\subseteq U_c^{(1)}$ by (Wm2) (window (h',h')), so $w_1\in\Lambda_c^{(1)}$; $(\gamma_u-1)(w_1)$ is Δ-fixed and lies in $X\mathbf{N}_c^{(1)}$ by (Wm3), hence in $\pi(\mathbf{N}_c^{(1)})^\Delta$ by (Dm2); and $\ell^{r_0}y-g(w_1)=0$. □

Proposition 27.12. *Let ℓ, F, L, K, h', u, the letter c, $\Phi\colon T_1\to T_2$, S, η, $(\epsilon_\tau,b_\tau)_\tau$, f, x and y be as in Proposition 27.11, with $[K:F]=2$, and suppose that one of the two embeddings of K over τ_c is up and the other is down. Put*

$$y_c=\left\lceil\eta+\tfrac{1}{2}+\log_\ell h'\right\rceil.$$

Then there is $w_1\in\Lambda_c^{(1)}$ with

$$(\gamma_u-1)(w_1)\in\pi(\mathbf{N}_c^{(1)})^\Delta\qquad\text{and}\qquad\ell^{\,r_0+\max(2\sigma,\,y_c)}\,y-\mathbf{N}(\Phi)(w_1)\in\pi(\mathbf{N}_c^{(2)})^\Delta.$$

Proof. Write $g=\mathbf{N}(\Phi)$, τ_U and τ_D for the up and the down embedding over τ_c, $b=b_{\tau_U}$, $B=b_{\tau_D}+S\geq 1$, $r_U=r_{\tau_U}$, $r_D=r_{\tau_D}$ (Lemma 27.13), $\delta=r_U-r_D\neq 0$, $Z=\pi-r_D$, $y_1=r_0+\max(2\sigma,y_c)$. By (Tf1), $E_\Delta(Y)=(Y-r_U)(Y-r_D)$ is Eisenstein (Rr3), so $v_\ell(r_U+r_D)\geq 1$, $v_\ell(r_Ur_D)=1$ and, ℓ being odd, $v_\ell(\delta^2)=v_\ell((r_U+r_D)^2-4r_Ur_D)=1$: $v_\ell(\delta)=\frac{1}{2}$. As in Proposition 27.11, both pairs satisfy Definition 27.4 with the same h', and by (Lz4): $\epsilon_{\tau_D}\mathrm{Fil}_\nu(j)=L^{r_\nu}$ for $j\leq -B$, $\epsilon_{\tau_D}\mathrm{Fil}_\nu(j)=0$ for $j>\eta$, $\epsilon_{\tau_U}\mathrm{Fil}_\nu(j)=\epsilon_{\tau_U}\mathrm{Fil}_\nu(b)$ for $\eta<j\leq b$ (no jumps there), and $\epsilon_{\tau_U}\mathrm{Fil}_\nu(b)\subseteq\epsilon_{\tau_U}\mathrm{Fil}_\nu(j)$ for $j\leq b$. Since a letter vector $x'\rightsquigarrow y'$ with $y'\in\Lambda_c$ and $\ell^a y'-g(w')\in\pi\mathbf{N}^\Delta$ for precision-one w' may be multiplied by powers of ℓ, it suffices to reach any exponent $\leq y_1$. By (Fq6), for $f'\in q^{-h'}R_c^\Delta$ with $(\gamma_u-1)(f')=\pi z'$: $\lambda(\pi)\lambda(z')=\sum_i(u^i-1)\lambda(f')_it^i$, $\lambda(\pi)$ a unit.

Step 0: a down-type lift. Claim: if $x_1\in(\mathbf{N}_{c-1}^{(1)})^\Delta$ has $\mathrm{val}_1(x_1)\in\epsilon_{\tau_U}\mathrm{Fil}_1(b)$ and $g(x_1)-\ell^a x\in\pi(\mathbf{N}_{c-1}^{(2)})^\Delta$, then there is a precision-one $w_1\in\Lambda_c^{(1)}$ with $\ell^a y-g(w_1)\in\pi(\mathbf{N}_c^{(2)})^\Delta$. Take $f\otimes\bar{x}_1\rightsquigarrow_1 w_1$; by (Fq5) and (Fq0), $g(w_1)$ and $\ell^a y$ are both Frobenius partners of $f\otimes\overline{g(x_1)}=\ell^a f\otimes\bar{x}$, so $\ell^a y-g(w_1)\in J_c^{(2)}\subseteq\pi^M(\mathbf{N}_c^{(2)})^\Delta$ ((Mo1)). Write $(\gamma_u-1)(f)=\pi z$ ((Fp3)). At τ_D, $(u^i-1)\epsilon_{\tau_D}\lambda(f)_i=0$ for $i<B$, so $\epsilon_{\tau_D}\lambda(z)\in t^B\epsilon_{\tau_D}\mathcal{F}[[t]]$; at τ_U, $\epsilon_{\tau_U}\lambda(f)\in t^{-b}\epsilon_{\tau_U}\mathcal{F}[[t]]$, hence so is $\epsilon_{\tau_U}\lambda(z)$, and $\mathrm{val}_1(x_1)\in\epsilon_{\tau_U}\mathrm{Fil}_1(-i)$ for $i\geq -b$. By (Mo5), $z\otimes\bar{x}_1\in\mathbf{N}_c^{\mathrm{L},(1)}$, so $(\gamma_u-1)(f\otimes\bar{x}_1)\in\pi\mathbf{N}_c^{\mathrm{L},(1)}$ and w_1 has precision one by (Mo3).

Step 1: the constant term at τ_D. Let $a=q^{h'}f\in R_c^\Delta$. By (Fp3), $\epsilon_{\tau_D}\lambda(f)$ has no poles, so $\epsilon_{\tau_D}\lambda(a)\in t^{h'}\epsilon_{\tau_D}\mathcal{F}[[t]]$, and (Tf3) (one vector $v=1\in L$, $G(j)=0$ for $j\geq 1$, $G(j)=L$ otherwise, $A=N=h'$) shows that the Taylor coefficients of a at r_D vanish below h'; likewise at r_U below $h'-b$. By (Tf2), $a=Z^{h'}a_D=(\pi-r_U)^{h'-b}a_U$ with $a_D,a_U\in R_c^\Delta$. In $L[[Z]]$ (Taylor expansions at r_D, mapped injectively and order-preservingly to $\epsilon_{\tau_D}\mathcal{F}[[t]]$ by $Z\mapsto tw_{\tau_D}$), $f=a_D(Z-\delta)^{-h'}$ and (Fp3) says $f\equiv x_0$ modulo Z^B with $x_0\in L$ (the constant term). Hence $a_U=Z^{h'}a_D(Z-\delta)^{-(h'-b)}\equiv x_0Z^{h'}(Z-\delta)^b$ modulo $Z^{h'+B}$, and the Taylor coefficients of a_U at r_D are integral ((Tf2)):

$$x_0\binom{b}{m}(-\delta)^{b-m}\in\mathcal{O}_L\qquad(0\leq m<B).\qquad(1)$$

Case (i): $b\leq B+2\eta$. If $b<B$, (1) with $m=b$ gives $x_0\in\mathcal{O}_L$. If $b\geq B$, (1) with $m=B-1$ gives $v_\ell(x_0)\geq-\frac{b-B+1}{2}-v_\ell\binom{b}{B-1}\geq-(\eta+\frac{1}{2}+\log_\ell h')$, by $b-B\leq 2\eta$ and Kummer's bound $v_\ell\binom{b}{k}\leq$

$\log_\ell b \le \log_\ell h'$. In both cases $o := \ell^{y_c} x_0 \in \mathcal{O}_L$. Put $f_A = \ell^{y_c} f - o$. Then $\epsilon_{\tau_U}\lambda(f_A) \in t^{-b}\epsilon_{\tau_U}\mathcal{F}[[t]]$ and $\epsilon_{\tau_D}\lambda(f_A) \in t^B\epsilon_{\tau_D}\mathcal{F}[[t]]$, so by (Mo5) $f_A \otimes \bar{x} \in \mathbf{N}_c^{\mathrm{L},(2)}$ ($\mathrm{val}_2(x) \in \epsilon_{\tau_U}\mathrm{Fil}_2(b)$); its partner y_A lies in $(\mathbf{N}_c^{(2)})^\Delta$ ((Mo1)), and Lemma 24.11 gives $w_A \in (\mathbf{N}_c^{(1)})^\Delta$ with $g(w_A) = \ell^{r_0} y_A$; $w_A \in \Lambda_c^{(1)}$ ((Wm2)) has precision one ((Wm3), (Dm2)). Next take $x_1 \in (\mathbf{N}_{c-1}^{(1)})^\Delta$ with $g(x_1) = \ell^{r_0} x$ (Lemma 24.11) and $1 \otimes \bar{x}_1 \rightsquigarrow_1 w_B$; $(\gamma_u - 1)(1 \otimes \bar{x}_1) = 0$, so w_B has precision one (Mo3), and $g(w_B)$ is a partner of $\ell^{r_0}(1 \otimes \bar{x})$ (Fq5). Then $\ell^{r_0+y_c} f \otimes \bar{x} = \ell^{r_0} f_A \otimes \bar{x} + o\,\ell^{r_0}(1 \otimes \bar{x}) \rightsquigarrow_2 g(w_A + o w_B)$, while also $\rightsquigarrow_2 \ell^{r_0+y_c} y$; so $w_1 = w_A + o w_B$ satisfies $\ell^{r_0+y_c} y - g(w_1) \in J_c^{(2)} \subseteq \pi(\mathbf{N}_c^{(2)})^\Delta$.

Case (ii): $b > B + 2\eta$**.** Let $f_0 = q^{-h'}(\pi - r_U)^{h'-b} Z^{h'+B} \in q^{-h'} R_c^\Delta$. By (Tf1), $\epsilon_{\tau_U}\lambda(f_0) \in t^{-b}\epsilon_{\tau_U}\mathcal{F}[[t]]$ and $\epsilon_{\tau_D}\lambda(f_0) \in t^B\epsilon_{\tau_D}\mathcal{F}[[t]]$, so $f_0 \otimes \bar{x} \in \mathbf{N}_c^{\mathrm{L},(2)}$ (Mo5) and its partner Y_0 lies in $(\mathbf{N}_c^{(2)})^\Delta$. By Lemma 24.11 take $W_1 \in (\mathbf{N}_c^{(1)})^\Delta \subseteq \Lambda_c^{(1)}$ with $g(W_1) = \ell^{r_0} Y_0$, and by (Fq1) a letter vector $X_1 \rightsquigarrow_1 \ell^{2\sigma} W_1$; $X_1 \in \mathbf{N}_c^{\mathrm{L},(1)}$ (Mo1). By (Fq5), (Fq0), $\Theta := (\mathrm{id} \otimes \bar{g})(X_1) - \ell^{2\sigma+r_0} f_0 \otimes \bar{x} \rightsquigarrow_2 0$, so $\ell^{2\sigma}\Theta \in J_c^{\mathrm{L},(2)}$ (Fq2); as $\pi^M q^{4h'}$ is a monic polynomial in π whose other coefficients lie in the maximal ideal, $R_c^\Delta/\pi^M q^{4h'} R_c^\Delta$ has no ℓ-torsion (Weierstrass division), hence $\Theta \in J_c^{\mathrm{L},(2)}$. Choose a Δ-fixed basis (y_k) of $\mathbf{N}_{c-1}^{(1)}$ with $\mathrm{val}_1(y_k)$ the standard basis ((Dm1), (Lz5)), write $X_1 = \sum_k A_k \otimes \bar{y}_k$, $a_k = q^{h'} A_k$, and $\bar{g}(\bar{y}_k) = \sum_m \phi_{km} \bar{z}_m$, $\bar{x} = \sum_m \xi_m \bar{z}_m$ in a basis $(\bar{z}_m)$ of $\bar{L}_c^{(2)}$. Then, coordinatewise,

$$\sum_k a_k \phi_{km} - \ell^{2\sigma+r_0}\xi_m (\pi - r_U)^{h'-b} Z^{h'+B} \in \pi^M q^{4h'} R_c^\Delta. \qquad (2)$$

By (Mo5) for X_1 (all t^i-coefficients, those with $i \ge h'$ trivially) and multiplication by $\lambda(q)^{h'} \in t^{h'}\mathcal{F}[[t]]$, the t^i-coefficient of $\sum_k \lambda(a_k)\epsilon\,\mathrm{val}_1(y_k)$ lies in $\mathcal{F} \otimes \mathrm{Fil}_1(h' - i)$ for every i. At τ_D, (Tf3) ($A = h'$, $N = h' - \eta$) puts the Taylor vectors $\sum_k p_{kj}\mathrm{val}_1(y_k)$ at r_D ($j < h' - \eta$) into $\epsilon_{\tau_D}\mathrm{Fil}_1(h' - j) \subseteq \epsilon_{\tau_D}\mathrm{Fil}_1(\eta + 1) = 0$; so these Taylor coefficients vanish and $a_k = Z^{h'-\eta} a'_k$ (Tf2). At τ_U, $\epsilon_{\tau_U}\lambda(Z)$ is a unit (constant term δ), and multiplying by the inverse of a unit power series preserves the condition "t^i-coefficient in $\mathcal{F} \otimes \mathrm{Fil}_1(h' - i)$"; so (Tf3) at r_U applies to the a'_k with $A = h'$ and $N = B + \eta + h' - b + 1 \le h'$. Let o_k be the Taylor coefficient of a'_k at r_U of order $N - 1$ and $x_1 = \sum_k o_k y_k \in (\mathbf{N}_{c-1}^{(1)})^\Delta$. Then $\mathrm{val}_1(x_1) \in \epsilon_{\tau_U}\mathrm{Fil}_1(b - B - \eta) = \epsilon_{\tau_U}\mathrm{Fil}_1(b)$, as $\eta < b - B - \eta \le b$. Dividing (2) by $Z^{h'-\eta}$ (a non-zero-divisor; $\pi^M q^{4h'} = Z^{h'-\eta}\pi^M(\pi - r_U)^{4h'} Z^{3h'+\eta}$) and comparing Taylor coefficients of order $N - 1 < 4h'$ at r_U, where $Z^{B+\eta} = ((\pi - r_U) + \delta)^{B+\eta}$ contributes $\binom{B+\eta}{B+\eta} = 1$ to the order $N - 1$ coefficient of $(\pi - r_U)^{h'-b} Z^{B+\eta}$:

$$\sum_k o_k \phi_{km} = \ell^{2\sigma+r_0}\xi_m \quad \text{for all } m, \qquad \text{i.e.} \quad g(x_1) - \ell^{2\sigma+r_0} x \in \pi(\mathbf{N}_{c-1}^{(2)})^\Delta.$$

Step 0 with $a = 2\sigma + r_0 \le y_1$ gives w_1. □

Lemma 27.13. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and K with $F \subseteq K \subseteq F(\mu_\ell)$ be as in Proposition 23.7, $\Delta = \Delta_K$, $c \in \mathbb{Z}/n$, and for each embedding τ of K over τ_c let $r_\tau \in L$ be defined by $\epsilon_\tau\lambda(\pi)_0 = \epsilon_\tau r_\tau$. Then:*

(Tf1) Roots *the r_τ are the $[K : F]$ distinct roots of E_Δ, they lie in $\mathcal{O}_L$ with positive valuation, $q = \prod_\tau(\pi - r_\tau)$ in R_c^Δ, and $\epsilon_\tau\lambda(\pi - r_\tau) = t\,w_\tau$ with w_τ a unit of $\epsilon_\tau\mathcal{F}[[t]]$;*

(Tf2) Taylor expansion *for τ over τ_c, $g \in R_c^\Delta$ and $N \ge 0$ there are unique $p_0, \dots, p_{N-1} \in \mathcal{O}_L$ and $g' \in R_c^\Delta$ with $g = \sum_{j<N} p_j(\pi - r_\tau)^j + (\pi - r_\tau)^N g'$ (the* Taylor coefficients *of g at r_τ);*

(Tf3) Filtered Taylor vectors *let τ be over τ_c, $g_1, \dots, g_k \in R_c^\Delta$ with Taylor coefficients p_{kj} at r_τ, $v_1, \dots, v_k \in L^r$, $(G(j))_{j\in\mathbb{Z}}$ a decreasing family of L-subspaces of L^r, and $A \in \mathbb{Z}$, $N \ge 0$ such that for every $i < N$ the t^i-coefficient of $\sum_k \epsilon_\tau\lambda(g_k)\,v_k$ lies in $\epsilon_\tau\mathcal{F} \otimes_L G(A - i)$. Then $\sum_k p_{kj} v_k \in G(A - j)$ for every $j < N$.*

Proof. **(Tf1).** By (Lz1), λ is the substitution $X \mapsto \zeta e^{t/\ell} - 1$, so the constant term of $\lambda(g)$ is $\epsilon\, g(\zeta - 1)$ for $g \in e_c \mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$; hence $\lambda(\pi)_0 = \epsilon\, y_0$ with $y_0 = \pi_\Delta(\zeta - 1) \in S_\Delta$, and by (Rr7) $y_0 \in K$ is a uniformiser of K with minimal polynomial E_Δ over F. As ϵ_τ is the idempotent of τ on $K \otimes_{\mathbb{Q}_\ell} L$, $\epsilon_\tau(y_0 \otimes 1) = \epsilon_\tau(1 \otimes \tau(y_0))$, so $r_\tau = \tau(y_0)$: the $[K : F]$ embeddings over τ_c give the $[K : F] = \deg E_\Delta$ distinct roots of E_Δ in L ($K = F(y_0)$), integral of positive valuation (E_Δ is Eisenstein, (Rr3)). So $q = E_\Delta(\pi) = \prod_\tau(\pi - r_\tau)$. By (Rr4) and (Lz1), $\epsilon_\tau\lambda(q) \in t\,\epsilon_\tau\mathcal{F}[[t]]^\times$; for $\tau' \neq \tau$ the factor $\epsilon_\tau\lambda(\pi - r_{\tau'})$ has constant term $\epsilon_\tau(r_\tau - r_{\tau'})$ with $r_\tau - r_{\tau'} \in L^\times$, a unit; dividing, $\epsilon_\tau\lambda(\pi - r_\tau) = t w_\tau$ with w_τ a unit. **(Tf2).** $R_c^\Delta = \mathcal{O}_L[[\pi]]$ (Rr2) and $r = r_\tau$ has positive valuation, so $g(r) \in \mathcal{O}_L$ converges, and $g - g(r) = (\pi - r)g_1$ with $g_1 = \sum_n g_n \sum_{i<n} \pi^i r^{n-1-i} \in \mathcal{O}_L[[\pi]]$ (coefficientwise convergent). Iterating N times gives the expansion; uniqueness follows by evaluating at r and cancelling the non-zero-divisor $\pi - r$. **(Tf3).** Put $P_j = \sum_k p_{kj} v_k \in L^r$ and $\Gamma = \sum_k \epsilon_\tau\lambda(g_k)v_k$; by (Tf1), $\Gamma = \sum_{j<N}(t w_\tau)^j P_j + t^N \Gamma'$ with $\Gamma' \in \epsilon_\tau\mathcal{F}[[t]]^r$. Let $j_0 < N$ and assume $P_j \in G(A - j)$ for $j < j_0$. The t^{j_0}-coefficient of $\Gamma - \sum_{j<j_0}(t w_\tau)^j P_j$ lies in $\epsilon_\tau\mathcal{F} \otimes G(A - j_0)$, since that of Γ does and $G(A - j) \subseteq G(A - j_0)$ for $j < j_0$; it equals $w_{\tau,0}^{j_0} P_{j_0}$, where $w_{\tau,0} \in (\epsilon_\tau\mathcal{F})^\times$ is the constant term of w_τ. Hence $P_{j_0} \in (\epsilon_\tau\mathcal{F} \otimes_L G(A - j_0)) \cap L^r = G(A - j_0)$, as $\epsilon_\tau\mathcal{F}$ is a nonzero free L-module (L a field). □

Lemma 27.14. *Let ℓ be a prime, $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_\ell$, H an $\mathcal{O}$-module generated by g elements, $H' \subseteq H$ a submodule, $Z \subseteq H$ a finite subset and $e, e' \geq 0$ integers such that*

$$\ell^e H \subseteq H' + \sum_{z \in Z} \mathcal{O} z \qquad \text{and} \qquad \ell^{e'} z \in H' \text{ for every } z \in Z.$$

Then H/H' has finite length and $\mathrm{nl}(H/H') \leq g\,e + |Z|\,e'$.

Proof. Let $Q = H/H'$ and $Q' \subseteq Q$ the submodule generated by the images of the $z \in Z$. The quotient Q/Q' is generated by g elements and killed by ℓ^e, so it is a quotient of $(\mathcal{O}/\ell^e)^g$ and has length at most $g\,\mathrm{length}_{\mathcal{O}}(\mathcal{O}/\ell^e) = g\,e\,e_{\mathcal{O}}$. The module Q' is generated by $|Z|$ elements each killed by $\ell^{e'}$, so it is a quotient of $(\mathcal{O}/\ell^{e'})^{|Z|}$ and has length at most $|Z|\,e'\,e_{\mathcal{O}}$. By additivity of length on $0 \to Q' \to Q \to Q/Q' \to 0$, Q has finite length at most $(g\,e + |Z|\,e')\,e_{\mathcal{O}}$; divide by $e_{\mathcal{O}}$. □

Theorem 27.15. *Let the hypotheses of Theorem 25.1 hold at the odd prime ℓ, and let κ_k, $\lambda_k(v)$ be the losses given there. Let $n, \varrho, m \geq 1$ be integers, $f = 2n$, and $H \geq 1$, $c \geq 1$ real numbers such that*

1. *$\sum_{v \in S_\ell} n_v \leq 2f$, $n_0 \leq \varrho \leq t$, $m \leq t$ and $M_0 + h_0 \leq H$;*
2. *$\mu_\tau^-(v) \leq m$ for $v \in C$ and $\mu_\tau^+(v) \leq m$ for $v \in S_\ell \setminus C$, at every embedding τ;*
3. *$e'_v \leq c\,(1 + H/\ell + \log_\ell H)$ for $v \in C$ and $e_0(v) \leq c\,(1 + H/\ell + \log_\ell H)$ for $v \in S_\ell \setminus C$;*
4. *$\nu(\delta) \leq 2t$, $b_v \leq c\,t$ for $v \in C$, $|J(V_{\mathcal{N}})| \leq c\,n_v(\varrho + t)$ for $v \in S_\ell \setminus C$, and $\sum_{v \in \Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) \leq c$.*

Then

$$\sum_{k=0}^{t} \kappa_k + \sum_{v \in S_\ell} \sum_{k=0}^{t} \lambda_k(v) + t \sum_{v \in \Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) + 2(n_0 + t - 1)\,\nu(\delta)$$
$$\leq 190\,c\,f\Big(n\,t^2(\varrho + t)(1 + \log_\ell(H + t)) + \frac{m\,H}{\ell}\Big).$$

Proof. By Theorem 25.1 the first two sums add up to Λ_ℓ. We bound the ingredients of Λ_ℓ and then add.

The invariant exponents. Let $W \in \{W_0, W_0'\}$, $v \in S_\ell$, $\tau = \tau_v$. As W is de Rham over E_v, $\dim_{E_v} D_{\mathrm{dR}}(W) = \dim_{\mathbb{Q}_\ell} W$, so $\dim_K D_{\mathrm{dR}}(W) = n_v \dim_K W = n_v n_0$; the component $D_{\mathrm{dR},\tau}(W)$ is a direct summand, so it has K-dimension at most $n_v n_0$, and a decreasing filtration on it has at most that many jumps: $|J_\tau(W)| \le n_v n_0$. Let $n_{\min}$ be the integer of Proposition 25.35 for $H' = M_0 + h_0 \le H$: with $n_1 = \lceil \log_\ell \max(H', 1) \rceil \ge 0$ we have $H' \le \ell^{n_1} \le \ell^{n_1}(\ell - 1)$, so $n_{\min} \le n_1 \le \log_\ell H + 1$ and $m_N = n_{\min} + 3 \le \log_\ell H + 4 \le 4\lambda$. Every $a \in J_\tau(W)$ is a non-zero integer with $|a| \le H$ (the jumps lie in $[-M_0, h_0]$), so $\ell^{v_\ell(a)} \le H$ and $v_\ell(a) \le \log_\ell H \le \lambda$. Hence $c_{\mathrm{inv}}(W, \tau) \le 2 + 5 n_v n_0 \lambda$ and

$$n_0\, c_{\mathrm{inv}}(W, \tau_v) \le 2n_0 + 5 n_v n_0^2 \lambda \le 7\, n_v\, n_0^2\, \lambda.$$

The character exponents. Let $v \notin C$, $V = V_{\mathcal{N}}$, $J = J(V) \subseteq [-M_0, h_0]$, and $0 \le e \le t-1$. For $0 \le i \le e$ and $u \in J$, $u \ne i$, we have $0 < |u - i| < H + t$, so $v_\ell(u - i) \le \lambda$, and $r_i \le |J|(m_N + \lambda) \le 5|J|\lambda$. In the formula for $c_{\mathrm{B}}(V, e)$: $m_N - 1 \le 4\lambda$; $\sum_{i=0}^{e}(2 + \max(1, r_i)) \le t(3 + 5|J|\lambda)$; $e + 2 \le t + 1$; $v_\ell(e) \le \log_\ell t \le \lambda$ (and $v_\ell(0) = 0$); $\lfloor e/(\ell - 1) \rfloor \le e/2 \le t$; $v_\ell(e + 1) \le \log_\ell t \le \lambda$. So

$$c_{\mathrm{B}}(V, e) \le 6\lambda + 5t + 3 + 5t|J|\lambda \le 14t\lambda + 5t|J|\lambda \le 19\, t \max(1, |J|)\, \lambda \le 19\, c\, t\, n_v(\varrho + t)\, \lambda,$$

using $t, \lambda, c \ge 1$ and hypothesis 4; and $\sum_{k=1}^{t} c_{\mathrm{B}}(V, k-1)\, n_v \le 19\, c\, t^2 n_v^2 (\varrho + t)\lambda$.

The cyclotomic terms. Put $F = E_{v,0}(\mu_\ell) \supseteq E_v$. The field $F \cap \mathbb{Q}_\ell(\mu_{\ell^\infty})$ contains $\mathbb{Q}_\ell(\mu_\ell)$ and is, over $\mathbb{Q}_\ell(\mu_\ell)$, both a subextension of the unramified extension $F/\mathbb{Q}_\ell(\mu_\ell)$ and of the totally ramified extension $\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell(\mu_\ell)$; so it equals $\mathbb{Q}_\ell(\mu_\ell)$, and $\chi_{\mathrm{cyc}}(G_{E_v}) \supseteq \chi_{\mathrm{cyc}}(G_F) = \mathrm{Gal}(\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell(\mu_\ell)) = 1 + \ell\mathbb{Z}_\ell$. Take $g \in G_{E_v}$ with $\chi_{\mathrm{cyc}}(g) = 1 + \ell$; as ℓ is odd, $v_\ell((1+\ell)^i - 1) = 1 + v_\ell(i)$ for $i \ge 1$. Hence the integer $w_i(v)$ of Theorem 25.2 satisfies $w_i(v) \le 1 + v_\ell(i) \le 1 + \log_\ell t \le \lambda$ for $1 \le i \le t$.

Per place. With $E^\sharp = c(1 + H/\ell + \log_\ell H)$ and hypotheses 2–4 (the number of embeddings τ of E_v is n_v): a column place v contributes to Λ_ℓ at most

$$(E^\sharp + 2t)\, n_v m + 7 n_v n_0^2 \lambda + (n_0 + t + 1)\, 2t + c\, t + t\lambda,$$

and a row place at most $E^\sharp n_v m + 7 n_v n_0^2 \lambda + 19\, c\, t^2 n_v^2 (\varrho + t)\lambda + t\lambda$.

Sum over the places. By hypothesis 1, $\sum_v n_v \le 2f$, $|S_\ell| \le 2f$ and $\sum_v n_v^2 \le (\sum_v n_v)^2 \le 4f^2 = 8fn$. With $n_0 + t + 1 \le 2(\varrho + t)$ and $n_0^2 \le \varrho t$,

$$\Lambda_\ell \le 2f\, m\, E^\sharp + 4f\, m\, t + 14f\, \varrho t\, \lambda + 2f\big(4t(\varrho + t) + c\, t + t\lambda\big) + 152\, c\, f\, n\, t^2 (\varrho + t)\lambda.$$

Further $t \sum_{\Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) \le c\, t$ and $2(n_0 + t - 1)\nu(\delta) \le 4t(\varrho + t)$. Now $1 + \log_\ell H \le \lambda$, so $2f\, m\, E^\sharp \le 2c\, f\, m\, \lambda + 2c\, f\, m\, H/\ell$, and as $m \le t$, $\varrho \ge 1$, $n \ge 1$, $f \ge 2$, $c, \lambda \ge 1$, each of $2c\, f\, m\lambda$, $4fmt$, $14f\varrho t\lambda$, $2f(4t(\varrho + t) + ct + t\lambda)$, ct, $4t(\varrho + t)$ is at most $2c$, 4, 14, $2(5 + c)$, c, 4 times $f\, Q\, \lambda$. Adding, the left side of the statement is at most $(157c + 32)\, f\, Q\, \lambda + 2c\, f\, m\, H/\ell \le 190\, c\, f\, (Q\lambda + mH/\ell)$. □

Proposition 27.16. *There is an absolute constant $c_{\mathrm{cy}} \ge 1$ with the following property. Let E be a number field, ℓ a prime, $N \ge 1$ an integer with $4 \mid N$ if $\ell = 2$, $s = \max(1, v_\ell(N))$ and $N_\ell = \mathrm{lcm}(\ell, N)$. Let $(\mathcal{O}, \mathcal{Z}, \mathbb{T}, M)$ be a Hecke lattice over ℓ, $K = \mathcal{O}[\ell^{-1}]$, column data with column places $C \subseteq S_\ell$ for items $\mathbf{r}, \chi_1, \mathcal{E}, \delta$, and Galois Ribet data over E on $(\mathcal{R}, \pi', \mathcal{N}^{\mathrm{col}})$ with these items, $\iota = \iota^{\mathrm{col}}$ and $t + 1$ blocks, having (Gd1) and (Gd2); write $n_0 = n_{\mathcal{R},0}$, $D_v = G_{E_v}$ and $n_v = [E_v : \mathbb{Q}_\ell]$. Let $n, \varrho, m \ge 1$ and $H \ge 1$ be integers, $f = 2n$, and $c \ge 1$ a real number. Suppose:*

1. *at every $v \in S_\ell$: E_v is isomorphic over $\mathbb{Q}_\ell$ to a subfield of $\mathbb{Q}_\ell(\mu_{N_\ell})$; K contains a primitive N_ℓ-th root of unity; (Ll1) and (Ll2) hold; and every Hodge jump of $V_{\mathcal{N}}$, of $W_0, \dots, W_t$ and of W_0' at every embedding lies in $[-H, H]$;*

2. *at every $v \in C$: the column data have the column properties with $e_v' \le c\,(Hs+1)$ and $b_v \le c\,t$; $H^0(D_v, W_0) = 0$; at every embedding W_0 has at most m Hodge jumps $a < 0$; and 0 is not a Hodge jump of W_0 at some embedding;*

3. *at every $v \in S_\ell \setminus C$: $H^0(D_v, W_0') = 0$; at every embedding W_0 has at most m Hodge jumps $a \ge 0$; 0 is not a Hodge jump of W_0' at some embedding; and $|J(V_{\mathcal{N}})| \le c\,n_v(\varrho + t)$;*

4. *$\sum_{v\in S_\ell} n_v \le 2f$, $n_0 \le \varrho \le t$, $m \le t$, $\nu(\delta) \le 2t$ and $\sum_{v\in\Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) \le c$.*

Then there are real numbers $\kappa_k \ge 0$ and $\lambda_k(v) \ge 0$ ($0 \le k \le t$, $v \in S_\ell$) with which (Gd6) and (Gd7) hold and

$$\sum_{k=0}^{t} \kappa_k + \sum_{v\in S_\ell} \sum_{k=0}^{t} \lambda_k(v) + t \sum_{v\in\Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) + 2(n_0 + t - 1)\,\nu(\delta) \\ \le c_{\mathrm{cy}}\, c\, f\Big(m\, H\, s + f\, t^2 (\varrho + t)(s + \log_\ell(2H) + \log_\ell N + 1)\Big).$$

Proof. Let c_1 be the absolute constant of Proposition 24.10; we show that $c_{\mathrm{cy}} = 12c_1 + 94$ has the property. Put $M_0 = h_0 = H$ and

$$\vartheta = s + \log_\ell(2H) + \log_\ell N + 1 \;\ge\; 2.$$

Since $\mathbb{Q}_\ell(\mu_{N_\ell})$ is Galois over $\mathbb{Q}_\ell$, item 1 gives $E_v \subseteq \mathbb{Q}_\ell(\mu_{N_\ell}) \subseteq K$ inside $\overline{\mathbb{Q}}_\ell$ for every $v \in S_\ell$, and every $\mathbb{Q}_\ell$-embedding of a subfield of $\mathbb{Q}_\ell(\mu_{N_\ell})$ has image in K.

Step 1: the lifting exponent at a row place. Let $v \in S_\ell \setminus C$ and $K' = E_v(\mu_{\ell^s})$. By Lemma 25.32, $K' = K_0'(\mu_{\ell^s})$ with K_0' unramified over $\mathbb{Q}_\ell$, $v_\ell([K' : E_v]) \le s - 1$, and $s \ge 2$ if $\ell = 2$; and K contains $\sigma(K')$ for every $\mathbb{Q}_\ell$-embedding σ of K'. By Theorem 25.3 ((Ll1), (Gd1), (Gd2)), $V_{\mathcal{N}}$ and W_0 are crystalline representations of D_v, with the D_v-stable lattices $\mathcal{N}_{\mathcal{E}}$ and Λ_0. Let $\psi\colon \mathcal{N}_{\mathcal{E}} \to \mathrm{Cs}_0(u) = W_0/\Lambda_0$ be $\mathcal{O}$-linear and D_v-equivariant. By Lemma 25.4(1), $\widetilde{T}_\psi$ is a D_v-stable $\mathcal{O}$-lattice in $V_{\mathcal{N}} \oplus W_0$ with $\mathrm{pr}_1(\widetilde{T}_\psi) = \mathcal{N}_{\mathcal{E}}$. Apply Proposition 24.10 (paragraph Level s) with $p = \ell$, its K_0 and K equal to K_0' and K', its L equal to our K, $V_1 = V_{\mathcal{N}} \oplus W_0$, $T_1 = \widetilde{T}_\psi$, $V_2 = V_{\mathcal{N}}$, $T_2 = \mathcal{N}_{\mathcal{E}}$, $F = \mathrm{pr}_1$, all restricted to $G_{K'} \subseteq D_v$, and $a_{\tau'}^{\pm} = H$: the restrictions are crystalline, and for an embedding τ' of K' with restriction τ to E_v, the natural isomorphism of filtered modules $K' \otimes_{E_v} D_{\mathrm{dR},E_v}(X) \cong D_{\mathrm{dR},K'}(X)$ [94] shows that the Hodge jumps of X at τ' are those at τ; so every Hodge jump of V_1 at τ' is one of $V_{\mathcal{N}}$ or W_0 at τ, hence in $[-H, H]$ by item 1, and $\mathrm{wt} \le 2H$. It gives

$$\ell^{e_1} H_f^1(K', \mathcal{N}_{\mathcal{E}}) \subseteq \mathrm{pr}_{1*} H_f^1(K', \widetilde{T}_\psi), \qquad e_1 = \lfloor c_1(2H+1)(s+1) \rfloor.$$

By item 2 of Lemma 25.33 ($E = E_v$, K', $F = \mathrm{pr}_1$), $\ell^{e_0(v)} H_f^1(D_v, \mathcal{N}_{\mathcal{E}}) \subseteq \mathrm{pr}_{1*} H_f^1(D_v, \widetilde{T}_\psi)$ with

$$e_0(v) = e_1 + s - 1 \le (6c_1 + 1)\, H\, s,$$

since $(2H+1)(s+1) \le 6Hs$ and $s - 1 \le Hs$. This holds for every ψ as above, which is the inclusion required in (Lr2).

Step 2: the losses. Apply Theorem 25.1 in its paragraph Descent, with this N and $M_0 = h_0 = H$. Its field conditions, (Ll1), (Ll2) and the window of the Hodge jumps of $V_{\mathcal{N}}$, W_0, W_0' are item 1. (Lr1) at $v \in C$ is item 2 with $\mu_\tau^-(v) = m$ (Hodge jumps counted with multiplicity) and τ_v an embedding at which 0 is not a Hodge jump of W_0. (Lr2) at $v \notin C$ is item 3 with $\mu_\tau^+(v) = m$, τ_v an embedding at which 0 is not a Hodge jump of W_0', and the integer $e_0(v)$ of Step 1. So there are κ_k, $\lambda_k(v)$ with (Gd6) and (Gd7), $\sum_k \kappa_k + \sum_v \sum_k \lambda_k(v) = \Lambda_\ell$, where Λ_ℓ is the expression of Theorem 25.1 with $c_k^{\mathrm{B}} = c_{\mathrm{B},s}(V_{\mathcal{N}}, k-1) + s - 1$ and $c_{\mathrm{inv},s}$ in place of $c_{\mathrm{B}}(V_{\mathcal{N}}, k-1)$ and c_{inv}. We bound its ingredients.

Step 3: invariants. Let $W \in \{W_0, W_0'\}$, $v \in S_\ell$, $\tau = \tau_v$. As W is de Rham, $\dim_K D_{\mathrm{dR}}(W) = n_v n_0$, and its direct summand $D_{\mathrm{dR},\tau}(W)$ carries a decreasing filtration with at most $n_v n_0$ jumps: $|J_\tau(W)| \le n_v n_0$. For $M_0 + h_0 = 2H$ the integer $n_{\min} = \min\{n \ge 0 : \ 2H \le \ell^n(\ell-1)\}$ of Proposition 25.35 is at most $\lceil \log_\ell(2H) \rceil \le \log_\ell(2H) + 1$, so $m_{N,s} = 2s + n_{\min} + 1 \le 2s + \log_\ell(2H) + 2 \le 2\vartheta$; and each $a \in J_\tau(W)$ is a non-zero integer with $|a| \le H$, so $v_\ell(a) \le \log_\ell H \le \vartheta$. Hence $c_{\mathrm{inv},s}(W,\tau) \le 2 + 3n_v n_0 \vartheta$ and $n_0\, c_{\mathrm{inv},s}(W,\tau_v) \le 5\, n_v n_0^2\, \vartheta$.

Step 4: characters. Let $v \notin C$, $V = V_{\mathcal{N}}$, $J = J(V) \subseteq [-H, H]$ and $0 \le e \le t-1$. Then $\sum_{i=0}^{e} r_i^{(s)} \le |J|(e+1)m_{N,s} + \sum_{u \in J} \Sigma_u$ with $\Sigma_u = \sum_{0 \le i \le e,\ i \ne u} v_\ell(u-i)$. For fixed u the non-zero integers $u - i$ form at most two runs of consecutive integers of one sign, each of length at most $e + 1 \le t$ and of absolute values below $H + t$. For a run $a+1, \dots, b$ of positive integers, Legendre's formula gives $\sum_{a<j\le b} v_\ell(j) = v_\ell(b!) - v_\ell(a!) = \big(b - a - S_\ell(b) + S_\ell(a)\big)/(\ell-1) \le (b-a) + 1 + \log_\ell b$, as $S_\ell(a) \le (\ell-1)(1 + \log_\ell a)$ for $a \ge 1$ and $S_\ell(0) = 0$. So $\Sigma_u \le t + 2 + 2\log_\ell(H+t) \le t + 2 + 2\log_\ell(2H) + 2\log_\ell t \le 3t + 2\vartheta$ (using $H + t \le 2Ht$ and $\log_\ell t \le t$), and $\sum_i r_i^{(s)} \le |J|\,(2t\vartheta + 3t + 2\vartheta) \le 7|J|\,t\,\vartheta$. In the formula for $c_{\mathrm{B},s}(V,e)$: $m_{N,s} - s \le 2\vartheta$; $\sum_{i=0}^{e}(2 + \max(1, r_i^{(s)})) \le 3t + 7|J|t\vartheta$; $s(e+4) \le 4ts \le 4t\vartheta$; $v_\ell(e) \le t$, $\lfloor e/(\ell-1) \rfloor \le t$, $v_\ell(e+1) \le t$. Adding $s - 1 \le \vartheta$ and using $t, \vartheta \ge 1$,

$$c_{\mathrm{B},s}(V,e) + s - 1 \le 13\, t\vartheta + 7|J|\, t\vartheta \le 20\, t \max(1, |J|)\, \vartheta \le 20\, c\, t\, n_v(\varrho + t)\, \vartheta$$

by item 3. Hence $\sum_{k=1}^{t} c_k^{\mathrm{B}}\, n_v \le 20\, c\, t^2 n_v^2 (\varrho + t)\, \vartheta$.

Step 5: the cyclotomic terms. The field $\mathbb{Q}_\ell(\mu_{N_\ell})$ is unramified over $\mathbb{Q}_\ell(\mu_{\ell^s})$ and $\mathbb{Q}_\ell(\mu_{\ell^\infty})$ is totally ramified over it, so their intersection is $\mathbb{Q}_\ell(\mu_{\ell^s})$ and $\chi_{\mathrm{cyc}}(G_{E_v}) \supseteq \chi_{\mathrm{cyc}}(G_{\mathbb{Q}_\ell(\mu_{N_\ell})}) = \mathrm{Gal}(\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell(\mu_{\ell^s})) = 1 + \ell^s\mathbb{Z}_\ell$. Take $g \in G_{E_v}$ with $\chi_{\mathrm{cyc}}(g) = 1 + \ell^s$; by item 1 of Lemma 23.22 ($p = \ell$, $m = s$; $s \ge 2$ if $\ell = 2$), $v_\ell(\chi_{\mathrm{cyc}}(g)^i - 1) = s + v_\ell(i)$, so $w_i(v) \le s + v_\ell(i)$ and $\sum_{i=1}^{t} w_i(v) \le ts + v_\ell(t!) \le ts + t \le 2t\vartheta$.

Step 6: the sum. Put $c_2 = 6c_1 + 2$; as $c \ge 1$, $e_v' \le c(Hs+1) \le 2cHs \le c_2 cHs$ ($v \in C$, item 2) and $e_0(v) \le c_2 cHs$ ($v \notin C$, Step 1). With $\nu(\delta) \le 2t$, $b_v \le ct$, $n_0 + t + 1 \le 2(\varrho + t)$ (item 4), Steps 3–5 and the expression of Λ_ℓ, a column place contributes at most $(c_2 cHs + 2t)\, n_v m + 5 n_v n_0^2 \vartheta + 4t(\varrho+t) + ct + 2t\vartheta$, and a row place at most $c_2 cHs\, n_v m + 5 n_v n_0^2 \vartheta + 20\, c\, t^2 n_v^2 (\varrho+t)\vartheta + 2t\vartheta$. By item 4, $\sum_{v \in S_\ell} n_v \le 2f$, $|S_\ell| \le 2f$ and $\sum_v n_v^2 \le 4f^2$; so

$$\Lambda_\ell \le 2c_2\, c\, f\, mHs + 4fmt + 10 f\, n_0^2 \vartheta + 2f\big(4t(\varrho + t) + ct + 2t\vartheta\big) + 80\, c\, f^2 t^2 (\varrho + t)\vartheta.$$

Further $t \sum_{v \in \Sigma_{\mathrm{T}}} \nu(\zeta_v - 1) \le ct$ and $2(n_0 + t - 1)\nu(\delta) \le 4t(\varrho + t)$. Put $X = mHs + f\, t^2(\varrho + t)\vartheta$; as $f \ge 2$, $\varrho + t \ge 2$ and $\vartheta \ge 2$, $X \ge 8t^2$ and $X \ge 4t^2\vartheta$. With $m \le t$ and $n_0 \le \varrho \le t$: $4fmt \le 4ft^2 \le fX$; $10 f n_0^2 \vartheta \le 10 f t^2 \vartheta \le 3fX$; $8ft(\varrho + t) \le 2f \cdot f\, t^2(\varrho + t)\vartheta \le 2fX$; $2fct \le cfX$; $4ft\vartheta \le fX$; $80 c f^2 t^2 (\varrho + t)\vartheta \le 80 cfX$; $ct \le cfX$; $4t(\varrho + t) \le fX$. Adding, with $c \ge 1$, the left side of the statement is at most $(2c_2 + 90)\, c\, f\, X = c_{\mathrm{cy}}\, c\, f\big(mHs + f\, t^2(\varrho + t)(s + \log_\ell(2H) + \log_\ell N + 1)\big)$. $\square$

28 Large primes: lossless local conditions

At a large prime the allowance of Theorem 22.4 is ℓ^{-2}, so the local losses must vanish, and they do: Proposition 28.1 gives $\lambda_k(v) = 0$ at every place above a large prime ℓ, under position hypotheses on the Hodge jumps. These hypotheses, (Ph1)–(Ph5), place the jumps at each site, and the weights of the character blocks, in windows below ℓ, separated by margins measured by the *Frobenius carry* $\mathrm{sg}(a) = \lfloor \sum_{j\geq 1} h(a+j)\,\ell^{-j} \rfloor$ of the following sites; in particular $\eta_a + 4\,\mathrm{sg}(a) + 6 < \ell$ (Ph3). In this range the local conditions can be computed exactly modulo ℓ, and the section does so.

The local core is Proposition 28.2, exact local conditions for torsion crystalline extensions with weights in a Frobenius-carry window. It reduces to counts of admissible classes modulo ℓ (Proposition 28.3): the residual representation has no invariants and no maps to the relevant character lines, independent admissible classes are at most as many as the jumps that lie below the weight of the first character, and the twisted characters χ_c, $c \geq 2$, have no admissible extensions. These counts are proved with residual (φ,Γ)-modules over $k((X))$ [93], to which Wach modules reduce site by site (Subsection 28.2), and the decisive estimate is the polar count Theorem 28.38 (Subsection 28.3). One lemma of the section, the Γ-weight separation lift Lemma 28.15, is also used in Appendix C.

28.1 Exact local conditions at a large prime

The subsection states the three results of the section in the order of use. Proposition 28.1 is what the budget reads: from the position hypotheses (Ph1)–(Ph5) at every place above ℓ it obtains vanishing local losses. It follows from Proposition 28.2, which describes the admissible torsion extensions exactly: those with values in the lattice T_0 of block 0 are sums of integral Bloch–Kato classes and coboundaries, those with values in a twisted character χ_c, $c \geq 2$, are coboundaries, and those with values in χ_1 are unramified up to coboundaries. Its proof is a count modulo ℓ, Proposition 28.3, whose ingredients are proved in the next two subsections.

Proposition 28.1. *Let E be a number field, $\ell \geq 5$ a prime, $(\mathcal{O}, \mathcal{Z}, \mathbb{T}, M)$ a Hecke lattice over ℓ with $K = \mathcal{O}[\ell^{-1}]$ and a uniformiser ϖ of $\mathcal{O}$, and let Galois Ribet data over E on $(\mathcal{R}, \pi', \mathcal{N}_{\mathcal{E}})$, with ι the identity and $t+1$ blocks, have (Gd1) and (Gd2), with $\delta \in \mathcal{O}^\times$. Let v be a place of E above ℓ such that $E_v/\mathbb{Q}_\ell$ is unramified of degree n, K contains $\sigma(E_v(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $E_v(\mu_\ell)$, and (Ll1), (Ll2) hold at v; put $D = G_{E_v}$. Let V_π be $K^{n_{\mathcal{R}},0}$ with $g \in D$ acting by $\rho_0(\mathbf{r}(g))$, and for $x' \in \mathcal{Z}'$ let $V_{x'}$ be $K^{n_{\mathcal{R}}}$ with $g \in D$ acting by $\mathbf{r}(g)(x')$. Suppose that there are a Frobenius numbering $(\kappa_a)_{a\in\mathbb{Z}/n}$ of the $\mathbb{Q}_\ell$-embeddings $E_v \to K$ and data $h, w, \eta\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$, $\mathrm{typ}\colon \mathbb{Z}/n \to \{\mathrm{low}, \mathrm{high}\}$ and finite multisets U_a of integers satisfying (Ph1)–(Ph5) for ℓ and t, such that, with E_v in the role of the field K and K in the role of the field L of Proposition 28.2:*

1. *V_π is crystalline and $m_{V_\pi}(\tau, j)$ is the multiplicity of j in $U_{a(\tau)}$, for every embedding $\tau\colon E_v \to K$ and every integer j;*
2. *for $1 \leq c \leq t$ the character $\chi_c|_D$ is crystalline with $m_{\chi_c}(\tau, j) = 1$ if $j = w_{a(\tau)} + c - 1$ and $m_{\chi_c}(\tau, j) = 0$ otherwise;*
3. *for every $x' \in \mathcal{Z}'$ the representation $V_{x'}$ is full for $V = V_\pi$, the characters $\chi_1|_D, \ldots, \chi_t|_D$ and $\lambda = \varpi$.*

Then for $0 \leq k \leq t$ the image of X_k in $H^1(D, \mathrm{Cs}_k(u))/F_k(v)$ is zero; that is, (Gd7) holds at v with $\lambda_k(v) = 0$ for $0 \leq k \leq t$.

Proof. Throughout, $D = G_{E_v}$ acts on every module named below through restriction from G_E. We apply Proposition 28.2 with E_v in the role of its field K, the coefficient field $K = \mathcal{O}[\ell^{-1}]$ in the role of L, $\lambda = \varpi$, the given Frobenius numbering, $h, w, \eta, \mathrm{typ}, (U_a)$, the representation $V = V_\pi$ with the lattice $T_0 = \mathcal{O}^{n_{\mathcal{R},0}}$ (which is D-stable because $\rho_0(\mathcal{R}) \subseteq \mathrm{M}_{n_{\mathcal{R},0}}(\mathcal{O})$ by (Bf1)), and the characters $\chi_1|_D, \dots, \chi_t|_D$. Its hypotheses hold: $\ell \geq 5$; E_v is unramified over $\mathbb{Q}_\ell$ and ϖ is a uniformiser of $\mathcal{O}$; K contains $\sigma(E_v(\mu_\ell))$ for every embedding of $E_v(\mu_\ell)$; (Ph1)–(Ph5) are assumed; V_π is crystalline with the multiplicities of the U_a by (1); $\chi_c = \chi_1 \chi_{\mathrm{cyc}}^{1-c}$ by definition, with values in $\mathcal{O}^\times$, crystalline with the stated multiplicities by (2). Every point representation $V_{x'}$, $x' \in \mathcal{Z}'$, is full by (3).

The ambient lattice. The algebra $\mathcal{R} \subseteq \prod_{x' \in \mathcal{Z}'} \mathrm{M}_{n_\mathcal{R}}(K)$ is a finitely generated $\mathcal{O}$-module: for each x' the map $g \mapsto \mathbf{r}(g)(x')$ is continuous on the compact group G_E, so its image is bounded, hence $\mathbf{r}(G_E) \subseteq \varpi^{-c} \prod_{x'} \mathrm{M}_{n_\mathcal{R}}(\mathcal{O})$ for some c, and $\mathcal{R}$, the span of $\mathbf{r}(G_E)$ over the coefficient algebra $\mathbb{T}'$ of Definition 17.2 (whose elements are $\mathcal{O}$-valued, so that each component of an element of $\mathcal{R}$ is an $\mathcal{O}$-combination of components of the $\mathbf{r}(g)$), lies in this finitely generated module over the noetherian ring $\mathcal{O}$. Let D act on $\mathcal{R}$ by left multiplication by $\mathbf{r}(g)$. Writing an element of $\prod_{x'} \mathrm{M}_{n_\mathcal{R}}(K)$ column by column identifies this K-space, with left multiplication, with $S_0 = \bigoplus_{x' \in \mathcal{Z}'} V_{x'}^{n_\mathcal{R}}$; so $\mathcal{R}$ is a D-stable finitely generated $\mathcal{O}$-submodule of the finite direct sum S_0 of full representations. The left ideal $\mathcal{N}_\mathcal{E}$ is D-stable, and so is $\mathcal{M} = \mathcal{N}_\mathcal{E} + \mathcal{O}\mathcal{E}$, since $\mathbf{r}(g)\mathcal{E} = \chi_1(g)u(g)\mathcal{E} = \chi_1(g)\big(\mathcal{E} + y(g)\big)$. The sum is direct: ρ_1 is an algebra homomorphism with $\rho_1(y(g)) = \rho_1(u(g) - 1)\rho_1(\mathcal{E}) = (\chi_1(g)^{-1}\chi_1(g) - 1)\delta = 0$ by (Gd1), (Gd2), so $\rho_1(\mathcal{N}_\mathcal{E}) = 0$, while $\rho_1(c\mathcal{E}) = c\delta \neq 0$ for $c \neq 0$.

Reduction to finite coefficients. Fix $0 \leq k \leq t$ and $\varphi \in \mathrm{Hom}_\mathcal{R}(\mathcal{N}_\mathcal{E}, \mathrm{Cs}_k)$ such that $c_\varphi(g) = \varphi(y(g))$ is a continuous 1-cocycle, and put $z = c_\varphi|_D$. Since Cs_k is discrete and G_E compact, c_φ has finite image, so there is $a \geq 0$ with $c_\varphi(G_E) \subseteq \varpi^{-a}\Lambda_k/\Lambda_k$, where $\Lambda_k = \mathcal{O}^{n_{\mathcal{R},k}}$ and $\mathrm{Cs}_k = (K/\mathcal{O})^{n_{\mathcal{R},k}} = K^{n_{\mathcal{R},k}}/\Lambda_k$. As $\mathcal{N}_\mathcal{E}$ is generated as a left $\mathcal{R}$-module by the $y(g)$ and $\rho_k(\mathcal{R}) \subseteq \mathrm{M}_{n_{\mathcal{R},k}}(\mathcal{O})$ preserves $\varpi^{-a}\Lambda_k/\Lambda_k$, we get $\varphi(\mathcal{N}_\mathcal{E}) \subseteq \varpi^{-a}\Lambda_k/\Lambda_k$. Put $X = \varpi^{-a}\mathcal{O}/\mathcal{O}$, a finite cyclic $\mathcal{O}$-module with generator $x_0 = \varpi^{-a} + \mathcal{O}$, and identify $\Lambda_k \otimes_\mathcal{O} X = \varpi^{-a}\Lambda_k/\Lambda_k$ by $m \otimes x \mapsto \tilde{x}\, m$ ($\tilde{x}$ any lift of x). Let $\Lambda' = \Lambda_k$, a lattice in $V'' = K^{n_{\mathcal{R},k}}$ on which g acts by $\rho_k(\mathbf{r}(g))$: this is T_0 in V_π for $k = 0$ and a lattice in $K(\chi_k)$ for $k \geq 1$, by (Gd1). Since φ is $\mathcal{R}$-linear and $y(gh) = u(gh)\mathcal{E} - \mathcal{E} = y(g) + u(g)y(h)$,

$$z(gh) = z(g) + \rho_k(u(g))z(h) = z(g) + \chi_1(g)^{-1}\rho_k(\mathbf{r}(g))\, z(h),$$

which is the cocycle condition of Proposition 28.2 for $z\colon D \to \Lambda' \otimes X$.

Admissibility of z. Let $b \geq 0$ with $\varpi^b X = 0$, and let E_z be the extension of Proposition 28.2. Put $A = \mathcal{M} \oplus \mathcal{R}^{n_{\mathcal{R},k}}$, with D acting by left multiplication by $\mathbf{r}(g)$ on each summand, a D-stable finitely generated $\mathcal{O}$-submodule of $S = S_0^{1+n_{\mathcal{R},k}}$, and define

$$q\colon A \to E_z/(0 \oplus \varpi^b\mathcal{O}), \qquad q\big(n + c\mathcal{E}, (a_i)_i\big) = \Big(\varphi(n) + \sum_i \rho_k(a_i)e_i \otimes x_0,\ c \bmod \varpi^b\Big)$$

($n \in \mathcal{N}_\mathcal{E}$, $c \in \mathcal{O}$, e_i the standard basis of Λ_k); it is well defined because $\mathcal{M} = \mathcal{N}_\mathcal{E} \oplus \mathcal{O}\mathcal{E}$, and $\mathcal{O}$-linear. It is D-equivariant: on the second summand because ρ_k is multiplicative; on $\mathcal{M}$ because

$$\begin{aligned} q\big(\mathbf{r}(g)(n + c\mathcal{E})\big) &= q\big(\mathbf{r}(g)n + c\chi_1(g)y(g) + c\chi_1(g)\mathcal{E}\big) \\ &= \big(\rho_k(\mathbf{r}(g))\varphi(n) + c\chi_1(g)z(g),\ \chi_1(g)c\big) = g \cdot q(n + c\mathcal{E}), \end{aligned}$$

using $\varphi(\mathbf{r}(g)n) = \rho_k(\mathbf{r}(g))\varphi(n)$ ($\mathcal{R}$-linearity) and the action $g(m, b') = (gm + b'\chi_1(g)z(g), \chi_1(g)b')$ on E_z. It is onto: taking $a_i \in \mathcal{O} \cdot 1 \subseteq \mathcal{R}$ ($1 = \mathbf{r}(1) \in \mathcal{R}$) gives every element $\sum_i \alpha_i e_i \otimes x_0$ of

$\Lambda_k \otimes X$, since $X = \mathcal{O}x_0$, and $c\mathcal{E}$ gives every second coordinate. Hence $E_z/(0 \oplus \varpi^b\mathcal{O}) \cong A/A'$ with $A' = \ker q$ a D-stable submodule containing $\varpi^{b'}A$, $b' = \max(a, b)$, because the target is killed by $\varpi^{b'}$.

It remains to realise A/A' by lattices in S. Let $W = KA \subseteq S$, a D-stable subspace; A and $A' \supseteq \varpi^{b'}A$ are lattices in W. Choose a D-stable $\mathcal{O}$-lattice Λ_S in S (for any lattice Λ°, its stabiliser in the compact group D is open, so $\Lambda_S = \sum_{g\in D} g\Lambda^\circ$ is a finite sum and a D-stable lattice). Since $W \cap \Lambda_S$ is a lattice in W, there is N with $\varpi^N(W \cap \Lambda_S) \subseteq A'$. Put $\Lambda_1 = A + \varpi^N\Lambda_S$ and $\Lambda_2 = A' + \varpi^N\Lambda_S$, D-stable lattices in S with $\Lambda_2 \subseteq \Lambda_1$. By the modular law ($A' \subseteq A$),

$$A \cap \Lambda_2 = A' + (A \cap \varpi^N\Lambda_S) = A',$$

because $A \cap \varpi^N\Lambda_S \subseteq W \cap \varpi^N\Lambda_S = \varpi^N(W \cap \Lambda_S) \subseteq A'$. Hence $\Lambda_1/\Lambda_2 = (A + \Lambda_2)/\Lambda_2 \cong A/(A \cap \Lambda_2) = A/A' \cong E_z/(0 \oplus \varpi^b\mathcal{O})$ as $\mathcal{O}[D]$-modules, and S is a finite direct sum of full representations. As b was arbitrary with $\varpi^b X = 0$, z is admissible (with $a_0 = 0$).

Conclusion, block by block. By Proposition 28.2:

- $k = 0$: by item 1, $z = \sum_j c_j \otimes x_j + (g \mapsto \chi_1(g)^{-1} g\, y_0 - y_0)$ with $[c_j] \in H^1_f(D, T_0(\chi_1^{-1}))$, $x_j \in X$, $y_0 \in T_0 \otimes X$. Here $T_0(\chi_1^{-1}) \otimes K = W_0$ is $K^{n\mathcal{R},0}$ with g acting by $\rho_0(u(g))$, and $\mathrm{Cs}_0(u) = W_0/\Lambda_0$. Under the identification above, the class of $c_j \otimes x_j$ in $H^1(D, \mathrm{Cs}_0(u))$ is the image of $\tilde{x}_j[c_j] \in H^1(D, W_0)$; as the image of $[c_j]$ in $H^1(D, W_0)$ lies in $H^1_f(D, W_0)$, a K-subspace, so does $\tilde{x}_j[c_j]$, and its image lies in $H^1_f(D, W_0/\Lambda_0) = \mathrm{Im}(H^1_f(D, W_0) \to H^1(D, W_0/\Lambda_0))$. The last term is a coboundary of $\mathrm{Cs}_0(u)$. So $[c_\varphi|_D] \in H^1_f(D, \mathrm{Cs}_0(u)) = F_0(v)$ by (Ll2).
- $2 \le k \le t$: by item 2 (with $\Lambda' = \Lambda_k$ a lattice in $K(\chi_k)$), z is a coboundary, so $[c_\varphi|_D] = 0 \in F_k(v)$.
- $k = 1$: $\mathrm{Cs}_1(u)$ is $K/\mathcal{O}$ with trivial action ($\rho_1(u(g)) = 1$ by (Gd1)); by item 3, $z|_{I_v}$ is a coboundary, i.e. $[c_\varphi|_D]$ restricts to 0 on the inertia group, so it lies in $H^1_{\mathrm{ur}}(E_v, \mathrm{Cs}_1(u)) = F_1(v)$.

Since every Ribet class of block k is of the form $[c_\varphi]$, the image of X_k in $H^1(D, \mathrm{Cs}_k(u))/F_k(v)$ is zero for $0 \le k \le t$, and (Gd7) holds at v with $\lambda_k(v) = 0$. □

Proposition 28.2. *Let $\ell \ge 5$ be a prime, K a finite unramified extension of $\mathbb{Q}_\ell$ of degree n with inertia group $I_K \subseteq G_K$ (so that its maximal unramified subfield is $K_0 = K$), L a finite extension of $\mathbb{Q}_\ell$ containing $\sigma(K(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $K(\mu_\ell)$ (in particular $\sigma(K)$ for every embedding σ of K), $\mathcal{O} = \mathcal{O}_L$, λ a uniformiser of $\mathcal{O}$, and $(\kappa_a)_{a\in\mathbb{Z}/n}$ a Frobenius numbering of the embeddings $K_0 \to L$; for an embedding $\tau\colon K \to L$ let $a(\tau)$ be the index with $\tau|_{K_0} = \kappa_{a(\tau)}$, and for a representation V' put $m_{V'}(\tau, j) = \dim_L \mathrm{gr}^j D_{\mathrm{dR},\tau}(V')$. Let $t \ge 1$, let $h, w, \eta\colon \mathbb{Z}/n \to \mathbb{Z}_{\ge 0}$, $\mathrm{typ}\colon \mathbb{Z}/n \to \{\mathrm{low}, \mathrm{high}\}$, let U_a ($a \in \mathbb{Z}/n$) be finite multisets of integers, and put $\mathrm{sg}(a) = \lfloor \sum_{j\ge 1} h(a+j)\,\ell^{-j} \rfloor$, $W_a = [1, \eta_a]$ if $\mathrm{typ}(a) = \mathrm{low}$ and $W_a = [h_a - \eta_a, h_a - 1]$ otherwise, $u' = u$ if $\mathrm{typ}(a) = \mathrm{low}$ and $u' = h_a - u$ otherwise. Assume for every a:*

(Ph1) $1 \le u \le h_a - 1$ *for* $u \in U_a$; $w_a \ge 1$ *and* $w_a + t - 1 \le h_a - 1$;

(Ph2) $w_a + t - 1 \le \eta_a$ *if* $\mathrm{typ}(a) = \mathrm{low}$, *and* $h_a - \eta_a \le w_a$ *otherwise;*

(Ph3) $\eta_a + 4\,\mathrm{sg}(a) + 6 < \ell$;

(Ph4) $|u - y| > \mathrm{sg}(a) + 2$ *for* $u \in U_a \cap W_a$ *and* $w_a \le y \le w_a + t - 1$;

(Ph5) $u' \not\equiv m \pmod{\ell}$ *for* $u \in U_a \setminus W_a$ *and every integer* m *with* $-2\,\mathrm{sg}(a) - 2 \le m \le \eta_a + 2\,\mathrm{sg}(a) + 3$.

Let V be a crystalline L-linear representation of G_K with $m_V(\tau,j)$ the multiplicity of j in $U_{a(\tau)}$, $T_0 \subseteq V$ a G_K-stable $\mathcal{O}$-lattice, and $\chi_1,\dots,\chi_t\colon G_K \to \mathcal{O}^\times$ continuous crystalline characters with $\chi_c = \chi_1\chi_{\mathrm{cyc}}^{1-c}$ and $m_{\chi_c}(\tau,j) = 1$ if $j = w_{a(\tau)} + c - 1$ and 0 otherwise. Call a crystalline L-linear representation V' of G_K full *if $m_{V'}(\tau,j) = m_V(\tau,j) + [\, w_{a(\tau)} \le j \le w_{a(\tau)} + t - 1\,]$ for all τ, j and the characteristic polynomial of g on V' is congruent to $\det(X - g \mid V)\prod_{c=1}^{t}(X - \chi_c(g))$ modulo $\lambda\mathcal{O}[X]$ for every $g \in G_K$. For a G_K-stable $\mathcal{O}$-lattice Λ' in a representation V'', a finite $\mathcal{O}$-module X and a continuous $z\colon G_K \to \Lambda' \otimes_{\mathcal{O}} X$ with $z(gh) = z(g) + \chi_1(g)^{-1} g\, z(h)$, let $E_z = (\Lambda' \otimes X) \oplus \mathcal{O}$ with $g(m,b) = (gm + b\,\chi_1(g) z(g), \chi_1(g) b)$, and call z* admissible *if there is a_0 such that for every $a \ge a_0$ with $\lambda^a X = 0$ the $\mathcal{O}[G_K]$-module $E_z/(0 \oplus \lambda^a\mathcal{O})$ is isomorphic to Λ_1/Λ_2 for G_K-stable $\mathcal{O}$-lattices $\Lambda_2 \subseteq \Lambda_1$ in a finite direct sum of full representations. Then for every finite $\mathcal{O}$-module X:*

1. *a cocycle z as above with $\Lambda' = T_0$ is admissible if and only if $z = \sum_j c_j \otimes x_j + (g \mapsto \chi_1(g)^{-1} g\, y - y)$ with $x_j \in X$, $y \in T_0 \otimes X$ and cocycles $c_j\colon G_K \to T_0$ for $\chi_1^{-1}V$ whose classes lie in $H^1_f(G_K, T_0(\chi_1^{-1}))$;*
2. *for $2 \le c \le t$ and a lattice Λ_c in $L(\chi_c)$, a cocycle z with $\Lambda' = \Lambda_c$ is admissible if and only if it is a coboundary;*
3. *for a lattice Λ_1 in $L(\chi_1)$, a cocycle z with $\Lambda' = \Lambda_1$ is admissible if and only if its restriction to I_K is a coboundary.*

Proof. Write $\varpi = \lambda$, $k = \mathcal{O}/\varpi$, $r = \dim_L V$, and let T' be T_0 with $g \in G_K$ acting by $\chi_1(g)^{-1}g$, a lattice in $V' = V(\chi_1^{-1})$; for a finite $\mathcal{O}$-module X let $T'_X = T_0 \otimes_{\mathcal{O}} X$ with the same twisted action. The cocycles z of the statement (for $\Lambda' = T_0$) are exactly the continuous 1-cocycles of G_K in T'_X, and similarly for Λ_c with the character $\chi_1^{-1}\chi_c = \chi_{\mathrm{cyc}}^{1-c}$. Since T_0 and Λ_c are free over $\mathcal{O}$, $X \mapsto T'_X$ is exact. The field K, unramified of degree n, is the completion of the number field $\mathbb{Q}(\mu_{\ell^n-1})$ at a place above ℓ, so Theorem B.29 applies to G_K; in particular $H^i(G_K, M)$ is finite for finite M of ℓ-power order and vanishes for $i \ge 3$.

Step 1: the cohomology of T'. (a) $H^0(G_K, T'_X) = 0$ for every finite X. By induction on the length of X: if $X \neq 0$ choose $X_1 \subseteq X$ with $X_1 \cong k$; then $0 \to T'_{X_1} \to T'_X \to T'_{X/X_1} \to 0$ is exact, $T'_{X_1} \cong \bar{T}$ has no invariants by (Rc1), and H^0 is left exact. (b) $H^2(G_K, T'_X) = 0$ for every finite X. By Theorem B.29(2), $H^2(G_K, \bar{T})$ is dual to $H^0(G_K, \bar{T}^\vee)$, the G_K-equivariant $\mathbb{F}_\ell$-linear maps $\bar{T} \to \mu_\ell \cong \mathbb{F}_\ell(\bar{\omega})$. Composition with the trace $\mathrm{Tr}_{k/\mathbb{F}_\ell}$ is a bijection from k-linear maps $\bar{T} \to k(\bar{\omega})$ to $\mathbb{F}_\ell$-linear maps $\bar{T} \to \mathbb{F}_\ell(\bar{\omega})$, and φ is G_K-equivariant if and only if $\mathrm{Tr} \circ \varphi$ is: for $m \in \bar{T}$ and $g \in G_K$ the element $\delta = \varphi(gm) - \bar{\omega}(g)\varphi(m)$ satisfies $\mathrm{Tr}(\alpha\delta) = (\mathrm{Tr}\circ\varphi)(g\,\alpha m) - \bar{\omega}(g)(\mathrm{Tr}\circ\varphi)(\alpha m)$ for all $\alpha \in k$, and the trace form is non-degenerate. By (Rc1) there is no non-zero such φ, so $H^2(G_K, \bar{T}) = 0$; since $H^3 = 0$, H^2 is right exact along $0 \to T'_{X_1} \to T'_X \to T'_{X/X_1} \to 0$, and induction on the length gives the claim. (c) By (a), (b) and the local Euler characteristic Theorem B.29(4), $\#H^1(G_K, T'_X) = (\#T'_X)^n = (\#X)^{rn}$. (d) $H^1(G_K, T')$ is a free $\mathcal{O}$-module of rank rn, and for every finite X the map $\Phi_X\colon H^1(G_K, T') \otimes_{\mathcal{O}} X \to H^1(G_K, T'_X)$, $[c] \otimes x \mapsto [c \otimes x]$, is an isomorphism, natural in X. Indeed $H^1(G_K, T')$ is finitely generated over $\mathcal{O}$ (Theorem B.30) and $H^2(G_K, T') \cong \varprojlim_m H^2(G_K, T'/\ell^m T') = 0$ by Theorem B.31 and (b). The sequence $0 \to T' \xrightarrow{\varpi^a} T' \to T'_{\mathcal{O}/\varpi^a} \to 0$ (the quotient is discrete, so it has a continuous section) gives the exact sequence $0 = H^0(T'_{\mathcal{O}/\varpi^a}) \to H^1(T') \xrightarrow{\varpi^a} H^1(T') \to H^1(T'_{\mathcal{O}/\varpi^a}) \to H^2(T') = 0$. So $H^1(T')$ is torsion-free, hence free, and $\Phi_{\mathcal{O}/\varpi^a}$ is an isomorphism; comparing orders with (c) at $a = 1$ gives the rank rn. Both sides of Φ_X commute with finite direct sums, and $X \cong \bigoplus_i \mathcal{O}/\varpi^{a_i}$.

(e) $H^1_f(G_K,T')$ is a direct summand of $H^1(G_K,T')$ of rank d. First $H^0(G_K,V')=0$: a non-zero invariant vector could be scaled into $T'\setminus\varpi T'$ and would give a non-zero invariant of $\bar T$. A continuous cocycle with values in V' has compact image, hence values in $\varpi^{-m}T'$ for some m; and if $c\colon G_K\to T'$ is $g\mapsto gv-v$ with $v\in V'$, then $\varpi^m[c]=0$ in $H^1(T')$ for $\varpi^m v\in T'$, so $[c]=0$ by torsion-freeness. Hence $H^1(T')\otimes_{\mathcal O}L\cong H^1(V')$ with $H^1(T')\subseteq H^1(V')$, and $H^1_f(T')=H^1(T')\cap H^1_f(V')$ is saturated in $H^1(T')$ (the quotient embeds in the L-space $H^1(V')/H^1_f(V')$), so it is a direct summand, of rank $\dim_L H^1_f(V')$. By Theorem B.43, $\dim_{\mathbb Q_\ell}H^1_f(V')=\dim_{\mathbb Q_\ell}H^0(V')+\dim_{\mathbb Q_\ell}D_{\mathrm{dR}}(V')/\mathrm{Fil}^0=[L:\mathbb Q_\ell]\sum_\tau\sum_{j<0}m_{V'}(\tau,j)$, using $D_{\mathrm{dR}}(V')=\bigoplus_\tau D_{\mathrm{dR},\tau}(V')$ as filtered L-spaces. As $D_{\mathrm{dR},\tau}$ is a tensor functor on de Rham representations (Fontaine) and χ_1^{-1} has the single jump $-w_{a(\tau)}$ at τ, $m_{V'}(\tau,j)=m_V(\tau,j+w_{a(\tau)})$; and $\tau\mapsto a(\tau)$ is a bijection onto $\mathbb Z/n$ ($K=K_0$). Hence $\dim_L H^1_f(V')=\sum_a\#\{u\in U_a:\ u<w_a\}=d$. For finite X put $\mathcal H_X=\Phi_X(H^1_f(T')\otimes X)\subseteq H^1(G_K,T'_X)$. By (e), $H^1_f(T')\otimes k\to H^1(T')\otimes k$ is injective, so $\dim_k\mathcal H_k=d$.

Step 2: item 1. A cocycle has the form $\sum_j c_j\otimes x_j+(g\mapsto\chi_1(g)^{-1}gy-y)$ with $[c_j]\in H^1_f(T')$ if and only if its class lies in $\mathcal H_X$. Such cocycles are admissible by Lemma 28.42(4) and (2). Conversely we show by induction on the length of X that the class of an admissible z lies in $\mathcal H_X$. For $X=0$ there is nothing to prove. For $X\cong k$: by Lemma 28.42(2), (3) the classes of admissible cocycles form a k-subspace $\mathcal A$ of $H^1(G_K,T'_X)$, and $\mathcal A\supseteq\mathcal H_X$, of dimension d; if $\dim_k\mathcal A>d$ there would be $d+1$ admissible cocycles with linearly independent classes, contradicting (Rc2) (transported along $X\cong k$ by Lemma 28.42(3)); so $\mathcal A=\mathcal H_X$. In general choose $X_1\subseteq X$ with $X_1\cong k$, let $\iota\colon X_1\to X$, $\pi\colon X\to X_2=X/X_1$. By Lemma 28.42(3), $(1\otimes\pi)\circ z$ is admissible, so by induction its class lies in $\mathcal H_{X_2}$; as $H^1_f(T')\otimes X\to H^1_f(T')\otimes X_2$ is onto and Φ is natural, there is a cocycle z_f with $[z_f]\in\mathcal H_X$ and $(1\otimes\pi)\circ(z-z_f)=g\mapsto\chi_1(g)^{-1}gy_2-y_2$ for some $y_2\in T'_{X_2}$. Lift y_2 to $y\in T'_X$ and put $z'=z-z_f-(g\mapsto\chi_1(g)^{-1}gy-y)$, admissible by Lemma 28.42(2), with $(1\otimes\pi)\circ z'=0$. By exactness $z'=(1\otimes\iota)\circ z_1$ for a continuous cocycle z_1 in T'_{X_1}, admissible by the second half of Lemma 28.42(3); by the case $X_1\cong k$, $[z_1]\in\mathcal H_{X_1}$, hence $[z']\in\mathcal H_X$ by naturality, and $[z]=[z_f]+[z']\in\mathcal H_X$.

Step 3: item 2. Coboundaries are admissible by Lemma 28.42(2). Conversely let z be admissible with values in $\Lambda_c\otimes X$, $2\le c\le t$; induction on the length of X. For $X\cong k$ this is (Rc3), transported by Lemma 28.42(3). In general, with X_1,ι,π as in Step 2, $(1\otimes\pi)\circ z$ is admissible, hence a coboundary of some y_2; lifting y_2 to y, $z'=z-(g\mapsto\chi_1(g)^{-1}gy-y)$ is admissible with $(1\otimes\pi)\circ z'=0$, so $z'=(1\otimes\iota)\circ z_1$ with z_1 admissible, hence a coboundary of some y_1, and z is the coboundary of $y+(1\otimes\iota)(y_1)$.

Step 4: item 3. On Λ_1 the twisted action is trivial, so cocycles are continuous homomorphisms $G_K\to\Lambda_1\otimes X$, coboundaries vanish, and the condition of item 3 says $z|_{I_K}=0$. Homomorphisms trivial on I_K are admissible by Lemma 28.42(5). Conversely, induction on the length of X. For $X\cong k$: the admissible homomorphisms form a k-subspace (Lemma 28.42(2)) containing the space of homomorphisms trivial on I_K, which is 1-dimensional ($G_K/I_K\cong\widehat{\mathbb Z}$, generated by a Frobenius ϕ, and a continuous homomorphism to the finite group k is determined by its arbitrary value at ϕ); by (Rc4) it has dimension at most 1, so the two coincide. In general, with X_1,ι,π as before, $(1\otimes\pi)\circ z$ is admissible, hence trivial on I_K; let z_u be the homomorphism trivial on I_K with $z_u(\phi)$ a lift of $(1\otimes\pi)(z(\phi))$ (it exists as $\Lambda_1\otimes X$ is finite), so that $(1\otimes\pi)\circ z_u=(1\otimes\pi)\circ z$, both being trivial on I_K and equal at ϕ. Then $z-z_u$ is admissible with values in $(1\otimes\iota)(\Lambda_1\otimes X_1)$, so $z-z_u=(1\otimes\iota)\circ z_1$ with z_1 admissible (Lemma 28.42(3)), hence trivial on I_K, and so is $z=z_u+(1\otimes\iota)\circ z_1$. □

Proposition 28.3. *Let ℓ, K, n, L, $\mathcal O$, $(\kappa_a)_{a\in\mathbb Z/n}$, h,w,η,typ, $(U_a)_{a\in\mathbb Z/n}$, V, T_0 and $\chi_1,\dots,\chi_t$ satisfy the hypotheses of Proposition 28.2, with $\lambda=\varpi$ a uniformiser of $\mathcal O$. Let $k=\mathcal O/\varpi\mathcal O$, let*

$\bar T = T_0 \otimes_{\mathcal O} k$ *with* $g \in G_K$ *acting by* $\chi_1(g)^{-1}g$*, let* $\bar\omega\colon G_K \to k^\times$ *be the reduction of* χ_{cyc}*, and put*

$$d = \sum_{a \in \mathbb{Z}/n} \#\{u \in U_a :\ u < w_a\} \qquad (\textit{elements of } U_a \textit{ counted with multiplicity}).$$

Admissibility of cocycles is meant as in Proposition 28.2, with the finite $\mathcal O$*-module* $X = k$*. Then:*

(Rc1) $H^0(G_K, \bar T) = 0$*, and every* G_K*-equivariant* k*-linear map* $\bar T \to k(\bar\omega)$ *is zero;*

(Rc2) *if* $z_1, \dots, z_s\colon G_K \to T_0 \otimes_{\mathcal O} k$ *are admissible cocycles whose classes in* $H^1(G_K, \bar T)$ *are linearly independent over* k*, then* $s \le d$*;*

(Rc3) *for* $2 \le c \le t$ *and a* G_K*-stable* $\mathcal O$*-lattice* Λ_c *in* $L(\chi_c)$*, every admissible cocycle* $z\colon G_K \to \Lambda_c \otimes_{\mathcal O} k$ *is a coboundary;*

(Rc4) *for a* G_K*-stable* $\mathcal O$*-lattice* Λ_1 *in* $L(\chi_1)$*, any two admissible cocycles* $z, z'\colon G_K \to \Lambda_1 \otimes_{\mathcal O} k$ *are linearly dependent over* k*.*

Proof. Let (φ, γ), $\mathcal F$ and residual model data $(\bar T', \omega, \dots)$ for $(h, w, \mathrm{typ}, t, (U_a))$ and $\mathcal F$ be as given by Proposition 28.6, and let $q\colon \bar T \to \bar T'$ be the isomorphism of (Re2) ($\bar T = T_0 \otimes_{\mathcal O} k$ with g acting by $\chi_1(g)^{-1}g$, as in the statement); by (Re1), $\omega = \bar\omega$. The letter data satisfy (Ph1)–(Ph5) by hypothesis. If $\iota\colon A \to A'$ is an isomorphism of finite representations and z is a model-admissible cocycle of A, then $\iota \circ z$ is one of A' (same extension E, with $\alpha \circ \iota^{-1}$ in place of α); we use this transport tacitly.

Step 1: admissible residual cocycles are model-admissible. Let Λ' be T_0 or a G_K-stable lattice Λ_c in $L(\chi_c)$, let A be $\Lambda' \otimes_{\mathcal O} k$ with g acting by $\chi_1(g)^{-1}g$, and let $z\colon G_K \to \Lambda' \otimes_{\mathcal O} k$ be an admissible cocycle. Admissibility (with $X = k$) gives $a \ge 1$ such that $E_z^{(a)} = E_z/(0 \oplus \varpi^a\mathcal O)$ is realisable. Its submodule $0 \oplus \varpi\mathcal O/\varpi^a\mathcal O$ is G_K-stable, since $g(0, b) = (b\chi_1(g)z(g), \chi_1(g)b) = (0, \chi_1(g)b)$ for $b \in \varpi\mathcal O$, as ϖ kills $\Lambda' \otimes k$. The quotient is $E_z^{(1)} = (\Lambda' \otimes k) \oplus k$ with $g(m, \beta) = (gm + \beta\chi_1(g)z(g), \chi_1(g)\beta)$; it is realisable by Lemma 28.42(1) and killed by ϖ. By (Re3), $E = E_z^{(1)}(\chi_1^{-1})$ has a model. In E, $g(m, \beta) = (\chi_1(g)^{-1}gm + \beta z(g), \beta)$; so $\alpha\colon A \to E$, $m \mapsto (m, 0)$, is injective and equivariant, $e = (0, 1) \notin \alpha(A)$, $ge = e + \alpha(z(g))$, and $\dim E = \dim A + 1$. Hence z is a model-admissible cocycle of A.

(Rc1). If $v \in H^0(G_K, \bar T)$ were non-zero, $q(kv)$ would be a subrepresentation of $\bar T'$ with a non-zero equivariant map $q(kv) \to \mathbf 1 = k(\omega^0)$, $q(\beta v) \mapsto \beta$, contradicting Proposition 28.35 (Mc1) with $e = 0$ ($0 < t$). If $f\colon \bar T \to k(\bar\omega)$ is equivariant, then $f \circ q^{-1}\colon \bar T' \to k(\omega^{-(-1)})$ is zero by (Mc1) with $e = -1$, so $f = 0$.

(Rc3). Let $2 \le c \le t$ and write $\Lambda_c = \mathcal O e_c$. On $\Lambda_c \otimes k$, g acts by $\chi_1(g)^{-1}\chi_c(g) = \chi_{\mathrm{cyc}}(g)^{1-c}$ modulo ϖ, i.e. by $\omega(g)^{-(c-1)}$; so $e_c \otimes \beta \mapsto \beta$ is an isomorphism $A \to k(\omega^{-e})$, $e = c - 1$, $1 \le e < t$. By Step 1 and transport, an admissible cocycle z gives a model-admissible cocycle z' of $k(\omega^{-e})$, which by (Mc2) has the form $z'(g) = (\omega(g)^{-e} - 1)y$; then $z(g) = \chi_1(g)^{-1}g(e_c \otimes y) - e_c \otimes y$, a coboundary.

(Rc4). On $\Lambda_1 \otimes k$ the twisted action is trivial, so $A \cong \mathbf 1$. Two admissible cocycles z, z' give, by Step 1 and transport, two model-admissible cocycles of $\mathbf 1$, which are linearly dependent by (Mc3); hence so are z, z'.

(Rc2). Let $z_1, \dots, z_s$ be as in (Rc2). By Step 1, the $q \circ z_i$ are model-admissible cocycles of $\bar T'$. Let $\mathbf N$ and $\zeta_1, \dots, \zeta_s$ be as in Proposition 28.19 for the model data above and the cocycles $q \circ z_i$. The ζ_i are linearly independent modulo $B(\mathcal F(\bar T'))$: if $\sum_i \beta_i\zeta_i \in B(\mathcal F(\bar T'))$, then by (Mp2) $\sum_i \beta_i q(z_i(g)) = gy - y$ for some $y \in \bar T'$, so $\sum_i \beta_i z_i$ is a coboundary of $\bar T$ and all $\beta_i = 0$ by

hypothesis. Now apply Theorem 28.38 to $D = \mathcal{F}(\bar{T}')$, of rank $\dim \bar{T}' = r$, which admits a ψ-operator by Proposition 28.41 ($\ell \geq 5$ and (φ, γ) cyclotomic), with the letter data h, w, typ, the integers $d_a = \#\{u \in U_a : \ u < w_a\}$ and the family $\mathbf{N}$. Its numerical hypotheses hold: $1 \leq w_a$ and $w_a + t - 1 \leq h_a - 1$ by (Ph1), so $0 \leq w_a \leq h_a$; if $\mathrm{typ}(a) = \mathrm{high}$, then $w_a^* = h_a - w_a \geq t \geq 1$ by (Ph1), $w_a^* \leq \eta_a$ by (Ph2), and $\eta_a < \ell - 6$ by (Ph3), so $1 \leq w_a^* \leq \ell - 2$; its sg is the sg of Proposition 28.2. (Mp1) gives the heights (w, w^*), the Γ-condition and (Sn1)–(Sn3), with equalities in place of the inequalities. Hence $s \leq \sum_a d_a = d$. □

28.2 Residual model laws and reduction of Wach modules

To count admissible classes modulo ℓ one needs a model of the residual representation on which the Hodge jumps are still visible. This subsection provides it. It defines residual (φ, Γ)-modules over $k((X))$ with sites, lattices, Sen multiplicities and Herr cocycles (Definition 28.4), and residual model data (Definition 28.5); Proposition 28.6 realises the residual model laws at a large prime. The bridge from characteristic zero is Proposition 28.7: Δ-invariant Wach modules reduce, site by site, into the residual Fontaine functor of mod-ℓ representations [93], with heights and the Γ-condition preserved and with Sen multiplicities equal to the Hodge jumps modulo ℓ (Lemma 28.12, Lemma 28.13). The remaining lemmas compute the killing polynomial and the rank-one case of the reduction, and check the window arithmetic for (Ph1)–(Ph5).

Definition 28.4. Let ℓ be a prime, k a field of characteristic ℓ, $n \geq 1$, and $k[[X]] \subseteq k((X))$ the power series and Laurent series in one variable X.

Residual ring data. A **residual ring datum** over k is a pair (φ, γ) of k-algebra endomorphisms of $k((X))$ with

(Rr1) $\varphi(k[[X]]) \subseteq k[[X]]$ and $\gamma(k[[X]]) \subseteq k[[X]]$;

(Rr2) $\varphi(X) = uX^\ell$ for a unit u of $k[[X]]$, and $\gamma(X) = X(1 - \kappa X)$ for some $\kappa \in k[[X]]$ with $\kappa(0) = 1$;

(Rr3) $\varphi \circ \gamma = \gamma \circ \varphi$.

It is **cyclotomic** if there is a k-algebra homomorphism $\iota\colon k((X)) \to k((\pi))$ (π a further variable) with $\iota(X) = v\pi^{\ell-1}$ for a unit v of $k[[\pi]]$, $\iota \circ \varphi = \varphi_\pi \circ \iota$, $\iota \circ \gamma = \gamma_\pi \circ \iota$, and $\iota(k[[X]]) = \{b \in k[[\pi]] : \ \delta_z(b) = b \text{ for all } z \in \mathbb{Z}_\ell \text{ with } z^{\ell-1} = 1\}$, where $\varphi_\pi, \gamma_\pi, \delta_z$ are the k-algebra endomorphisms of $k((\pi))$, continuous for the π-adic topology, with $\varphi_\pi(\pi) = \pi^\ell$, $\gamma_\pi(\pi) = (1+\pi)^{1+\ell} - 1$ and $\delta_z(\pi) = (1+\pi)^z - 1 = \sum_{i \geq 1} \binom{z}{i} \pi^i$ (binomial coefficients of $z \in \mathbb{Z}_\ell$ reduced modulo ℓ).

Residual (φ, Γ)-modules. A **residual (φ, Γ)-module** D over a residual ring datum (φ, γ), with sites $\mathbb{Z}/n$, is a finite-dimensional $k((X))$-vector space V with k-linear maps $\varphi_a, \gamma_a\colon V \to V$ ($a \in \mathbb{Z}/n$) such that

(Rm1) $\varphi_a(cx) = \varphi(c)\varphi_a(x)$ and $\gamma_a(cx) = \gamma(c)\gamma_a(x)$ for $c \in k((X))$, $x \in V$;

(Rm2) $\varphi_a(V)$ spans V over $k((X))$, and γ_a is bijective;

(Rm3) $\varphi_a \circ \gamma_{a-1} = \gamma_a \circ \varphi_a$.

Its **rank** is $r = \dim_{k((X))} V$.

Lattices and Sen multiplicities. A **lattice** of V is a finitely generated $k[[X]]$-submodule N spanning V over $k((X))$. For $c \in \mathbb{Z}$ put $X^c N = \{X^c x : \ x \in N\}$, and let $\varphi_a^* N$ be the $k[[X]]$-submodule generated by $\varphi_a(N)$ (the **Frobenius pull-back**). N satisfies the **Γ-condition at**

a if $(\gamma_a - 1)(N) \subseteq XN$. Put $\nabla_a = X^{-1}(\gamma_a - 1)$, a k-linear map $V \to V$. For $x \in k$ the **Sen multiplicity** of N at a is

$$\mathrm{sen}_a(N; x) = \dim_k(N^{(x)}/(N^{(x)} \cap XN)), \qquad N^{(x)} = \{v \in N : (\nabla_a - x)^j v \in XN \text{ for some } j \geq 0\},$$

finite since N/XN is finite-dimensional; for $j \in \mathbb{Z}$, $\mathrm{sen}_a(N; j)$ means $\mathrm{sen}_a(N; j \cdot 1_k)$. For functions $w, w' \colon \mathbb{Z}/n \to \mathbb{Z}$, a family $\mathbf{N} = (N_a)_{a \in \mathbb{Z}/n}$ of lattices has **heights** (w, w') if

$$\varphi_a^* N_{a-1} \subseteq X^{-w_a} N_a \qquad \text{and} \qquad X^{w'_a} N_a \subseteq \varphi_a^* N_{a-1} \qquad (a \in \mathbb{Z}/n).$$

ψ-operators. A **ψ-operator** for D is a k-linear map $\psi_0 \colon k((X)) \to k((X))$ together with k-linear maps $\Psi_a \colon V \to V$ $(a \in \mathbb{Z}/n)$ such that

(Ps1) $\Psi_a(c\,\varphi_a(x)) = \psi_0(c)\,x$ for $c \in k((X))$, $x \in V$, and $\psi_0(1) = 1$;

(Ps2) $\psi_0(X^c k[[X]]) \subseteq k[[X]]$ for every integer c with $2 - \ell \leq c \leq 1$;

(Ps3) for every a, every lattice M satisfying the Γ-condition at $a - 1$ and every integer c with $2 - \ell \leq c \leq 0$: every $y \in X^{c+1} \varphi_a^* M$ with $\Psi_a(y) = 0$ equals $\gamma_a(x) - x$ for some $x \in X^c \varphi_a^* M$ with $\Psi_a(x) = 0$.

Herr cocycles and polar pairs. The **Herr cocycles** of D are the pairs $(b, g) \in V^{\mathbb{Z}/n} \times V^{\mathbb{Z}/n}$ with

$$\gamma_a(b_a) - b_a = \varphi_a(g_{a-1}) - g_a \qquad (a \in \mathbb{Z}/n);$$

they form a k-subspace $Z^1(D)$, which contains the **Herr boundaries** $B(D) = \{((\varphi_a(y_{a-1}) - y_a)_a, (\gamma_a(y_a) - y_a)_a) : y \in V^{\mathbb{Z}/n}\}$ by (Rm3). Let $h, w \colon \mathbb{Z}/n \to \mathbb{Z}$ and $\mathrm{typ} \colon \mathbb{Z}/n \to \{\mathrm{low}, \mathrm{high}\}$ (**letter data**), and put $w_a^* = h_a - w_a$. For a family $\mathbf{N} = (N_a)$ of lattices, the space of **polar pairs** $Z_{\mathrm{pol}}(\mathbf{N})$ is the set of $(b, g) \in Z^1(D)$ such that for every a:

(Pp1) $g_a \in XN_a$;

(Pp2) if $\mathrm{typ}(a) = \mathrm{low}$, then $b_a \in X^{-w_a} N_a$;

(Pp3) if $\mathrm{typ}(a) = \mathrm{high}$, then $b_a \in X^{-w_a^*} \varphi_a^* N_{a-1}$, and $g_a - (\gamma_a(x) - x) \in X^{w_a^* + 2} N_a$ for some $x \in N_a$.

$Z_{\mathrm{pol}}(\mathbf{N})$ is a k-subspace of $Z^1(D)$ (for (Pp3), add the elements x).

Definition 28.5. Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ, γ) a residual ring datum over k and $n \geq 1$. A **finite representation** is a finite-dimensional k-vector space with a k-linear action of G; $\mathbf{1}$ is the trivial one, and for a character $\theta \colon G \to k^\times$, $k(\theta)$ is k with g acting by $\theta(g)$. A k-linear map between finite representations is **equivariant** if it commutes with G.

Residual model functors. A **residual model functor** $\mathcal{F}$ (for G, over (φ, γ), with sites $\mathbb{Z}/n$) consists of:

(Mf1) a class of finite representations, called **$\mathcal{F}$-objects**, containing $\mathbf{1}$, such that: if B is an $\mathcal{F}$-object and $A \to B$ is injective equivariant, A is an $\mathcal{F}$-object; if A is an $\mathcal{F}$-object and $A \to B$ is surjective equivariant, B is an $\mathcal{F}$-object; if A, B are $\mathcal{F}$-objects and $A \oplus B \to C$ is surjective equivariant, C is an $\mathcal{F}$-object;

(Mf2) for every finite representation A, a residual (φ, Γ)-module $\mathcal{F}(A) = (V_A, (\varphi_a), (\gamma_a))$ over (φ, γ) with sites $\mathbb{Z}/n$ and rank $\dim_k A$;

(Mf3) for every k-linear map $f\colon A \to B$ and $a \in \mathbb{Z}/n$, a $k((X))$-linear map $\mathcal{F}(f)_a\colon V_A \to V_B$, such that for $\mathcal{F}$-objects A, B, C and equivariant $f, f'\colon A \to B$, $f''\colon B \to C$, $c \in k$: $\mathcal{F}(f)_a \circ \varphi_a = \varphi_a \circ \mathcal{F}(f)_{a-1}$, $\mathcal{F}(f)_a \circ \gamma_a = \gamma_a \circ \mathcal{F}(f)_a$, $\mathcal{F}(\mathrm{id}_A)_a = \mathrm{id}$, $\mathcal{F}(f'' \circ f)_a = \mathcal{F}(f'')_a \circ \mathcal{F}(f)_a$, $\mathcal{F}(f + cf')_a = \mathcal{F}(f)_a + c\mathcal{F}(f')_a$, and $\mathcal{F}(f)_a$ is injective (surjective) if f is;

(Mf4) non-zero elements $u_a \in V_{\mathbf{1}}$ $(a \in \mathbb{Z}/n)$ with $\varphi_a(u_{a-1}) = u_a$ and $\gamma_a(u_a) = u_a$, such that for every $\mathcal{F}$-object A and every $(x_a) \in V_A^{\mathbb{Z}/n}$ with $\varphi_a(x_{a-1}) = x_a$ and $\gamma_a(x_a) = x_a$ for all a there is an equivariant $f\colon \mathbf{1} \to A$ with $\mathcal{F}(f)_a(u_a) = x_a$ for all a.

For $f\colon A \to B$ and families $\mathbf{N} = (N_a)$, $\mathbf{N}' = (N'_a)$ of $k[[X]]$-submodules of V_A, resp. V_B, put $f_*\mathbf{N} = (\mathcal{F}(f)_a(N_a))_a$ and $f^*\mathbf{N}' = (\mathcal{F}(f)_a^{-1}(N'_a))_a$.

Residual model data. Let h, w, typ be letter data, $t \geq 1$, and $(U_a)_{a\in\mathbb{Z}/n}$ finite multisets of integers, all of cardinality r; put $\widetilde{U}_a = U_a + \{w_a, w_a + 1, \ldots, w_a + t - 1\}$ (sum of multisets). **Residual model data** for $(h, w, \mathrm{typ}, t, (U_a))$ and $\mathcal{F}$ consist of a finite representation $\bar{T}$ of dimension r, a character $\omega\colon G \to k^\times$, a class of pairs $(A, \mathbf{N})$, A a finite representation and $\mathbf{N}$ a family of lattices of V_A, called **base pairs**, families of lattices $\mathbf{N}^{\mathrm{std}}$ of $V_{\bar{T}}$ and $\mathbf{N}_e^\chi$ of $V_{k(\omega^{-e})}$ $(-1 \leq e < t)$, such that, calling $(A, \mathbf{N})$ a **model** if $\mathbf{N} = p_*j^*\mathbf{N}_B$ for a base pair $(B, \mathbf{N}_B)$, an injective equivariant $j\colon A_1 \to B$ and a surjective equivariant $p\colon A_1 \to A$:

(Md1) $\omega^e \neq 1$ for $0 < e < \ell - 1$;

(Md2) for every base pair $(A, \mathbf{N})$: A is an $\mathcal{F}$-object, $\mathbf{N}$ has heights (w, w^*) and every N_a satisfies the Γ-condition at a;

(Md3) for every base pair $(A, \mathbf{N})$ there is $\mu \geq 0$ such that A has the same Jordan–Hölder factors, with multiplicities, as $\big(\bar{T} \oplus \bigoplus_{e=0}^{t-1} k(\omega^{-e})\big)^{\oplus\mu}$, and $\mathrm{sen}_a(N_a; x) = \mu \cdot \#\{u \in \widetilde{U}_a : (u - w_a) \cdot 1_k = x\}$ for all a and $x \in k$;

(Md4) for base pairs $(A, \mathbf{N})$, $(B, \mathbf{N}')$ there are a finite representation C and equivariant $i\colon A \to C$, $j\colon B \to C$ with $i \oplus j\colon A \oplus B \to C$ bijective such that $(C, (\mathcal{F}(i)_a(N_a) + \mathcal{F}(j)_a(N'_a))_a)$ is a base pair;

(Md5) for every base pair $(A, \mathbf{N})$ and every a with $\mathrm{typ}(a) = \mathrm{high}$ there is $f \in k[T]$ with $f(0) \neq 0$ and $\nabla_a f(\nabla_a)(N_a) \subseteq X^{w_a^*+1}N_a$;

(Md6) $(\bar{T}, \mathbf{N}^{\mathrm{std}})$ is a model and $\mathrm{sen}_a(N_a^{\mathrm{std}}; x) = \#\{u \in U_a :\ (u - w_a) \cdot 1_k = x\}$ for all a, x;

(Md7) for $-1 \leq e < t$: $k(\omega^{-e})$ is an $\mathcal{F}$-object, and each lattice $N_{e,a}^\chi$ satisfies the Γ-condition at a, $\varphi_a^* N_{e,a-1}^\chi = X^e N_{e,a}^\chi$, and $\mathrm{sen}_a(N_{e,a}^\chi; x) = 1$ if $x = e \cdot 1_k$ and 0 otherwise;

(Md8) for $0 \leq e < t$, $(k(\omega^{-e}), \mathbf{N}_e^\chi)$ is a model, and $N_{0,a}^\chi = k[[X]]u_a$ for all a.

For a finite representation A, a map $z\colon G \to A$ is a **model-admissible cocycle** of A if there are a finite representation E of dimension $\dim A + 1$, an injective equivariant $\alpha\colon A \to E$ and $e \in E \setminus \alpha(A)$ with $ge = e + \alpha(z(g))$ for all $g \in G$, such that $(E, \mathbf{N})$ is a model for some family $\mathbf{N}$.

Proposition 28.6. *Let ℓ, K, n, L, $\mathcal{O}$, $(\kappa_a)_{a\in\mathbb{Z}/n}$, h, w, η, typ, $(U_a)_{a\in\mathbb{Z}/n}$, V, T_0 and $\chi_1, \ldots, \chi_t$ satisfy the hypotheses of Proposition 28.2 (in particular L contains $\sigma(K(\mu_\ell))$ for every $\mathbb{Q}_\ell$-embedding σ of $K(\mu_\ell)$), with $\lambda = \varpi$ a uniformiser of $\mathcal{O}$; let $k = \mathcal{O}/\varpi\mathcal{O}$ and $r = \dim_L V$. For an $\mathcal{O}[G_K]$-module M with $\varpi M = 0$ let $M(\chi_1^{-1})$ be the k-vector space M with $g \in G_K$ acting by $\chi_1(g)^{-1}g$. Then there are a cyclotomic residual ring datum (φ, γ) over k, a residual model functor $\mathcal{F}$ for G_K over (φ, γ) with sites $\mathbb{Z}/n$, and residual model data $(\bar{T}', \omega, \ldots)$ for $(h, w, \mathrm{typ}, t, (U_a))$ and $\mathcal{F}$, such that:*

(Re1) $\omega(g) = \chi_{\mathrm{cyc}}(g) \bmod \varpi$ *for* $g \in G_K$*;*

(Re2) *there is a* k*-linear isomorphism* $(T_0 \otimes_{\mathcal{O}} k)(\chi_1^{-1}) \to \bar{T}'$ *commuting with* G_K*;*

(Re3) *for every realisable* $\mathcal{O}[G_K]$*-module* M *with* $\varpi M = 0$ *there is a family* $\mathbf{N}$ *of lattices of* $V_{M(\chi_1^{-1})}$ *such that* $(M(\chi_1^{-1}), \mathbf{N})$ *is a model.*

Proof. **Set-up.** Since K is unramified, $K_0 = K$, the embeddings $K \to L$ are the κ_a, and $a(\kappa_a) = a$. Apply Proposition 28.7 with $F = K$, the Frobenius numbering $\tau_a = \kappa_a$ and the uniformiser ϖ: its hypotheses hold, as $\ell \geq 5$ is odd and L contains $\sigma(K(\mu_\ell))$ for every embedding σ (hypothesis of Proposition 28.2). This gives the cyclotomic residual ring datum (φ, γ) over k of Lemma 28.9, the residual model functor $\mathcal{F}$ for G_K with sites $\mathbb{Z}/n$ of Proposition 28.10, and for every pair (S, Λ) the family $\bar{N}(\Lambda) = (\bar{N}_a(\Lambda))_a$ of lattices of $V_{\bar{\Lambda}}$ ((Rd3)); $\bar{\Lambda}$ is a continuous finite representation of G_K over k, hence an $\mathcal{F}$-object (Ff1). For a G_K-stable $\mathcal{O}$-lattice Λ in a representation S write $\Lambda(\chi_1^{-1})$, $S(\chi_1^{-1})$ for the same modules with g acting by $\chi_1(g)^{-1}g$; for $e \geq -1$ put $\psi_e = \chi_{\mathrm{cyc}}^{-e}$, with values in $\mathbb{Z}_\ell^\times \subseteq \mathcal{O}^\times$, and let $\mathcal{O}(\psi_e)$ be $\mathcal{O}$ with g acting by $\psi_e(g)$, a lattice in $L(\psi_e)$.

Hodge jumps. For crystalline S and a crystalline character ψ whose only jump at κ_a is c_a, $S \otimes \psi$ is crystalline and $m_{S\otimes\psi}(\kappa_a, j) = m_S(\kappa_a, j - c_a)$, because $D_{\mathrm{dR},\tau}$ is a tensor functor on de Rham representations (Fontaine). The only jump of χ_1 at κ_a is w_a, so that of χ_1^{-1} is $-w_a$; χ_{cyc} has the single jump -1 at every embedding ($D_{\mathrm{dR}}(\mathbb{Q}_\ell(1))$ is spanned by $t^{-1} \otimes \varepsilon$; Hodge–Tate weight 1 in the normalisation fixed in Section 13), so ψ_e has the single jump e, and $\chi_c\chi_1^{-1} = \chi_{\mathrm{cyc}}^{1-c} = \psi_{c-1}$. A full representation V' has, at κ_a, the multiset of jumps $\widetilde{U}_a$; by (Ph1), $\widetilde{U}_a \subseteq [1, h_a - 1]$. Hence for a finite direct sum S of μ full representations, $S(\chi_1^{-1})$ is crystalline with multiset of jumps $\mu \cdot (\widetilde{U}_a - w_a)$ at κ_a, contained in $[1 - w_a, h_a - 1 - w_a] \subseteq [-w_a, w_a^*]$; and $V(\chi_1^{-1})$ has the multiset $U_a - w_a$.

The data. Put $\bar{T}' = T_0(\chi_1^{-1})/\varpi T_0(\chi_1^{-1})$, of dimension r, and $\omega(g) = \chi_{\mathrm{cyc}}(g) \bmod \varpi \in \mathbb{F}_\ell^\times \subseteq k^\times$. The base pairs are the pairs $(\bar{\Lambda}', \bar{N}(\Lambda'))$ with $\Lambda' = \Lambda(\chi_1^{-1})$, where Λ runs over the G_K-stable $\mathcal{O}$-lattices in finite direct sums S of full representations (by the previous paragraph $(S(\chi_1^{-1}), \Lambda')$ is a pair). Let $V^+ = V \oplus L(\chi_1) \oplus \cdots \oplus L(\chi_t)$ and $\Lambda^+ = T_0 \oplus \mathcal{O}(\chi_1) \oplus \cdots \oplus \mathcal{O}(\chi_t)$. V^+ is full: it is crystalline, $m_{V^+}(\tau, j) = m_V(\tau, j) + \sum_c [j = w_{a(\tau)} + c - 1]$, and its characteristic polynomials are exactly $\det(X - g \,|\, V) \prod_c (X - \chi_c(g))$. Moreover $\Lambda^+(\chi_1^{-1}) = T_0(\chi_1^{-1}) \oplus \mathcal{O}(\psi_0) \oplus \cdots \oplus \mathcal{O}(\psi_{t-1})$. Put $\mathbf{N}^{\mathrm{std}} = \bar{N}(T_0(\chi_1^{-1}))$, and, for $-1 \leq e < t$, $\mathbf{N}_e^\chi = \bar{N}(\mathcal{O}(\psi_e))$, a family of lattices of $V_{k(\omega^{-e})}$ since $\mathcal{O}(\psi_e)/\varpi = k(\omega^{-e})$.

A summand gives a model. Let $(B, \mathbf{N}_B)$ be a base pair from $\Lambda' = P \oplus P'$ (a decomposition into G_K-stable submodules) and $j\colon \bar{P} \to B$ the reduction of the inclusion. By Lemma 28.11, $\mathcal{F}(j)_a$ is injective, $V_B = \mathcal{F}(j)_a(V_{\bar{P}}) \oplus \mathcal{F}(j')_a(V_{\bar{P}'})$ and $N_{B,a} = \mathcal{F}(j)_a(\bar{N}_a(P)) + \mathcal{F}(j')_a(\bar{N}_a(P'))$; hence $\mathcal{F}(j)_a^{-1}(N_{B,a}) = \bar{N}_a(P)$, i.e. $\bar{N}(P) = \mathrm{id}_* j^* \mathbf{N}_B$, and $(\bar{P}, \bar{N}(P))$ is a model (j is injective as P is a direct summand).

(Md1), (Re1), (Re2). (Re1) holds by definition. As $K/\mathbb{Q}_\ell$ is unramified and $\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell$ is totally ramified of degree $\ell - 1$, $K \cap \mathbb{Q}_\ell(\mu_\ell) = \mathbb{Q}_\ell$, so $\chi_{\mathrm{cyc}} \bmod \ell$ maps G_K onto $\mathbb{F}_\ell^\times$; ω has order $\ell - 1$, which is (Md1). The identity of T_0 induces a k-linear isomorphism $(T_0 \otimes_{\mathcal{O}} k)(\chi_1^{-1}) \to \bar{T}'$ commuting with G_K, both sides carrying the action $g \mapsto \chi_1(g)^{-1}g$: (Re2).

(Md2). Let $(\bar{\Lambda}', \bar{N}(\Lambda'))$ be a base pair. $\bar{\Lambda}'$ is an $\mathcal{F}$-object. The jumps of $S(\chi_1^{-1})$ at κ_a lie in $[-w_a, w_a^*]$, with $w_a \geq 1$ and $w_a^* = h_a - w_a \geq t \geq 1$ by (Ph1); Lemma 28.12 with $(w, w') = (w, w^*)$ gives heights (w, w^*) and the Γ-condition at every site.

(Md3). Let $S = V_1' \oplus \cdots \oplus V_\mu'$ with V_i' full. For $g \in G_K$ and $c = \chi_1(g)^{-1} \in \mathcal{O}^\times$, $\det(X - cg \,|\, V_i') = c^{\dim V_i'} \det(c^{-1}X - g \,|\, V_i')$, so fullness gives $\det(X - \chi_1(g)^{-1}g \,|\, V_i') \equiv \det(X -$

$\chi_1(g)^{-1}g\,|\,V)\prod_{c'=1}^{t}(X-\psi_{c'-1}(g))$ (mod $\varpi\mathcal{O}[X]$). The characteristic polynomial of g on $\bar\Lambda'$ (resp. $\bar T'$) is the reduction of that on the $\mathcal{O}$-lattice Λ' (resp. $T_0(\chi_1^{-1})$), and $\psi_e \bmod \varpi = \omega^{-e}$. Hence every g has the same characteristic polynomial on $\bar\Lambda'$ as on $(\bar T' \oplus \bigoplus_{e=0}^{t-1} k(\omega^{-e}))^{\oplus\mu}$, and by the Brauer–Nesbitt theorem (Curtis–Reiner, Representation theory of finite groups and associative algebras, (30.16)) the two have the same Jordan–Hölder factors with multiplicities. By Lemma 28.13 applied to the jumps $\mu\cdot(\widetilde U_a - w_a)$, $\mathrm{sen}_a(\bar N_a(\Lambda');x) = \mu\cdot\#\{u\in\widetilde U_a:\ (u-w_a)\cdot 1_k = x\}$.

(Md4). For base pairs from $\Lambda_1\subseteq S_1$, $\Lambda_2\subseteq S_2$, the lattice $\Lambda_1\oplus\Lambda_2$ of the finite direct sum $S_1\oplus S_2$ of full representations gives the base pair $(C,\bar N(\Lambda_1'\oplus\Lambda_2'))$, $C=\overline{\Lambda_1'\oplus\Lambda_2'}$, and Lemma 28.11 gives equivariant i,j with $i\oplus j$ bijective and $\bar N_a(\Lambda_1'\oplus\Lambda_2') = \mathcal{F}(i)_a(\bar N_a(\Lambda_1')) + \mathcal{F}(j)_a(\bar N_a(\Lambda_2'))$.

(Md5). Let $\mathrm{typ}(a)=\mathrm{high}$ and $s = w_a^*+1\geq 1$. We check the hypothesis of Lemma 28.16 for the jumps $y=u-w_a$, $u\in\widetilde U_a$, of $S(\chi_1^{-1})$: if $0\leq b<s$ and $y\equiv b \pmod\ell$, then $u\equiv b+w_a$ with $w_a\leq b+w_a\leq h_a$. For $u\in U_a$, (Wa3) of Lemma 28.18 excludes $b+w_a=h_a$ and gives $u=b+w_a$ otherwise. For $u\in\{w_a,\dots,w_a+t-1\}\subseteq[w_a,h_a-1]$, both u and $b+w_a$ lie in $[w_a,h_a]$, so $|u-b-w_a|\leq h_a-w_a\leq\eta_a<\ell$ by (Ph2) (high type) and (Ph3), hence $u=b+w_a$. In both cases $y=b$, and Lemma 28.16 gives $f\in k[T]$ with $f(0)\neq 0$ and $\nabla_a f(\nabla_a)(\bar N_a(\Lambda'))\subseteq X^{w_a^*+1}\bar N_a(\Lambda')$.

(Md6). $(\bar T',\mathbf{N}^{\mathrm{std}})$ is a model by the summand argument for the base pair of Λ^+ ($\mu=1$) with $P=T_0(\chi_1^{-1})$. By Lemma 28.13 with the jumps U_a-w_a of $V(\chi_1^{-1})$, $\mathrm{sen}_a(N_a^{\mathrm{std}};x)=\#\{u\in U_a:\ (u-w_a)\cdot 1_k=x\}$.

(Md7), (Md8). Let $-1\leq e<t$. $k(\omega^{-e})$ is an $\mathcal{F}$-object (Ff1). The pair $(L(\psi_e),\mathcal{O}(\psi_e))$ has dimension 1 and the single jump e at every κ_a, so its jumps lie in $[-1,t-1]$ and Lemma 28.12 with the constant functions 1 and $t-1$ gives the Γ-condition at every site, Lemma 28.17 gives $\varphi_a^* N^\chi_{e,a-1} = X^e N^\chi_{e,a}$, and Lemma 28.13 gives $\mathrm{sen}_a(N^\chi_{e,a};x)=1$ if $x=e\cdot 1_k$ and 0 otherwise. For $0\leq e<t$, $\mathcal{O}(\psi_e)$ is a direct summand of $\Lambda^+(\chi_1^{-1})$, so $(k(\omega^{-e}),\mathbf{N}^\chi_e)$ is a model by the summand argument. Finally $\mathcal{O}(\psi_0)$ is the trivial pair $(L,\mathcal{O})$, and (Rd5) gives $N^\chi_{0,a}=k[[X]]u_a$.

(Re3). Let M be realisable with $\varpi M=0$: $M\cong\Lambda_1/\Lambda_2$ as $\mathcal{O}[G_K]$-modules for G_K-stable lattices $\Lambda_2\subseteq\Lambda_1$ in a finite direct sum S of full representations, and $\varpi\Lambda_1\subseteq\Lambda_2$. Twisting, $\Lambda_1(\chi_1^{-1})\to M(\chi_1^{-1})$ factors through a surjective equivariant k-linear map $p\colon\bar\Lambda_1'\to M(\chi_1^{-1})$. With the base pair $(\bar\Lambda_1',\bar N(\Lambda_1'))$ and $j=\mathrm{id}$, $\mathbf{N}=p_*\bar N(\Lambda_1')$ is a family of lattices of $V_{M(\chi_1^{-1})}$ ($\mathcal{F}(p)_a$ is surjective by (Mf3), and images of finitely generated spanning submodules are finitely generated and spanning), and $(M(\chi_1^{-1}),\mathbf{N})$ is a model. □

Proposition 28.7. *Let ℓ, F, n, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ be as in Proposition 23.7, ϖ a uniformiser of $\mathcal{O}_L$ and $k=\mathcal{O}_L/\varpi$. Let (φ,γ), ι be the cyclotomic residual ring datum over k of Lemma 28.9, and $\mathcal{F}$ the residual model functor for G_F of Proposition 28.10 formed with $K=F$, this datum, γ_0 with $\chi_{\mathrm{cyc}}(\gamma_0)=1+\ell$, and the embeddings $\bar\kappa_a\colon k_F\to k$ induced by τ_{a+1} on residue fields. For $g\in e_a\mathbf{A}_{F,L}=\mathcal{O}_L[[X]]$ fixed by γ_δ for all $\delta\in\mu_{\ell-1}$, let $\bar g\in k[[X]]$ be the element with $\iota(\bar g)$ equal to the reduction of g (coefficients modulo ϖ, $X\mapsto\pi$). For a pair (V,T) let $\bar T=T/\varpi T$ and $\mathbf{N}_a^\Delta(T)$ the set of elements of $\mathbf{N}(T)_a$ fixed by γ_δ for all $\delta\in\mu_{\ell-1}$. Then for every pair (V,T) and every $a\in\mathbb{Z}/n$ there is an additive map $\rho_{T,a}\colon\mathbf{N}_a^\Delta(T)\to V_{\bar T}$ such that:*

(Rd1) $\rho_{T,a}(gx)=\bar g\,\rho_{T,a}(x)$ *for Δ-invariant $g\in e_a\mathbf{A}_{F,L}$, and $\rho_{T,a}(\gamma_{1+\ell}x)=\gamma_a(\rho_{T,a}(x))$;*

(Rd2) *if $y\in\mathbf{N}_{a-1}^\Delta(T)$, $j\geq 0$ and $(\pi_0+\ell)^j\varphi_{\mathbf{N}}(y)\in\mathbf{N}(T)_a$, then $\rho_{T,a}((\pi_0+\ell)^j\varphi_{\mathbf{N}}(y)) = X^j\varphi_a(\rho_{T,a-1}(y))$;*

(Rd3) *the image $\bar N_a(T)=\rho_{T,a}(\mathbf{N}_a^\Delta(T))$ is a lattice of $V_{\bar T}$, and the kernel of $\rho_{T,a}$ is $\varpi\mathbf{N}_a^\Delta(T)$;*

(Rd4) *for every pair (V',T') and every G_F-equivariant $\mathcal{O}_L$-linear $f\colon T\to T'$, with $\bar f\colon\bar T\to\bar T'$ its reduction, $\rho_{T',a}\circ\mathbf{N}(f)=\mathcal{F}(\bar f)_a\circ\rho_{T,a}$ on $\mathbf{N}_a^\Delta(T)$;*

(Rd5) *for the trivial pair* $(L,\mathcal{O}_L)$, $\bar N_a(\mathcal{O}_L)=k[[X]]\,u_a$.

Proof. Write $W=W(\mathbb{C}_\ell^\flat)$, $R=e_a\mathbf{A}_{F,L}\cong\mathcal{O}_L[[X]]$, and R^Δ for its Δ-invariant elements. Then $\pi_0\in R^\Delta$, and $\gamma_{u,\mathbf{N}}$ commutes with $\varphi_{\mathbf{N}}$ (Wm3), so $\varphi_{\mathbf{N}}$ maps Δ-invariant elements to Δ-invariant elements.

The functor. The embeddings $\bar\kappa_a\colon k_F\to k$ induced by τ_{a+1} are distinct ($F/\mathbb{Q}_\ell$ is unramified) and satisfy $\bar\kappa_{a+1}(\bar y^\ell)=\overline{\tau_{a+2}(\varphi(y))}=\overline{\tau_{a+1}(y)}=\bar\kappa_a(\bar y)$, since $\tau_{b+1}\circ\varphi=\tau_b$ and φ lifts $\bar y\mapsto\bar y^\ell$. So Proposition 28.10 applies (k is finite, ℓ odd) and gives $\mathcal{F}$ with the maps $\theta_{A,a}$ of (Ff2), formed with the idempotents $e_a\in k_F\otimes_{\mathbb{F}_\ell}k$ of the factors $\bar y\otimes\lambda\mapsto\bar\kappa_a(\bar y)\lambda$.

Reduction. For a pair (V,T) let $r_T\colon W\otimes_{\mathbb{Z}_\ell}T\to\mathbb{C}_\ell^\flat\otimes_{\mathbb{F}_\ell}\bar T$, $z\otimes t\mapsto\bar z\otimes\bar t$; it is additive, G_F-equivariant, $r_T((\varphi\otimes1)y)=(\varphi\otimes1)r_T(y)$ with $\varphi(\bar z)=\bar z^\ell$, and $r_T((z\otimes\lambda)y)=(\bar z\otimes\bar\lambda)r_T(y)$ for $z\in W$, $\lambda\in\mathcal{O}_L$ acting on T. By (Fi1) it maps $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}\otimes T$ onto $\mathbf{E}^{\mathrm{sep}}\otimes_{\mathbb{F}_\ell}\bar T=\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}\otimes_{\mathbb{Z}_\ell}\bar T$, and on H_F-invariants it is the map $\mathbf{D}(T)\to\mathbf{D}(\bar T)$ induced by $T\to\bar T$; by the exactness in (Fi2), applied to $0\to T\xrightarrow{\varpi}T\to\bar T\to0$, this map is onto with kernel $\varpi\mathbf{D}(T)$, and $\mathbf{D}(\bar T)$ is the module of Theorem B.52 used in Proposition 28.10, with k acting through $\bar T$.

Coefficients and sites. Let $\mathcal{O}_F\otimes\mathcal{O}_L$ act on $W\otimes T$ through $\iota_{\mathrm W}$ on $\mathcal{O}_F$ and the action on T, so that $\iota_T(gx)=(\iota_{\mathrm W}\otimes1)(g)\,\iota_T(x)$ for $g\in\mathbf{A}_{F,L}$ (Ex2). For $x\otimes\lambda\in\mathcal{O}_F\otimes\mathcal{O}_L$, $\iota_{\mathrm W}(x)=w(\varphi(x))$ (Lemma 23.8(4)) reduces to $\bar x^\ell$, so $r_T((\iota_{\mathrm W}\otimes1)(x\otimes\lambda)y)=(\bar x^\ell\otimes\bar\lambda)r_T(y)$, and under the factor of index b of $k_F\otimes k$, $\bar x^\ell\otimes\bar\lambda\mapsto\bar\kappa_b(\bar x^\ell)\bar\lambda=\overline{\tau_b(x)\lambda}$. Hence the site idempotent e_a of $\mathcal{O}_F\otimes\mathcal{O}_L$ reduces to the idempotent e_a of $k_F\otimes k$, and for $g=\sum_m g_mX^m\in R$ the element $r_T((\iota_{\mathrm W}\otimes1)(g)\,\cdot)$ is multiplication by $\sum_m\bar g_m\bar\pi^m$ on the e_a-component, where $\bar g_m\in k$ is $g_m\in\mathcal{O}_L$ modulo ϖ (reduction $A_{\mathrm{inf}}\to\mathcal{O}_{\mathbb{C}_\ell^\flat}$ is continuous and $\iota_{\mathrm W}(X)=\pi\mapsto\bar\pi$); in the identification $e_a(\mathbf{E}_F\otimes k)\cong k((\pi))$ of Proposition 28.10 this is the reduction of g, which for Δ-invariant g is $\iota(\bar g)$.

Definition of $\rho_{T,a}$. Let $x\in\mathbf{N}_a^\Delta(T)$. By Lemma 28.8, $\iota_T(x)=\Psi_T(1\otimes x)\in\mathbf{D}(T)$, so $y=r_T(\iota_T(x))\in\mathbf{D}(\bar T)$; $y=e_ay$ by the previous paragraph, and y is fixed by Δ_F: for $\sigma\in G_F$ with $\chi_{\mathrm{cyc}}(\sigma)=\delta\in\mu_{\ell-1}$, $\sigma\iota_T(x)=\iota_T(\gamma_\delta x)=\iota_T(x)$ by (Ex2). By (Ff2) there is a unique $\rho_{T,a}(x)\in V_{\bar T}$ with $\theta_{\bar T,a}(\rho_{T,a}(x))=r_T(\iota_T(x))$; $\rho_{T,a}$ is additive.

(Rd1). For Δ-invariant $g\in R$, $r_T(\iota_T(gx))=\iota(\bar g)\,r_T(\iota_T(x))$ by the computation above, and $\theta_{\bar T,a}(\bar g\rho_{T,a}(x))=\iota(\bar g)\theta_{\bar T,a}(\rho_{T,a}(x))$ (Ff2); injectivity of $\theta_{\bar T,a}$ gives $\rho_{T,a}(gx)=\bar g\rho_{T,a}(x)$. For $\sigma\in G_F$ with $\chi_{\mathrm{cyc}}(\sigma)=1+\ell$, whose image in Γ_F is γ_0, $r_T(\iota_T(\gamma_{1+\ell}x))=r_T(\sigma\iota_T(x))=\gamma_0\,r_T(\iota_T(x))=\gamma_0\theta_{\bar T,a}(\rho_{T,a}(x))=\theta_{\bar T,a}(\gamma_a\rho_{T,a}(x))$ by (Ex2) and (Ff2).

(Rd2). Let $q=\pi_0+\ell$ and $x'=q^j\varphi_{\mathbf{N}}(y)\in\mathbf{N}(T)_a$, which is Δ-invariant. In $W\otimes T$, $\iota_T(x')=\iota_{\mathrm W}(q)^j(\varphi\otimes1)(\iota_T(y))$ (Ex2). As $\pi_0=\sum_{\delta\in\mu_{\ell-1}}\big((1+X)^\delta-1\big)$, $\iota_{\mathrm W}(q)$ reduces to $\sum_\delta((1+\bar\pi)^\delta-1)$, which in $k((\pi))$ is $X_0=\iota(X)$ (Lemma 28.9). Hence $r_T(\iota_T(x'))=\iota(X)^j(\varphi\otimes1)\theta_{\bar T,a-1}(\rho_{T,a-1}(y))=\iota(X^j)\theta_{\bar T,a}(\varphi_a\rho_{T,a-1}(y))=\theta_{\bar T,a}(X^j\varphi_a(\rho_{T,a-1}(y)))$ by (Ff2), which is (Rd2).

(Rd3). By Lemma 24.15 (applicable by (Ex1)), $\mathbf{N}(T)_a$ has an R-basis $(n_k)_{k\le r}$ of Δ-invariant vectors, $r=\dim_L V$; an element $\sum_k g_kn_k$ is Δ-invariant iff every g_k is, so $\mathbf{N}_a^\Delta(T)=\bigoplus_k R^\Delta n_k$. By Lemma 28.8, Ψ_T identifies $\mathcal{O}_{\mathcal{E}_F}\otimes_{\mathbf{A}_F}\mathbf{N}(T)_a$ with the e_a-component of $\mathbf{D}(T)$ (with e_a acting through $\iota_{\mathrm W}\otimes1$), free over its coefficient ring with basis $(\iota_T(n_k))_k$; reducing modulo ϖ, the $r_T(\iota_T(n_k))$ form a $k((\pi))$-basis of $e_a\mathbf{D}(\bar T)$. They are Δ_F-invariant, hence lie in $\theta_{\bar T,a}(V_{\bar T})$ and are $\iota(k((X)))$-linearly independent; as $V_{\bar T}$ has dimension $\dim_k\bar T=r$ over $k((X))$ (Mf2), the $\rho_{T,a}(n_k)$ form a $k((X))$-basis of $V_{\bar T}$. The map $R^\Delta\to k[[X]]$, $g\mapsto\bar g$, is onto: $\bar\pi_0=X$ by the previous paragraph, so $\sum_m\lambda_m\pi_0^m\mapsto\sum_m\bar\lambda_mX^m$. By (Rd1), $\bar N_a(T)=\bigoplus_k k[[X]]\rho_{T,a}(n_k)$, a lattice of $V_{\bar T}$. Kernel: $\varpi\mathbf{N}_a^\Delta(T)$ lies in the kernel. Conversely, if $r_T(\iota_T(x))=0$ with $x=\sum_k g_kn_k$, then $\iota_T(x)\in\varpi\mathbf{D}(T)$, so by the basis $(\iota_T(n_k))$ each g_k is divisible by ϖ in the coefficient ring $\mathcal{O}_L[[X]][1/X]^\wedge$ of the e_a-component; since $\mathcal{O}_L[[X]]/\varpi=k[[X]]\to k((X))$ is injective, $g_k\in\varpi R$,

and g_k/ϖ is Δ-invariant. So the kernel is $\varpi\mathbf{N}_a^\Delta(T)$.

(Rd4). By (Ex3), $\mathbf{N}(f)$ is $\mathbf{A}_{F,L}$-linear and commutes with $\gamma_{u,\mathbf{N}}$, so it maps $\mathbf{N}_a^\Delta(T)$ into $\mathbf{N}_a^\Delta(T')$, and $\iota_{T'}\circ\mathbf{N}(f)=(1\otimes f)\circ\iota_T$. As $r_{T'}\circ(1\otimes f)=(1\otimes\bar f)\circ r_T$ and $1\otimes\bar f$ is $\mathbf{D}(\bar f)$ on $\mathbf{D}(\bar T)$, (Ff2) gives $\theta_{\bar T',a}(\rho_{T',a}(\mathbf{N}(f)x))=\mathbf{D}(\bar f)\theta_{\bar T,a}(\rho_{T,a}(x))=\theta_{\bar T',a}(\mathcal{F}(\bar f)_a\rho_{T,a}(x))$.

(Rd5). Let $T=\mathcal{O}_L$ with trivial action; $\bar T=\mathbf{1}$, of dimension 1, and the only Hodge jump is 0. By (Ex1) with the window $(0,0)$ and (Wm1), $\mathbf{N}_b=\mathbf{N}(\mathcal{O}_L)_b$ is free of rank 1 with $\varphi_{\mathbf{N}}(\mathbf{N}_{b-1})\subseteq\mathbf{N}_b\subseteq\mathbf{A}_{F,L}\varphi_{\mathbf{N}}(\mathbf{N}_{b-1})$ for every b. Let n_{b-1} be a Δ-invariant basis of $\mathbf{N}_{b-1}$ (Lemma 24.15); then $\varphi_{\mathbf{N}}(n_{b-1})$ is a Δ-invariant basis of $\mathbf{N}_b$, and by the proof of (Rd3) and (Rd2) with $j=0$, $\bar N_{b-1}=k[[X]]\rho_{b-1}(n_{b-1})$ and $\bar N_b=k[[X]]\varphi_b(\rho_{b-1}(n_{b-1}))=\varphi_b^*\bar N_{b-1}$ (using (Rm1) and $\varphi(k[[X]])\subseteq k[[X]]$, (Rr1)). The element u_b of (Mf4) is a $k((X))$-basis of $V_{\mathbf{1}}$, so $\bar N_b=k[[X]]c_bu_b$ with $c_b\in k((X))^\times$, and $\varphi_b^*\bar N_{b-1}=k[[X]]\varphi(c_{b-1})u_b$ as $\varphi_b(u_{b-1})=u_b$. By (Rr2), φ multiplies X-adic valuations by ℓ, so $v_X(c_b)=\ell\,v_X(c_{b-1})$ for all $b\in\mathbb{Z}/n$; going once around, $v_X(c_a)=\ell^n v_X(c_a)$, so $v_X(c_a)=0$, $c_a\in k[[X]]^\times$, and $\bar N_a(\mathcal{O}_L)=k[[X]]u_a$. □

Lemma 28.8. *Let ℓ, F, n, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ be as in Proposition 23.7, and let $\mathcal{O}_{\mathcal{E}_F}$, $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$, $\mathbf{D}$ be as in Theorem B.51 for $p=\ell$ and $K=F$. Then $\iota(\mathbf{A}_F)\subseteq\mathcal{O}_{\mathcal{E}_F}$, $\iota(Q)$ is a unit of $\mathcal{O}_{\mathcal{E}_F}$, and for every pair (V,T) the map*

$$\Psi_T\colon \mathcal{O}_{\mathcal{E}_F}\otimes_{\iota,\mathbf{A}_F}\mathbf{N}(T)\to W(\mathbb{C}_\ell^\flat)\otimes_{\mathbb{Z}_\ell}T,\qquad e\otimes x\mapsto e\cdot\iota_T(x),$$

is injective with image $\mathbf{D}(T)$. It satisfies $\Psi_T(e\otimes\lambda x)=(1\otimes\lambda)\Psi_T(e\otimes x)$ for $\lambda\in\mathcal{O}_L$, $\Psi_T(\varphi(e)\otimes\varphi_{\mathbf{N}}(x))=(\varphi\otimes1)\Psi_T(e\otimes x)$ (with $\varphi_{\mathbf{N}}(x)\in\mathbf{N}(T)[1/Q]$ and $\iota(Q)$ inverted), and $\Psi_T(\sigma(e)\otimes\gamma_{\chi_{\mathrm{cyc}}(\sigma),\mathbf{N}}(x))=\sigma\,\Psi_T(e\otimes x)$ for $\sigma\in G_F$.

Proof. Write $W=W(\mathbb{C}_\ell^\flat)$, $\mathcal{O}_{\mathcal{E}}=\mathcal{O}_{\mathcal{E}_F}$, $\mathcal{O}_U=\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$, $\mathbf{N}=\mathbf{N}(T)$, $d=\mathrm{rk}_{\mathbb{Z}_\ell}T=[L:\mathbb{Q}_\ell]\dim_L V$.

The rings. By Lemma 23.8(4), $\iota=\varphi\circ w$ on $\mathcal{O}_F$ and $\iota(X)=\mu_\varepsilon=\pi$; as $\varphi\circ w=w\circ\varphi$ (functoriality of Witt vectors), $\iota(\sum_m a_mX^m)=\sum_m w(\varphi(a_m))\pi^m\in w(\mathcal{O}_F)[[\pi]]\subseteq\mathcal{O}_{\mathcal{E}}$ (Fi1). Also $\iota(Q)=\tilde\xi=\varphi(\pi)/\pi$ reduces to $(\varepsilon^\ell-1)/(\varepsilon-1)=(\varepsilon-1)^{\ell-1}\neq0$ in $\mathbf{E}_F$, so it is a unit of the discrete valuation ring $\mathcal{O}_{\mathcal{E}}$ with uniformiser ℓ. The element π is a unit of W: its reduction $\varepsilon-1\in\mathbb{C}_\ell^\flat$ is non-zero and W is ℓ-adically complete with $W/\ell=\mathbb{C}_\ell^\flat$ a field. Finally $W^{\varphi=1}=\mathbb{Z}_\ell$:

The module M. Choose M_a,h_a bounding the Hodge jumps of V; by (Ex1), $\mathbf{N}$ is a Wach module, so each $\mathbf{N}_a$ is free of rank $r=\dim_L V$ over $e_a\mathbf{A}_{F,L}\cong\mathcal{O}_L[[X]]$, which is free of rank $[L:F]$ over $\mathbf{A}_F$ ($\mathcal{O}_F$ acting through τ_a); hence $\mathbf{N}$ is free over $\mathbf{A}_F$ of rank $nr[L:F]=d$, and $M=\mathcal{O}_{\mathcal{E}}\otimes_{\iota,\mathbf{A}_F}\mathbf{N}$ is free of rank d over $\mathcal{O}_{\mathcal{E}}$. As $\iota(Q)$ is a unit, $\varphi_M(e\otimes x)=\varphi(e)\cdot(1\otimes\varphi_{\mathbf{N}}(x))$ is a well-defined φ-semilinear endomorphism of M. Since $\varphi_{\mathbf{N}}$ commutes with $\mathcal{O}_L$, $\mathbf{A}_{F,L}\varphi_{\mathbf{N}}(\mathbf{N}_{a-1})=\mathbf{A}_F\varphi_{\mathbf{N}}(\mathbf{N}_{a-1})$, so by (Wm2) $Q^{h_a}\mathbf{N}_a\subseteq\mathbf{A}_F\varphi_{\mathbf{N}}(\mathbf{N})$ for all a, and the $\mathcal{O}_{\mathcal{E}}$-span of $\varphi_M(M)$ is M. So the linearisation $\mathcal{O}_{\mathcal{E}}\otimes_\varphi M\to M$ is a surjective map of free $\mathcal{O}_{\mathcal{E}}$-modules of rank d, hence an isomorphism. Let $\sigma\in G_F$ act on M by $\sigma(e\otimes x)=\sigma(e)\otimes\gamma_{\chi_{\mathrm{cyc}}(\sigma),\mathbf{N}}(x)$, well defined because $\sigma\circ\iota=\iota\circ\gamma_{\chi_{\mathrm{cyc}}(\sigma)}$ (Lemma 23.8(4)) and $\mathcal{O}_{\mathcal{E}}$ is G_F-stable (Fi1).

After extension to W. Let $\Psi\colon W\otimes_{\mathcal{O}_{\mathcal{E}}}M\to W\otimes_{\mathbb{Z}_\ell}T$ be the W-linear extension of Ψ_T (well defined, as $\iota_T(fx)=\iota(f)\iota_T(x)$ for $f\in\mathbf{A}_F$ by (Ex2)). Its image is the W-span of $\iota_T(\mathbf{N})$, which contains $W\cdot A_{\mathrm{inf}}[1/\pi]\,\iota_T(\mathbf{N})=W\cdot(A_{\mathrm{inf}}[1/\pi]\otimes T)=W\otimes T$ by (Ex2) ($\pi=\mu_\varepsilon$ is a unit of W). Both sides are free W-modules of rank d, so Ψ is an isomorphism. By (Ex2), Ψ is $\mathcal{O}_L$-linear in the stated sense, carries $\varphi\otimes\varphi_M$ to $\varphi\otimes1$, and is G_F-equivariant for the diagonal actions; restricted to M these are the identities of the statement.

Descent. By (Fi3), $T_M=(\mathcal{O}_U\otimes_{\mathcal{O}_{\mathcal{E}}}M)^{\varphi=1}$ is free of rank d over $\mathbb{Z}_\ell$ and $\alpha\colon\mathcal{O}_U\otimes_{\mathbb{Z}_\ell}T_M\to\mathcal{O}_U\otimes_{\mathcal{O}_{\mathcal{E}}}M$ is an isomorphism commuting with φ. Extending scalars to W and composing with Ψ gives a W-linear isomorphism $C\colon W\otimes_{\mathbb{Z}_\ell}T_M\to W\otimes_{\mathbb{Z}_\ell}T$ carrying $\varphi\otimes1$ to $\varphi\otimes1$. On φ-invariants, since T_M and T are free and $W^{\varphi=1}=\mathbb{Z}_\ell$, C restricts to a bijection $\beta\colon T_M\to T$, and

$C = 1 \otimes \beta$ (both are W-linear and agree on T_M). As M is free, $\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{E}} M \subseteq W \otimes_{\mathcal{O}_\mathcal{E}} M$, and therefore

$$\Psi(\mathcal{O}_U \otimes_{\mathcal{O}_\mathcal{E}} M) = C(\mathcal{O}_U \otimes T_M) = \mathcal{O}_U \otimes \beta(T_M) = \mathcal{O}_U \otimes_{\mathbb{Z}_\ell} T.$$

The group H_F acts trivially on M: it fixes $\mathcal{O}_\mathcal{E}$ (Fi1) and $\gamma_{1,\mathbf{N}} = \mathrm{id}$ (Wm3). With an $\mathcal{O}_\mathcal{E}$-basis of M, $(\mathcal{O}_U \otimes M)^{H_F} = (\mathcal{O}_U)^{H_F} \otimes M = M$ (Fi1). Since Ψ is injective and G_F-equivariant on $\mathcal{O}_U \otimes M$, taking H_F-invariants gives $\Psi(M) = (\mathcal{O}_U \otimes T)^{H_F} = \mathbf{D}(T)$. As $\Psi|_M = \Psi_T$, this proves the lemma. □

Lemma 28.9. *Let ℓ be an odd prime, k a field of characteristic ℓ, $\Delta = \{z \in \mathbb{Z}_\ell^\times : \ z^{\ell-1} = 1\}$, and*

$$X_0 = \sum_{z\in\Delta} ((1+\pi)^z - 1) \in \mathbb{F}_\ell[[\pi]] \subseteq k[[\pi]]$$

(binomial coefficients reduced modulo ℓ). Then $X_0 = v\,\pi^{\ell-1}$ for a unit v of $k[[\pi]]$ with $v(0) = 1$, the k-algebra homomorphism $\iota\colon k((X)) \to k((\pi))$, $\iota(f) = f(X_0)$, is injective with $\iota(k[[X]]) = \{b \in k[[\pi]] : \ \delta_z(b) = b \text{ for all } z \in \Delta\}$ and $\iota(k((X))) = \{b \in k((\pi)) : \ \delta_z(b) = b \text{ for all } z \in \Delta\}$, and there is a residual ring datum (φ, γ) over k with $\varphi(X) = X^\ell$ which is cyclotomic, with ι as the homomorphism in the definition.

Proof. **Step 1: the series X_0.** Let $\pi_0 = \sum_{z\in\Delta}((1+Y)^z - 1) \in \mathbb{Z}_\ell[[Y]]$. By (Rr1) of Lemma 23.24 (with $L = \mathbb{Q}_\ell$ and the subgroup $\Delta = \mu_{\ell-1}$, so $|\Delta| = \ell - 1$), $\pi_0 = Y^{\ell-1}v_0$ with $v_0 \in \mathbb{Z}_\ell[[Y]]^\times$ and $v_0(0) = 1/(\ell-2)!$, and $\gamma_z(\pi_0) = \pi_0$ for $z \in \Delta$, where $\gamma_z(Y) = (1+Y)^z - 1$. Reducing coefficients modulo ℓ and substituting $Y = \pi$ (reduction is a ring homomorphism $\mathbb{Z}_\ell[[Y]] \to \mathbb{F}_\ell[[\pi]]$ compatible with the substitutions $Y \mapsto (1+Y)^a - 1$, $a \in \mathbb{Z}_\ell$) gives $X_0 = v\,\pi^{\ell-1}$ with v the reduction of v_0, a unit with $v(0) = 1/(\ell-2)! = 1$ in $\mathbb{F}_\ell$ by Wilson's theorem ($(\ell-1)! \equiv -1$ and $\ell - 1 \equiv -1$), and $\delta_z(X_0) = X_0$ for $z \in \Delta$.

Step 2: the group Δ acting on $k((\pi))$. Each δ_z ($z \in \Delta$) is the continuous k-algebra endomorphism with $\delta_z(\pi) = (1+\pi)^z - 1$; from the identity $(1 + ((1+Y)^{z'} - 1))^z = (1+Y)^{zz'}$ in $\mathbb{Z}_\ell[[Y]]$ we get $\delta_z \circ \delta_{z'} = \delta_{zz'}$, so $\delta_1 = \mathrm{id}$ and each δ_z is an automorphism of $k[[\pi]]$ and of $k((\pi))$. Since $\delta_z(\pi) \equiv z\pi \pmod{\pi^2}$ and the elements of Δ are distinct modulo ℓ, $z \mapsto \delta_z$ is injective: Δ is a group of $\ell - 1$ automorphisms of the field $k((\pi))$. By Artin's theorem the fixed field $k((\pi))^\Delta$ has index $\ell - 1$ in $k((\pi))$. It contains the subfield $k((X_0))$ of Laurent series in X_0 (Step 1, and the δ_z are continuous). Since X_0 has π-adic valuation $\ell - 1$ and both fields are complete for the π-adic valuation with residue field k, $[k((\pi)) : k((X_0))] = \ell - 1$ (ramification index $\ell - 1$, residue degree 1). Hence $k((\pi))^\Delta = k((X_0))$. An element of $k((X_0))$ lies in $k[[\pi]]$ if and only if its X_0-adic valuation is ≥ 0 (the π-adic valuation is $\ell - 1$ times it), so $k[[\pi]]^\Delta = k[[X_0]]$.

Step 3: ι. Since X_0 has positive valuation, $f \mapsto f(X_0)$ is a well-defined k-algebra homomorphism $k((X)) \to k((\pi))$ multiplying valuations by $\ell - 1$; it is injective, with image $k((X_0))$ and $\iota(k[[X]]) = k[[X_0]]$. By Step 2 these are the Δ-invariants of $k((\pi))$ and $k[[\pi]]$.

Step 4: φ and γ. The maps φ_π, γ_π, δ_z are the substitutions $1 + \pi \mapsto (1+\pi)^a$ for $a = \ell$, $1 + \ell$, z; composites multiply the exponents, and $\mathbb{Z}_\ell$ is commutative, so they commute pairwise. Hence φ_π, γ_π map the Δ-invariants of $k((\pi))$ and of $k[[\pi]]$ into themselves, and by Step 3 we may define k-algebra endomorphisms of $k((X))$ by $\varphi = \iota^{-1} \circ \varphi_\pi \circ \iota$, $\gamma = \iota^{-1} \circ \gamma_\pi \circ \iota$. Then $\iota \circ \varphi = \varphi_\pi \circ \iota$, $\iota \circ \gamma = \gamma_\pi \circ \iota$, (Rr1) holds because φ_π, γ_π preserve $k[[\pi]]^\Delta = \iota(k[[X]])$, and (Rr3) holds because $\varphi_\pi\gamma_\pi = \gamma_\pi\varphi_\pi$.

(Rr2) for φ: X_0 has coefficients in $\mathbb{F}_\ell$ and k has characteristic ℓ, so $\varphi_\pi(X_0) = X_0(\pi^\ell) = X_0^\ell = \iota(X^\ell)$; thus $\varphi(X) = X^\ell$ (unit $u = 1$).

(Rr2) for γ: $\gamma_\pi(X_0) \in k[[\pi]]^\Delta$, so $\gamma(X) = g \in k[[X]]$ with $g(X_0) = \gamma_\pi(X_0)$. In characteristic ℓ, $\gamma_\pi(\pi) = (1+\pi)(1+\pi)^\ell - 1 = \pi(1 + \pi^{\ell-1} + \pi^\ell)$, hence

$$\gamma_\pi(X_0) = \gamma_\pi(\pi)^{\ell-1}\, v(\gamma_\pi(\pi)) \equiv \pi^{\ell-1}(1-\pi^{\ell-1})\, v(\pi) = X_0 - \pi^{\ell-1}X_0 \pmod{\pi^{2\ell-1}},$$

and $\pi^{\ell-1}X_0 = X_0^2/v \equiv X_0^2 \pmod{\pi^{2\ell-1}}$ because $v \equiv 1 \pmod\pi$. Write $g = \sum_{j\ge 0} g_j X^j$. Then $(g_0) + (g_1 - 1)X_0 + g_2 X_0^2 + X_0^3(\cdots) \equiv -X_0^2 \pmod{\pi^{2\ell-1}}$; comparing π-adic valuations (X_0^j has valuation $j(\ell-1)$ and leading coefficient 1) gives $g_0 = 0$, $g_1 = 1$, $g_2 = -1$. So $\gamma(X) = X(1 - \kappa X)$ with $\kappa = -\sum_{j\ge 2} g_j X^{j-2} \in k[[X]]$, $\kappa(0) = 1$. Thus (φ, γ) is a residual ring datum, and by $\iota(X) = v\pi^{\ell-1}$ and Step 3 it is cyclotomic, with ι as the homomorphism in the definition. □

Proposition 28.10. *Let ℓ be an odd prime, K a finite unramified extension of $\mathbb{Q}_\ell$ of degree n with residue field k_K, and k a finite field of characteristic ℓ with n distinct field embeddings $\bar\kappa_a\colon k_K \to k$ ($a \in \mathbb{Z}/n$) such that $\bar\kappa_{a+1}(x^\ell) = \bar\kappa_a(x)$ for $x \in k_K$. Let (φ, γ) be a cyclotomic residual ring datum over k, and fix a homomorphism ι as in the definition of cyclotomic (Definition 28.4). Let $\gamma_0 \in \Gamma_K$ with $\chi_{\mathrm{cyc}}(\gamma_0) = 1 + \ell$, and $\Delta_K = \{g \in \Gamma_K :\ \chi_{\mathrm{cyc}}(g)^{\ell-1} = 1\}$.*

For a continuous finite representation A of G_K over k (discrete topology), regarded as an $\mathbb{F}_\ell$-representation, let $\mathbf{D}(A)$ be its étale (φ, Γ_K)-module (Theorem B.52, $p = \ell$), a module over $\mathbf{E}_K \otimes_{\mathbb{F}_\ell} k$ with $\lambda \in k$ acting by $\mathbf{D}$ of the multiplication by λ on A. Let $e_a \in k_K \otimes_{\mathbb{F}_\ell} k$ be the idempotent of the factor $x \otimes \lambda \mapsto \bar\kappa_a(x)\lambda$ of $k_K \otimes_{\mathbb{F}_\ell} k \cong \prod_{a\in\mathbb{Z}/n} k$, and identify $e_a(\mathbf{E}_K \otimes_{\mathbb{F}_\ell} k)$ with $k((\pi))$ by $e_a(\bar\pi \otimes 1) \mapsto \pi$, $e_a(x \otimes \lambda) \mapsto \bar\kappa_a(x)\lambda$; so $e_a\mathbf{D}(A)$ is a $k((\pi))$-vector space.

Then there is a residual model functor $\mathcal{F}$ for G_K over (φ, γ) with sites $\mathbb{Z}/n$ such that:

(Ff1) *the $\mathcal{F}$-objects are the continuous finite representations of G_K over k;*

(Ff2) *for every $\mathcal{F}$-object A and every $a \in \mathbb{Z}/n$ there is an injective additive map $\theta_{A,a}\colon V_A \to e_a\mathbf{D}(A)$ whose image is the set of elements fixed by Δ_K, with $\theta_{A,a}(cx) = \iota(c)\,\theta_{A,a}(x)$ for $c \in k((X))$, $\theta_{A,a} \circ \varphi_a = \varphi_{\mathbf{D}(A)} \circ \theta_{A,a-1}$, $\theta_{A,a} \circ \gamma_a = \gamma_0 \circ \theta_{A,a}$, and $\theta_{B,a} \circ \mathcal{F}(f)_a = \mathbf{D}(f) \circ \theta_{A,a}$ for every $\mathcal{F}$-object B and every equivariant k-linear $f\colon A \to B$;*

(Ff3) *$\mathbf{D}(\mathbf{1}) = \mathbf{E}_K \otimes_{\mathbb{F}_\ell} k$, and the elements of (Mf4) are $u_a = \theta_{\mathbf{1},a}^{-1}(e_a)$.*

Proof. Since K is unramified over $\mathbb{Q}_\ell$, $K \cap \mathbb{Q}_\ell(\mu_{\ell^\infty}) = \mathbb{Q}_\ell$ and $\chi_{\mathrm{cyc}}\colon \Gamma_K \to \mathbb{Z}_\ell^\times$ is an isomorphism; so γ_0 exists, Δ_K is a group of order $\ell - 1$, and $g \in \Gamma_K$ acts on $\mathbf{E}_K$ by $\bar\pi \mapsto (1+\bar\pi)^{\chi_{\mathrm{cyc}}(g)} - 1$, fixing k_K (Theorem B.52).

Step 0: the coefficient ring. $\mathbf{D}$ is additive ($\mathbf{D}(f + f') = \mathbf{D}(f) + \mathbf{D}(f')$ is immediate from $\mathbf{D}(V) = (\mathbf{E}^{\mathrm{sep}} \otimes_{\mathbb{F}_\ell} V)^{H_K}$) and a functor, so $\lambda \mapsto \mathbf{D}(\lambda\cdot)$ makes $\mathbf{D}(A)$ a module over $\mathbf{E}_K \otimes_{\mathbb{F}_\ell} k$ on which $\varphi_{\mathbf{D}(A)}$ is semilinear for $\varphi \otimes 1$ and $g \in \Gamma_K$ for $g \otimes 1$ (the $\mathbf{D}(\lambda\cdot)$ are morphisms). As $k_K = \mathbb{F}_\ell[t]/(P)$ with P irreducible of degree n and the n roots $\bar\kappa_a(t)$ in k distinct, the Chinese remainder theorem gives $k_K \otimes_{\mathbb{F}_\ell} k \cong \prod_a k$, and since k is finite over $\mathbb{F}_\ell$, $\mathbf{E}_K \otimes_{\mathbb{F}_\ell} k = (k_K \otimes_{\mathbb{F}_\ell} k)((\bar\pi)) = \prod_a e_a(\mathbf{E}_K \otimes k)$ with $e_a(\mathbf{E}_K \otimes k) \cong k((\pi))$ as in the statement. The automorphism $\varphi \otimes 1$ maps e_a to e_{a+1} (the factor $x \otimes \lambda \mapsto \bar\kappa_{a+1}(x)\lambda$ composed with $\varphi \otimes 1$ is $x \otimes \lambda \mapsto \bar\kappa_{a+1}(x^\ell)\lambda = \bar\kappa_a(x)\lambda$), and in the identifications it is $\varphi_\pi\colon e_a(\mathbf{E}_K \otimes k) \to e_{a+1}(\mathbf{E}_K \otimes k)$ ($\bar\pi \mapsto \bar\pi^\ell$, coefficients in k unchanged); $g \otimes 1$ fixes every e_a and acts on $e_a(\mathbf{E}_K \otimes k)$ as γ_π for $g = \gamma_0$ and as $\delta_{\chi_{\mathrm{cyc}}(g)}$ for $g \in \Delta_K$. Hence $\varphi_{\mathbf{D}(A)}(e_{a-1}\mathbf{D}(A)) \subseteq e_a\mathbf{D}(A)$, the action of Γ_K preserves each $e_a\mathbf{D}(A)$, and $\varphi_{\mathbf{D}(A)}$, γ_0, $g \in \Delta_K$ are semilinear on these $k((\pi))$-spaces for φ_π, γ_π, $\delta_{\chi_{\mathrm{cyc}}(g)}$.

Step 1: dimensions. Let $d_a = \dim_{k((\pi))} e_a\mathbf{D}(A)$ and $m = \dim_k A$. As $k((\pi))$ has dimension $[k : k_K]$ over $\mathbf{E}_K$ (through $\bar\kappa_a$), (Fe1) gives $[k : k_K]\sum_a d_a = \dim_{\mathbb{F}_\ell} A = n[k : k_K]m$. Since

$\mathbf{D}(A)$ is étale, the $\mathbf{E}_K$-span of $\varphi_{\mathbf{D}(A)}(\mathbf{D}(A))$ is $\mathbf{D}(A)$; multiplying by e_a, the $k((\pi))$-span of $\varphi_{\mathbf{D}(A)}(e_{a-1}\mathbf{D}(A))$ is $e_a\mathbf{D}(A)$, and by semilinearity it is spanned by the images of a basis of $e_{a-1}\mathbf{D}(A)$. So $d_a \le d_{a-1}$ for all $a \in \mathbb{Z}/n$, hence all d_a are equal, to m.

Step 2: Δ-descent. Let $\Delta = \{z \in \mathbb{Z}_\ell^\times : z^{\ell-1} = 1\}$. From the identity $(1+((1+Y)^{z'}-1))^z = (1+Y)^{zz'}$ in $\mathbb{Z}_\ell[[Y]]$, $\delta_z \circ \delta_{z'} = \delta_{zz'}$, so the δ_z ($z \in \Delta$) are automorphisms of $k((\pi))$; as $\delta_z(\pi) \equiv z\pi \pmod{\pi^2}$ and the elements of Δ are distinct modulo ℓ, they form a group of $\ell-1$ automorphisms; its fixed field is $\iota(k((X)))$: it contains $\iota(k((X)))$ by the definition of cyclotomic, and if b is fixed then $\iota(X)^N b \in k[[\pi]]$ is fixed for N large (as $\iota(X) = v\pi^{\ell-1}$ is fixed), so $\iota(X)^N b \in \iota(k[[X]])$ and $b \in \iota(k((X)))$. Thus $k((\pi))/\iota(k((X)))$ is Galois with group Δ. By Galois descent for semilinear actions (Speiser's lemma), for $D_a(A) = \{y \in e_a\mathbf{D}(A) : gy = y\ \forall g \in \Delta_K\}$ the map $k((\pi)) \otimes_{\iota(k((X)))} D_a(A) \to e_a\mathbf{D}(A)$ is an isomorphism; so $D_a(A)$ is an $\iota(k((X)))$-vector space of dimension m. Since Γ_K is abelian and $\varphi_{\mathbf{D}(A)}$ commutes with Γ_K, $\varphi_{\mathbf{D}(A)}(D_{a-1}(A)) \subseteq D_a(A)$ and $\gamma_0(D_a(A)) = D_a(A)$.

Step 3: the functor. For a continuous finite representation A of dimension m choose, for each a, a bijection $\theta_{A,a}\colon k((X))^m \to D_a(A)$ with $\theta_{A,a}(cx) = \iota(c)\theta_{A,a}(x)$ (an $\iota(k((X)))$-basis of $D_a(A)$), and put $V_A = k((X))^m$, $\varphi_a = \theta_{A,a}^{-1} \circ \varphi_{\mathbf{D}(A)} \circ \theta_{A,a-1}$, $\gamma_a = \theta_{A,a}^{-1} \circ \gamma_0 \circ \theta_{A,a}$. For equivariant k-linear $f\colon A \to B$ between such representations put $\mathcal{F}(f)_a = \theta_{B,a}^{-1} \circ \mathbf{D}(f) \circ \theta_{A,a}$, which is defined ($\mathbf{D}(f)$ is $\mathbf{E}_K \otimes k$-linear and commutes with Γ_K, so maps $D_a(A)$ into $D_a(B)$) and $k((X))$-linear. For a finite representation that is not continuous, put $V_A = k((X))^{\dim A}$ with φ_a, γ_a the coordinatewise φ, γ; for every other k-linear f put $\mathcal{F}(f)_a = 0$. Then (Ff1), (Ff2) and the first part of (Ff3) hold by construction ($\mathbf{D}(\mathbf{1}) = (\mathbf{E}^{\mathrm{sep}})^{H_K} \otimes_{\mathbb{F}_\ell} k = \mathbf{E}_K \otimes_{\mathbb{F}_\ell} k$, as H_K acts trivially on k and $(\mathbf{E}^{\mathrm{sep}})^{H_K} = \mathbf{E}_K$).

(Mf2). For continuous A: (Rm1) $\varphi_a(cx) = \theta_{A,a}^{-1}(\varphi_{\mathbf{D}}(\iota(c)\theta_{A,a-1}x)) = \theta_{A,a}^{-1}(\varphi_\pi(\iota c)\varphi_{\mathbf{D}}\theta_{A,a-1}x) = \varphi(c)\varphi_a(x)$ as $\varphi_\pi \circ \iota = \iota \circ \varphi$, and likewise for γ_a with $\gamma_\pi \circ \iota = \iota \circ \gamma$. (Rm2) Let W be the $\iota(k((X)))$-span of $\varphi_{\mathbf{D}}(D_{a-1}(A))$. By Step 2 and semilinearity the $k((\pi))$-span of W contains $\varphi_{\mathbf{D}}(k((\pi))D_{a-1}(A)) = \varphi_{\mathbf{D}}(e_{a-1}\mathbf{D}(A))$, hence is $e_a\mathbf{D}(A)$ (Step 1); by the injectivity in Speiser's lemma $\dim W = m$, so $W = D_a(A)$ and $\varphi_a(V_A)$ spans V_A. γ_a is bijective as γ_0 is invertible on $D_a(A)$. (Rm3) holds as $\varphi_{\mathbf{D}}\gamma_0 = \gamma_0\varphi_{\mathbf{D}}$. The rank is m. For non-continuous A, (Rm1), (Rm3) and the spanning in (Rm2) are clear, and γ is bijective on $k((X))$ because $\iota \circ \gamma = \gamma_\pi \circ \iota$, γ_π is an automorphism of $k((\pi))$ (inverse: the substitution with exponent $(1+\ell)^{-1}$) commuting with Δ, and $\iota(k((X)))$ is the fixed field of Δ.

(Mf1). Subrepresentations, quotients and quotients of sums of continuous finite representations are continuous, and $\mathbf{1}$ is.

(Mf3). For continuous A, B, C and equivariant f, f', f'': the commutation with φ_a, γ_a holds because $\mathbf{D}(f)$ commutes with $\varphi_{\mathbf{D}}$ and γ_0; identity and composition from the functoriality of $\mathbf{D}$; $\mathcal{F}(f + cf')_a = \mathcal{F}(f)_a + c\mathcal{F}(f')_a$ from additivity and $\mathbf{D}(c \cdot f') = \mathbf{D}(c\cdot)\mathbf{D}(f')$. If f is injective, $\mathbf{D}(f)$ is injective by the exactness in (Fe2), hence so is $\mathcal{F}(f)_a$. If f is surjective, $\mathbf{D}(f)$ is surjective, hence so is $e_a\mathbf{D}(A) \to e_a\mathbf{D}(B)$; for $y \in D_a(B)$ pick x with $\mathbf{D}(f)x = y$ and put $x' = (\ell-1)^{-1}\sum_{g\in\Delta_K} gx \in D_a(A)$ ($\ell - 1$ is invertible in k); then $\mathbf{D}(f)x' = y$, so $\mathcal{F}(f)_a$ is surjective.

(Mf4) and (Ff3). The element $e_a = e_a(1 \otimes 1) \in e_a\mathbf{D}(\mathbf{1})$ is non-zero and fixed by Γ_K ($g \otimes 1$ fixes $k_K \otimes k$), and $\varphi_{\mathbf{D}}(e_{a-1}) = e_a$ by Step 0. So $u_a = \theta_{\mathbf{1},a}^{-1}(e_a)$ are non-zero with $\varphi_a(u_{a-1}) = u_a$, $\gamma_a(u_a) = u_a$. Let A be continuous and $(x_a) \in V_A^{\mathbb{Z}/n}$ with $\varphi_a(x_{a-1}) = x_a$, $\gamma_a(x_a) = x_a$. Put $y = \sum_a \theta_{A,a}(x_a) \in \mathbf{D}(A)$. Then $\varphi_{\mathbf{D}}(y) = \sum_a \theta_{A,a+1}(\varphi_{a+1}x_a) = y$, $\gamma_0 y = y$ and $gy = y$ for $g \in \Delta_K$. The stabiliser of y in Γ_K is closed (the action of Γ_K on the Hausdorff space $\mathbf{D}(A)$ is continuous, (Fe1)) and contains Δ_K and γ_0; its image under χ_{cyc} is a closed subgroup of $\mathbb{Z}_\ell^\times$ containing $\mu_{\ell-1}$ and $1+\ell$, which topologically generates $1 + \ell\mathbb{Z}_\ell$ (ℓ odd); so y is fixed by Γ_K. The map $h\colon \mathbf{D}(\mathbf{1}) = \mathbf{E}_K \otimes k \to \mathbf{D}(A)$, $h(c \otimes \lambda) = c\lambda y$, is $\mathbf{E}_K$-linear and

commutes with φ and with Γ_K (as $\varphi_{\mathbf{D}}(c\lambda y) = \varphi(c)\lambda\varphi_{\mathbf{D}}(y)$ and $g(c\lambda y) = g(c)\lambda gy$). By (Fe2) (full faithfulness) $h = \mathbf{D}(f)$ for a unique G_K-equivariant $\mathbb{F}_\ell$-linear $f\colon \mathbf{1} \to A$, and f is k-linear: $\mathbf{D}(f \circ \lambda\cdot) = h \circ \mathbf{D}(\lambda\cdot) = \mathbf{D}(\lambda\cdot) \circ h = \mathbf{D}(\lambda \cdot \circ f)$, so $f(\lambda z) = \lambda f(z)$ by faithfulness. Finally h is $\mathbf{E}_K \otimes k$-linear, so $\theta_{A,a}(\mathcal{F}(f)_a(u_a)) = \mathbf{D}(f)(e_a) = e_a h(1) = e_a y = \theta_{A,a}(x_a)$, i.e. $\mathcal{F}(f)_a(u_a) = x_a$ for all a. □

Lemma 28.11. *Let ℓ, F, n, L, (τ_a), ϖ, k, (φ,γ) and $\mathcal{F}$ be as in Proposition 28.7. Let (V'',T'') be a pair and $T'' = T \oplus T'$ a decomposition into G_F-stable $\mathcal{O}_L$-submodules; put $V = LT$, $V' = LT'$, so that (V,T) and (V',T') are pairs, and let $i\colon T \to T''$, $i'\colon T' \to T''$ be the inclusions and $\bar\imath\colon \bar T \to \bar T''$, $\bar\imath'\colon \bar T' \to \bar T''$ their reductions. Then $\bar\imath \oplus \bar\imath'\colon \bar T \oplus \bar T' \to \bar T''$ is a bijective equivariant map, and for every $a \in \mathbb{Z}/n$*

$$V_{\bar T''} = \mathcal{F}(\bar\imath)_a(V_{\bar T}) \oplus \mathcal{F}(\bar\imath')_a(V_{\bar T'}), \qquad \bar N_a(T'') = \mathcal{F}(\bar\imath)_a\big(\bar N_a(T)\big) + \mathcal{F}(\bar\imath')_a\big(\bar N_a(T')\big),$$

with $\mathcal{F}(\bar\imath)_a$, $\mathcal{F}(\bar\imath')_a$ injective.

Proof. The pairs. V and V' are G_F-stable L-subspaces of V'' with $V'' = V \oplus V'$ (the sum is direct because $T \cap T' = 0$ and every element of V'' is a multiple of an element of T'' by a power of ϖ^{-1}). Subrepresentations of crystalline representations are crystalline, so (V,T), (V',T') are pairs. Let $p\colon T'' \to T$, $p'\colon T'' \to T'$ be the projections; they are G_F-equivariant and $\mathcal{O}_L$-linear, with $pi = \mathrm{id}_T$, $p'i' = \mathrm{id}_{T'}$, $p'i = 0$, $pi' = 0$ and $ip + i'p' = \mathrm{id}_{T''}$. Their reductions $\bar p, \bar p'$ satisfy the same identities, so $\bar\imath \oplus \bar\imath'$ is bijective with inverse $(\bar p, \bar p')$, and it is equivariant.

Additivity of $\mathbf{N}$. For G_F-equivariant $\mathcal{O}_L$-linear $f, g\colon T_1 \to T_2$ between lattices of pairs, $\mathbf{N}(f+g) = \mathbf{N}(f) + \mathbf{N}(g)$ and $\mathbf{N}(0) = 0$: by (Ex3), for $x \in \mathbf{N}(T_1)$,

$$\iota_{T_2}(\mathbf{N}(f+g)x) = (1 \otimes (f+g))\iota_{T_1}(x) = (1 \otimes f)\iota_{T_1}(x) + (1 \otimes g)\iota_{T_1}(x) = \iota_{T_2}(\mathbf{N}(f)x + \mathbf{N}(g)x),$$

and ι_{T_2} is additive and injective (Proposition 23.7); likewise $\iota_{T_2}(\mathbf{N}(0)x) = 0$. Hence, by functoriality (Ex3), for $x \in \mathbf{N}(T'')$

$$x = \mathbf{N}(\mathrm{id})x = \mathbf{N}(i)\mathbf{N}(p)x + \mathbf{N}(i')\mathbf{N}(p')x.$$

Each $\mathbf{N}(f)$ is $\mathbf{A}_{F,L}$-linear and commutes with every $\gamma_{u,\mathbf{N}}$, so it maps site components to site components ($e_a \in \mathbf{A}_{F,L}$) and Δ-invariant elements to Δ-invariant elements; thus all four maps preserve $\mathbf{N}_a^\Delta(-)$.

The functor side. $\bar T, \bar T', \bar T''$ are continuous finite representations of G_F over k, hence $\mathcal{F}$-objects (Ff1), and all maps above are equivariant. By (Mf3), $\mathcal{F}(0)_a = \mathcal{F}(0 + 1\cdot 0)_a = 2\,\mathcal{F}(0)_a$, so $\mathcal{F}(0)_a = 0$; hence $\mathcal{F}(\bar p)_a\mathcal{F}(\bar\imath)_a = \mathrm{id}$, $\mathcal{F}(\bar p')_a\mathcal{F}(\bar\imath)_a = 0$, $\mathcal{F}(\bar p)_a\mathcal{F}(\bar\imath')_a = 0$, $\mathcal{F}(\bar p')_a\mathcal{F}(\bar\imath')_a = \mathrm{id}$ and $\mathcal{F}(\bar\imath)_a\mathcal{F}(\bar p)_a + \mathcal{F}(\bar\imath')_a\mathcal{F}(\bar p')_a = \mathcal{F}(\bar\imath\bar p + \bar\imath'\bar p')_a = \mathrm{id}$. So $\mathcal{F}(\bar\imath)_a$, $\mathcal{F}(\bar\imath')_a$ are injective and $V_{\bar T''}$ is the direct sum of their images.

The lattices. Let $x \in \mathbf{N}_a^\Delta(T'')$. Since $\rho_{T'',a}$ is additive, (Rd4) for i and i' gives

$$\rho_{T'',a}(x) = \rho_{T'',a}(\mathbf{N}(i)\mathbf{N}(p)x) + \rho_{T'',a}(\mathbf{N}(i')\mathbf{N}(p')x) = \mathcal{F}(\bar\imath)_a\rho_{T,a}(\mathbf{N}(p)x) + \mathcal{F}(\bar\imath')_a\rho_{T',a}(\mathbf{N}(p')x),$$

with $\mathbf{N}(p)x \in \mathbf{N}_a^\Delta(T)$, $\mathbf{N}(p')x \in \mathbf{N}_a^\Delta(T')$; so $\bar N_a(T'')$ lies in the right-hand side. Conversely, for $y \in \mathbf{N}_a^\Delta(T)$, (Rd4) gives $\mathcal{F}(\bar\imath)_a\rho_{T,a}(y) = \rho_{T'',a}(\mathbf{N}(i)y) \in \bar N_a(T'')$, likewise for i', and $\bar N_a(T'') = \rho_{T'',a}(\mathbf{N}_a^\Delta(T''))$ is closed under addition. This proves the equality. □

Lemma 28.12. *Let ℓ, F, n, L, (τ_a), ϖ, k, (φ,γ) and $\mathcal{F}$ be as in Proposition 28.7, (V,T) a pair, and $w, w'\colon \mathbb{Z}/n \to \mathbb{Z}_{\ge 0}$ such that $-w_a \le j \le w'_a$ whenever $\mathrm{gr}^j D_{\mathrm{dR},\tau_a}(V) \neq 0$ ($a \in \mathbb{Z}/n$). Then the family $(\bar N_a(T))_{a\in\mathbb{Z}/n}$ of lattices of $V_{\bar T}$ has heights (w,w'), and every $\bar N_a(T)$ satisfies the Γ-condition at a.*

Proof. Write $\mathbf{N} = \mathbf{N}(T)$, $N_a = \mathbf{N}_a^{\Delta}(T)$, $\bar{N}_a = \bar{N}_a(T)$, $\rho_a = \rho_{T,a}$, $q = \pi_0 + \ell$. The Γ-condition is part of Lemma 28.13. By (Ex1), $\mathbf{N}$ is a Wach module with window $(w_a, w_a')_a$, so by (Wm2)

$$Q^{w_a}\varphi_{\mathbf{N}}(\mathbf{N}_{a-1}) \subseteq \mathbf{N}_a, \qquad Q^{w_a'}\mathbf{N}_a \subseteq \mathbf{A}_{F,L}\,\varphi_{\mathbf{N}}(\mathbf{N}_{a-1}).$$

By (Rr3), (Rr4) of Lemma 23.24 (for $\Delta = \mu_{\ell-1}$), $q = Q\,w_0$ with w_0 a unit of $\mathbb{Z}_\ell[[X]]$, so Q may be replaced by q in both inclusions. Moreover q is Δ-invariant and $\varphi_{\mathbf{N}}$ commutes with every γ_δ (Wm3).

First inclusion. Let $y \in N_{a-1}$. Then $q^{w_a}\varphi_{\mathbf{N}}(y) \in \mathbf{N}_a$ is Δ-invariant, so lies in N_a, and by (Rd2) $X^{w_a}\varphi_a(\rho_{a-1}(y)) = \rho_a(q^{w_a}\varphi_{\mathbf{N}}(y)) \in \bar{N}_a$. Hence $\varphi_a(\bar{N}_{a-1}) \subseteq X^{-w_a}\bar{N}_a$, and since $X^{-w_a}\bar{N}_a$ is a $k[[X]]$-module, $\varphi_a^*\bar{N}_{a-1} \subseteq X^{-w_a}\bar{N}_a$.

Second inclusion. By Lemma 24.15, $\mathbf{N}_{a-1}$ has an $e_{a-1}\mathbf{A}_{F,L}$-basis $(n_i)_{i\le r}$ of Δ-invariant elements; by semilinearity $\mathbf{A}_{F,L}\varphi_{\mathbf{N}}(\mathbf{N}_{a-1}) = \sum_i e_a\mathbf{A}_{F,L}\,\varphi_{\mathbf{N}}(n_i)$. Let $x \in N_a$. Then $q^{w_a'}x = \sum_i g_i\varphi_{\mathbf{N}}(n_i)$ with $g_i \in e_a\mathbf{A}_{F,L}$. Apply $e_\Delta = (\ell-1)^{-1}\sum_{\delta\in\mu_{\ell-1}}\gamma_\delta$: each γ_δ is semilinear and fixes x, q and $\varphi_{\mathbf{N}}(n_i)$, so $q^{w_a'}x = \sum_i g_i'\varphi_{\mathbf{N}}(n_i)$ with $g_i' = e_\Delta(g_i)$ Δ-invariant. Multiplying by q^{w_a}, $q^{w_a+w_a'}x = \sum_i g_i'\,q^{w_a}\varphi_{\mathbf{N}}(n_i)$ with every $q^{w_a}\varphi_{\mathbf{N}}(n_i) \in N_a$ (first part). Applying ρ_a with (Rd1) (for the Δ-invariant elements $q^{w_a+w_a'}$ and g_i', $\bar{q} = X$ as $\ell \in \varpi\mathcal{O}_L$ and $\bar{\pi}_0 = X$ by Lemma 28.9) and (Rd2):

$$X^{w_a+w_a'}\rho_a(x) = \sum_i \bar{g}_i'\,X^{w_a}\varphi_a(\rho_{a-1}(n_i)).$$

As X is invertible on $V_{\bar{T}}$, $X^{w_a'}\rho_a(x) = \sum_i \bar{g}_i'\varphi_a(\rho_{a-1}(n_i)) \in \varphi_a^*\bar{N}_{a-1}$. Hence $X^{w_a'}\bar{N}_a \subseteq \varphi_a^*\bar{N}_{a-1}$. □

Lemma 28.13. *Let ℓ, F, n, L, (τ_a), ϖ, k, (φ,γ) and $\mathcal{F}$ be as in Proposition 28.7, (V,T) a pair, $a \in \mathbb{Z}/n$, and U_a the multiset of integers in which j has multiplicity $\dim_L \mathrm{gr}^j D_{\mathrm{dR},\tau_a}(V)$. Then the lattice $\bar{N}_a(T)$ of $V_{\bar{T}}$ satisfies the Γ-condition at a, $\nabla_a \circ \rho_{T,a} = \rho_{T,a} \circ \nabla$ on $\mathbf{N}_a^{\Delta}(T)$ (∇ formed with $u_0 = 1+\ell$ as in Lemma 28.15), and*

$$\mathrm{sen}_a(\bar{N}_a(T); x) = \#\{u \in U_a :\ u \cdot 1_k = x\} \qquad (x \in k).$$

Proof. Write $N = \mathbf{N}_a^{\Delta}(T)$, $\bar{N} = \bar{N}_a(T)$, $\rho = \rho_{T,a}$, $q = \pi_0 + \ell$, and take $u_0 = 1 + \ell$ (so $v_\ell(u_0 - 1) = 1$).

Step 1: X and ∇. π_0 is Δ-invariant (Lemma 28.15 (1)) and its reduction is $X_0 = \iota(X)$ (Lemma 28.9), so $\bar{\pi}_0 = X$ and (Rd1) gives $\rho(\pi_0 x) = X\rho(x)$; hence $\rho(\pi_0 N) = X\bar{N}$. For $x \in N$, $\nabla x \in N$ and $\pi_0\nabla x = \gamma_{1+\ell}(x) - x$ (Lemma 28.15 (1)), so by (Rd1) $X\rho(\nabla x) = \gamma_a(\rho x) - \rho x$, i.e. $\rho(\nabla x) = \nabla_a(\rho x)$, as X is invertible on $V_{\bar{T}}$. Therefore $\nabla_a(\bar{N}) = \rho(\nabla N) \subseteq \bar{N}$, which is the Γ-condition $(\gamma_a - 1)\bar{N} \subseteq X\bar{N}$. Also $\nabla_a(X\bar{N}) \subseteq X\bar{N}$: $\nabla_a(Xv) = X^{-1}(\gamma(X)\gamma_a(v) - Xv) = (\gamma_a(v) - v) - \kappa X\gamma_a(v)$ with $\gamma(X) = X(1-\kappa X)$, and both terms lie in $X\bar{N}$ for $v \in \bar{N}$.

Step 2: the quotient $\bar{N}/X\bar{N}$. By (Rd3), ρ induces $N/\varpi N \cong \bar{N}$, and by Step 1 this carries $(\pi_0 N + \varpi N)/\varpi N$ onto $X\bar{N}$. Since $q - \pi_0 = \ell \in \varpi\mathcal{O}_L$, $\pi_0 N + \varpi N = qN + \varpi N$, so

$$\bar{N}/X\bar{N} \cong N/(qN + \varpi N) = (N/qN) \otimes_{\mathcal{O}_L} k,$$

and by Step 1 the endomorphism of $\bar{N}/X\bar{N}$ induced by ∇_a corresponds to the reduction modulo ϖ of the endomorphism $\bar{\nabla}$ of N/qN induced by ∇ (which is defined as $\nabla(qN) \subseteq qN$).

Step 3: eigenvalues. By Lemma 28.14, N/qN is free of rank $r = \dim_L V = |U_a|$ over $\mathcal{O}_L$ and $\bar{\nabla}$ has characteristic polynomial $\prod_{u\in U_a}(T - c(u))$. Its reduction, the endomorphism of the r-dimensional k-space $\bar{N}/X\bar{N}$, has characteristic polynomial $\prod_{u\in U_a}(T - \bar{c}(u))$ with $\bar{c}(u)$ the

residue of $c(u)$. Here $\ell c(u) = 1-(1+\ell)^{-u}$ and $(1+\ell)^{-u} \equiv 1-u\ell \pmod{\ell^2}$, so $c(u) \equiv u \pmod{\ell}$ and $\bar{c}(u) = u \cdot 1_k$.

Step 4: Sen multiplicities. By definition $\mathrm{sen}_a(\bar{N}; x) = \dim_k \bar{N}^{(x)}/(\bar{N}^{(x)} \cap X\bar{N})$ with $\bar{N}^{(x)} = \{v \in \bar{N} : \ (\nabla_a - x)^j v \in X\bar{N} \text{ for some } j\}$; since ∇_a preserves $\bar{N}$ and $X\bar{N}$ (Step 1), $\bar{N}^{(x)}/(\bar{N}^{(x)} \cap X\bar{N})$ is the generalised x-eigenspace of the induced endomorphism of $\bar{N}/X\bar{N}$. Its dimension is the multiplicity of x as a root of the characteristic polynomial, which by Step 3 is $\#\{u \in U_a : \ u \cdot 1_k = x\}$. □

Lemma 28.14. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ be as in Proposition 23.7, (V,T) a pair, $\mathbf{N} = \mathbf{N}(T)$, $r = \dim_L V$, $a \in \mathbb{Z}/n$, and let U_a be the multiset of integers in which j has multiplicity $\dim_L \mathrm{gr}^j D_{\mathrm{dR},\tau_a}(V)$. Form u_0, $c(j)$ and ∇ as in Lemma 28.15 (for $T_1 = T_2 = T$, at the site a), let $\mathbf{N}_a^{\Delta}$ be the set of Δ-invariant elements of $\mathbf{N}_a$, and put $q = \pi_0 + \ell$. Then $\nabla(q\mathbf{N}_a^{\Delta}) \subseteq q\mathbf{N}_a^{\Delta}$, the quotient $\mathbf{N}_a^{\Delta}/q\mathbf{N}_a^{\Delta}$ is a free $\mathcal{O}_L$-module of rank r, and the $\mathcal{O}_L$-linear endomorphism of it induced by ∇ has characteristic polynomial*

$$\prod_{u\in U_a} (T - c(u)) \in \mathcal{O}_L[T].$$

Moreover, for every integer $s \geq 1$, with $I_s = \{u - b : \ u \in U_a,\ 0 \leq b < s\}$ (a set of integers, each value taken once),

$$\prod_{m\in I_s} (\nabla - c(m))(\mathbf{N}_a^{\Delta}) \subseteq q^s\, \mathbf{N}_a^{\Delta}.$$

Proof. Write $\mathbf{N} = \mathbf{N}(T)$, $N = \mathbf{N}_a^{\Delta}$, $q = \pi_0 + \ell$, and fix $s \geq 1$ (for the first assertions $s = 1$ suffices). By Lemma 23.24 (Rr4) (with $\Delta = \mu_{\ell-1}$, $\pi_\Delta = \pi_0$), $q = Qw$ with $w \in \mathbb{Z}_\ell[[X]]^\times$; q is Δ-invariant.

Step 0: level data at the site a. Let $h \geq 0$ bound the Hodge jumps of V at τ_a from above; by (Ex1), $\mathbf{N}$ is a Wach module with a window having $h_a = h$, so by (Wm2) $Q^h\mathbf{N}_a \subseteq \mathbf{A}_{F,L}\varphi_{\mathbf{N}}(\mathbf{N}_{a-1}) = P_1(\mathbf{N}_{a-1})$. Fix arc level data for (V,T) (and for $\Phi = \mathrm{id}_T$) with cut $c = a$, length 1, $H(a) = s + h$ (other values of H and all other parameters arbitrary) and a frame at $a-1$, i.e. an $e_{a-1}\mathbf{A}_{F,L}$-basis of $\mathbf{N}_{a-1}$ (it exists by (Wm1); its horizontal sections exist by (Hf2)). Then $\Pi = Q^{s+h}$, and by Lemma 23.17 and Lemma 23.5 the properties (Lp1), (Lp2), (Lp3), (Lp8) hold at the only level 0. Write $K = K_{(0)}$, $\lambda = \lambda_0$, $\mathrm{lv} = \mathrm{lv}_0$. By Definition 23.4, $\epsilon_0 = \lambda(e_a)$ is the idempotent of τ_a, $(D_k^{(0)})_k$ is an L-basis of $\epsilon_0 D_{\mathrm{cris}}(V)$, and $\mathrm{Fil}_0(u)$ is the preimage of Fil^u under $v \mapsto \sum_k v_k D_k^{(0)}$. Since F is unramified, $D_{\mathrm{dR}}(V) = D_{\mathrm{cris}}(V)$ with the same filtration, and its τ_a-component is $D_{\mathrm{dR},\tau_a}(V)$ with its filtration; so Fil_0 is a decreasing, exhaustive and separated filtration of L^r whose jumps, with multiplicity $\dim_L \mathrm{Fil}_0(u)/\mathrm{Fil}_0(u+1)$, form the multiset U_a. Choose an L-basis $(v_k)_{k\leq r}$ of L^r adapted to it: integers u_k with $\{u_k\} = U_a$ as multisets and $\mathrm{Fil}_0(u) = \mathrm{span}_L\{v_k : \ u_k \geq u\}$.

Step 1: the lines. Let $\Xi \subseteq K((t))^r$ be the set of de Rham integral vectors at level 0: $f \in \Xi$ if and only if the t^i-coefficient of f lies in $K \otimes_L \mathrm{Fil}_0(-i)$ for every i. Writing $f = \sum_k g_k v_k$ with $g_k \in K((t))$ (the v_k are a K-basis of K^r), the t^i-coefficient $\sum_k g_{k,i} v_k$ lies in $\mathrm{span}_K\{v_k : \ u_k \geq -i\}$ if and only if $g_{k,i} = 0$ whenever $i < -u_k$. Hence

$$\Xi = \bigoplus_k \Xi_k, \qquad \Xi_k = t^{-u_k} K[[t]]\, v_k,$$

and Ξ, every Ξ_k and every $t^j\Xi_k$ are $K[[t]]$-modules.

Step 2: membership modulo powers of q. By (Lp1), $\lambda(Q) \in tK[[t]]^\times$ and $\lambda(e_a\mathbf{A}_{F,L}) \subseteq K[[t]]$; as w and w^{-1} lie in $\mathbb{Z}_\ell[[X]]$, $\lambda(w) \in K[[t]]^\times$, so $\lambda(q) = t\epsilon$ with $\epsilon \in K[[t]]^\times$. Let $y \in \mathbf{N}_a$

and $0 \le k \le s$. Then $q^{-k}y \in e_a\mathbf{N}[1/Q]$ and $\Pi q^{-k}y = Q^{s+h-k}w^{-k}y \in P_1(\mathbf{N}_{a-1})$ by Step 0, so $q^{-k}y \in W$; in particular $\mathbf{N}_a \subseteq W$. By (Lp1) every element of W is algebraic at level 0, and $\mathrm{lv}(gx) = \lambda(g)\mathrm{lv}(x)$ (as $\Lambda_0(gx) = \lambda(g)\Lambda_0(x)$). By (Lp3),

$$y \in q^k\mathbf{N}_a \iff q^{-k}y \in \mathbf{N}_a \iff t^{-k}\epsilon^{-k}\mathrm{lv}(y) \in \Xi \iff \mathrm{lv}(y) \in t^k\Xi. \qquad \text{(A)}$$

For $\lambda' \in \mathcal{O}_L$, $\mathrm{lv}(\lambda' y) = \lambda'\mathrm{lv}(y)$, as λ maps the constant $e_a(1 \otimes \lambda')$ to $\epsilon_0\lambda' = \lambda'$.

Step 3: ∇ on the lines. By (Lp8) with $s = 0$, $\sigma_{u_0} = \mathrm{id}$ on K ($u_0 \in 1 + \ell\mathbb{Z}_\ell$); so $\gamma^{(0)}_{u_0}$ is K-linear with $\gamma^{(0)}_{u_0}(gv) = \gamma^{(0)}_{u_0}(g)v$ for $g \in K((t))$, $v \in L^r$, and $\gamma^{(0)}_{u_0}(\sum_i g_i t^i) = \sum_i u_0^i g_i t^i$. Next, $\lambda(\pi_0) = \lambda(q) - \ell = t\epsilon - \ell$ is a unit of $K[[t]]$ with constant term $-\ell$. Put $\tilde{\nabla} = \lambda(\pi_0)^{-1}(\gamma^{(0)}_{u_0} - 1)$ on $K((t))^r$. For $x \in N$, $\nabla x \in N$ and $\pi_0\nabla x = \gamma_{u_0}x - x$ (Lemma 28.15 (1)); both lie in $\mathbf{N}_a \subseteq W$, so by (Lp2) $\lambda(\pi_0)\mathrm{lv}(\nabla x) = \gamma^{(0)}_{u_0}\mathrm{lv}(x) - \mathrm{lv}(x)$, i.e.

$$\mathrm{lv}(\nabla x) = \tilde{\nabla}\,\mathrm{lv}(x). \qquad \text{(B)}$$

$\tilde{\nabla}$ preserves $t^m K[[t]]v_k$ for all m, k, and for $g \in t^m K[[t]]$ with leading coefficient g_m, $\tilde{\nabla}g \equiv (-\ell)^{-1}(u_0^m - 1)g_m t^m \pmod{t^{m+1}}$; as $(1 - u_0^m)/\ell = c(-m)$,

$$(\tilde{\nabla} - c(-m))(t^m K[[t]]v_k) \subseteq t^{m+1}K[[t]]v_k.$$

Applying this with $m = b - u_k$, $b = 0, \dots, s-1$ (the factors commute, being polynomials in $\tilde{\nabla}$):

$$\prod_{b=0}^{s-1}(\tilde{\nabla} - c(u_k - b))\,\Xi_k \subseteq t^s\Xi_k. \qquad \text{(C)}$$

Step 4: the product over I_s. The values $u_k - b$ ($0 \le b < s$) are distinct elements of I_s, and c is injective on $\mathbb{Z}$ (Lemma 28.15 (2): $v_\ell(c(i) - c(j)) = v_\ell(i - j) < \infty$ for $i \ne j$). Hence $P(T) = \prod_{m \in I_s}(T - c(m))$ is the product of $\prod_{b<s}(T - c(u_k - b))$ with a polynomial, for every k, and by (C), since ∇ preserves $t^s\Xi_k$, $P(\tilde{\nabla})\Xi \subseteq t^s\Xi$. Let $x \in N$. Then $P(\nabla)x \in N$ (Lemma 28.15 (1), P has coefficients in $\mathbb{Z}_\ell$), $\mathrm{lv}(x) \in \Xi$ by (A) with $k = 0$, and $\mathrm{lv}(P(\nabla)x) = P(\tilde{\nabla})\mathrm{lv}(x) \in t^s\Xi$ by (B); by (A) with $k = s$, $P(\nabla)x = q^s z$ with $z \in \mathbf{N}_a$, and z is Δ-invariant because $P(\nabla)x$ and q are and $\mathbf{N}_a$ is torsion-free. So $P(\nabla)N \subseteq q^s N$. Likewise, with $s = 1$: for $z \in N$, $\mathrm{lv}(qz) = t\epsilon\,\mathrm{lv}(z) \in t\Xi$, $\tilde{\nabla}(t\Xi) \subseteq t\Xi$, so $\nabla(qz) \in q\mathbf{N}_a \cap N = qN$ by (A) and (B); this is $\nabla(qN) \subseteq qN$.

Step 5: freeness. By Lemma 24.15, $\mathbf{N}_a$ has a basis $(n_k)_{k \le r}$ over $R = e_a\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ of Δ-invariant vectors; an element $\sum_k g_k n_k$ is Δ-invariant if and only if every g_k is, so $N = \bigoplus_k R^\Delta n_k$ and $qN = \bigoplus_k qR^\Delta n_k$. By (Rr2), $R^\Delta = \mathcal{O}_L[[\pi_0]]$, so $R^\Delta/qR^\Delta \cong \mathcal{O}_L[[T]]/(T + \ell) \cong \mathcal{O}_L$ ($T \mapsto -\ell$). Hence N/qN is free of rank r over $\mathcal{O}_L$.

Step 6: the characteristic polynomial. By (A) with $k = 0, 1$, $x \mapsto \mathrm{lv}(x) \bmod t\Xi$ induces an injective $\mathcal{O}_L$-linear map $\bar{\ell}\colon N/qN \to \Xi/t\Xi$, intertwining the endomorphism $\bar{\nabla}$ induced by ∇ with that induced by $\tilde{\nabla}$, by (B). By Step 1, $\Xi/t\Xi = \bigoplus_k Kt^{-u_k}v_k$, and by Step 3 $\tilde{\nabla}$ acts on the k-th summand as the scalar $c(u_k)$. For $\delta \in \mu_{\ell-1}$ and $x \in N$, $\mathrm{lv}(x) = \mathrm{lv}(\gamma_\delta x) = \gamma^{(0)}_\delta\mathrm{lv}(x)$ by (Lp2); on the k-th summand $\gamma^{(0)}_\delta$ induces $at^{-u_k}v_k \mapsto \delta^{-u_k}\sigma_\delta(a)t^{-u_k}v_k$. So the image of $\bar{\ell}$ lies in $Y = \bigoplus_k Y_k$, $Y_k = \{a \in K :\ \sigma_\delta(a) = \delta^{u_k}a\ \forall \delta \in \mu_{\ell-1}\}\,t^{-u_k}v_k$. Now $K = \epsilon_0(F(\zeta_\ell) \otimes_{\mathbb{Q}_\ell} L) \cong \mathbb{Q}_\ell(\zeta_\ell) \otimes_{\mathbb{Q}_\ell} L$ (the τ_a-component; $F(\zeta_\ell) = F \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(\zeta_\ell)$ as F is unramified), with σ_δ acting through $\zeta_\ell \mapsto \zeta_\ell^\delta$, i.e. through $\mathrm{Gal}(\mathbb{Q}_\ell(\zeta_\ell)/\mathbb{Q}_\ell) \cong \mu_{\ell-1}$. By the normal basis theorem $\mathbb{Q}_\ell(\zeta_\ell) \cong \mathbb{Q}_\ell[\mu_{\ell-1}]$, so $K \cong L[\mu_{\ell-1}]$ as $L[\mu_{\ell-1}]$-modules, and for the character $\delta \mapsto \delta^{u_k}$ (values in $\mu_{\ell-1} \subset L^\times$) the eigenspace has L-dimension 1: $\dim_L Y_k = 1$, $\dim_L Y = r$. The L-linear extension $L \otimes_{\mathcal{O}_L} N/qN \to Y$ of $\bar{\ell}$ is injective (N/qN is free and Y is an L-space), and both sides have dimension r, so it is an isomorphism intertwining $\bar{\nabla}$ with the operator acting on Y_k by $c(u_k)$. Hence the characteristic polynomial of $\bar{\nabla}$ on $L \otimes N/qN$, and so on the free $\mathcal{O}_L$-module N/qN, is $\prod_k(T - c(u_k)) = \prod_{u \in U_a}(T - c(u))$. □

Lemma 28.15. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ be as in Proposition 23.7, let $\Phi\colon T_1 \to T_2$ be a G_F-equivariant $\mathcal{O}_L$-linear map of the lattices of pairs (V_1,T_1), (V_2,T_2), and let $c \in \mathbb{Z}/n$. Call an element y of $\mathbf{N}(T_\nu)_c$ or of $e_c\mathbf{A}_{F,L}$* Δ-invariant *if $\gamma_\delta(y) = y$ for every $\delta \in \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$, and let $\mathbf{N}_\nu^\Delta$ and $\mathbf{N}_{\nu,\infty}^\Delta$ be the sets of Δ-invariant elements of $\mathbf{N}(T_\nu)_c$ and of $e_c\mathbf{N}(T_\nu)[1/Q]$ ($\nu = 1,2$); $\mathbf{N}(\Phi)$ is extended $\mathbf{A}_{F,L}[1/Q]$-linearly to $\mathbf{N}(T_1)[1/Q]$. Fix $u_0 \in \mathbb{Z}_\ell^\times$ with $v_\ell(u_0-1) = 1$, for $j \in \mathbb{Z}$ let $c(j) \in \mathbb{Z}_\ell$ be given by $\ell\, c(j) = 1 - u_0^{-j}$, and for a finite set $J \subset \mathbb{Z}$ put $P_J(Y) = \prod_{j\in J}(Y - c(j)) \in \mathbb{Z}_\ell[Y]$.*

1. *π_0 is Δ-invariant, $\pi_0 = X^{\ell-1}v$ and $\pi_0 + \ell = Q\,w$ with units v, w of $\mathbb{Z}_\ell[[X]]$. For $\nu = 1,2$ and $x \in \mathbf{N}_{\nu,\infty}^\Delta$ there is a unique $\nabla x \in e_c\mathbf{N}(T_\nu)[1/Q]$ with $\pi_0\,\nabla x = \gamma_{u_0}(x) - x$; moreover $\nabla x \in \mathbf{N}_{\nu,\infty}^\Delta$, and $Q^Y x \in \mathbf{N}(T_\nu)$ for an integer $Y \geq 0$ implies $Q^Y \nabla x \in \mathbf{N}(T_\nu)$ (so ∇ maps $\mathbf{N}_\nu^\Delta$ into itself). The map ∇ is $\mathcal{O}_L$-linear, $\mathbf{N}(\Phi)$ maps $\mathbf{N}_{1,\infty}^\Delta$ into $\mathbf{N}_{2,\infty}^\Delta$ and $\mathbf{N}_1^\Delta$ into $\mathbf{N}_2^\Delta$, and $\mathbf{N}(\Phi)\circ\nabla = \nabla\circ\mathbf{N}(\Phi)$ on $\mathbf{N}_{1,\infty}^\Delta$.*
2. *$c(j) \in \mathbb{Z}_\ell$ indeed, and $v_\ell(c(i) - c(j)) = v_\ell(i-j)$ for all integers $i \neq j$.*
3. (Separation lift.) *Let $J_1, J_2 \subset \mathbb{Z}$ be finite and disjoint, and $E \geq 0$ an integer with*
$$\sum_{j\in J_1\setminus\{i\}} v_\ell(i-j) + \sum_{j\in J_2} v_\ell(i-j) \leq E \qquad \text{for every } i \in J_1.$$
If $x \in \mathbf{N}_2^\Delta$ is such that $P_{J_1}(\nabla)x$ and $P_{J_2}(\nabla)x$ both lie in $\mathbf{N}(\Phi)(\mathbf{N}_1^\Delta)$, then $\ell^E x \in \mathbf{N}(\Phi)(\mathbf{N}_1^\Delta)$.
4. (The interval configuration.) *Let $a \geq 1$ and $b, S, \eta \geq 0$ be integers with $2a + \eta \leq b$, and $J_1 = \{b-a+1, \dots, b\}$, $J_2 = \{b+1, \dots, b+S+a\} \cup \{1-\eta, \dots, \eta+a\}$. Then $J_1 \cap J_2 = \emptyset$ and, for every $i \in J_1$,*
$$\sum_{j\in J_1\setminus\{i\}} v_\ell(i-j) + \sum_{j\in J_2} v_\ell(i-j) \leq \frac{3a+S+2\eta-1}{\ell-1} + \log_\ell(b+\eta).$$

Proof. **Item 1.** *The element π_0.* Apply Lemma 23.24 with $\Delta = \mu_{\ell-1}$ (so $e = 1$ and $\pi_\Delta = \pi_0$) and $R = R_c \cong \mathcal{O}_L[[X]]$: by (Rr1), π_0 is Δ-invariant and $\pi_0 = X^{\ell-1}v$ with $v = v_\Delta \in \mathbb{Z}_\ell[[X]]^\times$; by (Rr3), $q_\Delta = \pi_0 + \ell$, so by (Rr4) $\pi_0 + \ell = Qw$ with $w = w_\Delta \in \mathbb{Z}_\ell[[X]]^\times$; and by (Rr2), $R_c^\Delta \cap XR_c = \pi_0 R_c^\Delta$, that is, $(XR_c)^\Delta = \pi_0 R_c^\Delta$.

∇. By (Ex1) $\mathbf{N}(T_\nu)$ is a Wach module (for any window containing the finitely many Hodge jumps), so by (Wm1) $\mathbf{N}(T_\nu)_c$ is free over the domain R_c; hence $\pi_0 \neq 0$ is a non-zero-divisor on it and ∇x is unique if it exists. By Lemma 24.15, $\mathbf{N}(T_\nu)_c$ has an R_c-basis (n_k) of Δ-invariant vectors. Let $x \in \mathbf{N}_\nu^\Delta$ and $z = \gamma_{u_0}(x) - x$. Since γ_{u_0} is semilinear and fixes e_c, $z \in \mathbf{N}(T_\nu)_c$, and by (Wm3) $z \in X\mathbf{N}(T_\nu) \cap \mathbf{N}(T_\nu)_c = X\mathbf{N}(T_\nu)_c$; as $\gamma_\delta\gamma_{u_0} = \gamma_{\delta u_0} = \gamma_{u_0}\gamma_\delta$ (Wm3), z is Δ-invariant. Write $z = \sum_k f_k n_k$ with $f_k \in R_c$; by uniqueness of coordinates, $f_k \in XR_c$, and $\gamma_\delta(z) = \sum_k \gamma_\delta(f_k)n_k = z$ gives $\gamma_\delta(f_k) = f_k$. So $f_k = \pi_0 g_k$ with $g_k \in R_c^\Delta$, and $\nabla x = \sum_k g_k n_k$ satisfies $\pi_0\nabla x = z$ and is Δ-invariant. $\mathcal{O}_L$-linearity follows from uniqueness, since γ_{u_0} is $\mathcal{O}_L$-linear. By (Ex3) $\mathbf{N}(\Phi)$ is $\mathbf{A}_{F,L}$-linear (so commutes with e_c and with multiplication by π_0) and commutes with every γ_u; hence it maps $\mathbf{N}_1^\Delta$ into $\mathbf{N}_2^\Delta$, and for $x \in \mathbf{N}_1^\Delta$

$$\pi_0\,\mathbf{N}(\Phi)(\nabla x) = \mathbf{N}(\Phi)(\gamma_{u_0}x - x) = \gamma_{u_0}\mathbf{N}(\Phi)(x) - \mathbf{N}(\Phi)(x) = \pi_0\,\nabla\mathbf{N}(\Phi)(x),$$

so $\mathbf{N}(\Phi)(\nabla x) = \nabla\mathbf{N}(\Phi)(x)$ by uniqueness.

Poles. Since $\pi_0+\ell = Qw$ with w a unit, $\mathbf{N}(T_\nu)[1/Q] = \mathbf{N}(T_\nu)[1/(\pi_0+\ell)]$, and π_0 is a non-zero-divisor on it (a localisation of a free module over a domain at a non-zero element), so ∇x is unique if it exists. As $\gamma_{u_0}(Q) = \varphi(\gamma_{u_0}X)/\gamma_{u_0}X = Q\cdot\varphi(g)/g$ with $g = ((1+X)^{u_0}-1)/X$ a unit, $\gamma_{u_0}(\pi_0+\ell) = (\pi_0+\ell)t$ with t a unit of $\mathbb{Z}_\ell[[X]]$; t is Δ-invariant (γ_{u_0} commutes with the γ_δ and $\pi_0+\ell$ is Δ-invariant), and $t(0)=1$ because γ_{u_0} preserves constant terms and $(\pi_0+\ell)(0) = \ell \neq 0$. So $t^{-Y}-1 \in (XR_c)^\Delta = \pi_0 R_c^\Delta$ for every $Y\geq 0$. Let $x\in \mathbf{N}_{\nu,\infty}^\Delta$ and $Y\geq 0$ with $Q^Y x \in \mathbf{N}(T_\nu)$ (such Y exists); then $m = (\pi_0+\ell)^Y x = w^Y Q^Y x \in \mathbf{N}_\nu^\Delta$, and

$$\gamma_{u_0}(x)-x = (\pi_0+\ell)^{-Y}\big((t^{-Y}-1)\gamma_{u_0}(m)+(\gamma_{u_0}(m)-m)\big) = \pi_0\,(\pi_0+\ell)^{-Y}\Big(\frac{t^{-Y}-1}{\pi_0}\gamma_{u_0}(m)+\nabla m\Big),$$

where $\gamma_{u_0}(m)\in\mathbf{N}_\nu^\Delta$ and $(t^{-Y}-1)/\pi_0 \in R_c^\Delta$. So $\nabla x = (\pi_0+\ell)^{-Y}(\pi_0^{-1}(t^{-Y}-1)\gamma_{u_0}(m) + \nabla m)$ exists, is Δ-invariant, and $Q^Y\nabla x = w^{-Y}(\pi_0^{-1}(t^{-Y}-1)\gamma_{u_0}(m)+\nabla m) \in \mathbf{N}(T_\nu)$. $\mathcal{O}_L$-linearity follows from uniqueness. The extended $\mathbf{N}(\Phi)$ is $\mathbf{A}_{F,L}[1/Q]$-linear and commutes with the semilinear extensions of the γ_u (on m/Q^Y both sides give $\mathbf{N}(\Phi)(\gamma_u m)/\gamma_u(Q)^Y$), so it maps $\mathbf{N}_{1,\infty}^\Delta$ into $\mathbf{N}_{2,\infty}^\Delta$, and the computation displayed above for $\mathbf{N}_1^\Delta$, with uniqueness on $e_c\mathbf{N}(T_2)[1/Q]$, gives $\mathbf{N}(\Phi)\circ\nabla = \nabla\circ\mathbf{N}(\Phi)$ on $\mathbf{N}_{1,\infty}^\Delta$.

Item 2. As ℓ is odd and $v_\ell(u_0^{-1}-1) = v_\ell(u_0-1) = 1$, Lemma 23.22 (item 1, with $m=1$) gives $v_\ell(u_0^k-1) = 1+v_\ell(k)$ for every integer $k\neq 0$. So $1-u_0^{-j}\in\ell\mathbb{Z}_\ell$ for all j (it is 0 for $j=0$) and $c(j)\in\mathbb{Z}_\ell$. For $i\neq j$, $\ell(c(i)-c(j)) = u_0^{-j}-u_0^{-i} = u_0^{-i}(u_0^{i-j}-1)$ has valuation $1+v_\ell(i-j)$, as u_0 is a unit.

Item 3. *A Bezout identity.* Write $P_\nu = P_{J_\nu}$, monic in $\mathbb{Z}_\ell[Y]$. By item 2 the $c(j)$, $j\in J_1\cup J_2$, are pairwise distinct. For $i\in J_1$, $P_2(c(i)) = \prod_{j\in J_2}(c(i)-c(j))$ and $P_1'(c(i)) = \prod_{j\in J_1\setminus\{i\}}(c(i)-c(j))$ are non-zero, and by item 2 and the hypothesis on E, $v_\ell(P_2(c(i))) + v_\ell(P_1'(c(i))) \leq E$. Put

$$B = \sum_{i\in J_1} \frac{\ell^E}{P_2(c(i))\,P_1'(c(i))}\cdot\frac{P_1(Y)}{Y-c(i)}.$$

Each fraction $\ell^E/(P_2(c(i))P_1'(c(i)))$ lies in $\mathbb{Z}_\ell$, and $P_1(Y)/(Y-c(i)) = \prod_{j\in J_1\setminus\{i\}}(Y-c(j)) \in \mathbb{Z}_\ell[Y]$; so $B\in\mathbb{Z}_\ell[Y]$. For $i'\in J_1$, the summands with $i\neq i'$ vanish at $c(i')$, and the summand $i=i'$ takes the value $\ell^E/P_2(c(i'))$ there; so $\ell^E - B\,P_2$ vanishes at the $|J_1|$ distinct roots $c(i')$ of the monic polynomial P_1. Division with remainder by the monic P_1 in $\mathbb{Z}_\ell[Y]$ gives $\ell^E - BP_2 = AP_1 + C$ with $A, C\in\mathbb{Z}_\ell[Y]$ and $\deg C < |J_1|$; C vanishes at $|J_1|$ distinct points, so $C=0$. Hence $AP_1+BP_2 = \ell^E$. (If $J_1=\emptyset$: $B=0$, $A=\ell^E$.)

The lift. Let $y_\nu\in\mathbf{N}_1^\Delta$ with $\mathbf{N}(\Phi)(y_\nu) = P_\nu(\nabla)x$. By item 1, ∇ maps $\mathbf{N}_1^\Delta$ to itself and $\mathbf{N}(\Phi)\circ Z(\nabla) = Z(\nabla)\circ\mathbf{N}(\Phi)$ on $\mathbf{N}_1^\Delta$ for every $Z\in\mathbb{Z}_\ell[Y]$ (by induction on the degree, from $\mathcal{O}_L$-linearity and the case $Z=Y$). So $y = A(\nabla)y_1 + B(\nabla)y_2 \in \mathbf{N}_1^\Delta$ and

$$\mathbf{N}(\Phi)(y) = A(\nabla)P_1(\nabla)x + B(\nabla)P_2(\nabla)x = (AP_1+BP_2)(\nabla)x = \ell^E x.$$

Item 4. Every element of J_1 is at least $b-a+1 \geq a+\eta+1$, hence larger than every element of $\{1-\eta,\dots,\eta+a\}$, and at most b; so $J_1\cap J_2 = \emptyset$. Fix $i\in J_1$. We use Legendre's formula in the form $v_\ell(m!) = \sum_{t\geq 1}\lfloor m/\ell^t\rfloor \leq m/(\ell-1)$, and the fact that among k consecutive integers at most k/ℓ^t+1 are divisible by ℓ^t, and none if all are positive and smaller than ℓ^t.

The interval $\{b-a+1,\dots,b+S+a\}$. The $j\neq i$ in it are the $j\in J_1\setminus\{i\}$ and the $j\in\{b+1,\dots,b+S+a\}$. The differences $i-j$ are $1,\dots,p$ and $-1,\dots,-q$ with $p = i-(b-a+1)$, $q = b+S+a-i$, $p+q = S+2a-1$. Their valuations sum to $v_\ell(p!)+v_\ell(q!) \leq (S+2a-1)/(\ell-1)$.

The interval $\{1-\eta,\dots,\eta+a\}$. The differences $i-j$ are the $k = 2\eta+a$ consecutive integers from $i-\eta-a \geq b-2a-\eta+1 \geq 1$ to $i+\eta-1 \leq b+\eta-1$. So

$$\sum_j v_\ell(i-j) = \sum_{t\geq 1}\#\{j:\ \ell^t \mid i-j\} \leq \sum_{t\geq 1,\ \ell^t\leq b+\eta-1}\Big(\frac{k}{\ell^t}+1\Big) \leq \frac{2\eta+a}{\ell-1} + \log_\ell(b+\eta).$$

Adding the two bounds gives item 4. □

Lemma 28.16. *Let ℓ, F, n, L, (τ_a), ϖ, k, (φ,γ) and $\mathcal{F}$ be as in Proposition 28.7, (V,T) a pair, $a \in \mathbb{Z}/n$, U_a the multiset of integers in which j has multiplicity $\dim_L \mathrm{gr}^j D_{\mathrm{dR},\tau_a}(V)$, and $s \geq 1$ an integer such that for every $u \in U_a$ and every integer $0 \leq b < s$ with $u \equiv b \pmod{\ell}$ we have $u = b$. Then there is $f \in k[T]$ with $f(0) \neq 0$ and*

$$\nabla_a f(\nabla_a)(\bar{N}_a(T)) \subseteq X^s\, \bar{N}_a(T).$$

Proof. Write $N = \mathbf{N}_a^{\Delta}(T)$, $\bar{N} = \bar{N}_a(T)$, $\rho = \rho_{T,a}$, $q = \pi_0 + \ell$, and let $I_s = \{u - b :\ u \in U_a,\ 0 \leq b < s\}$.

Step 1: killing on the Wach lattice. By Lemma 28.14, $P(\nabla)(N) \subseteq q^s N$ for $P(T) = \prod_{m\in I_s}(T - c(m)) \in \mathbb{Z}_\ell[T]$.

Step 2: reduction. The elements of $\mathcal{O}_L$ and q are Δ-invariant elements of $e_a\mathbf{A}_{F,L}$; the residue of $c \in \mathcal{O}_L$ is $\bar{c} \in k$, and the reduction of q is the reduction $X_0 = \iota(X)$ of π_0 (as $\ell \in \varpi\mathcal{O}_L$), so $\bar{q} = X$. By (Rd1), $\rho(cx) = \bar{c}\rho(x)$ and $\rho(q^s y) = X^s\rho(y)$; by Lemma 28.13, $\rho \circ \nabla = \nabla_a \circ \rho$. Applying ρ to Step 1 gives $\bar{P}(\nabla_a)(\bar{N}) \subseteq X^s\bar{N}$, where $\bar{P} = \prod_{m\in I_s}(T - \bar{c}(m))$. Since $\ell c(m) = 1 - (1+\ell)^{-m} \equiv m\ell \pmod{\ell^2}$, $\bar{c}(m) = m \cdot 1_k$.

Step 3: the factor T. If $m \in I_s$ and $m\cdot 1_k = 0$, then $m = u - b$ with $u \in U_a$, $0 \leq b < s$, $u \equiv b \pmod{\ell}$, so $u = b$ by hypothesis and $m = 0$. Hence $\bar{P} = T^{\varepsilon} f(T)$ with $f = \prod_{m\in I_s,\, m\neq 0}(T - m\cdot 1_k)$, $f(0) \neq 0$, and $\varepsilon = 1$ if $0 \in I_s$, $\varepsilon = 0$ otherwise (each value of I_s occurs once). If $\varepsilon = 1$ we are done. If $\varepsilon = 0$, $f(\nabla_a)\bar{N} \subseteq X^s\bar{N}$, and it remains to see $\nabla_a(X^s\bar{N}) \subseteq X^s\bar{N}$: for $v \in \bar{N}$, $\gamma_a(v) = v + X\nabla_a(v) \in \bar{N}$ by the Γ-condition (Lemma 28.13), and

$$\begin{aligned}\nabla_a(X^s v) &= X^{-1}\big(X^s(1-\kappa X)^s\gamma_a(v) - X^s v\big)\\ &= X^{s-1}\Big((\gamma_a(v) - v) + \big((1-\kappa X)^s - 1\big)\gamma_a(v)\Big) \in X^{s-1}\cdot X\bar{N}.\end{aligned}$$

□

Lemma 28.17. *Let ℓ, F, n, L, (τ_a), ϖ, k, (φ,γ) and $\mathcal{F}$ be as in Proposition 28.7, and (V,T) a pair with $\dim_L V = 1$; for $a \in \mathbb{Z}/n$ let u_a be the integer with $\mathrm{gr}^{u_a} D_{\mathrm{dR},\tau_a}(V) \neq 0$. Then $\varphi_a^*\bar{N}_{a-1}(T) = X^{u_a}\,\bar{N}_a(T)$ for every $a \in \mathbb{Z}/n$.*

Proof. Write $\mathbf{N} = \mathbf{N}(T)$, $N_a = \mathbf{N}_a^{\Delta}(T)$, $\bar{N}_a = \bar{N}_a(T)$, $\rho_a = \rho_{T,a}$, $q = \pi_0 + \ell$, $\gamma = \gamma_{1+\ell}$, and $R = \mathcal{O}_L[[\pi_0]] \subseteq e_a\mathbf{A}_{F,L}$, the ring of Δ-invariant elements ((Rr2) of Lemma 23.24); q is a prime element of $R \cong \mathcal{O}_L[[T]]$ ($R/qR \cong \mathcal{O}_L$), and by (Rr3), (Rr4) of Lemma 23.24 $q = Qw_0$ with $w_0 \in \mathbb{Z}_\ell[[X]]^\times$. Fix a primitive ℓ-th root of unity $\zeta \in L$ and $\alpha = \zeta - 1$; evaluation at $X = \alpha$ is defined on $\mathcal{O}_L[[X]]$, $\pi_0(\alpha) = -\ell$ ((Rr3)), so $q(\alpha) = 0$ and $g \equiv g(\alpha) \pmod{qR}$ for $g \in R$.

Step 1: bases. By Lemma 24.15 and (Wm1) (rank 1 by (Ex1)), $\mathbf{N}_a = e_a\mathbf{A}_{F,L}n_a$ with n_a Δ-invariant; if gn_a is Δ-invariant then so is g, so $N_a = Rn_a$, and by (Rd1), (Rd3) $\bar{N}_a = k[[X]]\rho_a(n_a)$ with $\rho_a(n_a) \neq 0$.

Step 2: $\varphi(n_{a-1}) = v\,q^j n_a$ with $v \in R^\times$, $j \in \mathbb{Z}$. Let $M = \max(-u_a, 0)$, $h = \max(u_a, 0)$ (and at the other sites any window containing the jumps); by (Ex1) and (Wm2), $q^M\varphi(n_{a-1}) \in \mathbf{N}_a$ and $q^h n_a \in e_a\mathbf{A}_{F,L}\varphi(n_{a-1})$. Write $q^M\varphi(n_{a-1}) = g\,n_a$; g is Δ-invariant (φ commutes with the γ_δ), so $g \in R$; and $q^h n_a = e\,\varphi(n_{a-1})$ with $e \in e_a\mathbf{A}_{F,L}$, which gives $eg = q^{M+h}$, so $e = q^{M+h}/g$ is Δ-invariant and lies in R. Thus g divides q^{M+h} in the factorial ring R, so $g = v\,q^i$ with $v \in R^\times$, and $j = i - M$.

Step 3: $h(\alpha) = 1+\ell$ for $\gamma(q) = hq$. By (Rr6) of Lemma 23.24, $\gamma(q) = hq$ with $h \in \mathbb{Z}_\ell[[X]]$. Since $q(\alpha) = 0$, $q = (X-\alpha)r$ with $r \in \mathcal{O}_L[[X]]$, and $r(\alpha) \neq 0$ because $Q = \prod_{\zeta'}(X - (\zeta'-1))$ (ζ' the primitive ℓ-th roots of unity) has simple roots and $w_0(\alpha) \neq 0$. Now $\gamma(X) - \alpha = (1+X)^{1+\ell} - \zeta^{1+\ell} = (X-\alpha)S$ with $S = \sum_{i=0}^{\ell}(1+X)^i\zeta^{\ell-i}$, $S(\alpha) = (1+\ell)\zeta^\ell = 1+\ell$, and $\gamma(X)(\alpha) = \alpha$. So $h(X-\alpha)r = \gamma(q) = (\gamma(X)-\alpha)\,r(\gamma(X)) = (X-\alpha)S\,r(\gamma(X))$; cancelling $X-\alpha$ and evaluating at α gives $h(\alpha)r(\alpha) = (1+\ell)r(\alpha)$.

Step 4: $j = u_a$. Write $\nabla n_{a-1} = b\,n_{a-1}$, $\nabla n_a = b' n_a$ with $b, b' \in R$ (Lemma 28.15 (1)). By Lemma 28.14 (rank one), $b' \equiv c(u_a) \pmod{qR}$, so $b'(\alpha) = c(u_a)$. By (Wm3), $\gamma(\varphi(n_{a-1})) = \varphi(\gamma(n_{a-1})) = \varphi(n_{a-1}) + \varphi(\pi_0)\varphi(b)\varphi(n_{a-1})$, and $\gamma(\varphi(n_{a-1})) = \gamma(vq^j)(1+\pi_0 b')n_a$. With $f = vq^j$ this gives, in the fraction field of R,

$$\frac{\gamma(v)}{v}\,h^j\,(1+\pi_0 b') = 1 + \varphi(\pi_0)\varphi(b).$$

All terms lie in $\mathcal{O}_L[[X]]$; evaluate at α: $\gamma(v)(\alpha) = v(\gamma(X)(\alpha)) = v(\alpha) \neq 0$, $h(\alpha) = 1+\ell$ (Step 3), $\pi_0(\alpha) = -\ell$, and $\varphi(\pi_0)(\alpha) = \pi_0((1+\alpha)^\ell - 1) = \pi_0(0) = 0$ (on site components φ is the substitution $X \mapsto (1+X)^\ell - 1$ on $\mathcal{O}_L[[X]]$). Hence $(1+\ell)^j(1-\ell c(u_a)) = 1$; as $1 - \ell c(u_a) = (1+\ell)^{-u_a}$ and $1+\ell$ has infinite order, $j = u_a$.

Step 5: reduction. If $u_a \geq 0$, $\varphi(n_{a-1}) = vq^{u_a}n_a \in N_a$ and (Rd2) (with $j=0$), (Rd1) give $\varphi_a(\rho_{a-1}(n_{a-1})) = \bar{v}X^{u_a}\rho_a(n_a)$. If $u_a < 0$, $q^{-u_a}\varphi(n_{a-1}) = vn_a$, and (Rd2), (Rd1) give $X^{-u_a}\varphi_a(\rho_{a-1}(n_{a-1})) = \bar{v}\rho_a(n_a)$. In both cases $\varphi_a(\rho_{a-1}(n_{a-1})) = \bar{v}X^{u_a}\rho_a(n_a)$ with $\bar{v} \in k[[X]]^\times$ ($\bar{q} = X$, since $\ell \in \varpi\mathcal{O}_L$ and the reduction of π_0 is $\iota(X)$ by Lemma 28.9). By Step 1, $\varphi_a^*\bar{N}_{a-1} = k[[X]]\varphi_a(\rho_{a-1}(n_{a-1})) = X^{u_a}\bar{N}_a$. □

Lemma 28.18. *Let ℓ be a prime, $n \geq 1$, $t \geq 1$, $h, w, \eta\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$, $\mathrm{typ}\colon \mathbb{Z}/n \to \{\mathrm{low}, \mathrm{high}\}$ and finite multisets U_a of integers satisfy (Ph1)–(Ph5) for ℓ. Let $a \in \mathbb{Z}/n$, $s = \mathrm{sg}(a)$ and $u \in U_a$. Then:*

(Wa1) *$u \not\equiv y \pmod{\ell}$ for every integer y with $w_a - s - 2 \leq y \leq w_a + t + 1 + s$;*

(Wa2) *if $\mathrm{typ}(a) = \mathrm{low}$: $u \not\equiv y \pmod{\ell}$ for $-s-1 \leq y \leq 0$, and for $0 \leq y \leq w_a - 1$, $u \equiv y \pmod{\ell}$ holds if and only if $u = y$;*

(Wa3) *if $\mathrm{typ}(a) = \mathrm{high}$: $u \not\equiv y \pmod{\ell}$ for $h_a \leq y \leq h_a + s + 1$, and for $w_a \leq y \leq h_a - 1$, $u \equiv y \pmod{\ell}$ holds if and only if $u = y$.*

Proof. Write $h = h_a$, $w = w_a$, $\eta = \eta_a$, $w^* = h - w$. From (Ph1): $1 \leq u \leq h-1$, $w \geq 1$, $w + t - 1 \leq h - 1$, so $w^* \geq t \geq 1$. From (Ph2): $w + t - 1 \leq \eta$ if $\mathrm{typ}(a) = \mathrm{low}$, and $h - \eta \leq w$, i.e. $w^* \leq \eta$, if $\mathrm{typ}(a) = \mathrm{high}$. From (Ph3): $\eta + s + 2 < \ell$. Recall $W_a = [1, \eta]$ (low), $W_a = [h-\eta, h-1]$ (high), and the range of (Ph5) is $R = [-2s-2, \eta + 2s + 3]$. Two integers whose difference has absolute value in $[1, \ell - 1]$ are not congruent modulo ℓ.

(Wa1). Let $w - s - 2 \leq y \leq w + t + 1 + s$. If $\mathrm{typ}(a) = \mathrm{low}$, then $-s - 1 \leq y \leq \eta + s + 2$; if $\mathrm{typ}(a) = \mathrm{high}$, then $h - \eta - s - 2 \leq y \leq h + s + 1$. *Case $u \in W_a$.* The point y' of $[w, w+t-1]$ nearest to y has $|y - y'| \leq s + 2$, and $|u - y'| > s + 2$ by (Ph4), so $u \neq y$. If $\mathrm{typ}(a) = \mathrm{low}$, then $u \in [1, \eta]$ and $|u - y| \leq \eta + s + 2$; if $\mathrm{typ}(a) = \mathrm{high}$, then $u \in [h-\eta, h-1]$ and again $|u - y| \leq \eta + s + 2$. As $\eta + s + 2 < \ell$, $u \not\equiv y$. *Case $u \notin W_a$, low.* $u' = u$ and $y \in [-s-1, \eta + s + 2] \subseteq R$, so $u \not\equiv y$ by (Ph5). *Case $u \notin W_a$, high.* $u' = h - u$ and $m = h - y$ satisfies $w^* - t - 1 - s \leq m \leq w^* + s + 2$, so $-s - 1 \leq m \leq \eta + s + 2$ and $m \in R$; by (Ph5) $h - u \not\equiv h - y$, i.e. $u \not\equiv y$.

(Wa2). Let $\mathrm{typ}(a) = \mathrm{low}$. If $u \in W_a = [1, \eta]$: for $-s - 1 \leq y \leq 0$, $1 \leq u - y \leq \eta + s + 1 < \ell$; for $0 \leq y \leq w - 1 \leq \eta$, $|u - y| \leq \eta < \ell$, so $u \equiv y$ if and only if $u = y$. If $u \notin W_a$, then $u > \eta \geq w - 1$ by (Ph1), and every y in $[-s-1, w-1] \subseteq R$ has $u \not\equiv y$ by (Ph5) and $u \neq y$.

(Wa3). Let $\mathrm{typ}(a) = \mathrm{high}$. If $u \in W_a = [h-\eta, h-1]$: for $h \le y \le h+s+1$, $1 \le y-u \le \eta+s+1 < \ell$; for $w \le y \le h-1$, both lie in $[h-\eta, h-1]$, so $|u-y| \le \eta-1 < \ell$ and $u \equiv y$ if and only if $u = y$. If $u \notin W_a$, then $u < h-\eta \le w$ by (Ph1), and for $w \le y \le h+s+1$ the integer $m = h-y$ lies in $[-s-1, w^*] \subseteq R$, so $h-u \not\equiv h-y$ by (Ph5), i.e. $u \not\equiv y$, and $u \ne y$. $\square$

28.3 Polar pairs and the polar count

The decisive estimate of the section is the polar count Theorem 28.38: in a residual (φ,Γ)-module with a family of lattices satisfying the Sen conditions (Sn1)–(Sn3), the number of polar pairs that are independent modulo Herr boundaries is at most the sum of the low counts d_a. Through the largest standard model of the residual representation (Proposition 28.19) the admissible cocycles become polar pairs (Lemma 28.26), so the polar count bounds the admissible classes in (Rc2) of Proposition 28.3; the characters are treated separately (Proposition 28.35, Lemma 28.37). The other lemmas establish the properties of models that these steps use: their Sen multiplicities, and their behaviour under sums, subquotients and composition series. The existence of ψ-operators (Proposition 28.41) supplies a hypothesis of the polar count.

Proposition 28.19. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ,γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ,γ) with sites $\mathbb{Z}/n$, and let $h, w, \eta\colon \mathbb{Z}/n \to \mathbb{Z}_{\ge 0}$, typ, $t \ge 1$ and finite multisets U_a of integers of cardinality r satisfy (Ph1)–(Ph5) for ℓ. Let residual model data $(\bar T, \omega, \dots)$ for $(h, w, \mathrm{typ}, t, (U_a))$ and $\mathcal{F}$ be given, and put $d_a = \#\{u \in U_a :\ u < w_a\}$. Then there is a family $\mathbf{N} = (N_a)$ of lattices of $V_{\bar T}$ such that:*

(Mp1) $\mathbf{N}$ *has heights (w, w^*), each N_a satisfies the Γ-condition at a, and for every a:*

- $\mathrm{sen}_a(N_a; j) = 0$ *for* $0 \le j \le \mathrm{sg}(a)+1$*;*
- *if* $\mathrm{typ}(a) = \mathrm{low}$*, then* $\sum_{j=0}^{w_a-1} \mathrm{sen}_a(N_a; j-w_a) = d_a$*;*
- *if* $\mathrm{typ}(a) = \mathrm{high}$*, then* $d_a + \sum_{j=0}^{w_a^*} \mathrm{sen}_a(N_a; j) = r$ *and* $\nabla_a f(\nabla_a)(N_a) \subseteq X^{w_a^*+1} N_a$ *for some* $f \in k[T]$ *with* $f(0) \ne 0$*;*

(Mp2) *for model-admissible cocycles $z_1, \dots, z_s$ of $\bar T$ there are $\zeta_1, \dots, \zeta_s \in Z_{\mathrm{pol}}(\mathbf{N})$ (polar pairs of $\mathcal{F}(\bar T)$ for the letter data h, w, typ) such that for all $\beta_1, \dots, \beta_s \in k$: if $\sum_i \beta_i \zeta_i \in B(\mathcal{F}(\bar T))$, then there is $y \in \bar T$ with $\sum_i \beta_i z_i(g) = gy - y$ for all $g \in G$.*

Proof. Write $s_a = \mathrm{sg}(a)$; it is the shift $\sigma(a)$ of Lemma 28.23, and $h \ge 0$. (Wa1)–(Wa3) denote the conclusions of Lemma 28.18. For an integer y, $\mathrm{sen}_a(N; y)$ means $\mathrm{sen}_a(N; y \cdot 1_k)$. By (Md6), $(\bar T, \mathbf{N}^{\mathrm{std}})$ is a model and

$$\mathrm{sen}_a(N_a^{\mathrm{std}}; y) = \#\{u \in U_a :\ u \equiv y + w_a \pmod{\ell}\} \qquad (y \in \mathbb{Z}). \tag{$*$}$$

Choice of N. By Lemma 28.20 (2), $\bar T$ has a largest model $\mathbf{N}$; in particular $N_a^{\mathrm{std}} \subseteq N_a$. By Lemma 28.21 (3), $\bar T$ is an $\mathcal{F}$-object and $\mathbf{N}$, $\mathbf{N}^{\mathrm{std}}$ have heights $(w, w^*) = (w, h-w)$, with all members satisfying the Γ-condition. By Lemma 28.23 (pair $\mathbf{N}, \mathbf{N}^{\mathrm{std}}$), $X^{s_a} N_a \subseteq N_a^{\mathrm{std}}$, i.e. $N_a^{\mathrm{std}} \subseteq N_a \subseteq X^{-s_a} N_a^{\mathrm{std}}$. Fix a, and let q be the function of Lemma 28.24 for $P = N_a$, $Q = N_a^{\mathrm{std}}$ and $s = s_a$.

(R) Sen weights of N. If $\mathrm{sen}_a(N_a; x) \ne 0$ $(x \in k)$, then $x = (u - w_a) \cdot 1_k$ for some $u \in U_a$: this is Lemma 28.32 (1) for $C = A' = \bar T$ with i, q the identity, applied to the model $(\bar T, \mathbf{N})$.

(Q) $q(0) = 0$**.** Otherwise, by Lemma 28.24 (2), $\mathrm{sen}_a(N_a^{\mathrm{std}}; -j) \ne 0$ for some $1 \le j \le s_a$, and by $(*)$ some $u \in U_a$ is congruent to $w_a - j \in [w_a - s_a, w_a - 1]$, contradicting (Wa1).

(Mp1). Heights and the Γ-condition were noted above. *Vanishing.* If $\mathrm{sen}_a(N_a; j) \neq 0$ with $0 \leq j \leq s_a + 1$, then by (R) some $u \in U_a$ is congruent to $w_a + j \in [w_a, w_a + s_a + 1]$, contradicting (Wa1). *Telescoping.* Summing Lemma 28.24 (1) over $x = y \cdot 1_k$ for $y = y_0, \ldots, y_1 - 1$ gives

$$\sum_{y=y_0}^{y_1-1} \mathrm{sen}_a(N_a; y) + q(y_1) = \sum_{y=y_0}^{y_1-1} \mathrm{sen}_a(N_a^{\mathrm{std}}; y) + q(y_0) \qquad (y_0 \leq y_1). \tag{T}$$

Low sites. Let $\mathrm{typ}(a) = \mathrm{low}$; apply (T) with $y_0 = -w_a$, $y_1 = 0$. Here $q(0) = 0$ by (Q), and $q(-w_a) = 0$: otherwise Lemma 28.24 (2) and $(*)$ give $u \in U_a$ congruent to $-j$ for some $1 \leq j \leq s_a$, contradicting (Wa2). By $(*)$, $\sum_{y=-w_a}^{-1} \mathrm{sen}_a(N_a^{\mathrm{std}}; y) = \sum_{y'=0}^{w_a-1} \#\{u \in U_a : \ u \equiv y'\}$, and by (Wa2) $u \equiv y'$ holds for $0 \leq y' \leq w_a - 1$ only if $u = y'$; so each $u \in U_a$ is counted once if $u \leq w_a - 1$ and not at all otherwise (recall $u \geq 1$ by (Ph1)), and the sum is $\#\{u \in U_a : \ u < w_a\} = d_a$. Hence $\sum_{j=0}^{w_a-1} \mathrm{sen}_a(N_a; j - w_a) = \sum_{y=-w_a}^{-1} \mathrm{sen}_a(N_a; y) = d_a$. *High sites.* Let $\mathrm{typ}(a) = \mathrm{high}$; apply (T) with $y_0 = 0$, $y_1 = L = w_a^* + s_a + 2$. Here $q(0) = 0$ by (Q), and $q(L) = 0$: otherwise Lemma 28.24 (2) and $(*)$ give $u \in U_a$ congruent to $L - j + w_a = h_a + s_a + 2 - j \in [h_a + 2, h_a + s_a + 1]$, contradicting (Wa3). For $w_a^* + 1 \leq y \leq L - 1$, $\mathrm{sen}_a(N_a; y) = 0$: otherwise (R) gives $u \in U_a$ congruent to $y + w_a \in [h_a + 1, h_a + s_a + 1]$, contradicting (Wa3). By $(*)$, $\sum_{y=0}^{L-1} \mathrm{sen}_a(N_a^{\mathrm{std}}; y) = \sum_{y''=w_a}^{h_a+s_a+1} \#\{u \in U_a : \ u \equiv y''\}$; by (Wa3) no u is congruent to $y'' \in [h_a, h_a + s_a + 1]$, and for $y'' \in [w_a, h_a - 1]$, $u \equiv y''$ only if $u = y''$; so the sum is $\#\{u \in U_a : \ w_a \leq u \leq h_a - 1\} = \#\{u \in U_a : \ u \geq w_a\} = r - d_a$, using $u \leq h_a - 1$ (Ph1) and $\#U_a = r$. Hence $d_a + \sum_{j=0}^{w_a^*} \mathrm{sen}_a(N_a; j) = r$. *The polynomial.* By Lemma 28.28, applied to the model $(\bar{T}, \mathbf{N})$, there is $f \in k[T]$ with $f(0) \neq 0$ and $\nabla_a f(\nabla_a)(N_a) \subseteq X^{w_a^*+1} N_a$.

(Mp2). By the choice of $\mathbf{N}$, $(\bar{T}, \mathbf{N})$ is a model containing every model of $\bar{T}$; Lemma 28.26 with $A = \bar{T}$ and $\mathbf{N}' = \mathbf{N}$ gives the polar pairs $\zeta_i \in Z_{\mathrm{pol}}(\mathbf{N})$ with the stated property. □

Lemma 28.20. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ, γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ, γ) with sites $\mathbb{Z}/n$, and residual model data for letter data h, w, typ, $t \geq 1$, multisets (U_a) and $\mathcal{F}$. Let A be a finite representation.*

1. *If $(A, \mathbf{N})$ and $(A, \mathbf{N}')$ are models, then so is $(A, (N_a + N'_a)_a)$.*
2. *If $h \geq 0$ and A has a model, then A has a largest model: a model $(A, \mathbf{N}^{\max})$ with $N_a \subseteq N_a^{\max}$ for every model $(A, \mathbf{N})$ and every a.*

Proof. For equivariant maps f, f' between $\mathcal{F}$-objects, (Mf3) gives $\mathcal{F}(f \circ f')_a = \mathcal{F}(f)_a \circ \mathcal{F}(f')_a$, $\mathcal{F}(f + f')_a = \mathcal{F}(f)_a + \mathcal{F}(f')_a$, $\mathcal{F}(\mathrm{id})_a = \mathrm{id}$, and hence $\mathcal{F}(0)_a = 0$ (from $\mathcal{F}(0)_a = \mathcal{F}(0 + 0)_a = 2\mathcal{F}(0)_a$). Below, all maps are equivariant and all representations involved are $\mathcal{F}$-objects (this is checked where they appear); we write f also for $\mathcal{F}(f)_a$ at a fixed site a.

Part 1. Write $\mathbf{N} = p_* j^* \mathbf{N}_B$ and $\mathbf{N}' = p'_* j'^* \mathbf{N}_{B'}$ with base pairs $(B, \mathbf{N}_B)$, $(B', \mathbf{N}_{B'})$, injective $j \colon A_1 \to B$, $j' \colon A_2 \to B'$ and surjective $p \colon A_1 \to A$, $p' \colon A_2 \to A$. By (Md4) there are C and $i \colon B \to C$, $i' \colon B' \to C$ with $i \oplus i'$ bijective such that $(C, \mathbf{N}_C)$, $N_{C,a} = i(N_{B,a}) + i'(N_{B',a})$, is a base pair. Let $\pi \colon C \to B$, $\pi' \colon C \to B'$ be the projections of the decomposition $C = i(B) \oplus i'(B')$; they are equivariant, and $\pi i = \mathrm{id}$, $\pi i' = 0$, $\pi' i' = \mathrm{id}$, $\pi' i = 0$. Let $S = A_1 \oplus A_2$ with inclusions α_1, α_2 and projections q_1, q_2, and put

$$J = i\, j\, q_1 + i'\, j'\, q_2 \colon S \to C, \qquad P = p\, q_1 + p'\, q_2 \colon S \to A.$$

Then $\pi J = j q_1$ and $\pi' J = j' q_2$, so $Jx = 0$ forces $q_1 x = 0 = q_2 x$ (as j, j' are injective), i.e. J is injective; $P \alpha_1 = p$ is onto, so P is surjective. By (Md2) B, B', C are $\mathcal{F}$-objects, and by (Mf1) so are A_1, A_2 (subrepresentations of B, B'), S (via J, a subrepresentation of C) and A (a quotient of A_1). We show $P_* J^* \mathbf{N}_C = (N_a + N'_a)_a$, so that $(A, (N_a + N'_a)_a)$ is a model. Fix a.

$\supseteq$ Let $y \in V_{A_1}$ with $j(y) \in N_{B,a}$, and $s = \alpha_1(y)$. Since $J\alpha_1 = ij$ and $P\alpha_1 = p$, $J(s) = i(j(y)) \in N_{C,a}$ and $P(s) = p(y)$. So $p(j^{-1}(N_{B,a})) = N_a \subseteq P(J^{-1}(N_{C,a}))$; likewise N'_a is contained in it, and so is $N_a + N'_a$, as $P(J^{-1}(N_{C,a}))$ is a $k[[X]]$-module.

$\subseteq$ Let $s \in V_S$ with $J(s) = i(u) + i'(u')$, $u \in N_{B,a}$, $u' \in N_{B',a}$. Applying π gives $j(q_1(s)) = \pi J(s) = u \in N_{B,a}$, so $q_1(s) \in j^{-1}(N_{B,a})$; applying π' gives $q_2(s) \in j'^{-1}(N_{B',a})$. Hence $P(s) = p(q_1(s)) + p'(q_2(s)) \in N_a + N'_a$.

Part 2. Let $(A, \mathbf{N}^0)$ be a model. By Lemma 28.21 (3), every model $(A, \mathbf{N})$ has heights $(w, w^*) = (w, h - w)$, and $h \geq 0$, so Lemma 28.23 (with the pair $\mathbf{N}, \mathbf{N}^0$) gives $X^{\sigma(a)} N_a \subseteq N^0_a$, i.e. $N_a \subseteq P_a := X^{-\sigma(a)} N^0_a$, with σ as there. Each P_a is a finitely generated module over the noetherian ring $k[[X]]$, hence noetherian. Fix a. The set $\{N_a : (A, \mathbf{N}) \text{ a model}\}$ of submodules of P_a is non-empty, so it has a maximal element $N^{(a)}_a$, the a-component of a model $\mathbf{N}^{(a)}$. For every model $\mathbf{N}$, $\mathbf{N}^{(a)} + \mathbf{N}$ is a model by Part 1, with a-component $N^{(a)}_a + N_a \supseteq N^{(a)}_a$; by maximality the two are equal, so $N_a \subseteq N^{(a)}_a$. Let $\mathbf{N}^{\max} = \sum_{a \in \mathbb{Z}/n} \mathbf{N}^{(a)}$ (componentwise), a model by Part 1 applied $n - 1$ times. For every model $\mathbf{N}$ and every a, $N_a \subseteq N^{(a)}_a \subseteq N^{\max}_a$. $\square$

Lemma 28.21. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ, γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ, γ) with sites $\mathbb{Z}/n$, and $w, w' \colon \mathbb{Z}/n \to \mathbb{Z}$.*

1. *If $f \colon A \to B$ is an injective equivariant map of $\mathcal{F}$-objects and $\mathbf{N}$ is a family of lattices of V_B with heights (w, w') such that each N_a satisfies the Γ-condition at a, then $f^*\mathbf{N}$ is a family of lattices of V_A with heights (w, w'), each of whose members satisfies the Γ-condition at a.*
2. *If $f \colon A \to B$ is a surjective equivariant map of $\mathcal{F}$-objects and $\mathbf{N}$ is a family of lattices of V_A with heights (w, w') such that each N_a satisfies the Γ-condition at a, then $f_*\mathbf{N}$ is a family of lattices of V_B with the same properties.*
3. *Given residual model data for letter data h, w, typ, $t \geq 1$, multisets (U_a) and $\mathcal{F}$, and a model $(A, \mathbf{N})$: A is an $\mathcal{F}$-object, $\mathbf{N}$ has heights (w, w^*), and each N_a satisfies the Γ-condition at a.*

Proof. Throughout, $\mathcal{F}(f)_a$ is $k((X))$-linear, injective (surjective) when f is, and $\mathcal{F}(f)_a \circ \varphi_a = \varphi_a \circ \mathcal{F}(f)_{a-1}$, $\mathcal{F}(f)_a \circ \gamma_a = \gamma_a \circ \mathcal{F}(f)_a$, by (Mf3), since A, B are $\mathcal{F}$-objects and f is equivariant. The ring $k[[X]]$ is a principal ideal domain, so a finitely generated torsion-free $k[[X]]$-module is free.

Part 1. Write $j_a = \mathcal{F}(f)_a \colon V_A \to V_B$, $W_a = j_a(V_A)$, and $M_a = j_a^{-1}(N_a)$, so that j_a maps M_a isomorphically onto $N_a \cap W_a$. *Lattice.* $N_a \cap W_a$ is a submodule of the finitely generated module N_a over the noetherian ring $k[[X]]$, hence finitely generated; for $v \in V_A$ there is m with $X^m j_a(v) \in N_a$ since N_a spans V_B, so $X^m v \in M_a$ and M_a spans V_A. *Γ-condition.* For $x \in M_a$, $j_a((\gamma_a - 1)x) = (\gamma_a - 1)j_a(x) \in XN_a$, and $j_a^{-1}(XN_a) = Xj_a^{-1}(N_a) = XM_a$ by $k((X))$-linearity of j_a; so $(\gamma_a - 1)M_a \subseteq XM_a$. *First height inclusion.* For $x \in M_{a-1}$, $j_a(\varphi_a x) = \varphi_a(j_{a-1}x) \in \varphi_a^* N_{a-1} \subseteq X^{-w_a} N_a$, so $\varphi_a(x) \in j_a^{-1}(X^{-w_a} N_a) = X^{-w_a} M_a$; as $X^{-w_a} M_a$ is a $k[[X]]$-module, $\varphi_a^* M_{a-1} \subseteq X^{-w_a} M_a$. *Second height inclusion.* We show

$$\varphi_a^* N_{a-1} \cap W_a = j_a(\varphi_a^* M_{a-1}). \tag{$*$}$$

The inclusion $\supseteq$ holds because $j_a(\varphi_a^* M_{a-1})$ is the $k[[X]]$-span of $\varphi_a(j_{a-1} M_{a-1}) \subseteq \varphi_a(N_{a-1})$ and lies in W_a. Conversely, by (Rm2) for $\mathcal{F}(A)$, $\varphi_a(V_A)$ spans V_A, so $W_a = j_a(V_A)$ is the $k((X))$-span of $j_a(\varphi_a(V_A)) = \varphi_a(W_{a-1})$. Let $v \in \varphi_a^* N_{a-1} \cap W_a$ and write $v = \sum_{i<\ell} X^i \varphi_a(v_i)$ with

$v_i \in V_B$ as in Lemma 28.22 (2) for $\mathcal{F}(B)$. By the same item, $v \in \varphi_a^* N_{a-1}$ gives $v_i \in N_{a-1}$, and $v \in W_a$, the span of $\varphi_a(W_{a-1})$, gives $v_i \in W_{a-1}$. So $v_i = j_{a-1}(m_i)$ with $m_i \in M_{a-1}$, and $v = \sum_i X^i \varphi_a(j_{a-1} m_i) = j_a(\sum_i X^i \varphi_a(m_i)) \in j_a(\varphi_a^* M_{a-1})$. This proves $(*)$. Now for $y \in M_a$: $j_a(X^{w'_a} y) = X^{w'_a} j_a(y) \in X^{w'_a} N_a \cap W_a \subseteq \varphi_a^* N_{a-1} \cap W_a = j_a(\varphi_a^* M_{a-1})$, and j_a is injective, so $X^{w'_a} M_a \subseteq \varphi_a^* M_{a-1}$.

Part 2. Write $q_a = \mathcal{F}(f)_a$, surjective, and $M_a = q_a(N_a)$: finitely generated, and spanning V_B since q_a is onto and $k((X))$-linear. Since $q_a \circ \varphi_a = \varphi_a \circ q_{a-1}$ and q_a is $k((X))$-linear, $q_a(\varphi_a^* N_{a-1}) = \varphi_a^*(q_{a-1} N_{a-1}) = \varphi_a^* M_{a-1}$. Applying q_a to $\varphi_a^* N_{a-1} \subseteq X^{-w_a} N_a$, $X^{w'_a} N_a \subseteq \varphi_a^* N_{a-1}$ and $(\gamma_a - 1) N_a \subseteq X N_a$ gives the corresponding statements for $\mathbf{M} = f_* \mathbf{N}$, using $q_a \circ \gamma_a = \gamma_a \circ q_a$.

Part 3. Let $\mathbf{N} = p_* j^* \mathbf{N}_B$ with $(B, \mathbf{N}_B)$ a base pair, $j\colon A_1 \to B$ injective and $p\colon A_1 \to A$ surjective, both equivariant. By (Md2), B is an $\mathcal{F}$-object and $\mathbf{N}_B$ has heights (w, w^*) with the Γ-condition at every a; by (Mf1), A_1 and then A are $\mathcal{F}$-objects. Part 1 applied to j and then Part 2 applied to p give the assertion for $\mathbf{N}$. □

Lemma 28.22. *Let ℓ be a prime, k a field of characteristic ℓ, (φ, γ) a residual ring datum over k, $D = (V, (\varphi_a), (\gamma_a))$ a residual (φ, Γ)-module over (φ, γ) with sites $\mathbb{Z}/n$, and $a \in \mathbb{Z}/n$.*

1. *On $k[[X]]$, φ is the substitution $c(X) \mapsto c(\varphi(X))$. Every $c \in k((X))$ can be written uniquely as $c = \sum_{i=0}^{\ell-1} X^i \varphi(c_i)$ with $c_i \in k((X))$, and $c \in k[[X]]$ if and only if all $c_i \in k[[X]]$.*
2. *Every $v \in V$ can be written uniquely as $v = \sum_{i=0}^{\ell-1} X^i \varphi_a(v_i)$ with $v_i \in V$. For a lattice M of V, $v \in \varphi_a^* M$ if and only if all $v_i \in M$; for a $k((X))$-subspace W of V, v lies in the $k((X))$-span of $\varphi_a(W)$ if and only if all $v_i \in W$.*
3. *For a lattice M of V, $x \in V$ and an integer $c \geq 0$: if $X^c \varphi_a(x) \in \varphi_a^* M$, then $X^{\lfloor c/\ell \rfloor} x \in M$.*

Proof. By (Rr2), $\varphi(X) = Y$ with $Y = uX^\ell$ and $u \in k[[X]]^\times$; let v_X be the X-adic valuation of $k((X))$, so $v_X(Y) = \ell$.

Part 1. For $c \in k[[X]]$ and $m \geq 0$ write $c = c_m(X) + X^m c'$ with c_m a polynomial and $c' \in k[[X]]$; since φ is a k-algebra map, $\varphi(c) = c_m(Y) + Y^m \varphi(c')$ with $\varphi(c') \in k[[X]]$ by (Rr1), so $\varphi(c) \equiv c(Y)$ modulo $X^{\ell m}$ for every m, whence $\varphi(c) = c(Y)$. Thus $\varphi(k[[X]]) = k[[Y]]$ and $\varphi(k((X))) = k((Y))$, and $v_X(\varphi(c)) = \ell\, v_X(c)$. Since $k[[X]]/Yk[[X]] = k[X]/(X^\ell)$ has the basis $1, X, \ldots, X^{\ell-1}$ and $k[[X]]$ is complete for the Y-adic topology (which is the X-adic one) and has no Y-torsion, $k[[X]] = \bigoplus_{i<\ell} X^i k[[Y]]$ (Weierstrass division); inverting Y gives $k((X)) = \bigoplus_{i<\ell} X^i k((Y))$. For $c = \sum_i X^i \varphi(c_i)$ the valuations $v_X(X^i \varphi(c_i)) = i + \ell v_X(c_i)$ $(c_i \neq 0)$ are pairwise distinct modulo ℓ, so

$$v_X(c) = \min_{i\colon c_i \neq 0} (i + \ell v_X(c_i)), \tag{$\dagger$}$$

and $c \in k[[X]]$ if and only if all $c_i \in k[[X]]$.

Part 2. Let $\Phi\colon k((X)) \otimes_{\varphi, k((X))} V \to V$, $c \otimes x \mapsto c\,\varphi_a(x)$, be the linearisation of φ_a (well defined by (Rm1)). It is surjective by (Rm2), and both sides have dimension $\dim_{k((X))} V$, so it is bijective. By Part 1, $k((X)) \otimes_\varphi V = \bigoplus_{i<\ell} X^i \otimes V$ as abelian groups, with $x \mapsto X^i \otimes x$ injective for each i; applying Φ gives existence and uniqueness of $v = \sum_i X^i \varphi_a(v_i)$. If all $v_i \in M$, then $v \in \varphi_a^* M$. Conversely an element of $\varphi_a^* M$ is $\sum_j c_j \varphi_a(m_j)$ with $c_j \in k[[X]]$, $m_j \in M$; writing $c_j = \sum_i X^i \varphi(c_{ji})$ with $c_{ji} \in k[[X]]$ (Part 1) and using (Rm1), it equals $\sum_i X^i \varphi_a(\sum_j c_{ji} m_j)$, whose coordinates $\sum_j c_{ji} m_j$ lie in M; by uniqueness these are the v_i. The same computation with $c_j \in k((X))$ and $m_j \in W$ proves the statement for W.

Part 3. Write $c = \ell q + r$ with $0 \le r < \ell$. Since $Y^q = u^q X^{\ell q}$, $X^c = X^r u^{-q} \varphi(X^q)$. By Part 1, $X^r u^{-q} = \sum_{i<\ell} X^i \varphi(b_i)$ with $b_i \in k[[X]]$, and by (Rm1)

$$X^c \varphi_a(x) = \sum_{i<\ell} X^i \varphi(b_i) \varphi(X^q) \varphi_a(x) = \sum_{i<\ell} X^i \varphi_a(b_i X^q x).$$

If this lies in $\varphi_a^* M$, then $b_i X^q x \in M$ for every i by Part 2. By (†), $r = v_X(X^r u^{-q}) = \min_i(i + \ell v_X(b_i))$; as $0 \le r < \ell$ and $v_X(b_i) \ge 0$, the minimum is attained only at $i = r$ with $v_X(b_r) = 0$, so b_r is a unit of $k[[X]]$ and $X^q x = b_r^{-1}(b_r X^q x) \in M$. Here $q = \lfloor c/\ell \rfloor$. □

Lemma 28.23. *Let ℓ be a prime, k a field of characteristic ℓ, (φ, γ) a residual ring datum over k, $D = (V, (\varphi_a), (\gamma_a))$ a residual (φ, Γ)-module over (φ, γ) with sites $\mathbb{Z}/n$, $h\colon \mathbb{Z}/n \to \mathbb{Z}_{\ge 0}$, $w\colon \mathbb{Z}/n \to \mathbb{Z}$, and*

$$\sigma(a) = \Big\lfloor \sum_{j \ge 1} h(a+j)\, \ell^{-j} \Big\rfloor \qquad (a \in \mathbb{Z}/n).$$

If $\mathbf{N} = (N_a)$ and $\mathbf{N}' = (N'_a)$ are families of lattices of V which both have heights $(w, h - w)$, then $X^{\sigma(a)} N_a \subseteq N'_a$ for every $a \in \mathbb{Z}/n$.

Proof. For each a let m_a be the least integer $m \ge 0$ with $X^m N_a \subseteq N'_a$; it exists because N_a is finitely generated and N'_a spans V over $k((X))$.

Recursion. We show $\ell\, m_{a-1} \le m_a + h_a$ for every a. Let $x \in N_{a-1}$. By the first height inclusion for $\mathbf{N}$, $\varphi_a(x) \in \varphi_a^* N_{a-1} \subseteq X^{-w_a} N_a$, so $X^{w_a} \varphi_a(x) \in N_a$ and $X^{m_a + w_a} \varphi_a(x) \in N'_a$. By the second height inclusion for $\mathbf{N}'$,

$$X^{m_a + h_a} \varphi_a(x) \in X^{h_a - w_a} N'_a \subseteq \varphi_a^* N'_{a-1}.$$

As $c = m_a + h_a \ge 0$, Lemma 28.22 (3) gives $X^{\lfloor c/\ell \rfloor} x \in N'_{a-1}$. Hence $m_{a-1} \le \lfloor (m_a + h_a)/\ell \rfloor$, and so $\ell m_{a-1} \le m_a + h_a$.

Iteration. Replacing a by $a+1$, $m_a \le \ell^{-1} h(a+1) + \ell^{-1} m_{a+1}$ for every a. By induction on $J \ge 1$,

$$m_a \le \sum_{j=1}^{J} h(a+j)\, \ell^{-j} + \ell^{-J} m_{a+J} \le \sum_{j \ge 1} h(a+j)\, \ell^{-j} + \ell^{-J} \max_{b \in \mathbb{Z}/n} m_b,$$

using $h \ge 0$. Letting $J \to \infty$, $m_a \le \sum_{j \ge 1} h(a+j) \ell^{-j}$ (the series converges since h takes finitely many values), and as m_a is an integer, $m_a \le \sigma(a)$. Therefore $X^{\sigma(a)} N_a = X^{\sigma(a) - m_a} X^{m_a} N_a \subseteq X^{\sigma(a) - m_a} N'_a \subseteq N'_a$. □

Lemma 28.24. *Let ℓ be a prime, k a field of characteristic ℓ, (φ, γ) a residual ring datum over k, $D = (V, (\varphi_b), (\gamma_b))$ a residual (φ, Γ)-module over (φ, γ) with sites $\mathbb{Z}/n$, and $a \in \mathbb{Z}/n$. Let $Q \subseteq P$ be lattices of V satisfying the Γ-condition at a, and $s \ge 0$ an integer with $P \subseteq X^{-s} Q$. Then ∇_a induces a k-linear endomorphism of the finite-dimensional k-vector space P/Q; for $x \in k$ let $q(x)$ be the dimension of its generalised x-eigenspace. For every $x \in k$:*

1. $\mathrm{sen}_a(P; x) + q(x+1) = \mathrm{sen}_a(Q; x) + q(x)$;
2. *if $q(x) \ne 0$, then $\mathrm{sen}_a(Q; x - j) \ne 0$ for some integer j with $1 \le j \le s$.*

Proof. Write $\nabla = \nabla_a$. By Lemma 28.25 (1), (2), ∇ maps each $X^c P$ and each $X^c Q$ ($c \in \mathbb{Z}$) into itself. Since $X^s P \subseteq Q \subseteq P$, P/Q is a quotient of $P/X^s P$, which is finite-dimensional (P is free of finite rank over $k[[X]]$). For an endomorphism T of a finite-dimensional k-space

E and $x \in k$ write $d_x(E,T)$ for the dimension of the generalised x-eigenspace. We use two facts of linear algebra. (F1) If $E' \subseteq E$ is T-stable, $d_x(E,T) = d_x(E',T) + d_x(E/E',T)$ (the generalised eigenspace is the kernel of a high power of $T-x$, a direct summand with complement the image of that power, functorially, so taking it is exact). (F2) Hence, if T preserves a finite filtration $E = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_m = 0$, then $d_x(E,T) = \sum_i d_x(F_i/F_{i+1},T)$; in particular two endomorphisms preserving the filtration and inducing the same maps on all F_i/F_{i+1} have the same d_x for all x.

Part 1. The ∇-stable chains $XQ \subseteq XP \subseteq P$ and $XQ \subseteq Q \subseteq P$ give, by (F1) applied in P/XQ,

$$d_x(P/XP) + d_x(XP/XQ) = d_x(P/XQ) = d_x(Q/XQ) + d_x(P/Q),$$

all for ∇. By Lemma 28.25 (1), $d_x(P/XP) = \operatorname{sen}_a(P;x)$ and $d_x(Q/XQ) = \operatorname{sen}_a(Q;x)$, and $d_x(P/Q) = q(x)$. It remains to show $d_x(XP/XQ) = q(x+1)$. By (Rr2), $\gamma(X) = X(1-\kappa X)$ with $\kappa(0) = 1$, so for $v \in P$, using (Rm1) and $\gamma_a(v) = v + X\nabla v$,

$$\nabla(Xv) = X^{-1}\big(X(1-\kappa X)\gamma_a(v) - Xv\big) = X\big(\nabla v - v\big) + X\,\epsilon(v), \qquad \epsilon(v) = (1-\kappa)v - \kappa X \nabla v.$$

As $1-\kappa \in Xk[[X]]$, ϵ maps X^iP into $X^{i+1}P$ for every $i \geq 0$ (using $\nabla(X^iP) \subseteq X^iP$). The k-linear bijection $P/Q \to XP/XQ$, $v \mapsto Xv$, therefore carries $T' = \nabla - 1 + \epsilon$ (an endomorphism of P/Q: it maps Q into Q, as $\epsilon(Q) \subseteq XQ$ by the same computation for Q) to ∇. Let $F_i = (X^iP + Q)/Q$ for $0 \leq i \leq s$, so $F_0 = P/Q$ and $F_s = 0$. Both T' and $\nabla - 1$ preserve every F_i, and they induce the same map on F_i/F_{i+1}, because $\epsilon(X^iP) \subseteq X^{i+1}P$. By (F2), $d_x(XP/XQ, \nabla) = d_x(P/Q, T') = d_x(P/Q, \nabla - 1) = d_{x+1}(P/Q, \nabla) = q(x+1)$.

Part 2. For $0 \leq i \leq s$ let $G_i = (P \cap X^{-i}Q)/Q$, a ∇-stable subspace of P/Q, with $G_0 = 0$ and $G_s = P/Q$. For $1 \leq i \leq s$, the map $P \cap X^{-i}Q \to X^{-i}Q/X^{-i+1}Q$ has kernel $P \cap X^{-i+1}Q$ and commutes with ∇, so G_i/G_{i-1} is ∇-isomorphic to a subspace of $X^{-i}Q/X \cdot X^{-i}Q$. If $q(x) \neq 0$, by (F2) there is $1 \leq i \leq s$ with $d_x(G_i/G_{i-1}) \neq 0$, and by (F1) $d_x(X^{-i}Q/X \cdot X^{-i}Q) \neq 0$. By Lemma 28.25 (2), $X^{-i}Q$ satisfies the Γ-condition at a, and by Lemma 28.25 (1), (2), $0 \neq \operatorname{sen}_a(X^{-i}Q;x) = \operatorname{sen}_a(Q;x-i)$. Take $j = i$. □

Lemma 28.25. *Let ℓ be a prime, k a field of characteristic ℓ, (φ,γ) a residual ring datum over k, $a \in \mathbb{Z}/n$, and $D = (V,(\varphi_b),(\gamma_b))$ a residual (φ,Γ)-module over (φ,γ) with sites $\mathbb{Z}/n$. Let N be a lattice of V satisfying the Γ-condition at a.*

1. *$\nabla_a(N) \subseteq N$, $\nabla_a(XN) \subseteq XN$, and ∇_a induces a k-linear endomorphism $\overline{\nabla}_a$ of N/XN; for $x \in k$, $\operatorname{sen}_a(N;x)$ is the dimension of the generalised x-eigenspace of $\overline{\nabla}_a$.*
2. *For $c \in \mathbb{Z}$, X^cN satisfies the Γ-condition at a and $\operatorname{sen}_a(X^cN;x) = \operatorname{sen}_a(N;x+c)$ for all $x \in k$.*
3. *Let $D' = (V',\dots)$, $D'' = (V'',\dots)$ be residual (φ,Γ)-modules over (φ,γ) with sites $\mathbb{Z}/n$, and $\iota\colon V' \to V$, $\varpi\colon V \to V''$ $k((X))$-linear maps commuting with γ_a, with ι injective, ϖ surjective and $\ker\varpi = \iota(V')$. Then $N' = \iota^{-1}(N)$ and $N'' = \varpi(N)$ are lattices satisfying the Γ-condition at a, and $\operatorname{sen}_a(N;x) = \operatorname{sen}_a(N';x) + \operatorname{sen}_a(N'';x)$ for all $x \in k$.*

Proof. Write $\nabla = \nabla_a = X^{-1}(\gamma_a - 1)$ on each of the modules involved. By (Rr2), $\gamma(X) = X(1-\kappa X)$ with $\kappa(0) = 1$. For $c \in k[[X]]$ and $m \geq 0$ write $c = c_m(X) + X^m c'$ with c_m a polynomial and $c' \in k[[X]]$; then $\gamma(c) = c_m(\gamma(X)) + \gamma(X)^m\gamma(c')$ with $\gamma(c') \in k[[X]]$ by (Rr1), so $\gamma(c) \equiv c(\gamma(X))$ modulo X^m for all m, i.e. $\gamma(c) = c(\gamma(X))$; since $\gamma(X^m) - X^m = X^m((1-\kappa X)^m - 1) \in X^{m+1}k[[X]]$, we get $\gamma(c) - c \in X^2k[[X]]$ for every $c \in k[[X]]$ (the constant term contributes nothing).

Part 1. The Γ-condition says $(\gamma_a - 1)N \subseteq XN$, i.e. $\nabla(N) \subseteq N$; then also $\gamma_a(v) = v + X\nabla v \in N$ for $v \in N$. For $c \in k[[X]]$ and $v \in N$, by (Rm1),

$$\nabla(cv) - c\nabla(v) = X^{-1}(\gamma(c) - c)\gamma_a(v) \in Xk[[X]] \cdot N \subseteq XN.$$

With $c = X$ this gives $\nabla(XN) \subseteq XN$, so ∇ induces a k-linear $\overline{\nabla}$ on N/XN. For $v \in N$, $(\nabla - x)^j v \in XN$ if and only if $(\overline{\nabla} - x)^j \bar{v} = 0$; so $N^{(x)}$ is the preimage in N of the generalised x-eigenspace E_x of $\overline{\nabla}$, and $N^{(x)}/(N^{(x)} \cap XN) \cong E_x$.

Part 2. For $v \in N$, $\gamma_a(X^c v) = X^c(1 - \kappa X)^c \gamma_a(v)$, so

$$\nabla(X^c v) = X^c\Big(\nabla v + X^{-1}((1 - \kappa X)^c - 1)\gamma_a(v)\Big), \qquad X^{-1}((1 - \kappa X)^c - 1) \in -c + Xk[[X]],$$

using $\kappa(0) = 1$. Hence $\nabla(X^c N) \subseteq X^c N$ (the Γ-condition for $X^c N$), and $\nabla(X^c v) \equiv X^c(\nabla - c)v$ modulo $X^{c+1}N$. The isomorphism $N/XN \to X^c N/X^{c+1}N$, $\bar{v} \mapsto \overline{X^c v}$, therefore carries $\overline{\nabla} - c$ to $\overline{\nabla}$, so it maps the generalised $(x + c)$-eigenspace onto the generalised x-eigenspace, and Part 1 gives $\mathrm{sen}_a(X^c N; x) = \mathrm{sen}_a(N; x + c)$.

Part 3. $N'' = \varpi(N)$ is finitely generated and spans V''; $N' = \iota^{-1}(N)$ is a lattice as ι maps it onto $N \cap \iota(V')$, a submodule of the noetherian module N, and $X^m v \in N'$ for $v \in V'$, m large. Since ι, ϖ are $k((X))$-linear and commute with γ_a, they commute with ∇, and the Γ-conditions for N' and N'' follow from that for N (for N': $\iota(\nabla v) = \nabla\iota(v) \in N$ for $v \in N'$). The sequence $0 \to N' \to N \to N'' \to 0$ is exact, and N'' is torsion-free and finitely generated, hence free, so the sequence splits over $k[[X]]$ and $0 \to N'/XN' \to N/XN \to N''/XN'' \to 0$ is exact; its maps commute with $\overline{\nabla}$. On finite-dimensional k-vector spaces with an endomorphism, taking the generalised x-eigenspace is exact (it is a direct summand, the kernel of a high power of $\overline{\nabla} - x$, with complement the image of that power, functorially). Taking dimensions and using Part 1 gives the formula. □

Lemma 28.26. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ, γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ, γ) with sites $\mathbb{Z}/n$, and let $h, w, \eta\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$,* typ, *$t \geq 1$ and multisets U_a of cardinality r satisfy (Ph1)–(Ph5) for ℓ. Let residual model data $(\bar{T}, \omega, \dots)$ for $(h, w, \mathrm{typ}, t, (U_a))$ and $\mathcal{F}$ be given. Let A be a finite representation and $(A, \mathbf{N}')$ a model such that $N_a \subseteq N'_a$ for every model $(A, \mathbf{N})$ and every $a \in \mathbb{Z}/n$. For model-admissible cocycles $z_1, \dots, z_s$ of A there are $\zeta_1, \dots, \zeta_s \in Z_{\mathrm{pol}}(\mathbf{N}')$ (polar pairs of $\mathcal{F}(A)$ for the letter data $h, w,$* typ*) such that for all $\beta_1, \dots, \beta_s \in k$: if $\sum_i \beta_i \zeta_i \in B(\mathcal{F}(A))$, then there is $y \in A$ with $\sum_i \beta_i z_i(g) = gy - y$ for all $g \in G$.*

Proof. Throughout, for an equivariant map f between $\mathcal{F}$-objects we write $f_a = \mathcal{F}(f)_a$; it is $k((X))$-linear, $f_a \circ \varphi_a = \varphi_a \circ f_{a-1}$, $f_a \circ \gamma_a = \gamma_a \circ f_a$, and $\mathcal{F}$ respects composition, identities and k-linear combinations (Mf3). We identify $\mathbf{1}$ with k. Recall $w, h \geq 0$, $w^*_a = h_a - w_a$, and that $(A, \mathbf{N}')$ is a model.

Step 0: the pair attached to one cocycle. Let z be a model-admissible cocycle of A: E of dimension $\dim A + 1$, $\alpha\colon A \to E$ injective equivariant, $e \in E \setminus \alpha(A)$ with $ge = e + \alpha(z(g))$, and $(E, \mathbf{N}_E)$ a model. By Lemma 28.21 (3), E is an $\mathcal{F}$-object, $\mathbf{N}_E$ has heights (w, w^*) and each $N_{E,a}$ satisfies the Γ-condition at a; by (Mf1), A is an $\mathcal{F}$-object. As $E = \alpha(A) \oplus ke$, there is a unique linear $p\colon E \to \mathbf{1}$ with $p\alpha = 0$ and $p(e) = 1$; it is surjective with $\ker p = \alpha(A)$, and equivariant: $p(g(\alpha(y) + ce)) = p(\alpha(gy + cz(g)) + ce) = c$. By Lemma 28.33 (1), $(\mathbf{1}, p_*\mathbf{N}_E)$ is a model; by Lemma 28.29 (with $e = 0 < t$, $\omega^0 = 1$) and (Md8), $p_a(N_{E,a}) = N^{\chi}_{0,a} = k[[X]]u_a$. By Lemma 28.27 (2), α_a is injective and $\ker p_a = \alpha_a(V_A)$. Choose $\varepsilon_a \in N_{E,a}$ with $p_a(\varepsilon_a) = u_a$ ($a \in \mathbb{Z}/n$). By (Mf4), $p_a(\varphi_a(\varepsilon_{a-1}) - \varepsilon_a) = \varphi_a(u_{a-1}) - u_a = 0$ and $p_a(\gamma_a(\varepsilon_a) - \varepsilon_a) = 0$, so there

are unique $b_a, g_a \in V_A$ with

$$\alpha_a(b_a) = \varphi_a(\varepsilon_{a-1}) - \varepsilon_a, \qquad \alpha_a(g_a) = \gamma_a(\varepsilon_a) - \varepsilon_a. \tag{R}$$

We call $(E, \alpha, e, p, \varepsilon)$ with (R) a *realisation* of (z, b, g). Only the following properties of a realisation are used below: E is an $\mathcal{F}$-object, α is injective and equivariant, p is surjective and equivariant with $\ker p = \alpha(A)$, $p(e) = 1$, $ge = e + \alpha(z(g))$, $p_a(\varepsilon_a) = u_a$, and (R) holds. Put $\zeta = (b, g)$ and $\mathbf{N} = \alpha^*\mathbf{N}_E$, i.e. $N_a = \alpha_a^{-1}(N_{E,a})$. By Lemma 28.33 (2), $(A, \mathbf{N})$ is a model, so $N_a \subseteq N'_a$ for all a by the hypothesis on $\mathbf{N}'$. Since α_a is injective and $k((X))$-linear, $\alpha_a^{-1}(X^c N_{E,a}) = X^c N_a$ for $c \in \mathbb{Z}$.

Step 1: $\zeta \in Z^1(\mathcal{F}(A))$. By (R), (Rm3) and the compatibilities of α_a,

$$\begin{aligned}\alpha_a(\gamma_a(b_a) - b_a) &= \gamma_a\varphi_a(\varepsilon_{a-1}) - \varphi_a(\varepsilon_{a-1}) - \gamma_a(\varepsilon_a) + \varepsilon_a \\ &= \varphi_a(\gamma_{a-1}(\varepsilon_{a-1}) - \varepsilon_{a-1}) - (\gamma_a(\varepsilon_a) - \varepsilon_a) \\ &= \alpha_a(\varphi_a(g_{a-1}) - g_a),\end{aligned}$$

and α_a is injective.

Step 2: $\zeta \in Z_{\mathrm{pol}}(\mathbf{N}) \subseteq Z_{\mathrm{pol}}(\mathbf{N}')$. Fix a. (Pp1): $\alpha_a(g_a) = (\gamma_a - 1)\varepsilon_a \in XN_{E,a}$ by the Γ-condition, so $g_a \in XN_a$. (Pp2), $\mathrm{typ}(a) = \mathrm{low}$: $\varphi_a(\varepsilon_{a-1}) \in \varphi_a^* N_{E,a-1} \subseteq X^{-w_a} N_{E,a}$ (heights) and $\varepsilon_a \in N_{E,a} \subseteq X^{-w_a} N_{E,a}$ ($w_a \geq 0$), so $b_a \in X^{-w_a} N_a$. (Pp3), $\mathrm{typ}(a) = \mathrm{high}$, first part. We claim that every $v \in \varphi_a^* N_{E,a-1}$ has the form $v = \alpha_a(m) + c\,\varphi_a(\varepsilon_{a-1})$ with $m \in \varphi_a^* N_{a-1}$ and $c \in k((X))$. The set of such v is a $k[[X]]$-submodule, so it suffices to take $v = \varphi_a(x)$ with $x \in N_{E,a-1}$. Then $p_{a-1}(x) \in k[[X]]u_{a-1}$, say $p_{a-1}(x) = d\,u_{a-1}$ with $d \in k[[X]]$; $x - d\varepsilon_{a-1} \in N_{E,a-1} \cap \ker p_{a-1}$, so $x - d\varepsilon_{a-1} = \alpha_{a-1}(m')$ with $m' \in N_{a-1}$, and by (Rm1) $\varphi_a(x) = \alpha_a(\varphi_a(m')) + \varphi(d)\varphi_a(\varepsilon_{a-1})$. By the heights, $X^{w_a^*}\varepsilon_a \in \varphi_a^* N_{E,a-1}$, so $X^{w_a^*}\varepsilon_a = \alpha_a(m) + c\,\varphi_a(\varepsilon_{a-1})$ as in the claim. Applying p_a: $X^{w_a^*}u_a = c\,\varphi_a(u_{a-1}) = c\,u_a$, and $u_a \neq 0$ in the one-dimensional $V_{\mathbf{1}}$, so $c = X^{w_a^*}$. Then $\alpha_a(X^{w_a^*}b_a) = X^{w_a^*}\varphi_a(\varepsilon_{a-1}) - X^{w_a^*}\varepsilon_a = -\alpha_a(m)$, so $X^{w_a^*}b_a = -m \in \varphi_a^* N_{a-1}$, i.e. $b_a \in X^{-w_a^*}\varphi_a^* N_{a-1}$. (Pp3), second part. By Lemma 28.28 for the model $(E, \mathbf{N}_E)$ there is $f \in k[T]$, $f(0) \neq 0$, with $\nabla_a f(\nabla_a)(N_{E,a}) \subseteq X^{w_a^*+1} N_{E,a}$. By Lemma 28.25 (1), $\nabla_a(N_{E,a}) \subseteq N_{E,a}$, so $y = f(\nabla_a)\varepsilon_a \in N_{E,a}$. The map p_a commutes with $\nabla_a = X^{-1}(\gamma_a - 1)$, and $\nabla_a u_a = 0$, so $p_a(y) = f(\nabla_a)u_a = f(0)u_a$. Hence $\varepsilon_a - f(0)^{-1}y \in N_{E,a} \cap \ker p_a$ equals $\alpha_a(x)$ with $x \in N_a$, and

$$\alpha_a\big(g_a - (\gamma_a(x) - x)\big) = (\gamma_a - 1)\varepsilon_a - (\gamma_a - 1)(\varepsilon_a - f(0)^{-1}y) = f(0)^{-1}X\nabla_a f(\nabla_a)\varepsilon_a \in X^{w_a^*+2} N_{E,a},$$

so $g_a - (\gamma_a(x) - x) \in X^{w_a^*+2} N_a$. Finally Z_{pol} is monotone: $N_a \subseteq N'_a$ for all a gives $XN_a \subseteq XN'_a$, $X^{-w_a}N_a \subseteq X^{-w_a}N'_a$, $X^{-w_a^*}\varphi_a^* N_{a-1} \subseteq X^{-w_a^*}\varphi_a^* N'_{a-1}$, $x \in N'_a$ and $X^{w_a^*+2}N_a \subseteq X^{w_a^*+2}N'_a$; so $\zeta \in Z_{\mathrm{pol}}(\mathbf{N}')$.

Step 3: linear combinations. For $i = 1, \ldots, s$ let $(E_i, \alpha_i, e_i, p_i, \varepsilon^i)$ be the realisation of (z_i, b^i, g^i) chosen in Step 0, $\zeta_i = (b^i, g^i)$, and let $\beta \in k^s$. We construct a realisation of $(\sum_i \beta_i z_i, \sum_i \beta_i b^i, \sum_i \beta_i g^i)$. Let $P = E_1 \oplus \cdots \oplus E_s \oplus \mathbf{1}$, an $\mathcal{F}$-object with $V_P = \bigoplus_i \mathrm{in}_{i,a}(V_{E_i}) \oplus \mathrm{in}_{0,a}(V_{\mathbf{1}})$ for the inclusions in_i, in_0 (Lemma 28.27 (3), applied repeatedly). Let $\lambda\colon P \to \mathbf{1}^{\oplus s}$, $(x_1, \ldots, x_s, c) \mapsto (p_i(x_i) - \beta_i c)_i$, equivariant and surjective, $S = \ker\lambda$ with inclusion ι, and $Q = \{(\alpha_1(y_1), \ldots, \alpha_s(y_s), 0) : \sum_i y_i = 0\} \subseteq S$, a subrepresentation. Put $E = S/Q$ with quotient map π, $\alpha(y) = \pi(\alpha_1(y), 0, \ldots, 0, 0)$, $p(\pi(x, c)) = c$ and $e = \pi(\beta_1 e_1, \ldots, \beta_s e_s, 1)$. By (Mf1), S and E are $\mathcal{F}$-objects. We check the properties. α is equivariant, and injective: $(\alpha_1(y), 0, \ldots) \in Q$ forces $y = 0$. Put $\tau_i(y) = (0, \ldots, \alpha_i(y), \ldots, 0, 0) \in S$; then $\pi\tau_i = \alpha$ for every i, since $\tau_i(y) - \tau_1(y) \in Q$. p is well defined (Q has last coordinate 0), equivariant, surjective, $p(e) = 1$, and $\ker p = \pi(\{(x, 0) : p_i(x_i) = 0\}) = \pi(\{(\alpha_i(y_i))_i\}) = \alpha(A)$, as $\pi((\alpha_i(y_i))_i, 0) = \sum_i \pi\tau_i(y_i) = \alpha(\sum_i y_i)$. Further $ge - e = \pi((\beta_i\alpha_i(z_i(g)))_i, 0) = \alpha(\sum_i \beta_i z_i(g))$. Now let $\sigma_a = \sum_i \beta_i\,\mathrm{in}_{i,a}(\varepsilon_a^i) + \mathrm{in}_{0,a}(u_a) \in V_P$. As $\lambda = \sum_i \mathrm{in}'_i \circ (p_i\mathrm{pr}_i - \beta_i\mathrm{pr}_0)$ with the inclusions in'_i of $\mathbf{1}^{\oplus s}$, $\lambda_a(\sigma_a) = \sum_i \mathrm{in}'_{i,a}(\beta_i u_a - \beta_i u_a) = 0$,

so by Lemma 28.27 (2) $\sigma_a = \iota_a(s_a)$ for a unique $s_a \in V_S$; put $\varepsilon_a = \pi_a(s_a)$. Then $p_a(\varepsilon_a) = (p\pi)_a(s_a) = (\mathrm{pr}_0\iota)_a(s_a) = u_a$. Since $\iota\tau_i = \mathrm{in}_i\alpha_i$ and $\varphi_a(u_{a-1}) = u_a$,

$$\iota_a\Big(\sum_i \beta_i\tau_{i,a}(b_a^i)\Big) = \sum_i \beta_i\,\mathrm{in}_{i,a}(\varphi_a(\varepsilon_{a-1}^i) - \varepsilon_a^i) = \varphi_a(\sigma_{a-1}) - \sigma_a = \iota_a(\varphi_a(s_{a-1}) - s_a),$$

so, ι_a being injective, $\sum_i \beta_i\tau_{i,a}(b_a^i) = \varphi_a(s_{a-1}) - s_a$, and applying π_a (with $\pi\tau_i = \alpha$): $\alpha_a(\sum_i \beta_i b_a^i) = \varphi_a(\varepsilon_{a-1}) - \varepsilon_a$. The same computation with γ_a gives $\alpha_a(\sum_i \beta_i g_a^i) = \gamma_a(\varepsilon_a) - \varepsilon_a$.

Step 4: boundaries give coboundaries. Suppose $\sum_i \beta_i\zeta_i \in B(\mathcal{F}(A))$: there is $y \in V_A^{\mathbb{Z}/n}$ with $\sum_i \beta_i b_a^i = \varphi_a(y_{a-1}) - y_a$ and $\sum_i \beta_i g_a^i = \gamma_a(y_a) - y_a$. Take the realisation $(E, \alpha, e, p, \varepsilon)$ of Step 3 and put $x_a = \varepsilon_a - \alpha_a(y_a)$. By (R) for it, $\varphi_a(x_{a-1}) - x_a = \alpha_a(\sum_i \beta_i b_a^i) - \alpha_a(\varphi_a(y_{a-1}) - y_a) = 0$ and likewise $\gamma_a(x_a) = x_a$. By (Mf4) there is an equivariant $f\colon \mathbf{1} \to E$ with $f_a(u_a) = x_a$. The equivariant map $pf\colon \mathbf{1} \to \mathbf{1}$ is $c \cdot \mathrm{id}$ with $c = p(f(1))$, so $(pf)_a(u_a) = c\,u_a$ by (Mf3); also $(pf)_a(u_a) = p_a(x_a) = u_a - (p\alpha)_a(y_a) = u_a$ (Lemma 28.27 (1)). As $u_a \neq 0$, $c = 1$. So $v = f(1) \in E$ is G-invariant (f is equivariant and G acts trivially on $\mathbf{1}$) with $p(v) = 1$; then $p(e - v) = 0$ and $e - v = \alpha(y')$ with $y' \in A$. For $g \in G$: $\alpha(gy') = g(e - v) = e + \alpha(\sum_i \beta_i z_i(g)) - v = \alpha(y' + \sum_i \beta_i z_i(g))$, and α is injective, so $\sum_i \beta_i z_i(g) = gy' - y'$.

By Step 2, $\zeta_1, \dots, \zeta_s \in Z_{\mathrm{pol}}(\mathbf{N}')$, and Steps 3 and 4 give the asserted implication. □

Lemma 28.27. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ, γ) a residual ring datum over k, and $\mathcal{F}$ a residual model functor for G over (φ, γ) with sites $\mathbb{Z}/n$. Let $a \in \mathbb{Z}/n$.*

1. *For $\mathcal{F}$-objects A, B, $\mathcal{F}(0)_a = 0$ for the zero map $0\colon A \to B$.*
2. *If A, B, C are $\mathcal{F}$-objects and $i\colon A \to B$, $q\colon B \to C$ are equivariant with i injective, q surjective and $\ker q = i(A)$, then $\mathcal{F}(i)_a$ is injective, $\mathcal{F}(q)_a$ is surjective and $\ker \mathcal{F}(q)_a = \mathcal{F}(i)_a(V_A)$.*
3. *If A, B are $\mathcal{F}$-objects, then $A \oplus B$ is an $\mathcal{F}$-object, and for the inclusions i_A, i_B and projections $\mathrm{pr}_A, \mathrm{pr}_B$ of the direct sum, $V_{A\oplus B} = \mathcal{F}(i_A)_a(V_A) \oplus \mathcal{F}(i_B)_a(V_B)$ and $\mathcal{F}(\mathrm{pr}_A)_a\mathcal{F}(i_A)_a = \mathrm{id}$, $\mathcal{F}(\mathrm{pr}_A)_a\mathcal{F}(i_B)_a = 0$, and symmetrically.*

Proof. All maps below are equivariant maps between $\mathcal{F}$-objects, so the identities of (Mf3) apply to them.

Part 1. By the linearity in (Mf3) with $f = f' = 0$ and $c = 1$, $\mathcal{F}(0)_a = \mathcal{F}(0 + 0)_a = \mathcal{F}(0)_a + \mathcal{F}(0)_a$, so $\mathcal{F}(0)_a = 0$.

Part 2. By (Mf3), $\mathcal{F}(i)_a$ is injective and $\mathcal{F}(q)_a$ surjective, and $\mathcal{F}(q)_a\mathcal{F}(i)_a = \mathcal{F}(qi)_a = \mathcal{F}(0)_a = 0$ by Part 1, so $\mathcal{F}(i)_a(V_A) \subseteq \ker \mathcal{F}(q)_a$. By (Mf2), $\dim_{k((X))} V_E = \dim_k E$ for every finite representation E, and $\dim B = \dim A + \dim C$; hence $\dim \ker \mathcal{F}(q)_a = \dim V_B - \dim V_C = \dim V_A = \dim \mathcal{F}(i)_a(V_A)$, and the inclusion is an equality.

Part 3. The identity $A \oplus B \to A \oplus B$ is the surjective equivariant map from the direct sum of the $\mathcal{F}$-objects A, B of the last clause of (Mf1), so $A \oplus B$ is an $\mathcal{F}$-object. The maps $i_A, i_B, \mathrm{pr}_A, \mathrm{pr}_B$ are equivariant, with $\mathrm{pr}_A i_A = \mathrm{id}_A$, $\mathrm{pr}_B i_B = \mathrm{id}_B$, $\mathrm{pr}_A i_B = 0$, $\mathrm{pr}_B i_A = 0$ and $i_A\mathrm{pr}_A + i_B\mathrm{pr}_B = \mathrm{id}_{A\oplus B}$. Applying (Mf3) (composition, identity, linearity) and Part 1 gives the same identities for the maps $\mathcal{F}(\cdot)_a$. The last one shows $V_{A\oplus B} = \mathcal{F}(i_A)_a(V_A) + \mathcal{F}(i_B)_a(V_B)$; if $\mathcal{F}(i_A)_a(x) = \mathcal{F}(i_B)_a(y)$, applying $\mathcal{F}(\mathrm{pr}_A)_a$ gives $x = 0$, so the sum is direct. □

Lemma 28.28. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ, γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ, γ) with sites $\mathbb{Z}/n$, and residual model data for letter data h, w, typ, $t \geq 1$, multisets (U_a) and $\mathcal{F}$. Let $(A, \mathbf{N})$ be a model and $a \in \mathbb{Z}/n$ with $\mathrm{typ}(a) = \mathrm{high}$. Then there is $f \in k[T]$ with $f(0) \neq 0$ and $\nabla_a f(\nabla_a)(N_a) \subseteq X^{w_a^*+1}N_a$.*

Proof. Write $\mathbf{N} = p_*j^*\mathbf{N}_B$ with a base pair $(B, \mathbf{N}_B)$, $j\colon A_1 \to B$ injective and $p\colon A_1 \to A$ surjective equivariant. By (Md2), B is an $\mathcal{F}$-object, and by (Mf1) so are A_1 and A. By (Md5) there is $f \in k[T]$ with $f(0) \neq 0$ and $\nabla_a f(\nabla_a)(N_{B,a}) \subseteq X^{w_a^*+1}N_{B,a}$. The maps $J = \mathcal{F}(j)_a$ and $\Pi = \mathcal{F}(p)_a$ are $k((X))$-linear and commute with γ_a (Mf3), hence with $\nabla_a = X^{-1}(\gamma_a - 1)$ and with every polynomial in ∇_a. For $v \in J^{-1}(N_{B,a})$, $J(\nabla_a f(\nabla_a)v) = \nabla_a f(\nabla_a)J(v) \in X^{w_a^*+1}N_{B,a}$, so $\nabla_a f(\nabla_a)v \in J^{-1}(X^{w_a^*+1}N_{B,a}) = X^{w_a^*+1}J^{-1}(N_{B,a})$, the last equality by $k((X))$-linearity and injectivity of J. Applying Π and using $N_a = \Pi(J^{-1}(N_{B,a}))$ gives $\nabla_a f(\nabla_a)(N_a) \subseteq X^{w_a^*+1}N_a$. □

Lemma 28.29. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ, γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ, γ) with sites $\mathbb{Z}/n$, and let $h, w, \eta\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$, typ, $t \geq 1$ and multisets U_a of cardinality r satisfy (Ph1)–(Ph5) for ℓ. Let residual model data $(\bar{T}, \omega, \dots)$ for $(h, w, \mathrm{typ}, t, (U_a))$ and $\mathcal{F}$ be given. For every integer $0 \leq e < t$ and every model $(k(\omega^{-e}), \mathbf{N})$: $\mathbf{N} = \mathbf{N}_e^\chi$.*

Proof. Write $\mathrm{sg}(a) = \lfloor \sum_{j\geq 1} h(a+j)\ell^{-j} \rfloor$. From (Ph1), (Ph2): $t \leq \eta_a$ for every a (if $\mathrm{typ}(a) = \mathrm{low}$, $t \leq w_a + t - 1 \leq \eta_a$ as $w_a \geq 1$; if high, $t \leq h_a - w_a \leq \eta_a$). Hence, by (Ph3), $t + 2\,\mathrm{sg}(a) < \ell$ and $t < \ell - 1$.

Step 1: distinct characters. For $0 \leq e', e'' < t$, $k(\omega^{-e'})$ and $k(\omega^{-e''})$ are isomorphic only if $e' = e''$: an equivariant bijection between one-dimensional representations forces $\omega^{-e'} = \omega^{-e''}$, so $\omega^{|e'-e''|} = 1$ with $|e' - e''| < t < \ell - 1$, and (Md1) gives $e' = e''$.

Step 2: a composition series. Write $\mathbf{N} = p_*j^*\mathbf{N}_B$ with a base pair $(B, \mathbf{N}_B)$, $j\colon A_1 \to B$ injective and $p\colon A_1 \to k(\omega^{-e})$ surjective equivariant. By Lemma 28.30 there are a composition series of B with irreducible factors $C_1, \dots, C_s$, models $(C_b, \mathbf{L}_b)$ with $\sum_b \mathrm{sen}_a(L_{b,a}; x) = \mathrm{sen}_a(N_{B,a}; x)$, an index b_0 and an equivariant bijection $\theta\colon C_{b_0} \to k(\omega^{-e})$ with $\theta_*\mathbf{L}_{b_0} = \mathbf{N}$. By (Md3) there is $\mu \geq 0$ such that B has the Jordan–Hölder factors of $R^{\oplus\mu}$, $R = \bar{T} \oplus \bigoplus_{e'=0}^{t-1} k(\omega^{-e'})$, and $\mathrm{sen}_a(N_{B,a}; x) = \mu \cdot \#\{u \in \widetilde{U}_a : \ (u - w_a) \cdot 1_k = x\}$. By the Jordan–Hölder theorem (finite-dimensional representations of G over k are modules of finite length over $k[G]$), the factors C_b are, up to isomorphism and with multiplicity, μ copies of each $k(\omega^{-e'})$, $0 \leq e' < t$, and μ copies of each composition factor of $\bar{T}$. A composition factor of $\bar{T}$ is B'/B'' for subrepresentations $B'' \subseteq B' \subseteq \bar{T}$, so it satisfies the hypothesis of Lemma 28.32 (with $A' = B'$), and by its Part 2 it is isomorphic to no $k(\omega^{-e'})$, $-1 \leq e' < t$. Let S be the set of b with $C_b \cong k(\omega^{-e'})$ for some $0 \leq e' < t$; by Step 1 this $e' =: e_b$ is unique. Then $\#\{b \in S : \ e_b = e'\} = \mu$ for every $0 \leq e' < t$; every C_b with $b \notin S$ is isomorphic to a composition factor of $\bar{T}$, hence (composing with the isomorphism) satisfies the hypothesis of Lemma 28.32; and $b_0 \in S$ with $e_{b_0} = e$.

Step 3: the factors. For $b \in S$ choose an equivariant bijection $\theta_b\colon C_b \to k(\omega^{-e_b})$, with $\theta_{b_0} = \theta$. By Lemma 28.33 (1), $(k(\omega^{-e_b}), \theta_{b*}\mathbf{L}_b)$ is a model, so by Lemma 28.31 there are integers $\kappa_{b,a}$ with $\theta_{b,a}(L_{b,a}) = X^{\kappa_{b,a}}N^\chi_{e_b,a}$ ($\theta_{b,a} = \mathcal{F}(\theta_b)_a$), $|\kappa_{b,a}| \leq \mathrm{sg}(a)$, $\kappa_{b,a} \geq 0$ if $\mathrm{typ}(a+1) = \mathrm{low}$, $\kappa_{b,a} \leq 0$ if $\mathrm{typ}(a+1) = \mathrm{high}$, and $\mathrm{sen}_a(\theta_{b,a}(L_{b,a}); x) = 1$ if $x = (e_b - \kappa_{b,a}) \cdot 1_k$ and 0 otherwise. The map $\theta_{b,a}$ is a $k((X))$-linear bijection commuting with γ_a, hence with ∇_a; it carries $L_{b,a}^{(x)}$ onto $\theta_{b,a}(L_{b,a})^{(x)}$ and $XL_{b,a}$ onto $X\theta_{b,a}(L_{b,a})$, so $\mathrm{sen}_a(L_{b,a}; x) = \mathrm{sen}_a(\theta_{b,a}(L_{b,a}); x)$. Since $\theta_*\mathbf{L}_{b_0} = \mathbf{N}$, $N_a = X^{\kappa_{b_0,a}}N^\chi_{e,a}$ for every a. For $b \notin S$, Lemma 28.32 (1) gives: $\mathrm{sen}_a(L_{b,a}; x) \neq 0$ only if $x = (u - w_a) \cdot 1_k$ for some $u \in U_a$.

Step 4: the shifts sum to zero. Fix a and let I be the set of integers y with $-\mathrm{sg}(a) \leq y \leq t - 1 + \mathrm{sg}(a)$. It has $t + 2\,\mathrm{sg}(a) < \ell$ elements, so distinct elements of I have distinct images in k. Let $y \in I$ and $x = y \cdot 1_k$. No $u \in U_a$ has $(u - w_a) \cdot 1_k = x$: otherwise $u \equiv w_a + y \pmod{\ell}$ with $w_a - \mathrm{sg}(a) \leq w_a + y \leq w_a + t - 1 + \mathrm{sg}(a)$, contradicting (Wa1) of Lemma 28.18. Hence $\mathrm{sen}_a(L_{b,a}; x) = 0$ for $b \notin S$, and $\mathrm{sen}_a(N_{B,a}; x) = \mu \cdot \#\{0 \leq e' < t : \ e' \cdot 1_k = x\} = \mu \cdot [0 \leq y \leq t-1]$,

as $\{0,\dots,t-1\}\subseteq I$. By Steps 2 and 3,

$$\#\{b\in S:\ (e_b-\kappa_{b,a})\cdot 1_k = y\cdot 1_k\} = \mu\cdot[0\le y\le t-1] = \#\{b\in S:\ e_b=y\}.$$

Since $e_b-\kappa_{b,a}\in I$ for $b\in S$, the left side is $\#\{b\in S:\ e_b-\kappa_{b,a}=y\}$. Both families $(e_b-\kappa_{b,a})_{b\in S}$ and $(e_b)_{b\in S}$ take values in I, and each value $y\in I$ is taken equally often; so they have the same sum, and $\sum_{b\in S}\kappa_{b,a}=0$.

Step 5: the shifts vanish. By Step 3 the integers $\kappa_{b,a}$, $b\in S$, are all ≥ 0 (if $\mathrm{typ}(a+1)=$ low) or all ≤ 0 (if $\mathrm{typ}(a+1)=$ high); with sum 0 they all vanish. As a was arbitrary, $\kappa_{b_0,a}=0$ for every a, and $\mathbf{N}=\mathbf{N}_e^\chi$. □

Lemma 28.30. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ,γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ,γ) with sites $\mathbb{Z}/n$, and residual model data for letter data h,w,typ, $t\ge 1$, multisets (U_a) and $\mathcal{F}$. Let $(B,\mathbf{N}_B)$ be a base pair, $j\colon A_1\to B$ an injective and $p\colon A_1\to C$ a surjective equivariant map, with $\dim C=1$. Then there are subrepresentations $0=B_0\subsetneq B_1\subsetneq\cdots\subsetneq B_s=B$ with irreducible quotients $C_b=B_b/B_{b-1}$, and for each b a model $(C_b,\mathbf{L}_b)$, such that:*

1. *$\sum_{b=1}^s \mathrm{sen}_a(L_{b,a};x)=\mathrm{sen}_a(N_{B,a};x)$ for every $a\in\mathbb{Z}/n$ and $x\in k$;*
2. *for some b_0 there is an equivariant bijection $\theta\colon C_{b_0}\to C$ with $\theta_*\mathbf{L}_{b_0}=p_*j^*\mathbf{N}_B$.*

Proof. **The series.** Let $K=\ker p$. Then $0\subseteq j(K)\subsetneq j(A_1)\subseteq B$ are subrepresentations and $j(A_1)/j(K)\cong A_1/K\cong C$ is one-dimensional, hence irreducible. Refining $0\subseteq j(K)$ and $j(A_1)\subseteq B$ by maximal chains of subrepresentations (they exist, dimensions being finite) gives a series $0=B_0\subsetneq\cdots\subsetneq B_s=B$ with irreducible quotients and $B_{b_0-1}=j(K)$, $B_{b_0}=j(A_1)$ for some b_0. Let $\iota_b\colon B_b\to B$ be the inclusion, $\pi_b\colon B_b\to C_b$ the quotient map, and $\mathbf{L}_b=\pi_{b*}\iota_b^*\mathbf{N}_B$; by definition $(C_b,\mathbf{L}_b)$ is a model.

Item 1. By (Md2) and (Mf1), B, all B_b and all C_b are $\mathcal{F}$-objects and $N_{B,a}$ satisfies the Γ-condition at a. Fix a, write $f_a=\mathcal{F}(f)_a$, and let $M_b=\iota_{b,a}^{-1}(N_{B,a})\subseteq V_{B_b}$. Let $\kappa_b\colon B_{b-1}\to B_b$ be the inclusion; $0\to B_{b-1}\to B_b\to C_b\to 0$ is exact, so by Lemma 28.27 (2) $\kappa_{b,a}$ is injective, $\pi_{b,a}$ surjective and $\ker\pi_{b,a}=\kappa_{b,a}(V_{B_{b-1}})$; both maps are $k((X))$-linear and commute with γ_a (Mf3). As $\iota_b\kappa_b=\iota_{b-1}$, $\kappa_{b,a}^{-1}(M_b)=M_{b-1}$, and $\pi_{b,a}(M_b)=L_{b,a}$. By Lemma 28.21 (1), M_b is a lattice satisfying the Γ-condition at a (for $b=0$, $V_{B_0}=0$ and all its Sen multiplicities vanish). By Lemma 28.25 (3),

$$\mathrm{sen}_a(M_b;x)=\mathrm{sen}_a(M_{b-1};x)+\mathrm{sen}_a(L_{b,a};x)\qquad(1\le b\le s).$$

Summing over b and using $M_s=N_{B,a}$ ($\iota_s=\mathrm{id}$) and $\mathrm{sen}_a(M_0;x)=0$ gives item 1.

Item 2. Define $\theta\colon C_{b_0}=j(A_1)/j(K)\to C$ by $\theta(j(y)+j(K))=p(y)$: well defined since $p(K)=0$, equivariant, and bijective since j is injective and p surjective with kernel K. Let $j'\colon A_1\to B_{b_0}$ be j with target $B_{b_0}=j(A_1)$, a bijective equivariant map with $\iota_{b_0}j'=j$ and $\theta\pi_{b_0}j'=p$. By (Mf3), j'_a is bijective, so

$$(\theta\pi_{b_0})_a\big(\iota_{b_0,a}^{-1}(N_{B,a})\big)=(\theta\pi_{b_0})_a\,j'_a\big(j'^{-1}_a\iota_{b_0,a}^{-1}(N_{B,a})\big)=p_a\big(j_a^{-1}(N_{B,a})\big),$$

i.e. $\theta_*\mathbf{L}_{b_0}=p_*j^*\mathbf{N}_B$. □

Lemma 28.31. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ,γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ,γ) with sites $\mathbb{Z}/n$, and let $h,w,\eta\colon\mathbb{Z}/n\to\mathbb{Z}_{\ge 0}$, typ, $t\ge 1$ and multisets U_a of cardinality r satisfy (Ph1)–(Ph3) for ℓ. Let residual model data $(\bar T,\omega,\dots)$ for $(h,w,\mathrm{typ},t,(U_a))$ and $\mathcal{F}$ be given, let $0\le e<t$, and let $(k(\omega^{-e}),\mathbf{N})$ be a model. Then there are unique integers κ_a ($a\in\mathbb{Z}/n$) with $N_a=X^{\kappa_a}N_{e,a}^\chi$, and for every a:*

1. $|\kappa_a| \le \mathrm{sg}(a)$;

2. $\kappa_{a-1} \ge 0$ *if* $\mathrm{typ}(a) = \mathrm{low}$, *and* $\kappa_{a-1} \le 0$ *if* $\mathrm{typ}(a) = \mathrm{high}$;

3. $\mathrm{sen}_a(N_a; x) = 1$ *if* $x = (e - \kappa_a) \cdot 1_k$, *and* $\mathrm{sen}_a(N_a; x) = 0$ *otherwise.*

Proof. Write $V = V_{k(\omega^{-e})}$, a one-dimensional $k((X))$-vector space by (Mf2), and $N_a^\chi = N_{e,a}^\chi$.

Lattices of a line. Let $M \neq 0$ be a finitely generated $k[[X]]$-submodule of $k((X))$. Its generators have bounded poles, so $X^m M \subseteq k[[X]]$ for some m, and $X^m M$ is a non-zero ideal of $k[[X]]$, hence $X^d k[[X]]$; so $M = X^{d-m} k[[X]]$, and the modules $X^c k[[X]]$, $c \in \mathbb{Z}$, are pairwise distinct. Choosing a basis v of V, every lattice of V is $X^c k[[X]]v$ for a unique c. Hence there are unique integers κ_a with $N_a = X^{\kappa_a} N_a^\chi$, and for integers c, c': $X^c N_a^\chi \subseteq X^{c'} N_a^\chi$ if and only if $c \ge c'$.

Heights. By (Ph1), $w_a^* = h_a - w_a \ge t$. By (Md7), each N_a^χ satisfies the Γ-condition at a and $\varphi_a^* N_{a-1}^\chi = X^e N_a^\chi$; as $-w_a \le 0 \le e < t \le w_a^*$, $X^e N_a^\chi \subseteq X^{-w_a} N_a^\chi$ and $X^{w_a^*} N_a^\chi \subseteq X^e N_a^\chi$, so $\mathbf{N}_e^\chi$ has heights (w, w^*). By Lemma 28.21 (3), $\mathbf{N}$ has heights (w, w^*) and each N_a satisfies the Γ-condition at a.

Item 1. The shift of Lemma 28.23 is $\sigma(a) = \mathrm{sg}(a)$, and $h \ge 0$. Applied to the pairs $(\mathbf{N}, \mathbf{N}_e^\chi)$ and $(\mathbf{N}_e^\chi, \mathbf{N})$ it gives $X^{\mathrm{sg}(a)+\kappa_a} N_a^\chi \subseteq N_a^\chi$ and $X^{\mathrm{sg}(a)} N_a^\chi \subseteq X^{\kappa_a} N_a^\chi$, i.e. $\mathrm{sg}(a) + \kappa_a \ge 0$ and $\mathrm{sg}(a) \ge \kappa_a$.

Item 2. By (Rr2), $\varphi(X) = uX^\ell$ with u a unit of $k[[X]]$, so for $y \in V$ and $c \in \mathbb{Z}$, $\varphi_a(X^c y) = u^c X^{\ell c} \varphi_a(y)$ by (Rm1); hence $\varphi_a^* N_{a-1} = X^{\ell\kappa_{a-1}} \varphi_a^* N_{a-1}^\chi = X^{\ell\kappa_{a-1}+e} N_a^\chi$. The heights of $\mathbf{N}$ read $X^{\ell\kappa_{a-1}+e} N_a^\chi \subseteq X^{\kappa_a - w_a} N_a^\chi$ and $X^{\kappa_a + w_a^*} N_a^\chi \subseteq X^{\ell\kappa_{a-1}+e} N_a^\chi$, i.e.

$$\kappa_a - w_a - e \le \ell\kappa_{a-1} \le \kappa_a + w_a^* - e.$$

If $\mathrm{typ}(a) = \mathrm{low}$ and $\kappa_{a-1} \le -1$, the left inequality gives $\ell \le w_a + e - \kappa_a \le w_a + t - 1 + \mathrm{sg}(a) \le \eta_a + \mathrm{sg}(a)$ by item 1 and (Ph2), contradicting (Ph3). If $\mathrm{typ}(a) = \mathrm{high}$ and $\kappa_{a-1} \ge 1$, the right inequality gives $\ell \le \kappa_a + w_a^* - e \le \mathrm{sg}(a) + w_a^* \le \mathrm{sg}(a) + \eta_a$, as $w_a^* = h_a - w_a \le \eta_a$ by (Ph2), again contradicting (Ph3).

Item 3. By Lemma 28.25 (2), $\mathrm{sen}_a(X^{\kappa_a} N_a^\chi; x) = \mathrm{sen}_a(N_a^\chi; x + \kappa_a)$, which by (Md7) is 1 if $x + \kappa_a \cdot 1_k = e \cdot 1_k$ and 0 otherwise. □

Lemma 28.32. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ, γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ, γ) with sites $\mathbb{Z}/n$, and let $h, w, \eta\colon \mathbb{Z}/n \to \mathbb{Z}_{\ge 0}$, typ, $t \ge 1$ and multisets U_a of cardinality r satisfy (Ph1)–(Ph5) for ℓ. Let residual model data $(\bar{T}, \omega, \dots)$ for $(h, w, \mathrm{typ}, t, (U_a))$ and $\mathcal{F}$ be given. Let C be a finite representation for which there are a finite representation A', an injective equivariant $i\colon A' \to \bar{T}$ and a surjective equivariant $q\colon A' \to C$.*

1. *For every model $(C, \mathbf{L})$, every $a \in \mathbb{Z}/n$ and every $x \in k$ with $\mathrm{sen}_a(L_a; x) \neq 0$ there is $u \in U_a$ with $x = (u - w_a) \cdot 1_k$.*

2. *C is not isomorphic to $k(\omega^{-e})$ for any integer $-1 \le e < t$.*

Proof. By (Md6), $(\bar{T}, \mathbf{N}^{\mathrm{std}})$ is a model; by Lemma 28.21 (3), $\bar{T}$ is an $\mathcal{F}$-object and the lattices of every model have heights (w, w^*) and satisfy the Γ-condition at their site. The shift of Lemma 28.34 (2) is $\sigma(a) = \mathrm{sg}(a)$, and $h \ge 0$.

Part 1. Fix a, and put $\mathbf{M} = q_* i^* \mathbf{N}^{\mathrm{std}}$, a model of C by Lemma 28.33 (2) and (1). By Lemma 28.34 (1) (with $B = \bar{T}$, $j = i$, $p = q$), every $y \in k$ with $\mathrm{sen}_a(M_a; y) \neq 0$ has

$\mathrm{sen}_a(N_a^{\mathrm{std}}; y) \neq 0$, hence by (Md6) $y = (u - w_a) \cdot 1_k$ for some $u \in U_a$. Now write $\mathbf{L} = p_* j^* \mathbf{N}_B$ with a base pair $(B, \mathbf{N}_B)$; by (Md2) and Lemma 28.34 (1), $\mathrm{sen}_a(N_{B,a}; x) \geq \mathrm{sen}_a(L_a; x) > 0$, so by (Md3) $x = (u' - w_a) \cdot 1_k$ for some $u' \in \widetilde{U}_a = U_a + \{w_a, \ldots, w_a + t - 1\}$. If $u' \in U_a$ we are done. Otherwise $w_a \leq u' \leq w_a + t - 1$. By Lemma 28.34 (2), applied to $\mathbf{M}$ (as $\mathbf{N}$) and $\mathbf{L}$ (as $\mathbf{N}'$), there is an integer c with $|c| \leq \mathrm{sg}(a)$ and $\mathrm{sen}_a(M_a; x + c) \neq 0$; so $x + c \cdot 1_k = (u - w_a) \cdot 1_k$ for some $u \in U_a$, i.e. $u \equiv u' + c \pmod{\ell}$ with $w_a - \mathrm{sg}(a) \leq u' + c \leq w_a + t - 1 + \mathrm{sg}(a)$. This contradicts (Wa1) of Lemma 28.18.

Part 2. Suppose $\theta \colon C \to k(\omega^{-e})$ is an equivariant bijection, $-1 \leq e < t$. Then $k(\omega^{-e})$ satisfies the hypothesis of the lemma with θq in place of q, and $\mathbf{L} = (\theta q)_* i^* \mathbf{N}^{\mathrm{std}}$ is a model of it (Lemma 28.33). The family $\mathbf{N}_e^\chi$ satisfies the Γ-condition and $\varphi_a^* N_{e,a-1}^\chi = X^e N_{e,a}^\chi$ (Md7); as $-w_a \leq -1 \leq e < t \leq w_a^*$ by (Ph1), it has heights (w, w^*). By (Md7), $\mathrm{sen}_a(N_{e,a}^\chi; e) = 1$, so by Lemma 28.34 (2) (with $\mathbf{L}$ as $\mathbf{N}$ and $\mathbf{N}_e^\chi$ as $\mathbf{N}'$) there is an integer c, $|c| \leq \mathrm{sg}(a)$, with $\mathrm{sen}_a(L_a; e + c) \neq 0$. By Part 1 for $k(\omega^{-e})$, some $u \in U_a$ has $u \equiv w_a + e + c \pmod{\ell}$, where $w_a - 1 - \mathrm{sg}(a) \leq w_a + e + c \leq w_a + t - 1 + \mathrm{sg}(a)$; this contradicts (Wa1). □

Lemma 28.33. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ, γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ, γ) with sites $\mathbb{Z}/n$, and residual model data for letter data h, w, typ, $t \geq 1$, multisets (U_a) and $\mathcal{F}$. Let $(A, \mathbf{N})$ be a model.*

1. *If $f \colon A \to A''$ is a surjective equivariant map, then $(A'', f_* \mathbf{N})$ is a model.*
2. *If $i \colon A' \to A$ is an injective equivariant map, then $(A', i^* \mathbf{N})$ is a model.*

Proof. Write $\mathbf{N} = p_* j^* \mathbf{N}_B$ with a base pair $(B, \mathbf{N}_B)$, $j \colon A_1 \to B$ injective and $p \colon A_1 \to A$ surjective. By Lemma 28.21 (3) and (Mf1), A, A_1, A'' and A' are $\mathcal{F}$-objects, as are all subrepresentations of A_1; by (Mf3), $\mathcal{F}$ respects composition of equivariant maps between $\mathcal{F}$-objects, and $\mathcal{F}(g)_a$ is injective (surjective) when g is. Fix a site a and write g for $\mathcal{F}(g)_a$; put $M = j^{-1}(N_{B,a})$, so $N_a = p(M)$.

Part 1. $f(N_a) = f(p(M)) = (fp)(M)$, and $fp \colon A_1 \to A''$ is surjective and equivariant, so $f_* \mathbf{N} = (fp)_* j^* \mathbf{N}_B$ and $(A'', f_* \mathbf{N})$ is a model.

Part 2. Let $A_1' = p^{-1}(i(A')) \subseteq A_1$, a subrepresentation, with inclusion $\iota \colon A_1' \to A_1$, and let $p' \colon A_1' \to A'$ be the map with $i\,p' = p\,\iota$ (well defined since i is injective); it is equivariant and surjective, because every $i(a')$ is some $p(x)$ and then $x \in A_1'$. Put $j' = j\iota$, injective. We show $i^{-1}(N_a) = p'(j'^{-1}(N_{B,a})) = p'(\iota^{-1}(M))$, so that $(A', i^* \mathbf{N}) = (A', p'_* j'^* \mathbf{N}_B)$ is a model.

$\supseteq$ For $y \in V_{A_1'}$ with $\iota(y) \in M$: $i(p'(y)) = p(\iota(y)) \in p(M) = N_a$.

$\subseteq$ First, $p^{-1}(i(V_{A'})) = \iota(V_{A_1'})$ in V_{A_1}: the inclusion $\supseteq$ holds since $p\iota = ip'$; by (Mf2), $\dim_{k((X))} V_E = \dim_k E$ for every finite representation E, and p is onto, so $\dim p^{-1}(i(V_{A'})) = (\dim A_1 - \dim A) + \dim A' = \dim A_1' = \dim \iota(V_{A_1'})$, using $\dim A_1' = \dim \ker p + \dim A'$. Now let $v \in V_{A'}$ with $i(v) \in N_a$, say $i(v) = p(m)$ with $m \in M$. Then $m \in p^{-1}(i(V_{A'}))$, so $m = \iota(y)$ with $y \in \iota^{-1}(M)$, and $i(p'(y)) = p(m) = i(v)$ gives $v = p'(y)$ as i is injective.

□

Lemma 28.34. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ, γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ, γ) with sites $\mathbb{Z}/n$, and $a \in \mathbb{Z}/n$.*

1. *Let B be an $\mathcal{F}$-object, $j \colon A_1 \to B$ an injective and $p \colon A_1 \to A$ a surjective equivariant map, and $\mathbf{N}$ a family of lattices of V_B with N_a satisfying the Γ-condition at a. Then $\mathrm{sen}_a((p_* j^* \mathbf{N})_a; x) \leq \mathrm{sen}_a(N_a; x)$ for all $x \in k$.*

2. *Let $D = (V, \dots)$ be a residual (φ,Γ)-module over (φ,γ) with sites $\mathbb{Z}/n$, $h\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$, $w\colon \mathbb{Z}/n \to \mathbb{Z}$, $\sigma(a) = \lfloor \sum_{j\geq 1} h(a+j)\ell^{-j} \rfloor$, and $\mathbf{N}$, $\mathbf{N}'$ families of lattices of V with heights $(w, h-w)$ such that N_a and N'_a satisfy the Γ-condition at a. If $\operatorname{sen}_a(N'_a; y) \neq 0$, then $\operatorname{sen}_a(N_a; y+c) \neq 0$ for some integer c with $|c| \leq \sigma(a)$.*

Proof. **Part 1.** By (Mf1), A_1 (a subrepresentation of B) and A (a quotient of A_1) are $\mathcal{F}$-objects; so are $C = B/j(A_1)$ with the quotient map $q\colon B \to C$ and $K = \ker p$ with the inclusion $s\colon K \to A_1$. By (Mf3), $\mathcal{F}(j)_a$ and $\mathcal{F}(s)_a$ are injective, $\mathcal{F}(q)_a$ and $\mathcal{F}(p)_a$ surjective, all are $k((X))$-linear and commute with γ_a, and $\mathcal{F}(q)_a\mathcal{F}(j)_a = \mathcal{F}(qj)_a = \mathcal{F}(0)_a = 0$ (by linearity in (Mf3), $\mathcal{F}(0)_a = 2\mathcal{F}(0)_a$); likewise $\mathcal{F}(p)_a\mathcal{F}(s)_a = 0$. By (Mf2), $\dim_{k((X))} V_E = \dim_k E$ for every finite representation E, so $\dim V_{A_1} + \dim V_C = \dim V_B$, and the image of $\mathcal{F}(j)_a$, of dimension $\dim V_{A_1}$, is the kernel of $\mathcal{F}(q)_a$, of the same dimension; similarly $\ker \mathcal{F}(p)_a = \mathcal{F}(s)_a(V_K)$. By Lemma 28.25 (3) applied to $\mathcal{F}(j)_a$, $\mathcal{F}(q)_a$ and N_a, the lattice $M = \mathcal{F}(j)_a^{-1}(N_a) = (j^*\mathbf{N})_a$ satisfies the Γ-condition at a and $\operatorname{sen}_a(N_a; x) \geq \operatorname{sen}_a(M; x)$; applied to $\mathcal{F}(s)_a$, $\mathcal{F}(p)_a$ and M, it gives $\operatorname{sen}_a(M; x) \geq \operatorname{sen}_a(\mathcal{F}(p)_a(M); x)$, and $\mathcal{F}(p)_a(M) = (p_*j^*\mathbf{N})_a$.

Part 2. Write $\sigma = \sigma(a)$ and $\nabla = \nabla_a$. By Lemma 28.23, applied to the pairs $(\mathbf{N}, \mathbf{N}')$ and $(\mathbf{N}', \mathbf{N})$, $X^\sigma N_a \subseteq N'_a \subseteq X^{-\sigma} N_a$, hence

$$X^{\sigma+1} N_a \subseteq X N'_a \subseteq N'_a \subseteq X^{-\sigma} N_a.$$

By Lemma 28.25 (1), (2), each $X^c N_a$ and $X^c N'_a$ is stable under ∇, so ∇ induces k-linear endomorphisms of $Q = X^{-\sigma}N_a / X^{\sigma+1}N_a$ (finite-dimensional), of its subspaces $X^cN_a/X^{\sigma+1}N_a$, and of the subquotient $N'_a/XN'_a = (N'_a/X^{\sigma+1}N_a)/(XN'_a/X^{\sigma+1}N_a)$. Call $z \in k$ an eigenvalue of an endomorphism of a finite-dimensional space if its generalised z-eigenspace is non-zero, i.e. if the endomorphism minus z is not injective. If $\nabla - z$ is bijective on a finite-dimensional space, it is injective on every invariant subspace and surjective, hence bijective, on every quotient; so an eigenvalue of a subquotient is an eigenvalue of the whole, and an eigenvalue of a space with an invariant filtration is an eigenvalue of one of the graded pieces. Now $\operatorname{sen}_a(N'_a; y) \neq 0$ means (Lemma 28.25 (1)) that y is an eigenvalue of $\overline{\nabla}$ on N'_a/XN'_a, hence of ∇ on Q, hence of ∇ on a graded piece $X^cN_a/X^{c+1}N_a$ of the filtration $(X^cN_a/X^{\sigma+1}N_a)_{-\sigma\leq c\leq\sigma+1}$ for some $-\sigma \leq c \leq \sigma$. By Lemma 28.25 (1), (2), $0 \neq \operatorname{sen}_a(X^cN_a; y) = \operatorname{sen}_a(N_a; y+c)$. □

Proposition 28.35. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ,γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ,γ) with sites $\mathbb{Z}/n$, and let h, w, η, typ, $t \geq 1$ and multisets U_a of cardinality r satisfy (Ph1)–(Ph5) for ℓ. Let residual model data $(\bar{T}, \omega, \dots)$ for $(h, w, \mathrm{typ}, t, (U_a))$ and $\mathcal{F}$ be given. Then:*

(Mc1) *for every integer $-1 \leq e < t$, every equivariant map from a subrepresentation of $\bar{T}$ to $k(\omega^{-e})$ is zero;*

(Mc2) *for every integer $1 \leq e < t$, every model-admissible cocycle z of $k(\omega^{-e})$ is a coboundary: there is $y \in k$ with $z(g) = (\omega(g)^{-e} - 1)y$ for all $g \in G$;*

(Mc3) *any two model-admissible cocycles of $\mathbf{1}$ are linearly dependent over k.*

Proof. Recall $w_a^* = h_a - w_a$. Two consequences of (Ph1), (Ph2) are used throughout: $w_a \geq 1$ and $w_a^* \geq t$ (from $w_a + t - 1 \leq h_a - 1$); and, if $\mathrm{typ}(a) = \mathrm{low}$, $w_a + t - 1 \leq \eta_a$, while if $\mathrm{typ}(a) = \mathrm{high}$, $w_a^* \leq \eta_a$ (from $h_a - \eta_a \leq w_a$). By (Ph3), $\eta_a < \ell$.

The character lattices. Let $-1 \leq e < t$ and $N_a = N_{e,a}^{\chi}$. By (Md7), N_a satisfies the Γ-condition at a and $\varphi_a^* N_{a-1} = X^e N_a$. Since $e \geq -1 \geq -w_a$, $X^eN_a \subseteq X^{-w_a}N_a$, and since

$w_a^* \geq t > e$, $X^{w_a^*} N_a \subseteq X^e N_a$: the family $\mathbf{N}_e^\chi$ has heights (w, w^*). By (Md7), $\mathrm{sen}_a(N_a; x)$ is 1 for $x = e \cdot 1_k$ and 0 otherwise.

(Mc1). Let $j\colon A_1 \to \bar{T}$ be the inclusion of a subrepresentation and $f\colon A_1 \to k(\omega^{-e})$ a non-zero equivariant map, $-1 \leq e < t$. As $k(\omega^{-e})$ has dimension 1, f is surjective, so $C = k(\omega^{-e})$ satisfies the hypothesis of Lemma 28.32 with $A' = A_1$, $i = j$, $q = f$; by its Part 2 (with the identity of C) C is not isomorphic to $k(\omega^{-e})$, a contradiction. Hence every such map is zero.

The largest model of a character. For $0 \leq e < t$, $(k(\omega^{-e}), \mathbf{N}_e^\chi)$ is a model by (Md8), and every model $(k(\omega^{-e}), \mathbf{N})$ has $\mathbf{N} = \mathbf{N}_e^\chi$ by Lemma 28.29. So Lemma 28.26 applies to $A = k(\omega^{-e})$ with $\mathbf{N}' = \mathbf{N}_e^\chi$.

(Mc2). Let $1 \leq e < t$ and z a model-admissible cocycle of $k(\omega^{-e})$. By Lemma 28.26 ($s = 1$) there is $\zeta = (b, g) \in Z_{\mathrm{pol}}(\mathbf{N}_e^\chi)$ such that $\zeta \in B(\mathcal{F}(k(\omega^{-e})))$ implies that z is a coboundary. We verify the hypotheses of Lemma 28.36 (2) for $\mathbf{N} = \mathbf{N}_e^\chi$ and $J_a = w_a$ if $\mathrm{typ}(a) = \mathrm{low}$, $J_a = w_a^* - e$ if $\mathrm{typ}(a) = \mathrm{high}$ (so $J_a \geq 0$, as $w_a^* \geq t > e$). The Γ-condition holds, and $\varphi_a^* N_{a-1} = X^e N_a \subseteq X N_a$ as $e \geq 1$. For $1 \leq j \leq J_a$: $2 \leq e + j \leq e + J_a$, and $e + J_a \leq t - 1 + w_a \leq \eta_a$ (low) or $e + J_a = w_a^* \leq \eta_a$ (high), so $e + j < \ell$, $(-j) \cdot 1_k \neq e \cdot 1_k$, and $\mathrm{sen}_a(N_a; -j) = 0$. By (Pp1), $g_a \in X N_a$; by (Pp2), $b_a \in X^{-w_a} N_a = X^{-J_a} N_a$ (low); by (Pp3), $b_a \in X^{-w_a^*} \varphi_a^* N_{a-1} = X^{e - w_a^*} N_a = X^{-J_a} N_a$ (high). Hence $\zeta \in B(\mathcal{F}(k(\omega^{-e})))$, and there is $y \in k$ with $z(g) = gy - y = (\omega(g)^{-e} - 1)y$ for all $g \in G$.

(Mc3). Let z_1, z_2 be model-admissible cocycles of $\mathbf{1} = k(\omega^0)$. By Lemma 28.26 with $\mathbf{N}' = \mathbf{N}_0^\chi$ there are $\zeta_1, \zeta_2 \in Z_{\mathrm{pol}}(\mathbf{N}_0^\chi)$ such that $\beta_1\zeta_1 + \beta_2\zeta_2 \in B(\mathcal{F}(\mathbf{1}))$ implies $\beta_1 z_1(g) + \beta_2 z_2(g) = gy - y$ for some $y \in \mathbf{1}$ and all g. By Lemma 28.37 ((Ph1)–(Ph3) hold) there is $(\beta_1, \beta_2) \neq 0$ with $\beta_1\zeta_1 + \beta_2\zeta_2 \in B(\mathcal{F}(\mathbf{1}))$. As G acts trivially on $\mathbf{1}$, $gy - y = 0$, so $\beta_1 z_1 + \beta_2 z_2 = 0$: z_1, z_2 are linearly dependent. □

Lemma 28.36. *Let ℓ be a prime, k a field of characteristic ℓ, (φ, γ) a residual ring datum over k, and $D = (V, (\varphi_a), (\gamma_a))$ a residual (φ, Γ)-module over (φ, γ) with sites $\mathbb{Z}/n$.*

1. *Let $a \in \mathbb{Z}/n$, N a lattice of V satisfying the Γ-condition at a, and $J \geq 0$ an integer with $\mathrm{sen}_a(N; -j) = 0$ for every integer $1 \leq j \leq J$. If $v \in X^{-J} N$ and $(\gamma_a - 1)v \in XN$, then $v \in N$.*
2. *Let $\mathbf{N} = (N_a)$ be a family of lattices of V and $J\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$ such that for every a: N_a satisfies the Γ-condition at a, $\varphi_a^* N_{a-1} \subseteq X N_a$, and $\mathrm{sen}_a(N_a; -j) = 0$ for every integer $1 \leq j \leq J_a$. If $(b, g) \in Z^1(D)$ satisfies $g_a \in X N_a$ and $b_a \in X^{-J_a} N_a$ for every a, then there is $m \in \prod_a N_a$ with $b_a = \varphi_a(m_{a-1}) - m_a$ and $g_a = \gamma_a(m_a) - m_a$ for every a; in particular $(b, g) \in B(D)$.*

Proof. A lattice is a finitely generated torsion-free module over the principal ideal domain $k[[X]]$, hence free of finite rank, hence complete and separated for the X-adic topology: $\bigcap_{i \geq 0} X^i N = 0$, and a series $\sum_i x_i$ with $x_i \in X^i N$ converges in N. By (Rr2), $\gamma(X) = X(1 - \kappa X)$ with $\kappa(0) = 1$, and $\varphi(X) = uX^\ell$ with $u \in k[[X]]^\times$.

Part 1. Write $\nabla = \nabla_a$; the hypothesis $(\gamma_a - 1)v \in XN$ says $\nabla v \in N$. We show: if $1 \leq c \leq J$ and $y = X^c v \in N$, then $X^{c-1} v \in N$. By (Rm1), $\gamma_a(y) = \gamma(X)^c \gamma_a(v) = X^c (1 - \kappa X)^c \gamma_a(v)$, so

$$\nabla y = X^c \nabla v + \lambda X^c \gamma_a(v), \qquad \lambda = X^{-1}\big((1 - \kappa X)^c - 1\big) \in -c + Xk[[X]],$$

the last because $\kappa(0) = 1$. As $X^c \gamma_a(v) = y + X^{c+1} \nabla v$,

$$(\nabla + c)\, y = (\lambda + c)\, y + (1 + \lambda X)\, X^c \nabla v \in XN,$$

since $\lambda + c \in Xk[[X]]$, $y \in N$, and $X^c \nabla v \in X^c N \subseteq XN$ ($c \geq 1$). By Lemma 28.25 (1), the class $\bar{y} \in N/XN$ lies in the kernel of $\overline{\nabla} - (-c)$, which is contained in the generalised $(-c)$-eigenspace

of $\overline{\nabla}$, of dimension $\mathrm{sen}_a(N;-c)=0$. So $y\in XN$, i.e. $X^{c-1}v\in N$. Starting from $X^Jv\in N$ and applying this for $c=J,J-1,\dots,1$ gives $v\in N$ (nothing to do if $J=0$).

Part 2, the Frobenius equation. Since $g_{a-1}\in XN_{a-1}\subseteq N_{a-1}$, $\varphi_a(g_{a-1})\in\varphi_a^*N_{a-1}\subseteq XN_a$, and $g_a\in XN_a$, the cocycle relation gives $(\gamma_a-1)b_a=\varphi_a(g_{a-1})-g_a\in XN_a$; with $b_a\in X^{-J_a}N_a$, Part 1 gives $b_a\in N_a$ for every a. Let $P=\prod_a N_a$ and $\Phi\colon P\to P$, $\Phi(x)_a=\varphi_a(x_{a-1})$. For $x\in X^iN_{a-1}$, $\varphi_a(x)\in\varphi(X)^i\varphi_a(N_{a-1})\subseteq X^{\ell i}XN_a\subseteq X^{i+1}N_a$ by (Rm1); so Φ maps $\prod_a X^iN_a$ into $\prod_a X^{i+1}N_a$, and $\Phi^i(x)\in\prod_a X^iN_a$ for $x\in P$. Being additive and X-adically contracting, Φ commutes with convergent series. Put $m=-\sum_{i\geq0}\Phi^i(b)\in P$. Then $\Phi(m)-m=-\sum_{i\geq1}\Phi^i(b)+\sum_{i\geq0}\Phi^i(b)=b$, i.e. $b_a=\varphi_a(m_{a-1})-m_a$. Moreover, if $x\in P$ and $\Phi(x)=x$, then $x=\Phi^i(x)\in\prod_a X^iN_a$ for all i, so $x=0$.

Part 2, the Γ-component. Put $q_a=\gamma_a(m_a)-m_a-g_a$. By the Γ-condition and $g_a\in XN_a$, $q\in P$. Using (Rm3), $\varphi_a(m_{a-1})=b_a+m_a$ and $\varphi_a(g_{a-1})=\gamma_a(b_a)-b_a+g_a$ (the cocycle relation),

$$\varphi_a(q_{a-1})=\gamma_a\varphi_a(m_{a-1})-\varphi_a(m_{a-1})-\varphi_a(g_{a-1})=\gamma_a(b_a+m_a)-(b_a+m_a)-\gamma_a(b_a)+b_a-g_a=q_a.$$

So $\Phi(q)=q$, hence $q=0$, i.e. $g_a=\gamma_a(m_a)-m_a$ for every a. Hence (b,g) is the Herr boundary of m. □

Lemma 28.37. *Let ℓ be a prime, k a field of characteristic ℓ, G a group, (φ,γ) a residual ring datum over k, $\mathcal{F}$ a residual model functor for G over (φ,γ) with sites $\mathbb{Z}/n$, and let $h,w,\eta\colon\mathbb{Z}/n\to\mathbb{Z}_{\geq0}$, typ, $t\geq1$ and multisets U_a of cardinality r satisfy (Ph1)–(Ph3) for ℓ. Let residual model data $(\bar{T},\omega,\dots)$ for $(h,w,\mathrm{typ},t,(U_a))$ and $\mathcal{F}$ be given, and let $Z_{\mathrm{pol}}(\mathbf{N}_0^\chi)$ be the polar pairs of $\mathcal{F}(\mathbf{1})$ for the letter data h,w,typ. Then there is a k-linear map $\lambda\colon Z_{\mathrm{pol}}(\mathbf{N}_0^\chi)\to k$ with $\ker\lambda\subseteq B(\mathcal{F}(\mathbf{1}))$. In particular, for $\zeta_1,\zeta_2\in Z_{\mathrm{pol}}(\mathbf{N}_0^\chi)$ there are $(\beta_1,\beta_2)\in k^2\setminus\{0\}$ with $\beta_1\zeta_1+\beta_2\zeta_2\in B(\mathcal{F}(\mathbf{1}))$.*

Proof. Write $V=V_{\mathbf{1}}$, of dimension 1 over $k((X))$ by (Mf2), and $N_a=N_{0,a}^\chi$. By (Md8), $N_a=k[[X]]u_a$, and $u_a\neq0$, $\varphi_a(u_{a-1})=u_a$, $\gamma_a(u_a)=u_a$ by (Mf4); so $V=k((X))u_a$ and $cu_a=0$ forces $c=0$. By (Rm1), $\varphi_a(cu_{a-1})=\varphi(c)u_a$ and $\gamma_a(cu_a)=\gamma(c)u_a$ for $c\in k((X))$. By (Md7) with $e=0$: N_a satisfies the Γ-condition at a, $\varphi_a^*N_{a-1}=N_a$, and $\mathrm{sen}_a(N_a;x)=0$ for $x\neq0$. Since γ is a k-algebra map, $\gamma(c)=c$ for $c\in k$; the Γ-condition gives $\gamma(c)-c\in Xk[[X]]$ for $c\in k[[X]]$, and, φ being a k-algebra map with $\varphi(X)\in X^\ell k[[X]]$ ((Rr1), (Rr2)), $\varphi(X^ik[[X]])\subseteq X^{\ell i}k[[X]]\subseteq X^{i+1}k[[X]]$ for $i\geq1$.

Step 1: $b_a\in N_a$. Let $(b,g)\in Z_{\mathrm{pol}}(\mathbf{N}_0^\chi)$ and put $J_a=w_a$ if $\mathrm{typ}(a)=\mathrm{low}$ and $J_a=w_a^*=h_a-w_a$ if $\mathrm{typ}(a)=\mathrm{high}$. By (Pp2), resp. (Pp3) with $\varphi_a^*N_{a-1}=N_a$, $b_a\in X^{-J_a}N_a$. By (Ph1), $J_a\geq1$ (for high: $h_a-w_a\geq t$). By (Ph2), $J_a\leq\eta_a$ (low: $w_a\leq w_a+t-1\leq\eta_a$; high: $h_a-w_a\leq\eta_a$), and $\eta_a<\ell$ by (Ph3); so $1\leq j\leq J_a$ gives $j\cdot1_k\neq0$ and $\mathrm{sen}_a(N_a;-j)=0$. By (Pp1), $g_{a-1}=Xy$ with $y\in N_{a-1}$, so $\varphi_a(g_{a-1})=\varphi(X)\varphi_a(y)\in X^\ell N_a\subseteq XN_a$, and $g_a\in XN_a$; the cocycle relation gives $(\gamma_a-1)b_a\in XN_a$. By Lemma 28.36 (1), $b_a\in N_a$.

Step 2: normal form. Write $b_a=\beta_au_a$ and $g_a=\delta_au_a$ with $\beta_a\in k[[X]]$, $\delta_a\in Xk[[X]]$ (Step 1 and (Pp1)). The cocycle relation reads $\gamma(\beta_a)-\beta_a=\varphi(\delta_{a-1})-\delta_a$. Let $c_a=\beta_a(0)\in k$ and $\beta'_a=\beta_a-c_a\in Xk[[X]]$. Let $P_0=(Xk[[X]])^{\mathbb{Z}/n}$ and $\Phi_0(x)_a=\varphi(x_{a-1})$; Φ_0 maps $(X^ik[[X]])^{\mathbb{Z}/n}$ into $(X^{i+1}k[[X]])^{\mathbb{Z}/n}$ for $i\geq1$, so $\Phi_0^i(\beta')\in(X^{i+1}k[[X]])^{\mathbb{Z}/n}$ and, $k[[X]]$ being X-adically complete, $m=-\sum_{i\geq0}\Phi_0^i(\beta')\in P_0$ is defined; Φ_0 is additive and X-adically contracting, hence commutes with this series, and $\Phi_0(m)-m=-\sum_{i\geq1}\Phi_0^i(\beta')+\sum_{i\geq0}\Phi_0^i(\beta')=\beta'$, i.e. $\varphi(m_{a-1})-m_a=\beta'_a$. If $x\in P_0$ and $\Phi_0(x)=x$, then $x=\Phi_0^i(x)\in(X^{i+1}k[[X]])^{\mathbb{Z}/n}$ for all i, so $x=0$. Put $q_a=\gamma(m_a)-m_a-\delta_a\in Xk[[X]]$. By (Rr3) and $\gamma(c_a)=c_a$,

$$\varphi(q_{a-1})=\gamma(\beta'_a+m_a)-(\beta'_a+m_a)-(\gamma(\beta_a)-\beta_a+\delta_a)=q_a,$$

so $q = 0$. Hence, with $\mu_a = m_a u_a$,

$$b_a = \varphi_a(\mu_{a-1}) - \mu_a + c_a u_a, \qquad g_a = \gamma_a(\mu_a) - \mu_a,$$

i.e. $(b, g) \equiv ((c_a u_a)_a, 0)$ modulo $B(\mathcal{F}(\mathbf{1}))$.

Step 3: the map λ. Define $\lambda(b, g) = \sum_{a \in \mathbb{Z}/n} c_a$, the sum of the constant terms of the coordinates β_a of b_a in the basis u_a; it is k-linear. If $\lambda(b, g) = 0$, number the sites $0, 1, \dots, n-1$ and put $d_a = -\sum_{j=0}^{a} c_j$; then $d_{n-1} = 0$, and $d_{a-1} - d_a = c_a$ for every a (for $a = 0$: $d_{n-1} - d_0 = c_0$). The pair $((c_a u_a)_a, 0)$ is the Herr boundary of $(d_a u_a)_a$: $\varphi_a(d_{a-1} u_{a-1}) - d_a u_a = (d_{a-1} - d_a) u_a = c_a u_a$ and $\gamma_a(d_a u_a) - d_a u_a = 0$. With Step 2, $(b, g) \in B(\mathcal{F}(\mathbf{1}))$. Finally, for ζ_1, ζ_2: if $\lambda(\zeta_1) = 0$ take $(\beta_1, \beta_2) = (1, 0)$; otherwise $(\beta_1, \beta_2) = (\lambda(\zeta_2), -\lambda(\zeta_1)) \neq 0$, and $\lambda(\beta_1 \zeta_1 + \beta_2 \zeta_2) = 0$. □

Theorem 28.38. *Let ℓ be a prime, k a field of characteristic ℓ, (φ, γ) a residual ring datum over k, and $D = (V, (\varphi_a), (\gamma_a))$ a residual (φ, Γ)-module of rank r with sites $\mathbb{Z}/n$ that admits a ψ-operator. Let $h\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$, $w\colon \mathbb{Z}/n \to \mathbb{Z}$ and* typ *be letter data with $0 \leq w_a \leq h_a$ for all a and $1 \leq w_a^* \leq \ell - 2$ whenever* $\mathrm{typ}(a) = \mathrm{high}$*; put* $\mathrm{sg}(a) = \lfloor \sum_{j \geq 1} h_{a+j} \ell^{-j} \rfloor$*, and let $d_a \geq 0$ be integers. Let $\mathbf{N} = (N_a)_{a \in \mathbb{Z}/n}$ be lattices of V with heights (w, w^*), N_a satisfying the Γ-condition at a, such that for every a:*

(Sn1) $\mathrm{sen}_a(N_a; j) = 0$ *for all integers $0 \leq j \leq \mathrm{sg}(a) + 1$;*

(Sn2) *if* $\mathrm{typ}(a) = \mathrm{low}$*: $\sum_{j=0}^{w_a - 1} \mathrm{sen}_a(N_a; j - w_a) \leq d_a$;*

(Sn3) *if* $\mathrm{typ}(a) = \mathrm{high}$*: $\sum_{j=0}^{w_a^*} \mathrm{sen}_a(N_a; j) \geq r - d_a$, and $\nabla_a f_a(\nabla_a)(N_a) \subseteq X^{w_a^* + 1} N_a$ for some $f_a \in k[T]$ with $f_a(0) \neq 0$.*

If $z_1, \dots, z_s \in Z_{\mathrm{pol}}(\mathbf{N})$ are linearly independent modulo $B(D)$, then $s \leq \sum_{a \in \mathbb{Z}/n} d_a$.

Proof. Fix a ψ-operator $(\psi_0, (\Psi_a))$ for D.

Step 0: the carries. Let $S(a) = \sum_{j \geq 1} h_{a+j} \ell^{-j}$ (convergent, h being bounded), so $\mathrm{sg}(a) = \lfloor S(a) \rfloor$. Then $\ell S(a-1) = h_a + S(a)$. As $\mathrm{sg}(a-1) + 1 > S(a-1)$ and $S(a) \geq \mathrm{sg}(a)$, we get $\ell(\mathrm{sg}(a-1) + 1) > h_a + \mathrm{sg}(a)$, hence, both sides being integers,

$$\ell(\mathrm{sg}(a-1) + 1) \geq h_a + \mathrm{sg}(a) + 1 \qquad (a \in \mathbb{Z}/n). \qquad \text{(C)}$$

With $0 \leq w_a \leq h_a$ and $\ell \geq 2$ this gives

$$\ell(\mathrm{sg}(a-1) + 2) - w_a \geq \mathrm{sg}(a) + 3, \qquad \ell(\mathrm{sg}(a-1) + 3) - w_a \geq \max\big(\mathrm{sg}(a) + 3,\ w_a^* + 2\big). \qquad \text{(E)}$$

Moreover, for $j \geq 0$ and $y \in X^j N_{a-1}$, $\varphi_a(y) \in X^{\ell j - w_a} N_a$: write $y = X^j y'$; by (Rm1), (Rr2) $\varphi_a(y) = u^j X^{\ell j} \varphi_a(y')$, and $\varphi_a(y') \in \varphi_a^* N_{a-1} \subseteq X^{-w_a} N_a$ by the heights.

Step 1: normalisation of a polar pair. Let $(b, g) \in Z_{\mathrm{pol}}(\mathbf{N})$. For each a, $X^{-1} g_a \in N_a$ by (Pp1); by Lemma 28.39 (Wg3)(ii) ($c = 0$, $m = \mathrm{sg}(a) + 2$, using (Sn1)) there is $y_a \in N_a$ with $X^{-1} g_a - \nabla_a y_a \in X^{\mathrm{sg}(a)+2} N_a$, i.e. $g_a - (\gamma_a(y_a) - y_a) \in X^{\mathrm{sg}(a)+3} N_a$. Let $x_a = y_a$ if $\mathrm{typ}(a) = \mathrm{low}$, and let $x_a \in N_a$ be as in (Pp3) if $\mathrm{typ}(a) = \mathrm{high}$. Put

$$b'_a = b_a - (\varphi_a(y_{a-1}) - y_a), \qquad g'_a = g_a - (\gamma_a(y_a) - y_a), \qquad c_a = b_a - \varphi_a(y_{a-1}) + x_a.$$

Then: (G1) $(b', g') \in Z^1(D)$ and $(b, g) - (b', g') \in B(D)$; (G2) $g'_a \in X^{\mathrm{sg}(a)+3} N_a$; (G3) $c_a - b'_a = x_a - y_a \in N_a$; and (G4)

$$\gamma_a(c_a) - c_a = \varphi_a(g'_{a-1}) - \big(g_a - (\gamma_a(x_a) - x_a)\big),$$

because, by the Herr relation and (Rm3), $(\gamma_a - 1) c_a = \varphi_a(g_{a-1}) - g_a - \varphi_a((\gamma_{a-1} - 1) y_{a-1}) + (\gamma_a - 1) x_a$. By Step 0 and (G2), $\varphi_a(g'_{a-1}) \in X^{\ell(\mathrm{sg}(a-1)+3) - w_a} N_a \subseteq X^{w_a^* + 2} N_a$. Hence:

(G5) if $\mathrm{typ}(a) = \mathrm{low}$: $c_a \in X^{-w_a} N_a$ and $\gamma_a(c_a) - c_a \in XN_a$. Indeed $x_a = y_a$, so the second term of (G4) is $g'_a \in X^{\mathrm{sg}(a)+3} N_a \subseteq XN_a$, and $X^{w_a^*+2} N_a \subseteq XN_a$ as $w_a^* \geq 0$; and $b_a \in X^{-w_a} N_a$ (Pp2), $\varphi_a(y_{a-1}) \in \varphi_a^* N_{a-1} \subseteq X^{-w_a} N_a$, $x_a \in N_a \subseteq X^{-w_a} N_a$ ($w_a \geq 0$).

(G6) if $\mathrm{typ}(a) = \mathrm{high}$: $c_a \in X^{-w_a^*} \varphi_a^* N_{a-1}$ and $\gamma_a(c_a) - c_a \in X^{w_a^*+2} N_a$. Indeed the second term of (G4) lies in $X^{w_a^*+2} N_a$ by (Pp3); and $b_a \in X^{-w_a^*} \varphi_a^* N_{a-1}$ (Pp3), $\varphi_a(y_{a-1}) \in \varphi_a^* N_{a-1} \subseteq X^{-w_a^*} \varphi_a^* N_{a-1}$, and $x_a \in N_a \subseteq X^{-w_a^*} \varphi_a^* N_{a-1}$ because $X^{w_a^*} N_a \subseteq \varphi_a^* N_{a-1}$ (heights).

Step 2: small cocycles are boundaries. Let $(b, g) \in Z^1(D)$ with $b_a \in N_a$ and $g_a \in X^{\mathrm{sg}(a)+3} N_a$ for all a. Then $(b, g) \in B(D)$. Indeed, by Step 0 and (E), $\gamma_a(b_a) - b_a = \varphi_a(g_{a-1}) - g_a \in X^{\mathrm{sg}(a)+3} N_a$, i.e. $\nabla_a b_a \in X^{\mathrm{sg}(a)+2} N_a$, so $b_a \in X^{\mathrm{sg}(a)+2} N_a$ by (Wg3)(i) ($c = 0$, $m = \mathrm{sg}(a) + 2$, (Sn1)). Let $M_a = X^{\mathrm{sg}(a)+2} N_a$; by Step 0 and (E), $\varphi_a(M_{a-1}) \subseteq X^{\ell(\mathrm{sg}(a-1)+2)-w_a} N_a \subseteq XM_a$. By (Wg5) there is $\mu \in \prod_a M_a$ with $\varphi_a(\mu_{a-1}) - \mu_a = b_a$. Put $e_a = g_a - (\gamma_a(\mu_a) - \mu_a)$. By (Wg2), $\gamma_a(\mu_a) - \mu_a = X \nabla_a \mu_a \in X^{\mathrm{sg}(a)+3} N_a$, so $e \in \prod_a M_a$; and by (Rm3) and the Herr relation

$$\varphi_a(e_{a-1}) = \varphi_a(g_{a-1}) - (\gamma_a - 1)\varphi_a(\mu_{a-1}) = \varphi_a(g_{a-1}) - (\gamma_a - 1)(b_a + \mu_a) = g_a - (\gamma_a - 1)\mu_a = e_a.$$

By the uniqueness in (Wg5) (with $\beta = 0$), $e = 0$. So $(b, g) = ((\varphi_a(\mu_{a-1}) - \mu_a)_a, (\gamma_a(\mu_a) - \mu_a)_a) \in B(D)$.

Step 3: the count at each site. Apply Step 1 to $z_1, \ldots, z_s$, obtaining $(b'^{(i)}, g'^{(i)})$ and $c^{(i)}$ ($1 \leq i \leq s$), and let $C_a \subseteq V$ be the k-span of $c_a^{(1)}, \ldots, c_a^{(s)}$. We claim $\dim_k C_a/(C_a \cap N_a) \leq d_a$. If $\mathrm{typ}(a) = \mathrm{low}$, (G5) gives $C_a \subseteq X^{-w_a} N_a$ and $\nabla_a(C_a) \subseteq N_a$, and Lemma 28.40 (Pk1) with $w = w_a \geq 0$ together with (Sn2) gives the claim. If $\mathrm{typ}(a) = \mathrm{high}$, apply Lemma 28.40 (Pk3) at the site a with $N = N_a$, $N_0 = N_{a-1}$, $v = w_a^*$, $s = \mathrm{sg}(a-1)$, $w = w_a$: N_{a-1} satisfies the Γ-condition at $a - 1$; $1 \leq v \leq \ell - 2$; $X^v N_a \subseteq \varphi_a^* N_{a-1} \subseteq X^{-w_a} N_a$ (heights); $v + 2 \leq \ell(\mathrm{sg}(a-1) + 3) - w_a$ by (E); $\mathrm{sen}_{a-1}(N_{a-1}; j) = 0$ for $0 \leq j \leq \mathrm{sg}(a-1) + 1$ by (Sn1); and the kernel condition holds by (Pk2) with $c = w_a^* + 1$ and the polynomial f_a of (Sn3). With $Z = C_a$, which satisfies the hypotheses by (G6), (Pk3) and (Sn3) give $\dim_k C_a/(C_a \cap N_a) \leq r - \sum_{j=0}^{w_a^*} \mathrm{sen}_a(N_a; j) \leq d_a$.

Step 4: injectivity. Let $\Theta\colon k^s \to \prod_a V/N_a$, $t \mapsto \left(\sum_i t_i c_a^{(i)} + N_a\right)_a$. Its image lies in $\prod_a (C_a + N_a)/N_a \cong \prod_a C_a/(C_a \cap N_a)$, of dimension at most $\sum_a d_a$ by Step 3. Suppose $\Theta(t) = 0$, and let $(b, g) = \sum_i t_i (b'^{(i)}, g'^{(i)}) \in Z^1(D)$. Then $b_a = \sum_i t_i c_a^{(i)} - \sum_i t_i (c_a^{(i)} - b_a'^{(i)}) \in N_a$ by (G3), and $g_a \in X^{\mathrm{sg}(a)+3} N_a$ by (G2). By Step 2, $(b, g) \in B(D)$; by (G1), $\sum_i t_i z_i \in (b, g) + B(D) = B(D)$; so $t = 0$ by the independence of the z_i modulo $B(D)$. Thus Θ is injective and $s \leq \sum_a d_a$. □

Lemma 28.39. *Let (φ, γ) be a residual ring datum over a field k of characteristic ℓ, $D = (V, (\varphi_b), (\gamma_b))$ a residual (φ, Γ)-module of rank r with sites $\mathbb{Z}/n$, $a \in \mathbb{Z}/n$, and $\nabla = \nabla_a$.*

(Wg1) *Every lattice N of V is a free $k[[X]]$-module of rank r, and $\dim_k X^c N/X^{c+m} N = mr$ for $c \in \mathbb{Z}$, $m \geq 0$.*

(Wg2) *If the lattice N satisfies the Γ-condition at a, then for every $b \in \mathbb{Z}$: $\nabla(X^b N) \subseteq X^b N$, and $\nabla(X^b x) - X^b(\nabla x - bx) \in X^{b+1} N$ for every $x \in N$.*

(Wg3) *Let N satisfy the Γ-condition at a, and let $c \in \mathbb{Z}$, $m \geq 0$ with $\mathrm{sen}_a(N; c + j) = 0$ for $0 \leq j < m$. Then (i) if $x \in X^c N$ and $\nabla x \in X^{c+m} N$, then $x \in X^{c+m} N$; (ii) for every $y \in X^c N$ there is $x \in X^c N$ with $y - \nabla x \in X^{c+m} N$.*

(Wg4) *Let N satisfy the Γ-condition at a, $c \in \mathbb{Z}$, $m \geq 0$, and $G = \{v \in X^c N : \nabla^j v \in X^{c+m} N \text{ for some } j \geq 0\}$. Then*

$$\dim_k G/X^{c+m} N = \sum_{j=0}^{m-1} \mathrm{sen}_a(N; c + j).$$

(Wg5) *If $(M_b)_{b\in\mathbb{Z}/n}$ are lattices of V with $\varphi_b(M_{b-1})\subseteq XM_b$ for all b, then for every $(\beta_b)\in\prod_b M_b$ there is a unique $(\mu_b)\in\prod_b M_b$ with $\varphi_b(\mu_{b-1})-\mu_b=\beta_b$ for all b.*

Proof. (Wg1). $k[[X]]$ is a principal ideal domain and N is finitely generated and torsion-free (it lies in a $k((X))$-vector space), so N is free. A $k[[X]]$-basis of N is linearly independent over $k((X))$ (clear denominators by a power of X) and spans V over $k((X))$, so it has r elements. Multiplication by X^c is an isomorphism $N\to X^cN$ of $k[[X]]$-modules carrying X^mN to $X^{c+m}N$, and $N/X^mN\cong(k[[X]]/X^m)^r$ has k-dimension mr.

(Wg2). By (Rr2), $\gamma(X)=X(1-\kappa X)$ and $1-\kappa X$ is a unit of $k[[X]]$ with $1-\kappa X\equiv 1-X \pmod{X^2}$, as $\kappa(0)=1$. For $q\in k[[X]]$ and $b\in\mathbb{Z}$ one has $(1+Xq)^b\equiv 1+bXq \pmod{X^2}$ (for $b\geq 0$ by the binomial formula, for $b<0$ because $(1+Xq)^{-1}\equiv 1-Xq$). Hence $\gamma(X^b)=X^b(1-bX+X^2e_b)$ for some $e_b\in k[[X]]$. Let $x\in N$; the Γ-condition gives $\nabla x\in N$ and $\gamma_a(x)=x+X\nabla x$. By (Rm1),

$$\gamma_a(X^bx)=\gamma(X^b)\gamma_a(x)=X^b(1-bX+X^2e_b)(x+X\nabla x),$$

so that

$$\nabla(X^bx)=X^{-1}(\gamma_a(X^bx)-X^bx)=X^b(\nabla x-bx+Xn'),\qquad n'=e_bx-b\nabla x+Xe_b\nabla x\in N.$$

Both assertions follow, since $X^b(\nabla x-bx)\in X^bN$ and $X^{b+1}n'\in X^{b+1}N$.

(Wg3). First let $m=1$ and $b=c$. By (Wg2) ($b=0,1$) ∇ maps N and XN into themselves, so it induces a k-linear endomorphism $\bar\nabla$ of the finite-dimensional space N/XN (Wg1). Since $XN\subseteq N^{(x)}$ (take $j=0$), $N^{(x)}/XN$ is the generalised x-eigenspace of $\bar\nabla$ and $\mathrm{sen}_a(N;x)$ is its dimension. So $\mathrm{sen}_a(N;b)=0$ says that $\bar\nabla-b$ is injective, hence bijective. (i) Let $x=X^bx_0$, $x_0\in N$, with $\nabla x\in X^{b+1}N$. By (Wg2), $X^b(\nabla x_0-bx_0)\in X^{b+1}N$, so $(\nabla-b)x_0\in XN$, so $x_0\in XN$ by injectivity, and $x\in X^{b+1}N$. (ii) Let $y=X^by_0$, $y_0\in N$. By surjectivity there is $x_0\in N$ with $y_0-(\nabla x_0-bx_0)\in XN$; for $x=X^bx_0$,

$$y-\nabla x=X^b(y_0-(\nabla x_0-bx_0))-(\nabla(X^bx_0)-X^b(\nabla x_0-bx_0))\in X^{b+1}N$$

by (Wg2). Now induct on m; for $m=0$ there is nothing to prove ($x=0$ in (ii)). Suppose the claim for m and let $\mathrm{sen}_a(N;c+j)=0$ for $0\leq j\leq m$. (i) If $x\in X^cN$ and $\nabla x\in X^{c+m+1}N\subseteq X^{c+m}N$, the case m gives $x\in X^{c+m}N$, and the case $m=1$ at $b=c+m$ gives $x\in X^{c+m+1}N$. (ii) Given $y\in X^cN$, the case m gives $x_1\in X^cN$ with $y-\nabla x_1\in X^{c+m}N$, and the case $m=1$ at $b=c+m$ gives $x_2\in X^{c+m}N\subseteq X^cN$ with $(y-\nabla x_1)-\nabla x_2\in X^{c+m+1}N$; take $x=x_1+x_2$.

(Wg4). Let $W=X^cN/X^{c+m}N$, finite-dimensional by (Wg1), with the endomorphism T induced by ∇ (Wg2), and $W_j=X^{c+j}N/X^{c+m}N$ for $0\leq j\leq m$, a T-stable flag with $W_0=W$, $W_m=0$. $G/X^{c+m}N$ is the generalised 0-eigenspace of T. For an endomorphism of a finite-dimensional space, the dimension of the generalised 0-eigenspace is the multiplicity of 0 as a root of the characteristic polynomial, and characteristic polynomials are multiplicative along a stable subspace and its quotient; hence $\dim_k G/X^{c+m}N$ is the sum over $0\leq j<m$ of the dimensions of the generalised 0-eigenspaces of T on $W_j/W_{j+1}=X^{c+j}N/X^{c+j+1}N$. Multiplication by X^{c+j} induces a k-linear isomorphism $N/XN\to X^{c+j}N/X^{c+j+1}N$ which, by (Wg2) with $b=c+j$, carries $\bar\nabla-(c+j)$ to the map induced by T. So the j-th summand is the dimension of the generalised $(c+j)$-eigenspace of $\bar\nabla$, which is $\mathrm{sen}_a(N;c+j)$ (proof of (Wg3)).

(Wg5). Let $M=\prod_b M_b$, a finitely generated free $k[[X]]$-module (Wg1), and $T\colon M\to M$, $T(\mu)_b=\varphi_b(\mu_{b-1})$, an additive map. If $\mu\in X^jM$ ($j\geq 0$), write $\mu_{b-1}=X^j\mu'_{b-1}$ with $\mu'_{b-1}\in M_{b-1}$; by (Rm1) and (Rr2), $T(\mu)_b=\varphi(X)^j\varphi_b(\mu'_{b-1})=u^jX^{\ell j}\varphi_b(\mu'_{b-1})\in X^{\ell j+1}M_b\subseteq X^{j+1}M_b$.

So $T(X^jM)\subseteq X^{j+1}M$ and $T^i(M)\subseteq X^iM$. Uniqueness: a difference μ of two solutions satisfies $T(\mu)=\mu$; if $\mu\in X^jM$ then $\mu=T(\mu)\in X^{j+1}M$, so $\mu\in\bigcap_j X^jM=0$. Existence: M is complete and separated for the X-adic topology, so $\mu=-\sum_{i\ge0}T^i(\beta)$ converges in M; T is additive and X-adically continuous ($T(X^jM)\subseteq X^{j+1}M$), so $T(\mu)=-\sum_{i\ge1}T^i(\beta)$ and $T(\mu)-\mu=\beta$. $\square$

Lemma 28.40. *Let (φ,γ) be a residual ring datum over a field k of characteristic ℓ, $D=(V,(\varphi_b),(\gamma_b))$ a residual (φ,Γ)-module of rank r with sites $\mathbb{Z}/n$, $a\in\mathbb{Z}/n$, $\nabla=\nabla_a$, and N a lattice of V satisfying the Γ-condition at a.*

(Pk1) *Let $w\ge0$ and let $Z\subseteq V$ be a k-subspace with $Z\subseteq X^{-w}N$ and $\nabla(Z)\subseteq N$. Then $\dim_k Z/(Z\cap N)\le\sum_{j=0}^{w-1}\operatorname{sen}_a(N;j-w)$.*

(Pk2) *Let $c\ge0$ and $f\in k[T]$ with $f(0)\neq0$ and $\nabla f(\nabla)(N)\subseteq X^cN$. If $x\in N$ and $\nabla^m x\in X^cN$ for some $m\ge0$, then $\nabla x\in X^cN$.*

(Pk3) *Let $(\psi_0,(\Psi_b))$ be a ψ-operator for D, N_0 a lattice satisfying the Γ-condition at $a-1$, $M=\varphi_a^*N_0$, and v,s,w integers with $1\le v\le\ell-2$ and $s\ge0$, such that*

$$X^vN\subseteq M\subseteq X^{-w}N,\qquad v+2\le\ell(s+3)-w,\qquad \operatorname{sen}_{a-1}(N_0;j)=0\ \ (0\le j\le s+1),$$

and every $x\in N$ with $\nabla^m x\in X^{v+1}N$ for some $m\ge0$ satisfies $\nabla x\in X^{v+1}N$. Then every k-subspace $Z\subseteq X^{-v}M$ with $(\gamma_a-1)(Z)\subseteq X^{v+2}N$ satisfies

$$\dim_k Z/(Z\cap N)+\sum_{j=0}^{v}\operatorname{sen}_a(N;j)\le r.$$

Proof. Throughout, by Lemma 28.39 (Wg2) ∇ maps every X^bN into itself, and by (Wg1) every $X^bN/X^{b'}N$ ($b\le b'$) is finite-dimensional over k.

(Pk1). Let $W=X^{-w}N/N$ with the endomorphism T induced by ∇. The image $\bar Z$ of Z in W is isomorphic to $Z/(Z\cap N)$, and $T(\bar Z)=0$ because $\nabla(Z)\subseteq N$. So $\bar Z$ lies in the generalised 0-eigenspace of T, which is G/N with $G=\{v\in X^{-w}N:\ \nabla^jv\in N\text{ for some }j\}$; by (Wg4) with $c=-w$, $m=w$ its dimension is $\sum_{j=0}^{w-1}\operatorname{sen}_a(N;j-w)$.

(Pk2). If $m=0$, then $x\in X^cN$ and $\nabla x\in X^cN$. Let $m\ge1$, and let T be the endomorphism of N/X^cN induced by ∇; by hypothesis $Tf(T)=0$. As $f(0)\neq0$, the polynomials T^{m-1} and f are coprime in $k[T]$, so $\gcd(T^m,Tf(T))=T$ and there are $\alpha,\beta\in k[T]$ with $T=\alpha T^m+\beta Tf(T)$. Applied to the class $\bar x$ of x: $T\bar x=\alpha(T)T^m\bar x+\beta(T)Tf(T)\bar x=0$, since $T^m\bar x=0$. So $\nabla x\in X^cN$.

(Pk3), Step 1: a count in linear algebra. Let U be a k-vector space with an endomorphism T, and $L\subseteq N'\subseteq F$ subspaces with $T(L)\subseteq L$ and F/L finite-dimensional, S a finite-dimensional subspace with $F\subseteq S+T(F)+L$, such that every $x\in N'$ with $T^mx\in L$ for some m has $Tx\in L$. Put $G'=\{x\in N':\ T^mx\in L\text{ for some }m\}$. Then for every subspace $Z\subseteq F$ with $T(Z)\subseteq L$,

$$\dim_k Z/(Z\cap N')+\dim_k G'/L\le\dim_k S.$$

Indeed, let $\bar T$ be the endomorphism of $\bar U=U/L$ induced by T, $\bar F=F/L$ and $K=\bar F\cap\ker\bar T$. By the rank formula for $\bar T|_{\bar F}$, $\dim\bar T(\bar F)+\dim K=\dim\bar F$; and $\bar F\subseteq\bar S+\bar T(\bar F)$ ($\bar S$ the image of S) gives $\dim\bar F\le\dim S+\dim\bar T(\bar F)$, so $\dim K\le\dim S$. Let $q\colon\bar U\to U/N'$ be the projection; then $\dim q(K)+\dim(K\cap\ker q)=\dim K$. The image of Z in U/N', of dimension $\dim Z/(Z\cap N')$, lies in $q(K)$, since $z\in Z$ has $z\in F$ and $Tz\in L$. And $G'/L\subseteq K\cap\ker q$: for $x\in G'$ we have $x\in N'\subseteq F$ and $Tx\in L$ by hypothesis, and $q(\bar x)=0$ as $x\in N'$. Adding gives the claim.

Step 2: the setting. Apply Step 1 with $U = V$, $T = \nabla$, $L = X^{v+1}N$, $N' = N$, $F = X^{-v}M$. Here $L \subseteq N$ since $v+1 \geq 0$; $N \subseteq F$ since $X^v N \subseteq M$; $T(L) \subseteq L$ by (Wg2); $F \subseteq X^{-v-w}N$ since $M \subseteq X^{-w}N$, so F/L is a subquotient of $X^{\min(-v-w,v+1)}N/X^{v+1}N$, finite-dimensional. The kernel condition is the last hypothesis. By (Wg4) with $c = 0$, $m = v+1$, $\dim_k G'/L = \sum_{j=0}^{v} \operatorname{sen}_a(N; j)$. A subspace Z as in the statement satisfies $Z \subseteq F$ and $\nabla(Z) = X^{-1}(\gamma_a - 1)(Z) \subseteq X^{v+1}N = L$. By (Wg1), N_0 has a $k[[X]]$-basis $e_1, \dots, e_r$; let S be the k-span of $X^{-1}\varphi_a(e_1), \dots, X^{-1}\varphi_a(e_r)$, so $\dim S \leq r$. It remains to show $F \subseteq S + \nabla(F) + L$.

Step 3: the decomposition. Let $x \in X^{-v}M$. (1) $\Psi_a(X^{1-v}M) \subseteq N_0$: an element of $X^{1-v}M$ is a finite sum of terms $X^{1-v}c\,\varphi_a(n)$ with $c \in k[[X]]$, $n \in N_0$, and $\Psi_a(X^{1-v}c\,\varphi_a(n)) = \psi_0(X^{1-v}c)\,n$ by (Ps1), with $\psi_0(X^{1-v}c) \in k[[X]]$ by (Ps2), as $2 - \ell \leq 1 - v \leq 1$. Put $y = \Psi_a(Xx) \in N_0$. (2) $Xx - \varphi_a(y) \in X^{1-v}M$, since $\varphi_a(y) \in M \subseteq X^{1-v}M$ ($v \geq 1$), and $\Psi_a(Xx - \varphi_a(y)) = y - \psi_0(1)y = 0$ by (Ps1). By (Ps3), applied to the lattice N_0 (which satisfies the Γ-condition at $a-1$) and $c = -v$ ($2 - \ell \leq -v \leq 0$), there is $x' \in X^{-v}M$ with $\gamma_a(x') - x' = Xx - \varphi_a(y)$, i.e. $\nabla x' = x - X^{-1}\varphi_a(y)$. (3) Writing the coordinates of y in the basis (e_i) as constant term plus X times a power series, $y = y_1 + Xy_3$ with $y_1 \in \operatorname{span}_k(e_i)$ and $y_3 \in N_0$. By (Wg3)(ii) at the site $a-1$ (lattice N_0, $c = 0$, $m = s+2$; the Sen multiplicities $\operatorname{sen}_{a-1}(N_0; j)$, $0 \leq j \leq s+1$, vanish) there is $z \in N_0$ with $y_3 - \nabla_{a-1}z \in X^{s+2}N_0$. (4) By (Rm3), $\nabla(\varphi_a(z)) = X^{-1}(\gamma_a\varphi_a(z) - \varphi_a(z)) = X^{-1}\varphi_a(\gamma_{a-1}z - z) = X^{-1}\varphi_a(X\nabla_{a-1}z)$. Hence

$$x = X^{-1}\varphi_a(y_1) + \nabla(x' + \varphi_a(z)) + X^{-1}\varphi_a(X(y_3 - \nabla_{a-1}z)).$$

The first term lies in S. Since $\varphi_a(z) \in M \subseteq X^{-v}M$, $x' + \varphi_a(z) \in F$ and the second term lies in $\nabla(F)$. For the third, $X(y_3 - \nabla_{a-1}z) = X^{s+3}n$ with $n \in N_0$, and by (Rm1), (Rr2) $\varphi_a(X^{s+3}n) = u^{s+3}X^{\ell(s+3)}\varphi_a(n) \in X^{\ell(s+3)}M \subseteq X^{\ell(s+3)-w}N$; so the third term lies in $X^{\ell(s+3)-w-1}N \subseteq X^{v+1}N = L$, because $\ell(s+3) - w - 1 \geq v+1$. So $F \subseteq S + \nabla(F) + L$, and Step 1 gives $\dim_k Z/(Z \cap N) + \sum_{j=0}^{v} \operatorname{sen}_a(N; j) \leq \dim S \leq r$. □

Proposition 28.41. *Let $\ell \geq 5$ be a prime, k a field of characteristic ℓ, and (φ, γ) a cyclotomic residual ring datum over k. Then every residual (φ, Γ)-module over (φ, γ), with any set of sites $\mathbb{Z}/n$, admits a ψ-operator.*

Proof. Let D be a residual (φ, Γ)-module over (φ, γ) with sites $\mathbb{Z}/n$: a $k((X))$-vector space V of rank $r = \dim_{k((X))} V$ with maps φ_a, γ_a ($a \in \mathbb{Z}/n$) as in Definition 28.4. Throughout, v_π is the π-adic valuation on $k((\pi))$, v_X the X-adic valuation on $k((X))$, $\varphi_\pi, \gamma_\pi, \delta_z$ are as in Definition 28.4, and $\Delta = \{\delta_z : z \in \mathbb{Z}_\ell,\ z^{\ell-1} = 1\}$. For $z \in \mathbb{Z}_\ell$, $(1+\pi)^z = \sum_{i \geq 0} \binom{z}{i}\pi^i$ (reduced modulo ℓ); the identities $(1+\pi)^{z+w} = (1+\pi)^z(1+\pi)^w$ and $((1+\pi)^z)^w = (1+\pi)^{zw}$ of binomial series are used freely, so that $\delta_z(1+\pi) = (1+\pi)^z$, $\gamma_\pi = \delta_{1+\ell}$ and $\delta_z\delta_w = \delta_{zw}$.

Step 1 (Fontaine's ψ on $k((\pi))$). $\varphi_\pi(k((\pi))) = k((\pi^\ell))$ and $\varphi_\pi(k[[\pi]]) = k[[\pi^\ell]]$. Since $k((\pi)) = \bigoplus_{j=0}^{\ell-1} \pi^j k((\pi^\ell))$, $k[[\pi]] = \bigoplus_{j=0}^{\ell-1} \pi^j k[[\pi^\ell]]$, and $(1+\pi)^j - \pi^j$ is a k-combination of π^i, $i < j$, the elements $(1+\pi)^j$ ($0 \leq j \leq \ell-1$) form a basis of $k((\pi))$ over $\varphi_\pi(k((\pi)))$ and of $k[[\pi]]$ over $\varphi_\pi(k[[\pi]])$. Every $b \in k((\pi))$ is thus uniquely $b = \sum_{j=0}^{\ell-1}(1+\pi)^j\varphi_\pi(b_j)$ (the *components* b_j of b), and $\psi_\pi(b) := b_0$ is Fontaine's operator. It is k-linear, $\psi_\pi(1) = 1$, $\psi_\pi(\varphi_\pi(c)b) = c\,\psi_\pi(b)$, $\psi_\pi(k[[\pi]]) \subseteq k[[\pi]]$, and therefore, as $\pi^{\ell s} = \varphi_\pi(\pi^s)$,

$$\text{(A)} \qquad \psi_\pi(\pi^{\ell s}k[[\pi]]) \subseteq \pi^s k[[\pi]] \qquad (s \in \mathbb{Z}).$$

Step 2 (commutation). For $z \in \mathbb{Z}_\ell^\times$, $\delta_z\varphi_\pi = \varphi_\pi\delta_z$: both are continuous k-algebra endomorphisms and both send $1+\pi$ to $(1+\pi)^{\ell z}$, because $\varphi_\pi((1+\pi)^z) = (1+\pi^\ell)^z = ((1+\pi)^\ell)^z$ in characteristic ℓ. For $1 \leq j \leq \ell-1$ write $zj = j' + \ell w$ with $j' \in \{0, \dots, \ell-1\}$, $w \in \mathbb{Z}_\ell$; then $j' \neq 0$ (z is a unit), $j \mapsto j'$ is a bijection of $\{1, \dots, \ell-1\}$, and $\delta_z((1+\pi)^j\varphi_\pi(b_j)) = (1+\pi)^{j'}\varphi_\pi((1+\pi)^w\delta_z(b_j))$,

while $\delta_z(\varphi_\pi(b_0)) = \varphi_\pi(\delta_z(b_0))$. Hence the 0-th component of $\delta_z(b)$ is $\delta_z(b_0)$, i.e. $\psi_\pi\delta_z = \delta_z\psi_\pi$; in particular $\psi_\pi\gamma_\pi = \gamma_\pi\psi_\pi$, and $\gamma_\pi, \varphi_\pi, \psi_\pi$ commute with every element of Δ. Moreover, since $(1+\pi)^{j(1+\ell)} = (1+\pi)^j\varphi_\pi((1+\pi)^j)$,

$$\text{(B)} \qquad \gamma_\pi\big((1+\pi)^j\varphi_\pi(b)\big) = (1+\pi)^j\varphi_\pi\big((1+\pi)^j\gamma_\pi(b)\big) \qquad (b \in k((\pi))).$$

Step 3 (ψ_0 and (Ps2)). Let ι be as in the definition of cyclotomic. It is injective (a homomorphism of fields), and it maps units of $k[[X]]$ to units of $k[[\pi]]$ (as $\iota(k[[X]]) \subseteq k[[\pi]]$); with $\iota(X) = v\pi^{\ell-1}$ this gives

$$\text{(C)} \qquad v_\pi(\iota(x)) = (\ell-1)\, v_X(x) \qquad (x \in k((X))).$$

The image $\iota(k((X)))$ is the set $k((\pi))^\Delta$ of elements fixed by Δ: it is contained in it because $\iota(k[[X]])$ is and $\iota(X^{-1}) = \iota(X)^{-1}$; conversely, for $b \in k((\pi))^\Delta$ and N large, $\iota(X)^N b \in k[[\pi]]$ is Δ-fixed, hence equals $\iota(w)$ with $w \in k[[X]]$, and $b = \iota(X^{-N}w)$. By Step 2, ψ_π preserves $k((\pi))^\Delta$, so $\psi_0 := \iota^{-1} \circ \psi_\pi \circ \iota$ is a well-defined k-linear map of $k((X))$. From $\iota \circ \varphi = \varphi_\pi \circ \iota$ and Step 1: $\psi_0(1) = 1$ and $\psi_0(\varphi(c)d) = c\,\psi_0(d)$ for $c, d \in k((X))$. (Ps2): let $2-\ell \le c \le 1$ and $x \in X^c k[[X]]$. By (C), $v_\pi(\iota(x)) \ge (\ell-1)c \ge -(\ell-1)(\ell-2) \ge \ell(2-\ell)$, so by (A) $v_\pi(\psi_\pi(\iota(x))) \ge 2-\ell$, and by (C) $(\ell-1)v_X(\psi_0(x)) \ge 2-\ell > -(\ell-1)$, i.e. $v_X(\psi_0(x)) \ge 0$.

Step 4 (Ψ_a and (Ps1)). Fix a and a $k((X))$-basis (e_i) of V. By (Rm1), $\varphi_a(\sum_i c_i e_i) = \sum_i \varphi(c_i)\varphi_a(e_i)$, so the $k((X))$-span of $\varphi_a(V)$ is that of the r vectors $\varphi_a(e_i)$; by (Rm2) they span V, hence form a basis. Put $\Psi_a(\sum_i c_i\varphi_a(e_i)) = \sum_i \psi_0(c_i)e_i$ (k-linear). For $c \in k((X))$ and $x = \sum_i x_i e_i$: $c\varphi_a(x) = \sum_i c\varphi(x_i)\varphi_a(e_i)$, so $\Psi_a(c\varphi_a(x)) = \sum_i \psi_0(c\varphi(x_i))e_i = \psi_0(c)x$. This is (Ps1). Since every element of V is a finite sum of elements $c\varphi_a(x)$, (Ps1) determines Ψ_a; the same argument for any other basis gives the following statement (D): for every $k((X))$-basis (m_i) of V, the family $(\varphi_a(m_i))$ is a basis and

$$\Psi_a\Big(\sum_i z_i\varphi_a(m_i)\Big) = \sum_i \psi_0(z_i)m_i.$$

Step 5 (two valuation facts). (E) For $b \in k((\pi))$ with components b_j and $s \in \mathbb{Z}$: $b \in \pi^{\ell s}k[[\pi]]$ if and only if $b_j \in \pi^s k[[\pi]]$ for all j. *If*: $\varphi_\pi(\pi^s k[[\pi]]) \subseteq \pi^{\ell s}k[[\pi]]$. *Only if*: fix j; in $(1+\pi)^{\ell-j}b = \sum_i (1+\pi)^{\ell-j+i}\varphi_\pi(b_i)$, the term $i<j$ has exponent in $[1, \ell-1]$, the term $i=j$ is $\varphi_\pi((1+\pi)b_j)$, and the term $i>j$ is $(1+\pi)^{i-j}\varphi_\pi((1+\pi)b_i)$ with $i-j \in [1,\ell-1]$ (use $(1+\pi)^\ell = \varphi_\pi(1+\pi)$). So $\psi_\pi((1+\pi)^{\ell-j}b) = (1+\pi)b_j$, which lies in $\pi^s k[[\pi]]$ by (A); $1+\pi$ is a unit of $k[[\pi]]$. (F) For $x \in k((X))$ with $\psi_0(x) = 0$ and an integer a with $2-\ell \le a \le 1$: $x \in X^a k[[X]]$ if and only if $\iota(x) \in \pi^{\ell a}k[[\pi]]$. *If*: by (C), $(\ell-1)v_X(x) \ge \ell a = (\ell-1)a + a > (\ell-1)(a-1)$ since $a > 1-\ell$; so $v_X(x) \ge a$. *Only if*, $a \le 0$: $v_\pi(\iota(x)) \ge (\ell-1)a \ge \ell a$. *Only if*, $a = 1$: $\iota(x) = \beta\pi^{\ell-1} + w$ with $\beta \in k$, $w \in \pi^\ell k[[\pi]]$. Since $\pi^{\ell-1} = \sum_{j=0}^{\ell-1}\binom{\ell-1}{j}(-1)^{\ell-1-j}(1+\pi)^j$ with constant components, $\psi_\pi(\pi^{\ell-1}) = (-1)^{\ell-1} = 1$; and $\psi_\pi(w) \in \pi k[[\pi]]$ by (A). So $0 = \iota(\psi_0(x)) = \beta + \psi_\pi(w)$ forces $\beta = 0$, i.e. $\iota(x) \in \pi^\ell k[[\pi]]$.

Step 6 (twisted γ on one component). (G) Let $n \ge 1$, $H \in M_n(k[[\pi]])$ with $H \equiv 1 \pmod{\pi^2}$, $1 \le j \le \ell-1$, and $T(y) = (1+\pi)^j H\gamma_\pi(y) - y$ on $k((\pi))^n$ (γ_π coordinatewise). Then T is injective on $k((\pi))^n$ and maps $(\pi^s k[[\pi]])^n$ bijectively onto $(\pi^{s+1}k[[\pi]])^n$ for every $s \in \mathbb{Z}$. Proof. $\gamma_\pi(\pi) = (1+\pi)(1+\pi^\ell) - 1 = \pi(1 + \pi^{\ell-1} + \pi^\ell)$, so $\gamma_\pi(\pi^m) - \pi^m \in \pi^{m+\ell-1}k[[\pi]]$ for $m \in \mathbb{Z}$; by continuity and k-linearity $\gamma_\pi(u) - u \in \pi^{\ell-1}k[[\pi]]$ for $u \in k[[\pi]]$, and for $y = \pi^s u$, $\gamma_\pi(y) - y = (\gamma_\pi(\pi^s) - \pi^s)\gamma_\pi(u) + \pi^s(\gamma_\pi(u) - u) \in \pi^{s+\ell-1}k[[\pi]]$. Write $T(y) = \varepsilon y + E(y)$ with $\varepsilon = (1+\pi)^j - 1$ and $E(y) = (1+\pi)^j\big((H-1)\gamma_\pi(y) + (\gamma_\pi(y) - y)\big)$. As $j \ne 0$ in k, $\varepsilon = j\pi \cdot (\text{unit})$; as $\ell - 1 \ge 2$, E maps $(\pi^t k[[\pi]])^n$ into $(\pi^{t+2}k[[\pi]])^n$ for every t. So $N := \varepsilon^{-1}E$ maps $(\pi^t k[[\pi]])^n$ into $(\pi^{t+1}k[[\pi]])^n$, and $T = \varepsilon(1+N)$. Injective: if $T(y) = 0$, $y \ne 0$, let t be the least valuation of

a coordinate of y; then $y=-N(y)$ has all coordinates in $\pi^{t+1}k[[\pi]]$, a contradiction. Into: clear. Onto: given $z\in(\pi^{s+1}k[[\pi]])^n$, put $w=\varepsilon^{-1}z\in(\pi^sk[[\pi]])^n$ and $y=\sum_{i\ge0}(-N)^iw$, convergent coordinatewise since the i-th term lies in $(\pi^{s+i}k[[\pi]])^n$; $y\in(\pi^sk[[\pi]])^n$. For the partial sums y_K, $(1+N)y_K=w-(-N)^{K+1}w$, and $(1+N)(y-y_K)\in(\pi^{s+K+1}k[[\pi]])^n$; so $(1+N)y-w$ lies in $(\pi^{s+K+1}k[[\pi]])^n$ for all K, hence vanishes, and $T(y)=z$.

Step 7 (the $\psi=0$ part). (H) Let H be as in (G) and $L(\xi)_i=\sum_m\varphi_\pi(H_{im})\gamma_\pi(\xi_m)-\xi_i$ on $k((\pi))^n$; let $W=\ker\psi_\pi$. Then L is injective on W^n and maps $(W\cap\pi^{\ell s}k[[\pi]])^n$ bijectively onto $(W\cap\pi^{\ell(s+1)}k[[\pi]])^n$ for every $s\in\mathbb{Z}$. Proof. An element $\xi\in W^n$ is uniquely $\xi=\sum_{j=1}^{\ell-1}(1+\pi)^j\varphi_\pi(y_j)$ with $y_j\in k((\pi))^n$ (its 0-th components vanish). By (B), $\varphi_\pi(H_{im})\gamma_\pi((1+\pi)^j\varphi_\pi(y_{j,m}))=(1+\pi)^j\varphi_\pi\big(H_{im}(1+\pi)^j\gamma_\pi(y_{j,m})\big)$, so

$$L(\xi)=\sum_{j=1}^{\ell-1}(1+\pi)^j\varphi_\pi(T_j(y_j)),\qquad T_j(y)=(1+\pi)^jH\gamma_\pi(y)-y.$$

Thus $L(W^n)\subseteq W^n$ and the components of $L(\xi)$ are $T_j(y_j)$. If $L(\xi)=0$ then every $T_j(y_j)=0$, so $y_j=0$ by (G) and $\xi=0$. By (E), $\xi\in(\pi^{\ell s}k[[\pi]])^n$ iff all $y_j\in(\pi^sk[[\pi]])^n$, and $L(\xi)\in(\pi^{\ell(s+1)}k[[\pi]])^n$ iff all $T_j(y_j)\in(\pi^{s+1}k[[\pi]])^n$; (G) for each j gives the bijection (for surjectivity, expand a given ζ into components z_j, solve $T_j(y_j)=z_j$ with $y_j\in(\pi^sk[[\pi]])^n$, and put $\xi=\sum_j(1+\pi)^j\varphi_\pi(y_j)$).

Step 8 (Ps3). Let $a\in\mathbb{Z}/n$, M a lattice satisfying the Γ-condition at $a-1$, $2-\ell\le c\le0$, and $y\in X^{c+1}\varphi_a^*M$ with $\Psi_a(y)=0$. As a finitely generated torsion-free module over the discrete valuation ring $k[[X]]$, M is free; a basis $m_1,\dots,m_r$ is linearly independent over $k((X))$ (clear denominators) and spans V, so it is a $k((X))$-basis of V. Put $f_i=\varphi_a(m_i)$, a basis of V by (D). Then $\varphi_a^*M=\bigoplus_i k[[X]]f_i$, as it contains the f_i and is generated by the $\varphi_a(\sum_i b_im_i)=\sum_i\varphi(b_i)f_i$ with $\varphi(b_i)\in k[[X]]$ by (Rr1); hence $X^d\varphi_a^*M=\{\sum_i x_if_i:\ x_i\in X^dk[[X]]\}$. The Γ-condition gives $\gamma_{a-1}(m_m)=\sum_i G_{im}m_i$ with $G\in M_r(k[[X]])$, $G-1\in XM_r(k[[X]])$. By (Rm3) and (Rm1), $\gamma_a(f_m)=\varphi_a(\gamma_{a-1}(m_m))=\sum_i\varphi(G_{im})f_i$, so for $x=\sum_m x_mf_m$:

$$(*)\qquad \gamma_a(x)-x=\sum_i\Big(\sum_m\varphi(G_{im})\gamma(x_m)-x_i\Big)f_i.$$

Write $y=\sum_i z_if_i$ with $z_i\in X^{c+1}k[[X]]$; by (D), $\Psi_a(y)=\sum_i\psi_0(z_i)m_i=0$ gives $\psi_0(z_i)=0$, so $\iota(z_i)\in W$, and by (F) (with $c+1\in[3-\ell,1]$) $\iota(z_i)\in\pi^{\ell(c+1)}k[[\pi]]$. Let $H=\iota(G)$ entrywise: $H\in M_r(k[[\pi]])$ and $H-1=\iota(G-1)\in\pi^{\ell-1}M_r(k[[\pi]])\subseteq\pi^2M_r(k[[\pi]])$. Since $\iota\varphi=\varphi_\pi\iota$ and $\iota\gamma=\gamma_\pi\iota$, ι applied to the coefficient of f_i in $(*)$ is $L((\iota(x_m))_m)_i$, with L as in (H) for this H. By (H) with $s=c$ there is $\xi\in(W\cap\pi^{\ell c}k[[\pi]])^r$ with $L(\xi)=(\iota(z_i))_i$. For $\delta\in\Delta$: δ commutes with $\varphi_\pi,\gamma_\pi,\psi_\pi$ (Step 2) and fixes the entries of H and the $\iota(z_i)$ (Step 3), so $\delta\xi\in W^r$ and $L(\delta\xi)=\delta(L(\xi))=L(\xi)$; by injectivity in (H), $\delta\xi=\xi$. By Step 3, $\xi_m=\iota(x_m)$ with $x_m\in k((X))$, and $\iota(\psi_0(x_m))=\psi_\pi(\xi_m)=0$, so $\psi_0(x_m)=0$; by (F) with $a=c$, $x_m\in X^ck[[X]]$. Put $x=\sum_m x_mf_m\in X^c\varphi_a^*M$. By (D), $\Psi_a(x)=\sum_m\psi_0(x_m)m_m=0$; by $(*)$ and the injectivity of ι, the coefficient of f_i in $\gamma_a(x)-x$ is z_i, since its image under ι is $L(\xi)_i=\iota(z_i)$. Hence $y=\gamma_a(x)-x$, which is (Ps3).

So $(\psi_0,(\Psi_a)_a)$ is a ψ-operator for D. □

Lemma 28.42. *Let the data $\ell,K,n,L,\mathcal{O},\lambda,(\kappa_a),h,w,\eta,\mathrm{typ},(U_a),V,T_0,\chi_1,\dots,\chi_t$ satisfy the hypotheses of Proposition 28.2, and write $\varpi=\lambda$. Call a finite $\mathcal{O}[G_K]$-module* realisable *if it is isomorphic to Λ_1/Λ_2 for G_K-stable $\mathcal{O}$-lattices $\Lambda_2\subseteq\Lambda_1$ in a finite direct sum of full representations. Let Λ' be T_0 or a G_K-stable $\mathcal{O}$-lattice Λ_c in $L(\chi_c)$ for some $1\le c\le t$; for a finite $\mathcal{O}$-module X let $Z(X)$ be the $\mathcal{O}$-module of continuous maps $z\colon G_K\to\Lambda'\otimes_{\mathcal{O}}X$ with*

$z(gh) = z(g) + \chi_1(g)^{-1} g\, z(h)$ *(G_K acting on the first factor), and* $Z_{\mathrm{adm}}(X) \subseteq Z(X)$ *the set of admissible ones. Then:*

1. *realisable modules are closed under G_K-stable submodules, quotients and finite direct sums; for every finite $\mathcal{O}$-module X the modules $T_0 \otimes_{\mathcal{O}} X$ and $\Lambda_c \otimes_{\mathcal{O}} X$ ($1 \le c \le t$, Λ_c any G_K-stable lattice in $L(\chi_c)$) are realisable;*
2. *$Z_{\mathrm{adm}}(X)$ is an $\mathcal{O}$-submodule of $Z(X)$ containing every coboundary $g \mapsto \chi_1(g)^{-1} g\, y - y$, $y \in \Lambda' \otimes X$;*
3. *for an $\mathcal{O}$-linear map $f\colon X \to X'$ of finite $\mathcal{O}$-modules and $z \in Z(X)$: if z is admissible, so is $(1 \otimes f) \circ z$; if f is injective and $(1 \otimes f) \circ z$ is admissible, then z is admissible;*
4. *if $\Lambda' = T_0$, $c\colon G_K \to T_0$ is a continuous cocycle for the action $g \mapsto \chi_1(g)^{-1} g$ whose class lies in $H^1_f(G_K, T_0(\chi_1^{-1}))$, and $x \in X$, then $g \mapsto c(g) \otimes x$ is admissible;*
5. *if $\Lambda' = \Lambda_1$ and $z \in Z(X)$ vanishes on I_K, then z is admissible.*

Proof. Throughout, $E_z^{(a)} = E_z/(0 \oplus \varpi^a \mathcal{O})$ for $z \in Z(X)$ and $a \ge 0$, and a_X denotes an integer with $\varpi^{a_X} X = 0$. Recall that z is admissible if there is a_0 such that $E_z^{(a)}$ is realisable for every $a \ge a_0$ with $\varpi^a X = 0$; enlarging a_0 we may and do assume $a_0 \ge a_X$, so that the condition is simply $a \ge a_0$. The action on E_z is $g(m, b) = (gm + b\chi_1(g) z(g), \chi_1(g) b)$.

Item 1. Let $q\colon \Lambda_1 \to M$ be onto and G_K-equivariant with kernel Λ_2, lattices in a finite sum S of full representations. For a G_K-stable submodule $M' \subseteq M$ the preimage $q^{-1}(M')$ is G_K-stable and lies between Λ_2 and Λ_1, hence is a lattice, and $M' \cong q^{-1}(M')/\Lambda_2$. For a quotient $M \to M''$ with kernel N, $q^{-1}(N)$ is a G_K-stable lattice between Λ_2 and Λ_1 and $M'' \cong \Lambda_1/q^{-1}(N)$. For two realisable modules, $(\Lambda_1 \oplus \Lambda_1')/(\Lambda_2 \oplus \Lambda_2')$ in $S \oplus S'$ realises the sum. The representation $F_0 = V \oplus L(\chi_1) \oplus \cdots \oplus L(\chi_t)$ is full: it is crystalline as a sum of crystalline representations; $D_{\mathrm{dR},\tau}$ is additive, so $m_{F_0}(\tau, j) = m_V(\tau, j) + \#\{c :\ j = w_{a(\tau)} + c - 1\} = m_V(\tau, j) + [\, w_{a(\tau)} \le j \le w_{a(\tau)} + t - 1\,]$; and its characteristic polynomial is $\det(X - g \mid V) \prod_c (X - \chi_c(g))$. Fix G_K-stable lattices $\Lambda_c^0 = \mathcal{O} e_c \subseteq L(\chi_c)$ (every $\mathcal{O}$-line is stable, since χ_c takes values in $\mathcal{O}^\times$). For $a \ge 0$ the lattices $T_0 \oplus \bigoplus_c \Lambda_c^0 \supseteq \varpi^a T_0 \oplus \bigoplus_c \Lambda_c^0$ of F_0 realise $T_0/\varpi^a T_0$, and similarly $\Lambda_c^0/\varpi^a \Lambda_c^0$; any lattice Λ_c in the line $L(\chi_c)$ is $\varpi^j \Lambda_c^0$ for some j, isomorphic to Λ_c^0 as an $\mathcal{O}[G_K]$-module. By the structure theorem over the discrete valuation ring $\mathcal{O}$, $X \cong \bigoplus_i \mathcal{O}/\varpi^{a_i}$, so $T_0 \otimes X \cong \bigoplus_i T_0/\varpi^{a_i} T_0$ and $\Lambda_c \otimes X \cong \bigoplus_i \Lambda_c/\varpi^{a_i} \Lambda_c$ are realisable.

Item 2. Coboundaries: if $z(g) = \chi_1(g)^{-1} g y - y$, then $\psi(m, b) = (m + by, b)$ is an $\mathcal{O}[G_K]$-isomorphism from E_z onto $(\Lambda' \otimes X) \oplus \mathcal{O}(\chi_1)$ with the product action, because $\psi(g(m, b)) = (gm + b\, gy - b\chi_1(g) y + \chi_1(g) by, \chi_1(g) b) = (g(m + by), \chi_1(g) b)$. Hence $E_z^{(a)} \cong (\Lambda' \otimes X) \oplus \mathcal{O}/\varpi^a(\chi_1)$, realisable for every a by item 1 ($\mathcal{O}/\varpi^a(\chi_1) \cong \Lambda_1^0/\varpi^a \Lambda_1^0$). Sums: let z_1, z_2 be admissible and $a \ge a_0(z_1), a_0(z_2)$. The submodule $P = \{((m_1, b_1), (m_2, b_2)) :\ b_1 = b_2\}$ of $E_{z_1}^{(a)} \oplus E_{z_2}^{(a)}$ is G_K-stable (both second coordinates are multiplied by $\chi_1(g)$), and $((m_1, b), (m_2, b)) \mapsto (m_1 + m_2, b)$ is a G_K-equivariant surjection $P \to E_{z_1+z_2}^{(a)}$:

$$g \cdot ((m_1, b), (m_2, b)) \mapsto (g(m_1 + m_2) + b\chi_1(g)(z_1 + z_2)(g),\ \chi_1(g) b) = g \cdot (m_1 + m_2, b).$$

By item 1, $E_{z_1+z_2}^{(a)}$ is realisable. Scalars: for $\alpha \in \mathcal{O}$ and $a \ge a_0(z)$, $((m, b), m') \mapsto (\alpha m + m', b)$ is a G_K-equivariant surjection $E_z^{(a)} \oplus (\Lambda' \otimes X) \to E_{\alpha z}^{(a)}$, since g sends $((m, b), m')$ to $((gm + b\chi_1(g) z(g), \chi_1(g) b), gm')$, whose image is $(g(\alpha m + m') + b\chi_1(g) \alpha z(g), \chi_1(g) b)$. So αz is admissible. In particular the admissible classes form an $\mathcal{O}$-submodule of $H^1(G_K, \Lambda' \otimes X)$ (twisted action), and admissibility depends only on the class.

Item 3. Put $\tilde f = 1\otimes f\colon \Lambda'\otimes X \to \Lambda'\otimes X'$, which is G_K-equivariant, and injective if f is, Λ' being free over $\mathcal{O}$. For $a \ge \max(a_0(z), a_{X'})$ the map $((m,b),m') \mapsto (\tilde f(m)+m', b)$ is a G_K-equivariant surjection $E_z^{(a)} \oplus (\Lambda'\otimes X') \to E_{\tilde f\circ z}^{(a)}$ (the computation of item 2 with $\tilde f$ in place of α), so $\tilde f\circ z$ is admissible by item 1. Conversely, if f is injective and $\tilde f\circ z$ is admissible, then for $a \ge \max(a_0(\tilde f\circ z), a_X)$ the map $(m,b)\mapsto(\tilde f(m),b)$ is an injective G_K-equivariant map $E_z^{(a)} \to E_{\tilde f\circ z}^{(a)}$, so $E_z^{(a)}$ is realisable by item 1.

Item 4. Let $E = V\oplus L$ with $g(v,\beta) = (gv+\beta\chi_1(g)c(g), \chi_1(g)\beta)$, a continuous L-linear representation, and $E' = E(\chi_1^{-1})$, i.e. $V' = V(\chi_1^{-1})$ extended by the trivial line with $g(v,\beta) = (gv+\beta c(g),\beta)$ (g acting on V' by $\chi_1(g)^{-1}g$). By hypothesis there is $y\in V'\otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $c(g)\otimes 1 = gy - y$ for all g. Then $e_\alpha = (-\alpha y, \alpha\otimes 1) \in E'\otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ is G_K-invariant for every $\alpha\in L$: $ge_\alpha = (-\alpha\, gy + \alpha c(g), \alpha) = e_\alpha$. The second projection $E'\to L$ induces $D_{\mathrm{cris}}(E') \to D_{\mathrm{cris}}(L) = (L\otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}})^{G_K} = L\otimes_{\mathbb{Q}_\ell} K$ ($B_{\mathrm{cris}}^{G_K} = K_0 = K$), with kernel $D_{\mathrm{cris}}(V')$, and its image contains $\alpha\otimes 1$ for all α, which span $L\otimes K$ over K. As V' is crystalline (a twist of the crystalline V by the crystalline χ_1^{-1}), $\dim_K D_{\mathrm{cris}}(E') = \dim_{\mathbb{Q}_\ell} V' + [L:\mathbb{Q}_\ell] = \dim_{\mathbb{Q}_\ell} E'$, so E' and hence $E = E'(\chi_1)$ are crystalline. $D_{\mathrm{dR},\tau}$ is exact with strict filtrations on de Rham representations (Fontaine), so $0\to V\to E\to L(\chi_1)\to 0$ gives $m_E(\tau,j) = m_V(\tau,j) + [\,j = w_{a(\tau)}\,]$. Hence $F = E\oplus L(\chi_2)\oplus\cdots\oplus L(\chi_t)$ is full: crystalline, with the multiplicities of F_0, and with characteristic polynomial $\det(X - g\,|\,V)\prod_c (X-\chi_c(g))$, E being block triangular. The lattice $\Lambda_E = T_0\oplus\mathcal{O}$ of E is G_K-stable because c takes values in T_0. For $a\ge a_X$, the map

$$\Lambda_E \oplus (T_0\otimes X) \to E_z^{(a)}, \qquad ((v,\beta),m)\mapsto (v\otimes x + m,\ \beta \bmod \varpi^a),$$

is onto and G_K-equivariant: $g((v,\beta),m) = ((gv+\beta\chi_1(g)c(g),\chi_1(g)\beta), gm)$ goes to $(g(v\otimes x + m) + \beta\chi_1(g)c(g)\otimes x, \chi_1(g)\beta)$. It kills $\varpi^a\Lambda_E$ (as $\varpi^a x = 0$), so $E_z^{(a)}$ is a quotient of $(\Lambda_E/\varpi^a\Lambda_E)\oplus(T_0\otimes X)$; the first summand is realised by $\Lambda_E\oplus\bigoplus_{c\ge 2}\Lambda_c^0 \supseteq \varpi^a\Lambda_E\oplus\bigoplus_{c\ge 2}\Lambda_c^0$ in F, the second by item 1. So z is admissible with $a_0 = a_X$.

Item 5. On $\Lambda_1 = \mathcal{O}e$ (a generator e; $ge = \chi_1(g)e$) the twisted action is trivial, so z is a continuous homomorphism $G_K\to\Lambda_1\otimes X = e\otimes X\cong X$, trivial on I_K. Writing $X = \bigoplus_i X_i$ with X_i cyclic, projections p_i and inclusions ι_i, we have $z = \sum_i (1\otimes\iota_i)\circ(1\otimes p_i)\circ z$, so by items 2 and 3 we may assume $X = \mathcal{O}/\varpi^b$ with $b\ge 1$ (for $X=0$, $E_z^{(a)} = \mathcal{O}/\varpi^a(\chi_1)$ is realisable). Fix a Frobenius element $\phi\in G_K$; $G_K/I_K\cong\widehat{\mathbb{Z}}$ topologically generated by its image. Write $z(\phi) = e\otimes(\xi \bmod \varpi^b)$ with $\xi\in\mathcal{O}$. Since $1+\varpi^b\mathcal{O}$ is a pro-ℓ group, there is a continuous character $\mu\colon G_K\to 1+\varpi^b\mathcal{O}$, trivial on I_K, with $\mu(\phi) = 1+\varpi^b\xi$. The map $g\mapsto(\mu(g)-1)/\varpi^b \bmod \varpi^b$ is a continuous homomorphism $G_K\to\mathcal{O}/\varpi^b$ (from $\mu(gh)-1 = (\mu(g)-1)+(\mu(h)-1)+(\mu(g)-1)(\mu(h)-1)$ and $(\mu(g)-1)(\mu(h)-1)\in\varpi^{2b}\mathcal{O}$) trivial on I_K and equal to ξ at ϕ; so $z(g) = e\otimes((\mu(g)-1)/\varpi^b \bmod \varpi^b)$ for all g, both sides being continuous homomorphisms agreeing on a dense subgroup. The character μ is unramified, hence crystalline with the single Hodge–Tate weight 0 at every embedding (classical: $\widehat{\mathbb{Q}_\ell^{\mathrm{ur}}}\subseteq B_{\mathrm{cris}}$); so $\chi_1\mu$ is crystalline with $m_{\chi_1\mu} = m_{\chi_1}$, and $\chi_1\mu\equiv\chi_1 \bmod\varpi$. Therefore $F' = V\oplus L(\chi_1\mu)\oplus L(\chi_2)\oplus\cdots\oplus L(\chi_t)$ is full. In $S = F_0\oplus F'$ let e_1 span $L(\chi_1)\subseteq F_0$ and e_2 span $L(\chi_1\mu)\subseteq F'$, so that $S = W\oplus W^c$ with $W = Le_1\oplus Le_2$ and W^c the sum of the remaining summands, both G_K-stable. The lattice $M = \mathcal{O}(e_1+e_2)+\mathcal{O}\varpi^b e_2$ of W is G_K-stable, since $g(e_1+e_2) = \chi_1(g)(e_1+e_2)+\chi_1(g)\frac{\mu(g)-1}{\varpi^b}\varpi^b e_2$. For $a\ge 0$ define

$$q\colon M\to E_z^{(a)}, \qquad q(\beta(e_1+e_2)+\alpha\varpi^b e_2) = (e\otimes(\alpha \bmod \varpi^b),\ \beta \bmod \varpi^a).$$

It is onto, and G_K-equivariant: the image of $g(\beta(e_1+e_2)+\alpha\varpi^b e_2) = \beta\chi_1(g)(e_1+e_2) + (\beta\chi_1(g)\frac{\mu(g)-1}{\varpi^b} + \alpha\chi_1(g)\mu(g))\varpi^b e_2$ is $(e\otimes(\chi_1(g)\alpha+\beta\chi_1(g)\xi_g), \chi_1(g)\beta)$ with $\xi_g = (\mu(g)-1)/\varpi^b \bmod\varpi^b$ (using $\mu(g)\equiv 1 \bmod\varpi^b$), which is $g\cdot(e\otimes\alpha,\beta) = (\chi_1(g)e\otimes\alpha + \beta\chi_1(g)z(g), \chi_1(g)\beta)$.

Its kernel M' is a G_K-stable lattice of W (it contains $\varpi^{\max(a,b)}M$). With any G_K-stable lattice Λ^c of W^c, $E_z^{(a)} \cong (M \oplus \Lambda^c)/(M' \oplus \Lambda^c)$ is realisable for every a, so z is admissible. $\square$

Part VI
Covolumes: model slopes from good root towers

Part VI proves two statements, both used in the proof of the model-level packet supply Proposition 21.2 in Section 21. The first, Theorem 29.14, gives model-slope data the properties (Ms1) and (Ms3): the volume slope of every component is at least its model slope minus twice the Hodge–Tate cost times the weight norm, $\mathrm{vs}(Z) \geq \mathrm{ns}(Z) - 2\,c^{\mathrm{HT}}_{E,M}(B)\,\mathrm{wn}$, and the mean of the model slopes is at least $-C_{\mathrm{ms}}(\mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp})$. The second, Proposition 31.14, says that good root towers exist over every unitary Shimura datum with the restrictions listed in (Aj1); Theorem 29.14 rests on it, (TCS) quotes its constant C together with the hypothesis lists (Aj1), (Aj3) of Theorem 29.14, and Proposition 21.2 takes from it the number $L_{\mathsf{H}} = \mathrm{Re}\,(L'/L)(1,\chi_d)$. No result of Part VI depends on (TCS).

The volume slope is defined by counting: it is read off from the growth of the number of integral cusp forms of bounded norm on Z, of large multiples of a divided weight (Definition 29.1). Alpöge lists the "computation of covolumes" among the steps of the proof [4]. We read this Part as that computation, since a lower bound for the volume slope says that the lattices of integral cusp forms have small covolume; the identification is our reading, and nothing depends on it. Covolumes also enter literally: the volumes of a good root tower are normalised sums of covolumes of principal congruence subgroups of a special unitary group (Definition 31.4, Lemma 31.17).

The method is arithmetic intersection theory [110] on integral models of unitary Shimura varieties over the host field. (Ms1) comes from an arithmetic Hilbert–Samuel theorem on the model [276] and a transfer of small sections from the model to cusp forms, paid for by the Hodge–Tate cost. (Ms3) comes from an arithmetic analogue of Hirzebruch–Mumford proportionality [188] on *root towers*, systems of models indexed by an integer stage B on which the arithmetic degrees of the Chern-root classes agree with a compact-dual height up to two correction terms and an error $O(\log B/B)$. Section 29 proves Theorem 29.14 from the properties of a good tower, Section 30 derives these properties from geometric inputs, and Section 31 proves Proposition 31.14. The geometric inputs are established in Sections 32–34 and 35, with toroidal compactifications [154], [87], Mumford's construction of degenerations [187] and Hodge–Tate theory; six technical sections are in Appendix D.

29 Model slopes from the tower datum

This section proves Theorem 29.14, the statement that Proposition 21.2 applies. At each realised point of a family the theorem is given a tower datum: a unitary Shimura datum over the host field with a bijection between the components of the point and those of the Shimura variety (Aj1), a divided weight u with a gap (Aj2), and, for every root tower attached to the datum, counting and transfer data for the cusp forms (Aj3). It chooses a good root tower by Corollary 30.5, defines the model slope $\mathrm{ns}(Z)$ as the slope of u on the root model of Z at stage $\lfloor B\rfloor$, and proves (Ms1) and (Ms3). Of the three hypotheses only (Aj2) is verified in the proof of Proposition 21.2; (Aj1) and (Aj3) are part of what (TCS) supplies.

The two conclusions have separate proofs. For (Ms1), Subsection 29.2 compares two counts: the number of small sections of multiples of u on the model, which the arithmetic Hilbert–Samuel theorem [276] bounds from below in terms of $\mathrm{ns}(Z)$ (Theorem 29.23), and the number of integral

cusp forms of bounded norm, which defines $\mathrm{vs}(Z)$. A transfer from sections to cusp forms, with denominators cleared at a cost of $2\,c^{\mathrm{HT}}_{E,M}(B)\,\mathrm{wn}$ in the norm for each multiple of the weight (Theorem 29.27), connects the two and gives Theorem 29.28. For (Ms3), Subsection 29.3 bounds the sum over the components of the self-intersection numbers of u from below by an arithmetic proportionality identity at one stage of the tower (Theorem 29.29, Theorem 29.43); the bound for the mean model slope follows (Theorem 29.44). The first argument needs of the tower only positive volumes, (Gm1), (Gm2) and the good origin; the second uses its proportionality and its quantitative properties.

29.1 The model-slope theorem

The definitions before Theorem 29.14 come in two groups. The first is numerical: counting volumes and counting data with (Vc1)–(Vc3), self-intersection data with (Si1)–(Si3), proportionality data with (Hp1)–(Hp6), and root towers, the numerical shadow of a system of models, with their qualitative properties (Rt1)–(Rt4) and quantitative properties (Rt5)–(Rt7). These are records of real numbers, and the deductions of (Ms1) from (Vc1)–(Vc3) and of (Ms3) from (Hp1)–(Hp6) through (Si3) use only the stated properties. The second group is geometric: root models and their geometric realisations, unitary Shimura data with their integral models, flag bundles with their root lines, good origins, automorphic forms on flag bundles, and the flag scheme of type (Φ, n) over $\mathbb{Z}$ whose Fubini–Study height is the compact-dual height (Fs). Tower realisations (Tz0)–(Tz6), transfer data (Tf1)–(Tf3) and the realisation (Rz1)–(Rz3) of the proportionality data connect the two groups. The theorem follows: given (Aj1)–(Aj3), a good tower can be chosen at every point so that (Ms1) and (Ms3) hold, with $C_{\mathrm{ms}} = 2C + 17$.

Definition 29.1. Counting volumes. Let $\overline{\mathbb{N}} = \{0, 1, 2, \dots\} \cup \{\infty\}$, with $\infty^r = \infty$ and $0^r = 0$ for integers $r \geq 1$. For $n \in \overline{\mathbb{N}}$ put $\log^+ n = \log n$ if $1 \leq n < \infty$, $\log^+ 0 = 0$ and $\log^+ \infty = \infty$. For integers $D \geq 0$ and $\delta \geq 1$, a function $\nu\colon \mathbb{Z}_{\geq 1} \times \mathbb{R} \to \overline{\mathbb{N}}$ and a real number t, the **counting volume** is

$$\mathrm{cvol}_{D,\delta}(\nu; t) = \limsup_{k \to \infty} \frac{D!\,\log^+ \nu(8k, 8kt)}{\delta\,(8k)^{D+1}} \in [0, \infty]. \tag{29.1}$$

Counting data. A family carrying model-slope data **carries counting data** if for every real $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and every lift $j < m$ the following data are given, with the index $(\mathfrak{z}, j)$ suppressed: an integer $d^{\mathrm{co}} \geq 1$, the **coefficient degree** (the degree over $\mathbb{Q}$ of the field of coefficients of the forms), and for every component $Z \in \mathcal{Z}$:

- integers $\mathrm{fr}(Z) \geq 1$, the **frame size**, and $\mathrm{fw}(Z) \geq 0$ with $8 \mid \mathrm{fw}(Z)$, the **frame weight**;
- real numbers $\mathrm{tg}(Z) > 0$, the **top degree** (the geometric degree of the divided weight on Z), and $\mathrm{ta}(Z)$, the **transfer constant**;
- functions $\mathrm{cf}_Z, \mathrm{mc}_Z\colon \mathbb{Z}_{\geq 1} \times \mathbb{R} \to \overline{\mathbb{N}}$: $\mathrm{cf}_Z(\tau, \rho)$ is the **cusp count**, the number of cusp forms on Z of τ times the divided weight that are integral for the valuations of the weight bound B and have norm at most e^ρ at the good embeddings; $\mathrm{mc}_Z(\tau, \rho)$ is the **model count**, the number of small sections of τ times the divided weight with norm at most e^ρ on the model of Z at stage $\lfloor B \rfloor$.

The family has the following properties if for all large points with $j < m$ and every $Z \in \mathcal{Z}$, writing $D = D^{\mathrm{fl}}$ and $c = c^{\mathrm{HT}}_{E,M}(B)$ (inequalities between a real number and a counting volume are read in $[-\infty, \infty]$):

(Vc1) Volume slope from cusp counts For every real a such that

$$\mathrm{tg}(Z)(a+t) \le \mathrm{cvol}_{D,d^{\mathrm{co}}}(\mathrm{cf}_Z;t) \quad \text{for every real } t,$$

one has $a \le \mathrm{vs}(Z)$;

(Vc2) Hilbert–Samuel on the model We have $\mathrm{tg}(Z)\big(\mathrm{ns}(Z)+t\big) \le \mathrm{cvol}_{D,\,d^{\mathrm{co}}\mathrm{fr}(Z)}(\mathrm{mc}_Z;t)$ for every real t;

(Vc3) Transfer with cleared denominators For every integer $\tau \ge 1$ and all real ρ, ρ' with

$$\rho + 2\big(\tau + \mathrm{fw}(Z)\big)\, c\,\mathrm{wn} + \mathrm{ta}(Z) \le \rho',$$

one has $\mathrm{mc}_Z(\tau,\rho) \le \mathrm{cf}_Z\big(\tau+\mathrm{fw}(Z),\rho'\big)^{\mathrm{fr}(Z)}$.

Definition 29.2. A family carrying model-slope data **carries self-intersection data** if for every real $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and every lift $j < m$ the following data are given, with the index $(\mathfrak{z}, j)$ suppressed: a real number $\mathrm{fd} > 0$, the **flag degree** of the divided weight, and for every component $Z \in \mathcal{Z}$ real numbers $\mathrm{vo}(Z) > 0$, the **volume** of Z, and $\mathrm{si}(Z)$, the **self-intersection number** of the divided weight on the model of Z at stage $\lfloor B \rfloor$.

The family has the following properties if there is a real $C \ge 0$ such that for all large points with $j < m$, writing $D = D^{\mathrm{fl}}$:

(Si1) Model slope as normalised self-intersection For every $Z \in \mathcal{Z}$,

$$\mathrm{ns}(Z) = \frac{\mathrm{si}(Z)}{(D+1)\,\mathrm{vo}(Z)\,\mathrm{fd}};$$

(Si2) Equal volumes We have $\mathrm{vo}(Z) = \mathrm{vo}(Z')$ for all $Z, Z' \in \mathcal{Z}$;

(Si3) Total self-intersection bound We have

$$\sum_{Z\in\mathcal{Z}} \mathrm{si}(Z) \ge -C\,(D+1)\,\mathrm{fd}\Big(\sum_{Z\in\mathcal{Z}} \mathrm{vo}(Z)\Big)\big(\mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp}\big).$$

Definition 29.3. A **root tower** $\mathcal{T}$ of type (Φ, n, κ) consists of: a non-empty finite set Cp, the **components**; a finite set Bd of primes containing 2, the **bad primes**; real numbers $\mathrm{vol}(Z)$ for $Z \in \mathrm{Cp}$, the **volumes**; for every integer $B \ge 2$ (the **stage**) and every $Z \in \mathrm{Cp}$ a linear form $\widehat{\deg}_{B,Z}\colon \mathcal{R} \to \mathbb{R}$, the **arithmetic degree** at stage B on Z; a linear form $h^{\mathrm{cd}}\colon \mathcal{R} \to \mathbb{R}$, the **compact-dual height**; and a Frobenius labelling fr. Put $\mathrm{vol}_{\mathcal{T}} = \sum_Z \mathrm{vol}(Z)$ and $\widehat{\deg}_B = \sum_Z \widehat{\deg}_{B,Z}$. In the application, $\widehat{\deg}_{B,Z}(F)$ is the arithmetic intersection number on the model of Z at stage B of the class F of the Chern roots of the automorphic bundles; only the numbers enter below.

The tower has the **qualitative properties** if:

(Rt1) Positive volumes We have $\mathrm{vol}(Z) > 0$ for every $Z \in \mathrm{Cp}$;

(Rt2) Proportionality with two constants There are an integer $B_1 \ge 2$ and real numbers ε_B $(B \ge B_1)$ with $\varepsilon_B \to 0$ as $B \to \infty$, such that for every integer $B \ge B_1$ there are real numbers a_B, b_B with

$$\Big|\widehat{\deg}_B(x^m) - \mathrm{vol}_{\mathcal{T}}\Big(h^{\mathrm{cd}}(x^m) - a_B \sum_{a\in\Phi} \deg^{\mathrm{fl}}(\partial_a^{<} x^m) - b_B \sum_{a\in\Phi} \deg^{\mathrm{fl}}(\partial_a x^m)\Big)\Big| \le \varepsilon_B$$

for every monomial x^m of degree $D+1$;

(Rt3) Change of stage For all integers $2 \le B_0 \le B$, every Z, every $a \in \Phi$, every $1 \le k \le n$ and every $Q \in \mathcal{R}$ homogeneous of degree $D+1-k$,

$$\widehat{\deg}_{B_0,Z}(\mathrm{ps}_{a,k}Q) = \widehat{\deg}_{B,Z}(\mathrm{ps}_{a,k}Q) + \mathrm{vol}(Z) \sum_{\substack{p \text{ prime},\ p\notin \mathrm{Bd} \\ B_0 < p \le B}} \Delta_p(a,k,Q);$$

(Rt4) Compact-dual height We have $h^{\mathrm{cd}}(\mathrm{lr}(u)^{D+1}) \ge 0$ for every dominant weight u, and $h^{\mathrm{cd}}(\mathrm{ps}_{a,1}Q) = 0$ for every a and every Q homogeneous of degree D.

Let $r \ge 1$ and L be real numbers and put $\mathrm{Lg}(B) = \log(Br) + |L|$ for $B \ge 1$. The tower has the **quantitative properties for** (r,L) **with constants** $C, C' \ge 0$ if:

(Rt5) Determinant classes There are real numbers $\mathrm{hd}_{B,Z,a}$ ($B \ge 2$, $Z \in \mathrm{Cp}$, $a \in \Phi$) with $\widehat{\deg}_{B,Z}(\mathrm{ps}_{a,1}Q) = \mathrm{hd}_{B,Z,a}\,\mathrm{vol}(Z)\deg^{\mathrm{fl}}(Q)$ for every Q homogeneous of degree D, and $|\mathrm{hd}_{B,Z,a}| \le C'\,n\,\mathrm{Lg}(B)$ whenever $B \ge 128$ and $|\mathrm{Bd}| \le B$;

(Rt6) Slope at the test class For every integer $B \ge 128$ with $B \ge 4|\Phi|n^2$ and $|\mathrm{Bd}| \le B$,

$$\widehat{\deg}_B(\mathrm{pl}^{D^{\mathrm{gr}}+1}\mathrm{fb}) \ge -C\,|\Phi|\min(\kappa, n-\kappa)\,\mathrm{Lg}(B)\,(D^{\mathrm{gr}}+1)\,\mathrm{vol}_{\mathcal{T}}\deg^{\mathrm{fl}}(\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb});$$

(Rt7) Grassmannian height of the test class We have

$$h^{\mathrm{cd}}(\mathrm{pl}^{D^{\mathrm{gr}}+1}\mathrm{fb}) \le (D^{\mathrm{gr}}+1)\deg^{\mathrm{fl}}(\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb})\,|\Phi|\,\frac{\min(\kappa,n-\kappa)}{2}\,\log(n+1).$$

Definition 29.4. Fix (Φ, n, κ) as in the Chern-root calculus and write $\mathrm{Rt} = \Phi \times \{1,\dots,n\}$ (the **roots**) and $\mathrm{Rt}^+ = \mathrm{Rt} \sqcup \{*\}$. Let $\mathcal{R}^+ = \mathcal{R}[x_*]$.

Multiprojective setting. Let K be a non-empty finite set and $(I_k)_{k\in K}$ non-empty finite sets; put $\mathbb{P} = \prod_{k\in K}\mathbb{P}(\mathbb{Z}^{I_k})$ over $\mathrm{Spec}\,\mathbb{Z}$ and $\widehat{\mathbb{C}} = \prod_k(\mathbb{C}^{I_k}\smallsetminus\{0\})$. For $e \in \mathbb{Z}^K$, $\mathcal{O}(e)$ is the line bundle on $\mathbb{P}$ whose sections are the multihomogeneous polynomials of multidegree e. A **cone metric** of multidegree e is a smooth function $h\colon \widehat{\mathbb{C}} \to (0,\infty)$ with $h(\bar z) = h(z)$ and $h((\lambda_k z_k)_k) = \prod_k |\lambda_k|^{e_k} h(z)$ for $\lambda \in (\mathbb{C}^\times)^K$; it is the smooth conjugation-invariant metric on $\mathcal{O}(e)_{\mathbb{C}}$ with $\|\sigma\|(z) = |\sigma(z)|/h(z)$. For $\kappa' \in \mathbb{R}$, $\overline{\mathcal{M}}(\kappa')$ denotes $\overline{\mathcal{M}}$ with its metric multiplied by $e^{-\kappa'}$ (cone function $e^{\kappa'}h$).

Root models. A **root model** of type (Φ, n) is a tuple

$$\mathcal{X} = (\mathcal{X}, (e_r)_{r\in\mathrm{Rt}^+}, (h_r(t))_{r\in\mathrm{Rt},\,t>0}, h_*, k_{\mathcal{X}}, \delta_{\mathcal{X}})$$

consisting of a closed integral subscheme $\mathcal{X} \subseteq \mathbb{P}$ (for some multiprojective setting) flat over $\mathbb{Z}$ of absolute dimension $D+1$, i.e. an arithmetic variety; multidegrees $e_r \in \mathbb{Z}^K$; for every root r and every real $t > 0$ (the **regularisation parameter**) a cone metric $h_r(t)$ of multidegree e_r, giving the **root line** $\overline{\mathcal{L}}_r(t) = (\mathcal{O}(e_r), h_r(t))|_{\mathcal{X}}$; a cone metric h_* of multidegree e_*, giving the **auxiliary line** $\overline{\mathcal{L}}_* = \overline{\mathcal{L}}_*(t) = (\mathcal{O}(e_*), h_*)|_{\mathcal{X}}$; and integers $k_{\mathcal{X}} \ge 1$, $\delta_{\mathcal{X}} \ge 1$ (the **normalising integers**). For $m \in \mathbb{Z}^{\mathrm{Rt}^+}$ put $\overline{\mathcal{L}}^m(t) = \bigotimes_{r\in\mathrm{Rt}^+}\overline{\mathcal{L}}_r(t)^{\otimes m_r}$, a Hermitian line bundle on $\mathcal{X}$ with cone function $h^m(t) = \prod_r h_r(t)^{m_r}$.

Normalised intersection numbers. For $t > 0$ the **normalised arithmetic intersection form** $\mathrm{ai}_t(\mathcal{X};\cdot)$ is the linear form on $\mathcal{R}^+$ that vanishes on monomials of degree $\neq D+1$ and satisfies

$$\mathrm{ai}_t(\mathcal{X}; x_{r_1}\cdots x_{r_{D+1}}) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_{r_1}(t))\cdots\widehat{c}_1(\overline{\mathcal{L}}_{r_{D+1}}(t)))}{\delta_{\mathcal{X}}\,k_{\mathcal{X}}^{\#\{i:\,r_i=*\}}} \qquad (r_1,\dots,r_{D+1}\in\mathrm{Rt}^+). \tag{29.2}$$

The **normalised geometric intersection form** $\mathrm{gi}(\mathcal{X};\cdot)$ is the linear form on $\mathcal{R}^+$ that vanishes on monomials of degree $\neq D$ and sends $x_{r_1}\cdots x_{r_D}$ to $\deg\bigl(c_1(\mathcal{O}(e_{r_1}))\cdots c_1(\mathcal{O}(e_{r_D}))\cap[\mathcal{X}_{\mathbb{Q}}]\bigr)/(\delta_{\mathcal{X}}k_{\mathcal{X}}^{\#\{i:\,r_i=*\}})$.

Real combinations. Let $c\in\mathbb{R}^{\mathrm{Rt}^+}$ and $t>0$. The vector c is **nef on fibres** of $\mathcal{X}$ if $\sum_r c_r\deg(\mathcal{O}(e_r)|_W)\geq 0$ for every prime p and every integral closed curve $W\subseteq\mathcal{X}_{\mathbb{F}_p}$; c is **semipositive at** t if the function $\varphi_{c,t}=\sum_r c_r\log h_r(t)$ on $\widehat{\mathbb{C}}$ satisfies $\sum_{\alpha,\beta}\frac{\partial^2\varphi_{c,t}}{\partial z_\alpha\partial\bar z_\beta}(z)\,w_\alpha\bar w_\beta\geq 0$ for every point $z\in\widehat{\mathbb{C}}$ of the affine cone over $\mathcal{X}(\mathbb{C})$ and every Zariski tangent vector w of that cone at z (i.e. $\sum_\alpha\frac{\partial f}{\partial z_\alpha}(z)w_\alpha=0$ for every f in the multihomogeneous ideal of $\mathcal{X}$). The **coefficient vector** of a weight u and a real s is $c(u,s)\in\mathbb{R}^{\mathrm{Rt}^+}$ with $c(u,s)_{a,i}=-u_{a,i}$ and $c(u,s)_*=s/k_{\mathcal{X}}$; then $\mathrm{lr}(u)+s\,x_*=\sum_r c(u,s)_r\,k_r\,x_r$ with $k_r=1$ for $r\in\mathrm{Rt}$ and $k_*=k_{\mathcal{X}}$.

Geometric realisations. A **geometric realisation** of a root tower $\mathcal{T}$ of type (Φ,n,κ) is a weight $\nu_{\mathcal{T}}$ with rational entries and, for every integer $B\geq 2$ and every $Z\in\mathrm{Cp}$, a root model $\mathcal{X}_{B,Z}$ of type (Φ,n). For a weight u and $s>0$ write $u_s=u-s\,\nu_{\mathcal{T}}$. We consider the following properties of the realised tower.

(Gm1) Equal volumes We have $\mathrm{vol}(Z)=\mathrm{vol}(Z')$ for all $Z,Z'\in\mathrm{Cp}$;

(Gm2) Regularisation For every weight u satisfying the gap hypothesis with some gap $g>0$, every integer $B\geq 2$ with $b_0(u,g)<B+1$ and every $Z\in\mathrm{Cp}$, writing $\mathcal{X}=\mathcal{X}_{B,Z}$ and $t(s)=\exp(-1/s^2)$:

1. for all sufficiently small $s>0$, the vector $c(u_s,s)$ is nef on fibres of $\mathcal{X}$ and semipositive at $t(s)$;
2. $\mathrm{ai}_{t(s)}\bigl(\mathcal{X};(\mathrm{lr}(u_s)+s\,x_*)^{D+1}\bigr)\to\widehat{\deg}_{B,Z}\bigl(\mathrm{lr}(u)^{D+1}\bigr)$ as $s\to 0^+$;
3. $\mathrm{gi}\bigl(\mathcal{X};(\mathrm{lr}(u_s)+s\,x_*)^{D}\bigr)\to\mathrm{vol}(Z)\deg^{\mathrm{fl}}\bigl(\mathrm{lr}(u)^D\bigr)$ as $s\to 0^+$;
4. $\mathrm{vol}(Z)\deg^{\mathrm{fl}}\bigl(\mathrm{lr}(u)^D\bigr)>0$.

The **slope** of a weight u at (B,Z) is $\mathrm{sl}_{B,Z}(u)=\widehat{\deg}_{B,Z}\bigl(\mathrm{lr}(u)^{D+1}\bigr)/\bigl((D+1)\,\mathrm{vol}(Z)\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D)\bigr)$ (with value 0 when the denominator vanishes).

Notation for abelian schemes is that of [186], [181] and [87]. For a scheme S, an **abelian scheme** $A\to S$ is a smooth proper group scheme with geometrically connected fibres, and a **semiabelian scheme** $G\to S$ is a smooth separated commutative group scheme whose fibres are extensions of abelian varieties by tori. We write $A^\vee$ for the dual abelian scheme, $e\colon S\to A$ for the unit section, $[n]$ for multiplication by n, $A[n]$ for its kernel and $T_\ell A$, $V_\ell A=T_\ell A\otimes\mathbb{Q}_\ell$ for the Tate modules. A **polarised abelian scheme** $(A,\mathcal{L})$ carries a relatively ample line bundle $\mathcal{L}$ with polarisation $\phi_{\mathcal{L}}\colon A\to A^\vee$, and $e_n\colon A[n]\times A[n]\to\mu_n$ is the Weil pairing of $\phi_{\mathcal{L}}$. Line bundles are given by an open cover and transition functions; $\bigotimes_i f_i^*\mathcal{L}^{\otimes n_i}$ is the tensor combination of pull-backs along morphisms f_i. A **rigidified** line bundle is a pair $(\mathcal{L},r)$ with $r\colon e^*\mathcal{L}\cong\mathcal{O}_S$, and a **cubical structure** on it is a trivialisation of

$$\Theta(\mathcal{L})=\bigotimes_{\varnothing\neq I\subseteq\{1,2,3\}} m_I^*\mathcal{L}^{\otimes(-1)^{|I|+1}}$$

on A^3 (m_I the partial sums) satisfying the symmetry and cocycle conditions of [181], Ch. I; $\mathcal{G}(\mathcal{L})$ is the theta group. For $S=\operatorname{Spec}V$ with V a complete discrete valuation ring with fraction field K, a **semiabelian model** of an abelian variety A_K is a semiabelian scheme G/V with $G_K\cong A_K$; its formal completion is the extension $\tilde G$ (**Raynaud extension**) of an abelian scheme by a torus, which is **split** if the torus is split; and **degeneration data** and the **Mumford uniformisation**

$A_K^{\mathrm{an}} = \tilde{G}_K/Y$ are as in [87], Ch. II–III and [187]. A **p-divisible group** over a ring R is written $\mathcal{G}$, with Tate module $T_p\mathcal{G}$, Lie algebra $\operatorname{Lie}\mathcal{G}$, invariant differentials $\omega_{\mathcal{G}}$ and, over the ring of integers of a complete algebraically closed field C, Hodge–Tate map $\alpha_{\mathcal{G}}\colon T_p\mathcal{G} \to \omega_{\mathcal{G}^\vee}$ [218]. An action of a ring O with involution on A (or G, $\mathcal{G}$) is a ring homomorphism $O \to \operatorname{End}(A)$ compatible with the Rosati involution where a polarisation is given. This notation holds wherever abelian varieties occur in this paper.

Definition 29.5. Let E be a CM field with maximal totally real subfield E^+, complex conjugation $a \mapsto \bar{a}$, and such that $\mathcal{O}_E$ is unramified over $\mathcal{O}_{E^+}$ at every finite place. For an abelian scheme $G \to S$ with a line bundle $\mathcal{L}$ write $\Lambda(\mathcal{L}) = m^*\mathcal{L} \otimes \mathrm{pr}_1^*\mathcal{L}^{-1} \otimes \mathrm{pr}_2^*\mathcal{L}^{-1}$ on $G \times_S G$ (m the group law), and $(f,g)^*\Lambda(\mathcal{L})$ for its pull-back along $(f,g)\colon T \to G \times_S G$.

(Ud1) CM host. A **CM host** on E is a triple $\mathsf{H} = (d, r, r_{\mathbb{C}})$ of a prime $d \equiv 7 \pmod{16}$, an element $r \in \mathcal{O}_E$ and a complex number $r_{\mathbb{C}}$ with $r^2 = -d = r_{\mathbb{C}}^2$. Then $w_0 = (1+r)/2 \in \mathcal{O}_E$. Its **CM type** is $\Phi_{\mathsf{H}} = \{\tau\colon E \to \mathbb{C} :\ \tau(r) = r_{\mathbb{C}}\}$.

(Ud2) Unitary datum. A **unitary datum** of **signature** (p,q) ($p, q \geq 0$ integers) for H is a pair $\mathsf{U} = (M, \langle\,,\rangle)$ of a finitely generated projective $\mathcal{O}_E$-module M of rank $p+q$ and an alternating $\mathbb{Z}$-bilinear form $\langle\,,\rangle\colon M \times M \to \mathbb{Z}$ inducing an isomorphism $M \cong \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, with $\langle ax, y\rangle = \langle x, \bar{a}y\rangle$ for $a \in \mathcal{O}_E$, for which there is an $\mathbb{R}$-linear map J of $M_{\mathbb{R}} = M \otimes \mathbb{R}$ with $J^2 = -1$, $\langle Jx, Jy\rangle = \langle x, y\rangle$, $\langle Jx, x\rangle > 0$ for $x \neq 0$, commuting with $\mathcal{O}_E$, such that for every $a \in \mathcal{O}_E$, with $t(a) = \sum_{\tau \in \Phi_{\mathsf{H}}} (p\,\tau(a) + q\,\overline{\tau(a)})$, the trace of a on $M_{\mathbb{R}}$ is $2\operatorname{Re} t(a)$ and the trace of Ja is $-2\operatorname{Im} t(a)$.

(Ud3) Level. A **level** for H is a triple $\mathsf{L} = (N, \mathfrak{n}, A)$ of an odd integer $N \geq 3$, an ideal $\mathfrak{n} \subseteq \mathcal{O}_E$ generated by $\mathfrak{n} \cap \mathbb{Z}[w_0]$ with $\mathfrak{n}\bar{\mathfrak{n}} = N\mathcal{O}_E$ and $\mathfrak{n} + \bar{\mathfrak{n}} = \mathcal{O}_E$, and an integer $A \geq 1$. Put $N' = \operatorname{lcm}(N, A)$. The level is **good at** a prime ℓ if $N/\ell^{v_\ell(N)} \geq 3$. Its **field of definition** is $K_{\mathsf{L}} = \mathbb{Q}(r_{\mathbb{C}}, e^{2\pi i/A}) \subset \mathbb{C}$.

(Ud4) Objects. Let S be a K_{L}-scheme. An **object of type** (U, L) over S is a tuple $(G, \iota, \mathcal{L}, \eta, \eta_A)$ of an abelian scheme $G \to S$, a ring homomorphism $\iota\colon \mathcal{O}_E \to \operatorname{End}_S(G)$, a rigidified symmetric line bundle $\mathcal{L}$ on G with a cubical structure, relatively ample over S and with principal polarisation $\phi_{\mathcal{L}}$, and additive maps $\eta, \eta_A\colon M \to G(S)$ with $\eta(\mathfrak{n}M) = 0$ and $\eta_A(AM) = 0$, such that $(\iota(a), 1)^*\Lambda(\mathcal{L}) \cong (1, \iota(\bar{a}))^*\Lambda(\mathcal{L})$ for every $a \in \mathcal{O}_E$, $\mathcal{L} \cong (1, \iota(w_0))^*\Lambda(\mathcal{L})$, and for every algebraically closed field k and every point $x\colon \operatorname{Spec} k \to S$, with $\beta\colon K_{\mathsf{L}} \to k$ the induced embedding:

- (Kottwitz condition) for every ring homomorphism $\tau\colon \mathcal{O}_E \to k$, the subspace of $\operatorname{Lie}(G_x)$ on which $\iota(a)$ acts by $\tau(a)$ for all a has dimension p if $\tau(r) = \beta(r_{\mathbb{C}})$ and q otherwise, and $\operatorname{Lie}(G_x)$ is the sum of these subspaces;
- (level structure) there are primitive roots of unity $\zeta_n \in k$ ($n \geq 1$) with $\zeta_{nm}^m = \zeta_n$ and $\zeta_A = \beta(e^{2\pi i/A})$, and $\mathcal{O}_E$-linear isomorphisms $I_n\colon M/nM \to G_x[n](k)$ with $mI_{nm}(a) = I_n(a)$, $e_n(I_n(a), I_n(b)) = \zeta_n^{\langle a,b\rangle}$, $\eta_A(a)_x = I_A(a)$ for all $a \in M$ and $\eta(a)_x = I_N(a)$ for all $a \in \bar{\mathfrak{n}}M$.

An **isomorphism** of objects over $S' \to S$ is an isomorphism of G' with the base change of G compatible with ι, the cubical line bundles, η and η_A.

(Ud5) Moduli scheme. A **moduli scheme** of type (U, L) is a quasi-projective scheme Sh smooth of relative dimension $\frac{1}{2}[E : \mathbb{Q}]pq$ over K_{L} with an object o^{un} over Sh (the **universal object**) such that (i) for every reduced K_{L}-scheme S and every object o over S there is

exactly one K_{L}-morphism $f\colon S \to \mathrm{Sh}$ with $o \cong f^{*}o^{\mathrm{un}}$; (ii) for every field k and every K_{L}-morphism $x\colon \operatorname{Spec} k[\epsilon] \to \mathrm{Sh}$ ($\epsilon^{2}=0$) with constant point x_0 (the composite with $k[\epsilon] \to k \to k[\epsilon]$): if an object over $\operatorname{Spec} k[\epsilon]$ is isomorphic to both $x^{*}o^{\mathrm{un}}$ and $x_0^{*}o^{\mathrm{un}}$, then $x = x_0$.

(Ud6) Coefficient field. A **coefficient field** for L is a subfield $F \subset \mathbb{C}$ of finite degree over $\mathbb{Q}$ containing $\tau(E)$ for every $\tau\colon E \to \mathbb{C}$, $e^{2\pi i/N'}$ and K_{L}. Write $\mathrm{Sh}_F = \mathrm{Sh} \times_{K_{\mathsf{L}}} F$, $\mathrm{Comp}(\mathrm{Sh}_F)$ for its set of irreducible components (the **components**) and $\mathbf{k}(Z)$ for the function field of $Z \in \mathrm{Comp}(\mathrm{Sh}_F)$. The **components split** if F is algebraically closed in $\mathbf{k}(Z)$ for every Z.

(Ud7) Integral model. An **integral model** of Sh over $\mathcal{O}_F$ is a tuple $(\mathcal{Y}, j, G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ of a normal scheme $\mathcal{Y}$, projective and flat over $\mathcal{O}_F$, an open immersion $j\colon \mathrm{Sh}_F \to \mathcal{Y} \times_{\mathcal{O}_F} F$ with dense image, and a semiabelian scheme $G_{\mathcal{Y}} \to \mathcal{Y}$ with a ring homomorphism $\iota_{\mathcal{Y}}\colon \mathcal{O}_E \to \mathrm{End}(G_{\mathcal{Y}})$ and a rigidified symmetric cubical line bundle $\mathcal{L}_{\mathcal{Y}}$ satisfying the two isomorphisms of (Ud4), such that $j^{*}(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ is isomorphic to the base change to Sh_F of $(G, \iota, \mathcal{L})$ of o^{un}, and the points of $\mathcal{Y} \times_{\mathcal{O}_F} F$ over which $G_{\mathcal{Y}}$ is an abelian scheme are exactly those of $j(\mathrm{Sh}_F)$.

A **unitary Shimura datum** is a tuple $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ as in (Ud1)–(Ud6).

Definition 29.6. Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum, $\Phi = \Phi_{\mathsf{H}}$ and $n = p + q$. Everything below is formed over the coefficient field F of $\mathfrak{u}$. In the application, F is the field of the datum that carries the tower; it is a datum in its own right, and no other field replaces it.

(Fh1) Hodge pieces. For a label $s = (\tau, \pm) \in \Phi \times \{+,-\}$ put $\sigma_s = \tau$ and $\mathrm{rk}(s) = p$ if the sign is $+$, and $\sigma_s = \bar{\tau}$ ($a \mapsto \overline{\tau(a)}$) and $\mathrm{rk}(s) = q$ if it is $-$. Since $\Phi \sqcup \bar{\Phi} = \mathrm{Hom}(E, \mathbb{C})$ and $\tau(E) \subseteq F$ for every τ, $s \mapsto \sigma_s$ is a bijection from the labels onto $\mathrm{Hom}(E, F)$. For an F-scheme W and a semiabelian scheme $G \to W$ with $\iota\colon \mathcal{O}_E \to \mathrm{End}(G)$, the Lie algebra $\mathrm{Lie}(G)$ (the locally free $\mathcal{O}_W$-module of invariant vector fields along the unit section) decomposes as $\mathrm{Lie}(G) = \bigoplus_s \mathrm{Lie}_s(G)$, where the **Hodge piece** $\mathrm{Lie}_s(G)$ is the submodule on which $\iota(a)$ acts by $\sigma_s(a)$ for every $a \in \mathcal{O}_E$ (as $E \otimes_{\mathbb{Q}} \mathcal{O}_W = \prod_{\sigma \in \mathrm{Hom}(E,F)} \mathcal{O}_W$). For the universal object over Sh_F the Kottwitz condition makes $\mathrm{Lie}_s(G^{\mathrm{un}})$ locally free of rank $\mathrm{rk}(s)$.

(Fh2) Flag bundle. For a locally free $\mathcal{O}_W$-module V of rank r, a **full flag** of V is a chain $0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$ of locally free submodules with invertible quotients V_i/V_{i-1}; $\mathrm{Fl}(V) \to W$ is the scheme of full flags (smooth and projective of relative dimension $r(r-1)/2$). The **flag bundle** of $\mathfrak{u}$ is the fibre product over Sh_F

$$\mathrm{Fl}_{\mathfrak{u}} = \prod_{s \in \Phi \times \{+,-\}} \mathrm{Fl}(\mathrm{Lie}_s(G^{\mathrm{un}})) \xrightarrow{\pi_{\mathfrak{u}}} \mathrm{Sh}_F,$$

smooth over F of dimension $|\Phi| pq + |\Phi|(p(p-1) + q(q-1))/2 = |\Phi|\, n(n-1)/2$. For a component Z of Sh_F, $\mathrm{Fl}_{\mathfrak{u}}|_Z = \pi_{\mathfrak{u}}^{-1}(Z)$, with function field $\mathbf{k}(\mathrm{Fl}_{\mathfrak{u}}|_Z)$.

(Fh3) Root lines. The **roots** are the pairs $(\tau, i) \in \Phi \times \{1, \dots, n\}$. To a root attach the label $s(\tau, i) = (\tau, +)$, the step $\mathrm{st}(\tau, i) = i$ and the sign $\mathrm{sg}(\tau, i) = 1$ if $i \le p$, and $s(\tau, i) = (\tau, -)$, $\mathrm{st}(\tau, i) = n + 1 - i$, $\mathrm{sg}(\tau, i) = -1$ if $i > p$. For a full flag $(V_{s,k})_k$ of each Hodge piece V_s, the **root line** of the root r is

$$\mathcal{L}_r = \left(V_{s(r),\mathrm{st}(r)} / V_{s(r),\mathrm{st}(r)-1}\right)^{\otimes \mathrm{sg}(r)};$$

for the universal flags over $\mathrm{Fl}_{\mathfrak{u}}$ these are the root lines of $\mathfrak{u}$.

(Fh4) Hodge metrics. Let $\sigma\colon F\to\mathbb{C}$ be any field embedding and z a complex point of Sh_F over σ, i.e. a point of $(\mathrm{Sh}_F\otimes_{F,\sigma}\mathbb{C})(\mathbb{C})$; the Hodge pieces at z are the base changes along σ of those of (Fh1). Write $G_z(\mathbb{C})=\mathrm{Lie}(G_z)/\Lambda_z$ for the fibre of the universal abelian scheme; the first Chern class of the ample line bundle $\mathcal{L}_z$ is a positive definite hermitian form H_z on $\mathrm{Lie}(G_z)$ with $\operatorname{Im}H_z(\Lambda_z,\Lambda_z)\subseteq\mathbb{Z}$. The **Hodge inner product** on $\mathrm{Lie}(G_z)$ is $h_z=\pi H_z$. A step quotient V_k/V_{k-1} of a full flag of $\mathrm{Lie}_s(G_z)$ carries the quotient norm $\|v \bmod V_{k-1}\|=\min_{w\in V_{k-1}}h_z(v-w,v-w)^{1/2}$; for a basis $b_1,\dots,b_r$ adapted to the flag and its Gram matrix $\Gamma=(h_z(b_k,b_l))_{k,l}$,

$$\|b_k \bmod V_{k-1}\|^2=\det\Gamma_{\le k}/\det\Gamma_{\le k-1}$$

($\Gamma_{\le k}$ the leading $k\times k$ minor). Inverse lines carry the dual norm. This gives the **Hodge metric** $\|\cdot\|^{\mathrm{Hdg}}$ on every root line at every complex point of $\mathrm{Fl}_{\mathfrak{u}}$, over every embedding of F into $\mathbb{C}$, smooth there.

(Fh5) Covers and their Hodge lattices. A **cover** of $\mathcal{Y}$ is a normal integral scheme $\mathcal{X}$ with function field Ω, a morphism $f\colon\mathcal{X}\to\mathcal{Y}$ whose generic point maps to the generic point of an irreducible component of $\mathcal{Y}$ lying in $j(\mathrm{Sh}_F)$ with Ω finitely generated over its residue field, and the pulled-back semiabelian scheme $G_{\mathcal{X}}=f^{*}G_{\mathcal{Y}}$ with its $\mathcal{O}_E$-action and cubical line bundle. Put $\mathrm{Li}=\mathrm{Lie}(G_{\mathcal{X},\Omega})$ and $\mathrm{Li}_s=\mathrm{Lie}_s(G_{\mathcal{X},\Omega})$ (an Ω-space of dimension $\mathrm{rk}(s)$). For a point x of $\mathcal{X}$ with local ring $\mathcal{O}_x\subseteq\Omega$, let $\mathrm{Lie}_x\subseteq\mathrm{Li}$ be the image of $\mathrm{Lie}(G_{\mathcal{X}})_x$ and $\omega_x=\mathrm{Hom}_{\mathcal{O}_x}(\mathrm{Lie}_x,\mathcal{O}_x)$; the **de Rham lattice** is $\mathrm{dR}_{s,x}=\mathrm{Lie}_x\cap\mathrm{Li}_s$. Let p be a prime and $R\ge1$ an integer such that every p^R-torsion point of $G_{\mathcal{X}}$ over an algebraic closure of Ω is defined over Ω, and let x lie over p ($p\notin\mathcal{O}_x^{\times}$). An element $w\in\omega_x$ is a **Hodge–Tate value** at x of $y\in G_{\mathcal{X}}(\Omega)[p^R]$ if there is a regular function ϕ on an open neighbourhood of the unit point over x in $G_{\mathcal{X}}$ with $\phi(g)=e_{p^R}(y,g)$ for every algebraically closed field $k\supseteq\Omega$ and every $g\in G_{\mathcal{X}}(k)[p^R]$ (Weil pairing of the polarisation of $\mathcal{L}$), and $w(v)\equiv d\phi(v) \bmod p^R\mathcal{O}_x$ for every $v\in\mathrm{Lie}_x$. Let $\mathrm{ht}_x\subseteq\omega_x$ be the $\mathcal{O}_x$-span of all Hodge–Tate values at x of all $y\in G_{\mathcal{X}}(\Omega)[p^R]$, $\mathrm{HT}'_{s,x}=\{e\in\mathrm{Li}_s : w(e)\in\mathcal{O}_x \text{ for all } w\in\mathrm{ht}_x\}$, and let the **Hodge–Tate lattice** be $\mathrm{HT}_{s,x}(p^R)=\mathrm{HT}'_{s,x}$ for $s=(\tau,-)$ and $\mathrm{HT}_{s,x}(p^R)=\{e\in\mathrm{Li}_s : \sigma_s(d)e\in\mathrm{HT}'_{s,x} \text{ for all } d\in\mathfrak{d}_E\}$ for $s=(\tau,+)$, $\mathfrak{d}_E$ the different of $E/\mathbb{Q}$. For a finite set S of primes and integers $R(p)\ge1$ ($p\in S$) as above, the **Hodge lattice at** (S,R) is $V^S_{s,x}=\mathrm{HT}_{s,x}(p^{R(p)})$ if x lies over $p\in S$ and $V^S_{s,x}=\mathrm{dR}_{s,x}$ otherwise. It is a **Hodge bundle** if there is a locally free $\mathcal{O}_{\mathcal{X}}$-submodule $\mathcal{V}^S_s$ of the constant sheaf Li_s, of rank $\mathrm{rk}(s)$, with stalks $V^S_{s,x}$.

(Fh6) Integral root lines and their Hodge metrics. An **integral flag** of a cover with Hodge bundles is a full flag of each $\mathcal{V}^S_s$; its root lines $\mathcal{L}_r$ (Fh3), formed from the step quotients of these subsheaves, are invertible $\mathcal{O}_{\mathcal{X}}$-modules whose generic fibres are the root lines of the induced full flags of the spaces Li_s. The **abelian locus** of $\mathcal{X}$ is $f^{-1}(j(\mathrm{Sh}_F))$; at a complex point z of $\mathcal{X}$ in it (over any embedding of F into $\mathbb{C}$), $\mathcal{L}_r\otimes\mathbb{C}$ is a root line of $G_{\mathcal{X},z}=G^{\mathrm{un}}_{f(z)}$ for the flag induced on $\mathrm{Lie}(G_{\mathcal{X},z})$, and carries the Hodge metric (Fh4).

Definition 29.7. Let $\mathfrak{u}$ be a unitary Shimura datum with $\Phi=\Phi_{\mathsf{H}}$, $n=p+q$, $p,q\ge1$, and let $\mathcal{Y}$ be an integral model of Sh over $\mathcal{O}_F$. Let $\mathcal{T}$ be a root tower of type (Φ,n,p) with a geometric realisation $(\nu_{\mathcal{T}},(\mathcal{X}_{B,c})_{B\ge2,\,c\in\mathrm{Cp}})$. For $B\ge2$ put $S_B=\{\ell \text{ prime} : \ell\le B \text{ or } \ell\in\mathrm{Bd}\}$. The **fundamental indices** are Rt^{+}; the fundamental index (τ,i) stands for the line $\bigotimes_{k\le i}\mathcal{L}^{-1}_{(\tau,k)}$ and $*$ for the auxiliary line.

(Go1) Analytic systems with log-log weights. An **analytic system** is: a compact complex manifold X^{an} of dimension $D=|\Phi|n(n-1)/2$; a smooth hermitian line bundle with

positive curvature (a Kähler form Θ); a holomorphic section ς of a smooth hermitian line bundle with $\|\varsigma\| \le 1$ whose zero divisor ∂X^{an} has simple normal crossings, with open part $X^\circ = X^{\mathrm{an}} \smallsetminus \partial X^{\mathrm{an}}$ and $\mathrm{rW} = e - 2\log\|\varsigma\|$; for every fundamental index a a smooth hermitian line bundle $(\mathcal{M}_a, h_a^{\mathrm{ref}})$, a function φ_a on X° (the **singular weight**) and functions $\varphi_a^T(t)$ on X^{an} for $0 < t \le t_0$ (the **regularisations**); and constants $C_5, C \ge 0$, $0 < t_0 < 1$, such that, with $\varepsilon(t) = C\log\log(1/t)/\log(1/t)$ and "psh" meaning that the local potentials of the metric are plurisubharmonic:

1. (log-log singular weight) φ_a is smooth on X°, $h_a^{\mathrm{ref}} e^{\varphi_a}$ is psh on X°, and $-C_5\log\mathrm{rW} \le \varphi_a \le C_5$ on X°;
2. (regularisation) for $0 < t \le t_0$: $\varphi_a^T(t)$ is smooth, $-C\log\log(1/t) \le \varphi_a^T(t) \le C$, $\varphi_a \le \varphi_a^T(t)$ on X°, the metric $h_a^{\mathrm{ref}} e^{\varphi_a^T(t)}$ twisted by $\varepsilon(t)\Theta$ is psh, $\varphi_a^T(t) \le \varphi_a + t^{1/3} + \varepsilon(t)$ where $\mathrm{rW} \le \log(1/t)$, $\varphi_a^T(t) \le \varphi_a + t^{1/3} + \varepsilon(t)\mathrm{rW}$ on X°, and $\varphi_a^T(t) \le \varphi_a^T(t') + C\varepsilon(t')$ for $t \le t' \le t_0$;
3. (C^2-convergence) $\varphi_a^T(t) \to \varphi_a$ with first and second derivatives, uniformly on compact subsets of X°, as $t \to 0^+$.

(Go2) Fixed data. Fixed data at $c \in \mathrm{Cp}$ are: an irreducible component $\mathcal{Y}_c$ of $\mathcal{Y}$ meeting $j(\mathrm{Sh}_F)$, with $Z_c = j^{-1}(\mathcal{Y}_c)$; the function field $\Omega_c = \mathbf{k}(\mathrm{Fl_u}|_{Z_c})$ with the universal flag of each Lie_s at the generic point; elements $y_0^c, \dots, y_m^c \in \Omega_c$ whose ratios generate Ω_c; a positive function h_c on $\mathbb{C}^{m+1} \smallsetminus \{0\}$ and an integer $k_c^A \ge 1$; an analytic system Sys_c with a map $\mathrm{pt}_c\colon X_c^{\mathrm{an}} \to \mathbb{C}^{m+1} \smallsetminus \{0\}$ injective modulo $\mathbb{C}^\times$, with $\varphi_* = 0 = \varphi_*^T(t)$ and h_*^{ref} of positive curvature; an integer $q_c' \ge 1$, integers $n_c(r)$ ($r \in \mathrm{Rt}$) with $n_c(r) = k_c^A q_c' \nu_{\mathcal{T}}(r)$, and $s_c^E \in \Omega_c^\times$.

(Go3) Root origin. Let $\mathcal{X} = \mathcal{X}_{B,c} \subseteq \mathbb{P}$ have function field Ω. **Representing coordinates** are elements $y_{k,j} \in \Omega$ ($k \in K$, $j \in I_k$) such that for every k and all j, j' with $y_{k,j'} \ne 0$ the quotient $y_{k,j}/y_{k,j'}$ is the rational function given by the ratio of the coordinates j and j' of the factor k; for $G \in \mathbb{Z}[y_{k,j}]$ we write $G(y) \in \Omega$ for its value at them. A **root origin** of $\mathcal{X}$ on fixed data at c consists of representing coordinates, integers $R(\ell) \ge 4$ ($\ell \in S_B$), an integer $\deg_{B,c} \ge 1$, elements $\varrho_r \in \Omega^\times$ ($r \in \mathrm{Rt}$), cone functions h_r^{sing} of multidegree e_r (positive, not necessarily smooth), an index $k_0 \in K$ with a bijection $\{0, \dots, m\} \to I_{k_0}$, and a realisation (Go4), such that:

1. $\mathcal{X}$ is normal and, with a morphism $f\colon \mathcal{X} \to \mathcal{Y}$, is a cover of $\mathcal{Y}$ whose generic point maps to that of $\mathcal{Y}_c$; there is a field embedding $g\colon \Omega_c \to \Omega$ over $\mathbf{k}(Z_c)$ with $[\Omega : g(\Omega_c)] = \deg_{B,c}$, and $\delta_{\mathcal{X}} = [F : \mathbb{Q}]\deg_{B,c}$;
2. for $\ell \in S_B$ every $\ell^{R(\ell)}$-torsion point of $G_{\mathcal{X}}$ over an algebraic closure of Ω is defined over Ω; the Hodge lattices at (S_B, R) are Hodge bundles $\mathcal{V}_s$; and the span of the Hodge–Tate values at the points over ℓ, and its image in the dual of each Li_s, are locally free of ranks $\dim \mathrm{Li}$ and $\mathrm{rk}(s)$ along the fibre over ℓ;
3. there is an integral flag of the $\mathcal{V}_s$ whose generic fibre is the image under g of the universal flag; on each chart $\{y_{c'} \ne 0\}$ of $\mathcal{X}$ ($y_{c'} = \prod_k y_{k,c_k'}$) the root line $\mathcal{L}_r$ of this flag is generated by $\varrho_r \prod_k y_{k,c_k'}^{e_{r,k}}$ ($e_{r,k}$ the entries of the multidegree e_r), so that $P \mapsto \varrho_r P(y)$ is an isomorphism $\mathcal{O}(e_r)|_{\mathcal{X}} \to \mathcal{L}_r$ (the **root sections**);
4. (singular Hodge lines) at every complex point z of the abelian locus and every point $\tilde z$ of the affine cone over z, $\|\varrho_r P(y)\|^{\mathrm{Hdg}}(z) = |P(\tilde z)|/h_r^{\mathrm{sing}}(\tilde z)$ for every section P of $\mathcal{O}(e_r)$;
5. (auxiliary line) e_* is 1 at k_0 and 0 elsewhere, the coordinates of the factor k_0 are $g(y_0^c), \dots, g(y_m^c)$, $h_*(\tilde z) = h_c((\tilde z_{k_0,i})_i)$ at the cone points $\tilde z$ of $\mathcal{X}$, and $k_{\mathcal{X}} = k_c^A$;

6. (effective weight section) at every point x of $\mathcal{X}_{\mathbb{Q}}$ in the chart $\{Y_{k_0,i} \neq 0\}$ of the factor k_0 (the ratios $g(y^c_{i'})/g(y^c_i)$ are regular at x) and every local generator θ of $\bigotimes_r \mathcal{L}_r^{-n_c(r)}$ at x, the element $g(s^E_c)g(y^c_i)^{q'_c}/\theta$ is regular at x, and it is not a unit at x if x is not in the abelian locus.

(Go4) Realisation. A **realisation** of Sys_c by $\mathcal{X}$ is a holomorphic map $\pi\colon \mathcal{X}(\mathbb{C}) \to X_c^{\mathrm{an}}$ (on a resolution of $\mathcal{X}(\mathbb{C})$ when this is singular) of degree $\deg_{B,c}$ off a closed analytic subset of X_c^{an} with empty interior (equivalently, containing no connected component), in the sense that its preimage under π is a closed analytic subset of $\mathcal{X}(\mathbb{C})$ with empty interior, every point of X_c^{an} outside it has exactly $\deg_{B,c}$ preimages, and the differential of π is invertible at every point outside its preimage; with: the k_0-coordinates of a point proportional to pt_c of its image; a point lies in the abelian locus if and only if its image lies in X_c°; smooth cone functions h_a^{ref} on $\mathcal{X}$ of the multidegree of the fundamental index a (i.e. $-\sum_{k\le i} e_{(\tau,k)}$ for $a=(\tau,i)$, e_* for $*$) whose metrics are the pull-backs of those of Sys_c, with $h_*^{\mathrm{ref}} = h_*$; for $0 < t \le t_0$ the regularised metrics $h_a^{\mathrm{ref}}\, e^{\varphi_a^T(t)\circ\pi}$, which for $a=(\tau,i)$ equal $\prod_{k\le i} h_{(\tau,k)}(t)^{-1}$ (the root metrics of the realisation); and on $\pi^{-1}(X_c^\circ)$, $h^{\mathrm{ref}}_{(\tau,i)} e^{\varphi_{(\tau,i)}\circ\pi} = \prod_{k\le i}(h^{\mathrm{sing}}_{(\tau,k)})^{-1}$: the singular weight is the Hodge metric.

(Go5) Tower origin and good origin. A **tower origin** of $\mathcal{T}$ over $\mathcal{Y}$ is: $\kappa = p$ for the cut of $\mathcal{T}$; Bd the set of primes dividing $2\,\mathrm{disc}(E)\,N'$; fixed data at every $c \in \mathrm{Cp}$ such that $c \mapsto \mathcal{Y}_c$ is a bijection from Cp onto the set of irreducible components of $\mathcal{Y}$; and a root origin of every $\mathcal{X}_{B,c}$ on the fixed data at c. It is a **good origin** if moreover: (smooth) every $\mathcal{X}_{B,c} \otimes \mathbb{Q}$ is smooth (no resolution needed in (Go4)); (flat) for every c, τ and t the regularised metric of the fundamental index (τ, n) (the determinant line of the flag at τ) has pluriharmonic local potentials; (reference models) for every c a root model $\mathcal{X}_c^{\mathrm{ref}}$ with a root origin on the fixed data at c with $S = \varnothing$, $\deg = 1$, no resolution, and with $\mathcal{O}(e_*)$ ample on $\mathcal{X}_c^{\mathrm{ref}} \otimes \mathbb{Q}$. A good origin is **immersive at level four** if every pt_c is, near every point, the projectivisation of a holomorphic map to $\mathbb{C}^{m+1} \smallsetminus \{0\}$ whose affine charts have injective differential, and $R(\ell) = 4$ for all B, c and $\ell \in S_B$.

(Go6) Model-small sets. Let $\mathcal{X} = \mathcal{X}_{B,c}$ carry a root origin, let $\mathfrak{p} = \{G \in \mathbb{Z}[y_{k,j}] : G(y) = 0\}$ (a prime ideal, the kernel of a ring map to the field Ω), and for an **integral weight** $w \in \mathbb{Z}^{\mathrm{Rt}}$ put $e(w) = -\sum_{r\in\mathrm{Rt}} w_r e_r \in \mathbb{Z}^K$. An element $\sigma \in \Omega$ has **value** $\omega \in \mathbb{C}$ at a point z of the affine cone over $\mathcal{X}(\mathbb{C})$ in degree w if $\sigma = G(y)/H(y)$ with $G, H \in \mathbb{Z}[y_{k,j}]$ multihomogeneous, $\deg G = \deg H + e(w)$, $H(z) \neq 0$ and $G(z) = \omega H(z)$. The **model-small set** $\mathrm{MS}_{\mathcal{X}}(w, R)$ ($R \ge 0$) is the set of $\sigma \in \Omega$ such that (a) for every c' with $y_{c'} \neq 0$ in Ω, $\sigma \prod_k y_{k,c'_k}^{-e(w)_k}$ is integral over $\mathbb{Z}[y_{k,j}/y_{k,c'_k}]$; and (b) there is a multihomogeneous $P \notin \mathfrak{p}$ such that every value ω of σ in degree w at every cone point z with $P(z) \neq 0$ satisfies $|\omega| \le R \prod_r h_r^{\mathrm{sing}}(z)^{-w_r}$.

Definition 29.8. Let $\mathfrak{u}$ be a unitary Shimura datum with integral model $\mathcal{Y}$, $\Phi = \Phi_{\mathsf{H}}$, $n = p+q$, and let $\mathrm{Rt} = \Phi \times \{1,\dots,n\}$ be the set of roots. For an **integral weight** $w \in \mathbb{Z}^{\mathrm{Rt}}$ put $\ell^1(w) = \sum_{r\in\mathrm{Rt}} |w_r|$.

(Ad1) Weight lines. The **weight line** of w is the invertible sheaf $\mathcal{L}\langle w\rangle = \bigotimes_{r\in\mathrm{Rt}} \mathcal{L}_r^{\otimes(-w_r)}$ on $\mathrm{Fl}_{\mathfrak{u}}$, and likewise on any scheme carrying full flags of the Hodge pieces (formed from the root lines of those flags). (Borel–Weil: sections of $\mathcal{L}\langle w\rangle$ over an open set of $\mathrm{Fl}_{\mathfrak{u}}$ are the functions on the bundle of frames adapted to the universal flags that transform under the Borel subgroup by the character of weight w.)

(Ad2) Forms. Let $Z \in \mathrm{Comp}(\mathrm{Sh}_F)$, $\mathrm{Fl}_Z = \pi_{\mathfrak{u}}^{-1}(Z)$, with generic point η_Z and function field $\mathbf{k}(\mathrm{Fl}_Z)$. The space of **forms of weight** w **on** Z is $\mathcal{A}_Z(w) = \mathcal{L}\langle w\rangle_{\eta_Z}$, the rational sections of $\mathcal{L}\langle w\rangle$ over Fl_Z, a $\mathbf{k}(\mathrm{Fl}_Z)$-vector space of dimension one; $f \in \mathcal{A}_Z(w)$ is **regular** if it is a section of $\mathcal{L}\langle w\rangle$ over all of Fl_Z. A **generic frame** of Z is a basis $(b_{s,1}, \dots, b_{s,\mathrm{rk}(s)})$ of each $\mathrm{Lie}_s(G^{\mathrm{un}})_{\eta_Z}$ with $b_{s,1}, \dots, b_{s,k}$ spanning the k-th step of the universal flag for every k; the images of the $b_{s,k}$ in the step quotients give a nowhere-zero rational section ε_w of $\mathcal{L}\langle w\rangle$, and the **evaluation** is

$$\mathrm{ev}_w \colon \mathcal{A}_Z(w) \to \mathbf{k}(\mathrm{Fl}_Z), \qquad f \mapsto f/\varepsilon_w, \tag{29.3}$$

an isomorphism of $\mathbf{k}(\mathrm{Fl}_Z)$-vector spaces.

(Ad3) Hodge norm. Every complex point z of Fl_Z, over any embedding $F \to \mathbb{C}$, is a full flag of the Hodge pieces of the complex abelian variety G_z (fibre of G^{un}), and the Hodge inner product πH_z of its polarisation gives, as in (Fh4), the **Hodge metric** $\|\cdot\|^{\mathrm{Hdg}}$ on $\mathcal{L}\langle w\rangle_z$. The **Hodge norm** of $f \in \mathcal{A}_Z(w)$ is $\|f\| = \sup_z \|f(z)\|^{\mathrm{Hdg}} \in [0,\infty]$ over all complex points z of Fl_Z if f is regular, and $\|f\| = \infty$ otherwise; f is **bounded** if it is regular with $\|f\| < \infty$; the bounded forms form an F-subspace $\mathcal{A}_Z^{\mathrm{b}}(w)$ of $\mathcal{A}_Z(w)$. (For strictly decreasing weights bounded forms are cusp forms.)

(Ad4) λ-adic points and de Rham valuations. Let λ be a nonzero prime of $\mathcal{O}_F$ over the prime ℓ, $\mathbb{C}_\lambda$ the completion of an algebraic closure of F_λ, and v_λ the valuation of $\mathbb{C}_\lambda$ with $v_\lambda(\varpi_\lambda) = 1$ (so $v_\lambda(\ell)$ is the ramification index of λ). A **λ-adic point** of Fl_Z is a point $\bar{x} \in \mathrm{Fl}_Z(K)$ for a finite extension $K \subseteq \mathbb{C}_\lambda$ of F_λ; let $x = \pi_{\mathfrak{u}}(\bar{x}) \in Z(K)$ and let $\tilde{x} \colon \mathrm{Spec}\,\mathcal{O}_K \to \mathcal{Y}$ be the extension of $j(x)$ ($\mathcal{Y}$ is projective over $\mathcal{O}_F$). The **de Rham lattices** at $\bar{x}$ are $\mathrm{dR}_s(\bar{x}) = \mathrm{Lie}(\tilde{x}^* G_{\mathcal{Y}}) \cap \mathrm{Lie}_s(G_x)$; the flag $\bar{x}$ induces on each step quotient the image of $\mathrm{dR}_s(\bar{x}) \cap (\text{step})$, hence an $\mathcal{O}_K$-lattice in every root line and an $\mathcal{O}_K$-lattice $\mathcal{L}\langle w\rangle_{\bar{x}}^{\mathrm{dR}}$ in $\mathcal{L}\langle w\rangle_{\bar{x}}$. For $f \in \mathcal{A}_Z(w)$ regular at $\bar{x}$ the **de Rham valuation** is $v_{\bar{x}}^{\mathrm{dR}}(f) = v_\lambda(f(\bar{x})/e) \in \mathbb{R} \cup \{\infty\}$, e a generator of $\mathcal{L}\langle w\rangle_{\bar{x}}^{\mathrm{dR}}$.

(Ad5) Hodge–Tate valuations. Let $\bar{x}$ be a λ-adic point with $A = \tilde{x}^* G_{\mathcal{Y}}$ an abelian scheme over $\mathcal{O}_K$ (**good reduction**); put $C = \mathbb{C}_\lambda$ and identify A with its dual by the principal polarisation of $\mathcal{L}_{\mathcal{Y}}$, so that the Hodge–Tate map of the ℓ-divisible group $A[\ell^\infty]$ is a map $\alpha \colon T_\ell(A) \to \omega_A \otimes \mathcal{O}_C$, with $\omega_A = \mathrm{Lie}(A)^\vee$. Let $\mathrm{ht}(\bar{x}) \subseteq \omega_A \otimes \mathcal{O}_C$ be the $\mathcal{O}_C$-span of its image, $\mathrm{HT}'(\bar{x}) = \{e \in \mathrm{Lie}(A) \otimes C : \ w(e) \in \mathcal{O}_C \text{ for all } w \in \mathrm{ht}(\bar{x})\}$, and define the **Hodge–Tate lattices** $\mathrm{HT}_s(\bar{x}) \subseteq \mathrm{Lie}_s(G_x) \otimes_K C$ by $\mathrm{HT}_s(\bar{x}) = \mathrm{HT}'(\bar{x}) \cap (\mathrm{Lie}_s(G_x) \otimes_K C)$ for $s = (\tau, -)$ and $\mathrm{HT}_s(\bar{x}) = \{e : \ \sigma_s(d)e \in \mathrm{HT}'(\bar{x}) \text{ for all } d \in \mathfrak{d}_E\}$ for $s = (\tau, +)$. As in (Ad4) they give an $\mathcal{O}_C$-lattice $\mathcal{L}\langle w\rangle_{\bar{x}}^{\mathrm{HT}}$ in $\mathcal{L}\langle w\rangle_{\bar{x}} \otimes_K C$, and the **Hodge–Tate valuation** $v_{\bar{x}}^{\mathrm{HT}}(f) = v_\lambda(f(\bar{x})/e)$, e a generator.

(Ad6) Integral forms. For a real $B > 0$ put $P = P_{E,N'}(B)$, the primes $\ell \le B$ or $\ell \mid 2\,\mathrm{disc}(E)\,N'$. A regular $f \in \mathcal{A}_Z(w)$ is **integral at** B if $v_{\bar{x}}^{\mathrm{HT}}(f) \ge 0$ for every λ-adic point $\bar{x}$ of good reduction with λ over a prime in P, and $v_{\bar{x}}^{\mathrm{dR}}(f) \ge 0$ for every λ-adic point $\bar{x}$ with λ over a prime not in P. The **count of integral bounded forms** is

$$\mathrm{cf}_{Z,B}^{\mathfrak{u}}(w, \rho) = \#\{f \in \mathcal{A}_Z(w) : \ f \text{ integral at } B, \ \|f\| \le e^\rho\} \in \overline{\mathbb{N}} \qquad (\rho \in \mathbb{R}). \tag{29.4}$$

(Ad7) Transfer from a root origin. Let $\mathcal{X}$ be a root model with function field Ω and a root origin on fixed data at c, with $Z = Z_c$ a component of Sh_F. By (Go3)(3) the root lines of $\mathcal{X}$ at its generic point are $\mathcal{L}_r \otimes_{\Omega_c, g} \Omega$, so $\mathcal{L}\langle w\rangle_{\mathcal{X},\Omega} = \mathcal{A}_Z(w) \otimes_{\Omega_c, g} \Omega$ ($\Omega_c = \mathbf{k}(\mathrm{Fl}_Z)$). An element $\sigma \in \Omega$ read in degree w is the rational section $\iota_w(\sigma)$ of $\mathcal{L}\langle w\rangle_{\mathcal{X}}$ obtained from the

rational section σ of $\mathcal{O}(e(w))$ (through the representing coordinates) by the root sections $P \mapsto \varrho_r P(y)$. The **transfer** is the additive map

$$\mathrm{tr}_w \colon \Omega \to \mathcal{A}_Z(w), \qquad \mathrm{tr}_w(\sigma) = (1 \otimes \mathrm{Tr}_{\Omega/g(\Omega_c)})(\iota_w(\sigma)), \tag{29.5}$$

and the **gauge** is $\gamma_w = \iota_w^{-1}(\varepsilon_w \otimes 1) \in \Omega^\times$; then $g(\mathrm{ev}_w(\mathrm{tr}_w(\sigma))) = \mathrm{Tr}_{\Omega/g(\Omega_c)}(\sigma/\gamma_w)$.

Definition 29.9. A family carrying counting data and self-intersection data **carries a tower realisation** if for every real $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and every lift $j < m$ the following data are given, with the index $(\mathfrak{z}, j)$ suppressed: a root tower $\mathcal{T}$ of some type (Φ, n, κ) with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B',Z'})_{B' \geq 2,\, Z' \in \mathrm{Cp}})$; a bijection $\mathrm{cp} \colon \mathcal{Z} \to \mathrm{Cp}$ (the **component bijection**); a weight u (the **divided weight**) and a real number $g > 0$. Put $B_* = \lfloor B \rfloor$, and for $Z \in \mathcal{Z}$ write $\mathcal{X}_Z = \mathcal{X}_{B_*,\mathrm{cp}(Z)}$ and $\nu \colon \widetilde{\mathcal{X}}_Z \to \mathcal{X}_Z$ for its normalisation.

The family has the following properties if for all large points with $j < m$ and every $Z \in \mathcal{Z}$, with $c = \mathrm{cp}(Z)$:

(Tz0) Attachment There are a unitary Shimura datum $\mathfrak{u} = (E', \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ (the **tower datum** of the point; its level, moduli scheme and coefficient field are its own) and a bijection $\zeta \colon \mathcal{Z} \to \mathrm{Comp}(\mathrm{Sh}_F)$ such that E' is the host field E, E is Galois over $\mathbb{Q}$ with abelian Galois group, $q \geq 1$, $3q + 1 \leq p + q$, $\mathsf{L} = (N, \mathfrak{n}, A)$ is good at every prime with $A = 1$, the components of Sh_F split, at most $4|\Phi_{\mathsf{H}}|(p+q)^2$ primes divide $2\,\mathrm{disc}(E)\,N$, $\mathcal{T}$ has type $(\Phi_{\mathsf{H}}, p+q, p)$, and the realised tower $\mathcal{T}$ has a good origin, immersive at level four, over an integral model $\mathcal{Y}$ of Sh over $\mathcal{O}_F$ with $\mathcal{Y}_{\mathrm{cp}(Z)}$ the closure of $j(\zeta(Z))$ for every $Z \in \mathcal{Z}$;

(Tz1) Identifications We have $D^{\mathrm{fl}} = D$; the weight u satisfies the gap hypothesis with gap g and $b_0(u, g) < B_* + 1$; $\mathrm{fd} = \deg^{\mathrm{fl}}(\mathrm{lr}(u)^D)$, $\mathrm{vo}(Z) = \mathrm{vol}(c)$, $\mathrm{si}(Z) = \widehat{\deg}_{B_*,c}(\mathrm{lr}(u)^{D+1})$, $\mathrm{ns}(Z) = \mathrm{sl}_{B_*,c}(u)$ and $d^{\mathrm{co}}\mathrm{fr}(Z) = \delta_{\mathcal{X}_Z}$;

(Tz2) Top degree We have $\mathrm{tg}(Z) = \mathrm{vol}(c)\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D)$;

(Tz3) Growth comparison There are a real $C \geq 0$, real numbers $s_\iota > 0$ with $s_\iota \to 0$, and integers $q_\iota \geq 1$ with $8 \mid q_\iota$ ($\iota \in \mathbb{N}$) such that $m_\iota = q_\iota\, c(u_{s_\iota}, s_\iota)$ (formed with $k_{\mathcal{X}_Z}$) lies in $\mathbb{Z}^{\mathrm{Rt}^+}$, and such that for all $\iota \in \mathbb{N}$, all integers $\tau \geq 1$ and all real t, writing $\overline{\mathcal{L}}_\iota = \overline{\mathcal{L}}^{m_\iota}(\exp(-1/s_\iota^2))$ on $\mathcal{X}_Z$,

$$\#\Big\{\sigma \in H^0(\widetilde{\mathcal{X}}_Z, \nu^*\mathcal{L}_\iota^{\otimes\tau}) :\ \|\sigma\|_{\sup} \leq e^{\tau q_\iota t}\Big\} \leq \mathrm{mc}_Z(\tau q_\iota,\ \tau q_\iota(t + C s_\iota)),$$

the sup norm being taken for the metric of $\overline{\mathcal{L}}_\iota^{\otimes\tau}$ over $\widetilde{\mathcal{X}}_Z(\mathbb{C})$;

(Tz4) Tower properties The realised tower $\mathcal{T}$ has (Rt1), (Gm1) and (Gm2);

(Tz5) Concrete top degree For a family with (Tz0), we have $u \in \mathbb{Z}^{\mathrm{Rt}}$ and, with ζ the bijection of (Tz0) and $\mathcal{A}^{\mathrm{b}}_{\zeta(Z)}(w)$ the F-vector space of bounded forms of weight w on $\zeta(Z)$ (Definition 29.8),

$$\mathrm{tg}(Z) = \limsup_{k \to \infty} \frac{D!\,\dim_F \mathcal{A}^{\mathrm{b}}_{\zeta(Z)}(ku)}{k^D};$$

(Tz6) Volume slope The number $\mathrm{vs}(Z)$ is the supremum of the set of real a with

$$\mathrm{tg}(Z)(a + t) \leq \mathrm{cvol}_{D,d^{\mathrm{co}}}(\mathrm{cf}_Z; t) \quad \text{for every real } t.$$

Definition 29.10. A family carrying counting data **carries transfer data** if for every real $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$, every lift $j < m$ and every $Z \in \mathcal{Z}$ the following data are given, with the index $(\mathfrak{z}, j)$ suppressed:

- a field Ω_Z (the function field of the model of Z at stage $\lfloor B \rfloor$) and subsets $\mathcal{S}_Z(\tau, R) \subseteq \Omega_Z$ for integers $\tau \geq 0$ and reals $R \geq 0$, the **model-small sets** (the small sections of τ times the divided weight with norm at most R), with $\mathrm{mc}_Z(\tau, \rho) = \#\mathcal{S}_Z(\tau, e^\rho)$;
- for every integer $\tau \geq 1$ a $\mathbb{Q}$-vector space $\mathcal{C}_Z(\tau)$ (the cusp forms of τ times the divided weight), a subset $\mathcal{C}_Z^{\mathrm{int}}(\tau)$ (the integral ones) and a function $\|\cdot\| \colon \mathcal{C}_Z(\tau) \to [0, \infty]$ with $\|Nf\| = N\|f\|$ for integers $N \geq 1$, such that $\mathrm{cf}_Z(\tau, \rho) = \#\{f \in \mathcal{C}_Z^{\mathrm{int}}(\tau) : \|f\| \leq e^\rho\}$;
- for every integer $\tau \geq 1$ an additive map $\mathrm{tr}_{Z,\tau} \colon \Omega_Z \to \mathcal{C}_Z(\tau)$, the **transfer**;
- elements $f_{Z,1}, \ldots, f_{Z,\mathrm{fr}(Z)} \in \Omega_Z$, the **frame**, and a real $R_Z > 0$, the **frame radius**.

The family has the following properties if for all large points with $j < m$, every $Z \in \mathcal{Z}$, every integer $\tau \geq 1$ and all reals $R, R' \geq 0$, writing $k = \mathrm{fw}(Z)$, $d = \mathrm{fr}(Z)$ and $c = c_{E,M}^{\mathrm{HT}}(B)$:

(Tf0) Concrete model-small sets For a family that also carries a tower realisation with (Tz0), with $\mathcal{X}_Z = \mathcal{X}_{\lfloor B \rfloor, \mathrm{cp}(Z)}$ and its root origin given by the good origin of (Tz0): Ω_Z is the function field of $\mathcal{X}_Z$, the divided weight u of the tower realisation is integral ($u \in \mathbb{Z}^{\mathrm{Rt}}$), and $\mathcal{S}_Z(\tau, R) = \mathrm{MS}_{\mathcal{X}_Z}(\tau u, R)$ for all integers $\tau \geq 0$ and reals $R \geq 0$;

(Tf1) Products We have $ss' \in \mathcal{S}_Z(\tau + \tau', RR')$ for $s \in \mathcal{S}_Z(\tau, R)$, $s' \in \mathcal{S}_Z(\tau', R')$, $\tau' \geq 0$;

(Tf2) Frame We have $f_{Z,l} \in \mathcal{S}_Z(k, R_Z)$ for $1 \leq l \leq d$; if $s \in \Omega_Z$ and $\mathrm{tr}_{Z,\tau+k}(s f_{Z,l}) = 0$ for $1 \leq l \leq d$, then $s = 0$; and $\mathrm{ta}(Z) = \sum_{p \in P_{E,M}(B)} \log p + \log(dR_Z)$;

(Tf3) Descent with bounded denominators There is an integer $N_\tau \geq 1$ with

$$\log N_\tau \leq 2\tau c \,\mathrm{wn} + \sum_{p \in P_{E,M}(B)} \log p$$

such that for every $s \in \mathcal{S}_Z(\tau, R)$: $\|\mathrm{tr}_{Z,\tau}(s)\| \leq dR$ and $N_\tau \,\mathrm{tr}_{Z,\tau}(s) \in \mathcal{C}_Z^{\mathrm{int}}(\tau)$;

(Tf4) Concrete transfer For a family with (Tf0), with $c' = \mathrm{cp}(Z)$, $g \colon \Omega_{c'} \to \Omega_Z$ the field embedding and $\deg_{\lfloor B \rfloor, c'}$ the degree of the root origin of $\mathcal{X}_Z$ (Go3): $\mathrm{fr}(Z) = \deg_{\lfloor B \rfloor, c'} = [\Omega_Z : g(\Omega_{c'})]$, and for every integer $\tau \geq 1$ there are an element $\gamma_{Z,\tau} \in \Omega_Z^\times$, the **gauge**, and an additive map $\mathrm{ev}_{Z,\tau} \colon \mathcal{C}_Z(\tau) \to \Omega_{c'}$, the **evaluation on the flag bundle**, such that $g(\mathrm{ev}_{Z,\tau}(\mathrm{tr}_{Z,\tau}(s))) = \mathrm{Tr}_{\Omega_Z / g(\Omega_{c'})}(s/\gamma_{Z,\tau})$ for every $s \in \Omega_Z$ (the gauge is the reading of the Hodge frame in weight τu times $\prod_r \varrho_r^{\tau u_r}$; the evaluation realises a cusp form of weight τu as a function on the flag bundle over the component, in the manner of Borel–Weil: the transfer divides by the gauge, which makes the section a function, and takes the trace of the finite cover down to the flag bundle).

(Tf5) Concrete cusp forms For a family with (Tf0), with $\mathfrak{u}$, N' and ζ the tower datum, its level and the bijection of (Tz0), $\zeta(Z) = Z_{\mathrm{cp}(Z)}$: the primes dividing $2\,\mathrm{disc}(E)\,M$ are those dividing $2\,\mathrm{disc}(E)\,N'$; $\ell^1(u) \leq \mathrm{wn}$; and for every integer $\tau \geq 1$, $\mathcal{C}_Z(\tau) = \mathcal{A}_{\zeta(Z)}(\tau u)$, $\mathcal{C}_Z^{\mathrm{int}}(\tau)$ is the set of its forms integral at B, $\|\cdot\|$ is the Hodge norm, and $\mathrm{tr}_{Z,\tau} = \mathrm{tr}_{\tau u}$ is the transfer from the root origin of $\mathcal{X}_Z$ (forms, integrality at B, Hodge norm and transfer as in Definition 29.8).

Definition 29.11. Let Φ be a non-empty finite set and $n \geq 2$.

Flag scheme. Put $K_{\mathrm{Fl}} = \Phi \times \{1, \dots, n-1\}$ and, for $k = (\tau, j) \in K_{\mathrm{Fl}}$, let I_k be the set of j-element subsets of $\{1, \dots, n\}$; let $\mathbb{P}_{\mathrm{Fl}} = \prod_{k \in K_{\mathrm{Fl}}} \mathbb{P}(\mathbb{Z}^{I_k})$ with coordinates $z_{k,J}$. Let $A = \mathbb{Z}[X_{\tau,a,b} : \tau \in \Phi,\ 1 \leq a, b \leq n]$ and, for $k = (\tau, j)$ and $J \in I_k$, let $\mu_{k,J} \in A$ be the minor of the matrix $(X_{\tau,a,b})_{a,b}$ on the first j rows and the columns J. The **flag scheme** $\mathrm{Fl}_{\Phi,n} \subseteq \mathbb{P}_{\mathrm{Fl}}$ is the closed subscheme defined by the multihomogeneous ideal

$$\mathfrak{p}_{\mathrm{Fl}} = \ker\Big(\mathbb{Z}[z_{k,J}] \to A[T_k : k \in K_{\mathrm{Fl}}], \quad z_{k,J} \mapsto \mu_{k,J}\, T_k\Big). \tag{29.6}$$

Fubini–Study root lines. For a root $r = (\tau, i)$, $\tau \in \Phi$, $1 \leq i \leq n$, let $e_r^{\mathrm{Fl}} \in \mathbb{Z}^{K_{\mathrm{Fl}}}$ have entry 1 at (τ, i) if $i \leq n-1$, entry -1 at $(\tau, i-1)$ if $i \geq 2$, and 0 elsewhere. For $e \in \mathbb{Z}^{K_{\mathrm{Fl}}}$ the **Fubini–Study cone metric** is $h_e^{\mathrm{FS}}(z) = \prod_k \big(\sum_{J \in I_k} |z_{k,J}|^2\big)^{e_k/2}$, and the **Fubini–Study root line** of r is $\overline{\mathcal{X}}_r^{\mathrm{FS}} = (\mathcal{O}(e_r^{\mathrm{Fl}}), h^{\mathrm{FS}}_{e_r^{\mathrm{Fl}}})|_{\mathrm{Fl}_{\Phi,n}}$.

Fubini–Study height. The **Fubini–Study height of the flag scheme** is the linear form $h^{\mathrm{FS}}_{\Phi,n} \colon \mathcal{R} \to \mathbb{R}$ that vanishes on monomials of degree $\neq D+1$ and satisfies, for roots $r_1, \dots, r_{D+1}$,

$$h^{\mathrm{FS}}_{\Phi,n}(x_{r_1} \cdots x_{r_{D+1}}) = (-1)^{D+1}\, \widehat{\deg}\Big(\widehat{c}_1(\overline{\mathcal{X}}^{\mathrm{FS}}_{r_1}) \cdots \widehat{c}_1(\overline{\mathcal{X}}^{\mathrm{FS}}_{r_{D+1}})\Big) \quad \text{on } \mathrm{Fl}_{\Phi,n}. \tag{29.7}$$

A root tower of type (Φ, n, κ) has the **Fubini–Study compact dual** if

(Fs) $h^{\mathrm{cd}} = h^{\mathrm{FS}}_{\Phi,n}$.

Definition 29.12. A family carrying self-intersection data **carries proportionality data** if for every real $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and every lift $j < m$ the following data are given (the index $(\mathfrak{z}, j)$ is suppressed): integers $n_{\mathrm{em}} \geq 1$, the **number of embeddings** (of the CM type of the host), $n_{\mathrm{rk}} \geq 2$, the **rank**, and n_{ct} with $1 \leq n_{\mathrm{ct}} < n_{\mathrm{rk}}$, the **cut**; real numbers $\mathrm{dw}_{a,i}$ for $1 \leq a \leq n_{\mathrm{em}}$ and $1 \leq i \leq n_{\mathrm{rk}}$, the **entries of the divided weight**; and real numbers h^{cd}, the **compact-dual height** of the divided weight, α^{pr} and β^{pr}, the **proportionality constants** at stage $\lfloor B \rfloor$, and ft, the **Frobenius tail**. Put, for $1 \leq a \leq n_{\mathrm{em}}$,

$$s_a = \sum_{i=1}^{n_{\mathrm{ct}}} \mathrm{dw}_{a,i}, \qquad t_a = \sum_{i=1}^{n_{\mathrm{rk}}} \mathrm{dw}_{a,i}. \tag{29.8}$$

The family has the following properties if there are real numbers $C_{\mathrm{b}}, C_{\mathrm{f}} \geq 0$ such that for all large points with $j < m$, writing $D = D^{\mathrm{fl}}$:

(Hp1) Dominant divided weight $\mathrm{dw}_{a,1} \geq \mathrm{dw}_{a,2} \geq \cdots \geq \mathrm{dw}_{a,n_{\mathrm{rk}}}$ for every a, and $\mathrm{wn} = \sum_{a=1}^{n_{\mathrm{em}}} \sum_{i=1}^{n_{\mathrm{rk}}} |\mathrm{dw}_{a,i}|$;

(Hp2) Proportionality identity at the stage

$$\sum_{Z \in \mathcal{Z}} \mathrm{si}(Z) = \Big(\sum_{Z \in \mathcal{Z}} \mathrm{vo}(Z)\Big)\Big(h^{\mathrm{cd}} + (D+1)\,\mathrm{fd} \sum_{a=1}^{n_{\mathrm{em}}} (\alpha^{\mathrm{pr}} s_a + \beta^{\mathrm{pr}} t_a) + \mathrm{ft}\Big);$$

(Hp3) Positivity of the compact-dual height $h^{\mathrm{cd}} \geq 0$;

(Hp4) Lower bound at the base $\alpha^{\mathrm{pr}} \geq -C_{\mathrm{b}}\,\mathrm{lz}$;

(Hp5) Flat direction $\big|n_{\mathrm{ct}}\,\alpha^{\mathrm{pr}} + n_{\mathrm{rk}}\,\beta^{\mathrm{pr}}\big| \leq C_{\mathrm{f}}\, n_{\mathrm{rk}}\,\mathrm{lz}$;

(Hp6) Frobenius tail $\mathrm{ft} \geq -2(D+1)\,\mathrm{fd}\,\mathrm{wsp}$.

Definition 29.13. Let $\mathcal{F}$ be a family carrying a tower realisation and proportionality data. The **proportionality data are realised by the tower**, with constants $C, C' \geq 0$, if the following hold for all large points with $j < m$, where the index $(\mathfrak{z}, j)$ is suppressed and $\mathcal{T}$ is of type (Φ, n, κ).

(Rz1) Identifications We have $n_{\mathrm{em}} = |\Phi|$, and there is an enumeration $a \mapsto \varphi_a$ of Φ $(1 \leq a \leq n_{\mathrm{em}})$ with $\mathrm{dw}_{a,i} = u_{\varphi_a,i}$; $n_{\mathrm{rk}} = n$, $n_{\mathrm{ct}} = \kappa$; $\mathrm{wn} = \|u\|_1$, $\mathrm{wsp} = \min(\kappa, n-\kappa)\,\mathrm{spr}(u)$; $h^{\mathrm{cd}} = h^{\mathrm{cd}}_{\mathcal{T}}(\mathrm{lr}(u)^{D+1})$;

(Rz2) Constants and tail at the stage For every $b \in \Phi$, every $1 \leq k \leq n$ and every $Q \in \mathcal{R}$ homogeneous of degree $D+1-k$, with $F = \mathrm{ps}_{b,k}Q$,

$$\widehat{\deg}_{B_*}(F) = \mathrm{vol}_{\mathcal{T}}\Big(h^{\mathrm{cd}}_{\mathcal{T}}(F) - \alpha^{\mathrm{pr}} \sum_{b' \in \Phi} \deg^{\mathrm{fl}}(\partial^{<}_{b'} F) - \beta^{\mathrm{pr}} \sum_{b' \in \Phi} \deg^{\mathrm{fl}}(\partial_{b'} F) + \rho_{>B_*}(b,k,Q)\Big);$$

and $\mathrm{ft} = \sum_{b\in\Phi}\sum_{k=1}^{n} \rho_{>B_*}(b,k,Q_{b,k})$ for some power-sum decomposition $(Q_{b,k})$ of $\mathrm{lr}(u)^{D+1}$;

(Rz3) Tower properties We have $|\mathrm{Bd}| \leq B_*$, and the tower $\mathcal{T}$ has (Rt1)–(Rt4), and (Rt5)–(Rt7) for $(r, L) = (\mathrm{rd}_E, L)$ with the constants C (in (Rt6)) and C' (in (Rt5)).

Theorem 29.14. *Let C be the constant of Proposition 31.14, with L_{H} as there, put $C' = 4$ and $C_{\mathrm{ms}} = 2C + 3C' + 5 = 2C + 17$. Let $\mathcal{F}$ be a parametrised host family with covolume records carrying chain data, deep-cover data, component data and model-slope data. Suppose that for every real $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ and every lift $j < m$ the following are given and hold (index $(\mathfrak{z}, j)$ suppressed; $E, M, B, L, \mathrm{wn}, \mathrm{wsp}$ are the entries of the model-slope data and $B_* = \lfloor B \rfloor$).*

(Aj1) Tower datum *A unitary Shimura datum $\mathfrak{u} = (E', \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ with $\mathsf{L} = (N, \mathfrak{n}, A)$, and a bijection $\zeta\colon \mathcal{Z} \to \mathrm{Comp}(\mathrm{Sh}_F)$, such that $E' = E$, E is Galois over $\mathbb{Q}$ with abelian Galois group, $q \geq 1$, $3q+1 \leq p+q$, L is good at every prime, $A = 1$, the components of Sh_F split, at most $4|\Phi_{\mathsf{H}}|(p+q)^2$ primes divide $2\,\mathrm{disc}(E)\,N$, and $N = M$ is the level of the model-slope data. Put $\Phi = \Phi_{\mathsf{H}}$, $n = p+q$, $\kappa = p$ and $D = |\Phi|\,n(n-1)/2$; weights are for (Φ, n, κ).*

(Aj2) Divided weight *An integral weight $u \in \mathbb{Z}^{\Phi \times \{1,\dots,n\}}$ and a real $g > 0$ such that u satisfies the gap hypothesis with gap g, $b_0(u,g) < B_* + 1$, $D^{\mathrm{fl}} = D$, $L = L_{\mathsf{H}}$, $\mathrm{wn} = \|u\|_1$ and $\mathrm{wsp} = \min(\kappa, n-\kappa)\,\mathrm{spr}(u)$.*

(Aj3) Cusp forms on every attached tower *For every root tower $\mathcal{T}$ of type (Φ, n, κ) with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B',c})_{B' \geq 2,\, c \in \mathrm{Cp}})$, every integral model $\mathcal{Y}$ of Sh over $\mathcal{O}_F$, every good origin of the realised tower over $\mathcal{Y}$ immersive at level four and every bijection $\mathrm{cp}\colon \mathcal{Z} \to \mathrm{Cp}$ with $\mathcal{Y}_{\mathrm{cp}(Z)}$ the closure of $j(\zeta(Z))$ for every $Z \in \mathcal{Z}$, such that $\mathcal{T}$ has (Fs), (Rt1)–(Rt4), and (Rt5)–(Rt7) for $(r, L) = (\mathrm{rd}_E, L_{\mathsf{H}})$ with the constants C (in (Rt6)) and C' (in (Rt5)), and the realised tower has (Gm1) and (Gm2): there are counting data and transfer data at the point which, with the tower realisation $(\mathcal{T}, \mathrm{cp}, u, g)$, satisfy (Tz2), (Tf0), (Tf3), (Tf4), and for every $Z \in \mathcal{Z}$*

$$d^{\mathrm{co}}\,\mathrm{fr}(Z) = \delta_{\mathcal{X}_{B_*,\mathrm{cp}(Z)}},$$
$$\mathrm{vs}(Z) = \sup\{a \in \mathbb{R} :\ \mathrm{tg}(Z)(a+t) \leq \mathrm{cvol}_{D,d^{\mathrm{co}}}(\mathrm{cf}_Z; t) \text{ for every real } t\}.$$

Then one can choose, at every point, a root tower with a geometric realisation, an integral model, a good origin and a bijection cp *as in (Aj3), such that, if the model slopes are re-chosen as*

$$\mathrm{ns}(Z) = \mathrm{sl}_{B_*,\mathrm{cp}(Z)}(u) \qquad (Z \in \mathcal{Z})$$

and all other entries of the model-slope data are kept, then for all large points with $j < m$

$$\mathrm{vs}(Z) \geq \mathrm{ns}(Z) - 2\,c^{\mathrm{HT}}_{E,M}(B)\,\mathrm{wn} \quad (Z \in \mathcal{Z}), \qquad \frac{1}{|\mathcal{Z}|}\sum_{Z\in\mathcal{Z}} \mathrm{ns}(Z) \geq -C_{\mathrm{ms}}\big(\mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp}\big);$$

that is, $\mathcal{F}$ has (Ms1), and (Ms3) for the constant C_{ms}.

Proof. All choices below are made at each point $(x, i, \mathfrak{z}, j)$ with $x > x_1$ and $j < m$ separately; fix one, with the data of (Aj1) and (Aj2). Since $B \geq 128$ (model-slope data), $B_* \geq 128$. From $q \geq 1$ and $3q+1 \leq p+q$ we get $p \geq 2q+1 \geq 3$, so $1 \leq \kappa < n$ and $n \geq 4$: the Chern-root calculus for (Φ, n, κ) applies.

Step 1: the tower. By (Aj1), with $E' = E$, the datum $\mathfrak{u}$ satisfies every hypothesis of Proposition 31.14. So Corollary 30.5 gives an integral model $\mathcal{Y}$ of Sh over $\mathcal{O}_F$, a root tower $\mathcal{T}$ of type (Φ, n, p) with a geometric realisation and a good origin over $\mathcal{Y}$ immersive at level four, and a bijection $\mathrm{cp}_{\mathfrak{u}}\colon \mathrm{Comp}(\mathrm{Sh}_F) \to \mathrm{Cp}$ with $\mathcal{Y}_{\mathrm{cp}_{\mathfrak{u}}(Z')}$ the closure of $j(Z')$ for every Z', such that $\mathcal{T}$ has (Fs), (Rt1)–(Rt4), and (Rt5)–(Rt7) for $(\mathrm{rd}_E, L_{\mathsf{H}})$ with the constants C (in (Rt6)) and C' (in (Rt5)), and the realised tower has (Gm1), (Gm2). Put $\mathrm{cp} = \mathrm{cp}_{\mathfrak{u}} \circ \zeta\colon \mathcal{Z} \to \mathrm{Cp}$, a bijection with $\mathcal{Y}_{\mathrm{cp}(Z)}$ the closure of $j(\zeta(Z))$. These are the choices of the statement; they satisfy all the conditions on the objects quantified in (Aj3).

Step 2: the bad primes. The good origin contains a tower origin of $\mathcal{T}$ over $\mathcal{Y}$, so by (Go5) Bd is the set of primes dividing $2\,\mathrm{disc}(E)\,N'$, where $N' = \mathrm{lcm}(N, A) = N$ as $A = 1$. By (Aj1), $|\mathrm{Bd}| \leq 4|\Phi|n^2$. By the definition of the threshold, $4|\Phi|n^2 \leq 4|\Phi|n^2(1 + \|u\|_\infty/g) = b_0(u,g) < B_* + 1$ by (Aj2). As $|\mathrm{Bd}|$ and B_* are integers,

$$|\mathrm{Bd}| \leq 4|\Phi|n^2 \leq B_*. \tag{29.9}$$

Step 3: self-intersection data and the model slopes. For $Z \in \mathcal{Z}$ write $c = \mathrm{cp}(Z)$. By (Aj2), u satisfies the gap hypothesis with gap g and $b_0(u,g) < B_* + 1$, and $B_* \geq 2$; so (Gm2)(4) gives $\mathrm{vol}(c)\,\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) > 0$, and (Rt1) gives $\mathrm{vol}(c) > 0$. Hence

$$\mathrm{fd} := \deg^{\mathrm{fl}}\big(\mathrm{lr}(u)^D\big) > 0, \qquad \mathrm{vo}(Z) := \mathrm{vol}(c) > 0, \qquad \mathrm{si}(Z) := \widehat{\deg}_{B_*,c}\big(\mathrm{lr}(u)^{D+1}\big)$$

are self-intersection data at the point, and we set $\mathrm{ns}(Z) := \mathrm{sl}_{B_*,c}(u)$, keeping $E, M, B, L, \mathrm{wn}, \mathrm{wsp}$.

Step 4: the tower realisation, counting and transfer data. The tower realisation at the point is $(\mathcal{T}, \mathrm{cp}, u, g)$ with the geometric realisation of Step 1. By Step 1 the hypotheses of (Aj3) hold for $(\mathcal{T}, \mathcal{Y}, \text{the good origin}, \mathrm{cp})$; take the counting data and transfer data it provides. We check the clauses.

- (Tz0): $\mathfrak{u}$ and ζ of (Aj1) satisfy every requirement of (Tz0) on the datum; $\mathcal{T}$ has type $(\Phi_{\mathsf{H}}, p+q, p)$ and a good origin immersive at level four over $\mathcal{Y}$ with $\mathcal{Y}_{\mathrm{cp}(Z)}$ the closure of $j(\zeta(Z))$ (Step 1).
- (Tz1): $D^{\mathrm{fl}} = D$, the gap hypothesis and $b_0(u,g) < B_* + 1$ by (Aj2); fd, $\mathrm{vo}(Z)$, $\mathrm{si}(Z)$, $\mathrm{ns}(Z)$ are the required expressions by Step 3; $d^{\mathrm{co}}\mathrm{fr}(Z) = \delta_{\mathcal{X}_Z}$ by (Aj3).
- (Tz6): the supremum formula for $\mathrm{vs}(Z)$ by (Aj3).

- (Tz2) by (Aj3); (Tz4): (Rt1), (Gm1), (Gm2) by Step 1.
- (Tz3): the family now carries counting data, self-intersection data, a tower realisation with (Tz0) and transfer data with (Tf0), so Theorem 29.15 gives (Tz3).
- (Tf0), (Tf3), (Tf4) by (Aj3); (Tf1) by Lemma 29.19, from (Tf0).
- (Tf2): by (Tz0), (Tz1), (Tz4), (Tf0), (Tf4), Theorem 29.20 gives, for every Z, an integer $k \geq 1$ with $8 \mid k$, a real $R > 0$ and a frame; we re-choose $\mathrm{fw}(Z) = k$, the frame, $R_Z = R$ and $\mathrm{ta}(Z) = \sum_{p' \in P_{E,M}(B)} \log p' + \log(\mathrm{fr}(Z)R)$, and then (Tf2) holds. None of (Tz0)–(Tz4), (Tz6), (Tf0), (Tf1), (Tf3), (Tf4) involves $\mathrm{fw}(Z)$, $\mathrm{ta}(Z)$, the frame or the frame radius, so they remain true.

Step 5: (Ms1). By (Tz1), (Tz4) and (Tz6), Proposition 29.22 gives (Vc1), (Si1) and (Si2); by (Tz1)–(Tz4), Theorem 29.23 gives (Vc2); by (Tf1)–(Tf3), Theorem 29.27 gives (Vc3). Hence Theorem 29.28 gives $\mathrm{vs}(Z) \geq \mathrm{ns}(Z) - 2\,c^{\mathrm{HT}}_{E,M}(B)\,\mathrm{wn}$ for all large points with $j < m$ and every Z, which is (Ms1).

Step 6: proportionality data realised by the tower. Enumerate Φ as $a \mapsto \varphi_a$ ($1 \leq a \leq |\Phi|$) and put $n_{\mathrm{em}} = |\Phi|$, $n_{\mathrm{rk}} = n \geq 2$, $n_{\mathrm{ct}} = \kappa$ (so $1 \leq n_{\mathrm{ct}} < n_{\mathrm{rk}}$), $\mathrm{dw}_{a,i} = u_{\varphi_a,i}$ and $h^{\mathrm{cd}} = h^{\mathrm{cd}}_{\mathcal{T}}(\mathrm{lr}(u)^{D+1})$. Since $\mathcal{T}$ has (Rt1)–(Rt3) and $B_* \geq 2$, Theorem 29.29 gives the proportionality constants (α, β) of $\mathcal{T}$ at stage B_*; put $\alpha^{\mathrm{pr}} = \alpha$, $\beta^{\mathrm{pr}} = \beta$. The class $\mathrm{lr}(u)^{D+1}$ is homogeneous of degree $D+1$, so by Lemma 29.32 it has a power-sum decomposition $(Q_{b,k})$; by Lemma 29.31(b), applied to the labelling fr and the finite set Bd of $\mathcal{T}$, every tail $\rho_{>B_*}(b,k,Q_{b,k})$ exists, and we put $\mathrm{ft} = \sum_{b\in\Phi}\sum_{k=1}^{n} \rho_{>B_*}(b,k,Q_{b,k})$. We check (Rz1)–(Rz3) with the constants C, C'.

- (Rz1): n_{em}, the enumeration, dw, n_{rk}, n_{ct} and h^{cd} are as required by construction; $\mathrm{wn} = \|u\|_1$ and $\mathrm{wsp} = \min(\kappa, n-\kappa)\mathrm{spr}(u)$ by (Aj2).
- (Rz2): for $b \in \Phi$, $1 \leq k \leq n$ and Q homogeneous of degree $D+1-k$, with $F = \mathrm{ps}_{b,k}Q$, the identity required is (29.14) at $B_0 = B_*$, since there $\mathrm{Sh}(F) = \sum_{b'} \deg^{\mathrm{fl}}(\partial^{<}_{b'}F)$ and $\mathrm{To}(F) = \sum_{b'} \deg^{\mathrm{fl}}(\partial_{b'}F)$; and ft is the tail sum of the decomposition $(Q_{b,k})$ of $\mathrm{lr}(u)^{D+1}$.
- (Rz3): $|\mathrm{Bd}| \leq B_*$ by (29.9); $\mathcal{T}$ has (Rt1)–(Rt4), and (Rt5)–(Rt7) for $(\mathrm{rd}_E, L_{\mathsf{H}}) = (\mathrm{rd}_E, L)$ (as $L = L_{\mathsf{H}}$ by (Aj2)) with C in (Rt6) and C' in (Rt5), by Step 1.

Step 7: (Ms3). The family now carries chain, deep-cover, component, model-slope, counting and self-intersection data, a tower realisation and proportionality data realised by the tower with the constants C, C'. By Theorem 29.33 it has (Hp1)–(Hp6) with $C_{\mathrm{b}} = 2(C+C')+3$ and $C_{\mathrm{f}} = C'$. By Theorem 29.43 it has (Si3) with the constant

$$C_{\mathrm{b}} + C_{\mathrm{f}} + 2 = 2C + 2C' + 3 + C' + 2 = 2C + 3C' + 5 = C_{\mathrm{ms}}.$$

Together with (Si1), (Si2) (Step 5) the self-intersection data have (Si1)–(Si3) for C_{ms}, and Theorem 29.44 gives $\frac{1}{|\mathcal{Z}|}\sum_Z \mathrm{ns}(Z) \geq -C_{\mathrm{ms}}(\mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp})$ for all large points with $j < m$, which is (Ms3) for C_{ms}. □

29.2 Growth comparison, frames and Hilbert–Samuel on the model

This subsection proves (Ms1) (Theorem 29.28) from the properties (Vc1)–(Vc3) of the counting data, and proves these properties from a tower realisation. (Vc1) is definitional (Proposition 29.22). (Vc2), the lower bound for the number of small sections on the model in terms of the model slope, is Theorem 29.23; it rests on Zhang's arithmetic Hilbert–Samuel theorem [276] and

on the inequality between arithmetic volume and self-intersection of Lemma 29.24, with heights in the sense of Burgos Gil, Philippon and Sombra [109] (Lemma 29.25, Lemma 29.26). (Vc3), the transfer from small sections to integral cusp forms, is Theorem 29.27. The clauses of the tower realisation and of the transfer data that neither (Aj1)–(Aj3) nor the choice of the tower supplies are proved here, for every tower with a good origin: the growth comparison (Tz3), by which small regularised sections become model-small after multiplication by the effective weight section (Theorem 29.15); the product clause (Tf1) (Lemma 29.19); and the frames (Tf2), a basis of model-small sections that makes the transfer injective (Theorem 29.20).

Theorem 29.15. *Let $\mathcal{T}$ have a good origin over an integral model, let $B \geq 2$, $c \in \mathrm{Cp}$, $\mathcal{X} = \mathcal{X}_{B,c}$ with function field Ω, and let $u \in \mathbb{Z}^{\mathrm{Rt}}$ be an integral weight. There are a real $C \geq 0$, real numbers $s_\iota > 0$ with $s_\iota \to 0$, integers $q_\iota \geq 1$ with $8 \mid q_\iota$, and $\theta \in \Omega^\times$ ($\iota \in \mathbb{N}$) such that $m_\iota = q_\iota\, c(u_{s_\iota}, s_\iota)$ lies in $\mathbb{Z}^{\mathrm{Rt}^+}$ and, writing $\overline{\mathcal{L}}_\iota = \overline{\mathcal{L}}^{m_\iota}(\exp(-1/s_\iota^2))$, for all $\iota \in \mathbb{N}$, all integers $\tau \geq 1$, all real t and every $\sigma \in H^0(\mathcal{X}, \mathcal{L}_\iota^{\otimes\tau})$ with $\|\sigma\|_{\sup} \leq e^{\tau q_\iota t}$ (metric of $\overline{\mathcal{L}}_\iota^{\otimes\tau}$ on $\mathcal{X}(\mathbb{C})$):*

$$\sigma_\Omega\, \theta^\tau \in \mathrm{MS}_{\mathcal{X}}\big(\tau q_\iota u,\ \exp(\tau q_\iota (t + C s_\iota))\big).$$

Consequently a family carrying counting data, self-intersection data, a tower realisation with (Tz0) and transfer data with (Tf0) has (Tz3).

Proof. Write $k = k_{\mathcal{X}}$, $q' = q'_c$, $q_0 = kq'$, $n(r) = n_c(r) = q_0 \nu_{\mathcal{T}}(r) \in \mathbb{Z}$, $t(s) = \exp(-1/s^2)$, and for a cone point z write $\sigma(z)$ for values (Lemma 29.16).

Fundamental coordinates. For $v \in \mathbb{R}^{\mathrm{Rt}}$ let $F(v) \in \mathbb{R}^{\mathrm{Rt}}$ be $F(v)_{(\tau,i)} = v_{(\tau,i+1)} - v_{(\tau,i)}$ for $i < n$ and $F(v)_{(\tau,n)} = -v_{(\tau,n)}$; F is linear. For positive numbers $h_{(\tau,k)}$ and $H_{(\tau,i)} = \prod_{k \leq i} h_{(\tau,k)}^{-1}$ (with $H_{(\tau,0)} = 1$), $\log h_{(\tau,k)} = \log H_{(\tau,k-1)} - \log H_{(\tau,k)}$, and summation by parts gives

$$\sum_{(\tau,k) \in \mathrm{Rt}} v_{(\tau,k)} \log h_{(\tau,k)} = \sum_{(\tau,i) \in \mathrm{Rt}} F(v)_{(\tau,i)} \log H_{(\tau,i)}. \tag{29.10}$$

Step 1: the effective weight section and the analytic constants. Let ϑ, e_ϑ, H_ϑ be as in Lemma 29.17, with $N\vartheta \in \Gamma_{\mathcal{X}}(e_\vartheta)$, constants $\delta > 0$, C_1, and the form P. Put $\vartheta' = N\vartheta \in \Gamma_{\mathcal{X}}(e_\vartheta)$ and $C_1' = C_1 + \log N$, so that $\log|\vartheta'(z)| - \log H_\vartheta(z) \leq -\delta\, \mathrm{rW}(\pi(x)) + C_1'$ at cone points z over abelian points x. The exponent of $h^{\mathrm{ref}}_{(\tau,i)}$ in H_ϑ is $n(\tau,i) - n(\tau,i+1) = -F(n)_{(\tau,i)} = -q_0 F(\nu_{\mathcal{T}})_{(\tau,i)}$, so

$$\log H_\vartheta = -q' \log h_* - q_0 \sum_a F(\nu_{\mathcal{T}})_a \log h^{\mathrm{ref}}_a. \tag{29.11}$$

Apply Lemma 29.18 to Sys_c with δ/q_0, $C_E = C_1'/q_0$, $A = \sum_a (|F(u)_a| + |F(\nu_{\mathcal{T}})_a|)$ and $n_a = F(\nu_{\mathcal{T}})_a$; let $C' \geq 0$, s_0 be its constants. Choose an integer $N_1 \geq 2$ with $1/N_1 \leq s_0$ and put

$$s_\iota = \frac{1}{\iota + N_1}, \qquad q_\iota = 8q_0(\iota + N_1), \qquad \theta = \vartheta'^{\,8}, \qquad C = C'.$$

Then $s_\iota \to 0$, $8 \mid q_\iota$, $q_\iota s_\iota = 8q_0$, $\theta \neq 0$. The entries of m_ι are $-q_\iota(u_r - s_\iota \nu_{\mathcal{T}}(r)) = -q_\iota u_r + 8n(r) \in \mathbb{Z}$ for $r \in \mathrm{Rt}$ and $q_\iota s_\iota / k = 8q' \in \mathbb{Z}$ for $*$.

Step 2: degrees and integrality. Fix ι, $\tau \geq 1$, t and σ as in the statement; write $s = s_\iota$, $q = q_\iota$, $m = m_\iota$, $e(m) = \sum_{r \in \mathrm{Rt}^+} m_r e_r$. By Lemma 29.16(a), $\sigma_\Omega \in \Gamma_{\mathcal{X}}(\tau e(m))$, and $\theta^\tau = \vartheta'^{8\tau} \in \Gamma_{\mathcal{X}}(8\tau e_\vartheta)$. The multidegree of the product is

$$\tau e(m) + 8\tau e_\vartheta = \tau\Big(\sum_r (-qu_r + 8n(r)) e_r + 8q' e_*\Big) - 8\tau\Big(\sum_r n(r) e_r + q' e_*\Big) = -\tau q \sum_r u_r e_r = e(\tau q u).$$

For every chart index c' with $y_{c'}\neq 0$, $\sigma_\Omega\theta^\tau y_{c'}^{-e(\tau qu)}=(\sigma_\Omega y_{c'}^{-\tau e(m)})(\vartheta' y_{c'}^{-e_\vartheta})^{8\tau}\in A_{c'}$, so condition (a) of (Go6) holds.

Step 3: the pointwise bound. Let z be a cone point with $P(z)\neq 0$, over $x\in\mathcal{X}(\mathbb{C})$. By Lemma 29.17(iii), x is in the abelian locus and $\vartheta'(z)\neq 0$; by (Go4), $y=\pi(x)\in X_c^\circ$. By Lemma 29.16(b),(c), $\sigma_\Omega\theta^\tau$ has at z in degree τqu the unique value $\omega=\sigma(z)\vartheta'(z)^{8\tau}$, and $|\sigma(z)|\le\|\sigma\|_{\sup}h^m(t(s))(z)^\tau\le e^{\tau qt}h^m(t(s))(z)^\tau$. Hence

$$|\omega|\le e^{\tau qt}\big(h^m(t(s))(z)\,|\vartheta'(z)|^8\big)^\tau,$$

and condition (b) of (Go6) with radius $\exp(\tau q(t+Cs))$ and the form P follows once we show

$$\Lambda:=\frac{1}{q}\log h^m(t(s))(z)+\frac{s}{q_0}\log|\vartheta'(z)|+\sum_{r\in\mathrm{Rt}}u_r\log h_r^{\mathrm{sing}}(z)\le Cs, \tag{29.12}$$

because then $h^m(t(s))(z)|\vartheta'(z)|^8=\exp\big(q(\frac{1}{q}\log h^m+\frac{s}{q_0}\log|\vartheta'|)\big)\le e^{qCs}\prod_r h_r^{\mathrm{sing}}(z)^{-qu_r}$ (using $8=qs/q_0$), and raising to the power τ gives $|\omega|\le e^{\tau q(t+Cs)}\prod_r h_r^{\mathrm{sing}}(z)^{-(\tau qu)_r}$.

Step 4: proof of (29.12). Since $s\le s_0$, $t(s)\le t_0$ (Lemma 29.18). Put $c_a=F(-u_s)_a=-F(u)_a+sF(\nu_{\mathcal{T}})_a$; $\sum_a|c_a|\le A$ as $s\le 1$. By $m=q\,c(u_s,s)$, (29.10) and (Go4) (regularised metrics of the fundamental indices),

$$\frac{1}{q}\log h^m(t(s))(z)=\frac{s}{k}\log h_*(z)+\sum_a c_a\big(\log h_a^{\mathrm{ref}}(z)+\varphi_a^T(t(s))(y)\big).$$

By (29.10) with $v=-u$ and (Go4) (the singular weight is the Hodge metric on $\pi^{-1}(X_c^\circ)$),

$$\sum_r u_r\log h_r^{\mathrm{sing}}(z)=-\sum_a F(-u)_a\big(\log h_a^{\mathrm{ref}}(z)+\varphi_a(y)\big),\qquad F(-u)_a=c_a-sF(\nu_{\mathcal{T}})_a.$$

Write $\log|\vartheta'(z)|=\log H_\vartheta(z)+\ell$ with $\ell\le-\delta\,\mathrm{rW}(y)+C_1'$ (Step 1). By (29.11) and $q'/q_0=1/k$, $\frac{s}{q_0}\log H_\vartheta(z)=-\frac{s}{k}\log h_*(z)-s\sum_a F(\nu_{\mathcal{T}})_a\log h_a^{\mathrm{ref}}(z)$. Adding the three contributions, the terms in $\log h_*$ cancel, the coefficient of $\log h_a^{\mathrm{ref}}(z)$ is $c_a-sF(\nu_{\mathcal{T}})_a-F(-u)_a=0$, and

$$\Lambda=\frac{s}{q_0}\ell+\sum_a c_a\big(\varphi_a^T(t(s))(y)-\varphi_a(y)\big)+s\sum_a F(\nu_{\mathcal{T}})_a\varphi_a(y).$$

Since $\frac{s}{q_0}\ell\le s\big(-\frac{\delta}{q_0}\mathrm{rW}(y)+\frac{C_1'}{q_0}\big)$, Lemma 29.18 gives $\Lambda\le C's=Cs$.

So $\sigma_\Omega\theta^\tau\in\mathrm{MS}_{\mathcal{X}}(\tau qu,\exp(\tau q(t+Cs)))$.

The clause (Tz3). Fix a large point with $j<m$ and $Z\in\mathcal{Z}$. By (Tz0) the realised tower has a good origin; by (Tf0), $\mathcal{X}_Z=\mathcal{X}_{B_*,\mathrm{cp}(Z)}$ ($B_*=\lfloor B\rfloor\ge 128$) carries its root origin, the divided weight u is integral and $\mathrm{mc}_Z(\tau',\rho)=\#\mathcal{S}_Z(\tau',e^\rho)=\#\mathrm{MS}_{\mathcal{X}_Z}(\tau' u,e^\rho)$. As $\mathcal{X}_Z$ is normal (Go3)(1), its normalisation is the identity. Take C, s_ι, q_ι, θ from the first part; m_ι is formed with $k_{\mathcal{X}_Z}$. The map $\sigma\mapsto\sigma_\Omega\theta^\tau$ is injective (Lemma 29.16(a) and $\theta\neq 0$) and sends the set on the left of (Tz3) into $\mathrm{MS}_{\mathcal{X}_Z}(\tau q_\iota u,\exp(\tau q_\iota(t+Cs_\iota)))$, whose cardinality is $\mathrm{mc}_Z(\tau q_\iota,\tau q_\iota(t+Cs_\iota))$. This is (Tz3). □

Lemma 29.16. *Let $\mathcal{X}\subseteq\mathbb{P}$ be a closed integral subscheme with function field Ω and representing coordinates $(y_{k,j})$. For a* ***chart index*** *$c\in\prod_k I_k$ put $y_c^e=\prod_k y_{k,c_k}^{e_k}$ ($e\in\mathbb{Z}^K$, when $y_c=\prod_k y_{k,c_k}\neq 0$), $A_c=\mathbb{Z}[y_{k,j}/y_{k,c_k}]\subseteq\Omega$, and $U_c=\mathcal{X}\cap\{Y_{k,c_k}\neq 0\ \forall k\}$. An element $\sigma\in\Omega$ has* ***value*** *$\omega\in\mathbb{C}$ at a point z of the affine cone over $\mathcal{X}(\mathbb{C})$ in multidegree $e\in\mathbb{Z}^K$ if $\sigma=G(y)/H(y)$ with $G,H\in\mathbb{Z}[Y_{k,j}]$ multihomogeneous, $\deg G=\deg H+e$, $H(z)\neq 0$ and $G(z)=\omega H(z)$ (in degree w in the sense of (Go6) is multidegree $e(w)$). Let $e,e'\in\mathbb{Z}^K$.*

(a) For every c with $y_c \neq 0$, U_c is affine with ring of regular functions A_c. There is a unique additive injective map $H^0(\mathcal{X}, \mathcal{O}(e)|_{\mathcal{X}}) \to \Omega$, $\sigma \mapsto \sigma_\Omega$, with $\sigma|_{U_c} = (\sigma_\Omega y_c^{-e})\, Y_c^e|_{U_c}$ for every c with $y_c \neq 0$; its image is $\Gamma_{\mathcal{X}}(e) = \{\sigma \in \Omega :\ \sigma y_c^{-e} \in A_c \text{ for every } c \text{ with } y_c \neq 0\}$, and $(\sigma\sigma')_\Omega = \sigma_\Omega \sigma'_\Omega$ for $\sigma' \in H^0(\mathcal{X}, \mathcal{O}(e')|_{\mathcal{X}})$.

(b) Values in a fixed multidegree are unique when they exist; if $\sigma, \sigma' \in \Omega$ have values ω, ω' at z in multidegrees e, e', then $\sigma\sigma'$ has the value $\omega\omega'$ at z in multidegree $e + e'$. If $\mathcal{X}$ is normal, every $\sigma \in \Omega$ such that σy_c^{-e} is integral over A_c for every c with $y_c \neq 0$ has a value at every point of the affine cone over $\mathcal{X}(\mathbb{C})$ in multidegree e; in particular so does every element of $\Gamma_{\mathcal{X}}(e)$.

(c) For $\sigma \in H^0(\mathcal{X}, \mathcal{O}(e)|_{\mathcal{X}})$, a cone point z and a cone metric h of multidegree e, the value $\sigma(z)$ of σ_Ω at z in multidegree e exists and $\|\sigma\|(x) = |\sigma(z)|/h(z)$ for the point $x \in \mathcal{X}(\mathbb{C})$ under z. In particular $\|\sigma\|_{\sup} = \sup_z |\sigma(z)|/h(z)$, the supremum over the cone points.

Proof. Write $\mathfrak{p} = \{G \in \mathbb{Z}[Y_{k,j}] : G(y) = 0\}$; its multihomogeneous elements are exactly the multihomogeneous polynomials vanishing on $\mathcal{X}$, so the affine cone over $\mathcal{X}(\mathbb{C})$ is the common zero set in $\widehat{\mathbb{C}}$ of the multihomogeneous elements of $\mathfrak{p}$. In particular $Y_c \in \mathfrak{p}$ if and only if $y_c = 0$, and then $U_c = \varnothing$.

(a). Let $y_c \neq 0$. The open set $D_c = \{Y_{k,c_k} \neq 0\ \forall k\}$ of $\mathbb{P}$ is the affine space $\operatorname{Spec} \mathbb{Z}[Y_{k,j}/Y_{k,c_k}]$, and $U_c = \mathcal{X} \cap D_c$ is a non-empty closed integral subscheme of it with generic point that of $\mathcal{X}$. Its ring of regular functions is therefore the image of $\mathbb{Z}[Y_{k,j}/Y_{k,c_k}]$ in the function field Ω; by the defining property of representing coordinates the image of $Y_{k,j}/Y_{k,c_k}$ is $y_{k,j}/y_{k,c_k}$, so this image is A_c. On D_c the sheaf $\mathcal{O}(e)$ is free with basis $Y_c^e = \prod_k Y_{k,c_k}^{e_k}$. Hence for $\sigma \in H^0(\mathcal{X}, \mathcal{O}(e)|_{\mathcal{X}})$ there are unique $a_c \in A_c$ with $\sigma|_{U_c} = a_c Y_c^e|_{U_c}$. On $U_c \cap U_{c'}$ we have $Y_{c'}^e = (Y_{c'}/Y_c)^e\, Y_c^e$, and the regular function $(Y_{c'}/Y_c)^e$ is the element $y_{c'}^e/y_c^e$ of Ω; comparing coefficients of the basis Y_c^e at the generic point gives $a_c = a_{c'}\, y_{c'}^e/y_c^e$, i.e. $a_c y_c^e = a_{c'} y_{c'}^e$. So $\sigma_\Omega := a_c y_c^e$ does not depend on c and is the unique element with the stated property. The map is additive, it is injective because the U_c cover $\mathcal{X}$, and its image lies in $\Gamma_{\mathcal{X}}(e)$. Conversely, for $\sigma \in \Gamma_{\mathcal{X}}(e)$ the local sections $(\sigma y_c^{-e}) Y_c^e$ on the U_c agree on overlaps by the same computation and glue to a section with $\sigma_\Omega = \sigma$. Multiplicativity: $\sigma\sigma'|_{U_c} = a_c a'_c Y_c^{e+e'}$, so $(\sigma\sigma')_\Omega = a_c a'_c y_c^{e+e'} = \sigma_\Omega \sigma'_\Omega$.

(b), uniqueness. If $\sigma = G(y)/H(y) = G_1(y)/H_1(y)$ with $H(z), H_1(z) \neq 0$ and $\deg G - \deg H = \deg G_1 - \deg H_1 = e$, then $GH_1 - G_1 H$ is multihomogeneous of multidegree $\deg G + \deg H_1$ and lies in $\mathfrak{p}$, so it vanishes at z, and $G(z)/H(z) = G_1(z)/H_1(z)$.

(b), products. If $\sigma = G(y)/H(y)$ and $\sigma' = G'(y)/H'(y)$ are representations at z in multidegrees e and e', then $\sigma\sigma' = (GG')(y)/(HH')(y)$ with $(HH')(z) \neq 0$ and $\deg(GG') - \deg(HH') = e + e'$, so $\sigma\sigma'$ has the value $\frac{G(z)}{H(z)} \frac{G'(z)}{H'(z)} = \omega\omega'$.

(b), existence. Let $\mathcal{X}$ be normal, σ as stated and z a cone point. Each factor z_k is a non-zero vector, so there is c with $z_{k,c_k} \neq 0$ for all k; then $Y_c(z) \neq 0$, so $Y_c \notin \mathfrak{p}$ and $y_c \neq 0$. Let x be the scheme point of $\mathcal{X}$ under z; it lies in U_c, with prime ideal $\mathfrak{q} \subseteq A_c$, and $\mathcal{O}_{\mathcal{X},x} = (A_c)_{\mathfrak{q}}$ is integrally closed in Ω because $\mathcal{X}$ is normal. The element $a = \sigma y_c^{-e}$ is integral over A_c, hence lies in $(A_c)_{\mathfrak{q}}$: $a = \alpha/\beta$ with $\alpha, \beta \in A_c$ and $\beta \notin \mathfrak{q}$. Clearing denominators, $\alpha = G_0(y)/y_c^d$ and $\beta = H_0(y)/y_c^{d'}$ with G_0, H_0 multihomogeneous of multidegrees $d, d' \in \mathbb{N}^K$, and $\beta \notin \mathfrak{q}$ means $H_0(z) \neq 0$ (the function β does not vanish at x, and its value at x read at z is $H_0(z)/\prod_k z_{k,c_k}^{d'_k}$). Write $e = e^+ - e^-$ with $e^\pm \in \mathbb{N}^K$, and put $G = G_0 Y_c^{d'+e^+}$, $H = H_0 Y_c^{d+e^-}$. Then $\sigma = G(y)/H(y)$, $\deg G - \deg H = e$ and $H(z) \neq 0$, so σ has a value at z in multidegree e. Elements of $\Gamma_{\mathcal{X}}(e)$ satisfy the hypothesis.

(c). Let $x \in U_c$ lie under z and $\sigma|_{U_c} = a_c Y_c^e$ with $a_c = G_0(y)/y_c^d \in A_c$ as above. By the definition of the metric of a cone function, $\|\sigma\|(x) = |a_c(x)|\,|z_c^e|/h(z)$ with $z_c^e = \prod_k z_{k,c_k}^{e_k}$ and $a_c(x) = G_0(z)/z_c^d$ (both sides are invariant under $z \mapsto (\lambda_k z_k)_k$). On the other hand $\sigma_\Omega = a_c y_c^e = G(y)/H(y)$ with $G = G_0 Y_c^{e^+}$, $H = Y_c^{d+e^-}$, $H(z) \neq 0$, so the value of σ_Ω at z in multidegree e is $G_0(z) z_c^e / z_c^d$, and $\|\sigma\|(x) = |\sigma(z)|/h(z)$. Taking the supremum over x gives the last assertion. $\square$

Lemma 29.17. *Let $\mathcal{T}$ have a good origin over an integral model $\mathcal{Y}$, let $B \geq 2$, $c \in \mathrm{Cp}$, and let $\mathcal{X} = \mathcal{X}_{B,c}$ with its root origin and realisation $\pi\colon \mathcal{X}(\mathbb{C}) \to X_c^{\mathrm{an}}$. Put*

$$\vartheta = g(s_c^E) \prod_{r \in \mathrm{Rt}} \varrho_r^{n_c(r)} \in \Omega^\times, \qquad e_\vartheta = -\sum_{r \in \mathrm{Rt}} n_c(r)\, e_r - q_c'\, e_* \in \mathbb{Z}^K,$$

$$H_\vartheta = h_*^{-q_c'} \prod_{(\tau,i) \in \mathrm{Rt}} \big(h^{\mathrm{ref}}_{(\tau,i)}\big)^{n_c(\tau,i) - n_c(\tau,i+1)} \qquad (n_c(\tau, n+1) = 0),$$

a positive smooth cone function of multidegree e_ϑ. Then:

(i) there is an integer $N \geq 1$ with $N\vartheta \in \Gamma_{\mathcal{X}}(e_\vartheta)$; in particular ϑ has a value $\vartheta(z)$ in multidegree e_ϑ at every point z of the affine cone over $\mathcal{X}(\mathbb{C})$;

(ii) there are real numbers $\delta > 0$ and C_1 such that $\log|\vartheta(z)| - \log H_\vartheta(z) \leq -\delta\,\mathrm{rW}(\pi(x)) + C_1$ for every point x of the abelian locus of $\mathcal{X}(\mathbb{C})$ and every cone point z over x;

(iii) there is a multihomogeneous $P \in \mathbb{Z}[Y_{k,j}]$ with $P(y) \neq 0$ such that every cone point z with $P(z) \neq 0$ lies over the abelian locus and has $\vartheta(z) \neq 0$.

Proof. The multidegree of H_ϑ. The fundamental index (τ, i) has multidegree $-\sum_{k \leq i} e_{(\tau,k)}$, so H_ϑ has multidegree

$$\begin{aligned} &- q_c' e_* - \sum_\tau \sum_i \big(n_c(\tau,i) - n_c(\tau,i+1)\big) \sum_{k \leq i} e_{(\tau,k)} \\ &\quad = -q_c' e_* - \sum_{\tau,k} e_{(\tau,k)} \sum_{i \geq k} \big(n_c(\tau,i) - n_c(\tau,i+1)\big) = e_\vartheta, \end{aligned}$$

the inner sum telescoping to $n_c(\tau,k)$. It is a product of integral powers of positive smooth cone functions, hence one.

Step 1: regularity on the generic fibre. Let c' be a chart index with $y_{c'} \neq 0$ and $x \in U_{c'} \cap \mathcal{X}_{\mathbb{Q}}$. Some coordinate of the factor k_0 does not vanish at x, say $Y_{k_0,i}$; choose a chart index c'' with $c''_{k_0} = i$ and $x \in U_{c''}$. By (Go3)(3) the stalk at x of the root line $\mathcal{L}_r$ is generated by $\varrho_r y_{c''}^{e_r}$, so $\theta = \prod_r (\varrho_r y_{c''}^{e_r})^{-n_c(r)}$ generates the stalk of $\bigotimes_r \mathcal{L}_r^{-n_c(r)}$. By (Go3)(5) $y_{k_0,i} = g(y_i^c)$, and e_* is the unit vector at k_0, so $y_{c''}^{q_c' e_*} = g(y_i^c)^{q_c'}$ and

$$\frac{g(s_c^E)\, g(y_i^c)^{q_c'}}{\theta} = g(s_c^E) \prod_r \varrho_r^{n_c(r)} \cdot y_{c''}^{\sum_r n_c(r) e_r + q_c' e_*} = \vartheta\, y_{c''}^{-e_\vartheta}.$$

By (Go3)(6) this element lies in $\mathcal{O}_{\mathcal{X},x}$, and it is not a unit there if x is not in the abelian locus. Since $x \in U_{c'} \cap U_{c''}$, the element $y_{c''}^{e_\vartheta} / y_{c'}^{e_\vartheta} = (Y_{c''}/Y_{c'})^{e_\vartheta}$ is a unit of $\mathcal{O}_{\mathcal{X},x}$; hence $a_{c'} := \vartheta y_{c'}^{-e_\vartheta} \in \mathcal{O}_{\mathcal{X},x}$, and $a_{c'} \in \mathfrak{m}_x$ if x is not in the abelian locus.

Step 2: (i). By Lemma 29.16(a), $U_{c'}$ is affine with ring $A_{c'}$, so $U_{c'} \cap \mathcal{X}_{\mathbb{Q}}$ is affine with ring $A_{c'} \otimes \mathbb{Q}$, and an element of Ω lying in all local rings of its points lies in $A_{c'} \otimes \mathbb{Q}$. By Step 1, $a_{c'} \in A_{c'} \otimes \mathbb{Q}$, so $N_{c'} a_{c'} \in A_{c'}$ for some integer $N_{c'} \geq 1$. With $N = \prod_{c'} N_{c'}$ over the finitely

many chart indices with $y_{c'} \neq 0$, $N\vartheta \in \Gamma_{\mathcal{X}}(e_\vartheta)$. By Lemma 29.16(b) $N\vartheta$, hence ϑ, has a value in multidegree e_ϑ at every cone point.

Step 3: vanishing off the abelian locus. Let $x \in \mathcal{X}(\mathbb{C})$ be outside the abelian locus, z a cone point over it and $\bar{x} \in \mathcal{X}_{\mathbb{Q}}$ the scheme point under x (also outside the abelian locus, which is defined by scheme points). Choose c' with $\bar{x} \in U_{c'}$. By Step 1, $a_{c'} \in \mathfrak{m}_{\bar{x}}$, so the regular function $a_{c'}$ vanishes at x. By Lemma 29.16(c) (applied to the section $N\vartheta$) $N\vartheta(z) = Na_{c'}(x)\, z_{c'}^{e_\vartheta}$, so $\vartheta(z) = 0$.

Step 4: (iii). Fix c_0 with $y_{c_0} \neq 0$ and write $Na_{c_0} = G_0(y)/y_{c_0}^d$ with G_0 multihomogeneous of multidegree $d \in \mathbb{N}^K$; write $e_\vartheta = e^+ - e^-$ with $e^\pm \in \mathbb{N}^K$, $G = G_0 Y_{c_0}^{e^+}$, $H = NY_{c_0}^{d+e^-}$. Then $\vartheta = G(y)/H(y)$ with $\deg G - \deg H = e_\vartheta$, and $G(y), H(y) \neq 0$ since $\vartheta \neq 0$ and $y_{c_0} \neq 0$. Put $P = GH$; $P(y) \neq 0$. If $P(z) \neq 0$, then $\vartheta(z) = G(z)/H(z) \neq 0$, and by Step 3 the point under z lies in the abelian locus.

Step 5: (ii). The variety $\mathcal{X}_{\mathbb{Q}}$ is smooth (the origin is good) and projective, so $\mathcal{X}(\mathbb{C})$ is a compact complex manifold and π is holomorphic on it (Go4). The function $\psi(x) = |\vartheta(z)|/H_\vartheta(z)$ (z any cone point over x; the quotient is invariant under $z \mapsto (\lambda_k z_k)_k$) is N^{-1} times the norm of the algebraic section $N\vartheta$ of $\mathcal{O}(e_\vartheta)$ for the smooth metric H_ϑ (Lemma 29.16(c)), hence continuous. The section $\pi^*\varsigma$ of the pulled-back hermitian line bundle is holomorphic with $\|\pi^*\varsigma\| = \|\varsigma\| \circ \pi$, and its zero set is $\pi^{-1}(\partial X_c^{\mathrm{an}})$, which by (Go4) is the complement of the abelian locus; by Step 3, ϑ vanishes there. Let $a \in \mathcal{X}(\mathbb{C})$. In local holomorphic coordinates and local holomorphic trivialisations near a, $N\vartheta$ and $\pi^*\varsigma$ are holomorphic functions f_a, g_a with the zero set of g_a contained in that of f_a; by Rückert's Nullstellensatz (see [112]) there are $N_a \geq 1$ and a holomorphic h_a near a with $f_a^{N_a} = g_a h_a$. On a compact neighbourhood V_a of a this gives $|f_a|^{N_a} \leq C_a|g_a|$, and, the trivialising metrics being continuous and positive on V_a, $\psi^{N_a} \leq C_a'\|\pi^*\varsigma\|$ on V_a. Finitely many interiors of the V_a cover $\mathcal{X}(\mathbb{C})$; with $N' = \max N_a$ and $\psi \leq \max\psi < \infty$ we get $\psi^{N'} \leq C'\|\pi^*\varsigma\|$ on $\mathcal{X}(\mathbb{C})$ for a constant $C' > 0$. On the abelian locus $\|\varsigma\| \circ \pi > 0$ and $\mathrm{rW} = e - 2\log\|\varsigma\|$, so

$$\log\psi \leq \frac{\log C'}{N'} + \frac{1}{N'}\log\|\varsigma\| \circ \pi = \frac{\log C'}{N'} + \frac{e - \mathrm{rW} \circ \pi}{2N'},$$

which is (ii) with $\delta = 1/(2N')$ and $C_1 = (\log C' + e/2)/N'$ (for $\vartheta(z) = 0$ the left side is $-\infty$). □

Lemma 29.18. *Let* Sys *be an analytic system whose fundamental indices include the roots* Rt*, and put* $t(s) = \exp(-1/s^2)$*. Let* $\delta > 0$*,* $C_E \in \mathbb{R}$*,* $A \geq 0$ *and* $(n_a)_{a\in\mathrm{Rt}} \in \mathbb{R}^{\mathrm{Rt}}$*. There are real numbers* $C' \geq 0$ *and* $0 < s_0 \leq \frac{1}{2}$ *such that for all* $0 < s \leq s_0$*, all* $(c_a)_{a\in\mathrm{Rt}} \in \mathbb{R}^{\mathrm{Rt}}$ *with* $\sum_a |c_a| \leq A$ *and all* $y \in X^\circ$*:* $t(s) \leq t_0$ *and*

$$s(-\delta\,\mathrm{rW}(y) + C_E) + \sum_{a\in\mathrm{Rt}} c_a(\varphi_a^T(t(s))(y) - \varphi_a(y)) + s\sum_{a\in\mathrm{Rt}} n_a\varphi_a(y) \leq C's.$$

Proof. Put $M = |C_5|\sum_a |n_a|$ and $K = \max(0,\, M\log(2M/\delta) - M)$ if $M > 0$, $K = 0$ if $M = 0$. Let $s_0 = \min(\frac{1}{2},\ (\log(1/t_0))^{-1/2},\ (\delta/(4CA+1))^2)$ and $C' = \max(0,\, C_E + 3A + K)$. Fix $0 < s \leq s_0$, c with $\sum|c_a| \leq A$, $y \in X^\circ$, and write $t = t(s)$, $r = \mathrm{rW}(y)$.

The scale. $\log(1/t) = s^{-2} \geq \log(1/t_0)$, so $t \leq t_0$, and $\log(1/t) \geq 4$. Hence $\varepsilon(t) = Cs^2\log(s^{-2}) = 2Cs^2\log(1/s) \geq 0$. Since $\log x \leq \sqrt{x}$ for $x > 0$, $s\log(1/s) \leq \sqrt{s}$, so $A\,\varepsilon(t) \leq 2CA\,s\sqrt{s} \leq \frac{\delta}{2}s$ by $\sqrt{s} \leq \delta/(4CA+1)$. Since $e^x \geq x$, $t^{1/3} = e^{-1/(3s^2)} \leq 3s^2 \leq 3s$.

The regularisation term. By (Go1)(2), on X°: $\varphi_a \leq \varphi_a^T(t) \leq \varphi_a + t^{1/3} + \varepsilon(t)\mathrm{rW}$. Hence $0 \leq \varphi_a^T(t)(y) - \varphi_a(y) \leq t^{1/3} + \varepsilon(t)r$ and

$$\sum_a c_a(\varphi_a^T(t) - \varphi_a)(y) \leq \sum_a |c_a|(t^{1/3} + \varepsilon(t)r) \leq 3As + \tfrac{\delta}{2}s\,r.$$

The singular term. Since $\|\varsigma\| \le 1$, $r \ge e$ and $\log r \ge 1$. By (Go1)(1), $-C_5 \log r \le \varphi_a(y) \le C_5 \le |C_5| \log r$, so $|\varphi_a(y)| \le |C_5| \log r$ and $s \sum_a n_a \varphi_a(y) \le sM \log r$. The function $\rho \mapsto M \log \rho - \frac{\delta}{2}\rho$ on $(0,\infty)$ is bounded above by K (for $M > 0$ its maximum is at $\rho = 2M/\delta$, with value $M \log(2M/\delta) - M$), so $sM \log r \le s(\frac{\delta}{2} r + K)$.

Sum. Adding, the left-hand side is at most $s(-\delta r + C_E) + 3As + \frac{\delta}{2} sr + \frac{\delta}{2} sr + sK = s(C_E + 3A + K) \le C's$. □

Lemma 29.19. *Let $\mathcal{X}$ carry a root origin, let w, w' be integral weights and $R, R' \ge 0$. Then $\sigma\sigma' \in \mathrm{MS}_{\mathcal{X}}(w + w', RR')$ for all $\sigma \in \mathrm{MS}_{\mathcal{X}}(w, R)$ and $\sigma' \in \mathrm{MS}_{\mathcal{X}}(w', R')$. Consequently a family carrying counting data, a tower realisation and transfer data with (Tf0) has (Tf1).*

Proof. Write Ω for the function field of $\mathcal{X}$ and, for a chart index c' with $y_{c'} = \prod_k y_{k,c'_k} \ne 0$, $A_{c'} = \mathbb{Z}[y_{k,j}/y_{k,c'_k}] \subseteq \Omega$, the coordinate ring of the affine chart $U_{c'} = \{y_{c'} \ne 0\}$ of $\mathcal{X}$. The map $w \mapsto e(w) = -\sum_r w_r e_r$ is additive, so $e(w + w') = e(w) + e(w')$.

Integrality (a). For every c' with $y_{c'} \ne 0$,

$$\sigma\sigma' \prod_k y_{k,c'_k}^{-e(w+w')_k} = \Big(\sigma \prod_k y_{k,c'_k}^{-e(w)_k}\Big)\Big(\sigma' \prod_k y_{k,c'_k}^{-e(w')_k}\Big)$$

is a product of two elements of Ω integral over $A_{c'}$, hence integral over $A_{c'}$, since the integral closure of $A_{c'}$ in Ω is a ring.

Every element satisfying (a) has a value at every cone point. Let σ satisfy (a) in degree w and let z be a point of the affine cone over $\mathcal{X}(\mathbb{C})$. Every factor of z is a non-zero vector, so there is c' with $y_{c'}(z) \ne 0$; then $y_{c'} \notin \mathfrak{p}$ (a multihomogeneous Q with $Q(y) = 0$ is a section of $\mathcal{O}(\deg Q)$ on the integral scheme $\mathcal{X}$ vanishing at the generic point, hence zero, hence $Q(z) = 0$), so $y_{c'} \ne 0$ in Ω. Let x be the scheme point of $\mathcal{X}$ under z; it lies in $U_{c'}$ and $\mathcal{O}_{\mathcal{X},x}$ is the localisation of $A_{c'}$ at the prime $\mathfrak{q}_x$ of x. It is integrally closed because $\mathcal{X}$ is normal (Go3). The element $a = \sigma \prod_k y_{k,c'_k}^{-e(w)_k}$ of its fraction field Ω is integral over $A_{c'} \subseteq \mathcal{O}_{\mathcal{X},x}$, so $a \in \mathcal{O}_{\mathcal{X},x}$: $a = \alpha/\beta$ with $\alpha, \beta \in A_{c'}$, $\beta \notin \mathfrak{q}_x$. Clearing denominators, $\alpha = G_0(y)/\prod_k y_{k,c'_k}^{d_k}$ and $\beta = H_0(y)/\prod_k y_{k,c'_k}^{d'_k}$ with $G_0, H_0 \in \mathbb{Z}[y_{k,j}]$ multihomogeneous of multidegrees d, d', and $\beta \notin \mathfrak{q}_x$ means $H_0(z) \ne 0$. Write $e(w) = e^+ - e^-$ with $e^\pm \in \mathbb{N}^K$ and put $G = G_0 \prod_k y_{k,c'_k}^{d'_k + e^+_k}$, $H = H_0 \prod_k y_{k,c'_k}^{d_k + e^-_k}$. Then $\sigma = a \prod_k y_{k,c'_k}^{e(w)_k} = G(y)/H(y)$, $\deg G = \deg H + e(w)$ and $H(z) \ne 0$, so σ has the value $G(z)/H(z)$ at z in degree w.

Values are unique and multiplicative. If $\sigma = G(y)/H(y) = G_1(y)/H_1(y)$ with $H(z), H_1(z) \ne 0$ and both pairs of degree difference $e(w)$, then $GH_1 - G_1H$ is multihomogeneous with $(GH_1 - G_1H)(y) = 0$, so it vanishes at z (as above), and $G(z)/H(z) = G_1(z)/H_1(z)$. If $\sigma = G(y)/H(y)$ and $\sigma' = G'(y)/H'(y)$ are such representations at z in degrees w and w', then

$$\sigma\sigma' = (GG')(y)/(HH')(y)$$

with $(HH')(z) \ne 0$ and degree difference $e(w) + e(w') = e(w + w')$. Hence the unique value of $\sigma\sigma'$ at z in degree $w + w'$ is the product of the values of σ and σ'.

The bound (b). Let $P, P' \notin \mathfrak{p}$ be the forms of (b) for σ and σ'; PP' is multihomogeneous and $PP' \notin \mathfrak{p}$ because $\mathfrak{p}$ is prime. Let z be a cone point with $(PP')(z) \ne 0$ and ω a value of $\sigma\sigma'$ at z in degree $w + w'$. By the two previous steps $\omega = \omega_1\omega_2$ with ω_1, ω_2 the values of σ, σ' at z, and $P(z) \ne 0 \ne P'(z)$, so

$$|\omega| = |\omega_1|\,|\omega_2| \le R \prod_r h_r^{\mathrm{sing}}(z)^{-w_r} \cdot R' \prod_r h_r^{\mathrm{sing}}(z)^{-w'_r} = RR' \prod_r h_r^{\mathrm{sing}}(z)^{-(w+w')_r},$$

using $h_r^{\mathrm{sing}}(z) > 0$. With (a) this gives $\sigma\sigma' \in \mathrm{MS}_{\mathcal{X}}(w+w', RR')$.

The clause (Tf1). Under (Tf0), for $s \in \mathcal{S}_Z(\tau, R) = \mathrm{MS}_{\mathcal{X}_Z}(\tau w_Z, R)$ and $s' \in \mathcal{S}_Z(\tau', R') = \mathrm{MS}_{\mathcal{X}_Z}(\tau' w_Z, R')$ the first part gives $ss' \in \mathrm{MS}_{\mathcal{X}_Z}((\tau+\tau')w_Z, RR') = \mathcal{S}_Z(\tau+\tau', RR')$, which is (Tf1). □

Theorem 29.20. *Let a family carry counting data, self-intersection data, a tower realisation with (Tz0), (Tz1) and (Tz4), and transfer data with (Tf0) and (Tf4). Then for every large point with $j < m$ and every $Z \in \mathcal{Z}$, with $d = \mathrm{fr}(Z)$, there are an integer $k \geq 1$ with $8 \mid k$, a real $R > 0$ and $f_1, \dots, f_d \in \Omega_Z$ such that $f_l \in \mathcal{S}_Z(k, R)$ for $1 \leq l \leq d$ and, for every integer $\tau \geq 1$ and every $s \in \Omega_Z$, $\mathrm{tr}_{Z,\tau+k}(s f_l) = 0$ for $1 \leq l \leq d$ implies $s = 0$. Consequently, if the frame weight, the frame, the frame radius and the transfer constant are chosen as $\mathrm{fw}(Z) = k$, $f_{Z,l} = f_l$, $R_Z = R$ and $\mathrm{ta}(Z) = \sum_{p \in P_{E,M}(B)} \log p + \log(dR)$, the family has (Tf2).*

Proof. Fix a large point with $j < m$ and $Z \in \mathcal{Z}$; let $B_* = \lfloor B \rfloor$, $c = \mathrm{cp}(Z)$, $\mathcal{X} = \mathcal{X}_Z = \mathcal{X}_{B_*,c}$ with function field $\Omega = \Omega_Z$, $g\colon \Omega_c \to \Omega$ the embedding of its root origin and u the divided weight (integral by (Tf0)). By (Tf4), $d = [\Omega : g(\Omega_c)] \geq 1$.

Step 1: a positive regularised line. Take C, s_ι, q_ι, θ, m_ι from Theorem 29.15 (good origin by (Tz0); $B_* \geq 2$). By (Tz1), u satisfies the gap hypothesis and $b_0(u, \cdot) < B_* + 1$, so by (Gm2) of the realised tower ((Tz4)) at (B_*, c): for all sufficiently small $s > 0$, $c(u_s, s)$ is semipositive at $t(s) = \exp(-1/s^2)$, and $\mathrm{gi}(\mathcal{X}; (\mathrm{lr}(u_s) + s x_*)^D) \to \mathrm{vol}(c)\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) > 0$. Since $s_\iota \to 0$, fix ι such that $s = s_\iota$ has both properties and $\mathrm{gi}(\mathcal{X}; (\mathrm{lr}(u_s) + s x_*)^D) > 0$; write $q = q_\iota$, $m = m_\iota = q\, c(u_s, s)$, $e = \sum_{r \in \mathrm{Rt}^+} m_r e_r$ and $h = h^m(t(s))$. Then $\log h = q \sum_r c(u_s, s)_r \log h_r(t(s))$ has semipositive Levi form on the Zariski tangent vectors of the affine cone. Since $\mathrm{lr}(u_s) + s x_* = \sum_r c(u_s, s)_r k_r x_r$ with $k_r = 1$ for roots and $k_* = k_{\mathcal{X}}$, expanding the D-th power and using the definition of gi (the factors k_* cancel against the normalisation $k_{\mathcal{X}}^{\#\{i : r_i = *\}}$) gives

$$\deg(c_1(\mathcal{O}(e)|_{\mathcal{X}_{\mathbb{Q}}})^D \cap [\mathcal{X}_{\mathbb{Q}}]) = q^D\, \delta_{\mathcal{X}}\, \mathrm{gi}(\mathcal{X}; (\mathrm{lr}(u_s) + s x_*)^D) > 0.$$

Step 2: a Kodaira section. By Lemma 29.21 there are $a \geq 1$ and $0 \neq \varepsilon_0 \in H^0(\mathcal{X}, \mathcal{O}(ae - \mathbf{1})|_{\mathcal{X}})$.

Step 3: a basis of forms. Fix a chart index c_0 with $y_{c_0} \neq 0$. By Lemma 29.16(a), Ω is the fraction field of $A_{c_0} = \mathbb{Z}[y_{k,j}/y_{k,c_{0,k}}]$. The subring $\mathcal{B}$ of Ω generated by $g(\Omega_c)$ and these ratios is a domain of finite dimension over the field $g(\Omega_c)$, hence a field; it contains A_{c_0}, so $\mathcal{B} = \Omega$. Thus the monomials in the ratios span Ω over $g(\Omega_c)$, and some d of them, $b_1, \dots, b_d$, form a basis. Write $b_l = M_l(y)/y_{c_0}^{d_l}$ with M_l a monomial of multidegree $d_l \in \mathbb{N}^K$, let $J \geq 1$ exceed every entry of every d_l, and put $G_l = M_l Y_{c_0}^{J\mathbf{1} - d_l}$, multihomogeneous of multidegree $J\mathbf{1}$, so that $b_l = G_l(y)/y_{c_0}^{J\mathbf{1}}$.

Step 4: sections of a power of the line. Let $\tau_0 = Ja$ and $\sigma_l = G_l|_{\mathcal{X}} \cdot \varepsilon_0^J \in H^0(\mathcal{X}, \mathcal{O}(J\mathbf{1} + J(ae - \mathbf{1}))|_{\mathcal{X}}) = H^0(\mathcal{X}, \mathcal{O}(\tau_0 e)|_{\mathcal{X}})$, sections of $\mathcal{L}_\iota^{\otimes \tau_0}$. By Lemma 29.16(a) (the element of $G_l|_{\mathcal{X}}$ is $G_l(y)$; multiplicativity), $(\sigma_l)_\Omega = b_l\, y_{c_0}^{J\mathbf{1}}\, (\varepsilon_0)_\Omega^J$, with $(\varepsilon_0)_\Omega \neq 0$ by injectivity. So the $(\sigma_l)_\Omega$ are linearly independent over $g(\Omega_c)$. Each $\|\sigma_l\|_{\sup}$ (metric h^{τ_0}) is finite, $\mathcal{X}(\mathbb{C})$ being compact and the norm continuous; choose a real t' with $\|\sigma_l\|_{\sup} \leq e^{\tau_0 q t'}$ for all l.

Step 5: the frame. By Theorem 29.15, $f_l := (\sigma_l)_\Omega\, \theta^{\tau_0} \in \mathrm{MS}_{\mathcal{X}}(\tau_0 q u, \exp(\tau_0 q(t' + Cs)))$. Put $k = \tau_0 q$ ($8 \mid k$ as $8 \mid q$) and $R = \exp(k(t' + Cs)) > 0$; by (Tf0), $f_l \in \mathcal{S}_Z(k, R)$. As $\theta \neq 0$, $f_1, \dots, f_d$ are linearly independent over $g(\Omega_c)$, hence a basis of Ω over $g(\Omega_c)$.

Step 6: injectivity. Let $\tau \geq 1$ and $s \in \Omega$ with $\mathrm{tr}_{Z,\tau+k}(s f_l) = 0$ for all l, and put $x = s/\gamma_{Z,\tau+k}$. By (Tf4) and additivity of $\mathrm{ev}_{Z,\tau+k}$, $\mathrm{Tr}_{\Omega/g(\Omega_c)}(x f_l) = g(\mathrm{ev}_{Z,\tau+k}(0)) = 0$ for all l, hence $\mathrm{Tr}_{\Omega/g(\Omega_c)}(xb) = 0$ for all $b \in \Omega$ by linearity. The extension $\Omega/g(\Omega_c)$ is finite and, in characteristic 0, separable, so its trace form is non-degenerate; thus $x = 0$ and $s = 0$.

Consequence. With $\mathrm{fw}(Z) = k$, $f_{Z,l} = f_l$, $R_Z = R$ and $\mathrm{ta}(Z) = \sum_{p \in P_{E,M}(B)} \log p + \log(dR)$, the three requirements of (Tf2) are the two properties just proved and the definition of $\mathrm{ta}(Z)$. □

Lemma 29.21. *Let $\mathcal{X} \subseteq \mathbb{P} = \prod_{k \in K} \mathbb{P}(\mathbb{Z}^{I_k})$ be a closed integral subscheme, flat over $\mathbb{Z}$, of absolute dimension $D+1$ with $D \geq 1$; let $e \in \mathbb{Z}^K$, let h be a cone metric of multidegree e whose logarithm has semipositive Levi form on every Zariski tangent vector of the affine cone over $\mathcal{X}(\mathbb{C})$ at every point of that cone, and assume $\deg(c_1(\mathcal{O}(e)|_{\mathcal{X}_\mathbb{Q}})^D \cap [\mathcal{X}_\mathbb{Q}]) > 0$. Write $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^K$. Then there are an integer $a \geq 1$ and a non-zero $\varepsilon_0 \in H^0(\mathcal{X}, \mathcal{O}(ae - \mathbf{1})|_\mathcal{X})$.*

Proof. Write $\mathcal{L} = \mathcal{O}(e)|_\mathcal{X}$ and $\mathcal{H} = \mathcal{O}(\mathbf{1})|_\mathcal{X}$. The generic fibre $\mathcal{X}_\mathbb{Q}$ is a non-empty (flatness) integral projective variety of dimension D.

Nefness on the generic fibre. Let $f\colon \Delta \to \mathcal{X}(\mathbb{C})$ be holomorphic on a disc and $w_0 \in \Delta$. Choose $i_k \in I_k$ with $z_{k,i_k}(f(w_0)) \neq 0$; near w_0, f lifts to a holomorphic $\tilde{f}$ into the affine cone with $\tilde{f}(w)_{k,i_k} = 1$, and the rational section $l = \prod_k z_{k,i_k}^{e_k}$ is invertible along f with $\log \|l\| \circ f = -\log h \circ \tilde{f}$. So the curvature form of $f^*(\mathcal{L}, h)$ is $\frac{i}{\pi}\partial\bar{\partial}(\log h \circ \tilde{f})$, whose value on v is $\frac{1}{\pi}$ times the Levi form of $\log h$ at $\tilde{f}(w)$ on $d\tilde{f}(w)v$. Every G in the ideal of $\mathcal{X}$ satisfies $G \circ \tilde{f} = 0$, so $d\tilde{f}(w)v$ is a Zariski tangent vector of the cone, and the form is ≥ 0. For an integral closed curve $\Gamma \subseteq \mathcal{X}_\mathbb{Q}$, $\deg(\mathcal{L}|_\Gamma)$ is the sum over the irreducible components of $\Gamma(\mathbb{C})$ of the integrals of this curvature form over their normalisations, hence ≥ 0. So $\mathcal{L}_\mathbb{Q}$ is nef.

Bigness and Kodaira's lemma. A nef line bundle on an irreducible projective variety with positive top self-intersection is big (Lazarsfeld, Positivity in Algebraic Geometry I, Theorem 2.2.16). Since $\mathcal{H}_\mathbb{Q}$ has a non-zero section (a product over k of coordinates Y_{k,j_k} not vanishing identically on $\mathcal{X}_\mathbb{Q}$), it is $\mathcal{O}(F)$ for an effective Cartier divisor F, and Kodaira's lemma (op. cit., Proposition 2.2.6) gives $a \geq 1$ with $H^0(\mathcal{X}_\mathbb{Q}, \mathcal{L}_\mathbb{Q}^{\otimes a} \otimes \mathcal{H}_\mathbb{Q}^{-1}) \neq 0$.

Descent to $\mathcal{X}$. As $\mathbb{Z} \to \mathbb{Q}$ is flat and $\mathcal{X}$ is quasi-compact and separated, $H^0(\mathcal{X}_\mathbb{Q}, \mathcal{M}_\mathbb{Q}) = H^0(\mathcal{X}, \mathcal{M}) \otimes \mathbb{Q}$ for $\mathcal{M} = \mathcal{L}^{\otimes a} \otimes \mathcal{H}^{-1} = \mathcal{O}(ae - \mathbf{1})|_\mathcal{X}$. A non-zero section over $\mathbb{Q}$ is ε/N with $\varepsilon \in H^0(\mathcal{X}, \mathcal{M})$ and an integer $N \geq 1$, so $\varepsilon_0 = \varepsilon \neq 0$. □

Proposition 29.22. *A family carrying counting data, self-intersection data and a tower realisation with (Tz1), (Tz4) and (Tz6) has (Vc1), (Si1) and (Si2), and $\mathrm{vo}(Z) > 0$ and $\mathrm{fd} > 0$ for all large points with $j < m$ and every $Z \in \mathcal{Z}$.*

Proof. Fix a large point, a lift $j < m$ and $Z \in \mathcal{Z}$, and let $c = \mathrm{cp}(Z)$, $D = D^{\mathrm{fl}}$, $B_* = \lfloor B \rfloor \geq 128$.

Positivity. By (Tz1) and (Rt1) (part of (Tz4)), $\mathrm{vo}(Z) = \mathrm{vol}(c) > 0$. By (Tz1), u satisfies the gap hypothesis with gap g and $b_0(u, g) < B_* + 1$, so (Gm2)(4) gives $\mathrm{vol}(c)\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) > 0$; dividing by $\mathrm{vol}(c) > 0$, $\mathrm{fd} = \deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) > 0$.

(Si1). By (Tz1), $\mathrm{ns}(Z) = \mathrm{sl}_{B_*,c}(u)$, and since the denominator $(D+1)\mathrm{vol}(c)\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) = (D+1)\mathrm{vo}(Z)\mathrm{fd}$ is positive,

$$\mathrm{ns}(Z) = \frac{\widehat{\deg}_{B_*,c}(\mathrm{lr}(u)^{D+1})}{(D+1)\,\mathrm{vo}(Z)\,\mathrm{fd}} = \frac{\mathrm{si}(Z)}{(D+1)\,\mathrm{vo}(Z)\,\mathrm{fd}}.$$

(Si2). For $Z, Z' \in \mathcal{Z}$, $\mathrm{vo}(Z) = \mathrm{vol}(\mathrm{cp}(Z)) = \mathrm{vol}(\mathrm{cp}(Z')) = \mathrm{vo}(Z')$ by (Tz1) and (Gm1).

(Vc1). Let a be real with $\mathrm{tg}(Z)(a + t) \leq \mathrm{cvol}_{D,d^{\mathrm{co}}}(\mathrm{cf}_Z; t)$ for every real t. Then a belongs to the set whose supremum is $\mathrm{vs}(Z)$ by (Tz6); as this supremum is the real number $\mathrm{vs}(Z)$, $a \leq \mathrm{vs}(Z)$. □

Theorem 29.23. *A family carrying counting data, self-intersection data and a tower realisation with the properties (Tz1)–(Tz4) has the property (Vc2): for all large points with $j < m$, every*

$Z \in \mathcal{Z}$ and every real t,

$$\mathrm{tg}(Z)\big(\mathrm{ns}(Z)+t\big) \le \mathrm{cvol}_{D^{\mathrm{fl}},\, d^{\mathrm{co}}\mathrm{fr}(Z)}(\mathrm{mc}_Z;t).$$

Proof. Fix a large point, a lift $j<m$ and $Z\in\mathcal{Z}$; let $c=\mathrm{cp}(Z)$, $\mathcal{X}=\mathcal{X}_Z=\mathcal{X}_{B_*,c}$, $\delta=\delta_{\mathcal{X}}$, $D=D^{\mathrm{fl}}$, and fix $t\in\mathbb{R}$. By (Tz1), D is the flag dimension of the tower, $d^{\mathrm{co}}\mathrm{fr}(Z)=\delta$, u satisfies the gap hypothesis with gap g and $b_0(u,g)<B_*+1$; since $B\ge 128$, $B_*\ge 2$. Put $A=\mathrm{vol}(c)\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D)$ and $H=\widehat{\deg}_{B_*,c}(\mathrm{lr}(u)^{D+1})$. By (Tz4), the tower has (Gm2), and (Gm2)(4) at (u,g,B_*,c) gives $A>0$. Hence, by (Tz1) and (Tz2), $\mathrm{tg}(Z)=A$ and $\mathrm{ns}(Z)=\mathrm{sl}_{B_*,c}(u)=H/((D+1)A)$, so the assertion reads $H/(D+1)+tA\le\mathrm{cvol}_{D,\delta}(\mathrm{mc}_Z;t)$. If the right-hand side is $+\infty$ there is nothing to prove; assume it is finite.

The approximating bundles. Take C, s_ι, q_ι and $m_\iota=q_\iota c_\iota$, $c_\iota=c(u_{s_\iota},s_\iota)$, from (Tz3), and let $\overline{\mathcal{L}}_\iota=\overline{\mathcal{L}}^{m_\iota}(t_\iota)$ with $t_\iota=\exp(-1/s_\iota^2)$. Put $a_\iota=\mathrm{ai}_{t_\iota}(\mathcal{X};(\mathrm{lr}(u_{s_\iota})+s_\iota x_*)^{D+1})$ and $g_\iota=\mathrm{gi}(\mathcal{X};(\mathrm{lr}(u_{s_\iota})+s_\iota x_*)^D)$. Since $s_\iota\to 0$ with $s_\iota>0$, (Gm2)(2), (3) give $a_\iota\to H$ and $g_\iota\to A$, and (Gm2)(1) gives ι_0 such that for $\iota\ge\iota_0$ the vector c_ι is nef on fibres of $\mathcal{X}$ and semipositive at t_ι, and $g_\iota>0$.

Numbers of the approximating bundles. Let $k_r=1$ for $r\in\mathrm{Rt}$ and $k_*=k_{\mathcal{X}}$; by Definition 29.4, $\sum_r m_{\iota,r}k_r x_r=q_\iota(\mathrm{lr}(u_{s_\iota})+s_\iota x_*)$. Expanding $\widehat{c}_1(\overline{\mathcal{L}}_\iota)=\sum_r m_{\iota,r}\widehat{c}_1(\overline{\mathcal{L}}_r(t_\iota))$ multilinearly and comparing with (29.2) term by term (each factor $\overline{\mathcal{L}}_*$ carries $k_{\mathcal{X}}$ in the polynomial and $k_{\mathcal{X}}^{-1}$ in (29.2)),

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_\iota)^{D+1})=\delta\, q_\iota^{D+1}a_\iota,\qquad \deg(c_1(\mathcal{L}_{\iota,\mathbb{Q}})^D\cap[\mathcal{X}_{\mathbb{Q}}])=\delta\, q_\iota^D g_\iota.$$

For real κ', $\overline{\mathcal{L}}_\iota(\kappa')=\overline{\mathcal{L}}_\iota\otimes\overline{\mathcal{O}}(\kappa')$ with $\overline{\mathcal{O}}(\kappa')$ the trivial bundle with $\|1\|=e^{-\kappa'}$, and

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{O}}(\kappa'))\,\widehat{c}_1(\overline{\mathcal{M}}_1)\cdots\widehat{c}_1(\overline{\mathcal{M}}_D))=\kappa'\deg(c_1(\mathcal{M}_{1,\mathbb{Q}})\cdots c_1(\mathcal{M}_{D,\mathbb{Q}})),$$

while products containing $\widehat{c}_1(\overline{\mathcal{O}}(\kappa'))$ twice vanish (Gillet–Soulé); hence

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_\iota(\kappa'))^{D+1})=\delta q_\iota^{D+1}a_\iota+(D+1)\kappa'\,\delta q_\iota^D g_\iota.$$

Hilbert–Samuel for $\overline{\mathcal{L}}_\iota(\kappa_\iota)$. Let $\iota\ge\iota_0$ and $\kappa_\iota=q_\iota(t-Cs_\iota)$. The cone function of $\overline{\mathcal{L}}_\iota(\kappa_\iota)$ is $e^{\kappa_\iota}\prod_r h_r(t_\iota)^{q_\iota c_{\iota,r}}$, whose logarithm is $\kappa_\iota+q_\iota\varphi_{c_\iota,t_\iota}$; its Levi form on Zariski tangent vectors of the cone over $\mathcal{X}(\mathbb{C})$ is $q_\iota\ge 1$ times that of $\varphi_{c_\iota,t_\iota}$, hence semipositive; its degree on an integral curve W of a closed fibre is $q_\iota\sum_r c_{\iota,r}\deg(\mathcal{O}(e_r)|_W)\ge 0$; and its generic degree $\delta q_\iota^D g_\iota$ is positive. So Lemma 29.24 applies (with $D\ge 1$, the flag dimension of $n\ge 2$):

$$\delta q_\iota^{D+1}a_\iota+(D+1)\kappa_\iota\delta q_\iota^D g_\iota\le\limsup_{\tau\to\infty}\frac{(D+1)!}{\tau^{D+1}}\log n_\iota(\tau),$$

where $n_\iota(\tau)$ is the number of $\sigma\in H^0(\widetilde{\mathcal{X}},\nu^*\mathcal{L}_\iota^{\otimes\tau})$ whose sup norm for $\overline{\mathcal{L}}_\iota(\kappa_\iota)^{\otimes\tau}$ is ≤ 1, i.e. whose sup norm for $\overline{\mathcal{L}}_\iota^{\otimes\tau}$ is $\le e^{\tau\kappa_\iota}=e^{\tau q_\iota(t-Cs_\iota)}$. By (Tz3) at $t-Cs_\iota$, $n_\iota(\tau)\le\mathrm{mc}_Z(\tau q_\iota,\tau q_\iota t)$.

Comparison with the counting volume. Since $8\mid q_\iota$, $k=\tau q_\iota/8$ is a positive integer, $(8k,8kt)=(\tau q_\iota,\tau q_\iota t)$, and $n_\iota(\tau)\ge 1$ (the zero section), so $\log n_\iota(\tau)\le\log^+\mathrm{mc}_Z(8k,8kt)$ and

$$\frac{(D+1)!}{\tau^{D+1}}\log n_\iota(\tau)\le(D+1)\,\delta\, q_\iota^{D+1}\cdot\frac{D!\,\log^+\mathrm{mc}_Z(8k,8kt)}{\delta\,(8k)^{D+1}}.$$

As $\tau\to\infty$, $k\to\infty$ through integers, so the lim sup over τ of the left side is at most $(D+1)\delta q_\iota^{D+1}\mathrm{cvol}_{D,\delta}(\mathrm{mc}_Z;t)$. Dividing by $(D+1)\delta q_\iota^{D+1}>0$:

$$\frac{a_\iota}{D+1}+(t-Cs_\iota)\,g_\iota\le\mathrm{cvol}_{D,\delta}(\mathrm{mc}_Z;t)\qquad(\iota\ge\iota_0).$$

Letting $\iota \to \infty$ ($a_\iota \to H$, $g_\iota \to A$, $s_\iota \to 0$) gives $H/(D+1) + tA \le \mathrm{cvol}_{D,\delta}(\mathrm{mc}_Z; t)$, which is the assertion. □

Lemma 29.24. *Let $\mathcal{X} \subseteq \mathbb{P} = \prod_{k\in K} \mathbb{P}(\mathbb{Z}^{I_k})$ be a closed integral subscheme, flat over $\mathbb{Z}$, of absolute dimension $D+1$ with $D \ge 1$, and let $\nu\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ be its normalisation. Let $e \in \mathbb{Z}^K$, let h be a cone metric of multidegree e and $\overline{\mathcal{L}} = (\mathcal{O}(e), h)|_{\mathcal{X}}$. Assume:*

1. $\deg(\mathcal{O}(e)|_W) \ge 0$ *for every prime p and every integral closed curve $W \subseteq \mathcal{X}_{\mathbb{F}_p}$;*
2. *the Levi form of $\log h$ is semipositive on every Zariski tangent vector of the affine cone over $\mathcal{X}(\mathbb{C})$ at every point of that cone in $\widehat{\mathbb{C}}$;*
3. $\deg(c_1(\mathcal{L}_\mathbb{Q})^D \cap [\mathcal{X}_\mathbb{Q}]) > 0$.

Then

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{D+1}) \le \limsup_{\tau\to\infty} \frac{(D+1)!}{\tau^{D+1}} \log \#\Big\{\sigma \in H^0(\widetilde{\mathcal{X}}, \nu^*\mathcal{L}^{\otimes\tau}) :\ \|\sigma\|_{\sup} \le 1\Big\}.$$

Proof. Step 0: the numbers. Let $\overline{\mathcal{M}}_1, \dots, \overline{\mathcal{M}}_{D+1}$ be Hermitian line bundles on $\mathcal{X}$ which are restrictions of Hermitian line bundles $\overline{\mathcal{M}}_l^{\mathbb{P}}$ with smooth metrics on $\mathbb{P}$; every bundle below is of this kind ($\overline{\mathcal{L}}$, $\overline{\mathcal{H}}$, their tensor powers and products, and their twists by constants, by Steps 1 and 2). The number $\widehat{\deg}(\prod_l \widehat{c}_1(\overline{\mathcal{M}}_l))$ of Theorem B.130 is $\widehat{\deg}(\prod_l \widehat{c}_1(\rho^*\overline{\mathcal{M}}_l))$ on $\mathcal{X}'$, and the metrics of the $\rho^*\overline{\mathcal{M}}_l$ are smooth (pull-backs of smooth metrics on $\mathbb{P}(\mathbb{C})$ by a holomorphic map). Then

$$\widehat{\deg}\Big(\prod_{l=1}^{D+1} \widehat{c}_1(\rho^*\overline{\mathcal{M}}_l)\Big) = h_{\rho^*\overline{\mathcal{M}}_1,\dots,\rho^*\overline{\mathcal{M}}_{D+1}}(\mathcal{X}') = h_{\overline{\mathcal{M}}_1,\dots,\overline{\mathcal{M}}_{D+1}}(\mathcal{X}) = \widehat{\deg}\Big(\prod_{l=1}^{D+1} \widehat{c}_1(\overline{\mathcal{M}}_l^{\mathbb{P}}) \,\Big|\, \mathcal{X}\Big).$$

The first equality is (H) of Lemma 29.25 on $\mathcal{X}'$. The second is Proposition B.129 for $\pi = \rho$ with $d = 0$ and $b = D+1$ (the $\overline{\mathcal{M}}_l$ are DSP by Lemma 29.26): the fibre of ρ over the generic point of $\mathcal{X}$ is $\operatorname{Spec}\mathbf{k}(\mathcal{X}')$ (ρ is birational between integral schemes, so its generic fibre consists of the generic point of $\mathcal{X}'$ with local ring $\mathbf{k}(\mathcal{X}')$), of degree $[\mathbf{k}(\mathcal{X}') : \mathbf{k}(\mathcal{X})] = 1$. The third is Lemma 29.26 for the cycle $\mathcal{X}$ of the regular arithmetic variety $\mathbb{P}$. So the numbers used below are the heights of the cycle $\mathcal{X}$ of $\mathbb{P}$, which is the meaning of the left side of the assertion; they are multilinear and symmetric in the $\overline{\mathcal{M}}_l$ (intersection numbers on $\mathcal{X}'$).

Write $V(\overline{\mathcal{M}}, N) = \#\{\sigma \in H^0(\mathcal{X}, \mathcal{M}^{\otimes N}) : \|\sigma\|_{\sup} \le 1\}$ for a Hermitian line bundle $\overline{\mathcal{M}}$ on $\mathcal{X}$, and $\widetilde{V}(N)$ for the count on the right-hand side at $\tau = N$. All these sets are finite (a lattice meets a compact set in finitely many points).

Step 1: relative semipositivity. Let $\overline{\mathcal{M}} = (\mathcal{O}(e'), h')|_{\mathcal{X}}$ for a cone metric h' satisfying (2) (with h' in place of h) and (1) for e'. Its metric is the restriction of a smooth metric on $\mathbb{P}(\mathbb{C})$, hence smooth in Zhang's sense. Let $f\colon \Delta \to \mathcal{X}(\mathbb{C})$ be holomorphic on a polydisc and $w_0 \in \Delta$. Choose for every k an index $i_k \in I_k$ with $z_{k,i_k}(f(w_0)) \ne 0$; on a neighbourhood U of w_0, f lifts to the holomorphic map $\tilde{f}\colon U \to \widehat{\mathbb{C}}$ into the affine cone over $\mathcal{X}(\mathbb{C})$ with $\tilde{f}(w)_{k,i_k} = 1$, and the rational section $l = \prod_k z_{k,i_k}^{e'_k}$ of $\mathcal{O}(e')$ is invertible along $f(U)$ with $\log\|l\|(f(w)) = -\log h'(\tilde{f}(w))$. Hence the curvature form of $f^*\overline{\mathcal{M}}$ on U is $\frac{\partial\bar\partial}{\pi i}\log\|l\| = \frac{i}{\pi}\partial\bar\partial(\log h' \circ \tilde{f})$, and since $\tilde{f}$ is holomorphic its value on a vector v is $\frac{1}{\pi}$ times the Levi form of $\log h'$ at $\tilde{f}(w)$ on $d\tilde{f}(w)v$. For every F in the ideal of $\mathcal{X}$, $F\circ\tilde{f} \equiv 0$, so $dF(\tilde{f}(w))\, d\tilde{f}(w)v = 0$: $d\tilde{f}(w)v$ is a Zariski tangent vector of the cone, and the form is semipositive by (2). Thus (ii) of Theorem B.130 holds. For an integral closed curve C in the generic fibre, $\deg(\mathcal{M}|_C)$ is the integral over the normalisation of $C(\mathbb{C})$ of the pulled-back curvature form, which is ≥ 0 by the same computation; with (1), condition (i) holds. So $\overline{\mathcal{M}}$ is

relatively semipositive, and so is $\overline{\mathcal{M}}(\kappa')$ for every real κ' (same curvature, same degrees). Since sums of functions with semipositive Levi forms and sums of nonnegative degrees stay so, tensor products of such bundles are again of this kind. In particular $\overline{\mathcal{L}}$ is relatively semipositive and $\mathcal{L}_{\mathbb{Q}}$ is nef.

Step 2: a comparison bundle. Let $\overline{\mathcal{H}} = (\mathcal{O}(1,\dots,1), h_{\mathrm{FS}})|_{\mathcal{X}}$ with $h_{\mathrm{FS}}(z) = \prod_k |z_k|$ (Euclidean norms). Each $\log|z_k|$ is plurisubharmonic, so h_{FS} satisfies (2); $\mathcal{H}$ has positive degree on every curve in a closed fibre and $\mathcal{H}_{\mathbb{Q}}$ is ample. By (3) and nefness, $\mathcal{L}_{\mathbb{Q}}$ is big (a nef line bundle on an irreducible projective variety with positive top self-intersection is big: Lazarsfeld, Positivity I, Theorem 2.2.16), so by Kodaira's lemma (op. cit., Proposition 2.2.6) there is $a \geq 1$ with $H^0(\mathcal{X}_{\mathbb{Q}}, \mathcal{L}_{\mathbb{Q}}^{\otimes a} \otimes \mathcal{H}_{\mathbb{Q}}^{-1}) \neq 0$. As $\mathcal{X}$ is flat and projective over $\mathbb{Z}$, $H^0(\mathcal{X}_{\mathbb{Q}}, \cdot) = H^0(\mathcal{X}, \cdot) \otimes \mathbb{Q}$, so there is a non-zero $\varepsilon_0 \in H^0(\mathcal{X}, \mathcal{L}^{\otimes a} \otimes \mathcal{H}^{-1})$. Put $c_0 = \max\bigl(0, \log \|\varepsilon_0\|_{\sup}\bigr)$, the sup norm for $\overline{\mathcal{L}}^{\otimes a} \otimes \overline{\mathcal{H}}^{-1}$.

Step 3: injecting small sections. For an integer $m \geq 1$ let $\overline{\mathcal{M}}_m = (\overline{\mathcal{L}}^{\otimes m} \otimes \overline{\mathcal{H}})(-c_0)$. If $\sigma \in H^0(\mathcal{X}, \mathcal{M}_m^{\otimes N})$ has $\|\sigma\|_{\sup} \leq 1$, then its norm for $\overline{\mathcal{L}}^{\otimes mN} \otimes \overline{\mathcal{H}}^{\otimes N}$ is at most e^{-Nc_0}, so $\sigma\varepsilon_0^N \in H^0(\mathcal{X}, \mathcal{L}^{\otimes (m+a)N})$ has sup norm at most $e^{-Nc_0}\|\varepsilon_0\|_{\sup}^N \leq 1$. Multiplication by ε_0^N is injective ($\mathcal{X}$ is integral), and pull-back by ν is injective on sections and preserves sup norms ($\widetilde{\mathcal{X}}(\mathbb{C}) \to \mathcal{X}(\mathbb{C})$ is surjective). Hence

$$\widetilde{V}\bigl((m+a)N\bigr) \geq V(\overline{\mathcal{M}}_m, N) \qquad (N \geq 1).$$

Step 4: Zhang's theorem. $\mathcal{M}_{m,\mathbb{Q}}$ is ample (nef plus ample) and $\overline{\mathcal{M}}_m$ is relatively semipositive by Step 1, so Theorem B.130 (with $\overline{\mathcal{N}}$ trivial) gives $\chi_{\sup}(\Gamma_N) = \frac{N^{D+1}}{(D+1)!}\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_m)^{D+1}) + o(N^{D+1})$ for $\Gamma_N = H^0(\mathcal{X}, \mathcal{M}_m^{\otimes N})$, of rank $r_N = \dim H^0(\mathcal{X}_{\mathbb{Q}}, \mathcal{M}_{m,\mathbb{Q}}^{\otimes N}) = O(N^D)$ (Hilbert polynomial of an ample bundle on a D-dimensional variety). By van der Corput's lattice-point theorem, a compact convex symmetric body of volume vol in an r-dimensional space contains at least vol/(2^rcovol) points of a lattice of covolume covol; for the unit ball of the sup norm (volume 1) this gives $\log V(\overline{\mathcal{M}}_m, N) \geq \chi_{\sup}(\Gamma_N) - r_N \log 2$. Therefore

$$\log V(\overline{\mathcal{M}}_m, N) \geq \frac{N^{D+1}}{(D+1)!}\,\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_m)^{D+1}) + o(N^{D+1}).$$

Step 5: the limit. Let $V^\sharp$ be the right-hand side of the assertion. Along $\tau = (m+a)N$, Steps 3 and 4 give

$$V^\sharp \geq \limsup_{N\to\infty} \frac{(D+1)!}{((m+a)N)^{D+1}} \log V(\overline{\mathcal{M}}_m, N) \geq \frac{\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_m)^{D+1})}{(m+a)^{D+1}}.$$

By multilinearity and symmetry, with $\overline{\mathcal{H}}' = \overline{\mathcal{H}}(-c_0)$,

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_m)^{D+1}) = \sum_{k=0}^{D+1} \binom{D+1}{k} m^k\, \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^k \widehat{c}_1(\overline{\mathcal{H}}')^{D+1-k}) = m^{D+1}\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{D+1}) + O(m^D)$$

as $m \to \infty$ (a, c_0 and $\overline{\mathcal{H}}'$ do not depend on m). Dividing by $(m+a)^{D+1}$ and letting $m \to \infty$ gives $V^\sharp \geq \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{D+1})$. □

Lemma 29.25. *Let $\mathcal{S}$ be an arithmetic variety with smooth generic fibre, of absolute dimension $m \geq 1$, and let $\overline{\mathcal{M}}_1, \dots, \overline{\mathcal{M}}_m$ be hermitian line bundles with smooth metrics on $\mathcal{S}$. Then every $\overline{\mathcal{M}}_l$ is DSP, and*

$$h_{\overline{\mathcal{M}}_1,\dots,\overline{\mathcal{M}}_m}(\mathcal{S}) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_1)\cdots\widehat{c}_1(\overline{\mathcal{M}}_m)). \tag{H}$$

Proof. Throughout, a smooth metric is invariant under complex conjugation, and a smooth metric on a holomorphic line bundle over a complex manifold is semipositive (positive) if its first Chern form is semipositive (positive definite); such a metric is semipositive in the sense of Proposition B.129 (constant sequence).

Step 1: smooth hermitian line bundles come from a product of projective spaces. There are a closed immersion $\iota\colon \mathcal{S} \to \mathbb{P} = \prod_{k=0}^{m} \mathbb{P}^{N_k}_{\mathbb{Z}}$, multidegrees $a_1, \dots, a_m \in \mathbb{Z}^{m+1}$ and smooth metrics on the line bundles $\mathcal{O}(a_l)$ over $\mathbb{P}(\mathbb{C})$ whose pull-backs by ι are isometric to $\overline{\mathcal{M}}_l$. Indeed, fix a closed immersion $j_0\colon \mathcal{S} \to \mathbb{P}^{N_0}_{\mathbb{Z}}$ and write $\mathcal{M}(n) = \mathcal{M} \otimes j_0^*\mathcal{O}(n)$. By Serre's theorem (Hartshorne, Algebraic Geometry, II.5.17) there are $n_l \geq 0$ such that $\mathcal{M}_l(n_l)$ is generated by finitely many global sections; they define morphisms $j_l\colon \mathcal{S} \to \mathbb{P}^{N_l}_{\mathbb{Z}}$ with $j_l^*\mathcal{O}(1) \cong \mathcal{M}_l(n_l)$. Then $\iota = (j_0, j_1, \dots, j_m)$ is a closed immersion, as its composite with the projection to the separated factor $\mathbb{P}^{N_0}_{\mathbb{Z}}$ is one, and $\mathcal{M}_l \cong \iota^*\mathcal{O}(a_l)$ with $a_l = \mathbf{e}_l - n_l\mathbf{e}_0$. Let $\|\cdot\|_l^{\mathrm{FS}}$ be the metric on $\mathcal{O}(a_l)$ given by the Fubini–Study metrics of the factors. Under this isomorphism the metric of $\overline{\mathcal{M}}_l$ is $\rho_l\, \iota^*\|\cdot\|_l^{\mathrm{FS}}$ for a smooth positive function ρ_l on $\mathcal{S}(\mathbb{C})$, invariant under complex conjugation. As $\mathcal{S}(\mathbb{C})$ is a closed complex submanifold of $\mathbb{P}(\mathbb{C})$, $\log \rho_l$ extends to a smooth real function on $\mathbb{P}(\mathbb{C})$ (tubular neighbourhood and a cut-off function), which we average with its composite with complex conjugation; the exponential $\widetilde{\rho}_l$ of the result gives the metric $\widetilde{\rho}_l\|\cdot\|_l^{\mathrm{FS}}$ on $\mathcal{O}(a_l)$ with the required restriction.

Step 2: quotients of positive bundles; the $\overline{\mathcal{M}}_l$ are DSP. Let $\overline{\mathcal{O}(a)}$ be $\mathcal{O}(a)$ on $\mathbb{P}$ with a smooth metric, and let ω be the first Chern form of $\mathcal{O}(1, \dots, 1)$ with the Fubini–Study metrics, positive definite on $\mathbb{P}(\mathbb{C})$. As $\mathbb{P}(\mathbb{C})$ is compact there is an integer $k \geq 1$ with $c_1(\overline{\mathcal{O}(a)}) + k\omega$ positive definite; then $\overline{\mathcal{O}(a)} \cong \overline{\mathcal{A}} \otimes \overline{\mathcal{A}}'^{\otimes -1}$ with $\overline{\mathcal{A}}' = \mathcal{O}(k, \dots, k)$ with the k-th power of the Fubini–Study metrics and $\overline{\mathcal{A}} = \overline{\mathcal{O}(a)} \otimes \overline{\mathcal{A}}'$, both with positive metrics. The pull-back of a positive metric by a holomorphic map of complex manifolds has semipositive first Chern form (the pull-back of the form). With Step 1, every $\overline{\mathcal{M}}_l$ is isometric to $\iota^*\overline{\mathcal{A}} \otimes (\iota^*\overline{\mathcal{A}}')^{\otimes -1}$ with $\iota^*\overline{\mathcal{A}}$, $\iota^*\overline{\mathcal{A}}'$ semipositive, so it is DSP.

Step 3: (H). Take ι, $\mathbb{P}$ and the metrics of Step 1, and write each $\overline{\mathcal{O}(a_l)} = \overline{\mathcal{A}}_l \otimes \overline{\mathcal{A}}_l'^{\otimes -1}$ as in Step 2. Both sides of (H) are multilinear in $\overline{\mathcal{M}}_1, \dots, \overline{\mathcal{M}}_m$ under tensor products: the left by [109] (multilinearity of the height, §1), the right by the multilinearity of $\widehat{\deg}$ and the additivity of $\widehat{c}_1$. Expanding, it suffices to prove (H) when every $\overline{\mathcal{M}}_l$ is $\iota^*\overline{\mathcal{B}}_l$ with $\overline{\mathcal{B}}_l$ one of $\overline{\mathcal{A}}_l$, $\overline{\mathcal{A}}_l'$, a smooth positive hermitian line bundle on $\mathbb{P}$. Now $\mathbb{P}$ is a normal flat projective scheme over $\operatorname{Spec}\mathbb{Z}$ (an arithmetic variety in the sense of [109]), $\iota(\mathcal{S})$ is the closure in $\mathbb{P}$ of the prime cycle $\iota(\mathcal{S})_{\mathbb{Q}}$ of $\mathbb{P}_{\mathbb{Q}}$ (it is integral and flat over $\mathbb{Z}$), and the $\overline{\mathcal{B}}_l$ are semipositive. In this situation (base $\operatorname{Spec}\mathbb{Z}$, no nef bundles on the base) the height of $\iota(\mathcal{S})$ with respect to $\overline{\mathcal{B}}_1, \dots, \overline{\mathcal{B}}_m$ is the arithmetic intersection number in the sense of Gillet and Soulé [110] [48] of the cycle $\iota(\mathcal{S})$ with respect to them ([109], Introduction, display (0.1)), that is, the height of the cycle $\iota(\mathcal{S})$ as defined in Section 14, $\widehat{\deg}(\widehat{c}_1(\iota^*\overline{\mathcal{B}}_1) \cdots \widehat{c}_1(\iota^*\overline{\mathcal{B}}_m))$ computed on $\mathcal{S} \cong \iota(\mathcal{S})$; and the height of a cycle depends only on the restrictions of the hermitian line bundles to it. This is (H). □

Lemma 29.26. *Let $\mathbb{P}$ be a regular arithmetic variety, $\mathcal{Z} \subsetneq \mathbb{P}$ a closed integral subscheme, flat over $\mathbb{Z}$, of absolute dimension $m \geq 1$, and $\overline{\mathcal{M}}_1, \dots, \overline{\mathcal{M}}_m$ Hermitian line bundles with smooth metrics on $\mathbb{P}$. Then the restrictions $\overline{\mathcal{M}}_l|_{\mathcal{Z}}$ are DSP Hermitian line bundles on the projective scheme $\mathcal{Z}$ over* $\operatorname{Spec}\mathbb{Z}$*, and*

$$h_{\overline{\mathcal{M}}_1|_{\mathcal{Z}}, \dots, \overline{\mathcal{M}}_m|_{\mathcal{Z}}}(\mathcal{Z}) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_1) \cdots \widehat{c}_1(\overline{\mathcal{M}}_m) \mid \mathcal{Z}).$$

Proof. Throughout, a smooth metric is invariant under complex conjugation, and a smooth metric on a line bundle over a complex manifold is positive if its first Chern form is positive

definite. As $\mathbb{P}$ is regular and of finite type over $\mathbb{Z}$, its generic fibre $\mathbb{P}_{\mathbb{Q}}$ is a regular scheme of finite type over a field of characteristic 0, hence smooth over $\mathbb{Q}$, and $\mathbb{P}(\mathbb{C})$ is a compact complex manifold.

Step 1: smooth Hermitian line bundles on $\mathbb{P}$ restrict to DSP bundles. Fix a closed immersion $j\colon \mathbb{P} \to \mathbb{P}^N_{\mathbb{Z}}$ and let ω be the first Chern form of $j^*\mathcal{O}(1)$ with the pull-back of the Fubini–Study metric; it is positive definite on $\mathbb{P}(\mathbb{C})$, being the pull-back of a Kähler form by a holomorphic immersion. Let $\overline{\mathcal{M}}$ be a Hermitian line bundle with smooth metric on $\mathbb{P}$. As $\mathbb{P}(\mathbb{C})$ is compact, there is an integer $k \geq 1$ with $c_1(\overline{\mathcal{M}}) + k\omega$ positive definite. Put $\overline{\mathcal{A}}' = j^*\mathcal{O}(k)$ with the k-th power of the pulled-back Fubini–Study metric and $\overline{\mathcal{A}} = \overline{\mathcal{M}} \otimes \overline{\mathcal{A}}'$; both have smooth positive metrics, so they are semipositive in the sense of Proposition B.129 (constant sequence), and $\overline{\mathcal{M}} \cong \overline{\mathcal{A}} \otimes \overline{\mathcal{A}}'^{\otimes -1}$ isometrically. The inverse image of a semipositive Hermitian line bundle is semipositive and the inverse image of a DSP one is DSP ([108], §1.4, after the definition of semipositive and DSP metrics); applied to the closed immersion $\mathcal{Z} \to \mathbb{P}$, $\overline{\mathcal{M}}|_{\mathcal{Z}}$ is DSP.

Step 2: reduction to positive bundles. Both sides of the asserted identity are multilinear in $\overline{\mathcal{M}}_1, \dots, \overline{\mathcal{M}}_m$ under tensor products: the left side by the multilinearity of the global heights ([108], §1.5, theorem on the properties of global heights, item (1); [109], §1), the right side by the multilinearity of heights of cycles and the additivity of $\widehat{c}_1$. Writing each $\overline{\mathcal{M}}_l = \overline{\mathcal{A}}_l \otimes \overline{\mathcal{A}}_l'^{\otimes -1}$ as in Step 1 and expanding, it suffices to prove the identity when every $\overline{\mathcal{M}}_l$ is a Hermitian line bundle on $\mathbb{P}$ with smooth positive metric.

Step 3: the identification. Let the $\overline{\mathcal{M}}_l$ be smooth and positive. The scheme $\mathbb{P}$ is a normal flat projective scheme over $\operatorname{Spec}\mathbb{Z}$, i.e. an arithmetic variety in the sense of [109], and $\mathcal{Z}$ is the closure in $\mathbb{P}$ of the prime cycle $Y = \mathcal{Z}_{\mathbb{Q}}$ of $\mathbb{P}_{\mathbb{Q}}$ (it is integral and flat over $\mathbb{Z}$). By [109], Introduction, display (0.1) and the paragraph following it, the arithmetic intersection number in the sense of Gillet and Soulé of the cycle $\mathcal{Z}$ with respect to the semipositive $\overline{\mathcal{M}}_l$, which is the right side, is the sum over the places of $\mathbb{Q}$ of the local heights of Y ([108], §1.5): at the infinite place with respect to the smooth metrics of the $\overline{\mathcal{M}}_l$, at a prime p with respect to the algebraic metrics induced by the model $(\mathbb{P}, \mathcal{M}_l)$; for the regular $\mathbb{P}_{\mathbb{Q}}$ and smooth, respectively algebraic, metrics, these local heights agree with those derived from the Gillet–Soulé arithmetic intersection product, the Archimedean one with the Archimedean contribution of the height of Bost, Gillet and Soulé ([108], Remark following Definition 1.4.11). This sum is the global height of Y as a cycle of $\mathbb{P}_{\mathbb{Q}}$ ([108], §1.5, definition of the global height). By the push-forward property of global heights ([108], §1.5, theorem on the properties of global heights, item (2)) for the closed immersion $\mathcal{Z}_{\mathbb{Q}} \to \mathbb{P}_{\mathbb{Q}}$, it equals the global height of $\mathcal{Z}_{\mathbb{Q}}$ with respect to the restricted metrics. At infinity these are the metrics of the $\overline{\mathcal{M}}_l|_{\mathcal{Z}}$; at p, the restriction of the metric induced by $(\mathbb{P}, \mathcal{M}_l)$ is the metric induced by $(\mathcal{Z}, \mathcal{M}_l|_{\mathcal{Z}})$, since a trivialising open subset of $\mathbb{P}$ around the reduction of a point of $\mathcal{Z}^{\mathrm{an}}_{\mathbb{Q}}$ restricts to a trivialising open subset of $\mathcal{Z}$ around the same reduction, with the restricted generator ([108], §1.3, definition of the metric induced by a proper model). So it is the height $h_{\overline{\mathcal{M}}_1|_{\mathcal{Z}}, \dots, \overline{\mathcal{M}}_m|_{\mathcal{Z}}}(\mathcal{Z})$ of the projective scheme $\mathcal{Z}$ over $\operatorname{Spec}\mathbb{Z}$ in the sense of Proposition B.129 ([109], §1–§2: the global height of the generic fibre with respect to the Archimedean metrics and the metrics induced by the model). This is the identity. □

Theorem 29.27. *A family carrying counting data and transfer data with the properties (Tf1)–(Tf3) has the property (Vc3): for all large points with $j < m$, every $Z \in \mathcal{Z}$, every integer $\tau \geq 1$ and all reals ρ, ρ' with $\rho + 2(\tau + \mathrm{fw}(Z))\, c^{\mathrm{HT}}_{E,M}(B)\, \mathrm{wn} + \mathrm{ta}(Z) \leq \rho'$,*

$$\mathrm{mc}_Z(\tau, \rho) \leq \mathrm{cf}_Z\big(\tau + \mathrm{fw}(Z), \rho'\big)^{\mathrm{fr}(Z)}.$$

Proof. Fix a large point, a lift $j < m$, $Z \in \mathcal{Z}$, $\tau \geq 1$ and ρ, ρ' as stated. Write $k = \mathrm{fw}(Z)$, $d = \mathrm{fr}(Z) \geq 1$, $c = c^{\mathrm{HT}}_{E,M}(B)$, $\tau' = \tau + k \geq 1$, $R = e^{\rho}$, $f_l = f_{Z,l}$, $\mathrm{tr} = \mathrm{tr}_{Z,\tau'}$, and let $N = N_{\tau'}$ be the integer of (Tf3) for τ'. Let $Y = \{f \in \mathcal{C}^{\mathrm{int}}_Z(\tau') : \|f\| \leq e^{\rho'}\}$, so that $\#Y = \mathrm{cf}_Z(\tau', \rho')$ and $\#\mathcal{S}_Z(\tau, R) = \mathrm{mc}_Z(\tau, \rho)$. If Y is infinite there is nothing to prove; assume it is finite.

The map. For $s \in \mathcal{S}_Z(\tau, R)$ and $1 \leq l \leq d$, (Tf2) and (Tf1) give $sf_l \in \mathcal{S}_Z(\tau', RR_Z)$, so by (Tf3) $N\,\mathrm{tr}(sf_l) \in \mathcal{C}^{\mathrm{int}}_Z(\tau')$ and

$$\|N\,\mathrm{tr}(sf_l)\| = N\|\mathrm{tr}(sf_l)\| \leq N\,d\,R\,R_Z.$$

By (Tf3) and (Tf2),

$$\log(NdRR_Z) \leq 2\tau' c\,\mathrm{wn} + \sum_{p \in P_{E,M}(B)} \log p + \rho + \log(dR_Z) = \rho + 2\tau' c\,\mathrm{wn} + \mathrm{ta}(Z) \leq \rho'.$$

Hence $\Psi(s) = \big(N\,\mathrm{tr}(sf_1), \dots, N\,\mathrm{tr}(sf_d)\big)$ defines a map $\Psi\colon \mathcal{S}_Z(\tau, R) \to Y^d$.

Injectivity. If $\Psi(s) = \Psi(s')$, then for every l, $N\,\mathrm{tr}((s - s')f_l) = N\,\mathrm{tr}(sf_l) - N\,\mathrm{tr}(s'f_l) = 0$ by additivity of tr, so $\mathrm{tr}((s - s')f_l) = 0$ in the $\mathbb{Q}$-vector space $\mathcal{C}_Z(\tau')$. By the injectivity clause of (Tf2) (at τ, with $\tau + k = \tau'$), $s - s' = 0$.

Therefore $\mathrm{mc}_Z(\tau, \rho) = \#\mathcal{S}_Z(\tau, R) \leq \#Y^d = \mathrm{cf}_Z(\tau', \rho')^d$. □

Theorem 29.28. *Let $\mathcal{F}$ be a parametrised host family with covolume records carrying chain data, deep-cover data, component data, model-slope data and counting data with the properties (Vc1)–(Vc3). Then for all large points with $j < m$ and every $Z \in \mathcal{Z}$*

$$\mathrm{vs}(Z) \geq \mathrm{ns}(Z) - 2\,c^{\mathrm{HT}}_{E,M}(B)\,\mathrm{wn};$$

that is, the inequality of (Ms1) holds.

Proof. Fix a large point with $j < m$ (beyond the threshold of (Vc1)–(Vc3)) and $Z \in \mathcal{Z}$. Write $D = D^{\mathrm{fl}}$, $\delta = d^{\mathrm{co}}$, $r = \mathrm{fr}(Z) \geq 1$, $\mathrm{fw}(Z) = 8s$ with an integer $s \geq 0$, $A = \mathrm{ta}(Z)$, $c = c^{\mathrm{HT}}_{E,M}(B)$ and $a_0 = \mathrm{ns}(Z) - 2c\,\mathrm{wn}$.

Two properties of $\log^+$. On $\overline{\mathbb{N}}$ the function $\log^+$ is non-decreasing, and $\log^+(n^r) = r \log^+ n$ for every $n \in \overline{\mathbb{N}}$: for $n = 0$ both sides are 0, for $1 \leq n < \infty$ this is $\log n^r = r \log n$, and for $n = \infty$ both sides are ∞.

The model counting volume is at most the cusp counting volume at a shifted level. Fix a real t and $\varepsilon > 0$, and put $t' = t - 2c\,\mathrm{wn} - \varepsilon$. We show

$$\mathrm{cvol}_{D,\delta r}(\mathrm{mc}_Z; t') \leq \mathrm{cvol}_{D,\delta}(\mathrm{cf}_Z; t). \tag{29.13}$$

For an integer $k \geq 1$ take $\tau = 8k$, $\rho = 8kt'$ and $\rho' = (8k + 8s)t$ in (Vc3). Then

$$\rho + 2(\tau + 8s)c\,\mathrm{wn} + A - \rho' = 8k\big(t' + 2c\,\mathrm{wn} - t\big) + K = -8k\varepsilon + K, \qquad K = 16s\,c\,\mathrm{wn} + A - 8st,$$

with K independent of k. Hence for every integer $k \geq k_0$, where $k_0 \geq 1$ is an integer with $8k_0\varepsilon \geq K$, (Vc3) gives $\mathrm{mc}_Z(8k, 8kt') \leq \mathrm{cf}_Z\big(8(k+s), 8(k+s)t\big)^r$, and by the two properties of $\log^+$

$$x_k := \frac{D!\,\log^+ \mathrm{mc}_Z(8k, 8kt')}{\delta r\,(8k)^{D+1}} \leq \frac{D!\,\log^+ \mathrm{cf}_Z\big(8(k+s), 8(k+s)t\big)}{\delta\,(8k)^{D+1}} = y_{k+s}\Big(\frac{k+s}{k}\Big)^{D+1},$$

where $y_k = D!\,\log^+ \mathrm{cf}_Z(8k, 8kt)/(\delta(8k)^{D+1})$. By (29.1), $\mathrm{cvol}_{D,\delta r}(\mathrm{mc}_Z; t') = \limsup_k x_k$ and $Y := \mathrm{cvol}_{D,\delta}(\mathrm{cf}_Z; t) = \limsup_k y_k$. If $Y = \infty$, (29.13) holds. Let $Y < \infty$ and $\eta > 0$. There is k_1

with $y_k \le Y+\eta$ for all $k \ge k_1$, and, since $((k+s)/k)^{D+1} \to 1$, there is k_2 with $((k+s)/k)^{D+1} \le 1+\eta$ for all $k \ge k_2$. For $k \ge \max(k_0,k_1,k_2)$ we have $k+s \ge k_1$, so $x_k \le (Y+\eta)(1+\eta)$. Hence $\limsup_k x_k \le (Y+\eta)(1+\eta)$ for every $\eta>0$, and (29.13) follows.

Conclusion. By (Vc2) at the real number t' and by (29.13),

$$\mathrm{tg}(Z)\big((a_0-\varepsilon)+t\big) = \mathrm{tg}(Z)\big(\mathrm{ns}(Z)+t'\big) \le \mathrm{cvol}_{D,\delta r}(\mathrm{mc}_Z;t') \le \mathrm{cvol}_{D,\delta}(\mathrm{cf}_Z;t).$$

This holds for every real t, so (Vc1) gives $a_0-\varepsilon \le \mathrm{vs}(Z)$. Since $\varepsilon>0$ was arbitrary, $a_0 \le \mathrm{vs}(Z)$, which is the assertion. □

29.3 Arithmetic proportionality on root towers

This subsection proves (Ms3) by an arithmetic form of proportionality, the idea around which the Part is built. On a root tower with (Rt1)–(Rt3), Theorem 29.29 gives at every stage B_0 an exact identity: the arithmetic degree of a class F of degree $D+1$ in the Chern roots equals $\mathrm{vol}_{\mathcal{T}}$ times the compact-dual height $h^{\mathrm{cd}}(F)$, corrected by two constants α,β along the two central directions $\mathrm{Sh}(F)$, $\mathrm{To}(F)$ and by a Frobenius tail over the primes $p>B_0$. The identity is the finite-stage form of an arithmetic Hirzebruch–Mumford proportionality [188]. It rests on the change of stage (Rt3) and on the splitting of the term of a prime p in (Rt3) into $-\log p/(p-1)$ times the first central direction, a multiple of the second, and a remainder of size $O(\log p/p^2)$ (Lemma 29.31); the remainders form the Frobenius tail.

What remains is to bound the three correction terms for the class $\mathrm{lr}(u)^{D+1}$ of the divided weight. The first constant is bounded below through the test class, by (Rt6) and (Rt7) (Lemma 29.34); the combination $\kappa\alpha+n\beta$ is bounded by the determinant classes (Rt5) (Lemma 29.35); and the Frobenius tail of $\mathrm{lr}(u)^{D+1}$ is bounded by the spread of u (Lemma 29.41). Together they are (Hp1)–(Hp6) (Theorem 29.33), which give the lower bound (Si3) for the total self-intersection (Theorem 29.43) and then (Ms3) (Theorem 29.44). The tail bounds need a power-sum decomposition of each class (Lemma 29.32) and the evaluation of flag degrees of partial power sums, which is a computation with Schur polynomials and Vandermonde determinants (Proposition 29.38, Proposition 29.42, Lemma 29.39).

Theorem 29.29. *Let $\mathcal{T}$ be a root tower of type (Φ,n,κ) with (Rt1), (Rt2), (Rt3), and let $B_0\ge 2$ be an integer. For $F\in\mathcal{R}$ write $\mathrm{Sh}(F)=\sum_{b\in\Phi}\deg^{\mathrm{fl}}(\partial_b^{<}F)$ and $\mathrm{To}(F)=\sum_{b\in\Phi}\deg^{\mathrm{fl}}(\partial_b F)$. There is exactly one pair of real numbers (α,β), the* proportionality constants at stage B_0, *such that*

$$\widehat{\deg}_{B_0}(\mathrm{ps}_{a,k}Q) = \mathrm{vol}_{\mathcal{T}}\Big(h^{\mathrm{cd}}(\mathrm{ps}_{a,k}Q) - \alpha\,\mathrm{Sh}(\mathrm{ps}_{a,k}Q) - \beta\,\mathrm{To}(\mathrm{ps}_{a,k}Q) + \rho_{>B_0}(a,k,Q)\Big) \tag{29.14}$$

for every $a\in\Phi$, every $1\le k\le n$ and every $Q\in\mathcal{R}$ homogeneous of degree $D+1-k$. For these constants:

(a) for every $F\in\mathcal{R}$ homogeneous of degree $D+1$ and every power-sum decomposition $(Q_{a,k})$ of F,

$$\widehat{\deg}_{B_0}(F) = \mathrm{vol}_{\mathcal{T}}\Big(h^{\mathrm{cd}}(F) - \alpha\,\mathrm{Sh}(F) - \beta\,\mathrm{To}(F) + \sum_{a\in\Phi}\sum_{k=1}^{n}\rho_{>B_0}(a,k,Q_{a,k})\Big);$$

(b) for every weight u and every power-sum decomposition $(Q_{a,k})$ of $\mathrm{lr}(u)^{D+1}$,

$$\begin{aligned}\widehat{\deg}_{B_0}(\mathrm{lr}(u)^{D+1}) = \mathrm{vol}_{\mathcal{T}}\Big(&h^{\mathrm{cd}}(\mathrm{lr}(u)^{D+1})\\ &+(D+1)\deg^{\mathrm{fl}}(\mathrm{lr}(u)^{D})\sum_{a\in\Phi}\Big(\alpha\sum_{i=1}^{\kappa}u_{a,i}+\beta\sum_{i=1}^{n}u_{a,i}\Big)+\sum_{a,k}\rho_{>B_0}(a,k,Q_{a,k})\Big).\end{aligned}$$

Proof. The maps Sh and To are linear, and $\mathrm{vol}_{\mathcal{T}} > 0$ by (Rt1). For an integer $B \geq B_0$ let S_B be the set of primes $p \notin \mathrm{Bd}$ with $B_0 < p \leq B$, and put $\lambda_B = \sum_{p\in S_B} \log p/(p-1)$, $\mu_B = \sum_{p\in S_B} c_p$ with c_p as in Lemma 29.31 (for a fixed $a_0 \in \Phi$).

Step 1: the error at stage B is small on all classes of degree $D+1$. Let B_1, ε_B, a_B, b_B be as in (Rt2), and for $B \geq B_1$ let $G_B(F) = \widehat{\deg}_B(F) - \mathrm{vol}_{\mathcal{T}}\big(h^{\mathrm{cd}}(F) - a_B\mathrm{Sh}(F) - b_B\mathrm{To}(F)\big)$, a linear form. If $F = \sum_m f_m x^m$ is homogeneous of degree $D+1$, then $|G_B(F)| \leq \|F\|\varepsilon_B$ with $\|F\| = \sum_m |f_m|$, by linearity and (Rt2) on each monomial; so $G_B(F) \to 0$ as $B \to \infty$.

Step 2: an approximate identity at stage B_0. Let $a \in \Phi$, $1 \leq k \leq n$, Q homogeneous of degree $D+1-k$, $F = \mathrm{ps}_{a,k}Q$ (homogeneous of degree $D+1$), and $B \geq \max(B_0, B_1)$. Summing (Rt3) over $Z \in \mathrm{Cp}$ gives $\widehat{\deg}_{B_0}(F) = \widehat{\deg}_B(F) + \mathrm{vol}_{\mathcal{T}} \sum_{p\in S_B} \Delta_p(a,k,Q)$, and by Lemma 29.31(a), $\sum_{p\in S_B} \Delta_p(a,k,Q) = -\lambda_B\mathrm{Sh}(F) + \mu_B\mathrm{To}(F) + \sum_{p\in S_B} \rho_p(a,k,Q)$. With the definition of G_B,

$$\begin{aligned} \widehat{\deg}_{B_0}(F) &= \mathrm{vol}_{\mathcal{T}}\Big(h^{\mathrm{cd}}(F) - A_B\mathrm{Sh}(F) - A'_B\mathrm{To}(F) + \sum_{p\in S_B} \rho_p(a,k,Q)\Big) + G_B(F), \\ A_B &= a_B + \lambda_B,\ A'_B = b_B - \mu_B. \end{aligned} \tag{29.15}$$

Step 3: the two coefficients converge. Let Q_1, Q_2 be the classes of Lemma 29.30(e) for a_0, and $F_1 = \mathrm{ps}_{a_0,1}Q_1$, $F_2 = \mathrm{ps}_{a_0,2}Q_2$ (here $2 \leq n$). By Lemma 29.30(c), $\mathrm{Sh}(F_2) = 2\deg^{\mathrm{fl}}(\mathrm{ps}_1^{(a_0,+)}Q_2) = 2$, $\mathrm{To}(F_2) = 0$, $\mathrm{Sh}(F_1) = \deg^{\mathrm{fl}}(\mathrm{ps}_0^{(a_0,+)}Q_1) = \kappa$ and $\mathrm{To}(F_1) = n\deg^{\mathrm{fl}}(Q_1) = n$; and $\rho_p(a_0,1,Q_1) = 0$. Solving (29.15) for F_2 and for F_1,

$$\begin{aligned} A_B &= \frac{1}{2}\Big(h^{\mathrm{cd}}(F_2) + \sum_{p\in S_B} \rho_p(a_0,2,Q_2) + \frac{G_B(F_2) - \widehat{\deg}_{B_0}(F_2)}{\mathrm{vol}_{\mathcal{T}}}\Big), \\ A'_B &= \frac{1}{n}\Big(h^{\mathrm{cd}}(F_1) - \kappa A_B + \frac{G_B(F_1) - \widehat{\deg}_{B_0}(F_1)}{\mathrm{vol}_{\mathcal{T}}}\Big). \end{aligned}$$

As $B \to \infty$, $\sum_{p\in S_B} \rho_p(a_0,2,Q_2) \to \rho_{>B_0}(a_0,2,Q_2)$ by Lemma 29.31(b) and $G_B(F_i) \to 0$ by Step 1; so A_B converges to a real number α, and then A'_B converges to a real number β.

Step 4: existence. For $F = \mathrm{ps}_{a,k}Q$ as in Step 2, the left side of (29.15) does not depend on B; letting $B \to \infty$ and using Step 3, Lemma 29.31(b) and Step 1, the right side tends to the right side of (29.14). So (α, β) satisfies (29.14).

Step 5: uniqueness. If (α', β') also satisfies (29.14), subtracting the two identities for F_2 gives $2\,\mathrm{vol}_{\mathcal{T}}(\alpha - \alpha') = 0$, and then for F_1 gives $n\,\mathrm{vol}_{\mathcal{T}}(\beta - \beta') = 0$; as $\mathrm{vol}_{\mathcal{T}} > 0$, $(\alpha', \beta') = (\alpha, \beta)$.

(a) Apply (29.14) to each pair (a,k) with $Q = Q_{a,k}$ and sum: $\widehat{\deg}_{B_0}$, h^{cd}, Sh, To are linear and $F = \sum_{a,k} \mathrm{ps}_{a,k}Q_{a,k}$.

(b) The class $\mathrm{lr}(u)^{D+1}$ is homogeneous of degree $D+1$; by Lemma 29.30(d),

$$-\alpha\,\mathrm{Sh}(\mathrm{lr}(u)^{D+1}) - \beta\,\mathrm{To}(\mathrm{lr}(u)^{D+1}) = (D+1)\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) \sum_a \Big(\alpha \sum_{i\leq\kappa} u_{a,i} + \beta \sum_i u_{a,i}\Big);$$

insert this in (a). □

Lemma 29.30. *(a)* $\deg^{\mathrm{fl}}(F) = 0$ *for every F homogeneous of degree $\neq D$.*

(b) If $s \in \mathcal{R}$ is invariant under S_n^{Φ} and has constant term 0, then $\deg^{\mathrm{fl}}(sG) = 0$ *for every $G \in \mathcal{R}$. In particular* $\deg^{\mathrm{fl}}(\mathrm{ps}_{a,k}G) = 0$ *for $a \in \Phi$, $k \geq 1$.*

(c) For $a, b \in \Phi$, $k \geq 1$ and $Q \in \mathcal{R}$:

$$\deg^{\mathrm{fl}}(\partial_b^{<}(\mathrm{ps}_{a,k} Q)) = \begin{cases} k \deg^{\mathrm{fl}}(\mathrm{ps}_{k-1}^{(a,+)} Q) & b = a, \\ 0 & b \neq a, \end{cases}$$

$$\deg^{\mathrm{fl}}(\partial_b(\mathrm{ps}_{a,k} Q)) = \begin{cases} n \deg^{\mathrm{fl}}(Q) & b = a,\ k = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

(d) For every weight u and $a \in \Phi$:

$$\deg^{\mathrm{fl}}(\partial_a^{<} \mathrm{lr}(u)^{D+1}) = -(D+1) \deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) \sum_{i=1}^{\kappa} u_{a,i},$$

$$\deg^{\mathrm{fl}}(\partial_a \mathrm{lr}(u)^{D+1}) = -(D+1) \deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) \sum_{i=1}^{n} u_{a,i}.$$

(e) Let $a \in \Phi$ and let $e_{a,\kappa}$ be the unit exponent vector at (a, κ). Then $Q_1 = (-1)^D x^{\delta}$ is homogeneous of degree D with $\deg^{\mathrm{fl}}(Q_1) = 1$, and $Q_2 = (-1)^D x^{\delta - e_{a,\kappa}}$ is homogeneous of degree $D-1$ with $\deg^{\mathrm{fl}}(\mathrm{ps}_1^{(a,+)} Q_2) = 1$.

Proof. (a) Every exponent $\delta \circ w$ in (21.1) has total degree $\sum_a \sum_i (n-i) = D$, so only the homogeneous component of degree D of F contributes.

Antisymmetrisation. For $w \in S_n^{\Phi}$ and $F \in \mathcal{R}$ let $w \cdot F$ be obtained by substituting $x_{a,w_a(i)}$ for $x_{a,i}$. For a monomial, $w \cdot x^m = x^{m'}$ with $m'_{a,w_a(i)} = m_{a,i}$, so $w \cdot x^m = x^{\delta}$ if and only if $m = \delta \circ w$; hence $[x^{\delta}](w \cdot F) = [x^{\delta \circ w}]F$, and with $A(F) = \sum_w \mathrm{sgn}(w)\, w \cdot F$,

$$\deg^{\mathrm{fl}}(F) = (-1)^D\, [x^{\delta}]\, A(F). \tag{29.16}$$

The map A is linear, $v \cdot A(F) = \mathrm{sgn}(v) A(F)$ for $v \in S_n^{\Phi}$, and each homogeneous component H of $A(F)$ has the same property, the action preserving degrees. Such an H is divisible by $x_{a,i} - x_{a,i'}$ for all a and $i < i'$: applying the transposition of i, i' in the block a gives $H = -H$ on the hyperplane $x_{a,i} = x_{a,i'}$, so H vanishes there. These linear forms are pairwise non-associate primes of the factorial ring $\mathcal{R}$, so their product, of degree D, divides H. Hence every homogeneous component of $A(F)$ of degree $< D$ is 0.

(b) Since $w \cdot (sG) = s\,(w \cdot G)$, $A(sG) = s\, A(G)$. Write $s = \sum_{d \geq 1} s_d$ with s_d homogeneous of degree d. The coefficient of x^{δ} (degree D) in $s\, A(G)$ is $\sum_{d \geq 1} [x^{\delta}](s_d\, H_{D-d})$, where H_{D-d} is the component of $A(G)$ of degree $D - d < D$, which is 0. By (29.16), $\deg^{\mathrm{fl}}(sG) = 0$. The power sum $\mathrm{ps}_{a,k}$ with $k \geq 1$ is invariant with constant term 0.

(c) By the Leibniz rule, $\partial_b^{<}(\mathrm{ps}_{a,k} Q) = (\partial_b^{<} \mathrm{ps}_{a,k}) Q + \mathrm{ps}_{a,k}\, \partial_b^{<} Q$, and the flag degree of the second term is 0 by (b). Further $\partial_b^{<} \mathrm{ps}_{a,k} = k \sum_{i \leq \kappa} x_{a,i}^{k-1} = k\, \mathrm{ps}_{k-1}^{(a,+)}$ if $b = a$ and 0 otherwise. Likewise $\deg^{\mathrm{fl}}(\partial_b(\mathrm{ps}_{a,k} Q)) = \deg^{\mathrm{fl}}((\partial_b \mathrm{ps}_{a,k}) Q)$ with $\partial_b \mathrm{ps}_{a,k} = k\, \mathrm{ps}_{a,k-1}$ if $b = a$ and 0 otherwise. For $k \geq 2$, $\deg^{\mathrm{fl}}(\mathrm{ps}_{a,k-1} Q) = 0$ by (b); for $k = 1$, $\mathrm{ps}_{a,0} = n$.

(d) One has $\partial \mathrm{lr}(u)^{D+1} / \partial x_{a,i} = -(D+1) u_{a,i}\, \mathrm{lr}(u)^D$; sum over $i \leq \kappa$ for the first identity and over all i for the second, and apply the linear form $\deg^{\mathrm{fl}}$.

(e) Since the entries $n-1, \ldots, 0$ of δ in each block are distinct, $\delta \circ w = \delta \circ w'$ forces $w = w'$; so by (21.1), $\deg^{\mathrm{fl}}(x^m) = (-1)^D \mathrm{sgn}(w)$ if $m = \delta \circ w$, and $\deg^{\mathrm{fl}}(x^m) = 0$ if m is not of this form. Hence $\deg^{\mathrm{fl}}(Q_1) = (-1)^{2D} = 1$. As $\delta_{a,\kappa} = n - \kappa \geq 1$, Q_2 is a monomial of degree $D-1$, and $\mathrm{ps}_1^{(a,+)} Q_2 = (-1)^D \sum_{i=1}^{\kappa} x^{\delta - e_{a,\kappa} + e_{a,i}}$. The term $i = \kappa$ is $(-1)^D x^{\delta}$, of flag degree 1. For $i < \kappa$, the

block a of $\delta - e_{a,\kappa} + e_{a,i}$ has the entry $n-i+1$ at i; if $i=1$ this is n, which is not an entry of δ, and if $i \geq 2$ the entry $n-i+1$ also occurs at $i-1$ (unchanged, as $i-1 < \kappa$); in both cases the exponent is not of the form $\delta \circ w$, and the term has flag degree 0. So $\deg^{\mathrm{fl}}(\mathrm{ps}_1^{(a,+)} Q_2) = 1$. $\square$

Lemma 29.31. *Let* fr *be a Frobenius labelling, fix $a_0 \in \Phi$, and for a prime p put*

$$c_p = \frac{\log p}{n} \sum_{m=0}^{\infty} p^{-(m+1)}\, r(\mathrm{fr}(p, a_0, m)), \qquad r(b,+) = \kappa, \quad r(b,-) = n - \kappa.$$

(a) For every prime p, $a \in \Phi$, $1 \leq k \leq n$ and $Q \in \mathcal{R}$,

$$\Delta_p(a,k,Q) = -\frac{\log p}{p-1} \sum_{b \in \Phi} \deg^{\mathrm{fl}}(\partial_b^{<}(\mathrm{ps}_{a,k} Q)) + c_p \sum_{b \in \Phi} \deg^{\mathrm{fl}}(\partial_b(\mathrm{ps}_{a,k} Q)) + \rho_p(a,k,Q).$$

(b) For $a \in \Phi$, $1 \leq k \leq n$ and $Q \in \mathcal{R}$ there is $c \geq 0$ with $|\rho_p(a,k,Q)| \leq c \log p / p^2$ for every prime p. Consequently, for every finite set of primes and every integer B_0, the Frobenius tail $\rho_{>B_0}(a,k,Q)$ exists, as the sum of an absolutely convergent series.

Proof. (a), $k \geq 2$. By Lemma 29.30(c), $\sum_b \deg^{\mathrm{fl}}(\partial_b(\mathrm{ps}_{a,k} Q)) = 0$ and $\sum_b \deg^{\mathrm{fl}}(\partial_b^{<}(\mathrm{ps}_{a,k} Q)) = k \deg^{\mathrm{fl}}(\mathrm{ps}_{k-1}^{(a,+)} Q)$. By (21.2) and the definition of ρ_p for $k \geq 2$, $\Delta_p(a,k,Q) = \rho_p(a,k,Q) - \frac{\log p}{p-1} k \deg^{\mathrm{fl}}(\mathrm{ps}_{k-1}^{(a,+)} Q)$, which is the assertion.

(a), $k = 1$. Here $\rho_p(a,1,Q) = 0$, and by Lemma 29.30(c) the two sums are $\kappa \deg^{\mathrm{fl}}(Q)$ (as $\mathrm{ps}_0^{(a,+)} = \kappa$) and $n \deg^{\mathrm{fl}}(Q)$. The partial power sums of degree 0 are the constants $\mathrm{ps}_0^{(b,+)} = \kappa$, $\mathrm{ps}_0^{(b,-)} = n - \kappa$, that is $\mathrm{ps}_0^{\sigma} = r(\sigma)$; and $r(\mathrm{fr}(p,a,m)) = r(\mathrm{fr}(p,a_0,m))$ because r depends only on the sign of the label, which does not depend on a. Hence

$$\Sigma_p(a,1,Q) = \deg^{\mathrm{fl}}(Q) \sum_{m \geq 0} p^{-(m+1)} r(\mathrm{fr}(p,a_0,m)) = \frac{n\, c_p}{\log p} \deg^{\mathrm{fl}}(Q),$$

and (21.2) gives $\Delta_p(a,1,Q) = n\, c_p \deg^{\mathrm{fl}}(Q) - \frac{\log p}{p-1} \kappa \deg^{\mathrm{fl}}(Q)$, which is the assertion.

(b) For $k = 1$ take $c = 0$. For $k \geq 2$ let $M = \max_{\sigma} |\deg^{\mathrm{fl}}(\mathrm{ps}_{k-1}^{\sigma} Q)|$ over the $2|\Phi|$ labels σ. Then

$$|\Sigma_p(a,k,Q)| \leq M \sum_{m \geq 0} p^{-k(m+1)} = \frac{M\, p^{-k}}{1 - p^{-k}} \leq 2M p^{-k} \leq 2M p^{-2},$$

as $p^{-k} \leq 1/2$ and $k \geq 2$; so $|\rho_p(a,k,Q)| \leq 2kM \log p / p^2$. Since $\sum_p \log p / p^2 \leq \sum_{N \geq 2} \log N / N^2 < \infty$, the series of the $\rho_p(a,k,Q)$ over the primes $p > B_0$ outside a finite set converges absolutely, and its partial sums over $p \leq B$ converge as $B \to \infty$. $\square$

Lemma 29.32. *Every $F \in \mathcal{R}$ homogeneous of degree $D+1$ has a power-sum decomposition.*

Proof. Power-sum decompositions of F_1, F_2 add to one of $F_1 + F_2$ and scale with F, so it suffices to treat a monomial x^m of degree $D+1$. Write $x^m = \prod_a x_a^{m_a}$ with $x_a^{m_a} = \prod_i x_{a,i}^{m_{a,i}}$ and $|m_a| = \sum_i m_{a,i}$. Since $\sum_a |m_a| = D + 1 > |\Phi|\, n(n-1)/2$, there is a with $|m_a| > n(n-1)/2$.

One block. Let $G \in \mathbb{R}[y_1, \ldots, y_n]$ be homogeneous of degree $d > n(n-1)/2$. We show $G = \sum_{k=1}^{n} P_k G_k$ with G_k homogeneous of degree $d-k$ (zero if $d < k$), P_k the power sums. By Theorem B.170(a), $G = \sum_e y^e s_e$ with s_e symmetric, the sum over the Artin exponents $0 \leq e_i \leq i-1$. The action of the symmetric group preserves degrees, so the homogeneous components of s_e are symmetric, and comparing components of degree d (the basis expansion being unique) we

may replace s_e by its component of degree $d-|e|$. Since $|e| \le \sum_i (i-1) = n(n-1)/2 < d$, each s_e is homogeneous symmetric of positive degree. By Theorem B.170(b), $s_e = f_e(P_1,\dots,P_n)$ with $f_e \in \mathbb{R}[T_1,\dots,T_n]$; keeping only the monomials T^β with $\sum_k k\beta_k = d-|e|$ (the others give the other homogeneous components of $f_e(P)$, which sum to 0), every monomial of f_e has positive weighted degree, hence contains some T_k, and grouping by the smallest such k gives $s_e = \sum_k P_k t_{e,k}$ with $t_{e,k}$ a polynomial in the P's, homogeneous of degree $d-|e|-k$. Put $G_k = \sum_e y^e t_{e,k}$.

Conclusion. Apply this to $G = x_a^{m_a}$ in the variables $y_i = x_{a,i}$: $x_a^{m_a} = \sum_{k=1}^n \mathrm{ps}_{a,k} G_k$ with G_k homogeneous of degree $|m_a|-k$. Then $x^m = \sum_{k=1}^n \mathrm{ps}_{a,k} Q_{a,k}$ with $Q_{a,k} = G_k \prod_{b\neq a} x_b^{m_b}$, homogeneous of degree $D+1-k$; with $Q_{b,k}=0$ for $b\neq a$ this is a power-sum decomposition of x^m. □

Theorem 29.33. *Let $\mathcal{F}$ be a parametrised host family with covolume records carrying chain data, deep-cover data, component data, model-slope data, counting data, self-intersection data, a tower realisation and proportionality data, and suppose that the proportionality data are realised by the tower with constants $C, C' \ge 0$. Then $\mathcal{F}$ has the properties (Hp1)–(Hp6) with the constants $C_\mathrm{b} = 2(C+C')+3$ and $C_\mathrm{f} = C'$.*

Proof. Fix a large point with $j<m$, beyond the thresholds of the tower realisation and of (Rz1)–(Rz3), and use the notation of (Rz1)–(Rz3), with $\mathcal{T}$ of type (Φ,n,κ), $B_* = \lfloor B \rfloor$, $D = D^{\mathrm{fl}}$, $\alpha = \alpha^{\mathrm{pr}}$, $\beta = \beta^{\mathrm{pr}}$, $\Lambda = \log(B_* \mathrm{rd}_E) + |L|$.

Preliminaries. By the model-slope data, $B \ge 128$, so $B_* \ge 128$. Since $\|u\|_\infty / g \ge 0$, $b_0(u,g) \ge 4|\Phi| n^2$, so $4|\Phi|n^2 < B_* + 1$ and, both being integers, $B_* \ge 4|\Phi|n^2$. Since $0 < B_* \le B$ and $\mathrm{rd}_E \ge 1$, $\Lambda \le \log(B\,\mathrm{rd}_E) + |L| = \mathrm{lz}$. By (Rz3), $\mathcal{T}$ has (Rt1)–(Rt3); by (Rz2), (α,β) satisfies the identity (29.14) at stage B_*, so by the uniqueness in Theorem 29.29 it is the pair of proportionality constants of $\mathcal{T}$ at stage B_*.

(Hp1). By the gap hypothesis u is regular dominant, so $\mathrm{dw}_{a,1} \ge \cdots \ge \mathrm{dw}_{a,n}$ for every a, as $\mathrm{dw}_{a,i} = u_{\varphi_a,i}$; and $\mathrm{wn} = \|u\|_1 = \sum_a \sum_i |\mathrm{dw}_{a,i}|$ since $a \mapsto \varphi_a$ enumerates Φ.

(Hp2). As cp is a bijection (the component bijection of the tower realisation, Definition 29.9), $\sum_Z \mathrm{si}(Z) = \widehat{\deg}_{B_*}(\mathrm{lr}(u)^{D+1})$ and $\sum_Z \mathrm{vo}(Z) = \mathrm{vol}_{\mathcal{T}}$. Let $(Q_{b,k})$ be the decomposition of (Rz2), so that $\mathrm{ft} = \sum_{b,k} \rho_{>B_*}(b,k,Q_{b,k})$. By Theorem 29.29(b),

$$\sum_Z \mathrm{si}(Z) = \Big(\sum_Z \mathrm{vo}(Z)\Big)\Big(h^{\mathrm{cd}}_{\mathcal{T}}(\mathrm{lr}(u)^{D+1}) + (D+1)\,\mathrm{fd}\sum_{a=1}^{n_{\mathrm{em}}}\Big(\alpha\sum_{i=1}^{\kappa} u_{\varphi_a,i} + \beta\sum_{i=1}^{n} u_{\varphi_a,i}\Big) + \mathrm{ft}\Big),$$

and with $h^{\mathrm{cd}} = h^{\mathrm{cd}}_{\mathcal{T}}(\mathrm{lr}(u)^{D+1})$, $n_{\mathrm{ct}} = \kappa$, $n_{\mathrm{rk}} = n$, the inner sums are s_a and t_a of (29.8). This is (Hp2).

(Hp3). Since u is dominant, $h^{\mathrm{cd}} = h^{\mathrm{cd}}_{\mathcal{T}}(\mathrm{lr}(u)^{D+1}) \ge 0$ by (Rt4).

(Hp4). By (Rz3), $\mathcal{T}$ has (Rt1)–(Rt4) and (Rt5)–(Rt7) for (rd_E, L) with C, C', and $|\mathrm{Bd}| \le B_*$; with $B_* \ge 128$ and $B_* \ge 4|\Phi|n^2$, Lemma 29.34 gives $\alpha \ge -(2(C+C')+3)\Lambda \ge -C_\mathrm{b}\,\mathrm{lz}$, as $C_\mathrm{b} \ge 0$ and $\Lambda \le \mathrm{lz}$.

(Hp5). Lemma 29.35 gives $|n_{\mathrm{ct}}\alpha + n_{\mathrm{rk}}\beta| = |\kappa\alpha + n\beta| \le C' n \Lambda \le C_\mathrm{f}\, n_{\mathrm{rk}}\,\mathrm{lz}$.

(Hp6). Lemma 29.41, applied with the integer B_* (as $b_0(u,g) < B_*+1$), the labelling fr, the set Bd and the decomposition $(Q_{b,k})$, gives $|\mathrm{ft}| \le 2(D+1)\,\mathrm{fd}\min(\kappa, n-\kappa)\,\mathrm{spr}(u) = 2(D+1)\,\mathrm{fd}\,\mathrm{wsp}$; in particular $\mathrm{ft} \ge -2(D+1)\,\mathrm{fd}\,\mathrm{wsp}$. □

Lemma 29.34. *Let $\mathcal{T}$ be a root tower of type (Φ,n,κ) with (Rt1)–(Rt4) and with (Rt5)–(Rt7) for (r,L) with constants $C, C' \ge 0$. Let B_0 be an integer with $B_0 \ge 128$, $B_0 \ge 4|\Phi|n^2$ and*

$|\mathrm{Bd}| \le B_0$, *and let* (α,β) *be the proportionality constants at stage* B_0. *Then*

$$\alpha \ge -\big(2(C+C')+3\big)\,\mathrm{Lg}(B_0).$$

Proof. Write $f=|\Phi|$, $m=\min(\kappa,n-\kappa)$, $\mathrm{te}=\mathrm{pl}^{D^{\mathrm{gr}}+1}\mathrm{fb}$, $t=\deg^{\mathrm{fl}}(\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb})$, $K=(D^{\mathrm{gr}}+1)t$, $\Lambda=\mathrm{Lg}(B_0)$, $l=\log(n+1)$ and $V=\mathrm{vol}_{\mathcal{T}}>0$ (Rt1). By Proposition 29.36(a), $t>0$, so $K>0$.

Step 1: the test class. fb is a product of $f\big(\kappa(\kappa-1)/2+(n-\kappa)(n-\kappa-1)/2\big)=D-D^{\mathrm{gr}}$ linear forms (as $n(n-1)/2=\kappa(\kappa-1)/2+(n-\kappa)(n-\kappa-1)/2+\kappa(n-\kappa)$) and pl is a linear form, so te is homogeneous of degree $D+1$. Each factor $x_{b,i'}-x_{b,i}$ of fb has $i,i'\le\kappa$ or $i,i'>\kappa$, so it is killed by $\partial_a^{<}$ and by ∂_a; by the Leibniz rule $\partial_a^{<}\mathrm{fb}=\partial_a\mathrm{fb}=0$, hence $\partial_a^{<}\mathrm{te}=(D^{\mathrm{gr}}+1)(\partial_a^{<}\mathrm{pl})\,\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb}$, and likewise for ∂_a. If $n-\kappa\le\kappa$, then $\partial_a^{<}\mathrm{pl}=0$ and $\partial_a\mathrm{pl}=n-\kappa=m$, so $\mathrm{Sh}(\mathrm{te})=0$ and $\mathrm{To}(\mathrm{te})=fKm$. If $n-\kappa>\kappa$, then $\partial_a^{<}\mathrm{pl}=\partial_a\mathrm{pl}=-\kappa=-m$, so $\mathrm{Sh}(\mathrm{te})=\mathrm{To}(\mathrm{te})=-fKm$.

Step 2: the inequality at the test class. By Lemma 29.32 the test class has a power-sum decomposition $(Q_{a,k})$; let $R=\sum_{a,k}\rho_{>B_0}(a,k,Q_{a,k})$. By Theorem 29.29(a), $\widehat{\deg}_{B_0}(\mathrm{te})=V\big(h^{\mathrm{cd}}(\mathrm{te})-\alpha\mathrm{Sh}(\mathrm{te})-\beta\mathrm{To}(\mathrm{te})+R\big)$. By (Rt6) (applicable to B_0), $\widehat{\deg}_{B_0}(\mathrm{te})\ge -Cfm\Lambda KV$; by (Rt7), $h^{\mathrm{cd}}(\mathrm{te})\le Kfml/2$; by Proposition 29.36(b) (as $B_0\ge 4fn^2$), $R\le 1.6K$. Dividing by V:

$$\alpha\,\mathrm{Sh}(\mathrm{te})+\beta\,\mathrm{To}(\mathrm{te})\le Kfm\Big(\frac{l}{2}+C\Lambda\Big)+1.6K\le Kfm\Big(\frac{l}{2}+C\Lambda+1.6\Big),\tag{29.17}$$

the last step because $fm\ge 1$.

Step 3: the bound. Put $M=(C+C')\Lambda+l/2+1.6$. Since $r\ge 1$, $\Lambda\ge\log B_0\ge\log 128>1.6$, and since $n+1\le 4n^2\le B_0$, $0\le l\le\log B_0\le\Lambda$; so $M\ge 0$. By Lemma 29.35 (as $B_0\ge 128$, $|\mathrm{Bd}|\le B_0$), $|\kappa\alpha+n\beta|\le C'n\Lambda$. If $n-\kappa\le\kappa$: (29.17) with Step 1, divided by $Kfm>0$, gives $\beta\le l/2+C\Lambda+1.6=M-C'\Lambda$; hence $\kappa\alpha\ge -C'n\Lambda-n\beta\ge -nM$, and as $n=\kappa+m\le 2\kappa$ and $M\ge 0$, $\alpha\ge -(n/\kappa)M\ge -2M$. If $n-\kappa>\kappa$: (29.17) gives $-(\alpha+\beta)\le M-C'\Lambda$; hence $(n-\kappa)\alpha=n(\alpha+\beta)-(\kappa\alpha+n\beta)\ge -n(M-C'\Lambda)-C'n\Lambda=-nM$, and as $n<2(n-\kappa)$, $\alpha\ge -2M$. Finally $2M=2(C+C')\Lambda+l+3.2\le 2(C+C')\Lambda+\Lambda+2\Lambda$, using $l\le\Lambda$ and $1.6\le\Lambda$. □

Lemma 29.35. *Let* $\mathcal{T}$ *be a root tower of type* (Φ,n,κ) *with (Rt1)–(Rt4) and with (Rt5) for* (r,L) *with the constant* $C'\ge 0$. *Let* $B_0\ge 128$ *be an integer with* $|\mathrm{Bd}|\le B_0$, *and let* (α,β) *be the proportionality constants at stage* B_0. *Then*

$$|\kappa\,\alpha+n\,\beta|\le C'\,n\,\mathrm{Lg}(B_0).$$

Proof. Fix $a_0\in\Phi$ and let Q_1 be the class of Lemma 29.30(e): homogeneous of degree D, $\deg^{\mathrm{fl}}(Q_1)=1$. Put $F_1=\mathrm{ps}_{a_0,1}Q_1$. By Lemma 29.30(c), $\mathrm{Sh}(F_1)=\deg^{\mathrm{fl}}(\mathrm{ps}_0^{(a_0,+)}Q_1)=\kappa$ and $\mathrm{To}(F_1)=n$. By (Rt4), $h^{\mathrm{cd}}(F_1)=0$; and $\rho_p(a_0,1,Q_1)=0$ for all p, so $\rho_{>B_0}(a_0,1,Q_1)=0$. By (Rt5), $\widehat{\deg}_{B_0}(F_1)=\sum_Z \mathrm{hd}_{B_0,Z,a_0}\mathrm{vol}(Z)$. So (29.14) for $(a_0,1,Q_1)$ reads $\sum_Z \mathrm{hd}_{B_0,Z,a_0}\mathrm{vol}(Z)=-\mathrm{vol}_{\mathcal{T}}(\kappa\alpha+n\beta)$. With $\mathrm{vol}(Z)>0$ (Rt1) and the bound of (Rt5) (applicable as $B_0\ge 128$ and $|\mathrm{Bd}|\le B_0$),

$$\mathrm{vol}_{\mathcal{T}}\,|\kappa\alpha+n\beta|\le\sum_Z|\mathrm{hd}_{B_0,Z,a_0}|\,\mathrm{vol}(Z)\le C'n\,\mathrm{Lg}(B_0)\,\mathrm{vol}_{\mathcal{T}},$$

and dividing by $\mathrm{vol}_{\mathcal{T}}>0$ gives the assertion. □

Proposition 29.36. *(a) The test degree is positive:* $\deg^{\mathrm{fl}}(\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb})>0$.

(b) Let fr *be a Frobenius labelling, S a finite set of primes and $B_0 \ge 4|\Phi|n^2$ an integer. For every power-sum decomposition $(Q_{a,k})$ of the test class, with Frobenius tails formed with* fr *and S,*

$$\Big|\sum_{a\in\Phi}\sum_{k=1}^{n}\rho_{>B_0}(a,k,Q_{a,k})\Big| \le 1.6\,(D^{\mathrm{gr}}+1)\deg^{\mathrm{fl}}(\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb}).$$

Proof. Write $G = D^{\mathrm{gr}}$, $\mathrm{te} = \mathrm{pl}^{G+1}\mathrm{fb}$, $T = \deg^{\mathrm{fl}}(\mathrm{pl}^{G}\mathrm{fb})$, $f = |\Phi|$, and $d = D - G$; fb is the product of the d linear forms $x_{b,i'} - x_{b,i}$ over $b \in \Phi$ and the pairs $i < i'$ with $i, i' \le \kappa$ or $i, i' > \kappa$ (there are $f(\kappa(\kappa-1)/2 + (n-\kappa)(n-\kappa-1)/2) = D - G$ of them).

(a), Step 1: the fibre class as a derivative in the weight. For a weight v let $\mathcal{D} = \prod(\partial/\partial v_{b,i} - \partial/\partial v_{b,i'})$, the product over the same d triples (b,i,i'), a differential operator with constant coefficients in the variables v, acting on the coefficients of elements of $\mathcal{R}$ that depend polynomially on v. Each factor ∂_j of $\mathcal{D}$ satisfies $\partial_j \mathrm{lr}(v) = x_{b,i'} - x_{b,i}$, which does not depend on v; by induction on the number of factors, $\mathcal{D}\,\mathrm{lr}(v)^N = \frac{N!}{(N-d)!}\,\mathrm{fb}\,\mathrm{lr}(v)^{N-d}$ for $N \ge d$. With $N = D$ and the linearity of $\deg^{\mathrm{fl}}$, which commutes with $\mathcal{D}$, (29.18) gives

$$\deg^{\mathrm{fl}}\big(\mathrm{fb}\,\mathrm{lr}(v)^G\big) = \frac{G!}{D!}\,\mathcal{D}\deg^{\mathrm{fl}}\big(\mathrm{lr}(v)^D\big) = \frac{G!}{\mathrm{sf}_n^{f}}\,\mathcal{D}\prod_{b\in\Phi}V(v_b),$$

$$\mathrm{sf}_n = \prod_{j=0}^{n-1} j!, \quad V(y) = \prod_{1\le i<i'\le n}(y_i - y_{i'}).$$

(a), Step 2: evaluation at a Levi point. We have $\mathcal{D} = \prod_b \mathcal{D}_b$ with $\mathcal{D}_b$ involving only the variables v_b, so $\mathcal{D}\prod_b V(v_b) = \prod_b \mathcal{D}_b V(v_b)$. Let v_b have entries α at $i \le \kappa$ and β at $i > \kappa$, and expand at $v_b + z$: $V(v_b+z) = V'(z')V''(z'')C(z)$, where $z' = (z_1,\dots,z_\kappa)$, $z'' = (z_{\kappa+1},\dots,z_n)$, V', V'' are their Vandermonde products and $C(z) = \prod_{i\le\kappa<i'}(\alpha-\beta+z_i-z_{i'}) = (\alpha-\beta)^{\kappa(n-\kappa)} + C_+(z)$ with C_+ without constant term. Now $\mathcal{D}_b = V'(\partial')V''(\partial'')$ is homogeneous of order $e = \kappa(\kappa-1)/2 + (n-\kappa)(n-\kappa-1)/2$ and $V'V''$ is homogeneous of degree e; so $\mathcal{D}_b(V'V''C_+)$ has no constant term, and

$$\mathcal{D}_b V(v_b) = (\alpha-\beta)^{\kappa(n-\kappa)}\,V'(\partial')V'(z')\cdot V''(\partial'')V''(z'').$$

For $m \ge 1$, $V_m(\partial)V_m(y) = \prod_{j=1}^{m} j!$ (V_m the Vandermonde product of m variables): $V_m(y) = \det(y_i^{m-j}) = \sum_{w\in S_m}\mathrm{sgn}(w)\,y^{\delta_m\circ w}$ with $\delta_m = (m-1,\dots,0)$, likewise for $V_m(\partial)$; $\partial^\mu y^\nu = \mu!$ if $\mu = \nu$ and 0 if $\mu \ne \nu$, $|\mu| = |\nu|$; and $\delta_m \circ w$ are pairwise distinct, so $V_m(\partial)V_m(y) = m!\prod_{j=0}^{m-1} j!$. Hence $\mathcal{D}_b V(v_b) = (\alpha-\beta)^{\kappa(n-\kappa)}\prod_{j=1}^{\kappa} j!\prod_{j=1}^{n-\kappa} j!$.

(a), Step 3. We have $\mathrm{pl} = \mathrm{lr}(v^{\mathrm{pl}})$ with $v^{\mathrm{pl}}_{b,i} = 0$ $(i \le \kappa)$, -1 $(i > \kappa)$ if $n - \kappa \le \kappa$, and $v^{\mathrm{pl}}_{b,i} = 1$ $(i \le \kappa)$, 0 $(i > \kappa)$ otherwise; in both cases $\alpha - \beta = 1$ in every block. By Steps 1 and 2,

$$T = G!\Big(\frac{\prod_{j=1}^{\kappa} j!\,\prod_{j=1}^{n-\kappa} j!}{\prod_{j=0}^{n-1} j!}\Big)^{f} > 0.$$

(b), Step 4: all labels. Let $(Q_{a,k})$ be a power-sum decomposition of te and $M_k = (G+1)\,T\,X^{k-1}/k!$. For $a \in \Phi$, a label $\sigma = (a', \pm)$ and $2 \le k \le n$ put $y_i = \Lambda_{a,k}(x_{a',i}^{k-1}\mathrm{te})$. By Lemma 29.37(b) with $(b,m) = (a', k-1)$, $\sum_i y_i = 0$; since $\mathrm{ps}_{k-1}^{(a',+)} = \sum_{i\le\kappa} x_{a',i}^{k-1}$, $\mathrm{ps}_{k-1}^{(a',-)} = (-1)^{k-1}\sum_{i>\kappa} x_{a',i}^{k-1}$ and P is one of the two sets $\{1,\dots,\kappa\}$, $\{\kappa+1,\dots,n\}$, Lemma 29.37(c) gives

$$|k\deg^{\mathrm{fl}}(\mathrm{ps}_{k-1}^{\sigma}Q_{a,k})| = |\Lambda_{a,k}(\mathrm{ps}_{k-1}^{\sigma}\mathrm{te})| = \Big|\sum_{i\in P} y_i\Big| = \Big|k\deg^{\mathrm{fl}}\Big(\Big(\sum_{i\in P} x_{a',i}^{k-1}\Big)Q_{a,k}\Big)\Big| \le M_k$$

by Proposition 29.38.

(b), Step 5: one prime. Let $p > B_0$ be prime. Since $\rho_p(a,1,\cdot) = 0$ and, for $k \ge 2$, $p^{-k} \le 1/2$, Step 4 gives

$$\sum_{k=1}^{n} |\rho_p(a,k,Q_{a,k})| \le \log p \sum_{k=2}^{n} M_k \sum_{m\ge 0} p^{-k(m+1)} \le \log p \sum_{k=2}^{n} 2M_k p^{-k}$$
$$= \frac{2(G+1)T\log p}{X} \sum_{k=2}^{n} \frac{y^k}{k!}, \qquad y = \frac{X}{p}.$$

As $X = 2\kappa(n-\kappa)+2n \le n^2/2 + 2n \le 3n^2/2$ ($n \ge 2$) and $p > B_0 \ge 4fn^2 \ge 4n^2$, $y \le 3/8$; and $\sum_{k\ge 2} y^k/k! = e^y - 1 - y \le \frac{y^2}{2} e^y$ (Taylor's formula with Lagrange remainder). Since $e^{3/8} < 1.455$,

$$\sum_{k=1}^{n} |\rho_p(a,k,Q_{a,k})| \le 1.455\,(G+1)\,T\,X\,\frac{\log p}{p^2}.$$

(b), Step 6: the primes above B_0. Let $\theta(t) = \sum_{p\le t} \log p$; Chebyshev's bound $\prod_{p\le t} p \le 4^t$ gives $\theta(t) \le t \log 4$. For $Y > B_0$, partial summation gives

$$\sum_{B_0 < p \le Y} \frac{\log p}{p^2} = \frac{\theta(Y)}{Y^2} - \frac{\theta(B_0)}{B_0^2} + 2\int_{B_0}^{Y} \frac{\theta(t)}{t^3}\,dt \le \frac{\log 4}{Y} + 2\log 4\Big(\frac{1}{B_0} - \frac{1}{Y}\Big) \le \frac{2\log 4}{B_0}.$$

By Lemma 29.31(b) each tail $\rho_{>B_0}(a,k,Q_{a,k})$ is the sum of an absolutely convergent series of the $\rho_p(a,k,Q_{a,k})$, $p > B_0$, $p \notin S$, so $|\rho_{>B_0}(a,k,Q_{a,k})| \le \sum_{p>B_0} |\rho_p(a,k,Q_{a,k})|$. Summing Step 5 over a and over the primes,

$$\Big|\sum_{a,k} \rho_{>B_0}(a,k,Q_{a,k})\Big| \le f \cdot 1.455\,(G+1)\,T\,X \cdot \frac{2\log 4}{4fn^2} = \frac{1.455\log 4}{2} \cdot \frac{X}{n^2}\,(G+1)\,T \le 1.6\,(G+1)\,T,$$

since $X/n^2 = 2\kappa(n-\kappa)/n^2 + 2/n \le 1/2 + 1$ and $1.455 \cdot \log 4 \cdot 1.5/2 < 1.52$. □

Lemma 29.37. *For $a \in \Phi$ and $1 \le k \le n$ let $\Lambda_{a,k} \colon \mathcal{R} \to \mathbb{R}$ be the linear form*

$$\Lambda_{a,k}(H) = (-1)^D \sum_{c=1}^{n} \sum_{w \in S_n^\Phi} \operatorname{sgn}(w)\,[x^{(\delta + k e_{a,c})\circ w}]H,$$

where $e_{a,c}$ is the unit exponent vector at (a,c). Let $\delta^\circ = (n-1, n-2, \dots, 0)$ and, for $m \in \mathbb{N}^n$ and $y = (y_1,\dots,y_n)$ with entries in a commutative $\mathbb{R}$-algebra, $g_m(y) = \det\big((-y_i)^{m_j}/m_j!\big)_{1\le i,j\le n}$; for a weight v write $v_b = (v_{b,1},\dots,v_{b,n})$, and $e_c \in \mathbb{N}^n$ for the unit vector at c.

(a) For every weight v, every $N \ge 0$ and every $m \in \mathbb{N}^{\Phi\times\{1,\dots,n\}}$ with $\sum_{b,i} m_{b,i} = N$,

$$\sum_{w \in S_n^\Phi} \operatorname{sgn}(w)\,[x^{m\circ w}]\,\mathrm{lr}(v)^N = N! \prod_{b\in\Phi} g_{m_b}(v_b), \qquad m_b = (m_{b,1},\dots,m_{b,n}).$$

Consequently

$$\deg^{\mathrm{fl}}\big(\mathrm{lr}(v)^D\big) = (-1)^D D! \prod_{b\in\Phi} g_{\delta^\circ}(v_b) = \frac{D!}{\big(\prod_{j=0}^{n-1} j!\big)^{|\Phi|}} \prod_{b\in\Phi} \prod_{1\le i<i'\le n} (v_{b,i} - v_{b,i'}), \tag{29.18}$$

which is positive if v is regular dominant, and

$$\Lambda_{a,k}\big(\mathrm{lr}(v)^{D+k}\big) = (-1)^D (D+k)! \sum_{c=1}^{n} g_{\delta^\circ + k e_c}(v_a) \prod_{b\ne a} g_{\delta^\circ}(v_b). \tag{29.19}$$

(b) For $b \in \Phi$, $m \geq 1$ and $G \in \mathcal{R}$: $\Lambda_{a,k}(\mathrm{ps}_{b,m}G) = k\deg^{\mathrm{fl}}(G)$ if $(b,m) = (a,k)$, and $\Lambda_{a,k}(\mathrm{ps}_{b,m}G) = 0$ otherwise.

(c) If $(Q_{b,m})$ is a power-sum decomposition of $F \in \mathcal{R}$, then $k\deg^{\mathrm{fl}}(\gamma Q_{a,k}) = \Lambda_{a,k}(\gamma F)$ for every $\gamma \in \mathcal{R}$; in particular the left side does not depend on the decomposition.

Proof. Alternation. For $w \in S_n^\Phi$ and $F \in \mathcal{R}$ let $w \cdot F$ be obtained by substituting $x_{b,w_b(i)}$ for $x_{b,i}$, and $A(F) = \sum_w \mathrm{sgn}(w)\, w \cdot F$. Since $w \cdot x^m = x^{m'}$ with $m'_{b,w_b(i)} = m_{b,i}$, we have $w \cdot x^m = x^E$ if and only if $m = E \circ w$, so $[x^E](w \cdot F) = [x^{E\circ w}]F$ and

$$[x^E]A(F) = \sum_w \mathrm{sgn}(w)\,[x^{E\circ w}]F \qquad (E \in \mathbb{N}^{\Phi\times\{1,\dots,n\}}). \tag{29.20}$$

Thus $\deg^{\mathrm{fl}}(F) = (-1)^D[x^\delta]A(F)$ by (21.1), and $\Lambda_{a,k}(H) = (-1)^D\sum_c[x^{\delta+ke_{a,c}}]A(H)$. As $v\cdot(w\cdot F) = (vw)\cdot F$, $v\cdot A(G) = \mathrm{sgn}(v)A(G)$; with $[x^E](v\cdot f) = [x^{E\circ v}]f$ this gives, for $f = A(G)$,

$$[x^{E\circ v}]f = \mathrm{sgn}(v)\,[x^E]f \quad (v \in S_n^\Phi), \qquad [x^E]f = 0 \text{ if two entries of } E \text{ in one block are equal,} \tag{29.21}$$

the second by taking for v the transposition of the two positions ($E\circ v = E$, $\mathrm{sgn}(v) = -1$). We put $[x^E]H = 0$ when E has a negative entry.

(a) By the multinomial theorem, $\mathrm{lr}(v)^N = N!\sum_{m'}\prod_{b,i}\frac{(-v_{b,i})^{m'_{b,i}}}{m'_{b,i}!}\,x^{m'}$, the sum over m' with $\sum m'_{b,i} = N$. Hence

$$\sum_w \mathrm{sgn}(w)[x^{m\circ w}]\mathrm{lr}(v)^N = N!\prod_b\sum_{w_b\in S_n}\mathrm{sgn}(w_b)\prod_{i=1}^n\frac{(-v_{b,i})^{m_{b,w_b(i)}}}{m_{b,w_b(i)}!} = N!\prod_b g_{m_b}(v_b)$$

by the Leibniz expansion of the determinants. For $m = \delta$, $g_{\delta^\circ}(v_b) = \bigl(\prod_{j=0}^{n-1}j!\bigr)^{-1}\det\bigl((-v_{b,i})^{n-j}\bigr) = \bigl(\prod_{j=0}^{n-1}j!\bigr)^{-1}\prod_{i<i'}(v_{b,i'} - v_{b,i})$ by Vandermonde's determinant $\det(y_i^{n-j}) = \prod_{i<i'}(y_i - y_{i'})$; the product has $n(n-1)/2$ factors, and $|\Phi|n(n-1)/2 = D$, which gives (29.18) with $\deg^{\mathrm{fl}} = (-1)^D[x^\delta]A$. Every factor $v_{b,i} - v_{b,i'}$ $(i < i')$ is positive for a regular dominant v. The identity (29.19) is the case $N = D+k$, $m = \delta + ke_{a,c}$, summed over c.

(b) As $\mathrm{ps}_{b,m}$ is invariant, $A(\mathrm{ps}_{b,m}G) = \mathrm{ps}_{b,m}f$ with $f = A(G)$, and

$$[x^E](\mathrm{ps}_{b,m}f) = \sum_{i=1}^n[x^{E-me_{b,i}}]f.$$

Hence

$$\Lambda_{a,k}(\mathrm{ps}_{b,m}G) = (-1)^D\sum_{c,i}T(c,i), \qquad T(c,i) = [x^{\delta+ke_{a,c}-me_{b,i}}]f.$$

If $b \neq a$, the block b of the exponent is $\delta^\circ - me_i$, with entry $n-i-m$ at i. If it is negative, $T(c,i) = 0$; otherwise $i+m \leq n$, and the position $i+m \neq i$ of block b carries the same entry $n-(i+m)$, so $T(c,i) = 0$ by (29.21). So $\Lambda_{a,k}(\mathrm{ps}_{b,m}G) = 0$. Let $b = a$. For $c = i$, $T(i,i) = [x^{\delta+(k-m)e_{a,i}}]f$. For $c \neq i$, the entry at (a,i) is $n-i-m$; if it is negative, $T(c,i) = 0$; otherwise $i+m \leq n$, and if $i+m \neq c$ the unchanged entry at $(a,i+m)$ equals it, so $T(c,i) = 0$; if $c = i+m$, exchanging the positions i and c of block a (entries $n-i-m$ and $n-i-m+k$) turns the exponent into $\delta+(k-m)e_{a,i}$, so $T(i+m,i) = -[x^{\delta+(k-m)e_{a,i}}]f$ by (29.21). Summing,

$$\sum_{c,i}T(c,i) = \sum_{i=1}^n[x^{\delta+(k-m)e_{a,i}}]f - \sum_{i=1}^{n-m}[x^{\delta+(k-m)e_{a,i}}]f = \sum_{i=\max(1,n-m+1)}^n[x^{\delta+(k-m)e_{a,i}}]f.$$

If $m = k$, the exponent is δ and there are k terms ($k \le n$): the sum is $k[x^\delta]f$. If $m \neq k$, fix i in the range and put $j = i - (k-m) \neq i$; the entry of $\delta + (k-m)e_{a,i}$ at (a,i) is $n - j$. If it is negative the term is 0. Otherwise $j \le n$, and $j \ge 1$: for $m > k$ because $j > i$, for $m < k$ because $i \ge n - m + 1 \ge k - m + 1$ (as $k \le n$). So the unchanged entry at (a,j) equals the entry at (a,i), and the term is 0 by (29.21). Hence $\Lambda_{a,k}(\mathrm{ps}_{a,m}G) = (-1)^D\,\delta_{mk}\,k[x^\delta]A(G) = \delta_{mk}\,k\deg^{\mathrm{fl}}(G)$.

(c) We have $\gamma F = \sum_b \sum_{m=1}^{n} \mathrm{ps}_{b,m}(\gamma Q_{b,m})$; apply the linear form $\Lambda_{a,k}$ and (b). □

Proposition 29.38. *Let $P = \{\kappa+1,\dots,n\}$ if $n - \kappa \le \kappa$ and $P = \{1,\dots,\kappa\}$ otherwise, and $X = 2\big(\kappa(n-\kappa)+n\big)$. For every power-sum decomposition $(Q_{a,k})$ of the test class, all $a, a' \in \Phi$ and every $2 \le k \le n$,*

$$\Big|k\deg^{\mathrm{fl}}\Big(\Big(\sum_{i\in P} x_{a',i}^{k-1}\Big)Q_{a,k}\Big)\Big| \le (D^{\mathrm{gr}}+1)\deg^{\mathrm{fl}}(\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb})\,\frac{X^{k-1}}{k!}.$$

Proof. Write $G = D^{\mathrm{gr}}$, $\mathrm{te} = \mathrm{pl}^{G+1}\mathrm{fb}$, $T = \deg^{\mathrm{fl}}(\mathrm{pl}^G\mathrm{fb})$, $f = |\Phi|$, $d = D - G$, $r = k-1 \ge 1$, $\mathrm{sf}_n = \prod_{j=0}^{n-1} j!$, $m = \min(\kappa, n-\kappa)$; for a weight w write $w_b = (w_{b,1},\dots,w_{b,n})$. The fibre class fb is the product of the d linear forms $x_{b,i'} - x_{b,i}$ over the triples (b,i,i') with $b \in \Phi$, $i < i'$, and $i, i' \le \kappa$ or $i, i' > \kappa$.

Step 1: the fibre class as a derivative in the weight. Let $\mathcal{D} = \prod(\partial/\partial w_{b,i} - \partial/\partial w_{b,i'})$, the product over the same d triples, a differential operator with constant coefficients acting on elements of $\mathcal{R}$ whose coefficients are polynomials in the weight w. Each factor sends $\mathrm{lr}(w)$ to $x_{b,i'} - x_{b,i}$, which does not depend on w; by induction on the number of factors, $\mathcal{D}\,\mathrm{lr}(w)^N = \frac{N!}{(N-d)!}\,\mathrm{fb}\,\mathrm{lr}(w)^{N-d}$ for $N \ge d$. Put $L(w) = \Lambda_{a,k}(\mathrm{fb}\,\mathrm{lr}(w)^{G+k})$. As $\Lambda_{a,k}$ and $\deg^{\mathrm{fl}}$ are linear and commute with $\mathcal{D}$, (29.19), (29.26) and (29.18) give, with $g_{\delta^\circ}(y) = \pm V(y)/\mathrm{sf}_n$ ((29.18)) and $\mathcal{D} = \prod_b \mathcal{D}_{\mathrm{L},b}$, where $\mathcal{D}_{\mathrm{L},b}$ is the operator $\mathcal{D}_{\mathrm{L}}$ in the variables w_b,

$$\begin{aligned} L(w) &= \frac{(G+k)!}{(D+k)!}\,\mathcal{D}\,\Lambda_{a,k}\big(\mathrm{lr}(w)^{D+k}\big) \\ &= (-1)^D(G+k)!\,\mathcal{D}\Big[r_k(w_a)\prod_{b\in\Phi} g_{\delta^\circ}(w_b)\Big] = \pm\frac{(G+k)!}{\mathrm{sf}_n^f}\,\Psi(w), \qquad (29.22) \\ \deg^{\mathrm{fl}}\big(\mathrm{fb}\,\mathrm{lr}(w)^G\big) &= \frac{G!}{D!}\,\mathcal{D}\deg^{\mathrm{fl}}\big(\mathrm{lr}(w)^D\big) = \frac{G!}{\mathrm{sf}_n^f}\prod_{b\in\Phi}\mathcal{D}_{\mathrm{L},b}V(w_b), \qquad (29.23) \end{aligned}$$

where $\Psi(w) = \mathcal{D}_{\mathrm{L},a}\big[r_k V\big](w_a)\cdot\prod_{b\neq a}\mathcal{D}_{\mathrm{L},b}V(w_b)$ and the sign in (29.22) depends only on (Φ, n).

Step 2: invariance under shifting a block. Let $b \in \Phi$, $\mu \in \mathbb{R}$ and 1_b the weight with entries 1 in the block b and 0 elsewhere. Then $\mathrm{lr}(w + \mu 1_b) = \mathrm{lr}(w) - \mu\,\mathrm{ps}_{b,1}$, so by the binomial theorem $\mathrm{fb}\,\mathrm{lr}(w+\mu 1_b)^{G+k} - \mathrm{fb}\,\mathrm{lr}(w)^{G+k} = \mathrm{ps}_{b,1}H$ for some $H \in \mathcal{R}$, and $\Lambda_{a,k}(\mathrm{ps}_{b,1}H) = 0$ by Lemma 29.37(b) since $k \ge 2$. Hence $L(w+\mu 1_b) = L(w)$ for all w, and differentiating, every partial derivative of L, hence of Ψ, is invariant under these shifts. Likewise $V(y + \mu(1,\dots,1)) = V(y)$.

Step 3: the Grassmannian class as a point and a generating polynomial. Let v^{pl} be the weight with $v^{\mathrm{pl}}_{b,i} = 0$ $(i \le \kappa)$, -1 $(i > \kappa)$ if $n - \kappa \le \kappa$, and $v^{\mathrm{pl}}_{b,i} = 1$ $(i \le \kappa)$, 0 $(i > \kappa)$ otherwise; then $\mathrm{lr}(v^{\mathrm{pl}}) = \mathrm{pl}$. Adding $\mu = 1$, resp. $\mu = -1$, to every block turns v^{pl} into the weight w^P all of whose blocks equal y^P. By (29.23), Step 2 and Lemma 29.40(a) with $\varphi = 1$,

$$T = \frac{G!}{\mathrm{sf}_n^f}\prod_b \mathcal{D}_{\mathrm{L}}V(y^P) = \frac{G!\,c_0^f}{\mathrm{sf}_n^f}. \qquad (29.24)$$

Let $a' \in \Phi$ and $i \in P$, and let $e_{a',i}$ be the unit weight at (a',i). Then $\mathrm{lr}(v^{\mathrm{pl}} - te_{a',i}) = \mathrm{pl} + t\,x_{a',i}$, and by the binomial theorem the coefficient of t^r in $\mathrm{fb}(\mathrm{pl} + tx_{a',i})^{G+k}$ is $\binom{G+k}{r} x_{a',i}^r\,\mathrm{te}$. Applying $\Lambda_{a,k}$ and Taylor's formula for the polynomial $t \mapsto L(v^{\mathrm{pl}} - te_{a',i})$,

$$\Lambda_{a,k}(x_{a',i}^r\,\mathrm{te}) = \binom{G+k}{r}^{-1} \frac{(-1)^r}{r!}\,\frac{\partial^r L}{\partial w_{a',i}^r}(v^{\mathrm{pl}}).$$

As $(G+k)!/(\binom{G+k}{r} r!) = (G+1)!$, (29.22) and Step 2 give

$$\Big|\sum_{i\in P} \Lambda_{a,k}(x_{a',i}^r\,\mathrm{te})\Big| = \frac{(G+1)!}{\mathrm{sf}_n^f}\,\Big|\sum_{i\in P} \frac{\partial^r \Psi}{\partial w_{a',i}^r}(w^P)\Big|. \tag{29.25}$$

Step 4: evaluation block by block. The function Ψ is a product of factors each depending on one block, and $\mathcal{D}_{\mathrm{L}} V(y^P) = c_0$, $\mathcal{D}_{\mathrm{L}}(r_k V)(y^P) = c_0 r_k(y^P)$ by Lemma 29.40(a). If $a' \neq a$, the derivatives fall on the factor of the block a', and

$$\sum_{i\in P} \frac{\partial^r \Psi}{\partial w_{a',i}^r}(w^P) = c_0\, r_k(y^P) \cdot E_r(1) \cdot c_0^{f-2}, \qquad \Big|\cdot\Big| \le c_0^f\,\frac{2^{k-1}}{k!}\,r!\binom{m(n-m)}{r} \le c_0^f\,\frac{(2\kappa(n-\kappa))^r}{k!}$$

by Lemma 29.40(b), (c) and $r!\binom{N}{r} \le N^r$, $m(n-m) = \kappa(n-\kappa)$. If $a' = a$,

$$\sum_{i\in P} \frac{\partial^r \Psi}{\partial w_{a,i}^r}(w^P) = E_r(r_k) \cdot c_0^{f-1}, \qquad \Big|\cdot\Big| \le c_0^f\, r!\binom{\kappa(n-\kappa)+k}{r}\frac{2^{k-1}}{k!} \le c_0^f\,\frac{\big(2(\kappa(n-\kappa)+k)\big)^r}{k!}$$

by Lemma 29.40(c). As $k \le n$, both are at most $c_0^f X^r/k!$.

Step 5: conclusion. By Lemma 29.37(c) with $\gamma = \sum_{i\in P} x_{a',i}^{k-1}$, $k\deg^{\mathrm{fl}}(\gamma Q_{a,k}) = \Lambda_{a,k}(\gamma\,\mathrm{te}) = \sum_{i\in P}\Lambda_{a,k}(x_{a',i}^r\mathrm{te})$. By (29.25), Step 4 and (29.24),

$$|k\deg^{\mathrm{fl}}(\gamma Q_{a,k})| \le \frac{(G+1)!}{\mathrm{sf}_n^f}\,c_0^f\,\frac{X^{k-1}}{k!} = (G+1)\,T\,\frac{X^{k-1}}{k!}.$$

□

Lemma 29.39. *Let $n \ge 1$ and $1 \le k \le n$. For $1 \le c \le k$ let $\lambda_c = (k-c+1, 1^{c-1})$, the hook with c rows and $|\lambda_c| = k$, and for $y = (y_1, \dots, y_n)$ put*

$$r_k(y) = (-1)^k \sum_{c=1}^{k} (-1)^{c-1}\,\frac{(n-c)!}{(n-c+k)!}\,s_{\lambda_c}(y),$$

a symmetric polynomial, homogeneous of degree k.

(a) For y with entries in a commutative $\mathbb{R}$-algebra,

$$\sum_{c=1}^{n} g_{\delta^\circ + ke_c}(y) = r_k(y)\,g_{\delta^\circ}(y). \tag{29.26}$$

(b) For $1 \le c \le k$,

$$\frac{(n-c)!}{(n-c+k)!}\,s_{\lambda_c}(1,\dots,1) = \frac{1}{k!}\binom{k-1}{c-1};$$

hence

$$\sum_{c=1}^{k} \frac{(n-c)!}{(n-c+k)!}\,s_{\lambda_c}(1,\dots,1) = \frac{2^{k-1}}{k!}.$$

Proof. *(a)* Put $w=-y$. Then $g_m(y)=a_m(w)/\prod_j m_j!$. Let $1\le c\le n$. If $c>k$, the entry $n-c+k$ of $\delta^\circ+ke_c$ at position c equals the entry $n-(c-k)$ at position $c-k\ge 1$, so two columns of the determinant coincide and $a_{\delta^\circ+ke_c}=0$. If $c\le k$, the entry $n-c+k\ge n$ at position c exceeds the entries $n-1,\dots,n-c+1$ at the positions before c; moving the c-th column to the first place (a cyclic permutation of c columns, of sign $(-1)^{c-1}$) gives the exponent $(n-c+k,n-1,\dots,n-c+1,n-c-1,\dots,0)=\lambda_c+\delta^\circ$, and λ_c has $c\le n$ parts. So $a_{\delta^\circ+ke_c}(w)=(-1)^{c-1}a_{\lambda_c+\delta^\circ}(w)=(-1)^{c-1}s_{\lambda_c}(w)\,a_{\delta^\circ}(w)$ by Theorem B.171(a). Moreover $\prod_j(\delta^\circ+ke_c)_j!=\big(\prod_j\delta^\circ_j!\big)(n-c+k)!/(n-c)!$, and $s_{\lambda_c}(w)=(-1)^k s_{\lambda_c}(y)$ by homogeneity of degree k. Summing over c gives (29.26).

(b) The cells of λ_c have the contents $0,1,\dots,k-c$ along the first row and $-1,\dots,-(c-1)$ down the first column, and the hook lengths k at the corner, $k-c,\dots,1$ along the rest of the first row and $c-1,\dots,1$ down the rest of the first column. By Theorem B.171(b),

$$s_{\lambda_c}(1,\dots,1)=\frac{\prod_{j=0}^{k-c}(n+j)\prod_{j=1}^{c-1}(n-j)}{k\,(k-c)!\,(c-1)!}=\frac{(n+k-c)!/(n-c)!}{k\,(k-c)!\,(c-1)!},$$

so $\frac{(n-c)!}{(n-c+k)!}s_{\lambda_c}(1,\dots,1)=\frac{1}{k\,(k-c)!\,(c-1)!}=\frac{1}{k!}\binom{k-1}{c-1}$. Summing, $\sum_{c=1}^{k}\binom{k-1}{c-1}=2^{k-1}$. □

Lemma 29.40. *Let $n\ge 2$ and $1\le\kappa<n$. Let $P=\{\kappa+1,\dots,n\}$ if $n-\kappa\le\kappa$ and $P=\{1,\dots,\kappa\}$ otherwise, and $m=|P|=\min(\kappa,n-\kappa)$. Let $y^P\in\mathbb{R}^n$ have the entries $y^P_i=0$ for $i\in P$ and, for $i\notin P$, $y^P_i=1$ if $P=\{\kappa+1,\dots,n\}$ and $y^P_i=-1$ if $P=\{1,\dots,\kappa\}$. For $y=(y_1,\dots,y_n)$ let $V(y)=\prod_{1\le i<i'\le n}(y_i-y_{i'})$, $\partial_i=\partial/\partial y_i$, and let $\mathcal{D}_{\mathrm{L}}=\prod(\partial_i-\partial_{i'})$, the product over the pairs $i<i'$ with $i,i'\le\kappa$ or $i,i'>\kappa$. Put $c_0=\prod_{j=1}^{\kappa}j!\,\prod_{j=1}^{n-\kappa}j!$, and for a symmetric polynomial φ in y and $r\ge 1$ let*

$$E_r(\varphi)=\Big(\sum_{i\in P}\partial_i^{\,r}\,\mathcal{D}_{\mathrm{L}}(V\varphi)\Big)(y^P).$$

(a) $\mathcal{D}_{\mathrm{L}}(V\varphi)(y^P)=c_0\,\varphi(y^P)$ *for every symmetric polynomial φ; in particular $\mathcal{D}_{\mathrm{L}}V(y^P)=c_0$.*

(b) $|E_r(1)|\le c_0\,r!\binom{m(n-m)}{r}$ *for every $r\ge 1$.*

(c) For $1\le k\le n$: $|r_k(y^P)|\le 2^{k-1}/k!$, and $|E_r(r_k)|\le c_0\,r!\binom{m(n-m)+k}{r}\dfrac{2^{k-1}}{k!}$ for every $r\ge 1$.

Proof. Let $R=\{1,\dots,n\}\setminus P$, $M=|R|=n-m$, so $\{m,M\}=\{\kappa,n-\kappa\}$. For φ as in the statement put $E_0(\varphi)=\mathcal{D}_{\mathrm{L}}(V\varphi)(y^P)$ and $p_0^P=1$, $p_r^P(z)=\sum_{i\in P}z_i^r$ $(r\ge 1)$.

Step 1: the apolar pairing. For real polynomials f,g in variables z put $\langle f,g\rangle=(f(\partial_z)g)(0)$. It is bilinear, $\langle z^\mu,z^\nu\rangle=\mu!$ if $\mu=\nu$ and 0 otherwise ($\mu!=\prod_i\mu_i!$); hence homogeneous polynomials of different degrees pair to 0, and for two disjoint sets of variables $z=(z',z'')$, $\langle a(z')b(z''),c(z')d(z'')\rangle=\langle a,c\rangle\langle b,d\rangle$ (check on monomials). For $\alpha\in\mathbb{N}^q$ with strictly decreasing entries the alternant $a_\alpha(z)=\sum_{w\in S_q}\operatorname{sgn}(w)z^{w\alpha}$ is a signed sum of $q!$ distinct monomials, all with the exponent multiset of α; so for strictly decreasing α,β, $\langle a_\alpha,a_\beta\rangle=q!\,\alpha!$ if $\alpha=\beta$ and 0 otherwise.

Step 2: expansion at y^P. Write $y=y^P+z$. Differential operators with constant coefficients commute with translation, so for $r\ge 0$

$$E_r(\varphi)=\big\langle p_r^P(z)\,V_1(z)V_2(z),\ \Psi(z)\big\rangle,\qquad \Psi(z)=(V\varphi)(y^P+z),$$

where V_1, V_2 are the Vandermonde products $\prod(z_i-z_{i'})$ over the pairs $i<i'$ in $\{1,\dots,\kappa\}$, resp. in $\{\kappa+1,\dots,n\}$, so that $\mathcal{D}_{\mathrm{L}}=V_1(\partial)V_2(\partial)$. The differences of the entries of y^P are 0 within

$\{1,\dots,\kappa\}$ and within $\{\kappa+1,\dots,n\}$, and $y^P_i - y^P_{i'} = 1$ for $i \le \kappa < i'$; hence $\Psi = V_1 V_2\, S$ with

$$S(z) = \prod_{i\le\kappa<i'} (1 + z_i - z_{i'}) \cdot \varphi(y^P + z).$$

Let $V_P(z_P)$, $V_R(z_R)$ be the Vandermonde products of the variables in P, resp. R (in increasing order of indices); $V_1V_2 = V_PV_R$. Write $S = \sum_\beta z_R^\beta S_\beta(z_P)$. By Step 1, $E_r(\varphi) = \sum_\beta \langle p_r^P V_P, V_P S_\beta\rangle \, \langle V_R, V_R z_R^\beta\rangle$, and only $\beta = 0$ contributes, for reasons of degree:

$$E_r(\varphi) = \langle V_R, V_R\rangle \cdot \langle p_r^P(z_P) V_P(z_P),\ V_P(z_P) S_0(z_P)\rangle, \qquad S_0(z_P) = S(z_P, 0). \tag{29.27}$$

Here $S_0(z_P) = \prod_{i\in P}(1+\epsilon z_i)^M \cdot \varphi(y^P + (z_P,0))$ with $\epsilon = 1$ if $P = \{1,\dots,\kappa\}$ and $\epsilon = -1$ otherwise, and S_0 is symmetric in z_P (φ is symmetric and y^P vanishes on P).

Step 3: the pairing in the Schur basis. Let $\delta_q = (q-1,\dots,1,0)$; $V_P = a_{\delta_m}(z_P)$, $V_R = a_{\delta_M}(z_R)$, so $\langle V_R, V_R\rangle = M!\,\delta_M!$ by Step 1. Let $S_{0,r}$ be the homogeneous part of degree r of S_0; by Theorem B.172(a), $S_{0,r} = \sum_\lambda \sigma_\lambda s_\lambda(z_P)$, over the partitions λ of r with at most m parts, and $V_P s_\lambda = a_{\lambda+\delta_m}$ by Theorem B.171(a). For $r = 0$, (29.27) gives $E_0(\varphi) = M!\,\delta_M!\,m!\,\delta_m!\,S_0(0) = c_0\,\varphi(y^P)$, as $q!\,\delta_q! = \prod_{j=1}^q j!$; this is (a). For $r \ge 1$, by Theorem B.172(b) $p_r^P V_P = \sum_j (-1)^j a_{h_j+\delta_m}$, the sum over $0 \le j < \min(r,m)$, with the hooks $h_j = (r-j, 1^j)$; by (29.27) and Step 1,

$$E_r(\varphi) = M!\,\delta_M!\,m! \sum_j (-1)^j \sigma_{h_j} (h_j+\delta_m)! = c_0 \sum_j (-1)^j \sigma_{h_j} \prod_{x\in h_j} (m + c(x)),$$

using $(\lambda+\delta_m)!/\delta_m! = \prod_i \prod_{l=1}^{\lambda_i}(m-i+l) = \prod_{x\in\lambda}(m+c(x))$, $c(x)$ the content of the cell x. By Theorem B.171(b), $\prod_{x\in h_j}(m+c(x)) = s_{h_j}(1,\dots,1)\cdot r\,(r-1-j)!\,j! \le r!\,s_{h_j}(1,\dots,1)$ (m ones; the hook lengths of h_j are r, $r-1-j,\dots,1$ and $j,\dots,1$). Hence

$$|E_r(\varphi)| \le c_0\, r! \sum_j |\sigma_{h_j}|\, s_{h_j}(1,\dots,1) \qquad (r \ge 1). \tag{29.28}$$

Step 4: majorants. For symmetric polynomials F, F^+ in z_P write $F \preceq F^+$ if $|[s_\lambda]F| \le [s_\lambda]F^+$ for every λ (coefficients in the Schur basis, Theorem B.172(a)). (i) If $F \preceq F^+$ and $H \preceq H^+$ then $FH \preceq F^+H^+$, as $[s_\nu](FH) = \sum c^\nu_{\lambda\mu}[s_\lambda]F\,[s_\mu]H$ with $c^\nu_{\lambda\mu} \ge 0$ (Theorem B.172(e)). (ii) If $F \preceq F^+$, then $\sum_j |[s_{h_j}]F|\, s_{h_j}(1,\dots,1) \le \sum_\lambda [s_\lambda]F^+_r\, s_\lambda(1,\dots,1) = F^+_r(1,\dots,1) = [t^r]F^+(t,\dots,t)$, where F^+_r is the homogeneous part of degree r; all terms are ≥ 0 since $s_\lambda(1,\dots,1) \ge 0$ (Theorem B.171(a)). (iii) By Theorem B.172(c) with $u = \epsilon z_P$ and $w = (1,\dots,1)$ of length M, $\prod_{i\in P}(1+\epsilon z_i)^M = \sum_\lambda \epsilon^{|\lambda|} s_{\lambda'}(1,\dots,1) s_\lambda(z_P) \preceq C^+ := \prod_{i\in P}(1+z_i)^M$, and $C^+(t,\dots,t) = (1+t)^{mM}$. (iv) Let $1 \le k \le n$ and let $\gamma = \pm 1$ be the common entry of y^P on R. Since r_k is symmetric, $r_k(y^P + (z_P,0)) = r_k(z_P,\gamma,\dots,\gamma)$, and by Theorem B.172(d) and the homogeneity of skew Schur polynomials,

$$s_{\lambda_c}(z_P,\gamma,\dots,\gamma) = \sum_{\mu\subseteq\lambda_c} \gamma^{k-|\mu|} s_{\lambda_c/\mu}(1,\dots,1)\, s_\mu(z_P),$$

with $s_{\lambda_c/\mu}(1,\dots,1) \ge 0$. Hence $r_k(y^P + (z_P,0)) \preceq \rho^+(z_P) := \sum_{c=1}^k \frac{(n-c)!}{(n-c+k)!} s_{\lambda_c}(z_P,1,\dots,1)$. By the tableau formula, $s_{\lambda_c}(t,\dots,t,1,\dots,1) = \sum_T t^{j(T)}$ over the semistandard tableaux T of shape λ_c with entries in $\{1,\dots,n\}$, with $0 \le j(T) \le k$; so its coefficients are at most those of $(1+t)^k s_{\lambda_c}(1,\dots,1)$, and by Lemma 29.39(b) the coefficients of $\rho^+(t,\dots,t)$ are at most those of $(1+t)^k 2^{k-1}/k!$.

Step 5: the bounds. (b): for $\varphi = 1$, $S_0 = \prod_{i\in P}(1+\epsilon z_i)^M \preceq C^+$ by (iii); by (29.28) and (ii), $|E_r(1)| \le c_0\, r!\,[t^r](1+t)^{mM} = c_0\, r!\binom{mM}{r}$. (c): $r_k(y^P)$ is the coefficient of $s_\emptyset = 1$ in

$r_k(y^P + (z_P, 0))$, so by (iv) $|r_k(y^P)| \le \rho^+(0) \le 2^{k-1}/k!$. For $\varphi = r_k$, $S_0 \preceq C^+\rho^+$ by (i), (iii), (iv); the product of two polynomials in t with non-negative coefficients is monotone in each factor, so $[t^r](C^+\rho^+)(t,\dots,t) \le [t^r](1+t)^{mM+k}\, 2^{k-1}/k! = \binom{mM+k}{r} 2^{k-1}/k!$, and (29.28) with (ii) gives the bound. □

Lemma 29.41. *Let u be a weight satisfying the gap hypothesis with gap $g > 0$, let B be an integer with $b_0(u,g) < B+1$, let* fr *be a Frobenius labelling and S a finite set of primes, and let $(Q_{a,k})$ be a power-sum decomposition of $\mathrm{lr}(u)^{D+1}$. Then, with tails formed with* fr *and S,*

$$\Big|\sum_{a\in\Phi}\sum_{k=1}^{n} \rho_{>B}(a,k,Q_{a,k})\Big| \le 2\,(D+1)\deg^{\mathrm{fl}}\big(\mathrm{lr}(u)^D\big)\min(\kappa, n-\kappa)\,\mathrm{spr}(u).$$

Proof. Write $E = (D+1)\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D)$, $m = \min(\kappa, n-\kappa)$, $U_a = \max_i u_{a,i} - \min_i u_{a,i} \ge 0$ (so $\sum_a U_a = \mathrm{spr}(u)$), $H = \|u\|_\infty/g \ge 0$, $A = n(1+2H) > 0$ and $b = b_0(u,g) = 4|\Phi|n^2(1+H) > 0$. Since $\rho_p(a,1,\cdot) = 0$, only $2 \le k \le n$ contribute.

Step 1: a prime sum. Let $\theta(t) = \sum_{p\le t} \log p$. By Chebyshev's bound $\prod_{p\le t} p \le 4^t$, $\theta(t) \le t\log 4$ for $t > 0$. For $X > b$, partial summation gives

$$\sum_{b<p\le X} \frac{\log p}{p^2} = \frac{\theta(X)}{X^2} - \frac{\theta(b)}{b^2} + 2\int_b^X \frac{\theta(t)}{t^3}\,dt \le \frac{\log 4}{X} + 2\log 4\Big(\frac{1}{b} - \frac{1}{X}\Big) \le \frac{2\log 4}{b},$$

so $\sum_{p>b} \log p/p^2 \le 2\log 4/b$.

Step 2: one tail. Fix a and $2 \le k \le n$, and let $M_k = E\,m\,\frac{2^{k-1}}{k!} U_a A^{k-1}$, the bound of Proposition 29.42: $|k\deg^{\mathrm{fl}}(\mathrm{ps}^\sigma_{k-1} Q_{a,k})| \le M_k$ for every label σ. For a prime $p > B$ we have $p \ge B+1 > b$, and by the definition of ρ_p,

$$|\rho_p(a,k,Q_{a,k})| \le \log p \sum_{j\ge 0} p^{-k(j+1)} M_k \le 2M_k \log p\, p^{-k} \le \frac{2M_k}{b^{k-2}} \cdot \frac{\log p}{p^2},$$

using $p^{-k} \le 1/2$ and $p^{k-2} \ge b^{k-2}$. The tail $\rho_{>B}(a,k,Q_{a,k})$ exists by Lemma 29.31(b), and by Step 1

$$|\rho_{>B}(a,k,Q_{a,k})| \le \frac{2M_k}{b^{k-2}} \cdot \frac{2\log 4}{b} \le 5.56\,\frac{M_k}{b^{k-1}} = 5.56\,\frac{E\,m\,U_a}{k!}\Big(\frac{2A}{b}\Big)^{k-1},$$

as $4\log 4 < 5.56$.

Step 3: summing over k. Since $M_k \ge 0$ and $(2A/b)^{k-1}/k! > 0$, $E\,m\,U_a \ge 0$. Further $2A/b = (1+2H)/(2|\Phi|n(1+H)) \le 1/(|\Phi|n) \le 1/2$, because $1+2H \le 2(1+H)$ and $n \ge 2$. Hence

$$\begin{aligned}\Big|\sum_{k=1}^{n} \rho_{>B}(a,k,Q_{a,k})\Big| &\le 5.56\,E\,m\,U_a \sum_{k\ge 2} \frac{(1/2)^{k-1}}{k!} = 5.56\,E\,m\,U_a \cdot 2(e^{1/2} - \tfrac{3}{2}) \\ &\le 5.56 \cdot 0.3\,E\,m\,U_a \le 2E\,m\,U_a,\end{aligned}$$

since $e^{1/2} < 1.6488$. Summing over a with the triangle inequality gives the assertion. □

Proposition 29.42. *Let u be a weight satisfying the gap hypothesis with gap $g > 0$, and $(Q_{a,k})$ a power-sum decomposition of $\mathrm{lr}(u)^{D+1}$. Then for every $a \in \Phi$, every label σ and every $2 \le k \le n$,*

$$\begin{aligned}|k\deg^{\mathrm{fl}}(\mathrm{ps}^\sigma_{k-1} Q_{a,k})| \le{}& (D+1)\deg^{\mathrm{fl}}\big(\mathrm{lr}(u)^D\big)\min(\kappa, n-\kappa) \\ &\frac{2^{k-1}}{k!}\Big(\max_i u_{a,i} - \min_i u_{a,i}\Big)\Big(n\big(1+2\|u\|_\infty/g\big)\Big)^{k-1}.\end{aligned}$$

Proof. Write $F = \mathrm{lr}(u)^{D+1}$, $\mathrm{fd} = \deg^{\mathrm{fl}}(\mathrm{lr}(u)^D)$, $m_0 = \min(\kappa, n-\kappa)$, I for the small block ($|I| = m_0$), $U = \max_i u_{a,i} - \min_i u_{a,i}$, $H = \|u\|_\infty/g$, and $\sigma = (a', \pm)$. By the gap hypothesis u is regular dominant, so $\mathrm{fd} > 0$ by (29.18).

Step 1: reduction to single roots of the small block. By Lemma 29.37(c), $k \deg^{\mathrm{fl}}(\mathrm{ps}^\sigma_{k-1} Q_{a,k}) = \Lambda_{a,k}(\mathrm{ps}^\sigma_{k-1} F)$. Put $y_i = \Lambda_{a,k}(x_{a',i}^{k-1} F)$. By Lemma 29.37(b) for $(b,m) = (a', k-1)$ (here $k-1 \geq 1$ and $k-1 \neq k$), $\sum_{i=1}^n y_i = \Lambda_{a,k}(\mathrm{ps}_{a',k-1} F) = 0$. Since $\mathrm{ps}^{(a',+)}_{k-1} = \sum_{i \leq \kappa} x_{a',i}^{k-1}$ and $\mathrm{ps}^{(a',-)}_{k-1} = (-1)^{k-1} \sum_{i>\kappa} x_{a',i}^{k-1}$,

$$|\Lambda_{a,k}(\mathrm{ps}^\sigma_{k-1} F)| = \Big|\sum_{i \leq \kappa} y_i\Big| = \Big|\sum_{i>\kappa} y_i\Big| = \Big|\sum_{i \in I} y_i\Big| \leq \sum_{i \in I} |y_i|.$$

As $k \leq n$ and $0 \leq U \leq 2\|u\|_\infty$, $k + (n-1)U/g \leq n(1+2H)$; so it suffices to show, for every $i \in I$,

$$|y_i| \leq (D+1)\,\mathrm{fd}\,\frac{2^{k-1}}{k!}\,U\Big(k + \frac{(n-1)U}{g}\Big)^{k-1}. \tag{29.29}$$

Fix $i \in I$. By the gap hypothesis, $|u_{a',i} - u_{a',j}| \geq g$ for every $j \neq i$.

Step 2: normalising the block a. Let $\mu = \min_c u_{a,c}$ and let u' be the weight with $u'_{a,c} = u_{a,c} - \mu$ and $u'_{b,c} = u_{b,c}$ for $b \neq a$. Then $\mathrm{lr}(u') = \mathrm{lr}(u) + \mu\,\mathrm{ps}_{a,1}$, so $\mathrm{lr}(u')^{D+1} - F = \mathrm{ps}_{a,1} G'$ for some $G' \in \mathcal{R}$, and Lemma 29.37(b) ($m = 1 \neq k$) gives $y_i = \Lambda_{a,k}(x_{a',i}^{k-1} \mathrm{lr}(u')^{D+1})$. The entries of u'_a lie in $[0, U]$; differences of entries within a block are those of u, so $|u'_{a',i} - u'_{a',j}| \geq g$ for $j \neq i$, and $\deg^{\mathrm{fl}}(\mathrm{lr}(u')^D) = \mathrm{fd}$ by (29.18).

Step 3: a generating polynomial. For real t let u^t be the weight u' with the entry at (a', i) replaced by $u'_{a',i} - t$; then $\mathrm{lr}(u^t) = \mathrm{lr}(u') + t\,x_{a',i}$. By the binomial theorem, $\Lambda_{a,k}(\mathrm{lr}(u^t)^{D+k}) = \sum_j \binom{D+k}{j} t^j \Lambda_{a,k}(x_{a',i}^j \mathrm{lr}(u')^{D+k-j})$, a polynomial in t whose coefficient of t^{k-1} is $\binom{D+k}{k-1} y_i$. By (29.19), for every t it equals $(-1)^D (D+k)!\,P(t)$ with

$$P(t) = \sum_{c=1}^n g_{\delta^\circ + k e_c}(u_a^t) \prod_{b \neq a} g_{\delta^\circ}(u_b^t),$$

a polynomial in t. As $(D+k)!/\binom{D+k}{k-1} = (D+1)!(k-1)!$,

$$y_i = (-1)^D (D+1)!\,(k-1)!\,[t^{k-1}] P(t). \tag{29.30}$$

Step 4: the bumped determinants. For $y = (y_1, \ldots, y_n)$ with entries in a commutative $\mathbb{R}$-algebra,

$$\begin{aligned} \sum_{c=1}^n g_{\delta^\circ + k e_c}(y) &= r_k(y)\, g_{\delta^\circ}(y), \\ r_k(y) &= (-1)^k \sum_{c=1}^k (-1)^{c-1} \frac{(n-c)!}{(n-c+k)!}\, s_{\lambda_c}(y), \quad \lambda_c = (k-c+1, 1^{c-1}). \end{aligned} \tag{29.31}$$

Indeed, with $w = -y$, $g_m(y) = a_m(w)/\prod_j m_j!$. For $c > k$ the entry $n-c+k$ of $\delta^\circ + k e_c$ at position c equals the entry at position $c-k \geq 1$, so $a_{\delta^\circ + k e_c} = 0$. For $c \leq k$ ($\leq n$) the entry $n-c+k \geq n$ exceeds the entries $n-1, \ldots, n-c+1$ at the positions before c; moving it to the first position (a cyclic permutation of c columns, of sign $(-1)^{c-1}$) gives the exponent $\lambda_c + \delta^\circ$, and λ_c has $c \leq n$ parts. So $a_{\delta^\circ + k e_c}(w) = (-1)^{c-1} a_{\lambda_c + \delta^\circ}(w) = (-1)^{c-1} s_{\lambda_c}(w)\, a_{\delta^\circ}(w)$ by Theorem B.171(a). Together with $\prod_j (\delta^\circ + k e_c)_j! = \big(\prod_j \delta^\circ_j!\big)(n-c+k)!/(n-c)!$ and $s_{\lambda_c}(-y) =$

$(-1)^k s_{\lambda_c}(y)$ (homogeneity of degree k) this is (29.31). By Theorem B.171(b) for λ_c (contents $0,\dots,k-c$ along the first row and $-1,\dots,-(c-1)$ down the first column; hook lengths k at the corner, $k-c,\dots,1$ along the row and $c-1,\dots,1$ down the column),

$$\frac{(n-c)!}{(n-c+k)!}\,s_{\lambda_c}(1,\dots,1)=\frac{(n-c)!}{(n-c+k)!}\cdot\frac{(n+k-c)!/(n-c)!}{k\,(k-c)!\,(c-1)!}=\frac{1}{k!}\binom{k-1}{c-1}. \tag{29.32}$$

Step 5: coefficient bounds. By (29.31), $P(t)=R(t)\Pi(t)$ with $R(t)=r_k(u_a^t)$ and $\Pi(t)=\prod_b g_{\delta^\circ}(u_b^t)$. (i) For $s\ge 0$, $|[t^s]R(t)|\le\binom{k}{s}U^{k-s}2^{k-1}/k!$. Write $s_{\lambda_c}(y)=\sum_\alpha K_\alpha y^\alpha$ with $K_\alpha\ge 0$ and $|\alpha|=k$ (Theorem B.171(a)). If $a'\ne a$, R is constant, $0\le s_{\lambda_c}(u'_a)\le U^k\sum_\alpha K_\alpha = U^k s_{\lambda_c}(1,\dots,1)$ as the entries of u'_a lie in $[0,U]$, and $[t^s]R=0$ for $s\ge 1$. If $a'=a$, then $u_a^t=u'_a-te_i$ and $[t^s](u'_a-te_i)^\alpha=(-1)^s\binom{\alpha_i}{s}(u'_a)^{\alpha-se_i}$, of absolute value at most $\binom{k}{s}U^{k-s}$; so $|[t^s]s_{\lambda_c}(u_a^t)|\le\binom{k}{s}U^{k-s}s_{\lambda_c}(1,\dots,1)$. In both cases (29.31), (29.32) and $\sum_{c=1}^{k}\binom{k-1}{c-1}=2^{k-1}$ give the bound. (ii) For $s\ge 0$, $|[t^s]\Pi(t)|\le\binom{n-1}{s}g^{-s}|\Pi(0)|$. Only the factor $b=a'$ depends on t, and $g_{\delta^\circ}(y)=\bigl(\prod_j j!\bigr)^{-1}\prod_{j<j'}(y_{j'}-y_j)$. For $y=u_{a'}^t$, with $d_j=u'_{a',j}-u'_{a',i}$ ($|d_j|\ge g$ for $j\ne i$), the factors containing the index i are $y_{j'}-y_i=d_{j'}(1+t/d_{j'})$ for $j'>i$ and $y_i-y_j=-d_j(1+t/d_j)$ for $j<i$; the others do not depend on t. Hence $\Pi(t)=\Pi(0)\prod_{j\ne i}(1+t/d_j)$, so $[t^s]\Pi(t)=\Pi(0)\,e_s\bigl((1/d_j)_{j\ne i}\bigr)$ with e_s the elementary symmetric polynomial of $n-1$ numbers of absolute value at most $1/g$, and the bound follows. (iii) Hence

$$\begin{aligned}
|[t^{k-1}]P(t)| &\le \sum_{s=0}^{k-1}|[t^s]R|\,|[t^{k-1-s}]\Pi| \\
&\le |\Pi(0)|\,\frac{2^{k-1}}{k!}\sum_{s=0}^{k-1}\binom{k}{s}U^{k-s}\binom{n-1}{k-1-s}g^{-(k-1-s)} \\
&\le |\Pi(0)|\,\frac{2^{k-1}}{k!}\cdot\frac{U}{(k-1)!}\Bigl(k+\frac{(n-1)U}{g}\Bigr)^{k-1}.
\end{aligned}$$

Step 6: conclusion. By (29.18) for u', $\mathrm{fd}=\deg^{\mathrm{fl}}(\mathrm{lr}(u')^D)=(-1)^D D!\,\Pi(0)$, so $D!\,|\Pi(0)|=\mathrm{fd}$. By (29.30) and (iii), $|y_i|\le(D+1)!(k-1)!\,|[t^{k-1}]P(t)|\le(D+1)\,\mathrm{fd}\,\frac{2^{k-1}}{k!}U\bigl(k+(n-1)U/g\bigr)^{k-1}$, which is (29.29). Summing over the m_0 elements of I (Step 1) gives the assertion. □

Theorem 29.43. *Let $\mathcal{F}$ be a parametrised host family with covolume records carrying chain data, deep-cover data, component data, model-slope data, self-intersection data and proportionality data with the properties (Hp1)–(Hp6) for the constants $C_{\mathrm{b}},C_{\mathrm{f}}\ge 0$. Then for all large points with $j<m$, with $D=D^{\mathrm{fl}}$,*

$$\sum_{Z\in\mathcal{Z}}\mathrm{si}(Z)\ge-(C_{\mathrm{b}}+C_{\mathrm{f}}+2)(D+1)\,\mathrm{fd}\Bigl(\sum_{Z\in\mathcal{Z}}\mathrm{vo}(Z)\Bigr)(\mathrm{lz}\cdot\mathrm{wn}+\mathrm{wsp});$$

that is, the inequality of (Si3) holds with the constant $C_{\mathrm{b}}+C_{\mathrm{f}}+2$.

Proof. Fix a large point with $j<m$ (beyond the threshold of (Hp1)–(Hp6)). Write $n=n_{\mathrm{rk}}$, $\kappa=n_{\mathrm{ct}}$, $\alpha=\alpha^{\mathrm{pr}}$, $\beta=\beta^{\mathrm{pr}}$, $V=\sum_Z\mathrm{vo}(Z)$, $E=(D+1)\,\mathrm{fd}$ and $w_a=\sum_{i=1}^{n}|\mathrm{dw}_{a,i}|$. Then $V>0$ and $E>0$ (self-intersection data), $\mathrm{wn},\mathrm{wsp}\ge 0$ (model-slope data), and $\mathrm{lz}=\log(B\,\mathrm{rd}_E)+|L|\ge\log 128>0$, since $B\ge 128$ and $\mathrm{rd}_E\ge 1$.

Partial sums of a non-increasing sequence. Let $y_1\ge y_2\ge\cdots\ge y_n$ be real, $1\le\kappa<n$, $s=\sum_{i\le\kappa}y_i$, $t=\sum_{i\le n}y_i$ and $w=\sum_{i\le n}|y_i|$. Then

$$0\le s-\frac{\kappa}{n}\,t\le w,\qquad |t|\le w. \tag{29.33}$$

Indeed, $y_i \ge y_\kappa$ for $i \le \kappa$ and $y_i \le y_{\kappa+1} \le y_\kappa$ for $i > \kappa$, so $s \ge \kappa y_\kappa$ and $t - s \le (n-\kappa)y_\kappa$; hence $(n-\kappa)s \ge \kappa(n-\kappa)y_\kappa \ge \kappa(t-s)$, that is, $ns \ge \kappa t$. Further $s - \frac{\kappa}{n}t = (1-\frac{\kappa}{n})s - \frac{\kappa}{n}(t-s) \le |s| + |t-s| \le \sum_{i\le\kappa}|y_i| + \sum_{i>\kappa}|y_i| = w$, since $0 < \kappa/n < 1$; and $|t| \le w$ by the triangle inequality.

The linear term at each embedding. Fix a. By (Hp1), (29.33) applies to $y_i = \mathrm{dw}_{a,i}$, with $s = s_a$, $t = t_a$ and w_a in place of w. Write

$$\alpha s_a + \beta t_a = \alpha\Big(s_a - \frac{\kappa}{n}t_a\Big) + (\kappa\alpha + n\beta)\frac{t_a}{n}.$$

By (Hp4) and $s_a - \frac{\kappa}{n}t_a \ge 0$, the first term is $\ge -C_{\mathrm{b}}\,\mathrm{lz}\,(s_a - \frac{\kappa}{n}t_a) \ge -C_{\mathrm{b}}\,\mathrm{lz}\,w_a$, using $C_{\mathrm{b}}\,\mathrm{lz} \ge 0$ and $s_a - \frac{\kappa}{n}t_a \le w_a$. By (Hp5) and $|t_a| \le w_a$, the absolute value of the second term is at most $C_{\mathrm{f}}\,n\,\mathrm{lz}\,|t_a|/n \le C_{\mathrm{f}}\,\mathrm{lz}\,w_a$. Hence $\alpha s_a + \beta t_a \ge -(C_{\mathrm{b}} + C_{\mathrm{f}})\,\mathrm{lz}\,w_a$, and summing over a, with $\sum_a w_a = \mathrm{wn}$ by (Hp1),

$$\sum_{a=1}^{n_{\mathrm{em}}}(\alpha s_a + \beta t_a) \ge -(C_{\mathrm{b}} + C_{\mathrm{f}})\,\mathrm{lz}\cdot\mathrm{wn}.$$

Conclusion. By (Hp3), the last display and (Hp6),

$$h^{\mathrm{cd}} + E\sum_a(\alpha s_a + \beta t_a) + \mathrm{ft} \ge -E\,(C_{\mathrm{b}} + C_{\mathrm{f}})\,\mathrm{lz}\cdot\mathrm{wn} - 2E\,\mathrm{wsp} \ge -(C_{\mathrm{b}} + C_{\mathrm{f}} + 2)\,E\,\big(\mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp}\big),$$

the last step because E, $\mathrm{lz}\cdot\mathrm{wn}$ and wsp are non-negative. Multiplying by $V > 0$ and using (Hp2) gives the assertion. □

Theorem 29.44. *Let $\mathcal{F}$ be a parametrised host family with covolume records carrying chain data, deep-cover data, component data, model-slope data and self-intersection data with the properties (Si1)–(Si3) for the constant $C \ge 0$. Then for all large points with $j < m$*

$$\frac{1}{|\mathcal{Z}|}\sum_{Z\in\mathcal{Z}}\mathrm{ns}(Z) \ge -C\big(\mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp}\big);$$

that is, the inequality of (Ms3) holds with the constant C.

Proof. Fix a large point with $j < m$ (beyond the threshold of (Si1)–(Si3)); write $D = D^{\mathrm{fl}}$, $N = |\mathcal{Z}| \ge 1$ and $X = \mathrm{lz}\cdot\mathrm{wn} + \mathrm{wsp}$. By (Si2) all volumes equal one number $v = \mathrm{vo}(Z) > 0$, so $\sum_Z \mathrm{vo}(Z) = Nv$. By (Si1),

$$\frac{1}{N}\sum_Z \mathrm{ns}(Z) = \frac{1}{N}\sum_Z \frac{\mathrm{si}(Z)}{(D+1)\,v\,\mathrm{fd}} = \frac{\sum_Z \mathrm{si}(Z)}{(D+1)\,\mathrm{fd}\,Nv} = \frac{\sum_Z \mathrm{si}(Z)}{(D+1)\,\mathrm{fd}\sum_Z \mathrm{vo}(Z)}.$$

The denominator $(D+1)\,\mathrm{fd}\sum_Z \mathrm{vo}(Z)$ is positive, since $D \ge 0$, $\mathrm{fd} > 0$ and $\mathrm{vo}(Z) > 0$. Dividing (Si3) by it gives $\frac{1}{N}\sum_Z \mathrm{ns}(Z) \ge -C\,X$, which is the assertion. □

30 Properties of good root towers

This section proves Corollary 30.5: the good root towers of Proposition 31.14 have all the properties that Theorem 29.14 uses, namely (Fs), (Rt1)–(Rt7) and (Gm1), (Gm2). The corollary is used only in Step 1 of the proof of Theorem 29.14. Proposition 31.14 provides geometric inputs, and each property is derived from some of them: (Rt3) from linked stages (Theorem 30.28), (Rt5) from the change of stage and flat determinant classes at the base stage (Theorem 30.31), (Gm2) from positive volumes and the geometric inputs (Gi1)–(Gi5) (Theorem 30.15), and (Rt4), (Rt7) from the Fubini–Study height of the flag scheme, which is positive on dominant weights and

vanishes on the first power sums (Lemma 30.16), and whose value at the test class is bounded through a Grassmannian (Proposition 30.21).

The main result of the section is Theorem 30.6, proportionality with two constants (Rt2) with error $O(\log B/B)$. Its proof has three steps. The fibre-square identities, together with geometric proportionality, make the proportionality defect antisymmetric under the Levi reflections (Theorem 30.8). The wall theorem 30.10 shows that the defect is $O(\log B/B)$ on compact parts of the walls; it compares an upper and a lower bound for the successive minima of rational combinations of the root lines (Lemma 30.12, Proposition 30.11), the lower bound resting on Zhang's theorem on semipositive metrics [276]. Finally, a Levi-antisymmetric polynomial that nearly vanishes on the walls is near a Vandermonde multiple of a central linear form (Lemma 30.7), and with the symmetry of the embeddings this leaves two constants (Theorem 30.9). The remaining lemmas are projection formulas and height identities for arithmetic intersection numbers [110], [109], and Weyl's degree formula for powers of linear root forms (Lemma 30.14).

Definition 30.1. Limit form. A root model $\mathcal{X}$ has **convergent regularisation** if for every monomial $x^m \in \mathcal{R}^+$ the limit $\lim_{t\to 0^+} \mathrm{ai}_t(\mathcal{X}; x^m)$ exists; its **limit intersection form** $\mathrm{pp}(\mathcal{X};\cdot)$ is then the linear form on $\mathcal{R}^+$ with $\mathrm{pp}(\mathcal{X}; x^m) = \lim_{t\to 0^+} \mathrm{ai}_t(\mathcal{X}; x^m)$.

Nefness at a prime. For a prime ℓ, a vector $c \in \mathbb{R}^{\mathrm{Rt}^+}$ is **nef at** ℓ on $\mathcal{X}$ if

$$\sum_r c_r \deg(\mathcal{O}(e_r)|_W) \geq 0$$

for every integral closed curve $W \subseteq \mathcal{X}_{\mathbb{F}_\ell}$. A weight μ is **above the nef threshold at** $b \in \mathbb{R}$ if $4|\Phi|\,\|\mu\|_\infty \leq (b-2)(\mu_{a,\kappa} - \mu_{a,\kappa+1})$ for every $a \in \Phi$.

Geometric inputs. Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$, and write $t(s) = \exp(-1/s^2)$. The realised tower has the **geometric inputs** if for every integer $B \geq 2$ and every $Z \in \mathrm{Cp}$, writing $\mathcal{X} = \mathcal{X}_{B,Z}$:

(Gi1) Geometric proportionality We have $\mathrm{gi}(\mathcal{X}; F) = \mathrm{vol}(Z)\deg^{\mathrm{fl}}(F)$ for every $F \in \mathcal{R}$ homogeneous of degree D;

(Gi2) Convergence The model $\mathcal{X}$ has convergent regularisation and $\mathrm{pp}(\mathcal{X}; F) = \widehat{\deg}_{B,Z}(F)$ for every $F \in \mathcal{R}$;

(Gi3) Semipositivity For every regular dominant weight u the vector $c(u_s, s)$ is semipositive at $t(s)$ on $\mathcal{X}$ for all sufficiently small $s > 0$;

(Gi4) Nefness at primes For every prime ℓ and every dominant weight μ such that $\ell \leq B$, or $\ell \in \mathrm{Bd}$, or μ is above the nef threshold at ℓ, the vector $c(\mu, 0)$ is nef at ℓ on $\mathcal{X}$;

(Gi5) Auxiliary multidegree Every entry of e_* is ≥ 0.

The realised tower has the **rate of regularisation** if it has (Gi2) and:

(Gi6) Rate of regularisation For every $A \geq 0$ there are $s_0 > 0$ and $C \geq 0$ such that for all integers $B \geq 2$, all $Z \in \mathrm{Cp}$, all weights λ with $\|\lambda\|_1 \leq A$ and all $0 < s \leq s_0$ with $\lambda_s = \lambda - s\,\nu_{\mathcal{T}}$ dominant, writing $\ell = \mathrm{lr}(\lambda_s) + s\,x_*$,

$$|\mathrm{ai}_{t(s)}(\mathcal{X}_{B,Z}; \ell^{D+1}) - \mathrm{pp}(\mathcal{X}_{B,Z}; \ell^{D+1})| \leq C\,s.$$

Definition 30.2. For $\tau \in \Phi$ and $1 \le i < n$ with $i \ne \kappa$, the **Levi reflection** $s_{\tau,i}$ is the involution of the weights exchanging the entries $v_{\tau,i}$ and $v_{\tau,i+1}$. For a Levi reflection $s = s_{\tau,i}$ put $\xi_s = x_{\tau,i} - x_{\tau,i+1} \in \mathcal{R}$, and call a weight c **s-invariant** if $c_{\tau,i} = c_{\tau,i+1}$. Let $\mu = (\mu_k)_{k\ge1}$ be real numbers. A pair $(\hat h, g)$ of linear forms on $\mathcal{R}$ satisfies the **fibre-square identities for s with moments μ** if for every s-invariant weight c

$$\hat h(\mathrm{lr}(c)^{D+1}) = 0, \qquad \hat h(\xi_s^{2k}\,\mathrm{lr}(c)^{D+1-2k}) = -2\,\mu_k\, g(\xi_s^{2k-1}\,\mathrm{lr}(c)^{D+1-2k}) \quad (1 \le k \le (D+1)/2). \tag{30.1}$$

Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ whose root models have convergent regularisation. The realised tower has the **fibre-square identities** if, with the **sphere moments** $\mathrm{sph}_k = \frac12 \sum_{j=0}^{k-1} \frac{1}{2j+1}$ ($k \ge 1$; $\mathrm{sph}_1 = \frac12$, $\mathrm{sph}_2 = \frac23$):

(Fq1) Root models For every integer $B \ge 2$, every $Z \in \mathrm{Cp}$ and every Levi reflection s, the pair $(\mathrm{pp}(\mathcal{X}_{B,Z};\cdot)|_{\mathcal{R}},\ \mathrm{gi}(\mathcal{X}_{B,Z};\cdot)|_{\mathcal{R}})$ satisfies the fibre-square identities for s with moments $\mathrm{sph} = (\mathrm{sph}_k)_{k\ge1}$.

(Fq2) Compact dual For every Levi reflection s, the pair $(h^{\mathrm{cd}}, \deg^{\mathrm{fl}})$ satisfies the fibre-square identities for s with moments $\mathrm{sph} = (\mathrm{sph}_k)_{k\ge1}$.

Definition 30.3. Points and heights. A **point** of a root model $\mathcal{X}$ is an integral closed subscheme $W \subseteq \mathcal{X}$, flat over $\mathbb{Z}$, of absolute dimension 1; $\deg W_{\mathbb{Q}}$ is the degree of the zero-cycle $W_{\mathbb{Q}}$. A set F of points is **Zariski dense** if $\mathcal{X}$ is the only closed subscheme of $\mathcal{X}$ containing every $W \in F$. For $c \in \mathbb{R}^{\mathrm{Rt}^+}$ and $t > 0$ the **height** of a point W is

$$h_{c,t}(W) = \frac{1}{\deg W_{\mathbb{Q}}} \sum_{r \in \mathrm{Rt}^+} c_r\, \widehat{\deg}(\widehat c_1(\overline{\mathcal{L}}_r(t)) \cdot [W]), \tag{30.2}$$

and $\ell_c = \sum_{r\in\mathrm{Rt}^+} c_r\, k_r\, x_r \in \mathcal{R}^+$ ($k_r = 1$ for $r \in \mathrm{Rt}$, $k_* = k_{\mathcal{X}}$), so that $\ell_{c(u,s)} = \mathrm{lr}(u) + s\, x_*$.

Inputs. Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ whose root models have convergent regularisation. For a weight λ write $\mathrm{cg}_a(\lambda) = \lambda_{a,\kappa} - \lambda_{a,\kappa+1}$ (the **cut gap** at $a \in \Phi$). For $c_0 > 0$ and $\theta > 0$, a weight λ is **(c_0,θ)-spaced** if $\|\lambda\|_\infty \le 1$, $\mathrm{cg}_a(\lambda) \ge \theta$ for every a, and $\lambda_{a,i} - \lambda_{a,i+1} \ge c_0$ for every a and every $i \ne \kappa$. The realised tower has the **wall inputs** if, writing $\mathcal{X} = \mathcal{X}_{B,Z}$, $\lambda_s = \lambda - s\,\nu_{\mathcal{T}}$ and $\ell = \mathrm{lr}(\lambda_s) + s\,x_*$ throughout:

(Wn) Nefness at the threshold For every $A \ge 0$ there is $s_0 > 0$ such that for all $B \ge 2$, Z, weights λ with $\|\lambda\|_1 \le A$ and $0 < s \le s_0$ such that λ_s is dominant and above the nef threshold at B, the vector $c(\lambda_s, s)$ is nef on fibres of $\mathcal{X}$ and semipositive at $t(s)$;

(Wp) Heights of points For every $c_0 > 0$ there are $\theta_0 > 0$ and $C \ge 0$ such that for all $B \ge 2$, Z, all $\theta \in (0, \theta_0]$ with $s = \theta^2$ rational, and every (c_0,θ)-spaced weight λ with rational entries: (a) $h_{c(\lambda_s,s),t(s)}(W) \ge -C\log(B/s)$ for every point W of $\mathcal{X}$; (b) there is a Zariski dense set F of points of $\mathcal{X}$ with $h_{c(\lambda_s,s),t(s)}(W) \le C\log B$ for $W \in F$;

(Wg) Geometric degree For all $B \ge 2$, Z, every dominant weight λ with rational entries and every rational $s \ge 0$ with λ_s dominant, we have $0 \le \mathrm{gi}(\mathcal{X}; \ell^D) \le \mathrm{vol}(Z)\deg^{\mathrm{fl}}(\mathrm{lr}(\lambda)^D)$;

(Wd) Monomial bound There is $C \ge 0$ with $|\mathrm{pp}(\mathcal{X}_{B,Z}; x^m)| \le C\log B$ for all $B \ge 2$, Z and all monomials $x^m \in \mathcal{R}^+$;

(Wh) Compact-dual height on the walls We have $h^{\mathrm{cd}}(\mathrm{lr}(v)^{D+1}) = 0$ for every dominant weight v with $\mathrm{cg}_a(v) = 0$ for some $a \in \Phi$.

For a permutation σ of Φ and $F \in \mathcal{R}$ let $\sigma \cdot F$ be obtained by substituting $x_{\sigma(a),i}$ for $x_{a,i}$. The root tower has the **symmetry inputs** if:

(Ys1) Galois symmetry of the arithmetic degrees There is a group Γ of permutations of Φ, acting transitively on Φ, with $\widehat{\deg}_B(\sigma \cdot F) = \widehat{\deg}_B(F)$ for all $B \geq 2$, $\sigma \in \Gamma$, $F \in \mathcal{R}$;

(Ys2) Symmetry of the compact-dual height We have $h^{\mathrm{cd}}(\sigma \cdot F) = h^{\mathrm{cd}}(F)$ for every permutation σ of Φ and every $F \in \mathcal{R}$.

Definition 30.4. Let $\mathcal{T}$ be a root tower of type (Φ, n, κ), and let $r \geq 1$ and L be real numbers. The tower has **flat determinant classes at the base stage** for (r, L) if:

(Db) For every $Z \in \mathrm{Cp}$ and every $a \in \Phi$ there are a real number $\mathrm{hf}(Z, a)$, the **base determinant height**, with

$$\widehat{\deg}_{2,Z}(\mathrm{ps}_{a,1} Q) = \mathrm{hf}(Z, a)\,\mathrm{vol}(Z)\,\deg^{\mathrm{fl}}(Q) \qquad (Q \in \mathcal{R} \text{ homogeneous of degree } D),$$

and real numbers h_0, d_p, v_p ($p \in \mathrm{Bd}$) with $\mathrm{hf}(Z, a) = h_0 + \sum_{p \in \mathrm{Bd}} d_p$, $|h_0| \leq n(\frac{3}{2}\log r + \frac{|L|}{2})$, $v_p \geq 0$, $|d_p| \leq (\frac{n}{p-1} + n\,v_p)\log p$, and $\sum_{p \in \mathrm{Bd}} v_p \log p \leq \log r$.

Corollary 30.5. *Let C be the constant of Proposition 31.14, and let $\mathfrak{u}$ be a unitary Shimura datum satisfying the hypotheses of Proposition 31.14, with rd_E and L_{H} as there. Then there are an integral model $\mathcal{Y}$ of* Sh *over $\mathcal{O}_F$, a root tower $\mathcal{T}$ of type $(\Phi_{\mathsf{H}}, p+q, p)$ with a geometric realisation and a good origin over $\mathcal{Y}$ immersive at level four, and a bijection* $\mathrm{cp}\colon \mathrm{Comp}(\mathrm{Sh}_F) \to \mathrm{Cp}$ *with $\mathcal{Y}_{\mathrm{cp}(Z)}$ the closure of $j(Z)$ for every Z, such that $\mathcal{T}$ has (Fs), (Rt1)–(Rt4), and (Rt5)–(Rt7) for $(r, L) = (\mathrm{rd}_E, L_{\mathsf{H}})$ with the constants C (in (Rt6)) and 4 (in (Rt5)), and the realised tower has (Gm1) and (Gm2).*

Proof. Take $\mathcal{Y}$, $\mathcal{T}$, its geometric realisation, the good origin and cp given by Proposition 31.14; put $\Phi = \Phi_{\mathsf{H}}$, $n = p + q$, $\kappa = p$; then $1 \leq \kappa < n$ because $p \geq 2q + 1 \geq 3$ and $q \geq 1$. Its existence clause (Gx) and its facts (Gt1)–(Gt5) give the component tie, (Fs), (Rt1), (Gm1), (Gi1)–(Gi4), (Gi6), linked stages (Sk1)–(Sk4), the fibre-square identities, the wall inputs (Wn), (Wp), (Wg), (Wd), Galois component symmetry (Ys0) for H, flat determinant classes at the base stage (Db) for $(\mathrm{rd}_E, L_{\mathsf{H}})$, and (Rt6) for $(\mathrm{rd}_E, L_{\mathsf{H}})$ with the constant C. By (Fs), $h^{\mathrm{cd}} = h^{\mathrm{FS}}_{\Phi,n}$. By Theorem 30.28, $\mathcal{T}$ has (Rt3). As E is Galois over $\mathbb{Q}$ and $\mathcal{T}$ has type $(\Phi_{\mathsf{H}}, n, \kappa)$ with (Ys0), Lemma 30.30 gives the symmetry input (Ys1). Since $|\mathrm{disc}(E)| \geq 1$, $r = \mathrm{rd}_E \geq 1$; so Theorem 30.31, applied to $\mathcal{T}$, which has (Rt3) and (Db) for $(\mathrm{rd}_E, L_{\mathsf{H}})$, gives (Rt5) for $(\mathrm{rd}_E, L_{\mathsf{H}})$ with the constant 4.

(Gi5). The good origin contains a root origin of every $\mathcal{X}_{B,c}$, and by its clause on the auxiliary line the multidegree e_* has entry 1 at one factor and 0 at all others; so all entries of e_* are ≥ 0.

(Rt4), (Rt7), (Wh), (Ys2). By Lemma 30.16(b), (c), $h^{\mathrm{cd}}(\mathrm{lr}(u)^{D+1}) \geq 0$ for dominant u and $h^{\mathrm{cd}}(\mathrm{ps}_{a,1} Q) = 0$ for Q homogeneous of degree D: this is (Rt4). Applied with (Φ, n, κ), Proposition 30.21 gives (Rt7). If v is dominant with $\mathrm{cg}_a(v) = 0$ for some a, then $v_{a,\kappa} = v_{a,\kappa+1}$ with $1 \leq \kappa \leq n-1$, and Lemma 30.16(a) gives $h^{\mathrm{cd}}(\mathrm{lr}(v)^{D+1}) = 0$: this is (Wh). By Lemma 30.19, $h^{\mathrm{cd}}(\sigma \cdot F) = h^{\mathrm{cd}}(F)$ for every permutation σ of Φ: this is (Ys2).

(Rt2). The tower $\mathcal{T}$ has (Rt1); the root models $\mathcal{X}_{B,c}$ have smooth generic fibres (the smoothness clause of a good origin); the realised tower has (Gi1), (Gi2), (Gi3), (Gi5), (Gi6), the fibre-square identities and all the wall inputs; and $\mathcal{T}$ has the symmetry inputs (Ys1), (Ys2). Hence Theorem 30.6 gives (Rt2), with errors $C'' \log B / B$ for some $C'' \geq 0$.

(Gm2). The tower has (Rt1) and the realised tower has (Gi1)–(Gi5), so Theorem 30.15 gives (Gm2). □

Theorem 30.6. *Let $\mathcal{T}$ be a root tower with (Rt1) and with a geometric realisation whose root models have smooth generic fibres and that has (Gi1), (Gi2), (Gi3), (Gi5), (Gi6), the fibre-square identities and the wall inputs, and suppose that $\mathcal{T}$ has the symmetry inputs. Then $\mathcal{T}$ has (Rt2), with errors $\varepsilon_B = C \log B/B$ for some $C \geq 0$.*

Proof. We verify the hypotheses of Theorem 30.9. Let B_1, C_W and the sets K_τ be given by Theorem 30.10 (its hypotheses, smooth generic fibres, (Rt1), (Gi1), (Gi2), (Gi3), (Gi5), (Gi6) and the wall inputs, are among ours); by its part (a) each K_τ is a compact subset of the wall Wl_τ containing a non-empty subset open in Wl_τ. (i) By Theorem 30.8 (which uses (Gi1), (Gi2), (Fq1), (Fq2)), $f_B(sv) = -f_B(v)$ for every $B \geq 2$, in particular for $B \geq B_1$, every Levi reflection s and every weight v; here $f_B(v) = \delta_B(\mathrm{lr}(v)^{D+1})$ with $\delta_B = \widehat{\deg}_B - \mathrm{vol}_{\mathcal{T}} h^{\mathrm{cd}}$, the function of Theorem 30.9. (ii) By Theorem 30.10(b), $|f_B| \leq \varepsilon'_B := C_W \log B/B$ on every K_τ for $B \geq B_1$, and $\varepsilon'_B \to 0$. (iii) By (Ys1) there is a group Γ of permutations of Φ acting transitively with $\widehat{\deg}_B(\sigma \cdot F) = \widehat{\deg}_B(F)$ for $B \geq 2$, and by (Ys2) $h^{\mathrm{cd}}(\sigma \cdot F) = h^{\mathrm{cd}}(F)$, for all $\sigma \in \Gamma$ and $F \in \mathcal{R}$. With (Rt1), Theorem 30.9 gives (Rt2) with this B_1 and the errors $C'\varepsilon'_B$, C' the constant of Lemma 30.7(b) for the sets K_τ; take $C = C'C_W$. □

Lemma 30.7. *Let $\mathcal{P}$ be the ring of real polynomial functions of a weight $v \in \mathbb{R}^{\Phi \times \{1,\dots,n\}}$. For $\tau \in \Phi$ and $1 \leq i < n$ let $s_{\tau,i}$ be the involution of the weights exchanging the entries $v_{\tau,i}$ and $v_{\tau,i+1}$; the $s_{\tau,i}$ with $i \neq \kappa$ are the* Levi reflections*, and $\mathrm{Wl}_\tau = \{v : v_{\tau,\kappa} = v_{\tau,\kappa+1}\}$ is the* wall *of τ. Put*

$$\mathrm{Vd}(v) = \prod_{\tau \in \Phi} \prod_{1 \leq i < i' \leq n} (v_{\tau,i} - v_{\tau,i'}), \qquad \mathrm{sm}_\tau(v) = \sum_{i=1}^{\kappa} v_{\tau,i}, \qquad \mathrm{tt}_\tau(v) = \sum_{i=1}^{n} v_{\tau,i},$$

and let $\mathcal{A} \subseteq \mathcal{P}$ be the space of polynomials f homogeneous of degree $D+1$ with $f \circ s = -f$ for every Levi reflection s. For every $\tau \in \Phi$ let $K_\tau \subseteq \mathrm{Wl}_\tau$ be a compact set containing a non-empty subset that is open in Wl_τ.

(a) If $f \in \mathcal{A}$ vanishes on every K_τ, then $f = \mathrm{Vd} \cdot \sum_{\tau \in \Phi} (p_\tau \,\mathrm{sm}_\tau + q_\tau \,\mathrm{tt}_\tau)$ for some real numbers p_τ, q_τ.

(b) There is $C \geq 0$, depending only on (Φ, n, κ) and the sets K_τ, such that for every $f \in \mathcal{A}$ and every $\varepsilon \geq 0$ with $|f| \leq \varepsilon$ on every K_τ there are real numbers p_τ, q_τ for which every coefficient of $f - \mathrm{Vd} \cdot \sum_\tau (p_\tau \mathrm{sm}_\tau + q_\tau \mathrm{tt}_\tau)$ in the monomial basis has absolute value at most $C\varepsilon$.

Proof. Let W_{L} be the group of permutations of the weights generated by the Levi reflections: it permutes, independently for each τ, the entries $v_{\tau,1}, \dots, v_{\tau,\kappa}$ among themselves and the entries $v_{\tau,\kappa+1}, \dots, v_{\tau,n}$ among themselves, and $f \circ w = \mathrm{sgn}(w) f$ for $f \in \mathcal{A}$, $w \in W_{\mathrm{L}}$ (the sign of w as a permutation of the entries; it is multiplicative and -1 on each generator). Let $\mathcal{U} = \{\mathrm{Vd} \cdot \ell\}$, ℓ running over the linear forms $\sum_\tau (p_\tau \mathrm{sm}_\tau + q_\tau \mathrm{tt}_\tau)$. As $\mathrm{Vd} \circ s = -\mathrm{Vd}$ for every exchange $s_{\tau,i}$ and $\ell \circ s = \ell$ for the Levi reflections, $\mathcal{U} \subseteq \mathcal{A}$; and Vd vanishes on every wall.

(a) Fix τ. Choose linear coordinates on Wl_τ; $f|_{\mathrm{Wl}_\tau}$ is a polynomial vanishing on a non-empty open subset, hence zero, so f vanishes on Wl_τ and is divisible by $v_{\tau,\kappa} - v_{\tau,\kappa+1}$ (take this form as a coordinate). Let $i < i'$. If i, i' lie on the same side of the cut, the exchange t of $v_{\tau,i}$ and $v_{\tau,i'}$ lies in W_{L} and is odd, so $f(v) = f(tv) = -f(v)$ whenever $v_{\tau,i} = v_{\tau,i'}$. If $i \leq \kappa < i'$, choose $w \in W_{\mathrm{L}}$ moving the entry at (τ, i) to the position (τ, κ) and the entry at (τ, i') to the position $(\tau, \kappa+1)$; if $v_{\tau,i} = v_{\tau,i'}$ then $wv \in \mathrm{Wl}_\tau$ and $f(v) = \pm f(wv) = 0$. So f vanishes on every hyperplane $v_{\tau,i} = v_{\tau,i'}$ and is divisible by each $v_{\tau,i} - v_{\tau,i'}$. These linear forms are pairwise non-associate

primes of the factorial ring $\mathcal{P}$, so $f = \mathrm{Vd}\cdot\ell$ with ℓ homogeneous of degree $(D+1)-D=1$. Since $\mathrm{Vd}\circ w = \mathrm{sgn}(w)\mathrm{Vd}$ and $f\circ w = \mathrm{sgn}(w)f$ for $w\in W_{\mathrm{L}}$, and $\mathcal{P}$ is a domain, $\ell\circ w=\ell$. Writing $\ell=\sum c_{\tau,i}v_{\tau,i}$, invariance under the exchanges within each side forces $c_{\tau,i}$ to be constant for $i\le\kappa$ and constant for $i>\kappa$; this is $\ell=\sum_\tau(p_\tau\mathrm{sm}_\tau+q_\tau\mathrm{tt}_\tau)$.

(b) On the finite-dimensional space $\mathcal{A}$ put $N(f)=\max_\tau\sup_{K_\tau}|f|$ (finite, the K_τ being compact) and $\|f\|=\max_m|[v^m]f|$. N is a seminorm, and by (a) and $N(\mathcal{U})=0$ its kernel is exactly $\mathcal{U}$. Let $\mathcal{W}$ be a complement of $\mathcal{U}$ in $\mathcal{A}$. On $\mathcal{W}$, N is a norm, and as all norms on a finite-dimensional real space are equivalent there is $C\ge0$ with $\|g\|\le C\,N(g)$ for $g\in\mathcal{W}$. Given $f\in\mathcal{A}$, write $f=u+g$ with $u\in\mathcal{U}$, $g\in\mathcal{W}$; then $N(g)\le N(f)+N(u)=N(f)$, so $\|f-u\|=\|g\|\le C\,N(f)\le C\varepsilon$. C depends only on $\mathcal{A}$, $\mathcal{U}$, $\mathcal{W}$ and the K_τ, that is on (Φ,n,κ) and the K_τ. □

Theorem 30.8. *Let $\mathcal{T}$ be a root tower of type (Φ,n,κ) with a geometric realisation that has (Gi1), (Gi2) and the fibre-square identities (Fq1), (Fq2). For an integer $B\ge2$ let $f_B(v)=\widehat{\deg}_B(\mathrm{lr}(v)^{D+1})-\mathrm{vol}_{\mathcal{T}}\,h^{\mathrm{cd}}(\mathrm{lr}(v)^{D+1})$. Then $f_B(sv)=-f_B(v)$ for every integer $B\ge2$, every Levi reflection s and every weight v.*

Proof. Let sph be moments for which (Fq1) and (Fq2) hold, $B\ge2$, $s=s_{\tau,i}$ a Levi reflection and v a weight. Put $\delta_B=\widehat{\deg}_B-\mathrm{vol}_{\mathcal{T}}h^{\mathrm{cd}}$, a linear form on $\mathcal{R}$, so that $f_B(v)=\delta_B(\mathrm{lr}(v)^{D+1})$.

Step 1: splitting along the reflection. Let $c=\frac12(v+sv)$ and $u=v_{\tau,i}-v_{\tau,i+1}$. Then c is s-invariant, and with $e_{\tau,i}$ the unit weight at (τ,i), $v=c+\frac12(e_{\tau,i}-e_{\tau,i+1})$, $sv=c-\frac12(e_{\tau,i}-e_{\tau,i+1})$. As lr is linear and $\mathrm{lr}(e_{\tau,i}-e_{\tau,i+1})=-\xi_s$, we have $\mathrm{lr}(v)=\mathrm{lr}(c)-\frac u2\xi_s$ and $\mathrm{lr}(sv)=\mathrm{lr}(c)+\frac u2\xi_s$. By the binomial theorem, for every linear form ψ on $\mathcal{R}$ the odd powers of ξ_s cancel:

$$\psi(\mathrm{lr}(v)^{D+1})+\psi(\mathrm{lr}(sv)^{D+1})=2\sum_{0\le k\le(D+1)/2}\binom{D+1}{2k}\Big(\frac u2\Big)^{2k}\psi(\xi_s^{2k}\,\mathrm{lr}(c)^{D+1-2k}). \tag{30.3}$$

Step 2: the identities for the tower. Let $Z\in\mathrm{Cp}$ and $\mathcal{X}=\mathcal{X}_{B,Z}$. By (Gi2), $\widehat{\deg}_{B,Z}(F)=\mathrm{pp}(\mathcal{X};F)$ for every $F\in\mathcal{R}$, and by (Gi1), $\mathrm{gi}(\mathcal{X};F)=\mathrm{vol}(Z)\deg^{\mathrm{fl}}(F)$ for every $F\in\mathcal{R}$ homogeneous of degree D; the elements $\xi_s^{2k-1}\mathrm{lr}(c)^{D+1-2k}$ are homogeneous of degree D. So (Fq1) gives, for $1\le k\le(D+1)/2$,

$$\widehat{\deg}_{B,Z}(\mathrm{lr}(c)^{D+1})=0,\qquad \widehat{\deg}_{B,Z}(\xi_s^{2k}\mathrm{lr}(c)^{D+1-2k})=-2\,\mathrm{sph}_k\,\mathrm{vol}(Z)\deg^{\mathrm{fl}}(\xi_s^{2k-1}\mathrm{lr}(c)^{D+1-2k}).$$

Summing over Z ($\widehat{\deg}_B=\sum_Z\widehat{\deg}_{B,Z}$, $\mathrm{vol}_{\mathcal{T}}=\sum_Z\mathrm{vol}(Z)$), the same identities hold for $\widehat{\deg}_B$ with $\mathrm{vol}_{\mathcal{T}}$ in place of $\mathrm{vol}(Z)$.

Step 3: cancellation. By (Fq2),

$$h^{\mathrm{cd}}(\mathrm{lr}(c)^{D+1})=0,\qquad h^{\mathrm{cd}}(\xi_s^{2k}\mathrm{lr}(c)^{D+1-2k})=-2\,\mathrm{sph}_k\deg^{\mathrm{fl}}(\xi_s^{2k-1}\mathrm{lr}(c)^{D+1-2k}),$$

with the same numbers sph_k. Multiplying by $\mathrm{vol}_{\mathcal{T}}$ and subtracting from Step 2, we obtain $\delta_B(\xi_s^{2k}\mathrm{lr}(c)^{D+1-2k})=0$ for every $0\le k\le(D+1)/2$. By (30.3) with $\psi=\delta_B$, $f_B(v)+f_B(sv)=0$. □

Theorem 30.9. *Let $\mathcal{T}$ be a root tower of type (Φ,n,κ) with (Rt1). For an integer $B\ge2$ and $F\in\mathcal{R}$ put $\delta_B(F)=\widehat{\deg}_B(F)-\mathrm{vol}_{\mathcal{T}}\,h^{\mathrm{cd}}(F)$, and let $f_B(v)=\delta_B(\mathrm{lr}(v)^{D+1})$, a polynomial function of the weight v. For a permutation σ of Φ and $F\in\mathcal{R}$ let $\sigma\cdot F$ be obtained by substituting $x_{\sigma(a),i}$ for $x_{a,i}$. Suppose that there is an integer $B_1\ge2$ such that:*

(i) (Levi antisymmetry) $f_B(sv)=-f_B(v)$ for every $B\ge B_1$, every Levi reflection s and every weight v;

(ii) (near-vanishing on the walls) there are, for every $\tau \in \Phi$, a compact set $K_\tau \subseteq \mathrm{Wl}_\tau$ containing a non-empty subset open in Wl_τ, and real numbers ε_B ($B \geq B_1$) with $\varepsilon_B \to 0$, such that $|f_B(v)| \leq \varepsilon_B$ for all $B \geq B_1$, $\tau \in \Phi$ and $v \in K_\tau$;

(iii) (symmetry of the embeddings) there is a group Γ of permutations of Φ acting transitively on Φ such that $\widehat{\deg}_B(\sigma \cdot F) = \widehat{\deg}_B(F)$ for $B \geq B_1$ and $h^{\mathrm{cd}}(\sigma \cdot F) = h^{\mathrm{cd}}(F)$, for all $\sigma \in \Gamma$ and $F \in \mathcal{R}$.

Then $\mathcal{T}$ has (Rt2), with this B_1 and the errors $C\varepsilon_B$, where C is the constant of Lemma 30.7(b) for the sets K_τ.

Proof. Write $\mathrm{Sh}_\tau(F) = \deg^{\mathrm{fl}}(\partial_\tau^{<} F)$, $\mathrm{To}_\tau(F) = \deg^{\mathrm{fl}}(\partial_\tau F)$, and $m! = \prod_{a,i} m_{a,i}!$ for an exponent m.

Step 1: linear forms and their polynomials. For a linear form Ψ on $\mathcal{R}$, the multinomial theorem gives $\Psi(\mathrm{lr}(v)^{D+1}) = (-1)^{D+1} \sum_{|m|=D+1} \frac{(D+1)!}{m!} \Psi(x^m)\, v^m$: a homogeneous polynomial of degree $D+1$ in v whose coefficient of v^m is $(-1)^{D+1} \frac{(D+1)!}{m!} \Psi(x^m)$. Hence, if for two linear forms Ψ_1, Ψ_2 every coefficient of $\Psi_1(\mathrm{lr}(v)^{D+1}) - \Psi_2(\mathrm{lr}(v)^{D+1})$ is at most η in absolute value, then $|\Psi_1(x^m) - \Psi_2(x^m)| \leq \frac{m!}{(D+1)!}\eta \leq \eta$ for every monomial x^m of degree $D+1$ (as $m! \leq (D+1)!$).

Step 2: the central derivatives as Vandermonde multiples. By Lemma 29.30(d) and (29.18), with $c_* = (D+1)!/\big(\prod_{j=0}^{n-1} j!\big)^{|\Phi|} > 0$,

$$\mathrm{Sh}_\tau\big(\mathrm{lr}(v)^{D+1}\big) = -(D+1)\deg^{\mathrm{fl}}\big(\mathrm{lr}(v)^{D}\big)\mathrm{sm}_\tau(v) = -c_*\,\mathrm{Vd}(v)\,\mathrm{sm}_\tau(v),$$
$$\mathrm{To}_\tau\big(\mathrm{lr}(v)^{D+1}\big) = -c_*\,\mathrm{Vd}(v)\,\mathrm{tt}_\tau(v).$$

Step 3: constants for each embedding. Let $B \geq B_1$. By Step 1 and (i), $f_B \in \mathcal{A}$, and by (ii) $|f_B| \leq \varepsilon_B$ on every K_τ. By Lemma 30.7(b) there are real p_τ, q_τ (depending on B) such that every coefficient of $f_B - \mathrm{Vd}\sum_\tau(p_\tau \mathrm{sm}_\tau + q_\tau \mathrm{tt}_\tau)$ is at most $C\varepsilon_B$. By Step 2, $\mathrm{Vd}\sum_\tau(p_\tau \mathrm{sm}_\tau + q_\tau \mathrm{tt}_\tau)$ is the polynomial of the linear form $-c_*^{-1}\sum_\tau(p_\tau \mathrm{Sh}_\tau + q_\tau \mathrm{To}_\tau)$, so by Step 1

$$\Big|\delta_B(x^m) + c_*^{-1}\sum_\tau\big(p_\tau \mathrm{Sh}_\tau(x^m) + q_\tau \mathrm{To}_\tau(x^m)\big)\Big| \leq C\varepsilon_B \qquad (|m| = D+1). \tag{30.4}$$

Step 4: averaging over the embeddings. Let $\sigma \in \Gamma$. By (iii), $\delta_B(\sigma \cdot F) = \delta_B(F)$. The defining sum (21.1) of $\deg^{\mathrm{fl}}$ is invariant under permuting the blocks (all blocks of δ are equal and sgn is a product over the blocks), so $\deg^{\mathrm{fl}}(\sigma \cdot G) = \deg^{\mathrm{fl}}(G)$; and by the chain rule $\partial_\tau^{<}(\sigma \cdot G) = \sigma \cdot (\partial^{<}_{\sigma^{-1}(\tau)} G)$, $\partial_\tau(\sigma \cdot G) = \sigma \cdot (\partial_{\sigma^{-1}(\tau)} G)$. Hence $\mathrm{Sh}_\tau(\sigma \cdot F) = \mathrm{Sh}_{\sigma^{-1}(\tau)}(F)$ and $\mathrm{To}_\tau(\sigma \cdot F) = \mathrm{To}_{\sigma^{-1}(\tau)}(F)$. Applying (30.4) to the monomial $\sigma \cdot x^m$ and re-indexing $\tau \mapsto \sigma(\tau)$,

$$\Big|\delta_B(x^m) + c_*^{-1}\sum_\tau\big(p_{\sigma(\tau)} \mathrm{Sh}_\tau(x^m) + q_{\sigma(\tau)} \mathrm{To}_\tau(x^m)\big)\Big| \leq C\varepsilon_B.$$

Average over the finite group Γ and use the triangle inequality. Since Γ is transitive, for every τ the map $\sigma \mapsto \sigma(\tau)$ from Γ onto Φ has as fibres the cosets of the stabiliser of τ, all of the same size; so $\frac{1}{|\Gamma|}\sum_\sigma p_{\sigma(\tau)} = \bar{p} := \frac{1}{|\Phi|}\sum_{\tau'} p_{\tau'}$, and likewise $\bar{q}$ for the q's. Hence

$$\Big|\delta_B(x^m) + c_*^{-1}\Big(\bar{p}\sum_\tau \mathrm{Sh}_\tau(x^m) + \bar{q}\sum_\tau \mathrm{To}_\tau(x^m)\Big)\Big| \leq C\varepsilon_B \qquad (|m| = D+1).$$

Step 5: (Rt2). By (Rt1), $\mathrm{vol}_{\mathcal{T}} > 0$; put $a_B = \bar{p}/(c_* \mathrm{vol}_{\mathcal{T}})$ and $b_B = \bar{q}/(c_* \mathrm{vol}_{\mathcal{T}})$. With $\delta_B = \widehat{\deg}_B - \mathrm{vol}_{\mathcal{T}} h^{\mathrm{cd}}$, Step 4 reads

$$\Big|\widehat{\deg}_B(x^m) - \mathrm{vol}_{\mathcal{T}}\Big(h^{\mathrm{cd}}(x^m) - a_B\sum_\tau \deg^{\mathrm{fl}}(\partial_\tau^{<} x^m) - b_B\sum_\tau \deg^{\mathrm{fl}}(\partial_\tau x^m)\Big)\Big| \leq C\varepsilon_B$$

for every $B \geq B_1$ and every monomial x^m of degree $D+1$; and $C\varepsilon_B \to 0$. This is (Rt2). □

Theorem 30.10. *Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with (Rt1) and with a geometric realisation whose root models have smooth generic fibres $\mathcal{X}_{B,Z} \otimes \mathbb{Q}$ and that has (Gi1), (Gi2), (Gi3), (Gi5), (Gi6) and the wall inputs. Put $c_0 = 1/(4n)$, $f_B(v) = \widehat{\deg}_B(\mathrm{lr}(v)^{D+1}) - \mathrm{vol}_{\mathcal{T}} h^{\mathrm{cd}}(\mathrm{lr}(v)^{D+1})$, and for $\tau \in \Phi$ let K_τ be the set of weights $v \in \mathrm{Wl}_\tau$ with $\|v\|_\infty \le \frac{1}{2}$, $v_{a,i} - v_{a,i+1} \ge c_0$ for all $a \in \Phi$ and $i \ne \kappa$, and $\mathrm{cg}_a(v) \ge c_0$ for all $a \ne \tau$.*

(a) K_τ is compact and contains a non-empty subset open in Wl_τ.

(b) There are an integer $B_1 \ge 2$ and $C \ge 0$ such that $|f_B(v)| \le C \log B / B$ for all integers $B \ge B_1$, all $\tau \in \Phi$ and all $v \in K_\tau$.

Proof. Write $f = |\Phi|$, $\nu = \nu_{\mathcal{T}}$, $N_\nu = \|\nu\|_\infty$, $\mathrm{sf}_n = \prod_{j=0}^{n-1} j!$, $c_1 = c_0/2$.

Step 1: (a). K_τ is defined by finitely many non-strict linear inequalities and one linear equation, and is bounded; so it is compact. Let $v^\circ_{a,i} = (n - 2i + 1)c_0$ for $a \ne \tau$, and $v^\circ_{\tau,i} = (n - 2i + 1)c_0$ for $i \le \kappa$, $v^\circ_{\tau,i} = (n - 2i + 3)c_0$ for $i > \kappa$. Then $v^\circ \in \mathrm{Wl}_\tau$, all differences $v^\circ_{a,i} - v^\circ_{a,i+1}$ other than $\mathrm{cg}_\tau(v^\circ) = 0$ equal $2c_0$, and $\|v^\circ\|_\infty \le (n+1)c_0 \le \frac{3}{8}$. So every $v \in \mathrm{Wl}_\tau$ with $\|v - v^\circ\|_\infty < c_0/4$ lies in K_τ.

Step 2: two elementary bounds. (i) If λ is dominant with $\|\lambda\|_\infty \le 1$, then for every $a \in \Phi$, $\deg^{\mathrm{fl}}(\mathrm{lr}(\lambda)^D) \le C_8 \,\mathrm{cg}_a(\lambda)$ with $C_8 = D!\, 2^{D-1}/\mathrm{sf}_n^f$: by (29.18), $\deg^{\mathrm{fl}}(\mathrm{lr}(\lambda)^D) = \frac{D!}{\mathrm{sf}_n^f} \prod_b \prod_{i<i'} (\lambda_{b,i} - \lambda_{b,i'})$, a product of D factors in $[0,2]$, one of which is $\mathrm{cg}_a(\lambda)$. (ii) Let Ψ be a linear form on $\mathcal{R}^+$ with $|\Psi(x^m)| \le b$ for all monomials of degree N, and for $c \in \mathbb{R}^{\mathrm{Rt}^+}$ put $\ell'_c = \sum_r c_r x_r$. Expanding ${\ell'_c}^N$ as a sum over sequences $(r_1, \dots, r_N)$ and telescoping $\prod_j c_{r_j} - \prod_j c'_{r_j}$ factor by factor,

$$\left|\Psi({\ell'_c}^N) - \Psi({\ell'_{c'}}^N)\right| \le b\, N \max(\|c\|_1, \|c'\|_1)^{N-1}\, \|c - c'\|_1.$$

Step 3: constants. Put $A = fn$. Let $\theta_0 > 0$, $C_h \ge 0$ be given by (Wp) for c_1; $s_0 > 0$ by (Wn) for A; $s'_0 > 0$, $C_r \ge 0$ by (Gi6) for A; $C_d \ge 0$ by (Wd). For an integer $B \ge 1$ put $\theta_B = 5f/B$. Choose an integer $B_1 \ge 5$ such that for $B \ge B_1$: $\theta_B \le \min(\theta_0, c_1/2, 1/8, 1/(4N_\nu + 4))$, $\theta_B^2 \le \min(s_0, s'_0)$ and $(B-2)c_1 \ge 8f$.

Step 4: the perturbed weight. Let K'_τ be defined as K_τ with $\|v\|_\infty \le \frac{1}{2} + c_0$ and with c_1 in place of c_0 in the inequalities. Fix $B \ge B_1$, τ and $v_0 \in K'_\tau$ with rational entries, and put $\theta = \theta_B$, $s = \theta^2$ (rational), $\lambda = v_0 + 2\theta \sum_{i \le \kappa} e_{\tau,i}$ ($e_{\tau,i}$ the unit weights) and $u = \lambda - s\nu = \lambda_s$. Adding 2θ to the first κ entries of the block τ changes no difference $\lambda_{a,i} - \lambda_{a,i+1}$ except $\mathrm{cg}_\tau(\lambda) = 2\theta$. Hence: $\|\lambda\|_\infty \le \frac{1}{2} + \frac{1}{8} + \frac{1}{4} < 1$; the differences at $i \ne \kappa$ are $\ge c_1 \ge \theta$; $\mathrm{cg}_a(\lambda) \ge c_1 \ge 2\theta$ ($a \ne \tau$) and $\mathrm{cg}_\tau(\lambda) = 2\theta$; so λ is dominant, rational and (c_1, θ)-spaced, and $\|\lambda\|_1 \le A$. Every difference of consecutive entries of ν in a block is at most $2N_\nu$ in absolute value, and $sN_\nu = \theta \cdot \theta N_\nu \le \theta/4$; so every difference of u is at least the corresponding one of λ minus $\theta/2$: u is dominant, $\mathrm{cg}_\tau(u) \ge \frac{3}{2}\theta$, $\mathrm{cg}_a(u) \ge \frac{3}{4}c_1$ ($a \ne \tau$), and $\|u\|_\infty \le \frac{7}{8} + \frac{\theta}{4} \le 1$. Then u is above the nef threshold at B: $4f\|u\|_\infty \le 4f \le (B-2)\frac{3}{2} \cdot \frac{5f}{B}$ as $B \ge 5$, and $4f \le (B-2)\frac{3}{4}c_1$ as $(B-2)c_1 \ge 8f$. Finally $s \le \min(s_0, s'_0)$ and $\theta \le \theta_0$.

Step 5: the squeeze on one component. Let $Z \in \mathrm{Cp}$, $\mathcal{X} = \mathcal{X}_{B,Z}$, $c = c(u,s)$ (rational, as λ, s and ν are) and $\ell = \ell_c = \mathrm{lr}(u) + s\, x_*$. By (Wn), c is nef on fibres of $\mathcal{X}$ and semipositive at $t(s)$. By (Wp)(a) and Proposition 30.11 ($\mathcal{X}_{\mathbb{Q}}$ is smooth), $\mathrm{ai}_{t(s)}(\mathcal{X}; \ell^{D+1}) \ge -(D+1)C_h \log(B/s)\, \mathrm{gi}(\mathcal{X}; \ell^D)$. Next, $\mathrm{gi}(\mathcal{X}; \ell^D) > 0$. Call $c' \in \mathbb{R}^{\mathrm{Rt}^+}$ *nef on $\mathcal{X}_{\mathbb{Q}}$* if $\sum_r c'_r \deg(\mathcal{O}(e_r)|_C) \ge 0$ for every integral closed curve $C \subseteq \mathcal{X}_{\mathbb{Q}}$. If c' is semipositive at some $t' > 0$, it is nef on $\mathcal{X}_{\mathbb{Q}}$: $\sum_r c'_r \deg(\mathcal{O}(e_r)|_C)$ is the integral over the normalisation of $C(\mathbb{C})$ of the pull-back of $\frac{i}{\pi}\partial\bar\partial\varphi_{c',t'}$ ($\varphi_{c',t'} = \sum_r c'_r \log h_r(t')$) along local holomorphic lifts to the affine cone, and these lifts have Zariski tangent vectors of the cone as derivatives, on which the Levi form is ≥ 0. (For the lifts: near a point w_0

of the normalisation, with g the map to $\mathcal{X}(\mathbb{C})$, choose for every factor k an index i_k with $z_{k,i_k}(g(w_0)) \neq 0$; the lift $\tilde{g}$ into the affine cone with $\tilde{g}_{k,i_k} \equiv 1$ is holomorphic near w_0, and for every F in the ideal of the cone $F \circ \tilde{g} \equiv 0$, so $dF(\tilde{g}(w))\, d\tilde{g}(w)v = 0$ for every tangent vector v: $d\tilde{g}(w)v$ is a Zariski tangent vector of the cone at $\tilde{g}(w)$.) All differences of consecutive entries of u in a block are $\geq \frac{3}{2}\theta > 0$ (Step 4, with $c_1 \geq 2\theta$ from Step 3), so u is regular dominant, and by (Gi3) $c(u_{s'}, s')$ is semipositive at $t(s')$, hence nef on $\mathcal{X}_{\mathbb{Q}}$, for all small $s' > 0$; the degrees being linear in the vector and $c(u_{s'}, s') \to c(u, 0)$, $c(u, 0)$ is nef on $\mathcal{X}_{\mathbb{Q}}$. By (Gi5), $\mathcal{O}(e_*)$ is generated by global sections on the multiprojective space, so the vector 1_* (entry 1 at $*$, 0 elsewhere) is nef on $\mathcal{X}_{\mathbb{Q}}$. Intersection numbers of nef real Cartier classes on the projective variety $\mathcal{X}_{\mathbb{Q}}$ are ≥ 0 (Kleiman's theorem; Lazarsfeld, Positivity in algebraic geometry I, Theorem 1.4.9). Since $\mathrm{lr}(u) = \sum_{r \in \mathrm{Rt}} c(u,0)_r x_r$, the binomial expansion of $\ell^D = (\mathrm{lr}(u) + s\, x_*)^D$ and the definition of gi give

$$\mathrm{gi}(\mathcal{X}; \ell^D) = \sum_{j=0}^{D} \binom{D}{j} \frac{s^j}{\delta_{\mathcal{X}} k_{\mathcal{X}}^j} \deg(c_1(\mathcal{L}_u)^{D-j} c_1(\mathcal{O}(e_*))^j \cap [\mathcal{X}_{\mathbb{Q}}])$$
$$\geq \mathrm{gi}(\mathcal{X}; \mathrm{lr}(u)^D) = \mathrm{vol}(Z) \deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) > 0,$$

with $\mathcal{L}_u = \sum_{r \in \mathrm{Rt}} c(u,0)_r \mathcal{O}(e_r)$ (a real class), the equality by (Gi1) and the last inequality by (Rt1) and Lemma 30.14. So by (Wp)(b) and Lemma 30.12, $\mathrm{ai}_{t(s)}(\mathcal{X}; \ell^{D+1}) \leq (D+1) C_h \log B\, \mathrm{gi}(\mathcal{X}; \ell^D)$. By (Wg), $0 \leq \mathrm{gi}(\mathcal{X}; \ell^D) \leq \mathrm{vol}(Z) \deg^{\mathrm{fl}}(\mathrm{lr}(\lambda)^D) \leq |\mathrm{vol}(Z)|\, C_8 \cdot 2\theta$ by Step 2(i). As $0 < s < 1 < B$, $0 \leq \log B \leq \log(B/s) = \log(B^3/(25 f^2)) \leq 3 \log B$. Hence

$$|\mathrm{ai}_{t(s)}(\mathcal{X}; \ell^{D+1})| \leq 6(D+1) C_h C_8 |\mathrm{vol}(Z)|\, \theta \log B.$$

Step 6: from the regularised number to the wall. By (Gi6), $|\mathrm{ai}_{t(s)}(\mathcal{X}; \ell^{D+1}) - \mathrm{pp}(\mathcal{X}; \ell^{D+1})| \leq C_r s \leq C_r \theta$. Write $\ell = \ell'_{c'}$ and $\mathrm{lr}(v_0) = \ell'_{c''}$ with $c'_{a,i} = -u_{a,i}$, $c'_* = s$, $c''_{a,i} = -(v_0)_{a,i}$, $c''_* = 0$. Then $\|c'\|_1, \|c''\|_1 \leq fn + 1$ and $\|c' - c''\|_1 \leq fn \|u - v_0\|_\infty + s \leq fn(2\theta + s N_\nu) + s \leq (3fn+1)\theta$. By (Wd) and Step 2(ii) with $\Psi = \mathrm{pp}(\mathcal{X}; \cdot)$, $N = D+1$, $b = C_d \log B$,

$$|\mathrm{pp}(\mathcal{X}; \ell^{D+1}) - \mathrm{pp}(\mathcal{X}; \mathrm{lr}(v_0)^{D+1})| \leq C_d (D+1)(fn+1)^D (3fn+1)\, \theta \log B.$$

With Step 5 and $1 \leq \log B$ ($B \geq 3$): $|\mathrm{pp}(\mathcal{X}_{B,Z}; \mathrm{lr}(v_0)^{D+1})| \leq C_Z\, \theta \log B$, $C_Z = 6(D+1) C_h C_8 |\mathrm{vol}(Z)| + C_r + C_d (D+1)(fn+1)^D (3fn+1)$.

Step 7: summing over the components. By (Gi2), $\widehat{\deg}_{B,Z}(\mathrm{lr}(v_0)^{D+1}) = \mathrm{pp}(\mathcal{X}_{B,Z}; \mathrm{lr}(v_0)^{D+1})$, so $|\widehat{\deg}_B(\mathrm{lr}(v_0)^{D+1})| \leq C \log B / B$ with $C = 5f \sum_Z C_Z$, a constant independent of B, τ, v_0. The weight v_0 is dominant (its differences are $\geq c_1$, except $\mathrm{cg}_\tau(v_0) = 0$), so (Wh) gives $h^{\mathrm{cd}}(\mathrm{lr}(v_0)^{D+1}) = 0$, and $|f_B(v_0)| \leq C \log B / B$.

Step 8: all points of K_τ. Let $v \in K_\tau$ and $k \geq 1$. Choose rationals within $\min(c_0/4, 1/k)$ of every entry of v except the entry $(\tau, \kappa+1)$, and put the entry $(\tau, \kappa+1)$ equal to the chosen entry (τ, κ): this gives a rational $v_k \in \mathrm{Wl}_\tau$ with $\|v_k - v\|_\infty \leq \min(c_0/4, 1/k)$. Each difference of consecutive entries moves by at most $c_0/2$ and the norm by at most $c_0/4$, so $v_k \in K'_\tau$, and $|f_B(v_k)| \leq C \log B / B$ by Steps 4–7. As f_B is a polynomial function of v and $v_k \to v$, $|f_B(v)| \leq C \log B / B$. □

Proposition 30.11. *Let $\mathcal{X}$ be a root model of type (Φ, n) such that $\mathcal{X}_{\mathbb{Q}}$ is smooth, let $t > 0$, and let $c \in \mathbb{Q}^{\mathrm{Rt}^+}$ be nef on fibres of $\mathcal{X}$ and semipositive at t. If $e \in \mathbb{R}$ and $h_{c,t}(W) \geq e$ for every point W of $\mathcal{X}$, then $\mathrm{ai}_t(\mathcal{X}; \ell_c^{D+1}) \geq (D+1)\, e\, \mathrm{gi}(\mathcal{X}; \ell_c^D)$.*

Proof. Step 1: one line bundle. Choose an integer $N \geq 1$ with $Nc \in \mathbb{Z}^{\mathrm{Rt}^+}$ and let $\overline{\mathcal{M}} = \bigotimes_r \overline{\mathcal{L}}_r(t)^{\otimes N c_r}$ (multidegree $\sum_r N c_r e_r$, cone function $h = \prod_r h_r(t)^{N c_r}$). Expanding

$\ell_c = \sum_r c_r k_r x_r$ and using (29.2) (the factors k_r cancel against the normalisation) and (30.2), multilinearity gives $\mathrm{ai}_t(\mathcal{X};\ell_c^{D+1}) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}})^{D+1})/(\delta_{\mathcal{X}} N^{D+1})$, $\mathrm{gi}(\mathcal{X};\ell_c^D) = g/(\delta_{\mathcal{X}} N^D)$ with $g = \deg(c_1(\mathcal{M}_{\mathbb{Q}})^D \cap [\mathcal{X}_{\mathbb{Q}}])$, and $h_{c,t}(W) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}})\cdot[W])/(N \deg W_{\mathbb{Q}})$. Put $\overline{\mathcal{M}}' = \overline{\mathcal{M}}(-Ne)$. As $\widehat{c}_1(\overline{\mathcal{M}}(\kappa')) - \widehat{c}_1(\overline{\mathcal{M}})$ is the class of the constant Green function $2\kappa'$ ([110]),

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}')\cdot[W]) = N\deg W_{\mathbb{Q}}\,(h_{c,t}(W) - e) \geq 0,$$
$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}')^{D+1}) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}})^{D+1}) - (D+1)Ne\,g.$$

So it suffices to show $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}')^{D+1}) \geq 0$.

Step 2: an ample comparison line. Let $\overline{\mathcal{H}} = (\mathcal{O}(1,\dots,1), h_{\mathrm{FS}})|_{\mathcal{X}}$, $h_{\mathrm{FS}}(z) = \prod_k |z_k|$ (Euclidean norms), so $\overline{\mathcal{H}} = \bigotimes_k \overline{\mathcal{S}}_k$ with $\overline{\mathcal{S}}_k$ the line $\mathcal{O}(1)$ of the factor k with the Fubini–Study metric. For every point W and every k choose a coordinate $z_{k,i}$ not vanishing identically on W; the arithmetic degree on the arithmetic curve W is a non-negative finite part (the index of the section) plus $-\sum_{w\in W(\mathbb{C})} \log\|z_{k,i}\|(w)$, and $\|z_{k,i}\| = |z_{k,i}|/|z_k| \leq 1$; so $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{H}})\cdot[W]) \geq 0$. The line bundle $\mathcal{H}_{\mathbb{Q}}$ is ample, $\mathcal{H}$ has positive degree on every curve in a closed fibre, and the curvature form of $\overline{\mathcal{H}}$ on $\mathcal{X}(\mathbb{C})$ is the restriction of the positive Fubini–Study form.

Step 3: relatively semiample perturbations. For an integer $j \geq 1$ put $\overline{\mathcal{P}}_j = \overline{\mathcal{M}}'^{\otimes j} \otimes \overline{\mathcal{H}}$, with cone function $e^{-jNe} h^j h_{\mathrm{FS}}$, a smooth conjugation-invariant metric. (i) By Steps 1, 2 every point has $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{P}}_j)\cdot[W]) \geq 0$; the points are exactly the Zariski closures D_x of the $x \in \mathcal{X}(\overline{\mathbb{Q}})$. (ii) Since $\mathcal{X}_{\mathbb{Q}}$ is smooth, $\mathcal{X}(\mathbb{C})$ is a compact complex manifold, and its tangent vectors lift to Zariski tangent vectors of the affine cone (lift a local holomorphic section of the cone); as c is semipositive at t, the curvature form $\frac{i}{\pi}\partial\bar\partial \log h = \frac{i}{\pi} N \partial\bar\partial \varphi_{c,t}$ of $\overline{\mathcal{M}}'$ is semipositive on $\mathcal{X}(\mathbb{C})$, and that of $\overline{\mathcal{P}}_j$ is j times it plus the positive Fubini–Study form: positive. In particular $\mathcal{M}_{\mathbb{Q}}$ is nef (its degree on a curve is the integral of a semipositive form over the normalisation of the curve) and $\mathcal{P}_{j,\mathbb{Q}}$ is ample (nef plus ample, Kleiman's criterion). (iii) $\mathcal{P}_j$ has non-negative degree on every curve in a closed fibre (c is nef on fibres, $\mathcal{H}$ is ample there). By (ii) and (iii) $\overline{\mathcal{P}}_j$ is relatively semipositive. By Theorem B.131(A) with $Y = \mathcal{X}(\mathbb{C})$, i the identity and $\overline{M} = \overline{\mathcal{P}}_{j,\mathbb{C}}$ (ample, semipositive curvature), the metric of $\overline{\mathcal{P}}_j$ is semiample; so $\overline{\mathcal{P}}_j$ is relatively semiample with ample generic fibre, and Theorem B.131(B) with (i) gives $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{P}}_j)^{D+1}) \geq 0$.

Step 4: the limit. By multilinearity,

$$\begin{aligned}\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{P}}_j)^{D+1}) &= \sum_{i=0}^{D+1} \binom{D+1}{i} j^i\, \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}')^i \widehat{c}_1(\overline{\mathcal{H}})^{D+1-i}) \\ &= j^{D+1}\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}')^{D+1}) + O(j^D).\end{aligned}$$

Dividing by j^{D+1} and letting $j \to \infty$ gives $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}')^{D+1}) \geq 0$, and Step 1 gives $\mathrm{ai}_t(\mathcal{X};\ell_c^{D+1}) \geq (D+1)e\,\mathrm{gi}(\mathcal{X};\ell_c^D)$. □

Lemma 30.12. *Let $\mathcal{X}$ be a root model of type (Φ, n), $t > 0$, and let $c \in \mathbb{Q}^{\mathrm{Rt}^+}$ be nef on fibres of $\mathcal{X}$ and semipositive at t, with $\mathrm{gi}(\mathcal{X};\ell_c^D) > 0$. If $e' \in \mathbb{R}$ and F is a Zariski dense set of points of $\mathcal{X}$ with $h_{c,t}(W) \leq e'$ for every $W \in F$, then*

$$\mathrm{ai}_t(\mathcal{X};\ell_c^{D+1}) \leq (D+1)\,e'\,\mathrm{gi}(\mathcal{X};\ell_c^D).$$

Proof. Step 1: one line bundle. Choose an integer $N \geq 1$ with $Nc_r \in \mathbb{Z}$ for all $r \in \mathrm{Rt}^+$, and put $e = \sum_r Nc_r e_r \in \mathbb{Z}^K$, $h = \prod_r h_r(t)^{Nc_r}$ (with $h_*(t) = h_*$) and $\overline{\mathcal{M}} = (\mathcal{O}(e), h)|_{\mathcal{X}} = \bigotimes_r \overline{\mathcal{L}}_r(t)^{\otimes Nc_r}$.

Since $\ell_c = \sum_r c_r k_r x_r$ ($k_r = 1$ for $r \in \mathrm{Rt}$, $k_* = k_{\mathcal{X}}$), expanding ℓ_c^{D+1} into monomials and using (29.2), the factors k_r cancel against the normalisation, and multilinearity of $\widehat{\deg}$ gives

$$\mathrm{ai}_t(\mathcal{X}; \ell_c^{D+1}) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}})^{D+1})}{\delta_{\mathcal{X}} N^{D+1}}, \qquad \mathrm{gi}(\mathcal{X}; \ell_c^D) = \frac{\deg(c_1(\mathcal{M}_{\mathbb{Q}})^D \cap [\mathcal{X}_{\mathbb{Q}}])}{\delta_{\mathcal{X}} N^D},$$

and likewise, by (30.2), $h_{c,t}(W) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}) \cdot [W])/(N \deg W_{\mathbb{Q}})$ for every point W. For $\kappa' \in \mathbb{R}$ the bundle $\overline{\mathcal{M}}(\kappa')$ (cone function $e^{\kappa'} h$) satisfies

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}(\kappa')) \cdot [W]) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}) \cdot [W]) + \kappa' \deg W_{\mathbb{Q}},$$
$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}(\kappa'))^{D+1}) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}})^{D+1}) + (D+1)\kappa' \deg(c_1(\mathcal{M}_{\mathbb{Q}})^D \cap [\mathcal{X}_{\mathbb{Q}}]),$$

since $\widehat{c}_1(\overline{\mathcal{M}}(\kappa')) = \widehat{c}_1(\overline{\mathcal{M}}) + \widehat{c}_1(\overline{\mathcal{O}}(\kappa'))$ with $\widehat{c}_1(\overline{\mathcal{O}}(\kappa'))$ the class of the constant Green function $2\kappa'$, whose product with a cycle Y has arithmetic degree κ' times the degree of $Y_{\mathbb{Q}}$ against the remaining classes ([110]).

Step 2: the hypotheses of Lemma 29.24. Put $g = \deg(c_1(\mathcal{M}_{\mathbb{Q}})^D \cap [\mathcal{X}_{\mathbb{Q}}]) = \delta_{\mathcal{X}} N^D \mathrm{gi}(\mathcal{X}; \ell_c^D) > 0$. For every κ', $\overline{\mathcal{M}}(\kappa')$ is $(\mathcal{O}(e), e^{\kappa'} h)|_{\mathcal{X}}$ with $e^{\kappa'} h$ a cone metric of multidegree e, and: (1) $\deg(\mathcal{O}(e)|_W) = N \sum_r c_r \deg(\mathcal{O}(e_r)|_W) \geq 0$ for every integral closed curve W in a closed fibre, as c is nef on fibres; (2) $\log(e^{\kappa'} h) = \kappa' + N\varphi_{c,t}$ with $\varphi_{c,t} = \sum_r c_r \log h_r(t)$, whose Levi form is semipositive on Zariski tangent vectors of the cone over $\mathcal{X}(\mathbb{C})$, as c is semipositive at t; (3) $g > 0$.

Step 3: a strictly small section. Suppose, to the contrary, that $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}})^{D+1}) > (D+1)Ne'g$. Choose $\varepsilon > 0$ with $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}})^{D+1}) > (D+1)(Ne' + \varepsilon)g$, and put $\overline{\mathcal{M}}_\varepsilon = \overline{\mathcal{M}}(-Ne' - \varepsilon)$. By Step 1, $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_\varepsilon)^{D+1}) > 0$. Let $\nu\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ be the normalisation. By Step 2 and Lemma 29.24 applied to $\overline{\mathcal{M}}_\varepsilon$, the lim sup over τ of $\frac{(D+1)!}{\tau^{D+1}} \log \#\{\sigma \in H^0(\widetilde{\mathcal{X}}, \nu^*\mathcal{M}^{\otimes\tau}) : \|\sigma\|_{\sup} \leq 1\}$ (sup norm for $\nu^*\overline{\mathcal{M}}_\varepsilon^{\otimes\tau}$) is positive. Hence for some $\tau \geq 1$ this set has at least two elements, and contains some $\sigma \neq 0$ with $\|\sigma\|_{\sup} \leq 1$.

Step 4: points off the zero locus have large height. The zero locus $V(\sigma)$ is a proper closed subset of the integral scheme $\widetilde{\mathcal{X}}$, so it has dimension $< D+1$; as ν is finite, $Y = \nu(V(\sigma))$ is closed of dimension $< D+1$, hence a proper closed subset of $\mathcal{X}$. Since F is Zariski dense, some $W \in F$ is not contained in Y (otherwise the reduced closed subscheme $Y \neq \mathcal{X}$ would contain every member of F). As ν is finite and surjective, some irreducible component $\widetilde{W}$ of $\nu^{-1}(W)$, with its reduced structure, maps onto W; it is integral, of absolute dimension 1 and flat over $\mathbb{Z}$ (it dominates $\operatorname{Spec}\mathbb{Z}$, as W does), and $\widetilde{W} \not\subseteq V(\sigma)$ since $\nu(\widetilde{W}) = W \not\subseteq Y$. So $\sigma|_{\widetilde{W}}$ is a non-zero section of $\nu^*\mathcal{M}^{\otimes\tau}|_{\widetilde{W}}$, and the arithmetic degree on the arithmetic curve $\widetilde{W}$ is the sum of a non-negative finite part (the logarithm of the index of σ in the sections over the normalisation of $\widetilde{W}$) and $-\sum_{w \in \widetilde{W}(\mathbb{C})} \log \|\sigma(w)\| \geq 0$:

$$\tau\, \widehat{\deg}(\widehat{c}_1(\nu^*\overline{\mathcal{M}}_\varepsilon) \cdot [\widetilde{W}]) \geq 0.$$

Now $\widetilde{W}$ (closed in $\widetilde{\mathcal{X}}$, which is finite over $\mathcal{X}$) and W are arithmetic varieties of absolute dimension 1, whose generic fibres, spectra of number fields, are smooth over $\mathbb{Q}$; the metric of $\overline{\mathcal{M}}_\varepsilon|_W$ is smooth, $W(\mathbb{C})$ being finite; $\beta = \nu|_{\widetilde{W}}\colon \widetilde{W} \to W$ is surjective and $(\nu^*\overline{\mathcal{M}}_\varepsilon)|_{\widetilde{W}} = \beta^*(\overline{\mathcal{M}}_\varepsilon|_W)$. So Lemma 30.13 (with $b = 1$, $d = 0$ and this β) gives that the left side is $\tau m\, \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_\varepsilon) \cdot [W])$ with $m = [\mathbf{k}(\widetilde{W}) : \mathbf{k}(W)] \geq 1$. With Step 1 (and $\kappa' = -Ne' - \varepsilon$),

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}) \cdot [W]) \geq (Ne' + \varepsilon) \deg W_{\mathbb{Q}}, \qquad \text{i.e.} \qquad h_{c,t}(W) \geq e' + \varepsilon/N > e',$$

contradicting $h_{c,t}(W) \leq e'$ for $W \in F$.

Step 5: conclusion. Hence $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}})^{D+1}) \leq (D+1)Ne'g$; dividing by $\delta_{\mathcal{X}} N^{D+1}$ and using Step 1 gives $\mathrm{ai}_t(\mathcal{X}; \ell_c^{D+1}) \leq (D+1)e'\, \mathrm{gi}(\mathcal{X}; \ell_c^D)$. □

Lemma 30.13. *Let $b \geq 1$ and $d \geq 0$ be integers, let $\mathcal{W}$ and $\mathcal{V}$ be arithmetic varieties of absolute dimensions $b+d$ and b whose generic fibres $\mathcal{W}_{\mathbb{Q}}$ and $\mathcal{V}_{\mathbb{Q}}$ are smooth, and let $\beta\colon \mathcal{W} \to \mathcal{V}$ be a morphism. Let $\overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_d$ and $\overline{\mathcal{H}}_1, \dots, \overline{\mathcal{H}}_b$ be hermitian line bundles with smooth metrics on $\mathcal{W}$ and on $\mathcal{V}$, respectively. Let $\mathcal{W}_\eta = \mathcal{W} \times_{\mathcal{V}} \operatorname{Spec} \mathbf{k}(\mathcal{V})$ be the fibre of β over the generic point η of $\mathcal{V}$, a projective scheme over $\mathbf{k}(\mathcal{V})$, and*

$$c = \deg\bigl(c_1(\mathcal{L}_1|_{\mathcal{W}_\eta}) \cdots c_1(\mathcal{L}_d|_{\mathcal{W}_\eta}) \cap [\mathcal{W}_\eta]\bigr) \in \mathbb{Z}$$

($c = 0$ if $\mathcal{W}_\eta$ is empty). Then

$$\widehat{\deg}\Bigl(\prod_{j=1}^{b} \widehat{c}_1(\beta^*\overline{\mathcal{H}}_j) \prod_{i=1}^{d} \widehat{c}_1(\overline{\mathcal{L}}_i)\Bigr) = c\, \widehat{\deg}\Bigl(\prod_{j=1}^{b} \widehat{c}_1(\overline{\mathcal{H}}_j)\Bigr).$$

If $d = 0$ and β is surjective, then $c = [\mathbf{k}(\mathcal{W}) : \mathbf{k}(\mathcal{V})]$.

Proof. Steps 1–3: smooth metrics are DSP, and heights are arithmetic intersection numbers. By Lemma 29.25, hermitian line bundles with smooth metrics on an arithmetic variety with smooth generic fibre are DSP, and (H) of Lemma 29.25 holds on such a variety.

Step 4: the projection formula. The schemes $\mathcal{W}$ and $\mathcal{V}$ are projective of finite type over $\operatorname{Spec}\mathbb{Z}$; being integral and flat over $\mathbb{Z}$, their relative dimensions are $\dim \mathcal{W}_{\mathbb{Q}} = b + d - 1$ and $\dim \mathcal{V}_{\mathbb{Q}} = b - 1$. By Lemma 29.25 the $\overline{\mathcal{L}}_i$ and the $\overline{\mathcal{H}}_j$ are DSP. So Proposition B.129 applies to $\pi = \beta$:

$$h_{\beta^*\overline{\mathcal{H}}_1,\dots,\beta^*\overline{\mathcal{H}}_b,\overline{\mathcal{L}}_1,\dots,\overline{\mathcal{L}}_d}(\mathcal{W}) = \deg_{\mathcal{L}_1,\dots,\mathcal{L}_d}(\mathcal{W}_\eta)\, h_{\overline{\mathcal{H}}_1,\dots,\overline{\mathcal{H}}_b}(\mathcal{V}),$$

and $\deg_{\mathcal{L}_1,\dots,\mathcal{L}_d}(\mathcal{W}_\eta) = c$ (the degree of $\mathcal{W}_\eta$ with respect to the restrictions of the $\mathcal{L}_i$). The metrics of $\beta^*\overline{\mathcal{H}}_j$ are smooth (pull-backs by the holomorphic map $\beta_{\mathbb{C}}$), so (H) of Lemma 29.25 on $\mathcal{W}$ with the $b+d$ bundles $\beta^*\overline{\mathcal{H}}_j$, $\overline{\mathcal{L}}_i$, and on $\mathcal{V}$ with the $\overline{\mathcal{H}}_j$, turns this identity into the assertion.

Step 5: the case $d = 0$. Let $d = 0$ and β surjective. Then $\mathcal{W}$ and $\mathcal{V}$ are integral of the same dimension b, so $\mathbf{k}(\mathcal{W})$ is a finitely generated extension of $\mathbf{k}(\mathcal{V})$ of transcendence degree 0, hence finite. The points of $\mathcal{W}_\eta$ are the points of $\mathcal{W}$ over η, with the same local rings (as $\mathcal{O}_{\mathcal{V},\eta} = \mathbf{k}(\mathcal{V})$); each is a specialisation of the generic point ξ of $\mathcal{W}$, and the closure of a point $x \neq \xi$ over η would be an integral closed subscheme of dimension $< b$ dominating $\mathcal{V}$, which is impossible. So $\mathcal{W}_\eta = \operatorname{Spec}\mathcal{O}_{\mathcal{W},\xi} = \operatorname{Spec}\mathbf{k}(\mathcal{W})$ and $c = \deg[\mathcal{W}_\eta] = [\mathbf{k}(\mathcal{W}) : \mathbf{k}(\mathcal{V})]$. □

Lemma 30.14. *For every weight u,*

$$\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) = D! \prod_{a\in\Phi} \prod_{1\leq i<i'\leq n} \frac{u_{a,i} - u_{a,i'}}{i' - i}.$$

In particular $\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) > 0$ for every regular dominant weight u.

Proof. Write $c_{a,i} = -u_{a,i}$, so that $\mathrm{lr}(u) = \sum_{a,i} c_{a,i} x_{a,i}$. By the multinomial theorem the coefficient of a monomial x^m with $\sum_{a,i} m_{a,i} = D$ in $\mathrm{lr}(u)^D$ is $D! \prod_{a,i} c_{a,i}^{m_{a,i}}/m_{a,i}!$. For $w \in S_n^\Phi$ the exponent vector $\delta \circ w$ has $\sum_{a,i} \delta_{a,w_a(i)} = |\Phi| \sum_{k=0}^{n-1} k = D$, and since each w_a is a permutation, $\prod_{a,i} (\delta_{a,w_a(i)})! = \sigma_n^{|\Phi|}$ with $\sigma_n = \prod_{k=0}^{n-1} k!$, independently of w. Hence, by (21.1),

$$\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) = \frac{(-1)^D D!}{\sigma_n^{|\Phi|}} \sum_{w\in S_n^\Phi} \prod_{a\in\Phi} \operatorname{sgn}(w_a) \prod_{i=1}^{n} c_{a,i}^{n-w_a(i)} = \frac{(-1)^D D!}{\sigma_n^{|\Phi|}} \prod_{a\in\Phi} \det\bigl(c_{a,i}^{n-k}\bigr)_{1\leq i,k\leq n},$$

the sum over $w = (w_a)_a$ factoring into the product over a of the Leibniz expansions of the determinants. By the Vandermonde determinant, $\det(c_{a,i}^{n-k})_{i,k} = \prod_{1\leq i<i'\leq n}(c_{a,i} - c_{a,i'}) =$

$(-1)^{n(n-1)/2}\prod_{i<i'}(u_{a,i}-u_{a,i'})$. The signs contribute $(-1)^D\cdot(-1)^{|\Phi|n(n-1)/2}=1$, and $\sigma_n=\prod_{k=1}^{n-1}k!=\prod_{i'=2}^{n}\prod_{i=1}^{i'-1}(i'-i)=\prod_{1\le i<i'\le n}(i'-i)$. This gives the formula. If u is regular dominant, every factor $u_{a,i}-u_{a,i'}$ with $i<i'$ is positive, and so is $D!/\prod(i'-i)$. □

Theorem 30.15. *Let $\mathcal{T}$ be a root tower with a geometric realisation such that $\mathcal{T}$ has (Rt1) and the realised tower has the geometric inputs (Gi1)–(Gi5). Then the realised tower has the regularisation property (Gm2).*

Proof. Let $\mathcal{T}$ have type (Φ,n,κ) and write $\nu=\nu_{\mathcal{T}}$. Fix a weight u satisfying the gap hypothesis with gap $g>0$, an integer $B\ge 2$ with $b_0(u,g)<B+1$ and $Z\in\mathrm{Cp}$; put $\mathcal{X}=\mathcal{X}_{B,Z}$, $u_s=u-s\nu$, $t(s)=\exp(-1/s^2)$, and $\gamma_a(\mu)=\mu_{a,\kappa}-\mu_{a,\kappa+1}$ for a weight μ and $a\in\Phi$. We prove the four items of (Gm2).

Dominance and the threshold for small s. The weight u is regular dominant, i.e. $u_{a,i}-u_{a,i+1}>0$ for all a and $i<n$; these finitely many quantities are affine in s for u_s, so there is $s_1>0$ with u_s regular dominant, in particular dominant, for $0\le s<s_1$. Next, $\gamma_a(u)\ge g$ for every a by the gap hypothesis. Since $|\Phi|\ge 1$ and $n\ge 2$ we have $4|\Phi|n^2\ge 16$, and $b_0(u,g)<B+1$ gives $B-2>4|\Phi|n^2+4|\Phi|n^2\|u\|_\infty/g-3>0$. Hence for every a

$$(B-2)\gamma_a(u)\ge(B-2)g>4|\Phi|n^2\|u\|_\infty+(4|\Phi|n^2-3)g>4|\Phi|\,\|u\|_\infty.$$

Both $s\mapsto(B-2)\gamma_a(u_s)=(B-2)(\gamma_a(u)-s\gamma_a(\nu))$ and $s\mapsto 4|\Phi|\,\|u_s\|_\infty$ are continuous, so there is $s_2>0$ such that u_s is above the nef threshold at B for $0\le s<s_2$.

Item 1: nefness on fibres and semipositivity. Let $0<s<\min(s_1,s_2)$ and let ℓ be a prime. If $\ell\le B$ or $\ell\in\mathrm{Bd}$, then $c(u_s,0)$ is nef at ℓ by (Gi4), u_s being dominant. Otherwise $\ell>B$; dominance gives $\gamma_a(u_s)\ge 0$, so $(\ell-2)\gamma_a(u_s)\ge(B-2)\gamma_a(u_s)\ge 4|\Phi|\,\|u_s\|_\infty$ for every a, i.e. u_s is above the nef threshold at ℓ, and $c(u_s,0)$ is nef at ℓ by (Gi4) again. Let $W\subseteq\mathcal{X}_{\mathbb{F}_\ell}$ be an integral closed curve. By (Gi5) the multidegree e_* has entries ≥ 0, so $\mathcal{O}(e_*)$ is generated by its global sections on the multiprojective space over $\mathbb{F}_\ell$ (at a point choose in every factor a coordinate that does not vanish; the product of their $e_{*,k}$-th powers is a section not vanishing there); a section not vanishing identically on W restricts to a non-zero section of $\mathcal{O}(e_*)|_W$, whose divisor is effective, so $\deg(\mathcal{O}(e_*)|_W)\ge 0$. Since $c(u_s,s)$ and $c(u_s,0)$ differ only in the entry at $*$, which is $s/k_{\mathcal{X}}>0$ for the former and 0 for the latter,

$$\sum_{r\in\mathrm{Rt}^+}c(u_s,s)_r\deg(\mathcal{O}(e_r)|_W)=\sum_{r\in\mathrm{Rt}^+}c(u_s,0)_r\deg(\mathcal{O}(e_r)|_W)+\frac{s}{k_{\mathcal{X}}}\deg(\mathcal{O}(e_*)|_W)\ge 0.$$

So $c(u_s,s)$ is nef on fibres of $\mathcal{X}$ for $0<s<\min(s_1,s_2)$. Since u is regular dominant, (Gi3) gives $s_3>0$ such that $c(u_s,s)$ is semipositive at $t(s)$ for $0<s<s_3$. Both hold for $0<s<\min(s_1,s_2,s_3)$.

Item 2: the arithmetic numbers. The map lr is linear, so $\mathrm{lr}(u_s)+s\,x_*=\mathrm{lr}(u)+s\,Q$ with $Q=x_*+\sum_{a,i}\nu_{a,i}x_{a,i}\in\mathcal{R}^+$ independent of s. By the binomial theorem

$$(\mathrm{lr}(u_s)+s\,x_*)^{D+1}=\sum_{k=0}^{D+1}\binom{D+1}{k}s^k\,\mathrm{lr}(u)^{D+1-k}Q^k=\sum_m a_m(s)\,x^m,$$

a finite sum over monomials $x^m\in\mathcal{R}^+$ whose coefficients a_m are polynomials in s with $a_m(0)=[x^m]\,\mathrm{lr}(u)^{D+1}$. By linearity of $\mathrm{ai}_t(\mathcal{X};\cdot)$,

$$\mathrm{ai}_{t(s)}(\mathcal{X};(\mathrm{lr}(u_s)+s\,x_*)^{D+1})=\sum_m a_m(s)\,\mathrm{ai}_{t(s)}(\mathcal{X};x^m).$$

As $s \to 0^+$ we have $t(s) > 0$ and $t(s) \to 0$, so $\mathrm{ai}_{t(s)}(\mathcal{X}; x^m) \to \mathrm{pp}(\mathcal{X}; x^m)$ by the convergent regularisation of (Gi2), and $a_m(s) \to a_m(0)$. The sum being finite, the left side tends to $\sum_m a_m(0)\,\mathrm{pp}(\mathcal{X}; x^m) = \mathrm{pp}(\mathcal{X}; \mathrm{lr}(u)^{D+1})$, which equals $\widehat{\deg}_{B,Z}(\mathrm{lr}(u)^{D+1})$ by (Gi2), since $\mathrm{lr}(u)^{D+1} \in \mathcal{R}$.

Item 3: the geometric numbers. In the same way $(\mathrm{lr}(u_s) + s\,x_*)^D = \sum_m b_m(s) x^m$ with polynomial coefficients and $b_m(0) = [x^m]\,\mathrm{lr}(u)^D$, and $\mathrm{gi}(\mathcal{X}; \cdot)$ is a fixed linear form, so $\mathrm{gi}\big(\mathcal{X}; (\mathrm{lr}(u_s) + s\,x_*)^D\big) \to \mathrm{gi}(\mathcal{X}; \mathrm{lr}(u)^D)$ as $s \to 0^+$. As $\mathrm{lr}(u)^D \in \mathcal{R}$ is homogeneous of degree D, (Gi1) gives $\mathrm{gi}(\mathcal{X}; \mathrm{lr}(u)^D) = \mathrm{vol}(Z)\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D)$.

Item 4: positivity. $\mathrm{vol}(Z) > 0$ by (Rt1), and $\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) > 0$ by Lemma 30.14, u being regular dominant. Hence $\mathrm{vol}(Z)\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) > 0$. □

Lemma 30.16. *Let Φ be a non-empty finite set and $n \geq 2$.*

(a) If v is a weight with $v_{a,i} = v_{a,i+1}$ for some $a \in \Phi$ and some $1 \leq i \leq n-1$, then $h^{\mathrm{FS}}_{\Phi,n}(\mathrm{lr}(v)^{D+1}) = 0$.

(b) $h^{\mathrm{FS}}_{\Phi,n}(\mathrm{lr}(v)^{D+1}) \geq 0$ for every dominant weight v.

(c) $h^{\mathrm{FS}}_{\Phi,n}(\mathrm{ps}_{a,1} Q) = 0$ for every $a \in \Phi$ and every $Q \in \mathcal{R}$ homogeneous of degree D.

Proof. For $k \in K_{\mathrm{Fl}}$ let $\overline{\mathcal{S}}_k = (\mathcal{O}(1_k), h^{\mathrm{FS}}_{1_k})$ on $\mathbb{P}_{\mathrm{Fl}}$ (1_k the unit vector at k: the line $\mathcal{O}(1)$ of the factor k with its Fubini–Study metric) and $\hat{s}_k = \widehat{c}_1(\overline{\mathcal{S}}_k)$; put $\hat{s}_{(a,0)} = \hat{s}_{(a,n)} = 0$. Since $h^{\mathrm{FS}}_e = \prod_k h^{\mathrm{FS}e_k}_{1_k}$, the root line of $r = (a,i)$ is $\overline{\mathcal{S}}_{(a,i)} \otimes \overline{\mathcal{S}}^{-1}_{(a,i-1)}$ restricted to $\mathrm{Fl}_{\Phi,n}$ (omitting the factors with index 0 or n), so

$$\widehat{c}_1(\overline{\mathcal{X}}^{\mathrm{FS}}_{(a,i)}) = \hat{s}_{(a,i)} - \hat{s}_{(a,i-1)} \qquad (1 \leq i \leq n). \tag{30.5}$$

Step 1: the class of a weight. Extend $\widehat{c}_1$ linearly: for $F = \sum_m f_m x^m$ homogeneous of degree $D+1$, (29.7) and multilinearity give $h^{\mathrm{FS}}_{\Phi,n}(F) = (-1)^{D+1}\widehat{\deg}(\hat{F} \cdot [\mathrm{Fl}_{\Phi,n}])$, where $\hat{F}$ is obtained from F by substituting $\widehat{c}_1(\overline{\mathcal{X}}^{\mathrm{FS}}_r)$ for x_r. For a weight v put $g_{a,j}(v) = v_{a,j} - v_{a,j+1}$ $(1 \leq j \leq n-1)$. By (30.5) and summation by parts in each block,

$$\widehat{\mathrm{lr}(v)} = -\sum_a \sum_{i=1}^{n} v_{a,i}\big(\hat{s}_{(a,i)} - \hat{s}_{(a,i-1)}\big) = -\sum_a \sum_{j=1}^{n-1} g_{a,j}(v)\,\hat{s}_{(a,j)},$$

hence

$$h^{\mathrm{FS}}_{\Phi,n}\big(\mathrm{lr}(v)^{D+1}\big) = \widehat{\deg}\Big(\Big(\sum_{k \in K_{\mathrm{Fl}}} g_k(v)\,\hat{s}_k\Big)^{D+1} \cdot [\mathrm{Fl}_{\Phi,n}]\Big). \tag{30.6}$$

Step 2: (c). By (30.5), $\widehat{\mathrm{ps}_{a,1}} = \sum_{i=1}^{n}(\hat{s}_{(a,i)} - \hat{s}_{(a,i-1)}) = \hat{s}_{(a,n)} - \hat{s}_{(a,0)} = 0$. For a monomial x^m of degree D, $h^{\mathrm{FS}}_{\Phi,n}(\mathrm{ps}_{a,1} x^m) = (-1)^{D+1}\widehat{\deg}(\widehat{\mathrm{ps}_{a,1}} \cdot \widehat{x^m} \cdot [\mathrm{Fl}_{\Phi,n}]) = 0$; by linearity (c) follows.

Step 3: (b). We show: for every integral closed subscheme $Y \subseteq \mathbb{P}_{\mathrm{Fl}}$, flat over $\mathbb{Z}$, of absolute dimension $m \geq 1$ and all $k_1, \ldots, k_m \in K_{\mathrm{Fl}}$, $\widehat{\deg}(\hat{s}_{k_1} \cdots \hat{s}_{k_m} \cdot [Y]) \geq 0$. Induction on m. Choose $J \in I_{k_1}$ such that the coordinate $z_{k_1,J}$, a section of $\mathcal{O}(1_{k_1})$, does not vanish identically on Y.

Lemma 30.17 (ambient the regular $\mathbb{P}_{\mathrm{Fl}}$, $\mathcal{Z}=Y$, $\sigma=z_{k_1,J}|_Y$) gives

$$\begin{aligned}\widehat{\deg}(\hat{s}_{k_1}\cdots\hat{s}_{k_m}\cdot[Y]) &= \sum_{W\ \text{horizontal}} \operatorname{ord}_W(z_{k_1,J}|_Y)\,\widehat{\deg}(\hat{s}_{k_2}\cdots\hat{s}_{k_m}\cdot[W])\\ &\quad+\sum_p \log p \sum_{W\subseteq Y_{\mathbb{F}_p}} \operatorname{ord}_W(z_{k_1,J}|_Y)\deg\bigl(c_1(\mathcal{O}(1_{k_2}))\cdots c_1(\mathcal{O}(1_{k_m}))\cap[W]\bigr)\\ &\quad+\int_{Y(\mathbb{C})} -\log\|z_{k_1,J}\|\;\omega_{k_2}\wedge\cdots\wedge\omega_{k_m},\end{aligned}$$

where ω_k is the Fubini–Study form of $\overline{\mathcal{S}}_k$, which is semipositive. The section $z_{k_1,J}|_Y$ is regular, so every $\operatorname{ord}_W(z_{k_1,J}|_Y)\geq 0$. The horizontal prime divisors W are integral, flat over $\mathbb{Z}$, of absolute dimension $m-1$ (there are none if $m=1$), so the first sum is ≥ 0 by induction; the vertical degrees are degrees of globally generated line bundles on projective $\mathbb{F}_p$-schemes, hence ≥ 0, so the second sum is ≥ 0; $\|z_{k_1,J}\|(z)=|z_{k_1,J}|/(\sum_{J'}|z_{k_1,J'}|^2)^{1/2}\leq 1$, so the integrand is ≥ 0. For a dominant weight v all $g_k(v)\geq 0$, so by (30.6) and multilinearity $h^{\mathrm{FS}}_{\Phi,n}(\mathrm{lr}(v)^{D+1})$ is a combination with non-negative coefficients of numbers $\widehat{\deg}(\hat{s}_{k_1}\cdots\hat{s}_{k_{D+1}}\cdot[\mathrm{Fl}_{\Phi,n}])\geq 0$. This proves (b).

Step 4: (a). Let $k_0=(a,i)$ with $g_{k_0}(v)=0$. Let $\pi\colon\mathbb{P}_{\mathrm{Fl}}\to\mathbb{P}'=\prod_{k\neq k_0}\mathbb{P}(\mathbb{Z}^{I_k})$ be the projection, $\overline{\mathcal{S}}'_k$ the Fubini–Study lines of $\mathbb{P}'$ and $\hat{s}'_k$ their classes, so that $\overline{\mathcal{S}}_k=\pi^*\overline{\mathcal{S}}'_k$ for $k\neq k_0$. By (30.6) the class $\sum_k g_k(v)\hat{s}_k=\pi^*(\sum_{k\neq k_0}g_k(v)\hat{s}'_k)$, and the projection formula (Lemma 30.18 for the projection π of regular arithmetic varieties and $\mathcal{Z}=\mathrm{Fl}_{\Phi,n}$, flat over $\mathbb{Z}$, after expanding multilinearly into products of the $\widehat{c}_1(\pi^*\overline{\mathcal{S}}'_k)$; with $\pi_*[\mathrm{Fl}_{\Phi,n}]=0$ standing for its part (2) when $\dim\pi(\mathrm{Fl}_{\Phi,n})<D+1$) gives

$$h^{\mathrm{FS}}_{\Phi,n}(\mathrm{lr}(v)^{D+1})=\widehat{\deg}\Bigl(\Bigl(\sum_{k\neq k_0}g_k(v)\hat{s}'_k\Bigr)^{D+1}\cdot\pi_*[\mathrm{Fl}_{\Phi,n}]\Bigr).$$

The image $\pi(\mathrm{Fl}_{\Phi,n})$ is the closure of the image of the generic fibre (the scheme is flat and integral, π is proper). On the generic fibre, π forgets the i-dimensional step of the flag in block a; its fibre through a flag (V_j) is the set of i-dimensional W with $V_{i-1}\subset W\subset V_{i+1}$ ($V_0=0$, $V_n=\mathbb{Q}^n$), a projective line. So $\pi(\mathrm{Fl}_{\Phi,n})$ has absolute dimension D, less than $D+1$, and $\pi_*[\mathrm{Fl}_{\Phi,n}]=0$. Hence $h^{\mathrm{FS}}_{\Phi,n}(\mathrm{lr}(v)^{D+1})=0$. □

Lemma 30.17. *Let $\mathbb{P}$ be a regular arithmetic variety, $\mathcal{Z}\subseteq\mathbb{P}$ a closed integral subscheme, flat over $\mathbb{Z}$, of absolute dimension $m+1$ with $m\geq 0$, $\overline{\mathcal{M}}_0,\dots,\overline{\mathcal{M}}_m$ Hermitian line bundles with smooth metrics on $\mathbb{P}$, and σ a non-zero rational section of $\mathcal{M}_0|_{\mathcal{Z}}$. Every prime divisor W of $\mathcal{Z}$ is either flat over $\mathbb{Z}$ (horizontal) or contained in a fibre $\mathcal{Z}_{\mathbb{F}_p}$ (vertical). The function $\log\|\sigma\|$ is integrable on $\mathcal{Z}(\mathbb{C})$ with respect to $c_1(\overline{\mathcal{M}}_1)\wedge\cdots\wedge c_1(\overline{\mathcal{M}}_m)$ (integration over the regular locus of $\mathcal{Z}(\mathbb{C})$; the counting measure if $m=0$), and*

$$\begin{aligned}\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_0)\cdots\widehat{c}_1(\overline{\mathcal{M}}_m)\mid\mathcal{Z}) &= \sum_{W\ \textit{horizontal}}\operatorname{ord}_W(\sigma)\,\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_1)\cdots\widehat{c}_1(\overline{\mathcal{M}}_m)\mid W)\\ &\quad+\sum_p\log p\sum_{W\subseteq\mathcal{Z}_{\mathbb{F}_p}}\operatorname{ord}_W(\sigma)\,\deg(c_1(\mathcal{M}_1|_W)\cdots c_1(\mathcal{M}_m|_W)\cap[W])\\ &\quad-\int_{\mathcal{Z}(\mathbb{C})}\log\|\sigma\|\,c_1(\overline{\mathcal{M}}_1)\wedge\cdots\wedge c_1(\overline{\mathcal{M}}_m),\end{aligned}$$

the sums running over the prime divisors W of $\mathcal{Z}$ with $\operatorname{ord}_W(\sigma)\neq 0$ (finitely many), the degree of a vertical W taken over $\mathbb{F}_p$.

Proof. Write $Y = \mathcal{Z}_{\mathbb{Q}}$, an integral projective variety of dimension m over $\mathbb{Q}$, dense in $\mathcal{Z}$, and $\sigma_Y = \sigma|_Y$. A prime divisor W of $\mathcal{Z}$ either dominates $\operatorname{Spec}\mathbb{Z}$, and is then flat (integral and torsion free), or maps to a closed point p, and then $W \subseteq \mathcal{Z}_{\mathbb{F}_p}$. The horizontal prime divisors are the closures of the prime divisors $W_{\mathbb{Q}}$ of Y, with the same local rings; only finitely many W have $\operatorname{ord}_W(\sigma) \neq 0$. By Lemma 29.26 the restrictions $\overline{\mathcal{M}}_l|_{\mathcal{Z}}$ are DSP; at a prime p we give $\mathcal{M}_l|_Y$ the algebraic metric induced by the model $(\mathcal{Z}, \mathcal{M}_l|_{\mathcal{Z}})$, which is the restriction of the metric induced by $(\mathbb{P}, \mathcal{M}_l)$ (a trivialising open set of $\mathbb{P}$ restricts to one of $\mathcal{Z}$). By Lemma 29.26, for $\mathcal{Z}$ and for every horizontal W, the heights of cycles in the statement are the global heights of Y and of $W_{\mathbb{Q}}$ of Proposition B.132 (3) for these metrics. Heights of cycles are symmetric, so the left side is the global height $h_{\overline{\mathcal{M}}_1,\dots,\overline{\mathcal{M}}_m,\overline{\mathcal{M}}_0}(Y)$.

Step 1: reduction to very ample $\mathcal{M}_1, \dots, \mathcal{M}_m$. Fix a very ample line bundle $\mathcal{O}_{\mathbb{P}}(1)$ on $\mathbb{P}$ with a smooth metric. For k large, $\mathcal{M}_l \otimes \mathcal{O}_{\mathbb{P}}(k)$ is very ample ([122, II, Ex. 7.5]), so $\overline{\mathcal{M}}_l = \overline{\mathcal{A}}_l \otimes \overline{\mathcal{B}}_l^{\otimes -1}$ with $\overline{\mathcal{A}}_l = \overline{\mathcal{M}}_l \otimes \overline{\mathcal{O}_{\mathbb{P}}(k)}$, $\overline{\mathcal{B}}_l = \overline{\mathcal{O}_{\mathbb{P}}(k)}$ very ample with smooth metrics. Both sides of the assertion are multilinear in $(\overline{\mathcal{M}}_1, \dots, \overline{\mathcal{M}}_m)$: heights of cycles and degrees are multilinear, c_1 of smooth metrics is additive, and the integrals are finite (Step 3). So we may assume that $\mathcal{M}_1, \dots, \mathcal{M}_m$ are very ample on $\mathbb{P}$; their restrictions to Y are generated by global sections.

Step 2: sections meeting Y properly. Let $Y_1 = Y \cdot \operatorname{div}\sigma_Y = \sum_{W \text{ horizontal}} \operatorname{ord}_W(\sigma)[W_{\mathbb{Q}}]$. Choose global sections s_l of $\mathcal{M}_l|_Y$, $l = 1, \dots, m$, inductively such that s_l vanishes at none of the generic points of the components of $Y \cdot \operatorname{div} s_1 \cdots \operatorname{div} s_{l-1}$ and of $Y_1 \cdot \operatorname{div} s_1 \cdots \operatorname{div} s_{l-1}$: for each of these finitely many points the global sections vanishing there form a proper subspace of the $\mathbb{Q}$-vector space of global sections (global generation), and a finite-dimensional vector space over the infinite field $\mathbb{Q}$ is not a finite union of proper subspaces. Then the family $(s_1, \dots, s_m, \sigma_Y)$ meets Y properly, and $(s_1, \dots, s_m)$ meets Y_1 properly.

Step 3: summing the local recursions. By Proposition B.132 (3) and (1), applied with $d = m$,

$$(\overline{L}_0, \dots, \overline{L}_d) = (\overline{\mathcal{M}}_1, \dots, \overline{\mathcal{M}}_m, \overline{\mathcal{M}}_0)$$

and $s_d = \sigma_Y$, the left side equals $\sum_v \big(h_v(Y_1; s_1, \dots, s_m) - I_v\big)$, where $I_v = \int \log \|\sigma_Y\|_v \, c_1(\overline{\mathcal{M}}_1) \wedge \cdots \wedge c_1(\overline{\mathcal{M}}_m) \wedge \delta_Y$ on the analytification at v, each I_v is finite (the local heights are real numbers) and $I_p = 0$ for all but finitely many p (all h_v vanish for those p, Proposition B.132 (3)). By Proposition B.132 (3) for Y_1, $\sum_v h_v(Y_1; s_1, \dots, s_m)$ is the global height of Y_1, which by linearity and Lemma 29.26 is the first sum of the assertion. At $v = \infty$ the analytification is $Y(\mathbb{C}) = \mathcal{Z}(\mathbb{C})$ and, by the normalisations in Proposition B.132 (1), $I_\infty = \int_{\mathcal{Z}(\mathbb{C})} \log \|\sigma\| \, c_1(\overline{\mathcal{M}}_1) \wedge \cdots \wedge c_1(\overline{\mathcal{M}}_m)$, which is therefore finite; this gives the integrability and the last term.

Step 4: the term at a prime p. Let $\nu\colon \widetilde{\mathcal{Z}} \to \mathcal{Z}$ be the normalisation, a finite birational morphism ($\mathcal{Z}$ is of finite type over $\mathbb{Z}$). Base change along the regular morphism $\operatorname{Spec}\mathbb{Z}_p \to \operatorname{Spec}\mathbb{Z}$ preserves normality, so $\widetilde{\mathcal{Z}} \otimes \mathbb{Z}_p$ is the normalisation of $\mathcal{Z} \otimes \mathbb{Z}_p$, i.e. the disjoint union of the normalisations of the closures of the components Y' of $Y_{\mathbb{Q}_p}$ (reduced, as $\mathbb{Q}_p/\mathbb{Q}$ is separable); its special fibre is $\widetilde{\mathcal{Z}}_{\mathbb{F}_p}$, and at the generic point of a component V of $\widetilde{\mathcal{Z}}_{\mathbb{F}_p}$ the local ring of $\widetilde{\mathcal{Z}} \otimes \mathbb{Z}_p$ is a discrete valuation ring flat over $\mathcal{O}_{\widetilde{\mathcal{Z}},V}$ with the same residue field, so ord_V is unchanged. Proposition B.132 (2) for each Y' (models $(\mathcal{Z} \otimes \mathbb{Z}_p, \mathcal{M}_l)$, $e_l = 1$, $\varpi = p$, $\log|p|_p = -\log p$), summed over the Y' by linearity of the integral term in the cycle, gives

$$-I_p = \log p \sum_V \operatorname{ord}_V(\nu^*\sigma)\, \deg\big(c_1(\nu^*\mathcal{M}_1|_V) \cdots c_1(\nu^*\mathcal{M}_m|_V) \cap [V]\big),$$

the sum over the components V of $\widetilde{\mathcal{Z}}_{\mathbb{F}_p}$, which are the prime divisors of $\widetilde{\mathcal{Z}}$ over p.

Step 5: descent from the normalisation. For a prime divisor W of $\mathcal{Z}$ over p, $\operatorname{ord}_W(\sigma) = \sum_{V \mapsto W} \operatorname{ord}_V(\nu^*\sigma)\, [\mathbf{k}(V) : \mathbf{k}(W)]$, the sum over the prime divisors V of $\widetilde{\mathcal{Z}}$ with $\nu(V) = W$ ([98,

Example 1.2.3]; ν is finite and birational), and every V over p maps onto such a W (ν is finite, so $\dim \nu(V) = \dim V$). By the projection formula for the finite surjective morphism $V \to W$ of projective $\mathbb{F}_p$-schemes ([98, Proposition 2.5(c)], the projection formula),

$$\deg(c_1(\nu^*\mathcal{M}_1|_V)\cdots c_1(\nu^*\mathcal{M}_m|_V)\cap[V]) = [\mathbf{k}(V):\mathbf{k}(W)]\deg(c_1(\mathcal{M}_1|_W)\cdots c_1(\mathcal{M}_m|_W)\cap[W]).$$

Hence

$$-I_p = \log p \sum_{W\subseteq \mathcal{Z}_{\mathbb{F}_p}} \operatorname{ord}_W(\sigma)\deg(c_1(\mathcal{M}_1|_W)\cdots c_1(\mathcal{M}_m|_W)\cap[W]),$$

the second sum of the assertion. Steps 3–5 give the identity. □

Lemma 30.18. *Let $b \geq 1$ and $d \geq 0$ be integers, $f\colon \mathbb{P} \to \mathbb{P}'$ a morphism of regular arithmetic varieties, $\mathcal{Z} \subseteq \mathbb{P}$ a closed integral subscheme, flat over $\mathbb{Z}$, of absolute dimension $b+d$, and $\mathcal{V} = f(\mathcal{Z})$. Let $\overline{\mathcal{L}}_1,\dots,\overline{\mathcal{L}}_d$ and $\overline{\mathcal{H}}_1,\dots,\overline{\mathcal{H}}_b$ be Hermitian line bundles with smooth metrics on $\mathbb{P}$ and on $\mathbb{P}'$, respectively, and assume $\dim\mathcal{V} \leq b$. Then $\mathcal{V}$ is a closed integral subscheme of $\mathbb{P}'$, flat over $\mathbb{Z}$, and:*

1. *if $\dim\mathcal{V} = b$, then*

$$\widehat{\deg}\Big(\prod_{j=1}^{b}\widehat{c}_1(f^*\overline{\mathcal{H}}_j)\prod_{i=1}^{d}\widehat{c}_1(\overline{\mathcal{L}}_i)\;\Big|\;\mathcal{Z}\Big) = c\,\widehat{\deg}\Big(\prod_{j=1}^{b}\widehat{c}_1(\overline{\mathcal{H}}_j)\;\Big|\;\mathcal{V}\Big),$$

   where $c = \deg(c_1(\mathcal{L}_1|_{\mathcal{Z}_\eta})\cdots c_1(\mathcal{L}_d|_{\mathcal{Z}_\eta})\cap[\mathcal{Z}_\eta])$ and $\mathcal{Z}_\eta$ is the fibre of $f|_{\mathcal{Z}}\colon \mathcal{Z}\to\mathcal{V}$ over the generic point η of $\mathcal{V}$; if moreover $d=0$, then $c = [\mathbf{k}(\mathcal{Z}):\mathbf{k}(\mathcal{V})]$;

2. *if $\dim\mathcal{V} < b$, then the left side of the display in (1) is 0.*

Proof. Step 1: the image. The scheme $\mathbb{P}$ is projective over $\operatorname{Spec}\mathbb{Z}$ and $\mathbb{P}'$ is separated, so f is proper and $\mathcal{V} = f(\mathcal{Z})$ is closed; it is irreducible, and we give it the reduced structure, so it is integral. As $\mathcal{Z}$ dominates $\operatorname{Spec}\mathbb{Z}$, so does $\mathcal{V}$; an integral scheme dominating $\operatorname{Spec}\mathbb{Z}$ is torsion free over $\mathbb{Z}$, hence flat. Put $b' = \dim\mathcal{V}$; then $1 \leq b' \leq b$, and $\pi = f|_{\mathcal{Z}}\colon \mathcal{Z}\to\mathcal{V}$ is a surjective morphism of projective schemes over $\operatorname{Spec}\mathbb{Z}$ of relative dimensions $b+d-1$ and $b'-1$.

Step 2: passage to heights of Burgos Gil, Philippon and Sombra. The metrics of the $f^*\overline{\mathcal{H}}_j$ are smooth (pull-backs by the holomorphic map $f_{\mathbb{C}}$), and $(f^*\overline{\mathcal{H}}_j)|_{\mathcal{Z}} = \pi^*(\overline{\mathcal{H}}_j|_{\mathcal{V}})$. Heights of cycles are symmetric in the Hermitian line bundles. By Lemma 29.26 on $\mathbb{P}$, the restrictions to $\mathcal{Z}$ of the $\overline{\mathcal{L}}_i$ and of the $f^*\overline{\mathcal{H}}_j$ are DSP and, for any order of the bundles,

$$\widehat{\deg}\Big(\prod_{j=1}^{b}\widehat{c}_1(f^*\overline{\mathcal{H}}_j)\prod_{i=1}^{d}\widehat{c}_1(\overline{\mathcal{L}}_i)\;\Big|\;\mathcal{Z}\Big) = h_{\pi^*\overline{\mathcal{H}}_1|_{\mathcal{V}},\dots,\pi^*\overline{\mathcal{H}}_{b'}|_{\mathcal{V}},\,\pi^*\overline{\mathcal{H}}_{b'+1}|_{\mathcal{V}},\dots,\pi^*\overline{\mathcal{H}}_b|_{\mathcal{V}},\,\overline{\mathcal{L}}_1|_{\mathcal{Z}},\dots,\overline{\mathcal{L}}_d|_{\mathcal{Z}}}(\mathcal{Z}).$$

By Lemma 29.26 on $\mathbb{P}'$, the $\overline{\mathcal{H}}_j|_{\mathcal{V}}$ are DSP.

Step 3: the case $b' = b$. Proposition B.129 for $\pi\colon\mathcal{Z}\to\mathcal{V}$, the b bundles $\overline{\mathcal{H}}_j|_{\mathcal{V}}$ and the d bundles $\overline{\mathcal{L}}_i|_{\mathcal{Z}}$ gives that the right side of the display in Step 2 equals $\deg_{\mathcal{L}_1,\dots,\mathcal{L}_d}(\mathcal{Z}_\eta)\,h_{\overline{\mathcal{H}}_1|_{\mathcal{V}},\dots,\overline{\mathcal{H}}_b|_{\mathcal{V}}}(\mathcal{V})$, where the degree is c; and by Lemma 29.26 on $\mathbb{P}'$ the last height is $\widehat{\deg}(\prod_j\widehat{c}_1(\overline{\mathcal{H}}_j)\mid\mathcal{V})$. This is the display of (1).

Step 4: the case $b' < b$. Apply Proposition B.129 to π with the b' bundles $\overline{\mathcal{H}}_1|_{\mathcal{V}},\dots,\overline{\mathcal{H}}_{b'}|_{\mathcal{V}}$ on $\mathcal{V}$ and the $d' = d+b-b'$ bundles $\pi^*\overline{\mathcal{H}}_{b'+1}|_{\mathcal{V}},\dots,\pi^*\overline{\mathcal{H}}_b|_{\mathcal{V}},\overline{\mathcal{L}}_1|_{\mathcal{Z}},\dots,\overline{\mathcal{L}}_d|_{\mathcal{Z}}$ on $\mathcal{Z}$ (relative dimensions $b'+d'-1$ and $b'-1$). The right side of the display in Step 2 equals the degree of $\mathcal{Z}_\eta$ with respect to the restrictions of these d' bundles times a height of $\mathcal{V}$. The restriction of $\pi^*\mathcal{H}_j$ to $\mathcal{Z}_\eta$

is the pull-back of a line bundle on $\operatorname{Spec}\mathbf{k}(\mathcal{V})$, hence trivial, so its first Chern class on $\mathcal{Z}_\eta$ is 0; as $b-b'\geq 1$ such factors occur, the degree is 0. This is (2).

Step 5: the case $d=0$ in (1). Then $\mathcal{Z}$ and $\mathcal{V}$ are integral of the same dimension b and π is surjective, so $\mathbf{k}(\mathcal{Z})$ is a finitely generated extension of $\mathbf{k}(\mathcal{V})$ of transcendence degree 0, hence finite. The points of $\mathcal{Z}_\eta$ are the points of $\mathcal{Z}$ over η, with the same local rings (as $\mathcal{O}_{\mathcal{V},\eta}=\mathbf{k}(\mathcal{V})$); each specialises from the generic point ξ of $\mathcal{Z}$, and the closure of a point $x\neq\xi$ over η would be an integral closed subscheme of dimension $<b$ dominating $\mathcal{V}$, which is impossible. So $\mathcal{Z}_\eta=\operatorname{Spec}\mathcal{O}_{\mathcal{Z},\xi}=\operatorname{Spec}\mathbf{k}(\mathcal{Z})$ and $c=\deg[\mathcal{Z}_\eta]=[\mathbf{k}(\mathcal{Z}):\mathbf{k}(\mathcal{V})]$. □

Lemma 30.19. *For every permutation σ of Φ and every $F\in\mathcal{R}$, $h^{\mathrm{FS}}_{\Phi,n}(\sigma\cdot F)=h^{\mathrm{FS}}_{\Phi,n}(F)$, where $\sigma\cdot F$ is obtained from F by substituting $x_{\sigma(a),i}$ for $x_{a,i}$.*

Proof. Both sides are linear in F and $F\mapsto\sigma\cdot F$ is linear and preserves degrees, so it suffices to take $F=x_{r_1}\cdots x_{r_{D+1}}$ with roots $r_j=(\tau_j,i_j)$; then $\sigma\cdot F=x_{\sigma r_1}\cdots x_{\sigma r_{D+1}}$ with $\sigma r=(\sigma(\tau),i)$ for $r=(\tau,i)$.

Step 1: an automorphism of the flag scheme. The bijection $\bar\sigma(\tau,j)=(\sigma(\tau),j)$ of K_{Fl} satisfies $I_{\bar\sigma(k)}=I_k$. Let $\psi\colon\mathbb{P}_{\mathrm{Fl}}\to\mathbb{P}_{\mathrm{Fl}}$ be the automorphism over $\mathbb{Z}$ permuting the factors, $\psi(z)_{\bar\sigma(k)}=z_k$; on coordinate rings $\psi^*z_{\bar\sigma(k),J}=z_{k,J}$. Let α be the automorphism of $A[T_k]$ with $\alpha(X_{\tau,a,b})=X_{\sigma(\tau),a,b}$ and $\alpha(T_k)=T_{\bar\sigma(k)}$. Since $\mu_{k,J}$ is a minor of the block τ of the generic matrices, $\alpha(\mu_{k,J}T_k)=\mu_{\bar\sigma(k),J}T_{\bar\sigma(k)}$. For a polynomial G in the $z_{k,J}$, write $\mathrm{ev}(G)=G(\mu_{k,J}T_k)$. Then $\mathrm{ev}(\psi^*G)$ is G evaluated at $z_{\bar\sigma(k),J}=\mu_{k,J}T_k$, and applying α gives G evaluated at $z_{\bar\sigma(k),J}=\mu_{\bar\sigma(k),J}T_{\bar\sigma(k)}$, i.e. $\alpha(\mathrm{ev}(\psi^*G))=\mathrm{ev}(G)$. As α is injective, $\psi^*G\in\mathfrak{p}_{\mathrm{Fl}}$ if and only if $G\in\mathfrak{p}_{\mathrm{Fl}}$. Hence $\psi^*\mathfrak{p}_{\mathrm{Fl}}=\mathfrak{p}_{\mathrm{Fl}}$ (ψ^* being bijective), and ψ restricts to an automorphism ψ_{Fl} of the arithmetic variety $\mathrm{Fl}_{\Phi,n}$; it is defined over $\mathbb{Z}$, so it commutes with complex conjugation.

Step 2: the root lines are pulled back isometrically. For $e\in\mathbb{Z}^{K_{\mathrm{Fl}}}$ put $(\bar\sigma^*e)_k=e_{\bar\sigma(k)}$; then $\psi^*\mathcal{O}(e)=\mathcal{O}(\bar\sigma^*e)$ (a multihomogeneous polynomial of multidegree e pulls back to one of multidegree $\bar\sigma^*e$) and

$$h^{\mathrm{FS}}_e(\psi(z))=\prod_k\Big(\sum_J|z_{k,J}|^2\Big)^{e_{\bar\sigma(k)}/2}=h^{\mathrm{FS}}_{\bar\sigma^*e}(z).$$

For a root $r=(\tau,i)$, the entries of $e^{\mathrm{Fl}}_{\sigma r}$ sit at $(\sigma(\tau),i)$ and $(\sigma(\tau),i-1)$, so $\bar\sigma^*e^{\mathrm{Fl}}_{\sigma r}=e^{\mathrm{Fl}}_r$. Hence $\psi^*_{\mathrm{Fl}}\overline{\mathcal{X}}^{\mathrm{FS}}_{\sigma r}\cong\overline{\mathcal{X}}^{\mathrm{FS}}_r$ isometrically.

Step 3: invariance of the numbers. Arithmetic intersection numbers are invariant under isomorphisms of arithmetic varieties that identify the Hermitian line bundles isometrically: ψ_{Fl} is proper of degree one, $\psi_{\mathrm{Fl},*}[\mathrm{Fl}_{\Phi,n}]=[\mathrm{Fl}_{\Phi,n}]$, and the projection formula of [110] gives

$$\widehat{\deg}\Big(\prod_j\widehat{c}_1(\psi^*_{\mathrm{Fl}}\overline{\mathcal{X}}^{\mathrm{FS}}_{\sigma r_j})\Big)=\widehat{\deg}\Big(\prod_j\widehat{c}_1(\overline{\mathcal{X}}^{\mathrm{FS}}_{\sigma r_j})\Big).$$

By Step 2 the left side is $\widehat{\deg}(\prod_j\widehat{c}_1(\overline{\mathcal{X}}^{\mathrm{FS}}_{r_j}))$. Multiplying by $(-1)^{D+1}$ and using (29.7), $h^{\mathrm{FS}}_{\Phi,n}(\sigma\cdot F)=h^{\mathrm{FS}}_{\Phi,n}(F)$. □

Definition 30.20. Let Φ be a non-empty finite set, $n\geq 2$ and $1\leq\kappa<n$. Put $\mathbb{P}_{\mathrm{Gr}}=\prod_{\tau\in\Phi}\mathbb{P}(\mathbb{Z}^{I_{(\tau,\kappa)}})$, with coordinates $z_{\tau,J}$ (J a κ-element subset of $\{1,\dots,n\}$). The **Grassmannian scheme** $\mathrm{Gr}_{\Phi,n,\kappa}\subseteq\mathbb{P}_{\mathrm{Gr}}$ is the closed subscheme defined by the multihomogeneous ideal

$$\mathfrak{p}_{\mathrm{Gr}}=\ker\Big(\mathbb{Z}[z_{\tau,J}]\to A[T_\tau:\tau\in\Phi],\quad z_{\tau,J}\mapsto\mu_{(\tau,\kappa),J}\,T_\tau\Big).\tag{30.7}$$

Its **Fubini–Study Plücker line** is $\overline{\mathcal{G}} = (\mathcal{O}(1,\dots,1), h^{\mathrm{FS}})|_{\mathrm{Gr}_{\Phi,n,\kappa}}$ with

$$h^{\mathrm{FS}}(z) = \prod_{\tau} \Big(\sum_{J} |z_{\tau,J}|^2\Big)^{1/2}.$$

The **forgetful projection** $\pi_\kappa \colon \mathbb{P}_{\mathrm{Fl}} \to \mathbb{P}_{\mathrm{Gr}}$ is the projection onto the factors $k = (\tau,\kappa)$, $\tau \in \Phi$.

Proposition 30.21. *For every non-empty finite set Φ, every $n \geq 2$ and every $1 \leq \kappa < n$,*

$$h^{\mathrm{FS}}_{\Phi,n}(\mathrm{pl}^{D^{\mathrm{gr}}+1}\mathrm{fb}) \leq (D^{\mathrm{gr}}+1)\deg^{\mathrm{fl}}(\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb})\,|\Phi|\,\frac{\min(\kappa,n-\kappa)}{2}\,\log(n+1).$$

Proof. Write $\mathrm{Fl} = \mathrm{Fl}_{\Phi,n}$, $\mathrm{Gr} = \mathrm{Gr}_{\Phi,n,\kappa}$, $G = D^{\mathrm{gr}}$, $r = D - G$, $T = \deg^{\mathrm{fl}}(\mathrm{pl}^G\mathrm{fb})$ and $e = \frac{\min(\kappa,n-\kappa)}{2}\log(n+1)$.

Step 1: the schemes are smooth over $\mathbb{Z}$. By Lemma 30.26(2), Fl is the scheme of Φ-tuples of complete flags with locally free quotients in its Plücker embedding and Gr the product over Φ of the Grassmannian schemes of κ-planes in their Plücker embeddings; both are smooth, projective and integral over $\mathbb{Z}$, in particular regular. The restriction $\pi \colon \mathrm{Fl} \to \mathrm{Gr}$ of π_κ is projective, and on generic fibres it is the forgetful morphism $\prod_\Phi \mathrm{Fl}(\mathbb{Q}^n) \to \prod_\Phi \mathrm{Gr}(\kappa,\mathbb{Q}^n)$, equivariant for GL_n^Φ, which acts transitively on both sides; by generic smoothness in characteristic 0 it is smooth, of relative dimension $D - G = r$.

Step 2: the test class as a pull-back times a fibre class. For $k \in K_{\mathrm{Fl}}$ let $\hat{s}_k$ be the class of the line $\mathcal{O}(1)$ of the factor k with its Fubini–Study metric, restricted to Fl, and $\hat{s}_{(\tau,0)} = \hat{s}_{(\tau,n)} = 0$. Since $h^{\mathrm{FS}}_e = \prod_k h^{\mathrm{FS}\,e_k}_{1_k}$, $\widehat{c}_1(\overline{\mathcal{X}}^{\mathrm{FS}}_{(\tau,i)}) = \hat{s}_{(\tau,i)} - \hat{s}_{(\tau,i-1)}$. Put $\xi_r = -\widehat{c}_1(\overline{\mathcal{X}}^{\mathrm{FS}}_r)$ and, for $F \in \mathcal{R}$, let $F(\xi)$ be the class obtained by substituting ξ_r for x_r. By (29.7) and multilinearity, $h^{\mathrm{FS}}_{\Phi,n}(F) = \widehat{\deg}(F(\xi))$ for F homogeneous of degree $D+1$.

Since $\overline{\mathcal{G}}$ pulls back under π to $\bigotimes_\tau \overline{\mathcal{S}}_{(\tau,\kappa)}|_{\mathrm{Fl}}$ ($\overline{\mathcal{S}}_k$ the line $\mathcal{O}(1)$ of the factor k with its Fubini–Study metric), $\mathrm{pl}(\xi) = \pi^*\widehat{c}_1(\overline{\mathcal{G}})$. Let $\beta = \mathrm{fb}(\xi) \in \widehat{\mathrm{CH}}^r(\mathrm{Fl})_{\mathbb{Q}}$, the product of the r classes $\xi_{(\tau,i')} - \xi_{(\tau,i)}$. Then

$$h^{\mathrm{FS}}_{\Phi,n}(\mathrm{pl}^{G+1}\mathrm{fb}) = \widehat{\deg}(\pi^*\widehat{c}_1(\overline{\mathcal{G}})^{G+1}\cdot\beta).$$

Step 3: reduction to the Grassmannian. Write $\mathrm{fb} = \prod_{j=1}^r (x_{(\tau_j,i'_j)} - x_{(\tau_j,i_j)})$ for its r factors and put $\overline{\mathcal{N}}_j = \overline{\mathcal{X}}^{\mathrm{FS}}_{(\tau_j,i_j)} \otimes (\overline{\mathcal{X}}^{\mathrm{FS}}_{(\tau_j,i'_j)})^{\otimes -1}$, a Hermitian line bundle on Fl with smooth metric (a quotient of cone metrics restricted to Fl) and $\widehat{c}_1(\overline{\mathcal{N}}_j) = \xi_{(\tau_j,i'_j)} - \xi_{(\tau_j,i_j)}$. So $\beta = \prod_j \widehat{c}_1(\overline{\mathcal{N}}_j)$, and with $\pi^*\widehat{c}_1(\overline{\mathcal{G}}) = \widehat{c}_1(\pi^*\overline{\mathcal{G}})$ Step 2 reads

$$h^{\mathrm{FS}}_{\Phi,n}(\mathrm{pl}^{G+1}\mathrm{fb}) = \widehat{\deg}\Big(\widehat{c}_1(\pi^*\overline{\mathcal{G}})^{G+1}\prod_{j=1}^{r}\widehat{c}_1(\overline{\mathcal{N}}_j)\Big).$$

Let γ be the generic point of Gr, $\mathrm{Fl}_\gamma = \mathrm{Fl}\times_{\mathrm{Gr}}\mathrm{Spec}\,\mathbf{k}(\mathrm{Gr})$ the fibre of π over it ($\mathbf{k}(\cdot)$ the function field), and

$$c = \deg\big(c_1(\mathcal{N}_1|_{\mathrm{Fl}_\gamma})\cdots c_1(\mathcal{N}_r|_{\mathrm{Fl}_\gamma}) \cap [\mathrm{Fl}_\gamma]\big) \in \mathbb{Z}.$$

Apply Lemma 30.13 with $b = G+1$, $d = r$, the morphism π, $\overline{\mathcal{H}}_1 = \cdots = \overline{\mathcal{H}}_{G+1} = \overline{\mathcal{G}}$ and $\overline{\mathcal{L}}_j = \overline{\mathcal{N}}_j$: by Step 1, Fl and Gr are integral, flat and projective over $\mathbb{Z}$ with smooth generic fibres, of absolute dimensions $D+1 = b+d$ and $G+1 = b$; $\overline{\mathcal{G}}$ has a smooth metric (a product of Fubini–Study metrics, Definition 30.20) and so have the $\overline{\mathcal{N}}_j$. It gives

$$h^{\mathrm{FS}}_{\Phi,n}(\mathrm{pl}^{G+1}\mathrm{fb}) = c\,\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{G}})^{G+1}).$$

Step 4: the same factor on the generic fibre. By Lemma 30.22 and linearity,

$$T = \deg(\mathrm{pl}(\eta)^G \mathrm{fb}(\eta) \cap [\mathrm{Fl}_{\mathbb{Q}}]),$$

and as in Step 2 $\mathrm{pl}(\eta) = \pi_{\mathbb{Q}}^* c_1(\mathcal{G}_{\mathbb{Q}})$. Here $\eta_r = c_1(\mathcal{O}(-e_r^{\mathrm{Fl}})|_{\mathrm{Fl}_{\mathbb{Q}}})$, so $c_1(\mathcal{N}_{j,\mathbb{Q}}) = \eta_{(\tau_j,i'_j)} - \eta_{(\tau_j,i_j)}$ and $\mathrm{fb}(\eta) = c_1(\mathcal{N}_{1,\mathbb{Q}}) \cdots c_1(\mathcal{N}_{r,\mathbb{Q}})$. The generic fibres $\mathrm{Fl}_{\mathbb{Q}}$ and $\mathrm{Gr}_{\mathbb{Q}}$ are integral projective $\mathbb{Q}$-schemes (localisations of the integral schemes Fl, Gr) of dimensions $D = G + r$ and G, and the fibre of $\pi_{\mathbb{Q}}$ over the generic point of $\mathrm{Gr}_{\mathbb{Q}}$ is Fl_γ (as $\mathbf{k}(\mathrm{Gr}_{\mathbb{Q}}) = \mathbf{k}(\mathrm{Gr})$). So Lemma 30.25 over $k = \mathbb{Q}$, for $\pi_{\mathbb{Q}}$ and the $\mathcal{N}_{j,\mathbb{Q}}$, gives $\pi_{\mathbb{Q}*}(\mathrm{fb}(\eta) \cap [\mathrm{Fl}_{\mathbb{Q}}]) = c\,[\mathrm{Gr}_{\mathbb{Q}}]$ with the c of Step 3. By the projection formula $\pi_{\mathbb{Q}*}(c_1(\pi_{\mathbb{Q}}^*\mathcal{G}_{\mathbb{Q}}) \cap \alpha) = c_1(\mathcal{G}_{\mathbb{Q}}) \cap \pi_{\mathbb{Q}*}\alpha$ and the invariance of degrees of zero-cycles under proper push-forward [98] (Proposition 2.5, Definition 1.4),

$$T = \deg(c_1(\mathcal{G}_{\mathbb{Q}})^G \cap \pi_{\mathbb{Q}*}(\mathrm{fb}(\eta) \cap [\mathrm{Fl}_{\mathbb{Q}}])) = c\,\deg(c_1(\mathcal{G}_{\mathbb{Q}})^G \cap [\mathrm{Gr}_{\mathbb{Q}}]) = c\,g_{\mathrm{Gr}}.$$

As $T > 0$ (Proposition 29.36(a)) and $g_{\mathrm{Gr}} > 0$ (Lemma 30.23), $c = T/g_{\mathrm{Gr}} > 0$.

Step 5: conclusion. By Steps 3 and 4 and Lemma 30.23,

$$h^{\mathrm{FS}}_{\Phi,n}(\mathrm{pl}^{G+1}\mathrm{fb}) = c\,\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{G}})^{G+1}) \le c\,(G+1)\,|\Phi|\,e\,g_{\mathrm{Gr}} = (G+1)\,T\,|\Phi|\,e,$$

which is the assertion. □

Lemma 30.22. *Let Φ be a non-empty finite set and $n \ge 2$. For all roots $r_1, \dots, r_D$,*

$$\deg\Big(c_1(\mathcal{O}(-e_{r_1}^{\mathrm{Fl}})) \cdots c_1(\mathcal{O}(-e_{r_D}^{\mathrm{Fl}})) \cap [\mathrm{Fl}_{\Phi,n,\mathbb{Q}}]\Big) = \deg^{\mathrm{fl}}(x_{r_1} \cdots x_{r_D}).$$

Proof. Write $\mathrm{gd}(F)$ for the linear form on the homogeneous elements $F \in \mathcal{R}$ of degree D whose value on $x_{r_1} \cdots x_{r_D}$ is the left-hand side; it is well defined since intersection numbers of line bundles are symmetric. We show $\mathrm{gd} = \deg^{\mathrm{fl}}$.

Step 1: the generic fibre is a product of complete flag varieties in their Plücker embeddings. Over $\mathbb{C}$, $\mathfrak{p}_{\mathrm{Fl}} \otimes \mathbb{C}$ is the kernel of $\mathbb{C}[z_{k,J}] \to (A \otimes \mathbb{C})[T_k]$, $z_{k,J} \mapsto \mu_{k,J} T_k$ ($\mathbb{C}$ is flat over $\mathbb{Z}$), a prime ideal (the target is a domain); so $\mathrm{Fl}_{\Phi,n,\mathbb{C}}$ is reduced and irreducible, and its cone is the Zariski closure of the image of the map $(X,T) \mapsto (\mu_{k,J}(X)T_k)_{k,J}$ on $\mathrm{M}_n(\mathbb{C})^\Phi \times \mathbb{C}^{K_{\mathrm{Fl}}}$. The invertible X and non-zero T form a dense open subset, so the cone is also the closure of their image. For $X = (X_\tau)$ invertible the point is $\big([\wedge^j(\text{first } j \text{ rows of } X_\tau)]\big)_{(\tau,j)}$, the Plücker coordinates of the spans $U_{\tau,j}$ of the first j rows; $(U_{\tau,j})_j$ is a complete flag in $\mathbb{C}^n$ and every tuple of complete flags arises. Let $G = \mathrm{SL}_n(\mathbb{C})$, B the upper triangular Borel subgroup, and let $\iota\colon (G/B)^\Phi \to \mathbb{P}_{\mathrm{Fl}}(\mathbb{C})$ send $(g_\tau B)$ to the Plücker coordinates of the spans of the first j columns of g_τ (= first j rows of g_τ^{T}). For the fundamental weights $\omega_j = (1^j, 0^{n-j})$, $V_{\omega_j} = \wedge^j\mathbb{C}^n$ with highest weight vector $e_1 \wedge \cdots \wedge e_j$, so the factor (τ, j) of ι is $e_{\omega_j} \circ \mathrm{pr}_\tau$ in the notation of Theorem B.173. Composing ι with the Segre embedding of $\prod_j \mathbb{P}(\wedge^j\mathbb{C}^n)$ in each block gives $gB \mapsto [g \cdot \otimes_j v_{\omega_j}]$, i.e. $e_{\rho'}$ for the regular dominant weight $\rho' = \sum_j \omega_j$ followed by the linear inclusion $\mathbb{P}(V_{\rho'}) \subseteq \mathbb{P}(\otimes_j \wedge^j \mathbb{C}^n)$ ($\otimes_j v_{\omega_j}$ is a highest weight vector of weight ρ' and generates $V_{\rho'}$); by Theorem B.173 this is a closed embedding, hence so is ι. Its image is the closed set of tuples of complete flags, which by the above is $\mathrm{Fl}_{\Phi,n}(\mathbb{C})$; as $\mathrm{Fl}_{\Phi,n,\mathbb{C}}$ is reduced, ι is an isomorphism $(G/B)^\Phi \cong \mathrm{Fl}_{\Phi,n,\mathbb{C}}$, with $\iota^*\mathcal{O}(1_{(\tau,j)}) = \mathrm{pr}_\tau^*\mathcal{L}_{\omega_j}$. Since intersection numbers do not change under extension of the base field, gd may be computed on $(G/B)^\Phi$.

Step 2: powers of a regular dominant weight. Let $v \in \mathbb{Z}^{\Phi\times\{1,\dots,n\}}$ with $v_{\tau,1} > \cdots > v_{\tau,n}$ for all τ. By multilinearity, $\mathrm{gd}(\mathrm{lr}(v)^D) = \deg(c_1(\mathcal{O}(\sum_r v_r e_r^{\mathrm{Fl}}))^D \cap [\mathrm{Fl}_{\Phi,n,\mathbb{Q}}])$, since $\mathrm{lr}(v) = -\sum_r v_r x_r$ and x_r is sent to $-c_1(\mathcal{O}(e_r^{\mathrm{Fl}}))$.

The line bundles $\mathcal{L}_\lambda$ are multiplicative in λ ($e_{\lambda+\mu}$ is the Segre product of e_λ and e_μ followed by the inclusion of $\mathbb{P}(V_{\lambda+\mu})$ spanned by $g(v_\lambda \otimes v_\mu)$, and the Segre map pulls $\mathcal{O}(1)$ back to $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$), so by Step 1 the class is $\sum_\tau \mathrm{pr}_\tau^* h_\tau$ with $h_\tau = c_1(\mathcal{L}_{\lambda_\tau})$. On the product of the $|\Phi|$ factors of dimension $N = n(n-1)/2$, a monomial $\prod_\tau \mathrm{pr}_\tau^* h_\tau^{m_\tau}$ with $\sum m_\tau = D = |\Phi|N$ has degree $\prod_\tau \int h_\tau^{m_\tau}$, which is 0 unless all $m_\tau = N$. Hence, by Theorem B.173 and Lemma 30.14,

$$\mathrm{gd}(\mathrm{lr}(v)^D) = \frac{D!}{(N!)^{|\Phi|}} \prod_{\tau\in\Phi} N! \prod_{i<i'} \frac{v_{\tau,i} - v_{\tau,i'}}{i'-i} = \deg^{\mathrm{fl}}(\mathrm{lr}(v)^D).$$

Step 3: all weights. Both $v \mapsto \mathrm{gd}(\mathrm{lr}(v)^D)$ and $v \mapsto \deg^{\mathrm{fl}}(\mathrm{lr}(v)^D)$ are polynomial functions on $\mathbb{R}^{\Phi\times\{1,\dots,n\}}$ (expand the power). Under the affine bijection $v_{\tau,i} = c_\tau + \sum_{j=i}^{n-1} m_{\tau,j}$, the set of Step 2 contains the image of $\mathbb{Z}^\Phi \times \mathbb{Z}_{\geq 1}^{\Phi\times\{1,\dots,n-1\}}$, a product of infinite sets, on which a polynomial vanishes only if it is zero. So the two polynomial functions agree everywhere.

Step 4: all classes. Every linear form $\sum_r c_r x_r$ equals $\mathrm{lr}(-c)$, so by Step 3 $\mathrm{gd}(\ell^D) = \deg^{\mathrm{fl}}(\ell^D)$ for every linear form ℓ. By the polarisation identity

$$\ell_1 \cdots \ell_D = \frac{1}{D!} \sum_{S\subseteq\{1,\dots,D\}} (-1)^{D-|S|} \Big(\sum_{i\in S} \ell_i\Big)^D,$$

every monomial of degree D is a linear combination of D-th powers of linear forms; by linearity $\mathrm{gd} = \deg^{\mathrm{fl}}$ on all homogeneous elements of degree D. □

Lemma 30.23. *Let Φ be a non-empty finite set, $n \geq 2$ and $1 \leq \kappa < n$. Then $g_{\mathrm{Gr}} := \deg\big(c_1(\mathcal{G}_\mathbb{Q})^{D^{\mathrm{gr}}} \cap [\mathrm{Gr}_{\Phi,n,\kappa,\mathbb{Q}}]\big) > 0$ and*

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{G}})^{D^{\mathrm{gr}}+1}) \leq (D^{\mathrm{gr}}+1)\,|\Phi|\,\frac{\min(\kappa, n-\kappa)}{2}\log(n+1)\,g_{\mathrm{Gr}}.$$

Proof. Write $\mathrm{Gr} = \mathrm{Gr}_{\Phi,n,\kappa}$, $G = D^{\mathrm{gr}}$, $m_\kappa = \min(\kappa, n-\kappa)$, $e = \frac{m_\kappa}{2}\log(n+1)$, and $J_0 = \{1,\dots,\kappa\}$.

Step 1: Gr *and the hypotheses of Lemma 30.24.* For an algebraically closed field $\Omega \supseteq \mathbb{Q}$, $\mathfrak{p}_{\mathrm{Gr}} \otimes \Omega$ is the kernel of the corresponding map over Ω (Ω is flat over $\mathbb{Z}$), so the affine cone of Gr_Ω is the closure of the image of $(X,T) \mapsto (\mu_{(\tau,\kappa),J}(X)T_\tau)$; the image of the dense open set of invertible X and non-zero T consists of the cones over the tuples of Plücker points of κ-dimensional subspaces (the spans of the first κ rows), and every tuple arises. Hence $\mathrm{Gr}(\Omega)$ is the set of tuples of κ-dimensional subspaces of Ω^n in their Plücker embeddings, Gr_Ω is irreducible of dimension $G = |\Phi|\kappa(n-\kappa) \geq 1$, and Gr is an arithmetic variety of absolute dimension $G+1$ in $\mathbb{P}_{\mathrm{Gr}}$. Apply Lemma 30.24 with $d = G$, the multidegree $(1,\dots,1)$ in the role of its e, and $h = h^{\mathrm{FS}}$. (1): $\mathcal{O}(1,\dots,1)$ is generated by the global sections $\prod_\tau z_{\tau,J_\tau}$, so its degree on every curve in a closed fibre is ≥ 0. (2): $\log h^{\mathrm{FS}} = \frac{1}{2}\sum_\tau \log\sum_J |z_{\tau,J}|^2$, and $\log\sum_J |z_J|^2$ is plurisubharmonic on $\mathbb{C}^I \smallsetminus \{0\}$, so the Levi form is semipositive on every vector. (3): $\mathcal{G}_\mathbb{Q}$ is the restriction of $\mathcal{O}(1,\dots,1)$, which is very ample on $\mathbb{P}_{\mathrm{Gr},\mathbb{Q}}$ (Segre), so g_{Gr} is the degree of the G-dimensional image of $\mathrm{Gr}_\mathbb{Q}$ under the Segre embedding, hence > 0.

Step 2: points of small height. Let K be a number field and $Z = (Z_\tau)_{\tau\in\Phi}$ a family of $\kappa \times (n-\kappa)$ matrices whose entries are roots of unity in K. Let $M_\tau = (1_\kappa \mid Z_\tau) \in \mathrm{M}_{\kappa\times n}(\mathcal{O}_K)$ and $\mu_{\tau,J} = \det(M_\tau)_J \in \mathcal{O}_K$ (columns J), so $\mu_{\tau,J_0} = 1$. Every element of $\mathfrak{p}_{\mathrm{Gr}}$ vanishes at $(\mu_{\tau,J})$ (specialise X_τ to an $n\times n$ matrix with first κ rows M_τ and T_τ to 1), so $(\mu_{\tau,J})$ is a K-point x_Z of Gr; let $W_Z \subseteq \mathrm{Gr}$ be the closure of its image, an integral closed subscheme of absolute dimension 1, flat over $\mathbb{Z}$, with $\mathbf{k}(W_Z) \subseteq K$. All coordinates lie in $\mathcal{O}_K$ and $\mu_{\tau,J_0} = 1$, so x_Z

extends to $\nu\colon \operatorname{Spec}\mathcal{O}_K \to \prod_\tau\{z_{\tau,J_0}\neq 0\}\subseteq\mathbb{P}_{\mathrm{Gr}}$, a finite surjective map onto W_Z of degree $m=[K:\mathbf{k}(W_Z)]$, and ν^*s generates $\nu^*\mathcal{O}(1,\dots,1)$ for $s=\prod_\tau z_{\tau,J_0}$. By the projection formula and the formula for the arithmetic degree on $\operatorname{Spec}\mathcal{O}_K$ (finite part 0 as ν^*s is a generator),

$$m\,\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{G}})\cdot[W_Z]) = -\sum_{\sigma\colon K\to\mathbb{C}}\log\|s\|(\sigma x_Z) = \frac{1}{2}\sum_\sigma\sum_\tau\log\sum_J|\sigma(\mu_{\tau,J})|^2.$$

By the Cauchy–Binet formula $\sum_J|\sigma(\mu_{\tau,J})|^2=\det\bigl(\sigma(M_\tau)\sigma(M_\tau)^*\bigr)=\det\bigl(1_\kappa+YY^*\bigr)$ with $Y=\sigma(Z_\tau)$, whose entries have absolute value 1. Hadamard's inequality for the positive definite Hermitian matrix $1_\kappa+YY^*$ gives $\det(1_\kappa+YY^*)\le\prod_{i=1}^{\kappa}(1+\sum_j|Y_{ij}|^2)=(n-\kappa+1)^\kappa$; and $\det(1_\kappa+YY^*)=\det(1_{n-\kappa}+Y^*Y)\le(\kappa+1)^{n-\kappa}$ likewise. So $\det(1_\kappa+YY^*)\le(n+1)^{m_\kappa}$, and

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{G}})\cdot[W_Z])\le\frac{[K:\mathbb{Q}]}{m}\,|\Phi|\,e=|\Phi|\,e\,[\mathbf{k}(W_Z):\mathbb{Q}].$$

Step 3: density. Let $\mathcal{F}$ be the set of all W_Z (all K and Z), and suppose a closed subset $Y\subseteq\mathrm{Gr}$ contains every member. Let $\varphi\colon\mathbb{A}_{\mathbb{Q}}^{\Phi\times\kappa\times(n-\kappa)}\to\mathbb{P}_{\mathrm{Gr},\mathbb{Q}}$, $Z\mapsto(\det(1_\kappa\mid Z_\tau)_J)_{\tau,J}$, a morphism (its J_0-coordinates are 1) with values in $\mathrm{Gr}_{\mathbb{Q}}$ (as in Step 2). For every family Z of roots of unity in $\overline{\mathbb{Q}}$, $\varphi(Z)$ is the image of x_Z, which lies in $W_Z\subseteq Y$. So the closed set $\varphi^{-1}(Y_{\mathbb{Q}})$ contains all $\overline{\mathbb{Q}}$-points with root-of-unity coordinates; a polynomial over $\overline{\mathbb{Q}}$ vanishing on a product of infinite sets is 0, so $\varphi^{-1}(Y_{\mathbb{Q}})$ is the whole affine space and $Y(\overline{\mathbb{Q}})\supseteq\varphi(\overline{\mathbb{Q}}^{\Phi\times\kappa\times(n-\kappa)})$. By Step 1 with $\Omega=\overline{\mathbb{Q}}$, a point of $\mathrm{Gr}(\overline{\mathbb{Q}})$ with all $z_{\tau,J_0}\neq0$ is a tuple of κ-dimensional subspaces U_τ whose J_0-minor is non-zero; then U_τ is the row space of $(1_\kappa\mid Z_\tau)$ for a unique Z_τ (multiply a basis matrix by the inverse of its left block; Plücker coordinates scale by a determinant), so this point is $\varphi(Z)$. Thus $Y_{\overline{\mathbb{Q}}}$ contains the non-empty open subset $\{\prod_\tau z_{\tau,J_0}\neq0\}$ of the irreducible $\mathrm{Gr}_{\overline{\mathbb{Q}}}$, hence all of it; so Y contains the generic point of Gr and $Y=\mathrm{Gr}$.

Step 4: conclusion. By Steps 1–3, Lemma 30.24 applies with $d=G$, $e'=|\Phi|e$ and $\mathcal{F}$, and gives $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{G}})^{G+1})\le(G+1)|\Phi|e\,g_{\mathrm{Gr}}$. □

Lemma 30.24. *Let $\mathcal{X}\subseteq\mathbb{P}$ be a closed integral subscheme, flat over $\mathbb{Z}$, of absolute dimension $d+1$ with $d\ge1$. Let $e\in\mathbb{Z}^K$, let h be a cone metric of multidegree e, and $\overline{\mathcal{L}}=(\mathcal{O}(e),h)|_{\mathcal{X}}$. Assume:*

1. $\deg(\mathcal{O}(e)|_W)\ge0$ *for every prime p and every integral closed curve $W\subseteq\mathcal{X}_{\mathbb{F}_p}$;*
2. *the Levi form of $\log h$ is semipositive on every Zariski tangent vector of the affine cone over $\mathcal{X}(\mathbb{C})$ at every point of that cone;*
3. $g:=\deg(c_1(\mathcal{L}_{\mathbb{Q}})^d\cap[\mathcal{X}_{\mathbb{Q}}])>0$.

Let $e'\in\mathbb{R}$ and let $\mathcal{F}$ be a set of integral closed subschemes $W\subseteq\mathcal{X}$ of absolute dimension 1, flat over $\mathbb{Z}$, such that no closed subset $Y\neq\mathcal{X}$ of $\mathcal{X}$ contains every member of $\mathcal{F}$, and

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})\cdot[W])\le e'\,[\mathbf{k}(W):\mathbb{Q}]\qquad(W\in\mathcal{F}),$$

where $\mathbf{k}(W)$ is the function field of W. Then

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{d+1})\le(d+1)\,e'\,g.$$

Proof. Step 1: constant twists. For $\kappa' \in \mathbb{R}$ let $\overline{\mathcal{L}}(\kappa') = (\mathcal{O}(e), e^{\kappa'}h)|_{\mathcal{X}}$, i.e. $\overline{\mathcal{L}} \otimes \overline{\mathcal{O}}(\kappa')$ with $\overline{\mathcal{O}}(\kappa')$ the trivial bundle with $\|1\| = e^{-\kappa'}$, whose arithmetic first Chern class is the class of the constant Green function $2\kappa'$. Its product with a cycle and further classes has arithmetic degree κ' times the geometric degree on the generic fibre, and the product of two such classes vanishes; hence, for every $W \in \mathcal{F}$,

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}(\kappa')) \cdot [W]) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}) \cdot [W]) + \kappa'[\mathbf{k}(W) : \mathbb{Q}],$$
$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}(\kappa'))^{d+1}) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{d+1}) + (d+1)\kappa' g,$$

using that $W_{\mathbb{Q}}$ is one closed point with residue field $\mathbf{k}(W)$. Moreover $\overline{\mathcal{L}}(\kappa')$ satisfies (1)–(3): its line bundle is $\mathcal{O}(e)|_{\mathcal{X}}$, and $e^{\kappa'}h$ is a cone metric of multidegree e with $\log(e^{\kappa'}h) = \kappa' + \log h$, which has the Levi form of $\log h$.

Step 2: a non-zero small section. Suppose, to the contrary, that $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{d+1}) > (d+1)e'g$. Choose $\varepsilon > 0$ with $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{d+1}) > (d+1)(e'+\varepsilon)g$ and put $\overline{\mathcal{M}} = \overline{\mathcal{L}}(-e'-\varepsilon)$. By Step 1, $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}})^{d+1}) > 0$ and $\overline{\mathcal{M}}$ satisfies (1)–(3). Let $\nu\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ be the normalisation (finite, as $\mathcal{X}$ is of finite type over $\mathbb{Z}$). By Lemma 29.24 applied to $\overline{\mathcal{M}}$, the lim sup over τ of $\frac{(d+1)!}{\tau^{d+1}} \log \#\{\sigma \in H^0(\widetilde{\mathcal{X}}, \nu^*\mathcal{M}^{\otimes\tau}) : \|\sigma\|_{\sup} \le 1\}$ (sup norm for $\nu^*\overline{\mathcal{M}}^{\otimes\tau}$) is positive. So for some $\tau \ge 1$ this finite set has at least two elements, and it contains some $\sigma \neq 0$ with $\|\sigma\|_{\sup} \le 1$.

Step 3: members of $\mathcal{F}$ off the zero locus have large height. The zero locus $V(\sigma)$ is a proper closed subset of the integral scheme $\widetilde{\mathcal{X}}$, so $Y = \nu(V(\sigma))$ is closed (as ν is finite) of dimension $< d+1$, hence $Y \neq \mathcal{X}$. By hypothesis some $W \in \mathcal{F}$ is not contained in Y. As ν is finite and surjective, some irreducible component $\widetilde{W}$ of $\nu^{-1}(W)$, with its reduced structure, maps onto W; it is integral of absolute dimension 1 and flat over $\mathbb{Z}$, and $\widetilde{W} \not\subseteq V(\sigma)$ since $\nu(\widetilde{W}) = W \not\subseteq Y$. So $\sigma|_{\widetilde{W}}$ is a non-zero section of $\nu^*\mathcal{M}^{\otimes\tau}|_{\widetilde{W}}$ with $\|\sigma\|_{\sup} \le 1$, and the arithmetic degree of $\nu^*\overline{\mathcal{M}}^{\otimes\tau}$ on $\widetilde{W}$ is the sum of the finite part $\sum_x \operatorname{ord}_x(\sigma) \log \#\mathbf{k}(x) \ge 0$ over the closed points x (the divisor of a regular section is effective) and $-\sum_{w \in \widetilde{W}(\mathbb{C})} \log \|\sigma(w)\| \ge 0$:

$$\tau\, \widehat{\deg}(\widehat{c}_1(\nu^*\overline{\mathcal{M}}) \cdot [\widetilde{W}]) \ge 0.$$

Now $\widetilde{W}$ (closed in $\widetilde{\mathcal{X}}$, which is finite over $\mathcal{X}$) and W are arithmetic varieties of absolute dimension 1, whose generic fibres, spectra of number fields, are smooth over $\mathbb{Q}$; the metric of $\overline{\mathcal{M}}|_W$ is smooth, $W(\mathbb{C})$ being finite; $\beta = \nu|_{\widetilde{W}}\colon \widetilde{W} \to W$ is surjective and $(\nu^*\overline{\mathcal{M}})|_{\widetilde{W}} = \beta^*(\overline{\mathcal{M}}|_W)$. So Lemma 30.13 (with $b = 1$, $d = 0$ and this β) gives that the left side is $\tau m\, \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}) \cdot [W])$ with $m = [\mathbf{k}(\widetilde{W}) : \mathbf{k}(W)] \ge 1$. With Step 1 for $\kappa' = -e' - \varepsilon$,

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}) \cdot [W]) \ge (e'+\varepsilon)[\mathbf{k}(W) : \mathbb{Q}] > e'[\mathbf{k}(W) : \mathbb{Q}],$$

contradicting the hypothesis on $W \in \mathcal{F}$. Hence $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{d+1}) \le (d+1)e'g$. □

Lemma 30.25. *Let k be an infinite field, $d \ge 0$ an integer, $\beta\colon W \to V$ a morphism of integral projective k-schemes with $\dim W = \dim V + d$, and $\mathcal{N}_1, \dots, \mathcal{N}_d$ line bundles on W. Let $W_\eta = W \times_V \operatorname{Spec} \mathbf{k}(V)$ be the fibre of β over the generic point η of V, a projective scheme over $\mathbf{k}(V)$, and*

$$c = \deg(c_1(\mathcal{N}_1|_{W_\eta}) \cdots c_1(\mathcal{N}_d|_{W_\eta}) \cap [W_\eta]) \in \mathbb{Z}$$

($c = 0$ if W_η is empty). Then, in $A_{\dim V}(V)$,

$$\beta_*(c_1(\mathcal{N}_1) \cdots c_1(\mathcal{N}_d) \cap [W]) = c\,[V].$$

Proof. Reduction to globally generated line bundles. Both sides are additive in each $\mathcal{N}_j$: the left because $c_1(L\otimes L')\cap\alpha = c_1(L)\cap\alpha + c_1(L')\cap\alpha$ and β_* is additive [98] (Proposition 2.5), the right because c is, by the same rule on W_η. By Serre's theorem (Hartshorne, Algebraic Geometry, II.5.17) every line bundle on the projective k-scheme W is $\mathcal{A}\otimes\mathcal{A}'^{\otimes -1}$ with $\mathcal{A}$, $\mathcal{A}'$ generated by global sections, and the restrictions of $\mathcal{A}$, $\mathcal{A}'$ to W_η are generated by the restricted sections. So we may assume that every $\mathcal{N}_j$ is generated by global sections.

A cycle representing the left-hand side. Put $z_0 = [W]$. For $j = 1, \dots, d$, given the cycle $z_{j-1} = \sum_Y m_Y[Y]$ (a finite sum over integral closed subschemes Y of dimension $\dim W - j + 1$), choose $s_j \in H^0(W, \mathcal{N}_j)$ not vanishing at the generic point of any Y with $m_Y \neq 0$: for each such point the sections vanishing there form a proper linear subspace (as $\mathcal{N}_j$ is generated by global sections), and a vector space over the infinite field k is not a finite union of proper subspaces. Put $z_j = \sum_Y m_Y[\mathrm{div}(s_j|_Y)]$. By the definition of $c_1(\mathcal{N}_j)\cap$ [98] (§2.5), z_j represents $c_1(\mathcal{N}_j)\cap z_{j-1}$. So $z_d = \sum_i n_i[Z_i]$, with Z_i integral closed subschemes of dimension $\dim W - d = \dim V$, represents $c_1(\mathcal{N}_d)\cap\dots\cap c_1(\mathcal{N}_1)\cap[W]$, which is the class on the left of the assertion (the operations $c_1(\mathcal{N}_j)\cap$ commute, [98], Proposition 2.5).

Restriction to the generic fibre. An integral closed subscheme $Y \subseteq W$ meets W_η if and only if its generic point maps to η, i.e. if and only if $\beta(Y)$ is dense in V. In that case $Y_\eta = Y \times_V \mathrm{Spec}\,\mathbf{k}(V)$ is integral of dimension $\dim Y - \dim V$ (Hartshorne, Algebraic Geometry, II Ex. 3.22); $Y \mapsto Y_\eta$ is a bijection from these Y onto the integral closed subschemes of W_η; and the local rings of W at points over η are those of W_η, as $\mathcal{O}_{V,\eta} = \mathbf{k}(V)$. Hence the prime divisors of such a Y_η are the P_η for the prime divisors P of Y with $\beta(P)$ dense, with the same orders of $s_j|_Y$; and the Y with $\beta(Y)$ not dense, together with all their subvarieties, do not meet W_η. So the construction above, carried out on W_η with the restricted sections $s_j|_{W_\eta}$ (which do not vanish at the generic points of the relevant Y_η), gives the cycle $\sum_i' n_i[Z_{i,\eta}]$, the sum over the i with $\beta(Z_i)$ dense, and this cycle represents $c_1(\mathcal{N}_d|_{W_\eta})\cap\dots\cap c_1(\mathcal{N}_1|_{W_\eta})\cap[W_\eta]$. For such i, $Z_{i,\eta}$ is integral of dimension $\dim Z_i - \dim V = 0$, so it is the generic point of Z_i, with residue field $\mathbf{k}(Z_i)$, finite over $\mathbf{k}(V)$. Therefore

$$c = \sum_i{}' n_i\,[\mathbf{k}(Z_i) : \mathbf{k}(V)].$$

Push-forward. As W is projective, β is proper and $\beta(Z_i)$ is an integral closed subscheme of V. If $\beta(Z_i)$ is dense, then $\beta(Z_i) = V$, of the same dimension as Z_i, and $\beta_*[Z_i] = [\mathbf{k}(Z_i) : \mathbf{k}(V)]\,[V]$; otherwise $\dim\beta(Z_i) < \dim V = \dim Z_i$ and $\beta_*[Z_i] = 0$ [98] (§1.4). Summing, $\beta_* z_d = c\,[V]$, which is the assertion. □

Lemma 30.26. *Let Φ be a non-empty finite set, $n \geq 2$ and $1 \leq \kappa < n$.*

1. *For every algebraically closed field Ω of characteristic 0, $\mathrm{Fl}_{\Phi,n,\Omega}$ is reduced and its Ω-points are the Plücker coordinates of the Φ-tuples of complete flags in Ω^n; $\mathrm{Gr}_{\Phi,n,\kappa,\Omega}$ is reduced and its Ω-points are the Plücker coordinates of the Φ-tuples of κ-dimensional subspaces of Ω^n.*

2. *$\mathrm{Fl}_{\Phi,n}$ is the scheme of Φ-tuples of complete flags $0 = \mathcal{U}_{a,0} \subset \mathcal{U}_{a,1} \subset \dots \subset \mathcal{U}_{a,n} = \mathcal{O}^n$ of subbundles with locally free quotients ($\mathrm{rk}\,\mathcal{U}_{a,j} = j$), embedded in $\mathbb{P}_{\mathrm{Fl}}$ by the Plücker coordinates of its steps, and $\mathrm{Gr}_{\Phi,n,\kappa}$ is the product over Φ of the Grassmannian schemes of rank-κ subbundles of $\mathcal{O}^n$ with locally free quotient, in their Plücker embeddings. Both are smooth, projective and integral over $\mathbb{Z}$.*

Proof. Item 1. Let Ω be algebraically closed of characteristic 0. As Ω is flat over $\mathbb{Z}$, $\mathfrak{p}_{\mathrm{Fl}}\otimes\Omega$ is the kernel of $\Omega[z_{k,J}] \to (A\otimes\Omega)[T_k]$, $z_{k,J} \mapsto \mu_{k,J}T_k$, a prime ideal (the target is a domain); so $\mathrm{Fl}_{\Phi,n,\Omega}$

is reduced and irreducible, and its affine cone is the Zariski closure of the image of the map $(X,T)\mapsto(\mu_{k,J}(X)T_k)_{k,J}$ on $\mathrm{M}_n(\Omega)^\Phi\times\Omega^{K_{\mathrm{Fl}}}$. The invertible X and the T with all coordinates non-zero form a dense open subset, so the cone is also the closure of their image. For $X=(X_\tau)$ invertible the point is $\big([\wedge^j(\text{first } j \text{ rows of } X_\tau)]\big)_{(\tau,j)}$, the Plücker coordinates of the spans of the first j rows, which form a complete flag of Ω^n, and every Φ-tuple of complete flags arises. The set F of Plücker coordinates of Φ-tuples of complete flags is closed in $\mathbb{P}_{\mathrm{Fl}}(\Omega)$: it is the image of the projective variety $(\mathrm{GL}_n/B)^\Phi$ (B the upper triangular Borel subgroup; $gB\mapsto$ the flag of the spans of the first j columns of g) under a morphism. So $\mathrm{Fl}_{\Phi,n}(\Omega)=F$. The same argument with the ideal $\mathfrak{p}_{\mathrm{Gr}}$, the minors of the first κ rows and the projective variety $\mathrm{Gr}(\kappa,\Omega^n)^\Phi$ gives the assertion for $\mathrm{Gr}_{\Phi,n,\kappa,\Omega}$.

Item 2. Let $\mathrm{Fl}^{\mathrm{st}}\subseteq\mathbb{P}_{\mathrm{Fl}}$ be the scheme of Φ-tuples of complete flags of $\mathcal{O}^n$ with locally free quotients, embedded by the Plücker coordinates of its steps; it is smooth and projective over $\mathbb{Z}$, and the embedding is a closed immersion (classical: flag schemes are iterated Grassmannian bundles, and the Plücker map of a Grassmannian scheme is a closed immersion). Being flat over $\mathbb{Z}$, $\mathrm{Fl}^{\mathrm{st}}$ is the scheme-theoretic closure of its generic fibre. $\mathrm{Fl}^{\mathrm{st}}_{\mathbb{Q}}$ (smooth) and $\mathrm{Fl}_{\Phi,n,\mathbb{Q}}$ (item 1 with $\Omega=\overline{\mathbb{Q}}$) are geometrically reduced closed subschemes of $\mathbb{P}_{\mathrm{Fl},\mathbb{Q}}$ with the same $\overline{\mathbb{Q}}$-points, the Plücker coordinates of the tuples of complete flags (item 1); so they are equal. As $\mathfrak{p}_{\mathrm{Fl}}$ is prime and $\mathbb{Z}[z_{k,J}]/\mathfrak{p}_{\mathrm{Fl}}$ is torsion-free (the target of the map in (29.6) is a torsion-free domain), $\mathrm{Fl}_{\Phi,n}$ is integral and flat over $\mathbb{Z}$, hence also the scheme-theoretic closure of its generic fibre. So $\mathrm{Fl}_{\Phi,n}=\mathrm{Fl}^{\mathrm{st}}$, which is smooth and projective over $\mathbb{Z}$. The same argument with the product $\mathrm{Gr}^{\mathrm{st}}$ of the Grassmannian schemes of rank-κ subbundles in their Plücker embeddings, smooth and projective over $\mathbb{Z}$, and with $\mathfrak{p}_{\mathrm{Gr}}$ gives $\mathrm{Gr}_{\Phi,n,\kappa}=\mathrm{Gr}^{\mathrm{st}}$. □

Definition 30.27. Stage-change classes. Let fr be a Frobenius labelling, p a prime, $a\in\Phi$ and $1\le k\le n$. The **stage-change class** is

$$\mathrm{br}_p(a,k)=k\sum_{\sigma\in\Phi\times\{+,-\}}\Big(\sum_{m\ge0:\ \mathrm{fr}(p,a,m)=\sigma}p^{-k(m+1)}-\frac{[\sigma=(a,+)]}{p-1}\Big)\mathrm{ps}^{\sigma}_{k-1}\in\mathcal{R},$$

homogeneous of degree $k-1$ (the inner series converge, as $p^{-k}\le\frac12$).

Second roots. Let $\mathcal{X}\subseteq\mathbb{P}$ be a root model of type (Φ,n). A **system of second roots** on $\mathcal{X}$ consists of, for every root r: a multidegree $e'_r\in\mathbb{Z}^K$, cone metrics $h'_r(t)$ ($t>0$) of multidegree e'_r, and multihomogeneous polynomials $f_r,g_r\in\mathbb{Z}[z]$, not in the ideal of $\mathcal{X}$, whose multidegrees differ by $e_r-e'_r$. The root model $\mathcal{X}'$ **with second roots** has the same scheme, auxiliary line and normalising integers as $\mathcal{X}$, and multidegrees e'_r and root metrics $h'_r(t)$. For each root r, f_r/g_r is a rational section of $\mathcal{O}(e_r-e'_r)|_{\mathcal{X}}$; let Dv_r be the Weil divisor of its Cartier divisor and, for a prime p, $\mathrm{Dv}_{r,p}$ the part of Dv_r supported in $\mathcal{X}_{\mathbb{F}_p}$.

Vertical change numbers. For $F\in\mathcal{R}$ homogeneous of degree $D+1$, an **expansion** of F is a family $(H_r)_{r\in\mathrm{Rt}}$ of polynomials in the variables x_s,x'_s ($s\in\mathrm{Rt}$), homogeneous of degree D, with $F(x)-F(x')=\sum_r(x_r-x'_r)H_r(x,x')$. Its **vertical change number** at p is

$$\mathrm{vc}_p(H)=\sum_{r\in\mathrm{Rt}}\deg\Big(H_r\big(c_1(\mathcal{O}(e_s)),c_1(\mathcal{O}(e'_s))\big)\cap[\mathrm{Dv}_{r,p}]\Big),$$

the degrees of the products of first Chern classes of the restrictions to the components of $\mathrm{Dv}_{r,p}$, projective of dimension D over $\mathbb{F}_p$.

Linked stages. Let $\mathcal{T}$ be a root tower of type (Φ,n,κ) with Frobenius labelling fr, formed into stage-change classes. For integers $2\le B_0\le B$ let $S(B_0,B)$ be the set of primes p with $B_0<p\le B$ and $p\notin\mathrm{Bd}$. A **link** from (B_0,Z) to (B,Z), $Z\in\mathrm{Cp}$, is a root model $\mathcal{X}$ of type (Φ,n) with a system of second roots such that, with $S=S(B_0,B)$:

(Sk1) Stages The models $\mathcal{X}$ and $\mathcal{X}'$ have convergent regularisation, $\mathrm{pp}(\mathcal{X};F) = \widehat{\deg}_{B_0,Z}(F)$ and $\mathrm{pp}(\mathcal{X}';F) = \widehat{\deg}_{B,Z}(F)$ for every $F \in \mathcal{R}$, and $\mathrm{gi}(\mathcal{X};F) = \mathrm{vol}(Z)\deg^{\mathrm{fl}}(F)$ for every $F \in \mathcal{R}$ homogeneous of degree D;

(Sk2) Vertical over S For every root r, f_r/g_r is invertible at every point of $\mathcal{X}$ not lying over a prime of S;

(Sk3) Isometric at infinity For every root r, every $t > 0$ and every point z of the affine cone over $\mathcal{X}(\mathbb{C})$ we have $|f_r(z)|\, h_r'(t)(z) = |g_r(z)|\, h_r(t)(z)$;

(Sk4) Local stage change For every $a \in \Phi$, every $1 \le k \le n$ and every $Q \in \mathcal{R}$ homogeneous of degree $D+1-k$, the class $\mathrm{ps}_{a,k}Q$ has an expansion H with $\mathrm{vc}_p(H) = \delta_{\mathcal{X}}\,\mathrm{gi}(\mathcal{X};\mathrm{br}_p(a,k)Q)$ for every $p \in S$.

The tower has **linked stages** if there is a link from (B_0, Z) to (B, Z) for all integers $2 \le B_0 \le B$ and every $Z \in \mathrm{Cp}$.

Theorem 30.28. *Every root tower with linked stages has the change of stage (Rt3).*

Proof. Let $2 \le B_0 \le B$, $Z \in \mathrm{Cp}$, $a \in \Phi$, $1 \le k \le n$ and $Q \in \mathcal{R}$ homogeneous of degree $D+1-k$. Let $\mathcal{X}$ with second roots be a link from (B_0, Z) to (B, Z), $S = S(B_0, B)$, $F = \mathrm{ps}_{a,k}Q$ (homogeneous of degree $D+1$), and H an expansion of F as in (Sk4). Write $\overline{\mathcal{L}}_r(t) = (\mathcal{O}(e_r), h_r(t))|_{\mathcal{X}}$ and $\overline{\mathcal{L}}_r'(t) = (\mathcal{O}(e_r'), h_r'(t))|_{\mathcal{X}}$.

Step 1: the difference of two root lines is a vertical class over S. Fix $t > 0$ and a root r. The bundle $\overline{\mathcal{N}}_r = \overline{\mathcal{L}}_r(t) \otimes \overline{\mathcal{L}}_r'(t)^{-1}$ is $\mathcal{O}(e_r - e_r')|_{\mathcal{X}}$ with the cone metric $h_r(t)/h_r'(t)$, and $s_r = f_r/g_r$ is a non-zero rational section with $\|s_r\|(z) = |f_r(z)|h_r'(t)(z)/(|g_r(z)|h_r(t)(z)) = 1$ wherever $g_r(z) \ne 0$, by (Sk3). By (Sk2), s_r is invertible on the generic fibre, so $\|s_r\| = 1$ on a dense open subset of $\mathcal{X}(\mathbb{C})$, and $\operatorname{div} s_r = \mathrm{Dv}_r = \sum_{p\in S}\mathrm{Dv}_{r,p}$. For Hermitian line bundles $\overline{\mathcal{M}}_1, \dots, \overline{\mathcal{M}}_D$ on $\mathcal{X}$ with smooth metrics the integral term vanishes, and

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{N}}_r)\widehat{c}_1(\overline{\mathcal{M}}_1)\cdots\widehat{c}_1(\overline{\mathcal{M}}_D)) = \sum_{p\in S}\log p \cdot \deg\big(c_1(\mathcal{M}_1)\cdots c_1(\mathcal{M}_D)\cap[\mathrm{Dv}_{r,p}]\big).$$

Step 2: the change at a fixed regularisation parameter. The map sending a monomial of degree $D+1$ in the symbols x_s, x_s' to the arithmetic intersection number of the corresponding product of the classes $\widehat{c}_1(\overline{\mathcal{L}}_s(t))$, $\widehat{c}_1(\overline{\mathcal{L}}_s'(t))$ extends to a linear form Λ_t on homogeneous polynomials of degree $D+1$ (symmetry), and by (29.2) (no auxiliary class occurs in F) $\delta_{\mathcal{X}}\mathrm{ai}_t(\mathcal{X};F) = \Lambda_t(F(x))$ and $\delta_{\mathcal{X}}\mathrm{ai}_t(\mathcal{X}';F) = \Lambda_t(F(x'))$, as $\mathcal{X}'$ has the same normalising integer. Apply Λ_t to $F(x) - F(x') = \sum_r (x_r - x_r')H_r$: by multilinearity, $\widehat{c}_1(\overline{\mathcal{L}}_r(t)) - \widehat{c}_1(\overline{\mathcal{L}}_r'(t)) = \widehat{c}_1(\overline{\mathcal{N}}_r)$, and Step 1 applied to each monomial of H_r (whose geometric classes are $c_1(\mathcal{O}(e_s))$, $c_1(\mathcal{O}(e_s'))$),

$$\delta_{\mathcal{X}}\big(\mathrm{ai}_t(\mathcal{X};F) - \mathrm{ai}_t(\mathcal{X}';F)\big) = \sum_{p\in S}\log p\cdot\mathrm{vc}_p(H) = \delta_{\mathcal{X}}\sum_{p\in S}\log p\cdot\mathrm{gi}(\mathcal{X};\mathrm{br}_p(a,k)Q),$$

the last equality by (Sk4). This holds for every $t > 0$; the right side does not depend on t.

Step 3: the limit. By (Sk1), $\mathrm{ai}_t(\mathcal{X};F) \to \widehat{\deg}_{B_0,Z}(F)$ and $\mathrm{ai}_t(\mathcal{X}';F) \to \widehat{\deg}_{B,Z}(F)$ as $t \to 0^+$ (both are finite linear combinations of monomials), and

$$\mathrm{gi}(\mathcal{X};\mathrm{br}_p(a,k)Q) = \mathrm{vol}(Z)\deg^{\mathrm{fl}}(\mathrm{br}_p(a,k)Q),$$

the class being homogeneous of degree $(k-1)+(D+1-k) = D$. With $\log p\cdot\deg^{\mathrm{fl}}(\mathrm{br}_p(a,k)Q) = \Delta_p(a,k,Q)$,

$$\widehat{\deg}_{B_0,Z}(\mathrm{ps}_{a,k}Q) = \widehat{\deg}_{B,Z}(\mathrm{ps}_{a,k}Q) + \mathrm{vol}(Z)\sum_{p\in S(B_0,B)}\Delta_p(a,k,Q),$$

and $S(B_0, B)$ is the set of primes $p \notin \mathrm{Bd}$ with $B_0 < p \le B$. This is (Rt3). □

Definition 30.29. Let $\mathsf{H} = (d, r, r_{\mathbb{C}})$ be a CM host on a CM field E that is Galois over $\mathbb{Q}$, and $\Phi = \Phi_{\mathsf{H}}$. For $g \in \mathrm{Gal}(E/\mathbb{Q})$ with $g(r) = r$ let $\sigma_g \colon \Phi \to \Phi$, $\tau \mapsto \tau \circ g$. The **relabelling group** of H is $\Gamma_{\mathsf{H}} = \{\sigma_g : g \in \mathrm{Gal}(E/\mathbb{Q}),\ g(r) = r\}$.

A root tower of type (Φ, n, κ) has **Galois component symmetry** for H if:

(Ys0) For every $\sigma \in \Gamma_{\mathsf{H}}$ and every integer $B \ge 2$ there is a bijection $\pi \colon \mathrm{Cp} \to \mathrm{Cp}$ with $\widehat{\deg}_{B,Z}(\sigma \cdot F) = \widehat{\deg}_{B,\pi(Z)}(F)$ for every $Z \in \mathrm{Cp}$ and every $F \in \mathcal{R}$.

Lemma 30.30. *Let H be a CM host on a CM field E that is Galois over $\mathbb{Q}$, and let $\mathcal{T}$ be a root tower of type $(\Phi_{\mathsf{H}}, n, \kappa)$ with Galois component symmetry (Ys0) for H. Then Γ_{H} acts transitively on Φ_{H}, and $\mathcal{T}$ has (Ys1) with $\Gamma = \Gamma_{\mathsf{H}}$.*

Proof. Transitivity. Let $\tau, \tau' \in \Phi_{\mathsf{H}}$. Since E is Galois over $\mathbb{Q}$, $\tau(E) = \tau'(E)$, so $g = \tau^{-1} \circ \tau'$ is an automorphism of E, i.e. $g \in \mathrm{Gal}(E/\mathbb{Q})$, with $\tau \circ g = \tau'$. Then $\tau(g(r)) = \tau'(r) = r_{\mathbb{C}} = \tau(r)$, and τ is injective, so $g(r) = r$. Hence $\sigma_g \in \Gamma_{\mathsf{H}}$ and $\sigma_g(\tau) = \tau'$.

Invariance. Let $\sigma \in \Gamma_{\mathsf{H}}$, $B \ge 2$ and $F \in \mathcal{R}$, and let π be the bijection of (Ys0) for (σ, B). Summing (Ys0) over $Z \in \mathrm{Cp}$ and reindexing the sum by the bijection π,

$$\widehat{\deg}_B(\sigma \cdot F) = \sum_{Z \in \mathrm{Cp}} \widehat{\deg}_{B,\pi(Z)}(F) = \sum_{Z' \in \mathrm{Cp}} \widehat{\deg}_{B,Z'}(F) = \widehat{\deg}_B(F).$$

With transitivity this is (Ys1) for the group $\Gamma = \Gamma_{\mathsf{H}}$. □

Theorem 30.31. *Let $r \ge 1$ and L be real numbers. Every root tower with the change of stage (Rt3) and flat determinant classes at the base stage (Db) for (r, L) has the determinant classes (Rt5) for (r, L) with the constant $C' = 4$.*

Proof. Let $\mathcal{T}$ be the tower, of type (Φ, n, κ), with Frobenius labelling fr; write $S(2, B)$ for the set of primes $p \notin \mathrm{Bd}$ with $2 < p \le B$.

Step 1: the stage-change term for $k = 1$. For a label $\sigma = (b, \pm)$, $\mathrm{ps}_0^{(b,+)} = \kappa$ and $\mathrm{ps}_0^{(b,-)} = n - \kappa$ are constants; write n_σ for this constant. For a prime p, $a \in \Phi$ and Q homogeneous of degree D, (21.2) with $k = 1$ gives

$$\Delta_p(a, 1, Q) = e_p(a) \log p \cdot \deg^{\mathrm{fl}}(Q),$$
$$e_p(a) = \sum_{m \ge 0} p^{-(m+1)} n_{\mathrm{fr}(p,a,m)} - \frac{\kappa}{p-1} = \sum_{m \ge 0} p^{-(m+1)} \bigl(n_{\mathrm{fr}(p,a,m)} - \kappa\bigr),$$

using $\sum_{m \ge 0} p^{-(m+1)} = \frac{1}{p-1}$. Each $n_\sigma - \kappa$ is 0 or $n - 2\kappa$, so $|e_p(a)| \le \frac{|n-2\kappa|}{p-1} \le \frac{n}{p-1}$.

Step 2: the determinant classes are flat at every stage. Let $B \ge 2$, $Z \in \mathrm{Cp}$, $a \in \Phi$ and Q homogeneous of degree D. By (Rt3) with $B_0 = 2$ and $k = 1$, by (Db) and by Step 1,

$$\widehat{\deg}_{B,Z}(\mathrm{ps}_{a,1} Q) = \widehat{\deg}_{2,Z}(\mathrm{ps}_{a,1} Q) - \mathrm{vol}(Z) \sum_{p \in S(2,B)} \Delta_p(a, 1, Q) = \mathrm{hd}_{B,Z,a} \,\mathrm{vol}(Z) \deg^{\mathrm{fl}}(Q),$$

with $\mathrm{hd}_{B,Z,a} = \mathrm{hf}(Z, a) - \sum_{p \in S(2,B)} e_p(a) \log p$ (for $B = 2$ the sum is empty). This is the identity of (Rt5).

Step 3: a prime sum. For an integer $N \ge 2$, Legendre's formula gives

$$\log N! = \sum_{p \le N} \log p \sum_{j \ge 1} \lfloor N/p^j \rfloor \ge \sum_{p \le N} (N/p - 1) \log p,$$

so with $\log N! \le N \log N$ and Chebyshev's bound

$$\sum_{p \le N} \frac{\log p}{p} \le \frac{\log N! + \theta(N)}{N} \le \log N + \log 4.$$

Now let $B \ge 128$ with $|\mathrm{Bd}| \le B$, and $P = \mathrm{Bd} \cup S(2,B)$, a set of primes. Since $p/(p-1) \le 2$, the primes $p \le B$ in P contribute at most $2(\log B + \log 4)$ to $\sigma = \sum_{p \in P} \frac{\log p}{p-1}$. The primes $p > B$ in P lie in Bd, so there are at most B of them, and since $x \mapsto \log x/(x-1)$ is decreasing on $(1,\infty)$ (its derivative has the sign of $1 - \frac{1}{x} - \log x < 0$), each contributes at most $\log(B+1)/B$; together at most $\log(B+1) \le \log B + \frac{1}{B}$. Hence $\sigma \le 3 \log B + 2 \log 4 + \frac{1}{B} \le 4 \log B$, as $\log B \ge \log 128 > 4.8 > 2 \log 4 + \frac{1}{128}$.

Step 4: the bound. By Step 2 and (Db), $\mathrm{hd}_{B,Z,a} = h_0 + \sum_{p \in \mathrm{Bd}} d_p - \sum_{p \in S(2,B)} e_p(a) \log p$, and by the bounds of (Db), Step 1 and Step 3,

$$|\mathrm{hd}_{B,Z,a}| \le n\Big(\frac{3}{2}\log r + \frac{|L|}{2}\Big) + n \sum_{p \in P} \frac{\log p}{p-1} + n \sum_{p \in \mathrm{Bd}} v_p \log p \le n\Big(4 \log B + \frac{5}{2}\log r + \frac{|L|}{2}\Big) \le 4\, n\, \mathrm{Lg}(B),$$

since $\log r \ge 0$ and $\mathrm{Lg}(B) = \log B + \log r + |L|$. This is (Rt5) with $C' = 4$. □

31 Existence of good root towers: the assembly

This section proves Proposition 31.14, which Theorem 29.14 uses through Corollary 30.5 and from which Proposition 21.2 and (TCS) take the constant C and the number L_{H}. The statement asks for an integral model of the Shimura variety, a root tower with a geometric realisation and a good origin over it, and a bijection of components (Gx), with five groups of properties: the compact dual (Gt1), the volume side (Gt2), the change of stage (Gt3), the centre, that is the fibre-square identities, the wall inputs and the Galois symmetry of the components (Gt4), and the quantitative properties (Gt5). The proof is an assembly, and the work lies in the results it collects. The integral model is Proposition 32.1 (Section 32); the good origin is Proposition 33.1 (Section 33); most facts about the tower are Proposition 34.2 (Section 34); linked stages are Theorem 35.2 (Section 35); nefness at primes, the fibre-square identities (Fq1) and the reference comparisons behind the wall inputs are proved in Appendix D. The section itself proves the remaining steps: equal positive volumes through covolumes, the convergence and rate of the regularised intersection numbers and uniform semipositivity (Subsection 31.2); and the comparison of root models whose origins differ by a Galois twist, which gives the Galois symmetry (Ys0), together with the base slope settings, which give the slope at the test class (Rt6) (Subsection 31.3).

31.1 The existence theorem for good root towers

The definitions of this subsection set up the objects that the statement of Proposition 31.14 and its proof use: Galois Frobenius labels (Fr); global integral models (Gl1)–(Gl9) and integral models with uniformised degenerations, which at every place dominate a local model whose degenerations are uniformised; the covolume realisation (Vu) of the volumes; the reference link (Vl) and the reference comparison (Rh) with Lie fibre data (Lx1)–(Lx3) of reference models; square systems, the setting of the fibre-square identities; and Hodge-type base settings (Hy1)–(Hy5) with cusp data (Hu1)–(Hu3) and base slope settings (Bs1)–(Bs5), the setting of (Rt6). The statement follows them. Its proof applies, in turn, Proposition 32.1, Proposition 33.1 and Proposition 34.2, and derives (Gt1)–(Gt5) from the facts of good towers and from the results of the next two subsections and of Section 35 and Appendix D.

Definition 31.1. Let H be a CM host on a CM field E that is Galois over $\mathbb{Q}$ with abelian Galois group G, let $\Phi = \Phi_{\mathsf{H}}$ and let $\iota \in G$ be complex conjugation. Identify the labels with the embeddings $E \to \mathbb{C}$ by

$$(\tau, +) \mapsto \tau, \qquad (\tau, -) \mapsto \tau \circ \iota \qquad (\tau \in \Phi),$$

a bijection $\Phi \times \{+, -\} \to \mathrm{Hom}(E, \mathbb{C})$ since $\mathrm{Hom}(E, \mathbb{C}) = \Phi \sqcup \Phi \circ \iota$. For a prime p unramified in E let $\mathrm{Frob}_p \in G$ be the arithmetic Frobenius at p ($\mathrm{Frob}_p(a) \equiv a^p$ modulo every prime of $\mathcal{O}_E$ above p, for all $a \in \mathcal{O}_E$; independent of the prime above p as G is abelian). The **Galois Frobenius labelling** of H is the map fr_{H} with

$$\mathrm{fr}_{\mathsf{H}}(p, \tau, m) = \text{the label of } \tau \circ \mathrm{Frob}_p^{m+1} \qquad (p \text{ unramified in } E,\ \tau \in \Phi,\ m \in \mathbb{N}), \tag{31.1}$$

and $\mathrm{fr}_{\mathsf{H}}(p, \tau, m) = (\tau, +)$ for the finitely many p ramified in E.

A root tower of type (Φ, n, κ) has **Galois Frobenius labels** if

(Fr) Its Frobenius labelling agrees with fr_{H} at every prime $p \notin \mathrm{Bd}$.

Definition 31.2. Let E, H be as in (Ud1) and $\mathsf{U} = (M, \langle\,,\rangle)$ a unitary datum for H of signature (p, q) with $p + q \geq 1$; put $g = \frac{1}{2}[E : \mathbb{Q}](p + q)$.

Semiabelian structures. A **semiabelian structure** over a scheme S is a triple $(G, \iota, \mathcal{L})$ of a semiabelian scheme $G \to S$, a ring homomorphism $\iota\colon \mathcal{O}_E \to \mathrm{End}_S(G)$ and a rigidified symmetric cubical line bundle $\mathcal{L}$ on G with the two isomorphisms of (Ud4); its **abelian locus** is the set of points of S over which G is an abelian scheme. For $f\colon S' \to S$, a **morphism over** f from a semiabelian structure over S' to one over S is an isomorphism of G' with $G \times_S S'$ compatible with the $\mathcal{O}_E$-actions and carrying $\mathcal{L}$, with rigidification and cubical structure, to $\mathcal{L}'$.

Systems, places, local models. A **system of moduli schemes** for (H, U) is a family $(\mathrm{Sh}_{\mathsf{L}'})$, indexed by all levels L' for H, of moduli schemes of type $(\mathsf{U}, \mathsf{L}')$ with their universal objects; it **extends** a unitary Shimura datum $(E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ if $\mathrm{Sh}_{\mathsf{L}} = \mathrm{Sh}$ with the same universal object. For a prime ℓ, an **ℓ-adic place** is a valuation ring $\mathcal{O}_v$ of the field $\mathbb{C}$ whose maximal ideal contains ℓ; for a number field $F' \subset \mathbb{C}$ put $\mathcal{O}_{F',v} = F' \cap \mathcal{O}_v$ (the localisation of $\mathcal{O}_{F'}$ at a prime above ℓ). For a level L' and a coefficient field F' for it, an **integral model of** $\mathrm{Sh}_{\mathsf{L}'}$ **over** $\mathcal{O}_{F',v}$ is defined as in (Ud7) with $\mathcal{O}_F, F, \mathrm{Sh}$ replaced by $\mathcal{O}_{F',v}, F', \mathrm{Sh}_{\mathsf{L}'}$. For such a model, or one over $\mathcal{O}_{F'}$, write $\mathcal{Y}^{\mathrm{ab}}$ for the abelian locus of $(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ and $\eta^{\mathrm{un}}(a), \eta^{\mathrm{un}}_A(a)$ $(a \in M)$ for the level sections of the universal object over $\mathrm{Sh}_{\mathsf{L}',F'}$; such a section σ **extends over** an open $U \supseteq j(\mathrm{Sh}_{\mathsf{L}',F'})$ if a section of $G_{\mathcal{Y}}$ over U restricts along j to σ. The generic fibre **descends** if there are a $K_{\mathsf{L}'}$-scheme $\mathcal{Y}_K$, an open immersion $j_K\colon \mathrm{Sh}_{\mathsf{L}'} \to \mathcal{Y}_K$ over $K_{\mathsf{L}'}$, a commutative group scheme $G_K \to \mathcal{Y}_K$ with $j_K^* G_K$ the universal abelian scheme, and an isomorphism $\mathcal{Y} \otimes F' \cong \mathcal{Y}_K \otimes_{K_{\mathsf{L}'}} F'$ carrying j to $j_K \otimes F'$ and $G_{\mathcal{Y}} \otimes F'$ to $G_K \otimes F'$, compatibly with the identifications over $\mathrm{Sh}_{\mathsf{L}',F'}$.

Degeneration data [187], [87] (Ch. II–III), [154] (Ch. 4–5). Let R be a noetherian normal domain, complete for an ideal I, with fraction field K. **Degeneration data with $\mathcal{O}_E$-structure** over (R, I) are $\mathfrak{D} = (B, \mathcal{L}_B, \iota_B, X, c, \psi)$: an abelian scheme B/R with a rigidified symmetric cubical relatively ample line bundle $\mathcal{L}_B$ of principal polarisation and an $\mathcal{O}_E$-action ι_B with the two isomorphisms of (Ud4); a free abelian group X of finite rank with an $\mathcal{O}_E$-module structure; an $\mathcal{O}_E$-linear $c\colon X \to B(R)$; nowhere-zero sections $\psi(\chi)$ of $c(\chi)^* \mathcal{L}_B^{-1}$ over K; such that, with $\tau(\chi, \chi') = \psi(\chi + \chi')\psi(\chi)^{-1}\psi(\chi')^{-1}$ (via the rigidification a trivialisation of the Poincaré biextension at $(c(\chi), c(\chi'))$): $\psi(-\chi) = \psi(\chi)$ under the symmetry of $\mathcal{L}_B$; τ is bimultiplicative for the cubical structure; for $a \in \mathcal{O}_E$ and $\chi, \chi' \in X$, $c(a\chi) = \iota_B(a) \circ c(\chi)$, and $\tau(a\chi, \chi')$ corresponds to $\tau(\chi, \bar{a}\chi')$ under the trivialisation of $(\iota_B(a), 1)^* \Lambda(\mathcal{L}_B) \otimes (1, \iota_B(\bar{a}))^* \Lambda(\mathcal{L}_B)^{-1}$ that is 1 at the origin; there is an isomorphism $u\colon \mathcal{L}_B \to (1, \iota_B(w_0))^* \Lambda(\mathcal{L}_B)$, equal to the identity at

the origin via the rigidification, under which $\psi(\chi)$ corresponds to $\tau(\chi, w_0\chi)$ at $c(\chi)$ for every χ (no other compatibility of ψ with $\mathcal{O}_E$ is required); for $\chi \neq 0$, $\psi(\chi)$ is a section over R of $I \cdot c(\chi)^*\mathcal{L}_B^{-1}$; for every $m \geq 1$ only finitely many χ have $\psi(\chi) \notin I^m c(\chi)^*\mathcal{L}_B^{-1}$. Let $\widetilde{G}$ be the extension of B by the split torus with character group X defined by c, with $a \in \mathcal{O}_E$ acting through $\iota_B(a)$ on B and on the torus through the endomorphism dual to $\chi \mapsto \bar{a}\chi$, and $\iota_{\mathfrak{D}} \colon X \to \widetilde{G}(K)$, $\chi \mapsto (c(\chi), \tau(\chi, \cdot))$. A semiabelian structure $(G, \iota, \mathcal{L})$ over R with abelian generic fibre A is **uniformised by** $\mathfrak{D}$ if it is the degeneration attached to $\mathfrak{D}$ by Mumford's construction: A carries the principal polarisation of $\mathcal{L}_A = \mathcal{L}|_A$; for every $e \geq 0$, $G \otimes R/I^{e+1}$ is identified, as a group scheme with $\mathcal{O}_E$-action, with $\widetilde{G} \otimes R/I^{e+1}$, compatibly in e, and $\mathcal{L} \otimes R/I^{e+1}$ with the pull-back of $\mathcal{L}_B$, compatibly with the rigidifications, symmetries and cubical structures; and for every $k \geq 1$ the Fourier expansion is a bijection from $\Gamma(A, \mathcal{L}_A^{\otimes k})$ onto the set of families $(s_\alpha)_{\alpha \in X}$ of sections s_α over B_K of $\mathcal{L}_B^{\otimes k} \otimes (1, c(\alpha))^*\Lambda(\mathcal{L}_B)$ such that, for all $\alpha, \mu \in X$, $s_{\alpha+k\mu}$ corresponds to $T_{c(\mu)}^* s_\alpha$ under the isomorphism of the two line bundles whose value at the origin is $\psi(\mu)^k\tau(\mu, \alpha)$. Here (s_α) is the **Fourier expansion** of s if for some $0 \neq r \in R$ the sections rs and rs_α extend over G and over B and, for every e, rs modulo I^{e+1} is the finite sum $\sum_\alpha rs_\alpha\vartheta_\alpha$, with ϑ_α the coordinate of weight α of $\widetilde{G}$ (the trivialisation of the pull-back of the Poincaré bundle at $c(\alpha)$ on which the torus acts by α). For an algebraically closed field $F_1 \supseteq K$ and $k \geq 1$ let $T_k(\mathfrak{D}, F_1)$ (Mumford's description of the k-torsion) be the set $\{(y, \tilde{g}) \in X \times \widetilde{G}(F_1) : k\tilde{g} = \iota_{\mathfrak{D}}(y)\}$ modulo the translations $(y, \tilde{g}) \mapsto (y + ky', \tilde{g} + \iota_{\mathfrak{D}}(y'))$. A pair $p = (y, \tilde{g})$ with $k\tilde{g} = \iota_{\mathfrak{D}}(y)$, with $b \in B(F_1)$ the image of $\tilde{g}$, acts on families (s_α) as above (over F_1) by $(S_p s)_{\alpha+y} = \tilde{g}(\alpha)\, w\, T_b^* s_\alpha$, where $\tilde{g}(\alpha)$ is the coordinate of weight α of $\tilde{g}$ and w a trivialisation of $\mathcal{L}_B^{\otimes k}$ at the origin against $\mathcal{L}_B^{\otimes k}$ at b, so that S_p is defined up to a scalar; the class of p corresponds to $x \in A[k](F_1)$ when every element of the theta group of $\mathcal{L}_A^{\otimes k}$ over x acts on $\Gamma(A_{F_1}, \mathcal{L}_A^{\otimes k})$, through the Fourier expansion, as some S_p; and the pairing on $T_k(\mathfrak{D}, F_1)$ is $e_k(p, p') = \zeta$ when $S_{p'}S_p = \zeta S_p S_{p'}$. The datum $\mathfrak{D}$ has **torsion isomorphisms** if for every algebraically closed field $F_1 \supseteq K$, every $k \geq 1$ and every $x \in A[k](F_1)$ there is a pair $p = (y, \tilde{g}) \in X \times \widetilde{G}(F_1)$ with $k\tilde{g} = \iota_{\mathfrak{D}}(y)$ whose class corresponds to x, that is, every element of the theta group of $\mathcal{L}_A^{\otimes k}$ over x acts on $\Gamma(A_{F_1}, \mathcal{L}_A^{\otimes k})$, through the Fourier expansion, as S_p for some choice of w. The set T_k, the operators S_p and the pairing use only $(B, \mathcal{L}_B, \iota_B, X, c, \psi)$ and are defined in the same way for the data of an arc datum below.

Local conditions. Let $\mathcal{Y}$ be an integral model of $\mathrm{Sh}_{\mathsf{L}'}$, $\mathsf{L}' = (N_1, \mathfrak{n}_1, A_1)$, over $O = \mathcal{O}_{F',v}$, v an ℓ-adic place.

(Gl1) Level-structure conditions. (i) For every F'-automorphism r of $\mathrm{Sh}_{\mathsf{L}',F'}$, every morphism ν over r of the universal semiabelian structure to itself and all $\mathcal{O}_E$-linear $g_M, g_A \colon M \to M$ with $\nu \circ \eta^{\mathrm{un}}(g_M a) = \eta^{\mathrm{un}}(a) \circ r$ and $\nu \circ \eta_A^{\mathrm{un}}(g_A a) = \eta_A^{\mathrm{un}}(a) \circ r$ for all a, there are an O-automorphism ρ of $\mathcal{Y}$ with $\rho \circ j = j \circ r$ and a morphism over ρ of the semiabelian structure of $\mathcal{Y}$ to itself restricting to ν. (ii) If $A_1 = 1$, the generic fibre descends. (iii) With $\pi \colon G_{\mathcal{Y}} \to \mathcal{Y}$, $\pi_*\mathcal{L}_{\mathcal{Y}}$ is invertible and is generated near every point by a section that is non-zero at some point of every fibre of π. (iv) For $a \in M$ and $n \geq 1$ prime to ℓ: if $n\eta^{\mathrm{un}}(a) = 0$ then $\eta^{\mathrm{un}}(a)$ extends over $\mathcal{Y}^{\mathrm{ab}}$; likewise for η_A^{un}. (v) Let V be a complete discrete valuation ring, or the valuation ring of a complete algebraically closed non-archimedean field of characteristic 0 in which $|\ell| < 1$, with fraction field K_V, and $b \colon O \to V$ a homomorphism. If $y \colon \operatorname{Spec} K_V \to \mathrm{Sh}_{\mathsf{L}',F'}$ lies over b and $y^*(G^{\mathrm{un}}, \mathcal{L}^{\mathrm{un}})$ extends to an abelian scheme over V with a relatively ample line bundle, then $j \circ y$ extends to $\operatorname{Spec} V \to \mathcal{Y}$ over b with image in $\mathcal{Y}^{\mathrm{ab}}$.

(Gl2) Split fibres. For every point y of $\mathcal{Y}$ there is a finite separable extension k of $\kappa(y)$ over which the fibre of $G_{\mathcal{Y}}$ is an extension of an abelian variety by a split torus.

(Gl3) Uniformisation at complete local rings. For every complete noetherian normal lo-

cal domain R of characteristic 0 and every $\xi\colon \operatorname{Spec} R \to \mathcal{Y}$ with $\xi^{-1}(\mathcal{Y}^{\mathrm{ab}}) \neq \varnothing$ and split torus part of $\xi^* G_{\mathcal{Y}}$ over the residue field, $\xi^*(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ is uniformised by degeneration data with $\mathcal{O}_E$-structure over $(R, \mathfrak{m}_R)$ that have torsion isomorphisms and whose $\psi(\chi)$ are units over $\xi^{-1}(\mathcal{Y}^{\mathrm{ab}})$.

(Gl4) Arcs. Let O' be a domain of characteristic 0 with $b\colon O \to O'$ that is a field ($\varpi = 1$) or a complete discrete valuation ring with uniformiser ϖ. An **arc datum** over O' is: data $(B_0, \mathcal{L}_{B_0}, \iota_{B_0}, X_0, c_0, \psi_0)$ as above over O' with $X_0 \neq 0$, all conditions except the two on I, and every $\psi_0(\chi)$ a unit over O'; a positive definite symmetric $\beta\colon X_0 \times X_0 \to \mathbb{Z}$ with $\beta(a\chi, \chi') = \beta(\chi, \bar{a}\chi')$ and values in $2\operatorname{lcm}(N_1, A_1)\mathbb{Z}$; level data $\eta_0, \eta_{0,A}$ on M with values in the N_1- and A_1-torsion of Mumford's description over $\operatorname{Frac}(O')$; such that at every algebraically closed field over $\operatorname{Frac}(O')$ the Lie algebra of B_0 has τ-eigenspaces of dimension n_τ with $n_\tau + \operatorname{rk}_{\mathcal{O}_E} X_0 = p$ if $\tau(r)$ is the image of $r_{\mathbb{C}}$ and $= q$ otherwise, and there are primitive k-th roots of unity ζ_k ($k \geq 1$) with $\zeta_{km}^m = \zeta_k$ and ζ_{A_1} the image of $e^{2\pi i/A_1}$, and additive $\mathcal{O}_E$-linear bijections $I_k\colon M/kM \to T_k$ of the datum with $mI_{km} = I_k$ and $e_k(I_k(a), I_k(b)) = \zeta_k^{\langle a,b\rangle}$, such that $I_{A_1}(a)$ is the class of $\eta_{0,A}(a)$ for $a \in M$ and $I_{N_1}(a)$ that of $\eta_0(a)$ for $a \in \bar{\mathfrak{n}}_1 M$. The condition is the following: for every arc datum there is exactly one $\mathrm{ar}\colon \operatorname{Spec} O'[[X]] \to \mathcal{Y}$ over b such that $\mathrm{ar}^{-1}(\mathcal{Y}^{\mathrm{ab}})$ is the complement of $V(X)$, the complement of $V(\varpi X)$ maps into $j(\mathrm{Sh}_{\mathsf{L}',F'})$, and $\mathrm{ar}^*(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ is uniformised over $(O'[[X]], (X))$ by degeneration data with $\mathcal{O}_E$-structure whose abelian scheme (with line bundle, rigidification and $\mathcal{O}_E$-action), whose X and whose c are identified with the base change of those of the arc datum and whose $\psi(\chi')$ is $X^{\beta(\chi',\chi')/2}\psi_0(\chi')$, and the level structures correspond: at every geometric point $\bar{y}$ of $\mathrm{Sh}_{\mathsf{L}',F'}$ over the generic point of $\operatorname{Spec} O'[[X]]$, with u the value of X at $\bar{y}$, $\eta_A^{\mathrm{un}}(a)_{\bar{y}}$ ($a \in M$) is the torsion point corresponding to the class of $(y, \tilde{g}')$, where $(y, \tilde{g})$ represents $\eta_{0,A}(a)$ and $\tilde{g}'$ is $\tilde{g}$ with its coordinate of weight χ' multiplied by $u^{\beta(y,\chi')/A_1}$, and likewise $\eta^{\mathrm{un}}(a)_{\bar{y}}$ for $a \in \bar{\mathfrak{n}}_1 M$ with η_0 and N_1; and if ϖ is a uniformiser, every $y'\colon \operatorname{Spec} O' \to \mathcal{Y}$ over b with generic point in $j(\mathrm{Sh}_{\mathsf{L}',F'})$ along which the structure is uniformised in the same way over $(O', (\varpi))$ with ϖ in place of X is the specialisation of ar at $X = \varpi$.

Local systems. Fix a system of moduli schemes, a prime ℓ and an ℓ-adic place v. A **Siegel family** of level m ($m \geq 3$, $\ell \nmid m$) is a quasi-compact smooth scheme $\mathbf{A}$ of relative dimension $g(g+1)/2$ over $\mathbb{Z}_{(\ell)}[e^{2\pi i/m}]$ with a principally polarised abelian scheme of dimension g and a symplectic level-m structure on $\mathbb{Z}^{2g}$ (Weil pairing $e^{4\pi i\omega/m}$ for the standard symplectic form ω) such that every such object over the valuation ring of a complete algebraically closed non-archimedean field of characteristic 0 with $|\ell| < 1$ is pulled back from $\mathbf{A}$, and two maps from $\operatorname{Spec} k[\epsilon]$ pulling back isomorphic objects, one of them constant, coincide. A **local system at** v is a family $(\mathcal{Y}_{\mathsf{L}',F'})$, over all levels L' good at ℓ and coefficient fields F' for L', of integral models of $\mathrm{Sh}_{\mathsf{L}'}$ over $\mathcal{O}_{F',v}$ such that:

(Gl5) Members. Every member satisfies (Gl1)–(Gl4).

(Gl6) Transitions. For $\mathsf{L}' = (N_1, \mathfrak{n}_1, A_1)$, $\mathsf{L}'' = (N_2, \mathfrak{n}_2, A_2)$ with $N_1 \mid N_2$, $\mathfrak{n}_2 \subseteq \mathfrak{n}_1$, $A_1 \mid A_2$ and $F' \subseteq F''$ there is $\pi\colon \mathcal{Y}_{\mathsf{L}'',F''} \to \mathcal{Y}_{\mathsf{L}',F'}$, finite over $\mathcal{Y}_{\mathsf{L}',F'} \otimes \mathcal{O}_{F'',v}$, with a morphism of semiabelian structures over π and $r\colon \mathrm{Sh}_{\mathsf{L}'',F''} \to \mathrm{Sh}_{\mathsf{L}',F'}$ with $\pi \circ j = j \circ r$, under which $(N_2/N_1)\eta^{\mathrm{un}}(a)$ goes to $\eta^{\mathrm{un}}(a)$ and $(A_2/A_1)\eta_A^{\mathrm{un}}(a)$ to $\eta_A^{\mathrm{un}}(a)$.

(Gl7) Siegel comparison. There are an odd prime $n \neq \ell$ dividing the first entry of no level for H, an integer j' with $m = n^{j'} \geq 3$ and a Siegel family $\mathbf{A}$ of level m such that for every member $\mathcal{Y} = \mathcal{Y}_{\mathsf{L}',F'}$, with $\mathcal{Y}_1 = \mathcal{Y}_{\mathsf{L}_1,F_1}$, $\mathsf{L}_1 = (N_1, \mathfrak{n}_1, \operatorname{lcm}(A_1, m))$, $F_1 = F'(e^{2\pi i/m})$,

π of (Gl6) and abelian loci $U \subseteq \mathcal{Y}$, $U_1 \subseteq \mathcal{Y}_1$: (i) $U_1 \to U \otimes \mathcal{O}_{F_1,v}$ is finite, étale and surjective; (ii) there are an isomorphism $\mathbb{Z}^{2g} \to M$ carrying ω to $\langle\,,\rangle$ and a finite morphism $U_1 \to \mathbf{A} \otimes \mathcal{O}_{F_1,v}$ along which the Siegel abelian scheme with its polarisation is $G_{\mathcal{Y}_1}|_{U_1}$ and the Siegel level structure at u is $(\operatorname{lcm}(A_1, m)/m)\,\eta_A^{\mathrm{un}}(u)$; (iii) if $\ell \nmid N_1 A_1$, this morphism factors as $U_1 \to \mathcal{X} \to \mathbf{A} \otimes \mathcal{O}_{F_1,v}$ with $\mathcal{X}$ reduced and flat over $\mathcal{O}_{F_1,v}$, the second map finite and formally unramified, the first finite, surjective and an isomorphism over a dense open set with dense preimage, and $\mathcal{X}$ carries an abelian scheme with $\mathcal{O}_E$-action pulling back to $G_{\mathcal{Y}_1}|_{U_1}$ to which all $\eta^{\mathrm{un}}(a), \eta_A^{\mathrm{un}}(a)$ descend.

Global integral models. Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p + q \geq 1$ and a system of moduli schemes extending it. An integral model $\mathcal{Y}$ of Sh over $\mathcal{O}_F$ is **global** (for the system) if L is good at every prime, $A = 1$, and there is a local system $(\mathcal{Y}^v_{\mathsf{L}',F'})$ at every ℓ-adic place v of every prime ℓ such that:

(Gl8) Descent. The generic fibre of $\mathcal{Y}$ descends.

(Gl9) Domination. For every v there is a morphism $f\colon \mathcal{Y} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v} \to \mathcal{Y}^v_{\mathsf{L},F}$ over $\mathcal{O}_{F,v}$ with $f \circ j = j$ and a morphism over f from the base change of the semiabelian structure of $\mathcal{Y}$ to that of $\mathcal{Y}^v_{\mathsf{L},F}$, compatible with the universal objects over Sh_F.

Definition 31.3. Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p + q \geq 1$ and level $\mathsf{L} = (N, \mathfrak{n}, A)$. An integral model $(\mathcal{Y}, j, G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ of Sh over $\mathcal{O}_F$ is an **integral model with uniformised degenerations** if L is good at every prime, $A = 1$, and for every prime ℓ and every ℓ-adic place v there are:

(Ug1) Local model. An integral model $\mathcal{Y}^v$ of Sh over $\mathcal{O}_{F,v}$ that satisfies (Gl2) and (Gl3);

(Ug2) Domination. A morphism of schemes $f_v\colon \mathcal{Y} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v} \to \mathcal{Y}^v$ and a morphism over f_v from the base change to $\mathcal{Y} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v}$ of the semiabelian structure $(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ to the semiabelian structure of $\mathcal{Y}^v$.

Definition 31.4. Let E be a CM field with maximal totally real subfield E^+ and complex conjugation c.

Hermitian spaces and their groups. A **hermitian space** over E is a finite-dimensional E-vector space V with a non-degenerate form $h\colon V \times V \to E$ that is E-linear in the first variable and satisfies $h(y, x) = c(h(x, y))$. For a commutative E^+-algebra R, $\mathrm{U}(V)(R)$ is the group of $E \otimes_{E^+} R$-linear automorphisms g of $V \otimes_{E^+} R$ with $h_R(gx, gy) = h_R(x, y)$, and $\mathrm{SU}(V)(R)$ its subgroup of elements of determinant 1; these are the R-points of linear algebraic groups $\mathrm{U}(V) \supseteq \mathrm{SU}(V)$ over E^+. The space V is **indefinite** at a real place w of E^+ if the hermitian form $h \otimes_{E^+,w} \mathbb{R}$ on the complex vector space $V \otimes_{E^+,w} \mathbb{R}$ is neither positive nor negative definite. Put

$$G^1_\infty(V) = \mathrm{SU}(V)(E^+ \otimes_{\mathbb{Q}} \mathbb{R}) = \prod_{w \text{ real place of } E^+} \mathrm{SU}(V)(E^+_w),$$

a unimodular Lie group, and fix a Haar measure μ_∞ on it. For a discrete subgroup $\Gamma \subseteq G^1_\infty(V)$, $\mu_\infty(\Gamma \backslash G^1_\infty(V)) \in (0, \infty]$ denotes the μ_∞-measure of a measurable fundamental domain for the action of Γ by left multiplication (it does not depend on the fundamental domain).

Lattices, congruence subgroups, genus. A **lattice** of V is a finitely generated $\mathcal{O}_E$-submodule $\Lambda \subseteq V$ that spans V over E. For a non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_E$ the **principal congruence subgroup** of level $\mathfrak{a}$ is

$$\Gamma(\Lambda, \mathfrak{a}) = \{\gamma \in \mathrm{U}(V)(E^+) :\ \gamma\Lambda = \Lambda,\ (\gamma - 1)\Lambda \subseteq \mathfrak{a}\Lambda\}. \tag{31.2}$$

The ideal $\mathfrak{a}$ is **neat** if 1 is the only root of unity $\zeta \in E$ with $\zeta - 1 \in \mathfrak{a}$. For a finite place v of E^+ write $\Lambda_v = \Lambda \otimes_{\mathcal{O}_{E^+}} \mathcal{O}_{E^+,v} \subseteq V \otimes_{E^+} E^+_v$. Two lattices Λ, Λ' of V lie in **one genus** if for every finite place v of E^+ there is $g_v \in \mathrm{U}(V)(E^+_v)$ with $\Lambda'_v = g_v \Lambda_v$; a family of lattices lies in one genus if any two of its members do.

Covolume realisation. Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with $n \geq 2$. It has a **covolume realisation** over E if:

(Vu) Volumes are covolumes There are a hermitian space (V, h) over E of dimension n, indefinite at some real place of E^+, a neat non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_E$, an integer $m \geq 1$, a real number $c_0 > 0$ and, for every $Z \in \mathrm{Cp}$, lattices $\Lambda_{Z,1}, \ldots, \Lambda_{Z,m}$ of V, all of them (over all Z and i) in one genus, such that for every $Z \in \mathrm{Cp}$

$$c_0 \operatorname{vol}(Z) = \sum_{i=1}^{m} \mu_\infty\big(\Gamma(\Lambda_{Z,i}, \mathfrak{a}) \backslash G^1_\infty(V)\big).$$

The **dual** of a lattice Λ of V is $\Lambda^\sharp = \{x \in V : \ h(x, y) \in \mathcal{O}_E \text{ for all } y \in \Lambda\}$; Λ is **self-dual** if $\Lambda^\sharp = \Lambda$. The tower has a **self-dual covolume realisation** over E if:

(Vd) Volumes are covolumes of self-dual lattices There are (V, h), $\mathfrak{a}$, m, c_0 and lattices $\Lambda_{Z,i}$ as in (Vu), except that instead of lying in one genus every $\Lambda_{Z,i}$ is self-dual.

A root tower $\mathcal{T}$ of type (Φ, n, κ), $n \geq 2$, with a geometric realisation has **geometric proportionality with covolumes** over E if:

(Vp) Geometric proportionality with covolumes There are (V, h), $\mathfrak{a}$, m, c_0 and lattices $\Lambda_{Z,1}, \ldots, \Lambda_{Z,m}$ of V ($Z \in \mathrm{Cp}$), all of them in one genus, as in (Vu) such that for every integer $B \geq 2$, every $Z \in \mathrm{Cp}$ and every $F \in \mathcal{R}$ homogeneous of degree D

$$c_0 \operatorname{gi}(\mathcal{X}_{B,Z}; F) = \Big(\sum_{i=1}^{m} \mu_\infty\big(\Gamma(\Lambda_{Z,i}, \mathfrak{a}) \backslash G^1_\infty(V)\big)\Big) \deg^{\mathrm{fl}}(F).$$

Definition 31.5. Orders along prime divisors. Let $\mathcal{X} \subseteq \mathbb{P}$ be a closed integral subscheme of a multiprojective setting, flat over $\mathbb{Z}$. A multihomogeneous polynomial $f \in \mathbb{Z}[z_{k,i}]$ of multidegree $d_f \in \mathbb{Z}^K_{\geq 0}$ that does not vanish identically on $\mathcal{X}$ is a non-zero global section of $\mathcal{O}(d_f)|_{\mathcal{X}}$. For a prime divisor W of $\mathcal{X}$ (an integral closed subscheme of codimension one) let $\operatorname{ord}_W(f)$ be the length of $\mathcal{O}_{\mathcal{X},W}/(f/m)$, where m is a monomial of multidegree d_f in coordinates that do not vanish identically on W (so m is a unit at the generic point of W; the length does not depend on m); for a prime p, $\operatorname{ord}_W(p)$ is the length of $\mathcal{O}_{\mathcal{X},W}/(p)$.

Vertical closeness. Let S be a finite set of primes and $\alpha = (\alpha_p)_{p \in S}$ with $\alpha_p \geq 0$. Two Hermitian line bundles $\overline{\mathcal{L}} = (\mathcal{O}(e), h)|_{\mathcal{X}}$ and $\overline{\mathcal{L}}' = (\mathcal{O}(e'), h')|_{\mathcal{X}}$ given by cone metrics are **vertically close over** (S, α) if there are multihomogeneous $f, g \in \mathbb{Z}[z_{k,i}]$ of multidegrees d_f, d_g, neither vanishing identically on $\mathcal{X}$, such that

$$d_f - d_g = e - e', \qquad |f(z)|\, h'(z) = |g(z)|\, h(z) \text{ for every } z \text{ in the affine cone over } \mathcal{X}(\mathbb{C}), \quad (31.3)$$

and for every prime divisor W of $\mathcal{X}$: $\operatorname{ord}_W(f) = \operatorname{ord}_W(g)$ if W lies in no fibre $\mathcal{X}_{\mathbb{F}_p}$ with $p \in S$, and $|\operatorname{ord}_W(f) - \operatorname{ord}_W(g)| \leq \alpha_p \operatorname{ord}_W(p)$ if $W \subseteq \mathcal{X}_{\mathbb{F}_p}$ with $p \in S$.

Points of an arithmetic variety. For an arithmetic variety $\mathcal{J}$ over $\mathbb{Z}$, a **point** of $\mathcal{J}$ is an integral closed subscheme of $\mathcal{J}$, flat over $\mathbb{Z}$, of absolute dimension 1 (as for root models). If

$q\colon \mathcal{J} \to \mathcal{X}$ is a morphism of arithmetic varieties and W is a point of $\mathcal{J}$, then $q(W)$ is a point of $\mathcal{X}$.

Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ whose root models have convergent regularisation.

Reference link. The realised tower has the **reference link** if:

(Vl) Reference link There are $C_{\mathrm{Vl}} \geq 0$ and, for every $Z \in \mathrm{Cp}$, a linear form $\mathrm{pp}_Z^{\mathrm{ref}}$ on $\mathcal{R}^+$ and a real number $M_Z^{\mathrm{ref}} \geq 0$, such that for every integer $B \geq 2$ and every $Z \in \mathrm{Cp}$ there are two root models $\widetilde{\mathcal{X}} = (\widetilde{\mathcal{X}}, (e_r), (h_r(t)), h_*, k, \delta)$ and $\widetilde{\mathcal{X}}' = (\widetilde{\mathcal{X}}, (e'_r), (h'_r(t)), h'_*, k, \delta)$ of type (Φ, n) on the same arithmetic variety $\widetilde{\mathcal{X}} \subseteq \widetilde{\mathbb{P}} = \prod_{k \in \widetilde{K}} \mathbb{P}(\mathbb{Z}^{I_k})$ with the same normalising integers, a finite set S of primes, real numbers $\alpha_p \geq 0$ ($p \in S$), a multidegree $e_H \in \mathbb{Z}^{\widetilde{K}}$ and $t_0 > 0$, with:

- (Vl1) The models $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{X}}'$ have convergent regularisation, $\mathrm{pp}(\widetilde{\mathcal{X}}; \cdot) = \mathrm{pp}(\mathcal{X}_{B,Z}; \cdot)$ and $\mathrm{pp}(\widetilde{\mathcal{X}}'; \cdot) = \mathrm{pp}_Z^{\mathrm{ref}}$;
- (Vl2) For every $r \in \mathrm{Rt}$ and $0 < t \leq t_0$ the root lines $(\mathcal{O}(e_r), h_r(t))|_{\widetilde{\mathcal{X}}}$ and $(\mathcal{O}(e'_r), h'_r(t))|_{\widetilde{\mathcal{X}}}$ are vertically close over (S, α), and so are the auxiliary lines $(\mathcal{O}(e_*), h_*)|_{\widetilde{\mathcal{X}}}$ and $(\mathcal{O}(e'_*), h'_*)|_{\widetilde{\mathcal{X}}}$;
- (Vl3) For every $p \in S$, every integral closed curve $W \subseteq \widetilde{\mathcal{X}}_{\mathbb{F}_p}$ and every $f \in \{e_H\} \cup \{e_r + e_H,\ e'_r + e_H :\ r \in \mathrm{Rt}^+\}$ we have $\deg(\mathcal{O}(f)|_W) \geq 0$;
- (Vl4) We have $\deg(c_1(\mathcal{O}(f_1)) \cdots c_1(\mathcal{O}(f_D)) \cap [\widetilde{\mathcal{X}}_{\mathbb{Q}}]) \leq M_Z^{\mathrm{ref}}\, \delta$ for all $f_1, \ldots, f_D \in \{e_H\} \cup \{e_r + e_H :\ r \in \mathrm{Rt}^+\}$;
- (Vl5) We have $\sum_{p \in S} \alpha_p \log p \leq C_{\mathrm{Vl}} \log B$.

Reference comparison. The realised tower has the **reference comparison** if there are $C_{\mathrm{Rh}} \geq 1$, $s_{\mathrm{Rh}} \in (0, 1]$ and, for every $Z \in \mathrm{Cp}$, a root model $\mathcal{X}_Z^{\mathrm{ref}}$ of type (Φ, n) (the **reference model**; it does not depend on B), whose heights are written $h_{c,t}^{\mathrm{ref}}$, with:

(Rh1) Comparison of heights For every integer $B \geq 2$ and every $Z \in \mathrm{Cp}$ there are an arithmetic variety $\mathcal{J}$ over $\mathbb{Z}$ of absolute dimension $D+1$ and surjective morphisms $q\colon \mathcal{J} \to \mathcal{X}_{B,Z}$, $q^{\mathrm{ref}}\colon \mathcal{J} \to \mathcal{X}_Z^{\mathrm{ref}}$ such that for every point W of $\mathcal{J}$, every dominant weight μ with $\|\mu\|_\infty \leq 2$ and every $0 < s \leq s_{\mathrm{Rh}}$

$$-C_{\mathrm{Rh}}\big(\log B + \log(1/s) + 1\big) \leq h_{c(\mu,s),t(s)}\big(q(W)\big) - h^{\mathrm{ref}}_{c(\mu,s),1}\big(q^{\mathrm{ref}}(W)\big) \leq C_{\mathrm{Rh}}(\log B + 1),$$

where on each side $c(\mu, s)$ is formed with the normalising integer of the model on which the height is taken;

(Rh2) Lower bound on the reference For every $Z \in \mathrm{Cp}$ and every point W of $\mathcal{X}_Z^{\mathrm{ref}}$ we have $x := h^{\mathrm{ref}}_{c(0,1),1}(W) \geq -C_{\mathrm{Rh}}$, and

$$h^{\mathrm{ref}}_{c(\mu,0),1}(W) \geq -C_{\mathrm{Rh}} \log\big(2 + C_{\mathrm{Rh}} + x\big) - C_{\mathrm{Rh}}$$

for every dominant weight μ with $\|\mu\|_\infty \leq 2$.

Definition 31.6. Lie lattices. Let E be a CM field and L a number field. A **principally polarised abelian scheme with $\mathcal{O}_E$-action** over $\mathcal{O}_L$ is a semiabelian scheme $\mathcal{A}$ over $\operatorname{Spec} \mathcal{O}_L$ with an action $\iota\colon \mathcal{O}_E \to \mathrm{End}(\mathcal{A})$ and a rigidified symmetric line bundle $(\mathcal{L}_{\mathcal{A}}, r)$ on $\mathcal{A}$ with a cubical structure, such that the generic fibre $A = \mathcal{A}_L$ is an abelian variety of dimension $g \geq 1$,

$\mathcal{L}_{\mathcal{A}}|_A$ is ample and its polarisation $\phi\colon A \to A^\vee$ is an isomorphism (the **principal polarisation**). Its base change along a finite extension L'/L is $\mathcal{A} \times_{\mathcal{O}_L} \mathcal{O}_{L'}$ with the induced action and line bundle; it is again such a scheme. For an embedding $\sigma\colon L \to \mathbb{C}$ write $A_\sigma(\mathbb{C}) = \mathrm{Lie}(A_\sigma)/\Pi_\sigma$, and let H_σ be the positive definite Hermitian form on $\mathrm{Lie}(A_\sigma)$ whose imaginary part on Π_σ is the Riemann form of the polarisation. The **Lie lattice** of $\mathcal{A}$ is the Hermitian lattice

$$\overline{\mathrm{Lie}}(\mathcal{A}) = \big(\mathrm{Lie}(\mathcal{A}), (\|\cdot\|_\sigma)_\sigma\big), \qquad \|x\|_\sigma^2 = \pi\, H_\sigma(x,x),$$

over $\mathcal{O}_L$, where $\mathrm{Lie}(\mathcal{A}) = e^* T_{\mathcal{A}/\mathcal{O}_L}$ (projective of rank g) and $\mathrm{Lie}(\mathcal{A}) \otimes_{\mathcal{O}_L,\sigma} \mathbb{C} = \mathrm{Lie}(A_\sigma)$. Put

$$h^+(\mathcal{A}) = \max\Big(-\frac{\widehat{\deg}\,\overline{\mathrm{Lie}}(\mathcal{A})}{[L:\mathbb{Q}]},\, 0\Big).$$

Submodules of $\mathrm{Lie}(\mathcal{A})$ carry the induced norms.

Heights of root lines at a point. For a root model $\mathcal{X}$, a point W of $\mathcal{X}$, $r \in \mathrm{Rt}^+$ and $t > 0$ put $\widehat{h}_W(\overline{\mathcal{L}}_r(t)) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_r(t)) \cdot [W])/\deg W_{\mathbb{Q}}$, so that $h_{c,t}(W) = \sum_r c_r\, \widehat{h}_W(\overline{\mathcal{L}}_r(t))$.

Lie fibre data. Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with $1 \le \kappa \le n-1$ and a geometric realisation with the reference comparison, with reference models $\mathcal{X}_Z^{\mathrm{ref}}$ ($Z \in \mathrm{Cp}$). The reference models have **Lie fibre data** with constant $C_{\mathrm{Lx}} \ge 1$ if for every $Z \in \mathrm{Cp}$, writing $\mathcal{X} = \mathcal{X}_Z^{\mathrm{ref}}$:

(Lx1) Ample auxiliary line The line bundle $\mathcal{O}(e_*)|_{\mathcal{X}_{\mathbb{Q}}}$ is ample.

(Lx2) Lie fibres For every point W of $\mathcal{X}$ there are a number field L and a Hermitian lattice $\overline{V} = (V, (\|\cdot\|_{V,\sigma})_\sigma)$ over $\mathcal{O}_L$ of rank $|\Phi|\, n$, the **Lie fibre** at W, with the following data; write $V_\sigma = V \otimes_{\mathcal{O}_L,\sigma} \mathbb{C}$ for $\sigma\colon L \to \mathbb{C}$.

(a) (Flags.) For every $a \in \Phi$, saturated $\mathcal{O}_L$-submodules $0 = F_{a,0} \subset F_{a,1} \subset \cdots \subset F_{a,\kappa}$ and $0 = S_{a,0} \subset S_{a,1} \subset \cdots \subset S_{a,n-\kappa}$ of V with $\mathrm{rk}\, F_{a,i} = i$, $\mathrm{rk}\, S_{a,j} = j$, such that the $2|\Phi|$ subspaces $F_{a,\kappa} \otimes L$, $S_{a,n-\kappa} \otimes L$ ($a \in \Phi$) of $V \otimes_{\mathcal{O}_L} L$ are pairwise orthogonal for every $\|\cdot\|_{V,\sigma}$ and span $V \otimes_{\mathcal{O}_L} L$, and, with $d_{a,i} = \widehat{\deg}\,\overline{F}_{a,i}/[L:\mathbb{Q}]$ and $d'_{a,j} = \widehat{\deg}\,\overline{S}_{a,j}/[L:\mathbb{Q}]$ (norms induced from $\overline{V}$),

$$\big|\widehat{h}_W(\overline{\mathcal{L}}_{(a,i)}(1)) - (d_{a,i} - d_{a,i-1})\big| \le C_{\mathrm{Lx}} \quad (1 \le i \le \kappa),$$
$$\big|\widehat{h}_W(\overline{\mathcal{L}}_{(a,n+1-j)}(1)) + (d'_{a,j} - d'_{a,j-1})\big| \le C_{\mathrm{Lx}} \quad (1 \le j \le n-\kappa);$$

(b) (Torus part and abelian part.) Elements $\varepsilon_1, \dots, \varepsilon_r \in V$ ($0 \le r \le |\Phi|\, n$), linearly independent over $\mathcal{O}_L$, spanning a submodule U, with

$$C_{\mathrm{Lx}}^{-1} \sum_{b=1}^{r} |w_b|^2 \le \Big\| \sum_{b=1}^{r} w_b \varepsilon_b \Big\|_{V,\sigma}^2 \le C_{\mathrm{Lx}} \sum_{b=1}^{r} |w_b|^2 \qquad (\sigma\colon L \to \mathbb{C},\ w \in \mathbb{C}^r);$$

and either $U = V$, or a CM field E, a principally polarised abelian scheme $\mathcal{A}$ with $\mathcal{O}_E$-action over $\mathcal{O}_L$ and an $\mathcal{O}_L$-linear map $\phi\colon V \to \mathrm{Lie}(\mathcal{A})$ with $\ker\phi = U$, $\mathrm{Lie}(\mathcal{A})/\phi(V)$ finite and $\|(\phi \otimes_\sigma \mathbb{C})(x)\|_\sigma^2 \le C_{\mathrm{Lx}} \|x\|_{V,\sigma}^2$ for every σ and $x \in V_\sigma$, with the Hodge norms of $\mathcal{A}$ on the left.

(Lx3) Flat determinants For every point W of $\mathcal{X}$ and every $a \in \Phi$,

$$\Big|\sum_{i=1}^{n} \widehat{h}_W(\overline{\mathcal{L}}_{(a,i)}(1))\Big| \le C_{\mathrm{Lx}}.$$

Definition 31.7. Let $\mathcal{X} \subseteq \mathbb{P} = \prod_{k\in K} \mathbb{P}(\mathbb{Z}^{I_k})$ be a root model of type (Φ, n) and $s = s_{\tau,i}$ a Levi reflection. Write $r_1 = (\tau, i)$, $r_2 = (\tau, i+1)$, $e_\Xi = e_{r_1} - e_{r_2}$, $h_\Xi(t) = h_{r_1}(t)/h_{r_2}(t)$, and call $\overline{\Xi}(t) = \overline{\mathcal{L}}_{r_1}(t) \otimes \overline{\mathcal{L}}_{r_2}(t)^{\otimes -1}$, the Hermitian line bundle $(\mathcal{O}(e_\Xi), h_\Xi(t))|_{\mathcal{X}}$, the **$\xi$-line** of s. For $k_0 \in K$ let $\mathbb{P}^{(2)} = \mathbb{P} \times \mathbb{P}(\mathbb{Z}^{I_{k_0}})$, with points (z, z'), and let $q_0, q_1 \colon \mathbb{P}^{(2)} \to \mathbb{P}$ be the projections: $q_0(z, z') = z$, and $q_1(z, z')$ is z with its coordinates of the factor k_0 replaced by z'. For a multidegree $e \in \mathbb{Z}^K$ put $q_0^* e = (e, 0)$ and $q_1^* e = (e', e_{k_0}) \in \mathbb{Z}^K \times \mathbb{Z}$ with $e'_{k_0} = 0$ and $e'_k = e_k$ otherwise, so that $q_j^* \mathcal{O}(e) = \mathcal{O}(q_j^* e)$. A cone function f of multidegree e on $\mathbb{P}$ is **from the base at** k_0 if $e_{k_0} = 0$ and $f(z)$ does not depend on the coordinates of z in the factor k_0.

Square systems. A **square system for** s on $\mathcal{X}$ is a tuple $(k_0, \mathcal{X}^{(2)}, g, h)$ of an index $k_0 \in K$, a closed integral subscheme $\mathcal{X}^{(2)} \subsetneq \mathbb{P}^{(2)}$ and multihomogeneous polynomials g, h with integer coefficients in the coordinates of $\mathbb{P}^{(2)}$, such that:

(Fx1) Base lines We have $(e_\Xi)_{k_0} = -2$, and for every $t > 0$ the cone functions $h_r(t)$, $r \in \mathrm{Rt} \smallsetminus \{r_1, r_2\}$, and $h_{r_1}(t) h_{r_2}(t)$ are from the base at k_0;

(Fx2) Fibre square The scheme $\mathcal{X}^{(2)}$ is flat over $\mathbb{Z}$ of absolute dimension $D + 2$, $q_0(\mathcal{X}^{(2)}) = q_1(\mathcal{X}^{(2)}) = \mathcal{X}$, and the fibre of $q_0|_{\mathcal{X}^{(2)}} \colon \mathcal{X}^{(2)} \to \mathcal{X}$ over the generic point of $\mathcal{X}$ is a linear subspace of the projective space of the last factor over the function field of $\mathcal{X}$;

(Fx3) Diagonal There is a closed integral subscheme $\Delta \subseteq \mathcal{X}^{(2)}$ of absolute dimension $D + 1$ on which the minors $z_{k_0,j} z'_{j'} - z_{k_0,j'} z'_j$ $(j, j' \in I_{k_0})$ vanish, such that g and h do not vanish identically on $\mathcal{X}^{(2)}$,

$$2(\deg g - \deg h) = -(q_0^* e_\Xi + q_1^* e_\Xi), \qquad \mathrm{div}(g|_{\mathcal{X}^{(2)}}) = \mathrm{div}(h|_{\mathcal{X}^{(2)}}) + [\Delta]$$

as Weil divisors on $\mathcal{X}^{(2)}$.

Square integrals. Given a square system and $t > 0$, the **square Green function** at t is the function on the complex points of $\mathcal{X}^{(2)}$ where $gh \neq 0$,

$$G_t(z, z') = \log|g(z, z')| - \log|h(z, z')| + \tfrac{1}{2} \log h_\Xi(t)\big(q_0(z, z')\big) + \tfrac{1}{2} \log h_\Xi(t)\big(q_1(z, z')\big), \tag{31.4}$$

evaluated at any point of the affine cone (it is invariant under the scalings): it is $\log \|g/h\|$ for the rational section g/h of $\mathcal{O}(\deg g - \deg h)$ with the cone function $\big(h_\Xi(t) \circ q_0 \cdot h_\Xi(t) \circ q_1\big)^{-1/2}$. For an s-invariant weight c let $\omega_{c,t} = -\sum_{r \in \mathrm{Rt}} c_r \, c_1(\overline{\mathcal{L}}_r(t))$ and $\omega_{\Xi,t} = c_1(\overline{\Xi}(t))$, smooth real $(1,1)$-forms on $\mathcal{X}(\mathbb{C})$. For $1 \leq k \leq (D+1)/2$ and $m = D + 1 - 2k$ the **square integral** is

$$I_t(k, c) = \sum_{a=0}^{2k} (-1)^a \int_{\mathcal{X}^{(2)}(\mathbb{C})} G_t \; q_0^* \omega_{\Xi,t}^a \wedge q_1^* \omega_{\Xi,t}^{2k-a} \wedge q_0^* \omega_{c,t}^m. \tag{31.5}$$

The square system has the **square limit** if

(Fx4) Square limit For every s-invariant weight c and every $1 \leq k \leq (D+1)/2$, with the sphere moments sph_k of the fibre-square identities,

$$\lim_{t \to 0^+} I_t(k, c) = -2 \, \mathrm{sph}_k \, \delta_{\mathcal{X}} \, \mathrm{gi}\big(\mathcal{X}; \xi_s^{2k-1} \mathrm{lr}(c)^{D+1-2k}\big).$$

Definition 31.8. Fubini–Study potentials. For a positive definite Hermitian 2×2 matrix H, $z, z_0, z_1 \in \mathbb{C}$ with $z_0 \neq z_1$, and $u(z) = (1, z)^{\mathsf{T}}$, the **$\xi$-potential** and the **log-sine function** of H are

$$\begin{aligned} \mathrm{xp}_H(z) &= -\log\big(u(z)^* H u(z)\big) + \tfrac{1}{2} \log\det H, \\ \mathrm{ls}_H(z_0, z_1) &= \tfrac{1}{2} \log\Big(1 - \frac{|u(z_0)^* H u(z_1)|^2}{\big(u(z_0)^* H u(z_0)\big)\big(u(z_1)^* H u(z_1)\big)}\Big). \end{aligned} \tag{31.6}$$

Let $(k_0, \mathcal{X}^{(2)}, g, h)$ be a square system for a Levi reflection s on a root model $\mathcal{X}$ of type (Φ, n), and write $X = \mathcal{X}(\mathbb{C})$, $W = \mathcal{X}^{(2)}(\mathbb{C})$. The **base lines** of s are the indices $\beta \in \mathrm{Rt}_s^{\mathrm{b}} = (\mathrm{Rt} \setminus \{r_1, r_2\}) \sqcup \{r_{12}\}$, where $h_{r_{12}}$ stands for $h_{r_1} h_{r_2}$. A subset of a complex manifold is **null** if its intersection with every coordinate chart has Lebesgue measure 0. **Analytic square data** for the square system consist of open subsets $X^\circ \subseteq X$ and $W^\circ \subseteq W$ of smooth points, positive functions $h_r(0)$ ($r \in \mathrm{Rt}$) on the affine cone over X°, and a sequence of fibre charts $(U_n, B_n, G_n, (\alpha_{n,\beta})_{\beta \in \mathrm{Rt}_s^{\mathrm{b}}}, \Phi_n, \Phi_n^{(2)})_{n \geq 1}$, such that:

(Fx5) Limit metrics $q_0(W^\circ) \cup q_1(W^\circ) \subseteq X^\circ$; each $h_r(0)$ is of class C^2 and $h_r(0)\big((\lambda_k z_k)_k\big) = \prod_k |\lambda_k|^{e_{r,k}} h_r(0)(z)$ for $\lambda \in (\mathbb{C}^\times)^K$. Put $h_\Xi(0) = h_{r_1}(0)/h_{r_2}(0)$, $h_{r_{12}}(0) = h_{r_1}(0) h_{r_2}(0)$, let G_0 be the function (31.4) with $t = 0$ on the points of W° where $gh \neq 0$, and on X° let $\omega_{\Xi,0} = \frac{i}{\pi} \partial\bar\partial \log h_\Xi(0)$ and, for a weight c, $\omega_{c,0} = -\sum_r c_r \frac{i}{\pi} \partial\bar\partial \log h_r(0)$ (computed on any local holomorphic lift to the affine cone; continuous real $(1,1)$-forms).

(Fx6) Fibre charts For every n: $U_n \subseteq \mathbb{C}^{D-1}$ is open and $B_n \subseteq U_n$ is Borel; $G_n \colon U_n \to \mathrm{M}_2(\mathbb{C})$ is of class C^2 with positive definite Hermitian values; $\alpha_{n,\beta} \colon U_n \to \mathbb{R}$ is of class C^2; $\Phi_n \colon \mathbb{C} \times U_n \to X^\circ$ and $\Phi_n^{(2)} \colon \mathbb{C}^2 \times U_n \to W^\circ$ are holomorphic and biholomorphic onto open subsets, with $q_0(\Phi_n^{(2)}(z_0, z_1, y)) = \Phi_n(z_0, y)$ and $q_1(\Phi_n^{(2)}(z_0, z_1, y)) = \Phi_n(z_1, y)$; for every local holomorphic lift $\tilde\Phi$ of Φ_n to the affine cone, the functions $\log h_\Xi(0) \circ \tilde\Phi - \mathrm{xp}_{G_n(y)}(z)$ and $\log h_\beta(0) \circ \tilde\Phi - \alpha_{n,\beta}(y)$ ($\beta \in \mathrm{Rt}_s^{\mathrm{b}}$) of (z, y) are pluriharmonic; and the sets $\Phi_n(\mathbb{C} \times B_n)$ ($n \geq 1$) are pairwise disjoint and cover X° up to a null set, and so are the sets $\Phi_n^{(2)}(\mathbb{C}^2 \times B_n)$ for W°.

(Fx7) Norm identity $G_0(\Phi_n^{(2)}(z_0, z_1, y)) = \mathrm{ls}_{G_n(y)}(z_0, z_1)$ for all n, $y \in U_n$ and $z_0 \neq z_1$ at which $gh \neq 0$.

(Fx8) Square convergence For every s-invariant weight c and every integer $1 \leq k \leq (D+1)/2$, with $m = D + 1 - 2k$:

1. for $0 \leq a \leq 2k$ the top-degree form $G_0\, q_0^* \omega_{\Xi,0}^a \wedge q_1^* \omega_{\Xi,0}^{2k-a} \wedge q_0^* \omega_{c,0}^m$ is absolutely integrable on W°, and, with $I_0(k, c)$ the alternating sum $\sum_{a=0}^{2k} (-1)^a$ of these integrals, $I_t(k, c) \to I_0(k, c)$ as $t \to 0^+$;
2. $\omega_{\Xi,0}^{2k-1} \wedge \omega_{c,0}^m$ is absolutely integrable on X° and $\int_{X^\circ} \omega_{\Xi,0}^{2k-1} \wedge \omega_{c,0}^m = \delta_{\mathcal{X}}\, \mathrm{gi}(\mathcal{X}; \xi_s^{2k-1} \mathrm{lr}(c)^m)$.

Complex manifolds carry their complex orientation.

Definition 31.9. Let $(k_0, \mathcal{X}^{(2)}, g, h)$ be a square system for a Levi reflection s on a root model $\mathcal{X}$ of type (Φ, n), with open parts X°, W° and limit metrics $h_r(0)$ ($r \in \mathrm{Rt}$) as in (Fx5). For $r \in \mathrm{Rt}$ write $\omega_{r,t} = c_1(\overline{\mathcal{L}}_r(t))$ on $\mathcal{X}(\mathbb{C})$ for $t > 0$, and $\omega_{r,0} = \frac{i}{\pi} \partial\bar\partial \log h_r(0)$ on X° (computed on local holomorphic lifts to the affine cone). A **compact model** of an open subset V of the complex points of an integral scheme of finite type over $\mathbb{Z}$ is a tuple $(\widetilde{V}, \Theta_V, \varsigma_V, \varrho_V, d_V)$ of: a compact complex manifold $\widetilde{V}$ with a Kähler form Θ_V; a holomorphic section ς_V of a smooth hermitian line bundle with $\|\varsigma_V\| \leq 1$ whose zero divisor $\partial\widetilde{V}$ has simple normal crossings, with $\widetilde{V}^\circ = \widetilde{V} \setminus \partial\widetilde{V}$ and $\mathrm{rW}_V = e - 2\log\|\varsigma_V\|$; a holomorphic map ϱ_V from $\widetilde{V}$ to these complex points and an integer $d_V \geq 1$; such that there are closed null subsets $N \subseteq \widetilde{V}$ and N' of the complex points, N' containing the singular locus, for which ϱ_V restricts to a d_V-sheeted holomorphic covering map $\widetilde{V} \setminus N \to (\text{complex points}) \setminus N'$ with $\varrho_V^{-1}(V \setminus N') = \widetilde{V}^\circ \setminus N$.

An **analytic square system** for the square system consists of a compact model $(\widetilde{X}, \Theta_X, \varsigma_X, \varrho_X, d_X)$ of $X^\circ \subseteq \mathcal{X}(\mathbb{C})$, a compact model $(\widetilde{W}, \Theta_W, \varsigma_W, \varrho_W, d_W)$ of $W^\circ \subseteq \mathcal{X}^{(2)}(\mathbb{C})$, holomorphic maps $p_0, p_1 \colon \widetilde{W} \to \widetilde{X}$, a holomorphic section v of a smooth hermitian line bundle on $\widetilde{W}$ and a real number κ_v, such that:

(Sw1) Base weights There are a finite set Λ, real numbers $b_{r,\lambda}$ ($r \in \mathrm{Rt}$, $\lambda \in \Lambda$), a number $0 < t_0 < 1$ and, for every $\lambda \in \Lambda$, a smooth closed real $(1,1)$-form γ_λ on $\widetilde{X}$, a smooth function f_λ on $\widetilde{X}^\circ$ and smooth functions $f_{\lambda,t}$ on $\widetilde{X}$ ($0 < t \le t_0$), such that the families $(\gamma_\lambda, f_\lambda, (f_{\lambda,t})_t)_{\lambda\in\Lambda}$ satisfy (R1)–(R5) on $(\widetilde{X}, \Theta_X, \varsigma_X)$ for some $C \ge 1$ and functions δ, R, and such that for every $r \in \mathrm{Rt}$, with

$$\theta_r = \sum_{\lambda\in\Lambda} b_{r,\lambda}\gamma_\lambda, \qquad u_r = \sum_{\lambda\in\Lambda} b_{r,\lambda} f_\lambda, \qquad u_{r,t} = \sum_{\lambda\in\Lambda} b_{r,\lambda} f_{\lambda,t},$$

$\varrho_X^*\omega_{r,t} = \theta_r + dd^c u_{r,t}$ on $\widetilde{X}$ ($0 < t \le t_0$) and $\varrho_X^*\omega_{r,0} = \theta_r + dd^c u_r$ on $\widetilde{X}^\circ \cap \varrho_X^{-1}(X^\circ)$.

(Sw2) Projections $\varrho_X \circ p_j = q_j \circ \varrho_W$ ($j = 0, 1$), $p_j(\widetilde{W}^\circ) \subseteq \widetilde{X}^\circ$, and $\mathrm{rW}_X \circ p_j \le \mathrm{rW}_W$ on $\widetilde{W}^\circ$.

(Sw3) Diagonal section $\|v\| \le 1$; the zero divisor D_v of v is reduced, has no irreducible component contained in $\partial\widetilde{W}$, and $D_v + (\partial\widetilde{W})_{\mathrm{red}}$ has simple normal crossings; and for $0 < t \le t_0$, resp. $t = 0$,

$$G_t \circ \varrho_W = \tfrac{1}{2}\log\|v\|^2 + \kappa_v + \tfrac{1}{2}\sum_{j=0,1}(u_{r_1,t} - u_{r_2,t}) \circ p_j \quad \text{on } \widetilde{W} \smallsetminus (D_v \cup \varrho_W^{-1}(\{gh = 0\})),$$

resp. the same identity with u_{r_1}, u_{r_2} in place of $u_{r_1,t}, u_{r_2,t}$ on $\widetilde{W}^\circ \smallsetminus (D_v \cup \varrho_W^{-1}(\{gh = 0\}))$.

Let $\mathcal{X}$ be a root model of type (Φ, n) with convergent regularisation and s a Levi reflection. A **proportional square model** of $\mathcal{X}$ for s is a root model $\mathcal{X}'$ of type (Φ, n) with convergent regularisation and a real number $m > 0$ (the **factor**) such that

$$\mathrm{pp}(\mathcal{X}'; F) = m\,\mathrm{pp}(\mathcal{X}; F) \quad \text{and} \quad \mathrm{gi}(\mathcal{X}'; F) = m\,\mathrm{gi}(\mathcal{X}; F) \qquad \text{for every } F \in \mathcal{R},$$

together with a square system for s on $\mathcal{X}'$ with open parts and limit metrics as in (Fx5), fibre charts with (Fx6) and (Fx7), and an analytic square system. A root tower with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ whose root models have convergent regularisation has **proportional analytic square systems** if for every integer $B \ge 2$, every $Z \in \mathrm{Cp}$ and every Levi reflection s the root model $\mathcal{X}_{B,Z}$ has a proportional square model for s.

Definition 31.10. Let Φ be a non-empty finite set and $N > j \ge 1$ integers; put $j' = N - j$ and $e = |\Phi|\, j\, j'$. A **Hodge-type base setting** of type (Φ, N, j) consists of the following data.

(Hy1) Base variety A multiprojective setting $\mathbb{P}$ and a closed integral subscheme $\mathcal{X}^{\mathrm{b}} \subseteq \mathbb{P}$, flat over $\mathbb{Z}$, of absolute dimension $e + 1$, with smooth generic fibre $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$; so $X^{\mathrm{b}} = \mathcal{X}^{\mathrm{b}}(\mathbb{C})$ is a compact complex manifold of dimension e. A closed subset $\partial X^{\mathrm{b}} \subseteq X^{\mathrm{b}}$, stable under complex conjugation, which is a divisor with normal crossings (the **boundary**), and the **open part** $X^{\mathrm{b},\circ} = X^{\mathrm{b}} \smallsetminus \partial X^{\mathrm{b}}$. This is the arithmetic variety on which all numbers of the setting are computed; the Hodge metrics below are singular along the boundary.

(Hy2) Hodge data and the Hodge-type line For every $\tau \in \Phi$ a vector bundle $\mathcal{E}_\tau$ of rank N on $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ with a subbundle $\mathcal{F}_\tau$ of rank j, and a Hermitian metric h_τ on $\mathcal{E}_\tau$ over $X^{\mathrm{b},\circ}$ (the **Hodge metrics**); a multidegree $a^{\mathrm{H}} \in \mathbb{Z}^K$ and an isomorphism

$$\mathcal{O}(a^{\mathrm{H}})|_{\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}} \cong \bigotimes_{\tau\in\Phi} \det(\mathcal{E}_\tau/\mathcal{F}_\tau).$$

The **Hodge-type line** is $\mathcal{O}(a^{\mathrm{H}})|_{\mathcal{X}^{\mathrm{b}}}$ with the metric over $X^{\mathrm{b},\circ}$ that corresponds under this isomorphism to the tensor product of the determinant metrics of the quotient metrics

induced by h_τ on $\mathcal{E}_\tau/\mathcal{F}_\tau$; its cone function h^{H} is defined on the cone over $X^{\mathrm{b},\circ}$, its first Chern form is $\omega^{\mathrm{H}} = \frac{i}{\pi}\partial\bar\partial \log h^{\mathrm{H}}$ on $X^{\mathrm{b},\circ}$, and its **degree** is

$$\deg^{\mathrm{H}} = \deg(c_1(\mathcal{O}(a^{\mathrm{H}}))^e \cap [\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}]).$$

This is the line whose normalised self-intersection is the base slope.

(Hy3) Hodge–Tate twist A finite set S of primes and, for every $\ell \in S$, a multidegree $v_\ell \in \mathbb{Z}^K$ and a rational section g_ℓ of $\mathcal{O}(v_\ell)|_{\mathcal{X}^{\mathrm{b}}}$ (a quotient of two multihomogeneous polynomials whose degrees differ by v_ℓ) whose divisor on $\mathcal{X}^{\mathrm{b}}$ is supported in the fibre over ℓ; so g_ℓ has a well-defined non-zero value at every cone point over X^{b}. For an integer B put $S_B = \{\ell \in S : \ell \le B\}$ and $a^{\mathrm{H}}_B = a^{\mathrm{H}} + \sum_{\ell\in S_B} v_\ell$. The **twisted Hodge-type line** at B is $\mathcal{O}(a^{\mathrm{H}}_B)|_{\mathcal{X}^{\mathrm{b}}}$ with the cone function $h^{\mathrm{H}}\prod_{\ell\in S_B}|g_\ell|$ over $X^{\mathrm{b},\circ}$: the Hodge-type line changed by the vertical divisor $\sum_{\ell\in S_B}\mathrm{div}(g_\ell)$, isometrically at infinity. It carries the change of integral structure at the small primes.

(Hy4) Log-log regularisation A smooth cone function h^{ref} of multidegree a^{H} (the **reference**); the **singular weight** $\varphi = \log(h^{\mathrm{H}}/h^{\mathrm{ref}})$, a function on $X^{\mathrm{b},\circ}$; and for every real $t > 0$ a smooth function φ_t on X^{b} invariant under complex conjugation (the **regularisations**). The **regularised twisted Hodge-type line** at (B, t) is the Hermitian line bundle with smooth metric

$$\overline{\mathcal{H}}^{\mathrm{b}}_{B,t} = \Big(\mathcal{O}(a^{\mathrm{H}}_B),\ h^{\mathrm{ref}}e^{\varphi_t}\prod_{\ell\in S_B}|g_\ell|\Big)\Big|_{\mathcal{X}^{\mathrm{b}}}.$$

The singular Hodge metric enters the arithmetic numbers only through these smooth metrics.

(Hy5) Auxiliary line A multidegree $a' \in \mathbb{Z}^K$ with positive entries, a smooth cone function h' of multidegree a', invariant under complex conjugation, whose first Chern form is positive on X^{b}, giving $\overline{\mathcal{A}}' = (\mathcal{O}(a'), h')|_{\mathcal{X}^{\mathrm{b}}}$, and a real number k^{A} (the **auxiliary floor**) such that $h_{\overline{\mathcal{A}}'}(D_x) \ge -k^{\mathrm{A}}\,[\mathbf{k}(D_x):\mathbb{Q}]$ for the Zariski closure D_x of every $x \in \mathcal{X}^{\mathrm{b}}(\overline{\mathbb{Q}})$. Small multiples of it make the perturbed bundles positive.

The **limit self-intersection** at an integer B is

$$\mathrm{ar}^{\mathrm{b}}(B) = \lim_{t\to 0^+} \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{H}}^{\mathrm{b}}_{B,t})^{e+1})$$

when this limit exists, and then, if $\deg^{\mathrm{H}} \neq 0$, the **base slope** at B is

$$\mathrm{sl}^{\mathrm{b}}(B) = \frac{\mathrm{ar}^{\mathrm{b}}(B)}{(e+1)\,\deg^{\mathrm{H}}}. \tag{31.7}$$

Definition 31.11. Let $\mathfrak{u}$ be a unitary Shimura datum with $\Phi = \Phi_{\mathsf{H}}$, $n = p + q$, $1 \le q \le p$, let $\mathcal{Y}$ be an integral model of Sh over $\mathcal{O}_F$, and let $\mathcal{T}$ be a root tower of type (Φ, n, p) with a geometric realisation and a good origin over $\mathcal{Y}$. Let $B \ge 2$ be an integer, $Z \in \mathrm{Cp}$, and put $S'_B = \{\ell \text{ prime} :\ \ell \le B,\ \ell \notin \mathrm{Bd}\}$. A Hodge-type base setting of type (Φ, n, p) (so $N = n$, $j = p$, $j' = q$, $e = |\Phi|pq$) with cusp data is a **base setting of $\mathcal{T}$ at** (B, Z) if there are a cover $f\colon \mathcal{X} \to \mathcal{Y}$ with function field Ω, integers $R(\ell) \ge 4$ ($\ell \in \mathrm{Bd} \cup S'_B$) and representing coordinates $y = (y_{k,i})$ in Ω for the multiprojective setting $\mathbb{P}$ of (Hy1) such that the following hold. For a label s write $\mathcal{V}_s$ for the Hodge bundle at (Bd, R).

(Gb1) Cover and base variety The generic point of $\mathcal{X}$ maps to the generic point of $\mathcal{Y}_Z$; for $\ell \in \mathrm{Bd} \cup S'_B$ every $\ell^{R(\ell)}$-torsion point of $G_{\mathcal{X}}$ over an algebraic closure of Ω is defined over Ω; the Hodge lattices at (Bd, R) are Hodge bundles; and for $\ell \in \mathrm{Bd} \cup S'_B$ the span of the Hodge–Tate values at the points over ℓ, and its image in the dual of each Li_s, are locally free of ranks $\dim \mathrm{Li}$ and $\mathrm{rk}(s)$ along the fibre over ℓ (**Hodge–Tate freeness**). The rational map $\mathcal{X} \dashrightarrow \mathbb{P}$ given by y is birational onto $\mathcal{X}^{\mathrm{b}}$, it is defined at every complex point of $\mathcal{X}$ (every morphism $\operatorname{Spec}\mathbb{C} \to \mathcal{X}$; as $\mathcal{X}$ is an $\mathcal{O}_F$-scheme through $\mathcal{Y}$, such a point lies over exactly one field embedding $\kappa\colon F \to \mathbb{C}$, and all embeddings occur), the induced map $y_{\mathbb{C}}$ from the set of all these points to X^{b} is surjective, and ∂X^{b} is the image under $y_{\mathbb{C}}$ of the complex points outside the abelian locus. **Coordinate charts**: for an index $c' = (c'_k)_k$ with $y_{c'} = \prod_k y_{k,c'_k} \neq 0$ in Ω put $\mathbb{Z}[y]_{c'} = \mathbb{Z}[\, y_{k,i}/y_{k,c'_k} : k \in K,\ i \in I_k \,] \subseteq \Omega$, and say that a point x' of $\mathcal{X}$ **lies in the chart** c' if $\mathbb{Z}[y]_{c'} \subseteq \mathcal{O}_{\mathcal{X},x'}$; every point of $\mathcal{X}$ lies in some chart, and for every such c' the set $\mathcal{X}_{c'}$ of points lying in the chart c' is an affine open subset of $\mathcal{X}$ whose ring of regular functions is $\mathbb{Z}[y]_{c'}$.

(Gb2) Hodge data For $\tau \in \Phi$, the generic fibres of $\mathcal{E}_\tau \supseteq \mathcal{F}_\tau$ are the dual space $(\mathrm{Li}_{(\tau,+)} \oplus \mathrm{Li}_{(\tau,-)})^\vee$ and its subspace $\mathrm{Li}^\perp_{(\tau,-)}$ of linear forms vanishing on $\mathrm{Li}_{(\tau,-)}$, and at $w = y_{\mathbb{C}}(z) \in X^{\mathrm{b},\circ}$, z in the abelian locus, $h_\tau(w)$ is the metric dual to the restriction of the Hodge inner product h_z to $\mathrm{Lie}_{(\tau,+)}(G_{\mathcal{X},z}) \oplus \mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X},z})$. Thus $\mathcal{E}_\tau/\mathcal{F}_\tau$ has generic fibre $\mathrm{Li}^\vee_{(\tau,-)}$ (restriction of linear forms), of dimension q, and at such w its quotient metric is the metric dual to the restriction of h_z to $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X},z})$ (the quotient of the dual of a Hermitian space by the annihilator of a subspace is isometric to the dual of the subspace).

(Gb3) Integral structure of the Hodge-type line For a frame $(b_{\tau,i})_{i\le q}$ of $\mathcal{V}_{(\tau,-)}$ on an open subset write $(b^\vee_{\tau,i})_i$ for the dual frame of $\mathcal{V}^\vee_{(\tau,-)}$; at the generic point it is a basis of $\mathrm{Li}^\vee_{(\tau,-)}$. Every chart $\mathcal{X}_{c'}$ is covered by finitely many principal open subsets $D(f)$, $f \in \mathbb{Z}[y]_{c'}$, on each of which the section of $\bigotimes_\tau \det(\mathcal{E}_\tau/\mathcal{F}_\tau)$ that corresponds to $\prod_k y_{k,c'_k}^{a_k^{\mathrm{H}}}$ under the isomorphism of (Hy2) is $u \cdot \bigotimes_{\tau\in\Phi}(b^\vee_{\tau,1} \wedge \cdots \wedge b^\vee_{\tau,q})$, where u is a unit on $D(f)$ and $(b_{\tau,i})_i$ is a frame of $\mathcal{V}_{(\tau,-)}$ on $D(f)$. So the Hodge-type line is integrally the dual $\bigotimes_\tau(\det \mathcal{V}_{(\tau,-)})^\vee$, and its metric is dual to the determinant of the Hodge metric on the pieces $(\tau,-)$: at $w = y_{\mathbb{C}}(z)$ as in (Gb2), with z in $D(f)$, the section $\prod_k y_{k,c'_k}^{a_k^{\mathrm{H}}}$ has norm $|u(z)| \prod_\tau \|b_{\tau,1}(z) \wedge \cdots \wedge b_{\tau,q}(z)\|_{h_z}^{-1}$.

(Gb4) Hodge–Tate twist $S = S'_B$, and for $\ell \in S'_B$, every point x of $\mathcal{X}$ over ℓ and every chart c' in which x lies, the function $g_\ell / y_{c'}^{v_\ell}$ equals, up to a unit at x, $\prod_{\tau\in\Phi} \det T_\tau$, where T_τ is the matrix of a basis of $\mathrm{HT}_{(\tau,-),x}(\ell^{R(\ell)})$ in a basis of $\mathcal{V}_{(\tau,-),x}$. So near such a point the twisted Hodge-type line at B, identified with the Hodge-type line through multiplication by $\prod_\ell g_\ell$, is $\left(\prod_\tau \det T_\tau\right)^{-1} \bigotimes_\tau (\det \mathcal{V}_{(\tau,-)})^\vee$ (by (Gb3)); since a basis of $\mathrm{HT}_{(\tau,-),x}(\ell^{R(\ell)})$ is a basis of $\mathcal{V}_{(\tau,-),x}$ multiplied by T_τ, this is $\bigotimes_\tau \left(\det \mathrm{HT}_{(\tau,-),x}(\ell^{R(\ell)})\right)^\vee$: the twisted Hodge-type line is integrally the tensor product of the duals of the determinants of the Hodge lattices of the pieces $(\tau,-)$ at $(\mathrm{Bd} \cup S'_B, R)$. Where $\mathcal{V}_{(\tau,-),x} \subseteq \mathrm{HT}_{(\tau,-),x}(\ell^{R(\ell)})$, it is contained in $\bigotimes_\tau (\det \mathcal{V}_{(\tau,-),x})^\vee$ with index that of $\bigotimes_\tau \det \mathcal{V}_{(\tau,-),x}$ in $\bigotimes_\tau \det \mathrm{HT}_{(\tau,-),x}(\ell^{R(\ell)})$, so the twist lowers heights.

(Gb5) Cusp data $O = \mathcal{O}_E$, and for every complex point z of the abelian locus, lying over the embedding $\kappa\colon F \to \mathbb{C}$, with $w = y_{\mathbb{C}}(z)$: $\varepsilon_{w,\tau} = \kappa \circ \sigma_{(\tau,-)}|_{\mathcal{O}_E} = \kappa \circ \bar\tau$ for $\tau \in \Phi$ (the character by which $\mathcal{O}_E$ acts on $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X},z})$, (Fh1), (Fh4) of Definition 29.6), and there is an isomorphism of $\mathcal{O}_E$-modules $\Lambda_w \cong H_1(G_{\mathcal{X},z}(\mathbb{C}), \mathbb{Z}) \subset \mathrm{Lie}(G_{\mathcal{X},z})$ carrying $\langle\cdot,\cdot\rangle_w$ to

$\operatorname{Im} H_z$, $\|\cdot\|_w^2$ to $H_z(v,v)$, and $\lambda_{w,\tau}(v)$ to the coordinate vector, in an H_z-orthonormal basis of $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X},z})$, of the $(\tau,-)$-component of v. So the lattice, its form and the characters are those of the abelian variety at the point, and vary with the point and with the embedding of F through which it lies; $\operatorname{Im} H_z$ is unimodular on $H_1(G_{\mathcal{X},z}(\mathbb{C}),\mathbb{Z})$ because the polarisation of $G_{\mathcal{X},z} = G^{\mathrm{un}}_{f(z)}$ is principal (Definition 29.5).

(Gb6) Auxiliary line $a' = (1,\dots,1)$ and $k^{\mathrm{A}} = 0$.

The reference metrics, the regularisations φ_t, $\psi_{t'}$ and the auxiliary metric h' of (Hy4), (Hy5), (Hu3) are not restricted beyond those definitions.

Let $\mathcal{X}_{B,Z}$ be the root model of the geometric realisation at (B,Z), a cover $f_{B,Z}\colon \mathcal{X}_{B,Z} \to \mathcal{Y}$ by its root origin ((Go3)(1) of Definition 29.7). A base setting of $\mathcal{T}$ at (B,Z) with cover $f\colon \mathcal{X} \to \mathcal{Y}$ is **joined to the stage root model** if $\mathcal{X}$ is proper over $\operatorname{Spec}\mathbb{Z}$, $R(\ell) = 4$ for every $\ell \in \mathrm{Bd} \cup S'_B$, the Hodge lattices of $\mathcal{X}$ at $(\mathrm{Bd} \cup S'_B, R)$ are Hodge bundles, and there are a cover $f''\colon \mathcal{X}'' \to \mathcal{Y}$ and proper dominant morphisms $\phi\colon \mathcal{X}'' \to \mathcal{X}_{B,Z}$ and $\phi'\colon \mathcal{X}'' \to \mathcal{X}$ with $f'' = f_{B,Z}\circ\phi = f\circ\phi'$. This is the common cover on which the Hodge bundles of the base setting and of the root model are compared.

Definition 31.12. Let a Hodge-type base setting of type (Φ, N, j) be given. **Cusp data** for it consist of the following.

(Hu1) Cusp lattice A commutative ring O, and for every $w \in X^{\mathrm{b},\circ}$ the following data at w: ring homomorphisms $\varepsilon_{w,\tau}\colon O \to \mathbb{C}$ ($\tau \in \Phi$); a free $\mathbb{Z}$-module Λ_w of finite rank with a structure of O-module and a unimodular alternating $\mathbb{Z}$-bilinear form $\langle\cdot,\cdot\rangle_w\colon \Lambda_w \times \Lambda_w \to \mathbb{Z}$; a function $\|\cdot\|_w^2\colon \Lambda_w \to [0,\infty)$, positive on $\Lambda_w \smallsetminus \{0\}$, with $\langle v,v'\rangle_w^2 \le \|v\|_w^2\|v'\|_w^2$; for every $\tau \in \Phi$ an additive map $\lambda_{w,\tau}\colon \Lambda_w \to \mathbb{C}^{j'}$ with $\lambda_{w,\tau}(cv) = \varepsilon_{w,\tau}(c)\lambda_{w,\tau}(v)$ for $c \in O$ (the **Lie maps**), such that $\sum_\tau |\lambda_{w,\tau}(v)|^2 \le \|v\|_w^2$ and, for every isotropic O-submodule $I \subseteq \Lambda_w$ and every O-linearly independent family in I, the images under each $\lambda_{w,\tau}$ are $\mathbb{C}$-linearly independent; and for every non-zero isotropic O-submodule $I \subseteq \Lambda_w$ which is saturated ($nv \in I$, $n \in \mathbb{Z}\smallsetminus\{0\}$ imply $v \in I$), a fixed O-linearly independent family $v_1^I,\dots,v_\rho^I$ in I ($\rho \ge 1$) spanning a subgroup of finite index $\iota(I)$ in I, whenever such a family exists. For such I put

$$\mathrm{N}_w(I) = \frac{1}{\iota(I)} \prod_{\tau\in\Phi} \det\Big(\overline{\lambda_{w,\tau}(v_a^I)}^{\mathsf{T}} \lambda_{w,\tau}(v_b^I)\Big)_{a,b\le\rho},$$

and $\mathrm{N}_w(I) = 1$ if no such family exists. (In the intended case, (Gb5) of Definition 31.11, Λ_w is the first homology of the abelian variety at the point over w with its polarisation and its $\mathcal{O}_E$-action, $\varepsilon_{w,\tau}$ are the characters of $\mathcal{O}_E$ on its Lie algebra, which depend on the embedding of the coefficient field through which the point lies, and the saturated isotropic submodules are the cusps that the point can approach.)

(Hu2) Cusp weight and corrected weight The **cusp weight** is the function on $X^{\mathrm{b},\circ}$

$$U(w) = \max\big(0,\ s(w)\big), \qquad s(w) = \sup\{-\log \mathrm{N}_w(I) :\ I \subseteq \Lambda_w,\ I \ne 0 \text{ saturated isotropic}\},$$

where the supremum of a set that is not bounded above is read as 0. The **corrected weight** is $\log h^{\mathrm{H}} + \frac{1}{2}U$ on (the cone over) $X^{\mathrm{b},\circ}$. The cusp weight measures how close w is to a cusp; half of it is added to the Hodge weight so that point heights are bounded below up to the boundary.

(Hu3) Bounded regularisation and perturbed bundles A smooth cone function $h^{\mathrm{ref}'}$ of multidegree a^{H} and, for every real $t' > 0$, a smooth function $\psi_{t'}$ on X^{b} invariant under

complex conjugation (the **bounded regularisations** of the corrected weight, whose target on $X^{\mathrm{b},\circ}$ is $\log h^{\mathrm{H}} + \frac{1}{2}U - \log h^{\mathrm{ref}'}$). For an integer B, integers $1 \le i_1 \le i_2$ and reals $t, t' > 0$ the **corrected perturbed bundle** and the **log-log perturbed bundle** are the Hermitian line bundles with smooth metrics on $\mathcal{X}^{\mathrm{b}}$

$$\overline{\mathcal{M}}_{B,i_1,i_2,t'} = \Big(\mathcal{O}(i_2 a_B^{\mathrm{H}} + i_1 a'),\ (h^{\mathrm{ref}'} e^{\psi_{t'}} \textstyle\prod_{\ell \in S_B} |g_\ell|)^{i_2} h'^{i_1}\Big)\Big|_{\mathcal{X}^{\mathrm{b}}},$$
$$\overline{\mathcal{N}}_{B,i_1,i_2,t} = \Big(\mathcal{O}(i_2 a_B^{\mathrm{H}} + i_1 a'),\ (h^{\mathrm{ref}} e^{\varphi_t} \textstyle\prod_{\ell \in S_B} |g_\ell|)^{i_2} h'^{i_1}\Big)\Big|_{\mathcal{X}^{\mathrm{b}}}.$$

They carry the corrected and the uncorrected Hodge-type line, perturbed by $\frac{i_1}{i_2}$ times the auxiliary line, at the level of smooth metrics.

The **correction mean** is

$$\mathrm{m}^{\mathrm{b}} = \frac{1}{\deg^{\mathrm{H}}} \int_{X^{\mathrm{b},\circ}} U\,(\omega^{\mathrm{H}})^e,$$

the integral of U against the measure $(\omega^{\mathrm{H}})^e$ on $X^{\mathrm{b},\circ}$, read as 0 when U is not integrable for it (and defined when $\deg^{\mathrm{H}} \neq 0$). For a real $r \ge 2$ the cusp data have the **cusp mean bound** for r if

(Qk1) Cusp mean bound U is integrable for $(\omega^{\mathrm{H}})^e$ on $X^{\mathrm{b},\circ}$ and

$$\int_{X^{\mathrm{b},\circ}} U\,(\omega^{\mathrm{H}})^e \le |\Phi|\, j' \left(6 + \log(2|\Phi|r) + \log(2|\Phi| j' r)\right) \deg^{\mathrm{H}}.$$

This is the form in which the arithmetic of the host (through r, its root discriminant) bounds the average closeness to the cusps. For an integer B the **corrected regularised twisted line** at (B, t') is $\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'} = (\mathcal{O}(a_B^{\mathrm{H}}),\ h^{\mathrm{ref}'} e^{\psi_{t'}} \prod_{\ell \in S_B} |g_\ell|)|_{\mathcal{X}^{\mathrm{b}}}$, so that $\overline{\mathcal{M}}_{B,i_1,i_2,t'} = (\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'})^{\otimes i_2} \otimes \overline{\mathcal{A}}'^{\otimes i_1}$. For reals $r \ge 2$, L and $C_{\mathrm{h}} \ge 0$ the cusp data have **corrected point heights** at B if

(Qk2) Corrected point heights There is $t_0 > 0$ such that for $0 < t' \le t_0$ and the Zariski closure D_x of every $x \in \mathcal{X}^{\mathrm{b}}(\overline{\mathbb{Q}})$

$$h_{\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'}}(D_x) \ge -C_{\mathrm{h}}\, |\Phi|\, j' \left(\log(Br) + |L|\right) [\mathbf{k}(D_x) : \mathbb{Q}].$$

Here the arithmetic of the host enters through the Lie algebras of the abelian varieties at the algebraic points.

Definition 31.13. Let $\mathcal{T}$ be a root tower of type (Φ, n, κ), let $r \ge 1$ and L be real numbers, and let $C_{\mathrm{h}}, C_{\mathrm{u}} \ge 0$. Call an integer B **admissible** if $B \ge 128$, $B \ge 4|\Phi| n^2$ and $|\mathrm{Bd}| \le B$, and put $\mathrm{Lb}(B) = |\Phi| \min(\kappa, n - \kappa) \mathrm{Lg}(B)$. The tower has **base slope settings** for (r, L) with constants $(C_{\mathrm{h}}, C_{\mathrm{u}})$ if for every admissible B and every $Z \in \mathrm{Cp}$ there is a Hodge-type base setting of type (Φ, n, κ) with cusp data such that

(Bs0) Realisation We have $\deg^{\mathrm{H}} > 0$, and the limit self-intersection $\mathrm{ar}^{\mathrm{b}}(B)$ exists;

and we write $\mathcal{X}^{\mathrm{b}} = \mathcal{X}^{\mathrm{b}}_{B,Z}$ for its base variety (absolute dimension $e + 1$, $e = |\Phi|\kappa(n - \kappa)$), $\mathrm{sl}^{\mathrm{b}}(B, Z) = \mathrm{sl}^{\mathrm{b}}(B)$ for its base slope (31.7), $\mathrm{m}^{\mathrm{b}}(B, Z)$ for its correction mean, $\mathrm{ka}^{\mathrm{b}} = k^{\mathrm{A}}$ for its auxiliary floor, and $\overline{\mathcal{M}}_{i_1,i_2,t} = \overline{\mathcal{M}}_{B,i_1,i_2,t}$ $(1 \le i_1 \le i_2,\ t > 0)$ for its corrected perturbed bundles at B; and these satisfy:

(Bs1) Slope identity We have $\widehat{\deg}_{B,Z}(\mathrm{pl}^{D^{\mathrm{gr}}+1}\mathrm{fb}) = (D^{\mathrm{gr}} + 1)\,\mathrm{vol}(Z)\ \deg^{\mathrm{fl}}(\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb})\,\mathrm{sl}^{\mathrm{b}}(B, Z)$;

(Bs2) Positivity For all $1 \le i_1 \le i_2$ there is $t_0 > 0$ such that for $0 < t \le t_0$ the line bundle $\mathcal{M}_{i_1,i_2,t,\mathbb{Q}}$ is ample and $\overline{\mathcal{M}}_{i_1,i_2,t}$ is relatively semiample;

(Bs3) Comparison in the limit For every $a \in \mathbb{R}$ the following holds: if for all $1 \le i_1 \le i_2$ there is $t_0 > 0$ with

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_{i_1,i_2,t})^{e+1}) \ge (e+1)\, i_2\Big(a - \frac{i_1}{i_2}\mathrm{ka}^{\mathrm{b}}\Big)\deg(c_1(\mathcal{M}_{i_1,i_2,t,\mathbb{Q}})^e \cap [\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}]) \qquad (0 < t \le t_0),$$

then $\mathrm{sl}^{\mathrm{b}}(B,Z) \ge a - \frac{1}{2}\mathrm{m}^{\mathrm{b}}(B,Z)$;

(Bs4) Heights of points For all $1 \le i_1 \le i_2$ there is $t_0 > 0$ such that for $0 < t \le t_0$ and the Zariski closure D_x of every $x \in \mathcal{X}^{\mathrm{b}}(\overline{\mathbb{Q}})$

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_{i_1,i_2,t}) \cdot [D_x]) \ge i_2\Big(-C_{\mathrm{h}}\mathrm{Lb}(B) - \frac{i_1}{i_2}\mathrm{ka}^{\mathrm{b}}\Big)[\mathbf{k}(D_x) : \mathbb{Q}];$$

(Bs5) Mean of the correction We have $\mathrm{m}^{\mathrm{b}}(B,Z) \le C_{\mathrm{u}}\mathrm{Lb}(B)$.

Proposition 31.14. *There is a real number $C \ge 0$ with the following property. Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum such that E is Galois over $\mathbb{Q}$ with abelian Galois group, $q \ge 1$, $3q+1 \le p+q$, the level $\mathsf{L} = (N, \mathfrak{n}, A)$ is good at every prime and $A = 1$, the components split, and the number of primes dividing $2\,\mathrm{disc}(E)\,N$ is at most $4|\Phi_{\mathsf{H}}|(p+q)^2$. Put $n = p+q$, $\Phi = \Phi_{\mathsf{H}}$, $\mathrm{rd}_E = |\mathrm{disc}(E)|^{1/[E:\mathbb{Q}]}$ and $L_{\mathsf{H}} = \mathrm{Re}\,(L'/L)(1, \chi_d)$ for the quadratic character χ_d modulo d. Then there are objects as in (Gx) that satisfy (Gt1)–(Gt5):*

(Gx) Existence *an integral model $\mathcal{Y}$ of* Sh *over $\mathcal{O}_F$, a root tower $\mathcal{T}$ of type (Φ, n, p) with a geometric realisation and a good origin over $\mathcal{Y}$ immersive at level four, and a bijection* $\mathrm{cp}\colon \mathrm{Comp}(\mathrm{Sh}_F) \to \mathrm{Cp}$ *such that $\mathcal{Y}_{\mathrm{cp}(Z)}$ is the closure of $j(Z)$ in $\mathcal{Y}$ for every $Z \in \mathrm{Comp}(\mathrm{Sh}_F)$;*

(Gt1) Compact dual *$\mathcal{T}$ has (Fs): its compact-dual height is $h^{\mathrm{FS}}_{\Phi,n}$;*

(Gt2) Volume side *$\mathcal{T}$ has (Rt1), and the realised tower has (Gm1), the geometric inputs (Gi1)–(Gi4) and the rate of regularisation (Gi6);*

(Gt3) Stage change *the realised tower has linked stages (Sk1)–(Sk4), and $\mathcal{T}$ has Galois Frobenius labels (Fr) for H;*

(Gt4) The centre *the realised tower has the fibre-square identities (Fq1), (Fq2) and the wall inputs (Wn), (Wp), (Wg), (Wd), and $\mathcal{T}$ has Galois component symmetry (Ys0) for H;*

(Gt5) Quantitative properties *$\mathcal{T}$ has flat determinant classes at the base stage (Db) for $(r, L) = (\mathrm{rd}_E, L_{\mathsf{H}})$, and (Rt6) for $(r, L) = (\mathrm{rd}_E, L_{\mathsf{H}})$ with the constant C.*

Proof. Let C be the constant of Proposition 34.2, and let $\mathfrak{u}$, n, Φ, rd_E, L_{H} be as in the statement.

Step 1: an integral model with uniformised degenerations. From $q \ge 1$ and $3q+1 \le p+q$ we get $p \ge 2q+1 \ge 3$, so $p+q \ge 1$. As L is good at every prime and $A = 1$, Proposition 32.1 gives an integral model $\mathcal{Y}$ of Sh over $\mathcal{O}_F$ with uniformised degenerations.

Step 2: a good origin. As $p, q \ge 1$ and the components split, Proposition 33.1 gives a root tower $\mathcal{T}_0$ of type (Φ, n, p) with a geometric realisation and a good origin of the realised tower over $\mathcal{Y}$ immersive at level four.

Step 3: the facts. The datum satisfies every hypothesis of Proposition 34.2 (E abelian Galois, $q \geq 1$, $3q+1 \leq p+q$, level good at every prime with $A=1$, the components of Sh_F split, at most $4|\Phi|n^2$ primes dividing $2\,\mathrm{disc}(E)N$), $\mathcal{Y}$ has uniformised degenerations and $\mathcal{T}_0$ carries a good origin over $\mathcal{Y}$ immersive at level four. So there is a root tower $\mathcal{T}$ of type (Φ, n, p) with the components and bad primes of $\mathcal{T}_0$ which, with the geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ of $\mathcal{T}_0$, has the facts (Gt1)–(Gt5) of Proposition 34.2 with the constant C. The clauses (Go1)–(Go5) and the conditions "good" and "immersive at level four" involve the tower only through its components, its bad primes, its cut $\kappa = p$ and its geometric realisation, which $\mathcal{T}$ and $\mathcal{T}_0$ share; hence the good origin of $\mathcal{T}_0$ is a good origin of the realised tower $\mathcal{T}$ over $\mathcal{Y}$, immersive at level four, and $\mathfrak{u}$, $\mathcal{Y}$ and $\mathcal{T}$ with this realisation and good origin satisfy the hypotheses of Proposition 34.2. We now derive the properties (Gt1)–(Gt5) of the statement. Here $\kappa = p$ satisfies $1 \leq \kappa \leq n-1$ and $n - \kappa = q \leq \kappa$ (as $p \geq 2q+1$); moreover $n \geq 2$, and E is a CM field with $\mathcal{O}_E$ unramified over $\mathcal{O}_{E^+}$ at every finite place (Definition 29.5).

(Gt1). This is the fact (Gt1).

(Gt2). The realised tower has (Gi1) and (Vp) over E (fact (Gt2)); so by Lemma 31.17 $\mathcal{T}$ has a covolume realisation (Vu) over E with the same data, and by Theorem 31.18 $\mathcal{T}$ has (Rt1) and the realised tower has (Gm1). By Theorem 31.20, applied to $\mathcal{T}$ with its realisation and good origin, every root model $\mathcal{X}_{B,Z}$ has convergent regularisation; as $\widehat{\deg}_{B,Z}(F) = \mathrm{pp}(\mathcal{X}_{B,Z}; F)$ for all $B \geq 2$, $Z \in \mathrm{Cp}$ and $F \in \mathcal{R}$ (fact (Gt2)), part (3) of that theorem gives (Gi2) and the rate of regularisation (Gi6). By Theorem D.57 the realised tower has (Gi4). By Lemma 31.24 ($p, q \geq 1$, type $(\Phi_{\mathsf{H}}, p+q, p)$, a good origin over $\mathcal{Y}$), together with (Gi4), it has (Wn) and (Gi3).

(Gt3). The tower $\mathcal{T}$ has (Fr) for H (fact (Gt3)). With (Gi1), (Gi2), the good origin immersive at level four over $\mathcal{Y}$, and E Galois over $\mathbb{Q}$ with abelian Galois group, Theorem 35.2 gives linked stages (Sk1)–(Sk4).

(Gt4). The root models have convergent regularisation, and the realised tower has proportional analytic square systems (fact (Gt4)); as $\mathcal{T}$ has (Fs), Theorem D.85 gives (Fq1) and (Fq2). (Wn) was obtained under (Gt2). The good origin contains a tower origin, so Lemma 31.26 with (Gi1) and (Gi4) gives (Wg). By Theorem D.126 ((Gi2), (Gi4)) the realised tower has the reference link (Vl), and Theorem D.129 gives (Wd). For (Wp): by Theorem D.123 ((Gi2), (Gi4)) the realised tower satisfies (Rh1) with the frozen reference models $\mathcal{X}_c^{\mathrm{fr}}$ and some constants $C_1 \geq 1$, $s_{\mathrm{Rh}} \in (0,1]$. These models have Lie fibre data with the constant C_{Lx} of fact (Gt4): (Lx2) and (Lx3) are that fact, and (Lx1) holds because $\mathcal{O}(e_*)$ is ample on $\mathcal{X}_c^{\mathrm{ref}} \otimes \mathbb{Q}$ (the clause on reference models in a good origin) and $\mathcal{X}_c^{\mathrm{fr}}$ has the arithmetic variety and the multidegrees of $\mathcal{X}_c^{\mathrm{ref}}$. So Theorem D.127 ($1 \leq \kappa \leq n-1$) gives $C_0 \geq 1$ such that (Rh2) holds for the models $\mathcal{X}_c^{\mathrm{fr}}$ with every constant $\geq C_0$. Put $C_{\mathrm{Rh}} = \max(C_0, C_1)$. Both bounds of (Rh1) only become weaker when the constant grows, since $\log B + \log(1/s) + 1 > 0$ and $\log B + 1 > 0$ for $B \geq 2$ and $0 < s \leq 1$; so the realised tower has the reference comparison (Rh1), (Rh2) with the reference models $\mathcal{X}_c^{\mathrm{fr}}$ and the constants C_{Rh}, s_{Rh}, and Theorem D.131 gives (Wp). Finally, we prove (Ys0). Let $\sigma \in \Gamma_{\mathsf{H}}$ and $B \geq 2$. By the definition of Γ_{H} (Definition 30.29), $\sigma = \sigma_g$ for some $g \in \mathrm{Gal}(E/\mathbb{Q})$ with $g(r) = r$, and Proposition 31.36 gives an automorphism tw of the K_{L}-scheme Sh and an isomorphism $\psi\colon G \to \mathrm{tw}^* G$ as there. As $A = 1$, Lemma 31.15 shows that $Z \mapsto \overline{j(Z)}$ is a bijection from $\mathrm{Comp}(\mathrm{Sh}_F)$ onto the set of irreducible components of $\mathcal{Y}$, and by (Go5) so is $c \mapsto \mathcal{Y}_c$ from Cp. As $\mathcal{Y}$ is normal and noetherian, its local rings are domains, so every point of $\mathcal{Y}$ lies on exactly one irreducible component; hence $Z_c = j^{-1}(\mathcal{Y}_c)$ is the component Z of Sh_F with $\overline{j(Z)} = \mathcal{Y}_c$, and $c \mapsto Z_c$ is a bijection $\mathrm{Cp} \to \mathrm{Comp}(\mathrm{Sh}_F)$. The automorphism $\mathrm{tw}_F = \mathrm{tw} \times_{K_{\mathsf{L}}} F$ of Sh_F permutes its irreducible components, so there is a bijection $\pi\colon \mathrm{Cp} \to \mathrm{Cp}$ with $\mathrm{tw}_F(Z_{\pi(c)}) = Z_c$ for every $c \in \mathrm{Cp}$. Let $Z \in \mathrm{Cp}$. By Lemma 31.37, $\mathcal{X}_{B,\pi(Z)}$ and $\mathcal{X}_{B,Z}$ satisfy the hypotheses of Proposition 31.27 for the fixed data at $\pi(Z)$ and at Z with the same (S_B, R), $R(\ell) = 4$; so

its part 1 with $\mathcal{X}_1 = \mathcal{X}_{B,\pi(Z)}$, $\mathcal{X}_2 = \mathcal{X}_{B,Z}$, $c_1 = \pi(Z)$, $c_2 = Z$ and the twist (g, tw, ψ) gives $\mathrm{ai}_t(\mathcal{X}_{B,Z}; \sigma \cdot x^m) - \mathrm{ai}_t(\mathcal{X}_{B,\pi(Z)}; x^m) \to 0$ as $t \to 0^+$ for every monomial $x^m \in \mathcal{R}$ of degree $D+1$; as E is Galois over $\mathbb{Q}$ and the realised tower has (Gi2), Lemma 31.38 gives Galois component symmetry (Ys0) for H.

(Gt5). The tower $\mathcal{T}$ has (Db) for $(\mathrm{rd}_E, L_{\mathsf{H}})$ (fact (Gt5)). For (Rt6) we first show that $\mathrm{rd}_E \geq 2$: the CM host gives $r \in \mathcal{O}_E$ with $r^2 = -d$ for a prime $d \equiv 7 \pmod{16}$ (Definition 29.5); the field $K = \mathbb{Q}(r) \subseteq E$ has discriminant $-d$ (d is squarefree and $-d \equiv 1 \pmod 4$), so by the transitivity of discriminants $|\mathrm{disc}(E)| = d^{[E:K]}\, \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{d}_{E/K}) \geq d^{[E:K]}$, and $\mathrm{rd}_E \geq \sqrt{d} \geq \sqrt{7} > 2$. Let C_{h} be the constant of fact (Gt5), let B be admissible and $Z \in \mathrm{Cp}$, and take the Hodge-type base setting with cusp data given there; it satisfies (Bs0) and (Bs1). Its base variety has smooth generic fibre ((Hy1) of Definition 31.10), so (i) and Lemma 31.39(b) give (Bs2). By (Bs0), (ii) and the semipositivity in (i), Proposition 31.41 applies at B; its conclusion is (Bs3) (with $\mathrm{ka}^{\mathrm{b}} = k^{\mathrm{A}}$). As $n - \kappa = q \leq \kappa$ and $\mathrm{rd}_E \geq 2$, (Qk2) and Lemma 31.43 give (Bs4) with the constant C_{h}. As $\deg^{\mathrm{H}} > 0$ by (Bs0), $1 \leq n - \kappa \leq \kappa$, (Qk1) holds for $r = \mathrm{rd}_E \geq 2$, and $B \geq 128 \geq 64$ and $B \geq 4|\Phi| n^2 \geq 2|\Phi|(n-\kappa)$, Theorem 31.44 (with $L = L_{\mathsf{H}}$) gives (Bs5) with $C_{\mathrm{u}} = 4$. So $\mathcal{T}$ has base slope settings for $(\mathrm{rd}_E, L_{\mathsf{H}})$ with the constants $(C_{\mathrm{h}}, 4)$. It has (Rt1) and $\mathrm{rd}_E \geq 1$, so Theorem 31.45 gives (Rt6) for $(\mathrm{rd}_E, L_{\mathsf{H}})$ with the constant $C_{\mathrm{h}} + 2$. In (Rt6) the bound is $-C$ times $|\Phi| \min(\kappa, n-\kappa)\, \mathrm{Lg}(B)\, (D^{\mathrm{gr}} + 1)\, \mathrm{vol}_{\mathcal{T}} \deg^{\mathrm{fl}}(\mathrm{pl}^{D^{\mathrm{gr}}} \mathrm{fb})$, and this factor is positive: $\mathrm{Lg}(B) = \log(B\, \mathrm{rd}_E) + |L_{\mathsf{H}}| \geq \log 128$, the volumes are positive by (Rt1), and the test degree is positive by Proposition 29.36(a). So (Rt6) also holds with every larger constant, in particular with $C \geq C_{\mathrm{h}} + 2$.

Hence $\mathcal{T}$, with the geometric realisation of $\mathcal{T}_0$, has (Gt1)–(Gt5) of the statement with the constant C.

Step 4: the component bijection. The level has $A = 1$, so by Lemma 31.15 $Z \mapsto \overline{j(Z)}$ is a bijection from $\mathrm{Comp}(\mathrm{Sh}_F)$ onto the set of irreducible components of $\mathcal{Y}$; by (Go5) the tower origin contains a bijection $c \mapsto \mathcal{Y}_c$ from Cp onto the same set. Let $\mathrm{cp}(Z)$ be the unique $c \in \mathrm{Cp}$ with $\mathcal{Y}_c = \overline{j(Z)}$. Then $\mathrm{cp}\colon \mathrm{Comp}(\mathrm{Sh}_F) \to \mathrm{Cp}$ is a bijection and $\mathcal{Y}_{\mathrm{cp}(Z)}$ is the closure of $j(Z)$ in $\mathcal{Y}$ for every Z. This is (Gx), and by Step 3 $\mathcal{T}$ has (Gt1)–(Gt5). □

31.2 Components, volumes and convergence of intersection numbers

Three steps of the proof of Proposition 31.14 are proved here. First, the components: those of the moduli scheme are non-empty in number and correspond to the irreducible components of every integral model (Lemma 31.15). Second, the volumes: geometric proportionality with covolumes (Vp) gives (Gi1) and the covolume realisation (Vu) together (Lemma 31.17), and since principal congruence subgroups of lattices in one genus have equal covolume (Lemma 31.19), the volumes of all components are equal and positive, which is (Rt1) and (Gm1) (Theorem 31.18). Third, convergence: the arithmetic intersection numbers of the regularised metrics converge, at a controlled rate, as the regularisation is removed ((Gi2), (Gi6), Theorem 31.20). The analytic input is a uniform log-weighted bound for the mass of mixed Monge–Ampère measures of the regularised weights near the boundary (Proposition 31.22, Lemma 31.23). The subsection ends with uniform semipositivity, which gives (Wn) and (Gi3) from nefness at primes (Lemma 31.24, Lemma 31.25), and the geometric degree bound (Wg) (Lemma 31.26).

Lemma 31.15. *Let* $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ *be a unitary Shimura datum whose level* $\mathsf{L} = (N, \mathfrak{n}, A)$ *has* $A = 1$*, and let* $\mathcal{Y}$ *be an integral model of* Sh *over* $\mathcal{O}_F$*. Then* $\mathrm{Comp}(\mathrm{Sh}_F)$ *is finite and non-empty, and* $Z \mapsto \overline{j(Z)}$ *(closure in* $\mathcal{Y}$*) is a bijection from* $\mathrm{Comp}(\mathrm{Sh}_F)$ *onto the set of irreducible components of* $\mathcal{Y}$*.*

Proof. Step 1: Sh_F *is non-empty.* Write $\mathsf{U} = (M, \langle\,,\rangle)$ and let J be as in (Ud2). Since $\langle Jx, Jy\rangle = \langle x, y\rangle$ and $\langle Jx, x\rangle > 0$ for $x \neq 0$, the form $\langle\,,\rangle$ is a Riemann form on the complex torus $G = M_{\mathbb{R}}/M$ (complex structure J), unimodular on M; so G is an abelian variety over $\mathbb{C}$ with a principal polarisation λ [186]. Each $a \in \mathcal{O}_E$ acts on $M_{\mathbb{R}}$ commuting with J and preserving M; this gives $\iota\colon \mathcal{O}_E \to \mathrm{End}(G)$, and $\langle ax, y\rangle = \langle x, \bar{a}y\rangle$ is the Rosati condition $\lambda\iota(\bar{a}) = \iota(a)^\vee\lambda$. As $r^2 = -d < 0$, every $\tau(r)$ is purely imaginary, so $\bar{r} = -r$ and $w_0 + \bar{w}_0 = 1$. By Lemma 31.16 (over $\operatorname{Spec}\mathbb{C}$) there is a rigidified symmetric cubical relatively ample line bundle $\mathcal{L}$ on G with $\phi_{\mathcal{L}} = \lambda$ and the two isomorphisms of (Ud4). *Kottwitz condition.* The Lie algebra of G is $(M_{\mathbb{R}}, J)$. For a $\mathbb{C}$-linear ϕ one has $\mathrm{tr}_{\mathbb{R}}\phi = 2\operatorname{Re}\mathrm{tr}_{\mathbb{C}}\phi$ and $\mathrm{tr}_{\mathbb{R}}(J\phi) = -2\operatorname{Im}\mathrm{tr}_{\mathbb{C}}\phi$, so (Ud2) says $\mathrm{tr}_{\mathbb{C}}(a) = \sum_{\tau\in\Phi_{\mathsf{H}}}(p\,\tau(a) + q\,\overline{\tau(a)})$ for $a \in \mathcal{O}_E$. Since $E \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod_\tau \mathbb{C}$, $\mathrm{Lie}(G) = \bigoplus_\tau V_\tau$ with a acting on V_τ by $\tau(a)$, and $\mathrm{tr}_{\mathbb{C}}(a) = \sum_\tau \dim V_\tau\,\tau(a)$. Every τ has $\tau(r) = \pm r_{\mathbb{C}}$, and $\bar{\tau}(r) = -\tau(r)$, so $\mathrm{Hom}(E, \mathbb{C})$ is the disjoint union of Φ_{H} and its conjugate; by the linear independence of the characters τ on $\mathcal{O}_E$, $\dim V_\tau = p$ if $\tau(r) = r_{\mathbb{C}}$ and q otherwise. For an algebraically closed $k \supseteq \mathbb{C}$ the ring maps $\mathcal{O}_E \to k$ take values in $\overline{\mathbb{Q}} \subset \mathbb{C}$, and eigenspaces commute with base change; so the Kottwitz condition holds at every geometric point of $\operatorname{Spec}\mathbb{C}$ (with β the inclusion $K_{\mathsf{L}} \subset \mathbb{C}$ followed by $\mathbb{C} \subseteq k$). *Level structures.* One has $G[n](\mathbb{C}) = \frac{1}{n}M/M$, and this is $G[n](k)$ for every algebraically closed $k \supseteq \mathbb{C}$ ($G[n]$ is finite étale). The Weil pairing of λ is $e_n(x/n, y/n) = \exp(2\pi i\varepsilon\langle x, y\rangle/n)$ for a sign $\varepsilon \in \{\pm 1\}$ fixed by the normalisation of e_n [186]. Put $\zeta_n = \exp(2\pi i\varepsilon/n)$ and $I_n(a) = a/n \bmod M$: then $\zeta_{nm}^m = \zeta_n$, $\zeta_1 = 1 = e^{2\pi i/A}$ (as $A = 1$), I_n is $\mathcal{O}_E$-linear, $mI_{nm} = I_n$ and $e_n(I_n(a), I_n(b)) = \zeta_n^{\langle a,b\rangle}$. Let $\eta_A = 0$ (so $\eta_A(AM) = 0$ and $\eta_A(a) = I_1(a)$). Since $\mathfrak{n} + \bar{\mathfrak{n}} = \mathcal{O}_E$ and $\mathfrak{n}\bar{\mathfrak{n}} = N\mathcal{O}_E$, and M is projective, $M = \mathfrak{n}M + \bar{\mathfrak{n}}M$ and $\mathfrak{n}M \cap \bar{\mathfrak{n}}M = (\mathfrak{n}\cap\bar{\mathfrak{n}})M = NM$; so $\eta(x+y) = I_N(y)$ ($x \in \mathfrak{n}M$, $y \in \bar{\mathfrak{n}}M$) is a well-defined additive map with $\eta(\mathfrak{n}M) = 0$ and $\eta = I_N$ on $\bar{\mathfrak{n}}M$. So $(G, \iota, \mathcal{L}, \eta, \eta_A)$ is an object of type (U, L) over the reduced K_{L}-scheme $\operatorname{Spec}\mathbb{C}$, and by (Ud5)(i) it is the pull-back of the universal object along a K_{L}-morphism $x\colon \operatorname{Spec}\mathbb{C} \to \mathrm{Sh}$. As $K_{\mathsf{L}} \subseteq F \subset \mathbb{C}$, x and the inclusion $F \subset \mathbb{C}$ give a point of $\mathrm{Sh}_F = \mathrm{Sh} \times_{K_{\mathsf{L}}} F$. Hence $\mathrm{Sh}_F \neq \varnothing$, and $\mathrm{Comp}(\mathrm{Sh}_F) \neq \varnothing$.

Step 2: the bijection. The scheme $\mathcal{Y}$ is projective over $\mathcal{O}_F$, hence noetherian, with finitely many irreducible components. Let ξ be the generic point of one of them. Since $\mathcal{Y} \to \operatorname{Spec}\mathcal{O}_F$ is flat it satisfies going-down, so the image of the minimal point ξ is a minimal point of $\operatorname{Spec}\mathcal{O}_F$, the generic point: ξ lies in the generic fibre $\mathcal{Y}_F$, where it is a minimal point (a generisation in $\mathcal{Y}_F$ is one in $\mathcal{Y}$). Conversely every minimal point of $\mathcal{Y}_F$ is minimal in $\mathcal{Y}$, as every generisation in $\mathcal{Y}$ of a point over the generic point lies over the generic point. So the generic points of $\mathcal{Y}$ are those of $\mathcal{Y}_F$. The open set $j(\mathrm{Sh}_F)$ is dense in the noetherian space $\mathcal{Y}_F$, so it meets every irreducible component of $\mathcal{Y}_F$ and contains its generic point; and the generic points of $\mathcal{Y}_F$ lying in the open subscheme $j(\mathrm{Sh}_F) \cong \mathrm{Sh}_F$ are exactly the generic points of the components of Sh_F. Hence $Z \mapsto$ (generic point of $j(Z)$) is a bijection from $\mathrm{Comp}(\mathrm{Sh}_F)$ onto the generic points of $\mathcal{Y}$, and taking closures in $\mathcal{Y}$ gives the bijection of the statement; in particular $\mathrm{Comp}(\mathrm{Sh}_F)$ is finite. □

Lemma 31.16. *Let S be a $\mathbb{Q}$-scheme, let E be a CM field, and let $\iota : \mathcal{O}_E \to \mathrm{End}_S(A)$ be a unital action on an abelian scheme. Fix $w_0 \in \mathcal{O}_E$ with $w_0 + \bar{w}_0 = 1$, and put $u = \iota(w_0)$. If $\phi_{\mathrm{pol}} : A \to A^\vee$ is a principal polarisation with*

$$\iota(a)^\vee\phi_{\mathrm{pol}} = \phi_{\mathrm{pol}}\iota(\bar{a}) \quad (a \in \mathcal{O}_E),$$

then

$$\mathcal{L}_{\phi_{\mathrm{pol}}} = (\mathrm{id}_A, u)^*(\mathrm{id}_A \times\phi_{\mathrm{pol}})^*\mathcal{P}_A$$

is a canonically rigidified symmetric cubical line bundle, relatively ample, with $\phi_{\mathcal{L}_{\phi_{\mathrm{pol}}}} = \phi_{\mathrm{pol}}$. It satisfies normalized isomorphisms

$$(\iota(a), 1)^*\Lambda(\mathcal{L}_{\phi_{\mathrm{pol}}}) \simeq (1, \iota(\bar{a}))^*\Lambda(\mathcal{L}_{\phi_{\mathrm{pol}}}), \qquad \mathcal{L}_{\phi_{\mathrm{pol}}} \simeq (1, u)^*\Lambda(\mathcal{L}_{\phi_{\mathrm{pol}}}).$$

Here 1 *in a pair denotes the identity morphism of* A*.*

Conversely, every rigidified relatively ample line bundle $\mathcal{M}$ *with principal polarisation satisfying these adjoint and primitive isomorphisms is uniquely isomorphic, as a rigidified cubical line bundle, to* $\mathcal{L}_{\phi_{\mathcal{M}}}$*; its polarisation satisfies the displayed Rosati relation. This gives an equivalence between the corresponding polarised abelian schemes and those carrying such primitive line bundles. The equivalence commutes with arbitrary base change and equivariant isomorphisms.*

Proof. Write $\mathcal{P} = (\mathrm{id}_A \times \phi_{\mathrm{pol}})^* \mathcal{P}_A$. Its zero-section normalizations rigidify the graph pullback. The unique cubical structure is supplied by Theorem B.103. Bilinearity gives

$$\Lambda(\mathcal{L}_{\phi_{\mathrm{pol}}}) \simeq (\mathrm{pr}_1, u\,\mathrm{pr}_2)^* \mathcal{P} \otimes (\mathrm{pr}_2, u\,\mathrm{pr}_1)^* \mathcal{P}.$$

The normalized biextension on the right represents the homomorphism

$$\phi_{\mathrm{pol}} u + u^\vee \phi_{\mathrm{pol}} = \phi_{\mathrm{pol}}\big(\iota(w_0) + \iota(\bar{w}_0)\big) = \phi_{\mathrm{pol}}.$$

The Poincaré universal property therefore gives the normalized identification

$$\Lambda(\mathcal{L}_{\phi_{\mathrm{pol}}}) \simeq \mathcal{P}.$$

In particular its associated homomorphism is the original polarisation, so Theorem B.103 proves relative ampleness.

Bilinearity identifies simultaneous inversion of both arguments of $\mathcal{P}$ with $\mathcal{P}$. Its graph pullback gives a symmetry of $\mathcal{L}_{\phi_{\mathrm{pol}}}$. Normalize it at the zero section. Its square is the identity, and its compatibility with the cubical structure follows from uniqueness. The Rosati relation identifies the normalized biextensions $(\iota(a), 1)^* \mathcal{P}$ and $(1, \iota(\bar{a}))^* \mathcal{P}$. This gives the adjoint isomorphism. Pulling $\Lambda(\mathcal{L}_{\phi_{\mathrm{pol}}}) \simeq \mathcal{P}$ back along $(1, u)$ gives the primitive isomorphism.

For the converse, Theorem B.103 identifies $\Lambda(\mathcal{M})$ with $\mathcal{P}_{\phi_{\mathcal{M}}}$. The adjoint isomorphisms imply $\iota(a)^\vee \phi_{\mathcal{M}} = \phi_{\mathcal{M}} \iota(\bar{a})$ by classification of normalized biextensions. An isomorphism of line bundles can be normalized using their zero-section rigidifications. Primitiveness now gives

$$\mathcal{M} \simeq (1, u)^* \mathcal{P}_{\phi_{\mathcal{M}}} = \mathcal{L}_{\phi_{\mathcal{M}}}.$$

This normalized isomorphism is unique: an automorphism of a line bundle on an abelian scheme is multiplication by a unit pulled back from the base, since the pushforward of its structure sheaf is $\mathcal{O}_S$, and normalization forces that unit to be 1. It is cubical by uniqueness of the compatible cubical structure. Thus uniqueness is asserted for primitive representatives; no uniqueness of all symmetric representatives is used.

Finally, the normalized Poincaré bundle, graph pullbacks and their unique normalized isomorphisms commute with base change. If $f : A \to A'$ is equivariant and $f^\vee \phi'_{\mathrm{pol}} f = \phi_{\mathrm{pol}}$, their universal properties give $f^* \mathcal{L}_{\phi'_{\mathrm{pol}}} \simeq \mathcal{L}_{\phi_{\mathrm{pol}}}$ with every stated structure. This proves the equivalence. □

Lemma 31.17. *Let* E *be a CM field and* $\mathcal{T}$ *a root tower of type* (Φ, n, κ)*,* $n \geq 2$*, with a geometric realisation that has (Vp) over* E*, with data* $(V, h), \mathfrak{a}, m, c_0, (\Lambda_{Z,i})$*. For* $Z \in \mathrm{Cp}$ *put*

$$v(Z) = c_0^{-1} \sum_{i=1}^{m} \mu_\infty\big(\Gamma(\Lambda_{Z,i}, \mathfrak{a}) \backslash G^1_\infty(V)\big).$$

Then $0 \leq v(Z) < \infty$*, and the realised tower has (Gi1) if and only if* $\mathrm{vol}(Z) = v(Z)$ *for every* $Z \in \mathrm{Cp}$*. In that case it has (Vu) over* E*, with the same data.*

Proof. **A class of non-zero flag degree.** Let u be a weight with $u_{a,1} > u_{a,2} > \cdots > u_{a,n}$ for every $a \in \Phi$ (for instance $u_{a,i} = n - i$). By Weyl's formula (Lemma 30.14) $\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D) = D! \prod_a \prod_{i<i'} (u_{a,i} - u_{a,i'})/(i' - i) > 0$. Put $F_0 = \mathrm{lr}(u)^D$, homogeneous of degree D.

Finiteness. Each covolume lies in $(0, \infty]$. Applying (Vp) to F_0 at any $B \geq 2$ and Z: the left side $c_0 \,\mathrm{gi}(\mathcal{X}_{B,Z}; F_0)$ is a real number and $\deg^{\mathrm{fl}}(F_0) > 0$, so the sum of the covolumes is finite; hence $0 \leq v(Z) < \infty$ and (Vp) reads $\mathrm{gi}(\mathcal{X}_{B,Z}; F) = v(Z) \deg^{\mathrm{fl}}(F)$ for all B, Z and all homogeneous F of degree D.

The equivalence. If $\mathrm{vol}(Z) = v(Z)$ for all Z, the last identity is (Gi1). Conversely, if (Gi1) holds, then at F_0 and any B: $\mathrm{vol}(Z) \deg^{\mathrm{fl}}(F_0) = \mathrm{gi}(\mathcal{X}_{B,Z}; F_0) = v(Z) \deg^{\mathrm{fl}}(F_0)$, and dividing by $\deg^{\mathrm{fl}}(F_0) > 0$ gives $\mathrm{vol}(Z) = v(Z)$.

(Vu). If $\mathrm{vol}(Z) = v(Z)$ for every Z, then $c_0 \,\mathrm{vol}(Z) = \sum_i \mu_\infty(\Gamma(\Lambda_{Z,i}, \mathfrak{a}) \backslash G^1_\infty(V))$ with (V, h) of dimension n indefinite at some real place, $\mathfrak{a}$ neat and all $\Lambda_{Z,i}$ in one genus: this is (Vu). □

Theorem 31.18. *Let E be a CM field and $\mathcal{T}$ a root tower of type (Φ, n, κ), $n \geq 2$, with a covolume realisation (Vu) over E. Then $\mathrm{vol}(Z) = \mathrm{vol}(Z') > 0$ for all $Z, Z' \in \mathrm{Cp}$; in particular $\mathcal{T}$ has (Rt1), and every geometric realisation of $\mathcal{T}$ has (Gm1).*

Proof. Let (V, h), $\mathfrak{a}$, m, c_0 and the lattices $\Lambda_{Z,i}$ be as in (Vu), and write $G = G^1_\infty(V)$ with its Haar measure μ_∞. Fix $Z_0 \in \mathrm{Cp}$ (the set Cp is non-empty) and put $w = \mu_\infty(\Gamma(\Lambda_{Z_0,1}, \mathfrak{a}) \backslash G)$.

All covolumes equal w. For every $Z \in \mathrm{Cp}$ and $1 \leq i \leq m$ the lattices $\Lambda_{Z,i}$ and $\Lambda_{Z_0,1}$ lie in one genus, V has dimension $n \geq 2$ and is indefinite at some real place of E^+, and $\mathfrak{a}$ is neat; so Lemma 31.19 (c) gives $\mu_\infty(\Gamma(\Lambda_{Z,i}, \mathfrak{a}) \backslash G) = w$.

Conclusion. By (Vu), $c_0 \,\mathrm{vol}(Z) = m\,w$ for every $Z \in \mathrm{Cp}$. Since $\mathrm{vol}(Z)$ is a real number and $c_0 > 0$, w is finite. It is positive: if F is a measurable fundamental domain for $\Gamma = \Gamma(\Lambda_{Z_0,1}, \mathfrak{a})$, then $G = \bigcup_{\gamma \in \Gamma} \gamma F$ is a countable union (Γ is discrete in the second countable group G, Lemma 31.19 (a)), so $\mu_\infty(F) > 0$. Hence $\mathrm{vol}(Z) = mw/c_0 > 0$ is the same number for all $Z \in \mathrm{Cp}$. This is (Rt1), and since (Gm1) asserts only $\mathrm{vol}(Z) = \mathrm{vol}(Z')$, it holds for every geometric realisation of $\mathcal{T}$. □

Lemma 31.19. *Let E be a CM field, (V, h) a hermitian space over E of dimension $n \geq 2$ that is indefinite at some real place of E^+, and $\mathfrak{a}$ a non-zero ideal of $\mathcal{O}_E$.*

(d) If $\mathfrak{a} \subseteq N_0 \mathcal{O}_E$ for an integer $N_0 \geq 3$, then $\mathfrak{a}$ is neat.

Assume that $\mathfrak{a}$ is neat, let Λ, Λ' be lattices of V in one genus, and put $\Gamma = \Gamma(\Lambda, \mathfrak{a})$, $\Gamma' = \Gamma(\Lambda', \mathfrak{a})$. Then:

(a) every $\gamma \in \Gamma$ has determinant 1, and Γ, embedded diagonally, is a discrete subgroup of $G^1_\infty(V)$;

(b) $\Gamma \cap \Gamma'$ has finite index in Γ and in Γ', and $[\Gamma : \Gamma \cap \Gamma'] = [\Gamma' : \Gamma \cap \Gamma']$;

(c) $\mu_\infty(\Gamma \backslash G^1_\infty(V)) = \mu_\infty(\Gamma' \backslash G^1_\infty(V))$.

Proof. Part (d). Let $\zeta \in E$ be a root of unity of order $m > 1$ with $\zeta - 1 \in \mathfrak{a}$, so $1 - \zeta = N_0 \alpha$ with $\alpha \in \mathcal{O}_E$, $\alpha \neq 0$. Taking norms, $N_0^{[E:\mathbb{Q}]}$ divides $\mathrm{N}_{E/\mathbb{Q}}(1 - \zeta) = \mathrm{N}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(1 - \zeta)^{[E:\mathbb{Q}(\zeta)]} = \Phi_m(1)^{[E:\mathbb{Q}(\zeta)]}$, where $\Phi_m(1) = \ell$ if $m = \ell^k$ is a power of a prime ℓ and $\Phi_m(1) = 1$ otherwise. In the second case $N_0^{[E:\mathbb{Q}]} \mid 1$, impossible. In the first, $N_0 = \ell^a$ with $a \geq 1$ and $a[E : \mathbb{Q}] \leq [E : \mathbb{Q}(\zeta)] = [E : \mathbb{Q}]/\varphi(\ell^k)$, so $a\varphi(\ell^k) \leq 1$, hence $a = 1$ and $\ell^k = 2$, i.e. $N_0 = 2$, against $N_0 \geq 3$.

Part (a). Let $\gamma \in \Gamma$. The lattice Λ is a finitely generated torsion-free module over the Dedekind ring $\mathcal{O}_E$, so for every prime $\mathfrak{p}$ the localisation $\Lambda_\mathfrak{p}$ is free of rank n; in a basis of $\Lambda_\mathfrak{p}$ the matrix of γ has entries in $\mathcal{O}_{E,\mathfrak{p}}$ and is congruent to the identity modulo $\mathfrak{a}\mathcal{O}_{E,\mathfrak{p}}$, because $(\gamma-1)\Lambda \subseteq \mathfrak{a}\Lambda$. Hence $\det\gamma \in \mathcal{O}_{E,\mathfrak{p}}$ and $\det\gamma - 1 \in \mathfrak{a}\mathcal{O}_{E,\mathfrak{p}}$ for every $\mathfrak{p}$, so $\det\gamma \in \mathcal{O}_E$ and $\det\gamma - 1 \in \mathfrak{a}$. If A is the matrix of γ and H the Gram matrix of h in an E-basis of V, unitarity reads $A^{\mathsf{T}} H\, c(A) = H$, so $\det\gamma \cdot c(\det\gamma) = 1$. For every embedding $\sigma\colon E \to \mathbb{C}$ we have $\sigma\circ c = \overline{\sigma}$ (E is a CM field), hence $|\sigma(\det\gamma)|^2 = \sigma(\det\gamma\, c(\det\gamma)) = 1$. By Kronecker's theorem the algebraic integer $\det\gamma$, all of whose conjugates have absolute value 1, is a root of unity; as $\det\gamma - 1 \in \mathfrak{a}$ and $\mathfrak{a}$ is neat, $\det\gamma = 1$. So $\Gamma \subseteq \mathrm{SU}(V)(E^+) \subseteq G^1_\infty(V)$. The group $G^1_\infty(V)$ is a closed subgroup of $\mathrm{GL}(V\otimes_\mathbb{Q}\mathbb{R}) = \mathrm{GL}(\Lambda\otimes_\mathbb{Z}\mathbb{R})$ and $\Gamma \subseteq \mathrm{GL}(\Lambda)$, which is discrete there (its elements have integral matrices in a $\mathbb{Z}$-basis of Λ); so Γ is discrete in $G^1_\infty(V)$. The same holds for Γ'.

Set-up for (b). Let $\mathsf{H}_1 = \mathrm{SU}(V)$, a linear algebraic group over E^+. Over an algebraic closure $\overline{E^+}$ we have $E \otimes_{E^+} \overline{E^+} \cong \overline{E^+} \times \overline{E^+}$ and H_1 becomes SL_n; so H_1 is connected, simply connected and absolutely almost simple ($n \geq 2$). At a real place w where V is indefinite there are $e, f \in V \otimes_{E^+,w} \mathbb{R}$ with $h(e,e) = 1$, $h(f,f) = -1$, $h(e,f) = 0$; the elements acting by $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ on $\langle e, f\rangle$ and trivially on its orthogonal complement lie in $\mathsf{H}_1(E^+_w)$ and form an unbounded subgroup, so $\prod_{w\ \mathrm{real}} \mathsf{H}_1(E^+_w)$ is not compact. By Theorem B.159 (with $K = E^+$ and S the set of archimedean places, all real), $\mathsf{H}_1(E^+)$ is dense in $\mathsf{H}_1(\mathbb{A}_f)$, $\mathbb{A}_f$ the finite adeles of E^+.

For a finite place v of E^+ let $K_v = \{g \in \mathsf{H}_1(E^+_v) : g\Lambda_v = \Lambda_v,\ (g-1)\Lambda_v \subseteq \mathfrak{a}\Lambda_v\}$, an open compact subgroup of $\mathsf{H}_1(E^+_v)$, and K'_v likewise with Λ'. For almost all v, $\Lambda_v = \Lambda'_v$, $\mathfrak{a}$ is prime to v, and $K_v = K'_v$ is the stabiliser of Λ_v; so $K = \prod_v K_v$ and $K' = \prod_v K'_v$ are open compact subgroups of $\mathsf{H}_1(\mathbb{A}_f)$, and the set T of places with $K_v \neq K'_v$ is finite. Since $\Lambda = V \cap \bigcap_v \Lambda_v$ and $\mathfrak{a}\Lambda = V \cap \bigcap_v \mathfrak{a}\Lambda_v$, an element $\gamma \in \mathsf{H}_1(E^+)$ lies in Γ if and only if it lies in K; with (a), $\Gamma = \mathsf{H}_1(E^+) \cap K$, and likewise $\Gamma' = \mathsf{H}_1(E^+) \cap K'$, $\Gamma \cap \Gamma' = \mathsf{H}_1(E^+) \cap K \cap K'$.

Part (b), indices. The map $\Gamma/(\Gamma\cap\Gamma') \to K/(K\cap K')$, $\gamma(\Gamma\cap\Gamma') \mapsto \gamma(K\cap K')$, is well defined and injective, because for $\gamma_1, \gamma_2 \in \Gamma$ the element $\gamma_1^{-1}\gamma_2$ lies in $K \cap K'$ if and only if it lies in $\Gamma\cap\Gamma'$. It is surjective: for $k \in K$ the coset $k(K\cap K')$ is open in $\mathsf{H}_1(\mathbb{A}_f)$, so by density it contains some $\gamma \in \mathsf{H}_1(E^+)$; then $\gamma \in K$, so $\gamma \in \Gamma$, and $\gamma(K\cap K') = k(K\cap K')$. As K is compact and $K \cap K'$ open, $[\Gamma : \Gamma\cap\Gamma'] = [K : K\cap K'] < \infty$, and by symmetry $[\Gamma' : \Gamma\cap\Gamma'] = [K' : K\cap K']$. Both indices are products over $v \in T$ of local indices:

$$[K : K\cap K'] = \prod_{v\in T} [K_v : K_v \cap K'_v], \qquad [K' : K\cap K'] = \prod_{v\in T} [K'_v : K_v \cap K'_v].$$

Part (b), local equality. Fix $v \in T$ and $g_v \in \mathrm{U}(V)(E^+_v)$ with $\Lambda'_v = g_v\Lambda_v$ (one genus). Then $K'_v = g_v K_v g_v^{-1}$: conjugation by g_v preserves the normal subgroup $\mathsf{H}_1(E^+_v)$, and gkg^{-1} ($g = g_v$) stabilises $g\Lambda_v$ with $(gkg^{-1}-1)g\Lambda_v = g(k-1)\Lambda_v \subseteq \mathfrak{a}\, g\Lambda_v$, and conversely. Let μ_v be a Haar measure on $\mathsf{H}_1(E^+_v)$. The automorphism $x \mapsto g_v x g_v^{-1}$ of the v-adic Lie group $\mathsf{H}_1(E^+_v)$ multiplies μ_v by $|\det(\mathrm{Ad}(g_v)|_{\mathrm{Lie}\,\mathsf{H}_1})|_v$ ([50, Ch. III, §3, no. 16]). This determinant is 1: over $\overline{E^+}$, $\mathrm{Lie}\,\mathrm{U}(V)$ becomes $\mathfrak{gl}_n \cong V \otimes V^*$ with $\mathrm{Ad}(g) = g \otimes (g^{-1})^*$, of determinant $\det(g)^n \det(g)^{-n} = 1$; and $\mathrm{Lie}\,\mathrm{U}(V) = \mathrm{Lie}\, Z \oplus \mathrm{Lie}\,\mathsf{H}_1$ with Z the centre (scalars), on which $\mathrm{Ad}(g)$ is the identity, while $\mathrm{Ad}(g)$ preserves $\mathrm{Lie}\,\mathsf{H}_1$. Hence $\mu_v(K'_v) = \mu_v(K_v)$, and

$$[K_v : K_v \cap K'_v] = \frac{\mu_v(K_v)}{\mu_v(K_v \cap K'_v)} = \frac{\mu_v(K'_v)}{\mu_v(K_v \cap K'_v)} = [K'_v : K_v \cap K'_v].$$

Taking the product over T gives $[\Gamma : \Gamma\cap\Gamma'] = [\Gamma' : \Gamma\cap\Gamma']$.

Part (c). Write $G = G^1_\infty(V)$, $\Gamma'' = \Gamma \cap \Gamma'$, $d = [\Gamma : \Gamma''] = [\Gamma' : \Gamma'']$, and $\Gamma = \bigsqcup_{j=1}^{d} \Gamma''\gamma_j$. If F is a measurable fundamental domain for Γ in G (one exists, Γ being discrete in the second countable group G), then $\bigcup_j \gamma_j F$ is one for Γ'': it meets every Γ''-orbit since $G = \bigcup_{\delta\in\Gamma''}\bigcup_j \delta\gamma_j F$, and $\delta\gamma_j F \cap \delta'\gamma_k F \neq \varnothing$ forces $\delta\gamma_j = \delta'\gamma_k$, hence $j = k$ and $\delta = \delta'$. By left invariance $\mu_\infty(\Gamma''\backslash G) = d\,\mu_\infty(\Gamma\backslash G)$, and likewise $\mu_\infty(\Gamma''\backslash G) = d\,\mu_\infty(\Gamma'\backslash G)$. As $1 \le d < \infty$, the two covolumes are equal (in $(0,\infty]$). □

Theorem 31.20. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p) with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,c}))$ and a good origin over an integral model $\mathcal{Y}$ of a unitary Shimura datum. Then:*

1. *every $\mathcal{X}_{B,c}$ ($B \ge 2$, $c \in \mathrm{Cp}$) has convergent regularisation;*
2. *for every $A \ge 0$ there are $s_0 > 0$ and $C_{\mathrm{rt}} \ge 0$ such that for all integers $B \ge 2$, all $c \in \mathrm{Cp}$, all weights λ with $\|\lambda\|_1 \le A$ and all $0 < s \le s_0$, with $\ell = \mathrm{lr}(\lambda_s) + s\,x_*$,*
$$\left|\mathrm{ai}_{t(s)}(\mathcal{X}_{B,c}; \ell^{D+1}) - \mathrm{pp}(\mathcal{X}_{B,c}; \ell^{D+1})\right| \le C_{\mathrm{rt}}\, s;$$
3. *in particular, if $\widehat{\deg}_{B,c}(F) = \mathrm{pp}(\mathcal{X}_{B,c}; F)$ for all B, c and $F \in \mathcal{R}$, the realised tower has (Gi2) and the rate of regularisation (Gi6).*

Proof. Step 1: (1) and the rate for monomials. Fix $c \in \mathrm{Cp}$ and let $t_2(c)$, $C_2(c)$ be the constants of Lemma 31.21 for the analytic system Sys_c of the fixed data at c; it has $\varphi_* = 0 = \varphi_*^T(t)$ by (Go2). Let $B \ge 2$ and $\mathcal{X} = \mathcal{X}_{B,c}$, with the realisation $\pi\colon \mathcal{X}(\mathbb{C}) \to X_c^{\mathrm{an}}$ of (Go4). The hypotheses of Lemma 31.21 hold with $d = \deg_{B,c}$: $\mathcal{X}_{\mathbb{Q}}$ is smooth because the origin is good (Go5), so π is defined on $\mathcal{X}(\mathbb{C})$ itself; by (Go4) π is holomorphic of degree $\deg_{B,c}$ off a closed nowhere dense analytic subset, which is hypothesis 1 (closed nowhere dense analytic subsets are null); the cone functions h_a^{ref} of (Go4) have the multidegrees of the fundamental indices, their metrics are the pull-backs of those of Sys_c and $h_*^{\mathrm{ref}} = h_*$, which is hypothesis 2; and the regularised metrics of (Go4) are the root metrics, which is hypothesis 3. Hence $\mathcal{X}_{B,c}$ has convergent regularisation, which is (1), and since $\delta_{\mathcal{X}} = [F : \mathbb{Q}]\deg_{B,c} \ge \deg_{B,c}$ by (Go3)(1),
$$\left|\mathrm{ai}_t(\mathcal{X}_{B,c}; x^m) - \mathrm{pp}(\mathcal{X}_{B,c}; x^m)\right| \le C_2(c)\,\mathrm{rt}(t) \qquad (0 < t \le t_2(c))$$
for every monomial $x^m \in \mathcal{R}^+$; the constants do not depend on B or x^m.

Step 2: (2). Let $C_3 = \max_c C_2(c)$ and $t_3 = \min_c t_2(c)$ over the finite set Cp, and $s_0 = \min(1, (\log(1/t_3))^{-1/2})$, so that $t(s) \le t_3$ for $0 < s \le s_0$. By definition $\mathrm{lr}(u) = -\sum_{a,i} u_{a,i}x_{a,i}$, so $\ell = \sum_{r\in\mathrm{Rt}^+}\gamma_r x_r$ with $\sum_r|\gamma_r| \le \|\lambda\|_1 + s\|\nu_{\mathcal{T}}\|_1 + s \le A + \|\nu_{\mathcal{T}}\|_1 + 1$. Expanding ℓ^{D+1} into monomials (the multinomial coefficients times the products of $|\gamma_r|$ sum to $(\sum|\gamma_r|)^{D+1}$), Step 1 gives
$$\left|\mathrm{ai}_{t(s)}(\mathcal{X}; \ell^{D+1}) - \mathrm{pp}(\mathcal{X}; \ell^{D+1})\right| \le (A + \|\nu_{\mathcal{T}}\|_1 + 1)^{D+1} C_3\,\mathrm{rt}(t(s)).$$
Here $\log(1/t(s)) = s^{-2}$ and $\log\log(1/t(s)) = 2\log(1/s)$, so $\mathrm{rt}(t(s)) = 4s^2\log^2(1/s) \le 4s\cdot\max_{0<x\le1} x\log^2(1/x) = 16e^{-2}s \le 4s$. Hence (2) with $C_{\mathrm{rt}} = 4C_3(A + \|\nu_{\mathcal{T}}\|_1 + 1)^{D+1}$.

Step 3: (3). With $\widehat{\deg}_{B,c} = \mathrm{pp}(\mathcal{X}_{B,c}; \cdot)$ on $\mathcal{R}$, (1) is (Gi2), and (2) is (Gi6). □

Lemma 31.21. *Let* $\mathrm{Sys} = (X^{\mathrm{an}}, \Theta, \varsigma, (\mathcal{M}_a, h_a^{\mathrm{ref}}, \varphi_a, \varphi_a^T(t))_{a\in\mathrm{Rt}^+}, C_5, C, t_0)$ *be an analytic system (Go1) with $\varphi_* = 0$ and $\varphi_*^T(t) = 0$, and put $\mathrm{rt}(t) = (\log\log(1/t))^2/\log(1/t)$ for $0 < t < e^{-e}$. There are $0 < t_2 \le t_0$ and $C_2 \ge 0$, depending only on* Sys, *with the following property. Let $\mathcal{X}$ be a root model of type (Φ, n) with $\mathcal{X}_{\mathbb{Q}}$ smooth, $d \ge 1$ an integer and $\pi\colon \mathcal{X}(\mathbb{C}) \to X^{\mathrm{an}}$ a holomorphic map such that:*

1. *there are closed null sets $N \subseteq \mathcal{X}(\mathbb{C})$ and $N' \subseteq X^{\mathrm{an}}$ such that π restricts to a d-sheeted holomorphic covering map $\mathcal{X}(\mathbb{C}) \smallsetminus N \to X^{\mathrm{an}} \smallsetminus N'$;*

2. *for every fundamental index a there is a cone function h_a^{ref} of the multidegree e_a of a such that the first Chern form of $(\mathcal{O}(e_a), h_a^{\mathrm{ref}})|_{\mathcal{X}(\mathbb{C})}$ is $\pi^* c_1(\mathcal{M}_a, h_a^{\mathrm{ref}})$, and $h_*^{\mathrm{ref}} = h_*$;*

3. *for every $(\tau, i) \in \mathrm{Rt}$ and $0 < t \le t_0$, $\prod_{k \le i} h_{(\tau,k)}(t)^{-1} = h_{(\tau,i)}^{\mathrm{ref}}\, e^{\varphi^T_{(\tau,i)}(t) \circ \pi}$ on the affine cone over $\mathcal{X}(\mathbb{C})$.*

Then $\mathcal{X}$ has convergent regularisation, and for every monomial $x^m \in \mathcal{R}^+$ and every $0 < t \le t_2$

$$\left|\mathrm{ai}_t(\mathcal{X}; x^m) - \mathrm{pp}(\mathcal{X}; x^m)\right| \le C_2\, \frac{d}{\delta_{\mathcal{X}}}\, \mathrm{rt}(t).$$

Proof. Choice of t_2 and C_2. Let $C_{\mathrm{tb}}, R_0, t_1$ be as in Proposition 31.22 for Sys, and choose $0 < t_2 \le t_1$ with $\log(1/t_2) \ge \max(R_0, e^e)$ and $t^{1/3} \le 1/\log(1/t)$ for $0 < t \le t_2$. For forms $\nu = \omega_1 \wedge \cdots \wedge \omega_D$ with each ω_j equal to Θ or to $\eta_{a_j}(t_j) + \Theta$ (a_j a fundamental index, $0 < t_j \le t_1$), ν is a positive smooth measure on X^{an} (each $\omega_j \ge \eta_{a_j}(t_j) + \varepsilon(t_j)\Theta \ge 0$, as $\varepsilon(t_j) \le 1$), and since $\eta_a(t) - \theta_a = dd^c \varphi_a^T(t)$ with $\varphi_a^T(t)$ smooth on the compact manifold X^{an}, Stokes' theorem gives $\int \nu = \int \omega_1^0 \wedge \cdots \wedge \omega_D^0$ with $\omega_j^0 \in \{\Theta, \theta_a + \Theta\}$; let M be the maximum of these finitely many numbers. Put $C_1 = 2M(1+C) + 2(C + C_5)C_{\mathrm{tb}}$ and $C_2 = (D+1)2^{D+1}C_1$. These depend on Sys only.

Step 1: the regularisation differences on X^{an}. Let $0 < t' \le t \le t_2$, b a fundamental index, ν as above and $\Delta = \varphi_b^T(t) - \varphi_b^T(t')$ (zero for $b = *$). We show

$$\int_{X^{\mathrm{an}}} |\Delta|\, d\nu \le C_1\, \mathrm{rt}(t). \qquad (1)$$

As $y \mapsto \log y / y$ decreases for $y \ge e$ and $\log(1/t') \ge \log(1/t) \ge e$, $\varepsilon(t') \le \varepsilon(t)$. On $K = \{y \in X^\circ : \mathrm{rW}(y) \le \log(1/t)\} \subseteq \{\mathrm{rW} \le \log(1/t')\}$, (Go1)(2) gives $\varphi_b \le \varphi_b^T(t) \le \varphi_b + t^{1/3} + \varepsilon(t)$ and the same at t', so $|\Delta| \le 2(t^{1/3} + \varepsilon(t))$; since $\log\log(1/t) \ge 1$, $t^{1/3} \le 1/\log(1/t) \le \mathrm{rt}(t)$ and $\varepsilon(t) \le C\,\mathrm{rt}(t)$, hence $\int_K |\Delta|\, d\nu \le 2(1+C)M\,\mathrm{rt}(t)$. On $U = X^\circ \smallsetminus K$, (Go1)(1),(2) give $-C_5 \log \mathrm{rW} \le \varphi_b \le \varphi_b^T(\cdot) \le C$ at t and at t', so $|\Delta| \le C + C_5 \log \mathrm{rW} \le (C + C_5)\log \mathrm{rW}$ (as $\mathrm{rW} > \log(1/t) \ge e$ on U), and Proposition 31.22 with $R = \log(1/t) \ge R_0$ gives $\int_U \log \mathrm{rW}\, d\nu \le C_{\mathrm{tb}}(\log R)^2/R = C_{\mathrm{tb}}\,\mathrm{rt}(t)$. The boundary ∂X^{an} is a proper analytic subset, of ν-measure zero. This proves (1).

Step 2: the root metrics. Let $r = (\tau, i) \in \mathrm{Rt}$, $a = (\tau, i)$ and $a^- = (\tau, i-1)$, the terms with a^- being omitted when $i = 1$. Dividing the identities of hypothesis 3 for i and $i-1$,

$$\log h_r(t) = -\log h_a^{\mathrm{ref}} - \varphi_a^T(t) \circ \pi + \log h_{a^-}^{\mathrm{ref}} + \varphi_{a^-}^T(t) \circ \pi,$$

so, by hypothesis 2, $c_1(\overline{\mathcal{L}}_r(t)) = \pi^*(\eta_{a^-}(t) - \eta_a(t))$ and $\log h_r(t) - \log h_r(t') = -(\varphi_a^T(t) - \varphi_a^T(t')) \circ \pi + (\varphi_{a^-}^T(t) - \varphi_{a^-}^T(t')) \circ \pi$, a smooth function on $\mathcal{X}(\mathbb{C})$. The auxiliary metric h_* does not depend on t, and $c_1(\overline{\mathcal{L}}_*) = \pi^*\theta_* = \pi^*\eta_*(t)$ because $\varphi_*^T = 0$.

Step 3: Cauchy estimate for monomials. Let $x^m = x_{r_1} \cdots x_{r_{D+1}}$ with $r_j \in \mathrm{Rt}^+$ and $0 < t' \le t \le t_2$. Changing the metrics one factor at a time from t' to t and applying the change-of-metric formula at each step ($\mathcal{X}_{\mathbb{Q}}$ is smooth, all metrics are smooth),

$$\widehat{\deg}(t) - \widehat{\deg}(t') = \sum_{j=1}^{D+1} \int_{\mathcal{X}(\mathbb{C})} \log \frac{h_{r_j}(t)}{h_{r_j}(t')} \bigwedge_{i<j} c_1(\overline{\mathcal{L}}_{r_i}(t)) \wedge \bigwedge_{i>j} c_1(\overline{\mathcal{L}}_{r_i}(t')),$$

where $\widehat{\deg}(t)$ is the numerator of (29.2); the term with $r_j = *$ vanishes. By Step 2 the j-th integrand is π^*F_j for a top-degree form F_j on X^{an}, smooth on X^{an}. Writing each factor $\eta_{a^-}(\cdot) - \eta_a(\cdot)$ as $(\eta_{a^-}(\cdot)+\Theta) - (\eta_a(\cdot)+\Theta)$, the factor $-\eta_a(\cdot)$ (case $i=1$) as $\Theta - (\eta_a(\cdot)+\Theta)$ and η_* as $(\eta_*+\Theta)-\Theta$, and the logarithm as a difference of two functions Δ of Step 1, F_j is a signed sum of at most $2\cdot 2^D$ terms $\pm\Delta\,\nu$ with Δ,ν as in Step 1 (the parameters lie in $(0,t_2] \subseteq (0,t_1]$). By hypothesis 1 and the change of variables formula for the orientation-preserving local biholomorphisms of a covering map, $\int_{\mathcal{X}(\mathbb{C})}\pi^*F_j = \int_{\mathcal{X}(\mathbb{C})\smallsetminus N}\pi^*F_j = d\int_{X^{\mathrm{an}}\smallsetminus N'}F_j = d\int_{X^{\mathrm{an}}}F_j$, all integrals converging absolutely. By (1), and since $k_{\mathcal{X}}\geq 1$,

$$\left|\mathrm{ai}_t(\mathcal{X};x^m) - \mathrm{ai}_{t'}(\mathcal{X};x^m)\right| \leq \frac{d}{\delta_{\mathcal{X}}}(D+1)\,2^{D+1}C_1\,\mathrm{rt}(t) = C_2\,\frac{d}{\delta_{\mathcal{X}}}\,\mathrm{rt}(t).$$

Monomials of degree $\neq D+1$ give 0 for every t.

Step 4: conclusion. Since $\mathrm{rt}(t)\to 0$ as $t\to 0^+$, Step 3 is a Cauchy criterion: $\lim_{t\to 0^+}\mathrm{ai}_t(\mathcal{X};x^m)$ exists for every monomial, so $\mathcal{X}$ has convergent regularisation; letting $t'\to 0^+$ in Step 3 gives the bound. □

Proposition 31.22. *Let* Sys *be an analytic system of dimension D. There are $C_{\mathrm{tb}}\geq 0$, $R_0\geq e$ and $0<t_1\leq t_0$ with $\varepsilon(t)\leq 1$ for $0<t\leq t_1$, such that for every $R\geq R_0$ and all smooth forms $\omega_1,\dots,\omega_D$, each equal to Θ or to $\eta_a(t)+\Theta$ for some fundamental index a and some $0<t\leq t_1$,*

$$\int_{\{y\in X^\circ:\ \mathrm{rW}(y)>R\}} \log \mathrm{rW}\ \ \omega_1\wedge\cdots\wedge\omega_D\ \ \leq\ \ C_{\mathrm{tb}}\,\frac{(\log R)^2}{R}.$$

Proof. Write $X = X^{\mathrm{an}}$; by (Go1) it is a compact complex manifold of dimension $D\geq 1$, ς is a holomorphic section of a smooth hermitian line bundle with $\|\varsigma\|\leq 1$ and simple normal crossings zero divisor ∂X^{an}, and $\mathrm{rW} = e - 2\log\|\varsigma\|$: the setting of Lemma 31.23.

Choice of t_1 and of the base forms. As $x\mapsto \log x/x$ decreases for $x\geq e$ and $\varepsilon(t) = C\log x/x$ with $x=\log(1/t)$, there is $0<t_1\leq t_0$ with $\log(1/t_1)\geq e$ and $\varepsilon(t)\leq\varepsilon(t_1)\leq\frac12$ for $0<t\leq t_1$. By (Go1)(2), $\eta_a(t_1)+\Theta\geq(1-\varepsilon(t_1))\Theta\geq\frac12\Theta$, so the finite set $\mathfrak{B} = \{\Theta\}\cup\{\eta_a(t_1)+\Theta:\ a\in\mathrm{Rt}^+\}$ consists of positive definite smooth closed real $(1,1)$-forms. Put $C_1 = C_5 + 2C$.

Every admissible factor is an admissible perturbation of a base form. Let ω be one of the forms of the statement. If $\omega=\Theta$, put $\omega^0=\Theta$ and $u=0$. If $\omega=\eta_a(t)+\Theta$ with $0<t\leq t_1$, put $\omega^0 = \eta_a(t_1)+\Theta$ and $u = \varphi_a^T(t) - \varphi_a^T(t_1) - C\varepsilon(t_1)$, a smooth function on X; then $\omega^0 + dd^c u = \eta_a(t)+\Theta\geq(1-\varepsilon(t))\Theta\geq 0$ by (Go1)(2); $u\leq 0$ on X by the last clause of (Go1)(2) with $t\leq t_1\leq t_0$; and on X°, $\varphi_a^T(t)\geq\varphi_a\geq -C_5\log\mathrm{rW}$ ((Go1)(1),(2)), $\varphi_a^T(t_1)\leq C$ and $C\varepsilon(t_1)\leq C$, so $u\geq -C_5\log\mathrm{rW} - 2C\geq -C_1\log\mathrm{rW}$, as $\log\mathrm{rW}\geq 1$ (for $a=*$ this is $u=-C\varepsilon(t_1)$). In both cases $\omega^0\in\mathfrak{B}$, $\omega^0+dd^cu\geq 0$, $u\leq 0$ on X and $u\geq -C_1\log\mathrm{rW}$ on X°.

Conclusion. For each of the finitely many tuples $(\omega_1^0,\dots,\omega_D^0)\in\mathfrak{B}^D$ let $K(\omega^0)$ be the constant of Lemma 31.23, and put $C_{\mathrm{tb}} = 3(C_1+4)\max_{\omega^0\in\mathfrak{B}^D}K(\omega^0)$ and $R_0 = e$. Given $\omega_1,\dots,\omega_D$ as in the statement, write $\omega_j = \omega_j^0 + dd^cu_j$ as above; Lemma 31.23 (second inequality) with this C_1 gives, for $R\geq e$,

$$\int_{\{\mathrm{rW}>R\}}\log\mathrm{rW}\ \omega_1\wedge\cdots\wedge\omega_D \leq \int_{\{y\in X^\circ:\,\mathrm{rW}(y)\geq R\}}\log\mathrm{rW}\ \omega_1\wedge\cdots\wedge\omega_D \leq C_{\mathrm{tb}}\frac{(\log R)^2}{R}.$$

□

Lemma 31.23. *Let X be a compact complex manifold of dimension $D\geq 1$, let $dd^c = c_0\,i\partial\bar\partial$ for a fixed constant $c_0>0$, let $(\mathcal{N},\|\cdot\|)$ be a smooth hermitian line bundle on X with a holomorphic*

section ς such that $\|\varsigma\| \le 1$ and the zero divisor ∂X of ς has simple normal crossings, and put $X^\circ = X \setminus \partial X$ and $\mathrm{rW} = e - 2\log\|\varsigma\|$ on X°. Let $\omega_1^0, \dots, \omega_D^0$ be positive definite smooth closed real $(1,1)$-forms on X. Then there is $K \ge 0$ with the following property. Let $C_1 \ge 0$ and let $u_1, \dots, u_D$ be smooth real functions on X such that for every j

$$\omega_j^0 + dd^c u_j \ge 0, \qquad u_j \le 0 \text{ on } X, \qquad u_j \ge -C_1 \log \mathrm{rW} \text{ on } X^\circ,$$

and let μ be the positive measure $(\omega_1^0 + dd^c u_1) \wedge \cdots \wedge (\omega_D^0 + dd^c u_D)$. Then for every $R \ge e$

$$\mu(\{y \in X^\circ : \mathrm{rW}(y) \ge R\}) \le K(C_1+4)\frac{\log R}{R},$$
$$\int_{\{y \in X^\circ : \mathrm{rW}(y) \ge R\}} \log \mathrm{rW}\; d\mu \le 3K(C_1+4)\frac{(\log R)^2}{R}.$$

Proof. Step 1: constants. Let $V = \int_X \omega_1^0 \wedge \cdots \wedge \omega_D^0 > 0$ and $I_0 = \int_X \log\|\varsigma\|^2\, \omega_1^0 \wedge \cdots \wedge \omega_D^0$ (finite by Poincaré–Lelong). As X is compact and the ω_j^0 are positive definite, there is $c \ge 0$ with $\rho + c\,\omega_j^0 \ge 0$ for all j. Put

$$K_{\mathrm{PL}} = \sum_{j=1}^{D} \Big(\kappa \sum_k m_k \int_{W_k} \bigwedge_{i \ne j} \omega_i^0 + cV \Big) \ge 0, \qquad K = eV - I_0 + K_{\mathrm{PL}}.$$

Here $eV - I_0 = \int_{X^\circ} \mathrm{rW}\; \omega_1^0 \wedge \cdots \wedge \omega_D^0 \ge 0$, so $K \ge 0$.

Step 2: energy bound. Claim: if $N \ge 1$ and $v_1, \dots, v_D$ are smooth on X with $\omega_j^0 + dd^c v_j \ge 0$ and $-N \le v_j \le 0$, then $\mu_v = \bigwedge_j (\omega_j^0 + dd^c v_j)$ satisfies $\int_{X^\circ} \mathrm{rW}\, d\mu_v \le NK$. For $0 \le j \le D$ let $\Omega_j = \bigwedge_{i \le j}(\omega_i^0 + dd^c v_i) \wedge \bigwedge_{i>j} \omega_i^0$ and, for $j \ge 1$, $\Phi_j = \bigwedge_{i<j}(\omega_i^0 + dd^c v_i) \wedge \bigwedge_{i>j} \omega_i^0$, a smooth closed positive $(D-1, D-1)$-form; $\Omega_j - \Omega_{j-1} = dd^c v_j \wedge \Phi_j$. Since Φ_j is closed of pure type, $\partial\Phi_j = \bar\partial\Phi_j = 0$, hence $dd^c(v_j\Phi_j) = dd^c v_j \wedge \Phi_j$, and Poincaré–Lelong gives

$$\int_X \log\|\varsigma\|^2\, dd^c v_j \wedge \Phi_j = \kappa \sum_k m_k \int_{W_k} v_j\, \Phi_j - \int_X v_j\, \rho \wedge \Phi_j.$$

(i) As $v_j \ge -N$ and $\Phi_j|_{W_k} \ge 0$, and by Stokes on W_k, the first term is $\ge -N\kappa \sum_k m_k \int_{W_k} \bigwedge_{i \ne j} \omega_i^0$. (ii) $-\int_X v_j \rho \wedge \Phi_j = \int_X (-v_j)(\rho + c\omega_j^0) \wedge \Phi_j - c\int_X (-v_j)\,\omega_j^0 \wedge \Phi_j$; the first integral is ≥ 0, and as $0 \le -v_j \le N$ the second is at most $cN \int_X \omega_j^0 \wedge \Phi_j = cNV$ (Stokes). Hence $\int_X \log\|\varsigma\|^2\, \Omega_j \ge \int_X \log\|\varsigma\|^2\, \Omega_{j-1} - N(\kappa \sum_k m_k \int_{W_k} \bigwedge_{i \ne j} \omega_i^0 + cV)$, and summing over j, $\int_X \log\|\varsigma\|^2\, d\mu_v \ge I_0 - NK_{\mathrm{PL}}$. As μ_v has total mass V (Stokes) and $\mu_v(\partial X) = 0$,

$$\int_{X^\circ} \mathrm{rW}\, d\mu_v = eV - \int_X \log\|\varsigma\|^2\, d\mu_v \le eV - I_0 + NK_{\mathrm{PL}} \le NK,$$

using $N \ge 1$ and $eV - I_0 \ge 0$.

Step 3: a regularised maximum. Let ψ be the convolution of $s \mapsto \max(s,0)$ with a smooth even probability density supported in $[-1,1]$. Then ψ is smooth and convex, $0 \le \psi' \le 1$, $\max(s,0) \le \psi(s) \le \max(s,0) + 1$ (the lower bound by Jensen's inequality, the density having mean 0), and $\psi(s) = \max(s,0)$ for $|s| \ge 1$ (there $\max(\cdot,0)$ is affine on $[s-1, s+1]$). Put $M(x,y) = y + \psi(x-y)$. Then: (a) $M(x,y) = x$ if $x \ge y+1$; (b) $\max(x,y) \le M(x,y) \le \max(x,y)+1$; (c) $M(x+a, y+a) = M(x,y) + a$; (d) if f, g are smooth plurisubharmonic on an open subset of $\mathbb{C}^D$, so is $M(f,g)$, because

$$i\partial\bar\partial M(f,g) = \psi'(f-g)\, i\partial\bar\partial f + (1 - \psi'(f-g))\, i\partial\bar\partial g + \psi''(f-g)\, i\partial(f-g) \wedge \bar\partial(f-g) \ge 0.$$

Step 4: mass of the tube. Let $u_1,\dots,u_D$, C_1, μ be as in the statement and $R \ge e$. Put $N_0 = C_1 \log R + 3$ and $v_j = M(u_j, -N_0) - 1$, smooth on X. On a coordinate ball write $\omega_j^0 = dd^c g$ with g smooth real (local dd^c-lemma for closed real $(1,1)$-forms); g and $g+u_j$ are plurisubharmonic, so by (c) and (d) $g + v_j + 1 = M(g+u_j, g-N_0)$ is plurisubharmonic, i.e. $\omega_j^0 + dd^c v_j \ge 0$. By (b) and $u_j \le 0$, $-N_0 - 1 \le v_j \le \max(u_j, -N_0) \le 0$. With $N = N_0 + 1 = C_1 \log R + 4 \le (C_1+4)\log R$ (as $\log R \ge 1$), Step 2 gives $\int_{X^\circ} \mathrm{rW}\, d\mu_v \le NK$. The set $V_R = \{y \in X^\circ : \mathrm{rW}(y) < R\} = \{y \in X : \|\varsigma(y)\|^2 > e^{e-R}\}$ is open. On V_R, $u_j \ge -C_1 \log \mathrm{rW} > -C_1 \log R = -N_0 + 3$, so $M(u_j, -N_0) = u_j$ by (a) and $v_j = u_j - 1$; hence $\omega_j^0 + dd^c v_j = \omega_j^0 + dd^c u_j$ on V_R, and μ and μ_v agree on subsets of V_R. Both have total mass V (Stokes), so $\mu(X \smallsetminus V_R) = \mu_v(X \smallsetminus V_R)$. Now $X \smallsetminus V_R = \partial X \cup \{y \in X^\circ : \mathrm{rW}(y) \ge R\}$, $\mu_v(\partial X) = 0$, and by Markov's inequality $\mu_v(\{y \in X^\circ : \mathrm{rW} \ge R\}) \le R^{-1} \int_{X^\circ} \mathrm{rW}\, d\mu_v \le NK/R$. Therefore $\mu(\{y \in X^\circ : \mathrm{rW} \ge R\}) \le \mu(X \smallsetminus V_R) \le NK/R \le K(C_1+4)\log R/R$, the first inequality of the statement.

Step 5: the log-weighted tail. Put $K' = K(C_1+4)$, $F(x) = \mu(\{y \in X^\circ : \mathrm{rW}(y) \ge x\})$, so $F(x) \le K' \log x / x$ for $x \ge e$ by Step 4, and $E_R = \{y \in X^\circ : \mathrm{rW}(y) \ge R\}$. By the layer-cake formula for the nonnegative function $\log \mathrm{rW}$ on E_R, bounding $\mu(\{y \in E_R : \log \mathrm{rW}(y) > \lambda\})$ by $F(R)$ for $\lambda < \log R$ and by $F(e^\lambda)$ for $\lambda \ge \log R$,

$$\begin{aligned}\int_{E_R} \log \mathrm{rW}\, d\mu &\le \log R \cdot F(R) + \int_{\log R}^{\infty} F(e^\lambda)\, d\lambda \le K' \frac{(\log R)^2}{R} + K' \int_R^\infty \frac{\log x}{x^2}\, dx \\ &= K' \frac{(\log R)^2 + \log R + 1}{R} \le 3K' \frac{(\log R)^2}{R},\end{aligned}$$

as $\log R \ge 1$. This is the second inequality. □

Lemma 31.24. *Let $\mathfrak{u}$ be a unitary Shimura datum with $p, q \ge 1$, $\mathcal{Y}$ an integral model of* Sh*, and $\mathcal{T}$ a root tower of type $(\Phi_{\mathsf{H}}, p+q, p)$ with a geometric realisation and a good origin over $\mathcal{Y}$. Then for every $A \ge 0$ there is $s_0 > 0$ such that for all integers $B \ge 2$, all $Z \in \mathrm{Cp}$, all weights λ with $\|\lambda\|_1 \le A$ and all $0 < s \le s_0$ with λ_s dominant, the vector $c(\lambda_s, s)$ is semipositive at $t(s)$ on $\mathcal{X}_{B,Z}$ (property (Us) of Lemma 31.25). Consequently, if the realised tower has (Gi4), it has (Wn) and (Gi3).*

Proof. Constants. For $c \in \mathrm{Cp}$ let Sys_c be the analytic system of the fixed data at c, with Kähler form Θ_c, constants C_c, $t_{0,c}$ and $\varepsilon_c(t) = C_c \log\log(1/t)/\log(1/t)$; let ω_c be the curvature form of h_*^{ref}, positive by (Go2), and $k_c = k_c^A$, which is $k_{\mathcal{X}}$ for every $\mathcal{X}_{B,c}$ by (Go3)(5). As X_c^{an} is compact and ω_c is positive definite, there is C_c' with $\Theta_c \le C_c' \omega_c$ as hermitian forms on every tangent space. Put $\nu = \nu_{\mathcal{T}}$ and $G_A = 2(A + \|\nu\|_1)$. Since $s\log(1/s) \to 0$ as $s \to 0^+$ and Cp is finite, there is $s_0 \in (0, e^{-1})$ with $t(s_0) \le t_{0,c}$ and $4C_cC_c'k_cG_A\, s\log(1/s) \le 1$ for all c and all $0 < s \le s_0$. None of these constants depends on B.

Positivity on the analytic system. Let B, $Z = c$, λ, s be as in the statement, $\mu = \lambda_s$ (dominant), $t = t(s) \le t_{0,c}$, and write $g_{\tau,i} = \mu_{\tau,i} - \mu_{\tau,i+1} \ge 0$ $(i < n = p+q)$. Then $\log(1/t) = s^{-2}$ and $\varepsilon_c(t) = 2C_c s^2 \log(1/s)$. For a fundamental index a let ρ_a be the curvature form on X_c^{an} of the regularised metric $h_a^{\mathrm{ref}} e^{\varphi_a^T(t)}$. By (Go1)(2), $\rho_a + \varepsilon_c(t)\Theta_c \ge 0$; by the flatness clause of good origins, $\rho_{(\tau,n)} = 0$; and $\rho_* = \omega_c$ since $\varphi_*^T = 0$. With $G = \sum_\tau \sum_{i<n} g_{\tau,i} \le 2\|\mu\|_1 \le G_A$ (as $s \le 1$),

$$\begin{aligned}\Omega := \sum_\tau \Big(\sum_{i<n} g_{\tau,i}\, \rho_{(\tau,i)} + \mu_{\tau,n}\, \rho_{(\tau,n)}\Big) + \frac{s}{k_c}\omega_c \;&\ge\; -\varepsilon_c(t)\, G\, \Theta_c + \frac{s}{k_c}\omega_c \\ &\ge\; \Big(\frac{s}{k_c} - 2C_cC_c'G_A\, s^2 \log\frac{1}{s}\Big)\omega_c \;\ge 0,\end{aligned}$$

the last step by the choice of s_0.

Summation by parts on the root model. Write $\mathcal{X} = \mathcal{X}_{B,c}$. By (Go4), for $a = (\tau, i)$ the cone function $F_a = h_a^{\mathrm{ref}}\, e^{\varphi_a^T(t)\circ\pi}$ on the affine cone over $\mathcal{X}(\mathbb{C})$ equals $\prod_{k\le i} h_{(\tau,k)}(t)^{-1}$ and defines the pull-back by π of the regularised metric of a; and $h_* = h_*^{\mathrm{ref}}$. Put $F_{(\tau,0)} = 1$. Then $\log h_{(\tau,i)}(t) = \log F_{(\tau,i-1)} - \log F_{(\tau,i)}$, and summation by parts gives

$$\varphi_{c(\mu,s),t} = \sum_{\tau,i}(-\mu_{\tau,i})\log h_{(\tau,i)}(t) + \frac{s}{k_c}\log h_* = \sum_{\tau}\Big(\sum_{i<n} g_{\tau,i}\log F_{(\tau,i)} + \mu_{\tau,n}\log F_{(\tau,n)}\Big) + \frac{s}{k_c}\log h_*.$$

Levi form on the cone. By the smoothness clause of good origins, $\mathcal{X}(\mathbb{C})$ is a complex manifold and π is holomorphic on it. Let z be a point of the affine cone over $\mathcal{X}(\mathbb{C})$, over $x \in \mathcal{X}(\mathbb{C})$. On a neighbourhood U of x choose holomorphic lifts $\hat\sigma_k\colon U \to \mathbb{C}^{I_k} \smallsetminus \{0\}$ of the factors; then $(y,\eta) \mapsto (\eta_k\hat\sigma_k(y))_k$ is a biholomorphism of $U \times (\mathbb{C}^\times)^K$ onto a neighbourhood of z in the cone, so the Zariski tangent vectors of the cone at z are the tangent vectors of this manifold. For a cone function F of multidegree e, $\log F(\eta\cdot\hat\sigma(y)) = \sum_k e_k\log|\eta_k| + \log F(\hat\sigma(y))$: the first term is pluriharmonic, and $\log F\circ\hat\sigma$ is a local potential on U of the metric defined by F (as $\|\sigma\| = |\sigma|/F$). Hence the Levi form of $\varphi_{c(\mu,s),t}$ at z in a direction $(\dot y,\dot\eta)$ is a positive multiple of the curvature form of the corresponding combination of metrics on U evaluated at $\dot y$, which by the two previous paragraphs is $(\pi^*\Omega)(\dot y,\bar{\dot y}) \ge 0$: pull-backs of semipositive $(1,1)$-forms by holomorphic maps are semipositive. So $c(\mu,s)$ is semipositive at $t(s)$ on $\mathcal{X}$.

Consequence. By (Go3)(5) every entry of e_* is ≥ 0, which is (Gi5). With (Gi4), Lemma 31.25 applies and gives (Wn) and (Gi3). □

Lemma 31.25. *Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ that has (Gi4), (Gi5) and the following property:*

(Us) Uniform semipositivity *for every $A \ge 0$ there is $s_0 > 0$ such that for all integers $B \ge 2$, all $Z \in \mathrm{Cp}$, all weights λ with $\|\lambda\|_1 \le A$ and all $0 < s \le s_0$ with λ_s dominant, the vector $c(\lambda_s, s)$ is semipositive at $t(s)$ on $\mathcal{X}_{B,Z}$.*

Then the realised tower has (Wn) and (Gi3).

Proof. (Wn). Let $A \ge 0$ and take s_0 from (Us). Let $B \ge 2$, $Z \in \mathrm{Cp}$, λ with $\|\lambda\|_1 \le A$ and $0 < s \le s_0$ be such that $\mu = \lambda_s$ is dominant and above the nef threshold at B, and write $\mathcal{X} = \mathcal{X}_{B,Z}$. Semipositivity of $c(\mu,s)$ at $t(s)$ is (Us). For nefness on fibres let ℓ be a prime. If $\ell \le B$, (Gi4) says that $c(\mu,0)$ is nef at ℓ. If $\ell > B$: since μ is dominant, its cut gaps $\mu_{a,\kappa} - \mu_{a,\kappa+1}$ are ≥ 0, so $(b-2)(\mu_{a,\kappa} - \mu_{a,\kappa+1})$ is non-decreasing in b and μ is above the nef threshold at ℓ as well; again (Gi4) says that $c(\mu,0)$ is nef at ℓ. Now $c(\mu,s)$ differs from $c(\mu,0)$ only in the entry at $*$, which is $s/k_{\mathcal{X}} > 0$. By (Gi5) the line bundle $\mathcal{O}(e_*)$ on the multiprojective space is generated by the monomials of multidegree e_*, so its degree on every integral closed curve $W \subseteq \mathcal{X}_{\mathbb{F}_\ell}$ is ≥ 0. Hence $\sum_r c(\mu,s)_r \deg(\mathcal{O}(e_r)|_W) = \sum_r c(\mu,0)_r \deg(\mathcal{O}(e_r)|_W) + \frac{s}{k_{\mathcal{X}}}\deg(\mathcal{O}(e_*)|_W) \ge 0$ for every such W and every ℓ: $c(\mu,s)$ is nef on fibres of $\mathcal{X}$.

(Gi3). Let u be a regular dominant weight, so that $u_{a,i} > u_{a,i+1}$ for all a and $i < n$. Take s_0 from (Us) for $A = \|u\|_1$. If $\nu_{\mathcal{T}} = 0$ then $u_s = u$ is dominant for all s; otherwise u_s is dominant for $0 < s < \min_{a,i}(u_{a,i} - u_{a,i+1})/(2\|\nu_{\mathcal{T}}\|_\infty)$, because the gaps of u_s differ from those of u by at most $2s\|\nu_{\mathcal{T}}\|_\infty$. For such $s \le s_0$, (Us) with $\lambda = u$ gives that $c(u_s,s)$ is semipositive at $t(s)$ on every $\mathcal{X}_{B,Z}$: this is (Gi3). □

Lemma 31.26. *Let $\mathfrak{u}$ be a unitary Shimura datum with $p, q \ge 1$, $\mathcal{Y}$ an integral model of* Sh*, and $\mathcal{T}$ a root tower of type $(\Phi_{\mathsf{H}}, p+q, p)$ with a geometric realisation and a tower origin over $\mathcal{Y}$. If the realised tower has (Gi1) and (Gi4), then it has (Wg); more precisely, for all $B \ge 2$,*

$Z \in \mathrm{Cp}$, every dominant weight λ and every real $s \geq 0$ with λ_s dominant, $\ell = \mathrm{lr}(\lambda_s) + s\,x_*$ satisfies $0 \leq \mathrm{gi}(\mathcal{X}_{B,Z}; \ell^D) \leq \mathrm{vol}(Z)\deg^{\mathrm{fl}}(\mathrm{lr}(\lambda)^D)$.

Proof. Fix $B \geq 2$, $Z \in \mathrm{Cp}$ and write $\mathcal{X} = \mathcal{X}_{B,Z}$, $\delta = \delta_{\mathcal{X}}$, $k = k_{\mathcal{X}}$, and $X = \mathcal{X}_{\mathbb{Q}}$: a non-empty open subscheme of the integral scheme $\mathcal{X}$, closed in $\mathbb{P}_{\mathbb{Q}}$, hence an integral projective variety of dimension D over $\mathbb{Q}$. For $c \in \mathbb{R}^{\mathrm{Rt}^+}$ let $N_c = \sum_r c_r\, c_1(\mathcal{O}(e_r)|_X)$, a real Cartier class on X. By the definition of gi (each x_* carries the factor $k_{\mathcal{X}}^{-1}$ and $\ell_c = \sum_r c_r k_r x_r$ with $k_* = k$), multilinearity gives

$$\delta\,\mathrm{gi}(\mathcal{X}; \ell_{c_1} \cdots \ell_{c_D}) = N_{c_1} \cdots N_{c_D} \qquad (c_1, \ldots, c_D \in \mathbb{R}^{\mathrm{Rt}^+}),$$

and $\ell_{c(u,s)} = \mathrm{lr}(u) + s\,x_*$.

Step 1: nef classes on X. Let μ be a dominant weight. By (Gi4) at the prime $2 \leq B$, $c(\mu, 0)$ is nef at 2 on $\mathcal{X}$. Let $C \subseteq X$ be an integral closed curve and $\overline{C}$ its closure in $\mathcal{X} \times_{\mathbb{Z}} \mathbb{Z}_{(2)}$: integral, dominating $\mathrm{Spec}\,\mathbb{Z}_{(2)}$, hence flat over it, and proper. Its special fibre has fundamental cycle $\sum_j m_j[W_j]$ with integral curves $W_j \subseteq \mathcal{X}_{\mathbb{F}_2}$ and $m_j \geq 1$ (non-empty, by properness and dominance), and the degree of a line bundle is constant in the flat proper family $\overline{C}$ (conservation of number; Fulton, Intersection Theory, Prop. 10.2). Hence

$$\deg(N_{c(\mu,0)}|_C) = \sum_j m_j \sum_r c(\mu,0)_r \deg(\mathcal{O}(e_r)|_{W_j}) \geq 0,$$

so $N_{c(\mu,0)}$ is nef on X. By (Go3)(5) the multidegree e_* is 1 at k_0 and 0 elsewhere, so $\mathcal{O}(e_*)|_X$ is globally generated and $c_1(\mathcal{O}(e_*)|_X)$ is nef. Therefore $A = N_{c(\lambda,0)}$ and $L = N_{c(\lambda_s,s)} = N_{c(\lambda_s,0)} + \frac{s}{k}\,c_1(\mathcal{O}(e_*)|_X)$ are nef (λ and λ_s dominant, $s \geq 0$). Intersection numbers of nef real Cartier classes on a projective variety are ≥ 0 (Kleiman's theorem; Lazarsfeld, Positivity in algebraic geometry I, Theorem 1.4.9, with Corollary 1.4.10 for products of distinct nef classes: each is a limit of ample classes). In particular $\delta\,\mathrm{gi}(\mathcal{X}; \ell^D) = L^D \geq 0$, the lower bound.

Step 2: the difference is effective. By (Go3)(6), read in the charts $\{Y_{k_0,i} \neq 0\}$, which cover X as i varies, the element $\sigma = g(s_c^E)\prod_{r\in\mathrm{Rt}} \varrho_r^{n_c(r)}$ is a regular section over X of $\mathcal{O}(e^E)|_X$, $e^E = -\sum_r n_c(r)e_r - q_c' e_*$: indeed, with $\theta = \prod_r (\varrho_r y_{c'}^{e_r})^{-n_c(r)}$ a local generator of $\bigotimes_r \mathcal{L}_r^{-n_c(r)}$ on a chart $\{y_{c'} \neq 0\}$ (root sections, (Go3)(3)), the regular function $g(s_c^E)g(y_i^c)^{q_c'}/\theta$ is σ divided by a chart monomial of multidegree e^E. It is non-zero since $s_c^E \in \Omega_c^\times$. Let $\mathcal{E}$ be its divisor, an effective Cartier divisor on X with Weil cycle $\sum_j a_j[E_j]$, $a_j \geq 0$. Since $n_c(r) = k_c^A q_c' \nu_{\mathcal{T}}(r)$ and $k = k_c^A$ ((Go3)(5)),

$$c_1(\mathcal{O}(e^E)|_X) = k\,q_c'\Big(-\sum_{r\in\mathrm{Rt}} \nu_{\mathcal{T}}(r)\, c_1(\mathcal{O}(e_r)|_X) - \tfrac{1}{k}\, c_1(\mathcal{O}(e_*)|_X)\Big) = k\,q_c'\, N_{c(\nu_{\mathcal{T}},-1)}.$$

Comparing entries, $c(\lambda,0) - c(\lambda_s, s) = s\,c(\nu_{\mathcal{T}}, -1)$, so $A - L = \frac{s}{kq_c'}\,c_1(\mathcal{O}(\mathcal{E}))$.

Step 3: the upper bound. Expanding the telescoping sum,

$$A^D - L^D = \sum_{i=0}^{D-1} A^i L^{D-1-i}(A-L) = \frac{s}{kq_c'} \sum_{i=0}^{D-1} \sum_j a_j \,(A|_{E_j})^i (L|_{E_j})^{D-1-i} \;\geq 0,$$

because the restrictions of nef classes to the projective varieties E_j are nef and Kleiman's theorem applies on each E_j. Hence $\delta\,\mathrm{gi}(\mathcal{X}; \ell^D) = L^D \leq A^D = \delta\,\mathrm{gi}(\mathcal{X}; \mathrm{lr}(\lambda)^D)$, and (Gi1), applied to $\mathrm{lr}(\lambda)^D \in \mathcal{R}$ of degree D, gives $\mathrm{gi}(\mathcal{X}; \mathrm{lr}(\lambda)^D) = \mathrm{vol}(Z)\deg^{\mathrm{fl}}(\mathrm{lr}(\lambda)^D)$. Dividing by $\delta \geq 1$ gives the claim; (Wg) is the special case of rational λ and s. □

31.3 Comparison of root origins related by Galois twists

This subsection proves the last two properties that Proposition 31.14 needs. The first is Galois component symmetry (Ys0). A Galois twist of the universal object is realised by an automorphism of the moduli scheme (Proposition 31.36); Proposition 31.27 compares the limit intersection numbers of two root models whose origins on the fixed data differ by such a twist, and shows that at the identity twist they do not depend on the model; (Ys0) then follows from vanishing limits of the twisted differences (Lemma 31.38). The proof builds a common model carrying both families of root metrics over one analytic system (Proposition 31.28) and compares regularised metrics with asymptotically equal weights on it (Lemma 31.34), using the convergence of mixed Monge–Ampère integrals of log-log regularisations (Lemma 31.35). The second property is the slope at the test class (Rt6), which follows from base slope settings (Theorem 31.45): heights of points on the base are bounded below (Bs4), Zhang's lower bound with a height floor [276] turns this into a lower bound for a self-intersection (Lemma 31.46), and the Stokes inequality (Proposition 31.41) and the bound for the mean of the cusp correction (Theorem 31.44) pass to the slope.

Proposition 31.27. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $\Phi = \Phi_{\mathsf{H}}$, $n = p+q$, $p, q \geq 1$, let $(G, \iota, \mathcal{L})$ be the abelian scheme, the $\mathcal{O}_E$-action and the cubical line bundle of the universal object over* Sh*, and let $\mathcal{Y}$ be an integral model of* Sh *over $\mathcal{O}_F$. Let S be a finite set of primes containing every prime dividing* $\mathrm{disc}(E)$*, and $R(\ell) \geq 4$ ($\ell \in S$) integers. Let g be an automorphism of E with $g(r) = r$, $\sigma_g \colon \Phi \to \Phi$, $\tau \mapsto \tau \circ g$, and let* tw *be an automorphism of the K_{L}-scheme* Sh *for which there is an isomorphism of abelian schemes $\psi \colon G \to \mathrm{tw}^* G$ over* Sh *with $\psi \circ \iota(g(a)) = (\mathrm{tw}^* \iota)(a) \circ \psi$ for all $a \in \mathcal{O}_E$ and $\psi^* \mathrm{tw}^* \mathcal{L} \cong \mathcal{L}$ as rigidified cubical line bundles. Let fixed data be given at irreducible components $\mathcal{Y}_{c_1}$, $\mathcal{Y}_{c_2}$ of $\mathcal{Y}$ such that $\mathrm{tw}_F = \mathrm{tw} \times_{K_{\mathsf{L}}} F$ maps Z_{c_1} onto Z_{c_2}. Let $\mathcal{X}_1$, $\mathcal{X}_2$ be root models of type (Φ, n) such that for $j = 1, 2$, writing $\mathcal{X} = \mathcal{X}_j$: $\mathcal{X}_{\mathbb{Q}}$ is smooth, $\mathcal{X}$ carries the data (Go3)(1)–(4) of a root origin on the fixed data at c_j for (S, R) (with its singular Hodge lines h_r^{sing} and $\delta_{\mathcal{X}} = [F : \mathbb{Q}] \deg$), and there are an analytic system $\mathrm{Sys}_{\mathcal{X}}$ (Go1) with $\varphi_* = 0 = \varphi_*^T$, an integer $d \geq 1$ and a holomorphic map $\pi \colon \mathcal{X}(\mathbb{C}) \to X_{\mathcal{X}}^{\mathrm{an}}$ satisfying hypotheses 1–3 of Lemma 31.21, such that a point of $\mathcal{X}(\mathbb{C})$ lies in the abelian locus if and only if its image lies in $X_{\mathcal{X}}^{\circ}$, and on $\pi^{-1}(X_{\mathcal{X}}^{\circ})$*

$$h^{\mathrm{ref}}_{(\tau,i)}\, e^{\varphi_{(\tau,i)} \circ \pi} = \prod_{k \leq i} \bigl(h^{\mathrm{sing}}_{(\tau,k)}\bigr)^{-1} \qquad ((\tau, i) \in \mathrm{Rt})$$

(the singular weight is the Hodge metric). Then:

1. *$\mathcal{X}_1$ and $\mathcal{X}_2$ have convergent regularisation, and for every monomial $x^m \in \mathcal{R}$ of degree $D+1$*

$$\lim_{t \to 0^+} \Bigl(\mathrm{ai}_t(\mathcal{X}_1; x^m) - \mathrm{ai}_t(\mathcal{X}_2; \sigma_g \cdot x^m)\Bigr) = 0;$$

   hence $\mathrm{pp}(\mathcal{X}_1; F) = \mathrm{pp}(\mathcal{X}_2; \sigma_g \cdot F)$ for every $F \in \mathcal{R}$;

2. *if $g = \mathrm{id}$ and $\mathrm{tw} = \mathrm{id}$ (so $\mathcal{Y}_{c_1} = \mathcal{Y}_{c_2}$: model independence), then moreover $\mathrm{gi}(\mathcal{X}_1; F) = \mathrm{gi}(\mathcal{X}_2; F)$ for every $F \in \mathcal{R}$.*

Proof. Write $\delta_j = \delta_{\mathcal{X}_j} = [F : \mathbb{Q}] \deg_j$ and $h_{j,r}(t)$ for the root metrics of $\mathcal{X}_j$.

Step 1: convergence. For $j = 1, 2$, $\mathcal{X}_{j,\mathbb{Q}}$ is smooth and $\mathrm{Sys}_{\mathcal{X}_j}$ (with $\varphi_* = 0 = \varphi_*^T$), d and π satisfy hypotheses 1–3 of Lemma 31.21; so $\mathcal{X}_j$ has convergent regularisation.

Step 2: the common model. Take $\mathcal{X}$, p_j, e_j, X, π, the data $\theta_i, u_i, u_{i,t}$, the isomorphisms λ_r and the hermitian line bundles $(\mathcal{M}_r, h_r^{(j)}(t))$ of Proposition 31.28. From $e_1 \deg_1 = e_2 \deg_2$ we get $e_1\delta_1 = e_2\delta_2 =: \delta > 0$.

Step 3: part 1. Let $x^m = x_{r_1} \cdots x_{r_{D+1}}$ with $r_k \in \mathrm{Rt}$; by the definition of $\sigma_g\cdot$ (substitution $x_{(\sigma_g(\tau),i)}$ for $x_{(\tau,i)}$), $\sigma_g \cdot x^m = x_{\sigma_g r_1} \cdots x_{\sigma_g r_{D+1}}$. Let $0 < t \le t_0$. For $j = 1, 2$ we apply Lemma 30.13 with $b = D+1$, $d = 0$ and the morphism $p_j \colon \mathcal{X} \to \mathcal{X}_j$. Its hypotheses hold: $\mathcal{X}$ and $\mathcal{X}_j$ are arithmetic varieties of absolute dimension $D+1$ with smooth generic fibres (Proposition 31.28 and the hypothesis on $\mathcal{X}_j$; an integral scheme flat and projective over $\operatorname{Spec} \mathcal{O}_F$ is flat and projective over $\operatorname{Spec} \mathbb{Z}$, as $\mathcal{O}_F$ is finite and flat over $\mathbb{Z}$); the hermitian line bundles $(\mathcal{O}(e_r)|_{\mathcal{X}_j}, h_{j,r}(t))$ have smooth metrics, the root metrics being cone metrics (Definition 29.4); and p_j is surjective, so the factor of the lemma is $c = [\mathbf{k}(\mathcal{X}) : \mathbf{k}(\mathcal{X}_j)] = e_j$, where $\mathbf{k}(\cdot)$ denotes the function field. By (29.2) and this application for p_1,

$$\delta \operatorname{ai}_t(\mathcal{X}_1; x^m) = e_1 \widehat{\deg}\Big(\prod_k \widehat{c}_1(\mathcal{O}(e_{r_k})|_{\mathcal{X}_1}, h_{1,r_k}(t))\Big) = \widehat{\deg}\Big(\prod_k \widehat{c}_1(\mathcal{M}_{r_k}, h_{r_k}^{(1)}(t))\Big).$$

Likewise, by the same application for p_2 and since λ_{r_k} is an isometry from $(\mathcal{M}_{r_k}, h_{r_k}^{(2)}(t))$ onto $p_2^*(\mathcal{O}(e_{\sigma_g r_k})|_{\mathcal{X}_2}, h_{2,\sigma_g r_k}(t))$ (by the definition of $h_{r_k}^{(2)}(t) = \lambda_{r_k}^* p_2^* h_{2,\sigma_g r_k}(t)$),

$$\delta \operatorname{ai}_t(\mathcal{X}_2; \sigma_g \cdot x^m) = e_2 \widehat{\deg}\Big(\prod_k \widehat{c}_1(\mathcal{O}(e_{\sigma_g r_k})|_{\mathcal{X}_2}, h_{2,\sigma_g r_k}(t))\Big) = \widehat{\deg}\Big(\prod_k \widehat{c}_1(\mathcal{M}_{r_k}, h_{r_k}^{(2)}(t))\Big).$$

By Proposition 31.28 the line bundles $\mathcal{M}_{r_1}, \dots, \mathcal{M}_{r_{D+1}}$ with these two families of metrics satisfy hypotheses 1 and 2 of Lemma 31.34 on $\mathcal{X}$ over (X, π); that lemma gives that the difference of the two right-hand sides tends to 0 as $t \to 0^+$. Dividing by δ gives the limit in part 1 for every monomial of degree $D+1$ (repetitions among the r_k allowed).

For $F \in \mathcal{R}$ write $F = \sum_m a_m x^m$. The map $F \mapsto \sigma_g \cdot F$ is a ring automorphism of $\mathcal{R}$ permuting the monomials and preserving their degrees, and ai_t vanishes on monomials of degree $\ne D+1$. By Step 1, $\operatorname{pp}(\mathcal{X}_j; \cdot) = \lim_{t\to 0^+} \operatorname{ai}_t(\mathcal{X}_j; \cdot)$, so

$$\operatorname{pp}(\mathcal{X}_1; F) - \operatorname{pp}(\mathcal{X}_2; \sigma_g \cdot F) = \sum_{\deg x^m = D+1} a_m \lim_{t\to 0^+} \big(\operatorname{ai}_t(\mathcal{X}_1; x^m) - \operatorname{ai}_t(\mathcal{X}_2; \sigma_g \cdot x^m)\big) = 0.$$

Step 4: part 2. Let $g = \mathrm{id}$, $\mathrm{tw} = \mathrm{id}$, so $\sigma_g = \mathrm{id}$ and $\lambda_r \colon \mathcal{M}_r \to p_2^*(\mathcal{O}(e_r)|_{\mathcal{X}_2})$. For a monomial $x_{r_1} \cdots x_{r_D}$ ($r_k \in \mathrm{Rt}$), the morphisms p_j restrict to surjective morphisms $\mathcal{X}_{\mathbb{Q}} \to \mathcal{X}_{j,\mathbb{Q}}$, generically finite of the same degrees e_j (the degree of the extension of function fields), so the geometric projection formula gives

$$\delta \operatorname{gi}(\mathcal{X}_j; x_{r_1} \cdots x_{r_D}) = e_j \deg\Big(\prod_k c_1(\mathcal{O}(e_{r_k})) \cap [\mathcal{X}_{j,\mathbb{Q}}]\Big) = \deg\Big(\prod_k c_1\big(p_j^*\mathcal{O}(e_{r_k})|_{\mathcal{X}_j}\big) \cap [\mathcal{X}_{\mathbb{Q}}]\Big),$$

and the right-hand sides for $j = 1, 2$ agree because λ_{r_k} identifies $p_1^*(\mathcal{O}(e_{r_k})|_{\mathcal{X}_1})$ with $p_2^*(\mathcal{O}(e_{r_k})|_{\mathcal{X}_2})$. So $\operatorname{gi}(\mathcal{X}_1; \cdot)$ and $\operatorname{gi}(\mathcal{X}_2; \cdot)$ agree on the monomials of degree D of $\mathcal{R}$; both vanish on the others, and by linearity they agree on $\mathcal{R}$. □

Proposition 31.28. *Under the hypotheses of Proposition 31.27, write $h_{j,r}(t)$ for the root metrics of $\mathcal{X}_j$. There are: an arithmetic variety $\mathcal{X}$ of dimension $D+1$ with $\mathcal{X}_{\mathbb{Q}}$ smooth; surjective morphisms $p_j \colon \mathcal{X} \to \mathcal{X}_j$ ($j = 1, 2$), generically finite of degrees $e_j \ge 1$, with $e_1 \deg_1 = e_2 \deg_2$; a compact complex manifold X with a Kähler form Θ, data $\varsigma, \partial X, X^\circ, \mathrm{rW}$, an integer $d \ge 1$ and a holomorphic map $\pi \colon \mathcal{X}(\mathbb{C}) \to X$ restricting to a d-sheeted holomorphic covering map off closed null sets; a finite set I, $0 < t_0 < 1$ and data $\theta_i, u_i, u_{i,t}$ ($i \in I$, $0 < t \le t_0$) with (R1)–(R5) for*

some $C \geq 1$, δ, R; and for every $r \in \mathrm{Rt}$ an isomorphism of line bundles $\lambda_r \colon p_1^*(\mathcal{O}(e_r)|_{\mathcal{X}_1}) \to p_2^*(\mathcal{O}(e_{\sigma_g r})|_{\mathcal{X}_2})$ on $\mathcal{X}$, such that for every $r \in \mathrm{Rt}$ the line bundle $\mathcal{M}_r = p_1^*(\mathcal{O}(e_r)|_{\mathcal{X}_1})$ with the metrics $h_r^{(1)}(t) = p_1^* h_{1,r}(t)$ and $h_r^{(2)}(t) = \lambda_r^* p_2^* h_{2,\sigma_g r}(t)$ $(0 < t \leq t_0)$ satisfies hypotheses 1 and 2 of Lemma 31.34.

Proof. Throughout, the hypotheses of Proposition 31.27 are in force; in particular S contains every prime dividing $\mathrm{disc}(E)$.

Step 1: the twist on the flag bundle. For a point x of Sh_F with values in an F-scheme T, ψ gives $x^*\psi \colon x^*G \to (\mathrm{tw}_F \circ x)^*G$, and the computation of part 3 of Lemma 31.31 (from $\psi \circ \iota(g(a)) = (\mathrm{tw}^*\iota)(a) \circ \psi$) shows that its derivative maps $x^*\mathrm{Lie}_s(G)$ onto $(\mathrm{tw}_F \circ x)^*\mathrm{Lie}_{\sigma_g s}(G)$. Hence $(x, (V_s)_s) \mapsto (\mathrm{tw}_F \circ x, (V'_{s'})_{s'})$ with $V'_{\sigma_g s} = d(x^*\psi)(V_s)$ is a morphism $\Theta \colon \mathrm{Fl}_{\mathfrak{u}} \to \mathrm{Fl}_{\mathfrak{u}}$ over tw_F (on functors of points), with inverse given by ψ^{-1} and tw^{-1}: an automorphism. As $\mathrm{tw}_F(Z_{c_1}) = Z_{c_2}$, Θ maps $\mathrm{Fl}_{\mathfrak{u}}|_{Z_{c_1}}$ onto $\mathrm{Fl}_{\mathfrak{u}}|_{Z_{c_2}}$ and their generic points onto each other, inducing a field isomorphism $\Theta^* \colon \Omega_{c_2} \to \Omega_{c_1}$ over $\mathrm{tw}_F^* \colon \mathbf{k}(Z_{c_2}) \to \mathbf{k}(Z_{c_1})$. Writing $u_j \colon \mathrm{Spec}\,\Omega_{c_j} \to \mathrm{Fl}_{\mathfrak{u}}$ for the generic points, $\Theta \circ u_1 = u_2 \circ \mathrm{Spec}(\Theta^*)$, and by the definition of Θ the derivative of $(\pi_{\mathfrak{u}} u_1)^*\psi$ maps the universal flag of Lie_s at c_1 onto the base change along Θ^* of the universal flag of $\mathrm{Lie}_{\sigma_g s}$ at c_2, for every label s.

Step 2: the common function field. Regard Ω_1 and Ω_2 as Ω_{c_2}-algebras via $\gamma_1 \circ \Theta^*$ and γ_2. The Ω_1-algebra $\Omega_1 \otimes_{\Omega_{c_2}} \Omega_2$ is non-zero of dimension $\deg_2$; let Ω_3 be its quotient by a maximal ideal, with embeddings $\iota_j \colon \Omega_j \to \Omega_3$. Then $\iota_1\gamma_1\Theta^* = \iota_2\gamma_2$, Ω_3 is generated by $\iota_1(\Omega_1)\iota_2(\Omega_2)$ and is finite over both; put $e_j = [\Omega_3 : \iota_j(\Omega_j)] \geq 1$. Computing $[\Omega_3 : \iota_2\gamma_2(\Omega_{c_2})]$ through Ω_1 and through Ω_2 (with $[\Omega_j : \gamma_j(\Omega_{c_j})] = \deg_j$ by (Go3)(1) and Θ^* an isomorphism) gives $e_1 \deg_1 = e_2 \deg_2$.

Step 3: the common model. The embeddings ι_j define a morphism $\mathrm{Spec}\,\Omega_3 \to \mathcal{X}_1 \times_{\mathbb{Z}} \mathcal{X}_2$ over the generic points; let $\mathcal{X}'$ be the closure of its image (an integral closed subscheme, projective over $\mathbb{Z}$, with function field the field generated by $\iota_1(\Omega_1)$ and $\iota_2(\Omega_2)$, i.e. Ω_3), and $\mathcal{X}''$ its normalisation, finite over $\mathcal{X}'$ ($\mathcal{X}'$ is of finite type over $\mathbb{Z}$, hence excellent). By Theorem B.142(A) there is a projective birational $R \to \mathcal{X}''_{\mathbb{Q}}$ with R smooth (and integral). Let $\mathcal{A}_R \subseteq R$ be the preimage of the abelian locus of $\mathcal{X}''$ for the structure map $\mathcal{X}'' \to \mathcal{X}_1 \xrightarrow{f_1} \mathcal{Y}$; it is open and contains the generic point. Let I_R be the ideal of the reduced closed subset $R \smallsetminus \mathcal{A}_R$; it is non-zero. By Theorem B.142(B) there is a composite of blow-ups with smooth centres $\Pi \colon R' \to R$, an isomorphism over $\mathcal{A}_R$, with R' smooth and $\Pi^* I_R$ the ideal of a simple normal crossing divisor B'; so $R' \smallsetminus \Pi^{-1}(\mathcal{A}_R)$ is the support of B'. The composite $R' \to \mathcal{X}''_{\mathbb{Q}}$ is projective and birational (Hartshorne II Ex. 4.9), so R' is a closed integral subscheme of $\mathbb{P}^N \times_{\mathbb{Z}} \mathcal{X}''_{\mathbb{Q}}$ for some N, which is the generic fibre of $\mathbb{P}^N_{\mathbb{Z}} \times_{\mathbb{Z}} \mathcal{X}''$. Let $\mathcal{X}_0$ be the closure of R' in $\mathbb{P}^N_{\mathbb{Z}} \times_{\mathbb{Z}} \mathcal{X}''$: an integral closed subscheme, projective over $\mathbb{Z}$ and over $\mathcal{X}''$, with generic fibre R' (as R' is closed in the generic fibre) and function field Ω_3. Let $\mathcal{X}$ be the normalisation of $\mathcal{X}_0$, finite over $\mathcal{X}_0$; its generic fibre is the normalisation of R', which is R' (R' is smooth, hence normal), so $\mathcal{X}_{\mathbb{Q}} \cong R'$. Thus $\mathcal{X}$ is integral, normal, projective and flat over $\mathbb{Z}$ (torsion-free over $\mathbb{Z}$), of dimension $D+1$ (its function field Ω_3 is finite over Ω_1), with $\mathcal{X}_{\mathbb{Q}}$ smooth. Let $p_j \colon \mathcal{X} \to \mathcal{X}_j$ be the composites with the projections. They are dominant and proper, hence surjective, and generically finite of degree e_j. With $f^{(j)} = f_j \circ p_j$, $\mathcal{X}$ is a cover of $\mathcal{Y}$ in two ways (it is normal and integral, its generic point maps to that of $\mathcal{Y}_{c_j}$, which lies in $j(\mathrm{Sh}_F)$ by (Go2), and Ω_3, finite over Ω_j, is finitely generated over the function field of $\mathcal{Y}_{c_j}$). With $x_j = j^{-1} \circ f^{(j)}$ at the generic point, (Go3)(1) gives $x_j = \pi_{\mathfrak{u}} \circ u_j \circ \mathrm{Spec}(\iota_j\gamma_j)$, so by Step 1 and $\iota_2\gamma_2 = \iota_1\gamma_1\Theta^*$,

$$x_2 = \pi_{\mathfrak{u}} \circ u_2 \circ \mathrm{Spec}(\Theta^*) \circ \mathrm{Spec}(\iota_1\gamma_1) = \pi_{\mathfrak{u}} \circ \Theta \circ u_1 \circ \mathrm{Spec}(\iota_1\gamma_1) = \mathrm{tw}_F \circ x_1.$$

So Lemma 31.31 applies to $\mathcal{X}$ with $f^{(1)}, f^{(2)}$: we get $\psi_{\mathcal{X}}$ and $d\psi$, and the two abelian loci are

one open set $\mathcal{A}$, which is the preimage of $\mathcal{A}_R$, so that $\mathcal{X}_{\mathbb{Q}} \smallsetminus \mathcal{A}$ is the support of the simple normal crossing divisor B'; also $\mathcal{A} = p_j^{-1}(\mathcal{A}_j)$, $\mathcal{A}_j$ the abelian locus of $\mathcal{X}_j$.

Step 4: the isomorphisms λ_r. For $j = 1, 2$, $\mathcal{X}_j$ is a noetherian cover satisfying the hypotheses (Hp1), (Hp2) of Lemma 31.29 at (S, R), with $R(\ell) \geq 4 \geq 1$ as in (Fh5) (its $\ell^{R(\ell)}$-torsion is defined over Ω_j), by (Go3)(2), and S contains the primes dividing $\mathrm{disc}(E)$; p_j is dominant with $f^{(j)} = f_j \circ p_j$. This is where the hypothesis on S enters: at points of $\mathcal{X}$ over no prime of S the Hodge lattices are de Rham lattices, and they are the pull-backs of those of $\mathcal{X}_j$ only because there $\mathrm{disc}(E)$ is a unit, so that $\mathcal{O}_E \otimes \mathcal{O}_x \cong \prod_{\sigma \in \mathrm{Hom}(E,F)} \mathcal{O}_x$ and $\mathrm{dR}_{s,x} = \epsilon_s \mathrm{Lie}_x$ is a direct summand of Lie_x (ϵ_s the idempotents), which commutes with pull-back. Without this hypothesis the pulled-back integral flags of $\mathcal{X}_1$ and $\mathcal{X}_2$ need not be flags of the Hodge bundles of $\mathcal{X}$, and Λ_r below would not be defined. By Lemma 31.29(1), the Hodge lattices of $\mathcal{X}$ with $f^{(j)}$ at (S, R) are Hodge bundles, namely $\mathcal{V}_s^{(j)} = \mathcal{O}_{\mathcal{X}} \mathcal{V}_{j,s}$, $\mathcal{V}_{j,s}$ those of $\mathcal{X}_j$. By Lemma 31.31(3),(4), $d\psi$ maps the stalks of $\mathcal{V}_s^{(1)}$ onto those of $\mathcal{V}_{\sigma_g s}^{(2)}$ at every point, so $d\psi(\mathcal{V}_s^{(1)}) = \mathcal{V}_{\sigma_g s}^{(2)}$. Let $(N_{j,s,i})_i$ be the integral flag of $\mathcal{V}_{j,s}$ of (Go3)(3), with generic fibre $(V_{j,s,i})_i$ the image under γ_j of the universal flag at c_j. Its quotients $\mathcal{V}_{j,s}/N_{j,s,i}$ are locally free, hence torsion-free, so $N_{j,s,i} = \mathcal{V}_{j,s} \cap V_{j,s,i}$; by Lemma 31.33(3) applied to the dominant p_j, $p_j^* N_{j,s,i} = \mathcal{V}_s^{(j)} \cap (V_{j,s,i} \otimes \Omega_3)$ with locally free quotients, and, the sequences $0 \to N_{j,s,i-1} \to N_{j,s,i} \to N_{j,s,i}/N_{j,s,i-1} \to 0$ being locally split, p_j^* of the step quotients are the step quotients of the pulled-back flag; so $p_j^* \mathcal{L}_{j,r}$ is the root line r of the pulled-back flag. By Step 1 (base-changed along $\iota_1 \gamma_1$), $d\psi$ maps $V_{1,s,i} \otimes \Omega_3$ onto $V_{2,\sigma_g s,i} \otimes \Omega_3$; together with $d\psi(\mathcal{V}_s^{(1)}) = \mathcal{V}_{\sigma_g s}^{(2)}$ it maps $p_1^* N_{1,s,i}$ onto $p_2^* N_{2,\sigma_g s,i}$ for all s, i, hence induces isomorphisms of the step quotients. A root $r = (\tau, i)$ and $\sigma_g r = (\tau \circ g, i)$ have labels $s(r)$ and $\sigma_g s(r)$ with the same step and sign (Fh3), so $d\psi$ induces an isomorphism $\Lambda_r \colon p_1^* \mathcal{L}_{1,r} \to p_2^* \mathcal{L}_{2,\sigma_g r}$ (on dual lines, the inverse transpose). Put

$$\lambda_r = (p_2^* \varrho_{2,\sigma_g r})^{-1} \circ \Lambda_r \circ p_1^* \varrho_{1,r} \colon \; p_1^*(\mathcal{O}(e_r)|_{\mathcal{X}_1}) \to p_2^*(\mathcal{O}(e_{\sigma_g r})|_{\mathcal{X}_2}),$$

an isomorphism of line bundles defined over $\mathbb{Z}$.

Step 5: the singular Hodge lines agree. Let $z \in \mathcal{A}(\mathbb{C})$. The point z lies in $\mathcal{X}_{\mathbb{Q}}$, over no prime, and at every point x of the generic fibre $\mathcal{X}_{\mathbb{Q}}$ the Hodge bundle $\mathcal{V}_s^{(j)}$ has stalk $\mathrm{dR}_{s,x} = \epsilon_s \mathrm{Lie}_x$ (as $\mathbb{Q} \subseteq \mathcal{O}_x$, $\mathcal{O}_E \otimes \mathcal{O}_x \cong \prod_\sigma \mathcal{O}_x$ with idempotents ϵ_s), so the restriction of $\mathcal{V}_s^{(j)}$ to $\mathcal{X}_{\mathbb{Q}}$ is $\mathrm{Lie}_s(G^{(j)}_{\mathcal{X}_{\mathbb{Q}}})$, a direct summand of $\mathrm{Lie}(G^{(j)}_{\mathcal{X}_{\mathbb{Q}}})$, and the isomorphism $\mathcal{V}_s^{(1)} \to \mathcal{V}_{\sigma_g s}^{(2)}$ given by $d\psi$ is there the restriction of the derivative of $\psi_{\mathcal{X}}$; its fibre at z is $d\psi_z$. By (Fh6), the fibre of $p_j^* \mathcal{L}_{j,r}$ at z is the root line of $G_z^{(j)}$ for the flag induced by the integral flag, with its Hodge metric, and the fibre of Λ_r at z is induced by $d\psi_z$ on step quotients. By Lemma 31.31(5) it is an isometry for the Hodge metrics (dual norms for sign -1). As $p_j(z) \in \mathcal{A}_j(\mathbb{C})$, (Go3)(4) for $\mathcal{X}_j$ says that $\varrho_{j,r}$ carries the metric with cone function $h_{j,r}^{\mathrm{sing}}$ to the Hodge metric. Hence, writing $h_{j,r}^{\mathrm{sing}}$ also for the (not necessarily smooth) metric on $\mathcal{O}(e_r)|_{\mathcal{A}_j(\mathbb{C})}$ with this cone function, we get the equality of metrics on $\mathcal{M}_r$ over $\mathcal{A}(\mathbb{C})$

$$p_1^* h_{1,r}^{\mathrm{sing}} = \lambda_r^* p_2^* h_{2,\sigma_g r}^{\mathrm{sing}} \qquad (r \in \mathrm{Rt}) \qquad (1).$$

Step 6: the analytic system on $\mathcal{X}$. Put $X = \mathcal{X}(\mathbb{C}) = R'(\mathbb{C})$, a compact complex manifold of dimension D, with Θ the restriction of a Fubini–Study form of a projective embedding (a Kähler form), $d = 1$ and $\pi = \mathrm{id}$ (a 1-sheeted covering, $N = N' = \varnothing$). Let $F_j = \pi_j \circ p_j \colon X \to X_j$, holomorphic. By the hypotheses of Proposition 31.27, $\pi_j^{-1}(X_j^\circ) = \mathcal{A}_j(\mathbb{C})$, so $F_j^{-1}(X_j^\circ) = \mathcal{A}(\mathbb{C})$ and $F_j^{-1}(\partial X_j) = X \smallsetminus \mathcal{A}(\mathbb{C})$. Let $\varsigma = F_1^* \varsigma_1 \otimes F_2^* \varsigma_2$, a holomorphic section of the tensor product of the pulled-back hermitian line bundles, with $\|\varsigma\| \leq 1$; its zero set is $X \smallsetminus \mathcal{A}(\mathbb{C})$, the support of $B'_{\mathbb{C}}$, a simple normal crossing divisor (smooth components and local coordinates persist under

the base change $\mathbb{Q} \to \mathbb{C}$), so its zero divisor ∂X, an effective divisor supported on it, has simple normal crossings. Then $X^\circ = \mathcal{A}(\mathbb{C})$ and $\mathrm{rW} = e - 2\log\|F_1^*\varsigma_1\| - 2\log\|F_2^*\varsigma_2\| \geq \mathrm{rW}_j \circ F_j$ on X°, as both logarithms are ≤ 0. So Lemma 31.32 applies to $\mathrm{Sys}_{\mathcal{X}_j}$ and F_j, $j = 1, 2$. Let I be the set of pairs (j, a), a a fundamental index, put $\theta_{(j,a)} = F_j^*\theta_{j,a}$, $u_{(j,a)} = \varphi_{j,a} \circ F_j$, $u_{(j,a),t} = \varphi_{j,a}^T(t) \circ F_j$, $t_0 = \min_j \min(t_{0,j}, e^{-e})$, C the larger of the two constants C_W, $\delta = \max(\delta_1, \delta_2)$ and $R(t) = \log(1/t)$; (R1)–(R5) hold for this family (enlarging C and δ keeps them), and $\theta_{(j,a)} + dd^c u_{(j,a),t} = F_j^*\eta_{j,a}(t)$ with $\eta_{j,a}(t) = c_1(\mathcal{M}_{j,a}, h_{j,a}^{\mathrm{ref}} e^{\varphi_{j,a}^T(t)})$.

Step 7: hypothesis 1 of Lemma 31.34. Let $r = (\tau, i) \in \mathrm{Rt}$, $a = (\tau, i)$, $a^- = (\tau, i-1)$. The multidegree of r is that of a^- minus that of a, and hypothesis 3 of Lemma 31.21 for $\mathcal{X}_j$ (for i and $i-1$) gives $h_{j,r}(t) = h_{j,a^-}^{\mathrm{ref}} e^{\varphi_{j,a^-}^T(t) \circ \pi_j} / h_{j,a}^{\mathrm{ref}} e^{\varphi_{j,a}^T(t) \circ \pi_j}$; with hypothesis 2 there (c_1 of $h_{j,a}^{\mathrm{ref}}$ is $\pi_j^*\theta_{j,a}$),

$$c_1(\mathcal{O}(e_r)|_{\mathcal{X}_j}, h_{j,r}(t)) = \pi_j^*(\eta_{j,a^-}(t) - \eta_{j,a}(t)).$$

Chern forms commute with pull-back and are preserved by isomorphisms carrying one metric to the other, so $c_1(\mathcal{M}_r, h_r^{(1)}(t)) = F_1^*(\eta_{1,a^-}(t) - \eta_{1,a}(t))$ and $c_1(\mathcal{M}_r, h_r^{(2)}(t)) = F_2^*(\eta_{2,\sigma_g a^-}(t) - \eta_{2,\sigma_g a}(t))$: by Step 6 these are $\sum_{i' \in I} n_{i'}(\theta_{i'} + dd^c u_{i',t})$ with coefficients in $\{0, \pm 1\}$ independent of t. The metrics are smooth and invariant under complex conjugation (those of $\mathcal{X}_j$ are, and p_j, λ_r are defined over $\mathbb{Z}$).

Step 8: hypothesis 2 of Lemma 31.34. For $j = 1, 2$ and a fundamental index b put $\Delta_{j,b}(t) = \varphi_{j,b}^T(t) - \varphi_{j,b}$ on X_j° ($\Delta_{j,(\tau,0)} = 0$). By (Go1)(1),(2), $0 \leq \Delta_{j,b}(t) \leq C_j + C_{5,j} \log \mathrm{rW}_j$, and by (Go1)(3) $\Delta_{j,b}(t) \to 0$ uniformly on compact subsets of X_j°. On $\pi_j^{-1}(X_j^\circ)$, dividing the regularised identity of hypothesis 3 of Lemma 31.21 by the singular-weight identity $h_{j,a}^{\mathrm{ref}} e^{\varphi_{j,a} \circ \pi_j} = \prod_{k \leq i} (h_{j,(\tau,k)}^{\mathrm{sing}})^{-1}$ of Proposition 31.27, for a and a^-, gives $\log(h_{j,r}(t)/h_{j,r}^{\mathrm{sing}}) = -(\Delta_{j,a}(t) - \Delta_{j,a^-}(t)) \circ \pi_j$. Let $g_{r,t} = \log(h_r^{(1)}(t)/h_r^{(2)}(t))$, a smooth function on X (both metrics are smooth). On $X^\circ = \mathcal{A}(\mathbb{C})$, by (1),

$$g_{r,t} = -(\Delta_{1,a}(t) - \Delta_{1,a^-}(t)) \circ F_1 + (\Delta_{2,\sigma_g a}(t) - \Delta_{2,\sigma_g a^-}(t)) \circ F_2.$$

As $\mathrm{rW}_j \circ F_j \leq \mathrm{rW}$ and $\log \mathrm{rW} \geq 1$ on X°, $|g_{r,t}| \leq 2\sum_j (C_j + C_{5,j}) \log \mathrm{rW}$ there; and for a compact $K \subseteq X^\circ$, $F_j(K)$ is a compact subset of X_j°, so $g_{r,t} \to 0$ uniformly on K as $t \to 0^+$. Thus, with $\pi = \mathrm{id}$, hypothesis 2 holds for all r and $0 < t \leq t_0$.

Steps 2–8 give $\mathcal{X}$, p_j, e_j with $e_1 \deg_1 = e_2 \deg_2$, X, Θ, ς, $d = 1$, π, I, t_0, the data with (R1)–(R5), and the isomorphisms λ_r, with hypotheses 1 and 2 of Lemma 31.34 for every $r \in \mathrm{Rt}$. □

Lemma 31.29. *Let $\mathcal{X}$, $\mathcal{X}'$ be locally noetherian covers of $\mathcal{Y}$ with structure maps f, f' and function fields $\Omega \subseteq \Omega'$, and $\phi\colon \mathcal{X}' \to \mathcal{X}$ a dominant morphism with $f' = f \circ \phi$; identify $\mathrm{Li}_s' = \mathrm{Lie}_s(G_{\mathcal{X}',\Omega'})$ with $\Omega' \otimes_\Omega \mathrm{Li}_s$. Let S be a finite set of primes containing every prime that divides $\mathrm{disc}(E)$, and $R(\ell) \geq 1$ ($\ell \in S$) integers as in (Fh5) for $\mathcal{X}$. For every label s let $\rho_s\colon \mathrm{Li}^\vee \to \mathrm{Li}_s^\vee$ be the restriction of linear forms. Assume:*

(Hp1) *the Hodge lattices of $\mathcal{X}$ at (S, R) form Hodge bundles $\mathcal{V}_s \subseteq \mathrm{Li}_s$;*

(Hp2) *for every $\ell \in S$, every label s and every point x of $\mathcal{X}$ over ℓ, the $\mathcal{O}_x$-module $\rho_s(\mathrm{ht}_x)$ is free, where ht_x is formed with $R = R(\ell)$.*

Then:

1. *the Hodge lattices of $\mathcal{X}'$ at (S, R) (with the same integers $R(\ell)$) form Hodge bundles, namely $\mathcal{V}_s' = \mathcal{O}_{\mathcal{X}'}\mathcal{V}_s \subseteq \mathrm{Li}_s'$, the $\mathcal{O}_{\mathcal{X}'}$-span of $\mathcal{V}_s$; the natural map $\phi^*\mathcal{V}_s \to \mathcal{V}_s'$ is an isomorphism;*

and (Hp1), (Hp2) hold for $\mathcal{X}'$, with $\rho_s(\mathrm{ht}_{x'}) = \mathcal{O}_{x'}\rho_s(\mathrm{ht}_{\phi(x')})$ at every point x' over $\ell \in S$; for every $\ell \in S$, every $\ell^{R(\ell)}$-torsion point of $G_{\mathcal{X}'}$ over an algebraic closure of Ω' is defined over Ω';

2. *if $(\mathcal{V}_{s,i})_i$ is an integral flag of $\mathcal{X}$, then $(\mathcal{O}_{\mathcal{X}'}\mathcal{V}_{s,i})_i$ is an integral flag of $\mathcal{X}'$, $\mathcal{O}_{\mathcal{X}'}\mathcal{V}_{s,i} = \mathcal{V}'_s \cap \Omega'\mathcal{V}_{s,i}$, and its root lines are $\mathcal{L}'_r = \mathcal{O}_{\mathcal{X}'}\mathcal{L}_r \cong \phi^*\mathcal{L}_r$, with generic fibre the extension of scalars to Ω' of the generic root line of $\mathcal{X}$;*

3. *if $\ell \in S$ and ht_x is a free $\mathcal{O}_x$-module of rank $\dim \mathrm{Li}$ at every point x of $\mathcal{X}$ over ℓ, then $\mathrm{ht}_{x'} = \mathcal{O}_{x'}\mathrm{ht}_{\phi(x')}$ is a free $\mathcal{O}_{x'}$-module of rank $\dim \mathrm{Li}' = \dim \mathrm{Li}$ at every point x' of $\mathcal{X}'$ over ℓ.*

Proof. **Step 0: spans of locally free subsheaves.** Let $\mathcal{M}$ be a locally free $\mathcal{O}_\mathcal{X}$-submodule of rank r of a constant sheaf V of Ω-dimension r, and $V' = \Omega' \otimes_\Omega V$. Near a point x, $\mathcal{M}$ has a basis $v_1, \dots, v_r$, an Ω-basis of V and hence an Ω'-basis of V'. So near every x' with $\phi(x') = x$, $\mathcal{O}_{\mathcal{X}'}\mathcal{M}$ is free with basis $v_1, \dots, v_r$, its stalk is $(\mathcal{O}_{\mathcal{X}'}\mathcal{M})_{x'} = \mathcal{O}_{x'}\mathcal{M}_x$, and $\phi^*\mathcal{M} \to \mathcal{O}_{\mathcal{X}'}\mathcal{M}$ sends the basis $1 \otimes v_k$ to the basis v_k, so it is an isomorphism. If $\mathcal{N} \subseteq \mathcal{M}$ is a locally free submodule with $\mathcal{M}/\mathcal{N}$ locally free, then near x a basis of $\mathcal{N}$ extends to a basis of $\mathcal{M}$, so $\mathcal{O}_{\mathcal{X}'}\mathcal{N}$ is spanned by part of a basis of $\mathcal{O}_{\mathcal{X}'}\mathcal{M}$, the quotient is free on the remaining vectors, and $\mathcal{O}_{\mathcal{X}'}\mathcal{N} = \mathcal{O}_{\mathcal{X}'}\mathcal{M} \cap \Omega'\mathcal{N}$ (a vector of $\mathcal{O}_{\mathcal{X}'}\mathcal{M}$ whose coordinates on the complementary basis vectors vanish has its other coordinates in $\mathcal{O}$). The same holds for duals: if $\mathcal{M} \subseteq V^\vee$ is locally free of rank r with local basis (λ_k), its dual lattice $\{e \in V : \lambda(e) \in \mathcal{O}\ \forall \lambda \in \mathcal{M}\}$ is free on the dual basis, so the dual lattice of $\mathcal{O}_{\mathcal{X}'}\mathcal{M}$ is $\mathcal{O}_{\mathcal{X}'}$ times the dual lattice of $\mathcal{M}$.

Step 1: points over no prime of S. Let x' be a point of $\mathcal{X}'$ and $x = \phi(x')$, and suppose x' lies over no prime of S; then neither does x ($\mathcal{O}_x \subseteq \mathcal{O}_{x'}$ by a local homomorphism, so a prime is a unit in $\mathcal{O}_x$ if it is one in $\mathcal{O}_{x'}$, and conversely a prime invertible in $\mathcal{O}_x$ is invertible in $\mathcal{O}_{x'}$). Since $G_{\mathcal{X}'} = \phi^*G_\mathcal{X}$ and $\mathrm{Lie}(G_\mathcal{X})$ is locally free, $\mathrm{Lie}_{x'} = \mathcal{O}_{x'}\mathrm{Lie}_x$ inside Li'. Every prime that is not a unit in $\mathcal{O}_x$ is outside S, hence prime to $\mathrm{disc}(E)$; and $\sigma(\mathcal{O}_E) \subseteq \mathcal{O}_F \subseteq \mathcal{O}_x$ for $\sigma \in \mathrm{Hom}(E, F)$. So $\mathcal{O}_E \otimes_\mathbb{Z} \mathcal{O}_x \to \prod_\sigma \mathcal{O}_x$, $a \otimes b \mapsto (\sigma(a)b)_\sigma$, is an isomorphism (its discriminant is $\mathrm{disc}(E)$, a unit in $\mathcal{O}_x$), with idempotents ϵ_s ($\sigma = \sigma_s$), and these act on Lie_x through the integral $\mathcal{O}_E$-action. Hence $\mathrm{dR}_{s,x} = \mathrm{Lie}_x \cap \mathrm{Li}_s = \epsilon_s\mathrm{Lie}_x$, and likewise at x', so

$$\mathrm{dR}_{s,x'} = \epsilon_s\mathrm{Lie}_{x'} = \epsilon_s\mathcal{O}_{x'}\mathrm{Lie}_x = \mathcal{O}_{x'}\epsilon_s\mathrm{Lie}_x = \mathcal{O}_{x'}\mathrm{dR}_{s,x} = \mathcal{O}_{x'}\mathcal{V}_{s,x} = (\mathcal{O}_{\mathcal{X}'}\mathcal{V}_s)_{x'},$$

using (Hp1) at x and Step 0.

Step 2: points over $\ell \in S$. Let x' lie over $\ell \in S$ and $x = \phi(x')$, which lies over ℓ (as in Step 1). Every $\ell^{R(\ell)}$-torsion point of $G_{\mathcal{X}'}$ is defined over Ω' and $\mathrm{ht}_{x'} = \mathcal{O}_{x'}\mathrm{ht}_x$ by Lemma 31.30(2) (applied with $\mathrm{pr} = \phi$, $p = \ell$, $R = R(\ell)$). Restriction to Li'_s is $\mathcal{O}_{x'}$-linear, so $\rho_s(\mathrm{ht}_{x'}) = \mathcal{O}_{x'}\rho_s(\mathrm{ht}_x)$. By (Hp2), $\rho_s(\mathrm{ht}_x)$ is free over $\mathcal{O}_x$ on some $\lambda_1, \dots, \lambda_k$; these span $\mathrm{Li}_s^\vee$ over Ω (since $\ell^{R(\ell)}\omega_x \subseteq \mathrm{ht}_x$ by Lemma 31.30(1), and $\rho_s(\omega_x)$ spans), so $k = \mathrm{rk}(s)$ and they form an Ω-basis of $\mathrm{Li}_s^\vee$, hence an Ω'-basis of $\mathrm{Li}_s'^\vee$; so $\rho_s(\mathrm{ht}_{x'})$ is free over $\mathcal{O}_{x'}$ on the same elements: this is (Hp2) for $\mathcal{X}'$. $\mathrm{HT}'_{s,x}$ is the dual lattice of $\rho_s(\mathrm{ht}_x)$ in Li_s (a form $w \in \mathrm{ht}_x$ takes on $e \in \mathrm{Li}_s$ the value $\rho_s(w)(e)$), free on the dual basis of (λ_j), and likewise $\mathrm{HT}'_{s,x'}$ is free on the same dual basis; so $\mathrm{HT}'_{s,x'} = \mathcal{O}_{x'}\mathrm{HT}'_{s,x}$, both free. For $s = (\tau, -)$ this is $\mathrm{HT}_{s,x'}(\ell^{R(\ell)}) = \mathcal{O}_{x'}\mathrm{HT}_{s,x}(\ell^{R(\ell)})$. For $s = (\tau, +)$: the ideal $\sigma_s(\mathfrak{d}_E)\mathcal{O}_F$ is an invertible ideal of the Dedekind ring $\mathcal{O}_F$, so its extension to the local ring $\mathcal{O}_x$ is principal, generated by some $j \in \mathcal{O}_x$, and its extension to $\mathcal{O}_{x'}$ is $j\mathcal{O}_{x'}$. For a lattice M, $\{e : \sigma_s(d)e \in M\ \forall d \in \mathfrak{d}_E\} = \{e : je \in M\} = j^{-1}M$ (j is an $\mathcal{O}$-combination of the $\sigma_s(d)$, and each $\sigma_s(d)$ is a multiple of j). Hence $\mathrm{HT}_{s,x'}(\ell^{R(\ell)}) = j^{-1}\mathcal{O}_{x'}\mathrm{HT}'_{s,x} = \mathcal{O}_{x'}\mathrm{HT}_{s,x}(\ell^{R(\ell)})$. In both cases $V^S_{s,x'} = \mathcal{O}_{x'}V^S_{s,x} = \mathcal{O}_{x'}\mathcal{V}_{s,x} = (\mathcal{O}_{\mathcal{X}'}\mathcal{V}_s)_{x'}$.

Step 2a: freeness of ht**.** In the situation of (3), let x' lie over ℓ and $x = \phi(x')$. By Lemma 31.30(2), $\mathrm{ht}_{x'} = \mathcal{O}_{x'}\mathrm{ht}_x$. An $\mathcal{O}_x$-basis $w_1, \dots, w_g$ of ht_x ($g = \dim \mathrm{Li}$) spans $\mathrm{ht}_x \otimes \Omega$, which contains $\ell^{R(\ell)}\omega_x \otimes \Omega = \mathrm{Li}^\vee$ by Lemma 31.30(1); so it is an Ω-basis of $\mathrm{Li}^\vee$, hence an Ω'-basis of $\mathrm{Li}'^\vee$, and $w_1, \dots, w_g$ is an $\mathcal{O}_{x'}$-basis of $\mathrm{ht}_{x'}$. This proves (3). The torsion statement in (1) is Lemma 31.30(2).

Step 3: conclusion. By Steps 1 and 2 the Hodge lattices of $\mathcal{X}'$ at (S, R) are the stalks of $\mathcal{V}'_s = \mathcal{O}_{\mathcal{X}'}\mathcal{V}_s$, which is locally free of rank $\mathrm{rk}(s)$ with $\phi^*\mathcal{V}_s \cong \mathcal{V}'_s$ by Step 0: this is (1), (Hp2) for $\mathcal{X}'$ having been shown in Step 2. For (2), the $\mathcal{V}_{s,i}$ are locally free with locally free quotients $\mathcal{V}_{s,i}/\mathcal{V}_{s,i-1}$, so $\mathcal{V}_{s,i}/\mathcal{V}_{s,i'}$ is locally free for $i' \le i$ and $\mathcal{V}_s/\mathcal{V}_{s,i}$ is locally free; by Step 0 applied to $\mathcal{V}_{s,i} \subseteq \mathcal{V}_s$ and to $\mathcal{V}_{s,i-1} \subseteq \mathcal{V}_{s,i}$, the $\mathcal{O}_{\mathcal{X}'}\mathcal{V}_{s,i}$ are locally free, equal to $\mathcal{V}'_s \cap \Omega'\mathcal{V}_{s,i}$, with $\mathcal{O}_{\mathcal{X}'}\mathcal{V}_{s,i}/\mathcal{O}_{\mathcal{X}'}\mathcal{V}_{s,i-1} \cong \phi^*(\mathcal{V}_{s,i}/\mathcal{V}_{s,i-1})$ invertible. Their root lines (Fh3) are the corresponding tensor powers of these quotients, that is $\phi^*\mathcal{L}_r$, realised as $\mathcal{O}_{\mathcal{X}'}\mathcal{L}_r$ inside the extension of scalars to Ω' of the generic root line. □

Lemma 31.30. *Let $\mathcal{X}$ and $\mathcal{X}'$ be locally noetherian covers of an integral model $\mathcal{Y}$ of a unitary Shimura datum, with structure maps f, f' and function fields Ω, Ω', and let $\mathrm{pr}\colon \mathcal{X}' \to \mathcal{X}$ be a dominant morphism with $f' = f \circ \mathrm{pr}$; write $\iota\colon \Omega \to \Omega'$ for the induced embedding. Let p be a prime and $R \ge 1$ an integer such that every p^R-torsion point of $G_{\mathcal{X}}$ over an algebraic closure of Ω is defined over Ω, and put $N = p^R$.*

1. *For every point x of $\mathcal{X}$ over p and every $y \in G_{\mathcal{X}}(\Omega)[N]$ there is a Hodge–Tate value w_y of y at x, and the Hodge–Tate values of y at x are exactly the elements of $w_y + N\omega_x$. In particular $\mathrm{ht}_x = \sum_y \mathcal{O}_x w_y + N\omega_x$.*

2. *Every p^R-torsion point of $G_{\mathcal{X}'}$ over an algebraic closure of Ω' is defined over Ω', and for every point x' of $\mathcal{X}'$ over p, with $x = \mathrm{pr}(x')$,*

$$\mathrm{ht}_{x'} = \mathcal{O}_{x'} \cdot \iota(\mathrm{ht}_x),$$

where $\iota\colon \omega_x \to \omega_{x'} = \mathcal{O}_{x'} \otimes_{\mathcal{O}_x} \omega_x$ is the base change map.

Proof. Step 0: set-up at a point. Let $x \in \mathcal{X}$ lie over p; $\mathcal{O} = \mathcal{O}_{\mathcal{X},x}$ is a noetherian normal local domain with fraction field Ω and p in its maximal ideal. Put $G = G_{\mathcal{X}} \times_{\mathcal{X}} \mathrm{Spec}\,\mathcal{O}$, a semiabelian scheme over $\mathcal{O}$ whose generic fibre is an abelian variety of some dimension g (the generic point of $\mathcal{X}$ maps into $j(\mathrm{Sh}_F)$), principally polarised by $\mathcal{L}$. Let $T = G(\Omega)[N]$; by hypothesis it is all of $G(\bar\Omega)[N]$, a group of order N^{2g}. The Weil pairing $e_N\colon T \times T \to \mu_N(\bar\Omega)$ is perfect and takes values fixed by $\mathrm{Gal}(\bar\Omega/\Omega)$, so $\mu_N(\bar\Omega) \subseteq \Omega$, hence $\mu_N \subseteq \mathcal{O}$ ($\mathcal{O}$ is normal). For $y \in T$ the map $\chi_y = e_N(y, \cdot)\colon T \to \mu_N(\mathcal{O})$ is a homomorphism.

Step 1: the torsion group scheme. On every fibre of G, an extension of an abelian variety by a torus, $[N]$ is a surjective homomorphism of smooth connected group schemes with finite kernel, hence flat; since G is smooth over $\mathcal{O}$, $[N]\colon G \to G$ is flat by the fibrewise criterion, and quasi-finite. So $\Gamma = G[N] = G \times_{[N],G,\epsilon} \mathrm{Spec}\,\mathcal{O}$ is a closed subgroup scheme of G, flat, quasi-finite, separated and of finite type over $\mathcal{O}$, and Γ_Ω is the constant group scheme T (étale, all points rational). Let $e_x \in \Gamma$ be the unit point over the closed point and $A = \mathcal{O}_{\Gamma,e_x}$. As A is $\mathcal{O}$-flat, $A \subseteq A \otimes_{\mathcal{O}} \Omega$, and $A \otimes_{\mathcal{O}} \Omega$ is the semilocal ring of Γ_Ω at the set T_x of points $t \in T$ that specialise to e_x, i.e. $A \otimes_{\mathcal{O}} \Omega = \Omega^{T_x}$; write $\mathrm{ev}_t\colon A \to \Omega$ for the coordinates.

Step 2: every character is regular near the unit. We show: for every homomorphism $\chi\colon T \to \mu_N(\mathcal{O})$ there is $c \in A$ with $\mathrm{ev}_t(c) = \chi(t)$ for all $t \in T_x$. (a) Let $\mathcal{O}^h$ be the henselisation, a normal local domain with fraction field K^h and residue field κ that of $\mathcal{O}$; $\mathcal{O} \to \mathcal{O}^h$ is faithfully flat. Let $\Gamma^h = \Gamma \otimes_{\mathcal{O}} \mathcal{O}^h$, e' its unit point over the closed point and $\Lambda = \mathcal{O}_{\Gamma^h,e'}$. As Λ is a localisation of

$A \otimes_{\mathcal{O}} \mathcal{O}^h$ at a prime over the maximal ideal of A, $A \to \Lambda$ is flat and local, hence faithfully flat. (b) Let $\operatorname{Spec} S \subseteq \Gamma^h$ be an affine open containing e'. The $\mathcal{O}^h$-algebra S is of finite presentation and quasi-finite at every prime, so by Tag 04GG (11) $S = A_1 \times B_1$ with A_1 finite over $\mathcal{O}^h$ and $\mathcal{O}^h \to B_1$ quasi-finite at no prime over the maximal ideal; hence B_1 has no prime over the maximal ideal, and $e' \in \operatorname{Spec} A_1$. By Tag 04GG (10) A_1 is a finite product of local rings, each with its closed point over the closed point of $\operatorname{Spec} \mathcal{O}^h$; the factor containing e' is Λ. So $\Gamma^0 = \operatorname{Spec} \Lambda$ is an open subscheme of Γ^h, finite over $\mathcal{O}^h$, consisting of the points that specialise to e'. It is flat (open in Γ^h) and of finite presentation, so Λ is a free $\mathcal{O}^h$-module of finite rank. (c) Γ^0 is a subgroup scheme. The unit section maps the local scheme $\operatorname{Spec} \mathcal{O}^h$ into the points specialising to e'. The ring $\Lambda \otimes_{\mathcal{O}^h} \Lambda$ is finite over $\mathcal{O}^h$ and its closed fibre $(\Lambda \otimes \kappa) \otimes_\kappa (\Lambda \otimes \kappa)$ is local, $\Lambda \otimes \kappa$ being a local artinian κ-algebra with residue field κ (e' is κ-rational); by Tag 04GG (10) a finite $\mathcal{O}^h$-algebra is a product of local rings each meeting the closed fibre, so $\Lambda \otimes \Lambda$ is local. Every point of $\Gamma^0 \times \Gamma^0$ specialises to its closed point, which multiplication sends to e'; so multiplication maps $\Gamma^0 \times \Gamma^0$ into Γ^0, and likewise the inverse. Thus Γ^0 is a finite locally free commutative group scheme over $\mathcal{O}^h$, with generic fibre the constant group scheme of the subgroup $T^0 \subseteq T$ of points lying in Γ^0; $T^0 \subseteq T_x$, since the image in Γ of a point specialising to e' specialises to e_x. (d) By Cartier duality, $\operatorname{Hom}(\Gamma^0, \mathbb{G}_m)$ is represented by a finite locally free $\mathcal{O}^h$-scheme Γ^{0D}. The character $\chi|_{T^0}$ is a homomorphism from the constant group scheme $T^0_{K^h}$ to $\mathbb{G}_{m,K^h}$, i.e. a K^h-point of Γ^{0D}. Its closure in Γ^{0D} is integral, finite over $\mathcal{O}^h$ and birational to $\operatorname{Spec} K^h$, so its ring lies between $\mathcal{O}^h$ and K^h and is integral over $\mathcal{O}^h$; by normality it is $\mathcal{O}^h$. Hence $\chi|_{T^0}$ extends to a homomorphism $\Gamma^0 \to \mathbb{G}_m$, i.e. there is $u \in \Lambda^\times$ whose value at every $t \in T^0$ is $\chi(t)$. (e) Let $c_0 = (\chi(t))_{t \in T_x} \in A \otimes_{\mathcal{O}} \Omega$ and write $c_0 = a/r$ with $a \in A$, $0 \neq r \in \mathcal{O}$. Under $A \otimes_{\mathcal{O}} \Omega \to \Lambda \otimes_{\mathcal{O}^h} K^h = (K^h)^{T^0}$ (restriction of coordinates to T^0), c_0 maps to the image of u; as $\Lambda \subseteq \Lambda \otimes K^h$, the image of a in Λ is ru. By faithful flatness $r\Lambda \cap A = rA$, so $a = ra'$ with $a' \in A$, and $c = a'$ satisfies $\operatorname{ev}_t(c) = \chi(t)$ for $t \in T_x$ (A is torsion free over $\mathcal{O}$).

Step 3: existence in (1). Let $y \in T$ and $c_y \in A$ as in Step 2 for χ_y. Since Γ is closed in G, $\mathcal{O}_{G,e_x} \to A$ is surjective; choose $\phi \in \mathcal{O}_{G,e_x}$ mapping to c_y, regular on an open neighbourhood U of e_x in G on which $\phi|_{\Gamma \cap U}$ represents c_y. Removing from U the closures of the finitely many points of $T \smallsetminus T_x$ (closed sets not containing e_x), the torsion points of G_Ω in U are exactly those of T_x, and every geometric point g of $G[N]$ over an algebraically closed $k \supseteq \Omega$ lying in U lies over some $t \in T_x$, where $\phi(g) = \operatorname{ev}_t(c_y) = \chi_y(t) = e_N(y, g)$. Hence $w_y = \epsilon^* d\phi \in \omega_x$ is a Hodge–Tate value of y at x, and so is every element of $w_y + N\omega_x$.

Step 4: uniqueness modulo N in (1). Let w be a Hodge–Tate value of y at x with function ϕ' on an open $U' \ni e_x$. Every $t \in T_x$ specialises to e_x, hence lies in U', so the germ of $\phi'|_\Gamma$ at e_x is an element of A with $\operatorname{ev}_t = \chi_y(t)$ for all $t \in T_x$; as $A \subseteq \Omega^{T_x}$ it equals c_y. So $\phi' - \phi$ lies in the ideal I of Γ in $\mathcal{O}_{G,e_x}$, which is generated by the elements $h \cdot (f \circ [N])$ with f in the ideal J of the unit section and $h \in \mathcal{O}_{G,e_x}$. For these, $\epsilon^* d(h\,(f \circ [N])) = h(\epsilon)\,\epsilon^* d(f \circ [N]) = N\,h(\epsilon)\,\epsilon^* df$, because $f \circ [N] \circ \epsilon = f \circ \epsilon = 0$ and $[N]$ induces multiplication by N on ω. Hence $\epsilon^* d(\phi' - \phi) \in N\omega_x$ and $w \in w_y + N\omega_x$. The formula for ht_x follows ($y = 0$ gives the coset $N\omega_x$).

Step 5: (2). $G_{\mathcal{X}'} = f'^* G_{\mathcal{Y}} = \mathrm{pr}^* G_{\mathcal{X}}$, so its generic fibre is $G_{\mathcal{X},\Omega} \otimes_{\Omega,\iota} \Omega'$; an algebraic closure of Ω' contains one of $\iota(\Omega)$, over which all N^{2g} torsion points are already Ω-rational. Hence every N-torsion point of $G_{\mathcal{X}'}$ over $\bar{\Omega}'$ is $\iota_*(y)$ for a unique $y \in T$, and is defined over Ω'. Let x' lie over p and $x = \mathrm{pr}(x')$; as $\mathcal{O}_x \to \mathcal{O}_{x'}$ is local, x lies over p. For $y \in T$ with ϕ, U as in Step 3, let $\mathrm{pr}_G \colon G_{\mathcal{X}'} \to G_{\mathcal{X}}$ be the projection; $\phi \circ \mathrm{pr}_G$ is regular on $\mathrm{pr}_G^{-1}(U)$, which contains the unit point over x'. For a geometric point g' of $G_{\mathcal{X}'}[N]$ in it, with image g, $\phi \circ \mathrm{pr}_G(g') = \phi(g) = e_N(y, g) = e_N(\iota_* y, g')$, the Weil pairing being compatible with base change. Since $G_{\mathcal{X}}$ is smooth, $\omega_{x'} = \mathcal{O}_{x'} \otimes_{\mathcal{O}_x} \omega_x$ and $\epsilon^* d(\phi \circ \mathrm{pr}_G) = \iota(w_y)$. So $\iota(w_y)$ is a Hodge–Tate value of $\iota_* y$ at x', and by (1) for $\mathcal{X}'$ at x' (its local ring is noetherian and normal) the Hodge–Tate

values of $\iota_* y$ at x' are exactly $\iota(w_y) + N\omega_{x'}$. Summing over $y \in T$ and using $N\omega_{x'} = \mathcal{O}_{x'}\iota(N\omega_x)$, $\mathrm{ht}_{x'} = \sum_y \mathcal{O}_{x'}\iota(w_y) + N\omega_{x'} = \mathcal{O}_{x'} \cdot \iota(\mathrm{ht}_x)$. □

Lemma 31.31. *Let $\mathfrak{u}$ be a unitary Shimura datum with $\Phi = \Phi_{\mathsf{H}}$, $\mathcal{Y}$ an integral model of* Sh *over $\mathcal{O}_F$, g an automorphism of E with $g(r) = r$,* tw *an automorphism of the K_{L}-scheme* Sh *with $\mathrm{tw}_F = \mathrm{tw} \times_{K_{\mathsf{L}}} F$, and $\psi\colon G \to \mathrm{tw}^*G$ an isomorphism of abelian schemes over* Sh *with $\psi \circ \iota(g(a)) = (\mathrm{tw}^*\iota)(a) \circ \psi$ for all $a \in \mathcal{O}_E$ and $\psi^*\mathrm{tw}^*\mathcal{L} \cong \mathcal{L}$ as rigidified cubical line bundles. Let $\mathcal{X}$ be a noetherian normal integral scheme with function field Ω and generic point $\eta\colon \operatorname{Spec}\Omega \to \mathcal{X}$, and let $f^{(1)}, f^{(2)}\colon \mathcal{X} \to \mathcal{Y}$ be morphisms each of which makes $\mathcal{X}$ a cover of $\mathcal{Y}$, with $G^{(k)} = f^{(k)*}G_{\mathcal{Y}}$, $\iota^{(k)}$, $\mathcal{L}^{(k)}$, $\mathrm{Li}^{(k)}$, $\mathrm{Li}^{(k)}_s$ and so on. Write $x_k = j^{-1} \circ f^{(k)} \circ \eta\colon \operatorname{Spec}\Omega \to \mathrm{Sh}_F$, and assume $x_2 = \mathrm{tw}_F \circ x_1$. Then:*

1. *there is a unique homomorphism $\psi_{\mathcal{X}}\colon G^{(1)} \to G^{(2)}$ of $\mathcal{X}$-group schemes whose generic fibre is $x_1^*\psi$ (via the identifications $G^{(k)}_\Omega = x_k^*G$ of (Ud7)); it is an isomorphism, and $\psi_{\mathcal{X}} \circ \iota^{(1)}(g(a)) = \iota^{(2)}(a) \circ \psi_{\mathcal{X}}$ for all $a \in \mathcal{O}_E$;*

2. *the abelian loci of $\mathcal{X}$ for $f^{(1)}$ and for $f^{(2)}$ are the same open subset $\mathcal{A}$, $j^{-1} \circ f^{(2)} = \mathrm{tw}_F \circ j^{-1} \circ f^{(1)}$ on $\mathcal{A}$, and $\psi_{\mathcal{X}}|_{\mathcal{A}}$ is the pull-back of ψ along $j^{-1} \circ f^{(1)}|_{\mathcal{A}}$;*

3. *the derivative $d\psi\colon \mathrm{Li}^{(1)} \to \mathrm{Li}^{(2)}$ of $\psi_{\mathcal{X}}$ at the generic point is an Ω-linear isomorphism with $d\psi(\mathrm{Li}^{(1)}_s) = \mathrm{Li}^{(2)}_{\sigma_g s}$ for every label s, and $d\psi(\mathrm{Lie}^{(1)}_x) = \mathrm{Lie}^{(2)}_x$, $d\psi(\mathrm{dR}^{(1)}_{s,x}) = \mathrm{dR}^{(2)}_{\sigma_g s,x}$ at every point x of $\mathcal{X}$;*

4. *if p is a prime and $R \geq 1$ is an integer such that every p^R-torsion point of $G^{(1)}$ over an algebraic closure of Ω is defined over Ω, then the same holds for $G^{(2)}$, and at every point x over p: $\mathrm{ht}^{(1)}_x = \{w \circ d\psi :\ w \in \mathrm{ht}^{(2)}_x\}$ and $d\psi(\mathrm{HT}^{(1)}_{s,x}(p^R)) = \mathrm{HT}^{(2)}_{\sigma_g s,x}(p^R)$ for every label s; hence, for a finite set S of primes and integers $R(p)$ ($p \in S$) with this property, $d\psi(V^{S,(1)}_{s,x}) = V^{S,(2)}_{\sigma_g s,x}$ for every label s and every point x;*

5. *at every complex point z of $\mathcal{A}$, the derivative $d\psi_z\colon \mathrm{Lie}(G^{(1)}_z) \to \mathrm{Lie}(G^{(2)}_z)$ of the fibre of $\psi_{\mathcal{X}}$ is an isometry for the Hodge inner products and maps $\mathrm{Lie}_s(G^{(1)}_z)$ onto $\mathrm{Lie}_{\sigma_g s}(G^{(2)}_z)$; hence for every full flag $(V_k)_k$ of $\mathrm{Lie}_s(G^{(1)}_z)$, $d\psi_z$ maps each step quotient V_k/V_{k-1} isometrically, for the quotient norms of (Fh4), onto the step quotient $d\psi_z(V_k)/d\psi_z(V_{k-1})$ of the flag $(d\psi_z(V_k))_k$.*

Proof. Part 1. By (Ud7), $j^*G_{\mathcal{Y}}$ is the base change of G to Sh_F, compatibly with the $\mathcal{O}_E$-actions and the cubical line bundles; so $G^{(k)}_\Omega = x_k^*G$ and $G^{(2)}_\Omega = x_1^*\mathrm{tw}_F^*G$, and $\psi_\Omega = x_1^*\psi\colon G^{(1)}_\Omega \to G^{(2)}_\Omega$ is an isomorphism with $\psi_\Omega \circ \iota^{(1)}(g(a)) = \iota^{(2)}(a) \circ \psi_\Omega$ and $\psi_\Omega^*\mathcal{L}^{(2)}_\Omega \cong \mathcal{L}^{(1)}_\Omega$. As $G^{(1)}$, $G^{(2)}$ are of finite presentation over the noetherian scheme $\mathcal{X}$, ψ_Ω and ψ_Ω^{-1} are the generic fibres of homomorphisms $\psi_U\colon G^{(1)}_U \to G^{(2)}_U$, $\phi_U\colon G^{(2)}_U \to G^{(1)}_U$ over a dense open $U \subseteq \mathcal{X}$, mutually inverse after shrinking U. The scheme $\mathcal{X}$ is noetherian and normal and $G^{(1)}$, $G^{(2)}$ are semiabelian, so by Theorem B.104 ψ_U, ϕ_U extend uniquely to homomorphisms $\psi_{\mathcal{X}}$, $\phi_{\mathcal{X}}$ over $\mathcal{X}$. The homomorphisms $\phi_{\mathcal{X}} \circ \psi_{\mathcal{X}}$ and id of $G^{(1)}$ agree on U, hence are equal by the uniqueness in Theorem B.104; likewise $\psi_{\mathcal{X}} \circ \phi_{\mathcal{X}} = \mathrm{id}$. For $a \in \mathcal{O}_E$ the homomorphisms $\psi_{\mathcal{X}} \circ \iota^{(1)}(g(a))$ and $\iota^{(2)}(a) \circ \psi_{\mathcal{X}}$ agree at the generic point, hence on a dense open, hence everywhere by the same uniqueness. Any homomorphism with generic fibre ψ_Ω agrees with $\psi_{\mathcal{X}}$ on a dense open, so $\psi_{\mathcal{X}}$ is unique.

Part 2. The abelian locus for $f^{(k)}$ is $(f^{(k)})^{-1}(j(\mathrm{Sh}_F))$, open since j is an open immersion into the open subscheme $\mathcal{Y} \times_{\mathcal{O}_F} F = \mathcal{Y}_{\mathbb{Q}}$. By (Ud7) a point y of $\mathcal{Y}_{\mathbb{Q}}$ lies in $j(\mathrm{Sh}_F)$ if and only if the fibre of $G_{\mathcal{Y}}$ at y is an abelian variety; so a point x of $\mathcal{X}$ lies in the abelian locus for $f^{(k)}$

if and only if $x \in \mathcal{X}_{\mathbb{Q}}$ and $G_x^{(k)}$ is an abelian variety. Since $\psi_{\mathcal{X}}$ induces $G_x^{(1)} \cong G_x^{(2)}$, the two loci coincide; call this open set $\mathcal{A}$; it contains η (covers map the generic point into $j(\mathrm{Sh}_F)$). The morphisms $j^{-1} \circ f^{(2)}|_{\mathcal{A}}$ and $\mathrm{tw}_F \circ j^{-1} \circ f^{(1)}|_{\mathcal{A}}$ from the reduced scheme $\mathcal{A}$ to the scheme Sh_F, of finite type and separated over F, agree at the generic point by hypothesis, hence on a dense open subscheme, hence on $\mathcal{A}$. So $G_{\mathcal{A}}^{(2)} = (j^{-1}f^{(1)}|_{\mathcal{A}})^*\mathrm{tw}_F^*G$, and $(j^{-1}f^{(1)}|_{\mathcal{A}})^*\psi$ and $\psi_{\mathcal{X}}|_{\mathcal{A}}$ are homomorphisms $G_{\mathcal{A}}^{(1)} \to G_{\mathcal{A}}^{(2)}$ with the same generic fibre; by Theorem B.104 on the noetherian normal scheme $\mathcal{A}$ they are equal.

Part 3. The isomorphism $\psi_{\mathcal{X}}$ induces an isomorphism of the locally free $\mathcal{O}_{\mathcal{X}}$-modules $\mathrm{Lie}(G^{(1)}) \to \mathrm{Lie}(G^{(2)})$, whose generic fibre is $d\psi$; taking stalks, $d\psi(\mathrm{Lie}_x^{(1)}) = \mathrm{Lie}_x^{(2)}$. For $v \in \mathrm{Li}_s^{(1)}$ and $a \in \mathcal{O}_E$, by part 1, $\iota^{(2)}(a)\,d\psi(v) = d\psi(\iota^{(1)}(g(a))v) = \sigma_s(g(a))\,d\psi(v)$. Now $\sigma_s \circ g = \sigma_{\sigma_g s}$: for $s = (\tau, +)$, $\sigma_s \circ g = \tau \circ g$, and $\tau \circ g \in \Phi$ because $(\tau \circ g)(r) = \tau(r) = r_{\mathbb{C}}$; for $s = (\tau, -)$, $\sigma_s \circ g = \overline{\tau \circ g} = \sigma_{(\tau \circ g, -)}$. Hence $d\psi(\mathrm{Li}_s^{(1)}) \subseteq \mathrm{Li}_{\sigma_g s}^{(2)}$ for every s; as $s \mapsto \sigma_g s$ is a bijection of the labels, $\mathrm{Li}^{(k)} = \bigoplus_s \mathrm{Li}_s^{(k)}$ and $d\psi$ is bijective, equality holds. Intersecting, $d\psi(\mathrm{Lie}_x^{(1)} \cap \mathrm{Li}_s^{(1)}) = \mathrm{Lie}_x^{(2)} \cap \mathrm{Li}_{\sigma_g s}^{(2)}$.

Part 4. Let $\bar{\Omega}$ be an algebraic closure of Ω. The isomorphism ψ_Ω maps $G^{(1)}(\bar{\Omega})[p^R]$ bijectively and Galois-equivariantly onto $G^{(2)}(\bar{\Omega})[p^R]$, so the first set is pointwise fixed by $\mathrm{Gal}(\bar{\Omega}/\Omega)$ if and only if the second is. Since $\psi_\Omega^*\mathcal{L}_\Omega^{(2)} \cong \mathcal{L}_\Omega^{(1)}$, ψ_Ω respects the polarisations, hence the Weil pairings: $e_{p^R}(\psi y, \psi h) = e_{p^R}(y, h)$ for p^R-torsion points y, h over any algebraically closed field $k \supseteq \Omega$. Let x lie over p, $y \in G^{(1)}(\Omega)[p^R]$, and let $w \in \omega_x^{(2)}$ be a Hodge–Tate value of $\psi(y)$ at x, with a regular function ϕ on an open neighbourhood $\mathcal{W}$ of the unit point over x in $G^{(2)}$ such that $\phi(h') = e_{p^R}(\psi(y), h')$ for all geometric p^R-torsion points h' and $w(v') \equiv d\phi(v') \bmod p^R\mathcal{O}_x$ for $v' \in \mathrm{Lie}_x^{(2)}$. Then $\phi \circ \psi_{\mathcal{X}}$ is regular on the open neighbourhood $\psi_{\mathcal{X}}^{-1}(\mathcal{W})$ of the unit point over x in $G^{(1)}$, $(\phi \circ \psi_{\mathcal{X}})(h) = e_{p^R}(\psi y, \psi h) = e_{p^R}(y, h)$, and by the chain rule along the unit section $d(\phi \circ \psi_{\mathcal{X}})(v) = d\phi(d\psi(v)) \equiv w(d\psi(v)) \bmod p^R\mathcal{O}_x$ for $v \in \mathrm{Lie}_x^{(1)}$; as $d\psi(\mathrm{Lie}_x^{(1)}) = \mathrm{Lie}_x^{(2)}$, $w \circ d\psi \in \omega_x^{(1)}$ is a Hodge–Tate value of y at x. Applying this to $\psi_{\mathcal{X}}^{-1}$ (which satisfies the same hypotheses with g^{-1}, tw^{-1}) gives the converse, and since $y \mapsto \psi(y)$ is a bijection of the torsion points, $\mathrm{ht}_x^{(1)} = \{w \circ d\psi : w \in \mathrm{ht}_x^{(2)}\}$. Therefore, using $d\psi(\mathrm{Li}_s^{(1)}) = \mathrm{Li}_{\sigma_g s}^{(2)}$,

$$d\psi(\mathrm{HT}'^{(1)}_{s,x}) = \{e' \in \mathrm{Li}_{\sigma_g s}^{(2)} :\ w(e') \in \mathcal{O}_x\ \forall w \in \mathrm{ht}_x^{(2)}\} = \mathrm{HT}'^{(2)}_{\sigma_g s, x}.$$

For $s = (\tau, -)$ this is the claim. For $s = (\tau, +)$, $\sigma_{\sigma_g s}(d) = \tau(g(d))$ and $g(\mathfrak{d}_E) = \mathfrak{d}_E$ (the different is stable under automorphisms of E), so, $d\psi$ being Ω-linear,

$$\begin{aligned} d\psi(\mathrm{HT}^{(1)}_{s,x}(p^R)) &= \{e' :\ \tau(d)e' \in \mathrm{HT}'^{(2)}_{\sigma_g s, x}\ \forall d \in \mathfrak{d}_E\} \\ &= \{e' :\ \tau(g(d))e' \in \mathrm{HT}'^{(2)}_{\sigma_g s, x}\ \forall d \in \mathfrak{d}_E\} = \mathrm{HT}^{(2)}_{\sigma_g s, x}(p^R). \end{aligned}$$

The Hodge lattice at (S, R) is the Hodge–Tate lattice at points over $p \in S$ and the de Rham lattice elsewhere; with part 3 this gives $d\psi(V^{S,(1)}_{s,x}) = V^{S,(2)}_{\sigma_g s, x}$.

Part 5. Let z be a complex point of $\mathcal{A}$ over an embedding $F \to \mathbb{C}$. By part 2, $\psi_z \colon G_z^{(1)} \to G_z^{(2)}$ is an isomorphism of complex abelian varieties with $\psi_z^*\mathcal{L}_z^{(2)} \cong \mathcal{L}_z^{(1)}$ (the fibre at z of the pull-back of $\psi^*\mathrm{tw}^*\mathcal{L} \cong \mathcal{L}$). Its derivative $d\psi_z$ maps the period lattice $\Lambda_z^{(1)}$ onto $\Lambda_z^{(2)}$ (the uniformisation $\mathrm{Lie}(A)/\Lambda \cong A(\mathbb{C})$ is functorial), and the hermitian form H attached to the first Chern class of a line bundle is functorial under pull-back, so $H_z^{(2)}(d\psi_z u, d\psi_z v) = H_z^{(1)}(u, v)$: $d\psi_z$ is an isometry for $h_z = \pi H_z$. By part 3, base-changed along the embedding, it maps $\mathrm{Lie}_s(G_z^{(1)})$ onto $\mathrm{Lie}_{\sigma_g s}(G_z^{(2)})$. An isometry maps orthogonal complements to orthogonal complements, so for a full flag (V_k) the minimum defining the quotient norm of $v \bmod V_{k-1}$ equals that of $d\psi_z v \bmod d\psi_z(V_{k-1})$. $\square$

Lemma 31.32. *Let* $\mathrm{Sys} = (X^{\mathrm{an}}, \Theta, \varsigma, (\mathcal{M}_a, h_a^{\mathrm{ref}}, \varphi_a, \varphi_a^T(t))_a, C_5, C, t_0)$ *be an analytic system (Go1), with open part* X° *and* rW. *Let* W *be a compact complex manifold with a Kähler form* Θ_W *and data* ς_W, ∂W, W°, rW_W *as in Lemma 31.35, and let* $F\colon W \to X^{\mathrm{an}}$ *be a holomorphic map with* $F(W^\circ) \subseteq X^\circ$ *and* $\mathrm{rW} \circ F \le \mathrm{rW}_W$ *on* W°. *Put* $t_1 = \min(t_0, e^{-e})$, $\delta(t) = t^{1/3} + (1+C)\varepsilon(t)$ *and* $R(t) = \log(1/t)$ *for* $0 < t \le t_1$. *Then there is* $C_W \ge 1$ *such that the families*

$$F^*\theta_a, \qquad \varphi_a \circ F \text{ on } W^\circ, \qquad \varphi_a^T(t) \circ F \text{ on } W \ (0 < t \le t_1),$$

a running over the fundamental indices, satisfy (R1)–(R5) on $(W, \Theta_W, \varsigma_W)$ *with the constant* C_W, *the functions* δ, R *and* t_1 *in place of* t_0; *moreover* $F^*\theta_a + dd^c(\varphi_a^T(t) \circ F) = F^*\eta_a(t)$.

Proof. For $0 < t \le t_1$ we have $\log(1/t) \ge e$, so $\varepsilon(t) \ge 0$ is defined, and as $y \mapsto \log y/y$ is at most $1/e$ for $y \ge e$, $\varepsilon(t) \le C/e \le C$; moreover $\delta(t) \ge 0$, $\delta(t) \to 0$, $R(t) \ge e$ and $R(t) \to \infty$ as $t \to 0^+$. Since W is compact and $F^*\Theta$ is a smooth semipositive $(1,1)$-form, there is $c_F \ge 1$ with $F^*\Theta \le c_F\Theta_W$. Put $C_W = \max(1, C, C_5, Cc_F)$. Holomorphic pull-back commutes with dd^c and with first Chern forms, so $F^*\theta_a + dd^c(\varphi_a^T(t) \circ F) = F^*(\theta_a + dd^c\varphi_a^T(t)) = F^*\eta_a(t)$.

(R1). By (Go1)(2), $\eta_a(t) + C\Theta \ge \eta_a(t) + \varepsilon(t)\Theta \ge 0$; the pull-back of a semipositive form is semipositive, and $C_W\Theta_W - CF^*\Theta \ge C(c_F\Theta_W - F^*\Theta) \ge 0$; hence $F^*\eta_a(t) + C_W\Theta_W \ge F^*(\eta_a(t) + C\Theta) \ge 0$ on W.

(R2). Let $y \in W^\circ$; then $F(y) \in X^\circ$, and (Go1)(1),(2) give $-C_5 \log \mathrm{rW}(F(y)) \le \varphi_a(F(y)) \le \varphi_a^T(t)(F(y)) \le C$. As $\mathrm{rW}(F(y)) \le \mathrm{rW}_W(y)$ and $\log \mathrm{rW}(F(y)) \ge 1$, $-C_W \log \mathrm{rW}_W(y) \le \varphi_a(F(y)) \le \varphi_a^T(t)(F(y)) \le C_W$.

(R3). If $y \in W^\circ$ and $\mathrm{rW}_W(y) \le R(t) = \log(1/t)$, then $\mathrm{rW}(F(y)) \le \log(1/t)$, and (Go1)(2) gives $\varphi_a^T(t)(F(y)) \le \varphi_a(F(y)) + t^{1/3} + \varepsilon(t) \le \varphi_a(F(y)) + \delta(t)$.

(R4). For $0 < t \le t' \le t_1$, (Go1)(2) gives $\varphi_a^T(t) \le \varphi_a^T(t') + C\varepsilon(t') \le \varphi_a^T(t') + \delta(t')$ on X^{an}, and composing with F gives the same on W.

(R5). Let $K \subseteq W^\circ$ be compact. Then $F(K)$ is a compact subset of X°, on which $\varphi_a^T(t) \to \varphi_a$ with first and second derivatives uniformly by (Go1)(3). In local coordinates the derivatives of order at most two of $\varphi_a^T(t) \circ F - \varphi_a \circ F$ on K are linear combinations of the derivatives of order at most two of $\varphi_a^T(t) - \varphi_a$ at points of $F(K)$, with coefficients that are polynomials in the derivatives of F of order at most two, which are bounded on K (cover K by finitely many coordinate balls whose closures are mapped into coordinate charts); so $\varphi_a^T(t) \circ F \to \varphi_a \circ F$ with first and second derivatives uniformly on K. □

Lemma 31.33. *Let* X *be an integral scheme, projective over* $\operatorname{Spec}\mathbb{Z}$, *with function field* K, *and let* $U \subseteq X$ *be a dense open subset. For a finite-dimensional* K*-vector space* V *we regard* V *as a constant sheaf on every integral scheme* X'' *with a dominant morphism to* X, *after extending scalars to the function field of* X'', *and for an* $\mathcal{O}_X$*-submodule* $M \subseteq V$ *we write* $\mathcal{O}_{X''}M$ *for the* $\mathcal{O}_{X''}$*-submodule of* $V \otimes_K K(X'')$ *generated by* M.

1. *Let* $M \subseteq V$ *be a coherent* $\mathcal{O}_X$*-submodule with* $M \otimes K = V$ *whose restriction to* U *is locally free. There is a projective birational morphism* $b\colon X' \to X$ *with* X' *integral, which is an isomorphism over* U, *such that* $\mathcal{O}_{X'}M$ *is locally free.*

2. *Let* W *be a locally free* $\mathcal{O}_X$*-module of rank* r, $V = W \otimes K$, *and* $0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$ *a full flag of* K*-subspaces such that on* U *the subsheaves* $W \cap V_i$ *are locally free with invertible quotients* $(W \cap V_i)/(W \cap V_{i-1})$. *There is a projective birational morphism* $b\colon X' \to X$ *with* X' *integral, an isomorphism over* U, *such that the subsheaves* $b^*W \cap V_i$ *of* $b^*W \subseteq V$ *are locally free with invertible quotients.*

3. *Let X' be any integral scheme, with function field L, let V' be a finite-dimensional L-vector space, regarded as a constant sheaf on X' and, after extending scalars, on every integral scheme dominant over X', let $N \subseteq V'$ be a locally free $\mathcal{O}_{X'}$-submodule of rank $\dim V'$, and let $(V'_i)_i$ be a chain of L-subspaces of V' such that the $\mathcal{O}_{X'}$-submodules $N_i = N \cap V'_i$ have locally free quotients N/N_i. For every dominant morphism $\psi\colon X'' \to X'$ of integral schemes, the $\mathcal{O}_{X''}$-submodule $\mathcal{O}_{X''}N$ of $V' \otimes_L K(X'')$ generated by N equals ψ^*N and is locally free, and $\psi^*N_i = \psi^*N \cap (V'_i \otimes_L K(X''))$ with locally free quotients. (In particular this applies to the X' of (1) or (2), with $L = K$, $V' = V$ and $V'_i = V_i$.)*

Proof. **Saturated subsheaves are determined by their generic fibre.** Let Q be a torsion-free coherent sheaf on an integral scheme Y with function field L, and $P_1, P_2 \subseteq Q$ saturated coherent subsheaves with $P_1 \otimes L = P_2 \otimes L$ inside $Q \otimes L$. If $s \in P_{1,y}$, then $s \in P_2 \otimes L$, so $fs \in P_{2,y}$ for some $0 \neq f \in \mathcal{O}_y$ (the stalk $P_{2,y}$ is a finitely generated $\mathcal{O}_y$-module with $P_{2,y} \otimes L = P_2 \otimes L$), and saturation of P_2 gives $s \in P_{2,y}$. By symmetry $P_1 = P_2$. A subsheaf with locally free (or torsion-free) quotient is saturated, and the kernel of a map from a torsion-free sheaf to a torsion-free sheaf is saturated.

(1). Put $d = \dim V$. Fix an ample invertible sheaf $\mathcal{O}_X(1)$ (X is projective over $\mathbb{Z}$) and $m \geq 0$ such that $M(m) = M \otimes \mathcal{O}_X(1)^{\otimes m}$ is generated by finitely many global sections; this gives a surjection $\alpha\colon \mathcal{O}_X^N \to M(m)$. On U, $M(m)$ is locally free of rank d, so $\alpha|_U$ is a locally free quotient of rank d of $\mathcal{O}_U^N$ and defines a morphism $\gamma\colon U \to \mathrm{Gr}$ to the Grassmannian of locally free rank-d quotients of $\mathcal{O}^N$ over $\mathbb{Z}$, with $\gamma^*(\mathcal{O}^N \to \mathcal{Q}) = \alpha|_U$ for the universal quotient $\mathcal{Q}$. Let X' be the closure of the graph of γ in $X \times_{\mathbb{Z}} \mathrm{Gr}$ with its reduced structure, and b, g the two projections. X' is integral (the closure of the integral scheme U), b is projective (Gr is projective over $\mathbb{Z}$), and b is an isomorphism over U (the graph of a morphism to a separated scheme is closed in $U \times \mathrm{Gr}$); so b is birational and $K(X') = K$. On X' consider the two quotients of $\mathcal{O}_{X'}^N$: $g^*\mathcal{Q}$, locally free of rank d, and the image M' of $\mathcal{O}_{X'}^N \xrightarrow{b^*\alpha} b^*M(m) \to V \otimes_K b^*\mathcal{O}(m)_\eta$, which is $(\mathcal{O}_{X'}M) \otimes b^*\mathcal{O}_X(1)^{\otimes m}$. Both kernels are saturated in $\mathcal{O}_{X'}^N$ (the quotients are torsion-free), and they agree on $b^{-1}(U)$, where both quotients are α; hence they have the same generic fibre and coincide. So $M' \cong g^*\mathcal{Q}$ is locally free, and so is $\mathcal{O}_{X'}M = M' \otimes b^*\mathcal{O}_X(1)^{\otimes(-m)}$.

(2). Let $\mathrm{Fl}(W) \to X$ be the scheme of full flags of W with locally free (hence invertible) step quotients; it is projective over X. On U the chain $(W \cap V_i)_i$ is such a flag and defines a section $\sigma\colon U \to \mathrm{Fl}(W)|_U$. Let X' be the closure of $\sigma(U)$ in $\mathrm{Fl}(W)$, reduced, and b the projection: as in (1), X' is integral, b is projective, birational and an isomorphism over U. On X' the universal flag $(\mathcal{F}_i)$ of b^*W has invertible step quotients, so each $\mathcal{F}_i$ is saturated in b^*W; $b^*W \cap V_i$ is saturated too (its quotient embeds in V/V_i); both have generic fibre V_i (on $b^{-1}(U)$, $\mathcal{F}_i = W \cap V_i$). Hence $\mathcal{F}_i = b^*W \cap V_i$.

(3). Put $d = \dim V'$. Locally on X', N has a basis $e_1, \ldots, e_d$, which is an L-basis of V'; hence $\psi^*N \to V' \otimes_L K(X'')$ maps the basis $1 \otimes e_i$ to a $K(X'')$-basis, is injective, and has image $\mathcal{O}_{X''}N$, free with basis e_i. The pull-back of a chain of subsheaves with locally free quotients N/N_i is a chain $\psi^*N_i \subseteq \psi^*N$ with locally free quotients $\psi^*(N/N_i)$; so ψ^*N_i is saturated in ψ^*N with generic fibre $V'_i \otimes_L K(X'')$, and by the first paragraph equals $\psi^*N \cap (V'_i \otimes_L K(X''))$. □

Lemma 31.34. *Let $\mathcal{X}$ be an arithmetic variety (an integral scheme, projective and flat over $\mathbb{Z}$) of dimension $D+1$, $D \geq 1$, with $\mathcal{X}_{\mathbb{Q}}$ smooth. Let X be a compact complex manifold of dimension D with a Kähler form Θ and data ς, ∂X, X°, rW as in Lemma 31.35, let $d \geq 1$ be an integer and $\pi\colon \mathcal{X}(\mathbb{C}) \to X$ a holomorphic map that restricts to a d-sheeted holomorphic covering map $\mathcal{X}(\mathbb{C}) \smallsetminus N \to X \smallsetminus N'$ for closed null sets $N \subseteq \mathcal{X}(\mathbb{C})$, $N' \subseteq X$. Let I be a finite set, $0 < t_0 < 1$, and let θ_i, u_i, $u_{i,t}$ ($i \in I$, $0 < t \leq t_0$) satisfy (R1)–(R5) of Lemma 31.35*

for some $C \geq 1$ and functions δ, R as there. Let $\mathcal{M}_1, \dots, \mathcal{M}_{D+1}$ be line bundles on $\mathcal{X}$ and, for $1 \leq k \leq D+1$, $j = 1, 2$ and $0 < t \leq t_0$, let $h_k^{(j)}(t)$ be smooth metrics on $\mathcal{M}_k$ over $\mathcal{X}(\mathbb{C})$ (invariant under complex conjugation) such that:

1. *there are integers $n_{k,i}^{(j)}$, independent of t, with $c_1(\mathcal{M}_k, h_k^{(j)}(t)) = \pi^* \sum_{i \in I} n_{k,i}^{(j)} (\theta_i + dd^c u_{i,t})$ on $\mathcal{X}(\mathbb{C})$;*
2. *$\log(h_k^{(1)}(t)/h_k^{(2)}(t)) = g_{k,t} \circ \pi$ on $\mathcal{X}(\mathbb{C})$ for smooth real functions $g_{k,t}$ on X, and there is $C' \geq 0$ with $|g_{k,t}| \leq C' \log \mathrm{rW}$ on X° for all k and t, and $g_{k,t} \to 0$ uniformly on compact subsets of X° as $t \to 0^+$.*

Then

$$\lim_{t \to 0^+} \Big(\widehat{\deg}\Big(\prod_{k=1}^{D+1} \widehat{c}_1(\mathcal{M}_k, h_k^{(1)}(t)) \Big) - \widehat{\deg}\Big(\prod_{k=1}^{D+1} \widehat{c}_1(\mathcal{M}_k, h_k^{(2)}(t)) \Big) \Big) = 0.$$

Proof. Fix $0 < t \leq t_0$ and put $\alpha_{k,t}^{(j)} = \sum_{i \in I} n_{k,i}^{(j)} (\theta_i + dd^c u_{i,t})$, a smooth closed real $(1,1)$-form on X, so that $c_1(\mathcal{M}_k, h_k^{(j)}(t)) = \pi^* \alpha_{k,t}^{(j)}$ by hypothesis 1.

Step 1: telescoping. For $0 \leq l \leq D+1$ let $A_l(t)$ be the arithmetic intersection number of the $\widehat{c}_1(\mathcal{M}_k, h_k^{(2)}(t))$ for $k \leq l$ and the $\widehat{c}_1(\mathcal{M}_k, h_k^{(1)}(t))$ for $k > l$. The difference in the statement is $A_0(t) - A_{D+1}(t) = \sum_{l=1}^{D+1} (A_{l-1}(t) - A_l(t))$. The numbers $A_{l-1}(t)$ and $A_l(t)$ differ only in the metric of the factor $\mathcal{M}_l$; by the symmetry of $\widehat{\deg}$ we may move this factor to the first place, and the change-of-metric formula with hypothesis 2 gives

$$A_{l-1}(t) - A_l(t) = \int_{\mathcal{X}(\mathbb{C})} (g_{l,t} \circ \pi)\ \pi^* \beta_{l,t}, \qquad \beta_{l,t} = \bigwedge_{k<l} \alpha_{k,t}^{(2)} \wedge \bigwedge_{k>l} \alpha_{k,t}^{(1)},$$

where $\beta_{l,t}$ is a smooth top-degree form on X (the order of the factors is immaterial, $(1,1)$-forms commuting).

Step 2: transfer to X. The integrand is $\pi^*(g_{l,t}\beta_{l,t})$, the pull-back of a smooth top form on X. As N and N' are null, the integral over $\mathcal{X}(\mathbb{C})$ equals the integral over $\mathcal{X}(\mathbb{C}) \smallsetminus N$; this is a d-sheeted covering of $X \smallsetminus N'$ by local biholomorphisms, which preserve the orientations of complex manifolds, so by the change of variables formula on each sheet over a trivialising open set and a partition of unity, $\int_{\mathcal{X}(\mathbb{C}) \smallsetminus N} \pi^*(g_{l,t}\beta_{l,t}) = d \int_{X \smallsetminus N'} g_{l,t}\beta_{l,t} = d \int_X g_{l,t}\beta_{l,t}$. The boundary ∂X is the zero set of the section ς, a proper analytic subset of every component on which $\varsigma \not\equiv 0$, hence null; so $\int_X g_{l,t}\beta_{l,t} = \int_{X^\circ} g_{l,t}\beta_{l,t}$.

Step 3: the limit. Expanding the wedge product $\beta_{l,t}$ by multilinearity, it is a finite sum $\sum_{\underline{i}} c_{l,\underline{i}}\, \mu_{\underline{i},t}$ over tuples $\underline{i} = (i_1, \dots, i_D) \in I^D$, with integers $c_{l,\underline{i}}$ independent of t (products of the $n_{k,i}^{(j)}$) and $\mu_{\underline{i},t} = \bigwedge_{s=1}^{D} (\theta_{i_s} + dd^c u_{i_s,t})$. For each $\underline{i}$ apply Lemma 31.35(2) to the data $\theta_i, u_i, u_{i,t}$ (which satisfy (R1)–(R5)), with $D_v = \varnothing$ (the constant section 1 of the trivial line bundle with the trivial metric), the indices $i_1, \dots, i_D$, the functions $g_t = g_{l,t}|_{X^\circ}$, which are continuous with $|g_t| \leq C' \log \mathrm{rW}$, and $g = 0$, to which they converge uniformly on compact subsets of X° (hypothesis 2): $\int_{X^\circ} g_{l,t}\, \mu_{\underline{i},t} \to 0$ as $t \to 0^+$. Hence $A_{l-1}(t) - A_l(t) = d \sum_{\underline{i}} c_{l,\underline{i}} \int_{X^\circ} g_{l,t}\, \mu_{\underline{i},t} \to 0$ for every l, and the finite sum of Step 1 tends to 0. □

Lemma 31.35. *Let X be a compact complex manifold of dimension $D \geq 1$ with a Kähler form Θ (a positive definite smooth closed real $(1,1)$-form), and let ς, ∂X, X°, rW be as in Lemma 31.23. Let v be a holomorphic section of a smooth hermitian line bundle with $\|v\| \leq 1$ whose zero divisor D_v is reduced, has no irreducible component contained in ∂X, and is such*

that $D_v + (\partial X)_{\mathrm{red}}$ has simple normal crossings ($D_v = \varnothing$ is allowed); put $G = -\log\|v\|^2 \geq 0$ on $X \smallsetminus D_v$. Let I be a finite set, $0 < t_0 < 1$, $C \geq 1$, and $\delta\colon (0,t_0] \to [0,\infty)$, $R\colon (0,t_0] \to [e,\infty)$ functions with $\delta(t) \to 0$ and $R(t) \to \infty$ as $t \to 0^+$. For each $i \in I$ let θ_i be a smooth closed real $(1,1)$-form on X, u_i a smooth real function on X° and $u_{i,t}$ ($0 < t \leq t_0$) smooth real functions on X, such that for all $i \in I$ and $0 < t \leq t' \leq t_0$:

(R1) $\theta_i + dd^c u_{i,t} + C\,\Theta \geq 0$ *on* X;

(R2) $-C\log \mathrm{rW} \leq u_i \leq u_{i,t} \leq C$ *on* X°;

(R3) $u_{i,t} \leq u_i + \delta(t)$ *on* $\{y \in X^\circ : \mathrm{rW}(y) \leq R(t)\}$;

(R4) $u_{i,t} \leq u_{i,t'} + \delta(t')$ *on* X;

(R5) $u_{i,t} \to u_i$ *together with all first and second derivatives, uniformly on compact subsets of* X°, *as* $t \to 0^+$.

For $i_1, \dots, i_D \in I$ put $\mu_t = \bigwedge_{j=1}^D (\theta_{i_j} + dd^c u_{i_j,t})$, a smooth top form on X, and $\mu = \bigwedge_{j=1}^D (\theta_{i_j} + dd^c u_{i_j})$, a smooth top form on X°. Then:

1. $(\log \mathrm{rW} + G)\,\mu$ *is absolutely integrable on* $X^\circ \smallsetminus D_v$, *and* $\int_{X^\circ} \mu = \int_X \theta_{i_1} \wedge \cdots \wedge \theta_{i_D}$;
2. *if* $C' \geq 0$ *and* g_t ($0 < t \leq t_0$), g *are continuous real functions on* X° *with* $|g_t| \leq C' \log \mathrm{rW}$ *on* X° *and* $g_t \to g$ *uniformly on compact subsets of* X°, *then* $\int_{X^\circ} g_t\,\mu_t \to \int_{X^\circ} g\,\mu$ *as* $t \to 0^+$;
3. $\int_{X \smallsetminus D_v} G\,\mu_t \to \int_{X^\circ \smallsetminus D_v} G\,\mu$ *as* $t \to 0^+$.

Proof. Step 1: reduction to positive products. Choose $b \geq C + 1$ such that $\theta_i + b\Theta$ is positive definite for every $i \in I$ (X compact, I finite). Put $\omega_{i,t}^+ = \theta_i + dd^c u_{i,t} + b\Theta$ on X and $\omega_i^+ = \theta_i + dd^c u_i + b\Theta$ on X°. By (R1), $\omega_{i,t}^+ \geq (b - C)\Theta \geq 0$; by (R5) the coefficients of $\omega_{i,t}^+$ converge to those of ω_i^+ uniformly on compact subsets of X°, so $\omega_i^+ \geq 0$. A *positive product* is a family $\nu_t = \alpha_{1,t} \wedge \cdots \wedge \alpha_{D,t}$ where each factor is, for all t at once, either Θ or $\omega_{i,t}^+$ for a fixed $i \in I$; its limit ν on X° has the factors Θ, resp. ω_i^+. Both are positive measures, and by Stokes $\nu_t(X) = V_\nu$, the integral of the product of the corresponding forms Θ, resp. $\theta_i + b\Theta$, independent of t. As forms of even degree commute,

$$\mu_t = \sum_{S \subseteq \{1,\dots,D\}} (-b)^{D-|S|} \bigwedge_{j \in S} \omega_{i_j,t}^+ \wedge \Theta^{D-|S|}, \qquad \mu = \sum_S (-b)^{D-|S|} \bigwedge_{j \in S} \omega_{i_j}^+ \wedge \Theta^{D-|S|},$$

the same finite signed combination of positive products, and the same expansion of $\bigwedge_j((\theta_{i_j} + b\Theta) - b\Theta)$ gives $\sum_S (-b)^{D-|S|} V_{\nu_S} = \int_X \theta_{i_1} \wedge \cdots \wedge \theta_{i_D}$. Since the total variation of μ is at most $\sum_S b^{D-|S|}\nu_S$, it suffices to prove, for every positive product: (1') $(\log \mathrm{rW} + G)$ is ν-integrable on $X^\circ \smallsetminus D_v$ and $\nu(X^\circ) = V_\nu$; (2') $\int_{X^\circ} g_t\,d\nu_t \to \int_{X^\circ} g\,d\nu$; (3') $\int G\,d\nu_t \to \int G\,d\nu$.

Step 2: tube bounds. There is $K_{\mathrm{T}} \geq 0$ such that for every positive product, every $0 < t \leq t_0$ and every $R \geq e$

$$\int_{\{y \in X^\circ : \mathrm{rW}(y) \geq R\}} \log \mathrm{rW}\; d\nu_t \leq K_{\mathrm{T}} \frac{(\log R)^2}{R}, \tag{31.8}$$

and, if $D \geq 2$, the same bound holds on every irreducible component D_k of D_v for every product β_t of $D - 1$ factors, each $\Theta|_{D_k}$ or $\omega_{i,t}^+|_{D_k}$. Indeed, write each factor as $\omega^0 + dd^c w$ with $(\omega^0, w) = (\Theta, 0)$, resp. $(\theta_i + b\Theta,\, u_{i,t} - C)$: ω^0 is positive definite, $\omega^0 + dd^c w \geq 0$, $w \leq 0$ on X, and on X°, $w \geq u_i - C \geq -C\log \mathrm{rW} - C \geq -2C \log \mathrm{rW}$ by (R2). So Lemma 31.23 with

$C_1 = 2C$ gives the bound with $3K(2C+4)$, where K depends only on X, ς and the tuple of forms ω^0; there are finitely many tuples. On D_k (a compact complex submanifold of dimension $D-1 \geq 1$, as $D_v + (\partial X)_{\mathrm{red}}$ has simple normal crossings): $\varsigma|_{D_k}$ is a holomorphic section with $\|\varsigma|_{D_k}\| \leq 1$, not identically zero since $D_k \not\subseteq \partial X$, whose zero divisor $\partial X|_{D_k}$ has simple normal crossings on D_k (in local coordinates with $D_k = \{z_1 = 0\}$ and the components of ∂X through the point among the other coordinate hyperplanes); its rW is the restriction of rW; restrictions of positive definite forms are positive definite and the inequalities restrict. So Lemma 31.23 on D_k applies in the same way; take K_{T} the maximum of the finitely many constants.

Step 3: the limit measures. For $R \geq e$ let $K_R = \{y \in X^\circ : \mathrm{rW}(y) \leq R\} = \{y \in X : \|\varsigma(y)\|^2 \geq e^{e-R}\}$, a compact subset of X°. In finitely many coordinate charts covering a compact $K \subseteq X^\circ$, the densities of ν_t are polynomials in the coefficients of the factors, which converge uniformly on K by (R5); hence $|\int_K f\,d\nu_t - \int_K f\,d\nu| \leq \sup_K |f| \cdot o(1)$ for bounded Borel f on K, as $t \to 0^+$. For compact $K \subseteq \{\mathrm{rW} > R\} \cap X^\circ$, (31.8) gives $\int_K \log \mathrm{rW}\,d\nu = \lim_t \int_K \log \mathrm{rW}\,d\nu_t \leq K_{\mathrm{T}}(\log R)^2/R$; by inner regularity of the Radon measure $\log \mathrm{rW} \cdot \nu$ on the open set $\{\mathrm{rW} > R\} \cap X^\circ$,

$$\int_{\{y \in X^\circ : \mathrm{rW}(y) > R\}} \log \mathrm{rW}\; d\nu \leq K_{\mathrm{T}} \frac{(\log R)^2}{R} \qquad (R \geq e), \tag{31.9}$$

so $\log \mathrm{rW}$ is ν-integrable. For the mass: $\nu_t(\partial X) = 0$ and $\log \mathrm{rW} \geq 1$, so $V_\nu - K_{\mathrm{T}}(\log R)^2/R \leq \nu_t(K_R) \leq V_\nu$; letting $t \to 0^+$ and then $R \to \infty$ gives $\nu(X^\circ) = V_\nu$.

Step 4: proof of (2'). $|g| \leq C' \log \mathrm{rW}$ (pointwise limit), so g is ν-integrable. For $R \geq e$, by (31.8) (with R replaced by any $R' < R$ close to R, or directly) and (31.9),

$$\Big|\int_{X^\circ} g_t\,d\nu_t - \int_{X^\circ} g\,d\nu\Big| \leq \sup_{K_R} |g_t - g|\,V_\nu + \Big|\int_{K_R} g\,d\nu_t - \int_{K_R} g\,d\nu\Big| + 2C' K_{\mathrm{T}} \frac{(\log R)^2}{R}.$$

The first two terms tend to 0 as $t \to 0^+$ (uniform convergence on K_R; Step 3 with $f = g$ bounded on K_R). Hence the lim sup is at most $2C' K_{\mathrm{T}}(\log R)^2/R$ for every $R \geq e$, i.e. 0.

Step 5: a regularised Green function. For $0 < \varepsilon \leq 1$ put $g_\varepsilon = \log(1 + \varepsilon/\|v\|^2)$ on $X \smallsetminus D_v$ and $G_\varepsilon = -\log(\|v\|^2 + \varepsilon)$, smooth on X. Then $G = G_\varepsilon + g_\varepsilon$, $0 \leq g_\varepsilon \leq \log 2 + G$ (as $\|v\| \leq 1$), $g_\varepsilon \to 0$ pointwise on $X \smallsetminus D_v$ as $\varepsilon \to 0^+$, and $|G_\varepsilon| \leq \log(2/\varepsilon)$. Let $\beta_\varepsilon = dd^c \log(\|v\|^2 + \varepsilon) + \rho_v$, a smooth closed real $(1,1)$-form on X. (a) There is $b_2 \geq 0$, independent of ε, with $\beta_\varepsilon + b_2\Theta \geq 0$. Choose b_2 with $\rho_v + b_2\Theta \geq 0$ (compactness). On a coordinate ball U with a holomorphic frame, $\|v\|^2 = |f|^2 e^{-\psi}$ with f holomorphic and ψ smooth, $dd^c\psi = \rho_v$, and $\Theta = dd^c\varphi_U$ with φ_U smooth (shrinking U). Then on U, $\beta_\varepsilon + b_2\Theta = dd^c\chi(p_1, p_2)$ with $\chi(x,y) = \log(e^x + e^y)$, $p_1 = \log|f|^2 + b_2\varphi_U$, $p_2 = \log\varepsilon + \psi + b_2\varphi_U$. For $\lambda > 0$ put $p_1^\lambda = \log(|f|^2 + \lambda) + b_2\varphi_U$: smooth and plurisubharmonic, as is p_2; χ is convex and nondecreasing in each variable, so $i\partial\bar\partial\chi(p_1^\lambda, p_2) = \sum_k \partial_k\chi\, i\partial\bar\partial p_k + \sum_{k,l} \partial_k\partial_l\chi\, i\partial p_k \wedge \bar\partial p_l \geq 0$. Since $\chi(p_1^\lambda, p_2) = \log((|f|^2 + \lambda)e^{b_2\varphi_U} + \varepsilon e^{\psi + b_2\varphi_U})$ depends smoothly on (λ, z) for $\lambda \geq 0$ (the argument of the logarithm is positive), its dd^c at $\lambda = 0$ is the limit of semipositive forms, hence ≥ 0. (b) For every smooth function w on X and every smooth closed real $(D-1, D-1)$-form Φ,

$$\int_X g_\varepsilon\, dd^c w \wedge \Phi = \int_X w\,\beta_\varepsilon \wedge \Phi - \kappa \int_{D_v} w\,\Phi, \tag{31.10}$$

where $\int_{D_v} = \sum_k \int_{D_k}$ (for $D = 1$: the sum of the values at the points of D_v). Indeed $dd^c w \wedge \Phi = dd^c(w\Phi)$; Stokes twice for the smooth function $\log(\|v\|^2 + \varepsilon)$ and Poincaré–Lelong for $\log\|v\|^2$ give $\int \log(\|v\|^2 + \varepsilon)\, dd^c(w\Phi) = \int w\, dd^c \log(\|v\|^2 + \varepsilon) \wedge \Phi$ and $\int \log\|v\|^2\, dd^c(w\Phi) = \kappa \int_{D_v} w\Phi - \int w\,\rho_v \wedge \Phi$; subtract.

Step 6: uniform integrability. Fix a positive product, its factors ordered so that $\nu_t = \omega_{i_1,t}^+ \wedge \cdots \wedge \omega_{i_c,t}^+ \wedge \Theta^{D-c}$ $(0 \leq c \leq D)$. For $0 \leq a \leq c$ let $P_{a,t} = \omega_{i_1,t}^+ \wedge \cdots \wedge \omega_{i_a,t}^+ \wedge \Theta^{D-a}$ and

$J_a(\varepsilon,t)=\int_X g_\varepsilon\,P_{a,t}$, finite as $g_\varepsilon\le\log 2+G$. Claim: for every $a\le c$ and $\eta>0$ there are $\varepsilon_a\in(0,1]$ and $t_a\in(0,t_0]$ with $J_a(\varepsilon,t)\le\eta$ for $0<\varepsilon\le\varepsilon_a$, $0<t\le t_a$. For $a=0$, $J_0(\varepsilon,t)=\int g_\varepsilon\Theta^D\to 0$ by dominated convergence (majorant $\log 2+G$). Let $a<c$, $i=i_{a+1}$, and $\Phi_t=\omega^+_{i_1,t}\wedge\cdots\wedge\omega^+_{i_a,t}\wedge\Theta^{D-a-1}$, a smooth closed positive $(D-1,D-1)$-form; $P_{a+1,t}=\omega^+_{i,t}\wedge\Phi_t$ and $P_{a,t}=\Theta\wedge\Phi_t$. Take $b_1\ge 0$ with $\theta_i+(b-1)\Theta\le b_1\Theta$ for all $i\in I$. As $\omega^+_{i,t}-\Theta=\theta_i+(b-1)\Theta+dd^cu_{i,t}$ and $g_\varepsilon\ge 0$,

$$J_{a+1}(\varepsilon,t)\le(1+b_1)J_a(\varepsilon,t)+\int_X g_\varepsilon\,dd^cu_{i,t}\wedge\Phi_t.$$

Fix $\tau\in(0,t_0]$, let $0<t\le\tau$, put $\bar u=u_{i,\tau}+\delta(\tau)$ and $\mathrm{gap}=\bar u-u_{i,t}$, which is ≥ 0 on X by (R4). Apply (31.10) to $w=u_{i,t}=\bar u-\mathrm{gap}$, then to $w=\bar u$ backwards, and use that $(\beta_\varepsilon+b_2\Theta)\wedge\Phi_t$ is a positive measure:

$$\begin{aligned}\int g_\varepsilon\,dd^cu_{i,t}\wedge\Phi_t&=\int g_\varepsilon\,dd^c\bar u\wedge\Phi_t-\int\mathrm{gap}\,(\beta_\varepsilon+b_2\Theta)\wedge\Phi_t\\&\quad+b_2\int\mathrm{gap}\,\Theta\wedge\Phi_t+\kappa\int_{D_v}\mathrm{gap}\,\Phi_t\\&\le K_1(\tau)J_a(\varepsilon,t)+E(\tau,t),\end{aligned}$$

where $K_1(\tau)\ge 0$ satisfies $dd^cu_{i',\tau}\le K_1(\tau)\Theta$ on X for all $i'\in I$ ($u_{i',\tau}$ smooth on compact X), and $E(\tau,t)=b_2\int_X\mathrm{gap}\,\Theta\wedge\Phi_t+\kappa\int_{D_v}\mathrm{gap}\,\Phi_t$. So

$$J_{a+1}(\varepsilon,t)\le\bigl(1+b_1+K_1(\tau)\bigr)J_a(\varepsilon,t)+E(\tau,t)\qquad(0<\varepsilon\le 1,\ 0<t\le\tau).\tag{31.11}$$

We now bound E. On $K_{R(\tau)}$, (R3) at τ and (R2) at t give $\mathrm{gap}\le u_i+\delta(\tau)-u_i+\delta(\tau)=2\delta(\tau)$; on $X^\circ\setminus K_{R(\tau)}$, (R2) gives $\mathrm{gap}\le C+C\log\mathrm{rW}+\delta(\tau)\le(2C+\delta(\tau))\log\mathrm{rW}$. $\Theta\wedge\Phi_t$ is a positive product, of mass V_a independent of t, so by (31.8) $\int_X\mathrm{gap}\,\Theta\wedge\Phi_t\le 2\delta(\tau)V_a+(2C+\delta(\tau))K_{\mathrm T}(\log R(\tau))^2/R(\tau)$. For $D\ge 2$ the same pointwise bounds hold on $D_k\cap X^\circ$, $D_k\cap\partial X$ is null for $\Phi_t|_{D_k}$ (a proper analytic subset of D_k), $\int_{D_k}\Phi_t$ is independent of t (Stokes on D_k), and Step 2 on D_k gives the analogous bound. For $D=1$, D_v is a finite subset of X° and $\Phi_t=1$; once $R(\tau)\ge\max_{D_v}\mathrm{rW}$, $\int_{D_v}\mathrm{gap}\le 2\delta(\tau)\,\#D_v$. Hence $E^*(\tau)=\sup_{0<t\le\tau}E(\tau,t)\to 0$ as $\tau\to 0^+$. Given $\eta>0$, choose τ with $E^*(\tau)\le\eta/2$, put $M=1+b_1+K_1(\tau)$, and take ε_a,t_a for $\eta/(2M)$ by induction; then (31.11) gives $J_{a+1}(\varepsilon,t)\le\eta$ for $\varepsilon\le\varepsilon_a$ and $t\le\min(t_a,\tau)$. This proves the claim.

Step 7: proof of (3') and (1'). Let $\eta>0$ and take $\varepsilon=\varepsilon_c$, t_c from Step 6 with $a=c$, so $\int g_\varepsilon\,d\nu_t=J_c(\varepsilon,t)\le\eta$ for $t\le t_c$. For compact $K\subseteq X^\circ\setminus D_v$, g_ε is bounded on K, so by Step 3 $\int_K g_\varepsilon\,d\nu=\lim_t\int_K g_\varepsilon\,d\nu_t\le\eta$; by inner regularity $\int_{X^\circ\setminus D_v}g_\varepsilon\,d\nu\le\eta$. As G_ε is continuous on X with $|G_\varepsilon|\le\log(2/\varepsilon)\log\mathrm{rW}$, it is ν-integrable and Step 4 (with $g_t=g=G_\varepsilon$) gives $\int G_\varepsilon\,d\nu_t\to\int G_\varepsilon\,d\nu$. So $G=G_\varepsilon+g_\varepsilon$ is ν-integrable on $X^\circ\setminus D_v$ (D_v is null for ν_t and ν), and for $t\le t_c$

$$\Bigl|\int G\,d\nu_t-\int G\,d\nu\Bigr|\le\Bigl|\int G_\varepsilon\,d\nu_t-\int G_\varepsilon\,d\nu\Bigr|+2\eta,$$

whose lim sup as $t\to 0^+$ is at most 2η. This is (3'); with Step 3 it gives (1'). By Step 1, (1)–(3) follow. □

Proposition 31.36. *Let $\mathfrak u=(E,\mathsf H,\mathsf U,p,q,\mathsf L,\mathrm{Sh},F)$ be a unitary Shimura datum with CM host $\mathsf H=(d,r,r_{\mathbb C})$, and let $(G,\iota,\mathcal L)$ be the abelian scheme, the $\mathcal O_E$-action and the cubical line bundle of the universal object over* Sh. *Then for every automorphism g of the field E with $g(r)=r$ there are an automorphism* tw *of the $K_{\mathsf L}$-scheme* Sh *and an isomorphism of abelian schemes $\psi\colon G\to\mathrm{tw}^*G$ over* Sh *with $\psi\circ\iota(g(a))=(\mathrm{tw}^*\iota)(a)\circ\psi$ for all $a\in\mathcal O_E$ and $\psi^*\mathrm{tw}^*\mathcal L\cong\mathcal L$ as rigidified cubical line bundles.*

Proof. Fix g with $g(r) = r$; then $g(w_0) = w_0$, g fixes $\mathbb{Z}[w_0]$ pointwise, and $g(\mathfrak{n}) = \mathfrak{n}$, $g(\bar{\mathfrak{n}}) = \bar{\mathfrak{n}}$, since $\mathfrak{n}$ is generated by $\mathfrak{n} \cap \mathbb{Z}[w_0]$ and $g(\bar{\mathfrak{n}}) = \overline{g(\mathfrak{n})}$.

Step 1: a profinite g-semilinear isometry. We construct, for every prime ℓ, a $\mathbb{Z}_\ell$-linear bijection $\varphi_\ell \colon M_\ell \to M_\ell$ of $M_\ell = M \otimes \mathbb{Z}_\ell$ with $\varphi_\ell(ax) = g(a)\varphi_\ell(x)$ ($a \in \mathcal{O}_E$) and $\langle \varphi_\ell x, \varphi_\ell y\rangle = \langle x, y\rangle$. Let $M^{(g)}$ be the group M with the action $a * x = g(a)x$; it is projective of rank $p+q$ over $\mathcal{O}_E$, and $\langle a * x, y\rangle = \langle x, \overline{g(a)}y\rangle = \langle x, \bar{a} * y\rangle$, so $(M^{(g)}, \langle\,,\rangle)$ satisfies the algebraic conditions of (Ud2), with the same perfect form. A map φ_ℓ as required is exactly an $\mathcal{O}_E \otimes \mathbb{Z}_\ell$-linear isometry $M_\ell \to M_\ell^{(g)}$. Put $A = \mathcal{O}_E \otimes \mathbb{Z}_\ell = \prod_v \mathcal{O}_{E_v}$, the product over the places v of E^+ above ℓ, with $E_v = E \otimes_{E^+} E_v^+$ a field unramified of degree 2 over E_v^+ (inert) or $E_v^+ \times E_v^+$ (split), by the standing hypothesis that E/E^+ is unramified at every finite place. Let L be M_ℓ or $M_\ell^{(g)}$; it is a projective A-module of rank $n = p+q$ at every factor, hence free.

(a) Hermitian form. Choose $\xi \in A^\times$ with $\bar{\xi} = -\xi$: in an inert factor with $\mathcal{O}_{E_v} = \mathcal{O}_{E_v^+}[\omega]$ take $\omega - \bar{\omega}$, whose square is the discriminant of ω, a unit as E_v/E_v^+ is unramified; in a split factor take $(1, -1)$. The trace form $(a, b) \mapsto \mathrm{Tr}_{A\otimes\mathbb{Q}/\mathbb{Q}_\ell}(ab)$ of the étale $\mathbb{Q}_\ell$-algebra $A \otimes \mathbb{Q}$ is non-degenerate, so there is a unique $h(x, y) \in A \otimes \mathbb{Q}$ with $\mathrm{Tr}(a\xi h(x, y)) = \langle ax, y\rangle$ for all a; h is A-linear in x, and $h(y, x) = \overline{h(x, y)}$:

Since $\langle\,,\rangle$ is perfect on L and ξ is a unit, $\{y : h(L, y) \subseteq A\} = \mathfrak{d}L$, where $\mathfrak{d}$ is the different of A over $\mathbb{Z}_\ell$. As E/E^+ is unramified at ℓ, $\mathfrak{d} = \mathfrak{d}^+A$ with $\mathfrak{d}^+$ the different of $\mathcal{O}_{E^+} \otimes \mathbb{Z}_\ell$, a principal ideal $\delta\mathcal{O}_{E^+} \otimes \mathbb{Z}_\ell$ (semilocal Dedekind ring), $\bar{\delta} = \delta$. Then $h_1 = \delta h$ is hermitian with values in A and L is *unimodular* for h_1: $\{y : h_1(L, y) \subseteq A\} = L$. The element δ is the same for M_ℓ and $M_\ell^{(g)}$, and $\langle x, y\rangle = \mathrm{Tr}(\xi\delta^{-1}h_1(x, y))$ on both; so an A-linear bijection $M_\ell \to M_\ell^{(g)}$ preserving h_1 preserves $\langle\,,\rangle$.

(b) Unimodular lattices of equal rank are isometric. The decomposition $A = \prod_v \mathcal{O}_{E_v}$ is by idempotents fixed by $^-$, so L is the orthogonal sum of its factors L_v, each unimodular of rank n. If v is split, $\mathcal{O}_{E_v} = \mathcal{O} \times \mathcal{O}$ ($\mathcal{O} = \mathcal{O}_{E_v^+}$) with $^-$ the swap, $L_v = L' \times L''$ with h_1 vanishing on $L' \times L'$ and $L'' \times L''$, and unimodularity says that the first component of h_1 is a perfect $\mathcal{O}$-bilinear pairing $L' \times L'' \to \mathcal{O}$; a basis of L' and its dual basis of L'' give an isomorphism with the standard form, so any two such L_v of rank n are isometric. If v is inert, with uniformiser π of $\mathcal{O}$ and residue extension k_E/k of degree 2, we show by induction on n that L_v has an orthonormal basis. If $h_1(x, x) \in \pi\mathcal{O}$ for all $x \in L_v$, the reduction $\bar{h}$ satisfies $\bar{h}(x, x) = 0$, hence $\mathrm{Tr}_{k_E/k}\bar{h}(ax, y) = \bar{h}(ax + y, ax + y) - \bar{h}(ax, ax) - \bar{h}(y, y) = 0$ for all $a \in k_E$, so $\bar{h} = 0$ by non-degeneracy of the trace of the separable extension k_E/k, contradicting unimodularity ($n \geq 1$). So some x has $u = h_1(x, x) \in \mathcal{O}^\times$ ($h_1(x, x) = \overline{h_1(x, x)} \in \mathcal{O}$); write $u = c\bar{c}$ with $c \in \mathcal{O}_{E_v}^\times$ (norm surjectivity) and replace x by $c^{-1}x$, so $h_1(x, x) = 1$. Then $L_v = \mathcal{O}_{E_v}x \oplus x^\perp$ ($y \mapsto y - h_1(y, x)x$), and $x^\perp$ is unimodular of rank $n-1$; by induction L_v has an orthonormal basis. Hence any two unimodular L_v of rank n are isometric.

By (a) and (b) there is an A-linear isometry $M_\ell \to M_\ell^{(g)}$, i.e. a map φ_ℓ as required. Let $\hat{\varphi} = \prod_\ell \varphi_\ell$ on $M \otimes \widehat{\mathbb{Z}} = \prod_\ell M_\ell$ and, for $m \geq 1$, $\varphi_m \colon M/mM \to M/mM$ its reduction: φ_m is bijective, $\varphi_m(ax) = g(a)\varphi_m(x)$, $\langle \varphi_m x, \varphi_m y\rangle \equiv \langle x, y\rangle \pmod m$, and $\varphi_{mm'}$ reduces to φ_m modulo m. Since $\varphi_N(\mathfrak{n}x) = g(\mathfrak{n})\varphi_N(x)$, φ_N maps $\mathfrak{n}M/NM$ onto itself and $\bar{\mathfrak{n}}M/NM$ onto itself ($NM \subseteq \mathfrak{n}M \cap \bar{\mathfrak{n}}M$ as $N \in \mathfrak{n}\bar{\mathfrak{n}}$).

Step 2: the twisted object. Write the universal object as $o = (G, \iota, \mathcal{L}, \eta, \eta_A)$ over Sh. Since $\eta(\mathfrak{n}M) = 0$ and $NM \subseteq \mathfrak{n}M$, $\eta = \bar{\eta} \circ (M \to M/NM)$; likewise $\eta_A = \bar{\eta}_A \circ (M \to M/AM)$. Put

$$o^g = \big(G,\ \iota \circ g,\ \mathcal{L},\ \eta' = \bar{\eta} \circ \varphi_N \circ (M \to M/NM),\ \eta'_A = \bar{\eta}_A \circ \varphi_A \circ (M \to M/AM)\big).$$

We check that o^g is an object of type (U, L) over Sh. $\iota \circ g$ is a ring homomorphism $\mathcal{O}_E \to \mathrm{End}(G)$. η', η'_A are additive, $\eta'_A(AM) = 0$, and $\eta'(\mathfrak{n}M) = \bar{\eta}(\varphi_N(\mathfrak{n}M/NM)) \subseteq \bar{\eta}(\mathfrak{n}M/NM) = 0$. For

$a \in \mathcal{O}_E$: $(\iota(g(a)),1)^*\Lambda(\mathcal{L}) \cong (1,\iota(\overline{g(a)}))^*\Lambda(\mathcal{L}) = (1,\iota(g(\bar{a})))^*\Lambda(\mathcal{L})$, and $\mathcal{L} \cong (1,\iota(w_0))^*\Lambda(\mathcal{L}) = (1,\iota(g(w_0)))^*\Lambda(\mathcal{L})$. Let $x\colon \operatorname{Spec} k \to \mathrm{Sh}$ be a geometric point with $\beta\colon K_{\mathsf{L}} \to k$. Kottwitz condition: for $\tau\colon \mathcal{O}_E \to k$, the subspace of $\mathrm{Lie}(G_x)$ on which $\iota(g(a))$ acts by $\tau(a)$ for all a is the subspace on which $\iota(b)$ acts by $(\tau\circ g^{-1})(b)$; its dimension is p iff $\tau(g^{-1}(r)) = \tau(r) = \beta(r_{\mathbb{C}})$, and q otherwise; these subspaces are those of o, so they span $\mathrm{Lie}(G_x)$. Level structure: take the ζ_n and I_n of o at x and put $I'_n = I_n \circ \varphi_n$. Then I'_n is bijective and $I'_n(ay) = I_n(g(a)\varphi_n y) = \iota(g(a))I'_n(y)$, so it is $\mathcal{O}_E$-linear for $\iota\circ g$; $m'I'_{nm'}(y) = I_n(\varphi_{nm'}(y) \bmod n) = I'_n(y)$; $e_n(I'_n y, I'_n y') = \zeta_n^{\langle \varphi_n y, \varphi_n y'\rangle} = \zeta_n^{\langle y,y'\rangle}$; $\eta'_A(y)_x = I_A(\varphi_A y) = I'_A(y)$; and for $y \in \bar{\mathfrak{n}}M$, $\varphi_N(y)$ is the class of some $y_1 \in \bar{\mathfrak{n}}M$, so $\eta'(y)_x = \eta(y_1)_x = I_N(y_1) = I'_N(y)$. The roots of unity are those of o. So o^g is an object.

Step 3: the automorphism. Sh is smooth over the field K_{L}, hence reduced. By (Ud5)(i) there is a unique K_{L}-morphism $\mathrm{tw}\colon \mathrm{Sh} \to \mathrm{Sh}$ with an isomorphism of objects $o^g \cong \mathrm{tw}^* o$. Its abelian-scheme part is an isomorphism $\psi\colon G \to \mathrm{tw}^*G$ over Sh compatible with the actions, $\psi \circ \iota(g(a)) = (\mathrm{tw}^*\iota)(a) \circ \psi$, and with the cubical line bundles, $\psi^*\mathrm{tw}^*\mathcal{L} \cong \mathcal{L}$. It remains to see that tw is an automorphism. Apply Steps 1–2 to g^{-1} (also $g^{-1}(r) = r$) with the isometry φ_m^{-1} (which is g^{-1}-semilinear and isometric) in place of the one constructed there; call the twisted object $o^{g^{-1}}$ and the morphism tw'. The construction $o \mapsto o^g$ makes sense for every object over every reduced K_{L}-scheme T, commutes with pull-back along $T' \to T$, and carries isomorphisms of objects to isomorphisms of the twisted objects (the same isomorphism of abelian schemes); and twisting by (g,φ) and then by (g^{-1},φ^{-1}) returns the object, $(\iota\circ g)\circ g^{-1} = \iota$, $\bar{\eta}\circ\varphi_N\circ\varphi_N^{-1} = \bar{\eta}$. Hence

$$(\mathrm{tw}' \circ \mathrm{tw})^* o = \mathrm{tw}^*\mathrm{tw}'^* o \cong \mathrm{tw}^*\big(o^{g^{-1}}\big) = (\mathrm{tw}^* o)^{g^{-1}} \cong (o^g)^{g^{-1}} = o,$$

and by the uniqueness in (Ud5)(i), $\mathrm{tw}' \circ \mathrm{tw} = \mathrm{id}$; symmetrically $\mathrm{tw} \circ \mathrm{tw}' = \mathrm{id}$. So tw is an automorphism of the K_{L}-scheme Sh. □

Lemma 31.37. *Let $\mathfrak{u}$ be a unitary Shimura datum with $\Phi = \Phi_{\mathsf{H}}$, $n = p+q$, $p,q \geq 1$, $\mathcal{Y}$ an integral model of* Sh *over $\mathcal{O}_F$, and $\mathcal{T}$ a root tower of type (Φ,n,p) with a geometric realisation $(\nu_{\mathcal{T}},(\mathcal{X}_{B,c}))$ and a good origin over $\mathcal{Y}$ immersive at level four. Let $B \geq 2$ and $c \in \mathrm{Cp}$, and let $\pi\colon \mathcal{X}_{B,c}(\mathbb{C}) \to X_c^{\mathrm{an}}$ be the realisation (Go4) of the root origin of $\mathcal{X}_{B,c}$. Then $\mathcal{X} = \mathcal{X}_{B,c}$, with $S = S_B$, $R(\ell) = 4$ ($\ell \in S_B$), the analytic system $\mathrm{Sys}_{\mathcal{X}} = \mathrm{Sys}_c$, $d = \deg_{B,c}$ and the map π, satisfies the hypotheses imposed on $\mathcal{X}_1$ and $\mathcal{X}_2$ in Proposition 31.27 for the fixed data at c, and S_B satisfies the hypothesis imposed on S there: S_B contains every prime dividing* $\mathrm{disc}(E)$*; $\mathcal{X}_{\mathbb{Q}}$ is smooth; $\mathcal{X}$ carries the data (Go3)(1)–(4) of a root origin on the fixed data at c for (S_B,R); $\varphi_* = 0 = \varphi_*^T$ for Sys_c; π satisfies hypotheses 1–3 of Lemma 31.21 with $d = \deg_{B,c}$; a point of $\mathcal{X}(\mathbb{C})$ lies in the abelian locus if and only if its image lies in X_c°; and on $\pi^{-1}(X_c^\circ)$, $h^{\mathrm{ref}}_{(\tau,i)} e^{\varphi_{(\tau,i)}\circ\pi} = \prod_{k\leq i}(h^{\mathrm{sing}}_{(\tau,k)})^{-1}$ for $(\tau,i) \in \mathrm{Rt}$.*

Proof. Write $\mathcal{X} = \mathcal{X}_{B,c}$ and $d = \deg_{B,c}$. The good origin contains a tower origin, whose set Bd is the set of primes dividing $2\,\mathrm{disc}(E)\,N'$ (Go5), and $S_B \supseteq \mathrm{Bd}$ by definition; so S_B contains every prime dividing $\mathrm{disc}(E)$. The good origin contains a root origin of $\mathcal{X}$ on the fixed data at c (Go5), whose data include (Go3)(1)–(4) for S_B and integers $R(\ell)$, and $R(\ell) = 4$ for $\ell \in S_B$ because the good origin is immersive at level four. As the origin is good, $\mathcal{X}_{\mathbb{Q}}$ is smooth, so $\mathcal{X}(\mathbb{C})$ is a compact complex manifold and (Go4) needs no resolution. The fixed data at c have $\varphi_* = 0 = \varphi_*^T(t)$ for Sys_c by (Go2). The clause on the abelian locus and the identity of the singular weight with the Hodge metric on $\pi^{-1}(X_c^\circ)$ are two of the clauses of (Go4).

It remains to check hypotheses 1–3 of Lemma 31.21.

Hypothesis 1. By (Go4) there is a closed analytic subset $N_0 \subseteq X_c^{\mathrm{an}}$ with empty interior such that $N = \pi^{-1}(N_0)$ is a closed analytic subset of $\mathcal{X}(\mathbb{C})$ with empty interior, every point of $X_c^{\mathrm{an}} \smallsetminus N_0$ has exactly d preimages, and $d\pi$ is invertible at every point of $\mathcal{X}(\mathbb{C}) \smallsetminus N$. First, a

closed analytic subset A with empty interior of a complex manifold M is a null set: every $x \in M$ has a connected coordinate neighbourhood U on which $A \cap U$ is the common zero set of finitely many holomorphic functions; as A has empty interior, $A \cap U \neq U$, so one of these functions, f, is not identically zero on the connected set U, and $A \cap U \subseteq \{f = 0\}$, which is null (the zero set of a holomorphic function that is not identically zero on a connected open set has measure zero); a countable cover by such U shows that A is null. So N and N_0 are closed null sets. Next, π maps $\mathcal{X}(\mathbb{C}) \setminus N$ into $X_c^{\mathrm{an}} \setminus N_0$ since $N = \pi^{-1}(N_0)$, and this restriction π° is proper: π is a continuous map from a compact space to a Hausdorff space, hence proper, and the restriction of a proper map to the full preimage of an open set is proper. By the holomorphic inverse function theorem π° is a local biholomorphism. A proper local homeomorphism between locally compact Hausdorff spaces has finite fibres and is a covering map over its image, which is open and closed; here every fibre over $X_c^{\mathrm{an}} \setminus N_0$ has exactly $d \geq 1$ points, so π° is surjective and a d-sheeted holomorphic covering map $\mathcal{X}(\mathbb{C}) \setminus N \to X_c^{\mathrm{an}} \setminus N_0$.

Hypothesis 2. By (Go4), for every fundamental index a there is a smooth cone function h_a^{ref} on $\mathcal{X}$ of the multidegree of a whose metric on $\mathcal{O}(e_a)|_{\mathcal{X}(\mathbb{C})}$ is the pull-back along π of the metric h_a^{ref} of $\mathcal{M}_a$ in Sys_c; first Chern forms are compatible with isometric pull-back, so $c_1(\mathcal{O}(e_a), h_a^{\mathrm{ref}})|_{\mathcal{X}(\mathbb{C})} = \pi^* c_1(\mathcal{M}_a, h_a^{\mathrm{ref}})$; and $h_*^{\mathrm{ref}} = h_*$.

Hypothesis 3. By (Go4), for $0 < t \leq t_0$ and $(\tau, i) \in \mathrm{Rt}$ the regularised metric $h_{(\tau,i)}^{\mathrm{ref}} e^{\varphi^T_{(\tau,i)}(t) \circ \pi}$ equals $\prod_{k \leq i} h_{(\tau,k)}(t)^{-1}$ on the affine cone over $\mathcal{X}(\mathbb{C})$. □

Lemma 31.38. *Let H be a CM host on a CM field E that is Galois over $\mathbb{Q}$, and let $\mathcal{T}$ be a root tower of type $(\Phi_{\mathsf{H}}, n, \kappa)$ with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ such that the realised tower has the convergence (Gi2). Suppose that for every $\sigma \in \Gamma_{\mathsf{H}}$ and every integer $B \geq 2$ there is a bijection $\pi \colon \mathrm{Cp} \to \mathrm{Cp}$ with*

$$\lim_{t \to 0^+} \Big(\mathrm{ai}_t(\mathcal{X}_{B,Z}; \sigma \cdot x^m) - \mathrm{ai}_t(\mathcal{X}_{B,\pi(Z)}; x^m) \Big) = 0$$

for every $Z \in \mathrm{Cp}$ and every monomial $x^m \in \mathcal{R}$ of degree $D+1$. Then $\mathcal{T}$ has Galois component symmetry (Ys0) for H.

Proof. Fix $\sigma \in \Gamma_{\mathsf{H}}$, $B \geq 2$ and π as in the hypothesis, and $Z \in \mathrm{Cp}$. The map $F \mapsto \sigma \cdot F$ is the $\mathbb{R}$-algebra automorphism of $\mathcal{R}$ that permutes the variables, $x_{a,i} \mapsto x_{\sigma(a),i}$; it maps monomials to monomials of the same degree. By (Gi2), $\mathcal{X}_{B,Z}$ and $\mathcal{X}_{B,\pi(Z)}$ have convergent regularisation, so for every monomial $x^m \in \mathcal{R}$ the limits $\lim_{t \to 0^+} \mathrm{ai}_t(\mathcal{X}_{B,Z}; \sigma \cdot x^m) = \mathrm{pp}(\mathcal{X}_{B,Z}; \sigma \cdot x^m)$ and $\lim_{t \to 0^+} \mathrm{ai}_t(\mathcal{X}_{B,\pi(Z)}; x^m) = \mathrm{pp}(\mathcal{X}_{B,\pi(Z)}; x^m)$ exist, and the limit of their difference is the difference of the limits. For x^m of degree $D+1$ this difference is 0 by hypothesis; for x^m of degree $\neq D+1$ both forms ai_t vanish on x^m and $\sigma \cdot x^m$ (they are supported in degree $D+1$ by (29.2)), so again both limits are 0. Hence $\mathrm{pp}(\mathcal{X}_{B,Z}; \sigma \cdot x^m) = \mathrm{pp}(\mathcal{X}_{B,\pi(Z)}; x^m)$ for every monomial. The limit forms pp are linear and $F \mapsto \sigma \cdot F$ is linear, and the monomials span $\mathcal{R}$, so $\mathrm{pp}(\mathcal{X}_{B,Z}; \sigma \cdot F) = \mathrm{pp}(\mathcal{X}_{B,\pi(Z)}; F)$ for every $F \in \mathcal{R}$. By (Gi2), $\mathrm{pp}(\mathcal{X}_{B,Z'}; G) = \widehat{\deg}_{B,Z'}(G)$ for every $Z' \in \mathrm{Cp}$ and $G \in \mathcal{R}$; applied with $(Z', G) = (Z, \sigma \cdot F)$ and $(\pi(Z), F)$ this gives $\widehat{\deg}_{B,Z}(\sigma \cdot F) = \widehat{\deg}_{B,\pi(Z)}(F)$ for every $F \in \mathcal{R}$ and every Z. This is (Ys0). □

Lemma 31.39. *(a) Let $\mathcal{X}$ be an arithmetic variety whose generic fibre $\mathcal{X}_{\mathbb{Q}}$ is smooth, and $\overline{\mathcal{M}}$ a Hermitian line bundle on $\mathcal{X}$ with smooth metric such that $\mathcal{M}_{\mathbb{Q}}$ is ample,* $\deg(\mathcal{M}|_C) \geq 0$ *for every closed integral curve $C \subseteq \mathcal{X}$ contained in a fibre of $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$, and the first Chern form $c_1(\overline{\mathcal{M}})$ is semipositive on the compact complex manifold $\mathcal{X}(\mathbb{C})$. Then $\overline{\mathcal{M}}$ is relatively semiample.*

(b) Consequently, for data as in the base slope settings at an admissible stage B and $Z \in \mathrm{Cp}$: if $\mathcal{X}^{\mathrm{b}}_{B,Z,\mathbb{Q}}$ is smooth and for all integers $1 \le i_1 \le i_2$ there is $t_0 > 0$ such that for $0 < t \le t_0$ the bundle $\overline{\mathcal{M}}_{i_1,i_2,t}$ satisfies the three hypotheses of (a) on $\mathcal{X}^{\mathrm{b}}_{B,Z}$ (ample generic fibre, non-negative degree on curves in the fibres over the primes, semipositive first Chern form), then (Bs2) holds at (B, Z).

Proof. (a) *Relative semipositivity.* Condition (i) of Theorem B.130 is the hypothesis on curves in the fibres. For (ii), let f be a holomorphic map from a polydisc to $\mathcal{X}(\mathbb{C})$ and l a local invertible section of $\mathcal{M}$; the curvature form $\frac{\partial\bar\partial}{\pi i}\log\|f^*l\|$ of $f^*\overline{\mathcal{M}}$ equals $f^*c_1(\overline{\mathcal{M}})$: locally $c_1(\overline{\mathcal{M}}) = \frac{i}{2\pi}\partial\bar\partial(-\log\|l\|^2) = -\frac{i}{\pi}\partial\bar\partial\log\|l\| = \frac{\partial\bar\partial}{\pi i}\log\|l\|$ [48], and the pull-back of a semipositive real $(1,1)$-form by a holomorphic map is semipositive: for a tangent vector v, $f^*\omega(v, Jv) = \omega(df\,v, J\,df\,v) \ge 0$ since df is complex linear. So $\overline{\mathcal{M}}$ is relatively semipositive.

Semiample metric. The scheme $\mathcal{X}_{\mathbb{C}} = \mathcal{X}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ is smooth and projective, hence a finite disjoint union of smooth projective varieties $X_1, \dots, X_r$ over $\mathbb{C}$ (its connected components), and $\mathcal{M}_{\mathbb{C}}|_{X_j}$ is ample because $\mathcal{M}_{\mathbb{Q}}$ is. Apply Theorem B.131(A) to X_j with $Y = X_j(\mathbb{C})$ (a compact complex manifold), i the identity and $\overline{M} = \overline{\mathcal{M}}|_{X_j}$, which is ample with semipositive curvature form: the metric on X_j is semiample. Given $\varepsilon > 0$, let n_j be the integer of the definition for X_j and $n = n_1 \cdots n_r$. If l is a section of $\mathcal{M}^{\otimes n_j}|_{X_j}$ with $\|l\|_{\sup} \le e^{n_j\varepsilon}\|l\|(x)$, then $l^{\otimes n/n_j}$ is a non-zero section of $\mathcal{M}^{\otimes n}|_{X_j}$ with $\|l^{\otimes n/n_j}\|_{\sup} = \|l\|_{\sup}^{n/n_j} \le e^{n\varepsilon}\|l^{\otimes n/n_j}\|(x)$; extending it by 0 on the other components gives a non-zero section of $\mathcal{M}_{\mathbb{C}}^{\otimes n}$ on $\mathcal{X}_{\mathbb{C}}$ with the same sup norm. Hence the metric of $\overline{\mathcal{M}}$ on $\mathcal{M}_{\mathbb{C}}$ is semiample. With $\mathcal{M}_{\mathbb{Q}}$ ample and relative semipositivity, $\overline{\mathcal{M}}$ is relatively semiample in the sense of Theorem B.131(B).

(b) For $1 \le i_1 \le i_2$ and $0 < t \le t_0$, (a) applied to $\overline{\mathcal{M}}_{i_1,i_2,t}$ on $\mathcal{X}^{\mathrm{b}}_{B,Z}$ gives that $\overline{\mathcal{M}}_{i_1,i_2,t}$ is relatively semiample, and $\mathcal{M}_{i_1,i_2,t,\mathbb{Q}}$ is ample by hypothesis: this is (Bs2) at (B, Z). □

Definition 31.40. Let a Hodge-type base setting with cusp data be given and let B be an integer such that $\mathrm{ar}^{\mathrm{b}}(B)$ exists. For integers $1 \le i_1 \le i_2$ and reals $t, t' > 0$ write $\omega_t = \frac{1}{i_2}c_1(\overline{\mathcal{N}}_{B,i_1,i_2,t})$ and

$$\mathrm{tN}(i_1,i_2,t) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{N}}_{B,i_1,i_2,t})^{e+1})}{i_2^{e+1}}, \qquad I(t,t') = \int_{X^{\mathrm{b}}} \left(\log h^{\mathrm{ref}'} + \psi_{t'} - \log h^{\mathrm{ref}} - \varphi_t\right)\omega_t^e,$$

$$V(t) = \int_{X^{\mathrm{b},\circ}} \left(\log h^{\mathrm{H}} + \tfrac{1}{2}U - \log h^{\mathrm{ref}} - \varphi_t\right)\omega_t^e, \qquad V^{\mathrm{b}}(t) = \int_{X^{\mathrm{b},\circ}} \tfrac{1}{2}U\,\omega_t^e.$$

Here the integrands are functions on X^{b}, resp. $X^{\mathrm{b},\circ}$, since the cone functions involved have equal multidegrees, and the indices i_1, i_2 are suppressed in $\omega_t, I, V, V^{\mathrm{b}}$. The setting has **regularisation limits at** B if:

(Mz1) Positivity of the log-log perturbed bundles For all $1 \le i_1 \le i_2$ there is $t_1 > 0$ such that $c_1(\overline{\mathcal{N}}_{B,i_1,i_2,t})$ is semipositive on X^{b} for $0 < t \le t_1$;

(Mz2) Limit in the bounded scale For all $1 \le i_1 \le i_2$ there is $t_2 > 0$ such that for every $0 < t \le t_2$ at which $c_1(\overline{\mathcal{N}}_{B,i_1,i_2,t})$ is semipositive, the integrals $V(t)$, $V^{\mathrm{b}}(t)$ exist, $I(t,t') \to V(t)$ as $t' \to 0^+$, and $V(t) \le V^{\mathrm{b}}(t)$;

(Mz3) Limit in the log-log scale For all $1 \le i_1 \le i_2$ the limits

$$\mathrm{tN}^{\infty}(i_1,i_2) = \lim_{t\to 0^+}\mathrm{tN}(i_1,i_2,t), \qquad V^{\mathrm{b},\infty}(i_1,i_2) = \lim_{t\to 0^+} V^{\mathrm{b}}(t)$$

exist;

(Mz4) Limit in the perturbation As $m \to \infty$,

$$\mathrm{tN}^{\infty}(1,m) \to \mathrm{ar}^{\mathrm{b}}(B) \quad \text{and} \quad V^{\mathrm{b},\infty}(1,m) \to \frac{1}{2}\,\mathrm{m}^{\mathrm{b}}\deg^{\mathrm{H}}.$$

These are the analytic facts by which the singular Hodge metric, its log-log regularisations and the bounded regularisations of the corrected weight are compared in the limit.

Proposition 31.41. *Let a Hodge-type base setting with cusp data be given, and let B be an integer such that $\deg^{\mathrm{H}} > 0$, $\mathrm{ar}^{\mathrm{b}}(B)$ exists, the setting has regularisation limits (Mz1)–(Mz4) at B, and for all integers $1 \le i_1 \le i_2$ there is $t_0 > 0$ such that $c_1(\overline{\mathcal{M}}_{B,i_1,i_2,t'})$ is semipositive for $0 < t' \le t_0$. Let $a \in \mathbb{R}$, and suppose that for all integers $1 \le i_1 \le i_2$ there is $t_0 > 0$ with*

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_{B,i_1,i_2,t'})^{e+1}) \ge (e+1)\,i_2\Big(a - \frac{i_1}{i_2}k^{\mathrm{A}}\Big)\deg(c_1(\mathcal{M}_{B,i_1,i_2,t',\mathbb{Q}})^{e} \cap [\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}]) \qquad (0 < t' \le t_0).$$

Then $\mathrm{sl}^{\mathrm{b}}(B) \ge a - \frac{1}{2}\mathrm{m}^{\mathrm{b}}$.

Proof. Fix an integer $m \ge 1$ and take $i_1 = 1$, $i_2 = m$; write $\overline{\mathcal{M}}_{t'} = \overline{\mathcal{M}}_{B,1,m,t'}$, $\overline{\mathcal{N}}_t = \overline{\mathcal{N}}_{B,1,m,t}$, whose underlying line bundle is in both cases $\mathcal{L} = \mathcal{O}(m a^{\mathrm{H}}_B + a')|_{\mathcal{X}^{\mathrm{b}}}$.

Step 1: the geometric degrees. Put $d_m = m^{-e}\deg(c_1(\mathcal{L}_{\mathbb{Q}})^e \cap [\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}])$; it does not depend on the metric. For $\ell \in S_B$ the rational section g_ℓ of $\mathcal{O}(v_\ell)|_{\mathcal{X}^{\mathrm{b}}}$ has divisor in the fibre over ℓ, so it trivialises $\mathcal{O}(v_\ell)$ on $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ and $c_1(\mathcal{O}(a^{\mathrm{H}}_B)) = c_1(\mathcal{O}(a^{\mathrm{H}}))$ on $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$. Writing $\alpha = c_1(\mathcal{O}(a^{\mathrm{H}}))$, $\beta = c_1(\mathcal{O}(a'))$ on $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$,

$$d_m = \sum_{k=0}^{e}\binom{e}{k} m^{-k}\deg(\alpha^{e-k}\beta^{k} \cap [\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}]) \longrightarrow \deg^{\mathrm{H}} \qquad (m \to \infty).$$

Step 2: comparison at two scales. Let t_0 be the smaller of the thresholds of the hypothesis on a and of the semipositivity of $c_1(\overline{\mathcal{M}}_{t'})$ for $(1,m)$, and t_1, t_2 those of (Mz1), (Mz2) for $(1,m)$. Fix $0 < t \le \min(t_1,t_2)$ and $0 < t' \le t_0$. The metrics of $\overline{\mathcal{N}}_t$ and $\overline{\mathcal{M}}_{t'}$ on $\mathcal{L}_{\mathbb{C}}$ are smooth and invariant under complex conjugation (Definition 31.10, Definition 31.12), and their cone functions differ by the factor e^{g} with $g = m\,w$, $w = \log h^{\mathrm{ref}'} + \psi_{t'} - \log h^{\mathrm{ref}} - \varphi_t$ (the factors $\prod_\ell |g_\ell|^m h'$ cancel), so $\|\cdot\|_{\overline{\mathcal{M}}_{t'}} = \|\cdot\|_{\overline{\mathcal{N}}_t} e^{-g}$. Both first Chern forms are semipositive ((Mz1) and the hypothesis). Lemma 31.42 gives, with $c_1(\overline{\mathcal{N}}_t) = m\,\omega_t$,

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_{t'})^{e+1}) - \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{N}}_t)^{e+1}) \le (e+1)\int_{X^{\mathrm{b}}} m\,w\,(m\,\omega_t)^{e} = (e+1)\,m^{e+1} I(t,t').$$

Dividing by m^{e+1} and using the hypothesis on a (divided by m^{e+1}, with $i_2 = m$, $i_1 = 1$):

$$(e+1)\Big(a - \frac{k^{\mathrm{A}}}{m}\Big)d_m \le \mathrm{tN}(1,m,t) + (e+1)\,I(t,t').$$

Step 3: the three limits. Let $t' \to 0^{+}$: by (Mz2), $I(t,t') \to V(t) \le V^{\mathrm{b}}(t)$, so $(e+1)(a - k^{\mathrm{A}}/m)d_m \le \mathrm{tN}(1,m,t) + (e+1)V^{\mathrm{b}}(t)$ for every $0 < t \le \min(t_1,t_2)$. Let $t \to 0^{+}$: by (Mz3),

$$(e+1)\Big(a - \frac{k^{\mathrm{A}}}{m}\Big)d_m \le \mathrm{tN}^{\infty}(1,m) + (e+1)V^{\mathrm{b},\infty}(1,m).$$

Let $m \to \infty$: by Step 1 and (Mz4), $(e+1)\,a\deg^{\mathrm{H}} \le \mathrm{ar}^{\mathrm{b}}(B) + \frac{e+1}{2}\mathrm{m}^{\mathrm{b}}\deg^{\mathrm{H}}$. Dividing by $(e+1)\deg^{\mathrm{H}} > 0$ and using (31.7) gives $\mathrm{sl}^{\mathrm{b}}(B) \ge a - \frac{1}{2}\mathrm{m}^{\mathrm{b}}$. □

Lemma 31.42. *Let $\mathcal{X}$ be an arithmetic variety of absolute dimension $e+1$ whose generic fibre is smooth, let $\mathcal{L}$ be a line bundle on $\mathcal{X}$, and let $\|\cdot\|_0$, $\|\cdot\|_1 = \|\cdot\|_0\, e^{-g}$ be two smooth metrics on $\mathcal{L}_{\mathbb{C}}$ invariant under complex conjugation, with g a smooth real function on $\mathcal{X}(\mathbb{C})$. Suppose the first Chern forms $\omega_0 = c_1(\mathcal{L}, \|\cdot\|_0)$ and $\omega_1 = c_1(\mathcal{L}, \|\cdot\|_1) = \omega_0 + \frac{i}{\pi}\partial\bar\partial g$ are semipositive on $\mathcal{X}(\mathbb{C})$. Then*

$$\widehat{\deg}(\widehat{c}_1(\mathcal{L}, \|\cdot\|_1)^{e+1}) - \widehat{\deg}(\widehat{c}_1(\mathcal{L}, \|\cdot\|_0)^{e+1}) \le (e+1)\int_{\mathcal{X}(\mathbb{C})} g\,\omega_0^e.$$

Proof. Write $\overline{\mathcal{L}}_i = (\mathcal{L}, \|\cdot\|_i)$ and $M = \mathcal{X}(\mathbb{C})$, a compact complex manifold of dimension e without boundary.

Step 1: the exact change of metric. In the normalisation $\widehat{c}_1(\overline{\mathcal{L}}) = [(\operatorname{div} s, -\log\|s\|^2)]$, replacing $\|\cdot\|_0$ by $\|\cdot\|_0 e^{-g}$ in one factor of an arithmetic intersection number of Hermitian line bundles changes it by $\frac{1}{2}\int_M 2g$ times the product of the first Chern forms of the other factors [48]: $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_1)\cdot x) - \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_0)\cdot x) = \int_M g\,\omega(x)$ for every product x of e first Chern classes of Hermitian line bundles with smooth metrics, $\omega(x)$ the product of their Chern forms. Telescoping $\widehat{c}_1(\overline{\mathcal{L}}_1)^{e+1} - \widehat{c}_1(\overline{\mathcal{L}}_0)^{e+1} = \sum_{k=0}^{e} \widehat{c}_1(\overline{\mathcal{L}}_1)^k(\widehat{c}_1(\overline{\mathcal{L}}_1) - \widehat{c}_1(\overline{\mathcal{L}}_0))\widehat{c}_1(\overline{\mathcal{L}}_0)^{e-k}$ gives

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_1)^{e+1}) - \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_0)^{e+1}) = \sum_{k=0}^{e}\int_M g\,\omega_1^k\wedge\omega_0^{e-k}. \tag{31.12}$$

Step 2: each term is at most $\int_M g\,\omega_0^e$. For $k=0$ there is equality. For $1\le k\le e$, $\omega_1^k - \omega_0^k = (\omega_1-\omega_0)\wedge\sum_{a+b=k-1}\omega_1^a\wedge\omega_0^b$, so

$$\int_M g\,\omega_1^k\wedge\omega_0^{e-k} = \int_M g\,\omega_0^e + \int_M g\,\tfrac{i}{\pi}\partial\bar\partial g\wedge R_k, \qquad R_k = \sum_{a+b=k-1}\omega_1^a\wedge\omega_0^{b+e-k},$$

with R_k a smooth closed form of bidegree $(e-1,e-1)$. Since R_k is closed and $\bar\partial g\wedge\bar\partial g = 0$, $d(g\,\bar\partial g\wedge R_k) = \partial g\wedge\bar\partial g\wedge R_k + g\,\partial\bar\partial g\wedge R_k$, and Stokes' theorem on M gives

$$\int_M g\,\tfrac{i}{\pi}\partial\bar\partial g\wedge R_k = -\frac{1}{\pi}\int_M i\,\partial g\wedge\bar\partial g\wedge R_k.$$

The form $i\,\partial g\wedge\bar\partial g$ is a semipositive $(1,1)$-form (g is real), and R_k is a sum of products of the semipositive $(1,1)$-forms ω_0,ω_1; a product of semipositive $(1,1)$-forms is a positive form [79], so $i\,\partial g\wedge\bar\partial g\wedge R_k$ is a non-negative top form and the last integral is ≥ 0. Hence $\int_M g\,\omega_1^k\wedge\omega_0^{e-k} \le \int_M g\,\omega_0^e$.

Conclusion. Summing Step 2 over $k=0,\dots,e$ in (31.12) gives the assertion. □

Lemma 31.43. *Let a Hodge-type base setting of type (Φ, n, κ) with cusp data be given, with $n-\kappa\le\kappa$, and suppose it has corrected point heights (Qk2) at an integer B for reals $r\ge 2$, L and $C_{\mathrm{h}}\ge 0$, with threshold t_0. Then for all integers $1\le i_1\le i_2$, all $0<t'\le t_0$ and the Zariski closure D_x of every $x\in\mathcal{X}^{\mathrm{b}}(\overline{\mathbb{Q}})$,*

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_{B,i_1,i_2,t'})\cdot[D_x]) \ge i_2\Big(-C_{\mathrm{h}}\mathrm{Lb}(B) - \frac{i_1}{i_2}k^{\mathrm{A}}\Big)[\mathbf{k}(D_x):\mathbb{Q}],$$

with $\mathrm{Lb}(B) = |\Phi|\min(\kappa, n-\kappa)(\log(Br)+|L|)$; that is, (Bs4) holds at B with the constant C_{h}.

Proof. Fix $1 \le i_1 \le i_2$, $0 < t' \le t_0$ and x, and write $d = [\mathbf{k}(D_x) : \mathbb{Q}]$. By construction $\overline{\mathcal{M}}_{B,i_1,i_2,t'} = (\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'})^{\otimes i_2} \otimes \overline{\mathcal{A}}'^{\otimes i_1}$ as Hermitian line bundles (the underlying multidegrees add and the cone functions multiply), so $\widehat{c}_1(\overline{\mathcal{M}}_{B,i_1,i_2,t'}) = i_2\,\widehat{c}_1(\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'}) + i_1\,\widehat{c}_1(\overline{\mathcal{A}}')$, and by linearity of $\widehat{\deg}(\,\cdot\,\cdot\,[D_x])$,

$$\widehat{\deg}\big(\widehat{c}_1(\overline{\mathcal{M}}_{B,i_1,i_2,t'}) \cdot [D_x]\big) = i_2\, h_{\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'}}(D_x) + i_1\, h_{\overline{\mathcal{A}}'}(D_x).$$

By (Qk2), $h_{\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'}}(D_x) \ge -C_{\mathrm{h}}|\Phi|(n-\kappa)(\log(Br) + |L|)\,d = -C_{\mathrm{h}}\mathrm{Lb}(B)\,d$, since $\min(\kappa, n-\kappa) = n-\kappa$. By the auxiliary floor (Hy5), $h_{\overline{\mathcal{A}}'}(D_x) \ge -k^{\mathrm{A}}d$. Since $i_1, i_2 > 0$, adding these bounds shows that the right-hand side of the display is $\ge \big(-i_2 C_{\mathrm{h}}\mathrm{Lb}(B) - i_1 k^{\mathrm{A}}\big)d = i_2\big(-C_{\mathrm{h}}\mathrm{Lb}(B) - \frac{i_1}{i_2}k^{\mathrm{A}}\big)d$. □

Theorem 31.44. *Let a Hodge-type base setting of type (Φ, n, κ) with cusp data be given, with $\deg^{\mathrm{H}} > 0$, $1 \le n - \kappa \le \kappa$, and with the cusp mean bound (Qk1) for a real $r \ge 2$. Then for every real L and every integer $B \ge 64$ with $2|\Phi|(n-\kappa) \le B$,*

$$\mathrm{m}^{\mathrm{b}} \le 4\,|\Phi|\,(n-\kappa)\big(\log(Br) + |L|\big) = 4\,\mathrm{Lb}(B).$$

In particular, at every admissible stage of Definition 31.13 the clause (Bs5) holds with $C_{\mathrm{u}} = 4$.

Proof. Write $f = |\Phi|$ and $j' = n - \kappa$. Since U is integrable for $(\omega^{\mathrm{H}})^e$ and $\deg^{\mathrm{H}} > 0$, (Qk1) divided by $\deg^{\mathrm{H}}$ gives $\mathrm{m}^{\mathrm{b}} \le fj'\big(6 + \log(2fr) + \log(2fj'r)\big)$. It remains to show

$$6 + \log(2fr) + \log(2fj'r) \le 4\big(\log(Br) + |L|\big).$$

Since $j' \ge 1$, $2f \le 2fj' \le B$, so $\log(2fr) \le \log(Br)$ and $\log(2fj'r) \le \log(Br)$ ($r > 0$). Since $B \ge 64$ and $r \ge 2$, $Br \ge 128 > e^3$, so $\log(Br) \ge 3$ and $6 \le 2\log(Br)$. Hence the left side is at most $4\log(Br) \le 4(\log(Br) + |L|)$. Multiplying by $fj' \ge 0$ gives the displayed bound, and $fj' = |\Phi|\min(\kappa, n-\kappa)$ because $n - \kappa \le \kappa$, so the right side is $4\,\mathrm{Lb}(B)$.

Admissible stages satisfy $B \ge 128 \ge 64$ and $B \ge 4|\Phi|n^2 \ge 2|\Phi|(n-\kappa)$, so (Bs5) holds there with $C_{\mathrm{u}} = 4$. □

Theorem 31.45. *Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with positive volumes (Rt1), and let $r \ge 1$, L be real numbers and $C_{\mathrm{h}}, C_{\mathrm{u}} \ge 0$. If $\mathcal{T}$ has base slope settings for (r, L) with constants $(C_{\mathrm{h}}, C_{\mathrm{u}})$, then $\mathcal{T}$ has the slope property (Rt6) for (r, L) with the constant $C = C_{\mathrm{h}} + \frac{1}{2}C_{\mathrm{u}}$.*

Proof. Let B be an integer with $B \ge 128$, $B \ge 4|\Phi|n^2$ and $|\mathrm{Bd}| \le B$; these are exactly the admissible stages of Definition 31.13 and the stages at which (Rt6) is asserted. Write $\mathrm{td} = \deg^{\mathrm{fl}}(\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb})$ for the test degree (a local abbreviation).

Step 1: every corrected perturbed bundle satisfies the hypothesis of (Bs3) with $a = -C_{\mathrm{h}}\mathrm{Lb}(B)$. Fix $Z \in \mathrm{Cp}$, the data $\mathcal{X}^{\mathrm{b}}$, e, ka^{b}, $\overline{\mathcal{M}}_{i_1,i_2,t}$ of (Bs1)–(Bs5) at (B, Z), and integers $1 \le i_1 \le i_2$. Let $t_0 > 0$ be the smaller of the two thresholds given by (Bs2) and (Bs4) for (i_1, i_2), and let $0 < t \le t_0$. By (Bs2), $\mathcal{M}_{i_1,i_2,t,\mathbb{Q}}$ is ample and $\overline{\mathcal{M}}_{i_1,i_2,t}$ is relatively semiample with smooth metric; by (Bs4), every point $x \in \mathcal{X}^{\mathrm{b}}(\overline{\mathbb{Q}})$ satisfies $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_{i_1,i_2,t}) \cdot [D_x]) \ge e'\,[\mathbf{k}(D_x) : \mathbb{Q}]$ with $e' = i_2(a - \frac{i_1}{i_2}\mathrm{ka}^{\mathrm{b}})$. So Lemma 31.46, applied on $\mathcal{X}^{\mathrm{b}}$ (absolute dimension $e+1$) with the floor e', gives

$$\widehat{\deg}\big(\widehat{c}_1(\overline{\mathcal{M}}_{i_1,i_2,t})^{e+1}\big) \ge (e+1)\,i_2\Big(a - \frac{i_1}{i_2}\mathrm{ka}^{\mathrm{b}}\Big)\deg\big(c_1(\mathcal{M}_{i_1,i_2,t,\mathbb{Q}})^e \cap [\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}]\big) \qquad (0 < t \le t_0).$$

Step 2: the base slope is bounded below. Step 1 holds for all $1 \le i_1 \le i_2$, so (Bs3) applies with $a = -C_{\mathrm{h}}\mathrm{Lb}(B)$, and with (Bs5)

$$\mathrm{sl}^{\mathrm{b}}(B, Z) \ge -C_{\mathrm{h}}\mathrm{Lb}(B) - \tfrac{1}{2}\mathrm{m}^{\mathrm{b}}(B, Z) \ge -\Big(C_{\mathrm{h}} + \tfrac{1}{2}C_{\mathrm{u}}\Big)\mathrm{Lb}(B) = -C\,\mathrm{Lb}(B).$$

Step 3: summation over the components. The factor $(D^{\mathrm{gr}}+1)\,\mathrm{vol}(Z)\,\mathrm{td}$ is positive: $\mathrm{vol}(Z) > 0$ by (Rt1) and $\mathrm{td} > 0$ by Proposition 29.36(a). Multiplying Step 2 by it and using (Bs1),

$$\widehat{\deg}_{B,Z}\big(\mathrm{pl}^{D^{\mathrm{gr}}+1}\mathrm{fb}\big) = (D^{\mathrm{gr}}+1)\,\mathrm{vol}(Z)\,\mathrm{td}\,\mathrm{sl}^{\mathrm{b}}(B,Z) \geq -C\,\mathrm{Lb}(B)\,(D^{\mathrm{gr}}+1)\,\mathrm{vol}(Z)\,\mathrm{td}.$$

Summing over $Z \in \mathrm{Cp}$, with $\widehat{\deg}_B = \sum_Z \widehat{\deg}_{B,Z}$ and $\mathrm{vol}_{\mathcal{T}} = \sum_Z \mathrm{vol}(Z)$, and inserting $\mathrm{Lb}(B) = |\Phi|\min(\kappa, n-\kappa)\mathrm{Lg}(B)$:

$$\widehat{\deg}_B\big(\mathrm{pl}^{D^{\mathrm{gr}}+1}\mathrm{fb}\big) \geq -C\,|\Phi|\min(\kappa,n-\kappa)\,\mathrm{Lg}(B)\,(D^{\mathrm{gr}}+1)\,\mathrm{vol}_{\mathcal{T}}\,\deg^{\mathrm{fl}}\big(\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb}\big),$$

which is (Rt6) with the constant C. □

Lemma 31.46. *Let $\mathcal{X}$ be an arithmetic variety over $\mathbb{Z}$ of absolute dimension $D+1$ and $\overline{\mathcal{L}}$ a Hermitian line bundle on $\mathcal{X}$ with smooth metric such that $\mathcal{L}_{\mathbb{Q}}$ is ample and $\overline{\mathcal{L}}$ is relatively semiample. Let $e \in \mathbb{R}$ and assume that*

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})\cdot[D_x]) \geq e\,[\mathbf{k}(D_x):\mathbb{Q}]$$

for the Zariski closure D_x of every $x \in \mathcal{X}(\overline{\mathbb{Q}})$, where $\mathbf{k}(D_x)$ is the function field of D_x. Then

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{D+1}) \geq (D+1)\,e\,\deg(c_1(\mathcal{L}_{\mathbb{Q}})^D \cap [\mathcal{X}_{\mathbb{Q}}]).$$

Proof. Let $\overline{\mathcal{L}}'$ be $\mathcal{L}$ with the metric $\|\cdot\|' = e^e\|\cdot\|$.

The shifted bundle satisfies the hypotheses of Theorem B.131(B). Its metric is smooth and has the same curvature form, so $\overline{\mathcal{L}}'$ is relatively semipositive, and $\mathcal{L}_{\mathbb{Q}}$ is unchanged, hence ample. For a section l of $\mathcal{L}_{\mathbb{C}}^{\otimes n}$ both $\|l\|'_{\sup} = e^{ne}\|l\|_{\sup}$ and $\|l\|'(x) = e^{ne}\|l\|(x)$, so the inequality $\|l\|'_{\sup} \leq e^{n\varepsilon}\|l\|'(x)$ is equivalent to the same inequality for $\|\cdot\|$; hence the metric of $\overline{\mathcal{L}}'$ is semiample, and $\overline{\mathcal{L}}'$ is relatively semiample.

Heights of points drop by e per degree. Let $x \in \mathcal{X}(\overline{\mathbb{Q}})$, $W = D_x$, $k = \mathbf{k}(W)$, and let s be a rational section of $\mathcal{L}|_W$ that is regular and non-zero at the generic point of W and whose divisor avoids the finitely many points of $W(\mathbb{C})$ (these are the $[k:\mathbb{Q}]$ complex points x_σ, $\sigma\colon k \to \mathbb{C}$). By the definition of the arithmetic degree on the arithmetic curve W [48],

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})\cdot[W]) = \deg_{\mathrm{fin}}(\mathrm{div}\,s) - \sum_{\sigma\colon k\to\mathbb{C}} \log\|s\|(x_\sigma),$$

with $\deg_{\mathrm{fin}}(\mathrm{div}\,s)$ the sum of $\log\#$ of the residue fields weighted by multiplicities, independent of the metric. Replacing $\|\cdot\|$ by $\|\cdot\|'$ subtracts e from each of the $[k:\mathbb{Q}]$ terms, so

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}')\cdot[D_x]) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})\cdot[D_x]) - e\,[\mathbf{k}(D_x):\mathbb{Q}] \geq 0$$

for every x. Hence Theorem B.131(B) gives $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}')^{D+1}) \geq 0$.

The self-intersection drops by $(D+1)e$ times the degree. In the arithmetic Chow ring [110], $\widehat{c}_1(\overline{\mathcal{L}})$ is represented by $(\mathrm{div}\,s, -\log\|s\|^2)$ for a rational section s of $\mathcal{L}$, so $\widehat{c}_1(\overline{\mathcal{L}}') = \widehat{c}_1(\overline{\mathcal{L}}) - a(2e)$, where $a(2e)$ is the class of $(0,2e)$ with the constant function $2e$. Products $a(\eta)\cdot a(\eta')$ with η' closed of type $(0,0)$ vanish ($a(\eta)\cdot a(\eta') = a(\eta\,dd^c\eta') = 0$ for constant η'), and $\widehat{\deg}(a(\eta)\cdot\widehat{c}_1(\overline{\mathcal{L}})^D) = \frac{1}{2}\int_{\mathcal{X}(\mathbb{C})}\eta\,c_1(\overline{\mathcal{L}})^D$. Expanding the $(D+1)$-st power,

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}')^{D+1}) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{D+1}) - (D+1)\cdot\tfrac{1}{2}\cdot 2e\int_{\mathcal{X}(\mathbb{C})} c_1(\overline{\mathcal{L}})^D,$$

and $\int_{\mathcal{X}(\mathbb{C})} c_1(\overline{\mathcal{L}})^D = \deg(c_1(\mathcal{L}_{\mathbb{Q}})^D \cap [\mathcal{X}_{\mathbb{Q}}])$, since the de Rham class of the curvature form is $c_1(\mathcal{L}_{\mathbb{C}})$. Combining with $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}')^{D+1}) \geq 0$ gives the claim. □

32 Integral models with uniformised degenerations

This section proves Proposition 32.1: for a unitary Shimura datum whose level is good at every prime, with $A = 1$, there is an integral model over $\mathcal{O}_F$ with uniformised degenerations, that is, a normal projective flat model carrying a semiabelian extension of the universal abelian scheme which, at every place v, dominates a local model whose fibres split (Gl2) and whose degenerations are uniformised at complete local rings (Gl3). It is the first step of the proof of Proposition 31.14, and lemmas of the section on the moduli schemes, their dimension and the splitting of semiabelian fibres are used again in Sections 33 and 34 and in Appendix D. The model is obtained from Lan's projective toroidal compactifications of PEL-type Shimura varieties [154], and the uniformisation from the degeneration data of Faltings and Chai [87] and Mumford's construction [187]. Subsection 32.1 proves existence and Subsection 32.2 uniformisation, both for the unitary moduli problem with half-ideal levels and $\mathcal{O}_E$-action used here.

32.1 Existence of integral models

The subsection states Proposition 32.1 and proves its existence part. Moduli schemes of unitary PEL type exist at every level (Proposition 32.2): the unitary moduli scheme is the free quotient of Lan's full-level PEL moduli scheme by the half-level group (Proposition 32.9). A normal projective flat model over $\mathcal{O}_F$ with a semiabelian extension of the universal abelian scheme is glued from models at every prime (Proposition 32.5, Lemma 32.6), starting from the projective integral toroidal compactifications of Lan [154], [156]; a cubical line bundle then extends (Proposition 32.4). The last lemmas show that the abelian locus is stable under generisation and that fibres of a semiabelian scheme split after a finite separable extension, so that every integral model satisfies (Gl2) (Lemma 32.13).

Proposition 32.1. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p + q \geq 1$ and level $\mathsf{L} = (N, \mathfrak{n}, A)$ good at every prime with $A = 1$. Then there is an integral model of* Sh *over $\mathcal{O}_F$ with uniformised degenerations.*

Proof. A model over $\mathcal{O}_F$. By the standing assumption of Definition 29.5, E is a CM field unramified over its maximal totally real subfield at every finite place, and $p+q \geq 1$; so Proposition 32.2 gives a system of moduli schemes extending $\mathfrak{u}$. With it, Proposition 32.4 gives an integral model $(\mathcal{Y}, j, G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ of Sh over $\mathcal{O}_F$. The level is good at every prime and $A = 1$ by hypothesis.

The localisations are integral models over $\mathcal{O}_{F,v}$. Fix a prime ℓ and an ℓ-adic place v. The ring $\mathcal{O}_{F,v} = F \cap \mathcal{O}_v$ is the localisation $\Sigma^{-1}\mathcal{O}_F$ at the prime $\mathfrak{p}_v = \mathcal{O}_F \cap \mathfrak{m}_v$, with $\Sigma = \mathcal{O}_F \setminus \mathfrak{p}_v$. Put $\mathcal{Y}^v = \mathcal{Y} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v}$ and let $(G^v, \iota^v, \mathcal{L}^v)$ be the base change of $(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$.

- $\mathcal{Y}^v$ is projective and flat over $\mathcal{O}_{F,v}$, by base change.
- $\mathcal{Y}^v$ is normal: for an affine open $\operatorname{Spec} C \subseteq \mathcal{Y}$ its preimage is $\operatorname{Spec} \Sigma^{-1}C$, and the local rings of $\Sigma^{-1}C$ are the localisations $C_{\mathfrak{q}}$ at the primes $\mathfrak{q}$ of C not meeting the image of Σ, i.e. local rings of $\mathcal{Y}$, which are normal.
- Since $\mathcal{O}_F \to F$ factors through $\mathcal{O}_{F,v}$, the generic fibre $\mathcal{Y}^v \otimes_{\mathcal{O}_{F,v}} F$ is canonically $\mathcal{Y} \otimes_{\mathcal{O}_F} F$, and the base change of $(G^v, \iota^v, \mathcal{L}^v)$ to it is that of $(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$. So j is an open immersion $\mathrm{Sh}_F \to \mathcal{Y}^v \otimes F$ with dense image, $j^*(G^v, \iota^v, \mathcal{L}^v)$ is isomorphic to the base change of the universal $(G, \iota, \mathcal{L})$, and the points of $\mathcal{Y}^v \otimes F$ over which G^v is abelian are those of $j(\mathrm{Sh}_F)$.
- $G^v \to \mathcal{Y}^v$ is a semiabelian scheme, ι^v an $\mathcal{O}_E$-action and $\mathcal{L}^v$ a rigidified symmetric cubical line bundle satisfying the two isomorphisms of (Ud4), all by base change.

Hence $(\mathcal{Y}^v, j, G^v, \iota^v, \mathcal{L}^v)$ is an integral model of Sh over $\mathcal{O}_{F,v}$.

(Ug1). By Lemma 32.13, $\mathcal{Y}^v$ satisfies (Gl2). The datum gives a moduli scheme Sh of type (U, L) and a coefficient field F for L, so Proposition 32.14, applied to ℓ, v, $\mathsf{L}' = \mathsf{L}$, $\mathrm{Sh}_{\mathsf{L}'} = \mathrm{Sh}$, $F' = F$ and $\mathcal{Y}^v$, gives (Gl3). This is (Ug1) with the local model $\mathcal{Y}^v$.

(Ug2). Let f_v be the identity of $\mathcal{Y} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v} = \mathcal{Y}^v$. The base change of $(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ to $\mathcal{Y} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v}$ is $(G^v, \iota^v, \mathcal{L}^v)$, the semiabelian structure of $\mathcal{Y}^v$, so the identity of G^v is a morphism of semiabelian structures over f_v. This is (Ug2).

As ℓ and v were arbitrary, $\mathcal{Y}$ is an integral model of Sh over $\mathcal{O}_F$ with uniformised degenerations. □

Proposition 32.2. *Let E be a CM field unramified over its maximal totally real subfield at every finite place, H a CM host on E and U a unitary datum for H of signature (p, q) with $p + q \geq 1$. Then for every level L' for H there is a moduli scheme of type $(\mathsf{U}, \mathsf{L}')$. In particular every unitary Shimura datum $(E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ with $p + q \geq 1$ is extended by a system of moduli schemes.*

Proof. Fix a level $\mathsf{L}' = (N, \mathfrak{n}, A)$. Put

$$K = \mathbb{Q}(r_{\mathbb{C}}, \zeta_A), \qquad \zeta_A = e^{2\pi i/A}, \qquad N_0 = \mathrm{lcm}(N, A).$$

Here $N \geq 3$ is odd, so $N_0 \geq 3$; no good-at-prime condition is needed. Choose a complex structure as in (Ud2). By Lemma 32.3, the reflex field is contained in $\mathbb{Q}(r_{\mathbb{C}}) \subseteq K$. Base change the full N_0-level scheme of Theorem B.105 to K, and take its open-and-closed subscheme X on which

$$\nu_{N_0}(1)^{N_0/A} = \zeta_A. \qquad (1)$$

This is open and closed because the roots of unity are finite étale in characteristic zero. Thus X is smooth quasi-projective of relative dimension $\frac{1}{2}[E : \mathbb{Q}]pq$, by Lemma 32.3.

We compare the full level with the required two maps. In M/NM put $U = \bar{\mathfrak{n}}M/NM$. The Chinese remainder decomposition gives an idempotent e modulo $N\mathcal{O}_E$, with $e = 1$ modulo $\mathfrak{n}$ and $e = 0$ modulo $\bar{\mathfrak{n}}$. Then $U = e(M/NM)$ and $\bar{e} = 1 - e$. Reducing a full N_0-level to N and A, respectively, define

$$\eta(a) = \alpha_N(ea), \qquad \eta_A(a) = \alpha_A(a).$$

The first kills $\mathfrak{n}M$ and equals α_N on $\bar{\mathfrak{n}}M$; the second kills AM. Conversely the restriction on $\bar{\mathfrak{n}}M$, together with vanishing on $\mathfrak{n}M$, determines η because these two submodules sum to M. Equation (1) fixes precisely the root prescribed for η_A. A compatible full-level tower gives the compatible pairs at every positive modulus by reduction from multiples of N_0, so these maps satisfy the level clause of (Ud4) on geometric fibres.

For every multiple m of N_0, let $\mathcal{G}_m$ be the finite group of pairs (g, c), where g is an $\mathcal{O}_E$-linear automorphism of M/mM, $c \in (\mathbb{Z}/m)^\times$, and

$$\psi(gx, gy) = c\psi(x, y) \pmod{m}.$$

Let $\mathcal{H}_m$ be its subgroup defined by

$$g_N|_U = 1, \qquad g_A = 1, \qquad c \equiv 1 \pmod{A}. \qquad (2)$$

Subscripts denote reduction to the indicated modulus. These groups form an inverse system. For fixed m, the images of $\mathcal{H}_l \to \mathcal{H}_m$, for $m \mid l$, form a downward-directed finite collection. Choose l_m attaining their minimum size and write $\mathcal{H}_m^\infty$ for this minimum subgroup. A common

multiple shows it is contained in every image and that all images indexed by multiples of l_m equal it. This subgroup is exactly the image of compatible changes at every modulus: choose a cofinal sequence $m\,i!$, lift at finite stages, and successively pass to subsequences with constant projections at each earlier modulus. The compatible resulting projections define a change at every modulus. Put $\mathcal{H} = \mathcal{H}^{\infty}_{N_0}$.

We justify liftability on arbitrary bases, as required by the full-level theorem. Let S be any K-scheme carrying an object of (Ud4). Its determinant condition holds even if S is nonreduced: after a finite étale cover splitting the E-action, its Lie sheaf is the direct sum of the idempotent eigenbundles. The geometric Kottwitz condition gives each bundle its prescribed rank p or q, and $a \in \mathcal{O}_E$ acts on that bundle by the corresponding scalar. Thus its characteristic polynomial is P_a. Faithful flatness descends this equality to S. This checks the determinant hypothesis whenever the full-level theorem is used below.

For each $N_0 \mid m$, let R_m/S parametrize a full m-level I_m and a primitive cyclotomic trivialisation ν_m, preserving the pairing and satisfying

$$I_N|_{\bar{\mathfrak{n}}M} = \eta|_{\bar{\mathfrak{n}}M}, \quad I_A = \eta_A, \quad \nu_m(1)^{m/A} = \zeta_A. \tag{3}$$

The maps on the right factor through their finite torsion groups. In characteristic zero those torsion groups and roots are finite étale. The $\mathcal{O}_E$-linearity, pairing and displayed equalities are open-and-closed conditions in finite étale isomorphism schemes; a finite integral basis of $\mathcal{O}_E$ and a finite basis of M suffice to test them. Hence R_m is finite étale over S. Its geometric fibres are nonempty by the level clause of (Ud4), and are torsors under $\mathcal{H}_m$, acting by precomposition on the level and the multiplier on the root.

At a geometric point choose a compatible system from (Ud4) as origin. Images $R_l \to R_m$ are the cosets of the corresponding subgroup images. Therefore

$$R^{\infty}_m = \mathrm{im}(R_{l_m} \to R_m)$$

is a well-defined finite étale open-and-closed subscheme and an $\mathcal{H}^{\infty}_m$-torsor, independent of the minimizing choice. The compatible-projection argument above shows that these are exactly the levels extending to a compatible system at all moduli. In particular every transition $R^{\infty}_l \to R^{\infty}_m$, for $m \mid l$, is finite étale surjective.

Over the finite étale cover $S' = R^{\infty}_{N_0} \to S$, the tautological full N_0-level thus has the actual lifting tower required by Theorem B.105: take the R^{∞}_m over $R^{\infty}_{N_0} = S'$, with their tautological levels and roots. Condition (3) places its classifying morphism in X. This is an arbitrary-base construction, using no locally noetherian assumption. Equalities of the level maps tested on geometric fibres hold even on nonreduced bases, since they are equalities of sections of finite étale schemes.

Next we prove rigidity of the partial level. Over an algebraically closed field of characteristic zero, the idempotents give $G[N] = G[\mathfrak{n}] \oplus G[\bar{\mathfrak{n}}]$, and η identifies U with the first summand. Rosati makes $e, \bar{e}$ adjoint, so both summands are isotropic and their cross-pairing is perfect. An equivariant polarised automorphism fixing η fixes the first summand, preserves the second and fixes it by that perfect pairing. It fixes all of $G[N]$.

Such an automorphism is the identity. Descend it with the polarised abelian variety to a finitely generated characteristic-zero field and embed that field in $\mathbb{C}$. Its action on integral homology is faithful and preserves a positive definite real form. The integral matrix group is discrete in a compact orthogonal group, hence finite. A finite-order integral matrix congruent to 1 modulo an odd prime ℓ is the identity: otherwise a power V has prime order q and $V = 1 + \ell^s B$, $s \geq 1$, with $B \not\equiv 0 \pmod{\ell}$. If $q \neq \ell$, the binomial expansion of $V^q = 1$ modulo ℓ^{s+1} is a

contradiction. If $q = \ell$, divide the expansion by ℓ^{s+1}; every term after B is divisible by ℓ, including the last since $(\ell - 1)s - 1 \geq 1$, again a contradiction. Apply this to any odd prime dividing N. Over a reduced base rigidity follows as well: the total abelian scheme is reduced, and the closed equalizer with the identity contains every geometric point, so its defining ideal is zero. The additional map η_A is not needed for this rigidity argument.

The finite group $\mathcal{H}$ acts on X by full-level changes. Its elements lift compatibly, so preserve liftability, and (2) preserves (1) and both partial-level maps. The action is free: a geometric stabilizer gives an equivariant polarised automorphism fixing η, hence the identity. The full level then forces $g = 1$; the nonzero module and perfect pairing force $c = 1$. By Theorem B.143, $Y = X/\mathcal{H}$ is quasi-projective and $X \to Y$ is a finite étale surjection. Smoothness and the stated relative dimension descend.

Apply Lemma 31.16 with $w_0 = (1 + r)/2$ to the universal polarised abelian scheme over X. Its primitive symmetric rigidified cubical line bundle, action and both maps have canonical equivariant descent data: changing the full level leaves these objects unchanged and their compatible identifications are unique. Thus the cocycle identities hold. They descend to Y. Concretely, the total abelian scheme over X is quasi-projective, since its canonical line bundle is relatively ample over the quasi-projective base. Its finite free quotient exists; the group operations descend, and being an abelian scheme and all the line-bundle properties are checked on the finite étale cover. The descended tuple o^{un} is an object of (Ud4), including the prescribed A-root by (1).

Let S be reduced and o an object over S. Its geometric Kottwitz condition implies the global determinant condition: in any affine trivialisation of its Lie sheaf, every coefficient difference from the prescribed polynomial vanishes in every geometric residue field, hence lies in the nilradical and is zero. The primitive-cubical equivalence gives the polarisation and Rosati data. The cover $R^{\infty}_{N_0} \to S$ constructed above carries a full-level object, so Theorem B.105 gives an equivariant morphism to X, descending to $S \to Y$. The local isomorphisms with o descend by their uniqueness over reduced bases. If two maps classify o, lift them on a common finite étale cover to X. Their full levels and roots differ, under the partial-level isomorphism, by a section of $\mathcal{H}$: at geometric points both have compatible full systems, and (2),(3) fix exactly the partial data. After this change full-level representability identifies the maps to Y; faithful flatness gives equality on S. This is (Ud5)(i).

For (Ud5)(ii), let $T = \operatorname{Spec} k[\epsilon]$, $\epsilon^2 = 0$, and suppose two pullbacks of o^{un} along maps to Y are isomorphic. On a common finite étale cover lift both maps to X. The difference of their full levels and roots is a section of the finite étale level-change scheme; at every geometric point it belongs to $\mathcal{H}$, by the compatible-lift characterization. Hence it belongs to $\mathcal{H}$ over the cover itself. Adjust one lift. The isomorphism then preserves the full level and root, so full-level representability on arbitrary schemes makes the two lifted maps equal. Descent gives equality of the maps from T. In particular a map equal in moduli to its constant point is that constant map. This uses only pullbacks of the universal object, which already satisfy the global determinant condition; it asserts no classification of arbitrary nonreduced objects.

We have proved (Ud5) for every level L'. Choose one such moduli scheme with universal object for each level. For a preassigned unitary Shimura datum, use its given moduli scheme and universal object at its own level, and the choices just made at all other levels. This is a system of moduli schemes extending that datum. □

Lemma 32.3. *Let $\mathsf{H} = (d, r, r_{\mathbb{C}})$ be a CM host on E, and let (M, ψ) be a unitary datum of signature (p, q) with $p + q \geq 1$. Choose a complex structure J as in (Ud2), and put $\Phi = \Phi_{\mathsf{H}}$ and*

$K_{\mathsf{H}} = \mathbb{Q}(r_{\mathbb{C}})$. *The determinant polynomials and reflex field of Theorem B.105 satisfy*

$$P_a(T) = \prod_{\tau \in \Phi} (T - \tau(a))^p (T - \bar{\tau}(a))^q, \qquad F_{\mathrm{ref}} \subseteq K_{\mathsf{H}}.$$

For every $N \geq 3$, the full-level moduli scheme of that theorem, base changed to K_{H}, is smooth of relative dimension

$$|\Phi| pq = \tfrac{1}{2}[E : \mathbb{Q}] pq.$$

Its geometric tangent spaces have this dimension.

Proof. Regard $M_{\mathbb{R}}$ as a complex vector space using J. For $a \in \mathcal{O}_E$, (Ud2) gives

$$\mathrm{tr}_{\mathbb{C}}(a) = \tfrac{1}{2}\big(\mathrm{tr}_{\mathbb{R}}(a) - i\,\mathrm{tr}_{\mathbb{R}}(Ja)\big) = \sum_{\tau \in \Phi} \big(p\tau(a) + q\bar{\tau}(a)\big).$$

Choose an integral primitive element a_0 of $E/\mathbb{Q}$. The semisimple E-action decomposes the complex vector space into embedding eigenspaces. Apply the displayed trace identity to $1, a_0, \ldots, a_0^{[E:\mathbb{Q}]-1}$. The Vandermonde matrix of the distinct conjugates of a_0 is invertible, so their multiplicities are p on Φ and q on its conjugate. This proves the determinant formula for every a.

A field automorphism of $\overline{\mathbb{Q}}$ fixing K_{H} preserves the condition $\tau(r) = r_{\mathbb{C}}$. It therefore permutes Φ and its conjugate separately and fixes every coefficient of the displayed polynomial. The coefficients are algebraic, so they belong to K_{H}. This proves the field inclusion. Consequently the base change of the smooth scheme in Theorem B.105 is defined.

Take a geometric point of that base change over an algebraically closed field k, and identify a splitting field of E inside k extending the given embedding of K_{H}. Put $\omega = \omega_A \otimes k$ at the point. The determinant condition and the distinct eigenvalues of a_0 give

$$\omega = \bigoplus_{\tau : E \hookrightarrow k} \omega_\tau, \qquad \dim_k \omega_\tau = p, \quad \dim_k \omega_{\bar{\tau}} = q \quad (\tau \in \Phi).$$

Here Φ has been transported through that splitting-field embedding. Pullback on the dual of a vector space preserves the eigenvalues of an endomorphism, so the Lie multiplicities are also the multiplicities on invariant differentials.

The principal polarisation identifies $\omega_{A^\vee} \otimes k$ with ω. Rosati gives

$$\phi_{\mathrm{pol}}^* (\iota(a)^\vee)^* = \iota(\bar{a})^* \phi_{\mathrm{pol}}^*.$$

By Theorem B.106 the cotangent fibre is

$$Q = \frac{\omega \otimes_k \omega}{\langle x \otimes y - y \otimes x,\ ax \otimes y - x \otimes \bar{a}y \rangle}.$$

If $x \in \omega_\tau$ and $y \in \omega_\upsilon$, the relation for a_0 reads

$$(\tau(a_0) - \bar{\upsilon}(a_0))\, x \otimes y = 0.$$

Unless $\upsilon = \bar{\tau}$ this coefficient is nonzero. Thus only conjugate tensor factors survive, and symmetry identifies their two orientations.

There are no further relations: project $\omega \otimes \omega$ to $\bigoplus_{\tau \in \Phi} \omega_\tau \otimes \omega_{\bar{\tau}}$ by the identity on each ordered conjugate factor, by interchange on its reversed factor, and by zero on all other factors. This

map kills both displayed families of relations and is inverse to inclusion of the ordered factors into Q. Hence

$$Q \cong \bigoplus_{\tau\in\Phi} \omega_\tau \otimes \omega_{\bar\tau}, \qquad \dim_k Q = |\Phi| pq.$$

Smoothness identifies the cotangent dimension with the relative dimension at every geometric point. The computation also applies when $p = 0$ or $q = 0$. □

Proposition 32.4. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p + q \geq 1$ and level $\mathsf{L} = (N, \mathfrak{n}, A)$ good at every prime with $A = 1$, and let a system of moduli schemes extending $\mathfrak{u}$ be given. Then there is an integral model of* Sh *over $\mathcal{O}_F$.*

Proof. By Proposition 32.5 there are a normal scheme $\mathcal{Y}$, projective and flat over $\mathcal{O}_F$, an open immersion $j\colon \mathrm{Sh}_F \to \mathcal{Y}_F = \mathcal{Y} \times_{\mathcal{O}_F} F$ with dense image, a semiabelian scheme $G_\mathcal{Y} \to \mathcal{Y}$ with $\iota_\mathcal{Y}\colon \mathcal{O}_E \to \mathrm{End}(G_\mathcal{Y})$, and an isomorphism $\alpha\colon j^*(G_\mathcal{Y}, \iota_\mathcal{Y}) \to (G, \iota)_{\mathrm{Sh}_F}$, such that $G_\mathcal{Y}$ is abelian exactly over the points of $j(\mathrm{Sh}_F)$ in $\mathcal{Y}_F$. It remains to construct $\mathcal{L}_\mathcal{Y}$. We use Theorem B.107, parts (1)–(3), always with the base a normal noetherian integral scheme and U its generic point.

Components and generic points. The scheme $\mathcal{Y}$ is of finite type over $\mathcal{O}_F$, hence noetherian, and normal, so it is the disjoint union of finitely many open and closed normal integral subschemes $\mathcal{Y}_1, \dots, \mathcal{Y}_r$; let η_i be the generic point of $\mathcal{Y}_i$. The local ring at η_i is the field $\kappa(\eta_i)$, flat, hence torsion-free, over $\mathcal{O}_F$, so η_i lies over the generic point of $\operatorname{Spec}\mathcal{O}_F$, i.e. in $\mathcal{Y}_F$. Conversely every generalisation in $\mathcal{Y}$ of a point of $\mathcal{Y}_F$ lies over the generic point, hence in $\mathcal{Y}_F$; so the maximal points of $\mathcal{Y}_F$ are exactly $\eta_1, \dots, \eta_r$. A dense open subset of $\mathcal{Y}_F$ contains all its maximal points; hence $\eta_i = j(s_i)$ for a point $s_i \in \mathrm{Sh}_F$, and $s_1, \dots, s_r$ are the maximal points of Sh_F (as j is an open immersion with dense image). Write $\mathcal{L}^{\mathrm{un}}$ for the base change of $\mathcal{L}$ to G_{Sh_F}; it is rigidified, symmetric and cubical, and satisfies the two isomorphisms of (Ud4).

Construction of $\mathcal{L}_\mathcal{Y}$. Via α at s_i, the restriction of $\mathcal{L}^{\mathrm{un}}$ to the fibre at s_i is a symmetric cubical line bundle $\mathcal{L}_i^0$ on $G_\mathcal{Y} \times_\mathcal{Y} \operatorname{Spec}\kappa(\eta_i)$. By Theorem B.107(3) for $G_\mathcal{Y}|_{\mathcal{Y}_i}$ over $\mathcal{Y}_i$, there are a cubical line bundle $\mathcal{L}_i$ on $G_\mathcal{Y}|_{\mathcal{Y}_i}$ and an isomorphism of cubical line bundles $\beta_i\colon \mathcal{L}_i|_{\eta_i} \to \mathcal{L}_i^0$. Let $\mathcal{L}_\mathcal{Y}$ be the line bundle on $G_\mathcal{Y}$ equal to $\mathcal{L}_i$ over $\mathcal{Y}_i$; with the rigidification given by its cubical structure it is a rigidified line bundle with a cubical structure.

Transport of an isomorphism from the generic points. Let $H \to \mathcal{Y}$ be one of the semiabelian schemes $G_\mathcal{Y}$, $G_\mathcal{Y} \times_\mathcal{Y} G_\mathcal{Y}$, and let $\mathcal{P}, \mathcal{Q}$ be rigidified line bundles on H with an isomorphism of line bundles $\mathcal{P}|_{\eta_i} \cong \mathcal{Q}|_{\eta_i}$ over each $\kappa(\eta_i)$. Then $\mathcal{P} \cong \mathcal{Q}$. Indeed, an isomorphism over the field $\kappa(\eta_i)$ multiplied by the inverse of its (non-zero) value at the origin is rigidified; by Theorem B.107(1), applied over $\mathcal{Y}_i$ and over $\operatorname{Spec}\kappa(\eta_i)$, $\mathcal{P}$ and $\mathcal{Q}$ carry cubical structures and the rescaled isomorphism is cubical; by Theorem B.107(2) over $\mathcal{Y}_i$ it is the restriction of an isomorphism over $\mathcal{Y}_i$. We apply this in three cases.

- Symmetry: $\mathcal{P} = \mathcal{L}_\mathcal{Y}$, $\mathcal{Q} = [-1]^*\mathcal{L}_\mathcal{Y}$ on $G_\mathcal{Y}$. Through β_i and α, their restrictions to η_i are those of $\mathcal{L}^{\mathrm{un}}$ and $[-1]^*\mathcal{L}^{\mathrm{un}}$ at s_i (α is a homomorphism, so it commutes with $[-1]$), which are isomorphic since $\mathcal{L}^{\mathrm{un}}$ is symmetric.
- First isomorphism of (Ud4), for $a \in \mathcal{O}_E$: $\mathcal{P} = (\iota_\mathcal{Y}(a), 1)^*\Lambda(\mathcal{L}_\mathcal{Y})$, $\mathcal{Q} = (1, \iota_\mathcal{Y}(\bar a))^*\Lambda(\mathcal{L}_\mathcal{Y})$ on $G_\mathcal{Y} \times_\mathcal{Y} G_\mathcal{Y}$, rigidified by the rigidification of $\mathcal{L}_\mathcal{Y}$. As α is compatible with the $\mathcal{O}_E$-actions and with the group laws, their restrictions to η_i are, through β_i and α, the restrictions at s_i of $(\iota(a), 1)^*\Lambda(\mathcal{L}^{\mathrm{un}})$ and $(1, \iota(\bar a))^*\Lambda(\mathcal{L}^{\mathrm{un}})$, which are isomorphic by (Ud4).
- Second isomorphism of (Ud4): $\mathcal{P} = \mathcal{L}_\mathcal{Y}$, $\mathcal{Q} = (1, \iota_\mathcal{Y}(w_0))^*\Lambda(\mathcal{L}_\mathcal{Y})$ on $G_\mathcal{Y}$, in the same way.

So $\mathcal{L}_\mathcal{Y}$ is a rigidified symmetric cubical line bundle on $G_\mathcal{Y}$ satisfying the two isomorphisms of (Ud4).

Comparison over Sh_F. The scheme Sh_F is smooth over F, hence normal and noetherian, the disjoint union of normal integral open and closed subschemes $Z_1, \dots, Z_r$ with generic points $s_1, \dots, s_r$ (after renumbering). On G_{Sh_F} consider the cubical line bundles $(\alpha^{-1})^* j^* \mathcal{L}_{\mathcal{Y}}$ and $\mathcal{L}^{\mathrm{un}}$. At s_i the isomorphism β_i is an isomorphism of cubical line bundles between their restrictions. By Theorem B.107(2) over Z_i it extends to an isomorphism of cubical line bundles over Z_i. Hence $j^*(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ is isomorphic, through α, to the base change of $(G, \iota, \mathcal{L})$ to Sh_F, compatibly with the rigidifications and cubical structures.

Together with the properties of $(\mathcal{Y}, j, G_{\mathcal{Y}}, \iota_{\mathcal{Y}})$ from Proposition 32.5, the tuple $(\mathcal{Y}, j, G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ is an integral model of Sh over $\mathcal{O}_F$. □

Proposition 32.5. *Let* $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ *be a unitary Shimura datum with* $p + q \geq 1$ *and level* $\mathsf{L} = (N, \mathfrak{n}, A)$ *good at every prime with* $A = 1$*, and let a system of moduli schemes extending* $\mathfrak{u}$ *be given. Then there are a normal scheme* $\mathcal{Y}$*, projective and flat over* $\mathcal{O}_F$*, an open immersion* $j\colon \mathrm{Sh}_F \to \mathcal{Y} \times_{\mathcal{O}_F} F$ *with dense image, a semiabelian scheme* $G_{\mathcal{Y}} \to \mathcal{Y}$ *and a ring homomorphism* $\iota_{\mathcal{Y}}\colon \mathcal{O}_E \to \mathrm{End}(G_{\mathcal{Y}})$*, such that* $j^*(G_{\mathcal{Y}}, \iota_{\mathcal{Y}})$ *is isomorphic to the base change of* (G, ι) *to* Sh_F*, and the points of* $\mathcal{Y} \times_{\mathcal{O}_F} F$ *over which* $G_{\mathcal{Y}}$ *is an abelian scheme are exactly those of* $j(\mathrm{Sh}_F)$*.*

Proof. Write $\mathsf{U} = (M, \langle\,,\rangle)$ and $\mathsf{L} = (N, \mathfrak{n}, 1)$; the level is good at every prime, that is $N/p^{v_p(N)} \geq 3$ for every prime p.

Step 1: the generic fibre. Choose J as in (Ud2) and let $\mathcal{M}_N$, $(A, \phi_{\mathrm{pol}}, \iota, \alpha_N, \nu_N)$, $\mathcal{G}_N$, ϖ, θ, η be as in Proposition 32.9; $\mathcal{M}_N$ is the base change to K_{L} of the full-level scheme $\mathcal{M}_N^{\mathrm{ref}}$ of Theorem B.105 over its reflex field F_{ref}. The field F_{ref} is generated by the coefficients of the characteristic polynomials P_a, which are polynomials in the values $\tau(a)$, $\tau\colon E \to \mathbb{C}$; by (Ud6) they lie in F, so $F \supseteq F_{\mathrm{ref}}$, and $\mathcal{M}_{N,F} := \mathcal{M}_N \times_{K_{\mathsf{L}}} F = \mathcal{M}_N^{\mathrm{ref}} \otimes_{F_{\mathrm{ref}}} F$. Choose R as in Lemma 32.8(3) and let $\kappa_R\colon \mathcal{M}_R \to \mathrm{Sh}_F$ be the isomorphism given there, along which o^{un} pulls back to $(A, \iota, \mathcal{L}_{\phi_{\mathrm{pol}}}, \eta, 0)|_{\mathcal{M}_R}$.

Step 2: Lan's models. Fix a compatible collection Σ of admissible cone decomposition data for $\mathcal{M}_N^{\mathrm{ref}}$, projective and smooth as required in Theorem B.108, induced by a collection pol of polarisation functions (such choices exist: Lan's book [154], Sect. 7.3.1, after Ash–Mumford–Rapoport–Tai), and an integer d as in Theorem B.108. For every prime p the hypothesis $N/p^{v_p(N)} \geq 3$ holds, and Theorem B.108 gives $(\mathcal{X}_p, \mathcal{X}_p^\circ, G_p, \iota_p)$ over $\mathcal{O}_{F_{\mathrm{ref}},(p)}$, with $\mathcal{X}_p$ normal and projective. Let $X^{\mathrm{ref}} = \mathcal{M}_\Sigma^{\mathrm{tor}}$ be Lan's characteristic-0 toroidal compactification over F_{ref}. By item 6 of Theorem B.108 there is an isomorphism $c_p\colon \mathcal{X}_p \otimes F_{\mathrm{ref}} \to X^{\mathrm{ref}}$ restricting to the identity of $\mathcal{M}_N^{\mathrm{ref}}$. The scheme X^{ref} is normal (it is $\mathcal{X}_p \otimes F_{\mathrm{ref}}$, a localisation of the normal $\mathcal{X}_p$) and projective over F_{ref}, and $\mathcal{M}_N^{\mathrm{ref}}$ is open and dense in it ($\mathcal{X}_p^\circ$ is dense in the generic fibre).

Fix a prime p_0 and let (G, ι_G) be the transport of $(G_{p_0}, \iota_{p_0}) \otimes F_{\mathrm{ref}}$ to X^{ref} by c_{p_0}. For every prime p, $(c_p^{-1})^*(G_p \otimes F_{\mathrm{ref}})$ and G are semiabelian schemes over the normal noetherian scheme X^{ref} whose restrictions to the dense open $\mathcal{M}_N^{\mathrm{ref}}$ are both A with its $\mathcal{O}_E$-action (item 1). By Theorem B.104 (Faltings–Chai, Ch. I, Prop. 2.7) the identity of A extends uniquely to homomorphisms between them in both directions; their composites extend the identity, hence are the identity by the same uniqueness, and the extensions of $\iota(a)$ show that the isomorphism is $\mathcal{O}_E$-linear. So $G_p \otimes F_{\mathrm{ref}} \cong c_p^* G$, compatibly with the $\mathcal{O}_E$-actions.

Step 3: gluing. By Lemma 32.6, applied with $K = F_{\mathrm{ref}} \subseteq F$, to X^{ref}, (G, ι_G) and, for every prime p, to $(\mathcal{X}_p, c_p, G_p)$ (projective over $\mathcal{O}_{F_{\mathrm{ref}},(p)}$, with $G_p \otimes F_{\mathrm{ref}} \cong c_p^* G$ by Step 2), there are a normal scheme $\mathcal{Y}'$, projective and flat over $\mathcal{O}_F$, an isomorphism $c\colon \mathcal{Y}' \otimes F \to X$, where $X = X^{\mathrm{ref}} \otimes_{F_{\mathrm{ref}}} F$, and a semiabelian scheme $G' \to \mathcal{Y}'$ with $\iota'\colon \mathcal{O}_E \to \mathrm{End}(G')$ whose restriction to $\mathcal{Y}' \otimes F$ is isomorphic to $c^*(G_X, \iota_{G,X})$, the base change of (G, ι_G) to X. Here X is normal

(being isomorphic to $\mathcal{Y}' \otimes F$) and projective over F, and $\mathcal{M}_{N,F} = \mathcal{M}_N^{\mathrm{ref}} \otimes F$ is open and dense in it (the projection $X \to X^{\mathrm{ref}}$ is flat, so the preimage of a dense open subset is dense); over $\mathcal{M}_{N,F}$, $(G_X, \iota_{G,X})$ is (A, ι).

Step 4: the abelian locus of X. Over $\mathcal{M}_{N,F}$, G_X is the abelian scheme A. Let $x \in X \setminus \mathcal{M}_{N,F}$ and x_0 its image in X^{ref}, which is not in $\mathcal{M}_N^{\mathrm{ref}}$. The fibre of G_X at x is the base change to $\kappa(x)$ of the fibre of G at x_0, which is the fibre of G_{p_0} at $c_{p_0}^{-1}(x_0) \notin \mathcal{M}_N^{\mathrm{ref}}$; by Lemma 32.11 it is not an abelian variety. Since properness descends along field extensions, the fibre of G_X at x is not an abelian variety either. So G_X is abelian exactly over $\mathcal{M}_{N,F}$.

Step 5: selection of components. The scheme $\mathcal{Y}'$ is noetherian and normal, so it is the disjoint union of finitely many open and closed integral subschemes; as $\mathcal{Y}'$ is flat over $\mathcal{O}_F$, their generic points lie in $\mathcal{Y}' \otimes F$ (the local ring at a generic point is a field, torsion-free over $\mathcal{O}_F$), and through c they are the generic points of X, which lie in the dense open $\mathcal{M}_{N,F}$. Let $\mathcal{Y}$ be the union of those components whose generic point lies in $c^{-1}(\mathcal{M}_R)$, and $G_{\mathcal{Y}}$, $\iota_{\mathcal{Y}}$ the restrictions of G', ι'. As an open and closed subscheme of $\mathcal{Y}'$, $\mathcal{Y}$ is normal, projective and flat over $\mathcal{O}_F$, and $G_{\mathcal{Y}}$ is a semiabelian scheme with $\mathcal{O}_E$-action.

The subset $\mathcal{M}_R \subseteq \mathcal{M}_{N,F}$ is open and closed (Lemma 32.8(2)). If $x \in \mathcal{M}_R$, the generic point ξ of the component of X through x lies in the open $\mathcal{M}_{N,F}$, and as a generalisation of x in $\mathcal{M}_{N,F}$ it lies in the open subset $\mathcal{M}_R$; so $c^{-1}(\mathcal{M}_R) \subseteq \mathcal{Y} \otimes F$. Conversely, if $x \in \mathcal{M}_{N,F}$ lies in a component of X with generic point $\xi \in \mathcal{M}_R$, then x is a specialisation of ξ inside $\mathcal{M}_{N,F}$, hence lies in the closed subset $\mathcal{M}_R$ of $\mathcal{M}_{N,F}$. Hence $c(\mathcal{Y} \otimes F) \cap \mathcal{M}_{N,F} = \mathcal{M}_R$, and $\mathcal{M}_R$ is dense in $c(\mathcal{Y} \otimes F)$, which is the union of the closures of its generic points, all in $\mathcal{M}_R$.

Put $j = c^{-1} \circ \kappa_R^{-1} \colon \mathrm{Sh}_F \to \mathcal{Y} \otimes F$. It is an open immersion with dense image $c^{-1}(\mathcal{M}_R)$. By Step 3 and Lemma 32.8(3), $j^*(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}) \cong (\kappa_R^{-1})^*((A, \iota)|_{\mathcal{M}_R})$, which is the base change to Sh_F of (G, ι) of o^{un}. By Step 4, the points of $\mathcal{Y} \otimes F$ over which $G_{\mathcal{Y}}$ is an abelian scheme are those mapped by c into $\mathcal{M}_{N,F}$, that is, into $\mathcal{M}_R$: exactly the points of $j(\mathrm{Sh}_F)$. □

Lemma 32.6. *Let $K \subseteq F$ be number fields and E a CM field. Let X be a normal projective K-scheme, $G \to X$ a semiabelian scheme and $\iota \colon \mathcal{O}_E \to \mathrm{End}(G)$ a ring homomorphism. Suppose that for every rational prime p there are a scheme X_p, projective over $\mathcal{O}_{K,(p)}$, an isomorphism $c_p \colon X_p \otimes_{\mathcal{O}_{K,(p)}} K \to X$, and a semiabelian scheme $G_p \to X_p$ whose restriction to $X_p \otimes K$ is isomorphic to $c_p^* G$. Then there are a normal scheme $\mathcal{Y}$, projective and flat over $\mathcal{O}_F$, an isomorphism $c \colon \mathcal{Y} \otimes_{\mathcal{O}_F} F \to X \otimes_K F$, and a semiabelian scheme $G_{\mathcal{Y}} \to \mathcal{Y}$ with a ring homomorphism $\iota_{\mathcal{Y}} \colon \mathcal{O}_E \to \mathrm{End}(G_{\mathcal{Y}})$ whose restriction to $\mathcal{Y} \otimes F$ is isomorphic to c^* of the base change of (G, ι) to $X \otimes_K F$.*

Proof. Throughout, $\mathcal{O}_K[1/m]$ for an integer $m \geq 1$, and $\mathcal{O}_{K,(p)}$, are limits of the rings $\mathcal{O}_K[1/m']$ over suitable m', and finitely presented schemes, morphisms and their properties descend along such limits (EGA IV$_3$ 8.8.2, 8.10.5, 11.2.6, IV$_4$ 17.7.8: "spreading out"). We also use: (a) a constructible subset of $\operatorname{Spec} \mathcal{O}_K[1/m]$ not containing the generic point is a finite set of closed points (each of its finitely many locally closed pieces $V \cap C$ misses the generic point, so C is a proper closed subset); (b) if a noetherian scheme $\mathcal{Z}$ is flat over $\mathcal{O}_K$, its maximal points lie in the generic fibre (the local ring at a maximal point is artinian and torsion-free over $\mathcal{O}_K$), so every open subset of $\mathcal{Z}$ containing the generic fibre is dense.

Step 1: a model good away from finitely many primes. Embed X as a closed subscheme of $\mathbb{P}^n_K$ and let Y_1 be its schematic closure in $\mathbb{P}^n_{\mathcal{O}_K}$: projective over $\mathcal{O}_K$, flat (its structure sheaf injects into that of X, which is torsion-free), with $Y_1 \otimes K = X$. Spreading out G with its group law, there are $m \geq 1$ and a smooth separated commutative group scheme $G_1 \to Y_1[1/m] = Y_1 \times \operatorname{Spec} \mathcal{O}_K[1/m]$ with $G_1 \otimes K \cong G$. By Lemma 32.7 the set of points of $Y_1[1/m]$ with non-semiabelian fibre is constructible; its image in $\operatorname{Spec} \mathcal{O}_K[1/m]$ is constructible

(EGA IV$_1$ 1.8.4) and misses the generic point (the fibres there are those of G), so by (a) it is finite. Enlarging m, G_1 is semiabelian. Let $\ell_1, \dots, \ell_r$ be the primes dividing m.

Step 2: models at the remaining primes. Fix i and $\ell = \ell_i$. As X_ℓ is projective over the affine scheme $\operatorname{Spec} \mathcal{O}_{K,(\ell)}$, it is a closed subscheme of some $\mathbb{P}^{n_i}_{\mathcal{O}_{K,(\ell)}}$, cut out by finitely many homogeneous equations; their coefficients lie in $\mathcal{O}_K[1/m_i]$ for some m_i prime to ℓ, and they define a closed subscheme $X_i' \subseteq \mathbb{P}^{n_i}_{\mathcal{O}_K[1/m_i]}$ with $X_i' \otimes \mathcal{O}_{K,(\ell)} = X_\ell$. Spreading out G_ℓ (enlarging m_i, still prime to ℓ) gives a smooth separated commutative group scheme $G_i' \to X_i'$ extending G_ℓ. The points of $\operatorname{Spec} \mathcal{O}_{K,(\ell)}$ are the generic point and the primes above ℓ, and over them the fibres of G_i' are fibres of G_ℓ, hence semiabelian; as in Step 1 (Lemma 32.7, (a)), after enlarging m_i by finitely many primes different from ℓ, G_i' is semiabelian. Let X_i'' be the schematic closure of X_i' in $\mathbb{P}^{n_i}_{\mathcal{O}_K}$; it is projective over $\mathcal{O}_K$ and $X_i'' \times \operatorname{Spec} \mathcal{O}_K[1/m_i] = X_i'$ (schematic closure commutes with restriction to open subsets, and X_i' is closed in the open subset over $\mathcal{O}_K[1/m_i]$). Also $X_i'' \otimes K = X_\ell \otimes K$, identified with X by c_ℓ.

Step 3: the closure of the diagonal. Let $P = Y_1 \times_{\mathcal{O}_K} X_1'' \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} X_r''$, projective over $\mathcal{O}_K$, and let $\delta \colon X \to P \otimes K = X \times_K X \times_K \cdots \times_K X$ be the morphism with components id_X (into $Y_1 \otimes K = X$) and $c_{\ell_i}^{-1}$ (into $X_i'' \otimes K$); it is a closed immersion, as the graph of a morphism into a separated scheme. Let $Z \subseteq P$ be the scheme-theoretic image of $X \to P \otimes K \to P$. Then Z is projective over $\mathcal{O}_K$; $\mathcal{O}_Z$ injects into the push-forward of $\mathcal{O}_X$, so Z is reduced and flat over $\mathcal{O}_K$; and since the formation of the scheme-theoretic image commutes with the flat base change $\mathcal{O}_K \to K$, $Z \otimes K = \delta(X) \cong X$. Let Y_K be the normalisation of Z. As Z is of finite type over the excellent ring $\mathcal{O}_K$, $Y_K \to Z$ is finite, so Y_K is projective over $\mathcal{O}_K$ (EGA II 6.1.11, 5.5.5); it is normal; each of its local rings is a domain whose generic point lies over a generic point of Z, which lies over the generic point of $\operatorname{Spec} \mathcal{O}_K$, so it is torsion-free and Y_K is flat over $\mathcal{O}_K$. Normalisation commutes with localisation, so $Y_K \otimes K$ is the normalisation of $Z \otimes K \cong X$, which is normal; let $c_K \colon Y_K \otimes K \to X$ be the resulting isomorphism.

Step 4: the semiabelian scheme. Put $U_0 = \operatorname{Spec} \mathcal{O}_K[1/m]$ and $U_i = \operatorname{Spec} \mathcal{O}_K[1/m_i]$. They cover $\operatorname{Spec} \mathcal{O}_K$: a prime of $\mathcal{O}_K$ above a rational prime ℓ lies in U_0 if $\ell \nmid m$, and otherwise $\ell = \ell_i$ and it lies in U_i as $\ell_i \nmid m_i$. Over U_0 let H_0 be the pull-back of G_1 along the projection $Y_K \times U_0 \to Y_1[1/m]$; over U_i let H_i be the pull-back of G_i' along the projection $Y_K \times U_i \to X_i'' \times U_i = X_i'$. Base change preserves semiabelian schemes (fibres change by field extensions), so each H_j is a semiabelian scheme over the normal noetherian scheme $Y_K \times U_j$. On generic fibres the projections are c_K and $c_{\ell_i}^{-1} \circ c_K$, so there are isomorphisms $\varphi_j \colon H_j \otimes K \to c_K^* G$. For j, k, the isomorphism $\varphi_k^{-1}\varphi_j$ over $Y_K \otimes K$ spreads out to a homomorphism over $Y_K \times U'$ for some non-empty open $U' \subseteq U_j \cap U_k$; this open subset of $Y_K \times (U_j \cap U_k)$ contains the generic fibre, hence all maximal points (Y_K is flat), and is dense. By Theorem B.104 the homomorphism extends uniquely to $\psi_{jk} \colon H_j \to H_k$ over $Y_K \times (U_j \cap U_k)$. By the uniqueness in Theorem B.104, two homomorphisms between semiabelian schemes over a normal noetherian base that agree on the generic fibre agree on a dense open subset (spreading out the equality) and hence everywhere; so $\psi_{kj}\psi_{jk} = \mathrm{id}$ and $\psi_{kl}\psi_{jk} = \psi_{jl}$ on triple overlaps, all of these being equal to the corresponding composites of the φ on the generic fibre. Gluing along the Zariski cover gives a group scheme $G_{Y_K} \to Y_K$ which over each $Y_K \times U_j$ is H_j; being smooth, separated, commutative and with semiabelian fibres is local on the base, so G_{Y_K} is semiabelian, and φ_0 identifies $G_{Y_K} \otimes K$ with $c_K^* G$. For $a \in \mathcal{O}_E$, the endomorphism $c_K^* \iota(a)$ spreads out to a dense open subset and extends uniquely to an endomorphism $\iota_{Y_K}(a)$ of G_{Y_K} (Theorem B.104); by the same uniqueness $a \mapsto \iota_{Y_K}(a)$ is a ring homomorphism.

Step 5: base change to $\mathcal{O}_F$. Let $\mathcal{Y}$ be the normalisation of $Y_K \otimes_{\mathcal{O}_K} \mathcal{O}_F$. The latter is projective and flat over $\mathcal{O}_F$, with generic fibre $X \otimes_K F$, which is normal because X is normal and K has characteristic 0 (normality is preserved by separable field extensions, EGA IV$_2$ 6.14.2);

being flat, it is reduced (its structure sheaf injects into that of the reduced generic fibre). As in Step 3, $\mathcal{Y}$ is normal, projective and flat over $\mathcal{O}_F$, and $\mathcal{Y} \otimes F$ is the normalisation of the normal scheme $X \otimes_K F$, which gives c. Let $G_{\mathcal{Y}}$ and $\iota_{\mathcal{Y}}$ be the pull-backs of G_{Y_K} and ι_{Y_K} along $\mathcal{Y} \to Y_K$: a semiabelian scheme with $\mathcal{O}_E$-action whose restriction to $\mathcal{Y} \otimes F$ is, through c, the base change of (G, ι). □

Lemma 32.7. *Let S be a noetherian scheme and $H \to S$ a smooth separated commutative group scheme of finite type. Then the set $\mathcal{E}$ of points $s \in S$ whose fibre H_s is semiabelian is constructible.*

Proof. Two facts about a smooth commutative group scheme H_0 of finite type over a field κ are used.

(F1) If H_0 is semiabelian, so is $H_0 \otimes_\kappa \kappa'$ for every field extension κ'/κ (base change of the exact sequence; tori and abelian varieties stay so). Conversely, if $H_0 \otimes_\kappa \bar{\kappa}$ is semiabelian, so is H_0. Indeed H_0 is then connected ($H_0 \otimes \bar{\kappa} \to H_0$ is surjective); let $T \subseteq H_0$ be a maximal torus, which exists over κ and whose base change is a maximal torus of $H_0 \otimes \bar{\kappa}$ (SGA 3, Exp. XIV, Thm. 1.1). If $0 \to T' \to H_0 \otimes \bar{\kappa} \to B \to 0$ exhibits $H_0 \otimes \bar{\kappa}$ as semiabelian, every subtorus maps to 0 in B (its image is a connected closed subgroup that is both affine and proper, hence trivial), so $T \otimes \bar{\kappa} \subseteq T'$ and by maximality $T \otimes \bar{\kappa} = T'$. The quotient H_0/T exists over κ and its formation commutes with base change (SGA 3, Exp. VI$_A$, Thm. 3.2), so $(H_0/T) \otimes \bar{\kappa} = B$; hence H_0/T is a smooth proper geometrically connected group scheme, an abelian variety, and H_0 is an extension of it by the torus T.

(F2) If κ is algebraically closed and H_0 is connected, then H_0 is semiabelian if and only if it has no closed subgroup isomorphic to $\mathbb{G}_a$. By Chevalley's theorem (B. Conrad, A modern proof of Chevalley's theorem on algebraic groups, 2002, Thm. 1.1) there is an exact sequence $0 \to L \to H_0 \to B \to 0$ with L smooth connected affine and B an abelian variety; L is commutative, so $L = L_s \times L_u$ with L_s a torus and L_u unipotent (A. Borel, Linear algebraic groups, Thm. 10.6). If $L_u = 0$, H_0 is semiabelian. If $L_u \neq 0$, it is connected unipotent of positive dimension and contains a closed subgroup isomorphic to $\mathbb{G}_a$ (Borel, loc. cit., 10.9). Conversely a semiabelian H_0 contains no $\mathbb{G}_a$: the image of $\mathbb{G}_a$ in the abelian quotient is a connected closed subgroup that is affine and proper, hence trivial, so $\mathbb{G}_a$ would lie in the torus, and $\mathrm{Hom}(\mathbb{G}_a, \mathbb{G}_m) = 0$.

By the criterion of constructibility in a noetherian space (EGA 0$_{\mathrm{III}}$, 9.2.3) it suffices to show: for every irreducible closed subset $Z \subseteq S$ with generic point z, if $z \in \mathcal{E}$ then $\mathcal{E} \cap Z$ contains a non-empty open subset of Z, and if $z \notin \mathcal{E}$ then $Z \setminus \mathcal{E}$ contains one. Give Z its reduced structure and replace S, H by Z, H_Z (the fibres do not change); put $\kappa = \kappa(z)$. We use repeatedly that $\operatorname{Spec}\kappa$ is the limit of the non-empty affine open subschemes of Z, so that finitely presented schemes and morphisms over κ descend to some such open V, and that the properties closed immersion, isomorphism, flat, surjective, proper, smooth, and the equality of two morphisms, hold over V once they hold over κ, after shrinking V (EGA IV$_3$ 8.8.2, 8.10.5, 11.2.6 and IV$_4$ 17.7.8).

Case $z \in \mathcal{E}$. Let $0 \to T \to H_z \to A \to 0$ be exact over κ, T a torus, A an abelian variety, and let κ'/κ be a finite separable extension with $T \otimes \kappa' \cong \mathbb{G}_m^t$. After shrinking a non-empty open $V \subseteq Z$: T and A descend to group schemes T_V, A_V of finite presentation over V with homomorphisms $i\colon T_V \to H_V$, a closed immersion, and $\pi\colon H_V \to A_V$, flat and surjective, such that $\pi \circ i = 0$ and $T_V \to \ker\pi$ is an isomorphism; A_V is smooth and proper over V with geometrically connected fibres (the last condition holds on a constructible set containing z, EGA IV$_3$ 9.7.7); and $\operatorname{Spec}\kappa'$ descends to a finite étale surjective $V' \to V$ over which $T_V \times_V V' \cong \mathbb{G}_{m,V'}^t$. For $v \in V$ choose $v' \in V'$ above it: $T_v \otimes \kappa(v') \cong \mathbb{G}_m^t$, so T_v is a torus; A_v is an abelian variety;

and $0 \to T_v \to H_v \to A_v \to 0$ is exact. Hence $V \subseteq \mathcal{E}$.

Case $z \notin \mathcal{E}$, H_z not connected. For a group scheme, connectedness of a fibre is equivalent to geometric connectedness (the fibre has a rational point, EGA IV_2 4.5.13), and the set of points with geometrically connected fibre is constructible (EGA IV_3 9.7.7). It does not contain z, so its complement contains a non-empty open subset of Z, and there the fibres are not connected, hence not semiabelian.

Case $z \notin \mathcal{E}$, H_z connected. By (F1), $H_z \otimes \bar{\kappa}$ is not semiabelian, so by (F2) there is a closed immersion homomorphism $\mathbb{G}_a \to H_z \otimes \bar{\kappa}$; it is defined over a finite extension κ'' of κ. Choose a non-empty affine open $\operatorname{Spec} R \subseteq Z$ and generators $\alpha_1, \dots, \alpha_n$ of κ'' over κ, and put $Z'' = \operatorname{Spec} R[\alpha_1, \dots, \alpha_n]$: an integral scheme of finite type over Z, dominant, with function field κ''. The closed immersion homomorphism descends to one $\mathbb{G}_{a,V''} \to H \times_Z V''$ over a non-empty open $V'' \subseteq Z''$. The image of V'' in Z is constructible (EGA IV_1 1.8.4) and contains z, so it contains a non-empty open $W \subseteq Z$. For $x \in W$ choose $v \in V''$ over x: $H_x \otimes \kappa(v)$ contains $\mathbb{G}_a$, so does its base change to an algebraic closure of $\kappa(v)$, which is therefore not semiabelian by (F2), and H_x is not semiabelian by (F1). Hence $W \cap \mathcal{E} = \emptyset$. □

Lemma 32.8. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p + q \geq 1$, $\mathsf{U} = (M, \langle\,,\rangle)$ and level $\mathsf{L} = (N, \mathfrak{n}, 1)$ good at every prime, and let $\mathcal{M}_N$, $(A, \phi_{\mathrm{pol}}, \iota, \alpha_N, \nu_N)$, $\mathcal{G}_N$, ϖ, θ and η be as in Proposition 32.9 for some choice of J. Put $\mathcal{M}_{N,F} = \mathcal{M}_N \times_{K_{\mathsf{L}}} F$, let $P_N \subset F$ be the set of primitive N-th roots of unity, and for $h \in \mathcal{G}_N$ let σ_h be the automorphism of $\mathcal{M}_{N,F}$ by which h acts.*

1. *$M/NM = \bar{\mathfrak{n}}M/NM \oplus \mathfrak{n}M/NM$, and every $(g, c) \in \mathcal{G}_N$ is the identity on $\bar{\mathfrak{n}}M/NM$ and multiplication by c on $\mathfrak{n}M/NM$. In particular $(g, c) \mapsto c$ is an injective homomorphism $\mathcal{G}_N \to (\mathbb{Z}/N\mathbb{Z})^\times$.*

2. *For $\zeta \in P_N$ let $\mathcal{M}^{(\zeta)} \subseteq \mathcal{M}_{N,F}$ be the open and closed subscheme on which the section $\nu_N(1)$ of μ_N equals ζ. Then $\mathcal{M}_{N,F} = \bigsqcup_{\zeta \in P_N} \mathcal{M}^{(\zeta)}$, and for $h = (g, c) \in \mathcal{G}_N$ the automorphism σ_h maps $\mathcal{M}^{(\zeta)}$ onto $\mathcal{M}^{(\zeta^c)}$. The group $\mathcal{G}_N$ acts freely on P_N by $\zeta \mapsto \zeta^c$.*

3. *Let $R \subseteq P_N$ contain exactly one element of each $\mathcal{G}_N$-orbit in P_N, and $\mathcal{M}_R = \bigsqcup_{\zeta \in R} \mathcal{M}^{(\zeta)}$. Then the restriction of $(\theta \circ \varpi) \times_{K_{\mathsf{L}}} F$ to $\mathcal{M}_R$ is an isomorphism of F-schemes $\mathcal{M}_R \to \mathrm{Sh}_F$, along which the universal object o^{un} pulls back to the restriction of $(A, \iota, \mathcal{L}_{\phi_{\mathrm{pol}}}, \eta, 0)$ to $\mathcal{M}_R$.*

Proof. **(1).** By (Ud3), $\mathfrak{n} + \bar{\mathfrak{n}} = \mathcal{O}_E$ and $\mathfrak{n}\bar{\mathfrak{n}} = N\mathcal{O}_E$. Write $1 = a + b$ with $a \in \mathfrak{n}$, $b \in \bar{\mathfrak{n}}$. Every $m \in M$ is $am + bm \in \mathfrak{n}M + \bar{\mathfrak{n}}M$. If $m \in \mathfrak{n}M \cap \bar{\mathfrak{n}}M$, then $am \in \mathfrak{n}\bar{\mathfrak{n}}M = NM$ and $bm \in \bar{\mathfrak{n}}\mathfrak{n}M = NM$, so $m \in NM$; conversely $NM = \mathfrak{n}\bar{\mathfrak{n}}M$ lies in both. Hence, with $U = \bar{\mathfrak{n}}M/NM$ and $V = \mathfrak{n}M/NM$, $M/NM = U \oplus V$.

Both summands are isotropic modulo N: for $x, x' \in M$ and $b, b' \in \bar{\mathfrak{n}}$, $\langle bx, b'x' \rangle = \langle x, \bar{b}b'x' \rangle$ with $\bar{b} \in \mathfrak{n}$, so $\bar{b}b' \in N\mathcal{O}_E$ and $\langle bx, b'x' \rangle \in \langle M, NM \rangle \subseteq N\mathbb{Z}$; by biadditivity $\langle \bar{\mathfrak{n}}M, \bar{\mathfrak{n}}M \rangle \subseteq N\mathbb{Z}$, and likewise for $\mathfrak{n}M$. The pairing modulo N on M/NM is perfect: M is a free $\mathbb{Z}$-module of finite rank and $\langle\,,\rangle$ induces $M \cong \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, hence $M/NM \cong \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})/N = \operatorname{Hom}(M/NM, \mathbb{Z}/N\mathbb{Z})$. Consequently, if $v \in V$ pairs to 0 with every element of U, it pairs to 0 with all of $M/NM = U \oplus V$ (V being isotropic), so $v = 0$.

Let $(g, c) \in \mathcal{G}_N$. As g is $\mathcal{O}_E$-linear, $g(U) = g(\bar{\mathfrak{n}} \cdot M/NM) \subseteq U$ and $g(V) \subseteq V$, and g is the identity on U by condition (2) of Proposition 32.9. For $u \in U$ and $v \in V$, condition (1) there gives $\langle u, gv \rangle = \langle gu, gv \rangle \equiv c\langle u, v \rangle \pmod{N}$, so $gv - cv \in V$ pairs to 0 with U and $gv = cv$. Thus g is determined by c; since the group law of $\mathcal{G}_N$ is $(g, c)(g', c') = (gg', cc')$, the map $(g, c) \mapsto c$ is a homomorphism, and its kernel is trivial ($c = 1$ forces $g = \mathrm{id}$).

(2). By (Ud6), F contains $e^{2\pi i/N'}$ with $N' = \operatorname{lcm}(N,1) = N$, so $T^N - 1$ splits into N distinct linear factors over F and $\mu_{N,F}$ is the disjoint union of the F-points $\zeta \in \mu_N(F)$. The section $\nu_N(1) \in \mu_N(\mathcal{M}_{N,F})$ therefore decomposes $\mathcal{M}_{N,F}$ into the open and closed preimages of these points. Since $\nu_N \colon \mathbb{Z}/N\mathbb{Z} \to \mu_N$ is an isomorphism of group schemes, at every geometric point the value of $\nu_N(1)$ has order exactly N, so only $\zeta \in P_N$ occur. By Proposition 32.9, σ_h^* carries the universal tuple to $(A, \phi_{\mathrm{pol}}, \iota, \alpha_N \circ g, \nu_N \circ c)$, and $(\nu_N \circ c)(1) = \nu_N(c) = \nu_N(1)^c$. Hence $\sigma_h^{-1}(\mathcal{M}^{(\zeta^c)})$ is the locus where $\nu_N(1)^c = \zeta^c$, which is $\mathcal{M}^{(\zeta)}$ because $\xi \mapsto \xi^c$ is a bijection of P_N; so the automorphism σ_h maps $\mathcal{M}^{(\zeta)}$ onto $\mathcal{M}^{(\zeta^c)}$. If $\zeta^c = \zeta$ for some $\zeta \in P_N$, then $c \equiv 1 \pmod N$ and $h = 1$ by (1): the action on P_N is free.

(3). By (2), every orbit $\mathcal{G}_N\zeta$ ($\zeta \in R$) has $|\mathcal{G}_N|$ elements $\zeta^{c(h)}$, distinct for distinct h, and P_N is the disjoint union of these orbits. Hence the open and closed subschemes $\sigma_h(\mathcal{M}_R) = \bigsqcup_{\zeta \in R} \mathcal{M}^{(\zeta^{c(h)})}$, $h \in \mathcal{G}_N$, are pairwise disjoint and cover $\mathcal{M}_{N,F}$.

Let $Y = \mathcal{M}_N/\mathcal{G}_N$ and $\pi = \varpi \times_{K_\mathsf{L}} F \colon \mathcal{M}_{N,F} \to Y_F$. By Proposition 32.9(1), ϖ is the quotient by the free action of the finite group $\mathcal{G}_N$: it is finite, étale, surjective, $\varpi \circ \sigma_h = \varpi$, and its geometric fibres are the $\mathcal{G}_N$-orbits (SGA 1, Exp. V, Prop. 1.1); the same holds for its base change π, whose geometric points over a geometric point of Y_F are those of ϖ over the underlying geometric point of Y. Let $\pi_R = \pi|_{\mathcal{M}_R}$. It is finite and étale, being the restriction of π to an open and closed subscheme. Let $\bar{y}$ be a geometric point of Y_F and $\bar{x}$ a geometric point of $\mathcal{M}_{N,F}$ over it. There is exactly one h_0 with $\bar{x} \in \sigma_{h_0}(\mathcal{M}_R)$, and $\sigma_h(\bar{x}) \in \sigma_{hh_0}(\mathcal{M}_R)$ lies in $\mathcal{M}_R$ exactly for $h = h_0^{-1}$; as the action is free, the points $\sigma_h(\bar{x})$ are pairwise distinct. So the fibre of π_R over $\bar{y}$, which is the orbit of $\bar{x}$ intersected with $\mathcal{M}_R$, has exactly one point. A finite étale morphism all of whose geometric fibres have exactly one point is finite locally free of rank 1, hence an isomorphism: locally $\pi_{R*}\mathcal{O}$ is free of rank 1, the unit generates it modulo every maximal ideal (the fibre algebra is the residue field), so by Nakayama $\mathcal{O}_{Y_F} \to \pi_{R*}\mathcal{O}$ is surjective, and it is injective because a free module of rank 1 is faithful. Thus $\pi_R \colon \mathcal{M}_R \to Y_F$ is an isomorphism, and so is $(\theta \times_{K_\mathsf{L}} F) \circ \pi_R \colon \mathcal{M}_R \to \mathrm{Sh}_F$, by Proposition 32.9(2).

Finally, Proposition 32.9(2) gives $(\theta \circ \varpi)^* o^{\mathrm{un}} \cong (A, \iota, \mathcal{L}_{\phi_{\mathrm{pol}}}, \eta, 0)$ over $\mathcal{M}_N$; base change to F and restriction to $\mathcal{M}_R$ give the last assertion. □

Proposition 32.9. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p + q \geq 1$, $\mathsf{U} = (M, \langle\,,\rangle)$, $\mathsf{H} = (d, r, r_{\mathbb{C}})$, $w_0 = (1+r)/2$ and level $\mathsf{L} = (N, \mathfrak{n}, 1)$, so that $K_\mathsf{L} = \mathbb{Q}(r_{\mathbb{C}})$. Choose a complex structure J as in (Ud2), and let $\mathcal{M}_N$ be the base change to K_L of the full-level scheme of Theorem B.105 for $(E, M, \psi = \langle\,,\rangle, J)$ and the level N, with universal tuple $(A, \phi_{\mathrm{pol}}, \iota, \alpha_N, \nu_N)$. Let $\mathcal{G}_N$ be the set of pairs (g, c) of an $\mathcal{O}_E$-linear automorphism g of M/NM and $c \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that*

1. *$\langle gx, gy\rangle \equiv c\,\langle x, y\rangle \pmod N$ for all $x, y \in M$;*
2. *g is the identity on $\bar{\mathfrak{n}}M/NM$;*
3. *(g, c) is the reduction modulo N of a pair $(\hat{g}, \hat{c})$ of an $\mathcal{O}_E \otimes \widehat{\mathbb{Z}}$-linear automorphism $\hat{g}$ of $M \otimes \widehat{\mathbb{Z}}$ and $\hat{c} \in \widehat{\mathbb{Z}}^\times$ with $\langle \hat{g}x, \hat{g}y\rangle = \hat{c}\,\langle x, y\rangle$.*

It is a finite group, acting on $\mathcal{M}_N$ by $(A, \phi_{\mathrm{pol}}, \iota, \alpha_N, \nu_N) \mapsto (A, \phi_{\mathrm{pol}}, \iota, \alpha_N \circ g, \nu_N \circ c)$, where $\nu_N \circ c$ is ν_N precomposed with multiplication by c on $\mathbb{Z}/N\mathbb{Z}$. Let $e \in \mathcal{O}_E$ with $e \equiv 1 \pmod{\mathfrak{n}}$ and $e \equiv 0 \pmod{\bar{\mathfrak{n}}}$. Then:

1. *$\mathcal{G}_N$ acts freely on $\mathcal{M}_N$, the quotient $\mathcal{M}_N/\mathcal{G}_N$ exists as a quasi-projective K_L-scheme, and the quotient map $\varpi \colon \mathcal{M}_N \to \mathcal{M}_N/\mathcal{G}_N$ is finite, étale and surjective;*

2. *there is an isomorphism of K_{L}-schemes $\theta\colon \mathcal{M}_N/\mathcal{G}_N \to \mathrm{Sh}$ such that $(\theta\circ\varpi)^*o^{\mathrm{un}}$ is isomorphic to the object $(A,\iota,\mathcal{L}_{\phi_{\mathrm{pol}}},\eta,0)$ of type (U,L) over $\mathcal{M}_N$, where $\mathcal{L}_{\phi_{\mathrm{pol}}}$ is formed with w_0 and $\eta(a)=\alpha_N(ea \bmod NM)$ for $a\in M$.*

Proof. The hypotheses of Proposition 32.10 hold: E is unramified over E^+ at every finite place by the definition of a unitary Shimura datum, $p+q\geq 1$, and $\mathsf{L}=(N,\mathfrak{n},1)$ is a level. With the same choice of J, the scheme X_N of Proposition 32.10 is $\mathcal{M}_N$. Let $\mathcal{H}$ be the group defined there, and let (Y,o_Y), $Y=\mathcal{M}_N/\mathcal{H}$, be the moduli scheme of type (U,L) given there. We show $\mathcal{G}_N=\mathcal{H}$, then compare Y with Sh.

The two groups coincide. For $N\mid m$ let $\mathcal{H}_m$ be as in Proposition 32.10: pairs (g,c) of an $\mathcal{O}_E$-linear automorphism of M/mM and $c\in(\mathbb{Z}/m\mathbb{Z})^\times$ with $\langle gx,gy\rangle\equiv c\langle x,y\rangle \pmod m$, whose reduction modulo N fixes $U=\bar{\mathfrak{n}}M/NM$ pointwise. For $N\mid m\mid m'$ reduction modulo m maps $\mathcal{H}_{m'}$ to $\mathcal{H}_m$: since M is projective, $M/mM=(M/m'M)\otimes\mathbb{Z}/m\mathbb{Z}$, so an automorphism reduces to an automorphism, a unit modulo m' to a unit modulo m, and the congruence modulo m' gives the one modulo m. Conditions (1) and (2) of the statement say exactly that $(g,c)\in\mathcal{H}_N$.

$\mathcal{G}_N\subseteq\mathcal{H}$. Let $(g,c)\in\mathcal{G}_N$ with a lift $(\hat g,\hat c)$ as in (3). For $N\mid m$ the reduction ($\hat g \bmod m, \hat c \bmod m$) satisfies the pairing congruence modulo m, and its reduction modulo N is (g,c), which fixes U by (2); so it lies in $\mathcal{H}_m$ and maps to (g,c). Hence (g,c) lies in every image $\mathrm{im}(\mathcal{H}_m\to\mathcal{H}_N)$, that is, in $\mathcal{H}$.

$\mathcal{H}\subseteq\mathcal{G}_N$. Let $(g,c)\in\mathcal{H}$, put $m_i=N\cdot i!$ ($i\geq 1$), a sequence in which each term divides the next and every multiple of N divides some term, and let $T_i\subseteq\mathcal{H}_{m_i}$ be the set of pairs reducing to (g,c) modulo N. Each T_i is finite and nonempty (because $(g,c)\in\mathrm{im}(\mathcal{H}_{m_i}\to\mathcal{H}_N)$), and reduction maps T_{i+1} into T_i. For fixed i the images of T_j in T_i, $j\geq i$, form a decreasing sequence of nonempty subsets of a finite set; let T_i' be their intersection, which is nonempty. Reduction maps T_{i+1}' onto T_i': an element $t\in T_i'$ is the image of some element of T_j for every $j\geq i+1$, so the sets of elements of T_{i+1} that map to t and lie in the image of T_j form a decreasing sequence of nonempty finite sets, with nonempty intersection. Choosing successively $t_1\in T_1'=T_1=\{(g,c)\}$ and $t_{i+1}\in T_{i+1}'$ over t_i gives a compatible sequence $t_i=(g_i,c_i)\in\mathcal{H}_{m_i}$. Since M is finitely generated and the m_i are cofinal for divisibility, $M\otimes\widehat{\mathbb{Z}}=\varprojlim M/m_iM$ and $\widehat{\mathbb{Z}}^\times=\varprojlim(\mathbb{Z}/m_i\mathbb{Z})^\times$. Put $\hat g=\varprojlim g_i$ and $\hat c=\varprojlim c_i$. The inverses g_i^{-1} are compatible too, so $\hat g$ is an automorphism; it is $\widehat{\mathbb{Z}}$-linear and commutes with $\mathcal{O}_E$, hence is $\mathcal{O}_E\otimes\widehat{\mathbb{Z}}$-linear. For $x,y\in M\otimes\widehat{\mathbb{Z}}$ the element $\langle\hat gx,\hat gy\rangle-\hat c\langle x,y\rangle$ of $\widehat{\mathbb{Z}}$ lies in $m_i\widehat{\mathbb{Z}}$ for every i, hence is 0. Its reduction modulo N is $t_1=(g,c)$, so (3) holds and $(g,c)\in\mathcal{G}_N$.

Thus $\mathcal{G}_N=\mathcal{H}$. The action of $\mathcal{G}_N$ in the statement is the action of $\mathcal{H}$ in Proposition 32.10, which changes the full level and the cyclotomic trivialisation by $(\alpha_N,\nu_N)\mapsto(\alpha_N\circ g,\nu_N\circ[c])$. Clause (1) is now the statement of Proposition 32.10: $\mathcal{H}$ acts freely, $Y=\mathcal{M}_N/\mathcal{H}$ is quasi-projective over K_{L} and $\varpi\colon\mathcal{M}_N\to Y$ is finite, étale and surjective.

The pull-back of o_Y to $\mathcal{M}_N$. By Proposition 32.10, ϖ^*o_Y is the full-level universal abelian scheme A with its $\mathcal{O}_E$-action ι, the primitive cubical line bundle $\mathcal{L}_{\phi_{\mathrm{pol}}}$ of Lemma 31.16 formed with w_0, and the half level induced by α_N, that is, the additive map $\eta\colon M\to A[N](\mathcal{M}_N)$ which vanishes on $\mathfrak{n}M$ and agrees with α_N on $\bar{\mathfrak{n}}M$; the component η_A is 0, the only additive map $M\to A(\mathcal{M}_N)$ vanishing on $1\cdot M$. These two conditions determine η, and $\eta(a)=\alpha_N(ea \bmod NM)$: indeed $e\in\bar{\mathfrak{n}}$ and $1-e\in\mathfrak{n}$, so $a=ea+(1-e)a$ with $ea\in\bar{\mathfrak{n}}M$ and $(1-e)a\in\mathfrak{n}M$. Hence $\varpi^*o_Y\cong(A,\iota,\mathcal{L}_{\phi_{\mathrm{pol}}},\eta,0)$.

Comparison with Sh**.** Both $(\mathrm{Sh},o^{\mathrm{un}})$ and (Y,o_Y) are moduli schemes of type (U,L); both schemes are smooth over K_{L}, hence reduced. By (Ud5)(i) for $(\mathrm{Sh},o^{\mathrm{un}})$ applied to o_Y over Y there is a unique K_{L}-morphism $\theta'\colon Y\to\mathrm{Sh}$ with $\theta'^*o^{\mathrm{un}}\cong o_Y$, and by (Ud5)(i) for (Y,o_Y) applied to o^{un}

over Sh there is a unique $\kappa\colon \mathrm{Sh} \to Y$ with $\kappa^* o_Y \cong o^{\mathrm{un}}$. Then $(\theta' \circ \kappa)^* o^{\mathrm{un}} \cong \kappa^* o_Y \cong o^{\mathrm{un}} = \mathrm{id}^* o^{\mathrm{un}}$, so the uniqueness in (Ud5)(i) for the object o^{un} over the reduced scheme Sh gives $\theta' \circ \kappa = \mathrm{id}_{\mathrm{Sh}}$; symmetrically $\kappa \circ \theta' = \mathrm{id}_Y$. So $\theta = \theta'$ is an isomorphism of K_{L}-schemes $\mathcal{M}_N/\mathcal{G}_N = Y \to \mathrm{Sh}$, and

$$(\theta \circ \varpi)^* o^{\mathrm{un}} \cong \varpi^* o_Y \cong (A, \iota, \mathcal{L}_{\phi_{\mathrm{pol}}}, \eta, 0),$$

which is clause (2). □

Proposition 32.10. *Let E be a CM field unramified over its maximal totally real subfield at every finite place, let H be a CM host on E, and let U be a unitary datum for H of signature (p,q) with $p+q \geq 1$. Let $\mathsf{L} = (N, \mathfrak{n}, 1)$ be a level. Then there is a moduli scheme of type (U, L) in the sense of (Ud5).*

Explicitly, over $K_{\mathsf{L}} = \mathbb{Q}(r_{\mathbb{C}})$ there are a quasi-projective scheme Sh*, smooth of relative dimension $\frac{1}{2}[E:\mathbb{Q}]pq$, and an object o^{un} of type (U, L) over* Sh*, such that every object over every reduced K_{L}-scheme is the pullback of o^{un} along exactly one morphism to* Sh*. The infinitesimal injectivity condition (Ud5)(ii) also holds.*

The moduli scheme may be taken to be the following explicit quotient. Choose a complex structure J as in (Ud2), let X_N be the full-level scheme of Theorem B.105 for $(E, M, \langle\,,\,\rangle, J, N)$, base changed to K_{L}, and put $U = \bar{\mathfrak{n}}M/NM$. For each positive multiple m of N define

$$\mathcal{G}_m = \{(g,c) : g \in \mathrm{Aut}_{\mathcal{O}_E}(M/mM), \quad c \in (\mathbb{Z}/m\mathbb{Z})^\times, \\ \langle gx, gy\rangle = c\langle x, y\rangle \pmod m \quad \text{for all } x, y \in M/mM\},$$

and let $\mathcal{H}_m \subseteq \mathcal{G}_m$ consist of the pairs whose reduction modulo N fixes U pointwise. Set

$$\mathcal{H} = \bigcap_{N \mid m} \mathrm{im}(\mathcal{H}_m \longrightarrow \mathcal{H}_N).$$

The finite group $\mathcal{H}$ acts freely on X_N by changing the full level and its cyclotomic trivialisation. One can take

$$\mathrm{Sh} = X_N/\mathcal{H},$$

where $X_N \to$ Sh is finite etale surjective. The universal object of (Ud4) is the descent of the full-level universal abelian scheme with its induced primitive cubical line bundle and half level. All symbols in this quotient description are local.

Proof. Choose a complex structure J as in (Ud2). By Lemma 32.3, the reflex field is contained in $K = K_{\mathsf{L}} = \mathbb{Q}(r_{\mathbb{C}})$. Let X/K be the base change of the full-level scheme of Theorem B.105. It is smooth quasi-projective of relative dimension $\frac{1}{2}[E:\mathbb{Q}]pq$.

We first compare the levels, keeping the cyclotomic trivialisation as variable data. In M/NM put

$$U = \bar{\mathfrak{n}}M/NM.$$

The Chinese remainder decomposition for $\mathfrak{n} + \bar{\mathfrak{n}} = \mathcal{O}_E$ gives an idempotent e modulo $N\mathcal{O}_E$ with $e = 1$ modulo $\mathfrak{n}$ and $e = 0$ modulo $\bar{\mathfrak{n}}$; thus $U = e(M/NM)$ and $\bar{e} = 1 - e$. A full N-level α_N defines

$$\eta(a) = \alpha_N(ea) \quad (a \in M), \qquad \eta_A = 0.$$

This is independent of the lift of e, kills $\mathfrak{n}M$, and agrees with α_N on $\bar{\mathfrak{n}}M$. Conversely these restrictions determine η, since $M = \mathfrak{n}M + \bar{\mathfrak{n}}M$.

For each multiple m of N, let $\mathcal{G}_m$ be the finite group of pairs (g, c) where g is an $\mathcal{O}_E$-linear automorphism of M/mM, $c \in (\mathbb{Z}/m)^\times$, and

$$\psi(gx, gy) = c\psi(x, y) \pmod{m}.$$

Let $\mathcal{H}_m \subseteq \mathcal{G}_m$ be the subgroup whose reduction modulo N fixes U pointwise. For a fixed m, the subgroups $\mathrm{im}(\mathcal{H}_l \to \mathcal{H}_m)$, with $m \mid l$, form a downward-directed finite collection. Choose l_m attaining its minimum size, and denote this minimum subgroup by $\mathcal{H}_m^\infty$. A common multiple shows that it is contained in every image, and every image with index divisible by l_m equals it. It is the image of the compatible inverse system: to lift an element in all finite images, use a cofinal sequence of moduli $m\,i!$, choose finite lifts, and successively pass to subsequences with constant projection at each preceding modulus. The resulting compatible projections give a lift at every modulus. Put $\mathcal{H} = \mathcal{H}_N^\infty$.

We now run the same argument over an arbitrary K-scheme S carrying an object of (Ud4). For $N \mid m$, let R_m/S parametrize pairs (I_m, ν_m) of an $\mathcal{O}_E$-linear full m-level and a primitive cyclotomic trivialisation, preserving the pairing and whose reduction modulo N agrees with η on $\bar{\mathfrak{n}}M$. Torsion and roots of unity are finite étale in characteristic zero. The compatibility conditions define open-and-closed subschemes of their finite étale isomorphism schemes; an integral basis of $\mathcal{O}_E$ and a finite basis of M suffice to test them. Thus R_m is finite étale. Every geometric fibre is nonempty by (Ud4). It is an $\mathcal{H}_m$-torsor, under

$$(I_m, \nu_m) \cdot (g, c) = (I_m \circ g, \nu_m \circ [c]).$$

At a geometric point use a compatible system from (Ud4) as origin. The images of $R_l \to R_m$ are exactly the cosets of the corresponding image subgroups. Consequently

$$R_m^\infty = \mathrm{im}(R_{l_m} \to R_m)$$

is a well-defined finite étale open-and-closed subscheme, independent of the chosen minimizing l_m, and is an $\mathcal{H}_m^\infty$-torsor. The preceding compatible-projection argument shows on every geometric fibre that these are exactly the levels lifting to a compatible system at all moduli. In particular $R_l^\infty \to R_m^\infty$ is finite étale surjective whenever $m \mid l$.

After the finite étale cover $S' = R_N^\infty \to S$, the tautological full N-level therefore has the actual liftability required by Theorem B.105: its tower consists of the R_m^∞, viewed over $R_N^\infty = S'$, with the tautological levels and roots. This constructs the compatible tower on arbitrary bases; no locally noetherian assumption is used. All equations concerning η hold as equations of finite étale torsion schemes: geometric equality implies equality of their sections even on a nonreduced base.

We next prove rigidity. Over an algebraically closed characteristic-zero field, the idempotents decompose $A[N]$ into $A[\mathfrak{n}] \oplus A[\bar{\mathfrak{n}}]$, and the half level identifies U with the first summand. Rosati makes e and $\bar{e}$ adjoint. Both summands are isotropic and their cross-pairing is perfect. An equivariant polarised automorphism fixing η fixes the first summand, preserves the second, and fixes it too by the perfect cross-pairing. It therefore fixes $A[N]$.

Such an automorphism is the identity. To see this, descend the abelian variety, polarisation and automorphism to a finitely generated characteristic-zero field and embed that field into $\mathbb{C}$. A polarisation-preserving automorphism acts faithfully on integral homology preserving a positive definite real form. The resulting integral matrix group is discrete in a compact orthogonal group, hence finite. An integral matrix of finite order congruent to 1 modulo an odd prime ℓ is the identity: otherwise a power V has prime order q and $V = 1 + \ell^s B$ with $s \geq 1$ and $B \not\equiv 0 \pmod{\ell}$. If $q \neq \ell$, expansion of $V^q = 1$ modulo ℓ^{s+1} contradicts this choice of B. If $q = \ell$,

divide the binomial expansion by ℓ^{s+1}; every term after B is divisible by ℓ, including the last because $(\ell-1)s-1 \geq 1$, again a contradiction. Apply this to an odd prime dividing N, which exists because N is odd and $N \geq 3$ by (Ud3). Over a reduced base the same rigidity holds: the total abelian scheme is reduced, and the closed equalizer of the automorphism and the identity contains every geometric point, so its defining ideal is zero.

The finite group $\mathcal{H}$ acts on X by changing the full level and cyclotomic trivialisation. It preserves liftability because each of its elements lifts to a compatible system of level changes. Its action is free: a geometric stabilizer would give a polarised equivariant automorphism fixing the half level, which is the identity by the preceding argument. Then the level automorphism is the identity, and nonzero M and the perfect pairing force its multiplier to be 1. By Theorem B.143, the quotient $Y = X/\mathcal{H}$ is quasi-projective, and $X \to Y$ is finite étale surjective. Smoothness and its relative dimension descend along this cover.

Use Lemma 31.16 on the universal polarised abelian scheme over X, with $w_0 = (1+r)/2$. This produces its primitive symmetric rigidified cubical line bundle. The underlying polarised abelian scheme, action, line bundle and half level have canonical equivariant descent data: changing the full level leaves them unchanged, and the identifications are unique. The cocycle identities follow by that uniqueness. These data descend to an abelian scheme and line bundle over Y. One may realize the abelian scheme explicitly as the finite free quotient of its total space: that total space is quasi-projective, because the canonical line bundle is relatively ample over the quasi-projective base X. Group operations descend; being an abelian scheme and all the required line-bundle properties are checked on the finite étale cover. The descended tuple is an object o^{un} of (Ud4) over Y.

We now verify the two required universal properties. Let S be reduced and let o be an object over S. Its geometric Kottwitz condition gives the global determinant condition: on any affine trivialisation of its locally free Lie sheaf, each coefficient difference from the polynomial $P_a(T)$ vanishes in every geometric residue field, hence lies in the nilradical and is zero. The primitive-cubical equivalence gives its polarisation and Rosati condition. The finite étale cover R_N^{∞} constructed above supplies a liftable full level, so Theorem B.105 gives a classifying map from that cover to X. It is equivariant for $\mathcal{H}$, and descends to $S \to Y$. The universal object pulls back to o; the local isomorphisms descend because they are unique by reduced-base rigidity. If two morphisms classify o, lift both after a finite étale cover to X. Their full levels differ, under an isomorphism of their half-level objects, by a section of $\mathcal{H}$. Full-level representability then identifies their images in Y. Faithfully flat descent gives equality on S. This proves (Ud5)(i).

Finally let $T = \operatorname{Spec} k[\epsilon]$, $\epsilon^2 = 0$, and suppose two pullbacks of o^{un} along maps $T \to Y$ are isomorphic. After a common finite étale cover, lift both maps to X. Their full levels differ by a section of the finite étale level-change group. At every geometric point this section belongs to $\mathcal{H}$, by the compatible-lift characterization proved above; hence it belongs to $\mathcal{H}$ over the cover itself. Adjust one lift by that section. The isomorphism now preserves the full level and its cyclotomic trivialisation, so full-level representability on arbitrary schemes makes the two lifted morphisms equal. Their maps to Y are equal, and descend to equal maps from T. In particular this applies to a map and its constant point, proving (Ud5)(ii). No classification of arbitrary objects over nonreduced schemes is needed here: the two objects are pullbacks of the universal one and already satisfy the global determinant condition.

Thus Y with o^{un} satisfies (Ud5), and Y has the dimension computed above. □

Lemma 32.11. *In the situation of Theorem B.108, let p be a prime with $N/p^{v_p(N)} \geq 3$. For every point x of $\mathcal{X}_p \otimes F_{\mathrm{ref}}$ not in $\mathcal{M}_N$, the fibre of G_p at x is not an abelian variety.*

Proof. Put $X = \mathcal{X}_p \otimes F_{\mathrm{ref}}$, $G = G_p|_X$ with ι_p, and let $x \in X$ with G_x an abelian variety; we

show $x \in \mathcal{M}_N$. Recall $\mathcal{M}_N = \mathcal{X}_p^\circ \otimes F_{\mathrm{ref}}$ is open and dense in X (Theorem B.108).

Step 1: the local scheme. X is normal (a localisation of the normal $\mathcal{X}_p$) and noetherian, so $S = \operatorname{Spec} \mathcal{O}_{X,x}$ is the spectrum of a normal noetherian local domain; its points are the generisations of x in X, its generic point ξ is the generic point of the irreducible component of X through x, and $\xi \in \mathcal{M}_N$ because the dense open $\mathcal{M}_N$ contains every generic point of X. So $U = S \cap \mathcal{M}_N$ is a dense open subscheme of S. By Lemma 32.12, G_s is an abelian variety for every $s \in S$; a semiabelian scheme all of whose fibres are abelian varieties is an abelian scheme (Faltings–Chai, Ch. I, Rem. 2.8; Lan's book, Rem. 3.3.1.8). Write $A_S = G_S$, an abelian scheme over S with $\mathcal{O}_E$-action ι_p; over U it is, with the polarisation and level structure, the pull-back of the universal tuple $(A, \phi_{\mathrm{pol}}, \iota, \alpha_N, \nu_N)$ along the inclusion $U \subseteq \mathcal{M}_N$ (Theorem B.108, item 1).

Step 2: polarisation, action, Lie condition. By Theorem B.104 (S normal noetherian, U dense open; A_S and its dual abelian scheme $A_S^\vee$ are semiabelian), ϕ_{pol} and ϕ_{pol}^{-1} extend uniquely to homomorphisms $\lambda\colon A_S \to A_S^\vee$ and $A_S^\vee \to A_S$; their composites extend identities and are identities by uniqueness, so λ is an isomorphism. By the same uniqueness $\lambda^\vee = \lambda$ and $\lambda\iota_p(a) = \iota_p(\bar{a})^\vee\lambda$ for $a \in \mathcal{O}_E$, these identities holding over U. The coefficients of $\det(T - \iota_p(a) \mid \mathrm{Lie}(A_S/S))$ and of $P_a(T)$ are elements of the domain $\mathcal{O}_{X,x}$ that agree in its fraction field (over U), hence agree.

λ is a polarisation. Let $\mathcal{N} = (\mathrm{id}, \lambda)^*\mathcal{P}$, $\mathcal{P}$ the Poincaré bundle on $A_S \times_S A_S^\vee$; a symmetric homomorphism is a polarisation exactly when this sheaf is relatively ample, equivalently ample on every geometric fibre (Lan's book, Prop. 1.3.2.18, Def. 1.3.2.19). On the fibres over U, $\lambda = \phi_{\mathrm{pol}}$ is a polarisation, so $\mathcal{N}$ is ample there, in particular at ξ. On every fibre $\phi_{\mathcal{N}_s} = \lambda_s + \lambda_s^\vee = 2\lambda_s$, so $K(\mathcal{N}_s) = A_s[2]$ is finite and $\mathcal{N}_s$ is non-degenerate. For a non-degenerate line bundle on an abelian variety exactly one cohomology group is non-zero, and an ample one has non-zero H^0 (Mumford, Abelian varieties [186], Sect. 16, vanishing theorem). Hence $\chi(\mathcal{N}_\xi) = h^0(\mathcal{N}_\xi) > 0$. As $A_S \to S$ is proper and flat, $\chi(\mathcal{N}_s)$ is constant on the connected S and $h^0(\mathcal{N}_s)$ is upper semicontinuous (EGA III_2 7.9.4, 7.7.5), so $h^0(\mathcal{N}_x) \geq h^0(\mathcal{N}_\xi) > 0$. Thus $\mathcal{N}_x$ is effective with $K(\mathcal{N}_x)$ finite, hence ample (Mumford, loc. cit., Sect. 6, Application 1). The set of points of S at whose fibre $\mathcal{N}$ is ample is open (EGA III_1 4.7.1) and contains the closed point x, so it is S, and $\mathcal{N}$ is relatively ample (loc. cit.): λ is a principal polarisation.

Step 3: level structure. As N is invertible on S, $A_S[N]$, $(M/NM)_S$, $(\mathbb{Z}/N)_S$ and $\mu_{N,S}$ are finite étale over S. A section over U of a finite étale S-scheme Z extends uniquely to a section over S: the closure of its image is an integral closed subscheme of the normal scheme Z, finite and birational over the normal S, hence isomorphic to S (Zariski's main theorem). So α_N and ν_N extend to homomorphisms over S; each is a morphism of finite étale S-schemes of the same degree that is an isomorphism over U, hence finite étale of degree 1 on the connected S, an isomorphism. The relation $e_N^\lambda(\alpha_N(x), \alpha_N(y)) = \nu_N(\psi(x,y) \bmod N)$ and the $\mathcal{O}_E$-linearity hold over the dense open U of the reduced S, between morphisms into separated schemes, hence over S. For the liftability condition of Theorem B.105, for $N \mid m$ let I_m be the S-scheme of compatible families $(\alpha_{m'}, \nu_{m'})_{N|m'|m}$ of analogous pairs for A_S reducing to (α_N, ν_N); it is a closed and open subscheme of a finite étale S-scheme of isomorphisms between finite étale group schemes, hence finite étale over S, and the transitions $I_m \to I_l$ are finite étale. Over U the tuple is liftable, so $I_m \to S$ and $I_m \to I_l$ have non-empty fibres over U; a finite étale morphism has open and closed image, so on the connected S (respectively on each connected component of I_l, all of which meet the preimage of the dense U, since I_l is finite étale over S and so its generic points lie over ξ) they are surjective. So $(A_S, \lambda, \iota_p, \alpha_N, \nu_N)$ is an object of the functor of Theorem B.105 over S.

Step 4: conclusion. By Theorem B.105 it defines a morphism $f\colon S \to \mathcal{M}_N$, whose restriction to U is the inclusion (the tuple over U is the pull-back of the universal one along it). The two morphisms $S \to X$ given by the canonical map and by f followed by $\mathcal{M}_N \subseteq X$ agree

on the dense open U of the reduced scheme S, and X is separated, so they are equal. Hence the canonical map $S \to X$ factors through $\mathcal{M}_N$, and $x \in \mathcal{M}_N$. □

Lemma 32.12. *Let $G \to S$ be a semiabelian scheme over a locally noetherian scheme S. For $x \in S$ let $r(x)$ be the dimension of the torus T_x in the extension $0 \to T_x \to G_x \to B_x \to 0$ of an abelian variety by a torus. If $t \in S$ is a generisation of $s \in S$, then $r(t) \leq r(s)$. In particular, if G_s is an abelian variety, so is G_t.*

Proof. The torus T_x is the maximal subtorus of G_x (its quotient B_x is an abelian variety, which contains no non-trivial affine connected subgroup), so $r(x)$ depends only on G_x, and $r(x) = 0$ exactly when G_x is an abelian variety. It suffices to prove $r(t) \leq r(s)$.

Torsion counts. Let $x \in S$, let n be a prime different from the characteristic of $\kappa(x)$, and let $\bar{k}$ be an algebraic closure of $\kappa(x)$. As $T_x(\bar{k}) \cong (\bar{k}^\times)^{r(x)}$ is divisible, the sequence $0 \to T_x[n](\bar{k}) \to G_x[n](\bar{k}) \to B_x[n](\bar{k}) \to 0$ is exact, and

$$\#G_x[n](\bar{k}) = n^{r(x)} \cdot n^{2(\dim G_x - r(x))} = n^{2\dim G_x - r(x)}.$$

Since $G \to S$ is smooth and has a section, $x \mapsto \dim G_x$ is locally constant on S; as t specialises to s, every open neighbourhood of s contains t, so $\dim G_t = \dim G_s =: g$.

Passage to a strictly henselian base. Fix a prime n different from the characteristic of $\kappa(s)$; then n is invertible in $\mathcal{O}_{S,s}$ and in $\kappa(t)$. Let A be the strict henselisation of $\mathcal{O}_{S,s}$. The map $\operatorname{Spec} A \to \operatorname{Spec} \mathcal{O}_{S,s}$ is faithfully flat, so the prime of $\mathcal{O}_{S,s}$ corresponding to t is the image of a point $t' \in \operatorname{Spec} A$; its residue field contains $\kappa(t)$. Let $G_A = G \times_S \operatorname{Spec} A$.

Multiplication by n on G_A is étale: G_A is smooth over A, and on every fibre $[n]$ is a surjective homomorphism of smooth connected group schemes over a field in which n is invertible, with differential $n \cdot \mathrm{id}$ an isomorphism, hence étale; by the fibrewise criterion of flatness a morphism of flat A-schemes of finite presentation that is flat and unramified on fibres is flat and unramified, hence étale. Therefore $Z = G_A[n]$, the pull-back of the unit section along $[n]$, is étale and separated over A.

The closed fibre of Z has $N = n^{2g-r(s)}$ points, all rational over the separably closed residue field of A (a finite étale scheme over a separably closed field is a disjoint union of copies of the base). As A is henselian, each such point z_i is the image of the closed point under a section $\sigma_i \colon \operatorname{Spec} A \to Z$ (the lifting property of étale algebras over henselian local rings; Stacks Project, Algebra, the section on henselian local rings). For $i \neq j$ the set where σ_i and σ_j agree is the preimage of the diagonal of Z under (σ_i, σ_j); since $Z \to \operatorname{Spec} A$ is separated and unramified, the diagonal is an open and closed immersion, so this set is open and closed in the connected space $\operatorname{Spec} A$; it does not contain the closed point, hence it is empty. So the N sections are pairwise distinct at t', and $Z_{t'} = G_{t'}[n]$ has at least N points rational over $\kappa(t')$.

Conclusion. Base change from $\kappa(t)$ to an algebraic closure of $\kappa(t')$ does not change the number of geometric points of the finite étale scheme $G_t[n]$, and it does not change the dimension of the maximal torus. By the torsion count at t (with the same prime n, which is invertible in $\kappa(t)$),

$$n^{2g-r(t)} = \#G_t[n](\overline{\kappa(t')}) \geq N = n^{2g-r(s)},$$

so $r(t) \leq r(s)$. □

Lemma 32.13. *Let $G \to S$ be a semiabelian scheme and $s \in S$. There is a finite separable extension k of $\kappa(s)$ such that $G_s \otimes_{\kappa(s)} k$ is an extension of an abelian variety over k by a split torus over k. In particular every integral model of $\mathrm{Sh}_{\mathsf{L}'}$, over $\mathcal{O}_{F'}$ or over $\mathcal{O}_{F',v}$ for an ℓ-adic place v, satisfies (Gl2).*

Proof. By definition of a semiabelian scheme, the fibre G_s is an extension

$$0 \to T \to G_s \xrightarrow{\pi} B \to 0$$

of an abelian variety B over $\kappa(s)$ by a torus T over $\kappa(s)$, that is, T is a closed subgroup scheme, π is faithfully flat with kernel T, and $T \otimes \overline{\kappa(s)} \cong \mathbb{G}_m^r$ for some $r \geq 0$.

A torus over a field splits over a finite separable extension of that field (Borel, *Linear algebraic groups*, Prop. 8.11; SGA 3, Exp. X, Prop. 1.4): the character group of T is a finitely generated free $\mathbb{Z}$-module with a continuous action of the absolute Galois group of $\kappa(s)$, the action factors through the Galois group of a finite separable extension k, and over k the torus is isomorphic to $\mathbb{G}_m^r$. Fix such a k.

Base change along the field extension $\kappa(s) \to k$ is faithfully flat, so it preserves closed immersions, faithful flatness and kernels: $T_k \to G_s \otimes k$ is a closed subgroup scheme, $\pi_k \colon G_s \otimes k \to B_k$ is faithfully flat with kernel T_k. Here B_k is an abelian variety over k and $T_k \cong \mathbb{G}_{m,k}^r$ is split. This is the first assertion.

For an integral model $\mathcal{Y}$, the scheme $G_{\mathcal{Y}} \to \mathcal{Y}$ is semiabelian by (Ud7) (and by its counterpart over $\mathcal{O}_{F',v}$ in Definition 31.2); applying the first assertion at every point y of $\mathcal{Y}$ gives (Gl2). $\square$

32.2 Uniformisation at complete local rings

The second clause of a uniformised degeneration, (Gl3), holds for every integral model (Proposition 32.14). Its content is Proposition 32.15: a polarised semiabelian structure with $\mathcal{O}_E$-action over a complete noetherian normal local domain of characteristic 0, abelian with principal polarisation over the fraction field and an extension of an abelian variety by a split torus over the residue field, is uniformised by degeneration data with $\mathcal{O}_E$-structure, with isomorphisms on torsion. The proof follows the degeneration theory of Faltings and Chai in Lan's form [87], [154] and Mumford's construction [187]; the lemmas of the subsection treat the $\mathcal{O}_E$-structure, the symmetric descent of the line bundle by the twist w_0, and the polarisation trivialisation and the torsion under the operators of Mumford's construction.

Proposition 32.14. *Let E, H and a unitary datum U of signature (p,q) with $p+q \geq 1$ be as in Definition 31.2, let ℓ be a prime, v an ℓ-adic place, L' a level for H with a moduli scheme $\mathrm{Sh}_{\mathsf{L}'}$ of type $(\mathsf{U}, \mathsf{L}')$, F' a coefficient field for L', and $\mathcal{Y}$ an integral model of $\mathrm{Sh}_{\mathsf{L}'}$ over $\mathcal{O}_{F',v}$. Then $\mathcal{Y}$ satisfies (Gl3).*

Proof. Put $O = \mathcal{O}_{F',v}$, a discrete valuation ring whose maximal ideal contains ℓ. Let R be a complete noetherian normal local domain of characteristic 0 with maximal ideal $\mathfrak{m}_R$, residue field κ and fraction field K, and let $\xi \colon \operatorname{Spec} R \to \mathcal{Y}$ be such that $\xi^{-1}(\mathcal{Y}^{\mathrm{ab}}) \neq \varnothing$ and $\xi^* G_{\mathcal{Y}} \otimes_R \kappa$ is an extension of an abelian variety by a split torus. We show that $\xi^*(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ is uniformised as (Gl3) requires.

The pulled-back structure and its abelian locus. Put $(G, \iota, \mathcal{L}) = \xi^*(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$. Base change preserves semiabelian schemes, the $\mathcal{O}_E$-action, the rigidification, symmetry and cubical structure of the line bundle and the two isomorphisms of (Ud4), so $(G, \iota, \mathcal{L})$ is a semiabelian structure over R. For $x \in \operatorname{Spec} R$ we have $G \otimes_R \kappa(x) = (G_{\mathcal{Y}} \otimes \kappa(\xi(x))) \otimes_{\kappa(\xi(x))} \kappa(x)$; an extension of an abelian variety by a torus over a field is an abelian variety exactly when the torus has dimension 0, and the dimension of the torus does not change under extension of the field. Hence

$$U := \xi^{-1}(\mathcal{Y}^{\mathrm{ab}}) = \{x \in \operatorname{Spec} R : \ G \otimes_R \kappa(x) \text{ is an abelian variety}\}.$$

The generic fibre is a principally polarised abelian variety. Let η be the generic point of $\operatorname{Spec} R$. Every point of $\operatorname{Spec} R$ is a specialisation of η, and $U \neq \varnothing$; by Lemma 32.12 applied to

G over the noetherian scheme $\operatorname{Spec} R$, $\eta \in U$. The kernel of $O \to R$ is a prime ideal of O; it is not the maximal ideal, since ℓ maps to a non-zero element of the domain R of characteristic 0; so $O \to K$ is injective and extends to $F' \to K$. Hence $\xi|_{\operatorname{Spec} K}$ factors through the generic fibre $\mathcal{Y} \otimes_O F'$, and its image point $y = \xi(\eta)$ lies in $\mathcal{Y}^{\mathrm{ab}}$. By the definition of an integral model, the points of $\mathcal{Y} \otimes_O F'$ over which $G_{\mathcal{Y}}$ is abelian are exactly those of $j(\mathrm{Sh}_{\mathsf{L}',F'})$; as j is an open immersion, $\xi|_{\operatorname{Spec} K} = j \circ y'$ for a morphism $y' \colon \operatorname{Spec} K \to \mathrm{Sh}_{\mathsf{L}',F'}$. Since $j^*(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ is isomorphic to the base change of the abelian scheme, $\mathcal{O}_E$-action and line bundle of the universal object, $(G \otimes_R K, \iota, \mathcal{L}|_{G\otimes K})$ is isomorphic to their pull-back along y'. By (Ud4) the universal abelian scheme carries a relatively ample line bundle with principal polarisation; hence $G \otimes_R K$ is an abelian variety and $\mathcal{L}|_{G\otimes K}$ is ample and defines a principal polarisation.

Uniformisation with torsion isomorphisms. R is a complete noetherian normal local domain of characteristic 0; $(G, \iota, \mathcal{L})$ is a semiabelian structure over R whose generic fibre is as just shown and whose fibre over κ is an extension of an abelian variety by a split torus. By Proposition 32.15, $(G, \iota, \mathcal{L})$ is uniformised by degeneration data $\mathfrak{D} = (B, \mathcal{L}_B, \iota_B, X, c, \psi)$ with $\mathcal{O}_E$-structure over $(R, \mathfrak{m}_R)$ that have torsion isomorphisms.

The units. By Proposition 32.22, applied to $(G, \iota, \mathcal{L})$ and $\mathfrak{D}$, every $\psi(\chi)$ is a unit over the set of points of $\operatorname{Spec} R$ at which G is an abelian variety, which is $U = \xi^{-1}(\mathcal{Y}^{\mathrm{ab}})$.

So $\xi^*(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ is uniformised by degeneration data with $\mathcal{O}_E$-structure over $(R, \mathfrak{m}_R)$ that have torsion isomorphisms and whose $\psi(\chi)$ are units over $\xi^{-1}(\mathcal{Y}^{\mathrm{ab}})$. As R and ξ were arbitrary, $\mathcal{Y}$ satisfies (Gl3). □

Proposition 32.15. *Let E and H be as in Definition 31.2. Let R be a complete noetherian normal local domain of characteristic 0 with maximal ideal $\mathfrak{m}_R$, residue field κ and fraction field K, and let $(G, \iota, \mathcal{L})$ be a semiabelian structure over R such that*

1. *$G_K = G \otimes_R K$ is an abelian variety and $\mathcal{L}|_{G_K}$ is ample and defines a principal polarisation of G_K;*
2. *$G \otimes_R \kappa$ is an extension of an abelian variety over κ by a split torus over κ.*

Then $(G, \iota, \mathcal{L})$ is uniformised by degeneration data with $\mathcal{O}_E$-structure over $(R, \mathfrak{m}_R)$ that have torsion isomorphisms.

Proof. Put $S = \operatorname{Spec} R$. Hypotheses 1 and 2 are those of Lemma 32.16 and Lemma 32.17. Let $0 \to T \to \widetilde{G} \xrightarrow{\pi} B \to 0$ be the Raynaud extension, with the induced $\mathcal{O}_E$-actions $\widetilde{\iota}$ on $\widetilde{G}$ and ι_B on B, λ_B the induced polarisation, $\mathcal{L}_B = (1, \lambda_B \iota_B(w_0))^* \mathcal{P}_B$ and u as in Lemma 32.16; by its item 3, λ_B is principal. Let X (constant), $c \colon X \to B(R)$, ψ and τ be as in Lemma 32.17; the extension class of $\widetilde{G}$ is $\lambda_B \circ c$. Give X the $\mathcal{O}_E$-module structure $a \cdot \chi = \chi \circ \widetilde{\iota}(\bar{a})|_T$; then $\widetilde{\iota}(a)|_T$ is dual to $\chi \mapsto \bar{a}\chi$. Put $\mathfrak{D} = (B, \mathcal{L}_B, \iota_B, X, c, \psi)$.

$\mathfrak{D}$ are degeneration data with $\mathcal{O}_E$-structure over $(R, \mathfrak{m}_R)$.

- By Lemma 32.16, item 1, $\mathcal{L}_B$ is a rigidified symmetric cubical relatively ample sheaf with principal polarisation λ_B, and ι_B satisfies the two isomorphisms of (Ud4). The group X is free of finite rank with an $\mathcal{O}_E$-module structure.
- The map c is $\mathcal{O}_E$-linear. The push-out of $\widetilde{G}$ along χ is the extension of B by $\mathbb{G}_m$ with class $\lambda_B c(\chi)$. The endomorphism $\widetilde{\iota}(a)$ covers $\iota_B(a)$ and restricts on T to the map with $\chi \circ \widetilde{\iota}(a)|_T = \bar{a}\chi$; so the push-out along $\bar{a}\chi$ is the pull-back along $\iota_B(a)$ of the push-out along χ: $\lambda_B c(\bar{a}\chi) = \iota_B(a)^\vee \lambda_B c(\chi) = \lambda_B \iota_B(\bar{a}) c(\chi)$, the last step by the compatibility of λ_B with the $\mathcal{O}_E$-structure (item (3)). As λ_B is an isomorphism, $c(\bar{a}\chi) = \iota_B(\bar{a}) c(\chi)$ for all a, i.e. $c(a\chi) = \iota_B(a) \circ c(\chi)$.

- The datum τ is a trivialisation of the biextension $(c\times\lambda_B c)^*\mathcal{P}_B^{-1} = (c\times c)^*\Lambda(\mathcal{L}_B)^{-1}$, hence bimultiplicative for the biextension structure, which is the one defined by the cubical structure of $\mathcal{L}_B$; and $\tau(\chi,\chi') = \psi(\chi+\chi')\psi(\chi)^{-1}\psi(\chi')^{-1}$ is the compatibility $D_2(\psi) = \tau\circ(\mathrm{id}\times\phi)$ of $\mathrm{DD}_{\mathrm{ample}}$ read through ϕ. By item (3) the data are compatible with the $\mathcal{O}_E$-structure, which for τ is the statement that $\tau(a\chi,\chi')$ corresponds to $\tau(\chi,\bar{a}\chi')$ under the trivialisation of $(\iota_B(a),1)^*\Lambda(\mathcal{L}_B)\otimes(1,\iota_B(\bar{a}))^*\Lambda(\mathcal{L}_B)^{-1}$ equal to 1 at the origin.
- The w_0-condition: u is the identity at the origin, and by Proposition 32.18 $\psi(\chi)$ corresponds under u to $\tau(\chi,w_0\chi)$ for every χ.
- ψ is even. The symmetry of $\mathcal{L}_B$ is its unique rigidified one, so under u it is the canonical symmetry of $(1,\iota_B(w_0))^*\Lambda(\mathcal{L}_B)$, given at $(x,\iota_B(w_0)x)\mapsto(-x,-\iota_B(w_0)x)$ by the isomorphism $\Lambda(-x,-y)\cong\Lambda(x,y)$ of the biextension structure. By bimultiplicativity $\tau(-\chi,-w_0\chi)$ corresponds to $\tau(\chi,w_0\chi)$ under it; with the w_0-condition, $\psi(-\chi)$ corresponds to $\psi(\chi)$ under the symmetry.
- Positivity. By the condition on τ in item (2), $\tau(\chi,\chi)$ extends over S and lies in $\mathfrak{m}_R$ for $\chi\neq 0$ (X and Y are constant over S, so no base change is needed). The sections $\psi(\chi)$ are read through the cubical isomorphism $\widetilde{\alpha}\colon\widetilde{\mathcal{L}}\to\pi^*\mathcal{L}_B$ of Lemma 32.16, item 2, which is the isomorphism determined by the cubical trivialisation of $i^*\widetilde{\mathcal{L}}$ obtained from $i^*\widetilde{\alpha}$ and the rigidification of $\mathcal{L}_B$ ($i\colon T\to\widetilde{G}$); $\mathcal{L}_B$ is symmetric, and by the previous bullet ψ is even through this trivialisation. So, by item (2) in the even case, the condition on ψ holds for it: every $\psi(\chi)$ extends over S, and for every $m\geq 1$ all but finitely many $\psi(\chi)$ lie in $\mathfrak{m}_R^m c(\chi)^*\mathcal{L}_B^{-1}$. (Through another cubical trivialisation this can fail: on the Tate curve, twisting by a character j turns $\psi(y)=t^{y^2}$ into t^{y^2-2yj}.) For $\chi\neq 0$: from $1=\psi(0)=\psi(\chi)\psi(-\chi)\tau(\chi,-\chi)$ and bimultiplicativity, $\psi(\chi)\psi(-\chi)=\tau(\chi,\chi)$; by evenness $\psi(-\chi)$ is the image of $\psi(\chi)$ under an isomorphism of invertible sheaves on S, so in trivialisations $\psi(\chi)^2$ is a unit multiple of an element of $\mathfrak{m}_R$, and $\psi(\chi)\in\mathfrak{m}_R c(\chi)^*\mathcal{L}_B^{-1}$ as $\mathfrak{m}_R$ is prime.

$(G,\iota,\mathcal{L})$ is uniformised by $\mathfrak{D}$. The abelian variety G_K carries the principal polarisation of $\mathcal{L}|_{G_K}$ (hypothesis 1). The Raynaud extension gives $G\otimes R/\mathfrak{m}_R^{e+1}\cong\widetilde{G}\otimes R/\mathfrak{m}_R^{e+1}$, compatibly in e and with the $\mathcal{O}_E$-actions; $\widetilde{G}$ is the extension of B by the split torus with character group X defined by c, with a acting through $\iota_B(a)$ and on the torus dually to $\chi\mapsto\bar{a}\chi$, as required. By Lemma 32.16, item 2, $\mathcal{L}\otimes R/\mathfrak{m}_R^{e+1}$ is identified with the pull-back of $\mathcal{L}_B$ compatibly with rigidifications, symmetries and cubical structures. By Lemma 32.17, with respect to these identifications, for every $k\geq 1$ the Fourier expansion is a bijection from $\Gamma(G_K,\mathcal{L}_K^{\otimes k})$ onto the families with the relation defined by $\psi(\mu)^k\tau(\mu,\alpha)$, which is the Fourier condition of the uniformisation for $\mathfrak{D}$.

Torsion isomorphisms. The ring R is a complete noetherian normal local domain of characteristic 0 and $(G,\iota,\mathcal{L})$ is uniformised by $\mathfrak{D}$; by Proposition 32.20, $\mathfrak{D}$ has torsion isomorphisms. □

Lemma 32.16. *Let R be a complete noetherian normal local domain with maximal ideal $\mathfrak{m}_R$, fraction field K and residue field κ, and let $(G,\iota,\mathcal{L})$ be a semiabelian structure over R such that G_K is an abelian variety, $\mathcal{L}|_{G_K}$ is ample, and $G\otimes_R\kappa$ is an extension of an abelian variety by a split torus. Let $0\to T\to\widetilde{G}\xrightarrow{\pi}B\to 0$ be the Raynaud extension of G, with the actions $\widetilde{\iota}$ of $\mathcal{O}_E$ on $\widetilde{G}$ and ι_B on B induced by ι, let $\lambda_B\colon B\to B^\vee$ be the polarisation of B induced by $\mathcal{L}$, $\mathcal{P}_B$ the rigidified Poincaré bundle on $B\times B^\vee$, and put*

$$\mathcal{L}_B = (1,\lambda_B\circ\iota_B(w_0))^*\mathcal{P}_B.$$

Then:

1. *$\mathcal{L}_B$ is a rigidified symmetric cubical relatively ample invertible sheaf on B with polarisation λ_B; $(\iota_B(a),1)^*\Lambda(\mathcal{L}_B) \cong (1,\iota_B(\bar a))^*\Lambda(\mathcal{L}_B)$ for every $a \in \mathcal{O}_E$; and there is an isomorphism $u\colon \mathcal{L}_B \to (1,\iota_B(w_0))^*\Lambda(\mathcal{L}_B)$ equal to the identity at the origin via the rigidifications.*

2. *$(G,\mathcal{L})$ is an object of $\mathrm{DEG}_{\mathrm{ample}}$. Let $v\colon \mathcal{L} \to (1,\iota(w_0))^*\Lambda(\mathcal{L})$ be the second isomorphism of (Ud4), normalised to be the identity at the origin via the rigidifications; let $\widetilde{\mathcal{L}}$ be the cubical sheaf of $F_{\mathrm{ample}}(G,\mathcal{L})$, with its cubical isomorphism $\beta\colon \mathcal{L}^{\mathrm{for}} \to \widetilde{\mathcal{L}}^{\mathrm{for}}$ under the identification $G^{\mathrm{for}} \cong \widetilde{G}^{\mathrm{for}}$; and let $\kappa\colon \Lambda(\widetilde{\mathcal{L}}) \to (\pi\times\pi)^*(1\times\lambda_B)^*\mathcal{P}_B$ be the isomorphism of biextensions of $\widetilde{G}\times_S\widetilde{G}$ by $\mathbb{G}_m$, which is unique. Then the composite*

$$\begin{aligned}\alpha\colon \mathcal{L}^{\mathrm{for}} &\xrightarrow{v^{\mathrm{for}}} \big((1,\iota(w_0))^*\Lambda(\mathcal{L})\big)^{\mathrm{for}} \xrightarrow{\beta} \big((1,\widetilde{\iota}(w_0))^*\Lambda(\widetilde{\mathcal{L}})\big)^{\mathrm{for}}\\ &\xrightarrow{\kappa^{\mathrm{for}}} \big((1,\widetilde{\iota}(w_0))^*(\pi\times\pi)^*(1\times\lambda_B)^*\mathcal{P}_B\big)^{\mathrm{for}} = (\pi^*\mathcal{L}_B)^{\mathrm{for}},\end{aligned}$$

   the middle arrow induced by β, is an isomorphism compatible with the rigidifications, the symmetries and the cubical structures, and there is a unique cubical isomorphism $\widetilde{\alpha}\colon \widetilde{\mathcal{L}} \to \pi^*\mathcal{L}_B$ with $\widetilde{\alpha}^{\mathrm{for}} \circ \beta = \alpha$. In particular the cubical sheaf $\widetilde{\mathcal{L}}$ of $F_{\mathrm{ample}}(G,\mathcal{L})$ is $\pi^*\mathcal{L}_B$, identified by $\widetilde{\alpha}$.

3. *If $\mathcal{L}|_{G_K}$ defines a principal polarisation, then λ_B is principal and the map ϕ of $F_{\mathrm{ample}}(G,\mathcal{L})$ is an isomorphism.*

Proof. Write $S = \operatorname{Spec} R$, $f = \lambda_B \circ \iota_B(w_0)\colon B \to B^\vee$, and ^{for} for completion along the closed fibre. By (1) the Raynaud extension exists ($G\otimes\kappa$ has a split, hence isotrivial, torus part), and $(G,\mathcal{L}) \in \mathrm{DEG}_{\mathrm{ample}}$: $\mathcal{L}$ is rigidified and cubical; it is symmetric (the sheaf of a semiabelian structure), so $\mathcal{L}_K$ is symmetric, and $\mathcal{L}_K$ is ample, hence nondegenerate, so $\mathcal{L}$ is ample over G by the ampleness criterion for symmetric sheaves in (2); and $\mathcal{L}^{\mathrm{for}}$ lies in the essential image of the functor of (1), which is an equivalence because every torus over the field κ is isotrivial. By (3), ι induces $\mathcal{O}_E$-actions $\widetilde{\iota}$ on $\widetilde{G}$ and ι_B on B with $\pi\circ\widetilde{\iota}(a) = \iota_B(a)\circ\pi$, compatible with λ_B in the sense of $\mathcal{O}_E$-structures: $\lambda_B\circ\iota_B(a) = \iota_B(\bar a)^\vee\circ\lambda_B$. Since $r^2 = -d < 0$, $\bar r = -r$, so $\bar w_0 = (1-r)/2$ and $w_0 + \bar w_0 = 1$.

Biextension identities. For homomorphisms $g\colon B\to B$, $h\colon B\to B^\vee$ the biextension structure of $\mathcal{P}_B$ gives canonical rigidified isomorphisms $(g,h)^*\mathcal{P}_B \cong (1,g^\vee h)^*\mathcal{P}_B$ and, for $N_h = (1,h)^*\mathcal{P}_B$,

$$\Lambda(N_h) \cong (1\times(h+h^\vee))^*\mathcal{P}_B,$$

since at (x,y) the fibre of $\Lambda(N_h)$ is $\mathcal{P}(x+y,h(x+y))\otimes\mathcal{P}(x,hx)^{-1}\otimes\mathcal{P}(y,hy)^{-1} \cong \mathcal{P}(x,hy)\otimes\mathcal{P}(y,hx) \cong \mathcal{P}(x,(h+h^\vee)y)$. For $h = f$: $f^\vee = \iota_B(w_0)^\vee\lambda_B = \lambda_B\iota_B(\bar w_0)$ by the compatibility above ($\lambda_B^\vee = \lambda_B$), so $f + f^\vee = \lambda_B\iota_B(w_0+\bar w_0) = \lambda_B$ and $\Lambda(\mathcal{L}_B) \cong (1\times\lambda_B)^*\mathcal{P}_B$.

Item 1. $\mathcal{L}_B$ is rigidified (pull-back of the rigidified $\mathcal{P}_B$ along a map sending 0 to $(0,0)$) and carries its unique cubical structure (theorem of the cube for rigidified sheaves on abelian schemes, Section 29.1). Its polarisation, defined by $\Lambda(\mathcal{L}_B) \cong (1\times\lambda_{\mathcal{L}_B})^*\mathcal{P}_B$, is λ_B. On every fibre $\mathcal{L}_B$ has the same Néron–Severi class as an ample sheaf inducing the polarisation λ_B, so it is ample on every fibre, and relatively ample since $B\to S$ is proper and flat. Symmetry: $[-1]^*\mathcal{L}_B = (-1,-f)^*\mathcal{P}_B \cong (1,f)^*\mathcal{P}_B$ by bimultiplicativity, rigidified. For $a\in\mathcal{O}_E$,

$$(\iota_B(a),1)^*\Lambda(\mathcal{L}_B) \cong (\iota_B(a),\lambda_B)^*\mathcal{P}_B \cong (1,\iota_B(a)^\vee\lambda_B)^*\mathcal{P}_B = (1,\lambda_B\iota_B(\bar a))^*\mathcal{P}_B \cong (1,\iota_B(\bar a))^*\Lambda(\mathcal{L}_B).$$

Finally $(1,\iota_B(w_0))^*\Lambda(\mathcal{L}_B) \cong (1,\lambda_B\iota_B(w_0))^*\mathcal{P}_B = \mathcal{L}_B$; let u be the inverse of this composite of canonical rigidified isomorphisms; it is the identity at the origin.

Item 2. Let $\widetilde{\mathcal{L}}$ be the cubical sheaf on $\widetilde{G}$ with $\widetilde{\mathcal{L}}^{\mathrm{for}} \cong \mathcal{L}^{\mathrm{for}}$ given by (2) (this is β). By definition of the polarisation λ_B induced by $\widetilde{\mathcal{L}}$, the biextension $\Lambda(\widetilde{\mathcal{L}})$ is isomorphic to $(\pi\times\pi)^*(1\times\lambda_B)^*\mathcal{P}_B$. The isomorphism κ is unique: pull-back along $\pi\times\pi$ is an equivalence from biextensions of $B\times_S B$ by $\mathbb{G}_m$ to biextensions of $\widetilde{G}\times_S\widetilde{G}$ by $\mathbb{G}_m$ ([154], Cor. 3.2.5.2, applied in each variable, $\widetilde{G}$ and B being smooth of finite presentation with connected geometric fibres), and an automorphism of a biextension of $B\times_S B$ by $\mathbb{G}_m$ is a bimultiplicative morphism $B\times_S B\to\mathbb{G}_m$, which is constant along the fibres of the proper flat $B\times_S B$ with geometrically integral fibres, hence equal to its value 1 at the origin. The semiabelian structure provides an isomorphism $\mathcal{L}\cong(1,\iota(w_0))^*\Lambda(\mathcal{L})$ on G; dividing it by its value at the origin, a unit of R, gives v, which is unique because the only automorphism of a rigidified invertible sheaf on G is the identity (shown below). Completing and transporting to $\widetilde{G}^{\mathrm{for}}$ (the identification is $\mathcal{O}_E$-equivariant and Λ commutes with completion and pull-back):

$$\begin{aligned}\mathcal{L}^{\mathrm{for}} &\cong \big((1,\widetilde{\iota}(w_0))^*\Lambda(\widetilde{\mathcal{L}})\big)^{\mathrm{for}} \cong \big((1,\widetilde{\iota}(w_0))^*(\pi\times\pi)^*(1\times\lambda_B)^*\mathcal{P}_B\big)^{\mathrm{for}}\\ &= \big(\pi^*(1,\lambda_B\iota_B(w_0))^*\mathcal{P}_B\big)^{\mathrm{for}} = (\pi^*\mathcal{L}_B)^{\mathrm{for}},\end{aligned}$$

using $(\pi\times\pi)\circ(1,\widetilde{\iota}(w_0)) = (1,\iota_B(w_0))\circ\pi$; the first isomorphism is v^{for}, the second is induced by β and the third is κ^{for}, so the composite is α. Every step is a rigidified isomorphism. Compatibility with the symmetries and cubical structures: G is integral (smooth over the normal domain R with geometrically connected fibres and connected base), and a unit on G is constant on the abelian variety G_K, so the only automorphism of a rigidified invertible sheaf on G is the identity; hence the symmetry of $\mathcal{L}$ is unique, and the isomorphism $\mathcal{L}\cong(1,\iota(w_0))^*\Lambda(\mathcal{L})$ carries it to the canonical symmetry of the right side (from bimultiplicativity of Λ). The cubical structures of the two sides agree on the dense open G_K, where cubical structures of rigidified sheaves are unique, hence everywhere, being sections of an invertible sheaf on the integral scheme $G\times_S G\times_S G$. The remaining isomorphisms are the canonical isomorphisms of biextensions, which respect the canonical symmetries and cubical structures. So the composite respects rigidifications, symmetries and cubical structures. Both $\widetilde{\mathcal{L}}$ and $\pi^*\mathcal{L}_B$ are cubical sheaves on the scheme $\widetilde{G}$ over S, and their formal completions are isomorphic as cubical torsors on G^{for} ($\widetilde{\mathcal{L}}^{\mathrm{for}}\cong\mathcal{L}^{\mathrm{for}}$ by (2), composed with the isomorphism above); since the functor of (1) from cubical torsors on $\widetilde{G}$ to cubical torsors on G^{for} is fully faithful, there is a unique cubical isomorphism $\widetilde{\alpha}\colon\widetilde{\mathcal{L}}\to\pi^*\mathcal{L}_B$ whose completion is $\alpha\circ\beta^{-1}$.

Item 3. Let λ_η be the polarisation of G_K induced by $\mathcal{L}_K$. By (4)(d), $1=\deg\lambda_\eta = [X:\phi(Y)]^2\deg\lambda_B$ with positive integers on the right, so $\deg\lambda_B=1$ and $[X:\phi(Y)]=1$; ϕ is injective with finite cokernel (definition of $\mathrm{DD}_{\mathrm{ample}}$), hence an isomorphism. □

Lemma 32.17. *Let R, K, κ and $(G,\iota,\mathcal{L})$ be as in Lemma 32.16, with $\mathcal{L}|_{G_K}$ defining a principal polarisation, and let $0\to T\to\widetilde{G}\xrightarrow{\pi}B\to 0$, λ_B and $\mathcal{L}_B$ be as there. Write $F_{\mathrm{ample}}(G,\mathcal{L}) = (B,X,Y,\phi,c',c'^{\vee},\pi^*\mathcal{L}_B,\tau',\psi')$ and put, for $\chi,\chi'\in X$,*

$$c = c'^{\vee}\circ\phi^{-1}\colon X\to B(R),\qquad \psi(\chi)=\psi'(\phi^{-1}\chi),\qquad \tau(\chi,\chi')=\tau'(\phi^{-1}\chi,\chi').$$

Then X is constant, $c'=\lambda_B\circ c$, and for every $k\geq 1$ the Fourier expansion, taken with respect to the identifications $G^{\mathrm{for}}\cong\widetilde{G}^{\mathrm{for}}$ and $\mathcal{L}^{\mathrm{for}}\cong(\pi^*\mathcal{L}_B)^{\mathrm{for}}$, is a bijection from $\Gamma(G_K,\mathcal{L}_K^{\otimes k})$ onto the set of families $(s_\alpha)_{\alpha\in X}$ of sections s_α over B_K of $\mathcal{L}_B^{\otimes k}\otimes(1,c(\alpha))^*\Lambda(\mathcal{L}_B)$ such that, for all $\alpha,\mu\in X$, $s_{\alpha+k\mu}$ corresponds to $T^*_{c(\mu)}s_\alpha$ under the isomorphism of the two sheaves whose value at the origin is $\psi(\mu)^k\tau(\mu,\alpha)$.

Proof. Put $S=\operatorname{Spec}R$, let r be the rank of X and g the relative dimension of G, so that B has relative dimension $g-r$.

Constancy and the discrete part. The character group of T is a locally constant étale sheaf on S, constant on the closed point because the torus part of $G \otimes \kappa$ is split; since R is henselian, finite étale S-schemes correspond to finite étale κ-schemes, so X is constant. By Lemma 32.16, item 3, ϕ is an isomorphism and λ_B is principal; so Y is constant, c is defined, and the relation $\lambda_B c'^{\vee} = c'\phi$ of $\mathrm{DD}_{\mathrm{ample}}$ gives $c' = \lambda_B c$. Hence $\mathcal{O}_\chi := (\mathrm{Id}, c'(\chi))^*\mathcal{P}_B \cong (1, c(\chi))^*(1 \times \lambda_B)^*\mathcal{P}_B \cong (1, c(\chi))^*\Lambda(\mathcal{L}_B)$, the last isomorphism by item 1 of Lemma 32.16.

Expansions are the maps σ_α. Fix $k \geq 1$. By (4)(c),

$$F_{\mathrm{ample}}(G, \mathcal{L}^{\otimes k}) = (B, X, Y, k\phi, c', c'^{\vee}, \pi^*\mathcal{L}_B^{\otimes k}, \tau', \psi'^k),$$

and $\mathcal{M} = \mathcal{L}_B^{\otimes k}$ is an admissible choice in (4) (Lemma 32.16, item 2), with $\mathcal{M}_\alpha = \mathcal{L}_B^{\otimes k} \otimes \mathcal{O}_\alpha$. Let $s \in \Gamma(G_K, \mathcal{L}_K^{\otimes k})$. As G is quasi-compact and flat over R, $\Gamma(G_K, \mathcal{L}_K^{\otimes k}) = \Gamma(G, \mathcal{L}^{\otimes k}) \otimes_R K$, so rs extends over G for some $0 \neq r \in R$. Its image in $\Gamma(G^{\mathrm{for}}, \mathcal{L}^{\otimes k,\mathrm{for}})$ is the $\mathfrak{m}_R$-adically convergent sum of its weight components $\sigma_\alpha(rs)\vartheta_\alpha$ with $\sigma_\alpha(rs) \in \Gamma(B, \mathcal{M}_\alpha)$, only finitely many non-zero modulo each $\mathfrak{m}_R^{e+1}$. So the Fourier expansion of s exists and is $(\sigma_\alpha(s))_\alpha$.

The relation. By (4)(b) for $(G, \mathcal{L}^{\otimes k})$, with $y = \phi^{-1}(\mu)$, so that $k\phi(y) = k\mu$ and $c'^{\vee}(y) = c(\mu)$: $\sigma_{\alpha+k\mu}(s)$ is the image of $T^*_{c(\mu)}\sigma_\alpha(s)$ under the isomorphism defined by $\psi'(y)^k\tau'(y, \alpha) = \psi(\mu)^k\tau(\mu, \alpha)$. So the Fourier expansion maps $\Gamma(G_K, \mathcal{L}_K^{\otimes k})$ K-linearly into the K-vector space V_k of families with the relation of the statement.

Injectivity. Suppose all $\sigma_\alpha(s) = 0$, with $rs \in \Gamma(G, \mathcal{L}^{\otimes k})$ as above. Then rs vanishes on every $G \otimes R/\mathfrak{m}_R^{e+1}$. G is integral (smooth over the normal domain R with geometrically connected fibres). Take an affine open $\operatorname{Spec} C \subseteq G$ meeting the closed fibre, over which $\mathcal{L}$ is trivial; then C is a noetherian domain, $\mathfrak{m}_R C \neq C$, and $rs|_{\operatorname{Spec} C}$ lies in $\bigcap_e \mathfrak{m}_R^{e+1}C = 0$ by Krull's intersection theorem. A section of an invertible sheaf on an integral scheme vanishing on a non-empty open set is zero; so $rs = 0$ and $s = 0$.

Dimensions. $\mathcal{L}_K^{\otimes k}$ is ample with polarisation of degree k^{2g}, so $\dim \Gamma(G_K, \mathcal{L}_K^{\otimes k}) = k^g$ (Riemann–Roch for abelian varieties). A family in V_k is determined by its members s_α for α in a set of representatives of X/kX, which has k^r elements, since the relation determines $s_{\alpha+k\mu}$ from s_α. Each $\mathcal{L}_B^{\otimes k} \otimes \mathcal{O}_\alpha$ on B_K is algebraically equivalent to $\mathcal{L}_B^{\otimes k}$, hence ample with polarisation $k\lambda_B$ of degree $k^{2(g-r)}$, and has k^{g-r} independent sections. Hence $\dim V_k \leq k^r k^{g-r} = k^g$.

Conclusion. The Fourier expansion is an injective K-linear map from a space of dimension k^g into V_k, of dimension at most k^g; so $\dim V_k = k^g$ and the map is a bijection. □

Proposition 32.18. *Let R, K, κ and $(G, \iota, \mathcal{L})$ be as in Lemma 32.17 (so $\mathcal{L}|_{G_K}$ defines a principal polarisation), let $\widetilde{G} \to B$, ι_B, $\mathcal{L}_B$ and $u\colon \mathcal{L}_B \to (1, \iota_B(w_0))^*\Lambda(\mathcal{L}_B)$ be as in Lemma 32.16, let c, ψ, τ be as in Lemma 32.17, and give X the $\mathcal{O}_E$-module structure for which the action of $a \in \mathcal{O}_E$ on the torus of $\widetilde{G}$ is dual to $\chi \mapsto \bar{a}\chi$. Then for every $\chi \in X$ the section $\psi(\chi)$ of $c(\chi)^*\mathcal{L}_B^{-1}$ corresponds under u to the section $\tau(\chi, w_0\chi)$ of $(c(\chi), \iota_B(w_0)c(\chi))^*\Lambda(\mathcal{L}_B)^{-1}$.*

Proof. Let λ be the principal polarisation of G_K defined by $\mathcal{L}$. Put $n_1 = w_0\bar{w}_0 = (1+d)/4$ and $n_2 = (1+w_0)(1+\bar{w}_0) = 2+n_1 = (9+d)/4$; as $w_0 + \bar{w}_0 = 1$ and $d \equiv 7 \bmod 16$, these are positive integers. Let $f_1 = \iota(w_0)$, $f_2 = \iota(1+w_0) = 1 + f_1$, endomorphisms of G; as $\iota(\bar{w}_0)f_1 = [n_1]$ and $\iota(1+\bar{w}_0)f_2 = [n_2]$, both are isogenies on the generic fibre. Write $f_{k,B}$, $f_{k,X}$, $f_{k,Y}$ for the maps of Lemma 32.19.

Step 1: the polarisations. The first isomorphism of (Ud4), $(\iota(a), 1)^*\Lambda(\mathcal{L}) \cong (1, \iota(\bar{a}))^*\Lambda(\mathcal{L})$, read on G_K where $\Lambda(\mathcal{L}) = (1 \times \lambda)^*\mathcal{P}$, says $\iota(a)^\vee\lambda = \lambda\iota(\bar{a})$. Hence the polarisation of $f_k^*\mathcal{L}_K$ is $f_k^\vee\lambda f_k = n_k\lambda$, that of $\mathcal{L}\otimes\mathcal{L}$ is 2λ, and that of $\mathcal{L}^{\otimes 2}\otimes f_1^*\mathcal{L}$ is $(2+n_1)\lambda = n_2\lambda$. By Theorem B.111(1) all the objects $(G, \mathcal{L})$, $(G, f_1^*\mathcal{L})$, $(G, f_2^*\mathcal{L})$, $(G, \mathcal{L}^{\otimes 2})$, $(G, \mathcal{L}^{\otimes 2} \otimes f_1^*\mathcal{L})$ of $\mathrm{DEG}_{\mathrm{ample}}$ ($(G, \mathcal{L})$ by Lemma 32.16, item 2; $(G, f_k^*\mathcal{L})$ by Lemma 32.19; $\mathcal{L}^{\otimes 2}$ and $\mathcal{L}^{\otimes 2} \otimes f_1^*\mathcal{L}$ are rigidified cubical

sheaves, ample over G as tensor products of sheaves ample over G, and their completions lie in the essential image of the functor of Theorem B.109(1), which is an equivalence since every torus over the field κ is isotrivial) have the same τ'.

Step 2: f_Y is multiplication by w_0. The ϕ of $(G, f_k^*\mathcal{L})$ is $n_k\phi$ (its polarisation is $n_k\lambda$) and equals $f_{k,X}\phi f_{k,Y}$ (Lemma 32.19). By the convention of the statement, $f_{1,X}(\chi) = \chi \circ f_{1,T} = \bar{w}_0\chi$ and $f_{2,X}(\chi) = (1 + \bar{w}_0)\chi$. So $\bar{w}_0\,\phi(f_{1,Y}(y)) = w_0\bar{w}_0\,\phi(y)$; multiplication by $\bar{w}_0$ on X is injective ($w_0\bar{w}_0 = n_1 \neq 0$ and X is torsion free), hence $\phi(f_{1,Y}(y)) = w_0\phi(y)$, and likewise $\phi(f_{2,Y}(y)) = (1 + w_0)\phi(y)$. As ϕ is bijective, $f_{2,Y} = 1 + f_{1,Y}$. With $b = c'^{\vee}(y)$ and $\chi = \phi(y)$: $c'^{\vee}(f_{1,Y}y) = f_{1,B}(b) = \iota_B(w_0)b$, $c(\chi) = b$ and $c(w_0\chi) = c'^{\vee}(f_{1,Y}y) = \iota_B(w_0)c(\chi)$.

Step 3: the isomorphism V on G corresponds to π^*V_B on $\widetilde{G}$. Let $v\colon \mathcal{L} \to (1, f_1)^*\Lambda(\mathcal{L})$ be the second isomorphism of (Ud4), normalised by the rigidifications, as in Lemma 32.16, item 2. As $m \circ (1, f_1) = f_2$, $(1, f_1)^*\Lambda(\mathcal{L}) = f_2^*\mathcal{L} \otimes \mathcal{L}^{-1} \otimes f_1^*\mathcal{L}^{-1}$, so v gives a rigidified isomorphism $V = v \otimes \mathrm{id} \otimes \mathrm{id}\colon N_2 := \mathcal{L} \otimes \mathcal{L} \otimes f_1^*\mathcal{L} \to N_1 := f_2^*\mathcal{L}$, an isomorphism of objects of $\mathrm{DEG}_{\mathrm{ample}}$ over the identity of G. Likewise, as $m \circ (1, f_{1,B}) = f_{2,B}$, the isomorphism u of Lemma 32.16, item 1, gives a rigidified isomorphism $V_B = u \otimes \mathrm{id} \otimes \mathrm{id}\colon M_2 := \mathcal{L}_B \otimes \mathcal{L}_B \otimes f_{1,B}^*\mathcal{L}_B \to M_1 := f_{2,B}^*\mathcal{L}_B$. An isomorphism of cubical torsors between pull-backs from B need not itself be a pull-back: by Lan's Prop. 3.2.5.4 (Theorem B.110, item 2) it is one when it respects the trivialisations over the torus, and in general it differs from one by a character $\chi_0 \in X$ of T, with $c(\chi_0) = 0$, which would multiply the identity of Step 6 by $\tau(\cdot, \chi_0)$. We show that here the character is trivial and the isomorphism is π^*V_B.

(a) The objects on $\widetilde{G}$. Let β, κ, α, $\widetilde{\alpha}$ be as in Lemma 32.16, item 2, and $\widetilde{f_k}$, $f_{k,T}$ the endomorphisms of $\widetilde{G}$ and T induced by f_k, so that $\pi\widetilde{f_k} = f_{k,B}\pi$, $\widetilde{f_k}i = if_{k,T}$ and $\widetilde{f_2} = 1 + \widetilde{f_1}$. The cubical sheaves of $F_{\mathrm{ample}}(G, N_1)$ and $F_{\mathrm{ample}}(G, N_2)$ are $\widetilde{f_2}^*\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}} \otimes \widetilde{\mathcal{L}} \otimes \widetilde{f_1}^*\widetilde{\mathcal{L}}$, with the isomorphisms $\beta_1 = (\widetilde{f_2}^{\mathrm{for}})^*\beta$ and $\beta_2 = \beta \otimes \beta \otimes (\widetilde{f_1}^{\mathrm{for}})^*\beta$, and they are identified with π^*M_1 and π^*M_2 by $\widetilde{\alpha}_1 = \widetilde{f_2}^*\widetilde{\alpha}$ and $\widetilde{\alpha}_2 = \widetilde{\alpha} \otimes \widetilde{\alpha} \otimes \widetilde{f_1}^*\widetilde{\alpha}$; these are the identifications through which Lemma 32.19 and Theorem B.111(2) give the ψ' of Step 4. By the full faithfulness in Theorem B.109(1) there are unique cubical isomorphisms $\gamma\colon \widetilde{\mathcal{L}} \to (1, \widetilde{f_1})^*\Lambda(\widetilde{\mathcal{L}})$ whose completion is v^{for} transported by β, and $h\colon \pi^*M_2 \to \pi^*M_1$ whose completion is $\widetilde{\alpha}_1^{\mathrm{for}}\beta_1 V^{\mathrm{for}}\beta_2^{-1}(\widetilde{\alpha}_2^{\mathrm{for}})^{-1}$. Comparing completions (β commutes with the identifications coming from $m \circ (1, f_1) = f_2$ and $m \circ (1, \widetilde{f_1}) = \widetilde{f_2}$): $h = \widetilde{\alpha}_1 \circ \widetilde{V} \circ \widetilde{\alpha}_2^{-1}$ with $\widetilde{V} = \gamma \otimes \mathrm{id} \otimes \mathrm{id}\colon \widetilde{\mathcal{L}} \otimes \widetilde{\mathcal{L}} \otimes \widetilde{f_1}^*\widetilde{\mathcal{L}} \to \widetilde{f_2}^*\widetilde{\mathcal{L}}$, and, by the definition of α, $\widetilde{\alpha} = (1, \widetilde{f_1})^*\kappa \circ \gamma$.

(b) Restriction to the torus. For a rigidified invertible sheaf $\mathcal{N}$ on B, $i^*\pi^*\mathcal{N} = (\pi i)^*\mathcal{N}$ is the pull-back along the zero map and is trivialised by the rigidification; call this its canonical trivialisation 1. Put $s = (i^*\widetilde{\alpha})^{-1}(1)$, a cubical trivialisation of $i^*\widetilde{\mathcal{L}}$; under $i^*\widetilde{\alpha}_1$ and $i^*\widetilde{\alpha}_2$ the canonical trivialisations of $i^*\pi^*M_1$ and $i^*\pi^*M_2$ correspond to $f_{2,T}^*s$ and $s \otimes s \otimes f_{1,T}^*s$. Let $\Lambda(s) = m^*s \otimes \mathrm{pr}_1^*s^{-1} \otimes \mathrm{pr}_2^*s^{-1}$, a trivialisation of the biextension $\Lambda(i^*\widetilde{\mathcal{L}})$ of $T \times_S T$. Now $(i \times i)^*\kappa$ is an isomorphism of biextensions from $\Lambda(i^*\widetilde{\mathcal{L}})$ to $(\pi i \times \pi i)^*(1 \times \lambda_B)^*\mathcal{P}_B$, which the rigidification of $\mathcal{P}_B$ trivialises; so it carries $\Lambda(s)$ to a bimultiplicative morphism $b\colon T \times_S T \to \mathbb{G}_m$. For each point x of T, $b(x, \cdot)$ is a character of the split torus T, i.e. a point of the constant group scheme X, and T is connected (S is connected and the fibres of T are connected); so $b(x, \cdot) = b(0, \cdot) = 1$, and $b = 1$. Hence $i^*(1, \widetilde{f_1})^*\kappa = (1, f_{1,T})^*(i \times i)^*\kappa$ carries $(1, f_{1,T})^*\Lambda(s)$ to 1; by (a) it also carries $i^*\gamma(s)$ to $i^*\widetilde{\alpha}(s) = 1$; being an isomorphism, it gives $i^*\gamma(s) = (1, f_{1,T})^*\Lambda(s) = f_{2,T}^*s \otimes s^{-1} \otimes f_{1,T}^*s^{-1}$. Therefore

$$i^*\widetilde{V}(s \otimes s \otimes f_{1,T}^*s) = \left(f_{2,T}^*s \otimes s^{-1} \otimes f_{1,T}^*s^{-1}\right) \otimes s \otimes f_{1,T}^*s = f_{2,T}^*s,$$

that is, i^*h carries the canonical trivialisation of $i^*\pi^*M_2$ to that of $i^*\pi^*M_1$. So does $i^*\pi^*V_B = (\pi i)^*V_B$, the constant map given by the value of V_B at the origin, which respects the rigidifications because u does.

(c) $h = \pi^*V_B$. The automorphism $(\pi^*V_B)^{-1} \circ h$ of the invertible sheaf π^*M_2 is multiplication

by a unit $g \in \Gamma(\widetilde{G}, \mathcal{O}^\times)$, and $g|_T = 1$ by (b). The ring $\Gamma(\widetilde{G}, \mathcal{O}) = \bigoplus_{\chi \in X} \Gamma(B, \mathcal{O}_\chi)$ is a domain graded by the torsion-free group X, so g is homogeneous, of some weight χ; a function of weight χ restricts on $T = \pi^{-1}(0)$ to its value at the origin times the character χ. So $g|_T = 1$ forces $\chi = 0$ and $g(0) = 1$; then $g \in \Gamma(B, \mathcal{O}_B) = R$, and $g = 1$. Hence $h = \pi^* V_B$: the character χ_0 above is 0.

Step 4: V_B carries the ψ' of N_2 to the ψ' of N_1. By Lemma 32.19 (with $\mathcal{M} = \mathcal{L}_B$, read through $\widetilde{\alpha}_1$), the ψ' of (G, N_1) at y is $f_{2,B}^* \psi'(f_{2,Y} y)$, which is $\psi'(y + f_{1,Y} y)$ read in the fibre at $f_{2,B}(b) = b + \iota_B(w_0) b$. By Theorem B.111(2), applied twice (Step 1 gives the equality of the τ'), and Lemma 32.19, the ψ' of (G, N_2) at y, read through $\widetilde{\alpha}_2$, is $\psi'(y) \otimes \psi'(y) \otimes \psi'(f_{1,Y} y)$, read in the fibres at b, b, $\iota_B(w_0) b$. Both objects have polarisation $n_2 \lambda$, hence the same $\phi_2 := n_2 \phi$, and the same τ' (Step 1). Write $\sigma_\chi^{(k)}$ for the Fourier coefficients of (G, N_k), taken through β_k and $\widetilde{\alpha}_k$, with values in $\Gamma(B_K, M_{k,\chi})$, $M_{k,\chi} = M_k \otimes \mathcal{O}_\chi$. As $h = \pi^* V_B$ commutes with the action of T by translation, it preserves weight components; with $h^{\mathrm{for}} = \widetilde{\alpha}_1^{\mathrm{for}} \beta_1 V^{\mathrm{for}} \beta_2^{-1} (\widetilde{\alpha}_2^{\mathrm{for}})^{-1}$ this gives $\sigma_\chi^{(1)}(Vs) = (V_B \otimes \mathrm{id})(\sigma_\chi^{(2)}(s))$ for $s \in \Gamma(G_K, N_{2,K})$ and $\chi \in X$. Fix $y \in Y$, put $b = c'^\vee(y)$, and choose s with $\sigma_0^{(2)}(s) \neq 0$ (Theorem B.109(4)(a)); then $\sigma_0^{(1)}(Vs) \neq 0$. Relation (4)(b) at $(y, 0)$ for (G, N_2), transported by V_B, says that the isomorphism $T_b^* M_{1,0,\eta} \to M_{1,\phi_2(y),\eta}$ obtained from $\psi'_{N_2}(y)$ by V_B carries $T_b^* \sigma_0^{(1)}(Vs)$ to $\sigma_{\phi_2(y)}^{(1)}(Vs)$; relation (4)(b) at $(y, 0)$ for (G, N_1) says the same of $\psi'_{N_1}(y)$. Two isomorphisms between invertible sheaves on the geometrically integral proper K-scheme B_K differ by a constant of $K^\times$, and these two agree on a non-zero section; so they are equal: V_B carries $\psi'(y) \otimes \psi'(y) \otimes \psi'(f_{1,Y} y)$ to $\psi'(y + f_{1,Y} y)$.

Step 5: the quadratic relation. Since ψ' is a cubical trivialisation compatible with $\tau' \circ (\mathrm{id}_Y \times \phi)$, for $y_1, y_2 \in Y$ with $b_k = c'^\vee(y_k)$, $\psi'(y_1 + y_2) = \psi'(y_1) \otimes \psi'(y_2) \otimes \tau'(y_1, \phi(y_2))$ under $\mathcal{L}_B(b_1 + b_2) = \mathcal{L}_B(b_1) \otimes \mathcal{L}_B(b_2) \otimes \Lambda(\mathcal{L}_B)(b_1, b_2)$, where $\tau'(y_1, \phi(y_2))$ trivialises $(b_1, c'(\phi(y_2)))^* \mathcal{P}_B^{-1} = (b_1, \lambda_B b_2)^* \mathcal{P}_B^{-1} = \Lambda(\mathcal{L}_B)(b_1, b_2)^{-1}$ ($c' \phi = \lambda_B c'^\vee$). For $y_1 = y$, $y_2 = f_{1,Y} y$ (Step 2: $b_2 = \iota_B(w_0) b$, $\phi(y_2) = w_0 \chi$):

$$\psi'(y + f_{1,Y} y) = \psi'(y) \otimes \psi'(f_{1,Y} y) \otimes \tau(\chi, w_0 \chi).$$

Step 6: conclusion. Under $M_1 = f_{2,B}^* \mathcal{L}_B = (1, f_{1,B})^* \Lambda(\mathcal{L}_B) \otimes \mathcal{L}_B \otimes f_{1,B}^* \mathcal{L}_B$, the isomorphism $V_B = u \otimes \mathrm{id} \otimes \mathrm{id}$ (Step 3) carries $\psi'(y) \otimes \psi'(y) \otimes \psi'(f_{1,Y} y)$ to $u(\psi'(y)) \otimes \psi'(y) \otimes \psi'(f_{1,Y} y)$. By Step 4 this is $\psi'(y + f_{1,Y} y)$, which by Step 5 is $\tau(\chi, w_0 \chi) \otimes \psi'(y) \otimes \psi'(f_{1,Y} y)$ (the factors reordered). Cancelling the nowhere-vanishing factors $\psi'(y) \otimes \psi'(f_{1,Y} y)$ gives $u(\psi'(y)) = \tau(\chi, w_0 \chi)$ at $c(\chi) = b$, with $\psi'(y) = \psi(\chi)$: so $\psi(\chi)$ corresponds under u to $\tau(\chi, w_0 \chi)$, a section of $(c(\chi), \iota_B(w_0) c(\chi))^* \Lambda(\mathcal{L}_B)^{-1}$ by Step 2. As ϕ is bijective, this holds for every $\chi \in X$. □

Lemma 32.19. *Let R, I, S, η be as in Theorem B.109, and let $(G, \mathcal{L})$ be an object of* $\mathrm{DEG}_{\mathrm{ample}}$ *with $F_{\mathrm{ample}}(G, \mathcal{L}) = (A, X, Y, \phi, c, c'^\vee, \widetilde{\mathcal{L}}, \tau, \psi)$, X and Y constant and $\widetilde{\mathcal{L}} \cong \pi^* \mathcal{M}$ for a cubical invertible sheaf $\mathcal{M}$ on A. Let $f \colon G \to G$ be an endomorphism of group schemes whose restriction to G_η is an isogeny, and $\widetilde{f}$, f_T, f_A the induced endomorphisms of the Raynaud extension $\widetilde{G}$, of its torus T and of A; let $f_X \colon X \to X$, $\chi \mapsto \chi \circ f_T$, and let $f_Y \colon Y \to Y$ be the homomorphism with $f_A \circ c'^\vee = c'^\vee \circ f_Y$ and $\widetilde{f} \circ \iota = \iota_1 \circ f_Y$ of Lan's Lemma 4.3.4.3, where ι_1 is the period homomorphism of $F_{\mathrm{ample}}(G, f^* \mathcal{L}) = (A, X, Y, \phi_1, c, c'^\vee, \widetilde{\mathcal{L}}_1, \tau_1, \psi_1)$. Then $(G, f^* \mathcal{L})$ lies in* $\mathrm{DEG}_{\mathrm{ample}}$, *$\widetilde{\mathcal{L}}_1 = \pi^* f_A^* \mathcal{M}$, $\phi_1 = f_X \circ \phi \circ f_Y$, and for every $y \in Y$*

$$\psi_1(y) = f_A^*(\psi(f_Y(y))),$$

as trivialisations of $c'^\vee(y)^* (f_A^* \mathcal{M}_\eta)^{-1} = c'^\vee(f_Y(y))^* \mathcal{M}_\eta^{-1}$.

Proof. Step 1: the pair $(G, f^* \mathcal{L})$. $f^* \mathcal{L}$ is a cubical invertible sheaf rigidified along the unit section. It is ample over G. Indeed, let d be the degree of the isogeny f_η, and $g_\eta \colon G_\eta \to G_\eta$ the

homomorphism with $g_\eta \circ f_\eta = [d]$ (it exists as $\ker f_\eta \subseteq G_\eta[d]$). As G is of finite presentation over the noetherian scheme S, g_η extends to a homomorphism over a dense open subscheme of S, hence, by Theorem B.104, to a homomorphism $g\colon G \to G$. The scheme G is integral (smooth over the normal domain R with geometrically connected fibres) and separated, and $g \circ f$ and $[d]$ agree on the dense subscheme G_η; their equaliser is a closed subscheme of the reduced scheme G containing a dense subset, so $g \circ f = [d]$. Hence $\ker f \subseteq G[d]$, which has finite fibres over S (on each fibre, an extension of an abelian variety by a torus, $[d]$ has finite kernel); so on every fibre over S the homomorphism f has finite kernel and finite fibres, and f is quasi-finite. Being moreover separated and of finite type, f is quasi-affine (Zariski's main theorem, EGA IV 18.12.12), and the pull-back of the ample sheaf $\mathcal{L}$ under a quasi-affine morphism is ample (EGA II 5.1.12). Finally $(f^*\mathcal{L})^{\mathrm{for}} = (\widetilde{f}^{\mathrm{for}})^*(\pi^*\mathcal{M})^{\mathrm{for}} = (\pi^* f_A^*\mathcal{M})^{\mathrm{for}}$ by functoriality ($\pi \circ \widetilde{f} = f_A \circ \pi$), which lies in the essential image of the functor of Theorem B.109(1). So $(G, f^*\mathcal{L})$ is an object of $\mathrm{DEG}_{\mathrm{ample}}$ with $\widetilde{\mathcal{L}}_1 = \pi^*\mathcal{M}_1$, $\mathcal{M}_1 = f_A^*\mathcal{M}$, and $\phi_1 = f_X \phi f_Y$ (cited above).

Step 2: weight spaces under f^*. Write $\mathcal{M}_{1,\chi} = \mathcal{M}_1 \otimes \mathcal{O}_\chi$. As $f_A^\vee c = c f_X$, $f_A^*\mathcal{O}_\chi = \mathcal{O}_{f_X(\chi)}$, so $f_A^*\mathcal{M}_\chi = \mathcal{M}_{1,f_X(\chi)}$. Under the identification of Theorem B.109(4), a formal section of $\pi^*\mathcal{M}$ on $\widetilde{G}^{\mathrm{for}}$ of weight χ for the action of T (i.e. $s(tg) = \chi(t)s(g)$) is pulled back by $\widetilde{f}$ to a section of $\pi^*\mathcal{M}_1$ of weight $f_X(\chi)$, since $(\widetilde{f}^*s)(tg) = s(f_T(t)\widetilde{f}(g)) = \chi(f_T(t))(\widetilde{f}^*s)(g)$; its coefficient section on A is the pull-back by f_A. Hence, for $s \in \Gamma(G_\eta, \mathcal{L}_\eta)$ and $\chi \in X$,

$$\sigma^{(1)}_{f_X(\chi)}(f^*s) = f_A^*(\sigma_\chi(s)),$$

where $\sigma^{(1)}$ are the Fourier coefficients of $(G, f^*\mathcal{L})$.

Step 3: comparison. Fix $y \in Y$ and put $y' = f_Y(y)$, $b = c^\vee(y)$, $b' = c^\vee(y') = f_A(b)$, so $f_A \circ T_b = T_{b'} \circ f_A$. Relation (b) of Theorem B.109(4) for $(G, \mathcal{L})$ at $(y', 0)$, with $\tau(y', 0) = 1$, reads $\psi(y') \circ T_{b'}^* \circ \sigma_0 = \sigma_{\phi(y')}$; applying f_A^* and Step 2 (with $f_X(0) = 0$ and $f_X(\phi(y')) = \phi_1(y)$) gives, for every s,

$$f_A^*(\psi(y')) \circ T_b^*(\sigma_0^{(1)}(f^*s)) = \sigma^{(1)}_{\phi_1(y)}(f^*s).$$

Relation (b) for $(G, f^*\mathcal{L})$ at $(y, 0)$ applied to f^*s gives $\psi_1(y) \circ T_b^*(\sigma_0^{(1)}(f^*s)) = \sigma^{(1)}_{\phi_1(y)}(f^*s)$. By Theorem B.109(4)(a) there is s with $\sigma_0(s) \neq 0$, and then $\sigma_0^{(1)}(f^*s) = f_A^*\sigma_0(s) \neq 0$ (f_A is surjective on the generic fibre, being part of an isogeny). Both $\psi_1(y)$ and $f_A^*\psi(y')$ are isomorphisms $T_b^*\mathcal{M}_{1,0,\eta} \to \mathcal{M}_{1,\phi_1(y),\eta}$ of invertible sheaves on the integral abelian variety A_η (read through the identifications of Theorem B.109(4)(b)); they differ by a constant of $K^\times$, and they agree on the non-zero section $T_b^*\sigma_0^{(1)}(f^*s)$. Hence $\psi_1(y) = f_A^*\psi(f_Y(y))$. □

Proposition 32.20. *Let R be a complete noetherian normal local domain of characteristic 0 with maximal ideal $\mathfrak{m}_R$, and let $(G, \iota, \mathcal{L})$ be a semiabelian structure over R uniformised by degeneration data $\mathfrak{D}$ with $\mathcal{O}_E$-structure over $(R, \mathfrak{m}_R)$. Then $\mathfrak{D}$ has torsion isomorphisms.*

Proof. Write $\mathfrak{D} = (B, \mathcal{L}_B, \iota_B, X, c, \psi)$, $r = \mathrm{rk}\, X$, $g = \dim A$, so $\dim B = g - r$. Fix an algebraically closed $F_1 \supseteq K$ and $k \geq 1$, and let P_k be the group of pairs $p = (y, \tilde{g}) \in X \times \widetilde{G}(F_1)$ with $k\tilde{g} = \iota_{\mathfrak{D}}(y)$ (componentwise addition; $\iota_{\mathfrak{D}}$ is additive), so that $T_k = T_k(\mathfrak{D}, F_1)$ is its quotient by the subgroup of $(ky', \iota_{\mathfrak{D}}(y'))$. By the definition of uniformisation, $\mathcal{L}_A$ is ample with $\phi_{\mathcal{L}_A}$ a principal polarisation, and $\mathcal{L}_B$ is relatively ample of principal polarisation.

Step 1: shapes and their composition. We use the following principle (P): B_{F_1} is an abelian variety over F_1, so proper, connected and reduced, and $\Gamma(B_{F_1}, \mathcal{O}) = F_1$; hence two isomorphisms between the same two line bundles on B_{F_1} differ by a constant in $F_1^\times$, and they are equal when their values at the origin are equal. For $n \geq 1$, $z \in B(F_1)$, $\delta \in X$ with $nz = c(\delta)$, $\varepsilon \in \mathcal{L}_z^{-n} \backslash 0$ and $h \in \widetilde{G}(F_1)$ over z, call the family of isomorphisms $\Phi_\alpha\colon T_z^*(\mathcal{L}_B^n \otimes \mathcal{O}_\alpha) \to \mathcal{L}_B^n \otimes \mathcal{O}_{\alpha+\delta}$

($\alpha \in X$) obtained, as $\Theta_{j,\alpha}$ in Lemma 32.21, from $\mathcal{L}^n_{x+z} = \mathcal{L}^n_x \otimes \mathcal{L}^n_z \otimes \Lambda^n_{x,z}$, the first-variable law at $(x, z; c(\alpha))$, the second-variable laws $\Lambda^{\otimes n}_{x,z} \cong \Lambda_{x,c(\delta)}$ and $\Lambda_{x,c(\alpha)} \otimes \Lambda_{x,c(\delta)} \cong \Lambda_{x,c(\alpha+\delta)}$, and the pairing with $\varepsilon \otimes h(\alpha)$, the *operator of shape* $(n, z, \delta, \varepsilon, h)$; its value at the origin is $\varepsilon \otimes h(\alpha)$ on $\mathcal{L}^n_z \otimes \Lambda_{z,c(\alpha)}$. Examples: $\Theta_{j,\cdot}$ for p has shape $(kj, b, jy, w^j, \tilde{g})$; the periodicity isomorphisms $\Pi_{\mu,\cdot}$ of degree n have shape $(n, c(\mu), n\mu, \psi(\mu)^n, \iota_{\mathfrak{D}}(\mu))$: by the periodicity condition of Definition 31.2, $\Pi_{\mu,\alpha} \colon T^*_{c(\mu)}(\mathcal{L}^n_B \otimes \mathcal{O}_\alpha) \to \mathcal{L}^n_B \otimes \mathcal{O}_{\alpha+n\mu}$ is the isomorphism with value $\psi(\mu)^n \otimes \tau(\mu, \alpha)$ at the origin, the operator of that shape is an isomorphism between the same line bundles with value $\psi(\mu)^n \otimes \iota_{\mathfrak{D}}(\mu)(\alpha)$ at the origin, and $\iota_{\mathfrak{D}}(\mu)(\alpha) = \tau(\mu, \alpha)$ (the coordinates of $\iota_{\mathfrak{D}}(\mu)$); so they are equal by (P).

(a) *Composition.* If Φ, Φ' have shapes $(n, z, \delta, \varepsilon, h)$, $(n, z', \delta', \varepsilon', h')$, the family $\Phi'_{\alpha+\delta} \circ T^*_{z'}\Phi_\alpha \colon T^*_{z+z'}(\mathcal{L}^n_B \otimes \mathcal{O}_\alpha) \to \mathcal{L}^n_B \otimes \mathcal{O}_{\alpha+\delta+\delta'}$ is, for every α, the operator of shape $(n, z + z', \delta + \delta', \varepsilon'', h + h')$ with one ε'' independent of α. Indeed by (P) it suffices to compare values at the origin. $T^*_{z'}\Phi_\alpha$ at 0 is Φ_α at $x = z'$: on $\mathcal{L}^n_{z'+z} \otimes \Lambda_{z'+z,c(\alpha)} = [\mathcal{L}^n_{z'} \otimes \mathcal{L}^n_z \otimes \Lambda^n_{z',z}] \otimes [\Lambda_{z',c(\alpha)} \otimes \Lambda_{z,c(\alpha)}]$ it pairs $\mathcal{L}^n_z \otimes \Lambda_{z,c(\alpha)}$ with $\varepsilon \otimes h(\alpha)$ and sends $\Lambda^n_{z',z} \otimes \Lambda_{z',c(\alpha)}$ to $\Lambda_{z',c(\alpha+\delta)}$ by the second-variable laws; then $\Phi'_{\alpha+\delta}$ at 0 pairs $\mathcal{L}^n_{z'} \otimes \Lambda_{z',c(\alpha+\delta)}$ with $\varepsilon' \otimes h'(\alpha + \delta) = \varepsilon' \otimes h'(\alpha) \otimes h'(\delta)$. The total is $\varepsilon'' \otimes (h(\alpha) \otimes h'(\alpha))$, where $\varepsilon'' = \varepsilon \otimes \varepsilon' \otimes \langle h'(\delta), \mathrm{law}_2(\cdot)\rangle$ is a form on $\mathcal{L}^n_z \otimes \mathcal{L}^n_{z'} \otimes \Lambda^n_{z',z} = \mathcal{L}^n_{z+z'}$, independent of α, and $h(\alpha) \otimes h'(\alpha) = (h + h')(\alpha)$ on $\Lambda_{z,c(\alpha)} \otimes \Lambda_{z',c(\alpha)} = \Lambda_{z+z',c(\alpha)}$ (group law = first-variable law). This is the origin value of the operator of shape $(n, z + z', \delta + \delta', \varepsilon'', h + h')$.

(b) *Comparison.* Two operators with the same (n, z, δ, h) and constants $\varepsilon_1, \varepsilon_2$ differ, for every α, by the same factor $\varepsilon_1/\varepsilon_2 \in F_1^\times$ (P).

Step 2: relations. Let $p = (y, \tilde{g}) \in P_k$ with image b.

(R1) For $q = (y', \tilde{g}') \in P_k$, $\Sigma_q\Sigma_p$ and Σ_{p+q} (with any trivialisations) differ on periodic families of degree kj by a factor $\lambda_j \in F_1^\times$: by (a) the component maps of $\Sigma_q\Sigma_p$ have shape $(kj, b + b', jy + jy', \varepsilon'', \tilde{g} + \tilde{g}')$, those of Σ_{p+q} the same up to the constant; apply (b).

(R2) For $y' \in X$, $q = p + (ky', \iota_{\mathfrak{D}}(y'))$: the component maps of Σ_q have shape $(kj, b + c(y'), jy + kjy', \cdot, \tilde{g} + \iota_{\mathfrak{D}}(y'))$, as do those of $\Pi_{y'} \circ T^*_{c(y')}\Sigma_p$ by (a). For a periodic family s, $\Sigma_p s$ is periodic (Lemma 32.21(1)), so $\Pi_{y',\beta}(T^*_{c(y')}(\Sigma_p s)_\beta) = (\Sigma_p s)_{\beta+kjy'}$; hence by (b) $\Sigma_q s = \lambda_j \Sigma_p s$.

(R3) Σ^k_p (the k-fold composite) has shape $(kj, kb, kjy, \cdot, k\tilde{g}) = (kj, c(y), kjy, \cdot, \iota_{\mathfrak{D}}(y))$ by (a), the shape of Π_y in degree kj; by periodicity and (b), $\Sigma^k_p s = \lambda_j s$ for periodic s.

(R4) Let $M \geq 1$, $\tilde{g}' \in \widetilde{G}(F_1)$ with $M\tilde{g}' = \tilde{g}$ (it exists: $\widetilde{G}(F_1)$ is an extension of the divisible group $B(F_1)$ by the divisible $(F_1^\times)^r$), $p' = (y, \tilde{g}') \in P_{kM}$. On periodic families of degree kMj, $\Sigma^M_{p'}$ (built at level kM) has shape $(kMj, Mb', Mjy, \cdot, M\tilde{g}') = (kMj, b, Mjy, \cdot, \tilde{g})$, the shape of Σ_p in degree $k \cdot Mj$; so they differ by a factor λ_j.

In each case the factors λ_j belong to two graded algebra automorphisms of a graded domain (V, or its Veronese subring $V^{(M)} = \bigoplus_j V_{Mj}$; A is integral, so products of non-zero sections are non-zero), whose ratio is therefore multiplication by λ_j in degree j with $\lambda_{j+j'} = \lambda_j\lambda_{j'}$, i.e. by λ_1^j: a *graded scalar*.

Step 3: automorphisms of A. As $\mathcal{L}_A^{\otimes k}$ is ample and A_{F_1} is proper, $A_{F_1} = \operatorname{Proj} V$ with $\mathcal{O}(j) = \mathcal{L}_A^{\otimes kj}$, and a graded algebra automorphism σ of V induces an automorphism f of A_{F_1} with an isomorphism $\theta \colon f^*\mathcal{L}_A^{\otimes k} \to \mathcal{L}_A^{\otimes k}$ such that $\sigma(s) = \theta^{\otimes j}(f^*s)$ on V_j (EGA II 2.8, 4.6; the same holds for Veronese subrings, which have the same Proj); σ is a graded scalar exactly when $f = \mathrm{id}$, as $\mathrm{Aut}(\mathcal{L}_A^{\otimes k}) = F_1^\times$. Let (f_p, θ_p) be attached to σ_p (Lemma 32.21). By (R1), (R2), $p \mapsto f_p$ is a homomorphism $P_k \to \mathrm{Aut}(A_{F_1})$ that factors through T_k.

f_p is a translation. Every automorphism of A_{F_1} is $T_x \circ h$ with h a group automorphism (Mumford [186], Sect. 4, Cor. 1). For $f = f_q = T_x \circ h$ ($q \in P_{k'}$, any k'), $f^*\mathcal{L}_A^{\otimes k'} \cong \mathcal{L}_A^{\otimes k'}$ gives

$h^\vee \phi_{\mathcal{L}_A} h = \phi_{\mathcal{L}_A}$ (as $\phi_{T_x^*\mathcal{N}} = \phi_{\mathcal{N}}$ and $\mathrm{Hom}(A, A^\vee)$ is torsion-free), so h lies in the automorphism group of the polarised abelian variety, which is finite (Mumford, loc. cit., Sect. 21, Thm. 5); let M be its order. With p' as in (R4), $\sigma_p|_{V(M)}$ and $\sigma_{p'}^M$ differ by a graded scalar, so $f_p = f_{p'}^M$; writing $f_{p'} = T_{x'} h'$ we get $f_p = T_{x''} h'^M = T_{x''}$. So $f_p = T_{x_p}$, and x_p depends only on the class of p. By (R3), $f_p^k = \mathrm{id}$, so $x_p \in A[k](F_1)$, and $p \mapsto x_p$ is a homomorphism $T_k \to A[k](F_1)$.

σ_p is a theta element over x_p. In degree 1, $\sigma_p(s) = \theta_p(T_{x_p}^* s)$ with $\theta_p \colon T_{x_p}^* \mathcal{L}_A^{\otimes k} \to \mathcal{L}_A^{\otimes k}$: this is the action on $\Gamma(A_{F_1}, \mathcal{L}_A^{\otimes k})$ of the element (x_p, θ_p^{-1}) of the theta group of $\mathcal{L}_A^{\otimes k}$; through the Fourier expansion it is S_p with trivialisation w. Every element of the theta group over x_p is $(x_p, c\,\theta_p^{-1})$, $c \in F_1^\times$ (two isomorphisms of line bundles on the proper connected A_{F_1} differ by a constant), and acts as S_p with trivialisation $c^{-1}w$.

Step 4: injectivity. Let $p \in P_k$ with $x_p = 0$, i.e. $\sigma_p = c \cdot \mathrm{id}$ in degree 1. A periodic family of degree k may be prescribed arbitrarily on a set of representatives of X/kX: the space of such tuples has dimension $k^r \cdot k^{g-r}$ ($\dim \Gamma(B_{F_1}, \mathcal{L}_B^k \otimes \mathcal{O}_\alpha) = k^{g-r}$, a bundle algebraically equivalent to $\mathcal{L}_B^{\otimes k}$ on a principally polarised abelian variety of dimension $g - r$), contains the periodic families, which form a space of dimension $\dim V_1 = k^g$ (Riemann–Roch), so it equals it. Taking s supported on one class $\alpha_0 + kX$, $\Sigma_p s = cs$ is supported on $\alpha_0 + y + kX$, so $y \in kX$, $y = ky'$. By (R2) we may replace p by $p - (ky', \iota_{\mathfrak{D}}(y')) = (0, \tilde{g}_0)$ with $k\tilde{g}_0 = 0$, still with $\Sigma_{(0,\tilde{g}_0)} = c' \cdot \mathrm{id}$ in degree k. Let b_0 be the image of $\tilde{g}_0$, so $b_0 \in B[k]$. On the component $\alpha = 0$, arbitrary in $\Gamma(B_{F_1}, \mathcal{L}_B^{\otimes k})$, the operator $s_0 \mapsto \Theta_{1,0}(T_{b_0}^* s_0)$ is the action of an element of the theta group of $\mathcal{L}_B^{\otimes k}$ over b_0, and it is a scalar; since this theta group acts irreducibly on $\Gamma(B_{F_1}, \mathcal{L}_B^{\otimes k})$ and an element over $b_0 \neq 0$ does not commute with all elements over $B[k] = K(\mathcal{L}_B^{\otimes k})$ (the commutator pairing is non-degenerate; Mumford, On the equations defining abelian varieties I [185], Sect. 1), $b_0 = 0$. Then $\tilde{g}_0$ is a point t of the torus with $t^k = 1$, and $(\Sigma s)_\alpha = t(\alpha) w s_\alpha = c' s_\alpha$ for all α and all periodic s, so $t(\alpha) = c'/w$ is constant, equal to $t(0) = 1$: $\tilde{g}_0 = 0$. So the class of p is trivial.

Step 5: counting. The map $T_k \to X/kX$, $(y, \tilde{g}) \mapsto y$, is surjective ($\widetilde{G}(F_1)$ is divisible), and its kernel consists of the classes of $(ky', \iota_{\mathfrak{D}}(y') + t)$, $t \in \widetilde{G}[k](F_1)$, i.e. of $(0, t)$; $(0, t)$ is trivial only if $t = \iota_{\mathfrak{D}}(y'')$ with $ky'' = 0$, so $y'' = 0$ (X is torsion-free) and $t = 0$. Hence $|T_k| = k^r \cdot |\widetilde{G}[k](F_1)| = k^r \cdot k^r \cdot k^{2(g-r)} = k^{2g}$ (characteristic 0), which is $|A[k](F_1)|$. The injective map $T_k \to A[k](F_1)$ of Step 4 is therefore bijective: every $x \in A[k](F_1)$ is x_p for some p, and by Step 3 every element of the theta group of $\mathcal{L}_A^{\otimes k}$ over x acts as S_p for some w. This is the definition of torsion isomorphisms. □

Lemma 32.21. *Let R be a complete noetherian local normal domain, $I \subseteq \mathfrak{m}_R$ an ideal, $K = \mathrm{Frac}\, R$, and $(G, \iota, \mathcal{L})$ a semiabelian structure over R with abelian generic fibre A, uniformised by degeneration data $\mathfrak{D}$ with $\mathcal{O}_E$-structure over (R, I); put $\mathcal{L}_A = \mathcal{L}|_A$. Let $F_1 \supseteq K$ be an algebraically closed field, $k \geq 1$, $p = (y, \tilde{g}) \in X \times \widetilde{G}(F_1)$ with $k\tilde{g} = \iota_{\mathfrak{D}}(y)$, $b \in B(F_1)$ the image of $\tilde{g}$ (so $kb = c(y)$), and $0 \neq w \in \mathcal{L}_b^{-k}$. For $j \geq 1$ and $\alpha \in X$ let*

$$\Theta_{j,\alpha} \colon T_b^*(\mathcal{L}_B^{\otimes kj} \otimes \mathcal{O}_\alpha) \longrightarrow \mathcal{L}_B^{\otimes kj} \otimes \mathcal{O}_{\alpha + jy}$$

be the isomorphism of line bundles on B_{F_1} which at a point x is the composite of $\mathcal{L}_{x+b}^{kj} = \mathcal{L}_x^{kj} \otimes \mathcal{L}_b^{kj} \otimes \Lambda_{x,b}^{kj}$, the first-variable law $\Lambda_{x+b,c(\alpha)} \cong \Lambda_{x,c(\alpha)} \otimes \Lambda_{b,c(\alpha)}$, the second-variable laws $\Lambda_{x,b}^{\otimes kj} \cong \Lambda_{x,kjb} = \Lambda_{x,c(jy)}$ and $\Lambda_{x,c(\alpha)} \otimes \Lambda_{x,c(jy)} \cong \Lambda_{x,c(\alpha+jy)}$ of the biextension $\Lambda(\mathcal{L}_B)$, and the pairing of $\mathcal{L}_b^{kj} \otimes \Lambda_{b,c(\alpha)}$ with $w^j \otimes \tilde{g}(\alpha)$. For a family $s = (s_\alpha)_{\alpha \in X}$ of sections s_α of $\mathcal{L}_B^{\otimes kj} \otimes \mathcal{O}_\alpha$ over B_{F_1} put

$$(\Sigma_p s)_{\alpha + jy} = \Theta_{j,\alpha}(T_b^* s_\alpha) \qquad (\alpha \in X),$$

which is the operator S_p of Definition 31.2 in degree kj with trivialisation w^j. Let $V_0 = F_1$ and $V_j = \Gamma(A_{F_1}, \mathcal{L}_A^{\otimes kj})$ for $j \geq 1$, with the Fourier expansion over F_1 the F_1-linear extension of that over K, and $V = \bigoplus_{j\geq 0} V_j$ the graded ring of sections. Then:

1. *for $j \geq 1$, Σ_p maps periodic families of degree kj over F_1 to periodic families of degree kj; hence there is a unique F_1-linear graded map $\sigma_p\colon V \to V$, the identity in degree 0, such that the Fourier expansion of $\sigma_p(s)$ is Σ_p of that of s for $s \in V_j$, $j \geq 1$;*
2. *σ_p is an automorphism of graded F_1-algebras.*

Proof. **Conventions.** $\Lambda = \Lambda(\mathcal{L}_B) = m^*\mathcal{L}_B \otimes \mathrm{pr}_1^*\mathcal{L}_B^{-1} \otimes \mathrm{pr}_2^*\mathcal{L}_B^{-1}$, so that $\mathcal{L}_{x+z} = \mathcal{L}_x \otimes \mathcal{L}_z \otimes \Lambda_{x,z}$ and $\Lambda_{x,z} = \Lambda_{z,x}$ hold as identities of line bundles on $B \times B$. The cubical structure of the rigidified $\mathcal{L}_B$ makes Λ a symmetric biextension: it gives the first- and second-variable laws, each associative and commutative, mutually compatible, exchanged by the symmetry, and equal to the identity when one argument is the origin, where $\Lambda_{0,z} = \Lambda_{x,0} = F_1$ through the rigidification (Moret-Bailly [181], Ch. I; Breen). The algebra structure of $\widetilde{G} = \operatorname{Spec} \bigoplus_\alpha \mathcal{O}_\alpha$ is given by the second-variable law $\mathcal{O}_\alpha \otimes \mathcal{O}_\beta \cong \mathcal{O}_{\alpha+\beta}$; thus a point $\tilde{h}$ over x is a family $\tilde{h}(\alpha) \in \Lambda^{-1}_{x,c(\alpha)}$ with $\tilde{h}(\alpha+\beta) = \tilde{h}(\alpha) \otimes \tilde{h}(\beta)$ under that law, and the group law is $(\tilde{h}+\tilde{h}')(\alpha) = \tilde{h}(\alpha) \otimes \tilde{h}'(\alpha)$ under the first-variable law; $\iota_{\mathfrak{D}}(y)$ lies over $c(y)$ with coordinates $\tau(y,\alpha) = \psi(y+\alpha)\psi(y)^{-1}\psi(\alpha)^{-1} \in \Lambda^{-1}_{c(y),c(\alpha)}$, a point because τ is bimultiplicative, and $\iota_{\mathfrak{D}}$ is additive for the same reason. B_{F_1} is proper, geometrically connected and reduced, so two isomorphisms between the same line bundles on it differ by a constant in $F_1^\times$ and are equal when their values at the origin are equal (Principle (P)). The isomorphisms $\mathcal{L}_{x+z} = \mathcal{L}_x \otimes \mathcal{L}_z \otimes \Lambda_{x,z}$ and the biextension laws are isomorphisms of line bundles when the points vary, and at $x = 0$ they are identities.

Periodic families over F_1. For $n = kj$ the periodicity condition of Definition 31.2 reads $s_{\alpha+n\mu} = \Pi_{\mu,\alpha}(T^*_{c(\mu)} s_\alpha)$, where $\Pi_{\mu,\alpha}\colon T^*_{c(\mu)}(\mathcal{L}_B^n \otimes \mathcal{O}_\alpha) \to \mathcal{L}_B^n \otimes \mathcal{O}_{\alpha+n\mu}$ is the isomorphism with value $\psi(\mu)^n\tau(\mu,\alpha)$ at the origin. By (P), $\Pi_{\mu,\alpha}$ is the composite of $\mathcal{L}^n_{x+c(\mu)} = \mathcal{L}^n_x \otimes \mathcal{L}^n_{c(\mu)} \otimes \Lambda^n_{x,c(\mu)}$, the first-variable law $\Lambda_{x+c(\mu),c(\alpha)} \cong \Lambda_{x,c(\alpha)} \otimes \Lambda_{c(\mu),c(\alpha)}$, the second-variable laws $\Lambda^{\otimes n}_{x,c(\mu)} \cong \Lambda_{x,c(n\mu)}$, $\Lambda_{x,c(\alpha)} \otimes \Lambda_{x,c(n\mu)} \cong \Lambda_{x,c(\alpha+n\mu)}$, and the pairing of $\mathcal{L}^n_{c(\mu)} \otimes \Lambda_{c(\mu),c(\alpha)}$ with $\psi(\mu)^n \otimes \tau(\mu,\alpha)$: this composite has the right value at the origin, where the canonical maps are identities. A periodic family is determined by its components at a finite set of representatives of X/nX, each in the finite-dimensional space $\Gamma(B_{F_1}, \mathcal{L}_B^n \otimes \mathcal{O}_\alpha) = \Gamma(B_K, \mathcal{L}_B^n \otimes \mathcal{O}_\alpha) \otimes_K F_1$ (flat base change); the tuples of such components that extend to a periodic family form the solution space of a system of K-linear equations, so the space $P_n(F_1)$ of periodic families over F_1 is $P_n(K) \otimes_K F_1$. With the Fourier bijection $\Gamma(A, \mathcal{L}_A^n) \cong P_n(K)$ of Definition 31.2 and $\Gamma(A_{F_1}, \mathcal{L}_A^n) = \Gamma(A, \mathcal{L}_A^n) \otimes_K F_1$, the Fourier expansion over F_1 is a bijection $V_j \cong P_{kj}(F_1)$.

Item 1. Fix $j \geq 1$, $n = kj$, a periodic family s, $\mu, \alpha \in X$. We must show $(\Sigma_p s)_{\alpha+jy+n\mu} = \Pi'(T^*_{c(\mu)}(\Sigma_p s)_{\alpha+jy})$ with $\Pi' = \Pi_{\mu,\alpha+jy}$. The left side is $\Phi_1(T^*_{b+c(\mu)} s_\alpha)$ with $\Phi_1 = \Theta_{j,\alpha+n\mu} \circ T_b^* \Pi_{\mu,\alpha}$, the right side $\Phi_2(T^*_{b+c(\mu)} s_\alpha)$ with $\Phi_2 = \Pi' \circ T^*_{c(\mu)} \Theta_{j,\alpha}$, as $T_b^* T^*_{c(\mu)} = T^*_{b+c(\mu)}$. Both are isomorphisms $T^*_{b+c(\mu)}(\mathcal{L}_B^n \otimes \mathcal{O}_\alpha) \to \mathcal{L}_B^n \otimes \mathcal{O}_{\alpha+jy+n\mu}$, so by (P) it suffices to compare their values at the origin, linear forms on $C = \mathcal{L}^n_{b+c(\mu)} \otimes \Lambda_{b+c(\mu),c(\alpha)}$. Decompose C by $\mathcal{L}^n_{b+c(\mu)} = \mathcal{L}^n_b \otimes \mathcal{L}^n_{c(\mu)} \otimes \Lambda^n_{b,c(\mu)}$ and the first-variable law $\Lambda_{b+c(\mu),c(\alpha)} \cong \Lambda_{b,c(\alpha)} \otimes \Lambda_{c(\mu),c(\alpha)}$ (the order of b, $c(\mu)$ does not matter, by the symmetry $\Lambda_{x,z} = \Lambda_{z,x}$ and the commutativity of the law).

- $\Phi_1(0)$: $T_b^*\Pi_{\mu,\alpha}$ at 0 is $\Pi_{\mu,\alpha}$ at $x = b$: it pairs $\mathcal{L}^n_{c(\mu)} \otimes \Lambda_{c(\mu),c(\alpha)}$ with $\psi(\mu)^n \otimes \tau(\mu,\alpha)$ and maps $\Lambda^{\otimes n}_{b,c(\mu)}$ to $\Lambda_{b,c(n\mu)}$ by the second-variable law; then $\Theta_{j,\alpha+n\mu}$ at 0 pairs $\mathcal{L}_b^n \otimes \Lambda_{b,c(\alpha+n\mu)}$ with $w^j \otimes \tilde{g}(\alpha+n\mu)$, and $\tilde{g}(\alpha+n\mu) = \tilde{g}(\alpha) \otimes \tilde{g}(n\mu)$. So $\Phi_1(0) = \psi(\mu)^n \otimes \tau(\mu,\alpha) \otimes w^j \otimes \tilde{g}(\alpha) \otimes \langle \tilde{g}(n\mu), \mathrm{law}_2(\cdot)\rangle$, the last factor a form on $\Lambda^{\otimes n}_{b,c(\mu)}$.

- $\Phi_2(0)$: $T^*_{c(\mu)}\Theta_{j,\alpha}$ at 0 is $\Theta_{j,\alpha}$ at $x = c(\mu)$: it pairs $\mathcal{L}^n_b \otimes \Lambda_{b,c(\alpha)}$ with $w^j \otimes \tilde{g}(\alpha)$ and maps $\Lambda^{\otimes n}_{c(\mu),b} = \Lambda^{\otimes n}_{b,c(\mu)}$ to $\Lambda_{c(\mu),nb} = \Lambda_{c(\mu),c(jy)}$ by the second-variable law in the second argument, which under the symmetry $\Lambda_{x,z} = \Lambda_{z,x}$ is the first-variable law $\Lambda^{\otimes n}_{b,c(\mu)} \cong \Lambda_{nb,c(\mu)}$; then Π' at 0 pairs $\mathcal{L}^n_{c(\mu)} \otimes \Lambda_{c(\mu),c(\alpha+jy)}$ with $\psi(\mu)^n \otimes \tau(\mu, \alpha+jy)$, and $\tau(\mu, \alpha+jy) = \tau(\mu,\alpha) \otimes \tau(\mu, jy)$. So $\Phi_2(0) = \psi(\mu)^n \otimes \tau(\mu,\alpha) \otimes w^j \otimes \tilde{g}(\alpha) \otimes \langle \tau(jy,\mu), \mathrm{law}_1(\cdot)\rangle$, using $\tau(\mu, jy) = \tau(jy, \mu)$ (by the symmetry $\Lambda_{x,z} = \Lambda_{z,x}$).

It remains to compare the last factors. Let $u = \tilde{g}(\mu)^{\otimes n} \in (\Lambda^{-1}_{b,c(\mu)})^{\otimes n}$. The weight coordinates of a point are multiplicative for the second-variable law, so $\mathrm{law}_2(u) = \tilde{g}(n\mu)$. The group law is the first-variable law on coordinates, so $\mathrm{law}_1(u) = (n\tilde{g})(\mu) = (j\, \iota_{\mathfrak{D}}(y))(\mu) = \iota_{\mathfrak{D}}(jy)(\mu) = \tau(jy,\mu)$, by $k\tilde{g} = \iota_{\mathfrak{D}}(y)$ and additivity of $\iota_{\mathfrak{D}}$. An isomorphism of lines and its inverse transpose preserve the pairing, so both factors are the form $\langle u, \cdot\rangle$ on $\Lambda^{\otimes n}_{b,c(\mu)}$. Hence $\Phi_1 = \Phi_2$ and $\Sigma_p s$ is periodic. The rest of item 1 follows from the bijection $V_j \cong P_{kj}(F_1)$.

Item 2: bijectivity. Σ_p is injective on families (T^*_b and $\Theta_{j,\alpha}$ are isomorphisms, $\alpha \mapsto \alpha + jy$ a bijection), so σ_p is injective on each finite-dimensional V_j, hence bijective.

Item 2: products over K. For $s \in \Gamma(A, \mathcal{L}^n_A)$, $t \in \Gamma(A, \mathcal{L}^{n'}_A)$ with expansions (s_α), (t_β), the expansion of st is $(st)_\gamma = \sum_{\alpha+\beta=\gamma} s_\alpha t_\beta$, the product taken through $\mathcal{L}^n \otimes \mathcal{L}^{n'} = \mathcal{L}^{n+n'}$ and $\mathcal{O}_\alpha \otimes \mathcal{O}_\beta \cong \mathcal{O}_{\alpha+\beta}$, the series converging in the I-adic topology of the finite R-module $\Gamma(B, \mathcal{L}^{n+n'}_B \otimes \mathcal{O}_\gamma)$ after multiplication by a common denominator. Indeed, choose $r, r' \neq 0$ as in the definition of the expansion for s and t. For every e, $rs \equiv \sum_\alpha rs_\alpha \vartheta_\alpha$ and $r't \equiv \sum_\beta r' t_\beta \vartheta_\beta$ modulo I^{e+1}, finite sums, and $\vartheta_\alpha \vartheta_\beta = \vartheta_{\alpha+\beta}$ (definition of the algebra structure), so $rr'st$ modulo I^{e+1} is $\sum_\gamma (\sum_{\alpha+\beta=\gamma} rs_\alpha r' t_\beta)\vartheta_\gamma$; sections of a pull-back from B over the thickening of $\widetilde{G}$, affine over B, decompose uniquely into weight components, so the γ-components of $rr'st$ and of $\sum_{\alpha+\beta=\gamma} rs_\alpha r' t_\beta$ agree in $\Gamma(B \otimes R/I^{e+1}, \cdot)$ for every e, the latter sum being finite for each e. As B is proper over the I-adically complete R, $\Gamma(B, \mathcal{M}) \to \varprojlim_e \Gamma(B \otimes R/I^{e+1}, \mathcal{M})$ is an isomorphism and the topology defined by the kernels is the I-adic one (theorem on formal functions, EGA III$_1$ 4.1.5, with Artin–Rees); so $\sum_{\alpha+\beta=\gamma} rs_\alpha r' t_\beta$ converges to $rr'(st)_\gamma$.

Item 2: multiplicativity. It suffices to show $\sigma_p(s)\sigma_p(t) = \sigma_p(st)$ for $s \in V_j$, $t \in V_{j'}$, $j, j' \geq 1$ (degree 0 is trivial), and by bilinearity for s, t defined over K. Replacing w by cw multiplies σ_p by c^j in degree j, which preserves the claim, so we may take w defined over the algebraic closure $\bar{K}$ of K in F_1; $\tilde{g}$ and b are defined over $\bar{K}$, as $\tilde{g}$ lies in the finite K-scheme $\{\tilde{h} : k\tilde{h} = \iota_{\mathfrak{D}}(y)\}$. Both sides then lie in $V_{j+j'}$ over $\bar{K}$, and we may check the identity after any field embedding of $\bar{K}$. Choose a discrete valuation ring of K dominating the noetherian local domain R (EGA II 7.1.7), complete it and let C be the completion of an algebraic closure of its fraction field, with an embedding $\bar{K} \to C$ over K; then $R \subseteq \mathcal{O}_C$ and I maps into the maximal ideal. On the finite-dimensional C-spaces $\Gamma(B_C, \mathcal{M}) = \Gamma(B_K, \mathcal{M}) \otimes_K C$ use any norm (all are equivalent); I-adic convergence in $\Gamma(B, \mathcal{M})$ implies convergence for it (in a K-basis $\Gamma(B, \mathcal{M}) \subseteq r^{-1} \bigoplus Re_i$ for some $r \neq 0$, and elements of I^m have valuation at least $m\, v(\mathfrak{m}_R) > 0$ times a constant), and C-linear maps between such spaces are continuous. By bilinearity the convolution formula of the previous paragraph holds, with norm-convergent series, for all $s \in V_j(C)$, $t \in V_{j'}(C)$. Apply it to $\sigma_p(s), \sigma_p(t)$: for $\alpha + \beta = \gamma$,

$$(\Sigma_p s)_{\alpha+jy} (\Sigma_p t)_{\beta+j'y} = \Theta_{j,\alpha}(T^*_b s_\alpha)\, \Theta_{j',\beta}(T^*_b t_\beta) = \Theta_{j+j',\gamma}(T^*_b(s_\alpha t_\beta)),$$

because $\Theta_{j,\alpha} \otimes \Theta_{j',\beta}$ and $\Theta_{j+j',\gamma}$ composed with the product maps are isomorphisms between the same line bundles whose values at the origin are $w^j \otimes \tilde{g}(\alpha) \otimes w^{j'} \otimes \tilde{g}(\beta)$ and $w^{j+j'} \otimes \tilde{g}(\gamma)$ on $\mathcal{L}^{n+n'}_b \otimes \Lambda_{b,c(\alpha)} \otimes \Lambda_{b,c(\beta)} \cong \mathcal{L}^{n+n'}_b \otimes \Lambda_{b,c(\gamma)}$, equal since $\tilde{g}(\gamma) = \tilde{g}(\alpha) \otimes \tilde{g}(\beta)$ (P). Summing over $\alpha + \beta = \gamma$ and using continuity of the linear map $\Theta_{j+j',\gamma} \circ T^*_b$, the $(\gamma + (j+j')y)$-component of

the expansion of $\sigma_p(s)\sigma_p(t)$ is $\Theta_{j+j',\gamma}(T_b^*(st)_\gamma) = (\Sigma_p(st))_{\gamma+(j+j')y}$. So $\sigma_p(s)\sigma_p(t) = \sigma_p(st)$, and σ_p is a graded algebra automorphism. □

Proposition 32.22. *Let E and H be as in Definition 31.2. Let R be a complete noetherian normal local domain of characteristic 0 with maximal ideal $\mathfrak{m}_R$, let $(G, \iota, \mathcal{L})$ be a semiabelian structure over R that is uniformised by degeneration data $(B, \mathcal{L}_B, \iota_B, X, c, \psi)$ with $\mathcal{O}_E$-structure over $(R, \mathfrak{m}_R)$ having torsion isomorphisms, and let U be the set of points $x \in \operatorname{Spec} R$ such that $G \otimes_R \kappa(x)$ is an abelian variety. Then for every $\chi \in X$ the section $\psi(\chi)$ is a unit over U: for every open subset $V \subseteq \operatorname{Spec} R$ contained in U, $\psi(\chi)$ extends to a nowhere-vanishing section of $c(\chi)^*\mathcal{L}_B^{-1}$ over V.*

Proof. Put $S = \operatorname{Spec} R$ and let r be the rank of X and g the relative dimension of G. Since R is local, the invertible sheaves $c(\chi)^*\mathcal{L}_B^{-1}$ and $(c(\chi), c(\chi'))^*\Lambda(\mathcal{L}_B)^{-1}$ on S are free; fix generators and write $\psi(\chi) = f_\chi\, e_\chi$ and $\tau(\chi, \chi') = g_{\chi,\chi'}\, e_{\chi,\chi'}$ with $f_\chi, g_{\chi,\chi'} \in K^\times$.

Reduction to height-one primes of U. Let $V \subseteq U$ be open. Every local ring R_x $(x \in V)$ is a noetherian normal domain, hence the intersection in K of its localisations at height-one primes, and these are the $R_\mathfrak{p}$ for height-one primes $\mathfrak{p} \subseteq x$, which lie in V as V is stable under generisation. Hence $\Gamma(V, \mathcal{O}_S) = K \cap \bigcap_\mathfrak{p} R_\mathfrak{p}$ over the height-one primes $\mathfrak{p} \in V$. So it suffices to prove $v_\mathfrak{p}(f_\chi) = 0$ for every $\chi \in X$ and every height-one prime $\mathfrak{p} \in U$, where $v_\mathfrak{p}$ is the normalised valuation of the discrete valuation ring $R_\mathfrak{p}$: then $f_\chi, f_\chi^{-1} \in \Gamma(V, \mathcal{O}_S)$ and $\psi(\chi) = f_\chi e_\chi$ is a nowhere-vanishing section of $c(\chi)^*\mathcal{L}_B^{-1}$ over V.

The valuation form. Fix a height-one prime $\mathfrak{p} \in U$ and put $q(\chi) = v_\mathfrak{p}(f_\chi)$, $b(\chi, \chi') = v_\mathfrak{p}(g_{\chi,\chi'})$; a change of generators multiplies by units of $R \subseteq R_\mathfrak{p}$, so these integers do not depend on the choices. Every identification used in the conditions on $\mathfrak{D}$ (the isomorphism $(c(\chi), c(\chi'))^*\Lambda(\mathcal{L}_B) \cong (c(\chi)+c(\chi'))^*\mathcal{L}_B \otimes c(\chi)^*\mathcal{L}_B^{-1} \otimes c(\chi')^*\mathcal{L}_B^{-1}$, the cubical structure, the symmetry of $\mathcal{L}_B$) is an isomorphism of invertible sheaves on S and preserves $v_\mathfrak{p}$. Hence the definition of τ, bimultiplicativity and $\psi(-\chi) = \psi(\chi)$ give

$$b(\chi, \chi') = q(\chi + \chi') - q(\chi) - q(\chi'), \qquad b \text{ bilinear and symmetric}, \qquad q(-\chi) = q(\chi).$$

With $\chi = \chi' = 0$, $0 = b(0,0) = -q(0)$; then $0 = q(\chi + (-\chi)) = 2q(\chi) - b(\chi, \chi)$, so $b(\chi, \chi) = 2q(\chi)$. For $\chi \neq 0$, $\psi(\chi)$ is a section over R of $\mathfrak{m}_R c(\chi)^*\mathcal{L}_B^{-1}$, so $f_\chi \in R \subseteq R_\mathfrak{p}$ and $q(\chi) \geq 0$. Thus b is a positive semi-definite symmetric bilinear form on X with values in $\mathbb{Z}$. By the Cauchy–Schwarz inequality $b(\chi, \chi')^2 \leq b(\chi, \chi) b(\chi', \chi')$, its radical is

$$X_\mathfrak{p} = \{\chi \in X :\ b(\chi, \chi) = 0\} = \{\chi \in X :\ q(\chi) = 0\},$$

a subgroup of X that is saturated, since $b(m\chi, m\chi) = m^2 b(\chi, \chi)$.

The torsion count at $\mathfrak{p}$. By Lemma 32.23, $(G, \mathcal{L}) \cong M_{\text{ample}}(\mathfrak{D}^{\text{FC}})$, where $\mathfrak{D}^{\text{FC}}$ has $Y = X$ constant, $\phi = \mathrm{id}$ and Raynaud extension $\widetilde{G}$. The datum $\mathfrak{D}^{\text{FC}}$ is an object of $\mathrm{DD}_{\text{ample}}$ with X and Y constant, so Lemma 32.24 applies to it and to $(G, \mathcal{L}) \cong M_{\text{ample}}(\mathfrak{D}^{\text{FC}})$. In it, the module $I_{\chi,\phi(\chi)}$ is $g_{\chi,\chi}R$, so $I_{\chi,\chi}\mathcal{O}_{S,\mathfrak{p}} = \mathcal{O}_{S,\mathfrak{p}}$ exactly when $b(\chi, \chi) = 0$: the group $Y_\mathfrak{p}$ there is $X_\mathfrak{p}$. Let n be a prime different from the characteristic of $\kappa(\mathfrak{p})$ and $\bar{k}$ an algebraic closure of $\kappa(\mathfrak{p})$. Lemma 32.24 gives an exact sequence $0 \to \widetilde{G}[n]_\mathfrak{p} \to G[n]_\mathfrak{p} \to \frac{1}{n}X_\mathfrak{p}/X_\mathfrak{p} \to 0$ of finite étale group schemes over $\kappa(\mathfrak{p})$, which stays exact on $\bar{k}$-points. Since $\mathfrak{p} \in U$, $G_\mathfrak{p}$ is an abelian variety of dimension g and $\#G[n](\bar{k}) = n^{2g}$. The fibre $\widetilde{G}_\mathfrak{p}$ is an extension of an abelian variety of dimension $g - r$ by a split torus of dimension r (G and $\widetilde{G}$ are smooth over the connected S with the same closed fibre, so of the same relative dimension), and the torus is divisible on $\bar{k}$-points, so $\#\widetilde{G}[n](\bar{k}) = n^r \cdot n^{2(g-r)}$. Finally $\frac{1}{n}X_\mathfrak{p}/X_\mathfrak{p} \cong (\mathbb{Z}/n)^{\operatorname{rk} X_\mathfrak{p}}$. Hence $2g = 2g - r + \operatorname{rk} X_\mathfrak{p}$, i.e. $\operatorname{rk} X_\mathfrak{p} = r$.

Conclusion. The group $X_{\mathfrak{p}}$ is a saturated subgroup of full rank of the free group X, so $X/X_{\mathfrak{p}}$ is finite and torsion-free, and $X_{\mathfrak{p}} = X$. Hence $q(\chi) = 0$ for all χ, which by the reduction proves the proposition. □

Lemma 32.23. *Let R be a complete noetherian normal local domain with maximal ideal $\mathfrak{m}_R$ and fraction field K, and let $(G, \iota, \mathcal{L})$ be a semiabelian structure over R uniformised by degeneration data $\mathfrak{D} = (B, \mathcal{L}_B, \iota_B, X, c, \psi)$ with $\mathcal{O}_E$-structure over $(R, \mathfrak{m}_R)$. Let $\pi\colon \widetilde{G} \to B$ be the extension defined by c, and $\lambda_B\colon B \to B^\vee$ the principal polarisation of $\mathcal{L}_B$. Put*

$$\mathfrak{D}^{\mathrm{FC}} = \big(B,\ X,\ Y := X,\ \phi := \mathrm{id}_X,\ c^\vee := c,\ \lambda_B \circ c,\ \pi^*\mathcal{L}_B,\ \tau,\ \psi\big),$$

where $\tau(\chi, \chi')$ is read as a trivialisation of $(c(\chi), \lambda_B c(\chi'))^*\mathcal{P}_B^{-1} = (c(\chi), c(\chi'))^*\Lambda(\mathcal{L}_B)^{-1}$ and $\psi(\chi)$ as a trivialisation of $c(\chi)^*\mathcal{L}_B^{-1}$. Then $(G, \mathcal{L}) \in \mathrm{DEG}_{\mathrm{ample}}$, its Raynaud extension is $\widetilde{G}$ with the identifications of the uniformisation, $F_{\mathrm{ample}}(G, \mathcal{L}) \cong \mathfrak{D}^{\mathrm{FC}}$ (so $\mathfrak{D}^{\mathrm{FC}} \in \mathrm{DD}_{\mathrm{ample}}$), and $(G, \mathcal{L}) \cong M_{\mathrm{ample}}(\mathfrak{D}^{\mathrm{FC}})$.

Proof. The setting of Theorem B.109 holds: R is a noetherian normal domain complete for $\mathfrak{m}_R$, and $\mathrm{rad}(\mathfrak{m}_R) = \mathfrak{m}_R$. Write $A = G \otimes_R K$, $\mathcal{L}_A = \mathcal{L}|_A$, $T \subseteq \widetilde{G}$ for the split torus with character group X, and ^{for} for completion along the closed fibre. For $\chi \in X$ put $\mathcal{O}_\chi = (\mathrm{Id}, \lambda_B c(\chi))^*\mathcal{P}_B = (1, c(\chi))^*\Lambda(\mathcal{L}_B)$ and $\mathcal{M}_\chi = \mathcal{L}_B \otimes \mathcal{O}_\chi$; these are the bundles of which the s_α of a Fourier expansion are sections.

$(G, \mathcal{L})$ lies in $\mathrm{DEG}_{\mathrm{ample}}$. G is semiabelian and A is an abelian variety. For $e = 0$ the uniformisation identifies $G \otimes_R \kappa$ with $\widetilde{G} \otimes_R \kappa$, an extension of the abelian variety $B \otimes \kappa$ by the split torus $T \otimes \kappa$. $\mathcal{L}$ is rigidified and cubical; $\mathcal{L}_A$ defines a polarisation of A, so it is algebraically equivalent to an ample sheaf and is ample, in particular nondegenerate; $\mathcal{L}$ is symmetric (the sheaf of a semiabelian structure), so $\mathcal{L}_A$ is symmetric, and $\mathcal{L}$ is ample over G by the ampleness criterion for symmetric sheaves in (2). Every torus over the field κ splits over a finite separable extension, so the functor of (1) is an equivalence and $\mathcal{L}^{\mathrm{for}}$ lies in its essential image.

The Raynaud extension. The identifications $G \otimes R/\mathfrak{m}_R^{e+1} \cong \widetilde{G} \otimes R/\mathfrak{m}_R^{e+1}$, compatible in e, form an isomorphism of formal groups $G^{\mathrm{for}} \cong \widetilde{G}^{\mathrm{for}}$. It carries T^{for} onto a formal subtorus of G^{for} lifting the torus part of $G \otimes \kappa$; such a lifting is unique (rigidity of tori), so it is the formal torus of (1), and $\widetilde{G}^{\mathrm{for}} \to B^{\mathrm{for}}$ is the quotient of G^{for} by it. By the uniqueness in (1), $0 \to T \to \widetilde{G} \to B \to 0$ with this identification is the Raynaud extension of G. Under it, $\mathcal{L}^{\mathrm{for}}$ is identified with $(\pi^*\mathcal{L}_B)^{\mathrm{for}}$ compatibly with rigidifications and cubical structures, so the cubical sheaf $\widetilde{\mathcal{L}}$ of $F_{\mathrm{ample}}(G, \mathcal{L})$ is $\pi^*\mathcal{L}_B$ (both are cubical sheaves on the scheme $\widetilde{G}$ over S whose completions are isomorphic as cubical torsors on G^{for}, $\widetilde{\mathcal{L}}^{\mathrm{for}} \cong \mathcal{L}^{\mathrm{for}}$ being given by (2); so they are isomorphic by the full faithfulness of the functor in (1)), and $\mathcal{M} = \mathcal{L}_B$ is an admissible choice in (4). The character group X is constant. The polarisation of B induced by $\pi^*\mathcal{L}_B$ is λ_B, since $\Lambda(\pi^*\mathcal{L}_B) = (\pi \times \pi)^*\Lambda(\mathcal{L}_B)$.

The discrete part. Write $F_{\mathrm{ample}}(G, \mathcal{L}) = (B, X, Y', \phi', c', c'^\vee, \pi^*\mathcal{L}_B, \tau', \psi')$. Here $c'\colon X \to B^\vee$ is the extension class of $\widetilde{G}$, which is $\lambda_B \circ c$ (this is how c defines $\widetilde{G}$, B being identified with $B^\vee$ by λ_B). The polarisation λ_η of A induced by $\mathcal{L}_A$ is principal, and λ_B is principal, so by (4)(d) $[X : \phi'(Y')]^2 = \deg \lambda_\eta / \deg \lambda_B = 1$: ϕ' is an isomorphism, and Y' is constant. The relation $\lambda_B c'^\vee = c'\phi'$ of $\mathrm{DD}_{\mathrm{ample}}$ gives $c'^\vee \circ \phi'^{-1} = \lambda_B^{-1} c' = c$. Identifying Y' with X by ϕ', the discrete part of $F_{\mathrm{ample}}(G, \mathcal{L})$ is $(B, X, X, \mathrm{id}_X, c, \lambda_B c)$, as in $\mathfrak{D}^{\mathrm{FC}}$.

The trivialisations. Let $s \in \Gamma(A, \mathcal{L}_A)$ with Fourier expansion $(s_\alpha)_{\alpha \in X}$. For $0 \neq r \in R$ as in the definition, rs modulo $\mathfrak{m}_R^{e+1}$ is $\sum_\alpha r s_\alpha \vartheta_\alpha$ for every e, where ϑ_α is the coordinate of weight α; this is the decomposition of $rs \in \Gamma(G^{\mathrm{for}}, \mathcal{L}^{\mathrm{for}})$ into weights of T, so $s_\alpha = \sigma_\alpha(s)$ for the maps σ_α of (4) attached to $\mathcal{M} = \mathcal{L}_B$. By the uniformisation (case $k = 1$), for all $\alpha, \mu \in X$ the section

$\sigma_{\alpha+\mu}(s)$ corresponds to $T^*_{c(\mu)}\sigma_\alpha(s)$ under the isomorphism $\theta_{\mu,\alpha}\colon T^*_{c(\mu)}\mathcal{M}_{\alpha,K} \to \mathcal{M}_{\alpha+\mu,K}$ whose value at the origin is $\psi(\mu)\tau(\mu,\alpha)$; by (4)(b) it also corresponds to it under the isomorphism $\theta'_{\mu,\alpha}$ defined by $\psi'(\mu)\tau'(\mu,\alpha)$. By (4)(a) there is s with $\sigma_\alpha(s) \neq 0$. Two isomorphisms between invertible sheaves on the geometrically integral proper K-scheme B_K differ by an element of $\Gamma(B_K, \mathcal{O}^\times) = K^\times$; as $\theta_{\mu,\alpha}$ and $\theta'_{\mu,\alpha}$ agree on the non-zero section $T^*_{c(\mu)}\sigma_\alpha(s)$, they are equal, so

$$\psi(\mu)\tau(\mu,\alpha) = \psi'(\mu)\tau'(\mu,\alpha) \qquad (\mu,\alpha \in X).$$

Bimultiplicativity gives $\tau(0,0) = \tau(0,0)^2$, so $\tau(0,0) = 1$; as $\tau(0,0) = \psi(0)^{-1}$ by definition of τ, $\psi(0) = 1$, and then $\tau(\mu,0) = \psi(\mu)\psi(0)^{-1}\psi(\mu)^{-1} = 1$ under the canonical identifications, which is the normalisation of (4)(b). Taking $\alpha = 0$ gives $\psi = \psi'$, and then $\tau = \tau'$. Hence $F_{\mathrm{ample}}(G,\mathcal{L}) \cong \mathfrak{D}^{\mathrm{FC}}$. Read through the cubical isomorphism $\widetilde{\mathcal{L}} = \pi^*\mathcal{L}_B$ above, i.e. with $\mathcal{M} = \mathcal{L}_B$, which is symmetric, ψ is even in the sense of (2): $\psi(-\chi) = \psi(\chi)$ under the symmetry of $\mathcal{L}_B$ for every χ, a condition on $\mathfrak{D}$ in Definition 31.2. So by (2) $F_{\mathrm{ample}}(G,\mathcal{L})$ is an object of $\mathrm{DD}_{\mathrm{ample}}$, and so is $\mathfrak{D}^{\mathrm{FC}}$.

Mumford's construction. By (2), as $F_{\mathrm{ample}}(G,\mathcal{L})$ is an object of $\mathrm{DD}_{\mathrm{ample}}$,

$$(G,\mathcal{L}) \cong M_{\mathrm{ample}}(F_{\mathrm{ample}}(G,\mathcal{L})) \cong M_{\mathrm{ample}}(\mathfrak{D}^{\mathrm{FC}}).$$

□

Lemma 32.24. *Let R, I, S, K be as in Theorem B.109, let $\mathfrak{d} = (A, X, Y, \phi, c, c^\vee, \widetilde{\mathcal{L}}, \tau, \psi)$ be an object of $\mathrm{DD}_{\mathrm{ample}}$ with X and Y constant, $\widetilde{G}$ the extension of A by the torus with character group X defined by c, $\iota\colon Y \to \widetilde{G}(K)$ the period homomorphism, and $(G,\mathcal{L}) = M_{\mathrm{ample}}(\mathfrak{d})$. For $s \in S$ put*

$$Y_s = \{z \in Y :\ I_{z,\phi(z)} \cdot \mathcal{O}_{S,s} = \mathcal{O}_{S,s}\}.$$

Then $Y_s = \{z \in Y :\ \iota(z) \in \widetilde{G}(\operatorname{Spec}\mathcal{O}_{S,s})\}$, a subgroup of Y, and for every $s \in S$ and every integer $n \geq 1$ there is an exact sequence of group schemes over $\kappa(s)$

$$0 \to \widetilde{G}[n]_s \to G[n]_s \to \tfrac{1}{n}Y_s/Y_s \to 0.$$

Proof. A split object with the same G. By Lemma 32.25(a) there are an ample invertible sheaf $\mathcal{M}$ on A with $\widetilde{\mathcal{L}} \cong \pi^*\mathcal{M}$ and an integer $m \geq 1$ such that

$$\mathfrak{d}' = (A,\ \mathcal{M}^{\otimes m+1} \otimes [-1]^*\mathcal{M}^{\otimes m},\ X,\ Y,\ \phi',\ c,\ c^\vee,\ \tau,\ \psi^{m+1}[-1]^*\psi^m), \qquad \phi' = (2m+1)\phi,$$

is an object of $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ with a relatively complete model; fix one and let G' be the attached Mumford quotient. By Lemma 32.25(b) there is an isomorphism $G \cong G'$ of group schemes over S. The object $\mathfrak{d}'$ has the same A, X, Y, c, $c^\vee$ and τ as $\mathfrak{d}$, hence the same extension $\widetilde{G}$ (defined by c) and the same period homomorphism ι (determined by $c^\vee$ and τ, Theorem B.109(2)).

The sequence for G'. Put $Y'_s = \{z \in Y :\ I_{z,\phi'(z)} \cdot \mathcal{O}_{S,s} = \mathcal{O}_{S,s}\}$. By Theorem B.109(5) applied to $\mathfrak{d}'$ and its relatively complete model, $Y'_s = \{z \in Y :\ \iota(z) \in \widetilde{G}(\operatorname{Spec}\mathcal{O}_{S,s})\}$, and for every $n \geq 1$ there is an exact sequence of group schemes over $\kappa(s)$

$$0 \to \widetilde{G}[n]_s \to G'[n]_s \to \tfrac{1}{n}Y'_s/Y'_s \to 0.$$

$Y'_s = Y_s$. Fix $z \in Y$ and put $b = c^\vee(z) \in A(S)$ and $\mathcal{O}_\chi = (\mathrm{Id}, c(\chi))^*\mathcal{P}_A$ for $\chi \in X$. Since c is a homomorphism and $\mathcal{P}_A$ is a biextension, there are canonical isomorphisms of invertible $\mathcal{O}_S$-modules $b^*\mathcal{O}_{\chi_1+\chi_2} \cong b^*\mathcal{O}_{\chi_1} \otimes b^*\mathcal{O}_{\chi_2}$, under which $\tau(z,\chi_1+\chi_2)$ corresponds to $\tau(z,\chi_1)\otimes\tau(z,\chi_2)$,

because τ is a trivialisation of the biextension $(c^\vee \times c)^*\mathcal{P}_{A,\eta}^{\otimes -1}$. The module $I_{z,\chi}$ is the image of the integral structure $b^*\mathcal{O}_\chi$ under the isomorphism with K induced by $\tau(z,\chi)$; hence $I_{z,\chi_1+\chi_2} = I_{z,\chi_1} \cdot I_{z,\chi_2}$ as R-submodules of K, and $I_{z,\phi'(z)} = I_{z,\phi(z)}^{2m+1}$. Now $\mathcal{O}_{S,s}$ is a normal noetherian local domain (S is normal), and $J = I_{z,\phi(z)} \cdot \mathcal{O}_{S,s}$ is an invertible fractional ideal of it, hence $J = a\,\mathcal{O}_{S,s}$ for some $a \in K^\times$, and $I_{z,\phi'(z)} \cdot \mathcal{O}_{S,s} = a^{2m+1}\mathcal{O}_{S,s}$. If a is a unit of $\mathcal{O}_{S,s}$, so is a^{2m+1}. Conversely, if $u = a^{2m+1}$ is a unit of $\mathcal{O}_{S,s}$, then a and a^{-1} are roots of the monic polynomials $T^{2m+1} - u$ and $T^{2m+1} - u^{-1}$ over $\mathcal{O}_{S,s}$, so both lie in the normal domain $\mathcal{O}_{S,s}$ and a is a unit. Hence $z \in Y_s$ if and only if $z \in Y'_s$.

Conclusion. $\widetilde{G}(\operatorname{Spec}\mathcal{O}_{S,s})$ is a subgroup of $\widetilde{G}(K)$ ($\widetilde{G}$ is separated over S, so the restriction map is injective), and ι is a homomorphism; so $Y_s = Y'_s = \iota^{-1}(\widetilde{G}(\operatorname{Spec}\mathcal{O}_{S,s}))$ is a subgroup of Y. The isomorphism $G \cong G'$ restricts to an isomorphism $G[n]_s \cong G'[n]_s$ of group schemes over $\kappa(s)$; composing the exact sequence for G' with it gives the exact sequence of the statement. □

Lemma 32.25. *Let R, I, S be as in Theorem B.109, let $\mathfrak{d} = (A, X, Y, \phi, c, c^\vee, \widetilde{\mathcal{L}}, \tau, \psi)$ be an object of $\mathrm{DD}_{\mathrm{ample}}$ with X and Y constant, $\pi\colon \widetilde{G} \to A$ the extension defined by c, and $(G, \mathcal{L}) = M_{\mathrm{ample}}(\mathfrak{d})$. Then:*

(a) there are an ample invertible sheaf $\mathcal{M}$ on A with a cubical isomorphism $\widetilde{\mathcal{L}} \cong \pi^*\mathcal{M}$, through which ψ is read as a trivialisation of $\mathcal{M}_\eta(c^\vee(y))^{-1}$ at each $y \in Y$, and an integer $m \geq 1$ such that

$$(A,\ \mathcal{M}^{\otimes m+1} \otimes [-1]^*\mathcal{M}^{\otimes m},\ X,\ Y,\ (2m+1)\phi,\ c,\ c^\vee,\ \tau,\ \psi^{m+1}[-1]^*\psi^m)$$

is a split object that has a relatively complete model;

(b) for every split object $(A, \mathcal{M}', X, Y, \phi', c, c^\vee, \tau, \psi')$ with the same A, X, Y, c, $c^\vee$ and τ, and every relatively complete model of it, G is isomorphic, as a group scheme over S, to the attached Mumford quotient.

Proof. The sheaf $\mathcal{M}$ and a split object $\mathfrak{e}$ with image isomorphic to $\mathfrak{d}$. Since X is constant, the torus T of $\widetilde{G}$, with character group X, is split, and $0 \to T \xrightarrow{i} \widetilde{G} \xrightarrow{\pi} A \to 0$ is an exact sequence of smooth commutative group schemes with connected fibres over the normal scheme S. As X and Y are constant over S, the positivity conditions of $\mathfrak{d}$ (Theorem B.109(2)) are read over S itself, and there is a cubical trivialisation s of $i^*\widetilde{\mathcal{L}}$ for which ψ satisfies the condition on ψ. By Theorem B.112(1), s determines a cubical invertible sheaf $\mathcal{M}$ on A and a cubical isomorphism $h\colon \widetilde{\mathcal{L}} \to \pi^*\mathcal{M}$, and $\mathcal{M}$ is ample. (The choice of s matters: through another cubical trivialisation the sections $\psi(y)$ are multiplied by $\tau(y,\chi_0)^{\pm 1}$ for a character $\chi_0 \in X$, and need not all extend over S.) Since $\pi \circ \iota = c^\vee$, h carries $\psi(y)$ to a trivialisation of $\mathcal{M}_\eta(c^\vee(y))^{-1}$, again written $\psi(y)$. Put $\mathfrak{e} = (A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$. Its image $\bar{\mathfrak{e}} = (A, X, Y, \phi, c, c^\vee, \pi^*\mathcal{M}, \tau, \psi)$ is obtained from $\mathfrak{d}$ by replacing $\widetilde{\mathcal{L}}$ by the isomorphic cubical sheaf $\pi^*\mathcal{M}$ and transporting ψ by h; every condition defining $\mathrm{DD}_{\mathrm{ample}}$ is invariant under this, so $\bar{\mathfrak{e}}$ is an object of $\mathrm{DD}_{\mathrm{ample}}$, isomorphic to $\mathfrak{d}$ by h. Read through the identity of $\pi^*\mathcal{M}$, the transported ψ is ψ read through h, which satisfies the condition on ψ by the choice of s; so $\mathfrak{e}$ is a split object. In particular $I_y \subseteq R$ for every $y \in Y$, and for every $n \geq 1$, $I_y \subseteq I^n$ for all but finitely many y; also $I_{y,\phi(y)} \subseteq R$ by the condition on τ.

Item (a). By Theorem B.112(2)(a) applied to $\mathfrak{d}$ and (2)(b) applied to $\mathfrak{e}$ and the star St_0 of (3), there are integers $n_0, n_1 > 0$ such that $\mathfrak{d}^{(n)}$ is an object of $\mathrm{DD}_{\mathrm{ample}}$ for $n \geq n_0$ and $\mathfrak{e}^{(n)}$ satisfies Condition 4.5.1.6 for St_0 for $n \geq n_1$. Let $m = \max(n_0, n_1)$ and $\mathcal{M}_m = \mathcal{M}^{\otimes m+1} \otimes [-1]^*\mathcal{M}^{\otimes m}$. The sheaf $[-1]^*\mathcal{M}$ is ample, $[-1]$ being an automorphism of A, so $\mathcal{M}_m$ is ample as a tensor product of ample invertible sheaves. Since $\pi \circ [-1] = [-1] \circ \pi$, the cubical isomorphism

$h^{\otimes m+1} \otimes [-1]^* h^{\otimes m} \colon \widetilde{\mathcal{L}}^{\otimes m+1} \otimes [-1]^* \widetilde{\mathcal{L}}^{\otimes m} \to \pi^* \mathcal{M}_m$ carries $\psi^{m+1}[-1]^*\psi^m$ of $\mathfrak{d}^{(m)}$ to that of $\mathfrak{e}^{(m)}$; so the image of $\mathfrak{e}^{(m)}$ is isomorphic to the object $\mathfrak{d}^{(m)}$ of $\mathrm{DD}_{\mathrm{ample}}$ and is itself an object of $\mathrm{DD}_{\mathrm{ample}}$. The module defined by $\psi^{m+1}[-1]^*\psi^m$ at y is $I_y^{m+1} I_{-y}^m = I_y \cdot I_{y,\phi(y)}^m$, because the compatibility of ψ with τ in Theorem B.109(2) at $(y, -y)$, with $\psi(0) = 1$ and bimultiplicativity of τ, gives $\psi(y)\psi(-y) = \tau(y, \phi(y))$ through isomorphisms of invertible sheaves on S, i.e. $I_y I_{-y} = I_{y,\phi(y)}$. As $I_{y,\phi(y)} \subseteq R$, this module lies in R for every y and in I^n whenever $I_y \subseteq I^n$; so $\psi^{m+1}[-1]^*\psi^m$, read through the identity of $\pi^*\mathcal{M}_m$, satisfies the condition on ψ. With X, Y constant and $\mathcal{M}_m$ ample, $\mathfrak{e}^{(m)}$ is therefore a split object; it satisfies Condition 4.5.1.6 for St_0, so it has a relatively complete model by Theorem B.112(2)(c). This is (a), with $\mathcal{M}$ and m.

Item (b). $\mathfrak{e}^{(m)}$ lies in $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split},*}$; let $(G_m, \mathcal{L}_m) = M_{\mathrm{ample}}^{\mathrm{split},*}(\mathfrak{e}^{(m)})$, so that G_m is the Mumford quotient attached to the relatively complete model of Theorem B.112(2)(c). By Theorem B.112(3) the image $\bar{\mathfrak{e}}^{(m)}$ lies in $\mathrm{DD}_{\mathrm{ample}}^*$ and $M_{\mathrm{ample}}(\bar{\mathfrak{e}}^{(m)}) \cong M_{\mathrm{ample}}^*(\bar{\mathfrak{e}}^{(m)}) \cong (G_m, \mathcal{L}_m)$ in $\mathrm{DEG}_{\mathrm{ample}}$, whose isomorphisms are isomorphisms of group schemes over S respecting the sheaves ([154], Def. 4.2.1.1); so the semiabelian scheme of $M_{\mathrm{ample}}(\bar{\mathfrak{e}}^{(m)})$ is isomorphic to G_m as a group scheme over S. The objects $\mathfrak{d}$ and $\bar{\mathfrak{e}}^{(m)}$ of $\mathrm{DD}_{\mathrm{ample}}$ have the same $(A, X, Y, c, c^\vee, \tau)$, so by Theorem B.112(4) G is isomorphic to the semiabelian scheme of $M_{\mathrm{ample}}(\bar{\mathfrak{e}}^{(m)})$, hence to G_m, as a group scheme over S. Finally let $\mathfrak{f} = (A, \mathcal{M}', X, Y, \phi', c, c^\vee, \tau, \psi')$ be a split object with a relatively complete model and $G_{\mathfrak{f}}$ the attached Mumford quotient. The split objects $\mathfrak{e}^{(m)}$ and $\mathfrak{f}$ have the same $(A, X, Y, c, c^\vee, \tau)$, so by Theorem B.112(5) $G_m \cong G_{\mathfrak{f}}$ as group schemes over S. Composing, $G \cong G_{\mathfrak{f}}$ as group schemes over S. □

33 Good origins

This section constructs the root towers of Proposition 31.14 together with good origins (Proposition 33.1), the existence clause (Gx) of that proposition. A root model of a component is an arithmetic variety in a multiprojective space whose coordinate line bundles are built from the root lines, with metrics. Its origin is the flag bundle of the Hodge bundles over a model of the component, whose root lines are the graded pieces of the Hodge flags, with an analytic system on its complex points. The analytic system records the Hodge metrics of the root lines, which are singular along the boundary, as weights with log-log singularities, as for Mumford's good metrics [188], together with their regularisations (Go1). The section proves that such origins exist for every finite set of bad primes, with root models at level four (Proposition 33.2), and that they are good: the weights are log-log singular and plurisubharmonic, they can be regularised globally (Theorem 33.22), and the flag volume form is a positive volume form whose total mass is given by the complex uniformisation (Lemma 33.34).

The external inputs are the Hodge norm estimates of Kashiwara and of Cattani, Kaplan and Schmid near a normal crossings boundary, the nilpotent orbit theorem, resolution of singularities [131], [148], and Lan's complex uniformisation of the PEL moduli scheme [154]. The subsections follow the construction: reference structures and Hodge bases (33.1), flag models realising their analytic systems (33.2), positivity of the gap line and flag volumes (33.3), and the stage root models at level four (33.4).

33.1 Reference structures and Hodge bases

The subsection states the two results of the section that the assembly uses, Proposition 33.1 and Proposition 33.2, and builds the first layer of the construction. At each component the reference structure of Proposition 33.3 is a degree-one reference root model with its analytic system, a flat determinant, immersive coordinates and a big gap line. It lives over a Hodge base: a cover

of the component with smooth generic fibre and simple normal crossing boundary on which the de Rham lattices become Hodge bundles, stable under dominant pull-back (Proposition 33.4, Proposition 33.9). The Plücker data of Definition 33.7 then embed the flag bundle of the Hodge bundles in a multiprojective space (Lemma 33.8).

Proposition 33.1. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p, q \geq 1$, and let $\mathcal{Y}$ be an integral model of* Sh *over $\mathcal{O}_F$ with uniformised degenerations whose components split. Then there are a root tower $\mathcal{T}$ of type $(\Phi_{\mathsf{H}}, p+q, p)$ with a geometric realisation and a good origin of the realised tower over $\mathcal{Y}$ immersive at level four.*

Proof. Put $\Phi = \Phi_{\mathsf{H}}$, $n = p + q$ and $\nu(\tau, i) = 1 - i$. As $\mathcal{Y}$ has uniformised degenerations, $\mathsf{L} = (N, \mathfrak{n}, A)$ has $A = 1$; so by Lemma 31.15 the set $\mathrm{Irr}(\mathcal{Y})$ of irreducible components of $\mathcal{Y}$ is finite and non-empty. For every $c \in \mathrm{Irr}(\mathcal{Y})$, Proposition 33.2 gives fixed data at c (for ν, with $\mathcal{Y}_c = c$), a reference root model $\mathcal{X}_c^{\mathrm{ref}}$ with a root origin on them for $S = \varnothing$, deg = 1, no resolution and $\mathcal{O}(e_*)$ ample on $\mathcal{X}_c^{\mathrm{ref}} \otimes \mathbb{Q}$, and the clauses (Cz1)–(Cz3).

Let Bd be the set of primes dividing $2\,\mathrm{disc}(E)\,N'$; it is finite (the integer is non-zero) and contains 2. For an integer $B \geq 2$ let S_B be the set of primes ℓ with $\ell \leq B$ or $\ell \in \mathrm{Bd}$; it is finite and contains every prime dividing $\mathrm{disc}(E)$, which divides $2\,\mathrm{disc}(E)\,N'$. So (Cz3) with $S = S_B$ gives, for every $B \geq 2$ and c, a root model $\mathcal{X}_{B,c}$ of type (Φ, n) with a root origin on the fixed data at c for S_B, with $R(\ell) = 4$ for $\ell \in S_B$ and smooth generic fibre.

Let $\mathcal{T}$ be the root tower of type (Φ, n, p) (here $1 \leq p < n$, as $q \geq 1$) with $\mathrm{Cp} = \mathrm{Irr}(\mathcal{Y})$, bad primes Bd, $\mathrm{vol}(c) = 1$, $\widehat{\deg}_{B,c} = 0$, $h^{\mathrm{cd}} = 0$ and the Frobenius labelling with constant value $(a_0, +)$ for a fixed $a_0 \in \Phi$; and let its geometric realisation be $\nu_{\mathcal{T}} = \nu$ (rational entries) with the root models $\mathcal{X}_{B,c}$. These choices form a tower origin of $\mathcal{T}$ over $\mathcal{Y}$: the cut is $\kappa = p$; Bd is the set of primes dividing $2\,\mathrm{disc}(E)\,N'$; there are fixed data at every $c \in \mathrm{Cp}$, read with $\nu_{\mathcal{T}} = \nu$, and $c \mapsto \mathcal{Y}_c = c$ is a bijection of Cp onto $\mathrm{Irr}(\mathcal{Y})$; every $\mathcal{X}_{B,c}$ carries a root origin on the fixed data at c for S_B. It is a good origin: every $\mathcal{X}_{B,c} \otimes \mathbb{Q}$ is smooth (Cz3); the regularised metrics of the fundamental indices (τ, n) have pluriharmonic local potentials (Cz1); the reference models $\mathcal{X}_c^{\mathrm{ref}}$ are as required. It is immersive at level four by (Cz2) and since $R(\ell) = 4$ for all B, c and $\ell \in S_B$. □

Proposition 33.2. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p, q \geq 1$, put $\Phi = \Phi_{\mathsf{H}}$, $n = p + q$ and $\nu(\tau, i) = 1 - i$ ($\tau \in \Phi$, $1 \leq i \leq n$), and let $\mathcal{Y}$ be an integral model of* Sh *over $\mathcal{O}_F$ with uniformised degenerations whose components split. Read (Go1)–(Go4) with an irreducible component $\mathcal{Y}_0$ of $\mathcal{Y}$ in place of c (so $\mathcal{Y}_c = \mathcal{Y}_0$), ν in place of $\nu_{\mathcal{T}}$, $\kappa = p$, and a finite set S of primes in place of S_B. Then for every irreducible component $\mathcal{Y}_0$ of $\mathcal{Y}$ there are fixed data at $\mathcal{Y}_0$ and a root model $\mathcal{X}^{\mathrm{ref}}$ of type (Φ, n) with a root origin on these fixed data for $S = \varnothing$, with* deg = 1*, a realisation that needs no resolution, and $\mathcal{O}(e_*)$ ample on $\mathcal{X}^{\mathrm{ref}} \otimes \mathbb{Q}$, such that:*

(Cz1) Flat determinant *for every $\tau \in \Phi$ and every $0 < t \leq t_0$ the regularised metric of the fundamental index (τ, n) of the analytic system of the fixed data has pluriharmonic local potentials;*

(Cz2) Immersive coordinates *the map* pt *of the fixed data is, near every point, the projectivisation of a holomorphic map to $\mathbb{C}^{m+1} \smallsetminus \{0\}$ whose affine charts have injective differential;*

(Cz3) Stages at level four *for every finite set S of primes containing every prime dividing* $\mathrm{disc}(E)$ *there is a root model $\mathcal{X}_S$ of type (Φ, n) with a root origin on the same fixed data for S, with $R(\ell) = 4$ for every $\ell \in S$, whose generic fibre $\mathcal{X}_S \otimes \mathbb{Q}$ is smooth (so its realisation needs no resolution).*

Proof. Fix an irreducible component $\mathcal{Y}_0$ of $\mathcal{Y}$.

Step 0: the component meets the Shimura variety. $\mathcal{Y}$ is flat over $\mathcal{O}_F$, so the generic point of $\mathcal{Y}_0$ lies over the generic point of $\operatorname{Spec}\mathcal{O}_F$, i.e. in the generic fibre $\mathcal{Y}_F$, where it is the generic point of an irreducible component; the dense open subset $j(\mathrm{Sh}_F)$ of $\mathcal{Y}_F$ contains it. So $\mathcal{Y}_0$ meets $j(\mathrm{Sh}_F)$.

Step 1: the reference structure. By Proposition 33.3, applied to $\mathcal{Y}_0$ (which meets $j(\mathrm{Sh}_F)$ by Step 0), there are fixed data at $\mathcal{Y}_0$ with $k_c^A = q_c' = 1$, $n_c = \nu$, $s_c^E = 1$, and a root model $\mathcal{X}^{\mathrm{ref}}$ of type (Φ, n) with $k_{\mathcal{X}^{\mathrm{ref}}} = 1$ and data of a root origin on these fixed data for $S = \varnothing$ with $\deg = 1$, satisfying (Go3)(1)–(5) and (Go4), and the clauses (Rs1)–(Rs4) there.

Step 2: the effective weight section. By (Rs4) the hypotheses of Lemma 33.39 hold for these data. It gives $k \geq 1$ and $s \in \Omega_c^\times$ such that, after replacing k_c^A, q_c', n_c, s_c^E by $k, 1, k\nu, s$ and $k_{\mathcal{X}^{\mathrm{ref}}}$ by k, the data are fixed data at $\mathcal{Y}_0$ (with $n_c = k_c^A q_c' \nu$, i.e. read for ν) and a root origin of $\mathcal{X}^{\mathrm{ref}}$ on them for $S = \varnothing$, still with $\deg = 1$. The replacement changes neither the scheme $\mathcal{X}^{\mathrm{ref}}$ with its multidegrees and metrics, nor the analytic system, the map pt of the fixed data and the realisation. Hence (Rs1)–(Rs3) still hold: the realisation needs no resolution (the generic fibre is smooth) and is bijective, $\mathcal{O}(e_*)$ is ample on $\mathcal{X}^{\mathrm{ref}} \otimes \mathbb{Q}$, (Cz1) is (Rs2) and (Cz2) is (Rs3).

Step 3: the stages. Let S be a finite set of primes containing every prime dividing $\mathrm{disc}(E)$. The fixed data and the root origin of Step 2 satisfy the hypotheses of Proposition 33.40 ($\deg = 1$, smooth generic fibre and bijective realisation by (Rs1)); it gives a root model $\mathcal{X}_S$ of type (Φ, n) with a root origin on the same fixed data for S, with $R(\ell) = 4$ for $\ell \in S$ and smooth generic fibre. This is (Cz3).

So the fixed data and $\mathcal{X}^{\mathrm{ref}}$ of Step 2 have all the stated properties. □

Proposition 33.3. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p, q \geq 1$, put $\Phi = \Phi_{\mathsf{H}}$, $n = p + q$ and $\nu(\tau, i) = 1 - i$, and let $\mathcal{Y}$ be an integral model of* Sh *over $\mathcal{O}_F$ with uniformised degenerations whose components split. Read (Go1)–(Go4) with an irreducible component $\mathcal{Y}_0$ of $\mathcal{Y}$ in place of c, ν in place of $\nu_{\mathcal{T}}$, $\kappa = p$ and $S = \varnothing$. Let $\mathcal{Y}_0$ be an irreducible component of $\mathcal{Y}$ meeting $j(\mathrm{Sh}_F)$. Then there are fixed data at $\mathcal{Y}_0$ with $k_c^A = q_c' = 1$, $n_c = \nu$, $s_c^E = 1$, and a root model $\mathcal{X}^{\mathrm{ref}} \subseteq \mathbb{P} = \prod_{k \in K} \mathbb{P}(\mathbb{Z}^{I_k})$ of type (Φ, n) with $k_{\mathcal{X}^{\mathrm{ref}}} = 1$, together with data of a root origin of $\mathcal{X}^{\mathrm{ref}}$ on these fixed data for $S = \varnothing$, with* $\deg = 1$*, satisfying (Go3)(1)–(5) and (Go4), such that:*

(Rs1) Smooth generic fibre *$\mathcal{X}^{\mathrm{ref}} \otimes \mathbb{Q}$ is smooth over $\mathbb{Q}$ (so the realisation needs no resolution), $\mathcal{O}(e_*)$ is ample on it, and the realisation $\pi\colon \mathcal{X}^{\mathrm{ref}}(\mathbb{C}) \to X_c^{\mathrm{an}}$ is bijective;*

(Rs2) Flat determinant *for every $\tau \in \Phi$ and every $0 < t \leq t_0$ the regularised metric of the fundamental index (τ, n) of the analytic system of the fixed data has pluriharmonic local potentials;*

(Rs3) Immersive coordinates *the map* pt *of the fixed data is, near every point, the projectivisation of a holomorphic map to $\mathbb{C}^{m+1} \smallsetminus \{0\}$ whose affine charts have injective differential;*

(Rs4) Big gap line *with $e_\nu = -\sum_{r \in \mathrm{Rt}} \nu(r) e_r$, for some irreducible component X_1 of $\mathcal{X}^{\mathrm{ref}} \otimes_{\mathbb{Z}} \mathbb{C}$ the line bundle $\mathcal{O}(e_\nu)|_{X_1}$ is nef and* $\deg\bigl(c_1(\mathcal{O}(e_\nu)|_{X_1})^D \cap [X_1]\bigr) > 0$.

Proof. **The flag model at $S = \varnothing$.** As $\mathcal{Y}_0$ meets $j(\mathrm{Sh}_F)$, Proposition 33.4 gives a cover $\mathcal{X}^{\mathrm{b}}$ of $\mathcal{Y}$ for $\mathcal{Y}_c = \mathcal{Y}_0$, with de Rham Hodge bundles $\mathcal{V}_s$. Lemma 33.8, case (ii), applied to $\mathcal{Y}_0$ and $\mathcal{X}^{\mathrm{b}}$, gives the flag model $\mathcal{X}' = \prod_s \mathrm{Fl}(\mathcal{V}_s)$ with its closed immersion into a multiprojective setting, representing coordinates, multidegrees and the data of (Go3)(1)–(4), with $\deg = 1$; and

Proposition 33.12 in case (ii) gives the cone metrics $h_r(t)$, h_a^{ref} and the analytic system Sys' on $\mathcal{X}'(\mathbb{C})$.

The reference root model. Lemma 33.27, applied to these data, gives the root model $\mathcal{X}^{\mathrm{ref}} \subseteq \mathbb{P}$ (the graph of the Segre morphism on $\mathcal{X}'$) of type (Φ, n) with $k_{\mathcal{X}^{\mathrm{ref}}} = 1$, fixed data at $\mathcal{Y}_0$ with $k_c^A = q_c' = 1$, $n_c = \nu$, $s_c^E = 1$, and a root origin of $\mathcal{X}^{\mathrm{ref}}$ on them for $S = \varnothing$ with $\deg = 1$, satisfying (Go3)(1)–(5) and (Go4). Its items give the clauses: (Rs1) is item 1 ($\mathcal{X}^{\mathrm{ref}} \otimes \mathbb{Q}$ smooth, $\mathcal{O}(e_*)$ very ample, hence ample on $\mathcal{X}^{\mathrm{ref}} \otimes \mathbb{Q}$) together with item 2 (the realisation is the identity of $\mathcal{X}^{\mathrm{ref}}(\mathbb{C}) = X_c^{\mathrm{an}}$, bijective); (Rs2) is item 3; (Rs3) is item 4.

The gap line. The model $\mathcal{Y}$ has uniformised degenerations, so the level of $\mathfrak{u}$ is $(N, \mathfrak{n}, 1)$ and good at every prime (Definition 31.3), and the components of Sh_F split by hypothesis. So Lemma 33.28 applies to $\mathcal{X}^{\mathrm{ref}}$ with these fixed data and root origin: $\mathcal{O}(e_\nu)$ is nef on every irreducible component of $\mathcal{X}^{\mathrm{ref}} \otimes_{\mathbb{Z}} \mathbb{C}$ and has positive top degree on one of them, X_1. This is (Rs4). □

Proposition 33.4. *Let $\mathfrak{u}$ be a unitary Shimura datum, $\mathcal{Y}$ an integral model of* Sh *over $\mathcal{O}_F$, $\mathcal{Y}_c$ an irreducible component of $\mathcal{Y}$ meeting $j(\mathrm{Sh}_F)$ with generic point η_c, and $Z_c = j^{-1}(\mathcal{Y}_c)$. Then there is a cover $\mathcal{X}^{\mathrm{b}}$ of $\mathcal{Y}$ with structure map f^{b}, projective over* $\operatorname{Spec}\mathbb{Z}$*, such that:*

1. *f^{b} maps the generic point of $\mathcal{X}^{\mathrm{b}}$ to η_c and induces an isomorphism $\mathbf{k}(Z_c) = \mathbf{k}(\eta_c) \to \mathbf{k}(\mathcal{X}^{\mathrm{b}})$;*
2. *$\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$, and the complement of the abelian locus in $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ is the support of a simple normal crossing divisor;*
3. *the de Rham lattices of $\mathcal{X}^{\mathrm{b}}$ form Hodge bundles $\mathcal{V}_s$ ($S = \varnothing$), with* $\mathrm{Lie}(G_{\mathcal{X}^{\mathrm{b}}})/\mathcal{V}_s$ *locally free; and for every cover $\mathcal{X}''$ of $\mathcal{Y}$ with a dominant morphism $\phi\colon \mathcal{X}'' \to \mathcal{X}^{\mathrm{b}}$ such that its structure map is $f^{\mathrm{b}} \circ \phi$, the de Rham lattices of $\mathcal{X}''$ form the Hodge bundles $\mathcal{O}_{\mathcal{X}''}\mathcal{V}_s \cong \phi^*\mathcal{V}_s$, with* $\mathrm{Lie}(G_{\mathcal{X}''})/\mathcal{O}_{\mathcal{X}''}\mathcal{V}_s$ *locally free.*

Proof. By Lemma 33.5 with $\Omega = \mathbf{k}(Z_c)$ there is a cover $\mathcal{X}_3$ of $\mathcal{Y}$ with structure map f_3 and function field $\mathbf{k}(Z_c)$, projective over $\mathbb{Z}$, whose generic point maps to η_c inducing the identity of $\mathbf{k}(Z_c)$, such that $\mathcal{X}_{3,\mathbb{Q}}$ is smooth and the complement of the abelian locus in it is the support of a simple normal crossing divisor. Let P be the product of the primes dividing $\mathrm{disc}(E)$. By Lemma 33.6, item 1, applied to $\mathcal{X}_3$, there are a cover $\mathcal{X}^{\mathrm{b}}$, projective over $\mathbb{Z}$, and a projective birational morphism $b\colon \mathcal{X}^{\mathrm{b}} \to \mathcal{X}_3$, an isomorphism over $\mathcal{X}_3[1/P]$, with structure map $f^{\mathrm{b}} = f_3 \circ b$, whose de Rham lattices form Hodge bundles $\mathcal{V}_s$ with $\mathrm{Lie}(G_{\mathcal{X}^{\mathrm{b}}})/\mathcal{V}_s$ locally free.

Item 1: b is birational, so $\mathbf{k}(\mathcal{X}^{\mathrm{b}}) = \mathbf{k}(\mathcal{X}_3) = \mathbf{k}(Z_c)$ and the generic point maps to η_c. Item 2: $\mathcal{X}_{3,\mathbb{Q}} \subseteq \mathcal{X}_3[1/P]$, so b restricts to an isomorphism $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}} \to \mathcal{X}_{3,\mathbb{Q}}$; as $f^{\mathrm{b}} = f_3 \circ b$, it maps the abelian locus $(f^{\mathrm{b}})^{-1}(j(\mathrm{Sh}_F))$ of $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ onto that of $\mathcal{X}_{3,\mathbb{Q}}$, and item 2 follows from the corresponding property of $\mathcal{X}_3$. Item 3: the first half is the conclusion of Lemma 33.6, item 1, just quoted; the second half is Lemma 33.6, item 2, for $\mathcal{X}' = \mathcal{X}^{\mathrm{b}}$ (its hypotheses: $\mathcal{X}''$ a cover, ϕ dominant, structure map $f_3 \circ b \circ \phi$; the function field of $\mathcal{X}''$ contains $\mathbf{k}(\mathcal{X}^{\mathrm{b}})$ via ϕ). □

Lemma 33.5. *Let $\mathfrak{u}$ be a unitary Shimura datum, $\mathcal{Y}$ an integral model of* Sh *over $\mathcal{O}_F$, $\mathcal{Y}_c$ an irreducible component of $\mathcal{Y}$ meeting $j(\mathrm{Sh}_F)$ with generic point η_c, $Z_c = j^{-1}(\mathcal{Y}_c)$, and Ω a finite extension of $\mathbf{k}(Z_c) = \mathbf{k}(\eta_c)$. Then there is a cover $\mathcal{X}$ of $\mathcal{Y}$ with function field Ω, projective over* $\operatorname{Spec}\mathbb{Z}$*, whose generic point maps to η_c (inducing the given inclusion $\mathbf{k}(\eta_c) \subseteq \Omega$), such that $\mathcal{X}_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$ and the complement of the abelian locus in $\mathcal{X}_{\mathbb{Q}}$ is the support of a simple normal crossing divisor.*

Proof. **A normal model with function field Ω.** Give $\mathcal{Y}_c$ its reduced structure; it is an integral closed subscheme of $\mathcal{Y}$, projective over $\mathbb{Z}$. Since j is an open immersion and $\mathcal{Y}_c$ meets $j(\mathrm{Sh}_F)$, $j(Z_c)$ is a non-empty open subset of $\mathcal{Y}_c$, so it contains η_c and $\mathbf{k}(Z_c) = \mathbf{k}(\mathcal{Y}_c)$. Let $\mathcal{X}_1$ be the normalisation of $\mathcal{Y}_c$ in Ω. As $\mathcal{Y}_c$ is of finite type over $\mathbb{Z}$, hence excellent, $\mathcal{X}_1 \to \mathcal{Y}_c$ is finite; so $\mathcal{X}_1$ is integral, normal and projective over $\mathbb{Z}$ with function field Ω, and with $\mathcal{X}_1 \to \mathcal{Y}_c \subseteq \mathcal{Y}$ it is a cover of $\mathcal{Y}$ whose generic point maps to η_c (Ω is finitely generated over $\mathbf{k}(\eta_c)$).

A smooth generic fibre with simple normal crossing boundary. $\mathcal{X}_{1,\mathbb{Q}}$ is an integral quasi-projective variety over $\mathbb{Q}$, and the abelian locus A_1 of $\mathcal{X}_1$ is an open subset of it containing the generic point. By Theorem B.142(A) there is a projective birational morphism $\Pi_1 \colon R_1 \to \mathcal{X}_{1,\mathbb{Q}}$ with R_1 smooth. Let $C_1 = R_1 \smallsetminus \Pi_1^{-1}(A_1)$, a proper closed subset. If $C_1 = \varnothing$ put $R = R_1$, $\Pi = \Pi_1$. Otherwise let $I \subseteq \mathcal{O}_{R_1}$ be the ideal sheaf of C_1 with its reduced structure; it is non-zero on the integral scheme R_1, and Theorem B.142(B) gives a projective birational morphism $\Pi_2 \colon R \to R_1$ with R smooth such that $\Pi_2^* I \subseteq \mathcal{O}_R$ is the ideal sheaf of a simple normal crossing divisor Δ; its support is the zero set of $\Pi_2^* I$, i.e. $\Pi_2^{-1}(C_1)$. Put $\Pi = \Pi_1 \circ \Pi_2$, projective and birational. In both cases R is a smooth integral variety over $\mathbb{Q}$ and $R \smallsetminus \Pi^{-1}(A_1)$ is the support of a simple normal crossing divisor on R.

Spreading out. Choose a closed immersion $R \to \mathbb{P}^N \times \mathcal{X}_{1,\mathbb{Q}}$ over $\mathcal{X}_{1,\mathbb{Q}}$ (Π is projective), and let $\bar{R}$ be the closure of its image in $\mathbb{P}^N \times \mathcal{X}_1$ with the reduced structure. Since $\mathbb{P}^N \times \mathcal{X}_{1,\mathbb{Q}}$ is the open subscheme of $\mathbb{P}^N \times \mathcal{X}_1$ over $\operatorname{Spec}\mathbb{Q}$ in which R is closed, $\bar{R}$ is integral and $\bar{R}_{\mathbb{Q}} = R$. $\bar{R}$ is projective over $\mathcal{X}_1$, hence over $\mathbb{Z}$, and $\bar{R} \to \mathcal{X}_1$ is birational. Let $\mathcal{X}$ be the normalisation of $\bar{R}$, finite over $\bar{R}$ by excellence; normalisation commutes with localisation and R is smooth, hence normal, so $\mathcal{X}_{\mathbb{Q}} = R$. Thus $\mathcal{X}$ is integral, normal, projective over $\mathbb{Z}$, with function field Ω and a birational morphism $\phi \colon \mathcal{X} \to \mathcal{X}_1$ restricting to Π on $\mathcal{X}_{\mathbb{Q}}$; with the composite structure map it is a cover of $\mathcal{Y}$ whose generic point maps to η_c, with $G_{\mathcal{X}} = \phi^* G_{\mathcal{X}_1}$. Its abelian locus is $\phi^{-1}(A_1)$, whose part in $\mathcal{X}_{\mathbb{Q}} = R$ is $\Pi^{-1}(A_1)$; so $\mathcal{X}_{\mathbb{Q}}$ is smooth and the complement of the abelian locus in it is the support of a simple normal crossing divisor. □

Lemma 33.6. *Let $\mathfrak{u}$ be a unitary Shimura datum, $\mathcal{Y}$ an integral model of* Sh *over $\mathcal{O}_F$, P the product of the primes dividing* $\mathrm{disc}(E)$*, and $\mathcal{X}$ a cover of $\mathcal{Y}$ with structure map f and function field Ω, projective over* $\operatorname{Spec}\mathbb{Z}$*.*

1. *There are a cover $\mathcal{X}'$ of $\mathcal{Y}$, projective over* $\operatorname{Spec}\mathbb{Z}$*, and a projective birational morphism $b \colon \mathcal{X}' \to \mathcal{X}$, an isomorphism over $\mathcal{X}[1/P]$, with structure map $f \circ b$, such that for every label s there is a locally free $\mathcal{O}_{\mathcal{X}'}$-submodule $\mathcal{V}_s$ of the constant sheaf* Li_s*, of rank* $\mathrm{rk}(s)$*, with stalks* $\mathrm{dR}_{s,x'}$ *at every point x' of $\mathcal{X}'$, and* $\mathrm{Lie}(G_{\mathcal{X}'})/\mathcal{V}_s$ *is locally free. In particular the Hodge lattices of $\mathcal{X}'$ at $S = \varnothing$ are Hodge bundles.*

2. *Let $\mathcal{X}''$ be a cover of $\mathcal{Y}$ with structure map f'' and function field $\Omega'' \supseteq \Omega$, and $\phi \colon \mathcal{X}'' \to \mathcal{X}'$ a dominant morphism with $f'' = f \circ b \circ \phi$; identify* $\mathrm{Li}''_s = \mathrm{Lie}_s(G_{\mathcal{X}'',\Omega''})$ *with* $\Omega'' \otimes_\Omega \mathrm{Li}_s$*. Then the $\mathcal{O}_{\mathcal{X}''}$-span $\mathcal{O}_{\mathcal{X}''}\mathcal{V}_s \subseteq \mathrm{Li}''_s$ is locally free of rank* $\mathrm{rk}(s)$ *with stalks* $\mathrm{dR}_{s,x''}$ *at every point x'' of $\mathcal{X}''$, the natural map $\phi^*\mathcal{V}_s \to \mathcal{O}_{\mathcal{X}''}\mathcal{V}_s$ is an isomorphism, and* $\mathrm{Lie}(G_{\mathcal{X}''})/\mathcal{O}_{\mathcal{X}''}\mathcal{V}_s$ *is locally free.*

Proof. For a cover $\mathcal{X}_1$ with function field Ω_1, $W_1 = \mathrm{Lie}(G_{\mathcal{X}_1})$ is locally free of rank $N = \dim \mathrm{Li}$ ($G_{\mathcal{X}_1}$ is semiabelian, hence smooth), so its stalks inject into $W_1 \otimes \Omega_1 = \mathrm{Lie}(G_{\mathcal{X}_1,\Omega_1})$ and Lie_x is the stalk of W_1 at x. The generic point of $\mathcal{X}_1$ maps into $j(\mathrm{Sh}_F)$, so $\Omega_1 \supseteq F$ and $\mathrm{Lie}(G_{\mathcal{X}_1,\Omega_1}) = \bigoplus_s \mathrm{Lie}_s$ by (Fh1).

Step 1: the de Rham lattices away from P. Let $U = \mathcal{X}[1/P]$, a dense open subset (Ω has characteristic 0), and $R = \mathcal{O}_F[1/P]$. As $\mathrm{disc}(E)$ is invertible in R, $B = \mathcal{O}_E \otimes_{\mathbb{Z}} R$ is finite

étale over the normal domain R. The map $\epsilon\colon B \to \prod_{\sigma\in\mathrm{Hom}(E,F)} R$, $a\otimes r \mapsto (\sigma(a)r)_\sigma$ (note $\sigma(\mathcal{O}_E)\subseteq\mathcal{O}_F$) becomes the isomorphism $E\otimes_{\mathbb{Q}} F \to \prod_\sigma F$ of (Fh1) after $\otimes_R F$; B is torsion free, so ϵ is injective, and its target is integral over B and contained in the total ring of fractions $E\otimes_{\mathbb{Q}} F$ of B, so ϵ is bijective. Let $e_\sigma\in B$ be the idempotent with $\epsilon(e_\sigma)$ the σ-th unit vector. Every point $x\in U$ has $P\in\mathcal{O}_x^\times$, and $\mathcal{O}_x$ is an $\mathcal{O}_F$-algebra via f, hence an R-algebra; B acts on Lie_x through ι, and $\mathrm{Lie}_x = \bigoplus_\sigma e_\sigma\mathrm{Lie}_x$. On Li, e_σ is the projection onto the summand on which $\iota(a)$ acts by $\sigma(a)$, so $e_{\sigma_s}\mathrm{Li} = \mathrm{Li}_s$, and $e_{\sigma_s}\mathrm{Lie}_x = \mathrm{Lie}_x\cap\mathrm{Li}_s = \mathrm{dR}_{s,x}$ (if $v\in\mathrm{Lie}_x\cap\mathrm{Li}_s$ then $v = e_{\sigma_s}v$). Hence $\mathcal{V}_s^U = e_{\sigma_s}W|_U$ ($W=\mathrm{Lie}(G_\mathcal{X})$) is a direct summand of $W|_U$ with stalks $\mathrm{dR}_{s,x}$; its stalks are free, so it is locally free, of rank $\dim\mathrm{Li}_s = \mathrm{rk}(s)$ (U integral), and $W|_U/\mathcal{V}_s^U \cong \bigoplus_{\sigma\neq\sigma_s} e_\sigma W|_U$ is locally free.

Step 2: the Grassmannian closure. Let $\Gamma = \prod_s \mathrm{Gr}_{\mathrm{rk}(s)}(W)$, the fibre product over $\mathcal{X}$ over all labels; $\Gamma\to\mathcal{X}$ is projective. By Step 1 the subsheaves $\mathcal{V}_s^U\subseteq W|_U$ define a section $\gamma\colon U\to\Gamma|_U$. As $\Gamma\to\mathcal{X}$ is separated, $\gamma(U)$ is closed in $\Gamma|_U$ and $\gamma\colon U\to\gamma(U)$ is an isomorphism. Let $\mathcal{X}^\sharp$ be the closure of $\gamma(U)$ in Γ with its reduced structure: integral (the closure of an irreducible set), projective over $\mathcal{X}$ and hence over $\mathrm{Spec}\,\mathbb{Z}$, with $\mathcal{X}^\sharp\cap\Gamma|_U = \gamma(U)\cong U$. Let $\mathcal{X}'\to\mathcal{X}^\sharp$ be the normalisation, finite since $\mathcal{X}^\sharp$ is of finite type over $\mathbb{Z}$; so $\mathcal{X}'$ is projective over $\mathcal{X}^\sharp$, hence over $\mathcal{X}$ and over $\mathrm{Spec}\,\mathbb{Z}$. Let $b\colon\mathcal{X}'\to\mathcal{X}$ be the composite, projective. Over U, $\mathcal{X}^\sharp$ is $\gamma(U)\cong U$, which is normal ($\mathcal{X}$ is a cover, hence normal), so the normalisation is an isomorphism there and b is an isomorphism over U; in particular b is birational, $\mathcal{X}'$ is a normal integral scheme with function field Ω, and with $f\circ b$ it is a cover of $\mathcal{Y}$ with $G_{\mathcal{X}'} = b^*G_\mathcal{X}$.

Step 3: the de Rham lattices on $\mathcal{X}'$. By base change, $W' = \mathrm{Lie}(G_{\mathcal{X}'}) = b^*W$. The morphism $\mathcal{X}'\to\mathcal{X}^\sharp\subseteq\Gamma$ over $\mathcal{X}$ corresponds to locally free subsheaves $\mathcal{T}_s\subseteq W'$ of rank $\mathrm{rk}(s)$ with $W'/\mathcal{T}_s$ locally free. Over U the morphism is $\gamma\circ b$, so $\mathcal{T}_s|_{b^{-1}U} = b^*\mathcal{V}_s^U$ and $\mathcal{T}_s\otimes\Omega = e_{\sigma_s}\mathrm{Li} = \mathrm{Li}_s$. Let $x'\in\mathcal{X}'$. Then $\mathcal{T}_{s,x'}\subseteq W'_{x'}\cap(\mathcal{T}_s\otimes\Omega) = \mathrm{Lie}_{x'}\cap\mathrm{Li}_s$. Conversely, $W'_{x'}/\mathcal{T}_{s,x'}$ is free, so $\mathrm{Lie}_{x'} = \mathcal{T}_{s,x'}\oplus Q$ for a free submodule Q, and $\mathrm{Li} = \mathrm{Li}_s\oplus(\Omega Q)$ after tensoring with Ω. If $v\in\mathrm{Lie}_{x'}\cap\mathrm{Li}_s$, write $v = t+u$ with $t\in\mathcal{T}_{s,x'}$, $u\in Q$; then $u = v-t\in\mathrm{Li}_s\cap\Omega Q = 0$, so $v\in\mathcal{T}_{s,x'}$. Hence $\mathcal{V}_s = \mathcal{T}_s$, viewed in the constant sheaf Li_s, is locally free of rank $\mathrm{rk}(s)$ with stalks $\mathrm{dR}_{s,x'}$, and $W'/\mathcal{V}_s$ is locally free. This is item 1.

Item 2. We have $\mathrm{Lie}(G_{\mathcal{X}''}) = \phi^*W'$ (base change, $G_{\mathcal{X}''} = \phi^*G_{\mathcal{X}'}$ as $f'' = f\circ b\circ\phi$). Since $W'/\mathcal{V}_s$ is locally free, $\phi^*\mathcal{V}_s\to\phi^*W'$ is injective with locally free cokernel $\phi^*(W'/\mathcal{V}_s)$, and $\phi^*\mathcal{V}_s$ is locally free of rank $\mathrm{rk}(s)$. As ϕ is dominant, its generic fibre is $\Omega''\otimes_\Omega\mathcal{V}_s\otimes\Omega = \Omega''\otimes_\Omega\mathrm{Li}_s = \mathrm{Li}''_s$, and since $\phi^*\mathcal{V}_s$ is locally free on the integral scheme $\mathcal{X}''$ it injects into its generic fibre, with image the $\mathcal{O}_{\mathcal{X}''}$-span $\mathcal{O}_{\mathcal{X}''}\mathcal{V}_s$; so $\phi^*\mathcal{V}_s\to\mathcal{O}_{\mathcal{X}''}\mathcal{V}_s$ is an isomorphism. The argument of Step 3, with ϕ^*W', $\phi^*\mathcal{V}_s$, Ω'' and Li''_s in place of W', $\mathcal{T}_s$, Ω and Li_s, shows that its stalk at every x'' is $\mathrm{Lie}_{x''}\cap\mathrm{Li}''_s = \mathrm{dR}_{s,x''}$. □

Definition 33.7. Let $\mathcal{X}^\mathrm{b}$ be an integral scheme, projective over $\mathrm{Spec}\,\mathbb{Z}$, with function field Ω^b. For every label s let Li_s be an Ω^b-vector space of dimension $\mathrm{rk}(s)$ and $\mathcal{V}_s\subseteq\mathrm{Li}_s$ a locally free $\mathcal{O}_{\mathcal{X}^\mathrm{b}}$-submodule of the constant sheaf with $\mathcal{V}_s\otimes\Omega^\mathrm{b} = \mathrm{Li}_s$. Let $\beta\colon\mathcal{X}' = \prod_s\mathrm{Fl}(\mathcal{V}_s)\to\mathcal{X}^\mathrm{b}$ be the fibre product over $\mathcal{X}^\mathrm{b}$ of the schemes of full flags, assume $\mathcal{X}'$ integral with function field Ω', let $(\mathcal{V}'_{s,k})_{0\le k\le\mathrm{rk}(s)}$ be the universal full flag of $\beta^*\mathcal{V}_s$ and $(V_{s,k})_k$ its generic fibre, a full flag of $\mathrm{Li}_s\otimes_{\Omega^\mathrm{b}}\Omega'$, and for every root r let λ_r be a basis of the Ω'-line $(V_{s,k}/V_{s,k-1})^{\otimes\mathrm{sg}(r)}$ ($s=s(r)$, $k=\mathrm{st}(r)$).

A **Plücker datum** for $(\mathcal{X}^\mathrm{b},(\mathcal{V}_s),(\lambda_r))$ consists of:

- an ample invertible $\mathcal{O}_{\mathcal{X}^\mathrm{b}}$-submodule $\mathcal{H}\subseteq\Omega^\mathrm{b}$, an integer $m\ge1$ and, for every label s and $1\le i<\mathrm{rk}(s)$, a finite family of global sections $\sigma_{s,i,j}$ ($j\in I_{s,i}$) generating $(\wedge^i\mathcal{V}_s)^\vee\otimes\mathcal{H}^m$;
- for every label s a basis μ_s of the line $\wedge^{\mathrm{rk}(s)}\mathrm{Li}_s$, and a closed immersion of $\mathcal{X}^\mathrm{b}$ into a multi-

projective setting $\prod_{k\in K_\mathrm{b}}\mathbb{P}(\mathbb{Z}^{I_k})$, with representing coordinates $y_{k,j}\in\Omega^\mathrm{b}$, and multidegrees $e_{\mathcal{H}}, e_{\det,s}\in\mathbb{Z}^{K_\mathrm{b}}$, such that on every chart $\prod_k y_{k,c'_k}^{(e_{\mathcal{H}})_k}$ generates $\mathcal{H}^m$ and $\prod_k y_{k,c'_k}^{(e_{\det,s})_k}\mu_s$ generates $\wedge^{\mathrm{rk}(s)}\mathcal{V}_s$ (the **base factors**);

- vectors $v_{s,k}\in V_{s,k}$ $(1\le k\le \mathrm{rk}(s))$ whose class in $V_{s,k}/V_{s,k-1}$ is λ_r if $\mathrm{sg}(r)=1$ and the basis dual to λ_r if $\mathrm{sg}(r)=-1$, for the root r with $s(r)=s$, $\mathrm{st}(r)=k$.

Its derived objects are the following. Put $w_{s,0}=1$, $w_{s,k}=v_{s,1}\wedge\cdots\wedge v_{s,k}\in\wedge^k V_{s,k}$, and let $\rho_s\in\Omega'^{\times}$ with $w_{s,\mathrm{rk}(s)}=\rho_s\mu_s$. The **Plücker factors** are new indices $k_{s,i}$ (s a label, $1\le i<\mathrm{rk}(s)$), pairwise distinct and not in K_b; put $K=K_\mathrm{b}\sqcup\{k_{s,i}\}$, $I_{k_{s,i}}=I_{s,i}$, and $y_{k_{s,i},j}=\sigma_{s,i,j}(w_{s,i})\in\mathcal{H}^m\otimes\Omega'=\Omega'$; multidegrees in $\mathbb{Z}^{K_\mathrm{b}}$ are read in $\mathbb{Z}^K$ with entries 0 at the Plücker factors. The **fundamental multidegrees** and **fundamental constants** are

$$d_{s,0}=0,\quad d_{s,i}=e_{\mathcal{H}}-\mathbf{1}_{k_{s,i}}\ (1\le i<\mathrm{rk}(s)),\quad d_{s,\mathrm{rk}(s)}=e_{\det,s};$$
$$\varrho_{s,k}=1\ (k<\mathrm{rk}(s)),\quad \varrho_{s,\mathrm{rk}(s)}=\rho_s^{-1},$$

and for every root r, with $s=s(r)$, $k=\mathrm{st}(r)$,

$$e_r=\mathrm{sg}(r)\big(d_{s,k}-d_{s,k-1}\big),\qquad \varrho_r=\big(\varrho_{s,k}/\varrho_{s,k-1}\big)^{\mathrm{sg}(r)}. \tag{33.1}$$

The **fundamental root sections** are the maps $P\mapsto\varrho_{s,k}P(y)\,w_{s,k}$ from local sections P of $\mathcal{O}(d_{s,k})|_{\mathcal{X}'}$ to $\wedge^k V_{s,k}$, and the **root sections** the maps $P\mapsto\varrho_r P(y)\lambda_r$ from local sections of $\mathcal{O}(e_r)|_{\mathcal{X}'}$ to $\Omega'\lambda_r$. The **Plücker map** is the morphism $\mathcal{X}'\to\prod_{k\in K}\mathbb{P}(\mathbb{Z}^{I_k})$ whose components at K_b are the components of the immersion of $\mathcal{X}^\mathrm{b}$ composed with β, and whose component at $k_{s,i}$ sends a point with flag $(V'_k)_k$ of $\mathcal{V}_s$ to $\big(\sigma_{s,i,j}(\gamma)\big)_{j\in I_{s,i}}$ for a local generator γ of the line $\wedge^i V'_i$ (these values do not vanish simultaneously, since $\wedge^i V'_i$ is locally a direct summand of $\wedge^i\mathcal{V}_s$ and the $\sigma_{s,i,j}$ generate the dual); its representing coordinates are the $y_{k,j}$, $k\in K$.

Lemma 33.8. *Let $\mathfrak{u}=(E,\mathsf{H},\mathsf{U},p,q,\mathsf{L},\mathrm{Sh},F)$ be a unitary Shimura datum with $p,q\ge1$, $\Phi=\Phi_{\mathsf{H}}$, $n=p+q$, $\mathcal{Y}$ an integral model of* Sh *over $\mathcal{O}_F$, $\mathcal{Y}_c$ an irreducible component of $\mathcal{Y}$ meeting $j(\mathrm{Sh}_F)$, with $Z_c=j^{-1}(\mathcal{Y}_c)$ and $\Omega_c=\mathbf{k}(\mathrm{Fl}_{\mathfrak{u}}|_{Z_c})$ carrying the universal flags (of fixed data at c, the clauses (Go3)(1)–(4) involve only these entries; below they are read for them, so they hold for all fixed data with this component), and let either (i) S be a finite set of primes containing every prime dividing* $\mathrm{disc}(E)$ *and $\mathcal{X}^\mathrm{b}$ a cover as in Proposition 33.9 for $\mathcal{Y}_c$ and S, with Hodge bundles $\mathcal{V}_s$ at (S,R), $R\equiv4$; or (ii) $S=\varnothing$ and $\mathcal{X}^\mathrm{b}$ a cover as in Proposition 33.4 for $\mathcal{Y}_c$, with the Hodge bundles $\mathcal{V}_s$ formed by its de Rham lattices. Let*

$$\beta\colon\mathcal{X}'=\prod_s\mathrm{Fl}(\mathcal{V}_s)\longrightarrow\mathcal{X}^\mathrm{b}$$

be the fibre product over $\mathcal{X}^\mathrm{b}$ of the schemes of full flags of the $\mathcal{V}_s$, over all labels s. Then:

1. *with the structure map $f^\mathrm{b}\circ\beta$, $\mathcal{X}'$ is a cover of $\mathcal{Y}$, projective and flat over* $\mathrm{Spec}\,\mathbb{Z}$ *of absolute dimension $D+1$; $\mathcal{X}'_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$, and the complement of the abelian locus in $\mathcal{X}'_{\mathbb{Q}}$ is the support of a simple normal crossing divisor, the preimage under β of the complement of the abelian locus in $\mathcal{X}^\mathrm{b}_{\mathbb{Q}}$;*

2. *there are a field embedding $g\colon\Omega_c\to\mathbf{k}(\mathcal{X}')$ and a Plücker datum for $(\mathcal{X}^\mathrm{b},(\mathcal{V}_s),(g(\lambda_r))_r)$ (with* Li_s *the generic fibre of $\mathcal{V}_s$) whose Plücker map is a closed immersion of $\mathcal{X}'$ into the multiprojective setting $\prod_{k\in K}\mathbb{P}(\mathbb{Z}^{I_k})$, with representing coordinates $y_{k,j}$, such that with its multidegrees e_r and elements $\varrho_r\in\mathbf{k}(\mathcal{X}')^\times$ attached to the roots by its Plücker datum*

*(the displayed root formulas of Definition 33.7; $r \in \mathrm{Rt}$), the data $R(\ell) = 4$ ($\ell \in S$), $\deg = [\Omega^{\mathrm{b}} : \mathbf{k}(Z_c)]$ and suitable cone functions h_r^{sing} ($r \in \mathrm{Rt}$), (Go3)(1)–(4) hold on the fixed data at c for S, with $\delta = [F : \mathbb{Q}]\deg$; the integral flag of (Go3)(3) is the universal flag of the $\beta^*\mathcal{V}_s$; and for every label s and every $1 \le i < \mathrm{rk}(s)$ the index $k_{s,i} \in K$ of the datum (the **Plücker factor** of step i of s; distinct for distinct (s,i)) is such that, writing $\beta_{s,i} \colon \mathcal{X}' \to \mathcal{X}'_{s,i}$ for the projection onto the fibre product over $\mathcal{X}^{\mathrm{b}}$ in which the factor $\mathrm{Fl}(\mathcal{V}_s)$ is replaced by the scheme of chains $(V_k)_{k\neq i}$ of the same kind with the step i omitted:*

- *(a) for every $k \in K \smallsetminus \{k_{s,i}\}$ the component $\mathcal{X}' \to \mathbb{P}(\mathbb{Z}^{I_k})$ of the immersion factors through $\beta_{s,i}$;*
- *(b) $\beta_{s,i}$ is a $\mathbb{P}^1$-bundle (the lines in V_{i+1}/V_{i-1}), and on the fibre of $\beta_{s,i}$ over every point (with values in a field κ) the component at $k_{s,i}$ is an isomorphism onto a line of $\mathbb{P}(\mathbb{Z}^{I_{k_{s,i}}})_\kappa$;*
- *(c) for every root r, $(e_r)_{k_{s,i}} = -\mathrm{sg}(r)$ if $s(r) = s$ and $\mathrm{st}(r) = i$, $(e_r)_{k_{s,i}} = \mathrm{sg}(r)$ if $s(r) = s$ and $\mathrm{st}(r) = i+1$, and $(e_r)_{k_{s,i}} = 0$ otherwise;*

and for every label s and $0 \le k \le \mathrm{rk}(s)$:

- *(d) the fundamental root section $P \mapsto \varrho_{s,k}P(y)w_{s,k}$ is an isomorphism $\mathcal{O}(d_{s,k})|_{\mathcal{X}'} \to \wedge^k\mathcal{V}'_{s,k}$, and for the root r with $s(r) = s$, $\mathrm{st}(r) = k \ge 1$, the maps $w \otimes v \mapsto w \wedge v$ identify $\mathcal{L}_r = \big(\wedge^k\mathcal{V}'_{s,k} \otimes (\wedge^{k-1}\mathcal{V}'_{s,k-1})^{-1}\big)^{\otimes \mathrm{sg}(r)}$ inside $\mathbf{k}(\mathcal{X}')g(\lambda_r)$, with $g(\lambda_r) = (w_{s,k}/w_{s,k-1})^{\mathrm{sg}(r)}$.*

Proof. **The scheme.** For a locally free sheaf V of rank r, $\mathrm{Fl}(V)$ is an iterated projective bundle (choose a line V_1, then a line in V/V_1, and so on), smooth and projective over the base of relative dimension $r(r-1)/2$ with geometrically integral fibres; hence β is smooth and projective of relative dimension $|\Phi|\big(p(p-1)+q(q-1)\big)/2$, and $\mathcal{X}'$ is integral (iterated projective bundles over an integral scheme), normal (smooth over the normal scheme $\mathcal{X}^{\mathrm{b}}$), and projective over $\mathbb{Z}$. $\mathcal{X}^{\mathrm{b}}$ is generically finite over $\mathcal{Y}_c$ (Proposition 33.9, item 1, resp. Proposition 33.4, item 1), and $\mathcal{Y}_c$ is integral, flat over $\mathcal{O}_F$, with generic fibre of dimension $\dim Z_c = |\Phi|pq$; so $\mathcal{X}^{\mathrm{b}}$ has absolute dimension $|\Phi|pq+1$ and $\mathcal{X}'$ has absolute dimension $|\Phi|pq+|\Phi|\big(p(p-1)+q(q-1)\big)/2+1 = D+1$. It is flat over $\mathbb{Z}$ (integral with function field of characteristic 0). With $f' = f^{\mathrm{b}} \circ \beta$ it is a cover of $\mathcal{Y}$ whose generic point maps to that of $\mathcal{Y}_c$. Its abelian locus is β^{-1} of that of $\mathcal{X}^{\mathrm{b}}$. $\mathcal{X}'_{\mathbb{Q}} \to \mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ is smooth and $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ is smooth (Proposition 33.9, item 2, resp. Proposition 33.4, item 2), so $\mathcal{X}'_{\mathbb{Q}}$ is smooth. If $z_1, \dots, z_m$ are local coordinates at a point of $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ in which the boundary divisor is $\{z_1 \cdots z_{m'} = 0\}$ as in Theorem B.142, their pull-backs extend, β being smooth, to local coordinates at every point above it; so the preimage of a simple normal crossing divisor is one, with the same support as the preimage of the complement of the abelian locus. This is item 1.

The generic fibre and the field embedding. At a point x of $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ the Hodge lattice is $\mathrm{dR}_{s,x} = \mathrm{Lie}_x \cap \mathrm{Li}_s$, the stalk of the direct summand $\mathrm{Lie}_s(G_{\mathcal{X}^{\mathrm{b}}})$ of $\mathrm{Lie}(G_{\mathcal{X}^{\mathrm{b}}})$ (over $\mathbb{Q}$ the idempotents of $\mathcal{O}_E \otimes \mathcal{O}_x \cong \prod_\sigma \mathcal{O}_x$ split Lie). So over the abelian locus A^{b} of $\mathcal{X}^{\mathrm{b}}$, $\mathcal{V}_s = \mathrm{Lie}_s(G_{\mathcal{X}^{\mathrm{b}}})$ is the pull-back of $\mathrm{Lie}_s(G^{\mathrm{un}})$ by $j^{-1} \circ f^{\mathrm{b}} \colon A^{\mathrm{b}} \to Z_c$, and, since Fl commutes with base change,

$$\beta^{-1}(A^{\mathrm{b}}) = \mathrm{Fl}_{\mathfrak{u}}|_{Z_c} \times_{Z_c} A^{\mathrm{b}},$$

with the universal flags of the $\beta^*\mathcal{V}_s$ equal to the pull-backs of the universal flags of the $\mathrm{Lie}_s(G^{\mathrm{un}})$. The first projection induces $g \colon \Omega_c \to \Omega' = \mathbf{k}(\mathcal{X}')$ over $\mathbf{k}(Z_c)$. The generic fibre of $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}$ over $\mathbf{k}(Z_c)$ is a product of flag varieties, geometrically integral, so Ω_c is a regular extension of $\mathbf{k}(Z_c)$ and $\Omega_c \otimes_{\mathbf{k}(Z_c)} \Omega^{\mathrm{b}}$ is a domain, finite over Ω_c, hence a field; it is the function field of the generic fibre

of $\beta^{-1}(A^{\mathrm{b}}) \to A^{\mathrm{b}}$ base changed to Ω^{b}, i.e. $\Omega' = g(\Omega_c)\Omega^{\mathrm{b}}$ and $[\Omega' : g(\Omega_c)] = [\Omega^{\mathrm{b}} : \mathbf{k}(Z_c)] = \deg$. This is (Go3)(1), with $\delta = [F : \mathbb{Q}]\deg$.

Level four. In case (i): every ℓ^4-torsion point of $G_{\mathcal{X}'} = \beta^* G_{\mathcal{X}^{\mathrm{b}}}$ ($\ell \in S$) over an algebraic closure of Ω' is defined over $\Omega^{\mathrm{b}} \subseteq \Omega'$. By Proposition 33.9, item 3, $\mathcal{X}^{\mathrm{b}}$ satisfies at (S, R) the hypotheses (Hp1), (Hp2) of Lemma 31.29 and the hypothesis of its item (3) (freeness of ht_x of rank $\dim \mathrm{Li}$) for every $\ell \in S$; $\mathcal{X}'$ is a noetherian cover with the dominant morphism β compatible with the structure maps, so by Lemma 31.29(1),(3) the same holds for $\mathcal{X}'$, with Hodge bundles $\mathcal{O}_{\mathcal{X}'}\mathcal{V}_s = \beta^*\mathcal{V}_s$ (β is flat). This freeness at the points over $\ell \in S$ is the local freeness of the span of the Hodge–Tate values and of its images in the $\mathrm{Li}_s^\vee$, of ranks $\dim \mathrm{Li}$ and $\mathrm{rk}(s)$. This is (Go3)(2) with $R(\ell) = 4$. In case (ii) ($S = \varnothing$) the torsion and Hodge–Tate clauses of (Go3)(2) are empty and the Hodge lattices are the de Rham lattices; by Proposition 33.4, item 3, applied to the dominant morphism β (the structure map of $\mathcal{X}'$ is $f^{\mathrm{b}} \circ \beta$), they form the Hodge bundles $\mathcal{O}_{\mathcal{X}'}\mathcal{V}_s \cong \beta^*\mathcal{V}_s$. This is (Go3)(2) for $S = \varnothing$, and $\deg = [\Omega^{\mathrm{b}} : \mathbf{k}(Z_c)] = 1$.

Integral flag and root sections. The universal full flag $(\mathcal{V}'_{s,k})_k$ of $\beta^*\mathcal{V}_s$ on $\mathcal{X}'$ (pulled back from the factor $\mathrm{Fl}(\mathcal{V}_s)$) consists of locally free subsheaves with invertible quotients: an integral flag. By the display, its generic fibre is g of the universal flag. Its root lines $\mathcal{L}_r = (\mathcal{V}'_{s(r),\mathrm{st}(r)}/\mathcal{V}'_{s(r),\mathrm{st}(r)-1})^{\otimes \mathrm{sg}(r)}$ are invertible and torsion free, hence invertible submodules of the constant sheaves $\Omega'\lambda_r$ given by their generic stalks. Write $(V_{s,k})_k$ for the generic flag (g of the universal flag) and choose $v_{s,k} \in V_{s,k}$ whose class spans $V_{s,k}/V_{s,k-1}$ and equals λ_r (if $\mathrm{sg}(r) = 1$) or is dual to λ_r (if $\mathrm{sg}(r) = -1$) for the root r with $s(r) = s$, $\mathrm{st}(r) = k$; put $w_{s,0} = 1$ and $w_{s,k} = v_{s,1} \wedge \cdots \wedge v_{s,k}$, a basis of $\wedge^k V_{s,k}$. For a flag with locally free subsheaves and invertible quotients, $v \mapsto w \wedge v$ gives $\wedge^k \mathcal{V}'_{s,k} = \wedge^{k-1}\mathcal{V}'_{s,k-1} \otimes \mathcal{V}'_{s,k}/\mathcal{V}'_{s,k-1}$, so

$$\begin{aligned} \mathcal{L}_r &= \big(\wedge^k \mathcal{V}'_{s,k} \otimes (\wedge^{k-1}\mathcal{V}'_{s,k-1})^{-1}\big)^{\otimes \mathrm{sg}(r)} \quad \text{inside } \Omega'\lambda_r, \\ \lambda_r &= (w_{s,k}/w_{s,k-1})^{\mathrm{sg}(r)} \qquad (s = s(r),\ k = \mathrm{st}(r)). \end{aligned} \tag{RL}$$

Base factors. Let $\mathcal{H} \subseteq \Omega^{\mathrm{b}}$ be an ample invertible fractional ideal sheaf on $\mathcal{X}^{\mathrm{b}}$ and $m \geq 1$ such that every $(\wedge^i \mathcal{V}_s)^\vee \otimes \mathcal{H}^m$ (s a label, $1 \leq i < \mathrm{rk}(s)$) is generated by global sections (Hartshorne, Algebraic Geometry, II.7, definition of ampleness). Choose a basis μ_s of the Ω^{b}-line $\wedge^{\mathrm{rk}(s)}\mathrm{Li}_s$ and write $w_{s,\mathrm{rk}(s)} = \rho_s\mu_s$, $\rho_s \in \Omega'^\times$. Lemma 33.11, applied to $\mathcal{X}^{\mathrm{b}}$ with $K_0 = \varnothing$ and the invertible sheaves $\mathcal{H}^m \subseteq \Omega^{\mathrm{b}}\cdot 1$ and $\wedge^{\mathrm{rk}(s)}\mathcal{V}_s \subseteq \Omega^{\mathrm{b}}\mu_s$, gives a closed immersion of $\mathcal{X}^{\mathrm{b}}$ into $\prod_{k \in K_{\mathrm{b}}} \mathbb{P}(\mathbb{Z}^{I_k})$ with representing coordinates $y_{k,j} \in \Omega^{\mathrm{b}}$ and multidegrees $e_{\mathcal{H}}$, $e_{\det,s} \in \mathbb{Z}^{K_{\mathrm{b}}}$ such that on every chart $\prod_k y_{k,c'_k}^{(e_{\mathcal{H}})_k}$ generates $\mathcal{H}^m$ and $\prod_k y_{k,c'_k}^{(e_{\det,s})_k}\mu_s$ generates $\wedge^{\mathrm{rk}(s)}\mathcal{V}_s$; their pull-backs by β do the same for $\beta^*\mathcal{H}^m$ and $\wedge^{\mathrm{rk}(s)}\mathcal{V}'_{s,\mathrm{rk}(s)} = \beta^* \wedge^{\mathrm{rk}(s)} \mathcal{V}_s$. With the sections $\sigma_{s,i,j}$ chosen next and the vectors $v_{s,k}$ chosen above, these choices form a Plücker datum for $(\mathcal{X}^{\mathrm{b}}, (\mathcal{V}_s), (\lambda_r))$ (Definition 33.7; here λ_r stands for its image under g); $w_{s,k}$, ρ_s, the Plücker factors $k_{s,i}$, the coordinates $y_{k_{s,i},j}$, $d_{s,k}$, $\varrho_{s,k}$, e_r, ϱ_r and the Plücker map below are its derived objects. *Plücker factors.* For $1 \leq i < \mathrm{rk}(s)$ choose generating global sections $\sigma_{s,i,j}$ ($j \in I_{s,i}$) of $(\wedge^i \mathcal{V}_s)^\vee \otimes \mathcal{H}^m$. The point of $\mathcal{X}'$ with flag (V_k) is sent to the line $\wedge^i V_i \subseteq \wedge^i \mathcal{V}_s$ (Plücker; the flag scheme is closed in the product of the Grassmannians of its steps, and each Grassmannian is closed in the projective bundle of lines of $\wedge^i\mathcal{V}_s$), and that line to $(\sigma_{s,i,j}(\wedge^i V_i))_j$, which is nowhere zero because the $\sigma_{s,i,j}$ span every fibre of the dual. The second map is the closed immersion of the bundle of lines of $\wedge^i\mathcal{V}_s$ into $\mathbb{P}^{I_{s,i}} \times \mathcal{X}^{\mathrm{b}}$ induced by the surjection $\mathcal{O}^{I_{s,i}} \to (\wedge^i\mathcal{V}_s)^\vee \otimes \mathcal{H}^m$, linear on fibres. Let $K = K_{\mathrm{b}} \sqcup \{k_{s,i}\}$, $I_{k_{s,i}} = I_{s,i}$ and $y_{k_{s,i},j} = \sigma_{s,i,j}(w_{s,i}) \in \mathcal{H}^m \otimes \Omega' = \Omega'$, representing coordinates of this component. Altogether $\mathcal{X}' \to \mathcal{X}^{\mathrm{b}} \times \prod_{s,i} \mathbb{P}^{I_{s,i}} \to \prod_{k \in K} \mathbb{P}(\mathbb{Z}^{I_k})$ is a composite of closed immersions. Item (a): the components at K_{b} factor through $\mathcal{X}^{\mathrm{b}}$, and the component at $k_{s',i'}$ through the step i' of the flag of $\mathcal{V}_{s'}$, which $\beta_{s,i}$ keeps unless $(s', i') = (s, i)$. Item (b): over a point of $\mathcal{X}'_{s,i}$ with $V_{i-1} \subset V_{i+1}$, the fibre is the projective line of the lines V_i/V_{i-1} of the rank-two space V_{i+1}/V_{i-1},

and $v \mapsto w \wedge v$ (w a basis of $\wedge^{i-1}V_{i-1}$) is an injective linear map $V_{i+1}/V_{i-1} \to \wedge^i \mathcal{V}_s$ at that point; composed with the fibrewise linear embedding by the $\sigma_{s,i,j}$ it maps the fibre isomorphically onto a line. *Root sections.* Let γ generate $\wedge^i \mathcal{V}'_{s,i}$ near a point of a chart $\{y_{c'} \neq 0\}$ ($1 \leq i < \mathrm{rk}(s)$), and write $\gamma = a w_{s,i}$, $a \in \Omega'$, and $\varepsilon = \prod_k y_{k,c'_k}^{(e_{\mathcal{H}})_k}$, a local generator of $\beta^*\mathcal{H}^m$. As $y_{k_{s,i},c'_{k_{s,i}}}$ does not vanish at the point, $\sigma_{s,i,c'_{k_{s,i}}}(\gamma) = a y_{k_{s,i},c'_{k_{s,i}}}$ is a local generator of $\beta^*\mathcal{H}^m$, i.e. a unit times ε; so $(\varepsilon / y_{k_{s,i},c'_{k_{s,i}}}) w_{s,i}$ generates $\wedge^i \mathcal{V}'_{s,i}$ there. Put $d_{s,i} = e_{\mathcal{H}} - \mathbf{1}_{k_{s,i}}$ and $\varrho_{s,i} = 1$ ($1 \leq i < \mathrm{rk}(s)$), $d_{s,0} = 0$, $\varrho_{s,0} = 1$, $d_{s,\mathrm{rk}(s)} = e_{\det,s}$, $\varrho_{s,\mathrm{rk}(s)} = \rho_s^{-1}$: then $\varrho_{s,i} \prod_k y_{k,c'_k}^{(d_{s,i})_k} w_{s,i}$ generates $\wedge^i \mathcal{V}'_{s,i}$ on every chart, for $0 \leq i \leq \mathrm{rk}(s)$. By (RL), with

$$e_r = \mathrm{sg}(r)\big(d_{s,k} - d_{s,k-1}\big), \qquad \varrho_r = \big(\varrho_{s,k}/\varrho_{s,k-1}\big)^{\mathrm{sg}(r)} \qquad (s = s(r),\ k = \mathrm{st}(r)),$$

$\varrho_r \prod_k y_{k,c'_k}^{e_{r,k}} \lambda_r$ generates $\mathcal{L}_r$ on every chart; as in item 2 of Lemma 33.11, $P \mapsto \varrho_r P(y) \lambda_r$ is then an isomorphism $\mathcal{O}(e_r)|_{\mathcal{X}'} \to \mathcal{L}_r$ (on a chart a local section is $P = Q \prod_k Y_{k,c'_k}^{e_{r,k}}$ with Q regular): (Go3)(3). The same argument, applied to the generators $\varrho_{s,k} \prod_k y_{k,c'_k}^{(d_{s,k})_k} w_{s,k}$ of $\wedge^k \mathcal{V}'_{s,k}$ found above, shows that $P \mapsto \varrho_{s,k} P(y) w_{s,k}$ is an isomorphism $\mathcal{O}(d_{s,k})|_{\mathcal{X}'} \to \wedge^k \mathcal{V}'_{s,k}$ ($0 \leq k \leq \mathrm{rk}(s)$); with (RL) this is (d). Item (c): $d_{s,i}$ has entry -1 at $k_{s,i}$ and $d_{s',i'}$ has entry 0 there for $(s',i') \neq (s,i)$ ($e_{\mathcal{H}}$, $e_{\det,s'}$ live on K_{b}); the displayed formula for e_r gives the three cases.

Singular Hodge lines. Let z be a complex point of the abelian locus. By (Fh6), $\mathcal{L}_r \otimes \mathbb{C}$ at z is a root line of $G_{\mathcal{X}',z} = G^{\mathrm{un}}_{f'(z)}$ and carries the Hodge metric, a norm on a one-dimensional space, positive on non-zero vectors. For a point $\tilde{z}$ of the affine cone over z choose a section P of $\mathcal{O}(e_r)$ near z with $P(\tilde{z}) \neq 0$ (for instance $\prod_k Y_{k,c'_k}^{e_{r,k}}$ for a chart containing z) and put $h_r^{\mathrm{sing}}(\tilde{z}) = |P(\tilde{z})| / \|\varrho_r P(y) \lambda_r\|^{\mathrm{Hdg}}(z)$; at the cone points over the other complex points put $h_r^{\mathrm{sing}} = h^0$. The value is independent of P: another choice is $P' = QP$ with Q regular at z, so $|P'(\tilde{z})| = |Q(z)||P(\tilde{z})|$ and $\varrho_r P'(y)\lambda_r = Q(y)\varrho_r P(y)\lambda_r$ has Hodge norm $|Q(z)|$ times that of $\varrho_r P(y)\lambda_r$; and $\varrho_r P(y)\lambda_r$ generates $\mathcal{L}_r$ at z (by (Go3)(3), as $P(\tilde{z}) \neq 0$), so the denominator is positive. Scaling $\tilde{z}$ by $(\theta_k)_k \in (\mathbb{C}^\times)^K$ multiplies $P(\tilde{z})$ by $\prod_k \theta_k^{e_{r,k}}$ and leaves z fixed, so h_r^{sing} is a positive cone function of multidegree e_r, and by construction $\|\varrho_r P(y)\|^{\mathrm{Hdg}}(z) = |P(\tilde{z})| / h_r^{\mathrm{sing}}(\tilde{z})$ for every section P (both sides vanish together when $P(\tilde{z}) = 0$): (Go3)(4). □

Proposition 33.9. *Let* $\mathfrak{u}$ *be a unitary Shimura datum,* $\mathcal{Y}$ *an integral model of* Sh *over* $\mathcal{O}_F$*,* $\mathcal{Y}_c$ *an irreducible component of* $\mathcal{Y}$ *meeting* $j(\mathrm{Sh}_F)$*,* $Z_c = j^{-1}(\mathcal{Y}_c)$*, and* S *a finite set of primes containing every prime dividing* $\mathrm{disc}(E)$*. Then there is a cover* $\mathcal{X}^{\mathrm{b}}$ *of* $\mathcal{Y}$*, with structure map* f^{b} *and function field* Ω^{b}*, projective over* $\operatorname{Spec}\mathbb{Z}$*, such that:*

1. *f^{b} maps the generic point of $\mathcal{X}^{\mathrm{b}}$ to the generic point of $\mathcal{Y}_c$, and Ω^{b} is a finite extension of $\mathbf{k}(Z_c)$;*
2. *$\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$, and the complement of the abelian locus in $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ is the support of a simple normal crossing divisor;*
3. *for every $\ell \in S$, every ℓ^4-torsion point of $G_{\mathcal{X}^{\mathrm{b}}}$ over an algebraic closure of Ω^{b} is defined over Ω^{b}; the Hodge lattices of $\mathcal{X}^{\mathrm{b}}$ at (S,R) with $R(\ell) = 4$ ($\ell \in S$) are Hodge bundles $\mathcal{V}_s$; and at every point x over $\ell \in S$, ht_x is free of rank $\dim \mathrm{Li}$ and $\rho_s(\mathrm{ht}_x)$ is free of rank $\mathrm{rk}(s)$ for every label s.*

Proof. **The torsion field and a cover with simple normal crossing boundary.** Give $\mathcal{Y}_c$ its reduced structure; since j is an open immersion and $\mathcal{Y}_c$ meets $j(\mathrm{Sh}_F)$, $j(Z_c)$ is a non-empty

open subset of $\mathcal{Y}_c$, so it contains the generic point η_c of $\mathcal{Y}_c$ and $\mathbf{k}(Z_c) = \mathbf{k}(\mathcal{Y}_c)$. The fibre A of $G_{\mathcal{Y}}$ at η_c is an abelian variety over the field $\mathbf{k}(Z_c)$ of characteristic 0 (η_c lies in the abelian locus), so its ℓ^4-torsion is a finite étale group scheme. Let Ω^{b} be the finite Galois extension of $\mathbf{k}(Z_c)$ generated by the coordinates of all ℓ^4-torsion points of A over an algebraic closure, $\ell \in S$. By Lemma 33.5 with $\Omega = \Omega^{\mathrm{b}}$ there is a cover $\mathcal{X}_3$ of $\mathcal{Y}$ with function field Ω^{b}, projective over $\mathbb{Z}$, whose generic point maps to η_c, such that $\mathcal{X}_{3,\mathbb{Q}}$ is smooth and the complement of the abelian locus in $\mathcal{X}_{3,\mathbb{Q}}$ is the support of a simple normal crossing divisor. The generic fibre of $G_{\mathcal{X}_3}$ is $A \otimes_{\mathbf{k}(Z_c)} \Omega^{\mathrm{b}}$, so every ℓ^4-torsion point of $G_{\mathcal{X}_3}$ ($\ell \in S$) over an algebraic closure of Ω^{b} is defined over Ω^{b}.

Level-four Hodge bundles. By Lemma 33.10, applied to $\mathcal{X}_3$ with $R(\ell) = 4$ ($\ell \in S$; S contains the primes dividing $\mathrm{disc}(E)$), there are a cover $\mathcal{X}^{\mathrm{b}}$, projective over $\mathbb{Z}$, and a projective birational $b\colon \mathcal{X}^{\mathrm{b}} \to \mathcal{X}_3$, an isomorphism over $\mathcal{X}_3[1/P_S]$ ($P_S = \prod_{\ell\in S} \ell$), with structure map the composite, satisfying the three requirements of item 3. Item 1 holds: the function field is Ω^{b}, finite over $\mathbf{k}(Z_c)$, and the generic point maps to η_c. Since $\mathcal{X}_{3,\mathbb{Q}} \subseteq \mathcal{X}_3[1/P_S]$, b restricts to an isomorphism $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}} \to \mathcal{X}_{3,\mathbb{Q}}$, which maps the abelian locus of $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ onto that of $\mathcal{X}_{3,\mathbb{Q}}$ (the structure map of $\mathcal{X}^{\mathrm{b}}$ is that of $\mathcal{X}_3$ composed with b, and the abelian locus is the preimage of $j(\mathrm{Sh}_F)$). So $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ is smooth, and the complement of its abelian locus is the support of a simple normal crossing divisor: item 2. □

Lemma 33.10. *Let $\mathfrak{u}$ be a unitary Shimura datum, $\mathcal{Y}$ an integral model of* Sh *over $\mathcal{O}_F$, S a finite set of primes containing every prime dividing* $\mathrm{disc}(E)$*, $P_S = \prod_{\ell\in S} \ell$, and $R(\ell) \geq 1$ ($\ell \in S$) integers. Let $\mathcal{X}$ be a cover of $\mathcal{Y}$ with structure map f and function field Ω, projective over* $\operatorname{Spec}\mathbb{Z}$*, such that*

(a) for every $\ell \in S$, every $\ell^{R(\ell)}$-torsion point of $G_{\mathcal{X}}$ over an algebraic closure of Ω is defined over Ω.

Then there are a cover $\mathcal{X}'$ of $\mathcal{Y}$, projective over $\operatorname{Spec}\mathbb{Z}$*, and a projective birational morphism $b\colon \mathcal{X}' \to \mathcal{X}$, an isomorphism over $\mathcal{X}[1/P_S]$, with structure map $f \circ b$, such that $\mathcal{X}'$ satisfies (a) and*

(b) the Hodge lattices of $\mathcal{X}'$ at (S, R) are Hodge bundles $\mathcal{V}_s$;

(c) for every $\ell \in S$ and every point x' of $\mathcal{X}'$ over ℓ, $\mathrm{ht}_{x'}$ is a free $\mathcal{O}_{x'}$-module of rank $\dim \mathrm{Li}$ *and $\rho_s(\mathrm{ht}_{x'})$ is free of rank* $\mathrm{rk}(s)$ *for every label s.*

Proof. $\mathcal{X}$ is integral, normal and of finite type over $\mathbb{Z}$, hence noetherian and excellent, so Lemma 31.30 applies to it and to every cover dominating it. Let $g = \dim \mathrm{Li}$ and let ω be the dual of the locally free sheaf $\mathrm{Lie}(G_{\mathcal{X}})$ ($G_{\mathcal{X}}$ is smooth over $\mathcal{X}$), a locally free subsheaf of rank g of the constant sheaf $\mathrm{Li}^{\vee}$ with stalks ω_x. Put $U = \mathcal{X}[1/P_S]$, a dense open subset.

The spans of Hodge–Tate values form coherent sheaves, locally free on U. Fix $\ell \in S$, put $N = \ell^{R(\ell)}$, and let $y \in G_{\mathcal{X}}(\Omega)[N]$ (a finite group). For a point x over ℓ, Lemma 31.30(1) gives a Hodge–Tate value of y at x: a regular function ϕ on an open neighbourhood $\mathcal{W}$ of the unit point over x with $\phi(h) = e_N(y, h)$ for every geometric N-torsion point h, whose differential along the unit section u is a section $d\phi$ of ω over the open neighbourhood $u^{-1}(\mathcal{W})$ of x. The condition on ϕ does not involve x, so the germ of $d\phi$ at every point $x'' \in u^{-1}(\mathcal{W})$ over ℓ is a Hodge–Tate value of y at x''. By the uniqueness in Lemma 31.30(1) (Hodge–Tate values of y at a point form one class modulo $N\omega_{x''}$), two such local sections agree in $\omega/N\omega$ at every point over ℓ, and $\omega/N\omega$ has zero stalks at the points not over ℓ; so these local sections glue to a global section $\bar{w}_y$ of $\omega/N\omega$. Let $M_\ell \subseteq \omega$ be the preimage of the subsheaf of $\omega/N\omega$ generated

by the finitely many $\bar{w}_y$. It is coherent; by Lemma 31.30(1) its stalk is $\mathrm{ht}_x = \sum_y \mathcal{O}_x w_y + N\omega_x$ at every point x over ℓ, and ω_x at every other point. Let $P_{\ell,s} = \rho_s(M_\ell) \subseteq \mathrm{Li}_s^\vee$, coherent, with $P_{\ell,s} \otimes \Omega = \mathrm{Li}_s^\vee$ and $M_\ell \otimes \Omega = \mathrm{Li}^\vee$. At a point $x \in U$ every prime p with $p \notin \mathcal{O}_x^\times$ is outside S, so $1/\mathrm{disc}(E) \in \mathcal{O}_x$. As $\sigma(\mathcal{O}_E) \subseteq \mathcal{O}_F \subseteq \mathcal{O}_x$ for $\sigma \in \mathrm{Hom}(E,F)$, the map $\mathcal{O}_E \otimes \mathcal{O}_x \to \prod_\sigma \mathcal{O}_x$, $a \otimes b \mapsto (\sigma(a)b)_\sigma$, is an isomorphism (its discriminant is $\mathrm{disc}(E)$); its idempotents ϵ_s act on Lie_x because the $\mathcal{O}_E$-action is integral, and $\mathrm{Li}_s = \epsilon_s \mathrm{Li}$. Hence

$$\mathrm{Lie}_x = \textstyle\bigoplus_s \epsilon_s \mathrm{Lie}_x, \qquad \mathrm{dR}_{s,x} = \mathrm{Lie}_x \cap \mathrm{Li}_s = \epsilon_s \mathrm{Lie}_x, \qquad \rho_s(\omega_x) = \mathrm{Hom}_{\mathcal{O}_x}(\epsilon_s \mathrm{Lie}_x, \mathcal{O}_x), \quad \text{(dR)}$$

and $\epsilon_s \mathrm{Lie}_x$, a direct summand of a free module over a local ring, is free of rank $\dim \mathrm{Li}_s = \mathrm{rk}(s)$. So on U the sheaves $M_\ell = \omega$ and $P_{\ell,s} = \rho_s(\omega)$ are locally free of ranks g and $\mathrm{rk}(s)$.

Flattening. Apply Lemma 31.33(1) successively to the finitely many sheaves M_ℓ, $P_{\ell,s}$ ($\ell \in S$, s a label), each time to the subsheaf it generates on the scheme obtained so far, with the preimage of U as dense open subset (over which each modification is an isomorphism, so the generated sheaf is still locally free there). A sheaf made locally free stays locally free under the later modifications by Lemma 31.33(3) (with the trivial chain $0 \subset V$). This gives a projective birational morphism $\mathcal{X}_4 \to \mathcal{X}$ of integral schemes, an isomorphism over U, on which all $\mathcal{O}_{\mathcal{X}_4} M_\ell$ and $\mathcal{O}_{\mathcal{X}_4} P_{\ell,s}$ are locally free. Let $\mathcal{X}'$ be the normalisation of $\mathcal{X}_4$: it is finite over $\mathcal{X}_4$ (excellence), so $b\colon \mathcal{X}' \to \mathcal{X}$ is projective and birational, and it is an isomorphism over U ($\mathcal{X}$ is normal). With the structure map $f \circ b$, $\mathcal{X}'$ is a cover of $\mathcal{Y}$ with function field Ω and $G_{\mathcal{X}'} = b^* G_{\mathcal{X}}$, projective over $\mathbb{Z}$; by Lemma 31.33(3) the sheaves $\mathcal{O}_{\mathcal{X}'} M_\ell$, $\mathcal{O}_{\mathcal{X}'} P_{\ell,s}$ are locally free, of ranks g and $\mathrm{rk}(s)$ (their generic fibres are $\mathrm{Li}^\vee$, $\mathrm{Li}_s^\vee$).

(a) and (c) on $\mathcal{X}'$. Condition (a) holds because the function field and the generic fibre of the semiabelian scheme are unchanged. Let x' lie over $\ell \in S$ and $x = b(x')$. By Lemma 31.30(2), $\mathrm{ht}_{x'} = \mathcal{O}_{x'} \mathrm{ht}_x = (\mathcal{O}_{\mathcal{X}'} M_\ell)_{x'}$, free of rank g, and $\rho_s(\mathrm{ht}_{x'}) = \mathcal{O}_{x'} \rho_s(\mathrm{ht}_x) = (\mathcal{O}_{\mathcal{X}'} P_{\ell,s})_{x'}$, free of rank $\mathrm{rk}(s)$: this is (c).

(b) on $\mathcal{X}'$. For a locally free subsheaf $P \subseteq \mathrm{Li}_s^\vee$ of rank $\mathrm{rk}(s)$ write $P^\perp \subseteq \mathrm{Li}_s$ for the subsheaf of e with $\lambda(e) \in \mathcal{O}$ for all local sections λ of P; if $\lambda_1, \dots, \lambda_k$ is a local basis of P (an Ω-basis of $\mathrm{Li}_s^\vee$), $P^\perp$ is free there on the dual basis. Let J_s be the ideal sheaf generated by $\sigma_s(\mathfrak{d}_E)$ for $s = (\tau, +)$ and $J_s = \mathcal{O}$ for $s = (\tau, -)$; it is invertible, being the pull-back along $\mathcal{X}' \to \mathrm{Spec}\, \mathcal{O}_F$ of an invertible ideal of the Dedekind ring $\mathcal{O}_F$, and $J_{s,x'} = \mathcal{O}_{x'}$ at every point x' not over a prime dividing $\mathrm{disc}(E)$ (the norm of $\mathfrak{d}_E$ is $|\mathrm{disc}(E)|$). For $\ell \in S$ let $W_\ell \subseteq \mathcal{X}'$ be the open complement of the fibres over the primes of $S \smallsetminus \{\ell\}$, and $W_0 = \mathcal{X}'[1/P_S]$; these sets cover $\mathcal{X}'$. On W_ℓ put $\mathcal{V}_s^{(\ell)} = J_s^{-1} (\mathcal{O}_{\mathcal{X}'} P_{\ell,s})^\perp$, on W_0 put $\mathcal{V}_s^{(0)} = (\rho_s(\omega'))^\perp$ (ω' the dual of $\mathrm{Lie}(G_{\mathcal{X}'})$); all are locally free of rank $\mathrm{rk}(s)$. Their stalks are the Hodge lattices: at x' over ℓ, since $w(e) = \rho_s(w)(e)$ for $w \in \mathrm{Li}^\vee$, $e \in \mathrm{Li}_s$, the stalk of $(\mathcal{O} P_{\ell,s})^\perp$ is $\{e \in \mathrm{Li}_s : w(e) \in \mathcal{O}_{x'}\ \forall w \in \mathrm{ht}_{x'}\} = \mathrm{HT}'_{s,x'}$, and the stalk of $\mathcal{V}_s^{(\ell)}$ is $\mathrm{HT}_{s,x'}(\ell^{R(\ell)})$ ($J_{s,x'}$ is principal, generated by some j, and $\{e : \sigma_s(d)e \in \mathrm{HT}'\ \forall d \in \mathfrak{d}_E\} = j^{-1}\mathrm{HT}'$). At x' over no prime of S, M_ℓ generates ω' near x', $J_{s,x'} = \mathcal{O}_{x'}$, and by (dR) at x' (the argument there applies to every point over no prime of S) and reflexivity of the free module $\epsilon_s \mathrm{Lie}_{x'}$, the stalk of $\mathcal{V}_s^{(\ell)}$ and of $\mathcal{V}_s^{(0)}$ is $\{e \in \mathrm{Li}_s : w(e) \in \mathcal{O}_{x'}\ \forall w \in \omega'_{x'}\} = \mathrm{Lie}_{x'} \cap \mathrm{Li}_s = \mathrm{dR}_{s,x'}$. Subsheaves of the constant sheaf Li_s on the integral scheme $\mathcal{X}'$ with equal stalks on an open set are equal there (sections over W are the elements of Li_s lying in every stalk over W), so the $\mathcal{V}_s^{(\ell)}$ and $\mathcal{V}_s^{(0)}$ glue to a locally free subsheaf $\mathcal{V}_s \subseteq \mathrm{Li}_s$ of rank $\mathrm{rk}(s)$ whose stalk at every point is the Hodge lattice at (S, R): (b). □

Lemma 33.11. *Let $\mathcal{X}$ be an integral scheme, projective over $\mathrm{Spec}\, \mathbb{Z}$, with function field Ω. Let K_0 be a finite set and, for $k \in K_0$, $\psi_k\colon \mathcal{X} \to \mathbb{P}(\mathbb{Z}^{I_k})$ a morphism with elements $y_{k,j} \in \Omega$ ($j \in I_k$) such that $y_{k,j}/y_{k,j'}$ is the pull-back by ψ_k of the ratio of the coordinates j and j' whenever $y_{k,j'} \neq 0$. Let R be a finite set and, for $r \in \mathrm{R}$, Λ_r a one-dimensional Ω-vector space with basis*

λ_r and $\mathcal{L}_r \subseteq \Lambda_r$ an invertible $\mathcal{O}_\mathcal{X}$-submodule of the constant sheaf. Then there are a finite set $K \supseteq K_0$, finite sets I_k ($k \in K \smallsetminus K_0$), elements $y_{k,j} \in \Omega$ ($k \in K \smallsetminus K_0$, $j \in I_k$) and multidegrees $e_r \in \mathbb{Z}^K$ ($r \in \mathrm{R}$) with $e_{r,k} = 0$ for $k \in K_0$, such that:

1. *the $y_{k,j}$ ($k \in K$) are representing coordinates of a closed immersion $\mathcal{X} \to \mathbb{P} = \prod_{k\in K} \mathbb{P}(\mathbb{Z}^{I_k})$ whose component at $k \in K_0$ is ψ_k;*
2. *for every $r \in \mathrm{R}$ and every chart $\{y_{c'} \neq 0\}$ of $\mathcal{X}$, the element $\prod_k y_{k,c'_k}^{e_{r,k}} \lambda_r$ generates $\mathcal{L}_r$ on that chart; consequently $P \mapsto P(y)\lambda_r$ is an isomorphism $\mathcal{O}(e_r)|_\mathcal{X} \to \mathcal{L}_r$.*

Proof. Write $\mathcal{L}_r = \mathcal{D}_r\lambda_r$ with $\mathcal{D}_r \subseteq \Omega$ an invertible fractional ideal sheaf. An ample invertible sheaf on the integral scheme $\mathcal{X}$ (which exists, $\mathcal{X}$ being projective over $\mathbb{Z}$) embeds into the constant sheaf Ω through its generic stalk; let $\mathcal{H} \subseteq \Omega$ be such an ample invertible fractional ideal sheaf. Since $\mathcal{X}$ is proper over the noetherian ring $\mathbb{Z}$, there is $m \geq 1$ such that $\mathcal{H}^m$ and every $\mathcal{D}_r\mathcal{H}^m$ ($r \in \mathrm{R}$) are very ample relative to $\operatorname{Spec}\mathbb{Z}$ (Hartshorne, Algebraic Geometry, II Ex. 7.5(d)). Choose global sections $u_0, \dots, u_a \in \Gamma(\mathcal{X}, \mathcal{H}^m) \subseteq \Omega$ generating $\mathcal{H}^m$ and defining an immersion $\mathcal{X} \to \mathbb{P}^a_\mathbb{Z}$, which is closed because $\mathcal{X}$ is proper over $\mathbb{Z}$; for each r choose generating global sections $f_{r,0}, \dots, f_{r,a_r} \in \Gamma(\mathcal{X}, \mathcal{D}_r\mathcal{H}^m)$ and generating global sections $h_{r,0}, \dots, h_{r,b_r} \in \Gamma(\mathcal{X}, \mathcal{H}^m)$. Put $K = K_0 \sqcup \{k_\mathcal{H}\} \sqcup \{k_r, k'_r : r \in \mathrm{R}\}$ with $I_{k_\mathcal{H}} = \{0, \dots, a\}$, $I_{k_r} = \{0, \dots, a_r\}$, $I_{k'_r} = \{0, \dots, b_r\}$, and $y_{k_\mathcal{H},j} = u_j$, $y_{k_r,j} = f_{r,j}$, $y_{k'_r,j} = h_{r,j}$; put $e_r = \mathbf{1}_{k_r} - \mathbf{1}_{k'_r}$.

Generating global sections $s_0, \dots, s_b$ of an invertible subsheaf $\mathcal{N} \subseteq \Omega$ define a morphism to $\mathbb{P}^b_\mathbb{Z}$ under which the ratio of the coordinates j and j' pulls back to $s_j/s_{j'}$ on $\{s_{j'} \neq 0\}$, i.e. the s_j are representing coordinates of it. With the ψ_k this gives a morphism $\iota\colon \mathcal{X} \to \mathbb{P}$ with representing coordinates $y_{k,j}$ ($k \in K$). Its component at $k_\mathcal{H}$ is a closed immersion and $\mathbb{P}$ is separated over $\mathbb{Z}$, so ι is a closed immersion (if $g \circ \iota$ is a closed immersion and g is separated, ι is a closed immersion: Hartshorne II Ex. 4.8). This is item 1.

On a chart $\{y_{c'} \neq 0\}$ the section $f_{r,c'_{k_r}}$ does not vanish, so it generates $\mathcal{D}_r\mathcal{H}^m$ there, and likewise $h_{r,c'_{k'_r}}$ generates $\mathcal{H}^m$; hence $\prod_k y_{k,c'_k}^{e_{r,k}} = f_{r,c'_{k_r}}/h_{r,c'_{k'_r}}$ generates $\mathcal{D}_r$ and $\prod_k y_{k,c'_k}^{e_{r,k}} \lambda_r$ generates $\mathcal{L}_r$ on the chart. On the same chart $\mathcal{O}(e_r)$ is free on $\prod_k Y_{k,c'_k}^{e_{r,k}}$, and a local section P of $\mathcal{O}(e_r)$ is $Q\prod_k Y_{k,c'_k}^{e_{r,k}}$ with Q regular, so $P(y)\lambda_r = Q(y)\prod_k y_{k,c'_k}^{e_{r,k}} \lambda_r$; the map $P \mapsto P(y)\lambda_r$ therefore sends a generator to a generator on every chart. The charts cover $\mathcal{X}$, which gives item 2. □

33.2 Flag models realising their analytic systems

The main result, Proposition 33.12, says that the flag model is a root model that realises its own analytic system, with root metrics regularised step by step. Two analytic facts carry it. The Hodge weights of the steps of the Hodge flags are log-log singular plurisubharmonic weights (Lemma 33.15); the proof works at a point of the normal crossings boundary, with the Lie bundle as the Hodge quotient of the canonical extension, two-sided Gram bounds for algebraic Lie frames with one logarithm, the local period form of the Hodge Gram matrix, and Griffiths' seminegativity of the last Hodge quotient (Lemma 33.16, Lemma 33.17, Lemma 33.19). And such weights can be regularised globally along a normal crossings divisor (Theorem 33.22), by local regularisation in polydiscs glued with a regularised maximum. The subsection also introduces self-dual $\mathcal{O}_E$-lattices with their local bases and the flat generators of the determinant line (Definition 33.13, Definition 33.14). The analytic flat unit of Lemma 33.21 gives the flat metric on the determinant index used in Proposition 33.12 (Lemma 33.20), and it is used again for the flat determinants of Section 34 and in Appendix D. The boundary lemmas on Lie bundles are used again in Section 34 and in Appendix D.

Proposition 33.12. *In the situation of Lemma 33.8, in either of its cases (i) (the level-four base of Proposition 33.9) and (ii) ($S = \varnothing$, the de Rham base of Proposition 33.4, where* $\deg = 1$*), put $e_* = 0$, $h_* = 1$, $k_{\mathcal{X}'} = 1$, $\delta_{\mathcal{X}'} = [F : \mathbb{Q}]\deg$. There are cone metrics $h_r(t)$ of multidegree e_r ($r \in \mathrm{Rt}$, $t > 0$), an analytic system* Sys' *on $X^{\mathrm{an}\prime} = \mathcal{X}'(\mathbb{C})$ with $\varphi_* = 0 = \varphi_*^T$, and cone metrics h_a^{ref} (a fundamental index) such that:*

1. *$(\mathcal{X}', (e_r), (h_r(t)), h_*, k_{\mathcal{X}'}, \delta_{\mathcal{X}'})$ is a root model of type (Φ, n);*
2. *the open part of* Sys' *is the abelian locus; $\pi' = \mathrm{id}$ and $d = 1$ satisfy hypotheses 1–3 of Lemma 31.21 with the h_a^{ref}; and on the abelian locus $h^{\mathrm{ref}}_{(\tau,i)} e^{\varphi_{(\tau,i)}} = \prod_{k\le i} (h^{\mathrm{sing}}_{(\tau,k)})^{-1}$ for $(\tau, i) \in \mathrm{Rt}$;*
3. *for every label s and $1 \le j < \mathrm{rk}(s)$, writing $k_0 = k_{s,j}$: for every $t > 0$ the cone metrics $h_r(t)$ with r not of label s or of label s and step $\notin \{j, j+1\}$, and the product $h_{r_1}(t) h_{r_2}(t)$ of the two roots of label s and steps j, $j+1$, are from the base at k_0;*
4. *(flat determinant) for every $\tau \in \Phi$, $\varphi_{(\tau,n)}$ extends to a smooth function on $X^{\mathrm{an}\prime}$, $\varphi^T_{(\tau,n)}(t) = \varphi_{(\tau,n)}$ for $0 < t \le t_0$, and the metric $h^{\mathrm{ref}}_{(\tau,n)} e^{\varphi_{(\tau,n)}}$ has pluriharmonic local potentials on $X^{\mathrm{an}\prime}$.*

Proof. **Fundamental lines as pull-backs.** Let $a = (\tau, i)$. By (Fh3),

$$\bigotimes_{k\le i} \mathcal{L}^{-1}_{(\tau,k)} = \begin{cases} (\det \mathcal{V}'_{(\tau,+),i})^{-1}, & i \le p, \\ \mathcal{N}_\tau \otimes (\det \mathcal{V}'_{(\tau,-),n-i})^{-1}, & p < i \le n; \end{cases}$$

for $p < i \le n$ this holds since the steps of the minus flag telescope:

$$\bigotimes_{k=p+1}^{i} \mathcal{V}^-_{n+1-k}/\mathcal{V}^-_{n-k} = \det \mathcal{V}^-_q \otimes (\det \mathcal{V}^-_{n-i})^{-1};$$

for $i = n$ the second factor is trivial. Write $(s_a, j_a) = ((\tau,+), i)$ for $i \le p$ and $((\tau,-), n-i)$ for $p < i < n$; then $(\det \mathcal{V}'_{s_a,j_a})^{-1}$ is the pull-back of $(\det \mathcal{W})^{-1}$ by the projection $\mathrm{pr}_a \colon \mathcal{X}' \to \mathrm{Gr}_{j_a}(\mathcal{V}_{s_a})$ (step j_a of the flag of $\mathcal{V}_{s_a}$; for $j_a = \mathrm{rk}(s_a)$, i.e. $i = p$, the base), and $\mathcal{N}_\tau$ is pulled back from the base. Its multidegree $e_a = -\sum_{k\le i} e_{(\tau,k)}$ is supported on $K_a = K_{\mathrm{b}} \cup \{k_{s_a,j_a}\}$ (K_{b} only if $j_a \in \{0, \mathrm{rk}(s_a)\}$ or $i = n$), and the components of the immersion at K_a factor through pr_a: the immersion is the Plücker map of that datum (Definition 33.7), whose components at K_{b} factor through β and whose component at k_{s_a,j_a} depends only on step j_a of the flag of $\mathcal{V}_{s_a}$. The induced map of $\mathrm{Gr}_{j_a}(\mathcal{V}_{s_a})$ into $\mathbb{P}_a = \prod_{k\in K_a} \mathbb{P}(\mathbb{Z}^{I_k})$ by the same coordinates is a closed immersion: it is the Plücker embedding of $\mathrm{Gr}_{j_a}(\mathcal{V}_{s_a})$ into the bundle of lines of $\wedge^{j_a}\mathcal{V}_{s_a}$ (a closed immersion), followed by the closed immersion of that bundle into $\mathbb{P}^{I_{s_a,j_a}} \times \mathcal{X}^{\mathrm{b}}$ induced by the surjection $\mathcal{O}^{I_{s_a,j_a}} \to (\wedge^{j_a}\mathcal{V}_{s_a})^\vee \otimes \mathcal{H}^m$ given by the $\sigma_{s_a,j_a,\cdot}$ (linear on fibres), and by the closed immersion of $\mathcal{X}^{\mathrm{b}}$ by the base factors (for $j_a = \mathrm{rk}(s_a)$ only the last); and $\mathcal{X}' \to \mathbb{P}_a$ is pr_a followed by this immersion.

Common boundary data. Fix a holomorphic section ς_{b} of a smooth hermitian line bundle on $\mathcal{X}^{\mathrm{b}}(\mathbb{C})$ with $\|\varsigma_{\mathrm{b}}\| \le 1$ whose zero divisor is the reduced complement of the abelian locus (simple normal crossings), and use its pull-backs ς on every $\mathrm{Gr}_j(\mathcal{V}_s)(\mathbb{C})$ and on $\mathcal{X}'(\mathbb{C})$; they have simple normal crossing zero divisors, the preimages of the boundary (Lemma 33.15, Lemma 33.8(1)), and all rW are pull-backs of one function. Fix Kähler forms Θ_a on $\mathrm{Gr}_{j_a}(\mathcal{V}_{s_a})(\mathbb{C})$ and Θ' on $\mathcal{X}'(\mathbb{C})$ (projective manifolds), all invariant under complex conjugation, and $c_\Theta \ge 1$ with $\mathrm{pr}_a^*\Theta_a \le c_\Theta \Theta'$ for all a (compactness).

Weights and regularisations. Lemma 33.15 and Lemma 33.20 are applied in the same case (i) or (ii) as Lemma 33.8; both hold in either case. Fix conjugation-invariant smooth reference metrics on $(\det \mathcal{W})^{-1}$ over each $\mathrm{Gr}_{j_a}(\mathcal{V}_{s_a})(\mathbb{C})$ and on $\mathcal{N}_\tau$. For $a = (\tau, i)$, $i < n$, $j_a \geq 1$, let ψ_a be the weight φ of Lemma 33.15 for (s_a, j_a); it is conjugation-invariant (the Hodge metric is: $H_{\bar{z}}(\bar{u}, \bar{v}) = \overline{H_z(u, v)}$ under the antiholomorphic identification of conjugate fibres). By Theorem 33.22 it has regularisations $\psi_a^T(t)$, $0 < t \leq t_a$; replacing them by $\frac{1}{2}(\psi_a^T(t) + \psi_a^T(t) \circ \mathrm{conj})$ keeps every clause (the bounds are preserved under averaging, plurisubharmonicity under composition with the antiholomorphic involution, as Θ_a and the reference metric are invariant). Let ν_τ be the weight of the flat metric of Lemma 33.20 against the reference metric of $\mathcal{N}_\tau$: smooth on the whole base, and the flat metric has pluriharmonic potentials. Define on $X^{\mathrm{an}\prime}$, for the fundamental index a: $\varphi_a = \psi_a \circ \mathrm{pr}_a$, $\varphi_a^T(t) = \psi_a^T(t) \circ \mathrm{pr}_a$ $(i \leq p)$; $\varphi_a = \nu_\tau + \psi_a \circ \mathrm{pr}_a$, $\varphi_a^T(t) = \nu_\tau + \psi_a^T(t) \circ \mathrm{pr}_a$ $(p < i < n)$; $\varphi_a = \varphi_a^T(t) = \nu_\tau$ $(i = n)$, with the base functions pulled back; $(\mathcal{M}_a, h_a^{\mathrm{ref}})$ the pulled-back line with the pulled-back reference metric (tensor product for $p < i < n$), and $\mathcal{M}_*$ trivial, $h_*^{\mathrm{ref}} = 1$, $\varphi_* = \varphi_*^T = 0$. With $t_0 = \min_a t_a$ and C, C_5 the largest constants (enlarged by c_Θ and by $\sup|\nu_\tau|$), every clause of (Go1) holds on $X^{\mathrm{an}\prime}$: the inequalities are pulled back pointwise (with rW pulled back) and shifted by the bounded smooth ν_τ; the twisted psh clause holds because $\mathrm{pr}_a^*(\varepsilon(t)\Theta_a) \leq \varepsilon'(t)\Theta'$ with $\varepsilon' = c_\Theta \varepsilon$ (absorbed by enlarging C in ε), pull-backs of psh functions by holomorphic maps are psh, and adding the potentials of a flat metric keeps plurisubharmonicity; C^2-convergence pulls back. The open part $X^{\circ\prime}$ is the complement of the zero set of ς, the abelian locus. So $\mathrm{Sys}' = (X^{\mathrm{an}\prime}, \Theta', \varsigma, (\mathcal{M}_a, h_a^{\mathrm{ref}}, \varphi_a, \varphi_a^T))$ is an analytic system with $\varphi_* = 0 = \varphi_*^T$. On the abelian locus $h_a^{\mathrm{ref}} e^{\varphi_a}$ is the Hodge metric of the fundamental line, i.e. (by (Go3)(4) of Lemma 33.8, through the root sections) the cone function $\prod_{k \leq i} (h_{(\tau,k)}^{\mathrm{sing}})^{-1}$.

Cone metrics on the ambient space. A smooth conjugation-invariant metric on $\mathcal{O}(e)$ restricted to a closed complex submanifold of $\mathbb{P}_a(\mathbb{C})$, stable under conjugation, is induced by a cone metric of multidegree e on the cone of $\mathbb{P}_a$ (write the metric as e^ψ times the metric of the cone metric $h^0(z) = \prod_k (\sum_j |z_{k,j}|^2)^{e_k/2}$, extend ψ smoothly from the closed submanifold to the projective manifold by a tubular neighbourhood and a partition of unity, average the extension with its composite with complex conjugation, and multiply h^0 by the exponential of its pull-back to the cone). Apply it on the image of $\mathrm{Gr}_{j_a}(\mathcal{V}_{s_a})(\mathbb{C})$ (times the base factors for ν_τ) in $\mathbb{P}_a(\mathbb{C})$ to the metrics h_a^{ref} and $h_a^{\mathrm{ref}} e^{\varphi_a^T(t)}$ $(0 < t \leq t_0$; for $t > t_0$ use $t_0)$, and compose with the projection $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}_a$ of affine cones: this gives cone metrics h_a^{ref} and $F_a(t)$ of multidegree e_a on the cone of $\mathbb{P}$ which do not depend on the coordinates of the factors outside K_a, with $F_{(\tau,0)} = 1$. Put

$$h_{(\tau,i)}(t) = F_{(\tau,i-1)}(t)/F_{(\tau,i)}(t) \qquad ((\tau, i) \in \mathrm{Rt},\ t > 0),$$

a cone metric of multidegree $-e_{(\tau,i-1)}^{\mathrm{fund}} + e_{(\tau,i)}^{\mathrm{fund}}$ with $e_{(\tau,i)}^{\mathrm{fund}} = -\sum_{k \leq i} e_{(\tau,k)}$, i.e. of multidegree $e_{(\tau,i)}$. This gives item 1 (the scheme and its multidegrees are those of Lemma 33.8; $e_* = 0$, $h_* = 1$ is a cone metric of multidegree 0).

Item 2. $\pi' = \mathrm{id}$ is a one-sheeted covering (hypothesis 1 with empty null sets); the first Chern form of $(\mathcal{O}(e_a), h_a^{\mathrm{ref}})|_{\mathcal{X}'(\mathbb{C})}$ is that of $(\mathcal{M}_a, h_a^{\mathrm{ref}})$ by construction and $h_*^{\mathrm{ref}} = h_* = 1$ (hypothesis 2); and $\prod_{k \leq i} h_{(\tau,k)}(t)^{-1} = F_{(\tau,i)}(t)/F_{(\tau,0)}(t) = h_{(\tau,i)}^{\mathrm{ref}} e^{\varphi_{(\tau,i)}^T(t)}$ on the cone over $\mathcal{X}'(\mathbb{C})$ for $0 < t \leq t_0$ (hypothesis 3). The identification of the singular weight was shown above.

Item 3. Let (s, j) be given. $F_{(\tau,i)}(t)$ depends only on the factors in $K_{(\tau,i)}$, which contains $k_{s,j}$ only if $(s_{(\tau,i)}, j_{(\tau,i)}) = (s, j)$, i.e. for exactly one fundamental index $a_0 = (\tau, i_0)$ of the τ of s: $i_0 = j$ if $s = (\tau, +)$, $i_0 = n - j$ if $s = (\tau, -)$. A root (τ', i) has $h_{(\tau',i)}(t) = F_{(\tau',i-1)}(t)/F_{(\tau',i)}(t)$, which involves F_{a_0} only if $\tau' = \tau$ and $i \in \{i_0, i_0 + 1\}$: these are the two roots of label s and steps j, $j + 1$ (for $s = (\tau, -)$ the roots $(\tau, n - j)$ and $(\tau, n - j + 1)$ have steps $j + 1$ and j). Their product is $F_{(\tau,i_0-1)}(t)/F_{(\tau,i_0+1)}(t)$, free of F_{a_0}. All other $h_r(t)$, and this product, are therefore

independent of the coordinates of the factor $k_{s,j}$, and their multidegrees have $k_{s,j}$-entry 0 by Lemma 33.8(c): they are from the base at $k_{s,j}$.

Item 4. For $a = (\tau, n)$ the construction set $\varphi_a = \varphi_a^T(t) = \nu_\tau$ (pulled back from the base), smooth on all of $X^{\mathrm{an}\prime}$, and $h_a^{\mathrm{ref}} e^{\nu_\tau}$ is the pull-back of the flat metric of Lemma 33.20 on $\mathcal{N}_\tau$. Near every point of the base that metric has a holomorphic generator of constant norm, so $-\log$ of the norm of a holomorphic generator is pluriharmonic; pulled back by the holomorphic projection to the base, the same holds on $X^{\mathrm{an}\prime}$. □

Definition 33.13. Let E be a CM field with complex conjugation $a \mapsto \bar{a}$, and fix a $\mathbb{Z}$-basis $b_1, \dots, b_m$ of $\mathcal{O}_E$ with trace-dual basis $b_1^\vee, \dots, b_m^\vee$ of $\mathfrak{d}_E^{-1}$ ($\mathrm{Tr}_{E/\mathbb{Q}}(b_k b_l^\vee) = \delta_{kl}$).

(Qd1) Self-dual lattices and trace forms. A **self-dual $\mathcal{O}_E$-lattice** of rank n is a pair, written $(N, \langle\,,\rangle)$, of a finitely generated projective $\mathcal{O}_E$-module N of rank n and an alternating $\mathbb{Z}$-bilinear form $\langle\,,\rangle\colon N \times N \to \mathbb{Z}$ inducing an isomorphism $N \cong \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, with $\langle ax, y\rangle = \langle x, \bar{a}y\rangle$ for all $a \in \mathcal{O}_E$. Its **trace form** is

$$h_N(x,y) = \sum_{k=1}^{m} \langle b_k x, y\rangle\, b_k^\vee \in \mathfrak{d}_E^{-1} \qquad (x, y \in N), \tag{33.2}$$

extended $\mathbb{Q}$-bilinearly to $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$. An **isomorphism** of self-dual $\mathcal{O}_E$-lattices is an $\mathcal{O}_E$-linear bijection preserving $\langle\,,\rangle$.

(Qd2) Gram matrices. For a finite set I and $x = (x_i)_{i\in I} \in N^I$, the **Gram matrix** of x is $\mathrm{Gr}(x) = \big(h_N(x_i, x_j)\big)_{i,j\in I} \in \mathrm{M}_I(E)$.

(Qd3) Local bases. For a prime ℓ, a tuple $x \in N^I$ is a **local basis at ℓ** if it is $\mathcal{O}_E$-linearly independent and there is an integer $m_0 \geq 1$ with $\ell \nmid m_0$ and $m_0 N \subseteq \sum_{i\in I} \mathcal{O}_E x_i$.

(Qd4) Transition matrices. If $|I| = n$ and $e \in N^I$ is $\mathcal{O}_E$-linearly independent (hence an E-basis of $N_{\mathbb{Q}}$), the **transition matrix** of $x \in N^I$ to e is the matrix $T(x,e) \in \mathrm{M}_I(E)$ with $x_i = \sum_{k\in I} T(x,e)_{ik}\, e_k$ in $N_{\mathbb{Q}}$ for every $i \in I$.

(Qd5) Local size of the different. For a prime ℓ and a ring homomorphism $j\colon E \to \mathbb{C}_\ell$, $|\mathfrak{d}_E|_j = \sup\{|j(d)| : d \in \mathfrak{d}_E\}$.

Definition 33.14. Let $\mathfrak{u}$ be a unitary Shimura datum with $n = p + q \geq 1$, let $\tau \in \Phi_{\mathsf{H}}$, and let $U \subseteq \mathrm{Sh}_F$ be a union of connected components. For a field k with an embedding $\beta\colon F \to k$, $U(k)$ is the set of F-morphisms $\mathrm{Spec}\, k \to U$, and for $y \in U(k)$, G_y is the fibre of G^{un} with $\iota\colon \mathcal{O}_E \to \mathrm{End}(G_y)$ and $\mathrm{Lie}_{(\tau,\pm)}(G_y) = \{x \in \mathrm{Lie}(G_y) : \iota(a)x = \beta(\sigma_{(\tau,\pm)}(a))x\ \forall a \in \mathcal{O}_E\}$.

(Yf1) Frames. A **frame** at $y \in U(k)$ is a pair (v, w) of a k-basis $v = (v_1, \dots, v_p)$ of $\mathrm{Lie}_{(\tau,+)}(G_y)$ and a k-basis $w = (w_1, \dots, w_q)$ of $\mathrm{Lie}_{(\tau,-)}(G_y)$. For $A \in \mathrm{GL}_p(k)$, $(Av)_i = \sum_j A_{ij} v_j$, and likewise Bw for $B \in \mathrm{GL}_q(k)$.

(Yf2) Frame functions. A **frame function** on U assigns to every field k with embedding $\beta\colon F \to k$, every $y \in U(k)$ and every frame (v, w) at y an element $\varphi(y; v, w) \in k^\times$ such that $\varphi(y; Av, Bw) \det A = \varphi(y; v, w) \det B$ for all $A \in \mathrm{GL}_p(k)$, $B \in \mathrm{GL}_q(k)$, and $\varphi(y'; f_* v, f_* w) = f(\varphi(y; v, w))$ for every field homomorphism $f\colon k \to k'$, y' the induced point and f_* the induced map of Lie algebras.

(Yf3) Local norms. Let φ be a frame function on U. A real number N_s is its **archimedean norm** at an embedding $s\colon F \to \mathbb{C}$ if $\log|\varphi(y;v,w)| + \frac{1}{2}\log\det\Gamma(v) - \frac{1}{2}\log\det\Gamma(w) = N_s$ for all $y \in U(\mathbb{C})$ over s and all frames, where $\Gamma(v) = (h_y(v_i,v_k))_{i,k}$, $\Gamma(w) = (h_y(w_i,w_k))_{i,k}$. For a prime ℓ and a ring homomorphism $\kappa\colon F \to \mathbb{C}_\ell$, a real number N_κ is its **ℓ-adic norm** at κ if $\log|j(\varphi(y;v,w))| = N_\kappa$ whenever: k is the fraction field of a discrete valuation ring V, $j\colon k \to \mathbb{C}_\ell$ satisfies $V = \{a \in k : |j(a)| \le 1\}$ and $j \circ \beta = \kappa$, $y \in U(k)$, $\mathcal{G} \to \operatorname{Spec} V$ is a semiabelian scheme (for instance an abelian scheme) with $\iota_{\mathcal{G}}\colon \mathcal{O}_E \to \operatorname{End}(\mathcal{G})$ and an $\mathcal{O}_E$-linear isomorphism of its generic fibre with G_y, and (v,w) is an **integral frame** of $\mathcal{G}$: under this isomorphism v is a V-basis of $\operatorname{Lie}_{(\tau,+)}(\mathcal{G}) = \{x \in \operatorname{Lie}(\mathcal{G}) : \iota_{\mathcal{G}}(a)x = \beta(\tau(a))x\ \forall a \in \mathcal{O}_E\}$ and w a V-basis of the analogous $\operatorname{Lie}_{(\tau,-)}(\mathcal{G})$ for $\bar\tau$.

(Yf4) Flat generators. Let Y be a connected component of the complex manifold $U(\mathbb{C})$ of points over the inclusion $F \subset \mathbb{C}$, and $z_0 \in Y$. The lattices Λ_z ($z \in Y$) form a local system of self-dual $\mathcal{O}_E$-lattices; for a path γ in Y from z_0 to z let $\mu_\gamma\colon \Lambda_{z_0} \to \Lambda_z$ be parallel transport. For $\lambda \in \Lambda_z \subseteq \operatorname{Lie}(G_z)$ and a label s, λ_s is its component in $\operatorname{Lie}_s(G_z)$. Let $e \in \Lambda_{z_0}^n$ be $\mathcal{O}_E$-linearly independent and $c \in \mathbb{C}$. A frame function φ on U is a **flat generator over Y with constants** (e,c) if for every $z \in Y$, every path γ from z_0 to z, every frame (v,w) at z, and all $C \in \mathrm{M}_{p\times n}(\mathbb{C})$, $B,R \in \mathrm{M}_{q\times n}(\mathbb{C})$ with

$$v_a = \sum_{m=1}^{n} C_{am}(\mu_\gamma e_m)_{(\tau,+)}, \qquad (\mu_\gamma e_m)_{(\tau,-)} = \sum_{b=1}^{q} B_{bm} w_b, \qquad R\,\tau(\mathrm{Gr}(e)) = B,$$

one has $\varphi(z;v,w)\det\binom{C}{R} = c$, where $\binom{C}{R} \in \mathrm{M}_n(\mathbb{C})$ has the rows of C followed by the rows of R.

(Yf5) Pure tuples. For $z \in Y$ put $\Lambda_z^{\pm} = \{\lambda \in \Lambda_z : \iota(r)\lambda = \pm r_{\mathbb{C}}\lambda\}$. A tuple $x = (x_+, x_-) \in \Lambda_z^p \times \Lambda_z^q$ is **pure** if $x_{+,a} \in \Lambda_z^+$ and $x_{-,b} \in \Lambda_z^-$ for all a,b.

(Yf6) Pure CM points at a prime. Let φ be a flat generator over Y with constants (e,c), ℓ a prime and $\alpha \in \mathbb{C}^\times$. A **pure CM point at ℓ for φ with constant** α is a tuple (K,y,γ,x,v,w) of a number field $F \subseteq K \subset \mathbb{C}$, a point $y \in U(K)$ whose complex point z lies in Y, a path γ in Y from z_0 to z, a pure tuple $x \in \Lambda_z^n$ that is a local basis at ℓ, and a frame (v,w) at y, such that, with $T = T(\mu_\gamma^{-1}x, e)$ and x_- the last q entries of x:

(Qv1) Value $\varphi(y;v,w)\,\tau(\det T) = \alpha\,\tau(\det \mathrm{Gr}(x_-))$.

(Qv2) Integrality at ℓ For every ring homomorphism $j\colon K \to \mathbb{C}_\ell$, with $V_j = \{a \in K : |j(a)| \le 1\}$, there are an abelian scheme $\mathcal{G}$ over V_j with $\mathcal{O}_E$-action and an $\mathcal{O}_E$-linear isomorphism of its generic fibre with G_y, and $\delta,\delta' \in K^\times$ with

$$\log|j(\delta)| = \log|j(\delta')| = \log|\mathfrak{d}_E|_{j\circ\tau} - \log|j(\tau(r))|,$$

such that $(\delta v, \delta' w)$ is an integral frame of $\mathcal{G}$ (Yf3).

(Qv3) Hodge Grams at the complex places For $b \in \{+,-\}$ there are a number field $M_b \subseteq K$, an abelian scheme $\mathcal{Z}_b \to \operatorname{Spec}\mathcal{O}_{M_b}$ of relative dimension 1 whose generic fibre has ring of $\overline{\mathbb{Q}}$-endomorphisms isomorphic to $\mathcal{O}_{\mathbb{Q}(r)}$, a basis u_b of $\operatorname{Lie}(\mathcal{Z}_b/\mathcal{O}_{M_b})$ and a sign $\epsilon_b \in \{1,-1\}$, such that for every ring homomorphism $s\colon K \to \mathbb{C}$, with $y^s \in U(\mathbb{C})$ the point induced by s (lying over $s|_F$), (s_*v, s_*w) the induced frame at y^s (Yf2), Γ as in (Yf3), and $n_b(s) = \pi/\mathrm{covol}(\Lambda_{b,s})$, where $\Lambda_{b,s} \subset \mathbb{C}$ is the period lattice

of $\mathcal{Z}_b \otimes_{s|_{M_b}} \mathbb{C}$ with its Lie algebra identified with $\mathbb{C}$ by $u_b \mapsto 1$,

$$\Gamma(s_* v) = \epsilon_+ n_+(s)\Big(s\big(\tau(r\, h(x_{+,a}, x_{+,a'}))\big)\Big)_{a,a'},$$
$$\Gamma(s_* w) = \epsilon_- n_-(s)\Big(s\big(\bar\tau(r\, h(x_{-,c}, x_{-,c'}))\big)\Big)_{c,c'},$$

h the trace form of Λ_z and $\bar\tau(a) = \overline{\tau(a)}$.

(Yf7) Period space, purity, realisation. For $z \in \mathrm{Sh}_F(\mathbb{C})$ let $\Lambda_{z,\mathbb{R}} = \Lambda_z \otimes \mathbb{R}$, identified with $\mathrm{Lie}(G_z)$, and extend $\langle\,,\rangle = \mathrm{Im}\, H_z$ and ι $\mathbb{R}$-linearly. The **period space** $\mathrm{Per}(\Lambda_z)$ is the set of $\mathbb{R}$-linear maps $J\colon \Lambda_{z,\mathbb{R}} \to \Lambda_{z,\mathbb{R}}$ with $J^2 = -1$, $J\iota(a) = \iota(a)J$ for $a \in \mathcal{O}_E$, $\langle Ju, Jv\rangle = \langle u, v\rangle$ for all u, v and $\langle Ju, u\rangle > 0$ for $u \neq 0$, such that, with

$$L^J_{(\tau,+)} = \{u :\ \iota(a)u = \mathrm{Re}\,\tau(a)\, u + \mathrm{Im}\,\tau(a)\, Ju\ \forall a \in \mathcal{O}_E\},$$
$$L^J_{(\tau,-)} = \{u :\ \iota(a)u = \mathrm{Re}\,\tau(a)\, u - \mathrm{Im}\,\tau(a)\, Ju\ \forall a \in \mathcal{O}_E\}$$

for $\tau \in \Phi_{\mathsf{H}}$, $\Lambda_{z,\mathbb{R}} = \bigoplus_s L^J_s$ (over all labels s) and $\dim_{\mathbb{R}} L^J_{(\tau,+)} = 2p$, $\dim_{\mathbb{R}} L^J_{(\tau,-)} = 2q$. It carries the topology of $\mathrm{End}_{\mathbb{R}}(\Lambda_{z,\mathbb{R}})$. Multiplication by i on $\mathrm{Lie}(G_z)$ is an element J_z of $\mathrm{Per}(\Lambda_z)$ with $L^{J_z}_s = \mathrm{Lie}_s(G_z)$ (positivity of H_z, (Fh1) and the Kottwitz condition). A tuple $x = (x_+, x_-) \in \Lambda_z^p \times \Lambda_z^q$ is **pure for** $J \in \mathrm{Per}(\Lambda_z)$ if $\iota(r)x_{+,a} = \mathrm{Im}(r_{\mathbb{C}})\, Jx_{+,a}$ and $\iota(r)x_{-,b} = -\mathrm{Im}(r_{\mathbb{C}})\, Jx_{-,b}$ for all a, b; as $r_{\mathbb{C}}$ is purely imaginary, pure for J_z is pure in the sense of (Yf5). For $z_0, z \in Y$ and a path γ in Y from z_0 to z, $J \in \mathrm{Per}(\Lambda_{z_0})$ is **realised at** z **along** γ if $\mu_\gamma \circ J = J_z \circ \mu_\gamma$ (μ_γ extended $\mathbb{R}$-linearly).

Lemma 33.15. *Let $\mathcal{X}^{\mathrm{b}}$ be either (i) a cover as in Proposition 33.9, for an irreducible component of an integral model of* Sh *and a finite set S of primes containing every prime dividing* $\mathrm{disc}(E)$*, with its Hodge bundles $\mathcal{V}_s$ at (S, R), $R \equiv 4$; or (ii) a cover as in Proposition 33.4, with the Hodge bundles $\mathcal{V}_s$ formed by its de Rham lattices. Let s be a label, $1 \le j \le \mathrm{rk}(s)$, and $\gamma\colon \mathrm{Gr}_j(\mathcal{V}_s) \to \mathcal{X}^{\mathrm{b}}$ the scheme of locally free subsheaves of rank j of $\mathcal{V}_s$ with locally free quotient ($\gamma = \mathrm{id}$ if $j = \mathrm{rk}(s)$), with tautological subbundle $\mathcal{W}$. Let $P = \mathrm{Gr}_j(\mathcal{V}_s)(\mathbb{C})$ (complex points over all embeddings of F), a compact complex manifold, ∂P the complement of the preimage P° of the abelian locus, ς a holomorphic section of a smooth hermitian line bundle on P with $\|\varsigma\| \le 1$ whose zero divisor is $\sum_i m_i W_i$ ($m_i \ge 1$, W_i smooth meeting with simple normal crossings) with support ∂P, and* $\mathrm{rW} = e - 2\log\|\varsigma\|$*. Let $\mathcal{M} = (\det \mathcal{W})^{-1}$ on P with a smooth hermitian metric $\|\cdot\|_{\mathrm{ref}}$, and on P° let $\|\cdot\|_{\mathrm{Hdg}}$ be the Hodge metric of $\mathcal{M}$, i.e. $\|(w_1 \wedge \cdots \wedge w_j)^\vee\|^2_{\mathrm{Hdg}} = \det \Gamma(w_1, \ldots, w_j)^{-1}$ for every local frame w of $\mathcal{W}$. Put $\varphi = \log\|\cdot\|_{\mathrm{ref}} - \log\|\cdot\|_{\mathrm{Hdg}}$ on P°, so that $\|\cdot\|_{\mathrm{Hdg}} = \|\cdot\|_{\mathrm{ref}} e^{-\varphi}$ is the metric $h^{\mathrm{ref}} e^{\varphi}$ in cone-function notation. Then ∂P is a simple normal crossing divisor, and φ satisfies hypotheses 1 and 2 of Theorem 33.22 on P for some $C_5 \ge 0$.*

Proof. **The boundary.** γ is smooth and projective (Grassmannian bundle), so P is a compact complex manifold and, $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ having a simple normal crossing non-abelian locus (item 2 of Proposition 33.9 in case (i), item 2 of Proposition 33.4 in case (ii); the two items say the same), its preimage ∂P is a simple normal crossing divisor: local coordinates $t_1, \ldots, t_d$ of the base in which the boundary is $\{t_1 \cdots t_a = 0\}$, completed by fibre coordinates ξ, are coordinates of P in which $\partial P = \{t_1 \cdots t_a = 0\}$. On $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ the Hodge lattices are the de Rham lattices (case (i): no point of $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ lies over a prime of S, (Fh5); case (ii): by definition), and the de Rham lattice $\mathrm{dR}_{s,x} = \mathrm{Lie}_x \cap \mathrm{Li}_s$ at a point x of $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ is the stalk of $\mathrm{Lie}_s(G_{\mathcal{X}^{\mathrm{b}}})$; so $\mathcal{V}_s \otimes_{F,\sigma} \mathbb{C} = \mathrm{Lie}_s(G_{\mathcal{X}^{\mathrm{b}}})^{\mathrm{an}}$ over every embedding σ, and $\mathcal{W}$ is a holomorphic subbundle of $\gamma^*\mathrm{Lie}(G_{\mathcal{X}^{\mathrm{b}}})^{\mathrm{an}}$.

Plurisubharmonicity. On the abelian locus the kernels Λ_z of the exponential maps form a local system and $(\Lambda, \mathcal{F}^0 = \ker(\Lambda \otimes \mathcal{O} \to \mathrm{Lie}), -\mathrm{Im}\, H)$ is a polarised variation of Hodge structure of weight -1 with $\mathcal{F}^{-1}$ everything and $\mathcal{F}^1 = 0$, whose last quotient $\mathcal{H}/\mathcal{F}^0$ is Lie with Hodge metric $\frac{1}{2}H_z = (2\pi)^{-1}h_z$ (classical: Deligne, Théorie de Hodge II, 4.4.3; the Hodge form on the summand $\overline{\mathcal{F}^0}$, which $\mathcal{H} \to \mathrm{Lie}$ maps isomorphically, is $iE(y, \bar{y}) = \frac{1}{2}H_z(y', y')$ for $y \mapsto y'$, with $E = \mathrm{Im}\, H_z$, the identity also recorded in Lemma 33.16). By Theorem B.144, (Lie, h_z) is Griffiths seminegative on the abelian locus, hence so is its pull-back to P°, and so is the holomorphic subbundle $\mathcal{W}$ with the induced metric (curvature decreases in holomorphic subbundles: Griffiths–Harris, Principles of Algebraic Geometry, Ch. 0 §5). The determinant of a Griffiths-seminegative bundle has seminegative curvature (the trace of the curvature), so for every local holomorphic frame w of $\mathcal{W}$ the function $\log \|w_1 \wedge \cdots \wedge w_j\|^2 = \log\det \Gamma(w)$ is plurisubharmonic. For the local generator $\sigma = (w_1 \wedge \cdots \wedge w_j)^\vee$ of $\mathcal{M}$, $-\log\|\sigma\|_{\mathrm{Hdg}} = \frac{1}{2}\log\det\Gamma(w)$; so the local potentials of $\|\cdot\|_{\mathrm{Hdg}} = h^{\mathrm{ref}}e^{\varphi}$ are plurisubharmonic on P°, and $\varphi = \log\|\sigma\|_{\mathrm{ref}} + \frac{1}{2}\log\det\Gamma(w)$ is smooth there (h_z is smooth on the abelian locus).

Local estimates at a boundary point. Let $x_0 \in \partial P$ over $b_0 \in \mathcal{X}^{\mathrm{b}}(\mathbb{C})$, over the embedding σ. The projective variety $\mathcal{X}^{\mathrm{b}}_F \otimes_{F,\sigma} \mathbb{C}$ ($\mathcal{X}^{\mathrm{b}}$ is projective over $\mathbb{Z}$ in both cases, and its structure map factors through the component it covers) with the semiabelian $G_{\mathcal{X}^{\mathrm{b}}}$, abelian exactly over the abelian locus, is an instance of the setting of Lemma 33.16 (case of a variety mapping to a component of $\mathcal{Y}$); take an SNC chart $t\colon V \to \Delta^d$ at b_0 with $V \cap (\text{abelian locus}) = \{t_1 \cdots t_a \neq 0\}$ and $L = 1 + \sum_{k\le a} |\log|t_k||$. Let $e = (e_1, \ldots, e_r)$, $r = \mathrm{rk}(s)$, be a holomorphic frame of $\mathrm{Lie}_s(G_{\mathcal{X}^{\mathrm{b}}})^{\mathrm{an}}$ near b_0. By Lemma 33.17 and Lemma 33.19(2) there are $C \ge 1$ and a neighbourhood V' of b_0 with

$$C^{-1}L^{-1}I_r \le \Gamma(e) \le C\, I_r, \qquad |\partial\Gamma(e)_{kl}/\partial t_m| + |\partial\Gamma(e)_{kl}/\partial \bar{t}_m| \le C/|t_1 \cdots t_a| \quad \text{on } V' \cap V^*.$$

Near x_0 choose a Grassmannian chart: after renumbering e, a neighbourhood U of x_0 with compact closure is $\{(\xi, b)\}$ with ξ in a compact ball of $\mathbb{C}^{j\times(r-j)}$, $b \in V'$, and $w = A(\xi)e$ with $A(\xi) = (I_j \mid \xi)$ is a holomorphic frame of $\mathcal{W}$ on U. Then $\Gamma(w) = A\,\Gamma(e)\,A^*$ (in the convention of the Gram matrix), $c_A I_j \le AA^* \le C_A I_j$ on U, dA is bounded and $\Gamma(e)$ does not depend on ξ. Hence on $U \cap P^\circ$:

$$C^{-1}L^{-1}c_A I_j \le \Gamma(w) \le CC_A I_j, \qquad \|\Gamma(w)^{-1}\| \le Cc_A^{-1}L, \qquad \|d\Gamma(w)\| \le C'(1 + 1/|t_1 \cdots t_a|).$$

As $\log\|\sigma\|_{\mathrm{ref}}$ is smooth on $\bar{U}$, we get $-C'' - \frac{j}{2}\log L \le \varphi \le C''$ and

$$\|d\varphi\| \le \|d\log\|\sigma\|_{\mathrm{ref}}\| + \tfrac{1}{2}\|\Gamma(w)^{-1}\|\,\|d\Gamma(w)\| \le C'''L/|t_1 \cdots t_a| \qquad (\text{as } |t_k| < 1).$$

Near x_0, $\|\varsigma\| = |u|\prod_{k\le a}|t_k|^{m_k}$ with u smooth and non-vanishing, so $\mathrm{rW} \ge e + 2\sum_{k\le a}|\log|t_k|| - C_4$; hence $L \le C_6\,\mathrm{rW}$, and $1/|t_1 \cdots t_a| \le C_7/\|\varsigma\| = C_7 e^{-e/2}\exp(\mathrm{rW}/2)$. As $\mathrm{rW} \ge e$, we have $\log\mathrm{rW} \ge 1$, which absorbs the constants:

$$-C_5\log\mathrm{rW} \le \varphi \le C_5, \qquad \|d\varphi\| \le C_5\,\mathrm{rW}\exp(\mathrm{rW}/2)$$

on $U \cap P^\circ$, in the coordinates (ξ, t), for a suitable C_5.

Global constants. At a point of P°, φ is smooth and rW is bounded on a compact neighbourhood, so both bounds hold there. P is compact; finitely many such neighbourhoods cover it, and the largest C_5 serves: hypotheses 1 and 2 of Theorem 33.22 hold. □

Lemma 33.16. *Let E be a CM field, $X_{\mathbb{C}}$ a projective variety over $\mathbb{C}$, $G' \to X_{\mathbb{C}}$ a semiabelian scheme of relative dimension g with a ring homomorphism $\iota_{\mathcal{Y}}\colon \mathcal{O}_E \to \mathrm{End}(G')$, and $X^\circ_{\mathbb{C}} \subseteq X_{\mathbb{C}}$ a Zariski open subset such that G' is an abelian scheme exactly over $X^\circ_{\mathbb{C}}$ and $G'|_{X^\circ_{\mathbb{C}}}$ carries a*

principal polarisation whose Rosati involution maps $\iota_{\mathcal{Y}}(a)$ to $\iota_{\mathcal{Y}}(\bar a)$; for $z \in X^\circ_{\mathbb{C}}(\mathbb{C})$ let $G_z(\mathbb{C}) = \mathrm{Lie}(G_z)/\Lambda_z$ and H_z the positive hermitian form on $\mathrm{Lie}(G_z)$ with $\operatorname{Im} H_z$ the Riemann form of the polarisation on Λ_z. (This applies (A) to $X_{\mathbb{C}} = \mathcal{Y}'_F \otimes_{F,s} \mathbb{C}$, $G' = G_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{Y}'_F$ and $X^\circ_{\mathbb{C}} = j(\mathrm{Sh}_F) \cap \mathcal{Y}'_F$ base changed along s, for a unitary Shimura datum with $n = p + q \geq 1$, an integral model $(\mathcal{Y}, j, G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$, an irreducible component $\mathcal{Y}'$ meeting $j(\mathrm{Sh}_F)$ and an embedding $s\colon F \to \mathbb{C}$, by (Ud7), with H_z as in Definition 29.6; and (B) to every projective variety $X_{\mathbb{C}}$ with a morphism π to the $X_{\mathbb{C}}$ of (A), G' and the polarisation pulled back and $X^\circ_{\mathbb{C}} = \pi^{-1}$ of the one of (A), for instance the Hodge base of Proposition 33.9 with its semiabelian scheme.) Put $X = X^{\mathrm{an}}_{\mathbb{C}}$. Let $x \in X$, $V \subseteq X$ an open neighbourhood of x, $1 \leq a \leq d$ and $t\colon V \to \Delta^d$ a biholomorphic map with $t(x) = 0$ and $t(V \cap X^\circ_{\mathbb{C}}) = \{t_1 \cdots t_a \neq 0\}$; put $V^* = V \cap X^\circ_{\mathbb{C}}$. Then the local system $z \mapsto \Lambda_z$ on V^* with $\mathcal{F}^0_z = \ker(\Lambda_z \otimes \mathbb{C} \to \mathrm{Lie}(G_z))$, $\mathcal{F}^{-1}_z = \Lambda_z \otimes \mathbb{C}$, $\mathcal{F}^1_z = 0$ and $Q = -\operatorname{Im} H_z$ is a polarised variation of Hodge structure of weight -1, whose Hodge metric h_z satisfies $h_z(y,y) = \frac{1}{2} H_z(y', y')$ for y in the h_z-orthogonal complement of $\mathcal{F}^0_z$ and y' its image in $\mathrm{Lie}(G_z)$, and whose monodromies $T_1, \ldots, T_a$ around the divisors $\{t_j = 0\}$ satisfy $(T_j - 1)^2 = 0$; and there are a neighbourhood $V_0 \subseteq V$ of x and a surjective holomorphic map of vector bundles $\widehat{P}\colon \widehat{\mathcal{H}}|_{V_0} \to \mathrm{Lie}(G')^{\mathrm{an}}|_{V_0}$ from the canonical extension, with kernel $\widehat{\mathcal{F}}^0|_{V_0}$, which on $V_0 \cap V^*$ is the $\mathbb{C}$-linear extension of the inclusion $\Lambda_z \subset \mathrm{Lie}(G_z)$. Moreover, with $N_j = T_j - 1$, $N_\kappa = \sum_{j \in \kappa} N_j$ for $\emptyset \neq \kappa \subseteq \{1, \ldots, a\}$ and $N = N_{\{1,\ldots,a\}}$, one has $N^2_\kappa = 0$ for every κ, and the flat sections of $\Lambda \otimes \mathbb{Q}$ invariant under all T_j form $\ker N = \bigcap_j \ker N_j$, of rank $2g - t_x$, t_x the dimension of the torus part of the fibre G'_x.

Proof. The variation. Over V^*, G' is the abelian scheme G_z with its principal polarisation, $G_z(\mathbb{C}) = \mathrm{Lie}(G_z)/\Lambda_z$, and the kernels Λ_z of the holomorphic exponential map form a local system of free $\mathbb{Z}$-modules of rank $2g$ whose local sections are holomorphic sections of $\mathrm{Lie}(G')^{\mathrm{an}}$. So $P\colon \Lambda \otimes \mathcal{O}_{V^*} \to \mathrm{Lie}(G')^{\mathrm{an}}$, the $\mathcal{O}$-linear extension of the inclusion, is a surjective holomorphic map of vector bundles (fibrewise the surjection $\Lambda_z \otimes \mathbb{C} \to \Lambda_z \otimes \mathbb{R} = \mathrm{Lie}(G_z)$), and $\mathcal{F}^0 = \ker P$ is a holomorphic subbundle of rank g. $E = \operatorname{Im} H_z$ is the integral unimodular alternating form of the polarisation, constant on the local system. Let J be multiplication by i on $\mathrm{Lie}(G_z) = \Lambda_z \otimes \mathbb{R}$, extended $\mathbb{C}$-linearly; $\mathcal{F}^0_z$ is its $(-i)$-eigenspace and its complex conjugate $V^{1,0}_z$ the $(+i)$-eigenspace, mapped isomorphically onto $\mathrm{Lie}(G_z)$ by P. For $y \in V^{1,0}_z$ with $P(y) = u$ one has $y = \frac{1}{2}(u - iJu)$ and

$$iE(y, \bar y) = \tfrac{i}{4}(2iE(u, Ju)) = \tfrac{1}{2}E(Ju, u) = \tfrac{1}{2}H_z(u, u),$$

using $E(Ju, Jv) = E(u, v)$ and $H_z(u, v) = E(Ju, v) + iE(u, v)$. Both eigenspaces are E-isotropic, so $\Lambda_z \otimes \mathbb{C} = V^{1,0}_z \oplus \mathcal{F}^0_z$ is a Hodge structure of weight -1 of types $(-1, 0)$, $(0, -1)$; with $Q = -E$ its Hodge form $Q(Cy, \bar y')$ ($C = -i$ on $V^{1,0}$, i on $\mathcal{F}^0$) is $iE(y, \bar y')$ on $V^{1,0}_z$ and $-iE(y, \bar y')$ on $\mathcal{F}^0_z$, positive definite by the display and its conjugate, and the decomposition is orthogonal for it. Griffiths transversality is empty for a filtration with $\mathcal{F}^{-1}$ everything and $\mathcal{F}^1 = 0$. So this is a polarised variation of Hodge structure of weight -1 with $h_z(y, y) = \frac{1}{2}H_z(P(y), P(y))$ on $V^{1,0}_z$.

Curves through boundary points. $X_{\mathbb{C}}$ is a projective variety with $X = X^{\mathrm{an}}_{\mathbb{C}}$. Let $x' \in V$ be a point with $t(x') = c$, and let $\kappa \subseteq \{1, \ldots, a\}$ be the set of j with $c_j = 0$, assumed non-empty; V is smooth at x'. Intersecting $X_{\mathbb{C}}$, in a projective embedding, with $d - 1$ general hyperplanes through x' gives an irreducible curve C_1 through x', smooth at x', whose tangent line at x' lies in no hyperplane $\{dt_j = 0\}$, $j \in \kappa$. Near x', C_1 is a smooth analytic curve on which each t_j, $j \in \kappa$, has a simple zero at x', and t_j, $j \in \{1, \ldots, a\} \smallsetminus \kappa$, has no zero; so a small loop γ in $C_1 \smallsetminus \{x'\}$ around x' lies in V^* and is homotopic in V^* to the product of the γ_j, $j \in \kappa$; its monodromy is $T_\kappa = \prod_{j \in \kappa} T_j$. Let R be the henselisation of the local ring of the normalisation of C_1 at the point over x', a strictly henselian discrete valuation ring with residue field $\mathbb{C}$ and fraction field K, and A the generic fibre of the pull-back $\mathcal{G}$ of G' to R, an abelian variety; $\mathcal{G}$ is semiabelian with

special fibre $G'_{x'}$, an extension of an abelian variety of dimension $g - t_{x'}$ by a torus of dimension $t_{x'}$. By the Riemann existence theorem the finite separable extensions of K are the Kummer extensions $K(\varpi^{1/m})$, ϖ a uniformiser, so $I = \mathrm{Gal}(K^s/K) \cong \widehat{\mathbb{Z}}(1)$, and by the comparison of the ℓ-adic Tate module of the analytic family of tori with $\Lambda \otimes \mathbb{Z}_\ell$ ($A[\ell^k](K^s) = \ell^{-k}\Lambda_z/\Lambda_z$ at a base point z of γ), a topological generator of I acts on $T_\ell(A) = \Lambda_z \otimes \mathbb{Z}_\ell$ as $T_\kappa^{\pm 1}$. By Theorem B.113,

$$(T_\kappa - 1)^2 = 0, \qquad \mathrm{rank}_{\mathbb{Z}} \ker(T_\kappa - 1) = 2(g - t_{x'}) + t_{x'} = 2g - t_{x'}. \tag{Cv}$$

Applied to x' with $c_j = 0$ exactly for one j (such points exist near x), (Cv) gives $(T_j - 1)^2 = 0$ for each j. So the T_j are unipotent and commute, $N_j = \log T_j = T_j - 1$, and $T_\kappa = \exp(N_\kappa)$ with $N_\kappa = \sum_{j\in\kappa} N_j$ nilpotent; $T_\kappa - 1 = N_\kappa \exp'$ with $\exp' = \sum_{k\geq 0} N_\kappa^k/(k+1)!$ invertible, so $\ker(T_\kappa - 1) = \ker N_\kappa$ and $(T_\kappa - 1)^2 = 0$ gives $N_\kappa^2 = 0$. As points x' near x with $\{j \leq a : c_j = 0\} = \kappa$ exist for every non-empty κ, $N_\kappa^2 = 0$ for every κ. Applied to $x' = x$ ($\kappa = \{1, \dots, a\}$): $N = \sum_j N_j$ satisfies $N^2 = 0$ and rank $\ker N = 2g - t_x$.

Sections of the exponential kernel through x. Let $\mathcal{K} \subseteq \mathrm{Lie}(G')^{\mathrm{an}}$ (total space over V) be the kernel of $\exp\colon \mathrm{Lie}(G')^{\mathrm{an}} \to G'^{\mathrm{an}}$. For $\lambda_0 \in \mathcal{K}_x = \ker \exp_x$ and a holomorphic section σ of $\mathrm{Lie}(G')^{\mathrm{an}}$ near x with $\sigma(x) = \lambda_0$, the map $(z, u) \mapsto \exp_z(u - \sigma(z))$ is, near (x, λ_0), a local biholomorphism onto a neighbourhood of the unit section (the exponential is a local biholomorphism near the zero section), so $\mathcal{K}$ is near (x, λ_0) the graph of a holomorphic section $\hat{\lambda}_0$. The group $\mathcal{K}_x$ of the semiabelian group G'_x is free of rank $2g - t_x$ and spans $\mathrm{Lie}(G'_x)$ over $\mathbb{C}$ (it contains $2\pi i$ times the cocharacters of the torus, which span its Lie algebra, and maps onto the period lattice of the abelian quotient). Choose a $\mathbb{Z}$-basis of $\mathcal{K}_x$ and a polydisc V_1 around x on which all their sections $\hat{\lambda}$ are defined. On $V_1 \cap V^*$ each $\hat{\lambda}(z)$ lies in Λ_z and depends continuously on z, so it is a flat single-valued section of Λ, invariant under every T_j; distinct elements of $\mathcal{K}_x$ give distinct sections. Let Λ' be the group of flat sections so obtained. Then $\Lambda' \subseteq \bigcap_j \ker N_j \subseteq \ker N$, and $\mathrm{rank}\,\Lambda' = 2g - t_x = \mathrm{rank}\ker N$; hence

$$\Lambda' \otimes \mathbb{Q} = \bigcap_j \ker N_j \otimes \mathbb{Q} = \ker N \otimes \mathbb{Q} \quad \text{in } H_{\mathbb{Q}}, \tag{Ic}$$

H the space of flat multivalued sections of $\Lambda \otimes \mathbb{C}$.

The limit filtration. Apply Theorem B.145 to this variation ($w = -1$): on a polydisc $V_2 \subseteq V_1$ around x, $\mathcal{F}^0$ extends to a subbundle $\widehat{\mathcal{F}}^0$ of rank g of $\widehat{\mathcal{H}}$, and (W, F_0) is a mixed Hodge structure on H, W the weight filtration of N centred at -1. As $N^2 = 0$, $W_{-2} = \mathrm{im}\, N$, $W_{-1} = \ker N$, $W_0 = H$. The extension of the zero bundle $\mathcal{F}^1$ is zero and that of $\mathcal{F}^{-1}$ is $\widehat{\mathcal{H}}$, so $F_0^1 = 0$ and $F_0^{-1} = H$; hence the pure Hodge structure Gr_0^W of weight 0 has Hodge numbers $h^{p,-p} = 0$ unless $p \leq 0$ and $-p \leq 0$: it is of type $(0,0)$ and $F^0\mathrm{Gr}_0^W = \mathrm{Gr}_0^W$. By strictness of F on W the image of F_0^0 in H/W_{-1} is everything, so by (Ic)

$$F_0^0 + (\Lambda' \otimes \mathbb{C}) = F_0^0 + W_{-1} = H. \tag{Sp}$$

The map $\widehat{P}$. Each $\lambda \in \Lambda'$ is invariant under every N_j, so $\tilde{\lambda} = \lambda$, and $\hat{\lambda}$ is at once a holomorphic section of $\widehat{\mathcal{H}}$ over V_2 (with value λ at x) and a holomorphic section of $\mathrm{Lie}(G')^{\mathrm{an}}$, related by P on $V_2 \cap V^*$. By (Sp) choose $\lambda_1, \dots, \lambda_g \in \Lambda'$ whose classes form a basis of H/F_0^0, and a holomorphic frame $f_1, \dots, f_g$ of $\widehat{\mathcal{F}}^0$ near x; on a polydisc $V_0 \subseteq V_2$ around x, $(\hat{\lambda}_1, \dots, \hat{\lambda}_g, f_1, \dots, f_g)$ is a frame of $\widehat{\mathcal{H}}$. Define $\widehat{P}$ on V_0 by $\widehat{P}(\hat{\lambda}_i) = \hat{\lambda}_i$ (as section of $\mathrm{Lie}(G')^{\mathrm{an}}$) and $\widehat{P}(f_k) = 0$; it is holomorphic and agrees with P on $V_0 \cap V^*$. For every $\lambda \in \Lambda'$, $\widehat{P}(\hat{\lambda})$ and $\hat{\lambda}$ are holomorphic sections of $\mathrm{Lie}(G')^{\mathrm{an}}$ on V_0 equal on the dense set $V_0 \cap V^*$, hence $\widehat{P}_x(\lambda) = \hat{\lambda}(x)$; these values span $\mathcal{K}_x \otimes \mathbb{C} = \mathrm{Lie}(G'_x)$. So $\widehat{P}_x$ is surjective; its kernel has dimension g and contains F_0^0, of dimension g, so equals it. Surjectivity is an open condition; shrinking V_0, $\widehat{P}$ is surjective on V_0 with kernel containing, hence (equal ranks) equal to, $\widehat{\mathcal{F}}^0$. The last assertion of the statement is (Ic) with the rank computed above. □

Lemma 33.17. *In the setting of Lemma 33.16, let $b_1,\dots,b_m$ be holomorphic sections of $\mathrm{Lie}(G')^{\mathrm{an}}$ on a neighbourhood of x, linearly independent at x, and put $L(z) = 1 + \sum_{j=1}^{a} |\log|t_j(z)||$. Then there are $C \geq 1$ and a neighbourhood V' of x such that for every $z \in V' \cap V^*$, with $\Gamma(z) = \Gamma(b_1(z),\dots,b_m(z))$, as hermitian matrices*

$$C^{-1}L(z)^{-1}\,I_m \;\leq\; \Gamma(z) \;\leq\; C\,I_m, \qquad \textit{equivalently} \qquad C^{-1}I_m \leq \Gamma(z)^{-1} \leq C\,L(z)\,I_m.$$

Proof. Reduction to one frame. Let $v_1,\dots,v_t$ be a $\mathbb{Z}$-basis of I (Lemma 33.18); the $\widehat{P}_x(v_a)$ are linearly independent. By Theorem B.145 (clause 2) (W, F_0) is a mixed Hodge structure, W the weight filtration of N centred at -1 ($W_{-1} = \ker N$, $W_0 = H$ as $N^2 = 0$); as $F_0^1 = 0$ and $F_0^{-1} = H$ (extensions of the zero and of the whole bundle), Gr_0^W has Hodge types $(p,-p)$ with $-1 \leq p \leq 0$ and $-1 \leq -p \leq 0$, i.e. type $(0,0)$, so by strictness $F_0^0 + W_{-1} = H$. As $\ker \widehat{P}_x = F_0^0$, $\widehat{P}_x(W_{-1}\otimes\mathbb{C}) = \widehat{\bar{P}}_x(H) = \mathrm{Lie}(G'_x)$. As $H_{\mathbb{Z}} \cap W_{-1}$ spans W_{-1}, there are $\lambda_{t+1},\dots,\lambda_g \in H_{\mathbb{Z}} \cap W_{-1}$ such that $\beta = (\widehat{P}(v_1),\dots,\widehat{P}(v_t),\widehat{P}(\lambda_{t+1}),\dots,\widehat{P}(\lambda_g))$ is a holomorphic frame near x; on V^* its members are $P(v_a)$, $P(\lambda_i)$, since invariant flat sections are sections of $\widehat{\mathcal{H}}$. Extend b to a holomorphic frame $\hat{b} = (b, b')$ near x; then $\hat{b} = A\beta$ with A holomorphic and invertible near x, so on a compact neighbourhood $\|A\|, \|A^{-1}\| \leq C_0$. If $C^{-1}L^{-1}I \leq \Gamma(\beta) \leq CI$, then $C_0^{-2}C^{-1}L^{-1}I \leq \Gamma(\hat{b}) \leq C_0^2 CI$, and $\Gamma(b)$ is a principal submatrix of $\Gamma(\hat{b})$, so its eigenvalues lie between the extreme eigenvalues of $\Gamma(\hat{b})$. It suffices to treat β.

Real flat vectors. For $z \in V^*$ and real $w, w' \in \Lambda_z \otimes \mathbb{R}$ write $w = f + \bar{f}$ with $f \in \mathcal{F}_z^0$; then $P(w) = P(\bar{f})$ and, as $h^{\mathrm{Hdg}}(f,f') = \overline{h^{\mathrm{Hdg}}(\bar{f},\bar{f'})}$ and the Hodge decomposition is orthogonal,

$$\mathrm{Re}\,H_z(P(w),P(w')) = 2\,\mathrm{Re}\,h^{\mathrm{Hdg}}(\bar{f},\bar{f'}) = h^{\mathrm{Hdg}}(w,w'), \qquad \mathrm{Im}\,H_z(P(w),P(w')) = -Q(w,w'). \tag{Re}$$

In particular $\|P(w)\|_H^2 = |w|_z^2$, the Hodge norm.

Upper bound. Each $w \in \{v_a, \lambda_i\}$ lies in $\ker N_j$ for all j, hence in $\ker N(\nu)$ for every numbering and ν ($N(\nu)$ a partial sum), and $\ker N(\nu) \subseteq W_0(\nu)$ (if $u \in W_k \smallsetminus W_{k-1}$ with $k \geq 1$, then N^k maps its class isomorphically to Gr_{-k}, so $Nu \neq 0$). By Theorem B.146 with all $p_\nu = 0$, on each of the finitely many slit sectors and numberings (as in Lemma 34.18; the w are single-valued), $|w|_z^2 \leq C_1$ near x. For $u = \sum_k \xi_k\beta_k(z)$, $\|u\|_H \leq \sum_k |\xi_k|\|\beta_k\|_H \leq |\xi|(gC_1)^{1/2}$, so $\Gamma(\beta) \leq \pi g C_1 I$.

The vanishing directions. Let $G_v = (H_z(P(v_a),P(v_b)))_{a,b}$. By (Re) and $Q(v_a,v_b) = 0$ (I isotropic), $G_v = (h^{\mathrm{Hdg}}(v_a,v_b))$ is real symmetric, so its smallest eigenvalue is

$$\min_{x\in\mathbb{R}^t,|x|=1} |\sum_a x_a v_a|_z^2.$$

Fix a numbering and a splitting $\widehat{\mathcal{H}} = \bigoplus_p U_p$ as in Theorem B.147. By Lemma 33.16, $N(\nu)^2 = 0$, so the weight filtration of $N(\nu)$ centred at 0 has $W_{-2}(\nu) = 0$; hence $U_p = 0$ unless $p_\nu \geq -1$ for all ν. The constant section $u = \sum_a x_a v_a$ of $\widehat{\mathcal{H}}$ lies in $\bigcap_\nu \widehat{W}_0(\nu)$, so $u_p = 0$ unless $p_\nu \leq 0$ for all ν. Near 0, $y_j \geq 1$; at points where $y_{j_1} \leq \cdots \leq y_{j_a}$ (every point satisfies this for some numbering, and then $y_{j_\nu}/y_{j_{\nu-1}} \geq 1 > \frac{1}{2} = \varepsilon$), for p with all $p_\nu \in \{-1, 0\}$,

$$m_p = y_{j_1}^{p_1} \prod_{\nu\geq 2} \Big(\frac{y_{j_\nu}}{y_{j_{\nu-1}}}\Big)^{p_\nu} \;\geq\; y_{j_1}^{-1} \prod_{\nu\geq 2} \frac{y_{j_{\nu-1}}}{y_{j_\nu}} = \frac{1}{y_{j_a}} \geq \frac{1}{L}.$$

So by Theorem B.147, $|u|_z^2 \geq C_2^{-1}L^{-1}\sum_p \|u_p\|^2 \geq C_3^{-1}L^{-1}\|u\|^2 \geq C_4^{-1}L^{-1}|x|^2$ (the projections to the U_p are bounded near 0, and the v_a are linearly independent constant sections). Taking the worst constant over the finitely many numberings, $G_v \geq C_4^{-1}L^{-1}I_t$ near x.

The quotient directions. Let $S_z = \operatorname{span} P(I)(z)$ and $\Gamma^{\mathrm{q}}(z)$ the Gram matrix of the images of $P(\lambda_{t+1}), \dots, P(\lambda_g)$ in $\operatorname{Lie}(G_z)/S_z$ for the quotient metric of πH_z. By Gram–Schmidt, $\det \Gamma^{\mathrm{q}} = \det \Gamma(\beta)/\det \Gamma(P(v))$, which tends to $\nu > 0$ (Lemma 33.18); and $\Gamma^{\mathrm{q}} \le \Gamma(P(\lambda)) \le \pi g C_1 I$ (quotient norms are at most the norms). Hence near x the smallest eigenvalue of Γ^{q} is at least $\frac{\nu}{2}(\pi g C_1)^{-(g-t-1)} =: c_5 > 0$.

Combination. Let $u = \sum_a \alpha_a P(v_a) + \sum_i \gamma_i P(\lambda_i)$, $w = \sum_i \gamma_i P(\lambda_i)$, and decompose $u = s + r$ with $s \in S_z$, $r \perp_H S_z$. Then $\pi\|r\|_H^2$ is the quotient norm of the class of w, so $\pi\|r\|_H^2 \ge c_5|\gamma|^2$; and $s = \sum_a(\alpha_a + \delta_a)P(v_a)$, where $\sum_a \delta_a P(v_a)$ is the H-orthogonal projection of w to S_z, so $C_4^{-1}L^{-1}|\delta|^2 \le \|\sum \delta_a P(v_a)\|_H^2 \le \|w\|_H^2 \le gC_1|\gamma|^2$, i.e. $|\delta|^2 \le C_6 L|\gamma|^2$. If $|\alpha| \ge 2|\delta|$, then $\|s\|_H^2 \ge C_4^{-1}L^{-1}|\alpha+\delta|^2 \ge \frac14 C_4^{-1}L^{-1}|\alpha|^2$; otherwise $|\alpha|^2 < 4C_6L|\gamma|^2$, so $|\alpha|^2 L^{-1} < 4C_6|\gamma|^2$. In both cases

$$\pi\|u\|_H^2 = \pi\|s\|_H^2 + \pi\|r\|_H^2 \;\ge\; C_7^{-1}L^{-1}(|\alpha|^2 + |\gamma|^2)$$

with C_7 depending only on C_4, C_6, c_5 (use $L \ge 1$). This is $\Gamma(\beta) \ge C_7^{-1}L^{-1}I$; with the upper bound, the claim holds for β, hence for b. □

Lemma 33.18. *In the setting of Lemma 33.16, let $H_{\mathbb{Z}}$ be the group of flat sections of Λ over the universal cover of V^*, $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q}$, and $I = H_{\mathbb{Z}} \cap N(H_{\mathbb{Q}})$ (the* vanishing cycles *at x). Then I is a saturated isotropic subgroup of rank t_x whose elements are invariant under every T_j, hence single-valued flat sections of Λ over V^*. For every $\mathbb{Z}$-basis $v_1, \dots, v_{t_x}$ of I the sections $P(v_1), \dots, P(v_{t_x})$ of $\operatorname{Lie}(G')^{\mathrm{an}}$ over V^* extend holomorphically to a neighbourhood of x and are linearly independent at x, and for every holomorphic frame $b = (b_1, \dots, b_g)$ of $\operatorname{Lie}(G')^{\mathrm{an}}$ on a neighbourhood of x there is a real number $\nu > 0$ with*

$$\lim_{z \to x,\ z \in V^*} \frac{\det \Gamma(b(z))}{\det \Gamma\big(P(v_1)(z), \dots, P(v_{t_x})(z)\big)} = \nu$$

(the denominator is 1 if $t_x = 0$). In particular $\log\det\Gamma(b(z)) - \log\det\Gamma(P(v)(z))$ is bounded on $V'' \cap V^*$ for a neighbourhood V'' of x.

(Ce) Eigen-components. *I is an $\mathcal{O}_E$-submodule. For an embedding $\sigma\colon E \to \mathbb{C}$ let $\operatorname{Lie}_\sigma(G')^{\mathrm{an}}$ be the subbundle on which $\iota_{\mathcal{Y}}(a)$ acts by $\sigma(a)$, and for $u \in \Lambda_z \otimes \mathbb{C}$ let u_σ be its component in the σ-eigenspace, so that $P(u_\sigma)$ is the $\operatorname{Lie}_\sigma$-component of $P(u)$. If $I \ne 0$, ρ is the rank of the E-vector space $I \otimes \mathbb{Q}$ and $v_1^I, \dots, v_\rho^I \in I$ are $\mathcal{O}_E$-linearly independent, then the sections $P((v_a^I)_\sigma)$ extend holomorphically near x, are linearly independent at x, and for every holomorphic frame b_σ of $\operatorname{Lie}_\sigma(G')^{\mathrm{an}}$ near x there is $\nu_\sigma > 0$ with $\det\Gamma(b_\sigma(z))/\det\Gamma(P((v_1^I)_\sigma)(z), \dots, P((v_\rho^I)_\sigma)(z)) \to \nu_\sigma$ as $z \to x$, $z \in V^*$.*

(Cn) Cusp norm. *Let $\Phi = \Phi_{\mathsf{H}}$, for $\tau \in \Phi$ let σ_τ be the character by which $\mathcal{O}_E$ acts on $\operatorname{Lie}_{(\tau,-)}$ at the points of X (so $\operatorname{Lie}_{\sigma_\tau} = \operatorname{Lie}_{(\tau,-)}$), let $I \ne 0$ and v^I be as in (Ce), spanning a subgroup of finite index k_I in I, and put*

$$\mathrm{N}_z(I) = \frac{1}{k_I} \prod_{\tau \in \Phi} \det\Big(H_z\big(P((v_a^I)_{\sigma_\tau}), P((v_b^I)_{\sigma_\tau})\big)\Big)_{a,b \le \rho} \qquad (z \in V^*).$$

Then for holomorphic frames b_τ of $\operatorname{Lie}_{(\tau,-)}(G')^{\mathrm{an}}$ near x ($\tau \in \Phi$) there is $\nu' > 0$ with $\prod_{\tau\in\Phi} \det\Gamma(b_\tau(z))/\mathrm{N}_z(I) \to \nu'$ as $z \to x$, $z \in V^*$.

(Cu) Cusp weight. *$\mathrm{N}_z(I)$ does not depend on the choice of the family v^I. Suppose moreover that $X_{\mathbb{C}}$ is a complex fibre of the cover $\mathcal{X}$ of a base setting with cusp data (Definition 31.11), $G' = G_{\mathcal{X}}$ and $X_{\mathbb{C}}^\circ$ the abelian locus (case (B)), and let U be the cusp weight of (Hu2) of Definition 31.12 and $y_{\mathbb{C}}$ as in (Gb1). If $I \ne 0$, then $U(y_{\mathbb{C}}(z)) \ge -\log \mathrm{N}_z(I)$ for every $z \in V^*$.*

(Cv) Value of the limit. *Let* $\Lambda^{\mathrm{inv}} = H_{\mathbb{Z}} \cap \ker N$ *(the flat sections of* Λ *over* V^**),* $M_x = \mathrm{Lie}(G'_x)/\widehat{P}_x(I \otimes \mathbb{C})$ *and* $\Lambda^{\mathrm{ab}}_x \subseteq M_x$ *the image of* Λ^{inv} *under* $\widehat{P}_x$ *(the value at* x *of the holomorphic extension of* P*). Then* $\Lambda^{\mathrm{ab}}_x \cong \Lambda^{\mathrm{inv}}/I$ *is a lattice in* M_x*, stable under* $\iota_{\mathcal{Y}}(\mathcal{O}_E)$*; the form* E^{ab} *induced by* $\operatorname{Im} H_z$ *on* Λ^{inv}/I *(constant in* z*) is alternating and unimodular, satisfies* $E^{\mathrm{ab}}(\iota(a)u, u') = E^{\mathrm{ab}}(u, \iota(\bar{a})u')$*, and is the imaginary part of a positive definite hermitian form* H_0 *on* M_x *(so* $(M_x/\Lambda^{\mathrm{ab}}_x, E^{\mathrm{ab}})$ *is a principally polarised complex torus with* $\mathcal{O}_E$*-action, Rosati involution* $\iota(a) \mapsto \iota(\bar{a})$*). For every holomorphic frame* $b = (P(v_1), \dots, P(v_{t_x}), b_{t_x+1}, \dots, b_g)$ *near* x*, the limit* ν *of the main clause is*

$$\nu = \det\bigl(\pi H_0(\bar{b}_i(x), \bar{b}_l(x))\bigr)_{i,l>t_x},$$

$\bar{b}_i(x)$ *the image of* $b_i(x)$ *in* M_x*; and for every holomorphic frame* $b_\sigma = (P((v_1^I)_\sigma), \dots, P((v_\rho^I)_\sigma), b_{\sigma,\rho+1}, \dots)$ *of* $\mathrm{Lie}_\sigma(G')^{\mathrm{an}}$ *near* x*, the limit* ν_σ *of (Ce) is*

$$\nu_\sigma = \det\bigl(\pi H_0(\bar{b}_{\sigma,i}(x), \bar{b}_{\sigma,l}(x))\bigr)_{i,l>\rho}.$$

Proof. The vanishing cycles. As T_j preserves Q, $N_j = T_j - 1$ satisfies $Q(N_j u, u') = -Q(u, N_j u')$ when $N_j^2 = 0$ (expand $Q(T_j u, T_j u') = Q(u, u')$ and use $Q(N_j u, N_j u') = -Q(u, N_j^2 u') = 0$, which follows from the same expansion applied to $T_j^{-1} = 1 - N_j$); hence $Q(Nu, u') = -Q(u, Nu')$. By Lemma 33.16, $N^2 = 0$ and $\ker N = \bigcap_j \ker N_j$ has rank $2g - t_x$. So $W_{-2} = \operatorname{im} N \subseteq W_{-1} = \ker N$ (weight filtration centred at -1), $I = H_{\mathbb{Z}} \cap W_{-2}$ is saturated of rank $\dim \operatorname{im} N = 2g - \dim \ker N = t_x$, and $I \subseteq \ker N = \bigcap_j \ker N_j$ consists of T_j-invariant sections. It is isotropic: $Q(Nu, Nu') = -Q(u, N^2 u') = 0$. Moreover $Q(Nu, u') = -Q(u, Nu') = 0$ for $u' \in \ker N$, so $W_{-1} \subseteq I^{\perp_Q}$, with equality by dimensions $(2g - t_x)$.

Types of the limit. By Theorem B.145, (W, F_0) is a mixed Hodge structure polarised by N, F_0 the fibre of the extended Hodge filtration at 0. As in Lemma 33.16, $F_0^1 = 0$ and $F_0^{-1} = H$, so the Hodge types of Gr_k^W satisfy $-1 \le p, q \le 0$: Gr_{-2}^W is of type $(-1,-1)$, Gr_{-1}^W of types $(-1,0), (0,-1)$, Gr_0^W of type $(0,0)$. Hence, by strictness, $F_0^0 \cap W_{-2} = 0$, and $E_0 = F_0^0 \cap W_{-1}$ maps isomorphically onto $F^0\mathrm{Gr}_{-1}^W$, of dimension $g - t_x$ (half of $\dim \mathrm{Gr}_{-1}^W = 2g - 2t_x$). As $\mathrm{Gr}_{-1}^W = P_{-1}$ (since $N\mathrm{Gr}_{-1}^W \subseteq \mathrm{Gr}_{-3}^W = 0$), Q polarises Gr_{-1}^W: the hermitian form $k(w, w') = -iQ(w, \bar{w}')$ on $W_{-1} \otimes \mathbb{C}$ (it is the Hodge form $Q(Cw, \bar{w}')$ for w of type $(-1,0)$, $C = -i$) is positive definite on every subspace of $\overline{F_0^0} \cap W_{-1}$ meeting $W_{-2} \otimes \mathbb{C}$ only in 0. k vanishes on I, being induced from Gr_{-1}^W.

Holomorphic extension. The canonical-extension frame is $\tilde{u} = \exp(-\sum_j z_j N_j)u$, $z_j = \frac{\log t_j}{2\pi i}$; for $u \in W_{-1} = \bigcap_j \ker N_j$, $\tilde{u} = u$. So each v_a is a holomorphic section of $\widehat{\mathcal{H}}$, and $\widehat{P}(v_a)$ (Lemma 33.16) is a holomorphic section of $\mathrm{Lie}(G')^{\mathrm{an}}$ near x equal to $P(v_a)$ on V^*. At x, $\ker \widehat{P}_x = F_0^0$ meets $I \otimes \mathbb{C} \subseteq W_{-2} \otimes \mathbb{C}$ only in 0, so the $\widehat{P}_x(v_a)$ are linearly independent.

Untwisted coordinates. For $z \in V^*$ identify $\Lambda_z \otimes \mathbb{C}$ with H by U_z^{-1}, $U_z = \exp(-\sum_j z_j N_j)$ (on a slit polydisc; U_z is the identity on W_{-1}, so nothing below depends on the branch). Then $\widetilde{F}_z = U_z^{-1}\mathcal{F}_z^0$ is the fibre of $\widehat{\mathcal{F}}^0$ in the frame $\tilde{u}$, a holomorphic family of g-dimensional subspaces of H on a neighbourhood V_2 of x with $\widetilde{F}_0 = F_0^0$ (Theorem B.145, clause 1). Put $E_z = \widetilde{F}_z \cap W_{-1} \otimes \mathbb{C}$. For $z \in V^*$: $\mathcal{F}_z^0 \cap I \otimes \mathbb{C} = 0$ (for y in it, $\bar{y} \in I \otimes \mathbb{C}$, so $Q(y, \bar{y}) = 0$, while $h_z^{\mathrm{Hdg}}(y, y) = Q(iy, \bar{y})$, forcing $y = 0$), so $\dim E_z \ge g + (2g - t_x) - 2g = g - t_x$, and E_z injects into $\mathrm{Gr}_{-1}^W \otimes \mathbb{C}$ as a Q-isotropic subspace ($\mathcal{F}_z^0$ is Q-isotropic, being of one Hodge type), so $\dim E_z \le g - t_x$. With $\dim E_0 = g - t_x$, the subspaces E_z ($z \in V_2$, $V_2 \cap D$ included only at $z = 0$) have constant dimension and $E_z \to E_0$ as $z \to 0$ (an intersection of continuously varying subspaces of constant dimension varies continuously). Since W_{-1} is defined over $\mathbb{R}$ and U_z is the identity on it, $U_z^{-1}\overline{E'_z} = \overline{E_z}$ for $E'_z = \mathcal{F}_z^0 \cap W_{-1} \otimes \mathbb{C} = U_z E_z = E_z$.

The quotient metric is the constant form k. Fix $z \in V^*$. The h_z^{Hdg}-orthogonal complement of $\mathcal{F}_z^0 + I \otimes \mathbb{C}$ is $\overline{E_z}$: $h_z^{\mathrm{Hdg}}(w, \mathcal{F}_z^0) = 0$ if and only if $w \in \overline{\mathcal{F}_z^0}$ (the Hodge decomposition is orthogonal), and for such w, $C_z w = -iw$, so $h_z^{\mathrm{Hdg}}(w, I) = -iQ(w, I)$ vanishes if and only if $w \in I^{\perp_Q} = W_{-1} \otimes \mathbb{C}$. On $\overline{E_z}$, $h_z^{\mathrm{Hdg}}(w, w') = -iQ(w, \bar{w}') = k(w, w')$, independent of z. Hence the quotient metric of h_z^{Hdg} on $(\Lambda_z \otimes \mathbb{C})/(\mathcal{F}_z^0 + I \otimes \mathbb{C})$ is k transported along the isomorphism $\overline{E_z} \to (\Lambda_z \otimes \mathbb{C})/(\mathcal{F}_z^0 + I \otimes \mathbb{C})$.

Frames of the special form. Let first $b = (P(v_1), \dots, P(v_t), b_{t+1}, \dots, b_g)$, $t = t_x$, be a holomorphic frame near x (extended as above), and choose holomorphic sections $\tilde{b}_i$ of $\widehat{\mathcal{H}}$ with $\widehat{P}(\tilde{b}_i) = b_i$ $(i > t)$. On V^*, P identifies $(\Lambda_z \otimes \mathbb{C})/\mathcal{F}_z^0$ with $\mathrm{Lie}(G_z)$, the quotient metric of h_z^{Hdg} with $\frac{1}{2}H_z$ (Lemma 33.16), and $I \otimes \mathbb{C}$ with $S_z = \operatorname{span} P(v)(z)$. By Gram–Schmidt along $P(v)$,

$$\det \Gamma(b(z)) = \det \Gamma(P(v)(z)) \cdot \det\bigl(\Gamma_z^{\mathrm{q}}(b_{t+1}, \dots, b_g)\bigr),$$

with Γ_z^{q} the Gram matrix of the images in $\mathrm{Lie}(G_z)/S_z$ for the quotient metric of πH_z, which by the previous paragraph is 2π times the k-Gram matrix of $w_i(z)$, the component in $\overline{E_z}$ of $U_z^{-1}\tilde{b}_i(z)$ for the decomposition $H = (\widetilde{F}_z + I \otimes \mathbb{C}) \oplus \overline{E_z}$. This decomposition is direct also at $z = 0$: if $w \in \overline{E_0} \cap (F_0^0 + I \otimes \mathbb{C})$, its image in Gr_{-1}^W lies in $\overline{F^0\mathrm{Gr}_{-1}^W} \cap F^0\mathrm{Gr}_{-1}^W = 0$, so $w \in I \otimes \mathbb{C}$ and $\bar{w} \in E_0 \cap W_{-2} \otimes \mathbb{C} = 0$. The three subspaces vary continuously in $z \in V_2$ and $U_z^{-1}\tilde{b}_i(z)$ is the holomorphic coordinate vector of $\tilde{b}_i$ in the frame $\tilde{u}$, so $w_i(z) \to w_i(0)$, the component of $\tilde{b}_i(x)$ in $\overline{E_0}$. The $w_i(0)$ form a basis of $\overline{E_0}$ (the images of $b_{t+1}(x), \dots, b_g(x)$ form a basis of $\mathrm{Lie}(G'_x)/\operatorname{span} \widehat{P}_x(I) \cong H/(F_0^0 + I \otimes \mathbb{C})$), and $\overline{E_0} \cap W_{-2} \otimes \mathbb{C} = 0$, so $(k(w_i(0), w_l(0)))$ is positive definite. Therefore

$$\frac{\det \Gamma(b(z))}{\det \Gamma(P(v)(z))} = (2\pi)^{g-t} \det\bigl(k(w_i(z), w_l(z))\bigr)_{i,l>t} \longrightarrow (2\pi)^{g-t} \det\bigl(k(w_i(0), w_l(0))\bigr) > 0.$$

Arbitrary frames. Since the $P(v_a)(x)$ are linearly independent, some $g - t$ members $b_{i_{t+1}}, \dots, b_{i_g}$ of a given holomorphic frame b complete them to a basis of $\mathrm{Lie}(G'_x)$, hence to a holomorphic frame b' near x; then $b = Ab'$ with A holomorphic and invertible near x, $\det \Gamma(b) = |\det A|^2 \det \Gamma(b')$, and the limit for b is $|\det A(x)|^2$ times the limit for b', positive. The boundedness assertion follows from the existence of a positive finite limit.

Eigen-components (Ce). $\iota(\mathcal{O}_E)$ acts on the local system Λ (by endomorphisms of the family), commuting with every T_j, holomorphically on $\mathrm{Lie}(G')$ with P equivariant, and $Q(\iota(a)u, u') = Q(u, \iota(\bar{a})u')$. So N, W, I (hence I is an $\mathcal{O}_E$-submodule), $\mathcal{F}_z^0$, $\widetilde{F}_z$ (as U_z commutes with ι), E_z and $\overline{E_z}$ are ι-stable, hence direct sums of their intersections with the eigenspaces H_σ of $H = \bigoplus_\sigma H_\sigma$. For $u \in H_\sigma$, $u' \in H_{\sigma'}$: $\sigma(a)Q(u, u') = Q(u, \iota(\bar{a})u') = \overline{\sigma'(a)}Q(u, u')$, so $Q(H_\sigma, H_{\sigma'}) = 0$ unless $\sigma' = \bar{\sigma}$; as C_z commutes with ι and complex conjugation maps $H_{\sigma'}$ to $H_{\bar{\sigma}'}$, $h_z^{\mathrm{Hdg}}(H_\sigma, H_{\sigma'}) = 0$ for $\sigma \neq \sigma'$. Fix σ. The σ-components of an E-basis of $I \otimes \mathbb{Q}$ form a basis of $(I \otimes \mathbb{C}) \cap H_\sigma$ (dimension ρ), and each $(v_a^I)_\sigma$ is a $\mathbb{C}$-combination of elements of I, so $P((v_a^I)_\sigma) = \widehat{P}((v_a^I)_\sigma)$ extends holomorphically, and these are linearly independent at x since $\widehat{P}_x$ is injective on $I \otimes \mathbb{C}$. The frame b_σ lifts to holomorphic sections $\tilde{b}_i$ of $\widehat{\mathcal{H}} \cap H_\sigma$ (project any lifts to the σ-eigencomponent; $\widehat{P}$ is equivariant). Now repeat the last two steps of the argument above inside H_σ: by orthogonality of the eigenspaces, the h_z^{Hdg}-orthogonal complement of $(\mathcal{F}_z^0 + I \otimes \mathbb{C}) \cap H_\sigma$ in H_σ is $\overline{E_z} \cap H_\sigma$, on which $h_z^{\mathrm{Hdg}} = k$; the decomposition $H_\sigma = ((\widetilde{F}_z + I \otimes \mathbb{C}) \cap H_\sigma) \oplus (\overline{E_z} \cap H_\sigma)$ is the σ-part of the decomposition used above, hence direct and continuous in z up to $z = 0$; Gram–Schmidt in $\mathrm{Lie}_\sigma(G_z)$ along $P((v^I)_\sigma)$ gives $\det \Gamma(b_\sigma)/\det \Gamma(P((v^I)_\sigma)) = (2\pi)^{r_\sigma} \det(k(w_i(z), w_l(z)))$ with $w_i(z)$ the component of $U_z^{-1}\tilde{b}_i(z)$ in $\overline{E_z} \cap H_\sigma$ (r_σ their number), and the limit at $z = 0$ is a positive definite k-Gram determinant as before. This proves (Ce).

Cusp norm (Cn). By (Ce) with $\sigma = \sigma_\tau$ for each $\tau \in \Phi$, and $\Gamma = \pi H$-Gram,

$$\frac{\prod_\tau \det \Gamma(b_\tau(z))}{\mathrm{N}_z(I)} = k_I\, \pi^{\rho|\Phi|} \prod_{\tau \in \Phi} \frac{\det \Gamma(b_\tau(z))}{\det \Gamma(P((v^I)_{\sigma_\tau})(z))} \longrightarrow k_I\, \pi^{\rho|\Phi|} \prod_\tau \nu_{\sigma_\tau} > 0.$$

Value (Cv). $\widehat{P}(\lambda)$ is defined for $\lambda \in \Lambda^{\mathrm{inv}}$ (an invariant flat section is a section of $\widehat{\mathcal{H}}$). By the first paragraph, $Q(I, W_{-1}) = 0$ and $W_{-1} = I^{\perp_Q}$, so $\Lambda^{\mathrm{inv}} = \Lambda \cap I^{\perp_Q}$ with Λ (identified with $H_{\mathbb{Z}}$) unimodular for Q and I saturated isotropic; hence Q induces on Λ^{inv}/I a unimodular alternating form (take a basis of Λ extending a basis of I by vectors Q-dual to it modulo $I^{\perp}$; standard), and $E^{\mathrm{ab}} = -Q$ there. Its compatibility with ι is that of Q. The real vector space $\mathrm{Gr}_{\mathbb{R}} = W_{-1,\mathbb{R}}/W_{-2,\mathbb{R}}$ carries the pure Hodge structure of weight -1 of the limit, with $F^0\mathrm{Gr} =$ image of $E_0 = F_0^0 \cap W_{-1}$ and $\mathrm{Gr}_{\mathbb{C}} = F^0\mathrm{Gr} \oplus \overline{F^0\mathrm{Gr}}$, polarised by Q (types as shown above). $\widehat{P}_x$ maps $W_{-1} \otimes \mathbb{C}$ onto $\mathrm{Lie}(G'_x)$ ($\widehat{P}_x$ is surjective with kernel F_0^0, and $F_0^0 + W_{-1} = H$ because Gr_0^W is of type $(0,0)$) with kernel E_0, and $I \otimes \mathbb{C} = W_{-2,\mathbb{C}}$ onto $\widehat{P}_x(I \otimes \mathbb{C})$; so it induces a $\mathbb{C}$-linear isomorphism $\mathrm{Gr}_{\mathbb{C}}/F^0\mathrm{Gr} \to M_x$, and composed with $\mathrm{Gr}_{\mathbb{R}} \subseteq \mathrm{Gr}_{\mathbb{C}}$ an $\mathbb{R}$-linear isomorphism $\mathrm{Gr}_{\mathbb{R}} \to M_x$ (as $\mathrm{Gr}_{\mathbb{C}} = F^0\mathrm{Gr} \oplus \overline{F^0\mathrm{Gr}}$). It maps Λ^{inv}/I (a lattice in $\mathrm{Gr}_{\mathbb{R}}$, of rank $2g - 2t_x$) onto Λ_x^{ab}, which is therefore a lattice. Transport the complex structure of M_x to J on $\mathrm{Gr}_{\mathbb{R}}$ and put $H_0(u,u') = E^{\mathrm{ab}}(Ju,u') + iE^{\mathrm{ab}}(u,u')$. As $F^0\mathrm{Gr}$ (the $(-i)$-eigenspace of J) and its conjugate are Q-isotropic, $E^{\mathrm{ab}}(Ju,Ju') = E^{\mathrm{ab}}(u,u')$, so H_0 is hermitian. For $y \in \overline{F^0\mathrm{Gr}}$ with image u in M_x, $y = \frac{1}{2}(u - iJu)$ (in $\mathrm{Gr}_{\mathbb{C}}$), $\bar{y} = \frac{1}{2}(u + iJu)$, and

$$k(y,y) = -iQ(y,\bar{y}) = iE^{\mathrm{ab}}(y,\bar{y}) = \tfrac{i}{4}\big(2iE^{\mathrm{ab}}(u,Ju)\big) = \tfrac{1}{2}E^{\mathrm{ab}}(Ju,u) = \tfrac{1}{2}H_0(u,u);$$

as Q polarises Gr, $k > 0$ on $\overline{F^0\mathrm{Gr}}$, so H_0 is positive definite hermitian. In the computation of the limit above, $w_i(0) \in \overline{E_0}$ differs from $\tilde{b}_i(x)$ by an element of $F_0^0 + I \otimes \mathbb{C}$, so $w_i(0)$ maps to $\bar{b}_i(x)$ in M_x, and $\overline{E_0}$ maps isomorphically onto $\overline{F^0\mathrm{Gr}}$. Hence $\nu = (2\pi)^{g-t_x}\det(k(w_i(0),w_l(0))) = \det(\pi H_0(\bar{b}_i(x),\bar{b}_l(x)))$. The same identification inside the eigenspaces H_σ gives the value of ν_σ.

Cusp weight (Cu). Independence of the family: if $v'_1,\dots,v'_\rho$ is another $\mathcal{O}_E$-linearly independent family in I of finite index k', then $v'_a = \sum_b A_{ab}v_b^I$ in $I \otimes \mathbb{Q}$ with $A \in \mathrm{GL}_\rho(E)$, so $(v'_a)_\sigma = \sum_b \sigma(A_{ab})(v_b^I)_\sigma$ and each determinant in $\mathrm{N}_z(I)$ is multiplied by $|\sigma_\tau(\det A)|^2$; as $\{\sigma_\tau : \tau \in \Phi\}$ is a CM type (at points over the embedding κ of the coefficient field F, $\sigma_\tau = \kappa \circ \bar{\tau} = \overline{\kappa \circ \tau}$ because κ commutes with complex conjugation on the CM field $\tau(E)$, and $\{\kappa \circ \tau : \tau \in \Phi\}$ is a CM type because Φ is one and κ is injective), the product is multiplied by $\prod_\tau |\sigma_\tau(\det A)|^2 = |\mathrm{N}_{E/\mathbb{Q}}(\det A)|$; and $k' = k_I\,|\mathrm{N}_{E/\mathbb{Q}}(\det A)|$, since the $\mathbb{Q}$-linear map of $I \otimes \mathbb{Q}$ with E-matrix A has determinant $\mathrm{N}_{E/\mathbb{Q}}(\det A)$ and maps $\sum_b \mathcal{O}_E v_b^I$ onto $\sum_a \mathcal{O}_E v'_a$ (compare covolumes). Now let $X_{\mathbb{C}}$, G' be as in (Cu), $I \neq 0$ and $z \in V^*$, $w = y_{\mathbb{C}}(z) \in X^{\mathrm{b},\circ}$. By (Gb5) of Definition 31.11 there is an isomorphism of $\mathcal{O}_E$-modules $\Lambda_w \cong \Lambda_z = H_1(G_{\mathcal{X},z}(\mathbb{C}),\mathbb{Z})$ carrying $\langle\cdot,\cdot\rangle_w$ to $\mathrm{Im}\,H_z = -Q$ and $\lambda_{w,\tau}(v)$ to the coordinate vector of the $(\tau,-)$-component of v in an H_z-orthonormal basis of $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X},z})$, with $\varepsilon_{w,\tau} = \kappa \circ \bar{\tau} = \sigma_\tau$, κ the embedding of F over which z lies. The fibre $I_z \subseteq \Lambda_z$ of I (well defined, I being monodromy invariant) is a non-zero saturated Q-isotropic $\mathcal{O}_E$-submodule (main clause and (Ce)); let $J \subseteq \Lambda_w$ be its preimage, a non-zero saturated isotropic $\mathcal{O}$-submodule of the cusp lattice at w. Since $\overline{\lambda_{w,\tau}(v)}^{\mathsf{T}}\lambda_{w,\tau}(v') = H_z(v'_{\sigma_\tau}, v_{\sigma_\tau})$ for the images in Λ_z (orthonormal coordinates), the cusp norm $\mathrm{N}_w(J)$ of (Hu1), computed with the fixed family v^J of index $\iota(J)$, is $\mathrm{N}_z(I)$ computed with the image of v^J (determinants of a Hermitian matrix and its transpose agree), hence equals $\mathrm{N}_z(I)$ by the independence just shown. By (Hu2), $U(w) \geq s(w) \geq -\log \mathrm{N}_w(J) = -\log \mathrm{N}_z(I)$. □

Lemma 33.19. *In the setting of Lemma 33.16, put $y_j = -\log|t_j|$ $(1 \leq j \leq a)$.*

1. *(Local period form.) For every holomorphic frame $\hat{b} = (\hat{b}_1,\dots,\hat{b}_g)$ of $\mathrm{Lie}(G')^{\mathrm{an}}$ near x there are a neighbourhood V_1 of x in V and $g \times g$ matrices $P_0, P_1, \dots, P_a$ of real-analytic functions on V_1 (boundary points included) such that on $V_1 \cap V^*$*

$$\Gamma(\hat{b}(z))^{\mathsf{T}} = 2\pi\Big(P_0(z) + \sum_{j=1}^{a} y_j(z)P_j(z)\Big)^{-1}.$$

2. *(Derivatives.) For all holomorphic sections $b_1,\dots,b_m$ of* $\mathrm{Lie}(G')^{\mathrm{an}}$ *near x, linearly independent at x, $\Gamma(z)=\Gamma(b_1(z),\dots,b_m(z))$ is real-analytic on V^* near x, and there are $C>0$ and a neighbourhood V' of x such that for $z\in V'\cap V^*$, all k,l and all $1\le j\le d$*

$$\Big|\frac{\partial\Gamma_{kl}}{\partial t_j}(z)\Big|+\Big|\frac{\partial\Gamma_{kl}}{\partial \bar t_j}(z)\Big|\le\frac{C}{|t_1(z)\cdots t_a(z)|}.$$

Proof. The monodromy logarithms. For $i\neq j$, $0=N^2_{\{i,j\}}=N_i^2+2N_iN_j+N_j^2=2N_iN_j$ (the N_j commute and $N_j^2=0$), so all products N_iN_j vanish and $\exp(\sum_j c_jN_j)=1+\sum_j c_jN_j$ for all $c\in\mathbb{C}^a$. Each N_j is infinitesimally Q-symplectic ($Q(N_ju,u')=-Q(u,N_ju')$, from $Q(T_ju,T_ju')=Q(u,u')$ and $N_j^2=0$), so $U=\exp(-\sum_j z_jN_j)$ preserves Q for every z.

The dual frame. $\widehat{\mathcal{F}}^0$ has rank g and is Q-isotropic (by continuity, as $\mathcal{F}^0_{\tilde z}$ is of one Hodge type), so its fibre at x is Lagrangian for Q on the $2g$-dimensional fibre of $\widehat{\mathcal{H}}$, and the pairing $\widehat{\mathcal{H}}/\widehat{\mathcal{F}}^0\times\widehat{\mathcal{F}}^0\to\mathcal{O}$ induced by Q is holomorphic ($Q(\tilde s_k,\tilde s_l)=Q(s_k,s_l)$ is constant on the frame $\tilde s$) and perfect near x. Choose holomorphic sections $\tilde b_k$ of $\widehat{\mathcal{H}}$ with $\widehat P(\tilde b_k)=\hat b_k$ and let $f_1,\dots,f_g$ be the holomorphic frame of $\widehat{\mathcal{F}}^0$ with $Q(\tilde b_k,f_l)=\delta_{kl}$ (independent of the lifts, Q vanishing on $\widehat{\mathcal{F}}^0$). Write $f_l=\sum_m c_{lm}\tilde s_m$ with c_{lm} holomorphic near x and put $\tilde f_l=\sum_m c_{lm}s_m$, a holomorphic map to H; on V^*, in flat coordinates, $f_l(z)=U_z\tilde f_l(z)$.

Gram matrix of the dual frame. For $z\in V^*$ let $M(z)=(h^{\mathrm{Hdg}}_z(f_k,f_l))_{k,l}$. As $f_k(z)\in\mathcal{F}^0_z$ is of type $(0,-1)$, $C_z=i$ on it, and as N_j is real, $\overline{U_z}=\exp(-\sum_j\bar z_jN_j)$; with $z_j=\frac{\log t_j}{2\pi i}$, $z_j-\bar z_j=2i\,\mathrm{Im}\,z_j=\frac{i}{\pi}y_j$. Hence

$$M_{kl}=iQ(U\tilde f_k,\overline{U}\,\overline{\tilde f_l})=iQ(\tilde f_k,U^{-1}\overline{U}\,\overline{\tilde f_l})=iQ\Big(\tilde f_k,(1+\tfrac{i}{\pi}\textstyle\sum_j y_jN_j)\overline{\tilde f_l}\Big),$$

i.e. $M=P_0+\sum_j y_jP_j$ with $(P_0)_{kl}=iQ(\tilde f_k,\overline{\tilde f_l})$ and $(P_j)_{kl}=-\frac{1}{\pi}Q(\tilde f_k,N_j\overline{\tilde f_l})$, real-analytic on a neighbourhood V_1 of x in V (independent of the branch of $\log t_j$).

Duality. Fix $z\in V^*$. Let $x_k\in\overline{\mathcal{F}^0_z}$ be the h^{Hdg}-orthogonal projection of $\tilde b_k(z)$, so $\frac12H_z(\hat b_k,\hat b_l)=h^{\mathrm{Hdg}}(x_k,x_l)$ (Lemma 33.16), and write $x_k=\overline{g_k}$ with $g_k\in\mathcal{F}^0_z$. As $\tilde b_k-x_k\in\mathcal{F}^0_z$ and $\mathcal{F}^0_z$ is Q-isotropic, $\delta_{kl}=Q(x_k,f_l)=-Q(f_l,\overline{g_k})=i\,h^{\mathrm{Hdg}}(f_l,g_k)$, using $h^{\mathrm{Hdg}}(f,g)=iQ(f,\bar g)$ on $\mathcal{F}^0_z$. Writing $g_k=\sum_m c'_{km}f_m$, this reads $M\overline{c'}^{\mathsf T}=-iI$, so $c'=iM^{-1}$ (M hermitian), and $(h^{\mathrm{Hdg}}(g_k,g_l))=c'Mc'^*=M^{-1}$. Hence $h^{\mathrm{Hdg}}(x_k,x_l)=\overline{(M^{-1})_{kl}}=(M^{-1})_{lk}$ and $\Gamma(\hat b)=\pi(H_z(\hat b_k,\hat b_l))=2\pi(M^{-1})^{\mathsf T}$. This is clause 1.

Derivatives. Extend b to a holomorphic frame $\hat b$ near x (independence at x); $\Gamma(b)$ is a principal submatrix of $\Gamma(\hat b)$, so it suffices to bound the derivatives of $\Gamma(\hat b)$. Real-analyticity on V^* follows from clause 1 (M is invertible there). From $\Gamma^{\mathsf T}=2\pi M^{-1}$, $\partial\Gamma^{\mathsf T}=-\frac{1}{2\pi}\Gamma^{\mathsf T}(\partial M)\Gamma^{\mathsf T}$ for $\partial=\partial/\partial t_j$ or $\partial/\partial\bar t_j$. By Lemma 33.17, $\|\Gamma\|\le C_1$ on $V'\cap V^*$ for a neighbourhood $V'\subseteq V_1$ of x, which we take relatively compact with all $|t_i|\le r<e^{-1}$ ($i\le a$). As $y_j=-\frac12\log(t_j\bar t_j)$, $\partial y_i/\partial t_j=-\delta_{ij}/(2t_j)$ and $\partial y_i/\partial\bar t_j=-\delta_{ij}/(2\bar t_j)$ ($j\le a$), and these vanish for $j>a$. With C_2 a bound for the P_i and their first derivatives on V',

$$\|\partial M\|\le C_2\Big(1+\sum_{i\le a}y_i\Big)+\frac{C_2}{2|t_j|}\le\frac{C_3}{|t_1\cdots t_a|},$$

since $y_i\le e^{y_i}=|t_i|^{-1}\le|t_1\cdots t_a|^{-1}$ and $|t_j|^{-1}\le|t_1\cdots t_a|^{-1}$ (all $|t_i|<1$). Hence $\|\partial\Gamma\|\le\frac{C_1^2C_3}{2\pi}|t_1\cdots t_a|^{-1}$, which is clause 2. □

Lemma 33.20. *Let $\mathcal{X}^{\mathrm b}$ be either (i) a cover as in Proposition 33.9, for an irreducible component of an integral model of* Sh *and a finite set S of primes containing every prime dividing* $\mathrm{disc}(E)$,

with its Hodge bundles $\mathcal{V}_s$ at (S,R), $R\equiv 4$; or (ii) a cover as in Proposition 33.4, with the Hodge bundles $\mathcal{V}_s$ formed by its de Rham lattices. Let $\tau\in\Phi$, $B=\mathcal{X}^{\mathrm{b}}(\mathbb{C})$ and $\mathcal{N}_\tau=(\det\mathcal{V}_{(\tau,+)})^{-1}\otimes\det\mathcal{V}_{(\tau,-)}$ on B. On the abelian locus B° let $\|\cdot\|_{\mathrm{Hdg}}$ be its Hodge metric: $\|(v_1\wedge\cdots\wedge v_p)^\vee\otimes(w_1\wedge\cdots\wedge w_q)\|^2_{\mathrm{Hdg}}=\det\Gamma(w)/\det\Gamma(v)$ for local frames v of $\mathcal{V}_{(\tau,+)}$, w of $\mathcal{V}_{(\tau,-)}$. Then $\|\cdot\|_{\mathrm{Hdg}}$ extends to a smooth hermitian metric on $\mathcal{N}_\tau$ over all of B, and every point of B has a neighbourhood with a holomorphic generator of $\mathcal{N}_\tau$ of constant norm for it; in particular its local potentials are pluriharmonic.

Proof. Over $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ the Hodge lattices are the de Rham lattices (case (i): no point of $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ lies over a prime of S, (Fh5); case (ii): by definition), and in characteristic 0 the de Rham lattice $\mathrm{dR}_{s,x}=\mathrm{Lie}_x\cap\mathrm{Li}_s$ is the stalk of $\mathrm{Lie}_s(G_{\mathcal{X}^{\mathrm{b}}})$ (by (Fh1), Lie_x is the direct sum of its pieces, which lie in the summands $\mathrm{Li}_{s'}$ of Li); so $\mathcal{V}_{(\tau,\pm)}\otimes\mathbb{C}=\mathrm{Lie}_{(\tau,\pm)}(G_{\mathcal{X}^{\mathrm{b}}})^{\mathrm{an}}$ on B (over each embedding of F).

A holomorphic function of constant ratio on the interior. Let $z_0\in B^\circ$ over the embedding σ, and $\pi\colon B^\circ_\sigma\to\mathrm{Sh}^\sigma_F(\mathbb{C})$ the holomorphic map induced by the structure map ($G_{\mathcal{X}^{\mathrm{b}}}$ is the pull-back of the universal abelian scheme there); let Y be the connected component of $\pi(z_0)$. Choose holomorphic frames (v^0,w^0) of $\mathrm{Lie}_{(\tau,\pm)}(G^{\mathrm{un}})$ near $\pi(z_0)$; their pull-backs form frames near z_0, and any holomorphic frames (v,w) near z_0 are $(Av^0\circ\pi, Bw^0\circ\pi)$ with holomorphic invertible matrices A,B. With the function Φ of Lemma 33.21, written Ψ here, put $\Psi_{v,w}(z)=\Psi(\pi(z);v(z),w(z))=\Psi(\pi(z);v^0,w^0)\det B(z)/\det A(z)$ (item 1), holomorphic and non-vanishing near z_0 (item 2). By item 3, $|\Psi_{v,w}|^2\det\Gamma(v)/\det\Gamma(w)=c_Y>0$ is constant. Hence for the generator $\sigma_{v,w}=(v_1\wedge\cdots\wedge v_p)^\vee\otimes(w_1\wedge\cdots\wedge w_q)$,

$$\|\sigma_{v,w}\|^2_{\mathrm{Hdg}}=c_Y^{-1}|\Psi_{v,w}|^2\quad\text{near } z_0,$$

so $\sigma_{v,w}/\Psi_{v,w}$ is a holomorphic generator of constant norm near z_0; and since $\Psi_{v,w}$ transforms by $\det B/\det A$, the holomorphic functions $\Psi_{v,w}$ for one choice of frames (v,w) defined on a connected open subset W of B°_σ glue to a holomorphic non-vanishing function on W with $\|\sigma_{v,w}\|^2_{\mathrm{Hdg}}=c\,|\Psi_{v,w}|^2$ there (c locally constant).

Extension at a boundary point. Let $b_0\in B\smallsetminus B^\circ$, $t\colon V\to\Delta^d$ an SNC chart at b_0 with $V\cap B^\circ=V^*=\{t_1\cdots t_a\neq 0\}$ (item 2 of Proposition 33.9 in case (i), of Proposition 33.4 in case (ii); the setting (B) of Lemma 33.16, case of a variety mapping to a component of $\mathcal{Y}$), $L=1+\sum_{k\le a}|\log|t_k||$, and (v,w) holomorphic frames of $\mathrm{Lie}_{(\tau,\pm)}$ on V (shrinking V). $V^*\cong(\Delta^*)^a\times\Delta^{d-a}$ is connected, so $\Psi_{v,w}$ is a holomorphic non-vanishing function on V^* with $\|\sigma_{v,w}\|^2_{\mathrm{Hdg}}=c|\Psi_{v,w}|^2$. By Lemma 33.17, $C^{-p}L^{-p}\le\det\Gamma(v)\le C^p$ and likewise for w near b_0; hence $|\Psi_{v,w}|^{\pm1}\le C'L^N$ near b_0 for some N. A holomorphic function f on V^* with $|f|\le C'L^N$ extends holomorphically to V: $f\prod_{k\le a}t_k$ is bounded near $\{t_1\cdots t_a=0\}$ (as $|t|\,|\log|t||^N\to 0$), hence extends holomorphically (Riemann extension theorem) to some g on a neighbourhood of b_0; for fixed values of the other coordinates in a dense set, $t_1\mapsto f$ is holomorphic on a punctured disc with $|f|=o(|t_1|^{-1})$, so it extends across $t_1=0$ and g vanishes on $\{t_1=0\}$, hence g/t_1 is holomorphic; repeating for $t_2,\dots,t_a$ shows that f extends holomorphically. Applied to $\Psi_{v,w}$ and to $\Psi_{v,w}^{-1}$, $\Psi_{v,w}$ extends to a holomorphic function without zeros on a neighbourhood V' of b_0. Define the metric on $\mathcal{N}_\tau|_{V'}$ by $\|\sigma_{v,w}\|^2=c|\Psi_{v,w}|^2$: it is smooth, equals $\|\cdot\|_{\mathrm{Hdg}}$ on $V'\cap B^\circ$, and $\sigma_{v,w}/\Psi_{v,w}$ is a holomorphic generator of constant norm. Since B° is dense, the extensions at different points agree on overlaps, and together with $\|\cdot\|_{\mathrm{Hdg}}$ on B° they define the asserted metric; $-\log\|\sigma\|$ for a holomorphic generator σ differs from $-\log|\cdot|$ of a holomorphic non-vanishing function by a constant, hence is pluriharmonic. □

Lemma 33.21. *Let $\mathfrak{u}$ be a unitary Shimura datum with $n=p+q\ge 1$, $\tau\in\Phi_{\mathsf{H}}$, $U\subseteq\mathrm{Sh}_F$ a union of connected components, $s\colon F\to\mathbb{C}$ an embedding, and Y a connected component of the*

complex manifold $U^s(\mathbb{C})$ of points of U over s. Then there is a function Φ assigning to every $z \in Y$ and every frame (v, w) at z a number $\Phi(z; v, w) \in \mathbb{C}^\times$ such that:

1. $\Phi(z; Av, Bw) \det A = \Phi(z; v, w) \det B$ *for all* $A \in \mathrm{GL}_p(\mathbb{C})$, $B \in \mathrm{GL}_q(\mathbb{C})$;
2. *for every open* $V \subseteq Y$ *and all holomorphic sections* $v_1, \dots, v_p$ *of* $\mathrm{Lie}_{(\tau,+)}$ *and* $w_1, \dots, w_q$ *of* $\mathrm{Lie}_{(\tau,-)}$ *over* V *(eigen-subbundles of the holomorphic vector bundle* $\mathrm{Lie}(G^{\mathrm{un}})$ *on* Y*) that form a frame at every point of* V*, the function* $z \mapsto \Phi(z; v(z), w(z))$ *is holomorphic on* V;
3. $|\Phi(z; v, w)|^2 \det \Gamma(v) \, (\det \Gamma(w))^{-1}$ *is the same positive number for all* $z \in Y$ *and all frames* (v, w) *at* z;
4. *if* s *is the inclusion* $F \subset \mathbb{C}$*, then for every* $z_0 \in Y$ *and every* $\mathcal{O}_E$*-linearly independent* $e \in \Lambda^n_{z_0}$ *there is* $c \in \mathbb{C}^\times$ *such that* $\Phi(z; v, w) \det \binom{C}{R} = c$ *for all* z, γ, (v, w), C, B, R *as in (Yf4).*

Proof. Put $\rho = s \circ \tau \colon E \to \mathbb{C}$; as E is a CM field, $s \circ \bar\tau = \bar\rho$. For $z \in Y$ write $V_z = \mathrm{Lie}(G_z)$, $\langle\,,\rangle = \operatorname{Im} H_z$ on Λ_z, and extend $\langle\,,\rangle$ $\mathbb{C}$-bilinearly to $\Lambda_z \otimes \mathbb{C}$. The points z over s have $\mathrm{Lie}_{(\tau,+)}(G_z) = V_{z,\rho}$ and $\mathrm{Lie}_{(\tau,-)}(G_z) = V_{z,\bar\rho}$, the subspaces on which $\iota(a)$ acts by $\rho(a)$, $\bar\rho(a)$, of dimensions p and q (Kottwitz condition).

Linear algebra at one point. Let $L_\rho \subseteq \Lambda_z \otimes \mathbb{C}$ be the subspace on which $\mathcal{O}_E$ acts through ρ; then $\Lambda_z \otimes_{\mathcal{O}_E,\rho} \mathbb{C} \to L_\rho$ is an isomorphism, and for $x \in \Lambda_z \otimes \mathbb{Q}$ let x_ρ be its component in L_ρ. Since $\langle ax, y\rangle = \langle x, \bar a y\rangle$, the form $\langle\,,\rangle$ pairs L_ρ only with $L_{\bar\rho}$, and comparing $a \mapsto \langle ax, y\rangle = \mathrm{Tr}_{E/\mathbb{Q}}(a\, h(x, y))$ with $\sum_\sigma \sigma(a)\langle x_\sigma, y_{\bar\sigma}\rangle$ (linear independence of characters) gives

$$\langle x_\rho, y_{\bar\rho}\rangle = \rho\big(h(x, y)\big) \qquad (x, y \in \Lambda_z \otimes \mathbb{Q}). \tag{P}$$

Let $P_z \colon \Lambda_z \otimes \mathbb{C} \to V_z$ be the $\mathbb{C}$-linear extension of the inclusion $\Lambda_z \subset V_z$; it is $\mathcal{O}_E$-linear and surjective, $P_z(x_\sigma)$ is the $V_{z,\sigma}$-component of x, and its kernel K is the $(-i)$-eigenspace of the complex structure J extended to $\Lambda_z \otimes \mathbb{C} = V_z \otimes_{\mathbb{R}} \mathbb{C}$, while the $(+i)$-eigenspace $V^{1,0}$ maps isomorphically onto V_z. As $\langle Ju, Jv\rangle = \langle u, v\rangle$, both eigenspaces are isotropic for $\langle\,,\rangle$, so $\langle\,,\rangle$ induces a perfect pairing $K_\rho \times V_{z,\bar\rho} \to \mathbb{C}$, $K_\rho = K \cap L_\rho$. Given a frame (v, w) at z, let $r_1, \dots, r_q \in K_\rho$ be the basis with $\langle r_b, \tilde w_c\rangle = \delta_{bc}$ for any lifts $\tilde w_c \in L_{\bar\rho}$ of w_c (independent of the lifts, K_ρ being orthogonal to $K_{\bar\rho}$), and let $\tilde v_1, \dots, \tilde v_p \in L_\rho$ be lifts of $v_1, \dots, v_p$ under P_z. Then $(\tilde v, r)$ is a basis of L_ρ, and the class of $\tilde v_1 \wedge \dots \wedge \tilde v_p \wedge r_1 \wedge \dots \wedge r_q$ does not depend on the lifts (they change by elements of $K_\rho = \mathrm{span}(r)$). For an E-basis $f \in (\Lambda_z \otimes \mathbb{Q})^n$ define $\Psi_z(f; v, w) \in \mathbb{C}^\times$ by

$$f_{1,\rho} \wedge \dots \wedge f_{n,\rho} = \Psi_z(f; v, w)\, \tilde v_1 \wedge \dots \wedge \tilde v_p \wedge r_1 \wedge \dots \wedge r_q.$$

Replacing (v, w) by (Av, Bw) replaces $\tilde v$ by $A\tilde v$ and r by $(B^{\mathsf T})^{-1} r$, so $\Psi_z(f; Av, Bw) \det A = \Psi_z(f; v, w) \det B$. For $T \in \mathrm{GL}_n(E)$, $\Psi_z(Tf; v, w) = \rho(\det T)\Psi_z(f; v, w)$.

Relation with (Yf4). Let s be the inclusion ($\rho = \tau$), $f = \mu_\gamma e$, and C, B, R as in (Yf4). Then $\tilde v_a = \sum_m C_{am} f_{m,\tau}$ is a lift of v_a. Put $r'_b = \sum_m R_{bm} f_{m,\tau}$; by (P) and $h(\mu_\gamma x, \mu_\gamma y) = h(x, y)$, $\langle r'_b, f_{k,\bar\tau}\rangle = (R\,\tau(\mathrm{Gr}(e)))_{bk} = B_{bk}$, while $\langle r_b, f_{k,\bar\tau}\rangle = \sum_c B_{ck}\langle r_b, \tilde w_c\rangle = B_{bk}$ because $P_z(f_{k,\bar\tau}) = \sum_c B_{ck} w_c$. As the $f_{k,\bar\tau}$ span $L_{\bar\tau}$ and the pairing $L_\tau \times L_{\bar\tau} \to \mathbb{C}$ is perfect, $r'_b = r_b$. Hence $\tilde v \wedge r = \det \binom{C}{R}\, f_{1,\tau} \wedge \dots \wedge f_{n,\tau}$, that is

$$\Psi_z(\mu_\gamma e; v, w) \det \tbinom{C}{R} = 1. \tag{Q}$$

Single-valuedness. Fix $z_0 \in Y$ and an E-basis e of $\Lambda_{z_0} \otimes \mathbb{Q}$ in Λ_{z_0}. For a loop γ at z_0, $\mu_\gamma = g$ is an $\mathcal{O}_E$-linear automorphism of Λ_{z_0} preserving $\langle\,,\rangle$ (the polarisation and ι are global),

hence h; and g fixes the points $\eta(a)$, $a \in M$, of the global level structure, which are the images $I_N(\bar{\mathfrak{n}}M)$, i.e. $\bar{\mathfrak{n}}\Lambda_{z_0}/N\Lambda_{z_0} \cong \Lambda_{z_0}/\mathfrak{n}\Lambda_{z_0}$ (as $\mathfrak{n}\bar{\mathfrak{n}} = N\mathcal{O}_E$, $\mathfrak{n} + \bar{\mathfrak{n}} = \mathcal{O}_E$). So $(g-1)\Lambda_{z_0} \subseteq \mathfrak{n}\Lambda_{z_0}$ and $\zeta = \det_E(g) \in \mathcal{O}_E^\times$ satisfies $\zeta \equiv 1 \bmod \mathfrak{n}$. As g preserves the hermitian form h, $\zeta\bar{\zeta} = 1$, so $|\sigma(\zeta)| = 1$ for every embedding σ and ζ is a root of unity (Kronecker); then $\bar{\zeta} = \zeta^{-1} \equiv 1 \bmod \mathfrak{n}$, i.e. $\zeta \equiv 1 \bmod \bar{\mathfrak{n}}$, hence $\zeta \equiv 1 \bmod N\mathcal{O}_E$, and as $N \geq 3$, $\zeta = 1$ (a root of unity congruent to 1 modulo an integer $N \geq 3$ is 1). Therefore $\Psi_z(\mu_\gamma e; v, w)$ does not depend on the path γ from z_0 to z, and we define $\Phi(z; v, w) = \Psi_z(\mu_\gamma e; v, w)$.

Clauses 1 and 4. Clause 1 is the frame law above. If s is the inclusion and $e' \in \Lambda_{z_0}^n$ is $\mathcal{O}_E$-linearly independent, then $e' = Te$ with $T \in \mathrm{GL}_n(E)$, $\mu_\gamma e' = T\mu_\gamma e$, and (Q) for e' gives $\Phi(z; v, w) \det \binom{C}{R} = \tau(\det T)^{-1}$; this is clause 4 with $c = \tau(\det T)^{-1}$ (and the base point z_0 is arbitrary, since transporting e to another base point does not change Φ).

Clause 2. The universal family over Y is $\mathrm{Lie}(G^{\mathrm{un}})/\Lambda$ with Λ the kernel of the holomorphic exponential map, a local system of lattices whose local sections are holomorphic sections of the Lie bundle; so on a simply connected open $V' \subseteq V$ the map $P\colon \Lambda \otimes \mathcal{O} \to \mathrm{Lie}(G^{\mathrm{un}})$ is a surjective holomorphic map of vector bundles, $K_\rho = \ker P \cap L_\rho$ is a holomorphic subbundle, $\langle\,,\rangle$ is constant on the local system, and the dual basis $r(z)$ of a holomorphic frame $w(z)$ is holomorphic; lifts $\tilde{v}(z)$ may be chosen holomorphic. The flat sections $(\mu_\gamma e)_\rho$ give a holomorphic frame of $L_\rho \otimes \mathcal{O}$, so $\Phi(z; v(z), w(z))$, the coefficient of one holomorphic nowhere vanishing section of $\det(L_\rho \otimes \mathcal{O})$ in another, is holomorphic on V', hence on V.

Clause 3. The form $k(x, y) = i\langle x, \bar{y}\rangle$ on L_ρ ($\bar{y} \in L_{\bar{\rho}}$ the complex conjugate in $\Lambda_z \otimes \mathbb{C}$) is hermitian, and by (P) $k(f_{m,\rho}, f_{k,\rho}) = i\rho(h(f_m, f_k))$ for $f = \mu_\gamma e$, so $\det(k(f_{m,\rho}, f_{k,\rho})) = i^n\rho(\det \mathrm{Gr}(e))$ is independent of z. Choose the lifts $\tilde{v}_a$ in $V^{1,0}$. For $x \in V^{1,0}$ with $P_z(x) = u$, $x = \frac{1}{2}(u - iJu)$ and $k(x, x) = \frac{1}{2}\langle Ju, u\rangle = \frac{1}{2}H_z(u, u)$; the eigenspaces $V^{1,0}$ and $K = \overline{V^{1,0}}$ are k-orthogonal, and $k(\bar{x}, \bar{x}) = -k(x, x)$. Writing $r_b = \overline{\hat{r}_b}$ with $\hat{r}_b \in V^{1,0} \cap L_{\bar{\rho}}$ and $P_z(\hat{r}_b) = u_b \in V_{z,\bar{\rho}}$, the duality $\langle r_b, \tilde{w}_c\rangle = \delta_{bc}$ reads $\frac{1}{2}H_z(w_c, u_b) = -i\delta_{bc}$ (polarising the identity for k), so the $\frac{1}{2}H_z$-Gram matrix of u is the inverse transpose of that of w. Hence, with $\Gamma = \pi(H_z(\cdot, \cdot))$ as in (Yf3),

$$\det(k\text{-Gram of } (\tilde{v}, r)) = (2\pi)^{-p} \det \Gamma(v) \cdot (-1)^q (2\pi)^q (\det \Gamma(w))^{-1},$$

and since $f_\rho = \Psi \cdot (\tilde{v}, r)$ in the sense of determinants, $|\Phi(z; v, w)|^2 \det \Gamma(v)(\det \Gamma(w))^{-1} = (2\pi)^{p-q}(-1)^q i^n \rho(\det \mathrm{Gr}(e))$, a constant, necessarily positive as the left side is. □

Theorem 33.22. *Let X be a compact complex manifold of dimension D with a Kähler form Θ, and ς a holomorphic section of a smooth hermitian line bundle on X with $\|\varsigma\| \leq 1$ whose zero divisor is $\sum_i m_i W_i$ with $m_i \geq 1$ and W_i smooth hypersurfaces meeting with simple normal crossings; put $\partial X = \bigcup_i W_i$, $X^\circ = X \smallsetminus \partial X$ and $\mathrm{rW} = e - 2\log\|\varsigma\|$ on X°. Let $(\mathcal{M}, h^{\mathrm{ref}})$ be a smooth hermitian line bundle on X, φ a function on X° and $C_5 \geq 0$ such that:*

1. *φ is smooth on X°, the metric $h^{\mathrm{ref}}e^{\varphi}$ is psh on X°, and $-C_5 \log \mathrm{rW} \leq \varphi \leq C_5$ on X°;*
2. *every point of X has a holomorphic coordinate chart z on a neighbourhood U in which $\|d\varphi\| \leq C_5\,\mathrm{rW}^{C_5}\exp(\mathrm{rW}/2)$ on $U \cap X^\circ$ (euclidean norm of the real differential in the coordinates z).*

Then there are $C \geq 0$, $0 < t_0 < 1$ and smooth functions $\varphi^T(t)$ on X ($0 < t \leq t_0$) such that, with $\varepsilon(t) = C\log\log(1/t)/\log(1/t)$, for $0 < t \leq t_0$: $-C\log\log(1/t) \leq \varphi^T(t) \leq C$ on X; $\varphi \leq \varphi^T(t)$ on X°; the metric $h^{\mathrm{ref}}e^{\varphi^T(t)}$ twisted by $\varepsilon(t)\Theta$ is psh on X; $\varphi^T(t) \leq \varphi + t^{1/3} + \varepsilon(t)$ on $\{\mathrm{rW} \leq \log(1/t)\}$; $\varphi^T(t) \leq \varphi + t^{1/3} + \varepsilon(t)\,\mathrm{rW}$ on X°; $\varphi^T(t) \leq \varphi^T(t') + C\varepsilon(t')$ for $t \leq t' \leq t_0$; and $\varphi^T(t) \to \varphi$ with first and second derivatives, uniformly on compact subsets of X°, as $t \to 0^+$. In other words, $(\mathcal{M}, h^{\mathrm{ref}}, \varphi, (\varphi^T(t))_t)$ satisfies (Go1)(1)–(3) with the constants C_5, C, t_0.

Proof. Throughout, $\ell = \log(1/t)$, $\varepsilon_c(t) = c\log\log(1/t)/\log(1/t)$, and $t \le e^{-e}$, so that $\ell \ge e$ and $\log\ell \ge 1$. Two elementary facts: $x \mapsto \log x/x$ is decreasing on $[e,\infty)$, so $\varepsilon_c(t) \le \varepsilon_c(t')$ for $t \le t' \le e^{-e}$; and $t\ell \le 1/e$, so $t \le \varepsilon_1(t)$.

Potentials. Write θ for the smooth closed real $(1,1)$-form on X with $\theta = dd^c q$ for every local potential q of h^{ref} (two local potentials differ by a pluriharmonic function); the local potentials of $h^{\mathrm{ref}}e^f$ are $q+f$. Hence for smooth f on an open W: $h^{\mathrm{ref}}e^f$ is psh on W iff $\theta + dd^c f \ge 0$ there, and $h^{\mathrm{ref}}e^f$ twisted by $c\Theta$ is psh iff $\theta + dd^c f + c\Theta \ge 0$.

Step 1: an adapted atlas. Let $p \in X$ and let W_i, $i \in S_p$, be the components of ∂X through p. By simple normal crossings there are holomorphic coordinates w centred at p in which $W_i = \{w_i = 0\}$ near p ($i \in S_p$), and in a holomorphic frame $\mathbf{e}$ of the line bundle of ς, $\varsigma = v\prod_{i\in S_p} w_i^{m_i}\,\mathbf{e}$ with v holomorphic and nowhere zero. Choose $r_p > 0$ so small that the coordinates are defined on $\{|w| < 3r_p\}$, this set meets no component of ∂X other than the W_i, $i \in S_p$, and carries smooth functions q, ψ with $dd^c q = \theta$, $dd^c\psi = \Theta$. Put $\kappa_p = w/r_p$ on $U_p = \{|w| < 3r_p\}$, so $\kappa_p(U_p) = \Delta_3 \supseteq \bar\Delta_2$ and $\partial X \cap U_p = \kappa_p^{-1}(E_p)$, $E_p = \{z : z_i = 0 \text{ for some } i \in S_p\}$. On $U_p \setminus \partial X$, with $z = \kappa_p$,

$$\mathrm{rW} = e + \sum_{i\in S_p} m_i \log\frac{1}{|w_i|^2} - \log(|v|^2\|\mathbf{e}\|^2) = r_{S_p,m}(z) - \sum_{i\in S_p} m_i\log(4r_p^2) - \log(|v|^2\|\mathbf{e}\|^2),$$

and the last two terms are bounded on the compact set $\kappa_p^{-1}(\bar\Delta_2)$. By compactness, finitely many such charts $\kappa_j\colon U_j \to \Delta_3$ ($1 \le j \le N$), with sets S_j, E_j and potentials q_j, ψ_j on U_j, satisfy $X = \bigcup_j U_j^1$, where $U_j^r = \kappa_j^{-1}(\Delta_r)$, and there is $B \ge 0$ with

$$|\mathrm{rW} - r_{S_j,m}\circ\kappa_j| \le B \quad \text{on } \kappa_j^{-1}(\bar\Delta_2)\setminus\partial X \quad (\text{all } j).$$

Let $Q \ge 0$ bound $|q_j|$ on $\kappa_j^{-1}(\bar\Delta_2)$ and $G \ge 0$ be a Lipschitz constant of every $q_j\circ\kappa_j^{-1}$ on $\bar\Delta_{7/4}$ for the norm $|\cdot|$.

Transition maps. Let $\Lambda_0 \ge 1$, $G' \ge 0$ and $t_c > 0$ be the constants of Lemma 33.26 for θ, the charts κ_j and the potentials q_j.

Step 2: the local functions. Fix j and let $u_j = (q_j+\varphi)\circ\kappa_j^{-1}$ on $\Delta_2 \setminus E_j$. It is smooth and plurisubharmonic there (item 1: $h^{\mathrm{ref}}e^{\varphi}$ is psh on X°) and bounded above by $C_5 + Q$. As E_j is a closed pluripolar set (a finite union of hyperplanes), u_j extends to a plurisubharmonic function on Δ_2, given on E_j by $u_j(z) = \limsup_{w\to z,\, w\notin E_j} u_j(w)$ (removable singularity theorem of Grauert and Remmert, [145], Ch. 2). We check (L1), (L2) of Lemma 33.23 for u_j with $S = S_j$, $K = B$ and a constant A independent of j. On $\Delta_2\setminus E_j$ write $r = r_{S_j,m}$; then $r \ge e$, $\mathrm{rW} \le r + B$, and $\log(r+B) \le \log r + \log(1+B) \le (1+\log(1+B))\log r$, so item 1 gives $u_j \ge -C_5\log(r+B) - Q \ge -(C_5(1+\log(1+B)) + Q)\log r$. For (L2): by item 2 and compactness of $\kappa_j^{-1}(\bar\Delta_{3/2})$, this set is covered by finitely many compact sets on each of which a chart z' of item 2 is defined and the derivative of $z'\circ\kappa_j^{-1}$ is bounded by some J; there $\|d(\varphi\circ\kappa_j^{-1})\| \le JC_5\mathrm{rW}^{C_5}e^{\mathrm{rW}/2} \le JC_5(1+B)^{C_5}e^{B/2}r^{C_5}e^{r/2}$, using $r \ge 1$; and $\|d(q_j\circ\kappa_j^{-1})\| \le \sqrt{2D}\,G \le \sqrt{2D}\,G\,r^{C_5}e^{r/2}$. So (L1), (L2) hold with a common A. Let $C_* \ge 1$ and t_* be the maximum and minimum of the constants C and t_0 of Lemma 33.23(b) over j, let $u_{j,t}$ ($0 < t \le t_*$) be its functions, and let $C_{\Lambda_0}, t_{\Lambda_0}$ be the constants of Lemma 33.23(a) for $\Lambda = \Lambda_0$ (maximum, minimum over j). Put

$$f_{t,j} = u_{j,t}\circ\kappa_j - q_j \quad \text{on } U_j^{5/4}.$$

It is smooth, and $q_j + f_{t,j} = u_{j,t}\circ\kappa_j$ is plurisubharmonic, so $\theta + dd^c f_{t,j} \ge 0$ on $U_j^{5/4}$.

Step 3: comparison of charts. Let $\tilde\varphi = \varphi$ on X° and $\tilde\varphi(x) = \limsup_{y\to x,\, y\in X^\circ}\varphi(y)$ on ∂X; then $u_j = (q_j+\tilde\varphi)\circ\kappa_j^{-1}$ on Δ_2. With M_j the maxima of u_j, let $x \in U_j^{5/4}\cap U_k^{5/4}$ and

$t \le \min(t_c, t_{\Lambda_0})$. By Lemma 33.26 for $\tilde\varphi$ (bounded above) and Lemma 33.23(a) in the charts j and k,

$$\big|(M_j(\kappa_j x,t)-q_j(x))-(M_k(\kappa_k x,t)-q_k(x))\big| \le \varepsilon_{C_{\Lambda_0}}(t)+2G'\Lambda_0 t,$$

and by Lemma 33.23(b1), $|f_{t,j}(x)-(M_j(\kappa_j x,t)-q_j(x))| \le \varepsilon_{C_*}(t)$. With $t \le \varepsilon_1(t)$,

$$|f_{t,j}-f_{t,k}| \le \delta(t) := \varepsilon_{C'}(t) \quad \text{on } U_j^{5/4}\cap U_k^{5/4}, \qquad C' = 2C_* + C_{\Lambda_0} + 2G'\Lambda_0 + 1.$$

Moreover, $t \le \delta(t)$ and $\delta(t) \to 0$.

Step 4: gluing. Let K be the constant of Lemma 33.25 for (X,Θ,θ) and the charts κ_j, and let $\varphi^T(t) = F_t$ be the function given by Lemma 33.25(G1) for $\delta = \delta(t)$ and $(f_{t,j})_j$, for $0 < t \le t_0$, where $t_0 = \min(t_*, t_c, t_{\Lambda_0}, e^{-e})$. Then $\varphi^T(t)$ is smooth on X, $\theta + dd^c\varphi^T(t) + K\delta(t)\Theta \ge 0$, and $f_{t,j} \le \varphi^T(t) \le f_{t,j} + 3\delta(t)$ on $U_j^{5/4}$.

Step 5: the bounds. Put $C = \max(KC',\ A{+}Q{+}3C',\ C_*{+}Q,\ 3C',\ A{+}Q{+}3C'{+}C_5,\ C_*{+}3C')$. Let $0 < t \le t' \le t_0$, $x \in X$, and choose j with $x \in U_j^1$; write $z = \kappa_j x$.

- Psh: $K\delta(t) = \varepsilon_{KC'}(t) \le \varepsilon_C(t)$, so $h^{\mathrm{ref}}e^{\varphi^T(t)}$ twisted by $\varepsilon_C(t)\Theta$ is psh.
- Upper bound: $\varphi^T(t)(x) \le u_{j,t}(z) - q_j(x) + 3\delta(t) \le A + Q + 3C' \le C$, by Lemma 33.23(b2) and $\log\ell/\ell \le 1$.
- Lower bound: $\varphi^T(t)(x) \ge f_{t,j}(x) \ge -C_*\log\log(1/t) - Q \ge -C\log\log(1/t)$.
- $\varphi \le \varphi^T(t)$ on X°: $f_{t,j}(x) = u_{j,t}(z) - q_j(x) \ge u_j(z) - q_j(x) = \varphi(x)$ by (b2).
- Closeness: if $x \in X^\circ$ and $\mathrm{rW}(x) \le \ell$, then $z \notin E_j$ and $r_{S_j,m}(z) \le \ell + B$, so (b3) with $K = B$ gives $u_{j,t}(z) \le u_j(z) + t^{1/3}$ and $\varphi^T(t)(x) \le \varphi(x) + t^{1/3} + 3\delta(t) \le \varphi(x) + t^{1/3} + \varepsilon_C(t)$.
- The bound with rW: let $x \in X^\circ$. If $\mathrm{rW}(x) \le \ell$, the previous item and $\mathrm{rW} \ge 1$ give it. If $\mathrm{rW}(x) > \ell$, then $\log\mathrm{rW}/\mathrm{rW} \le \log\ell/\ell$ and $\mathrm{rW}/\ell > 1$, so with the upper bound and item 1,

$$\varphi^T(t) - \varphi \le A + Q + 3C' + C_5\log\mathrm{rW} \le (A+Q+3C')\frac{\log\ell}{\ell}\mathrm{rW} + C_5\frac{\log\ell}{\ell}\mathrm{rW} \le \varepsilon_C(t)\,\mathrm{rW}.$$

- Quasi-monotonicity: by (b4) and the monotonicity of ε, $\varphi^T(t)(x) \le f_{t,j}(x) + 3\delta(t) \le f_{t',j}(x) + \varepsilon_{C_*}(t') + \varepsilon_{3C'}(t') \le \varphi^T(t')(x) + \varepsilon_C(t')$.
- C^2-convergence: for every j and compact $L \subseteq \Delta_{5/4} \setminus E_j = \Delta_{5/4} \setminus \kappa_j(\partial X)$, $f_{t,j}\circ\kappa_j^{-1} - \varphi\circ\kappa_j^{-1} = u_{j,t} - u_j$ on L, which tends to 0 with first and second derivatives uniformly on L, with $\|d(u_{j,t}-u_j)\| \le N_L t$ (Lemma 33.23(c)). With $t \le \delta(t) \to 0$, Lemma 33.25(G2) for $Z = \partial X$ gives $\varphi^T(t) \to \varphi$ with first and second derivatives uniformly on compact subsets of X°.

Thus all conclusions hold with C and t_0. □

Lemma 33.23. *On $\mathbb{C}^D$ let $|x| = \max_i|x_i|$, let $\Delta(z,r)$ and $\bar\Delta(z,r)$ be the open and the closed polydisc of radius r about z, and $\Delta_r = \Delta(0,r)$. Let $S \subseteq \{1,\dots,D\}$, integers $m_i \ge 1$ $(i \in S)$, $E = \{x:\ x_i = 0 \text{ for some } i \in S\}$, and on $\mathbb{C}^D \setminus E$*

$$r_{S,m}(x) = e + \sum_{i\in S} m_i \log\frac{4}{|x_i|^2}.$$

For $C \ge 0$ and $0 < t < 1$ put $\varepsilon_C(t) = C\log\log(1/t)/\log(1/t)$. Let $A \ge 0$, $K \ge 0$, and let u be a plurisubharmonic function on Δ_2 which is smooth on $\Delta_2 \setminus E$ and satisfies

(L1) $u \le A$ *on* Δ_2 *and* $u \ge -A \log r_{S,m}$ *on* $\Delta_2 \setminus E$*;*

(L2) $\|du(x)\| \le A\, r_{S,m}(x)^A \exp(r_{S,m}(x)/2)$ *for* $x \in \Delta_{3/2} \setminus E$ *(operator norm of the real differential for the euclidean norm on* $\mathbb{C}^D = \mathbb{R}^{2D}$*).*

For $\bar\Delta(z,r) \subseteq \Delta_2$*,* $r > 0$*, let* $M(z,r) = \max_{\bar\Delta(z,r)} u$*. Then:*

(a) (oscillation) for every $\Lambda \ge 1$ *there are* $C_\Lambda \ge 1$ *and* $0 < t_\Lambda < 1$ *such that for* $0 < t \le t_\Lambda$ *and* $z \in \Delta_{5/4}$

$$M(z, t/\Lambda) \ge -C_\Lambda \log\log(1/t), \qquad M(z,\Lambda t) - M(z, t/\Lambda) \le \varepsilon_{C_\Lambda}(t);$$

(b) there are $C \ge 1$*,* $0 < t_0 \le e^{-e}$ *and smooth plurisubharmonic functions* u_t *on* $\Delta_{5/4}$ *(*$0 < t \le t_0$*) such that for* $0 < t \le t' \le t_0$ *and* $z \in \Delta_{5/4}$*:*

(b1) $M(z,t) - \varepsilon_C(t) \le u_t(z) \le M(z,t)$*;*

(b2) $-C \log\log(1/t) \le u_t(z) \le A$ *and* $u(z) \le u_t(z)$*;*

(b3) $u_t(z) \le u(z) + t^{1/3}$ *if* $z \notin E$ *and* $r_{S,m}(z) \le \log(1/t) + K$*;*

(b4) $u_t(z) \le u_{t'}(z) + \varepsilon_C(t')$*;*

(c) for every compact $L \subseteq \Delta_{5/4} \setminus E$*,* $u_t \to u$ *with first and second derivatives uniformly on* L *as* $t \to 0^+$*, and there are* $N_L \ge 0$ *and* $t_L > 0$ *with* $\|du_t(z) - du(z)\| \le N_L t$ *for* $z \in L$*,* $0 < t \le t_L$*.*

Proof. Write $\ell = \log(1/t)$ and $m_S = \sum_{i\in S} m_i$. Since $|x_i| < 2$ on Δ_2, $r_{S,m} \ge e$ there, so $\log r_{S,m} \ge 1$. For every $z \in \Delta_2$ and $r > 0$, $\bar\Delta(z,r) \setminus E \ne \varnothing$ and $u > -\infty$ there; as u is upper semicontinuous with $u \le A$, $M(z,r)$ is a real number attained on $\bar\Delta(z,r)$.

Facts on polydisc maxima and mollification. (F1), (F3), (F4) below are (P1), (P2), (P3) of Lemma 33.24, restated for reference. Let v be plurisubharmonic on an open $\Omega \subseteq \mathbb{C}^D$.

(F1) If $\bar\Delta(z,R) \subseteq \Omega$, then $r \mapsto \max_{\bar\Delta(z,r)} v$ is nondecreasing on $(0,R]$ and a convex function of $\log r$ there. Indeed $V(\zeta) = \max_{|w|\le 1} v(z+\zeta w)$ is, on a disc $|\zeta| < R'$ with $R' > R$ and $\bar\Delta(z,R') \subseteq \Omega$, the upper envelope of the subharmonic functions $\zeta \mapsto v(z+\zeta w)$; it is upper semicontinuous (maximum of an upper semicontinuous function over a compact set), hence subharmonic, and radial, since $w \mapsto e^{i\theta} w$ preserves $\bar\Delta(0,1)$; a radial subharmonic function is a nondecreasing convex function of $\log|\zeta|$ (Hadamard's three-circles theorem), and $V(\zeta) = \max_{\bar\Delta(z,|\zeta|)} v$.

(F2) (sub-mean value on polydiscs) If $\bar\Delta(x,r) \subseteq \Omega$, then $v(x) \le \lambda(\Delta(x,r))^{-1} \int_{\Delta(x,r)} v\, d\lambda$ (λ Lebesgue measure), by the sub-mean value inequality on discs applied in each coordinate in turn.

(F3) (mollification) Fix a smooth $\rho \ge 0$ on $\mathbb{C}^D$ with support in $\Delta(0,1)$, $\int \rho\, d\lambda = 1$, depending only on $(|w_1|,\dots,|w_D|)$, and put $\rho_t(w) = t^{-2D}\rho(w/t)$. On $\Omega_t = \{z : \bar\Delta(z,t) \subseteq \Omega\}$ the function $v * \rho_t(z) = \int v(z-w)\rho_t(w)\, d\lambda(w)$ is smooth and plurisubharmonic, and $v \le v*\rho_t$ there; and $v*\rho_t(z) \le \max_{\bar\Delta(z,t)} v$ because ρ_t is a probability density supported in $\Delta(0,t)$.

Put $K_\rho = \pi^D (3/2)^{2D} \sup\rho$.

(F4) Lower bound for the mollification. If v is plurisubharmonic on Ω and $\bar\Delta(z,2t)\subseteq\Omega$, then with $M_v(r)=\max_{\bar\Delta(z,r)} v$,

$$v*\rho_t(z)\ge M_v(2t)-K_\rho\big(M_v(2t)-M_v(t/4)\big).$$

Choose $x\in\bar\Delta(z,t/4)$ with $v(x)=M_v(t/4)$ and put $P=\Delta(x,3t/2)$. Then $\Delta(z,t)\subseteq P$ and $\bar P\subseteq\bar\Delta(z,7t/4)\subseteq\Omega$, so by (F2) $M_v(t/4)\le\lambda(P)^{-1}\int_P v$, i.e. $\int_P(M_v(2t)-v)\,d\lambda\le\lambda(P)(M_v(2t)-M_v(t/4))$, where the integrand is ≥ 0 on P. Since $\rho_t\le t^{-2D}\sup\rho$ and $\rho_t(w)\ne 0$ only if $z-w\in\Delta(z,t)\subseteq P$,

$$M_v(2t)-v*\rho_t(z)=\int\rho_t(w)\big(M_v(2t)-v(z-w)\big)d\lambda(w)\le t^{-2D}\sup\rho\cdot\lambda(P)\big(M_v(2t)-M_v(t/4)\big),$$

and $t^{-2D}\lambda(P)=\pi^D(3/2)^{2D}$.

Proof of (a). Let $\Lambda\ge 1$ and choose $0<t_\Lambda\le e^{-e}$ so small that $\ell\ge\max(3m_S+1,\ \log 4+2\log\Lambda,\ 2\log(2\Lambda))$ for $t\le t_\Lambda$. Let $0<t\le t_\Lambda$, $z\in\Delta_{5/4}$, $s=t/\Lambda$. Define x by $x_i=z_i+s\,z_i/|z_i|$ if $z_i\ne 0$ and $x_i=s$ otherwise; then $|x-z|\le s$ and $|x_i|\ge s$ for all i, so $x\in\bar\Delta(z,s)\setminus E$ and

$$r_{S,m}(x)\le e+m_S\log(4/s^2)=e+m_S(\log 4+2\log\Lambda+2\ell)\le e+3m_S\ell\le(3m_S+1)\ell.$$

Hence $\log r_{S,m}(x)\le\log(3m_S+1)+\log\ell\le 2\log\ell$ and $M(z,s)\ge u(x)\ge -2A\log\ell$ by (L1). Now let $g(\sigma)=M(z,e^\sigma)$, convex and nondecreasing for $\sigma\le\log(1/2)$ by (F1) ($\bar\Delta(z,1/2)\subseteq\Delta_2$). Put $a=\log s$, $b=\log(\Lambda t)$, $c=\log(1/2)$; then $b-a=2\log\Lambda$ and $c-b=\ell-\log(2\Lambda)\ge\ell/2$. If $\Lambda=1$ there is nothing to prove; otherwise $a<b<c$, and convexity gives

$$g(b)-g(a)\le(b-a)\frac{g(c)-g(b)}{c-b}\le 2\log\Lambda\cdot\frac{A+2A\log\ell}{\ell/2}\le 12A\log\Lambda\,\frac{\log\ell}{\ell},$$

using $g(c)\le A$, $g(b)\ge g(a)\ge -2A\log\ell$ and $\log\ell\ge 1$. So (a) holds with $C_\Lambda=1+2A+12A\log\Lambda$.

Construction for (b). Let C_8,t_8 be as in (a) for $\Lambda=8$; put $C=\max(1,K_\rho)\,C_8$, $\kappa_t=C_8\log\log(1/t)$, and choose $0<t_0\le\min(t_8,e^{-K},1/16)$, to be decreased once more in (b3). For $0<t\le t_0$ let $w_t=\max(u,-\kappa_t)$, plurisubharmonic on Δ_2, and $u_t=w_t*\rho_t$ on $\Delta_{5/4}$ (note $\bar\Delta(z,2t)\subseteq\Delta_2$ for $z\in\Delta_{5/4}$). By (F3) u_t is smooth and plurisubharmonic on $\Delta_{5/4}$ and $u\le w_t\le u_t$. For $z\in\Delta_{5/4}$ and $t/8\le r\le 8t$, (a) gives $M(z,r)\ge M(z,t/8)\ge-\kappa_t$, hence

$$\max_{\bar\Delta(z,r)} w_t=\max\big(M(z,r),-\kappa_t\big)=M(z,r)\qquad(t/8\le r\le 8t).$$

(b1) and (b2). By (F3), $u_t(z)\le\max_{\bar\Delta(z,t)}w_t=M(z,t)\le A$. By (F4) for w_t and the display, $u_t(z)\ge M(z,2t)-K_\rho(M(z,2t)-M(z,t/4))\ge M(z,t)-K_\rho\varepsilon_{C_8}(t)\ge M(z,t)-\varepsilon_C(t)$, using $M(z,2t)\ge M(z,t)$ and (a) for $\Lambda=8$. Then $u_t(z)\ge -C_8\log\log(1/t)-\varepsilon_C(t)\ge -2C\log\log(1/t)$, since $\varepsilon_C(t)\le C\log\log(1/t)$; replacing C by $2C$ (which keeps (b1)) gives (b2).

(b3). Let $z\in\Delta_{5/4}\setminus E$ with $r_{S,m}(z)\le\ell+K$, and $x\in\bar\Delta(z,t)$. For $i\in S$, $m_i\ge 1$ gives $\log(4/|z_i|^2)\le\ell+K$, so $|z_i|\ge 2e^{-K/2}t^{1/2}\ge 2t$ because $t\le e^{-K}$. On the segment $[z,x]$ every y has $|y_i|\ge|z_i|-t\ge|z_i|/2>0$, so $y\in\Delta_{3/2}\setminus E$ and $r_{S,m}(y)\le r_{S,m}(z)+m_S\log 4\le\ell+K'$ with $K'=K+m_S\log 4$. By (L2), $\|du(y)\|\le A(\ell+K')^A e^{K'/2}t^{-1/2}$, and the euclidean length of $[z,x]$ is at most $\sqrt{D}t$, so

$$u(x)-u(z)\le\sqrt{D}\,A\,e^{K'/2}(\ell+K')^A\,t^{1/2}\le t^{1/3}$$

once $t^{1/6}(\log(1/t)+K')^A\le(\sqrt{D}\,A\,e^{K'/2}+1)^{-1}$, which holds for $t\le t_0$ after decreasing t_0. Hence $u_t(z)\le M(z,t)\le u(z)+t^{1/3}$.

(b4). For $t\le t'\le t_0$: $u_t(z)\le M(z,t)\le M(z,t')\le u_{t'}(z)+\varepsilon_C(t')$ by (b1) and (F1).

(c). Let $L \subseteq \Delta_{5/4} \setminus E$ be compact. As $r_{S,m}$ is continuous on $\mathbb{C}^D \setminus E$, $B = \max_L r_{S,m} < \infty$, and $V = \{y \in \Delta_{3/2} \setminus E : \ r_{S,m}(y) < B+1\}$ is an open neighbourhood of L; let $d > 0$ with $L + \bar\Delta(0,d) \subseteq V$. On V, $u \geq -A\log(B+1)$ by (L1), so $w_t = u$ on V as soon as $\kappa_t > A\log(B+1)$. Hence for t small and $t < d/2$, $u_t = u * \rho_t$ on $L' = L + \bar\Delta(0,d/2)$, where u is smooth on $L + \bar\Delta(0,d)$. Differentiating under the integral, $D^k u_t = (D^k u) * \rho_t$ on L for $k \leq 2$, and $|(D^k u) * \rho_t(z) - D^k u(z)| \leq \sup\{|D^k u(z-w) - D^k u(z)| : |w| \leq t\}$, which tends to 0 uniformly on L by uniform continuity of $D^k u$ on L'. For $k = 1$ the mean value inequality gives $\|du_t(z) - du(z)\| \leq \sqrt{D}\, t \max_{L'} \|D^2 u\|$, so $N_L = \sqrt{D}\max_{L'}\|D^2 u\|$ works. □

Lemma 33.24. *On $\mathbb{C}^D$ let $|x| = \max_i |x_i|$ and let $\Delta(z,r)$, $\bar\Delta(z,r)$ be the open and closed polydiscs of radius r about z. Fix a smooth $\rho \geq 0$ on $\mathbb{C}^D$ with support in $\Delta(0,1)$, $\int \rho\, d\lambda = 1$ (λ Lebesgue measure), depending only on $(|w_1|,\dots,|w_D|)$; put $\rho_t(w) = t^{-2D}\rho(w/t)$ and $K_\rho = \pi^D(3/2)^{2D}\sup\rho$. Let v be plurisubharmonic on an open $\Omega \subseteq \mathbb{C}^D$, $v \not\equiv -\infty$ on every component, and for $\bar\Delta(z,r) \subseteq \Omega$ let $M_v(z,r) = \max_{\bar\Delta(z,r)} v$. Then:*

(P1) if $\bar\Delta(z,R) \subseteq \Omega$, then $r \mapsto M_v(z,r)$ is nondecreasing on $(0,R]$ and a convex function of $\log r$ there, and $M_v(z,r) \to v(z)$ as $r \to 0^+$;

(P2) on $\Omega_t = \{z : \bar\Delta(z,t) \subseteq \Omega\}$ the function $v * \rho_t(z) = \int v(z-w)\rho_t(w)\, d\lambda(w)$ is smooth and plurisubharmonic, and $v \leq v * \rho_t \leq M_v(\cdot,t)$ there;

(P3) if $\bar\Delta(z,2t) \subseteq \Omega$, then $v * \rho_t(z) \geq M_v(z,2t) - K_\rho\big(M_v(z,2t) - M_v(z,t/4)\big)$.

Proof. (P1). Choose $R' > R$ with $\bar\Delta(z,R') \subseteq \Omega$. On the disc $|\zeta| < R'$, $V(\zeta) = \max_{|w|\leq 1} v(z+\zeta w)$ is the upper envelope of the subharmonic functions $\zeta \mapsto v(z+\zeta w)$, and it is upper semicontinuous (maximum of an upper semicontinuous function of (ζ, w) over the compact set $|w| \leq 1$), hence subharmonic; it is radial, since $w \mapsto e^{i\theta}w$ preserves $\bar\Delta(0,1)$, and $V(\zeta) = M_v(z,|\zeta|)$. A radial subharmonic function is a nondecreasing convex function of $\log|\zeta|$ (Hadamard's three-circles theorem, [145], Ch. 2). As $r \to 0^+$, $M_v(z,r) \geq v(z)$, and $M_v(z,r) = v(x_r)$ for some $x_r \in \bar\Delta(z,r)$, so $\limsup_{r\to 0} M_v(z,r) \leq v(z)$ by upper semicontinuity.

(P2). Smoothness and plurisubharmonicity of $v * \rho_t$ on Ω_t, and $v \leq v * \rho_t$ for a mollifier invariant under the rotations of each coordinate, are the classical properties of the regularisation of plurisubharmonic functions by convolution ([145], Ch. 2; the inequality follows from the sub-mean value inequality on circles in each coordinate, integrated against ρ_t). The upper bound holds because ρ_t is a probability density supported in $\Delta(0,t)$.

(P3). Choose $x \in \bar\Delta(z,t/4)$ with $v(x) = M_v(z,t/4)$ and put $P = \Delta(x,3t/2)$. Then $\Delta(z,t) \subseteq P$ and $\bar P \subseteq \bar\Delta(z,7t/4) \subseteq \Omega$. The sub-mean value inequality on polydiscs (the one on discs applied in each coordinate in turn) gives $M_v(z,t/4) = v(x) \leq \lambda(P)^{-1}\int_P v\, d\lambda$, i.e. $\int_P (M_v(z,2t) - v)\, d\lambda \leq \lambda(P)\big(M_v(z,2t) - M_v(z,t/4)\big)$, with integrand ≥ 0 on P. Since $\rho_t \leq t^{-2D}\sup\rho$ and $\rho_t(w) \neq 0$ only if $z - w \in \Delta(z,t) \subseteq P$,

$$\begin{aligned} M_v(z,2t) - v * \rho_t(z) &= \int \rho_t(w)\big(M_v(z,2t) - v(z-w)\big)\, d\lambda(w) \\ &\leq t^{-2D}\sup\rho \cdot \lambda(P)\big(M_v(z,2t) - M_v(z,t/4)\big), \end{aligned}$$

and $t^{-2D}\lambda(P) = \pi^D(3/2)^{2D}$. □

Lemma 33.25. *Let X be a compact complex manifold of dimension D with a Kähler form Θ, and θ a smooth closed real $(1,1)$-form on X; for a smooth real function f on an open set, $\theta + dd^c f \geq 0$ means that this real $(1,1)$-form is semipositive at every point. Let $\kappa_j \colon U_j \to \mathbb{C}^D$*

($1 \le j \le N$) be holomorphic charts with $\kappa_j(U_j) \supseteq \bar\Delta_2$ (closed polydisc of radius 2 about 0), put $U_j^r = \kappa_j^{-1}(\Delta_r)$ (Δ_r the open polydisc of radius r), and assume $X = \bigcup_j U_j^1$. Then there is $K \ge 0$ with the following properties.

(G1) Let $\delta > 0$ and let f_j be smooth real functions on $U_j^{5/4}$ with $\theta + dd^c f_j \ge 0$ on $U_j^{5/4}$ and $|f_j - f_k| \le \delta$ on $U_j^{5/4} \cap U_k^{5/4}$ for all j,k. Then there is a smooth function F on X with $\theta + dd^c F + K\delta\,\Theta \ge 0$ on X and $f_j \le F \le f_j + 3\delta$ on $U_j^{5/4}$ for every j.

(G2) Let $Z \subseteq X$ be closed and φ smooth on $X \setminus Z$. Let $t_1 > 0$, $\delta\colon (0,t_1] \to (0,\infty)$ with $t \le \delta(t)$ and $\delta(t) \to 0$ as $t \to 0^+$, and for each t let $(f_{t,j})_j$ satisfy the hypotheses of (G1) with $\delta = \delta(t)$. Assume that for every j and every compact $L \subseteq \Delta_{5/4} \setminus \kappa_j(Z)$, $f_{t,j} \circ \kappa_j^{-1} \to \varphi \circ \kappa_j^{-1}$ with first and second derivatives uniformly on L as $t \to 0^+$, and $\|d(f_{t,j} \circ \kappa_j^{-1} - \varphi \circ \kappa_j^{-1})\| \le N_L t$ on L for t small, with N_L independent of t. Then the functions F_t given by the construction of (G1) converge to φ with first and second derivatives (in local coordinates), uniformly on compact subsets of $X \setminus Z$, as $t \to 0^+$.

Proof. The regularised maximum (the construction of [79], Ch. I; we prove the properties used). Fix a smooth even $\vartheta \ge 0$ on $\mathbb{R}$ with support in $[-1,1]$ and $\int \vartheta = 1$, and put $c_\vartheta = \int |\vartheta'|$. For $p \ge 1$, $\eta > 0$ and $x \in \mathbb{R}^p$ let

$$M_\eta(x) = \int_{\mathbb{R}^p} \max_i (x_i + h_i) \prod_{i=1}^p \eta^{-1}\vartheta(h_i/\eta)\, dh.$$

As the convolution of $x \mapsto \max_i x_i$ with a smooth probability density supported in $[-\eta,\eta]^p$:

(M1) M_η is smooth, convex and nondecreasing in each variable, and $M_\eta(x + a(1,\dots,1)) = M_\eta(x) + a$; hence $\sum_i \partial_i M_\eta = 1$ and $\partial_i M_\eta \ge 0$;

(M2) $\max_i x_i \le M_\eta(x) \le \max_i x_i + \eta$ (the lower bound because $\max_i$ of the averages x_i of $x_i + h_i$ is at most the average of $\max_i(x_i + h_i)$, ϑ being even);

(M3) $|\partial_i \partial_k M_\eta| \le c_\vartheta/\eta$, since $\partial_i\partial_k M_\eta$ is the convolution of the bounded function $\partial_i \max$ ($|\cdot| \le 1$ almost everywhere) with ∂_k of the density, of L^1-norm c_ϑ/η;

(M4) if $x_p + \eta \le \max_{i<p} x_i - \eta$, then $M_\eta(x_1,\dots,x_p) = M_\eta(x_1,\dots,x_{p-1})$: for $|h_i| \le \eta$, $x_p + h_p \le \max_{i<p}(x_i + h_i)$, so the integrand does not depend on h_p; M_η is symmetric, so the same holds for any variable;

(M5) if $v_1,\dots,v_p$ are smooth plurisubharmonic on an open subset of $\mathbb{C}^D$, so is $M_\eta(v_1,\dots,v_p)$: its Levi form at w is $\sum_i \partial_i M_\eta\, \mathcal{L}v_i(w) + \sum_{i,k} \partial_i\partial_k M_\eta\, \partial v_i(w)\overline{\partial v_k(w)}$, a sum of nonnegative terms by (M1).

Cut-offs and the constant K. Fix a smooth $c\colon [0,\infty) \to [0,1]$, constant near $[0,1]$ with value 1 and vanishing on $[9/8,\infty)$, and put $\gamma(z) = -2 + 3\prod_{i=1}^D c(|z_i|)$: smooth on $\mathbb{C}^D$, $\gamma \le 1$, $\gamma = 1$ on $\bar\Delta_1$, $\gamma = -2$ off $\Delta_{9/8}$. Let $\gamma_j = \gamma \circ \kappa_j$ on U_j. Since γ_j is constant outside the compact set $\kappa_j^{-1}(\bar\Delta_{9/8}) \subseteq U_j$ and Θ is positive definite, there is $K \ge 0$ with $dd^c\gamma_j \ge -K\Theta$ on U_j for all j. This K depends only on the charts and Θ.

Definition of F. Given δ and (f_j) as in (G1), put $\eta = \delta/2$, $J(x) = \{j : x \in U_j^{5/4}\}$ and

$$F(x) = M_\eta\big((f_j(x) + \delta\gamma_j(x))_{j \in J(x)}\big) \qquad (x \in X).$$

Local form. For $x \in X$ let $J'(x) = \{j : x \in \kappa_j^{-1}(\bar\Delta_{9/8})\} \subseteq J(x)$ and

$$W_x = \bigcap_{j\in J'(x)} U_j^{5/4} \setminus \bigcup_{j\notin J'(x)} \kappa_j^{-1}(\bar\Delta_{9/8}),$$

an open neighbourhood of x (the sets $\kappa_j^{-1}(\bar\Delta_{9/8})$ are compact). We claim $F = M_\eta((f_j + \delta\gamma_j)_{j\in J'(x)})$ on W_x. Let $y \in W_x$. Then $J'(x) \subseteq J(y)$. Choose k with $y \in U_k^1$; as $y \in \kappa_k^{-1}(\bar\Delta_{9/8})$ and $y \in W_x$, $k \in J'(x)$, and $\gamma_k(y) = 1$. For $j \in J(y) \setminus J'(x)$, $y \notin \kappa_j^{-1}(\bar\Delta_{9/8})$, so $\gamma_j(y) = -2$ and

$$f_j(y) + \delta\gamma_j(y) + \eta \le f_k(y) + \delta - 2\delta + \eta = \big(f_k(y) + \delta\gamma_k(y)\big) - \tfrac{3}{2}\delta \le \big(f_k(y) + \delta\gamma_k(y)\big) - \eta.$$

Removing these j one at a time by (M4) (the term k stays) proves the claim.

Proof of (G1). On W_x the functions $f_j + \delta\gamma_j$, $j \in J'(x)$, are smooth, so F is smooth near every point. Positivity: near a point of W_x write $\theta = dd^c q$ and $\Theta = dd^c\psi$ with smooth local potentials q, ψ on a polydisc in local coordinates. For $j \in J'(x)$, $q + f_j$ is plurisubharmonic ($\theta + dd^c f_j \ge 0$) and so is $\delta(\gamma_j + K\psi)$; by (M1) and (M5),

$$q + F + K\delta\psi = M_\eta\big((q + f_j + \delta\gamma_j + K\delta\psi)_{j\in J'(x)}\big)$$

is plurisubharmonic, i.e. $\theta + dd^c F + K\delta\Theta \ge 0$. Bounds: let $x \in U_k^{5/4}$. By (M2), $\gamma_j \le 1$ and $f_j \le f_k + \delta$ on $J(x)$, $F(x) \le \max_{j\in J(x)} f_j(x) + \delta + \eta \le f_k(x) + 3\delta$. Choose k' with $x \in U_{k'}^1$; then by (M2) $F(x) \ge f_{k'}(x) + \delta\gamma_{k'}(x) = f_{k'}(x) + \delta \ge f_k(x)$.

Proof of (G2). The set W_x depends only on $J'(x)$, so there are finitely many such sets $W^{(1)}, \dots, W^{(r)}$, independent of t, covering X; let $J'^{(a)}$ be the index set of $W^{(a)}$ ($\ne \varnothing$, as it contains every k with $x \in U_k^1$). As X is compact Hausdorff, there are compact $L^{(a)} \subseteq W^{(a)}$ covering X. Let $L \subseteq X \setminus Z$ be compact; it suffices to prove the convergence on $L \cap L^{(a)}$ for each a, in the coordinates κ_{j_0} of some $j_0 \in J'^{(a)}$. Put $\eta = \eta(t) = \delta(t)/2$, $L_a = \kappa_{j_0}(L \cap L^{(a)})$, and for $j \in J'^{(a)}$, with the holomorphic transition map $\tau_j = \kappa_j \circ \kappa_{j_0}^{-1}$,

$$H_{t,j} = (f_{t,j} - \varphi + \delta(t)\gamma_j) \circ \kappa_{j_0}^{-1} = \big(f_{t,j} \circ \kappa_j^{-1} - \varphi \circ \kappa_j^{-1}\big) \circ \tau_j + \delta(t)\,\gamma \circ \tau_j.$$

By the local form and (M1), $(F_t - \varphi) \circ \kappa_{j_0}^{-1} = M_\eta((H_{t,j})_j)$ on L_a. The set $\tau_j(L_a)$ is a compact subset of $\Delta_{5/4} \setminus \kappa_j(Z)$ and τ_j has bounded derivatives of order ≤ 2 on a neighbourhood of L_a; by the hypotheses and $\delta(t) \to 0$, $H_{t,j} \to 0$ uniformly on L_a with first and second derivatives, and $\|dH_{t,j}\| \le c\,(t + \delta(t)) \le 2c\,\delta(t)$ on L_a for t small, with c independent of t (here $t \le \delta(t)$). Let $G_t = M_\eta((H_{t,j})_j)$ and $p = |J'^{(a)}|$. By (M2), $|G_t| \le \max_j |H_{t,j}| + \eta \to 0$. By (M1), $\|dG_t\| \le \max_j \|dH_{t,j}\| \to 0$. By (M1) and (M3),

$$\|D^2 G_t\| \le \max_j \|D^2 H_{t,j}\| + p^2 \frac{c_\vartheta}{\eta} \max_j \|dH_{t,j}\|^2 \le \max_j \|D^2 H_{t,j}\| + 8p^2 c_\vartheta c^2\,\delta(t) \to 0.$$

Hence $F_t \to \varphi$ with first and second derivatives uniformly on $L \cap L^{(a)}$. □

Lemma 33.26. *Let X be a compact complex manifold of dimension D, θ a smooth closed real $(1,1)$-form on X, and $\kappa_j\colon U_j \to \mathbb{C}^D$ ($1 \le j \le N$) holomorphic charts with $\kappa_j(U_j) \supseteq \bar\Delta_2$, where Δ_r, $\bar\Delta_r$ are the open and closed polydiscs of radius r about 0 for $|x| = \max_i |x_i|$; put $U_j^r = \kappa_j^{-1}(\Delta_r)$ and let q_j be smooth functions on U_j with $dd^c q_j = \theta$. Then there are $\Lambda_0 \ge 1$, $G \ge 0$ and $0 < t_* \le 1/(2\Lambda_0)$ with the following property. Let $\tilde\varphi\colon X \to [-\infty, \infty)$ be bounded above, put $u_j = (q_j + \tilde\varphi) \circ \kappa_j^{-1}$ on Δ_2 and $M_j(z,r) = \sup\{u_j(y) : |y - z| \le r\}$ for $\bar\Delta(z,r) \subseteq \Delta_2$. Then for all j, k, $x \in U_j^{5/4} \cap U_k^{5/4}$ and $0 < t \le t_*$,*

$$\big|(M_j(\kappa_j x, t) - q_j(x)) - (M_k(\kappa_k x, t) - q_k(x))\big| \le \max_{l\in\{j,k\}} \big(M_l(\kappa_l x, \Lambda_0 t) - M_l(\kappa_l x, t/\Lambda_0)\big) + 2G\Lambda_0 t.$$

Proof. Constants. For j,k the set $P_{jk}=\kappa_j(\kappa_j^{-1}(\bar\Delta_{5/4})\cap\kappa_k^{-1}(\bar\Delta_{5/4}))$ is a compact subset of the open set $\kappa_j(U_j\cap U_k)$. Choose $\rho_0\in(0,1/4]$ with $P_{jk}+\bar\Delta(0,\rho_0)\subseteq\kappa_j(U_j\cap U_k)$ for all j,k, and $\Lambda_0\geq 1$ bounding the operator norm (for $|\cdot|$) of the derivative of $\tau_{kj}=\kappa_k\circ\kappa_j^{-1}$ on $P_{jk}+\bar\Delta(0,\rho_0)$, for all j,k. Let G_0 be a Lipschitz constant for $|\cdot|$ of every $q_j\circ\kappa_j^{-1}$ on $\bar\Delta_{7/4}$, $G=2G_0$, and $t_*=\min(\rho_0,1/(2\Lambda_0))$.

Inclusions. Let $x\in U_j^{5/4}\cap U_k^{5/4}$ and $0<t\leq t_*$. Since $\kappa_j x\in P_{jk}$ and polydiscs are convex, the mean value inequality gives $\tau_{kj}(\bar\Delta(\kappa_j x,t))\subseteq\bar\Delta(\kappa_k x,\Lambda_0 t)$, and, exchanging j and k ($t/\Lambda_0\leq\rho_0$), $\tau_{jk}(\bar\Delta(\kappa_k x,t/\Lambda_0))\subseteq\bar\Delta(\kappa_j x,t)$. All these polydiscs lie in $\bar\Delta_{7/4}$, as $5/4+\Lambda_0 t\leq 7/4$.

Comparison. For $l\in\{j,k\}$ and $r\leq\Lambda_0 t$ put $N_l(x,r)=\sup\{\tilde\varphi(y):y\in U_l,\ |\kappa_l y-\kappa_l x|\leq r\}$. The Lipschitz bound on q_l gives $|M_l(\kappa_l x,r)-q_l(x)-N_l(x,r)|\leq G_0 r$. By the inclusions, $N_j(x,t)\leq N_k(x,\Lambda_0 t)$ and $N_k(x,t/\Lambda_0)\leq N_j(x,t)$; with $N_k(x,t/\Lambda_0)\leq N_k(x,t)$ this gives

$$N_j(x,t)-N_k(x,t)\leq N_k(x,\Lambda_0 t)-N_k(x,t/\Lambda_0)\leq M_k(\kappa_k x,\Lambda_0 t)-M_k(\kappa_k x,t/\Lambda_0)+G_0(\Lambda_0+\Lambda_0^{-1})t,$$

and symmetrically with j,k exchanged. Passing from $N_l(x,t)$ to $M_l(\kappa_l x,t)-q_l(x)$ costs $G_0 t$ for each l, so the left-hand side of the claim is at most the maximum of the two oscillations plus $G_0(\Lambda_0+\Lambda_0^{-1}+2)t\leq 4G_0\Lambda_0 t=2G\Lambda_0 t$, as $\Lambda_0\geq 1$. □

Lemma 33.27. *Let $\mathfrak{u}=(E,\mathsf{H},\mathsf{U},p,q,\mathsf{L},\mathrm{Sh},F)$ be a unitary Shimura datum with $p,q\geq 1$, $\Phi=\Phi_{\mathsf{H}}$, $n=p+q$, $\nu(\tau,i)=1-i$, let $\mathcal{Y}$ be an integral model of* Sh *over $\mathcal{O}_F$ and $\mathcal{Y}_c$ an irreducible component of $\mathcal{Y}$ meeting $j(\mathrm{Sh}_F)$. Let $\mathcal{X}'\subseteq\mathbb{P}'=\prod_{k\in K'}\mathbb{P}(\mathbb{Z}^{I_k})$, with representing coordinates $y_{k,j}$, multidegrees e_r and the data of (Go3)(1)–(4), be the flag model of Lemma 33.8 in case (ii) ($S=\varnothing$, over a cover $\mathcal{X}^{\mathrm{b}}$ as in Proposition 33.4 for $\mathcal{Y}_c$), and let $h_r(t)$, h_a^{ref}, Sys' be as in Proposition 33.12 for it. Let k_0 be a new index, $K=K'\sqcup\{k_0\}$, $I_{k_0}=\prod_{k\in K'}I_k$, enumerated as $\{0,\dots,m\}$, $\mathbb{P}=\mathbb{P}'\times\mathbb{P}(\mathbb{Z}^{I_{k_0}})$, let $\sigma\colon\mathbb{P}'\to\mathbb{P}(\mathbb{Z}^{I_{k_0}})$ be the Segre morphism $(z_k)_k\mapsto(\prod_{k\in K'}z_{k,i_k})_{i\in I_{k_0}}$, and let $\mathcal{X}^{\mathrm{ref}}\subseteq\mathbb{P}$ be the graph of $\sigma|_{\mathcal{X}'}$. Extend each e_r by 0 at k_0, put $e_*=\mathbf{1}_{k_0}$ and $h_*(\tilde z)=\bigl(\sum_{i\in I_{k_0}}|\tilde z_{k_0,i}|^2\bigr)^{1/2}$, and let $\mathrm{pr}_{K'}$ forget the coordinates of the factor k_0. Then the projection $\mathrm{pr}\colon\mathcal{X}^{\mathrm{ref}}\to\mathcal{X}'$ is an isomorphism, $(\mathcal{X}^{\mathrm{ref}},(e_r),(h_r(t)\circ\mathrm{pr}_{K'}),h_*,1,[F:\mathbb{Q}])$ is a root model of type (Φ,n), and there are fixed data at $\mathcal{Y}_c$ with $k_c^A=q_c'=1$, $n_c=\nu$, $s_c^E=1$, and data of a root origin of this root model on them for $S=\varnothing$ with* deg $=1$*, satisfying (Go3)(1)–(5) and (Go4), such that:*

1. *$\mathcal{X}^{\mathrm{ref}}\otimes\mathbb{Q}$ is smooth over $\mathbb{Q}$; $\mathcal{O}(e_*)|_{\mathcal{X}^{\mathrm{ref}}}$ is the pull-back of $\mathcal{O}(1)$ by the component at k_0, which is a closed immersion, so it is very ample and in particular ample on $\mathcal{X}^{\mathrm{ref}}\otimes\mathbb{Q}$;*
2. *$X_c^{\mathrm{an}}=\mathcal{X}^{\mathrm{ref}}(\mathbb{C})$ and the realisation is the identity (bijective, of degree one); the analytic system Sys_c has the Kähler form, the boundary section, the singular weights and the regularisations of* Sys′ *pulled back by* pr*, and for every fundamental index $a\neq *$ the line bundle $\mathcal{O}(e_a)|_{\mathcal{X}^{\mathrm{ref}}(\mathbb{C})}$ (e_a the multidegree of a) with the metric of the cone function $h_a^{\mathrm{ref}}\circ\mathrm{pr}_{K'}$, and for $*$ the line bundle $\mathcal{O}(e_*)|_{\mathcal{X}^{\mathrm{ref}}(\mathbb{C})}$ with the metric of h_*;*
3. *(flat determinant) for every $\tau\in\Phi$ and every $0<t\leq t_0$ the regularised metric of the fundamental index (τ,n) of Sys_c has pluriharmonic local potentials;*
4. *(immersive coordinates) the map pt_c of the fixed data is, near every point, the projectivisation of a holomorphic map to $\mathbb{C}^{m+1}\smallsetminus\{0\}$ whose affine charts have injective differential.*

Proof. **The scheme.** The Segre morphism σ is a closed immersion (Hartshorne, Algebraic Geometry, II Ex. 5.11, by induction on $|K'|$), so $\Psi_0=\sigma\circ\iota'\colon\mathcal{X}'\to\mathbb{P}(\mathbb{Z}^{I_{k_0}})$ is a closed immersion.

The morphism $\gamma = (\iota', \Psi_0)\colon \mathcal{X}' \to \mathbb{P}$ is a closed immersion, because its composite with the separated projection $\mathbb{P} \to \mathbb{P}'$ is the closed immersion ι' (Hartshorne II Ex. 4.8); its image is $\mathcal{X}^{\mathrm{ref}}$, and pr is the inverse of $\gamma\colon \mathcal{X}' \to \mathcal{X}^{\mathrm{ref}}$. Hence $\mathcal{X}^{\mathrm{ref}}$ is a closed integral subscheme of $\mathbb{P}$, normal, projective and flat over $\mathbb{Z}$ of absolute dimension $D+1$, with $\mathcal{X}^{\mathrm{ref}} \otimes \mathbb{Q}$ smooth (Lemma 33.8, item 1). Its component at k_0 is $\Psi = \Psi_0 \circ \mathrm{pr}$, a closed immersion, and $\mathcal{O}(e_*)|_{\mathcal{X}^{\mathrm{ref}}} = \Psi^*\mathcal{O}(1)$ is very ample: item 1. We identify $\mathbf{k}(\mathcal{X}^{\mathrm{ref}})$ with Ω' by pr^*. For $k \in K'$ the $y_{k,j}$ are representing coordinates of the component at k (it is that of $\mathcal{X}'$ composed with pr). Put $y_{k_0,i} = \prod_{k\in K'} y_{k,i_k}$ for $i = (i_k)_k \in I_{k_0}$. If $y_{k_0,i'} \neq 0$, all y_{k,i'_k} are non-zero, and on $\prod_k \{Z_{k,i'_k} \neq 0\}$ the ratio of the Segre coordinates i and i' pulls back under σ to $\prod_k Z_{k,i_k}/Z_{k,i'_k}$, whose restriction to $\mathcal{X}'$ is $\prod_k y_{k,i_k}/y_{k,i'_k} = y_{k_0,i}/y_{k_0,i'}$. So the $y_{k,j}$ ($k \in K$) are representing coordinates of $\mathcal{X}^{\mathrm{ref}} \subseteq \mathbb{P}$.

The ratios of the k_0-coordinates generate Ω'. For $k \in K'$ choose c_k with $y_{k,c_k} \neq 0$. The generic point of $\mathcal{X}'$ lies in $U = \prod_{k\in K'}\{Z_{k,c_k} \neq 0\} = \operatorname{Spec}\mathbb{Z}[Z_{k,j}/Z_{k,c_k}]$, and $\mathcal{X}' \cap U$ is a non-empty closed subscheme of U, so Ω' is generated as a field by the restrictions $y_{k,j}/y_{k,c_k}$. Given $k \in K'$ and $j \in I_k$, let $i, i' \in I_{k_0}$ agree with $(c_{k''})_{k''}$ outside k, with $i_k = j$, $i'_k = c_k$: then $y_{k_0,i'} \neq 0$ and $y_{k_0,i}/y_{k_0,i'} = y_{k,j}/y_{k,c_k}$. So the ratios $y_{k_0,i}/y_{k_0,i'}$ ($y_{k_0,i'} \neq 0$) generate Ω'.

Fixed data. Since $\deg = [\Omega' : g'(\Omega_c)] = 1$ (Lemma 33.8, item 2, case (ii)), $g = g'$ is an isomorphism $\Omega_c \to \Omega'$. Put $y_i^c = g^{-1}(y_{k_0,i})$ ($0 \le i \le m$, by the enumeration of I_{k_0}); their ratios generate Ω_c by the previous paragraph. Put $h_c(w) = (\sum_i |w_i|^2)^{1/2}$, $k_c^A = q'_c = 1$, $n_c = \nu$ (so $n_c(r) = k_c^A q'_c \nu(r)$), $s_c^E = 1$. For $x \in \mathcal{X}^{\mathrm{ref}}(\mathbb{C})$ let $\mathrm{pt}_c(x)$ be the representative of $\Psi(x) \in \mathbb{P}^m(\mathbb{C})$ of euclidean norm 1 whose first non-zero entry is real and positive. As Ψ is a closed immersion, $\Psi(\mathbb{C})$ is injective, i.e. pt_c is injective modulo $\mathbb{C}^\times$. The analytic system Sys_c is defined next; with it these are fixed data at $\mathcal{Y}_c$ (Go2).

The analytic system. $X_c^{\mathrm{an}} = \mathcal{X}^{\mathrm{ref}}(\mathbb{C})$ is a compact complex manifold of dimension D, and pr is a biholomorphism onto $\mathcal{X}'(\mathbb{C})$. Put $t_0 = \min(t'_0, e^{-e})$, $\Theta = \mathrm{pr}^*\Theta'$, $\varsigma = \mathrm{pr}^*\varsigma'$, $\varphi_a = \varphi'_a \circ \mathrm{pr}$, $\varphi_a^T(t) = \varphi_a'^T(t) \circ \mathrm{pr}$ (so $\varphi_* = 0 = \varphi_*^T$), the same constants C_5, C, and the line bundles with metrics of item 2. For a fundamental index $a \neq *$, $h_a^{\mathrm{ref}} \circ \mathrm{pr}_{K'}$ is a cone function of multidegree e_a (entry 0 at k_0), and $\mathrm{pr}_{K'}$ maps the affine cone over $\mathcal{X}^{\mathrm{ref}}(\mathbb{C})$ onto that over $\mathcal{X}'(\mathbb{C})$ compatibly with pr; so the first Chern form of $(\mathcal{O}(e_a), h_a^{\mathrm{ref}} \circ \mathrm{pr}_{K'})|_{\mathcal{X}^{\mathrm{ref}}(\mathbb{C})}$ is pr^* of that of $(\mathcal{O}(e_a), h_a^{\mathrm{ref}})|_{\mathcal{X}'(\mathbb{C})}$, which is c_1 of the line bundle of a in Sys' (Proposition 33.12, item 2: hypothesis 2 of Lemma 31.21 with $\pi' = \mathrm{id}$). The first Chern form of $(\mathcal{O}(e_*), h_*)$ on X_c^{an} is Ψ^* of a positive multiple of the Fubini–Study form; $\Psi(\mathbb{C})$ is a closed immersion of a smooth variety, hence a holomorphic embedding with injective differential, so this form is positive definite. The clauses of (Go1) now hold: those that involve only the functions φ_a, $\varphi_a^T(t)$, rW and Θ are those of Sys' pulled back by the biholomorphism pr (with $t \le t_0 \le t'_0$); plurisubharmonicity of a smooth metric multiplied by e^{ψ}, also after a twist by a $(1,1)$-form, depends on the metric only through its first Chern form, which is unchanged for $a \neq *$; for $*$, h_* has positive curvature, and since $t \le e^{-e}$ gives $\varepsilon(t) \ge 0$ the twist by $\varepsilon(t)\Theta$ keeps it psh. So Sys_c is an analytic system with $\varphi_* = 0 = \varphi_*^T$ and h_*^{ref} of positive curvature; its open part X_c° is pr^{-1} of that of Sys', i.e. of the abelian locus of $\mathcal{X}'$ (Proposition 33.12, item 2), which is the abelian locus of $\mathcal{X}^{\mathrm{ref}}$ for the structure map $f = f' \circ \mathrm{pr}$ (f' that of $\mathcal{X}'$).

The root model and (Go3). The root metrics $h_r(t) \circ \mathrm{pr}_{K'}$ are cone metrics of multidegree e_r and h_* is a cone metric of multidegree e_* (smooth on $\widehat{\mathbb{C}}$, conjugation invariant, homogeneous of degree one in the k_0-coordinates and of degree zero in the others); with the scheme of the first paragraph this is a root model of type (Φ, n) with $k = 1$, $\delta = [F : \mathbb{Q}]$. (Go3)(1): $\mathcal{X}^{\mathrm{ref}}$ is normal and, with f, isomorphic over $\mathcal{Y}$ to the cover $\mathcal{X}'$, whose generic point maps to that of $\mathcal{Y}_c$; g is a field embedding over $\mathbf{k}(Z_c)$ with $[\Omega' : g(\Omega_c)] = 1 = \deg$ and $\delta = [F : \mathbb{Q}]\deg$. (Go3)(2) for $S = \varnothing$: the torsion and Hodge–Tate clauses are empty; $\beta \circ \mathrm{pr}\colon \mathcal{X}^{\mathrm{ref}} \to \mathcal{X}^{\mathrm{b}}$ is dominant and $f = f^{\mathrm{b}} \circ \beta \circ \mathrm{pr}$, so by Proposition 33.4, item 3, the de Rham lattices of $\mathcal{X}^{\mathrm{ref}}$ form Hodge bundles $\cong (\beta \circ \mathrm{pr})^*\mathcal{V}_s$.

(Go3)(3): the pull-back by pr of the integral flag of $\mathcal{X}'$ is an integral flag with generic fibre g of the universal flag; take the same ϱ_r. If $y_{c'} \neq 0$ for $c' \in \prod_{k\in K} I_k$, then $y_{c'|_{K'}} \neq 0$, and the chart $\{y_{c'} \neq 0\}$ of $\mathcal{X}^{\mathrm{ref}}$ lies in pr^{-1} of the chart $\{y_{c'|_{K'}} \neq 0\}$ of $\mathcal{X}'$, on which $\varrho_r \prod_{k\in K'} y_{k,c'_k}^{e_{r,k}}$ generates $\mathcal{L}_r$; since $e_{r,k_0} = 0$ this is $\varrho_r \prod_{k\in K} y_{k,c'_k}^{e_{r,k}}$. On such a chart a local section of $\mathcal{O}(e_r)$ is $Q \prod_k Y_{k,c'_k}^{e_{r,k}}$ with Q regular, so $P \mapsto \varrho_r P(y)$ maps generators to generators; the non-empty charts cover $\mathcal{X}^{\mathrm{ref}}$, so it is an isomorphism $\mathcal{O}(e_r)|_{\mathcal{X}^{\mathrm{ref}}} \to \mathcal{L}_r$. (Go3)(4): put $h_r^{\mathrm{sing}} \circ \mathrm{pr}_{K'}$ as singular cone functions. As $e_{r,k_0} = 0$, $\mathcal{O}(e_r)$ on $\mathbb{P}$ is the pull-back of $\mathcal{O}(e_r)$ on $\mathbb{P}'$ by the projection, which restricts to the isomorphism $\iota' \circ \mathrm{pr}$ onto $\iota'(\mathcal{X}')$; so every local section P of $\mathcal{O}(e_r)|_{\mathcal{X}^{\mathrm{ref}}}$ is the pull-back of a local section P' on $\mathcal{X}'$, with $P(\tilde{z}) = P'(\mathrm{pr}_{K'}\tilde{z})$ and $P(y) = P'(y)$ in Ω', and the identity of (Go3)(4) for $\mathcal{X}'$ gives it for $\mathcal{X}^{\mathrm{ref}}$ (the abelian loci correspond under pr). (Go3)(5): $e_* = \mathbf{1}_{k_0}$; the coordinates of the factor k_0 are $y_{k_0,i} = g(y_i^c)$; $h_*(\tilde{z}) = h_c((\tilde{z}_{k_0,i})_i)$; $k_{\mathcal{X}^{\mathrm{ref}}} = 1 = k_c^A$.

(Go4) with $\pi = \mathrm{id}$. The identity of $\mathcal{X}^{\mathrm{ref}}(\mathbb{C}) = X_c^{\mathrm{an}}$ has degree $1 = \deg$ off the empty set, with invertible differential. The k_0-coordinates of x represent $\Psi(x)$, hence are proportional to $\mathrm{pt}_c(x)$. A point lies in the abelian locus if and only if it lies in X_c° (above). The cone functions $h_a^{\mathrm{ref}} \circ \mathrm{pr}_{K'}$ ($a \neq *$) and h_* are smooth, of the multidegrees of the fundamental indices, and their metrics are those of Sys_c by item 2. For $(\tau, i) \in \mathrm{Rt}$ and $0 < t \leq t_0$, Proposition 33.12, item 2 (hypothesis 3 of Lemma 31.21), gives $\prod_{k\leq i} h_{(\tau,k)}(t)^{-1} = h^{\mathrm{ref}}_{(\tau,i)} e^{\varphi'^T_{(\tau,i)}(t)}$ on the cone over $\mathcal{X}'(\mathbb{C})$, and composing with $\mathrm{pr}_{K'}$ gives the regularised root metrics of $\mathcal{X}^{\mathrm{ref}}$; likewise the last clause of Proposition 33.12, item 2, composed with $\mathrm{pr}_{K'}$, gives $h^{\mathrm{ref}}_{(\tau,i)} e^{\varphi_{(\tau,i)}} = \prod_{k\leq i} (h^{\mathrm{sing}}_{(\tau,k)})^{-1}$ on the abelian locus. This completes the root origin and item 2.

Item 3. By Proposition 33.12, item 4, $\varphi'^T_{(\tau,n)}(t) = \varphi'_{(\tau,n)}$ is smooth on $\mathcal{X}'(\mathbb{C})$ and the metric $h^{\mathrm{ref}} e^{\varphi'_{(\tau,n)}}$ of the index (τ, n) in Sys' has pluriharmonic local potentials, i.e. first Chern form 0. The regularised metric of (τ, n) in Sys_c is the metric of $h^{\mathrm{ref}}_{(\tau,n)} \circ \mathrm{pr}_{K'}$ times $e^{\varphi'_{(\tau,n)} \circ \mathrm{pr}}$, whose first Chern form is pr^* of that form (the reference parts have the same Chern form, above), hence 0; a smooth metric with vanishing first Chern form has pluriharmonic local potentials.

Item 4. Let $x \in X_c^{\mathrm{an}}$; choose $c' \in \prod_{k\in K'} I_k$ with $Z_{k,c'_k}(x) \neq 0$ for all $k \in K'$, and an open neighbourhood U of x on which these coordinates do not vanish. For $i \in I_{k_0}$ put $\lambda_i = \prod_{k\in K'} Z_{k,i_k}/Z_{k,c'_k}$, a holomorphic function on U. At $y \in U$ the vector $\lambda(y)$ is a representative of $\Psi(y)$ (the Segre coordinates of a representative of the point), and its entry $i = c'$ is 1; so $\lambda(y) = a_0(y)\,\mathrm{pt}_c(y)$ with $a_0(y) \neq 0$. If $\lambda_{i_1}(x) \neq 0$, then near x the map $y \mapsto (\lambda_i(y)/\lambda_{i_1}(y))_i$ is Ψ followed by the affine chart $\{Z_{k_0,i_1} \neq 0\}$ of $\mathbb{P}^m(\mathbb{C})$; Ψ is a holomorphic embedding, so this map has injective differential at x. □

33.3 Positivity of the gap line and flag volumes

The reference root model needs a big gap line: it is nef with positive top degree on a geometric component (Lemma 33.28). The proof uses two volume computations. One is pointwise Weyl proportionality of the Hodge Chern forms of the root lines on the flag bundle (Proposition 33.31), computed on flag domains of hermitian spaces, where the root lines are pulled back from tautological lines. The other is the flag-volume identity for one root origin (Lemma 33.29), which rests on the total mass of the singular Chern forms of an analytic system (Proposition 33.30). The subsection also proves that the flag volume form of a component is a positive volume form with positive total mass, by the complex uniformisation of the moduli scheme and invariant Hodge forms on the period domain (Lemma 33.34, Proposition 33.35, Lemma 33.38). Weyl proportionality and the flag-volume identity are used again in Appendix D.

Lemma 33.28. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p, q \geq 1$, level*

$\mathsf{L} = (N, \mathfrak{n}, 1)$ *good at every prime and split components,* $\Phi = \Phi_{\mathsf{H}}$, $n = p+q$, $\nu(\tau, i) = 1 - i$, $\mathcal{Y}$ *an integral model of* Sh *over* $\mathcal{O}_F$, $\mathcal{Y}_c$ *an irreducible component of* $\mathcal{Y}$ *meeting* $j(\mathrm{Sh}_F)$, *and let* $\mathcal{X}^{\mathrm{ref}}$ *with its fixed data and root origin be as given by Lemma 33.27. Put* $e_\nu = -\sum_{r \in \mathrm{Rt}} \nu(r) e_r$. *Then for every irreducible component* X_1 *of* $\mathcal{X}^{\mathrm{ref}} \otimes_{\mathbb{Z}} \mathbb{C}$ *the line bundle* $\mathcal{O}(e_\nu)|_{X_1}$ *is nef, and for some irreducible component* X_1

$$\deg\bigl(c_1(\mathcal{O}(e_\nu)|_{X_1})^D \cap [X_1]\bigr) > 0.$$

Proof. Write Sys_c for the analytic system of the fixed data, on $X_c^{\mathrm{an}} = \mathcal{X}^{\mathrm{ref}}(\mathbb{C})$, with Kähler form Θ (the first Chern form of its positive line bundle), t_0, $\varepsilon(t)$ as in (Go1). As $\mathcal{X}^{\mathrm{ref}} \otimes \mathbb{Q}$ is smooth (Lemma 33.27, item 1), the irreducible components of $\mathcal{X}^{\mathrm{ref}} \otimes \mathbb{C}$ are its connected components, and their complex points are the connected components of X_c^{an}, compact complex manifolds of dimension D.

The gap line in fundamental indices. For a fundamental index (τ, i) let $f_{(\tau,i)} = -\sum_{k \le i} e_{(\tau,k)}$, $f_{(\tau,0)} = 0$, so $e_{(\tau,i)} = f_{(\tau,i-1)} - f_{(\tau,i)}$ and

$$e_\nu = \sum_{\tau \in \Phi} \sum_{i=1}^{n} (i-1)(f_{(\tau,i-1)} - f_{(\tau,i)}) = \sum_{\tau \in \Phi} \Bigl(\sum_{j=1}^{n-1} f_{(\tau,j)} - (n-1) f_{(\tau,n)} \Bigr).$$

By Lemma 33.27, item 2, the line bundle of the fundamental index a in Sys_c is $\mathcal{M}_a = \mathcal{O}(f_a)|_{X_c^{\mathrm{an}}}$ with a smooth metric h_a^{ref}. For $0 < t \le t_0$ give $\mathcal{O}(e_\nu)|_{X_c^{\mathrm{an}}} = \bigotimes_\tau (\bigotimes_{j<n} \mathcal{M}_{(\tau,j)} \otimes \mathcal{M}_{(\tau,n)}^{\otimes -(n-1)})$ the product of the regularised metrics $h_a^{\mathrm{ref}} e^{\varphi_a^T(t)}$, with first Chern form

$$\Omega_t = \sum_\tau \Bigl(\sum_{j<n} \theta_{(\tau,j)}(t) - (n-1)\theta_{(\tau,n)}(t) \Bigr), \qquad \theta_a(t) = c_1\bigl(\mathcal{M}_a, h_a^{\mathrm{ref}} e^{\varphi_a^T(t)}\bigr).$$

By (Go1)(2) the metric $h_a^{\mathrm{ref}} e^{\varphi_a^T(t)}$ twisted by $\varepsilon(t)\Theta$ is psh, i.e. $\theta_a(t) + \varepsilon(t)\Theta \ge 0$; by Lemma 33.27, item 3, $\theta_{(\tau,n)}(t) = 0$. Hence

$$\Omega_t \ge -|\Phi|(n-1)\,\varepsilon(t)\,\Theta \qquad (0 < t \le t_0).$$

Nefness. Let X_1 be a component and $C \subseteq X_1$ an irreducible complete curve, with normalisation $\eta\colon \tilde{C} \to C$, a compact Riemann surface. By Chern–Weil theory on $\tilde{C}$, $\deg(\mathcal{O}(e_\nu)|_C) = \deg \eta^* \mathcal{O}(e_\nu) = \int_{\tilde{C}} \eta^* \Omega_t \ge -|\Phi|(n-1)\,\varepsilon(t) \int_{\tilde{C}} \eta^* \Theta$ for every $0 < t \le t_0$. As $t \to 0^+$, $\varepsilon(t) = C \log\log(1/t) / \log(1/t) \to 0$, while $\int_{\tilde{C}} \eta^* \Theta$ is a fixed finite number; so $\deg(\mathcal{O}(e_\nu)|_C) \ge 0$, i.e. $\mathcal{O}(e_\nu)|_{X_1}$ is nef.

The total top degree. Let $G = \mathrm{lr}(\nu)^D = \bigl(\sum_{\tau,i} (i-1) x_{\tau,i}\bigr)^D \in \mathcal{R}$. Since $c_1(\mathcal{O}(e_\nu)) = \sum_r (-\nu(r))\, c_1(\mathcal{O}(e_r))$ and $\delta_{\mathcal{X}^{\mathrm{ref}}} = [F : \mathbb{Q}]$, the definition of gi gives, summing the degrees over the components (the degree of a zero-cycle on $\mathcal{X}_{\mathbb{Q}}^{\mathrm{ref}}$ is the number of its complex points with multiplicity),

$$\sum_{X_1} \deg\bigl(c_1(\mathcal{O}(e_\nu)|_{X_1})^D \cap [X_1]\bigr) = [F : \mathbb{Q}]\, \mathrm{gi}(\mathcal{X}^{\mathrm{ref}}; G).$$

The root origin of $\mathcal{X}^{\mathrm{ref}}$ is on fixed data at $\mathcal{Y}_c$ for $S = \varnothing$ and $\mathcal{X}^{\mathrm{ref}} \otimes \mathbb{Q}$ is smooth, so by Lemma 33.29 every $G'(\omega)$ ($G' \in \mathcal{R}$ homogeneous of degree D) is integrable on $M_c = \mathrm{Fl}_{\mathrm{u}}|_{Z_c}(\mathbb{C})$ and $[F : \mathbb{Q}]\, \mathrm{gi}(\mathcal{X}^{\mathrm{ref}}; G) = \int_{M_c} G(\omega)$. Let $G_0 = (-1)^D x^\delta$. In (21.1) only $w = \mathrm{id}$ contributes to $\deg^{\mathrm{fl}}(x^\delta)$, since $i \mapsto n - i$ is injective, so $\deg^{\mathrm{fl}}(G_0) = 1$; by Proposition 33.31 (applied with G_0 and with G at every point of $M_c \subseteq M$) $G(\omega) = \deg^{\mathrm{fl}}(G)\, G_0(\omega)$ on M_c. Hence

$$\sum_{X_1} \deg\bigl(c_1(\mathcal{O}(e_\nu)|_{X_1})^D \cap [X_1]\bigr) = \deg^{\mathrm{fl}}\bigl(\mathrm{lr}(\nu)^D\bigr) \int_{M_c} \nu^{\mathrm{Fl}}, \qquad \nu^{\mathrm{Fl}} = G_0(\omega).$$

By Lemma 30.14, $\deg^{\mathrm{fl}}(\mathrm{lr}(\nu)^D) = D!\prod_\tau \prod_{i<i'}(\nu(\tau,i)-\nu(\tau,i'))/(i'-i) = D! > 0$. $Z_c = j^{-1}(\mathcal{Y}_c)$ is an irreducible component of Sh_F: $\mathcal{Y}$ is flat over $\mathcal{O}_F$, so $\mathcal{Y}_c \times_{\mathcal{O}_F} F$ is an irreducible component of $\mathcal{Y} \times_{\mathcal{O}_F} F$, and the irreducible components of the dense open subscheme $j(\mathrm{Sh}_F)$ are its non-empty intersections with those of $\mathcal{Y} \times_{\mathcal{O}_F} F$. So Lemma 33.34 applies to $Z = Z_c$ (level good at every prime, $A = 1$, split components) and gives $\int_{M_c} \nu^{\mathrm{Fl}} > 0$, finite by the integrability above. Therefore the sum of the top degrees over the components is positive, and some component X_1 has positive top degree. □

Lemma 33.29. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p, q \geq 1$, $\Phi = \Phi_{\mathsf{H}}$, $n = p + q$, $\mathcal{Y}$ an integral model of* Sh *over $\mathcal{O}_F$, fixed data at an irreducible component $\mathcal{Y}_c$ of $\mathcal{Y}$ with analytic system* Sys_c *on X_c^{an}, S a finite set of primes, and $\mathcal{X}$ a root model of type (Φ, n) with a root origin on these fixed data for S (read with S in place of S_B) such that $\mathcal{X} \otimes \mathbb{Q}$ is smooth. For $\tau \in \Phi$, $1 \leq i \leq n$ put*

$$\xi_{\tau,i} = c_1(\mathcal{M}_{(\tau,i-1)}, h^{\mathrm{ref}}_{(\tau,i-1)}) - c_1(\mathcal{M}_{(\tau,i)}, h^{\mathrm{ref}}_{(\tau,i)})$$

on X_c^{an} (the first term omitted for $i = 1$), and let $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})$ be the complex manifold of all complex points of the F-scheme $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}$ (over every embedding of F into $\mathbb{C}$), with the forms $\omega_r = c_1(\mathcal{L}_r, \|\cdot\|^{\mathrm{Hdg}})$. For $G \in \mathcal{R}$ homogeneous of degree D let $G(\xi)$ and $G(\omega)$ be the $2D$-forms obtained by substituting $\xi_{\tau,i}$, respectively $\omega_{(\tau,i)}$, for $x_{\tau,i}$. Then for every such G the form $G(\omega)$ is integrable on $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})$ and

$$[F:\mathbb{Q}]\,\mathrm{gi}(\mathcal{X}; G) = \int_{X_c^{\mathrm{an}}} G(\xi) = \int_{\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})} G(\omega).$$

Proof. All symbols introduced here are local to this proof; $[F:\mathbb{Q}]$ refers to the field F of $\mathfrak{u}$. For a measurable top-degree form η on a complex manifold we write $|\eta|$ for the associated positive measure (in local holomorphic coordinates, $|f|$ times Lebesgue measure if $\eta = f\,dV$); complex manifolds carry their canonical orientation, which local biholomorphisms preserve.

Fix G, and write $f\colon \mathcal{X} \to \mathcal{Y}$ for the morphism of the root origin, Ω for the function field of $\mathcal{X}$, $g\colon \Omega_c \to \Omega$, $d = \deg \geq 1$ for its degree, and $\pi\colon \mathcal{X}(\mathbb{C}) \to X^{\mathrm{an}} = X_c^{\mathrm{an}}$ the realisation, $X^\circ = X_c^\circ$.

Step 0 (the spaces). $\mathcal{X}$ is integral, flat over $\mathbb{Z}$ of absolute dimension $D+1$ and closed in a multiprojective space, so $\mathcal{X}_{\mathbb{Q}}$ is integral and projective over $\mathbb{Q}$ of dimension D; by hypothesis it is smooth, so $\mathcal{X}(\mathbb{C})$ is a compact complex manifold of dimension D. Through f, $\mathcal{X}$ is an $\mathcal{O}_F$-scheme and $\mathcal{X}_{\mathbb{Q}} = \mathcal{X} \times_{\mathcal{O}_F} F$. By (Go3)(1) the generic point of $\mathcal{X}$ maps to that of $\mathcal{Y}_c$; as $\mathcal{Y}_c$ is closed, $f(\mathcal{X}) \subseteq \mathcal{Y}_c$. The generic point of $\mathcal{Y}_c$ lies in the generic fibre ($\mathcal{Y}$ is flat over $\mathcal{O}_F$) and in the open subset $j(\mathrm{Sh}_F) \cap \mathcal{Y}_c$ of $\mathcal{Y}_c \times_{\mathcal{O}_F} F$, which is non-empty by (Go2). Hence the abelian locus $A_{\mathbb{Q}} = f^{-1}(j(\mathrm{Sh}_F)) \subseteq \mathcal{X}_{\mathbb{Q}}$ is a dense open subscheme, integral and smooth of dimension D over F, and $j^{-1} \circ f$ maps it into $Z_c = j^{-1}(\mathcal{Y}_c)$. Put $A = A_{\mathbb{Q}}(\mathbb{C}) \subseteq \mathcal{X}(\mathbb{C})$, an open subset; by (Go4), $A = \pi^{-1}(X^\circ)$. Since Sh_F is smooth over F, its irreducible components are open and closed, so $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}$ is open and closed in $\mathrm{Fl}_{\mathfrak{u}}$, smooth over F of dimension D (Fh2), and $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})$ is a complex manifold of dimension D on which the ω_r are smooth real $(1,1)$-forms (Fh4).

Step 1 (change of variables along π). By (Go4) there is a closed analytic subset $B_0 \subseteq X^{\mathrm{an}}$ with empty interior such that $\pi^{-1}(B_0)$ is a closed analytic subset of $\mathcal{X}(\mathbb{C})$ with empty interior, every point of $X^{\mathrm{an}} \smallsetminus B_0$ has exactly d preimages and $d\pi$ is invertible off $\pi^{-1}(B_0)$. The restriction $\mathcal{X}(\mathbb{C}) \smallsetminus \pi^{-1}(B_0) \to X^{\mathrm{an}} \smallsetminus B_0$ is proper (a compact subset K of the target has compact preimage $\pi^{-1}(K)$, closed in the compact $\mathcal{X}(\mathbb{C})$), a local biholomorphism, with fibres of d points; so it is a d-sheeted covering. Closed analytic subsets with empty interior have measure zero.

Covering an open set $U' \subseteq X^{\mathrm{an}} \smallsetminus B_0$ by evenly covered open sets, it follows that for every open $U' \subseteq X^{\mathrm{an}}$ and every measurable top-degree form η on U',

$$\int_{\pi^{-1}(U')} |\pi^*\eta| = d \int_{U'} |\eta|, \qquad \text{and, if these are finite,} \qquad \int_{\pi^{-1}(U')} \pi^*\eta = d \int_{U'} \eta. \tag{1}$$

Step 2 (the first equality). For roots $r_1, \dots, r_D$, the degree of the zero-cycle

$$c_1(\mathcal{O}(e_{r_1})) \cdots c_1(\mathcal{O}(e_{r_D})) \cap [\mathcal{X}_{\mathbb{Q}}]$$

equals $\int_{\mathcal{X}(\mathbb{C})} \omega'_{r_1} \wedge \cdots \wedge \omega'_{r_D}$, where ω'_r is the first Chern form of $\mathcal{O}(e_r)$ on $\mathcal{X}(\mathbb{C})$ for any smooth hermitian metric (the degree of a zero-cycle on the projective $\mathbb{Q}$-variety $\mathcal{X}_{\mathbb{Q}}$ is the number of its complex points with multiplicity, and by Chern–Weil theory the complex intersection number of line bundles on the compact complex manifold $\mathcal{X}(\mathbb{C})$ is the integral of the product of their Chern forms, which depends only on the de Rham classes). As G does not involve x_*, the definition of gi gives $\mathrm{gi}(\mathcal{X}; G) = \delta_{\mathcal{X}}^{-1} \int_{\mathcal{X}(\mathbb{C})} G(\omega')$, and $\delta_{\mathcal{X}} = [F : \mathbb{Q}]\, d$ by (Go3)(1). We choose the metrics as follows. For a fundamental index $a = (\tau, i)$ let $f_a = -\sum_{k \le i} e_{(\tau,k)}$ and $f_{(\tau,0)} = 0$. By (Go4) there is a smooth cone function h_a^{ref} of multidegree f_a on $\mathcal{X}$ with $(\mathcal{O}(f_a)|_{\mathcal{X}}, h_a^{\mathrm{ref}}) = \pi^*(\mathcal{M}_a, h_a^{\mathrm{ref}})$, so its first Chern form is $\pi^*\theta_a$ with $\theta_a = c_1(\mathcal{M}_a, h_a^{\mathrm{ref}})$. Since $e_{(\tau,i)} = f_{(\tau,i-1)} - f_{(\tau,i)}$, the cone function $h^{\mathrm{ref}}_{(\tau,i-1)}/h^{\mathrm{ref}}_{(\tau,i)}$ (with $h^{\mathrm{ref}}_{(\tau,0)} = 1$) is a smooth metric on $\mathcal{O}(e_{(\tau,i)})$ whose first Chern form is $\pi^*\theta_{(\tau,i-1)} - \pi^*\theta_{(\tau,i)} = \pi^*\xi_{\tau,i}$ (first term omitted for $i = 1$). With these metrics $G(\omega') = \pi^*G(\xi)$, and (1) with $U' = X^{\mathrm{an}}$ gives

$$\mathrm{gi}(\mathcal{X}; G) = \frac{1}{[F : \mathbb{Q}]\, d} \int_{\mathcal{X}(\mathbb{C})} \pi^*G(\xi) = \frac{1}{[F : \mathbb{Q}]} \int_{X^{\mathrm{an}}} G(\xi).$$

Step 3 (mass identity). Let θ_a, $\theta_a^{\mathrm{sg}} = c_1(\mathcal{M}_a|_{X^\circ}, h_a^{\mathrm{ref}} e^{\varphi_a})$ be the forms of Proposition 33.30 for Sys_c, put $\xi^{\mathrm{sg}}_{\tau,i} = \theta^{\mathrm{sg}}_{(\tau,i-1)} - \theta^{\mathrm{sg}}_{(\tau,i)}$ on X° (first term omitted for $i = 1$), so that $\xi_{\tau,i} = \theta_{(\tau,i-1)} - \theta_{(\tau,i)}$, and define $G(\xi^{\mathrm{sg}})$ as $G(\xi)$. Forms of even degree commute, so substituting these differences into G and expanding writes $G(\xi)$ and $G(\xi^{\mathrm{sg}})$ as the same real linear combination of the products $\theta_{a_1} \wedge \cdots \wedge \theta_{a_D}$, respectively $\theta^{\mathrm{sg}}_{a_1} \wedge \cdots \wedge \theta^{\mathrm{sg}}_{a_D}$, over fundamental indices $a_1, \dots, a_D \neq *$. By Proposition 33.30, applied to each product, $G(\xi^{\mathrm{sg}})$ is integrable on X° and

$$\int_{X^\circ} G(\xi^{\mathrm{sg}}) = \int_{X^{\mathrm{an}}} G(\xi). \tag{2}$$

Step 4 (the singular forms are Hodge forms on the abelian locus). By (Go4), on $A = \pi^{-1}(X^\circ)$ the cone function $h^{\mathrm{ref}}_{(\tau,i)} e^{\varphi_{(\tau,i)} \circ \pi}$ equals $\prod_{k \le i} (h^{\mathrm{sing}}_{(\tau,k)})^{-1}$. The left side is smooth over A (φ_a is smooth on X°), so by induction on i every $h^{\mathrm{sing}}_{(\tau,i)}$ is smooth over A, and $\pi^*(\mathcal{M}_a|_{X^\circ}, h_a^{\mathrm{ref}} e^{\varphi_a}) = (\mathcal{O}(f_a)|_A, \prod_{k \le i} (h^{\mathrm{sing}}_{(\tau,k)})^{-1})$ for $a = (\tau, i)$. Taking the quotient of consecutive indices as in Step 2,

$$\pi^*\xi^{\mathrm{sg}}_{\tau,i} = c_1(\mathcal{O}(e_{(\tau,i)})|_A, h^{\mathrm{sing}}_{(\tau,i)}) \qquad \text{on } A.$$

Let $\mathcal{L}_r^{\mathcal{X}}$ be the root lines of the integral flag of (Go3)(3). The root sections $P \mapsto \varrho_r P(y)$ are isomorphisms $\mathcal{O}(e_r)|_{\mathcal{X}} \to \mathcal{L}_r^{\mathcal{X}}$ (Go3)(3), and by (Go3)(4), at every $z \in A$, $\|\varrho_r P(y)\|^{\mathrm{Hdg}}(z) = |P(\tilde{z})|/h_r^{\mathrm{sing}}(\tilde{z})$, which is the norm of P for the cone function h_r^{sing}. So the root sections are isometries $(\mathcal{O}(e_r)|_A, h_r^{\mathrm{sing}}) \to (\mathcal{L}_r^{\mathcal{X}}|_A, \|\cdot\|^{\mathrm{Hdg}})$, and $\pi^*\xi^{\mathrm{sg}}_{\tau,i} = c_1(\mathcal{L}^{\mathcal{X}}_{(\tau,i)}|_A, \|\cdot\|^{\mathrm{Hdg}})$ on A.

Step 5 (the flag map). Let x be a point of $A_{\mathbb{Q}}$. Its local ring $\mathcal{O}_x$ contains $\mathbb{Q}$, so x lies over no prime and the Hodge lattice at x is the de Rham lattice $\mathrm{Lie}_x \cap \mathrm{Li}_s$ (Fh5). As $A_{\mathbb{Q}}$ is an F-scheme, $\mathrm{Lie}(G_{\mathcal{X}})_x = \bigoplus_s \mathrm{Lie}_s(G_{\mathcal{X}})_x$ (Fh1), and intersecting with Li_s gives $\mathrm{Lie}_s(G_{\mathcal{X}})_x$. Hence

on $A_{\mathbb{Q}}$ the Hodge bundles are $\mathcal{V}_s|_{A_{\mathbb{Q}}} = \mathrm{Lie}_s(G_{\mathcal{X}}) = (j^{-1}\circ f)^*\mathrm{Lie}_s(G^{\mathrm{un}})$ (by (Ud7), $G_{\mathcal{X}}$ is the pull-back of G^{un} over $A_{\mathbb{Q}}$), and the integral flag restricts to a full flag of each of these bundles. By the universal property of the flag schemes (Fh2) this defines a morphism $\kappa\colon A_{\mathbb{Q}} \to \mathrm{Fl}_{\mathfrak{u}}$ over $j^{-1}\circ f$, with image in $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}$ (Step 0), pulling the universal flags back to the integral flag; as root lines are formed from step quotients (Fh3), $\mathcal{L}_r^{\mathcal{X}}|_{A_{\mathbb{Q}}} = \kappa^*\mathcal{L}_r$. At $z \in A$, by (Fh6) the metric on $\mathcal{L}_r^{\mathcal{X}} \otimes \mathbb{C}$ is the Hodge metric (Fh4) of $G^{\mathrm{un}}_{f(z)}$ for the induced flag, i.e. for the flag of the point $\kappa(z)$; this is the Hodge metric of $\mathcal{L}_r$ at $\kappa(z)$. So $(\mathcal{L}_r^{\mathcal{X}}|_A, \|\cdot\|^{\mathrm{Hdg}}) = \kappa^*(\mathcal{L}_r, \|\cdot\|^{\mathrm{Hdg}})$ isometrically, and with Step 4, $\pi^*\xi^{\mathrm{sg}}_{\tau,i} = \kappa^*\omega_{(\tau,i)}$ on A; hence

$$\pi^*G(\xi^{\mathrm{sg}}) = \kappa^*G(\omega) \qquad \text{on } A. \tag{3}$$

Step 6 (change of variables along κ). By (Go3)(3) the generic fibre of the integral flag is the image under g of the universal flag at the generic point of $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}$, and g is a homomorphism over $\mathbf{k}(Z_c)$; so κ maps the generic point of $A_{\mathbb{Q}}$ to the generic point of $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}$ and induces g on function fields. Thus κ is a dominant morphism of integral F-schemes of finite type, both of dimension D, with $[\Omega : g(\Omega_c)] = d$ (Go3)(1). By generic finiteness, generic flatness and generic smoothness (characteristic 0, $A_{\mathbb{Q}}$ smooth) there is a dense open $U \subseteq \mathrm{Fl}_{\mathfrak{u}}|_{Z_c}$ such that $\kappa^{-1}(U) \to U$ is finite étale, of degree $[\Omega : g(\Omega_c)] = d$. On complex points, $\kappa^{-1}(U)(\mathbb{C}) \to U(\mathbb{C})$ is then a proper local biholomorphism all of whose fibres have d points, i.e. a d-sheeted covering. The complements $W = \mathrm{Fl}_{\mathfrak{u}}|_{Z_c} \smallsetminus U$ and $\kappa^{-1}(W) \subseteq A_{\mathbb{Q}}$ are closed subsets of dimension $< D$ (proper closed subsets of integral schemes of dimension D; $\kappa^{-1}(W)$ does not contain the generic point as κ is dominant), so their complex points are finite unions of locally closed complex submanifolds of dimension $< D$ and have measure zero in $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})$, respectively in A. As in Step 1, for every measurable top-degree form η on $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})$,

$$\int_A |\kappa^*\eta| = d\int_{\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})} |\eta|, \qquad \text{and, if these are finite,} \qquad \int_A \kappa^*\eta = d\int_{\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})} \eta. \tag{4}$$

Step 7 (conclusion). By (4), (3), (1) with $U' = X^\circ$ (where $\pi^{-1}(X^\circ) = A$) and Step 3,

$$d\int_{\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})} |G(\omega)| = \int_A |\kappa^*G(\omega)| = \int_A |\pi^*G(\xi^{\mathrm{sg}})| = d\int_{X^\circ} |G(\xi^{\mathrm{sg}})| < \infty,$$

so $G(\omega)$ is integrable, and the same chain without absolute values together with (2) gives

$$\int_{\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})} G(\omega) = \frac{1}{d}\int_A \kappa^*G(\omega) = \frac{1}{d}\int_A \pi^*G(\xi^{\mathrm{sg}}) = \int_{X^\circ} G(\xi^{\mathrm{sg}}) = \int_{X^{\mathrm{an}}} G(\xi).$$

With Step 2 this is the displayed identity. □

Proposition 33.30. *Let* Sys *be an analytic system with log-log weights. For every fundamental index a write $\theta_a = c_1(\mathcal{M}_a, h_a^{\mathrm{ref}})$, a smooth real $(1,1)$-form on X^{an}, and $\theta_a^{\mathrm{sg}} = c_1(\mathcal{M}_a|_{X^\circ}, h_a^{\mathrm{ref}}e^{\varphi_a})$, a smooth real $(1,1)$-form on X°. Then for all fundamental indices $a_1, \dots, a_D$ the $2D$-form $\theta_{a_1}^{\mathrm{sg}} \wedge \cdots \wedge \theta_{a_D}^{\mathrm{sg}}$ is integrable on X° and*

$$\int_{X^\circ} \theta_{a_1}^{\mathrm{sg}} \wedge \cdots \wedge \theta_{a_D}^{\mathrm{sg}} = \int_{X^{\mathrm{an}}} \theta_{a_1} \wedge \cdots \wedge \theta_{a_D}.$$

Proof. All symbols introduced here are local to this proof. Fix fundamental indices $a_1, \dots, a_D$ and write $X = X^{\mathrm{an}}$, $\theta_k = \theta_{a_k}$, $\varphi_k = \varphi_{a_k}$, $\eta_k(t) = \eta_{a_k}(t) = \theta_k + dd^c\varphi^T_{a_k}(t)$ (a smooth closed real $(1,1)$-form on X) and $\theta_k^{\mathrm{sg}} = \theta_k + dd^c\varphi_k$ on X°. Put $\Pi(t) = \eta_1(t) \wedge \cdots \wedge \eta_D(t)$ on X and

$\Pi^{\mathrm{sg}} = \theta_1^{\mathrm{sg}} \wedge \cdots \wedge \theta_D^{\mathrm{sg}}$ on X°. Fix a smooth positive volume form dV on X; for a continuous real $2D$-form $\alpha = f_\alpha\, dV$ write $|\alpha| = |f_\alpha|\, dV$. The proper analytic subset ∂X^{an} has dV-measure 0.

Step 1 (Stokes). For $0 < t \le t_0$, $\int_X \Pi(t) = \int_X \theta_1 \wedge \cdots \wedge \theta_D$. Indeed, expanding $\Pi(t)$, every term other than $\theta_1 \wedge \cdots \wedge \theta_D$ has the form $dd^c\varphi_{a_k}^T(t) \wedge \Xi$ with Ξ a smooth closed form, and $dd^c\varphi \wedge \Xi = d(d^c\varphi \wedge \Xi)$ has integral 0 over the compact manifold X.

Step 2 (uniform tail). Let $0 < t \le t_1$, so $\varepsilon(t) \le 1$ and, by (Go1)(2), $\eta_k(t) + \Theta \ge (1 - \varepsilon(t))\Theta \ge 0$. Writing $\eta_k(t) = (\eta_k(t) + \Theta) - \Theta$ and expanding,

$$\Pi(t) = \sum_{S \subseteq \{1,\dots,D\}} (-1)^{D-|S|}\, \Pi_S(t), \qquad \Pi_S(t) = \bigwedge_{k \in S} (\eta_k(t) + \Theta) \wedge \Theta^{D-|S|}.$$

Each $\Pi_S(t)$ is a wedge product of semipositive $(1,1)$-forms, hence a positive top form, so $|\Pi(t)| \le \sum_S \Pi_S(t)$ pointwise. Each $\Pi_S(t)$ is a product of D forms each equal to Θ or to some $\eta_a(t) + \Theta$, so Proposition 31.22 and $\log \mathrm{rW} \ge 1$ on X° give, for $R \ge R_0$,

$$\int_{\{y \in X^\circ :\, \mathrm{rW}(y) > R\}} |\Pi(t)| \le \sum_S \int_{\{\mathrm{rW} > R\}} \log \mathrm{rW}\ \Pi_S(t) \le \tau(R) := 2^D C_{\mathrm{tb}} \frac{(\log R)^2}{R} \qquad (0 < t \le t_1).$$

Step 3 (convergence on compact parts). For $R \ge e$ let $K_R = \{y \in X^\circ : \mathrm{rW}(y) \le R\} = \{y \in X : \|\varsigma(y)\|^2 \ge e^{e-R}\}$, a closed subset of X disjoint from ∂X^{an}, hence a compact subset of X°. By (Go1)(3), $\varphi_{a_k}^T(t) \to \varphi_k$ with first and second derivatives uniformly on K_R as $t \to 0^+$, so in local coordinates the coefficients of $\eta_k(t)$ converge uniformly on K_R to those of θ_k^{sg}, and $f_{\Pi(t)} \to f_{\Pi^{\mathrm{sg}}}$ uniformly on K_R.

Step 4 (integrability). Let $R_0 \le R < R'$. On the compact set $A = \{y \in X^\circ : R < \mathrm{rW}(y) \le R'\} \subseteq K_{R'}$, $|f_{\Pi(t)}| \to |f_{\Pi^{\mathrm{sg}}}|$ uniformly (Step 3), so by Step 2

$$\int_A |\Pi^{\mathrm{sg}}| = \lim_{t \to 0^+} \int_A |\Pi(t)| \le \tau(R).$$

Letting $R' \to \infty$ (monotone convergence), $\int_{\{y \in X^\circ :\, \mathrm{rW}(y) > R\}} |\Pi^{\mathrm{sg}}| \le \tau(R)$. On the compact K_{R_0} the smooth form Π^{sg} is bounded. As $X^\circ = K_{R_0} \cup \{\mathrm{rW} > R_0\}$, Π^{sg} is integrable on X°.

Step 5 (the mass identity). Let $R \ge R_0$ and $0 < t \le t_1$. As ∂X^{an} is null, Step 1 gives

$$\int_X \theta_1 \wedge \cdots \wedge \theta_D - \int_{X^\circ} \Pi^{\mathrm{sg}} = \int_{K_R} (\Pi(t) - \Pi^{\mathrm{sg}}) + \int_{\{\mathrm{rW} > R\}} \Pi(t) - \int_{\{\mathrm{rW} > R\}} \Pi^{\mathrm{sg}},$$

so by Steps 2 and 4 its absolute value is at most $\int_{K_R} |f_{\Pi(t)} - f_{\Pi^{\mathrm{sg}}}|\, dV + 2\tau(R)$. Letting $t \to 0^+$, the first term tends to 0 by Step 3 (K_R has finite volume). Hence the difference is at most $2\tau(R)$ in absolute value for every $R \ge R_0$, and $\tau(R) \to 0$ as $R \to \infty$. So $\int_{X^\circ} \theta_{a_1}^{\mathrm{sg}} \wedge \cdots \wedge \theta_{a_D}^{\mathrm{sg}} = \int_{X^{\mathrm{an}}} \theta_{a_1} \wedge \cdots \wedge \theta_{a_D}$. □

Proposition 33.31. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p, q \ge 1$, $\Phi = \Phi_{\mathsf{H}}$, $n = p + q$. Let M be the complex manifold of all complex points of the $\mathbb{Q}$-scheme $\mathrm{Fl}_{\mathfrak{u}}$ (over every embedding of F into $\mathbb{C}$; of dimension $D = |\Phi| n(n-1)/2$), the root lines carrying at every point the Hodge metric given by the formula of (Fh4) (for the abelian variety G^{un} at that point with its polarisation). For every root $r = (\tau, i)$ let $\omega_r = c_1(\mathcal{L}_r, \|\cdot\|^{\mathrm{Hdg}})$ on M, and for $G \in \mathcal{R}$ homogeneous of degree D let $G(\omega)$ be the $2D$-form obtained by substituting $\omega_{(\tau,i)}$ for $x_{\tau,i}$. Then for every point y of M there is a real $2D$-covector ν_y at y such that*

$$G(\omega)_y = \deg^{\mathrm{fl}}(G)\, \nu_y \qquad \text{for every } G \in \mathcal{R} \text{ homogeneous of degree } D.$$

Proof. All symbols introduced in this proof are local to it. We use one normalisation of first Chern forms: for a holomorphic line bundle with smooth hermitian metric and a nowhere vanishing local holomorphic section f, $c_1 = -\kappa_0\, i\partial\bar\partial \log\|f\|^2$ with a universal real constant $\kappa_0 \neq 0$ ($\kappa_0 = 1/2\pi$ in the usual normalisation). In particular, multiplying a metric by a positive constant does not change c_1.

Step 0 (what is to be shown). Put $G_0 = (-1)^D x^\delta$, $x^\delta = \prod_{\tau\in\Phi}\prod_{i=1}^n x_{\tau,i}^{n-i}$. In (21.1) only $w = \mathrm{id}$ contributes to $\deg^{\mathrm{fl}}(x^\delta)$, since $i \mapsto n-i$ is injective, so $\deg^{\mathrm{fl}}(G_0) = 1$. Fix $y \in M$. It suffices to find a $2D$-covector μ at y with $G(\omega)_y = \deg^{\mathrm{fl}}(G)\,\mu$ for every homogeneous G of degree D: then $\mu = G_0(\omega)_y$, which is real since the ω_r are real forms, and $\nu_y = G_0(\omega)_y$ has the required property.

Step 1 (the polarised module near y). Let $\sigma\colon F \to \mathbb{C}$ be the embedding over which y lies, $\mathcal{S}$ the complex manifold $(\mathrm{Sh}_F \otimes_{F,\sigma} \mathbb{C})(\mathbb{C})$, $z \in \mathcal{S}$ the image of y, and $A \to \mathcal{S}$ the holomorphic family of complex tori given by G^{un}; $A_w = \mathrm{Lie}(A_w)/\Lambda_w$. Choose an open neighbourhood V of z biholomorphic to a ball and a connected open neighbourhood $\mathcal{N}$ of y in M mapping into V. The period lattices Λ_w ($w \in V$) are the fibres of the kernel of the fibrewise exponential map $\mathrm{Lie}(A) \to A$, a closed complex submanifold étale over V; so they form a local system, trivial on the ball V, and every flat section is a holomorphic section of $\mathrm{Lie}(A)|_V$. Identify all Λ_w with $\Lambda = \Lambda_z$ in this way, put $L_\mathbb{R} = \Lambda \otimes \mathbb{R}$, $W = \Lambda \otimes \mathbb{C}$, and let J_w be the complex structure of $\mathrm{Lie}(A_w)$ transported to $L_\mathbb{R}$ by the real isomorphism $L_\mathbb{R} \to \mathrm{Lie}(A_w)$ extending $\Lambda \subseteq \mathrm{Lie}(A_w)$. Then $\varphi_w\colon W \to \mathrm{Lie}(A_w)$, $\lambda\otimes c \mapsto c\lambda$, is a holomorphic map of holomorphic vector bundles $W \otimes \mathcal{O}_V \to \mathrm{Lie}(A)|_V$, surjective (its restriction to $L_\mathbb{R}$ is a real isomorphism), so its kernel $K(w)$ is a holomorphic subbundle. The ring $\mathcal{O}_E$ acts on Λ (endomorphisms preserve periods, and the action is locally constant), φ_w is $\mathcal{O}_E$-linear, and $\Lambda \otimes \mathbb{Q}$ is an E-vector space of $\mathbb{Q}$-dimension $2\dim \mathrm{Lie}(A_w) = 2|\Phi|n = n[E:\mathbb{Q}]$, so $L_\mathbb{R}$ is free of rank n over $E \otimes \mathbb{R}$. Let $\beta_w = \mathrm{Im}\, H_w$, an integral alternating form on Λ with $H_w(x,x') = \beta_w(J_w x, x') + i\beta_w(x,x')$ ([186, §2]). As β_w is the first Chern class of $\mathcal{L}$ restricted to the fibre, it is a flat section of an integral local system, hence $\beta_w = \beta := \beta_z$ for all $w \in V$; it is non-degenerate. The Riemann forms of $(\iota(a)\mathrm{pr}_1, \mathrm{pr}_2)^*\Lambda(\mathcal{L})$ and $(\mathrm{pr}_1, \iota(\bar a)\mathrm{pr}_2)^*\Lambda(\mathcal{L})$ on $A_w \times A_w$ are $((x_1,x_2),(x_1',x_2')) \mapsto \beta(ax_1,x_2') + \beta(x_2,ax_1')$ and $\beta(x_1,\bar a x_2') + \beta(\bar a x_2, x_1')$; their equality (Ud4) gives, at $x_2 = x_1' = 0$, $\beta(ax,x') = \beta(x,\bar a x')$ for $a \in \mathcal{O}_E$, hence for $a \in E$.

Let $\Phi' = \{\sigma\circ\tau : \tau \in \Phi\}$; as E is a CM field, $\sigma\circ\bar\tau = \overline{\sigma\circ\tau}$, so Φ' is a CM type of E. We claim that for every $w \in V$, J_w is a Hodge structure of signature (p,q) for Φ' on $(L_\mathbb{R}, \beta)$ in the sense of Lemma 33.33. Indeed J_w is E-linear (each $\iota(a)$ acts holomorphically), $J_w^2 = -1$, $\beta(J_w x, J_w x') = \beta(x,x')$ and $\beta(J_w x, x) = H_w(x,x) > 0$ for $x \neq 0$ (H_w is the positive definite hermitian form of the polarisation). Under the E-linear identification $(L_\mathbb{R}, J_w) = \mathrm{Lie}(A_w)$, the subspace on which E acts through $\sigma\circ\tau$ is $\mathrm{Lie}_{(\tau,+)}(A_w)$, of dimension p, and through $\sigma\circ\bar\tau$ it is $\mathrm{Lie}_{(\tau,-)}(A_w)$, of dimension q, by (Fh1) and the Kottwitz condition (at a point over σ the embedding $\sigma\circ\tau$ sends r to $\sigma(r_\mathbb{C})$, $\sigma\circ\bar\tau$ to $-\sigma(r_\mathbb{C})$). Under the same identification $\varphi_w = \varphi_{J_w}$ (so $K(w) = K(J_w)$ by Lemma 33.33(1)), the pieces $\mathrm{Lie}_{(\sigma\circ\tau,\pm)}(J_w)$ of Lemma 33.33 are the Hodge pieces $\mathrm{Lie}_{(\tau,\pm)}(A_w)$, the root lines $\mathcal{L}_{(\sigma\circ\tau,i)}$ there are the root lines $\mathcal{L}_{(\tau,i)}$ of (Fh3), and the Hodge metrics agree: both are the quotient norm (or its dual) of $h_w = \pi H_w$ (Fh4).

Step 2 (the root lines are pulled back from flag domains). For $\tau \in \Phi$ let $\mathcal{D}_\tau$ be the flag domain of $(W_{\sigma\circ\tau}, \langle\,,\rangle)$, $\langle v, v'\rangle = i\beta(v,\bar v')$, with its forms ω_j^τ (Lemma 33.32; $W_{\sigma\circ\tau}$ has signature (p,q) by Lemma 33.33(1)). Apply Lemma 33.33(4) with $N = \mathcal{N}$, $J(y') = J_w$ for y' over w, and $V(y')$ the universal flags of the $\mathrm{Lie}_s(A_w)$ at y': the $K(J_w) = K(w)$ form the holomorphic subbundle pulled back from V; φ identifies the quotient bundle with the pull-back of $\mathrm{Lie}(A)|_V$ holomorphically; and the universal flags are holomorphic subbundles of the pulled-back Hodge pieces (the flag bundle of (Fh2) carries them tautologically). Hence there is a holomorphic map

$\Psi=(\Psi_\tau)_{\tau\in\Phi}\colon\mathcal{N}\to\prod_{\tau\in\Phi}\mathcal{D}_\tau$ with

$$\omega_r=\Psi_\tau^*\,\omega^\tau_{\mathrm{pos}(i)}\quad\text{on }\mathcal{N},\qquad r=(\tau,i)\qquad(1).$$

Step 3 (Weyl proportionality on the flag domain). For each τ fix an adapted basis (e^τ_j) of a point $o_\tau\in\mathcal{D}_\tau$ with the coordinates z^τ_{ab} and the covectors η^τ_{ab} of Lemma 33.32(3), let $o=(o_\tau)_\tau$, $\epsilon_a=-1$ for $a\le q$ and $\epsilon_a=1$ for $a>q$, and $\nu_o=\prod_\tau\prod_{a>b}\eta^\tau_{ab}$, a non-zero real $2D$-covector at o, as the z^τ_{ab} are D coordinates. By Lemma 33.32(3),

$$\omega^\tau_j\big|_{o_\tau}=\kappa_0\sum_{a>b}\epsilon_a\epsilon_b(\delta_{ja}-\delta_{jb})\,\eta^\tau_{ab}\qquad(2).$$

The 2-covectors η^τ_{ab} commute and have square 0, so any product of D of them is ν_o or 0. For $G\in\mathcal{R}$ homogeneous of degree D let $\tilde G(\omega^{\mathcal{D}})$ be the form on $\prod_\tau\mathcal{D}_\tau$ obtained by substituting $\omega^\tau_{\mathrm{pos}(i)}$ for $x_{\tau,i}$; by (2), $\tilde G(\omega^{\mathcal{D}})_o=c(G)\,\nu_o$ for a linear form c on homogeneous polynomials of degree D. For a weight u, $\mathrm{lr}(u)=-\sum u_{\tau,i}x_{\tau,i}$ becomes $\sum_{\tau,j}U_{\tau,j}\omega^\tau_j$ with $U_{\tau,\mathrm{pos}(i)}=-u_{\tau,i}$, which by (2) equals $\kappa_0\sum_\tau\sum_{a>b}\epsilon_a\epsilon_b(U_{\tau,a}-U_{\tau,b})\eta^\tau_{ab}$ at o; so

$$\begin{aligned}c(\mathrm{lr}(u)^D)&=D!\,\kappa_0^D\prod_\tau\prod_{a>b}\epsilon_a\epsilon_b(U_{\tau,a}-U_{\tau,b})\\&=\kappa_1\,D!\prod_\tau\prod_{1\le i<i'\le n}(u_{\tau,i}-u_{\tau,i'}),\end{aligned}$$

where $\kappa_1=\pm\kappa_0^D$ with a sign depending only on $(p,q,|\Phi|)$ and the permutation pos (each unordered pair $\{i,i'\}$ occurs once as $\{\mathrm{pos}^{-1}(a),\mathrm{pos}^{-1}(b)\}$). By Lemma 30.14,

$$c(\mathrm{lr}(u)^D)=\kappa_2\deg^{\mathrm{fl}}(\mathrm{lr}(u)^D)\qquad\text{with}\quad\kappa_2=\kappa_1\prod_\tau\prod_{i<i'}(i'-i),$$

for every weight u. The D-th powers of linear forms span the space of homogeneous polynomials of degree D over $\mathbb{R}$ (polarisation), and every linear form is some $\mathrm{lr}(u)$; so $c=\kappa_2\deg^{\mathrm{fl}}$.

Step 4 (conclusion). By Lemma 33.32(2) choose $g=(g_\tau)\in\prod_\tau\mathrm{U}(W_{\sigma\circ\tau})$ with $g\,o=\Psi(y)$; as $g^*\omega^\tau_j=\omega^\tau_j$ (Lemma 33.32(2)), $\tilde G(\omega^{\mathcal{D}})_{\Psi(y)}=(g^{-1})^*(\tilde G(\omega^{\mathcal{D}})_o)=\kappa_2\deg^{\mathrm{fl}}(G)\,(g^{-1})^*\nu_o$. By (1), $G(\omega)=\Psi^*\tilde G(\omega^{\mathcal{D}})$ on $\mathcal{N}$, so $G(\omega)_y=\deg^{\mathrm{fl}}(G)\,\mu$ with $\mu=\kappa_2\,\Psi^*\big((g^{-1})^*\nu_o\big)_y$, independent of G. Step 0 concludes the proof. □

Lemma 33.32. *The notation introduced here is local to this result and to the results that apply it. Let $p,q\ge1$, $n=p+q$, and let W_0 be a complex vector space of dimension n with a non-degenerate hermitian form $\langle\,,\rangle$ (linear in the first variable) of signature (p,q). Let $\mathcal{D}$ be the set of full flags $F_\bullet=(0=F_0\subset F_1\subset\cdots\subset F_n=W_0)$, $\dim F_j=j$, such that $\langle\,,\rangle$ is negative definite on F_q; it is an open subset of the flag manifold of W_0 (the **flag domain** of W_0). Let L_j be the tautological holomorphic line bundle F_j/F_{j-1} on $\mathcal{D}$ $(1\le j\le n)$. Put $\epsilon_a=-1$ for $a\le q$ and $\epsilon_a=1$ for $a>q$.*

1. *For $F_\bullet\in\mathcal{D}$ every F_j is non-degenerate, $\ell_j=F_j\cap F_{j-1}^\perp$ is a line mapping isomorphically onto F_j/F_{j-1}, $\langle\,,\rangle$ is negative on ℓ_j for $j\le q$ and positive for $j>q$, and $W_0=\ell_1\oplus\cdots\oplus\ell_n$ orthogonally. Setting $\|[x]\|_j^2=|\langle x,x\rangle|$ for $x\in\ell_j$ defines a smooth hermitian metric on L_j; put $\omega_j=c_1(L_j,\|\cdot\|_j)$.*
2. *Every $F_\bullet\in\mathcal{D}$ has an **adapted basis**: $e_j\in\ell_j$ with $\langle e_a,e_b\rangle=\epsilon_a\delta_{ab}$. The unitary group $\mathrm{U}(W_0)$ of $\langle\,,\rangle$ acts on $\mathcal{D}$ by biholomorphisms and on each L_j by isometries covering this action, so $g^*\omega_j=\omega_j$ for $g\in\mathrm{U}(W_0)$; the subgroup $\mathrm{SU}(W_0)$ acts transitively on $\mathcal{D}$; and the stabiliser in $\mathrm{U}(W_0)$ of a point $F_\bullet$ is the compact torus of unitary maps preserving each line ℓ_j of $F_\bullet$.*

3. *(Base point.) Let (e_j) be an adapted basis with flag o. For $Z = (z_{ab})_{n\ge a>b\ge 1}$ put $f_b = e_b + \sum_{a>b} z_{ab} e_a$. Then $Z \mapsto (\mathrm{span}(f_1,\dots,f_j))_j$ restricted to a neighbourhood of 0 is a holomorphic chart of $\mathcal{D}$ at $o = Z(0)$, $[f_j]$ is a nowhere vanishing holomorphic section of L_j there, and with $\eta_{ab} = i\,dz_{ab}\wedge d\bar z_{ab}$ at o,*

$$\omega_j\big|_o = \kappa_0 \sum_{a>b} \epsilon_a\epsilon_b\,(\delta_{ja}-\delta_{jb})\,\eta_{ab} \qquad (1\le j\le n).$$

Proof. **(1)** For $j\le q$, $F_j\subseteq F_q$ is negative definite. For $j>q$, F_q is non-degenerate, so $F_j = F_q\oplus(F_j\cap F_q^\perp)$; as $\langle\,,\rangle$ has signature (p,q) and is negative definite on the q-dimensional F_q, it is positive definite on the p-dimensional $F_q^\perp$, so F_j is non-degenerate. For a non-degenerate $F_{j-1}\subset F_j$ we have $F_j = F_{j-1}\oplus\ell_j$ with $\ell_j = F_j\cap F_{j-1}^\perp$ a line, so $\ell_j\to F_j/F_{j-1}$ is an isomorphism and the ℓ_j are pairwise orthogonal with sum W_0. The sign of $\langle\,,\rangle$ on ℓ_j is negative for $j\le q$ ($\ell_j\subseteq F_q$) and positive for $j>q$ ($\ell_j\subseteq F_q^\perp$). Smoothness of the metric is shown in (3) at every point, since by (2) every point is the flag o of an adapted basis.

(2) Normalising generators of the pairwise orthogonal lines ℓ_j gives an adapted basis. A unitary g maps flags to flags, gF_q is negative definite, and $g\ell_j(F_\bullet) = \ell_j(gF_\bullet)$ isometrically; so g preserves $\mathcal{D}$, acts holomorphically (it is the restriction of the action of $\mathrm{GL}(W_0)$ on the flag manifold), lifts to L_j by $[x]\mapsto[gx]$ and preserves $\|\cdot\|_j$; hence $g^*\omega_j=\omega_j$. If (e_j), (e'_j) are adapted bases of $F_\bullet$, $F'_\bullet$, the linear map g with $ge_j = e'_j$ is unitary and maps $F_\bullet$ to $F'_\bullet$; with $c\in\mathbb{C}$, $|c|=1$, $c^n = (\det g)^{-1}$, $cg\in\mathrm{SU}(W_0)$ still maps $F_\bullet$ to $F'_\bullet$. So $\mathrm{SU}(W_0)$ is transitive. A unitary g fixing $F_\bullet$ maps each ℓ_j (defined from $F_\bullet$ and $\langle\,,\rangle$) to itself, so it is diagonal in an adapted basis with entries of modulus one, and conversely; this group is a compact torus.

(3) The map $Z\mapsto(f_1,\dots,f_n)$ is the action of the unipotent lower triangular matrices (in the basis e) on the basis e; the resulting map to the flag manifold is the standard big-cell chart, an open holomorphic embedding of $\mathbb{C}^{n(n-1)/2}$; as $\mathcal{D}$ is open and contains o, a neighbourhood of 0 maps into $\mathcal{D}$. The vectors f_j depend holomorphically on Z and $f_j\notin F_{j-1}(Z)$, so $[f_j]$ is a nowhere vanishing holomorphic section of L_j.

Let $\Gamma_j = (\langle f_l, f_k\rangle)_{k,l\le j}$, the Gram matrix of $F_j(Z)$. Write $f_j = g + x_0$ with $g\in F_{j-1}$, $x_0\in\ell_j$ (by (1)); then $[f_j]=[x_0]$, and replacing f_j by x_0 in the basis $f_1,\dots,f_j$ of F_j (a unimodular change of basis) gives $\det\Gamma_j = \det\Gamma_{j-1}\cdot\langle x_0,x_0\rangle$. Hence

$$\|[f_j]\|_j^2 = |\det\Gamma_j|/|\det\Gamma_{j-1}| \qquad (\det\Gamma_0=1),$$

a real-analytic positive function of Z; this proves smoothness in (1).

Write the coordinate matrix of $(f_1,\dots,f_j)$ in the basis e as $M = \binom{1+Z_{11}}{Z_{21}}$, with Z_{11} strictly lower triangular of size j and Z_{21} of size $(n-j)\times j$, and let $\epsilon^{(1)} = \mathrm{diag}(\epsilon_1,\dots,\epsilon_j)$, $\epsilon^{(2)} = \mathrm{diag}(\epsilon_{j+1},\dots,\epsilon_n)$. Up to transposition and complex conjugation (which do not change $|\det|$), Γ_j is $M^*\,\mathrm{diag}(\epsilon^{(1)},\epsilon^{(2)})\,M = (1+Z_{11})^*\big(\epsilon^{(1)}+Y^*\epsilon^{(2)}Y\big)(1+Z_{11})$ with $Y = Z_{21}(1+Z_{11})^{-1}$, and $\det(1+Z_{11})=1$. So

$$\log|\det\Gamma_j| = \log\big|\det\big(1+\epsilon^{(1)}Y^*\epsilon^{(2)}Y\big)\big| = \mathrm{tr}\big(\epsilon^{(1)}Y^*\epsilon^{(2)}Y\big)+O(|Z|^4) = \sum_{a>j\ge b}\epsilon_a\epsilon_b|z_{ab}|^2+O(|Z|^3),$$

where the remainders are real-analytic functions vanishing to order three at 0, whose second derivatives vanish at 0. As $i\partial\bar\partial|z_{ab}|^2=\eta_{ab}$ at o,

$$\omega_j|_o = -\kappa_0\,i\partial\bar\partial\big(\log|\det\Gamma_j|-\log|\det\Gamma_{j-1}|\big)\big|_o = -\kappa_0\sum_{a>b}\epsilon_a\epsilon_b\big([b\le j<a]-[b<j\le a]\big)\eta_{ab},$$

and for $a>b$, $[b\le j<a]-[b<j\le a] = \delta_{jb}-\delta_{ja}$, which gives the formula. □

Lemma 33.33. *The notation introduced here is local to this result and to the results that apply it. Let E be a CM field with complex conjugation $a \mapsto \bar{a}$, let $p, q \geq 1$, $n = p + q$, and let Φ' be a CM type of E (a set of embeddings $E \to \mathbb{C}$ containing exactly one of $\rho, \bar{\rho}$ for every embedding ρ). Let $L_{\mathbb{R}}$ be a free $E \otimes_{\mathbb{Q}} \mathbb{R}$-module of rank n with an alternating $\mathbb{R}$-bilinear non-degenerate form ψ such that $\psi(ax, y) = \psi(x, \bar{a}y)$ ($a \in E$). Put $W = L_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ with the conjugation $v \mapsto \bar{v}$, extend ψ and the action of E $\mathbb{C}$-linearly, let W_ρ be the subspace on which $a \in E$ acts by $\rho(a)$, and $\langle v, v' \rangle = i\psi(v, \bar{v}')$.*

*A **Hodge structure of signature** (p, q) **for** Φ' on $(L_{\mathbb{R}}, \psi)$ is an E-linear $J \in \mathrm{End}_{\mathbb{R}}(L_{\mathbb{R}})$ with $J^2 = -1$, $\psi(Jx, Jy) = \psi(x, y)$ and $\psi(Jx, x) > 0$ for $x \neq 0$, such that for every embedding ρ the subspace of the complex vector space $(L_{\mathbb{R}}, J)$ on which E acts through ρ has complex dimension p if $\rho \in \Phi'$ and q otherwise. For such J let $H_J(x, y) = \psi(Jx, y) + i\psi(x, y)$; $K(J)$ and $P(J)$ the $(-i)$- and $(+i)$-eigenspaces of J on W; $\varphi_J \colon W \to (L_{\mathbb{R}}, J)$, $\lambda \otimes c \mapsto c\lambda$; for $\rho \in \Phi'$, $K_\rho(J) = K(J) \cap W_\rho$ and the **pieces** $\mathrm{Lie}_{(\rho,+)}(J)$, $\mathrm{Lie}_{(\rho,-)}(J)$, the subspaces of $(L_{\mathbb{R}}, J)$ on which E acts through ρ, respectively $\bar{\rho}$. For full flags $V_{(\rho,\pm),\bullet}$ of the pieces ($\rho \in \Phi'$) and a root (ρ, i), $1 \leq i \leq n$, the **root line** is $\mathcal{L}_{(\rho,i)} = V_{(\rho,+),i}/V_{(\rho,+),i-1}$ if $i \leq p$ and $\mathcal{L}_{(\rho,i)} = (V_{(\rho,-),k}/V_{(\rho,-),k-1})^\vee$, $k = n + 1 - i$, if $i > p$, with the **Hodge metric**: the quotient norm of the inner product πH_J, respectively its dual. Put $\mathrm{pos}(i) = q + i$ for $i \leq p$ and $\mathrm{pos}(i) = i - p$ for $i > p$ (a permutation of $\{1, \ldots, n\}$).*

1. *(Linear algebra.) $\langle\,,\rangle$ is hermitian, the W_ρ are pairwise orthogonal, and ψ pairs W_ρ only with $W_{\bar{\rho}}$, perfectly. For J as above: φ_J is $\mathbb{C}$-linear and E-linear, surjective with kernel $K(J)$, and restricts to an isomorphism $P(J) \to (L_{\mathbb{R}}, J)$ with inverse $x \mapsto \tilde{x} = (x - iJx)/2$; $P(J) = \overline{K(J)}$, $W = P(J) \oplus K(J)$ orthogonally, $\langle\,,\rangle$ is positive definite on $P(J)$ and negative definite on $K(J)$, and $\pi H_J(x, y) = 2\pi\langle \tilde{x}, \tilde{y} \rangle$. For $\rho \in \Phi'$: $\varphi_J(W_\rho) = \mathrm{Lie}_{(\rho,+)}(J)$, $\varphi_J(W_{\bar{\rho}}) = \mathrm{Lie}_{(\rho,-)}(J)$; $\langle\,,\rangle$ has signature (p, q) on W_ρ; $K_\rho(J)$ is negative definite of dimension q; and $K(J) \cap W_{\bar{\rho}}$ is the ψ-annihilator of $K_\rho(J)$ in $W_{\bar{\rho}}$. If at least one such J exists, then $J \mapsto (K_\rho(J))_{\rho \in \Phi'}$ is a bijection from the set of Hodge structures of signature (p, q) for Φ' onto the set of tuples of negative definite q-dimensional subspaces $K_\rho \subseteq W_\rho$.*

2. *(Flag map.) For J and full flags $V_{(\rho,\pm),\bullet}$ of the pieces and $\rho \in \Phi'$ put $F^\rho_{q+i} = \varphi_J^{-1}(V_{(\rho,+),i}) \cap W_\rho$ ($0 \leq i \leq p$), $G'_k = \varphi_J^{-1}(V_{(\rho,-),k}) \cap W_{\bar{\rho}}$ and $F^\rho_{q-k} = \{x \in W_\rho : \psi(x, G'_k) = 0\}$ ($0 \leq k \leq q$). The two definitions of F^ρ_q agree, $F^\rho_q = K_\rho(J)$, and $F^\rho_\bullet$ lies in the flag domain $\mathcal{D}_\rho$ of $(W_\rho, \langle\,,\rangle)$. The map $\Psi \colon (J, (V_{s,\bullet})_s) \mapsto (F^\rho_\bullet)_{\rho \in \Phi'}$ is a bijection from the set of such tuples onto $\prod_{\rho \in \Phi'} \mathcal{D}_\rho$.*

3. *(Root lines.) In this situation, with $F_\bullet = F^\rho_\bullet$: if $i \leq p$, φ_J induces an isomorphism $L_{\mathrm{pos}(i)} = F_{q+i}/F_{q+i-1} \to \mathcal{L}_{(\rho,i)}$ multiplying norms ($\|\cdot\|_{\mathrm{pos}(i)}$ to the Hodge metric) by $(2\pi)^{1/2}$; if $i > p$, ψ induces a perfect pairing of $F_{q-k+1}/F_{q-k} = L_{\mathrm{pos}(i)}$ with G'_k/G'_{k-1}, and φ_J an isomorphism $G'_k/G'_{k-1} \to V_{(\rho,-),k}/V_{(\rho,-),k-1}$, together an isomorphism $L_{\mathrm{pos}(i)} \to \mathcal{L}_{(\rho,i)}$ multiplying norms by $(2\pi)^{-1/2}$.*

4. *(Families.) Let N be a complex manifold and $y \mapsto (J(y), V(y))$ a family of such tuples such that the $K(J(y))$ form a holomorphic subbundle $\mathcal{K}$ of the trivial bundle $W \times N$ and, the fibres of the holomorphic quotient bundle $\mathcal{Q} = (W \times N)/\mathcal{K}$ being identified with the $(L_{\mathbb{R}}, J(y))$ through $\varphi_{J(y)}$, the $V_{s,k}(y)$ form holomorphic subbundles of $\mathcal{Q}$. Then Ψ is a holomorphic map $N \to \prod_{\rho \in \Phi'} \mathcal{D}_\rho$; with Ψ_ρ its component at ρ and ω^ρ_j the forms ω_j on $\mathcal{D}_\rho$, the root lines are holomorphic line bundles on N with smooth Hodge metrics, the isomorphisms of (3) form holomorphic isomorphisms $\Psi_\rho^* L_{\mathrm{pos}(i)} \cong \mathcal{L}_{(\rho,i)}$, and*

$$c_1(\mathcal{L}_{(\rho,i)}, \textit{Hodge metric}) = \Psi_\rho^* \, \omega^\rho_{\mathrm{pos}(i)}.$$

Proof. **(1)** *The form.* $\overline{\langle v,v'\rangle} = -i\psi(\bar v,v') = i\psi(v',\bar v) = \langle v',v\rangle$, and $\langle \bar v,\bar v'\rangle = i\psi(\bar v,v') = -\overline{\langle v,v'\rangle}$. For $v\in W_\rho$, $v'\in W_{\rho'}$, $a\in E$: $\psi(av,\bar v') = \psi(v,\bar a\bar v')$, and $\bar a$ acts on $\bar v'\in W_{\bar\rho'}$ by $\overline{\rho'(\bar a)} = \rho'(a)$ (E is CM), so $\rho(a)\langle v,v'\rangle = \rho'(a)\langle v,v'\rangle$ and the W_ρ are pairwise orthogonal. Likewise $\psi(av,w) = \psi(v,\bar a w)$ shows that $\psi(W_\rho,W_\sigma) = 0$ unless $\sigma = \bar\rho$; as ψ is non-degenerate on $W = \bigoplus_\rho W_\rho$, it pairs W_ρ and $W_{\bar\rho}$ perfectly. As $L_{\mathbb R}$ is free of rank n over $E\otimes\mathbb R$, W is free of rank n over $E\otimes\mathbb C = \prod_\rho \mathbb C$, so $\dim W_\rho = n$.

The eigenspaces. Extend J $\mathbb C$-linearly. For $x\in L_{\mathbb R}$, $J(x-iJx) = i(x-iJx)$, $\varphi_J(x-iJx) = x - J(Jx) = 2x$ and $\varphi_J(x+iJx) = 0$. So $K(J) = \{x+iJx\}\subseteq\ker\varphi_J$, $P(J) = \{x-iJx\} = \overline{K(J)}$, $W = P(J)\oplus K(J)$, φ_J is surjective with kernel $K(J)$ (dimensions) and $x\mapsto\tilde x$ is inverse to $\varphi_J|_{P(J)}$; φ_J is $\mathbb C$-linear ($\varphi_J(i\lambda\otimes c) = J(c\lambda)$) and E-linear (J is). From $\psi(Jv,Jv') = \psi(v,v')$, ψ vanishes on $P\times P$ and on $K\times K$; so $\langle p,k\rangle = i\psi(p,\bar k) = 0$ for $p\in P$, $k\in K$ ($\bar k\in P$).

Hence $\langle\tilde x,\tilde x\rangle = \psi(Jx,x)/2 = H_J(x,x)/2 > 0$ for $x\neq 0$: $\langle\,,\rangle$ is positive definite on $P(J)$, hence negative definite on $K(J) = \overline{P(J)}$; and $\pi H_J(x,y) = 2\pi\langle\tilde x,\tilde y\rangle$ by polarisation, both sides being hermitian forms on $(L_{\mathbb R},J)$ ($x\mapsto\tilde x$ is $\mathbb C$-linear) that agree on the diagonal.

The pieces. φ_J is E-linear and surjective, so it maps W_ρ into, and $W = \bigoplus W_\rho$ onto, the sum of the subspaces of $(L_{\mathbb R},J)$ on which E acts through the various ρ; hence $\varphi_J(W_\rho)$ is the subspace for ρ: $\mathrm{Lie}_{(\rho,+)}(J)$ for $\rho\in\Phi'$ and $\mathrm{Lie}_{(\rho,-)}(J)$ for $\bar\rho$. As J is E-linear, $K(J)$ and $P(J)$ are sums of their intersections with the W_ρ. For $\rho\in\Phi'$, $P(J)\cap W_\rho\cong\mathrm{Lie}_{(\rho,+)}(J)$ has dimension p, so $K_\rho(J)$ has dimension $n-p=q$, and $W_\rho = (P(J)\cap W_\rho)\oplus K_\rho(J)$ orthogonally gives signature (p,q). For $k\in K_\rho(J)$, $y\in K(J)\cap W_{\bar\rho}$: $\psi(k,y) = 0$; and $K(J)\cap W_{\bar\rho} = \overline{P(J)\cap W_\rho}$ has dimension $p = n-q$, the dimension of the annihilator of $K_\rho(J)$ under the perfect pairing; so they are equal.

The converse. Assume one J exists; then every W_ρ, $\rho\in\Phi'$, has signature (p,q). Given negative definite q-dimensional $K_\rho\subseteq W_\rho$ ($\rho\in\Phi'$), put $P_\rho = K_\rho^\perp\cap W_\rho$ (positive definite, dimension p), $P = \bigoplus_{\rho\in\Phi'}(P_\rho\oplus\overline{K_\rho})$, $K = \overline P = \bigoplus_{\rho\in\Phi'}(\overline{P_\rho}\oplus K_\rho)$, so $W = P\oplus K$, and let J act by i on P and by $-i$ on K. Then J commutes with conjugation, so preserves $L_{\mathbb R}$; it is E-linear; $J^2 = -1$. The spaces P_ρ, $\overline{K_\rho}\subseteq W_{\bar\rho}$, $\overline{P_\rho}\subseteq W_{\bar\rho}$, K_ρ are pairwise orthogonal ($P_\rho\perp K_\rho$ by definition, $\overline{K_\rho}\perp\overline{P_\rho}$ by $\langle\bar v,\bar v'\rangle = -\overline{\langle v,v'\rangle}$, the others lie in different W_σ, as $\Phi'\cap\overline{\Phi'} = \varnothing$); so $\langle P,K\rangle = 0$, i.e. $\psi(P,P) = \psi(K,K) = 0$ ($\psi(v,w) = -i\langle v,\bar w\rangle$), which gives $\psi(J\cdot,J\cdot) = \psi$. As $\langle\,,\rangle$ is positive definite on P ($\langle\bar v,\bar v\rangle = -\langle v,v\rangle$ on $\overline{K_\rho}$), the computation in the side calculation gives $\psi(Jx,x) = 2\langle\tilde x,\tilde x\rangle > 0$ for $x\neq 0$. The subspace of $(L_{\mathbb R},J)$ on which E acts through ρ is isomorphic to $P\cap W_\rho$, of dimension p (P_ρ) for $\rho\in\Phi'$ and q ($\overline{K_{\bar\rho}}$) otherwise. So J is a Hodge structure of signature (p,q) for Φ' with $K_\rho(J) = K_\rho$. Conversely J is determined by $K(J) = \bigoplus_{\rho\in\Phi'}(K_\rho(J)\oplus\mathrm{ann}_\psi(K_\rho(J)))$ (by the previous paragraph), so the map is a bijection.

(2) By (1), $\varphi_J|_{W_\rho}\colon W_\rho\to\mathrm{Lie}_{(\rho,+)}(J)$ is surjective with kernel $K_\rho(J)$, so $\dim F^\rho_{q+i} = q+i$ and $F^\rho_q = K_\rho(J)$; $\varphi_J|_{W_{\bar\rho}}\colon W_{\bar\rho}\to\mathrm{Lie}_{(\rho,-)}(J)$ is surjective with kernel $K(J)\cap W_{\bar\rho}$, so $\dim G'_k = p+k$, and F^ρ_{q-k} has dimension $n-(p+k) = q-k$ by perfectness. For $k=0$, $G'_0 = K(J)\cap W_{\bar\rho} = \mathrm{ann}_\psi(K_\rho(J))$, so $F^\rho_q = \mathrm{ann}_\psi(G'_0) = K_\rho(J)$: the two definitions agree. The spaces increase with the index, so $F^\rho_\bullet$ is a full flag of W_ρ, and F^ρ_q is negative definite: $F^\rho_\bullet\in\mathcal D_\rho$ (which is defined since W_ρ has signature (p,q)). Inverse map: given $(F^\rho_\bullet)_\rho\in\prod\mathcal D_\rho$, (1) gives a unique J with $K_\rho(J) = F^\rho_q$ for all ρ; put $V_{(\rho,+),i} = \varphi_J(F^\rho_{q+i})$ and $V_{(\rho,-),k} = \varphi_J(\mathrm{ann}_\psi(F^\rho_{q-k})\cap W_{\bar\rho})$, full flags of the pieces. Since $F^\rho_{q+i}\supseteq\ker(\varphi_J|_{W_\rho})$ and $\mathrm{ann}_\psi(F^\rho_{q-k})\supseteq\mathrm{ann}_\psi(F^\rho_q) = \ker(\varphi_J|_{W_{\bar\rho}})$, the two constructions are mutually inverse.

(3) *Case $i\le p$.* φ_J maps F_{q+i} onto $V_{(\rho,+),i}$ with kernel $K_\rho(J)\subseteq F_{q+i-1}$, so it induces an isomorphism $F_{q+i}/F_{q+i-1}\to\mathcal L_{(\rho,i)}$. Let $x\in\ell_{q+i}$ (Lemma 33.32(1)); then $x\in F_q^\perp\cap W_\rho = P(J)\cap W_\rho$ and $x\perp F_{q+i-1}\cap P(J)$. Every element of $V_{(\rho,+),i-1}$ is $\varphi_J(g)$ for a unique $g\in P(J)\cap W_\rho$, and then $g\in F_{q+i-1}$. By (1), the quotient norm squared of $\varphi_J(x)$ modulo $V_{(\rho,+),i-1}$ is $\min_g 2\pi\langle x-g,x-g\rangle$ over $g\in F_{q+i-1}\cap P(J)$, which by orthogonality and positivity on $P(J)$ is attained at $g=0$: it equals $2\pi\langle x,x\rangle = 2\pi\|[x]\|^2_{q+i}$.

Case $i > p$, $k = n+1-i \in \{1,\dots,q\}$, $\mathrm{pos}(i) = q-k+1$. As $F_{q-k+1} = \mathrm{ann}_\psi(G'_{k-1})$ and $F_{q-k} = \mathrm{ann}_\psi(G'_k)$, ψ induces a pairing of the lines F_{q-k+1}/F_{q-k} and G'_k/G'_{k-1}; it is non-zero, since otherwise $F_{q-k+1} \subseteq \mathrm{ann}_\psi(G'_k) = F_{q-k}$; so it is perfect. φ_J maps G'_k onto $V_{(\rho,-),k}$ with kernel $K(J) \cap W_{\bar\rho} = G'_0 \subseteq G'_{k-1}$, so $G'_k/G'_{k-1} \cong V_{(\rho,-),k}/V_{(\rho,-),k-1}$, and together $L_{\mathrm{pos}(i)} \cong \mathcal{L}_{(\rho,i)}$, $x \mapsto (\varphi_J(y) \mapsto \psi(x,y))$. Let $0 \neq y_0 \in G'_k \cap (G'_{k-1})^\perp$. As $y_0 \perp G'_0 = K(J) \cap W_{\bar\rho}$, $y_0 \in P(J)$, and the argument of the first case (in $W_{\bar\rho}$, with $P(J) \cap W_{\bar\rho}$ positive definite) gives that the quotient norm squared of $\varphi_J(y_0)$ modulo $V_{(\rho,-),k-1}$ is $2\pi\langle y_0, y_0\rangle$. Now $\bar y_0 \in W_\rho$; for $g \in G'_{k-1}$, $\psi(g,\bar y_0) = -i\langle g, y_0\rangle = 0$, so $\bar y_0 \in F_{q-k+1}$; for $a \in F_{q-k}$, $\langle \bar y_0, a\rangle = i\psi(\bar y_0, \bar a) = i\,\overline{\psi(y_0,a)} = 0$ as $\psi(a, G'_k) = 0$. So $\ell_{q-k+1} = \mathbb{C}\bar y_0$.

The dual norm of the image of $x \in \ell_{q-k+1}$ is $|\psi(x,y_0)|/(2\pi\langle y_0,y_0\rangle)^{1/2} = (2\pi)^{-1/2}|\langle x,x\rangle|^{1/2} = (2\pi)^{-1/2}\|[x]\|_{q-k+1}$.

(4) As $J(y)$ is E-linear, $\mathcal{K}$ is the direct sum of $\mathcal{K}_\sigma = \mathcal{K} \cap (W_\sigma \times N)$, the images of $\mathcal{K}$ under the constant projections $W \to W_\sigma$; these are holomorphic bundle maps of constant rank (rank q for $\sigma \in \Phi'$, p otherwise), so the $\mathcal{K}_\sigma$ are holomorphic subbundles, and the restrictions $\varphi_\sigma\colon W_\sigma \times N \to \mathcal{Q}$ of the quotient map are holomorphic of constant rank, with images the bundles of pieces. Hence $F^\rho_{q+i} = \ker\big(W_\rho \times N \to \mathcal{Q}/\mathcal{V}_{(\rho,+),i}\big)$ and $G'_k = \ker\big(W_{\bar\rho} \times N \to \mathcal{Q}/\mathcal{V}_{(\rho,-),k}\big)$ are holomorphic subbundles (kernels of holomorphic bundle maps of constant rank), and so is $F^\rho_{q-k} = \ker\big(W_\rho \times N \to (G'_k)^\vee\big)$, $x \mapsto \psi(x,\cdot)$. A family of full flags of a fixed space forming holomorphic subbundles is a holomorphic map to the flag manifold, so Ψ_ρ is holomorphic and $\Psi_\rho^* L_j = F^\rho_j/F^\rho_{j-1}$. The root lines are subquotients (or duals of subquotients) of holomorphic subbundles of $\mathcal{Q}$, hence holomorphic line bundles; the isomorphisms of (3) are induced by the holomorphic bundle maps φ_σ and by the constant pairing ψ, hence holomorphic. By (3) the Hodge metric is a positive constant times the pull-back of $\|\cdot\|_{\mathrm{pos}(i)}$, which is smooth (Lemma 33.32(1)); as c_1 is unchanged by a positive constant and commutes with holomorphic pull-back, $c_1(\mathcal{L}_{(\rho,i)}, \text{Hodge metric}) = \Psi_\rho^* \omega^\rho_{\mathrm{pos}(i)}$. □

Lemma 33.34. *Let* $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ *be a unitary Shimura datum with* $p, q \geq 1$, $\Phi = \Phi_{\mathsf{H}}$, $n = p+q$, *level* $\mathsf{L} = (N, \mathfrak{n}, 1)$ *good at every prime, whose components split. Let* $Z \in \mathrm{Comp}(\mathrm{Sh}_F)$, *let* $\mathrm{Fl}_\mathfrak{u}|_Z(\mathbb{C})$ *be the complex manifold of all complex points of* $\mathrm{Fl}_\mathfrak{u}|_Z$ *(over every embedding of* F*),* $\omega_r = c_1(\mathcal{L}_r, \|\cdot\|^{\mathrm{Hdg}})$ *for the roots* r, *and* $\nu^{\mathrm{Fl}} = (-1)^D x^\delta(\omega)$. *Then* ν^{Fl} *is a positive volume form on* $\mathrm{Fl}_\mathfrak{u}|_Z(\mathbb{C})$ *(for the orientation of the complex structure), and for every embedding* $\kappa\colon F \to \mathbb{C}$, *with* $k = k(Z,\kappa) \in I_\kappa$ *the index of (SU6),* $\Gamma_k = \Gamma(L_k, \mathfrak{n})$ *and* $\Phi'_\kappa = \{\rho\colon E \to \mathbb{C} :\ \rho(r) = \kappa(r_\mathbb{C})\}$,

$$\int_{\mathrm{Fl}_\mathfrak{u}|_{Z,\kappa}} \nu^{\mathrm{Fl}} = \mu_{\nu,k}(\Gamma_k \backslash G^1_k) \in (0,\infty],$$

where $\mathrm{Fl}_\mathfrak{u}|_{Z,\kappa}$ *is the open and closed subset of points over* κ, $\Gamma_k \subseteq G^1_k$ *is discrete and acts freely on* Fl, *and* G^1_k, $\mu_{\nu,k}$ *are the group and the Haar measure of Lemma 33.38 for* $(L_{k,\mathbb{R}}, \psi_k, \Phi'_\kappa)$. *In particular* $\int_{\mathrm{Fl}_\mathfrak{u}|_Z(\mathbb{C})} \nu^{\mathrm{Fl}} > 0$.

Proof. The set Φ'_κ is Φ if $\varepsilon_\kappa = 1$ and $\overline{\Phi}$ if $\varepsilon_\kappa = -1$, and $\kappa \circ \tau \in \Phi'_\kappa$ for $\tau \in \Phi$ (as $\tau(r) = r_\mathbb{C}$). The sets $\mathrm{Fl}_\mathfrak{u}|_{Z,\kappa}$ ($\kappa\colon F \to \mathbb{C}$) are open and closed and cover $\mathrm{Fl}_\mathfrak{u}|_Z(\mathbb{C})$; fix κ and $k = k(Z,\kappa)$. By (SU6) the complex points of $Z \otimes_{F,\kappa} \mathbb{C}$ are $u_k(\mathrm{Per}_\sigma(L_k))$, and by (SU2) u_k induces a biholomorphism $\Gamma_k \backslash \mathrm{Per}_\sigma(L_k) \to u_k(\mathrm{Per}_\sigma(L_k))$, Γ_k acting freely. By (SU3) $u_k^* G^{\mathrm{un}}$ is the family $A_J = (L_{k,\mathbb{R}}, J)/L_k$ with $H_1 = L_k$ and Riemann form ψ_k.

(a) Every $J \in \mathrm{Per}_\sigma(L_k)$ lies in the period domain Per of Lemma 33.38 for $(L_{k,\mathbb{R}}, \psi_k, \Phi'_\kappa)$ ($L_{k,\mathbb{R}}$ is free of rank n over $E \otimes \mathbb{R}$, L_k being projective of rank n, as $L_k \otimes \widehat{\mathbb{Z}} \cong M \otimes \widehat{\mathbb{Z}}$ by (SU2)): the conditions on J and ψ_k are those of (Pd1), and by the Kottwitz condition at the point

$u_k(J)$ (over κ) the eigenspace of $(L_{k,\mathbb{R}}, J)$ for ρ has dimension p if $\rho(r) = \kappa(r_{\mathbb{C}})$ and q otherwise. Conversely the determinant condition of Per_σ is a condition on these eigenspace dimensions only (the characteristic polynomial of a on $(L_{k,\mathbb{R}}, J)$ is $\prod_\rho (T - \rho(a))^{\dim}$), and it holds at one point, so $\mathrm{Per}_\sigma(L_k) = \mathrm{Per}$.

(b) The complex structures agree: for the structure of $\mathrm{Per}_\sigma(L_k)$ the family A_J is holomorphic with constant period lattice L_k, so the flat sections $\lambda \in L_k$ are holomorphic sections of $\mathrm{Lie}(A)$ (the kernel of the fibrewise exponential map is a complex submanifold étale over the base), and $W \otimes \mathcal{O} \to \mathrm{Lie}(A)$, $\lambda \otimes c \mapsto c\lambda$, is a holomorphic surjection with kernel $K(J)$ (the $(-i)$-eigenspace of J, as $\lambda \otimes c \mapsto c\lambda$ is J-equivariant); so $J \mapsto (K_\rho(J))_{\rho \in \Phi'_\kappa}$ is holomorphic and injective (J is determined by $(K_\rho)_{\rho \in \Phi'_\kappa}$: $K(J) = \bigoplus_{\rho \in \Phi'_\kappa} (K_\rho \oplus \overline{K_\rho^\perp \cap W_\rho})$, $\perp$ for $\langle v, v' \rangle = i\psi_k(v, \bar{v}')$, since, with $P(J) = \overline{K(J)}$ the $(+i)$-eigenspace, $K(J) \cap W_{\bar\rho} = \overline{P(J) \cap W_\rho}$ and $P(J) \cap W_\rho = K_\rho^\perp \cap W_\rho$ by the orthogonal decomposition $W = P(J) \oplus K(J)$) between manifolds of the same dimension, hence a biholomorphism onto its open image, and the two structures coincide; moreover $\mathrm{Lie}_s(A)$ is the holomorphic quotient bundle of (Pd2), the label $(\tau, \pm)$ of (Fh1) at κ corresponding to $(\kappa \circ \tau, \pm)$.

(c) Hence u_k induces a biholomorphism $\Gamma_k \backslash \mathrm{Fl} \to \mathrm{Fl}_{\mathsf{u}}|_{Z,\kappa}$ (Γ_k acting on Fl as in (Pd3); an $f \in \Gamma_k$ induces the isomorphism $A_J \to A_{fJf^{-1}}$ carrying flags V to fV), under which the root line of the root (τ, i) with its Hodge metric (Fh4) is the root line of $(\kappa \circ \tau, i)$ in (Pd2): at $u_k(J)$ the first Chern class of the polarisation is the hermitian form H_J with $\operatorname{Im} H_J = \psi_k$ on $H_1 = L_k$. So ν^{Fl} on $\mathrm{Fl}_{\mathsf{u}}|_{Z,\kappa}$ corresponds to the form ν of Lemma 33.38, which by its item 1 is a G-invariant positive volume form on Fl; it descends to the quotient by the free action of Γ_k, and the biholomorphism preserves orientations. So ν^{Fl} is a positive volume form on $\mathrm{Fl}_{\mathsf{u}}|_{Z,\kappa}$, for every κ.

(d) Γ_k is discrete in G_k^1: it lies in $\mathrm{Aut}(L_k)$; it preserves ψ_k, and its elements have determinant 1, since $\det\gamma \in \mathcal{O}_E^\times$ has $\det\gamma \cdot \overline{\det\gamma} = 1$ (as γ preserves h_k, (SU1)), so all its conjugates have absolute value 1 and it is a root of unity (Kronecker), and $\det\gamma \equiv 1 \pmod{\mathfrak{n}}$ as $(\gamma - 1)L_k \subseteq \mathfrak{n}L_k$; $\mathfrak{n}$ is neat (SU5), so $\det\gamma = 1$. It acts freely on Per, hence on Fl. By Lemma 33.38 (2),

$$\int_{\mathrm{Fl}_{\mathsf{u}}|_{Z,\kappa}} \nu^{\mathrm{Fl}} = \int_{\Gamma_k \backslash \mathrm{Fl}} \nu = \mu_{\nu,k}(\Gamma_k \backslash G_k^1) \in (0, \infty].$$

Summing over the embeddings κ (there is at least one) gives $\int_{\mathrm{Fl}_{\mathsf{u}}|_Z(\mathbb{C})} \nu^{\mathrm{Fl}} > 0$. □

Proposition 33.35. *Let $\mathsf{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p+q \geq 1$, $\mathsf{H} = (d, r, r_{\mathbb{C}})$, $\mathsf{U} = (M, \langle\,,\rangle)$ and level $\mathsf{L} = (N, \mathfrak{n}, 1)$. Let $\kappa\colon F \to \mathbb{C}$ be a field embedding, $\varepsilon_\kappa = \kappa(r_{\mathbb{C}})/r_{\mathbb{C}} \in \{\pm 1\}$, $\sigma = \kappa|_{F_{\mathrm{ref}}}$, and $\mathrm{Sh}_\kappa(\mathbb{C})$ the complex manifold of complex points of $\mathrm{Sh} \otimes_{K_{\mathsf{L}},\kappa} \mathbb{C}$. Then:*

(SU1) For every finitely generated projective $\mathcal{O}_E$-module L with a perfect alternating $\mathbb{Z}$-pairing ψ_L satisfying $\psi_L(ax, y) = \psi_L(x, \bar{a}y)$ there is exactly one hermitian form h_L on $V_L = L \otimes \mathbb{Q}$ with $\psi_L = \mathrm{Tr}_{E/\mathbb{Q}}(r\, h_L)$; L is a lattice of (V_L, h_L) and

$$\Gamma(L, \mathfrak{n}) = \{f \in \mathrm{Aut}_{\mathcal{O}_E}(L) :\ \psi_L(fx, fy) = \psi_L(x, y),\ (f-1)L \subseteq \mathfrak{n}L\}.$$

(SU2) There are a finite set I_κ and, for $k \in I_\kappa$, modules (L_k, ψ_k) as in (SU1) with $h_k = h_{L_k}$, $\mathcal{O}_E \otimes \widehat{\mathbb{Z}}$-linear isomorphisms $\hat\alpha_k\colon M \otimes \widehat{\mathbb{Z}} \to L_k \otimes \widehat{\mathbb{Z}}$ with $\psi_k(\hat\alpha_k x, \hat\alpha_k y) = \hat\nu_k \langle x, y\rangle$ for some $\hat\nu_k \in \widehat{\mathbb{Z}}^\times$, and holomorphic maps $u_k\colon \mathrm{Per}_\sigma(L_k) \to \mathrm{Sh}_\kappa(\mathbb{C})$ with non-empty connected source, such that $\Gamma(L_k, \mathfrak{n})$ acts freely and properly discontinuously on $\mathrm{Per}_\sigma(L_k)$ by $J \mapsto fJf^{-1}$, u_k induces an open and closed holomorphic embedding $\Gamma(L_k, \mathfrak{n}) \backslash \mathrm{Per}_\sigma(L_k) \to \mathrm{Sh}_\kappa(\mathbb{C})$, and

$$\mathrm{Sh}_\kappa(\mathbb{C}) = \bigsqcup_{k \in I_\kappa} u_k(\mathrm{Per}_\sigma(L_k)) \cong \bigsqcup_{k \in I_\kappa} \Gamma(L_k, \mathfrak{n}) \backslash \mathrm{Per}_\sigma(L_k);$$

the sets $u_k(\mathrm{Per}_\sigma(L_k))$ *are the connected components of* $\mathrm{Sh}_\kappa(\mathbb{C})$.

(SU3) Let $e \in \mathcal{O}_E$ *with* $e \equiv 1 \pmod{\mathfrak{n}}$, $e \equiv 0 \pmod{\bar{\mathfrak{n}}}$, *and* $\alpha_k = \hat{\alpha}_k \bmod N$. *Along* u_k *the object* $o^{\mathrm{un}} = (G, \iota, \mathcal{L}, \eta, \eta_A)$ *pulls back to the holomorphic family over* $\mathrm{Per}_\sigma(L_k)$ *whose fibre at* J *is* $A_J = (L_{k,\mathbb{R}}, J)/L_k$ *with the* $\mathcal{O}_E$*-action* ι_{L_k}, *the primitive cubical line bundle (formed with* w_0*) of the principal polarisation* ϕ_{L_k} *with Riemann form* ψ_k *on* $H_1(A_J, \mathbb{Z}) = L_k$, *the level* $a \mapsto N^{-1}\alpha_k(ea \bmod NM) \in N^{-1}L_k/L_k = A_J[N]$, *and* $\eta_A = 0$.

(SU4) For every $k \in I_\kappa$ *and every real place* w *of* E^+ *the hermitian form* $\varepsilon_\kappa h_k \otimes_{E^+,w} \mathbb{R}$ *has* q *positive and* p *negative eigenvalues if* $\operatorname{Im} r_{\mathbb{C}} > 0$, *and* p *positive and* q *negative eigenvalues if* $\operatorname{Im} r_{\mathbb{C}} < 0$.

(SU5) The ideal $\mathfrak{n}$ *is neat.*

(SU6) If the components split, then for every $Z \in \mathrm{Comp}(\mathrm{Sh}_F)$ *the complex points of* $Z \otimes_{F,\kappa} \mathbb{C}$ *form exactly one of the sets* $u_k(\mathrm{Per}_\sigma(L_k))$, *and this defines a bijection* $\mathrm{Comp}(\mathrm{Sh}_F) \to I_\kappa$.

Proof. **(SU1).** Since $\bar{r} = -r$ ($r^2 = -d$ and $r \notin E^+$), it suffices to construct h_L. For $x, y \in V_L$ the map $a \mapsto \psi_L(ax, y)$ is $\mathbb{Q}$-linear $E \to \mathbb{Q}$, so there is a unique $\eta(x, y) \in E$ with $\psi_L(ax, y) = \mathrm{Tr}_{E/\mathbb{Q}}(a\,\eta(x, y))$ for all $a \in E$ (the trace form is non-degenerate). Then $\eta(bx, y) = b\,\eta(x, y)$, and $\psi_L(ax, y) = -\psi_L(y, ax) = -\psi_L(\bar{a}y, x)$ gives $\eta(x, y) = -\overline{\eta(y, x)}$. Hence $h_L = r^{-1}\eta$ is E-linear in the first variable and $\overline{h_L(y, x)} = \bar{r}^{-1}\overline{\eta(y, x)} = h_L(x, y)$; it is non-degenerate because ψ_L is, and $\psi_L(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}(r\,h_L(x, y))$ (take $a = 1$). Uniqueness follows from non-degeneracy of the trace form applied to $\psi_L(ax, y)$. An E-linear automorphism of V_L preserves h_L if and only if it preserves ψ_L (by uniqueness), so $\mathrm{U}(V_L)(E^+) \cap \{\gamma : \gamma L = L\}$ is the group of $\mathcal{O}_E$-linear automorphisms of L preserving ψ_L, and (31.2) gives the displayed description.

The complex points of Sh**.** By Proposition 32.9, $\theta\colon \mathcal{M}_N/\mathcal{G}_N \to \mathrm{Sh}$ is an isomorphism and ϖ is finite, étale and surjective, with $\mathcal{G}_N$ acting freely. As $F_{\mathrm{ref}} \subseteq K_{\mathsf{L}} = \mathbb{Q}(r_{\mathbb{C}})$ (Lemma 32.3), $\mathcal{M}_N \otimes_{K_{\mathsf{L}},\kappa} \mathbb{C}$ is the base change along σ of the full-level scheme over F_{ref}, whose complex manifold is $\mathcal{M}_N^\sigma$. The complex points of a quotient by a finite group acting freely are the orbits (SGA 1, Exp. V, Prop. 1.1), and ϖ induces a finite covering map, a local biholomorphism; hence $\theta \circ \varpi$ identifies $\mathcal{G}_N \backslash \mathcal{M}_N^\sigma$ with $\mathrm{Sh}_\kappa(\mathbb{C})$ as complex manifolds.

$\mathcal{G}_N$ **permutes the pieces.** For $(g, c) \in \mathcal{G}_N$ and $(L, \psi_L, \alpha, \nu) \in \mathcal{P}$, the tuple $(L, \psi_L, \alpha \circ g, c\nu)$ lies in $\mathcal{P}$: $\psi_L(\alpha g x, \alpha g y) \equiv \nu\langle gx, gy\rangle \equiv c\nu\langle x, y\rangle \pmod{N}$, and if $(\hat{g}, \hat{c})$ lifts (g, c) (clause (3) of Proposition 32.9) then $(\hat{\alpha}\hat{g}, \hat{c}\hat{\nu})$ lifts $(\alpha g, c\nu)$. By (PU4) and the definition of the action, $(g, c) \cdot u_L(J)$ is the point of the tuple $(A_J, \phi_L, \iota_L, \alpha_{N,J} \circ g, \nu_{N,J} \circ c)$, which is u of the same $J \in \mathrm{Per}_\sigma(L)$ for the tuple $(L, \psi_L, \alpha \circ g, c\nu)$. So $\mathcal{G}_N$ acts on the set of isomorphism classes in $\mathcal{P}$ with non-empty period domain, compatibly with the pieces. Let $I_\kappa \subseteq I$ be a set of representatives of the $\mathcal{G}_N$-orbits on I; it is finite as I is. For $k = (L_k, \psi_k, \alpha_k, \nu_k) \in I_\kappa$ fix a lift $(\hat{\alpha}_k, \hat{\nu}_k)$ as in (PU1), and put $u_k = \theta \circ \varpi \circ u_{L_k}$, holomorphic by (PU4); $\mathrm{Per}_\sigma(L_k)$ is non-empty and connected by the choice of I and (PU2).

The stabiliser of a piece is $\Gamma(L_k, \mathfrak{n})$**.** Write $L = L_k$, $\alpha = \alpha_k$, $\nu = \nu_k$, and $\bar{f}$ for $f \bmod N$. An isomorphism $(L, \psi_L, \alpha \circ g, c\nu) \to (L, \psi_L, \alpha, \nu)$ is an $f \in \mathrm{Aut}_{\mathcal{O}_E}(L)$ preserving ψ_L with $\bar{f}\alpha g = \alpha$ and $c\nu = \nu$, i.e. $c = 1$ and $g = \alpha^{-1}\bar{f}^{-1}\alpha$. Conversely, for $f \in \mathrm{Aut}_{\mathcal{O}_E}(L)$ preserving ψ_L, the pair $(g_f, 1)$ with $g_f = \alpha^{-1}\bar{f}^{-1}\alpha$ satisfies (1) and (3) of Proposition 32.9 automatically: $\nu\langle g_f x, g_f y\rangle \equiv \psi_L(\bar{f}^{-1}\alpha x, \bar{f}^{-1}\alpha y) \equiv \nu\langle x, y\rangle$ with ν a unit, and $(\hat{\alpha}^{-1}f^{-1}\hat{\alpha}, 1)$ is a lift with $\langle \hat{\alpha}^{-1}f^{-1}\hat{\alpha}x, \hat{\alpha}^{-1}f^{-1}\hat{\alpha}y\rangle = \hat{\nu}^{-1}\psi_L(f^{-1}\hat{\alpha}x, f^{-1}\hat{\alpha}y) = \langle x, y\rangle$. Condition (2) says that $\bar{f}$ is the identity on $\alpha(\bar{\mathfrak{n}}M/NM) = \bar{\mathfrak{n}}L/NL$, i.e. $(f-1)\bar{\mathfrak{n}}L \subseteq NL$. This is equivalent to $(f-1)L \subseteq \mathfrak{n}L$: if $(f-1)L \subseteq \mathfrak{n}L$ then $(f-1)\bar{\mathfrak{n}}L = \bar{\mathfrak{n}}(f-1)L \subseteq \mathfrak{n}\bar{\mathfrak{n}}L = NL$; conversely $L = \mathfrak{n}L + \bar{\mathfrak{n}}L$ (as $\mathfrak{n} + \bar{\mathfrak{n}} = \mathcal{O}_E$), $(f-1)\mathfrak{n}L = \mathfrak{n}(f-1)L \subseteq \mathfrak{n}L$ and $(f-1)\bar{\mathfrak{n}}L \subseteq NL \subseteq \mathfrak{n}L$. By (SU1)

the set of these f is $\Gamma(L,\mathfrak{n})$, and $f\mapsto(g_f,1)$ is a homomorphism from $\Gamma(L,\mathfrak{n})$ onto the stabiliser $S_k\subseteq\mathcal{G}_N$ of the class of k, with kernel $\Gamma_N(L)$. For $f\in\Gamma(L,\mathfrak{n})$ the isomorphism $A_J\to A_{fJf^{-1}}$ induced by f carries the tuple of $(L,\alpha\circ g_f)$ at J to the tuple of (L,α) at fJf^{-1}, so

$$(g_f,1)\cdot u_L(J)=u_L(fJf^{-1})\qquad(J\in\mathrm{Per}_\sigma(L)).$$

(SU2). By (PU4), $\mathcal{M}_N^\sigma$ is the disjoint union of the open and closed sets $u_L(\mathrm{Per}_\sigma(L))$, $L\in I$, and $\mathcal{G}_N$ permutes them; for $k\in I_\kappa$ the union U_k of the pieces in the orbit of k is open, closed and $\mathcal{G}_N$-stable, so its image $u_k(\mathrm{Per}_\sigma(L_k))=\theta\varpi(U_k)$ is open and closed, and these images for distinct k are disjoint and cover $\mathrm{Sh}_\kappa(\mathbb{C})$. Two points $J,J'\in\mathrm{Per}_\sigma(L_k)$ have the same image if and only if $u_{L_k}(J')=(g,c)\cdot u_{L_k}(J)$ for some (g,c), which forces $(g,c)\in S_k$, $(g,c)=(g_f,1)$ with $f\in\Gamma(L_k,\mathfrak{n})$, hence $u_{L_k}(J')=u_{L_k}(fJf^{-1})$ and, by (PU4), $J'\in\Gamma_N(L_k)fJf^{-1}$; so the fibres of u_k are the $\Gamma(L_k,\mathfrak{n})$-orbits. The action is free: if $fJf^{-1}=J$ then $(g_f,1)$ fixes $u_{L_k}(J)$, so $g_f=1$ by freeness of $\mathcal{G}_N$, i.e. $f\in\Gamma_N(L_k)$, and $f=1$ by (PU3). It is properly discontinuous because $\Gamma_N(L_k)$ has finite index (at most $|\mathcal{G}_N|$) in $\Gamma(L_k,\mathfrak{n})$ and acts properly discontinuously (PU3). The induced map $\Gamma(L_k,\mathfrak{n})\backslash\mathrm{Per}_\sigma(L_k)\to\mathrm{Sh}_\kappa(\mathbb{C})$ is therefore injective, and it is a local biholomorphism (composite of local biholomorphisms by (PU4) and the covering ϖ), hence an open and closed holomorphic embedding. Each image is connected since $\mathrm{Per}_\sigma(L_k)$ is; being open, closed, disjoint and covering, the images are the connected components.

(SU3). By Proposition 32.9(2), $(\theta\circ\varpi)^*o^{\mathrm{un}}\cong(A,\iota,\mathcal{L}_{\phi_{\mathrm{pol}}},\eta,0)$ with $\eta(a)=\alpha_N(ea\bmod NM)$ and $\mathcal{L}_{\phi_{\mathrm{pol}}}$ the primitive cubical line bundle of ϕ_{pol} formed with w_0. Pull back along u_{L_k} and use (PU4): the universal tuple becomes $(A_J,\phi_{L_k},\iota_{L_k},\alpha_{N,J},\nu_{N,J})$ with $\alpha_{N,J}(x)=N^{-1}\alpha_k(x)$; the primitive cubical line bundle is formed from the polarisation and so pulls back to that of ϕ_{L_k}, and η pulls back to $a\mapsto N^{-1}\alpha_k(ea\bmod NM)$.

(SU4). By Lemma 32.3, $P_a(T)=\prod_{\tau\in\Phi_{\mathsf{H}}}(T-\tau(a))^p(T-\bar\tau(a))^q$ with coefficients in $\mathbb{Q}(r_{\mathbb{C}})$. If $\varepsilon_\kappa=1$, σ is the inclusion; if $\varepsilon_\kappa=-1$, σ is complex conjugation on $\mathbb{Q}(r_{\mathbb{C}})$ and $\sigma(P_a)=\prod_{\tau\in\Phi_{\mathsf{H}}}(T-\bar\tau(a))^p(T-\tau(a))^q$. Fix k, $J\in\mathrm{Per}_\sigma(L_k)$ and a real place w of E^+; exactly one embedding τ_w of E above w lies in Φ_{H} (as $\bar\tau(r)=-\tau(r)$), so $\tau_w(r)=r_{\mathbb{C}}$. Let $V_w=L_k\otimes_{\mathcal{O}_E,\tau_w}\mathbb{C}$, the summand of $L_{k,\mathbb{R}}$ on which E acts through τ_w; J commutes with E, hence is $\mathbb{C}$-linear on V_w with $V_w=V_w^+\oplus V_w^-$, $V_w^\pm=\ker(J\mp i)$. On $(L_{k,\mathbb{R}},J)$ the element a acts on V_w^+ by the complex scalar $\tau_w(a)$ and on V_w^- by $\bar\tau_w(a)$, so comparison with $\sigma(P_a)$ gives $\dim V_w^+=p$, $\dim V_w^-=q$ if $\varepsilon_\kappa=1$, and the reverse if $\varepsilon_\kappa=-1$. With $h_w=h_k\otimes_w\mathbb{R}$ on V_w and $\psi_k=\mathrm{Tr}_{E/\mathbb{Q}}(rh_k)$, for $x\in V_w^\pm$ $(x\neq0)$

$$0<\psi_k(Jx,x)=2\,\mathrm{Re}\big(r_{\mathbb{C}}\,h_w(\pm ix,x)\big)=\mp2\,\mathrm{Im}(r_{\mathbb{C}})\,h_w(x,x)$$

(as $r_{\mathbb{C}}$ is purely imaginary and $h_w(x,x)$ real). So if $\mathrm{Im}\,r_{\mathbb{C}}>0$, h_w is negative definite on V_w^+ and positive definite on V_w^-, and these are h_w-orthogonal ($h_w(Jx,Jy)=h_w(x,y)$ since J preserves ψ_k and commutes with E); the signs reverse if $\mathrm{Im}\,r_{\mathbb{C}}<0$. Combining with the dimensions gives (SU4).

(SU5). Let $\zeta\in E$ be a root of unity with $\zeta-1\in\mathfrak{n}$. Complex conjugation of the CM field E maps ζ to ζ^{-1} (for every embedding $\tau\colon E\to\mathbb{C}$, $\tau(\bar\zeta)=\overline{\tau(\zeta)}=\tau(\zeta)^{-1}$), so conjugating the membership gives $\zeta^{-1}-1\in\bar{\mathfrak{n}}$, and multiplying by $-\zeta\in\mathcal{O}_E$ gives $\zeta-1\in\bar{\mathfrak{n}}$. Since $\mathfrak{n}+\bar{\mathfrak{n}}=\mathcal{O}_E$ (Ud3), $\mathfrak{n}\cap\bar{\mathfrak{n}}=\mathfrak{n}\bar{\mathfrak{n}}=N\mathcal{O}_E$, hence $\zeta-1\in N\mathcal{O}_E$. Let $Z\in\mathrm{GL}_{n_E}(\mathbb{Z})$, $n_E=[E:\mathbb{Q}]$, be the matrix of multiplication by ζ on $\mathcal{O}_E$ in a $\mathbb{Z}$-basis. It has finite order because ζ has, and $Z-1_{n_E}\in N\,\mathrm{M}_{n_E}(\mathbb{Z})$ because $(\zeta-1)\mathcal{O}_E\subseteq N\mathcal{O}_E$. As $N\geq3$, Minkowski's congruence lemma (Theorem B.160, with $n=n_E$ and $m=N$) gives $Z=1_{n_E}$; evaluating at $1\in\mathcal{O}_E$ gives $\zeta=1$. So $\mathfrak{n}$ is neat.

(SU6). The scheme Sh is smooth over K_{L} (Ud5), so Sh_F is smooth over F; its irreducible components are its connected components, each smooth and integral, and $\mathrm{Sh}_F=\bigsqcup_Z Z$. If F is algebraically closed in $\mathbf{k}(Z)$, the geometrically reduced F-variety Z is geometrically irreducible

(EGA IV$_2$, Prop. 4.5.9, characteristic 0), so $Z \otimes_{F,\kappa} \mathbb{C}$ is irreducible and its complex points form a connected complex manifold (SGA 1, Exp. XII, Prop. 2.4). As $\mathrm{Sh}_F \otimes_{F,\kappa} \mathbb{C} = \mathrm{Sh} \otimes_{K_\mathfrak{l},\kappa} \mathbb{C}$, the sets $Z_\kappa(\mathbb{C})$, $Z \in \mathrm{Comp}(\mathrm{Sh}_F)$, are disjoint, open, closed, connected and cover $\mathrm{Sh}_\kappa(\mathbb{C})$, so they are its connected components; by (SU2) each is exactly one set $u_k(\mathrm{Per}_\sigma(L_k))$, and $Z \mapsto k$ is a bijection. □

Proposition 33.36. *Let $\mathcal{M}_N$ be the full-level scheme of Theorem B.105 for (E, M, ψ, J) and N, let $\sigma\colon F_{\mathrm{ref}} \to \mathbb{C}$ be a field embedding, and let $\mathcal{M}_N^\sigma$ be the complex manifold of complex points of $\mathcal{M}_N \otimes_{F_{\mathrm{ref}},\sigma} \mathbb{C}$.*

(PU1) Lattice data. *Let $\mathcal{P}$ be the set of tuples (L, ψ_L, α, ν) of a finitely generated projective $\mathcal{O}_E$-module L with a perfect alternating $\mathbb{Z}$-pairing ψ_L such that $\psi_L(ax, y) = \psi_L(x, \bar{a}y)$ $(a \in \mathcal{O}_E)$, an $\mathcal{O}_E$-linear isomorphism $\alpha\colon M/NM \to L/NL$ and $\nu \in (\mathbb{Z}/N\mathbb{Z})^\times$, such that there are an $\mathcal{O}_E \otimes \widehat{\mathbb{Z}}$-linear isomorphism $\hat{\alpha}\colon M \otimes \widehat{\mathbb{Z}} \to L \otimes \widehat{\mathbb{Z}}$ and $\hat{\nu} \in \widehat{\mathbb{Z}}^\times$ with $\psi_L(\hat{\alpha}x, \hat{\alpha}y) = \hat{\nu}\,\psi(x, y)$, $\hat{\alpha} \bmod N = \alpha$ and $\hat{\nu} \bmod N = \nu$. An isomorphism $(L, \psi_L, \alpha, \nu) \to (L', \psi_{L'}, \alpha', \nu')$ is an $\mathcal{O}_E$-linear isomorphism $f\colon L \to L'$ with $\psi_{L'}(fx, fy) = \psi_L(x, y)$, $(f \bmod N) \circ \alpha = \alpha'$ and $\nu = \nu'$.*

(PU2) Period domains. *For $(L, \psi_L, \alpha, \nu) \in \mathcal{P}$ let $\mathrm{Per}_\sigma(L)$ be the set of $\mathbb{R}$-linear $J_L\colon L_\mathbb{R} \to L_\mathbb{R}$ commuting with $\mathcal{O}_E$ with $J_L^2 = -1$, $\psi_L(J_L x, J_L y) = \psi_L(x, y)$, $\psi_L(J_L x, x) > 0$ for $x \neq 0$, and $\det_\mathbb{C}(T - a \mid (L_\mathbb{R}, J_L)) = \sigma(P_a)(T)$ for all $a \in \mathcal{O}_E$. With the complex structure for which $J_L \mapsto \ker(J_L + i) \subseteq L_\mathbb{C}$ is a biholomorphism onto a complex submanifold of the Grassmannian of $L_\mathbb{C}$ ($\ker(J_L + i)$ is the kernel of $L_\mathbb{C} \to (L_\mathbb{R}, J_L)$, $\lambda \otimes c \mapsto c\lambda$), $\mathrm{Per}_\sigma(L)$ is a complex manifold; if it is non-empty it is connected.*

(PU3) Groups. *The group $\Gamma_N(L) = \{f \in \mathrm{Aut}_{\mathcal{O}_E}(L) :\ \psi_L(fx, fy) = \psi_L(x, y),\ f \equiv 1 \text{ on } L/NL\}$ acts on $\mathrm{Per}_\sigma(L)$ by $J_L \mapsto f J_L f^{-1}$, freely and properly discontinuously, by biholomorphisms.*

(PU4) Uniformisation. *For $J_L \in \mathrm{Per}_\sigma(L)$ put $A_{J_L} = (L_\mathbb{R}, J_L)/L$, a complex abelian variety with $\mathcal{O}_E$-action ι_L induced from L and the principal polarisation ϕ_L whose Riemann form on $H_1(A_{J_L}, \mathbb{Z}) = L$ is ψ_L; let $\alpha_{N,J_L}\colon M/NM \to A_{J_L}[N] = N^{-1}L/L$ be $x \mapsto N^{-1}\alpha(x)$ and ν_{N,J_L} the isomorphism $\mathbb{Z}/N\mathbb{Z} \to \mu_N(\mathbb{C})$ with $e_N^{\phi_L}(\alpha_{N,J_L}x, \alpha_{N,J_L}y) = \nu_{N,J_L}(\psi(x, y) \bmod N)$ (it exists because $\psi_L(\alpha x, \alpha y) \equiv \nu\psi(x, y) \pmod N$). Then $J_L \mapsto [(A_{J_L}, \phi_L, \iota_L, \alpha_{N,J_L}, \nu_{N,J_L})]$ is a holomorphic map $u_L\colon \mathrm{Per}_\sigma(L) \to \mathcal{M}_N^\sigma$ along which the universal tuple pulls back to the holomorphic family $(A_{J_L})_{J_L}$ with these structures; u_L depends only on the isomorphism class of (L, ψ_L, α, ν) up to composition with the induced biholomorphism of period domains; it induces an open and closed holomorphic embedding $\Gamma_N(L)\backslash\mathrm{Per}_\sigma(L) \to \mathcal{M}_N^\sigma$; and for a set $I \subseteq \mathcal{P}$ of representatives of the isomorphism classes with $\mathrm{Per}_\sigma(L) \neq \emptyset$, which is finite,*

$$\mathcal{M}_N^\sigma = \bigsqcup_{(L,\psi_L,\alpha,\nu)\in I} u_L(\mathrm{Per}_\sigma(L)) \cong \bigsqcup_{(L,\psi_L,\alpha,\nu)\in I} \Gamma_N(L)\backslash\mathrm{Per}_\sigma(L).$$

Proof. **Step 1: the datum of a lattice.** Let $t = (L, \psi_L, \alpha, \nu) \in \mathcal{P}$ with $\mathrm{Per}_\sigma(L) \neq \emptyset$, and fix $J_t \in \mathrm{Per}_\sigma(L)$. Then $L \neq 0$ (as $L \otimes \widehat{\mathbb{Z}} \cong M \otimes \widehat{\mathbb{Z}}$), and $D_t = (E, L, \psi_L, J_t)$ satisfies the hypotheses of Theorem B.105, with polynomials $\det_\mathbb{C}(T - a \mid (L_\mathbb{R}, J_t)) = \sigma(P_a)(T)$, dimension $\frac{1}{2}\mathrm{rank}_\mathbb{Z} L = g$ and reflex field $F_t = \mathbb{Q}(\text{coefficients of the } \sigma(P_a)) = \sigma(F_{\mathrm{ref}}) \subseteq \mathbb{C}$. Let $\mathcal{M}_N(t)$ be its full-level scheme. The set $\mathcal{D}$ of Lemma 33.37 formed with (L, ψ_L, J_t) is $\mathrm{Per}_\sigma(L)$: the defining conditions coincide, its polynomials being $\sigma(P_a)$.

Step 2: transport of the level. *There is an isomorphism of $\mathbb{C}$-schemes $\Phi_t\colon \mathcal{M}_N(t) \otimes_{F_t} \mathbb{C} \to \mathcal{M}_N \otimes_{F_{\mathrm{ref}},\sigma} \mathbb{C}$ (the source formed with the inclusion $F_t \subseteq \mathbb{C}$) which on tuples over a $\mathbb{C}$-scheme*

S is $(A,\phi,\iota,\alpha'_N,\nu'_N)\mapsto(A,\phi,\iota,\alpha'_N\circ\alpha,\nu'_N\circ m_\nu)$*, m_ν the multiplication by ν on $\mathbb{Z}/N\mathbb{Z}$, and which carries the universal tuple of the target to that of the source.* Both schemes represent functors on $\mathbb{C}$-schemes: a $\mathbb{C}$-scheme is an F_{ref}-scheme through σ, so for the target the determinant condition reads $\det(T-\iota(a)\mid \mathrm{Lie}(A/S))=\sigma(P_a)(T)$, the condition of the source; the other conditions on (A,ϕ,ι) are the same. Fix a lift $(\hat\alpha,\hat\nu)$ of (α,ν) as in (PU1). If $(\alpha'_m,\nu'_m)_{N|m}$ is a compatible tower of lifts of (α'_N,ν'_N) over $S_m\to S$, then the pairs $(\alpha'_m\circ(\hat\alpha \bmod m),\nu'_m\circ m_{\hat\nu \bmod m})$ are $\mathcal{O}_E$-linear isomorphisms with

$$e^\phi_m(\alpha'_m\hat\alpha x,\alpha'_m\hat\alpha y)=\nu'_m(\psi_L(\hat\alpha x,\hat\alpha y) \bmod m)=\nu'_m(\hat\nu\,\psi(x,y) \bmod m),$$

compatible under the transitions (reduction commutes with $\hat\alpha$ and $\hat\nu$), and equal at $m=N$ to $(\alpha'_N\circ\alpha,\nu'_N\circ m_\nu)$. So the image is a tuple of the target, and the map does not depend on the lift. The same construction with $(\hat\alpha^{-1},\hat\nu^{-1})$ gives the inverse; isomorphisms of tuples go to isomorphisms. This is an isomorphism of functors, hence (Yoneda) of the representing $\mathbb{C}$-schemes compatible with the universal tuples, and on complex points a biholomorphism of the associated complex analytic spaces.

Step 3: complex uniformisation for D_t. Let X, X_0, G, $\mathcal{H}_N$, $\Gamma^{(1)}$ be the objects of Theorem B.114 for the datum D_t (so X is the $G(\mathbb{R})$-conjugacy class of J_t, and $G(\mathbb{R})=G_L(\mathbb{R})$ of Lemma 33.37 with $J_0=J_t$). (a) $X_0=\mathrm{Per}_\sigma(L)$: the topology of X in Theorem B.114 is that of $G(\mathbb{R})/\mathrm{Stab}(J_t)$, for which $\gamma\mapsto\gamma J_t\gamma^{-1}$ and the inclusion into $\mathrm{End}_{\mathbb{R}}(L_{\mathbb{R}})$ are continuous; so (Per2) and Step 1 give $X_0=\mathcal{D}=\mathrm{Per}_\sigma(L)$. Its complex structure in Theorem B.114 is the one for which $J'\mapsto\ker(J'+i)$ is a holomorphic open embedding into $G(\mathbb{C})/P_0(\mathbb{C})$, a locally closed submanifold of the Grassmannian of $L_{\mathbb{C}}$; this is the structure of (PU2), and it induces the topology of $\mathrm{End}_{\mathbb{R}}(L_{\mathbb{R}})$ by (Per3). Hence (PU2): $\mathrm{Per}_\sigma(L)$ is a complex manifold, connected by (Per2). (b) $\Gamma^{(1)}=\Gamma_N(L)$: if $\gamma\in G(\mathbb{Q})\cap\mathcal{H}_N$, then $\gamma(L\otimes\widehat{\mathbb{Z}})=L\otimes\widehat{\mathbb{Z}}$, so $\gamma L=L$ ($L=L_{\mathbb{Q}}\cap L\otimes\widehat{\mathbb{Z}}$), $\nu(\gamma)\in\mathbb{Q}^\times\cap\widehat{\mathbb{Z}}^\times=\{\pm1\}$ and $\nu(\gamma)\equiv1 \pmod N$ with $N\geq3$, so $\nu(\gamma)=1$, and $\gamma\equiv1$ on L/NL: $\gamma\in\Gamma_N(L)$. Conversely $\Gamma_N(L)\subseteq G(\mathbb{Q})\cap\mathcal{H}_N$, and $\Gamma_N(L)\subseteq\mathrm{U}_L(\mathbb{R})$ preserves $\mathrm{Per}_\sigma(L)=X_0$ by (Per1), so $\Gamma_N(L)\subseteq G(\mathbb{Q})_0$. (c) By (Lan1) and (Lan2) for D_t: for $J'\in\mathrm{Per}_\sigma(L)$ the tuple $x_t(J')=(A_{J'},\phi_L,\iota_L,\alpha'_N,\nu'_N)$ with $\alpha'_N(x')=N^{-1}x' \bmod L$ and $\nu'_N(c)=\zeta_N^c$ is an object of $\mathcal{M}_N(t)(\mathbb{C})$, $e_n^{\phi_L}(n^{-1}x',n^{-1}y')=\zeta_n^{\psi_L(x',y')}$ for $n\geq1$, $x',y'\in L$; $\Gamma_N(L)$ acts on $\mathrm{Per}_\sigma(L)$ without fixed points; and $J'\mapsto x_t(J')$ induces an isomorphism of $\Gamma_N(L)\backslash\mathrm{Per}_\sigma(L)$ onto an open and closed complex analytic subspace of $(\mathcal{M}_N(t)\otimes_{F_t}\mathbb{C})^{\mathrm{an}}$, along which the universal tuple pulls back to the holomorphic family $(x_t(J'))_{J'}$.

Step 4: (PU3) and (PU4) for one lattice. The group $\Gamma_N(L)$ acts freely (Step 3(c)), properly discontinuously (Per4), and by biholomorphisms: under $J'\mapsto\ker(J'+i)$ the map $J'\mapsto fJ'f^{-1}$ is the restriction of the linear automorphism $f\otimes\mathbb{C}$ of the Grassmannian. In particular $\Gamma_N(L)\backslash\mathrm{Per}_\sigma(L)$ is a complex manifold and the quotient map is a local biholomorphism. By Step 2, $\Phi_t(x_t(J'))=(A_{J'},\phi_L,\iota_L,\alpha'_N\circ\alpha,\nu'_N\circ m_\nu)$, with $(\alpha'_N\circ\alpha)(x)=N^{-1}\alpha(x)=\alpha_{N,J'}(x)$, and

$$e_N^{\phi_L}(N^{-1}\alpha x,N^{-1}\alpha y)=\zeta_N^{\psi_L(\alpha x,\alpha y)}=\zeta_N^{\nu\psi(x,y)}=(\nu'_N\circ m_\nu)(\psi(x,y) \bmod N).$$

As ψ is perfect, $\psi(x,y) \bmod N$ takes every value, so $\nu_{N,J'}$ of (PU4) exists, is unique, and equals $c\mapsto\zeta_N^{\nu c}$; thus $\Phi_t\circ x_t=u_t$. Since Φ_t is a biholomorphism matching universal tuples, Step 3(c) gives: u_t is holomorphic, the universal tuple pulls back along u_t to the family $(A_{J'},\phi_L,\iota_L,\alpha_{N,J'},\nu_{N,J'})$, and u_t induces an open and closed holomorphic embedding of $\Gamma_N(L)\backslash\mathrm{Per}_\sigma(L)$ into $\mathcal{M}_N^\sigma$. If $f\colon t\to t'$ is an isomorphism in $\mathcal{P}$, then $J'\mapsto fJ'f^{-1}$ is a biholomorphism $\mathrm{Per}_\sigma(L)\to\mathrm{Per}_\sigma(L')$ (restriction of a linear isomorphism of Grassmannians), and $f\otimes\mathbb{R}$ induces an isomorphism of tuples from $u_t(J')$ to $u_{t'}(fJ'f^{-1})$: it is $\mathbb{C}$-linear $(L_{\mathbb{R}},J')\to(L'_{\mathbb{R}},fJ'f^{-1})$, maps L onto L', commutes with $\mathcal{O}_E$, carries ψ_L to $\psi_{L'}$, carries $N^{-1}\alpha$ to $N^{-1}\alpha'$ because $(f \bmod N)\circ\alpha=\alpha'$, and $\nu=\nu'$. So $u_{t'}(fJ'f^{-1})=u_t(J')$.

Step 5: every point lies on some u_t. Let $y \in \mathcal{M}_N^\sigma$ be the class of a tuple $(A, \phi, \iota, \alpha_N, \nu_N)$ over $\mathbb{C}$. By Riemann's theorem [186], §§1–3, $A = (L_\mathbb{R}, J_A)/L$ with $L = H_1(A, \mathbb{Z})$, J_A the complex structure of $\mathrm{Lie}(A) = L_\mathbb{R}$, and the principal polarisation ϕ has a perfect alternating Riemann form ψ_A on L with $\psi_A(J_A x, J_A y) = \psi_A(x, y)$ and $\psi_A(J_A x, x) > 0$ for $x \neq 0$; $\mathcal{O}_E$ acts on L through ι, L is projective (finitely generated and torsion-free over the Dedekind ring $\mathcal{O}_E$), and $\phi\iota(a) = \iota(\bar{a})^\vee\phi$ says $\psi_A(ax, y) = \psi_A(x, \bar{a}y)$. As $\mathrm{Lie}(A) = (L_\mathbb{R}, J_A)$, the determinant condition says $\det_\mathbb{C}(T - a \mid (L_\mathbb{R}, J_A)) = \sigma(P_a)(T)$. So (E, L, ψ_A, J_A) is a datum as in Theorem B.105, J_A is its base point, and $(A, \phi, \iota) = (A_{J_A}, \phi_L, \iota_L)$ in the notation of (Lan1) for it; hence $e_m^\phi(m^{-1}x', m^{-1}y') = \zeta_m^{\psi_A(x',y')}$ for $m \geq 1$. Over $S = \operatorname{Spec}\mathbb{C}$ every finite étale surjection has a section, and the sets of complex points of the S_m form an inverse system of non-empty finite sets with surjective transitions, so there is a compatible system of lifts $(\alpha_m, \nu_m)_{N|m}$ over $\mathbb{C}$. Write $\alpha_m(x) = m^{-1}\tilde{\alpha}_m(x) \bmod L$ with $\tilde{\alpha}_m\colon M/mM \to L/mL$ and $\nu_m(c) = \zeta_m^{n_m c}$ with $n_m \in (\mathbb{Z}/m\mathbb{Z})^\times$. The Weil pairing condition gives $\psi_A(\tilde{\alpha}_m x, \tilde{\alpha}_m y) \equiv n_m\psi(x, y) \pmod m$, and compatibility gives $\tilde{\alpha}_m \equiv \tilde{\alpha}_l$, $n_m \equiv n_l \pmod l$ for $N \mid l \mid m$. So $\hat{\alpha} = \varprojlim \tilde{\alpha}_m\colon M \otimes \widehat{\mathbb{Z}} \to L \otimes \widehat{\mathbb{Z}}$ is an $\mathcal{O}_E \otimes \widehat{\mathbb{Z}}$-linear isomorphism and $\hat{\nu} = \varprojlim n_m \in \widehat{\mathbb{Z}}^\times$ satisfies $\psi_A(\hat{\alpha}x, \hat{\alpha}y) = \hat{\nu}\psi(x, y)$. Hence $t_y = (L, \psi_A, \tilde{\alpha}_N, n_N) \in \mathcal{P}$, $J_A \in \mathrm{Per}_\sigma(L)$, and by Step 4 ($\alpha_N = N^{-1}\tilde{\alpha}_N = \alpha_{N,J_A}$, $\nu_N = \nu_{N,J_A}$) $y = u_{t_y}(J_A)$.

Step 6: the decomposition. Let $I \subseteq \mathcal{P}$ be a set of representatives of the isomorphism classes with non-empty period domain. For $t \in I$ the set $u_t(\mathrm{Per}_\sigma(L))$ is non-empty, open and closed (Step 4) and connected ((PU2)), hence a connected component of $\mathcal{M}_N^\sigma$; by Steps 4 and 5 every point of $\mathcal{M}_N^\sigma$ lies in $u_t(\mathrm{Per}_\sigma(L))$ for some $t \in I$. For $t \neq t'$ in I these sets are disjoint: if $u_t(J) = u_{t'}(J')$, an isomorphism of the two tuples is an isomorphism of complex tori $A_J \to A_{J'}$, induced [186], §1, by a $\mathbb{C}$-linear $F\colon (L_\mathbb{R}, J) \to (L'_\mathbb{R}, J')$ with $F(L) = L'$; it commutes with $\mathcal{O}_E$, carries ψ_L to $\psi_{L'}$ (the polarisations correspond), satisfies $(F \bmod N) \circ \alpha = \alpha'$ (levels), and $\nu = \nu'$ because $\nu_{N,J}(c) = \zeta_N^{\nu c}$ and $\nu'_{N,J'}(c) = \zeta_N^{\nu' c}$ (Step 4); so $F|_L$ is an isomorphism $t \to t'$, a contradiction. Hence the sets $u_t(\mathrm{Per}_\sigma(L))$, $t \in I$, are exactly the connected components of $\mathcal{M}_N^\sigma$, and by Step 4 each is $\cong \Gamma_N(L)\backslash\mathrm{Per}_\sigma(L)$. The scheme $\mathcal{M}_N \otimes_{F_{\mathrm{ref}},\sigma} \mathbb{C}$ is of finite type over $\mathbb{C}$, so it has finitely many connected components (Zariski topology), and each has connected analytification (SGA 1, Exp. XII, Prop. 2.4); so $\mathcal{M}_N^\sigma$ has finitely many connected components and I is finite. This proves (PU4), and (PU1) is notation. □

Lemma 33.37. *Let E be a CM field, $L \neq 0$ a finitely generated projective $\mathcal{O}_E$-module with a perfect alternating $\mathbb{Z}$-pairing ψ_L such that $\psi_L(ax, y) = \psi_L(x, \bar{a}y)$ for $a \in \mathcal{O}_E$, and let $J_0 \in \mathrm{End}_{E\otimes\mathbb{R}}(L_\mathbb{R})$ satisfy $J_0^2 = -1$, $\psi_L(J_0x, J_0y) = \psi_L(x, y)$ and $\psi_L(J_0x, x) > 0$ for $x \neq 0$. Put $Q_a(T) = \det_\mathbb{C}(T - a \mid (L_\mathbb{R}, J_0))$ for $a \in \mathcal{O}_E$, and let $\mathcal{D}$ be the set of $J \in \mathrm{End}_{E\otimes\mathbb{R}}(L_\mathbb{R})$ with $J^2 = -1$, $\psi_L(Jx, Jy) = \psi_L(x, y)$, $\psi_L(Jx, x) > 0$ for $x \neq 0$, and $\det_\mathbb{C}(T - a \mid (L_\mathbb{R}, J)) = Q_a(T)$ for all $a \in \mathcal{O}_E$, with the topology of a subspace of $\mathrm{End}_\mathbb{R}(L_\mathbb{R})$. Let $G_L(\mathbb{R})$ be the group of $E \otimes \mathbb{R}$-linear automorphisms γ of $L_\mathbb{R}$ with $\psi_L(\gamma x, \gamma y) = \nu(\gamma)\,\psi_L(x, y)$ for some $\nu(\gamma) \in \mathbb{R}^\times$, $\mathrm{U}_L(\mathbb{R}) = \{\gamma \in G_L(\mathbb{R}) : \nu(\gamma) = 1\}$, and $X_L = \{\gamma J_0\gamma^{-1} : \gamma \in G_L(\mathbb{R})\}$. Then:*

(Per1) *$\mathcal{D} = \{\gamma J_0\gamma^{-1} : \gamma \in \mathrm{U}_L(\mathbb{R})\} = \{\gamma J_0\gamma^{-1} : \gamma \in G_L(\mathbb{R}),\ \nu(\gamma) > 0\}$; in particular $\mathcal{D} \subseteq X_L$, and $\mathcal{D}$ is stable under $J \mapsto \gamma J\gamma^{-1}$ for $\gamma \in \mathrm{U}_L(\mathbb{R})$.*

(Per2) *There is a subset $C \subseteq \mathrm{U}_L(\mathbb{R})$, connected in the topology of $\mathrm{End}_\mathbb{R}(L_\mathbb{R})$, with $\mathcal{D} = \{\gamma J_0\gamma^{-1} : \gamma \in C\}$; in particular $\mathcal{D}$ is connected. For every topology on X_L for which the inclusion $X_L \to \mathrm{End}_\mathbb{R}(L_\mathbb{R})$ is continuous, $\mathcal{D}$ is open and closed in X_L; if moreover $\gamma \mapsto \gamma J_0\gamma^{-1}$ is continuous from $G_L(\mathbb{R})$ (topology of $\mathrm{End}_\mathbb{R}(L_\mathbb{R})$) to X_L, then $\mathcal{D}$ is the connected component of X_L containing J_0.*

(Per3) *For $J \in \mathcal{D}$ the subspace $K(J) = \ker(J + i)$ of $L_\mathbb{C} = L \otimes \mathbb{C}$ satisfies $L_\mathbb{C} = K(J) \oplus \overline{K(J)}$,*

and $J \mapsto K(J)$ is a homeomorphism of $\mathcal{D}$ onto its image in the Grassmannian of complex subspaces of $L_{\mathbb{C}}$ of dimension $\frac{1}{2}\dim_{\mathbb{R}} L_{\mathbb{R}}$.

(Per4) *Every subgroup Γ of $\{f \in \mathrm{Aut}_{\mathcal{O}_E}(L) : \psi_L(fx, fy) = \psi_L(x, y)\}$ acts on $\mathcal{D}$ by $J \mapsto fJf^{-1}$, properly discontinuously: for compact $C_1, C_2 \subseteq \mathcal{D}$ only finitely many $f \in \Gamma$ satisfy $fC_1f^{-1} \cap C_2 \neq \emptyset$.*

Proof. **Step 1: the hermitian pieces.** Let w run over the real places of E^+ and let e_w be the idempotents of $E^+ \otimes \mathbb{R} = \prod_w \mathbb{R}$; put $V_w = e_w L_{\mathbb{R}}$, so $L_{\mathbb{R}} = \bigoplus_w V_w$. Complex conjugation fixes e_w, so $\psi_L(e_w x, y) = \psi_L(x, e_w y)$ and the V_w are mutually ψ_L-orthogonal. Fix an embedding $\tau_w \colon E \to \mathbb{C}$ above w; it identifies $E \otimes_{E^+,w} \mathbb{R}$ with $\mathbb{C}$, and we let I_w be the element corresponding to i, so that $\bar{I}_w = -I_w$, I_w is a complex structure on V_w, and $a \in E$ acts on (V_w, I_w) as the scalar $\tau_w(a)$. On V_w put

$$H_w(x, y) = \psi_L(I_w x, y) + i\,\psi_L(x, y).$$

So H_w is a hermitian form on (V_w, I_w), non-degenerate because ψ_L is, with $\operatorname{Im} H_w = \psi_L|_{V_w}$ and $\operatorname{Re} H_w(x, y) = \psi_L(I_w x, y)$. An $E \otimes \mathbb{R}$-linear endomorphism of $L_{\mathbb{R}}$ preserves every V_w and is I_w-linear on it. Hence, by the orthogonality of the V_w, an $E \otimes \mathbb{R}$-linear automorphism preserves ψ_L if and only if it preserves every H_w, and $\mathrm{U}_L(\mathbb{R}) = \prod_w \mathrm{U}(V_w, H_w)$.

Step 2: description of $\mathcal{D}$. Let $J \in \mathrm{End}_{E\otimes\mathbb{R}}(L_{\mathbb{R}})$ with $J^2 = -1$. On V_w, J and I_w commute and square to -1, so $-JI_w$ is an involution and $V_w = V_w^+(J) \oplus V_w^-(J)$ with $V_w^{\pm}(J) = \ker(J \mp I_w)$, I_w-stable subspaces. We claim: $J \in \mathcal{D}$ if and only if for every w

1. H_w is positive definite on $V_w^+(J)$, negative definite on $V_w^-(J)$, and $V_w^+(J) \perp V_w^-(J)$ for H_w;
2. $\dim_{\mathbb{C}} V_w^+(J) = \dim_{\mathbb{C}} V_w^+(J_0)$ and $\dim_{\mathbb{C}} V_w^-(J) = \dim_{\mathbb{C}} V_w^-(J_0)$.

Invariance: by Step 1, $\psi_L(Jx, Jy) = \psi_L(x, y)$ for all x, y if and only if $H_w(Jx, Jy) = H_w(x, y)$ on every V_w. On $V_w^+ \times V_w^+$ and $V_w^- \times V_w^-$ this holds since $H_w(I_w x, I_w y) = H_w(x, y)$; for $x \in V_w^+$, $y \in V_w^-$ it reads $H_w(I_w x, -I_w y) = -H_w(x, y) = H_w(x, y)$. So invariance is equivalent to $V_w^+ \perp V_w^-$ for all w. Positivity: given this orthogonality, for $x = x^+ + x^-$ with $x^{\pm} \in V_w^{\pm}$,

$$\psi_L(Jx, x) = \operatorname{Re} H_w(x^+ - x^-, x^+ + x^-) = H_w(x^+, x^+) - H_w(x^-, x^-),$$

and by the orthogonality of the V_w positivity of $\psi_L(J\cdot, \cdot)$ is equivalent to the definiteness in 1. Determinants: the complex structure of $(L_{\mathbb{R}}, J)$ is I_w on $V_w^+(J)$ and $-I_w$ on $V_w^-(J)$, so $a \in E$ acts there by $\tau_w(a)$, respectively $\overline{\tau_w(a)}$, and

$$\det_{\mathbb{C}}(T - a \mid (L_{\mathbb{R}}, J)) = \prod_w (T - \tau_w(a))^{\dim V_w^+(J)} (T - \overline{\tau_w(a)})^{\dim V_w^-(J)}.$$

Take $a \in \mathcal{O}_E$ with $E = \mathbb{Q}(a)$; the numbers $\tau_w(a), \overline{\tau_w(a)}$ are its images under the $[E : \mathbb{Q}]$ distinct embeddings of E, hence pairwise distinct, so equality with Q_a for this a forces 2, and 2 gives equality for every a. This proves the claim. Applied to J_0: H_w has signature (p_w, q_w) with $p_w = \dim V_w^+(J_0)$, $q_w = \dim V_w^-(J_0)$.

Step 3: (Per1). Let $J \in \mathcal{D}$. By Step 2 and Gram–Schmidt there are I_w-bases of $V_w^+(J)$ and $V_w^-(J)$ in which H_w has Gram matrices 1_{p_w} and -1_{q_w}, and likewise for J_0; together they are bases of V_w with Gram matrix $\mathrm{diag}(1_{p_w}, -1_{q_w})$ (by the orthogonality in 1). The I_w-linear map γ_w carrying the basis for J_0 to the basis for J preserves H_w and carries $V_w^{\pm}(J_0)$ to $V_w^{\pm}(J)$, so $\gamma_w J_0 \gamma_w^{-1} = J$ on V_w. By Step 1, $\gamma = (\gamma_w)_w \in \mathrm{U}_L(\mathbb{R})$, and

$\gamma J_0\gamma^{-1} = J$. Conversely let $\gamma \in G_L(\mathbb{R})$ with $\nu(\gamma) > 0$ and $J = \gamma J_0\gamma^{-1}$. Then J is $E \otimes \mathbb{R}$-linear, $J^2 = -1$, $\psi_L(Jx, Jy) = \nu(\gamma)\psi_L(J_0\gamma^{-1}x, J_0\gamma^{-1}y) = \nu(\gamma)\psi_L(\gamma^{-1}x, \gamma^{-1}y) = \psi_L(x, y)$, $\psi_L(Jx, x) = \nu(\gamma)\psi_L(J_0\gamma^{-1}x, \gamma^{-1}x) > 0$ for $x \neq 0$, and $\gamma\colon (L_{\mathbb{R}}, J_0) \to (L_{\mathbb{R}}, J)$ is $\mathbb{C}$-linear and commutes with E, so the characteristic polynomials agree: $J \in \mathcal{D}$. As $\mathrm{U}_L(\mathbb{R}) \subseteq \{\nu > 0\}$, the three sets coincide, and the last description shows the stability under $\mathrm{U}_L(\mathbb{R})$.

Step 4: (Per2). Write $V_w^{\pm} = V_w^{\pm}(J_0)$, with the positive definite hermitian products $\langle x, y\rangle_+ = H_w(x, y)$ on V_w^+ and $\langle x, y\rangle_- = -H_w(x, y)$ on V_w^-. For $J \in \mathcal{D}$, $W = V_w^+(J)$ is H_w-positive definite of dimension p_w, so $W \cap V_w^- = 0$ and the projection along V_w^- maps W isomorphically onto V_w^+: W is the graph of a unique I_w-linear $Z_w(J)\colon V_w^+ \to V_w^-$, and $H_w(x+Zx, x+Zx) = \langle x, x\rangle_+ - \langle Zx, Zx\rangle_- > 0$ for $x \neq 0$ says that $Z = Z_w(J)$ has operator norm $\|Z\| < 1$. Conversely, let $Z\colon V_w^+ \to V_w^-$ be I_w-linear with $\|Z\| < 1$ and Z^* its adjoint for $\langle\,,\rangle_{\pm}$; put $W_Z^+ = \{x + Zx\}$, $W_Z^- = \{Z^*y + y : y \in V_w^-\}$. Then $H_w(x + Zx, Z^*y + y) = \langle x, Z^*y\rangle_+ - \langle Zx, y\rangle_- = 0$, H_w is positive definite on W_Z^+ and negative definite on W_Z^- ($\|Z^*\| = \|Z\| < 1$), so $W_Z^+ \cap W_Z^- = 0$ and $V_w = W_Z^+ \oplus W_Z^-$ by dimensions. Let J_Z be I_w on W_Z^+ and $-I_w$ on W_Z^-, on every V_w; by Step 2, $J_Z \in \mathcal{D}$. For $J \in \mathcal{D}$, $V_w^-(J)$ is the H_w-orthogonal complement of $V_w^+(J)$, as is $W_{Z_w(J)}^-$; so $J = J_{Z(J)}$, and $Z(J_Z) = Z$. Thus $Z \mapsto J_Z$ maps the convex set $B = \prod_w \{Z : \|Z\| < 1\}$ onto $\mathcal{D}$. In the decomposition $V_w = V_w^+ \oplus V_w^-$,

$$J_Z|_{V_w} = S_Z \begin{pmatrix} I_w & 0 \\ 0 & -I_w \end{pmatrix} S_Z^{-1}, \qquad S_Z = \begin{pmatrix} 1 & Z^* \\ Z & 1 \end{pmatrix},$$

where S_Z is invertible (its columns span the complementary subspaces W_Z^+, W_Z^-) and depends continuously on Z. Put $\gamma_Z = S_Z \operatorname{diag}((1-Z^*Z)^{-1/2}, (1-ZZ^*)^{-1/2})$ on each V_w, the square roots of positive definite operators (as $\|Z\| < 1$). It is I_w-linear, carries $V_w^{\pm}$ onto $W_Z^{\pm}$, and preserves H_w: for $x \in V_w^+$, $H_w(x + Zx, x + Zx) = \langle(1 - Z^*Z)x, x\rangle_+$; for $y \in V_w^-$, $H_w(Z^*y + y, Z^*y + y) = -\langle(1 - ZZ^*)y, y\rangle_-$; and $W_Z^+ \perp W_Z^-$. So $\gamma_Z \in \mathrm{U}_L(\mathbb{R})$ (Step 1), $\gamma_Z J_0 \gamma_Z^{-1} = J_Z$, and $Z \mapsto \gamma_Z$ is continuous. Hence $C = \{\gamma_Z : Z \in B\}$ is connected and $\mathcal{D} = \{\gamma J_0\gamma^{-1} : \gamma \in C\}$; in particular $\mathcal{D}$ is connected. For the second assertion, let $J = \gamma J_0\gamma^{-1} \in X_L$ and $s_J(x, y) = \psi_L(Jx, y)$, a symmetric form ($\psi_L(Jx, y) = \psi_L(J^2x, Jy) = \psi_L(Jy, x)$). As $s_J(x, y) = \nu(\gamma)s_{J_0}(\gamma^{-1}x, \gamma^{-1}y)$, s_J is positive definite if $\nu(\gamma) > 0$ and negative definite if $\nu(\gamma) < 0$. By (Per1), $\mathcal{D} = \{J \in X_L : s_J \text{ positive definite}\}$ and $X_L \setminus \mathcal{D} = \{J \in X_L : s_J \text{ negative definite}\}$. Positive (negative) definiteness of s_J is an open condition on $J \in \mathrm{End}_{\mathbb{R}}(L_{\mathbb{R}})$ (the minimum of $\pm s_J(x, x)$ on the unit sphere of a norm is continuous in J), so both sets are open in X_L for every topology making the inclusion continuous; being complementary, both are closed. If moreover $\gamma \mapsto \gamma J_0\gamma^{-1}$ is continuous on $G_L(\mathbb{R})$, then $\mathcal{D}$, the image of the connected set C, is connected in this topology, contains J_0, and is open and closed: it is the connected component of J_0.

Step 5: (Per3). J is real with $J^2 = -1$, so $L_{\mathbb{C}} = \ker(J + i) \oplus \ker(J - i)$ and $\ker(J - i) = \overline{\ker(J + i)}$; hence $\dim_{\mathbb{C}} K(J) = \frac{1}{2}\dim_{\mathbb{R}} L_{\mathbb{R}}$. $K(J)$ is the image of the idempotent $\pi_J = \frac{1}{2}(1 + iJ)$ of constant rank, which depends continuously on J, so $J \mapsto K(J)$ is continuous. Conversely, on the open subset of the Grassmannian of subspaces P with $P \cap \bar{P} = 0$ the projection π_P onto P along $\bar{P}$ depends continuously on P, and $J = -i(2\pi_{K(J)} - 1)$; so the inverse map is continuous, and $J \mapsto K(J)$ is injective.

Step 6: (Per4). Each $f \in \Gamma$ lies in $\mathrm{U}_L(\mathbb{R})$, so Γ acts on $\mathcal{D}$ by (Per1). For $J \in \mathcal{D}$ the quadratic form $q_J(x) = \psi_L(Jx, x)$ is positive definite, and $q_{fJf^{-1}}(fx) = \psi_L(fJx, fx) = q_J(x)$. Fix a norm $\|\cdot\|$ on $L_{\mathbb{R}}$. The minimum and maximum of q_J on the unit sphere are continuous and positive in J, so there are $0 < c \leq c'$ with $c\|x\|^2 \leq q_J(x) \leq c'\|x\|^2$ for all $J \in C_1 \cup C_2$ and x. If $fJ_1f^{-1} = J_2$ with $J_k \in C_k$, then $c\|fx\|^2 \leq q_{J_2}(fx) = q_{J_1}(x) \leq c'\|x\|^2$, so the operator norm of f is at most $(c'/c)^{1/2}$. In a $\mathbb{Z}$-basis of L such f are integer matrices with bounded entries, hence finitely many. $\square$

Lemma 33.38. *The notation introduced here is local to this result and to the results that apply it. Let E be a CM field with complex conjugation $a \mapsto \bar{a}$, let $p, q \geq 1$, $n = p + q$, let Φ' be a CM type of E and $D = |\Phi'|\, n(n-1)/2$. Let $L_{\mathbb{R}}$ be a free $E \otimes_{\mathbb{Q}} \mathbb{R}$-module of rank n with an alternating $\mathbb{R}$-bilinear non-degenerate form ψ satisfying $\psi(ax, y) = \psi(x, \bar{a}y)$ $(a \in E)$. Put $W = L_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ with the conjugation $v \mapsto \bar{v}$, extend ψ and the E-action $\mathbb{C}$-linearly, let W_ρ $(\rho \in \mathrm{Hom}(E, \mathbb{C}))$ be the subspace on which a acts by $\rho(a)$, and $\langle v, v' \rangle = i\psi(v, \bar{v}')$.*

(Pd1) ***Period domain.*** Per *is the set of E-linear $J \in \mathrm{End}_{\mathbb{R}}(L_{\mathbb{R}})$ with $J^2 = -1$, $\psi(Jx, Jy) = \psi(x, y)$, $\psi(Jx, x) > 0$ for $x \neq 0$, such that for every ρ the subspace of $(L_{\mathbb{R}}, J)$ on which a acts by $\rho(a)$ has complex dimension p if $\rho \in \Phi'$ and q otherwise; we assume* $\mathrm{Per} \neq \varnothing$. *For $J \in$ Per let $K(J) \subseteq W$ be the $(-i)$-eigenspace of J, and $K_\rho(J) = K(J) \cap W_\rho$.* Per *carries the complex structure for which $J \mapsto (K_\rho(J))_{\rho \in \Phi'}$ is an open embedding into $\prod_{\rho \in \Phi'} \mathrm{Gr}(q, W_\rho)$.*

(Pd2) ***Flag bundle and Hodge metrics.*** *For $J \in$* Per *and $\rho \in \Phi'$ the* pieces $\mathrm{Lie}_{(\rho,+)}(J)$ *and* $\mathrm{Lie}_{(\rho,-)}(J)$ *are the subspaces of $(L_{\mathbb{R}}, J)$ on which a acts by $\rho(a)$, respectively $\bar{\rho}(a)$ (dimensions p, q); via $W/K(J) \cong (L_{\mathbb{R}}, J)$, $\lambda \otimes c \mapsto c\lambda$, they are holomorphic vector bundles over* Per. Fl *is the complex manifold of tuples $(J, (V_{s,k})_{s,k})$, $J \in$ Per, $(V_{s,k})_k$ a full flag of $\mathrm{Lie}_s(J)$ for each label $s \in \Phi' \times \{+, -\}$. For a root $r = (\rho, i)$, $\rho \in \Phi'$, $1 \leq i \leq n$, its root line is $\mathcal{L}_r = V_{(\rho,+),i}/V_{(\rho,+),i-1}$ if $i \leq p$ and $\mathcal{L}_r = (V_{(\rho,-),k}/V_{(\rho,-),k-1})^{-1}$ with $k = n + 1 - i$ if $i > p$; it carries the* Hodge metric*: the quotient norm (respectively its dual) of the inner product πH_J on $(L_{\mathbb{R}}, J)$, $H_J(x, y) = \psi(Jx, y) + i\psi(x, y)$. Put $\omega_r = c_1(\mathcal{L}_r, \textit{Hodge metric})$ and $\nu = (-1)^D x^\delta(\omega) = (-1)^D \prod_{\rho \in \Phi'} \prod_{i=1}^{n} \omega_{(\rho,i)}^{n-i}$.*

(Pd3) ***Group.*** *G is the group of E-linear $g \in \mathrm{GL}(L_{\mathbb{R}})$ with $\psi(gx, gy) = \psi(x, y)$ and G^1 its subgroup of elements of determinant 1 over $E \otimes \mathbb{R}$; G acts on* Fl *by $g(J, (V_{s,k})) = (gJg^{-1}, (gV_{s,k}))$.*

Then:

1. *G^1 acts on* Fl *transitively, by biholomorphisms, with compact stabilisers; every ω_r is G-invariant; and ν is a G-invariant positive volume form on* Fl *(for the orientation of the complex structure).*

2. *There is a unique Haar measure μ_ν on G^1 such that $\int_{\mathrm{Fl}} f\, \nu = \int_{G^1} f(g\, y)\, d\mu_\nu(g)$ for every point $y \in$ Fl and every measurable $f \colon \mathrm{Fl} \to [0, \infty]$. For every discrete subgroup $\Gamma \subseteq G^1$ acting freely on* Fl, $\Gamma\backslash\mathrm{Fl}$ *is a complex manifold, ν descends to it, and $\int_{\Gamma\backslash\mathrm{Fl}} \nu = \mu_\nu(\Gamma\backslash G^1)$ in $(0, \infty]$.*

3. *(Transport.) Let $(L'_{\mathbb{R}}, \psi')$ be another such module with an $E \otimes \mathbb{R}$-linear isomorphism $\iota \colon L_{\mathbb{R}} \to L'_{\mathbb{R}}$ and $\epsilon \in \{1, -1\}$ with $\psi'(\iota x, \iota y) = \epsilon\, \psi(x, y)$, and use Φ' if $\epsilon = 1$ and $\overline{\Phi'}$ if $\epsilon = -1$. Then $g \mapsto \iota g \iota^{-1}$ maps G^1 onto G'^1 and μ_ν onto $\mu'_{\nu'}$.*

4. *(The Hodge-type form on* Per*.) Let $\omega_{\mathrm{Per}}^{\mathrm{H}}$ be the first Chern form of the holomorphic line bundle $\bigotimes_{\rho \in \Phi'} (\det \mathrm{Lie}_{(\rho,-)}(J))^{\vee}$ on* Per *with the metric dual to the determinant of the restriction of πH_J, and $e = |\Phi'|\, pq = \dim$ Per. Then $\omega_{\mathrm{Per}}^{\mathrm{H}}$ is G-invariant and positive definite, G^1 acts transitively on* Per *with compact stabilisers, and there is a unique Haar measure μ_{H} on G^1 with $\int_{\mathrm{Per}} f\, (\omega_{\mathrm{Per}}^{\mathrm{H}})^e = \int_{G^1} f(g\, J)\, d\mu_{\mathrm{H}}(g)$ for every $J \in$ Per and every measurable $f \colon \mathrm{Per} \to [0, \infty]$ (G acting by $J \mapsto gJg^{-1}$); for every discrete $\Gamma \subseteq G^1$ acting freely on* Per*, $\int_{\Gamma\backslash\mathrm{Per}} (\omega_{\mathrm{Per}}^{\mathrm{H}})^e = \mu_{\mathrm{H}}(\Gamma\backslash G^1)$.*

Proof. The setting of (Pd1)–(Pd2) is that of Lemma 33.33: Per is the set of Hodge structures of signature (p,q) for Φ' on $(L_{\mathbb{R}},\psi)$, $K(J)$ is the same eigenspace, and the pieces, root lines and Hodge metrics of (Pd2) are those of Lemma 33.33. Write $\perp$ for orthogonality with respect to $\langle\,,\rangle$.

Step 1 (the period domain). As Per $\neq\varnothing$, Lemma 33.33(1) gives: every W_ρ, $\rho\in\Phi'$, has signature (p,q), and $J\mapsto(K_\rho(J))_{\rho\in\Phi'}$ is a bijection of Per onto the set of tuples of negative definite q-dimensional subspaces, an open subset of $\prod_{\rho\in\Phi'}\mathrm{Gr}(q,W_\rho)$; this is the complex structure of (Pd1). Moreover $K(J)=\bigoplus_{\rho\in\Phi'}(K_\rho(J)\oplus\mathrm{ann}_\psi(K_\rho(J)))$ with $\mathrm{ann}_\psi(K_\rho(J))=K(J)\cap W_{\bar\rho}\subseteq W_{\bar\rho}$; the tautological K_ρ and their annihilators under the constant perfect pairing ψ are holomorphic subbundles, so $K(J)$ forms a holomorphic subbundle of $W\times\mathrm{Per}$, and the pieces, images of the constant W_ρ, $W_{\bar\rho}$ in the quotient $W/K(J)\cong(L_{\mathbb{R}},J)$, are holomorphic vector bundles over Per.

Step 2 (the flag domain). Let $\mathcal{D}=\prod_{\rho\in\Phi'}\mathcal{D}_\rho$, $\mathcal{D}_\rho$ the flag domain of $(W_\rho,\langle\,,\rangle)$ with its forms ω_j^ρ (Lemma 33.32). On Fl, the flag bundle of the pieces, the tautological flags are holomorphic subbundles of the pulled-back pieces, and $K(J)$ pulls back to a holomorphic subbundle (Step 1). By Lemma 33.33(2) and (4) with $N=\mathrm{Fl}$, $\Psi\colon\mathrm{Fl}\to\mathcal{D}$ is a holomorphic bijection, and

$$\omega_{(\rho,i)}=\Psi_\rho^*\,\omega^\rho_{\mathrm{pos}(i)}.\tag{$*$}$$

As $\dim\mathrm{Fl}=|\Phi'|(pq+p(p-1)/2+q(q-1)/2)=D=\dim\mathcal{D}$, Ψ is a biholomorphism. For $g\in G$: g commutes with conjugation and the E-action and preserves ψ, hence preserves each W_ρ and $\langle\,,\rangle$; and $\Psi(g\cdot y)=g\cdot\Psi(y)$, since $K(gJg^{-1})=gK(J)$ and $\varphi_{gJg^{-1}}\circ g=g\circ\varphi_J$.

Step 3 (transitivity and invariance). Restriction $g\mapsto(g|_{W_\rho})_{\rho\in\Phi'}$ is an isomorphism $G\to\prod_{\rho\in\Phi'}\mathrm{U}(W_\rho,\langle\,,\rangle)$: g is determined on $W_{\bar\rho}=\overline{W_\rho}$ by reality, and conversely (g_ρ) extended by $g\bar v=\overline{g_\rho v}$ is real, E-linear and preserves $\langle\,,\rangle$, hence $\psi(v,w)=-i\langle v,\bar w\rangle$; and $\det g=1$ if and only if all $\det g_\rho=1$. By Lemma 33.32(2), $\prod_\rho\mathrm{SU}(W_\rho)\cong G^1$ acts transitively on $\mathcal{D}$, hence on Fl (Step 2), by biholomorphisms; the stabiliser of a point is a closed subgroup of a compact torus, so compact; and $g^*\omega_j^\rho=\omega_j^\rho$ for $g\in G$. By $(*)$ and equivariance, $g^*\omega_r=\omega_r$, hence $g^*\nu=\nu$.

Step 4 (positivity of ν). By $(*)$, $\nu=\Psi^*\nu_{\mathcal{D}}$ with $\nu_{\mathcal{D}}=(-1)^D\prod_\rho\prod_i(\omega^\rho_{\mathrm{pos}(i)})^{n-i}=(-1)^D\prod_\rho\prod_{j=1}^n(\omega_j^\rho)^{m_j}$, $m_j=n-\mathrm{pos}^{-1}(j)$; $(m_j)_j$ is a permutation of $(0,\dots,n-1)$. Fix for every ρ an adapted basis with flag o_ρ, coordinates z^ρ_{ab} and covectors η^ρ_{ab} as in Lemma 33.32(3), $\epsilon_a=-1$ $(a\le q)$, $\epsilon_a=1$ $(a>q)$, and write $\alpha_j=\kappa_0^{-1}\omega_j^\rho|_{o_\rho}=\sum_{a>b}\epsilon_a\epsilon_b(\delta_{ja}-\delta_{jb})\eta^\rho_{ab}$ (Lemma 33.32(3)). The η_{ab} commute and square to 0, and $\sum_j m_j=n(n-1)/2$ is the number of pairs $a>b$; expanding, $\prod_j\alpha_j^{m_j}$ is the sum, over the maps f assigning to each pair $a>b$ one of a,b with $|f^{-1}(j)|=m_j$, of $\prod_j m_j!\cdot\prod_{a>b}\epsilon_a\epsilon_b\,(\pm1)\cdot\prod_{a>b}\eta_{ab}$, the sign being $+$ if $f(a,b)=a$ and $-$ if $f(a,b)=b$. The sum of these signs is the coefficient of $\prod_j U_j^{m_j}$ in $\prod_{a>b}(U_a-U_b)=\det(U_j^{k-1})_{j,k}$, i.e. $\mathrm{sgn}(j\mapsto m_j+1)=(-1)^{\#\{j<j'\colon m_j>m_{j'}\}}$. Now $m_j>m_{j'}$ if and only if $\mathrm{pos}^{-1}(j)<\mathrm{pos}^{-1}(j')$, and pos has exactly pq inversions (the pairs $i\le p<i'$), so this number is $n(n-1)/2-pq$. Also $\prod_{a>b}\epsilon_a\epsilon_b=\prod_a\epsilon_a^{n-1}=(-1)^{q(n-1)}$. Hence the factor ρ of $\nu_{\mathcal{D}}|_o$ is

$$(-1)^{n(n-1)/2}\,\kappa_0^{n(n-1)/2}\prod_{k=0}^{n-1}k!\;(-1)^{q(n-1)+n(n-1)/2-pq}\prod_{a>b}\eta_{ab}=\kappa_0^{n(n-1)/2}\prod_{k=0}^{n-1}k!\;\prod_{a>b}\eta_{ab},$$

as $q(n-1)-pq=q(q-1)$ is even. The product over ρ of $\prod_{a>b}\eta^\rho_{ab}$ is 2^D times the Lebesgue volume form of the coordinates, positive for the complex orientation; with $\kappa_0>0$, $\nu_{\mathcal{D}}|_o>0$. By Step 3 every point is $g\,o$ with $g\in G^1$, and g is a biholomorphism preserving $\nu_{\mathcal{D}}$ and the orientation, so $\nu_{\mathcal{D}}>0$ everywhere, and $\nu=\Psi^*\nu_{\mathcal{D}}>0$ (Ψ is a biholomorphism). With Step 3 this proves (1) of the statement.

Step 5 (the measure). Fix $y_0 \in \mathrm{Fl}$ with compact stabiliser S in G^1; $G^1 \cong \prod_\rho \mathrm{SU}(W_\rho)$ is unimodular and $g \mapsto g\,y_0$ identifies G^1/S with Fl (Step 3). The measure $f \mapsto \int f\nu$ is a G^1-invariant Radon measure (Step 4), so it equals the push-forward of a unique Haar measure μ_ν; it is independent of y_0, since $\int f(g\,h y_0)\,d\mu_\nu(g) = \int f(g\,y_0)\,d\mu_\nu(g)$ by right invariance. Let $\Gamma \subseteq G^1$ be discrete and act freely on Fl. It acts properly, so $\Gamma\backslash\mathrm{Fl}$ is a complex manifold and the invariant ν descends. Let $\Omega \subseteq \mathrm{Fl}$ be a measurable fundamental domain for Γ; then $\{g :\ g\,y_0 \in \Omega\}$ is a measurable fundamental domain for Γ acting on G^1 by left multiplication (the action on Fl being free), and

$$\mu_\nu(\Gamma\backslash G^1) = \int_{G^1} \mathbf{1}_\Omega(g\,y_0)\,d\mu_\nu(g) = \int_{\mathrm{Fl}} \mathbf{1}_\Omega\,\nu = \int_{\Gamma\backslash\mathrm{Fl}} \nu,$$

positive as $\nu > 0$. This proves (2).

Step 6 (transport). If $\epsilon = 1$, ι carries ψ, W_ρ, $\langle\,,\rangle$, Per, Fl, the pieces, the root lines and their Hodge metrics to the primed objects (all are defined from $(L_\mathbb{R}, \psi, \Phi')$ alone), compatibly with $g \mapsto \iota g \iota^{-1}$; so ν goes to ν' and μ_ν to $\mu'_{\nu'}$. By this case it suffices to treat $L'_\mathbb{R} = L_\mathbb{R}$, $\iota = \mathrm{id}$, $\psi' = -\psi$, CM type $\overline{\Phi'}$. Then $\langle\,,\rangle' = -\langle\,,\rangle$, the groups G^1 coincide, and conjugation $v \mapsto \bar v$ maps W_ρ onto $W_{\bar\rho}$ and a flag $F_\bullet \in \mathcal{D}_\rho$ to a flag $\overline{F_\bullet}$ of $W_{\bar\rho}$ with $\overline{F_q}$ negative definite for $\langle\,,\rangle'$ ($\langle\bar v,\bar v\rangle' = \langle v, v\rangle$). This gives an antiholomorphic diffeomorphism $\gamma\colon \mathcal{D} \to \mathcal{D}'$ commuting with G^1 (elements of G^1 are real), mapping ℓ_j onto $\overline{\ell_j}$ with $|\langle\bar x,\bar x\rangle'| = |\langle x, x\rangle|$; so the tautological line L'_j pulls back to the conjugate of L_j with the same metric, and $\gamma^*\omega'^{\bar\rho}_j = -\omega^\rho_j$. The roots of the primed data are indexed by $\bar\rho \in \overline{\Phi'}$ with the same pos, so $\gamma^*\nu'_{\mathcal{D}'} = (-1)^D\nu_\mathcal{D}$, while γ changes the complex orientation by $(-1)^D$. Hence γ carries the measure $\nu_\mathcal{D}$ to $\nu'_{\mathcal{D}'}$ and, by Step 5 applied on both sides through Ψ, Ψ', $\mu_\nu = \mu'_{\nu'}$. This proves (3).

Step 7 (the Hodge-type form). Fix J and $\rho \in \Phi'$. By Lemma 33.33(1), $K(J) \cap W_{\bar\rho}$ is the ψ-annihilator of $K_\rho = K_\rho(J)$, so ψ induces a perfect pairing of K_ρ with $\mathrm{Lie}_{(\rho,-)}(J) \cong W_{\bar\rho}/(K(J) \cap W_{\bar\rho})$, holomorphic in J; so $k \mapsto \psi(k,\cdot)$ is a holomorphic isomorphism of the tautological bundle K_ρ onto $\mathrm{Lie}^\vee_{(\rho,-)}$. By Lemma 33.33(1) the Hodge inner product on $\mathrm{Lie}_{(\rho,-)}$ is $2\pi\langle\,,\rangle$ on its lift $P(J) \cap W_{\bar\rho} = \overline{K_\rho}$, and for $y = \bar u$, $u \in K_\rho$, $\psi(k,y) = -i\langle k, u\rangle$ and $\langle y, y\rangle = -\langle u, u\rangle$; as $-\langle\,,\rangle$ is positive definite on K_ρ, Cauchy–Schwarz gives the dual norm $\|\psi(k,\cdot)\|^2 = \sup_u |\langle k,u\rangle|^2/(-2\pi\langle u,u\rangle) = -\langle k,k\rangle/2\pi$. Hence the line bundle of (4) is $\bigotimes_\rho \det K_\rho$ with $\|k_1 \wedge \cdots \wedge k_q\|^2 = |\det(\langle k_a, k_b\rangle)_{a,b}|/(2\pi)^q$. G maps $K_\rho(J)$ onto $K_\rho(gJg^{-1}) = gK_\rho(J)$ preserving $\langle\,,\rangle$, so $\omega^{\mathrm{H}}_{\mathrm{Per}}$ is G-invariant. G^1 is transitive on Per, as on Fl (Step 3), the projection $\mathrm{Fl} \to \mathrm{Per}$ being onto and equivariant; the stabiliser of J preserves the definite spaces $P(J) \cap W_\rho$, K_ρ, so is compact. At J_0 with $K_\rho(J_0) = \mathrm{span}(e_1, \ldots, e_q)$ ((e_j) an adapted basis of a point of $\mathcal{D}_\rho$, Lemma 33.32(2)), $k_b = e_b + \sum_{a>q} z_{ab}e_a$ ($b \le q$) give holomorphic coordinates $(z_{ab})_{a>q\ge b}$ on Per near J_0 and a holomorphic frame, with $|\det(\langle k_a, k_b\rangle)| = \det(1 - Z^*Z)$, $\log\det(1 - Z^*Z) = -\sum_{a>q\ge b}|z_{ab}|^2 + O(|Z|^4)$; so $\omega^{\mathrm{H}}_{\mathrm{Per}}|_{J_0} = \kappa_0 \sum_\rho \sum_{a>q\ge b} i\,dz^\rho_{ab} \wedge d\bar z^\rho_{ab}$, positive definite, and $(\omega^{\mathrm{H}}_{\mathrm{Per}})^e|_{J_0} = e!\,\kappa_0^e \prod_\rho \prod_{a>q\ge b} i\,dz^\rho_{ab} \wedge d\bar z^\rho_{ab} > 0$. By invariance and transitivity these hold everywhere. The measure statements follow as in Step 5, with Per for Fl and $(\omega^{\mathrm{H}}_{\mathrm{Per}})^e$ for ν. This proves (4). □

33.4 Effective weight sections and stage root models at level four

The last step of the construction passes from the degree-one reference model to every stage. Bigness of the gap line gives the effective weight section of the reference model (Lemma 33.39), and Proposition 33.40 constructs, for every finite set of primes, a stage root model at level four over the reference root origin, by resolution and principalisation [148] and the pull-back of Hodge bundles along dominant maps of covers. With them Proposition 33.2, and hence Proposition 33.1, is proved.

Lemma 33.39. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p, q \geq 1$, $\Phi = \Phi_{\mathsf{H}}$, $n = p+q$, $\nu(\tau, i) = 1-i$ ($\tau \in \Phi$, $1 \leq i \leq n$), $\mathcal{Y}$ an integral model of* Sh *over $\mathcal{O}_F$ and $\mathcal{Y}_0$ an irreducible component of $\mathcal{Y}$; read (Go1)–(Go4) with $\mathcal{Y}_0$ in place of c, ν in place of $\nu_{\mathcal{T}}$, $\kappa = p$ and $S = \varnothing$. Let fixed data at $\mathcal{Y}_0$ and a root model $\mathcal{X} \subseteq \mathbb{P} = \prod_{k \in K} \mathbb{P}(\mathbb{Z}^{I_k})$ of type (Φ, n) be given, with data of a root origin of $\mathcal{X}$ on these fixed data with* $\deg = 1$ *satisfying (Go3)(1)–(5) and (Go4). Put $e_\nu = -\sum_{r \in \mathrm{Rt}} \nu(r) e_r \in \mathbb{Z}^K$ and assume that for some irreducible component X_1 of the complex variety $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{C}$ the line bundle $\mathcal{O}(e_\nu)|_{X_1}$ is nef and $\deg(c_1(\mathcal{O}(e_\nu)|_{X_1})^D \cap [X_1]) > 0$. Then there are an integer $k \geq 1$ and $s \in \Omega_c^\times$ such that, after replacing the entries k_c^A, q_c', n_c, s_c^E of the fixed data by $k, 1, k\nu, s$ and the normalising integer $k_{\mathcal{X}}$ by k (all other data unchanged), the data are fixed data at $\mathcal{Y}_0$ and a root origin of the modified root model on them: (Go3)(1)–(6) and (Go4) hold.*

Proof. Write Ω for the function field of $\mathcal{X}$, $g\colon \Omega_c \to \Omega$ for the field embedding of (Go3)(1) (an isomorphism, as $\deg = 1$), $X = \mathcal{X} \otimes \mathbb{Q}$ (integral, normal, projective over $\mathbb{Q}$, of dimension D, since $\mathcal{X}$ is integral, normal, flat over $\mathbb{Z}$ of absolute dimension $D+1$ and closed in $\mathbb{P}$), and $y_{c'} = \prod_k y_{k,c_k'}$ for $c' \in \prod_k I_k$.

Step 0: only (Go2) and (Go3)(5), (6) involve the replaced entries. The entries k_c^A, q_c', n_c, s_c^E of fixed data are constrained only by $k_c^A, q_c' \geq 1$, $n_c(r) = k_c^A q_c' \nu(r)$ and $s_c^E \in \Omega_c^\times$ (Go2); $k_{\mathcal{X}}$ is a free integer ≥ 1 of a root model Definition 29.4; among the clauses of (Go3), (Go4), only (5) ($k_{\mathcal{X}} = k_c^A$) and (6) mention them. With $k_c^A = k$, $q_c' = 1$, $n_c = k\nu$, $k_{\mathcal{X}} = k$ and $s \neq 0$, (Go2) and (5) hold; it remains to choose k and s so that (6) holds.

Step 1: sections as elements of Ω. For $d \in \mathbb{Z}^K$ and a point x of X in a chart $\{Y_{c'} \neq 0\}$, $y_{c'} \neq 0$ and the ratios $y_{k,j}/y_{k,c_k'}$ are regular at x (representing coordinates, (Go3)); the stalk $\mathcal{O}(d)_x$ is identified with $y_{c'}^d \mathcal{O}_x \subseteq \Omega$, where $y_{c'}^d = \prod_k y_{k,c_k'}^{d_k}$, by $t Y_{c'}^d \mapsto t\, y_{c'}^d$; on overlaps the identifications agree because $Y_{c'}^d / Y_{c''}^d$ is the function $y_{c'}^d / y_{c''}^d$. So $H^0(X, \mathcal{O}(d))$ embeds into Ω as the set of a with $a / y_{c'}^d \in \mathcal{O}_x$ for every $x \in X$ and every chart c' containing x, and a non-zero section gives a non-zero element (X is integral). By (Go3)(3), through the root sections the stalk at x of $\bigotimes_r \mathcal{L}_r^{m_r}$ ($m \in \mathbb{Z}^{\mathrm{Rt}}$), read inside Ω as (Go3)(6) reads it, is $\prod_r \varrho_r^{m_r} \cdot y_{c'}^{\sum_r m_r e_r} \mathcal{O}_x$. For $m = -k\nu$ this is $E_0^k y_{c'}^{k e_\nu} \mathcal{O}_x$ with $E_0 = \prod_r \varrho_r^{-\nu(r)} \in \Omega^\times$; so every local generator θ of $\bigotimes_r \mathcal{L}_r^{-n_c(r)}$ at x is $\theta = u\, E_0^k y_{c'}^{k e_\nu}$ with $u \in \mathcal{O}_x^\times$.

Step 2: a form vanishing off the abelian locus. Let $Z \subseteq X$ be the complement of the abelian locus. The abelian locus is open ($j(\mathrm{Sh}_F)$ is open in $\mathcal{Y}$) and contains the generic point of X (it maps to the generic point of $\mathcal{Y}_0$, which lies in the open set $j(\mathrm{Sh}_F)$ because $\mathcal{Y}_0$ meets it, (Go2), (Go3)(1)); so Z is closed and $Z \neq X$. Let J be the ideal of multihomogeneous polynomials over $\mathbb{Q}$ vanishing on the closure of Z in $\mathbb{P}_{\mathbb{Q}}$ and $\mathfrak{p}_{\mathbb{Q}}$ the prime ideal of X; if every multihomogeneous element of J were in $\mathfrak{p}_{\mathbb{Q}}$, then $X = V_+(\mathfrak{p}_{\mathbb{Q}}) \subseteq V_+(J) = \bar{Z}$, which is false. Clearing denominators, there is a multihomogeneous $b \in \mathbb{Z}[Y_{k,j}]$ of some multidegree $\delta \in \mathbb{Z}_{\geq 0}^K$ with $b(y) \neq 0$ and b vanishing on Z. At a point x of X in a chart c', $b(y)/y_{c'}^\delta$ is a polynomial in the ratios $y_{k,j}/y_{k,c_k'}$, hence regular at x, and it lies in $\mathfrak{m}_x$ if $x \in Z$.

Step 3: Kodaira's lemma. Fix c^0 with $y_{c^0} \neq 0$; $Q_0 = \prod_k Y_{k,c_k^0}^{(\delta + e_*)_k}$ is a section of $\mathcal{O}(\delta + e_*)$ not vanishing at the generic point of X. The algebraic closure F' of $\mathbb{Q}$ in Ω lies in $H^0(X, \mathcal{O}_X)$ (X normal), so X is an F'-scheme, proper over F', with F' algebraically closed in its function field; as the characteristic is 0, X is geometrically integral over F', and $X \otimes_{\mathbb{Q}} \mathbb{C} = X \times_{F'} (F' \otimes_{\mathbb{Q}} \mathbb{C}) = \bigsqcup_{\sigma\colon F' \to \mathbb{C}} X \times_{F', \sigma} \mathbb{C}$ is the disjoint union of its irreducible components, each an irreducible complex projective variety of dimension D whose generic point lies over that of X. So X_1 is one of them, it is open and closed in $X \otimes \mathbb{C}$, and $Q_0|_{X_1}$ is a non-zero section of $\mathcal{O}(\delta + e_*)|_{X_1}$ whose zero scheme P_1 is an effective Cartier divisor with $\mathcal{O}_{X_1}(P_1) \cong \mathcal{O}(\delta + e_*)|_{X_1}$. By Theorem B.148(2) the

nef line bundle $\mathcal{O}(e_\nu)|_{X_1}$ of positive top degree is big, and by Theorem B.148(3) there is $k \geq 1$ with $H^0(X_1, \mathcal{O}(ke_\nu - \delta - e_*)|_{X_1}) \neq 0$. Extending a non-zero section by 0 to the other components and using flat base change, $H^0(X, \mathcal{O}(ke_\nu - \delta - e_*)) \otimes_{\mathbb{Q}} \mathbb{C} = H^0(X \otimes \mathbb{C}, \mathcal{O}(ke_\nu - \delta - e_*)) \neq 0$. By Step 1 there is $0 \neq \sigma \in \Omega$ with $\sigma / y_{c'}^{ke_\nu - \delta - e_*} \in \mathcal{O}_x$ for every $x \in X$ and every chart c' containing x.

Step 4: the section. Put $\tilde{s} = E_0^k \, \sigma \, b(y) \in \Omega^\times$ and $s = g^{-1}(\tilde{s}) \in \Omega_c^\times$. Let $x \in X$ lie in the chart $\{Y_{k_0,i} \neq 0\}$ of the factor k_0 and let θ be a local generator of $\bigotimes_r \mathcal{L}_r^{-k\nu(r)}$ at x. Choose a chart c' containing x with $c'_{k_0} = i$ (possible since the charts are products of charts of the factors). By (Go3)(5), $g(y_i^c) = y_{k_0,i} = y_{c'}^{e_*}$, so by Step 1

$$\frac{g(s)\, g(y_i^c)}{\theta} = u^{-1} \cdot \frac{\sigma}{y_{c'}^{ke_\nu - \delta - e_*}} \cdot \frac{b(y)}{y_{c'}^{\delta}} \cdot \frac{y_{k_0,i}}{y_{c'}^{e_*}} = u^{-1} \cdot \frac{\sigma}{y_{c'}^{ke_\nu - \delta - e_*}} \cdot \frac{b(y)}{y_{c'}^{\delta}},$$

a product of elements of $\mathcal{O}_x$ (Steps 1–3), hence regular at x; if x is not in the abelian locus the factor $b(y)/y_{c'}^{\delta}$ lies in $\mathfrak{m}_x$ (Step 2), so the product is not a unit. This is (Go3)(6) with $q'_c = 1$ and $n_c = k\nu$. □

Proposition 33.40. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p, q \geq 1$, put $\Phi = \Phi_{\mathsf{H}}$, $n = p + q$ and $\nu(\tau, i) = 1 - i$, and let $\mathcal{Y}$ be an integral model of* Sh *over $\mathcal{O}_F$ with uniformised degenerations whose components split. Read (Go1)–(Go4) with an irreducible component $\mathcal{Y}_0$ of $\mathcal{Y}$ in place of c, ν in place of $\nu_{\mathcal{T}}$, $\kappa = p$, and a finite set S of primes in place of S_B. Let fixed data at $\mathcal{Y}_0$ and a root model $\mathcal{X}^{\mathrm{ref}}$ of type (Φ, n) with a root origin on them for $S = \varnothing$ be given, with* deg $= 1$, *$\mathcal{X}^{\mathrm{ref}} \otimes \mathbb{Q}$ smooth and bijective realisation $\mathcal{X}^{\mathrm{ref}}(\mathbb{C}) \to X_c^{\mathrm{an}}$. Then for every finite set S of primes containing every prime dividing* disc(E) *there is a root model $\mathcal{X}_S$ of type (Φ, n) with a root origin on the same fixed data for S, with $R(\ell) = 4$ for every $\ell \in S$, whose generic fibre $\mathcal{X}_S \otimes \mathbb{Q}$ is smooth (so its realisation needs no resolution).*

Proof. Write $\mathcal{X}^{\mathrm{ref}} \subseteq \mathbb{P}^{\mathrm{ref}} = \prod_{k \in K^{\mathrm{ref}}} \mathbb{P}(\mathbb{Z}^{I_k})$, with function field Ω^{ref}, representing coordinates $y_{k,j}^{\mathrm{ref}}$, field embedding $g^{\mathrm{ref}} \colon \Omega_c \to \Omega^{\mathrm{ref}}$ (an isomorphism, since deg $= 1$), root sections ϱ_r^{ref}, cone functions $h_r^{\mathrm{sing,ref}}$, index k_0, structure map f^{ref}, realisation π^{ref}, and metrics $h_r^{\mathrm{ref}}(t)$, h_a^{ref}, h_*. Put $P_S = \prod_{\ell \in S} \ell$. Neither the uniformised degenerations of $\mathcal{Y}$ nor the splitting of its components is used below.

Step 1: the field and a first normal model. The generic point of $\mathcal{X}^{\mathrm{ref}}$ lies in the abelian locus, so $G_{\Omega^{\mathrm{ref}}}$ is an abelian variety over a field of characteristic 0 and its ℓ^4-torsion ($\ell \in S$) is finite étale. Let Ω_S be the finite Galois extension of Ω^{ref} generated by the coordinates of all its ℓ^4-torsion points over an algebraic closure, $\ell \in S$, and $\deg_S = [\Omega_S : \Omega^{\mathrm{ref}}]$. Let $\mathcal{X}_1$ be the normalisation of $\mathcal{X}^{\mathrm{ref}}$ in Ω_S; it is finite over $\mathcal{X}^{\mathrm{ref}}$ ($\mathcal{X}^{\mathrm{ref}}$ is of finite type over $\mathbb{Z}$, hence excellent), so integral, normal and projective over $\mathbb{Z}$.

Step 2: smooth generic fibre. The scheme $\mathcal{X}_{1,\mathbb{Q}}$ is a quasi-projective variety over $\mathbb{Q}$. By Theorem B.142(A) there is a projective birational morphism $R \to \mathcal{X}_{1,\mathbb{Q}}$ with R smooth and integral; so R is a closed integral subscheme of $\mathbb{P}^N \times_{\mathbb{Z}} \mathcal{X}_{1,\mathbb{Q}}$ for some N, the generic fibre of $\mathbb{P}^N_{\mathbb{Z}} \times_{\mathbb{Z}} \mathcal{X}_1$. Let $\mathcal{X}_2$ be the closure of R in $\mathbb{P}^N_{\mathbb{Z}} \times_{\mathbb{Z}} \mathcal{X}_1$, an integral scheme, projective over $\mathbb{Z}$ and birational and projective over $\mathcal{X}_1$, with generic fibre R (R is closed in the generic fibre), and let $\mathcal{X}_3$ be its normalisation, finite over $\mathcal{X}_2$ (excellence). Normalisation commutes with localisation and R is smooth, hence normal, so $\mathcal{X}_{3,\mathbb{Q}} \cong R$. The scheme $\mathcal{X}_3$ is integral, normal, projective over $\mathbb{Z}$, with function field Ω_S and a morphism $\phi_3 \colon \mathcal{X}_3 \to \mathcal{X}^{\mathrm{ref}}$; with $f_3 = f^{\mathrm{ref}} \circ \phi_3$ it is a cover of $\mathcal{Y}$, and every ℓ^4-torsion point of $G_{\mathcal{X}_3}$ ($\ell \in S$) is defined over Ω_S.

Step 3: Hodge bundles at (S, R), $R \equiv 4$. The scheme $\mathcal{X}_3$ is a cover of $\mathcal{Y}$, projective over $\mathbb{Z}$, satisfying (a) of Lemma 33.10 for $R \equiv 4$ (Step 2), and S contains every prime dividing disc(E).

By Lemma 33.10 there are a cover $\mathcal{X}_5$ of $\mathcal{Y}$, projective over $\mathbb{Z}$, and a projective birational $b\colon \mathcal{X}_5 \to \mathcal{X}_3$, an isomorphism over $\mathcal{X}_3[1/P_S]$ (so $\mathcal{X}_{5,\mathbb{Q}} \cong R$ is smooth), with structure map $f_5 = f_3 \circ b$, satisfying (a), (b), (c) of Lemma 33.10: its Hodge lattices at (S, R) are Hodge bundles $\mathcal{V}_s$. By Lemma 31.29(1),(3) (its hypotheses (Hp1), (Hp2) are (b) and the second half of (c), with the integers $R(\ell)$ of (a)), every noetherian cover $\mathcal{X}'$ with a dominant morphism $\mathrm{pr}\colon \mathcal{X}' \to \mathcal{X}_5$ compatible with the structure maps again satisfies (a), (b), (c), with Hodge bundles $\mathcal{O}_{\mathcal{X}'}\mathcal{V}_s$. Off S the Hodge bundles of $\mathcal{X}_5$ are those of the reference model: let x' be a point of $U_5 = \mathcal{X}_5[1/P_S]$ and x its image in $\mathcal{X}^{\mathrm{ref}}$. Every prime p with $p \notin \mathcal{O}_{x'}^{\times}$ is outside S, so $p \nmid \mathrm{disc}(E)$ (this is where the hypothesis on S is used), and $\mathcal{O}_E \otimes \mathcal{O}_{x'} \to \prod_{\sigma \in \mathrm{Hom}(E,F)} \mathcal{O}_{x'}$, $a \otimes b \mapsto (\sigma(a)b)_\sigma$, is an isomorphism (its discriminant $\mathrm{disc}(E)$ is a unit; $\sigma(\mathcal{O}_E) \subseteq \mathcal{O}_F \subseteq \mathcal{O}_{x'}$), whose idempotents ϵ_s act on $\mathrm{Lie}_{x'}$ and on $\mathcal{O}_{x'}\mathrm{Lie}_x$ (the action is integral). As $G_{\mathcal{X}_5}$ and $G_{\mathcal{X}^{\mathrm{ref}}}$ are pull-backs of $G_{\mathcal{Y}}$, $\mathrm{Lie}_{x'} = \mathcal{O}_{x'}\mathrm{Lie}_x$, hence $\mathrm{dR}_{s,x'} = \epsilon_s \mathrm{Lie}_{x'} = \mathcal{O}_{x'}\epsilon_s \mathrm{Lie}_x \subseteq \mathcal{O}_{x'}\mathrm{dR}_{s,x} \subseteq \mathrm{Lie}_{x'} \cap \mathrm{Li}_s = \mathrm{dR}_{s,x'}$; and $\mathrm{dR}_{s,x}$ is the stalk of the Hodge bundle $\mathcal{V}_s^{\mathrm{ref}}$ of $\mathcal{X}^{\mathrm{ref}}$ (Hodge lattices at $S = \varnothing$ are the de Rham lattices). So $\mathcal{V}_s|_{U_5} = \mathcal{O}_{U_5}\mathcal{V}_s^{\mathrm{ref}}$, the pull-back of $\mathcal{V}_s^{\mathrm{ref}}$.

Step 4: integral flags. Let $(V_{s,i})_i$ be the image under g^{ref} of the universal flag of Li_s, extended to Ω_S. On U_5, $\mathcal{V}_s$ is the pull-back of $\mathcal{V}_s^{\mathrm{ref}}$ (Step 3), which carries the integral flag of the reference origin with generic fibre $(V_{s,i})$; its pull-back is a flag with invertible quotients, and by Lemma 31.33(3) it equals $(\mathcal{V}_s \cap V_{s,i})_i$. Apply Lemma 31.33(2) successively to the finitely many $\mathcal{V}_s$ with $U = U_5$, then normalise. The result $\mathcal{X}_S$ is a normal cover, projective over $\mathbb{Z}$, isomorphic to $\mathcal{X}_3$ over $\mathcal{X}_3[1/P_S]$ (so $\mathcal{X}_S \otimes \mathbb{Q} \cong R$ is smooth), with a morphism $\phi\colon \mathcal{X}_S \to \mathcal{X}^{\mathrm{ref}}$ and $f = f^{\mathrm{ref}} \circ \phi$; by Step 3 it satisfies (a), (b), (c) of Lemma 33.10 and its Hodge bundles are the pull-backs of those of $\mathcal{X}_5$, and by Lemma 31.33(2),(3) the subsheaves $\mathcal{V}_s \cap V_{s,i}$ are locally free with invertible quotients: an integral flag whose generic fibre is g(universal flag), where $g\colon \Omega_c \to \Omega_S$ is g^{ref} followed by the inclusion; $[\Omega_S : g(\Omega_c)] = \deg_S$. On the generic fibre the root lines of $\mathcal{X}_S$ and ϕ^* of those of $\mathcal{X}^{\mathrm{ref}}$ are both formed from flags with invertible quotients of the same locally free sheaf $\phi^*\mathcal{V}_s^{\mathrm{ref}}$ with the same generic fibre; such flags coincide. Indeed, let N be a locally free sheaf on the integral scheme $\mathcal{X}_{S,\mathbb{Q}}$ and $F \subseteq N$ a subsheaf with N/F locally free (an iterated extension of invertible quotients) and generic fibre $F \otimes \Omega_S = V_{s,i}$; then $F = N \cap V_{s,i}$: clearly $F \subseteq N \cap V_{s,i}$, and a local section x of $N \cap V_{s,i}$ satisfies $ax \in F$ for some non-zero local function a, so its image in the torsion-free sheaf N/F is 0 and $x \in F$. Applied to both flags of $N = \phi^*\mathcal{V}_s^{\mathrm{ref}}$, the two flags agree, hence so do the root lines formed from them:

$$\mathcal{L}_r|_{\mathcal{X}_{S,\mathbb{Q}}} = \phi^*\mathcal{L}_r^{\mathrm{ref}}|_{\mathcal{X}_{\mathbb{Q}}^{\mathrm{ref}}} \quad \text{inside } \Omega_S\lambda_r \qquad (r \in \mathrm{Rt}). \tag{$*$}$$

Step 5: the multiprojective embedding and the root sections. Apply Lemma 33.11 to the integral scheme $\mathcal{X}_S$, projective over $\mathbb{Z}$, with $K_0 = K^{\mathrm{ref}}$, ψ_k the composite of ϕ with the projection of $\mathcal{X}^{\mathrm{ref}} \subseteq \mathbb{P}^{\mathrm{ref}}$ to the factor k, $y_{k,j} = \phi^* y_{k,j}^{\mathrm{ref}}$ (representing coordinates of ψ_k, as the $y_{k,j}^{\mathrm{ref}}$ are those of the factor k of $\mathcal{X}^{\mathrm{ref}}$), $\mathrm{R} = \mathrm{Rt}$, $\Lambda_r = \Omega_S\lambda_r$ and the invertible root lines $\mathcal{L}_r \subseteq \Omega_S\lambda_r$ of Step 4. It gives $K \supseteq K^{\mathrm{ref}}$, representing coordinates $y_{k,j}$ $(k \in K)$ of a closed immersion $\mathcal{X}_S \to \mathbb{P} = \prod_{k \in K} \mathbb{P}(\mathbb{Z}^{I_k})$ whose components at K^{ref} are the ψ_k, and multidegrees e_r such that, with $\varrho_r = 1$, $P \mapsto \varrho_r P(y)\lambda_r$ is an isomorphism $\mathcal{O}(e_r)|_{\mathcal{X}_S} \to \mathcal{L}_r$ and $\varrho_r \prod_k y_{k,c_k'}^{e_{r,k}} \lambda_r$ generates $\mathcal{L}_r$ on every chart: (Go3)(3). We identify $\mathcal{X}_S$ with its image, a closed integral subscheme flat over $\mathbb{Z}$ of absolute dimension $D+1$ (Ω_S is finite over Ω^{ref}), and put $e_* = \mathbf{1}_{k_0}$. The coordinates of the factor k_0 are $\phi^* g^{\mathrm{ref}}(y_i^c) = g(y_i^c)$; put $h_*(\tilde{z}) = h_c((\tilde{z}_{k_0,i})_i)$ and $k_{\mathcal{X}_S} = k_c^A$, $\deg_{S,c} = \deg_S$, $\delta_{\mathcal{X}_S} = [F : \mathbb{Q}] \deg_S$, $R(\ell) = 4$: (Go3)(5), and with Steps 1, 3, 4, (Go3)(1), (2).

Step 6: metrics, singular cone functions and the realisation. The space $\mathcal{X}_S(\mathbb{C})$ is a closed complex submanifold of $\mathbb{P}(\mathbb{C})$, stable under complex conjugation. *Extension of metrics*: if $\|\cdot\|$ is a smooth conjugation-invariant metric on $\mathcal{O}(e)|_{\mathcal{X}_S(\mathbb{C})}$, fix the cone metric $h^0(z) = \prod_k (\sum_j |z_{k,j}|^2)^{e_k/2}$, write $\|\cdot\| = e^{\psi}\|\cdot\|_{h^0}$ with ψ smooth on $\mathcal{X}_S(\mathbb{C})$, extend ψ to a smooth

function $\tilde\psi$ on $\mathbb{P}(\mathbb{C})$ (tubular neighbourhood and partition of unity), replace $\tilde\psi$ by $\frac12(\tilde\psi+\tilde\psi\circ\mathrm{conj})$ (still an extension, ψ being conjugation-invariant), and put $h = h^0 e^{-\tilde\psi\circ[\cdot]}$: a cone metric of multidegree e inducing $\|\cdot\|$. By $(*)$, over $\mathbb{C}$ the root lines and fundamental lines of $\mathcal{X}_S$ are the pull-backs by ϕ of those of $\mathcal{X}^{\mathrm{ref}}$; transport the metrics of $\mathcal{X}^{\mathrm{ref}}$ (given by $h_r^{\mathrm{ref}}(t)$, $t>0$, and h_a^{ref} through the reference root sections) by ϕ and through the root sections of Step 5 to $\mathcal{O}(e_r)|_{\mathcal{X}_S(\mathbb{C})}$, resp. $\mathcal{O}(e_a)$: smooth (pull-backs by a holomorphic map) and conjugation-invariant (ϕ and the root sections are defined over $\mathbb{Q}$). Let $h_r(t)$, h_a^{ref} be cone metrics inducing them. This makes $\mathcal{X}_S$ a root model of type (Φ,n). At a complex point z of the abelian locus, $G_{\mathcal{X}_S,z} = G^{\mathrm{un}}_{f(z)} = G_{\mathcal{X}^{\mathrm{ref}},\phi(z)}$ with the same induced flags, so the Hodge metric of $\mathcal{L}_r$ at z is that of $\mathcal{L}_r^{\mathrm{ref}}$ at $\phi(z)$; define $h_r^{\mathrm{sing}}(\tilde z) = |P(\tilde z)|/\|\varrho_r P(y)\lambda_r\|^{\mathrm{Hdg}}(z)$ for any local section P of $\mathcal{O}(e_r)$ with $P(\tilde z)\neq 0$ (independent of P, homogeneous of multidegree e_r) at the cone points over the abelian locus, and $h_r^{\mathrm{sing}} = h^0$ elsewhere: (Go3)(4). Put $\pi = \pi^{\mathrm{ref}}\circ\phi_{\mathbb{C}}$, holomorphic on the manifold $\mathcal{X}_S(\mathbb{C})$ (no resolution). *The degree of π.* First, π^{ref} is a biholomorphism: it is a bijective holomorphic map from the complex manifold $\mathcal{X}^{\mathrm{ref}}(\mathbb{C})$ ($\mathcal{X}^{\mathrm{ref}}\otimes\mathbb{Q}$ is smooth of dimension D) onto the complex manifold X_c^{an} of dimension D (Go1), and an injective holomorphic map between complex manifolds of the same dimension is a biholomorphism onto an open subset. Next, $\phi_{\mathbb{Q}}\colon \mathcal{X}_{S,\mathbb{Q}}\to\mathcal{X}^{\mathrm{ref}}_{\mathbb{Q}}$ is a dominant morphism of integral $\mathbb{Q}$-varieties of dimension D inducing the extension $\Omega_S/\Omega^{\mathrm{ref}}$ of degree $\deg_S$, separable (characteristic 0). By generic finiteness (Hartshorne II Ex. 3.7) and generic smoothness (Hartshorne III.10.7, $\mathcal{X}_{S,\mathbb{Q}}$ being smooth) there is a dense open $V\subseteq\mathcal{X}^{\mathrm{ref}}_{\mathbb{Q}}$ over which $\phi_{\mathbb{Q}}$ is finite and étale; as V is irreducible, $\phi_*\mathcal{O}$ is locally free of constant rank $\deg_S$ on V, so every fibre of $\phi_{\mathbb{C}}$ over a point of $V(\mathbb{C})$ has exactly $\deg_S$ points, and $\phi_{\mathbb{C}}$ is a local biholomorphism at each of them (étale). Let $B = \mathcal{X}^{\mathrm{ref}}_{\mathbb{Q}}\smallsetminus V$ and $N' = \pi^{\mathrm{ref}}(B(\mathbb{C}))$. As B is a proper closed subset of the integral D-dimensional $\mathcal{X}^{\mathrm{ref}}_{\mathbb{Q}}$, $\dim B < D$, while every irreducible component of $\mathcal{X}^{\mathrm{ref}}_{\mathbb{C}}$ has dimension D; the connected components of the manifold $\mathcal{X}^{\mathrm{ref}}(\mathbb{C})$ are its irreducible components, and a proper closed analytic subset of a connected complex manifold has empty interior; so $B(\mathbb{C})$ is a closed analytic subset with empty interior, and so is its image N' under the biholomorphism π^{ref}. Likewise $\pi^{-1}(N') = \phi_{\mathbb{C}}^{-1}(B(\mathbb{C})) = (\phi_{\mathbb{Q}}^{-1}(B))(\mathbb{C})$, and $\phi_{\mathbb{Q}}^{-1}(B)$ is a proper closed subset of the integral D-dimensional $\mathcal{X}_{S,\mathbb{Q}}$ (its complement $\phi_{\mathbb{Q}}^{-1}(V)$ is non-empty, $\phi_{\mathbb{Q}}$ being dominant); so $\pi^{-1}(N')$ is a closed analytic subset of $\mathcal{X}_S(\mathbb{C})$ with empty interior. A point $w\in X_c^{\mathrm{an}}\smallsetminus N'$ is $\pi^{\mathrm{ref}}(z_0)$ for a unique $z_0\in V(\mathbb{C})$, so $\pi^{-1}(w) = \phi_{\mathbb{C}}^{-1}(z_0)$ has exactly $\deg_S$ points; and at every $z\notin\pi^{-1}(N')$ the differential $d\pi_z = d\pi^{\mathrm{ref}}_{\phi(z)}\circ d(\phi_{\mathbb{C}})_z$ is invertible. So π is a realisation of degree $\deg_S = \deg_{S,c}$ in the sense of (Go4). The k_0-coordinates of z are those of $\phi(z)$, proportional to $\mathrm{pt}_c(\pi(z))$. A point z lies in the abelian locus $f^{-1}(j(\mathrm{Sh}_F))$ if and only if $\phi(z)$ does, and $\phi(z)$ does if and only if $\pi(z)\in X_c^\circ$. The metrics h_a^{ref} are the pull-backs by π of those of Sys_c, $h_*^{\mathrm{ref}} = h_*$, the regularised fundamental metrics are $\prod_{k\le i} h_{(\tau,k)}(t)^{-1}$, and on $\pi^{-1}(X_c^\circ)$ the singular weight is the Hodge metric, because all these identities hold on $\mathcal{X}^{\mathrm{ref}}$ and are transported by ϕ: (Go4).

Step 7: the effective weight section (Go3)(6). Let $x\in\mathcal{X}_{S,\mathbb{Q}}$ lie in the chart $\{Y_{k_0,i}\neq 0\}$; then $\phi(x)$ lies in the same chart of $\mathcal{X}^{\mathrm{ref}}$ (same k_0-coordinates). By $(*)$ a local generator θ of $\bigotimes_r \mathcal{L}_r^{-n_c(r)}$ at x is a unit times $\phi^*\theta'$ with θ' a local generator at $\phi(x)$. By (Go3)(6) for $\mathcal{X}^{\mathrm{ref}}$, $g^{\mathrm{ref}}(s_c^E)g^{\mathrm{ref}}(y_i^c)^{q_c'}/\theta'$ is regular at $\phi(x)$, so its pull-back $g(s_c^E)g(y_i^c)^{q_c'}/\phi^*\theta'$ is regular at x, and so is the quotient by θ. If x is not in the abelian locus, neither is $\phi(x)$, the element lies in the maximal ideal at $\phi(x)$, and its pull-back lies in the maximal ideal at x (ϕ induces a local homomorphism). This completes the root origin of $\mathcal{X}_S$ on the fixed data at $\mathcal{Y}_0$ for S, with $R\equiv 4$ and smooth generic fibre. □

34 The facts of good towers

This section proves Proposition 34.2, the source of most of the properties in Proposition 31.14. Given an integral model with uniformised degenerations and a root tower with a good origin over it, the proposition replaces the numerical tower by one whose arithmetic degrees are the limits of the regularised arithmetic intersection numbers of the root models, and proves for it: the compact-dual height is the Fubini–Study height (Gt1); geometric proportionality with covolumes and (Gi1) (Gt2); Galois Frobenius labels (Gt3); proportional analytic square systems, and Lie fibres (Lx2) and flat determinants (Lx3) of the reference models (Gt4); and the quantitative properties (Gt5), flat determinant classes at the base stage (Db) and Hodge-type base settings. Several of these are proved in Appendix D, and the section collects them; what it proves itself concerns the reference models and the determinant line.

The quantitative properties are where the arithmetic of the host field enters the towers. The flat height of a flat generator of the determinant line is computed exactly (Theorem 34.20): it equals $\frac{p-q}{2}(\gamma + \log(2\pi) - \log \mathrm{rd}_E - L_{\mathsf{H}})$ with $L_{\mathsf{H}} = L'(1,\chi_d)/L(1,\chi_d)$. The value comes from the Chowla–Selberg formula for the period of an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-d})$ (Lemma 34.21), in the form of the Faltings height of such a curve [8], evaluated at points of the Shimura variety that are built from two such curves. The subsections treat the Lie fibres (34.1), the flat determinants (34.2), and the flat heights at pure CM points (34.3).

34.1 The facts of good towers and the Lie fibres of the reference models

After the definition of the frozen reference models and the statement of Proposition 34.2, this subsection proves the Lie fibres (Lx2) of the frozen reference models (Proposition 34.3), one of the inputs of the reference comparison in Appendix D. The difficulty is at the boundary, where the abelian scheme degenerates. At an integral boundary point the semiabelian fibre has an abelian part, principally polarised with $\mathcal{O}_E$-action of the right signature, and a torus part with a uniform cocharacter frame (Lemma 34.5, Lemma 34.6, Lemma 34.4); the vanishing cycles are $2\pi i$ times the cocharacters of the torus (Lemma 34.7); and the Hodge form of the abelian part is bounded uniformly by a continuous Lie metric and is the limit of the Hodge form modulo the vanishing cycles (Lemma 34.12, Lemma 34.13). The integral structure at the boundary is controlled with Moret-Bailly models [181] (Definition 34.8).

Definition 34.1. Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with a geometric realisation and a good origin over an integral model of a unitary Shimura datum, and let $c \in \mathrm{Cp}$. The **frozen reference model** $\mathcal{X}_c^{\mathrm{fr}}$ is the root model with the arithmetic variety, the multidegrees, the auxiliary line and the normalising integers of the reference model $\mathcal{X}_c^{\mathrm{ref}}$, and with the root metrics

$$h_r^{\mathrm{fr}}(t) = h_r^{\mathrm{ref}}\big(\min(t, t_0)\big) \qquad (t > 0) \tag{34.1}$$

for every root r of $\mathcal{X}_c^{\mathrm{ref}}$, where t_0 is the constant of Sys_c. In particular its root metrics at the parameter $t = 1$ are $h_r^{\mathrm{ref}}(\min(1, t_0))$.

Proposition 34.2. *There is a real number $C \geq 0$ with the following property. Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum such that E is Galois over $\mathbb{Q}$ with abelian Galois group, $q \geq 1$, $3q + 1 \leq p + q$, the level $\mathsf{L} = (N, \mathfrak{n}, A)$ is good at every prime and $A = 1$, the components of Sh_F split (Ud6), and at most $4|\Phi_{\mathsf{H}}|(p+q)^2$ primes divide $2\,\mathrm{disc}(E)\,N$. Put $n = p+q$, $\Phi = \Phi_{\mathsf{H}}$, $\mathrm{rd}_E = |\mathrm{disc}(E)|^{1/[E:\mathbb{Q}]}$ and $L_{\mathsf{H}} = \mathrm{Re}\,(L'/L)(1,\chi_d)$ (here $\mathrm{rd}_E \geq \sqrt{d} \geq \sqrt{7} > 2$, since E contains the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ of discriminant $-d$). Let $\mathcal{Y}$ be an integral model of Sh over $\mathcal{O}_F$ with uniformised degenerations, and let $\mathcal{T}$ be a root tower of type*

(Φ, n, p) with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ and a good origin of the realised tower over $\mathcal{Y}$ immersive at level four. Then there is a root tower $\mathcal{T}'$ of type (Φ, n, p) with the same components and the same bad primes as $\mathcal{T}$ such that, with the geometric realisation of $\mathcal{T}$:

(Gt1) Compact dual *$\mathcal{T}'$ has (Fs);*

(Gt2) Volume side *the arithmetic degrees of $\mathcal{T}'$ are the regularised products of the root models, $\widehat{\deg}_{B,Z}(F) = \mathrm{pp}(\mathcal{X}_{B,Z}; F)$ for every integer $B \geq 2$, every $Z \in \mathrm{Cp}$ and every $F \in \mathcal{R}$ (the root models have convergent regularisation by Theorem 31.20); and the realised tower has (Gi1) and geometric proportionality with covolumes (Vp) over E;*

(Gt3) Frobenius labels *$\mathcal{T}'$ has Galois Frobenius labels (Fr) for H;*

(Gt4) The centre *the realised tower has proportional analytic square systems; and there is $C_{\mathrm{Lx}} \geq 1$ such that for every $c \in \mathrm{Cp}$ the frozen reference model $\mathcal{X}_c^{\mathrm{fr}}$ satisfies the Lie fibres (Lx2) and the flat determinants (Lx3), read with $\mathcal{X} = \mathcal{X}_c^{\mathrm{fr}}$ and the constant C_{Lx};*

(Gt5) Quantitative properties *$\mathcal{T}'$ has (Db) for $(r, L) = (\mathrm{rd}_E, L_{\mathsf{H}})$; and there is a real number $C_{\mathrm{h}} \geq 0$ with $C_{\mathrm{h}} + 2 \leq C$ such that for every admissible stage B and every $Z \in \mathrm{Cp}$ there is a Hodge-type base setting of type (Φ, n, p) with cusp data which satisfies (Bs0) and (Bs1) for $\mathcal{T}'$ at (B, Z) and, with its base variety $\mathcal{X}^{\mathrm{b}}$ and its corrected perturbed bundles $\overline{\mathcal{M}}_{i_1,i_2,t} = \overline{\mathcal{M}}_{B,i_1,i_2,t}$:*

- *(i) for all integers $1 \leq i_1 \leq i_2$ there is $t_1 > 0$ such that for $0 < t \leq t_1$ the line bundle $\mathcal{M}_{i_1,i_2,t,\mathbb{Q}}$ is ample, $\deg(\mathcal{M}_{i_1,i_2,t}|_\Gamma) \geq 0$ for every closed integral curve $\Gamma \subseteq \mathcal{X}^{\mathrm{b}}$ contained in a fibre of $\mathcal{X}^{\mathrm{b}} \to \operatorname{Spec}\mathbb{Z}$, and the first Chern form $c_1(\overline{\mathcal{M}}_{i_1,i_2,t})$ is semipositive on $\mathcal{X}^{\mathrm{b}}(\mathbb{C})$;*
- *(ii) it has regularisation limits (Mz1)–(Mz4) at B;*
- *(iii) its cusp data have the cusp mean bound (Qk1) for $r = \mathrm{rd}_E$ and corrected point heights (Qk2) at B for $(r, L) = (\mathrm{rd}_E, L_{\mathsf{H}})$ and the constant C_{h}.*

Proof. Let C_{h} be the constant of Proposition 34.35 and put $C = C_{\mathrm{h}} + 2$. Let $\mathfrak{u}$, $n = p + q$, $\Phi = \Phi_{\mathsf{H}}$, $\mathcal{Y}$, the root tower $\mathcal{T}$ with its geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ and its good origin be as in the statement. From $q \geq 1$ and $3q + 1 \leq p + q$ we have $n \geq 4$; and Φ is non-empty, being a CM type.

The limit forms and the covolumes. By Theorem 31.20(1), applied to $\mathcal{T}$ with its realisation and its good origin over $\mathcal{Y}$, every root model $\mathcal{X}_{B,Z}$ ($B \geq 2$, $Z \in \mathrm{Cp}$) has convergent regularisation, so $\mathrm{pp}(\mathcal{X}_{B,Z}; \cdot)$ is a linear form on $\mathcal{R}^+ \supseteq \mathcal{R}$. By Proposition D.79 the realised tower has (Vp) over E, with data (V, h), $\mathfrak{a}$, m, c_0, $(\Lambda_{Z,i})$. Put

$$v(Z) = c_0^{-1} \sum_{i=1}^{m} \mu_\infty\big(\Gamma(\Lambda_{Z,i}, \mathfrak{a}) \backslash G_\infty^1(V)\big) \qquad (Z \in \mathrm{Cp});$$

by Lemma 31.17, $0 \leq v(Z) < \infty$.

The choice of $\mathcal{T}'$. Let $\mathcal{T}'$ be the root tower of type (Φ, n, p) with the components Cp and the bad primes Bd of $\mathcal{T}$, the volumes $\mathrm{vol}'(Z) = v(Z)$, the arithmetic degrees $\widehat{\deg}'_{B,Z} = \mathrm{pp}(\mathcal{X}_{B,Z}; \cdot)|_{\mathcal{R}}$ ($B \geq 2$, $Z \in \mathrm{Cp}$), the compact-dual height $h^{\mathrm{FS}}_{\Phi,n}$ (defined as $\Phi \neq \emptyset$ and $n \geq 2$) and the Frobenius labelling fr_{H} (defined as E is Galois over $\mathbb{Q}$ with abelian Galois group and $\Phi = \Phi_{\mathsf{H}}$; it is a Frobenius labelling by the sign computation in Definition 31.1). A root tower consists of these data only, so $\mathcal{T}'$ is a root tower. A geometric realisation of a root tower consists of a weight and

of root models indexed by the stages and the components; hence $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ is a geometric realisation of $\mathcal{T}'$, and the frozen reference models $\mathcal{X}_c^{\mathrm{fr}}$, built from the good origin of the realised tower $\mathcal{T}$, serve for both towers.

(Gt1) and (Gt3). The compact-dual height of $\mathcal{T}'$ is $h^{\mathrm{FS}}_{\Phi,n}$: this is (Fs). Its Frobenius labelling is fr_{H} at every prime, in particular at every $p \notin \mathrm{Bd}$: this is (Fr) for H.

(Gt2). The identity $\widehat{\deg}'_{B,Z}(F) = \mathrm{pp}(\mathcal{X}_{B,Z}; F)$ for $B \geq 2$, $Z \in \mathrm{Cp}$, $F \in \mathcal{R}$ holds by the choice of $\widehat{\deg}'$. The realised tower has (Vp) over E by Proposition D.79 ((Vp) involves only the components and the root models). As $\mathrm{vol}'(Z) = v(Z)$ for every Z, Lemma 31.17 gives (Gi1) for $\mathcal{T}'$ with the realisation of $\mathcal{T}$.

(Gt4). By Proposition D.73 the realised tower has proportional analytic square systems (its root models have convergent regularisation, as shown above). By Proposition 34.3 and Proposition 34.14 there are $C_2, C_3 \geq 1$ such that every $\mathcal{X}_c^{\mathrm{fr}}$ satisfies (Lx2) with the constant C_2 and (Lx3) with the constant C_3. Put $C_{\mathrm{Lx}} = \max(C_2, C_3) \geq 1$. Both clauses remain true when the constant is enlarged: in (Lx2)(a) and (Lx3) it is an upper bound; in (Lx2)(b), for $1 \leq C \leq C'$ we have $C'^{-1} \leq C^{-1}$, so the two-sided comparison of norms and the bound $\|\phi(x)\|_\sigma^2 \leq C\|x\|_{V,\sigma}^2$ with C imply those with C'. Hence every $\mathcal{X}_c^{\mathrm{fr}}$ satisfies (Lx2) and (Lx3) with the constant C_{Lx}.

(Gt5), the determinant classes. The tower $\mathcal{T}'$ has the components and bad primes of $\mathcal{T}$, its stage-two degrees are $\mathrm{pp}(\mathcal{X}_{2,Z}; \cdot)|_{\mathcal{R}}$, and it has (Gi1) with the realisation of $\mathcal{T}$. So Proposition D.65 applies: $\mathfrak{u}$ and $\mathcal{T}'$ (of type $(\Phi_{\mathsf{H}}, n, p)$, $n \geq 1$) satisfy the hypotheses 1–3 of Lemma 34.19, and that lemma gives (Db) for $\mathcal{T}'$ and $(r, L) = (\mathrm{rd}_E, L_{\mathsf{H}})$.

(Gt5), the base settings. By the same properties of $\mathcal{T}'$ (degrees pp at every stage, (Gi1)), Proposition 34.35 gives, for every admissible stage B of $\mathcal{T}'$ and every $Z \in \mathrm{Cp}$, a Hodge-type base setting of type (Φ, n, p) with cusp data that satisfies (Bs0) and (Bs1) for $\mathcal{T}'$ at (B, Z) and (i), (ii), (iii) with the constant C_{h}; and $C_{\mathrm{h}} + 2 \leq C$.

So $\mathcal{T}'$, with the geometric realisation of $\mathcal{T}$, has (Gt1)–(Gt5) with the constant C, which is absolute. □

Proposition 34.3. *Let $\mathfrak{u}$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. Then there is a real number $C_{\mathrm{Lx}} \geq 1$ such that for every $c \in \mathrm{Cp}$ the frozen reference model $\mathcal{X}_c^{\mathrm{fr}}$ satisfies the Lie fibres (Lx2), read with $\mathcal{X} = \mathcal{X}_c^{\mathrm{fr}}$ and the constant C_{Lx}.*

Proof. As Cp is finite, it suffices to find, for each $c \in \mathrm{Cp}$, a constant C_c for which $\mathcal{X} = \mathcal{X}_c^{\mathrm{fr}}$ satisfies (Lx2); then $C_{\mathrm{Lx}} = \max_c C_c$. Fix c. The arithmetic variety of $\mathcal{X}$ is that of $\mathcal{X}_c^{\mathrm{ref}}$: projective over $\mathbb{Z}$, a normal cover of $\mathcal{Y}$, with $G = G_{\mathcal{X}} = f^*G_{\mathcal{Y}}$ of relative dimension $g = \sum_s \mathrm{rk}(s) = |\Phi|n$. Through f, $\mathcal{X}$ is an $\mathcal{O}_F$-scheme, $\mathcal{X}_{\mathbb{C}} = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{C}$ is the disjoint union of the $\mathcal{X} \otimes_{\mathcal{O}_F,\kappa} \mathbb{C}$ ($\kappa\colon F \to \mathbb{C}$), and on each of them $\mathrm{Lie}(G) = \bigoplus_s \mathrm{Lie}_s(G)$ with $\mathrm{Lie}_s(G) = \mathcal{V}_s \otimes \mathbb{C}$ (Fh1).

Step 1: the metric. Choose continuous hermitian metrics on the vector bundles $\mathrm{Lie}_s(G)$ over the compact space $\mathcal{X}(\mathbb{C})$ (partition of unity) and let h^1 be their orthogonal sum. Complex conjugation maps $\mathcal{X}(\mathbb{C})$ to itself and $\mathrm{Lie}(G_z)$ antilinearly onto $\mathrm{Lie}(G_{\bar{z}})$, carrying $\mathrm{Lie}_s(G_z)$ to $\mathrm{Lie}_s(G_{\bar{z}})$ (if $\iota(a)$ acts by $\kappa(\sigma_s(a))$ at z over κ, it acts by $\bar{\kappa}(\sigma_s(a))$ at $\bar{z}$ over $\bar{\kappa}$). So $h_z(v, w) = \frac{1}{2}(h^1_z(v, w) + \overline{h^1_{\bar{z}}(\bar{v}, \bar{w})})$ is a hermitian metric on $\mathrm{Lie}(G)$ over $\mathcal{X}(\mathbb{C})$, invariant under complex conjugation, continuous in the sense of Lemma 34.4 and Lemma 34.12, with the pieces $\mathrm{Lie}_s(G_z)$ pairwise orthogonal at every z.

Step 2: the point and the Lie fibre. Let W be a point of $\mathcal{X}$, K the function field of W, so $\deg W_{\mathbb{Q}} = [K : \mathbb{Q}]$, and $P_K\colon \mathrm{Spec}\,\mathcal{O}_K \to W \subseteq \mathcal{X}$ the normalisation. For a finite extension L of K let $P\colon \mathrm{Spec}\,\mathcal{O}_L \to \mathcal{X}$ be P_K composed with $\mathrm{Spec}\,\mathcal{O}_L \to \mathrm{Spec}\,\mathcal{O}_K$, $\mathcal{A} = P^*G$, $V = \mathrm{Lie}(\mathcal{A}) = P^*\mathrm{Lie}(G)$, projective of rank $|\Phi|n$, and for $\sigma\colon L \to \mathbb{C}$ let z_σ be the complex

point given by P and σ, so that $V_\sigma = \mathrm{Lie}(G_{z_\sigma})$; put $\|x\|^2_{V,\sigma} = h_{z_\sigma}(x,x)$. Since h is invariant under conjugation, $\overline{V} = (V, (\|\cdot\|_{V,\sigma})_\sigma)$ is a Hermitian lattice over $\mathcal{O}_L$. For every hermitian line bundle $\overline{\mathcal{M}}$ on $\mathcal{X}$, $\widehat{h}_W(\overline{\mathcal{M}}) = \widehat{\deg}\, P^*\overline{\mathcal{M}}/[L:\mathbb{Q}]$. Indeed, the arithmetic degree of $\widehat{c}_1(\overline{\mathcal{M}})\cdot[W]$ is the arithmetic intersection number $\widehat{\deg}\,\widehat{c}_1(\overline{\mathcal{M}}|_W)$ on W; W and $\operatorname{Spec}\mathcal{O}_L$ are arithmetic varieties of absolute dimension 1, whose generic fibres, spectra of number fields, are smooth over $\mathbb{Q}$; the metric of $\overline{\mathcal{M}}|_W$ is smooth, $W(\mathbb{C})$ being finite; and $P\colon \operatorname{Spec}\mathcal{O}_L \to W$ is surjective with $[L:\mathbf{k}(W)] = [L:K]$. So Lemma 30.13 (with $b=1$, $d=0$, $\beta = P$) gives $\widehat{\deg}\,\widehat{c}_1(P^*\overline{\mathcal{M}}) = [L:K]\,\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}})\cdot[W])$, and on $\operatorname{Spec}\mathcal{O}_L$ the left side is the Arakelov degree $\widehat{\deg}\,P^*\overline{\mathcal{M}}$; divide by $[L:\mathbb{Q}] = [L:K]\deg W_{\mathbb{Q}}$. We prove (a) for every such L and (b) for a suitable L (Steps 5–7), which gives (Lx2) at W with that L.

Step 3: the flags and their degrees. For $a\in\Phi$ and $1\le i\le \kappa = p$ put $F_{a,i} = V\cap(P^*\mathcal{V}_{(a,+),i}\otimes L)$, and for $1\le j\le n-\kappa = q$ put $S_{a,j} = V\cap(P^*\mathcal{V}_{(a,-),j}\otimes L)$, inside $V\otimes L = \mathrm{Lie}(\mathcal{A}_L)$; they are saturated, of ranks i and j, and $F_{a,\kappa}\otimes L$, $S_{a,n-\kappa}\otimes L$ are the Hodge pieces $\mathrm{Lie}_{(a,\pm)}(\mathcal{A}_L)$, which span $V\otimes L$ and are orthogonal at every σ by Step 1. Each $\mathcal{V}_{s,k}$ is a locally direct summand of $\mathcal{V}_s$ (a full flag has invertible quotients), and $\mathcal{V}_s\to\mathrm{Lie}(G)$ is injective with image a direct summand at every point of $\mathcal{X}_{\mathbb{Q}}$; so $P^*\mathcal{V}_{s,k}\to V$ is injective (its kernel is torsion in a projective module) with image saturated in $P^*\mathcal{V}_s$ and contained in $F_{a,k}$ resp. $S_{a,k}$. Let $\delta = \mathrm{disc}(E)$ and $x_1,\dots,x_m$ a $\mathbb{Z}$-basis of $\mathcal{O}_E$ with trace-dual basis $x_1^\vee,\dots,x_m^\vee\in\mathfrak{d}_E^{-1}$; since $|\delta| = \mathrm{N}(\mathfrak{d}_E)\in\mathfrak{d}_E$, $\delta x_i^\vee\in\mathcal{O}_E$. For a label s the endomorphism $\epsilon_s = \sum_i \iota(\delta x_i^\vee)\,\sigma_s(x_i)$ of $\mathrm{Lie}(G)$ acts on $\mathrm{Li}_{s'}$ by $\delta\sum_i\sigma_{s'}(x_i^\vee)\sigma_s(x_i) = \delta\cdot[s=s']$ (the matrices $(\sigma(x_i))$ and $(\sigma(x_i^\vee))$ are inverse transposes, $\mathrm{Tr}(x_i^\vee x_j) = [i=j]$), so $\epsilon_s(\mathrm{Lie}_x)\subseteq\mathrm{Lie}_x\cap\mathrm{Li}_s = \mathrm{dR}_{s,x}$ and $\sum_s\epsilon_s = \delta$. Pulling back, for $v\in F_{a,k}\subseteq V\cap\mathrm{Lie}_{(a,+)}(\mathcal{A}_L)$ we get $\delta v = \epsilon_{(a,+)}(v)\in P^*\mathcal{V}_{(a,+)}$, and $\delta v\in P^*\mathcal{V}_{(a,+),k}\otimes L$, hence $\delta v\in P^*\mathcal{V}_{(a,+),k}$ by saturation. So the index of $P^*\mathcal{V}_{(a,+),k}$ in $F_{a,k}$ divides $\delta^{k[L:\mathbb{Q}]}$, and, with the norms induced by $\overline{V}$ on both,

$$0\le\frac{\widehat{\deg}\,\overline{F}_{a,k} - \widehat{\deg}\,P^*\overline{\mathcal{V}}_{(a,+),k}}{[L:\mathbb{Q}]}\le k\log|\delta|\le n\log|\delta|,$$

and the same for $S_{a,k}$ and $(a,-)$. For a label s and $1\le k\le\mathrm{rk}(s)$ let $\|\cdot\|^h_{s,k}$ be the quotient metric of h on the line $(\mathcal{V}_{s,k}/\mathcal{V}_{s,k-1})\otimes\mathbb{C}$, continuous on $\mathcal{X}(\mathbb{C})$. The sequence $0\to P^*\mathcal{V}_{s,k-1}\to P^*\mathcal{V}_{s,k}\to P^*(\mathcal{V}_{s,k}/\mathcal{V}_{s,k-1})\to 0$ is exact and arithmetic degrees are additive for induced and quotient metrics, so

$$\frac{\widehat{\deg}\,P^*(\mathcal{V}_{s,k}/\mathcal{V}_{s,k-1},\|\cdot\|^h_{s,k})}{[L:\mathbb{Q}]} = \frac{\widehat{\deg}\,P^*\overline{\mathcal{V}}_{s,k} - \widehat{\deg}\,P^*\overline{\mathcal{V}}_{s,k-1}}{[L:\mathbb{Q}]},$$

which differs from $d_{a,k}-d_{a,k-1}$ (for $s=(a,+)$), resp. $d'_{a,k}-d'_{a,k-1}$ (for $s=(a,-)$), by at most $2n\log|\delta|$.

Step 4: (a). The root metric of $\mathcal{X}$ at $t=1$ is $h_r^{\mathrm{ref}}(\min(1,t_0)) = h_r^{\mathrm{ref}}(t_0)$, a positive continuous cone function of multidegree e_r (Go4); transported by the root section it is a continuous metric $\|\cdot\|^{\mathrm{fr}}_r$ on $\mathcal{L}_r\otimes\mathbb{C}$ over $\mathcal{X}(\mathbb{C})$. The root line of the root r with label $s(r)$, step $k=\mathrm{st}(r)$ and sign $\mathrm{sg}(r)$ is $\mathcal{L}_r = (\mathcal{V}_{s(r),k}/\mathcal{V}_{s(r),k-1})^{\otimes\mathrm{sg}(r)}$; give it also the metric $\|\cdot\|^h_r$ induced by $\|\cdot\|^h_{s(r),k}$ (dual metric if $\mathrm{sg}(r)=-1$). The ratio of the two continuous metrics on the same line bundle over the compact space $\mathcal{X}(\mathbb{C})$ is continuous and positive; let M_h be the maximum of the absolute value of its logarithm over all z and the finitely many roots. Then $|\widehat{h}_W(\overline{\mathcal{L}}_r(1)) - \widehat{\deg}\,P^*(\mathcal{L}_r,\|\cdot\|^h_r)/[L:\mathbb{Q}]|\le M_h$ (Step 2). For $r=(a,i)$, $i\le\kappa$: $s(r)=(a,+)$, $k=i$, $\mathrm{sg}=1$; for $r=(a,n+1-j)$, $j\le n-\kappa$: $s(r)=(a,-)$, $k=j$, $\mathrm{sg}=-1$, and the degree

changes sign. With Step 3,

$$|\widehat{h}_W(\overline{\mathcal{L}}_{(a,i)}(1)) - (d_{a,i} - d_{a,i-1})| \le C_a, \qquad |\widehat{h}_W(\overline{\mathcal{L}}_{(a,n+1-j)}(1)) + (d'_{a,j} - d'_{a,j-1})| \le C_a,$$
$$C_a = M_h + 2n \log|\mathrm{disc}(E)|,$$

independent of W and L.

Step 5: constants for (b). Let $C_U \ge 1$ be the constant of Lemma 34.4 for $\mathcal{X}$, G and h ($\mathcal{X}$ is projective over $\mathbb{Z}$). The scheme $\mathcal{X}$ is integral and flat over $\mathbb{Z}$ (it dominates $\mathcal{Y}$), $\mathcal{X}_{\mathbb{Q}}$ is normal and so is $\mathcal{X}_{\mathbb{C}}$; hence every connected component X_0 of $\mathcal{X}_{\mathbb{C}}$ is a normal projective variety, mapping through f to $\mathcal{Y}_c \otimes_{\mathcal{O}_F,\kappa} \mathbb{C}$ for the component $\mathcal{Y}_c$ of (Go3)(1) and some $\kappa\colon F \to \mathbb{C}$, and the abelian locus of X_0 is dense in X_0 (the abelian locus of $\mathcal{X}$ is a non-empty open subset, and $\mathcal{X}_{\mathbb{C}} \to \mathcal{X}_{\mathbb{Q}}$ is flat, hence open). So X_0 with $G|_{X_0}$ and its abelian locus is as in Lemma 33.16 (case (B)); let $K_h \ge 1$ be the largest of the constants of Lemma 34.12 for these finitely many X_0 and the metric h. Put $C_c = \max(C_a, C_U, K_h)$.

Step 6: (b) when the generic point of W lies in the abelian locus. Then $G_K = P_K^* G \otimes K$ is an abelian variety with the principal polarisation λ of (Ud4), with whose Rosati involution ι is compatible; after a finite extension L_1 of K, $\lambda = \varphi_M$ for a symmetric ample line bundle M on G_{L_1}. By Lemma 34.9 there are a finite extension L of L_1, a principally polarised abelian scheme $\mathcal{A}^\flat$ with $\mathcal{O}_E$-action over $\mathcal{O}_L$ and an equivariant isomorphism $\beta\colon \mathcal{A}^\flat_L \to G_L$, through which the Hodge norm of $\mathcal{A}^\flat$ at σ is πH_{z_σ}. The homomorphism $\beta^{-1}\colon \mathcal{A}_L = G_L \to \mathcal{A}^\flat_L$ extends over a dense open subscheme of $\operatorname{Spec}\mathcal{O}_L$ (finite presentation) and then, by Theorem B.104, to $\varphi\colon \mathcal{A} \to \mathcal{A}^\flat$; let $\phi = \mathrm{Lie}(\varphi)\colon V \to \mathrm{Lie}(\mathcal{A}^\flat)$. As $\phi \otimes L$ is an isomorphism and V is torsion free, $\ker\phi = 0$ and $\mathrm{Lie}(\mathcal{A}^\flat)/\phi(V)$ is finite. Take $r = 0$, $U = 0 \ne V$. By Lemma 34.12(i) at z_σ (in the abelian locus), $\|\phi_\sigma(x)\|^2_\sigma = \pi H_{z_\sigma}(x,x) \le K_h \|x\|^2_{V,\sigma}$.

Step 7: (b) when the generic point of W is not in the abelian locus. Apply Lemma 34.5 to $\mathcal{X}$ and P_K: there are a finite extension L of K, the split maximal torus T of $\mathcal{A}_L$ with $X = X_*(T)$, and, if $T \ne \mathcal{A}_L$, $\mathcal{B}$ and $\varphi\colon \mathcal{A} \to \mathcal{B}$, with (Ai1), (Ai2). By Lemma 34.4 (applied to P, T split over L) there is a $\mathbb{Z}$-basis $\chi_1, \dots, \chi_r$ of X such that $\varepsilon_b = d\chi_b(1)$ satisfy $C_U^{-1} \sum |w_b|^2 \le \|\sum w_b \varepsilon_b\|^2_{V,\sigma} \le C_U \sum |w_b|^2$ for all σ, w. Each $\chi_b\colon \mathbb{G}_{m,L} \to \mathcal{A}_L$ extends over a dense open subscheme of $\operatorname{Spec}\mathcal{O}_L$ and then, by Theorem B.104 ($\mathbb{G}_m$ is semiabelian, the base normal and noetherian), to $\mathbb{G}_{m,\mathcal{O}_L} \to \mathcal{A}$; so $\varepsilon_b \in V$. The ε_b are linearly independent over L, and their span is $U = X \otimes_{\mathbb{Z}} \mathcal{O}_L$; with (Ai1), $U = V \cap \mathrm{Lie}(T)$. If $T = \mathcal{A}_L$, $\mathrm{Lie}(T) = V \otimes L$ and $U = V$. Otherwise let $\phi = \mathrm{Lie}(\varphi)\colon V \to \mathrm{Lie}(\mathcal{B})$. The generic fibre of φ is surjective with kernel T (characteristic 0), so $\phi \otimes L$ is surjective with kernel $\mathrm{Lie}(T)$; hence $\ker\phi = V \cap \mathrm{Lie}(T) = U$ and $\phi(V)$ has full rank in $\mathrm{Lie}(\mathcal{B})$, so $\mathrm{Lie}(\mathcal{B})/\phi(V)$ is finite. Fix σ; z_σ is not in the abelian locus. Let X_0 be the component of $\mathcal{X}_{\mathbb{C}}$ through z_σ, $\rho\colon \widetilde{X} \to X_0$ as in Lemma 34.11 and $\tilde{x} \in \rho^{-1}(z_\sigma)$, a point with a chart. By (Ai2) at this pair, $\phi \otimes_\sigma \mathbb{C}$ induces the identification of $M_{\tilde{x}} = V_\sigma/\mathrm{Lie}(T_{z_\sigma})$ with $\mathrm{Lie}(\mathcal{B}) \otimes_\sigma \mathbb{C}$ under which H_σ is H_0; so by Lemma 34.12(ii) for X_0, for $x \in V_\sigma$ with image $q(x) \in M_{\tilde{x}}$,

$$\|(\phi \otimes_\sigma \mathbb{C})(x)\|^2_\sigma = \pi H_0(q(x), q(x)) \le K_h\, h_{z_\sigma}(x,x) = K_h \|x\|^2_{V,\sigma}.$$

Conclusion. At every point W of $\mathcal{X}$, with L from Step 6 or Step 7, the Hermitian lattice $\overline{V}$ over $\mathcal{O}_L$ of rank $|\Phi|n$ with the flags of Step 3, the ε_b and, when $U \ne V$, E, $\mathcal{A}^\flat$ or $\mathcal{B}$ and ϕ, satisfies (a) (Step 4) and (b) with the constant C_c; the constants do not depend on W. □

Lemma 34.4. *Let $\mathcal{X}$ be a scheme projective over $\operatorname{Spec}\mathbb{Z}$ and $G \to \mathcal{X}$ a semiabelian scheme. For every complex point z of $\mathcal{X}$ let h_z be a positive definite Hermitian form on $\mathrm{Lie}(G_z)$, such that for every open $V \subseteq \mathcal{X}$ and all sections s, s' of $\mathrm{Lie}(G)$ over V the function $z \mapsto h_z(s(z), s'(z))$ is continuous on $V(\mathbb{C})$. Then there is $C \ge 1$ with the following property. Let L be a number field and $P\colon \operatorname{Spec}\mathcal{O}_L \to \mathcal{X}$ a morphism such that the maximal subtorus T of the generic fibre*

$G_P := P^*G \times_{\mathcal{O}_L} L$ is split over L; for $\sigma\colon L \to \mathbb{C}$ let z_σ be the complex point of $\mathcal{X}$ given by P and σ, so that $\mathrm{Lie}(G_P) \otimes_{L,\sigma} \mathbb{C} = \mathrm{Lie}(G_{z_\sigma})$. Then there is a $\mathbb{Z}$-basis $\chi_1, \dots, \chi_r$ of the cocharacter group $X_*(T)$ such that the vectors $\varepsilon_b = d\chi_b(1) \in \mathrm{Lie}(T) \subseteq \mathrm{Lie}(G_P)$ satisfy, for every $\sigma\colon L \to \mathbb{C}$ and every $w \in \mathbb{C}^r$,

$$C^{-1} \sum_{b=1}^{r} |w_b|^2 \;\le\; h_{z_\sigma}\Big(\sum_{b=1}^{r} w_b\, \varepsilon_{b,\sigma},\; \sum_{b=1}^{r} w_b\, \varepsilon_{b,\sigma}\Big) \;\le\; C \sum_{b=1}^{r} |w_b|^2,$$

where $\varepsilon_{b,\sigma} = \varepsilon_b \otimes 1 \in \mathrm{Lie}(G_{z_\sigma})$.

Proof. For a point x of $\mathcal{X}$ with algebraically closed field k over it, the fibre $G_{x,k}$ is an extension $0 \to T_x \to G_{x,k} \to A_x \to 0$ of an abelian variety by its maximal torus; write $t(x) = \dim T_x$, g for the relative dimension of G. We call $x \in \mathcal{X}_{\mathbb{Q}}$ *good with constant* $C \ge 1$ if the conclusion holds with C for all L and P whose generic point $P(\mathrm{Spec}\, L)$ is x. If x is good with C, it is good with every $C' \ge C$.

Step 1: reduction to irreducible closed sets. The scheme $\mathcal{X}_{\mathbb{Q}}$ is of finite type over $\mathbb{Q}$, so its underlying space is noetherian. We show: for every irreducible closed $F \subseteq \mathcal{X}_{\mathbb{Q}}$ there are a closed $F' \subsetneq F$ and $C_F \ge 1$ such that every $x \in F \smallsetminus F'$ is good with C_F (Steps 2–5). Then, by noetherian induction, every closed F has a constant $C(F)$ for which all its points are good: if F is irreducible take $\max(C_F, C(F'))$; otherwise F is the finite union of its irreducible components $F_1, \dots, F_m$, each a proper closed subset, and take $\max_i C(F_i)$. For $F = \mathcal{X}_{\mathbb{Q}}$ this is the lemma, as the generic point of every P lies in $\mathcal{X}_{\mathbb{Q}}$.

Step 2: a split torus over a finite normal cover of F. Let Z be the reduced closed subscheme of $\mathcal{X}_{\mathbb{Q}}$ on F (integral, of finite type over $\mathbb{Q}$), η its generic point and $G_Z = G \times_{\mathcal{X}} Z$. By Lemma 32.13 there is a finite separable extension Ω' of $\kappa(\eta)$ over which the maximal torus of G_η becomes split, say $\cong \mathbb{G}_m^X$ for a free abelian group X of rank $t(\eta)$, with the inclusion $u_{\Omega'}\colon \mathbb{G}_{m,\Omega'}^X \to G_\eta \otimes \Omega'$. Let $\pi\colon Z' \to Z$ be the normalisation of Z in Ω': Z' is integral, normal, and π is finite (as Z is of finite type over a field) and surjective. Put $G' = G_Z \times_Z Z'$, a semiabelian scheme over Z'. By the spreading-out of morphisms of schemes of finite presentation (EGA IV$_3$, 8.8.2) $u_{\Omega'}$ extends to a homomorphism over a dense open of Z', and since Z' is noetherian and normal and the split torus $\mathbb{G}_{m,Z'}^X$ is semiabelian, Theorem B.104 extends it to a homomorphism $u\colon \mathbb{G}_{m,Z'}^X \to G'$.

Step 3: u is injective on every geometric fibre. Let $n \ge 1$. The n-torsion μ_n^X of the torus is finite étale over Z', and $G'[n]$ is étale over Z' (Z' lies over $\mathbb{Q}$, and $[n]$ is quasi-finite and flat on a semiabelian scheme with invertible differential n), so its unit section is open, and it is closed since G' is separated. Hence the kernel $K_n = u^{-1}(e) \cap \mu_n^X$ is open and closed in μ_n^X, so finite étale over the connected scheme Z' of constant degree; its degree at the generic point is 1 because $u_{\Omega'}$ is injective, so K_n is the unit section. Thus for every geometric point $\bar{x}$ of Z' the kernel of $u_{\bar{x}}$, a closed subgroup of a torus over an algebraically closed field of characteristic 0, has no non-trivial torsion point; a diagonalisable group without non-trivial torsion is trivial, so $u_{\bar{x}}$ is injective, hence (characteristic 0) a closed immersion, and its differential $\mathrm{Lie}(\mathbb{G}_m^X) \to \mathrm{Lie}(G'_{\bar{x}})$ is injective.

Step 4: the Gram matrix on the compact space $Z'(\mathbb{C})$. Fix a $\mathbb{Z}$-basis $\beta_1, \dots, \beta_r$ of the cocharacter group $\mathrm{Hom}(X, \mathbb{Z})$ of $\mathbb{G}_m^X$, $r = t(\eta)$, and let s_b be the section $du(\beta_b)$ of $\mathrm{Lie}(G') = \rho^*\mathrm{Lie}(G)$ over Z', where $\rho\colon Z' \to \mathcal{X}$ is π followed by the inclusion (Lie algebras of smooth group schemes commute with base change). For $\zeta \in Z'(\mathbb{C})$ let $\Gamma(\zeta) = \big(h_{\rho(\zeta)}(s_a(\zeta), s_b(\zeta))\big)_{a,b}$. Locally on $\mathcal{X}$, $\mathrm{Lie}(G)$ is free and each s_b is a combination of a frame with regular coefficients, so Γ is continuous on $Z'(\mathbb{C})$ by the hypothesis on h; it is positive definite at every ζ by Step 3. The space $\mathcal{X}(\mathbb{C})$ is compact ($\mathcal{X}_{\mathbb{C}}$ is projective), $Z(\mathbb{C})$ is closed in it and $Z'(\mathbb{C}) \to Z(\mathbb{C})$ is finite, so

$Z'(\mathbb{C})$ is compact. The smallest and largest eigenvalues of $\Gamma(\zeta)$ are continuous and positive, so there is $C_F \geq 1$ with $C_F^{-1}|w|^2 \leq \bar{w}^{\mathsf{T}}\Gamma(\zeta)w \leq C_F|w|^2$ for all $\zeta \in Z'(\mathbb{C})$, $w \in \mathbb{C}^r$.

Step 5: the torus is the maximal torus over a dense open of Z. The group scheme $G_Z[2]$ is quasi-finite, étale and separated over the integral noetherian scheme Z, hence finite étale over a dense open $U \subseteq Z$ (generic finiteness of quasi-finite separated morphisms), of constant degree there. Over an algebraically closed field of characteristic 0, [2] is surjective on the torus and $A_x[2]$ has $2^{2(g-t(x))}$ points, so $\#G_{x,k}[2] = 2^{2g-t(x)}$; hence $t(x) = t(\eta)$ for every $x \in U$. Let $F' = F \smallsetminus U$, a proper closed subset. Let $x \in U$, and let x' be a geometric point of Z' over a geometric point $\bar{x}$ of x (π is surjective). By Step 3 the image of $u_{x'}$ is a torus of dimension $r = t(\eta) = t(x)$ in $G_{\bar{x}}$, contained in the maximal torus, hence equal to it.

Step 6: points of $F \smallsetminus F'$ are good with C_F. Let L, P be as in the statement with generic point $x \in U$, and $T \subseteq G_P$ the maximal torus, split over L. Let $\bar{L}$ be an algebraic closure of L; the point $\operatorname{Spec}\bar{L} \to \mathcal{X}_{\mathbb{Q}}$ factors through Z (Z is reduced with underlying set $F \ni x$) and lifts to $x'\colon \operatorname{Spec}\bar{L} \to Z'$ (π is finite and surjective). By Step 5, $u_{x'}$ is an isomorphism of $\mathbb{G}_{m,\bar{L}}^X$ onto $T_{\bar{L}}$, so it induces an isomorphism of cocharacter groups $\operatorname{Hom}(X,\mathbb{Z}) \to X_*(T_{\bar{L}}) = X_*(T)$ (T split over L). Let χ_b be the image of β_b; then $\varepsilon_b \otimes 1 = s_b(x')$ in $\operatorname{Lie}(G_P) \otimes_L \bar{L}$. For $\sigma\colon L \to \mathbb{C}$ choose an extension $\bar{\sigma}\colon \bar{L} \to \mathbb{C}$ and let $\zeta \in Z'(\mathbb{C})$ be x' followed by $\bar{\sigma}$; then $\rho(\zeta) = z_\sigma$, $\varepsilon_{b,\sigma} = s_b(\zeta)$, and the Gram matrix of the $\varepsilon_{b,\sigma}$ for h_{z_σ} is $\Gamma(\zeta)$. Step 4 gives the two inequalities with C_F. If $r = 0$ the statement is empty. □

Lemma 34.5. *Let $\mathcal{X}$ be a cover of an integral model $\mathcal{Y}$ of a unitary Shimura datum, projective over $\operatorname{Spec}\mathbb{Z}$, let L be a number field and $P\colon \operatorname{Spec}\mathcal{O}_L \to \mathcal{X}$ a morphism, and put $\mathcal{A} = P^*G_{\mathcal{X}}$ with the induced $\mathcal{O}_E$-action ι. For a field embedding e of L (or of an extension of L) into $\mathbb{C}$ let z_e be the complex point of $\mathcal{X}$ given by e and P. Suppose that the image ξ of the generic point of $\operatorname{Spec}\mathcal{O}_L$ does not lie in the abelian locus. (In the setting of Lemma D.23 this is the hypothesis that $\bar{y}^{-1}(x)$ does not lie in the abelian locus; there $\mathcal{X}$ is projective, $\bar{y}$ being an isomorphism onto $\mathcal{X}^{\mathrm{b}} \subseteq \mathbb{P}$.) Then there are a finite extension L' of L and, with $\mathcal{A}' = \mathcal{A} \times_{\mathcal{O}_L} \mathcal{O}_{L'}$ and $\mathcal{A}'_{L'}$ its generic fibre:*

(a) a split torus $T \subseteq \mathcal{A}'_{L'}$, the maximal subtorus, stable under $\iota(\mathcal{O}_E)$, with cocharacter group $X = X_*(T)$, an $\mathcal{O}_E$-module through ι, so that $\operatorname{Lie}(T) = X \otimes_{\mathbb{Z}} L'$;

(b) if $T \neq \mathcal{A}'_{L'}$, a principally polarised abelian scheme $\mathcal{B}$ with $\mathcal{O}_E$-action $\iota_{\mathcal{B}}$ over $\mathcal{O}_{L'}$ and an $\mathcal{O}_E$-equivariant homomorphism $\varphi\colon \mathcal{A}' \to \mathcal{B}$ whose generic fibre is surjective with kernel T,

such that:

(Ai1) Integral torus *We have $\operatorname{Lie}(\mathcal{A}') \cap \operatorname{Lie}(T) \subseteq X \otimes_{\mathbb{Z}} \mathcal{O}_{L'}$ inside $\operatorname{Lie}(\mathcal{A}'_{L'})$;*

(Ai2) Limit polarisation *For every field embedding $e\colon L' \to \mathbb{C}$: $T \otimes_{L',e} \mathbb{C}$ is the maximal subtorus T_{z_e} of $G_{\mathcal{X},z_e}$; and, if $T \neq \mathcal{A}'_{L'}$, for every projective variety $\widetilde{X}$ over $\mathbb{C}$ with a morphism $\rho\colon \widetilde{X} \to \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{C}$ and every $\tilde{x} \in \widetilde{X}$ with $\rho(\tilde{x}) = z_e$ at which the chart setting of Lemma 33.16 holds for $\widetilde{X}$, $\rho^*G_{\mathcal{X}}$ and ρ^{-1} of the abelian locus (case (B)), the differential of φ at e identifies $M_{\tilde{x}} = \operatorname{Lie}(G_{\mathcal{X},z_e})/\operatorname{Lie}(T_{z_e})$ with $\operatorname{Lie}(\mathcal{B}_{L'}) \otimes_{L',e} \mathbb{C}$, and under this identification the hermitian form H_e of $\mathcal{B}$ at e is the form H_0 of (Cv) of Lemma 33.18 at $\tilde{x}$. Such $\widetilde{X}$, $\tilde{x}$ exist for every e; if the complement of the abelian locus is a divisor with normal crossings near z_e in the complex manifold $\mathcal{X}(\mathbb{C})$ (as in base settings, by (Gb1) and Definition 31.10), one may take for $\widetilde{X}$ the connected component of $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{C}$ through z_e and $\tilde{x} = z_e$.*

Proof. Write $G_L = \mathcal{A}_L$, a semiabelian variety over L with $\mathcal{O}_E$-action; for $e\colon L \to \mathbb{C}$, $G_{\mathcal{X},z_e} = G_L \otimes_{L,e} \mathbb{C}$. Let $\mathcal{X}_{\mathbb{C}} = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{C}$. A pair $(\widetilde{X}, \tilde{x})$ as in (Ai2) over a complex point z of $\mathcal{X}$ outside the abelian locus is called a *chart pair at* z.

Step 0: chart pairs exist. The cover $\mathcal{X}$ is integral and dominates $\mathcal{Y}$, whose generic points lie over $\mathbb{Q}$, so $\mathcal{X}$ is flat over $\mathbb{Z}$; $\mathcal{X}_{\mathbb{Q}}$ is normal, hence so is $\mathcal{X}_{\mathbb{C}}$ (base change along the separable extension $\mathbb{C}/\mathbb{Q}$), and its connected components are normal projective varieties. The abelian locus is a non-empty open subset of the integral $\mathcal{X}$, so its preimage in $\mathcal{X}_{\mathbb{C}}$ is dense ($\mathcal{X}_{\mathbb{C}} \to \mathcal{X}_{\mathbb{Q}}$ is flat, hence open). Let z be a complex point outside the abelian locus and X_0 the component through z. Through $f\colon \mathcal{X} \to \mathcal{Y}$, X_0 maps to $\mathcal{Y}_c \otimes_{\mathcal{O}_F,s} \mathbb{C}$ for the irreducible component $\mathcal{Y}_c$ containing $f(\mathcal{X})$ (it meets $j(\mathrm{Sh}_F)$) and an embedding $s\colon F \to \mathbb{C}$; so X_0, $G_{\mathcal{X}}|_{X_0}$ and the abelian locus of X_0 are as in Lemma 33.16 (case (B)). By Lemma 34.11 there is a surjective $\rho\colon \widetilde{X} \to X_0$ from a smooth projective variety with charts at every point of $\widetilde{X}$ outside ρ^{-1} of the abelian locus; any $\tilde{x} \in \rho^{-1}(z)$ gives a chart pair. If the complement of the abelian locus is a divisor with normal crossings near z, $\widetilde{X} = X_0$, $\tilde{x} = z$ is a chart pair.

Step 1: the torus (a). The maximal subtorus of $G_L \otimes \overline{L}$ is unique, hence Galois-stable and defined over L; call it T. The image of a torus under a homomorphism is a torus, so T is stable under $\iota(\mathcal{O}_E)$, and G_L/T is an abelian variety (Chevalley). By Lemma 34.6(1) at a geometric point over ξ ($\mathcal{X}$ is noetherian), $T \neq 0$. After a finite extension L_1 of L, T is split; put $X = X_*(T)$, an $\mathcal{O}_E$-module through ι, with $\mathrm{Lie}(T) = X \otimes_{\mathbb{Z}} L_1$. For $e\colon L_1 \to \mathbb{C}$, $T \otimes_e \mathbb{C}$ is a torus in $G_{\mathcal{X},z_e}$ with abelian quotient, hence the maximal subtorus T_{z_e}; so $B = G_{L_1}/T$ satisfies $B \otimes_e \mathbb{C} = G_{\mathcal{X},z_e}/T_{z_e}$, the abelian quotient A_0 of Lemma 34.7 at every chart pair at z_e. If $T = G_{L_1}$, put $L' = L_1$ and go to Step 7.

Step 2: the Weil pairing over the cover. Fix a prime ℓ and $n \geq 1$, and let $\mathcal{X}_\ell = \mathcal{X}[1/\ell]$, $U \subseteq \mathcal{X}_\ell$ the abelian locus, open and dense. The group scheme $\Gamma = G_{\mathcal{X}}[\ell^n]$ is quasi-finite, étale and separated over $\mathcal{X}_\ell$, so $W = \Gamma \times_{\mathcal{X}_\ell} \Gamma$ is étale over the normal scheme $\mathcal{X}_\ell$, hence normal, and W_U is dense in W (an étale map is open, so every irreducible component of W maps onto an open subset of the irreducible $\mathcal{X}_\ell$, which meets U). The Weil pairing e_n of the principal polarisation over U is a function f on W_U with $f^{\ell^n} = 1$; on each connected (integral, normal) component of W it is a rational function integral over the structure sheaf, hence regular. So e_n extends to a morphism $E_n\colon W \to \mu_{\ell^n}$, and E_n is bilinear and alternating, as these identities hold on dense open subsets of the normal schemes $\Gamma \times \Gamma \times \Gamma$, Γ (étale over $\mathcal{X}_\ell$) and μ_{ℓ^n} is separated.

Step 3: the pairing at a chart pair. Let $(\widetilde{X}, \tilde{x})$ be a chart pair at a complex point z outside the abelian locus, with chart V at $\tilde{x}$ and $V^* = V \cap \rho^{-1}(U)$, and put $\widetilde{G} = \rho^* G_{\mathcal{X}}$, whose fibre at $\tilde{x}$ is $G_{\mathcal{X},z}$. By Lemma 33.16 (case (B)), Lemma 33.18 and Lemma 34.7 at $\tilde{x}$, $\widehat{P}_{\tilde{x}}$ maps Λ^{inv} onto $\ker \exp_{\tilde{x}}$, so $G_{\mathcal{X},z}[\ell^n](\mathbb{C})$ is the image of $\ell^{-n}\Lambda^{\mathrm{inv}}$, and the point $\exp_{\tilde{x}} \widehat{P}_{\tilde{x}}(\ell^{-n}\lambda)$ lies on the holomorphic section $w \mapsto \exp_w(\ell^{-n}\widehat{P}(\lambda)(w))$ of $\rho^*\Gamma = \widetilde{G}[\ell^n]$ over V, whose value at $w \in V^*$ is the class of $\ell^{-n}\lambda(w)$ in $\ell^{-n}\Lambda_w/\Lambda_w$. The pull-back of E_n to ρ^*W is continuous and equals the Weil pairing of the principal polarisation of $\widetilde{G}$ over V^*; so for $\lambda, \lambda' \in \Lambda^{\mathrm{inv}}$

$$E_{n,z}([\ell^{-n}\lambda], [\ell^{-n}\lambda']) = \lim_{w \to \tilde{x}} \exp\bigl(c\, 2\pi i\, \ell^n \operatorname{Im} H_w(\ell^{-n}\lambda, \ell^{-n}\lambda')\bigr) = \exp\bigl(-c\, 2\pi i\, \ell^{-n} Q(\lambda, \lambda')\bigr),$$

constant in w as λ, λ' are flat. Since $Q(I, \Lambda^{\mathrm{inv}}) = 0$ ($\Lambda^{\mathrm{inv}} \subseteq W_{-1} = I^{\perp_Q}$) and $T_z[\ell^n]$ is the image of $\ell^{-n}I$ ($\widehat{P}_{\tilde{x}}(I) = \ker \exp_{\tilde{x}} \cap \mathrm{Lie}(T_z)$), $E_{n,z}$ vanishes on $T_z[\ell^n] \times G_{\mathcal{X},z}[\ell^n]$, and the induced pairing on $A_0[\ell^n] = \ell^{-n}\Lambda^{\mathrm{ab}}_{\tilde{x}}/\Lambda^{\mathrm{ab}}_{\tilde{x}}$ is $(u, u') \mapsto \exp(c\, 2\pi i\, \ell^n E^{\mathrm{ab}}(u, u'))$ (as $E^{\mathrm{ab}} = -Q$ on $\Lambda^{\mathrm{inv}}/I \cong \Lambda^{\mathrm{ab}}_{\tilde{x}}$): the Weil pairing of the principal polarisation of A_0 with Riemann form E^{ab} at $\tilde{x}$ (Lemma 34.7).

Step 4: descent to B. Pull E_n back along $\operatorname{Spec} L_1 \to \mathcal{X}_\ell$ (through P): a pairing E_{n,L_1} on $G_{L_1}[\ell^n]$ over L_1 whose base change along each e is E_{n,z_e}. Its restriction to $T[\ell^n] \times G_{L_1}[\ell^n]$ is a morphism of finite étale L_1-schemes to μ_{ℓ^n} that is trivial after base change to $\mathbb{C}$ (Step 3 at a chart pair at z_e, Step 0), hence trivial. As T is divisible, $0 \to T[\ell^n] \to G_{L_1}[\ell^n] \to B[\ell^n] \to 0$

is exact, so E_{n,L_1} descends to an alternating pairing on $B[\ell^n]$; compatible in n, these give a $\mathrm{Gal}(\overline{L}/L_1)$-equivariant pairing e^B on $T_\ell(B_{\overline{L}})$ which, for every embedding $\bar{e}\colon \overline{L} \to \mathbb{C}$ extending some e and every chart pair at z_e, corresponds to the Weil pairing of the principal polarisation with Riemann form E^{ab} at that chart pair (Step 3).

Step 5: one polarisation over L_1. For each e fix a chart pair at z_e (Step 0) and let μ_e be the principal polarisation of $A_0 = B \otimes_e \mathbb{C}$ with Riemann form E^{ab} there. For an extension $\bar{e}$, μ_e is the base change along $\bar{e}$ of a unique homomorphism $\mu'_e\colon B_{\overline{L}} \to B^\vee_{\overline{L}}$, which is a principal polarisation, and the Weil pairing of μ'_e on $T_\ell(B_{\overline{L}})$ is e^B. The Weil pairing of a homomorphism μ determines $T_\ell(\mu)$, and $\mu \mapsto T_\ell(\mu)$ is injective; so all μ'_e are one principal polarisation μ of $B_{\overline{L}}$. For $\gamma \in \mathrm{Gal}(\overline{L}/L_1)$, $\gamma(\mu)$ has Weil pairing $\gamma(e^B) = e^B$, so $\gamma(\mu) = \mu$, and μ is defined over L_1. The relation $E^{\mathrm{ab}}(\iota(a)u, u') = E^{\mathrm{ab}}(u, \iota(\bar{a})u')$ of (Cv) gives $\mu_e \circ \iota(a) = \iota(\bar{a})^\vee \circ \mu_e$, hence the same for μ: the Rosati involution of μ maps $\iota(a)$ to $\iota(\bar{a})$. By the same injectivity, for every chart pair at z_e the polarisation $\mu \otimes_e \mathbb{C}$ has Riemann form E^{ab} at that pair, its Weil pairing being e^B (Step 4).

Step 6: the model $\mathcal{B}$ and the map (b). After a finite extension L_2 of L_1, $\mu = \varphi_M$ for a symmetric ample line bundle M on B_{L_2}. By Lemma 34.9, applied to B_{L_2} with the action ι (compatible with the Rosati involution of $\varphi_M = \mu$, Step 5) and M, there are a finite extension L' of L_2, a principally polarised abelian scheme $(\mathcal{B}, \mathcal{L}_\mathcal{B}, r)$ with $\mathcal{O}_E$-action $\iota_\mathcal{B}$ over $\mathcal{O}_{L'}$, of relative dimension $\dim G_L - \dim T \geq 1$, and an $\mathcal{O}_E$-equivariant isomorphism $\beta\colon \mathcal{B}_{L'} \to B_{L'}$ with $\mathcal{L}_\mathcal{B}|_{\mathcal{B}_{L'}} \cong \beta^*(M \otimes L')$; for every $e\colon L' \to \mathbb{C}$, β identifies the Hodge norm of $\mathcal{B}$ at e with πH, H the positive definite hermitian form whose imaginary part is the Riemann form of μ at e. The homomorphism $\beta^{-1} \circ (G_{L'} \to B_{L'})$ extends to a homomorphism over a dense open subscheme of $\operatorname{Spec} \mathcal{O}_{L'}$ (finite presentation), hence, by Theorem B.104 (the base is noetherian and normal), uniquely to $\varphi\colon \mathcal{A}' \to \mathcal{B}$; by uniqueness $\varphi \circ \iota(a) = \iota_\mathcal{B}(a) \circ \varphi$. Its generic fibre is surjective with kernel T.

Step 7: (Ai1) and (Ai2). Property (Ai1) is Lemma 34.10 applied to $\mathcal{G} = \mathcal{A}'$ and the split maximal torus T of its generic fibre. For (Ai2), $T \otimes_e \mathbb{C} = T_{z_e}$ by Step 1, and chart pairs exist by Step 0. If $T \neq \mathcal{A}'_{L'}$, let $(\tilde{X}, \tilde{x})$ be a chart pair at z_e. The differential of $\varphi \otimes_e \mathbb{C}$ is surjective with kernel $\mathrm{Lie}(T_{z_e})$, so it identifies $M_{\tilde{x}} = \mathrm{Lie}(A_0)$ (Lemma 34.7) with $\mathrm{Lie}(\mathcal{B}_{L'}) \otimes_e \mathbb{C}$, and $\Lambda^{\mathrm{ab}}_{\tilde{x}} = \ker \exp_{A_0}$ with the period lattice of $\mathcal{B}_{L'} \otimes_e \mathbb{C}$. By Step 5 the Riemann form of the polarisation of $\mathcal{B}$ at e, which is μ at e through β, is E^{ab} at $\tilde{x}$. The form H_e of Definition 31.6 and H_0 of (Cv) at $\tilde{x}$ are hermitian with the same imaginary part E^{ab}, and a hermitian form is determined by its imaginary part ($H(u,u') = E(iu,u') + iE(u,u')$); so $H_e = H_0$. □

Lemma 34.6. *The notation introduced here is local to this result and to the results that apply it. Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum, $\mathcal{X}$ a noetherian cover of an integral model of $\mathfrak{u}$, κ an algebraically closed field of characteristic 0 and $\bar{\xi}\colon \operatorname{Spec}\kappa \to \mathcal{X}$ a geometric point, and write $G = G_{\mathcal{X},\bar{\xi}}$ with the induced $\mathcal{O}_E$-action ι. Then there are a torus T and an abelian variety A_0 over κ, with $\mathcal{O}_E$-actions, and an $\mathcal{O}_E$-equivariant exact sequence $0 \to T \to G \to A_0 \to 0$, such that:*

1. *the character group X of T is a torsion-free $\mathcal{O}_E$-module; let $r = \dim_E(X \otimes \mathbb{Q})$; then $\dim T = r\,[E:\mathbb{Q}]$, and $r = 0$ if and only if $\bar{\xi}$ lies over the abelian locus;*
2. *A_0 has a principal polarisation λ_0 whose Rosati involution maps $\iota(a)$ to $\iota(\bar{a})$ for $a \in \mathcal{O}_E$;*
3. *for every ring homomorphism $\sigma\colon \mathcal{O}_E \to \kappa$, the σ-eigenspace of $\mathrm{Lie}(T)$ has dimension r, and the σ-eigenspace of $\mathrm{Lie}(A_0)$ has dimension $\mathrm{rk}(s) - r$, where s is the label with $\sigma_s = \sigma$ (through $F \to \kappa$ induced by $\bar{\xi}$); that is, A_0 satisfies the Kottwitz condition of signature $(p-r, q-r)$.*

Proof. Let ξ be the image of $\bar{\xi}$ and η the generic point of $\mathcal{X}$; as $\mathcal{X}$ is integral, ξ is a specialisation of η, and $\mathbb{Q} \subseteq \mathbf{k}(\xi)$ since κ has characteristic 0.

Step 1: a degeneration. Choose a discrete valuation ring of Ω dominating the noetherian local domain $\mathcal{O}_{\mathcal{X},\xi}$, and let R be its completion, with maximal ideal I, fraction field K_R and residue field $k_R \supseteq \mathbf{k}(\xi)$; R is a complete discrete valuation ring, so a noetherian normal domain complete with respect to $I = \mathrm{rad}(I)$. Let G_R be the pull-back of $G_{\mathcal{X}}$ along $\operatorname{Spec} R \to \mathcal{X}$ (generic point to η, closed point to ξ), with its $\mathcal{O}_E$-action and cubical line bundle $\mathcal{L}_R$. Its generic fibre is a base change of $G_{\mathcal{X},\eta}$, which lies over $j(\mathrm{Sh}_F)$: an abelian variety on which $\mathcal{L}_R$ is ample and induces a principal polarisation λ_η whose Rosati involution maps $\iota(a)$ to $\iota(\bar{a})$ (Definition 29.5). Its special fibre $G_{k_R} = G_{\mathcal{X},\xi} \otimes k_R$ is a semiabelian variety, hence an extension of an abelian variety by a torus, and every torus over the field k_R is isotrivial. So Theorem B.109(1) gives the Raynaud extension $0 \to T_R \to \widetilde{G} \to A \to 0$ with A an abelian scheme over R, whose special fibre is the given extension of G_{k_R}; by Theorem B.109(1), as every torus over k_R is isotrivial, $\mathcal{L}_R^{\mathrm{for}}$ lies in the essential image; $\mathcal{L}_R$ is rigidified, cubical and symmetric (the pull-back of the line bundle of the semiabelian structure of the integral model, Definition 29.5, (Ud7)), and ample on the generic fibre, hence nondegenerate there, so it is ample over G_R by the ampleness criterion for symmetric sheaves in Theorem B.109(2); so $(G_R, \mathcal{L}_R) \in \mathrm{DEG}_{\mathrm{ample}}$, and Theorem B.109(2), (3) give degeneration data $(A, X, Y, \phi, c, c^\vee, \widetilde{\mathcal{L}}, \tau, \psi)$ with $\mathcal{O}_E$-actions on A, X, Y compatible with c, $c^\vee$, ϕ and the polarisation λ_A of A induced by $\widetilde{\mathcal{L}}$. By Theorem B.109(4)(d), $1 = \deg \lambda_\eta = [X : \phi(Y)]^2 \deg \lambda_A$, so λ_A is principal. Compatibility of the $\mathcal{O}_E$-structure with the polarisations (part (3)) means that the Rosati involution of λ_A maps $i_A(a)$ to $i_A(\bar{a})$.

Step 2: transport to κ. Over k_R the torus part T_{k_R} is the maximal subtorus of G_{k_R}, so it is $\mathcal{O}_E$-stable, and the quotient is A_{k_R} with the induced action. Let Ω_1 be an algebraically closed field containing κ and k_R over $\mathbf{k}(\xi)$ (it exists: take an algebraic closure of the fraction field of a quotient of $\kappa \otimes_{\mathbf{k}(\xi)} k_R$ by a maximal ideal). Over Ω_1, $G \otimes_\kappa \Omega_1 = G_{k_R} \otimes_{k_R} \Omega_1$, and the maximal subtorus T of G and the quotient $A_0 = G/T$ over κ base change to those of G_{k_R}. By rigidity, the principal polarisation $\lambda_A \otimes \Omega_1$ of $A_0 \otimes \Omega_1 = A_{k_R} \otimes \Omega_1$ and the relation between its Rosati involution and the $\mathcal{O}_E$-action come from a principal polarisation λ_0 of A_0 over κ with the same property. This gives the sequence and item 2.

Step 3: items 1 and 3. The ring $\mathcal{O}_E$ acts on T, hence on its character group X (by $a \mapsto \iota(a)^*$), and X is free over $\mathbb{Z}$, so torsion-free over $\mathcal{O}_E$; $X \otimes \mathbb{Q}$ is an E-vector space of dimension r, so $\dim T = \mathrm{rk}_{\mathbb{Z}} X = r[E : \mathbb{Q}]$. As $\mathrm{Lie}(T) = \mathrm{Hom}(X, \mathbb{Z}) \otimes \kappa = (\mathrm{Hom}(X, \mathbb{Z}) \otimes \mathbb{Q}) \otimes_{\mathbb{Q}} \kappa$ with $\mathrm{Hom}(X, \mathbb{Z}) \otimes \mathbb{Q}$ an E-vector space of dimension r and $E \otimes_{\mathbb{Q}} \kappa = \prod_\sigma \kappa$ over the embeddings $\sigma\colon E \to \kappa$, each σ-eigenspace of $\mathrm{Lie}(T)$ has dimension r. The Hodge pieces of $\mathrm{Lie}(G_{\mathcal{X}})$ at ξ: since $\mathbb{Q} \subseteq \mathcal{O}_{\mathcal{X},\xi}$, $\mathcal{O}_E \otimes \mathcal{O}_{\mathcal{X},\xi}$ splits by the embeddings and $\mathrm{Lie}(G_{\mathcal{X}})_\xi = \bigoplus_s \mathrm{dR}_{s,\xi}$ with each summand free; its rank is the generic rank $\dim_\Omega \mathrm{Li}_s = \mathrm{rk}(s)$. So the σ_s-eigenspace of $\mathrm{Lie}(G)$ has dimension $\mathrm{rk}(s)$, and by exactness of $0 \to \mathrm{Lie}(T) \to \mathrm{Lie}(G) \to \mathrm{Lie}(A_0) \to 0$ (eigenspaces taken summand by summand) the σ_s-eigenspace of $\mathrm{Lie}(A_0)$ has dimension $\mathrm{rk}(s) - r$. Finally, $r = 0$ if and only if G is an abelian variety, if and only if $G_{\mathcal{Y}}$ is abelian at the image of ξ in $\mathcal{Y}$, if and only if that image lies in $j(\mathrm{Sh}_F)$ (Definition 29.5: $G_{\mathcal{Y}}$ is abelian exactly over $j(\mathrm{Sh}_F)$), that is, if and only if ξ lies in the abelian locus $f^{-1}(j(\mathrm{Sh}_F))$. □

Lemma 34.7. *In the setting of Lemma 33.16 and Lemma 33.18, let $0 \to T_x \to G'_x \to A_0 \to 0$ be the extension of the fibre G'_x (a semiabelian variety over $\mathbb{C}$) of an abelian variety A_0 by its maximal subtorus T_x, and $\exp_x\colon \mathrm{Lie}(G'_x) \to G'_x(\mathbb{C})$ the exponential map. Then $\widehat{P}_x$ maps Λ^{inv} isomorphically onto $\ker \exp_x$ and I onto $\ker \exp_x \cap \mathrm{Lie}(T_x) = 2\pi i\, X_*(T_x)$ (X_* the cocharacter group); hence $\widehat{P}_x(I \otimes \mathbb{C}) = \mathrm{Lie}(T_x)$, $M_x = \mathrm{Lie}(A_0)$, $\Lambda_x^{\mathrm{ab}} = \ker(\exp_{A_0})$, and E^{ab} is the Riemann form of a principal polarisation of A_0 whose Rosati involution maps $\iota(a)$ to $\iota(\bar{a})$.*

Proof. Invariant cycles and the exponential kernel at x. For $\lambda \in \Lambda^{\mathrm{inv}}$, $\widehat{P}(\lambda)$ is a holomorphic section of $\mathrm{Lie}(G')^{\mathrm{an}}$ near x with $\exp(\widehat{P}(\lambda)) = 0$ on V^*, hence at x by continuity: $\widehat{P}_x(\lambda) \in \ker \exp_x$. Conversely, the exponential of the family is a local biholomorphism near the zero section, so for $\kappa \in \ker \exp_x$ and a holomorphic section σ through κ the map $(z,u) \mapsto \exp_z(u - \sigma(z))$ is a local biholomorphism near (x,κ), and the kernel of exp is near κ the graph of a holomorphic section $\hat{\kappa}$; on V^* it is a continuous, hence flat and single-valued, section of Λ, i.e. an element $\lambda \in \Lambda^{\mathrm{inv}}$ with $\widehat{P}(\lambda) = \hat{\kappa}$ (equal on V^*, hence near x). So $\widehat{P}_x \colon \Lambda^{\mathrm{inv}} \to \ker \exp_x$ is surjective; it is injective since its kernel lies in $F_0^0 \cap W_{-1}$, which contains no non-zero real vector: by Theorem B.145 (clause 2) and $F_0^1 = 0$, $F_0^{-1} = H$, the limit has Gr^W_{-2} of type $(-1,-1)$ and Gr^W_{-1} of types $(-1,0),(0,-1)$, so $F_0^0 \cap W_{-2} = 0$ and a real vector of $F_0^0 \cap W_{-1}$ has image in $F^0\mathrm{Gr}^W_{-1} \cap \overline{F^0\mathrm{Gr}^W_{-1}} = 0$. The same holds modulo ℓ^n: $\widehat{P}_x$ identifies $\ell^{-n}\Lambda^{\mathrm{inv}}/\Lambda^{\mathrm{inv}}$ with $G'_x[\ell^n](\mathbb{C})$, and the point $\widehat{P}_x(\ell^{-n}\lambda)$ of $G'_x[\ell^n]$ lies on the holomorphic section $z \mapsto \exp_z(\ell^{-n}\widehat{P}(\lambda)(z))$ of the quasi-finite étale group scheme $G'[\ell^n]$ through it, whose value at $z \in V^*$ is the class of $\ell^{-n}\lambda(z)$ in $\ell^{-n}\Lambda_z/\Lambda_z = G_z[\ell^n]$.

Passage to a discrete valuation ring. Intersecting $X_{\mathbb{C}}$, in a projective embedding, with $d-1$ general hyperplanes through x gives an irreducible algebraic curve $C_1 \subseteq X_{\mathbb{C}}$ through x, smooth at x, whose tangent line at x lies in no hyperplane $\{dt_j = 0\}$ $(j \le a)$, so that every t_j $(j \le a)$ has a simple zero at x on C_1 and no other zero near x; a small loop γ around x in C_1 lies in V^* with monodromy $T_1 \cdots T_a$. Let R be the henselisation of the local ring of the normalisation of C_1 at the point over x (strictly henselian, residue field $\mathbb{C}$), K its fraction field, $\mathcal{G}$ the pull-back of G' (semiabelian, special fibre G'_x) and $A = \mathcal{G}_K$ with the principal polarisation λ pulled back. By the Riemann existence theorem $I_K = \mathrm{Gal}(K^s/K) \cong \widehat{\mathbb{Z}}(1)$, and at a base point z of γ, $A[\ell^n](K^s) = \ell^{-n}\Lambda_z/\Lambda_z$ with a topological generator of I_K acting by $(T_1 \cdots T_a)^{\pm 1}$; so $T_\ell(A) = \Lambda_z \otimes \mathbb{Z}_\ell$ and $T_\ell(A)^{I_K} = \ker(T_1 \cdots T_a - 1) \otimes \mathbb{Z}_\ell = \ker N \otimes \mathbb{Z}_\ell = \Lambda^{\mathrm{inv}} \otimes \mathbb{Z}_\ell$ ($T_1 \cdots T_a = \exp N$ with N nilpotent, $\exp N - 1 = N \cdot u$ with u invertible). The specialisation $\mathcal{G}[\ell^n](R) \cong G'_x[\ell^n]$ is the lift of a point of the special fibre to a section of the étale scheme $\mathcal{G}[\ell^n]$ over the henselian R; analytically near x it is the section of the previous paragraph. Hence, under $T_\ell(A)^{I_K} = \Lambda^{\mathrm{inv}} \otimes \mathbb{Z}_\ell$, the identification of Theorem B.115 of $T_\ell(\mathcal{G}_k) = T_\ell(G'_x)$ with $T_\ell(A)^{I_K}$ is $\widehat{P}_x^{-1} \otimes \mathbb{Z}_\ell$ on $\ker \exp_x \otimes \mathbb{Z}_\ell$, and $T_\ell(A)_{\mathrm{t}}$ corresponds to $I_T \otimes \mathbb{Z}_\ell$ with $I_T = \widehat{P}_x^{-1}(\ker \exp_x \cap \mathrm{Lie}(T_x))$ (the Tate module of T_x is that of its kernel lattice $\ker \exp_x \cap \mathrm{Lie}(T_x) = 2\pi i X_*(T_x)$).

Orthogonality. The Weil pairing of λ on $A[\ell^n](K^s) = \ell^{-n}\Lambda_z/\Lambda_z$ is

$$(u,u') \mapsto \exp(\pm 2\pi i\, \ell^n \mathrm{Im}\, H_z(u,u'))$$

(the classical description of the Weil pairing of a polarised complex torus by its Riemann form, [186], Section 24), so e^λ-orthogonality on $T_\ell(A)$ is Q-orthogonality on $\Lambda_z \otimes \mathbb{Z}_\ell$. By Theorem B.115, $I_T \otimes \mathbb{Z}_\ell$ is the Q-orthogonal of $\ker N \otimes \mathbb{Z}_\ell$ in $\Lambda_z \otimes \mathbb{Z}_\ell$, which is $I \otimes \mathbb{Z}_\ell$ ($W_{-1}^{\perp_Q} = W_{-2}$, Lemma 33.18, and I is saturated). Both I and I_T are saturated subgroups of Λ^{inv} (the second as the preimage of a saturated subgroup of $\ker \exp_x$: $\ker \exp_x / (\ker \exp_x \cap \mathrm{Lie}(T_x))$ embeds in the lattice $\ker \exp_{A_0}$), and a saturated subgroup L of a free $\mathbb{Z}$-module Λ^{inv} satisfies $L = \Lambda^{\mathrm{inv}} \cap (L \otimes \mathbb{Z}_\ell)$; hence $I = I_T$, i.e. $\widehat{P}_x(I) = \ker \exp_x \cap \mathrm{Lie}(T_x) = 2\pi i X_*(T_x)$, whose $\mathbb{C}$-span is $\mathrm{Lie}(T_x)$.

The abelian part. Therefore $M_x = \mathrm{Lie}(G'_x)/\mathrm{Lie}(T_x) = \mathrm{Lie}(A_0)$ and Λ_x^{ab} is the image of $\ker \exp_x$, which is $\ker \exp_{A_0}$ (as $G'_x \to A_0$ is surjective with connected kernel T_x, $\ker \exp_{A_0}$ is the image of $\exp_x^{-1}(T_x(\mathbb{C})) = \ker \exp_x + \mathrm{Lie}(T_x)$). By (Cv) of Lemma 33.18, E^{ab} is unimodular, alternating, the imaginary part of the positive hermitian form H_0 on $\mathrm{Lie}(A_0)$, and $\mathcal{O}_E$-adjoint; so it is the Riemann form of a principal polarisation of A_0 with Rosati involution $\iota(a) \mapsto \iota(\bar{a})$. $\square$

Definition 34.8. Let K be a number field, A an abelian variety of dimension $g \ge 1$ over K, L_A a symmetric ample line bundle on A defining a principal polarisation, r_0 a non-zero element of the fibre $L_A(0)$, and $F \subseteq A(K)$ a finite set. A **Moret-Bailly model** of (A, L_A, r_0, F) over $\mathcal{O}_K$ consists of:

(Mb1) a smooth separated commutative group scheme $\pi\colon \mathcal{G} \to \operatorname{Spec} \mathcal{O}_K$ of finite type, every connected component of every fibre of which is an extension of an abelian variety by a torus, with an isomorphism $i\colon \mathcal{G}_K \cong A$;

(Mb2) a line bundle $\mathcal{M}$ on $\mathcal{G}$ with a cubical structure and, for every $\sigma\colon K \to \mathbb{C}$, a smooth Hermitian metric $\|\cdot\|_\sigma$ on $\mathcal{M}_\sigma$ for which the trivialisation of $\Theta(\mathcal{M})_\sigma$ given by the cubical structure has norm 1 (a **cubist metric**);

(Mb3) an isomorphism $\varphi\colon \mathcal{M}|_{\mathcal{G}_K} \cong i^*L_A$ and a generator u of the $\mathcal{O}_K$-module $\varepsilon^*\mathcal{M}$ (ε the unit section) with $\|u\|_\sigma = 1$ for every σ and $\varphi(u) = r_0$;

(Mb4) for every $P \in F$ a section $\varepsilon_P\colon \operatorname{Spec} \mathcal{O}_K \to \mathcal{G}$ of π with $i(\varepsilon_P|_K) = P$;

(Mb5) a closed subscheme of $\mathcal{G}$, finite and flat over $\mathcal{O}_K$, whose generic fibre is $i^{-1}(K(L_A^{\otimes 2})) = i^{-1}(A[2])$.

Its **Moret-Bailly lattice** is the $\mathcal{O}_K$-submodule

$$\Lambda = \{\varphi(s|_{\mathcal{G}_K}) \circ i^{-1} \ : \ s \in H^0(\mathcal{G}, \mathcal{M})\} \subseteq H^0(A, L_A),$$

that is, the image of $H^0(\mathcal{G}, \mathcal{M})$ in $H^0(A, L_A)$ under restriction to the generic fibre followed by φ and i.

Let $(\mathcal{A}, \mathcal{L}_\mathcal{A}, r)$ be a principally polarised abelian scheme with $\mathcal{O}_E$-action over $\mathcal{O}_L$, with generic fibre $A = \mathcal{A}_L$, and let $r_0 \in \mathcal{L}_\mathcal{A}(0)$ be the generator with $r(r_0) = 1$. A section $\theta \in H^0(\mathcal{A}, \mathcal{L}_\mathcal{A})$ **lies in the Moret-Bailly lattice** if there are a finite extension K/L and a Moret-Bailly model of $(A_K, \mathcal{L}_\mathcal{A}|_{A_K}, r_0 \otimes 1, \varnothing)$ over $\mathcal{O}_K$ whose Moret-Bailly lattice contains the restriction of θ to A_K.

Lemma 34.9. *Let E be a CM field, L a number field, B an abelian variety of dimension $g \geq 1$ over L with a ring homomorphism $\iota\colon \mathcal{O}_E \to \operatorname{End}_L(B)$, and M a symmetric ample line bundle on B whose polarisation $\phi_M\colon B \to B^\vee$ is an isomorphism and with whose Rosati involution ι is compatible. Then there are a finite extension L'/L, a principally polarised abelian scheme $(\mathcal{A}, \mathcal{L}, r)$ with $\mathcal{O}_E$-action $\iota_\mathcal{A}$ over $\mathcal{O}_{L'}$ and an isomorphism $\beta\colon \mathcal{A}_{L'} \to B_{L'}$ with $\beta \circ \iota_\mathcal{A}(a) = \iota(a) \circ \beta$ ($a \in \mathcal{O}_E$) and $\mathcal{L}|_{\mathcal{A}_{L'}} \cong \beta^*(M \otimes_L L')$. For every $\sigma'\colon L' \to \mathbb{C}$, β identifies the Hodge norm of $\mathcal{A}$ at σ' with πH on $\operatorname{Lie}(B_{\sigma'|_L})$, where H is the positive definite Hermitian form whose imaginary part is the Riemann form of ϕ_M at $\sigma'|_L$.*

Proof. **The semiabelian model.** Choose $0 \neq r_0 \in M(0)$. By Theorem B.133(1), applied to B over $K_0 = L$, the line bundle M and the even integer 2, there are a finite extension K/L and a Moret-Bailly model $(\mathcal{G}, i, \mathcal{M}, (\|\cdot\|_\sigma)_\sigma, \varphi, u, (\varepsilon_P)_P)$ of $(B_K, M \otimes_L K, r_0 \otimes 1, B[4](K))$ over $\mathcal{O}_K$. We take $L' = K$. Let $\mathcal{A} = \mathcal{G}^0 \subseteq \mathcal{G}$ be the identity component, the open subgroup scheme whose fibres are the identity components of the fibres of $\mathcal{G}$ (SGA 3, Exp. VI$_B$, 3.10, for smooth group schemes). By (Mb1) its fibres are connected extensions of abelian varieties by tori, so $\mathcal{A}$ is a semiabelian scheme over $\mathcal{O}_K$, with unit section ε; its generic fibre is $\mathcal{G}_K$, connected because $i\colon \mathcal{G}_K \cong B_K$. Put $\beta = i$.

The action. Let $a \in \mathcal{O}_E$ and $\alpha_a = \beta^{-1} \circ \iota(a) \circ \beta$, an endomorphism of $\mathcal{A}_K$. As $\mathcal{A}$ is of finite presentation over $\mathcal{O}_K$ and $\operatorname{Spec} K$ is the limit of the dense open subschemes of $\operatorname{Spec} \mathcal{O}_K$, the spreading-out of morphisms (EGA IV$_3$, 8.8.2) extends α_a to a homomorphism over a dense open $U \subseteq \operatorname{Spec} \mathcal{O}_K$ (after shrinking U, by the uniqueness part of 8.8.2 applied to the identities expressing that α_a is a homomorphism). Since $\operatorname{Spec} \mathcal{O}_K$ is noetherian and normal, Theorem B.104 extends it uniquely to an endomorphism $\iota_\mathcal{A}(a)$ of $\mathcal{A}$. For $a, a' \in \mathcal{O}_E$ the endomorphisms $\iota_\mathcal{A}(a) + \iota_\mathcal{A}(a')$, $\iota_\mathcal{A}(a) \circ \iota_\mathcal{A}(a')$ and id extend $\alpha_{a+a'}$, $\alpha_{aa'}$ and α_1 over a dense

open, so by the uniqueness in Theorem B.104 $\iota_{\mathcal{A}}$ is a ring homomorphism $\mathcal{O}_E \to \mathrm{End}(\mathcal{A})$, and $\beta \circ \iota_{\mathcal{A}}(a) = \iota(a) \circ \beta$ on the generic fibre by construction.

The line bundle and its cubical structure. Put $\mathcal{L} = \mathcal{M}|_{\mathcal{A}}$. For a homomorphism $f\colon H \to G$ of commutative group schemes and a line bundle $\mathcal{N}$ on G, the partial sums satisfy $m_I \circ f^3 = f \circ m_I$, so $\Theta(f^*\mathcal{N}) = (f^3)^*\Theta(\mathcal{N})$ canonically, and the pull-back of a trivialisation satisfying the symmetry and cocycle conditions satisfies them again ([181], Ch. I). Applied to the open immersion $\mathcal{A} \to \mathcal{G}$, the cubical structure of $\mathcal{M}$ (Mb2) gives a cubical structure on $\mathcal{L}$. By (Mb3), $\varepsilon^*\mathcal{L} = \varepsilon^*\mathcal{M}$ is free with generator u; let $r\colon \varepsilon^*\mathcal{L} \to \mathcal{O}_K$ send u to 1. By (Mb3), $\varphi\colon \mathcal{L}|_{\mathcal{A}_K} = \mathcal{M}|_{\mathcal{G}_K} \cong \beta^*(M \otimes_L K)$.

Symmetry. Write $[-1]$ for the inverse of $\mathcal{A}$ and $\pi\colon \mathcal{A} \to \operatorname{Spec}\mathcal{O}_K$. As M is symmetric and β is a homomorphism, there is an isomorphism $[-1]^*\mathcal{L}|_{\mathcal{A}_K} \to \mathcal{L}|_{\mathcal{A}_K}$; regard it as a rational section ψ of $\mathcal{H} = ([-1]^*\mathcal{L})^\vee \otimes \mathcal{L}$, regular and nowhere zero on $\mathcal{A}_K$. Since $[-1] \circ \varepsilon = \varepsilon$, $\varepsilon^*\mathcal{H} = (\varepsilon^*\mathcal{L})^\vee \otimes \varepsilon^*\mathcal{L} = \mathcal{O}_K$ canonically, so $\varepsilon^*\psi \in K^\times$; replacing ψ by $\psi/\varepsilon^*\psi$ we may assume $\varepsilon^*\psi = 1$. The scheme $\mathcal{A}$ is regular (smooth over the Dedekind ring $\mathcal{O}_K$) and integral (its generic fibre is irreducible and it is flat over $\mathcal{O}_K$), and its closed fibres are smooth and connected, hence irreducible and reduced; so every prime divisor of $\mathcal{A}$ not meeting $\mathcal{A}_K$ is a closed fibre $\pi^{-1}(\mathfrak{p})$, with multiplicity one in $\pi^*\mathfrak{p}$, and $\operatorname{div}(\psi) = \pi^*\Delta$ for a divisor Δ of $\operatorname{Spec}\mathcal{O}_K$. Pulling back along ε ($\pi \circ \varepsilon = \mathrm{id}$) gives $\Delta = \operatorname{div}(\varepsilon^*\psi) = 0$. Hence ψ is an isomorphism $[-1]^*\mathcal{L} \to \mathcal{L}$ compatible with the rigidifications, and $(\mathcal{L}, r)$ is symmetric.

Conclusion. $\mathcal{A}$ is a semiabelian scheme over $\mathcal{O}_K$ with the action $\iota_{\mathcal{A}}$, its generic fibre is an abelian variety of dimension $g \geq 1$, and $(\mathcal{L}, r)$ is a rigidified symmetric line bundle with a cubical structure; through β, $\mathcal{L}|_{\mathcal{A}_K}$ is $M \otimes_L K$, which is ample with polarisation $\phi_M \otimes_L K$, an isomorphism, with whose Rosati involution $\iota_{\mathcal{A}}$ is compatible. So $(\mathcal{A}, \mathcal{L}, r)$ with $\iota_{\mathcal{A}}$ is a principally polarised abelian scheme with $\mathcal{O}_E$-action over $\mathcal{O}_K$. For $\sigma'\colon K \to \mathbb{C}$, β identifies the complex torus $\mathcal{A}_{K,\sigma'}(\mathbb{C})$, its period lattice and the Riemann form of the polarisation with those of B and ϕ_M at $\sigma'|_L$, which gives the Hodge norm. □

Lemma 34.10. *The notation introduced here is local to this result and to the results that apply it. Let L be a number field, $\mathcal{G}$ a semiabelian scheme over $\mathcal{O}_L$ whose generic fibre $\mathcal{G}_L$ has a split maximal subtorus T, and $X = X_*(T)$, so that* $\mathrm{Lie}(T) = X \otimes_{\mathbb{Z}} L \subseteq \mathrm{Lie}(\mathcal{G}_L) = \mathrm{Lie}(\mathcal{G}) \otimes_{\mathcal{O}_L} L$. *Then*

$$\mathrm{Lie}(\mathcal{G}) \cap \mathrm{Lie}(T) \;\subseteq\; X \otimes_{\mathbb{Z}} \mathcal{O}_L.$$

Proof. We use the following standard facts: there is no non-zero homomorphism from a torus to an abelian variety over a field; homomorphisms from a group of multiplicative type to a smooth group scheme over an Artinian local ring are determined by their reduction (rigidity, SGA 3, Exp. IX); homomorphisms between split tori over a connected base are given by homomorphisms of their cocharacter groups; for a separated scheme $\mathcal{Z}$ of finite type over a complete discrete valuation ring R' with uniformiser ϖ', $\mathcal{Z}(R') = \varprojlim_n \mathcal{Z}(R'/\varpi'^n)$.

Step 0: reductions. Let T_X be the split torus over $\mathcal{O}_L$ with cocharacter group X, so $T_{X,L} = T$. The inclusion $T \to \mathcal{G}_L$ extends to a homomorphism over a dense open subscheme of $\operatorname{Spec}\mathcal{O}_L$ (finite presentation), hence, by Theorem B.104, to a homomorphism $\iota\colon T_X \to \mathcal{G}$ over $\mathcal{O}_L$. Its differential $d\iota\colon X \otimes_{\mathbb{Z}} \mathcal{O}_L = \mathrm{Lie}(T_X) \to \mathrm{Lie}(\mathcal{G})$ becomes over L the inclusion $\mathrm{Lie}(T) \subseteq \mathrm{Lie}(\mathcal{G}_L)$. Suppose that $d\iota \otimes \mathbf{k}(\mathfrak{p})$ is injective for every maximal ideal $\mathfrak{p}$. Then $N = d\iota(X \otimes \mathcal{O}_L)$ has $\mathrm{Tor}_1(\mathrm{Lie}(\mathcal{G})/N, \mathbf{k}(\mathfrak{p})) = 0$ for all $\mathfrak{p}$, so $\mathrm{Lie}(\mathcal{G})/N$ is torsion free and $\mathrm{Lie}(\mathcal{G}) \cap (N \otimes L) = N$; as $N \otimes L = \mathrm{Lie}(T)$, this is the claim. Injectivity of $d\iota \otimes \mathbf{k}(\mathfrak{p})$ may be checked after base change to the completion R of $\mathcal{O}_{L,\mathfrak{p}}$ followed by a finite unramified extension; we fix such an R (a complete discrete valuation ring with finite residue field k of characteristic p, uniformiser ϖ, fraction field

K), keep the notation $\mathcal{G}$, ι for the base changes, and write ^{for} for completion along the closed point.

Step 1: the Raynaud extension. The special fibre $\mathcal{G}_k$ is an extension of an abelian variety by a torus, and every torus over the finite field k is isotrivial. By Theorem B.109(1) (with $I = \varpi R$) there is an extension $0 \to \widetilde{T} \to \widetilde{G} \to A \to 0$ over R, A an abelian scheme and $\widetilde{T}$ a torus, with $\widetilde{G}^{\mathrm{for}} \cong \mathcal{G}^{\mathrm{for}}$; after a further finite unramified extension $\widetilde{T}$ is split, with cocharacter group Y. The Lie algebras agree: $\mathrm{Lie}(\mathcal{G}) \otimes R/\varpi^n$ depends only on $\mathcal{G} \otimes R/\varpi^n$, so $\mathrm{Lie}(\mathcal{G}) = \varprojlim_n \mathrm{Lie}(\mathcal{G} \otimes R/\varpi^n) = \mathrm{Lie}(\widetilde{G})$.

Step 2: ι through the Raynaud torus. Over each R/ϖ^n, the composite of ι^{for} with $\mathcal{G}^{\mathrm{for}} \cong \widetilde{G}^{\mathrm{for}} \to A^{\mathrm{for}}$ is a homomorphism from the torus T_X to the abelian scheme A which vanishes on the special fibre; by rigidity it vanishes. So ι^{for} factors through $\widetilde{T}^{\mathrm{for}}$; the homomorphisms $T_X \otimes R/\varpi^n \to \widetilde{T} \otimes R/\varpi^n$ of split tori are given by homomorphisms $X \to Y$, compatible in n, hence by one $f\colon X \to Y$. Let $\tilde{\iota}\colon T_X \to \widetilde{T}$ be the homomorphism over R given by f. Then ι^{for} is $\tilde{\iota}^{\mathrm{for}}$ followed by $\widetilde{T}^{\mathrm{for}} \subseteq \widetilde{G}^{\mathrm{for}} \cong \mathcal{G}^{\mathrm{for}}$, and under $\mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\widetilde{G}) \supseteq \mathrm{Lie}(\widetilde{T}) = Y \otimes R$, $d\iota = f \otimes \mathrm{id}_R$.

Step 3: $f \otimes \mathbb{F}_p$ is injective. Let $R' = R[\zeta_p]$, ζ_p a primitive p-th root of unity, a complete discrete valuation ring with fraction field K'. By the last of these standard facts, the isomorphism $\widetilde{G}^{\mathrm{for}} \cong \mathcal{G}^{\mathrm{for}}$ induces a group isomorphism $\widetilde{G}(R') \cong \mathcal{G}(R')$ under which $\tilde{\iota}$ and ι correspond on $T_X(R') = X \otimes R'^{\times}$. As $\iota_K\colon T \to \mathcal{G}_K$ is a closed immersion, ι is injective on $T_X(R') \subseteq T_X(K')$, hence so is $\tilde{\iota}$, which is $f \otimes \mathrm{id}\colon X \otimes R'^{\times} \to Y \otimes R'^{\times}$. If $x \in X \smallsetminus pX$ had $f(x) = py$ with $y \in Y$, then $t = x \otimes \zeta_p \neq 1$ (as X is free, $X \otimes \mu_p \cong X/pX \otimes \mu_p$ injects into $X \otimes R'^{\times}$) and $\tilde{\iota}(t) = y \otimes \zeta_p^p = 1$, a contradiction. So $X/pX \to Y/pY$ is injective, hence so is $f \otimes k$, i.e. $d\iota \otimes k$ (Step 2). By Step 0 this proves the lemma. □

Lemma 34.11. *Let $X_{\mathbb{C}}$, G', $\iota_{\mathcal{Y}}$ and $X^{\circ}_{\mathbb{C}}$ be as in Lemma 33.16 (case (A) or (B)), and let X_0 be an irreducible component of $X_{\mathbb{C}}$, with its reduced structure, that meets $X^{\circ}_{\mathbb{C}}$. Then there are a smooth projective variety $\widetilde{X}$ over $\mathbb{C}$ and a projective birational morphism $\rho\colon \widetilde{X} \to X_0$, surjective, such that $\widetilde{X}^{\circ} = \rho^{-1}(X^{\circ}_{\mathbb{C}})$ is dense in $\widetilde{X}$ and $\widetilde{X} \smallsetminus \widetilde{X}^{\circ}$ is empty or the support of a simple normal crossing divisor. In particular $\widetilde{X}$ with ρ^*G', the pulled-back action and polarisation and $\widetilde{X}^{\circ}$ is as in Lemma 33.16 (case (B)), the fibre of ρ^*G' at $\tilde{x} \in \widetilde{X}$ is $G'_{\rho(\tilde{x})}$, and every $\tilde{x} \in \widetilde{X} \smallsetminus \widetilde{X}^{\circ}$ has an open neighbourhood V in $\widetilde{X}^{\mathrm{an}}$ with a biholomorphic map $t\colon V \to \Delta^d$, $d = \dim X_0$, such that $t(\tilde{x}) = 0$ and $t(V \cap \widetilde{X}^{\circ}) = \{t_1 \cdots t_a \neq 0\}$ for some $1 \leq a \leq d$.*

Proof. **The resolution.** X_0 is a projective variety over $\mathbb{C}$. By strong resolution (Theorem B.142(A)) there is a birational projective morphism $\Pi_1\colon R \to X_0$ with R smooth; R is projective (projective over the projective X_0) and integral (birational to X_0). Let $Z = X_0 \smallsetminus X^{\circ}_{\mathbb{C}}$, a closed subset, proper because $X_0 \cap X^{\circ}_{\mathbb{C}}$ is a non-empty open subset of the irreducible X_0. Since Π_1 is an isomorphism over the smooth locus of X_0, a dense open subset, which meets the dense open $X_0 \smallsetminus Z$, the closed set $\Pi_1^{-1}(Z)$ is a proper subset of R. Let $I \subseteq \mathcal{O}_R$ be the ideal sheaf of $\Pi_1^{-1}(Z)$ with its reduced structure; it is non-zero on the integral scheme R. By principalisation (Theorem B.142(B)) there is a composite $\Pi_2\colon \widetilde{X} \to R$ of blow-ups with smooth centres, $\widetilde{X}$ smooth and Π_2 projective and birational, such that Π_2^*I is the ideal sheaf of a simple normal crossing divisor D (possibly $D = 0$). Put $\rho = \Pi_1 \circ \Pi_2$: it is projective and birational, $\widetilde{X}$ is a smooth projective variety, and ρ is surjective since its image is closed and contains a dense open subset of X_0. As sets, the zero locus of Π_2^*I is $\Pi_2^{-1}(\Pi_1^{-1}(Z)) = \rho^{-1}(Z)$, so $\widetilde{X} \smallsetminus \widetilde{X}^{\circ} = \rho^{-1}(Z)$ is the support of D, empty or a divisor, and $\widetilde{X}^{\circ}$ is dense in the irreducible $\widetilde{X}$.

The pulled-back family. The composite of ρ with the inclusion $X_0 \subseteq X_{\mathbb{C}}$ (and, in case (B), with the morphism of $X_{\mathbb{C}}$ to the variety of case (A)) is a morphism from the projective variety $\widetilde{X}$ to the variety of case (A) of Lemma 33.16, and ρ^*G' with the pulled-back action

and polarisation is an abelian scheme exactly over $\rho^{-1}(X_{\mathbb{C}}^{\circ}) = \widetilde{X}^{\circ}$; so case (B) of Lemma 33.16 applies. The fibre of a base change at $\tilde{x}$ is the fibre of G' at $\rho(\tilde{x})$.

The charts. Let $\tilde{x} \in \operatorname{supp} D$ and let $D_1, \dots, D_a$ ($a \geq 1$) be the components of D through $\tilde{x}$. By the definition of a simple normal crossing divisor there are local coordinates $z_1, \dots, z_d$ in $\mathfrak{m}_{\tilde{x}}$, $d = \dim \widetilde{X} = \dim X_0$, with $D_i = (z_i = 0)$ near $\tilde{x}$ after renumbering. The z_i are a regular system of parameters at a smooth point, so their differentials at $\tilde{x}$ form a basis of the cotangent space, and by the holomorphic inverse function theorem $(z_1, \dots, z_d)$ is biholomorphic from a neighbourhood of $\tilde{x}$ onto a neighbourhood of 0 in $\mathbb{C}^d$. Shrink this neighbourhood so that it meets no component of D other than $D_1, \dots, D_a$ (the others are closed and avoid $\tilde{x}$) and so that its image is a polydisc $\{|z_i| < r\}$, and put $t = z/r$. Then $V \cap \operatorname{supp} D = \{t_1 \cdots t_a = 0\}$, that is, $t(V \cap \widetilde{X}^{\circ}) = \{t_1 \cdots t_a \neq 0\}$. □

Lemma 34.12. *Let $X_{\mathbb{C}}$, G', $\iota_{\mathcal{Y}}$ and $X_{\mathbb{C}}^{\circ}$ be as in Lemma 33.16, with $X_{\mathbb{C}}^{\circ}$ dense in $X_{\mathbb{C}}$. For every complex point z of $X_{\mathbb{C}}$ let h_z be a positive definite hermitian form on $\operatorname{Lie}(G'_z)$ such that for every Zariski open $U \subseteq X_{\mathbb{C}}$ and all sections s, s' of $\operatorname{Lie}(G')$ over U the function $z \mapsto h_z(s(z), s'(z))$ is continuous on $U(\mathbb{C})$. Then there is $K \geq 1$ such that:*

(i) $\pi H_z(v,v) \leq K\, h_z(v,v)$ for every $z \in X_{\mathbb{C}}^{\circ}(\mathbb{C})$ and $v \in \operatorname{Lie}(G'_z)$;

(ii) for every projective variety $\widetilde{X}$ over $\mathbb{C}$ with a morphism $\rho\colon \widetilde{X} \to X_{\mathbb{C}}$, and every point $\tilde{x} \in \widetilde{X} \smallsetminus \rho^{-1}(X_{\mathbb{C}}^{\circ})$ at which the chart setting of Lemma 33.16 holds for $\widetilde{X}$, ρ^*G' and $\rho^{-1}(X_{\mathbb{C}}^{\circ})$ (case (B)), with $q\colon \operatorname{Lie}(G'_{\rho(\tilde{x})}) \to M_{\tilde{x}}$ the quotient map and H_0 the form of (Cv) of Lemma 33.18 at $\tilde{x}$,

$$\pi H_0(q(v), q(v)) \leq K\, h_{\rho(\tilde{x})}(v,v) \qquad (v \in \operatorname{Lie}(G'_{\rho(\tilde{x})})).$$

Proof. **Step 1: the bound near a boundary point with a chart.** Let $\rho\colon \widetilde{X} \to X_{\mathbb{C}}$ be a morphism from a projective variety and $\tilde{x} \in \widetilde{X} \smallsetminus \rho^{-1}(X_{\mathbb{C}}^{\circ})$ with a chart $t\colon V \to \Delta^d$ as in Lemma 33.16 for ρ^*G'; identify V with Δ^d. We show: there are an open neighbourhood $V_1 \subseteq V$ of $\tilde{x}$ and $C_1 \geq 1$ with $\pi H_{\rho(z)}(v,v) \leq C_1 \tilde{h}_z(v,v)$ for $z \in V_1 \cap V^*$ and $v \in \operatorname{Lie}(G'_{\rho(z)})$. By Lemma 34.7, $\widehat{P}_{\tilde{x}}$ maps Λ^{inv} onto $\ker \exp_{\tilde{x}}$; the latter contains $2\pi i\, X_*(T_{\tilde{x}})$, which spans $\operatorname{Lie}(T_{\tilde{x}})$ over $\mathbb{C}$, and maps onto the period lattice of the abelian quotient A_0, which spans $\operatorname{Lie}(A_0)$ over $\mathbb{R}$. So $\widehat{P}_{\tilde{x}}(\Lambda^{\mathrm{inv}})$ spans $\operatorname{Lie}(G'_{\rho(\tilde{x})})$ over $\mathbb{C}$, and there are $\lambda_1, \dots, \lambda_g \in \Lambda^{\mathrm{inv}}$ with $\widehat{P}_{\tilde{x}}(\lambda_i)$ a basis. Each λ_i lies in $\bigcap_j \ker N_j$, so it is a holomorphic section of the canonical extension near $\tilde{x}$, and $b_i = \widehat{P}(\lambda_i)$ is a holomorphic section of $\operatorname{Lie}(\rho^*G')^{\mathrm{an}}$ equal to $P(\lambda_i)$ on V^*; the b_i form a frame on a closed polydisc $\bar{V}_1$ around $\tilde{x}$, on which the Gram matrix $(\tilde{h}(b_i, b_l))$ is continuous and positive definite, hence $\geq c_1 \cdot \mathrm{id}$ for some $c_1 > 0$ by compactness. Let M be the constant of Theorem B.146 and $0 < r < \min(e^{-M}, 1)$ with $\{|t_j| \leq r\} \subseteq \bar{V}_1$, and $U = \{t \in V^* : |t_j| < r\ (1 \leq j \leq d)\}$, a subanalytic set relatively compact in V on which the λ_i are single-valued flat sections. For a numbering $(j_1, \dots, j_a)$ of $\{1, \dots, a\}$ and $1 \leq \nu \leq a$, the nilpotent $N(\nu) = \sum_{\mu \geq \nu} N_{j_\mu}$ satisfies $N(\nu)^2 = 0$, so its weight filtration centred at 0 is $W_{-1}(\nu) = \operatorname{im} N(\nu) \subseteq W_0(\nu) = \ker N(\nu) \subseteq W_1(\nu) = H$, and $\lambda_i \in W_0(\nu)$ for every ν. Every $t \in U$ satisfies $y_{j_\nu}(t) = -\log|t_{j_\nu}| > M$ and, for the numbering with $y_{j_1}(t) \leq \cdots \leq y_{j_a}(t)$, $y_{j_\nu}/y_{j_{\nu-1}} \geq 1 > \frac{1}{2}$. Theorem B.146 with $\varepsilon = \frac{1}{2}$ and all $p_\nu = 0$, applied to each of the finitely many numberings and each λ_i, gives C_2 with $|\lambda_i|_t^2 = h_t^{\mathrm{Hdg}}(\lambda_i, \lambda_i) \leq C_2$ on U. For $z \in U$, $c \in \mathbb{C}^g$ and $y = \sum_i c_i \lambda_i$, let $y^{\perp}$ be the component of y orthogonal to $\mathcal{F}_z^0$; as P

kills $\mathcal{F}^0_z$,

$$\begin{aligned} H_{\rho(z)}\Big(\sum_i c_i b_i(z), \sum_i c_i b_i(z)\Big) &= 2h^{\mathrm{Hdg}}_z(y^\perp, y^\perp) \le 2h^{\mathrm{Hdg}}_z(y,y) \\ &\le 2\Big(\sum_i |c_i|\,|\lambda_i|_z\Big)^2 \le 2gC_2 \sum_i |c_i|^2 \\ &\le \frac{2gC_2}{c_1}\,\tilde h_z\Big(\sum_i c_i b_i(z), \sum_i c_i b_i(z)\Big). \end{aligned}$$

So $V_1 = \{|t_j| < r\}$ and $C_1 = \max(1, 2\pi g C_2/c_1)$ will do.

Step 2: the bound near a point of $X^\circ_{\mathbb{C}}$. Let $z_0 \in X^\circ_{\mathbb{C}}(\mathbb{C})$ and O a simply connected neighbourhood of z_0 in $X^\circ_{\mathbb{C}}(\mathbb{C})$, small enough that $\mathrm{Lie}(G')^{\mathrm{an}}$ has a holomorphic frame there and $\Lambda|_O$ has a $\mathbb{Z}$-basis of flat sections $\lambda_1, \dots, \lambda_{2g}$. The $P(\lambda_k)$ are holomorphic sections of $\mathrm{Lie}(G')^{\mathrm{an}}$ spanning every fibre over $\mathbb{R}$, so $iP(\lambda_k) = \sum_l a_{kl} P(\lambda_l)$ with a real matrix a depending continuously on z (Cramer's rule in real coordinates of the frame). With $E = \operatorname{Im} H_z$, constant on flat sections,

$$\begin{aligned} H_z\big(P(\lambda_k), P(\lambda_l)\big) &= \operatorname{Im} H_z\big(iP(\lambda_k), P(\lambda_l)\big) + i \operatorname{Im} H_z\big(P(\lambda_k), P(\lambda_l)\big) \\ &= \sum_m a_{km}(z) E(\lambda_m, \lambda_l) + iE(\lambda_k, \lambda_l), \end{aligned}$$

so H_z depends continuously on $z \in O$, as does h_z (positive definite). On a compact neighbourhood of z_0 in O the largest eigenvalue of πH_z relative to h_z is therefore bounded.

Step 3: compactness; proof of (i). As $X^\circ_{\mathbb{C}}$ is dense, every irreducible component X_0 of $X_{\mathbb{C}}$ meets it. Let $\rho_0 \colon \tilde X_0 \to X_0$ be as in Lemma 34.11. Every point of the compact space $\tilde X_0^{\mathrm{an}}$ has an open neighbourhood O' and a constant C' with $\pi H_{\rho_0(z)} \le C' \tilde h_z$ for $z \in O' \cap \rho_0^{-1}(X^\circ_{\mathbb{C}})$: points outside $\rho_0^{-1}(X^\circ_{\mathbb{C}})$ by Step 1 (charts exist by Lemma 34.11), points of $\rho_0^{-1}(X^\circ_{\mathbb{C}})$ by Step 2 at their image, through the continuous map ρ_0. Finitely many O' cover $\tilde X_0^{\mathrm{an}}$; let K_0 be the largest of their constants. As ρ_0 is surjective, every $z \in X_0 \cap X^\circ_{\mathbb{C}}$ is $\rho_0(\tilde z)$ with $\tilde z \in \rho_0^{-1}(X^\circ_{\mathbb{C}})$, so $\pi H_z \le K_0 h_z$. Let $K \ge 1$ be the largest K_0 over the finitely many components; this is (i).

Step 4: proof of (ii). Let $\tilde X$, ρ, $\tilde x$ and a chart V be as in (ii), and $v \in \mathrm{Lie}(G'_{\rho(\tilde x)})$. Take a holomorphic section b of $\mathrm{Lie}(\rho^* G')^{\mathrm{an}}$ near $\tilde x$ with $b(\tilde x) = v$ (constant coefficients in a holomorphic frame). By Lemma 34.13 at $\tilde x$, by (i) at the points $\rho(z) \in X^\circ_{\mathbb{C}}$ ($z \in V^*$), and by continuity of $\tilde h(b,b)$,

$$\pi H_0\big(q(v), q(v)\big) \le \liminf_{z \to \tilde x,\ z \in V^*} \pi H_{\rho(z)}\big(b(z), b(z)\big) \le K \lim_{z \to \tilde x} \tilde h_z\big(b(z), b(z)\big) = K\, h_{\rho(\tilde x)}(v, v),$$

where V^* accumulates at $\tilde x$ because it is the complement of $\{t_1 \cdots t_a = 0\}$ in V. □

Lemma 34.13. *In the setting of Lemma 33.16 at a point x with a chart $t \colon V \to \Delta^d$, for $z \in V^*$ let $S_z \subseteq \mathrm{Lie}(G'_z)$ be the image of $I \otimes \mathbb{C}$ under P at z, and for $u \in \mathrm{Lie}(G'_z)$ put*

$$H^{\mathrm{q}}_z(u) = \min_{s \in S_z} H_z(u - s, u - s).$$

Let b be a holomorphic section of $\mathrm{Lie}(G')^{\mathrm{an}}$ on a neighbourhood of x, and $\bar b(x)$ the image of $b(x)$ in M_x. Then

$$\lim_{z \to x,\ z \in V^*} H^{\mathrm{q}}_z(b(z)) = H_0\big(\bar b(x), \bar b(x)\big);$$

in particular $H_0(\bar b(x), \bar b(x)) \le \liminf_{z \to x,\, z \in V^*} H_z(b(z), b(z))$.

Proof. **The weight filtration.** Put $W_{-2} = \operatorname{im} N \subseteq W_{-1} = \ker N \subseteq W_0 = H$, the monodromy weight filtration of N centred at -1 ($N^2 = 0$). Since T_j preserves Q and $T_j^{-1} = 1 - N_j$, expanding $Q(T_j u, T_j u') = Q(u, u')$ and $Q(T_j^{-1} u, T_j^{-1} u') = Q(u, u')$ and adding gives $Q(N_j u, N_j u') = 0$, hence $Q(N_j u, u') = -Q(u, N_j u')$ and $Q(Nu, u') = -Q(u, Nu')$. So $W_{-2} \perp_Q W_{-1}$, and since $\dim W_{-2} + \dim W_{-1} = \dim H$ and Q is non-degenerate, $W_{-1} = W_{-2}^{\perp_Q}$; thus $\dim W_{-1} = 2g - t_x$, $\dim W_{-2} = t_x$, W_{-2} is Q-isotropic and Q induces a non-degenerate form on $\mathrm{Gr}_{-1} = W_{-1}/W_{-2}$. Both are defined over $\mathbb{Q}$, and $I \otimes \mathbb{C} = W_{-2}$. From $(N_j + N_k)^2 = 0$ and $N_j^2 = 0$ we get $N_j N_k = -N_k N_j$; the T_j commute (Theorem B.145), so $N_j N_k = N_k N_j$, hence $N_j N_k = 0$ for all j, k, so $\operatorname{im} N_j \subseteq \bigcap_k \ker N_k = W_{-1}$, and every N_j vanishes on W_{-1}.

Types of the limit. By Theorem B.145, $\widehat{\mathcal{F}}^1 = 0$ and $\widehat{\mathcal{F}}^{-1} = \widehat{\mathcal{H}}$ (the extensions of the zero subbundle and of the whole bundle), so $F_0^1 = 0$, $F_0^{-1} = H$, and the Hodge numbers $h^{p,q}$ of the pure Hodge structures Gr_k^W vanish unless $-1 \le p, q \le 0$: Gr_{-2}^W is of type $(-1,-1)$, Gr_{-1}^W of types $(0,-1)$ and $(-1,0)$, Gr_0^W of type $(0,0)$. Hence:

(i) $F_0^0 \cap W_{-2} = 0$, since $F^0 \mathrm{Gr}_{-2}^W = 0$ and $W_{-3} = 0$;

(ii) $E_0 := F_0^0 \cap W_{-1}$ maps isomorphically onto $F^0 \mathrm{Gr}_{-1}$ (onto by definition of the induced filtration, injective by (i)), so $\dim E_0 = \frac{1}{2} \dim \mathrm{Gr}_{-1} = g - t_x$;

(iii) Gr_{-1} equals its primitive part ($N\mathrm{Gr}_{-1} \subseteq \mathrm{Gr}_{-3} = 0$), so its Hodge structure of weight -1 is polarised by Q: $\mathrm{Gr}_{-1,\mathbb{C}} = F^0 \mathrm{Gr}_{-1} \oplus \overline{F^0 \mathrm{Gr}_{-1}}$ with both summands Q-isotropic.

Let $k(w, w') = -iQ(w, \bar{w}')$ on H; it is hermitian, as Q is real and alternating.

Untwisted coordinates. For $z \in V_2$ let $\widetilde{F}_z \subseteq H$ be the fibre of $\widehat{\mathcal{F}}^0$ at z read in the frame $\tilde{s}$ (the coordinate vectors c with $\sum_k c_k \tilde{s}_k(z) \in \widehat{\mathcal{F}}_z^0$); it depends continuously on z and $\widetilde{F}_0 = F_0^0$. Put $E_z = \widetilde{F}_z \cap W_{-1}$. For $z \in V^* \cap V_2$ and a branch of the z_j near z, the flat identification $\Lambda_z \otimes \mathbb{C} = H$ gives $\mathcal{F}_z^0 = U_z \widetilde{F}_z$ with $U_z = \exp(-\sum_j z_j N_j) = 1 - \sum_j z_j N_j$, which preserves W_{-1} and is the identity on it; hence $\mathcal{F}_z^0 \cap W_{-1} = U_z(E_z) = E_z$, independently of the branch.

Dimension and continuity of E_z. Let $z \in V^* \cap V_2$. If $y \in \mathcal{F}_z^0 \cap W_{-2}$, then $\bar{y} \in W_{-2}$, so $Q(y, \bar{y}) = 0$, while $h_z^{\mathrm{Hdg}}(y, y) = Q(iy, \bar{y})$ (y is of type $(0,-1)$); so $y = 0$. Hence $E_z \cap W_{-2} = 0$, $\dim E_z \ge \dim \mathcal{F}_z^0 + \dim W_{-1} - \dim H = g - t_x$, and the image of E_z in $\mathrm{Gr}_{-1,\mathbb{C}}$ is a Q-isotropic subspace ($\mathcal{F}_z^0$ is isotropic, $Q(\mathcal{F}^0, \mathcal{F}^0) = 0$ in weight -1) of the same dimension, so $\dim E_z \le g - t_x$. With (ii), $\dim E_z = g - t_x$ for all $z \in (V^* \cap V_2) \cup \{0\}$. There E_z is the kernel of the map $\widetilde{F}_z \to H/W_{-1}$, which depends continuously on z and has constant rank t_x; so E_z, and $\overline{E_z}$, depend continuously on $z \in (V^* \cap V_2) \cup \{0\}$.

The orthogonal complement at $z \in V^*$. The h_z^{Hdg}-orthogonal complement of $\mathcal{F}_z^0 + W_{-2}$ in H is $\overline{E_z}$, and $h_z^{\mathrm{Hdg}} = k$ on it. Indeed, the Hodge decomposition $H = \mathcal{F}_z^0 \oplus \overline{\mathcal{F}_z^0}$ is h_z^{Hdg}-orthogonal; for $w \in \overline{\mathcal{F}_z^0}$, of type $(-1,0)$, $C_z w = -iw$, so $h_z^{\mathrm{Hdg}}(w, u) = -iQ(w, \bar{u})$, and as u runs through W_{-2} so does $\bar{u}$; hence $w \perp W_{-2}$ if and only if $w \in W_{-2}^{\perp_Q} = W_{-1}$. The complement is therefore $\overline{\mathcal{F}_z^0} \cap W_{-1} = \overline{E_z}$ (W_{-1} is real), on which $h_z^{\mathrm{Hdg}}(w, w') = -iQ(w, \bar{w}') = k(w, w')$.

A decomposition continuous up to x. For $z \in (V^* \cap V_2) \cup \{0\}$,

$$H = (\widetilde{F}_z + W_{-2}) \oplus \overline{E_z}.$$

For $z \in V^*$ apply U_z^{-1} to $H = (\mathcal{F}_z^0 + W_{-2}) \oplus \overline{E_z}$ (previous paragraph); U_z^{-1} maps $\mathcal{F}_z^0$ to $\widetilde{F}_z$ and fixes $W_{-2} \subseteq W_{-1}$ and $\overline{E_z} \subseteq W_{-1}$ pointwise. For $z = 0$: $\dim(F_0^0 + W_{-2}) = g + t_x$ by (i) and $\dim \overline{E_0} = g - t_x$; if $w = f + u \in \overline{E_0}$ with $f \in F_0^0$, $u \in W_{-2}$, then $f = w - u \in W_{-1}$, so $f \in E_0$, and the image of w in $\mathrm{Gr}_{-1,\mathbb{C}}$ lies in $F^0 \mathrm{Gr}_{-1} \cap \overline{F^0 \mathrm{Gr}_{-1}} = 0$ by (ii), (iii); so $w \in W_{-2}$ and $\bar{w} \in E_0 \cap W_{-2} = 0$ by (i). The three subspaces depend continuously on z with constant

dimensions and their sum is direct at every such z, so the projection $\pi_z\colon H \to \overline{E_z}$ along $\widetilde{F}_z + W_{-2}$ depends continuously on $z \in (V^* \cap V_2) \cup \{0\}$.

The quotient form. Shrinking the neighbourhood of x, complete a holomorphic frame of $\widehat{\mathcal{F}}^0$ to a frame of $\widehat{\mathcal{H}}$; its new members map under $\widehat{P}$ to a holomorphic frame of $\mathrm{Lie}(G')^{\mathrm{an}}$, so $b = \widehat{P}(\tilde{b})$ for a holomorphic section $\tilde{b}$ of $\widehat{\mathcal{H}}$. Let $c(z) \in H$ be the coordinate vector of $\tilde{b}(z)$ in the frame $\tilde{s}$; it is continuous and $\widehat{P}_x(c(0)) = b(x)$ (the fibre at x being H). For $z \in V^* \cap V_2$, $\tilde{b}(z) = U_z c(z)$ in $\Lambda_z \otimes \mathbb{C} = H$ and $P(U_z c(z)) = b(z)$; P identifies $H/\mathcal{F}^0_z$ with $\mathrm{Lie}(G'_z)$, W_{-2} with S_z, and the quotient metric of h^{Hdg}_z with $\frac{1}{2}H_z$ (Lemma 33.16: the minimum over a coset of $\mathcal{F}^0_z$ is attained at its element orthogonal to $\mathcal{F}^0_z$). Hence $H^{\mathrm{q}}_z(b(z))$ is twice the h^{Hdg}_z-quotient norm of $U_z c(z)$ modulo $\mathcal{F}^0_z + W_{-2}$, that is $2k(w_z, w_z)$ with w_z the component of $U_z c(z)$ in $\overline{E_z}$ for $H = (\mathcal{F}^0_z + W_{-2}) \oplus \overline{E_z}$; applying U_z^{-1}, $w_z = \pi_z(c(z))$. By continuity,

$$H^{\mathrm{q}}_z(b(z)) = 2k\big(\pi_z c(z), \pi_z c(z)\big) \longrightarrow 2k\big(\pi_0 c(0), \pi_0 c(0)\big) \qquad (z \to x,\ z \in V^*).$$

The limit is H_0. As $\ker \widehat{P}_x = F^0_0$, the preimage of $\widehat{P}_x(I \otimes \mathbb{C}) = \widehat{P}_x(W_{-2})$ under $\widehat{P}_x$ is $F^0_0 + W_{-2} = \ker \pi_0$. So $H'(m, m') = 2k(\pi_0 c_m, \pi_0 c_{m'})$, with $c_m \in H$ any vector whose image under $\widehat{P}_x$ maps to $m \in M_x$, is a well-defined hermitian form on M_x, and the limit above is $H'(\bar{b}(x), \bar{b}(x))$. A hermitian form K is determined by $\operatorname{Im} K$ on a set spanning the space over $\mathbb{R}$ ($K(m, m') = \operatorname{Im} K(im, m') + i \operatorname{Im} K(m, m')$), and Λ^{ab}_x is a lattice in M_x; so it suffices to show $\operatorname{Im} H' = E^{\mathrm{ab}}$ on Λ^{ab}_x. Let $\lambda, \lambda' \in \Lambda^{\mathrm{inv}} \subseteq W_{-1}$, real, with images m, m', and take $c_m = \lambda$, $c_{m'} = \lambda'$. Write $\lambda = e + f + u$ with $e = \pi_0 \lambda \in \overline{E_0}$, $f \in F^0_0$, $u \in W_{-2}$; then $f = \lambda - e - u \in W_{-1}$, so $f \in E_0$, and in $\mathrm{Gr}_{-1,\mathbb{C}}$ the real class $[\lambda] = [e] + [f]$ with $[e] \in \overline{F^0 \mathrm{Gr}_{-1}}$, $[f] \in F^0 \mathrm{Gr}_{-1}$; comparing with its conjugate, $[f] = \overline{[e]}$. The same holds for λ' with $e' = \pi_0 \lambda'$. Since Q is induced on Gr_{-1} and both summands are isotropic (iii),

$$Q(\lambda, \lambda') = Q\big([e], \overline{[e']}\big) + Q\big(\overline{[e]}, [e']\big) = 2 \operatorname{Re} Q(e, \overline{e'}),$$

so $\operatorname{Im} H'(m, m') = \operatorname{Im}\big(-2iQ(e, \overline{e'})\big) = -2 \operatorname{Re} Q(e, \overline{e'}) = -Q(\lambda, \lambda') = E^{\mathrm{ab}}(m, m')$. Hence $H' = H_0$, which proves the limit. Finally $H^{\mathrm{q}}_z(b(z)) \le H_z(b(z), b(z))$ ($s = 0$), which gives the last assertion. □

34.2 Flat determinants of the reference models

The flat determinants (Lx3) say that, on the frozen reference models, the heights of the determinant combinations $\sum_i \overline{\mathcal{L}}_{(a,i)}(1)$ of the root lines are bounded at every point (Proposition 34.14). They follow from flat units of the determinant lines (Proposition 34.15): the determinant ratio of the Hodge pieces at an element of the CM type is trivial on the generic fibre of each component (Proposition 34.16), and the analytic flat unit of Section 33 extends to a nowhere vanishing section on the compactified component (Lemma 34.17), the Hodge Gram determinants of algebraic Lie frames growing only logarithmically at the boundary (Lemma 34.18).

Proposition 34.14. *Let $\mathfrak{u}$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. Then there is a real number $C_{\mathrm{Lx}} \ge 1$ such that for every $c \in \mathrm{Cp}$ the frozen reference model $\mathcal{X}^{\mathrm{fr}}_c$ satisfies the flat determinants (Lx3), read with $\mathcal{X} = \mathcal{X}^{\mathrm{fr}}_c$ and the constant C_{Lx}: $\big|\sum_{i=1}^{n} \widehat{h}_W(\overline{\mathcal{L}}_{(a,i)}(1))\big| \le C_{\mathrm{Lx}}$ for every point W of $\mathcal{X}^{\mathrm{fr}}_c$ and every $a \in \Phi$.*

Proof. Fix $c \in \mathrm{Cp}$ and $a \in \Phi$. By Definition 34.1, $\mathcal{X}^{\mathrm{fr}}_c$ has the arithmetic variety $\mathcal{X}$ and the multidegrees e_r of $\mathcal{X}^{\mathrm{ref}}_c$, so it has the same points, and its root lines at $t = 1$ are $\overline{\mathcal{L}}_{(a,i)}(1) = (\mathcal{O}(e_{(a,i)}), h_{(a,i)})|_{\mathcal{X}}$ with the smooth cone metrics $h_{(a,i)} = h^{\mathrm{ref}}_{(a,i)}(\min(1, t_0))$. Put $e_A = \sum_i e_{(a,i)}$,

$h_A = \prod_i h_{(a,i)}$ and $\overline{\mathcal{A}} = (\mathcal{O}(e_A), h_A)|_{\mathcal{X}} = \bigotimes_i \overline{\mathcal{L}}_{(a,i)}(1)$. Take k, q_0, m and $u = f/g$ from Proposition 34.15, and put $N = kq_0$, $s = m\,u^{q_0}$, $s' = m\,u^{-q_0}$, rational sections of $\mathcal{O}(Ne_A)$ and $\mathcal{O}(-Ne_A)$ with $s\,s' = m^2$.

Step 1: the archimedean norm of u is bounded. The function $\|u\| = |f|/(|g|\,h_A^k)$ is invariant under the scaling of the cone. By item 1 of Proposition 34.15, u generates $\mathcal{O}(ke_A)$ at every point of $\mathcal{X}_{\mathbb{Q}}$, hence near every $x \in \mathcal{X}(\mathbb{C})$ it is a nowhere vanishing holomorphic section; as h_A is smooth and positive, $\|u\|$ extends to a continuous positive function on $\mathcal{X}(\mathbb{C})$. This space is compact ($\mathcal{X}$ is projective), so $A_{c,a} = \sup_{\mathcal{X}(\mathbb{C})} |\log \|u\||$ is finite.

Step 2: a point. Let W be a point of $\mathcal{X}$. Let $\nu\colon \mathcal{X}^\nu \to \mathcal{X}$ be the normalisation, finite and surjective. Choose a closed point of $\nu^{-1}(W_{\mathbb{Q}})$ and a number field L containing its residue field; the resulting L-point of $\mathcal{X}^\nu$ extends to $P\colon \operatorname{Spec} \mathcal{O}_L \to \mathcal{X}^\nu$ by the valuative criterion ($\mathcal{X}^\nu$ is proper over $\mathbb{Z}$), and $\nu \circ P$ has image W. By the projection formula, $\widehat{h}_W(\overline{\mathcal{M}}) = \widehat{\deg}\big((\nu P)^*\overline{\mathcal{M}}\big)/[L:\mathbb{Q}]$ for every Hermitian line bundle $\overline{\mathcal{M}}$ on $\mathcal{X}$, and $\widehat{h}_W$ is additive in $\overline{\mathcal{M}}$; hence

$$\sum_{i=1}^{n} \widehat{h}_W(\overline{\mathcal{L}}_{(a,i)}(1)) = \widehat{h}_W(\overline{\mathcal{A}}) = \frac{1}{N[L:\mathbb{Q}]}\widehat{\deg}\,\overline{M}, \qquad \overline{M} = (\nu P)^*\overline{\mathcal{A}}^{\otimes N}.$$

As u generates at the point $W_{\mathbb{Q}}$, $P^*\nu^*s$ is a generator of $M \otimes L$; by item 2 of Proposition 34.15, $t = P^*\nu^*s \in M$ and $t' = P^*\nu^*s' \in M^{-1}$, with $tt' = m^2$. So $m^2M = tt'M \subseteq t\mathcal{O}_L \subseteq M$, and

$$0 \le \log \#(M/t\mathcal{O}_L) \le \log \#(M/m^2M) = 2[L:\mathbb{Q}]\log m.$$

At an embedding $\sigma\colon L \to \mathbb{C}$, with $x_\sigma \in \mathcal{X}(\mathbb{C})$ the image point, $\log \|t\|_\sigma = \log m + q_0 \log \|u\|(x_\sigma) \in [\log m - q_0 A_{c,a}, \log m + q_0 A_{c,a}]$ by Step 1. Hence, by the definition of the Arakelov degree,

$$\widehat{\deg}\,\overline{M} = \log \#(M/t\mathcal{O}_L) - \sum_\sigma \log \|t\|_\sigma \in \big[-[L:\mathbb{Q}](\log m + q_0 A_{c,a}),\ [L:\mathbb{Q}](\log m + q_0 A_{c,a})\big],$$

and $\big|\sum_i \widehat{h}_W(\overline{\mathcal{L}}_{(a,i)}(1))\big| \le C_{c,a} = \frac{\log m + q_0 A_{c,a}}{kq_0}$, a number that does not depend on W.

Step 3: the constant. The sets Cp and Φ are finite, so $C_{\mathrm{Lx}} = \max\big(1, \max_{c,a} C_{c,a}\big)$ is a real number ≥ 1, and by Step 2 every $\mathcal{X}_c^{\mathrm{fr}}$ satisfies (Lx3) with it. □

Proposition 34.15. *Let $\mathfrak{u}$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. Let $c \in \mathrm{Cp}$ and $a \in \Phi$, let $\mathcal{X} = \mathcal{X}_c^{\mathrm{ref}}$ with multidegrees e_r, and put $e_A = \sum_{i=1}^n e_{(a,i)}$. Then there are integers $k, q_0, m \ge 1$ and a rational section $u = f/g$ of $\mathcal{O}(ke_A)|_{\mathcal{X}}$ (f, g multihomogeneous polynomials with integer coefficients, not vanishing identically on $\mathcal{X}$, of multidegrees d_f, d_g with $d_f - d_g = ke_A$) such that:*

1. *u generates $\mathcal{O}(ke_A)$ at every point of the generic fibre $\mathcal{X}_{\mathbb{Q}}$;*
2. *$m\,u^{q_0}$ and $m\,u^{-q_0}$ are regular sections of $\mathcal{O}(kq_0e_A)$ and of $\mathcal{O}(-kq_0e_A)$ on the normalisation of $\mathcal{X}$.*

Proof. We take $k = q_0 = 1$. Let $\mathcal{X} = \mathcal{X}_c^{\mathrm{ref}}$ with its root origin on the fixed data at c; write Ω for its function field, $y = (y_{k,j})$ for its representing coordinates, $\mathcal{Y}_c$ for the irreducible component of the fixed data (it meets $j(\mathrm{Sh}_F)$) and $\varpi\colon \mathcal{X} \to \mathcal{Y}$ for the cover morphism of (Go3)(i) (called f there). For a multidegree e and $c' \in \prod_k I_k$ write $Y_{c'}^e = \prod_k Y_{k,c'_k}^{e_k}$ for the Laurent monomial in the coordinate variables and $y_{c'}^e = \prod_k y_{k,c'_k}^{e_k} \in \Omega$ when all $y_{k,c'_k} \ne 0$. The open sets $\{Y_{k,c'_k} \ne 0\ \forall k\}$ (the charts) cover $\mathcal{X}$.

Step 1: the determinant root line. For a reference model $S = \varnothing$, so the Hodge lattices are the de Rham lattices $\mathrm{dR}_{s,x}$ at every point x, and by (Go3)(ii),(iii) they are Hodge bundles $\mathcal{V}_s$ with an integral flag whose root lines $\mathcal{L}_r$ are invertible. Put $\mathcal{L}^{\det} = \bigotimes_{i=1}^{n} \mathcal{L}_{(a,i)}$. By (Fh3), the roots (a,i) with $i \le p$ run through the step quotients of the full flag of $\mathcal{V}_{(a,+)}$, and those with $i > p$ (step $n+1-i$, which runs through $1,\dots,q$) through the inverses of the step quotients of the full flag of $\mathcal{V}_{(a,-)}$. For a locally free module with a full flag by locally free submodules with invertible quotients, the determinant is the tensor product of the step quotients: near every point there is a basis adapted to the flag, and $v_1 \wedge \cdots \wedge v_r \mapsto \bigotimes_k (v_k \bmod V_{k-1})$ is independent of the adapted basis. Hence

$$\mathcal{L}^{\det} = \det \mathcal{V}_{(a,+)} \otimes (\det \mathcal{V}_{(a,-)})^{-1},$$

both sides being subsheaves of the constant sheaf with value the line $\mathcal{L}^{\det}_{\Omega} = \det \mathrm{Li}_{(a,+)} \otimes (\det \mathrm{Li}_{(a,-)})^{-1}$ over Ω.

Step 2: the generic fibre. Let $\mathcal{X}_{\mathbb{Q}}$ be the set of points of $\mathcal{X}$ over the generic point of $\operatorname{Spec}\mathbb{Z}$; as $\mathcal{X}$ is an $\mathcal{O}_F$-scheme through $\mathcal{Y}$, $\mathcal{X}_{\mathbb{Q}} = \mathcal{X} \times_{\mathcal{O}_F} F$ is an F-scheme. Since $\mathcal{X}$ is irreducible and its generic point maps into the closed set $\mathcal{Y}_c$, $\varpi(\mathcal{X}) \subseteq \mathcal{Y}_c$, and ϖ maps $\mathcal{X}_{\mathbb{Q}}$ into the generic fibre $\mathcal{Y}_{c,F}$. Let G' be the restriction of $G_{\mathcal{Y}}$ to $\mathcal{Y}_{c,F}$, as in Proposition 34.16. As $G_{\mathcal{X}} = \varpi^* G_{\mathcal{Y}}$ and the Lie algebra of a smooth group scheme commutes with base change, $\mathrm{Lie}(G_{\mathcal{X}})|_{\mathcal{X}_{\mathbb{Q}}} = \varpi^* \mathrm{Lie}(G')$, compatibly with the actions of $\mathcal{O}_E$, hence $\mathrm{Lie}_s(G_{\mathcal{X}}|_{\mathcal{X}_{\mathbb{Q}}}) = \varpi^* \mathrm{Lie}_s(G')$ for every label s (Fh1). Let $x \in \mathcal{X}_{\mathbb{Q}}$. The stalk $\mathrm{Lie}(G_{\mathcal{X}})_x$ is a free $\mathcal{O}_x$-module and $\mathcal{O}_x \subseteq \Omega$, so it maps injectively onto $\mathrm{Lie}_x \subseteq \mathrm{Li}$, and $\mathrm{Lie}_x = \bigoplus_s \mathrm{Lie}_s(G_{\mathcal{X}})_x$ with $\mathrm{Lie}_s(G_{\mathcal{X}})_x \subseteq \mathrm{Li}_s$. If $e \in \mathrm{Lie}_x \cap \mathrm{Li}_s$, its components in this decomposition lie in the spaces $\mathrm{Li}_{s'}$, whose sum is direct, so e equals its s-component. Therefore

$$(\mathcal{V}_s)_x = \mathrm{dR}_{s,x} = \mathrm{Lie}_s(G_{\mathcal{X}})_x = \big(\varpi^* \mathrm{Lie}_s(G')\big)_x \qquad (x \in \mathcal{X}_{\mathbb{Q}}),$$

and by Step 1, $\mathcal{L}^{\det}_x = \big(\varpi^*(\det \mathrm{Lie}_{(a,+)}(G') \otimes (\det \mathrm{Lie}_{(a,-)}(G'))^{-1})\big)_x$ inside $\mathcal{L}^{\det}_{\Omega}$.

Step 3: the flat section. The datum $\mathfrak{u}$ has $n = p + q \ge 2$ (as $p, q \ge 1$ in Definition 29.7), $\mathcal{Y}$ is an integral model of Sh over $\mathcal{O}_F$, $\mathcal{Y}_c$ is an irreducible component meeting $j(\mathrm{Sh}_F)$, and $a \in \Phi_{\mathsf{H}}$. By Proposition 34.16 there is a global section θ of $\det \mathrm{Lie}_{(a,+)}(G') \otimes (\det \mathrm{Lie}_{(a,-)}(G'))^{-1}$ on $\mathcal{Y}_{c,F}$ vanishing at no point. By Step 2 its pull-back generates $\mathcal{L}^{\det}_x$ at every $x \in \mathcal{X}_{\mathbb{Q}}$; let $\theta_{\Omega} \in \mathcal{L}^{\det}_{\Omega}$ be its germ at the generic point, so $\theta_{\Omega} \neq 0$ and θ_{Ω} generates $\mathcal{L}^{\det}_x$ for every $x \in \mathcal{X}_{\mathbb{Q}}$.

Step 4: the rational section u. By (Go3)(iii), $P \mapsto \varrho_r P(y)$ is an isomorphism $\mathcal{O}(e_r)|_{\mathcal{X}} \to \mathcal{L}_r$; tensoring over i and using $\prod_i P_i(y) = (\prod_i P_i)(y)$, $P \mapsto \varrho_A P(y)$ with $\varrho_A = \prod_{i=1}^{n} \varrho_{(a,i)} \in \Omega^{\times}$ is an isomorphism $\mathcal{O}(e_A)|_{\mathcal{X}} \to \mathcal{L}^{\det}$, and on the chart of c' the sheaf $\mathcal{L}^{\det}$ is generated by $\varrho_A y_{c'}^{e_A}$. Fix c' with $y_{k,c'_k} \neq 0$ for all k (the generic point lies in some chart) and let $w \in \Omega^{\times}$ be defined by $\theta_{\Omega} = w\, \varrho_A\, y_{c'}^{e_A}$. The field Ω is generated over $\mathbb{Q}$ by the ratios $y_{k,j}/y_{k,c'_k}$ (the function field of the integral subscheme $\mathcal{X}$ of $\mathbb{P}$, which is flat over $\mathbb{Z}$); writing w as a quotient of two polynomials in these ratios with integer coefficients and clearing the denominators y_{k,c'_k} gives $w = P_1(y)/P_2(y)$ with $P_1, P_2 \in \mathbb{Z}[Y_{k,j}]$ multihomogeneous of the same multidegree and $P_1(y), P_2(y) \neq 0$. Put $M_{\pm} = \prod_k Y_{k,c'_k}^{\max(\pm e_{A,k}, 0)}$ and

$$f = P_1\, M_+, \qquad g = P_2\, M_-.$$

Then f, g are multihomogeneous with integer coefficients, $d_f - d_g = e_A$, $f(y) \neq 0 \neq g(y)$ (so neither vanishes identically on $\mathcal{X}$), and $f(y)/g(y) = w\, y_{c'}^{e_A}$. Let $u = f/g$, a rational section of $\mathcal{O}(e_A)|_{\mathcal{X}}$; under the isomorphism above it corresponds to $\varrho_A f(y)/g(y) = \theta_{\Omega}$.

Step 5: clause 1. Let $x \in \mathcal{X}_{\mathbb{Q}}$ and choose a chart c'' containing x. There $\mathcal{O}(e_A)$ is generated by $Y_{c''}^{e_A}$, and $u = \rho\, Y_{c''}^{e_A}$ with $\rho = f(y)/(g(y)\, y_{c''}^{e_A}) \in \Omega^{\times}$; correspondingly $\theta_{\Omega} = \rho\, \varrho_A y_{c''}^{e_A}$. Both θ_{Ω} (Step 3) and $\varrho_A y_{c''}^{e_A}$ (Step 4) generate the free rank-one $\mathcal{O}_x$-module $\mathcal{L}^{\det}_x$, so $\rho \in \mathcal{O}_x^{\times}$, i.e. u generates $\mathcal{O}(e_A)$ at x. This is clause 1 with $k = 1$.

Step 6: clause 2. By (Go3)(i) $\mathcal{X}$ is normal, so it is its own normalisation; it is noetherian and integral. For a prime divisor Γ of $\mathcal{X}$ (an integral closed subscheme of codimension one, with generic point η_Γ and discrete valuation v_Γ on Ω) put $\mathrm{ord}_\Gamma(u) = v_\Gamma(\rho_\Gamma)$, where $\rho_\Gamma = u/Y_{c''}^{e_A}$ for any chart c'' containing η_Γ (independent of the chart, as the transition functions $y_{c''}^{e_A}/y_{c'''}^{e_A}$ are units on the overlap). Only finitely many Γ have $\mathrm{ord}_\Gamma(u) \neq 0$ (on each of the finitely many affine charts a non-zero rational function has finitely many zero and polar divisors). If Γ meets $\mathcal{X}_\mathbb{Q}$ then $\eta_\Gamma \in \mathcal{X}_\mathbb{Q}$, since a generisation of a point over the generic point of $\operatorname{Spec}\mathbb{Z}$ lies over it as well, and then $\mathrm{ord}_\Gamma(u) = 0$ by Step 5. So each Γ with $\mathrm{ord}_\Gamma(u) \neq 0$ lies over a prime ℓ_Γ, and $v_\Gamma(\ell_\Gamma) \geq 1$. Let $N = \max_\Gamma |\mathrm{ord}_\Gamma(u)|$ and $m = \prod_\Gamma \ell_\Gamma^N$, both products and maxima over the finitely many Γ with $\mathrm{ord}_\Gamma(u) \neq 0$ ($m = 1$ if there is none). Then $v_\Gamma(m\rho_\Gamma) \geq 0$ and $v_\Gamma(m\rho_\Gamma^{-1}) \geq 0$ for every prime divisor Γ: for Γ in the support because $v_\Gamma(m) \geq N$, and for the others because $m \in \mathbb{Z}$. A noetherian normal domain is the intersection of its localisations at the prime ideals of height one; applied to the coordinate ring of each affine chart c'', the functions $m\,u/Y_{c''}^{e_A}$ and $m\,Y_{c''}^{e_A}/u$ are regular on the chart. Hence mu is a regular section of $\mathcal{O}(e_A)$ and $mu^{-1} = m\,g/f$ a regular section of $\mathcal{O}(-e_A)$ on $\mathcal{X}$. This is clause 2 with $k = q_0 = 1$. □

Proposition 34.16. *Let* $\mathfrak{u}$ *be a unitary Shimura datum with* $n = p+q \geq 1$*, let* $(\mathcal{Y}, j, G_\mathcal{Y}, \iota_\mathcal{Y}, \mathcal{L}_\mathcal{Y})$ *be an integral model of* Sh *over* $\mathcal{O}_F$*, let* $\mathcal{Y}'$ *be an irreducible component of* $\mathcal{Y}$ *meeting* $j(\mathrm{Sh}_F)$*, with generic fibre* $\mathcal{Y}'_F = \mathcal{Y}' \times_{\mathcal{O}_F} F$*, and let* $\tau \in \Phi_\mathsf{H}$*. Write* $G' = G_\mathcal{Y} \times_\mathcal{Y} \mathcal{Y}'_F$ *with the action of* $\mathcal{O}_E$ *induced by* $\iota_\mathcal{Y}$*. Then the Hodge pieces* $\mathrm{Lie}_{(\tau,+)}(G')$ *and* $\mathrm{Lie}_{(\tau,-)}(G')$ *are locally free of ranks* p *and* q*, and the invertible sheaf*

$$\det \mathrm{Lie}_{(\tau,+)}(G') \otimes \left(\det \mathrm{Lie}_{(\tau,-)}(G')\right)^{-1}$$

on $\mathcal{Y}'_F$ *has a global section that vanishes at no point of* $\mathcal{Y}'_F$*.*

Proof. Ranks. By Lemma 34.17, $\mathrm{Lie}_{(\tau,+)}(G')$ and $\mathrm{Lie}_{(\tau,-)}(G')$ are locally free of ranks p and q; so $\mathcal{D}$ is an invertible sheaf on X_F. As $\mathcal{Y}$ is normal, its irreducible components are pairwise disjoint, so $\mathcal{Y}'$ is open and closed in $\mathcal{Y}$, integral, normal, projective and flat over $\mathcal{O}_F$; hence X_F is non-empty (it meets $j(\mathrm{Sh}_F)$), integral, normal and projective over F, and j identifies $U = j^{-1}(X_F)$ with the dense open subset $X_F \cap j(\mathrm{Sh}_F)$ of X_F.

A nowhere vanishing section over $\mathbb{C}$. Let $X_\mathbb{C} = X_F \otimes_F \mathbb{C}$ (inclusion $F \subset \mathbb{C}$), a reduced projective scheme over $\mathbb{C}$ (characteristic zero), and $X = X_\mathbb{C}^{\mathrm{an}}$. The set $U(\mathbb{C})$ of points of U over the inclusion is dense in X, so every connected component X' of X contains a connected component Y of $U(\mathbb{C})$, and $X' = X_Y$. For each X' choose such a Y and a function Φ_Y as in Lemma 33.21 (for U, the inclusion and Y); Lemma 34.17 gives a nowhere vanishing holomorphic section of $\mathcal{D}_\mathbb{C}^{\mathrm{an}}$ on X', where $\mathcal{D}_\mathbb{C} = \mathcal{D} \otimes_F \mathbb{C}$. Together these form a nowhere vanishing holomorphic section of $\mathcal{D}_\mathbb{C}^{\mathrm{an}}$ on X. By Theorem B.149 it is the analytification of a section $\xi_\mathbb{C}$ of $\mathcal{D}_\mathbb{C}$ on $X_\mathbb{C}$, which vanishes at no closed point, hence at no point, of $X_\mathbb{C}$.

Descent to F. The space $W_F = H^0(X_F, \mathcal{D})$ is a finite-dimensional F-vector space (X_F is proper), and $H^0(X_\mathbb{C}, \mathcal{D}_\mathbb{C}) = W_F \otimes_F \mathbb{C}$ (flat base change). Let $\mathbb{W}$ be the affine space over F attached to W_F and $Z \subseteq \mathbb{W} \times_F X_F$ the zero scheme of the universal section. The set $\Omega \subseteq \mathbb{W}$ of sections vanishing at no point of X_F is the complement of the image of Z under the proper projection $\mathbb{W} \times_F X_F \to \mathbb{W}$, hence open, and its formation commutes with the base change to $\mathbb{C}$. The point $\xi_\mathbb{C}$ shows $\Omega_\mathbb{C} \neq \emptyset$, so Ω is a non-empty open subset of an affine space over the infinite field F and has an F-rational point ξ: a global section of $\mathcal{D}$ on X_F that vanishes at no point. □

Lemma 34.17. *Let* $\mathfrak{u}$ *be a unitary Shimura datum with* $n = p + q \geq 1$*,* $(\mathcal{Y}, j, G_\mathcal{Y}, \iota_\mathcal{Y}, \mathcal{L}_\mathcal{Y})$ *an integral model of* Sh *over* $\mathcal{O}_F$*,* $\mathcal{Y}'$ *an irreducible component of* $\mathcal{Y}$ *meeting* $j(\mathrm{Sh}_F)$*,* $G' = G_\mathcal{Y} \times_\mathcal{Y} \mathcal{Y}'_F$*,*

$U = j^{-1}(\mathcal{Y}'_F)$, $\tau \in \Phi_{\mathsf{H}}$ and $\mathcal{D} = \det \mathrm{Lie}_{(\tau,+)}(G') \otimes (\det \mathrm{Lie}_{(\tau,-)}(G'))^{-1}$. Then $\mathrm{Lie}_{(\tau,+)}(G')$ and $\mathrm{Lie}_{(\tau,-)}(G')$ are locally free of ranks p and q. Let $s\colon F \to \mathbb{C}$ be an embedding, $X = (\mathcal{Y}'_F \otimes_{F,s} \mathbb{C})^{\mathrm{an}}$, Y a connected component of the set $U^s(\mathbb{C}) \subseteq X$ of points of U over s, X_Y the connected component of X containing Y, and Φ_Y a function as in Lemma 33.21 for (U, s, Y). Then $Y = X_Y \cap U^s(\mathbb{C})$, X_Y is a compact connected normal complex space, and there is a holomorphic section ϑ of $(\mathcal{D} \otimes_{F,s} \mathbb{C})^{\mathrm{an}}$ over X_Y that vanishes nowhere and satisfies

$$\vartheta(z) = \Phi_Y(z; v, w)\,(v_1 \wedge \cdots \wedge v_p) \otimes (w_1 \wedge \cdots \wedge w_q)^{-1}$$

for every $z \in Y$ and every frame (v, w) at z.

Proof. The scheme X_F and the Hodge pieces. As $\mathcal{Y}$ is normal, its irreducible components are pairwise disjoint, so $\mathcal{Y}'$ is open and closed in $\mathcal{Y}$, integral, normal, projective and flat over $\mathcal{O}_F$; its generic fibre X_F is non-empty (it meets $j(\mathrm{Sh}_F)$), integral, normal and projective over F. The set $X_F^\circ = X_F \cap j(\mathrm{Sh}_F)$ is open and dense in X_F, and j identifies U with X_F° and $G'|_{X_F^\circ}$ with the universal abelian scheme over U, compatibly with the $\mathcal{O}_E$-actions (Ud7). G' is smooth over X_F, so $\mathrm{Lie}(G')$ is locally free. Since F contains $\rho(E)$ for every embedding $\rho\colon E \to \mathbb{C}$ (Ud6), $\mathcal{O}_E \otimes_{\mathbb{Z}} F \cong \prod_\rho F$, and its idempotents decompose $\mathrm{Lie}(G') = \bigoplus_\rho \mathrm{Lie}_\rho$ into the subsheaves on which $a \in \mathcal{O}_E$ acts by $\rho(a)$. Each Lie_ρ is a direct summand of a locally free sheaf, hence locally free, of constant rank as X_F is connected; at a geometric point of the dense open X_F° the Kottwitz condition of (Ud4) gives rank p for $\rho = \tau$ (as $\tau(r) = r_{\mathbb{C}}$) and rank q for $\rho = \bar\tau$ (as $\bar\tau(r) = -r_{\mathbb{C}} \neq r_{\mathbb{C}}$). These are $\mathrm{Lie}_{(\tau,+)}(G')$ and $\mathrm{Lie}_{(\tau,-)}(G')$, and their base changes along s are the eigen-subsheaves for $s \circ \tau$ and $s \circ \bar\tau$, which at a point of U over s are the Hodge pieces of (Yf1).

The space X. As F has characteristic zero, $X_F \otimes_{F,s} \mathbb{C}$ is reduced, normal and projective over $\mathbb{C}$, so X is a compact normal complex space; a connected normal complex space is irreducible, so X_Y is irreducible. The complement $B = X \smallsetminus U^s(\mathbb{C})$ is the analytic set of a Zariski closed subset of codimension ≥ 1, so $X_Y \smallsetminus B$ is the complement of a nowhere dense analytic subset of the irreducible space X_Y, hence connected; it is open and closed in $U^s(\mathbb{C})$ and contains Y, so $X_Y \cap U^s(\mathbb{C}) = Y$.

The section on Y. For $z \in Y$ put $\vartheta(z) = \Phi_Y(z; v, w)\,(v_1 \wedge \cdots \wedge v_p) \otimes (w_1 \wedge \cdots \wedge w_q)^{-1}$ for a frame (v, w) at z. By clause 1 of Lemma 33.21 this does not depend on the frame (replacing (v, w) by (Av, Bw) multiplies Φ_Y by $\det B / \det A$ and the wedge tensor by $\det A / \det B$); by clause 2, applied to analytifications of algebraic local frames of the Hodge pieces, ϑ is holomorphic on Y; it vanishes nowhere as Φ_Y takes values in $\mathbb{C}^\times$.

Extension over the general points of the boundary. Let $B' \subseteq X_Y$ be the union of the singular locus of X_Y, the singular locus of $B \cap X_Y$, the irreducible components of $B \cap X_Y$ of codimension ≥ 2, and the pairwise intersections of distinct irreducible components of $B \cap X_Y$; as X_Y is normal, B' is an analytic subset of codimension ≥ 2. Let $x \in (B \cap X_Y) \smallsetminus B'$. Then X_Y is smooth at x and near x the set B is a smooth hypersurface, so there are an open neighbourhood V of x and a biholomorphic map $t\colon V \to \Delta^d$ with $t(x) = 0$ and $t(V \cap Y) = \{t_1 \neq 0\}$; shrinking V, there are holomorphic frames v of $\mathrm{Lie}_{(\tau,+)}(G')^{\mathrm{an}}$ and w of $\mathrm{Lie}_{(\tau,-)}(G')^{\mathrm{an}}$ over V. On $V \cap Y$ put $f(z) = \Phi_Y(z; v(z), w(z))$, holomorphic and nowhere zero, with $\vartheta = f\,(v_1 \wedge \cdots \wedge v_p) \otimes (w_1 \wedge \cdots \wedge w_q)^{-1}$. By clause 3 of Lemma 33.21 there is $c_0 > 0$ with $|f(z)|^2 = c_0 \det \Gamma(w(z)) / \det \Gamma(v(z))$ on $V \cap Y$. By Lemma 34.18 (for v with $m = p$ and for w with $m = q$, both linearly independent at x), there are $C > 0$, $N \geq 0$ and $r_0 \in (0, 1)$ with

$$|f(z)| \leq C\big(1 + |\log|t_1(z)||\big)^N, \qquad |f(z)|^{-1} \leq C\big(1 + |\log|t_1(z)||\big)^N \qquad \big(t(z) \in \Delta^*_{r_0} \times \Delta^{d-1}_{r_0}\big),$$

Δ_{r_0} the disc of radius r_0. Let $g = f \circ t^{-1}$ on $\Delta^*_{r_0} \times \Delta^{d-1}_{r_0}$. For fixed $(t_2, \ldots, t_d)$, $t_1 g \to 0$ as $t_1 \to 0$, so g extends holomorphically in t_1 across 0 (Riemann's removable singularity theorem), and for

$|t_1| < r_0/2$ the extension is the Cauchy integral of g over $|t_1| = r_0/2$, hence holomorphic in all variables on $\Delta_{r_0/2} \times \Delta_{r_0}^{d-1}$. The same holds for $1/g$, and the product of the two extensions is 1 on the dense set $\{t_1 \neq 0\}$, hence everywhere. So ϑ extends to a nowhere vanishing holomorphic section near x. Two such extensions agree on the dense set Y of their common domain; hence ϑ extends to a nowhere vanishing holomorphic section on $X_Y \smallsetminus B'$.

Extension in codimension two. In a local trivialisation of $(\mathcal{D} \otimes_{F,s} \mathbb{C})^{\mathrm{an}}$ on an open $W \subseteq X_Y$, ϑ and ϑ^{-1} are holomorphic functions on $W \smallsetminus B'$; since X_Y is normal and B' has codimension ≥ 2, both extend holomorphically to W (Riemann's second extension theorem on normal complex spaces), with product 1. Hence ϑ extends to a nowhere vanishing holomorphic section on X_Y. $\square$

Lemma 34.18. *Let $\mathfrak{u}$ be a unitary Shimura datum with $n = p + q \geq 1$, $(\mathcal{Y}, j, G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ an integral model of* Sh *over $\mathcal{O}_F$, $\mathcal{Y}'$ an irreducible component of $\mathcal{Y}$ meeting $j(\mathrm{Sh}_F)$, $G' = G_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{Y}'_F$, $\tau \in \Phi_{\mathsf{H}}$, $s\colon F \to \mathbb{C}$ an embedding and $X = (\mathcal{Y}'_F \otimes_{F,s} \mathbb{C})^{\mathrm{an}}$. Let $x \in X$, let $V \subseteq X$ be an open neighbourhood of x, $1 \leq a \leq d$, and $t\colon V \to \Delta^d$ (Δ the open unit disc) a biholomorphic map with $t(x) = 0$ such that t maps $V \cap j(\mathrm{Sh}_F)$ onto $\{(t_1, \dots, t_d) \in \Delta^d : t_1 \cdots t_a \neq 0\}$. Let $u_1, \dots, u_m$ be holomorphic sections over V of the analytification of $\mathrm{Lie}_{(\tau,+)}(G')$ or of $\mathrm{Lie}_{(\tau,-)}(G')$ (the same for all i) that are linearly independent at x, and put $L(z) = 1 + \sum_{j=1}^{a} |\log|t_j(z)||$. Then there are $K \geq 1$, an integer $N \geq 0$ and a neighbourhood $V' \subseteq V$ of x such that for every $z \in V' \cap j(\mathrm{Sh}_F)$, with $u(z) = (u_1(z), \dots, u_m(z))$ in $\mathrm{Lie}(G_z)$,*

$$K^{-1} L(z)^{-N} \;\leq\; \det \Gamma(u(z)) \;\leq\; K\, L(z)^{N}.$$

For $a = 1$, $L(z) = 1 + |\log|t_1(z)||$. More generally, the same conclusion holds in the setting of Lemma 33.16 (any projective variety $X_{\mathbb{C}}$ with G', $X^{\circ}_{\mathbb{C}}$ as there, in place of $j(\mathrm{Sh}_F)$), for holomorphic sections $u_1, \dots, u_m$ of $\mathrm{Lie}(G')^{\mathrm{an}}$ linearly independent at x, with $\Gamma = \pi H_z$-Gram.

Proof. Lifting to the canonical extension. By Lemma 33.16 there are a polydisc V_0 around x and a surjective holomorphic map $\widehat{P}\colon \widehat{\mathcal{H}} \to \mathrm{Lie}(G')^{\mathrm{an}}$ on V_0 with kernel $\widehat{\mathcal{F}}^0$, equal on $V_0 \cap j(\mathrm{Sh}_F)$ to the extension P of the inclusions $\Lambda_z \subset \mathrm{Lie}(G_z)$. Extend $u_1, \dots, u_m$ by $u_{m+1}, \dots, u_g$ to a holomorphic frame of $\mathrm{Lie}(G')^{\mathrm{an}}$ near x (g the relative dimension). Shrinking V_0, choose a holomorphic frame $f_1, \dots, f_g$ of $\widehat{\mathcal{F}}^0$ and holomorphic sections $\tilde{u}_1, \dots, \tilde{u}_g$ of $\widehat{\mathcal{H}}$ with $\widehat{P}(\tilde{u}_i) = u_i$; then $e = (f, \tilde{u})$ is a frame of $\widehat{\mathcal{H}}$ on V_0. Let $s_1, \dots, s_{2g}$ be a $\mathbb{Z}$-basis of the flat multivalued sections of Λ and $\tilde{s}_k = \exp(-\sum_j z_j N_j) s_k$, $z_j = \frac{\log t_j}{2\pi i}$, the frame of $\widehat{\mathcal{H}}$ of Theorem B.145; then $e = A\tilde{s}$ with A a holomorphic invertible matrix on V_0, and shrinking V_0 there is $c > 0$ with $|A_{ik}| \leq c^{-1}$ and $|\det A| \geq c$ on V_0.

Upper bounds. Each $N_\kappa = \sum_{j \in \kappa} N_j$ is a nilpotent endomorphism of the $2g$-dimensional space H, so $N_\kappa^{2g} = 0$ and its weight filtration centred at 0 has $W_{2g} = H$: every flat section lies in W_{2g}. Let M be as in Theorem B.146, $r < \min(e^{-M}, 1)$, and cover $\Delta_r^d \cap \{t_1 \cdots t_a \neq 0\}$ by the 2^a simply connected subanalytic sets $U_\theta = \{t \in \Delta_r^d : t_j \neq 0,\ \arg t_j \in \theta_j\ (j \leq a)\}$, $\theta_j \in \{(-\pi, \pi), (0, 2\pi)\}$, relatively compact in V, on which the s_k and the $N^\beta s_k$ are single-valued flat sections. Every t lies in a region of Theorem B.146 with $\varepsilon = \frac{1}{2}$: number the indices so that $y_{j_1}(t) \leq \cdots \leq y_{j_a}(t)$. With all $p_\nu = 2g$ the bound there is $C(y_{j_1} \prod_{\nu \geq 2} y_{j_\nu}/y_{j_{\nu-1}})^{2g} = C y_{j_a}^{2g} \leq C L^{2g}$. As there are finitely many U_θ, numberings and flat sections $u = N^\beta s_k$ ($N^\beta = \prod_j N_j^{\beta_j}$, $\beta_j \in \{0,1\}$), there is C_1 with $|N^\beta s_k|_t^2 \leq C_1 L(t)^{2g}$ on $\Delta_r^d \cap V^*$. On U_θ, $|z_j| \leq (y_j + 2\pi)/2\pi \leq L$, and, as $N_j^2 = 0$ ($(T_j - 1)^2 = 0$ by Lemma 33.16) and the N_j commute, $\tilde{s}_k = \prod_j (1 - z_j N_j) s_k = \sum_\beta \prod_j (-z_j)^{\beta_j} N^\beta s_k$, so $|\tilde{s}_k|_t \leq 2^a L^a C_1^{1/2} L^g$ and, with $E_0 = 2a + 2g$,

$$h_z(e_i, e_i) \leq C_2 L^{E_0} \qquad (z \in \Delta_r^d \cap V^*), \tag{Up}$$

$C_2 = (2g\, c^{-1} 2^a)^2 C_1$.

Lower bound. At $z \in V^*$, $\tilde{s}(z) = U_z s$ with $U_z = \exp(-\sum_j z_j N_j)$ unipotent, so the Gram matrices satisfy $\det \mathcal{G}(\tilde{s}) = |\det U_z|^2 \det \mathcal{G}(s) = \det \mathcal{G}(s)$, $\mathcal{G}(v) = (h_z(v_i, v_k))$. For the real basis s, $h_z(s_i, s_k) = Q(C_z s_i, s_k)$, so $\mathcal{G}(s) = C_z^{\mathsf{T}}(Q(s_l, s_k))_{l,k}$ with C_z the matrix of the Weil operator, whose eigenvalues $\pm i$ have modulus one; as $\mathcal{G}(s)$ is positive definite, $\det \mathcal{G}(s) = |\det(Q(s_l, s_k))| = 1$ (E is unimodular). Hence

$$\det \mathcal{G}(e) = |\det A|^2 \det \mathcal{G}(\tilde{s}) \geq c^2. \tag{Lo}$$

Gram determinants of the quotient. Fix $z \in \Delta_r^d \cap V_0 \cap V^*$ and let π_z be the h_z-orthogonal projection of $\Lambda_z \otimes \mathbb{C}$ onto the h_z-orthogonal complement of $\mathcal{F}_z^0 = \operatorname{span}(f(z))$. As P kills $\mathcal{F}_z^0$, $P(\pi_z \tilde{u}_i) = u_i(z)$, so by $h_z(y, y) = \frac{1}{2} H_z(P(y), P(y))$ (polarised) and Γ the πH_z-Gram matrices, $\Gamma(u_I(z)) = 2\pi\, \mathcal{G}(\pi_z \tilde{u}_I)$ for every subtuple u_I of u. The projection does not increase h_z-norms, so by Hadamard's inequality and (Up), $\det \Gamma(u_I(z)) \leq (2\pi C_2 L^{E_0})^{|I|}$. Gram–Schmidt along f gives $\det \mathcal{G}(f, \tilde{u}) = \det \mathcal{G}(f) \det \mathcal{G}(\pi_z \tilde{u})$, and $\det \mathcal{G}(f) \leq (C_2 L^{E_0})^g$ by Hadamard, so by (Lo)

$$\det \Gamma(u(z)) = (2\pi)^g \frac{\det \mathcal{G}(f, \tilde{u})}{\det \mathcal{G}(f)} \geq (2\pi)^g c^2 (C_2 L^{E_0})^{-g}.$$

Fischer's inequality for the positive definite matrix $\Gamma(u(z))$ with the blocks $(u_1, \dots, u_m)$ and $(u_{m+1}, \dots, u_g)$ gives

$$\det \Gamma(u_1, \dots, u_m) \geq \frac{\det \Gamma(u(z))}{\det \Gamma(u_{m+1}, \dots, u_g)} \geq \frac{(2\pi)^g c^2 (C_2 L^{E_0})^{-g}}{(2\pi C_2 L^{E_0})^{g-m}}.$$

With the upper bound for $I = \{1, \dots, m\}$, the statement holds on $V' = \Delta_r^d \cap V_0$ with $N = 2gE_0$ and $K = (2\pi C_2)^{2g} c^{-2}$, as $L \geq 1$ and we may take $C_2 \geq 1$, $c \leq 1$. □

34.3 Flat heights of determinant lines and pure CM points

This subsection proves the flat determinant classes at the base stage (Db) and states the Hodge-type base settings of good towers. By Lemma 34.19, (Db) follows from the flat height of a flat generator of the determinant line, which Theorem 34.20 computes. The flat height is the sum of the local norms of the generator over all places (Theorem 34.22), and these are evaluated at *pure CM points*: points with a pure local basis at a prime, which receive an isogeny of degree prime to that prime from Serre tensor products of two fixed elliptic curves with complex multiplication (Lemma 34.31). Such points exist at every prime on every component (Proposition 34.26), because the periods realised on a component contain an open subset of the period space and CM splittings that are local bases are dense there (Proposition 34.29, Proposition 34.27). At such a point the ℓ-adic and archimedean norms are computed explicitly (Theorem 34.23, Theorem 34.25), and their sum is evaluated with the Chowla–Selberg period (Lemma 34.21). The subsection ends with Proposition 34.35, the Hodge-type base settings of good towers, whose clauses are proved in Sections D.1–D.3 of Appendix D.

Lemma 34.19. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $n = p + q \geq 1$; put $\mathrm{rd}_E = |\mathrm{disc}(E)|^{1/[E:\mathbb{Q}]}$ and $L_{\mathsf{H}} = \mathrm{Re}\,(L'/L)(1, \chi_d)$. Let $\mathcal{T}$ be a root tower of type $(\Phi_{\mathsf{H}}, n, m)$ for some m. Suppose that for every $Z \in \mathrm{Cp}$ and every $a \in \Phi_{\mathsf{H}}$ there are:*

1. *a real number $\mathrm{hf}(Z, a)$ with $\widehat{\deg}_{2,Z}(\mathrm{ps}_{a,1} Q) = \mathrm{hf}(Z, a)\,\mathrm{vol}(Z)\,\deg^{\mathrm{fl}}(Q)$ for every $Q \in \mathcal{R}$ homogeneous of degree D;*
2. *$\tau \in \Phi_{\mathsf{H}}$, a union $U \subseteq \mathrm{Sh}_F$ of connected components, a connected component Y of $U(\mathbb{C})$, $z_0 \in Y$, and a flat generator φ over Y with constants (e, c) that has an archimedean norm N_s at every embedding $s\colon F \to \mathbb{C}$ and an ℓ-adic norm N_κ at every ring homomorphism $\kappa\colon F \to \mathbb{C}_\ell$, for every prime ℓ;*

3. *real numbers* d_ℓ, v_ℓ ($\ell \in \mathrm{Bd}$) *with* $v_\ell \geq 0$, $|d_\ell| \leq (n\,c_\ell + n\,v_\ell)\log \ell$, *where* $c_2 = 2$ *and* $c_\ell = \frac{1}{\ell-1}$ *for* $\ell \geq 3$, $\sum_{\ell\in\mathrm{Bd}} v_\ell \log \ell \leq \log \mathrm{rd}_E$, *and*

$$\mathrm{hf}(Z,a) = -\frac{1}{[F:\mathbb{Q}]}\Big(\sum_{s\colon F\to\mathbb{C}} N_s + \sum_{\ell}\sum_{\kappa\colon F\to\mathbb{C}_\ell} N_\kappa\Big) + \sum_{\ell\in\mathrm{Bd}} d_\ell$$

(the sum over all primes ℓ *has finitely many non-zero terms, by Theorem 34.20).*

Then $\mathcal{T}$ *has flat determinant classes at the base stage (Db) for* $(r,L) = (\mathrm{rd}_E, L_{\mathsf{H}})$.

Proof. Fix $Z \in \mathrm{Cp}$ and $a \in \Phi_{\mathsf{H}}$, and take the data of 1–3. The datum $\mathfrak{u}$, τ, U, Y, z_0 and φ with its norms satisfy the hypotheses of Theorem 34.20. By (Fd1) only finitely many N_κ are non-zero, so $h_0 = -\frac{1}{[F:\mathbb{Q}]}(\sum_s N_s + \sum_\ell \sum_\kappa N_\kappa)$ is a real number (it is the flat height $h^{\mathrm{fl}}(\varphi)$ of that theorem). By (Fd3), $\log \mathrm{rd}_E \geq \frac{1}{2}\log 7 > 0$, so $r = \mathrm{rd}_E \geq 1$, and

$$|h_0| + n\log 2 \leq n\Big(\frac{3}{2}\log \mathrm{rd}_E + \frac{|L_{\mathsf{H}}|}{2}\Big).$$

Moving $n\log 2$ *from the prime* 2. If $2 \notin \mathrm{Bd}$ put $e = 0$. If $2 \in \mathrm{Bd}$ put $e = n\log 2$ when $d_2 \geq 0$ and $e = -n\log 2$ when $d_2 < 0$; then $|d_2 - e| \leq (n + n\,v_2)\log 2$: for $d_2 \geq 0$, $-n\log 2 \leq d_2 - e \leq (2n + n\,v_2)\log 2 - n\log 2$, and $n\log 2 \leq (n + n\,v_2)\log 2$ as $v_2 \geq 0$; the case $d_2 < 0$ is symmetric. Put $h_0' = h_0 + e$, $d_2' = d_2 - e$ (if $2 \in \mathrm{Bd}$) and $d_\ell' = d_\ell$ for $\ell \in \mathrm{Bd}$, $\ell \geq 3$. Then $\mathrm{hf}(Z,a) = h_0' + \sum_{\ell\in\mathrm{Bd}} d_\ell'$ by 3, $|h_0'| \leq |h_0| + n\log 2 \leq n(\frac{3}{2}\log \mathrm{rd}_E + \frac{|L_{\mathsf{H}}|}{2})$, and $|d_\ell'| \leq (\frac{n}{\ell-1} + n\,v_\ell)\log \ell$ for every $\ell \in \mathrm{Bd}$ (for $\ell = 2$ this is the bound just shown, as $\frac{n}{2-1} = n$; for $\ell \geq 3$ it is 3).

Conclusion. With $\mathrm{hf}(Z,a)$ from 1, h_0', d_ℓ' and v_ℓ ($\ell \in \mathrm{Bd}$), every requirement of (Db) for $(r,L) = (\mathrm{rd}_E, L_{\mathsf{H}})$ at (Z,a) holds: the degree identity is 1; the decomposition, the bound on $|h_0'|$ and the bounds on d_ℓ' were just shown; $v_\ell \geq 0$ and $\sum_{\ell\in\mathrm{Bd}} v_\ell \log \ell \leq \log r$ are 3 with $r = \mathrm{rd}_E$. As Z and a were arbitrary, $\mathcal{T}$ has (Db). □

Theorem 34.20. *Let* $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ *be a unitary Shimura datum with* $n = p+q \geq 1$, $\tau \in \Phi_{\mathsf{H}}$, $U \subseteq \mathrm{Sh}_F$ *a union of connected components,* Y *a connected component of* $U(\mathbb{C})$, $z_0 \in Y$, *and* φ *a flat generator over* Y *with constants* (e,c). *Suppose that* φ *has an archimedean norm* N_s *at every embedding* $s\colon F \to \mathbb{C}$ *and an* ℓ*-adic norm* N_κ *at every ring homomorphism* $\kappa\colon F \to \mathbb{C}_\ell$, *for every prime* ℓ. *Put* $L_{\mathsf{H}} = \mathrm{Re}\,(L'/L)(1,\chi_d)$. *Then:*

(Fd1) Finiteness $N_\kappa = 0$ *for all* κ *above all but finitely many primes* ℓ, *so that the* ***flat height***

$$h^{\mathrm{fl}}(\varphi) = -\frac{1}{[F:\mathbb{Q}]}\Big(\sum_{s\colon F\to\mathbb{C}} N_s + \sum_{\ell}\sum_{\kappa\colon F\to\mathbb{C}_\ell} N_\kappa\Big)$$

is a finite sum;

(Fd2) Value $L_{\mathsf{H}} = L'(1,\chi_d)/L(1,\chi_d)$ *and*

$$h^{\mathrm{fl}}(\varphi) = \frac{p-q}{2}(\gamma + \log(2\pi) - \log \mathrm{rd}_E - L_{\mathsf{H}}),$$

γ *Euler's constant;*

(Fd3) Base bound $\frac{1}{2}\log 7 \leq \frac{1}{2}\log d \leq \log \mathrm{rd}_E$, *and*

$$|h^{\mathrm{fl}}(\varphi)| + n\log 2 \leq n\Big(\frac{3}{2}\log \mathrm{rd}_E + \frac{|L_{\mathsf{H}}|}{2}\Big).$$

Proof. (Fd1). This is the first conclusion of Theorem 34.22.

(Fd2). The character χ_d is real-valued and non-principal, so $L(s,\chi_d)$ extends to an entire function, real for real $s>1$ (as a Dirichlet series with real coefficients); the entire function $s\mapsto \overline{L(\bar{s},\chi_d)}-L(s,\chi_d)$ vanishes on $(1,\infty)$, hence identically, so $L(s,\chi_d)$ and $L'(s,\chi_d)$ are real at $s=1$; and $L(1,\chi_d)\neq 0$ (Dirichlet). So $(L'/L)(1,\chi_d)$ is a real number and equals L_{H}. Multiplying the identity of Theorem 34.22 by -1 gives

$$h^{\mathrm{fl}}(\varphi)=-\frac{p-q}{2}\Big(\log \mathrm{rd}_E-\gamma-\log(2\pi)+L_{\mathsf{H}}\Big)=\frac{p-q}{2}(\gamma+\log(2\pi)-\log \mathrm{rd}_E-L_{\mathsf{H}}).$$

(Fd3), the discriminant. As d is a prime with $d\equiv 7 \pmod{16}$, $d\geq 7$, which gives the first inequality. The subfield $K=\mathbb{Q}(r)\subseteq E$ is imaginary quadratic with $r^2=-d$; since d is squarefree and $-d\equiv 1 \pmod 4$, $\mathrm{disc}(K)=-d$. By the transitivity of discriminants, $|\mathrm{disc}(E)|=|\mathrm{disc}(K)|^{[E:K]}\,\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{d}_{E/K})\geq d^{[E:K]}$, the relative discriminant $\mathfrak{d}_{E/K}$ being a non-zero integral ideal. Taking the power $1/[E:\mathbb{Q}]=1/(2[E:K])$ gives $\mathrm{rd}_E\geq\sqrt{d}$.

(Fd3), the constant. We show $\gamma+\log(2\pi)+2\log 2<2\log 7$, that is $8\pi e^{\gamma}<49$. With $\gamma<0.6$ and $\pi<3.15$, and $e^{0.3}=\sum_{k\geq 0}0.3^k/k!\leq 1+0.3+0.045+0.0045+0.0003375+\frac{0.3^5}{5!}\cdot\frac{1}{1-0.3}<1.35$, we get $8\pi e^{\gamma}<8\cdot 3.15\cdot 1.35^2<45.93<49$. (Computer check C1, listed in Appendix B, also certifies this inequality in exact rational arithmetic from $\gamma<0.5773$ and $\pi<3.1416$, and checks it by the route just given.)

(Fd3), the bound. Put $l=\log \mathrm{rd}_E$, $L=L_{\mathsf{H}}$, $c=\gamma+\log(2\pi)$ and $b=\frac{3}{2}l+\frac{|L|}{2}-\log 2$. By the first part, $4l\geq 2\log d\geq 2\log 7$. Then

$$\frac{c-l-L}{2}\leq b \iff c-L\leq 4l+|L|-2\log 2,$$

which holds because $-L\leq|L|$ and $c+2\log 2<2\log 7\leq 4l$; and

$$\frac{l+L-c}{2}\leq b \iff L+2\log 2-c\leq 2l+|L|,$$

which holds because $L\leq|L|$, $l\geq 0$ and $c>2\log 2$ (as $\gamma>0$ and $2\pi>4$). Hence $|c-l-L|\leq 2b$, and as $|p-q|\leq n$,

$$|h^{\mathrm{fl}}(\varphi)|=\frac{|p-q|}{2}\,|c-l-L|\leq n\,b,$$

which is the stated bound after adding $n\log 2$.

Check. For $E=\mathbb{Q}(\sqrt{-d})$ one has $\mathrm{rd}_E=\sqrt{d}$ and $h^{\mathrm{fl}}(\varphi)=\frac{p-q}{2}(\gamma+\log\frac{2\pi}{\sqrt{d}}-L_{\mathsf{H}})$, $\frac{p-q}{2}$ times the Chowla–Selberg average of Lemma 34.21; computer check C1 confirms this numerically against computer check C2, a check of the Chowla–Selberg formula (both listed in Appendix B), and confirms the bound (Fd3), for $d\in\{7,23,71,103,199\}$ and $p,q\leq 12$. This is a numerical check only; the proof does not use it. □

Lemma 34.21. *Let $d\geq 7$ be a squarefree integer with $d\equiv 3 \bmod 4$, $K=\mathbb{Q}(\sqrt{-d})$ and χ_d the quadratic Dirichlet character of K (odd and primitive modulo d). Let $M\subseteq\mathbb{C}$ be a number field, $\mathcal{A}\to\operatorname{Spec}\mathcal{O}_M$ an abelian scheme of relative dimension 1 whose generic fibre A has ring of $\overline{\mathbb{Q}}$-endomorphisms isomorphic to $\mathcal{O}_K$, and v a basis of the $\mathcal{O}_M$-module $\mathrm{Lie}(\mathcal{A}/\mathcal{O}_M)$. For an embedding $\sigma\colon M\to\mathbb{C}$ identify $\mathrm{Lie}(A^{\sigma}_{\mathbb{C}})$ with $\mathbb{C}$ so that $v\mapsto 1$, let $\Lambda_\sigma\subseteq\mathbb{C}$ be the period lattice, $A^{\sigma}(\mathbb{C})=\mathbb{C}/\Lambda_\sigma$, and put*

$$n_\sigma=\frac{\pi}{\mathrm{covol}(\Lambda_\sigma)}=\pi\,H_\sigma(1,1),$$

where covol *is the area of a fundamental parallelogram and* $H_\sigma(z,w) = z\bar{w}/\mathrm{covol}(\Lambda_\sigma)$ *is the Riemann form of the principal polarisation. Then*

$$-\frac{1}{[M:\mathbb{Q}]}\sum_{\sigma\colon M\to\mathbb{C}} \log n_\sigma = \gamma + \log\frac{2\pi}{\sqrt{d}} - \frac{L'(1,\chi_d)}{L(1,\chi_d)} = -2h^{\mathrm{Falt}}(A) - \log(2\pi), \tag{34.2}$$

with γ *Euler's constant and* h^{Falt} *normalised as in Theorem B.134.*

Proof. Write x for the left side of (34.2) and $m = [M:\mathbb{Q}]$.

Step 1: $x = -2h^{\mathrm{Falt}}(A) - \log(2\pi)$. For an abelian scheme $p\colon \mathcal{A}\to S$ with unit section e, $p_*\Omega^1_{\mathcal{A}/S} = e^*\Omega^1_{\mathcal{A}/S}$ is the $\mathcal{O}_S$-dual of $\mathrm{Lie}(\mathcal{A}/S)$; let ω be the basis dual to v. Then ω is a global section generating $p_*\Omega^1$, so $\mathrm{ord}_{\mathfrak{p}}(\omega) = 0$ for every $\mathfrak{p}$, and $\mathcal{A}$ is the Néron model of A. At σ, ω^σ is the invariant form dz dual to $\partial/\partial z$, and $\int_{\mathbb{C}/\Lambda_\sigma} dz\wedge d\bar{z} = -2i\,\mathrm{covol}(\Lambda_\sigma)$. By the definition in Theorem B.134 with $s=\omega$,

$$h^{\mathrm{Falt}}(A) = -\frac{1}{2m}\sum_\sigma \log\bigl(2\,\mathrm{covol}(\Lambda_\sigma)\bigr) = -\frac{1}{2m}\sum_\sigma\bigl(\log(2\pi) - \log n_\sigma\bigr) = -\frac{1}{2}\log(2\pi) - \frac{x}{2}.$$

(That H_σ is the Riemann form of the principal polarisation: $\operatorname{Im} H_\sigma$ is the alternating form on Λ_σ with $\mathrm{Im}H_\sigma(\lambda_1,\lambda_2) = \pm 1$ for an oriented basis, because $\mathrm{Im}(\lambda_1\bar{\lambda}_2) = \mp\mathrm{covol}(\Lambda_\sigma)$; it is unimodular, so it is the unique principal polarisation of $\mathbb{C}/\Lambda_\sigma$.)

Step 2: the Faltings height. Since $d\equiv 3 \bmod 4$ is squarefree, $D_K = -d$ and $\chi_K = \chi_d$. By Theorem B.134, $h^{\mathrm{Falt}}(A) = -\frac{1}{2}\frac{L'}{L}(0,\chi_d) - \frac{1}{4}\log d - \frac{1}{2}\log(2\pi)$, so by Step 1

$$x = \frac{L'}{L}(0,\chi_d) + \frac{1}{2}\log d.$$

Step 3: the functional equation. For the odd primitive real character χ_d of conductor d, $\Lambda(s) = (d/\pi)^{(s+1)/2}\Gamma(\frac{s+1}{2})L(s,\chi_d)$ is entire and satisfies $\Lambda(1-s) = \Lambda(s)$ (the root number of a real odd primitive character is 1, as the Gauss sum is $i\sqrt{d}$). Also $L(1,\chi_d) = \pi h_K/\sqrt{d} \neq 0$ (class number formula, $w_K = 2$ for $d\geq 7$), so $L(0,\chi_d)\neq 0$ by the functional equation. Taking logarithmic derivatives, $\frac{\Lambda'}{\Lambda}(s) = -\frac{\Lambda'}{\Lambda}(1-s)$ with $\frac{\Lambda'}{\Lambda}(s) = \frac{1}{2}\log\frac{d}{\pi} + \frac{1}{2}\psi(\frac{s+1}{2}) + \frac{L'}{L}(s,\chi_d)$; at $s=0$, with $\psi(\frac{1}{2}) = -\gamma - 2\log 2$ and $\psi(1) = -\gamma$,

$$\frac{L'}{L}(0,\chi_d) = -\log\frac{d}{\pi} + \gamma + \log 2 - \frac{L'}{L}(1,\chi_d) = \gamma + \log(2\pi) - \log d - \frac{L'}{L}(1,\chi_d).$$

Inserting this in Step 2 gives $x = \gamma + \log\frac{2\pi}{\sqrt{d}} - \frac{L'}{L}(1,\chi_d)$, which with Step 1 is (34.2). The normalisations of Steps 1–3 were checked numerically for seven discriminants against the classical eta-quotient form of the Chowla–Selberg formula and against a published Faltings height, in computer check C2, listed in Appendix B. □

Theorem 34.22. *Let* $\mathfrak{u} = (E,\mathsf{H},\mathsf{U},p,q,\mathsf{L},\mathrm{Sh},F)$ *be a unitary Shimura datum with* $n = p+q\geq 1$, $\tau\in\Phi_{\mathsf{H}}$, $U\subseteq \mathrm{Sh}_F$ *a union of connected components,* Y *a connected component of* $U(\mathbb{C})$, $z_0\in Y$, *and* φ *a flat generator over* Y *with constants* (e,c). *Suppose that* φ *has an archimedean norm* N_s *at every embedding* $s\colon F\to\mathbb{C}$ *and an* ℓ*-adic norm* N_κ *at every ring homomorphism* $\kappa\colon F\to\mathbb{C}_\ell$, *for every prime* ℓ. *Then* $N_\kappa = 0$ *for all* κ *above all but finitely many primes* ℓ, *and, with* $\mathrm{rd}_E = |\mathrm{disc}(E)|^{1/[E:\mathbb{Q}]}$, γ *Euler's constant and* χ_d *the quadratic Dirichlet character of* $\mathbb{Q}(\sqrt{-d})$,

$$\frac{1}{[F:\mathbb{Q}]}\Bigl(\sum_{s\colon F\to\mathbb{C}} N_s + \sum_\ell \sum_{\kappa\colon F\to\mathbb{C}_\ell} N_\kappa\Bigr) = \frac{p-q}{2}\Bigl(\log\mathrm{rd}_E - \gamma - \log(2\pi) + \frac{L'(1,\chi_d)}{L(1,\chi_d)}\Bigr).$$

Proof. Write $m_X = [X:\mathbb{Q}]$ for a number field X, $G_e = \det \mathrm{Gr}(e) \in E^\times$ ((Jz1) of Lemma 34.24), and $x_d = \gamma + \log\frac{2\pi}{\sqrt{d}} - \frac{L'(1,\chi_d)}{L(1,\chi_d)}$. As $d \equiv 7 \pmod{16}$ is a prime, $d \geq 7$ is squarefree and $d \equiv 3 \bmod 4$, and $\mathbb{Q}(r) \cong \mathbb{Q}(\sqrt{-d})$.

Step 0: the CM points. By Proposition 34.26 there is $\alpha \in \mathbb{C}^\times$ and, for every prime ℓ, a pure CM point $P_\ell = (K_\ell, y_\ell, \gamma_\ell, x_\ell, v_\ell, w_\ell)$ at ℓ for φ with constant α. By Theorem 34.23, $\alpha \in K_\ell$ and $t_\ell = \det T(\mu_{\gamma_\ell}^{-1} x_\ell, e) \in E^\times$. Put $L = F(\alpha) \subseteq K_2$, a number field.

Step 1: the finite places. Let ℓ be a prime and $\kappa\colon F \to \mathbb{C}_\ell$. Every extension $j\colon K_\ell \to \mathbb{C}_\ell$ of κ satisfies $j \circ \tau = \kappa \circ \tau$ (as $\tau(E) \subseteq F$), so (Qn3) gives

$$N_\kappa = A_\kappa - \log|\kappa\tau(t_\ell)| - q\log|\mathfrak{d}_E|_{\kappa\tau} + (q-p)\big(\log|\mathfrak{d}_E|_{\kappa\tau} - \log|\kappa\tau(r)|\big),$$

where $A_\kappa = \log|j(\alpha)|$ does not depend on j. Every embedding $\lambda\colon L \to \mathbb{C}_\ell$ extending κ extends to K_ℓ ($\mathbb{C}_\ell$ is algebraically closed), so $\log|\lambda(\alpha)| = A_\kappa$ for all $[L:F]$ such λ.

Step 2: the archimedean places. Let $s\colon F \to \mathbb{C}$. By Theorem 34.25 at P_2, for each of the $[K_2:F]$ extensions $s'\colon K_2 \to \mathbb{C}$ of s, $N_s = \log|s'(\alpha)| + \frac{p}{2}(\log n_+(s') + \frac{1}{2}\log d) - \frac{q}{2}(\log n_-(s') + \frac{1}{2}\log d) + \frac{1}{2}\log|s\tau(G_e)|$; we average over these s'.

Step 3: finiteness. Let S_0 be the set of primes dividing $d \cdot \mathrm{disc}(E)$, the primes at which e is not a local basis, and the primes ℓ with $|\lambda(\alpha)| \neq 1$ for some $\lambda\colon L \to \mathbb{C}_\ell$; it is finite by (Qn2) and because $\alpha \in L^\times$ is a unit at almost all primes. Let $\ell \notin S_0$ and $\varepsilon = \kappa \circ \tau$. Then $|\varepsilon(t_\ell)| = 1$ (Qn2), $A_\kappa = 0$, $|\varepsilon(r)|^2 = |d|_\ell = 1$, and $|\mathfrak{d}_E|_\varepsilon = 1$ because $\mathfrak{d}_E \subseteq \mathcal{O}_E$ contains its norm $|\mathrm{disc}(E)|$, which is prime to ℓ. So $N_\kappa = 0$.

Step 4: sums at one prime. Each embedding $E \to \mathbb{C}_\ell$ is $\kappa\circ\tau$ for exactly m_F/m_E embeddings κ of F, so $\sum_\kappa f(\kappa\circ\tau) = \frac{m_F}{m_E}\sum_\varepsilon f(\varepsilon)$, ε running over $\mathrm{Hom}(E,\mathbb{C}_\ell)$. For $a \in E^\times$, $\sum_\varepsilon \log|\varepsilon(a)| = \log|\mathrm{N}_{E/\mathbb{Q}}(a)|_\ell$; let $S = \mathbb{Z} \smallsetminus \ell\mathbb{Z}$; $S^{-1}\mathcal{O}_E$ is a semilocal Dedekind domain, hence principal, so $S^{-1}\mathfrak{d}_E = \delta\, S^{-1}\mathcal{O}_E$ with $\delta \in \mathfrak{d}_E$ (clear the denominator, an element of S). Every $d \in \mathfrak{d}_E$ is $\delta a/s$ with $a \in \mathcal{O}_E$, $s \in S$, so $|\varepsilon(d)| = |\varepsilon(\delta)|\,|\varepsilon(a)| \leq |\varepsilon(\delta)|$ ($|s| = 1$, $\varepsilon(a)$ integral); with $\delta \in \mathfrak{d}_E$ this gives $|\mathfrak{d}_E|_\varepsilon = |\varepsilon(\delta)|$ (Qd5). The ideals $\delta\mathcal{O}_E$ and $\mathfrak{d}_E$ agree after inverting S, so their norms have the same ℓ-adic absolute value, and with $\mathrm{N}(\mathfrak{d}_E) = |\mathrm{disc}(E)|$ we get $\sum_\varepsilon \log|\mathfrak{d}_E|_\varepsilon = \log|\mathrm{N}_{E/\mathbb{Q}}(\delta)|_\ell = \log|\mathrm{disc}(E)|_\ell$. Also $\mathrm{N}_{E/\mathbb{Q}}(r) = d^{m_E/2}$. Summing (Qn1) over ε and using that $\varepsilon \mapsto \varepsilon(\bar{\cdot})$ permutes $\mathrm{Hom}(E,\mathbb{C}_\ell)$: $2\log|\mathrm{N}(t_\ell)|_\ell = -n\log|\mathrm{disc}(E)|_\ell - \log|\mathrm{N}(G_e)|_\ell$. Hence

$$\sum_\kappa N_\kappa = \sum_\kappa A_\kappa - \frac{m_F}{m_E}\Big(\tfrac{1}{2}\big(-n\log|\mathrm{disc}(E)|_\ell - \log|\mathrm{N}(G_e)|_\ell\big) + q\log|\mathrm{disc}(E)|_\ell\Big) + \frac{m_F}{m_E}(q-p)\Big(\log|\mathrm{disc}(E)|_\ell - \tfrac{m_E}{2}\log|d|_\ell\Big).$$

Step 5: sum over the primes. By the product formula over $\mathbb{Q}$, $\sum_\ell \log|\mathrm{disc}(E)|_\ell = -m_E \log \mathrm{rd}_E$, $\sum_\ell \log|d|_\ell = -\log d$, and $\sum_\ell \log|\mathrm{N}(G_e)|_\ell = -\sum_{\sigma\colon E\to\mathbb{C}} \log|\sigma(G_e)|$ (all sums finite). Therefore

$$\frac{1}{m_F}\sum_\ell\sum_\kappa N_\kappa = \frac{1}{m_F}\sum_\ell\sum_\kappa A_\kappa + \frac{p-q}{2}\log\mathrm{rd}_E + \frac{q-p}{2}\log d - \frac{1}{2m_E}\sum_\sigma \log|\sigma(G_e)|,$$

the coefficient of $\log\mathrm{rd}_E$ being $-\frac{n}{2} + q - (q-p) = \frac{p-q}{2}$.

Step 6: sum over the complex embeddings. Averaging Step 2 over s' and summing over s,

$$\sum_s N_s = \frac{1}{[K_2:F]}\sum_{s'\colon K_2\to\mathbb{C}}\Big(\log|s'(\alpha)| + \frac{p}{2}\log n_+(s') - \frac{q}{2}\log n_-(s')\Big) + m_F\frac{p-q}{4}\log d + \frac{m_F}{2m_E}\sum_\sigma \log|\sigma(G_e)|,$$

using $\sum_s \log|s\tau(G_e)| = \frac{m_F}{m_E}\sum_\sigma \log|\sigma(G_e)|$. For $b=\pm$, $n_b(s')$ is the number n_σ of Lemma 34.21 for the abelian scheme $\mathcal{Z}_b$ over $\mathcal{O}_{M_b}$, its basis u_b and $\sigma = s'|_{M_b}$, and each $\sigma\colon M_b \to \mathbb{C}$ has $[K_2 : M_b]$ extensions; so by the period formula of Lemma 34.21, $\sum_{s'} \log n_b(s') = [K_2 : M_b]\sum_\sigma \log n_\sigma = -m_{K_2} x_d$. The $n_\pm$-terms thus contribute $\frac{1}{[K_2:F]} \cdot \frac{q-p}{2} m_{K_2} x_d = -m_F \frac{p-q}{2} x_d$.

Step 7: the constant α and the total. By Steps 1 and 2, $\sum_\kappa A_\kappa = \frac{1}{[L:F]}\sum_{\lambda\colon L\to\mathbb{C}_\ell} \log|\lambda(\alpha)|$ and $\frac{1}{[K_2:F]}\sum_{s'} \log|s'(\alpha)| = \frac{1}{[L:F]}\sum_{\sigma\colon L\to\mathbb{C}} \log|\sigma(\alpha)|$; their total over all primes and the complex embeddings is $\frac{1}{[L:F]}\sum_v \log|\mathrm{N}_{L/\mathbb{Q}}(\alpha)|_v = 0$ by the product formula. Adding Steps 5 and 6, the terms in G_e cancel, and

$$\begin{aligned}\frac{1}{m_F}\Big(\sum_s N_s + \sum_\ell \sum_\kappa N_\kappa\Big) &= \frac{p-q}{2}\Big(\log \mathrm{rd}_E - \frac{1}{2}\log d - x_d\Big)\\ &= \frac{p-q}{2}\Big(\log \mathrm{rd}_E - \gamma - \log(2\pi) + \frac{L'(1,\chi_d)}{L(1,\chi_d)}\Big).\end{aligned}$$

□

Theorem 34.23. *Let $\mathfrak{u}$ be a unitary Shimura datum with $n = p+q \geq 1$, $\tau \in \Phi_{\mathsf{H}}$, $U \subseteq \mathrm{Sh}_F$ a union of connected components, Y a connected component of $U(\mathbb{C})$, $z_0 \in Y$, and φ a flat generator over Y with constants (e,c). Let ℓ be a prime, $\alpha \in \mathbb{C}^\times$, and (K, y, γ, x, v, w) a pure CM point at ℓ for φ with constant α. Put $t = \det T(\mu_\gamma^{-1}x, e)$. Then $t \in E^\times$, $\alpha \in K$, and:*

(Qn1) *for every ring homomorphism $j\colon E \to \mathbb{C}_\ell$,*

$$\log|j(t)| + \log|j(\bar{t})| + \log|j(\det \mathrm{Gr}(e))| = -n\log|\mathfrak{d}_E|_j;$$

(Qn2) *if e is a local basis at ℓ, then $|j(t)| = 1$ for every ring homomorphism $j\colon E \to \mathbb{C}_\ell$; and e is a local basis at all but finitely many primes;*

(Qn3) *if φ has ℓ-adic norm N_κ at a ring homomorphism $\kappa\colon F \to \mathbb{C}_\ell$, then for every ring homomorphism $j\colon K \to \mathbb{C}_\ell$ with $j|_F = \kappa$,*

$$N_\kappa = \log|j(\alpha)| - \log|j(\tau(t))| - q\log|\mathfrak{d}_E|_{j\circ\tau} + (q-p)\big(\log|\mathfrak{d}_E|_{j\circ\tau} - \log|j(\tau(r))|\big).$$

(Qn4) *with $x = (x_+, x_-)$ and h the trace form of Λ_z: $h(x_{+,a}, x_{-,b}) = 0$ for all a, b, and $\det \mathrm{Gr}(x_-) \neq 0$.*

Proof. Write $N = \Lambda_{z_0}$, $x^\circ = \mu_\gamma^{-1}x \in N^n$ and $T = T(x^\circ, e)$. Parallel transport is an isomorphism of self-dual $\mathcal{O}_E$-lattices, so x° is a local basis at ℓ, it is the transport of a pure tuple, and Gram matrices are preserved: $\mathrm{Gr}(x^\circ) = \mathrm{Gr}(x)$, $\mathrm{Gr}(x^\circ_-) = \mathrm{Gr}(x_-)$.

$t \neq 0$. Both x° and e are $\mathcal{O}_E$-linearly independent of length n, hence E-bases of $N_\mathbb{Q}$, and $T(x^\circ, e)\,T(e, x^\circ) = 1$; so $t \in E^\times$.

(Qn1). By (Jz1), $\mathrm{Gr}(x^\circ) = T\,\mathrm{Gr}(e)\,\overline{T}^{\,t}$, and $\det \overline{T}^{\,t} = \bar{t}$ since conjugation is a ring automorphism of E; so $\det \mathrm{Gr}(x^\circ) = t\,\bar{t}\,\det \mathrm{Gr}(e)$, all three factors non-zero ((Jz1) for $\det \mathrm{Gr}(e)$). By (Jz2), $|j(\det \mathrm{Gr}(x^\circ))| = |\mathfrak{d}_E|_j^{-n}$. Taking logarithms gives (Qn1).

(Qn2). If e is a local basis at ℓ, (Jz4) applied to x° and e gives $|j(t)| = 1$. By (Jz1), e is a local basis at all but finitely many primes.

The two pure parts are orthogonal. Let $\lambda \in \Lambda_z^+$ and $\mu \in \Lambda_z^-$. Write u_s for the component of $u \in \mathrm{Lie}(G_z)$ in $\mathrm{Lie}_s(G_z)$. On $\mathrm{Lie}_s(G_z)$, $\iota(r)$ acts by $\sigma_s(r)$, which is $\tau'(r) = r_\mathbb{C}$ for $s = (\tau', +)$ and $\overline{\tau'(r)} = -r_\mathbb{C}$ for $s = (\tau', -)$ ($\tau' \in \Phi_{\mathsf{H}}$; $r_\mathbb{C}^2 = -d < 0$, so $r_\mathbb{C}$ is purely imaginary and non-zero).

Hence $\iota(r)\lambda = r_{\mathbb{C}}\lambda$ forces $\lambda_s = 0$ for every label of sign $-$, and $\iota(r)\mu = -r_{\mathbb{C}}\mu$ forces $\mu_s = 0$ for every label of sign $+$. For $u \in \mathrm{Lie}_s(G_z)$, $u' \in \mathrm{Lie}_{s'}(G_z)$ and $a \in \mathcal{O}_E$, the Rosati property gives $\sigma_s(a)H_z(u,u') = H_z(\iota(a)u,u') = H_z(u,\iota(\bar a)u') = \overline{\sigma_{s'}(\bar a)}\,H_z(u,u') = \sigma_{s'}(a)H_z(u,u')$; if $s \neq s'$ then $\sigma_s \neq \sigma_{s'}$ (Definition 29.6, (Fh1)), and choosing a with $\sigma_s(a) \neq \sigma_{s'}(a)$ gives $H_z(u,u') = 0$. As $\iota(a)$ commutes with $\iota(r)$, $\iota(a)\lambda \in \Lambda_z^+$ for $a \in \mathcal{O}_E$; so $H_z(\iota(b_k)\lambda,\mu) = 0$ for every k, the alternating form $\operatorname{Im} H_z$ vanishes on these pairs, and $h_{\Lambda_z}(\lambda,\mu) = 0$ by the formula for the trace form in (Qd1) of Definition 33.13. Transporting, $h_N(x^\circ_{+,a}, x^\circ_{-,b}) = 0$ for all a, b.

The Gram determinant of the negative part. By the previous step and (Jz3) applied to the local basis $x^\circ = (x^\circ_+, x^\circ_-)$, $\det \mathrm{Gr}(x_-) = \det \mathrm{Gr}(x^\circ_-) \neq 0$ and $|j'(\det \mathrm{Gr}(x_-))| = |\mathfrak{d}_E|_{j'}^{-q}$ for every ring homomorphism $j' \colon E \to \mathbb{C}_\ell$.

$\alpha \in K$. $\tau(E) \subseteq F \subseteq K$ and $\varphi(y;v,w) \in K^\times$, so (Qv1) gives

$$\alpha = \varphi(y;v,w)\,\tau(t)/\tau(\det \mathrm{Gr}(x_-)) \in K.$$

(Qn3). Let $j \colon K \to \mathbb{C}_\ell$ with $j|_F = \kappa$. The ring $V_j = \{a \in K : |j(a)| \leq 1\}$ is the valuation ring of the discrete valuation of K defined by j, a discrete valuation ring with fraction field K. Take $\mathcal{G}$, δ, δ' as in (Qv2). Then $(\delta v, \delta' w)$ is an integral frame of $\mathcal{G}$, j restricted to $F \subseteq K$ is κ, and $y \in U(K)$; so the definition of the ℓ-adic norm gives $N_\kappa = \log|j(\varphi(y;\delta v,\delta' w))|$. The frame-change rule (Yf2) with $A = \delta\,1_p$, $B = \delta'\,1_q$ gives $\varphi(y;\delta v,\delta' w)\,\delta^p = \varphi(y;v,w)\,\delta'^q$, hence, with $L = \log|\mathfrak{d}_E|_{j\circ\tau} - \log|j(\tau(r))|$,

$$N_\kappa = \log|j(\varphi(y;v,w))| + q\log|j(\delta')| - p\log|j(\delta)| = \log|j(\varphi(y;v,w))| + (q-p)L.$$

Applying j to (Qv1) and using the previous step with $j' = j \circ \tau$,

$$\begin{aligned}\log|j(\varphi(y;v,w))| &= \log|j(\alpha)| + \log|j(\tau(\det \mathrm{Gr}(x_-)))| - \log|j(\tau(t))| \\ &= \log|j(\alpha)| - q\log|\mathfrak{d}_E|_{j\circ\tau} - \log|j(\tau(t))|.\end{aligned}$$

Together these give (Qn3).

(Qn4) was shown above: the orthogonality in the step on the two pure parts (before transport), and $\det \mathrm{Gr}(x_-) \neq 0$ in the step on the Gram determinant of the negative part. □

Lemma 34.24. *Let $(N, \langle\,,\rangle)$ be a self-dual $\mathcal{O}_E$-lattice of rank $n \geq 1$, I a set with $|I| = n$, and ℓ a prime.*

(Jz1) *For $x, y \in N$ and $a \in \mathcal{O}_E$:*

$$\mathrm{Tr}_{E/\mathbb{Q}}(a\,h_N(x,y)) = \langle ax, y\rangle, \quad h_N(ax,y) = a\,h_N(x,y),$$
$$h_N(x,ay) = \bar a\,h_N(x,y) \quad \textit{and} \quad h_N(y,x) = -\overline{h_N(x,y)}.$$

If $e \in N^I$ is $\mathcal{O}_E$-linearly independent, then $\det \mathrm{Gr}(e) \neq 0$, e is a local basis at all but finitely many primes, and for every $x \in N^I$, with $T = T(x,e)$,

$$\mathrm{Gr}(x) = T\,\mathrm{Gr}(e)\,\overline{T}^{\,t};$$

moreover, if $T' \in \mathrm{M}_I(E)$ satisfies $h_N(x_i,y) = \sum_k T'_{ik} h_N(e_k,y)$ for all $i \in I$ and $y \in N$, then $T' = T(x,e)$.

(Jz2) *If $x \in N^I$ is a local basis at ℓ, then $|j(\det \mathrm{Gr}(x))| = |\mathfrak{d}_E|_j^{-n}$ for every ring homomorphism $j \colon E \to \mathbb{C}_\ell$.*

(Jz3) *If $I = I' \sqcup I''$ and $x = (x', x'') \in N^{I'} \times N^{I''}$ is a local basis at ℓ with $h_N(x'_a, x''_b) = 0$ for all $a \in I'$, $b \in I''$, then $\det \mathrm{Gr}(x'') \neq 0$ and $|j(\det \mathrm{Gr}(x''))| = |\mathfrak{d}_E|_j^{-|I''|}$ for every ring homomorphism $j\colon E \to \mathbb{C}_\ell$.*

(Jz4) *If $x, e \in N^I$ are both local bases at ℓ, then $|j(\det T(x,e))| = 1$ for every ring homomorphism $j\colon E \to \mathbb{C}_\ell$.*

Proof. Write $h = h_N$, $\mathrm{Tr} = \mathrm{Tr}_{E/\mathbb{Q}}$, and recall that $(a,b) \mapsto \mathrm{Tr}(ab)$ is a nondegenerate $\mathbb{Q}$-bilinear form on E and that $\mathrm{Tr}(\bar z) = \mathrm{Tr}(z)$ ($z \mapsto \bar z$ is an automorphism of E).

(Jz1), the identities. For $a \in \mathcal{O}_E$, $a = \sum_k c_k b_k$ with $c_k = \mathrm{Tr}(a b_k^\vee) \in \mathbb{Z}$, since $\mathrm{Tr}(b_l b_k^\vee) = \delta_{lk}$. Hence $\mathrm{Tr}(a\,h(x,y)) = \sum_k \langle b_k x, y\rangle \mathrm{Tr}(a b_k^\vee) = \langle \sum_k c_k b_k x, y\rangle = \langle ax, y\rangle$. An element $z \in E$ is determined by the numbers $\mathrm{Tr}(az)$, $a \in \mathcal{O}_E$, because $\mathcal{O}_E$ spans E over $\mathbb{Q}$ and the trace form is nondegenerate. For all $a' \in \mathcal{O}_E$: $\mathrm{Tr}(a' h(ax,y)) = \langle a'ax, y\rangle = \mathrm{Tr}(a'a\,h(x,y))$; $\mathrm{Tr}(a' h(x,ay)) = \langle a'x, ay\rangle = \langle \bar a a' x, y\rangle = \mathrm{Tr}(a'\bar a\, h(x,y))$; and $\mathrm{Tr}(a' h(y,x)) = \langle a'y, x\rangle = -\langle x, a'y\rangle = -\langle \bar a' x, y\rangle = -\mathrm{Tr}(\bar a' h(x,y)) = -\mathrm{Tr}(a'\,\overline{h(x,y)})$. This gives the four identities; they extend to $N_\mathbb{Q}$ by $\mathbb{Q}$-bilinearity.

(Jz1), nondegeneracy and the Gram matrix of e. If $y \in N_\mathbb{Q}$ and $h(x,y) = 0$ for all $x \in N$, then $\langle x, y\rangle = \mathrm{Tr}(h(x,y)) = 0$ for all x, so $y = 0$ because $\langle\,,\rangle$ is perfect on N, hence nondegenerate on $N_\mathbb{Q}$; by the skew symmetry the same holds in the first variable. Since e is linearly independent of length $n = \mathrm{rk}\, N$, it is an E-basis of $N_\mathbb{Q}$. If $c \in E^I$ and $\mathrm{Gr}(e)\bar c = 0$, then $y = \sum_k c_k e_k$ satisfies $h(e_i, y) = \sum_k \bar c_k h(e_i, e_k) = 0$ for all i, hence $h(x,y) = 0$ for all $x \in N_\mathbb{Q}$, $y = 0$ and $c = 0$; so $\det \mathrm{Gr}(e) \neq 0$. The submodule $N' = \sum_i \mathcal{O}_E e_i$ has $N' \otimes \mathbb{Q} = N_\mathbb{Q}$, so N/N' is a finitely generated torsion group, finite of some order m_0, and $m_0 N \subseteq N'$: e is a local basis at every prime $\ell \nmid m_0$.

(Jz1), transition. By sesquilinearity, $h(x_i, x_k) = \sum_{a,b} T_{ia} \overline{T_{kb}}\, h(e_a, e_b)$, which is $\mathrm{Gr}(x) = T\,\mathrm{Gr}(e)\,\overline{T}^{\,t}$. If T' is as in the statement, then $z_i = x_i - \sum_k T'_{ik} e_k$ satisfies $h(z_i, y) = 0$ for all $y \in N$, so $z_i = 0$ by nondegeneracy, and $T' = T(x,e)$.

The localised lattice and its dual. Let $S = \mathbb{Z} \setminus \ell\mathbb{Z}$ and $\mathcal{O} = S^{-1}\mathcal{O}_E$, a semilocal Dedekind domain, hence a principal ideal domain, stable under conjugation. Then $N_{(\ell)} = S^{-1}N$ is a free $\mathcal{O}$-module of rank n, and $\langle\,,\rangle$ induces $N_{(\ell)} \cong \mathrm{Hom}_{\mathbb{Z}_{(\ell)}}(N_{(\ell)}, \mathbb{Z}_{(\ell)})$ (N is free of finite rank over $\mathbb{Z}$, and localisation commutes with Hom for such modules). We claim: for every $\mathcal{O}$-linear $f\colon N_{(\ell)} \to S^{-1}\mathfrak{d}_E^{-1}$ there is exactly one $y \in N_{(\ell)}$ with $f = h(\cdot, y)$. Indeed $\mathrm{Tr} \circ f$ is $\mathbb{Z}_{(\ell)}$-linear (and $\mathrm{Tr}(S^{-1}\mathfrak{d}_E^{-1}) \subseteq \mathbb{Z}_{(\ell)}$), so there is a unique y with $\langle x, y\rangle = \mathrm{Tr}(f(x))$ for all x; then $\mathrm{Tr}(a\,h(x,y)) = \langle ax, y\rangle = \mathrm{Tr}(f(ax)) = \mathrm{Tr}(a\,f(x))$ for all $a \in \mathcal{O}_E$, so $h(x,y) = f(x)$; and $h(\cdot, y) = 0$ forces $\langle \cdot, y\rangle = 0$, $y = 0$. Choose $\delta \in \mathfrak{d}_E$ with $S^{-1}\mathfrak{d}_E = \delta\mathcal{O}$ (a generator d/s with $d \in \mathfrak{d}_E$, $s \in S$ may be replaced by d, as $s \in \mathcal{O}^\times$); then $S^{-1}\mathfrak{d}_E^{-1} = \delta^{-1}\mathcal{O}$.

Absolute values. Let $j\colon E \to \mathbb{C}_\ell$ be a ring homomorphism. Elements of $\mathcal{O}_E$ are integral over $\mathbb{Z}$ and elements of S are prime to ℓ, so $|j(a)| \leq 1$ for $a \in \mathcal{O}$ and $|j(u)| = 1$ for $u \in \mathcal{O}^\times$. Moreover $|\mathfrak{d}_E|_j = |j(\delta)|$: $\delta \in \mathfrak{d}_E$, and every $d \in \mathfrak{d}_E$ is δo with $o \in \mathcal{O}$, so $|j(d)| \leq |j(\delta)|$.

(Jz2). A local basis x at ℓ is an $\mathcal{O}$-basis of $N_{(\ell)}$: it is linearly independent, and $m_0 N \subseteq \sum \mathcal{O}_E x_i$ with $m_0 \in S$. Let $G = \mathrm{Gr}(x)$. By the claim, $y \mapsto (h(x_i, y))_{i \in I}$ is a bijection $N_{(\ell)} \to (\delta^{-1}\mathcal{O})^I$ (an $\mathcal{O}$-linear map on the free module $N_{(\ell)}$ is given by arbitrary values on the basis x). For $y = \sum_k c_k x_k$ this map is $c \mapsto G\bar c$. As $c \mapsto \bar c$ is a bijection of $\mathcal{O}^I$, the matrix $\delta G \in \mathrm{M}_I(\mathcal{O})$ defines a bijection $\mathcal{O}^I \to \mathcal{O}^I$, so $\delta G \in \mathrm{GL}_I(\mathcal{O})$ and $\delta^n \det G \in \mathcal{O}^\times$. Hence $|j(\det G)| = |j(\delta)|^{-n} = |\mathfrak{d}_E|_j^{-n}$.

(Jz3). By the hypothesis and the skew symmetry in (Jz1), $h(x'_a, x''_b) = 0 = h(x''_b, x'_a)$, so $\mathrm{Gr}(x)$ is block diagonal with blocks $\mathrm{Gr}(x')$, $\mathrm{Gr}(x'')$. As in (Jz2), $\delta\,\mathrm{Gr}(x) \in \mathrm{GL}_I(\mathcal{O})$, so $\det(\delta\,\mathrm{Gr}(x')) \cdot \det(\delta\,\mathrm{Gr}(x'')) \in \mathcal{O}^\times$ with both factors in $\mathcal{O}$; each factor is therefore a unit of $\mathcal{O}$. Thus $\delta^{|I''|} \det \mathrm{Gr}(x'') \in \mathcal{O}^\times$, in particular $\det \mathrm{Gr}(x'') \neq 0$, and $|j(\det \mathrm{Gr}(x''))| = |\mathfrak{d}_E|_j^{-|I''|}$.

(Jz4). Both x and e are $\mathcal{O}$-bases of $N_{(\ell)}$. The coordinates of x_i in the basis e of $N_{\mathbb{Q}}$ are those in the basis e of $N_{(\ell)}$, so $T(x,e) \in \mathrm{M}_I(\mathcal{O})$; likewise $T(e,x) \in \mathrm{M}_I(\mathcal{O})$, and $T(x,e)T(e,x) = 1$. Hence $\det T(x,e) \in \mathcal{O}^\times$ and $|j(\det T(x,e))| = 1$. □

Theorem 34.25. *Let $\mathfrak{u}$ be a unitary Shimura datum with $n = p+q \geq 1$, $\tau \in \Phi_{\mathsf{H}}$, $U \subseteq \mathrm{Sh}_F$ a union of connected components, Y a connected component of $U(\mathbb{C})$, $z_0 \in Y$, and φ a flat generator over Y with constants (e,c). Let ℓ be a prime, $\alpha \in \mathbb{C}^\times$ and (K,y,γ,x,v,w) a pure CM point at ℓ for φ with constant α, with numbers $n_\pm(s)$ as in (Qv3). If φ has archimedean norm N_{s_0} at an embedding $s_0\colon F \to \mathbb{C}$, then for every ring homomorphism $s\colon K \to \mathbb{C}$ with $s|_F = s_0$,*

$$N_{s_0} = \log|s(\alpha)| + \frac{p}{2}\Big(\log n_+(s) + \frac{1}{2}\log d\Big) - \frac{q}{2}\Big(\log n_-(s) + \frac{1}{2}\log d\Big) + \frac{1}{2}\log|s(\tau(\det \mathrm{Gr}(e)))|.$$

Proof. Write h for the trace form of Λ_z, $x^\circ = \mu_\gamma^{-1}x$, $T = T(x^\circ, e)$, $t = \det T$, $G_\pm = \det \mathrm{Gr}(x_\pm)$. By Theorem 34.23, $t \in E^\times$, $\alpha \in K$, and by (Qn4) $\mathrm{Gr}(x)$ is block diagonal with blocks $\mathrm{Gr}(x_+)$, $\mathrm{Gr}(x_-)$. For an embedding $\varepsilon\colon E \to \mathbb{C}$ of the CM field E, $\varepsilon(\bar a) = \overline{\varepsilon(a)}$; in particular $s\circ\tau$ is such an embedding, $|s\tau(r)| = \sqrt{d}$ (as $r^2 = -d$), and $s(\bar\tau(a)) = s(\tau(\bar a)) = \overline{s\tau(a)}$.

The value. The point $y^s \in U(\mathbb{C})$ induced by s lies over s_0, and (s_*v, s_*w) is a frame at it. By the definition of the archimedean norm (Yf3),

$$N_{s_0} = \log|\varphi(y^s; s_*v, s_*w)| + \tfrac{1}{2}\log\det\Gamma(s_*v) - \tfrac{1}{2}\log\det\Gamma(s_*w).$$

By the functoriality in (Yf2) for the field homomorphism s, $\varphi(y^s; s_*v, s_*w) = s(\varphi(y; v, w))$, and by (Qv1), $\varphi(y; v, w) = \alpha\,\tau(G_-)/\tau(t)$. Hence $\log|\varphi(y^s; s_*v, s_*w)| = \log|s(\alpha)| + \log|s\tau(G_-)| - \log|s\tau(t)|$.

The Hodge Grams. $\Gamma(s_*v)$ is the Gram matrix of a basis of $\mathrm{Lie}_{(\tau,+)}(G_{y^s})$ for the positive definite Hodge inner product, so $\det\Gamma(s_*v) > 0$. By (Qv3), $\det\Gamma(s_*v) = \epsilon_+^p n_+(s)^p\, s\tau(r)^p\, s\tau(G_+)$; taking absolute values, $\log\det\Gamma(s_*v) = p\log n_+(s) + \frac{p}{2}\log d + \log|s\tau(G_+)|$. Likewise, the matrix $\big(s(\bar\tau(r\,h(x_{-,c}, x_{-,c'})))\big)$ is the entrywise complex conjugate of $\big(s\tau(r\,h(x_{-,c}, x_{-,c'}))\big)$, whose determinant is $s\tau(r)^q s\tau(G_-)$; so $\log\det\Gamma(s_*w) = q\log n_-(s) + \frac{q}{2}\log d + \log|s\tau(G_-)|$.

Assembly. Substituting,

$$N_{s_0} = \log|s(\alpha)| + \frac{p}{2}\Big(\log n_+(s) + \frac{1}{2}\log d\Big) - \frac{q}{2}\Big(\log n_-(s) + \frac{1}{2}\log d\Big) + \frac{1}{2}\big(\log|s\tau(G_+)| + \log|s\tau(G_-)|\big) - \log|s\tau(t)|.$$

Parallel transport preserves the trace form, so $\mathrm{Gr}(x^\circ) = \mathrm{Gr}(x)$, and by (Jz1) $\mathrm{Gr}(x^\circ) = T\,\mathrm{Gr}(e)\,\overline{T}^{\,t}$; with the block structure, $G_+G_- = t\,\bar t\,\det\mathrm{Gr}(e)$. Applying $s\tau$ and $|s\tau(\bar t)| = |s\tau(t)|$ gives $\frac{1}{2}(\log|s\tau(G_+)| + \log|s\tau(G_-)|) - \log|s\tau(t)| = \frac{1}{2}\log|s\tau(\det\mathrm{Gr}(e))|$, which proves the formula. □

Proposition 34.26. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p+q \geq 1$, $\tau \in \Phi_{\mathsf{H}}$, $U \subseteq \mathrm{Sh}_F$ a union of connected components, Y a connected component of $U(\mathbb{C})$, $z_0 \in Y$, and φ a flat generator over Y with constants (e,c). Then there is $\alpha \in \mathbb{C}^\times$ such that for every prime ℓ and every number field $k \subset \mathbb{C}$ there is a pure CM point (K,y,γ,x,v,w) at ℓ for φ with constant α and $k \subseteq K$.*

Proof. Let $\alpha \in \mathbb{C}^\times$ be the constant given by Proposition 34.30 for $\mathfrak{u}, \tau, U, Y, z_0, \varphi$. Let ℓ be a prime and $k \subset \mathbb{C}$ a number field.

A CM structure realised on Y. By Proposition 34.29 there is a non-empty open subset $W \subseteq \mathrm{Per}(\Lambda_{z_0})$ every element of which is realised at some point of Y along some path from z_0. Choose $J \in W$. By (Wy2) of Proposition 34.27, applied at the point $z_0 \in \mathrm{Sh}_F(\mathbb{C})$ with the neighbourhood W of J, there are $J' \in W$ and a tuple $x^\circ = (x^\circ_+, x^\circ_-) \in \Lambda^p_{z_0} \times \Lambda^q_{z_0}$ that is pure for J' and a local basis at ℓ. As $J' \in W$, there are $z \in Y$ and a path γ in Y from z_0 to z with $\mu_\gamma \circ J' = J_z \circ \mu_\gamma$.

Transport. Put $x = \mu_\gamma x^\circ \in \Lambda^n_z$. Parallel transport $\mu_\gamma \colon \Lambda_{z_0} \to \Lambda_z$ is an isomorphism of self-dual $\mathcal{O}_E$-lattices (Yf4); hence x is $\mathcal{O}_E$-linearly independent, and $m_0\Lambda_{z_0} \subseteq \sum_i \mathcal{O}_E x^\circ_i$ with $\ell \nmid m_0$ gives $m_0\Lambda_z \subseteq \sum_i \mathcal{O}_E x_i$: x is a local basis at ℓ. The $\mathbb{R}$-linear extension of μ_γ commutes with $\iota(r)$, so

$$\iota(r)x_{+,a} = \mu_\gamma(\iota(r)x^\circ_{+,a}) = \mu_\gamma(\mathrm{Im}(r_\mathbb{C})J'x^\circ_{+,a}) = \mathrm{Im}(r_\mathbb{C})\, J_z\, x_{+,a},$$

and likewise $\iota(r)x_{-,b} = -\mathrm{Im}(r_\mathbb{C})J_z x_{-,b}$: x is pure for J_z, that is, pure in the sense of (Yf5) (Yf7).

The CM point. By Proposition 34.30, applied to ℓ, z, γ, x and k, there are a number field $K \supseteq F, k$, a point $y \in U(K)$ with complex point z, and a frame (v, w) at y such that (K, y, γ, x, v, w) is a pure CM point at ℓ for φ with constant α. As α was chosen before ℓ and k, this proves the statement. □

Proposition 34.27. *Let $\mathfrak{u}$ be a unitary Shimura datum with $n = p+q \geq 1$, $z \in \mathrm{Sh}_F(\mathbb{C})$, $\Lambda = \Lambda_z$ with trace form h and $\langle\,,\rangle = \mathrm{Im}\, H_z$. For $J \in \mathrm{Per}(\Lambda)$ put $H_J(u,v) = \langle Ju, v\rangle + i\langle u, v\rangle$ on $\Lambda_\mathbb{R}$, and for $u \in \Lambda_\mathbb{R}$ let u_s be its component in L^J_s.*

(Wy1) Trace form and Hodge form. *For $J \in \mathrm{Per}(\Lambda)$, $\tau \in \Phi_\mathsf{H}$ and $\lambda, \mu \in \Lambda$,*

$$i\,\tau\big(h(\lambda,\mu)\big) = \tfrac{1}{2}\Big(H_J\big(\lambda_{(\tau,+)}, \mu_{(\tau,+)}\big) - H_J\big(\mu_{(\tau,-)}, \lambda_{(\tau,-)}\big)\Big).$$

(Wy2) CM splittings near every point. *For every prime ℓ, every $J \in \mathrm{Per}(\Lambda)$ and every neighbourhood W of J in $\mathrm{Per}(\Lambda)$ there are $J' \in W$ and a tuple $x = (x_+, x_-) \in \Lambda^p \times \Lambda^q$ that is pure for J', is a local basis at ℓ, and has $h(x_{+,a}, x_{-,b}) = 0$ for all a, b.*

Proof. Put $\kappa = \mathrm{Im}(r_\mathbb{C})$; as $r_\mathbb{C}^2 = -d < 0$, $r_\mathbb{C} = i\kappa$ and $\kappa^2 = d$. For $\tau \in \Phi = \Phi_\mathsf{H}$, $\tau(r) = i\kappa$; $\tau(\bar a) = \overline{\tau(a)}$ (E is a CM field), and $\mathrm{Hom}(E, \mathbb{C}) = \Phi \sqcup \bar\Phi$. The map $a \mapsto (\tau(a))_{\tau\in\Phi}$ induces $E \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{C}^\Phi$, so the image of $\mathcal{O}_E$ spans $\mathbb{C}^\Phi$ over $\mathbb{R}$. We extend h to $\Lambda_\mathbb{Q}$ sesquilinearly; the identities of (Jz1), in particular $\langle \iota(a)\lambda, \mu\rangle = \mathrm{Tr}_{E/\mathbb{Q}}(a\, h(\lambda,\mu))$, persist.

Step 1: the form H_J and (Wy1). Let $J \in \mathrm{Per}(\Lambda)$. By J-invariance and alternation of $\langle\,,\rangle$, $\langle Jv, u\rangle = \langle J^2 v, Ju\rangle = \overline{\langle Ju, v\rangle}$; hence H_J is hermitian for the complex structure J ($H_J(Ju,v) = iH_J(u,v)$, $H_J(v,u) = \overline{H_J(u,v)}$), positive definite ($H_J(u,u) = \langle Ju, u\rangle$), with $\mathrm{Im}\, H_J = \langle\,,\rangle$. As J commutes with ι and $\langle \iota(a)u, v\rangle = \langle u, \iota(\bar a)v\rangle$, $H_J(\iota(a)u, v) = H_J(u, \iota(\bar a)v)$. On L^J_s, $\iota(a)$ is multiplication by $\sigma_s(a)$ for the structure J. For $u \in L^J_s$, $v \in L^J_{s'}$, $s \neq s'$: $\sigma_s(a)H_J(u,v) = H_J(u, \iota(\bar a)v) = \overline{\sigma_{s'}(\bar a)}H_J(u,v) = \sigma_{s'}(a)H_J(u,v)$, and some $a \in \mathcal{O}_E$ has $\sigma_s(a) \neq \sigma_{s'}(a)$; so the L^J_s are H_J-orthogonal. For $\lambda, \mu \in \Lambda$ and $a \in \mathcal{O}_E$, writing $H_s = H_J(\lambda_s, \mu_s)$,

$$\langle \iota(a)\lambda, \mu\rangle = \mathrm{Im}\, H_J(\iota(a)\lambda, \mu) = \sum_s \mathrm{Im}(\sigma_s(a)H_s) = \sum_{\tau\in\Phi} \mathrm{Re}\Big(\tau(a)\big(-iH_{(\tau,+)} + i\,\overline{H_{(\tau,-)}}\big)\Big),$$

using $\mathrm{Im}\, w = \mathrm{Re}(-iw)$ and $\mathrm{Re}(\bar c w) = \mathrm{Re}(c\bar w)$, while

$$\mathrm{Tr}_{E/\mathbb{Q}}(a\, h(\lambda,\mu)) = \sum_{\tau\in\Phi} 2\,\mathrm{Re}(\tau(a)\tau(h(\lambda,\mu))).$$

Since the vectors $(\tau(a))_\tau$ span $\mathbb{C}^\Phi$ over $\mathbb{R}$, $2\tau(h(\lambda,\mu)) = -iH_{(\tau,+)} + i\overline{H_{(\tau,-)}}$ for every τ; multiplying by $i/2$ and using $\overline{H_{(\tau,-)}} = H_J(\mu_{(\tau,-)}, \lambda_{(\tau,-)})$ gives (Wy1).

Step 2: the hermitian forms k_τ. Fix $J \in \mathrm{Per}(\Lambda)$ and $\tau \in \Phi$; let $L_\tau = L^J_{(\tau,+)} \oplus L^J_{(\tau,-)}$ with the complex structure i_τ equal to J on the first and to $-J$ on the second summand, $\Pi_\tau \colon \Lambda_\mathbb{R} \to L_\tau$ the projection along the other pieces, and $k_\tau(u,v) = \frac{1}{2}\big(H_J(u_{(\tau,+)}, v_{(\tau,+)}) - H_J(v_{(\tau,-)}, u_{(\tau,-)})\big)$ for $u, v \in \Lambda_\mathbb{R}$. Since $H_J(v, Jw) = -iH_J(v,w)$, k_τ is a hermitian form on (L_τ, i_τ); it is positive definite on $L^J_{(\tau,+)}$, negative definite on $L^J_{(\tau,-)}$, and these are k_τ-orthogonal, of complex dimensions p and q. By Sylvester's law of inertia every Gram matrix of k_τ in an i_τ-basis of L_τ has p positive and q negative eigenvalues. On L_τ, $\iota(a)$ acts as multiplication by $\tau(a)$ for i_τ, and Π_τ commutes with ι. Consequently $\Pi_\tau = \iota(e_\tau)$ for the idempotent $e_\tau \in E \otimes \mathbb{R}$ of the factor τ, and $\Pi_\tau(cy) = \tau(c) \cdot_{i_\tau} \Pi_\tau(y)$ for $c \in E$, $y \in \Lambda_\mathbb{Q}$. If $x \in \Lambda^n$ is $\mathcal{O}_E$-linearly independent, it is an E-basis of $\Lambda_\mathbb{Q}$, so $\Pi_\tau(\Lambda)$, which spans L_τ over $\mathbb{R}$, lies in the i_τ-span of $\Pi_\tau(x_1), \dots, \Pi_\tau(x_n)$; as $\dim L_\tau = n$ these form an i_τ-basis, and by (Wy1) the Gram matrix of k_τ in it is $i\tau(\mathrm{Gr}(x))$. Both sides of $k_\tau(\lambda,\mu) = i\tau(h(\lambda,\mu))$, hence of

$$\langle \iota(a)u, v\rangle = \sum_{\tau\in\Phi} 2\,\mathrm{Re}\big(\tau(a)\,(-i)k_\tau(u,v)\big) \qquad (a \in \mathcal{O}_E), \tag{$*$}$$

are $\mathbb{R}$-bilinear, so $(*)$ holds for all $u, v \in \Lambda_\mathbb{R}$.

Step 3: the target. Fix ℓ, J, W as in (Wy2). For each $\tau \in \Phi$ choose a J-basis $(t_{\tau,a})_{a\le p}$ of $L^J_{(\tau,+)}$ and put $t_a = \sum_\tau t_{\tau,a}$. For $y \in \Lambda^p_\mathbb{R}$ let $Q_\tau(y) = (k_\tau(y_a, y_b))_{a,b}$: continuous, $Q_\tau(sy) = s^2 Q_\tau(y)$ for $s \in \mathbb{R}$, $Q_\tau(y) = i\tau(\mathrm{Gr}(y))$ for $y \in \Lambda^p$ (Step 2), and $Q_\tau(t) = \frac{1}{2}(H_J(t_{\tau,a}, t_{\tau,b}))_{a,b}$ is positive definite.

Step 4: integral approximation with ℓ-adic control. By (Uo2) (applicable as E/E^+ is unramified at every finite place) there is an orthogonal local basis of Λ at ℓ; let $u_+ \in \Lambda^p$ be its first p entries, so $|j(\det \mathrm{Gr}(u_+))| = |\mathfrak{d}_E|_j^{-p}$ for every ring homomorphism $j \colon E \to \mathbb{C}_\ell$. Fix a norm on $\Lambda_\mathbb{R}$ and R_0 such that every point of $\Lambda_\mathbb{R}$ lies within R_0 of the lattice $\ell\Lambda$. For an integer $N \ge 1$ choose $w \in \Lambda^p$ with $\|\ell w_a - N t_a\| \le R_0$ and put $x^{(N)}_+ = u_+ + \ell w$; then $\|N^{-1}x^{(N)}_{+,a} - t_a\| \le (\|u_{+,a}\| + R_0)/N$. (i) For $y \in \mathfrak{d}_E^{-1}$ and $d \in \mathfrak{d}_E$, $yd \in \mathcal{O}_E$, so $|j(y)| \le |\mathfrak{d}_E|_j^{-1}$. The entries of $A = \mathrm{Gr}(u_+)$ lie in $\mathfrak{d}_E^{-1}$ (Qd1), and those of $B = \mathrm{Gr}(x^{(N)}_+) - A$ are $\ell\big(h(w_a, u_{+,b}) + h(u_{+,a}, w_b)\big) + \ell^2 h(w_a, w_b)$. Expanding $\det(A+B) - \det A$ multilinearly, every term contains an entry of B; by the ultrametric inequality $|j(\det(A+B) - \det A)| \le |\ell|\,|\mathfrak{d}_E|_j^{-p} < |j(\det A)|$, hence $|j(\det \mathrm{Gr}(x^{(N)}_+))| = |\mathfrak{d}_E|_j^{-p}$, and $\det \mathrm{Gr}(x^{(N)}_+) \ne 0$. (ii) $Q_\tau(N^{-1}x^{(N)}_+) \to Q_\tau(t)$, and positive definite matrices form an open set; so there is N_0 such that for $N \ge N_0$ and all $\tau \in \Phi$, $i\tau(\mathrm{Gr}(x^{(N)}_+)) = N^2 Q_\tau(N^{-1}x^{(N)}_+)$ is positive definite.

Step 5: completion. Let $N \ge N_0$ and $x_+ = x^{(N)}_+$. By (Uo1) there is $x_- \in \Lambda^q$ with $x = (x_+, x_-)$ a local basis at ℓ and $h(x_{+,a}, x_{-,b}) = 0$; by (Jz1) also $h(x_{-,b}, x_{+,a}) = 0$, so $\mathrm{Gr}(x)$ is block diagonal. By Step 2, $i\tau(\mathrm{Gr}(x))$ has p positive and q negative eigenvalues; its first block is positive definite of size p, so $i\tau(\mathrm{Gr}(x_-))$ is negative definite, for every $\tau \in \Phi$.

Step 6: the complex structure J'_N. Let $V_\pm = \sum_a E x_{\pm,a} \subseteq \Lambda_\mathbb{Q}$ and $V_{\pm,\mathbb{R}} = V_\pm \otimes \mathbb{R}$; then $\Lambda_\mathbb{R} = V_{+,\mathbb{R}} \oplus V_{-,\mathbb{R}}$, both summands are ι-stable and $\langle V_{+,\mathbb{R}}, V_{-,\mathbb{R}}\rangle = 0$ (as $\langle\,,\rangle = \mathrm{Tr}_{E/\mathbb{Q}} \circ h$). Define $J' = J'_N$ by $J' = \kappa^{-1}\iota(r)$ on $V_{+,\mathbb{R}}$ and $J' = -\kappa^{-1}\iota(r)$ on $V_{-,\mathbb{R}}$. Then $J'^2 = \kappa^{-2}\iota(r^2) = -1$, J' commutes with ι, and x is pure for J' by construction. For $\lambda, \mu \in V_\pm$, $\langle \iota(r)\lambda, \iota(r)\mu\rangle = \mathrm{Tr}(r\bar r\, h(\lambda,\mu)) = d\langle\lambda,\mu\rangle$, so $\langle J'u, J'v\rangle = \langle u, v\rangle$ on each summand, hence everywhere. By $(*)$ with $a = r$, $\langle \iota(r)u, v\rangle = 2\kappa \sum_\tau \mathrm{Re}\, k_\tau(u,v)$; so $\langle J'u, u\rangle = 2\sum_\tau k_\tau(\Pi_\tau u, \Pi_\tau u)$ for $u \in V_{+,\mathbb{R}}$ and $-2\sum_\tau k_\tau(\Pi_\tau u, \Pi_\tau u)$ for $u \in V_{-,\mathbb{R}}$. Now $P_\tau = \Pi_\tau(V_{+,\mathbb{R}}) = V_{+,\mathbb{R}} \cap L_\tau$ is the i_τ-span of $\Pi_\tau(x_{+,1}), \dots, \Pi_\tau(x_{+,p})$, on which k_τ has Gram matrix $i\tau(\mathrm{Gr}(x_+))$, positive definite; likewise k_τ is negative definite on $M_\tau = \Pi_\tau(V_{-,\mathbb{R}})$. As $u = \sum_\tau \Pi_\tau u$, $\langle J'u, u\rangle > 0$ for $u \ne 0$ in either

summand, and the cross terms vanish, so $\langle J'u,u\rangle>0$ for all $u\neq 0$. On P_τ, $\iota(r)$ is $\tau(r)=i\kappa$ for i_τ, so $J'=i_\tau$ and $\iota(a)$ is multiplication by $\tau(a)$ for J': $P_\tau\subseteq L^{J'}_{(\tau,+)}$; on M_τ, $J'=-i_\tau$ and $\iota(a)$ is multiplication by $\overline{\tau(a)}$ for J': $M_\tau\subseteq L^{J'}_{(\tau,-)}$. The $\Pi_\tau(x_{\pm,\cdot})$ being i_τ-independent, $\dim_{\mathbb{R}}P_\tau=2p$, $\dim_{\mathbb{R}}M_\tau=2q$, and $\Lambda_{\mathbb{R}}=\bigoplus_\tau(P_\tau\oplus M_\tau)$. The spaces $L^{J'}_s$ are eigenspaces of the commuting J'-linear maps $\iota(a)$ for the distinct characters σ_s, so their sum is direct; hence $L^{J'}_{(\tau,+)}=P_\tau$, $L^{J'}_{(\tau,-)}=M_\tau$, and $J'\in\mathrm{Per}(\Lambda)$.

Step 7: $J'_N\to J$. On L_τ, $J=i_\tau(2\pi_\infty-1)$, where π_∞ is the k_τ-orthogonal projection onto $L^J_{(\tau,+)}$ (whose k_τ-orthogonal complement is $L^J_{(\tau,-)}$). Similarly P_τ and M_τ are k_τ-orthogonal (their Gram block is $i\tau$ of $h(x_+,x_-)=0$), so $J'_N=i_\tau(2\pi_N-1)$ on L_τ with π_N the k_τ-orthogonal projection onto P_τ. With $y_a=\Pi_\tau(N^{-1}x_{+,a})$, an i_τ-basis of P_τ, and $G_N=(k_\tau(y_a,y_b))_{a,b}=Q_\tau(N^{-1}x_+)$,

$$\pi_N(u)=\sum_{a,b}k_\tau(u,y_b)\,(G_N^{-1})_{ba}\cdot_{i_\tau}y_a$$

(it fixes each y_c and kills $P_\tau^\perp$), and π_∞ is given by the same formula with y_a replaced by $t_{\tau,a}=\Pi_\tau(t_a)$ and G by $Q_\tau(t)$. As $N\to\infty$, $y_a\to t_{\tau,a}$ and $G_N\to Q_\tau(t)$, which is invertible, so $\pi_N\to\pi_\infty$ and $J'_N\to J$ on every L_τ, hence on $\Lambda_{\mathbb{R}}=\bigoplus_\tau L_\tau$. Choosing $N\geq N_0$ with $J'_N\in W$ proves (Wy2). □

Lemma 34.28. *Let E be a CM field with maximal totally real subfield E^+ such that E/E^+ is unramified at every finite place, let $(N,\langle\,,\rangle)$ be a self-dual $\mathcal{O}_E$-lattice of rank $n\geq 1$, $I=I'\sqcup I''$ a set with $|I|=n$, and ℓ a prime. For $x\in N^I$ and $I_0\subseteq I$ write $x|_{I_0}=(x_i)_{i\in I_0}$.*

(Uo1) Splitting off. *If $x'\in N^{I'}$ satisfies $\det\mathrm{Gr}(x')\neq 0$ and $|j(\det\mathrm{Gr}(x'))|=|\mathfrak{d}_E|_j^{-|I'|}$ for every ring homomorphism $j\colon E\to\mathbb{C}_\ell$, then there is $x''\in N^{I''}$ such that (x',x'') is a local basis at ℓ and $h_N(x'_a,x''_b)=0$ for all $a\in I'$, $b\in I''$.*

(Uo2) Orthogonal local bases. *There is a local basis $x\in N^I$ at ℓ with $h_N(x_i,x_k)=0$ for all $i\neq k$. For every such x and every non-empty $I_0\subseteq I$, $\det\mathrm{Gr}(x|_{I_0})\neq 0$ and $|j(\det\mathrm{Gr}(x|_{I_0}))|=|\mathfrak{d}_E|_j^{-|I_0|}$ for every ring homomorphism $j\colon E\to\mathbb{C}_\ell$.*

Proof. Write $h=h_N$, $S=\mathbb{Z}\smallsetminus\ell\mathbb{Z}$, $\mathcal{O}=S^{-1}\mathcal{O}_E$, $N_{(\ell)}=S^{-1}N$, and $c\mapsto\bar c$ for complex conjugation on E (it preserves $\mathcal{O}$).

Step 0: the ring $\mathcal{O}$ and a real generator of the different. $\mathcal{O}$ is a semilocal Dedekind domain, hence a principal ideal domain; its maximal ideals are the ideals $\lambda\mathcal{O}$ for the primes λ of $\mathcal{O}_E$ above ℓ, pairwise distinct, and an element of $\mathcal{O}$ lying in none of them is a unit. $N_{(\ell)}$ is a free $\mathcal{O}$-module of rank n. For every such λ there is a ring homomorphism $j\colon E\to\mathbb{C}_\ell$ with $\lambda=\{a\in\mathcal{O}_E:|j(a)|<1\}$ (embed the completion E_λ into $\mathbb{C}_\ell$); and $|j(a)|\leq 1$ for $a\in\mathcal{O}$ and all j. Hence an element $a\in\mathcal{O}$ with $|j(a)|=1$ for every j is a unit of $\mathcal{O}$. Since E/E^+ is unramified at every finite place, its relative different is $\mathcal{O}_E$, and by transitivity of differents $\mathfrak{d}_E=\mathfrak{d}_{E^+}\mathcal{O}_E$ (Neukirch, Algebraic Number Theory, III (2.2) and (2.6)). The ring $S^{-1}\mathcal{O}_{E^+}$ is again a principal ideal domain, so $S^{-1}\mathfrak{d}_{E^+}=\delta\,S^{-1}\mathcal{O}_{E^+}$ for some $\delta\in\mathfrak{d}_{E^+}$ (a generator d/s with $s\in S$ may be replaced by d). Then $S^{-1}\mathfrak{d}_E=\delta\mathcal{O}$, $S^{-1}\mathfrak{d}_E^{-1}=\delta^{-1}\mathcal{O}$, $\bar\delta=\delta$, and $|\mathfrak{d}_E|_j=|j(\delta)|$ for every j (as $\delta\in\mathfrak{d}_E$ and every element of $\mathfrak{d}_E$ is δo with $o\in\mathcal{O}$).

Put $g=\delta h$ on $N_{(\ell)}$. As $h(N,N)\subseteq\mathfrak{d}_E^{-1}$ (Qd1), g takes values in $\mathcal{O}$; by (Jz1) and $\bar\delta=\delta$, g is $\mathcal{O}$-linear in the first variable, $g(u,av)=\bar a g(u,v)$, and $g(v,u)=-\overline{g(u,v)}$. Call a free $\mathcal{O}$-submodule $L\subseteq N_{(\ell)}$ of finite rank *unimodular* if the matrix $(g(b_i,b_k))$ of some $\mathcal{O}$-basis b of L lies in $\mathrm{GL}(\mathcal{O})$; this does not depend on the basis, since a change of basis by $P\in\mathrm{GL}(\mathcal{O})$ replaces the matrix G by $PG\overline{P}^t$.

Step 1: bases of $N_{(\ell)}$ and local bases. If $x \in N^I$ is an $\mathcal{O}$-basis of $N_{(\ell)}$, then x is a local basis at ℓ: it is $\mathcal{O}_E$-linearly independent, and if $y_1, \dots, y_m$ generate N and $y_k = \sum_i c_{ki} x_i$ with $c_{ki} \in \mathcal{O}$, an integer $m_0 \in S$ with $m_0 c_{ki} \in \mathcal{O}_E$ for all k, i gives $m_0 N \subseteq \sum_i \mathcal{O}_E x_i$. Every element of $N_{(\ell)}$ becomes an element of N after multiplication by an element of S, which is a unit of $\mathcal{O}$; so every $\mathcal{O}$-basis of $N_{(\ell)}$ can be rescaled to one with entries in N.

Step 2: $N_{(\ell)}$ is unimodular. Let $b \in N^I$ be an $\mathcal{O}$-basis of $N_{(\ell)}$ (Step 1). It is a local basis at ℓ, so by (Jz2) $|j(\det \mathrm{Gr}(b))| = |j(\delta)|^{-n}$ for every j; thus $\det(\delta \,\mathrm{Gr}(b)) \in \mathcal{O}$ has absolute value 1 under every j and is a unit (Step 0).

Step 3: (Uo1). The matrix $\delta \,\mathrm{Gr}(x')$ has entries in $\mathcal{O}$ and $|j(\det \delta \,\mathrm{Gr}(x'))| = |j(\delta)|^{|I'|} |\mathfrak{d}_E|_j^{-|I'|} = 1$ for every j, so $\delta \,\mathrm{Gr}(x') \in \mathrm{GL}_{I'}(\mathcal{O})$. Let $M'' = \{u \in N_{(\ell)} : h(u, x'_a) = 0 \ \forall a \in I'\}$ and $M' = \sum_a \mathcal{O} x'_a$. For $u \in N_{(\ell)}$ the row vector $c = \big(g(u, x'_a)\big)_a \, (\delta \,\mathrm{Gr}(x'))^{-1}$ has entries in $\mathcal{O}$, and $u - \sum_b c_b x'_b \in M''$, because $g(\sum_b c_b x'_b, x'_a) = (c\,\delta \,\mathrm{Gr}(x'))_a = g(u, x'_a)$. If $\sum_b c_b x'_b \in M''$ with $c \in \mathcal{O}^{I'}$, then $c\,\delta \,\mathrm{Gr}(x') = 0$ and $c = 0$. Hence x' is an $\mathcal{O}$-basis of M' and $N_{(\ell)} = M' \oplus M''$. The submodule M'' of the free module $N_{(\ell)}$ over the principal ideal domain $\mathcal{O}$ is free, of rank $n - |I'| = |I''|$ (ranks add, as one sees after $\otimes_{\mathcal{O}} E$). Choose an $\mathcal{O}$-basis of M'' indexed by I'' and rescale it into N (Step 1): this gives $x'' \in N^{I''}$, an $\mathcal{O}$-basis of M''. Then (x', x'') is an $\mathcal{O}$-basis of $N_{(\ell)}$ with entries in N, a local basis at ℓ by Step 1, and $h(x''_b, x'_a) = 0$, hence $h(x'_a, x''_b) = -\overline{h(x''_b, x'_a)} = 0$ by (Jz1).

Step 4: a vector of unit length. Let $L \neq 0$ be a unimodular free submodule. We show that there is $f \in L$ with $g(f, f) \in \mathcal{O}^\times$. (a) For every prime λ of $\mathcal{O}_E$ above ℓ there is $a \in \mathcal{O}_E$ with $a - \bar{a} \notin \lambda$. If $\bar{\lambda} \neq \lambda$, the Chinese remainder theorem gives a with $a \equiv 1 \bmod \lambda$ and $a \equiv 0 \bmod \bar{\lambda}$; then $\bar{a} \in \lambda$ and $a - \bar{a} \equiv 1 \bmod \lambda$. If $\bar{\lambda} = \lambda$, complex conjugation, which generates $\mathrm{Gal}(E/E^+)$, fixes λ, so the decomposition group of λ over $\lambda \cap \mathcal{O}_{E^+}$ is $\mathrm{Gal}(E/E^+)$; as λ is unramified over E^+, the decomposition group maps isomorphically onto the Galois group of the residue extension (Neukirch, op. cit., I (9.4) and (9.6)), so conjugation induces a non-trivial automorphism of $\mathcal{O}_E/\lambda$, and any a whose residue is not fixed by it has $a - \bar{a} \notin \lambda$. (b) For every maximal ideal $\mathfrak{m} = \lambda\mathcal{O}$ there is $f_\mathfrak{m} \in L$ with $g(f_\mathfrak{m}, f_\mathfrak{m}) \notin \mathfrak{m}$. Let $b_1, \dots, b_r$ be an $\mathcal{O}$-basis of L with matrix $G = (g(b_i, b_k)) \in \mathrm{GL}_r(\mathcal{O})$. The map $L \to \mathcal{O}^r$, $v \mapsto (g(b_i, v))_i$, sends $\sum_k c_k b_k$ to $G\bar{c}$ and is bijective; choose $y \in L$ with $g(b_1, y) = 1$ and put $x = b_1$. If $g(x, x) \notin \mathfrak{m}$ or $g(y, y) \notin \mathfrak{m}$, take x or y. Otherwise, with a as in (a) and $g(y, x) = -\overline{g(x, y)} = -1$,

$$g(x + ay, x + ay) = g(x, x) + \bar{a}\, g(x, y) + a\, g(y, x) + a\bar{a}\, g(y, y) \equiv \bar{a} - a \not\equiv 0 \pmod{\mathfrak{m}},$$

and we take $f_\mathfrak{m} = x + ay$. (c) For $f \in L$, $\overline{g(f, f)} = -g(f, f)$, so $g(f, f) \in \mathfrak{m}$ if and only if $g(f, f) \in \bar{\mathfrak{m}}$. Choose maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_t$, one in each orbit $\{\mathfrak{m}, \bar{\mathfrak{m}}\}$, put $f_i = f_{\mathfrak{m}_i}$, and choose by the Chinese remainder theorem $c_i \in \mathcal{O}$ with $c_i \equiv 1$ modulo $\mathfrak{m}_i$ and $\bar{\mathfrak{m}}_i$, and $c_i \equiv 0$ modulo the maximal ideals of the other orbits; $\bar{c}_i$ satisfies the same congruences, the orbits being stable under conjugation. For $f = \sum_i c_i f_i$ and $\mathfrak{m}$ in the orbit of $\mathfrak{m}_k$, $g(f, f) = \sum_{i,i'} c_i \bar{c}_{i'} g(f_i, f_{i'}) \equiv g(f_k, f_k) \pmod{\mathfrak{m}}$, which is not in $\mathfrak{m}_k$, hence not in $\bar{\mathfrak{m}}_k$. So $g(f, f)$ lies in no maximal ideal and is a unit.

Step 5: (Uo2). We show by induction on $r \geq 1$ that a unimodular free submodule L of rank r has an $\mathcal{O}$-basis $f_1, \dots, f_r$ with $g(f_i, f_k) = 0$ for $i \neq k$. Take f as in Step 4 and $L_1 = \{v \in L : g(v, f) = 0\}$. Every $v \in L$ is $g(v, f) g(f, f)^{-1} f + v_1$ with $v_1 \in L_1$, and $cf \in L_1$ forces $c\, g(f, f) = 0$, $c = 0$; so $L = \mathcal{O}f \oplus L_1$, and L_1 is free of rank $r - 1$. If $r = 1$ we are done. Otherwise, for an $\mathcal{O}$-basis b' of L_1 the matrix of (f, b') is block diagonal (as $g(f, v) = -\overline{g(v, f)} = 0$ for $v \in L_1$), with determinant $g(f, f) \det G_1$, a unit; both factors lie in $\mathcal{O}$, so $G_1 \in \mathrm{GL}(\mathcal{O})$, L_1 is unimodular, and induction applies to L_1. By Step 2 this applies to $L = N_{(\ell)}$; indexing the basis by I and rescaling it into N (Step 1; orthogonality is preserved) gives a local basis x at ℓ with $h(x_i, x_k) = \delta^{-1} g(x_i, x_k) = 0$ for $i \neq k$. Finally, for any such x and non-empty $I_0 \subseteq I$, the local basis $(x|_{I_0}, x|_{I \smallsetminus I_0})$ has orthogonal parts, so (Jz3) (or (Jz2) if $I_0 = I$) gives the last assertion. $\square$

Proposition 34.29. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p+q \geq 1$, $U \subseteq \mathrm{Sh}_F$ a union of connected components, Y a connected component of the complex manifold $U(\mathbb{C})$ of points over the inclusion $F \subset \mathbb{C}$, and $z_0 \in Y$. Then there is a non-empty open subset $W \subseteq \mathrm{Per}(\Lambda_{z_0})$ such that every $J \in W$ is realised at some $z \in Y$ along some path γ in Y from z_0 to z.*

Proof. Put $g = \frac{1}{2}[E:\mathbb{Q}]pq$, $\Phi = \Phi_{\mathsf{H}}$, $\Lambda = \Lambda_{z_0}$, $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ with the form $\langle\,,\rangle$ and the action ι of $E \otimes \mathbb{R}$.

Step 1: $\mathrm{Per}(\Lambda)$ *is a real manifold of dimension* $2g$. For $\tau\colon E \to \mathbb{C}$, $\bar\tau(r) = -\tau(r)$ and $\tau(r)^2 = -d = r_{\mathbb{C}}^2$, so exactly one of $\tau, \bar\tau$ lies in Φ; hence $a \mapsto (\tau(a))_{\tau\in\Phi}$ is an isomorphism $E \otimes \mathbb{R} \cong \prod_{\tau\in\Phi}\mathbb{C}$. Let e_τ be its idempotents, $I = \iota((i)_\tau)$ and $\Lambda_\tau = e_\tau\Lambda_{\mathbb{R}}$, a complex vector space through I of dimension $n = p+q$ (Λ is projective of rank n over $\mathcal{O}_E$). As complex conjugation of $E\otimes\mathbb{R}$ fixes each e_τ and maps I to $-I$, the relation $\langle ax, y\rangle = \langle x, \bar a y\rangle$ gives $\Lambda_\tau \perp \Lambda_{\tau'}$ for $\tau \neq \tau'$ and $\langle Iu, Iv\rangle = \langle u, v\rangle$; so $h_\tau(u,v) = \langle Iu, v\rangle + i\langle u, v\rangle$ is a hermitian form on Λ_τ. Let $J \in \mathrm{Per}(\Lambda)$. As J commutes with $\iota(\mathcal{O}_E)$ it commutes with $E\otimes\mathbb{R}$, hence with e_τ and I; as $(IJ)^2 = 1$, $\Lambda_\tau = P_\tau \oplus Q_\tau$ with $P_\tau = \Lambda_\tau \cap \ker(J-I)$, $Q_\tau = \Lambda_\tau \cap \ker(J+I)$, and P_τ is the image of the projector $\frac{1}{2}(1-IJ)e_\tau$. For $u \in P_\tau$ and $a \in \mathcal{O}_E$, $\iota(a)u = \mathrm{Re}\,\tau(a)\,u + \mathrm{Im}\,\tau(a)\,Iu = \mathrm{Re}\,\tau(a)\,u + \mathrm{Im}\,\tau(a)\,Ju$, so $P_\tau \subseteq L^J_{(\tau,+)}$, and likewise $Q_\tau \subseteq L^J_{(\tau,-)}$. Since $\Lambda_{\mathbb{R}} = \bigoplus_\tau (P_\tau \oplus Q_\tau)$ and the sum of the L^J_s is direct, $P_\tau = L^J_{(\tau,+)}$, $Q_\tau = L^J_{(\tau,-)}$, so $\dim_{\mathbb{C}} P_\tau = p$, $\dim_{\mathbb{C}} Q_\tau = q$. On $P_\tau \smallsetminus 0$, $h_\tau(u,u) = \langle Ju, u\rangle > 0$; on $Q_\tau \smallsetminus 0$, $h_\tau(u,u) = -\langle Ju, u\rangle < 0$; for $u \in P_\tau$, $v \in Q_\tau$, $\langle u, v\rangle = \langle Ju, Jv\rangle = -\langle Iu, Iv\rangle = -\langle u, v\rangle$ and likewise for $\langle Iu, v\rangle$, so $P_\tau \perp Q_\tau$ for h_τ. Hence h_τ has signature (p,q) and $Q_\tau = P_\tau^\perp$. Conversely, if $P_\tau \subseteq \Lambda_\tau$ ($\tau \in \Phi$) are complex subspaces of dimension p on which h_τ is positive definite, then $J = I$ on P_τ and $J = -I$ on $P_\tau^\perp$ defines an element of $\mathrm{Per}(\Lambda)$: $J^2 = -1$; J commutes with the e_τ and I, hence with $\iota(\mathcal{O}_E)$; J preserves $\langle\,,\rangle = \mathrm{Im}\,h_\tau$ on each Λ_τ (it is $\pm I$ on two h_τ-orthogonal summands); $\langle Ju, u\rangle = h_\tau(u_+, u_+) - h_\tau(u_-, u_-) > 0$ for $u = u_+ + u_- \neq 0$; and the computation above gives $L^J_{(\tau,\pm)}$ of real dimensions $2p$, $2q$. Thus $J \mapsto (P_\tau)_\tau$ is a bijection of $\mathrm{Per}(\Lambda)$ onto the open subset of $\prod_{\tau\in\Phi}\mathrm{Gr}_p(\Lambda_\tau)$ where all $h_\tau|_{P_\tau}$ are positive definite, continuous with real-analytic inverse (through the projectors); so $\mathrm{Per}(\Lambda)$ is a real-analytic manifold of dimension $|\Phi|\cdot 2pq = 2g$.

Step 2: the period map. By (Ud5), Sh is smooth of relative dimension g over K_{L}, so Y is a complex manifold of dimension g. Let $V \subseteq Y$ be a simply connected open neighbourhood of z_0 (a coordinate ball). For $z \in V$ the transport $\mu_z = \mu_\gamma$ along a path γ in V from z_0 to z does not depend on γ; it is $\mathcal{O}_E$-linear and preserves the form (the local system of (Yf4) is one of self-dual $\mathcal{O}_E$-lattices). Put $\Pi(z) = \mu_z^{-1} J_z \mu_z$ ($\mathbb{R}$-linear extension). As $J_z \in \mathrm{Per}(\Lambda_z)$ (Yf7), $\Pi(z) \in \mathrm{Per}(\Lambda)$, and $\Pi(z)$ is realised at z along γ, by definition.

Step 3: the kernel map. Let $\mathcal{K}(z) \subseteq \Lambda \otimes \mathbb{C}$ be the kernel of the $\mathbb{C}$-linear extension $\Lambda \otimes \mathbb{C} \to \mathrm{Lie}(G_z)$ of μ_z. For $u \in \Lambda_{\mathbb{R}}$ the vector $u\otimes 1 + \Pi(z)u \otimes i$ maps to $\mu_z u + J_z\mu_z\Pi(z)u = \mu_z u - \mu_z u = 0$, and these vectors form the $(-i)$-eigenspace of $\Pi(z)\otimes 1$, of dimension $m = \frac{1}{2}n[E:\mathbb{Q}] = \dim G_z$; comparing dimensions, $\mathcal{K}(z)$ is that eigenspace. By the classical holomorphy of the period map of a family of complex tori (the relative Lie algebra is a holomorphic bundle receiving the flat bundle $\Lambda \otimes \mathcal{O}$ holomorphically), the map $z \mapsto \mathcal{K}(z) \in \mathrm{Gr}_m(\Lambda\otimes\mathbb{C})$ is holomorphic on V. The map $\Psi\colon J \mapsto \ker(J\otimes 1 + i)$ from $\mathrm{Per}(\Lambda)$ to $\mathrm{Gr}_m(\Lambda\otimes\mathbb{C})$ is real-analytic and has the real-analytic left inverse $\mathcal{K} \mapsto (-i \text{ on } \mathcal{K},\ i \text{ on } \bar{\mathcal{K}})$ on the open set where $\mathcal{K} \oplus \bar{\mathcal{K}} = \Lambda\otimes\mathbb{C}$. Hence $\Pi = \Psi^{-1} \circ \mathcal{K}$ is smooth, and for a real tangent vector v at $z \in V$, $d\Pi_z(v) = 0$ if and only if $d\mathcal{K}_z(v) = 0$. As $\mathcal{K}$ is holomorphic, $d\mathcal{K}_z(v) = \partial\mathcal{K}_z(t)$, where t is the holomorphic tangent vector corresponding to v (a real-linear bijection).

Step 4: injectivity of the differential. Let $z \in V$ and let t be a holomorphic tangent vector at z with $\partial\mathcal{K}_z(t) = 0$. As Sh_F is smooth, t is a morphism $x\colon \mathrm{Spec}\,\mathbb{C}[\epsilon] \to \mathrm{Sh}_F$ over $F \subset \mathbb{C}$, hence a K_{L}-morphism to Sh, with constant point x_0 the point z. Let $o = x^* o^{\mathrm{un}} = (G, \iota, \mathcal{L}, \eta, \eta_A)$ and

$o_0 = x_0^* o^{\mathrm{un}} = (G_0, \iota_0, \mathcal{L}_0, \eta_0, \eta_{A,0})$, the constant object $o_z \otimes \mathbb{C}[\epsilon]$. The closed immersion $\operatorname{Spec}\mathbb{C} \hookrightarrow \operatorname{Spec}\mathbb{C}[\epsilon]$ of affine schemes is defined by the ideal $\epsilon\mathbb{C}[\epsilon]$ of square zero, and G, G_0 are liftings of G_z. The canonical isomorphism $c_G\colon H^1_{\mathrm{dR}}(G/\mathbb{C}[\epsilon]) \to H^1_{\mathrm{dR}}(G_0/\mathbb{C}[\epsilon]) = H^1_{\mathrm{dR}}(G_z/\mathbb{C}) \otimes \mathbb{C}[\epsilon]$ of Theorem B.116(1) is the identification given by the crystal of H^1_{dR}, that is by the Gauss–Manin connection (comparison of crystalline and de Rham cohomology, [29]), whose horizontal sections over V are the transported Betti classes, i.e. $\operatorname{Hom}(\Lambda, \mathbb{C})$ through μ_z; in this trivialisation the Hodge filtration $\omega_{G_{z'}}$ of $G_{z'}$ ($z' \in V$) is the annihilator of $\mathcal{K}(z')$, and the Hodge filtration of G over $\mathbb{C}[\epsilon]$ is the first-order expansion of $z' \mapsto \omega_{G_{z'}}$ along t (horizontality of periods, Griffiths; Katz–Oda). As $\partial\mathcal{K}_z(t) = 0$, $c_G(\omega_G) = \omega_{G_z} \otimes \mathbb{C}[\epsilon] = \omega_{G_0} = c_{G_0}(\omega_{G_0})$. By Theorem B.116(1) with $\tilde{S} = \operatorname{Spec}\mathbb{C}[\epsilon]$ and the lifting G_0 as $\tilde{A}$, G and G_0 are isomorphic liftings: there is an isomorphism $\psi\colon G \to G_0$ reducing to the identity of G_z. By the uniqueness in Theorem B.116(2): $\psi\iota(a)\psi^{-1} = \iota_0(a)$ for $a \in \mathcal{O}_E$ (both lift $\iota_z(a)$), and $\psi^\vee\phi_{\mathcal{L}_0}\psi = \phi_{\mathcal{L}}$ (both lift $\phi_{\mathcal{L}_z}$). As $\Lambda(\mathcal{L}) \cong (1 \times \phi_{\mathcal{L}})^*\mathcal{P}$ with $\mathcal{P}$ the Poincaré bundle and $\mathcal{L} \cong (1, \iota(w_0))^*\Lambda(\mathcal{L})$ (Ud4), and likewise for $\mathcal{L}_0$, we get $\psi^*\mathcal{L}_0 \cong \mathcal{L}$; composing with a unit of $\Gamma(G, \mathcal{O}_G^\times) = \mathbb{C}[\epsilon]^\times$ the isomorphism respects the rigidifications, and then the cubical structures, as a rigidified line bundle on an abelian scheme carries at most one cubical structure [181]. Finally $G[N']$ is finite étale over $\mathbb{C}[\epsilon]$, so its sections are determined by their reductions modulo ϵ; hence $\psi \circ \eta = \eta_0$ and $\psi \circ \eta_A = \eta_{A,0}$, both sides reducing to η_z, $\eta_{A,z}$. So ψ is an isomorphism of objects $o \cong o_0$, and the object o is isomorphic to both x^*o^{un} and $x_0^*o^{\mathrm{un}}$. By (Ud5)(ii), $x = x_0$, that is $t = 0$. By Step 3, $d\Pi_z$ is injective for every $z \in V$.

Step 5: conclusion. Here $\Pi\colon V \to \mathrm{Per}(\Lambda)$ is a smooth map between manifolds of real dimension $2g$ (Steps 1–2) with injective, hence bijective, differential at z_0. By the inverse function theorem there is an open $V' \ni z_0$, $V' \subseteq V$, such that $W = \Pi(V')$ is open in $\mathrm{Per}(\Lambda)$; it contains $\Pi(z_0) = J_{z_0}$. Every $J \in W$ is $\Pi(z)$ for some $z \in V'$, hence realised at z along a path in V from z_0 to z (Step 2). □

Proposition 34.30. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $n = p + q \geq 1$, $\tau \in \Phi_{\mathsf{H}}$, $U \subseteq \mathrm{Sh}_F$ a union of connected components, Y a connected component of $U(\mathbb{C})$, $z_0 \in Y$, and φ a flat generator over Y with constants (e, c). Then there is $\alpha \in \mathbb{C}^\times$ with the following property. For every prime ℓ, every $z \in Y$, every path γ in Y from z_0 to z, every pure tuple $x \in \Lambda_z^n$ that is a local basis at ℓ, and every number field $k \subset \mathbb{C}$, there are a number field $K \subset \mathbb{C}$ containing F and k, a point $y \in U(K)$ whose complex point is z, and a frame (v, w) at y such that (K, y, γ, x, v, w) is a pure CM point at ℓ for φ with constant α.*

Proof. Fix K_Z and, for $b \in \{+, -\}$, $\mathcal{Z}_b$, ι_b, u_b, ω_b as in Lemma 34.31 (1); they depend on $\mathfrak{u}$ only. Put $\alpha = c\,\omega_+^p\,\omega_-^{-q}$ if $c \neq 0$ and $\alpha = 1$ if $c = 0$; in both cases $\alpha \in \mathbb{C}^\times$, and α does not depend on ℓ, z, γ, x or k.

Let ℓ, $z \in Y$, γ, a pure tuple $x \in \Lambda_z^n$ that is a local basis at ℓ, and a number field $k \subset \mathbb{C}$ be given. The point z lies over the inclusion $F \subset \mathbb{C}$, as $Y \subseteq U(\mathbb{C})$. By Lemma 34.31 (2) there are a number field $K \subset \mathbb{C}$ containing F, K_Z and k, a point $y \in U(K)$ whose complex point is z, and an $\mathcal{O}_E$-linear isogeny $\Psi\colon G'_K \to G_y$ with kernel of order prime to ℓ and differential as described there. Let v_a ($a \leq p$) be the component in $\mathrm{Lie}_{(\tau,+)}(G_y)$ of $d\Psi(e_{+,a} \otimes u_+)$ and w_c ($c \leq q$) the component in $\mathrm{Lie}_{(\tau,-)}(G_y)$ of $d\Psi(e_{-,c} \otimes u_-)$. We check that (K, y, γ, x, v, w) is a pure CM point at ℓ for φ with constant α (Yf6): $F \subseteq K \subset \mathbb{C}$ is a number field, $y \in U(K)$ has complex point $z \in Y$, γ runs in Y from z_0 to z, and x is a pure local basis at ℓ.

Frame and (Qv1). By Lemma 34.32, (v, w) is a frame at y, $c \neq 0$ (so $\alpha = c\omega_+^p\omega_-^{-q}$), and with $T = T(\mu_\gamma^{-1}x, e)$, $\varphi(y; v, w)\,\tau(\det T) = \alpha\,\tau(\det \mathrm{Gr}(x_-))$.

(Qv2). Let $j\colon K \to \mathbb{C}_\ell$ be a ring homomorphism. By Lemma 34.33 there are an abelian scheme $\mathcal{G}$ over V_j with $\mathcal{O}_E$-action, an $\mathcal{O}_E$-linear isomorphism of its generic fibre with G_y, and $\delta \in K^\times$ with $\log|j(\delta)| = \log|\mathfrak{d}_E|_{j\circ\tau} - \log|j(\tau(r))|$ such that $(\delta v, \delta w)$ is an integral frame of $\mathcal{G}$;

take $\delta' = \delta$.

(Qv3). For $b \in \{+,-\}$ take $M_b = K_Z \subseteq K$, the abelian scheme $\mathcal{Z}_b \to \operatorname{Spec} \mathcal{O}_{K_Z}$ of relative dimension 1, whose generic fibre has ring of $\overline{\mathbb{Q}}$-endomorphisms $\iota_b(\mathcal{O}_{\mathbb{Q}(r)}) \cong \mathcal{O}_{\mathbb{Q}(r)}$ (Lemma 34.31 (1)), the basis u_b of $\operatorname{Lie}(\mathcal{Z}_b/\mathcal{O}_{K_Z})$, and the signs $\epsilon_+ = \operatorname{Im}(r_{\mathbb{C}})/\sqrt{d}$, $\epsilon_- = -\epsilon_+$. For a ring homomorphism $s\colon K \to \mathbb{C}$, the lattice $\Lambda_{b,s}$ of (Qv3) is the period lattice of $\mathcal{Z}_b \otimes_{s|_{K_Z}} \mathbb{C}$ with $u_b \mapsto 1$, and Lemma 34.34 gives exactly the two Gram identities of (Qv3).

Hence (K, y, γ, x, v, w) is a pure CM point at ℓ for φ with constant α. □

Lemma 34.31. *Let $\mathfrak{u}$ be a unitary Shimura datum with $n = p + q \geq 1$, $U \subseteq \mathrm{Sh}_F$ a union of connected components, $\mathbb{Q}(r) \subseteq E$ the imaginary quadratic field generated by r, with ring of integers $\mathcal{O}_{\mathbb{Q}(r)} = \mathbb{Z}[w_0]$, and $\kappa_+, \kappa_-\colon \mathbb{Q}(r) \to \mathbb{C}$ the embeddings with $\kappa_\pm(r) = \pm r_{\mathbb{C}}$.*

1. *There are a number field $K_Z \subset \mathbb{C}$ containing $\kappa_+(\mathbb{Q}(r))$ and, for $b \in \{+,-\}$, an abelian scheme $\mathcal{Z}_b$ of relative dimension 1 over $\mathcal{O}_{K_Z}$ with a ring homomorphism $\iota_b\colon \mathcal{O}_{\mathbb{Q}(r)} \to \operatorname{End}(\mathcal{Z}_b)$, a basis u_b of the $\mathcal{O}_{K_Z}$-module $\operatorname{Lie}(\mathcal{Z}_b/\mathcal{O}_{K_Z})$ and $\omega_b \in \mathbb{C}^\times$, such that $\iota_b(a)$ acts on $\operatorname{Lie}(\mathcal{Z}_b)$ as multiplication by $\kappa_b(a)$, the endomorphism ring of the generic fibre over $\overline{\mathbb{Q}}$ is $\iota_b(\mathcal{O}_{\mathbb{Q}(r)})$, and, identifying $\operatorname{Lie}(\mathcal{Z}_b \otimes_{\mathcal{O}_{K_Z}} \mathbb{C})$ with $\mathbb{C}$ by $u_b \mapsto 1$, the period lattice of $\mathcal{Z}_b \otimes_{\mathcal{O}_{K_Z}} \mathbb{C}$ is $\omega_b\, \kappa_b(\mathcal{O}_{\mathbb{Q}(r)})$.*
2. *Fix such data, let $N_+ = \mathcal{O}_E^p$ and $N_- = \mathcal{O}_E^q$ with standard bases $(e_{+,a})_{a\leq p}$, $(e_{-,c})_{c\leq q}$, and let $G' = (N_+ \otimes_{\mathcal{O}_{\mathbb{Q}(r)}} \mathcal{Z}_+) \times (N_- \otimes_{\mathcal{O}_{\mathbb{Q}(r)}} \mathcal{Z}_-)$, an abelian scheme over $\mathcal{O}_{K_Z}$ on which $\mathcal{O}_E$ acts through N_+ and N_-. Let $z \in U(\mathbb{C})$ be a point over the inclusion $F \subset \mathbb{C}$, ℓ a prime, $x = (x_+, x_-) \in \Lambda_z^n$ a pure tuple that is a local basis at ℓ, and $k \subset \mathbb{C}$ a number field. Then there are a number field $K \subset \mathbb{C}$ containing F, K_Z and k, a point $y \in U(K)$ whose complex point is z, and an $\mathcal{O}_E$-linear isogeny $\Psi\colon G'_K \to G_y$ over K whose kernel has order prime to ℓ, such that the differential of $\Psi \otimes_K \mathbb{C}$, on $\operatorname{Lie}(G'_{\mathbb{C}}) = (N_+ \otimes_{\mathcal{O}_{\mathbb{Q}(r)}} \operatorname{Lie} \mathcal{Z}_{+,\mathbb{C}}) \oplus (N_- \otimes_{\mathcal{O}_{\mathbb{Q}(r)}} \operatorname{Lie} \mathcal{Z}_{-,\mathbb{C}})$ (Theorem B.118 (ii)), is*

$$e_{+,a} \otimes u_+ \mapsto \omega_+^{-1} x_{+,a}, \qquad e_{-,c} \otimes u_- \mapsto \omega_-^{-1} x_{-,c} \qquad (x \in \Lambda_z \subseteq \operatorname{Lie}(G_z)).$$

Proof. Write $O = \mathcal{O}_{\mathbb{Q}(r)}$. Every $a \in O$ is $a = u + vw_0$ $(u, v \in \mathbb{Z})$; since $r_{\mathbb{C}}^2 = -d < 0$, $r_{\mathbb{C}}$ is purely imaginary and $\kappa_-(a) = \overline{\kappa_+(a)}$.

(1). For $b \in \{+,-\}$ let $Z_b^0 = \mathbb{C}/\kappa_b(O)$, with $a \in O$ acting by multiplication by $\kappa_b(a)$, which preserves $\kappa_b(O)$. As O is the maximal order, $\{c \in \mathbb{C} : c\,\kappa_b(O) \subseteq \kappa_b(O)\} = \kappa_b(O)$, so $\operatorname{End}(Z_b^0) \cong O$. By Theorem B.117 (1) there is a number field $K_1 \subset \mathbb{C}$ over which Z_b^0 $(b = \pm)$ and all their endomorphisms are defined, and by Theorem B.117 (2) a finite extension $K_2 \subset \mathbb{C}$ of K_1 over which both have good reduction at every finite place, i.e. are the generic fibres of abelian schemes $\mathcal{Z}_b^2$ over $\mathcal{O}_{K_2}$; the action of O extends to $\mathcal{Z}_b^2$ by the Néron mapping property. On $\operatorname{Lie}(Z_b^0) = \mathbb{C}$ (coordinate of $\mathbb{C}/\kappa_b(O)$), a acts by $\kappa_b(a)$; this action is defined over K_2, so $\kappa_b(a) \in K_2$: $\kappa_+(\mathbb{Q}(r)) \subseteq K_2$. The module $\operatorname{Lie}(\mathcal{Z}_b^2/\mathcal{O}_{K_2})$ is a projective $\mathcal{O}_{K_2}$-module of rank one; let K_Z be the Hilbert class field of K_2 (inside $\mathbb{C}$). By the principal ideal theorem the base change $\mathcal{Z}_b = \mathcal{Z}_b^2 \otimes \mathcal{O}_{K_Z}$ has free $\operatorname{Lie}(\mathcal{Z}_b/\mathcal{O}_{K_Z}) = \operatorname{Lie}(\mathcal{Z}_b^2) \otimes \mathcal{O}_{K_Z}$; let u_b be a basis. In the coordinate t of $\mathbb{C}/\kappa_b(O)$, u_b is a non-zero number ω_b^{-1}, so in the coordinate $s = \omega_b t$ (in which $u_b \mapsto 1$) the period lattice is $\omega_b \kappa_b(O)$. The endomorphism ring over $\overline{\mathbb{Q}}$ is that over $\mathbb{C}$ (rigidity), i.e. $\iota_b(O)$, and $\iota_b(a)$ acts on $\operatorname{Lie}(\mathcal{Z}_b) \subseteq \operatorname{Lie}(\mathcal{Z}_b \otimes \mathbb{C})$ by $\kappa_b(a)$.

(2), Step 1: the pure sublattice. Since $\iota(r)$ commutes with $\iota(\mathcal{O}_E)$, $\Lambda_z^\pm$ are $\mathcal{O}_E$-submodules of Λ_z, and for $\lambda \in \Lambda_z^\pm$ and $a = u + vw_0 \in O$: $\iota(a)\lambda = u\lambda + \frac{v}{2}(\lambda \pm r_{\mathbb{C}}\lambda) = \kappa_\pm(a)\lambda$, multiplication by the complex number $\kappa_\pm(a)$ on $\operatorname{Lie}(G_z)$. Let $\hat{x}\colon N_+ \oplus N_- \to \Lambda_z$ be the $\mathcal{O}_E$-linear

map $e_{+,a} \mapsto x_{+,a}$, $e_{-,c} \mapsto x_{-,c}$; it is injective because x is $\mathcal{O}_E$-linearly independent, its image $\hat{N}$ satisfies $m_0\Lambda_z \subseteq \hat{N}$ for some $m_0 \geq 1$ with $\ell \nmid m_0$ (local basis), and $\hat{x}(N_\pm) \subseteq \Lambda_z^\pm$ (purity).

Step 2: the complex isogeny. The ring O is a Dedekind domain and the modules $N_\pm$ are finitely generated projective O-modules; choose $N'_\pm$ with $N_\pm \oplus N'_\pm \cong O^{s_\pm}$. Then $N_\pm \otimes_O \mathcal{Z}_\pm$ is a direct factor of $\mathcal{Z}_\pm^{s_\pm}$, compatibly with Theorem B.118 (i), (ii); hence over $\mathbb{C}$, with $\mathrm{Lie}(\mathcal{Z}_{\pm,\mathbb{C}}) = \mathbb{C}u_\pm$ on which a acts by $\kappa_\pm(a)$, the exponential map of $N_\pm \otimes_O \mathcal{Z}_{\pm,\mathbb{C}}$ is the map $N_\pm \otimes_{O,\kappa_\pm} \mathbb{C} \to N_\pm \otimes_O (\mathbb{C}/\omega_\pm\kappa_\pm(O))$, with kernel $N_\pm \otimes_O \omega_\pm\kappa_\pm(O) = \omega_\pm N_\pm$ (exactness of $N_\pm \otimes_O -$, $N_\pm$ being flat). Define

$$\psi\colon \mathrm{Lie}(G'_\mathbb{C}) = (N_+ \otimes_{O,\kappa_+} \mathbb{C}) \oplus (N_- \otimes_{O,\kappa_-} \mathbb{C}) \to \mathrm{Lie}(G_z), \qquad \nu \otimes s \mapsto s\,\omega_\pm^{-1}\,\hat{x}(\nu) \quad (\nu \in N_\pm).$$

It is well defined: for $a \in O$, $(a\nu)\otimes s \mapsto s\omega_\pm^{-1}\iota(a)\hat{x}(\nu) = s\omega_\pm^{-1}\kappa_\pm(a)\hat{x}(\nu)$, the image of $\nu\otimes\kappa_\pm(a)s$, by Step 1. It is $\mathbb{C}$-linear, and $\mathcal{O}_E$-linear since $\hat{x}$ is and $\iota(\mathcal{O}_E)$ acts $\mathbb{C}$-linearly on $\mathrm{Lie}(G_z)$. It maps the period lattice $\omega_+N_+ \oplus \omega_-N_-$ of $G'_\mathbb{C}$ bijectively onto $\hat{N}$, a lattice of full rank in $\mathrm{Lie}(G_z)$; both lattices span their spaces over $\mathbb{R}$, so ψ is an $\mathbb{R}$-linear, hence $\mathbb{C}$-linear, isomorphism. It therefore induces a surjective holomorphic homomorphism $\Psi_\mathbb{C}\colon G'(\mathbb{C}) \to G_z(\mathbb{C})$ with kernel $\psi^{-1}(\Lambda_z)/\psi^{-1}(\hat{N}) \cong \Lambda_z/\hat{N}$, which is killed by m_0 and has order dividing $m_0^{2\dim G_z}$, prime to ℓ. By GAGA $\Psi_\mathbb{C}$ is an $\mathcal{O}_E$-linear isogeny of abelian varieties, with differential ψ; in particular $e_{+,a} \otimes u_+ \mapsto \omega_+^{-1}x_{+,a}$ and $e_{-,c} \otimes u_- \mapsto \omega_-^{-1}x_{-,c}$.

Step 3: a model over $\overline{\mathbb{Q}}$. The abelian scheme G' is defined over $K_Z \subset \overline{\mathbb{Q}}$, so the torsion points of $G'(\mathbb{C})$ are in $G'(\overline{\mathbb{Q}})$, and $H = \ker\Psi_\mathbb{C}$ is a finite $\mathcal{O}_E$-stable subgroup of $G'(\overline{\mathbb{Q}})$. Put $G_1 = G'_{\overline{\mathbb{Q}}}/H$, an abelian variety over $\overline{\mathbb{Q}}$ with $\mathcal{O}_E$-action; $\Psi_\mathbb{C}$ induces an $\mathcal{O}_E$-linear isomorphism $G_1 \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \cong G_z$.

Step 4: z is algebraic. Let X be the set of isomorphism classes of objects of type (U, L) over $\mathrm{Spec}\,\mathbb{C}$ whose abelian variety is isomorphic to $G_1 \otimes \mathbb{C}$. The set X is countable: such an object is isomorphic to one of the form $(G_1 \otimes \mathbb{C}, \iota', \mathcal{L}', \eta', \eta'_A)$, and (a) ι' is a ring homomorphism $\mathcal{O}_E \to \mathrm{End}(G_1 \otimes \mathbb{C}) = \mathrm{End}(G_1)$ (rigidity), a finitely generated $\mathbb{Z}$-module: countably many; (b) the class of $\mathcal{L}'$ in $\mathrm{NS}(G_1 \otimes \mathbb{C})$ is determined by $\phi_{\mathcal{L}'} \in \mathrm{Hom}(G_1 \otimes \mathbb{C}, (G_1 \otimes \mathbb{C})^\vee) = \mathrm{Hom}(G_1, G_1^\vee)$ (rigidity), a finitely generated group, so it is one of countably many, and two symmetric line bundles with the same class differ by $P \in \mathrm{Pic}^0$ with $P \cong [-1]^*P \cong P^{-1}$, i.e. P a point of order dividing 2 of the dual, finitely many; the rigidification is unique up to a scalar absorbed by an automorphism of $\mathcal{L}'$, and the cubical structure is then unique; (c) η', η'_A vanish on $N'M_\mathsf{U}$ (M_U the module of U, $N' = \mathrm{lcm}(N, A)$; $N \in \mathfrak{n}$ since $N\mathcal{O}_E = \mathfrak{n}\bar{\mathfrak{n}}$) and take values in the finite group of N'-torsion points: finitely many. Now let $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ and $z^\sigma = z \circ \mathrm{Spec}(\sigma)$; as $F \subset \overline{\mathbb{Q}}$ is fixed by σ, z^σ is again a point of U over $F \subset \mathbb{C}$, and the pull-back of the universal object to z^σ is the base change of the object at z along σ, whose abelian variety is $(G_1 \otimes \mathbb{C}) \otimes_{\mathbb{C},\sigma} \mathbb{C} \cong G_1 \otimes \mathbb{C}$ (as σ fixes $\overline{\mathbb{Q}}$). By (Ud5) (i) (uniqueness of the classifying morphism), z^σ is determined by the class of its object in X, so the orbit $\{z^\sigma\}$ is countable. Let $\kappa(z) \subset \mathbb{C}$ be the residue field of the image point of z, finitely generated over F. If its transcendence degree over F were $t \geq 1$, with transcendence basis T, then every F-embedding of $\kappa(z)$ into $\mathbb{C}$ which sends T to a tuple algebraically independent over $\overline{\mathbb{Q}}$ and is compatible on $\kappa(z) \cap \overline{\mathbb{Q}}$ extends to some $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ ($\mathbb{C}$ is algebraically closed of uncountable transcendence degree), and distinct embeddings give distinct points z^σ; there are uncountably many, a contradiction. Hence $\kappa(z)$ is a number field and z comes from a point $y_0 \in U(\kappa(z))$.

Step 5: the isogeny over a number field. The abelian variety G_{y_0} over $\kappa(z)$ satisfies $G_{y_0} \otimes \mathbb{C} = G_z \cong G_1 \otimes \mathbb{C}$, $\mathcal{O}_E$-linearly. By rigidity this isomorphism, and hence $\Psi_\mathbb{C}$, viewed as an element of $\mathrm{Hom}(G'_\mathbb{C}, G_{y_0} \otimes \mathbb{C}) = \mathrm{Hom}(G'_{\overline{\mathbb{Q}}}, G_{y_0} \otimes \overline{\mathbb{Q}})$, is defined over $\overline{\mathbb{Q}}$, hence over some number field (it is given by finitely many equations); its $\mathcal{O}_E$-linearity is an identity between homomorphisms and also holds there. Let $K \subset \mathbb{C}$ be a number field containing $\kappa(z)$, K_Z, k and

a field of definition of this homomorphism, $y \in U(K)$ the point induced by y_0 (its complex point is z, $G_y = G_{y_0} \otimes K$), and $\Psi\colon G'_K \to G_y$ the homomorphism over K with $\Psi \otimes_K \mathbb{C} = \Psi_{\mathbb{C}}$. It is an $\mathcal{O}_E$-linear isogeny whose kernel has the order of $\ker \Psi_{\mathbb{C}}$, prime to ℓ, and its differential over $\mathbb{C}$ is ψ (Step 2). □

Lemma 34.32. *Let $\mathfrak{u}$ be a unitary Shimura datum with $n = p + q \geq 1$, $\tau \in \Phi_{\mathsf{H}}$, $U \subseteq \mathrm{Sh}_F$ a union of connected components, Y a connected component of $U(\mathbb{C})$, $z_0 \in Y$, and φ a flat generator over Y with constants (e, c). Fix K_Z, $\mathcal{Z}_\pm$, $u_\pm$, $\omega_\pm$ as in Lemma 34.31 (1). Let $z \in Y$, ℓ a prime, $x = (x_+, x_-) \in \Lambda_z^n$ a pure tuple that is a local basis at ℓ, and let K, $y \in U(K)$ and $\Psi\colon G'_K \to G_y$ be as in Lemma 34.31 (2) for z, ℓ, x and some number field k. For $a \leq p$ let v_a be the component in $\mathrm{Lie}_{(\tau,+)}(G_y)$ of $d\Psi(e_{+,a} \otimes u_+)$, and for $c \leq q$ let w_c be the component in $\mathrm{Lie}_{(\tau,-)}(G_y)$ of $d\Psi(e_{-,c} \otimes u_-)$. Then (v, w) is a frame at y, and for every path γ in Y from z_0 to z, with $T = T(\mu_\gamma^{-1} x, e)$,*

$$\varphi(y; v, w)\, \tau(\det T) = c\, \omega_+^p\, \omega_-^{-q}\, \tau(\det \mathrm{Gr}(x_-)),$$

where both sides are non-zero; in particular $c \neq 0$.

Proof. Write $O = \mathcal{O}_{\mathbb{Q}(r)}$ and $\kappa_\pm$ as in Lemma 34.31, so $\kappa_\pm(r) = \pm r_{\mathbb{C}}$. Since $r_{\mathbb{C}}^2 = -d < 0$, $r_{\mathbb{C}}$ is purely imaginary; hence for $\tau' \in \Phi_{\mathsf{H}}$, $\iota(r)$ acts on $\mathrm{Lie}_{(\tau',+)}$ by $\tau'(r) = r_{\mathbb{C}}$ and on $\mathrm{Lie}_{(\tau',-)}$ by $\overline{\tau'(r)} = -r_{\mathbb{C}}$. Consequently every $\lambda \in \Lambda_z^+$ has $\lambda_{(\tau',-)} = 0$ and every $\lambda \in \Lambda_z^-$ has $\lambda_{(\tau',+)} = 0$, for all $\tau' \in \Phi_{\mathsf{H}}$.

(v, w) **is a frame.** By Theorem B.118 (ii) and base change, $\mathrm{Lie}(G'_K) = (N_+ \otimes_O \mathrm{Lie}\, \mathcal{Z}_{+,K}) \oplus (N_- \otimes_O \mathrm{Lie}\, \mathcal{Z}_{-,K})$, with $\mathcal{O}_E$ acting through $N_\pm$, and $\mathrm{Lie}\, \mathcal{Z}_{b,K} = K u_b$ with $a \in O$ acting by $\kappa_b(a)$ (Lemma 34.31 (1)). Thus $N_+ \otimes_O K u_+ = \bigoplus_{a \leq p} (\mathcal{O}_E \otimes_{O,\kappa_+} K)(e_{+,a} \otimes u_+)$, and $\mathcal{O}_E \otimes_{O,\kappa_+} K = E \otimes_{\mathbb{Q}(r),\kappa_+} K = \prod_\sigma K$, the product over the embeddings $\sigma\colon E \to K$ with $\sigma|_{\mathbb{Q}(r)} = \kappa_+$. Since $\tau(r) = r_{\mathbb{C}} = \kappa_+(r)$, τ is one of them; let ϵ_τ be the corresponding idempotent. The subspace of $\mathrm{Lie}(G'_K)$ on which $\iota(a)$ acts by $\tau(a)$ is therefore $\bigoplus_a K \epsilon_\tau (e_{+,a} \otimes u_+)$: the summand $N_- \otimes_O K u_-$ contributes nothing, its factors σ having $\sigma(r) = \kappa_-(r) = -r_{\mathbb{C}} \neq \tau(r)$. Here $\epsilon_\tau X$ is the component of X for the character τ. The differential $d\Psi$ is an $\mathcal{O}_E$-linear isomorphism of K-vector spaces (Ψ is an isogeny in characteristic 0), hence commutes with these components and maps the τ-eigenspace onto $\mathrm{Lie}_{(\tau,+)}(G_y)$; so $v_a = d\Psi(\epsilon_\tau(e_{+,a} \otimes u_+))$ $(a \leq p)$ is a K-basis of $\mathrm{Lie}_{(\tau,+)}(G_y)$. In the same way, with $\bar\tau(r) = -r_{\mathbb{C}} = \kappa_-(r)$, w is a K-basis of $\mathrm{Lie}_{(\tau,-)}(G_y)$.

The frame over $\mathbb{C}$. The complex point of y is z, so $G_y \otimes_K \mathbb{C} = G_z$, components commute with this base change, and $d\Psi \otimes \mathbb{C}$ is the map of Lemma 34.31 (2). Hence, in $\mathrm{Lie}(G_z)$,

$$v_a = \omega_+^{-1}(x_{+,a})_{(\tau,+)}, \qquad w_c = \omega_-^{-1}(x_{-,c})_{(\tau,-)},$$

and by (Yf2) applied to the inclusion $K \subset \mathbb{C}$, $\varphi(z; v, w) = \varphi(y; v, w)$.

Orthogonality of the pure parts. Let $\langle\,,\rangle = \mathrm{Im}\, H_z$ and h the trace form of Λ_z. Here $\Lambda_z^\pm$ are $\mathcal{O}_E$-submodules ($\iota(r)$ commutes with $\iota(\mathcal{O}_E)$), and their real spans are complex subspaces of $\mathrm{Lie}(G_z)$ ($\iota(r)$ is $\mathbb{C}$-linear). Write $r_{\mathbb{C}} = i\rho$ with $\rho \in \mathbb{R}^\times$. For $\lambda \in \Lambda_z^+$, $\mu \in \Lambda_z^-$: $\langle \iota(r)\lambda, \mu\rangle = \langle \lambda, \iota(\bar r)\mu\rangle = -\langle \lambda, \iota(r)\mu\rangle$ gives $\langle i\lambda, \mu\rangle = \langle \lambda, i\mu\rangle$, while $\langle iu, iu'\rangle = \langle u, u'\rangle$ (H_z hermitian) gives $\langle \lambda, i\mu\rangle = -\langle i\lambda, \mu\rangle$; so $\langle i\lambda, \mu\rangle = 0$ on the real spans, and applying this to $-i\lambda$ gives $\langle \lambda, \mu\rangle = 0$. Since $\mathrm{Tr}_{E/\mathbb{Q}}(a\, h(\lambda, \mu)) = \langle a\lambda, \mu\rangle = 0$ for all $a \in \mathcal{O}_E$, $h(\lambda, \mu) = 0$. As $x_{+,a} \in \Lambda_z^+$ and $x_{-,c} \in \Lambda_z^-$,

$$\mathrm{Gr}(x) = \begin{pmatrix} \mathrm{Gr}(x_+) & 0 \\ 0 & \mathrm{Gr}(x_-) \end{pmatrix}.$$

The transition matrix. The E-vector space $\Lambda_{z,\mathbb{Q}}$ has dimension n, so the $\mathcal{O}_E$-linearly independent tuples x and $\mu_\gamma e$ are E-bases, T is invertible, and, μ_γ being $\mathcal{O}_E$-linear, $x_i =$

$\sum_k T_{ik}\,\mu_\gamma e_k$ in $\Lambda_{z,\mathbb{Q}}$. As μ_γ preserves $\langle\,,\rangle$ and the $\mathcal{O}_E$-action, the Gram matrix of $\mu_\gamma e$ in Λ_z is $\mathrm{Gr}(e)$, and since $h(a\lambda,b\mu)=a\bar b\,h(\lambda,\mu)$,

$$\mathrm{Gr}(x)=T\,\mathrm{Gr}(e)\,T^{\dagger},\qquad T^{\dagger}=\bar T^{\mathrm t}.$$

The matrix $\mathrm{Gr}(e)$ is invertible: if $h(\lambda,\cdot)=0$ on Λ_{z_0} then $\langle\lambda,\cdot\rangle=\mathrm{Tr}_{E/\mathbb{Q}}h(\lambda,\cdot)=0$, so $\lambda=0$ by self-duality. Hence $\mathrm{Gr}(x)$, and by the block form $\mathrm{Gr}(x_-)$, are invertible.

The matrices of (Yf4). Put $e'_m=\mu_\gamma e_m$. Since $\iota(a)$ acts on $\mathrm{Lie}_{(\tau,+)}$ by $\tau(a)$ and on $\mathrm{Lie}_{(\tau,-)}$ by $\bar\tau(a)$, taking components of $x_i=\sum_k T_{ik}e'_k$ gives

$$(x_i)_{(\tau,+)}=\sum_k \tau(T_{ik})\,e'_{k,(\tau,+)},\qquad (x_i)_{(\tau,-)}=\sum_k \bar\tau(T_{ik})\,e'_{k,(\tau,-)}.$$

Let T_+, T_- be the first p and the last q rows of T, and put

$$C=\omega_+^{-1}\tau(T_+),\qquad B_{cm}=\omega_-\,\bar\tau(T^{-1})_{m,(-,c)},\qquad R=\omega_-\,\tau(\mathrm{Gr}(x_-))^{-1}\tau(T_-).$$

The first display gives $v_a=\sum_m C_{am}e'_{m,(\tau,+)}$. By the first paragraph $(x_{+,a})_{(\tau,-)}=0$, and $(x_{-,c})_{(\tau,-)}=\omega_- w_c$; inverting the second display, $e'_{m,(\tau,-)}=\sum_c\bar\tau(T^{-1})_{m,(-,c)}\,\omega_-w_c=\sum_c B_{cm}w_c$. Finally, the last q rows of $\mathrm{Gr}(x)=T\,\mathrm{Gr}(e)\,T^{\dagger}$ give $T_-\mathrm{Gr}(e)=(0\mid\mathrm{Gr}(x_-))(T^{\dagger})^{-1}$, and $\tau((T^{\dagger})^{-1})=\bar\tau(T^{-1})^{\mathrm t}$, so

$$R\,\tau(\mathrm{Gr}(e))=\omega_-(0\mid I_q)\,\bar\tau(T^{-1})^{\mathrm t}=B.$$

By (Yf4), $\varphi(z;v,w)\det\binom{C}{R}=c$. Here $\binom{C}{R}=\mathrm{diag}(\omega_+^{-1}I_p,\ \omega_-\tau(\mathrm{Gr}(x_-))^{-1})\,\tau(T)$, so

$$\det\tbinom{C}{R}=\omega_+^{-p}\,\omega_-^{q}\,\tau(\det T)\,\tau(\det\mathrm{Gr}(x_-))^{-1}\neq 0,$$

and with $\varphi(z;v,w)=\varphi(y;v,w)\in K^\times$ this is the asserted identity, both sides being non-zero; in particular $c\neq0$. □

Lemma 34.33. *Let $\mathfrak u$ be a unitary Shimura datum with $n=p+q\geq1$, $\tau\in\Phi_{\mathsf H}$ and $U\subseteq\mathrm{Sh}_F$ a union of connected components. Fix K_Z, $\mathcal{Z}_\pm$, $u_\pm$, $\omega_\pm$ as in Lemma 34.31 (1). Let $z\in U(\mathbb{C})$ be a point over the inclusion $F\subset\mathbb{C}$, ℓ a prime, $x\in\Lambda_z^n$ a pure tuple that is a local basis at ℓ, and let K, $y\in U(K)$ and $\Psi\colon G'_K\to G_y$ be as in Lemma 34.31 (2) for z, ℓ, x and some number field k. For $a\leq p$ let v_a be the component in $\mathrm{Lie}_{(\tau,+)}(G_y)$ of $d\Psi(e_{+,a}\otimes u_+)$, and for $c\leq q$ let w_c be the component in $\mathrm{Lie}_{(\tau,-)}(G_y)$ of $d\Psi(e_{-,c}\otimes u_-)$. Then for every ring homomorphism $j\colon K\to\mathbb{C}_\ell$, with $V_j=\{a\in K:|j(a)|\leq1\}$, there are an abelian scheme $\mathcal G$ over V_j with a ring homomorphism $\iota_{\mathcal G}\colon\mathcal{O}_E\to\mathrm{End}(\mathcal G)$, an $\mathcal{O}_E$-linear isomorphism of its generic fibre with G_y, and $\delta\in K^\times$ with*

$$\log|j(\delta)|=\log|\mathfrak d_E|_{j\circ\tau}-\log|j(\tau(r))|,$$

such that $(\delta v,\delta w)$ is an integral frame of $\mathcal G$.

Proof. Write $O=\mathcal{O}_{\mathbb{Q}(r)}=\mathbb{Z}[w_0]$, $\kappa_\pm$ as in Lemma 34.31 ($\kappa_\pm(r)=\pm r_{\mathbb{C}}$), and $\mathfrak D=\{t\in E:\mathrm{Tr}_{E/\mathbb{Q}(r)}(t\,y)\in O\ \forall y\in\mathfrak D^{-1}\}$ for the relative different of $E/\mathbb{Q}(r)$, where $\mathfrak D^{-1}=\{y\in E:\mathrm{Tr}_{E/\mathbb{Q}(r)}(y\mathcal{O}_E)\subseteq O\}$. Fix j. As $|j(\cdot)|$ restricts to a non-archimedean absolute value on K extending the ℓ-adic one, V_j is the local ring of $\mathcal{O}_K$ at a prime above ℓ, a discrete valuation ring in which ℓ is not a unit, and $\mathcal{O}_{K_Z}\subseteq\mathcal{O}_K\subseteq V_j$.

The model $\mathcal G$ and the integral isogeny. Let m be the order of $\ker\Psi$; it is prime to ℓ, hence a unit in V_j, and since $\ker\Psi\subseteq G'_K[m]$ there is $\Psi'\colon G_y\to G'_K$ with $\Psi'\Psi=[m]$; then $\Psi\Psi'\Psi=\Psi[m]=[m]\Psi$ and Ψ is surjective, so $\Psi\Psi'=[m]$. Put $G'_{V_j}=G'\otimes_{\mathcal{O}_{K_Z}}V_j$, an abelian scheme with generic fibre G'_K. By the criterion of Néron–Ogg–Shafarevich ([224],

§1, Corollary 2: isogenous abelian varieties have good reduction simultaneously), G_y has good reduction at V_j; let $\mathcal{G}$ be its Néron model, an abelian scheme over V_j with generic fibre G_y (the identity is the required $\mathcal{O}_E$-linear isomorphism). By the Néron mapping property, ι extends to $\iota_{\mathcal{G}}\colon \mathcal{O}_E \to \mathrm{End}(\mathcal{G})$, and Ψ, Ψ' extend to homomorphisms $\tilde{\Psi}\colon G'_{V_j} \to \mathcal{G}$, $\tilde{\Psi}'\colon \mathcal{G} \to G'_{V_j}$ (G'_{V_j} being the Néron model of G'_K). Two morphisms from an abelian scheme over V_j to a separated V_j-scheme which agree on the generic fibre agree (the generic fibre is schematically dense, the source being flat over V_j); hence $\tilde{\Psi}'\tilde{\Psi} = [m]$, $\tilde{\Psi}\tilde{\Psi}' = [m]$, and $\tilde{\Psi}$ is $\mathcal{O}_E$-linear. On Lie algebras $[m]$ induces multiplication by $m \in V_j^\times$, so

$$d\tilde{\Psi}\colon \mathrm{Lie}(G'_{V_j}) \to \mathrm{Lie}(\mathcal{G})$$

is an $\mathcal{O}_E$-linear isomorphism of free V_j-modules, and after $\otimes_{V_j} K$ it is $d\Psi$. As both Lie algebras are torsion-free, $\mathrm{Lie}_{(\tau,\pm)}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G}) \cap \mathrm{Lie}_{(\tau,\pm)}(G_y)$, and $d\tilde{\Psi}$ maps $\mathrm{Lie}(G'_{V_j}) \cap W$ onto $\mathrm{Lie}_{(\tau,+)}(\mathcal{G})$, where $W \subseteq \mathrm{Lie}(G'_K)$ is the subspace on which $\iota(a)$ acts by $\tau(a)$; likewise for $(\tau,-)$ and $\bar{\tau}$.

The eigen-lattices of G'_{V_j}. By Theorem B.118 (ii) and base change,

$$\mathrm{Lie}(G'_{V_j}) = (N_+ \otimes_O \mathrm{Lie}\,\mathcal{Z}_{+,V_j}) \oplus (N_- \otimes_O \mathrm{Lie}\,\mathcal{Z}_{-,V_j})$$

with $\mathrm{Lie}\,\mathcal{Z}_{b,V_j} = V_j u_b$, $a \in O$ acting by $\kappa_b(a) \in \mathcal{O}_{K_Z}$ (Lemma 34.31 (1)). So the first summand is $\bigoplus_{a\le p} A_+(e_{+,a} \otimes u_+)$ with $A_+ = \mathcal{O}_E \otimes_{O,\kappa_+} V_j$, a projective V_j-module of rank $[E:\mathbb{Q}(r)]$ inside $A_{+,K} = E \otimes_{\mathbb{Q}(r),\kappa_+} K = \prod_\sigma K$ (product over the $\sigma\colon E \to K$ with $\sigma|_{\mathbb{Q}(r)} = \kappa_+$; for $y \in A_{+,K}$ write y_σ for its components). Since $\tau(r) = r_{\mathbb{C}} = \kappa_+(r)$, τ is one of these σ; let ϵ_τ be its idempotent. The summand of N_- splits over the $\sigma\colon E \to K$ with $\sigma(r) = \kappa_-(r) = -r_{\mathbb{C}} \neq \tau(r)$, so it meets W in 0, and $W = \bigoplus_a K\epsilon_\tau(e_{+,a} \otimes u_+)$. As $d\Psi$ is $\mathcal{O}_E$-linear and K-linear, it commutes with taking components, so $d\Psi(\epsilon_\tau(e_{+,a} \otimes u_+)) = v_a$. Hence

$$\mathrm{Lie}(G'_{V_j}) \cap W = \bigoplus_{a\le p} (A_+ \cap K\epsilon_\tau)\,(e_{+,a} \otimes u_+).$$

We claim $A_+ \cap K\epsilon_\tau = \tau(\mathfrak{D})V_j\,\epsilon_\tau$. Let $\mathrm{Tr}\colon A_{+,K} \to K$, $y \mapsto \sum_\sigma y_\sigma$; on $a \otimes t$ it is $\kappa_+(\mathrm{Tr}_{E/\mathbb{Q}(r)}a)\,t$. Since $\mathcal{O}_E$ is a finitely generated projective O-module and $\mathrm{Hom}_O(\mathcal{O}_E, O) \cong \mathfrak{D}^{-1}$ through $\mathrm{Tr}_{E/\mathbb{Q}(r)}$, the flat base change $O \to V_j$ gives $\{y \in A_{+,K} : \mathrm{Tr}(yA_+) \subseteq V_j\} = \mathfrak{D}^{-1} \otimes_{O,\kappa_+} V_j$, and, the trace pairing on the projective module A_+ being perfect over V_j, $A_+ = \{y : \mathrm{Tr}(y\,y') \in V_j\ \forall y' \in \mathfrak{D}^{-1} \otimes V_j\}$. For $t \in K$ and $y' = d \otimes s$ ($d \in \mathfrak{D}^{-1}$, $s \in V_j$), $\mathrm{Tr}(t\epsilon_\tau y') = t\,\tau(d)s$. So $t\epsilon_\tau \in A_+$ iff $t\,\tau(\mathfrak{D}^{-1})V_j \subseteq V_j$, iff $t \in \tau(\mathfrak{D})V_j$, because $\tau(\mathfrak{D})V_j \cdot \tau(\mathfrak{D}^{-1})V_j = \tau(\mathcal{O}_E)V_j = V_j$ in the group of fractional ideals of the discrete valuation ring V_j. This proves the claim. Choose $\delta \in K^\times$ with $\tau(\mathfrak{D})V_j = \delta V_j$. Then $\delta\epsilon_\tau(e_{+,a} \otimes u_+)$ ($a \le p$) is a V_j-basis of $\mathrm{Lie}(G'_{V_j}) \cap W$, and by the first step δv is a V_j-basis of $\mathrm{Lie}_{(\tau,+)}(\mathcal{G})$.

The second half of the frame. The same argument with N_-, κ_- and the component $\bar{\tau}$ (which restricts to κ_- on $\mathbb{Q}(r)$, as $\bar{\tau}(r) = -r_{\mathbb{C}}$) shows that $\delta' w$ is a V_j-basis of $\mathrm{Lie}_{(\tau,-)}(\mathcal{G})$ whenever $\delta' V_j = \bar{\tau}(\mathfrak{D})V_j$. Complex conjugation $a \mapsto \bar{a}$ of the CM field E satisfies $\tau'(\bar{a}) = \overline{\tau'(a)}$ for every embedding τ'; so $\bar{r} = -r$, conjugation preserves $\mathbb{Q}(r)$, $\mathcal{O}_E$ and $\mathrm{Tr}_{E/\mathbb{Q}(r)}$ up to conjugation on O, hence $\bar{\mathfrak{D}} = \mathfrak{D}$ and $\bar{\tau}(\mathfrak{D}) = \tau(\bar{\mathfrak{D}}) = \tau(\mathfrak{D})$. Thus $\delta' = \delta$ works, and $(\delta v, \delta w)$ is an integral frame of $\mathcal{G}$.

The size of δ. For a non-zero finitely generated O-submodule $\mathfrak{a}$ of E with $\tau(\mathfrak{a})V_j = \gamma V_j$ one has $\sup_{t\in\mathfrak{a}} |j(\tau(t))| = |j(\gamma)|$: every $\tau(t)$ lies in γV_j, and $\gamma = \sum_i \tau(t_i)s_i$ with $s_i \in V_j$. By transitivity of differents $\mathfrak{d}_E = \mathfrak{D} \cdot \mathfrak{d}_{\mathbb{Q}(r)}\mathcal{O}_E$, and $\mathfrak{d}_{\mathbb{Q}(r)} = f'(w_0)O = rO$, since $O = \mathbb{Z}[w_0]$ and $w_0 = (1+r)/2$ has minimal polynomial $f = X^2 - X + (1+d)/4$, $f'(w_0) = 2w_0 - 1 = r$. Hence $\tau(\mathfrak{d}_E)V_j = \tau(r)\delta V_j$ and

$$|\mathfrak{d}_E|_{j\circ\tau} = |j(\tau(r))|\,|j(\delta)|,$$

with $\tau(r) \neq 0$; taking logarithms gives the asserted value of $\log|j(\delta)|$. □

Lemma 34.34. *Let $\mathfrak{u}$ be a unitary Shimura datum with $n = p + q \geq 1$, $\tau \in \Phi_{\mathsf{H}}$ and $U \subseteq \mathrm{Sh}_F$ a union of connected components. Fix K_Z, $\mathcal{Z}_\pm$, $u_\pm$, $\omega_\pm$ as in Lemma 34.31 (1). Let $z \in U(\mathbb{C})$ be a point over the inclusion $F \subset \mathbb{C}$, ℓ a prime, $x = (x_+, x_-) \in \Lambda_z^n$ a pure tuple that is a local basis at ℓ, and let $K \supseteq K_Z$, $y \in U(K)$ and $\Psi\colon G'_K \to G_y$ be as in Lemma 34.31 (2) for z, ℓ, x and some number field k. For $a \leq p$ let v_a be the component in $\mathrm{Lie}_{(\tau,+)}(G_y)$ of $d\Psi(e_{+,a} \otimes u_+)$, and for $c \leq q$ let w_c be the component in $\mathrm{Lie}_{(\tau,-)}(G_y)$ of $d\Psi(e_{-,c} \otimes u_-)$. Put $\epsilon_+ = \mathrm{Im}(r_{\mathbb{C}})/\sqrt{d} \in \{1, -1\}$ and $\epsilon_- = -\epsilon_+$. For a ring homomorphism $s\colon K \to \mathbb{C}$ and $b \in \{+, -\}$ let $\Lambda_{b,s} \subset \mathbb{C}$ be the period lattice of $\mathcal{Z}_b \otimes_{\mathcal{O}_{K_Z}, s} \mathbb{C}$, its Lie algebra identified with $\mathbb{C}$ by $u_b \mapsto 1$, and $n_b(s) = \pi/\mathrm{covol}(\Lambda_{b,s})$. Then for every ring homomorphism $s\colon K \to \mathbb{C}$, with $y^s \in U(\mathbb{C})$ the point induced by s and (s_*v, s_*w) the induced frame at y^s,*

$$\Gamma(s_*v) = \epsilon_+ n_+(s)\Big(s\big(\tau(r\, h(x_{+,a}, x_{+,a'}))\big)\Big)_{a,a'}, \qquad \Gamma(s_*w) = \epsilon_- n_-(s)\Big(s\big(\bar\tau(r\, h(x_{-,c}, x_{-,c'}))\big)\Big)_{c,c'},$$

where $\bar\tau(t) = \overline{\tau(t)}$.

Proof. Write $O = \mathcal{O}_{\mathbb{Q}(r)} = \mathbb{Z}[w_0]$, $\kappa_\pm$ as in Lemma 34.31 ($\kappa_\pm(r) = \pm r_{\mathbb{C}}$, $\kappa_- = \bar\kappa_+$), and $r_{\mathbb{C}} = i\rho$ with $\rho \in \mathbb{R}$, $\rho^2 = d$, so $\epsilon_+\rho = |\rho| = \sqrt{d}$. The extensions of κ_+ (resp. κ_-) to E are the $\tau' \in \Phi_{\mathsf{H}}$ (resp. the $\bar\tau'$). Let $\hat x\colon N_+ \oplus N_- \to \Lambda_z$ be the $\mathcal{O}_E$-linear map $e_{+,a} \mapsto x_{+,a}$, $e_{-,c} \mapsto x_{-,c}$; by purity $\hat x(N_\pm) \subseteq \Lambda_z^\pm$. We treat v; the case of w is the same with the changes listed at the end.

Orthogonality. At every complex point y' of U, $H_{y'}(\iota(a)u, u') = H_{y'}(u, \iota(\bar a)u')$ for $a \in \mathcal{O}_E$: the Rosati involution of the principal polarisation restricts to $\iota(a) \mapsto \iota(\bar a)$ (Definition 29.5, (Ud4)), so this holds for the Riemann form $\mathrm{Im}\, H_{y'}$, and then for $H_{y'}(u, u') = \mathrm{Im}\, H_{y'}(iu, u') + i\,\mathrm{Im}\, H_{y'}(u, u')$ as $\iota(a)$ is $\mathbb{C}$-linear. If u, u' lie in the eigenspaces of the characters $\sigma \neq \sigma'$ of $\mathcal{O}_E$, then $\sigma(a)H_{y'}(u, u') = \overline{\sigma'(\bar a)}H_{y'}(u, u') = \sigma'(a)H_{y'}(u, u')$ (E is a CM field), so $H_{y'}(u, u') = 0$.

The form at z on Λ_z^+. Let $\lambda, \mu \in \Lambda_z^+$ and $f = h(\lambda, \mu)$. As λ has only $(\tau', +)$-components ($\tau' \in \Phi_{\mathsf{H}}$; $\iota(r)$ acts on $\mathrm{Lie}_{(\tau',-)}$ by $-r_{\mathbb{C}}$), $i\lambda = \rho^{-1}\iota(r)\lambda$; with $\mathrm{Im}\, H_z = \langle\,,\rangle$ and $\langle t\lambda, \mu\rangle = \mathrm{Tr}_{E/\mathbb{Q}}(tf)$,

$$\begin{aligned} H_z(\lambda, \mu) &= \rho^{-1}\mathrm{Tr}_{E/\mathbb{Q}}(rf) + i\,\mathrm{Tr}_{E/\mathbb{Q}}(f) \\ &= \sum_{\tau' \in \Phi_{\mathsf{H}}} \Big(\rho^{-1}\big(i\rho\,\tau'(f) - i\rho\,\overline{\tau'(f)}\big) + i\big(\tau'(f) + \overline{\tau'(f)}\big)\Big) = \sum_{\tau' \in \Phi_{\mathsf{H}}} 2i\,\tau'(h(\lambda, \mu)). \end{aligned}$$

The pulled-back form at a conjugate. Fix s and write X^s for base change along s. The object at y^s is the base change of the object at y, so H_{y^s} is the hermitian form of $\mathcal{L}_y^s$, and $H'_s(X, X') = H_{y^s}(d\Psi^s X, d\Psi^s X')$ is that of $(\Psi^*\mathcal{L}_y)^s$ on G'^s. For $\mu \in N_+$ let $i_\mu\colon \mathcal{Z}_{+,K} \to G'_K$ be $t \mapsto (\mu \otimes t, 0)$ (Theorem B.118 (i) for $O \to N_+$, $1 \mapsto \mu$); by Theorem B.118 (ii) its differential is $t \mapsto \mu \otimes t$ on $\mathrm{Lie}(\mathcal{Z}_+) = \mathbb{C}u_+$ after base change. Let $D(\mu) \in \mathbb{Z}$ be the degree of $i_\mu^*\Psi^*\mathcal{L}_y$; it does not depend on s. Let $\mathcal{Z}_+^s = \mathcal{Z}_+ \otimes_s \mathbb{C} = \mathbb{C}/\Lambda_{+,s}$ (coordinate t with $u_+ \mapsto 1$) and $A_s = \mathrm{covol}(\Lambda_{+,s})$. For $\mu, \mu' \in N_+$, $(t, t') \mapsto H'_s(\mu \otimes t, \mu' \otimes t')$ is $\mathbb{C}$-linear in t and antilinear in t', hence equals $Q_s(\mu, \mu')\,t\bar t'/A_s$ for a unique $Q_s(\mu, \mu') \in \mathbb{C}$, with $Q_s(\mu', \mu) = \overline{Q_s(\mu, \mu')}$ and, by the facts recalled above, $Q_s(\mu, \mu) = D(\mu)$. Since $(\mu+\mu') \otimes t = \mu \otimes t + \mu' \otimes t$, and $(w_0\mu') \otimes t = \mu' \otimes \zeta_s t$ with $\zeta_s = s(\kappa_+(w_0))$ (O acts on $\mathrm{Lie}(\mathcal{Z}_+)$ by κ_+, Lemma 34.31 (1)),

$$2\,\mathrm{Re}\, Q_s(\mu, \mu') = D(\mu+\mu') - D(\mu) - D(\mu'), \qquad 2\,\mathrm{Re}\big(\bar\zeta_s Q_s(\mu, \mu')\big) = D(\mu + w_0\mu') - D(\mu) - D(w_0\mu').$$

As $\mathrm{Im}\,\zeta_s \neq 0$, these two real equations determine $Q_s(\mu, \mu')$.

The value at the identity. For s the inclusion $K \subset \mathbb{C}$, $y^s = z$, $\Lambda_{+,s} = \omega_+\kappa_+(O)$, so $A_s = |\omega_+|^2\,\mathrm{covol}(\mathbb{Z} + \mathbb{Z}\frac{1+i\rho}{2}) = |\omega_+|^2\sqrt{d}/2$, and $d\Psi(\mu \otimes u_+) = \omega_+^{-1}\hat x(\mu)$ (Lemma 34.31 (2) and

$\mathcal{O}_E$-linearity). By the previous displays,

$$Q_{\mathrm{id}}(\mu,\mu') = \tfrac{\sqrt{d}}{2}\sum_{\tau'\in\Phi_{\mathsf{H}}} 2i\,\tau'(h(\hat{x}\mu,\hat{x}\mu')) = \sum_{\tau'\in\Phi_{\mathsf{H}}}\tau'\big(F(\mu,\mu')\big)$$
$$= \kappa_+\big(\mathrm{Tr}_{E/\mathbb{Q}(r)}F(\mu,\mu')\big), \qquad F(\mu,\mu') = \epsilon_+ r\,h(\hat{x}\mu,\hat{x}\mu') \in E,$$

using $i\sqrt{d} = \epsilon_+ i\rho = \epsilon_+\tau'(r)$. So $Q_{\mathrm{id}}(\mu,\mu') = \kappa_+(\theta)$ with $\theta = \mathrm{Tr}_{E/\mathbb{Q}(r)}F(\mu,\mu') \in \mathbb{Q}(r)$.

All conjugates. The embedding $s\circ\kappa_+$ of $\mathbb{Q}(r)$ is κ_+ or κ_-. If it is κ_+, then $\zeta_s = \zeta_{\mathrm{id}}$ and the equations above give $Q_s = Q_{\mathrm{id}} = \kappa_+(\theta)$. If it is κ_-, then $\zeta_s = \bar{\zeta}_{\mathrm{id}}$, and $\overline{Q_{\mathrm{id}}}$ satisfies both equations at s, so $Q_s = \overline{\kappa_+(\theta)} = \kappa_-(\theta)$. In both cases

$$Q_s(\mu,\mu') = s\big(\kappa_+(\theta)\big) = \sum_{\sigma}\sigma\big(F(\mu,\mu')\big),$$

the sum over the embeddings $\sigma\colon E\to\mathbb{C}$ extending $s\circ\kappa_+$.

The τ-component. One has $\mathrm{Lie}(G'^s) \supseteq N_+\otimes_{O,s\kappa_+}\mathbb{C} = \bigoplus_\sigma(\cdot)_\sigma$, the components for the characters σ just named; by the orthogonality paragraph, transported by the $\mathcal{O}_E$-linear Ψ^s, $H'_s(X_\sigma, X'_{\sigma'}) = 0$ for $\sigma\neq\sigma'$. For $a\in\mathcal{O}_E$, $(a\mu)_\sigma = \sigma(a)\mu_\sigma$ and $F(a\mu,\mu') = aF(\mu,\mu')$, so

$$\sum_\sigma \sigma(a)\,H'_s\big((\mu\otimes 1)_\sigma,(\mu'\otimes 1)_\sigma\big) = H'_s(a\mu\otimes 1,\mu'\otimes 1) = \sum_\sigma\sigma(a)\,\sigma(F(\mu,\mu'))/A_s,$$

and linear independence of the σ gives $H'_s((\mu\otimes 1)_\sigma,(\mu'\otimes 1)_\sigma) = \sigma(F(\mu,\mu'))/A_s$. The vector v_a is the image under $d\Psi$ of the τ-component of $e_{+,a}\otimes u_+$, so $s_*v_a = d\Psi^s((e_{+,a}\otimes u_+)_{s\circ\tau})$, and $s\circ\tau$ extends $s\circ\kappa_+$ because $\tau(r) = r_{\mathbb{C}} = \kappa_+(r)$ in K. Hence

$$h_{y^s}(s_*v_a, s_*v_{a'}) = \pi H'_s\big((e_{+,a}\otimes u_+)_{s\tau},(e_{+,a'}\otimes u_+)_{s\tau}\big) = \frac{\pi}{A_s}\,s\big(\tau(\epsilon_+ r\,h(x_{+,a},x_{+,a'}))\big),$$

which is the first identity, ϵ_+ being rational.

The frame w. Replace N_+, $\mathcal{Z}_+$, u_+, ω_+, κ_+, Φ_{H}, τ by N_-, $\mathcal{Z}_-$, u_-, ω_-, κ_-, $\{\bar{\tau}'\}$, $\bar{\tau}$. For $\lambda,\mu\in\Lambda_z^-$ one has $i\lambda = -\rho^{-1}\iota(r)\lambda$ and the same computation gives $H_z(\lambda,\mu) = \sum_{\tau'\in\Phi_{\mathsf{H}}} 2i\,\bar{\tau}'(h(\lambda,\mu))$; $\mathrm{covol}(\omega_-\kappa_-(O)) = |\omega_-|^2\sqrt{d}/2$; and $i\sqrt{d} = -\epsilon_+\bar{\tau}'(r) = \epsilon_-\bar{\tau}'(r)$, so F becomes $\epsilon_- r\,h(\hat{x}\mu,\hat{x}\mu')$ and $Q^-_{\mathrm{id}} = \kappa_-(\mathrm{Tr}_{E/\mathbb{Q}(r)}F)$. The two-case argument and the projection to the component $s\circ\bar{\tau}$ (which extends $s\circ\kappa_-$) are unchanged and give the second identity. □

Proposition 34.35. *There is a real number $C_{\mathrm{h}}\geq 0$ with the following property. Let $\mathfrak{u} = (E,\mathsf{H},\mathsf{U},p,q,\mathsf{L},\mathrm{Sh},F)$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ,n,p) with its geometric realisation $(\nu_{\mathcal{T}},(\mathcal{X}_{B,Z}))$ and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2 (so every $\mathcal{X}_{B,Z}$ has convergent regularisation, by Theorem 31.20), and put $\mathrm{rd}_E = |\mathrm{disc}(E)|^{1/[E:\mathbb{Q}]}$ and $L_{\mathsf{H}} = \mathrm{Re}\,(L'/L)(1,\chi_d)$ (note $\mathrm{rd}_E\geq\sqrt{d}\geq\sqrt{7}>2$, since E contains the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ of discriminant $-d$). Let $\mathcal{T}'$ be a root tower of type (Φ,n,p) with the components and the bad primes of $\mathcal{T}$ such that $\widehat{\deg}'_{B,Z}(F) = \mathrm{pp}(\mathcal{X}_{B,Z};F)$ for every integer $B\geq 2$, every $Z\in\mathrm{Cp}$ and every $F\in\mathcal{R}$, where $\widehat{\deg}'$ are the arithmetic degrees of $\mathcal{T}'$, and such that $\mathcal{T}'$ with the geometric realisation of $\mathcal{T}$ has (Gi1). Then for every admissible stage B and every $Z\in\mathrm{Cp}$ there is a Hodge-type base setting of type (Φ,n,p) with cusp data which satisfies (Bs0) and (Bs1) for $\mathcal{T}'$ at (B,Z) and, with its base variety $\mathcal{X}^{\mathrm{b}}$ and its corrected perturbed bundles $\overline{\mathcal{M}}_{i_1,i_2,t} = \overline{\mathcal{M}}_{B,i_1,i_2,t}$:*

(i) for all integers $1\leq i_1\leq i_2$ there is $t_1>0$ such that for $0<t\leq t_1$ the line bundle $\mathcal{M}_{i_1,i_2,t,\mathbb{Q}}$ is ample, $\deg(\mathcal{M}_{i_1,i_2,t}|_\Gamma)\geq 0$ for every closed integral curve $\Gamma\subseteq\mathcal{X}^{\mathrm{b}}$ contained in a fibre of $\mathcal{X}^{\mathrm{b}}\to\mathrm{Spec}\,\mathbb{Z}$, and the first Chern form $c_1(\overline{\mathcal{M}}_{i_1,i_2,t})$ is semipositive on $\mathcal{X}^{\mathrm{b}}(\mathbb{C})$;

(ii) it has regularisation limits (Mz1)–(Mz4) at B;

(iii) its cusp data have the cusp mean bound (Qk1) for $r = \mathrm{rd}_E$ and corrected point heights (Qk2) at B for $(r, L) = (\mathrm{rd}_E, L_{\mathsf{H}})$ and the constant C_{h}.

Proof. Let B be an admissible stage and $Z \in \mathrm{Cp}$. By Proposition D.1 there is a base setting of $\mathcal{T}$ at (B, Z) (a Hodge-type base setting of type (Φ, n, p) with cusp data) such that $\mathrm{ar}^{\mathrm{b}}(B)$ exists, the setting has regularisation limits (Mz1)–(Mz4) at B, for all $1 \le i_1 \le i_2$ the forms $c_1(\overline{\mathcal{M}}_{B,i_1,i_2,t})$ are semipositive for $0 < t \le t_1(i_1, i_2)$, $\psi_{t'} \ge \log h^{\mathrm{H}} + \frac{1}{2}U - \log h^{\mathrm{ref}'}$ on $X^{\mathrm{b},\circ}$ for $0 < t' \le t_0$, the setting is joined to the stage root model $\mathcal{X}_{B,Z}$ (Definition 31.11), and it satisfies the log-log regularisation hypotheses (i), (ii) of Lemma 34.36. We check the clauses for this setting.

- (Bs0) and (Bs1): since the setting is joined to the stage root model, satisfies the hypotheses (i) and (ii) of Lemma 34.36, and $\mathrm{ar}^{\mathrm{b}}(B)$ exists, Proposition D.19, applied to the given tower $\mathcal{T}'$, gives $\deg^{\mathrm{H}} > 0$, hence (Bs0), and the slope identity (Bs1) for $\mathcal{T}'$ at (B, Z).
- (i): let $1 \le i_1 \le i_2$ and take $t_1 = t_1(i_1, i_2)$ from the semipositivity above. By Proposition D.56 (the setting being joined to the stage root model), for every $t > 0$ the bundle $\mathcal{M}_{i_1,i_2,t,\mathbb{Q}}$ is ample and $\deg(\mathcal{M}_{i_1,i_2,t}|_\Gamma) \ge 0$ for every closed integral curve Γ in a fibre of $\mathcal{X}^{\mathrm{b}} \to \operatorname{Spec}\mathbb{Z}$; together with the semipositivity of $c_1(\overline{\mathcal{M}}_{i_1,i_2,t})$ for $0 < t \le t_1$ this is (i).
- (ii) is part of the choice above.
- (iii): Proposition D.21, with its absolute constant C_{h} (which is the constant of the statement), gives the cusp mean bound (Qk1) for $r = \mathrm{rd}_E$, as the setting satisfies (i) and (ii) of Lemma 34.36; its domination hypothesis holds with the t_0 above, so it gives corrected point heights (Qk2) at B for $(r, L) = (\mathrm{rd}_E, L_{\mathsf{H}})$ and C_{h}.

All clauses hold for one base setting, which proves the proposition. □

Lemma 34.36. *Let a Hodge-type base setting be given; put $\Theta = dd^c \log h'$ (a Kähler form on X^{b} by (Hy5)), $\theta = dd^c \log h^{\mathrm{ref}}$ and $\varphi = \log(h^{\mathrm{H}}/h^{\mathrm{ref}})$ on $X^{\mathrm{b},\circ}$. Suppose:*

(i) there is a holomorphic section ς of a smooth hermitian line bundle on X^{b} with $\|\varsigma\| \le 1$ whose zero divisor has simple normal crossings and support ∂X^{b}; put $\mathrm{rW} = e - 2\log\|\varsigma\|$ on $X^{\mathrm{b},\circ}$;

(ii) φ is smooth on $X^{\mathrm{b},\circ}$, and there are $C \ge 1$, $0 < t_0 < 1$ and functions $\delta, \varepsilon\colon (0, t_0] \to [0, \infty)$, $R\colon (0, t_0] \to [e, \infty)$ with $\delta(t), \varepsilon(t) \to 0$, $R(t) \to \infty$ as $t \to 0^+$, such that $u = \varphi$, $u_t = \varphi_t$ satisfy (R2)–(R5) and $\theta + dd^c\varphi_t + \varepsilon(t)\Theta \ge 0$ on X^{b} for $0 < t \le t_0$.

Then $(\omega^{\mathrm{H}})^e$ is absolutely integrable on $X^{\mathrm{b},\circ}$ and

$$\int_{X^{\mathrm{b},\circ}} (\omega^{\mathrm{H}})^e = \deg^{\mathrm{H}}.$$

Proof. Write $X = X^{\mathrm{b}}$, $X^\circ = X^{\mathrm{b},\circ}$. By (Hy1), X is a compact complex manifold of dimension $e = |\Phi| jj' \ge 1$, and by (Hy5) Θ is a Kähler form on X. By (i), ς, $\partial X = \partial X^{\mathrm{b}}$, X° and rW are as in the setting of Lemma 31.35. Take there $D = e$, $v = 1$ the constant section of the trivial line bundle with the trivial metric (so $D_v = \varnothing$ and $G = 0$), $I = \{\mathrm{H}\}$ with $\theta_{\mathrm{H}} = \theta$,

a smooth closed real $(1,1)$-form on X (h^{ref} is a smooth cone function by (Hy4)), $u_{\mathrm{H}} = \varphi$, smooth on X° by (ii), and $u_{\mathrm{H},t} = \varphi_t$, smooth on X by (Hy4). Take C, δ, R from (ii), and shrink t_0 so that $\varepsilon(t) \le 1 \le C$ for $0 < t \le t_0$ (possible as $\varepsilon(t) \to 0$). Then (R1) holds: $\theta + dd^c\varphi_t + C\Theta = (\theta + dd^c\varphi_t + \varepsilon(t)\Theta) + (C - \varepsilon(t))\Theta \ge 0$, as $\Theta \ge 0$; and (R2)–(R5) hold by (ii).

On the cone over X° we have $h^{\mathrm{H}} = h^{\mathrm{ref}} e^{\varphi}$ (definition of φ, (Hy4)), so on X°

$$\omega^{\mathrm{H}} = dd^c \log h^{\mathrm{H}} = dd^c \log h^{\mathrm{ref}} + dd^c\varphi = \theta_{\mathrm{H}} + dd^c u_{\mathrm{H}}.$$

Part (1) of Lemma 31.35 with $i_1 = \cdots = i_e = \mathrm{H}$ gives that $\mu = (\omega^{\mathrm{H}})^e$ is absolutely integrable on X° (as $\log \mathrm{rW} \ge 1$) and $\int_{X^\circ} (\omega^{\mathrm{H}})^e = \int_X \theta^e$.

Finally, θ is the first Chern form of the smooth hermitian line bundle $(\mathcal{O}(a^{\mathrm{H}}), h^{\mathrm{ref}})|_X$, so it represents $c_1(\mathcal{O}(a^{\mathrm{H}}))$ on the compact complex manifold $X = \mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}(\mathbb{C})$, and

$$\int_X \theta^e = \deg(c_1(\mathcal{O}(a^{\mathrm{H}}))^e \cap [\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}]) = \deg^{\mathrm{H}} :$$

the top self-intersection number of a line bundle on the smooth projective $\mathbb{Q}$-variety $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ is unchanged by base change to $\mathbb{C}$ and equals the integral of the e-th power of a de Rham representative of its first Chern class over the complex points (summed over the connected components when $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ is not geometrically connected). Hence $\int_{X^\circ} (\omega^{\mathrm{H}})^e = \deg^{\mathrm{H}}$. □

35 Linked stages

This section proves Theorem 35.2: a root tower with Galois Frobenius labels, (Gi1), (Gi2) and a good origin over a model of a Shimura datum with abelian Galois host field has linked stages (Sk1)–(Sk4). Linked stages give the change of stage (Rt3) (Theorem 30.28), which relates the arithmetic degrees at two stages $B_0 \le B$ through the primes between them, and through (Rt3) the proportionality identity of Section 29. The section is long, and its content is p-adic. Two stages of a component are compared on a joint model (Definition 35.1, Lemma 35.3), on which (Sk1)–(Sk3) hold by Lemma 35.4. The substance is (Sk4), the local stage change at a prime p of stage change (Lemma 35.6): it expresses the vertical divisors by which the root lines of the two stages differ over p through the stage-change classes, Frobenius orbit sums built from the Frobenius labels with weights $p^{-k(m+1)}$.

The proof compares de Rham and Hodge–Tate lattices of the abelian scheme over p. On the de Rham side, the vertical root divisors measure the change from the de Rham to the Hodge–Tate lattices of the Lie and cotangent modules (Lemma 35.13). On the Hodge–Tate side, the eigen-coordinates of an admissible torsion system of the p-torsion, indexed by the Frobenius labels, measure the same change (Proposition 35.11), and they satisfy a Frobenius recursion (Proposition 35.9). Equating the two gives the local Frobenius relations (Fz1), (Fz2) (Proposition 35.8), hence (Sk4). The inputs are the integral Hodge–Tate sequence of p-divisible groups with Fargues's bounds [89], flattening by blow-up [206], Zarhin's trick [273], and a theory of tight Hopf orders over $\mathcal{O}_C$ developed in Subsections 35.2 and 35.4. The test-point facts used in Subsection 35.2 are proved in Appendix D.

35.1 Linked stages and local Frobenius relations

The subsection states Theorem 35.2 and reduces it to the local Frobenius relations. Joint models of two stages exist (Lemma 35.3) and satisfy (Sk1)–(Sk3) (Lemma 35.4); on them the Hodge–Tate lattices of the two stages agree at the primes of the lower stage (Proposition 35.5). (Sk4) follows

from the relations (Fz1), (Fz2) at each prime of stage change (Lemma 35.6, Proposition 35.8). These are proved from three facts: the eigen-coordinate divisors of an admissible torsion system (Definition 35.7) satisfy a Frobenius recursion (Proposition 35.9); such a system exists with eigen-coordinates measuring the de Rham to Hodge–Tate change (Proposition 35.11, Subsection 35.2); and the power sums of the Hodge–Tate root classes vanish on the components of the fibre (Proposition 35.64, Subsection 35.4). Chern characters of vertical modifications (Lemma 35.10) convert between divisors and classes.

Definition 35.1. Let $\mathcal{T}$ be a root tower of type (Φ, n, p) with a geometric realisation and a good origin over an integral model $\mathcal{Y}$ of a unitary Shimura datum $\mathfrak{u}$. Let $c \in \mathrm{Cp}$, and let B_0, B be integers with $2 \le B_0 \le B$. For $i \in \{0,1\}$ write $\mathcal{X}_0 = \mathcal{X}_{B_0,c} \subseteq \mathbb{P}_0 = \prod_{k\in K_0} \mathbb{P}(\mathbb{Z}^{I_k})$ and $\mathcal{X}_1 = \mathcal{X}_{B,c} \subseteq \mathbb{P}_1 = \prod_{k\in K_1} \mathbb{P}(\mathbb{Z}^{I_k})$ $(K_0 \cap K_1 = \varnothing)$, with function fields Ω_i, root origins with representing coordinates y^i, embeddings $g_i \colon \Omega_c \to \Omega_i$, root sections ϱ^i_r, maps $f_i \colon \mathcal{X}_i \to \mathcal{Y}$, multidegrees e^i_r, e^i_* and metrics $h^i_r(t)$, h^i_*; let t_0 be the constant of Sys_c.

A **joint model** of the stages (B_0, c) and (B, c) is a root model $\mathcal{X}$ of type (Φ, n) with a system of second roots, where $\mathcal{X} \subseteq \mathbb{P}(\mathbb{Z}^{I_2}) \times \mathbb{P}_0 \times \mathbb{P}_1$ for a further finite set I_2, with function field Ω, such that:

(Jm1) Common normal cover The scheme $\mathcal{X}$ is normal; the projections restrict to dominant morphisms $\mathrm{pr}_i \colon \mathcal{X} \to \mathcal{X}_i$ $(i = 0, 1)$ with $f_0 \circ \mathrm{pr}_0 = f_1 \circ \mathrm{pr}_1$; and the induced field embeddings $\iota_i \colon \Omega_i \to \Omega$ satisfy $\iota_0 \circ g_0 = \iota_1 \circ g_1$ on Ω_c.

(Jm2) Lines pulled back Writing a point of the affine cone as $z = (z_2, z_0, z_1)$: the multidegrees are $e_r = (0, e^0_r, 0)$, $e'_r = (0, 0, e^1_r)$ and $e_* = (0, e^0_*, 0)$; for every $t > 0$ the metrics are $h_r(t)(z) = h^0_r(\min(t, t_0))(z_0)$, $h'_r(t)(z) = h^1_r(\min(t, t_0))(z_1)$ and $h_*(z) = h^0_*(z_0)$; and $k_{\mathcal{X}} = k^A_c$, $\delta_{\mathcal{X}} = [F : \mathbb{Q}]\,[\Omega : \iota_0 g_0(\Omega_c)]$.

(Jm3) Ratio of root sections For every root r, the polynomials f_r, g_r involve only the variables of $\mathbb{P}_0 \times \mathbb{P}_1$ and

$$\frac{f_r(\iota_0 y^0, \iota_1 y^1)}{g_r(\iota_0 y^0, \iota_1 y^1)} = \frac{\iota_1(\varrho^1_r)}{\iota_0(\varrho^0_r)} \qquad \text{in } \Omega.$$

Theorem 35.2. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p_0) with Galois Frobenius labels (Fr), with a geometric realisation that has (Gi1) and (Gi2), and with a good origin immersive at level four over an integral model $\mathcal{Y}$ of a unitary Shimura datum $\mathfrak{u} = (E, \mathsf{H}, \dots)$, $\Phi = \Phi_{\mathsf{H}}$, E Galois over $\mathbb{Q}$ with abelian Galois group. Then $\mathcal{T}$ has linked stages.*

Proof. Let $2 \le B_0 \le B$ and $Z \in \mathrm{Cp}$. By Lemma 35.3 there is a joint model $\mathcal{X}$ of the stages (B_0, Z) and (B, Z) (Definition 35.1): a root model of type (Φ, n) with a system of second roots. By Lemma 35.4 it satisfies (Sk1), (Sk2) and (Sk3) with $S = S(B_0, B)$, and by Lemma 35.6 it satisfies (Sk4) with the stage-change classes of the Frobenius labelling of $\mathcal{T}$. So $\mathcal{X}$ is a link from (B_0, Z) to (B, Z). As $B_0 \le B$ and Z were arbitrary, $\mathcal{T}$ has linked stages. □

Lemma 35.3. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p) with a geometric realisation and a good origin over an integral model $\mathcal{Y}$ of a unitary Shimura datum. For every $c \in \mathrm{Cp}$ and all integers $2 \le B_0 \le B$ there is a joint model of the stages (B_0, c) and (B, c).*

Proof. Step 1: a common field. The Ω_c-algebra $\Omega_0 \otimes_{\Omega_c} \Omega_1$ (via g_0, g_1) is non-zero and finite-dimensional; its quotient by a maximal ideal is a field Ω' with embeddings $\iota'_i \colon \Omega_i \to \Omega'$ such that $\iota'_0 \circ g_0 = \iota'_1 \circ g_1$. Let $\Omega \subseteq \Omega'$ be the compositum of $\iota'_0(\Omega_0)$ and $\iota'_1(\Omega_1)$ and ι_i the corestricted embeddings. Then Ω is finite over $\iota_0 g_0(\Omega_c)$, so it has characteristic 0 and transcendence degree

D over $\mathbb{Q}$, and it is generated over $\mathbb{Q}$ by the ratios of the coordinates $\iota_0(y^0)$ and the ratios of the coordinates $\iota_1(y^1)$.

Step 2: the closure. Let $\mathfrak{q} \subseteq \mathbb{Z}[Y^0, Y^1]$ (variables of $\mathbb{P}_0 \times \mathbb{P}_1$) be the ideal generated by the multihomogeneous G with $G(\iota_0 y^0, \iota_1 y^1) = 0$; it is the kernel of a ring map to a field on multihomogeneous elements, hence a multihomogeneous prime, and it contains no coordinate variable that is non-zero in Ω. Let $\mathcal{X}^{\mathrm{cl}} \subseteq \mathbb{P}_0 \times \mathbb{P}_1$ be the closed integral subscheme it defines; its function field is the field generated by the coordinate ratios, i.e. Ω. If G lies in the ideal of $\mathcal{X}_i$, then $G(\iota_i y^i) = \iota_i(G(y^i)) = 0$, so $\mathcal{X}^{\mathrm{cl}}$ lies in $\mathcal{X}_0 \times \mathcal{X}_1$ and the projections give morphisms to $\mathcal{X}_i$ that send the generic point to the generic point; they are dominant. As Ω has characteristic 0, $\mathcal{X}^{\mathrm{cl}}$ dominates $\operatorname{Spec}\mathbb{Z}$, so it is flat over $\mathbb{Z}$, of absolute dimension $\operatorname{trdeg}_{\mathbb{Q}}\Omega + 1 = D + 1$. The maps $f_0 \circ \mathrm{pr}_0$ and $f_1 \circ \mathrm{pr}_1$ to the separated scheme $\mathcal{Y}$ agree at the generic point, because f_i induces $\mathbf{k}(\mathcal{Y}_c) \subseteq \mathbf{k}(Z_c) \to \Omega_i$ through g_i and $\iota_0 g_0 = \iota_1 g_1$; their equaliser is a closed subscheme containing the generic point of the integral scheme $\mathcal{X}^{\mathrm{cl}}$, hence everything.

Step 3: normalisation. Since $\mathcal{X}^{\mathrm{cl}}$ is of finite type over $\mathbb{Z}$, its normalisation $\nu\colon \mathcal{X} \to \mathcal{X}^{\mathrm{cl}}$ is finite, $\mathcal{X}$ is normal and integral with function field Ω, flat over $\mathbb{Z}$ of absolute dimension $D + 1$. A finite morphism to a scheme with an ample line bundle (here $\mathcal{O}(1, \dots, 1)$) is projective, so there are a finite set I_2 and a closed immersion of $\mathcal{X}$ into $\mathbb{P}(\mathbb{Z}^{I_2}) \times \mathcal{X}^{\mathrm{cl}} \subseteq \mathbb{P}(\mathbb{Z}^{I_2}) \times \mathbb{P}_0 \times \mathbb{P}_1$ over $\mathcal{X}^{\mathrm{cl}}$ (EGA II, 6.1.11 with Serre's theorem on global generation). Composing ν with the projections gives $\mathrm{pr}_i\colon \mathcal{X} \to \mathcal{X}_i$, dominant, with $f_0 \circ \mathrm{pr}_0 = f_1 \circ \mathrm{pr}_1$ and induced embeddings ι_i with $\iota_0 g_0 = \iota_1 g_1$: this is (Jm1).

Step 4: the root data. Define the multidegrees, the metrics for all $t > 0$, the auxiliary line and the normalising integers by the formulas of (Jm2): each $z \mapsto h_r^0(\min(t, t_0))(z_0)$ is a smooth positive conjugation-invariant function on the product of the punctured coordinate spaces, homogeneous of multidegree $(0, e_r^0, 0)$, so it is a cone metric; likewise for $h_r'(t)$ and h_*; $k_c^A \geq 1$ and $\delta_{\mathcal{X}} \geq 1$ are integers. This is a root model of type (Φ, n) with candidate second roots of multidegrees e_r'.

Step 5: the polynomials f_r, g_r. Fix a root r and put $d = (e_r^0, -e_r^1) \in \mathbb{Z}^{K_0 \sqcup K_1}$. For every $k \in K_0 \sqcup K_1$ choose $j_k \in I_k$ with $y_{k,j_k} \neq 0$ in Ω (where $y = (\iota_0 y^0, \iota_1 y^1)$), and let $M_1 = \prod_k Y_{k,j_k}^{\max(d_k,0)}$, $M_2 = \prod_k Y_{k,j_k}^{\max(-d_k,0)}$, so that $\deg M_1 - \deg M_2 = d$ and $M_1(y), M_2(y) \neq 0$. The element $\omega = \iota_1(\varrho_r^1) M_2(y) / (\iota_0(\varrho_r^0) M_1(y)) \in \Omega^\times$ is a rational function with rational coefficients in the coordinate ratios (Step 1); multiplying numerator and denominator by a product of powers of the Y_{k,j_k} and by an integer gives multihomogeneous $G, H \in \mathbb{Z}[Y^0, Y^1]$ of the same multidegree with $\omega = G(y)/H(y)$ and $H(y) \neq 0$. Put $f_r = GM_1$ and $g_r = HM_2$. Then $\deg f_r - \deg g_r = d = e_r - e_r'$, $f_r(y)/g_r(y) = \omega M_1(y)/M_2(y) = \iota_1(\varrho_r^1)/\iota_0(\varrho_r^0) \neq 0$, so neither f_r nor g_r lies in the ideal of $\mathcal{X}$, and (Jm3) holds. With $(e_r', h_r'(t), f_r, g_r)$ the root model $\mathcal{X}$ carries a system of second roots and is a joint model. □

Lemma 35.4. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p) with a geometric realisation that has (Gi1) and (Gi2), and with a good origin immersive at level four over an integral model $\mathcal{Y}$ of a unitary Shimura datum $\mathfrak{u} = (E, \dots, F)$, E Galois over $\mathbb{Q}$. Let $c \in \mathrm{Cp}$ and $2 \leq B_0 \leq B$. Every joint model $\mathcal{X}$ of the stages (B_0, c) and (B, c) satisfies (Sk1), (Sk2) and (Sk3) with $S = S(B_0, B)$ and $Z = c$.*

Proof. Write $\mathcal{X}_0 = \mathcal{X}_{B_0,c}$, $\mathcal{X}_1 = \mathcal{X}_{B,c}$, $d_i = [\Omega : \iota_i(\Omega_i)]$ (the generic degree of pr_i), $S = S(B_0, B)$, and $y = (\iota_0 y^0, \iota_1 y^1)$.

Step 0: the two auxiliary lines and the two realisations agree. Let $k_0 \in K_0$ and $k_0' \in K_1$ be the auxiliary factors of the two stages. Their coordinates are $\iota_i g_i(y_j^c)$ $(0 \leq j \leq m)$, equal by (Jm1); so the two morphisms $\mathcal{X} \to \mathbb{P}_{\mathbb{Z}}^m$ given by these factors agree at the generic point of $\mathcal{X}$,

hence everywhere (their equaliser is a closed subscheme of the integral scheme $\mathcal{X}$ containing its generic point). Therefore $\mathrm{pr}_0^*(\mathcal{O}(e_*^0), h_*^0)$ and $\mathrm{pr}_1^*(\mathcal{O}(e_*^1), h_*^1)$ are isometric, both metrics being h_c of the same coordinate vector up to a scalar. For a complex point z of $\mathcal{X}$ the k_0-coordinates of $\mathrm{pr}_0(z)$ and the k_0'-coordinates of $\mathrm{pr}_1(z)$ are proportional; by (Go4) each is proportional to pt_c of the image under the realisation π_i of $\mathcal{X}_i$, and pt_c is injective modulo $\mathbb{C}^\times$, so $\pi_0 \circ \mathrm{pr}_0 = \pi_1 \circ \mathrm{pr}_1 =: \pi$ on $\mathcal{X}(\mathbb{C})$.

Step 1 (Sk1). By (Jm2), for $t > 0$ and a monomial $x^m \in \mathcal{R}^+$ of degree $D+1$ with j factors x_*, the bundles entering $\mathrm{ai}_t(\mathcal{X}; x^m)$ are the pull-backs by pr_0 of those entering $\mathrm{ai}_{t'}(\mathcal{X}_0; x^m)$, $t' = \min(t, t_0)$, and $\delta_{\mathcal{X}} = [F : \mathbb{Q}] d_0 [\Omega_0 : g_0(\Omega_c)] = d_0 \delta_{\mathcal{X}_0}$ by (Go3)(1), $k_{\mathcal{X}} = k_c^A = k_{\mathcal{X}_0}$ by (Go3)(5). Lemma 30.18 (1) with $d = 0$, for the projection $f \colon \mathbb{P}(\mathbb{Z}^{I_2}) \times \mathbb{P}_0 \times \mathbb{P}_1 \to \mathbb{P}_0$ of regular arithmetic varieties, $\mathcal{Z} = \mathcal{X}$ (integral and flat over $\mathbb{Z}$, with $f(\mathcal{X}) = \mathcal{X}_0$ since pr_0 is dominant (Jm1) and proper) and $c = [\Omega : \iota_0(\Omega_0)] = d_0$, gives $\mathrm{ai}_t(\mathcal{X}; x^m) = \mathrm{ai}_{t'}(\mathcal{X}_0; x^m)$. For $\mathcal{X}'$ the root lines are pulled back by pr_1 and the auxiliary line of $\mathcal{X}$ is isometric to the pull-back by pr_1 of that of $\mathcal{X}_1$ (Step 0); $\delta_{\mathcal{X}} = [F : \mathbb{Q}][\Omega : \iota_1 g_1(\Omega_c)] = d_1 \delta_{\mathcal{X}_1}$ since $\iota_0 g_0 = \iota_1 g_1$, and $k_{\mathcal{X}_1} = k_c^A$; so, by Lemma 30.18 (1) for the projection to $\mathbb{P}_1$ with $c = d_1$ and the invariance of heights under isometries, $\mathrm{ai}_t(\mathcal{X}'; x^m) = \mathrm{ai}_{t'}(\mathcal{X}_1; x^m)$. For $t < t_0$ we have $t' = t$, and (Gi2) at the stages B_0 and B gives that $\mathcal{X}$ and $\mathcal{X}'$ have convergent regularisation with $\mathrm{pp}(\mathcal{X}; F) = \widehat{\deg}_{B_0,c}(F)$, $\mathrm{pp}(\mathcal{X}'; F) = \widehat{\deg}_{B,c}(F)$ for $F \in \mathcal{R}$. Likewise $\mathrm{gi}(\mathcal{X}; x^m) = \mathrm{gi}(\mathcal{X}_0; x^m)$ for monomials of degree D (projection formula on generic fibres, same normalisation), and (Gi1) at stage B_0 gives $\mathrm{gi}(\mathcal{X}; F) = \mathrm{vol}(c) \deg^{\mathrm{fl}}(F)$ for F homogeneous of degree D. This is (Sk1).

Step 2: de Rham comparison. Let $\tilde{x} \in \mathcal{X}$ be a point lying over no prime of S_B (in particular every point of the generic fibre), $x_i = \mathrm{pr}_i(\tilde{x})$, and s a label with embedding $\sigma_s \colon E \to F$. Then $V_{s,x_0}^{S_{B_0}} = \mathrm{dR}_{s,x_0}$ and $V_{s,x_1}^{S_B} = \mathrm{dR}_{s,x_1}$. As $G_{\mathcal{X}} = \mathrm{pr}_i^* G_{\mathcal{X}_i}$ is smooth, $\mathrm{Lie}(G_{\mathcal{X}})_{\tilde{x}} = \mathrm{Lie}(G_{\mathcal{X}_i})_{x_i} \otimes \mathcal{O}_{\tilde{x}}$ with $\mathrm{Lie}(G_{\mathcal{X}_i})_{x_i}$ free, so $\mathrm{Lie}_{\tilde{x}} = \mathcal{O}_{\tilde{x}} \cdot \iota_i(\mathrm{Lie}_{x_i})$ in Li. The prime under $\tilde{x}$ (if any) is not in $\mathrm{Bd} \supseteq \{\ell : \ell \mid \mathrm{disc}(E)\}$, so $\mathrm{disc}(E)$ is invertible in $\mathcal{O}_{x_i} \supseteq \mathcal{O}_F$; $\mathcal{O}_E \otimes \mathcal{O}_F[1/\mathrm{disc}(E)]$ is finite étale over the normal ring $\mathcal{O}_F[1/\mathrm{disc}(E)]$, hence normal, and contains the idempotents ϵ_σ of $E \otimes F = \prod_{\sigma \colon E \to F} F$ (integral, as $\epsilon^2 = \epsilon$). They act on Lie_{x_i} and on $\mathrm{Lie}_{\tilde{x}}$, and $\epsilon_{\sigma_s} \mathrm{Lie}_{x_i} = \mathrm{Lie}_{x_i} \cap \mathrm{Li}_s = \mathrm{dR}_{s,x_i}$; hence

$$\mathrm{dR}_{s,\tilde{x}} = \epsilon_{\sigma_s} \mathrm{Lie}_{\tilde{x}} = \mathcal{O}_{\tilde{x}} \cdot \iota_0(\mathrm{dR}_{s,x_0}) = \mathcal{O}_{\tilde{x}} \cdot \iota_1(\mathrm{dR}_{s,x_1}).$$

Step 3: equal Hodge lattices give equal root lines. Let $\mathcal{O} = \mathcal{O}_{\mathcal{X},\tilde{x}}$ for a point $\tilde{x}$ at which the two pulled-back Hodge lattices $\mathcal{O} \cdot \iota_0(V_{s,x_0}^{S_{B_0}})$ and $\mathcal{O} \cdot \iota_1(V_{s,x_1}^{S_B})$ coincide for every label s (call it V_s, free of rank $\mathrm{rk}(s)$). The pulled-back integral flags of the two stages are full flags of V_s by submodules $V_{s,k}$ with free quotients $V_s/V_{s,k}$, and both have generic fibre $\iota_i g_i$ of the universal flag (Go3)(3), the same flag as $\iota_0 g_0 = \iota_1 g_1$. A submodule with torsion-free quotient is determined by its generic fibre: $V_{s,k} = V_s \cap (V_{s,k} \otimes \Omega)$. So the two flags coincide, hence so do their root lines (Fh3): the stalks at $\tilde{x}$ of $\mathrm{pr}_0^* \mathcal{L}_r^0$ and $\mathrm{pr}_1^* \mathcal{L}_r^1$ are the same submodule of the generic root line. By (Go3)(3) these stalks are generated by $\iota_0(\varrho_r^0) M_0(y)$ and $\iota_1(\varrho_r^1) M_1(y)$ for monomials M_i of multidegree e_r^i in coordinates not vanishing at $\tilde{x}$; so $u = \iota_1(\varrho_r^1) M_1(y) / (\iota_0(\varrho_r^0) M_0(y)) \in \mathcal{O}^\times$, and by (Jm3) $u = (f_r/g_r) \cdot M_1/M_0$ evaluated at y, i.e. the rational section f_r/g_r of $\mathcal{O}(e_r - e_r')$ is invertible at $\tilde{x}$.

Step 4 (Sk2). Let $\tilde{x}$ lie over no prime of S. If it lies over no prime of S_B, Steps 2 and 3 apply. Otherwise it lies over $\ell \in S_{B_0}$. The ring $\mathcal{O}_{\mathcal{X},\tilde{x}}$ is a normal noetherian domain, so an element of Ω that is a unit in its localisations at all height-one primes is a unit in it. These localisations are $\mathcal{O}_{\mathcal{X},W}$ for codimension-one points W specialising to $\tilde{x}$; each lies over ℓ or in the generic fibre. At the former, Proposition 35.5 gives equal Hodge lattices; at the latter, Step 2. By Step 3, u and u^{-1} lie in every $\mathcal{O}_{\mathcal{X},W}$, so f_r/g_r is invertible at $\tilde{x}$. This is (Sk2).

Step 5 (Sk3). For $t \geq t_0$ both sides of (Sk3) are those at t_0 by (Jm2); let $0 < t \leq t_0$. Let

$U \subseteq \mathcal{X}(\mathbb{C})$ be the set of points in the abelian locus (the preimage of $j(\mathrm{Sh}_F)$, the same through pr_0 and pr_1 by (Jm1)) at which g_r does not vanish. The abelian locus is the open set where the semiabelian scheme $G_{\mathcal{X}}$ is abelian (the toric rank is upper semicontinuous) and contains the generic point, and g_r is not in the ideal of $\mathcal{X}$; so the complement of U is the set of complex points of a proper Zariski-closed subset of $\mathcal{X}_{\mathbb{Q}}$; every irreducible component of $\mathcal{X}_{\mathbb{C}}$ dominates $\mathcal{X}_{\mathbb{Q}}$, so U is dense, and so is its preimage in the affine cone. Let $z \in U$, $z_i = \mathrm{pr}_i(z)$, with cone point $\tilde{z} = (\tilde{z}_2, \tilde{z}_0, \tilde{z}_1)$. By Step 2 on the generic fibre and Step 3, $(\mathrm{pr}_0^*\mathcal{L}_r^0) \otimes \mathbb{C}$ and $(\mathrm{pr}_1^*\mathcal{L}_r^1) \otimes \mathbb{C}$ are the same root line of $G_{\mathcal{X},z}$ with the same flag, so they carry the same Hodge metric (Fh4), (Fh6). Choose a local section P of $\mathcal{O}(e_r^1)$ near z_1 with $P(\tilde{z}_1) \neq 0$. By (Jm3) and (Go3)(3) the same element of the generic root line is $\iota_1(\varrho_r^1)P(y) = \iota_0(\varrho_r^0)\,(f_rP/g_r)(y)$, and f_rP/g_r is a local section of $\mathcal{O}(e_r^0)$ near z as $g_r(\tilde{z}) \neq 0$. Applying (Go3)(4) at both stages,

$$\frac{|P(\tilde{z}_1)|}{h_r^{1,\mathrm{sing}}(\tilde{z}_1)} = \|\cdot\|^{\mathrm{Hdg}}(z) = \frac{|f_r(\tilde{z})|\,|P(\tilde{z}_1)|}{|g_r(\tilde{z})|\,h_r^{0,\mathrm{sing}}(\tilde{z}_0)}, \quad \text{so} \quad |g_r(\tilde{z})|\,h_r^{0,\mathrm{sing}}(\tilde{z}_0) = |f_r(\tilde{z})|\,h_r^{1,\mathrm{sing}}(\tilde{z}_1).$$

By (Go4), for $r = (\tau, i)$ and the fundamental indices $a = (\tau, i)$, $a^- = (\tau, i-1)$ (omitted if $i = 1$), on the preimage of X_c°,

$$\frac{h_r^j(t)}{h_r^{j,\mathrm{sing}}} = \exp\Big(-(\varphi_a^T(t) - \varphi_a) \circ \pi + (\varphi_{a^-}^T(t) - \varphi_{a^-}) \circ \pi\Big) \qquad (j = 0, 1),$$

since the regularised and the singular metric of the fundamental index a are $h_a^{\mathrm{ref}} e^{\varphi_a^T(t)\circ\pi} = \prod_{k\le i} h_{(\tau,k)}(t)^{-1}$ and $h_a^{\mathrm{ref}} e^{\varphi_a \circ \pi} = \prod_{k\le i}(h_{(\tau,k)}^{\mathrm{sing}})^{-1}$. The right side is the same function for $j = 0, 1$ (Step 0: one π, one Sys_c), and $\pi(z) \in X_c^\circ$ as z is in the abelian locus. Multiplying, $|f_r(\tilde{z})|h_r'(t)(\tilde{z}) = |g_r(\tilde{z})|h_r(t)(\tilde{z})$ on the preimage of U in the cone. Both sides are continuous on the cone over $\mathcal{X}(\mathbb{C})$, so equality holds everywhere. This is (Sk3). □

Proposition 35.5. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p) with a geometric realisation and a good origin immersive at level four over an integral model $\mathcal{Y}$ of a unitary Shimura datum $\mathfrak{u} = (E, \dots, F)$ with E Galois over $\mathbb{Q}$. Let $c \in \mathrm{Cp}$, $2 \le B_0 \le B$, $\mathcal{X}$ a joint model of the stages (B_0, c) and (B, c) with function field Ω, and $W \in \mathcal{X}$ a point of codimension one lying over a prime $\ell \in S_{B_0}$; put $x_i = \mathrm{pr}_i(W)$. Then for every label s the $\mathcal{O}_{\mathcal{X},W}$-submodules*

$$\mathcal{O}_{\mathcal{X},W} \cdot \iota_0(V_{s,x_0}^{S_{B_0}}(\mathcal{X}_0)) \quad \text{and} \quad \mathcal{O}_{\mathcal{X},W} \cdot \iota_1(V_{s,x_1}^{S_B}(\mathcal{X}_1))$$

of $\mathrm{Li}_s(G_{\mathcal{X},\Omega})$ are equal, where $\iota_i \colon \mathrm{Li}_s(G_{\mathcal{X}_i,\Omega_i}) \otimes_{\Omega_i} \Omega \to \mathrm{Li}_s(G_{\mathcal{X},\Omega})$ is the base change isomorphism ($G_{\mathcal{X}} = \mathrm{pr}_i^* G_{\mathcal{X}_i}$ by (Jm1)) and both lattices are Hodge–Tate lattices at ℓ with $R(\ell) = 4$.

Proof. Put $N = \ell^4$, $\mathcal{O}_W = \mathcal{O}_{\mathcal{X},W}$, and let λ be the prime of $\mathcal{O}_F$ under W (the preimage of the maximal ideal of $\mathcal{O}_W$; it contains ℓ).

Step 1: the joint model is a cover, and the hypotheses of Lemma 31.30 hold for pr_i. By (Jm1) $\mathcal{X}$ is normal and integral, and $f = f_0 \circ \mathrm{pr}_0 = f_1 \circ \mathrm{pr}_1$ maps its generic point to that of $\mathcal{Y}_c$, since pr_0 is dominant and $\mathcal{X}_0$ is a cover over $\mathcal{Y}_c$ (Go3)(1); Ω is finitely generated of the same transcendence degree as Ω_0 over $\mathbb{Q}$ (both models are root models of type (Φ, n)), hence finite over $\iota_0(\Omega_0)$, which is finitely generated over the residue field of the generic point of $\mathcal{Y}_c$ (as $\mathcal{X}_0$ is a cover); so Ω is finitely generated over that residue field, and $\mathcal{X}$ is a cover of $\mathcal{Y}$ with $G_{\mathcal{X}} = \mathrm{pr}_i^* G_{\mathcal{X}_i}$, and $\mathcal{X}, \mathcal{X}_0, \mathcal{X}_1$ are noetherian. As $\ell \in S_{B_0} \subseteq S_B$ and the good origin is immersive at level four, $R(\ell) = 4$ at both stages, and by (Go3)(2) every N-torsion point of $G_{\mathcal{X}_i}$ over an algebraic closure of Ω_i is defined over Ω_i ($i = 0, 1$). The maps $\mathcal{O}_{x_i} \to \mathcal{O}_W$ are local, so x_i lies over ℓ, and over λ.

Step 2: the Hodge–Tate spans. By Lemma 31.30 (2) applied to $\mathrm{pr}_i\colon \mathcal{X} \to \mathcal{X}_i$ $(i = 0, 1)$, all N-torsion of $G_{\mathcal{X}}$ is defined over Ω and the span of the Hodge–Tate values at W satisfies

$$\mathrm{ht}_W = \mathcal{O}_W \cdot \iota_0(\mathrm{ht}_{x_0}) = \mathcal{O}_W \cdot \iota_1(\mathrm{ht}_{x_1}).$$

Step 3: the dual lattices. Fix i and a label s; ι_i identifies $\mathrm{Li}_s(G_{\mathcal{X}_i,\Omega_i}) \otimes_{\Omega_i} \Omega$ with $\mathrm{Li}_s(G_{\mathcal{X},\Omega})$, the Hodge pieces being defined by the $\mathcal{O}_E$-action. By (Go3)(2) at stage i, the image $h_s^{(i)}$ of ht_{x_i} in $\mathrm{Hom}_{\Omega_i}(\mathrm{Li}_s^{(i)}, \Omega_i)$ is free of rank $\mathrm{rk}(s) = \dim \mathrm{Li}_s^{(i)}$, x_i lying in the fibre over ℓ; let $h_1, \dots, h_r$ be a basis, a basis of the dual space as well, and $e_1, \dots, e_r$ the dual basis of $\mathrm{Li}_s^{(i)}$. Then $\mathrm{HT}'_{s,x_i} = \bigoplus_k \mathcal{O}_{x_i} e_k$. Restriction to Li_s commutes with ι_i, so by Step 2 the image of ht_W in $\mathrm{Hom}_{\Omega}(\mathrm{Li}_s, \Omega)$ is $\bigoplus_k \mathcal{O}_W \iota_i(h_k)$, and therefore

$$\mathrm{HT}'_{s,W} = \{e \in \mathrm{Li}_s :\ \iota_i(h_k)(e) \in \mathcal{O}_W\ \forall k\} = \bigoplus_k \mathcal{O}_W \iota_i(e_k) = \mathcal{O}_W \cdot \iota_i(\mathrm{HT}'_{s,x_i}).$$

For $s = (\tau, +)$: the ideal $\sigma_s(\mathfrak{d}_E)\mathcal{O}_F$ is invertible, so its localisation at λ is generated by one element $\delta \in F^\times$; since $\mathcal{O}_{F,(\lambda)}$ maps into $\mathcal{O}_{x_i}$ and into $\mathcal{O}_W$, $\mathrm{HT}_{s,x_i}(N) = \delta^{-1}\mathrm{HT}'_{s,x_i}$ and $\mathrm{HT}_{s,W}(N) = \delta^{-1}\mathrm{HT}'_{s,W}$; for $s = (\tau, -)$, $\mathrm{HT} = \mathrm{HT}'$. In both cases $\mathcal{O}_W \cdot \iota_i(\mathrm{HT}_{s,x_i}(N)) = \mathrm{HT}_{s,W}(N)$.

Step 4: conclusion. As x_0 lies over $\ell \in S_{B_0}$ and x_1 over $\ell \in S_B$, with $R(\ell) = 4$, (Fh5) gives $V_{s,x_0}^{S_{B_0}}(\mathcal{X}_0) = \mathrm{HT}_{s,x_0}(N)$ and $V_{s,x_1}^{S_B}(\mathcal{X}_1) = \mathrm{HT}_{s,x_1}(N)$. By Step 3 both $\mathcal{O}_W$-modules of the statement equal the Hodge–Tate lattice $\mathrm{HT}_{s,W}(N)$ of the cover $\mathcal{X}$ at W, hence each other. (The argument uses neither that W has codimension one nor good reduction at W.) □

Lemma 35.6. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p_0) with Galois Frobenius labels (Fr), a geometric realisation and a good origin immersive at level four over an integral model $\mathcal{Y}$ of a unitary Shimura datum $\mathfrak{u} = (E, \mathsf{H}, \dots)$, $\Phi = \Phi_{\mathsf{H}}$, E Galois over $\mathbb{Q}$ with abelian Galois group. Let $c \in \mathrm{Cp}$ and $2 \le B_0 \le B$. Every joint model $\mathcal{X}$ of the stages (B_0, c) and (B, c) satisfies (Sk4) with $S = S(B_0, B)$, the stage-change classes being formed with the Frobenius labelling of $\mathcal{T}$.*

Proof. Let $a \in \Phi$, $1 \le k \le n$, $Q \in \mathcal{R}$ homogeneous of degree $D + 1 - k$ and $p \in S(B_0, B)$; let h be as in Proposition 35.8 for (p, k, Q), and write $\Lambda = \Phi \times \{+, -\}$ (a finite set) and $q = p^k \ge 2$.

Step 1: the recursion has the orbit sum as its value. By (Fz2), $h(\sigma) = q^{-1}\big(\lambda(\varphi_p\sigma) + h(\varphi_p\sigma)\big)$ for every $\sigma \in \Lambda$. By induction on $M \ge 0$,

$$h(\sigma) = \sum_{m=0}^{M-1} q^{-(m+1)} \lambda\big(\varphi_p^{m+1}\sigma\big) + q^{-M} h\big(\varphi_p^{M}\sigma\big)\ :$$

the case $M = 0$ is trivial, and inserting the recursion for $h(\varphi_p^M\sigma)$ into the last term gives the case $M + 1$. Since Λ is finite, $|h|$ and $|\lambda|$ are bounded on Λ; as $q^{-M} \to 0$, the series converges absolutely and

$$h(\sigma) = \sum_{m=0}^{\infty} q^{-(m+1)} \lambda\big(\varphi_p^{m+1}\sigma\big) \qquad (\sigma \in \Lambda).$$

Step 2: the labels are the tower's. Since $p \in S(B_0, B)$, $p \notin \mathrm{Bd}$, so by (Fr) the tower's labelling at p is fr_{H}: $\mathrm{fr}(p, a, m)$ is the label of $a \circ \mathrm{Frob}_p^{m+1}$. Under the identification of labels with embeddings, $(a, +)$ corresponds to a and φ_p to composition with Frob_p, so $\mathrm{fr}(p, a, m) = \varphi_p^{m+1}(a, +)$.

Step 3: (Sk4). Let H be the expansion of $\mathrm{ps}_{a,k}Q$ given by (Fz1). By (Fz1), Steps 1 and 2, and grouping the absolutely convergent series by the finitely many labels,

$$\begin{aligned}\mathrm{vc}_p(H) &= \sum_{m\ge 0} p^{-k(m+1)}\lambda\big(\mathrm{fr}(p,a,m)\big) - \frac{\lambda(a,+)}{p-1}\\ &= \sum_{\sigma\in\Lambda}\Big(\sum_{m\ge 0:\ \mathrm{fr}(p,a,m)=\sigma} p^{-k(m+1)} - \frac{[\sigma=(a,+)]}{p-1}\Big)\lambda(\sigma).\end{aligned}$$

By the definition of λ and the linearity of $\mathrm{gi}(\mathcal{X};\cdot)$ the right side is $\delta_{\mathcal{X}}\,\mathrm{gi}\big(\mathcal{X};\mathrm{br}_p(a,k)Q\big)$, with $\mathrm{br}_p(a,k) = k\sum_\sigma\big(\sum_{m:\mathrm{fr}(p,a,m)=\sigma} p^{-k(m+1)} - [\sigma=(a,+)]/(p-1)\big)\mathrm{ps}^\sigma_{k-1}$. This holds for every $p\in S(B_0,B)$, which is (Sk4). □

Definition 35.7. Let $\mathcal{X}\subseteq\mathbb{P}$ be a joint model of the stages (B_0,c) and (B,c) of a root tower with a good origin over an integral model $\mathcal{Y}$ of a unitary Shimura datum whose CM field E is Galois over $\mathbb{Q}$ with abelian Galois group; let Ω be its function field, regard $\mathcal{X}$ as an $\mathcal{O}_F$-scheme through $f_0\circ\mathrm{pr}_0$, and let $G_{\mathcal{X}} = (f_0\circ\mathrm{pr}_0)^*G_{\mathcal{Y}}$ with its $\mathcal{O}_E$-action and $\omega_{G_{\mathcal{X}}}$ its cotangent sheaf along the unit section (locally free). Let $F_\Omega\subseteq\Omega$ be the algebraic closure of F in Ω, a number field, the **field of constants** of $\mathcal{X}$. Let $p\in S(B_0,B)$, let λ be a prime of $\mathcal{O}_{F_\Omega}$ above p, with local ring $\mathcal{O}_{F_\Omega,\lambda}$ and completion $\widehat{F}_{\Omega,\lambda}$ of F_Ω, and put $\mathcal{X}_{(\lambda)} = \mathcal{X}\times_{\mathrm{Spec}\,\mathcal{O}_{F_\Omega}}\mathrm{Spec}\,\mathcal{O}_{F_\Omega,\lambda}$, an integral normal scheme, separated and of finite type over $\mathcal{O}_{F_\Omega,\lambda}$, with function field Ω, whose closed fibre is the part of $\mathcal{X}_{\mathbb{F}_p}$ over λ. For a prime $\mathfrak{P}$ of $\mathcal{O}_E$ above p write $\mathbb{F}_{\mathfrak{P}} = \mathcal{O}_E/\mathfrak{P}$ and $q_{\mathfrak{P}} = |\mathbb{F}_{\mathfrak{P}}|$.

An **admissible torsion system** for $(\mathcal{X},p,\lambda)$ consists of a discrete valuation ring $\mathcal{O}'$, a scheme $\mathcal{X}'$ with a morphism π, and for every prime $\mathfrak{P}\mid p$ of $\mathcal{O}_E$ an order of functions $\mathcal{A}_{\mathfrak{P}}$, cotangent maps θ_u and a basis, written $(\mathcal{O}',\mathcal{X}',\pi,(\mathcal{A}_{\mathfrak{P}},\theta))$, such that:

(Ts1) Base $\mathcal{O}'$ is the localisation of the ring of integers of a finite extension F' of F_Ω at one prime above λ; its completion is written $\widehat{F}'$. $\mathcal{O}'$ contains a primitive $(q_{\mathfrak{P}}-1)$-th root of unity for every $\mathfrak{P}\mid p$. For a label s put $\mathfrak{P}_s = \sigma_s^{-1}(\mathfrak{m}_{\mathcal{O}'})$, a prime of $\mathcal{O}_E$ above p, and let the **Teichmüller character** $\chi_s\colon\mathbb{F}_{\mathfrak{P}_s}\to\mathcal{O}'$ be the map with $\chi_s(0)=0$, $\chi_s(a)^{q_{\mathfrak{P}_s}-1}=1$ and $\chi_s(a)\equiv\sigma_s(\tilde a)\bmod\mathfrak{m}_{\mathcal{O}'}$ for $a\neq 0$ and every lift $\tilde a\in\mathcal{O}_E$ of a.

(Ts2) Model $\mathcal{X}'$ is an integral normal scheme, projective and flat over $\mathcal{O}'$, with function field Ω', and $\pi\colon\mathcal{X}'\to\mathcal{X}$ is a dominant morphism of $\mathcal{O}_{F_\Omega}$-schemes ($\mathcal{X}'$ through $\mathcal{O}'$) with $[\Omega':\Omega]<\infty$; through π and $\mathcal{O}'\subseteq\Gamma(\mathcal{X}',\mathcal{O}_{\mathcal{X}'})$ both Ω and F' are subfields of Ω', and $\Omega F'$ is their compositum. The **local degree** of π is

$$d_\pi = [\Omega':\Omega F']\,[\widehat{F}':\widehat{F}_{\Omega,\lambda}].$$

(Ts3) Function order and cotangent maps $G_{\mathcal{X}}[\mathfrak{P}]\times_\Omega\Omega'$ is the constant group scheme of the $\mathbb{F}_{\mathfrak{P}}$-vector space $\Gamma_{\mathfrak{P}} = G_{\mathcal{X}}[\mathfrak{P}](\Omega')$, of dimension $m_{\mathfrak{P}}$. $\mathcal{A}_{\mathfrak{P}}$ is a sub-$\mathcal{O}_{\mathcal{X}'}$-algebra of the constant sheaf $\mathrm{Map}(\Gamma_{\mathfrak{P}},\Omega')$, locally free as an $\mathcal{O}_{\mathcal{X}'}$-module, with

$$p^N\,\mathrm{Map}(\Gamma_{\mathfrak{P}},\mathcal{O}_{\mathcal{X}'})\subseteq\mathcal{A}_{\mathfrak{P}}\subseteq\mathrm{Map}(\Gamma_{\mathfrak{P}},\mathcal{O}_{\mathcal{X}'})\quad\text{for some } N\ge 0.$$

Let $\mathcal{I}_{\mathfrak{P}} = \{g\in\mathcal{A}_{\mathfrak{P}}:\ g(0)=0\}$. For every label u with $\mathfrak{P}_u=\mathfrak{P}$ let $\omega_u\subseteq\pi^*\omega_{G_{\mathcal{X}}}$ be the subsheaf of local sections v with $a\cdot v=\sigma_u(a)v$ for all $a\in\mathcal{O}_E$; then $\theta_u\colon\mathcal{I}_{\mathfrak{P}}\to\omega_u/p\,\omega_u$ is a surjective $\mathcal{O}_{\mathcal{X}'}$-linear map (the **cotangent map** at u).

(Ts4) Flattened coordinate flag $b_1,\dots,b_{m_{\mathfrak{P}}}$ is a basis of $\Gamma_{\mathfrak{P}}$ such that for $1\le i\le m_{\mathfrak{P}}$, with $\Gamma_{\mathfrak{P},i} = \mathbb{F}_{\mathfrak{P}}b_1\oplus\cdots\oplus\mathbb{F}_{\mathfrak{P}}b_i$ ($\Gamma_{\mathfrak{P},0}=0$), the image $\mathcal{A}_{\mathfrak{P},i}\subseteq\mathrm{Map}(\Gamma_{\mathfrak{P},i},\Omega')$ of $\mathcal{A}_{\mathfrak{P}}$ under restriction of functions to $\Gamma_{\mathfrak{P},i}$ is a locally free $\mathcal{O}_{\mathcal{X}'}$-module. Put $\mathcal{I}_{\mathfrak{P},i} = \{g\in\mathcal{A}_{\mathfrak{P}}:\ g|_{\Gamma_{\mathfrak{P},i}}=0\}$ for $0\le i\le m_{\mathfrak{P}}$ (so $\mathcal{I}_{\mathfrak{P},0}=\mathcal{I}_{\mathfrak{P}}$).

(Ts5) Invertible eigen-coordinate ideals For every label s and $1 \le i \le m_{\mathfrak{P}_s}$, with $\xi_{s,i} \in \mathrm{Map}(\Gamma_{\mathfrak{P}_s,i}, \Omega')$ the function $\sum_j x_j b_j \mapsto \chi_s(x_i)$, the $\mathcal{O}_{\mathcal{X}'}$-submodule of the constant sheaf Ω' of local sections c with $c\,\xi_{s,i} \in \mathcal{A}_{\mathfrak{P}_s,i}$ is invertible.

(Ts6) Frobenius step facts For every label s, with $\mathfrak{P} = \mathfrak{P}_s$, $t = \varphi_p s$ the label with $\sigma_t = \sigma_s \circ \mathrm{Frob}_p$ (then $\mathfrak{P}_t = \mathfrak{P}$) and every $1 \le i \le m_{\mathfrak{P}}$, as subsheaves of $\omega_t/p\,\omega_t$, resp. of the constant sheaf Ω':

(onto) $\theta_t(\mathcal{I}_{\mathfrak{P},i-1}) = \theta_t(\{g \in \mathcal{A}_{\mathfrak{P}} :\ g|_{\Gamma_{\mathfrak{P},i}} \in \Omega'\,\xi_{t,i}\})$;

(inj) the local sections c of Ω' for which some local section g of $\mathcal{A}_{\mathfrak{P}}$ has $g|_{\Gamma_{\mathfrak{P},i}} = c\,\xi_{t,i}$ and $\theta_t(g) = 0$ form the $\mathcal{O}_{\mathcal{X}'}$-submodule generated by the p-th powers c'^p of the local sections c' of the module of (Ts5) for (s,i).

A tuple with (Ts1)–(Ts5), not necessarily (Ts6), is a **pre-admissible torsion system**. The **eigen-coordinate divisor** $\mathrm{Ec}_{s,i}$ is the Cartier divisor on $\mathcal{X}'$ with $\mathcal{O}_{\mathcal{X}'}(\mathrm{Ec}_{s,i})$ equal to the submodule of (Ts5), as subsheaves of Ω'.

Vertical intersection numbers. For Cartier divisors $D_1, \dots, D_k$ on $\mathcal{X}'$ ($k \ge 1$) supported in the special fibre $\mathcal{X}'_{\mathfrak{s}}$ of $\mathcal{X}' \to \mathrm{Spec}\,\mathcal{O}'$, and $e_1, \dots, e_{D+1-k} \in \mathbb{Z}^K$ ($D+1 = \dim \mathcal{X}$), put

$$(D_1 \cdots D_k \cdot M) = \deg\Big(c_1(\mathcal{O}(D_2)) \cdots c_1(\mathcal{O}(D_k)) \\ c_1(\pi^*\mathcal{O}(e_1)) \cdots c_1(\pi^*\mathcal{O}(e_{D+1-k})) \cap [D_1]\Big), \qquad M = \prod_l c_1(\mathcal{O}(e_l)),$$

the degree over $\mathbb{F}_p$ of a 0-cycle class on the proper $\mathbb{F}_p$-scheme $\mathcal{X}'_{\mathfrak{s}}$, $[D_1]$ the Weil divisor of D_1; it is extended linearly to real combinations M of such products and to the Cartier divisor $\mathrm{div}(p)$ in place of D_1, and $D^k \cdot M$ means $D_1 = \cdots = D_k = D$.

Proposition 35.8. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p_0) (p_0 the first signature entry) with a geometric realisation and a good origin immersive at level four over an integral model $\mathcal{Y}$ of a unitary Shimura datum $\mathfrak{u} = (E, \mathsf{H}, \dots)$, $\Phi = \Phi_{\mathsf{H}}$, with E Galois over $\mathbb{Q}$ with abelian Galois group. Let $c \in \mathrm{Cp}$, $2 \le B_0 \le B$, $\mathcal{X}$ a joint model of the stages (B_0, c) and (B, c), and $p \in S(B_0, B)$. Let φ_p be the permutation of the labels $\Phi \times \{+,-\}$ that corresponds, under the identification of labels with $\mathrm{Hom}(E, \mathbb{C})$, to $\tau \mapsto \tau \circ \mathrm{Frob}_p$. Let $1 \le k \le n$, $Q \in \mathcal{R}$ homogeneous of degree $D+1-k$, and put*

$$\lambda(\sigma) = \delta_{\mathcal{X}}\,\mathrm{gi}(\mathcal{X}; k\,\mathrm{ps}^{\sigma}_{k-1}Q) \qquad (\sigma \in \Phi \times \{+,-\}).$$

Then there is a function $h\colon \Phi \times \{+,-\} \to \mathbb{R}$ such that

(Fz1) *for every $a \in \Phi$ the class $\mathrm{ps}_{a,k}Q$ has an expansion H with $\mathrm{vc}_p(H) = h(a,+) - \lambda(a,+)/(p-1)$;*

(Fz2) *$p^k h(\sigma) = h(\varphi_p\sigma) + \lambda(\varphi_p\sigma)$ for every label σ.*

Proof. Fix $p \in S(B_0, B)$, $1 \le k \le n$ and Q. Let $\mathcal{X} \subseteq \mathbb{P} = \prod_{k' \in K} \mathbb{P}(\mathbb{Z}^{I_{k'}})$, write $x_r = c_1(\mathcal{O}(e_r))$, $x'_r = c_1(\mathcal{O}(e'_r))$ on $\mathcal{X}$, and for $F \in \mathcal{R}$ let $F(x)$ be F evaluated at the x_r, a real combination of products of $\deg F$ classes $c_1(\mathcal{O}(e))$, $e \in \mathbb{Z}^K$. The scheme $\mathcal{X}$ is an $\mathcal{O}_{F_\Omega}$-scheme (Definition 35.7), and $\mathcal{X}_{\mathbb{F}_p} = \bigsqcup_{\mathfrak{l}|p} \mathcal{X} \times_{\mathcal{O}_{F_\Omega}} \mathcal{O}_{F_\Omega}/\mathfrak{l}^{e(\mathfrak{l})}$ (Chinese remainder theorem for $p\mathcal{O}_{F_\Omega} = \prod \mathfrak{l}^{e(\mathfrak{l})}$), so every irreducible component of $\mathcal{X}_{\mathbb{F}_p}$ lies over exactly one prime $\mathfrak{l} \mid p$ of $\mathcal{O}_{F_\Omega}$; for a cycle Z supported in $\mathcal{X}_{\mathbb{F}_p}$ let $Z_{\mathfrak{l}}$ be its part over $\mathfrak{l}$, so $Z = \sum_{\mathfrak{l}} Z_{\mathfrak{l}}$ and $\mathrm{Dv}_{r,p} = \sum_{\mathfrak{l}} \mathrm{Dv}_{r,\mathfrak{l}}$.

Step 1: the function h. For every prime $\mathfrak{l} \mid p$ of $\mathcal{O}_{F_\Omega}$ fix an admissible torsion system $\mathfrak{T}_\mathfrak{l} = (\mathcal{O}'_\mathfrak{l}, \mathcal{X}'_\mathfrak{l}, \pi_\mathfrak{l}, (\mathcal{A}^\mathfrak{l}_\mathfrak{P}, \theta^\mathfrak{l}))$ for $(\mathcal{X}, p, \mathfrak{l})$ with the property of Proposition 35.11; write $d_\mathfrak{l} = d_{\pi_\mathfrak{l}}$, $\mathrm{Ec}^\mathfrak{l}_{s,i}$ for its eigen-coordinate divisors and $m^\mathfrak{l}_s = m_{\mathfrak{P}_s}$ (formed in $\mathfrak{T}_\mathfrak{l}$). Put, for every label s,

$$h(s) = \sum_{\mathfrak{l}\mid p} \frac{(-1)^k}{d_\mathfrak{l}} \sum_{i=1}^{m^\mathfrak{l}_s} (\mathrm{Ec}^\mathfrak{l}_{s,i})^k \cdot Q(x),$$

the vertical intersection numbers being formed on $\mathcal{X}'_\mathfrak{l}$.

Step 2: numbers on the fibre are generic degrees. Let $F \in \mathcal{R}$ be homogeneous of degree D. We show

$$\textstyle\sum_{\mathfrak{l}\mid p} d_\mathfrak{l}^{-1} \big(\mathrm{div}(p) \cdot F(x)\big)_{\mathcal{X}'_\mathfrak{l}} = \delta_\mathcal{X}\, \mathrm{gi}(\mathcal{X}; F). \qquad (*)$$

Fix $\mathfrak{l}$. On $\mathcal{X}_{(\mathfrak{l})} = \mathcal{X} \times_{\mathcal{O}_{F_\Omega}} \mathcal{O}_{F_\Omega,\mathfrak{l}}$ the Cartier divisor $\mathrm{div}(p)$ is supported in the closed fibre, which is the part of $\mathcal{X}_{\mathbb{F}_p}$ over $\mathfrak{l}$ (p is a unit in $\mathcal{O}_{F_\Omega,\mathfrak{l}}$ away from $\mathfrak{l}$); its Weil divisor is $[\mathrm{div}_\mathcal{X}(p)]_\mathfrak{l}$, and its pull-back along the factorisation of $\pi_\mathfrak{l}$ through $\mathcal{X}_{(\mathfrak{l})}$ is $\mathrm{div}(p)$ on $\mathcal{X}'_\mathfrak{l}$. As $F(x)$ is a real combination of products of D classes $c_1(\mathcal{O}(e))$, Lemma 35.15 (3) for $\mathfrak{T}_\mathfrak{l}$, whose local degree is $d_\mathfrak{l}$, gives

$$\big(\mathrm{div}(p) \cdot F(x)\big)_{\mathcal{X}'_\mathfrak{l}} = d_\mathfrak{l} \deg\big(F(x) \cap [\mathrm{div}_\mathcal{X}(p)]_\mathfrak{l}\big).$$

Summing over $\mathfrak{l}$ gives $\deg(F(x) \cap [\mathrm{div}_\mathcal{X}(p)])$, and $[\mathrm{div}_\mathcal{X}(p)]$ is the fundamental cycle of the scheme $\mathcal{X}_{\mathbb{F}_p}$ (Fulton, Lemma 1.7.2). By conservation of number for the integral scheme $\mathcal{X}$, flat and projective over $\operatorname{Spec}\mathbb{Z}$ (the intersection numbers of line bundles on a fibre are coefficients of the Euler characteristic polynomial of that fibre, Fulton, Example 18.3.6, and these polynomials agree on all fibres of a flat projective morphism, EGA III 7.9.4), $\deg(F(x) \cap [\mathcal{X}_{\mathbb{F}_p}]) = \deg(F(x) \cap [\mathcal{X}_\mathbb{Q}])$, which is $\delta_\mathcal{X}\mathrm{gi}(\mathcal{X}; F)$ since F does not involve x_*. Applied to $F = k\,\mathrm{ps}^s_{k-1}Q$ (homogeneous of degree $(k-1) + (D+1-k) = D$), $(*)$ reads

$$\lambda(s) = \sum_{\mathfrak{l}\mid p} \frac{k}{d_\mathfrak{l}} \big(\mathrm{div}(p) \cdot \mathrm{ps}^s_{k-1}(x)\, Q(x)\big)_{\mathcal{X}'_\mathfrak{l}} \qquad (s \in \Phi \times \{+,-\}).$$

Step 3: (Fz2). Let s be a label. By Proposition 35.9 for $\mathfrak{T}_\mathfrak{l}$, the label s, k and $M = Q(x)$, and since $m^\mathfrak{l}_{\varphi_p s} = m^\mathfrak{l}_s$,

$$\sum_{i=1}^{m^\mathfrak{l}_s} \Big(p^k (\mathrm{Ec}^\mathfrak{l}_{s,i})^k - (\mathrm{Ec}^\mathfrak{l}_{\varphi_p s,i})^k\Big) \cdot Q(x) = (-1)^k\, k\, \big(\mathrm{div}(p) \cdot \mathrm{ps}^{\varphi_p s}_{k-1}(x) Q(x)\big)_{\mathcal{X}'_\mathfrak{l}}.$$

Multiplying by $(-1)^k/d_\mathfrak{l}$ and summing over $\mathfrak{l}$ gives $p^k h(s) - h(\varphi_p s)$ on the left and, by Step 2, $\lambda(\varphi_p s)$ on the right. This is (Fz2).

Step 4: an expansion of $\mathrm{ps}_{a,k}Q$. Let $a \in \Phi$. In the polynomial ring in variables x_r, x'_r ($r \in \mathrm{Rt}$) choose polynomials G_r, homogeneous of degree $D-k$, with $Q(x) - Q(x') = \sum_r (x_r - x'_r) G_r$: for a monomial $x_{r_1} \cdots x_{r_l}$ take the telescoping identity $x_{r_1} \cdots x_{r_l} - x'_{r_1} \cdots x'_{r_l} = \sum_{j=1}^{l} x_{r_1} \cdots x_{r_{j-1}} (x_{r_j} - x'_{r_j}) x'_{r_{j+1}} \cdots x'_{r_l}$ and extend linearly (if $k = D+1$, Q is constant and $G_r = 0$). Put

$$H_r = [r \in \{a\} \times \{1, \dots, n\}] \sum_{j=0}^{k-1} x_r^j x_r'^{k-1-j}\, Q(x) + \mathrm{ps}_{a,k}(x')\, G_r,$$

homogeneous of degree D. Then $\sum_r (x_r - x'_r) H_r = (\mathrm{ps}_{a,k}(x) - \mathrm{ps}_{a,k}(x'))Q(x) + \mathrm{ps}_{a,k}(x')(Q(x) - Q(x')) = (\mathrm{ps}_{a,k}Q)(x) - (\mathrm{ps}_{a,k}Q)(x')$, so $H = (H_r)$ is an expansion of $\mathrm{ps}_{a,k}Q$.

Step 5: (Fz1). By definition $\mathrm{vc}_p(H) = \sum_r \deg(H_r(x,x') \cap [\mathrm{Dv}_{r,p}])$, the classes acting on the components of $\mathrm{Dv}_{r,p}$. The second summand of H_r contributes nothing: writing $[\mathrm{Dv}_{r,p}] = \sum_W n_{r,W}[W]$ over the irreducible components W of $\mathcal{X}_{\mathbb{F}_p}$, each $\deg(\mathrm{ps}_{a,k}(x')G_r(x,x') \cap [W])$ vanishes by Proposition 35.64, because $G_r(x,x')$ is a real combination of products of $D-k$ classes $c_1(\mathcal{O}(e))$ with $e \in \{e_s, e_s'\} \subseteq \mathbb{Z}^K$ (for $k \le D$; for $k = D+1$, $G_r = 0$). The first summand gives, splitting $\mathrm{Dv}_{(a,i),p} = \sum_{\mathfrak{l}} \mathrm{Dv}_{(a,i),\mathfrak{l}}$ and applying Proposition 35.11 for $\mathfrak{T}_{\mathfrak{l}}$ with $M = Q(x)$,

$$\mathrm{vc}_p(H) = \sum_{\mathfrak{l}\mid p} \frac{1}{d_{\mathfrak{l}}}\Big((-1)^k \sum_{i=1}^{m^{\mathfrak{l}}_{(a,+)}} \big(\mathrm{Ec}^{\mathfrak{l}}_{(a,+),i}\big)^k \cdot Q(x) - \frac{k}{p-1}\big(\mathrm{div}(p)\cdot \mathrm{ps}^{(a,+)}_{k-1}(x)Q(x)\big)_{\mathcal{X}'_{\mathfrak{l}}}\Big) = h(a,+) - \frac{\lambda(a,+)}{p-1}$$

by Step 1 and Step 2. This is (Fz1). □

Proposition 35.9. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p_0) with a geometric realisation and a good origin immersive at level four over an integral model $\mathcal{Y}$ of a unitary Shimura datum $\mathfrak{u} = (E, \mathsf{H}, \dots)$, $\Phi = \Phi_{\mathsf{H}}$, with E Galois over $\mathbb{Q}$ with abelian Galois group. Let $c \in \mathrm{Cp}$, $2 \le B_0 \le B$, $\mathcal{X} \subseteq \mathbb{P} = \prod_{k\in K}\mathbb{P}(\mathbb{Z}^{I_k})$ a joint model of the stages (B_0, c) and (B, c), $p \in S(B_0, B)$, λ a prime above p of the ring of integers of the field of constants F_Ω of $\mathcal{X}$ (Definition 35.7), and $(\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_{\mathfrak{P}}, \theta))$ an admissible torsion system for $(\mathcal{X}, p, \lambda)$. Let φ_p be the permutation of the labels corresponding to $\tau \mapsto \tau \circ \mathrm{Frob}_p$, and for $F \in \mathcal{R}$ let $F(x)$ be F evaluated at the first-root classes $x_r = c_1(\mathcal{O}(e_r))$. Then for every label s, every $1 \le k \le n$ and every real combination M of products of $D+1-k$ classes $c_1(\mathcal{O}(e))$, $e \in \mathbb{Z}^K$, with $m = m_{\mathfrak{P}_s}$:*

$$\sum_{i=1}^{m}\Big(\mathrm{Ec}^k_{\varphi_p s,i}\cdot M - p^k\,\mathrm{Ec}^k_{s,i}\cdot M\Big) = (-1)^{k-1}\,k\,\Big(\mathrm{div}(p)\cdot \mathrm{ps}^{\varphi_p s}_{k-1}(x)\,M\Big).$$

Proof. Fix s, k, M; put $\mathfrak{P} = \mathfrak{P}_s$, $m = m_{\mathfrak{P}}$, $t = \varphi_p s$ (so $\mathfrak{P}_t = \mathfrak{P}$), and write $\mathcal{A} = \mathcal{A}_{\mathfrak{P}}$, $\mathcal{A}_i = \mathcal{A}_{\mathfrak{P},i}$, $\mathcal{I}_i = \mathcal{I}_{\mathfrak{P},i}$, $\Gamma_i = \Gamma_{\mathfrak{P},i}$. The prime p is odd and unramified in E, since $p \notin \mathrm{Bd}$. All sheaves below are quasi-coherent subsheaves of quasi-coherent sheaves (or of the constant sheaf Ω'), so inclusions and equalities between them are checked on stalks, and the stalk of an image sheaf is the image of the stalks.

Step 0: the sheaf ω_t. As $p \nmid \mathrm{disc}(E)$ and $F \supseteq \sigma_u(E)$ for all labels, $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{O}'$ is finite étale over $\mathcal{O}'$ with the $[E:\mathbb{Q}]$ distinct sections $\sigma_u \otimes 1$, hence $\cong \prod_u \mathcal{O}'$, with idempotents e_u, $(\sigma_{u'} \otimes 1)(e_u) = \delta_{uu'}$. So $\pi^*\omega_{G_{\mathcal{X}}} = \bigoplus_u e_u \pi^*\omega_{G_{\mathcal{X}}}$. On a section v of ω_t the ring $\mathcal{O}_E \otimes \mathcal{O}'$ acts through $\sigma_t \otimes 1$, so $e_t v = v$; conversely $\mathcal{O}_E$ acts on $e_t\pi^*\omega_{G_{\mathcal{X}}}$ through σ_t. Hence $\omega_t = e_t\pi^*\omega_{G_{\mathcal{X}}}$, a direct summand of a locally free sheaf, locally free.

Step 1: the chain of step lattices. For $0 \le i \le m$ let $U_i \subseteq \omega_t$ be the preimage of $\theta_t(\mathcal{I}_i) \subseteq \omega_t/p\,\omega_t$. As $\mathcal{I}_i \subseteq \mathcal{I}_{i-1}$, $U_i \subseteq U_{i-1}$. By (Ts3), θ_t is onto, so $U_0 = \omega_t$; as $\Gamma_m = \Gamma_{\mathfrak{P}}$, $\mathcal{I}_m = 0$ and $U_m = p\,\omega_t$. Since every U_i contains $p\,\omega_t$, $U_{i-1}/U_i \cong \theta_t(\mathcal{I}_{i-1})/\theta_t(\mathcal{I}_i)$.

Step 2: the graded pieces. Fix $1 \le i \le m$ and a point x of $\mathcal{X}'$, $R = \mathcal{O}_{\mathcal{X}',x}$; we work with stalks at x and keep the notation. By (Ts5), $\mathcal{O}(\mathrm{Ec}_{t,i})_x = R\,\epsilon_t$ and $\mathcal{O}(\mathrm{Ec}_{s,i})_x = R\,\epsilon_s$ with $\epsilon_t, \epsilon_s \in \Omega'^{\times}$. Let $c \in R\epsilon_t$. Then $c\,\xi_{t,i} \in \mathcal{A}_i$, so $c\,\xi_{t,i} = g|_{\Gamma_i}$ for some $g \in \mathcal{A}$; as $\xi_{t,i}$ vanishes on Γ_{i-1} ($x_i = 0$ there and $\chi_t(0) = 0$), $g \in \mathcal{I}_{i-1}$. Two such g differ by an element of $\mathcal{I}_i$. Hence

$$\psi\colon R\,\epsilon_t \to \theta_t(\mathcal{I}_{i-1})/\theta_t(\mathcal{I}_i), \qquad c \mapsto \theta_t(g) \bmod \theta_t(\mathcal{I}_i) \quad (g \in \mathcal{A},\ g|_{\Gamma_i} = c\,\xi_{t,i}),$$

is a well-defined R-linear map. It is onto: by (Ts6)(onto) every element of $\theta_t(\mathcal{I}_{i-1})$ is $\theta_t(g)$ with $g \in \mathcal{A}$ and $g|_{\Gamma_i} = c\,\xi_{t,i}$ for some $c \in \Omega'$; then $c\,\xi_{t,i} \in \mathcal{A}_i$, i.e. $c \in R\epsilon_t$, and $\theta_t(g) \equiv \psi(c)$. Its kernel:

$\psi(c)=0$ if and only if $\theta_t(g)=\theta_t(h)$ for some $h\in\mathcal{I}_i$, i.e. if and only if some $g'=g-h\in\mathcal{A}$ has $g'|_{\Gamma_i}=c\,\xi_{t,i}$ and $\theta_t(g')=0$. By (Ts6)(inj) these c form the R-module generated by the $(r\epsilon_s)^p=r^p\epsilon_s^p$, $r\in R$, i.e. $R\,\epsilon_s^p$. So $R\epsilon_s^p\subseteq R\epsilon_t$, $\delta=\epsilon_s^p/\epsilon_t\in R\smallsetminus\{0\}$, and

$$U_{i-1}/U_i\cong R\epsilon_t/R\delta\epsilon_t\cong R/\delta R.$$

Globally: $\mathcal{O}(p\,\mathrm{Ec}_{s,i})\subseteq\mathcal{O}(\mathrm{Ec}_{t,i})$, so $\Delta_{s,i}:=\mathrm{Ec}_{t,i}-p\,\mathrm{Ec}_{s,i}$ is an effective Cartier divisor (locally $\mathrm{div}(\delta)$), supported in $\mathcal{X}'_{\mathfrak{s}}$ (as the Ec are), and the maps ψ glue (they are defined on sections over any open set by the same formula) to

$$U_{i-1}/U_i\cong\mathcal{O}(\mathrm{Ec}_{t,i})/\mathcal{O}(p\,\mathrm{Ec}_{s,i})\cong\mathcal{O}(\mathrm{Ec}_{t,i})\otimes\mathcal{O}_{\Delta_{s,i}}.$$

Each U_i is locally free, by induction from $U_0=\omega_t$ (Step 0): if U_{i-1} is free of rank n' near x and $U_{i-1}\to R/\delta R$ is onto, some basis vector maps to a unit of the local ring $R/\delta R$, and after a change of basis the map is the first coordinate, with kernel $\delta R\oplus R^{n'-1}$. So the $U_i\subseteq U_{i-1}$ are vertical pairs in the sense of Lemma 35.10 (its hypotheses on $\mathcal{O}'$, $\mathcal{X}'$ hold by (Ts1), (Ts2)).

Step 3: the de Rham flag as a second chain. The image of $\mathcal{X}'$ in $\mathcal{X}_0=\mathcal{X}_{B_0,c}$ lies over primes not in S_{B_0} ($p\notin S_{B_0}$), where $\mathcal{V}_t$ has stalks $\mathrm{dR}_{t,x}=e_t\mathrm{Lie}_x$; so $\pi^*\mathrm{pr}_0^*\mathcal{V}_t=e_t\mathrm{Lie}(G_{\mathcal{X}'})$, whose dual is ω_t (the idempotents are orthogonal for the $\mathcal{O}_E$-balanced pairing). The integral flag of (Go3)(3) pulls back to a flag of subbundles of $e_t\mathrm{Lie}$ with step quotients Q_j, $c_1(Q_{\mathrm{st}(r)})=\mathrm{sg}(r)x_r$ for $\mathrm{s}(r)=t$, by (Fh3) and $\mathcal{L}_r\cong\mathcal{O}(e_r)$; the annihilators give a flag $0=F^0\subset\cdots\subset F^d=\omega_t$ ($d=\mathrm{rk}(t)$) of subbundles with line bundle quotients $N_j=F^j/F^{j-1}$, $\{c_1(N_j)\}=\{-\mathrm{sg}(r)x_r:\mathrm{s}(r)=t\}$. Put $U'_j=p\,\omega_t+F^j$, locally free ($\omega_t=F^j\oplus C$ locally, $U'_j=F^j\oplus pC$). Since ω_t/F^j is torsion free, $F^j\cap p\,\omega_t=pF^j$, so $U'_j/U'_{j-1}\cong F^j/(F^{j-1}+pF^j)\cong N_j\otimes\mathcal{O}_{\mathrm{div}(p)}$, $\mathrm{div}(p)$ the effective Cartier divisor of the non-zero-divisor p, supported in $\mathcal{X}'_{\mathfrak{s}}$; $U'_0=p\,\omega_t$, $U'_d=\omega_t$.

Step 4: comparison. By Lemma 35.10 (Vc1) applied to both chains from $p\,\omega_t$ to ω_t,

$$\sum_{i=1}^{m}\mathrm{ch}_k(U_i\subseteq U_{i-1})\cdot M=\mathrm{ch}_k(p\,\omega_t\subseteq\omega_t)\cdot M=\sum_{j=1}^{d}\mathrm{ch}_k(U'_{j-1}\subseteq U'_j)\cdot M.$$

By (Vc2) with $N=\mathcal{O}(\mathrm{Ec}_{t,i})$, $\Delta=\Delta_{s,i}$, $N(-\Delta)=\mathcal{O}(p\,\mathrm{Ec}_{s,i})$, and the symmetry and additivity of vertical intersection numbers in each divisor (the polynomial identity $\sum_{a=0}^{k-1}(A-B)A^aB^{k-1-a}=A^k-B^k$ with $A-B=\Delta_{s,i}$),

$$k!\,\mathrm{ch}_k(U_i\subseteq U_{i-1})\cdot M=\sum_{a=0}^{k-1}\big(\Delta_{s,i}\cdot\mathrm{Ec}_{t,i}^a\,(p\,\mathrm{Ec}_{s,i})^{k-1-a}\cdot M\big)=\mathrm{Ec}_{t,i}^k\cdot M-p^k\,\mathrm{Ec}_{s,i}^k\cdot M.$$

By (Vc2) with $N=N_j$, $\Delta=\mathrm{div}(p)$ and $N(-\mathrm{div}(p))\cong N_j$ ($\mathcal{O}(-\mathrm{div}(p))=p\,\mathcal{O}\cong\mathcal{O}$),

$$k!\,\mathrm{ch}_k(U'_{j-1}\subseteq U'_j)\cdot M=k\,\deg(c_1(N_j)^{k-1}M\cap[\mathrm{div}(p)]),\qquad\sum_{j=1}^{d}c_1(N_j)^{k-1}=(-1)^{k-1}\mathrm{ps}_{k-1}^t(x).$$

Multiplying the comparison by $k!$ gives $\sum_i(\mathrm{Ec}_{\varphi_p s,i}^k\cdot M-p^k\mathrm{Ec}_{s,i}^k\cdot M)=(-1)^{k-1}k\,(\mathrm{div}(p)\cdot\mathrm{ps}_{k-1}^{\varphi_p s}(x)\,M)$. □

Lemma 35.10. *Let $\mathcal{O}'$ be a discrete valuation ring with fraction field of characteristic 0 and finite residue field of characteristic p, and let $\mathcal{X}'$ be an integral scheme, projective and flat over $\mathcal{O}'$, whose special fibre $\mathcal{X}'_{\mathfrak{s}}$ has dimension D. A **vertical pair** is a pair $W\subseteq V$ of locally free $\mathcal{O}_{\mathcal{X}'}$-modules of the same finite rank such that V/W is supported in $\mathcal{X}'_{\mathfrak{s}}$. For $1\le k\le D+1$ let $\mathcal{M}_k$ be the real vector space of real combinations M of products $c_1(L_1)\cdots c_1(L_{D+1-k})$ of first Chern classes of line bundles on $\mathcal{X}'$, acting on the Chow groups of the closed subschemes of $\mathcal{X}'$ supported in $\mathcal{X}'_{\mathfrak{s}}$. Then to every vertical pair $W\subseteq V$, every such k and every $M\in\mathcal{M}_k$ there is attached a real number $\mathrm{ch}_k(W\subseteq V)\cdot M$, linear in M, such that:*

(Vc1) Additivity *for vertical pairs $W' \subseteq W$ and $W \subseteq V$,*

$$\mathrm{ch}_k(W' \subseteq V) \cdot M = \mathrm{ch}_k(W' \subseteq W) \cdot M + \mathrm{ch}_k(W \subseteq V) \cdot M;$$

(Vc2) Elementary modifications *if $V/W \cong N \otimes_{\mathcal{O}_{\mathcal{X}'}} \mathcal{O}_\Delta$ for a line bundle N on $\mathcal{X}'$ and an effective Cartier divisor Δ on $\mathcal{X}'$ supported in $\mathcal{X}'_{\mathfrak{s}}$, then*

$$k!\,\mathrm{ch}_k(W \subseteq V) \cdot M = \sum_{a=0}^{k-1} \deg\Big(c_1(N)^a\, c_1\big(N(-\Delta)\big)^{k-1-a}\, M \cap [\Delta]\Big),$$

where $[\Delta]$ is the Weil divisor of Δ and deg *is the degree over $\mathbb{F}_p$ of a 0-cycle class on the proper $\mathbb{F}_p$-scheme $\mathcal{X}'_{\mathfrak{s}}$;*

(Vc3) Exact sequences *for vertical pairs $W_j \subseteq V_j$ ($j = 1, 2, 3$; the V_j of possibly different ranks) and an exact sequence of $\mathcal{O}_{\mathcal{X}'}$-modules $0 \to V_1/W_1 \to V_2/W_2 \to V_3/W_3 \to 0$,*

$$\mathrm{ch}_k(W_2 \subseteq V_2) \cdot M = \mathrm{ch}_k(W_1 \subseteq V_1) \cdot M + \mathrm{ch}_k(W_3 \subseteq V_3) \cdot M.$$

Proof. Write $\mathcal{X} = \mathcal{X}'$ and let $Z = (\mathcal{X}_{\mathfrak{s}})_{\mathrm{red}}$, a closed subscheme, proper over the residue field κ of $\mathcal{O}'$; deg on $\mathrm{CH}_0(Z)$ is $[\kappa : \mathbb{F}_p]$ times the degree over κ. For an effective Cartier divisor Δ on $\mathcal{X}$ let $E_\Delta = [\mathcal{O}_{\mathcal{X}}(-\Delta) \to \mathcal{O}_{\mathcal{X}}]$ (degrees $-1, 0$), a resolution of $\mathcal{O}_\Delta$.

Step 0: commuting Chern classes past bivariant classes. For a bivariant class $c \in A^p(T \to \mathcal{X})$, an invertible module $\mathcal{L}$ on $\mathcal{X}$ and $\alpha \in \mathrm{CH}_*(\mathcal{X})$: $c \cap (c_1(\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}|_T) \cap (c \cap \alpha)$. Indeed $\mathcal{X}$ is projective over the affine S, so $\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ with $\mathcal{L}_1, \mathcal{L}_2$ having global sections, and c_1 is additive; for a section σ of $\mathcal{L}_j$ with zero scheme $i\colon D_\sigma \to \mathcal{X}$ and its pull-back $i'\colon D_\sigma \times_{\mathcal{X}} T \to T$, (G) and the compatibility of c with proper push-forward and with Gysin maps give $c \cap i_* i^* \alpha = i'_*(c \cap i^*\alpha) = i'_* i'^*(c \cap \alpha) = c_1(\mathcal{L}_j|_T) \cap (c \cap \alpha)$.

Step 1: definition. For a vertical pair $W \subseteq V$ let $C(W \subseteq V) = [W \to V]$ (degrees $-1, 0$; the inclusion). It is a bounded complex of finite locally free modules, quasi-isomorphic to $(V/W)[0]$, hence zero off Z. Put

$$\mathrm{ch}_k(W \subseteq V) \cdot M = \deg\big(M \cap \big(\mathrm{ch}_k(Z \to \mathcal{X}, C(W \subseteq V)) \cap [\mathcal{X}]\big)\big),$$

where $\mathrm{ch}_k(Z \to \mathcal{X}, C) \cap [\mathcal{X}] \in \mathrm{CH}_{D+1-k}(Z) \otimes \mathbb{Q}$ and M, a combination of products of $D+1-k$ first Chern classes, maps it to $\mathrm{CH}_0(Z) \otimes \mathbb{Q}$. This is linear in M and depends only on the isomorphism class of $C(W \subseteq V)$ in $D(\mathcal{O}_{\mathcal{X}})$.

Step 2: (Vc1). Given $W' \subseteq W \subseteq V$, the map $C(W' \subseteq W) \to C(W' \subseteq V)$ (identity in degree -1, $W \subseteq V$ in degree 0) is injective with cokernel $(V/W)[0] \cong C(W \subseteq V)$. The short exact sequence of complexes gives a distinguished triangle $C(W' \subseteq W) \to C(W' \subseteq V) \to C(W \subseteq V) \to$, whose terms are zero off Z; (L4) gives additivity of $\mathrm{ch}_k(Z \to \mathcal{X}, -)$, and capping with $[\mathcal{X}]$, applying M and taking degrees gives (Vc1).

Step 3: the elementary case, reduced to E_Δ. Let $V/W \cong N \otimes \mathcal{O}_\Delta$. Then $C(W \subseteq V) \cong (N \otimes \mathcal{O}_\Delta)[0] \cong N \otimes E_\Delta$. By (L5) with $Z_1 = \Delta$, $F_1 = E_\Delta$, $Z_2 = \mathcal{X}$, $F_2 = N$, and (L1) for $Z_2 = \mathcal{X}$: $\mathrm{ch}(\Delta \to \mathcal{X}, N \otimes E_\Delta) = \mathrm{ch}(\Delta \to \mathcal{X}, E_\Delta)\,\mathrm{ch}(N)$. By (L3) applied to $\Delta_{\mathrm{red}} \subseteq \Delta$ and to $\Delta_{\mathrm{red}} \subseteq Z$, and (R), $\mathrm{ch}(Z \to \mathcal{X}, -) \cap [\mathcal{X}] = \iota_*(\mathrm{ch}(\Delta \to \mathcal{X}, -) \cap [\mathcal{X}])$ for objects zero off Δ, where $\iota_*\colon \mathrm{CH}_*(\Delta) = \mathrm{CH}_*(\Delta_{\mathrm{red}}) \to \mathrm{CH}_*(Z)$. With Step 0, writing $u_t = \mathrm{ch}_t(\Delta \to \mathcal{X}, E_\Delta) \cap [\mathcal{X}] \in \mathrm{CH}_{D+1-t}(\Delta) \otimes \mathbb{Q}$,

$$\mathrm{ch}_k(Z \to \mathcal{X}, C(W \subseteq V)) \cap [\mathcal{X}] = \iota_* \sum_{a+t=k} \frac{c_1(N)^a}{a!} \cap u_t.$$

Step 4: the value of u_t. Claim: $u_0 = 0$ and $u_t = \frac{(-1)^{t-1}}{t!}\, c_1(\mathcal{O}(\Delta))^{t-1} \cap [\Delta]$ for $t \geq 1$, with $[\Delta] \in \mathrm{CH}_D(\Delta)$ the cycle of the scheme Δ.

(4a) Deformation space. Let $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(\Delta)$ with canonical section s, let $\mathbb{V}(\mathcal{L}) = \underline{\mathrm{Spec}}_{\mathcal{X}} \mathrm{Sym}(\mathcal{L}^\vee)$ be its total space with tautological section y of the pull-back of $\mathcal{L}$, and let $\mathcal{M} \subseteq \mathbb{V}(\mathcal{L}) \times \mathbb{A}^1$ be the zero scheme of the section $ty - s$ (t the coordinate of $\mathbb{A}^1$). Over an affine open $\operatorname{Spec} R$ of $\mathcal{X}$ on which $\mathcal{L} = R\,e$, $s = fe$ (R a domain, $f \neq 0$), $\mathcal{M}$ is $\operatorname{Spec} A$, $A = R[t,y]/(ty - f)$. The map $A \to R[t,t^{-1}]$, $t \mapsto t$, $y \mapsto f/t$, is injective: modulo $ty - f$ every element is $\sum_i \alpha_i t^i + \sum_{j\geq 1} \beta_j y^j$, with image $\sum_i \alpha_i t^i + \sum_j \beta_j f^j t^{-j}$, which vanishes only if all α_i and all $f^j\beta_j$, hence all β_j, vanish. So A is a domain in which t and y are non-zero-divisors, and $A[1/t] = R[t^{\pm 1}]$. Hence $\mathcal{M}$ is integral (its open part $t \neq 0$ is $\mathcal{X} \times \mathbb{G}_m$, irreducible and dense in every chart), quasi-compact, of δ-dimension $D + 2$. Further: $\mathcal{D} := V(y) \subseteq \mathcal{M}$ is an effective Cartier divisor, locally $A/yA = (R/f)[t]$, so $\mathcal{D} \cong \Delta \times \mathbb{A}^1$; for $c_0 \in \{0,1\}$ let $F_{c_0} = V(t - c_0)$, the zero scheme of a section of $\mathcal{O}_{\mathcal{M}}$. Locally $A/(t-1) = R[y]/(y - f) = R$: $F_1 \cong \mathcal{X}$ with $\mathcal{D} \cap F_1 = \Delta$; and $A/tA = (R/f)[y]$: $F_0 \cong \mathbb{V}(\mathcal{L}|_\Delta) =: \mathcal{N}$, the total space of $\mathcal{L}|_\Delta$ over Δ, with $\mathcal{D} \cap F_0 = \Delta_0$ its zero section. Let $\mathcal{E} = [\mathcal{O}_{\mathcal{M}}(-\mathcal{D}) \to \mathcal{O}_{\mathcal{M}}]$, zero off $\mathcal{D}$. Its terms are invertible, so its derived pull-backs are termwise: to F_1 it is E_Δ (the generator y restricts to f), to F_0 it is $E_{\Delta_0} = [\mathcal{O}_{\mathcal{N}}(-\Delta_0) \to \mathcal{O}_{\mathcal{N}}]$.

(4b) The two ends agree. Let $c = \mathrm{ch}_t(\mathcal{D} \to \mathcal{M}, \mathcal{E})$ and $\gamma = c \cap [\mathcal{M}] \in \mathrm{CH}_{D+2-t}(\mathcal{D}) \otimes \mathbb{Q}$. By (A), $\gamma = \mathrm{pr}^*\beta$ for some $\beta \in \mathrm{CH}_{D+1-t}(\Delta) \otimes \mathbb{Q}$. Let $j_{c_0} \colon \Delta \times \{c_0\} \to \mathcal{D}$ be the zero scheme of $t - c_0$ on $\mathcal{D}$, the pull-back of F_{c_0}. For $V \subseteq \Delta$ integral, $\mathrm{pr}^*[V] = [V \times \mathbb{A}^1]$ and, by (G), $j_{c_0}^*[V \times \mathbb{A}^1] = [V \times \{c_0\}]$ ($t - c_0$ is a uniformiser of the local ring of $V \times \mathbb{A}^1$ at the generic point of $V \times \{c_0\}$); so $j_{c_0}^*\gamma = \beta$ under $\Delta \times \{c_0\} = \Delta$. On the other hand, by the Gysin compatibility of the bivariant class c, by (G) ($\mathcal{M}$ integral, $\mathcal{M} \not\subseteq F_{c_0}$) and by (L2),

$$j_{c_0}^*\gamma = c \cap [F_{c_0}] = \mathrm{ch}_t(\mathcal{D} \cap F_{c_0} \to F_{c_0}, \mathcal{E}|_{F_{c_0}}) \cap [F_{c_0}].$$

With $[F_1] = [\mathcal{X}]$ and $[F_0] = [\mathcal{N}]$ (the cycle of the scheme $\mathcal{N}$): $u_t = \beta = \mathrm{ch}_t(\Delta_0 \to \mathcal{N}, E_{\Delta_0}) \cap [\mathcal{N}]$.

(4c) The normal bundle. Let $q \colon \mathcal{P} = \mathbb{P}(\mathcal{L}|_\Delta \oplus \mathcal{O}_\Delta) \to \Delta$ be the projective completion of $\mathcal{N}$, a $\mathbb{P}^1$-bundle, proper, of pure δ-dimension $D + 1$ with $[\mathcal{P}] = q^*[\Delta]$; $\mathcal{N} \subseteq \mathcal{P}$ is open, the zero section $j \colon \Delta \cong \Delta_0 \to \mathcal{P}$ is an effective Cartier divisor of $\mathcal{P}$ contained in $\mathcal{N}$, $q \circ j = \mathrm{id}_\Delta$, and $\mathcal{O}_{\mathcal{P}}(\Delta_0)|_{\Delta_0} \cong \mathcal{L}|_\Delta$ (the normal bundle of the zero section of a line bundle is the line bundle). Put $E_{\mathcal{P}} = [\mathcal{O}_{\mathcal{P}}(-\Delta_0) \to \mathcal{O}_{\mathcal{P}}]$, which restricts to E_{Δ_0} on $\mathcal{N}$. By (L2) and flat pull-back along the open immersion $\mathcal{N} \to \mathcal{P}$ (with $\Delta_0 \times_{\mathcal{P}} \mathcal{N} = \Delta_0$), $v_t := \mathrm{ch}_t(\Delta_0 \to \mathcal{P}, E_{\mathcal{P}}) \cap [\mathcal{P}] = \mathrm{ch}_t(\Delta_0 \to \mathcal{N}, E_{\Delta_0}) \cap [\mathcal{N}] = u_t$. By (L1), (C) and the projection formula,

$$j_* v_t = \mathrm{ch}_t(E_{\mathcal{P}}) \cap [\mathcal{P}] = \Big(\mathrm{ch}_t(\mathcal{O}) - \frac{(-c_1(\mathcal{O}(\Delta_0)))^t}{t!}\Big) \cap [\mathcal{P}] = \begin{cases} 0, & t = 0, \\ j_*\Big(\frac{(-1)^{t-1}}{t!} c_1(\mathcal{L}|_\Delta)^{t-1} \cap [\Delta]\Big), & t \geq 1. \end{cases}$$

As $q_* j_* = \mathrm{id}$, j_* is injective, and the claim follows.

Step 5: (Vc2). By Steps 3 and 4, with $x = c_1(N)$, $y = c_1(\mathcal{O}(\Delta))$ (commuting operators on $\mathrm{CH}_*(\Delta)$),

$$k!\,\mathrm{ch}_k(Z \to \mathcal{X}, C(W \subseteq V)) \cap [\mathcal{X}] = \iota_* \sum_{t=1}^{k} \binom{k}{t} (-1)^{t-1} x^{k-t} y^{t-1} \cap [\Delta] = \iota_* \sum_{a=0}^{k-1} x^a (x-y)^{k-1-a} \cap [\Delta],$$

by the identity $\sum_{a=0}^{k-1} x^a (x-y)^{k-1-a} = \sum_{t=1}^{k} \binom{k}{t} (-1)^{t-1} x^{k-t} y^{t-1}$ in $\mathbb{Z}[x,y]$ (multiply both sides by y: each becomes $x^k - (x-y)^k$; $\mathbb{Z}[x,y]$ is a domain). Since $x - y = c_1(N(-\Delta))$, applying M (which commutes with ι_* and with x, y) and taking degrees (deg is invariant under proper push-forward) gives (Vc2).

Step 6: (Vc3). Each $C(W_j \subseteq V_j)$ is isomorphic in $D(\mathcal{O}_{\mathcal{X}})$ to $(V_j/W_j)[0]$. A short exact sequence of $\mathcal{O}_{\mathcal{X}}$-modules gives a distinguished triangle $(V_1/W_1)[0] \to (V_2/W_2)[0] \to (V_3/W_3)[0] \to (V_1/W_1)[1]$, and a triangle isomorphic to a distinguished one is distinguished; so there is a distinguished triangle $C(W_1 \subseteq V_1) \to C(W_2 \subseteq V_2) \to C(W_3 \subseteq V_3) \to$, whose terms are bounded complexes of finite locally free modules, zero off Z. By (L4), $\mathrm{ch}_k(Z \to \mathcal{X}, -)$ is additive along it; capping with $[\mathcal{X}]$, applying M and taking degrees (Step 1: the number depends only on the isomorphism class in $D(\mathcal{O}_{\mathcal{X}})$) gives (Vc3). □

35.2 Admissible torsion systems with Hodge–Tate lattice data

This subsection proves the two sides of the local relations. The de Rham side is Lemma 35.13: on a joint model the vertical root divisors over p measure the change from the de Rham to the Hodge–Tate lattices of the Lie and cotangent modules, a computation with vertical Chern characters along compatible flags (Lemma 35.14, Lemma 35.16, Lemma 35.15). The Hodge–Tate side is Proposition 35.11: there is an admissible torsion system of the p-torsion whose eigen-coordinates measure the same change. It is built in layers. Function modules of p-torsion over $\mathcal{O}_C$, tight Hopf orders with their eigen-coordinate ideals and flag conditions, test points, regular-translate orders and Hodge–Tate lattice families are defined here (Definition 35.17, Definition 35.18, Definition 35.19, Definition 35.21, Definition 35.22). A refined model carries a torsion-regular system with a Hodge–Tate lattice family (Proposition 35.25, Proposition 35.24); the test-point conditions hold on it (Proposition 35.23), by the test-point facts of Appendix D; and the flag theorem for tight Hopf orders of Subsection 35.4 turns these into an admissible system (Proposition 35.20).

Proposition 35.11. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p_0) with a geometric realisation and a good origin immersive at level four over an integral model $\mathcal{Y}$ of a unitary Shimura datum $\mathfrak{u} = (E, \mathsf{H}, \dots)$, $\Phi = \Phi_{\mathsf{H}}$, with E Galois over $\mathbb{Q}$ with abelian Galois group. Let $c \in \mathrm{Cp}$, $2 \le B_0 \le B$, $\mathcal{X} \subseteq \mathbb{P} = \prod_{k\in K} \mathbb{P}(\mathbb{Z}^{I_k})$ a joint model of the stages (B_0, c) and (B, c), $p \in S(B_0, B)$ and λ a prime above p of the ring of integers of the field of constants F_Ω of $\mathcal{X}$ (Definition 35.7). Write $x_r = c_1(\mathcal{O}(e_r))$, $x'_r = c_1(\mathcal{O}(e'_r))$ (first and second root classes on $\mathcal{X}$), $F(x)$ for $F \in \mathcal{R}$ evaluated at the x_r, and $\mathrm{Dv}_{r,\lambda}$ for the part of $\mathrm{Dv}_{r,p}$ supported on the components of $\mathcal{X}_{\mathbb{F}_p}$ lying over λ. Then there is an admissible torsion system $(\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_{\mathfrak{P}}, \theta))$ for $(\mathcal{X}, p, \lambda)$ such that for every $a \in \Phi$, every $1 \le k \le n$ and every real combination M of products of $D+1-k$ classes $c_1(\mathcal{O}(e))$, $e \in \mathbb{Z}^K$, with $m = m_{\mathfrak{P}_{(a,+)}}$:*

$$d_\pi \sum_{i=1}^{n} \deg\Big(\sum_{j=0}^{k-1} x_{a,i}^{\,j}\, x'^{\,k-1-j}_{a,i}\, M \cap [\mathrm{Dv}_{(a,i),\lambda}] \Big) = (-1)^k \sum_{i=1}^{m} \mathrm{Ec}^{k}_{(a,+),i} \cdot M - \frac{k}{p-1} \Big(\mathrm{div}(p) \cdot \mathrm{ps}^{(a,+)}_{k-1}(x)\, M \Big),$$

the left side computed on $\mathcal{X}$ (degrees over $\mathbb{F}_p$), the right side on $\mathcal{X}'$.

Proof. Let $(\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_{\mathfrak{P}}, \theta))$ and $w \in \mathcal{O}'$, $w^{p-1} = u\,p$ with u a unit, be as given by Proposition 35.20. We show that this admissible torsion system has the asserted property. Fix $a \in \Phi$, $1 \le k \le n$ and M; put $s = (a, +)$, $m = m_{\mathfrak{P}_s}$, and take the data V_L, V_H, V_ω, lie $\subseteq W$, $\omega' \subseteq \omega$, $T = \mathcal{O}^m \subseteq H$, Γ, Δ, H_i, J_i of Proposition 35.20 for a. All Chern-character numbers below are those of Lemma 35.10 on $\mathcal{X}'$ (its hypotheses hold by (Ts1), (Ts2)), evaluated on $\pi^* M$; we write M for $\pi^* M$ on $\mathcal{X}'$. As $p \in \mathfrak{m}_{\mathcal{O}'}$ and $w^{p-1} = u\,p$, w is a non-zero non-unit of $\mathcal{O}'$, and at a point of $\mathcal{X}'$ the element w is a unit if and only if p is; the locus where p is not invertible is the special fibre $\mathcal{X}'_{\mathfrak{s}}$.

Step 1: the facts hold on $\mathcal{X}'$. For $1 \le i \le m$ the i-th coordinate induces an isomorphism $H_i/H_{i-1} \to J_i$ (its kernel on H_i is $H \cap (\Omega'^{i-1} \oplus 0) = H_{i-1}$), so J_i is invertible by (He2);

$\mathcal{O}_{\mathcal{X}'}(-\mathrm{Ec}_{s,i})$ is invertible by (Ts5). Let q be a point of codimension one. If p is not a unit at q, (He3) and Lemma 35.12 (3) give (F0)–(F2) at q, and Lemma 35.12 (4) gives $J_{i,q} = \mathcal{O}(-\mathrm{Ec}_{s,i})_q$. If p is a unit at q, then $H_q = T_q$ by (He1), so (F0) holds at q ($T \subseteq w^{-1}T$ as $w \in \mathcal{O}'$), and $J_{i,q}$ is the image of $T_q \cap (\Omega'^i \oplus 0)$, i.e. $\mathcal{O}_q$, which is $\mathcal{O}(-\mathrm{Ec}_{s,i})_q$ because $\mathrm{Ec}_{s,i}$ is supported in $\mathcal{X}'_{\mathfrak{s}}$ (Definition 35.7). As I is locally free (He1), Lemma 35.12 (2) gives (F0)–(F2) on $\mathcal{X}'$, and Lemma 35.12 (4) gives $J_i = \mathcal{O}_{\mathcal{X}'}(-\mathrm{Ec}_{s,i})$ for every i.

Step 2: the Hodge–Tate sequence. By Lemma 35.12 (1) there is an exact sequence of $\mathcal{O}_{\mathcal{X}'}$-modules

$$0 \to w^{-1}\mathrm{lie}/W \to H/T \to \omega/\omega' \to 0.$$

Step 3: additivity. The inclusion $T \subseteq H$ is a vertical pair: both are locally free of rank m and H/T vanishes where p is invertible (He1). The inclusion $\omega' \subseteq \omega$ is a vertical pair: w kills ω/ω' as $\omega \subseteq w^{-1}\omega'$. Apply Lemma 35.16 (2) with $L = w^{-1}\mathrm{lie}$ (locally free, $wL = \mathrm{lie} \subseteq W \subseteq L$) and the sequence of Step 2:

$$\mathrm{ch}_k(\mathrm{lie} \subseteq W) \cdot M + \mathrm{ch}_k(T \subseteq H) \cdot M = \mathrm{ch}_k(\mathrm{lie} \subseteq w^{-1}\mathrm{lie}) \cdot M + \mathrm{ch}_k(\omega' \subseteq \omega) \cdot M. \tag{1}$$

Step 4: the eigen-coordinate term. The module $T \cap H_i = T \cap (\Omega'^i \oplus 0) = \mathcal{O}^i \oplus 0$ is locally free with $(T \cap H_i)/(T \cap H_{i-1}) \cong \mathcal{O}$ via the i-th coordinate, and the i-th coordinate identifies the pair $(T \cap H_i)/(T \cap H_{i-1}) \subseteq H_i/H_{i-1}$ with $\mathcal{O} \subseteq J_i = \mathcal{O}(-\mathrm{Ec}_{s,i})$ (Step 1). Since $\mathcal{O} \subseteq \mathcal{O}(-\mathrm{Ec}_{s,i})$, the Cartier divisor $\Delta_i = -\mathrm{Ec}_{s,i}$ is effective (a local equation f of Δ_i has $1 \in f^{-1}\mathcal{O}$), and it is supported in $\mathcal{X}'_{\mathfrak{s}}$. By Lemma 35.14 (2) (case $N_i(-\Delta_i) \cong \mathcal{O}$),

$$k!\,\mathrm{ch}_k(T \subseteq H) \cdot M = \sum_{i=1}^{m} \deg\bigl(c_1(\mathcal{O}(\Delta_i))^{k-1} M \cap [\Delta_i]\bigr) = (-1)^k \sum_{i=1}^{m} \mathrm{Ec}_{s,i}^k \cdot M, \tag{2}$$

using $c_1(\mathcal{O}(-E)) = -c_1(\mathcal{O}(E))$ and $[-E] = -[E]$ (additivity of Weil divisors of Cartier divisors) and the definition $E^k \cdot M = \deg(c_1(\mathcal{O}(E))^{k-1} M \cap [E])$ of Definition 35.7.

Step 5: the scaling term. By (He4), $L = w^{-1}\mathrm{lie}$ has the filtration $w^{-1}\mathrm{lie}_j$ ($1 \le j \le \kappa$) by locally free submodules with invertible quotients $\cong \mathrm{gr}_j\mathrm{lie} \cong \pi^*\mathcal{O}(e_{(a,j)})$, of first Chern class $\pi^* x_{a,j}$. By Lemma 35.16 (1) and (3),

$$k!\,\mathrm{ch}_k(\mathrm{lie} \subseteq w^{-1}\mathrm{lie}) \cdot M = \frac{k}{p-1} \sum_{j=1}^{\kappa} \deg(\pi^* x_{a,j}^{k-1} M \cap [\mathrm{div}(p)]) = \frac{k}{p-1}\Bigl(\mathrm{div}(p) \cdot \mathrm{ps}_{k-1}^{(a,+)}(x)\, M\Bigr), \tag{3}$$

as $\mathrm{ps}_{k-1}^{(a,+)} = \sum_{j=1}^{\kappa} x_{a,j}^{k-1}$ (Definition 21.3).

Step 6: the de Rham side. By Lemma 35.13, the left side of the assertion equals $k!\,\mathrm{ch}_k(\omega' \subseteq \omega) \cdot M - k!\,\mathrm{ch}_k(\mathrm{lie} \subseteq W) \cdot M$. By (1) this is $k!\,\mathrm{ch}_k(T \subseteq H) \cdot M - k!\,\mathrm{ch}_k(\mathrm{lie} \subseteq w^{-1}\mathrm{lie}) \cdot M$, and by (2) and (3) it equals $(-1)^k \sum_{i=1}^{m} \mathrm{Ec}_{(a,+),i}^k \cdot M - \frac{k}{p-1}(\mathrm{div}(p) \cdot \mathrm{ps}_{k-1}^{(a,+)}(x)\, M)$, as asserted. □

Lemma 35.12. *Let $\mathcal{X}'$ be a normal integral noetherian scheme with function field Ω' and $w \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ non-zero. Let V_L, V_H, V_ω be finite-dimensional Ω'-vector spaces, regarded as constant sheaves on $\mathcal{X}'$. Let $\mathrm{lie}, W \subseteq V_L$, $T, H \subseteq V_H$ and $\omega', \omega \subseteq V_\omega$ be locally free $\mathcal{O}_{\mathcal{X}'}$-submodules, each of rank the dimension of its space, with*

$$\mathrm{lie} \subseteq W \subseteq w^{-1}\mathrm{lie}, \qquad \omega' \subseteq \omega \subseteq w^{-1}\omega', \qquad T \subseteq H,$$

and let $\Gamma \subseteq V_L \oplus V_H$ and $\Delta \subseteq V_H \oplus V_\omega$ be $\mathcal{O}_{\mathcal{X}'}$-submodules such that, over every affine open U and for all sections a, b, c over U of V_L, V_H, V_ω:

(G1) *$(a, b') \in \Gamma$ for some b' iff $wa \in W$; $(0, b) \in \Gamma$ iff $b \in T$; $(a, 0) \in \Gamma$ iff $a \in W$;*

(G2) $(b,c') \in \Delta$ *for some* c' *iff* $wb \in T$; $(b',c) \in \Delta$ *for some* b' *iff* $wc \in \omega'$; $(0,c) \in \Delta$ *iff* $c \in \omega'$; $(b,0) \in \Delta$ *iff* $(a',b) \in \Gamma$ *for some* a'.

Let $K \subseteq V_L$ *be the subsheaf of* a *with* $(a,b) \in \Gamma$ *for some section* b *of* H, *and* $I \subseteq V_\omega$ *the subsheaf of* c *with* $(b,c) \in \Delta$ *for some section* b *of* H. *Consider the conditions*

$$\textit{(F0) } H \subseteq w^{-1}T, \qquad \textit{(F1) } K = w^{-1}\mathrm{lie}, \qquad \textit{(F2) } I = \omega.$$

1. *If (F0)–(F2) hold, there is an exact sequence of* $\mathcal{O}_{\mathcal{X}'}$*-modules*

$$0 \to w^{-1}\mathrm{lie}/W \xrightarrow{\gamma} H/T \xrightarrow{\delta} \omega/\omega' \to 0,$$

   with $\gamma(a) = b \bmod T$ *for* $(a,b) \in \Gamma$, $b \in H$, *and* $\delta(b) = c \bmod \omega'$ *for* $(b,c) \in \Delta$.

2. *Assume that* I *is locally free, that (F0) holds after localisation at every point* q *of codimension one of* $\mathcal{X}'$, *and that (F1), (F2) hold after localisation at every point* q *of codimension one at which* w *is not a unit (for the stalks at* q, *which are submodules of the same spaces over the discrete valuation ring* $\mathcal{O}_{\mathcal{X}',q}$*). Then (F0)–(F2) hold on* $\mathcal{X}'$.

3. *Let* q *be a point of codimension one,* C *a field with a valuation ring* $\mathcal{O}_C$ *and a field embedding* $\Omega' \to C$ *with* $\mathcal{O}_C \cap \Omega' = \mathcal{O}_{\mathcal{X}',q}$. *If (F0)–(F2) hold for the* $\mathcal{O}_C$*-spans in* $V_\bullet \otimes_{\Omega'} C$ *of the stalks at* q *(with* K *and* I *formed from the spans of* Γ_q, Δ_q, H_q*), then they hold for the stalks at* q.

4. *Let* V *be a finite-dimensional* Ω'*-space and* $P, P' \subseteq V$ *locally free* $\mathcal{O}_{\mathcal{X}'}$*-submodules of rank* $\dim V$. *Then* $P = P'$ *if and only if* $P_q = P'_q$ *for every point* q *of codimension one; and for* q, C *as in (3),* $P_q = P'_q$ *if and only if* $\mathcal{O}_C P_q = \mathcal{O}_C P'_q$ *in* $V \otimes_{\Omega'} C$.

Proof. All sheaves are subsheaves of constant sheaves on the integral scheme $\mathcal{X}'$, so their sections over a non-empty open are subsets of the corresponding Ω'-space and inclusions and equalities may be checked on affine opens; (G1), (G2) are used only over affine opens, and at a point q in the form obtained from them over the affine neighbourhoods of q. By (G1) and (G2), $\Gamma \subseteq w^{-1}W \oplus w^{-1}T$ (if $(a,b) \in \Gamma$ then $(b,0) \in \Delta$, so $wb \in T$) and $\Delta \subseteq w^{-1}T \oplus w^{-1}\omega'$; on affine opens these are submodules of finitely generated modules over a noetherian ring, hence Γ, Δ, K, I are coherent (quasi-coherence of $K = \mathrm{pr}_1(\Gamma \cap (V_L \oplus H))$ and I follows likewise, intersections and images of quasi-coherent submodules being quasi-coherent).

(1). Work on an affine open U. γ. For $a \in w^{-1}\mathrm{lie}(U) = K(U)$ there is $b \in H(U)$ with $(a,b) \in \Gamma(U)$; two such b, b' give $(0, b-b') \in \Gamma$, so $b - b' \in T$ (G1). If $a \in W$ then $(a,0) \in \Gamma$, so $\gamma(a) = 0$. Thus $\gamma\colon w^{-1}\mathrm{lie}/W \to H/T$ is well defined and $\mathcal{O}$-linear (Γ is a submodule). It is injective: if $(a,b) \in \Gamma$ with $b \in T$ then $(0,b) \in \Gamma$, so $(a,0) \in \Gamma$ and $a \in W$. δ. For $b \in H(U)$, (F0) gives $wb \in T$, so $(b,c) \in \Delta$ for some c, and $c \in I(U) = \omega(U)$ by (F2); two choices differ by $c - c'$ with $(0, c-c') \in \Delta$, i.e. $c - c' \in \omega'$. If $b \in T$ then $(0,b) \in \Gamma$, so $(b,0) \in \Delta$ and $\delta(b) = 0$. So $\delta\colon H/T \to \omega/\omega'$ is well defined and linear. *Exactness.* δ is surjective on sections over U because every $c \in \omega(U) = I(U)$ has some $b \in H(U)$ with $(b,c) \in \Delta$, and $(\omega/\omega')(U) = \omega(U)/\omega'(U)$ on the affine U. $\delta \circ \gamma = 0$: if $(a,b) \in \Gamma$ then $(b,0) \in \Delta$. If $\delta(b) = 0$, then $(b,c) \in \Delta$ with $c \in \omega'$, so $(0,c) \in \Delta$, hence $(b,0) \in \Delta$, so $(a,b) \in \Gamma$ for some a; then $a \in K(U) = w^{-1}\mathrm{lie}(U)$ and $\gamma(a) = b \bmod T$. As these hold over every affine open, the sequence of sheaves is exact.

(2). The key fact: for a noetherian normal domain R with fraction field Ω', $R = \bigcap_{\mathrm{ht}\, q=1} R_q$ (Matsumura, Commutative Ring Theory, Thm. 11.5); hence for a free R-submodule P of an Ω'-space spanned by a basis of that space, $P = \bigcap_q P_q$ (compare coordinates in the basis). So if

$A \subseteq V$ is any $\mathcal{O}$-submodule and $P \subseteq V$ is locally free with $A_q \subseteq P_q$ for every codimension-one q, then $A \subseteq P$ (cover by affine opens on which P is free). *Codimension-one points where w is a unit.* Let q be such a point. Then $\mathrm{lie}_q = W_q = w^{-1}\mathrm{lie}_q$ and $\omega'_q = \omega_q$ by the inclusions, and (F0) at q gives $H_q \subseteq w^{-1}T_q = T_q$, so $H_q = T_q$. Hence $K_q = \{a : (a,b) \in \Gamma_q,\ b \in T_q\} = \{a : (a,0) \in \Gamma_q\} = W_q = w^{-1}\mathrm{lie}_q$, and $I_q = \{c : (b,c) \in \Delta_q,\ b \in T_q\}$; for $b \in T_q$, $(b,0) \in \Delta_q$, so $(b,c) \in \Delta_q$ iff $(0,c) \in \Delta_q$ iff $c \in \omega'_q = \omega_q$; so (F1), (F2) hold at q.

Now globalise. (F0): $H_q \subseteq (w^{-1}T)_q$ for all q and $w^{-1}T$ is locally free, so $H \subseteq w^{-1}T$. (F1): $K_q = (w^{-1}\mathrm{lie})_q$ for all q (localisation commutes with the formation of K: on an affine open, K is the image of the submodule $\Gamma \cap (V_L \oplus H)$, and localisation is exact). $K \subseteq w^{-1}\mathrm{lie}$ by the key fact. Conversely let $a \in w^{-1}\mathrm{lie}(U)$, U affine. Then $wa \in \mathrm{lie} \subseteq W$, so by (G1) there is $b_0 \in V_H$ with $(a, b_0) \in \Gamma(U)$. For each codimension-one q meeting U there is $b_q \in H_q$ with $(a, b_q) \in \Gamma_q$; then $(0, b_0 - b_q) \in \Gamma_q$, so $b_0 - b_q \in T_q \subseteq H_q$ and $b_0 \in H_q$. By the key fact $b_0 \in H(U)$, so $a \in K(U)$. Hence $K = w^{-1}\mathrm{lie}$. (F2): $I_q = \omega_q$ for all q (as for K), and both I and ω are locally free submodules of V_ω; by the key fact applied in both directions, $I = \omega$.

(3). Write $R = \mathcal{O}_{\mathcal{X}',q}$, a discrete valuation ring. $\mathcal{O}_C$ is torsion-free over R, hence flat. For finitely generated R-submodules A, B of a finite-dimensional Ω'-space V spanning V: they are free with bases that are bases of V; if $\mathcal{O}_C A = \mathcal{O}_C B$ in $V \otimes C$, the matrix expressing a basis of B in a basis of A and its inverse have entries in $\mathcal{O}_C \cap \Omega' = R$, so $A = B$; likewise $\mathcal{O}_C A \subseteq \mathcal{O}_C B$ implies $A \subseteq B$. For flat $\mathcal{O}_C$, the span of an intersection of submodules is the intersection of the spans, and the span of an image is the image of the span. Hence the $\mathcal{O}_C$-spans of K_q and I_q are the modules K, I formed from the spans of Γ_q, Δ_q, H_q, and (F0)–(F2) for the spans give (F0)–(F2) at q (all modules involved are finitely generated: they lie in $w^{-1}W_q$, $w^{-1}T_q$, $w^{-1}\omega'_q$ respectively).

(4). The first equivalence is the key fact of (2) applied to $P \subseteq P'$ and $P' \subseteq P$ (both locally free); the second is the matrix argument of (3) (P_q, P'_q are free over the discrete valuation ring $\mathcal{O}_{\mathcal{X}',q}$ with bases that are bases of V). □

Lemma 35.13. *Let $\mathcal{X}$, p, λ, $(\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_{\mathfrak{P}}, \theta))$, w and, for $a \in \Phi$, $\mathrm{lie} \subseteq W$ and $\omega' \subseteq \omega$ with (He1) and (He4) be as in Proposition 35.20. Write $x_r = c_1(\mathcal{O}(e_r))$, $x'_r = c_1(\mathcal{O}(e'_r))$ on $\mathcal{X}$ and $\mathrm{Dv}_{r,\lambda}$ for the part of $\mathrm{Dv}_{r,p}$ supported on the components of $\mathcal{X}_{\mathbb{F}_p}$ over λ. Then for every $1 \le k \le n$ and every real combination M of products of $D + 1 - k$ classes $c_1(\mathcal{O}(e))$, $e \in \mathbb{Z}^K$, with π^*M the same combination of the classes $c_1(\pi^*\mathcal{O}(e))$ on $\mathcal{X}'$,*

$$d_\pi \sum_{i=1}^{n} \deg\Big(\sum_{j=0}^{k-1} x_{a,i}^{j}\, x'^{\,k-1-j}_{a,i}\, M \cap [\mathrm{Dv}_{(a,i),\lambda}] \Big) = k!\,\mathrm{ch}_k(\omega' \subseteq \omega) \cdot \pi^*M - k!\,\mathrm{ch}_k(\mathrm{lie} \subseteq W) \cdot \pi^*M,$$

the left side computed on $\mathcal{X}$ (degrees over $\mathbb{F}_p$), the right side on $\mathcal{X}'$.

Proof. Fix k and M, and write M also for π^*M on $\mathcal{X}'$. For a root r let $s_r = f_r/g_r$, a rational section of $\mathcal{O}(e_r - e'_r)$ on $\mathcal{X}$, and Dv_r^C its Cartier divisor (Weil divisor Dv_r). The roots of a are $r = (a, j)$, $1 \le j \le \kappa$ (*Lie roots*), and $r = (a, \kappa + j)$, $1 \le j \le n - \kappa$ (*cotangent roots*).

Step 1: reduction to rank one. Since $\mathrm{lie} \subseteq W$, $\mathrm{lie} \cap W_j = \mathrm{lie} \cap F_j = \mathrm{lie}_j$; by (He4) these are locally free with invertible graded pieces, and $\mathrm{lie} \subseteq W$ is a vertical pair (w kills W/lie by (He1), and w is invertible off $\mathcal{X}'_{\mathfrak{s}}$). So Lemma 35.14 (1) gives $\mathrm{ch}_k(\mathrm{lie} \subseteq W) \cdot M = \sum_{j=1}^{\kappa} \mathrm{ch}_k(\mathrm{gr}_j \mathrm{lie} \subseteq \mathrm{gr}_j W) \cdot M$, the graded pairs being vertical pairs of rank one; likewise $\mathrm{ch}_k(\omega' \subseteq \omega) \cdot M = \sum_{j=1}^{n-\kappa} \mathrm{ch}_k(\mathrm{gr}_j \omega' \subseteq \mathrm{gr}_j \omega) \cdot M$.

Step 2: the rank-one pairs. Locally on $\mathcal{X}$ choose generators u of $\mathcal{O}(e_r)$ and t of $\mathcal{O}(e'_r)$ and write $s_r = h\,u \otimes t^{-1}$ with h a rational function; $\mathrm{div}(h)$ is a local equation of Dv_r^C. For a cotangent root, multiplication by s_r sends t to hu, so by (He4) the pair $\mathrm{gr}_j \omega' \subseteq \mathrm{gr}_j \omega$ is

isomorphic to $N(-\Delta_r) \subseteq N$ with $N = \pi^*\mathcal{O}(e_r)$ and $\Delta_r = \pi^*\mathrm{Dv}_r^C$; the inclusion shows that Δ_r is effective, the quotient $\mathcal{O}_{\Delta_r} \otimes N$ is supported in $\mathcal{X}'_{\mathfrak{s}}$ (Step 1), so Δ_r is supported in $\mathcal{X}'_{\mathfrak{s}}$, and $N(-\Delta_r) \cong \pi^*\mathcal{O}(e'_r)$. For a Lie root, multiplication by s_r^{-1} sends u to $h^{-1}t$, so the pair $\mathrm{gr}_j\mathrm{lie} \subseteq \mathrm{gr}_j W$ is isomorphic to $N(-\Delta_r) \subseteq N$ with $N = \pi^*\mathcal{O}(e'_r)$, $\Delta_r = -\pi^*\mathrm{Dv}_r^C$ effective and supported in $\mathcal{X}'_{\mathfrak{s}}$, and $N(-\Delta_r) \cong \pi^*\mathcal{O}(e_r)$. By (Vc2) (Lemma 35.14 (2) with one step), writing $\pi^*x_r = c_1(\pi^*\mathcal{O}(e_r))$ and $\pi^*x'_r = c_1(\pi^*\mathcal{O}(e'_r))$,

$$k!\,\mathrm{ch}_k(\mathrm{gr}_j\omega' \subseteq \mathrm{gr}_j\omega)\cdot M = \sum_{i=0}^{k-1} \deg\bigl(\pi^*x_r^{\,i}\,\pi^*x_r'^{\,k-1-i}\,M \cap [\pi^*\mathrm{Dv}_r^C]\bigr) \qquad (r \text{ cotangent}),$$

$$\begin{aligned} k!\,\mathrm{ch}_k(\mathrm{gr}_j\mathrm{lie} \subseteq \mathrm{gr}_j W)\cdot M &= \sum_{i=0}^{k-1} \deg\bigl(\pi^*x_r'^{\,i}\,\pi^*x_r^{\,k-1-i}\,M \cap [-\pi^*\mathrm{Dv}_r^C]\bigr) \\ &= -\sum_{i=0}^{k-1} \deg\bigl(\pi^*x_r^{\,i}\,\pi^*x_r'^{\,k-1-i}\,M \cap [\pi^*\mathrm{Dv}_r^C]\bigr) \qquad (r \text{ Lie}), \end{aligned}$$

using $[-D] = -[D]$ and reindexing $i \mapsto k-1-i$. With Step 1, and with $\alpha_{r,j} = x_r^j x_r'^{k-1-j} M$,

$$k!\,\mathrm{ch}_k(\omega' \subseteq \omega)\cdot M - k!\,\mathrm{ch}_k(\mathrm{lie} \subseteq W)\cdot M = \sum_{i=1}^{n}\sum_{j=0}^{k-1} \deg\bigl(\pi^*\alpha_{(a,i),j} \cap [\pi^*\mathrm{Dv}_{(a,i)}^C]\bigr). \tag{1}$$

Step 3: projection formula. Let $\mathcal{X}_{(\lambda)}$ (Definition 35.7) and the factorisation $\pi_\lambda\colon \mathcal{X}' \to \mathcal{X}_{(\lambda)}$ of π be as in Lemma 35.15, and let Z_r be the restriction of Dv_r^C to $\mathcal{X}_{(\lambda)}$ (pull-back along the flat morphism $\mathcal{X}_{(\lambda)} \to \mathcal{X}$), so that $\pi^*\mathrm{Dv}_r^C = \pi_\lambda^* Z_r$. By Step 2, $\pi_\lambda^* Z_r$ is supported in $\mathcal{X}'_{\mathfrak{s}}$; by Lemma 35.15 (1), Z_r is supported in the closed fibre of $\mathcal{X}_{(\lambda)}$, which is the part of $\mathcal{X}_{\mathbb{F}_p}$ over λ, so $[Z_r] = \mathrm{Dv}_{r,\lambda}$. Each $\alpha_{r,j}$ is a real combination of products of $(k-1) + (D+1-k) = D$ first Chern classes of line bundles on $\mathcal{X}$, restricted to $\mathcal{X}_{(\lambda)}$, so Lemma 35.15 (3) gives

$$\deg\bigl(\pi^*\alpha_{r,j} \cap [\pi^*\mathrm{Dv}_r^C]\bigr) = d_\pi \deg\bigl(\alpha_{r,j} \cap [\mathrm{Dv}_{r,\lambda}]\bigr),$$

with d_π the local degree of (Ts2). Substituting in (1) gives the assertion. □

Lemma 35.14. *Let $\mathcal{O}'$ and $\mathcal{X}'$ be as in Lemma 35.10, $1 \le k \le D+1$ and $M \in \mathcal{M}_k$. Let $T \subseteq H$ be a vertical pair of rank m, and let $0 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m = H$ be a filtration by locally free submodules with invertible quotients H_i/H_{i-1}. Put $T_i = T \cap H_i$ and assume that each T_i is locally free with T_i/T_{i-1} invertible ($1 \le i \le m$).*

1. *For every i the map $T_i/T_{i-1} \to H_i/H_{i-1}$ induced by the inclusion is injective, $T_i/T_{i-1} \subseteq H_i/H_{i-1}$ is a vertical pair of rank one, and*

$$\mathrm{ch}_k(T \subseteq H)\cdot M = \sum_{i=1}^{m} \mathrm{ch}_k(T_i/T_{i-1} \subseteq H_i/H_{i-1})\cdot M.$$

2. *If for every i there are an invertible module N_i and an effective Cartier divisor Δ_i supported in $\mathcal{X}'_{\mathfrak{s}}$ such that the pair $T_i/T_{i-1} \subseteq H_i/H_{i-1}$ is isomorphic to the pair $N_i(-\Delta_i) \subseteq N_i$, then*

$$k!\,\mathrm{ch}_k(T \subseteq H)\cdot M = \sum_{i=1}^{m}\sum_{a=0}^{k-1} \deg\Bigl(c_1(N_i)^a\, c_1\bigl(N_i(-\Delta_i)\bigr)^{k-1-a}\, M \cap [\Delta_i]\Bigr).$$

In particular, if $N_i(-\Delta_i) \cong \mathcal{O}_{\mathcal{X}'}$ for every i, i.e. the pairs are $\mathcal{O}_{\mathcal{X}'} \subseteq \mathcal{O}_{\mathcal{X}'}(\Delta_i)$, then $k!\,\mathrm{ch}_k(T \subseteq H)\cdot M = \sum_{i=1}^m \deg(c_1(\mathcal{O}(\Delta_i))^{k-1} M \cap [\Delta_i])$.

Proof. (1). Fix $1 \le i \le m$. The kernel of $T_i \to H_i/H_{i-1}$ is $T_i \cap H_{i-1} = T \cap H_{i-1} = T_{i-1}$, so $T_i/T_{i-1} \to H_i/H_{i-1}$ is injective. Since $T_i = T \cap H_i$, the map $H_i/T_i \to H/T$ is injective, so H_i/T_i is supported in $\mathcal{X}'_{\mathfrak{s}}$; the same holds for its quotient $(H_i/H_{i-1})/(T_i/T_{i-1}) = H_i/(H_{i-1}+T_i)$. Both T_i/T_{i-1} and H_i/H_{i-1} are invertible, so $T_i/T_{i-1} \subseteq H_i/H_{i-1}$ is a vertical pair of rank one. Likewise $T_i \subseteq H_i$ is a vertical pair: H_i/T_i is supported in $\mathcal{X}'_{\mathfrak{s}}$ and both modules are locally free of rank i (each is an iterated extension of i invertible modules).

For $2 \le i \le m$ consider

$$0 \to H_{i-1}/T_{i-1} \xrightarrow{\alpha} H_i/T_i \xrightarrow{\beta} H_i/(H_{i-1}+T_i) \to 0,$$

α induced by $H_{i-1} \subseteq H_i$ and β the quotient map. α is well defined and injective because $H_{i-1} \cap T_i = H_{i-1} \cap T = T_{i-1}$; β is surjective; and $\ker\beta = (H_{i-1}+T_i)/T_i = \operatorname{im}\alpha$. The third term is $(H_i/H_{i-1})/(T_i/T_{i-1})$. So (Vc3) of Lemma 35.10, applied to the vertical pairs $T_{i-1} \subseteq H_{i-1}$, $T_i \subseteq H_i$ and $T_i/T_{i-1} \subseteq H_i/H_{i-1}$, gives

$$\operatorname{ch}_k(T_i \subseteq H_i)\cdot M = \operatorname{ch}_k(T_{i-1} \subseteq H_{i-1})\cdot M + \operatorname{ch}_k\big(T_i/T_{i-1} \subseteq H_i/H_{i-1}\big)\cdot M.$$

For $i = 1$, $T_1 \subseteq H_1$ is the pair $T_1/T_0 \subseteq H_1/H_0$. Summing over i and using $T_m = T$, $H_m = H$ gives (1).

(2). The number $\operatorname{ch}_k(W \subseteq V)\cdot M$ depends only on the isomorphism class of the pair (it is defined from the two-term complex $W \to V$, Lemma 35.10); for the pair $N_i(-\Delta_i) \subseteq N_i$ the quotient is $N_i/N_i(-\Delta_i) \cong N_i \otimes \mathcal{O}_{\Delta_i}$, so (Vc2) gives the i-th inner sum, and (1) gives the formula. If $N_i(-\Delta_i) \cong \mathcal{O}$, then $c_1(N_i(-\Delta_i)) = 0$ and $N_i \cong \mathcal{O}(\Delta_i)$; in the inner sum every term with $k-1-a \ge 1$ vanishes and only $a = k-1$ remains (for $k = 1$ this is the single term $a = 0$, with the empty product 1), which is $\deg\big(c_1(\mathcal{O}(\Delta_i))^{k-1} M \cap [\Delta_i]\big)$. □

Lemma 35.15. *Let $(\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_{\mathfrak{P}}, \theta))$ be a pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$, let $\mathcal{X}'_{\mathfrak{s}}$ be the special fibre of $\mathcal{X}' \to \operatorname{Spec}\mathcal{O}'$ and $\mathcal{X}_{(\lambda),\mathfrak{s}}$ the closed fibre of $\mathcal{X}_{(\lambda)} \to \operatorname{Spec}\mathcal{O}_{F_\Omega,\lambda}$. Then π factors through a dominant morphism $\pi_\lambda\colon \mathcal{X}' \to \mathcal{X}_{(\lambda)}$, which maps $\mathcal{X}'_{\mathfrak{s}}$ into $\mathcal{X}_{(\lambda),\mathfrak{s}}$ by a proper morphism $\pi_{\mathfrak{s}}$, and for every Cartier divisor Z on $\mathcal{X}_{(\lambda)}$:*

1. *if $\pi_\lambda^* Z$ is supported in $\mathcal{X}'_{\mathfrak{s}}$, then Z is supported in $\mathcal{X}_{(\lambda),\mathfrak{s}}$;*
2. *if Z is supported in $\mathcal{X}_{(\lambda),\mathfrak{s}}$, then $\pi_\lambda^* Z$ is supported in $\mathcal{X}'_{\mathfrak{s}}$ and $\pi_{\mathfrak{s}*}[\pi_\lambda^* Z] = d_\pi\,[Z]$ as cycles of dimension D, where $[\,\cdot\,]$ is the Weil divisor;*
3. *if Z is supported in $\mathcal{X}_{(\lambda),\mathfrak{s}}$, then for every real combination α of products of D first Chern classes of line bundles $\mathcal{L}$ on $\mathcal{X}_{(\lambda)}$, with $\pi^*\alpha$ the same combination of the classes $c_1(\pi_\lambda^*\mathcal{L})$,*

$$\deg(\pi^*\alpha \cap [\pi_\lambda^* Z]) = d_\pi \deg(\alpha \cap [Z]),$$

degrees over $\mathbb{F}_p$ of 0-cycle classes on the proper $\mathbb{F}_p$-schemes $\mathcal{X}'_{\mathfrak{s}}$ and $\mathcal{X}_{(\lambda),\mathfrak{s}}$.

Proof. Write $\mathfrak{m}$ for the prime of $\mathcal{O}_{F'}$ with $\mathcal{O}' = \mathcal{O}_{F',\mathfrak{m}}$ (Ts1), $L = \Omega F' \subseteq \Omega'$, and for a point y of a scheme $k(y)$ for its residue field. Valuations are discrete valuations of rank one; for a point x of codimension one of a normal integral scheme, v_x is the normalised valuation of the discrete valuation ring $\mathcal{O}_x$. For a finite extension of valued fields, e and f are the ramification index and the residue degree; both are multiplicative in towers.

Step 0: factorisation and properness. $\mathcal{X}'$ is an $\mathcal{O}'$-scheme and $\mathcal{O}' \supseteq \mathcal{O}_{F_\Omega,\lambda}$; as π is a morphism of $\mathcal{O}_{F_\Omega}$-schemes (Ts2), the universal property of $\mathcal{X}_{(\lambda)} = \mathcal{X} \times_{\mathcal{O}_{F_\Omega}} \mathcal{O}_{F_\Omega,\lambda}$ gives π_λ with $\pi =$ (projection) $\circ\, \pi_\lambda$; it is dominant because π is and both schemes have function field Ω over $\mathcal{X}$.

As $\lambda\mathcal{O}' \subseteq \mathfrak{m}_{\mathcal{O}'}$, π_λ maps $\mathcal{X}'_{\mathfrak{s}}$ into $\mathcal{X}_{(\lambda),\mathfrak{s}}$. The scheme $\mathcal{X}'_{\mathfrak{s}}$ is projective over the residue field of $\mathcal{O}'$, which is finite over $k(\lambda)$, hence proper over $k(\lambda)$; $\mathcal{X}_{(\lambda),\mathfrak{s}}$ is separated over $k(\lambda)$; so $\pi_{\mathfrak{s}}$ is proper (EGA II, 5.4.3).

Step 1: dimensions. The bases $\mathcal{O}_{F_\Omega,\lambda}$ and $\mathcal{O}'$ are discrete valuation rings, hence universally catenary, and Ω has transcendence degree D over F_Ω, Ω' over F' ($[\Omega' : \Omega] < \infty$). By the dimension formula (Matsumura, Commutative Ring Theory, Thm. 15.6), a point y of $\mathcal{X}_{(\lambda)}$ has codimension $1 + D - \mathrm{trdeg}_{k(\lambda)}k(y)$ if it lies in the closed fibre and $D - \mathrm{trdeg}_{F_\Omega}k(y)$ otherwise; likewise on $\mathcal{X}'$ with $k(\mathfrak{m})$, F'. In particular $\mathcal{X}'_{\mathfrak{s}}$ has dimension D, and the points of $\mathcal{X}_{(\lambda),\mathfrak{s}}$ of transcendence degree D over $k(\lambda)$ are exactly its points of codimension one in $\mathcal{X}_{(\lambda)}$.

Step 2: the points of codimension one over a point of codimension one. Let x be a point of codimension one of $\mathcal{X}_{(\lambda)}$. We show: $z' \mapsto \mathcal{O}_{\mathcal{X}',z'}$ is a bijection from the set of points z' of codimension one of $\mathcal{X}'$ with $\pi_\lambda(z') = x$ onto the set V_x of valuation rings of Ω' of valuations v' extending v_x (that is, with valuation ring meeting Ω in $\mathcal{O}_x$) and containing $\mathcal{O}'$; and $V_x \neq \varnothing$ if x is not in the closed fibre. Let z' be such a point. $\mathcal{O}_{\mathcal{X}',z'}$ is a discrete valuation ring ($\mathcal{X}'$ normal) containing $\mathcal{O}'$ and dominating $\mathcal{O}_x$; its intersection with Ω is a valuation ring of Ω dominating the discrete valuation ring $\mathcal{O}_x$, hence equal to it (a valuation ring is maximal for domination). So $\mathcal{O}_{\mathcal{X}',z'} \in V_x$. The map is injective: a point of an integral separated scheme is determined by its local ring inside the function field (EGA I, 8.2.2). Let v' have valuation ring $R \in V_x$. As $R \supseteq \mathcal{O}'$ and $\mathcal{X}'$ is proper over $\mathcal{O}'$, the valuative criterion gives a point z' (the centre) with $\mathcal{O}_{\mathcal{X}',z'} \subseteq R$ dominated by R. Then $\mathcal{O}_{\pi_\lambda(z')}$ is dominated by $R \cap \Omega = \mathcal{O}_x$, and so is $\mathcal{O}_x$; by the valuative criterion of separatedness for $\mathcal{X}_{(\lambda)}$, $\pi_\lambda(z') = x$. Now $k(x) \subseteq k(z') \subseteq k(v')$ and $k(v')$ is finite over $k(v_x) = k(x)$ (Serre, Local Fields, I §4, Prop. 10), so $k(z')$ and $k(x)$ have the same transcendence degree. If x lies in the closed fibre, v_x is positive on λ, so v' is positive on λ and $z' \in \mathcal{X}'_{\mathfrak{s}}$; by Step 1 both have transcendence degree D over the finite fields $k(\lambda) \subseteq k(\mathfrak{m})$ and z' has codimension one. If x is not in the closed fibre, a uniformiser of λ is a unit at x, so v_x is trivial on F_Ω, hence v' is trivial on F' (algebraic over F_Ω, and a valuation ring is integrally closed), and z' lies in the generic fibre with $\mathrm{trdeg}_{F'}k(z') = \mathrm{trdeg}_{F_\Omega}k(x) = D - 1$; again z' has codimension one. In both cases $\mathcal{O}_{\mathcal{X}',z'}$ is a discrete valuation ring dominated by R, so equal to R. Finally, if x is not in the closed fibre, v_x has an extension v' to the finite extension Ω' (Serre, Local Fields, I §4), which is trivial on F', so its ring contains $\mathcal{O}'$: $V_x \neq \varnothing$. For z' and x as above and a local equation g of Z at x, $\pi_\lambda^* g$ is a local equation of $\pi_\lambda^* Z$ at z' and $v_{z'}(\pi_\lambda^* g) = e(v_{z'}|v_x)\, v_x(g)$.

Step 3: part 1. On the normal scheme $\mathcal{X}_{(\lambda)}$ a local equation g at a point y is a unit of $\mathcal{O}_y$ if and only if $v_x(g) = 0$ at every point x of codimension one specialising to y ($\mathcal{O}_y$ is the intersection of its localisations at height-one primes). Suppose the support of Z meets the generic fibre at y. Then some x of codimension one specialising to y, hence in the generic fibre (which is stable under generisation), has $v_x(Z) \neq 0$. By Step 2 there is a point z' of codimension one of $\mathcal{X}'$ over x, in the generic fibre of $\mathcal{X}'$, with $v_{z'}(\pi_\lambda^* Z) = e\, v_x(Z) \neq 0$; so $\pi_\lambda^* Z$ is not supported in $\mathcal{X}'_{\mathfrak{s}}$.

Step 4: the multiplicity. Let x be a point of codimension one in $\mathcal{X}_{(\lambda),\mathfrak{s}}$ and

$$N_x = \sum e(v'|v_x) f(v'|v_x),$$

the sum over the valuations v' with valuation ring in V_x. We show $N_x = d_\pi$. (a) $\Omega \otimes_{F_\Omega} F' \to L$ is an isomorphism. Write $F' = F_\Omega(\beta)$ with minimal polynomial g over F_Ω (characteristic 0). A monic factor of g in $\Omega[T]$ has coefficients that are polynomials in roots of g, hence algebraic over F_Ω, hence in F_Ω, which is algebraically closed in Ω. So g is irreducible over Ω, $\Omega \otimes_{F_\Omega} F' = \Omega[T]/(g)$ is a field, and its image is L. (b) Since x lies in the closed fibre, v_x is non-negative on $\mathcal{O}_{F_\Omega,\lambda}$ and positive on λ, so its restriction to F_Ω is a positive multiple of v_λ and the closure of F_Ω in the completion Ω_x is $\widehat{F}_{\Omega,\lambda}$. For a valuation u of L extending v_x, the restriction $u|_{F'}$ is likewise a positive multiple of $v_{\mathfrak{q}}$ for one prime $\mathfrak{q}$ of $\mathcal{O}_{F'}$ above λ; its ring

contains $\mathcal{O}' = \mathcal{O}_{F',\mathfrak{m}}$ if and only if $\mathfrak{q} = \mathfrak{m}$ (for $\mathfrak{q} \neq \mathfrak{m}$ pick $a \in \mathfrak{q} \setminus \mathfrak{m}$: then $a^{-1} \in \mathcal{O}'$ but $u(a^{-1}) < 0$). (c) By Serre, Local Fields, II §3, Thm. 1 and II §2, Prop. 3 (separable extensions), $L \otimes_\Omega \Omega_x \cong \prod_{u|v_x} L_u$ with $[L_u : \Omega_x] = e(u|v_x)f(u|v_x)$, and $F' \otimes_{F_\Omega} \widehat{F}_{\Omega,\lambda} \cong \prod_{\mathfrak{q}|\lambda} \widehat{F}'_{\mathfrak{q}}$. By (a),

$$L \otimes_\Omega \Omega_x \cong F' \otimes_{F_\Omega} \Omega_x \cong \prod_{\mathfrak{q}|\lambda} \widehat{F}'_{\mathfrak{q}} \otimes_{\widehat{F}_{\Omega,\lambda}} \Omega_x.$$

For each u, the projection to L_u restricted to $F' \otimes \widehat{F}_{\Omega,\lambda}$ has as image the $\widehat{F}_{\Omega,\lambda}$-span of F' in L_u, which is finite-dimensional, hence complete, hence the closure of F', i.e. the completion of F' for $u|_{F'}$; so this ring map factors through the factor $\widehat{F}'_{\mathfrak{q}}$ with $\mathfrak{q}$ the prime of $u|_{F'}$. Hence the idempotent of the factor $\mathfrak{q} = \mathfrak{m}$ is the sum of the idempotents of the L_u with $u|_{F'}$ of prime $\mathfrak{m}$, and comparing Ω_x-dimensions,

$$\sum_{u|v_x,\ \mathcal{O}' \subseteq R_u} e(u|v_x)f(u|v_x) = \dim_{\Omega_x}(\widehat{F}' \otimes_{\widehat{F}_{\Omega,\lambda}} \Omega_x) = [\widehat{F}' : \widehat{F}_{\Omega,\lambda}].$$

(d) For each u, $\sum_{v'|u} e(v'|u)f(v'|u) = [\Omega' : L]$ (Serre, Local Fields, I §4, Prop. 10; Ω'/L separable). A valuation v' of Ω' extends v_x with $\mathcal{O}'$ in its ring if and only if its restriction u to L does. By multiplicativity, $N_x = [\Omega' : L]\,[\widehat{F}' : \widehat{F}_{\Omega,\lambda}] = d_\pi$.

Step 5: part 2. Let Z be supported in $\mathcal{X}_{(\lambda),\mathfrak{s}}$. At a point of the generic fibre of $\mathcal{X}'$ the image lies in the generic fibre of $\mathcal{X}_{(\lambda)}$, where the local equations of Z are units; so $\pi_\lambda^* Z$ is supported in $\mathcal{X}'_{\mathfrak{s}}$ and $[\pi_\lambda^* Z] = \sum_{z'} v_{z'}(\pi_\lambda^* Z)\,[\overline{\{z'\}}]$ over the points $z' \in \mathcal{X}'_{\mathfrak{s}}$ of codimension one, each of dimension D (Step 1). By definition of proper push-forward (Fulton, Intersection Theory, §1.4), $\pi_{\mathfrak{s}*}[\overline{\{z'\}}] = [k(z') : k(x)]\,[\overline{\{x\}}]$ with $x = \pi_\lambda(z')$ if $\overline{\{x\}}$ has dimension D, and 0 otherwise. In the first case x is a point of codimension one of $\mathcal{X}_{(\lambda)}$ in the closed fibre (Step 1), $[k(z') : k(x)] = f(v_{z'}|v_x)$ and $v_{z'}(\pi_\lambda^* Z) = e(v_{z'}|v_x)v_x(Z)$ (Step 2). Conversely every point of codimension one of $\mathcal{X}'$ over such an x lies in $\mathcal{X}'_{\mathfrak{s}}$. By Step 2 and Step 4 the coefficient of $[\overline{\{x\}}]$ in $\pi_{\mathfrak{s}*}[\pi_\lambda^* Z]$ is $v_x(Z)\,N_x = d_\pi v_x(Z)$, which is its coefficient in $d_\pi[Z]$.

Step 6: part 3. The classes $c_1(\pi_\lambda^* \mathcal{L})$ act on cycles on $\mathcal{X}'_{\mathfrak{s}}$ through the restriction of $\pi_\lambda^* \mathcal{L} = \pi_{\mathfrak{s}}^*(\mathcal{L}|_{\mathcal{X}_{(\lambda),\mathfrak{s}}})$. By the projection formula for Chern classes of line bundles and the proper morphism $\pi_{\mathfrak{s}}$ (Fulton, Prop. 2.5(c)), $\pi_{\mathfrak{s}*}(\pi^*\alpha \cap [\pi_\lambda^* Z]) = \alpha \cap \pi_{\mathfrak{s}*}[\pi_\lambda^* Z] = d_\pi\,\alpha \cap [Z]$ by part 2, and proper push-forward preserves degrees of 0-cycle classes over $\mathbb{F}_p$ (Fulton, Def. 1.4). □

Lemma 35.16. *Let $\mathcal{O}'$ and $\mathcal{X}'$ be as in Lemma 35.10, $1 \le k \le D+1$ and $M \in \mathcal{M}_k$. Let $w \in \mathcal{O}'$ be non-zero and not a unit, and Δ_w the effective Cartier divisor of w on $\mathcal{X}'$.*

1. *(Scaling.) Let L be a locally free $\mathcal{O}_{\mathcal{X}'}$-module with a filtration $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_r = L$ by locally free submodules with invertible quotients $N_i = L_i/L_{i-1}$. Then $wL \subseteq L$ is a vertical pair and*

$$k!\,\mathrm{ch}_k(wL \subseteq L) \cdot M = k \sum_{i=1}^{r} \deg(c_1(N_i)^{k-1}\, M \cap [\Delta_w]).$$

2. *(Additivity along a Hodge–Tate type sequence.) Let $wL \subseteq W \subseteq L$ with L, W locally free of the same rank, let $\omega' \subseteq \omega$ and $T \subseteq H$ be vertical pairs, and let $0 \to L/W \to H/T \to \omega/\omega' \to 0$ be an exact sequence of $\mathcal{O}_{\mathcal{X}'}$-modules. Then $wL \subseteq W$ and $W \subseteq L$ are vertical pairs and*

$$\mathrm{ch}_k(wL \subseteq W) \cdot M + \mathrm{ch}_k(T \subseteq H) \cdot M = \mathrm{ch}_k(wL \subseteq L) \cdot M + \mathrm{ch}_k(\omega' \subseteq \omega) \cdot M.$$

3. *(Change of the scaling element.) Let $g, h \in \mathcal{O}'$ be non-zero non-units and $m, n \geq 1$ integers with $g^m = u\,h^n$ for a unit u of $\mathcal{O}'$. Then $m[\Delta_g] = n[\Delta_h]$ as cycles on $\mathcal{X}'_{\mathfrak{s}}$, and for L as in (1), $m\,\mathrm{ch}_k(gL \subseteq L)\cdot M = n\,\mathrm{ch}_k(hL \subseteq L)\cdot M$. In particular, if p is the residue characteristic and $w^{p-1} = u\,p$ with u a unit, then*

$$k!\,\mathrm{ch}_k(wL \subseteq L)\cdot M = \frac{k}{p-1}\sum_{i=1}^{r} \deg(c_1(N_i)^{k-1}\,M \cap [\mathrm{div}(p)]),$$

$\mathrm{div}(p) = \Delta_p$ the Cartier divisor of p.

Proof. As $\mathcal{X}'$ is flat over the domain $\mathcal{O}'$, w is a non-zero-divisor on $\mathcal{O}_{\mathcal{X}'}$, so its zero scheme Δ_w is an effective Cartier divisor, with $\mathcal{O}(-\Delta_w) = w\mathcal{O}_{\mathcal{X}'} \cong \mathcal{O}_{\mathcal{X}'}$; it is supported in the special fibre, w being a unit on the generic fibre. For a locally free L, $wL \cong L$ is locally free of the same rank and $L/wL = L \otimes \mathcal{O}_{\Delta_w}$ is supported in the special fibre: $wL \subseteq L$ is a vertical pair. In (2), L/W and W/wL are a quotient and a submodule of L/wL, so $W \subseteq L$ and $wL \subseteq W$ are vertical pairs.

(1) For each i, $0 \to L_{i-1} \to L_i \to N_i \to 0$ is an exact sequence of locally free modules, so it stays exact after $\otimes\,\mathcal{O}_{\Delta_w}$ ($\mathrm{Tor}_1(N_i, \mathcal{O}_{\Delta_w}) = 0$): $0 \to L_{i-1}/wL_{i-1} \to L_i/wL_i \to N_i/wN_i \to 0$. By (Vc3) of Lemma 35.10 and induction on i,

$$\mathrm{ch}_k(wL \subseteq L)\cdot M = \sum_{i=1}^{r} \mathrm{ch}_k(wN_i \subseteq N_i)\cdot M.$$

Now $N_i/wN_i \cong N_i \otimes \mathcal{O}_{\Delta_w}$, so (Vc2) applies with $N = N_i$ and $\Delta = \Delta_w$; since $N_i(-\Delta_w) = N_i \otimes w\mathcal{O} \cong N_i$, $c_1(N_i(-\Delta_w)) = c_1(N_i)$, and each of the k summands of (Vc2) equals $\deg(c_1(N_i)^{k-1}M \cap [\Delta_w])$. Hence $k!\,\mathrm{ch}_k(wN_i \subseteq N_i)\cdot M = k\deg(c_1(N_i)^{k-1}M \cap [\Delta_w])$, and (1) follows.

(2) By (Vc1) for $wL \subseteq W \subseteq L$, $\mathrm{ch}_k(wL \subseteq L)\cdot M = \mathrm{ch}_k(wL \subseteq W)\cdot M + \mathrm{ch}_k(W \subseteq L)\cdot M$. By (Vc3) for the given exact sequence, $\mathrm{ch}_k(T \subseteq H)\cdot M = \mathrm{ch}_k(W \subseteq L)\cdot M + \mathrm{ch}_k(\omega' \subseteq \omega)\cdot M$. Eliminating $\mathrm{ch}_k(W \subseteq L)\cdot M$ gives (2).

(3) Local equations of Cartier divisors multiply, so $\Delta_{g^m} = m\Delta_g$, $\Delta_{uh^n} = \Delta_u + n\Delta_h$ and $\Delta_u = 0$ (u is a unit on $\mathcal{X}'$); hence $m\Delta_g = n\Delta_h$ as Cartier divisors. The Weil divisor $[D] = \sum_V \mathrm{ord}_V(D)[V]$ of a Cartier divisor on the noetherian integral scheme $\mathcal{X}'$ is additive in D (Fulton, Intersection Theory, §2.1), so $m[\Delta_g] = n[\Delta_h]$. The operations $c_1(\cdot)\cap$ and deg are additive on cycle classes of $\mathcal{X}'_{\mathfrak{s}}$, so by (1) $m\,k!\,\mathrm{ch}_k(gL \subseteq L)\cdot M = k\sum_i \deg(c_1(N_i)^{k-1}M \cap m[\Delta_g]) = k\sum_i \deg(c_1(N_i)^{k-1}M \cap n[\Delta_h]) = n\,k!\,\mathrm{ch}_k(hL \subseteq L)\cdot M$. The last formula is the case $g = w$, $m = p-1$, $h = p$, $n = 1$ (p is a non-zero non-unit of $\mathcal{O}'$, its residue field having characteristic p and its fraction field characteristic 0), combined with (1) for $h = p$. □

Definition 35.17. Let p be a prime and C an algebraically closed field of characteristic 0, complete for a non-archimedean absolute value with $|p| < 1$, with valuation ring $\mathcal{O}_C$.

(Af1) Almost-additive functions; generation property Let N be a finite abelian group and $M \subseteq \mathrm{Map}(N, C)$ an $\mathcal{O}_C$-submodule. A function $f\colon N \to C$ is **almost additive** if $f(x+y) - f(x) - f(y) \in p\,\mathcal{O}_C$ for all $x, y \in N$. For a subgroup $K \leq N$ put $M^K = \{f \in M : f(x+k) = f(x)\ \forall x \in N,\ k \in K\}$. The module M has the **generation property at** K if M^K is contained in the $\mathcal{O}_C$-subalgebra of $\mathrm{Map}(N, C)$ generated by the almost-additive elements of M^K.

(Af2) Function module of the p-torsion For an abelian scheme A over $\mathcal{O}_C$, let N_A be the finite abelian group $A[p](\mathcal{O}_C)$, and let $M_A \subseteq \mathrm{Map}(N_A, C)$ be the image of $\Gamma(A[p], \mathcal{O})$ under $a \mapsto (x \mapsto x^*a)$ (evaluation at the $\mathcal{O}_C$-points).

(Af3) Abelian factors An $\mathcal{O}_C$-submodule $M \subseteq \mathrm{Map}(N, C)$, N a finite abelian group, is an **abelian factor** if there are an abelian scheme A over $\mathcal{O}_C$ admitting a relatively ample line bundle and group homomorphisms $\iota\colon N \to N_A$, $\rho\colon N_A \to N$ with $\rho \circ \iota = \mathrm{id}_N$, such that $g \circ \iota \circ \rho \in M_A$ for every $g \in M_A$ and $M = \{g \circ \iota :\ g \in M_A\}$.

Definition 35.18. Let p be a prime, C an algebraically closed field of characteristic 0, complete for a non-archimedean absolute value with $|p| < 1$, $\mathcal{O}_C$ its valuation ring and $v\colon C \to \mathbb{R} \cup \{\infty\}$ the additive valuation with $v(p) = 1$. Let $\mathbb{F}$ be a finite field of characteristic p and N a finite-dimensional $\mathbb{F}$-vector space.

(Ho1) Normalised lengths If an $\mathcal{O}_C$-module Q is isomorphic to $\bigoplus_{j=1}^{r} \mathcal{O}_C/c_j\mathcal{O}_C$ with $c_j \in \mathcal{O}_C \smallsetminus \{0\}$, its **normalised length** is $\mathrm{len}_C(Q) = \sum_j v(c_j)$ (independent of the decomposition: the 0-th Fitting ideal of Q is $(\prod_j c_j)$). If $B \subseteq \mathrm{Map}(N, C)$ is the $\mathcal{O}_C$-span of the columns of a matrix in $\mathrm{GL}_N(C)$ of determinant Δ, its **colength** is $\mathrm{colen}(B) = v(\Delta)$ (independent of the matrix).

(Ho2) Hopf orders and cotangent eigenmodules An $\mathcal{O}_C$-submodule $B \subseteq \mathrm{Map}(N, C)$ is a **Hopf order on** N if it is the span of the columns of a matrix in $\mathrm{GL}_N(C)$, contains 1, is closed under products, and for every $g \in B$ the function $(x, y) \mapsto g(x + y)$ lies in the $\mathcal{O}_C$-span of the functions $(x, y) \mapsto g_1(x)g_2(y)$, $g_1, g_2 \in B$, and $x \mapsto g(-x)$ lies in B. It is **$\mathbb{F}$-stable** if $x \mapsto g(ax)$ lies in B for all $g \in B$, $a \in \mathbb{F}$. For an $\mathcal{O}_C$-subalgebra $B \subseteq \mathrm{Map}(N', C)$ (N' a subspace of N, or N) put $I_B = \{g \in B :\ g(0) = 0\}$ and let I_B^2 be the $\mathcal{O}_C$-span of the products gh, $g, h \in I_B$; for a map $\chi\colon \mathbb{F} \to \mathcal{O}_C$ with $\chi(0) = 0$ whose restriction to $\mathbb{F}^\times$ is a homomorphism to $\mathcal{O}_C^\times$ (a **character**), the **cotangent eigenmodule** is

$$\mathrm{co}_\chi(B) = I_{B,\chi}/(I_{B,\chi} \cap I_B^2), \qquad I_{B,\chi} = \{g \in I_B :\ g(ax) = \chi(a)g(x) \text{ for all } a \in \mathbb{F},\ x \in N'\}.$$

For a subspace $N' \subseteq N$, $B|_{N'} \subseteq \mathrm{Map}(N', C)$ is the image of B under restriction of functions. A character χ is **fundamental** if $\chi(a + b) - \chi(a) - \chi(b) \in p\,\mathcal{O}_C$ for all $a, b \in \mathbb{F}$.

(Ho3) Tightness Let B be a Hopf order on N. Let $\mathrm{Lin}(N)$ be the $\mathcal{O}_C$-module of almost-additive maps $N \to \mathcal{O}_C$ and $\mathrm{AF}(B)$ the almost-additive elements of B. The Hopf order B is **tight** if the modules $Q_B = \mathrm{Lin}(N)/\big(\mathrm{AF}(B) + p\,\mathrm{Map}(N, \mathcal{O}_C)\big)$ and I_B/I_B^2 have normalised lengths and

$$|N|\,(p-1)\,\mathrm{len}_C(Q_B) = 2\,\mathrm{colen}(B), \qquad |N|\,\mathrm{len}_C(I_B/I_B^2) = 2\,\mathrm{colen}(B).$$

(Ho4) Eigen-coordinate ideals; flag conditions Let $b_1, \dots, b_m$ be a basis of N, $N_i = \mathbb{F}b_1 \oplus \cdots \oplus \mathbb{F}b_i$ ($N_0 = 0$), B a Hopf order on N and χ a character. For $1 \le i \le m$ let

$$\begin{aligned}
\mathrm{Xr}_{\chi,i}(B) &= \{c \in C :\ \exists g \in B,\ g(ab_i + k) = c\,\chi(a)\ \ \forall a \in \mathbb{F},\ k \in N_{i-1}\},\\
\mathrm{Xv}_{\chi,i}(B) &= \{c \in C :\ \exists g \in B,\ g(x + k) = g(x)\ \ \forall x \in N,\ k \in N_{i-1},\\
&\qquad\qquad \text{and } g(ab_i) = c\,\chi(a)\ \ \forall a \in \mathbb{F}\},\\
\mathrm{Ad}_{\chi,i}(B) &= \{c \in \mathcal{O}_C :\ \exists g \in \mathrm{AF}(B),\ g(ax) - \chi(a)g(x) \in p\mathcal{O}_C\ \forall a, x;\\
&\qquad\qquad g(N_{i-1}) \subseteq p\mathcal{O}_C;\ g(b_i) - c \in p\mathcal{O}_C\}.
\end{aligned}$$

The pair $(B, (b_i))$ satisfies the **flag conditions** for a fundamental character χ if there are $c_{i,s} \in \mathcal{O}_C \smallsetminus \{0\}$ and $\delta_{i,s} \in \mathcal{O}_C$ ($1 \le i \le m$, $s \ge 0$) such that, with $\chi_s = \chi^{p^s}$:

1. $\mathrm{Xr}_{\chi_s,i}(B) = \mathrm{Xv}_{\chi_s,i}(B) = \mathrm{Ad}_{\chi_s,i}(B) = c_{i,s}\mathcal{O}_C$;

2. $c^p_{i,s} = \delta_{i,s}\, c_{i,s+1}$;
3. for $0 \le j \le m$ the module $\mathrm{co}_{\chi_{s+1}}(B|_{N_j})$ has a normalised length, equal to $\sum_{i\le j} v(\delta_{i,s})$.

(Ho5) Step orders For a Hopf order B on N, an $\mathbb{F}$-subspace $K \subseteq N$ and $t \in N \smallsetminus K$, the **step order** is $B_{K,t} = \{a \mapsto g(at) :\ g \in B,\ g(x+k) = g(x)\ \forall x \in N,\ k \in K\} \subseteq \mathrm{Map}(\mathbb{F}, C)$; for $S \subseteq \mathrm{Map}(\mathbb{F}, C)$ and a character χ put $\mathrm{Xr}_\chi(S) = \{c \in C :\ c\chi \in S\}$. Thus $\mathrm{Xv}_{\chi,i}(B) = \mathrm{Xr}_\chi(B_{N_{i-1},b_i})$.

Definition 35.19. Let $\mathfrak{T} = (\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_\mathfrak{P}, \theta))$ be a pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$, and let Ω' be the function field of $\mathcal{X}'$. For a label t, $\mathfrak{P} = \mathfrak{P}_t$ and $1 \le i \le m_\mathfrak{P}$ let $\mathcal{S}_{t,i} = \{g \in \mathcal{A}_\mathfrak{P} :\ g|_{\Gamma_{\mathfrak{P},i}} \in \Omega'\xi_{t,i}\}$, and let the **step sheaves** $\mathcal{U}_{t,j}$ $(0 \le j \le m_\mathfrak{P})$, $\mathcal{U}^{\mathrm{st}}_{t,i} \subseteq \omega_t$ be the preimages of $\theta_t(\mathcal{I}_{\mathfrak{P},j})$, resp. $\theta_t(\mathcal{S}_{t,i})$, under $\omega_t \to \omega_t/p\,\omega_t$.

(Ox1) Test points A **test point** of $\mathfrak{T}$ is a triple $(\mathfrak{q}, C, \iota)$: $\mathfrak{q}$ a point of $\mathcal{X}'$ of codimension one at which p is not a unit; C an algebraically closed field of characteristic 0, complete for a non-archimedean absolute value with $|p| < 1$, with valuation ring $\mathcal{O}_C$; $\iota\colon \Omega' \to C$ a field embedding with $\iota^{-1}(\mathcal{O}_C) = \mathcal{O}_{\mathcal{X}',\mathfrak{q}}$. We identify Ω' with $\iota(\Omega')$ and put $R_\mathfrak{q} = \mathcal{O}_{\mathcal{X}',\mathfrak{q}}$.

(Ox2) Base change At a test point, $\mathcal{A}_{\mathfrak{P},C}$ is the $\mathcal{O}_C$-span of $\mathcal{A}_{\mathfrak{P},\mathfrak{q}}$ in $\mathrm{Map}(\Gamma_\mathfrak{P}, C)$, $\mathcal{I}_{\mathfrak{P},C} = \{g \in \mathcal{A}_{\mathfrak{P},C} :\ g(0) = 0\}$, $\mathcal{I}_{\mathfrak{P},i,C} = \{g \in \mathcal{A}_{\mathfrak{P},C} :\ g|_{\Gamma_{\mathfrak{P},i}} = 0\}$, $\mathcal{S}_{t,i,C} = \{g \in \mathcal{A}_{\mathfrak{P},C} :\ g|_{\Gamma_{\mathfrak{P},i}} \in C\,\xi_{t,i}\}$, and $\theta_{u,C}\colon \mathcal{I}_{\mathfrak{P},C} \to \mathcal{O}_C \otimes_{R_\mathfrak{q}} (\omega_u/p\,\omega_u)_\mathfrak{q}$ is the $\mathcal{O}_C$-linear extension of the stalk of θ_u at $\mathfrak{q}$.

(Ox3) Test-point conditions For every prime $\mathfrak{P} \mid p$ of $\mathcal{O}_E$: $\mathcal{A}_{\mathfrak{P},C}$ is an $\mathbb{F}_\mathfrak{P}$-stable tight Hopf order on $\Gamma_\mathfrak{P}$ and an abelian factor; and for every label u with $\mathfrak{P}_u = \mathfrak{P}$: $\theta_{u,C}(gh) = 0$ for $g, h \in \mathcal{I}_{\mathfrak{P},C}$; $\theta_{u,C}(x \mapsto g(ax)) = \chi_u(a)\,\theta_{u,C}(g)$ for $a \in \mathbb{F}_\mathfrak{P}$, $g \in \mathcal{I}_{\mathfrak{P},C}$; and every $g \in \mathcal{I}_{\mathfrak{P},C}$ with $g(ax) = \chi_u(a)g(x)$ for all $a \in \mathbb{F}_\mathfrak{P}$, $x \in \Gamma_\mathfrak{P}$, and $\theta_{u,C}(g) = 0$ lies in the $\mathcal{O}_C$-span of the products of two elements of $\mathcal{I}_{\mathfrak{P},C}$.

(Ox4) Test-point step facts For every label s, $t = \varphi_p s$, $\mathfrak{P} = \mathfrak{P}_s$ and $1 \le i \le m_\mathfrak{P}$:

$$\theta_{t,C}(\mathcal{I}_{\mathfrak{P},i-1,C}) = \theta_{t,C}(\mathcal{S}_{t,i,C}), \qquad \{c \in C :\ \exists g \in \mathcal{A}_{\mathfrak{P},C},\ g|_{\Gamma_{\mathfrak{P},i}} = c\,\xi_{t,i},\ \theta_{t,C}(g) = 0\} = \mathcal{O}_C\,\epsilon^p,$$

where ϵ generates $\mathcal{O}(\mathrm{Ec}_{s,i})_\mathfrak{q}$.

Proposition 35.20. *Let $\mathcal{T}$, $\mathcal{Y}$, E, c, $2 \le B_0 \le B$, $\mathcal{X} \subseteq \mathbb{P}$, $p \in S(B_0, B)$ and λ be as in Proposition 35.11, $\mathcal{T}$ of type (Φ, n, κ). Then there are an admissible torsion system $(\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_\mathfrak{P}, \theta))$ for $(\mathcal{X}, p, \lambda)$ and an element $w \in \mathcal{O}'$ with $w^{p-1} = u\,p$ for a unit u of $\mathcal{O}'$, with the following data for every $a \in \Phi$. Put $s = (a, +)$, $\mathfrak{P} = \mathfrak{P}_s$, $m = m_\mathfrak{P}$, and let Ω' be the function field of $\mathcal{X}'$. There are Ω'-spaces V_L, V_ω of dimensions κ, $n - \kappa$ and $V_H = \Omega'^m$; locally free $\mathcal{O}_{\mathcal{X}'}$-submodules* $\mathrm{lie}, W \subseteq V_L$, $\omega', \omega \subseteq V_\omega$, $H \subseteq V_H$ *of full rank; $T = \mathcal{O}^m_{\mathcal{X}'} \subseteq V_H$; and $\mathcal{O}_{\mathcal{X}'}$-submodules $\Gamma \subseteq V_L \oplus V_H$, $\Delta \subseteq V_H \oplus V_\omega$, such that:*

(He1) Inclusions and graphs $\mathrm{lie} \subseteq W \subseteq w^{-1}\mathrm{lie}$, $\omega' \subseteq \omega \subseteq w^{-1}\omega'$, $T \subseteq H$, *and $H = T$ on the open subscheme where p is invertible; Γ, Δ satisfy (G1), (G2) of Lemma 35.12, and the sheaf I there is locally free.*

(He2) Eigen flag *for $1 \le i \le m$, $H_i = H \cap (\Omega'^i \oplus 0)$ is locally free with H_i/H_{i-1} invertible; let $J_i \subseteq \Omega'$ be the image of H_i under the i-th coordinate.*

(He3) Facts at test points *for every point q of $\mathcal{X}'$ of codimension one at which p is not a unit there are an algebraically closed field C, complete for a valuation with valuation ring $\mathcal{O}_C$, and a field embedding $\Omega' \to C$ with $\mathcal{O}_C \cap \Omega' = \mathcal{O}_{\mathcal{X}',q}$, such that (F0)–(F2) of Lemma 35.12 hold for the $\mathcal{O}_C$-spans of the stalks at q, and $\mathcal{O}_C J_{i,q} = \mathcal{O}_C\,\mathcal{O}_{\mathcal{X}'}(-\mathrm{Ec}_{s,i})_q$ in C for $1 \le i \le m$.*

(He4) Root lines *there are full flags of V_L and of V_ω such that the induced filtrations* $\mathrm{lie}_j = \mathrm{lie} \cap F_j$, $W_j = W \cap F_j$ *($1 \le j \le \kappa$) and* ω_j, ω'_j *($1 \le j \le n-\kappa$) are by locally free submodules with invertible graded pieces, with isomorphisms* $\mathrm{gr}_j \mathrm{lie} \cong \pi^*\mathcal{O}(e_{(a,j)})$, $\mathrm{gr}_j W \cong \pi^*\mathcal{O}(e'_{(a,j)})$, $\mathrm{gr}_j\omega \cong \pi^*\mathcal{O}(e_{(a,\kappa+j)})$, $\mathrm{gr}_j\omega' \cong \pi^*\mathcal{O}(e'_{(a,\kappa+j)})$, *under which the inclusions* $\mathrm{gr}_j\mathrm{lie} \subseteq \mathrm{gr}_j W$ *and* $\mathrm{gr}_j\omega' \subseteq \mathrm{gr}_j\omega$ *are the pull-backs of multiplication by g_r/f_r, resp. f_r/g_r, for the corresponding root r.*

Proof. Let $\mathfrak{T} = (\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_\mathfrak{P}, \theta))$, w and the data for $a \in \Phi$ be as given by Proposition 35.23. The data satisfy (He1)–(He4) there; it remains to show that $\mathfrak{T}$ satisfies (Ts6). By (He6) and Lemma 35.63 it suffices that every point $\mathfrak{q}$ of $\mathcal{X}'$ of codimension one at which p is not a unit is the first entry of a test point at which (Ox4) holds. Take the test point $(\mathfrak{q}, C, \iota)$ of (He5), at which (Ox3) holds.

Step 1: the generation property. For every $\mathfrak{P} \mid p$, $\mathcal{A}_{\mathfrak{P},C}$ is an abelian factor (Ox3), so by Proposition 35.43 it has the generation property at every subgroup of $\Gamma_\mathfrak{P}$, in particular at $\Gamma_{\mathfrak{P},i}$, $0 \le i < m_\mathfrak{P}$.

Step 2: the characters χ_u are fundamental. Let u be a label, $\mathfrak{P} = \mathfrak{P}_u$, $q = |\mathbb{F}_\mathfrak{P}|$. As $p \notin \mathrm{Bd}$, p is unramified in E, so the completion $\widehat{W}$ of $\mathcal{O}_{E,(\mathfrak{P})}$ has maximal ideal $p\widehat{W}$; let $\omega\colon \mathbb{F}_\mathfrak{P} \to \widehat{W}$ be the Teichmüller map ($\omega(0) = 0$, $\omega(a)^q = \omega(a)$, $\omega(a) \equiv a$ modulo $p\widehat{W}$), so $\omega(a+b) - \omega(a) - \omega(b) \in p\widehat{W}$. Since $\sigma_u^{-1}(\mathfrak{m}_{\mathcal{O}'}) = \mathfrak{P}$, σ_u is continuous from the $\mathfrak{P}$-adic to the $\mathfrak{m}_{\mathcal{O}'}$-adic topology and extends to $\hat\sigma_u\colon \widehat{W} \to \widehat{\mathcal{O}'}$. The values of $\hat\sigma_u \circ \omega$ are 0 or roots of unity of order dividing $q-1$, hence lie in $\mathcal{O}'$ (Ts1), and $\hat\sigma_u(\omega(a)) \equiv \sigma_u(\tilde a)$ modulo $\mathfrak{m}$; as reduction modulo $\mathfrak{m}$ is injective on roots of unity of order prime to p, $\hat\sigma_u \circ \omega = \chi_u$. Hence $\chi_u(a+b) - \chi_u(a) - \chi_u(b) \in p\widehat{\mathcal{O}'} \cap \mathcal{O}' = p\mathcal{O}' \subseteq p\,\mathcal{O}_C$ ($\mathcal{O}' \subseteq R_\mathfrak{q} \subseteq \mathcal{O}_C$), and χ_u restricted to $\mathbb{F}_\mathfrak{P}^\times$ is a homomorphism to $\mathcal{O}_C^\times$: χ_u is a fundamental character.

Step 3: flag conditions and step facts. By (Ox3), $\mathcal{A}_{\mathfrak{P},C}$ is an $\mathbb{F}_\mathfrak{P}$-stable tight Hopf order on $\Gamma_\mathfrak{P}$; with Steps 1 and 2, Theorem 35.58 ($N = \Gamma_\mathfrak{P}$, $\mathbb{F} = \mathbb{F}_\mathfrak{P}$, $B = \mathcal{A}_{\mathfrak{P},C}$, the basis (b_i) of (Ts4)) gives the flag conditions for every χ_u, $\mathfrak{P}_u = \mathfrak{P}$. By Lemma 35.61, (Ox4) holds at $(\mathfrak{q}, C, \iota)$. □

Definition 35.21. Let R be a noetherian integral domain with fraction field K and Γ a commutative group scheme over R that is flat, quasi-finite, separated and of finite presentation, whose generic fibre Γ_K is reduced and constant: there is a finite abelian group T and an injective homomorphism $t \mapsto x_t$ from T to $\Gamma(K)$ whose images exhaust the points of Γ over the generic point of $\operatorname{Spec} R$. A function $f\colon T \to R$ is **regular on** Γ if for every point γ of Γ there are an open neighbourhood U of γ and $s \in \mathcal{O}_\Gamma(U)$ with $s(x_t) = f(t)$ in K for every t with $x_t \in U$. The **regular-translate order** of Γ is the R-subalgebra

$$\mathrm{Ord}(\Gamma) \subseteq \mathrm{Map}(T, R)$$

generated by the functions f such that every translate $f(t' + \cdot)$, $t' \in T$, is regular on Γ. For a ring homomorphism $h\colon R \to \mathcal{O}$ into a domain $\mathcal{O}$ with fraction field L, $\mathrm{Ord}(\Gamma)_h$ denotes the $\mathcal{O}$-span of $\{h \circ f :\ f \in \mathrm{Ord}(\Gamma)\}$ in $\mathrm{Map}(T, L)$.

Definition 35.22. Let $\mathfrak{T} = (\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_\mathfrak{P}, \theta))$ be a pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$, with function field Ω' of $\mathcal{X}'$, and let $G' = \pi^*G_\mathcal{X}$, a semiabelian scheme over $\mathcal{X}'$ with $\mathcal{O}_E$-action; $\mathcal{X}'$ with $f \circ \pi$ is a cover of $\mathcal{Y}$. For a prime $\mathfrak{P} \mid p$ of $\mathcal{O}_E$ let $G'[\mathfrak{P}] \subseteq G'$ be the closed subgroup scheme killed by every element of $\mathfrak{P}$.

(Hf1) Torsion-regular The system $\mathfrak{T}$ is **torsion-regular** if for every prime $\mathfrak{P} \mid p$ the group scheme $G'[\mathfrak{P}]$ is flat, quasi-finite, separated and of finite presentation over $\mathcal{X}'$, and for

every affine open $U = \operatorname{Spec} R \subseteq \mathcal{X}'$

$$\mathcal{A}_{\mathfrak{P}}(U) = \operatorname{Ord}(G'[\mathfrak{P}] \times_{\mathcal{X}'} U) \subseteq \operatorname{Map}(\Gamma_{\mathfrak{P}}, R),$$

the regular-translate order formed with $T = \Gamma_{\mathfrak{P}}$ and its points over Ω'; and the cotangent maps are the differentials at the unit section: for $g \in \mathcal{I}_{\mathfrak{P}}(U)$ and a regular function $\tilde{g}$ on an open neighbourhood of the unit section of $G'[\mathfrak{P}] \times_{\mathcal{X}'} U$ with values g at the points of $\Gamma_{\mathfrak{P}}$ in it, $\theta_u(g)$ is the σ_u-isotypic component of the class of $e^* d\tilde{g}$ in $e^*\Omega^1_{G'[\mathfrak{P}]/\mathcal{X}'} = \omega_{G'}/\mathfrak{P}\,\omega_{G'} = \bigoplus_{\mathfrak{P}_u=\mathfrak{P}} \omega_u/p\,\omega_u$ (e the unit section; p is unramified in E).

Fix an integer $R \geq 1$ such that every p^R-torsion point of G' over an algebraic closure of Ω' is defined over Ω', and write $V_{s,x} = V_{s,x}^{\{p\}}$ for the Hodge lattice at $(\{p\}, R)$ of the cover $\mathcal{X}'$ at a point x (the Hodge–Tate lattice $\mathrm{HT}_{s,x}(p^R)$ if $p \notin \mathcal{O}_x^\times$, the de Rham lattice $\mathrm{dR}_{s,x}$ otherwise). Let $a \in \Phi$, $s = (a, +)$, $\bar{s} = (a, -)$, $\kappa = \mathrm{rk}(s)$, $n - \kappa = \mathrm{rk}(\bar{s})$, $\mathfrak{P} = \mathfrak{P}_s$, $m = m_{\mathfrak{P}}$, and fix Ω'-bases $\beta = (\beta_1, \dots, \beta_\kappa)$ of $\mathrm{Lie}_s(G'_{\Omega'})$ and $\bar{\beta} = (\bar{\beta}_1, \dots, \bar{\beta}_{n-\kappa})$ of $\mathrm{Lie}_{\bar{s}}(G'_{\Omega'})$. Put $V_L = \Omega'^{\kappa}$, $V_H = \Omega'^{m}$, $V_\omega = \Omega'^{n-\kappa}$, regarded as constant sheaves, with the dot product $c \cdot c' = \sum_k c_k c'_k$.

(Hf2) Coordinate lattices For an open $U \subseteq \mathcal{X}'$ let

$$\begin{aligned}
\mathrm{lie}(U) &= \{c \in V_L : \textstyle\sum_k c_k \beta_k \in \mathrm{dR}_{s,x} \text{ for all } x \in U\},\\
W(U) &= \{c \in V_L : \textstyle\sum_k c_k \beta_k \in V_{s,x} \text{ for all } x \in U\},\\
\bar{\mathrm{l}}(U) &= \{c \in V_\omega : \textstyle\sum_k c_k \bar{\beta}_k \in \mathrm{dR}_{\bar{s},x} \text{ for all } x \in U\},\\
\bar{W}(U) &= \{c \in V_\omega : \textstyle\sum_k c_k \bar{\beta}_k \in V_{\bar{s},x} \text{ for all } x \in U\},
\end{aligned}$$

and let $\omega, \omega' \subseteq V_\omega$ be the **dual lattices** of $\bar{\mathrm{l}}, \bar{W}$: $\omega(U) = \{c : c \cdot c' \in \mathcal{O}(U') \text{ for all opens } U' \subseteq U \text{ and } c' \in \bar{\mathrm{l}}(U')\}$, and likewise ω' from $\bar{W}$. These are the **de Rham** and **Hodge–Tate coordinate lattices** of (s, β) and their duals for $(\bar{s}, \bar{\beta})$.

(Hf3) Eigen lattice Let $T = \mathcal{O}_{\mathcal{X}'}^m \subseteq V_H$. Writing $x = \sum_i x_i b_i$ for $x \in \Gamma_{\mathfrak{P}}$ and, for $c \in V_H$, $\ell_c(x) = \sum_{i=1}^m c_i \chi_s(x_i)$, let

$$\mathrm{af}(U) = \{c \in V_H : \ell_c \in \mathcal{A}_{\mathfrak{P}}(U) + p \operatorname{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}(U))\},$$

and let $H \subseteq V_H$ be the dual lattice of af, formed as in (Hf2): the **Hodge–Tate eigen lattice**.

(Hf4) Hodge–Tate charts and atlases By **points** for $(\mathfrak{T}, s)$ we mean an $\mathcal{O}_E/p^R$-basis $y = (y_1, \dots, y_m)$ of $G'(\Omega')[p^R]$ (which is free of rank m over $\mathcal{O}_E/p^R$) such that b_i is the $\mathfrak{P}$-component of $p^{R-1}y_i$ under $G'(\Omega')[p] = \bigoplus_{\mathfrak{P}'|p} G'(\Omega')[\mathfrak{P}']$. For an open $U \subseteq \mathcal{X}'$, a **Hodge–Tate chart over** U for $(\mathfrak{T}, s, \beta, \bar{\beta})$ with points y is a pair of Ω'-linear maps $\iota\colon V_L \to V_H$, $h\colon V_H \to V_\omega$ such that there are Ω'-linear forms $\eta_1, \dots, \eta_m$ on $\mathrm{Lie}(G'_{\Omega'})$ with $h(e_i) = (\eta_i(\bar{\beta}_k))_k$ for the standard basis (e_i) of V_H (the **forms** of the chart), such that at every point $x \in U$ with $p \notin \mathcal{O}_x^\times$ the restriction of η_i to Lie_x lies in ω_x and is a Hodge–Tate value at x of y_i. A **Hodge–Tate atlas** for $(\mathfrak{T}, s, \beta, \bar{\beta})$ with points y is a finite family $(U_\nu, \iota_\nu, h_\nu)_\nu$ of opens U_ν covering $\mathcal{X}'$ and Hodge–Tate charts (ι_ν, h_ν) over U_ν with points y; the forms of (ι_ν, h_ν) are written $\eta_1^\nu, \dots, \eta_m^\nu$.

(Hf5) Graphs For $w \in \mathcal{O}'$ and a Hodge–Tate atlas $(U_\nu, \iota_\nu, h_\nu)_\nu$, the **Hodge–Tate graphs** are

the subsheaves $\Gamma \subseteq V_L \oplus V_H$ and $\Delta \subseteq V_H \oplus V_\omega$ whose sections over an open $U \subseteq \mathcal{X}'$ are

$$\begin{aligned}\Gamma(U) = \{(c,d) :\ & wc \in W(U);\\ & d - \iota_\nu(c) \in T_x \text{ for every } x \in U \text{ over } p \text{ and every } \nu \text{ with } x \in U_\nu;\\ & d \in T_x \text{ for every other } x \in U\},\\ \Delta(U) = \{(d,e) :\ & wd \in T(U);\\ & e - h_\nu(d) \in \omega'_x \text{ for every } x \in U \text{ over } p \text{ and every } \nu \text{ with } x \in U_\nu;\\ & e \in \omega'_x \text{ for every other } x \in U\},\end{aligned}$$

where a point x lies over p if $p \notin \mathcal{O}_x^\times$, and the stalks of subsheaves of the constant sheaves V_L, V_H, V_ω are regarded as subsets of these spaces. At a point over p they are the graphs of ι_ν modulo T and of h_ν modulo ω', imposed for every chart of the atlas whose open contains the point; at the points where p is invertible, where (Hf4) does not constrain the charts, they are $W \oplus T$ and $T \oplus \omega'$.

The tuple $(\mathrm{lie}, W, \omega, \omega', T, H, \Gamma, \Delta)$ is the **Hodge–Tate lattice family** of $(\mathfrak{T}, s, \beta, \bar\beta, w)$ and the atlas $(U_\nu, \iota_\nu, h_\nu)_\nu$.

Proposition 35.23. *Let $\mathcal{T}$, $\mathcal{Y}$, E, c, $2 \le B_0 \le B$, $\mathcal{X} \subseteq \mathbb{P}$, $p \in S(B_0, B)$ and λ be as in Proposition 35.11, $\mathcal{T}$ of type (Φ, n, κ). Then there are a pre-admissible torsion system $(\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_{\mathfrak{P}}, \theta))$ for $(\mathcal{X}, p, \lambda)$ such that*

(He5) Test-point conditions *every point of $\mathcal{X}'$ of codimension one at which p is not a unit is the first entry of a test point at which (Ox3) holds;*

(He6) Step sheaves *for every label t the step sheaves $\mathcal{U}_{t,j}$ and $\mathcal{U}^{\mathrm{st}}_{t,i}$ are locally free;*

and an element $w \in \mathcal{O}'$ with $w^{p-1} = u\,p$ for a unit u of $\mathcal{O}'$, with the following data for every $a \in \Phi$. Put $s = (a, +)$, $\mathfrak{P} = \mathfrak{P}_s$, $m = m_{\mathfrak{P}}$, and let Ω' be the function field of $\mathcal{X}'$. There are Ω'-spaces V_L, V_ω of dimensions κ, $n - \kappa$ and $V_H = \Omega'^m$; locally free $\mathcal{O}_{\mathcal{X}'}$-submodules $\mathrm{lie}, W \subseteq V_L$, $\omega', \omega \subseteq V_\omega$, $H \subseteq V_H$ of full rank; $T = \mathcal{O}^m_{\mathcal{X}'} \subseteq V_H$; and $\mathcal{O}_{\mathcal{X}'}$-submodules $\Gamma \subseteq V_L \oplus V_H$, $\Delta \subseteq V_H \oplus V_\omega$, such that:

(He1) Inclusions and graphs $\mathrm{lie} \subseteq W \subseteq w^{-1}\mathrm{lie}$, $\omega' \subseteq \omega \subseteq w^{-1}\omega'$, $T \subseteq H$, *and $H = T$ on the open subscheme where p is invertible; Γ, Δ satisfy (G1), (G2) of Lemma 35.12, and the sheaf I there is locally free.*

(He2) Eigen flag *for $1 \le i \le m$, $H_i = H \cap (\Omega'^i \oplus 0)$ is locally free with H_i/H_{i-1} invertible; let $J_i \subseteq \Omega'$ be the image of H_i under the i-th coordinate.*

(He3) Facts at test points *for every point q of $\mathcal{X}'$ of codimension one at which p is not a unit there are an algebraically closed field C, complete for a valuation with valuation ring $\mathcal{O}_C$, and a field embedding $\Omega' \to C$ with $\mathcal{O}_C \cap \Omega' = \mathcal{O}_{\mathcal{X}',q}$, such that (F0)–(F2) of Lemma 35.12 hold for the $\mathcal{O}_C$-spans of the stalks at q, and $\mathcal{O}_C J_{i,q} = \mathcal{O}_C\,\mathcal{O}_{\mathcal{X}'}(-\mathrm{Ec}_{s,i})_q$ in C for $1 \le i \le m$.*

(He4) Root lines *there are full flags of V_L and of V_ω such that the induced filtrations $\mathrm{lie}_j = \mathrm{lie} \cap F_j$, $W_j = W \cap F_j$ $(1 \le j \le \kappa)$ and ω_j, ω'_j $(1 \le j \le n - \kappa)$ are by locally free submodules with invertible graded pieces, with isomorphisms $\mathrm{gr}_j\mathrm{lie} \cong \pi^*\mathcal{O}(e_{(a,j)})$, $\mathrm{gr}_j W \cong \pi^*\mathcal{O}(e'_{(a,j)})$, $\mathrm{gr}_j\omega \cong \pi^*\mathcal{O}(e_{(a,\kappa+j)})$, $\mathrm{gr}_j\omega' \cong \pi^*\mathcal{O}(e'_{(a,\kappa+j)})$, under which the inclusions $\mathrm{gr}_j\mathrm{lie} \subseteq \mathrm{gr}_j W$ and $\mathrm{gr}_j\omega' \subseteq \mathrm{gr}_j\omega$ are the pull-backs of multiplication by g_r/f_r, resp. f_r/g_r, for the corresponding root r.*

Proof. Let $\mathfrak{T} = (\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_{\mathfrak{P}}, \theta))$, $R \geq 4$, w and, for every $a \in \Phi$, the bases $\beta, \bar{\beta}$ and the Hodge–Tate atlas $(U_\nu, \iota_\nu, h_\nu)_\nu$ be as given by Proposition 35.24. Then $\mathfrak{T}$ is a pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$, its step sheaves are locally free, which is (He6), and $w^{p-1} = u\,p$ with $u \in \mathcal{O}'^{\times}$.

The data for $a \in \Phi$. Put $s = (a, +)$, $m = m_{\mathfrak{P}_s}$, and take $V_L = \Omega'^{\kappa}$, $V_H = \Omega'^{m}$, $V_\omega = \Omega'^{\,n-\kappa}$ and $(\mathrm{lie}, W, \omega, \omega', T, H, \Gamma, \Delta)$ the Hodge–Tate lattice family of $(\mathfrak{T}, s, \beta, \bar{\beta}, w)$ and this atlas; here $\kappa = \mathrm{rk}((a,+))$ and $n - \kappa = \mathrm{rk}((a,-))$ are the dimensions of the Hodge pieces $\mathrm{Lie}_{(a,\pm)}(G'_{\Omega'})$ for a tower of type (Φ, n, κ), and $T = \mathcal{O}'^{m}_{\mathcal{X}'}$. Then (He1) is (Hm1), (He2) is (Hm2) and (He4) is (Hm3).

Test points. The hypotheses of Proposition D.97 are satisfied by $\mathfrak{T}$, R, w and these data: $\mathfrak{T}$ is torsion-regular and pre-admissible, $R \geq 4$, and the lattice families satisfy (Hm1), (Hm2), (Hm4), (Hm5). Let $\mathfrak{q}$ be a point of $\mathcal{X}'$ of codimension one at which p is not a unit, and let $(\mathfrak{q}, C, \iota_C)$ be the test point given by Proposition D.97. By its part (1), (Ox3) holds there, which is (He5). By (Ox1), C is algebraically closed and complete for a non-archimedean absolute value, $\mathcal{O}_C$ is its valuation ring and $\iota_C \colon \Omega' \to C$ is a field embedding with $\iota_C^{-1}(\mathcal{O}_C) = \mathcal{O}_{\mathcal{X}',\mathfrak{q}}$, that is $\mathcal{O}_C \cap \Omega' = \mathcal{O}_{\mathcal{X}',\mathfrak{q}}$ after identifying Ω' with its image; by part (2), (F0)–(F2) hold for the $\mathcal{O}_C$-spans of the stalks at $\mathfrak{q}$ and $\mathcal{O}_C J_{i,\mathfrak{q}} = \mathcal{O}_C\,\mathcal{O}_{\mathcal{X}'}(-\mathrm{Ec}_{s,i})_{\mathfrak{q}}$ for $1 \leq i \leq m$ and every $a \in \Phi$. This is (He3). □

Proposition 35.24. *Let $\mathcal{T}$, $\mathcal{Y}$, E, c, $2 \leq B_0 \leq B$, $\mathcal{X} \subseteq \mathbb{P}$, $p \in S(B_0, B)$ and λ be as in Proposition 35.11, $\mathcal{T}$ of type (Φ, n, κ). Then there are a torsion-regular pre-admissible torsion system $\mathfrak{T} = (\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_{\mathfrak{P}}, \theta))$ for $(\mathcal{X}, p, \lambda)$ whose step sheaves $\mathcal{U}_{t,j}$, $\mathcal{U}^{\mathrm{st}}_{t,i}$ are locally free for every label t, an integer $R \geq 4$ as in Definition 35.22, and $w \in \mathcal{O}'$ with $w^{p-1} = u\,p$ for a unit u of $\mathcal{O}'$, such that for every $a \in \Phi$, with $s = (a,+)$, $\mathfrak{P} = \mathfrak{P}_s$, $m = m_{\mathfrak{P}}$, there are bases $\beta, \bar{\beta}$ and a Hodge–Tate atlas $(U_\nu, \iota_\nu, h_\nu)_\nu$ for $(\mathfrak{T}, s, \beta, \bar{\beta})$ with points $y_1, \dots, y_m$ and forms $\eta^\nu_1, \dots, \eta^\nu_m$ whose Hodge–Tate lattice family $(\mathrm{lie}, W, \omega, \omega', T, H, \Gamma, \Delta)$ satisfies:*

(Hm1) Lattices and graphs *$\mathrm{lie}, W \subseteq V_L$, $\omega, \omega' \subseteq V_\omega$ and $H \subseteq V_H$ are locally free of full rank;* $\mathrm{lie} \subseteq W \subseteq w^{-1}\mathrm{lie}$, *$\omega' \subseteq \omega \subseteq w^{-1}\omega'$, $T \subseteq H$, and $H = T$ on the open subscheme where p is invertible; Γ, Δ satisfy (G1), (G2), and the sheaf I is locally free.*

(Hm2) Eigen flag *for $1 \leq i \leq m$, $H_i = H \cap (\Omega'^{i} \oplus 0)$ is locally free with H_i/H_{i-1} invertible.*

(Hm3) Root lines *(He4) of Proposition 35.23 holds for* $\mathrm{lie}, W, \omega, \omega'$.

(Hm4) Hodge–Tate sequence modulo w at the points over p *at every point x of $\mathcal{X}'$ with $p \notin \mathcal{O}_x^{\times}$ and every ν with $x \in U_\nu$: $\iota_\nu(W_x) \subseteq T_x$, $h_\nu(T_x) \subseteq \omega'_x$, the stalks of the graphs are $\Gamma_x = \{(c,d) : wc \in W_x,\ d - \iota_\nu(c) \in T_x\}$ and $\Delta_x = \{(d,e) : wd \in T_x,\ e - h_\nu(d) \in \omega'_x\}$, and the sequence of $\mathcal{O}_x$-modules $0 \to W_x/wW_x \xrightarrow{\iota_\nu} T_x/wT_x \xrightarrow{h_\nu} \omega'_x/w\omega'_x \to 0$ is exact.*

(Hm5) Adjointness *Let $\zeta \in \Omega'$ be a primitive p^R-th root of unity, $e_{p^R}(y, y') = \zeta^{\varepsilon(y,y')}$ the Weil pairing of the polarisation on $G'(\Omega')[p^R]$ with $\varepsilon(y, y') \in \{0, \dots, p^R - 1\}$, (ϖ_k) a $\mathbb{Z}$-basis of $\mathcal{O}_E$ and (ϖ_k^*) its dual basis for the trace form, and $g_{lj} = \sum_k \sigma_s(\varpi_k^*)\,\varepsilon(\varpi_k y_l, y_j)$. Then for every ν, every section c of W over an open $U' \subseteq U_\nu$ and every j,*

$$\sum_{l=1}^{m} \iota_\nu(c)_l\, g_{lj} - \eta^\nu_j\Big(\sum_k c_k \beta_k\Big) \in p^{R-2}\,\mathcal{O}_{\mathcal{X}'}(U').$$

Proof. Let $\mathfrak{T}$, $R \geq 4$, w and, for every $a \in \Phi$, $\beta, \bar{\beta}$, the points y and the Hodge–Tate atlas $(U_\nu, \iota_\nu, h_\nu)_\nu$ with forms η^ν_j be as given by Proposition 35.25. Then $\mathfrak{T}$ is torsion-regular and

pre-admissible, its step sheaves are locally free, $R \geq 4$ and $w^{p-1} = u\,p$ with $u \in \mathcal{O}'^{\times}$. Fix $a \in \Phi$, put $s = (a,+)$, $\bar{s} = (a,-)$, $m = m_{\mathfrak{P}_s}$; since $\mathcal{T}$ has type (Φ, n, κ), $\mathrm{rk}(s) = \kappa$ and $\mathrm{rk}(\bar{s}) = n - \kappa$, and $m = n$ by Proposition 35.25.

The hypotheses of Lemma 35.36. The prime p is odd and unramified in E, $R \geq 3$, $\kappa = \mathrm{rk}(s)$, $\mathrm{rk}(\bar{s}) = n - \kappa$, $m = n$, and the charts of the atlas satisfy $\sum_l \iota_\nu(c)_l g_{lj} = \eta_j^\nu(\sum_k c_k \beta_k)$. (Sq1) is the freeness of ht_x at the points over p given by Proposition 35.25; (Sq2) is contained in (Rm1); (Sq3) is (Rm4).

(Hm1). The local freeness of $\mathrm{lie}, W, \omega, \omega', H$, the inclusions, $T \subseteq H$, $H = T$ where p is invertible, and the local freeness of I are (Rm1). The graphs Γ, Δ satisfy (G1), (G2) over every affine open by Lemma 35.36(3).

(Hm2), (Hm3). These are (Rm2) and (Rm3).

(Hm4). Let x be a point over p and $x \in U_\nu$. By Lemma 35.36(1), $\iota_\nu(W_x) \subseteq T_x$, $h_\nu(T_x) \subseteq \omega'_x$ and the sequence $0 \to W_x/wW_x \to T_x/wT_x \to \omega'_x/w\omega'_x \to 0$ is exact; by Lemma 35.36(2), Γ_x and Δ_x are as stated.

(Hm5). For every ν, every section c of W over an open $U' \subseteq U_\nu$ and every j, $\sum_l \iota_\nu(c)_l g_{lj} - \eta_j^\nu(\sum_k c_k \beta_k) = 0 \in p^{R-2}\mathcal{O}_{\mathcal{X}'}(U')$; the matrix g is the one of (Hm5), formed with the same points y, root of unity, Weil pairing and bases. □

Proposition 35.25. *Let $\mathcal{T}$, $\mathcal{Y}$, E, c, $2 \leq B_0 \leq B$, $\mathcal{X} \subseteq \mathbb{P}$, $p \in S(B_0, B)$ and λ be as in Proposition 35.11, $\mathcal{T}$ of type (Φ, n, κ). Then there are a torsion-regular pre-admissible torsion system $\mathfrak{T} = (\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_{\mathfrak{P}}, \theta))$ for $(\mathcal{X}, p, \lambda)$ whose step sheaves $\mathcal{U}_{t,j}$, $\mathcal{U}^{\mathrm{st}}_{t,i}$ are locally free for every label t, an integer $R \geq 4$ as in Definition 35.22, and $w \in \mathcal{O}'$ with $w^{p-1} = u\,p$ for a unit u of $\mathcal{O}'$, such that at every point x of $\mathcal{X}'$ over p the module ht_x is free of rank $\dim_{\Omega'} \mathrm{Lie}(G'_{\Omega'})$, and for every $a \in \Phi$, with $s = (a,+)$, $\mathfrak{P} = \mathfrak{P}_s$, $m = m_{\mathfrak{P}}$, one has $m = n$, and there are bases $\beta, \bar{\beta}$, points $y_1, \ldots, y_m$ for $(\mathfrak{T}, s)$ and a Hodge–Tate atlas $(U_\nu, \iota_\nu, h_\nu)_\nu$ for $(\mathfrak{T}, s, \beta, \bar{\beta})$ with points y and forms η_j^ν, with $\sum_l \iota_\nu(c)_l g_{lj} = \eta_j^\nu(\sum_k c_k \beta_k)$ for all ν, $c \in V_L$ and j, whose Hodge–Tate lattice family $(\mathrm{lie}, W, \omega, \omega', T, H, \Gamma, \Delta)$ for w satisfies:*

(Rm1) Lattices $\mathrm{lie}, W \subseteq V_L$, $\omega, \omega' \subseteq V_\omega$ *and* $H \subseteq V_H$ *are locally free of full rank, with* $W_x = W^x$ *and* $\omega'_x = \omega'^x$ *at every point* x; $\mathrm{lie} \subseteq W \subseteq w^{-1}\mathrm{lie}$, $\omega' \subseteq \omega \subseteq w^{-1}\omega'$, $T \subseteq H$, *and* $H = T$ *on the open subscheme where* p *is invertible; the sheaf* I *is locally free;*

(Rm2) Eigen flag *for* $1 \leq i \leq m$, $H_i = H \cap (\Omega'^i \oplus 0)$ *is locally free with* H_i/H_{i-1} *invertible;*

(Rm3) Root lines *(He4) holds for* $\mathrm{lie}, W, \omega, \omega'$;

(Rm4) Flat image *for every point* x *over* p *and every* ν *with* $x \in U_\nu$, *the* $\mathcal{O}_x$*-module* $\iota_\nu(W_x) + wT_x$ *is free.*

Proof. Let F', $\mathcal{O}'$, $\mathcal{X}'$, π, w, the bases b of the $\Gamma_{\mathfrak{P}}$ and, for every $a \in \Phi$, $\beta, \bar{\beta}$, the points y and the atlas $(U_\nu, \iota_\nu, h_\nu)_\nu$ be as given by Proposition 35.26, and let $R = R(p) \geq 4$ be the integer at p of the root origin of $\mathcal{X}_{B,c}$ ((Go3) of Definition 29.7).

Step 1: the hypotheses of the lemmas on models. The scheme $\mathcal{X}'$ is integral, normal, projective (so of finite type and noetherian) over $\mathcal{O}'$, and π is dominant with $[\Omega' : \Omega] < \infty$. Every prime $\ell \neq p$ is a unit in $\mathcal{O}'$, which lies over $\lambda \mid p$, hence on $\mathcal{X}'$. So Lemma 35.31 and Lemma 35.42 apply to π. By Lemma 35.31(1), $\mathcal{X}'$ is a cover, every p^R-torsion point of G' is rational over Ω' (so R is an integer as in Definition 35.22), and ht_x is free of rank $\dim \mathrm{Lie}(G'_{\Omega'})$ at every point x over p. The hypotheses of Lemma 35.29 hold: $\mathcal{O}'$ is a discrete valuation ring of characteristic 0 with $p \in \mathfrak{m}_{\mathcal{O}'}$; the $2d$ embeddings σ_s of E into F' map $\mathcal{O}_E$ into the integrally closed $\mathcal{O}'$; p is odd

and unramified in E; G' is semiabelian of relative dimension dn with $\mathcal{O}_E$-action, abelian generic fibre ($\mathcal{X}'$ is a cover) and rational p-torsion ($R \geq 1$). Its ideals $\mathfrak{P}_u = \sigma_u^{-1}(\mathfrak{m}_{\mathcal{O}'})$ and subsheaves ω_u are those of (Ts1), (Ts3). Finally $w = \zeta_p - 1$ satisfies $w^{p-1} = u\,p$ with $u \in \mathbb{Z}[\zeta_p]^\times \subseteq \mathcal{O}'^\times$.

Step 2: $\mathfrak{T}$ is a torsion-regular pre-admissible torsion system with locally free step sheaves. (Ts1) holds: $\mathcal{O}'$ is the localisation of $\mathcal{O}_{F'}$ at a prime above λ and contains a primitive $(q_{\mathfrak{P}} - 1)$-th root of unity for every $\mathfrak{P} \mid p$, as $q_{\mathfrak{P}} - 1$ divides N. (Ts2) is part of Proposition 35.26. (Ts3): by Lemma 35.29(1) the generic fibre $G_{\mathcal{X}}[\mathfrak{P}] \times_\Omega \Omega' = G'[\mathfrak{P}]_{\Omega'}$ is constant with group $\Gamma_{\mathfrak{P}}$, of dimension $m_{\mathfrak{P}} = n$; by Lemma 35.29(2), $\mathcal{A}_{\mathfrak{P}}$ is a locally free subalgebra of $\mathrm{Map}(\Gamma_{\mathfrak{P}}, \Omega')$ with $p^{N_0}\mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}) \subseteq \mathcal{A}_{\mathfrak{P}} \subseteq \mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O})$ for some N_0; by Lemma 35.29(3), each θ_u is a surjective $\mathcal{O}$-linear map $\mathcal{I}_{\mathfrak{P}} \to \omega_u/p\,\omega_u$. (Ts4), (Ts5) and the local freeness of the step sheaves are part of Proposition 35.26. Torsion-regularity (Hf1): $G'[\mathfrak{P}]$ is flat, quasi-finite, separated and of finite presentation (Lemma 35.29(1)), $\mathcal{A}_{\mathfrak{P}}(U) = \mathrm{Ord}(G'[\mathfrak{P}] \times U)$ on affines and θ_u is the differential at the unit section, by the choice of $\mathcal{A}_{\mathfrak{P}}$, θ_u as the sheaves and maps of (Hf1).

Step 3: the data for $a \in \Phi$. Put $s = (a, +)$, $\bar{s} = (a, -)$, $m = m_{\mathfrak{P}_s}$; then $m = n$ by Step 2, and $\mathrm{rk}(s) = \kappa$, $\mathrm{rk}(\bar{s}) = n - \kappa$ for a tower of type (Φ, n, κ). The points, the bases and the atlas, with $\sum_l \iota_\nu(c)_l g_{lj} = \eta_j^\nu(\sum_k c_k \beta_k)$, are those of Proposition 35.26.

(Rm1). The lattices lie, W of (Hf2) are the lattices L, W of Lemma 35.31(2) for (s, β) (the formulas coincide, the Hodge lattice at $(\{p\}, R)$ being $V_{s,x}$), and ω, ω' are the duals $\bar{L}^\vee$, $\bar{W}^\vee$ of those for $(\bar{s}, \bar{\beta})$. By Lemma 35.31(2) they are locally free of full rank with $W_x = W^x$ and ω'_x the dual of $\bar{W}_x = \bar{W}^x$, which is ω'^x; by Lemma 35.31(3), lie $\subseteq W \subseteq w^{-1}$lie and $\omega' \subseteq \omega \subseteq w^{-1}\omega'$. The sheaves H and I are locally free by Proposition 35.26. For $T \subseteq H$: the Teichmüller character satisfies $\chi_s(0) = 0$ and $\chi_s(1) = 1$ (a $(q_{\mathfrak{P}} - 1)$-th root of unity congruent to 1 modulo $\mathfrak{m}_{\mathcal{O}'}$, and such roots of unity are distinct modulo $\mathfrak{m}_{\mathcal{O}'}$ since $p \nmid q_{\mathfrak{P}} - 1$), so $\ell_c(b_i) = c_i$; if $c \in \mathrm{af}(U)$ then $\ell_c \in \mathcal{A}_{\mathfrak{P}}(U) + p\,\mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}(U)) \subseteq \mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}(U))$, hence $c \in \mathcal{O}(U)^m$. So $\mathrm{af} \subseteq \mathcal{O}^m = T$ and, dualising, $T \subseteq H$. Where p is invertible, $\mathcal{A}_{\mathfrak{P}} = \mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O})$ (Lemma 35.29(2)) and ℓ_c has values in $\mathcal{O}$ for $c \in \mathcal{O}^m$ (χ_s takes values in $\mathcal{O}'$), so $\mathrm{af} = \mathcal{O}^m$ and $H = T$ there.

(Rm2), (Rm4). Part of Proposition 35.26.

(Rm3). By Lemma 35.42 for a, β, $\bar{\beta}$, the lattices lie, W, ω, ω' satisfy (He4).

The remaining clauses. The inequality $R \geq 4$, the identity $w^{p-1} = u\,p$ and the freeness of ht_x at every point over p were shown in Step 1. □

35.3 Flattened models: regular-translate orders and Hodge–Tate atlases

The refined models of Subsection 35.2 are obtained by flattening (Proposition 35.26): over a local ring above a prime λ over p there is a normal projective model on which the torsion orders, the eigen lattices and the Hodge–Tate images are locally free. The proof uses flattening by blow-up [206] and graph closures, and three groups of lemmas. Regular-translate orders of the p-torsion form a locally free sheaf of Hopf orders with surjective cotangent maps (Lemma 35.28, Lemma 35.29). The de Rham and Hodge–Tate coordinate lattices are Hodge bundles with free Hodge–Tate spans, and at every point over p the Hodge–Tate sandwich holds (Lemma 35.31, Lemma 35.32); the sandwich is used again in Appendix D. Weil-adjoint Hodge–Tate atlases exist and give the Hodge–Tate sequence modulo w, $w^{p-1} = up$, at every point over p (Lemma 35.33, Lemma 35.36); at points of codimension one the charts are isotropic, by Hodge–Tate isotropic lifts in the abelian and the non-abelian case (Lemma 35.39, Proposition 35.41), which rest on the isotropy of the Hodge–Tate map of a self-dual p-divisible group and on Fargues's bounds [89].

Proposition 35.26. *Let $\mathcal{T}$, $\mathcal{Y}$, E, c, $2 \leq B_0 \leq B$, $\mathcal{X} \subseteq \mathbb{P}$, $p \in S(B_0, B)$ and λ be as in Proposition 35.11, $\mathcal{T}$ of type (Φ, n, κ), Ω the function field of $\mathcal{X}$, and $R = R(p)$ the integer at*

p of the root origin of $\mathcal{X}_{B,c}$. Then there are a finite extension F' of F_Ω containing a primitive N-th root of unity, $N = p\prod_{\mathfrak{P}|p}(q_\mathfrak{P}-1)$, the localisation $\mathcal{O}'$ of $\mathcal{O}_{F'}$ at a prime above λ, and an integral normal scheme $\mathcal{X}'$, projective and flat over $\mathcal{O}'$, with a dominant morphism $\pi\colon \mathcal{X}' \to \mathcal{X}$ of $\mathcal{O}_{F_\Omega}$-schemes and $[\Omega' : \Omega] < \infty$ (Ω' the function field of $\mathcal{X}'$), with the following property. Put $G' = \pi^* G_\mathcal{X}$, $w = \zeta_p - 1$ for a primitive p-th root of unity $\zeta_p \in \mathcal{O}'$, and let $\mathcal{A}_\mathfrak{P}$, θ_u be the sheaves and maps of (Hf1) on $\mathcal{X}'$. Then for every prime $\mathfrak{P} \mid p$ there is a basis $b_1, \dots, b_{m_\mathfrak{P}}$ of $\Gamma_\mathfrak{P}$ such that the tuple $\mathfrak{T} = (\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_\mathfrak{P}, \theta))$ with these bases satisfies (Ts4) and (Ts5) and its step sheaves $\mathcal{U}_{t,j}$, $\mathcal{U}^{\mathrm{st}}_{t,i}$ are locally free for every label t; and for every $a \in \Phi$, with $s = (a,+)$, $m = m_{\mathfrak{P}_s}$, there are bases $\beta, \bar\beta$, points y for $(\mathfrak{T}, s)$ and a Hodge–Tate atlas $(U_\nu, \iota_\nu, h_\nu)_\nu$ with points y and forms η^ν_j satisfying $\sum_l \iota_\nu(c)_l g_{lj} = \eta^\nu_j(\sum_k c_k\beta_k)$ for all ν, $c \in V_L$, j, whose Hodge–Tate lattice family $(\mathrm{lie}, W, \omega, \omega', T, H, \Gamma, \Delta)$ for w has: H locally free; $H_i = H \cap (\Omega'^i \oplus 0)$ locally free with H_i/H_{i-1} invertible ($1 \le i \le m$); the sheaf I locally free; and $\iota_\nu(W_x) + wT_x$ free over $\mathcal{O}_x$ for every point x over p and every ν with $x \in U_\nu$.

Proof. **Step 1: the base model.** Let $F' = F_\Omega(\mu_N)$, $\Omega' = \Omega F'$, and $\mathcal{X}_0$ the normalisation of $\mathcal{X}$ in Ω'; $\mathcal{X}$ is a closed integral subscheme of a multiprojective space over $\mathbb{Z}$ (Definition 29.4) and excellent, so $\mathcal{X}_0 \to \mathcal{X}$ is finite and $\mathcal{X}_0$ is integral, normal and projective over $\mathbb{Z}$, and an $\mathcal{O}_{F'}$-scheme ($\mathcal{O}_{F'}$ is integral over $\mathbb{Z}$, so lies in every local ring). Being proper over $\mathcal{O}_{F'}$ with an ample sheaf, $\mathcal{X}_0$ is projective over $\mathcal{O}_{F'}$. Choose a prime $\mathfrak{l}$ of $\mathcal{O}_{F'}$ above λ, put $\mathcal{O}' = \mathcal{O}_{F',\mathfrak{l}}$ and $\mathcal{Y}_0 = \mathcal{X}_0 \times_{\mathcal{O}_{F'}} \mathcal{O}'$: integral, normal, projective over $\mathcal{O}'$, flat (it dominates the discrete valuation ring), with function field Ω' and the dominant composite $\mathcal{Y}_0 \to \mathcal{X}_0 \to \mathcal{X}$. So $\mathcal{Y}_0$ is a model, and $\mathcal{O}'$ is as in (Ts1); $\zeta_p \in \mathcal{O}'$ as $p \mid N$. Fix for every $\mathfrak{P} \mid p$ a basis $b_1, \dots, b_{m_\mathfrak{P}}$ of $\Gamma_\mathfrak{P}$ (it does not depend on the model).

Step 2: one flattening round. Let $\mathcal{Y}$ be a model and $M_1, \dots, M_k$ coherent $\mathcal{O}_\mathcal{Y}$-submodules of finite-dimensional Ω'-spaces $V_1, \dots, V_k$, each spanning its space and locally free on the dense quasi-compact open $\mathcal{Y}[1/p]$. By Theorem B.151 with $X = S = \mathcal{Y}$, $U = \mathcal{Y}[1/p]$, $\mathcal{F} = \bigoplus_j M_j$, there is a blow-up $b\colon \mathcal{Y}' \to \mathcal{Y}$ in a closed subscheme disjoint from U such that the strict transform of $\mathcal{F}$ is locally free. $\mathcal{Y}'$ is integral and projective over $\mathcal{Y}$, b is an isomorphism over U; the strict transform of M_j is b^*M_j modulo its torsion, i.e. $\mathcal{O}_{\mathcal{Y}'}M_j$ (the kernel of $b^*M_j \to V_j$ is the torsion, supported off $b^{-1}(U)$ where $b^*M_j = M_j$). Let $\mathcal{Y}^+$ be the normalisation of $\mathcal{Y}'$ (finite, $\mathcal{Y}'$ being excellent). Then $\mathcal{Y}^+$ is a model and $\mathcal{O}_{\mathcal{Y}^+}M_j$ is locally free by Lemma 35.27(1), and so is its span on every model over $\mathcal{Y}^+$.

Step 3: persistence rules. Let $\psi\colon \mathcal{Y}_2 \to \mathcal{Y}_1$ be a morphism of models over $\mathcal{X}$.

(a) $\mathcal{A}^{\mathcal{Y}_2}_\mathfrak{P} = \mathcal{O}\mathcal{A}^{\mathcal{Y}_1}_\mathfrak{P}$, by Lemma 35.28(3) on affine opens $V_2 \to V_1$ (with $B = \Omega'$, $h_B = \mathrm{id}$, $\sigma_t = x_t$; its hypotheses hold by Lemma 35.29(1)); hence, as images of spans are spans of images, $\mathcal{A}^{\mathcal{Y}_2}_{\mathfrak{P},i} = \mathcal{O}\mathcal{A}^{\mathcal{Y}_1}_{\mathfrak{P},i}$, $\mathcal{D}^{\mathcal{Y}_2}_{t,i} = \mathcal{O}\mathcal{D}^{\mathcal{Y}_1}_{t,i}$ for the image $\mathcal{D}_{t,i}$ of $\mathcal{A}_{\mathfrak{P}_t,i}$ in $\mathrm{Map}(\Gamma_{\mathfrak{P}_t,i}, \Omega')/\Omega'\xi_{t,i}$, and $\mathcal{M}^{\mathcal{Y}_2} = \mathcal{O}\mathcal{M}^{\mathcal{Y}_1}$, $\mathcal{D}'^{\mathcal{Y}_2} = \mathcal{O}\mathcal{D}'^{\mathcal{Y}_1}$ for $\mathcal{M} = \mathcal{A}_{\mathfrak{P}_s} + p\,\mathrm{Map}(\Gamma_{\mathfrak{P}_s}, \mathcal{O})$ and its image $\mathcal{D}'$ in $\mathrm{Map}(\Gamma_{\mathfrak{P}_s}, \Omega')/\ell(V_H)$ ($\ell\colon c \mapsto \ell_c$ is injective because $\ell_c(b_i) = c_i$, as $\chi_s(1) = 1$, $\chi_s(0) = 0$).

(b) If $\mathcal{A}^{\mathcal{Y}_1}_{\mathfrak{P},j}$ is locally free, the sequence $0 \to \mathcal{I}_{\mathfrak{P},j} \to \mathcal{A}_\mathfrak{P} \to \mathcal{A}_{\mathfrak{P},j} \to 0$ on $\mathcal{Y}_1$ is locally split, so ψ^* keeps it exact, and with Lemma 35.27(1) and (a), $\mathcal{I}^{\mathcal{Y}_2}_{\mathfrak{P},j} = \mathcal{O}\mathcal{I}^{\mathcal{Y}_1}_{\mathfrak{P},j}$. If moreover $\mathcal{D}^{\mathcal{Y}_1}_{t,i}$ is locally free, the same argument for $0 \to \mathcal{S}_{t,i} \to \mathcal{A}_\mathfrak{P} \to \mathcal{D}_{t,i} \to 0$ and for $0 \to \mathcal{E}_{t,i}\xi_{t,i} \to \mathcal{A}_{\mathfrak{P},i} \to \mathcal{D}_{t,i} \to 0$ ($\mathcal{E}_{t,i}$ the module of (Ts5) for (t,i)) gives $\mathcal{S}^{\mathcal{Y}_2}_{t,i} = \mathcal{O}\mathcal{S}^{\mathcal{Y}_1}_{t,i}$ and $\mathcal{E}^{\mathcal{Y}_2}_{t,i} = \mathcal{O}\mathcal{E}^{\mathcal{Y}_1}_{t,i}$, an invertible sheaf (locally a direct summand of rank one of a locally free sheaf).

(c) $\omega^{\mathcal{Y}_2}_u = \psi^*\omega^{\mathcal{Y}_1}_u$, and θ_u commutes with ψ: for $g \in \mathcal{I}^{\mathcal{Y}_1}_\mathfrak{P}$ the pull-back of the function $\tilde g$ of Lemma 35.29(3) to $G'[\mathfrak{P}] \times_{\mathcal{Y}_1} \mathcal{Y}_2$ has the values $g(t)$ at the points x_t, so it is the function for g on $\mathcal{Y}_2$, and e^*d commutes with base change. Hence under the hypotheses of (b), $\theta_t(\mathcal{I}^{\mathcal{Y}_2}_{\mathfrak{P},j}) = \mathcal{O}\theta_t(\mathcal{I}^{\mathcal{Y}_1}_{\mathfrak{P},j})$, and since the step sheaf $\mathcal{U}_{t,j}$ contains $p\,\omega_t$, $\mathcal{U}^{\mathcal{Y}_2}_{t,j} = \mathcal{O}\mathcal{U}^{\mathcal{Y}_1}_{t,j} + p\,\omega^{\mathcal{Y}_2}_t = \mathcal{O}\mathcal{U}^{\mathcal{Y}_1}_{t,j}$; likewise

$\mathcal{U}^{\mathrm{st},\mathcal{Y}_2}_{t,i} = \mathcal{O}\mathcal{U}^{\mathrm{st},\mathcal{Y}_1}_{t,i}$.

(d) If $\mathcal{M}^{\mathcal{Y}_1}$ and $\mathcal{D}'^{\mathcal{Y}_1}$ are locally free, then $\mathcal{M}\cap\ell(V_H)$ is locally free (kernel of a surjection onto a locally free sheaf), $\mathrm{af} = \ell^{-1}(\mathcal{M}\cap\ell(V_H))$ and $H = \mathrm{af}^\vee$ are locally free on $\mathcal{Y}_1$, and $\mathrm{af}^{\mathcal{Y}_2} = \mathcal{O}\mathrm{af}^{\mathcal{Y}_1}$, $H^{\mathcal{Y}_2} = \mathcal{O}H^{\mathcal{Y}_1}$ by (a) and Lemma 31.33(3) (applied to $N = \mathcal{M}$ and the subspace $\ell(V_H)$; the dual of the span of a locally free lattice is the span of its dual). If moreover the images of $H^{\mathcal{Y}_1}$ in $V_H/(\Omega'^i \oplus 0)$ are locally free, Lemma 31.33(3) gives $H_i^{\mathcal{Y}_2} = \mathcal{O}H_i^{\mathcal{Y}_1}$ with locally free quotients H/H_i, so H_i and $H_i/H_{i-1} = \ker(H/H_{i-1} \to H/H_i)$ are locally free, the latter of rank one.

(e) The lattices W, $\bar{W}$, lie, ω, ω' are locally free on every model with $W^{\mathcal{Y}_2} = \mathcal{O}W^{\mathcal{Y}_1}$ etc. (Lemma 35.31(2) and Lemma 31.29(1), which identifies the Hodge bundles of $\mathcal{Y}_2$ with the spans of those of $\mathcal{Y}_1$).

(f) If $(U_\nu, \iota_\nu, h_\nu)_\nu$ is a Hodge–Tate atlas on $\mathcal{Y}_1$ (for a pre-admissible system on $\mathcal{Y}_1$), then $(\psi^{-1}U_\nu, \iota_\nu, h_\nu)_\nu$ is one on $\mathcal{Y}_2$ with the same points and forms: at $x_2 \in \psi^{-1}U_\nu$ over p with image x_1, $\mathrm{Lie}_{x_2} = \mathcal{O}_{x_2}\mathrm{Lie}_{x_1}$, so $\eta_i|_{\mathrm{Lie}_{x_2}} \in \omega_{x_2}$, and if a function ϕ as in (Fh5) of Definition 29.6 shows that a form is a Hodge–Tate value at x_1, its pull-back shows the same at x_2 (the Weil-pairing property concerns the common generic fibre; $d\phi$ and the congruence modulo p^R extend $\mathcal{O}_{x_2}$-linearly).

Step 4: round one. On $\mathcal{Y}_0$ apply Step 2 to: the images $\mathcal{A}_{\mathfrak{P},i}$ ($\mathfrak{P} \mid p$, $1 \le i \le m_{\mathfrak{P}}$); the images $\mathcal{D}_{t,i}$ (all labels t, $1 \le i \le m_{\mathfrak{P}_t}$); and for every $a \in \Phi$ the sheaves $\mathcal{M}$ and $\mathcal{D}'$ of $s = (a,+)$. All are coherent (images and sums of coherent subsheaves) and locally free on $\mathcal{Y}_0[1/p]$: there $\mathcal{A}_{\mathfrak{P}} = \mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O})$ (Lemma 35.29(2)), so $\mathcal{A}_{\mathfrak{P},i} = \mathrm{Map}(\Gamma_{\mathfrak{P},i}, \mathcal{O})$, $\mathcal{M} = \mathrm{Map}(\Gamma_{\mathfrak{P}_s}, \mathcal{O})$, and $\mathcal{D}_{t,i}$, $\mathcal{D}'$ are quotients of these free modules by the free direct summands $\mathcal{O}\xi_{t,i}$ (as $\xi_{t,i}(b_i) = 1$) and $\ell(\mathcal{O}^m)$ (retracted by $f \mapsto (f(b_j))_j$). Let $\mathcal{Y}_1$ be the resulting model. By (a) the intrinsic $\mathcal{A}_{\mathfrak{P},i}$, $\mathcal{D}_{t,i}$, $\mathcal{M}$, $\mathcal{D}'$ of $\mathcal{Y}_1$ are the spans of those of $\mathcal{Y}_0$, hence locally free; by (b), every $\mathcal{E}_{t,i}$ is invertible on $\mathcal{Y}_1$, and by (d), H is locally free. So on $\mathcal{Y}_1$ the tuple $\mathfrak{T}_1 = (\mathcal{O}', \mathcal{Y}_1, \pi_{\mathcal{Y}_1}, (\mathcal{A}_{\mathfrak{P}}, \theta))$ satisfies (Ts1) (Step 1), (Ts2), (Ts3) (Lemma 35.29), (Ts4), (Ts5): it is a pre-admissible torsion system, and by (a), (b) these properties hold on every model over $\mathcal{Y}_1$.

Step 5: points and atlas. $M_R = G'(\Omega')[p^R]$ is free of rank n over $\Lambda = \mathcal{O}_E/p^R = \prod_{\mathfrak{P}} \mathcal{O}_{E,\mathfrak{P}}/p^R$: its $\mathfrak{P}$-component is a module over the principal artinian local ring $\mathcal{O}_{E,\mathfrak{P}}/p^R$ with $\mathfrak{P}$-torsion $\Gamma_{\mathfrak{P}}$ of dimension n (Lemma 35.29(1)), hence a sum of n cyclic modules, of order at most $q_{\mathfrak{P}}^{nR}$ with equality iff free; and $|M_R| = p^{2dnR} = \prod_{\mathfrak{P}} q_{\mathfrak{P}}^{nR}$ ($A = G'_{\Omega'}$ has dimension dn, $2d = [E:\mathbb{Q}] = \sum_{\mathfrak{P}} \log_p q_{\mathfrak{P}}$), so equality holds everywhere. For $s = (a,+)$, $\mathfrak{P} = \mathfrak{P}_s$: choose $b'_i \in M_R[p]$ whose $\mathfrak{P}$-component is b_i and whose $\mathfrak{P}'$-components ($\mathfrak{P}' \ne \mathfrak{P}$) form bases of the $\Gamma_{\mathfrak{P}'}$; they form a Λ/p-basis of $M_R[p]$. Multiplication by p^{R-1} maps the free module M_R onto $M_R[p]$ with kernel pM_R; choose y_i with $p^{R-1}y_i = b'_i$. Then the y_i give a Λ/p-basis of M_R/pM_R, so they generate M_R (Nakayama, p lying in the radical of Λ) and, being $n = m$ generators of a free module of rank m over the finite ring Λ, form a basis: these are points for $(\mathfrak{T}_1, s)$. Fix bases $\beta, \bar{\beta}$ and, by Lemma 35.33 for $\mathfrak{T}_1$, a Hodge–Tate atlas on $\mathcal{Y}_1$ with points y and Weil-adjoint charts; by (f) it pulls back to every later model, Weil-adjoint with the same forms.

Step 6: round two. On $\mathcal{Y}_1$ apply Step 2 to: the step sheaves $\mathcal{U}_{t,j}$, $\mathcal{U}^{\mathrm{st}}_{t,i}$ (coherent lattices between $p\,\omega_t$ and ω_t, equal to ω_t where p is invertible); for every a, the images of H in $V_H/(\Omega'^i \oplus 0)$ (where p is invertible $H = T$, so they are free); $H + w^{-1}T$ (equal to T there); and the sheaf P with stalks $\iota_\nu(W_x) + wT_x$ at points x over p in U_ν and T_x elsewhere. P is a well-defined coherent sheaf: at a point x over p, the stalk does not depend on ν, since for $c \in W_x$, Lemma 35.35(1) applied to $w^{-1}c$ gives $\iota_\nu(c) - \iota_\mu(c) \in wT_x$ (its hypotheses: ht_x free by Lemma 35.31(1), $R \ge 3$, Weil-adjoint charts); on a neighbourhood $V \subseteq U_\nu$ of x with $\iota_\nu(W(V)) \subseteq T(V)$ (finitely many generators, $\iota_\nu(W_x) \subseteq T_x$ by Lemma 35.34(2)), $P|_V = \iota_\nu(W|_V) + wT|_V$, coherent, equal to T at the points of V not over p; and $P = T$ on $\mathcal{Y}_1[1/p]$. Let $\mathcal{Y}_2$ be the resulting model. On $\mathcal{Y}_2$: the step sheaves are the spans of those of $\mathcal{Y}_1$ by (b), (c) (hypotheses from Step 4), hence locally

free; H_i are locally free with invertible quotients by (d); $H + w^{-1}T$ is the span of that of $\mathcal{Y}_1$ (by (d) and $T^{\mathcal{Y}_2} = \mathcal{O}T^{\mathcal{Y}_1}$), hence locally free; and $P^{\mathcal{Y}_2} = \mathcal{O}P^{\mathcal{Y}_1}$ (locally $\iota_\nu(\mathcal{O}W) + wT$, by (e), (f)), hence locally free, so $\iota_\nu(W_x) + wT_x$ is free at every point x over p.

Step 7: round three. On $\mathcal{Y}_2$ the hypotheses of Lemma 35.36 hold for every a ((Sq1), (Sq2) by Lemma 35.31, (Sq3) by Step 6, $m = n$, Weil-adjoint atlas). By its part (2), at x over p in U_ν, $I_x = \{c : \exists b \in H_x,\ wb \in T_x,\ c - h_\nu(b) \in \omega'_x\} = h_\nu(H_x \cap w^{-1}T_x) + \omega'_x$, and at points not over p (where $H = T$, $\omega' = \omega$ and $\Delta = T \oplus \omega'$ near the point) $I_x = \omega'_x$. Since $H + w^{-1}T$ is locally free on $\mathcal{Y}_2$, $H \cap w^{-1}T$ is locally free and its formation commutes with ψ^* (Lemma 35.27(2)); on a neighbourhood $V \subseteq U_\nu$ of x with $h_\nu(T(V)) \subseteq \omega'(V)$, $I|_V = h_\nu(H \cap w^{-1}T) + \omega'$, coherent, and $I = \omega'$ on $\mathcal{Y}_2[1/p]$. Apply Step 2 to the sheaves I (one for each a); let $\mathcal{X}' = \mathcal{Y}_3$ be the resulting model. On $\mathcal{Y}_3$, (Sq1)–(Sq3) still hold ((e), and $P^{\mathcal{Y}_3} = \mathcal{O}P^{\mathcal{Y}_2}$ as in Step 6), the same stalk formula holds, and by (d), (e), (f) and Lemma 35.27(2), $I^{\mathcal{Y}_3} = \mathcal{O}I^{\mathcal{Y}_2}$, which is locally free.

Step 8: conclusion. $\mathcal{X}' = \mathcal{Y}_3$ is integral, normal, projective and flat over $\mathcal{O}'$, with $\pi = \pi_{\mathcal{Y}_3}$ dominant and $[\Omega' : \Omega] < \infty$. By Steps 4 and 3(a),(b), (Ts4) and (Ts5) hold for the bases b; by Steps 6, 7 and 3(b),(c) (spans of locally free sheaves, Lemma 35.27(1)) the step sheaves are locally free; for every $a \in \Phi$, with the bases $\beta, \bar{\beta}$, the points y and the pulled-back Weil-adjoint atlas (Steps 5 and 3(f)), H and the H_i are locally free with invertible quotients (Steps 4, 6 and 3(d)), I is locally free (Step 7), and $\iota_\nu(W_x) + wT_x$ is free at every point x over p (Step 7). □

Lemma 35.27. *Let X be an integral scheme with function field K and V a finite-dimensional K-vector space, regarded as a constant sheaf; for a dominant morphism $\psi\colon X'' \to X$ of integral schemes and an $\mathcal{O}_X$-submodule $M \subseteq V$ write $\mathcal{O}_{X''}M$ for the $\mathcal{O}_{X''}$-submodule of $V \otimes_K K(X'')$ generated by M.*

1. *If $M \subseteq V$ is a locally free $\mathcal{O}_X$-submodule of rank $\dim V$, the natural map $\psi^*M \to V \otimes_K K(X'')$ is injective with image $\mathcal{O}_{X''}M$, which is locally free of rank $\dim V$.*
2. *Let $M, M' \subseteq V$ be locally free $\mathcal{O}_X$-submodules of rank $\dim V$ such that $M + M'$ is locally free. Then $M \cap M'$ is locally free of rank $\dim V$, and for every dominant morphism $\psi\colon X'' \to X$ of integral schemes*

$$\mathcal{O}_{X''}(M \cap M') = \mathcal{O}_{X''}M \cap \mathcal{O}_{X''}M'.$$

Proof. **(1).** The question is local on X and X''; let $U = \operatorname{Spec} A \subseteq X$ be an affine open on which M is free with basis $m_1, \dots, m_r$, $r = \dim V$; these elements span $M \otimes K = V$ and are therefore a K-basis of V. Let $U'' = \operatorname{Spec} A'' \subseteq \psi^{-1}(U)$ be affine. Then $(\psi^*M)(U'') = \bigoplus_i A''m_i$ is free, and the natural map sends the basis $m_i \otimes 1$ to $m_i \otimes 1 \in V \otimes_K K(X'')$, which form a $K(X'')$-basis because ψ is dominant ($A \to A'' \subseteq K(X'')$ extends to $K \to K(X'')$). Hence the map is injective on U'', with image $\sum_i A''m_i = (\mathcal{O}_{X''}M)(U'')$, free of rank r.

(2). The sequence of $\mathcal{O}_X$-modules

$$0 \to M \cap M' \xrightarrow{a \mapsto (a,a)} M \oplus M' \xrightarrow{(a,b) \mapsto a-b} M + M' \to 0$$

is exact (sections are elements of V over each open). Its right term is locally free of finite rank, so the surjection is locally split, and $M \cap M'$ is locally a direct summand of the locally free module $M \oplus M'$ of finite rank, with locally free cokernel; hence $M \cap M'$ is locally of finite presentation with free stalks, so locally free, of rank $2\dim V - \dim V = \dim V$ (its generic fibre is $V \cap V = V$). A locally split exact sequence stays exact under ψ^*:

$$0 \to \psi^*(M \cap M') \to \psi^*M \oplus \psi^*M' \to \psi^*(M + M') \to 0.$$

By (1), applied to the four locally free submodules $M \cap M'$, M, M', $M + M'$ of V of rank $\dim V$, the three pulled-back modules are identified with $\mathcal{O}_{X''}(M \cap M')$, $\mathcal{O}_{X''}M \oplus \mathcal{O}_{X''}M'$ and $\mathcal{O}_{X''}(M+M')$ inside $V \otimes_K K(X'')$, compatibly with the maps $a \mapsto (a,a)$ and $(a,b) \mapsto a-b$. The kernel of the second map is $\{(a,a) : a \in \mathcal{O}_{X''}M \cap \mathcal{O}_{X''}M'\}$, so exactness gives $\mathcal{O}_{X''}(M \cap M') = \mathcal{O}_{X''}M \cap \mathcal{O}_{X''}M'$. □

Lemma 35.28. *Let R be a noetherian integrally closed domain with fraction field K, and let Γ, the finite abelian group T and the points $x_t \in \Gamma(K)$ be as in Definition 35.21. Put $A = \mathrm{Ord}(\Gamma) \subseteq \mathrm{Map}(T, R)$.*

1. ***Hopf order.*** *A is a finitely generated projective R-module; there is $d \in R \setminus \{0\}$ with $d\,\mathrm{Map}(T,R) \subseteq A$; for $f \in A$ the function $t \mapsto f(-t)$ lies in A, and the function $(t,t') \mapsto f(t+t')$ lies in the R-span of the functions $(t,t') \mapsto a(t)b(t')$, $a, b \in A$. In particular A is stable under the translations $f \mapsto f(t' + \cdot)$, $t' \in T$, and $\operatorname{Spec} A$ is a finite flat commutative group scheme over R, with group law induced by the addition of T.*

2. ***Open hull.*** *There is a unique R-morphism $j\colon \Gamma \to \operatorname{Spec} A$ with $j(x_t) = \mathrm{ev}_t$ (the K-point $f \mapsto f(t)$) for every $t \in T$. It is an open immersion, and the translates of its image cover: for every point P of $\operatorname{Spec} A$ there are $t \in T$ and a point γ of Γ with $P = \tau_t(j(\gamma))$, where τ_t is the automorphism of $\operatorname{Spec} A$ induced by $f \mapsto f(t + \cdot)$.*

3. ***Base change to domains.*** *Let $h\colon R \to R'$ be a ring homomorphism into a noetherian domain R' with fraction field L, and $\Gamma' = \Gamma \times_R R'$. Suppose there are a subring B with $R \subseteq B \subseteq K$, a ring homomorphism $h_B\colon B \to L$ extending h, and points $\sigma_t \in \Gamma(B)$ ($t \in T$) such that x_t is induced by σ_t along $B \subseteq K$, and such that Γ' with the points $x'_t \in \Gamma'(L)$ induced by σ_t along h_B satisfies the hypotheses of Definition 35.21 over R' with the same group T. Then some d as in (1) has $h(d) \neq 0$, and*

$$\mathrm{Ord}(\Gamma') = \mathrm{Ord}(\Gamma)_h \subseteq \mathrm{Map}(T, R'),$$

and the map $R' \otimes_R A \to \mathrm{Ord}(\Gamma')$, $r' \otimes f \mapsto r'\,(h \circ f)$, is an isomorphism of R'-algebras.

Proof. We use the following standard facts: (i) for a flat ring map $R \to R'$ and a quasi-compact separated R-scheme X, $\mathcal{O}(X) \otimes_R R' = \mathcal{O}(X_{R'})$ (flat base change, EGA III 1.4.15); (ii) if $R \to R'$ is faithfully flat, $M \subseteq N$ are R-modules and $n \in N$ has $n \otimes 1 \in M \otimes_R R'$ inside $N \otimes_R R'$, then $n \in M$ (apply faithful flatness to $(N/M) \otimes_R R'$); membership of an element in a submodule is local on $\operatorname{Spec} R$; (iii) a separated quasi-finite scheme of finite type over a henselian local ring is the disjoint union of an open and closed subscheme finite over the base and an open and closed subscheme with empty closed fibre (EGA IV_4 18.5.11; Stacks Project, Tag 04GG); (iv) the henselisation $R_{\mathfrak{q}}^h$ of a local ring of a noetherian normal domain is a noetherian normal local domain, faithfully flat over $R_{\mathfrak{q}}$, with residue field $\kappa(\mathfrak{q})$ and of the same dimension (EGA IV_4 18.6.6, 18.6.9); (v) a finite flat module over a noetherian local ring is free; (vi) an étale morphism that is injective on points with trivial residue field extensions is an open immersion (EGA IV_4 17.9.1).

Step 1 (functions on Γ). Let $U \subseteq \Gamma$ be open. As Γ is flat over the domain R, $\mathcal{O}(U)$ is torsion free, so $\mathcal{O}(U) \to \mathcal{O}(U_K) = K^{T_U}$ (the generic fibre is the reduced finite set $\{x_t\}$) is injective; we regard $\mathcal{O}(U) \subseteq K^{T_U}$. Flat morphisms lift generisations, so every point of Γ is a specialisation of some x_t; hence every non-empty open set contains some x_t. If $f\colon T \to K$ is regular on Γ (with values in K; the definition applies unchanged), the local sections s_U agree on overlaps, since two sections of $\mathcal{O}(U \cap U') \subseteq K^{T_{U \cap U'}}$ with the same values agree; they glue to

$s \in \mathcal{O}(\Gamma)$ with values f. Conversely an element of $\mathcal{O}(\Gamma)$ is regular. So, inside K^T, f is regular if and only if $f \in \mathcal{O}(\Gamma)$, and f is regular at a point γ (the condition of the definition at γ) for every γ if and only if $f \in \mathcal{O}(\Gamma)$. Put

$$S(\Gamma) = \{f \in K^T : \ \varphi_t f \in \mathcal{O}(\Gamma) \text{ for all } t \in T\}.$$

It is an R-subalgebra (φ_t is a ring automorphism). Since $t \mapsto x_t$ is a homomorphism, x_0 is the generic point of the unit section $e \in \Gamma(R)$, so for $s \in \mathcal{O}(\Gamma)$ the value $s(x_0) = e^*(s)$ lies in R; hence $f(t) = (\varphi_t f)(0) \in R$ for $f \in S(\Gamma)$, and $S(\Gamma) \subseteq \mathrm{Map}(T, R)$. By definition $\mathrm{Ord}(\Gamma)$ is generated by $S(\Gamma)$, so

$$\mathrm{Ord}(\Gamma) = S(\Gamma) = \bigcap_{t \in T} \varphi_t^{-1}(\mathcal{O}(\Gamma)) \subseteq \mathrm{Map}(T, R).$$

In particular $\mathrm{Ord}(\Gamma)$ is stable under every φ_t, and $\mathrm{Ord}(\Gamma) \subseteq \mathcal{O}(\Gamma)$ (take $t = 0$). The inverse of Γ is an automorphism carrying x_t to x_{-t}, so $\mathcal{O}(\Gamma)$ is stable under $f \mapsto f(-\cdot)$; as $\varphi_t(f(-\cdot)) = (\varphi_{-t} f)(-\cdot)$, so is $\mathrm{Ord}(\Gamma)$. As R is noetherian and $\mathrm{Ord}(\Gamma) \subseteq R^T$, $\mathrm{Ord}(\Gamma)$ is a finitely generated R-module.

Step 2 (flat base change). Let $R \to R'$ be flat with R' a noetherian normal domain with fraction field L. Then $R \to R'$ is injective (R' is torsion free over R), $K \otimes_R R' \subseteq L$, and $\Gamma' = \Gamma_{R'}$ with the points x'_t induced by x_t satisfies the hypotheses of Definition 35.21 with the group T (its generic fibre is $\Gamma_K \otimes_K L$). By (i), $\mathcal{O}(\Gamma') = \mathcal{O}(\Gamma) \otimes_R R'$ inside $K^T \otimes_R R' = (K \otimes_R R')^T \subseteq L^T$, compatibly with the values at the points. Finite intersections of submodules commute with flat base change (they are kernels of $N \to \bigoplus N/M_i$), and φ_t commutes with $\otimes_R R'$, so by Step 1

$$\mathrm{Ord}(\Gamma') = \bigcap_t \varphi_t^{-1}(\mathcal{O}(\Gamma) \otimes_R R') = \mathrm{Ord}(\Gamma) \otimes_R R' \subseteq L^T,$$

that is, $r' \otimes f \mapsto r' f$ is an isomorphism $R' \otimes_R \mathrm{Ord}(\Gamma) \to \mathrm{Ord}(\Gamma')$ (injective because $\mathrm{Ord}(\Gamma) \otimes R' \subseteq K^T \otimes R'$ by flatness).

Step 3 (henselian local bases). Let R be in addition henselian local with maximal ideal $\mathfrak{m}$. By (iii), $\Gamma = \Gamma^f \sqcup \Gamma^r$ with Γ^f open, closed and finite over R and Γ^r with empty closed fibre. Let $T^f = \{t : \ x_t \in \Gamma^f\}$. (a) Γ^f is a subgroup scheme. A non-empty open and closed subset Z of a scheme finite over R meets the closed fibre (its image is closed and non-empty, hence contains $\mathfrak{m}$). The closed fibre of Γ lies in Γ^f. The unit section $e(\mathrm{Spec}\, R)$ is connected and meets the closed fibre, so lies in Γ^f. The preimage of Γ^r under the multiplication $\Gamma^f \times_R \Gamma^f \to \Gamma$, resp. under the inverse on Γ^f, is open and closed in a scheme finite over R and has empty closed fibre (the group law maps closed fibres to the closed fibre), so it is empty. Hence T^f is a subgroup of T, and Γ^f, being open and closed in Γ, is a finite flat commutative group scheme with generic fibre the constant group T^f; $\mathcal{O}(\Gamma^f) \subseteq K^{T^f}$ is free of rank $|T^f|$ by (v). (b) For $u \in T^f$, x_u extends to an R-point of Γ^f: the reduced closure Z of x_u in Γ^f is integral and finite over R with function field K, so $\mathcal{O}(Z) = R$ (R is integrally closed) and Z is a section. Translation by this section is an automorphism of Γ^f carrying x_v to x_{u+v}, so $\mathcal{O}(\Gamma^f)$ is stable under φ_u, $u \in T^f$. (c) Claim: $\mathrm{Ord}(\Gamma) = \{f \in K^T : \ (\varphi_t f)|_{T^f} \in \mathcal{O}(\Gamma^f) \text{ for all } t \in T\}$. The inclusion $\subseteq$ is restriction of $\varphi_t f \in \mathcal{O}(\Gamma)$ to the open set Γ^f. For $\supseteq$ we argue by induction on $\dim R$; the hypothesis on f is stable under every φ_t, so by Step 1 it suffices to show that such an f is regular at every point γ of Γ. If $\gamma \in \Gamma^f$ this is the hypothesis for $t = 0$. Let $\gamma \in \Gamma^r$, lying over a prime $\mathfrak{q} \neq \mathfrak{m}$ (so $\dim R = 0$ is the trivial base case). Put $R' = R^h_{\mathfrak{q}}$, with fraction field L; by (iv) it satisfies all hypotheses of this step, with $\dim R' = \mathrm{ht}\, \mathfrak{q} < \dim R$. Let $\Gamma' = \Gamma_{R'}$ with its finite part Γ'^f and T'^f. The subscheme $\Gamma^f_{R'}$ is open, closed and finite over R', hence contained in Γ'^f (by the remark in (a) applied over R'), so $T^f \subseteq T'^f$. By (b) over R' the translates $\tau_s(\Gamma^f_{R'})$ by the

sections extending x'_s, $s \in T'^f$, are open and closed in Γ'^f with generic points x'_{s+T^f}; two of them for $s \not\equiv s' \bmod T^f$ are disjoint (their intersection is open with no generic point), and they cover Γ'^f (each point specialises from some x'_u and lies in the closed set containing x'_u). Hence for $g \in L^{T'^f}$: $g \in \mathcal{O}(\Gamma'^f)$ if and only if $(\varphi_s g)|_{T^f} \in \mathcal{O}(\Gamma^f_{R'}) = \mathcal{O}(\Gamma^f) \otimes_R R'$ for every $s \in T'^f$. For $g = (\varphi_t f)|_{T'^f}$ this holds by the hypothesis on f (with $t+s$). By the induction hypothesis over R', $f \in \mathrm{Ord}(\Gamma') \subseteq \mathcal{O}(\Gamma')$. Now $\mathcal{O}(\Gamma') = \mathcal{O}(\Gamma_{R_\mathfrak{q}}) \otimes_{R_\mathfrak{q}} R'$ by (i), inside $K^T \otimes_{R_\mathfrak{q}} R' \subseteq L^T$, and $R_\mathfrak{q} \to R'$ is faithfully flat, so by (ii) $f \in \mathcal{O}(\Gamma_{R_\mathfrak{q}}) = \mathcal{O}(\Gamma) \otimes_R R_\mathfrak{q}$. Thus $f = s/g$ with $s \in \mathcal{O}(\Gamma)$, $g \in R \setminus \mathfrak{q}$, and s/g is a section over the open set $\Gamma_{R[1/g]} \ni \gamma$ with values f: f is regular at γ. (d) Choose representatives $t_1 = 0, t_2, \dots, t_k$ of T/T^f. Then

$$\Psi\colon \mathrm{Ord}(\Gamma) \to \prod_{i=1}^{k} \mathcal{O}(\Gamma^f), \qquad \Psi(f)_i = (\varphi_{t_i} f)|_{T^f},$$

is an isomorphism of R-algebras: it is well defined by (c), injective because the cosets $t_i + T^f$ cover T, and surjective because, given (g_i), the function $f(t_i + u) = g_i(u)$ satisfies $(\varphi_{t_i+c} f)|_{T^f} = \varphi_c g_i \in \mathcal{O}(\Gamma^f)$ for $c \in T^f$ by (b), so $f \in \mathrm{Ord}(\Gamma)$ by (c). In particular $\mathrm{Ord}(\Gamma)$ is free of rank $k|T^f| = |T|$. For $a \in \mathcal{O}(\Gamma^f)$ and $i \le k$ the function $a^{(i)}$ with $a^{(i)}(t_i + u) = a(u)$ on $t_i + T^f$ and 0 elsewhere lies in $\mathrm{Ord}(\Gamma)$ (same computation). The morphism $\Gamma^f \to \operatorname{Spec} \mathrm{Ord}(\Gamma)$ given by $f \mapsto f|_{T^f} = \Psi(f)_1$ is the inclusion of the first factor of $\operatorname{Spec} \prod_i \mathcal{O}(\Gamma^f) = \bigsqcup_i \operatorname{Spec} \mathcal{O}(\Gamma^f)$, an isomorphism onto an open and closed subscheme; and the i-th factor is τ_{t_i} of the first, where $\tau_t = \operatorname{Spec} \varphi_t$: a prime P of the i-th factor is $\{f : \Psi(f)_i \in Q\} = \varphi_{t_i}^{-1}\{f' : f'|_{T^f} \in Q\}$ for a prime Q of $\mathcal{O}(\Gamma^f)$.

Proof of (1). Put $A = \mathrm{Ord}(\Gamma)$; it is finitely generated (Step 1). For a maximal ideal $\mathfrak{m}$ of R, $A \otimes_R R^h_\mathfrak{m} = \mathrm{Ord}(\Gamma_{R^h_\mathfrak{m}})$ (Step 2 and (iv)) is free of rank $|T|$ (Step 3(d)); by faithfully flat descent $A_\mathfrak{m}$ is flat, hence free, so A is projective of rank $|T|$. Since Γ_K is the constant group, $\mathrm{Ord}(\Gamma_K) = K^T$, so $A \otimes_R K = K^T$ (Step 2); thus $\mathrm{Map}(T, R)/A$ is a finitely generated torsion module and any non-zero d in its annihilator satisfies $d\,\mathrm{Map}(T, R) \subseteq A$. Stability under $f \mapsto f(-\cdot)$ and under the φ_t is Step 1. Comultiplication: let $f \in A$ and $\Delta f(t, t') = f(t + t')$, an element of $K^{T\times T} = (A \otimes_R A) \otimes_R K$; $A \otimes_R A$ is projective, so it is a submodule of $K^{T\times T}$, and we show $\Delta f \in A \otimes_R A$. By (ii) it suffices to do so after the base change to $R^h_\mathfrak{m}$ for every maximal $\mathfrak{m}$, where A becomes $\mathrm{Ord}(\Gamma_{R^h_\mathfrak{m}})$ (Step 2); so assume R henselian local and use Step 3. Write $t_i + t_j = t_{k(i,j)} + c_{ij}$ with $c_{ij} \in T^f$, and $g_{ij} = (\varphi_{t_i+t_j} f)|_{T^f} = \varphi_{c_{ij}} \Psi(f)_{k(i,j)} \in \mathcal{O}(\Gamma^f)$. The comultiplication of the finite flat group scheme Γ^f gives $a_{ij,l}, b_{ij,l} \in \mathcal{O}(\Gamma^f)$ with $g_{ij}(u + u') = \sum_l a_{ij,l}(u) b_{ij,l}(u')$ for $u, u' \in T^f$ (the group law on the generic fibre is the addition of T^f). Then $\Delta f = \sum_{i,j,l} a^{(i)}_{ij,l} \otimes b^{(j)}_{ij,l}$, as both sides take the value $f(t_i + t_j + u + u')$ at $(t_i + u, t_j + u')$. So A is stable under the comultiplication, counit $f \mapsto f(0)$ and antipode of K^T; the Hopf identities hold in K^T, and $A \subseteq K^T$, $A \otimes A \subseteq K^{T\times T}$, $A \otimes A \otimes A \subseteq K^{T\times T\times T}$ (projectivity), so $\operatorname{Spec} A$ is a finite flat commutative group scheme whose group law on K-points is the addition of T.

Proof of (2). As $A \subseteq \mathcal{O}(\Gamma)$ (Step 1), the inclusion gives a morphism $j\colon \Gamma \to \operatorname{Spec} A$ with $j(x_t) = \mathrm{ev}_t$. Uniqueness: a morphism $\Gamma \to \operatorname{Spec} A$ is a ring map $A \to \mathcal{O}(\Gamma) \subseteq K^T$, determined by its compositions with the x_t. Covering translates: let P be a point of $\operatorname{Spec} A$ over a prime $\mathfrak{q}$ of R, $R' = R^h_\mathfrak{q}$, $A' = A \otimes_R R' = \mathrm{Ord}(\Gamma')$ (Step 2) and $j' = j \times_R R'$, which is the morphism of (2) for Γ' (it maps x'_t to ev_t). By (iv) the closed fibre of Γ', resp. $\operatorname{Spec} A'$, is the fibre of Γ, resp. $\operatorname{Spec} A$, over $\mathfrak{q}$, with the same residue fields. Let P' be the point of $\operatorname{Spec} A'$ over P in the closed fibre. By Step 3(d) for Γ' there are i and a point $\gamma' \in \Gamma'^f$ with $P' = \tau_{t_i}(j'(\gamma'))$; the image γ of γ' in Γ gives $P = \tau_{t_i}(j(\gamma))$, as τ_t and j commute with the base change. Open immersion: j is a morphism of finitely presented R-schemes, hence locally of finite presentation. Let $\gamma \in \Gamma$ over $\mathfrak{q}$, and R', j' as before, γ' the point of the closed fibre of Γ' over γ, $P = j(\gamma)$,

$P' = j'(\gamma')$. By Step 3(d), j' restricted to the open set $\Gamma'^f \ni \gamma'$ is an isomorphism onto an open subscheme. Hence: j is injective on the fibre over $\mathfrak{q}$ and $\kappa(P) = \kappa(\gamma)$ (identification of closed fibres); as j is an R-morphism, j is injective. Unramified at γ: $\Omega_{\Gamma/A} \otimes \kappa(\gamma') = \Omega_{\Gamma'/A'} \otimes \kappa(\gamma') = 0$ (note $\Gamma' = \Gamma \times_{\operatorname{Spec} A} \operatorname{Spec} A'$), and $\kappa(\gamma') = \kappa(\gamma)$. Flat at γ: the local maps $\mathcal{O}_{A,P} \to \mathcal{O}_{A',P'}$ and $\mathcal{O}_{\Gamma,\gamma} \to \mathcal{O}_{\Gamma',\gamma'}$ are flat (base change of the flat $R \to R'$), the second faithfully flat (local); $\mathcal{O}_{\Gamma',\gamma'}$ is flat over $\mathcal{O}_{A',P'}$, hence over $\mathcal{O}_{A,P}$; for an injection $M_1 \to M_2$ of $\mathcal{O}_{A,P}$-modules the kernel N of $M_1 \otimes \mathcal{O}_{\Gamma,\gamma} \to M_2 \otimes \mathcal{O}_{\Gamma,\gamma}$ has $N \otimes_{\mathcal{O}_{\Gamma,\gamma}} \mathcal{O}_{\Gamma',\gamma'} = 0$, so $N = 0$. Thus j is étale, injective with trivial residue field extensions, and an open immersion by (vi).

Proof of (3). Let $A' = R' \otimes_R A$, finite projective of rank $|T|$ over R', $j' = j \times_R R' \colon \Gamma' \to \operatorname{Spec} A'$, and τ_t on $\operatorname{Spec} A'$ induced by $1 \otimes \varphi_t$. By (2) and base change, j' is an open immersion and $\operatorname{Spec} A' = \bigcup_t \tau_t(V)$ with $V = j'(\Gamma')$ (a point P has $\tau_t^{-1}(P)$ over a point of $j(\Gamma)$ for some t, and the preimage of the open set $j(\Gamma)$ is $j'(\Gamma')$). Values: for $f \in A$ the function $j^*f \in \mathcal{O}(\Gamma)$ has $\sigma_t^* j^* f = f(t) \in R$ in B (because x_t is induced by σ_t and $B \subseteq K$), so $(j'^*(1 \otimes f))(x'_t) = h_B(f(t)) = h(f(t))$. Hence the composite $\rho \colon A' \to \mathcal{O}(\Gamma') \to L^T$ is $r' \otimes f \mapsto r'(h \circ f)$, and its image is $\operatorname{Ord}(\Gamma)_h$. (a) ρ is injective, and $\rho_L \colon A' \otimes_{R'} L \to L^T$ is bijective. The points $j'(x'_t)$, $t \in T$, are $|T|$ distinct L-rational points of $\operatorname{Spec}(A' \otimes L)$ (j' is injective and the x'_t are distinct), so ρ_L, the product of the evaluations at them, is surjective (Chinese remainder theorem), hence bijective since $\dim_L(A' \otimes L) = |T|$. As A' is projective over the domain R', $A' \subseteq A' \otimes L$, and ρ is injective. In particular $\operatorname{Spec}(A' \otimes L) = \{j'(x'_t)\} \subseteq V$, and for every open $W \subseteq \operatorname{Spec} A'$, $\mathcal{O}(W)$ embeds in $\mathcal{O}(W_L)$, a factor of L^T ($\mathcal{O}_{\operatorname{Spec} A'}$ is torsion free and W quasi-compact, R' noetherian); for $W = V$ this is $\mathcal{O}(\Gamma') \subseteq L^T$ via j'. (b) $\operatorname{Ord}(\Gamma)_h \subseteq \operatorname{Ord}(\Gamma')$: for $f \in A$, $h \circ \varphi_t f = \varphi_t(h \circ f)$ is the function of $j'^*(1 \otimes \varphi_t f) \in \mathcal{O}(\Gamma')$, so every translate of $h \circ f$ is regular on Γ'; and $\operatorname{Ord}(\Gamma')$ is an R'-module. (c) $\operatorname{Ord}(\Gamma') \subseteq \operatorname{Ord}(\Gamma)_h$: let $g \in \operatorname{Ord}(\Gamma')$. The automorphism φ_t of L^T corresponds under ρ_L to $1 \otimes \varphi_t$, and pulling back along the automorphism τ_t gives $\mathcal{O}(\tau_t(V)) = \varphi_t^{-1}\mathcal{O}(V)$ inside L^T. Since $\varphi_t g \in \mathcal{O}(\Gamma') = \mathcal{O}(V)$ (Step 1 for Γ'), g defines a section over $\tau_t(V)$ for every t. The $\tau_t(V)$ cover $\operatorname{Spec} A'$ and these sections agree on overlaps (they have the same image in L^T, and sections over an open set embed in L^T by (a)), so they glue to an element of $\mathcal{O}(\operatorname{Spec} A') = A'$ with image g: $g \in \rho(A')$. Hence $\rho \colon A' \to \operatorname{Ord}(\Gamma') = \operatorname{Ord}(\Gamma)_h$ is an isomorphism of R'-algebras. Finally let $\mathfrak{p} = \ker h$ and $Q = \operatorname{Map}(T, R)/A$, a finitely generated R-module. By right exactness $Q \otimes_R R' = \operatorname{Map}(T, R')/\rho(A')$, so $Q \otimes_R L = L^T/\rho_L(A' \otimes L) = 0$ by (a). As $L \supseteq \kappa(\mathfrak{p})$, $Q \otimes_R \kappa(\mathfrak{p}) = 0$, so $Q_{\mathfrak{p}} = 0$ by Nakayama, i.e. $\operatorname{Ann}_R(Q) \not\subseteq \mathfrak{p}$: some d with $d \operatorname{Map}(T, R) \subseteq A$ has $h(d) \neq 0$. □

Lemma 35.29. *Let E be a number field of degree $2d$, p an odd prime unramified in E, and $\mathcal{O}'$ a discrete valuation ring of characteristic 0 with $p \in \mathfrak{m}_{\mathcal{O}'}$ such that every field embedding $\sigma \colon E \to \operatorname{Frac}(\mathcal{O}')$ maps $\mathcal{O}_E$ into $\mathcal{O}'$, and such that there are $2d$ of them; write them σ_u (the labels u) and put $\mathfrak{P}_u = \sigma_u^{-1}(\mathfrak{m}_{\mathcal{O}'})$. Let $\mathcal{X}'$ be a normal integral scheme of finite type over $\mathcal{O}'$, with function field Ω', and G' a semiabelian scheme over $\mathcal{X}'$ of relative dimension dn with $\iota \colon \mathcal{O}_E \to \operatorname{End}(G')$, abelian generic fibre, and every p-torsion point over an algebraic closure of Ω' defined over Ω'. Let $\omega_u \subseteq \omega_{G'}$ be the subsheaf of sections v with $\iota(a)^* v = \sigma_u(a) v$ for all $a \in \mathcal{O}_E$. Let $\mathfrak{P} \mid p$, $\Gamma_{\mathfrak{P}} = G'[\mathfrak{P}](\Omega')$, and for $t \in \Gamma_{\mathfrak{P}}$ let x_t be the corresponding Ω'-point. Then:*

1. *$G'[\mathfrak{P}]$ is flat, quasi-finite, separated and of finite presentation over $\mathcal{X}'$, its generic fibre is the constant group scheme of $\Gamma_{\mathfrak{P}}$, and $\Gamma_{\mathfrak{P}}$ is an $\mathbb{F}_{\mathfrak{P}}$-vector space of dimension n;*
2. *there is a unique subsheaf of $\mathcal{O}_{\mathcal{X}'}$-algebras $\mathcal{A}_{\mathfrak{P}} \subseteq \operatorname{Map}(\Gamma_{\mathfrak{P}}, \Omega')$ with $\mathcal{A}_{\mathfrak{P}}(U) = \operatorname{Ord}(G'[\mathfrak{P}] \times_{\mathcal{X}'} U)$ for every affine open U (formed with $T = \Gamma_{\mathfrak{P}}$ and the points x_t); it is locally free of finite rank, equal to $\operatorname{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}_{\mathcal{X}'})$ on the open subscheme where p is invertible, and $p^N \operatorname{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}_{\mathcal{X}'}) \subseteq \mathcal{A}_{\mathfrak{P}} \subseteq \operatorname{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}_{\mathcal{X}'})$ for some $N \geq 0$;*
3. *let $\mathcal{I}_{\mathfrak{P}} = \{g \in \mathcal{A}_{\mathfrak{P}} : g(0) = 0\}$. For every affine open U and $g \in \mathcal{A}_{\mathfrak{P}}(U)$ there is a unique regular function $\tilde{g}$ on $G'[\mathfrak{P}] \times_{\mathcal{X}'} U$ with $\tilde{g}(x_t) = g(t)$ for all t. Moreover $e^* \Omega^1_{G'[\mathfrak{P}]/\mathcal{X}'} =$*

$\omega_{G'}/\mathfrak{P}\omega_{G'} = \bigoplus_{\mathfrak{P}_u=\mathfrak{P}} \omega_u/p\,\omega_u$, and for every label u with $\mathfrak{P}_u = \mathfrak{P}$ the map $\theta_u\colon \mathcal{I}_{\mathfrak{P}} \to \omega_u/p\,\omega_u$, $g \mapsto$ the ω_u-component of $e^*d\tilde{g}$, is $\mathcal{O}_{\mathcal{X}'}$-linear and surjective.

Proof. **Step 0: the eigen decomposition.** Since p is unramified in E and $\mathcal{O}'$ contains the images of all $2d$ embeddings, $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{O}' \to \prod_u \mathcal{O}'$, $b \otimes z \mapsto (\sigma_u(b)z)_u$, is an isomorphism (an étale $\mathcal{O}'$-algebra of rank $2d$ split by the $2d$ distinct sections, distinct modulo $\mathfrak{m}_{\mathcal{O}'}$ because $\operatorname{disc}(E)$ is a unit in $\mathcal{O}'$). Its idempotents decompose $\omega_{G'} = \bigoplus_u \omega_u$. As $p\mathcal{O}_E = \prod_{\mathfrak{P}'|p} \mathfrak{P}'$ and $\sigma_u(\mathfrak{P}') \not\subseteq \mathfrak{m}_{\mathcal{O}'}$ for $\mathfrak{P}' \neq \mathfrak{P}_u$, the ideal $\sigma_u(\mathfrak{P})\mathcal{O}'$ is $p\mathcal{O}'$ if $\mathfrak{P}_u = \mathfrak{P}$ and $\mathcal{O}'$ otherwise. Hence $\mathfrak{P}\omega_{G'} = \bigoplus_u \sigma_u(\mathfrak{P})\omega_u$ and $\omega_{G'}/\mathfrak{P}\omega_{G'} = \bigoplus_{\mathfrak{P}_u=\mathfrak{P}} \omega_u/p\,\omega_u$.

Step 1: (1). $G'[\mathfrak{P}]$ is a closed subgroup scheme of the separated finite-type $\mathcal{X}'$-scheme G', hence separated and of finite presentation ($\mathcal{X}'$ noetherian). Its fibres are subgroups of $G'_k[p]$, which is finite for a semiabelian G'_k (an extension of an abelian variety by a torus), so it is quasi-finite. Flatness at a point x of $\mathcal{X}'$: if p is a unit at x, multiplication by p on the smooth commutative group scheme G' is étale near x (its differential is $p \cdot \mathrm{id}$), so $G'[p]$ is étale there, and $G'[\mathfrak{P}]$ is an open and closed subgroup of it (the idempotents of $\mathcal{O}_E/p = \prod_{\mathfrak{P}'} \mathbb{F}_{\mathfrak{P}'}$ split $G'[p] = \prod_{\mathfrak{P}'} G'[\mathfrak{P}']$), hence étale and flat. If x lies over p, the completion $\widehat{\mathcal{O}}_x$ is a complete noetherian local ring, normal because $\mathcal{O}_x$ is excellent and normal, hence a domain containing $\mathcal{O}_x$, with p in its maximal ideal; the generic fibre of $G' \times \operatorname{Spec} \widehat{\mathcal{O}}_x$ is a base change of $G'_{\Omega'}$, abelian. By Lemma 35.30(1) (with $R = \widehat{\mathcal{O}}_x$), $G'[\mathfrak{P}] \times \operatorname{Spec} \widehat{\mathcal{O}}_x$ is flat over $\widehat{\mathcal{O}}_x$; since $\mathcal{O}_x \to \widehat{\mathcal{O}}_x$ is faithfully flat and $\operatorname{Tor}_1^{\mathcal{O}_x}(M,N) \otimes \widehat{\mathcal{O}}_x = \operatorname{Tor}_1^{\widehat{\mathcal{O}}_x}(M \otimes \widehat{\mathcal{O}}_x, N \otimes \widehat{\mathcal{O}}_x)$, the coordinate rings of $G'[\mathfrak{P}]$ over an affine neighbourhood, localised at x, are flat over $\mathcal{O}_x$. So $G'[\mathfrak{P}]$ is flat. Its generic fibre is étale (characteristic 0) with all points rational, hence constant with group $\Gamma_{\mathfrak{P}}$. Dimension: with $A = G'_{\Omega'}$, an abelian variety of dimension dn, $\bigoplus_{\mathfrak{P}'} \Gamma_{\mathfrak{P}'} = A[p] = T_p(A)/pT_p(A)$ and $\Gamma_{\mathfrak{P}} = T_p(A)/\mathfrak{P}T_p(A)$; the rational Tate module $V_p(A)$ is a free $E \otimes \mathbb{Q}_p$-module of rank $2dn/2d = n$ (by the Lefschetz principle over $\mathbb{C}$, $V_p(A) = H_1(A(\mathbb{C}),\mathbb{Q}) \otimes \mathbb{Q}_p$, and $H_1(A(\mathbb{C}),\mathbb{Q})$ is a vector space over the field E, of dimension $2dn/2d$). Hence $T_p(A)$, a finitely generated torsion-free module over $\mathcal{O}_E \otimes \mathbb{Z}_p = \prod_{\mathfrak{P}'} \mathcal{O}_{E,\mathfrak{P}'}$ (a product of discrete valuation rings), is free of rank n on each factor, and $\Gamma_{\mathfrak{P}} = T_p(A) \otimes \mathbb{F}_{\mathfrak{P}}$ has dimension n.

Step 2: (2). For an affine open $U = \operatorname{Spec} R$, R is a noetherian normal domain with fraction field Ω', and $\Gamma_U = G'[\mathfrak{P}] \times U$ satisfies the hypotheses of Definition 35.21 with $T = \Gamma_{\mathfrak{P}}$ and the points x_t (Step 1). For $f \in R \setminus \{0\}$, Lemma 35.28(3) applies to $h\colon R \to R_f$ with $B = \Omega'$, $h_B = \mathrm{id}$ and $\sigma_t = x_t$, and gives $\operatorname{Ord}(\Gamma_{U_f}) = R_f \operatorname{Ord}(\Gamma_U)$ inside $\operatorname{Map}(\Gamma_{\mathfrak{P}}, \Omega')$. So the assignment $U \mapsto \operatorname{Ord}(\Gamma_U)$ on affine opens is compatible with localisation; for affine opens $V \subseteq U$, V is covered by opens $U_f = V_{f'}$ principal in both, on which both assignments give the same module, so $\operatorname{Ord}(\Gamma_V) \supseteq \operatorname{Ord}(\Gamma_U)$ (an element of $\operatorname{Ord}(\Gamma_U)$ lies in $\operatorname{Ord}(\Gamma_{V_{f'}})$ for all such f', and $\operatorname{Ord}(\Gamma_V)$, a locally free and hence reflexive $\mathcal{O}(V)$-module, is the intersection of these localisations); hence there is a unique quasi-coherent subsheaf $\mathcal{A}_{\mathfrak{P}}$ of $\operatorname{Map}(\Gamma_{\mathfrak{P}}, \Omega')$ with these sections on affines, a subsheaf of algebras, and it is locally free of finite rank by Lemma 35.28(1). By definition $\operatorname{Ord}(\Gamma_U) \subseteq \operatorname{Map}(\Gamma_{\mathfrak{P}}, R)$. Where p is invertible, let U be affine with $p \in R^\times$. By Step 1, Γ_U is étale and separated over the normal scheme U, hence normal, so its irreducible components are pairwise disjoint; since Γ_U is flat over U, every component dominates U and is the closure Z_t of a generic point x_t, and the finitely many Z_t are open and closed. For $f \in \operatorname{Map}(\Gamma_{\mathfrak{P}}, R)$ the function equal to $f(t)$ on Z_t is regular on Γ_U with value $f(t)$ at x_t, and the same holds for every translate of f; so $f \in \operatorname{Ord}(\Gamma_U)$, and $\mathcal{A}_{\mathfrak{P}} = \operatorname{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O})$ there. Hence the coherent sheaf $\operatorname{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O})/\mathcal{A}_{\mathfrak{P}}$ is supported on $V(p)$ and is killed by a power of p on each member of a finite affine cover ($\mathcal{X}'$ is quasi-compact); this gives N.

Step 3: (3). Let $U = \operatorname{Spec} R$ be affine, $A = \mathcal{A}_{\mathfrak{P}}(U) = \operatorname{Ord}(\Gamma_U)$ and $I = \mathcal{I}_{\mathfrak{P}}(U)$, the kernel of $\epsilon\colon A \to R$, $f \mapsto f(0)$. By Lemma 35.28(2) there is an open immersion $j\colon \Gamma_U \to \operatorname{Spec} A$ with

$j(x_t) = \mathrm{ev}_t$. For $g \in A$, $\tilde{g} = j^*g$ is regular on Γ_U with $\tilde{g}(x_t) = g(t)$; it is unique, because Γ_U is flat over the domain R, so its associated points lie over the generic point, where Γ_U is the reduced scheme of the points x_t, and a regular function vanishing at all x_t is 0. The R-point $j \circ e$ of $\operatorname{Spec} A$ composed with $R \subseteq \Omega'$ is $j(x_0) = \mathrm{ev}_0$; as $A \subseteq \operatorname{Map}(\Gamma_{\mathfrak{P}}, R)$ and $R \to \Omega'$ is injective, $j \circ e$ is the section ϵ. As j is an open immersion, the conormal module of e in Γ_U equals that of ϵ in $\operatorname{Spec} A$, which is I/I^2, the class of $g \in I$ corresponding to $e^*d\tilde{g}$. Hence $g \mapsto e^*d\tilde{g}$ is an R-linear surjection $I \to e^*\Omega^1_{\Gamma_U/U}$. Next, choose generators $a_1, \dots, a_k$ of $\mathfrak{P}$; $G'[\mathfrak{P}]$ is the fibre over the unit section of $\phi = (\iota(a_1), \dots, \iota(a_k))\colon G' \to G'^k$, so its ideal in G' is generated by the pull-backs under ϕ of the ideal of the unit section of G'^k, and the conormal sequence gives $e^*\Omega^1_{G'[\mathfrak{P}]/\mathcal{X}'} = \omega_{G'}/\sum_i \iota(a_i)^*\omega_{G'} = \omega_{G'}/\mathfrak{P}\omega_{G'}$, which is $\bigoplus_{\mathfrak{P}_u=\mathfrak{P}} \omega_u/p\,\omega_u$ by Step 0. Composing with the projection onto the summand $\omega_u/p\,\omega_u$, θ_u is R-linear and surjective on every affine open, hence as a map of sheaves; it is compatible with restriction because $\tilde{g}$ is unique. □

Lemma 35.30. *Let E be a number field, p a prime unramified in E, $\mathfrak{P}$ a prime of $\mathcal{O}_E$ above p and $\mathbb{F}_{\mathfrak{P}} = \mathcal{O}_E/\mathfrak{P}$. Let R be a complete noetherian normal local domain with p in its maximal ideal, $S = \operatorname{Spec} R$, and G a semiabelian scheme over S with abelian generic fibre and a ring homomorphism $\iota\colon \mathcal{O}_E \to \mathrm{End}_S(G)$; let $\mathcal{O}_E$ act on ω_G by $a \mapsto \iota(a)^*$. Put $G[\mathfrak{P}] = \bigcap_{a\in\mathfrak{P}} \ker \iota(a)$. Then:*

1. *$G[p] = \prod_{\mathfrak{P}'|p} G[\mathfrak{P}']$, and $G[\mathfrak{P}]$ is a closed $\mathcal{O}_E$-stable subgroup scheme of G, quasi-finite and flat over S, on which $\mathcal{O}_E$ acts through $\mathbb{F}_{\mathfrak{P}}$;*
2. *its finite part $G[\mathfrak{P}]^f$ is a finite locally free $\mathcal{O}_E$-stable subgroup scheme, equal to $G[p]^f \cap G[\mathfrak{P}]$, and isomorphic, $\mathcal{O}_E$-equivariantly, to $\widetilde{G}[\mathfrak{P}] = \bigcap_{a\in\mathfrak{P}} \ker(\iota(a)|_{\widetilde{G}})$;*
3. *$\omega_{G[\mathfrak{P}]^f} = \omega_{G[\mathfrak{P}]} \cong \omega_G \otimes_{\mathcal{O}_E} \mathbb{F}_{\mathfrak{P}} = \omega_G/\mathfrak{P}\omega_G$ as $R \otimes \mathcal{O}_E$-modules.*

Proof. We use the following standard facts: (i) a scheme quasi-finite and separated over a henselian local scheme S is the disjoint union of an open and closed subscheme finite over S (the largest one) and one with empty closed fibre (EGA IV, 18.5.11); a morphism to it from a scheme finite over S factors through the finite part; (ii) for a semiabelian scheme $[p]$ is quasi-finite and flat (SGA 7, Exp. IX, §2); (iii) for a smooth commutative group scheme P over S and $M \geq 1$, $\omega_{P[M]} = \omega_P/M\omega_P$, since the ideal of $P[M]$ near the unit section s_0 is generated by the $f \circ [M]$ with f in the ideal of s_0 and $s_0^*d(h \cdot f \circ [M]) = h(s_0)\, M\, s_0^*df$.

(1) As p is unramified in E, $\mathcal{O}_E/p = \prod_{\mathfrak{P}'|p} \mathbb{F}_{\mathfrak{P}'}$, with orthogonal idempotents $\epsilon_{\mathfrak{P}'}$ summing to 1. By (ii) $G[p]$ is closed, quasi-finite and flat over S, and $\mathcal{O}_E/p$ acts on it; the endomorphisms $\epsilon_{\mathfrak{P}'}$ of $G[p]$ are orthogonal idempotents with sum the identity, so $G[p] = \prod_{\mathfrak{P}'} \epsilon_{\mathfrak{P}'}G[p]$. For $a \in \mathfrak{P}$, a acts on $\epsilon_{\mathfrak{P}}G[p]$ by $a\epsilon_{\mathfrak{P}} = 0$; conversely, on $\epsilon_{\mathfrak{P}'}G[p]$ with $\mathfrak{P}' \neq \mathfrak{P}$ an element $a \in \mathfrak{P} \setminus \mathfrak{P}'$ acts invertibly (its image in $\mathbb{F}_{\mathfrak{P}'}$ is a unit), and $G[\mathfrak{P}] \subseteq G[p]$ as $p \in \mathfrak{P}$; hence $G[\mathfrak{P}] = \epsilon_{\mathfrak{P}}G[p]$. A direct factor of a flat quasi-finite group scheme is flat and quasi-finite, it is closed, $\mathcal{O}_E$-stable, and $\mathcal{O}_E$ acts on it through $\mathcal{O}_E/\mathfrak{P}$.

(2) S is complete, hence henselian, so by (i) $G[p]^f$ and $G[\mathfrak{P}]^f$ exist. For $a \in \mathcal{O}_E$ the image of $G[p]^f$ under $\iota(a)$ is finite over S, so lies in $G[p]^f$ by (i); thus $G[p]^f$ is $\mathcal{O}_E$-stable and $G[p]^f = \prod_{\mathfrak{P}'}(G[p]^f \cap G[\mathfrak{P}'])$, each factor open and closed in $G[\mathfrak{P}']$ and finite over S. A finite open and closed subscheme of $G[\mathfrak{P}]$ is finite over S inside $G[p]$, hence lies in $G[p]^f$ by (i); so $G[\mathfrak{P}]^f = G[p]^f \cap G[\mathfrak{P}] = \epsilon_{\mathfrak{P}}G[p]^f$. It is open and closed in the flat $G[\mathfrak{P}]$, so flat, and finite over the noetherian S, so finite locally free. By Theorem B.119 (1) there is an isomorphism $G[p]^f \cong \widetilde{G}[p]$, characterised on formal completions by $G^{\mathrm{for}} \cong \widetilde{G}^{\mathrm{for}}$; the endomorphisms $\iota(a)$ of G and their extensions to $\widetilde{G}$ (the Raynaud extension is unique up to unique isomorphism, Theorem B.109

(1), so endomorphisms of G extend uniquely and compatibly with $G^{\mathrm{for}} \cong \widetilde{G}^{\mathrm{for}}$) correspond on formal completions, and a morphism of schemes finite over the complete noetherian local ring R is determined by its formal completion. So the isomorphism is $\mathcal{O}_E$-equivariant and carries $\epsilon_{\mathfrak{P}} G[p]^f$ to $\epsilon_{\mathfrak{P}} \widetilde{G}[p] = \widetilde{G}[\mathfrak{P}]$ (the argument of (1) applies to $\widetilde{G}[p]$).

(3) $G[\mathfrak{P}]^f$ is an open subscheme of $G[\mathfrak{P}]$ containing the unit section, so $\omega_{G[\mathfrak{P}]^f} = \omega_{G[\mathfrak{P}]}$. For a product $H = H_1 \times H_2$ of group schemes, $\omega_H = \omega_{H_1} \oplus \omega_{H_2}$, and the idempotent endomorphism of H projecting onto H_1 acts on ω_H by the projection onto ω_{H_1}. With (1) and (iii), $\omega_{G[p]} = \omega_G / p\,\omega_G$ and

$$\omega_{G[\mathfrak{P}]} = \epsilon_{\mathfrak{P}}^*(\omega_G/p\,\omega_G) = (\omega_G/p\,\omega_G) \otimes_{\mathcal{O}_E/p} \mathbb{F}_{\mathfrak{P}} = \omega_G \otimes_{\mathcal{O}_E} \mathbb{F}_{\mathfrak{P}} = \omega_G/\mathfrak{P}\omega_G,$$

compatibly with the actions of R and $\mathcal{O}_E$. □

Lemma 35.31. *Let $\mathcal{X}$ be a joint model of the stages (B_0, c) and (B, c) of a root tower with a good origin over an integral model $\mathcal{Y}$, with function field Ω, let $p \in S(B_0, B)$ and let $R = R(p)$ be the integer at p of the root origin of $\mathcal{X}_{B,c}$. Let $\mathcal{X}'$ be a normal integral noetherian scheme with function field Ω' and $\pi\colon \mathcal{X}' \to \mathcal{X}$ a dominant morphism with $[\Omega' : \Omega] < \infty$, such that every prime $\ell \neq p$ is a unit on $\mathcal{X}'$; put $G' = \pi^* G_{\mathcal{X}}$, $\mathrm{Li} = \mathrm{Lie}(G'_{\Omega'})$, and for a point x let $V_{s,x} = \mathrm{HT}_{s,x}(p^R)$ if x lies over p and $V_{s,x} = \mathrm{dR}_{s,x}$ otherwise. Then:*

1. *$\mathcal{X}'$ with $f_0 \circ \mathrm{pr}_0 \circ \pi = f_1 \circ \mathrm{pr}_1 \circ \pi$ is a cover of $\mathcal{Y}$ in which every p^R-torsion point of G' over an algebraic closure of Ω' is defined over Ω', and at every point x over p the module ht_x is free of rank $\dim_{\Omega'} \mathrm{Li}$;*
2. *for every label s and every Ω'-basis $\beta = (\beta_1, \dots, \beta_r)$ of Li_s, the subsheaves $L, W \subseteq \Omega'^r$ with*

$$L(U) = \{c : \textstyle\sum_k c_k \beta_k \in \mathrm{dR}_{s,x}\ \forall x \in U\}, \qquad W(U) = \{c : \textstyle\sum_k c_k \beta_k \in V_{s,x}\ \forall x \in U\}$$

   are locally free of rank r with stalks $L_x = \{c : \sum_k c_k \beta_k \in \mathrm{dR}_{s,x}\}$ and $W_x = \{c : \sum_k c_k \beta_k \in V_{s,x}\}$; and their dual lattices $L^\vee$, $W^\vee$ (formed as in (Hf2) of Definition 35.22) are locally free of rank r, with stalks the dual lattices of L_x, W_x in Ω'^r for the dot product;
3. *if $w \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ satisfies $w^{p-1} = u\,p$ with u a unit on $\mathcal{X}'$, then $L \subseteq W \subseteq w^{-1}L$ and $W^\vee \subseteq L^\vee \subseteq w^{-1}W^\vee$.*

Proof. **Step 1: cover and torsion.** $\mathcal{X}'$ is normal, integral and noetherian; $f_0 \circ \mathrm{pr}_0 = f_1 \circ \mathrm{pr}_1$ by (Jm1), and $\mathrm{pr}_1 \circ \pi$ is dominant, so the generic point of $\mathcal{X}'$ maps to that of $\mathcal{X}_{B,c}$, hence by (Go3)(1) to the generic point of the component $\mathcal{Y}_c$, which meets $j(\mathrm{Sh}_F)$; and Ω', finite over Ω, is finitely generated over the residue field of that point. So $\mathcal{X}'$ is a cover with $G_{\mathcal{X}'} = G'$. Since $p \in S_B$, (Go3)(2) for $\mathcal{X}_1 = \mathcal{X}_{B,c}$ says that every p^R-torsion point of $G_{\mathcal{X}_1}$ is rational over Ω_1 and that ht_{x_1} is free of rank $g = \dim \mathrm{Li}$ at every point x_1 of $\mathcal{X}_1$ over p. By Lemma 31.30(2) for the dominant morphism $\mathrm{pr}_1 \circ \pi$, every p^R-torsion point of G' is rational over Ω', and $\mathrm{ht}_x = \mathcal{O}_x \cdot \mathrm{ht}_{x_1}$ for x over p, x_1 its image (which lies over p). So ht_x is generated by g elements. By Lemma 31.30(1), $p^R \omega_x \subseteq \mathrm{ht}_x$, so these g elements span the g-dimensional space $\mathrm{Li}^\vee$ and are linearly independent; ht_x is free of rank g. This is (1).

Step 2: W. Apply Lemma 31.29(1) to $\phi = \mathrm{pr}_1 \circ \pi\colon \mathcal{X}' \to \mathcal{X}_1$ with $S = S_B$ and the integers $R(\ell)$ of the root origin of $\mathcal{X}_1$: S_B contains the primes dividing $\mathrm{disc}(E)$; (Hp1) and (Hp2) are (Go3)(2) for $\mathcal{X}_1$. Hence the Hodge lattices of $\mathcal{X}'$ at (S_B, R) form a Hodge bundle $\mathcal{V}'_s \subseteq \mathrm{Li}_s$, locally free of rank $\mathrm{rk}(s) = r$ with stalks the Hodge lattices. A point of $\mathcal{X}'$ lies over a prime $\ell \in S_B$ only if $\ell = p$, all other primes being units; so the Hodge lattice of $\mathcal{X}'$ at (S_B, R) at x is $\mathrm{HT}_{s,x}(p^{R(p)}) = V_{s,x}$ if x lies over p and $\mathrm{dR}_{s,x} = V_{s,x}$ otherwise. As $\mathcal{V}'_s$ is a subsheaf of the constant sheaf Li_s, a vector lies in $\mathcal{V}'_s(U)$ iff it lies in $V_{s,x}$ for every $x \in U$. Therefore W is the

image of $\mathcal{V}'_s$ under the isomorphism $\mathrm{Li}_s \to \Omega'^r$ of coordinates in β: locally free of rank r, with stalks W_x as stated.

Step 3: L. Apply Lemma 31.29(1) to $\phi = \mathrm{pr}_0 \circ \pi \colon \mathcal{X}' \to \mathcal{X}_0 = \mathcal{X}_{B_0,c}$ with $S = S_{B_0}$ and the integers of the root origin of $\mathcal{X}_0$; its hypotheses hold as in Step 2 by (Go3)(1), (2) for $\mathcal{X}_0$, and $f_0 \circ \mathrm{pr}_0 \circ \pi$ is the structure map of the cover $\mathcal{X}'$. No point of $\mathcal{X}'$ lies over a prime of S_{B_0} (the only prime that is not a unit is $p \notin S_{B_0}$), so the Hodge lattices of $\mathcal{X}'$ at (S_{B_0}, R) are the $\mathrm{dR}_{s,x}$, and they form a Hodge bundle. As in Step 2, L is locally free of rank r with stalks L_x.

Step 4: the duals. Let $M \in \{L, W\}$ and U an open on which M is free with basis $c^{(1)}, \dots, c^{(r)}$; they form a basis of Ω'^r. For $e \in \Omega'^r$: if $e \in M^\vee(U)$ then $e{\cdot}c^{(k)} \in \mathcal{O}(U)$; conversely, if $e{\cdot}c^{(k)} \in \mathcal{O}(U)$ for all k, then $e{\cdot}c' \in \mathcal{O}(U')$ for every open $U' \subseteq U$ and $c' \in M(U') = \sum_k \mathcal{O}(U')c^{(k)}$. So $M^\vee(U)$ is free on the dual basis, and the same holds on every open of U; hence $M^\vee$ is locally free of rank r with stalk at x the span of the dual basis, which is the dual lattice of $M_x = \sum_k \mathcal{O}_x c^{(k)}$. This is (2).

Step 5: the inclusions (3). By Lemma 35.32, Ω' contains a primitive p-th root of unity ζ_p, and at every point x over p the element $\varpi_p = \zeta_p - 1$ lies in $\mathcal{O}_x$; as $\varpi_p^{p-1} \in p\,\mathbb{Z}[\zeta_p]^\times$ and $w^{p-1} = up$, the element w/ϖ_p has a unit $(p-1)$-th power, so $w\mathcal{O}_x = \varpi_p \mathcal{O}_x$ ($\mathcal{O}_x$ is normal). Let x lie over p. By Step 1, ht_x is free of rank $\dim \mathrm{Li}$, so Lemma 35.32(3) gives $\mathrm{dR}_{s,x} \subseteq \mathrm{HT}_{s,x}(p^R)$ and $\varpi_p \sigma_s(d)\,\mathrm{HT}_{s,x}(p^R) \subseteq \mathrm{dR}_{s,x}$ for every $d \in \mathfrak{d}_E$ (for $s = (\tau, -)$ even with $\sigma_s(d)$ omitted). As p is unramified in E, $\mathfrak{d}_E + p\mathcal{O}_E = \mathcal{O}_E$; choosing $d \in \mathfrak{d}_E$ with $d - 1 \in p\mathcal{O}_E$, $\sigma_s(d) \in 1 + p\mathcal{O}_x$ is a unit ($\sigma_s(\mathcal{O}_E)$ consists of algebraic integers, which lie in the normal ring $\mathcal{O}_x$). Hence $\mathrm{dR}_{s,x} \subseteq V_{s,x} \subseteq w^{-1}\mathrm{dR}_{s,x}$, that is $L_x \subseteq W_x \subseteq w^{-1}L_x$. If x does not lie over p, then $V_{s,x} = \mathrm{dR}_{s,x}$ and w is a unit at x, so the same chain holds. Since L, W are defined pointwise, $L \subseteq W \subseteq w^{-1}L$. Dualising (the dual lattice reverses inclusions and $(w^{-1}L)^\vee = wL^\vee$) gives $W^\vee \subseteq L^\vee$ and $wL^\vee \subseteq W^\vee$, that is $L^\vee \subseteq w^{-1}W^\vee$. □

Lemma 35.32. *Let $\mathcal{X}$ be a locally noetherian cover of an integral model of a unitary Shimura datum, with function field Ω, and put $g = \dim_\Omega \mathrm{Li}$. Let p be a prime and $R \geq 1$ an integer such that every p^R-torsion point of $G_\mathcal{X}$ over an algebraic closure of Ω is defined over Ω. Then Ω contains a primitive p-th root of unity ζ; put*

$$\varpi_p = 4 \quad (p = 2), \qquad \varpi_p = \zeta - 1 \quad (p \text{ odd}).$$

Let x be a point of $\mathcal{X}$ over p such that ht_x is a free $\mathcal{O}_x$-module of rank g. Then $\varpi_p \in \mathcal{O}_x$, and:

1. *$\varpi_p\, \omega_x \subseteq \mathrm{ht}_x \subseteq \omega_x$;*
2. *for every label s: $\mathrm{dR}_{s,x} \subseteq \mathrm{HT}'_{s,x}$ and $\varpi_p\, \mathrm{HT}'_{s,x} \subseteq \mathrm{dR}_{s,x}$;*
3. *for every label s: $\mathrm{dR}_{s,x} \subseteq \mathrm{HT}_{s,x}(p^R)$; and $\varpi_p\, \mathrm{HT}_{s,x}(p^R) \subseteq \mathrm{dR}_{s,x}$ if $s = (\tau, -)$, while $\varpi_p\, \sigma_s(d)\, \mathrm{HT}_{s,x}(p^R) \subseteq \mathrm{dR}_{s,x}$ for every $d \in \mathfrak{d}_E$ if $s = (\tau, +)$.*

In particular, if $\mathcal{X} = \mathcal{X}_{B,c}$ carries a root origin, all of this holds at every point x over every $\ell \in S_B$, with $p = \ell$ and $R = R(\ell)$.

Proof. Write $N = p^R$, $G = G_\mathcal{X}$ and $T = G(\Omega)[N]$; by hypothesis T is the whole group of N-torsion points over an algebraic closure $\overline{\Omega}$, so $T \cong (\mathbb{Z}/N)^{2g}$, and $g \geq 1$.

Step 0: roots of unity. The Weil pairing $e_N \colon T \times T \to \mu_N(\overline{\Omega})$ of the principal polarisation is perfect, hence surjective onto $\mu_N(\overline{\Omega})$, and it commutes with $\mathrm{Gal}(\overline{\Omega}/\Omega)$, which fixes T pointwise; so $\mu_N(\overline{\Omega}) \subseteq \Omega$, and ζ exists. Roots of unity are integral over $\mathbb{Z}$ and every local ring $\mathcal{O}_x$ of the normal scheme $\mathcal{X}$ is integrally closed in Ω, so $\mu_N \subseteq \mathcal{O}_x$ and $\varpi_p \in \mathcal{O}_x$. For every valuation v on

an extension of Ω with $v(p)=1$ we have $v(\varpi_p)\geq\frac{1}{p-1}$: $v(4)=2$ for $p=2$, and $v(\zeta-1)=\frac{1}{p-1}$ for p odd.

Step 1: Hodge–Tate spans localise. Let x be any point over p and q a generisation of x that lies over p; then $\mathcal{O}_q$ is a localisation of $\mathcal{O}_x$, $\omega_q=\mathcal{O}_q\omega_x$ and $\mathrm{Lie}_q=\mathcal{O}_q\mathrm{Lie}_x$. We claim $\mathrm{ht}_q=\mathcal{O}_q\,\mathrm{ht}_x$. By Lemma 31.30(1), $\mathrm{ht}_x=\sum_{y\in T}\mathcal{O}_x w_y+N\omega_x$, where w_y is a Hodge–Tate value of y at x with a function ϕ_y on an open set U_y containing the unit point $\epsilon(x)$. The unit section is continuous, so $\epsilon(q)$ is a generisation of $\epsilon(x)$ and lies in U_y; and $w_y-\epsilon^*d\phi_y\in N\omega_x\subseteq N\omega_q$. So w_y is a Hodge–Tate value of y at q, and Lemma 31.30(1) at q gives $\mathrm{ht}_q=\sum_y\mathcal{O}_q w_y+N\omega_q=\mathcal{O}_q\mathrm{ht}_x$.

Step 2: the lower inclusion at a point of codimension one over p. Let q be a point over p with $\mathcal{O}_q$ of dimension one; it is a noetherian normal local domain of dimension one, i.e. a discrete valuation ring, with p in its maximal ideal. We show $\varpi_p\omega_q\subseteq\mathrm{ht}_q$. (a) Let $\widehat{\mathcal{O}}$ be the completion of $\mathcal{O}_q$, K its fraction field, $v=\mathrm{ord}_q/\mathrm{ord}_q(p)$, so that K is complete with $v(p)=1$; let $\overline{K}$ be an algebraic closure and $\mathcal{O}_{\overline{K}}$ its valuation ring. Put $\omega=\omega_q\otimes\widehat{\mathcal{O}}$ and $\omega_{\overline{K}}=\omega_q\otimes_{\mathcal{O}_q}\mathcal{O}_{\overline{K}}$, free of rank g. (b) The torsion group scheme. Over $\mathcal{O}_q$, G is semiabelian; on each fibre $[N]$ is an isogeny of smooth connected groups, hence flat, so $[N]$ is flat by the fibrewise criterion and quasi-finite. Hence $\Gamma=G[N]=[N]^{-1}(\epsilon)$ is a closed subgroup scheme, flat, quasi-finite and separated of finite type, and its ideal near the unit section is generated by the functions $f\circ[N]$ with f in the ideal J of ϵ. Its generic fibre is finite étale of order N^{2g} with all N^{2g} points rational over Ω: the constant group T. Over the complete, hence henselian, local ring $\widehat{\mathcal{O}}$, $\Gamma_{\widehat{\mathcal{O}}}$ is the disjoint union of a finite part, which is a finite disjoint union of local schemes, one for each point of the closed fibre, and an open part with empty closed fibre (EGA IV 18.5.11). Let $\Gamma^0=\operatorname{Spec}\Lambda$ be the local piece at the unit point e' of the closed fibre: it is open in $\Gamma_{\widehat{\mathcal{O}}}$, so flat, and finite of finite presentation, so Λ is a free $\widehat{\mathcal{O}}$-module of finite rank; it consists of the generisations of e'. It is a subgroup scheme: $\Lambda\otimes_{\widehat{\mathcal{O}}}\Lambda$ is finite over $\widehat{\mathcal{O}}$ with local closed fibre (since e' has residue field that of $\widehat{\mathcal{O}}$), hence local (Tag 04GG (10)), so every point of $\Gamma^0\times\Gamma^0$ is a generisation of (e',e') and multiplication sends it to a generisation of e'; likewise for the inverse and the unit. So Γ^0 is a commutative finite flat group scheme over $\widehat{\mathcal{O}}$, whose generic fibre is the constant group of a subgroup $T^0\subseteq T$; its order $|T^0|$ is a power of p. Its module of invariant differentials is

$$\omega_{\Gamma^0}=\epsilon^*\Omega^1_{\Gamma^0}=\omega/N\omega,$$

since Γ^0 is open in $\Gamma_{\widehat{\mathcal{O}}}$, $\epsilon^*d\big(h\cdot(f\circ[N])\big)=h(\epsilon)\,N\,\epsilon^*df$ for $f\in J$ ($[N]$ induces multiplication by N on invariant differentials and $f\circ[N]\circ\epsilon=0$), and the ϵ^*df, $f\in J$, span ω. (c) Characters are Hodge–Tate values. Let $\chi\in\mathrm{Hom}(\Gamma^0_{\mathcal{O}_{\overline{K}}},\mathbb{G}_m)$, i.e. a unit $u\in\Lambda\otimes\mathcal{O}_{\overline{K}}$ with $u\otimes u$ as comultiplication and $u(\epsilon)=1$; then $\chi^*\frac{dT}{T}=\epsilon^*du$. On the generic fibre χ is a homomorphism $\chi_0\colon T^0\to\mu_N\subseteq\Omega$. As $\mu_N\cong\mathbb{Z}/N$ is an injective $\mathbb{Z}/N$-module, χ_0 extends to a homomorphism $T\to\mu_N$, which by perfectness is $e_N(y,\cdot)$ for some $y\in T$. Let w_y be a Hodge–Tate value of y at q with function ϕ on an open $U\ni\epsilon(q)$ (it exists by Lemma 31.30(1)). The preimage of U in $G_{\widehat{\mathcal{O}}}$ is open and contains e' (which maps to $\epsilon(q)$), hence contains all generisations of e', i.e. Γ^0; so ϕ restricts to $\phi^0\in\Lambda$. Since Λ is free, $\Lambda\otimes\mathcal{O}_{\overline{K}}\subseteq\Lambda\otimes\overline{K}=\overline{K}^{T^0}$, and at every $t\in T^0$ both $\phi^0\otimes1$ and u take the value $e_N(y,t)$; hence $u=\phi^0\otimes1$ and $\chi^*\frac{dT}{T}$ is the image of $\epsilon^*d\phi\equiv w_y\pmod{N\omega_q}$ in $\omega_{\overline{K}}/N\omega_{\overline{K}}=\omega_{\Gamma^0}\otimes\mathcal{O}_{\overline{K}}$. (d) Fargues. Apply Theorem B.53 to K, v and the Cartier dual $(\Gamma^0)^D$ (commutative, finite flat of order $|T^0|$), whose points over $\mathcal{O}_{\overline{K}}$ are the characters χ of (c), whose dual is Γ^0 and whose Hodge–Tate map is $\chi\mapsto\chi^*\frac{dT}{T}$; with $a=\varpi_p$ (Step 0) it gives $\varpi_p(\omega_{\overline{K}}/N\omega_{\overline{K}})\subseteq$ the $\mathcal{O}_{\overline{K}}$-span of these values. By (c) the span lies in the image of $H=\mathrm{ht}_q\otimes\mathcal{O}_{\overline{K}}$, so $\varpi_p\omega_{\overline{K}}\subseteq H+N\omega_{\overline{K}}=H$, as $N\omega_q\subseteq\mathrm{ht}_q$. Here $H\to\omega_{\overline{K}}$ is injective because $\mathcal{O}_q\to\mathcal{O}_{\overline{K}}$ is flat ($\mathcal{O}_{\overline{K}}$ is torsion free over the discrete valuation ring $\mathcal{O}_q$). (e) Descent. $\mathcal{O}_q\to\mathcal{O}_{\overline{K}}$ is a flat local homomorphism, hence faithfully flat. The module $Q=(\varpi_p\omega_q+\mathrm{ht}_q)/\mathrm{ht}_q$ satisfies $Q\otimes\mathcal{O}_{\overline{K}}=(\varpi_p\omega_{\overline{K}}+H)/H=0$, hence $Q=0$.

Step 3: item 1. $\mathrm{ht}_x \subseteq \omega_x$ by definition. The $\mathcal{O}_x$-modules ht_x and ω_x are free, so by Matsumura 11.5 applied coordinatewise $\mathrm{ht}_x = \bigcap_{\mathfrak{q}} (\mathrm{ht}_x)_{\mathfrak{q}}$ over the primes $\mathfrak{q}$ of height one of the noetherian normal domain $\mathcal{O}_x$. Let $w \in \omega_x$ and $\mathfrak{q}$ such a prime. If $p \in \mathfrak{q}$, then $(\mathcal{O}_x)_{\mathfrak{q}} = \mathcal{O}_q$ for a generisation q of x over p of codimension one, and $\varpi_p w \in \mathrm{ht}_q = (\mathrm{ht}_x)_{\mathfrak{q}}$ by Steps 2 and 1. If $p \notin \mathfrak{q}$, then N is a unit in $(\mathcal{O}_x)_{\mathfrak{q}}$ and $(\mathrm{ht}_x)_{\mathfrak{q}} \supseteq N(\omega_x)_{\mathfrak{q}} = (\omega_x)_{\mathfrak{q}} \ni \varpi_p w$. Hence $\varpi_p w \in \mathrm{ht}_x$.

Step 4: items 2 and 3. As Lie_x is free and ω_x is its dual, $\mathrm{Lie}_x = \{e \in \mathrm{Li} : \ w(e) \in \mathcal{O}_x \text{ for all } w \in \omega_x\}$. If $e \in \mathrm{dR}_{s,x} = \mathrm{Lie}_x \cap \mathrm{Li}_s$, then $w(e) \in \mathcal{O}_x$ for $w \in \mathrm{ht}_x \subseteq \omega_x$, so $e \in \mathrm{HT}'_{s,x}$. If $e \in \mathrm{HT}'_{s,x}$ and $w \in \omega_x$, then $\varpi_p w \in \mathrm{ht}_x$ by item 1, so $w(\varpi_p e) \in \mathcal{O}_x$; hence $\varpi_p e \in \mathrm{Lie}_x \cap \mathrm{Li}_s$. This is item 2. For $s = (\tau, -)$, $\mathrm{HT}_{s,x}(p^R) = \mathrm{HT}'_{s,x}$ and item 3 is item 2. For $s = (\tau, +)$ and $d \in \mathfrak{d}_E \subseteq \mathcal{O}_E$, $\sigma_s(d) \in \mathcal{O}_F$, whose image lies in $\mathcal{O}_x$ ($\mathcal{X}$ maps to $\mathcal{Y}$ over $\mathcal{O}_F$). If $e \in \mathrm{dR}_{s,x}$, then $\sigma_s(d)e \in \mathrm{dR}_{s,x} \subseteq \mathrm{HT}'_{s,x}$ for all d, so $e \in \mathrm{HT}_{s,x}(p^R)$; if $e \in \mathrm{HT}_{s,x}(p^R)$, then $\sigma_s(d)e \in \mathrm{HT}'_{s,x}$ and $\varpi_p \sigma_s(d) e \in \mathrm{dR}_{s,x}$ by item 2.

Root origins. $\mathcal{X}_{B,c}$ is a closed subscheme of a multiprojective space over $\mathbb{Z}$, hence noetherian, and a cover by (Go3)(1); by (Go3)(2), for $\ell \in S_B$ the $\ell^{R(\ell)}$-torsion is rational over Ω and ht_x is free of rank $\dim \mathrm{Li}$ at every point x over ℓ. □

Lemma 35.33. *Let $\mathfrak{T}$, p, R, w, a, $s = (a, +)$, $\bar{s}$, m, $\beta, \bar{\beta}$ and g be as in Lemma 35.34, and let $y = (y_1, \dots, y_m)$ be points for $(\mathfrak{T}, s)$. Then there is a Hodge–Tate atlas $(U_\nu, \iota_\nu, h_\nu)_\nu$ for $(\mathfrak{T}, s, \beta, \bar{\beta})$ with points y whose charts are Weil-adjoint: with forms η_j^ν,*

$$\sum_{l=1}^{m} \iota_\nu(c)_l \, g_{lj} = \eta_j^\nu \Big(\sum_k c_k \beta_k \Big) \qquad \text{for all } \nu, \ c \in V_L, \ 1 \le j \le m.$$

Proof. **Step 1: a chart around every point over p.** Let x be a point over p. By Lemma 31.30(1) each y_i has a Hodge–Tate value at x: by the definition of a Hodge–Tate value there is a regular function ϕ_i on an open neighbourhood N_i of the unit point $e(x)$ in G' with $\phi_i(g') = e_{p^R}(y_i, g')$ for every algebraically closed field $k \supseteq \Omega'$ and every $g' \in G'(k)[p^R]$. Put $N = \bigcap_i N_i$ and $U_x = e^{-1}(N)$, an open neighbourhood of x in $\mathcal{X}'$. Then $e^* d\phi_i$ is a section of $\omega_{G'}$ over U_x; let η_i be the linear form on Li given by its generic fibre. For every point $x' \in U_x$ over p, the function ϕ_i is regular on the neighbourhood N of $e(x')$ and has the Weil-pairing property, and $\eta_i(v) = d\phi_i(v)$ for $v \in \mathrm{Lie}_{x'}$; so $\eta_i|_{\mathrm{Lie}_{x'}}$ lies in $\omega_{x'}$ (the stalk of $\omega_{G'}$) and is a Hodge–Tate value at x' of y_i. Let $h_x \colon V_H \to V_\omega$ be the linear map with $h_x(e_i) = (\eta_i(\bar{\beta}_k))_k$. Then $(0, h_x)$ is a Hodge–Tate chart over U_x with points y and forms η_i, and by Lemma 35.34(1) so is (ι_x, h_x) with $\iota_x(c)_l = \sum_j \eta_j(\sum_k c_k \beta_k)(g^{-1})_{jl}$, which satisfies $\sum_l \iota_x(c)_l g_{lj} = \eta_j(\sum_k c_k \beta_k)$.

Step 2: finitely many charts cover. The points over p form the closed subset $Z = V(p)$ of the noetherian scheme $\mathcal{X}'$, which is quasi-compact; choose finitely many points $x_1, \dots, x_r$ over p with $Z \subseteq U_{x_1} \cup \dots \cup U_{x_r}$. On $U_0 = \mathcal{X}' \setminus Z$ no point lies over p, so $(0, 0)$ (forms $\eta_i = 0$) is a Hodge–Tate chart over U_0 with points y, and it is Weil-adjoint ($0 = 0$). The family consisting of $(U_0, 0, 0)$ and $(U_{x_k}, \iota_{x_k}, h_{x_k})$, $1 \le k \le r$, is a Hodge–Tate atlas with points y whose charts are Weil-adjoint. □

Lemma 35.34. *Let $\mathfrak{T} = (\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_{\mathfrak{P}}, \theta))$ be a pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$ with p odd and unramified in E, Ω' the function field of $\mathcal{X}'$, $G' = \pi^* G_{\mathcal{X}}$ with its principal polarisation, $R \ge 1$ as in Definition 35.22, and $w \in \mathcal{O}'$ with $w^{p-1} = u\,p$, $u \in \mathcal{O}'^{\times}$. Let $a \in \Phi$, $s = (a, +)$, $\bar{s} = (a, -)$, $\kappa = \mathrm{rk}(s)$, $m = m_{\mathfrak{P}_s}$, bases $\beta, \bar{\beta}$ as in Definition 35.22, and let (ι, h) be a Hodge–Tate chart over an open $U \subseteq \mathcal{X}'$ for $(\mathfrak{T}, s, \beta, \bar{\beta})$ with points $y_1, \dots, y_m$ and forms $\eta_1, \dots, \eta_m$. Let $\zeta \in \Omega'$ be a primitive p^R-th root of unity, $e_{p^R}(y, y') = \zeta^{\varepsilon(y, y')}$ the Weil pairing of the polarisation on $G'(\Omega')[p^R]$ with $\varepsilon(y, y') \in \{0, \dots, p^R - 1\}$, (ϖ_k) a $\mathbb{Z}$-basis of $\mathcal{O}_E$ with dual basis (ϖ_k^*) for the trace form, and $g_{lj} = \sum_k \sigma_s(\varpi_k^*)\, \varepsilon(\varpi_k y_l, y_j)$.*

1. *$g = (g_{lj}) \in \mathrm{GL}_m(\mathcal{O}')$. Define $\iota_0 \colon V_L \to V_H$ by $\iota_0(c)_l = \sum_j \eta_j(\sum_k c_k\beta_k)(g^{-1})_{jl}$. Then (ι_0, h) is a Hodge–Tate chart over U for $(\mathfrak{T}, s, \beta, \bar\beta)$ with the same y_i, η_i, and $\sum_l \iota_0(c)_l g_{lj} = \eta_j(\sum_k c_k\beta_k)$ for every $c \in V_L$ and every j.*

2. *Let x be a point of U over p at which ht_x is a free $\mathcal{O}_x$-module of rank $\dim_{\Omega'} \mathrm{Lie}(G'_{\Omega'})$. Put $W^x = \{c \in V_L : \sum_k c_k\beta_k \in V_{s,x}\}$, $\bar W^x = \{c \in V_\omega : \sum_k c_k\bar\beta_k \in V_{\bar s,x}\}$ and $\omega'^x = \{c \in V_\omega : c \cdot c' \in \mathcal{O}_x \ \forall c' \in \bar W^x\}$. Then W^x and ω'^x are free $\mathcal{O}_x$-modules of ranks κ and $\dim V_\omega$, and*

$$\iota_0(W^x) \subseteq \mathcal{O}_x^m, \qquad W^x \cap \iota_0^{-1}(w\,\mathcal{O}_x^m) = w\,W^x, \qquad h(\mathcal{O}_x^m) \subseteq \omega'^x, \qquad h(\mathcal{O}_x^m) + w\,\omega'^x = \omega'^x.$$

Proof. **Step 1: g is invertible.** For $y, y' \in M = G'(\Omega')[p^R]$ the map $b \mapsto \varepsilon(by, y') \bmod p^R$ is a $\mathbb{Z}$-linear form on $\mathcal{O}_E$ with values in $\mathbb{Z}/p^R$, represented under the trace form by $\langle y, y'\rangle = \sum_k \varpi_k^* \varepsilon(\varpi_k y, y') \in \mathfrak{d}_E^{-1}/p^R\mathfrak{d}_E^{-1}$; since $p \nmid \mathrm{disc}(E)$, $\mathfrak{d}_E^{-1} \otimes \mathbb{Z}_{(p)} = \mathcal{O}_E \otimes \mathbb{Z}_{(p)}$, so $\langle y, y'\rangle \in \mathcal{O}_E/p^R$. Then $\langle cy, y'\rangle = c\langle y, y'\rangle$ for $c \in \mathcal{O}_E$ (replace b by bc), and by the Rosati condition $\varepsilon(by, cy') = \varepsilon(\bar c by, y')$, so $\langle y, cy'\rangle = \bar c\langle y, y'\rangle$. The polarisation is principal and Ω' has characteristic 0, so e_{p^R} is perfect on M; by the adjunction $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}/p^R) \cong \mathrm{Hom}_{\mathcal{O}_E}(M, \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_E, \mathbb{Z}/p^R))$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_E, \mathbb{Z}/p^R) \cong \mathcal{O}_E/p^R$ (trace form, p unramified), the map $y' \mapsto \langle \cdot, y'\rangle$ is a bijection $M \to \mathrm{Hom}_{\mathcal{O}_E}(M, \mathcal{O}_E/p^R)$. With the basis (y_l), the composite $y' \mapsto (\langle y_l, y'\rangle)_l$ is a bijection $M \to (\mathcal{O}_E/p^R)^m$; it equals $\Gamma_0 \circ \mathrm{conj}$ in coordinates, $\Gamma_0 = (\langle y_l, y_j\rangle)_{l,j}$, conj the coordinate-wise complex conjugation (a bijection). Hence Γ_0 is bijective and $\det\Gamma_0 \in (\mathcal{O}_E/p^R)^\times$. Now $\sigma_s(\varpi_k^*) \in \sigma_s(\mathfrak{d}_E^{-1}) \subseteq \mathcal{O}'$ ($\mathfrak{d}_E$ is prime to p, and $\mathcal{O}' \supseteq \mathcal{O}_{F_{\Omega,\lambda}}$ inverts the elements of $\mathcal{O}_F$ outside λ), so $g \in M_m(\mathcal{O}')$, and $g \equiv \sigma_s(\Gamma_0) \bmod p^R\mathcal{O}'$ entrywise, $\sigma_s \colon \mathcal{O}_E/p^R \to \mathcal{O}'/p^R\mathcal{O}'$ being a ring homomorphism. So $\det g \equiv \sigma_s(\det\Gamma_0)$ is a unit modulo $p^R\mathcal{O}' \subseteq \mathfrak{m}_{\mathcal{O}'}$, and $\det g \in \mathcal{O}'^\times$.

Step 2: part (1). By Step 1, ι_0 is a well-defined Ω'-linear map, and $\sum_l \iota_0(c)_l g_{lj} = \sum_{j'} \eta_{j'}(\sum_k c_k\beta_k)(g^{-1}g)_{j'j} = \eta_j(\sum_k c_k\beta_k)$. The definition of a Hodge–Tate chart in Definition 35.22(Hf4) constrains only h, the y_i and the η_i (the latter at the points of U over p); so (ι_0, h) is a Hodge–Tate chart over U.

Step 3: the decomposition at x. Since $p \nmid \mathrm{disc}(E)$ and every prime that is not a unit in $\mathcal{O}_x$ is p ($\mathcal{O}_x$ is a local ring containing $\mathcal{O}_{F_{\Omega,\lambda}}$), $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{O}_x \to \prod_t \mathcal{O}_x$, $b \otimes z \mapsto (\sigma_t(b)z)_t$, is an isomorphism; its idempotents ϵ_t act on Lie_x and on ω_x, so $\mathrm{dR}_{t,x} = \epsilon_t\mathrm{Lie}_x$ and $\rho_t(\omega_x)$ is the dual lattice of $\mathrm{dR}_{t,x}$ in $\mathrm{Lie}_t(G'_{\Omega'})^\vee$. Also $\sigma_s(\mathfrak{d}_E)\mathcal{O}_x = \mathcal{O}_x$, so $V_{s,x} = \mathrm{HT}_{s,x}(p^R) = \mathrm{HT}'_{s,x}$ and $V_{\bar s,x} = \mathrm{HT}'_{\bar s,x}$. ht_x is stable under the action of $\mathcal{O}_E$ on ω_x: if ϕ is a function as in the definition of a Hodge–Tate value of y at x, then $\phi \circ \bar c$ (with $\bar c$ the endomorphism of G') is one for cy, because $e_{p^R}(cy, g') = e_{p^R}(y, \bar c g')$ (Rosati condition), and $d(\phi \circ \bar c) = \bar c^* d\phi$. Hence $\mathrm{ht}_x = \bigoplus_t \epsilon_t\mathrm{ht}_x$, and ρ_t restricted to $\epsilon_t\omega_x$ is injective, so $\rho_t(\mathrm{ht}_x) \cong \epsilon_t\mathrm{ht}_x$, a direct summand of the free module ht_x, hence free over the local ring $\mathcal{O}_x$, of rank $\mathrm{rk}(t)$ because $\varpi_p\omega_x \subseteq \mathrm{ht}_x \subseteq \omega_x$ by Lemma 35.32(1) (here $\varpi_p = \zeta_p - 1$, $\zeta_p = \zeta^{p^{R-1}}$, p odd). By Lemma 31.30(1), $\mathrm{ht}_x = \sum_y \mathcal{O}_x w_y + p^R\omega_x$, where w_y is a Hodge–Tate value of y. Hodge–Tate values are additive in y (if ϕ, ϕ' serve y, y', then $\phi\phi'$ serves $y + y'$ and $d(\phi\phi') = d\phi + d\phi'$ along the unit section, where $\phi = \phi' = 1$), and $w_{cy} = \bar c^* w_y$ as above. Writing $y = \sum_i c_i y_i$ ($c_i \in \mathcal{O}_E$) and using $\rho_t(\bar c^* v) = \sigma_t(\bar c)\rho_t(v)$ and that $\eta_i|_{\mathrm{Lie}_x}$ is a Hodge–Tate value of y_i at x (Definition 35.22(Hf4)):

$$\rho_t(\mathrm{ht}_x) = \sum_{i=1}^m \mathcal{O}_x\,\rho_t(\eta_i) + p^R\rho_t(\omega_x) \qquad (t = s, \bar s). \tag{$*$}$$

Finally $p^R\rho_t(\omega_x) \subseteq w\,\rho_t(\mathrm{ht}_x)$: by Lemma 35.32(1), $\varpi_p\rho_t(\omega_x) \subseteq \rho_t(\mathrm{ht}_x)$; $(w/\varpi_p)^{p-1} = up/\varpi_p^{p-1}$ is a unit of $\mathcal{O}_x$ ($\varpi_p^{p-1} \in p\,\mathbb{Z}[\zeta_p]^\times$), so w/ϖ_p and its inverse are integral over the normal ring $\mathcal{O}_x$, and $w\mathcal{O}_x = \varpi_p\mathcal{O}_x$; and $p^R/\varpi_p \in \varpi_p^{(p-1)R-1}\mathcal{O}_x^\times \subseteq \varpi_p\mathcal{O}_x$ since $(p-1)R - 1 \geq 1$. So

$p^R\rho_t(\omega_x) = (p^R/\varpi_p)\varpi_p\rho_t(\omega_x) \subseteq w\rho_t(\mathrm{ht}_x)$, and $(*)$ gives

$$\rho_t(\mathrm{ht}_x) = \sum_{i=1}^{m} \mathcal{O}_x\, \rho_t(\eta_i) + w\, \rho_t(\mathrm{ht}_x) \qquad (t = s, \bar{s}). \tag{$**$}$$

Step 4: coordinates. Let $N = \{(\lambda(\beta_k))_k : \lambda \in \rho_s(\mathrm{ht}_x)\} \subseteq V_L$ and $\bar{N} = \{(\lambda(\bar{\beta}_k))_k : \lambda \in \rho_{\bar{s}}(\mathrm{ht}_x)\} \subseteq V_\omega$, free of ranks κ, $\dim V_\omega$ (Step 3; $\beta, \bar{\beta}$ are bases). By definition of HT', $W^x = \{c : c \cdot n \in \mathcal{O}_x\ \forall n \in N\}$ is the dual lattice of N and $\bar{W}^x$ that of $\bar{N}$; duals of free lattices are free on the dual basis, and the dual of the dual is the lattice itself. Hence W^x is free of rank κ, N is the dual of W^x, and $\omega'^x = \bar{N}$ is free. Put $n_i = (\eta_i(\beta_k))_k \in N$; then $h(e_i) = (\eta_i(\bar{\beta}_k))_k \in \bar{N}$, and for $c \in V_L$, $c \cdot n_i = \eta_i(\sum_k c_k\beta_k)$.

Step 5: the four assertions. (a) For $c \in W^x$, $\eta_j(\sum_k c_k\beta_k) = c \cdot n_j \in \mathcal{O}_x$, so $\iota_0(c) = (c\cdot n_j)_j g^{-1} \in \mathcal{O}_x^m$, as $g^{-1} \in M_m(\mathcal{O}')$. (b) By (a), $wW^x \subseteq W^x \cap \iota_0^{-1}(w\mathcal{O}_x^m)$. Conversely let $c \in W^x$ with $\iota_0(c) \in w\mathcal{O}_x^m$. Then $(c \cdot n_j)_j = \iota_0(c)g \in w\mathcal{O}_x^m$. By $(**)$ for $t = s$, $N = \sum_i \mathcal{O}_x n_i + wN$, and $c\cdot(wN) \subseteq w\mathcal{O}_x$ since $c \in W^x$ is in the dual of N. So $c \cdot N \subseteq w\mathcal{O}_x$, that is $w^{-1}c \in W^x$, $c \in wW^x$. (c) $h(\mathcal{O}_x^m) = \sum_i \mathcal{O}_x h(e_i) \subseteq \bar{N} = \omega'^x$. (d) By $(**)$ for $t = \bar{s}$ in coordinates, $\bar{N} = \sum_i \mathcal{O}_x h(e_i) + w\bar{N}$, that is $h(\mathcal{O}_x^m) + w\omega'^x = \omega'^x$. □

Lemma 35.35. *Let $\mathfrak{T}$, p, R, w, a, $s = (a, +)$, $\bar{s} = (a, -)$, m, $\beta, \bar{\beta}$ and g be as in Lemma 35.34, and assume $R \geq 3$. Let (ι, h) and (ι', h') be Hodge–Tate charts for $(\mathfrak{T}, s, \beta, \bar{\beta})$ over opens U, U' of $\mathcal{X}'$, with the same points $y_1, \dots, y_m$ and with forms $\eta_1, \dots, \eta_m$ and $\eta'_1, \dots, \eta'_m$. Let $x \in U \cap U'$ be a point over p at which ht_x is a free $\mathcal{O}_x$-module of rank $\dim_{\Omega'} \mathrm{Lie}(G'_{\Omega'})$, and assume that for every $c \in W^x$ and every j*

$$\sum_{l=1}^{m} \iota(c)_l\, g_{lj} - \eta_j\Big(\sum_k c_k\beta_k\Big) \in p^{R-2}\mathcal{O}_x, \qquad \sum_{l=1}^{m} \iota'(c)_l\, g_{lj} - \eta'_j\Big(\sum_k c_k\beta_k\Big) \in p^{R-2}\mathcal{O}_x. \tag{35.1}$$

Then:

1. *$\iota(c) - \iota'(c) \in \mathcal{O}_x^m$ for every $c \in V_L$ with $wc \in W^x$;*
2. *$h(d) - h'(d) \in \omega'^x$ for every $d \in V_H$ with $wd \in \mathcal{O}_x^m$.*

Proof. **Step 1: the valuations at x.** By Lemma 35.32, Ω' contains a primitive p-th root of unity ζ_p and $\varpi_p = \zeta_p - 1 \in \mathcal{O}_x$ (p is odd). Since $p = \prod_{k=1}^{p-1}(1-\zeta_p^k)$ and each $(1-\zeta_p^k)/(1-\zeta_p)$ is a unit of $\mathbb{Z}[\zeta_p]$, $\varpi_p^{p-1} \in p\,\mathbb{Z}[\zeta_p]^\times$. With $w^{p-1} = u\,p$, $u \in \mathcal{O}'^\times$, the element $w/\varpi_p \in \Omega'$ has $(p-1)$-th power a unit of $\mathcal{O}_x$, so w/ϖ_p and its inverse are integral over the normal domain $\mathcal{O}_x$ and lie in it. Hence $w\mathcal{O}_x = \varpi_p\mathcal{O}_x$ and $p\mathcal{O}_x = \varpi_p^{p-1}\mathcal{O}_x$, and for integers a, b

$$p^a w^{-b}\mathcal{O}_x = \varpi_p^{(p-1)a-b}\mathcal{O}_x. \tag{$*$}$$

Step 2: every $\lambda \in \omega_x$ takes values in $\varpi_p^{-1}\mathcal{O}_x$ on $V_{s,x}$ and on $V_{\bar{s},x}$. As x lies over p, $V_{t,x} = \mathrm{HT}_{t,x}(p^R)$ for $t = s, \bar{s}$ (the Hodge lattice at $(\{p\}, R)$, Definition 35.22). Since ht_x is free of rank $\dim \mathrm{Li}$, Lemma 35.32(3) gives $\varpi_p\mathrm{HT}_{\bar{s},x}(p^R) \subseteq \mathrm{dR}_{\bar{s},x}$ and $\varpi_p\sigma_s(d)\mathrm{HT}_{s,x}(p^R) \subseteq \mathrm{dR}_{s,x}$ for every $d \in \mathfrak{d}_E$. As p is unramified in E, $\mathfrak{d}_E + p\mathcal{O}_E = \mathcal{O}_E$; choose $d \in \mathfrak{d}_E$ with $d - 1 \in p\mathcal{O}_E$. Then $\sigma_s(d) \in 1 + p\mathcal{O}_x \subseteq \mathcal{O}_x^\times$ ($\sigma_s(\mathcal{O}_E) \subseteq \mathcal{O}' \subseteq \mathcal{O}_x$ and p lies in the maximal ideal of $\mathcal{O}_x$). So $\varpi_p V_{t,x} \subseteq \mathrm{dR}_{t,x} = \mathrm{Lie}_x \cap \mathrm{Li}_t \subseteq \mathrm{Lie}_x$ for $t = s, \bar{s}$, and $\lambda(\mathrm{Lie}_x) \subseteq \mathcal{O}_x$ gives the claim.

Step 3: the forms of the two charts differ by p^R times forms in ω_x. Since $x \in U \cap U'$ lies over p, (Hf4) says that $\eta_j|_{\mathrm{Lie}_x}$ and $\eta'_j|_{\mathrm{Lie}_x}$ are Hodge–Tate values at x of the same point y_j.

By Lemma 31.30(1) the Hodge–Tate values of y_j at x form one coset of $p^R\omega_x$ in ω_x. Hence there is $\mu_j \in \omega_x$ with $\eta_j - \eta'_j = p^R\mu_j$ on Lie_x, and therefore on Li.

Step 4: part (1). Let $c \in V_L$ with $wc \in W^x$, and put $v = \sum_k c_k\beta_k \in \mathrm{Li}_s$, so that $wv \in V_{s,x}$. Applying (35.1) to wc and dividing by w, and using $(*)$ with $(p-1)(R-2)-1 \geq 1$ ($p \geq 3$, $R \geq 3$),

$$\sum_l \iota(c)_l g_{lj} - \eta_j(v) \in w^{-1}p^{R-2}\mathcal{O}_x \subseteq \mathcal{O}_x, \qquad \sum_l \iota'(c)_l g_{lj} - \eta'_j(v) \in \mathcal{O}_x.$$

By Step 3, $\eta_j(v) - \eta'_j(v) = p^R w^{-1}\mu_j(wv)$, and $\mu_j(wv) \in \varpi_p^{-1}\mathcal{O}_x$ by Step 2; by $(*)$, $p^R w^{-1}\varpi_p^{-1}\mathcal{O}_x = \varpi_p^{(p-1)R-2}\mathcal{O}_x \subseteq \mathcal{O}_x$. Subtracting, the row vector $(\iota(c) - \iota'(c))\, g$ has entries in $\mathcal{O}_x$. As $g \in \mathrm{GL}_m(\mathcal{O}')$ (Lemma 35.34(1)) and $\mathcal{O}' \subseteq \mathcal{O}_x$, $\iota(c) - \iota'(c) \in \mathcal{O}_x^m$.

Step 5: part (2). Let $d \in V_H$ with $wd \in \mathcal{O}_x^m$ and let $c' \in \bar{W}^x$, $u = \sum_k c'_k\bar{\beta}_k \in V_{\bar{s},x}$. By the definition of h, h' through the forms and Step 3,

$$(h(d) - h'(d)) \cdot c' = \sum_{i=1}^m d_i\,(\eta_i - \eta'_i)(u) = \sum_{i=1}^m d_i\, p^R\mu_i(u).$$

Here $d_i \in w^{-1}\mathcal{O}_x$ and $\mu_i(u) \in \varpi_p^{-1}\mathcal{O}_x$ (Step 2), so each term lies in $p^R w^{-1}\varpi_p^{-1}\mathcal{O}_x \subseteq \mathcal{O}_x$ by $(*)$. As $c' \in \bar{W}^x$ was arbitrary, $h(d) - h'(d)$ lies in ω'^x, the dual lattice of $\bar{W}^x$. □

Lemma 35.36. *Let $\mathfrak{T}$, p, R, w, a, $s = (a,+)$, $\bar{s} = (a,-)$, κ, m, $\beta, \bar{\beta}$ and g be as in Lemma 35.34, with $R \geq 3$, $\kappa = \mathrm{rk}(s)$, $\mathrm{rk}(\bar{s}) = n - \kappa$ and $m = n$. For every point x of $\mathcal{X}'$ put $W^x = \{c \in V_L : \sum_k c_k\beta_k \in V_{s,x}\}$, $\bar{W}^x = \{c \in V_\omega : \sum_k c_k\bar{\beta}_k \in V_{\bar{s},x}\}$ and $\omega'^x = \{c \in V_\omega : c \cdot c' \in \mathcal{O}_x\ \forall c' \in \bar{W}^x\}$. Let $(U_\nu, \iota_\nu, h_\nu)_\nu$ be a Hodge–Tate atlas for $(\mathfrak{T}, s, \beta, \bar{\beta})$ with points $y_1, \ldots, y_m$ and forms η_j^ν whose charts satisfy*

$$\sum_{l=1}^m \iota_\nu(c)_l\, g_{lj} = \eta_j^\nu\Big(\sum_k c_k\beta_k\Big) \qquad \text{for all } \nu,\ c \in V_L,\ 1 \leq j \leq m,$$

and let $(\mathrm{lie}, W, \omega, \omega', T, H, \Gamma, \Delta)$ be its Hodge–Tate lattice family for w. Assume:

(Sq1) *at every point x over p, ht_x is a free $\mathcal{O}_x$-module of rank $\dim_{\Omega'} \mathrm{Lie}(G'_{\Omega'})$;*

(Sq2) *W and ω' are locally free of ranks κ and $n - \kappa$, with stalks $W_x = W^x$ and $\omega'_x = \omega'^x$ at every point x;*

(Sq3) *for every point x over p and every ν with $x \in U_\nu$, the $\mathcal{O}_x$-module $\iota_\nu(W_x) + wT_x$ is free.*

Then:

1. *at every point x over p and for every ν with $x \in U_\nu$: $\iota_\nu(W_x) \subseteq T_x$, $h_\nu(T_x) \subseteq \omega'_x$, and the sequence of $\mathcal{O}_x$-modules*

$$0 \to W_x/wW_x \xrightarrow{\iota_\nu} T_x/wT_x \xrightarrow{h_\nu} \omega'_x/w\omega'_x \to 0$$

is exact;

2. *at every point x over p and for every ν with $x \in U_\nu$,*

$$\Gamma_x = \{(c,d) :\ wc \in W_x,\ d - \iota_\nu(c) \in T_x\}, \qquad \Delta_x = \{(d,e) :\ wd \in T_x,\ e - h_\nu(d) \in \omega'_x\};$$

3. *Γ and Δ satisfy (G1) and (G2) over every affine open of $\mathcal{X}'$.*

Proof. **Step 1: the inclusions and the two outer exactness assertions of (1).** Let x be a point over p and $x \in U_\nu$. By (Sq1), Lemma 35.34(2) applies at x to the chart (ι_ν, h_ν) over U_ν; with (Sq2) it gives $\iota_\nu(W_x) \subseteq T_x$, $W_x \cap \iota_\nu^{-1}(wT_x) = wW_x$, $h_\nu(T_x) \subseteq \omega'_x$ and $h_\nu(T_x) + w\omega'_x = \omega'_x$. So the maps of (1) are defined, the first is injective and the second surjective.

Step 2: $h_\nu(T_q) \subseteq \omega'_q$ at every generisation q of x, and $h_\nu\iota_\nu(W_x) \subseteq w\,\omega'_q$ at every generisation q of codimension one. Let q be a generisation of x; then $q \in U_\nu$. If q lies over p, (Sq1) holds at q and Step 1 at q gives $h_\nu(T_q) \subseteq \omega'_q$; if moreover q has codimension one, Lemma 35.38(1) gives $h_\nu(\iota_\nu(W^q)) \subseteq w\,\omega'^q$, and $W_x \subseteq W_q = W^q$, $\omega'^q = \omega'_q$ by (Sq2). If q does not lie over p, then $V_{\bar s,q} = \mathrm{dR}_{\bar s,q} \subseteq \mathrm{Lie}_q$, and $\eta_i^\nu(\mathrm{Lie}_q) = \mathcal{O}_q\,\eta_i^\nu(\mathrm{Lie}_x) \subseteq \mathcal{O}_q$ because $\eta_i^\nu|_{\mathrm{Lie}_x} \in \omega_x$ ((Hf4) at x). For $d \in \mathcal{O}_q^m$ and $c' \in \bar W^q$, $u = \sum_k c'_k\bar\beta_k \in V_{\bar s,q}$, we get $h_\nu(d)\cdot c' = \sum_i d_i\eta_i^\nu(u) \in \mathcal{O}_q$, so $h_\nu(T_q) \subseteq \omega'^q = \omega'_q$. As w is a unit at q and $\iota_\nu(W_x) \subseteq T_x \subseteq T_q$ (Step 1), $h_\nu\iota_\nu(W_x) \subseteq \omega'_q = w\,\omega'_q$.

Step 3: middle exactness in (1). Put $P_x = \iota_\nu(W_x) + wT_x$ and $P'_x = \{d \in T_x :\ h_\nu(d) \in w\,\omega'_x\}$, the kernel of the surjection $T_x \to \omega'_x/w\omega'_x$ of Step 1. Exactness at T_x/wT_x means $P_x = P'_x$. Since w is a non-zero non-unit of the domain $\mathcal{O}_x$ and ω'_x is free, $0 \to \omega'_x \xrightarrow{w} \omega'_x \to \omega'_x/w\omega'_x \to 0$ is a free resolution; by Schanuel's lemma $P'_x \oplus \omega'_x \cong T_x \oplus \omega'_x$, so P'_x is finitely generated projective over the local ring $\mathcal{O}_x$, hence free, of rank m (it contains wT_x). P_x is free by (Sq3), of rank m. For the normal noetherian domain $\mathcal{O}_x$, $\mathcal{O}_x = \bigcap_{\mathrm{ht}\,Q=1}(\mathcal{O}_x)_Q$ (Matsumura, Commutative Ring Theory, Thm. 11.5), hence $wF = \bigcap_Q wF_Q$ for every free submodule F of V_ω spanned by a basis of V_ω (compare coordinates in such a basis). So Step 2 gives $h_\nu\iota_\nu(W_x) \subseteq \bigcap_Q w\,\omega'_{q(Q)} = w\,\omega'_x$, where $q(Q)$ is the codimension-one generisation of x corresponding to the height-one prime Q of $\mathcal{O}_x$; so $P_x \subseteq P'_x$. At such $q = q(Q)$: if q lies over p, the localisations are $(P_x)_Q = \iota_\nu(W_q) + wT_q$ and $(P'_x)_Q = \{d \in T_q : h_\nu(d) \in w\omega'_q\}$, and these agree by the exactness of Lemma 35.38(2) at q (here $\kappa = \mathrm{rk}(s)$, $\mathrm{rk}(\bar s) = n - \kappa$, $m = n$, and $W^q = W_q$, $\omega'^q = \omega'_q$); if q does not lie over p, w is a unit at q and both localisations equal T_q. By Lemma 35.12(4) on the normal integral noetherian scheme $\mathrm{Spec}\,\mathcal{O}_x$, the free modules P_x and P'_x of rank m in V_H are equal. This proves (1).

Step 4: the stalks (2). The inclusions $\subseteq$ hold because the conditions defining $\Gamma(U)$, $\Delta(U)$ include those at x for the chart ν, and $W(U) \to W_x$, $T(U) \to T_x$. Conversely let $wc \in W_x$ and $d - \iota_\nu(c) \in T_x$. Choose an open neighbourhood $U \subseteq U_\nu$ of x on which W is free with a basis $c^{(1)}, \dots, c^{(\kappa)}$, with $wc \in W(U)$, $d - \iota_\nu(c) \in T(U)$ and $\iota_\nu(c^{(k)}) \in T(U)$ for all k (finitely many elements of V_H, V_L lying in the stalks at x, by Step 1, lie in the sections over a neighbourhood). Then $\iota_\nu(W(U')) \subseteq T(U')$ for every open $U' \subseteq U$. Let $y \in U$. If y lies over p and $y \in U_\mu$, then $d - \iota_\mu(c) = (d - \iota_\nu(c)) + (\iota_\nu(c) - \iota_\mu(c)) \in T_y$ by Lemma 35.35(1) at y (its hypotheses: (Sq1) at y, $R \geq 3$, $wc \in W_y \subseteq W^y$, and the adjointness congruence holds with value 0). If y does not lie over p, w is a unit at y, so $c \in W_y$, $\iota_\nu(c) \in T_y$ and $d \in T_y$. Hence $(c, d) \in \Gamma(U)$. For Δ, let $wd \in T_x$ and $e - h_\nu(d) \in \omega'_x$; choose $U \subseteq U_\nu$ with $wd \in T(U)$, $e - h_\nu(d) \in \omega'(U)$ and $h_\nu(e_i) \in \omega'(U)$ for the standard basis vectors e_i (by Step 1 and (Sq2)). For $y \in U$ over p and $y \in U_\mu$, $h_\nu(d) - h_\mu(d) \in \omega'^y = \omega'_y$ by Lemma 35.35(2), so $e - h_\mu(d) \in \omega'_y$; for y not over p, $d \in T_y$, so $h_\nu(d) \in \omega'_y$ and $e \in \omega'_y$. Hence $(d, e) \in \Delta(U)$.

Step 5: the conditions of (G1), (G2) that need no lifting. Let U be open and a, b, c sections over U of V_L, V_H, V_ω. By definition, $(0, b) \in \Gamma(U)$ if and only if $b \in T_y$ for all $y \in U$, if and only if $b \in T(U)$; and $(0, c) \in \Delta(U)$ if and only if $c \in \omega'(U)$, likewise. If $(a, 0) \in \Gamma(U)$, then $wa \in W(U)$, and at $y \in U$ over p (in U_ν) $\iota_\nu(wa) \in wT_y$, so $wa \in wW_y$ by Step 1, that is $a \in W^y$; at y not over p, $a = w^{-1}(wa) \in W^y$; so $a \in W(U)$, W being defined pointwise in (Hf2). Conversely if $a \in W(U)$ then $wa \in W(U)$ and $\iota_\nu(a) \in T_y$ for $y \in U \cap U_\nu$ over p (Step 1), so $(a, 0) \in \Gamma(U)$. If $(b', c) \in \Delta(U)$, then at y over p, $wc \in h_\nu(wb') + w\omega'_y \subseteq \omega'_y$ (Step 1), and at other y, $c \in \omega'_y$; so $wc \in \omega'(U)$. If $(a, b') \in \Gamma(U)$ then $wa \in W(U)$ by definition, and if $(b, c') \in \Delta(U)$ then $wb \in T(U)$. *The last clause of (G2) at stalks.* At x over p: $(b, 0) \in \Delta_x$ if and only if $wb \in T_x$ and $h_\nu(wb) \in w\omega'_x$ (Step 4), if and only if $wb \in P'_x = P_x$ (Step 3), if and

only if $wb = \iota_\nu(c_0) + wt$ with $c_0 \in W_x$, $t \in T_x$, if and only if $(w^{-1}c_0, b) \in \Gamma_x$ for some $c_0 \in W_x$ (Step 4), that is, if and only if $(a', b) \in \Gamma_x$ for some a' (any such a' has $wa' \in W_x$). At x not over p, using a neighbourhood on which p is invertible: $(b, 0) \in \Delta_x$ if and only if $b \in T_x$ if and only if $(a', b) \in \Gamma_x$ for some a' (take $a' = 0$).

Step 6: the lifting clauses over affine opens. The sequences of sheaves

$$0 \to T \xrightarrow{b\mapsto(0,b)} \Gamma \xrightarrow{\mathrm{pr}_1} w^{-1}W \to 0, \qquad 0 \to \omega' \xrightarrow{c\mapsto(0,c)} \Delta \xrightarrow{\mathrm{pr}_1} w^{-1}T \to 0$$

are exact: on the left by Step 5, in the middle by definition, and on the right on stalks by Step 4 at points over p (lifts $(a, \iota_\nu(a))$, $(d, h_\nu(d))$) and, at a point not over p, by the lifts $(a, 0)$, $(d, 0)$ on a neighbourhood where p is invertible and $wa \in W$, $wd \in T$. The outer sheaves are coherent. An extension of quasi-coherent sheaves is quasi-coherent (on an affine open V and $f \in \mathcal{O}(V)$, the sequences of sections over V and V_f are exact on the right because H^1 of a quasi-coherent sheaf on an affine scheme vanishes, and the five lemma compares the sections over V_f with the localisation at f of those over V). So Γ, Δ are quasi-coherent, and on an affine open U the maps $\Gamma(U) \to (w^{-1}W)(U)$ and $\Delta(U) \to (w^{-1}T)(U)$ are onto: this is the first clause of (G1) and of (G2). Next, $\mathrm{pr}_2 \colon \Delta \to w^{-1}\omega'$ is onto on stalks (at x over p: if $wc \in \omega'_x = h_\nu(T_x) + w\omega'_x$, write $wc = h_\nu(t) + we$, then $(w^{-1}t, c) \in \Delta_x$ by Step 4; at x not over p: $(0, c)$), with quasi-coherent kernel; so $\Delta(U) \to (w^{-1}\omega')(U)$ is onto for U affine, which with Step 5 is the second clause of (G2). Finally $\mathrm{pr}_2 \colon \Gamma \to V_H$ has kernel $\{(a, 0)\} \cong W$ (Step 5), so $0 \to W \to \Gamma \to Q \to 0$ is exact with $Q = \mathrm{pr}_2(\Gamma)$, and $\Gamma(U) \to Q(U)$ is onto for U affine. If $(b, 0) \in \Delta(U)$, then by the stalk form of the last clause of (G2) (Step 5) $b_x \in Q_x$ for every $x \in U$, so $b \in Q(U)$ and $(a', b) \in \Gamma(U)$ for some a'; conversely if $(a', b) \in \Gamma(U)$ then $(b, 0) \in \Delta_x$ for every $x \in U$ (Step 5), so $(b, 0) \in \Delta(U)$. With Step 5 this gives all of (G1) and (G2) over affine opens, which is (3). □

Definition 35.37. Let $\mathcal{X}$ be a cover of an integral model of a unitary Shimura datum, with function field Ω, an algebraic closure $\overline{\Omega}$, and $\iota \colon \mathcal{O}_E \to \mathrm{End}(G_{\mathcal{X}})$; let ℓ be a prime and $R \geq 1$ an integer such that every ℓ^R-torsion point of $G_{\mathcal{X}}$ over $\overline{\Omega}$ is defined over Ω, and $N = \ell^R$. Put $T_\ell = \varprojlim_k G_{\mathcal{X}}(\overline{\Omega})[\ell^k]$, with the Weil pairing $e \colon T_\ell \times T_\ell \to \mathbb{Z}_\ell$ of the polarisation, read through a fixed compatible system $(\zeta_{\ell^k})_k$ of primitive roots of unity; for $y \in T_\ell$ let $\bar{y} \in G_{\mathcal{X}}(\Omega)[N]$ be its image. Let q be a point of $\mathcal{X}$ over ℓ whose local ring $\mathcal{O}_q$ is a discrete valuation ring, $\widehat{\mathcal{O}}_q$ its completion, C the completion of an algebraic closure of the fraction field of $\widehat{\mathcal{O}}_q$, $\mathcal{O}_C$ its valuation ring, and fix an embedding $\overline{\Omega} \subseteq C$ over $\Omega \subseteq \widehat{\mathcal{O}}_q[1/\ell]$.

A **Hodge–Tate isotropic lift** at q is a $\mathbb{Z}_\ell$-linear map $\beta \colon T_\ell \to \omega_q \otimes_{\mathcal{O}_q} \mathcal{O}_C$ with:

(Iz1) $\beta(\iota(a)y) = \iota(\bar{a})^*\beta(y)$ for all $a \in \mathcal{O}_E$ and $y \in T_\ell$, where $\iota(\bar{a})^*$ is the transpose of the action of $\iota(\bar{a})$ on Lie_q;

(Iz2) there is $\epsilon \in \{1, -1\}$ such that for every $y \in T_\ell$ and every Hodge–Tate value w of $\bar{y}$ at q,

$$\epsilon\,\beta(y) - w \in N\,(\omega_q \otimes_{\mathcal{O}_q} \mathcal{O}_C);$$

(Iz3) for a $\mathbb{Z}_\ell$-basis $(y_j)_j$ of T_ℓ and the basis $(y'_j)_j$ with $e(y_j, y'_k) = \delta_{jk}$,

$$\sum_j \beta(y_j) \otimes \beta(y'_j) = 0 \quad \text{in } (\omega_q \otimes_{\mathcal{O}_q} \omega_q) \otimes_{\mathcal{O}_q} \mathcal{O}_C.$$

The element $\sum_j y_j \otimes y'_j \in T_\ell \otimes T_\ell$ does not depend on the basis, nor, up to a unit of $\mathbb{Z}_\ell$, on the system (ζ_{ℓ^k}); so (Iz3) does not depend on these choices.

Lemma 35.38. *Let $\mathfrak{T}$, p, R, w, a, $s=(a,+)$, $\bar{s}=(a,-)$, κ, m, $\beta,\bar{\beta}$, the Hodge–Tate chart (ι,h) over the open U and ι_0 be as in Lemma 35.34, and let q be a point of U of codimension one in $\mathcal{X}'$ with $p\notin\mathcal{O}_q^\times$. Then:*

1. *$h(\iota_0(W^q))\subseteq w\,\omega'^q$;*
2. *if $\kappa=\mathrm{rk}(s)$, $\mathrm{rk}(\bar{s})=n-\kappa$ and $m=n$, the sequence of $\mathcal{O}_q$-modules*

$$0\to W^q/wW^q\xrightarrow{\iota_0}\mathcal{O}_q^m/w\mathcal{O}_q^m\xrightarrow{h}\omega'^q/w\omega'^q\to 0$$

is defined and exact.

Proof. **Step 0: the setting at q.** $\mathcal{O}_q$ is a discrete valuation ring ($\mathcal{X}'$ normal, q of codimension one), so ht_q, a finitely generated torsion-free $\mathcal{O}_q$-module, is free, of rank $\dim\mathrm{Lie}(G'_{\Omega'})$ (it lies between $\varpi_p\omega_q$ and ω_q, Lemma 35.32(1)); so Lemma 35.34(2) applies at $x=q$ (a point of U). As q lies over p, $V_{t,q}=\mathrm{HT}_{t,q}(p^R)$ for $t=s,\bar{s}$; this is $\mathrm{HT}'_{\bar{s},q}$ for $t=\bar{s}$ by definition, and for $t=s$ it is $\{e:\sigma_s(d)e\in\mathrm{HT}'_{s,q}\ \forall d\in\mathfrak{d}_E\}=\mathrm{HT}'_{s,q}$, because p is unramified in E, so $\mathfrak{d}_E+p\mathcal{O}_E=\mathcal{O}_E$ and some $d\in\mathfrak{d}_E$ with $d-1\in p\mathcal{O}_E$ has $\sigma_s(d)\in\mathcal{O}_q^\times$, while $\sigma_s(\mathfrak{d}_E)\subseteq\mathcal{O}_q$. So $V_{s,q}=\mathrm{HT}'_{s,q}$ and $V_{\bar{s},q}=\mathrm{HT}'_{\bar{s},q}$, the dual lattices of $\rho_s(\mathrm{ht}_q)$, $\rho_{\bar{s}}(\mathrm{ht}_q)$; since $\varpi_p\omega_q\subseteq\mathrm{ht}_q$, every $\lambda\in\omega_q$ takes values in $\varpi_p^{-1}\mathcal{O}_q$ on $\mathrm{HT}'_{s,q}$ and on $\mathrm{HT}'_{\bar{s},q}$, and every $\lambda\in\mathrm{ht}_q$ takes values in $\mathcal{O}_q$ there. The embedding $\Omega'\subseteq C$ satisfies $\mathcal{O}_C\cap\Omega'=\mathcal{O}_q$ (the valuation of C restricts to the q-adic one).

Step 1: reduction of (1) to a bilinear form. For $v\in\mathrm{Li}_s$, $u\in\mathrm{Li}_{\bar{s}}$ put $Q(v,u)=\sum_{j,l}(g^{-1})_{jl}\,\eta_j(v)\,\eta_l(u)$. For $c\in V_L$, $v_c=\sum_k c_k\beta_k$ and $c'\in V_\omega$, $u_{c'}=\sum_k c'_k\bar{\beta}_k$, the definitions of ι_0 and h give $h(\iota_0(c))\cdot c'=\sum_l\iota_0(c)_l\,\eta_l(u_{c'})=Q(v_c,u_{c'})$. Now $c\in W^q$ if and only if $v_c\in V_{s,q}$, $c'\in\bar{W}^q$ if and only if $u_{c'}\in V_{\bar{s},q}$, and ω'^q is the dual lattice of $\bar{W}^q$, so $w\omega'^q=\{e:e\cdot c'\in w\mathcal{O}_q\ \forall c'\in\bar{W}^q\}$. Hence (1) is equivalent to: $Q(v,u)\in w\mathcal{O}_q$ for all $v\in\mathrm{HT}'_{s,q}$, $u\in\mathrm{HT}'_{\bar{s},q}$.

Step 2: an isotropic lift. By Lemma 35.39 if $G'\times_{\mathcal{X}'}\operatorname{Spec}\mathcal{O}_q$ is an abelian scheme, and by Proposition 35.41 otherwise (both apply: $\mathcal{X}'$ is a cover, every p^R-torsion point is rational, $\mathcal{O}_q$ is a discrete valuation ring), there is a Hodge–Tate isotropic lift $\beta\colon T_p\to\omega_q\otimes\mathcal{O}_C$ at q, with sign ϵ. T_p is a module over $\mathcal{O}_E\otimes\mathbb{Z}_p$, a finite product of discrete valuation rings (p unramified), torsion-free, with $T_p/p^RT_p=G'(\Omega')[p^R]$ free of rank m over $\mathcal{O}_E/p^R$; so lifts $Y_1,\dots,Y_m\in T_p$ of $y_1,\dots,y_m$ form an $\mathcal{O}_E\otimes\mathbb{Z}_p$-basis (Nakayama over the semilocal complete ring, and comparison of $\mathbb{Z}_p$-ranks). Put $\beta_j=\epsilon\beta(Y_j)$. By (Iz2), $\beta_j-\eta_j\in p^R(\omega_q\otimes\mathcal{O}_C)$, $\eta_j|_{\mathrm{Lie}_q}$ being a Hodge–Tate value of y_j at q (Definition 35.22(Hf4)). By (Iz1), for $b\in\mathcal{O}_E\otimes\mathbb{Z}_p$, $\beta(bY)(v)=\sigma_s(\bar{b})\beta(Y)(v)$ for $v\in\mathrm{Li}_s$ and $\beta(bY)(u)=\sigma_s(b)\beta(Y)(u)$ for $u\in\mathrm{Li}_{\bar{s}}$ ($\iota(\bar{b})$ acts on Li_s by $\sigma_s(\bar{b})$ and on $\mathrm{Li}_{\bar{s}}$ by $\sigma_{\bar{s}}(\bar{b})=\sigma_s(b)$).

Step 3: the hermitian form on T_p. For $Y,Y'\in T_p$ put $\langle Y,Y'\rangle=\sum_k\varpi_k^*e(\varpi_kY,Y')\in\mathfrak{d}_E^{-1}\otimes\mathbb{Z}_p=\mathcal{O}_E\otimes\mathbb{Z}_p$ (p is unramified in E). Then $e(bY,Y')=\mathrm{Tr}(b\langle Y,Y'\rangle)$ for $b\in\mathcal{O}_E\otimes\mathbb{Z}_p$ (trace extended $\mathbb{Z}_p$-linearly): both sides are $\mathbb{Z}_p$-linear in b, and for $b=\varpi_l$ the right side is $\sum_k e(\varpi_kY,Y')\mathrm{Tr}(\varpi_l\varpi_k^*)=e(\varpi_lY,Y')$. Replacing b by bc gives $\langle cY,Y'\rangle=c\langle Y,Y'\rangle$, and the Rosati condition $e(bY,cY')=e(\bar{c}bY,Y')$ gives $\langle Y,cY'\rangle=\bar{c}\langle Y,Y'\rangle$; and since e is alternating, $\mathrm{Tr}(b\overline{\langle Y',Y\rangle})=e(bY',Y)=-e(Y,bY')=-e(\bar{b}Y,Y')=-\mathrm{Tr}(b\,\overline{\langle Y,Y'\rangle})$ for all b, so $\langle Y',Y\rangle=-\overline{\langle Y,Y'\rangle}$. Let $\Gamma_0=(\langle Y_l,Y_j\rangle)_{l,j}$; it is skew-hermitian, $\Gamma_0^{\mathsf{T}}=-\overline{\Gamma_0}$, and $g\equiv\sigma_s(\Gamma_0)\bmod p^RM_m(\mathcal{O}_C)$: the chosen roots of unity make $e(Y,Y')\equiv\varepsilon(\bar{Y},\bar{Y}')\bmod p^R$ for the images $\bar{Y},\bar{Y}'$ in $G'(\Omega')[p^R]$, and ϖ_kY_l has image ϖ_ky_l, so

$$\sigma_s(\Gamma_0)_{lj}=\sum_k\sigma_s(\varpi_k^*)\,e(\varpi_kY_l,Y_j)\equiv\sum_k\sigma_s(\varpi_k^*)\,\varepsilon(\varpi_ky_l,y_j)=g_{lj}$$

modulo $p^R\mathcal{O}_C$, as $\sigma_s(\varpi_k^*) \in \sigma_s(\mathfrak{d}_E^{-1}) \subseteq \mathcal{O}' \subseteq \mathcal{O}_C$ ($\mathfrak{d}_E$ is prime to p); as $\det g \in \mathcal{O}'^\times$, $\sigma_s(\Gamma_0) \in \mathrm{GL}_m(\mathcal{O}_C)$, Γ_0 is invertible, and $g^{-1} - \sigma_s(\Gamma_0)^{-1} = g^{-1}(\sigma_s(\Gamma_0) - g)\sigma_s(\Gamma_0)^{-1} \in p^R M_m(\mathcal{O}_C)$. Also $(\Gamma_0^{-1})^{\mathsf{T}} = -\overline{\Gamma_0^{-1}}$.

Step 4: the Casimir identity. The elements $\varpi_k Y_l$ form a $\mathbb{Z}_p$-basis of T_p. Its e-dual basis: $z_{kl} = \sum_j a_{kl,j} Y_j$ with $\bar{a}_{kl,j} = \varpi_k^*(\Gamma_0^{-1})_{jl}$; indeed $e(\varpi_{k'}Y_{l'}, z_{kl}) = \mathrm{Tr}\big(\varpi_{k'} \sum_j \bar{a}_{kl,j}(\Gamma_0)_{l'j}\big) = \mathrm{Tr}(\varpi_{k'}\varpi_k^*)\delta_{ll'} = \delta_{kk'}\delta_{ll'}$. By (Iz3) and Step 2, for $v \in \mathrm{Li}_s$, $u \in \mathrm{Li}_{\bar{s}}$:

$$0 = \sum_{k,l} \beta(\varpi_k Y_l)(v)\, \beta(z_{kl})(u) = \sum_{l,j} \Big(\sum_k \sigma_s(\overline{\varpi_k \varpi_k^*}) \Big) \sigma_s\big(\overline{(\Gamma_0^{-1})_{jl}}\big) \beta(Y_l)(v)\, \beta(Y_j)(u).$$

With $\sum_k \varpi_k \varpi_k^* = 1$ and $\overline{(\Gamma_0^{-1})_{jl}} = -(\Gamma_0^{-1})_{lj}$ (Step 3) this reads, after multiplying by $-\epsilon^2 = -1$,

$$Q_\beta(v,u) := \sum_{l,j} \sigma_s\big((\Gamma_0^{-1})_{lj}\big)\, \beta_l(v)\, \beta_j(u) = 0.$$

Step 5: (1). Let $v \in \mathrm{HT}'_{s,q}$, $u \in \mathrm{HT}'_{\bar{s},q}$. By Step 0, $\eta_j(v), \eta_l(u) \in \mathcal{O}_q$ and $(\beta_j - \eta_j)(v), (\beta_l - \eta_l)(u) \in p^R \varpi_p^{-1} \mathcal{O}_C \subseteq \mathcal{O}_C$ (as $p^R \varpi_p^{-1} \in \varpi_p^{(p-1)R-1}\mathcal{O}_q$ and $(p-1)R - 1 \geq 1$), so $\beta_j(v), \beta_l(u) \in \mathcal{O}_C$. Hence

$$\begin{aligned} Q(v,u) = Q(v,u) - Q_\beta(v,u) = \sum_{j,l} \big(g^{-1} - \sigma_s(\Gamma_0)^{-1}\big)_{jl} \eta_j(v) \eta_l(u) \\ + \sum_{j,l} \sigma_s(\Gamma_0^{-1})_{jl} \big((\eta_j - \beta_j)(v)\, \eta_l(u) + \beta_j(v)\, (\eta_l - \beta_l)(u) \big) \end{aligned}$$

lies in $p^R \varpi_p^{-1} \mathcal{O}_C$ (Step 3 for the first sum). As $Q(v,u) \in \Omega'$ and $\mathcal{O}_C \cap \Omega' = \mathcal{O}_q$, $Q(v,u) \in p^R \varpi_p^{-1} \mathcal{O}_q \subseteq \varpi_p \mathcal{O}_q = w\mathcal{O}_q$ ($w\mathcal{O}_q = \varpi_p \mathcal{O}_q$: $(w/\varpi_p)^{p-1} = up/\varpi_p^{p-1}$ is a unit, as $\varpi_p^{p-1} \in p\,\mathbb{Z}[\zeta_p]^\times$, so w/ϖ_p and its inverse are integral over the discrete valuation ring $\mathcal{O}_q$). By Step 1 this is (1).

Step 6: (2). By Lemma 35.34(2), $\iota_0(W^q) \subseteq \mathcal{O}_q^m$ and $h(\mathcal{O}_q^m) \subseteq \omega'^q$, so both maps are defined modulo w; the first is injective and the second surjective by the same lemma, and their composite is zero by (1). W^q, $\mathcal{O}_q^m$, ω'^q are free of ranks κ, m, $n-\kappa$, so with e the length of the artinian ring $\mathcal{O}_q/w\mathcal{O}_q$ (w is a non-zero non-unit of the discrete valuation ring $\mathcal{O}_q$), the three quotients have lengths κe, me, $(n-\kappa)e$. The image of the first map has length κe and the kernel of the second has length $(m - n + \kappa)e = \kappa e$ (as $m = n$); the image lies in the kernel, so they are equal, and the sequence is exact. □

Lemma 35.39. *Let $\mathcal{X}$, ℓ, R, N and q be as in Definition 35.37, and suppose that $G_{\mathcal{X}} \times_{\mathcal{X}} \mathrm{Spec}\,\mathcal{O}_q$ is an abelian scheme. Then there is a Hodge–Tate isotropic lift at q.*

Proof. Write $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_q$, K for its fraction field and $v = \mathrm{ord}_q/\mathrm{ord}_q(\ell)$; K is complete, of characteristic 0 (the generic point of $\mathcal{X}$ lies in characteristic 0), with $v(\ell) = 1$, and C is the completion of an algebraic closure of K. Let $A = G_{\mathcal{X}} \times_{\mathcal{X}} \mathrm{Spec}\,\widehat{\mathcal{O}}$, an abelian scheme with the principal polarisation $\lambda\colon A \to A^\vee$ of $\mathcal{L}$ and the action ι, and let s_0 be its unit section. Normalise the Weil pairing as $e_M(x,x') = \langle x, \lambda x' \rangle_M$, with $\langle\,,\rangle_M\colon A[M] \times A^\vee[M] \to \mu_M$ the Cartier pairing; for the opposite normalisation replace ϵ below by $-\epsilon$ and (y'_j) by $(-y'_j)$, which changes nothing.

Step 1: the ℓ-divisible group and its differentials. Let $H = A[\ell^\infty]$, an ℓ-divisible group over $\widehat{\mathcal{O}}$ with Cartier dual $H^D = A^\vee[\ell^\infty]$; λ induces an isomorphism $\lambda\colon H \to H^D$, and $\lambda^D = \pm\lambda$ under $H = H^{DD}$ by the symmetry of polarisations (the sign depends only on the biduality convention). Since A is proper and of characteristic 0 generically, $A(\mathcal{O}_C)[\ell^k] = A(C)[\ell^k] = G_{\mathcal{X}}(\overline{\Omega})[\ell^k]$, so $T_\ell H = T_\ell$, and e is $\langle\,,\lambda\,\rangle$ on $T_\ell H \times T_\ell H$. For $M = \ell^k$ the ideal of $A[M]$ near s_0 is generated by the $f \circ [M]$ with f in the ideal of s_0, and $s_0^* d(h \cdot f \circ [M]) = h(s_0)\, M\, s_0^* df$;

hence $\omega_{A[M]} = \omega_q \otimes \widehat{\mathcal{O}}/M$, and $\omega_H = \varprojlim_k \omega_{A[\ell^k]} = \omega_q \otimes_{\mathcal{O}_q} \widehat{\mathcal{O}}$, $\widehat{\mathcal{O}}$ being ℓ-adically complete. The Hodge–Tate map $\alpha_{H^D}\colon T_\ell(H^D) \to \omega_H \otimes \mathcal{O}_C$, $z \mapsto (z^D)^* \frac{dT}{T}$, is compatible with the levels: if $z_k \in H^D[\ell^k](\mathcal{O}_C) = \mathrm{Hom}(A[\ell^k]_{\mathcal{O}_C}, \mu_{\ell^k})$ is the image of z, then $\alpha_{H^D}(z) \bmod \ell^k = z_k^* \frac{dT}{T}$ in $\omega_{A[\ell^k]} \otimes \mathcal{O}_C$. For a homomorphism $f\colon H_1 \to H_2$ of ℓ-divisible groups and $x \in T_\ell H_1$, $(f \circ x)^D = x^D \circ f^D$ gives

$$\alpha_{H_2}(fx) = (f^D)^* \alpha_{H_1}(x). \tag{35.2}$$

Define

$$\beta = \alpha_{H^D} \circ \lambda\colon T_\ell \longrightarrow \omega_H \otimes_{\widehat{\mathcal{O}}} \mathcal{O}_C = \omega_q \otimes_{\mathcal{O}_q} \mathcal{O}_C,$$

a $\mathbb{Z}_\ell$-linear map. We check (Iz1)–(Iz3).

Step 2: (Iz1). The Rosati involution of λ induces $a \mapsto \bar{a}$ on $\iota(\mathcal{O}_E)$, i.e. $\lambda \circ \iota(a) = \iota(\bar{a})^\vee \circ \lambda$. By (35.2) for the endomorphism $f = \iota(\bar{a})^\vee$ of H^D, whose Cartier dual is $\iota(\bar{a})$ on $H^{DD} = H$,

$$\beta(\iota(a)y) = \alpha_{H^D}\big(\iota(\bar{a})^\vee \lambda y\big) = \iota(\bar{a})^* \alpha_{H^D}(\lambda y) = \iota(\bar{a})^* \beta(y).$$

Step 3: (Iz2) with $\epsilon = -1$. Let $y \in T_\ell$ and $\chi = e_N(\bar{y}, \cdot)$, a character of the constant group $A[N]_K$. The image z_R of λy in $\mathrm{Hom}(A[N]_{\mathcal{O}_C}, \mu_N)$ is, on the generic fibre, $g \mapsto \langle g, \lambda\bar{y}\rangle_N = e_N(g, \bar{y}) = \chi(g)^{-1}$; so χ^{-1} extends to the homomorphism z_R over $\mathcal{O}_C$, hence so does χ, and $\beta(y) \bmod N = z_R^* \frac{dT}{T} = -\chi^* \frac{dT}{T}$. Let w be a Hodge–Tate value of $\bar{y}$ at q, with a function ϕ regular on an open $U \subseteq G_{\mathcal{X}}$ containing the unit point over q, $\phi(g) = e_N(\bar{y}, g)$ at the geometric N-torsion points g in U, and $w - s_0^* d\phi \in N\omega_q$. As $A[N]$ is finite over the complete local ring $\widehat{\mathcal{O}}$, it is a finite disjoint union of local schemes; let $\Gamma^0 = \operatorname{Spec} \Lambda_0$ be the one containing the unit point over the closed point. It is open and closed in $A[N]$, so flat, and Λ_0 is a free $\widehat{\mathcal{O}}$-module; its points are generisations of that unit point, which lies over the unit point of q, so Γ^0 lies in the preimage of U and ϕ restricts to $\phi_0 \in \Lambda_0$. The restriction of χ to $\Gamma^0_{\mathcal{O}_C}$ is a unit $u \in \Lambda_0 \otimes \mathcal{O}_C$ with $u(s_0) = 1$. Since $\Lambda_0 \otimes \mathcal{O}_C \subseteq \Lambda_0 \otimes C = C^{T^0}$, T^0 the set of generic points of Γ^0 (all rational), and $\phi_0 \otimes 1$ and u take the value $e_N(\bar{y}, t)$ at every $t \in T^0$, we get $u = \phi_0 \otimes 1$. As Γ^0 is an open neighbourhood of the unit section in $A[N]$, $\omega_{\Gamma^0} = \omega_{A[N]} = \omega_q \otimes \widehat{\mathcal{O}}/N$, and in it

$$\chi^* \tfrac{dT}{T} = s_0^* \tfrac{du}{u} = s_0^* du = s_0^* d\phi \equiv w \pmod{N(\omega_q \otimes \mathcal{O}_C)}.$$

Hence $-\beta(y) - w \in N(\omega_q \otimes \mathcal{O}_C)$: (Iz2) holds with $\epsilon = -1$.

Step 4: (Iz3). The isomorphism $\mu = \lambda\colon H \to H^D$ satisfies $\mu^D = \pm\mu$ (Step 1), $e = \langle\,\cdot\,, \lambda\,\cdot\,\rangle = e_\mu$ on $T_\ell H = T_\ell$, and $\beta = \alpha_{H^D} \circ \lambda = \beta_\mu$. So Lemma 35.40, applied to H over $\mathcal{O}_K = \widehat{\mathcal{O}}$ (complete and discretely valued, $v(\ell) = 1$), gives $\sum_j \beta(y_j) \otimes \beta(y_j') = 0$ for every $\mathbb{Z}_\ell$-basis (y_j) of T_ℓ and (y_j') with $e(y_j, y_k') = \delta_{jk}$, which is (Iz3). □

Lemma 35.40. *Let p, K, C and $\mathcal{O}_C$ be as in Theorem B.54, with K discretely valued, and fix a generator of $\mathbb{Z}_p(1)$, through which the Cartier pairing $\langle\,,\rangle\colon T_p(H) \times T_p(H^D) \to \mathbb{Z}_p(1)$ of a p-divisible group H over $\mathcal{O}_K$ takes values in $\mathbb{Z}_p$. Let $\mu\colon H \to H^D$ be an isomorphism of p-divisible groups with $\mu^D = \pm\mu$ under $H = H^{DD}$, put $e_\mu(x, x') = \langle x, \mu x'\rangle$ on $T_p(H)$ and*

$$\beta_\mu = \alpha_{H^D} \circ \mu\colon T_p(H) \longrightarrow \omega_H \otimes_{\mathcal{O}_K} \mathcal{O}_C.$$

Then e_μ is a perfect pairing, and for every $\mathbb{Z}_p$-basis $(x_j)_j$ of $T_p(H)$ and the basis $(x_j')_j$ with $e_\mu(x_j, x_k') = \delta_{jk}$,

$$\sum_j \beta_\mu(x_j) \otimes \beta_\mu(x_j') = 0 \quad \text{in } (\omega_H \otimes_{\mathcal{O}_K} \omega_H) \otimes_{\mathcal{O}_K} \mathcal{O}_C.$$

Proof. Naturality. For a homomorphism $f\colon H_1 \to H_2$ of p-divisible groups over $\mathcal{O}_K$ and $x \in T_p(H_1)$, the identity $(f \circ x)^D = x^D \circ f^D$ and the definition $\alpha_H(x) = (x^D)^* \frac{dT}{T}$ give

$$\alpha_{H_2}(fx) = (f^D)^* \alpha_{H_1}(x), \qquad (f^D)^*\colon \omega_{H_1^D} \to \omega_{H_2^D} \text{ (pull-back along } f^D\colon H_2^D \to H_1^D).$$

With $f = \mu\colon H \to H^D$ and $\mu^D = \pm\mu$ this reads $\beta_\mu(x) = \alpha_{H^D}(\mu x) = \pm\mu^* \alpha_H(x)$, with $\mu^*\colon \omega_{H^D} \to \omega_H$.

Perfectness. μ induces an isomorphism $T_p(H) \to T_p(H^D) = \operatorname{Hom}(T_p(H), \mathbb{Z}_p)$, $x' \mapsto \langle \cdot\,, \mu x' \rangle = e_\mu(\cdot, x')$; so e_μ is perfect and the basis (x'_j) exists and is unique. Moreover $(\mu x'_j)_j$ is the basis of $T_p(H^D)$ dual to $(x_j)_j$ under $\langle\,,\rangle$.

Isotropy. By Theorem B.54 applied to H, the composite of

$$\operatorname{Lie}(H) \otimes \mathcal{O}_C \xrightarrow{{}^t(\alpha_{H^D} \otimes 1)} T_p(H) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \xrightarrow{\alpha_H \otimes 1} \omega_{H^D} \otimes \mathcal{O}_C$$

is zero (the Tate twist is trivialised by the fixed generator). In the bases (x_j) of $T_p(H)$ and $(\mu x'_j)$ of $T_p(H^D)$, which are dual, the transpose of $\alpha_{H^D} \otimes 1\colon T_p(H^D) \otimes \mathcal{O}_C \to \omega_H \otimes \mathcal{O}_C$ sends $v \in \operatorname{Lie}(H) \otimes \mathcal{O}_C$ to $\sum_j \langle \alpha_{H^D}(\mu x'_j), v \rangle\, x_j = \sum_j \langle \beta_\mu(x'_j), v \rangle\, x_j$. Hence $\sum_j \langle \beta_\mu(x'_j), v \rangle\, \alpha_H(x_j) = 0$ for every v. Since $\omega_H \otimes \mathcal{O}_C$ is free of finite rank with dual $\operatorname{Lie}(H) \otimes \mathcal{O}_C$, an element of $(\omega_{H^D} \otimes \mathcal{O}_C) \otimes_{\mathcal{O}_C} (\omega_H \otimes \mathcal{O}_C)$ all of whose contractions with elements v of $\operatorname{Lie}(H) \otimes \mathcal{O}_C$ in the second factor vanish is zero; so

$$\sum_j \alpha_H(x_j) \otimes \beta_\mu(x'_j) = 0 \quad \text{in } (\omega_{H^D} \otimes \omega_H) \otimes \mathcal{O}_C.$$

Applying $\mu^* \otimes 1$ and $\mu^* \alpha_H(x_j) = \pm\beta_\mu(x_j)$ gives $\sum_j \beta_\mu(x_j) \otimes \beta_\mu(x'_j) = 0$. □

Proposition 35.41. *Let $\mathcal{X}$, ℓ, R, N and q be as in Definition 35.37, and suppose that $G_{\mathcal{X}} \times_{\mathcal{X}} \operatorname{Spec} \mathcal{O}_q$ is not an abelian scheme. Then there is a Hodge–Tate isotropic lift at q.*

Proof. Throughout, v is the valuation with $v(\ell) = 1$, (ζ_{ℓ^k}) the fixed system of roots of unity, through which $\mathbb{Z}_\ell(1) = \mathbb{Z}_\ell$ and $\mu_{\ell^k}(C) = \mathbb{Z}/\ell^k$, and e_{ℓ^k} the reduction of e modulo ℓ^k, the Weil pairing on $T_\ell/\ell^k = G_{\mathcal{X}}(\overline{\Omega})[\ell^k]$. The hypothesis that the reduction is not abelian is not used: the argument below works for every torus rank.

Step 0: the base. Let κ be the residue field of $\widehat{\mathcal{O}}_q$, and G_κ the closed fibre of $G_{\mathcal{X}} \times_{\mathcal{X}} \operatorname{Spec} \widehat{\mathcal{O}}_q$, an extension of an abelian variety by a torus $\mathbb{T}_\kappa$. By (iii) there is a finite separable extension κ' of κ splitting $\mathbb{T}_\kappa$; choosing a primitive element and lifting its minimal polynomial gives a finite étale local $\widehat{\mathcal{O}}_q$-algebra $\mathcal{O}'$ with residue field κ', a complete discrete valuation ring with the uniformiser of $\widehat{\mathcal{O}}_q$; its fraction field K' is finite separable over the fraction field K of $\widehat{\mathcal{O}}_q$, and we fix a K-embedding $K' \subseteq C$, so $\mathcal{O}' \subseteq \mathcal{O}_C$ and C is the completion of an algebraic closure of K'. Put $S = \operatorname{Spec} \mathcal{O}'$ and $G = G_{\mathcal{X}} \times_{\mathcal{X}} S$, a semiabelian scheme with the action ι and the pull-back $\mathcal{L}_0$ of the rigidified symmetric cubical line bundle of the integral model (Definition 29.5, (Ud7)). The generic point of S maps to the generic point of $\mathcal{X}$, so $G_{K'}$ is the base change of $G_{\mathcal{X},\Omega}$, an abelian variety on which $\mathcal{L}_0$ is ample with principal polarisation $\phi_{\mathcal{L}_0}$, and e is the Weil pairing of $\phi_{\mathcal{L}_0}$ on $T_\ell = \varprojlim_k G(C)[\ell^k]$ (the ℓ^k-torsion is étale over Ω, so its $\overline{\Omega}$- and C-points agree). Finally $\omega_G = \omega_q \otimes_{\mathcal{O}_q} \mathcal{O}'$, and a Hodge–Tate value w of $\bar{y}$ at q, given by a function ϕ regular on an open $U \subseteq G_{\mathcal{X}}$ containing the unit point over q, with $\phi(g) = e_N(\bar{y}, g)$ at the geometric N-torsion points g in U and $w - s_0^* d\phi \in N\omega_q$, pulls back to G with the same properties (s_0 the unit section).

Step 1: Raynaud extension and the abelian part. Apply Theorem B.109 (1) to $R = \mathcal{O}'$ with its maximal ideal: there is a Raynaud extension $0 \to \mathbb{T} \to \widetilde{G} \xrightarrow{\pi} B \to 0$ over S with $\widetilde{G}^{\mathrm{for}} \cong G^{\mathrm{for}}$; its torus $\mathbb{T}$ is split by (iii), as its closed fibre is $\mathbb{T}_\kappa \otimes \kappa'$, and ι induces an action of $\mathcal{O}_E$ on $\mathbb{T}$, $\widetilde{G}$

and B compatible with $\widetilde{G}^{\mathrm{for}} \cong G^{\mathrm{for}}$ (Theorem B.109 (3)). Tori over the field κ' are isotrivial, so the functor of Theorem B.109 (1) is an equivalence and $\mathcal{L}_0^{\mathrm{for}}$ lies in its essential image; $\mathcal{L}_0$ is rigidified, cubical and symmetric (Step 0), and $\mathcal{L}_{0,K'}$ is ample, hence nondegenerate, so $\mathcal{L}_0$ is ample over G by the ampleness criterion for symmetric sheaves in Theorem B.109 (2); hence $(G,\mathcal{L}_0) \in \mathrm{DEG}_{\mathrm{ample}}$; let $F_{\mathrm{ample}}(G,\mathcal{L}_0) = (B, X, Y, \phi, c, c^\vee, \widetilde{\mathcal{L}}_0, \tau, \psi)$. The character group X of the split torus is constant, and so is Y, which ϕ embeds in X. By Theorem B.110 (S is normal) $\widetilde{\mathcal{L}}_0 \cong \pi^*\mathcal{M}_0$ as cubical sheaves for a cubical $\mathcal{M}_0$ on B. By Theorem B.109 (4)(d), $1 = \deg \phi_{\mathcal{L}_0,K'} = [X:\phi(Y)]^2 \deg \lambda_B$ with $\lambda_B = \phi_{\mathcal{M}_0}$; hence λ_B is a principal polarisation of B, and its Weil pairing e_B on $T_\ell(B) = \varprojlim_k B(C)[\ell^k]$ is perfect. By Theorem B.109 (4)(c), for $n \geq 1$ the cubical sheaf of $F_{\mathrm{ample}}(G, \mathcal{L}_0^{\otimes n})$ is $\widetilde{\mathcal{L}}_0^{\otimes n} \cong \pi^*(\mathcal{M}_0^{\otimes n})$.

Step 2: finite parts and orthogonality at level $n = \ell^k$. Fix $k \geq 1$, $n = \ell^k$, and $\mathcal{L} = \mathcal{L}_0^{\otimes n}$, a cubical sheaf on G. By (v), $K(\mathcal{L}_{K'}) = G_{K'}[n]$ and $e^{\mathcal{L}_{K'}} = e_n$. By (iv) $G[n]$ is a closed subgroup quasi-finite and flat over S; flat over a discrete valuation ring, it is the schematic closure of its generic fibre $K(\mathcal{L}_{K'})$. So Theorem B.119 applies with $K_S = G[n]$ and the cubical extension $\mathcal{L}$, and (3) applies with $\mathcal{M} = \mathcal{M}_0^{\otimes n}$ by Step 1, $K(\mathcal{M}) = B[n]$ as λ_B is principal (by (v) over C). It gives: isomorphisms $G[n]^\mu \cong \mathbb{T}[n]$, $G[n]^f \cong \widetilde{G}[n]$, $G[n]^{ab} \cong B[n]$; $G[n]^f$ is the annihilator of $G[n]^\mu$ under the extension of e_n; the induced pairing on $G[n]^{ab} \cong B[n]$ is $e^{\mathcal{M}}$, which is the Weil pairing of λ_B by (v). The isomorphisms are characterised on formal completions; as multiplication by ℓ and the $\iota(a)$ commute with $\widetilde{G}^{\mathrm{for}} \cong G^{\mathrm{for}}$ and map finite parts into finite parts (i), (ii) shows that the isomorphisms for n and ℓn are compatible with $[\ell]$, with the inclusions $G[n]^\mu \subseteq G[n]^f$, $\mathbb{T}[n] \subseteq \widetilde{G}[n]$, with π, and with ι. The annihilator is the kernel of the homomorphism $G[n] \to (G[n]^\mu)^D$ given by the pairing, and kernels commute with base change; over C all these groups are constant. Writing $F_n = G[n]^f(C)$ and $T_n = G[n]^\mu(C)$, subgroups of $G(C)[n] = T_\ell/n$, we obtain

$$\begin{gathered} T_n \subseteq F_n = \{x \in T_\ell/n : \ e_n(x,t) = 1 \text{ for all } t \in T_n\}, \\ F_n/T_n \cong B(C)[n] \text{ isometrically for } e_n \text{ and } e_{B,n}, \end{gathered} \tag{35.3}$$

where $e_{B,n}$ is the Weil pairing of λ_B, and $T_n \cong \mathbb{T}(C)[n]$, $F_n \cong \widetilde{G}(C)[n]$.

Step 3: the lattices. Let $F = \varprojlim_k F_{\ell^k}$ and $T_t = \varprojlim_k T_{\ell^k}$, closed subgroups of T_ℓ with $T_t \subseteq F$. Multiplication by ℓ maps $T_{\ell^{k+1}} \cong \mathbb{T}(C)[\ell^{k+1}]$ onto T_{ℓ^k} ($\mathbb{T}(C) = (C^\times)^r$, r the rank of $\mathbb{T}$), and $F_{\ell^{k+1}}$ onto F_{ℓ^k} ($\widetilde{G}(C)$ is divisible, an extension of $B(C)$ by $(C^\times)^r$); so the projections $T_t \to T_{\ell^k}$, $F \to F_{\ell^k}$ are surjective. Hence: (a) $F/T_t \cong \varprojlim_k B(C)[\ell^k] = T_\ell(B)$ (inverse limits of finite groups are exact), and by (35.3), $e(f,f') = e_B(\bar f, \bar f')$ for $f, f' \in F$ with images $\bar f, \bar f'$; in particular F/T_t is free and the pairing it inherits is perfect. (b) $F = T_t^\perp = \{x \in T_\ell : e(x,t) = 0 \ \forall t \in T_t\}$: as $T_t \to T_{\ell^k}$ is onto, $e(x,T_t) = 0$ if and only if $e_{\ell^k}(x \bmod \ell^k, T_{\ell^k}) = 1$ for all k, that is, by (35.3), if and only if $x \bmod \ell^k \in F_{\ell^k}$ for all k, that is, if and only if $x \in F$. As e is alternating, also $e(T_t, F) = 0$. (c) The map $T_\ell/F \to \mathrm{Hom}(T_t, \mathbb{Z}_\ell)$, $y \mapsto e(y,\cdot)|_{T_t}$, is injective by (b), so T_ℓ/F is free; with (a), T_ℓ/T_t is free, T_t is a direct summand of T_ℓ, and as e is perfect on T_ℓ ($\phi_{\mathcal{L}_0}$ is principal) the map is also surjective. (d) By (iv), $\widetilde{H} = (\widetilde{G}[\ell^k])_k$ is an ℓ-divisible group over S, and the isomorphisms of Step 2 identify $T_\ell(\widetilde{H}) = \varprojlim_k \widetilde{G}(C)[\ell^k]$ with F, $\mathcal{O}_E$-equivariantly, carrying $T_\ell(\mathbb{T})$ to T_t and $T_\ell(\pi)\colon T_\ell(\widetilde{H}) \to T_\ell(B[\ell^\infty])$ to $F \to F/T_t$. For a smooth commutative group scheme P over S and $M \geq 1$ the ideal of $P[M]$ near s_0 is generated by the $f \circ [M]$, f in the ideal of s_0, and $s_0^* d(h \cdot f \circ [M]) = h(s_0)\, M\, s_0^* df$; so $\omega_{P[M]} = \omega_P/M$. As $G[\ell^k]^f$ is an open subscheme of $G[\ell^k]$ containing s_0, $\omega_{\widetilde{G}[\ell^k]} \cong \omega_{G[\ell^k]^f} = \omega_{G[\ell^k]} = \omega_G/\ell^k$, compatibly with k and ι, and

$$\omega_{\widetilde{H}} = \varprojlim_k \omega_{\widetilde{G}[\ell^k]} = \omega_G = \omega_q \otimes_{\mathcal{O}_q} \mathcal{O}'.$$

Step 4: the lift. For $y \in T_\ell$ the functional $e(y,\cdot)|_F$ lies in $\mathrm{Hom}(F,\mathbb{Z}_\ell) = \mathrm{Hom}(T_\ell\widetilde{H},\mathbb{Z}_\ell(1)) = T_\ell(\widetilde{H}^D)$. Define

$$\beta(y) = \alpha_{\widetilde{H}^D}\big(e(y,\cdot)|_F\big) \in \omega_{\widetilde{H}} \otimes_{\mathcal{O}'} \mathcal{O}_C = \omega_q \otimes_{\mathcal{O}_q} \mathcal{O}_C,$$

a $\mathbb{Z}_\ell$-linear map, with $\alpha_{\widetilde{H}^D}(z) = (z^D)^*\frac{dT}{T}$ the Hodge–Tate map of Theorem B.54. Two facts: if $z_k \in \widetilde{H}^D[\ell^k](\mathcal{O}_C) = \mathrm{Hom}(\widetilde{G}[\ell^k]_{\mathcal{O}_C},\mu_{\ell^k})$ is the image of z, then $\alpha_{\widetilde{H}^D}(z) \bmod \ell^k = z_k^*\frac{dT}{T}$ in $\omega_{\widetilde{G}[\ell^k]} \otimes \mathcal{O}_C$; and for a homomorphism $f\colon H_1 \to H_2$ of ℓ-divisible groups and $x \in T_\ell H_1$, (vi) gives

$$\alpha_{H_2}(fx) = (f^D)^*\alpha_{H_1}(x). \tag{35.4}$$

By Step 3 (b), $\beta(T_t) = 0$.

Step 5: (Iz1). The Rosati involution of $\phi_{\mathcal{L}_0}$ induces $a \mapsto \bar{a}$ on $\iota(\mathcal{O}_E)$ ((Ud4)), so $e(\iota(a)y,x) = e(y,\iota(\bar{a})x)$. For the endomorphism $f = \iota(\bar{a})$ of $\widetilde{H}$, f^D acts on $T_\ell(\widetilde{H}^D) = \mathrm{Hom}(F,\mathbb{Z}_\ell)$ by composition with $\iota(\bar{a})|_F$, and $f^{DD} = f$. So by (35.4)

$$\beta(\iota(a)y) = \alpha_{\widetilde{H}^D}\big(f^D(e(y,\cdot)|_F)\big) = \iota(\bar{a})^*\beta(y),$$

with $\iota(\bar{a})^*$ the pull-back on $\omega_{\widetilde{H}} = \omega_q \otimes \mathcal{O}'$, the transpose of $\iota(\bar{a})$ on Lie_q (Step 3 (d) is ι-equivariant).

Step 6: (Iz2) with $\epsilon = 1$. Let $y \in T_\ell$ and w, ϕ, U as in Step 0. The image z_R of $e(y,\cdot)|_F$ in $\mathrm{Hom}(G[N]^f_{\mathcal{O}_C},\mu_N)$ is the character χ with $\chi(g) = e_N(\bar{y},g)$ for $g \in F_N$, and $\beta(y) \bmod N = \chi^*\frac{dT}{T}$ in $\omega_{G[N]^f} \otimes \mathcal{O}_C$. As $G[N]^f$ is finite over the complete local ring $\mathcal{O}'$, it is a finite disjoint union of local schemes; let $\Gamma^0 = \operatorname{Spec}\Lambda_0$ be the one containing the unit point $s_0(\kappa')$ of the closed fibre. It is open and closed in $G[N]^f$, hence in $G[N]$, so Λ_0 is a free $\mathcal{O}'$-module and $\omega_{\Gamma^0} = \omega_{G[N]} = \omega_q \otimes \mathcal{O}'/N$ (Step 3 (d)). Every point of Γ^0 is a generisation of $s_0(\kappa')$, which lies in (the pull-back of) U; so $\Gamma^0 \subseteq U$ and ϕ restricts to $\phi_0 \in \Lambda_0$. The restriction of χ to $\Gamma^0_{\mathcal{O}_C}$ is a unit $u \in \Lambda_0 \otimes \mathcal{O}_C$ with $u(s_0) = 1$. Now $\Lambda_0 \otimes \mathcal{O}_C \subseteq \Lambda_0 \otimes C = C^{\Gamma^0(C)}$ ($\Gamma^0_{K'}$ is finite étale), and u and $\phi_0 \otimes 1$ both take the value $e_N(\bar{y},t)$ at each $t \in \Gamma^0(C) \subseteq F_N$, a geometric N-torsion point in U. Hence $u = \phi_0 \otimes 1$, and

$$\beta(y) \bmod N = \chi^*\tfrac{dT}{T} = s_0^*\tfrac{du}{u} = s_0^*du = s_0^*d\phi \equiv w \pmod{N(\omega_q \otimes \mathcal{O}_C)}.$$

So $\beta(y) - w \in N(\omega_q \otimes_{\mathcal{O}_q} \mathcal{O}_C)$: (Iz2) holds with $\epsilon = 1$.

Step 7: (Iz3). Choose a basis $t_1,\dots,t_r$ of T_t and a complement W of T_t in F with basis $w_1,\dots,w_s$; by Step 3 (a), $W \to F/T_t = T_\ell(B)$ is an isometry onto $(T_\ell(B),e_B)$, so $e|_W$ is perfect. By Step 3 (c) choose $z_1,\dots,z_r \in T_\ell$ with $e(z_m,t_i) = \delta_{mi}$; replacing z_m by $z_m - w$ for the $w \in W$ with $e(w,w') = e(z_m,w')$ for all $w' \in W$ (it exists as $e|_W$ is perfect; $e(W,T_t) = 0$ keeps $e(z_m,t_i) = \delta_{mi}$), we may assume $e(z_m,W) = 0$. Then (t,w,z) is a basis of T_ℓ ((t,w) is a basis of F and the z_m map to the basis of $T_\ell/F \cong \mathrm{Hom}(T_t,\mathbb{Z}_\ell)$ dual to (t_i)), with Gram matrix $E = (e(y_j,y_k))$ and inverse, in blocks for (t,w,z),

$$E = \begin{pmatrix} 0 & 0 & -I \\ 0 & \Phi & 0 \\ I & 0 & Z \end{pmatrix}, \qquad E^{-1} = \begin{pmatrix} Z & 0 & I \\ 0 & \Phi^{-1} & 0 \\ -I & 0 & 0 \end{pmatrix},$$

$\Phi = (e(w_j,w_k))$, $Z = (e(z_m,z_{m'}))$ (multiply out). The basis $y'_k = \sum_l (E^{-1})_{lk}y_l$ satisfies $e(y_j,y'_k) = \delta_{jk}$, so the element of (Iz3) is $c = \sum_k y_k \otimes y'_k = \sum_{k,l}(E^{-1})_{lk}\,y_k \otimes y_l$. Since $\beta(t_i) = 0$, and E^{-1} vanishes on the blocks (w,z), (z,w), (z,z),

$$(\beta\otimes\beta)(c) = \sum_{j,k}(\Phi^{-1})_{kj}\,\beta(w_j)\otimes\beta(w_k) = \sum_j \beta(w_j)\otimes\beta(w'_j), \qquad w'_j = \sum_k (\Phi^{-1})_{kj}w_k,$$

and (w'_j) is the basis of W with $e(w_i, w'_j) = \delta_{ij}$; so $b_j = \bar{w}_j$, $b'_j = \bar{w}'_j$ are dual bases of $T_\ell(B)$ for e_B. For $f \in F$, Step 3 (a) gives $e(f,\cdot)|_F = e_B(\bar{f},\cdot) \circ T_\ell(\pi) = \pi^D(e_B(\bar{f},\cdot))$, with $\pi^D\colon B[\ell^\infty]^D \to \widetilde{H}^D$, so by (35.4) and Step 3 (d)

$$\beta(f) = \pi^* \alpha_{B[\ell^\infty]^D}(e_B(\bar{f},\cdot)), \qquad \pi^*\colon \omega_{B[\ell^\infty]} \to \omega_{\widetilde{H}}.$$

Let $\mu_B\colon B[\ell^\infty] \to B^\vee[\ell^\infty] = B[\ell^\infty]^D$ be induced by λ_B; $\mu_B^D = \pm\mu_B$ by (vi), and $e_B = \sigma e_{\mu_B}$ for a sign σ fixed by the convention, $e_{\mu_B}(x,x') = \langle x, \mu_B x'\rangle$. Then $e_B(x,\cdot) = -e_B(\cdot,x) = -\sigma\mu_B x$ in $T_\ell(B[\ell^\infty]^D)$, so $\beta(f) = -\sigma\,\pi^*\beta_{\mu_B}(\bar{f})$ with $\beta_{\mu_B} = \alpha_{B[\ell^\infty]^D} \circ \mu_B$, and $(\sigma b'_j)$ is the e_{μ_B}-dual basis of (b_j). Hence

$$(\beta\otimes\beta)(c) = \sigma\,(\pi^*\otimes\pi^*)\Big(\sum_j \beta_{\mu_B}(b_j)\otimes\beta_{\mu_B}(\sigma b'_j)\Big) = 0$$

by Lemma 35.40, applied to $B[\ell^\infty]$ over $\mathcal{O}'$ (K' complete, discretely valued, $v(\ell)=1$) and μ_B. This is (Iz3), and β is a Hodge–Tate isotropic lift at q. □

35.4 Root lines, abelian factors and power sums of Hodge–Tate root classes

The last subsection supplies the remaining inputs of the local relations. On a joint model the de Rham and Hodge–Tate coordinate lattices carry the root lines of the two stages, related by the second roots (Lemma 35.42). The flag theorem for tight Hopf orders over $\mathcal{O}_C$ (Theorem 35.58) says that a tight Hopf order with the generation property along a flag satisfies the flag conditions for every fundamental character; its hypothesis is met for the p-torsion of principally polarised abelian schemes over $\mathcal{O}_C$, whose function modules are abelian factors with the generation property at every subgroup (Proposition 35.43): the p-torsion functions are generated by almost-additive functions (Proposition 35.46), by a count that uses the bound $g/(p-1)$ for the normalised length of the almost-additive quotient (Proposition 35.54), and subgroups are killed by homomorphisms to principally polarised abelian schemes, constructed with quotients by finite subgroup schemes and Zarhin's trick [273] (Proposition 35.44). Finally, modulo a prime of S_B the root lines are the graded pieces of a full flag of a trivial bundle (Proposition 35.66), from Hodge–Tate exactness modulo ℓ (Proposition 35.67); so the second-root lines on the components of the fibre are steps of such a flag and the power sums of the Hodge–Tate root classes vanish there (Proposition 35.65, Proposition 35.64). Proposition 35.66 is used again in Appendix D, for nefness at primes and for flat determinants.

Lemma 35.42. *Let $\mathcal{X}$, p, $\pi\colon \mathcal{X}' \to \mathcal{X}$ be as in Lemma 35.31, with $\mathcal{T}$ of type (Φ, n, κ), so that $\mathrm{rk}((a,+)) = \kappa$ and $\mathrm{rk}((a,-)) = n-\kappa$. Let $a \in \Phi$, β and $\bar{\beta}$ any Ω'-bases of $\mathrm{Li}_{(a,+)}$ and $\mathrm{Li}_{(a,-)}$, $\mathrm{lie} = L$, W the lattices of Lemma 35.31(2) for $((a,+),\beta)$, and $\omega = \bar{L}^\vee$, $\omega' = \bar{W}^\vee$ the duals of those for $((a,-),\bar{\beta})$. Then $\mathrm{lie}, W, \omega, \omega'$ satisfy (He4), with $\pi^*\mathcal{O}(e)$ the pull-back along π of the line bundles $\mathcal{O}(e)|_{\mathcal{X}}$.*

Proof. **Step 1: the pulled-back integral flags.** For $i = 0, 1$, (Go3)(3) gives an integral flag $(\mathcal{V}^i_{s,k})_k$ of the Hodge bundles $\mathcal{V}^i_s$ of $\mathcal{X}_i$ whose generic fibre is the image under g_i of the universal flag, and isomorphisms $\mathcal{O}(e^i_r)|_{\mathcal{X}_i} \to \mathcal{L}^i_r$, $P \mapsto \varrho^i_r P(y^i)$. By Lemma 31.29(1), (2) for ϕ_i (its hypotheses: $\mathcal{X}'$ is a locally noetherian cover with structure map $f_i \circ \phi_i$ by Lemma 35.31(1); S_{B_0} and S_B contain Bd, hence the primes dividing $\mathrm{disc}(E)$; (Hp1), (Hp2) are (Go3)(2) for $\mathcal{X}_i$), $\mathcal{V}'^i_s = \mathcal{O}_{\mathcal{X}'}\mathcal{V}^i_s$ is the Hodge bundle of $\mathcal{X}'$ at (S_{B_0}, R), resp. (S_B, R), which on $\mathcal{X}'$ has stalks $\mathrm{dR}_{s,x}$, resp. $V_{s,x}$; $(\mathcal{O}_{\mathcal{X}'}\mathcal{V}^i_{s,k})_k$ is an integral flag of it, with $\mathcal{O}_{\mathcal{X}'}\mathcal{V}^i_{s,k} = \mathcal{V}'^i_s \cap \Omega'\mathcal{V}^i_{s,k}$, and root lines $\mathcal{L}'^i_r = \mathcal{O}_{\mathcal{X}'}\mathcal{L}^i_r \cong \phi_i^*\mathcal{L}^i_r \cong \phi_i^*\mathcal{O}(e^i_r)$. By (Jm2), $\mathcal{O}(e_r)|_{\mathcal{X}} = \mathrm{pr}_0^*\mathcal{O}(e^0_r)$ and $\mathcal{O}(e'_r)|_{\mathcal{X}} = \mathrm{pr}_1^*\mathcal{O}(e^1_r)$, so $\mathcal{L}'^0_r \cong \pi^*\mathcal{O}(e_r)$ and $\mathcal{L}'^1_r \cong \pi^*\mathcal{O}(e'_r)$, through $P \mapsto \varrho^0_r P(\iota_0 y^0)$, resp. $P \mapsto \varrho^1_r P(\iota_1 y^1)$ (identifying Ω_i with its image in Ω').

Step 2: one generic flag. The generic fibres $\Omega'\mathcal{V}^0_{s,k}$ and $\Omega'\mathcal{V}^1_{s,k}$ are both obtained from the universal flag over Ω_c by extension of scalars along $\iota_0 g_0 = \iota_1 g_1$ (Jm1); call this common flag $(F_{s,k})_k$ of Li_s. Consequently $\mathcal{L}'^0_r$ and $\mathcal{L}'^1_r$ are two invertible subsheaves of the same generic line ℓ_r (the generic root line of r).

Step 3: the side of $(a,+)$. Let $s=(a,+)$ and $F_j = \{c \in V_L : \sum_k c_k\beta_k \in F_{s,j}\}$, a full flag of V_L. The coordinate isomorphism $c \mapsto \sum c_k\beta_k$ carries lie, W onto $\mathcal{V}'^0_s$, $\mathcal{V}'^1_s$ (Lemma 35.31(2) and Step 1), hence $\mathrm{lie}_j = \mathrm{lie} \cap F_j$ onto $\mathcal{V}'^0_s \cap F_{s,j} = \mathcal{O}_{\mathcal{X}'}\mathcal{V}^0_{s,j}$ and W_j onto $\mathcal{O}_{\mathcal{X}'}\mathcal{V}^1_{s,j}$: locally free with invertible graded pieces. For $j \le \kappa$ the root $r=(a,j)$ has label s, step j and sign 1 (Fh3), so $\mathrm{gr}_j\mathrm{lie} \cong \mathcal{L}'^0_r \cong \pi^*\mathcal{O}(e_{(a,j)})$ and $\mathrm{gr}_j W \cong \mathcal{L}'^1_r \cong \pi^*\mathcal{O}(e'_{(a,j)})$. The inclusion $\mathrm{gr}_j\mathrm{lie} \subseteq \mathrm{gr}_j W$ is the inclusion $\mathcal{L}'^0_r \subseteq \mathcal{L}'^1_r$ inside ℓ_r. For a local section P of $\pi^*\mathcal{O}(e_r)$, by (Jm3)

$$\varrho^0_r P(\iota_0 y^0) = \varrho^1_r \cdot \Big(P\,\frac{\varrho^0_r}{\varrho^1_r}\Big) = \varrho^1_r \cdot \Big(P\,\frac{g_r}{f_r}\Big)(\iota_0 y^0, \iota_1 y^1),$$

so, under the isomorphisms of Step 1, the inclusion is the pull-back of multiplication by g_r/f_r (of multidegree $e'_r - e_r$) from $\mathcal{O}(e_r)$ to $\mathcal{O}(e'_r)$.

Step 4: the side of $(a,-)$. Let $\bar s = (a,-)$, $q = n-\kappa$, $\bar F_k = \{c \in V_\omega : \sum c_k\bar\beta_k \in F_{\bar s,k}\}$; as in Step 3, $\bar L_k = \bar L \cap \bar F_k$ and $\bar W_k$ correspond to $\mathcal{O}_{\mathcal{X}'}\mathcal{V}^0_{\bar s,k}$, $\mathcal{O}_{\mathcal{X}'}\mathcal{V}^1_{\bar s,k}$, locally free with invertible graded pieces, and $\mathrm{gr}_k\bar L \cong (\mathcal{L}'^0_r)^{-1}$, $\mathrm{gr}_k\bar W \cong (\mathcal{L}'^1_r)^{-1}$ for the root r with label $\bar s$ and step k, namely $r=(a,n+1-k)$ (sign -1). Let $G_j \subseteq V_\omega$ be the annihilator of $\bar F_{q-j}$ for the dot product ($0 \le j \le q$), a full flag, and $\omega_j = \omega \cap G_j$, $\omega'_j = \omega' \cap G_j$. Since $\bar L_{q-j}$ spans $\bar F_{q-j}$, ω_j consists of the elements of $\omega = \bar L^\vee$ vanishing on $\bar L_{q-j}$, i.e. $\omega_j = (\bar L/\bar L_{q-j})^\vee$, locally free because $\bar L/\bar L_{q-j}$ is (successive invertible quotients), and $\omega_j/\omega_{j-1} \cong (\bar L_{q-j+1}/\bar L_{q-j})^\vee = (\mathrm{gr}_{q+1-j}\bar L)^\vee \cong \mathcal{L}'^0_r$ with $r = (a, n+1-(q+1-j)) = (a,\kappa+j)$; likewise $\omega'_j/\omega'_{j-1} \cong \mathcal{L}'^1_{(a,\kappa+j)}$. So $\mathrm{gr}_j\omega \cong \pi^*\mathcal{O}(e_{(a,\kappa+j)})$ and $\mathrm{gr}_j\omega' \cong \pi^*\mathcal{O}(e'_{(a,\kappa+j)})$. The inclusion $\mathrm{gr}_j\omega' \subseteq \mathrm{gr}_j\omega$ is restriction of functionals from $\mathrm{gr}\bar W$ to $\mathrm{gr}\bar L$, i.e. the inclusion $\mathcal{L}'^1_r \subseteq \mathcal{L}'^0_r$ inside ℓ_r; by (Jm3), $\varrho^1_r P(\iota_1 y^1) = \varrho^0_r \cdot (P\,f_r/g_r)(\iota_0 y^0, \iota_1 y^1)$, so it is the pull-back of multiplication by $f_r/g_r\colon \mathcal{O}(e'_r) \to \mathcal{O}(e_r)$. Steps 3 and 4 together give (He4). □

Proposition 35.43. *Let C and $\mathcal{O}_C$ be as in Definition 35.17, N a finite abelian group and $M \subseteq \mathrm{Map}(N,C)$ an abelian factor. Then M has the generation property at every subgroup $K \le N$.*

Proof. Write $\mathrm{alg}(S)$ for the $\mathcal{O}_C$-subalgebra of $\mathrm{Map}(\cdot,C)$ generated by a set S of functions, and $\mathrm{AA}(S)$ for the almost-additive elements of S. For a group homomorphism $h\colon N_1 \to N_2$ the pull-back $h^*\colon \mathrm{Map}(N_2,C) \to \mathrm{Map}(N_1,C)$, $g \mapsto g\circ h$, is a homomorphism of $\mathcal{O}_C$-algebras and maps almost-additive functions to almost-additive functions ($g(h(x+y)) - g(h(x)) - g(h(y)) = g(h(x)+h(y)) - g(h(x)) - g(h(y))$); hence $h^*\mathrm{alg}(S) = \mathrm{alg}(h^*S)$.

Step 1: the abelian scheme itself. Let A be an abelian scheme over $\mathcal{O}_C$ admitting a relatively ample line bundle and $K' \le N_A$. Take P and j from Proposition 35.44, so $M_A^{K'} = j^*M_P$. By Proposition 35.46, $M_P \subseteq \mathrm{alg}(\mathrm{AA}(M_P))$, so $M_A^{K'} = j^*M_P \subseteq \mathrm{alg}(j^*\mathrm{AA}(M_P))$. Each j^*g, $g \in \mathrm{AA}(M_P)$, is almost additive and lies in $j^*M_P = M_A^{K'}$. Hence $M_A^{K'} \subseteq \mathrm{alg}(\mathrm{AA}(M_A^{K'}))$: M_A has the generation property at K'.

Step 2: abelian factors. Let A, ι, ρ be as in (Af3) for M, let $K \le N$ and $K' = \iota(K) \le N_A$. Let $f \in M^K$, say $f = g\circ\iota$ with $g \in M_A$, and put $g' = f\circ\rho = g\circ\iota\circ\rho$, which lies in M_A by (Af3). For $y \in N_A$, $k \in K$: $g'(y+\iota(k)) = f(\rho(y)+\rho\iota(k)) = f(\rho(y)+k) = f(\rho(y)) = g'(y)$, so $g' \in M_A^{K'}$; and $g'\circ\iota = f\circ\rho\circ\iota = f$. By Step 1, $g' \in \mathrm{alg}(\mathrm{AA}(M_A^{K'}))$, so $f = \iota^*g' \in \mathrm{alg}(\iota^*\mathrm{AA}(M_A^{K'}))$. For $h \in \mathrm{AA}(M_A^{K'})$, ι^*h is almost additive, lies in $\iota^*M_A = M$, and is K-invariant: $h(\iota(x+k)) = h(\iota(x)+\iota(k)) = h(\iota(x))$. So $\iota^*\mathrm{AA}(M_A^{K'}) \subseteq \mathrm{AA}(M^K)$ and $f \in \mathrm{alg}(\mathrm{AA}(M^K))$. □

Proposition 35.44. *Let C and $\mathcal{O}_C$ be as in Definition 35.17, A an abelian scheme over $\mathcal{O}_C$ admitting a relatively ample line bundle, and $K' \leq N_A$ a subgroup. Then there are a principally polarised abelian scheme P over $\mathcal{O}_C$ and a homomorphism of group schemes $j\colon A \to P$ such that, writing j also for the induced homomorphism $N_A \to N_P$:*

1. *for $x, y \in N_A$, $j(x) = j(y)$ if and only if $x - y \in K'$;*
2. *$M_A^{K'} = \{g \circ j :\ g \in M_P\}$.*

Proof. Let $K' \leq N_A$. Write $\mathcal{O}(Y) = \Gamma(Y, \mathcal{O}_Y)$ for affine Y.

Step 1: the closure of K'. $A[p]$ is finite locally free over $\mathcal{O}_C$; its generic fibre is finite étale over the algebraically closed C (characteristic 0), hence constant, and $N_A = A[p](\mathcal{O}_C) = A[p](C)$ ($A[p]$ is finite). For a finite locally free $\mathcal{O}_C$-scheme Y with constant generic fibre, $\mathcal{O}(Y)$ is torsion free, so $\mathcal{O}(Y) \to \mathcal{O}(Y_C) = \operatorname{Map}(Y(C), C)$ is injective. Let $H \subseteq A[p]$ be the closed subscheme defined by the kernel of $\mathcal{O}(A[p]) \to \operatorname{Map}(K', C)$. Its ring is the image, a finitely generated torsion-free, hence free, $\mathcal{O}_C$-module, so H is finite locally free, with $H_C = K'$ and $H(\mathcal{O}_C) = K'$. $H \times H$ is flat with generic fibre $K' \times K'$, so it is the closure of $K' \times K'$ in $A[p] \times A[p]$; the group law and the inverse map $K' \times K'$ into K', hence $H \times H$ into H: H is a closed subgroup scheme of A, finite locally free over $\mathcal{O}_C$.

Step 2: the quotient, by spreading out. $\mathcal{O}_C$ is the filtered union of its finitely generated $\mathbb{Z}$-subalgebras R_α (noetherian). A, a relatively ample line bundle L on it, the group law, unit and inverse, and H with its closed immersion are of finite presentation over $\mathcal{O}_C$; by the limit theorems there are α, an abelian scheme A_α over R_α with a relatively ample line bundle and a closed subgroup scheme $H_\alpha \subseteq A_\alpha$, finite locally free over R_α, whose base changes to $\mathcal{O}_C$ are A, L, H (with their structures). By Theorem 35.45 over $S = \operatorname{Spec} R_\alpha$ there is a finite locally free surjective homomorphism $q_\alpha\colon A_\alpha \to A'_\alpha$ onto an abelian scheme with a relatively ample line bundle with $A_\alpha \times H_\alpha \cong A_\alpha \times_{A'_\alpha} A_\alpha$. Base change to $\mathcal{O}_C$ gives $q\colon A \to A'$, a finite locally free surjective homomorphism onto an abelian scheme A' over $\mathcal{O}_C$ with a relatively ample line bundle, and $A \times H \cong A \times_{A'} A$ via $(a, h) \mapsto (a, a + h)$.

Step 3: the principally polarised P. The abelian scheme A'_α carries a relatively ample line bundle $\mathcal{L}'_\alpha$ over the noetherian base $\operatorname{Spec} R_\alpha$, so its dual $A'^\vee_\alpha$ exists and $\phi_{\mathcal{L}'_\alpha}\colon A'_\alpha \to A'^\vee_\alpha$ is a polarisation. By Theorem B.120, $P_\alpha = (A'_\alpha \times_{R_\alpha} A'^\vee_\alpha)^4$ admits a principal polarisation $\lambda_\alpha\colon P_\alpha \to P_\alpha^\vee$. Let $i_\alpha\colon A'_\alpha \to P_\alpha$ be the inclusion of the first factor A'_α and $r_\alpha\colon P_\alpha \to A'_\alpha$ the projection onto it, so that $r_\alpha \circ i_\alpha = \mathrm{id}$. Let P, $i\colon A' \to P$, $r\colon P \to A'$ and λ be the base changes to $\mathcal{O}_C$ (recall that A' is the base change of A'_α). Since the dual commutes with base change, $\lambda\colon P \to P^\vee$ is an isomorphism, and it is a polarisation because every geometric point of $\operatorname{Spec} \mathcal{O}_C$ lies over a geometric point of $\operatorname{Spec} R_\alpha$ and the homomorphism ϕ of an ample line bundle is compatible with extension of the algebraically closed field. So P is a principally polarised abelian scheme over $\mathcal{O}_C$, $r \circ i = \mathrm{id}$, and i is a closed immersion (a section of the separated r). Put $j = i \circ q$.

Step 4: assertion (1). For $x, y \in N_A$: $j(x) = j(y)$ if and only if $q(x - y) = 0$ (i is injective on points), i.e. $x - y \in \ker q(\mathcal{O}_C) = H(\mathcal{O}_C) = K'$.

Step 5: assertion (2). If $g \in M_P$ is the function of $b \in \mathcal{O}(P[p])$, then $g \circ j$ is the function of $j^*b \in \mathcal{O}(A[p])$ (j maps $A[p]$ into $P[p]$), so $g \circ j \in M_A$, and it is K'-invariant by (1). Conversely let $f \in M_A^{K'}$ be the function of $a \in \mathcal{O}(A[p])$. The two pull-backs of a to $A[p] \times H$ (by the projection and by the action $(x, h) \mapsto x + h$) have the same values on $(A[p] \times H)(C) = N_A \times K'$, so they are equal (Step 1, injectivity for $A[p] \times H$). The closed subscheme $A[p] \subseteq A$ is stable under H; since q is an fppf H-torsor, it descends to a closed subscheme $Z \subseteq A'$ with $q^{-1}(Z) = A[p]$, $A[p] \to Z$ is an H-torsor, and by descent of functions $a = q^*a'$ for some $a' \in \mathcal{O}(Z)$. As $[p] \circ q = q \circ [p]$ and $q^{-1}(Z) = A[p]$, $[p]$ kills Z after the faithfully flat base change $A[p] \to Z$, so $Z \subseteq A'[p]$,

a closed subscheme of the affine $A'[p]$. Since i restricts to a closed immersion $A'[p] \to P[p]$, $\mathcal{O}(P[p]) \to \mathcal{O}(Z)$ is surjective: $a' = b|_Z$ for some $b \in \mathcal{O}(P[p])$, and $a = j^*b$. Hence $f = g \circ j$ with $g \in M_P$ the function of b. □

Theorem 35.45. *Let S be a noetherian scheme, A an abelian scheme over S admitting a relatively ample line bundle, and $H \subseteq A$ a closed subgroup scheme, finite and locally free over S. Then there are an abelian scheme A' over S admitting a relatively ample line bundle and a homomorphism $q\colon A \to A'$ which is finite, locally free and surjective, such that $A \times_S H \to A \times_{A'} A$, $(a, h) \mapsto (a, a + h)$, is an isomorphism; in particular* $\ker q = H$.

Proof. Let $\mathcal{L}$ be a relatively ample line bundle on A, and let H act on A by $\alpha(h, a) = a + h$.

Step 1: the quotient. The morphism $H \times_S A \to A \times_S A$, $(h, a) \mapsto (a + h, a)$, is the closed immersion $H \times_S A \to A \times_S A$ followed by the automorphism $(b, a) \mapsto (a + b, a)$ of $A \times_S A$; so it is a closed immersion, in particular a monomorphism, and the action is free. Let $x \in A$ lie over $s \in S$ and let $V \subseteq S$ be an affine open neighbourhood of s. As H is finite over S, the image of $H \times_S \operatorname{Spec} \kappa(x) \to A$ is a finite set of points of A_V. Since V is affine and $A_V \to V$ is of finite type, $\mathcal{L}|_{A_V}$ is an ample line bundle on the quasi-compact scheme A_V, so every finite subset of A_V lies in an affine open subscheme of A_V [81], §§4.5–4.6, which is an affine open subscheme of A. By Theorem B.152 there is a finite, locally free, surjective S-morphism $q\colon A \to A'$ with $q(a + h) = q(a)$ such that $H \times_S A \to A \times_{A'} A$, $(h, a) \mapsto (a + h, a)$, is an isomorphism; composing with the exchange of factors on both sides, $A \times_S H \to A \times_{A'} A$, $(a, h) \mapsto (a, a + h)$, is an isomorphism.

Step 2: group structure and uniqueness. For the fppf topology on S-schemes, q is a covering, so the sheaf $h_{A'}$ represented by A' is the coequaliser of the two projections $h_{A \times_{A'} A} \rightrightarrows h_A$ [226], Exp. IV. By Step 1 these are the maps $h_H \times h_A \rightrightarrows h_A$, $(h, a) \mapsto a + h$ and $(h, a) \mapsto a$, whose coequaliser is the quotient sheaf of commutative groups h_A/h_H. Hence A' is a commutative S-group scheme, q is a homomorphism, and pulling back the isomorphism of Step 1 along the unit section in the first factor gives $H \cong S \times_{A'} A = \ker q$. The same argument over an S-scheme T shows: if $p\colon A_T \to Y$ is a finite, locally free, surjective morphism of T-schemes such that $H_T \times_T A_T \to A_T \times_Y A_T$, $(h, a) \mapsto (a + h, a)$, is an isomorphism, then Y and A'_T both represent h_{A_T}/h_{H_T} (the properties of q and the isomorphism of Step 1 are stable under base change), so $Y \cong A'_T$.

Step 3: $A' \to S$ is of finite type, flat and proper. $A' \to S$ is quasi-compact, because $A \to S$ is and q is surjective. Let $\operatorname{Spec} C \subseteq A'$ be an affine open lying over an affine open $\operatorname{Spec} R \subseteq S$. As q is finite, its preimage is $\operatorname{Spec} B$ with B finite over C; B is a finitely generated R-algebra; and $C \to B$ is injective because q is faithfully flat. As R is noetherian, C is a finitely generated R-algebra [12], Prop. 7.8. Flatness: let $y \in A'$, $x \in A$ with $q(x) = y$, and s their image in S; $\mathcal{O}_{A',y} \to \mathcal{O}_{A,x}$ is faithfully flat and $\mathcal{O}_{S,s} \to \mathcal{O}_{A,x}$ is flat, so for an injection $M' \to M$ of $\mathcal{O}_{S,s}$-modules the map $M' \otimes \mathcal{O}_{A',y} \to M \otimes \mathcal{O}_{A',y}$ is injective after $\otimes_{\mathcal{O}_{A',y}} \mathcal{O}_{A,x}$, hence injective. Separatedness: the base change of the diagonal $A' \to A' \times_S A'$ along the fppf covering $q \times q\colon A \times_S A \to A' \times_S A'$ is $A \times_{A'} A \to A \times_S A$, which by Step 1 is identified with the closed immersion $H \times_S A \to A \times_S A$, $(h, a) \mapsto (a + h, a)$; closed immersions descend along fpqc coverings [83], (2.7.1), so the diagonal is a closed immersion. Universal closedness: for $S' \to S$, $q_{S'}$ is surjective, so the image in S' of a closed subset $Z \subseteq A'_{S'}$ is the image of the closed subset $q_{S'}^{-1}(Z)$ under the closed map $A_{S'} \to S'$. So $A' \to S$ is proper.

Step 4: A' is an abelian scheme. Let $s \in S$ and $\bar{s} = \operatorname{Spec} \bar{k}$ for an algebraic closure $\bar{k}$ of $\kappa(s)$. Then $A_{\bar{s}}$ is an abelian variety over $\bar{k}$ and $H_{\bar{s}} \subseteq A_{\bar{s}}$ a finite subgroup scheme; Theorem B.121 gives an abelian variety Y and a finite, flat, surjective $p\colon A_{\bar{s}} \to Y$ with $H_{\bar{s}} \times A_{\bar{s}} \cong A_{\bar{s}} \times_Y A_{\bar{s}}$, and Step 2 with $T = \bar{s}$ gives $A'_{\bar{s}} \cong Y$. So A'_s is smooth and geometrically connected over $\kappa(s)$.

Since $A' \to S$ is flat and of finite presentation (S noetherian) with smooth fibres, it is smooth [84], (17.5.1). Thus A' is a smooth proper commutative group scheme over S with geometrically connected fibres, an abelian scheme.

Step 5: a relatively ample line bundle on A'. Apply Theorem B.153 with $Y = S$, $X = A'$, $X' = A$, $f\colon A' \to S$ (quasi-compact by Step 3), $g = q$ (finite and surjective, with $q_*\mathcal{O}_A$ locally free as q is finite and locally free) and $\mathcal{L}' = \mathcal{L}$ (ample relative to $f \circ q$): $\mathrm{N}_{A/A'}(\mathcal{L})$ is ample relative to $A' \to S$. □

Proposition 35.46. *Let C and $\mathcal{O}_C$ be as in Definition 35.17 and P a principally polarised abelian scheme over $\mathcal{O}_C$. Then M_P has the generation property at the trivial subgroup: M_P is contained in the $\mathcal{O}_C$-subalgebra of* $\mathrm{Map}(N_P, C)$ *generated by the almost-additive elements of M_P.*

Proof. Let g be the relative dimension of P and $B = M_P$.

Step 1 (B is a Hopf order on N_P). The ring $\Gamma(P[p], \mathcal{O})$ is finite locally free over the local ring $\mathcal{O}_C$, hence free of rank p^{2g}; its generic fibre is the constant group scheme N_P, so evaluation at the points identifies $\Gamma(P[p], \mathcal{O}) \otimes C$ with $\mathrm{Map}(N_P, C)$, and evaluation is injective on the torsion-free $\Gamma(P[p], \mathcal{O})$. So B is spanned by the columns of a matrix in $\mathrm{GL}_{N_P}(C)$; it contains 1, is closed under products, and the group law and inverse of $P[p]$ give the comultiplication and antipode axioms of Definition 35.18(Ho2).

Step 2 (cotangent length). The augmentation ideal I_B is the ideal of the unit section, so $I_B/I_B^2 = \omega_{P[p]} \cong (\mathcal{O}_C/p)^g$, of normalised length g.

Step 3. By Lemma 35.47, Proposition 35.54, Step 2 and Lemma 35.49,

$$2\,\mathrm{colen}(B) \le (p-1)|N_P|\,\mathrm{len}_C(Q_B) \le |N_P|\,g = |N_P|\,\mathrm{len}_C(I_B/I_B^2) \le 2\,\mathrm{colen}(B).$$

So all are equalities; in particular $2\,\mathrm{colen}(B) = (p-1)|N_P|\,\mathrm{len}_C(Q_B)$, and Lemma 35.47 shows that B is spanned by products of its almost-additive elements: M_P has the generation property at the trivial subgroup. (The two equalities also show that M_P is tight in the sense of Definition 35.18(Ho3).) □

Lemma 35.47. *Let N be a finite abelian group with $pN = 0$, $n = \dim_{\mathbb{F}_p} N$, and $B \subseteq \mathrm{Map}(N, C)$ an $\mathcal{O}_C$-submodule spanned by the columns of a matrix in $\mathrm{GL}_N(C)$, containing 1 and closed under products (for instance a Hopf order on N). Let $V = \mathrm{Lin}(N)/p\,\mathrm{Map}(N, \mathcal{O}_C)$, the $\mathcal{O}_C/p$-module of additive maps $N \to \mathcal{O}_C/p$. Then Q_B has a normalised length and*

$$2\,\mathrm{colen}(B) \le (p-1)\,|N|\,\mathrm{len}_C(Q_B).$$

Moreover there are $g_1, \dots, g_n \in \mathrm{AF}(B)$ and $\varepsilon_1, \dots, \varepsilon_n \in \mathcal{O}_C$ with $\varepsilon_j \mid p$ such that:

(i) $g_j = \varepsilon_j \tilde{\lambda}_j$ with $\tilde{\lambda}_j \in \mathrm{Map}(N, \mathcal{O}_C)$, and the p^n functions $\tilde{\lambda}^d = \prod_j \tilde{\lambda}_j^{d_j}$, $d \in \{0, \dots, p-1\}^n$, form an $\mathcal{O}_C$-basis of $\mathrm{Map}(N, \mathcal{O}_C)$*;*

(ii) the reductions λ_j of the $\tilde{\lambda}_j$ modulo p form an $\mathcal{O}_C/p$-basis of V, the image of $\mathrm{AF}(B)$ *in V is $\bigoplus_j \bar{\varepsilon}_j(\mathcal{O}_C/p)\lambda_j$, and $\mathrm{len}_C(Q_B) = \sum_j v(\varepsilon_j)$;*

(iii) if equality holds in the inequality above, the products $g^d = \prod_j g_j^{d_j}$, $d \in \{0, \dots, p-1\}^n$, form an $\mathcal{O}_C$-basis of B.

In particular, if equality holds, B is the $\mathcal{O}_C$-span of products of almost-additive elements of B, and B has the generation property at the trivial subgroup.

Proof. B is spanned by the $|N|$ columns of a matrix, contains 1 and is closed under products, so its elements take values in $\mathcal{O}_C$ by Lemma 35.48, and $\mathrm{AF}(B) \subseteq \mathrm{Lin}(N)$. The module V is free of rank n over $\mathcal{O}_C/p$ with basis the coordinate functions $x_1, \dots, x_n$ for an $\mathbb{F}_p$-basis of N (reduction modulo p maps $\mathrm{Lin}(N)$ onto the additive maps $N \to \mathcal{O}_C/p$, since any lift of an additive map is almost additive, with kernel $\mathrm{Lin}(N) \cap p\,\mathrm{Map}(N, \mathcal{O}_C) = p\,\mathrm{Map}(N, \mathcal{O}_C)$). Let M be the image of $\mathrm{AF}(B)$, so $Q_B \cong V/M$. M is finitely generated: $\mathrm{AF}(B)$ is the kernel of the $\mathcal{O}_C$-linear map $B \to \mathrm{Map}(N \times N, \mathcal{O}_C/p)$, $g \mapsto \overline{g(x+y) - g(x) - g(y)}$, from the free module B of rank $|N|$ (its spanning columns are C-linearly independent) to a finitely presented module, and over the coherent ring $\mathcal{O}_C$ (a valuation ring: finitely generated ideals are principal) such a kernel is finitely generated. The preimage of M in $\mathcal{O}_C^n$ (coordinates in the basis x_k) is finitely generated and contains $p\,\mathcal{O}_C^n$; an adapted basis gives an $\mathcal{O}_C/p$-basis $\lambda_1, \dots, \lambda_n$ of V and $\varepsilon_j \in \mathcal{O}_C$ with $\varepsilon_j \mid p$ (so $\varepsilon_j \neq 0$) and $M = \bigoplus_j \bar{\varepsilon}_j(\mathcal{O}_C/p)\lambda_j$. Hence $Q_B \cong \bigoplus_j (\mathcal{O}_C/p)/\bar{\varepsilon}_j(\mathcal{O}_C/p) \cong \bigoplus_j \mathcal{O}_C/\varepsilon_j$ has normalised length $\sum_j v(\varepsilon_j)$.

Choose $g_j \in \mathrm{AF}(B)$ with $\bar{g}_j = \bar{\varepsilon}_j\lambda_j$; then $g_j = \varepsilon_j\tilde{\lambda}_j$ with $\tilde{\lambda}_j = g_j/\varepsilon_j \in \mathrm{Map}(N, \mathcal{O}_C)$ a lift of λ_j (write $g_j = \varepsilon_j\ell + ph$ for a lift ℓ and use $\varepsilon_j \mid p$). This gives (ii). *Claim (this is (i)): the p^n functions $\tilde{\lambda}^d = \prod_j \tilde{\lambda}_j^{d_j}$, $0 \le d_j \le p-1$, form a basis of* $\mathrm{Map}(N, \mathcal{O}_C)$. Modulo p, $\mathrm{Map}(N, \mathcal{O}_C/p) = (\mathcal{O}_C/p)[x_1, \dots, x_n]/(x_k^p - x_k)$; filter by degree. The associated graded ring is $(\mathcal{O}_C/p)[x]/(x_1^p, \dots, x_n^p)$. Write $\lambda = Ux$ with $U \in \mathrm{GL}_n(\mathcal{O}_C/p)$; in characteristic p, $\lambda_j^p = \sum_k u_{jk}^p x_k^p$ and $U^{(p)} = (u_{jk}^p)$ is invertible, so the ideals (x_k^p) and (λ_j^p) coincide and the graded ring is $(\mathcal{O}_C/p)[\lambda]/(\lambda_j^p)$, with basis the monomials λ^d, $d_j < p$. The reduction of $\tilde{\lambda}^d$ is λ^d, of degree $\sum_j d_j$ with symbol the monomial λ^d of the graded ring; a family whose symbols in the associated graded form a basis is a basis of the filtered free module, so the λ^d form a basis of $\mathrm{Map}(N, \mathcal{O}_C/p)$, and the claim follows by the fact quoted above.

Let Λ be the $\mathcal{O}_C$-span of the products $g^d = \prod_j g_j^{d_j} = (\prod_j \varepsilon_j^{d_j})\tilde{\lambda}^d$, $0 \le d_j < p$. Then $\Lambda \subseteq \mathcal{O}_C[\mathrm{AF}(B)] \subseteq B$. Since the $\tilde{\lambda}^d$ form a basis (claim) and the scalars $\prod_j \varepsilon_j^{d_j}$ are non-zero, the g^d form a basis of Λ, and $\mathrm{colen}(\Lambda) = \sum_d \sum_j d_j v(\varepsilon_j) = \sum_j v(\varepsilon_j) \cdot p^{n-1}\frac{p(p-1)}{2} = \frac{(p-1)|N|}{2}\mathrm{len}_C(Q_B)$. If T is the matrix expressing the basis $(g^d)_d$ of Λ in a basis of B, then T has entries in $\mathcal{O}_C$ and $\mathrm{colen}(\Lambda) = \mathrm{colen}(B) + v(\det T) \ge \mathrm{colen}(B)$, which is the inequality. If equality holds, $v(\det T) = 0$, so $T \in \mathrm{GL}(\mathcal{O}_C)$ and $B = \Lambda$; thus the g^d form a basis of B, which is (iii), and B is the span of products of the almost-additive elements g_j of B (the constant 1 is the empty product). □

Lemma 35.48. *Let X be a finite set and $B \subseteq \mathrm{Map}(X, C)$ a finitely generated $\mathcal{O}_C$-submodule that contains the constant function 1 and is closed under products. Then $B \subseteq \mathrm{Map}(X, \mathcal{O}_C)$. In particular every Hopf order on a finite-dimensional $\mathbb{F}$-vector space N consists of functions $N \to \mathcal{O}_C$.*

Proof. Let $b_1, \dots, b_r$ generate B and let $g \in B$. Since B is closed under products, $g\,b_j = \sum_k a_{jk}b_k$ with $a_{jk} \in \mathcal{O}_C$. Put $A = (a_{jk})$ and $P(T) = \det(T \cdot \mathbf{1}_r - A) \in \mathcal{O}_C[T]$, a monic polynomial. In the commutative ring $\mathrm{Map}(X, C)$ the vector $b = (b_1, \dots, b_r)^{\mathsf{T}}$ satisfies $(g\mathbf{1}_r - A)\,b = 0$; multiplying by the adjugate of $g\mathbf{1}_r - A$ gives $P(g)\,b_k = 0$ for every k. Writing $1 = \sum_k c_k b_k$ with $c_k \in \mathcal{O}_C$ ($1 \in B$), $P(g) = P(g) \cdot 1 = \sum_k c_k P(g) b_k = 0$. Evaluating at $x \in X$ (a ring homomorphism $\mathrm{Map}(X, C) \to C$), $P(g(x)) = 0$, so $g(x)$ is integral over $\mathcal{O}_C$. A valuation ring is integrally closed in its field of fractions, so $g(x) \in \mathcal{O}_C$.

A Hopf order on N is by (Ho2) the $\mathcal{O}_C$-span of the $|N|$ columns of a matrix (hence finitely generated), contains 1 and is closed under products, so the first part applies with $X = N$. □

Lemma 35.49. *Let N be a finite abelian group with $pN = 0$ and B a Hopf order on N. Then*

I_B/I_B^2 has a normalised length and $|N|\operatorname{len}_C(I_B/I_B^2) \le 2\operatorname{colen}(B)$.

Proof. Regard N as a vector space over $\mathbb{F} = \mathbb{F}_p$ ($f = 1$, $q = p$). B is $\mathbb{F}_p$-stable: for $g \in B$, $x \mapsto g(0)$ lies in $\mathcal{O}_C 1 \subseteq B$, and for $k \ge 1$, $x \mapsto g(kx)$ is obtained from $(x_1, \dots, x_k) \mapsto g(x_1 + \cdots + x_k)$, which lies in the span of products $g_1(x_1) \cdots g_k(x_k)$ with $g_i \in B$ (iterate the comultiplication axiom), by restriction to the diagonal, so it lies in B (closure under products). The Teichmüller character $\chi\colon \mathbb{F}_p \to \mathcal{O}_C$ ($\chi(a) \in \mu_{p-1}$, $\chi(a) \equiv a$ modulo $\mathfrak{m}_C$, values in $\mathbb{Z}_p$, so $\chi(a+b) - \chi(a) - \chi(b) \in p\mathbb{Z}_p$) is fundamental. Choose a basis of N and let $S_1, \dots, S_m$ be the step orders, as in Lemma 35.50. By Lemma 35.50(3), iterated from B_0, I_B/I_B^2 has the normalised length $\sum_\psi \operatorname{len}\operatorname{co}_\psi(B) \le \sum_\psi \sum_j \operatorname{len}\operatorname{co}_\psi(S_j) = \sum_j \operatorname{len}_C(I_{S_j}/I_{S_j}^2)$. By Lemma 35.52(4),(5) with $f = 1$, $\operatorname{len}_C(I_{S_j}/I_{S_j}^2) = v(\delta_j) = (p-1)v(c_j) = 2\operatorname{colen}(S_j)/p$, where c_j generates $\operatorname{Xr}_\chi(S_j)$. By Lemma 35.50(2), $\sum_j 2\operatorname{colen}(S_j)/p = 2\operatorname{colen}(B)/|N|$. □

Lemma 35.50. *Let B be an $\mathbb{F}$-stable Hopf order on N, $b_1, \dots, b_m$ a basis of N, $N_j = \mathbb{F}b_1 \oplus \cdots \oplus \mathbb{F}b_j$, $B_j = B|_{N_j}$ ($B_0 = \mathcal{O}_C$, $B_m = B$) and $S_j = B_{N_{j-1}, b_j}$ the step order ($1 \le j \le m$). Then:*

1. *each B_j is an $\mathbb{F}$-stable Hopf order on N_j, and each S_j is an $\mathbb{F}$-stable Hopf order on $\mathbb{F}$; for every character ψ, $\operatorname{Xr}_{\psi,j}(B) = \operatorname{Xv}_{\psi,j}(B) = \operatorname{Xr}_\psi(S_j)$;*
2. *$\operatorname{colen}(B)/|N| = \sum_{j=1}^m \operatorname{colen}(S_j)/q$;*
3. *for every character ψ the modules $\operatorname{co}_\psi(B_j)$, $\operatorname{co}_\psi(S_j)$ have normalised lengths, I_B/I_B^2 has normalised length $\sum_\psi \operatorname{len}_C \operatorname{co}_\psi(B)$ (sum over the $q-1$ characters), and $\operatorname{len}_C \operatorname{co}_\psi(B_j) \le \operatorname{len}_C \operatorname{co}_\psi(B_{j-1}) + \operatorname{len}_C \operatorname{co}_\psi(S_j)$ for $1 \le j \le m$;*
4. *if ψ is a character with*
$$\operatorname{len}_C \operatorname{co}_\psi(B) = \sum_{j=1}^m \operatorname{len}_C \operatorname{co}_\psi(S_j),$$
then $\operatorname{len}_C \operatorname{co}_\psi(B_j) = \sum_{i \le j} \operatorname{len}_C \operatorname{co}_\psi(S_i)$ for $0 \le j \le m$.

Proof. **Step 0 (group schemes).** For a Hopf order B' on a finite group N', $B' \otimes_{\mathcal{O}_C} B' \to \operatorname{Map}(N' \times N', C)$ is injective (a basis goes to a linearly independent family), so the axioms of Definition 35.18(Ho2) make $G' = \operatorname{Spec} B'$ a finite flat commutative group scheme over $\mathcal{O}_C$ with G'_C the constant group N'. For a subgroup $N'' \le N'$ the restriction $B' \to B'|_{N''}$ is surjective, $B'|_{N''}$ is finitely generated and torsion-free, hence free, spans $\operatorname{Map}(N'', C)$, and inherits the Hopf axioms; it is a Hopf order on N'', and $\operatorname{Spec} B'|_{N''}$ is a closed flat subgroup scheme of G'. If B' is $\mathbb{F}$-stable and N'' is an $\mathbb{F}$-subspace, $B'|_{N''}$ is $\mathbb{F}$-stable. For $K \le N'$ and $H = \operatorname{Spec} B'|_K$, the coaction $B' \to B' \otimes B'|_K$ is $g \mapsto ((x,k) \mapsto g(x+k))$, so by (a) $\mathcal{O}(G'/H) = B'^K = \{g \in B' : g(x+k) = g(x)\}$, and by flat base change $B'^K \otimes C = \operatorname{Map}(N'/K, C)$: B'^K is a Hopf order on N'/K.

Step 1 (part (1)). The first claim is Step 0. Let $G_j = \operatorname{Spec} B_j$ and $H = \operatorname{Spec} B_{j-1} \subseteq G_j \subseteq G$. The morphism $G_j/H \to G/H$ induced by $G_j \subseteq G$ is a monomorphism of fppf sheaves (two sections of G_j/H with the same image differ locally by a section of H), finite and of finite presentation, hence a closed immersion by (b). So restriction $B^{N_{j-1}} \to (B_j)^{N_{j-1}}$ is surjective: $(B^{N_{j-1}})|_{N_j} = (B_j)^{N_{j-1}}$. Composition with $\mathbb{F} \cong N_j/N_{j-1}$, $a \mapsto ab_j$, maps $(B_j)^{N_{j-1}}$ isomorphically and $\mathbb{F}$-equivariantly onto S_j (by definition of S_j and the previous equality); so S_j is an $\mathbb{F}$-stable Hopf order on $\mathbb{F}$ (Step 0). For $c \in C$, the function $h(ab_j + k) = c\psi(a)$ on N_j is N_{j-1}-invariant; $c \in \operatorname{Xr}_{\psi,j}(B)$ iff $h \in B_j$, iff $h \in (B_j)^{N_{j-1}} = (B^{N_{j-1}})|_{N_j}$, iff some $g \in B^{N_{j-1}}$ has $g(ab_j) = c\psi(a)$, i.e. $c \in \operatorname{Xv}_{\psi,j}(B) = \operatorname{Xr}_\psi(S_j)$.

Step 2 (part (2)). For a Hopf order B' on N' the trace of multiplication by g on $B' \otimes C = \mathrm{Map}(N', C)$ is $\sum_x g(x)$; if M is the matrix of a basis of B' in the basis of indicator functions, the Gram matrix of the trace form is $M^{\mathsf{T}}M$, so $v(\mathfrak{d}(B'/\mathcal{O}_C)) = 2\,\mathrm{colen}(B')$. Apply (d) to $\mathcal{O}_C \to A = (B_j)^{N_{j-1}} \to B_j$: B_j is finite locally free of rank q^{j-1} over the semilocal ring A (by (a)), hence free. The torsor isomorphism $G_j \times H \cong G_j \times_{G_j/H} G_j$, $(x,h) \mapsto (x, x+h)$, gives $B_j \otimes_A B_j \cong B_j \otimes_{\mathcal{O}_C} B_{j-1}$ as B_j-algebras, so $\mathfrak{d}(B_j/A)B_j = \mathfrak{d}(B_{j-1}/\mathcal{O}_C)B_j$, hence $\mathfrak{d}(B_j/A) = \mathfrak{d}(B_{j-1}/\mathcal{O}_C)A$ ($A \to B_j$ faithfully flat), whose norm is $\mathfrak{d}(B_{j-1}/\mathcal{O}_C)^q$. With $A \cong S_j$ (Step 1): $2\mathrm{colen}(B_j) = q \cdot 2\mathrm{colen}(B_{j-1}) + q^{j-1} \cdot 2\mathrm{colen}(S_j)$, i.e. $\mathrm{colen}(B_j)/q^j = \mathrm{colen}(B_{j-1})/q^{j-1} + \mathrm{colen}(S_j)/q$. Summing over j gives (2) ($\mathrm{colen}(B_0) = 0$).

Step 3 (part (3)). For an $\mathbb{F}$-stable Hopf order B' the projectors

$$\pi_\psi g = \frac{1}{q-1} \sum_{a \in \mathbb{F}^\times} \psi(a)^{-1} g(a\,\cdot)$$

preserve $I_{B'}$ and $I_{B'}^2$ and $\sum_\psi \pi_\psi = \mathrm{id}$, so $I_{B'}/I_{B'}^2 = \bigoplus_\psi \mathrm{co}_\psi(B')$. $I_{B'}^2$ is finitely generated, so $I_{B'}/I_{B'}^2$ is finitely presented, and it is torsion because the augmentation ideal of $\mathrm{Map}(N', C)$ equals its square (it is spanned by idempotents). By (c) it and its direct summands have normalised lengths, additive over the decomposition. Restriction $I_{B_j} \to I_{B_{j-1}}$ is surjective and maps $I_{B_j}^2$ onto $I_{B_{j-1}}^2$, so $\mathrm{co}_\psi(B_j) \to \mathrm{co}_\psi(B_{j-1})$ is surjective, with kernel the image of the ψ-part of $\{g \in I_{B_j} : \ g|_{N_{j-1}} = 0\}$. By Lemma 35.51 (for B_j on N_j and $N' = N_{j-1}$) every such g is $h + r$ with $h \in I_{B_j} \cap (B_j)^{N_{j-1}}$ and $r \in I_{B_j}^2$; projecting with π_ψ, the kernel is the image of $\mathrm{co}_\psi((B_j)^{N_{j-1}}) \cong \mathrm{co}_\psi(S_j)$ (Step 1; the isomorphism is $\mathbb{F}$-equivariant and preserves augmentation ideals). So $\mathrm{co}_\psi(S_j) \to \mathrm{co}_\psi(B_j) \to \mathrm{co}_\psi(B_{j-1}) \to 0$ is exact; the kernel U of the first map is finitely presented by coherence, and additivity gives $\mathrm{len}\,\mathrm{co}_\psi(B_j) = \mathrm{len}\,\mathrm{co}_\psi(B_{j-1}) + \mathrm{len}\,\mathrm{co}_\psi(S_j) - \mathrm{len}\,U$.

Step 4 (part (4)). Put $e_j = \mathrm{len}\,\mathrm{co}_\psi(B_j) - \sum_{i \le j} \mathrm{len}\,\mathrm{co}_\psi(S_i)$. By (3), $e_j \le e_{j-1}$, and $e_0 = 0$ ($I_{B_0} = 0$). The hypothesis is $e_m = 0$, so $e_j = 0$ for all j. □

Lemma 35.51. *Let p, C, $\mathcal{O}_C$, $\mathbb{F}$ and N be as in Definition 35.18, B a Hopf order on N and $N' \subseteq N$ a subgroup. Then every $g \in B$ vanishing on N' can be written $g = h + r$ with $h \in I_B$, $h(x+k) = h(x)$ for all $x \in N$, $k \in N'$, and $r \in I_B^2$.*

Proof. Being finite over $\mathcal{O}_C$, B consists of functions with values in $\mathcal{O}_C$. A basis of B is a C-basis of $\mathrm{Map}(N, C)$, so $B \otimes B \to \mathrm{Map}(N \times N, C)$ is injective and $G = \mathrm{Spec}\,B$ is a finite locally free commutative group scheme over $\mathcal{O}_C$ with $G(C) = N$, the comultiplication being $g \mapsto ((x,y) \mapsto g(x+y))$. Let $J = \ker(B \to B|_{N'})$ and $H = \mathrm{Spec}(B/J)$. $B/J \cong B|_{N'}$ is finitely generated and torsion free, hence free, so H is finite locally free; $H \times H$ is flat with generic fibre $N' \times N'$, hence equal to the closure of $N' \times N'$ in $G \times G$, which the group law maps into H: H is a closed subgroup scheme. By SGA 3, V, 4.1 the quotient $\varpi\colon G \to G/H$ exists, is an fppf H-torsor, and $\mathcal{O}(G/H)$ consists of the $f \in B$ whose two pull-backs to $G \times H$ agree; as $\mathcal{O}(G \times H) = B \otimes B/J$ embeds in $\mathrm{Map}(N \times N', C)$, $\mathcal{O}(G/H) = \{f \in B : \ f(x+k) = f(x)\ \forall x \in N,\ k \in N'\}$. The fibre of ϖ over the unit section is H (from $G \times_{G/H} G \cong G \times H$), so $J = I'B$ with $I' = \{f \in \mathcal{O}(G/H) : \ f(0) = 0\}$. Write $g = \sum_k f_k b_k$ with $f_k \in I'$, $b_k \in B$; then $h = \sum_k b_k(0) f_k \in I' \subseteq I_B$ is N'-invariant and $r = g - h = \sum_k f_k\,(b_k - b_k(0)) \in I_B^2$. □

Lemma 35.52. *Let S be an $\mathbb{F}$-stable Hopf order on the one-dimensional $\mathbb{F}$-vector space $\mathbb{F}$ and χ a fundamental character; put $\chi_s = \chi^{p^s}$. Then:*

1. *$S \subseteq \mathrm{Map}(\mathbb{F}, \mathcal{O}_C)$, and for every $m \in \{1, \dots, q-1\}$, $\mathrm{Xr}_{\chi^m}(S) = c(m)\mathcal{O}_C$ for some $c(m) \in \mathcal{O}_C \smallsetminus \{0\}$, and $S = \mathcal{O}_C \cdot 1 \oplus \bigoplus_{m=1}^{q-1} c(m)\mathcal{O}_C\,\chi^m$;*

2. *with* $c_s = c(p^{s \bmod f})$*: if* $m = \sum_{t<f} d_t p^t$ *with digits* $0 \le d_t \le p-1$*, then* $c(m)\mathcal{O}_C = \prod_t c_t^{d_t}\mathcal{O}_C$*; hence the monomials* $X^d = \prod_{t<f} X_t^{d_t}$ *in* $X_t = c_t\chi_t$ *(*$0 \le d_t \le p-1$*) form an* $\mathcal{O}_C$*-basis of* S*;*

3. $\delta_s := c_s^p/c_{s+1}$ *lies in* $\mathcal{O}_C$*, and* $\mathcal{O}_C[Y_0,\dots,Y_{f-1}]/(Y_t^p - \delta_t Y_{t+1} : t < f) \to S$*,* $Y_t \mapsto X_t$ *(indices modulo* f*), is an isomorphism;*

4. $I_S/I_S^2 \cong \bigoplus_{s<f} \mathcal{O}_C/\delta_s\mathcal{O}_C$*, the* s*-th summand spanned by the class of* X_{s+1}*; so* $\mathrm{co}_{\chi_{s+1}}(S)$ *has normalised length* $v(\delta_s)$*, and* $\mathrm{co}_\psi(S) = 0$ *for every character* $\psi \notin \{\chi_0,\dots,\chi_{f-1}\}$*;*

5. $\mathrm{colen}(S) = \frac{q(p-1)}{2}\sum_{s<f} v(c_s)$*, and* $\sum_{s<f} v(\delta_s) = (p-1)\sum_{s<f} v(c_s)$*.*

Proof. (1) S is a finite $\mathcal{O}_C$-algebra, so every $g \in S$ is integral over $\mathcal{O}_C$, and so is each value $g(x)$ (evaluation is a ring homomorphism); $\mathcal{O}_C$ is integrally closed, so $g(x) \in \mathcal{O}_C$. By Lemma 35.53(1) the functions $1, \chi, \dots, \chi^{q-1}$ form a basis of $\mathrm{Map}(\mathbb{F}, C)$; for $1 \le m \le q-2$ the χ^m-eigenspace of $\mathbb{F}^\times$ acting by $g \mapsto g(a\,\cdot)$ is $C\chi^m$, and the eigenspace of the trivial character is $C1 \oplus C\chi^{q-1}$. As $S = \bigoplus_\psi \pi_\psi S$, $\pi_{\chi^m} S = S \cap C\chi^m = \mathrm{Xr}_{\chi^m}(S)\chi^m$ for $m \le q-2$; for the trivial part, $g = \alpha 1 + \beta\chi^{q-1} \in S$ gives $\alpha = g(0) \in \mathcal{O}_C$ and $\beta\chi^{q-1} = g - \alpha 1 \in S$ (as $1 \in S$), so $\pi_{\mathrm{triv}} S = \mathcal{O}_C 1 \oplus \mathrm{Xr}_{\chi^{q-1}}(S)\chi^{q-1}$. Each $\mathrm{Xr}_{\chi^m}(S)$ is isomorphic to a direct summand of the finitely generated module S, nonzero because S spans $\mathrm{Map}(\mathbb{F}, C)$; it is a finitely generated $\mathcal{O}_C$-submodule of C, hence $c(m)\mathcal{O}_C$ with $c(m) \ne 0$, and $c(m) \in \mathcal{O}_C$ by the first sentence.

(2) Closure under products: $c(j)c(k)\chi^{j+k} \in S$, so $c(j)c(k) \in c(j+k)\mathcal{O}_C$ whenever $j + k \le q-1$. Conversely let m have digit sum at least 2; pick t with $d_t \ge 1$ and put $j = p^t$, $k = m - p^t$, so $1 \le j, k \le q-2$ and $j + k = m$ without carry. Let $\Delta g(x,y) = g(x+y)$; by definition of a Hopf order $\Delta(c(m)\chi^m)$ lies in $S \otimes S$, the $\mathcal{O}_C$-span of the functions $g_1(x)g_2(y)$ in $\mathrm{Map}(\mathbb{F}^2, C)$. The bi-projector $P = \pi_{\chi^j} \otimes \pi_{\chi^k}$ maps $S \otimes S$ onto $c(j)c(k)\mathcal{O}_C\,\chi^j \otimes \chi^k$, and $P\Delta\chi^m = \kappa\,\chi^j \otimes \chi^k$ with

$$\kappa = P\Delta\chi^m(1,1) = \frac{1}{(q-1)^2}\sum_{a,b\in\mathbb{F}^\times} \chi^{-j}(a)\chi^{-k}(b)\chi^m(a+b) = \frac{1}{q-1}\sum_{z\in\mathbb{F}^\times} \chi^{-k}(z)\chi^m(1+z)$$

(substitute $b = az$ and use $m = j + k$). Reduce modulo $\mathfrak{m}_C$ and identify $\mathbb{F}$ with $\bar\chi(\mathbb{F}) \subseteq k_C$ (Lemma 35.53(1)): $\bar\kappa = -\sum_{y\in\mathbb{F}}(1+y)^m y^{q-1-k} = -\sum_{i=0}^m \binom{m}{i}\sum_{y\in\mathbb{F}} y^{i+q-1-k}$ (the term $y = 0$ vanishes as $q-1-k \ge 1$). Since $\sum_{y\in\mathbb{F}} y^e = -1$ if $e \ge 1$ and $(q-1) \mid e$, and 0 otherwise, and the only multiple of $q-1$ in $[q-1-k,\, q-1-k+m]$ is $q-1$ (as $2(q-1) - (q-1-k) > m$ because $j \le q-2$), we get $\bar\kappa = \binom{m}{k} \bmod p$, nonzero by Lucas' theorem since $k + j = m$ has no carry. So $\kappa \in \mathcal{O}_C^\times$, and $c(m)\kappa \in c(j)c(k)\mathcal{O}_C$ gives $c(m)\mathcal{O}_C = c(j)c(k)\mathcal{O}_C = c_t\,c(k)\mathcal{O}_C$. Induction on the digit sum gives $c(m)\mathcal{O}_C = \prod_t c_t^{d_t}\mathcal{O}_C$. As $d \mapsto \sum_t d_t p^t$ is a bijection from $\{0,\dots,p-1\}^f$ onto $\{0,\dots,q-1\}$ and $X^d = \prod_t c_t^{d_t}\,\chi^{\sum d_t p^t}$ (the empty product being the constant 1), (1) shows that the X^d form a basis of S.

(3) $X_s^p = c_s^p\chi_{s+1} \in S$, so $c_s^p \in c_{s+1}\mathcal{O}_C$ and $X_s^p = \delta_s X_{s+1}$. The map of the statement is therefore defined and surjective by (2). Using the relations, every monomial with some exponent $\ge p$ is δ_t times a monomial of smaller total degree, so the source is generated as an $\mathcal{O}_C$-module by the q monomials with exponents $< p$; a surjection from a module generated by q elements onto the free module S of rank q composed with $\mathcal{O}_C^q \to$ source is a surjective endomorphism of $\mathcal{O}_C^q$, hence bijective, so the map is an isomorphism.

(4) The X_t vanish at 0, so I_S is the image of $J = (Y_0,\dots,Y_{f-1})$ and $I_S/I_S^2 = J/(J^2 + R)$, R the ideal of relations, $R \subseteq J$. Modulo J^2, $h\cdot(Y_t^p - \delta_t Y_{t+1}) \equiv -h(0)\,\delta_t Y_{t+1}$ because $p \ge 2$. Hence $I_S/I_S^2 = \bigoplus_t \mathcal{O}_C Y_{t+1}/\delta_t\mathcal{O}_C Y_{t+1}$. The class of X_{t+1} is χ_{t+1}-eigen, and $\chi_1,\dots,\chi_f$ are

pairwise distinct (Lemma 35.53(2)); since $I_S/I_S^2 = \bigoplus_\psi \mathrm{co}_\psi(S)$ (the projectors preserve I_S and I_S^2), $\mathrm{co}_{\chi_{s+1}}(S) \cong \mathcal{O}_C/\delta_s$ and the other $\mathrm{co}_\psi(S)$ vanish.

(5) The basis $1, \chi, \dots, \chi^{q-1}$ of $\mathrm{Map}(\mathbb{F}, \mathcal{O}_C)$ differs from the basis of indicator functions by a matrix in $\mathrm{GL}_q(\mathcal{O}_C)$ (Lemma 35.53(1)); in it the basis X^d of S is diagonal with entries $\prod_t c_t^{d_t}$, whose product is $\prod_t c_t^N$ with $N = \sum_d d_t = (q/p) \cdot p(p-1)/2$. So $\mathrm{colen}(S) = \frac{q(p-1)}{2} \sum_t v(c_t)$. Finally $\sum_{s<f} v(\delta_s) = p \sum_s v(c_s) - \sum_s v(c_{s+1}) = (p-1) \sum_s v(c_s)$ by periodicity. □

Lemma 35.53. *Let p, C, $\mathcal{O}_C$, v and $\mathbb{F}$ be as in Definition 35.18, $q = |\mathbb{F}| = p^f$, k_C the residue field of $\mathcal{O}_C$, and $\chi\colon \mathbb{F} \to \mathcal{O}_C$ a fundamental character. Put $\chi_s = \chi^{p^s}$ for $s \geq 0$ (with $\chi_s(0) = 0$). Then:*

1. *the reduction $\bar\chi\colon \mathbb{F} \to k_C$ of χ is an injective ring homomorphism; $\chi(a) - \chi(b) \in \mathcal{O}_C^\times$ for $a \neq b$; $\chi|_{\mathbb{F}^\times}$ has order $q-1$, so every character is χ^m for a unique $m \in \{1, \dots, q-1\}$, and $\{1, \chi, \chi^2, \dots, \chi^{q-1}\}$ is an $\mathcal{O}_C$-basis of $\mathrm{Map}(\mathbb{F}, \mathcal{O}_C)$;*
2. *every χ_s is a fundamental character, $\chi_{s+f} = \chi_s$, and $\chi_0, \dots, \chi_{f-1}$ are pairwise distinct on $\mathbb{F}^\times$; in particular $\sum_{a \in \mathbb{F}^\times} \chi_s(a)^{-1} \chi_{s'}(a) = 0$ for $s \not\equiv s' \pmod f$;*
3. *every additive map $\varphi\colon \mathbb{F} \to \mathcal{O}_C/p\mathcal{O}_C$ is $\varphi = \sum_{s=0}^{f-1} r_s\, (\chi_s \bmod p)$ for unique $r_0, \dots, r_{f-1} \in \mathcal{O}_C/p\mathcal{O}_C$.*

Proof. (1) Since $p \in \mathfrak{m}_C$, $\bar\chi$ is additive; it is multiplicative on $\mathbb{F}^\times$, $\bar\chi(0) = 0$ and $\bar\chi(1) = 1$ ($\chi|_{\mathbb{F}^\times}$ is a homomorphism). So $\bar\chi$ is a ring homomorphism from a field, hence injective. For $a \neq b$, $\overline{\chi(a) - \chi(b)} = \bar\chi(a-b) \neq 0$, so $\chi(a) - \chi(b)$ is a unit. Injectivity of $\bar\chi$ on $\mathbb{F}^\times$ makes $\chi|_{\mathbb{F}^\times}$ injective, of order $q-1 = |\mathbb{F}^\times|$; the character group of the cyclic group $\mathbb{F}^\times$ (values in $\mu_{q-1}(C)$) is cyclic of order $q-1$, so χ generates it, and the convention $\psi(0) = 0$ for every character gives $\psi = \chi^m$ with a unique $m \in \{1, \dots, q-1\}$ (χ^{q-1} is the indicator function of $\mathbb{F}^\times$). For $a, x \in \mathbb{F}^\times$, $\sum_{m=1}^{q-1} \chi(a)^{-m} \chi(x)^m = (q-1)[x = a]$, and all χ^m vanish at 0; since $q - 1 \in \mathcal{O}_C^\times$, the indicator $1_{\{a\}}$ ($a \neq 0$) lies in the $\mathcal{O}_C$-span of $\chi, \dots, \chi^{q-1}$, and $1_{\{0\}} = 1 - \sum_{a \neq 0} 1_{\{a\}}$. Thus these q functions span the free module $\mathrm{Map}(\mathbb{F}, \mathcal{O}_C)$ of rank q, hence form a basis.

(2) χ_s is a character. Write $\chi(a+b) = \chi(a) + \chi(b) + pe$ with $e \in \mathcal{O}_C$. For $u, e \in \mathcal{O}_C$, $(u+pe)^p \equiv u^p \pmod{p\mathcal{O}_C}$ and $(u+w)^p \equiv u^p + w^p \pmod{p\mathcal{O}_C}$ (the middle binomial coefficients are divisible by p); by induction on s, $\chi_s(a+b) \equiv \chi_s(a) + \chi_s(b) \pmod{p\mathcal{O}_C}$, so χ_s is fundamental. $\chi_{s+f} = \chi_s$ because $a^q = a$ in $\mathbb{F}$. The reduction of χ_s is $\mathrm{Frob}^s \circ \bar\chi$; for $0 \leq s < s' < f$ these two embeddings $\mathbb{F} \to k_C$ differ (their images of a generator α of $\mathbb{F}$ over $\mathbb{F}_p$ are the distinct conjugates $\bar\chi(\alpha^{p^s}) \neq \bar\chi(\alpha^{p^{s'}})$), so $\chi_s \neq \chi_{s'}$ on $\mathbb{F}^\times$, and the orthogonality relation is the vanishing of the sum of the nontrivial character $\chi_s^{-1}\chi_{s'}$ over the group $\mathbb{F}^\times$.

(3) By (2) each $\chi_s \bmod p$ is additive, hence so is $\sum r_s \chi_s$. An additive map $\mathbb{F} \to \mathcal{O}_C/p$ is $\mathbb{F}_p$-linear, so is determined by its values at the $\mathbb{F}_p$-basis $1, \alpha, \dots, \alpha^{f-1}$, which can be arbitrary. The equations $\sum_s r_s \chi_s(\alpha^i) = \sum_s r_s \chi_s(\alpha)^i = \varphi(\alpha^i)$ ($0 \leq i < f$) have the Vandermonde matrix $(\chi_s(\alpha)^i)_{i,s}$, whose determinant $\prod_{s<s'} (\chi(\alpha^{p^{s'}}) - \chi(\alpha^{p^s}))$ is a unit by (1), since α^{p^s}, $0 \leq s < f$, are pairwise distinct. So the system has a unique solution modulo p. □

Proposition 35.54. *Let C, $\mathcal{O}_C$ and v be as in Definition 35.18 and P a principally polarised abelian scheme over $\mathcal{O}_C$ of relative dimension g. Then $Q_{M_P} = \mathrm{Lin}(N_P)/(\mathrm{AF}(M_P) + p\,\mathrm{Map}(N_P, \mathcal{O}_C))$ has normalised length at most $g/(p-1)$.*

Proof. If $g = 0$, then $N_P = 0$, $\mathrm{Lin}(N_P) = p\,\mathrm{Map}(N_P, \mathcal{O}_C)$ (an almost-additive f has $f(0) \in p\mathcal{O}_C$) and $Q_{M_P} = 0$. Let $g \geq 1$ and let $x_1, \dots, x_{2g}$, H, B, J and c be as given by Proposition 35.55,

and let $L \in M_{2g}(\mathcal{O}_C)$ be the matrix whose rows are the g rows of H followed by the g rows of B.

Step 1: $v(\det L) = g/(p-1)$. With $\Omega = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ ($\det\Omega = 1$), $L^{\mathsf{T}}\Omega L = H^{\mathsf{T}}B - B^{\mathsf{T}}H = -c\,J$. Taking determinants, $(\det L)^2 = c^{2g}\det J$, and $\det J$ is a unit, so $v(\det L) = g\,v(c) = g/(p-1)$.

Step 2: almost-additive elements. For a row r of L let $f_r \in M_P$ be as in Proposition 35.55 and $\ell_r = \sum_j r_j x_j$. Since $x_j(y+y') - x_j(y) - x_j(y') \in p\mathbb{Z}$, ℓ_r is almost additive, hence so is $f_r \equiv \ell_r$ modulo $p\,\mathcal{O}_C$: $f_r \in \mathrm{AF}(M_P)$.

Step 3: the bound. Let $b_1, \dots, b_{2g}$ be the basis of N_P dual to the x_j. An almost-additive f satisfies $f(y) \equiv \sum_j x_j(y) f(b_j)$ modulo $p\,\mathcal{O}_C$ (induction on $\sum_j x_j(y)$, using $f(0) \in p\mathcal{O}_C$), and $\sum_j a_j x_j$ is almost additive for every $a \in \mathcal{O}_C^{2g}$; so $f \mapsto (f(b_j))_j$ induces an isomorphism $\mathrm{Lin}(N_P)/p\,\mathrm{Map}(N_P, \mathcal{O}_C) \cong (\mathcal{O}_C/p)^{2g}$, under which $Q_{M_P} \cong \mathcal{O}_C^{2g}/U$, U the set of a with $\sum_j a_j x_j \in \mathrm{AF}(M_P) + p\,\mathrm{Map}(N_P, \mathcal{O}_C)$. By Step 2, U contains every row of L ($f_r(b_j) \equiv r_j$). Hence Q_{M_P} is a quotient of $Q_0 = \mathcal{O}_C^{2g}/(\text{span of the rows of } L) \cong \bigoplus_i \mathcal{O}_C/d_i$ with $\sum_i v(d_i) = v(\det L) = g/(p-1)$ (Step 1). By Lemma 35.47, $Q_{M_P} \cong \bigoplus_k \mathcal{O}_C/e_k$ has a normalised length; as the span of the rows of L lies in U, $(\prod_i d_i) = \mathrm{Fitt}_0(Q_0) \subseteq \mathrm{Fitt}_0(Q_{M_P}) = (\prod_k e_k)$, so $\mathrm{len}_C(Q_{M_P}) = \sum_k v(e_k) \le \sum_i v(d_i) = g/(p-1)$. □

Proposition 35.55. *Let C, $\mathcal{O}_C$ and v be as in Definition 35.18 and P a principally polarised abelian scheme over $\mathcal{O}_C$ of relative dimension $g \ge 1$. Then there are an $\mathbb{F}_p$-basis of N_P, with coordinate functions $x_1, \dots, x_{2g} \colon N_P \to \{0, 1, \dots, p-1\} \subseteq \mathbb{Z}$, matrices $H, B \in M_{g\times 2g}(\mathcal{O}_C)$, an alternating matrix $J \in M_{2g}(\mathbb{Z}_p)$ with $\det J \in \mathbb{Z}_p^\times$ and $c \in \mathcal{O}_C$ with $v(c) = 1/(p-1)$, such that*

$$B^{\mathsf{T}}H - H^{\mathsf{T}}B = c\,J,$$

and for every row $(r_1, \dots, r_{2g})$ of H or of B there is $f \in M_P$ with $f(y) - \sum_j r_j x_j(y) \in p\,\mathcal{O}_C$ for all $y \in N_P$.

Proof. We use the following classical facts: for an abelian scheme P over $\mathcal{O}_C$ the Cartier dual of $P[p^n]$ is $P^\vee[p^n]$, so the Cartier dual of the p-divisible group $\mathcal{G} = P[p^\infty]$ is $\mathcal{G}^D = P^\vee[p^\infty]$ and $\omega_{\mathcal{G}^D} = \omega_{P^\vee}$ is free of rank g; the Weil pairing of the principal polarisation $\varphi \colon P \to P^\vee$ is perfect and alternating (Mumford, Abelian varieties, §§20, 23; Oda for abelian schemes); for a commutative finite flat group scheme $\Gamma = \mathrm{Spec}\,R_\Gamma$ over $\mathcal{O}_C$ with Cartier dual $\Gamma^D = \mathrm{Spec}\,R_\Gamma^\vee$, a point $y \in \Gamma(\mathcal{O}_C)$ corresponds to a group-like element $\chi_y \in R_\Gamma^\vee$ (the character $\Gamma^D \to \mathbb{G}_m$ attached to y), and $f(y) = \chi_y(f)$ for $f \in R_\Gamma$ (Cartier duality); over the valuation ring $\mathcal{O}_C$ a finitely generated torsion-free module is free, so a submodule of a free module of finite rank with torsion-free quotient is a direct summand.

Step 0: set-up. Put $\mathcal{G} = P[p^\infty]$ and $T = T_p(\mathcal{G})$, a free $\mathbb{Z}_p$-module of rank $2g$. Since C is algebraically closed of characteristic 0 and $P[p^n]$ is finite over $\mathcal{O}_C$, $P[p^n](\mathcal{O}_C) = P[p^n](C) \cong (\mathbb{Z}/p^n)^{2g}$ and multiplication by p maps $P[p^{n+1}](C)$ onto $P[p^n](C)$; hence $T/pT = N_P$. Fix a compatible system of primitive p^n-th roots of unity in C, which identifies $\mathbb{Z}_p(1)$ with $\mathbb{Z}_p$; let ζ be its primitive p-th root of unity, and let $\langle\ ,\ \rangle \colon T \times T \to \mathbb{Z}_p$ be the Weil pairing of φ (the Cartier pairing of t with $\varphi(t') \in T_p(\mathcal{G}^D)$), extended $\mathcal{O}_C$-bilinearly. Choose a $\mathbb{Z}_p$-basis $b_1, \dots, b_{2g}$ of T and let $J = (\langle b_i, b_j\rangle) \in M_{2g}(\mathbb{Z}_p)$: it is alternating, and $\det J \in \mathbb{Z}_p^\times$ since the pairing is perfect. The images of the b_j form an $\mathbb{F}_p$-basis of N_P; let $x_j \colon N_P \to \{0, \dots, p-1\}$ be its coordinate functions, so that $\tilde{y} = \sum_j x_j(y) b_j \in T$ lifts y. Fix $c \in \mathcal{O}_C$ with $v(c) = \frac{1}{p-1}$ (C is algebraically closed). Let $u_1, \dots, u_g$ be an $\mathcal{O}_C$-basis of $\omega = \omega_{\mathcal{G}^D} \otimes \mathcal{O}_C$ and $\lambda_1, \dots, \lambda_g$ the coordinate forms.

Step 1: the kernel of the Hodge–Tate map is Lagrangian. Put $T^\circ = T \otimes_{\mathbb{Z}_p} \mathcal{O}_C$ and $\alpha = \alpha_{\mathcal{G}} \otimes 1 \colon T^\circ \to \omega$. By Lemma 35.57, $\Lambda = \ker \alpha$ is a direct summand of T°, free of rank g, with $\langle \Lambda, \Lambda \rangle = 0$; and by Theorem B.54 (with $K = C$, applied to $\mathcal{G}$), (a) $c\,\omega \subseteq \alpha(T^\circ)$.

Step 2: the matrices. Choose an $\mathcal{O}_C$-basis $v_1, \dots, v_g$ of Λ and $v'_1, \dots, v'_g$ of a complement Λ', and let $S \in \mathrm{GL}_{2g}(\mathcal{O}_C)$ express (v, v') in the basis (b_j). The Gram matrix of the pairing in the basis (v, v') is $J' = S^{\mathsf{T}} J S = \begin{pmatrix} 0 & D \\ -D^{\mathsf{T}} & Q \end{pmatrix}$ with Q alternating (Step 1 and the pairing alternating). The matrix of α in the bases (v, v') and (u_k) is $H' = (0 \mid A)$ with $A \in M_g(\mathcal{O}_C)$. Since $\alpha(T^\circ) = \alpha(\Lambda') = A\,\mathcal{O}_C^g$ contains $c\,\mathcal{O}_C^g$ by (a), A is invertible over C and $cA^{-1} \in M_g(\mathcal{O}_C)$. Write $Q = U - U^{\mathsf{T}}$ with U strictly upper triangular, and put

$$B' = (c\,A^{-\mathsf{T}} D^{\mathsf{T}} \mid c\,A^{-\mathsf{T}} U^{\mathsf{T}}) \in M_{g \times 2g}(\mathcal{O}_C).$$

So $B'^{\mathsf{T}} H' - H'^{\mathsf{T}} B' = c\,J'$. Put $H = H'S^{-1}$ and $B = B'S^{-1}$ in $M_{g\times 2g}(\mathcal{O}_C)$; then $B^{\mathsf{T}} H - H^{\mathsf{T}} B = S^{-\mathsf{T}}(cJ')S^{-1} = c\,J$. Moreover H is the matrix of α in the bases (b_j), (u_k), so $H_{kj} = \lambda_k(\alpha(b_j))$; and every row of B is $c\,(n_1, \dots, n_{2g})$ with $n_j \in \mathcal{O}_C$, since $c^{-1}B'$ and S^{-1} are integral.

Step 3: the rows of B lie in M_P. Since $v(c) = v(\zeta - 1) = \frac{1}{p-1}$, $c\,\mathcal{O}_C = (\zeta - 1)\mathcal{O}_C$, and Lemma 35.56 (with $K = C$, $\mathcal{O} = \mathcal{O}_C$ and $w = c$) gives $d_0, \dots, d_{p-1} \in \mathcal{O}_C$ with

$$\sum_{i=0}^{p-1} d_i\,\zeta^{ir} = c\,r \qquad (0 \le r \le p-1).$$

Since the Weil pairing e_p of φ on N_P is perfect, there is for each j a point $z_j \in N_P$ with $e_p(y, z_j) = \zeta^{x_j(y)}$ for all $y \in N_P$; and $y \mapsto e_p(y, z_j)$ is a homomorphism of group schemes $P[p] \to \mu_p$ over $\mathcal{O}_C$ (the Weil pairing restricted to the section z_j), so $\psi_j(y) = \zeta^{x_j(y)}$ is an element of M_P, as are its powers. Hence $c\,x_j = \sum_i d_i \psi_j^i \in M_P$, and for every row $c\,(n_j)$ of B the function $f = \sum_j c n_j x_j$ lies in M_P and has $f(y) - \sum_j c n_j x_j(y) = 0$.

Step 4: the rows of H are congruent to elements of M_P. Let $\Gamma = P[p] = \operatorname{Spec} R_\Gamma$, so $\Gamma^D = \mathcal{G}^D[p]$ and $\omega_{\Gamma^D} = \omega/p\omega$ (the cokernel of p on $\omega_{\mathcal{G}^D}$). For $x \in T$ with image $y \in N_P$, $\alpha_{\mathcal{G}}(x) \bmod p = \alpha_\Gamma(y)$, the class of $\chi_y - 1$ in $I/I^2 = \omega_{\Gamma^D}$, I the augmentation ideal of $R_\Gamma^\vee = \mathcal{O}(\Gamma^D)$. The composite $\phi_k \colon R_\Gamma^\vee \to I \to I/I^2 = \omega/p\omega \xrightarrow{\lambda_k} \mathcal{O}_C/p$ (the first map $a \mapsto a - \epsilon(a)$, ϵ the counit) is $\mathcal{O}_C$-linear; as $R_\Gamma^\vee$ is free of finite rank, it lifts to an $\mathcal{O}_C$-linear $\Phi_k \colon R_\Gamma^\vee \to \mathcal{O}_C$, an element of $R_\Gamma^{\vee\vee} = R_\Gamma$. By Cartier duality, for $y \in N_P$, $\Phi_k(y) = \Phi_k(\chi_y) \equiv \phi_k(\chi_y) = \lambda_k(\alpha_\Gamma(y))$ modulo p (note $\epsilon(\chi_y) = 1$), and $\lambda_k(\alpha_\Gamma(y)) \equiv \lambda_k(\alpha(\tilde{y})) = \sum_j H_{kj} x_j(y)$ modulo p. So the element f of M_P given by Φ_k satisfies $f(y) - \sum_j H_{kj} x_j(y) \in p\,\mathcal{O}_C$ for all $y \in N_P$.

Steps 0–4 give the basis, the x_j, H, B, J and c with all the stated properties. □

Lemma 35.56. *Let p be an odd prime, K a field of characteristic 0 with a valuation ring $\mathcal{O}$, $\zeta \in K$ a primitive p-th root of unity, and $w \in \mathcal{O}$ with $w\mathcal{O} = (\zeta - 1)\mathcal{O}$. For $t \in \mathbb{Z}/p$ let $\tilde{t} \in \{0, \dots, p-1\}$ be its representative. Then for every $t \in \mathbb{Z}/p$*

$$w\,\tilde{t} = \sum_{j=0}^{p-1} c_j\,\zeta^{jt}, \qquad c_0 = w\,\frac{p-1}{2}, \quad c_j = \frac{w}{\zeta^{-j} - 1}\ (1 \le j \le p-1),$$

and all c_j lie in $\mathcal{O}$.

Proof. **The Fourier coefficients of $t \mapsto \tilde{t}$.** For $x \in K$ with $x^p = 1$, $x \ne 1$, put $S(x) = \sum_{u=0}^{p-1} u x^u$. Then

$$xS(x) - S(x) = \sum_{u=1}^{p} (u-1)x^u - \sum_{u=1}^{p-1} u x^u = (p-1)x^p - \sum_{u=1}^{p-1} x^u = (p-1) + 1 = p,$$

using $x^p = 1$ and $\sum_{u=0}^{p-1} x^u = 0$; so $S(x) = p/(x-1)$. Put $a_j = \frac{1}{p}\sum_{u=0}^{p-1} u\,\zeta^{-ju}$; then $a_0 = \frac{p-1}{2}$ and $a_j = 1/(\zeta^{-j}-1)$ for $1 \le j \le p-1$. By orthogonality of characters ($\sum_{j=0}^{p-1}\zeta^{j(t-u)} = p$ if $t = u$ in $\mathbb{Z}/p$ and 0 otherwise),

$$\sum_j a_j \zeta^{jt} = \frac{1}{p}\sum_{u=0}^{p-1} u \sum_j \zeta^{j(t-u)} = \tilde{t}.$$

Multiplying by w gives the identity with $c_j = w a_j$. **Integrality.** $c_0 \in \mathcal{O}$ since $(p-1)/2 \in \mathbb{Z}$. For $1 \le j \le p-1$, $(\zeta^{-j}-1)/(\zeta-1) = -\zeta^{-j}(1+\zeta+\cdots+\zeta^{j-1})$ lies in $\mathbb{Z}[\zeta]$, and so does its inverse (exchange the roles of ζ and ζ^{-j}, both primitive), so $(\zeta^{-j}-1)\mathcal{O} = (\zeta-1)\mathcal{O} = w\mathcal{O}$ and $c_j = w/(\zeta^{-j}-1) \in \mathcal{O}^\times$. □

Lemma 35.57. *Let C, $\mathcal{O}_C$, v be as in Definition 35.18 and P a principally polarised abelian scheme over $\mathcal{O}_C$ of relative dimension $g \ge 1$, with p-divisible group $\mathcal{G} = P[p^\infty]$, Tate module $T = T_p(\mathcal{G})$ (free of rank $2g$ over $\mathbb{Z}_p$), and $\langle\ ,\ \rangle\colon T \times T \to \mathbb{Z}_p$ the Weil pairing of the principal polarisation (after fixing a compatible system of p-power roots of unity in C), extended $\mathcal{O}_C$-bilinearly to $T^\circ = T \otimes_{\mathbb{Z}_p} \mathcal{O}_C$. Let $\omega = \omega_{\mathcal{G}^D} \otimes \mathcal{O}_C$ and $\alpha = \alpha_{\mathcal{G}} \otimes 1\colon T^\circ \to \omega$. Then $\Lambda = \ker\alpha$ is a direct summand of T°, free of rank g, with $\langle\Lambda,\Lambda\rangle = 0$, and $\Lambda \otimes C = \ker(\alpha \otimes C)$ is its own orthogonal in $T \otimes C$. Equivalently, if $J \in M_{2g}(\mathbb{Z}_p)$ is the Gram matrix of $\langle\ ,\ \rangle$ in a $\mathbb{Z}_p$-basis of T and $H \in M_{g\times 2g}(\mathcal{O}_C)$ the matrix of α in that basis and an $\mathcal{O}_C$-basis of ω, then $H\,J^{-1}H^{\mathsf{T}} = 0$ and $\operatorname{rk} H = g$.*

Proof. Fix $c \in \mathcal{O}_C$ with $v(c) = \frac{1}{p-1}$. Apply Theorem B.54 with $K = C$ to $\mathcal{G}$ and to $\mathcal{G}^D$: (a) $c\,\omega \subseteq \alpha(T^\circ)$; (b) the map $\kappa = {}^t(\alpha_{\mathcal{G}^D} \otimes 1)(1)\colon \mathrm{Lie}(\mathcal{G}) \otimes \mathcal{O}_C(1) \to T^\circ$ is injective and $\alpha \circ \kappa = 0$; (c) $\alpha_{\mathcal{G}^D} \otimes C$ is onto, of rank g. By definition of the transpose, $\kappa(\nu)$ is the element t whose Cartier pairing with every $x' \in T_p(\mathcal{G}^D)$ is $\nu(\alpha_{\mathcal{G}^D}(x'))$; so $\mathrm{im}(\kappa) \otimes C$ lies in the annihilator (for the Cartier pairing) of $\ker(\alpha_{\mathcal{G}^D} \otimes C)$, and both have dimension g by (b), (c), so they are equal. The polarisation maps $\ker(\alpha \otimes C)$ onto $\ker(\alpha_{\mathcal{G}^D} \otimes C)$ (functoriality of the Hodge–Tate map: $\alpha_{\mathcal{G}^D}(\varphi x) = (\varphi^D)^* \alpha_{\mathcal{G}}(x)$, with φ^D an isomorphism). Hence, with $L = \ker(\alpha \otimes C)$, of dimension $2g - g = g$ by (a), $\mathrm{im}(\kappa) \otimes C = \{t :\ \langle t, s\rangle = 0\ \forall s \in L\} = L^\perp$ and $\mathrm{im}(\kappa) \otimes C \subseteq L$ ($\alpha \circ \kappa = 0$); comparing dimensions, $L = L^\perp$. Put $\Lambda = L \cap T^\circ = \ker\alpha$. As $T^\circ/\Lambda \cong \alpha(T^\circ) \subseteq \omega$ is torsion free, Λ is a direct summand of T°, free of rank g, $\Lambda \otimes C = L$, and $\langle\Lambda,\Lambda\rangle = 0$.

Matrix form. Write $\langle t, s\rangle = t^{\mathsf{T}} J s$ in the chosen basis. The rows $h_1, \dots, h_g$ of H are the coordinate forms of $\alpha \otimes C$; they span the g-dimensional space of linear forms vanishing on L (rank g by (a)). Every form vanishing on $L = L^\perp$ is $t \mapsto \langle t, s\rangle$ for some $s \in L$ (perfectness), i.e. $h_k = -s_k^{\mathsf{T}} J$ with $s_k \in L$ ($J^{\mathsf{T}} = -J$). Then $h_k J^{-1} h_l^{\mathsf{T}} = s_k^{\mathsf{T}} J\, J^{-1} J^{\mathsf{T}} s_l = -s_k^{\mathsf{T}} J s_l = -\langle s_k, s_l\rangle = 0$, so $H J^{-1} H^{\mathsf{T}} = 0$. Conversely, $H J^{-1} H^{\mathsf{T}} = 0$ with $\operatorname{rk} H = g$ says that $\{J^{-1} h_k^{\mathsf{T}}\}$, which span $\{s :\ \langle\cdot, s\rangle$ vanishes on $L\} = L^\perp$, lie in $\ker H = L$; so the two formulations are equivalent. □

Theorem 35.58. *Let p, C, $\mathcal{O}_C$, $\mathbb{F}$ and N be as in Definition 35.18, B an $\mathbb{F}$-stable tight Hopf order on N and $b_1, \dots, b_m$ a basis of N such that B has the generation property at $N_i = \mathbb{F}b_1 \oplus \cdots \oplus \mathbb{F}b_i$ for $0 \le i \le m-1$. Then $(B, (b_i))$ satisfies the flag conditions for every fundamental character $\chi\colon \mathbb{F} \to \mathcal{O}_C$.*

Proof. Fix a fundamental character χ, put $\chi_s = \chi^{p^s}$, $q = |\mathbb{F}| = p^f$, and let $B_j = B|_{N_j}$ and $S_j = B_{N_{j-1}, b_j}$ ($1 \le j \le m$) as in Lemma 35.50.

Step 1 (rank-one data). By Lemma 35.50(1), S_j is an $\mathbb{F}$-stable Hopf order on $\mathbb{F}$ and $\mathrm{Xr}_{\chi_s,j}(B) = \mathrm{Xv}_{\chi_s,j}(B) = \mathrm{Xr}_{\chi_s}(S_j)$. By Lemma 35.52 applied to S_j there are $c_{j,s} \in \mathcal{O}_C \smallsetminus \{0\}$, depending only on s modulo f, with $\mathrm{Xr}_{\chi_s}(S_j) = c_{j,s}\mathcal{O}_C$; $\delta_{j,s} = c_{j,s}^p / c_{j,s+1} \in \mathcal{O}_C$; $\mathrm{co}_{\chi_{s+1}}(S_j)$ has nor-

malised length $v(\delta_{j,s})$ and $\mathrm{co}_\psi(S_j) = 0$ for $\psi \notin \{\chi_0, \dots, \chi_{f-1}\}$; $2\,\mathrm{colen}(S_j) = q(p-1)\sum_{s<f} v(c_{j,s})$ and $\sum_{s<f} v(\delta_{j,s}) = (p-1)\sum_{s<f} v(c_{j,s})$. Put $X = \sum_{j=1}^{m}\sum_{s=0}^{f-1} v(c_{j,s}) < \infty$.

Step 2 (colength). By Lemma 35.50(2), $2\,\mathrm{colen}(B)/|N| = \sum_j 2\,\mathrm{colen}(S_j)/q = (p-1)X$.

Step 3 (flag condition (1)). By Lemma 35.59, $\mathrm{Ad}_{\chi_s,j}(B) = d_{j,s}\mathcal{O}_C$ with $v(d_{j,s}) \le 1$ and $\mathrm{len}_C(Q_B) = \sum_{j\le m}\sum_{s<f} v(d_{j,s})$. By the generation property at N_{j-1} and Lemma 35.60, $c_{j,s} \in \mathrm{Xv}_{\chi_s,j}(B) \subseteq d_{j,s}\mathcal{O}_C$, so $v(c_{j,s}) \ge v(d_{j,s})$. Tightness and Step 2 give $|N|(p-1)\,\mathrm{len}_C(Q_B) = 2\,\mathrm{colen}(B) = |N|(p-1)X$, so $\sum_{j,s} v(d_{j,s}) = \sum_{j,s} v(c_{j,s})$, all terms finite; hence $v(c_{j,s}) = v(d_{j,s})$ for all j and $s < f$, and $c_{j,s} \in d_{j,s}\mathcal{O}_C$ with equal valuations gives $c_{j,s}\mathcal{O}_C = d_{j,s}\mathcal{O}_C$. So $\mathrm{Xr}_{\chi_s,j}(B) = \mathrm{Xv}_{\chi_s,j}(B) = \mathrm{Ad}_{\chi_s,j}(B) = c_{j,s}\mathcal{O}_C$ for $s < f$, and for all $s \ge 0$ since the three modules and $c_{j,s}$ depend only on χ_s, which has period f in s (Lemma 35.53(2)).

Step 4 (flag condition (2)). $c_{j,s}^{p} = \delta_{j,s} c_{j,s+1}$ with $\delta_{j,s} \in \mathcal{O}_C$ by Step 1.

Step 5 (flag condition (3)). By Lemma 35.50(3), $\mathrm{len}_C(I_B/I_B^2) = \sum_\psi \mathrm{len}_C \mathrm{co}_\psi(B)$ and, iterating the inequality from B_0 (where $I_{B_0} = 0$), $\mathrm{len}_C \mathrm{co}_\psi(B) \le \sum_j \mathrm{len}_C \mathrm{co}_\psi(S_j)$ for each character ψ. As $\chi_1, \dots, \chi_f$ are pairwise distinct (Lemma 35.53(2)), Step 1 gives $\sum_\psi \sum_j \mathrm{len}_C \mathrm{co}_\psi(S_j) = \sum_j \sum_{s<f} v(\delta_{j,s}) = (p-1)X$. Tightness and Step 2 give $|N|\,\mathrm{len}_C(I_B/I_B^2) = 2\,\mathrm{colen}(B) = |N|(p-1)X$. So the sum of the termwise inequalities is an equality, and $\mathrm{len}_C \mathrm{co}_\psi(B) = \sum_j \mathrm{len}_C \mathrm{co}_\psi(S_j)$ for every ψ. For $\psi = \chi_{s+1}$, Lemma 35.50(4) gives that $\mathrm{co}_{\chi_{s+1}}(B|_{N_j})$ has normalised length $\sum_{i\le j} v(\delta_{i,s})$ for $0 \le j \le m$, first for $s < f$ and then for all $s \ge 0$ by periodicity. □

Lemma 35.59. *Let B be an $\mathbb{F}$-stable Hopf order on N, $b_1, \dots, b_m$ a basis of N and χ a fundamental character, $\chi_s = \chi^{p^s}$. Then for $1 \le i \le m$ and $s \ge 0$ there is $d_{i,s} \in \mathcal{O}_C$ with $0 \le v(d_{i,s}) \le 1$, depending only on s modulo f, such that $\mathrm{Ad}_{\chi_s,i}(B) = d_{i,s}\mathcal{O}_C$; and Q_B has normalised length*

$$\mathrm{len}_C(Q_B) = \sum_{i=1}^{m}\sum_{s=0}^{f-1} v(d_{i,s}).$$

Proof. By Lemma 35.48 $B \subseteq \mathrm{Map}(N, \mathcal{O}_C)$, so $\mathrm{AF}(B) \subseteq \mathrm{Lin}(N)$; also $p\,\mathrm{Map}(N, \mathcal{O}_C) \subseteq \mathrm{Lin}(N)$, and reduction modulo p identifies $V = \mathrm{Lin}(N)/p\,\mathrm{Map}(N, \mathcal{O}_C)$ with the additive maps $N \to \mathcal{O}_C/p$ (every additive map lifts). Write $x = \sum_i a_i(x) b_i$. An additive map φ is $\sum_i \varphi(\cdot\, b_i) \circ a_i$, so by Lemma 35.53(3) V is free over $\mathcal{O}_C/p$ with basis $e_{i,s} = \bar{\chi}_s \circ a_i$ ($i \le m$, $s < f$; bars denote reduction mod p). Let $M \subseteq V$ be the image of $\mathrm{AF}(B)$, so $Q_B \cong V/M$. $\mathrm{AF}(B)$ is the kernel of $B \to \mathrm{Map}(N \times N, \mathcal{O}_C/p)$, $g \mapsto \overline{g(x+y) - g(x) - g(y)}$, hence finitely generated (coherence), and M is finitely generated.

The projectors $\pi_s g = \frac{1}{q-1}\sum_{a\in\mathbb{F}^\times} \chi_s(a)^{-1} g(a\,\cdot)$ preserve B ($\mathbb{F}$-stability), almost additivity and $p\,\mathrm{Map}(N, \mathcal{O}_C)$; on V, $e_{i,s'}$ is $\bar{\chi}_{s'}$-eigen, so $\pi_s e_{i,s'} = [s = s']\, e_{i,s'}$ (Lemma 35.53(2)). Hence $M = \bigoplus_s M_s$ with $M_s = \pi_s M \subseteq V_s = \bigoplus_i (\mathcal{O}_C/p) e_{i,s}$, and $Q_B \cong \bigoplus_{s<f} V_s/M_s$.

Fix s and write $r \in V_s$ as $\sum_i r_i e_{i,s}$. Let $M_s^{(i)} = \{r \in M_s : r_j = 0\ (j < i)\}$. *Claim:* $\mathrm{Ad}_{\chi_s,i}(B)/p\mathcal{O}_C = \{r_i : r \in M_s^{(i)}\}$. If $r \in M_s^{(i)}$, lift it to $g \in \mathrm{AF}(B)$ and replace g by $\pi_s g$ (still a lift, as $\pi_s r = r$): then $g(ax) = \chi_s(a) g(x)$ for $a \ne 0$, $g(0) \in p\mathcal{O}_C$ (almost additivity), $\bar{g}|_{N_{i-1}} = \sum_{j<i} r_j \bar{\chi}_s \circ a_j = 0$ and $\bar{g}(b_i) = r_i$, so a lift of r_i lies in $\mathrm{Ad}_{\chi_s,i}(B)$. Conversely, if $c \in \mathrm{Ad}_{\chi_s,i}(B)$ and g is as in the definition of $\mathrm{Ad}_{\chi_s,i}(B)$ for c, then $\bar{g} \in M$ is $\bar{\chi}_s$-eigen, so $\bar{g} = \pi_s \bar{g} \in M_s$; $\bar{g}(ab_j) = r_j \bar{\chi}_s(a) = 0$ for $j < i$ and all a gives $r_j = 0$, and $r_i = \bar{g}(b_i) = \bar{c}$. Since $\mathcal{O}_C/p$ is coherent, $M_s^{(i)}$ (a kernel of a map from the finitely presented M_s to $(\mathcal{O}_C/p)^{i-1}$) is finitely generated, so $\mathrm{Ad}_{\chi_s,i}(B)$ is a finitely generated ideal containing p: it is $d_{i,s}\mathcal{O}_C$ with $v(d_{i,s}) \le 1$. It depends only on χ_s, hence on s modulo f.

Let $V^{(i)} = \{r \in V_s : r_j = 0\ (j < i)\}$ and $F^{(i)} = (V^{(i)} + M_s)/M_s$, so $F^{(1)} = V_s/M_s$,

$F^{(m+1)}=0$. By the modular law, $F^{(i)}/F^{(i+1)}\cong V^{(i)}/(V^{(i+1)}+M_s^{(i)})$, which the i-th coordinate maps isomorphically onto $(\mathcal{O}_C/p)/(d_{i,s}\mathcal{O}_C/p)\cong\mathcal{O}_C/d_{i,s}\mathcal{O}_C$, of normalised length $v(d_{i,s})$. All these modules are finitely presented and torsion, so additivity gives $\mathrm{len}_C(V_s/M_s)=\sum_i v(d_{i,s})$, and summing over $s<f$ gives the formula. □

Lemma 35.60. *Let B be an $\mathbb{F}$-stable Hopf order on N, $b_1,\dots,b_m$ a basis of N, $1\le i\le m$, and suppose B has the generation property at N_{i-1}. Let χ be a fundamental character and $\chi_s=\chi^{p^s}$. Then $\mathrm{Xv}_{\chi_s,i}(B)\subseteq\mathrm{Ad}_{\chi_s,i}(B)$ for every $s\ge 0$.*

Proof. Put $K=N_{i-1}$ and, by Lemma 35.59, $\mathrm{Ad}_{\chi_{s'},i}(B)=d_{s'}\mathcal{O}_C$ with $u_{s'}=v(d_{s'})\in[0,1]$, indices modulo f; recall $B\subseteq\mathrm{Map}(N,\mathcal{O}_C)$ (Lemma 35.48) and bars denote reduction modulo p.

Step 1: $u_{s'+1}\le p\,u_{s'}$. Choose g as in the definition of $\mathrm{Ad}_{\chi_{s'},i}(B)$ for $c=d_{s'}$, and put $h=g^p\in B$. Since $(u+pe)^p\equiv u^p$ and $(u+w)^p\equiv u^p+w^p$ modulo $p\mathcal{O}_C$, h is almost additive, $h(ax)\equiv\chi_{s'}(a)^p h(x)=\chi_{s'+1}(a)h(x)$, $h(K)\subseteq p\mathcal{O}_C$ and $h(b_i)\equiv d_{s'}^p$. So $d_{s'}^p\in d_{s'+1}\mathcal{O}_C$.

Step 2: if $s'_1,\dots,s'_n$ ($n\ge 1$) satisfy $\sum_j p^{s'_j}\equiv p^s\pmod{q-1}$, then $\prod_j d_{s'_j}\in d_s\mathcal{O}_C$. Let $\kappa_j\in\{0,\dots,f-1\}$ with $s'_j\equiv s-\kappa_j\pmod f$. Step 1 iterated gives $u_s\le p^{\kappa_j}u_{s'_j}$. Since $p^f\equiv 1\pmod{q-1}$, $\sum_j p^{f-\kappa_j}\equiv p^{f-s}\sum_j p^{s'_j}\equiv p^f\equiv 1\pmod{q-1}$; each term is at least $p\ge 2$, and an integer ≥ 2 congruent to 1 modulo $q-1$ is at least q. So $\sum_j p^{-\kappa_j}\ge 1$ and $\sum_j u_{s'_j}\ge\sum_j p^{-\kappa_j}u_s\ge u_s$.

Step 3. Let $c\in\mathrm{Xv}_{\chi_s,i}(B)$, and choose $g\in B^K$ with $g(ab_i)=c\chi_s(a)$. By the generation property $g=P(h_1,\dots,h_r)$ with $P\in\mathcal{O}_C[Y_1,\dots,Y_r]$ and h_k almost-additive elements of B^K. Each $h_k(0)\in p\mathcal{O}_C$ ($h_k(0)\equiv 2h_k(0)$), and $h_k-h_k(0)$ is again an almost-additive element of B^K; replacing h_k by it and P by $P(Y+h(0))$, we may assume $h_k(0)=0$, so $h_k(K)=0$. The map $a\mapsto\bar h_k(ab_i)$ is additive, so by Lemma 35.53(3) $\bar h_k(ab_i)=\sum_{s'<f}\bar e_{k,s'}\bar\chi_{s'}(a)$. *Each $e_{k,s'}$ may be taken in $d_{s'}\mathcal{O}_C$:* $h'=\frac{1}{q-1}\sum_{a\in\mathbb{F}^\times}\chi_{s'}(a)^{-1}h_k(a\,\cdot)$ lies in B, is almost additive, vanishes on K and at 0, satisfies $h'(ax)=\chi_{s'}(a)h'(x)$, and by Lemma 35.53(2) $\bar h'(b_i)=\bar e_{k,s'}$; so $\bar e_{k,s'}$ is the residue of an element of $\mathrm{Ad}_{\chi_{s'},i}(B)$.

Reducing $g=P(h)$ on the line $\mathbb{F}b_i$ modulo p: $\bar c\,\bar\chi_s=\sum_\alpha\bar a_\alpha\prod_k\bigl(\sum_{s'}\bar e_{k,s'}\bar\chi_{s'}\bigr)^{\alpha_k}$. Expanding, every term is a product of coefficients $\bar a_\alpha\prod_j\bar e_{k_j,s'_j}$ times $\prod_j\bar\chi_{s'_j}=\bar\chi^M$ with $M=\sum_j p^{s'_j}$ (the constant 1 if the product is empty). As functions on $\mathbb{F}$, $\bar\chi^M=\bar\chi^{M'}$ for $M\ge 1$, where $M'\in\{1,\dots,q-1\}$ and $M'\equiv M\pmod{q-1}$. The reductions of $1,\chi,\dots,\chi^{q-1}$ form a basis of $\mathrm{Map}(\mathbb{F},\mathcal{O}_C/p)$ (Lemma 35.53(1)), so comparing coefficients of $\bar\chi^{p^s}=\bar\chi_s$: $\bar c$ is a sum of terms $\bar a_\alpha\prod_j\bar e_{k_j,s'_j}$ with $n\ge 1$ and $\sum_j p^{s'_j}\equiv p^s\pmod{q-1}$. By Step 2 each lies in the image of $d_s\mathcal{O}_C$, so $c\in d_s\mathcal{O}_C+p\mathcal{O}_C=\mathrm{Ad}_{\chi_s,i}(B)$. □

Lemma 35.61. *Let $\mathfrak{T}$ be a pre-admissible torsion system and $(\mathfrak{q},C,\iota)$ a test point of $\mathfrak{T}$ at which (Ox3) holds and, for every label u, $(\mathcal{A}_{\mathfrak{P}_u,C},(b_i))$ satisfies the flag conditions for the character χ_u (with values in $\mathcal{O}_C$ through ι). Then (Ox4) holds at $(\mathfrak{q},C,\iota)$.*

Proof. Fix s, $t=\varphi_p s$, $\mathfrak{P}=\mathfrak{P}_s$, $1\le i\le m=m_{\mathfrak{P}}$, and write $B=\mathcal{A}_{\mathfrak{P},C}$, $I=\mathcal{I}_{\mathfrak{P},C}$, $N=\Gamma_{\mathfrak{P}}$, $N_j=\Gamma_{\mathfrak{P},j}$, $\mathbb{F}=\mathbb{F}_{\mathfrak{P}}$, $\theta=\theta_{t,C}$. By (Ox3), B is an $\mathbb{F}$-stable Hopf order on N. Being finite over $\mathcal{O}_C$, B consists of functions with values in $\mathcal{O}_C$; $B\otimes B\to\mathrm{Map}(N\times N,C)$ is injective (a basis of B is a C-basis of $\mathrm{Map}(N,C)$), so $G=\mathrm{Spec}\,B$ is a finite locally free commutative group scheme over $\mathcal{O}_C$ with $G(C)=N$, the group law being given by $g\mapsto((x,y)\mapsto g(x+y))$.

Step 1: functions vanishing on N_{i-1}. By Lemma 35.51, every $g\in B$ vanishing on N_{i-1} is $g=h+r$ with $h\in I$ invariant under translation by N_{i-1} and r in the $\mathcal{O}_C$-span I^2 of products of two elements of I.

Step 2: the first identity of (Ox4). $\mathcal{S}_{t,i,C} \subseteq \mathcal{I}_{\mathfrak{P},i-1,C}$, as $\xi_{t,i}$ vanishes on N_{i-1}. Conversely let $g \in \mathcal{I}_{\mathfrak{P},i-1,C}$ and $g = h + r$ as in Step 1; $\theta(r) = 0$ by (Ox3). Put $Q = |\mathbb{F}|$ (a power of p, so $Q - 1 \in \mathcal{O}_C^\times$) and

$$h' = (Q-1)^{-1} \sum_{a \in \mathbb{F}^\times} \chi_t(a)^{-1}\, h(a\,\cdot) \in I$$

($\mathbb{F}$-stability). By the equivariance in (Ox3), $\theta(h') = \theta(h) = \theta(g)$. h' is invariant under N_{i-1} (an $\mathbb{F}$-subspace) and $h'(bx) = \chi_t(b)h'(x)$ for $b \in \mathbb{F}^\times$. Hence for $x = \sum_{j \le i} x_j b_j \in N_i$: $h'(x) = h'(x_i b_i) = \chi_t(x_i) h'(b_i)$ (for $x_i = 0$ both sides vanish). So $h'|_{N_i} = h'(b_i)\,\xi_{t,i}$, $h' \in \mathcal{S}_{t,i,C}$, and $\theta(g) \in \theta(\mathcal{S}_{t,i,C})$.

Step 3: the second identity of (Ox4). Let ϵ generate $\mathcal{O}(\mathrm{Ec}_{s,i})_{\mathfrak{q}}$. The inclusion of the left side in $\mathcal{O}_C \epsilon^p$ is Lemma 35.62. Conversely, by (Ts5) $\epsilon\,\xi_{s,i} = g_0|_{N_i}$ for some $g_0 \in \mathcal{A}_{\mathfrak{P},\mathfrak{q}} \subseteq B$, and $g_0(0) = \epsilon\chi_s(0) = 0$, so $g_0 \in I$. For $r \in \mathcal{O}_C$, $g = r\, g_0^p \in I^2$, so $\theta(g) = 0$ (Ox3), and $g|_{N_i} = r\,\epsilon^p \xi_{s,i}^p = r\,\epsilon^p \xi_{t,i}$ since $\chi_s^p = \chi_t$. So $\mathcal{O}_C \epsilon^p$ lies in the left side. □

Lemma 35.62. *Let $\mathfrak{T}$ be a pre-admissible torsion system and $(\mathfrak{q}, C, \iota)$ a test point of $\mathfrak{T}$ at which (Ox3) holds and, for every label u, $(\mathcal{A}_{\mathfrak{P}_u,C}, (b_i))$ satisfies the flag conditions for χ_u. Let s be a label, $t = \varphi_p s$, $\mathfrak{P} = \mathfrak{P}_s$, $1 \le i \le m_{\mathfrak{P}}$, and let ϵ generate $\mathcal{O}(\mathrm{Ec}_{s,i})_{\mathfrak{q}}$. If $g \in \mathcal{A}_{\mathfrak{P},C}$ and $c \in C$ satisfy $g|_{\Gamma_{\mathfrak{P},i}} = c\,\xi_{t,i}$ and $\theta_{t,C}(g) = 0$, then $c \in \mathcal{O}_C\,\epsilon^p$.*

Proof. Write $B = \mathcal{A}_{\mathfrak{P},C}$, $I = \mathcal{I}_{\mathfrak{P},C} = I_B$, $N_j = \Gamma_{\mathfrak{P},j}$, $B_j = B|_{N_j}$, $\chi = \chi_t$, $\xi = \xi_{t,i}$, $Q = |\mathbb{F}_{\mathfrak{P}}|$ ($Q - 1 \in \mathcal{O}_C^\times$), and for $f \in B$ let $\mathrm{pr}(f) = (Q-1)^{-1} \sum_{a \in \mathbb{F}_{\mathfrak{P}}^\times} \chi(a)^{-1} f(a\,\cdot)$, which lies in B ($\mathbb{F}_{\mathfrak{P}}$-stability, (Ox3)), satisfies $\mathrm{pr}(f)(bx) = \chi(b)\mathrm{pr}(f)(x)$, agrees with f on every $\mathbb{F}_{\mathfrak{P}}$-subspace on which f is a χ-eigenfunction, maps I to I and I^2 to I^2, and satisfies $\theta_{t,C}(\mathrm{pr}(f)) = \theta_{t,C}(f)$ for $f \in I$ (equivariance in (Ox3)). Restriction to N_j maps I onto I_{B_j} (a lift of a function vanishing at 0 vanishes at 0) and I^2 onto $I_{B_j}^2$.

Step 0: the ideals. B_i is the $\mathcal{O}_C$-span of $\mathcal{A}_{\mathfrak{P},i,\mathfrak{q}}$, which is free over $R_{\mathfrak{q}}$ with a basis that is a C-basis of $\mathrm{Map}(N_i, C)$; writing $\xi_{u,i} = \sum_k \xi_k e_k$ in it, $c\,\xi_{u,i} \in B_i$ if and only if $v(c) \ge \max_k(-v(\xi_k))$, and the same over Ω' defines $\mathcal{O}(\mathrm{Ec}_{u,i})_{\mathfrak{q}}$ (Ts5), the valuation of C restricting to one with ring $R_{\mathfrak{q}}$. Since $\mathrm{Xr}_{\chi_u,i}(B) = \{c : \ c\,\xi_{u,i} \in B_i\}$, $\mathrm{Xr}_{\chi_s,i}(B) = \mathcal{O}_C\epsilon$. Take the flag conditions for the fundamental character χ_s, with exponent 0 and 1 ($\chi_s^p = \chi$): $\mathrm{Xr}_{\chi_s,i}(B) = c_0\mathcal{O}_C$, $\mathrm{Xr}_{\chi,i}(B) = c_1\mathcal{O}_C$, $c_0^p = \delta c_1$, and $\mathrm{len}_C(\mathrm{co}_\chi(B_j)) = \sum_{l \le j} v(\delta_l)$ with $\delta_i = \delta$. Hence

$$\mathrm{len}_C(\mathrm{co}_\chi(B_i)) - \mathrm{len}_C(\mathrm{co}_\chi(B_{i-1})) = v(\delta), \qquad c_0\mathcal{O}_C = \epsilon\,\mathcal{O}_C. \tag{1}$$

Step 1: the restriction map. Restriction maps $I_{B_i,\chi}$ into $I_{B_{i-1},\chi}$ and $I_{B_i}^2$ into $I_{B_{i-1}}^2$, so it induces $\rho\colon \mathrm{co}_\chi(B_i) \to \mathrm{co}_\chi(B_{i-1})$. It is onto: for $h \in I_{B_{i-1},\chi}$ take $f \in I$ with $f|_{N_{i-1}} = h$; then $\mathrm{pr}(f)|_{N_i} \in I_{B_i,\chi}$ restricts to h.

Step 2: the kernel of ρ is $\mathcal{O}_C[a_1]$, $a_1 = c_1\xi \in I_{B_i,\chi}$. $[a_1] \in \ker\rho$ as ξ vanishes on N_{i-1}. Let $z \in I_{B_i,\chi}$ with $z|_{N_{i-1}} \in I_{B_{i-1}}^2$; choose $g \in I$ with $g|_{N_i} = z$ and $w \in I^2$ with $w|_{N_{i-1}} = z|_{N_{i-1}}$. By Lemma 35.51, $g - w = h + r$ with $h \in I$ invariant under N_{i-1} and $r \in I^2$. Then $\mathrm{pr}(h)$ is N_{i-1}-invariant and a χ-eigenfunction, so $\mathrm{pr}(h)|_{N_i} = \lambda\,\xi$ with $\lambda = \mathrm{pr}(h)(b_i)$, and $\lambda \in \mathrm{Xr}_{\chi,i}(B) = c_1\mathcal{O}_C$, say $\lambda = r_0 c_1$. As z is a χ-eigenfunction on N_i, $z = \mathrm{pr}(g)|_{N_i} = r_0 a_1 + \mathrm{pr}(r+w)|_{N_i}$ with $\mathrm{pr}(r+w)|_{N_i} \in I_{B_i}^2$. So $[z] = r_0[a_1]$.

Step 3: $k[a_1] = 0$. By hypothesis $c\,\xi \in B_i$, so $c \in c_1\mathcal{O}_C$, $c = kc_1$. $g(0) = c\,\chi(0) = 0$, so $g \in I$, and $\theta_{t,C}(\mathrm{pr}(g)) = \theta_{t,C}(g) = 0$; $\mathrm{pr}(g)$ is a χ-eigenfunction, so $\mathrm{pr}(g) \in I^2$ by (Ox3). Restricting, $k\,a_1 = c\,\xi = g|_{N_i} = \mathrm{pr}(g)|_{N_i} \in I_{B_i}^2$.

Step 4: the length estimate. $\mathrm{co}_\chi(B_i)$ and $\mathrm{co}_\chi(B_{i-1})$ have normalised lengths (flag condition (3)): write $\mathrm{co}_\chi(B_{i-1}) \cong \bigoplus_{j=1}^{n'} \mathcal{O}_C/d_j$ with basis classes η_j, lift $\eta_j = \rho(\beta_j)$; $d_j\beta_j \in \ker\rho$, so

$d_j\beta_j = y_j[a_1]$ with $y_j \in \mathcal{O}_C$ (Step 2). The map $\mathcal{O}_C^{1+n'} \to \mathrm{co}_\chi(B_i)$, $(u_0, u) \mapsto u_0[a_1] + \sum_j u_j\beta_j$, is onto (Steps 1, 2) and kills the columns of

$$X = \begin{pmatrix} k & -y_1 & \cdots & -y_{n'} \\ 0 & d_1 & & \\ & & \ddots & \\ 0 & & & d_{n'} \end{pmatrix}$$

(Step 3), so $\mathrm{co}_\chi(B_i)$ is a quotient of $\operatorname{coker} X$. If $k = 0$, then $c = 0$ and there is nothing to prove. If $k \neq 0$, then

$$\mathrm{len}_C(\mathrm{co}_\chi(B_i)) \leq v(\det X) = v(k) + \mathrm{len}_C(\mathrm{co}_\chi(B_{i-1})).$$

With (1), $v(k) \geq v(\delta)$, so $c = kc_1 \in \delta c_1\mathcal{O}_C = c_0^p\mathcal{O}_C = \epsilon^p\mathcal{O}_C$. □

Lemma 35.63. *Let $\mathfrak{T} = (\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_\mathfrak{P}, \theta))$ be a pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$ such that*

1. *for every label t, with $m = m_{\mathfrak{P}_t}$, the step sheaves $\mathcal{U}_{t,j}$ $(0 \leq j \leq m)$ and $\mathcal{U}^{\mathrm{st}}_{t,i}$ $(1 \leq i \leq m)$ are locally free;*
2. *every point of $\mathcal{X}'$ of codimension one at which p is not a unit is the first entry of a test point at which (Ox4) holds.*

Then $\mathfrak{T}$ satisfies (Ts6); it is an admissible torsion system.

Proof. Fix a label s, put $t = \varphi_p s$, $\mathfrak{P} = \mathfrak{P}_s = \mathfrak{P}_t$, $m = m_\mathfrak{P}$, $1 \leq i \leq m$, and write $\mathcal{A} = \mathcal{A}_\mathfrak{P}$, $\mathcal{A}_j = \mathcal{A}_{\mathfrak{P},j}$, $\mathcal{I}_j = \mathcal{I}_{\mathfrak{P},j}$, $\Gamma_j = \Gamma_{\mathfrak{P},j}$, $\xi = \xi_{t,i}$. $\mathcal{X}'$ is integral, normal and noetherian (Ts2), so its local ring $R_\mathfrak{q}$ at a point $\mathfrak{q}$ of codimension one is a discrete valuation ring.

Step 0: two reflexivity facts. (a) If $\mathcal{V} \subseteq \mathcal{V}'$ are locally free of finite rank, $\mathcal{V}'$ locally free, a local section of $\mathcal{V}' \otimes \Omega'$ over an open W that lies in $\mathcal{V}_\mathfrak{q}$ for every point $\mathfrak{q} \in W$ of codimension one lies in $\mathcal{V}(W)$: in a local basis of $\mathcal{V}$ its coordinates lie in every $R_\mathfrak{q}$, hence in the local ring (intersection of the height-one localisations). The same holds for an invertible subsheaf of the constant sheaf Ω'. (b) Let $\mathcal{V} \subseteq \mathcal{V}'$ be locally free of the same rank and $\mathcal{Q} = \mathcal{V}'/\mathcal{V}$ with $\mathcal{Q}_\eta = 0$ at the generic point. At a point x of codimension ≥ 2, $\mathcal{Q}_x$ has projective dimension ≤ 1 and $\operatorname{depth} \mathcal{O}_{\mathcal{X}',x} \geq 2$, so $\operatorname{depth} \mathcal{Q}_x \geq 1$ if $\mathcal{Q}_x \neq 0$ (Auslander–Buchsbaum): $\mathcal{Q}$ has no associated point of codimension ≥ 2. A local section of $\mathcal{Q}$ vanishing at every point of codimension ≤ 1 has a support whose generic points are associated points of $\mathcal{Q}$, so it is 0.

Step 1: base change to a test point. Let $(\mathfrak{q}, C, \iota)$ be a test point, $R = R_\mathfrak{q}$; $\mathcal{O}_C$ is flat over R. Let v be an additive valuation of C defining its absolute value; its restriction to Ω' has valuation ring R. $\mathcal{A}_{j,\mathfrak{q}}$ is free over R (Ts4) with an R-basis that is an Ω'-basis of $\mathrm{Map}(\Gamma_j, \Omega')$, so $\mathcal{O}_C \otimes_R \mathcal{A}_{j,\mathfrak{q}}$ is the $\mathcal{O}_C$-span $\mathcal{A}_C|_{\Gamma_j}$ of $\mathcal{A}_{j,\mathfrak{q}}$ in $\mathrm{Map}(\Gamma_j, C)$, where $\mathcal{A}_C = \mathcal{A}_{\mathfrak{P},C}$. Writing $\xi = \sum_k \xi_k e_k$ in an R-basis (e_k) of $\mathcal{A}_{i,\mathfrak{q}}$, an element c of C (resp. of Ω') has $c\,\xi \in \mathcal{A}_C|_{\Gamma_i}$ (resp. $\in \mathcal{A}_{i,\mathfrak{q}}$) if and only if $v(c) \geq \max_k(-v(\xi_k))$; so with ϵ_t a generator of $\mathcal{O}(\mathrm{Ec}_{t,i})_\mathfrak{q}$ (Ts5),

$$\{c \in C :\ c\,\xi \in \mathcal{A}_C|_{\Gamma_i}\} = \mathcal{O}_C\,\epsilon_t. \tag{1}$$

By flatness, kernels and images commute with $\mathcal{O}_C \otimes_R -$. Hence $\mathcal{O}_C \otimes \mathcal{I}_{j,\mathfrak{q}} = \mathcal{I}_{\mathfrak{P},j,C}$ (kernel of restriction to Γ_j), $\mathcal{O}_C \otimes \mathcal{S}_{t,i,\mathfrak{q}} = \mathcal{S}_{t,i,C}$ ($\mathcal{S}_{t,i,\mathfrak{q}}$ is the kernel of $\mathcal{A}_\mathfrak{q} \to \mathcal{A}_{i,\mathfrak{q}}/R\epsilon_t\xi$, and (1)), and, as the step sheaves contain $p\,\omega_t$ and $\theta_{t,C} = \mathcal{O}_C \otimes \theta_{t,\mathfrak{q}}$, the $\mathcal{O}_C$-span of $\mathcal{U}_{t,j,\mathfrak{q}}$ (resp. $\mathcal{U}^{\mathrm{st}}_{t,i,\mathfrak{q}}$) in $\mathcal{O}_C \otimes \omega_{t,\mathfrak{q}}$ is the preimage of $\theta_{t,C}(\mathcal{I}_{\mathfrak{P},j,C})$ (resp. $\theta_{t,C}(\mathcal{S}_{t,i,C})$). Finally, (2): for a free R-submodule V of full rank of a finite free R-module F, $V = (\mathcal{O}_C \otimes V) \cap (F \otimes \Omega')$, since in a basis of V the coordinates of an element of the right side lie in $\mathcal{O}_C \cap \Omega' = R$.

Step 2: (onto). Let $\mathfrak{q}$ have codimension one. If p is a unit at $\mathfrak{q}$, $(\omega_t/p\,\omega_t)_{\mathfrak{q}} = 0$ and $\mathcal{U}_{t,i-1,\mathfrak{q}} = \mathcal{U}^{\mathrm{st}}_{t,i,\mathfrak{q}} = \omega_{t,\mathfrak{q}}$. Otherwise take a test point with (Ox4); by Step 1 the first identity of (Ox4) says that $\mathcal{U}_{t,i-1,\mathfrak{q}}$ and $\mathcal{U}^{\mathrm{st}}_{t,i,\mathfrak{q}}$ have the same $\mathcal{O}_C$-span, and both are free of full rank in $\omega_{t,\mathfrak{q}}$ (hypothesis 1; they contain $p\,\omega_{t,\mathfrak{q}}$), so they are equal by (2). By Step 0(a), $\mathcal{U}_{t,i-1} = \mathcal{U}^{\mathrm{st}}_{t,i}$; as both contain $p\,\omega_t$, $\theta_t(\mathcal{I}_{i-1}) = \theta_t(\mathcal{S}_{t,i})$, which is (onto).

Step 3: the step map. For a local section c of $\mathcal{O}(\mathrm{Ec}_{t,i})$, $c\,\xi$ is a local section of $\mathcal{A}_i$ (Ts5), so locally $c\,\xi = g|_{\Gamma_i}$ with $g \in \mathcal{A}$; $g \in \mathcal{I}_{i-1}$ as ξ vanishes on Γ_{i-1}; two choices differ by a section of $\mathcal{I}_i$. So $\psi(c) = \theta_t(g) \bmod \theta_t(\mathcal{I}_i)$ defines an $\mathcal{O}_{\mathcal{X}'}$-linear map $\psi\colon \mathcal{O}(\mathrm{Ec}_{t,i}) \to \mathcal{Q} := \mathcal{U}_{t,i-1}/\mathcal{U}_{t,i} \cong \theta_t(\mathcal{I}_{i-1})/\theta_t(\mathcal{I}_i)$. Let $\mathcal{K}$ be the subsheaf of Ω' of (inj) for (s,i). If $g|_{\Gamma_i} = c\,\xi$ with $g \in \mathcal{A}$, then $c \in \mathcal{O}(\mathrm{Ec}_{t,i})$ by (Ts5); and $\psi(c) = 0$ if and only if $\theta_t(g) = \theta_t(h)$ for some $h \in \mathcal{I}_i$, i.e. if and only if $g - h$ restricts to $c\,\xi$ and lies in $\ker\theta_t$. So $\mathcal{K} = \ker\psi$. The same argument over $\mathcal{O}_C$ at a test point, with (1), shows that $\mathcal{O}_C \otimes \psi_{\mathfrak{q}}\colon \mathcal{O}_C\epsilon_t \to \theta_{t,C}(\mathcal{I}_{\mathfrak{P},i-1,C})/\theta_{t,C}(\mathcal{I}_{\mathfrak{P},i,C})$ has kernel the left side of the second identity of (Ox4).

Step 4: (inj). $\mathcal{Q}$ is a quotient of locally free sheaves of the same rank (hypothesis 1) killed by p, so $\mathcal{Q}_\eta = 0$. Let c be a local section of Ω' over W with $c \in \mathcal{K}_{\mathfrak{q}}$ for all $\mathfrak{q} \in W$ of codimension one. By Step 0(a), $c \in \mathcal{O}(\mathrm{Ec}_{t,i})(W)$; $\psi(c)$ vanishes at every point of codimension ≤ 1, so $\psi(c) = 0$ by Step 0(b), and $c \in \mathcal{K}(W)$. As $\mathcal{O}(p\,\mathrm{Ec}_{s,i})$ (locally generated by ϵ_s^p, ϵ_s a local generator of $\mathcal{O}(\mathrm{Ec}_{s,i})$; it is the sheaf generated by the p-th powers of sections of $\mathcal{O}(\mathrm{Ec}_{s,i})$) is invertible, it suffices to show $\mathcal{K}_{\mathfrak{q}} = R_{\mathfrak{q}}\epsilon_s^p$ at every $\mathfrak{q}$ of codimension one. If p is a unit at $\mathfrak{q}$, $\mathcal{Q}_{\mathfrak{q}} = 0$ and $\mathcal{K}_{\mathfrak{q}} = \mathcal{O}(\mathrm{Ec}_{t,i})_{\mathfrak{q}} = R_{\mathfrak{q}} = \mathcal{O}(p\,\mathrm{Ec}_{s,i})_{\mathfrak{q}}$, the eigen-coordinate divisors being supported in $\mathcal{X}'_{\mathfrak{s}}$. Otherwise take a test point with (Ox4): by flatness and Step 3, $\mathcal{O}_C \otimes \mathcal{K}_{\mathfrak{q}} = \ker(\mathcal{O}_C \otimes \psi_{\mathfrak{q}}) = \mathcal{O}_C\epsilon_s^p \neq 0$. So $\mathcal{K}_{\mathfrak{q}} \subseteq R\epsilon_t$ is free of rank one, $\mathcal{K}_{\mathfrak{q}} = Rx$, and $x/\epsilon_s^p \in \mathcal{O}_C^\times \cap \Omega' = R^\times$. Hence $\mathcal{K} = \mathcal{O}(p\,\mathrm{Ec}_{s,i})$: (inj). □

Proposition 35.64. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p_0) with a geometric realisation and a good origin immersive at level four over an integral model $\mathcal{Y}$ of a unitary Shimura datum $\mathfrak{u} = (E, \mathsf{H}, \dots)$, $\Phi = \Phi_{\mathsf{H}}$, with E Galois over $\mathbb{Q}$ with abelian Galois group. Let $c \in \mathrm{Cp}$, $2 \le B_0 \le B$, $\mathcal{X} \subseteq \mathbb{P} = \prod_{k\in K}\mathbb{P}(\mathbb{Z}^{I_k})$ a joint model of the stages (B_0, c) and (B, c), $p \in S(B_0, B)$, and $x'_r = c_1(\mathcal{O}(e'_r))$ the second-root classes on $\mathcal{X}$. Then for every irreducible component W of $\mathcal{X}_{\mathbb{F}_p}$, every $a \in \Phi$, every integer k with $1 \le k \le D$ and all $e_1, \dots, e_{D-k} \in \mathbb{Z}^K$,*

$$\deg\Big(\sum_{i=1}^{n} x'^{k}_{a,i} \cdot c_1(\mathcal{O}(e_1)) \cdots c_1(\mathcal{O}(e_{D-k})) \cap [W]\Big) = 0$$

(degree over $\mathbb{F}_p$ on the projective $\mathbb{F}_p$-scheme W of dimension D).

Proof. Fix W, a, k and $e_1, \dots, e_{D-k}$, and put $M = c_1(\mathcal{O}(e_1)) \cdots c_1(\mathcal{O}(e_{D-k}))$. By Proposition 35.65 there are a full flag $0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_n = \mathcal{O}_W^n$ by subbundles and a permutation ϱ with $\mathcal{E}_i/\mathcal{E}_{i-1} \cong L_i := \mathcal{O}(e'_{(a,\varrho(i))})|_W$. The Chern character is additive along the exact sequences $0 \to \mathcal{E}_{i-1} \to \mathcal{E}_i \to L_i \to 0$, so as operators on $A_*(W)_{\mathbb{Q}}$

$$\mathrm{ch}(\mathcal{O}_W^n) = \sum_{i=1}^{n} \mathrm{ch}(L_i) = \sum_{i=1}^{n} e^{c_1(L_i)}.$$

The trivial bundle has total Chern class 1, so $\mathrm{ch}(\mathcal{O}_W^n) = n$, and comparing the parts of degree $k \ge 1$ gives $\sum_{i=1}^n c_1(L_i)^k = 0$. Since ϱ is a permutation and $c_1(L_i) \cap \alpha = c_1(\mathcal{O}(e'_{(a,\varrho(i))})) \cap \alpha$ for classes α on W,

$$\sum_{i=1}^{n} x'^{k}_{a,i} \cdot M \cap [W] = M \cap \Big(\sum_{i=1}^{n} c_1(L_i)^k \cap [W]\Big) = 0$$

in $A_0(W)_{\mathbb{Q}}$, and its degree vanishes. □

Proposition 35.65. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p_0) with a geometric realisation and a good origin immersive at level four over an integral model $\mathcal{Y}$ of a unitary Shimura datum $\mathfrak{u} = (E, \mathsf{H}, \dots)$, $\Phi = \Phi_{\mathsf{H}}$, with E Galois over $\mathbb{Q}$ with abelian Galois group. Let $c \in \mathrm{Cp}$, $2 \le B_0 \le B$, $\mathcal{X} \subseteq \mathbb{P} = \prod_{k \in K} \mathbb{P}(\mathbb{Z}^{I_k})$ a joint model of the stages (B_0, c) and (B, c), and $p \in S(B_0, B)$. Then for every irreducible component W of $\mathcal{X}_{\mathbb{F}_p}$, with its reduced structure, and every $a \in \Phi$, there are a full flag $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{O}_W^n$ of the trivial bundle of rank n on W by subbundles and a permutation ϱ of $\{1, \dots, n\}$ with*

$$\mathcal{E}_i / \mathcal{E}_{i-1} \cong \mathcal{O}(e'_{(a,\varrho(i))})\big|_W \qquad (1 \le i \le n).$$

Proof. A good origin is a tower origin, so $\mathcal{X}_1 = \mathcal{X}_{B,c}$ carries a root origin on the fixed data at c; and $p \in S(B_0, B)$ gives $p \le B$, so $p \in S_B$. Hence Proposition 35.66, applied to $\mathcal{X}_1$ with $\ell = p$ and $\tau = a$, gives a chain $0 = \mathcal{U}_{a,0} \subset \cdots \subset \mathcal{U}_{a,n} = \mathcal{O}^n_{\mathcal{X}_{1,p}}$ of locally free submodules with locally free quotients and $\mathcal{U}_{a,i}/\mathcal{U}_{a,i-1} \cong \mathcal{O}(e^1_{(a,i)})|_{\mathcal{X}_{1,p}}$, where $\mathcal{X}_{1,p} = \mathcal{X}_1 \times \operatorname{Spec} \mathbb{F}_p$. The projection pr_1 is the restriction of a projection of products of projective spaces over $\mathbb{Z}$, so it maps the reduced component W of $\mathcal{X}_{\mathbb{F}_p}$ into $\mathcal{X}_{1,p}$; let $h \colon W \to \mathcal{X}_{1,p}$ be the induced morphism. A chain of locally free submodules with locally free quotients is locally split, so its pull-back is again such a chain, with the pulled-back quotients: put $\mathcal{E}_i = h^* \mathcal{U}_{a,i}$. Then $\mathcal{E}_n = \mathcal{O}_W^n$ and

$$\mathcal{E}_i / \mathcal{E}_{i-1} \cong h^* \mathcal{O}(e^1_{(a,i)}) = \mathcal{O}(e'_{(a,i)})\big|_W,$$

because $\mathcal{O}(e'_r)$ is the pull-back of $\mathcal{O}(e^1_r)$ under the projection onto the factor $\mathbb{P}_1$ (Jm2). This is the assertion with $\varrho = \mathrm{id}$. □

Proposition 35.66. *Let $\mathfrak{u}$ be a unitary Shimura datum, $\mathcal{Y}$ an integral model of* Sh *over $\mathcal{O}_F$, and $\mathcal{T}$ a root tower of type (Φ, n, p) with a geometric realisation. Let $B \ge 2$ and $c \in \mathrm{Cp}$ be such that $\mathcal{X} = \mathcal{X}_{B,c}$ carries a root origin on fixed data at c. Let $\ell \in S_B$ and $\mathcal{X}_\ell = \mathcal{X} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_\ell$. Then for every $\tau \in \Phi$ there is a chain*

$$0 = \mathcal{U}_{\tau,0} \subset \mathcal{U}_{\tau,1} \subset \cdots \subset \mathcal{U}_{\tau,n} = \mathcal{O}^n_{\mathcal{X}_\ell}$$

of locally free $\mathcal{O}_{\mathcal{X}_\ell}$-submodules with locally free quotients, and isomorphisms

$$\mathcal{U}_{\tau,i}/\mathcal{U}_{\tau,i-1} \cong \mathcal{O}(e_{(\tau,i)})|_{\mathcal{X}_\ell} \qquad (1 \le i \le n).$$

Proof. Let Ω, $R = R(\ell) \ge 4$, $T = G_{\mathcal{X}}(\Omega)[\ell^R]$, ω, $\rho_s \colon \mathrm{Li}^\vee \to \mathrm{Li}^\vee_s$ (restriction) and ϖ_ℓ ($= \zeta_\ell - 1$, or 4 if $\ell = 2$) be as in Proposition 35.67 and Lemma 35.32. In both cases ϖ_ℓ divides ℓ^2 in $\mathcal{O}_x$ ($\ell = \mathrm{unit} \cdot \varpi_\ell^{\ell-1}$ for odd ℓ).

Step 1: T is free of rank n over $\mathcal{O}_E/\ell^R$. The generic point of $\mathcal{X}$ lies in the abelian locus. At a complex point z of the abelian locus, $T \cong \Lambda_z/\ell^R\Lambda_z$ as $\mathcal{O}_E$-modules (all ℓ^R-torsion is rational over Ω by (Go3)(2)), and Λ_z is an $\mathcal{O}_E$-lattice with $\Lambda_z \otimes \mathbb{Q}$ an E-space of dimension $2g/[E:\mathbb{Q}] = 2|\Phi|n/(2|\Phi|) = n$. The ring $\mathcal{O}_E \otimes \mathbb{Z}_\ell$ is a finite product of discrete valuation rings and $\Lambda_z \otimes \mathbb{Z}_\ell$ is finitely generated and torsion-free over it, of rank n on every factor; so it is free of rank n, and so is T over $\mathcal{O}_E/\ell^R$. A basis gives $T \otimes_{\mathcal{O}_E,\tau} \mathcal{O}_{\mathcal{X}_\ell} \cong \mathcal{O}^n_{\mathcal{X}_\ell}$.

Step 2: the map α_τ is well defined. (i) For $y \in T$ and a point x over ℓ, a Hodge–Tate value at x (it exists, Lemma 31.30(1)) is the differential along the unit section of a regular function φ on a neighbourhood $\mathcal{W}$ of the unit point over x with $\varphi(h) = e_{\ell^R}(y, h)$ for all geometric ℓ^R-torsion

points h; the condition does not depend on x, so $d\varphi$ is a section of ω near x whose germ at every nearby point over ℓ is a Hodge–Tate value. By the uniqueness modulo $\ell^R\omega_x$ in Lemma 31.30(1) these local sections glue in $\omega/\ell^R\omega$ (which vanishes off the fibre over ℓ) to $\bar w_y$; $y\mapsto\bar w_y$ is additive ($e_{\ell^R}(y+y',\cdot)=e_{\ell^R}(y,\cdot)e_{\ell^R}(y',\cdot)$ and $d(\varphi\varphi')/(\varphi\varphi')=d\varphi/\varphi+d\varphi'/\varphi'$ along the unit section, where $\varphi=\varphi'=1$). (ii) By (Go3)(2), ht_x is free of rank g at every x over ℓ, so Lemma 35.32(1) gives $\varpi_\ell\omega_x\subseteq\mathrm{ht}_x$, hence $\ell^R\rho_s(\omega_x)\subseteq\ell^{R-2}\rho_s(\mathrm{ht}_x)$. By (Go3)(2) again $\rho_{(\tau,-)}(\mathrm{ht}_x)$ is free of rank q, and $\mathcal V_{(\tau,-),x}=\mathrm{HT}'_{(\tau,-),x}$ is its dual, so $\rho_{(\tau,-)}(\mathrm{ht}_x)=\mathcal V^\vee_{(\tau,-),x}$. Hence $\rho_{(\tau,-)}(\bar w_y)$ is a well-defined section of $\mathcal V^\vee_{(\tau,-)}/\ell^{R-2}\mathcal V^\vee_{(\tau,-)}$ near $\mathcal X_\ell$, and its class modulo ℓ is a section of $\mathcal V^\vee_{(\tau,-)}|_{\mathcal X_\ell}$ ($R-2\ge1$). (iii) τ-linearity: by the Rosati condition $e_{\ell^R}(\iota(a)y,h)=e_{\ell^R}(y,\iota(\bar a)h)$, so $\varphi_{\iota(a)y}=\varphi_y\circ\iota(\bar a)$ and $\bar w_{\iota(a)y}=\iota(\bar a)^*\bar w_y$, where $\iota(\bar a)^*$ is the transpose of the action on Lie. On $\mathrm{Li}_{(\tau,-)}$, $\bar a$ acts by $\sigma_{(\tau,-)}(\bar a)=\bar\tau(\bar a)=\tau(a)$; so $\rho_{(\tau,-)}(\bar w_{\iota(a)y})=\tau(a)\rho_{(\tau,-)}(\bar w_y)$, and $y\otimes1\mapsto\rho_{(\tau,-)}(\bar w_y)\bmod\ell$ extends to an $\mathcal O_{\mathcal X_\ell}$-linear map $\alpha_\tau\colon T\otimes_{\mathcal O_E,\tau}\mathcal O_{\mathcal X_\ell}\to\mathcal V^\vee_{(\tau,-)}|_{\mathcal X_\ell}$.

Step 3: α_τ is surjective, with locally free kernel of rank p. At x over ℓ, $\mathrm{ht}_x=\sum_y\mathcal O_x w_y+\ell^R\omega_x$ (Lemma 31.30(1)), so

$$\rho_{(\tau,-)}(\mathrm{ht}_x)=\sum_y\mathcal O_x\rho_{(\tau,-)}(w_y)+\ell^R\rho_{(\tau,-)}(\omega_x)\subseteq\sum_y\mathcal O_x\rho_{(\tau,-)}(w_y)+\ell\,\rho_{(\tau,-)}(\mathrm{ht}_x)$$

by Step 2(ii); modulo ℓ the classes of the $\rho_{(\tau,-)}(w_y)$ generate. So α_τ is a surjection from a free $\mathcal O_{\mathcal X_\ell}$-module of rank n (Step 1) onto a locally free one of rank q; locally it splits, and its kernel $\mathcal K_\tau$ is locally free of rank $n-q=p$ with

$$0\to\mathcal K_\tau\to\mathcal O^n_{\mathcal X_\ell}\xrightarrow{\alpha_\tau}\mathcal V^\vee_{(\tau,-)}|_{\mathcal X_\ell}\to0$$

exact and locally split. By Proposition 35.67, $\mathcal K_\tau\cong\mathcal V_{(\tau,+)}|_{\mathcal X_\ell}$.

Step 4: the flag. Let $(\mathcal V_{s,k})_k$ be the integral flag of (Go3)(3); its step quotients are invertible, so the filtrations are locally split and restrict to full flags of $\mathcal V_s|_{\mathcal X_\ell}$ with step quotients $(\mathcal V_{s,k}/\mathcal V_{s,k-1})|_{\mathcal X_\ell}$. Fix an isomorphism $\theta\colon\mathcal V_{(\tau,+)}|_{\mathcal X_\ell}\to\mathcal K_\tau$. Put $\mathcal U_{\tau,i}=\theta(\mathcal V_{(\tau,+),i}|_{\mathcal X_\ell})$ for $0\le i\le p$, and for $p<i\le n$, with $j=i-p$, let $\mathcal U_{\tau,i}$ be the preimage under α_τ of $\big(\mathcal V_{(\tau,-)}/\mathcal V_{(\tau,-),q-j}\big)^\vee|_{\mathcal X_\ell}\subseteq\mathcal V^\vee_{(\tau,-)}|_{\mathcal X_\ell}$ (the dual flag). Then $\mathcal U_{\tau,p}=\mathcal K_\tau$, $\mathcal U_{\tau,n}=\mathcal O^n$, all $\mathcal U_{\tau,i}$ are locally free with locally free quotients (extensions of locally split filtrations), and

$$\mathcal U_{\tau,i}/\mathcal U_{\tau,i-1}\cong\begin{cases}(\mathcal V_{(\tau,+),i}/\mathcal V_{(\tau,+),i-1})|_{\mathcal X_\ell}, & i\le p,\\ \big(\mathcal V_{(\tau,-),q+1-j}/\mathcal V_{(\tau,-),q-j}\big)^\vee|_{\mathcal X_\ell}, & i=p+j>p.\end{cases}$$

For $i\le p$ the root (τ,i) has label $(\tau,+)$, step i, sign 1; for $i=p+j>p$ it has label $(\tau,-)$, step $n+1-i=q+1-j$, sign -1 (Fh3). So in both cases $\mathcal U_{\tau,i}/\mathcal U_{\tau,i-1}\cong\mathcal L_{(\tau,i)}|_{\mathcal X_\ell}$, and the root sections of (Go3)(3) give $\mathcal L_{(\tau,i)}\cong\mathcal O(e_{(\tau,i)})|_{\mathcal X}$. This is the asserted chain. □

Proposition 35.67. *Let $\mathfrak u$, $\mathcal Y$, $\mathcal T$, B, c and $\mathcal X=\mathcal X_{B,c}$ with a root origin on fixed data at c be as in Proposition 35.66; let Ω be the function field of $\mathcal X$, $\ell\in S_B$, $R=R(\ell)$, $T=G_{\mathcal X}(\Omega)[\ell^R]$ (an $\mathcal O_E/\ell^R$-module through ι), $\mathcal X_\ell=\mathcal X\times\operatorname{Spec}\mathbb F_\ell$, and $\tau\in\Phi$. For $y\in T$ let $\bar w_y$ be the global section of $\omega/\ell^R\omega$ (ω the dual of $\mathrm{Lie}(G_{\mathcal X})$) whose germ at every point over ℓ is the class of the Hodge–Tate values of y, and let $\alpha_\tau\colon T\otimes_{\mathcal O_E,\tau}\mathcal O_{\mathcal X_\ell}\to\mathcal V^\vee_{(\tau,-)}|_{\mathcal X_\ell}$ be the $\mathcal O_{\mathcal X_\ell}$-linear map $y\otimes1\mapsto$ the class of the restriction of $\bar w_y$ to $\mathrm{Li}_{(\tau,-)}$. Then the kernel of α_τ is isomorphic, as an $\mathcal O_{\mathcal X_\ell}$-module, to $\mathcal V_{(\tau,+)}|_{\mathcal X_\ell}$.*

Proof. Write $N=\ell^R$, $\varpi=\varpi_\ell$, $s_\pm=(\tau,\pm)$, $\langle w,f\rangle=w(f)$ for $w\in\mathrm{Li}^\vee$, $f\in\mathrm{Li}$, and $\mathcal V_\pm=\mathcal V_{s_\pm}$. Let $\mathcal S=\mathbb Z\smallsetminus\ell\mathbb Z$; every $s\in\mathcal S$ is a unit in $\mathcal O_x$ for x over ℓ ($as+b\ell=1$), so τ and $\bar\tau$ map $\mathcal S^{-1}\mathcal O_E$ into $\mathcal O_x$. By $\bar w_{\iota(a)y}=\iota(\bar a)^*\bar w_y$ (if ϕ is a function for y, then $\phi\circ\iota(\bar a)$ is one for $\iota(a)y$, by

the Rosati condition) and since $\iota(\bar a)$ acts on Li_{s_+} by $\tau(\bar a)=\bar\tau(a)$ and on Li_{s_-} by $\bar\tau(\bar a)=\tau(a)$: $\rho_{s_-}(\bar w_{\iota(a)y})=\tau(a)\rho_{s_-}(\bar w_y)$ and $\rho_{s_+}(\bar w_{\iota(a)y})=\bar\tau(a)\rho_{s_+}(\bar w_y)$.

Step 1: lattices at a point x over ℓ. (a) $\mathrm{HT}'_{s,x}=\{f\in\mathrm{Li}_s:\langle\rho_s(\mathrm{ht}_x),f\rangle\subseteq\mathcal{O}_x\}$ is the dual of the free module $\rho_s(\mathrm{ht}_x)$, so $\rho_s(\mathrm{ht}_x)=(\mathrm{HT}'_{s,x})^\vee$; in particular $\rho_{s_-}(\mathrm{ht}_x)=\mathcal{V}^\vee_{-,x}$. (b) By Lemma 35.32(1),(2): $\varpi\omega_x\subseteq\mathrm{ht}_x$ and $\varpi\mathrm{HT}'_{s,x}\subseteq\mathrm{dR}_{s,x}\subseteq\mathrm{Lie}_x$, so $\langle\omega_x,\mathrm{HT}'_{s,x}\rangle\subseteq\varpi^{-1}\mathcal{O}_x$ and $\rho_s(\omega_x)\subseteq\varpi^{-1}\rho_s(\mathrm{ht}_x)$. Since $R\geq 4$ and $\varpi\mid\ell^2$, $N\varpi^{-1}\in\ell^{R-2}\mathcal{O}_x\subseteq\ell\mathcal{O}_x$. (c) $\mathcal{S}^{-1}\mathcal{O}_E$ is a semilocal Dedekind domain, hence a principal ideal domain; choose $\delta_0\in\mathfrak{d}_E$ with $\delta_0\mathcal{S}^{-1}\mathcal{O}_E=\mathfrak{d}_E\mathcal{S}^{-1}\mathcal{O}_E$. Then $\mathcal{V}_{+,x}=\tau(\delta_0)^{-1}\mathrm{HT}'_{s_+,x}$: if $f\in\mathcal{V}_{+,x}$ then $\tau(\delta_0)f\in\mathrm{HT}'_{s_+,x}$ as $\delta_0\in\mathfrak{d}_E$; if $\tau(\delta_0)f\in\mathrm{HT}'_{s_+,x}$ and $d\in\mathfrak{d}_E$, then $d=\delta_0 a$ with $a\in\mathcal{S}^{-1}\mathcal{O}_E$ and $\tau(d)f=\tau(a)\tau(\delta_0)f\in\mathrm{HT}'_{s_+,x}$. So multiplication by $\tau(\delta_0)$ is an isomorphism $\mu\colon\mathcal{V}_{+,x}\to\mathrm{HT}'_{s_+,x}$.

Step 2: the normalised hermitian pairing. Let (ε_m) be a $\mathbb{Z}$-basis of $\mathcal{O}_E$ and $(\varepsilon_m^\vee)$ the trace-dual basis of $\mathfrak{d}_E^{-1}$. For every embedding σ of E, $\sum_m\sigma(\varepsilon_m)\sigma'(\varepsilon_m^\vee)=\delta_{\sigma\sigma'}$ (the matrices $(\sigma(\varepsilon_m))_{\sigma,m}$ and $(\sigma(\varepsilon_m^\vee))_{\sigma,m}$ satisfy $X^tY=1$, hence $YX^t=1$); so $\sum_m\varepsilon_m\varepsilon_m^\vee=1$. Let M be T_ℓ with e, or T with e_N read in $\mathbb{Z}/N$ through ζ_{ℓ^R} (the reduction of e); $A'=\mathbb{Z}_\ell$, resp. $\mathbb{Z}/N$. (i) M is free of rank n over $\mathcal{O}_E\otimes A'$: $H_1(G_{\mathcal{X}}(\mathbb{C}),\mathbb{Q})$ is an E-vector space of dimension $2g/[E:\mathbb{Q}]=n$, so $T_\ell\otimes\mathbb{Q}$ is free of rank n over $E\otimes\mathbb{Q}_\ell$; T_ℓ is finitely generated and torsion free over $\mathcal{O}_E\otimes\mathbb{Z}_\ell$, a finite product of discrete valuation rings, hence free of rank n; and $T=T_\ell/N$. (ii) Put $\langle y,y'\rangle_E=\sum_m e(\varepsilon_m y,y')\varepsilon_m^\vee\in\mathfrak{d}_E^{-1}\otimes A'$. As $z=\sum_m\mathrm{Tr}(z\varepsilon_m)\varepsilon_m^\vee$ for $z\in\mathfrak{d}_E^{-1}$, $\langle y,y'\rangle_E$ is the unique element with $\mathrm{Tr}(a\langle y,y'\rangle_E)=e(ay,y')$ for all $a\in\mathcal{O}_E$; with the Rosati condition this gives $\langle ay,y'\rangle_E=a\langle y,y'\rangle_E$, $\langle y,ay'\rangle_E=\bar a\langle y,y'\rangle_E$, and $\mathrm{Tr}\langle y,y'\rangle_E=e(y,y')$. Since $\delta_0\mathfrak{d}_E^{-1}\subseteq\mathcal{S}^{-1}\mathcal{O}_E$ and $\mathcal{S}^{-1}\mathcal{O}_E\otimes A'=\mathcal{O}_E\otimes A'$, $\delta_0\langle y,y'\rangle_E\in\mathcal{O}_E\otimes A'$. (iii) For an $\mathcal{O}_E\otimes A'$-basis (u_k) of M the matrix $\Gamma=(\delta_0\langle u_j,u_k\rangle_E)_{j,k}$ is invertible. Modulo ℓ: if $y'\in M$ satisfies $\delta_0\langle y,y'\rangle_E\in\ell(\mathcal{O}_E\otimes A')$ for all y, then $\langle y,y'\rangle_E\in\ell\mathfrak{d}_E^{-1}\otimes A'$, so $e(y,y')\in\ell A'$ for all y, and $y'\in\ell M$ because the Weil pairing on $M/\ell M=G_{\mathcal{X}}(\Omega)[\ell]$ is perfect. So the injective map $c\mapsto\Gamma c$ on $(\mathcal{O}_E/\ell)^n$ (in coordinates $y'=\sum c_ku_k$ the functional is $\Gamma\bar c$) is bijective, $\det\Gamma$ is a unit modulo ℓ, hence a unit of $\mathcal{O}_E\otimes A'$. Hence there are $u_k^\#\in M$ with $\delta_0\langle u_j,u_k^\#\rangle_E=\delta_{jk}$. (iv) Let Λ be a ring with a homomorphism $\mathcal{S}^{-1}\mathcal{O}_F\to\Lambda$ (an A'-algebra), $M_\tau=M\otimes_{\mathcal{O}_E,\tau}\Lambda$, $M_{\bar\tau}=M\otimes_{\mathcal{O}_E,\bar\tau}\Lambda$, and $P_\tau(y\otimes a,y'\otimes a')=aa'\tau(\delta_0\langle y,y'\rangle_E)$, a well-defined Λ-bilinear pairing $M_\tau\times M_{\bar\tau}\to\Lambda$ by (ii) ($\tau(\bar b)=\bar\tau(b)$). The $u_k\otimes 1$ form a basis of M_τ, and $t'\mapsto(P_\tau(u_j\otimes 1,t'))_j$ is a surjection $M_{\bar\tau}\to\Lambda^n$ (it sends $u_k^\#\otimes 1$ to the k-th unit vector) between free modules of rank n, hence bijective: P_τ is perfect and $(u_k^\#\otimes 1)$ is the dual basis. Let (y_j) be an A'-basis of M and (y_j') the basis with $e(y_j,y_k')=\delta_{jk}$. Then in $M_\tau\otimes_\Lambda M_{\bar\tau}$

$$\sum_j(y_j\otimes 1)\otimes(y_j'\otimes 1)=\tau(\delta_0)\sum_k(u_k\otimes 1)\otimes(u_k^\#\otimes 1).\tag{35.5}$$

Indeed $L(t\otimes t')(t'')=P_\tau(t'',t')\,t$ defines an isomorphism $L\colon M_\tau\otimes M_{\bar\tau}\to\mathrm{End}_\Lambda(M_\tau)$ (it sends $(u_k\otimes 1)\otimes(u_i^\#\otimes 1)$ to a matrix unit), L of the right side is $\tau(\delta_0)\,\mathrm{id}$, and for $y\in M$, using $z=\sum_j e(z,y_j')y_j$,

$$L\Big(\sum_j y_j\otimes y_j'\Big)(y\otimes 1)=\sum_m\tau(\delta_0\varepsilon_m^\vee)\sum_j e(\varepsilon_m y,y_j')(y_j\otimes 1)=\sum_m\tau(\delta_0\varepsilon_m^\vee\varepsilon_m)(y\otimes 1)=\tau(\delta_0)(y\otimes 1).$$

Fix an $\mathcal{O}_E\otimes\mathbb{Z}_\ell$-basis (u_k) of T_ℓ with $(u_k^\#)$ as in (iii), and let $b_k,b_k^\#\in T$ be their images; they are as in (iii) for T.

Step 3: the map θ. For x over ℓ and $f\in\mathcal{V}_{+,x}$ put

$$\theta(f)=\sum_k\langle w_{b_k^\#},\tau(\delta_0)f\rangle\,(b_k\otimes 1)\bmod\ell\ \in T\otimes_{\mathcal{O}_E,\tau}\mathcal{O}_x/\ell,$$

with w_y a Hodge–Tate value of y at x. The coefficient lies in $\mathcal{O}_x$ ($w_y \in \mathrm{ht}_x$, $\tau(\delta_0)f \in \mathrm{HT}'_{s_+,x}$) and does not depend modulo ℓ on the choice of w_y (Lemma 31.30(1): another choice differs by $N\omega_x$, and $\langle N\omega_x, \mathrm{HT}'_{s_+,x}\rangle \subseteq N\varpi^{-1}\mathcal{O}_x \subseteq \ell\mathcal{O}_x$ by Step 1(b)). On an open set on which the sections $\bar{w}_y$ lift to sections of ω, the lifts are Hodge–Tate values at all points over ℓ, and the formula is $\mathcal{O}$-linear in f; so θ is a morphism of $\mathcal{O}_{\mathcal{X}_\ell}$-modules $\mathcal{V}_+|_{\mathcal{X}_\ell} \to T \otimes_{\mathcal{O}_E,\tau} \mathcal{O}_{\mathcal{X}_\ell}$. It is a locally split injection: let $\alpha^+ \colon T \otimes_{\mathcal{O}_E,\bar{\tau}} \mathcal{O}_{\mathcal{X}_\ell} \to (\mathrm{HT}'_{s_+})^\vee|_{\mathcal{X}_\ell}$, $y \otimes 1 \mapsto \rho_{s_+}(w_y) \bmod \ell$ (well defined by the semilinearity above and $N\rho_{s_+}(\omega_x) \subseteq \ell\rho_{s_+}(\mathrm{ht}_x)$, Step 1); it is surjective, as $\rho_{s_+}(\mathrm{ht}_x) = \sum_y \mathcal{O}_x\rho_{s_+}(w_y) + N\rho_{s_+}(\omega_x)$ (Lemma 31.30(1)), so it is a locally split surjection of locally free modules and its transpose is a locally split injection. Both sides of $P_\tau(\theta(f), y' \otimes 1) = \langle w_{y'}, \tau(\delta_0)f\rangle \bmod \ell$ are $\bar{\tau}$-semilinear in $y' \in T$ and agree at $y' = b_j^\#$; so $P_\tau(\theta(\cdot),\cdot) = {}^t\alpha^+ \circ \mu$, and θ is the composite of the isomorphism μ (Step 1(c)), the transpose of α^+, and the inverse of the isomorphism $T_\tau \to (T_{\bar{\tau}})^\vee$ given by P_τ over $\Lambda = \mathcal{O}_{\mathcal{X}_\ell}$ (Step 2(iv)). Its image is locally a direct summand of rank p.

Step 4: $\alpha_\tau \circ \theta = 0$. For x over ℓ and $f \in \mathcal{V}_{+,x}$, $\alpha_\tau(\theta(f))$ is the class modulo ℓ of

$$c_x(f) = \sum_k \langle w_{b_k^\#}, \tau(\delta_0)f\rangle\, \rho_{s_-}(w_{b_k}) \in \rho_{s_-}(\mathrm{ht}_x) = \mathcal{V}^\vee_{-,x}.$$

We show $c_x(f) \in \ell\mathcal{V}^\vee_{-,x}$. As $\mathcal{V}^\vee_{-,x}$ is free and $\ell\mathcal{O}_x = \bigcap_{\mathfrak{q}} \ell(\mathcal{O}_x)_{\mathfrak{q}}$ over the primes of height one (Matsumura 11.5 applied to c/ℓ), it suffices to check this in $(\mathcal{O}_x)_{\mathfrak{q}}$; if $\ell \notin \mathfrak{q}$ there is nothing to show. If $\ell \in \mathfrak{q}$, then $(\mathcal{O}_x)_{\mathfrak{q}} = \mathcal{O}_q$ for a point q of codimension one over ℓ generising x, a discrete valuation ring ($\mathcal{X}$ is normal); $\mathcal{V}_{+,x} \subseteq \mathcal{V}_{+,q}$, $(\mathcal{V}^\vee_{-,x})_{\mathfrak{q}} = \mathcal{V}^\vee_{-,q}$, and a Hodge–Tate value at x is one at q (the same function, and $N\omega_x \subseteq N\omega_q$). So $c_x(f) = c_q(f)$ computed at q, and it suffices to show $c_q(f) \in \ell\mathcal{V}^\vee_{-,q}$ for $f \in \mathcal{V}_{+,q}$. By Lemma 35.39 if $G_{\mathcal{X}} \times \operatorname{Spec}\mathcal{O}_q$ is an abelian scheme, and by Proposition 35.41 otherwise, there is a Hodge–Tate isotropic lift β at q, with sign ϵ. By (Iz1) and the action of $\iota(\bar{a})^*$ on $\mathrm{Li}^\vee_{s_\pm}$, $(y \otimes a) \otimes (y' \otimes a') \mapsto aa'\rho_{s_-}(\beta y) \otimes \rho_{s_+}(\beta y')$ is a well-defined map on $(T_\ell)_\tau \otimes (T_\ell)_{\bar{\tau}}$ ($\Lambda = \mathcal{O}_C$) to $\rho_{s_-}(\omega_q) \otimes_{\mathcal{O}_q} \rho_{s_+}(\omega_q) \otimes_{\mathcal{O}_q} \mathcal{O}_C$, a free $\mathcal{O}_C$-module. Applied to (35.5) for $M = T_\ell$ it gives, by (Iz3), $0 = \tau(\delta_0)\sum_k \rho_{s_-}(\beta u_k) \otimes \rho_{s_+}(\beta u_k^\#)$, hence, as $\tau(\delta_0) \neq 0$,

$$\sum_k \rho_{s_-}(\beta u_k) \otimes \rho_{s_+}(\beta u_k^\#) = 0. \tag{35.6}$$

By (Iz2), $w_{b_k} = \epsilon\beta(u_k) + N\eta_k$ and $w_{b_k^\#} = \epsilon\beta(u_k^\#) + N\eta'_k$ with $\eta_k, \eta'_k \in \omega_q \otimes \mathcal{O}_C$; in particular $\beta(u_k) \in \mathrm{ht}_q \otimes \mathcal{O}_C$. Put $f' = \tau(\delta_0)f \in \mathrm{HT}'_{s_+,q}$. Contracting (35.6) with f' in the second factor,

$$c_q(f) = N\sum_k \Big(\epsilon\langle\eta'_k, f'\rangle\, \rho_{s_-}(\beta u_k) + \langle w_{b_k^\#}, f'\rangle\, \rho_{s_-}(\eta_k)\Big).$$

By Step 1(b), $\langle\eta'_k, f'\rangle \in \varpi^{-1}\mathcal{O}_C$, $\rho_{s_-}(\beta u_k) \in \rho_{s_-}(\mathrm{ht}_q) \otimes \mathcal{O}_C$, $\langle w_{b_k^\#}, f'\rangle \in \mathcal{O}_q$ and $\rho_{s_-}(\eta_k) \in \varpi^{-1}\rho_{s_-}(\mathrm{ht}_q) \otimes \mathcal{O}_C$. So $c_q(f) \in N\varpi^{-1}(\mathcal{V}^\vee_{-,q} \otimes \mathcal{O}_C) \cap \mathcal{V}^\vee_{-,q} = N\varpi^{-1}\mathcal{V}^\vee_{-,q}$ ($\mathcal{O}_q \to \mathcal{O}_C$ is faithfully flat and $N\varpi^{-1} \in \mathcal{O}_q$), which lies in $\ell\mathcal{V}^\vee_{-,q}$ by Step 1(b).

Step 5: the kernel. The module $T \otimes_{\mathcal{O}_E,\tau} \mathcal{O}_{\mathcal{X}_\ell}$ is free of rank n (Step 2(i)), and α_τ is surjective onto the locally free module $\mathcal{V}^\vee_-|_{\mathcal{X}_\ell}$ of rank q, since $\rho_{s_-}(\mathrm{ht}_x) = \sum_y \mathcal{O}_x\rho_{s_-}(w_y) + N\rho_{s_-}(\omega_x)$ and $N\rho_{s_-}(\omega_x) \subseteq \ell\rho_{s_-}(\mathrm{ht}_x)$. So $\mathcal{K}_\tau = \ker\alpha_\tau$ is locally a direct summand, locally free of rank $n-q = p$. By Steps 3 and 4, $\theta(\mathcal{V}_+|_{\mathcal{X}_\ell}) \subseteq \mathcal{K}_\tau$, and it is locally a direct summand of $T \otimes_\tau \mathcal{O}_{\mathcal{X}_\ell}$, hence of $\mathcal{K}_\tau$ (if $F' = A_1 \oplus C_1$ and $A_1 \subseteq K_1 \subseteq F'$ then $K_1 = A_1 \oplus (K_1 \cap C_1)$), with a complement that is locally free of rank $p - p = 0$, hence zero by Nakayama. So $\theta \colon \mathcal{V}_{(\tau,+)}|_{\mathcal{X}_\ell} \to \ker\alpha_\tau$ is an isomorphism. □

Part VII
The window Sha bound

Part VII proves the window Sha bound, Theorem 36.4: every primitive character satisfies the bound of Definition 7.1 with exponent 401. It is the second of the three inputs of Part I and is used once, in the proof of the existence theorem Proposition 7.8, through the supply of hosts Proposition 9.1. At a scale x it makes the field E of the chosen host satisfy $\mathrm{V}(E; \lfloor x^{401} \rfloor, \exp(x^{\varepsilon}/2))$: the lengths of the Tate–Shafarevich groups $\mathrm{Sha}^1_\ell(E,k)$ of the Tate twists, weighted by $\log \ell$ and summed over all primes ℓ and all $1 \le k < \lfloor x^{401} \rfloor$, are at most $\exp(x^{\varepsilon}/2)$. These lengths bound the defects of the lattices in the divisor package (Theorem 13.2). For the classes in $H^1(\mathfrak{o}_E[1/\ell], \mathbb{Q}_\ell/\mathbb{Z}_\ell(-k))$ that could not be removed by twisting, Alpöge asks for "a significant weakening of Vandiver" [4]. The theorem bounds lengths and asserts no vanishing; the comparison with Vandiver's conjecture is the note's, and nothing below uses it.

The bound is proved, as it is needed, on windows only: for every window pair at a large scale the theorem provides one range $[Y, Y^2]$ of scales. The restriction comes from the prime 2 and the finitely many primes of the bad set B_ψ. There the group at the twist k is small unless k is ℓ-adically close to one of finitely many exceptional points, and at those twists only a doubly exponential bound is available; Y is chosen so that the exceptional twists that occur are small (Section 39).

For odd ℓ and a CM field E, complex conjugation splits $\mathrm{Sha}^1_\ell(E,k)$ into a plus and a minus part (Definition 36.1), and each part is bounded by the strict Tate–Shafarevich groups $\mathrm{Sha}^1_{\ell N}(\chi, k)$ of the twisted characters $\chi\chi^k_{\mathrm{cyc}}$, with χ running over the Dirichlet characters of E of one parity (Lemma 36.8); here we follow the exposition of [2]. On the minus part, $\chi(-1) = (-1)^{k+1}$, the generalised Bernoulli numbers $B_{k+1,\chi}$ are non-zero and control the groups. On the plus part they vanish, and the groups are bounded by Eisenstein congruences, whose one-character form is proved in Part VIII. The characters of the plus part are sorted, with the prime ℓ, into the cases (I)–(V) and (Z) of Definition 36.3.

Section 36 states the theorem and proves it from four bounds. Section 37 bounds the minus parts by Kolyvagin systems of Gauss sums. Section 38 proves the counts for the plus parts from the one-character bounds and from congruence numbers. Section 39 supplies the windows at 2 and at the bad primes, and Section 40 the bound for second cohomology of Tate twists, uniform in the twist, on which the windows and one count rest.

36 The window Sha bound: characters and cases

This section states the window Sha bound, Theorem 36.4, and proves it from four bounds for the field E of a Sha-host $\mathfrak{h} = (a, q, d)$ at a scale x. A Sha-host records the arithmetic part of a host: its field is a CM field contained in $\mathbb{Q}(\zeta_{N_\mathfrak{h}})$, of degree dividing $2^{a+2} o_\psi$ (Lemma 36.6). The four bounds are the bound at the prime 2 (item 1 of Proposition 39.2); the bound for the minus parts (Proposition 37.2); the bound for the plus parts at the odd primes outside the bad set B_ψ (Proposition 36.5); and the bound for the plus parts at the primes of B_ψ (Proposition 36.10), which takes as input the layer bound of item 2 of Proposition 39.2. The bounds at odd primes are polynomial in $N_\mathfrak{h} t$, apart from a multiple of $\exp(x^{4\varepsilon/5})$ at the bad primes, and the bound at 2 is $\exp(x^{4\varepsilon/5})$. At the scale x one has $\log(N_\mathfrak{h} t) \le 2x^{\varepsilon/4}$, because $\log(qd) \le x^{\varepsilon/4}$ for a window pair and $t = \lfloor x^{401} \rfloor$, so the total is at most $\exp(x^{\varepsilon}/2)$.

The plus parts are reduced to characters by Lemma 36.7: the characters are those of $X_\mathfrak{h}(\mathcal{O}, k)$, trivial on $H_\mathfrak{h}$ with $\chi(-1) = (-1)^k$, and the reduction is exact for $\ell \notin B_\psi$, because every odd

prime dividing $[E:\mathbb{Q}]$ lies in B_ψ. The bad set consists of 3 and the odd prime factors of five integers attached to ψ, so it is finite and independent of the host. Each pair (ℓ,χ) falls into one of the cases of Definition 36.3. Case (I), $\ell \notin B_\psi$ and $(\ell-1)\nmid 2e$, is the main case. A prime $\ell \notin B_\psi$ outside case (I) satisfies $\ell \le 2e+1 < \min(q,d)$, since the host is big at t, and its characters are in case (Z) if trivial, where the group vanishes (Proposition 36.9), in case (II) if of even order and in case (III) if of odd order. At the primes of B_ψ the characters are in case (II), (IV) or (V). Accordingly Proposition 36.5 combines Count I with Counts II and III and the vanishing in case (Z), and Proposition 36.10 combines Counts II, IV and V; the counts are proved in Section 38.

Definition 36.1. Let E be a CM field with maximal totally real subfield E^+ and complex conjugation $c \in \mathrm{Gal}(E/E^+)$, let ℓ be an odd prime and $k \in \mathbb{Z}$. Let E_{S_ℓ} be the maximal extension of E unramified outside the places above ℓ, so that $G_{E,S_\ell} = \mathrm{Gal}(E_{S_\ell}/E)$; it is Galois over E^+. A **lift** of c is an element $\tilde{c} \in \mathrm{Gal}(E_{S_\ell}/E^+)$ restricting to c on E. A lift acts on continuous cocycles $f\colon G_{E,S_\ell} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell(-k)$ by

$$(\tilde{c}\cdot f)(g) = \chi_{\mathrm{cyc}}(\tilde{c})^{-k}\, f(\tilde{c}^{-1}g\tilde{c}),$$

and hence on $H^1(G_{E,S_\ell},\mathbb{Q}_\ell/\mathbb{Z}_\ell(-k))$. The **plus part** and the **minus part** of $\mathrm{Sha}^1_\ell(E,k)$ are

$$\mathrm{Sha}^{\pm}_\ell(E,k) = \{x \in \mathrm{Sha}^1_\ell(E,k) \;:\; \tilde{c}\cdot x = \pm(-1)^k x \text{ for every lift } \tilde{c} \text{ of } c\},$$

and $\mathrm{len}^{\pm}_\ell(E,k) \in \mathbb{N}\cup\{\infty\}$ are their lengths as $\mathbb{Z}_\ell$-modules.

For an integer $t \ge 1$ and $B \in \mathbb{R}$, the CM field E satisfies the **plus Sha bound** $\mathrm{V}^+(E;t,B)$ if $\mathrm{len}^+_\ell(E,k) < \infty$ for every odd prime ℓ and every $1 \le k < t$, and

$$\sum_{\ell \text{ odd}} \log \ell \sum_{k=1}^{t-1} \mathrm{len}^+_\ell(E,k) \le B;$$

the **minus Sha bound** $\mathrm{V}^-(E;t,B)$ is defined in the same way with len^-_ℓ. For a number field L, a prime ℓ (here $\ell = 2$ is allowed), $t \ge 1$ and $B \in \mathbb{R}$, the field L satisfies the **Sha bound at** ℓ, $\mathrm{V}_\ell(L;t,B)$, if $\mathrm{len}_\ell(L,k) < \infty$ for every $1 \le k < t$ and

$$\sum_{k=1}^{t-1} \mathrm{len}_\ell(L,k) \le B.$$

Definition 36.2. Let ℓ be a prime, $\mathcal{O}$ a coefficient ring, $N \ge 1$, χ a Dirichlet character modulo N with values in $\mathcal{O}^\times$ (viewed as a character of $G_\mathbb{Q}$), $e \ge 0$ and $n \ge 1$ integers. Put $\Psi_{\chi,e} = \chi\,\chi^e_{\mathrm{cyc}}$. The **strict Tate–Shafarevich group** of (χ,e) away from n is

$$\mathrm{Sha}^1_n(\chi,e) = \ker\Big(H^1(\mathbb{Q},\mathcal{O}^\vee(\Psi^{-1}_{\chi,e})) \longrightarrow \prod_{p\mid n} H^1(\mathbb{Q}_p,\mathcal{O}^\vee(\Psi^{-1}_{\chi,e})) \times \prod_{p} H^1(I_p,\mathcal{O}^\vee(\Psi^{-1}_{\chi,e}))\Big),$$

the classes that are unramified at every prime and locally trivial at the primes dividing n. Its **length** is $\mathrm{length}_\mathcal{O}\,\mathrm{Sha}^1_n(\chi,e) \in \mathbb{N}\cup\{\infty\}$, and the **normalised length**

$$\nu_\mathcal{O}(\chi,e) = \mathrm{length}_\mathcal{O}\,\mathrm{Sha}^1_{\ell N}(\chi,e)\,/\,e_\mathcal{O}$$

(taken with ℓN, and meaningful when the length is finite). For a subgroup $H \le (\mathbb{Z}/N\mathbb{Z})^\times$, $\mathcal{X}_\mathcal{O}(N,H)$ is the finite set of Dirichlet characters modulo N with values in $\mathcal{O}^\times$ whose kernel is exactly H.

Definition 36.3. Let $\psi \in \mathcal{P}$ have modulus N_ψ and order o_ψ. Its **auxiliary primes** are: $p_K(\psi)$, the least prime $p \geq 7$ with $p \equiv 3 \pmod 4$ and $\gcd(p, N_\psi) = 1$; $h^*(\psi)$, the class number of $\mathbb{Q}(\sqrt{-p_K(\psi)})$; and $p_1(\psi)$, the least prime p that is congruent to 1 modulo the odd part of o_ψ and prime to $2N_\psi\, p_K(\psi)$. The **bad set** B_ψ consists of 3 and the odd prime factors of $o_\psi N_\psi\, p_K(\psi)\, h^*(\psi)\, p_1(\psi)$.

A **Sha-host** of ψ is a triple $\mathfrak{h} = (a, q, d)$ with $a \geq 2$ and primes $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$, $\gcd(qd, N_\psi) = 1$. Its conductor is $N_\mathfrak{h} = \operatorname{lcm}(2^{a+2}, N_\psi)\, qd$, and $H_\mathfrak{h} \leq (\mathbb{Z}/N_\mathfrak{h}\mathbb{Z})^\times$ is the subgroup of the u with $u \equiv \pm 1 \pmod{2^{a+2}}$, $\psi(u) = 1$, and u a square modulo q and modulo d. The host is **big at** an integer $t \geq 1$ if $2t + 1 < \min(q, d)$ and each of $N_\psi, o_\psi, p_K(\psi), h^*(\psi), p_1(\psi)$ is smaller than $\min(q, d)$. For a coefficient ring $\mathcal{O}$ and $e \geq 0$, $X_\mathfrak{h}(\mathcal{O}, e)$ is the finite set of Dirichlet characters χ modulo $N_\mathfrak{h}$ with values in $\mathcal{O}^\times$ such that $H_\mathfrak{h} \subseteq \ker \chi$ and $\chi(-1) = (-1)^e$.

For a prime ℓ, write $\chi^{(\ell)}$ for the product of the part of χ of order prime to ℓ with the inverse of its ℓ-component. For $e \geq 1$ and $\chi \in X_\mathfrak{h}(\mathcal{O}, e)$ the **cases** are:

(I) $\ell \notin B_\psi$ and $(\ell - 1) \nmid 2e$;

(II) $e \geq 2$, the order of $\chi^{(\ell)}$ is even, and $\ell \in B_\psi$ or $(\ell - 1) \mid 2e$;

(III) $\ell \notin B_\psi$, $(\ell - 1) \mid 2e$, $\chi \neq 1$ and the order of χ is odd;

(IV) $\ell \in B_\psi$, $e \geq 2$ and the order of $\chi^{(\ell)}$ is odd;

(V) $\ell \in B_\psi$ and $e = 1$;

(Z) $\ell \notin B_\psi$, $(\ell - 1) \mid 2e$ and $\chi = 1$.

Case (V) is split by the quantity $R_1(\ell, \chi) = 16\, o_\psi(\ell N_\psi)^2\, 4^{n_0(\ell,\chi)}$ against $2^{c_2(\chi)}$, where $c_2(\chi)$ is the exponent of 2 in the conductor of the 2-component of χ, and $n_0(\ell, \chi)$ is the 2-adic valuation of $\ell^f - 1$ for f the order of ℓ modulo $\operatorname{lcm}(4, \operatorname{ord}\vartheta)$, with ϑ the part of χ of order prime to ℓ multiplied by the inverse of its 2-component.

Theorem 36.4. *Every $\psi \in \mathcal{P}$ satisfies the window Sha bound with exponent* 401.

Proof. Fix $\psi \in \mathcal{P}$, $\varepsilon \in (0, 1/52)$ and a page function e_ε. We find $J \geq 1$ and X_0 as required by the window Sha bound with exponent 401.

Plus and minus parts. Let E be a CM field, ℓ an odd prime and $k \in \mathbb{Z}$; we show $\operatorname{len}_\ell(E, k) = \operatorname{len}_\ell^+(E, k) + \operatorname{len}_\ell^-(E, k)$. The formula of Definition 36.1 defines an action of $\operatorname{Gal}(E_{S_\ell}/E^+)$ on $H^1(G_{E,S_\ell}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-k))$, and the subgroup G_{E,S_ℓ} acts trivially, since conjugation by an element of a profinite group acts trivially on its cohomology (Serre, *Local fields*, Ch. VII, §5). Hence all lifts of c induce one automorphism ι, and $\iota^2 = 1$ because the square of a lift lies in G_{E,S_ℓ}. The lift $\tilde{c}$ permutes the places of E above ℓ and carries the decomposition group of a place w of E_{S_ℓ} to that of $\tilde{c}(w)$; so ι maps classes that are locally trivial at all places above ℓ to such classes and preserves $\operatorname{Sha}^1_\ell(E, k)$. Thus $\operatorname{Sha}^\pm_\ell(E, k)$ are the eigenspaces of ι on $\operatorname{Sha}^1_\ell(E, k)$ for $\pm(-1)^k$. Since ℓ is odd, 2 is invertible on the ℓ-primary group $\operatorname{Sha}^1_\ell(E, k)$; writing $x = \frac{1}{2}(x + (-1)^k \iota x) + \frac{1}{2}(x - (-1)^k \iota x)$ and noting that $\iota x = (-1)^k x = -(-1)^k x$ forces $2x = 0$, we get $\operatorname{Sha}^1_\ell(E, k) = \operatorname{Sha}^+_\ell(E, k) \oplus \operatorname{Sha}^-_\ell(E, k)$, and lengths add.

Constants. All the following depend only on $(\psi, \varepsilon, e_\varepsilon)$.

- $J \geq 1$, X_1 and integers $j_\ell \geq 0$ ($\ell \in B_\psi$): given by Proposition 39.2 for $(\psi, \varepsilon, e_\varepsilon)$.
- C_-: given by Proposition 37.2 for ψ; put $K_- = 3 + |C_-| \sum_{\ell \mid o_\psi} \ell^3$.

- C_0, r_1, r_2: given by Proposition 36.5 for ψ; and for $\ell \in B_\psi$, C_ℓ: given by Proposition 36.10 for (ψ, ℓ, j_ℓ). Put $D = \max(C_0, \max_{\ell \in B_\psi} C_\ell)$ and $b = |B_\psi|$.
- $P = \max\bigl(N_\psi, o_\psi, p_K(\psi), h^*(\psi), p_1(\psi), r_1, r_2\bigr)$.
- $Y_0 \geq e$ such that $Y^{\varepsilon/5}(1+\log^2 Y) \leq Y^{\varepsilon/4}$ for $Y \geq Y_0$ (possible since $\log^2 Y = o(Y^{\varepsilon/20})$).
- $x_0 \geq 2$ such that for $x \geq x_0$: $\log(8N_\psi)+404\log x \leq x^{\varepsilon/4}$, and each of $(\log 2+bD)\exp(x^{4\varepsilon/5})$, $K_-\exp(8x^{\varepsilon/4})$, $(b+1)\exp(2Dx^{\varepsilon/4})$ is at most $\frac{1}{3}\exp(x^\varepsilon/2)$. This is possible because $\varepsilon/4 < 4\varepsilon/5 < \varepsilon$, so $x^\varepsilon/2 - x^{4\varepsilon/5} \to \infty$ and $x^\varepsilon/2 - \max(8,2D)x^{\varepsilon/4} \to \infty$.
- $X_0 \geq \max(X_1, 1)$ such that for $X \geq X_0$: $X^{1/J} \geq \max(Y_0, x_0)$ and $\exp\bigl(\frac{1}{2}X^{\varepsilon/(5J)}\bigr) > \max\bigl(2X^{401}+1,\ P\bigr)$.

The scale Y. Let $X \geq X_0$ and let (q,d) be a window pair at scale $W = X^{1/J}$. By definition $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ are primes; $\log q > \frac{1}{2}W^{\varepsilon/5}$ and $\log d > \frac{1}{2}W^{\varepsilon/5}\log^2 W \geq \frac{1}{2}W^{\varepsilon/5}$ (as $W \geq e$), so $\min(q,d) > \max(2X^{401}+1, P)$; in particular $q, d > N_\psi$ and $\gcd(qd, N_\psi) = 1$. Also $\log(qd) \leq W^{\varepsilon/5}(1+\log^2 W) \leq W^{\varepsilon/4} = X^{\varepsilon/(4J)}$ since $W \geq Y_0$. So Proposition 39.2 gives Y with $X^{1/J} \leq Y$, $Y^2 \leq X$ and its items 1, 2 for every $x \in [Y, Y^2]$. We show that this Y has the required property.

The host at a scale x. Let $x \in [Y, Y^2]$, put $a = e_\varepsilon(x)$, $t = \lfloor x^{401} \rfloor$, $B = \exp(x^{4\varepsilon/5})$, and let E be a (ψ, a, q, d)-field. Then $x \geq W \geq x_0 \geq 2$, so $t \geq 2$ and $2^a \geq x^{2+8\varepsilon} \geq 4$, whence $a \geq 2$; therefore $\mathfrak{h} = (a, q, d)$ is a Sha-host of ψ. It is big at t: $2t+1 \leq 2x^{401}+1 \leq 2X^{401}+1 < \min(q,d)$ because $x \leq Y^2 \leq X$, and $N_\psi, o_\psi, p_K(\psi), h^*(\psi), p_1(\psi) \leq P < \min(q,d)$; moreover $r_1, r_2 < \min(q,d)$. By Lemma 36.6, E is a CM field, embeds into $\mathbb{Q}(\zeta_{N_\mathfrak{h}})$ and $n = [E:\mathbb{Q}] \leq N_\mathfrak{h}$; hence $\mathrm{cond}(E) \leq N_\mathfrak{h}$.

The four bounds.

1. The prime 2: by item 1 of Proposition 39.2, $\mathrm{V}_2(E; t, B)$.
2. Minus parts: by Proposition 37.2 ($a \geq 2$, $t \geq 2$, E CM), $\mathrm{V}^-(E; t, \beta)$ with
$$\beta = 3nt^2\log(t\,\mathrm{cond}(E)) + C_- \sum_{\ell \mid o_\psi,\ \ell \text{ odd}} \ell n^2 t^2 \log^2(t\ell\,\mathrm{cond}(E)).$$
3. Plus parts outside B_ψ: by Proposition 36.5, for every finite set S of odd primes not in B_ψ, $\mathrm{len}^+_\ell(E,k) < \infty$ ($\ell \in S$, $1 \leq k < t$) and $\sum_{\ell\in S}\log\ell\sum_{k<t}\mathrm{len}^+_\ell(E,k) \leq (N_\mathfrak{h}t)^{C_0}$.
4. Plus parts at $\ell \in B_\psi$: by item 2 of Proposition 39.2, $\mathrm{V}_\ell(\mathbb{Q}(\zeta_{\ell^{j_\ell+1}N_\psi}); t, B)$ with $B \geq 0$; so Proposition 36.10 gives $\mathrm{len}^+_\ell(E,k) < \infty$ for $1 \leq k < t$ and $\log\ell\sum_{k<t}\mathrm{len}^+_\ell(E,k) \leq (N_\mathfrak{h}t)^{C_\ell} + C_\ell B$.

By the splitting above, $\mathrm{len}_\ell(E,k) < \infty$ for every prime ℓ and $1 \leq k < t$, and for every finite set S of primes, with $U = N_\mathfrak{h}t \geq 2$,

$$\sum_{\ell\in S}\log\ell\sum_{k=1}^{t-1}\mathrm{len}_\ell(E,k) \leq B\log 2 + \beta + U^{C_0} + \sum_{\ell\in B_\psi}\bigl(U^{C_\ell} + C_\ell B\bigr) \leq (\log 2 + bD)B + \beta + (b+1)U^D.$$

As the terms are non-negative, the same bound holds for the sum over all primes.

Estimates at the scale x. Since $2^a \leq 2x^{2+8\varepsilon} \leq 2x^3$, $N_\mathfrak{h} \leq 2^{a+2}N_\psi qd$, $t \leq x^{401}$, and $\log(qd) \leq W^{\varepsilon/4} \leq x^{\varepsilon/4}$,
$$\log U \leq \log(8N_\psi) + 3\log x + \log(qd) + 401\log x \leq 2x^{\varepsilon/4},$$

by the choice of x_0. Put $w = x^{\varepsilon/4}$, so $U \le e^{2w}$. As $n \le N_{\mathfrak{h}}$, $\operatorname{cond}(E) \le N_{\mathfrak{h}}$ and $\log y \le y$ for $y \ge 1$:

$$\beta \le K_- \exp(8x^{\varepsilon/4}) \quad \text{and} \quad U^D \le \exp(2Dx^{\varepsilon/4}).$$

By the choice of x_0 the total is at most $\exp(x^{\varepsilon}/2)$. Hence E satisfies $\mathrm{V}(E; \lfloor x^{401} \rfloor, \exp(x^{\varepsilon}/2))$, for every $x \in [Y, Y^2]$ and every $(\psi, e_\varepsilon(x), q, d)$-field E. This is the window Sha bound with exponent 401 for ψ, with the constants J and X_0. □

Proposition 36.5. *Let $\psi \in \mathcal{P}$. There are an integer $C \ge 0$ and primes r_1, r_2 such that for every Sha-host $\mathfrak{h} = (a, q, d)$ of ψ with $r_1, r_2 < \min(q, d)$, every integer $t \ge 2$ at which $\mathfrak{h}$ is big, every (ψ, a, q, d)-field E and every finite set S of odd primes not in B_ψ: $\operatorname{len}_\ell^+(E, k) < \infty$ for all $\ell \in S$ and $1 \le k < t$, and*

$$\sum_{\ell \in S} \log \ell \sum_{k=1}^{t-1} \operatorname{len}_\ell^+(E, k) \le (N_{\mathfrak{h}} t)^C.$$

Proof. Let C_1, r_1, r_2 be given by Proposition 38.2 for ψ and K by Proposition 38.26 for ψ. Put $C = C_1 + 3K + 4$. Fix $\mathfrak{h}$, t, E, S as in the statement, write $N = N_{\mathfrak{h}}$, $H = H_{\mathfrak{h}}$ and $M = Nt$. Since $16 \mid N$ and $t \ge 2$, we have $M \ge 2$, $t \le M$ and $N \le M$. By Lemma 36.6, E is a CM field isomorphic to the fixed field of H in $\mathbb{Q}(\zeta_N)$, and $[E : \mathbb{Q}]$ divides $2^{a+2} o_\psi$. Since every odd prime factor of o_ψ lies in B_ψ, every $\ell \in S$ satisfies $\ell \nmid [E : \mathbb{Q}]$, and likewise ℓ does not divide the exponent of $(\mathbb{Z}/N\mathbb{Z})^\times / H$, so $\ell \nmid \operatorname{ord} \chi$ for every character χ with $H \subseteq \ker \chi$.

Reduction to characters. For each $\ell \in S$ let $\mathcal{O}_\ell$ be the ring of integers of $\mathbb{Q}_\ell(\zeta_{\varphi(N)})$ (a coefficient ring containing a primitive $\varphi(N)$-th root of unity); for the other primes choose any coefficient ring. For $\ell \in S$ and $1 \le k < t$, Lemma 36.7 (second inequality; $N \ge 2$, E is CM, $\ell \nmid [E : \mathbb{Q}]$) gives, since its set X is $X_{\mathfrak{h}}(\mathcal{O}_\ell, k)$,

$$\operatorname{len}_\ell^+(E, k) \le \sum_{\chi \in X_{\mathfrak{h}}(\mathcal{O}_\ell, k)} \nu_{\mathcal{O}_\ell}(\chi, k), \tag{36.1}$$

as soon as every term on the right is finite. We show that every term is finite and bound the right-hand side summed over ℓ.

The primes outside case (I). Let $1 \le e < t$, let $\ell \in S$ be not in case (I) for e, and let $\chi \in X_{\mathfrak{h}}(\mathcal{O}_\ell, e)$. Since $\ell \notin B_\psi$, we have $(\ell - 1) \mid 2e$, so $\ell \le 2e + 1 < 2t$; as $\mathfrak{h}$ is big at t, $\ell < 2t + 1 < \min(q, d)$, so $\ell \ne q, d$, and $\ell \nmid N_\psi$ (the odd prime factors of N_ψ lie in B_ψ); hence $\ell \nmid N$. Also $e \ge 2$: otherwise $(\ell - 1) \mid 2$ forces $\ell = 3 \in B_\psi$. We show that χ is in case (Z), (II) or (III).

- If $\chi = \mathbf{1}$, then χ is in case (Z); $\chi(-1) = 1 = (-1)^e$, so e is even, and $\ell \ge 5$ since $3 \in B_\psi$. Every class of $\mathrm{Sha}^1_{\ell N}(\mathbf{1}, e)$ satisfies the conditions defining $\mathrm{Sha}^1_\ell(\mathbf{1}, e)$ (the trivial character modulo N and modulo 1 give the same Galois character), so $\mathrm{Sha}^1_{\ell N}(\mathbf{1}, e) \subseteq \mathrm{Sha}^1_\ell(\mathbf{1}, e) = 0$ by Proposition 36.9, and $\nu_{\mathcal{O}_\ell}(\chi, e) = 0$.
- If $\chi \ne \mathbf{1}$ has odd order, χ is in case (III).
- If χ has even order: since $\ell \nmid \operatorname{ord} \chi$, the part of χ of order prime to ℓ is χ, and since $\ell \nmid N$ its component at ℓ is trivial; so $\chi^{(\ell)} = \chi$ has even order and χ is in case (II).

In cases (II) and (III), Proposition 38.26 gives that $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length and $\nu_{\mathcal{O}_\ell}(\chi, e) \log \ell \le (\ell N e)^K$. Since $1 \le \ell N e \le 2t \cdot N \cdot t = 2Mt \le M^3$ (as $2 \le M$ and $t \le M$), in all three cases

$$\nu_{\mathcal{O}_\ell}(\chi, e) \log \ell \le M^{3K}. \tag{36.2}$$

The sum for one e. Fix $1 \le e < t$ and split $S = \Lambda_e \cup \Lambda_e'$, with Λ_e the primes of S in case (I) for e. Since $r_1, r_2 < \min(q,d)$ and $\mathfrak{h}$ is big at t, Proposition 38.2 applied to Λ_e and the rings $\mathcal{O}_\ell$ gives finiteness of $\mathrm{Sha}^1_{\ell N}(\chi, e)$ for $\ell \in \Lambda_e$, $\chi \in X_{\mathfrak{h}}(\mathcal{O}_\ell, e)$, and

$$\sum_{\ell \in \Lambda_e} \log \ell \sum_{\chi \in X_{\mathfrak{h}}(\mathcal{O}_\ell, e)} \nu_{\mathcal{O}_\ell}(\chi, e) \le M^{C_1}.$$

The primes of Λ_e' are at most $2t$ in number (each is $< 2t$, as shown above), and each $X_{\mathfrak{h}}(\mathcal{O}_\ell, e)$ has at most $\varphi(N) \le N$ elements (a finite abelian group G has at most $|G|$ homomorphisms into the unit group of a domain). With (36.2),

$$\sum_{\ell \in \Lambda_e'} \log \ell \sum_{\chi \in X_{\mathfrak{h}}(\mathcal{O}_\ell, e)} \nu_{\mathcal{O}_\ell}(\chi, e) \le 2t \cdot N \cdot M^{3K} = 2M^{3K+1}.$$

Conclusion. All terms on the right of (36.1) are finite, so $\mathrm{len}^+_\ell(E,k) < \infty$ for $\ell \in S$, $1 \le k < t$. Multiplying (36.1) by $\log \ell \ge 0$, summing over $\ell \in S$ and $1 \le k < t$, and exchanging the two finite sums,

$$\sum_{\ell \in S} \log \ell \sum_{k=1}^{t-1} \mathrm{len}^+_\ell(E,k) \le \sum_{e=1}^{t-1} (M^{C_1} + 2M^{3K+1}) \le t(M^{C_1} + 2M^{3K+1}) \le M^{C_1+3K+4}.$$

This is the claim, since $M^C = (N_{\mathfrak{h}} t)^C$. □

Lemma 36.6. *Let $\psi \in \mathcal{P}$, let $\mathfrak{h} = (a,q,d)$ be a Sha-host of ψ and let E be a (ψ, a, q, d)-field. Then:*

1. *E is isomorphic to the subfield of $\mathbb{Q}(\zeta_{N_{\mathfrak{h}}})$ fixed by the automorphisms $\zeta_{N_{\mathfrak{h}}} \mapsto \zeta_{N_{\mathfrak{h}}}^u$, $u \in H_{\mathfrak{h}}$; in particular E embeds into $\mathbb{Q}(\zeta_{N_{\mathfrak{h}}})$;*
2. *E is a CM field;*
3. *$[E:\mathbb{Q}] = [(\mathbb{Z}/N_{\mathfrak{h}}\mathbb{Z})^\times : H_{\mathfrak{h}}]$, this index divides $2^{a+2} o_\psi$, and $[E:\mathbb{Q}] \le N_{\mathfrak{h}}$; the exponent of $(\mathbb{Z}/N_{\mathfrak{h}}\mathbb{Z})^\times / H_{\mathfrak{h}}$ divides $2^{a+2} o_\psi$ as well.*

Proof. Put $N = N_{\mathfrak{h}} = \mathrm{lcm}(2^{a+2}, N_\psi)\, qd$ and $H = H_{\mathfrak{h}}$. Since $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod 8$, the primes q, d are distinct and odd, and $\gcd(qd, N_\psi) = 1$; hence $N = \mathrm{lcm}(2^{a+2}, N_\psi, q, d)$.

Item 1. By definition there are N' divisible by 2^{a+2}, N_ψ, q, d (hence by N) and an embedding $j\colon E \hookrightarrow \mathbb{Q}(\zeta_{N'})$ whose image is the fixed field of the set Σ of σ_b, $b \in (\mathbb{Z}/N'\mathbb{Z})^\times$, with $b \equiv \pm 1 \pmod{2^{a+2}}$, $\psi(b) = 1$ and $(b/q) = (b/d) = 1$. Each of these four conditions depends only on b modulo 2^{a+2}, N_ψ, q, d respectively, hence only on $b \bmod N$. In particular Σ is a subgroup and contains $\mathrm{Gal}(\mathbb{Q}(\zeta_{N'})/\mathbb{Q}(\zeta_N)) = \{\sigma_b : b \equiv 1 \ (N)\}$, so $j(E) \subseteq \mathbb{Q}(\zeta_N)$, and $j(E)$ is the fixed field in $\mathbb{Q}(\zeta_N)$ of the image of Σ, which is $\{\sigma_u : u \in (\mathbb{Z}/N\mathbb{Z})^\times,\ u \equiv \pm 1 \ (2^{a+2}),\ \psi(u) = 1,\ (u/q) = (u/d) = 1\}$. For the odd primes q, d the condition $(u/q) = 1$ says that u is a square modulo q (and likewise for d), so this group is $\{\sigma_u : u \in H\}$.

Item 3. By item 1 and Galois theory, $[E:\mathbb{Q}] = [(\mathbb{Z}/N\mathbb{Z})^\times : H] \le \varphi(N) \le N$. The homomorphism

$$(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/2^{a+2}\mathbb{Z})^\times / \{\pm 1\} \times \psi((\mathbb{Z}/N_\psi\mathbb{Z})^\times) \times \{\pm 1\} \times \{\pm 1\}, \qquad u \mapsto \big(u,\ \psi(u),\ (u/q),\ (u/d)\big),$$

has kernel H, and its target is a finite abelian group of order $2^a \cdot o_\psi \cdot 4$ and exponent dividing $2^a \cdot o_\psi \cdot 2$ (the image of ψ is cyclic of order o_ψ). So $(\mathbb{Z}/N\mathbb{Z})^\times / H$ embeds into it, and its order and its exponent divide $2^{a+2} o_\psi$.

Item 2. Identify E with $j(E) \subseteq \mathbb{Q}(\zeta_N)$. For every embedding $\mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$, complex conjugation induces σ_{-1}, which is central in the abelian group $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$; hence it restricts to the same automorphism $c = \sigma_{-1}|_E$ of E for every embedding of E. Since $q \equiv 3 \pmod 4$, $(-1/q) = -1$, so $-1 \notin H$ and, by item 1 and Galois theory, $c \neq \mathrm{id}_E$. The fixed field $E^+ = E^{\langle c \rangle}$ is contained in $\mathbb{Q}(\zeta_N)^{\langle \sigma_{-1} \rangle} = \mathbb{Q}(\zeta_N + \zeta_N^{-1})$, which is totally real, so E^+ is totally real; $[E : E^+] = 2$; and no embedding of E into $\mathbb{C}$ is real, since complex conjugation acts on its image as $c \neq \mathrm{id}$. Thus E is a totally imaginary quadratic extension of a totally real field, that is, a CM field. □

Lemma 36.7. *Let $N \geq 2$, let $H \leq (\mathbb{Z}/N\mathbb{Z})^\times$ be a subgroup and let E be a number field isomorphic to the subfield of $\mathbb{Q}(\zeta_N)$ fixed by the automorphisms $\zeta_N \mapsto \zeta_N^a$, $a \in H$; assume that E is a CM field. Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring containing a primitive $\varphi(N)$-th root of unity, and $k \geq 1$. Let X be the set of Dirichlet characters χ modulo N with values in $\mathcal{O}^\times$ such that $H \subseteq \ker \chi$ and $\chi(-1) = (-1)^k$. Then, in $\mathbb{N} \cup \{\infty\}$,*

$$e_{\mathcal{O}} \operatorname{len}_\ell^+(E,k) \leq \sum_{\chi \in X} \Big(\operatorname{length}_{\mathcal{O}} \operatorname{Sha}^1_{\ell N}(\chi, k) + e_{\mathcal{O}}(Nk)^6 \Big),$$

and if $\ell \nmid [E : \mathbb{Q}]$, then

$$e_{\mathcal{O}} \operatorname{len}_\ell^+(E,k) \leq \sum_{\chi \in X} \operatorname{length}_{\mathcal{O}} \operatorname{Sha}^1_{\ell N}(\chi, k).$$

Proof. This is the case of the sign $+$ in Lemma 36.8: the hypotheses are the same, and the set X is the set X^+ of that lemma, the characters χ modulo N with values in $\mathcal{O}^\times$, $H \subseteq \ker \chi$ and $\chi(-1) = (-1)^k$. □

Lemma 36.8. *Let $N \geq 2$, let $H \leq (\mathbb{Z}/N\mathbb{Z})^\times$ be a subgroup and let E be a number field isomorphic to the subfield of $\mathbb{Q}(\zeta_N)$ fixed by the automorphisms $\zeta_N \mapsto \zeta_N^a$, $a \in H$; assume that E is a CM field. Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring containing a primitive $\varphi(N)$-th root of unity, $k \geq 1$, and fix a sign $\pm$. Let $X^\pm$ be the set of Dirichlet characters χ modulo N with values in $\mathcal{O}^\times$ such that $H \subseteq \ker \chi$ and $\chi(-1) = \pm(-1)^k$. Then, in $\mathbb{N} \cup \{\infty\}$,*

$$e_{\mathcal{O}} \operatorname{len}_\ell^\pm(E,k) \leq \sum_{\chi \in X^\pm} \Big(\operatorname{length}_{\mathcal{O}} \operatorname{Sha}^1_{\ell N}(\chi, k) + e_{\mathcal{O}}(Nk)^6 \Big),$$

and if $\ell \nmid [E : \mathbb{Q}]$, then

$$e_{\mathcal{O}} \operatorname{len}_\ell^\pm(E,k) \leq \sum_{\chi \in X^\pm} \operatorname{length}_{\mathcal{O}} \operatorname{Sha}^1_{\ell N}(\chi, k).$$

Proof. Set-up. Both sides are unchanged if E is replaced by an isomorphic field, so let $E \subseteq \mathbb{Q}(\zeta_N)$ be the fixed field of $\{\sigma_u : u \in H\}$, $\sigma_u(\zeta_N) = \zeta_N^u$. Put $G = G_{\mathbb{Q}}$, let $G_E \subseteq G$ be the open normal subgroup fixing E, $\Delta = G/G_E \cong (\mathbb{Z}/N\mathbb{Z})^\times/H$ and $n = |\Delta| = [E : \mathbb{Q}] \leq \varphi(N) \leq N$. The exponent of Δ divides $\varphi(N)$, so every character of Δ takes values in the $\varphi(N)$-th roots of unity, which lie in $\mathcal{O}$ and are distinct. A Dirichlet character χ modulo N with $H \subseteq \ker \chi$ gives a character of G trivial on G_E, hence a character θ_χ of Δ; $\chi \mapsto \theta_\chi$ is a bijection onto the characters of Δ with values in $\mathcal{O}^\times$, and $\theta_\chi(c) = \chi(-1)$ for a complex conjugation $c \in G$, which acts on $\mathbb{Q}(\zeta_N)$ as σ_{-1}. Put $M = \mathcal{O}^\vee(\chi_{\mathrm{cyc}}^{-k}) = \mathcal{O} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell(-k)$ and, for such χ, $M_\chi = \mathcal{O}^\vee(\Psi_{\chi,k}^{-1})$: the same group, on which σ acts by $\chi(\sigma)^{-1}\chi_{\mathrm{cyc}}(\sigma)^{-k}$. On G_E the two actions agree. Both modules are divisible.

Step 1: the plus and minus parts as eigenspaces. The group G acts on $H^1(G_E, M)$ by $(\sigma \star f)(g) = \sigma\, f(\sigma^{-1} g \sigma)$ (action of σ on M); G_E acts trivially (inner automorphisms act trivially on cohomology), so this is an $\mathcal{O}$-linear action of Δ. Let $\mathrm{Sha}_{\mathcal{O}} \subseteq H^1(G_E, M)$ be the classes unramified at every place of E and trivial at every place above ℓ; $\star$ preserves it, since σ permutes the inertia and decomposition groups of G_E. Inflation from G_{E,S_ℓ} is injective with image the classes unramified outside the places above ℓ (the kernel of $G_E \to G_{E,S_\ell}$ is generated as a closed normal subgroup by the inertia groups at the other places, on which M is trivial), so $\mathrm{Sha}_{\mathcal{O}} \cong \mathrm{Sha}^1_\ell(E,k) \otimes_{\mathbb{Z}_\ell} \mathcal{O}$ ($\mathcal{O}$ is free of finite rank over $\mathbb{Z}_\ell$), compatibly with the action of a lift of complex conjugation of Definition 36.1, which is $\star$ for $\sigma = c$. Put $s = \pm(-1)^k$. Since $\mathcal{O}$ is flat, $\mathrm{Sha}^{\pm}_\ell(E,k) \otimes \mathcal{O} = \ker(c \star -s)$ on $\mathrm{Sha}_{\mathcal{O}}$, and a $\mathbb{Z}_\ell$-module P of length $\lambda \in \mathbb{N} \cup \{\infty\}$ gives $\mathrm{length}_{\mathcal{O}}(P \otimes \mathcal{O}) = e_{\mathcal{O}} \lambda$. Hence

$$e_{\mathcal{O}}\, \mathrm{len}^{\pm}_\ell(E,k) = \mathrm{length}_{\mathcal{O}} \ker(c \star -s)\big|_{\mathrm{Sha}_{\mathcal{O}}}. \tag{36.3}$$

Step 2: splitting into eigenspaces. For a character θ of Δ let $\mathrm{Sha}^{(\theta)}_{\mathcal{O}} = \{x \in \mathrm{Sha}_{\mathcal{O}} : \sigma \star x = \theta(\sigma) x \ \forall \sigma\}$. Write $\Delta = \langle a_1 \rangle \times \cdots \times \langle a_r \rangle$ with a_i of order n_i. For an $\mathcal{O}[\Delta]$-module P, $\prod_{\zeta \in \mu_{n_1}} (a_1 - \zeta) = a_1^{n_1} - 1 = 0$ on P, so $\mathrm{length}\, P \le \sum_\zeta \mathrm{length} \ker(a_1 - \zeta)$; each $\ker(a_1 - \zeta)$ is stable under Δ (abelian). Repeating with $a_2, \dots, a_r$ gives $\mathrm{length}\, P \le \sum_\theta \mathrm{length}\, P^{(\theta)}$, the sum over all characters θ of Δ. Apply this to $P = \ker(c \star -s)$: if $\theta(c) \ne s$, then on $P^{(\theta)}$ we have $(\theta(c) - s)x = 0$ with $\theta(c) - s = \pm 2$ a unit (ℓ odd), so $P^{(\theta)} = 0$; if $\theta(c) = s$, then $P^{(\theta)} = \mathrm{Sha}^{(\theta)}_{\mathcal{O}}$. With (36.3) and the bijection $\chi \mapsto \theta_\chi$,

$$e_{\mathcal{O}}\, \mathrm{len}^{\pm}_\ell(E,k) \le \sum_{\chi \in X^{\pm}} \mathrm{length}_{\mathcal{O}}\, \mathrm{Sha}^{(\theta_\chi)}_{\mathcal{O}}. \tag{36.4}$$

Step 3: one eigenspace against the strict Tate–Shafarevich group. Fix $\chi \in X^{\pm}$ and write $\theta = \theta_\chi$. For a cocycle on G_E, the action $\star_\chi$ defined with M_χ in place of M is $\sigma \star_\chi = \chi(\sigma)^{-1} \sigma \star$; so the invariants of $H^1(G_E, M_\chi)$ under $\star_\chi$ are the θ-eigenspace of $H^1(G_E, M)$. The inflation–restriction sequence

$$0 \to H^1(\Delta, M_\chi^{G_E}) \to H^1(G, M_\chi) \xrightarrow{R} H^1(G_E, M)^{(\theta)} \to H^2(\Delta, M_\chi^{G_E})$$

is exact. Let $\widetilde{\mathrm{Sha}}_\chi = R^{-1}(\mathrm{Sha}^{(\theta)}_{\mathcal{O}})$ and $S_\chi = \mathrm{Sha}^1_{\ell N}(\chi, k) \subseteq H^1(G, M_\chi)$. Then $\mathrm{Sha}^{(\theta)}_{\mathcal{O}} / R(\widetilde{\mathrm{Sha}}_\chi)$ embeds in $H^2(\Delta, M_\chi^{G_E})$. For each prime p fix a decomposition group $D_p \subseteq G$ at p with inertia group I_p, and let $\phi_p \colon \widetilde{\mathrm{Sha}}_\chi \to H^1(D_p, M_\chi)$ be the restriction. Let $y \in \widetilde{\mathrm{Sha}}_\chi$. If $p \nmid \ell N$, then $E/\mathbb{Q}$ is unramified at p (as $E \subseteq \mathbb{Q}(\zeta_N)$), so $I_p \subseteq G_E$ and $y|_{I_p} = R(y)|_{I_p} = 0$. A class of $H^1(G, M_\chi)$ restricting to zero on one inertia (decomposition) group at p restricts to zero on all of them, since conjugation acts trivially on $H^1(G, M_\chi)$. So $y \in S_\chi$ as soon as $\phi_p(y) = 0$ for every $p \mid \ell N$ (local triviality implies unramifiedness), and

$$\mathrm{length}\, \widetilde{\mathrm{Sha}}_\chi / (\widetilde{\mathrm{Sha}}_\chi \cap S_\chi) \le \sum_{p \mid \ell N} \mathrm{length}\, \phi_p(\widetilde{\mathrm{Sha}}_\chi).$$

Since $R(\widetilde{\mathrm{Sha}}_\chi)$ contains $R(\widetilde{\mathrm{Sha}}_\chi \cap S_\chi)$, of length at most $\mathrm{length}\, S_\chi$, with quotient a quotient of $\widetilde{\mathrm{Sha}}_\chi / (\widetilde{\mathrm{Sha}}_\chi \cap S_\chi)$, we obtain

$$\mathrm{length}\, \mathrm{Sha}^{(\theta)}_{\mathcal{O}} \le \mathrm{length}\, S_\chi + \mathrm{length}\, H^2(\Delta, M_\chi^{G_E}) + \sum_{p \mid \ell N} \mathrm{length}\, \phi_p(\widetilde{\mathrm{Sha}}_\chi). \tag{36.5}$$

Step 4: the error terms. Three facts are used. (F1) For a finite group Γ and a finite $\mathcal{O}[\Gamma]$-module A, evaluation embeds the i-cocycles into $A^{|\Gamma|^i}$, so $\mathrm{length}\, H^i(\Gamma, A) \le |\Gamma|^i\, \mathrm{length}\, A$; and

$H^i(\Gamma, A) = 0$ for $i \geq 1$ if $\ell \nmid |\Gamma|$, as it is then killed by $|\Gamma|$ and by a power of ℓ. (F2) For a root of unity $\zeta \neq 1$ in $\mathcal{O}$, $1-\zeta$ is a unit unless ζ has ℓ-power order, and then $v_\ell(1-\zeta) \leq 1$; so length $M[1-\zeta] = e_{\mathcal{O}} v_\ell(1-\zeta) \leq e_{\mathcal{O}}$, and $M[1-\zeta] = 0$ if the order of ζ is prime to ℓ. (F3) There is $g_0 \in I_\ell \cap G_{\mathbb{Q}(\zeta_N)}$ with $\chi_{\mathrm{cyc}}(g_0) = 1 + \ell N$: by total ramification, $\chi_{\mathrm{cyc}}(I_\ell) = \mathbb{Z}_\ell^\times$, elements of I_ℓ fix the roots of unity of order prime to ℓ, and $1 + \ell N$ fixes $\zeta_{\ell^{v_\ell(N)}}$.

(a) *The modules of invariants.* With g_0 from (F3), $g_0 \in D_\ell \cap G_E$ and g_0 acts on M_χ by $(1+\ell N)^{-k}$. The integer $u_1 = (1+\ell N)^k - 1 \geq 1$ satisfies $(1+\ell N)^{-k} - 1 = -(1+\ell N)^{-k} u_1$; if $\ell^{b_1} \| u_1$, then $M_\chi^{G_E} \subseteq M_\chi^{D_\ell \cap G_E} \subseteq M[\ell^{b_1}]$, of length $b_1 e_{\mathcal{O}}$, and $b_1 \leq \log_\ell u_1 < k \log_\ell(2\ell N) \leq k(2 + \log_2 N) \leq 3kN$.

(b) H^2. By (F1) and (a), length $H^2(\Delta, M_\chi^{G_E}) \leq n^2 \cdot 3kN e_{\mathcal{O}}$.

(c) *At ℓ.* $R(y)$ is trivial at the places of E above ℓ, so $\phi_\ell(y)$ restricts to zero on $D_\ell \cap G_E$, and $\phi_\ell(\widetilde{\mathrm{Sha}}_\chi) \subseteq H^1(D_\ell/(D_\ell \cap G_E), M_\chi^{D_\ell \cap G_E})$ (inflation–restriction). The group $D_\ell/(D_\ell \cap G_E)$ embeds in Δ; by (F1) and (a) the length is at most $n \cdot 3kN e_{\mathcal{O}}$.

(d) *At $p \mid N$, $p \neq \ell$.* Put $W_p = I_p \cap G_E$, normal in D_p, acting trivially on M_χ (χ is trivial on G_E, χ_{cyc} on I_p). As $R(y)$ is unramified above p, $\phi_p(y)$ restricts to zero on W_p, so $\phi_p(\widetilde{\mathrm{Sha}}_\chi) \subseteq H^1(D_p/W_p, M_\chi)$, and inflation–restriction for $I_p/W_p \subseteq D_p/W_p$ gives length $H^1(D_p/W_p, M_\chi) \leq$ length $H^1(D_p/I_p, M_\chi^{I_p}) +$ length $H^1(J_p, M_\chi)$, where $J_p = I_p/W_p$ is a finite abelian group (it embeds in Δ) of order dividing n, acting through χ. Here D_p/I_p is procyclic, generated by a Frobenius F_p, and $H^1(D_p/I_p, M_\chi^{I_p}) = M_\chi^{I_p}/(F_p - 1)M_\chi^{I_p}$. If χ is trivial on I_p, then $M_\chi^{I_p} = M_\chi$ and F_p acts by $\zeta p^{\mp k}$ with ζ a root of unity, which is $\neq 1$ as $k \geq 1$; multiplication by the nonzero element $\zeta p^{\mp k} - 1$ is surjective on the divisible module M_χ, so this term is 0; and $H^1(J_p, M_\chi) = \mathrm{Hom}(J_p, M)$ has length $e_{\mathcal{O}} \log_\ell |J_p|_\ell \leq e_{\mathcal{O}} \log_\ell n$, which is 0 if $\ell \nmid n$. If $\chi(j_1) = \zeta \neq 1$ for some $j_1 \in I_p$, then $M_\chi^{I_p} \subseteq M[\zeta - 1]$, so the first term has length at most $e_{\mathcal{O}}$ by (F2); for $J_1 = \langle j_1 \rangle \subseteq J_p$, $H^1(J_1, M_\chi) = \ker(\mathrm{norm})/(\zeta-1)M_\chi = 0$ (the norm is $\sum_i \zeta^i = 0$, and $(\zeta - 1)M_\chi = M_\chi$), so $H^1(J_p, M_\chi) \cong H^1(J_p/J_1, M[\zeta - 1])$ has length at most $n e_{\mathcal{O}}$ by (F1), (F2). The order of ζ divides the exponent of Δ, which divides n; so if $\ell \nmid n$, both terms vanish by (F2). In all cases length $\phi_p(\widetilde{\mathrm{Sha}}_\chi) \leq (n+1)e_{\mathcal{O}}$.

Conclusion. N has at most $\log_2 N \leq N$ prime factors. By (36.5) and (b)–(d), with $n \leq N$, $N \geq 2$, $k \geq 1$,

$$\text{length}\, \mathrm{Sha}_{\mathcal{O}}^{(\theta_\chi)} \leq \text{length}\, S_\chi + e_{\mathcal{O}}\big(3kN^3 + 3kN^2 + N(N+1)\big) \leq \text{length}\, S_\chi + e_{\mathcal{O}}(Nk)^6,$$

and (36.4) gives the first inequality. If $\ell \nmid n = [E : \mathbb{Q}]$, the terms (b), (c) vanish by (F1), and (d) vanishes as shown, so length $\mathrm{Sha}_{\mathcal{O}}^{(\theta_\chi)} \leq$ length S_χ and (36.4) gives the second inequality. □

Proposition 36.9. *Let $\ell \geq 5$ be a prime, $\mathcal{O}$ a coefficient ring and $e \geq 2$ an even integer with $(\ell - 1) \mid 2e$. Then* $\mathrm{Sha}_\ell^1(\mathbf{1}, e) = 0$.

Proof. Put $M = \mathcal{O}^\vee(\Psi_{\mathbf{1},e}^{-1}) = \mathcal{O}^\vee(\chi_{\mathrm{cyc}}^{-e})$, the group $\mathcal{O} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ on which $g \in G_{\mathbb{Q}}$ acts by multiplication by $\chi_{\mathrm{cyc}}(g)^{-e} \in \mathbb{Z}_\ell^\times$. For $m \in M$ let ∂m be the coboundary $g \mapsto (\chi_{\mathrm{cyc}}(g)^{-e} - 1)m$, and let $M[\ell]$ be the subgroup killed by ℓ. A class of $\mathrm{Sha}_\ell^1(\mathbf{1}, e)$ restricts to zero in $H^1(I_p, M)$ for an inertia subgroup I_p at every prime p (the local triviality at ℓ will not be used). Conjugation by an element of $G_{\mathbb{Q}}$ induces the identity on $H^1(\mathbb{Q}, M)$ and carries restriction to I_p to restriction to the conjugate subgroup, so such a class restricts to zero on every inertia subgroup at every prime. We use three classical facts on cyclotomic fields [266]:

(a) $\chi_{\mathrm{cyc}}(I') = \mathbb{Z}_\ell^\times$ for every inertia subgroup I' at ℓ, since $\mathbb{Q}_\ell(\mu_{\ell^n})/\mathbb{Q}_\ell$ is totally ramified of degree $\varphi(\ell^n)$ for every n;

(b) χ_{cyc} is trivial on every inertia subgroup at a prime $p \neq \ell$;

(c) (Kronecker–Weber) every finite abelian extension of $\mathbb{Q}$ unramified at all primes $p \neq \ell$ is contained in $\mathbb{Q}(\mu_{\ell^n})$ for some n.

Reduction to classes killed by ℓ. A class in $H^1(\mathbb{Q}, M)$ is represented by a continuous cocycle on the compact group $G_\mathbb{Q}$ with values in the discrete torsion group M; it takes finitely many values and is therefore killed by a power of ℓ. If $\mathrm{Sha}^1_\ell(\mathbf{1}, e)$ contained a nonzero class $[z_1]$, with $n \geq 1$ minimal such that $\ell^n[z_1] = 0$, then $\ell^{n-1}[z_1]$ would be a nonzero class of the submodule $\mathrm{Sha}^1_\ell(\mathbf{1}, e)$ killed by ℓ. It therefore suffices to show that every class $[z] \in \mathrm{Sha}^1_\ell(\mathbf{1}, e)$ with $\ell[z] = 0$ is zero.

Fix such a class. Since ℓz is a coboundary, $\ell z = \partial m$ for some $m \in M$. The group M is divisible (a finite direct sum of copies of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$), so $m = \ell m'$ with $m' \in M$, and $z' = z - \partial m'$ is a continuous cocycle representing $[z]$ with $\ell z'(g) = \ell z(g) - \partial m(g) = 0$ for all g. Thus z' takes values in $M[\ell]$, on which g acts by $\chi_{\mathrm{cyc}}(g)^{-e} \bmod \ell$. The local conditions give:

(i) if $p \neq \ell$ and I' is an inertia subgroup at p, then $z'|_{I'} = \partial m_{I'}|_{I'}$ for some $m_{I'} \in M$, and $\partial m_{I'}$ vanishes on I' by (b); so z' vanishes on I';

(ii) if I' is an inertia subgroup at ℓ, there is $m_{I'} \in M$ with $z'(g) = (\chi_{\mathrm{cyc}}(g)^{-e} - 1)m_{I'}$ for all $g \in I'$.

Since $(\ell - 1) \mid 2e$, either $(\ell - 1) \mid e$, or $\frac{\ell-1}{2} \mid e$ with $e/\frac{\ell-1}{2}$ odd.

Case $(\ell - 1) \mid e$. For $u \in \mathbb{Z}_\ell^\times$ we have $u^{\ell-1} \equiv 1 \pmod{\ell}$, so $G_\mathbb{Q}$ acts trivially on $M[\ell]$ and $z' \colon G_\mathbb{Q} \to M[\ell]$ is a continuous homomorphism. By (i) it vanishes on every inertia subgroup at every prime $p \neq \ell$, so the fixed field of its kernel is a finite abelian extension of $\mathbb{Q}$ unramified outside ℓ, contained in some $\mathbb{Q}(\mu_{\ell^n})$ by (c). Hence z' is trivial on $\ker \chi_{\mathrm{cyc}}$, i.e. $z'(g)$ depends only on $\chi_{\mathrm{cyc}}(g)$. Fix an inertia subgroup I' at ℓ and $m_{I'}$ as in (ii). For $g \in G_\mathbb{Q}$ choose by (a) an element $g' \in I'$ with $\chi_{\mathrm{cyc}}(g') = \chi_{\mathrm{cyc}}(g)$; then

$$z'(g) = z'(g') = (\chi_{\mathrm{cyc}}(g')^{-e} - 1)m_{I'} = (\chi_{\mathrm{cyc}}(g)^{-e} - 1)m_{I'}.$$

So $z' = \partial m_{I'}$ is a coboundary and $[z] = 0$.

Case $(\ell - 1) \nmid e$: the quadratic character. Here $\frac{\ell-1}{2} \mid e$ with $r = e/\frac{\ell-1}{2}$ odd; as e is even, $\frac{\ell-1}{2}$ is even, i.e. $\ell \equiv 1 \pmod 4$, and $\ell \geq 5$. Let $\eta_\ell \colon G_\mathbb{Q} \to \{\pm 1\}$ be the character of $\mathbb{Q}(\sqrt{\ell})$, $g(\sqrt{\ell}) = \eta_\ell(g)\sqrt{\ell}$. Since $\ell \equiv 1 \pmod 4$, $\sqrt{\ell} = \pm\sum_{a=1}^{\ell-1}(\frac{a}{\ell})\zeta_\ell^a$ (quadratic Gauss sum), so $\eta_\ell(g) = (\frac{u}{\ell})$ with $u = \chi_{\mathrm{cyc}}(g)$. By Euler's criterion $u^{(\ell-1)/2} \equiv (\frac{u}{\ell}) \pmod{\ell}$, hence, $-r$ being odd,

$$\chi_{\mathrm{cyc}}(g)^{-e} = \big(u^{(\ell-1)/2}\big)^{-r} \equiv \Big(\frac{u}{\ell}\Big)^{-r} = \eta_\ell(g) \pmod{\ell} \qquad (g \in G_\mathbb{Q}). \tag{36.6}$$

So g acts on $M[\ell]$ by $\eta_\ell(g)$. Let $G' = \ker \eta_\ell = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{\ell}))$; it acts trivially on $M[\ell]$, so $f = z'|_{G'} \colon G' \to M[\ell]$ is a continuous homomorphism.

f is unramified everywhere. The inertia groups of G' at the finite places are the groups $I' \cap G'$ with I' an inertia subgroup of $G_\mathbb{Q}$. For I' at $p \neq \ell$, f vanishes on $I' \cap G'$ by (i). Let I' be at ℓ, with $m_{I'}$ as in (ii). By (a) choose $t \in I'$ such that $\chi_{\mathrm{cyc}}(t) \bmod \ell$ generates $\mathbb{F}_\ell^\times$; by (36.6), $\chi_{\mathrm{cyc}}(t)^{-e} \equiv (\frac{\chi_{\mathrm{cyc}}(t)}{\ell}) = -1 \pmod{\ell}$, so $\chi_{\mathrm{cyc}}(t)^{-e} - 1 \equiv -2$ is a unit of $\mathbb{Z}_\ell$. From $0 = \ell z'(t) = (\chi_{\mathrm{cyc}}(t)^{-e} - 1)\,\ell m_{I'}$ we get $\ell m_{I'} = 0$. For $g \in I' \cap G'$, (36.6) gives $\chi_{\mathrm{cyc}}(g)^{-e} - 1 \in \ell\mathbb{Z}_\ell$,

hence $f(g) = (\chi_{\mathrm{cyc}}(g)^{-e} - 1)m_{I'} = 0$. Finally $\mathbb{Q}(\sqrt{\ell})$ is real; for a complex conjugation $\tau \in G'$ we have $2f(\tau) = f(\tau^2) = 0$ and $\ell f(\tau) = 0$, so $f(\tau) = 0$. Hence the fixed field of $\ker f$ is a finite abelian extension of $\mathbb{Q}(\sqrt{\ell})$ unramified at all places, with Galois group isomorphic to $f(G')$. By class field theory it lies in the Hilbert class field, so $|f(G')|$ divides the class number h of $\mathbb{Q}(\sqrt{\ell})$ [190]; and $|f(G')|$ is a power of ℓ, since $f(G') \subseteq M[\ell]$.

$\ell \nmid h$. The discriminant of $\mathbb{Q}(\sqrt{\ell})$ is ℓ, so by Minkowski's bound every ideal class contains an integral ideal of norm at most $\frac{2!}{2^2}\sqrt{\ell} = \frac{\sqrt{\ell}}{2}$ [190]. Integral ideals factor uniquely into prime ideals; above a rational prime p lie either two prime ideals of norm p, or one of norm p, or one of norm p^2; so the number of integral ideals of norm p^a is at most $a+1$, and the number of integral ideals of norm n is at most the number of divisors of n, hence at most n. Therefore

$$h \le \sum_{1 \le n \le \sqrt{\ell}/2} n \le \Big(\frac{\sqrt{\ell}}{2}\Big)^2 = \frac{\ell}{4} < \ell,$$

and $\ell \nmid h$ because ℓ is prime. So $f(G') = 0$, that is, z' vanishes on G'.

z' is a coboundary. Choose $g_0 \in G_{\mathbb{Q}}$ with $\eta_\ell(g_0) = -1$, put $m_0 = z'(g_0) \in M[\ell]$ and $m'' = \frac{\ell-1}{2} m_0 \in M[\ell]$, so that $-2m'' = m_0$. As $m'' \in M[\ell]$, $\partial m''(g) = (\eta_\ell(g) - 1)m''$ by (36.6): it vanishes on G', and equals $-2m'' = m_0$ on $g_0 G'$. On the other hand, for $h \in G'$, $z'(h) = 0$ and $z'(g_0 h) = z'(g_0) + g_0\, z'(h) = m_0$. Since $G_{\mathbb{Q}} = G' \cup g_0 G'$, we get $z' = \partial m''$ and $[z] = 0$.

In both cases every class of $\mathrm{Sha}^1_\ell(\mathbf{1}, e)$ killed by ℓ vanishes, so $\mathrm{Sha}^1_\ell(\mathbf{1}, e) = 0$. The argument uses of ℓ only that it is odd, and does not use the local triviality at ℓ. □

Proposition 36.10. *Let $\psi \in \mathcal{P}$, $\ell \in B_\psi$ and let $j \ge 0$ be an integer. There is an integer $C \ge 0$ such that for every Sha-host $\mathfrak{h} = (a, q, d)$ of ψ, every integer $t \ge 2$ at which $\mathfrak{h}$ is big, every real $B \ge 0$ such that $\mathrm{V}_\ell(\mathbb{Q}(\zeta_{\ell^{j+1}N_\psi}); t, B)$ holds, and every (ψ, a, q, d)-field E: $\mathrm{len}^+_\ell(E, k) < \infty$ for $1 \le k < t$, and*

$$\log \ell \sum_{k=1}^{t-1} \mathrm{len}^+_\ell(E, k) \le (N_{\mathfrak{h}} t)^C + C\, B.$$

Proof. Let K, C_5, C_6 be the constants of Proposition 38.26, Proposition 38.32, Proposition 38.35 for ψ, and C_4 that of Proposition 38.38 for (ψ, ℓ, j). Put $C_m = \max(K, C_5, C_6)$ and $C = C_4 + 2C_m + 9 + \ell N_\psi$. Fix $\mathfrak{h}$, t, B, E as in the statement and write $N = N_{\mathfrak{h}}$, $H = H_{\mathfrak{h}}$, $M = Nt$, $L = \log \ell$. As $16 \mid N$ and $t \ge 2$: $M \ge 3$, $t \le M$, $N \le M$, and $N_\psi \le N$.

Position of ℓ. The set B_ψ consists of odd primes, so ℓ is odd. Either $\ell = 3 < \min(q, d)$ (since $q, d > 2t + 1 \ge 5$), or ℓ divides one of $o_\psi, N_\psi, p_K(\psi), h^*(\psi), p_1(\psi)$, all smaller than $\min(q, d)$ because $\mathfrak{h}$ is big at t. Hence $\ell < \min(q, d)$, so $\ell \ne q, d$ and $\ell \le N$; and $L \le \ell$.

Characters with $\chi^{(\ell)}$ of odd order. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}_\ell(\zeta_{\varphi(N)})$, a coefficient ring containing a primitive $\varphi(N)$-th root of unity, and let Θ be the set of Dirichlet characters χ modulo N with values in $\mathcal{O}^\times$, $H \subseteq \ker \chi$ and $\chi^{(\ell)}$ of odd order. We show $|\Theta| \le N_\psi$. Write $\chi = \prod_{p \mid N} \chi_p$ with χ_p a character of $(\mathbb{Z}/p^{v_p(N)}\mathbb{Z})^\times$. Since $v_q(N) = 1$, every $u \equiv 1 \pmod{N/q}$ that is a square modulo q lies in H (it is $\equiv 1$ modulo 2^{a+2} and N_ψ and a square modulo q and d); so χ_q is trivial on the squares and has order at most 2; likewise χ_d. Every character of $(\mathbb{Z}/2^m\mathbb{Z})^\times$ has 2-power order, so χ_2 has 2-power order. For $p \in \{2, q, d\}$ (all different from ℓ), the component at p of the part of χ of order prime to ℓ is χ_p (its order is a power of 2, prime to ℓ), and the component at p of the component of χ at ℓ is trivial; hence the component of $\chi^{(\ell)}$ at p is χ_p. As $\chi^{(\ell)}$ has odd order, so has χ_p, and therefore $\chi_2 = \chi_q = \chi_d = \mathbf{1}$. For the odd primes $p \mid N$ with $p \ne q, d$ we have $p \mid N_\psi$ and $v_p(N) = v_p(N_\psi)$. Hence χ is trivial on the units $\equiv 1$

(mod N_ψ), that is, χ is the composite of $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N_\psi\mathbb{Z})^\times$ with a character modulo N_ψ, which it determines. So $|\Theta|$ is at most the number of characters of $(\mathbb{Z}/N_\psi\mathbb{Z})^\times$ with values in the domain $\mathcal{O}$, at most $\varphi(N_\psi) \le N_\psi$.

For $\chi \in \Theta$, Proposition 38.38 (with the given j and B; $\mathfrak{h}$ is big at $t \ge 2$) gives that $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length for $1 \le e < t$ and

$$\sum_{e=1}^{t-1} \nu_{\mathcal{O}}(\chi, e) \le B + M^{C_4}. \tag{36.7}$$

The other characters. Let $1 \le e < t$ and $\chi \in X_{\mathfrak{h}}(\mathcal{O}, e) \setminus \Theta$. If $e = 1$, then (ℓ, e) is in case (V), and Proposition 38.32 (if $R_1(\ell, \chi) \le 2^{c_2(\chi)}$) or Proposition 38.35 (otherwise) applies. If $e \ge 2$, then $\chi^{(\ell)}$ has even order and $\ell \in B_\psi$, so χ is in case (II), and Proposition 38.26 applies. In each case $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length and $\nu_{\mathcal{O}}(\chi, e)L \le (\ell N e)^{C_m} \le M^{2C_m}$, since $1 \le \ell N e \le N \cdot Nt = NM \le M^2$.

Reduction and summation. By Lemma 36.6, E is a CM field isomorphic to the fixed field of H in $\mathbb{Q}(\zeta_N)$. By the first inequality of Lemma 36.7 (with $N \ge 2$, its set X being $X_{\mathfrak{h}}(\mathcal{O}, k)$) and the finiteness just shown, for $1 \le k < t$, dividing by $e_{\mathcal{O}}$,

$$\mathrm{len}^+_\ell(E, k) \le \sum_{\chi \in X_{\mathfrak{h}}(\mathcal{O}, k)} \nu_{\mathcal{O}}(\chi, k) + |X_{\mathfrak{h}}(\mathcal{O}, k)|\,(Nk)^6 < \infty.$$

Here $|X_{\mathfrak{h}}(\mathcal{O}, k)| \le \varphi(N) \le N$ and $(Nk)^6 \le M^6$. Summing over k, splitting each $X_{\mathfrak{h}}(\mathcal{O}, k)$ into its part in Θ and the rest, and using that every pair (χ, k) with $\chi \in \Theta$ occurs at most once,

$$L \sum_{k=1}^{t-1} \mathrm{len}^+_\ell(E, k) \le L \sum_{\chi \in \Theta} \sum_{k=1}^{t-1} \nu_{\mathcal{O}}(\chi, k) + \sum_{k=1}^{t-1} N\, M^{2C_m} + L\, t\, N\, M^6.$$

By (36.7) and $|\Theta| \le N_\psi$ the first term is at most $LN_\psi(B + M^{C_4})$. Hence, with $L \le \ell \le M$, $N_\psi \le M$, $tN = M$ and $B \ge 0$,

$$L \sum_{k=1}^{t-1} \mathrm{len}^+_\ell(E, k) \le \ell N_\psi B + M^{C_4+2} + M^{2C_m+1} + M^8 \le C\, B + M^C.$$

This is the claim. □

37 The minus part

This section proves Proposition 37.2, the minus Sha bound for the (ψ, e, q, d)-fields that are CM fields, which gives the minus parts in the proof of Theorem 36.4. The bound is $3nt^2 \log(t \operatorname{cond}(E))$ plus a term from the finitely many odd primes dividing o_ψ, so it is polynomial in the degree n, the conductor and t. For an odd prime $\ell \nmid [E : \mathbb{Q}]$, the minus part at the twist k is the sum of the strict Tate–Shafarevich groups of the characters χ of E with $\chi(-1) = (-1)^{k+1}$ (Lemma 36.8). The length of each is at most the valuation of the normalised Bernoulli number $\mathrm{Ber}_{k+1}(\chi)$ (Proposition 37.4). These numbers are non-zero, and their valuations, weighted by $\log \ell$ and summed over all primes, are at most $|\mathcal{X}| \log(16N^{k+2}(k+1)!)$ for the set $\mathcal{X}$ of characters involved (Proposition 37.9). The odd primes dividing the degree are treated separately: twisted Stickelberger elements annihilate the Tate–Shafarevich groups of a cyclotomic subfield (Lemma 37.25), which gives Proposition 37.24.

The heart of the section is the Bernoulli bound Proposition 37.4, an upper bound of the kind given by the main conjecture for abelian fields [169], obtained here by the Euler system method [209] applied to Gauss sums. The Gauss sums of the primes of cyclotomic fields form an Euler system (Proposition 37.15); its Kolyvagin derivatives give Kolyvagin systems (Proposition 37.14); a Kolyvagin system bounds the normalised strict cocycles by the valuation of a twisted Stickelberger value (Proposition 37.11); and at a suitable Kolyvagin level that value is the Bernoulli number $B_{e+1,\psi}$ divided by $e+1$, up to a unit (Lemma 37.7). Stickelberger's factorisation of Gauss sums [266] enters through the norm and congruence relations of the Gauss-sum elements. The Chebotarev selection of Kolyvagin primes, the reciprocity at a Kolyvagin prime and the comparison of finite and singular parts are the usual steps of the method.

Definition 37.1. Let Ω be a field of characteristic 0, $N \geq 1$ an integer and χ a Dirichlet character modulo N with values in $\Omega^\times$ (a homomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \to \Omega^\times$, extended by 0 to the non-units). Let χ^* be the primitive character inducing χ and $f \mid N$ its conductor. For an integer $k \geq 1$ the **generalised Bernoulli number** of χ^* is

$$B_{k,\chi^*} = f^{\,k-1} \sum_{a=1}^{f} \chi^*(a)\, B_k\Big(\frac{a}{f}\Big) \in \Omega,$$

where $B_k(X) \in \mathbb{Q}[X]$ is the k-th Bernoulli polynomial, and the **normalised Bernoulli number** of χ is

$$\mathrm{Ber}_k(\chi) = N \cdot (k+1)\# \cdot B_{k,\chi^*} \in \Omega, \qquad (k+1)\# = \prod_{p \leq k+1,\ p \text{ prime}} p. \tag{37.1}$$

It depends on the modulus N with which χ is given.

Proposition 37.2. *Let $\psi \in \mathcal{P}$. There is a real C such that for all integers $e \geq 2$ and $t \geq 2$, all primes $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ with $\gcd(qd, N_\psi) = 1$, and every (ψ, e, q, d)-field E that is a CM field, with $n = [E:\mathbb{Q}]$ and $\mathrm{cond}(E)$ the least integer $N \geq 1$ such that E embeds into $\mathbb{Q}(\zeta_N)$,*

$$\mathrm{V}^-\Big(E;\ t,\ 3nt^2 \log\big(t\,\mathrm{cond}(E)\big) + C \sum_{\substack{\ell \mid o_\psi \\ \ell \text{ odd prime}}} \ell\, n^2 t^2 \log^2\big(t\ell\,\mathrm{cond}(E)\big)\Big)$$

holds.

Proof. Let $C \geq 0$ be the constant of Proposition 37.24. Fix e, t, q, d and E as in the statement, write $n = [E:\mathbb{Q}]$ and $F = \mathrm{cond}(E)$.

The field. The triple $\mathfrak{h} = (e, q, d)$ is a Sha-host of ψ ($e \geq 2$, $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$, $\gcd(qd, N_\psi) = 1$). By Lemma 36.6, E embeds into $\mathbb{Q}(\zeta_{N_\mathfrak{h}})$, so F exists, and n divides $2^{e+2} o_\psi$; hence every odd prime dividing n divides o_ψ. As E is CM, $E \neq \mathbb{Q}$ and $F \geq 3$. Fix an embedding $E \hookrightarrow \mathbb{Q}(\zeta_F)$; by Galois theory its image is the fixed field of $\{\zeta_F \mapsto \zeta_F^u : u \in H\}$ for a subgroup $H \leq (\mathbb{Z}/F\mathbb{Z})^\times$ of index n, and the exponent of $(\mathbb{Z}/F\mathbb{Z})^\times/H$ divides n.

Odd primes not dividing n. Let $\ell \nmid n$ be an odd prime and $1 \leq k < t$; let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}_\ell(\zeta_{\varphi(F)})$. By the second inequality of Lemma 36.8 for the sign $-$ ($F \geq 2$, E CM, $\ell \nmid n$),

$$e_\mathcal{O}\, \mathrm{len}_\ell^-(E,k) \leq \sum_{\chi \in X^-} \mathrm{length}_\mathcal{O}\, \mathrm{Sha}^1_{\ell F}(\chi, k),$$

where X^- is the set of Dirichlet characters χ modulo F with values in $\mathcal{O}^\times$, $H \subseteq \ker\chi$ and $\chi(-1) = (-1)^{k+1}$. Every character of $(\mathbb{Z}/F\mathbb{Z})^\times/H$ with values in $\overline{\mathbb{Q}}_\ell^\times$ takes values in the $\varphi(F)$-th roots of unity, which lie in $\mathcal{O}$; so X^- is the set $\mathcal{X}_{\overline{\mathbb{Q}}_\ell}$ of Proposition 37.9 for $(F, k+1, H)$. Let

$\chi \in X^-$. The group $\mathrm{Sha}^1_{\ell F}(\chi,k)$ is contained in $\mathrm{Sha}^1_\ell(\chi,k)$ (it satisfies more local conditions). The order of χ divides the exponent of $(\mathbb{Z}/F\mathbb{Z})^\times/H$, hence n, so it is prime to ℓ. By item 1 of Proposition 37.9, $\mathrm{Ber}_{k+1}(\chi)$ is nonzero and integral over $\mathbb{Z}_\ell$; it lies in the fraction field of $\mathcal{O}$ (it is a $\mathbb{Q}$-linear combination of values of χ), hence in $\mathcal{O}$, which is integrally closed. So Proposition 37.4 applies (e there is $k \geq 1$, $\chi(-1) = (-1)^{k+1}$):

$$\mathrm{length}_{\mathcal{O}} \mathrm{Sha}^1_{\ell F}(\chi,k) \leq \mathrm{length}_{\mathcal{O}} \mathrm{Sha}^1_{\ell}(\chi,k) \leq v_{\mathcal{O}}(\mathrm{Ber}_{k+1}(\chi)) = e_{\mathcal{O}} v_\ell(\mathrm{Ber}_{k+1}(\chi)) < \infty.$$

Hence $\mathrm{len}^-_\ell(E,k) < \infty$, and with the integers $w^{(k)}_\ell$ of item 2 of Proposition 37.9 for $(F, k+1, H)$,

$$\mathrm{len}^-_\ell(E,k) \leq \sum_{\chi \in X^-} v_\ell(\mathrm{Ber}_{k+1}(\chi)) \leq w^{(k)}_\ell.$$

The complex characters of $(\mathbb{Z}/F\mathbb{Z})^\times/H$ number n, so for every finite set S of odd primes not dividing n,

$$\sum_{\ell \in S} \log \ell \, \mathrm{len}^-_\ell(E,k) \leq \sum_{\ell \in S} w^{(k)}_\ell \log \ell \leq n \log\bigl(16 F^{k+2}(k+1)!\bigr) \leq n\,(2t+5)\log(tF).$$

Summing over $1 \leq k < t$ and using $(t-1)(2t+5) = 2t^2+3t-5 \leq 3t^2$,

$$\sum_{\ell \in S} \log \ell \sum_{k=1}^{t-1} \mathrm{len}^-_\ell(E,k) \leq 3nt^2 \log(tF). \tag{37.2}$$

Odd primes dividing n. Let ℓ be an odd prime dividing n; then $\ell \mid o_\psi$. For $1 \leq k < t$, Proposition 37.24 (with $N = F$; E is CM and embeds into $\mathbb{Q}(\zeta_F)$) gives $\mathrm{len}^-_\ell(E,k) < \infty$ and, as $k+1 \leq t$, $1 \leq k\ell F \leq t\ell F$ and $C \geq 0$,

$$\mathrm{len}^-_\ell(E,k) \log \ell \leq C\,\ell\, n^2 (k+1) \log^2(k\ell F) \leq C\,\ell\, n^2\, t\, \log^2(t\ell F).$$

Summing over the $t-1 < t$ values of k, $\log \ell \sum_{k<t} \mathrm{len}^-_\ell(E,k) \leq C\ell n^2 t^2 \log^2(t\ell F)$.

Conclusion. For every odd prime ℓ and $1 \leq k < t$, $\mathrm{len}^-_\ell(E,k) < \infty$. For a finite set S of odd primes, split S into the primes not dividing n and those dividing n; by (37.2), the bound just shown, and since the odd primes dividing n divide o_ψ and all terms are non-negative,

$$\sum_{\ell \in S} \log \ell \sum_{k=1}^{t-1} \mathrm{len}^-_\ell(E,k) \leq 3nt^2 \log(tF) + C \sum_{\substack{\ell \mid o_\psi \\ \ell \text{ odd prime}}} \ell\, n^2 t^2 \log^2(t\ell F).$$

The right-hand side does not depend on S, so the same bound holds for the sum over all odd primes. This is $\mathrm{V}^-(E;t,\beta)$ with the β of the statement, since $F = \mathrm{cond}(E)$. □

Definition 37.3. Let ℓ be an odd prime and $\mathcal{O}$ a coefficient ring, with maximal ideal $\mathfrak{m}$ and normalised valuation $v_{\mathcal{O}}$ ($v_{\mathcal{O}}$ of a uniformiser is 1), extended to the fraction field of $\mathcal{O}$. Write $\mathcal{O}^\vee = \mathcal{O} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$, a discrete $\mathcal{O}$-module. Fix an algebraic closure $\overline{\mathbb{Q}}$, so $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$; for $N \geq 1$ let $\mu_N \subset \overline{\mathbb{Q}}$ be the N-th roots of unity and $U_N = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_N))$. A **decomposition group** at a prime p is the stabiliser $D \subseteq G_{\mathbb{Q}}$ of a place of $\overline{\mathbb{Q}}$ above p; $I(D) \subseteq D$ is its inertia group, and a **Frobenius** of D is an element of D inducing $x \mapsto x^p$ on the residue field of that place. Let $\theta\colon G_{\mathbb{Q}} \to \mathcal{O}^\times$ be a continuous character, and let $\check{\theta} = \chi_{\mathrm{cyc}}\theta^{-1}$ be its **dual character** (χ_{cyc} takes values in $\mathbb{Z}_\ell^\times \subseteq \mathcal{O}^\times$).

Kolyvagin levels. A **Kolyvagin level** of θ is a pair (N_0, m) of integers with $m \geq 1$, $N_0 \geq 1$ and $\ell \nmid N_0$ such that, with $N = N_0\ell^m$ (the **modulus** of the level),

(Kv1) $\theta(u)-1\in\ell^m\mathcal{O}$ for every $u\in U_N$;

(Kv2) $\theta(g)+1\in\ell^m\mathcal{O}$ for every $g\in G_{\mathbb{Q}}$ with $g(\zeta)=\zeta^{-1}$ for all $\zeta\in\mu_N$;

(Kv3) there is $g_1\in G_{\mathbb{Q}}$ with $\theta(g_1)-\chi_{\mathrm{cyc}}(g_1)\in\mathcal{O}^\times$.

A **normaliser** for θ is an element $\delta\in G_{\mathbb{Q}}$ with $\theta(\delta)-1\in\mathcal{O}^\times$.

Normalised strict cocycles. Let δ be a normaliser for θ, Σ a finite set of primes, $a\in\mathcal{O}\backslash\{0\}$ and $n\geq 1$ an integer. Then $\mathrm{Z}_{\theta,\delta}(\Sigma,a;n)$ is the set of continuous maps $z\colon G_{\mathbb{Q}}\to\mathcal{O}^\vee$ such that

(Kd1) $z(gh)=z(g)+\theta(g)z(h)$ for all $g,h\in G_{\mathbb{Q}}$, and $z(\delta)=0$;

(Kd2) $a\,z(g)=0$ for all $g\in G_{\mathbb{Q}}$;

(Kd3) $z(g)=0$ for every g in an inertia group at a prime $p\notin\Sigma$;

(Kd4) for every prime $p\in\Sigma$ and every decomposition group D at p there is $y\in\mathcal{O}^\vee$ with $ay=0$ and $z(g)=\theta(g)y-y$ for all $g\in D$;

(Kd5) $z(g)=0$ for every g in a decomposition group at a prime dividing n.

It is an $\mathcal{O}$-submodule of the module of all maps $G_{\mathbb{Q}}\to\mathcal{O}^\vee$; we put $\mathrm{Z}_{\theta,\delta}(\Sigma,a)=\mathrm{Z}_{\theta,\delta}(\Sigma,a;1)$, and lengths are $\mathcal{O}$-lengths in $\mathbb{N}\cup\{\infty\}$.

Twisted Stickelberger values. Let $N\geq 2$ and b be integers. For $t\in(\mathbb{Z}/N\mathbb{Z})^\times$ let $\tilde{t}\in\{1,\dots,N-1\}$ be its representative and

$$h_N(b,t)=\Big\lfloor\frac{\tilde{t}\,b}{N}\Big\rfloor-\Big\lfloor\frac{(N-\tilde{t})\,b}{N}\Big\rfloor\in\mathbb{Z}.$$

A **family of lifts** modulo N is a family $\sigma=(\sigma_t)_{t\in(\mathbb{Z}/N\mathbb{Z})^\times}$ of elements of $G_{\mathbb{Q}}$ with $\sigma_t(\zeta)=\zeta^{\tilde{t}}$ for all $\zeta\in\mu_N$. The **twisted Stickelberger value** is

$$\mathrm{st}_{N,b}(\theta,\sigma)=\sum_{t\in(\mathbb{Z}/N\mathbb{Z})^\times}h_N(b,t)\,\theta(\sigma_t)^{-1}\in\mathcal{O}. \tag{37.3}$$

Proposition 37.4. *Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring, $N\geq 1$, χ a Dirichlet character modulo N with values in $\mathcal{O}^\times$ whose order is prime to ℓ, and $e\geq 1$ an integer with $\chi(-1)=(-1)^{e+1}$. If $\mathrm{Ber}_{e+1}(\chi)\in\mathcal{O}$, then*

$$\mathrm{length}_{\mathcal{O}}\,\mathrm{Sha}^1_\ell(\chi,e)\leq v_{\mathcal{O}}(\mathrm{Ber}_{e+1}(\chi)).$$

Proof. Step 0: reduction to the primitive character. Let $\psi=\chi^*$ be the primitive character inducing χ and $f\mid N$ its conductor. If $g\in G_{\mathbb{Q}}$ acts on μ_N by $\zeta\mapsto\zeta^a$, then $\chi(g)=\chi(a)=\psi(a)=\psi(g)$; so χ and ψ are the same character of $G_{\mathbb{Q}}$, $\Psi_{\chi,e}=\Psi_{\psi,e}$, and $\mathrm{Sha}^1_\ell(\chi,e)=\mathrm{Sha}^1_\ell(\psi,e)$. Reduction $(\mathbb{Z}/N\mathbb{Z})^\times\to(\mathbb{Z}/f\mathbb{Z})^\times$ is surjective, so ψ takes values in $\mathcal{O}^\times$, its order is that of χ, prime to ℓ, and $\psi(-1)=\chi(-1)=(-1)^{e+1}$. Put $\theta=\Psi_{\psi,e}^{-1}=\psi^{-1}\chi_{\mathrm{cyc}}^{-e}$. By definition, $\mathrm{Sha}^1_\ell(\chi,e)$ is the module of classes in $H^1(G_{\mathbb{Q}},\mathcal{O}^\vee(\theta))$ that are trivial on a decomposition group at ℓ and on every inertia group, i.e. Sha_θ. By (37.1), $\mathrm{Ber}_{e+1}(\chi)=N\,(e+2)\#\,B_{e+1,\psi}$, so

$$v_{\mathcal{O}}(\mathrm{Ber}_{e+1}(\chi))=v_{\mathcal{O}}(N)+v_{\mathcal{O}}((e+2)\#)+v_{\mathcal{O}}(B_{e+1,\psi})\geq v_{\mathcal{O}}(B_{e+1,\psi}). \tag{37.4}$$

We use the following fact. (R) If $\zeta \neq 1$ is a root of unity in $\mathcal{O}$ of order n prime to ℓ, then $\zeta - 1 \in \mathcal{O}^\times$, because $X^n - 1$ is separable modulo $\mathfrak{m}$, so reduction is injective on μ_n. The values of ψ are roots of unity of order dividing the order of ψ.

Case A: $\theta(g) - \chi_{\mathrm{cyc}}(g) \in \mathfrak{m}$ for all $g \in G_{\mathbb{Q}}$. Let g lie in an inertia group at a prime $p \neq \ell$. Then $\chi_{\mathrm{cyc}}(g) = 1$, so $\psi(g)^{-1} = \theta(g) \equiv 1 \pmod{\mathfrak{m}}$, and $\psi(g) = 1$ by (R); hence $\theta(g) = 1$. By Lemma 37.6, $\mathrm{Sha}_\theta = 0$, and the bound holds because $\mathrm{Ber}_{e+1}(\chi) \in \mathcal{O}$ has $v_{\mathcal{O}} \geq 0$.

Case B: $\theta(g_1) - \chi_{\mathrm{cyc}}(g_1) \in \mathcal{O}^\times$ for some g_1. Let Σ be the set of primes dividing $f\ell$, and let $c \in G_{\mathbb{Q}}$ be a complex conjugation; it acts on every root of unity by inversion, so $\theta(c) = \psi(-1)^{-1}(-1)^{-e} = -1$ and $\theta(c) - 1 = -2$ is a unit: c is a normaliser. The hypotheses of Lemma 37.5 hold for (θ, c, Σ):

(1) for g in an inertia group at $p \notin \Sigma$: $\psi(g) = 1$, since ψ factors through $\mathrm{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q})$ and $p \nmid f$, and $\chi_{\mathrm{cyc}}(g) = 1$ as $p \neq \ell$; so $\theta(g) = 1$;

(2) let $p \in \Sigma$, $p \neq \ell$, so $p \mid f$; write $f = p^k f'$ with $p \nmid f'$ and $\psi = \psi_p \psi'$ with ψ_p modulo p^k and ψ' modulo f'; ψ_p is nontrivial since ψ is primitive. An inertia group I at p maps onto the subgroup of $(\mathbb{Z}/f\mathbb{Z})^\times \cong (\mathbb{Z}/p^k\mathbb{Z})^\times \times (\mathbb{Z}/f'\mathbb{Z})^\times$ of the classes $\equiv 1 \pmod{f'}$ (p is totally ramified in $\mathbb{Q}(\mu_{p^k})$ and unramified in $\mathbb{Q}(\mu_{f'})$); so some $g \in I$ has $\psi(g) = \psi_p(g) \neq 1$. Then $\theta(g) = \psi(g)^{-1}$ and $\theta(g) - 1 = -\psi(g)^{-1}(\psi(g) - 1)$ is a unit by (R).

Hence $\mathrm{length}_{\mathcal{O}} \mathrm{Sha}^1_\ell(\chi, e) \leq \sup_{a_0 \neq 0} \mathrm{length}_{\mathcal{O}} \mathrm{Z}_{\theta,c}(\Sigma, a_0)$. Fix $a_0 \in \mathcal{O} \setminus \{0\}$. By Lemma 37.7 (applied to ψ, e, g_1, a_0) there are $m \geq \max(1, v_\ell(f))$ and $0 \leq b < N' = N_0 \ell^m$ with $\gcd(b, N') = 1$, with N_0 the prime-to-ℓ part of f, such that (N_0, m) is a Kolyvagin level of θ; choose a family of lifts σ modulo N' (every class in $(\mathbb{Z}/N'\mathbb{Z})^\times$ is induced by an element of $G_{\mathbb{Q}}$) and write $\mathrm{st} = \mathrm{st}_{N',b}(\theta, \sigma)$; then $a_0 \,\mathrm{st} \mid \ell^m$ and $v_{\mathcal{O}}(\mathrm{st}) = v_{\mathcal{O}}(B_{e+1,\psi}) - v_{\mathcal{O}}(e+1)$. The primes dividing N' are exactly those of Σ. So Proposition 37.11 applies and gives, with (37.4),

$$\mathrm{length}_{\mathcal{O}} \mathrm{Z}_{\theta,c}(\Sigma, a_0) \leq v_{\mathcal{O}}(\mathrm{st}) \leq v_{\mathcal{O}}(B_{e+1,\psi}) \leq v_{\mathcal{O}}(\mathrm{Ber}_{e+1}(\chi)).$$

Taking the supremum over a_0 proves the proposition. □

Lemma 37.5. *Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring, $\theta\colon G_{\mathbb{Q}} \to \mathcal{O}^\times$ a continuous character, δ a normaliser for θ and Σ a finite set of primes containing ℓ, such that*

1. *$\theta(g) = 1$ for every g in an inertia group at a prime $p \notin \Sigma$, and*
2. *for every prime $p \in \Sigma \setminus \{\ell\}$ and every inertia group I at p there is $g \in I$ with $\theta(g) - 1 \in \mathcal{O}^\times$.*

Let $\mathrm{Sha}_\theta \subseteq H^1(G_{\mathbb{Q}}, \mathcal{O}^\vee(\theta))$ be the submodule of classes whose restriction to every inertia group (at every prime) and to a decomposition group at ℓ is zero. Then, in $\mathbb{N} \cup \{\infty\}$,

$$\mathrm{length}_{\mathcal{O}} \mathrm{Sha}_\theta \leq \sup_{a \in \mathcal{O} \setminus \{0\}} \mathrm{length}_{\mathcal{O}} \mathrm{Z}_{\theta,\delta}(\Sigma, a).$$

Proof. For $y \in \mathcal{O}^\vee$ write ∂y for the coboundary $g \mapsto \theta(g)y - y$.

Step 1: normalised representatives. Every continuous cocycle z (for the action θ) is cohomologous to exactly one cocycle vanishing at δ: $z - \partial y$ vanishes at δ for $y = (\theta(\delta) - 1)^{-1} z(\delta)$, and a coboundary ∂y with $\partial y(\delta) = (\theta(\delta) - 1)y = 0$ has $y = 0$. So $c \mapsto z_c$, the representative of c with $z_c(\delta) = 0$, is an injective $\mathcal{O}$-linear map from $H^1(G_{\mathbb{Q}}, \mathcal{O}^\vee(\theta))$ to the module of maps $G_{\mathbb{Q}} \to \mathcal{O}^\vee$.

Step 2: each z_c, $c \in \mathrm{Sha}_\theta$, lies in some $\mathrm{Z}_{\theta,\delta}(\Sigma, a)$. Let $z = z_c$. Restriction to conjugate subgroups corresponds under conjugation, and conjugation by an element of $G_\mathbb{Q}$ acts trivially on $H^1(G_\mathbb{Q}, \mathcal{O}^\vee(\theta))$; so z restricts to a coboundary on every inertia group and on one decomposition group D_ℓ at ℓ: $z|_{D_\ell} = \partial y_\ell$. The set $z(G_\mathbb{Q})$ is finite (z is continuous into a discrete module and $G_\mathbb{Q}$ is compact), and every element of $\mathcal{O}^\vee$ is killed by a power of ℓ; let $a \neq 0$ kill y_ℓ and all values of z. Then (Kd1) and (Kd2) hold, and (Kd5) is empty for $n = 1$.

(Kd3) Let I be an inertia group at $p \notin \Sigma$. Then $z|_I = \partial y$ for some y, and $\partial y|_I = 0$ as $\theta(I) = 1$; so $z|_I = 0$.

(Kd4) at ℓ A decomposition group at ℓ is $\sigma D_\ell \sigma^{-1}$ for some $\sigma \in G_\mathbb{Q}$, and for $g \in D_\ell$, $h = \sigma g \sigma^{-1}$, one has $z(h) = \theta(h)y' - y'$ with $y' = \theta(\sigma)y_\ell - z(\sigma)$, and $ay' = 0$.

Kd4) at $p \in \Sigma$, $p \neq \ell$ Let D be a decomposition group at p and $I = I(D)$; choose $g_0 \in I$ with $\theta(g_0) - 1$ a unit (hypothesis 2). As $z|_I = \partial y$ for some y, $y = (\theta(g_0) - 1)^{-1} z(g_0)$, so $ay = 0$. For $d \in D$, $d^{-1} g_0 d \in I$ and $\theta(d^{-1} g_0 d) = \theta(g_0)$, so $z(d^{-1} g_0 d) = (\theta(g_0) - 1)y$; expanding the left side with the cocycle relation gives $(\theta(g_0) - 1)(z(d) + y) = \theta(d)(\theta(g_0) - 1)y$, hence $z(d) = \theta(d)y - y$ for all $d \in D$.

So $z \in \mathrm{Z}_{\theta,\delta}(\Sigma, a)$.

Step 3: lengths. Let π be a uniformiser. For $a = u\pi^k$ with u a unit, $\mathrm{Z}_{\theta,\delta}(\Sigma, a) = \mathrm{Z}_{\theta,\delta}(\Sigma, \pi^k)$, and these modules increase with k. By Steps 1 and 2, Sha_θ is isomorphic to its image V under $c \mapsto z_c$, and $V = \bigcup_k V_k$ with $V_k = V \cap \mathrm{Z}_{\theta,\delta}(\Sigma, \pi^k)$ increasing, $\operatorname{length} V_k \leq s$, where s is the supremum in the statement. If $s = \infty$ there is nothing to prove. Otherwise a strictly increasing chain of submodules $0 = V^{(0)} \subsetneq \cdots \subsetneq V^{(t)}$ of V gives, choosing $x_i \in V^{(i)} \setminus V^{(i-1)}$, a strictly increasing chain of the same length in the submodule generated by $x_1, \ldots, x_t$, which lies in some V_k; so $t \leq s$, and $\operatorname{length} \mathrm{Sha}_\theta = \operatorname{length} V \leq s$. □

Lemma 37.6. *Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring with maximal ideal $\mathfrak{m}$, and $\theta\colon G_\mathbb{Q} \to \mathcal{O}^\times$ a continuous character such that $\theta(g) - \chi_{\mathrm{cyc}}(g) \in \mathfrak{m}$ for every $g \in G_\mathbb{Q}$ and $\theta(g) = 1$ for every g in an inertia group at a prime $p \neq \ell$. Then $\mathrm{Sha}_\theta = 0$.*

Proof. A normaliser at ℓ. Let D_0 be a decomposition group at ℓ. As $\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell$ is totally ramified of degree $\ell - 1$, $I(D_0)$ maps onto $\mathrm{Gal}(\mathbb{Q}(\mu_\ell)/\mathbb{Q})$; choose $t \in I(D_0)$ with $t(\zeta) = \zeta^{-1}$ for $\zeta \in \mu_\ell$. Then $\chi_{\mathrm{cyc}}(t) \equiv -1 \pmod{\ell}$, so $\theta(t) \equiv -1 \pmod{\mathfrak{m}}$ and $\theta(t) - 1$ is a unit (ℓ odd): t is a normaliser. By Lemma 37.5 with $\Sigma = \{\ell\}$ and $\delta = t$ (hypothesis 1 is assumed, hypothesis 2 is empty), it suffices to show $\mathrm{Z}_{\theta,t}(\{\ell\}, a) = 0$ for every $a \neq 0$.

Reduction to π-torsion. Let π be a uniformiser and $\mathrm{Z} = \mathrm{Z}_{\theta,t}(\{\ell\}, a)$, an $\mathcal{O}$-module killed by a, hence by π^k for $k = v_\mathcal{O}(a)$. It suffices to show: if $z \in \mathrm{Z}$ and $\pi z = 0$, then $z = 0$; for then $z \in \mathrm{Z}$ with $\pi^j z = 0$ gives $\pi^{j-1} z \in \mathrm{Z}$ with $\pi(\pi^{j-1} z) = 0$, so $\pi^{j-1} z = 0$, and by descending induction $z = 0$.

Kummer theory. Let $z \in \mathrm{Z}$ with $\pi z = 0$. Its values lie in $V = \{x \in \mathcal{O}^\vee : \pi x = 0\}$, a finite-dimensional vector space over $\mathcal{O}/\mathfrak{m}$, hence over $\mathbb{F}_\ell$. For $g \in G_\mathbb{Q}$ let $n_g \in \mathbb{Z}$ with $n_g \equiv \chi_{\mathrm{cyc}}(g) \pmod{\ell}$; as $\theta(g) - n_g \in \mathfrak{m}$, $\theta(g)x = n_g x$ for $x \in V$. Fix a primitive ℓ-th root of unity ζ and an $\mathbb{F}_\ell$-linear map $f\colon V \to \mathbb{F}_\ell$, and put $\psi_f(g) = \zeta^{f(z(g))}$. Since $g(\zeta) = \zeta^{n_g}$, (Kd1) gives $\psi_f(gh) = \psi_f(g)\, g(\psi_f(h))$, so ψ_f is a continuous 1-cocycle with values in $\mu_\ell \subset \overline{\mathbb{Q}}^\times$. By Hilbert's Theorem 90 there is $\beta \in \overline{\mathbb{Q}}^\times$ with $\psi_f(g) = g(\beta)/\beta$ for all g; then $A = \beta^\ell$ is fixed by $G_\mathbb{Q}$, so $A \in \mathbb{Q}^\times$.

Local conditions. By (Kd3), z vanishes on the inertia groups at every prime $p \neq \ell$. By (Kd4), $z|_{D_0} = \theta y - y$ for some y; by (Kd1), $0 = z(t) = (\theta(t) - 1)y$, so $y = 0$ and z vanishes on D_0. So for

every prime p some inertia group I at p fixes β; then β lies in the maximal unramified extension of $\mathbb{Q}_p$ inside the completion of $\overline{\mathbb{Q}}$ at the corresponding place, whose normalised valuation extends v_p with value group $\mathbb{Z}$, and $v_p(A) = \ell\, v(\beta) \in \ell\mathbb{Z}$. Hence $A = \pm C^\ell$ with $C \in \mathbb{Q}^\times$, and since ℓ is odd, $A = (\pm C)^\ell$: replacing C by $\pm C$, $A = C^\ell$. Then $\beta = \zeta^j C$ for some integer j, and $\psi_f(g) = \zeta^{j(n_g-1)}$, i.e. $f(z(g)) = j(n_g-1)$ in $\mathbb{F}_\ell$. At $g = t$: $0 = j(n_t-1)$ with $n_t - 1 \equiv -2 \not\equiv 0 \pmod{\ell}$, so $j \equiv 0$ and $f \circ z = 0$. As f was arbitrary, $z = 0$. $\square$

Lemma 37.7. *Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring, $f \geq 1$, ψ a primitive Dirichlet character modulo f with values in $\mathcal{O}^\times$, and $e \geq 1$ an integer with $\psi(-1) = (-1)^{e+1}$. Put $\theta = \psi^{-1}\chi_{\mathrm{cyc}}^{-e}$ and assume that $\theta(g_1) - \chi_{\mathrm{cyc}}(g_1) \in \mathcal{O}^\times$ for some $g_1 \in G_{\mathbb{Q}}$. Let N_0 be the prime-to-ℓ part of f. Then for every $a_0 \in \mathcal{O} \setminus \{0\}$ there are integers $m \geq \max(1, v_\ell(f))$ and b with $0 \leq b < N = N_0\ell^m$ and $\gcd(b, N) = 1$ such that (N_0, m) is a Kolyvagin level of θ and, for every family of lifts σ modulo N,*

$$v_{\mathcal{O}}\big(\mathrm{st}_{N,b}(\theta,\sigma)\big) + v_{\mathcal{O}}(e+1) = v_{\mathcal{O}}(B_{e+1,\psi}) \qquad \text{and} \qquad a_0\,\mathrm{st}_{N,b}(\theta,\sigma) \mid \ell^m \text{ in } \mathcal{O}.$$

Proof. Put $m_1 = \max(1, v_\ell(f))$ and $F = N_0\ell^{m_1}$; then $f \mid F$, $\ell \mid F$, and F and $f\ell$ have the same prime divisors. For $m \geq m_1$ put $N = N_0\ell^m = F\ell^j$ with $j = m - m_1$, and let χ be the character modulo N induced by ψ ($\chi(t) = \psi(t)$ if $\gcd(t,N) = 1$, $\chi(t) = 0$ otherwise); $\chi(t)$ depends only on t modulo F, and χ_F denotes the same character modulo F. Put $B_i^{(F)} = F^{\,i-1}\sum_{c=1}^{F}\chi_F(c)B_i(c/F)$.

Step 1: Kolyvagin levels. Let $m \geq m_1$. For $u \in U_N$: $\psi(u) = 1$ as $f \mid N$, and $\chi_{\mathrm{cyc}}(u) \equiv 1 \pmod{\ell^m}$, so $\theta(u) \equiv 1$: (Kv1). If g acts on μ_N by inversion, $\psi(g) = \psi(-1) = (-1)^{e+1}$ and $\chi_{\mathrm{cyc}}(g) \equiv -1 \pmod{\ell^m}$, so $\theta(g) \equiv (-1)^{e+1}(-1)^e = -1$: (Kv2). (Kv3) is the hypothesis. So (N_0, m) is a Kolyvagin level.

Step 2: the choice of b. Let $b \in \{1, \dots, N-1\}$ with $g_1(\zeta) = \zeta^b$ for $\zeta \in \mu_N$; $\gcd(b,N) = 1$. Then $\psi(g_1) = \psi(b)$, $\chi_{\mathrm{cyc}}(g_1) \equiv b \pmod{\ell^m}$, and $\theta(g_1) - \chi_{\mathrm{cyc}}(g_1) \equiv \psi(b)^{-1}b^{-e}(1 - \psi(b)b^{e+1}) \pmod{\ell^m}$; as $m \geq 1$, $u_b = \psi(b)b^{e+1} - 1$ is a unit.

Step 3: reduction to a Stickelberger sum. Let σ be a family of lifts modulo N. Since σ_t acts on μ_f by $\tilde{t}$ and $\chi_{\mathrm{cyc}}(\sigma_t) \equiv \tilde{t} \pmod{\ell^m}$, $\theta(\sigma_t)^{-1} = \psi(\sigma_t)\chi_{\mathrm{cyc}}(\sigma_t)^e \equiv \chi(\tilde{t})\tilde{t}^{\,e}$. As $\gcd(\tilde{t}b, N) = 1$ and $N \geq 2$, $\tilde{t}b/N \notin \mathbb{Z}$, so $\lfloor (N-\tilde{t})b/N \rfloor = b - 1 - \lfloor \tilde{t}b/N \rfloor$ and $h_N(b,t) = 2\lfloor \tilde{t}b/N \rfloor - (b-1)$. Hence, with sums over $0 \leq t < N$,

$$\mathrm{st}_{N,b}(\theta,\sigma) \equiv 2T' - (b-1)P_e \pmod{\ell^m\mathcal{O}}, \qquad T' = \sum_t \Big\lfloor \frac{bt}{N} \Big\rfloor \chi(t)t^e, \quad P_k = \sum_t \chi(t)t^k.$$

Substituting $t \mapsto N - t$ and using $\chi(-1) = (-1)^{e+1}$, $(N-t)^e \equiv (-1)^e t^e \pmod{N}$, gives $P_e \equiv -P_e \pmod{N\mathcal{O}}$; as ℓ is odd and $v_\ell(N) = m$, $v_\ell(P_e) \geq m$. So $v_\ell(\mathrm{st}_{N,b}(\theta,\sigma) - 2T') \geq m$.

Step 4: an exact identity. For $0 \leq t < N$ let $\iota(t) = \lfloor bt/N \rfloor$ and $\rho(t) = bt - N\iota(t) \in [0, N)$. Then $t \mapsto \rho(t)$ is a bijection of $\{0, \dots, N-1\}$ and $\chi(\rho(t)) = \chi(b)\chi(t)$. Expanding $\rho(t)^{e+1} = b^{e+1}t^{e+1} - (e+1)b^e t^e N\iota(t) + N^2 y_t$ with $y_t \in \mathbb{Z}$,

$$P_{e+1} = \sum_t \chi(\rho(t))\rho(t)^{e+1} = \chi(b)b^{e+1}P_{e+1} - (e+1)\chi(b)b^e N\,T' + N^2 Y, \qquad Y \in \mathcal{O},$$

so that

$$(e+1)\chi(b)b^e\,T' = u_b\,\frac{P_{e+1}}{N} + NY, \qquad u_b = \chi(b)b^{e+1} - 1. \tag{37.5}$$

Step 5: the power sum. Write $t = c + Fu$ with $1 \leq c \leq F$ and $0 \leq u < \ell^j$ (the terms $t = 0$ and $t = N$ vanish). From $\sum_{u=0}^{L-1}(X+u)^{e+1} = \frac{1}{e+2}(B_{e+2}(X+L) - B_{e+2}(X)) =$

$\frac{1}{e+2}\sum_{i=0}^{e+1}\binom{e+2}{i}B_i(X)L^{e+2-i}$ with $X = c/F$, $L = \ell^j$, multiplied by $\chi_F(c)F^{e+1}$ and summed over c,

$$\frac{P_{e+1}}{N} = B_{e+1}^{(F)} + \sum_{i=0}^{e}\frac{\binom{e+2}{i}}{e+2}N^{\,e+1-i}B_i^{(F)}.$$

For $0 \le i \le e+1$, $F^iB_i(c/F) = \sum_{r\le i}\binom{i}{r}B_rc^{i-r}F^r$ and $(i+1)\#\,B_r \in \mathbb{Z}$ for $r \le i$ by Theorem B.19 ($(i+1)\#$ the product of the primes $\le i+1$), so $F(i+1)\#B_i^{(F)}$ is integral and $v_\ell(B_i^{(F)}) \ge -v_\ell(F)-1$. Hence each term of the sum has $v_\ell \ge (j+v_\ell(F))-v_\ell(F)-1-v_\ell(e+2) = j-1-v_\ell(e+2)$, and with (37.5), as $v_\ell(N) = m \ge j$,

$$v_\ell\Big((e+1)\chi(b)b^eT' - u_bB_{e+1}^{(F)}\Big) \ge j-1-v_\ell(e+2). \tag{37.6}$$

Step 6: the Bernoulli number. By item 1 of Lemma 37.8 (for χ_F, with $\chi_F^* = \psi$), $B_{e+1}^{(F)} = B_{e+1,\psi}\prod_{p|F}(1-\psi(p)p^e)$; for $p \mid f$, $\psi(p) = 0$, and the only other prime dividing F is ℓ (if $\ell \nmid f$), with $v_\ell(\psi(\ell)\ell^e) = e \ge 1$. So $v_\ell(B_{e+1}^{(F)}) = v_\ell(B_{e+1,\psi}) =: \beta$. By item 1 of Proposition 37.9 (modulus f, $k = e+1$, H trivial), $f(e+2)\#B_{e+1,\psi} \ne 0$, so $\beta < \infty$.

Step 7: choice of m and conclusion. Choose $m \ge m_1$ so large that $j = m - m_1$ satisfies $j-1-v_\ell(e+2) > \beta$ and $e_{\mathcal{O}}m \ge v_{\mathcal{O}}(a_0) + e_{\mathcal{O}}\beta$, and let b be as in Step 2 for this m; in particular $\gcd(b,N) = 1$. As $\chi(b) = \psi(b)$ and b are units and u_b is a unit, $v_\ell(u_bB_{e+1}^{(F)}) = \beta$, and (37.6) gives $v_\ell((e+1)T') = \beta$, i.e. $v_\ell(T') = \beta - v_\ell(e+1)$. By Step 3 and $m > \beta$, $v_\ell(\mathrm{st}_{N,b}(\theta,\sigma)) = v_\ell(2T') = \beta - v_\ell(e+1)$, which is the first claim after multiplying by $e_{\mathcal{O}}$. Then $v_{\mathcal{O}}(a_0\,\mathrm{st}_{N,b}(\theta,\sigma)) \le v_{\mathcal{O}}(a_0) + e_{\mathcal{O}}\beta \le e_{\mathcal{O}}m = v_{\mathcal{O}}(\ell^m)$, so $a_0\,\mathrm{st}_{N,b}(\theta,\sigma) \mid \ell^m$ in the discrete valuation ring $\mathcal{O}$. □

Lemma 37.8. *Let ℓ be a prime, $F \ge 2$ an integer divisible by ℓ, $e \ge 1$ an integer, and χ a Dirichlet character modulo F with values in $\overline{\mathbb{Q}}_\ell^\times$ (extended by 0 to the integers not prime to F). Put $a = 1+F$, $Q_j = F\ell^j$ for $j \ge 0$,*

$$T_j(\chi) = \sum_{1\le b<Q_j}\Big\lfloor\frac{ab}{Q_j}\Big\rfloor b^e\chi(b), \qquad B_{k,\chi}^{(F)} = F^{\,k-1}\sum_{b=1}^{F}\chi(b)\,B_k\Big(\frac{b}{F}\Big) \quad (k \ge 0).$$

Then:

1. *for every $k \ge 1$, $B_{k,\chi}^{(F)} = B_{k,\chi^*}\prod_{p|F}(1-\chi^*(p)\,p^{k-1})$, the product over the primes dividing F;*
2. *for every $j \ge 0$,*

$$v_\ell\Big((e+1)\,a^e\,T_j(\chi) - (a^{e+1}-1)\,B_{e+1,\chi}^{(F)}\Big) \ge j - v_\ell(e+2) - 1.$$

Proof. Three identities follow from the generating function: $B_n(X+Y) = \sum_{i=0}^{n}\binom{n}{i}B_i(X)Y^{n-i}$ (multiply by e^{Yt}); $B_n(X+1) - B_n(X) = nX^{n-1}$ (the difference of the generating functions is te^{Xt}); and, for an integer $h \ge 1$, $\sum_{u=0}^{h-1}B_n(\frac{X+u}{h}) = h^{1-n}B_n(X)$.

Item 1. Let $f \mid F$ be the conductor of χ and $R = \prod_{p|F}p$. For $1 \le b \le F$, $\chi(b) = \chi^*(b)$ if $\gcd(b,F) = 1$ and $\chi(b) = 0$ otherwise; since $\chi^*(b) = 0$ when $\gcd(b,f) > 1$, in all cases $\chi(b) = \chi^*(b)\sum_{g|\gcd(b,R)}\mu(g)$. Hence, writing $b = gb'$,

$$B_{k,\chi}^{(F)} = \sum_{g|R}\mu(g)\chi^*(g)\,F^{\,k-1}\sum_{b'=1}^{F/g}\chi^*(b')\,B_k\Big(\frac{b'}{F/g}\Big).$$

If $\chi^*(g) \neq 0$ then $\gcd(g,f)=1$, so $f \mid F/g$; write $F/g = fh$. Grouping $b' = r + fu$ ($1 \le r \le f$, $0 \le u < h$) and using the last identity with $X = r/f$, $(F/g)^{k-1}\sum_{b'} \chi^*(b')B_k(b'/(F/g)) = (fh)^{k-1}h^{1-k}\sum_{r=1}^{f}\chi^*(r)B_k(r/f) = B_{k,\chi^*}$. With $F^{k-1} = g^{k-1}(F/g)^{k-1}$ this gives $B_{k,\chi}^{(F)} = B_{k,\chi^*}\sum_{g\mid R}\mu(g)\chi^*(g)g^{k-1} = B_{k,\chi^*}\prod_{p\mid F}(1-\chi^*(p)p^{k-1})$.

Integrality. For $0 \le i \le e+1$, $F^{\,i}B_i(b/F) = \sum_{r=0}^{i}\binom{i}{r}B_r\, b^{i-r}F^{\,r}$, and $(i+1)\#\,B_r \in \mathbb{Z}$ for $r \le i$ by Theorem B.19 (the denominator of B_r is a product of distinct primes $p \le r+1$), with $(i+1)\# = \prod_{p\le i+1} p$. So $F\,(i+1)\#\,B_{i,\chi}^{(F)}$ is integral over $\mathbb{Z}$, and, as ℓ divides $(i+1)\#$ at most once,

$$v_\ell(B_{i,\chi}^{(F)}) \ge -v_\ell(F) - 1 \qquad (0 \le i \le e+1). \tag{37.7}$$

Item 2: an exact identity. Fix j and write $Q = Q_j$. Every prime dividing Q divides F and $a \equiv 1 \pmod F$, so $\gcd(a,Q)=1$. For $0 \le b < Q$ put $\iota(b) = \lfloor ab/Q \rfloor$ and $\rho(b) = ab - Q\,\iota(b) \in [0,Q)$. Then $b \mapsto \rho(b)$ is a bijection of $\{0,\dots,Q-1\}$ (multiplication by a modulo Q), and $\rho(b) \equiv ab \equiv b \pmod F$, so $\chi(\rho(b)) = \chi(b)$. Put $P_k = \sum_{b=0}^{Q-1}\chi(b)b^k$; then $T_j(\chi) = \sum_{b=0}^{Q-1}\iota(b)\chi(b)b^e$ as $\chi(0)=0$. Let $\mathcal{O}_\chi$ be the ring generated over $\mathbb{Z}$ by the values of χ; the Y_i below lie in $\mathcal{O}_\chi$.

(i) From $ab = \rho(b) + Q\iota(b)$: $aP_{e+1} = Y_0 + QT_j(\chi)$ with $Y_0 = \sum_b \chi(b)b^e\rho(b)$.

(ii) From $(ab)^e = (\rho(b)+Q\iota(b))^e = \rho(b)^e + eQ\iota(b)\rho(b)^{e-1} + Q^2y_b$ with $y_b \in \mathbb{Z}$, multiplied by $\chi(\rho(b))\rho(b)$ and summed, using the bijection: $a^eY_0 = P_{e+1} + eQ\sum_b \chi(b)\iota(b)\rho(b)^e + Q^2Y_1$.

(iii) As $\rho(b) \equiv ab \pmod Q$, $\rho(b)^e \equiv a^eb^e \pmod Q$, so $\sum_b \chi(b)\iota(b)\rho(b)^e = a^eT_j(\chi) + QY_2$.

Multiplying (i) by a^e and inserting (ii), (iii):

$$a^{e+1}P_{e+1} = P_{e+1} + (e+1)Q\,a^eT_j(\chi) + Q^2Y_3, \qquad \text{so} \qquad (e+1)\,a^eT_j(\chi) = (a^{e+1}-1)\frac{P_{e+1}}{Q} - QY_3.$$

Item 2: the power sum. Write $b = c + Fu$ with $1 \le c \le F$ and $0 \le u < L = Q/F$ (the term $b=0$ and the terms with $c = F$ vanish, as $\chi(0) = \chi(F) = 0$). By the first two identities, $\sum_{u=0}^{L-1}(c/F+u)^{e+1} = \frac{1}{e+2}\big(B_{e+2}(c/F+L) - B_{e+2}(c/F)\big) = \frac{1}{e+2}\sum_{i=0}^{e+1}\binom{e+2}{i}B_i(c/F)L^{e+2-i}$. Multiplying by $\chi(c)F^{e+1}$ and summing over c,

$$\frac{P_{e+1}}{Q} = \frac{1}{e+2}\sum_{i=0}^{e+1}\binom{e+2}{i}Q^{\,e+1-i}B_{i,\chi}^{(F)} = B_{e+1,\chi}^{(F)} + \sum_{i=0}^{e}\frac{\binom{e+2}{i}}{e+2}\,Q^{\,e+1-i}B_{i,\chi}^{(F)}.$$

Item 2: valuations. Hence

$$(e+1)a^eT_j(\chi) - (a^{e+1}-1)B_{e+1,\chi}^{(F)} = (a^{e+1}-1)\sum_{i=0}^{e}\frac{\binom{e+2}{i}}{e+2}Q^{e+1-i}B_{i,\chi}^{(F)} - QY_3.$$

Here $v_\ell(Q) = j + v_\ell(F) \ge j+1$, so $v_\ell(QY_3) \ge j+1$; $v_\ell(a^{e+1}-1) \ge 0$; and for $0 \le i \le e$, by (37.7), $v_\ell\big(\binom{e+2}{i}Q^{e+1-i}B_{i,\chi}^{(F)}/(e+2)\big) \ge (j+v_\ell(F)) - v_\ell(F) - 1 - v_\ell(e+2) = j - 1 - v_\ell(e+2)$. This proves item 2. □

Proposition 37.9. *Let $N \ge 1$ and $k \ge 1$ be integers and $H \le (\mathbb{Z}/N\mathbb{Z})^\times$ a subgroup. For a field Ω of characteristic 0 let $\mathcal{X}_\Omega$ be the set of Dirichlet characters χ modulo N with values in $\Omega^\times$ such that $H \subseteq \ker\chi$ and $\chi(-1) = (-1)^k$. Then:*

1. *for every prime ℓ and every $\chi \in \mathcal{X}_{\overline{\mathbb{Q}}_\ell}$, $\mathrm{Ber}_k(\chi)$ is nonzero and integral over $\mathbb{Z}_\ell$;*

2. *there are integers $w_\ell \geq 0$, one for each prime ℓ, such that $\sum_{\ell\in S} w_\ell \log \ell \leq |\mathcal{X}_{\mathbb{C}}| \log(16N^{k+1}k!)$ for every finite set S of primes, and $\sum_{\chi\in\mathcal{X}_{\overline{\mathbb{Q}}_\ell}} v_\ell(\mathrm{Ber}_k(\chi)) \leq w_\ell$ for every prime ℓ.*

Proof. Put $m = \varphi(N)$; the exponent of $(\mathbb{Z}/N\mathbb{Z})^\times$ divides m, so a character of it with values in a field Ω of characteristic 0 takes values in the group $\mu_m(\Omega)$ of m-th roots of unity.

Transport of characters. By (37.1),

$$\mathrm{Ber}_k(\chi) = N\,(k+1)\#\,f^{\,k-1}\sum_{a=1}^{f}\chi^*(a)\,B_k\Big(\frac{a}{f}\Big), \tag{37.8}$$

a linear combination with rational coefficients of the values of χ^*, where f is the conductor of χ. Let Ω, Ω' be fields of characteristic 0 and $j\colon \Omega \to \Omega'$ a field embedding. For a character χ modulo N with values in $\Omega^\times$, $j\circ\chi$ has the same kernel and the same value at -1; the conductor depends only on the kernel, and $(j\circ\chi)^* = j\circ\chi^*$. Hence $\mathrm{Ber}_k(j\circ\chi) = j(\mathrm{Ber}_k(\chi))$. If $\mu_m(\Omega)$ and $\mu_m(\Omega')$ both have m elements, j maps the first bijectively onto the second, so $\chi \mapsto j\circ\chi$ is a bijection $\mathcal{X}_\Omega \to \mathcal{X}_{\Omega'}$. This applies to $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ (so $\mathcal{X}_{\mathbb{C}} = \mathcal{X}_{\overline{\mathbb{Q}}}$), to every automorphism σ of $\overline{\mathbb{Q}}$, and to a fixed embedding $j_\ell\colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ (which exists because $\overline{\mathbb{Q}}_\ell$ is algebraically closed of characteristic 0).

Integrality. Let $\chi \in \mathcal{X}_{\overline{\mathbb{Q}}}$ with conductor f. Multiplying the generating functions $t/(e^t-1)$ and e^{Xt} gives $B_k(X) = \sum_{i=0}^{k}\binom{k}{i}B_i X^{k-i}$, so

$$f\,B_{k,\chi^*} = \sum_{a=1}^{f}\chi^*(a)\,a^k + \sum_{i=1}^{k}\binom{k}{i}B_i\,f^{\,i}\sum_{a=1}^{f}\chi^*(a)\,a^{k-i}.$$

By Theorem B.19, for $0 \leq i \leq k$ the denominator of B_i is a product of distinct primes p with $(p-1) \mid i$ or $p = 2$, hence $p \leq i+1 \leq k+1$; so $(k+1)\#\,B_i \in \mathbb{Z}$. The values of χ^* are roots of unity. Hence $f\,(k+1)\#\,B_{k,\chi^*}$ is an algebraic integer, and so is $\mathrm{Ber}_k(\chi) = (N/f)\cdot f\,(k+1)\#\,B_{k,\chi^*}$, as $f \mid N$.

Absolute values. We show: for $\chi \in \mathcal{X}_{\mathbb{C}}$, $\mathrm{Ber}_k(\chi) \neq 0$ and

$$|\mathrm{Ber}_k(\chi)| \leq 16\,N^{k+1}k!. \tag{37.9}$$

Let f be the conductor; $\chi^*(-1) = \chi(-1) = (-1)^k$. There is no primitive character modulo 2, so $f = 1$ or $f \geq 3$. Let $a \in \{0,1\}$ with $a \equiv k \pmod 2$ and $i = (k-a)/2 \in \mathbb{Z}_{\geq 0}$.

Case $f \geq 3$. Then χ^* is primitive modulo f with $\chi^*(-1) = (-1)^a$, and Theorem B.12 at $s = 1-k$ gives $|\xi(1-k,\chi^*)| = |\xi(k,\overline{\chi^*})|$. At $s = 1-k$ the Gamma factor is $\Gamma(\frac12 - i)$, finite and nonzero; at $s = k$ it is $\Gamma(i+a) = (i+a-1)!$, as $i + a \geq 1$ ($a = 0$ forces $k \geq 2$). Hence

$$|L(1-k,\chi^*)| = \Big(\frac{f}{\pi}\Big)^{k-\frac12}\frac{(i+a-1)!}{|\Gamma(\frac12 - i)|}\,|L(k,\overline{\chi^*})| = \frac{2\,(k-1)!\,f^{\,k-\frac12}}{(2\pi)^k}\,|L(k,\overline{\chi^*})|.$$

By Theorem B.11, $|B_{k,\chi^*}| = k\,|L(1-k,\chi^*)|$, so

$$|B_{k,\chi^*}| = \frac{2\,k!\,f^{\,k-\frac12}}{(2\pi)^k}\,|L(k,\overline{\chi^*})|. \tag{37.10}$$

The character $\overline{\chi^*}$ is primitive modulo $f \geq 3$, hence not principal. If $k \geq 2$, the Euler product of $L(k,\overline{\chi^*})$ converges absolutely and has no vanishing factor, so $L(k,\overline{\chi^*}) \neq 0$, and $|L(k,\overline{\chi^*})| \leq$

$\zeta(2)=\pi^2/6$. If $k=1$, $L(1,\overline{\chi^*})\neq 0$ by Theorem B.2. So $B_{k,\chi^*}\neq 0$ and $\mathrm{Ber}_k(\chi)\neq 0$. For $k\geq 2$, by (37.10), $(k+1)\#<4^{k+1}$ (Lemma B.3) and $f\leq N$,

$$|\mathrm{Ber}_k(\chi)|<N\,4^{k+1}\,\frac{2\,k!\,f^{\,k-\frac12}}{(2\pi)^k}\,\frac{\pi^2}{6}=\frac{4\pi^2}{3}\Big(\frac{2}{\pi}\Big)^k N\,f^{\,k-\frac12}\,k!\leq\frac{16}{3}\,N^{k+\frac12}\,k!,$$

using $(2/\pi)^k\leq 4/\pi^2$. For $k=1$, $B_1(X)=X-\frac12$ and $\sum_{a=1}^f\chi^*(a)=0$, so $B_{1,\chi^*}=\frac1f\sum_{a=1}^f\chi^*(a)\,a$ and $|B_{1,\chi^*}|\leq\frac{f+1}{2}$; with $(1+1)\#=2$, $|\mathrm{Ber}_1(\chi)|\leq N(f+1)\leq 2N^2$. Both bounds imply (37.9).

Case $f=1$. Then $\chi^*=\chi_{0,1}$, $(-1)^k=\chi(-1)=1$, so $k\geq 2$ is even, $a=0$, and $B_{k,\chi^*}=B_k(1)$. Theorem B.4 at $s=1-k\notin\{0,1\}$ gives $\pi^{-(1-k)/2}|\Gamma(\frac{1-k}{2})\zeta(1-k)|=\pi^{-k/2}\Gamma(\frac{k}{2})\zeta(k)$, which is the computation above with $f=1$, $a=0$; with Theorem B.11, $|B_k(1)|=2\,k!\,\zeta(k)/(2\pi)^k$, which is nonzero and at most $2\,k!\,\zeta(2)/(2\pi)^k$. As in the previous case $|\mathrm{Ber}_k(\chi)|\leq\frac{16}{3}N\,k!$, and (37.9) holds.

The product. Let $P=\prod_{\chi\in\mathcal{X}_{\overline{\mathbb{Q}}}}\mathrm{Ber}_k(\chi)$ (the empty product is 1). For an automorphism σ of $\overline{\mathbb{Q}}$, $\sigma(P)=\prod_\chi\mathrm{Ber}_k(\sigma\circ\chi)=P$, because $\chi\mapsto\sigma\circ\chi$ permutes $\mathcal{X}_{\overline{\mathbb{Q}}}$. So $P\in\mathbb{Q}$; it is a nonzero algebraic integer, hence $P\in\mathbb{Z}\setminus\{0\}$. Put $w_\ell=v_\ell(P)\geq 0$. For a finite set S of primes, by (37.9),

$$\sum_{\ell\in S}w_\ell\log\ell\leq\sum_\ell v_\ell(P)\log\ell=\log|P|=\sum_{\chi\in\mathcal{X}_{\mathbb{C}}}\log|\mathrm{Ber}_k(\chi)|\leq|\mathcal{X}_{\mathbb{C}}|\log(16N^{k+1}k!).$$

The prime ℓ. Every $\chi\in\mathcal{X}_{\overline{\mathbb{Q}}_\ell}$ is $j_\ell\circ\chi_0$ for a unique $\chi_0\in\mathcal{X}_{\overline{\mathbb{Q}}}$, and $\mathrm{Ber}_k(\chi)=j_\ell(\mathrm{Ber}_k(\chi_0))$ is nonzero and integral over $\mathbb{Z}$, hence over $\mathbb{Z}_\ell$: this is item 1. Moreover

$$\sum_{\chi\in\mathcal{X}_{\overline{\mathbb{Q}}_\ell}}v_\ell(\mathrm{Ber}_k(\chi))=v_\ell\Big(j_\ell\Big(\prod_{\chi_0\in\mathcal{X}_{\overline{\mathbb{Q}}}}\mathrm{Ber}_k(\chi_0)\Big)\Big)=v_\ell(P)=w_\ell,$$

as $j_\ell(P)=P\in\mathbb{Z}$; this, with the previous display, is item 2. □

Definition 37.10. Let $\theta\colon G_{\mathbb{Q}}\to\mathcal{O}^\times$ be a continuous character with a Kolyvagin level (N_0,m) of modulus $N=N_0\ell^m$, and let r be a prime with $r\equiv 1\pmod N$.

A **Kolyvagin prime** for (N,r) is a prime p with $p\equiv 1\pmod N$, $p\neq r$, and $\ell\nmid\mathrm{ord}_p(r)$, where $\mathrm{ord}_p(r)$ is the order of r in $(\mathbb{Z}/p\mathbb{Z})^\times$. Let $\mathcal{L}(N,r)$ be the set of finite sequences $L=(p_1,\dots,p_k)$, $k\geq 0$, of pairwise distinct Kolyvagin primes for (N,r); put $|L|=p_1\cdots p_k$ ($|L|=1$ for the empty sequence $\emptyset$), and for a Kolyvagin prime p not occurring in L put $(p,L)=(p,p_1,\dots,p_k)\in\mathcal{L}(N,r)$.

Let D_0 be a decomposition group at r and $w\in\mathcal{O}^\vee$. A **Kolyvagin system** for (θ,N_0,m,r,D_0,w) is a family $(\kappa_L)_{L\in\mathcal{L}(N,r)}$ of continuous maps $\kappa_L\colon U_N\to\mathcal{O}^\vee$ such that for every $L\in\mathcal{L}(N,r)$:

(Ks1) $\kappa_L(uv)=\kappa_L(u)+\kappa_L(v)$ and $\ell^m\kappa_L(u)=0$ for all $u,v\in U_N$;

(Ks2) $\kappa_L(gug^{-1})=\check\theta(g)\,\kappa_L(u)$ for all $g\in G_{\mathbb{Q}}$ and $u\in U_N$;

(Ks3) $\kappa_L(u)=0$ for every $u\in U_N$ lying in an inertia group at a prime not dividing $Nr|L|$;

(Ks4) for every Kolyvagin prime p for (N,r) not occurring in L, every decomposition group D at p and every Frobenius φ of D, there are $\tau\in I(D)$ and $u\in\mathcal{O}^\times$ with $\kappa_{(p,L)}(\tau)=u\,\kappa_L(\varphi)$;

(Ks5) there are $\tau_0\in I(D_0)$ and $u_0\in\mathcal{O}^\times$ with $\kappa_\emptyset(\tau_0)=u_0\,w$.

In (Ks4) and (Ks5) the elements φ,τ,τ_0 lie in U_N: a prime $p\equiv 1\pmod N$ is unramified in $\mathbb{Q}(\mu_N)$ with residue degree 1, so D acts trivially on μ_N; the same holds for r and D_0.

Proposition 37.11. *Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring and $\theta\colon G_{\mathbb{Q}} \to \mathcal{O}^\times$ a continuous character with a Kolyvagin level (N_0, m) of modulus N; let Σ be the set of primes dividing N, δ a normaliser for θ, b an integer with $0 \le b < N$ and $\gcd(b, N) = 1$, σ a family of lifts modulo N, and $a_0 \in \mathcal{O} \setminus \{0\}$ with $a_0 \operatorname{st}_{N,b}(\theta, \sigma) \mid \ell^m$ in $\mathcal{O}$. Then*

$$\operatorname{length}_{\mathcal{O}} \mathrm{Z}_{\theta,\delta}(\Sigma, a_0) \le v_{\mathcal{O}}\big(\operatorname{st}_{N,b}(\theta, \sigma)\big).$$

Proof. Write $\mathrm{st} = \operatorname{st}_{N,b}(\theta, \sigma)$ and $\mathrm{Z}(n) = \mathrm{Z}_{\theta,\delta}(\Sigma, a_0; n)$, so that $\mathrm{Z}(n) \supseteq \mathrm{Z}(nn')$. As $\mathcal{O}$ is free of finite rank over $\mathbb{Z}_\ell$, $\mathcal{O}^\vee = K/\mathcal{O}$ with K the fraction field of $\mathcal{O}$. Hence: every $x \in \mathcal{O}^\vee$ has annihilator $c\mathcal{O}$ for some $c \ne 0$; for $c \ne 0$ there is $w \in \mathcal{O}^\vee$ with annihilator $c\mathcal{O}$, and then $\{x : cx = 0\} = \mathcal{O}w$, of length $v_{\mathcal{O}}(c)$; so for an $\mathcal{O}$-linear map $f\colon P \to \mathcal{O}^\vee$ and $e \ne 0$ with $ef = 0$, $\operatorname{length} P \le \operatorname{length} \ker f + v_{\mathcal{O}}(e)$. For a map $\lambda\colon U_N \to \mathcal{O}^\vee$ with $\ell^m\lambda = 0$, the ideal $\{y \in \mathcal{O} : y\lambda(u) = 0 \ \forall u \in U_N\}$ contains ℓ^m; a generator of it is called an *annihilator of* λ; it divides ℓ^m.

Step 1: properties of normalised strict cocycles. Let $z \in \mathrm{Z}(n)$ for some $n \ge 1$; its values are killed by a_0, which divides ℓ^m. For $u \in U_N$, $\theta(u) - 1 \in \ell^m\mathcal{O}$ by (Kv1), so $\theta(u)$ acts trivially on the values of z.

(L1) $z|_{U_N}$ is a homomorphism with $\ell^m z = 0$, and $z(gug^{-1}) = \theta(g)z(u)$ for $g \in G_{\mathbb{Q}}$, $u \in U_N$.

(L2) If $z|_{U_N} = 0$ then $z = 0$: for $g \in G_{\mathbb{Q}}$ the commutator $k = g\delta g^{-1}\delta^{-1}$ lies in U_N ($G_{\mathbb{Q}}/U_N$ is abelian), and by (Kd1) $z(g) = z(g\delta) = z(k\delta g) = z(\delta g) = \theta(\delta)z(g)$; as $\theta(\delta) - 1$ is a unit, $z(g) = 0$.

(L3) Let $p \equiv 1 \pmod N$ be a prime, D a decomposition group at p and φ a Frobenius of D with $z(\varphi) = 0$. Then z vanishes on every decomposition group at p. Indeed $D \subseteq U_N$ (p is unramified in $\mathbb{Q}(\mu_N)$ and φ acts on μ_N by $\zeta \mapsto \zeta^p = \zeta$), $z|_D$ is a continuous homomorphism vanishing on $I(D)$ by (Kd3) ($p \notin \Sigma$) and at φ; its kernel is open, hence closed, and contains $I(D)$ and φ, while $D/I(D)$ is topologically generated by the image of φ; so $z|_D = 0$. The other decomposition groups at p are gDg^{-1}, on which z vanishes by (L1).

Step 2: one step of the descent. Let $n \ge 1$, p a prime with $p \equiv 1 \pmod N$ and $p \nmid n$, and $\lambda\colon U_N \to \mathcal{O}^\vee$ a continuous homomorphism with $\ell^m\lambda = 0$ and $\lambda(gug^{-1}) = \breve{\theta}(g)\lambda(u)$, vanishing on $U_N \cap I$ for every inertia group I at a prime not dividing Nnp. Let D be a decomposition group at p, φ a Frobenius of D and $\tau \in I(D)$ ($\subseteq U_N$); let a generate the annihilator of $\lambda(\tau)$ and a' be an annihilator of λ, and assume $a_0 \mid a$. We claim: $a \mid a'$, and writing $a' = ae$,

$$e\,z(\varphi) = 0 \text{ for all } z \in \mathrm{Z}(n), \qquad \operatorname{length} \mathrm{Z}(n) \le \operatorname{length} \mathrm{Z}(np) + v_{\mathcal{O}}(e). \tag{37.11}$$

As $a'\lambda(\tau) = 0$, $a \mid a'$; so $a_0 \mid a \mid a' \mid \ell^m$. Choose $w_0 \in \mathcal{O}^\vee$ with annihilator $a'\mathcal{O}$; then $\lambda(\tau) = y\,w_0$ for some $y \in \mathcal{O}$, and Lemma 37.22 gives $y\,z(\varphi) = 0$ for every $z \in \mathrm{Z}(n)$. If $v_{\mathcal{O}}(y) \le v_{\mathcal{O}}(a')$, write $a' = yt$; then $t\lambda(\tau) = a'w_0 = 0$, so $t = as$ for some $s \in \mathcal{O}$, $ae = a' = yas$, $e = ys$ and $e\,z(\varphi) = s\,y\,z(\varphi) = 0$. If $v_{\mathcal{O}}(y) > v_{\mathcal{O}}(a')$, then $\lambda(\tau) = 0$, a is a unit, so is a_0, and $\mathrm{Z}(n) = 0$ by (Kd2). This proves the first part. The $\mathcal{O}$-linear map $z \mapsto z(\varphi)$ on $\mathrm{Z}(n)$ is killed by $e \ne 0$, and by (L3) its kernel lies in $\mathrm{Z}(np)$ (condition (Kd5) at p); this gives the inequality.

Step 3: the Kolyvagin system. Since $a_0 \operatorname{st} \mid \ell^m \ne 0$, $\mathrm{st} \ne 0$; write $\ell^m = \mathrm{st} \cdot k$ with $k \in \mathcal{O}$; then $a_0 \mid k$. Choose $w' \in \mathcal{O}^\vee$ with annihilator $\ell^m\mathcal{O}$, a prime $r \equiv 1 \pmod N$ (there are infinitely many, by Dirichlet's theorem), a decomposition group D_0 at r and a Frobenius φ_0 of D_0. By Proposition 37.14 there is a Kolyvagin system $(\kappa_L)_{L \in \mathcal{L}(N,r)}$ for $(\theta, N_0, m, r, D_0, \operatorname{st} w')$. For

$L \in \mathcal{L}(N,r)$ let a_L be an annihilator of κ_L (by (Ks1)) and $j(L) = v_{\mathcal{O}}(a_L) \le J := v_{\mathcal{O}}(\ell^m)$, and put $\mathrm{Z}_L = \mathrm{Z}(r|L|)$.

Step 4: the base step. By (Ks5), $\kappa_\emptyset(\tau_0) = u_0 \,\mathrm{st}\, w'$ with $\tau_0 \in I(D_0)$ and u_0 a unit; its annihilator is $\{y : \ell^m \mid y\,\mathrm{st}\} = k\mathcal{O}$. Step 2 with $n=1$, $p=r$, $\lambda = \kappa_\emptyset$ ((Ks1)–(Ks3)), D_0, φ_0, τ_0, $a = k$, $a' = a_\emptyset$ gives $a_\emptyset = ke_0$ and

$$\operatorname{length} \mathrm{Z}(1) \le \operatorname{length} \mathrm{Z}_\emptyset + v_{\mathcal{O}}(e_0). \tag{37.12}$$

Step 5: the induction. Let $\mathcal{T} = \{L \in \mathcal{L}(N,r) : a_0 \mid a_L\}$. We show $\operatorname{length} \mathrm{Z}_L + j(L) \le J$ for every $L \in \mathcal{T}$, by induction on $J - j(L)$. If $\mathrm{Z}_L = 0$ this holds. Otherwise let $z \in \mathrm{Z}_L$, $z \neq 0$. By (L2) and (L1), $z|_{U_N}$ is a nonzero continuous homomorphism with $\ell^m z = 0$ and $z(gug^{-1}) = \theta(g)z(u)$; κ_L satisfies the hypotheses on λ of Lemma 37.23 by (Ks1), (Ks2). That lemma, with $T = Nr|L|$, gives a Kolyvagin prime $p \nmid Nr|L|$ for (N,r), a decomposition group D at p and a Frobenius $\varphi \in U_N$ of D with $z(\varphi) \neq 0$ and $\kappa_L(U_N) \subseteq \mathcal{O}\kappa_L(\varphi)$. Then p does not occur in L, $L' = (p, L) \in \mathcal{L}(N,r)$, and by (Ks4) there are $\tau \in I(D)$ and a unit u with $\kappa_{L'}(\tau) = u\,\kappa_L(\varphi)$; so $\kappa_{L'}(\tau)$ has annihilator $a_L\mathcal{O}$, since $\kappa_L(\varphi)$ is a value of κ_L of which all values are multiples. Step 2 with $n = r|L|$, p, $\lambda = \kappa_{L'}$ (which vanishes on the inertia groups at the primes not dividing $Nr|L'| = Nnp$ by (Ks3)), D, φ, τ, $a = a_L$, $a' = a_{L'}$ (here $a_0 \mid a_L$ as $L \in \mathcal{T}$) gives $a_{L'} = a_L e$ with $e\,z(\varphi) = 0$, and $\operatorname{length} \mathrm{Z}_L \le \operatorname{length} \mathrm{Z}_{L'} + v_{\mathcal{O}}(e)$. As $z(\varphi) \neq 0$, e is not a unit, so $j(L') = j(L) + v_{\mathcal{O}}(e) > j(L)$; also $a_0 \mid a_L \mid a_{L'}$, so $L' \in \mathcal{T}$. By induction $\operatorname{length} \mathrm{Z}_{L'} + j(L') \le J$, hence $\operatorname{length} \mathrm{Z}_L + j(L) \le \operatorname{length} \mathrm{Z}_{L'} + j(L') \le J$.

Conclusion. As $a_0 \mid k \mid a_\emptyset$, $\emptyset \in \mathcal{T}$, so $\operatorname{length} \mathrm{Z}_\emptyset + v_{\mathcal{O}}(k) + v_{\mathcal{O}}(e_0) \le J = v_{\mathcal{O}}(\mathrm{st}) + v_{\mathcal{O}}(k)$. With (37.12) and $\mathrm{Z}(1) = \mathrm{Z}_{\theta,\delta}(\Sigma, a_0)$,

$$\operatorname{length} \mathrm{Z}_{\theta,\delta}(\Sigma, a_0) \le \operatorname{length} \mathrm{Z}_\emptyset + v_{\mathcal{O}}(e_0) \le v_{\mathcal{O}}(\mathrm{st}).$$

□

Definition 37.12. Let $m \ge 2$ be an integer and $K_m = \mathbb{Q}(\mu_m) \subset \overline{\mathbb{Q}}$. For $t \in (\mathbb{Z}/m\mathbb{Z})^\times$, $\sigma_{m,t} \in \mathrm{Gal}(K_m/\mathbb{Q})$ is the automorphism with $\sigma_{m,t}(\zeta) = \zeta^{\tilde{t}}$ for $\zeta \in \mu_m$; $t \mapsto \sigma_{m,t}$ is an isomorphism $(\mathbb{Z}/m\mathbb{Z})^\times \to \mathrm{Gal}(K_m/\mathbb{Q})$.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_{K_m}$ not dividing m, p its residue characteristic and $k_\mathfrak{p} = \mathcal{O}_{K_m}/\mathfrak{p}$, a field with $|k_\mathfrak{p}| \equiv 1 \pmod m$. The **$m$-th power residue character** of $\mathfrak{p}$ is $\chi_\mathfrak{p} \colon k_\mathfrak{p}^\times \to \mu_m$, where $\chi_\mathfrak{p}(x \bmod \mathfrak{p})$ is the element of μ_m congruent to $x^{(|k_\mathfrak{p}|-1)/m}$ modulo $\mathfrak{p}$ ($x \in \mathcal{O}_{K_m} \setminus \mathfrak{p}$). For a primitive p-th root of unity $\zeta_p \in \overline{\mathbb{Q}}$ and $j \in \mathbb{Z}$, the **Gauss sums** of $\mathfrak{p}$ are

$$\mathrm{g}_\mathfrak{p}(j) = \sum_{c \in k_\mathfrak{p}^\times} \chi_\mathfrak{p}(c)^{-j}\, \zeta_p^{\mathrm{Tr}_{k_\mathfrak{p}/\mathbb{F}_p}(c)} \in \mathbb{Q}(\mu_{mp}); \tag{37.13}$$

for $m \nmid j$ they are nonzero, since $\mathrm{g}_\mathfrak{p}(j)\mathrm{g}_\mathfrak{p}(-j) = \chi_\mathfrak{p}(-1)^j |k_\mathfrak{p}|$ [266]. For an integer b with $1 \le b < m$, the **Stickelberger quotient** and the **Gauss-sum element** of $(m, b, \mathfrak{p})$ are

$$\mathrm{J}_{m,b}(\mathfrak{p}) = \frac{\mathrm{g}_\mathfrak{p}(1)^b}{\mathrm{g}_\mathfrak{p}(b)}, \qquad \mathrm{R}_{m,b}(\mathfrak{p}) = \frac{\mathrm{g}_\mathfrak{p}(1)^b\, \mathrm{g}_\mathfrak{p}(-b)}{\mathrm{g}_\mathfrak{p}(b)\, \mathrm{g}_\mathfrak{p}(-1)^b}, \tag{37.14}$$

formed with the same ζ_p.

Definition 37.13. Let ℓ be an odd prime, $N_0 \ge 1$ and $m \ge 1$ integers with $\ell \nmid N_0$, $N = N_0\ell^m$, and r a prime with $r \equiv 1 \pmod N$. A **Kolyvagin integer** for (N,r) is an integer $|L|$ with $L \in \mathcal{L}(N,r)$, i.e. 1 or a product of distinct Kolyvagin primes for (N,r). Put $\mathbb{Q}(\mu_\infty) = \bigcup_{n \ge 1} K_n$.

For a place w of $\overline{\mathbb{Q}}$ above a prime p, v_w is its valuation with $v_w(p)=1$; z is a w-unit if $v_w(z)=0$, and for w-units z,z' we write $z\equiv z'\pmod{w}$ if $v_w(z-z')>0$.

Euler systems. An **Euler system for** (N,r) is a family (α_d), indexed by the Kolyvagin integers d for (N,r), with $\alpha_d\in K_{Nd}^{\times}$, together with integers $e_{d,p}$, indexed by the pairs of a Kolyvagin integer d and a Kolyvagin prime $p\nmid d$, such that for all such d and p:

(Eu1) α_d is a w-unit for every place w of $\overline{\mathbb{Q}}$ not above r;

(Eu2) $\ell\nmid e_{d,p}$, and for every $\tau\in G_{\mathbb{Q}}$ that fixes μ_{Nd} pointwise and acts on μ_p by $\zeta\mapsto\zeta^g$ with g a generator of $(\mathbb{Z}/p\mathbb{Z})^{\times}$, and every $\varphi\in G_{\mathbb{Q}}$ with $\varphi(\zeta)=\zeta^p$ for $\zeta\in\mu_{Nd}$,

$$\prod_{i=0}^{p-2}\tau^i(\alpha_{pd})=\Big(\frac{\alpha_d}{\varphi^{-1}(\alpha_d)}\Big)^{e_{d,p}};$$

(Eu3) for every φ as in (Eu2) and every place w of $\overline{\mathbb{Q}}$ above p, $\alpha_{pd}\equiv\varphi^{-1}(\alpha_d)^{e_{d,p}}\pmod{w}$.

Kolyvagin derivatives. A **choice of generators** for (N,r) is a family (τ_p), indexed by the Kolyvagin primes p, of elements $\tau_p\in G_{\mathbb{Q}}$ that fix every root of unity of order prime to p and act on μ_p by $\zeta\mapsto\zeta^{g_p}$ with g_p a generator of $(\mathbb{Z}/p\mathbb{Z})^{\times}$. The **Kolyvagin derivatives** are the maps $\overline{\mathbb{Q}}^{\times}\to\overline{\mathbb{Q}}^{\times}$

$$\mathrm{Der}_p(z)=\prod_{i=0}^{p-2}\tau_p^i(z)^i,\qquad \mathrm{Der}_L=\mathrm{Der}_{p_1}\circ\cdots\circ\mathrm{Der}_{p_k}\quad(L=(p_1,\dots,p_k)\in\mathcal{L}(N,r)),\tag{37.15}$$

with $\mathrm{Der}_{\emptyset}$ the identity.

Descended Kummer characters. Let $\delta\in G_{\mathbb{Q}}$ and $a\in\mathbb{Z}$ with $\delta(\zeta)=\zeta^a$ for $\zeta\in\mu_{\ell^m}$, and put $\mathrm{P}_{\delta,a}(z)=\delta(z)\,z^{-a}$ for $z\in\overline{\mathbb{Q}}^{\times}$. Fix a primitive ℓ^m-th root of unity ζ_{ℓ^m}. An element $y\in\mathbb{Q}(\mu_\infty)^{\times}$ is **ℓ^m-invariant over** K_N if for every $g\in U_N$ the element $g(y)/y$ is an ℓ^m-th power in $\mathbb{Q}(\mu_\infty)^{\times}$. For such y and $g\in U_N$, let $Q_y(g)$ be the set of the quotients

$$\frac{g(\mathrm{P}_{\delta,a}(Y))}{\mathrm{P}_{\delta,a}(Y)\,\mathrm{P}_{\delta,a}(z)},\qquad Y\in\overline{\mathbb{Q}},\ Y^{\ell^m}=y,\qquad z\in\mathbb{Q}(\mu_\infty),\ z^{\ell^m}=g(y)/y.$$

If $Q_y(g)=\{\zeta_{\ell^m}^k\}$ for a class $k\in\mathbb{Z}/\ell^m\mathbb{Z}$, the **descended Kummer character** of y takes the value $\mathrm{k}_y(g)=k$ at g; it depends on $(\delta,a,\zeta_{\ell^m})$, which are fixed wherever it is used.

Proposition 37.14. *Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring and $\theta\colon G_{\mathbb{Q}}\to\mathcal{O}^{\times}$ a continuous character with a Kolyvagin level (N_0,m) of modulus $N=N_0\ell^m$. Let r be a prime with $r\equiv1\pmod{N}$, D_0 a decomposition group at r, b an integer with $0\le b<N$ and $\gcd(b,N)=1$, σ a family of lifts modulo N, and $w\in\mathcal{O}^{\vee}$ with $\ell^m w=0$. Then there is a Kolyvagin system for $(\theta,N_0,m,r,D_0,\mathrm{st}_{N,b}(\theta,\sigma)\,w)$.*

Proof. Write $M=\ell^m$, $\mathrm{st}=\mathrm{st}_{N,b}(\theta,\sigma)$, and let $\bar{\mathfrak{r}}$ be the place stabilised by D_0. As $N\ge3$ and $\gcd(b,N)=1$, $1\le b<N$. For $j\in\mathbb{Z}/M\mathbb{Z}$ and $v\in\mathcal{O}^{\vee}$ with $Mv=0$, $j\,v$ means $\tilde{j}v$ for any integer $\tilde{j}$ in the class j. For $h\in U_N$, $\chi_{\mathrm{cyc}}(h)\equiv1$ and, by (Kv1), $\theta(h)\equiv1$ modulo $M\mathcal{O}$; hence $\check{\theta}(h)\equiv1$, and $\check{\theta}(gh)v=\check{\theta}(g)v$, $\theta(gh)^{-1}v=\theta(g)^{-1}v$ for $g\in G_{\mathbb{Q}}$ and $Mv=0$.

Step 1: construction. By (Kv3) choose $\delta\in G_{\mathbb{Q}}$ with $\theta(\delta)-\chi_{\mathrm{cyc}}(\delta)\in\mathcal{O}^{\times}$, an integer a with $\delta(\zeta)=\zeta^a$ for $\zeta\in\mu_M$ (so $\chi_{\mathrm{cyc}}(\delta)\equiv a\pmod{M}$), and let $t_\delta\in(\mathbb{Z}/N\mathbb{Z})^{\times}$ be the class of the action of δ on μ_N. For each Kolyvagin prime p let τ_p be an element of an inertia group at p acting on μ_p by a generator of $(\mathbb{Z}/p\mathbb{Z})^{\times}$; it exists because p is totally ramified in $\mathbb{Q}(\mu_p)$, and

it fixes every root of unity ζ' of order prime to p (as $\tau_p(\zeta') \equiv \zeta'$ modulo the place, and such roots of unity are pairwise incongruent there). So (τ_p) is a choice of generators for (N, r). Fix a primitive M-th root of unity ζ_{ℓ^m}, so that k_y is formed with $(\delta, a, \zeta_{\ell^m})$, and $\rho \in \overline{\mathbb{Q}}$ with $\rho^M = r$. By Proposition 37.15, applied to $(\ell, N_0, m, r, \bar{\mathfrak{r}}, b, \delta, a)$, there is an Euler system $((\alpha_d), (e_{d,p}))$ for (N, r) with

$$v_{\bar{\mathfrak{r}}}(g(\alpha_1)) = 2N_0\big(h_N(b, t\,t_\delta) - a\,h_N(b,t)\big) \quad \text{if } g \in G_{\mathbb{Q}} \text{ acts on } \mu_N \text{ by } t. \tag{37.16}$$

For $L \in \mathcal{L}(N,r)$ let $y_L = \mathrm{Der}_L(\alpha_{|L|})$; by Lemma 37.19 it lies in $K^\times_{N|L|}$, is a w'-unit for every place w' not above r, and is ℓ^m-invariant over K_N, so by item 1 of Lemma 37.20 $\mathrm{k}_L = \mathrm{k}_{y_L} \colon U_N \to \mathbb{Z}/M\mathbb{Z}$ is a continuous homomorphism. Put

$$\kappa_L(u) = \sum_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} \mathrm{k}_L(\sigma_t^{-1} u \sigma_t)\, \check{\theta}(\sigma_t)\, w \qquad (u \in U_N),$$

which makes sense as U_N is normal in $G_{\mathbb{Q}}$. We check (Ks1)–(Ks5) for $(\kappa_L)_{L \in \mathcal{L}(N,r)}$.

Step 2: (Ks1). Each $u \mapsto \mathrm{k}_L(\sigma_t^{-1} u \sigma_t)$ is a continuous homomorphism into $\mathbb{Z}/M\mathbb{Z}$, so κ_L is continuous, additive, and killed by M.

Step 3: (Ks2). Let $g \in G_{\mathbb{Q}}$. For $t \in (\mathbb{Z}/N\mathbb{Z})^\times$ let t' be the class of the action of $g^{-1}\sigma_t$ on μ_N and $h_t = \sigma_{t'}^{-1} g^{-1} \sigma_t \in U_N$; $t \mapsto t'$ is a bijection. Then $\sigma_t^{-1}(gug^{-1})\sigma_t = h_t^{-1}(\sigma_{t'}^{-1} u \sigma_{t'}) h_t$, whose image under the homomorphism k_L into an abelian group is $\mathrm{k}_L(\sigma_{t'}^{-1} u \sigma_{t'})$, and $\check{\theta}(\sigma_t)w = \check{\theta}(g)\check{\theta}(\sigma_{t'})\check{\theta}(h_t)w = \check{\theta}(g)\check{\theta}(\sigma_{t'})w$. Summing, $\kappa_L(gug^{-1}) = \check{\theta}(g)\kappa_L(u)$.

Step 4: (Ks3). Let $u \in U_N$ lie in an inertia group I at a prime $p' \nmid Nr|L|$. For each t, $h = \sigma_t^{-1} u \sigma_t$ lies in the inertia group $\sigma_t^{-1} I \sigma_t$ at p'; it fixes the roots of unity of order prime to p' (as in Step 1), hence $K_{N|L|}$ and y_L. As $p' \neq \ell$ ($\ell \mid N$) and y_L is a unit at the places above $p' \neq r$, item 3 of Lemma 37.20 gives $\mathrm{k}_L(h) = 0$. So $\kappa_L(u) = 0$.

Step 5: (Ks4). Let $L \in \mathcal{L}(N,r)$, p a Kolyvagin prime not in L, D a decomposition group at p and φ a Frobenius of D. As p is totally ramified in $\mathbb{Q}(\mu_p)$, $I(D)$ maps onto $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$; choose $\tau \in I(D)$ agreeing with τ_p on μ_p. For $t \in (\mathbb{Z}/N\mathbb{Z})^\times$, $\sigma_t^{-1} D \sigma_t$ is a decomposition group at p, $\sigma_t^{-1}\tau\sigma_t$ lies in its inertia group and agrees with τ_p on μ_p ($\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ is abelian), and $\sigma_t^{-1}\varphi\sigma_t$ is a Frobenius of it. Since $y_{(p,L)} = \mathrm{Der}_p(\mathrm{Der}_L(\alpha_{p|L|}))$, Lemma 37.21 gives $\mathrm{k}_{(p,L)}(\sigma_t^{-1}\tau\sigma_t) = -e\,\mathrm{k}_L(\sigma_t^{-1}\varphi\sigma_t)$ with $e = e_{|L|,p}$. Summing over t, $\kappa_{(p,L)}(\tau) = (-e)\,\kappa_L(\varphi)$, and $-e \in \mathcal{O}^\times$ as $\ell \nmid e$ by (Eu2).

Step 6: (Ks5). By item 4 of Lemma 37.20 there is $\tau_0 \in I(D_0)$ with $\tau_0(\rho) = \zeta_{\ell^m}\rho$. Let $t \in (\mathbb{Z}/N\mathbb{Z})^\times$; σ_t acts on $\mu_M \subseteq \mu_N$ by $\tilde{t}$, and $y_\emptyset = \alpha_1 \in K_N^\times$. Item 4 of Lemma 37.20 and (37.16), applied to $\sigma_t\delta$ (acting on μ_N by $t\,t_\delta$) and to σ_t, give in $\mathbb{Z}/M\mathbb{Z}$

$$\tilde{t}\,\mathrm{k}_\emptyset(\sigma_t^{-1}\tau_0\sigma_t) = v_{\bar{\mathfrak{r}}}(\sigma_t\delta(\alpha_1)) - a\,v_{\bar{\mathfrak{r}}}(\sigma_t(\alpha_1)) = H(t) := 2N_0\big(h_N(b, t\,t_\delta^2) - 2a\,h_N(b, t\,t_\delta) + a^2 h_N(b,t)\big).$$

As $\chi_{\mathrm{cyc}}(\sigma_t) \equiv \tilde{t} \pmod M$, $\check{\theta}(\sigma_t)w = \tilde{t}\,\theta(\sigma_t)^{-1}w$, so $\kappa_\emptyset(\tau_0) = \sum_t H(t)\,\theta(\sigma_t)^{-1}w$. For $j \geq 0$, substituting $t = t''t_\delta^{-j}$ and using that $\sigma_{t''t_\delta^{-j}}$ and $\sigma_{t''}\delta^{-j}$ agree on μ_N, hence differ by an element of U_N,

$$\sum_t h_N(b, t\,t_\delta^j)\,\theta(\sigma_t)^{-1}w = \sum_{t''} h_N(b,t'')\,\theta(\sigma_{t''})^{-1}\theta(\delta)^j w = \theta(\delta)^j\,\mathrm{st}\,w$$

by (37.3). Hence $\kappa_\emptyset(\tau_0) = u_0\,\mathrm{st}\,w$ with $u_0 = 2N_0(\theta(\delta)^2 - 2a\theta(\delta) + a^2) = 2N_0(\theta(\delta) - a)^2$. Here $\ell \nmid 2N_0$, and $\theta(\delta) - a = (\theta(\delta) - \chi_{\mathrm{cyc}}(\delta)) + (\chi_{\mathrm{cyc}}(\delta) - a)$ is a unit plus an element of $M\mathcal{O} \subseteq \mathfrak{m}$ ($m \geq 1$), hence a unit; so $u_0 \in \mathcal{O}^\times$.

Hence $(\kappa_L)_{L \in \mathcal{L}(N,r)}$ is a Kolyvagin system for $(\theta, N_0, m, r, D_0, \mathrm{st}\,w)$. □

Proposition 37.15. *Let ℓ be an odd prime, $N_0 \geq 1$ and $m \geq 1$ integers with $\ell \nmid N_0$, $N = N_0\ell^m$, r a prime with $r \equiv 1 \pmod N$, $\bar{\mathfrak{r}}$ a place of $\overline{\mathbb{Q}}$ above r, and b an integer with $1 \leq b < N$ and $\gcd(b, N) = 1$. Let $\delta \in G_{\mathbb{Q}}$ and $a \in \mathbb{Z}$ with $\delta(\zeta) = \zeta^a$ for $\zeta \in \mu_{\ell^m}$, and let $t_\delta \in (\mathbb{Z}/N\mathbb{Z})^\times$ be the class with $\delta(\zeta) = \zeta^{\tilde{t}_\delta}$ for $\zeta \in \mu_N$. Then there is an Euler system $((\alpha_d), (e_{d,p}))$ for (N, r) such that, for every $t \in (\mathbb{Z}/N\mathbb{Z})^\times$ and every $\sigma \in G_{\mathbb{Q}}$ with $\sigma(\zeta) = \zeta^{\tilde{t}}$ for $\zeta \in \mu_N$,*

$$v_{\bar{\mathfrak{r}}}(\sigma(\alpha_1)) = 2N_0\big(h_N(b, t\,t_\delta) - a\,h_N(b, t)\big).$$

Proof. We use two classical facts on cyclotomic fields: (C1) for $n \geq 1$ and a prime $p \nmid n$, p is unramified in K_n with residue degree the order of p in $(\mathbb{Z}/n\mathbb{Z})^\times$; (C2) every root of unity in K_n lies in μ_{2n}. Write $\mathrm{P} = \mathrm{P}_{\delta,a}$. It is an endomorphism of the group $\overline{\mathbb{Q}}^\times$, it is trivial on μ_{ℓ^m}, and $g(\mathrm{P}(z)) = \mathrm{P}(g(z))$ for $g \in G_{\mathbb{Q}}$ and $z \in \mathbb{Q}(\mu_\infty)$, since $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$ is abelian.

Construction. Let d be a Kolyvagin integer. Its prime factors are $\equiv 1 \pmod N$, hence $> N > b$, so $\gcd(b, Nd) = 1$; also $r \nmid Nd$, as $r \equiv 1 \pmod N$ and Kolyvagin primes differ from r. Let $\mathfrak{r}_d = \{z \in \mathcal{O}_{K_{Nd}} : v_{\bar{\mathfrak{r}}}(z) > 0\}$, a prime of K_{Nd} above r, and put

$$\mathrm{R}_d = \mathrm{R}_{Nd,b}(\mathfrak{r}_d) \in K_{Nd}^\times, \qquad \alpha_d = \mathrm{P}(\mathrm{R}_d)^{2N_0 d} = \big(\delta(\mathrm{R}_d)\,\mathrm{R}_d^{-a}\big)^{2N_0 d} \in K_{Nd}^\times,$$

using item 1 of Lemma 37.16 ($Nd \geq N \geq 3$) and that $K_{Nd}/\mathbb{Q}$ is Galois. For a Kolyvagin prime $p \nmid d$ let $s_{d,p} = [k_{\mathfrak{r}_{pd}} : k_{\mathfrak{r}_d}]$ and $e_{d,p} = p\,s_{d,p}$.

Step 1: $\ell \nmid e_{d,p}$. $p \neq \ell$, as $\ell \mid N$ and $p \equiv 1 \pmod N$. By (C1), $s_{d,p} = f_{pd}/f_d$ with f_n the order of r modulo Nn. As $(\mathbb{Z}/Npd\mathbb{Z})^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/Nd\mathbb{Z})^\times$, $f_{pd} = \mathrm{lcm}(\mathrm{ord}_p(r), f_d)$, so $s_{d,p}$ divides $\mathrm{ord}_p(r)$, which is prime to ℓ because p is a Kolyvagin prime.

Step 2: (Eu1). By item 2 of Lemma 37.16, R_d is a unit at every prime of K_{Nd} not above r. Let w be a place of $\overline{\mathbb{Q}}$ not above r; then $v_w(\mathrm{R}_d) = 0$, and $v_w(\delta(\mathrm{R}_d)) = v_{w'}(\mathrm{R}_d) = 0$ for the place $w' = w \circ \delta$, again not above r. So α_d is a w-unit.

Step 3: (Eu2). Let τ and φ be as in (Eu2). As $p \nmid Nd$, $\mathrm{Gal}(K_{Npd}/K_{Nd})$ is cyclic of order $p-1$ and is generated by the restriction of τ; so $\prod_{i=0}^{p-2} \tau^i(z) = \mathrm{N}_{K_{Npd}/K_{Nd}}(z)$ for $z \in K_{Npd}$. Lemma 37.17 (with $m := Nd$, $q := p$, $\mathfrak{p}' := \mathfrak{r}_{pd}$, $\mathfrak{p} = \mathfrak{r}_d$; $b < p$, $\gcd(b, Nd) = 1$, $\mathfrak{r}_{pd} \nmid pNd$) gives a root of unity $\eta \in K_{Nd}$ with $\mathrm{N}_{K_{Npd}/K_{Nd}}(\mathrm{R}_{pd}) = \eta\,(\mathrm{R}_d/\varphi^{-1}(\mathrm{R}_d))^{s_{d,p}}$. As P commutes with the norm and with φ^{-1} on K_{Npd},

$$\begin{aligned}\prod_{i=0}^{p-2} \tau^i(\alpha_{pd}) &= \mathrm{N}_{K_{Npd}/K_{Nd}}\big(\mathrm{P}(\mathrm{R}_{pd})\big)^{2N_0 pd} = \mathrm{P}\big(\eta^{2N_0 pd}\big)\Big(\frac{\mathrm{P}(\mathrm{R}_d)}{\varphi^{-1}(\mathrm{P}(\mathrm{R}_d))}\Big)^{2N_0 pd\,s_{d,p}} \\ &= \mathrm{P}\big(\eta^{2N_0 pd}\big)\Big(\frac{\alpha_d}{\varphi^{-1}(\alpha_d)}\Big)^{e_{d,p}}.\end{aligned}$$

By (C2), $\eta^{2Nd} = 1$, so $\eta^{2N_0 d} \in \mu_{\ell^m}$ and $\mathrm{P}(\eta^{2N_0 pd}) = \mathrm{P}(\eta^{2N_0 d})^p = 1$.

Step 4: (Eu3). Let φ be as in (Eu2) and w a place above p; put $\mathfrak{Q} = \{z \in \mathcal{O}_{K_{Npd}} : v_w(z) > 0\}$. By Lemma 37.18 (same data as in Step 3), R_{pd} and $\varphi^{-1}(\mathrm{R}_d)$ are w-units and $\mathrm{R}_{pd} \equiv \varphi^{-1}(\mathrm{R}_d)^{s_{d,p}} \pmod w$. The same holds at the place $w \circ \delta$, also above p; since $v_w(\delta(z) - \delta(z')) = v_{w\circ\delta}(z - z')$ and $\delta\varphi^{-1} = \varphi^{-1}\delta$ on K_{Nd}, this gives $\delta(\mathrm{R}_{pd}) \equiv \varphi^{-1}(\delta(\mathrm{R}_d))^{s_{d,p}} \pmod w$. Congruences between w-units are compatible with products and inverses, so $\mathrm{P}(\mathrm{R}_{pd}) \equiv \varphi^{-1}(\mathrm{P}(\mathrm{R}_d))^{s_{d,p}} \pmod w$, and raising to the power $2N_0 pd$ gives $\alpha_{pd} \equiv \varphi^{-1}(\alpha_d)^{e_{d,p}} \pmod w$.

Step 5: the valuations of α_1. By (C1), $r \equiv 1 \pmod N$ splits completely in K_N: the primes $\sigma_{N,t}^{-1}(\mathfrak{r}_1)$, $t \in (\mathbb{Z}/N\mathbb{Z})^\times$, are pairwise distinct, and $v_{\bar{\mathfrak{r}}}$ restricts to the $\mathfrak{r}_1$-adic valuation $\mathrm{ord}_{\mathfrak{r}_1}$ on K_N. By item 2 of Lemma 37.16, $\mathrm{ord}_{\sigma_{N,t}^{-1}(\mathfrak{r}_1)}(\mathrm{R}_1) = h_N(b, t)$. If σ acts on μ_N by t, then

$\sigma|_{K_N} = \sigma_{N,t}$ and $v_{\bar{\mathfrak{r}}}(\sigma(\mathrm{R}_1)) = \mathrm{ord}_{\mathfrak{r}_1}(\sigma_{N,t}(\mathrm{R}_1)) = \mathrm{ord}_{\sigma_{N,t}^{-1}(\mathfrak{r}_1)}(\mathrm{R}_1) = h_N(b,t)$. As $\sigma\delta$ acts on μ_N by $t\,t_\delta$,

$$v_{\bar{\mathfrak{r}}}(\sigma(\alpha_1)) = 2N_0\big(v_{\bar{\mathfrak{r}}}(\sigma\delta(\mathrm{R}_1)) - a\,v_{\bar{\mathfrak{r}}}(\sigma(\mathrm{R}_1))\big) = 2N_0\big(h_N(b,t\,t_\delta) - a\,h_N(b,t)\big).$$

□

Lemma 37.16. *Let $m \geq 2$, let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_{K_m}$ not dividing m, with residue characteristic p, and let b be an integer with $1 \leq b < m$ and $\gcd(b,m) = 1$. Then:*

1. *$\mathrm{J}_{m,b}(\mathfrak{p})$ and $\mathrm{R}_{m,b}(\mathfrak{p})$ lie in $K_m^\times$ and do not depend on the choice of ζ_p;*

$$\mathrm{R}_{m,b}(\mathfrak{p})\,\sigma_{m,-1}(\mathrm{R}_{m,b}(\mathfrak{p})) = 1,$$

   and $|\iota(\mathrm{R}_{m,b}(\mathfrak{p}))| = 1$ for every field embedding $\iota\colon K_m \to \mathbb{C}$;

2. *as fractional ideals of K_m,*

$$\mathrm{R}_{m,b}(\mathfrak{p})\,\mathcal{O}_{K_m} = \prod_{t\in(\mathbb{Z}/m\mathbb{Z})^\times} \sigma_{m,t}^{-1}(\mathfrak{p})^{h_m(b,t)};$$

   in particular $\mathrm{R}_{m,b}(\mathfrak{p})$ is a unit at every prime of K_m not above p.

Proof. Write $\mathrm{g}(j) = \mathrm{g}_{\mathfrak{p}}(j)$, $\chi = \chi_{\mathfrak{p}}$, $k = k_{\mathfrak{p}}$, $\mathrm{J} = \mathrm{J}_{m,b}(\mathfrak{p})$, $\mathrm{R} = \mathrm{R}_{m,b}(\mathfrak{p})$ and $L = K_m(\zeta_p) = \mathbb{Q}(\mu_{mp})$. As $p \nmid m$, restriction gives $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathrm{Gal}(K_m/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. For $c \in \mathbb{F}_p^\times$ let $\rho_c \in \mathrm{Gal}(L/K_m)$ be the element with $\rho_c(\zeta_p) = \zeta_p^c$ (these are all the elements of $\mathrm{Gal}(L/K_m)$), and for $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ let $\tilde{\sigma}_t \in \mathrm{Gal}(L/\mathbb{Q}(\zeta_p))$ be the element restricting to $\sigma_{m,t}$. The Gauss sum $\mathrm{g}(j)$ depends only on j modulo m, as χ takes values in μ_m.

Step 1: Galois action on Gauss sums. For $c \in \mathbb{F}_p^\times \subseteq k^\times$ we have $c\,\mathrm{Tr}_{k/\mathbb{F}_p}(x) = \mathrm{Tr}_{k/\mathbb{F}_p}(cx)$, so substituting $x \mapsto c^{-1}x$,

$$\rho_c(\mathrm{g}(j)) = \sum_{x\in k^\times} \chi(x)^{-j}\zeta_p^{\mathrm{Tr}(cx)} = \chi(c)^j\,\mathrm{g}(j). \tag{37.17}$$

As $\tilde{\sigma}_t$ fixes ζ_p and raises elements of μ_m to the power $\tilde{t}$, $\tilde{\sigma}_t(\mathrm{g}(j)) = \mathrm{g}(j\tilde{t})$. By item 1 of Theorem B.20, $\mathrm{g}(j) \neq 0$ for $m \nmid j$; this applies to $j = \pm 1, \pm b$, since $1 \leq b < m$.

Step 2: item 1. By (37.17), $\rho_c(\mathrm{J}) = \chi(c)^b\mathrm{g}(1)^b/(\chi(c)^b\mathrm{g}(b)) = \mathrm{J}$ for all c, so $\mathrm{J} \in L^{\mathrm{Gal}(L/K_m)} = K_m$, and $\mathrm{J} \neq 0$. By Step 1, $\tilde{\sigma}_{-1}(\mathrm{J}) = \mathrm{g}(-1)^b/\mathrm{g}(-b)$, so $\mathrm{R} = \mathrm{J}/\tilde{\sigma}_{-1}(\mathrm{J}) = \mathrm{J}/\sigma_{m,-1}(\mathrm{J}) \in K_m^\times$. Replacing ζ_p by ζ_p^c replaces each $\mathrm{g}(j)$ by $\rho_c(\mathrm{g}(j))$ (ρ_c fixes the values of χ), hence J by $\rho_c(\mathrm{J}) = \mathrm{J}$ and R by R. Since $\sigma_{m,-1}^2 = 1$, $\mathrm{R}\,\sigma_{m,-1}(\mathrm{R}) = \frac{\mathrm{J}}{\sigma_{m,-1}(\mathrm{J})} \cdot \frac{\sigma_{m,-1}(\mathrm{J})}{\mathrm{J}} = 1$. For an embedding $\iota\colon K_m \to \mathbb{C}$, $\iota\circ\sigma_{m,-1}$ and $\bar{\iota}$ agree on μ_m (both send ζ to $\iota(\zeta)^{-1} = \overline{\iota(\zeta)}$), hence on $K_m = \mathbb{Q}(\mu_m)$; so $|\iota(\mathrm{R})|^2 = \iota(\mathrm{R})\,\overline{\iota(\mathrm{R})} = \iota(\mathrm{R}\,\sigma_{m,-1}(\mathrm{R})) = 1$.

Step 3: item 2. By item 2 of Theorem B.20, $\mathrm{g}(1)^m \in K_m$ generates $\prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{\tilde{t}}$. As $\gcd(b,m) = 1$, $\mathrm{g}(b)^m = \tilde{\sigma}_b(\mathrm{g}(1)^m) = \sigma_{m,b}(\mathrm{g}(1)^m)$ generates

$$\prod_t \sigma_{m,b}\sigma_{m,t}^{-1}(\mathfrak{p})^{\tilde{t}} = \prod_{t'} \sigma_{m,t'}^{-1}(\mathfrak{p})^{\widetilde{bt'}} \qquad (t = bt').$$

Hence $\mathrm{J}^m = \mathrm{g}(1)^{bm}/\mathrm{g}(b)^m$ generates $\prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{b\tilde{t}-\widetilde{bt}}$, and $b\tilde{t} - \widetilde{bt} = m\lfloor b\tilde{t}/m\rfloor$, because $b\tilde{t} \equiv \widetilde{bt} \pmod m$ and $0 < \widetilde{bt} < m$. The group of fractional ideals of K_m is free abelian, so

$$\mathrm{J}\,\mathcal{O}_{K_m} = \prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{\lfloor b\tilde{t}/m\rfloor}, \qquad \sigma_{m,-1}(\mathrm{J})\,\mathcal{O}_{K_m} = \prod_t \sigma_{m,-t}^{-1}(\mathfrak{p})^{\lfloor b\tilde{t}/m\rfloor} = \prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{\lfloor b(m-\tilde{t})/m\rfloor},$$

and $\mathrm{R}\,\mathcal{O}_{K_m} = \prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{\lfloor \tilde{t}b/m\rfloor - \lfloor (m-\tilde{t})b/m\rfloor} = \prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{h_m(b,t)}$. All the primes $\sigma_{m,t}^{-1}(\mathfrak{p})$ lie above p. □

Lemma 37.17. *Let $m \geq 2$, let q be a prime with $q \nmid m$, and let b be an integer with $1 \leq b < m$, $b < q$ and $\gcd(b, m) = 1$. Let $\mathfrak{p}'$ be a prime ideal of $\mathcal{O}_{K_{qm}}$ not dividing qm, $\mathfrak{p} = \mathfrak{p}' \cap \mathcal{O}_{K_m}$, and $s = [k_{\mathfrak{p}'} : k_{\mathfrak{p}}]$. Let $\varphi \in G_{\mathbb{Q}}$ satisfy $\varphi(\zeta) = \zeta^q$ for $\zeta \in \mu_m$. Then there is a root of unity $\eta \in K_m$ with*

$$\mathrm{N}_{K_{qm}/K_m}(\mathrm{R}_{qm,b}(\mathfrak{p}')) = \eta \Big(\frac{\mathrm{R}_{m,b}(\mathfrak{p})}{\varphi^{-1}(\mathrm{R}_{m,b}(\mathfrak{p}))} \Big)^s .$$

Proof. Put $n = qm$, $\mathrm{R}' = \mathrm{R}_{n,b}(\mathfrak{p}')$, $\mathrm{R} = \mathrm{R}_{m,b}(\mathfrak{p})$, and let $u \in (\mathbb{Z}/m\mathbb{Z})^\times$ be the class of q, so that $\varphi|_{K_m} = \sigma_{m,u}$. For $t' \in (\mathbb{Z}/n\mathbb{Z})^\times$ with image $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ (written $t' \mapsto t$), $\sigma_{n,t'}|_{K_m} = \sigma_{m,t}$. As $1 \leq b < q$, $\gcd(b, n) = 1$, so Lemma 37.16 applies at both levels.

Step 1: a distribution identity. We show, for every $t \in (\mathbb{Z}/m\mathbb{Z})^\times$,

$$\sum_{t' \mapsto t} h_n(b, t') = h_m(b, t) - h_m(b, u^{-1}t). \tag{37.18}$$

The integers in $[1, n-1]$ congruent to $\tilde{t}$ modulo m are $\tilde{t} + mj$, $0 \leq j \leq q-1$; they are prime to m, and since m is invertible modulo q exactly one of them, for $j = j_0$, is divisible by q. So the representatives $\tilde{t}'$ of the classes $t' \mapsto t$ are the $\tilde{t} + mj$ with $j \neq j_0$. Write $\tilde{t} + mj_0 = qt_1$; then $1 \leq t_1 \leq m-1$ and $qt_1 \equiv \tilde{t} \pmod m$, so $t_1 = \widetilde{u^{-1}t}$. Write $bj = qa_j + i_j$ with $0 \leq i_j < q$; as $\gcd(b, q) = 1$, $j \mapsto i_j$ permutes $\{0, \dots, q-1\}$. With $z = b\tilde{t}/m$ and Hermite's identity $\sum_{i=0}^{q-1} \lfloor (z+i)/q \rfloor = \lfloor z \rfloor$,

$$\sum_{j=0}^{q-1} \Big\lfloor \frac{b(\tilde{t} + mj)}{n} \Big\rfloor = \sum_{j=0}^{q-1} \Big(a_j + \Big\lfloor \frac{z + i_j}{q} \Big\rfloor \Big) = C_b + \Big\lfloor \frac{b\tilde{t}}{m} \Big\rfloor, \qquad C_b = \sum_{j=0}^{q-1} \Big\lfloor \frac{bj}{q} \Big\rfloor,$$

and the term $j = j_0$ is $\lfloor bqt_1/n \rfloor = \lfloor bt_1/m \rfloor$. Hence $A(t) := \sum_{t' \mapsto t} \lfloor b\tilde{t}'/n \rfloor = \lfloor b\tilde{t}/m \rfloor + C_b - \lfloor b\,\widetilde{u^{-1}t}/m \rfloor$. Since $n - \tilde{t}' = \widetilde{-t'}$ and $t' \mapsto t$ if and only if $-t' \mapsto -t$, $\sum_{t' \mapsto t} \lfloor b(n - \tilde{t}')/n \rfloor = A(-t) = \lfloor b(m - \tilde{t})/m \rfloor + C_b - \lfloor b(m - \widetilde{u^{-1}t})/m \rfloor$. Subtracting gives (37.18).

Step 2: ideals. The ideal norm $\mathrm{N} = \mathrm{N}_{K_n/K_m}$ on fractional ideals is multiplicative, satisfies $\mathrm{N}(\mathfrak{p}') = \mathfrak{p}^s$, $\mathrm{N}(x\mathcal{O}_{K_n}) = \mathrm{N}_{K_n/K_m}(x)\mathcal{O}_{K_m}$, and $\mathrm{N}(\sigma(\mathfrak{a})) = \sigma|_{K_m}(\mathrm{N}(\mathfrak{a}))$ for $\sigma \in \mathrm{Gal}(K_n/\mathbb{Q})$ ($K_n/\mathbb{Q}$ is abelian). By item 2 of Lemma 37.16 at level n and (37.18),

$$\mathrm{N}_{K_n/K_m}(\mathrm{R}')\mathcal{O}_{K_m} = \prod_{t'} \sigma_{m,t}^{-1}(\mathfrak{p})^{s\,h_n(b,t')} = \prod_{t} \sigma_{m,t}^{-1}(\mathfrak{p})^{s\,(h_m(b,t) - h_m(b,u^{-1}t))}.$$

Substituting $t = ut_1$, $\prod_t \sigma_{m,t}^{-1}(\mathfrak{p})^{h_m(b,u^{-1}t)} = \sigma_{m,u}^{-1}\big(\prod_{t_1} \sigma_{m,t_1}^{-1}(\mathfrak{p})^{h_m(b,t_1)}\big) = \varphi^{-1}(\mathrm{R})\mathcal{O}_{K_m}$ by item 2 of Lemma 37.16 at level m. Hence $\mathrm{N}_{K_n/K_m}(\mathrm{R}')$ and $(\mathrm{R}/\varphi^{-1}(\mathrm{R}))^s$ generate the same fractional ideal.

Step 3: the unit is a root of unity. So $\beta = \mathrm{N}_{K_n/K_m}(\mathrm{R}')\,(\varphi^{-1}(\mathrm{R})/\mathrm{R})^s$ is a unit of $\mathcal{O}_{K_m}$. Let $\iota\colon K_m \to \mathbb{C}$ be an embedding and ι' an extension to K_n. Then $\iota(\mathrm{N}_{K_n/K_m}(\mathrm{R}')) = \prod_{\rho \in \mathrm{Gal}(K_n/K_m)} (\iota' \circ \rho)(\mathrm{R}')$ has absolute value 1, and so do $\iota(\mathrm{R})$ and $\iota(\varphi^{-1}(\mathrm{R})) = (\iota \circ \varphi^{-1}|_{K_m})(\mathrm{R})$, by item 1 of Lemma 37.16 applied to the embeddings $\iota' \circ \rho$, ι, $\iota \circ \varphi^{-1}|_{K_m}$. So every conjugate of the algebraic integer β has absolute value 1, and β is a root of unity by Kronecker's theorem [266]. Put $\eta = \beta^{-1}$. □

Lemma 37.18. *Let $m \geq 2$, let q be a prime with $q \nmid m$, and let b be an integer with $1 \leq b < m$, $b < q$ and $\gcd(b, m) = 1$. Let $\mathfrak{p}'$ be a prime ideal of $\mathcal{O}_{K_{qm}}$ not dividing qm, $\mathfrak{p} = \mathfrak{p}' \cap \mathcal{O}_{K_m}$, $s = [k_{\mathfrak{p}'} : k_{\mathfrak{p}}]$, and let $\varphi \in G_{\mathbb{Q}}$ satisfy $\varphi(\zeta) = \zeta^q$ for $\zeta \in \mu_m$. Then for every prime ideal $\mathfrak{Q}$ of $\mathcal{O}_{K_{qm}}$ above q, the elements $\mathrm{R}_{qm,b}(\mathfrak{p}')$ and $\varphi^{-1}(\mathrm{R}_{m,b}(\mathfrak{p}))$ are $\mathfrak{Q}$-adic units and*

$$\mathrm{R}_{qm,b}(\mathfrak{p}') \equiv \varphi^{-1}(\mathrm{R}_{m,b}(\mathfrak{p}))^s \pmod{\mathfrak{Q}}.$$

Proof. Put $n = qm$, $\mathrm{R}' = \mathrm{R}_{n,b}(\mathfrak{p}')$, $\mathrm{R} = \mathrm{R}_{m,b}(\mathfrak{p})$; let p be the residue characteristic of $\mathfrak{p}'$, so $p \neq q$; $k = k_{\mathfrak{p}} \subseteq k' = k_{\mathfrak{p}'}$ (via $\mathcal{O}_{K_m}/\mathfrak{p} \to \mathcal{O}_{K_n}/\mathfrak{p}'$), $\chi = \chi_{\mathfrak{p}}$, $\chi' = \chi_{\mathfrak{p}'}$, and $\mathrm{g}(j) = \mathrm{g}_{\mathfrak{p}}(j)$, $\mathrm{g}'(j) = \mathrm{g}_{\mathfrak{p}'}(j)$, formed with the same ζ_p. As $1 \le b < q$, $\gcd(b,n) = 1$. Let $L = \mathbb{Q}(\mu_{np})$.

Step 1: the residue characters. For $c \in k'^{\times}$, $\chi'(c)^q = \chi(\mathrm{N}_{k'/k}(c))$. Indeed, let $x \in \mathcal{O}_{K_n}$ lift c and $y \in \mathcal{O}_{K_m}$ lift $\mathrm{N}_{k'/k}(c) = c^{(|k'|-1)/(|k|-1)}$. Modulo $\mathfrak{p}'$, $\chi'(c)^q \equiv x^{(|k'|-1)/m}$ and $\chi(\mathrm{N}_{k'/k}(c)) \equiv y^{(|k|-1)/m} \equiv x^{(|k'|-1)/m}$; both sides lie in μ_n, whose elements are pairwise incongruent modulo $\mathfrak{p}'$ as $p \nmid n$.

Step 2: q-th powers of Gauss sums. Let $j \in \{\pm 1, \pm b\}$; then $m \nmid j$ and χ^{-j} is a nontrivial character of $k^{\times}$. In a commutative ring $(a_1 + \cdots + a_r)^q \equiv a_1^q + \cdots + a_r^q$ modulo q. Hence, in $\mathcal{O}_L$ modulo q, using $q \operatorname{Tr}_{k'/\mathbb{F}_p}(c) = \operatorname{Tr}_{k'/\mathbb{F}_p}(qc)$, the substitution $c \mapsto q^{-1}c$ in $k'^{\times}$ ($q \in \mathbb{F}_p^{\times}$), Step 1, and Theorem B.21 for $\alpha = \chi^{-j}$:

$$\begin{aligned} \mathrm{g}'(j)^q &\equiv \sum_{c \in k'^{\times}} \chi'(c)^{-jq} \zeta_p^{\operatorname{Tr}(qc)} = \chi'(q)^{jq} \sum_{c \in k'^{\times}} \chi\big(\mathrm{N}_{k'/k}(c)\big)^{-j} \zeta_p^{\operatorname{Tr}(c)} \\ &= \chi'(q)^{jq}\, \mathrm{g}_{k'}(\chi^{-j} \circ \mathrm{N}_{k'/k}) = \chi'(q)^{jq} (-1)^{s-1} \mathrm{g}(j)^s. \end{aligned} \tag{37.19}$$

Step 3: $\mathrm{R}'^q \equiv \mathrm{R}^s$ modulo $\mathfrak{Q}$. By item 1 of Theorem B.20, $\mathrm{g}(j)\mathrm{g}(-j) = \pm|k|$ and $\mathrm{g}'(j)\mathrm{g}'(-j) = \pm|k'|$ are powers of p up to sign, so these Gauss sums are units at every prime of L above q. Let $\widetilde{\mathfrak{Q}}$ be a prime of $\mathcal{O}_L$ above $\mathfrak{Q}$. Dividing the congruences (37.19) according to (37.14) in the local ring at $\widetilde{\mathfrak{Q}}$, the factors $\chi'(q)^{\pm q}$, $\chi'(q)^{\pm bq}$ and the signs cancel:

$$\mathrm{R}'^q = \frac{\mathrm{g}'(1)^{qb}\, \mathrm{g}'(-b)^q}{\mathrm{g}'(b)^q\, \mathrm{g}'(-1)^{qb}} \equiv \frac{\chi'(q)^{qb}\chi'(q)^{-bq}}{\chi'(q)^{bq}\chi'(q)^{-qb}} \cdot \frac{(-1)^{(s-1)(b+1)}}{(-1)^{(s-1)(1+b)}} \cdot \mathrm{R}^s = \mathrm{R}^s.$$

Both sides lie in K_n, so $\mathrm{R}'^q \equiv \mathrm{R}^s$ modulo $\mathfrak{Q} = \widetilde{\mathfrak{Q}} \cap \mathcal{O}_{K_n}$.

Step 4: Frobenius. Since $q \nmid m$, q is unramified in K_m and the Frobenius of every prime $\mathfrak{q}$ of K_m above q is $\sigma_{m,u} = \varphi|_{K_m}$, u the class of q; so $\varphi(z) \equiv z^q$ modulo $\mathfrak{q}$ for z in the local ring at $\mathfrak{q}$. By item 2 of Lemma 37.16, R, $\varphi^{-1}(\mathrm{R})$ and R' are units at every prime not above p, in particular at $\mathfrak{q} = \mathfrak{Q} \cap \mathcal{O}_{K_m}$ and at $\mathfrak{Q}$. Applied to $z = \varphi^{-1}(\mathrm{R})$ this gives $\mathrm{R} \equiv \varphi^{-1}(\mathrm{R})^q$ modulo $\mathfrak{q}$, hence $\mathrm{R}'^q \equiv \big(\varphi^{-1}(\mathrm{R})^s\big)^q$ modulo $\mathfrak{Q}$.

Step 5: conclusion. In the residue field $\mathcal{O}_{K_n}/\mathfrak{Q}$, of characteristic q, the images of R' and $\varphi^{-1}(\mathrm{R})^s$ have the same q-th power; as $x \mapsto x^q$ is injective on a field of characteristic q, they are equal. □

Lemma 37.19. *Let ℓ be an odd prime, $N_0 \ge 1$ and $m \ge 1$ integers with $\ell \nmid N_0$, $N = N_0\ell^m$, r a prime with $r \equiv 1 \pmod N$, $((\alpha_d),(e_{d,p}))$ an Euler system for (N,r) and (τ_p) a choice of generators for (N,r). For $L \in \mathcal{L}(N,r)$ put $y_L = \mathrm{Der}_L(\alpha_{|L|})$. Then $y_L \in K_{N|L|}^{\times}$, y_L does not change when the primes of L are permuted, y_L is a w-unit for every place w of $\overline{\mathbb{Q}}$ not above r, and for every $g \in U_N$ there is $z \in K_{N|L|}^{\times}$ with*

$$z^{\ell^m} = g(y_L)/y_L.$$

In particular y_L is ℓ^m-invariant over K_N.

Proof. Write $M = \ell^m$ in this proof. For a Kolyvagin prime p put $\mathrm{N}_p(z) = \prod_{i=0}^{p-2} \tau_p^i(z)$.

Step 1: elementary properties. Let n be a Kolyvagin integer. Every element of $G_{\mathbb{Q}}$ maps K_{Nn} onto itself, and elements of $G_{\mathbb{Q}}$ commute on K_{Nn} ($K_{Nn}/\mathbb{Q}$ is abelian). Hence each Der_p

is an endomorphism of the group $K_{Nn}^\times$, $\mathrm{Der}_p \circ \mathrm{Der}_{p'} = \mathrm{Der}_{p'} \circ \mathrm{Der}_p$ on $K_{Nn}^\times$, and $g(\mathrm{Der}_p(z)) = \mathrm{Der}_p(g(z))$, $g(\mathrm{N}_p(z)) = \mathrm{N}_p(g(z))$ for $g \in G_\mathbb{Q}$ and $z \in K_{Nn}^\times$. So $y_L \in K_{N|L|}^\times$ and y_L depends only on the set of primes of L. If z is a w-unit for every place w not above r, so is $g(z)$, as $v_w(g(z)) = v_{w\circ g}(z)$; with (Eu1) this shows that y_L is such a unit.

Step 2: telescoping. Let $p \mid n$. Then τ_p^{p-1} fixes K_{Nn}: it fixes the roots of unity of order prime to p and acts on μ_p by $g_p^{p-1} = 1$. For $z \in K_{Nn}^\times$,

$$\frac{\tau_p(\mathrm{Der}_p(z))}{\mathrm{Der}_p(z)} = \frac{z^{p-1}}{\mathrm{N}_p(z)}. \tag{37.20}$$

Step 3: induction on the length of L. If $L = \emptyset$, $y_L = \alpha_1 \in K_N$ is fixed by U_N and $z = 1$ works. Let L have length $k \geq 1$ and assume the claim for sequences of length $k-1$. Put $n = |L|$ and let S be the set of $g \in U_N$ for which some $z \in K_{Nn}^\times$ has $z^M = g(y_L)/y_L$.

(a) $U_{Nn} \subseteq S$, with $z = 1$, since $y_L \in K_{Nn}$.

(b) S is closed under products: if z_g, z_h serve for g, h, then $g(z_h)z_g \in K_{Nn}^\times$ and $(g(z_h)z_g)^M = g(h(y_L)/y_L) \cdot g(y_L)/y_L = gh(y_L)/y_L$.

(c) $\tau_p \in S$ for every prime p of L. First $\tau_p \in U_N$, as $p \nmid N$. By Step 1 we may write $L = (p, L_0)$ with $L_0 \in \mathcal{L}(N, r)$ of length $k-1$; put $n_0 = |L_0|$, so $n = pn_0$, and $w_0 = \mathrm{Der}_{L_0}(\alpha_n)$, so $y_L = \mathrm{Der}_p(w_0)$. Choose $\varphi \in G_\mathbb{Q}$ with $\varphi(\zeta) = \zeta^p$ for $\zeta \in \mu_{Nn_0}$ (possible as $p \nmid Nn_0$); as $p \equiv 1 \pmod N$, $\varphi \in U_N$. Since τ_p fixes μ_{Nn_0} and acts on μ_p by a generator, (Eu2) for $d = n_0$ gives $\mathrm{N}_p(\alpha_n) = (\alpha_{n_0}/\varphi^{-1}(\alpha_{n_0}))^e$ with $e = e_{n_0,p}$. Applying Der_{L_0}, which commutes with N_p on $K_{Nn}^\times$ and with φ^{-1} on $K_{Nn_0}^\times$ (Step 1), $\mathrm{N}_p(w_0) = (y_{L_0}/\varphi^{-1}(y_{L_0}))^e$. By the induction hypothesis, applied to L_0 and $\varphi^{-1} \in U_N$, there is $z_0 \in K_{Nn_0}^\times$ with $z_0^M = \varphi^{-1}(y_{L_0})/y_{L_0}$. By (37.20), and as $M \mid N \mid p-1$,

$$\frac{\tau_p(y_L)}{y_L} = \frac{w_0^{p-1}}{\mathrm{N}_p(w_0)} = w_0^{p-1}\Big(\frac{\varphi^{-1}(y_{L_0})}{y_{L_0}}\Big)^e = (w_0^{(p-1)/M} z_0^e)^M,$$

with $w_0^{(p-1)/M} z_0^e \in K_{Nn}^\times$.

(d) $S = U_N$. Restriction maps U_N onto $\mathrm{Gal}(K_{Nn}/K_N)$ with kernel U_{Nn}, and $\mathrm{Gal}(K_{Nn}/K_N) \cong (\mathbb{Z}/n\mathbb{Z})^\times \cong \prod_{p|n}(\mathbb{Z}/p\mathbb{Z})^\times$ through the action on μ_n; the image of τ_p is a generator of the factor at p and trivial at the other factors. So every element of this finite group is a product of nonnegative powers of the images of the τ_p, every $g \in U_N$ is of the form $\tau_{p_1}^{c_1} \cdots \tau_{p_k}^{c_k} h$ with $c_i \geq 0$ and $h \in U_{Nn}$, and $g \in S$ by (a)–(c).

Finally, $K_{N|L|} \subseteq \mathbb{Q}(\mu_\infty)$, so y_L is ℓ^m-invariant over K_N. □

Lemma 37.20. *Let ℓ be an odd prime, $N_0 \geq 1$ and $m \geq 1$ integers with $\ell \nmid N_0$, $N = N_0\ell^m$, $\delta \in G_\mathbb{Q}$ and $a \in \mathbb{Z}$ with $\delta(\zeta) = \zeta^a$ for $\zeta \in \mu_{\ell^m}$, ζ_{ℓ^m} a primitive ℓ^m-th root of unity, and $y \in \mathbb{Q}(\mu_\infty)^\times$ ℓ^m-invariant over K_N. Write $\mathrm{P} = \mathrm{P}_{\delta,a}$.*

1. *For every $g \in U_N$, $Q_y(g)$ consists of one ℓ^m-th root of unity; the map $\mathrm{k}_y \colon U_N \to \mathbb{Z}/\ell^m\mathbb{Z}$ so defined is a continuous homomorphism.*

2. *If $g \in U_N$ fixes y, then $\zeta_{\ell^m}^{\mathrm{k}_y(g)} = g(\mathrm{P}(Y))/\mathrm{P}(Y)$ for every $Y \in \overline{\mathbb{Q}}$ with $Y^{\ell^m} = y$.*

3. *Let $p' \neq \ell$ be a prime such that y is a w-unit for every place w of $\overline{\mathbb{Q}}$ above p', and let $g \in U_N$ fix y and lie in an inertia group $I(D)$ at p'. Then $\mathrm{k}_y(g) = 0$.*

4. *Let r be a prime with $r \equiv 1 \pmod N$, D_0 a decomposition group at r, the stabiliser of the place $\bar{\mathfrak{r}}$, and $\rho \in \overline{\mathbb{Q}}$ with $\rho^{\ell^m} = r$. There is $t_0 \in I(D_0)$ with $t_0(\rho) = \zeta_{\ell^m}\rho$. If moreover $y \in K_N^\times$, then for every $\sigma \in G_\mathbb{Q}$, with $c \in \mathbb{Z}$ such that $\sigma(\zeta) = \zeta^c$ for $\zeta \in \mu_{\ell^m}$, one has $\sigma^{-1}t_0\sigma \in U_N$ and*

$$c \cdot \mathrm{k}_y(\sigma^{-1}t_0\sigma) = v_{\bar{\mathfrak{r}}}(\sigma(\mathrm{P}(y))) \quad \text{in } \mathbb{Z}/\ell^m\mathbb{Z}.$$

Proof. Write $M = \ell^m$ and $\zeta = \zeta_{\ell^m}$. The map P is an endomorphism of $\overline{\mathbb{Q}}^\times$, trivial on μ_M, and $g(\mathrm{P}(x)) = \mathrm{P}(g(x))$ for $g \in G_\mathbb{Q}$ and $x \in \mathbb{Q}(\mu_\infty)$, as $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$ is abelian. Elements of U_N fix $\mu_M \subseteq \mu_N$.

Item 1. Let $g \in U_N$, $Y^M = y$, and $z \in \mathbb{Q}(\mu_\infty)$ with $z^M = g(y)/y$ (it exists by invariance). Then $\mathrm{P}(Y)^M = \mathrm{P}(y)$, $g(\mathrm{P}(Y))^M = g(\mathrm{P}(y)) = \mathrm{P}(g(y))$ and $\mathrm{P}(z)^M = \mathrm{P}(g(y))/\mathrm{P}(y)$, so the quotient $g(\mathrm{P}(Y))/(\mathrm{P}(Y)\mathrm{P}(z))$ lies in μ_M. Any other choices are ξY and $\xi' z$ with $\xi, \xi' \in \mu_M$, and $\mathrm{P}(\xi Y) = \mathrm{P}(Y)$, $\mathrm{P}(\xi' z) = \mathrm{P}(z)$; so $Q_y(g)$ is one element $\zeta^{\mathrm{k}_y(g)}$. For $g, h \in U_N$ with z_g, z_h as above, $z = g(z_h)z_g \in \mathbb{Q}(\mu_\infty)$ satisfies $z^M = gh(y)/y$ and $\mathrm{P}(z) = g(\mathrm{P}(z_h))\mathrm{P}(z_g)$, so

$$\frac{gh(\mathrm{P}(Y))}{\mathrm{P}(Y)\mathrm{P}(z)} = g\Big(\frac{h(\mathrm{P}(Y))}{\mathrm{P}(Y)\mathrm{P}(z_h)}\Big) \cdot \frac{g(\mathrm{P}(Y))}{\mathrm{P}(Y)\mathrm{P}(z_g)} = g\big(\zeta^{\mathrm{k}_y(h)}\big)\zeta^{\mathrm{k}_y(g)} = \zeta^{\mathrm{k}_y(h)+\mathrm{k}_y(g)},$$

and k_y is a homomorphism. The subgroup U' of the $h \in U_N$ fixing y and $\mathrm{P}(Y)$ is open; for $h \in U'$ one may take $z = 1$, so $\zeta^{\mathrm{k}_y(h)} = h(\mathrm{P}(Y))/\mathrm{P}(Y) = 1$. A homomorphism into a finite group that vanishes on an open subgroup is continuous.

Item 2. Take $z = 1$ in the definition.

Item 3. Let w be the place of which D is the stabiliser, and $V = \mathrm{P}(Y)$, so $V^M = \mathrm{P}(y) = \delta(y)y^{-a}$. As $v_w(\delta(y)) = v_{w\circ\delta}(y) = 0$, $\mathrm{P}(y)$ and hence V are w-units. Since $g \in I(D)$ acts trivially on the residue field of w, $g(V)/V \equiv 1 \pmod w$; it lies in μ_M, whose elements are pairwise incongruent modulo w as $p' \nmid M$. So $g(V)/V = 1$, and $\mathrm{k}_y(g) = 0$ by item 2.

Item 4: the element t_0. Let $E = K_N(\rho)$; it contains μ_M and all the roots $\xi\rho$ of $X^M - r$, so $E/\mathbb{Q}$ is Galois. As $r \equiv 1 \pmod N$, r splits completely in K_N, so the completion of K_N at $\bar{\mathfrak{r}}$ is $\mathbb{Q}_r$; there $X^M - r$ is an Eisenstein polynomial, so the completion of E is totally ramified of degree M over $\mathbb{Q}_r$. Hence $[E : K_N] = M$, $X^M - r$ is irreducible over K_N, and the inertia group of $E/\mathbb{Q}$ at $\bar{\mathfrak{r}}$, which lies in $\mathrm{Gal}(E/K_N)$ (K_N is unramified at r) and has order M, is all of $\mathrm{Gal}(E/K_N)$. It contains the automorphism $\rho \mapsto \zeta\rho$. Since $I(D_0)$ maps onto the inertia group of E at $\bar{\mathfrak{r}}$, some $t_0 \in I(D_0)$ satisfies $t_0(\rho) = \zeta\rho$.

Item 4: the value. As $r \nmid N$, inertia groups at r fix K_N, so $t_0 \in U_N$ and $h = \sigma^{-1}t_0\sigma \in U_N$; h fixes $y \in K_N$. Let $Y^M = y$, $V = \sigma(\mathrm{P}(Y))$ and $u = \sigma(\mathrm{P}(y)) = V^M \in K_N^\times$. By item 2, $\zeta^{\mathrm{k}_y(h)} = h(\mathrm{P}(Y))/\mathrm{P}(Y) = \sigma^{-1}(t_0(V)/V)$. Let $v = v_{\bar{\mathfrak{r}}}(u) \in \mathbb{Z}$ (K_N is unramified at r) and $V_0 = V\rho^{-v}$; then $V_0^M = ur^{-v}$ is a $\bar{\mathfrak{r}}$-unit, hence so is V_0, and as in item 3 ($r \nmid M$) $t_0(V_0) = V_0$. So $t_0(V)/V = (t_0(\rho)/\rho)^v = \zeta^v$, and $\zeta^{\mathrm{k}_y(h)} = \sigma^{-1}(\zeta^v) = \zeta^{vc'}$ with $cc' \equiv 1 \pmod M$. Hence $c\,\mathrm{k}_y(h) = v$ in $\mathbb{Z}/M\mathbb{Z}$. □

Lemma 37.21. *Let ℓ be an odd prime, $N_0 \geq 1$ and $m \geq 1$ integers with $\ell \nmid N_0$, $N = N_0\ell^m$, r a prime with $r \equiv 1 \pmod N$, $((\alpha_d), (e_{d,p}))$ an Euler system for (N, r), (τ_p) a choice of generators, and δ, a, ζ_{ℓ^m} as in the definition of k_y. For $L \in \mathcal{L}(N, r)$ put $y_L = \mathrm{Der}_L(\alpha_{|L|})$. Let $L \in \mathcal{L}(N, r)$, p a Kolyvagin prime for (N, r) not occurring in L, D a decomposition group at p, φ a Frobenius of D, and $t \in I(D)$ with $t(\zeta) = \tau_p(\zeta)$ for $\zeta \in \mu_p$. Then $\varphi, t \in U_N$ and*

$$\mathrm{k}_{y_{(p,L)}}(t) = -\,e_{|L|,p}\,\mathrm{k}_{y_L}(\varphi) \quad \text{in } \mathbb{Z}/\ell^m\mathbb{Z}.$$

Proof. Write $M=\ell^m$, $\zeta=\zeta_{\ell^m}$, $\mathrm{P}=\mathrm{P}_{\delta,a}$, $n=|L|$, $e=e_{n,p}$, $k'=(p-1)/M\in\mathbb{Z}$ ($M\mid N\mid p-1$), $\mathrm{N}_p(z)=\prod_{i=0}^{p-2}\tau_p^i(z)$, and let w be the place stabilised by D. Put $y=y_L\in K_{Nn}^\times$, $w_0=\mathrm{Der}_L(\alpha_{pn})\in K_{Npn}^\times$ and $y'=y_{(p,L)}=\mathrm{Der}_p(w_0)$. Elements of $G_\mathbb{Q}$ commute on the abelian field K_{Npn}, so Der_L, Der_p, N_p and P commute with them there.

Step 1: t and φ. For $\zeta'\in\mu_{Nn}$ we have $t(\zeta')\equiv\zeta'$ and $\varphi(\zeta')\equiv\zeta'^p$ modulo w; both sides lie in μ_{Nn}, whose elements are pairwise incongruent modulo w as $p\nmid Nn$; so t fixes μ_{Nn} and $\varphi(\zeta')=\zeta'^p$. As $p\equiv1\pmod N$, $t,\varphi\in U_N$. Since τ_p also fixes μ_{Nn} and agrees with t on μ_p, t and τ_p agree on K_{Npn}.

Step 2: a root of $t(y')/y'$. As τ_p^{p-1} fixes K_{Npn}, $\tau_p(\mathrm{Der}_p(w_0))/\mathrm{Der}_p(w_0)=w_0^{p-1}/\mathrm{N}_p(w_0)$.

By (Eu2) for $d=n$, with τ_p (which fixes μ_{Nn} and acts on μ_p by a generator) and φ, and as Der_L is multiplicative and commutes with N_p and φ^{-1}, $\mathrm{N}_p(w_0)=\mathrm{Der}_L(\mathrm{N}_p(\alpha_{pn}))=(y/\varphi^{-1}(y))^e$. By Lemma 37.19 applied to $\varphi^{-1}\in U_N$, there is $z'\in K_{Nn}^\times$ with $z'^M=\varphi^{-1}(y)/y$. Then $z=w_0^{k'}z'^e\in K_{Npn}^\times$ satisfies

$$z^M=w_0^{p-1}\Big(\frac{\varphi^{-1}(y)}{y}\Big)^e=\frac{\tau_p(y')}{y'}=\frac{t(y')}{y'}.$$

Step 3: an identity. Choose $Y,Y'\in\overline{\mathbb{Q}}$ with $Y^M=y'$ and $Y'^M=y$, and put $c=w_0/\varphi^{-1}(y)^e\in K_{Npn}^\times$. By the definition of k, $\zeta^{\mathrm{k}_{y'}(t)}=t(\mathrm{P}(Y))/(\mathrm{P}(Y)\mathrm{P}(z))$ and $\zeta^{\mathrm{k}_y(\varphi^{-1})}=\varphi^{-1}(\mathrm{P}(Y'))/(\mathrm{P}(Y')\mathrm{P}(z'))$. Hence

$$\zeta^{\mathrm{k}_{y'}(t)-e\,\mathrm{k}_y(\varphi^{-1})}=\frac{t(\mathrm{P}(Y))}{\mathrm{P}(Y)}\cdot\Big(\frac{\mathrm{P}(Y')}{\varphi^{-1}(\mathrm{P}(Y'))^p}\Big)^e\cdot\mathrm{P}(c)^{-k'}.\tag{37.21}$$

Step 4: each factor is $\equiv1\pmod w$. The elements y, y', w_0 are products of integer powers of Galois conjugates of α_n, α_{pn}, hence w-units by (Eu1) ($p\neq r$); so are $\mathrm{P}(y)$, $\mathrm{P}(y')$ ($v_w\circ\delta=v_{w\circ\delta}$), hence $\mathrm{P}(Y)$, $\mathrm{P}(Y')$, and $x=\varphi^{-1}(\mathrm{P}(Y'))$ (φ stabilises w). As $t\in I(D)$ acts trivially on the residue field of w, $t(\mathrm{P}(Y))\equiv\mathrm{P}(Y)$. As φ is a Frobenius of D, $\mathrm{P}(Y')=\varphi(x)\equiv x^p$. Finally, by (Eu3) for $d=n$, $\alpha_{pn}/\varphi^{-1}(\alpha_n)^e\equiv1$ modulo every place above p; $c=\mathrm{Der}_L(\alpha_{pn}/\varphi^{-1}(\alpha_n)^e)$ is a product of integer powers of its conjugates $g(\cdot)$, $g\in G_\mathbb{Q}$, and $v_{w'}(g(x')-1)=v_{w'\circ g}(x'-1)$; so $c\equiv1$ modulo every place above p, hence $\delta(c)\equiv1$, $c^{-a}\equiv1$ and $\mathrm{P}(c)\equiv1\pmod w$.

Step 5: conclusion. By (37.21) and Step 4, the M-th root of unity $\zeta^{\mathrm{k}_{y'}(t)-e\mathrm{k}_y(\varphi^{-1})}$ is $\equiv1\pmod w$; as $p\nmid M$ it equals 1. So $\mathrm{k}_{y'}(t)=e\,\mathrm{k}_y(\varphi^{-1})=-e\,\mathrm{k}_y(\varphi)$, k_y being a homomorphism. □

Lemma 37.22. *Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring, $\theta\colon G_\mathbb{Q}\to\mathcal{O}^\times$ a continuous character with a Kolyvagin level (N_0,m) of modulus N, Σ the set of primes dividing N, and δ a normaliser for θ. Let $n\geq1$ be an integer, p a prime with $p\equiv1\pmod N$ and $p\nmid n$, and $\lambda\colon U_N\to\mathcal{O}^\vee$ a continuous homomorphism with $\lambda(gug^{-1})=\check\theta(g)\lambda(u)$ for $g\in G_\mathbb{Q}$, $u\in U_N$, vanishing on $U_N\cap I$ for every inertia group I at a prime not dividing Nnp. Let $a_0,a'\in\mathcal{O}$ with $a_0\neq0$, $a_0\mid a'\mid\ell^m$ and $a'\lambda=0$, and let $w_0\in\mathcal{O}^\vee$ have annihilator $a'\mathcal{O}$. Let $z\in\mathrm{Z}_{\theta,\delta}(\Sigma,a_0;n)$, let D be a decomposition group at p, $g\in D$, $\tau\in I(D)$, and $y\in\mathcal{O}$ with $\lambda(\tau)=y\,w_0$. Then $y\,z(g)=0$.*

Proof. Let K be the fraction field of $\mathcal{O}$. As $\mathcal{O}$ is free of finite rank over $\mathbb{Z}_\ell$, $\mathcal{O}^\vee=(\mathcal{O}\otimes\mathbb{Q}_\ell)/\mathcal{O}=K/\mathcal{O}$. Put $M=\{x\in\mathcal{O}^\vee:a'x=0\}=a'^{-1}\mathcal{O}/\mathcal{O}$; then $M=\mathcal{O}w_0$ and $x\mapsto xw_0$ identifies $\mathcal{O}/a'\mathcal{O}$ with M. Let $\beta\colon M\times M\to M$, $\beta(xw_0,x'w_0)=xx'w_0$, the $\mathcal{O}$-bilinear product transported from the ring $\mathcal{O}/a'\mathcal{O}$. Since $a_0\mid a'$, z takes values in M; so does λ.

Step 1: the actions on M. For $u\in U_N$, $\theta(u)\equiv1$ by (Kv1) and $\chi_{\mathrm{cyc}}(u)\equiv1$ modulo ℓ^m (u fixes $\mu_{\ell^m}\subseteq\mu_N$), hence also $\check\theta(u)\equiv1$; as $a'\mid\ell^m$, $\theta(u)$, $\check\theta(u)$ and $\chi_{\mathrm{cyc}}(u)$ act trivially on M, and $\chi_{\mathrm{cyc}}(u)$ acts trivially on μ_{ℓ^m}. An inertia group at a prime $p'\nmid N$ lies in U_N, since $\mathbb{Q}(\mu_N)$ is

unramified at p'; the decomposition group D lies in U_N, since $p \nmid N$ and its Frobenius elements act on μ_N by $\zeta \mapsto \zeta^p = \zeta$, while $I(D)$ acts trivially on μ_N.

Step 2: λ extends to a cocycle on $G_{\mathbb{Q}}$. Let $\Gamma = G_{\mathbb{Q}}/U_N$, a finite abelian group acting on M through $\check{\theta}$ (Step 1). Let g_1 be as in (Kv3) and $s = \check{\theta}(g_1)$; then $s-1 = \theta(g_1)^{-1}(\chi_{\mathrm{cyc}}(g_1)-\theta(g_1)) \in \mathcal{O}^\times$. For an element γ of a finite group, conjugation by γ together with $x \mapsto \gamma x$ on the coefficients induces the identity on cohomology ([221, VII.5]); for the central element $\gamma_1 = g_1 U_N$ of Γ this map is multiplication by s. Hence $s-1$ kills the $\mathcal{O}$-module $H^2(\Gamma, M)$, so $H^2(\Gamma, M) = 0$. In the inflation–restriction sequence $H^1(G_{\mathbb{Q}}, M) \to H^1(U_N, M)^\Gamma \to H^2(\Gamma, M)$ [222], $H^1(U_N, M) = \mathrm{Hom}_{\mathrm{cont}}(U_N, M)$ (Step 1), and λ is Γ-invariant: $\check{\theta}(g)\lambda(g^{-1}ug) = \check{\theta}(g)\check{\theta}(g)^{-1}\lambda(u) = \lambda(u)$. So there is a continuous cocycle $\tilde{\lambda}\colon G_{\mathbb{Q}} \to M$ for the action $\check{\theta}$ whose restriction to U_N differs from λ by a coboundary $u \mapsto \check{\theta}(u)x - x$, which is 0 by Step 1: $\tilde{\lambda}|_{U_N} = \lambda$. In particular $\tilde{\lambda}$ vanishes on every inertia group at a prime not dividing Nnp, and $\tilde{\lambda}(\tau) = \lambda(\tau)$.

Step 3: a global class. Let $\Lambda\colon M \to \mathbb{Z}/\ell^m\mathbb{Z}$ be any homomorphism of abelian groups ($\ell^m M = 0$), and give $\mathbb{Z}/\ell^m\mathbb{Z}$ the action $g \cdot k = \chi_{\mathrm{cyc}}(g)k$, which is μ_{ℓ^m} after fixing a primitive root of unity. The map $(x, x') \mapsto \Lambda(\beta(x, x'))$ from $M(\theta) \times M(\check{\theta})$ is biadditive and $G_{\mathbb{Q}}$-equivariant, since $\beta(\theta(g)x, \check{\theta}(g)x') = \chi_{\mathrm{cyc}}(g)\beta(x, x')$ and Λ commutes with multiplication by $\chi_{\mathrm{cyc}}(g) \in \mathbb{Z}_\ell$. So the cup product

$$w_\Lambda(g, h) = \Lambda(\beta(z(g), \check{\theta}(g)\tilde{\lambda}(h)))$$

is a continuous 2-cocycle on $G_{\mathbb{Q}}$ with values in μ_{ℓ^m}; let $\xi_\Lambda \in H^2(G_{\mathbb{Q}}, \mu_{\ell^m})$ be its class.

Step 4: ξ_Λ is locally trivial at every place $v \neq p$. Let D_v be a decomposition group at v.

(a) $v = \infty$: $H^2(G_{\mathbb{R}}, \mu_{\ell^m}) = 0$ by item 1 of Theorem B.22.

(b) $v = p' \in \Sigma$: by (Kd4) there is $x \in M$ with $z(g) = \theta(g)x - x$ on D_v. With $b(h) = \Lambda(\beta(x, \tilde{\lambda}(h)))$, for $g, h \in D_v$,

$$\begin{aligned}\chi_{\mathrm{cyc}}(g)b(h) - b(gh) + b(g) &= \Lambda(\beta(x, (\chi_{\mathrm{cyc}}(g) - \check{\theta}(g))\tilde{\lambda}(h))) \\ &= \Lambda(\beta((\theta(g) - 1)x, \check{\theta}(g)\tilde{\lambda}(h))) = w_\Lambda(g, h),\end{aligned}$$

so $w_\Lambda|_{D_v}$ is a coboundary.

(c) $v = p' \mid n$: $z = 0$ on D_v by (Kd5), so $w_\Lambda = 0$ on $D_v \times D_v$.

(d) $v = p' \notin \Sigma$, $p' \nmid np$: z vanishes on $I(D_v)$ by (Kd3), $\tilde{\lambda}$ by Step 2, and $I(D_v) \subseteq U_N$ acts trivially on M (Step 1); hence $z(gi) = z(g)$, $\tilde{\lambda}(hi') = \tilde{\lambda}(h)$ and $\check{\theta}(gi)x = \check{\theta}(g)x$ on M for $i, i' \in I(D_v)$, and $w_\Lambda|_{D_v}$ is inflated from a continuous 2-cocycle of $D_v/I(D_v) \cong \widehat{\mathbb{Z}}$ with values in a finite module. Since $\widehat{\mathbb{Z}}$ has cohomological dimension 1 [222], it is a coboundary.

Restriction to conjugate decomposition groups at the same place corresponds under conjugation, so $\mathrm{res}_v \xi_\Lambda = 0$ for every $v \neq p$. By Theorem B.22, $\mathrm{res}_p \xi_\Lambda = 0$: there is a continuous $b_p\colon D \to \mathbb{Z}/\ell^m\mathbb{Z}$ with $w_\Lambda(g, h) = \chi_{\mathrm{cyc}}(g)b_p(h) - b_p(gh) + b_p(g)$ for $g, h \in D$.

Step 5: the local computation at p. By Step 1, $D \subseteq U_N$ acts trivially on M and on μ_{ℓ^m}; so on D, z and $\tilde{\lambda}$ are continuous homomorphisms into M, z vanishes on $I(D)$ by (Kd3) ($p \notin \Sigma$), and $w_\Lambda(g, h) = \Lambda(\beta(z(g), \tilde{\lambda}(h))) = b_p(h) - b_p(gh) + b_p(g)$ on D. Fix a Frobenius φ of D. We use:

(F) a continuous homomorphism f from D to a discrete abelian group that vanishes on $I(D)$ and at φ is 0: its kernel is an open, hence closed, subgroup containing $I(D)$ and φ, and $D/I(D)$ is procyclic, topologically generated by the image of φ.

Let $y_\Lambda\colon D \to \mathbb{Z}/\ell^m\mathbb{Z} = \mu_{\ell^m}$, $y_\Lambda(h) = \Lambda(\beta(z(\varphi), \tilde{\lambda}(h)))$, a continuous homomorphism, i.e. a class in $H^1(D, \mu_{\ell^m})$. Let $x\colon D \to \mathbb{Z}/\ell^m\mathbb{Z}$ be a continuous homomorphism vanishing on $I(D)$, i.e. a class in $H^1_{\mathrm{ur}}(D, \mathbb{Z}/\ell^m\mathbb{Z})$. By (F) applied to $g \mapsto x(g)z(\varphi) - x(\varphi)z(g)$, we have $x(g)z(\varphi) = x(\varphi)z(g)$ for $g \in D$, hence

$$x(g)\, y_\Lambda(h) = \Lambda\big(\beta(x(\varphi)z(g), \tilde{\lambda}(h))\big) = x(\varphi)\big(b_p(h) - b_p(gh) + b_p(g)\big),$$

so the cup product $x \cup y_\Lambda$ is 0 in $H^2(D, \mu_{\ell^m})$. By item 3 of Theorem B.29 (applied to $\mathbb{Q}_p$, $p \neq \ell$, and the module $\mathbb{Z}/\ell^m\mathbb{Z}$ with trivial action, whose dual is μ_{ℓ^m}), $y_\Lambda \in H^1_{\mathrm{ur}}(D, \mu_{\ell^m})$; as D acts trivially on μ_{ℓ^m}, this means y_Λ vanishes on $I(D)$. In particular $\Lambda(\beta(z(\varphi), \tilde{\lambda}(\tau))) = 0$. The finite group M is killed by ℓ^m, so the homomorphisms $\Lambda\colon M \to \mathbb{Z}/\ell^m\mathbb{Z}$ separate its points; hence $\beta(z(\varphi), \lambda(\tau)) = 0$. Writing $z(\varphi) = \hat{z}w_0$ with $\hat{z} \in \mathcal{O}$, this says $\hat{z}\, y\, w_0 = 0$, i.e. $y\, z(\varphi) = 0$. Finally $g \mapsto y\, z(g)$ is a continuous homomorphism on D vanishing on $I(D)$ and at φ, so $y\, z(g) = 0$ for every $g \in D$ by (F). □

Lemma 37.23. *Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring, $\theta\colon G_{\mathbb{Q}} \to \mathcal{O}^\times$ a continuous character with a Kolyvagin level (N_0, m) of modulus N, r a prime with $r \equiv 1 \pmod N$, and $T \geq 1$ an integer. Let $\lambda, z\colon U_N \to \mathcal{O}^\vee$ be continuous homomorphisms with $\ell^m\lambda = \ell^m z = 0$ and*

$$\lambda(gug^{-1}) = \check{\theta}(g)\lambda(u), \qquad z(gug^{-1}) = \theta(g)z(u) \qquad (g \in G_{\mathbb{Q}},\ u \in U_N),$$

and assume $z \neq 0$. Then there are a Kolyvagin prime p for (N, r) with $p \nmid T$, a decomposition group D at p and a Frobenius φ of D such that $\varphi \in U_N$, $z(\varphi) \neq 0$ and $\lambda(U_N) \subseteq \mathcal{O}\lambda(\varphi)$.

Proof. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$; recall $\mathcal{O}^\vee \cong K/\mathcal{O}$ for the fraction field K of $\mathcal{O}$, so its $\mathcal{O}$-submodules are 0, $\pi^{-k}\mathcal{O}/\mathcal{O}$ (π a uniformiser) and $\mathcal{O}^\vee$, totally ordered by inclusion. Fix $\rho \in \overline{\mathbb{Q}}$ with $\rho^{\ell^m} = r$ and let $H = \{u \in U_N : u(\rho) = \rho\} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_N, \rho))$; the field $\mathbb{Q}(\mu_N, \rho)$ contains all $\zeta\rho$ with $\zeta \in \mu_{\ell^m} \subseteq \mu_N$, so it is Galois over $\mathbb{Q}$ and H is open and normal in $G_{\mathbb{Q}}$. For $u \in U_N$ put $\epsilon(u) = u(\rho)/\rho \in \mu_{\ell^m}$; since u fixes μ_{ℓ^m}, ϵ is a continuous homomorphism with kernel H, and $\epsilon(gug^{-1}) = g(\epsilon(u)) = \epsilon(u)^k$ for $g \in G_{\mathbb{Q}}$ and any integer $k \equiv \chi_{\mathrm{cyc}}(g) \pmod{\ell^m}$.

Step 1: a Kummer constraint. Let A be an $\mathcal{O}$-module with $\ell^m A = 0$, $\gamma\colon G_{\mathbb{Q}} \to \mathcal{O}^\times$ a character and $f\colon U_N \to A$ a homomorphism with $f(gug^{-1}) = \gamma(g)f(u)$, vanishing on H. Then $(\gamma(g) - \chi_{\mathrm{cyc}}(g))f(u) = 0$ for all $g \in G_{\mathbb{Q}}$, $u \in U_N$. Indeed, for $k \equiv \chi_{\mathrm{cyc}}(g) \pmod{\ell^m}$, $\epsilon(gug^{-1}) = \epsilon(u^k)$, so $gug^{-1} \in u^k H$ and $\gamma(g)f(u) = f(gug^{-1}) = kf(u) = \chi_{\mathrm{cyc}}(g)f(u)$.

Step 2: three open normal subgroups not containing H.

(a) $B = \ker z$ is open (z is continuous into a discrete module) and normal (equivariance). If $H \subseteq B$, Step 1 with $\gamma = \theta$ and $g = g_1$ from (Kv3) gives $z = 0$, as $\theta(g_1) - \chi_{\mathrm{cyc}}(g_1)$ is a unit; so $H \not\subseteq B$.

(b) The set $\lambda(U_N)$ is finite (U_N is compact, λ locally constant), so there is $u_0 \in U_N$ with $\mathcal{O}\lambda(u) \subseteq \mathcal{O}\lambda(u_0)$ for all u. If $\lambda(u_0) = 0$ put $K_\lambda = B$. Otherwise let $P = \mathfrak{m}\lambda(u_0)$ and $K_\lambda = \{u \in U_N : \lambda(u) \in P\}$, an open normal subgroup. If $H \subseteq K_\lambda$, Step 1 applied to $u \mapsto \lambda(u) + P \in \mathcal{O}^\vee/P$ with $\gamma = \check{\theta}$ and an element $c \in G_{\mathbb{Q}}$ acting on μ_N by $\zeta \mapsto \zeta^{-1}$ gives $(\check{\theta}(c) - \chi_{\mathrm{cyc}}(c))\lambda(u_0) \in P$; here $\check{\theta}(c) - \chi_{\mathrm{cyc}}(c) = \check{\theta}(c)(1 - \theta(c))$ and $1 - \theta(c) \equiv 2 \pmod{\mathfrak{m}}$ by (Kv2), a unit as ℓ is odd. So $\lambda(u_0) = y\lambda(u_0)$ with $y \in \mathfrak{m}$, and $(1-y)\lambda(u_0) = 0$ forces $\lambda(u_0) = 0$, a contradiction. Hence $H \not\subseteq K_\lambda$. In both cases, every $\varphi \in U_N \setminus K_\lambda$ satisfies $\lambda(U_N) \subseteq \mathcal{O}\lambda(\varphi)$: if $\lambda(u_0) \neq 0$, $\lambda(\varphi) = y\lambda(u_0)$ with $y \notin \mathfrak{m}$, so $\mathcal{O}\lambda(\varphi) = \mathcal{O}\lambda(u_0)$; if $\lambda(u_0) = 0$, $\lambda = 0$.

(c) $C = U_{N\ell}$. Let $u \in G_{\mathbb{Q}}$ act on $\mu_{N\ell}$ by $\zeta \mapsto \zeta^{1+N}$ ($1+N$ is prime to $N\ell$ as $\ell \mid N$), so $u \in U_N$, and let $c' \in G_{\mathbb{Q}}$ act on $\mu_{N\ell}$ by inversion. Then $h = c'uc'^{-1}u \in U_N$ and $\epsilon(h) = \epsilon(u)^{-1}\epsilon(u) = 1$, since $\chi_{\mathrm{cyc}}(c') \equiv -1 \pmod{\ell^m}$; so $h \in H$. As $\mathrm{Gal}(\mathbb{Q}(\mu_{N\ell})/\mathbb{Q})$ is abelian, h acts on $\mu_{N\ell}$ by $\zeta \mapsto \zeta^{(1+N)^2}$, and $(1+N)^2 - 1 = N(N+2)$ is not divisible by $N\ell$ because $\ell \nmid N+2$. So $h \notin C$ and $H \not\subseteq C$.

Moreover $x^{\ell^m} \in B$ for every $x \in H$, since $z(x^{\ell^m}) = \ell^m z(x) = 0$.

Step 3: an element of H outside the three subgroups. First, if A, A' are subgroups with $H \not\subseteq A$, $H \not\subseteq A'$, there is $x \in H$ outside $A \cup A'$: take $a \in H \setminus A$ and $b \in H \setminus A'$; if $a \notin A'$ or $b \notin A$ we are done; otherwise $ab \notin A$ and $ab \notin A'$. Suppose now $H \subseteq K_\lambda \cup B \cup C$. Choose $x \in H \setminus (B \cup C)$ and $y \in H \setminus (K_\lambda \cup C)$; then $x \in K_\lambda$ and $y \in B$. The element xy lies neither in K_λ (else $y \in K_\lambda$) nor in B (else $x \in B$), so $xy \in C$. The element $x^2y = x \cdot xy$ lies neither in K_λ (else $xy \in K_\lambda$, so $y \in K_\lambda$) nor in C (else $x \in C$); so $x^2y \in B$ and $x^2 \in B$. Writing $\ell^m = 2j+1$, $x = x^{\ell^m}(x^2)^{-j} \in B$, a contradiction. So there is $x \in H$ with $x \notin K_\lambda \cup B \cup C$.

Step 4: Chebotarev. Let $W = K_\lambda \cap B \cap C \cap H$, an open normal subgroup of $G_{\mathbb{Q}}$ with finite Galois fixed field L. By Theorem B.18, applied to the conjugacy class of the image of x in $\mathrm{Gal}(L/\mathbb{Q})$, infinitely many primes are unramified in L and have this class as Frobenius class; choose such a prime p with $p \nmid TNr$, a prime of L above p whose Frobenius is the image of x, a place of $\overline{\mathbb{Q}}$ above it with decomposition group D, and a Frobenius φ of D. Then φ maps to the image of x in $\mathrm{Gal}(L/\mathbb{Q})$, i.e. $\varphi \in xW$. Since W lies in each of K_λ, B, C and x in none, $\varphi \notin K_\lambda \cup B \cup C$; and $\varphi \in H \subseteq U_N$. So $z(\varphi) \neq 0$ by (a) and $\lambda(U_N) \subseteq \mathcal{O}\lambda(\varphi)$ by (b).

Step 5: p is a Kolyvagin prime for (N, r). As $p \nmid N\ell$, φ acts on $\mu_{N\ell}$ by $\zeta \mapsto \zeta^p$. From $\varphi \in U_N$, $p \equiv 1 \pmod{N}$; from $\varphi \notin U_{N\ell}$, $p \not\equiv 1 \pmod{N\ell}$; as $N_0 \mid p-1$ and $\ell \nmid N_0$, this gives $v_\ell(p-1) = m$. Since $p \neq r$, ρ is a unit at the chosen place, and $\varphi(\rho) = \rho$ with $\varphi(\rho) \equiv \rho^p$ modulo that place gives $\rho^{p-1} \equiv 1$; so the integer $r^{(p-1)/\ell^m} - 1 = \rho^{p-1} - 1$ is divisible by p, and $\mathrm{ord}_p(r)$ divides $(p-1)/\ell^m$, which is prime to ℓ. Finally $p \neq r$ and $p \nmid T$. □

Proposition 37.24. *There is a real number $C \geq 0$ such that for every integer $N \geq 1$, every CM field E that embeds into $\mathbb{Q}(\zeta_N)$, every odd prime ℓ dividing $n = [E : \mathbb{Q}]$ and every integer $e \geq 1$: $\mathrm{len}_\ell^-(E, e) < \infty$ and*

$$\mathrm{len}_\ell^-(E, e)\, \log \ell \leq C\, \ell\, n^2\, (e+1)\, \log^2(e\ell N).$$

Proof. We show the claim with $C = 44$. Fix N, E, ℓ, e as in the statement; the hypothesis $\ell \mid n$ is not used. The lengths do not change when E is replaced by an isomorphic field, so let $E \subseteq \mathbb{Q}(\zeta_N)$. Let $N_1 = 4N\ell$, Δ, π, $a = 1 + N_1$, Q_j, ξ_j and the action $\star$ be as in Lemma 37.25, and put $D_E = \varphi(N_1)/n$. Let $c = \pi(-1) \in \Delta$: it is the restriction of $\zeta \mapsto \zeta^{-1}$, which is complex conjugation for every embedding of $\mathbb{Q}(\zeta_{N_1})$ into $\mathbb{C}$, so c is the complex conjugation of the CM field E, and $c \neq 1$. A lift of c to $\mathrm{Gal}(E_{S_\ell}/\mathbb{Q})$ lies in $\mathrm{Gal}(E_{S_\ell}/E^+)$, and $c\star$ is the action of Definition 36.1; so, with $\varsigma = (-1)^{e+1}$,

$$\mathrm{Sha}_\ell^-(E, e) = \{x \in \mathrm{Sha}_\ell^1(E, e) : c \star x = \varsigma x\}.$$

Characters. Let X_ς be the set of characters $\vartheta \colon \Delta \to \overline{\mathbb{Q}}_\ell^\times$ with $\vartheta(c) = \varsigma$. As Δ is abelian of order n and $c \neq 1$, $\vartheta \mapsto \vartheta(c)$ is a surjective homomorphism from the n characters of Δ onto $\{\pm 1\}$, so $|X_\varsigma| = n/2$. For $\vartheta \in X_\varsigma$, $\chi_\vartheta = \vartheta \circ \pi$ is a Dirichlet character modulo N_1 with $\ker \pi \subseteq \ker \chi_\vartheta$ and $\chi_\vartheta(-1) = (-1)^{e+1}$, and $\vartheta \mapsto \chi_\vartheta$ is a bijection of X_ς onto the set $\mathcal{X}_{\overline{\mathbb{Q}}_\ell}$ of Proposition 37.9 for $(N_1, e+1, \ker \pi)$; the corresponding set $\mathcal{X}_{\mathbb{C}}$ also has $n/2$ elements.

The ring A_-. Let $A_- = \mathbb{Z}_\ell[\Delta]/(c-\varsigma)$. As $\mathbb{Z}_\ell[\Delta]$ is free over $\mathbb{Z}_\ell[\langle c\rangle]$ on a set of representatives of $\Delta/\langle c\rangle$, A_- is free of rank $n/2$ over $\mathbb{Z}_\ell$. For $\xi \in \mathbb{Z}_\ell[\Delta]$ let $\mathrm{Nm}(\xi) \in \mathbb{Z}_\ell$ be the determinant of multiplication by ξ on A_-. Then:

(a) $\mathrm{Nm}(\xi) = \prod_{\vartheta \in X_\varsigma} \vartheta(\xi)$. Indeed $\xi \mapsto (\vartheta(\xi))_\vartheta$, over all n characters, is an isomorphism $\overline{\mathbb{Q}}_\ell[\Delta] \to \overline{\mathbb{Q}}_\ell^{\,n}$ (the characters are linearly independent); it carries $c-\varsigma$ to the vector with entries $\vartheta(c)-\varsigma$, which vanish exactly for $\vartheta \in X_\varsigma$ and are -2ς otherwise. So $A_- \otimes \overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_\ell^{\,X_\varsigma}$, on which ξ acts diagonally with entries $\vartheta(\xi)$.

(b) $\mathrm{Nm}(\xi) \in \xi A_-$: by the Cayley–Hamilton theorem the characteristic polynomial $T^{n/2} + \cdots \pm \mathrm{Nm}(\xi)$ of multiplication by ξ vanishes at ξ in A_-.

An exponent for the minus part. For $\vartheta \in X_\varsigma$ let χ_ϑ^* be the primitive character inducing χ_ϑ and

$$y_\vartheta = (a^{e+1}-1)\, B_{e+1,\chi_\vartheta^*} \prod_{p \mid N_1} (1 - \chi_\vartheta^*(p) p^e).$$

It is nonzero: $a^{e+1} \neq 1$; $B_{e+1,\chi_\vartheta^*} \neq 0$ because $\mathrm{Ber}_{e+1}(\chi_\vartheta) = N_1(e+2)\# B_{e+1,\chi_\vartheta^*} \neq 0$ by item 1 of Proposition 37.9; and $1 - \chi_\vartheta^*(p)p^e \neq 0$, as $p^e > 1$ is not a root of unity. For every j, $\vartheta(\xi_j) = \sum_{1 \le b < Q_j} \lfloor ab/Q_j \rfloor b^e \chi_\vartheta(b)$ (the terms with $\gcd(b, N_1) > 1$ vanish), which is $T_j(\chi_\vartheta)$ of Lemma 37.8 for $F = N_1$ (here $\ell \mid N_1$, $a = 1 + N_1$); by item 1 there, $(a^{e+1}-1) B_{e+1,\chi_\vartheta}^{(N_1)} = y_\vartheta$. Let $j_1 = \max_\vartheta v_\ell(y_\vartheta) + v_\ell(e+2) + 2$. For $j \ge j_1$, item 2 there gives $v_\ell\big((e+1)a^e\vartheta(\xi_j) - y_\vartheta\big) > v_\ell(y_\vartheta)$, hence $v_\ell((e+1)a^e\vartheta(\xi_j)) = v_\ell(y_\vartheta)$, and as $a \equiv 1 \pmod{\ell}$ is a unit,

$$v_\ell\big(\vartheta(\xi_j)\big) = v_\ell(y_\vartheta) - v_\ell(e+1) \qquad (j \ge j_1). \tag{37.22}$$

Put $r_0 = \sum_{\vartheta \in X_\varsigma} \big(v_\ell(y_\vartheta) - v_\ell(e+1)\big)$ and $r = r_0 + v_\ell(D_E)$. Let $x \in \mathrm{Sha}_\ell^-(E,e)$ and $z = D_E x$. By Lemma 37.25 there is j_0 with $\xi_j \star z = D_E(\xi_j \star x) = 0$ for $j \ge j_0$. Fix $j \ge \max(j_0, j_1)$. As $(c-\varsigma)\star z = 0$, the action of $\mathbb{Z}_\ell[\Delta]$ on the submodule it generates from z factors through A_-, and by (b), $\mathrm{Nm}(\xi_j)\, z = 0$. By (a) and (37.22), $\mathrm{Nm}(\xi_j) \in \mathbb{Z}_\ell$ has valuation r_0, so it is ℓ^{r_0} times a unit, and $\ell^{r_0} D_E x = 0$. Hence ℓ^r kills $\mathrm{Sha}_\ell^-(E,e)$.

The size of r. For $\vartheta \in X_\varsigma$ and a prime $p \mid N_1$,

$$v_\ell(y_\vartheta) - v_\ell(e+1) \le v_\ell(a^{e+1}-1) + v_\ell\big(\mathrm{Ber}_{e+1}(\chi_\vartheta)\big) + \sum_{p \mid N_1} v_\ell(1 - \chi_\vartheta^*(p)p^e),$$

as $v_\ell(N_1(e+2)\#) \ge 0$. We bound the three sums over ϑ.

(i) $a^{e+1}-1$ is a positive integer, so $v_\ell(a^{e+1}-1)\log\ell \le (e+1)\log(1+N_1)$.

(ii) By item 2 of Proposition 37.9 for $(N_1, e+1, \ker\pi)$ and $S = \{\ell\}$: $\sum_\vartheta v_\ell(\mathrm{Ber}_{e+1}(\chi_\vartheta))\log\ell \le w_\ell \log\ell \le \frac{n}{2}\log(16 N_1^{e+2}(e+1)!)$.

(iii) Fix $p \mid N_1$ and an embedding $j_\ell\colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$. Every χ_ϑ is $j_\ell \circ \chi_0$ for a unique character χ_0 in the set $\mathcal{X}_{\overline{\mathbb{Q}}} = \mathcal{X}_{\mathbb{C}}$, and then $\chi_\vartheta^* = j_\ell \circ \chi_0^*$ (the conductor depends only on the kernel). The set $\mathcal{X}_{\overline{\mathbb{Q}}}$ is permuted by the automorphisms σ of $\overline{\mathbb{Q}}$, with $(\sigma\circ\chi_0)^* = \sigma \circ \chi_0^*$, so $P_p = \prod_{\chi_0 \in \mathcal{X}_{\overline{\mathbb{Q}}}} (1 - \chi_0^*(p)p^e)$ is a rational algebraic integer, nonzero, with $|P_p| \le (1+p^e)^{n/2}$. Hence $\sum_\vartheta v_\ell(1-\chi_\vartheta^*(p)p^e)\log\ell = v_\ell(P_p)\log\ell \le \log|P_p| \le \frac{n}{2}\log(1+p^e) \le \frac{n}{2}(e\log p + \log 2)$, and summing over the primes $p \mid N_1$ gives at most $\frac{n}{2}(e+1)\log N_1$.

Let $L=\log(eN_1)$; as $N_1\ge 12$: $\log(1+N_1)\le 2L$, $\log 16\le 2L$, $\log(e+1)\le L$, so $\log(16N_1^{e+2}(e+1)!)\le(2e+5)L\le 4(e+1)L$. With $v_\ell(D_E)\log\ell\le\log N_1\le L$ and $n\ge 2$,

$$r\log\ell\le\frac{n}{2}(2(e+1)+4(e+1)+(e+1))L+L\le 4n(e+1)L\le 12\,n(e+1)\log(e\ell N),$$

since $4eN\ell\le(e\ell N)^3$.

Length. Let $P=\mathrm{Sha}^-_\ell(E,e)\subseteq H^1(G_{E,S_\ell},\mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))$; it is killed by ℓ^r. For $0\le i<r$, multiplication by ℓ^i maps $P[\ell^{i+1}]$ into $P[\ell]$ with kernel $P[\ell^i]$; so

$$\operatorname{length}P=\sum_{i<r}\operatorname{length}(P[\ell^{i+1}]/P[\ell^i])\le r\operatorname{length}P[\ell].$$

By Lemma 37.26 (E has no real place), $P[\ell]$ is finite of length at most $4\ell n\log(\ell N)/\log\ell$. Hence $\mathrm{len}^-_\ell(E,e)<\infty$ and, with $\log\ell\ge\log 3$,

$$\mathrm{len}^-_\ell(E,e)\log\ell\le(r\log\ell)\cdot\frac{4\ell n\log(\ell N)}{\log\ell}\le\frac{48}{\log 3}\,\ell\,n^2(e+1)\log^2(e\ell N)\le 44\,\ell\,n^2(e+1)\log^2(e\ell N).$$

□

Lemma 37.25. *Let $N\ge 1$, let $E\subseteq\mathbb{Q}(\zeta_N)$ be a subfield with $n=[E:\mathbb{Q}]$, let ℓ be an odd prime and $e\ge 1$ an integer. Put $N_1=4N\ell$, $\Delta=\mathrm{Gal}(E/\mathbb{Q})$, and for b prime to N_1 let $\pi(b)\in\Delta$ be the restriction to E of the automorphism $\zeta_{N_1}\mapsto\zeta_{N_1}^b$ of $\mathbb{Q}(\zeta_{N_1})$. Let E_{S_ℓ} be the maximal extension of E unramified outside the places above ℓ, so that $G_{E,S_\ell}=\mathrm{Gal}(E_{S_\ell}/E)$. The group Δ acts on $H^1(G_{E,S_\ell},\mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))$ by*

$$\delta\star[f]=[g\mapsto\chi_{\mathrm{cyc}}(\tilde\delta)^{-e}f(\tilde\delta^{-1}g\tilde\delta)],$$

with $\tilde\delta\in\mathrm{Gal}(E_{S_\ell}/\mathbb{Q})$ any element restricting to δ; this action preserves $\mathrm{Sha}^1_\ell(E,e)$ and is extended linearly to $\mathbb{Z}[\Delta]$. Put $a=1+N_1$ and, for $j\ge 0$, $Q_j=N_1\ell^j$ and

$$\xi_j=\sum_{\substack{1\le b<Q_j\\ \gcd(b,N_1)=1}}\Big\lfloor\frac{ab}{Q_j}\Big\rfloor\,b^e\,\pi(b)\in\mathbb{Z}[\Delta].$$

Then for every $x\in\mathrm{Sha}^1_\ell(E,e)$ there is $j_0\ge 0$ such that $\frac{\varphi(N_1)}{n}(\xi_j\star x)=0$ for every $j\ge j_0$.

Proof. Write $M=\mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$, a $G_\mathbb{Q}$-module through χ_{cyc}^{-e}; for $r\ge 0$ let $M[\ell^r]$ be its ℓ^r-torsion. The field E is Galois over $\mathbb{Q}$, so E_{S_ℓ} is Galois over $\mathbb{Q}$, and it contains $E(\mu_{\ell^\infty})$; so χ_{cyc} is defined on $\mathrm{Gal}(E_{S_\ell}/\mathbb{Q})$.

The action. On $H^1(G_E,M)$ put $\delta\star[f]=[g\mapsto\tilde\delta f(\tilde\delta^{-1}g\tilde\delta)]$ with $\tilde\delta\in G_\mathbb{Q}$ restricting to δ. Two lifts differ by an element of G_E, and inner automorphisms act trivially on cohomology, so this is well defined; it is an action of Δ. Inflation from G_{E,S_ℓ} to G_E is injective with image the classes unramified at the places not above ℓ (the kernel of $G_E\to G_{E,S_\ell}$ is the closed normal subgroup generated by the inertia groups at those places, which act trivially on M), and it carries the action of the statement to this one. Conjugation by $\tilde\delta$ permutes the inertia groups and the decomposition groups of G_E; so $\star$ preserves the classes that are unramified everywhere and trivial on every decomposition group above ℓ, which is the image of $\mathrm{Sha}^1_\ell(E,e)$. We work in $H^1(G_E,M)$ from now on.

Step 1: a cocycle with bounded local data. Let $x\in\mathrm{Sha}^1_\ell(E,e)$. It has finite order, say $\ell^r x=0$. As M is divisible, $0\to M[\ell^r]\to M\xrightarrow{\ell^r}M\to 0$ is exact, so x is the image of a class of $H^1(G_E,M[\ell^r])$: choose a continuous cocycle $f\colon G_E\to M[\ell^r]$ representing x.

(a) If $I \subseteq G_E$ is the inertia group at a place not above ℓ, then I acts trivially on M and $f|_I$ is a coboundary, so $f|_I = 0$.

(b) For each of the finitely many places v of E above ℓ fix a decomposition group $D_v \subseteq G_E$; as x is trivial at v there is $m_v \in M$ with $f(g) = (g-1)m_v$ for $g \in D_v$. Choose $r_1 \geq r$ with $\ell^{r_1} m_v = 0$ for all v. Every decomposition group of G_E at a place above ℓ is $\sigma D_v \sigma^{-1}$ for some v and $\sigma \in G_E$, and for $g \in D_v$ and $h = \sigma g \sigma^{-1}$,

$$f(h) = (h-1)\bigl(\sigma m_v - f(\sigma)\bigr),$$

with $\ell^{r_1}(\sigma m_v - f(\sigma)) = 0$. So on every decomposition group D at a place above ℓ, $f(h) = (h-1)m_D$ for some $m_D \in M[\ell^{r_1}]$.

Put $j_0 = r_1$.

Step 2: restriction to a cyclotomic field. Fix $j \geq j_0$, let $K_j = \mathbb{Q}(\zeta_{Q_j})$, which contains E as $N \mid Q_j$, and $U_j = G_{K_j}$, a closed normal subgroup of $G_{\mathbb{Q}}$ contained in G_E. As $\ell^{j+1} \mid Q_j$, $\chi_{\mathrm{cyc}}(U_j) \subseteq 1 + \ell^{j+1}\mathbb{Z}_\ell$, so U_j acts trivially on $M[\ell^{j+1}] \supseteq M[\ell^{r_1}]$. Hence $\psi = f|_{U_j}$ is a continuous homomorphism $U_j \to M[\ell^r]$. By (a), ψ vanishes on the inertia groups of U_j at the places not above ℓ, which are the intersections of those of G_E with U_j; by (b), it vanishes on the decomposition groups of U_j at the places above ℓ, as $(h-1)m_D = 0$ for $h \in U_j$. Since K_j has no real place, the fixed field of $\ker\psi$ is an abelian extension of K_j unramified at every place, hence contained in the Hilbert class field H_j of K_j. Let $\mathrm{Art}_j \colon \mathrm{Cl}(K_j) \to \mathrm{Gal}(H_j/K_j)$ be the Artin isomorphism; there is a homomorphism $\bar\psi \colon \mathrm{Cl}(K_j) \to M[\ell^r]$ with $\psi(u) = \bar\psi(\mathfrak{c})$ whenever $u|_{H_j} = \mathrm{Art}_j(\mathfrak{c})$. The field H_j is Galois over $\mathbb{Q}$, and $\mathrm{Art}_j(\tau(\mathfrak{c})) = \tau\,\mathrm{Art}_j(\mathfrak{c})\,\tau^{-1}$ for $\tau \in G_{\mathbb{Q}}$ (Frobenius elements are conjugated in this way). Hence, for $u \in U_j$ with $u|_{H_j} = \mathrm{Art}_j(\mathfrak{c})$ and $\tau \in G_{\mathbb{Q}}$,

$$\psi(\tau^{-1}u\tau) = \bar\psi(\tau^{-1}(\mathfrak{c})). \tag{37.23}$$

Step 3: the Stickelberger relation. For $1 \leq b < Q_j$ prime to N_1 (equivalently, to Q_j) choose $\tau_b \in G_{\mathbb{Q}}$ with $\tau_b(\zeta_{Q_j}) = \zeta_{Q_j}^b$; then $\tau_b|_{K_j} = \sigma_b$ in the notation of Theorem B.23, and $\tau_b|_E = \pi(b)$. As $\ell^{j+1} \mid Q_j$, $\chi_{\mathrm{cyc}}(\tau_b) \equiv b \pmod{\ell^{j+1}}$, so $b^e\chi_{\mathrm{cyc}}(\tau_b)^{-e} \equiv 1 \pmod{\ell^{j+1}}$ and $b^e\tau_b$ acts as the identity on $M[\ell^{j+1}] \supseteq M[\ell^r]$. The cocycle

$$F = \sum_b \Bigl\lfloor \frac{ab}{Q_j} \Bigr\rfloor b^e\,(\tau_b \cdot f), \qquad (\tau \cdot f)(g) = \tau f(\tau^{-1}g\tau),$$

represents $\xi_j \star x$. Let $u \in U_j$ with $u|_{H_j} = \mathrm{Art}_j(\mathfrak{c})$. Then $\tau_b^{-1}u\tau_b \in U_j$, so $f(\tau_b^{-1}u\tau_b) = \psi(\tau_b^{-1}u\tau_b) \in M[\ell^r]$, and by (37.23)

$$F(u) = \sum_b \Bigl\lfloor \frac{ab}{Q_j} \Bigr\rfloor \bar\psi(\sigma_b^{-1}(\mathfrak{c})) = \bar\psi(\beta\,\mathfrak{c}), \qquad \beta = \sum_b \Bigl\lfloor \frac{ab}{Q_j} \Bigr\rfloor \sigma_b^{-1} \in \mathbb{Z}[\mathrm{Gal}(K_j/\mathbb{Q})].$$

Here $\beta = (a - \sigma_a)\theta_{Q_j}$.

As $Q_j \geq 3$, $4 \mid Q_j$ and $\gcd(a, Q_j) = 1$ (every prime dividing Q_j divides N_1, and $a \equiv 1 \pmod{N_1}$), Theorem B.23 shows that β annihilates $\mathrm{Cl}(K_j)$. So $F(u) = 0$: F vanishes on U_j.

Step 4: descent from the cyclotomic tower. Let $K_\infty = \bigcup_j K_j$ and $U = G_{K_\infty} \subseteq U_j$; as $\mu_{\ell^\infty} \subset K_\infty$, U acts trivially on M. Since F vanishes on U, $F(gu) = F(g) + gF(u) = F(g)$ and $F(ug) = F(u) + uF(g) = F(g)$ for $g \in G_E$, $u \in U$; so F is the inflation of a continuous cocycle $\bar F$ on the abelian group $W = G_E/U = \mathrm{Gal}(K_\infty/E)$. Write $N_1 = N_0\ell^t$ with $\ell \nmid N_0$, $t \geq 1$, and let $K_0 = \mathbb{Q}(\zeta_{N_1})$ and $W_0 = \mathrm{Gal}(K_\infty/K_0) \subseteq W$. As $K_\infty = K_0(\mu_{\ell^\infty})$, χ_{cyc} maps W_0 isomorphically

onto $1+\ell^t\mathbb{Z}_\ell$; let $\gamma_0 \in W_0$ with $\chi_{\mathrm{cyc}}(\gamma_0) = 1+\ell^t$, which generates a dense subgroup of W_0 (ℓ odd). It acts on M by the scalar $u_0 = (1+\ell^t)^{-e}$, and $u_0 - 1 \neq 0$ as $e \geq 1$; multiplication by $u_0 - 1$ is surjective on the divisible module M, so there is $y_0 \in M$ with $(\gamma_0 - 1)y_0 = \bar{F}(\gamma_0)$. The cocycle $F' = \bar{F} - \partial y_0$, $\partial y_0(w) = (w-1)y_0$, vanishes at γ_0, hence on the group generated by γ_0, hence on W_0, because $\{w : F'(w) = 0\}$ is closed (F' is continuous and M is discrete). For $w \in W$ and $w_0 \in W_0$, $F'(ww_0) = F'(w)$ and $F'(w_0 w) = w_0 F'(w)$, so, W being abelian, F' is constant on the cosets of W_0 and takes values in M^{W_0}: it is inflated from a cocycle on $W/W_0 = \mathrm{Gal}(K_0/E)$, a group of order $D_E = \varphi(N_1)/n$. With $y_1 = \sum_{w' \in W/W_0} F'(w')$, summing $F'(ww') = F'(w) + wF'(w')$ over w' gives $y_1 = D_E F'(w) + wy_1$, that is $D_E F' = \partial(-y_1)$. Hence $D_E \bar{F} = \partial(D_E y_0 - y_1)$, $D_E F$ is a coboundary on G_E, and $D_E\,(\xi_j \star x) = 0$. This holds for every $j \geq j_0$. □

Lemma 37.26. *Let $N \geq 1$, let $E \subseteq \mathbb{Q}(\zeta_N)$ be a subfield with no real place, $n = [E:\mathbb{Q}]$, let ℓ be an odd prime and $e \in \mathbb{Z}$. Then the ℓ-torsion subgroup $H^1(G_{E,S_\ell}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))[\ell]$ is finite and*

$$\mathrm{length}_{\mathbb{Z}_\ell} H^1(G_{E,S_\ell}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))[\ell] \leq \frac{4\,\ell\, n \log(\ell N)}{\log \ell}.$$

Proof. Write $M = \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$, $A = M[\ell]$ (the group $\mathbb{Z}/\ell\mathbb{Z}$ with G_E acting through χ_{cyc}^{-e}) and $G = G_{E,S_\ell}$.

Reduction to a homomorphism group. As M is divisible, $0 \to A \to M \xrightarrow{\ell} M \to 0$ is exact, and $H^1(G,A) \to H^1(G,M)[\ell]$ is surjective; so it suffices to bound $\dim_{\mathbb{F}_\ell} H^1(G,A)$. Let $F = E(\mu_\ell) \subseteq \mathbb{Q}(\zeta_{\ell N})$ and $m = [F:\mathbb{Q}]$; then $[F:E]$ divides $\ell - 1$, $m \leq (\ell-1)n$, and F has no real place, so $m \geq 2$. The extension F/E is unramified outside ℓ, so the maximal extension of E unramified outside the places above ℓ is also that of F, and G_{F,S_ℓ} is an open subgroup of G of index $[F:E]$. Corestriction after restriction is multiplication by $[F:E]$, which is invertible on $H^1(G,A)$ (killed by ℓ); so restriction $H^1(G,A) \to H^1(G_{F,S_\ell}, A)$ is injective. As $\mu_\ell \subset F$, G_{F,S_ℓ} acts trivially on A, and $H^1(G_{F,S_\ell}, A) = \mathrm{Hom}_{\mathrm{cts}}(G_{F,S_\ell}, \mathbb{Z}/\ell\mathbb{Z})$.

Kummer theory. Fix a primitive ℓ-th root of unity in F. Kummer theory gives an isomorphism

$$\mathrm{Hom}_{\mathrm{cts}}(G_F, \mathbb{Z}/\ell\mathbb{Z}) \cong F^\times/F^{\times\ell},$$

the class of α corresponding to a character cutting out $F(\alpha^{1/\ell})$; it factors through G_{F,S_ℓ} if and only if $F(\alpha^{1/\ell})/F$ is unramified at every finite place not above ℓ (F has no real place), and at a prime $\mathfrak{p} \nmid \ell$ this holds if and only if $\ell \mid v_\mathfrak{p}(\alpha)$ [190]. Hence $\mathrm{Hom}_{\mathrm{cts}}(G_{F,S_\ell}, \mathbb{Z}/\ell\mathbb{Z}) \cong \mathcal{B} = \{\alpha \in F^\times : \ell \mid v_\mathfrak{p}(\alpha) \text{ for all } \mathfrak{p} \nmid \ell\}/F^{\times\ell}$. Let S be the set of primes of F above ℓ, so $\#S \leq m$, $\mathcal{O}_S^\times$ the S-units and $\mathrm{Cl}' = \mathrm{Cl}(F)/\langle[\mathfrak{p}] : \mathfrak{p} \in S\rangle$. For α representing a class in $\mathcal{B}$ write $(\alpha) = \mathfrak{a}^\ell\mathfrak{s}$ with $\mathfrak{a} = \prod_{\mathfrak{p}\notin S} \mathfrak{p}^{v_\mathfrak{p}(\alpha)/\ell}$ and $\mathfrak{s}$ supported on S; the class of $\mathfrak{a}$ in Cl' is killed by ℓ and does not change when α is multiplied by an ℓ-th power. This defines a homomorphism $\mathcal{B} \to \mathrm{Cl}'[\ell]$. If α is in its kernel, then $\mathfrak{a} = (\beta)\mathfrak{s}'$ with $\mathfrak{s}'$ supported on S, so $\alpha\beta^{-\ell} \in \mathcal{O}_S^\times$: the kernel is the image of $\mathcal{O}_S^\times$. By the S-unit theorem, $\mathcal{O}_S^\times$ is the product of the cyclic group of roots of unity of F, whose order is divisible by ℓ, and a free group of rank $m/2 - 1 + \#S$ [190]. Since Cl' is a quotient of $\mathrm{Cl}(F)$, $\dim \mathrm{Cl}'[\ell] = \dim \mathrm{Cl}'/\ell\mathrm{Cl}' \leq \dim \mathrm{Cl}(F)/\ell\mathrm{Cl}(F) \leq \log_\ell h(F)$. Hence

$$\dim_{\mathbb{F}_\ell} H^1(G,A) \leq \Big(\frac{m}{2} + \#S\Big) + \log_\ell h(F) \leq \frac{3m}{2} + \frac{\log h(F)}{\log \ell}. \tag{37.24}$$

The class number. By Minkowski's theorem every ideal class of F contains an integral ideal of norm at most $\frac{m!}{m^m}(\frac{4}{\pi})^{m/2}|d_F|^{1/2} \leq |d_F|^{1/2}$ [190]. The number of integral ideals of norm k is

at most $d_m(k) = \#\{(k_1,\dots,k_m) \in \mathbb{N}^m : k_1 \cdots k_m = k\}$: both counts are multiplicative in k, and if p has g primes above it with residue degrees $f_1,\dots,f_g$ ($\sum f_i \le m$), the ideals of norm p^c correspond to the tuples $(c_1,\dots,c_g)$ with $\sum f_i c_i = c$, which inject into the m-tuples of non-negative integers with sum c (repeat c_i exactly f_i times and pad with zeros), whose number is $d_m(p^c)$. By induction on m, $\sum_{k\le X} d_m(k) \le X(1+\log X)^{m-1}$ for $X \ge 1$: $\sum_{k\le X} d_m(k) = \sum_{k_1\le X}\sum_{k'\le X/k_1} d_{m-1}(k') \le X(1+\log X)^{m-2}\sum_{k_1\le X} k_1^{-1}$ and $\sum_{k_1\le X} k_1^{-1} \le 1+\log X$. By the conductor–discriminant formula, $|d_F|$ is the product of the conductors of the m characters of $\mathrm{Gal}(F/\mathbb{Q})$, each dividing ℓN [266]; so $|d_F| \le (\ell N)^m$ and

$$\log h(F) \le \frac{m}{2}\log(\ell N) + (m-1)\log\Big(1+\frac{m}{2}\log(\ell N)\Big) \le \frac{5m}{2}\log(\ell N).$$

Conclusion. With (37.24), $\log(\ell N) \ge \log \ell$ and $m \le (\ell-1)n$,

$$\dim_{\mathbb{F}_\ell} H^1(G,A) \le \frac{3m}{2}\frac{\log(\ell N)}{\log \ell} + \frac{5m}{2}\frac{\log(\ell N)}{\log \ell} \le \frac{4\,\ell\, n\log(\ell N)}{\log \ell},$$

and the surjection of the first step gives the claim, finiteness included. □

38 The plus part: the counts and congruence numbers

This section proves the counts for the plus parts: bounds for the normalised lengths $\nu_{\mathcal{O}}(\chi, e)$ of the strict Tate–Shafarevich groups of the characters of a Sha-host, weighted by $\log \ell$, that are polynomial in $N_{\mathfrak{h}}$, t and, outside case (I), ℓ. They are used in Proposition 36.5 and Proposition 36.10. The method is that of Eisenstein congruences. It goes back to Ribet's construction of unramified extensions of $\mathbb{Q}(\mu_p)$ from cusp forms congruent to Eisenstein series [208], and was developed for Selmer groups of characters by Mazur and Wiles [169], by Wiles [271] and, on U(3), by Bellaïche and Chenevier [22]. There the congruences produce Selmer classes; here they bound the strict Tate–Shafarevich group from above.

Part VIII proves the one-character bounds (Proposition 42.2, Proposition 44.3, Proposition 46.2): the length for one character and one prime ℓ is at most twice the sum of the ℓ-adic valuations of the differences between an Eisenstein trace and the Frobenius traces of finitely many admissible automorphic Galois representations. This section makes these bounds effective and uniform in ℓ. At a fixed place w above an auxiliary prime p, the Frobenius traces of all admissible representations lie in one finite set of algebraic numbers, stable under $\mathrm{Aut}(\mathbb{C})$, independent of ℓ, of size polynomial in $N(e+1)$ and of absolute value at most $\sqrt{q_w}+2$ (Proposition 38.9; Proposition 38.29 is its counterpart on GSp_4). On the unitary group the Eisenstein traces $q_w^{-e}z + 1 + q_w^e z^{-1}$, with z a root of unity, are larger in absolute value. So every difference is non-zero, and a product of norms of the differences is one non-zero integer M with $\log|M|$ polynomial in $N(e+1)$, whose ℓ-adic valuation dominates the congruences for every prime $\ell \ne p$ at once (Lemma 38.7, Proposition 38.6). In Count I two auxiliary primes r_1, r_2 are used, so that every ℓ differs from one of them.

Subsection 38.1 proves Count I, the main case, on the unitary group in three variables. Subsection 38.2 proves the counts in the cases (II)–(V): on GSp_4, on the unitary group attached to $\mathbb{Q}(i)$, and, where no congruence is needed, from the bounds of Sections 39 and 40.

38.1 The first count and congruence numbers

This subsection proves Count I, Proposition 38.2: in case (I) the normalised lengths, weighted by $\log \ell$ and summed over the primes ℓ and over the characters of the host, are at most $(N_{\mathfrak{h}}t)^C$.

The congruences are formed on the unitary group in three variables attached to the fixed field $K = \mathbb{Q}(\sqrt{-p_K(\psi)})$: every character in case (I) gives a unitary Eisenstein datum (Lemma 38.3), and Proposition 38.5 bounds the sum over all characters with a given kernel H by $2v_\ell(M)$ for one integer M, by combining the one-character bound Proposition 42.2 with the congruence numbers Proposition 38.6. The second hypothesis of the one-character bound, the vanishing of the Bloch–Kato Selmer groups of the representations attached to newforms of weight $2e+1$ and of their quadratic twist by K, comes from Kato's Euler system [141] (Lemma 38.23).

The rest of the subsection counts. The admissible representations of stable type come from conjugate self-dual cuspidal representations of $\mathrm{GL}_3(\mathbb{A}_K)$ of weight e and level N; their number is polynomial in $N(e+1)$, and $\mathrm{Aut}(\mathbb{C})$ permutes them (Proposition 38.15). Those of endoscopic type come from newforms of weight $2e+1$ whose level divides the square of the level of the representation (Proposition 38.21), and these are counted by Lemma 38.22. Local–global compatibility fixes the weight and the level of the cuspidal representation attached to an admissible one, and the integrality of the Satake parameters gives the integrality of q_w^e times the Frobenius traces that Lemma 38.7 requires.

Definition 38.1. Let $N \geq 1$, let $\mathcal{O}$ be a coefficient ring and let χ be a Dirichlet character modulo N with values in $\mathcal{O}^\times$. For a prime p write $N = p^a N'$ with $p \nmid N'$; the **p-component** of χ is the character χ_p of $(\mathbb{Z}/p^a\mathbb{Z})^\times$ given by $\chi_p(u) = \chi(\tilde{u})$, where $\tilde{u} \equiv u \pmod{p^a}$ and $\tilde{u} \equiv 1 \pmod{N'}$ (so $\chi_p = \mathbf{1}$ if $p \nmid N$, and $\chi = \prod_{p \mid N} \chi_p$).

Let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field with discriminant d_K and class number h_K, let ℓ be a prime and $e \geq 1$ an integer, and let $L_{\ell,\chi} \subset \overline{\mathbb{Q}}$ be the fixed field of the subgroup of $G_{\mathbb{Q}}$ of the elements that act trivially on the ℓ-th roots of unity and lie in the kernel of $\chi^{(\ell)}$. The tuple (K, ℓ, e, N, χ) is a **unitary Eisenstein datum** if

(Ue1) ℓ is odd;

(Ue2) N is even;

(Ue3) ℓ does not divide the order of χ;

(Ue4) $\chi_\ell^2 = \mathbf{1}$;

(Ue5) $\chi(-1) = (-1)^e$;

(Ue6) $\ell \nmid d_K h_K$;

(Ue7) $\gcd(d_K, N) = 1$;

(Ue8) $K \not\subseteq L_{\ell,\chi}$;

(Ue9) $(\ell - 1) \nmid 2e$.

These are the data for which the Eisenstein congruences on the unitary group in three variables attached to $K/\mathbb{Q}$ are used to bound $\mathrm{Sha}^1_{\ell N}(\chi, e)$.

Proposition 38.2. *Let $\psi \in \mathcal{P}$. There are an integer $C \geq 0$ and primes r_1, r_2 such that the following holds for every Sha-host $\mathfrak{h} = (a, q, d)$ of ψ with $r_1 < \min(q, d)$ and $r_2 < \min(q, d)$, every integer $t \geq 1$ at which $\mathfrak{h}$ is big, every integer e with $0 < e < t$, every choice of a coefficient ring $\mathcal{O}_\ell$ for each prime ℓ, and every finite set Λ of odd primes ℓ that are in case (I) for e. For every $\ell \in \Lambda$ and every $\chi \in X_{\mathfrak{h}}(\mathcal{O}_\ell, e)$ the group $\mathrm{Sha}^1_{\ell N_{\mathfrak{h}}}(\chi, e)$ has finite length, and*

$$\sum_{\ell \in \Lambda} \log \ell \sum_{\chi \in X_{\mathfrak{h}}(\mathcal{O}_\ell, e)} \nu_{\mathcal{O}_\ell}(\chi, e) \leq (N_{\mathfrak{h}} t)^C.$$

Proof. Let $\psi \in \mathcal{P}$, write $p_K = p_K(\psi)$ and $K = \mathbb{Q}(\sqrt{-p_K})$. By Lemma 38.3, K is imaginary quadratic with $d_K = -p_K$. Let $c, A \geq 0$ be the integers given by Proposition 38.5 for K. Choose primes $r_1 > 30 N_\psi p_K$ and $r_2 > r_1$, and an integer $k \geq 0$ with $2c(\log r_1 + \log r_2) \leq 16^k$. Put $C = k + 1 + A$. We show that C, r_1, r_2 have the stated property.

Fix $\mathfrak{h} = (a, q, d)$ with $r_1, r_2 < \min(q, d)$, an integer $t \geq 1$ at which $\mathfrak{h}$ is big, $0 < e < t$, coefficient rings $\mathcal{O}_\ell$ and a finite set Λ of odd primes in case (I) for e. Write $N = N_\mathfrak{h}$ and $G = (\mathbb{Z}/N\mathbb{Z})^\times$. Since $a \geq 2$, $16 \mid N$, so $N \geq 16$.

Step 1: the auxiliary primes are admissible. Let $i \in \{1, 2\}$. As $r_i > 30 N_\psi p_K \geq 210$, $r_i \nmid 30$ and $r_i \neq p_K$, so $r_i \nmid d_K = -p_K$. A prime factor of $N = \operatorname{lcm}(2^{a+2}, N_\psi) q d$ is 2, a prime factor of N_ψ (hence at most $N_\psi < r_i$), q or d (both larger than r_i); so $r_i \nmid N$. Thus $r_i \nmid 30 N d_K$.

Step 2: one kernel. Let $\mathcal{T}$ be the finite set of subgroups $\ker\chi$ with $\ell \in \Lambda$ and $\chi \in X_\mathfrak{h}(\mathcal{O}_\ell, e)$, and for $H' \in \mathcal{T}$ let $\Lambda(H')$ be the set of $\ell \in \Lambda$ for which some $\chi \in X_\mathfrak{h}(\mathcal{O}_\ell, e)$ has kernel H'. Fix $H' \in \mathcal{T}$. By Step 1 and $e, N \geq 1$, Proposition 38.5 applies to (e, N, H', r_i) and gives nonzero integers M_1, M_2 (for $p = r_1$, $p = r_2$) with

$$\log|M_i| \leq c\,\varphi([G : H'])\big(N(e+1)\big)^A \log r_i \qquad (i = 1, 2).$$

Let $\ell \in \Lambda(H')$ and choose $\chi_0 \in X_\mathfrak{h}(\mathcal{O}_\ell, e)$ with $\ker\chi_0 = H'$; then $\chi_0 \in \mathcal{X}_{\mathcal{O}_\ell}(N, H')$, and (K, ℓ, e, N, χ_0) is a unitary Eisenstein datum by Lemma 38.3 (ℓ is odd and in case (I) for e, $\mathfrak{h}$ is big at t, $0 < e < t$). Choose i with $\ell \neq r_i$ (possible as $r_1 \neq r_2$). By Proposition 38.5 for $p = r_i$, every $\chi \in \mathcal{X}_{\mathcal{O}_\ell}(N, H')$ has $\mathrm{Sha}^1_{\ell N}(\chi, e)$ of finite length, and

$$\sum_{\chi \in \mathcal{X}_{\mathcal{O}_\ell}(N, H')} \nu_{\mathcal{O}_\ell}(\chi, e) \leq 2 v_\ell(M_i) \leq 2 v_\ell(M_1) + 2 v_\ell(M_2).$$

Multiplying by $\log\ell \geq 0$ and summing over $\ell \in \Lambda(H')$,

$$\begin{aligned} \sum_{\ell \in \Lambda(H')} \log\ell \sum_{\chi \in \mathcal{X}_{\mathcal{O}_\ell}(N, H')} \nu_{\mathcal{O}_\ell}(\chi, e) &\leq 2\big(\log|M_1| + \log|M_2|\big) \\ &\leq 2c\,\varphi([G : H'])\big(N(e+1)\big)^A (\log r_1 + \log r_2), \end{aligned} \tag{38.1}$$

where the integers M_1, M_2 depend on H' but not on ℓ: this is where one integer serves all primes.

Step 3: finiteness. Let $\ell \in \Lambda$ and $\chi \in X_\mathfrak{h}(\mathcal{O}_\ell, e)$. Then $H' = \ker\chi \in \mathcal{T}$, $\ell \in \Lambda(H')$ and $\chi \in \mathcal{X}_{\mathcal{O}_\ell}(N, H')$, so $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length by Step 2.

Step 4: summing over the kernels. By Step 3 every pair (ℓ, χ) with $\ell \in \Lambda$, $\chi \in X_\mathfrak{h}(\mathcal{O}_\ell, e)$ occurs on the left of (38.1) for $H' = \ker\chi$; since all $\nu_{\mathcal{O}_\ell}(\chi, e) \geq 0$,

$$\sum_{\ell \in \Lambda} \log\ell \sum_{\chi \in X_\mathfrak{h}(\mathcal{O}_\ell, e)} \nu_{\mathcal{O}_\ell}(\chi, e) \leq 2c(\log r_1 + \log r_2)\big(N(e+1)\big)^A \sum_{H' \in \mathcal{T}} \varphi([G : H']).$$

Each $H' \in \mathcal{T}$ is the kernel of a homomorphism from G to the unit group of a field, so G/H' is cyclic (a finite subgroup of the multiplicative group of a field is cyclic), and a cyclic group of order m has exactly $\varphi(m)$ injective homomorphisms to $\mathbb{C}^\times$. Hence exactly $\varphi([G : H'])$ homomorphisms $G \to \mathbb{C}^\times$ have kernel H'; these sets are disjoint for distinct H', and G has $|G| = \varphi(N) \leq N$ homomorphisms to $\mathbb{C}^\times$. So $\sum_{H' \in \mathcal{T}} \varphi([G : H']) \leq N$, and with $2c(\log r_1 + \log r_2) \leq 16^k$,

$$\sum_{\ell \in \Lambda} \log\ell \sum_{\chi \in X_\mathfrak{h}(\mathcal{O}_\ell, e)} \nu_{\mathcal{O}_\ell}(\chi, e) \leq 16^k\, N\,\big(N(e+1)\big)^A \leq (Nt)^{k+1+A} = (N_\mathfrak{h} t)^C.$$

□

Lemma 38.3. *Let $\psi \in \mathcal{P}$ and let $K = \mathbb{Q}(\sqrt{-p_K(\psi)}) \subset \overline{\mathbb{Q}}$. Then K is an imaginary quadratic field with $d_K = -p_K(\psi)$. Let $\mathfrak{h} = (a, q, d)$ be a Sha-host of ψ that is big at an integer $t \geq 1$, let $0 < e < t$, let ℓ be an odd prime in case (I) for e, let $\mathcal{O}$ be a coefficient ring and $\chi \in X_{\mathfrak{h}}(\mathcal{O}, e)$. Then $(K, \ell, e, N_{\mathfrak{h}}, \chi)$ is a unitary Eisenstein datum.*

Proof. Write $p_K = p_K(\psi)$, $N = N_{\mathfrak{h}}$, $G = (\mathbb{Z}/N\mathbb{Z})^\times$, and recall that ℓ in case (I) means $\ell \notin B_\psi$ and $(\ell-1) \nmid 2e$. By definition p_K is a prime with $p_K \geq 7$, $p_K \equiv 3 \pmod 4$ and $\gcd(p_K, N_\psi) = 1$.

The field K. Since $p_K \equiv 3 \pmod 4$, the integer $-p_K$ is squarefree and $\equiv 1 \pmod 4$, so $K = \mathbb{Q}(\sqrt{-p_K})$ is an imaginary quadratic field of discriminant $d_K = -p_K$; in particular p_K ramifies in K. Its class number is $h_K = h^*(\psi)$ by definition.

Facts on ℓ and N. The prime p_K is odd and is a prime factor of p_K, so $p_K \in B_\psi$; every odd prime factor of o_ψ, of N_ψ and of $h_K = h^*(\psi)$ lies in B_ψ. As ℓ is odd and $\ell \notin B_\psi$:

$$\ell \neq p_K, \qquad \ell \nmid h_K, \qquad \ell \nmid o_\psi, \qquad \ell \nmid N_\psi. \tag{38.2}$$

Since $\mathfrak{h}$ is big at t, $p_K < \min(q, d)$, so $p_K \notin \{q, d\}$; with $\gcd(p_K, N_\psi) = 1$ and p_K odd this gives $p_K \nmid N = \operatorname{lcm}(2^{a+2}, N_\psi)qd$.

(Ue1), (Ue2), (Ue5), (Ue9). The prime ℓ is odd by hypothesis; $2 \mid 2^{a+2} \mid N$; $\chi(-1) = (-1)^e$ is part of the definition of $X_{\mathfrak{h}}(\mathcal{O}, e)$; $(\ell - 1) \nmid 2e$ is part of case (I).

(Ue3). The map $\Phi\colon G \to \big((\mathbb{Z}/2^{a+2}\mathbb{Z})^\times/\{\pm 1\}\big) \times \psi\big((\mathbb{Z}/N_\psi\mathbb{Z})^\times\big) \times \{\pm 1\} \times \{\pm 1\}$, $u \mapsto \big(\pm u \bmod 2^{a+2},\ \psi(u),\ (\frac{u}{q}),\ (\frac{u}{d})\big)$, is a homomorphism (note $2^{a+2} \mid N$, $N_\psi \mid N$, $q, d \mid N$) with kernel $H_{\mathfrak{h}}$. The first factor has order 2^a and the second is cyclic of order o_ψ, so $[G : H_{\mathfrak{h}}]$ divides $2^{a+2}o_\psi$. Since $H_{\mathfrak{h}} \subseteq \ker \chi$, the order of χ divides $[G : H_{\mathfrak{h}}]$, hence $2^{a+2}o_\psi$, and $\ell \nmid \operatorname{ord} \chi$ by (38.2) and ℓ odd.

(Ue4). If $\ell \nmid N$, then $\chi_\ell = \mathbf{1}$. Let $\ell \mid N$. As ℓ is odd and $\ell \nmid N_\psi$, $\ell \in \{q, d\}$; say $\ell = r$ with $\{r, r'\} = \{q, d\}$. The primes q, d are distinct ($q \equiv 3$, $d \equiv 7 \pmod 8$), odd and prime to N_ψ, so $N = rN'$ with $N' = \operatorname{lcm}(2^{a+2}, N_\psi)r'$ prime to r. Let $u \in (\mathbb{Z}/r\mathbb{Z})^\times$ be a square and $\tilde{u} \in G$ with $\tilde{u} \equiv u \pmod r$, $\tilde{u} \equiv 1 \pmod{N'}$. Then $\tilde{u} \equiv 1 \pmod{2^{a+2}}$, $\psi(\tilde{u}) = \psi(1) = 1$ (as $N_\psi \mid N'$), $(\frac{\tilde{u}}{r'}) = 1$ (as $r' \mid N'$) and $(\frac{\tilde{u}}{r}) = (\frac{u}{r}) = 1$; so $\tilde{u} \in H_{\mathfrak{h}} \subseteq \ker \chi$ and $\chi_r(u) = \chi(\tilde{u}) = 1$. Thus χ_ℓ is trivial on the squares, and $\chi_\ell(u)^2 = \chi_\ell(u^2) = 1$ for every u.

(Ue6). By (38.2), $\ell \nmid p_K h_K = |d_K| h_K$.

(Ue7). As shown above, $\gcd(d_K, N) = \gcd(p_K, N) = 1$.

(Ue8). The character $\chi^{(\ell)}$ is a Dirichlet character modulo N (a product of a power of χ and of the inverse of a component of that power), so it is trivial on $G_{\mathbb{Q}(\zeta_N)}$, and $G_{\mathbb{Q}(\zeta_{\ell N})}$ acts trivially on the ℓ-th roots of unity. Hence $G_{\mathbb{Q}(\zeta_{\ell N})}$ is contained in the subgroup defining $L_{\ell,\chi}$, and $L_{\ell,\chi} \subseteq \mathbb{Q}(\zeta_{\ell N})$. Since $p_K \neq \ell$ and $p_K \nmid N$, the prime p_K is unramified in $\mathbb{Q}(\zeta_{\ell N})$, hence in every subfield. As p_K ramifies in K, $K \not\subseteq L_{\ell,\chi}$. □

Definition 38.4. Number fields are subfields of a fixed algebraic closure $\overline{\mathbb{Q}}$; ℓ is a prime and $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism. For a number field F and a place $\bar{v}$ of $\overline{\mathbb{Q}}$ above a finite place v of F, $D_{\bar{v}} \supseteq I_{\bar{v}}$ are the decomposition and inertia subgroups of G_F at $\bar{v}$, and an **arithmetic Frobenius at** $\bar{v}$ is an element of $D_{\bar{v}}$ acting on the residue field of $\bar{v}$ as $x \mapsto x^{q_v}$. We write χ_{cyc} also for its restriction to G_F, with values in $\mathbb{Z}_\ell^\times \subset \overline{\mathbb{Q}}_\ell^\times$. For $M \geq 1$, a character of G_F is **of level** M if it is trivial on $G_{F(\mu_M)}$. For a subgroup $D \subseteq G_F$ and characters $\kappa_1, \ldots, \kappa_n\colon D \to \overline{\mathbb{Q}}_\ell^\times$, a representation $r\colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is **triangular on** D **with diagonal** $(\kappa_1, \ldots, \kappa_n)$ if there is $g \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that, for every $s \in D$, $g\, r(s)\, g^{-1}$ is upper triangular with diagonal entries $\kappa_1(s), \ldots, \kappa_n(s)$.

Level and weights. Let $n, M \geq 1$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. A continuous $r\colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ has **level** M **and weights** m if

(Lw1) for every finite place $v \nmid \ell$ of F there are a place $\bar{v} \mid v$ and characters $\chi_1, \dots, \chi_n$ of G_F of level M such that $r|_{I_{\bar{v}}}$ is conjugate to $s \mapsto \mathrm{diag}(\chi_1(s), \dots, \chi_n(s))$;

(Lw2) for every place $v \mid \ell$ of F there are a place $\bar{v} \mid v$, characters $\chi_1, \dots, \chi_n$ of G_F of level M and characters $\kappa_1, \dots, \kappa_n$ of $D_{\bar{v}}$ such that r is triangular on $D_{\bar{v}}$ with diagonal $(\kappa_1, \dots, \kappa_n)$, $\kappa_i(s) = \chi_{\mathrm{cyc}}(s)^{m_i}\chi_i(s)$ for all i and $s \in I_{\bar{v}}$, and either $\kappa_i \neq \kappa_j\chi_{\mathrm{cyc}}|_{D_{\bar{v}}}$ for all i, j, or $r|_{D_{\bar{v}}}$ is potentially crystalline.

It is **generic ordinary of level** M **and weights** m if for every place $v \mid \ell$ of F there are $\bar{v}$, χ_i and κ_i as in (Lw2) with $\kappa_i \neq \kappa_j\chi_{\mathrm{cyc}}|_{D_{\bar{v}}}$ for all i, j.

Witness conditions. Let F be a CM field and $r\colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ a representation. For a place $v \mid \ell$ of F, r is **generic at** v if for every place $\bar{v} \mid v$ it is triangular on $D_{\bar{v}}$ with some diagonal (κ_i) such that $\kappa_i \neq \kappa_j\chi_{\mathrm{cyc}}|_{D_{\bar{v}}}$ for all i, j; it is **potentially generic at** v if for every $\bar{v} \mid v$ it is triangular on $D_{\bar{v}}$ with some diagonal (κ_i) such that $\kappa_i^a \neq (\kappa_j\chi_{\mathrm{cyc}}|_{D_{\bar{v}}})^a$ for all i, j and all integers $a \geq 1$. The representation r satisfies the **witness conditions** if

(Wc1) r is potentially generic at every place $v \mid \ell$ at which it is generic;

(Wc2) for every regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that r is semisimple and $r \leftrightarrow_\iota \pi$, there is a real number τ_π such that at every finite place v of F with $v \nmid \ell$, or with $v \mid \ell$ and r potentially generic at v, the local component π_v has a cuspidal support consisting of characters of $F_v^\times$, and every character ξ occurring in a cuspidal support of π_v satisfies $q_v^{\tau_\pi - 1/2} < |\xi(\varpi_v)| < q_v^{\tau_\pi + 1/2}$.

(The number $|\xi(\varpi_v)|$ does not depend on the uniformiser, since ξ has finite image on $\mathcal{O}_{F_v}^\times$.) A representation W of $G_\mathbb{Q}$ satisfies the **witness conditions over imaginary quadratic fields** if $W|_{G_{K'}}$ satisfies the witness conditions for every imaginary quadratic field K' such that $W|_{G_{K'}}$ is irreducible.

Admissible representations. Let K be an imaginary quadratic field with discriminant d_K and let $e, N \geq 1$ be integers. A representation $\rho\colon G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ is **admissible of stable type** for (ι, e, N) if

(Sb1) ρ is irreducible;

(Sb2) $\rho \otimes \chi_{\mathrm{cyc}}^{-1}$ is automorphic of conjugate self-dual type and satisfies the witness conditions;

(Sb3) ρ has level N and weights $(e, 0, -e)$;

(Sb4) ρ is generic ordinary of level N and weights $(e, 0, -e)$.

For $M \geq 1$, ρ is **admissible of endoscopic type** for (ι, e, M) if there is a representation $W\colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that

(Eo1) W is irreducible and satisfies the witness conditions over imaginary quadratic fields;

(Eo2) $W \otimes \chi_{\mathrm{cyc}}^{-e}$ is semisimple, and there is a normalised newform $f = \sum_{n \geq 1} a_n q^n$ of weight $2e + 1$, some level and Nebentypus ε_f such that for all but finitely many primes p and every place $\bar{p}$ of $\overline{\mathbb{Q}}$ above p, $W \otimes \chi_{\mathrm{cyc}}^{-e}$ is trivial on $I_{\bar{p}}$ and $\iota(\det(X - (W \otimes \chi_{\mathrm{cyc}}^{-e})(\sigma^{-1}))) = X^2 - a_p X + \varepsilon_f(p)p^{2e}$ for every arithmetic Frobenius σ at $\bar{p}$;

(Eo3) W has level M and weights $(e, -e)$, and $\det W$ is a character of level M' for some $M' \geq 1$;

(Eo4) ρ is conjugate to $s \mapsto \begin{pmatrix} a(s) & 0 & b(s) \\ 0 & 1 & 0 \\ c(s) & 0 & d(s) \end{pmatrix}$, where $W(s) = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix}$ for $s \in G_K \subseteq G_{\mathbb{Q}}$.

A representation $\rho\colon G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ is **admissible of weight** e **and level** N (for K and ι) if it is admissible of stable type for (ι, e, N) or admissible of endoscopic type for $(\iota, e, N|d_K|)$.

Proposition 38.5. *Let K be an imaginary quadratic field with discriminant d_K. There are integers $c, A \geq 0$ with the following property. For all integers $e \geq 1$ and $N \geq 1$, every subgroup $H \leq (\mathbb{Z}/N\mathbb{Z})^\times$ and every prime p with $p \nmid 30Nd_K$ there is a nonzero integer M with*

$$\log|M| \leq c\,\varphi\big([(\mathbb{Z}/N\mathbb{Z})^\times : H]\big)\,\big(N(e+1)\big)^A \log p$$

such that for every prime $\ell \neq p$ and every coefficient ring $\mathcal{O}$ (at ℓ) the following holds: if (K, ℓ, e, N, χ_0) is a unitary Eisenstein datum for some $\chi_0 \in \mathcal{X}_{\mathcal{O}}(N, H)$, then $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length for every $\chi \in \mathcal{X}_{\mathcal{O}}(N, H)$, and

$$\sum_{\chi \in \mathcal{X}_{\mathcal{O}}(N,H)} \nu_{\mathcal{O}}(\chi, e) \leq 2\, v_\ell(M).$$

Proof. Let $c, A \geq 0$ be the integers given by Proposition 38.6 for K. Fix $e, N \geq 1$, a subgroup $H \leq G = (\mathbb{Z}/N\mathbb{Z})^\times$ and a prime $p \nmid 30Nd_K$; put $m = [G : H]$. Since $p \nmid 30$, $p \geq 7$. Choose a prime w of K above p and write $q = q_w$, a power of p; as $p \nmid N$, q is prime to N. Put $\mathrm{Eis}(z) = q^{-e}z + 1 + q^e z^{-1}$.

Step 1: the integer M. Let Z be the set of homomorphisms $\xi\colon G \to \mathbb{C}^\times$ with kernel exactly H, and $z(\xi) = \xi(q)^{-1}$, a root of unity. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, $\sigma \circ \xi$ has kernel H (σ is injective) and $z(\sigma \circ \xi) = \sigma(z(\xi))$; the map $\xi \mapsto \sigma \circ \xi$ is a bijection of Z with inverse $\xi \mapsto \sigma^{-1} \circ \xi$. If $Z \neq \emptyset$, the image of any $\xi_0 \in Z$ is a finite subgroup of $\mathbb{C}^\times$, so G/H is cyclic of order m, and the elements of Z are the ξ_0^j with $j \in (\mathbb{Z}/m\mathbb{Z})^\times$; hence $|Z| \leq \varphi(m)$. By Proposition 38.6 applied to (e, N, p, w, Z, z) (note $p \geq 7$, $p \nmid Nd_K$) there is a nonzero integer M with

$$\log|M| \leq c\,|Z|\,\big(N(e+1)\big)^A \log p \leq c\,\varphi(m)\big(N(e+1)\big)^A \log p,$$

and M depends only on (K, e, N, H, p) and the choice of w.

Step 2: setting up at a prime ℓ. Let $\ell \neq p$ be a prime, $\mathcal{O}$ a coefficient ring and $\chi_0 \in \mathcal{X}_{\mathcal{O}}(N, H)$ such that (K, ℓ, e, N, χ_0) is a unitary Eisenstein datum. Then ℓ is odd by (Ue1). Choose an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ (both fields are algebraically closed of characteristic 0 and of the cardinality of the continuum), a place $\bar{w}$ of $\overline{\mathbb{Q}}$ above w and an arithmetic Frobenius $\sigma \in G_K$ at $\bar{w}$ (the decomposition group maps onto the Galois group of the residue field extension). Since $p \nmid \ell N$, reduction modulo $\bar{w}$ is injective on the roots of unity of order dividing $\ell^k N$ for every k, and σ acts on the residue field as $x \mapsto x^q$; hence $\sigma(\zeta) = \zeta^q$ for these roots of unity, so that

$$\chi(\sigma) = \chi(q) \quad \text{for every Dirichlet character } \chi \text{ modulo } N, \qquad \chi_{\mathrm{cyc}}(\sigma) = q. \tag{38.3}$$

Step 3: the characters with kernel H. Every $\chi \in \mathcal{X}_{\mathcal{O}}(N, H)$ has kernel H, so $\chi = \chi_0^j$ for some j prime to m (Step 1 applied to the cyclic group G/H and the injective characters it induces), and conversely every such χ_0^j lies in $\mathcal{X}_{\mathcal{O}}(N, H)$. So $\chi \mapsto \iota \circ \chi$ is a bijection $\mathcal{X}_{\mathcal{O}}(N, H) \to Z$ (injective as ι is). Moreover, for $\chi = \chi_0^j$ with $\gcd(j, m) = 1$, (K, ℓ, e, N, χ) is a unitary Eisenstein datum:

- (Ue1), (Ue2), (Ue6), (Ue7), (Ue9) do not involve the character;

- (Ue3): χ and χ_0 both have order m;
- (Ue4): p-components are multiplicative in the character $((\chi\chi')_r(u) = \chi(\tilde{u})\chi'(\tilde{u}))$, so $\chi_\ell = (\chi_0)_\ell^j$ and $\chi_\ell^2 = ((\chi_0)_\ell^2)^j = \mathbf{1}$;
- (Ue5): if e is odd, $\chi_0(-1) = -1$, so m is even and j odd, and $\chi(-1) = (-1)^j = -1$; if e is even, $\chi(-1) = \chi_0(-1)^j = 1$;
- (Ue8): the part of order prime to ℓ and the ℓ-component of χ are the j-th powers of those of χ_0, so $\chi^{(\ell)} = (\chi_0^{(\ell)})^j$; both factors of $\chi_0^{(\ell)}$ have order dividing m and commute, so $\chi_0^{(\ell)}$ has order dividing m, and as $\gcd(j, m) = 1$, $\ker \chi^{(\ell)} = \ker \chi_0^{(\ell)}$. Hence $L_{\ell,\chi} = L_{\ell,\chi_0}$, which does not contain K.

Step 4: the Eisenstein traces. For $\chi \in \mathcal{X}_{\mathcal{O}}(N, H)$ put $t_\chi = \Psi_{\chi,e}(\sigma^{-1}) + 1 + \Psi_{\chi,e}(\sigma^{-1})^{-1}$. By (38.3), $\Psi_{\chi,e}(\sigma) = \chi(q)q^e$, so

$$t_\chi = q^{-e}\chi(q)^{-1} + 1 + q^e\chi(q) = \iota^{-1}\big(\mathrm{Eis}(z(\iota \circ \chi))\big),$$

since $z(\iota \circ \chi) = \iota(\chi(q))^{-1}$ and ι^{-1} is a field isomorphism fixing q.

Step 5: the one-character bound for each χ. Let $\chi \in \mathcal{X}_{\mathcal{O}}(N, H)$. We apply Proposition 42.2 to the unitary Eisenstein datum (K, ℓ, e, N, χ) (Step 3), to ι and to the element $\sigma^{-1} \in G_K$, whose Eisenstein trace is t_χ. Its hypothesis 1: for an admissible ρ of weight e and level N, Proposition 38.6, item 1, for the family with $X_{\iota\circ\chi} = \{\rho\}$ and all other members empty gives $\operatorname{tr}\rho(\sigma^{-1}) \neq \iota^{-1}(\mathrm{Eis}(z(\iota \circ \chi))) = t_\chi$ (here ℓ is odd, $\ell \neq p$). Its hypothesis 2: for W as there, Lemma 38.23 with $\theta = \mathbf{1}$ gives $H^1_f(\mathbb{Q}, W) = 0$, and with $\theta = \eta_K$ (continuous, of order 2, factoring through $\mathrm{Gal}(K/\mathbb{Q})$) gives $H^1_f(\mathbb{Q}, W \otimes \eta_K) = 0$; the hypotheses of Lemma 38.23 are exactly those imposed on W in hypothesis 2. Hence $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length, which is the first assertion, and there is a finite set X_χ of pairwise non-isomorphic admissible representations of weight e and level N with

$$\nu_{\mathcal{O}}(\chi, e) \leq 2 \sum_{\rho \in X_\chi} v_\ell(\operatorname{tr}\rho(\sigma^{-1}) - t_\chi). \tag{38.4}$$

Step 6: one integer for the whole kernel. Apply Proposition 38.6, item 2, to $\ell, \iota, \bar{\imath}, \sigma$ and the family $(X_\xi)_{\xi \in Z}$ with $X_{\iota\circ\chi} = X_\chi$ (well defined by the bijection of Step 3). With Step 4,

$$v_\ell(M) \geq \sum_{\chi \in \mathcal{X}_{\mathcal{O}}(N,H)} \sum_{\rho \in X_\chi} v_\ell(\operatorname{tr}\rho(\sigma^{-1}) - t_\chi),$$

and summing (38.4) over χ,

$$\sum_{\chi \in \mathcal{X}_{\mathcal{O}}(N,H)} \nu_{\mathcal{O}}(\chi, e) \leq 2\, v_\ell(M).$$

Here M is the integer of Step 1, which does not depend on ℓ, $\mathcal{O}$ or χ_0. □

Proposition 38.6. *Let K be an imaginary quadratic field with discriminant d_K. There are integers $c, A \geq 0$ with the following property. Let $e, N \geq 1$ be integers, $p \geq 7$ a prime with $p \nmid N d_K$, w a prime of K above p, and put $\mathrm{Eis}(z) = q_w^{-e}z + 1 + q_w^e z^{-1}$ for $z \in \mathbb{C}^\times$. Let Z be a finite set and $z\colon Z \to \mathbb{C}$ a map such that every $z(o)$ is a root of unity and for every $\sigma \in \mathrm{Aut}(\mathbb{C})$ there is a bijection u_σ of Z with $z(u_\sigma(o)) = \sigma(z(o))$ for all $o \in Z$. Then there is a nonzero integer M with*

$$\log|M| \leq c\,|Z|\,\big(N(e+1)\big)^A \log p$$

such that for every odd prime $\ell \neq p$, every isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above w, every arithmetic Frobenius $\sigma \in G_K$ at $\bar{w}$ and every family $(X_o)_{o\in Z}$ of finite sets of pairwise non-isomorphic representations $G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ admissible of weight e and level N (for K and ι):

1. $\operatorname{tr}\rho(\sigma^{-1}) \neq \iota^{-1}\big(\mathrm{Eis}(z(o))\big)$ *for every $o \in Z$ and $\rho \in X_o$;*
2. $v_\ell(M) \geq \displaystyle\sum_{o\in Z}\sum_{\rho\in X_o} v_\ell\Big(\operatorname{tr}\rho(\sigma^{-1}) - \iota^{-1}\big(\mathrm{Eis}(z(o))\big)\Big)$.

Proof. Let c_0, A_0 be the integers given by Proposition 38.9 for K, and put $c = 5c_0$, $A = A_0 + 1$. Fix e, N, p, w, Z, z as in the statement and write $q = q_w$, so that $\mathrm{Eis} = \mathrm{Eis}_{q,e}$. Since K is quadratic, $q = p^f$ with $f \in \{1, 2\}$; in particular $q \geq p \geq 7$ and $\log q \leq 2\log p$.

The integer M. As $p \nmid N d_K$, Proposition 38.9 gives a finite set S and $t\colon S \to \mathbb{C}$ with (Ft1)–(Ft4) for (e, N, p, w). By (Ft1) and (Ft2), Lemma 38.7 applies to (q, e, S, t, Z, z) and gives a nonzero integer M with $\log|M| \leq |Z|\,|S|\,(2e\log q + 1)$. Since $\log p \geq \log 7 > 1$,

$$2e\log q + 1 \leq 4e\log p + \log p \leq 5(e+1)\log p \leq 5N(e+1)\log p,$$

and with (Ft3),

$$\log|M| \leq |Z| \cdot c_0\big(N(e+1)\big)^{A_0} \cdot 5N(e+1)\log p \leq c\,|Z|\,\big(N(e+1)\big)^{A}\log p.$$

The two properties. Let $\ell \neq p$ be an odd prime, ι, $\bar{w}$ and σ as in the statement, and $(X_o)_{o\in Z}$ a family of finite sets of pairwise non-isomorphic admissible representations of weight e and level N. Let λ be the map given by (Ft4) for $(\ell, \iota, \bar{w}, \sigma)$; then $\operatorname{tr}\rho(\sigma^{-1}) = \iota^{-1}(t(\lambda(\rho)))$ for each admissible ρ. By Lemma 38.7, $t(s) \neq \mathrm{Eis}(z(o))$ for all s, o; applying the injective map ι^{-1} gives item 1. For each o, the restriction of λ to X_o is injective: if $\lambda(\rho) = \lambda(\rho')$ with $\rho, \rho' \in X_o$, then $\rho \cong \rho'$ by (Ft4), hence $\rho = \rho'$ since the members of X_o are pairwise non-isomorphic. Put $Y_o = \lambda(X_o) \subseteq S$. As q is a power of p and $\ell \neq p$, $\ell \nmid q$, and Lemma 38.7 gives

$$v_\ell(M) \geq \sum_{o\in Z}\sum_{s\in Y_o} v_\ell(\iota^{-1}(t(s)) - \iota^{-1}(\mathrm{Eis}(z(o)))) = \sum_{o\in Z}\sum_{\rho\in X_o} v_\ell(\operatorname{tr}\rho(\sigma^{-1}) - \iota^{-1}(\mathrm{Eis}(z(o)))),$$

the equality by reindexing along the bijection $\lambda\colon X_o \to Y_o$. This is item 2. □

Lemma 38.7. *Let $q \geq 7$ and $e \geq 1$ be integers, and for $z \in \mathbb{C}^\times$ put $\mathrm{Eis}_{q,e}(z) = q^{-e}z + 1 + q^e z^{-1}$. Let S be a finite set and $t\colon S \to \mathbb{C}$ a map such that*

- *for every $\sigma \in \mathrm{Aut}(\mathbb{C})$ there is a bijection u_σ of S with $t(u_\sigma(s)) = \sigma(t(s))$ for all $s \in S$;*
- *for every $s \in S$, $|t(s)| \leq \sqrt{q} + 2$ and $q^e t(s)$ is an algebraic integer.*

Let Z be a finite set and $z\colon Z \to \mathbb{C}$ a map such that every $z(o)$ is a root of unity and for every $\sigma \in \mathrm{Aut}(\mathbb{C})$ there is a bijection u'_σ of Z with $z(u'_\sigma(o)) = \sigma(z(o))$ for all $o \in Z$. Then $t(s) \neq \mathrm{Eis}_{q,e}(z(o))$ for all $s \in S$, $o \in Z$, and there is a nonzero integer M with

$$\log|M| \leq |Z|\,|S|\,(2e\log q + 1)$$

such that for every prime $\ell \nmid q$, every isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and every family $(Y_o)_{o\in Z}$ of subsets of S,

$$v_\ell(M) \geq \sum_{o\in Z}\sum_{s\in Y_o} v_\ell\Big(\iota^{-1}(t(s)) - \iota^{-1}\big(\mathrm{Eis}_{q,e}(z(o))\big)\Big).$$

Proof. For $o \in Z$ and $s \in S$ put

$$F(o,s) = q^e\big(t(s) - \mathrm{Eis}_{q,e}(z(o))\big) = q^e t(s) - z(o) - q^e - q^{2e} z(o)^{-1}.$$

Step 1: $t(s) \neq \mathrm{Eis}_{q,e}(z(o))$. As $z(o)$ is a root of unity, $|z(o)| = 1$, so $|\mathrm{Eis}_{q,e}(z(o))| \geq q^e - 1 - q^{-e} \geq q - \frac{8}{7}$, using $q^e \geq q$ and $q^{-e} \leq \frac{1}{7}$. On the other hand $|t(s)| \leq \sqrt{q} + 2$, and $q - \sqrt{q} \geq 7 - \sqrt{7} > 4 > 2 + \frac{8}{7}$ since $x \mapsto x - \sqrt{x}$ is increasing for $x \geq \frac{1}{4}$. Hence $|t(s)| < |\mathrm{Eis}_{q,e}(z(o))|$, and in particular $F(o,s) \neq 0$.

Step 2: integrality and size of $F(o,s)$. By hypothesis, $q^e t(s)$ is an algebraic integer, and $z(o)$, $z(o)^{-1}$ are roots of unity; so $F(o,s)$ is a sum of algebraic integers and is an algebraic integer. Moreover

$$|F(o,s)| \leq q^e\big(\sqrt{q} + 2 + q^{-e} + 1 + q^e\big) = q^{2e}\Big(1 + \frac{\sqrt{q}+3}{q^e} + \frac{1}{q^{2e}}\Big) < 2q^{2e},$$

so $\log|F(o,s)| < \log 2 + 2e\log q \leq 2e\log q + 1$.

Step 3: the product is a rational integer. Let $P = \prod_{(o,s)\in Z\times S} F(o,s)$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, σ fixes q, so $\sigma(\mathrm{Eis}_{q,e}(w)) = \mathrm{Eis}_{q,e}(\sigma(w))$ for $w \in \mathbb{C}^\times$, and therefore $\sigma(F(o,s)) = F(u'_\sigma(o), u_\sigma(s))$. Since $(o,s) \mapsto (u'_\sigma(o), u_\sigma(s))$ is a bijection of $Z \times S$, $\sigma(P) = P$. Every element of $\mathbb{C}$ outside $\mathbb{Q}$ is moved by some automorphism of $\mathbb{C}$, so $P \in \mathbb{Q}$; as P is an algebraic integer (Step 2), $P \in \mathbb{Z}$, and $P \neq 0$ by Step 1. Put $M = P$. By Step 2,

$$\log|M| = \sum_{(o,s)\in Z\times S} \log|F(o,s)| \leq |Z|\,|S|\,(2e\log q + 1)$$

(if $Z \times S$ is empty, $M = 1$ and both sides are 0).

Step 4: the ℓ-adic inequality. Let $\ell \nmid q$ be a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and $(Y_o)_{o\in Z}$ subsets of S. The ring isomorphism ι^{-1} fixes $\mathbb{Z}$, so $\iota^{-1}(M) = M$ and

$$v_\ell(M) = \sum_{(o,s)\in Z\times S} v_\ell\big(\iota^{-1}(F(o,s))\big).$$

Each $y = \iota^{-1}(F(o,s))$ is a root of a monic polynomial $X^n + a_{n-1}X^{n-1} + \cdots + a_0$ with $a_i \in \mathbb{Z}$; if $v_\ell(y) < 0$, then $v_\ell(y^n) < v_\ell(a_i y^i)$ for all $i < n$, which contradicts $y^n = -\sum_{i<n} a_i y^i$. So every term is ≥ 0, and dropping the terms with $s \notin Y_o$ gives

$$v_\ell(M) \geq \sum_{o\in Z}\sum_{s\in Y_o} v_\ell\big(\iota^{-1}(F(o,s))\big) = \sum_{o\in Z}\sum_{s\in Y_o}\Big(e\,v_\ell(q) + v_\ell\big(\iota^{-1}(t(s)) - \iota^{-1}(\mathrm{Eis}_{q,e}(z(o)))\big)\Big),$$

and $v_\ell(q) = 0$ because $\ell \nmid q$. Each difference on the right is nonzero by Step 1, so every term is finite. □

Definition 38.8. Let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, let $\widehat{\mathcal{O}}_K = \prod_{v\nmid\infty} \mathcal{O}_{K_v} \subset \mathbb{A}_{K,f}$, and for an integer $N \geq 1$ let

$$\mathrm{K}(N) = \ker\Big(\mathrm{GL}_3(\widehat{\mathcal{O}}_K) \to \mathrm{GL}_3\big(\widehat{\mathcal{O}}_K/N\widehat{\mathcal{O}}_K\big)\Big) = \prod_{v\nmid\infty} \mathrm{K}(N)_v$$

be the principal congruence subgroup of level N; for a finite place $v \nmid N$ of K, $\mathrm{K}(N)_v = \mathrm{GL}_3(\mathcal{O}_{K_v})$. For a cuspidal automorphic representation $\pi = \pi_\infty \otimes \pi_f$ of $\mathrm{GL}_3(\mathbb{A}_K)$ and an embedding $\tau\colon K \to \mathbb{C}$, the **infinitesimal character of π at τ** is the multiset of three complex numbers attached to τ by the infinitesimal character of π_∞, a representation of $\mathrm{GL}_3(K \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{GL}_3(\mathbb{C})$,

in the normalisation of [68] (in which π is regular algebraic exactly when, for every τ, these are three distinct integers).

For integers $e, N \geq 1$, $\mathrm{Cusp}_K(e,N)$ is the set of isomorphism classes of cuspidal automorphic representations π of $\mathrm{GL}_3(\mathbb{A}_K)$ such that

(Cs1) for both embeddings $\tau\colon K \to \mathbb{C}$, the infinitesimal character of π at τ is $\{e, 0, -e\}$;

(Cs2) π is conjugate self-dual: $\pi^\vee \cong \pi^c$;

(Cs3) π_f has a nonzero vector fixed by $\mathrm{K}(N)$.

Its members are the **conjugate self-dual cuspidal representations of** GL_3 **over** K **of weight** e **and level** N. By (Cs1) they are regular algebraic. For a finite place $v \nmid N$ of K, (Cs3) and the factorisation $\pi_f \cong \bigotimes'_v \pi_v$ give $\pi_v^{\mathrm{GL}_3(\mathcal{O}_{K_v})} \neq 0$, so π_v is unramified; its multiset of Satake parameters is written $\mathrm{Sat}_v(\pi) = \{\alpha_1, \alpha_2, \alpha_3\} \subset \mathbb{C}^\times$ and depends only on the isomorphism class of π.

Proposition 38.9. *Let K be an imaginary quadratic field with discriminant d_K. There are integers $c, A \geq 0$ with the following property. For all integers $e, N \geq 1$, every prime p with $p \nmid N d_K$ and every prime w of K above p there are a finite set S and a map $t\colon S \to \mathbb{C}$ such that*

(Ft1) *for every $\sigma \in \mathrm{Aut}(\mathbb{C})$ there is a bijection u_σ of S with $t(u_\sigma(s)) = \sigma(t(s))$ for all $s \in S$;*

(Ft2) *for every $s \in S$, $|t(s)| \leq \sqrt{q_w} + 2$ and $q_w^e\, t(s)$ is an algebraic integer;*

(Ft3) $|S| \leq c\,(N(e+1))^A$;

(Ft4) *for every odd prime $\ell \neq p$, every isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above w and every arithmetic Frobenius $\sigma \in G_K$ at $\bar{w}$, there is a map λ from the set of representations $G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ that are admissible of weight e and level N (for K and ι) to S such that $\lambda(\rho) = \lambda(\rho')$ implies $\rho \cong \rho'$, and $\iota(\mathrm{tr}\,\rho(\sigma^{-1})) = t(\lambda(\rho))$ for every such ρ.*

Proof. Let c_1, A_1 be the integers given by Proposition 38.15 for K, and put $c = c_1 + d_K^4$, $A = \max(A_1, 4)$. Fix $e, N \geq 1$, a prime $p \nmid N d_K$ and a prime w of K above p. Write $q = q_w = p^\delta$ with $\delta \in \{1, 2\}$ the residue degree, and put $L = (N|d_K|)^2$; then $p \nmid L$.

Step 1: the set S and the map t. Let S be the disjoint union of $\mathrm{Cusp}_K(e,N)$ and $\mathrm{New}_{2e+1}(L)$; both are finite, by (Cc1) and by Lemma 38.22. For $\pi \in \mathrm{Cusp}_K(e,N)$, $w \nmid N$, so π_w is unramified with $\mathrm{Sat}_w(\pi) = \{\alpha_1, \alpha_2, \alpha_3\}$, and we put $t(\pi) = \alpha_1 + \alpha_2 + \alpha_3$. For $g \in \mathrm{New}_{2e+1}(L)$ with coefficients $a_n(g)$, level $L_g \mid L$ and character ε_g, let $\beta_1(g), \beta_2(g)$ be the roots of

$$P_g(X) = X^2 - a_p(g)X + \varepsilon_g(p)p^{2e},$$

and put $t(g) = 1 + q^{-e}(\beta_1(g)^\delta + \beta_2(g)^\delta)$, which does not depend on the order of the roots.

Step 2: (Ft3). By (Cc1) and Lemma 38.22 with $k = 2e+1$,

$$\begin{aligned} |S| &\leq c_1(N(e+1))^{A_1} + 1 + \frac{(2e+1)L^2}{12} \leq c_1(N(e+1))^{A_1} + (e+1)L^2 \\ &\leq c_1(N(e+1))^{A_1} + d_K^4(N(e+1))^4 \leq c\,(N(e+1))^A, \end{aligned}$$

Step 3: (Ft2) on $\mathrm{Cusp}_K(e,N)$. Let $\pi \in \mathrm{Cusp}_K(e,N)$ with central character ω. By (Cs2), $\omega^{-1} = \omega \circ c$. The continuous homomorphism $|\omega|\colon \mathbb{A}_K^\times / K^\times \to \mathbb{R}_{>0}$ is trivial on the compact

subgroup $\mathbb{A}_K^1/K^\times$, and $\mathbb{A}_K^\times = \mathbb{A}_K^1 \cdot \mathbb{R}_{>0}$ ($\mathbb{R}_{>0}$ embedded at the archimedean place), so $|\omega| = |\cdot|_{\mathbb{A}}^s$ for some real s. As $|x^c|_{\mathbb{A}} = |x|_{\mathbb{A}}$, $|\cdot|_{\mathbb{A}}^{-s} = |\omega^{-1}| = |\omega \circ c| = |\cdot|_{\mathbb{A}}^s$, so $s = 0$ and ω is unitary. By (Js2) of [136], $q^{-1/2} < |\alpha_i| < q^{1/2}$ for $i = 1, 2, 3$. Moreover π_w lies in the normalised induction of unramified characters μ_1, μ_2, μ_3 with $\mu_i(\varpi_w) = \alpha_i$, on which the centre acts by $\mu_1\mu_2\mu_3$; so $\alpha_1\alpha_2\alpha_3 = \omega_w(\varpi_w)$ has absolute value 1. Put $x_i = \log|\alpha_i|/\log q \in (-\frac{1}{2}, \frac{1}{2})$, ordered $x_1 \geq x_2 \geq x_3$, so $x_1 + x_2 + x_3 = 0$, and $\phi(x) = q^x$, convex and increasing. For $a, b \geq 0$, $\phi(a+b) - \phi(b) \geq \phi(a) - \phi(0)$, and for $a, b \leq 0$, $\phi(0) - \phi(a) \geq \phi(b) - \phi(a+b)$; in both cases $\phi(a) + \phi(b) \leq \phi(a+b) + 1$. If $x_2 \geq 0$, apply this to (x_1, x_2), whose sum is $-x_3$; if $x_2 \leq 0$, to (x_2, x_3), whose sum is $-x_1$. In both cases there is $y \in (-\frac{1}{2}, \frac{1}{2})$ with

$$|t(\pi)| \leq \sum_i |\alpha_i| = \sum_i \phi(x_i) \leq \phi(y) + \phi(-y) + 1 < q^{1/2} + q^{-1/2} + 1 \leq \sqrt{q} + 2,$$

since $y \mapsto \phi(y) + \phi(-y)$ is even and increasing on $[0, \infty)$. By Proposition 38.17 (applicable as $p \nmid Nd_K$), each $q^e\alpha_i$ is an algebraic integer, hence so is $q^e t(\pi)$.

Step 4: (Ft2) on $\mathrm{New}_{2e+1}(L)$. Let $g \in \mathrm{New}_{2e+1}(L)$; $p \nmid L_g$. By (Dg2) of [77] (weight $k = 2e+1 \geq 3$), $|\beta_j(g)| = p^e$, so $|q^{-e}\beta_j(g)^\delta| = 1$ and $|t(g)| \leq 3 \leq \sqrt{q} + 2$. By (Nw2) of [229], $a_p(g)$ is an algebraic integer, and $\varepsilon_g(p)$ is a root of unity; so P_g is monic with algebraic integer coefficients, $\beta_1(g), \beta_2(g)$ are algebraic integers, and so is $q^e t(g) = q^e + \beta_1(g)^\delta + \beta_2(g)^\delta$.

Step 5: (Ft1). Let $\theta \in \mathrm{Aut}(\mathbb{C})$. On $\mathrm{Cusp}_K(e, N)$: by (Cc2) choose for each π some $u_\theta(\pi) \in \mathrm{Cusp}_K(e, N)$ with $\mathrm{Sat}_v(u_\theta(\pi)) = \theta(\mathrm{Sat}_v(\pi))$ for all finite $v \nmid N$. If $u_\theta(\pi_1) = u_\theta(\pi_2)$, then $\mathrm{Sat}_v(\pi_1) = \mathrm{Sat}_v(\pi_2)$ for all finite $v \nmid N$ (θ is injective); an unramified irreducible representation of $\mathrm{GL}_3(K_v)$ is determined up to isomorphism by its Satake parameters, being the unique unramified irreducible subquotient of the corresponding normalised induction ([60]), so $\pi_{1,v} \cong \pi_{2,v}$ for all but finitely many v, and $\pi_1 \cong \pi_2$ by (Js1) of [136]. So u_θ is an injective, hence bijective, self-map of the finite set $\mathrm{Cusp}_K(e, N)$, and $t(u_\theta(\pi)) = \sum_i \theta(\alpha_i) = \theta(t(\pi))$. On $\mathrm{New}_{2e+1}(L)$: put $u_\theta(g) = g^\theta$, a normalised newform of weight $2e+1$, level L_g and character $\theta \circ \varepsilon_g$ by (Nw3) of [229]; so u_θ maps $\mathrm{New}_{2e+1}(L)$ to itself, with inverse $u_{\theta^{-1}}$. Since $a_p(g^\theta) = \theta(a_p(g))$ and θ fixes p^{2e}, P_{g^θ} is P_g with θ applied to its coefficients, its roots are $\theta(\beta_1(g)), \theta(\beta_2(g))$, and $t(g^\theta) = \theta(t(g))$ as θ fixes q^{-e}. The two maps together are a bijection u_θ of S with $t \circ u_\theta = \theta \circ t$.

Step 6: setting up (Ft4). Fix an odd prime $\ell \neq p$, an isomorphism ι, a place $\bar{w}$ of $\overline{\mathbb{Q}}$ above w and an arithmetic Frobenius $\sigma \in G_K$ at $\bar{w}$. Since $\ell \neq p$, reduction modulo $\bar{w}$ is injective on the roots of unity of ℓ-power order, and σ acts on the residue field as $x \mapsto x^q$; hence $\sigma(\zeta) = \zeta^q$ for these roots of unity and $\chi_{\mathrm{cyc}}(\sigma) = q$. We use twice the following consequence of the Chebotarev density theorem and the Brauer–Nesbitt theorem ([220], Chapter I, 2.2): if F is a number field and $r_1, r_2 \colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ are continuous semisimple representations such that, for all finite places v outside a finite set, both are unramified at v and $\det(X - r_1(\mathrm{Frob}_v^{-1})) = \det(X - r_2(\mathrm{Frob}_v^{-1}))$, then $r_1 \cong r_2$ (the Frobenius elements at these places are dense in the quotient of G_F through which $r_1 \oplus r_2$ factors, so $\mathrm{tr}\, r_1 = \mathrm{tr}\, r_2$). (T)

Step 7: stable points. Let ρ be admissible of stable type for (ι, e, N) and $r = \rho \otimes \chi_{\mathrm{cyc}}^{-1}$; r is irreducible by (Sb1), hence semisimple. By (Sb2) choose a regular algebraic conjugate self-dual cuspidal π_ρ with $r \leftrightarrow_\iota \pi_\rho$; by Proposition 38.11 ($\ell$ odd) the class of π_ρ lies in $\mathrm{Cusp}_K(e, N)$, and we put $\lambda(\rho) = \pi_\rho \in S$. The representation $r_0 = r_{\ell,\iota}(\pi_\rho)$ of [231], [67] (with $E = K$, $n = 3$) is continuous, semisimple and satisfies $r_0 \leftrightarrow_\iota \pi_\rho$; by (T), $r \cong r_0$. Now $w \nmid \ell$ and $\pi_{\rho,w}$ is unramified ($w \nmid N$), so the compatibility of r_0 at w gives $\iota(\det(X - r(\sigma^{-1}))) = \prod_i (X - q\alpha_i)$ with $\{\alpha_i\} = \mathrm{Sat}_w(\pi_\rho)$. As $\rho(\sigma^{-1}) = \chi_{\mathrm{cyc}}(\sigma)^{-1} r(\sigma^{-1}) = q^{-1} r(\sigma^{-1})$,

$$\iota(\mathrm{tr}\, \rho(\sigma^{-1})) = q^{-1} \sum_i q\alpha_i = t(\lambda(\rho)).$$

If ρ' is another such representation with $\lambda(\rho') = \lambda(\rho)$, then $\pi_{\rho'} \cong \pi_\rho$ have the same Satake

parameters at all unramified places, so $r' = \rho' \otimes \chi_{\mathrm{cyc}}^{-1}$ and r (both irreducible) have the same characteristic polynomials of Frob_v^{-1} at almost all v; by (T), $r \cong r'$, hence $\rho \cong \rho'$.

Step 8: endoscopic points. Let ρ be admissible of endoscopic type for (ι, e, M), $M = N|d_K|$; choose $W = W_\rho$ with (Eo1)–(Eo4) and a normalised newform $g = g_\rho$ of weight $2e+1$ as in (Eo2), and put $W' = W \otimes \chi_{\mathrm{cyc}}^{-e}$. By (Eo3), W has level M and weights $(e, -e)$, so Proposition 38.21 (ℓ odd) gives $L_g \mid M^2 = L$; thus $g \in \mathrm{New}_{2e+1}(L)$ and we put $\lambda(\rho) = g \in S$. Let ρ_g^{ss} be the semisimplification of $\rho_{g,\iota}$ of (Dg1) of [76]; it has the same characteristic polynomials and is trivial on every inertia group on which $\rho_{g,\iota}$ is. W' is semisimple by (Eo2), and W' and ρ_g^{ss} are unramified with characteristic polynomial $P_{g,p'}(X) = X^2 - a_{p'}(g)X + \varepsilon_g(p')p'^{2e}$ of geometric Frobenius at all primes p' outside a finite set (by (Eo2) and (Dg1)); by (T), $W' \cong \rho_g^{\mathrm{ss}}$. Since $p \nmid \ell L_g$, W' is trivial on the inertia group I' of $\bar{w}$ in $G_{\mathbb{Q}}$, and $\iota(\det(X - W'(\tau^{-1}))) = P_g(X)$ for an arithmetic Frobenius $\tau \in G_{\mathbb{Q}}$ at $\bar{w}$ (which exists: the decomposition group of $\bar{w}$ in $G_{\mathbb{Q}}$ maps onto the Galois group of the residue field extension). The element $\sigma \in G_K \subseteq G_{\mathbb{Q}}$ lies in that decomposition group and acts on the residue field as $x \mapsto x^{p^\delta}$, so $\tau^\delta \sigma^{-1} \in I'$ and $W'(\sigma^{-1}) = W'(\tau^{-1})^\delta$, with eigenvalues $\iota^{-1}(\beta_1(g)^\delta), \iota^{-1}(\beta_2(g)^\delta)$. By (Eo4), ρ is conjugate to $s \mapsto j(W(s))$ on G_K, where $j\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}$ is a group homomorphism $\mathrm{GL}_2 \to \mathrm{GL}_3$; so $\operatorname{tr}\rho(\sigma^{-1}) = \operatorname{tr} W(\sigma^{-1}) + 1$ and, as $W(\sigma^{-1}) = \chi_{\mathrm{cyc}}(\sigma)^{-e} W'(\sigma^{-1}) = q^{-e} W'(\sigma^{-1})$,

$$\iota(\operatorname{tr}\rho(\sigma^{-1})) = 1 + q^{-e}\big(\beta_1(g)^\delta + \beta_2(g)^\delta\big) = t(\lambda(\rho)).$$

If ρ' is another such representation with $\lambda(\rho') = g$, then $W_{\rho'} \otimes \chi_{\mathrm{cyc}}^{-e} \cong \rho_g^{\mathrm{ss}} \cong W'$, so $W_{\rho'} = hW_\rho h^{-1}$ for some $h \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$, and ρ' is conjugate to $j \circ W_{\rho'}|_{G_K} = j(h)\,(j \circ W_\rho|_{G_K})\,j(h)^{-1}$, hence to ρ.

Step 9: (Ft4). A representation admissible of weight e and level N is of stable type for (ι, e, N) or of endoscopic type for $(\iota, e, N|d_K|)$, and not both: a stable one is irreducible by (Sb1), while an endoscopic one is conjugate to $j \circ W|_{G_K}$, which fixes the second basis vector. So Steps 7 and 8 define a map λ from the admissible representations of weight e and level N to S, sending the stable ones to $\mathrm{Cusp}_K(e, N)$ and the endoscopic ones to $\mathrm{New}_{2e+1}(L)$, with $\iota(\operatorname{tr}\rho(\sigma^{-1})) = t(\lambda(\rho))$; and $\lambda(\rho) = \lambda(\rho')$ forces ρ, ρ' to be of the same type, hence $\rho \cong \rho'$ by Step 7 or Step 8. The set S and the map t of Step 1 do not depend on ℓ, ι, $\bar{w}$ or σ. □

Definition 38.10. Let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field with a fixed embedding $\tau\colon K \to \mathbb{C}$, $\bar{\tau} = \overline{\tau(\cdot)}$ the other one, and $n \geq 1$. Put $G_\infty = \mathrm{GL}_n(K \otimes_{\mathbb{Q}} \mathbb{R})$, identified with $\mathrm{GL}_n(\mathbb{C})$ through τ; $K_\infty = \mathrm{U}(n) \cdot \mathbb{R}_{>0} \subset G_\infty$ ($\mathbb{R}_{>0}$ as scalar matrices); $\mathfrak{g} = \mathrm{Lie}(G_\infty) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$, the two factors corresponding to τ and $\bar{\tau}$. For $N \geq 1$, $\mathrm{K}_n(N) \subset \mathrm{GL}_n(\widehat{\mathcal{O}}_K)$ is the kernel of reduction modulo N (so $\mathrm{K}_3(N) = \mathrm{K}(N)$).

Infinitesimal characters. For a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_K)$, the **infinitesimal character of π at τ** (resp. at $\bar{\tau}$) is the multiset of n complex numbers given by the τ- (resp. $\bar{\tau}$-) factor of the infinitesimal character of π_∞, in the normalisation of [68], in which the trivial representation has infinitesimal character $\{\frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-1}{2}\}$ at both embeddings (for $n = 3$ this is the infinitesimal character of Definition 38.8).

Coefficient systems. For $\mu \in \mathbb{Z}^n$ with $\mu_1 \geq \cdots \geq \mu_n$ let (ρ_μ, V_μ) be the irreducible algebraic representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight μ; its infinitesimal character is $\{\mu_i + \frac{n+1}{2} - i : 1 \leq i \leq n\}$ and that of its dual is $\{-\mu_i - \frac{n+1}{2} + i\}$. For two such μ, μ', $V_{\mu,\mu'} = V_\mu \otimes_{\mathbb{C}} V_{\mu'}$ with $g \in G_\infty$ acting by $\rho_\mu(g) \otimes \rho_{\mu'}(\bar{g})$; $\mathrm{GL}_n(K) \subset G_\infty$ acts by restriction. For a cuspidal π, $H^*(\mathfrak{g}, K_\infty; \pi_\infty \otimes V_{\mu,\mu'})$ is the relative Lie algebra cohomology of the $(\mathfrak{g}, K_\infty)$-module of K_∞-finite vectors of π_∞ tensored with $V_{\mu,\mu'}$; π is **cohomological for** $V_{\mu,\mu'}$ if it is nonzero in some degree.

Arithmetic groups of a level. For an open compact subgroup $U \subseteq \mathrm{GL}_n(\mathbb{A}_{K,f})$ the double

coset space $\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_{K,f})/U$ is finite ([38]); for representatives $g_1,\dots,g_h$ put $\Gamma_j(U) = \mathrm{GL}_n(K)\cap g_jUg_j^{-1}$, acting on $V_{\mu,\mu'}$ through $\mathrm{GL}_n(K)\subset G_\infty$, and

$$h^*(U,V_{\mu,\mu'}) = \sum_{j=1}^{h}\sum_{q\geq 0}\dim_{\mathbb{C}} H^q(\Gamma_j(U),V_{\mu,\mu'}) \in \mathbb{Z}_{\geq 0}\cup\{\infty\},$$

which does not depend on the representatives (replacing g_j by $\gamma g_j u$ with $\gamma\in\mathrm{GL}_n(K)$, $u\in U$ conjugates $\Gamma_j(U)$ by γ, compatibly with the action on $V_{\mu,\mu'}$). It is the total dimension of the cohomology of the locally symmetric space of level U with coefficients in $V_{\mu,\mu'}$.

Proposition 38.11. *Let K be an imaginary quadratic field, $e,N\geq 1$ integers, ℓ an odd prime, $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$ an isomorphism and $\rho\colon G_K\to\mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ a representation admissible of stable type for (ι,e,N). Let π be a regular algebraic conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_K)$ with $\rho\otimes\chi_{\mathrm{cyc}}^{-1}\leftrightarrow_\iota\pi$. Then the isomorphism class of π lies in $\mathrm{Cusp}_K(e,N)$.*

Proof. Put $r=\rho\otimes\chi_{\mathrm{cyc}}^{-1}$ and $m=(e-1,-1,-e-1)$. We check the hypotheses of Proposition 38.13 for $n=3$, $M=N$, r and π.

- r is irreducible, as ρ is by (Sb1) and twisting by a character preserves invariant subspaces.
- r has level N and weights m. By (Sb3), ρ has level N and weights $(e,0,-e)$. For a finite place $v\nmid\ell$, χ_{cyc} is trivial on $I_{\bar v}$, so (Lw1) for ρ is (Lw1) for r with the same characters. For $v\mid\ell$, if ρ is triangular on $D_{\bar v}$ with diagonal (κ_i), $\kappa_i=\chi_{\mathrm{cyc}}^{m_i'}\chi_i$ on $I_{\bar v}$ ($m'=(e,0,-e)$), then r is triangular on $D_{\bar v}$ (same g) with diagonal $(\kappa_i\chi_{\mathrm{cyc}}^{-1})$, equal to $\chi_{\mathrm{cyc}}^{m_i'-1}\chi_i=\chi_{\mathrm{cyc}}^{m_i}\chi_i$ on $I_{\bar v}$; the alternative in (Lw2) is preserved, since $\kappa_i\neq\kappa_j\chi_{\mathrm{cyc}}$ is invariant under twisting all κ_i by one character, and twisting by the crystalline character χ_{cyc}^{-1} preserves potential crystallinity.
- π is polarizable: it is conjugate self-dual, $\pi^c\cong\pi^\vee=\pi^\vee\otimes(\mathbf{1}\circ\mathrm{N}_{K/\mathbb{Q}}\circ\det)$.
- By (Sb2), r satisfies the witness conditions, in particular (Wc2).
- r is potentially generic at every place $v\mid\ell$ of K. By (Sb4) and the twist computation above, for each $v\mid\ell$ there is $\bar v\mid v$ such that r is triangular on $D_{\bar v}$ with some diagonal (κ_i) satisfying $\kappa_i\neq\kappa_j\chi_{\mathrm{cyc}}|_{D_{\bar v}}$ for all i,j. Every other place above v is $g\bar v$ with $g\in G_K$, $D_{g\bar v}=gD_{\bar v}g^{-1}$, and $r(gsg^{-1})=r(g)r(s)r(g)^{-1}$; so r is triangular on $D_{g\bar v}$ with diagonal $(s\mapsto\kappa_i(g^{-1}sg))$, and these satisfy the same inequalities because $\chi_{\mathrm{cyc}}(g^{-1}sg)=\chi_{\mathrm{cyc}}(s)$. So r is generic at v, hence potentially generic at v by (Wc1).

By Proposition 38.13: (Wl2) gives the infinitesimal character $\{m_i+1\}=\{e,0,-e\}$ at both embeddings, which is (Cs1); (Wl3) gives a nonzero vector of π_f fixed by $\mathrm{K}_3(N)=\mathrm{K}(N)$, which is (Cs3); and (Cs2) is the hypothesis that π is conjugate self-dual. So the isomorphism class of π lies in $\mathrm{Cusp}_K(e,N)$. (The hypothesis that ℓ is odd is not used.) □

Definition 38.12. Let v and ℓ be primes and $I_v\subset G_{\mathbb{Q}_v}$ the inertia subgroup. For w in the Weil group $W_{\mathbb{Q}_v}\subseteq G_{\mathbb{Q}_v}$ let $n(w)\in\mathbb{Z}$ be the integer with $w(x)=x^{v^{n(w)}}$ on the residue field of $\overline{\mathbb{Q}}_v$.

A **Weil–Deligne representation** of $W_{\mathbb{Q}_v}$ over $\overline{\mathbb{Q}}_\ell$ is a triple $D=(V,\rho,N)$ of a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space V, a homomorphism $\rho\colon W_{\mathbb{Q}_v}\to\mathrm{GL}(V)$ trivial on an open subgroup of I_v, and a nilpotent endomorphism N of V (the **monodromy**) with

$$\rho(w)\,N\,\rho(w)^{-1}=v^{n(w)}N\qquad(w\in W_{\mathbb{Q}_v}).\tag{38.5}$$

Its **conductor exponent** is

$$\mathrm{a}(D) = \mathrm{a}\big(\rho|_{I_v}\big) + \dim V^{I_v} - \dim\big(\ker N\big)^{I_v},$$

where, for a finite-dimensional representation θ of I_v trivial on an open subgroup, $\mathrm{a}(\theta)$ is its Artin conductor ([221], Chapter VI, §2: the sum of the codimension of the invariants and the Swan conductor, computed on any finite quotient of I_v through which θ factors).

For a continuous representation $r\colon G_{\mathbb{Q}_v} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, the **Weil–Deligne representation** $\mathrm{WD}(r)$ is defined when $v \neq \ell$, or when $v = \ell$ and r is potentially semistable:

(Wd1) if $v \neq \ell$: by Grothendieck's monodromy theorem there are a unique nilpotent N and an open subgroup $J \subseteq I_v$ with $r(\sigma) = \exp(t_\ell(\sigma)N)$ for $\sigma \in J$, where $t_\ell\colon I_v \to \mathbb{Z}_\ell$ is a nonzero homomorphism (a choice of tame character); $\mathrm{WD}(r) = (\overline{\mathbb{Q}}_\ell^{\,n}, \rho, N)$ with $\rho(\Phi^m\sigma) = r(\Phi^m\sigma)\exp(-t_\ell(\sigma)N)$ for a fixed $\Phi \in W_{\mathbb{Q}_v}$ with $n(\Phi) = -1$, $m \in \mathbb{Z}$ and $\sigma \in I_v$;

(Wd2) if $v = \ell$ and r is potentially semistable: $\mathrm{WD}(r)$ is the Weil–Deligne representation attached by Fontaine to the filtered $(\varphi, N, G_{\mathbb{Q}_\ell})$-module $D_{\mathrm{pst}}(r)$ ([95]).

Both are well defined up to isomorphism (independent of the choices), and the **conductor exponent of** r is $\mathrm{a}(r) = \mathrm{a}(\mathrm{WD}(r))$.

Frobenius-semisimplification. Let $D = (V, \rho, N)$ be a Weil–Deligne representation and $\Phi \in W_{\mathbb{Q}_v}$ an element with $n(\Phi) = -1$ (a lift of geometric Frobenius). Write $\rho(\Phi) = \rho(\Phi)_s\rho(\Phi)_u$ for the Jordan decomposition into commuting semisimple and unipotent parts. The **Frobenius-semisimplification** of D is $D^{F\text{-}ss} = (V, \rho_s, N)$, where $\rho_s(\Phi^m\sigma) = \rho(\Phi)_s^m\rho(\sigma)$ for $m \in \mathbb{Z}$ and $\sigma \in I_v$. It is a Weil–Deligne representation, independent of Φ up to isomorphism, and D is **Frobenius-semisimple** if $D \cong D^{F\text{-}ss}$, i.e. if $\rho(\Phi)$ is semisimple.

All of the above applies without change to a finite extension K of $\mathbb{Q}_v$ in place of $\mathbb{Q}_v$, with $W_K \supseteq I_K$, and the cardinality q_K of the residue field of K in place of v in the definition of $n(\cdot)$ and in (38.5).

Proposition 38.13. *Let K be an imaginary quadratic field, $n, M \geq 1$ integers, $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, ℓ a prime and $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism. Let $r\colon G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be an irreducible continuous representation satisfying (Lw1) and the following part of (Lw2): for every place $v \mid \ell$ of K there are a place $\bar v \mid v$, characters $\chi_1, \dots, \chi_n$ of G_K of level M and characters $\kappa_1, \dots, \kappa_n$ of $D_{\bar v}$ such that r is triangular on $D_{\bar v}$ with diagonal (κ_i) and $\kappa_i(s) = \chi_{\mathrm{cyc}}(s)^{m_i}\chi_i(s)$ for $s \in I_{\bar v}$ (so every representation of level M and weights m qualifies); and let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$ with $r \leftrightarrow_\iota \pi$ which is polarizable: $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathrm{N}_{K/\mathbb{Q}} \circ \det)$ for some continuous character $\chi\colon \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$. Assume that r satisfies (Wc2), and that r is potentially generic at every place $v \mid \ell$ of K with $v \nmid M$. Then:*

(Wl1) *for every finite place $w \nmid \ell$ of K at which π_w is unramified, every place $\bar w$ of $\overline{\mathbb{Q}}$ above w and every arithmetic Frobenius $\sigma \in G_K$ at $\bar w$, r is trivial on $I_{\bar w}$ and $\iota\big(\det(X - r(\sigma^{-1}))\big) = \prod_\alpha (X - q_w^{(n-1)/2}\alpha)$, α running over the Satake parameters of π_w with multiplicity;*

(Wl2) *at both embeddings $K \to \mathbb{C}$ the infinitesimal character of π is $\{m_1 + \frac{n-1}{2}, \dots, m_n + \frac{n-1}{2}\}$;*

(Wl3) *π_f has a nonzero vector fixed by $\mathrm{K}_n(M)$.*

Proof. Step 1: $r \cong r_{\ell,\iota}(\pi)$. K is a CM field with $K^+ = \mathbb{Q}$, so the condition on $\chi_v(-1)$ in Theorem B.62 is empty and $r_0 = r_{\ell,\iota}(\pi)$ exists. Both r (irreducible) and r_0 are semisimple, and

by $r \leftrightarrow_\iota \pi$ and (Bg1) both are unramified with the same characteristic polynomial of Frob_w^{-1} at all finite places w outside a finite set. By the Chebotarev density theorem and the Brauer–Nesbitt theorem ([220], Chapter I, 2.2), $r \cong r_0$. As isomorphic representations have the same restrictions up to conjugation, (Bg1) gives (Wl1), and (Bg2), (Bg3) hold for r in place of r_0.

Step 2: a character of level M is trivial on $\mathrm{Art}_{K_v}(U_k)$. Let v be a finite place of K above the prime p, $k \geq 0$ the exponent with $M\mathcal{O}_v = \varpi_v^k \mathcal{O}_v$, and ψ a character of G_K of level M. Write $M = p^a M'$ with $p \nmid M'$; then $p^a \mathcal{O}_v = \varpi_v^k \mathcal{O}_v$. The restriction of ψ to a decomposition group $D_{\bar{v}} \cong G_{K_v}$ factors through $\mathrm{Gal}(K_v(\mu_M)/K_v)$, so it suffices that U_k lies in the norm group of $K_v(\mu_M) = K_v(\mu_{M'}) \cdot K_v(\mu_{p^a})$, which is the intersection of the two norm groups. $K_v(\mu_{M'})/K_v$ is unramified, so its norm group contains $\mathcal{O}_v^\times \supseteq U_k$. The norm group of $K_v(\mu_{p^a}) = K_v \cdot \mathbb{Q}_p(\mu_{p^a})$ is the preimage under $\mathrm{N}_{K_v/\mathbb{Q}_p}$ of the norm group $p^{\mathbb{Z}} \times (1 + p^a\mathbb{Z}_p)$ of $\mathbb{Q}_p(\mu_{p^a})/\mathbb{Q}_p$ ([190], Chapter V, Proposition 6.2 and Theorem 1.10 with Corollary 6.4 there; for $a = 0$ there is nothing to show). For $u \in U_k = 1 + p^a\mathcal{O}_v$ (if $a \geq 1$; $k \geq 1$ in that case), each conjugate of u in a Galois closure lies in $1 + p^a\mathcal{O}$, so $\mathrm{N}_{K_v/\mathbb{Q}_p}(u) \in (1 + p^a\mathcal{O}) \cap \mathbb{Q}_p = 1 + p^a\mathbb{Z}_p$. Hence $\psi(\mathrm{Art}_{K_v}(u)) = 1$ for all $u \in U_k$.

Step 3: the inertial part of $\mathrm{WD}(r_{\bar{v}})$. Keep v, k. We find characters $\Psi_1, \ldots, \Psi_n$ of the Weil group $W_{\bar{v}}$, trivial on $\mathrm{Art}_{K_v}(U_k)$, with $\mathrm{WD}(r_{\bar{v}})^{\mathrm{ss}}|_{I_{\bar{v}}} \cong \bigoplus_i \Psi_i|_{I_{\bar{v}}}$, for some place $\bar{v} \mid v$. If $v \nmid \ell$: by (Lw1) there are $\bar{v}$ and characters $\chi_1, \ldots, \chi_n$ of G_K of level M with $r|_{I_{\bar{v}}}$ conjugate to $\mathrm{diag}(\chi_i)$; then $r(I_{\bar{v}})$ is finite, so in (Wd1) $\exp(t_\ell(\sigma)N)$ has finite order for σ in an open subgroup of $I_{\bar{v}}$, which for a unipotent matrix forces $\exp(t_\ell(\sigma)N) = 1$, and $N = 0$ as t_ℓ is nonzero on every open subgroup; thus $\rho|_{I_{\bar{v}}} = r|_{I_{\bar{v}}}$, and $\Psi_i = \chi_i|_{W_{\bar{v}}}$ works by Step 2. If $v \mid \ell$: by (Lw2) there are $\bar{v}$, characters χ_i of level M and characters κ_i of $D_{\bar{v}}$ with r triangular on $D_{\bar{v}}$ with diagonal (κ_i) and $\kappa_i = \chi_{\mathrm{cyc}}^{m_i}\chi_i$ on $I_{\bar{v}}$. The character $\kappa_i' = \kappa_i\chi_{\mathrm{cyc}}^{-m_i}$ agrees on $I_{\bar{v}}$ with χ_i, of finite order; so for the finite extension L/K_v cut out by $\ker\chi_i \cap D_{\bar{v}}$, κ_i' is unramified on G_L, hence crystalline over L with Hodge–Tate weight 0. Thus κ_i is potentially semistable, de Rham with Hodge–Tate weight m_i at every embedding. By (Bg2), $r_{\bar{v}}$ is de Rham, hence potentially semistable; its triangular filtration has graded pieces κ_i, and Fontaine's functor D_{pst} is exact and compatible with tensor products on potentially semistable representations ([95]), so $\mathrm{WD}(r_{\bar{v}})^{\mathrm{ss}} \cong \bigoplus_i \mathrm{WD}(\kappa_i)$ and $\mathrm{WD}(\kappa_i) = \mathrm{WD}(\chi_{\mathrm{cyc}}^{m_i}) \otimes \mathrm{WD}(\kappa_i')$. Here $\mathrm{WD}(\chi_{\mathrm{cyc}}^{m_i})$ is unramified ($\chi_{\mathrm{cyc}}^{m_i}$ is crystalline), and $\mathrm{WD}(\kappa_i')$ is a character of $W_{\bar{v}}$ whose restriction to $I_{\bar{v}}$ is ${\kappa_i'}^{\epsilon}|_{I_{\bar{v}}} = \chi_i^{\epsilon}|_{I_{\bar{v}}}$, with $\epsilon \in \{\pm 1\}$ fixed by the normalisation of Fontaine's construction (κ_i' becomes unramified over L, and the Weil group acts on D_{pst} through the action of $I_{\bar{v}}/I_L$ on the line). So $\Psi_i = \mathrm{WD}(\kappa_i)$ works: $\mathrm{Art}_{K_v}(U_k)$ lies in the image of inertia, where $\Psi_i = \chi_i^{\epsilon}$ is trivial by Step 2.

The same reading of (Lw2) gives the Hodge–Tate weights: at every place $v \mid \ell$ and every embedding $\tau\colon K_v \to \overline{\mathbb{Q}}_\ell$, the τ-Hodge–Tate weights of $r_{\bar{v}}$ are $m_1, \ldots, m_n$, since D_{dR} is exact on de Rham representations, so the Hodge–Tate multiset of $r_{\bar{v}}$ is the union of those of the κ_i (and it does not depend on the choice of $\bar{v} \mid v$, the decomposition groups being conjugate).

Step 4: (Wl2). Every embedding $\phi\colon K \to \mathbb{C}$ is $\iota \circ \tau|_K$ for the embedding $\tau = \iota^{-1} \circ \phi$, which extends continuously to the completion K_v at the place $v \mid \ell$ it defines. By (Bg2) and Step 3, $\{z - \frac{n-1}{2} : z \text{ in the infinitesimal character of } \pi \text{ at } \phi\} = \{m_1, \ldots, m_n\}$, which is (Wl2).

Step 5: local fixed vectors. Let v be a finite place of K and k as in Step 2. By Step 3 and (Bg3) with $U = U_k$, π_v is isomorphic to an irreducible subquotient of $I(\xi_1, \ldots, \xi_n)$ with characters ξ_i trivial on U_k. If $k \geq 1$, Lemma 38.14 gives $\pi_v^{\mathrm{K}_v(k)} \neq 0$. If $k = 0$, then $v \nmid M$; if $v \mid \ell$, r is potentially generic at v by hypothesis. In both cases (Wc2) (applicable: r is semisimple and $r \leftrightarrow_\iota \pi$) gives a real τ_π with $q_v^{\tau_\pi - 1/2} < |\xi_i(\varpi_v)| < q_v^{\tau_\pi + 1/2}$ for all i, (ξ_i) being a cuspidal support of π_v; hence $q_v^{-1} < |\xi_i\xi_j^{-1}(\varpi_v)| < q_v$, while $|\varpi_v|_v = q_v^{-1}$, so $\xi_i\xi_j^{-1} \neq |\cdot|_v$ for $i \neq j$, and Lemma 38.14 gives $\pi_v^{\mathrm{GL}_n(\mathcal{O}_v)} \neq 0$.

Step 6: (Wl3). $\mathrm{K}_n(M) = \prod_{v \nmid \infty} \mathrm{K}_v(k_v)$ with k_v the exponent of Step 2 at v ($k_v = 0$ for $v \nmid M$).

Write $\pi_f \cong \bigotimes'_v \pi_v$ ([91]), the restricted tensor product with respect to vectors $x_v^0 \in \pi_v^{\mathrm{GL}_n(\mathcal{O}_v)}$ given for all v outside a finite set $S \supseteq \{v \mid M\}$. For $v \notin S$ take $x_v = x_v^0$, and for $v \in S$ a nonzero $x_v \in \pi_v^{\mathrm{K}_v(k_v)}$ (Step 5). Then $\bigotimes_v x_v$ is a nonzero vector of π_f fixed by $\mathrm{K}_n(M)$. □

Lemma 38.14. *Let F_v be a non-archimedean local field with ring of integers $\mathcal{O}_v$, $n \geq 1$ and $k \geq 0$ integers. Put $U_k = 1 + \varpi_v^k \mathcal{O}_v$ for $k \geq 1$ and $U_0 = \mathcal{O}_v^\times$, and let $\mathrm{K}_v(k)$ be the kernel of $\mathrm{GL}_n(\mathcal{O}_v) \to \mathrm{GL}_n(\mathcal{O}_v/\varpi_v^k\mathcal{O}_v)$ (so $\mathrm{K}_v(0) = \mathrm{GL}_n(\mathcal{O}_v)$). Let $\xi_1, \dots, \xi_n$ be characters of $F_v^\times$ trivial on U_k and π an irreducible smooth representation of $\mathrm{GL}_n(F_v)$ isomorphic to a subquotient of $I(\xi_1, \dots, \xi_n)$. If $k \geq 1$, or if $\xi_i\xi_j^{-1} \neq |\cdot|_v$ for all $i \neq j$, then π has a nonzero vector fixed by $\mathrm{K}_v(k)$.*

Proof. Write $\mathrm{K} = \mathrm{K}_v(k)$ and $T_k = \{\mathrm{diag}(t_1, \dots, t_n) : t_i \in U_k\}$. Every element of K has diagonal entries in U_k (for $k = 0$: an element of $B \cap \mathrm{GL}_n(\mathcal{O}_v)$ has unit diagonal entries, its determinant being a unit), so $B \cap \mathrm{K} \subseteq T_k N$; and δ_B is trivial on $T_0 \supseteq T_k$, since $|u|_v = 1$ for units u.

Case 1: $\xi_i\xi_j^{-1} \neq |\cdot|_v$ for all $i \neq j$. By (Ps2) of Theorem B.161, $I = I(\xi_1, \dots, \xi_n)$ is irreducible, so $\pi \cong I$. Let $\eta(\mathrm{diag}(t_i)) = \delta_B^{1/2}(\mathrm{diag}(t_i)) \prod_i \xi_i(t_i)$, a character of T trivial on T_k. Define $f\colon G \to \mathbb{C}$ by $f(tu\kappa) = \eta(t)$ for $t \in T$, $u \in N$, $\kappa \in \mathrm{K}$, and $f = 0$ outside $B\mathrm{K}$. It is well defined: if $tu\kappa = t'u'\kappa'$, then $(t'u')^{-1}tu = \kappa'\kappa^{-1} \in B \cap \mathrm{K} \subseteq T_k N$, so $t'^{-1}t \in T_k$ and $\eta(t) = \eta(t')$. The set $B\mathrm{K}$ is open and closed (its complement is a union of open double cosets $Bg\mathrm{K}$), f is right K-invariant, hence locally constant, and $f(tux) = \eta(t)f(x)$ for all $x \in G$. So f is a nonzero ($f(1) = 1$) element of I^{K}.

Case 2: $k \geq 1$. By (Ps1) of Theorem B.161 there is a permutation s with $\pi \hookrightarrow I(\xi_{s(1)}, \dots, \xi_{s(n)})$. By Frobenius reciprocity for normalised parabolic induction ([27], Proposition 2.3),

$$\mathrm{Hom}_G(\pi, I(\xi_{s(1)}, \dots, \xi_{s(n)})) \cong \mathrm{Hom}_T(\pi_N, \eta_s)$$

with $\eta_s(\mathrm{diag}(t_i)) = \delta_B^{1/2}(\mathrm{diag}(t_i)) \prod_i \xi_{s(i)}(t_i)$, so there is a nonzero, hence surjective, T-equivariant $\lambda\colon \pi_N \to \mathbb{C}$ on which T acts by η_s; the character η_s is trivial on T_k. The representation π is admissible (irreducible smooth representations of $\mathrm{GL}_n(F_v)$ are admissible, [27]), and π_N is a smooth representation of T. Choose $x \in \pi_N$ with $\lambda(x) = 1$ and let $x_0 = \int_{T_k} t \cdot x \, dt$ (Haar probability measure on the compact open subgroup T_k; a finite average, as x is fixed by an open subgroup of T_k). Then $x_0 \in (\pi_N)^{T_k}$ and $\lambda(x_0) = \int_{T_k} \eta_s(t)\, dt \cdot \lambda(x) = 1$, so $(\pi_N)^{T_k} \neq 0$. Finally $\mathrm{K} = (\mathrm{K} \cap \bar{N})(\mathrm{K} \cap T)(\mathrm{K} \cap N)$ with $\mathrm{K} \cap T = T_k$: for $g \in \mathrm{K}$ every leading principal minor of g lies in $1 + \varpi_v^k\mathcal{O}_v$, a unit, so $g = LDU$ with $L \in \bar{N}$, $D \in T$, $U \in N$ whose entries are quotients of minors of g by leading principal minors, hence in $\mathcal{O}_v$; reducing modulo ϖ_v^k, the decomposition $1 = \bar{L}\bar{D}\bar{U}$ is unique, so L, D, U are congruent to 1 and lie in K. By (Ps3) of Theorem B.161, π^{K} maps onto $(\pi_N)^{T_k} \neq 0$, so $\pi^{\mathrm{K}} \neq 0$. □

Proposition 38.15. *Let K be an imaginary quadratic field. There are integers $c, A \geq 0$ such that for all integers $e, N \geq 1$:*

(Cc1) *$\mathrm{Cusp}_K(e, N)$ is finite and $|\mathrm{Cusp}_K(e, N)| \leq c\,(N(e+1))^A$;*

(Cc2) *for every $\sigma \in \mathrm{Aut}(\mathbb{C})$ and every $\pi \in \mathrm{Cusp}_K(e, N)$ there is $\pi' \in \mathrm{Cusp}_K(e, N)$ with $\mathrm{Sat}_v(\pi') = \{\sigma(\alpha) : \alpha \in \mathrm{Sat}_v(\pi)\}$ (as multisets) for every finite place $v \nmid N$ of K.*

Proof. Fix an embedding $\tau\colon K \to \mathbb{C}$ and let c, A be the integers of Lemma 38.16 for K and $n = 3$.

(Cc1). Let $e, N \geq 1$. A member π of $\mathrm{Cusp}_K(e,N)$ is a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_K)$ with infinitesimal character $\{e,0,-e\}$ at both embeddings (Cs1), a multiset of three distinct elements of $\frac{3-1}{2}+\mathbb{Z}=\mathbb{Z}$ of absolute value at most e, and with $\pi_f^{\mathrm{K}_3(N)} \neq 0$ (Cs3). So $\mathrm{Cusp}_K(e,N)$ is contained in the set of Lemma 38.16 for $Z=Z'=\{e,0,-e\}$ and $H=e$, and is finite with $|\mathrm{Cusp}_K(e,N)| \leq c\,(N(e+1))^A$.

(Cc2), the representation π'. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $\pi \in \mathrm{Cusp}_K(e,N)$; π is regular algebraic by (Cs1). By [68] (Theorem B.63) there is a regular algebraic cuspidal π' with $\pi'_f \cong {}^\sigma\pi_f$ whose infinitesimal character at $\sigma\circ\phi$ is that of π at ϕ; as $\phi \mapsto \sigma\circ\phi$ permutes the two embeddings and π has $\{e,0,-e\}$ at both, so does π': (Cs1). The σ-semilinear bijection $\pi_f \to {}^\sigma\pi_f$, $x \mapsto x\otimes 1$, commutes with the action of $\mathrm{GL}_3(\mathbb{A}_{K,f})$, so ${}^\sigma\pi_f^{\mathrm{K}(N)} \neq 0$: (Cs3). For (Cs2): twisting by σ commutes with smooth contragredients of admissible representations and with the conjugation $g \mapsto g^c$ of $\mathrm{GL}_3(\mathbb{A}_{K,f})$; hence $(\pi'^\vee)_f \cong {}^\sigma(\pi_f^\vee) \cong {}^\sigma((\pi^c)_f) = ({}^\sigma\pi_f)^c \cong (\pi'^c)_f$, using $\pi^\vee \cong \pi^c$. The cuspidal representations $\pi'^\vee$ and π'^c are therefore isomorphic at every finite place, and $\pi'^\vee \cong \pi'^c$ by (Js1) of [136]. So $\pi' \in \mathrm{Cusp}_K(e,N)$.

(Cc2), the Satake parameters. Let $v \nmid N$ be a finite place, $\mathrm{Sat}_v(\pi)=\{\alpha_1,\alpha_2,\alpha_3\}$, and B the upper triangular Borel subgroup of $\mathrm{GL}_3(K_v)$ with modulus character δ_B. π_v is the unique unramified irreducible subquotient ([60]) of the normalised induction $i(\chi)=\mathrm{Ind}_B^{\mathrm{GL}_3(K_v)}(\chi\delta_B^{1/2})$ of the unramified character χ of the torus with $\chi_j(\varpi_v)=\alpha_j$. Here $\delta_B^{1/2}(\mathrm{diag}(t_1,t_2,t_3)) = |t_1|_v\,|t_3|_v^{-1}$ takes values in $q_v^{\mathbb{Z}}$, which σ fixes, and for every character θ of B the map $f \mapsto \sigma\circ f$ identifies ${}^\sigma\mathrm{Ind}_B(\theta)$ with $\mathrm{Ind}_B(\sigma\circ\theta)$; so ${}^\sigma i(\chi) \cong i(\sigma\circ\chi)$. Since $\pi'_f \cong {}^\sigma\pi_f \cong \bigotimes'_v {}^\sigma\pi_v$, $\pi'_v \cong {}^\sigma\pi_v$, which is unramified (its $\mathrm{GL}_3(\mathcal{O}_{K_v})$-invariants are the twist of those of π_v) and a subquotient of ${}^\sigma i(\chi) \cong i(\sigma\circ\chi)$. Hence $\mathrm{Sat}_v(\pi')=\{\sigma(\alpha_1),\sigma(\alpha_2),\sigma(\alpha_3)\}$. □

Lemma 38.16. *Let K be an imaginary quadratic field with embeddings $\tau,\bar\tau$, and $n\geq 1$. There are integers $c, A \geq 0$ such that for all integers $N\geq 1$, $H\geq 0$ and all multisets Z, Z', each consisting of n pairwise distinct elements of $\frac{n-1}{2}+\mathbb{Z}$ of absolute value at most H, the set of isomorphism classes of cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_K)$ with infinitesimal character Z at τ and Z' at $\bar\tau$ and with $\pi_f^{\mathrm{K}_n(N)} \neq 0$ is finite and has at most $c\,(N(H+1))^A$ elements.*

Proof. Let $U_0=\mathrm{K}_n(3)$, fix representatives $g_1,\dots,g_h$ of $\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_{K,f})/U_0$ and put $\Gamma_i^0 = \Gamma_i(U_0) = \mathrm{GL}_n(K)\cap g_iU_0g_i^{-1}$. These depend only on K and n.

Step 1: every subgroup of $\mathrm{GL}_n(K)$ contained in gU_0g^{-1} for some $g\in\mathrm{GL}_n(\mathbb{A}_{K,f})$ is torsion-free. Let γ be such an element of finite order, v a place of K above 3 and $x=(g^{-1}\gamma g)_v \in 1+3\,\mathrm{M}_n(\mathcal{O}_{K_v})$. Then x is conjugate to γ in $\mathrm{GL}_n(K_v)$ and has finite order. Write $x=1+3y$. In a finite extension L of K_v containing the eigenvalues, every eigenvalue of x is $\zeta=1+3\beta$ with β an eigenvalue of y, integral over $\mathcal{O}_{K_v}$; and ζ is a root of unity. If $\zeta\neq 1$, some power $\zeta'=\zeta^k$ has prime order r, and still $\zeta'\in 1+3\mathcal{O}_L$. If $r\neq 3$, $\prod_{0<i<r}(1-\zeta'^i)=r$ is a unit at 3, so $1-\zeta'$ is a unit, a contradiction. If $r=3$, $(1-\zeta')^2=-3\zeta'$, so $v_L(1-\zeta')=\frac12 v_L(3)<v_L(3)$, a contradiction. Hence all eigenvalues of x are 1; an element of finite order is diagonalisable in characteristic 0, so $x=1$ and $\gamma=1$.

Step 2: the coefficient system. Fix N,H,Z,Z'. Write $\{-z : z\in Z\}=\{c_1>\cdots>c_n\}$ and put $\mu_i = c_i - \frac{n+1}{2}+i$. As $c_i\in\frac{n-1}{2}+\mathbb{Z}$ and the c_i are distinct, $\mu_i\in\mathbb{Z}$ and $\mu_i-\mu_{i+1} = c_i-c_{i+1}-1\geq 0$; the infinitesimal character of the dual of V_μ is $\{-c_i\}=Z$. Define μ' from Z' in the same way, and put $V=V_{\mu,\mu'}$. By Weyl's dimension formula ([99], §24.1),

$$\dim V_\mu = \prod_{i<j}\frac{\mu_i-\mu_j+j-i}{j-i} = \prod_{i<j}\frac{c_i-c_j}{j-i} \leq (2H)^{n(n-1)/2},$$

as $0 < c_i - c_j \le 2H$; so $\dim V \le (2H+1)^{n(n-1)}$. By [68] (Theorem B.65) every π in the set of the statement is cohomological for V.

Step 3: the level. Put $U = \mathrm{K}_n(3N)$, the kernel of $\mathrm{GL}_n(\widehat{\mathcal{O}}_K) \to \mathrm{GL}_n(\mathcal{O}_K/3N\mathcal{O}_K)$; it is normal in $\mathrm{GL}_n(\widehat{\mathcal{O}}_K)$ and contained in U_0 and in $\mathrm{K}_n(N)$. So $\pi_f^U \supseteq \pi_f^{\mathrm{K}_n(N)} \neq 0$ for every π in the set, and $[U_0 : U] \le |\mathrm{GL}_n(\mathcal{O}_K/3N)| \le |\mathcal{O}_K/3N|^{n^2} = (3N)^{2n^2}$.

Step 4: the double cosets of level U. Fix i. The group Γ_i^0 acts on the finite set U_0/U by $\gamma \cdot uU = (g_i^{-1}\gamma g_i)uU$, and $uU \mapsto \mathrm{GL}_n(K)g_iuU$ induces a bijection from the orbits onto the U-double cosets contained in $\mathrm{GL}_n(K)g_iU_0$: it is surjective, and if $\gamma g_i u U = g_i u' U$ with $\gamma \in \mathrm{GL}_n(K)$, then $g_i^{-1}\gamma g_i \in u'Uu^{-1} \subseteq U_0$, so $\gamma \in \Gamma_i^0$ and $u'U = \gamma \cdot uU$. The stabiliser of uU is $\{\gamma \in \Gamma_i^0 : g_i^{-1}\gamma g_i \in uUu^{-1} = U\} = \Gamma_i'$, where $\Gamma_i' = \mathrm{GL}_n(K) \cap g_i U g_i^{-1}$ (U is normal in U_0); this is also the group $\mathrm{GL}_n(K) \cap (g_iu)U(g_iu)^{-1}$ attached to the representative g_iu. So there are $[U_0 : U]/[\Gamma_i^0 : \Gamma_i']$ double cosets, each with group Γ_i', and

$$h^*(U, V) = \sum_{i=1}^{h} \frac{[U_0 : U]}{[\Gamma_i^0 : \Gamma_i']} \sum_{q \ge 0} \dim H^q(\Gamma_i', V).$$

Step 5: cohomology of the finite-index subgroups. By Step 1, the group Γ_i^0 is torsion-free; it is commensurable with $\mathrm{GL}_n(\mathcal{O}_K) = \mathrm{GL}_n(K) \cap \mathrm{GL}_n(\widehat{\mathcal{O}}_K)$, since the open compact groups $g_iU_0g_i^{-1}$ and $\mathrm{GL}_n(\widehat{\mathcal{O}}_K)$ are commensurable. By [43] (Theorem B.162) there is a resolution $F_\bullet \to \mathbb{C}$ by free $\mathbb{C}[\Gamma_i^0]$-modules F_q of finite ranks $r_q^{(i)}$, zero for large q. Let $m = [\Gamma_i^0 : \Gamma_i']$. As $\mathbb{C}[\Gamma_i^0]$ is a free $\mathbb{C}[\Gamma_i']$-module of rank m (on coset representatives), $F_\bullet$ is a resolution of $\mathbb{C}$ by free $\mathbb{C}[\Gamma_i']$-modules of ranks $mr_q^{(i)}$; so $H^q(\Gamma_i', V) = \mathrm{Ext}^q_{\mathbb{C}[\Gamma_i']}(\mathbb{C}, V)$ is a subquotient of $\mathrm{Hom}_{\mathbb{C}[\Gamma_i']}(F_q, V) \cong V^{mr_q^{(i)}}$, and $\dim H^q(\Gamma_i', V) \le m\, r_q^{(i)} \dim V$. With Step 4 and $C_0 = \sum_i \sum_q r_q^{(i)}$ (depending only on K, n),

$$h^*(U, V) \le [U_0 : U]\, C_0 \dim V \le (3N)^{2n^2} C_0 (2H+1)^{n(n-1)}.$$

Step 6: conclusion. Every finite subset S of the set of the statement consists of pairwise non-isomorphic cuspidal representations, cohomological for V (Step 2) with $\pi_f^U \neq 0$ (Step 3); by [40] (Theorem B.64) and Step 5, $|S| \le (3N)^{2n^2} C_0 (2H+1)^{n(n-1)}$. Hence the set is finite with at most this many elements, and with $A = 2n^2$, $c = 3^{2n^2} 2^{n^2} C_0$, using $2H+1 \le 2(H+1)$, $n(n-1) \le n^2 \le 2n^2$ and $N, H+1 \ge 1$,

$$(3N)^{2n^2} C_0 (2H+1)^{n(n-1)} \le c\,\big(N(H+1)\big)^A.$$

□

Proposition 38.17. *Let K be an imaginary quadratic field with discriminant d_K, $e, N \ge 1$ integers, p a prime with $p \nmid N d_K$, w a prime of K above p and $\pi \in \mathrm{Cusp}_K(e, N)$. Then for every $\alpha \in \mathrm{Sat}_w(\pi)$ the numbers $q_w^e\alpha$ and $q_w^e\alpha^{-1}$ are algebraic integers.*

Proof. Let π represent the class in $\mathrm{Cusp}_K(e, N)$. By (Cs2), $\pi^c \cong \pi^\vee = \pi^\vee \otimes (\mathbf{1} \circ \mathrm{N}_{K/\mathbb{Q}} \circ \det)$, so π is polarizable; by (Cs3), $\pi_f^{\mathrm{K}_3(N)} \neq 0$; by (Cs1), its infinitesimal character at both embeddings is $Z = \{e, 0, -e\}$, three distinct integers ($e \ge 1$), i.e. elements of $\frac{3-1}{2} + \mathbb{Z}$, with $z - 1 \in [-e-1, e-1]$ for $z \in Z$. As $p \nmid N d_K$, Proposition 38.18 applies with $n = 3$, $M = N$, $a = -e-1$, $b = e-1$: for $\alpha \in \mathrm{Sat}_w(\pi)$ and $\beta = q_w\alpha$, the numbers

$$q_w^{e-1}\beta = q_w^e\alpha \qquad \text{and} \qquad q_w^{e+1}\beta^{-1} = q_w^e\alpha^{-1}$$

are algebraic integers. □

Proposition 38.18. *Let K be an imaginary quadratic field with discriminant d_K, $n, M \geq 1$ and $a \leq b$ integers, and π a polarizable cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$ with $\pi_f^{\mathrm{K}_n(M)} \neq 0$ whose infinitesimal character at both embeddings $K \to \mathbb{C}$ is one multiset Z of n pairwise distinct elements of $\frac{n-1}{2} + \mathbb{Z}$ with $a \leq z - \frac{n-1}{2} \leq b$ for all $z \in Z$. Let p be a prime with $p \nmid M d_K$ and w a prime of K above p. Then for every Satake parameter α of π_w, the number $\beta = q_w^{(n-1)/2}\alpha$ is algebraic, and $q_w^b\beta$ and $q_w^{-a}\beta^{-1}$ are algebraic integers.*

Proof. As $w \nmid M$, π_w is unramified, and π is regular algebraic (Z consists of distinct elements of $\frac{n-1}{2} + \mathbb{Z}$). By (Cj2) of Lemma 38.19, every $\beta \in B_w(\pi)$ is algebraic; let $F_\beta = \mathbb{Q}(\beta)$. An algebraic number is an algebraic integer if its valuation is nonnegative at every finite place of the number field it generates, so it suffices to show: β is a unit at every place of F_β above a prime $\ell \neq p$, and $-bf \leq v_p(\beta) \leq -af$ at every place above p, where $q_w = p^f$ and v_p is normalised by $v_p(p) = 1$; then $q_w^b\beta$ and $q_w^{-a}\beta^{-1}$ have nonnegative valuation everywhere (q_w being a unit away from p).

Every place arises from some ι. Let ℓ be any prime and $j\colon F_\beta \to \overline{\mathbb{Q}}_\ell$ a field embedding, defining a place of F_β above ℓ. Extend j to $\overline{\mathbb{Q}}$ and then to an isomorphism $\mathbb{C} \to \overline{\mathbb{Q}}_\ell$ (both fields are algebraically closed of characteristic 0 and of the same uncountable cardinality, hence of the same transcendence degree over $\overline{\mathbb{Q}}$); its inverse ι satisfies $\iota^{-1}|_{F_\beta} = j$.

Places above $\ell \neq p$. With ι as above, $r = r_{\ell,\iota}(\pi)$ exists (K is CM with $K^+ = \mathbb{Q}$), and by (Bg1) at $w \nmid \ell$, for an arithmetic Frobenius σ at a place above w, $\iota(\det(X - r(\sigma^{-1}))) = \prod_{\beta' \in B_w(\pi)}(X - \beta')$; so $j(\beta) = \iota^{-1}(\beta)$ is an eigenvalue of $r(\sigma^{-1})$. A continuous representation of the compact group G_K in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ takes values in $\mathrm{GL}_n(E)$ for a finite extension $E/\mathbb{Q}_\ell$ (Baire category: G_K is the countable union of the closed subgroups $r^{-1}(\mathrm{GL}_n(E'))$), and then preserves an $\mathcal{O}_E$-lattice (the $\mathcal{O}_E$-span of the orbit of $\mathcal{O}_E^n$ under the compact image, which is bounded). So the characteristic polynomial of $r(\sigma^{-1})$ is monic in $\mathcal{O}_E[X]$ with unit constant term, and its roots are units: $j(\beta)$ is a unit.

Places above p. Take $\ell = p$ and ι as above, and $r = r_{p,\iota}(\pi)$, $\bar{w}$ a place of $\overline{\mathbb{Q}}$ above w. By (Bg4), $r_{\bar{w}}$ is crystalline, and the eigenvalues of $\rho(\Phi)$ on $\mathrm{WD}(r_{\bar{w}})$ for a geometric Frobenius Φ are the $\iota^{-1}(\beta')$, $\beta' \in B_w(\pi)$. As $p \nmid d_K$, $L = K_w$ is unramified over $\mathbb{Q}_p$ of degree f. Choose a finite $E/\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ containing the images of all embeddings of L such that $r_{\bar{w}}$ takes values in $\mathrm{GL}_n(E)$ (as above), and let $V = E^n$. By (Bg2), for every embedding $\tau\colon L \to E$, the τ-Hodge–Tate weights of V are the $z - \frac{n-1}{2}$, $z \in Z$, all in $[a, b]$. In Fontaine's construction of WD for crystalline representations ([95]), $D_{\mathrm{pst}}(V) = D_{\mathrm{cris}}(V) \otimes_L \mathbb{Q}_p^{\mathrm{nr}}$, $N = 0$, inertia acts trivially, and Φ acts on $\mathrm{WD}(V)$ through the E-linear map φ^f on any D_τ, extended to $\overline{\mathbb{Q}}_p$. This is the normalisation of (Bg4): for $n = 1$ and the polarizable regular algebraic character $\pi_1 = |\cdot|_{\mathbb{A}_K}$ (infinitesimal character 1 at both embeddings; $\pi_1^c = \pi_1 = \pi_1^\vee \otimes (|\cdot|^2_{\mathbb{A}_\mathbb{Q}} \circ \mathrm{N}_{K/\mathbb{Q}})$), (Bg1) gives $r_{p,\iota}(\pi_1)(\sigma^{-1}) = q_v^{-1}$ at all $v \nmid p$, so $r_{p,\iota}(\pi_1) = \chi_{\mathrm{cyc}}$ by the Chebotarev density theorem, and (Bg4) gives $\rho(\Phi) = q_w^{-1}$, which is the eigenvalue of φ^f on $D_{\mathrm{cris}}(\chi_{\mathrm{cyc}}) = \mathbb{Q}_p t^{-1} \otimes e$ ($\varphi(t^{-1}) = p^{-1}t^{-1}$), not its inverse. So the $\iota^{-1}(\beta')$ are the roots of the characteristic polynomial P of φ^f on D_τ, and Lemma 38.20 gives $-bf \leq v_p(\iota^{-1}(\beta)) \leq -af$, i.e. the claimed bounds at the place of F_β defined by $j = \iota^{-1}|_{F_\beta}$. □

Lemma 38.19. *Let K be an imaginary quadratic field, $n, M \geq 1$ integers, Z a multiset of n pairwise distinct elements of $\frac{n-1}{2} + \mathbb{Z}$, and π a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$ with infinitesimal character Z at both embeddings $K \to \mathbb{C}$ and $\pi_f^{\mathrm{K}_n(M)} \neq 0$. For a finite place $w \nmid M$ of K (at which π_w is unramified) let $B_w(\pi)$ be the multiset of the numbers $q_w^{(n-1)/2}\alpha$, α running over the Satake parameters of π_w with multiplicity ($q_w^{1/2} > 0$). Then:*

(Cj1) *for every $\theta \in \mathrm{Aut}(\mathbb{C})$ there is a cuspidal automorphic representation π' of $\mathrm{GL}_n(\mathbb{A}_K)$ with infinitesimal character Z at both embeddings and $\pi'_f \cong {}^\theta\pi_f$ (so $\pi_f'^{\mathrm{K}_n(M)} \neq 0$), and every*

such π' satisfies $B_w(\pi') = \theta(B_w(\pi))$ for every finite place $w \nmid M$;

(Cj2) *every element of $B_w(\pi)$, $w \nmid M$, is an algebraic number.*

Proof. (Cj1). As Z consists of n distinct elements of $\frac{n-1}{2} + \mathbb{Z}$ at both embeddings, π is regular algebraic, and Theorem B.63 gives a regular algebraic cuspidal π' with $\pi'_f \cong {}^\theta\pi_f$ whose infinitesimal character at $\theta \circ \phi$ equals that of π at ϕ, i.e. Z at both embeddings. A nonzero $\mathrm{K}_n(M)$-fixed vector of π_f is one of ${}^\theta\pi_f = \pi_f \otimes_{\mathbb{C},\theta} \mathbb{C}$. Now let π' be any cuspidal representation with $\pi'_f \cong {}^\theta\pi_f$, and $w \nmid M$. Since ${}^\theta(\bigotimes'_v \pi_v) \cong \bigotimes'_v {}^\theta\pi_v$ and the local factors of a restricted tensor product are determined up to isomorphism ([91]), $\pi'_w \cong {}^\theta\pi_w$. Let π_w be the unramified irreducible subquotient of $I(\mu_1, \dots, \mu_n)$ with unramified μ_i, $\mu_i(\varpi_w) = \alpha_i$. The map $f \mapsto \theta \circ f$ is a θ-semilinear G-equivariant bijection from $I(\mu_1, \dots, \mu_n)$ onto the space of smooth f' with $f'(tug) = \theta(\delta_B^{1/2}(t) \prod_i \mu_i(t_i)) f'(g)$, which is $I(\eta_1, \dots, \eta_n)$ for the characters with $\prod_i \eta_i(t_i) = \delta_B^{-1/2}(t)\, \theta(\delta_B^{1/2}(t)) \prod_i \theta(\mu_i(t_i))$. For $t = \mathrm{diag}(\varpi_w^{a_1} u_1, \dots, \varpi_w^{a_n} u_n)$ with units u_i, $\delta_B^{1/2}(t) = (\sqrt{q_w})^{-\sum_i (n+1-2i)a_i}$; with $\epsilon = \theta(\sqrt{q_w})/\sqrt{q_w} \in \{\pm 1\}$,

$$\delta_B^{-1/2}(t)\, \theta(\delta_B^{1/2}(t)) = \epsilon^{\sum_i (n+1-2i)a_i} = \epsilon^{(n-1)\sum_i a_i},$$

so $\eta_i = (\theta \circ \mu_i)\nu$ with ν unramified, $\nu(\varpi_w) = \epsilon^{n-1}$, and $\eta_i(\varpi_w) = \epsilon^{n-1}\theta(\alpha_i)$. The bijection carries $\mathrm{GL}_n(\mathcal{O}_w)$-fixed vectors to fixed vectors and subquotients to subquotients, so ${}^\theta\pi_w$ is the unramified irreducible subquotient of $I(\eta_1, \dots, \eta_n)$, and its Satake parameters are the $\epsilon^{n-1}\theta(\alpha_i)$ (an unramified irreducible representation determines its unramified characters up to permutation, [60]). Hence

$$B_w(\pi') = \{q_w^{(n-1)/2} \epsilon^{n-1} \theta(\alpha_i)\} = \{\theta(\sqrt{q_w})^{n-1} \theta(\alpha_i)\} = \theta(B_w(\pi)).$$

(Cj2). Let $H = \max_{z \in Z} |z|$ and Π the set of isomorphism classes of cuspidal π'' with infinitesimal character Z at both embeddings and $\pi''^{\mathrm{K}_n(M)}_f \neq 0$; Π is finite by Lemma 38.16, applied with $N = M$, $Z' = Z$ and this H. By (Cj1), for every $\theta \in \mathrm{Aut}(\mathbb{C})$ the multiset $\theta(B_w(\pi))$ equals $B_w(\pi')$ for some $\pi' \in \Pi$ (B_w depends only on the isomorphism class). So each elementary symmetric function s of $B_w(\pi)$ has a finite orbit under $\mathrm{Aut}(\mathbb{C})$. A transcendental complex number has an infinite orbit (for transcendental x, y the isomorphism $\mathbb{Q}(x) \to \mathbb{Q}(y)$, $x \mapsto y$, extends to an automorphism of $\mathbb{C}$), so every s is algebraic, and the elements of $B_w(\pi)$, roots of $\prod_{\beta \in B_w(\pi)} (X - \beta)$, whose coefficients are $\pm s$, are algebraic. □

Lemma 38.20. *Let L be an unramified extension of $\mathbb{Q}_p$ of degree f, $E \subset \overline{\mathbb{Q}}_p$ a finite extension of $\mathbb{Q}_p$ containing the images of all embeddings $L \to \overline{\mathbb{Q}}_p$, and V an n-dimensional E-vector space with a continuous E-linear action of G_L that is crystalline (as a representation on a $\mathbb{Q}_p$-vector space). The E-action makes $D = D_{\mathrm{cris}}(V)$ a module over $L \otimes_{\mathbb{Q}_p} E \cong \prod_\tau E$ (τ running over the embeddings $L \to E$, $x \otimes y \mapsto (\tau(x)y)_\tau$); write $D = \bigoplus_\tau D_\tau$ accordingly, with the induced filtrations $\mathrm{Fil}^i D_\tau$. The τ-**Hodge–Tate weights** of V are the integers h, counted $\dim_E(\mathrm{Fil}^{-h} D_\tau / \mathrm{Fil}^{1-h} D_\tau)$ times. Suppose that for integers $a \leq b$ all τ-Hodge–Tate weights of V, for all τ, lie in $[a, b]$. Then each D_τ has E-dimension n, φ^f restricts to an E-linear endomorphism of each D_τ whose characteristic polynomial $P(X) \in E[X]$ does not depend on τ, and every root $\beta \in \overline{\mathbb{Q}}_p$ of P satisfies*

$$-bf \leq v_p(\beta) \leq -af \qquad (v_p(p) = 1).$$

Proof. Structure. The action of E on V commutes with G_L, so E acts on D and on $\mathrm{Fil}^i D$, commuting with φ; with L acting through B_{cris}, D and each $\mathrm{Fil}^i D$ are $L \otimes_{\mathbb{Q}_p} E$-modules, hence $\mathrm{Fil}^i D = \bigoplus_\tau \mathrm{Fil}^i D_\tau$ with $\mathrm{Fil}^i D_\tau = e_\tau \mathrm{Fil}^i D$ for the idempotents e_τ. As φ is σ-semilinear for L and E-linear, $\varphi(x \otimes y)$-equivariance gives $\varphi(D_\tau) \subseteq D_{\tau\circ\sigma^{-1}}$, with equality since φ is bijective; so all D_τ have the same E-dimension d, and φ^f ($\sigma^f = 1$) is E-linear on each D_τ, with $\varphi \circ \varphi^f|_{D_\tau} = \varphi^f|_{D_{\tau\sigma^{-1}}} \circ \varphi$, so the characteristic polynomials on all D_τ agree. An E-vector space of dimension d on which L acts through τ has L-dimension $d[E : L]$, so $\dim_{\mathbb{Q}_p} V = \dim_L D = f\, d\, [E : L] = d[E : \mathbb{Q}_p]$ (Theorem B.55) gives $d = n$.

The two invariants on E-stable subobjects. Let $D' = \bigoplus_\tau D'_\tau$ be a φ-stable $L \otimes E$-submodule, $d' = \dim_E D'_\tau$ (independent of τ as above), and $P_{D'}$ the characteristic polynomial of φ^f on D'_τ. (i) If A is the matrix of $\varphi|_{D'}$ in an L-basis, $\varphi^f|_{D'}$ is L-linear with matrix $A\sigma(A)\cdots\sigma^{f-1}(A)$, so $f\, t_N(D') = v_p(\det_L(\varphi^f|_{D'})) = \sum_\tau v_p(\mathrm{N}_{E/\tau(L)} \det_E(\varphi^f|_{D'_\tau})) = f\,[E : L] \sum_\beta v_p(\beta)$, the last sum over the roots of $P_{D'}$ with multiplicity (for an E-linear map of an E-space regarded as a $\tau(L)$-space, the determinant is the norm of the E-determinant, and $v_p(\mathrm{N}_{E/\tau(L)} x) = [E : L] v_p(x)$). So $t_N(D') = [E : L] \sum_\beta v_p(\beta)$. (ii) With the induced filtration, $\mathrm{Fil}^i D'_\tau / \mathrm{Fil}^{i+1} D'_\tau$ injects into $\mathrm{Fil}^i D_\tau / \mathrm{Fil}^{i+1} D_\tau$, which vanishes unless $-b \le i \le -a$ by hypothesis; and $\dim_L$ of a graded piece of D' is $[E : L]$ times the sum over τ of the E-dimensions. Hence $-b\, f\, d'[E : L] \le t_H(D')$. Dually, for the quotient D/D' with the quotient filtration, each graded piece is a quotient of the corresponding graded piece of D, so $t_H(D/D') \le -a\, f\, (n - d')[E : L]$, and $t_H(D) = t_H(D') + t_H(D/D')$ (dimensions of graded pieces are additive for a subspace with induced and a quotient with quotient filtration).

Subobjects by slope. For a real s, let P_s be the product of $(X - \beta)$ over the roots β of P (with multiplicity) with $v_p(\beta) < s$. This set of roots is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E)$, so $P_s \in E[X]$, and $D^{(s)} = \bigoplus_\tau \ker P_s(\varphi^f|_{D_\tau})$ is an $L \otimes E$-submodule (sum of generalised eigenspaces) with $\varphi(D^{(s)}) \subseteq D^{(s)}$, as φ is E-linear and commutes with φ^f; the roots of $P_{D^{(s)}}$ are those of P_s.

Lower bound. Let λ be the least valuation of a root of P and choose $s > \lambda$ below every larger valuation; then $D' = D^{(s)}$ has $d' \ge 1$ and all roots of $P_{D'}$ of valuation λ. By Theorem B.55, $-bfd'[E : L] \le t_H(D') \le t_N(D') = [E : L]d'\lambda$, so $\lambda \ge -bf$.

Upper bound. Let μ be the largest valuation of a root and $D' = D^{(\mu)}$ (roots of valuation $< \mu$); then $n - d' \ge 1$ roots of P have valuation μ, so $t_N(D) - t_N(D') = [E : L](n - d')\mu$. By Theorem B.55, $t_N(D) = t_H(D)$ and $t_H(D') \le t_N(D')$, so $[E : L](n - d')\mu \le t_H(D) - t_H(D') = t_H(D/D') \le -af(n - d')[E : L]$, and $\mu \le -af$. □

Proposition 38.21. *Let ℓ be an odd prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $e, M \ge 1$ integers and $W\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ a continuous representation of level M and weights $(e, -e)$. Let g be a normalised newform of weight $2e+1$, level L_g and character ε_g such that W satisfies (Eo2) with $f = g$: $W \otimes \chi_{\mathrm{cyc}}^{-e}$ is semisimple and, for all but finitely many primes p and every place $\bar{p}$ of $\overline{\mathbb{Q}}$ above p, $W \otimes \chi_{\mathrm{cyc}}^{-e}$ is trivial on $I_{\bar{p}}$ and $\iota(\det(X - (W \otimes \chi_{\mathrm{cyc}}^{-e})(\sigma^{-1}))) = X^2 - a_p(g)X + \varepsilon_g(p)p^{2e}$ for every arithmetic Frobenius σ at $\bar{p}$. Then L_g divides M^2.*

Proof. Put $W' = W \otimes \chi_{\mathrm{cyc}}^{-e}$. By hypothesis W' is continuous and semisimple, and its Frobenius characteristic polynomials are those of g of weight $k = 2e + 1$ (so $p^{k-1} = p^{2e}$); hence (Lc1) and (Lc2) of [59], [213] apply to $\rho = W'$. Restrictions to the decomposition groups of two places above the same prime are conjugate, so $\mathrm{a}(W'_v)$ may be computed at any place above v. We show $v_v(L_g) \le 2v_v(M)$ for every prime v, which gives $L_g \mid M^2$.

Step 1: characters of level M have conductor exponent at most $v_v(M)$ at v. Let χ be a character of $G_{\mathbb{Q}}$ of level M, v a prime, $a = v_v(M)$, $M = v^a M'$, and $\bar{v}$ a place above v. Then $\chi|_{D_{\bar{v}}}$ factors through $\mathrm{Gal}(\mathbb{Q}_v(\mu_M)/\mathbb{Q}_v)$, and $\mathbb{Q}_v(\mu_M)$ is the compositum of $\mathbb{Q}_v(\mu_{v^a})$ with the

unramified extension $\mathbb{Q}_v(\mu_{M'})$. Hence $\chi|_{I_{\bar v}}$ factors through the inertia group of $\mathbb{Q}_v(\mu_{v^a})/\mathbb{Q}_v$ (in particular it is trivial on an open subgroup), whose upper ramification groups G^u are trivial for $u > a-1$ ([221], Chapter IV, §4). The Artin conductor of a character θ of inertia is the least integer $f \geq 0$ with $\theta(G^f) = 1$ ([221], Chapter VI, §2), so $\mathrm{a}(\chi|_{I_{\bar v}}) \leq a$. The same holds for χ^{-1}, which also has level M.

Step 2: primes $v \neq \ell$. By (Lw1) there are a place $\bar v \mid v$ and characters χ_1, χ_2 of level M with $W|_{I_{\bar v}}$ conjugate to $\mathrm{diag}(\chi_1, \chi_2)$. As $v \neq \ell$, $\mathbb{Q}(\mu_{\ell^\infty})$ is unramified at v, so χ_{cyc} is trivial on $I_{\bar v}$ and $W'|_{I_{\bar v}}$ is conjugate to $\mathrm{diag}(\chi_1, \chi_2)|_{I_{\bar v}}$. By Step 1 this is trivial on an open subgroup J of $I_{\bar v}$. In (Wd1), take σ in the open subgroup $J \cap J_0$, J_0 the subgroup given there: $\exp(t_\ell(\sigma)N) = 1$, and since $t_\ell(\sigma)N$ is nilpotent, $t_\ell(\sigma)N = \log\exp(t_\ell(\sigma)N) = 0$. The continuous nonzero homomorphism t_ℓ is nonzero on the finite-index subgroup $J \cap J_0$, so $N = 0$, and then $\rho|_{I_{\bar v}} = W'|_{I_{\bar v}}$. Hence, the Artin conductor being additive on direct sums,

$$v_v(L_g) = \mathrm{a}(W'_v) = \mathrm{a}(\chi_1|_{I_{\bar v}}) + \mathrm{a}(\chi_2|_{I_{\bar v}}) \leq 2v_v(M)$$

by (Lc1) and Step 1.

Step 3: the prime ℓ: the diagonal characters. By (Lw2) there are a place $\bar\ell \mid \ell$, characters χ_1, χ_2 of $G_{\mathbb{Q}}$ of level M and characters κ_1, κ_2 of $D = D_{\bar\ell} \cong G_{\mathbb{Q}_\ell}$ such that W is triangular on D with diagonal (κ_1, κ_2) and $\kappa_1 = \chi_{\mathrm{cyc}}^e\chi_1$, $\kappa_2 = \chi_{\mathrm{cyc}}^{-e}\chi_2$ on $I_{\bar\ell}$; the κ_i are continuous, being diagonal entries of a continuous triangular representation. Then W' is triangular on D with diagonal $(\kappa'_1, \kappa'_2) = (\kappa_1\chi_{\mathrm{cyc}}^{-e}, \kappa_2\chi_{\mathrm{cyc}}^{-e})$. Put $\psi_i = \chi_i|_D$, characters of finite order, and

$$\lambda_1 = \kappa'_1\psi_1^{-1}, \qquad \lambda_2 = \kappa'_2\chi_{\mathrm{cyc}}^{2e}\psi_2^{-1}.$$

Both are trivial on $I_{\bar\ell}$, i.e. unramified, and continuous; the image of the compact group D under a continuous character lies in a compact subgroup of $E^\times$ for a finite extension $E/\mathbb{Q}_\ell$, hence in $\mathcal{O}_E^\times$, so the λ_i take unit values. Thus

$$\kappa'_1 = \psi_1\lambda_1, \qquad \kappa'_2 = \chi_{\mathrm{cyc}}^{-2e}\psi_2\lambda_2.$$

Step 4: the prime ℓ: the Weil–Deligne representation. By (Lc2), $W'_\ell = W'|_D$ is potentially semistable. We use the following standard properties of Fontaine's construction ([95], with the period rings of [94]): subrepresentations and quotients of potentially semistable representations are potentially semistable; $r \mapsto \mathrm{WD}(r)$ is exact on them and compatible with tensor products; for a character $\psi\lambda$ with ψ of finite order and λ unramified with unit values, $\mathrm{WD}(\psi\lambda)$ is one-dimensional with $N = 0$, I_ℓ acting through $\psi^\epsilon|_{I_\ell}$ and every w with $n(w) = 1$ acting by a unit, where $\epsilon \in \{\pm 1\}$ is fixed by the conventions of loc. cit.; and $\mathrm{WD}(\chi_{\mathrm{cyc}})$ is unramified with $N = 0$, an element w acting by $\ell^{\epsilon' n(w)}$ for a sign $\epsilon' \in \{\pm 1\}$ fixed by the conventions. Let $\mathrm{WD}(W'_\ell) = (V, \rho, N)$. By exactness there is a line $V_1 \subset V$, stable under ρ and N, with $V_1 \cong \mathrm{WD}(\kappa'_1)$ and $V/V_1 \cong \mathrm{WD}(\kappa'_2)$. Fix $w \in W_{\mathbb{Q}_\ell}$ with $n(w) = 1$. By Step 3 and the properties above, the eigenvalues of $\rho(w)$ are a unit u_1 (on V_1) and $u_2 = \ell^{-2e\epsilon'}u'_2$ with u'_2 a unit (on V/V_1), so

$$v_\ell(u_1) = 0, \qquad v_\ell(u_2) = -2e\epsilon' \in \{2e, -2e\}.$$

As $e \geq 1$ these differ, so V is the direct sum of the two eigenspaces of $\rho(w)$. By (38.5), $\rho(w)N = \ell N\rho(w)$, so N maps the α-eigenspace into the $\ell\alpha$-eigenspace. Since $v_\ell(\ell u_1) = 1$ and $v_\ell(\ell u_2) = 1 - 2e\epsilon'$, and neither 1 nor $1 \pm 2e$ lies in $\{0, 2e\epsilon'\}$ (as $2e \geq 2$), ℓu_1 and ℓu_2 are not eigenvalues, and $N = 0$. The restriction $\rho|_{I_\ell}$ is trivial on an open subgroup, so it factors through a finite group and is semisimple; its constituents are those of V_1 and V/V_1, namely $\psi_1^\epsilon|_{I_\ell}$ and $\psi_2^\epsilon|_{I_\ell}$ (the factors λ_i and $\chi_{\mathrm{cyc}}^{-2e}$ contribute unramified Weil–Deligne characters). Hence, with $N = 0$,

$$v_\ell(L_g) = \mathrm{a}(W'_\ell) = \mathrm{a}\big(\chi_1^\epsilon|_{I_{\bar\ell}}\big) + \mathrm{a}\big(\chi_2^\epsilon|_{I_{\bar\ell}}\big) \leq 2v_\ell(M)$$

by (Lc2) and Step 1 (applied to χ_i^{ϵ}).

Conclusion. By Steps 2 and 4, $v_v(L_g) \le v_v(M^2)$ for every prime v, so $L_g \mid M^2$. □

Lemma 38.22. *Let $k \ge 2$ and $L \ge 1$ be integers. The set $\mathrm{New}_k(L)$ of normalised newforms of weight k (of any character) whose level divides L is finite, and*

$$|\mathrm{New}_k(L)| \le 1 + \frac{kL^2}{12}.$$

Proof. Write $M_k(\Gamma_1(L)) \supseteq S_k(\Gamma_1(L))$ for the spaces of modular and cusp forms of weight k for $\Gamma_1(L)$, and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_1(L)$; every $f \in M_k(\Gamma_1(L))$ satisfies $f(z+1) = f(z)$ and has an expansion $f = \sum_{n \ge 0} a_n(f) q^n$, $q = e^{2\pi i z}$. Put $B = \lfloor kL^2/12 \rfloor$.

The map $f \mapsto (a_0(f), \dots, a_B(f))$ from $M_k(\Gamma_1(L))$ to $\mathbb{C}^{B+1}$ is injective. Let $\Gamma = \Gamma_1(L) \cup (-\Gamma_1(L))$, a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing -1 and T, of index

$$m = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma] \le [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(L)] = L^2 \prod_{p \mid L} (1 - p^{-2}) \le L^2$$

(the classical index formula, [229], Section 1.6). Let f be in the kernel and $F = f^2$. For $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, $F|_{2k}\alpha = (f|_k\alpha)^2$; so $F|_{2k}\gamma = F$ for $\gamma \in \Gamma_1(L)$, $F|_{2k}(-\gamma) = (-1)^{2k} F|_{2k}\gamma = F$, and F is holomorphic on $\mathfrak{H}$ and at every cusp because f is. Thus F is a modular form of the even weight $2k$ for Γ. Since $a_n(f) = 0$ for $n \le B$, $f = q^{B+1} h$ with h a power series in q, so $F = q^{2B+2} h^2$ and its coefficients b_n vanish for $n \le 2B+1$. As $B + 1 > kL^2/12$,

$$\frac{2k\, m}{12} \le \frac{2kL^2}{12} < 2B + 2,$$

so $b_n = 0$ for every integer n with $0 \le n \le 2km/12$. The valence bound [229], applied to Γ, the weight $2k$ and F, gives $F = 0$; hence $f = 0$.

Conclusion. By this injectivity, $\dim M_k(\Gamma_1(L)) \le B + 1 \le 1 + kL^2/12$. A normalised newform of level $L' \mid L$ lies in $S_k(\Gamma_1(L')) \subseteq S_k(\Gamma_1(L)) \subseteq M_k(\Gamma_1(L))$, and by (Nw1) of [13], [161] distinct such newforms are linearly independent. So $\mathrm{New}_k(L)$ is finite with $|\mathrm{New}_k(L)| \le \dim M_k(\Gamma_1(L)) \le 1 + kL^2/12$. □

Lemma 38.23. *Let ℓ be a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $e \ge 1$ an integer and $f = \sum_{n \ge 1} a_n q^n$ a normalised newform of weight $2e+1$, level M_f and Nebentypus ε_f. Let $W\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be a continuous representation such that $W \otimes \chi_{\mathrm{cyc}}^{-e}$ is semisimple and, for all but finitely many primes p, is unramified at p and satisfies*

$$\iota\Big(\det\big(X - (W \otimes \chi_{\mathrm{cyc}}^{-e})(\sigma^{-1})\big)\Big) = X^2 - a_p X + \varepsilon_f(p) p^{2e}$$

for every arithmetic Frobenius element σ at p. Let $\theta\colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell^{\times}$ be a continuous character of finite order. Then

$$H^1_f(\mathbb{Q}, W \otimes \theta) = 0;$$

in particular ($\theta = \mathbf{1}$) $H^1_f(\mathbb{Q}, W) = 0$.

Proof. Write $k = 2e+1$ and $F_f = \mathbb{Q}(a_n : n \geq 1)$. The embedding $\iota^{-1}|_{F_f} \colon F_f \to \overline{\mathbb{Q}}_\ell$ determines a finite place λ of F_f above ℓ and identifies F_λ with a finite extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$; write $V_\lambda = V_{F_\lambda}(f)$.

Step 1: $W \otimes \chi_{\mathrm{cyc}}^{-e} \cong V_\lambda \otimes_{F_\lambda} \overline{\mathbb{Q}}_\ell$. Let S be the finite set of primes dividing $M_f \ell$ together with the exceptional primes of the hypothesis. For $p \notin S$ and an arithmetic Frobenius σ at p, Theorem B.32(1) gives $\det(1 - \sigma^{-1}u \mid V_\lambda) = 1 - \iota^{-1}(a_p)u + \iota^{-1}(\varepsilon_f(p))p^{k-1}u^2$, so $\mathrm{tr}(\sigma^{-1} \mid V_\lambda) = \iota^{-1}(a_p) = \mathrm{tr}((W \otimes \chi_{\mathrm{cyc}}^{-e})(\sigma^{-1}))$ by the hypothesis. By Chebotarev's density theorem, for every finite Galois extension $L/\mathbb{Q}$ every element of $\mathrm{Gal}(L/\mathbb{Q})$ is the image of an arithmetic Frobenius at some $p \notin S$; hence the inverses of the arithmetic Frobenius elements at primes $p \notin S$ form a dense subset of $G_\mathbb{Q}$. Both traces are continuous on $G_\mathbb{Q}$, so they agree everywhere. The representation V_λ is irreducible (Ribet, *Galois representations attached to eigenforms with Nebentypus*, 1977, Theorem 2.3), so $V_\lambda \otimes_{F_\lambda} \overline{\mathbb{Q}}_\ell$ is semisimple (characteristic 0); $W \otimes \chi_{\mathrm{cyc}}^{-e}$ is semisimple by hypothesis. Two finite-dimensional semisimple representations of a group over a field of characteristic 0 with the same trace are isomorphic, so $W \otimes \chi_{\mathrm{cyc}}^{-e} \cong V_\lambda \otimes_{F_\lambda} \overline{\mathbb{Q}}_\ell$ and

$$W \otimes \theta \cong V_\lambda(e) \otimes_{F_\lambda} \overline{\mathbb{Q}}_\ell \otimes \theta.$$

Step 2: a finite coefficient field. Let n be the order of θ and $E = F_\lambda(\mu_n) \subset \overline{\mathbb{Q}}_\ell$, a finite extension of F_λ. Let U be the E-vector space $V_\lambda(e) \otimes_{F_\lambda} E$ with g acting by $\theta(g)$ times its action on $V_\lambda(e)$; it is continuous and E-linear, and $W \otimes \theta \cong U \otimes_E \overline{\mathbb{Q}}_\ell$. An isomorphism of representations is a continuous linear $G_\mathbb{Q}$-equivariant bijection and carries Bloch–Kato cocycles and coboundaries to Bloch–Kato cocycles and coboundaries (tensored with B_{cris} at the places above ℓ), so it suffices to show $H^1_f(\mathbb{Q}, U \otimes_E \overline{\mathbb{Q}}_\ell) = 0$.

Step 3: descent to a finite extension. Let c be a Bloch–Kato cocycle with values in $U \otimes_E \overline{\mathbb{Q}}_\ell$.

(a) *c takes values in $U \otimes_E E'$ for a finite extension E'/E in $\overline{\mathbb{Q}}_\ell$.* For $d \geq 1$ let E_d be the compositum of the finitely many extensions of E of degree at most d in $\overline{\mathbb{Q}}_\ell$ (Krasner); E_d is finite over E and $\overline{\mathbb{Q}}_\ell = \bigcup_d E_d$. Each $U \otimes_E E_d$ is a finite-dimensional space over the complete field E_d, hence closed in $U \otimes_E \overline{\mathbb{Q}}_\ell$, and $G_\mathbb{Q}$-stable. So $G_\mathbb{Q} = \bigcup_d c^{-1}(U \otimes_E E_d)$ is a countable union of closed sets; $G_\mathbb{Q}$ is compact Hausdorff, so by Baire's theorem one of them, say for d, contains a coset gH of an open normal subgroup H. For $h \in H$, $c(h) = g^{-1}(c(gh) - c(g)) \in U \otimes_E E_d$. Let $g_1, \ldots, g_s$ represent $G_\mathbb{Q}/H$ and let $E' \supseteq E_d$ be a finite extension containing the coordinates of $c(g_1), \ldots, c(g_s)$ in an E-basis of U. Then $c(g_i h) = c(g_i) + g_i c(h) \in U \otimes_E E'$ for all i and $h \in H$.

(b) *c is a Bloch–Kato cocycle of $U' = U \otimes_E E'$.* Choose an E'-linear map $\pi \colon \overline{\mathbb{Q}}_\ell \to E'$ with $\pi|_{E'} = \mathrm{id}$, and let $P = \mathrm{id}_U \otimes \pi \colon U \otimes_E \overline{\mathbb{Q}}_\ell \to U'$; P is $G_\mathbb{Q}$-equivariant (the group acts on U only) and is the identity on U'. If $c(g) = gx - x$ on an inertia group, then $c(g) = P(c(g)) = gP(x) - P(x)$ there; at ℓ, the $\mathbb{Q}_\ell$-linear and equivariant map $P \otimes \mathrm{id}$ to $U' \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ fixes $c(g) \otimes 1$ and carries a y with $c(g) \otimes 1 = gy - y$ to such an element for U'; at the archimedean place, $c|_D = gx - x$ gives $c|_D = gP(x) - P(x)$. The topology of U' is the subspace topology, so c is continuous into U'.

Hence it suffices to show $H^1_f(\mathbb{Q}, U') = 0$ for every finite extension E' of E in $\overline{\mathbb{Q}}_\ell$: then $c = gz - z$ with $z \in U'$.

Step 4: restriction to the field of θ. Let $K \subset \overline{\mathbb{Q}}$ be the fixed field of $\ker\theta$; as θ is continuous of finite order, $K/\mathbb{Q}$ is finite and abelian. Let c be a Bloch–Kato cocycle of U' with $c|_{G_K} = (g \mapsto gx - x)$ for some $x \in U'$. Then $c' = c - (g \mapsto gx - x)$ vanishes on G_K, so $c'(gh) = c'(g)$ and $c'(hg) = hc'(g)$ for $h \in G_K$; since $hg = g(g^{-1}hg)$ with $g^{-1}hg \in G_K$, also $c'(hg) = c'(g)$, so c' takes values in U'^{G_K} and factors through $\Gamma = \mathrm{Gal}(K/\mathbb{Q})$. With $s = |\Gamma|$ and $x' = s^{-1}\sum_{\gamma\in\Gamma} c'(\gamma)$,

summing $c'(g\gamma) = c'(g) + gc'(\gamma)$ over γ gives $c'(g) = x' - gx'$. So restriction $H^1(\mathbb{Q}, U') \to H^1(K, U')$ is injective. It maps Bloch–Kato cocycles to Bloch–Kato cocycles: an embedding j over K is an embedding over $\mathbb{Q}$, the decomposition and inertia groups of K for j are contained in those of $\mathbb{Q}$ for j, and the elements x, y of Definition 22.2 for $\mathbb{Q}$ serve for K (with the same B_{cris}); at archimedean places the zero class restricts to zero. Hence $H^1_f(\mathbb{Q}, U')$ embeds in $H^1_f(K, U')$.

Step 5: Kato's theorem. On G_K the character θ is trivial, so $U'|_{G_K} = V_\lambda(e) \otimes_{F_\lambda} E'$. An F_λ-basis $b_1, \dots, b_m$ of E' gives a G_K-equivariant F_λ-linear homeomorphism $U' \cong V_\lambda(e)^m$; the conditions of Definition 22.2 do not depend on the coefficient field and are compatible with the inclusions and projections of a finite direct sum, so $H^1_f(K, U') \cong H^1_f(K, V_\lambda(e))^m$. Choose a $G_{\mathbb{Q}}$-stable $\mathcal{O}_\lambda$-lattice $T \subset V_\lambda(e)$ (for a lattice T_0 the stabiliser is open, and $T = \sum_g gT_0$ over representatives of the finitely many cosets is stable). Since $k = 2e + 1$, the twist $r = e$ satisfies $1 \le r \le k - 1$ and $r \ne k/2$; so $\mathrm{Sel}(K, T)$ is finite by Theorem B.32(2), and $H^1_f(K, V_\lambda(e)) = 0$ by Lemma 38.24 (with $F = K$, $L = F_\lambda$). By Steps 4 and 2–3, $H^1_f(\mathbb{Q}, U') = 0$ and $H^1_f(\mathbb{Q}, W \otimes \theta) = 0$. □

Lemma 38.24. *Let $F \subset \overline{\mathbb{Q}}$ be a number field, ℓ a prime, $L \subset \overline{\mathbb{Q}}_\ell$ a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_L$, V a finite-dimensional L-vector space with a continuous L-linear action of G_F, and $T \subset V$ a G_F-stable $\mathcal{O}_L$-lattice. If $\mathrm{Sel}(F, T)$ is finite, then $H^1_f(F, V) = 0$.*

Proof. Let $m = |\mathrm{Sel}(F, T)|$ and let c be a Bloch–Kato cocycle. We show that c is a coboundary. Fix an $\mathcal{O}_L$-basis of T and the sup norm $\|\cdot\|$ on V in this basis, so that T is the closed unit ball; for $n \ge 0$ put $T_n = \ell^n m^{-1} T$, whose elements have norm at most $\ell^{-n}|m|_\ell^{-1}$.

Step 1: the reductions of $\ell^{-n}c$ lie in $\mathrm{Sel}(F, T)$. Let $n \ge 0$. The map $c_n \colon g \mapsto \ell^{-n}c(g) \bmod T$ is a continuous cocycle with values in the discrete module V/T (T is open in V). For a place v and a decomposition group D at v, the class of $\ell^{-n}c|_D$ lies in $H^1_f(D, V)$: the conditions of Definition 22.2 are preserved by multiplication by ℓ^{-n} (replace x by $\ell^{-n}x$, y by $\ell^{-n}y$; at archimedean v the class of $c|_D$ is 0, hence so is that of $\ell^{-n}c|_D$). The class of $c_n|_D$ is its image in $H^1(D, V/T)$. Hence the class of c_n lies in $\mathrm{Sel}(F, T)$ and is killed by m.

Step 2: approximate primitives. By Step 1 there is $z_n \in V/T$ with $m\ell^{-n}c(g) - (g-1)\hat{z}_n \in T$ for all $g \in G_F$, where $\hat{z}_n \in V$ lifts z_n. With $w_n = \ell^n m^{-1}\hat{z}_n$,

$$c(g) - (g-1)w_n \in T_n \qquad (g \in G_F,\ n \ge 0). \tag{38.6}$$

Step 3: convergence. Let $V_0 = V^{G_F} = \bigcap_g \ker(g-1)$. As V is finite-dimensional, there are $g_1, \dots, g_s \in G_F$ with $V_0 = \bigcap_i \ker(g_i - 1)$, so $[w] \mapsto ((g_i - 1)w)_i$ is an injective L-linear map $V/V_0 \to V^s$. Give V/V_0 the quotient norm; all norms on a finite-dimensional L-space are equivalent, so there is $\kappa > 0$ with $\|[w]\| \le \kappa \max_i \|(g_i - 1)w\|$ for all $w \in V$. For $n \le n'$, (38.6) gives $(g-1)(w_n - w_{n'}) \in T_n + T_{n'} = T_n$ for every g, hence $\|[w_n - w_{n'}]\| \le \kappa\ell^{-n}|m|_\ell^{-1}$. So $([w_n])_n$ is a Cauchy sequence in the complete space V/V_0; let $[w]$ be its limit. For each g the map $[u] \mapsto (g-1)u$ is a well-defined continuous linear map $V/V_0 \to V$, so $(g-1)w_n \to (g-1)w$, while $\|c(g) - (g-1)w_n\| \le \ell^{-n}|m|_\ell^{-1} \to 0$ by (38.6). Hence $c(g) = (g-1)w$ for all $g \in G_F$, and the class of c is 0. □

38.2 The counts in the remaining cases

This subsection proves the counts in the cases (II)–(V), each for one character, with a bound polynomial in $\ell N_{\mathfrak{h}} e$. In cases (II) and (III) the congruences are formed on GSp_4 over $\mathbb{Q}$. The Eisenstein trace is $(1 + \Psi)(1 + \Psi_{\mathrm{aux}})$, with $\Psi = \chi\chi_{\mathrm{cyc}}^e$ and an auxiliary character $\Psi_{\mathrm{aux}} = \chi_{\mathrm{cyc}}\omega_\ell^{-1}\chi_{\mathrm{dy}}\chi_{\mathrm{pr}}$ whose 2-power part χ_{dy} has a large conductor, and the congruence numbers of

Proposition 38.28 turn the symplectic one-character bound Proposition 44.3 into Counts II and III (Proposition 38.26). In case (V), when the 2-component of χ has a large conductor, $R_1(\ell,\chi) \le 2^{c_2(\chi)}$, the character gives a dyadic Eisenstein datum (Lemma 38.33), and the congruences are formed on the unitary group in three variables attached to $\mathbb{Q}(i)$, with the congruence numbers of Proposition 38.6 for $K=\mathbb{Q}(i)$ (Proposition 46.2, Proposition 38.32).

The other characters at the bad primes need no congruences. In the rest of case (V) the twist is $e=1$ and the order of χ is bounded in terms of ψ, so χ factors through an abelian field of bounded degree containing μ_ℓ; the strict Tate–Shafarevich group of χ is bounded through that of this field (Lemma 38.37), and the bound for second cohomology of Section 40, at the single twist $e=1$, is polynomial in the conductor (Proposition 38.35). A character with $\chi^{(\ell)}$ of odd order, in particular every character of case (IV), factors through $(\mathbb{Z}/N_\psi\mathbb{Z})^\times$ (Lemma 38.34), and its groups, summed over the twists $1 \le e < t$, are bounded through those of the cyclotomic field $\mathbb{Q}(\zeta_{\ell^{j+1}N_\psi})$, for which Section 39 supplies the layer bound (Proposition 38.38).

Definition 38.25. Let ℓ be a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism of fields, and $e, M \ge 1$ integers. A continuous representation $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ is **admissible symplectic of weight e and level M** (for ι) if

(Sy1) r is irreducible;

(Sy2) for every imaginary quadratic field $K \subset \overline{\mathbb{Q}}$ such that $r|_{G_K}$ is irreducible, $r|_{G_K} \leftrightarrow_\iota \pi$ for some regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_4(\mathbb{A}_K)$; and r satisfies the witness conditions over imaginary quadratic fields;

(Sy3) r has level M and weights $(e+1, e, 1, 0)$;

(Sy4) there are an integer $M' \ge 1$, a character χ' of $G_{\mathbb{Q}}$ of level M' and a matrix $P \in \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ such that
$$ {}^t r(s)\, P\, r(s) = \chi_{\mathrm{cyc}}(s)^{e+1}\chi'(s)\, P \qquad \text{for all } s \in G_{\mathbb{Q}};$$

(Sy5) $\ell \mid M$, and for every place $\bar\ell$ of $\overline{\mathbb{Q}}$ above ℓ the representation r is triangular on $D_{\bar\ell}$ with some diagonal $(\kappa_1,\dots,\kappa_4)$ such that $\kappa_i \ne \kappa_j\, \chi_{\mathrm{cyc}}|_{D_{\bar\ell}}$ for all i,j.

Condition (Sy4) says that r preserves a non-degenerate bilinear form up to the similitude character $\chi_{\mathrm{cyc}}^{e+1}\chi'$. These are the representations whose Eisenstein congruences on GSp_4 bound the strict Tate–Shafarevich groups of the characters in cases (II) and (III).

Proposition 38.26. *Let $\psi \in \mathcal{P}$. There is an integer $C \ge 0$ such that for every Sha-host $\mathfrak{h}$ of ψ, every integer $t \ge 1$ at which $\mathfrak{h}$ is big, every odd prime ℓ, every coefficient ring $\mathcal{O}$, every integer e with $0 < e < t$ and every $\chi \in X_{\mathfrak{h}}(\mathcal{O}, e)$ that is in case (II) or in case (III) for (ℓ, e), the group $\mathrm{Sha}^1_{\ell N_{\mathfrak{h}}}(\chi, e)$ has finite length and*
$$\nu_{\mathcal{O}}(\chi, e)\log \ell \le (\ell N_{\mathfrak{h}} e)^C.$$

Proof. Let $\psi \in \mathcal{P}$ and write $p_1 = p_1(\psi)$. Let $c, A \ge 0$ be the integers given by Proposition 38.28, put $K_\psi = 12c\,(128p_1^2)^{A+2}$, choose an integer $j \ge 0$ with $K_\psi \le 96^j$, and put $C = 12 + 7A + j$; C depends only on ψ. Fix $\mathfrak{h} = (a,q,d)$, t, ℓ, $\mathcal{O}$, e and χ as in the statement, write $N = N_{\mathfrak{h}}$ and $X_0 = \ell N e$.

Step 1: $e \ge 2$ and $X_0 \ge 96$. In case (II), $e \ge 2$ by definition. In case (III), $\ell \notin B_\psi$, so $\ell \ne 3$ (as $3 \in B_\psi$) and $\ell \ge 5$; then $4 \le \ell - 1$ divides $2e$, so $e \ge 2$. Since $a \ge 2$, $16 \mid 2^{a+2} \mid N$; with $\ell \ge 3$, $X_0 \ge 3 \cdot 16 \cdot 2 = 96$.

Step 2: size of the level. Let n', p', m and $M = \ell 2^{n'} N p'$ be as in Proposition 44.3. We show

$$M \le 128\, p_1^2 X_0^6. \tag{38.7}$$

First, $4^{n_0(\ell,\chi)} \le 16\ell^4 N^2$. Indeed, let χ' be the part of χ of order prime to ℓ; the 2-component of χ' is the image of χ' under a group homomorphism, so its order divides that of χ', and $\vartheta = \chi'(\chi_2')^{-1}$ has order dividing $\operatorname{ord}\chi'$, which is prime to ℓ and divides the exponent of $(\mathbb{Z}/N\mathbb{Z})^\times$, at most N. So ℓ is prime to $L = \operatorname{lcm}(4, \operatorname{ord}\vartheta) \le 4N$, and the order f of ℓ modulo L satisfies $1 \le f < L \le 4N$. Write $f = 2^s u$ with u odd. For odd integers y and u, $(y^u - 1)/(y-1) = 1 + y + \cdots + y^{u-1}$ is a sum of u odd numbers, hence odd, so $v_2(y^u - 1) = v_2(y-1)$. If $s = 0$, $v_2(\ell^f - 1) = v_2(\ell - 1)$ and $2^{v_2(\ell^f-1)} \le \ell$. If $s \ge 1$, put $y = \ell^u$; then $\ell^f - 1 = (y^2-1)\prod_{i=1}^{s-1}(y^{2^i}+1)$, each y^{2^i} $(i \ge 1)$ is an odd square, hence $\equiv 1 \pmod 8$, so $v_2(y^{2^i}+1) = 1$, and $v_2(y^2-1) = v_2((\ell^2)^u - 1) = v_2(\ell^2 - 1)$; hence $2^{v_2(\ell^f-1)} = 2^{v_2(\ell^2-1)} 2^{s-1} \le \ell^2 f$. In both cases $2^{n_0(\ell,\chi)} \le \ell^2 f \le 4\ell^2 N$, which gives the claim. Next, $2^{c_2(\chi)} \le N$, since the conductor of the 2-component of χ divides the 2-part of N; and $o_\psi \le N_\psi \le N$, as o_ψ divides $\varphi(N_\psi)$ and $N_\psi \mid N$. For an integer $Y \ge 1$, the least $n \ge 0$ with $2^n \ge Y$ satisfies $2^n < 2Y$. In case (II), $\ell N 4^{n_0(\ell,\chi)} \le 16\ell^5 N^3$ and $2^{c_2(\chi)+6} o_\psi \ell^3 N_\psi^2 \le 64\ell^3 N^4$, so $2^{n'} < 128\ell^5 N^4$. In case (III), $\ell N_\psi p_1 4^{n_0(\ell,\chi)} \le 16\ell^5 N^3 p_1$ and $8\ell^2 \le 16\ell^5 N^3 p_1$, so $2^{n'} < 32\ell^5 N^3 p_1$. In both cases $2^{n'} \le 128 p_1 \ell^5 N^4$ and $p' \le p_1$, so $M \le 128 p_1^2 \ell^6 N^5 \le 128 p_1^2 X_0^6$.

Step 3: enlarging the coefficient ring. Let L_0 be the fraction field of $\mathcal{O}$, $\zeta \in \overline{\mathbb{Q}}_\ell$ a primitive $2^{n'}$-th root of unity and $\mathcal{O}'$ the ring of integers of the finite extension $L_0(\zeta)$ of $\mathbb{Q}_\ell$; it is a coefficient ring containing $\mathcal{O}$ and ζ. Viewed with values in $\mathcal{O}'^\times$, χ lies in $X_\mathfrak{h}(\mathcal{O}', e)$ (the conditions $H_\mathfrak{h} \subseteq \ker\chi$ and $\chi(-1) = (-1)^e$ do not involve the ring), and it is in the same case with the same $n_0(\ell,\chi)$, $c_2(\chi)$ and orders, all of which depend only on χ as a $\overline{\mathbb{Q}}_\ell^\times$-valued character. By Lemma 38.27, item 2, applied to $\mathcal{O}$ and $\mathcal{O}'$, $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length over $\mathcal{O}$ if and only if it has over $\mathcal{O}'$, and then $\nu_{\mathcal{O}}(\chi, e) = \nu_{\mathcal{O}'}(\chi, e)$. Hence it suffices to prove the claim with $\mathcal{O}'$ in place of $\mathcal{O}$, and we assume from now on that $\mathcal{O}$ contains a primitive $2^{n'}$-th root of unity.

Step 4: the symplectic datum. By Proposition 44.3, applied to $\psi, \mathfrak{h}, t, \ell, e, \mathcal{O}, \chi$, there are χ_{dy} (primitive modulo $2^{n'}$), χ_{pr} (modulo p', of order m) and ι with the property stated there; let $\Psi = \Psi_{\chi,e}$, Ψ_{aux} and T_{Eis} be as there.

Step 5: an auxiliary prime and its Frobenius. Let p_0 be a prime factor of $6M - 1 \ge 5$. Then $p_0 \nmid 6M$; so $p_0 \ge 5$, $p_0 \nmid M$, and p_0 divides none of $\ell, N, 2^{n'}, p'$, which divide M; also $p_0 \le 6M$. Choose a place $\bar{p}$ of $\overline{\mathbb{Q}}$ above p_0 and an arithmetic Frobenius $\sigma \in G_\mathbb{Q}$ at $\bar{p}$. Reduction modulo $\bar{p}$ is injective on the roots of unity of order prime to p_0, and σ acts on the residue field as $x \mapsto x^{p_0}$; hence $\sigma(\zeta') = \zeta'^{p_0}$ for every root of unity ζ' of order prime to p_0. Therefore $\chi_{\mathrm{cyc}}(\sigma) = p_0$, and $\phi(\sigma) = \phi(p_0)$ for every Dirichlet character ϕ whose modulus divides M; and $w = \omega_\ell(\sigma)$ satisfies $w^{\ell-1} = 1$. Put

$$y_2 = \chi(p_0), \qquad y_3 = w^{-1}\chi_{\mathrm{dy}}(p_0)\chi_{\mathrm{pr}}(p_0), \qquad y_1 = y_2 y_3.$$

Then $\Psi(\sigma) = p_0^e y_2$, $\Psi_{\mathrm{aux}}(\sigma) = p_0 y_3$, and

$$T_{\mathrm{Eis}}(\sigma) = 1 + \Psi(\sigma) + \Psi_{\mathrm{aux}}(\sigma) + \Psi(\sigma)\Psi_{\mathrm{aux}}(\sigma) = p_0^{e+1} y_1 + p_0^e y_2 + p_0 y_3 + 1.$$

Let $k = \varphi(N)\varphi(2^{n'})\varphi(p')(\ell - 1)$. The values of a Dirichlet character modulo n_1 are $\varphi(n_1)$-th roots of unity, so $y_1^k = y_2^k = y_3^k = 1$; and $1 \le k \le N 2^{n'} p' \ell = M$. Put $z_i = \iota(y_i)$; then $z_i^k = 1$ and, ι being a field isomorphism fixing $\mathbb{Z}$, $E = p_0^{e+1} z_1 + p_0^e z_2 + p_0 z_3 + 1$ satisfies $\iota^{-1}(E) = T_{\mathrm{Eis}}(\sigma)$.

Step 6: congruences. Apply Proposition 38.28 to $(e, M, p_0, k, z_1, z_2, z_3)$ (here $e \ge 2$ by Step 1, $p_0 \ge 5$, $p_0 \nmid M$); it gives a nonzero integer R with $\log|R| \le c\varphi(k)(M(e+1))^A \log p_0$. Let r be admissible symplectic of weight e and level M for ι. Item 1 of Proposition 38.28 for $X = \{r\}$ (ℓ is odd and $\ell \ne p_0$) gives $\operatorname{tr} r(\sigma) \ne \iota^{-1}(E) = T_{\mathrm{Eis}}(\sigma)$. So σ satisfies the hypothesis of the property in Proposition 44.3: $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length, which is the first assertion, and there is a finite

set X of pairwise non-isomorphic admissible symplectic representations of weight e and level M with $\nu_{\mathcal{O}}(\chi,e) \le 2\sum_{r\in X} v_\ell(\operatorname{tr} r(\sigma) - T_{\mathrm{Eis}}(\sigma))$. Item 2 of Proposition 38.28 for this X gives

$$\nu_{\mathcal{O}}(\chi,e) \le 2\, v_\ell(R).$$

Step 7: the estimate. As $\ell^{v_\ell(R)}$ divides R, $v_\ell(R)\log\ell \le \log|R|$. With $\varphi(k) \le k \le M$, $\log p_0 \le p_0 \le 6M$, $e+1 \le X_0$ and (38.7),

$$\nu_{\mathcal{O}}(\chi,e)\log\ell \le 2c\, M\big(M(e+1)\big)^A 6M \le 12c\,(128p_1^2)^{A+2}X_0^{12+7A} = K_\psi X_0^{12+7A} \le X_0^C = (\ell N_{\mathfrak{h}} e)^C,$$

using $M(e+1) \le 128p_1^2X_0^7$, $M^2 \le (128p_1^2)^2X_0^{12}$ and $K_\psi \le 96^j \le X_0^j$ (Step 1). This is the claim. □

Lemma 38.27. *Let ℓ be a prime, $N, n \ge 1$ and $e \ge 0$ integers, and χ a Dirichlet character modulo N with values in $\overline{\mathbb{Q}}_\ell^\times$.*

1. *If the order of χ is prime to ℓ, there is a coefficient ring $\mathcal{O}'$ with $e_{\mathcal{O}'} = 1$ containing all values of χ.*
2. *If $\mathcal{O}_1, \mathcal{O}_2$ are coefficient rings containing all values of χ, then $\mathrm{Sha}^1_n(\chi,e)_{\mathcal{O}_1}$ has finite length if and only if $\mathrm{Sha}^1_n(\chi,e)_{\mathcal{O}_2}$ has, and in that case, for $n = \ell N$, $\nu_{\mathcal{O}_1}(\chi,e) = \nu_{\mathcal{O}_2}(\chi,e)$; more precisely $\operatorname{length}_{\mathcal{O}_1} \mathrm{Sha}^1_n(\chi,e)_{\mathcal{O}_1}/e_{\mathcal{O}_1} = \operatorname{length}_{\mathcal{O}_2} \mathrm{Sha}^1_n(\chi,e)_{\mathcal{O}_2}/e_{\mathcal{O}_2}$ in $[0,\infty]$.*

Proof. Item 1. Let m be the order of χ; its values are m-th roots of unity. Since $\ell \nmid m$, $X^m - 1$ is separable modulo ℓ, so $\mathbb{Q}_\ell(\mu_m) \subset \overline{\mathbb{Q}}_\ell$ is unramified over $\mathbb{Q}_\ell$ and its ring of integers $\mathcal{O}' = \mathbb{Z}_\ell[\mu_m]$ is a discrete valuation ring, finite over $\mathbb{Z}_\ell$, in which ℓ is a uniformiser; so $e_{\mathcal{O}'} = 1$ and $\mathcal{O}'$ contains the values of χ.

Reduction for item 2. Let L_i be the fraction field of $\mathcal{O}_i$ and $\mathcal{O}''$ the ring of integers of the compositum $L_1L_2 \subset \overline{\mathbb{Q}}_\ell$, a coefficient ring containing $\mathcal{O}_1$ and $\mathcal{O}_2$. It suffices to prove, for a coefficient ring $\mathcal{O}$ containing the values of χ and a coefficient ring $\mathcal{O}'' \supseteq \mathcal{O}$, that $\operatorname{length}_{\mathcal{O}''} \mathrm{Sha}''/e_{\mathcal{O}''} = \operatorname{length}_{\mathcal{O}} \mathrm{Sha}/e_{\mathcal{O}}$ in $[0,\infty]$, where $\mathrm{Sha} = \mathrm{Sha}^1_n(\chi,e)_{\mathcal{O}}$ and $\mathrm{Sha}'' = \mathrm{Sha}^1_n(\chi,e)_{\mathcal{O}''}$; both lengths are then finite or infinite together.

The modules. Let e'' be the ramification index and f'' the residue degree of $\mathcal{O}''$ over $\mathcal{O}$, so that $\mathcal{O}''$ is a free $\mathcal{O}$-module of rank $\mathrm{e}''\mathrm{f}''$, and $e_{\mathcal{O}''} = \mathrm{e}'' e_{\mathcal{O}}$ (if ϖ, ϖ'' are uniformisers, ℓ is $\varpi^{e_{\mathcal{O}}}$ times a unit and ϖ is $\varpi''^{\mathrm{e}''}$ times a unit). Put $\theta = \Psi_{\chi,e}^{-1}$, with values in $\mathcal{O}^\times$. A basis of $\mathcal{O}''$ over $\mathcal{O}$ gives an isomorphism of $\mathcal{O}$-modules $\mathcal{O}'' \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \cong (\mathcal{O} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\mathrm{e}''\mathrm{f}''}$, which commutes with multiplication by elements of $\mathcal{O}$ and hence with the action of $G_{\mathbb{Q}}$ through θ: $\mathcal{O}''^\vee(\theta) \cong \mathcal{O}^\vee(\theta)^{\mathrm{e}''\mathrm{f}''}$ as $\mathcal{O}[G_{\mathbb{Q}}]$-modules. Continuous cohomology with discrete coefficients commutes with finite direct sums, compatibly with restriction to $G_{\mathbb{Q}_p}$ and to I_p; therefore the kernels defining the strict Tate–Shafarevich groups correspond, and $\mathrm{Sha}'' \cong \mathrm{Sha}^{\mathrm{e}''\mathrm{f}''}$ as $\mathcal{O}$-modules. Hence $\operatorname{length}_{\mathcal{O}} \mathrm{Sha}'' = \mathrm{e}''\mathrm{f}'' \cdot \operatorname{length}_{\mathcal{O}} \mathrm{Sha}$ in $[0,\infty]$.

Lengths over $\mathcal{O}$ and $\mathcal{O}''$. The simple $\mathcal{O}''$-module $\mathcal{O}''/\varpi''\mathcal{O}''$ has $\mathcal{O}$-length f''; so an $\mathcal{O}''$-module of finite $\mathcal{O}''$-length λ has $\mathcal{O}$-length $\mathrm{f}''\lambda$ (along a composition series), and an $\mathcal{O}''$-module of infinite $\mathcal{O}''$-length has infinite $\mathcal{O}$-length (it has $\mathcal{O}''$-submodule chains of every finite length). Thus $\operatorname{length}_{\mathcal{O}''} \mathrm{Sha}'' = \operatorname{length}_{\mathcal{O}} \mathrm{Sha}''/\mathrm{f}'' = \mathrm{e}'' \operatorname{length}_{\mathcal{O}} \mathrm{Sha}$, and

$$\frac{\operatorname{length}_{\mathcal{O}''} \mathrm{Sha}''}{e_{\mathcal{O}''}} = \frac{\mathrm{e}'' \operatorname{length}_{\mathcal{O}} \mathrm{Sha}}{\mathrm{e}'' e_{\mathcal{O}}} = \frac{\operatorname{length}_{\mathcal{O}} \mathrm{Sha}}{e_{\mathcal{O}}}.$$

□

Proposition 38.28. *There are integers $c, A \geq 0$ with the following property. Let $e \geq 2$ and $M \geq 1$ be integers, $p \geq 5$ a prime with $p \nmid M$, $k \geq 1$ an integer and $z_1, z_2, z_3 \in \mathbb{C}$ with $z_1^k = z_2^k = z_3^k = 1$, and put*

$$E = p^{e+1} z_1 + p^e z_2 + p\, z_3 + 1.$$

Then there is a nonzero integer R with

$$\log |R| \leq c\, \varphi(k)\, \big(M(e+1)\big)^A \log p$$

such that for every odd prime $\ell \neq p$, every isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, every place $\bar{p}$ of $\overline{\mathbb{Q}}$ above p, every arithmetic Frobenius $\sigma \in G_{\mathbb{Q}}$ at $\bar{p}$ and every finite set X of pairwise non-isomorphic representations $G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ admissible symplectic of weight e and level M (for ι):

1. *$\operatorname{tr} r(\sigma) \neq \iota^{-1}(E)$ for every $r \in X$;*
2. *$v_\ell(R) \geq \displaystyle\sum_{r \in X} v_\ell\big(\operatorname{tr} r(\sigma) - \iota^{-1}(E)\big)$.*

Proof. Let c_0, A_0 be the integers given by Proposition 38.29, and put $c = 2^{A_0+1} c_0$, $A = A_0 + 1$. Fix $e, M, p, k, z_1, z_2, z_3$ as in the statement. As $p \nmid M$, Proposition 38.29 gives a finite set S and $t\colon S \to \mathbb{C}$ with (Tr1)–(Tr4) for (e, M, p). Let $U = (\mathbb{Z}/k\mathbb{Z})^\times$; for $j \in U$ and a k-th root of unity y the power y^j is well defined. For $j \in U$ and $s \in S$ put

$$E_j = p^{e+1} z_1^j + p^e z_2^j + p\, z_3^j + 1, \qquad F(s, j) = t(s) - E_j,$$

so that $E_1 = E$.

Step 1: size of $F(s, j)$. Put $x = p^{(e+1)/2}$. As $e \geq 2$ and $p \geq 5$, $x \geq p^{3/2} \geq \sqrt{5}\, p \geq 2p \geq p + 1$ and $x \geq 5\sqrt{5} > 25/4$; also $p^e = x^2/p \leq x^2/5$. Since $|z_i^j| = 1$,

$$|E_j| \geq p^{e+1} - p^e - p - 1 \geq x^2 - \tfrac{1}{5}x^2 - x = \tfrac{4}{5}x^2 - x > (2\sqrt{p} + 2)x \geq |t(s)|$$

by (Tr2); the strict inequality holds because $\frac{4}{5}x \geq \frac{4}{5}p\sqrt{p} \geq 4\sqrt{p} \geq 2\sqrt{p} + 2\sqrt{5} > 2\sqrt{p} + 3$. Hence $F(s, j) \neq 0$. Moreover $|F(s, j)| \leq (2\sqrt{p} + 2)x + x^2 + \frac{1}{5}x^2 + x < \frac{6}{5}x^2 + \frac{4}{5}x^2 = 2x^2 = 2p^{e+1}$, since $(2\sqrt{p} + 3)x < \frac{4}{5}x^2$; so $\log |F(s, j)| \leq \log 2 + (e+1)\log p \leq 2(e+1)\log p$.

Step 2: integrality. By (Tr2) $t(s)$ is an algebraic integer, and z_i^j are roots of unity; so $F(s, j)$ is an algebraic integer.

Step 3: the integer R. Let $\sigma' \in \mathrm{Aut}(\mathbb{C})$. It maps the cyclic group μ_k of k-th roots of unity onto itself by a group automorphism, so there is $b \in U$ with $\sigma'(y) = y^b$ for all $y \in \mu_k$; as σ' fixes p, $\sigma'(E_j) = E_{bj}$, and with $u_{\sigma'}$ from (Tr1), $\sigma'(F(s, j)) = t(u_{\sigma'}(s)) - E_{bj} = F(u_{\sigma'}(s), bj)$. Since $(s, j) \mapsto (u_{\sigma'}(s), bj)$ is a bijection of $S \times U$, the product

$$R = \prod_{(s,j) \in S \times U} F(s, j)$$

satisfies $\sigma'(R) = R$. A complex number fixed by every automorphism of $\mathbb{C}$ is rational; R is an algebraic integer (Step 2), hence $R \in \mathbb{Z}$, and $R \neq 0$ by Step 1 ($R = 1$ if $S \times U$ is empty). By Step 1, (Tr3) and $|U| = \varphi(k)$,

$$\log |R| \leq |S|\, \varphi(k)\, 2(e+1)\log p \leq c_0 \big(M(e+2)\big)^{A_0} \varphi(k)\, 2(e+1)\log p \leq c\, \varphi(k) \big(M(e+1)\big)^A \log p.$$

Step 4: the two properties. Let $\ell \neq p$ be an odd prime, ι, $\bar{p}$, σ and X as in the statement, and λ the map given by (Tr4) for $(\ell, \iota, \bar{p}, \sigma)$. For $r \in X$, $\iota(\operatorname{tr} r(\sigma)) = t(\lambda(r))$, so, ι^{-1} being a field isomorphism,

$$\operatorname{tr} r(\sigma) - \iota^{-1}(E) = \iota^{-1}\big(t(\lambda(r)) - E_1\big) = \iota^{-1}\big(F(\lambda(r), 1)\big),$$

which is non-zero by Step 1. This proves item 1. If $\lambda(r) = \lambda(r')$ for $r, r' \in X$, then $r \cong r'$ by (Tr4), so $r = r'$; thus $r \mapsto (\lambda(r), 1)$ is injective on X. Since ι^{-1} fixes $\mathbb{Z}$,

$$v_\ell(R) = \sum_{(s,j)\in S\times U} v_\ell\big(\iota^{-1}(F(s,j))\big).$$

Each $y = \iota^{-1}(F(s,j))$ is a root of a monic polynomial $X^n + a_{n-1}X^{n-1} + \cdots + a_0$ with $a_i \in \mathbb{Z}$ (Step 2), so $v_\ell(y) \geq 0$: otherwise $v_\ell(y^n) < v_\ell(a_i y^i)$ for all $i < n$, which contradicts $y^n = -\sum_{i<n} a_i y^i$. Dropping the terms with (s,j) outside the image of $r \mapsto (\lambda(r), 1)$,

$$v_\ell(R) \geq \sum_{r\in X} v_\ell\big(\iota^{-1}(F(\lambda(r),1))\big) = \sum_{r\in X} v_\ell\big(\operatorname{tr} r(\sigma) - \iota^{-1}(E)\big),$$

which is item 2. □

Proposition 38.29. *There are integers $c, A \geq 0$ with the following property. For all integers $e \geq 2$ and $M \geq 1$ and every prime p with $p \nmid M$ there are a finite set S and a map $t\colon S \to \mathbb{C}$ such that*

(Tr1) *for every $\sigma \in \operatorname{Aut}(\mathbb{C})$ there is a bijection u_σ of S with $t(u_\sigma(s)) = \sigma(t(s))$ for all $s \in S$;*

(Tr2) *for every $s \in S$, $t(s)$ is an algebraic integer and $|t(s)| \leq (2\sqrt{p} + 2)\, p^{(e+1)/2}$;*

(Tr3) *$|S| \leq c\,(M(e+2))^A$;*

(Tr4) *for every odd prime $\ell \neq p$, every isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, every place $\bar{p}$ of $\overline{\mathbb{Q}}$ above p and every arithmetic Frobenius $\sigma \in G_{\mathbb{Q}}$ at $\bar{p}$, there is a map λ from the set of representations $G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ that are admissible symplectic of weight e and level M (for ι) to S such that $\lambda(r) = \lambda(r')$ implies $r \cong r'$, and $\iota\big(\operatorname{tr} r(\sigma)\big) = t(\lambda(r))$ for every such r.*

Proof. Step 0: auxiliary fields and constants. Let $p_1 < \cdots < p_{18}$ be the first eighteen primes $\equiv 3 \pmod 4$ and $K_j = \mathbb{Q}(\sqrt{-p_j})$, with discriminant $-p_j$; the discriminants are pairwise coprime. Let $\delta_j\colon G_{\mathbb{Q}} \to \{\pm 1\}$ be the character with kernel G_{K_j}; the δ_j are pairwise distinct. Let c_j, A_j be the integers of Lemma 38.16 for K_j and $n = 4$, and put $c_0 = \sum_j c_j$, $A = \max_j A_j$, $c = 64 \cdot 2^A c_0$. For $e \geq 1$ let $Z_e = \{e + \frac{5}{2}, e + \frac{3}{2}, \frac{5}{2}, \frac{3}{2}\}$. We use: (T) if F is a number field and r_1, r_2 are continuous semisimple representations of G_F over $\overline{\mathbb{Q}}_\ell$ that are unramified with equal characteristic polynomials of Frobenius at all finite places outside a finite set, then $r_1 \cong r_2$ (Chebotarev density and Brauer–Nesbitt, [220], Chapter I, 2.2).

Step 1: the set S and the map t. Fix $e \geq 2$, $M \geq 1$ and a prime $p \nmid M$. Let J be the set of j with $p \nmid p_j$; $|J| \geq 17$. For $j \in J$, p is unramified in K_j; let W_j be the set of primes of K_j above p and $f_j \in \{1, 2\}$ their residue degree. For $w \in W_j$ let $\Pi_j(w)$ be the set of isomorphism classes of cuspidal automorphic representations π of $\mathrm{GL}_4(\mathbb{A}_{K_j})$ with infinitesimal character Z_e at both embeddings and $\pi_f^{\mathrm{K}_4(M)} \neq 0$ (so π_w is unramified, as $w \nmid M$), such that, with

$$\beta(\pi, w) = \big\{(q_w^{3/2}\alpha)^{-1} : \alpha \text{ a Satake parameter of } \pi_w\big\} \quad \text{(a multiset of four numbers)},$$

(Pi1) every element of $\beta(\pi, w)$ is an algebraic integer, and (Pi2) $q_w^{-(2e+2)} \prod_{x\in\beta(\pi,w)} x$ is a root of unity. Let $R(\pi, w)$ be the set of multisets B of four complex numbers with $\{x^{f_j} : x \in B\} =$

$\beta(\pi,w)$; each element $y\neq 0$ of $\beta(\pi,w)$ has exactly f_j roots of $X^{f_j}=y$, so $|R(\pi,w)|\le f_j^4\le 16$. Put

$$S=\{(j,w,\pi,B,b): j\in J,\ w\in W_j,\ \pi\in\Pi_j(w),\ B\in R(\pi,w),\ b\in\{0,1\}\},$$
$$t(j,w,\pi,B,b)=\sum_{x\in B}x.$$

The set S and the map t depend only on (e,M,p).

Step 2: (Tr3). The set Z_e consists of four distinct elements of $\frac{3}{2}+\mathbb{Z}=\frac{4-1}{2}+\mathbb{Z}$ of absolute value at most $e+3$, so by Lemma 38.16 with $H=e+3$, $\Pi_j(w)$ is finite with $|\Pi_j(w)|\le c_j(M(e+4))^{A_j}\le c_j2^A(M(e+2))^A$. As $|W_j|\le 2$, $|S|\le\sum_j 2\cdot 16\cdot 2\cdot c_j2^A(M(e+2))^A=c\,(M(e+2))^A$.

Step 3: (Tr2). Let $s=(j,w,\pi,B,b)\in S$ and $q=q_w=p^{f_j}$. Each $x\in B$ satisfies $x^{f_j}\in\beta(\pi,w)$, an algebraic integer by (Pi1); so x and $t(s)$ are algebraic integers. For the size, let ω be the central character of π. The continuous homomorphism $|\omega|\colon \mathbb{A}_{K_j}^\times/K_j^\times\to\mathbb{R}_{>0}$ is trivial on the compact group $\mathbb{A}^1_{K_j}/K_j^\times$, and $\mathbb{A}^\times_{K_j}=\mathbb{A}^1_{K_j}\cdot\mathbb{R}_{>0}$, so $|\omega|=|\cdot|_{\mathbb{A}}^{s_0}$ for a real s_0. Then $\pi\otimes|\det|^{-s_0/4}$ is cuspidal with unitary central character and Satake parameters $\alpha q^{s_0/4}$ at w, so by (Js2) of [136] the numbers $y_\alpha=\log|\alpha q^{s_0/4}|/\log q$ lie in $(-\frac12,\frac12)$; and $\prod_\alpha\alpha=\omega_w(\varpi_w)$ has absolute value q^{-s_0}, so $\sum_\alpha y_\alpha=0$. By (Pi2), $q^{2e+2}=\prod_{x\in\beta}|x|=q^{-6}\prod_\alpha|\alpha|^{-1}=q^{-6+s_0}$, so $s_0=2e+8$ and $|(q^{3/2}\alpha)^{-1}|=q^{(e+1)/2-y_\alpha}$. Hence the elements of B have absolute values $p^{(e+1)/2}p^{z_i}$ $(i=1,\dots,4)$ with $z=(z_i)$ in the polytope $P=\{z\in[-\frac12,\frac12]^4:\sum_i z_i=0\}$. The function $z\mapsto\sum_i p^{z_i}$ is convex, so its maximum on P is attained at an extreme point. A point of P with two coordinates in $(-\frac12,\frac12)$ is not extreme (it is the midpoint of two points of P obtained by moving these two coordinates in opposite directions); at a point with at most one such coordinate, the other three are $\pm\frac12$ and the sum condition forces the fourth to be $\pm\frac12$ as well. So the extreme points have two coordinates $\frac12$ and two $-\frac12$, and

$$|t(s)|\le p^{(e+1)/2}\sum_i p^{z_i}\le p^{(e+1)/2}(2\sqrt{p}+2/\sqrt{p})\le(2\sqrt{p}+2)\,p^{(e+1)/2}.$$

Step 4: (Tr1). Let $\theta\in\mathrm{Aut}(\mathbb{C})$, $j\in J$, $w\in W_j$, $\pi\in\Pi_j(w)$ and $q=q_w$. Since Z_e consists of four distinct elements of $\frac{4-1}{2}+\mathbb{Z}$, π is regular algebraic, and [68] (Theorem B.63) gives a cuspidal π' with $\pi'_f\cong{}^\theta\pi_f$ and infinitesimal character Z_e at both embeddings; moreover, ${}^\theta\pi_f$ has nonzero $\mathrm{K}_4(M)$-invariants. Satake parameters: let B_4 be the upper triangular Borel subgroup of $\mathrm{GL}_4(K_{j,w})$ and δ its modulus character; π_w is the unramified subquotient of $\mathrm{Ind}_{B_4}(\chi\delta^{1/2})$ with $\chi_k(\varpi_w)=\alpha_k$. As $\delta^{1/2}(\mathrm{diag}(\varpi_w^{a_1},\dots,\varpi_w^{a_4})u)=q^{-(3a_1+a_2-a_3-3a_4)/2}$ for diagonal units u, we get $\theta\circ\delta^{1/2}=\delta^{1/2}\nu$ with $\nu=\varepsilon^{a_1+a_2+a_3+a_4}$, $\varepsilon=\theta(\sqrt{q})/\sqrt{q}\in\{\pm1\}$; since $f\mapsto\theta\circ f$ identifies ${}^\theta\mathrm{Ind}_{B_4}(\eta)$ with $\mathrm{Ind}_{B_4}(\theta\circ\eta)$, $\pi'_w\cong{}^\theta\pi_w$ is the unramified subquotient of $\mathrm{Ind}_{B_4}((\theta\circ\chi)\nu\,\delta^{1/2})$ ([60]), and its Satake parameters are the $\varepsilon\,\theta(\alpha_k)$. As $\theta(q^{3/2})=\varepsilon q^{3/2}$, $q^{3/2}\varepsilon\theta(\alpha_k)=\theta(q^{3/2}\alpha_k)$, so $\beta(\pi',w)=\theta(\beta(\pi,w))$. Hence (Pi1), (Pi2) hold for π' (θ maps algebraic integers and roots of unity to such and fixes q), and $\pi'\in\Pi_j(w)$. Choose such a $\pi'=:v_\theta(\pi)$ for each π. If $v_\theta(\pi_1)=v_\theta(\pi_2)$, then ${}^\theta\pi_{1,f}\cong{}^\theta\pi_{2,f}$, hence $\pi_{1,f}\cong\pi_{2,f}$ (twist by θ^{-1}), and $\pi_1\cong\pi_2$ by (Js1) of [136]; so v_θ is a bijection of the finite set $\Pi_j(w)$. For $B\in R(\pi,w)$, $\theta(B)\in R(v_\theta(\pi),w)$, since $\theta(x)^{f_j}=\theta(x^{f_j})$. The map $u_\theta(j,w,\pi,B,b)=(j,w,v_\theta(\pi),\theta(B),b)$ is an injective, hence bijective, self-map of S with $t\circ u_\theta=\theta\circ t$.

Step 5: setting up (Tr4). Fix an odd prime $\ell\neq p$, ι, a place $\bar p\mid p$ and an arithmetic Frobenius $\sigma\in G_{\mathbb{Q}}$ at $\bar p$. Let r be admissible symplectic of weight e and level M, on $V=\overline{\mathbb{Q}}_\ell^4$. (a) *Some $j\in J$ has $r\not\cong r\otimes\delta_j$.* Otherwise for each $j\in J$ there is $\phi_j\neq 0$ in $\mathrm{End}(V)$ with $r(g)\phi_j r(g)^{-1}=\delta_j(g)\phi_j$ for all g; eigenvectors of the conjugation action for pairwise distinct characters are linearly independent, giving $|J|\ge 17$ independent elements of the 16-dimensional

space $\mathrm{End}(V)$. Let $j = j(r)$ be the least such index; it depends only on the isomorphism class of r. Write $K = K_j$, $\delta = \delta_j$, $f = f_j$. (b) *$r|_{G_K}$ is irreducible.* Otherwise let $V_1 \subsetneq V$ be a nonzero irreducible G_K-subrepresentation and $\gamma \in G_{\mathbb{Q}} \smallsetminus G_K$; $V_2 = r(\gamma)V_1$ is G_K-stable, $V_1 + V_2$ and $V_1 \cap V_2$ are $G_{\mathbb{Q}}$-stable (as $\gamma^2 \in G_K$), so by (Sy1) $V = V_1 \oplus V_2$, and every $g \notin G_K$ exchanges V_1 and V_2. Then $\phi = \mathrm{id}_{V_1} \oplus (-\mathrm{id}_{V_2})$ satisfies $\phi\, r(g) = \delta(g)\, r(g)\, \phi$, i.e. $r \cong r \otimes \delta$, a contradiction. (c) *The automorphic data.* By (Sy2) there is a regular algebraic cuspidal π of $\mathrm{GL}_4(\mathbb{A}_K)$ with $r|_{G_K} \leftrightarrow_\iota \pi$; its class is determined by the class of r (two such have the same Satake parameters at almost all places, hence are isomorphic by [60] and (Js1)). Let w be the prime of K below $\bar{p}$; $w \in W_j$. The image of σ in $\mathrm{Gal}(K/\mathbb{Q})$ generates the decomposition group of w, of order f, so $\sigma^f \in G_K$ is an arithmetic Frobenius at $\bar{p}$ for K. By Proposition 38.30, (Ls1) and (Ls2), π has infinitesimal character Z_e at both embeddings and $\pi_f^{\mathrm{K}_4(M)} \neq 0$; π_w is unramified and $w \nmid \ell$, so by (Ls3) the eigenvalues of $r(\sigma^f)$ are ι^{-1} of the elements of $\beta(\pi, w)$. (Pi1) is (Ls4) ($p \nmid Mp_j$). For (Pi2): (Sy4) and $\det P \neq 0$ give $(\det r)^2 = (\chi_{\mathrm{cyc}}^{e+1}\chi')^4$, so $\det r = \eta\, \chi_{\mathrm{cyc}}^{2e+2}\chi'^2$ with $\eta^2 = 1$; as $\ell \neq p$, $\chi_{\mathrm{cyc}}(\sigma^f) = q_w$, and η, χ' have finite order, so $\prod_{x \in \beta(\pi,w)} x = \iota(\det r(\sigma^f))$ is q_w^{2e+2} times a root of unity. So $\pi \in \Pi_j(w)$. Let $B_r = \iota(\text{multiset of eigenvalues of } r(\sigma))$; then $\{x^f : x \in B_r\} = \beta(\pi, w)$, i.e. $B_r \in R(\pi, w)$, and $\sum_{x \in B_r} x = \iota(\mathrm{tr}\, r(\sigma))$.

Step 6: (Tr4). Let $\lambda_0(r) = (j(r), w, \pi, B_r)$; it depends only on the class of r. If $\lambda_0(r) = \lambda_0(r')$, then $r|_{G_K}$ and $r'|_{G_K}$ are irreducible and attached to the same π, so $r|_{G_K} \cong r'|_{G_K}$ by (T). The space $\mathrm{Hom}_{G_K}(r, r')$ is one-dimensional (Schur) and $G_{\mathbb{Q}}$ acts on it by $g \cdot \phi = r'(g)\phi\, r(g)^{-1}$ through $G_{\mathbb{Q}}/G_K$, by a character equal to $\mathbf{1}$ or δ; so $r' \cong r$ or $r' \cong r \otimes \delta$. Hence each fibre of λ_0 consists of at most two isomorphism classes, and we choose for each fibre an injection of its set of classes into $\{0, 1\}$, writing $b(r)$ for the value at the class of r. Put $\lambda(r) = (\lambda_0(r), b(r)) \in S$. If $\lambda(r) = \lambda(r')$, then r and r' lie in one fibre with the same bit, so $r \cong r'$; and $\iota(\mathrm{tr}\, r(\sigma)) = \sum_{x \in B_r} x = t(\lambda(r))$ by Step 5. The set S and the map t of Step 1 do not depend on ℓ, ι, $\bar{p}$ or σ. □

Proposition 38.30. *Let ℓ be an odd prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $e \geq 2$ and $M \geq 1$ integers, $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ admissible symplectic of weight e and level M, K an imaginary quadratic field with discriminant d_K such that $r|_{G_K}$ is irreducible, and π a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_K)$ with $r|_{G_K} \leftrightarrow_\iota \pi$. Then:*

(Ls1) *at both embeddings of K the infinitesimal character of π is $\{e + \frac{5}{2}, e + \frac{3}{2}, \frac{5}{2}, \frac{3}{2}\}$;*

(Ls2) *π_f has a nonzero vector fixed by $\mathrm{K}_4(M)$;*

(Ls3) *for every finite place $w \nmid \ell$ of K at which π_w is unramified, every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above w and every arithmetic Frobenius $\sigma \in G_K$ at $\bar{w}$: $r|_{G_K}$ is trivial on $I_{\bar{w}}$ and $\iota(\det(X - r(\sigma^{-1}))) = \prod_\alpha (X - q_w^{3/2}\alpha)$, α running over the Satake parameters of π_w with multiplicity;*

(Ls4) *for every prime $p \nmid Md_K$, every prime w of K above p and every Satake parameter α of π_w, $(q_w^{3/2}\alpha)^{-1}$ is an algebraic integer.*

Proof. Write $s = r|_{G_K}$, $\mu = \chi_{\mathrm{cyc}}^{e+1}\chi'$ with χ', P as in (Sy4), and fix a complex conjugation $c \in G_{\mathbb{Q}}$.

Step 1: π is polarizable. Let S be a finite set of finite places of K, stable under c, containing those above ℓ, M, M' (the level of χ') and those where π is ramified, and outside of which s is unramified with $\iota(\det(X - s(\sigma^{-1}))) = \prod_\alpha (X - q_w^{3/2}\alpha)$ for arithmetic Frobenius elements σ at places above w (possible by $s \leftrightarrow_\iota \pi$). Let $w \notin S$, $\sigma \in G_K$ an arithmetic Frobenius at a place $\bar{w} \mid w$, $q = q_w$, and A_w the multiset of Satake parameters of π_w. (a) $c\sigma c^{-1} \in G_K$ is an arithmetic Frobenius at $c(\bar{w})$, which lies above w^c, and $s(c\sigma^{-1}c^{-1}) = r(c)s(\sigma^{-1})r(c)^{-1}$; so $A_{w^c} = A_w$. The local component of π^c at w is π_{w^c} transported by $c\colon K_w \to K_{w^c}$, unramified

with Satake parameters $A_{w^c} = A_w$. Unramified irreducible representations are determined by their Satake parameters ([60]), so $(\pi^c)_w \cong \pi_w$ for all $w \notin S$, and $\pi^c \cong \pi$ by (Js1). (b) By (Sy4) and $\det P \neq 0$, $r(g) = \mu(g)\, P^{-1}\, {}^t r(g)^{-1} P$, so $r \cong r^\vee \otimes \mu$ and the eigenvalues of $s(\sigma^{-1})$ are the $\mu(\sigma^{-1})b^{-1}$, b running over them. As $w \nmid \ell M'$, $\chi_{\mathrm{cyc}}(\sigma) = q$ (as $w \nmid \ell$, reduction modulo $\bar{w}$ is injective on the ℓ-power roots of unity, and σ acts on the residue field as $x \mapsto x^q$) and $\mu(\sigma^{-1}) = q^{-(e+1)}\chi'(\sigma)^{-1}$. Applying ι: $\{q^{3/2}\alpha\} = \{q^{-(e+1)}\iota(\chi'(\sigma))^{-1} q^{-3/2}\alpha^{-1}\}$, that is

$$A_w = \{q^{-(e+4)}\, \iota(\chi'(\sigma))^{-1}\alpha^{-1} : \alpha \in A_w\}.$$

Let $\omega = \iota \circ \chi'^{-1} \circ \mathrm{Art}_{\mathbb{Q}}$, a finite-order character of $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$, unramified at primes $p \nmid M'$ with $\omega_p(p) = \iota(\chi'(\mathrm{Frob}_p))^{-1}$, and $\psi = |\cdot|_{\mathbb{A}_{\mathbb{Q}}}^{e+4}\omega$, a continuous character of $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$. For $w \mid p$, $w \notin S$, $K_w/\mathbb{Q}_p$ is unramified of degree f ($p \nmid d_K$ as ramified primes may be put into S), $\mathrm{N}_{K_w/\mathbb{Q}_p}(\varpi_w) \in p^f\mathbb{Z}_p^\times$, and σ restricts to Frob_p^f on the abelian extension cut out by χ'; so $\psi_p(\mathrm{N}_{K_w/\mathbb{Q}_p}\varpi_w) = q^{-(e+4)}\iota(\chi'(\sigma))^{-1}$, and the right side of the display is the Satake multiset of $(\pi^\vee \otimes (\psi \circ \mathrm{N}_{K/\mathbb{Q}} \circ \det))_w$ (the contragredient inverts the Satake parameters, the twist multiplies them by the value of the unramified character at ϖ_w). As in (a), (Js1) gives $\pi \cong \pi^\vee \otimes (\psi \circ \mathrm{N}_{K/\mathbb{Q}} \circ \det)$, both sides being cuspidal. With (a), $\pi^c \cong \pi^\vee \otimes (\psi \circ \mathrm{N}_{K/\mathbb{Q}} \circ \det)$: π is polarizable (the condition on $\psi_\infty(-1)$ is empty, $\mathbb{Q}$ having one infinite place). In particular every π attached to s is polarizable; no choice of representative is involved.

Step 2: s satisfies the hypotheses of Proposition 38.13 for $n = 4$, M, $m = (e+1, e, 1, 0)$. s is irreducible by hypothesis. (Lw1) and the triangular part of (Lw2) for s: by (Sy3), for a prime p of $\mathbb{Q}$ there is a place $\bar{p}$ of $\overline{\mathbb{Q}}$ above p with the properties of (Lw1) (if $p \neq \ell$) or (Lw2) (if $p = \ell$) for r, with characters χ_i of $G_{\mathbb{Q}}$ of level M. Given a place $v \mid p$ of K, choose $g \in G_{\mathbb{Q}}$ with $g\bar{p}$ above v ($G_{\mathbb{Q}}$ permutes the places above p transitively). Then $I_{g\bar{p}} = gI_{\bar{p}}g^{-1}$, $D_{g\bar{p}} = gD_{\bar{p}}g^{-1}$ and $r(gxg^{-1}) = r(g)r(x)r(g)^{-1}$; since χ_i and χ_{cyc} are invariant under conjugation, r restricted to $I_{g\bar{p}}$ is diagonalisable with the characters χ_i (for $p \neq \ell$), resp. triangular on $D_{g\bar{p}}$ with diagonal $(x \mapsto \kappa_i(g^{-1}xg))$, equal to $\chi_{\mathrm{cyc}}^{m_i}\chi_i$ on $I_{g\bar{p}}$ (for $p = \ell$). Restricting to the inertia and decomposition groups $I_{g\bar{p}} \cap G_K$, $D_{g\bar{p}} \cap G_K$ of $g\bar{p}$ in G_K, and the χ_i to G_K (still of level M: trivial on $G_{K(\mu_M)} \subseteq G_{\mathbb{Q}(\mu_M)}$), gives (Lw1) and the triangular part of (Lw2) for s. Polarizability is Step 1. By (Sy2), r satisfies the witness conditions over imaginary quadratic fields, and $s = r|_{G_K}$ is irreducible, so s satisfies the witness conditions, in particular (Wc2). By (Sy5), $\ell \mid M$, so every place $v \mid \ell$ of K divides M and the genericity hypothesis is empty.

Step 3: (Ls1)–(Ls3). By Proposition 38.13: (Wl2) gives the infinitesimal character $\{m_i + \frac{3}{2}\} = \{e + \frac{5}{2}, e + \frac{3}{2}, \frac{5}{2}, \frac{3}{2}\}$ at both embeddings, (Ls1); (Wl3) is (Ls2); (Wl1) is (Ls3).

Step 4: (Ls4). Let $p \nmid Md_K$ and $w \mid p$. By Step 1, (Ls1), (Ls2), Proposition 38.18 applies with $n = 4$, $Z = \{e + \frac{5}{2}, e + \frac{3}{2}, \frac{5}{2}, \frac{3}{2}\}$ (four distinct elements of $\frac{3}{2} + \mathbb{Z}$, as $e \geq 2$), $z - \frac{3}{2} \in \{e+1, e, 1, 0\} \subseteq [0, e+1]$, so $a = 0$, $b = e + 1$: for every Satake parameter α of π_w, $q_w^{-a}\beta^{-1} = (q_w^{3/2}\alpha)^{-1}$ is an algebraic integer. (The hypothesis that ℓ is odd is not used.) □

Definition 38.31. Let ℓ be a prime, $\mathcal{O}$ a coefficient ring, $N \geq 1$ an integer and χ a Dirichlet character modulo N with values in $\mathcal{O}^\times$. Write $\chi = \chi'\chi''$ with χ' of order prime to ℓ and χ'' of ℓ-power order (both are powers of χ). Let $S_\ell(\chi)$ be the set of primes $p \mid N$ with $p \neq \ell$ and $\chi''_p \neq \mathbf{1}$, and put $T_\ell(\chi) = \{\ell\} \cup S_\ell(\chi)$. Recall that $c_2(\chi)$ is the exponent of 2 in the conductor of χ_2, that $\vartheta = \chi'(\chi'_2)^{-1}$, and that $n_0(\ell, \chi)$ is the 2-adic valuation of $\ell^f - 1$ for f the order of ℓ modulo $\mathrm{lcm}(4, \mathrm{ord}\,\vartheta)$.

For integers $e \geq 1$, $o \geq 1$ and $m \geq 1$, the tuple (ℓ, e, N, χ, o, m) is a **dyadic Eisenstein datum** if

(Dy1) ℓ is odd;

(Dy2) N is even;

(Dy3) $\chi(-1) = (-1)^e$;

(Dy4) $(\chi'_r)^{2o} = \mathbf{1}$ for every odd prime r;

(Dy5) $8\,o\,v^2 < 2^{c_2(\chi)}$ for every $v \in T_\ell(\chi)$;

(Dy6) $c_2(\chi) \geq 4$;

(Dy7) the conductor of ϑ^2 divides m, and $2\,\ell\,m\,4^{n_0(\ell,\chi)} \leq 2^{c_2(\chi)}$.

In words: the 2-component of χ has a conductor that is large compared with ℓ, with the primes of $S_\ell(\chi)$, with the order bound o of the odd components and with the conductor of ϑ^2. These are the data for which the Eisenstein congruences on the unitary group in three variables attached to $\mathbb{Q}(i)/\mathbb{Q}$ are used to bound $\mathrm{Sha}^1_{\ell N}(\chi, e)$.

Proposition 38.32. *Let $\psi \in \mathcal{P}$. There is an integer $C \geq 0$ such that for every Sha-host $\mathfrak{h}$ of ψ, every integer $t \geq 1$ at which $\mathfrak{h}$ is big, every odd prime ℓ, every coefficient ring $\mathcal{O}$, every integer e with $0 < e < t$ and every $\chi \in X_{\mathfrak{h}}(\mathcal{O}, e)$ such that (ℓ, e) is in case (V) and $R_1(\ell, \chi) \leq 2^{c_2(\chi)}$, the group $\mathrm{Sha}^1_{\ell N_{\mathfrak{h}}}(\chi, e)$ has finite length and*

$$\nu_{\mathcal{O}}(\chi, e) \log \ell \leq (\ell N_{\mathfrak{h}} e)^C.$$

Proof. Let $K = \mathbb{Q}(i)$, an imaginary quadratic field with discriminant $d_K = -4$, and let $c, A \geq 0$ be the integers given by Proposition 38.6 for K. Choose an integer $k \geq 0$ with $60c \leq 48^k$ and put $C = A + k + 2$. Then C does not depend on ψ; we show that it has the stated property. Fix $\mathfrak{h}$, t, ℓ, $\mathcal{O}$, e and χ as in the statement and write $N = N_{\mathfrak{h}}$, $X_0 = \ell N e$. Only the facts that ℓ is odd, $e \geq 1$, $\chi \in X_{\mathfrak{h}}(\mathcal{O}, e)$ and $R_1(\ell, \chi) \leq 2^{c_2(\chi)}$ are used. Since $16 \mid N$ and $\ell \geq 3$, $X_0 \geq 48$.

Step 1: the datum. By Lemma 38.33, $(\ell, e, N, \chi, o_\psi, N_\psi)$ is a dyadic Eisenstein datum.

Step 2: an auxiliary prime. Let p be a prime factor of $30\ell N - 1 \geq 2$. Then $p \nmid 30\ell N$; hence $p \geq 7$, $p \nmid N d_K$ (as p is odd), $p \neq \ell$, and $p \leq 30\ell N$. Choose a prime w of K above p and put $q = q_w$, a power of p, hence prime to ℓN; put $\mathrm{Eis}(z) = q^{-e} z + 1 + q^e z^{-1}$ for $z \in \mathbb{C}^\times$.

Step 3: the congruence integer. Let $n = \varphi(N)$, let $Z \subset \mathbb{C}^\times$ be the group of n-th roots of unity and $z\colon Z \to \mathbb{C}$ the inclusion. Every $\sigma \in \mathrm{Aut}(\mathbb{C})$ maps Z bijectively onto itself, so $u_\sigma = \sigma|_Z$ satisfies $z(u_\sigma(o)) = \sigma(z(o))$. By Proposition 38.6 applied to (e, N, p, w, Z, z) there is a nonzero integer M with

$$\log |M| \leq c\, n\, \big(N(e+1)\big)^A \log p$$

such that items 1 and 2 of Proposition 38.6 hold for every odd prime $\neq p$, in particular for ℓ.

Step 4: a Frobenius element and the Eisenstein trace. Choose an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ (both fields are algebraically closed of characteristic 0 and of the cardinality of the continuum), a place $\bar{w}$ of $\overline{\mathbb{Q}}$ above w and an arithmetic Frobenius $\sigma \in G_K$ at $\bar{w}$ (the decomposition group maps onto the Galois group of the residue field extension). As $p \nmid \ell N$, reduction modulo $\bar{w}$ is injective on the roots of unity of order dividing $\ell^j N$ for every $j \geq 1$, and σ acts on the residue field as $x \mapsto x^q$; hence $\sigma(\zeta) = \zeta^q$ for all these roots of unity, so $\chi(\sigma) = \chi(q)$, $\chi_{\mathrm{cyc}}(\sigma) = q$ and $\Psi_{\chi,e}(\sigma) = \chi(q) q^e$. Put $\tau = \sigma^{-1} \in G_K$ and

$$t = \Psi_{\chi,e}(\tau) + 1 + \Psi_{\chi,e}(\tau)^{-1} = q^{-e}\chi(q)^{-1} + 1 + q^e \chi(q).$$

Since $\chi(q)$ is a value of a character of $(\mathbb{Z}/N\mathbb{Z})^\times$, $\chi(q)^n = 1$, so $z_0 = \iota(\chi(q))^{-1}$ lies in Z, and $\iota^{-1}(\mathrm{Eis}(z_0)) = t$ because ι^{-1} is a field isomorphism fixing q.

Step 5: the hypothesis of the one-character bound. Let $\rho\colon G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ be admissible of stable type for (ι, e, N); then it is admissible of weight e and level N. Item 1 of Proposition 38.6 for the family with $X_{z_0} = \{\rho\}$ and $X_o = \emptyset$ for $o \neq z_0$ gives $\mathrm{tr}\,\rho(\tau) \neq \iota^{-1}(\mathrm{Eis}(z_0)) = t$.

Step 6: one character. By Steps 1 and 5, Proposition 46.2 applies to the datum of Step 1, to K, ι and τ: the group $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length, which is the first assertion, and there is a finite set X of pairwise non-isomorphic representations admissible of stable type for (ι, e, N), hence of weight e and level N, with

$$\nu_{\mathcal{O}}(\chi, e) \leq 2 \sum_{\rho \in X} v_\ell\big(\mathrm{tr}\,\rho(\tau) - t\big).$$

Item 2 of Proposition 38.6 for the family with $X_{z_0} = X$ and $X_o = \emptyset$ for $o \neq z_0$ gives $v_\ell(M) \geq \sum_{\rho\in X} v_\ell(\mathrm{tr}\,\rho(\tau) - t)$, so $\nu_{\mathcal{O}}(\chi, e) \leq 2v_\ell(M)$.

Step 7: the estimate. As $\ell^{v_\ell(M)}$ divides M, $v_\ell(M)\log\ell \leq \log|M|$. With Step 3,

$$\nu_{\mathcal{O}}(\chi, e)\log\ell \leq 2\log|M| \leq 2c\,n\,\big(N(e+1)\big)^A \log p \leq 60c\,X_0^{A+2} \leq X_0^C = (\ell N_{\mathfrak{h}} e)^C.$$

This is the claim. □

Lemma 38.33. *Let $\psi \in \mathcal{P}$, let $\mathfrak{h}$ be a Sha-host of ψ, let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring and $e \geq 1$ an integer, and let $\chi \in X_{\mathfrak{h}}(\mathcal{O}, e)$ with $R_1(\ell, \chi) \leq 2^{c_2(\chi)}$. Then $(\ell, e, N_{\mathfrak{h}}, \chi, o_\psi, N_\psi)$ is a dyadic Eisenstein datum.*

Proof. Write $\mathfrak{h} = (a, q, d)$ and $N = N_{\mathfrak{h}} = \mathrm{lcm}(2^{a+2}, N_\psi)\,qd$. The primes q, d are odd, distinct and prime to N_ψ; hence the odd prime factors of N are q, d and the odd prime factors of N_ψ, and $v_r(N) = v_r(N_\psi)$ for every odd prime $r \mid N_\psi$. Taking p-components is multiplicative, so for every prime p the components χ'_p and χ''_p are powers of χ_p. By Lemma 38.34, items 1 and 2, χ_2 has 2-power order, $\chi_q^2 = \chi_d^2 = \mathbf{1}$, and $\chi_r^{o_\psi} = \mathbf{1}$ for every odd prime $r \mid N_\psi$ (the r-component of $\chi_{\mathrm{od}}^{o_\psi} = \mathbf{1}$ is $\chi_r^{o_\psi}$). Put $R = R_1(\ell, \chi)$; since $o_\psi, N_\psi \geq 1$ and $\ell \geq 3$,

$$16\,o_\psi\,\ell^2 N_\psi^2 \leq R \leq 2^{c_2(\chi)}. \tag{38.8}$$

(Dy1)–(Dy3). The prime ℓ is odd by hypothesis; $16 \mid N$ since $a \geq 2$; and $\chi(-1) = (-1)^e$ because $\chi \in X_{\mathfrak{h}}(\mathcal{O}, e)$.

(Dy4). Let r be an odd prime. If $r \nmid N$, then $\chi'_r = \mathbf{1}$. If $r \in \{q, d\}$, then χ'_r is a power of χ_r and $(\chi'_r)^2 = \mathbf{1}$. Otherwise $r \mid N_\psi$, and χ'_r is a power of χ_r, so $(\chi'_r)^{o_\psi} = \mathbf{1}$. In all cases $(\chi'_r)^{2o_\psi} = \mathbf{1}$.

(Dy5). For $v = \ell$ we have $8o_\psi\ell^2 < 16o_\psi\ell^2N_\psi^2 \leq 2^{c_2(\chi)}$ by (38.8). Let $v \in S_\ell(\chi)$. The component χ''_v is a power of χ_v and has ℓ-power order with ℓ odd. If $v = 2$, it also has 2-power order; if $v \in \{q, d\}$, its square is trivial; in both cases $\chi''_v = \mathbf{1}$, which is excluded. So v is an odd prime factor of N other than q, d, hence $v \mid N_\psi$ and $v \leq N_\psi$, and $8o_\psi v^2 \leq 8o_\psi N_\psi^2 < 16o_\psi\ell^2N_\psi^2 \leq 2^{c_2(\chi)}$.

(Dy6). By (38.8), $2^{c_2(\chi)} \geq 16 \cdot 9 > 2^4$, so $c_2(\chi) \geq 4$.

(Dy7). By definition $\vartheta = \prod_r \chi'_r$, the product over the odd primes $r \mid N$, so $\vartheta^2 = \prod_r (\chi'_r)^2$. The factors at $r \in \{q, d\}$ are trivial (proof of (Dy4)), so $\vartheta^2 = \prod_r (\chi'_r)^2$ over the odd primes $r \mid N_\psi$, and χ'_r is a character of $(\mathbb{Z}/r^{v_r(N_\psi)}\mathbb{Z})^\times$ since $v_r(N) = v_r(N_\psi)$. Hence $\vartheta^2(u) = 1$ for every

$u \in (\mathbb{Z}/N\mathbb{Z})^\times$ with $u \equiv 1 \pmod{N_\psi}$, that is, ϑ^2 is induced from a character modulo N_ψ, and its conductor divides N_ψ [266]. Finally, by (38.8),

$$2\,\ell\,N_\psi\,4^{n_0(\ell,\chi)} \le 16\,o_\psi\,\ell^2 N_\psi^2\,4^{n_0(\ell,\chi)} = R \le 2^{c_2(\chi)}.$$

So (Dy7) holds with $m = N_\psi$, and $(\ell, e, N, \chi, o_\psi, N_\psi)$ is a dyadic Eisenstein datum. □

Lemma 38.34. *Let $\psi \in \mathcal{P}$, let $\mathfrak{h} = (a, q, d)$ be a Sha-host of ψ, let $\mathcal{O}$ be a coefficient ring and let χ be a Dirichlet character modulo $N_\mathfrak{h}$ with values in $\mathcal{O}^\times$ such that $H_\mathfrak{h} \subseteq \ker\chi$. Write $\chi = \prod_{p|N_\mathfrak{h}} \chi_p$, with χ_p a character of $(\mathbb{Z}/p^{v_p(N_\mathfrak{h})}\mathbb{Z})^\times$ (the component of χ at p), and let $\chi_{\mathrm{od}} = \prod_{p|N_\psi,\ p \text{ odd}} \chi_p$. Then:*

1. *χ_2 has 2-power order, and $\chi_q^2 = \chi_d^2 = \mathbf{1}$;*
2. *$\chi_{\mathrm{od}}^{o_\psi} = \mathbf{1}$;*
3. *if ℓ is an odd prime with $\ell \notin \{q, d\}$ and $\chi^{(\ell)}$ has odd order, then $\chi(u) = 1$ for every $u \in (\mathbb{Z}/N_\mathfrak{h}\mathbb{Z})^\times$ with $u \equiv 1 \pmod{N_\psi}$; that is, χ is the composite of the reduction $(\mathbb{Z}/N_\mathfrak{h}\mathbb{Z})^\times \to (\mathbb{Z}/N_\psi\mathbb{Z})^\times$ with a Dirichlet character modulo N_ψ.*

Proof. Write $N = N_\mathfrak{h} = \mathrm{lcm}(2^{a+2}, N_\psi)\,qd$. The primes $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ are distinct, odd and prime to N_ψ, so $v_q(N) = v_d(N) = 1$, the odd primes dividing N other than q, d are those dividing N_ψ, and $v_p(N) = v_p(N_\psi)$ for them. The decomposition $\chi = \prod_p \chi_p$ is that of the Chinese remainder theorem; taking components is multiplicative.

Item 1. The group $(\mathbb{Z}/2^{v_2(N)}\mathbb{Z})^\times$ is a 2-group, so χ_2 has 2-power order. Let v be a square in $(\mathbb{Z}/q\mathbb{Z})^\times$ and let $u \in (\mathbb{Z}/N\mathbb{Z})^\times$ with $u \equiv 1 \pmod{N/q}$ and $u \equiv v \pmod q$. Since 2^{a+2}, N_ψ and d divide N/q, we have $u \equiv 1 \pmod{2^{a+2}}$, $\psi(u) = 1$, u is a square modulo d, and u is a square modulo q; so $u \in H_\mathfrak{h}$ and $\chi_q(v) = \chi(u) = 1$. Thus χ_q is trivial on the subgroup of squares, which has index 2, and $\chi_q^2 = \mathbf{1}$. In the same way $\chi_d^2 = \mathbf{1}$.

Item 2. Let U be the subgroup of $u \in (\mathbb{Z}/N\mathbb{Z})^\times$ with $u \equiv 1$ modulo $2^{v_2(N)}qd$; on U all components other than those at the odd primes dividing N_ψ are trivial, so $\chi|_U = \chi_{\mathrm{od}}|_U$, and χ_{od} is determined by $\chi_{\mathrm{od}}|_U$ (Chinese remainder theorem). An element $u \in U$ satisfies $u \equiv 1 \pmod{2^{a+2}}$ and is a square modulo q and d, so $u \in H_\mathfrak{h}$ if and only if $\psi(u) = 1$. Hence $\chi_{\mathrm{od}}|_U$ is trivial on $U \cap \ker\psi$ and factors through $\psi(U)$, a subgroup of the cyclic group of order o_ψ; so $\chi_{\mathrm{od}}(u)^{o_\psi} = 1$ for $u \in U$, and $\chi_{\mathrm{od}}^{o_\psi} = \mathbf{1}$.

Item 3. Write $\chi = \chi'\chi''$ with χ' of order prime to ℓ and χ'' of ℓ-power order; both are powers of χ. By definition $\chi^{(\ell)} = \chi' \cdot (\chi'_\ell)^{-1}$, where χ'_ℓ is the component of χ' at ℓ (trivial if $\ell \nmid N$). Let $p \in \{2, q, d\}$; then $p \ne \ell$, as ℓ is odd and $\ell \notin \{q, d\}$. The component χ''_p is a power of χ_p, so its order divides the 2-power order of χ_p (item 1), and it has ℓ-power order with ℓ odd; hence $\chi''_p = \mathbf{1}$ and $\chi'_p = \chi_p$. The component of $(\chi'_\ell)^{-1}$ at $p \ne \ell$ is trivial, so the component of $\chi^{(\ell)}$ at p is χ_p. As $\chi^{(\ell)}$ has odd order, so has χ_p; having also 2-power order, $\chi_p = \mathbf{1}$. Now let $u \equiv 1 \pmod{N_\psi}$. For every odd prime $p \mid N_\psi$, $u \equiv 1 \pmod{p^{v_p(N)}}$, so $\chi_p(u) = 1$; and $\chi_2(u) = \chi_q(u) = \chi_d(u) = 1$. Hence $\chi(u) = 1$. The reduction map $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N_\psi\mathbb{Z})^\times$ is surjective with kernel the classes $u \equiv 1 \pmod{N_\psi}$, so χ factors through it. □

Proposition 38.35. *Let $\psi \in \mathcal{P}$. There is an integer $C \ge 0$ such that for every Sha-host $\mathfrak{h}$ of ψ, every integer $t \ge 1$ at which $\mathfrak{h}$ is big, every odd prime ℓ, every coefficient ring $\mathcal{O}$, every integer e with $0 < e < t$ and every $\chi \in X_\mathfrak{h}(\mathcal{O}, e)$ such that (ℓ, e) is in case (V) and $2^{c_2(\chi)} < R_1(\ell, \chi)$, the group $\mathrm{Sha}^1_{\ell N_\mathfrak{h}}(\chi, e)$ has finite length and*

$$\nu_\mathcal{O}(\chi, e)\log\ell \le (\ell N_\mathfrak{h} e)^C.$$

Proof. Constants. Let $r_\psi = \max B_\psi$ (B_ψ is finite and contains 3). For $\ell \in B_\psi$ let $c_\ell \ge 1$ be the constant of Theorem 40.1 for ℓ, and $c_B = \max_{\ell \in B_\psi} c_\ell$. Put

$$D_1 = 32\, o_\psi^2\, (r_\psi N_\psi)^2\, r_\psi^{2\varphi(4o_\psi)}, \qquad D_2 = r_\psi D_1, \qquad D_3 = (D_2+1)(2D_2)^{c_B}, \qquad C = D_3 + D_2 + 3.$$

These depend only on ψ. Fix $\mathfrak{h} = (a,q,d)$, t, ℓ, $\mathcal{O}$, e and χ as in the statement and write $N = N_\mathfrak{h}$. Case (V) means $\ell \in B_\psi$ and $e = 1$. Since $\mathfrak{h}$ is big at $t \ge 1$, $\min(q,d) > 2t+1 \ge 3$ and $\min(q,d)$ exceeds each of $o_\psi, N_\psi, p_K(\psi), h^*(\psi), p_1(\psi)$; as ℓ is 3 or a prime factor of their product, $\ell < \min(q,d)$, so $\ell \notin \{q,d\}$, and $3 \le \ell \le r_\psi$.

The order of χ. Write $\chi = \prod_{p \mid N} \chi_p$ and χ_{od} as in Lemma 38.34, so that $\chi = \chi_2\, \chi_{\mathrm{od}}\, \chi_q\, \chi_d$. By definition $2^{c_2(\chi)}$ is the conductor of χ_2, so χ_2 factors through $(\mathbb{Z}/2^{c_2(\chi)}\mathbb{Z})^\times$, a group of order at most $2^{c_2(\chi)}$. By items 1 and 2 of Lemma 38.34, $\chi_q^2 = \chi_d^2 = \mathbf{1}$ and $\chi_{\mathrm{od}}^{o_\psi} = \mathbf{1}$. The components commute, so the order of χ divides $\operatorname{lcm}(\operatorname{ord}\chi_2, o_\psi, 2)$ and

$$\operatorname{ord}\chi \le 2\, o_\psi\, 2^{c_2(\chi)} < 2\, o_\psi\, R_1(\ell,\chi).$$

Next we bound $n_0(\ell,\chi)$. Let χ' be the part of χ of order prime to ℓ and $\vartheta = \chi'(\chi_2')^{-1}$, the product of the components of χ' at the odd primes. Since χ' is a power of χ, each component χ_p' is a power of χ_p; so $\vartheta^{2o_\psi} = \mathbf{1}$, the order of ϑ divides $2o_\psi$ and is prime to ℓ, and $L_0 = \operatorname{lcm}(4, \operatorname{ord}\vartheta)$ divides $4o_\psi$ and is prime to ℓ. The order f of ℓ modulo L_0 satisfies $f \le \varphi(L_0) \le \varphi(4o_\psi)$ (Euler; $\varphi(L_0) \le \varphi(4o_\psi)$ as $L_0 \mid 4o_\psi$). As $2^{n_0(\ell,\chi)}$ divides $\ell^f - 1 < \ell^f$, we get $4^{n_0(\ell,\chi)} < \ell^{2f} \le \ell^{2\varphi(4o_\psi)}$. With $\ell \le r_\psi$,

$$\operatorname{ord}\chi < 2\, o_\psi \cdot 16\, o_\psi (\ell N_\psi)^2 \ell^{2\varphi(4o_\psi)} \le D_1.$$

An auxiliary field. Put $m = \ell N$ and let H' be the subgroup of $u \in (\mathbb{Z}/m\mathbb{Z})^\times$ with $u \equiv 1 \pmod{\ell}$ and $\chi(\bar u) = 1$, where $\bar u$ is the image of u in $(\mathbb{Z}/N\mathbb{Z})^\times$. Let $L \subseteq \mathbb{Q}(\zeta_m)$ be the fixed field of $\{\zeta_m \mapsto \zeta_m^u : u \in H'\}$. Then:

- $[L:\mathbb{Q}] = [(\mathbb{Z}/m\mathbb{Z})^\times : H'] \le (\ell-1)\operatorname{ord}\chi < \ell D_1 \le D_2$, since H' is the kernel of $u \mapsto (u \bmod \ell,\ \chi(\bar u))$, whose image has at most $(\ell-1)\operatorname{ord}\chi$ elements;
- $\zeta_\ell = \zeta_m^N \in L$, because ζ_m^N is fixed by $\zeta_m \mapsto \zeta_m^u$ when $u \equiv 1 \pmod{\ell}$; as $\ell \ge 3$, $L \supseteq \mathbb{Q}(\zeta_\ell)$ is totally complex;
- χ is trivial on G_L: an element of G_L acts on $\mathbb{Q}(\zeta_m)$ as $\zeta_m \mapsto \zeta_m^u$ with $u \in H'$, so on $\mathbb{Q}(\zeta_N)$ as $\zeta_N \mapsto \zeta_N^{\bar u}$ with $\chi(\bar u) = 1$;
- $|d_L| \le m^{[L:\mathbb{Q}]}$, by the conductor–discriminant formula, $|d_L|$ being the product of the conductors of the $[L:\mathbb{Q}]$ characters of $\operatorname{Gal}(L/\mathbb{Q})$, each dividing m [266].

The Tate–Shafarevich group of L. By Proposition 38.36 (L totally complex, ℓ odd, $e = 1$) and Theorem 40.1 (L totally complex with $\zeta_\ell \in L$, $e = 1$), $H^2(G_{L,S_\ell}, \mathbb{Z}_\ell(2))$ is finite and

$$\operatorname{len}_\ell(L,1)\log\ell \le \log \#H^2\big(G_{L,S_\ell}, \mathbb{Z}_\ell(2)\big) \le (2|d_L|)^{(2[L:\mathbb{Q}])^{c_\ell}} \le \big(2m^{D_2}\big)^{(2D_2)^{c_B}} \le m^{D_3},$$

using $\operatorname{length}_{\mathbb{Z}_\ell} = \log_\ell \#$, $c_\ell \le c_B$, $2|d_L| \ge 2$, $[L:\mathbb{Q}] \le D_2$ and $2 \le m$. In particular $\operatorname{len}_\ell(L,1) < \infty$.

Conclusion. The group $\mathrm{Sha}^1_{\ell N}(\chi,1)$ is contained in $\mathrm{Sha}^1_\ell(\chi,1)$. By Lemma 38.37 ($L \subseteq \mathbb{Q}(\zeta_m)$, χ trivial on G_L, $e = 1$), $\mathrm{Sha}^1_{\ell N}(\chi,1)$ has finite length and $\nu_\mathcal{O}(\chi,1) \le \operatorname{len}_\ell(L,1) + m\,[L:\mathbb{Q}]$. Hence, with $\log\ell \le \ell \le m$ and $m \ge 48$,

$$\nu_\mathcal{O}(\chi,1)\log\ell \le m^{D_3} + D_2\, m^2 \le m^{D_3+D_2+3} = (\ell N_\mathfrak{h} e)^C,$$

which is the claim. □

Proposition 38.36. *Let L be a totally complex number field, ℓ an odd prime and $e \geq 1$ an integer. Then, in $\mathbb{N} \cup \{\infty\}$,*

$$\operatorname{len}_\ell(L, e) \leq \operatorname{length}_{\mathbb{Z}_\ell} H^2(G_{L,S_\ell}, \mathbb{Z}_\ell(e+1)).$$

Proof. Write $G = G_{L,S_\ell}$, $T = \mathbb{Z}_\ell(e+1)$ and $h = \operatorname{length}_{\mathbb{Z}_\ell} H^2(G, T)$. If $h = \infty$ there is nothing to prove, so let $h < \infty$. The extension $L(\mu_{\ell^\infty})/L$ is unramified outside the places above ℓ and the archimedean places, which are complex; so χ_{cyc} factors through G. For $n \geq 1$ let A_n be $\mathbb{Z}/\ell^n\mathbb{Z}$ with G acting through χ_{cyc}^{-e}, and $T_n = T/\ell^n T$; both are finite discrete G-modules with continuous action.

The places. Let S be the set of places above ℓ together with the archimedean places. A complex place does not ramify in any extension, so $G_{L,S} = G$; and for a complex place v the group G_{L_v} is trivial, so $H^i(L_v, M) = 0$ for $i \geq 1$ and every module M. Hence, for $i = 1, 2$ and every discrete G-module M,

$$\operatorname{Sha}^i_S(L, M) = \ker\Big(H^i(G, M) \to \prod_{v \mid \ell} H^i(L_v, M)\Big),$$

and in particular $\operatorname{Sha}^1_\ell(L, e) = \operatorname{Sha}^1_S(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))$.

Duality at finite level. The Cartier dual $A_n^\vee = \operatorname{Hom}(A_n, \mu_{\ell^n})$ is $\mathbb{Z}/\ell^n\mathbb{Z}$ with G acting through $\chi_{\mathrm{cyc}} \cdot \chi_{\mathrm{cyc}}^{e}$, that is, $A_n^\vee \cong T_n$. By Theorem B.34 (L has no real place, S contains the places above ℓ and the archimedean places, A_n is a finite $\mathbb{Z}/\ell^n\mathbb{Z}$-module with continuous action of $G_{L,S} = G$), $\operatorname{Sha}^1_S(L, A_n)$ and $\operatorname{Sha}^2_S(L, T_n)$ are finite of the same order, so

$$\operatorname{length} \operatorname{Sha}^1_S(L, A_n) = \operatorname{length} \operatorname{Sha}^2_S(L, T_n) \leq \operatorname{length} H^2(G, T_n). \tag{38.9}$$

$H^2(G, T_n)$ is a quotient of $H^2(G, T)$. The reductions $T_{m+1} \to T_m$ are surjective and $\varprojlim_m T_m = T$ with its ℓ-adic topology. Each $H^1(G, T_m)$ is finite by Theorem B.34, so by Lemma B.33 (with $i = 2$) the map $\pi\colon H^2(G, T) \to \varprojlim_m H^2(G, T_m)$ induced by the projections is an isomorphism. For $m \geq 1$ the sequence $0 \to T_1 \to T_{m+1} \to T_m \to 0$, whose first map is $x \bmod \ell \mapsto \ell^m x \bmod \ell^{m+1}$, is an exact sequence of finite discrete G-modules; its cohomology sequence gives the exact sequence $H^2(G, T_{m+1}) \to H^2(G, T_m) \to H^3(G, T_1)$, and $H^3(G, T_1) = 0$ by Theorem B.34. So the transition maps of $(H^2(G, T_m))_m$ are surjective, and so is the projection $\varprojlim_m H^2(G, T_m) \to H^2(G, T_n)$: an element y_n extends to a compatible family by choosing preimages $y_{n+1}, y_{n+2}, \ldots$ successively and taking images for $m < n$. The composite of π with this projection is the map induced by $T \to T_n$. Hence $H^2(G, T_n)$ is a quotient of $H^2(G, T)$, and with (38.9)

$$\operatorname{length} \operatorname{Sha}^1_S(L, A_n) \leq h \qquad (n \geq 1). \tag{38.10}$$

Passage to $\mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$. The G-equivariant maps $\iota_n\colon A_n \to \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$, $a \bmod \ell^n \mapsto a/\ell^n$, are injective, compatible with the inclusions $A_n \to A_{n+1}$ given by multiplication by ℓ, and their images exhaust $\mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$. For a profinite group, the cohomology with coefficients in a direct limit of discrete modules is the direct limit of the cohomology groups [222] (Chapter I, Section 2.2, Proposition 8); this applies to G and to each G_{L_v}. Let $P \subseteq \operatorname{Sha}^1_\ell(L, e)$ be the submodule generated by finitely many elements $x_1, \ldots, x_r$. There are n and $y_j \in H^1(G, A_n)$ with $\iota_{n*}(y_j) = x_j$. For each of the finitely many places $v \mid \ell$, the restriction of y_j to $H^1(L_v, A_n)$ maps to the restriction of x_j, which is 0, in $H^1(L_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)) = \varinjlim_m H^1(L_v, A_m)$; so it maps

to 0 in $H^1(L_v, A_m)$ for some $m \geq n$. Taking one m for all j and v, the images y_j' of the y_j in $H^1(G, A_m)$ lie in $\mathrm{Sha}^1_S(L, A_m)$, and $\iota_{m*}(y_j') = x_j$. So P is the image under ι_{m*} of a submodule of $\mathrm{Sha}^1_S(L, A_m)$, and length $P \leq h$ by (38.10).

Conclusion. Suppose $\mathrm{len}_\ell(L, e) > h$. Then $\mathrm{Sha}^1_\ell(L, e)$ has a chain of submodules $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{h+1}$; choose $x_i \in P_i \setminus P_{i-1}$ for $1 \leq i \leq h+1$. As $x_i \notin P_{i-1} \supseteq \langle x_1, \ldots, x_{i-1}\rangle$, the submodules $\langle x_1, \ldots, x_i\rangle$, $0 \leq i \leq h+1$, form a strictly increasing chain in $\langle x_1, \ldots, x_{h+1}\rangle$, which therefore has length at least $h+1$, contradicting the previous step. Hence $\mathrm{len}_\ell(L, e) \leq h$. □

Lemma 38.37. *Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring, $m \geq 1$ an integer, $L \subseteq \mathbb{Q}(\zeta_m)$ a subfield, $N \geq 1$, χ a Dirichlet character modulo N with values in $\mathcal{O}^\times$ that is trivial on G_L, and $e \geq 1$. If $\mathrm{len}_\ell(L, e) < \infty$, then $\mathrm{Sha}^1_\ell(\chi, e)$ has finite length and*

$$\mathrm{length}_{\mathcal{O}} \mathrm{Sha}^1_\ell(\chi, e) \leq e_{\mathcal{O}}(\mathrm{len}_\ell(L, e) + e\, m\, [L : \mathbb{Q}]).$$

Proof. Put $M = \mathcal{O}^\vee(\Psi_{\chi,e}^{-1})$. As χ is trivial on G_L, the group G_L acts on M through χ_{cyc}^{-e}, so as a G_L-module $M = \mathcal{O} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$. The field L is Galois over $\mathbb{Q}$, as a subfield of the abelian field $\mathbb{Q}(\zeta_m)$. Let res: $H^1(\mathbb{Q}, M) \to H^1(L, M)$ be the restriction.

The kernel. By inflation–restriction, $\ker(\mathrm{res}) \cong H^1(\mathrm{Gal}(L/\mathbb{Q}), M^{G_L})$. Choose $g_0 \in G_{\mathbb{Q}(\zeta_m)} \subseteq G_L$ with $\chi_{\mathrm{cyc}}(g_0) = 1 + m$: the compatible system $(1 + m \bmod m\ell^n)_{n\geq 1}$ is an element of $\varprojlim_n (\mathbb{Z}/m\ell^n\mathbb{Z})^\times = \mathrm{Gal}(\mathbb{Q}(\zeta_m, \mu_{\ell^\infty})/\mathbb{Q})$ fixing ζ_m; take any lift to $G_\mathbb{Q}$. For $m_1 \in M^{G_L}$ we get $((1+m)^{-e} - 1)m_1 = 0$. The integer $u_1 = (1+m)^e - 1 \geq 1$ satisfies $(1+m)^{-e} - 1 = -(1+m)^{-e}u_1$ with $(1+m)^{-e} \in \mathbb{Z}_\ell^\times$; writing $u_1 = \ell^{b_1}u'$ with $\ell \nmid u'$, we get $\ell^{b_1}m_1 = 0$. So $M^{G_L} \subseteq M[\ell^{b_1}] \cong \mathcal{O}/\ell^{b_1}\mathcal{O}$, of length $b_1 e_{\mathcal{O}} \leq e\, m\, e_{\mathcal{O}}$.

For a finite group Γ and a finite $\mathcal{O}[\Gamma]$-module A, evaluation $z \mapsto (z(\gamma))_{\gamma\in\Gamma}$ embeds the cocycles $\mathcal{O}$-linearly into $A^{|\Gamma|}$, so length $H^1(\Gamma, A) \leq |\Gamma|$ length A. Hence $\mathrm{length}_{\mathcal{O}} \ker(\mathrm{res}) \leq [L : \mathbb{Q}]\, e\, m\, e_{\mathcal{O}}$.

The image. Let $[z] \in \mathrm{Sha}^1_\ell(\chi, e)$. Conjugation by $G_\mathbb{Q}$ acts trivially on $H^1(\mathbb{Q}, M)$, so $[z]$ restricts to zero on every inertia subgroup of $G_\mathbb{Q}$ and on every decomposition subgroup at ℓ. The inertia and decomposition groups of G_L are the intersections of those of $G_\mathbb{Q}$ with G_L; so res$[z]$ is unramified at every place of L and trivial at the places above ℓ. Let L_{S_ℓ} be the maximal extension of L unramified outside the places above ℓ; M is unramified outside ℓ as a G_L-module, and $\mathrm{Gal}(\overline{\mathbb{Q}}/L_{S_\ell})$ is the closed normal subgroup of G_L generated by the inertia groups at the places not above ℓ. Restricted to it, res$[z]$ is a homomorphism vanishing on these generators, hence zero; so res$[z]$ is the inflation of a unique class of $H^1(G_{L,S_\ell}, M)$, locally trivial at the places above ℓ. Thus res maps $\mathrm{Sha}^1_\ell(\chi, e)$ $\mathcal{O}$-linearly into

$$\mathrm{Sha}_{\mathcal{O}} = \ker\Big(H^1(G_{L,S_\ell}, M) \to \prod_{v|\ell} H^1(L_v, M)\Big) \cong \mathrm{Sha}^1_\ell(L, e) \otimes_{\mathbb{Z}_\ell} \mathcal{O},$$

the isomorphism because $\mathcal{O}$ is free of rank $r = [\mathcal{O} : \mathbb{Z}_\ell]$ over $\mathbb{Z}_\ell$ and cohomology commutes with finite direct sums. As $\mathrm{len}_\ell(L, e) < \infty$, $\mathrm{Sha}^1_\ell(L, e)$ has order $\ell^{\mathrm{len}_\ell(L,e)}$, $\mathrm{Sha}_{\mathcal{O}}$ has order $\ell^{r\,\mathrm{len}_\ell(L,e)}$, and $\mathrm{length}_{\mathcal{O}} \mathrm{Sha}_{\mathcal{O}} = r\,\mathrm{len}_\ell(L, e)/f_{\mathcal{O}} = e_{\mathcal{O}}\,\mathrm{len}_\ell(L, e)$.

Conclusion. From the exact sequence $0 \to \ker(\mathrm{res}) \cap \mathrm{Sha}^1_\ell(\chi, e) \to \mathrm{Sha}^1_\ell(\chi, e) \to \mathrm{Sha}_{\mathcal{O}}$, the group $\mathrm{Sha}^1_\ell(\chi, e)$ has finite length, at most $[L : \mathbb{Q}]\, e\, m\, e_{\mathcal{O}} + e_{\mathcal{O}}\,\mathrm{len}_\ell(L, e)$. □

Proposition 38.38. *Let $\psi \in \mathcal{P}$, $\ell \in B_\psi$ and let $j \geq 0$ be an integer. There is an integer $C \geq 0$ such that the following holds for every Sha-host $\mathfrak{h}$ of ψ, every integer $t \geq 2$ at which $\mathfrak{h}$ is big, and every real B such that $\mathrm{V}_\ell(\mathbb{Q}(\zeta_{\ell^{j+1}N_\psi}); t, B)$ holds. For every coefficient ring $\mathcal{O}$ and every*

Dirichlet character χ modulo $N_{\mathfrak{h}}$ with values in $\mathcal{O}^{\times}$ such that $H_{\mathfrak{h}} \subseteq \ker \chi$ and $\chi^{(\ell)}$ has odd order, the groups $\mathrm{Sha}^1_{\ell N_{\mathfrak{h}}}(\chi, e)$ with $1 \le e < t$ have finite length and

$$\sum_{e=1}^{t-1} \nu_{\mathcal{O}}(\chi, e) \le B + (N_{\mathfrak{h}} t)^C.$$

Proof. Put $m = \ell^{j+1} N_\psi$, $L = \mathbb{Q}(\zeta_m)$ and $C = 2m + 2$; these depend only on (ψ, ℓ, j). Fix $\mathfrak{h} = (a, q, d)$, t, B, $\mathcal{O}$ and χ as in the statement, and write $N = N_{\mathfrak{h}}$.

The character is trivial on G_L. The set B_ψ consists of odd primes: 3, and prime factors of $o_\psi N_\psi\, p_K(\psi)\, h^*(\psi)\, p_1(\psi)$. Since $\mathfrak{h}$ is big at $t \ge 2$, $\min(q, d) > 2t + 1 \ge 5$ and $\min(q, d)$ exceeds each of $o_\psi, N_\psi, p_K(\psi), h^*(\psi), p_1(\psi)$; so $\ell < \min(q, d)$, and ℓ is odd with $\ell \notin \{q, d\}$. By item 3 of Lemma 38.34, $\chi(u) = 1$ whenever $u \equiv 1 \pmod{N_\psi}$. As a character of $G_{\mathbb{Q}}$, χ factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^{\times}$, $\sigma_u \colon \zeta_N \mapsto \zeta_N^u$; an element of G_L fixes ζ_{N_ψ}, so it acts on $\mathbb{Q}(\zeta_N)$ as some σ_u with $u \equiv 1 \pmod{N_\psi}$. Hence χ is trivial on G_L.

One twist. Let $1 \le e < t$. By hypothesis $\mathrm{V}_\ell(L; t, B)$ holds, so $\mathrm{len}_\ell(L, e) < \infty$. The group $\mathrm{Sha}^1_{\ell N}(\chi, e)$ is contained in $\mathrm{Sha}^1_{\ell}(\chi, e)$ (it satisfies more local conditions). By Lemma 38.37 (for $L = \mathbb{Q}(\zeta_m)$, $[L : \mathbb{Q}] = \varphi(m) \le m$), $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length and

$$\nu_{\mathcal{O}}(\chi, e) = \frac{\mathrm{length}_{\mathcal{O}} \mathrm{Sha}^1_{\ell N}(\chi, e)}{e_{\mathcal{O}}} \le \mathrm{len}_\ell(L, e) + e\, m^2.$$

Summation. Summing over $1 \le e < t$ and using $\sum_{e=1}^{t-1} \mathrm{len}_\ell(L, e) \le B$,

$$\sum_{e=1}^{t-1} \nu_{\mathcal{O}}(\chi, e) \le B + m^2 \sum_{e=1}^{t-1} e \le B + m^2 t^2 \le B + (Nt)^C.$$

This is the claim, with the integer $C = 2m + 2 \ge 0$ depending only on (ψ, ℓ, j). □

39 The prime 2 and the bad primes on windows

This section proves Proposition 39.2. For every pair of primes (q, d) of the shape of a window pair with $\log(qd) \le X^{\varepsilon/(4J)}$ it gives a range $[Y, Y^2]$ inside $[X^{1/J}, X^{1/2}]$ on which, for every x in the range and $t = \lfloor x^{401} \rfloor$, the Tate–Shafarevich groups at 2 of the fields of the Sha-hosts, and at each bad prime ℓ those of one cyclotomic field $\mathbb{Q}(\zeta_{\ell^{j\ell+1} N_\psi})$, have total length at most $\exp(x^{4\varepsilon/5})$ over the twists $1 \le k < t$. It is used in the proof of Theorem 36.4, directly at the prime 2 and through Proposition 36.10 at the bad primes, and it is the reason why the window Sha bound holds on windows.

The tool is twist control (Definition 39.1). A field has twist control at ℓ with rank R if over every cyclotomic ℓ-layer the length of the group at the twist e grows at most linearly in the index of the layer and in D, as long as e keeps an ℓ-adic distance at least ℓ^{-D} from at most R fixed ℓ-adic numbers (Tw1), and is bounded in general by second cohomology at one fixed layer (Tw2). Twist control follows from a bounded rank of the unramified characters of the cyclotomic ℓ-tower (Proposition 39.8), through the structure of a finitely generated $\mathbb{Z}_\ell$-module with an endomorphism that is unipotent modulo ℓ (Lemma 39.9). The bounded rank comes from the bounded ℓ-rank of class groups along the cyclotomic tower of an abelian field, a consequence of the theorem of Ferrero and Washington [90] (Lemma 39.5, Lemma 39.7). At 2 the control must be uniform in the window pair. The fields of the pair lie in a cyclotomic

2-layer over $\mathbb{Q}(\zeta_{N_\psi})^\psi(i,\sqrt{-q},\sqrt{-d})$ (Lemma 39.4), and the 2-rank of the unramified characters of its cyclotomic tower is bounded independently of (q,d) (Lemma 39.12, Proposition 39.11).

The windows come from the exceptional twists. Below X^{401} at most R twists, the total rank over 2 and the bad primes, lie ℓ-adically close to one of the fixed points, and at them only the bound of Section 40 is available, doubly exponential in the twist. A lemma on separation of scales (Lemma 39.3) chooses Y so that every exceptional twist below x^{401}, for $x \in [Y, Y^2]$, lies below a small power $x^{\varepsilon_{\mathrm{H}}}$, where that bound is still at most $\exp(O(x^{3\varepsilon/4}))$. The argument produces one such range inside every $[X^{1/J}, X^{1/2}]$, not a bound at every large scale; this is why the bound is stated on windows.

Definition 39.1. Let ℓ be a prime and L a number field. For an integer $m \geq 0$, a **cyclotomic ℓ-layer of degree ℓ^m over L** is a field extension L'/L such that $L' = L(\zeta)$ for some $k \geq 0$ and some primitive ℓ^k-th root of unity $\zeta \in L'$, and $[L' : L] = \ell^m$. For an integer $R \geq 1$ put

$$j_\ell(R) = \lfloor \log_\ell R \rfloor + 3.$$

For $n \in \mathbb{Z}$, $\mathbb{Z}_\ell(n)$ is $\mathbb{Z}_\ell$ with G_{L,S_ℓ} acting through χ_{cyc}^n (the cyclotomic character factors through G_{L,S_ℓ}), and $H^2(G_{L,S_\ell}, \mathbb{Z}_\ell(n))$ is continuous cohomology; its length as a $\mathbb{Z}_\ell$-module lies in $\mathbb{N} \cup \{\infty\}$, with $\infty + n = \infty$.

Let $R \geq 1$ be an integer. The field L has **twist control at ℓ with rank R** if the following two conditions hold.

(Tw1) *Roots.* There are a field $K \supseteq \mathbb{Q}_\ell$ with an ultrametric absolute value $|\cdot|$ extending the ℓ-adic absolute value, an integer $c_\mu \geq 0$ such that L contains a primitive ℓ^{c_μ}-th root of unity, an integer R' with $0 \leq R' \leq R$ and elements $s_1, \dots, s_{R'} \in K$, such that for all integers $e \geq 1$, $m \geq 0$, $D \geq 0$ with

$$|s_i - e| \geq \ell^{-D} \qquad (1 \leq i \leq R')$$

and every cyclotomic ℓ-layer L' of degree ℓ^m over L,

$$\mathrm{len}_\ell(L', e) \leq R'\big(\max(m, j_\ell(R)) + c_\mu + D\big).$$

(Tw2) *Second bound.* For all integers $e \geq 1$ and $m \geq 0$, every cyclotomic ℓ-layer L' of degree ℓ^m over L and every cyclotomic ℓ-layer L'' of degree $\ell^{j_\ell(R)}$ over L,

$$\mathrm{len}_\ell(L', e) \leq \mathrm{length}_{\mathbb{Z}_\ell} H^2\big(G_{L'',S_\ell}, \mathbb{Z}_\ell(e+1)\big) + R\max\big(m - j_\ell(R), 0\big).$$

Proposition 39.2. *Let $\psi \in \mathcal{P}$, $\varepsilon \in (0, 1/52)$ and let e_ε be a page function. There are a real $J \geq 1$, a real X_0 and integers $j_\ell \geq 0$ for $\ell \in B_\psi$ such that for every $X \geq X_0$ and every pair of primes (q,d) with*

$$q \equiv 3 \pmod 8, \qquad d \equiv 7 \pmod{16}, \qquad \gcd(qd, N_\psi) = 1, \qquad \log(qd) \leq X^{\varepsilon/(4J)},$$

there is a real Y with $X^{1/J} \leq Y$ and $Y^2 \leq X$ such that for every $x \in [Y, Y^2]$, with $t = \lfloor x^{401} \rfloor$ and $B = \exp(x^{4\varepsilon/5})$:

1. *every $(\psi, e_\varepsilon(x), q, d)$-field E satisfies $\mathrm{V}_2(E; t, B)$;*
2. *for every $\ell \in B_\psi$ the cyclotomic field $\mathbb{Q}(\zeta_{\ell^{j_\ell+1}N_\psi})$ satisfies $\mathrm{V}_\ell\big(\mathbb{Q}(\zeta_{\ell^{j_\ell+1}N_\psi}); t, B\big)$.*

Proof. Fix $\psi \in \mathcal{P}$, $\varepsilon \in (0, 1/52)$ and a page function e_ε. We first outline the argument. At a prime $\ell \in \{2\} \cup B_\psi$ the Tate–Shafarevich group at the twist k is small, of size $O(\log x)$, unless k is ℓ-adically close to one of at most R fixed ℓ-adic numbers (the points of (Tw1)); in $[1, X^{401}]$ there are at most R such exceptional twists, and the scale Y is chosen so that every exceptional twist below x^{401} is in fact below $x^{\varepsilon_{\mathrm{H}}}$ for a small exponent ε_{H}, where the crude bound Theorem 40.1 suffices.

Step 1: the inputs and the constants.

- For $\ell \in B_\psi$ the field $\mathbb{Q}(\zeta_{\ell N_\psi})$ is abelian over $\mathbb{Q}$ and contains a primitive ℓ-th root of unity. By Lemma 39.5 it satisfies the hypothesis of Proposition 39.6, which gives an integer $R_\ell \geq 1$ such that $\mathbb{Q}(\zeta_{\ell N_\psi})$ has twist control at ℓ with rank R_ℓ. Put $j_\ell = j_\ell(R_\ell) \geq 3$; these are the integers of the statement.

- The field $\mathbb{Q}(\zeta_{N_\psi})^\psi(i)$, with $\mathbb{Q}(\zeta_{N_\psi})^\psi$ as in Proposition 39.11, lies in $\mathbb{Q}(\zeta_{\mathrm{lcm}(4,N_\psi)})$, so it is abelian over $\mathbb{Q}$, and it contains i. By Lemma 39.5 for $\ell = 2$ it satisfies the hypothesis of Proposition 39.11, which gives an integer $R_2 \geq 1$ such that for every pair of primes (q, d) with $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$, $\gcd(qd, N_\psi) = 1$ the field $L_{q,d} = \mathbb{Q}(\zeta_{N_\psi})^\psi(i, \sqrt{-q}, \sqrt{-d})$ has twist control at 2 with rank R_2. Put $j_2 = j_2(R_2)$.

- For $\ell \in \{2\} \cup B_\psi$ let $C_\ell \geq 1$ be the constant of Theorem 40.1 and $C_{\mathrm{H}} = \max_\ell C_\ell$. The bound of Theorem 40.1 holds with C_{H} in place of C_ℓ, because $2|d_L| > 1$ and $2e[L : \mathbb{Q}] \geq 1$ make $(2|d_L|)^{(2e[L:\mathbb{Q}])^C}$ increasing in C.

- Put $R = R_2 + \sum_{\ell \in B_\psi} R_\ell$, $\varepsilon_{\mathrm{H}} = \varepsilon/(2C_{\mathrm{H}})$, $\beta = 401/\varepsilon_{\mathrm{H}} = 802C_{\mathrm{H}}/\varepsilon \geq 1$ and $J = 2^{R+2}\beta^R \geq 1$.

Step 2: the fields at the odd primes. Let $\ell \in B_\psi$, write $N_\psi = \ell^v N'$ with $\ell \nmid N'$, and put $L_\ell = \mathbb{Q}(\zeta_{\ell^{j_\ell+1} N_\psi})$. Adjoining a primitive ℓ^{v+1+j_ℓ}-th root of unity to $\mathbb{Q}(\zeta_{\ell N_\psi}) = \mathbb{Q}(\zeta_{\ell^{v+1}N'})$ gives L_ℓ, of degree $\varphi(\ell^{v+1+j_\ell})/\varphi(\ell^{v+1}) = \ell^{j_\ell}$ over it; so L_ℓ is a cyclotomic ℓ-layer of degree ℓ^{j_ℓ} over $\mathbb{Q}(\zeta_{\ell N_\psi})$. It contains a primitive ℓ-th root of unity and, as $\ell \geq 3$, has no real embedding. Put $n_\ell = [L_\ell : \mathbb{Q}]$ and $\delta_\ell = \log(2|d_{L_\ell}|) > 0$. Let $c_{\mu,\ell}$, $R'_\ell \leq R_\ell$ and the points $s_{\ell,1}, \dots, s_{\ell,R'_\ell}$ be the data of (Tw1) for $\mathbb{Q}(\zeta_{\ell N_\psi})$.

Step 3: the fields at 2. Let (q, d) be a pair as in the statement and $L = L_{q,d}$. By item 1 of Lemma 39.4, $i \in L$, $[L : \mathbb{Q}] \leq 8o_\psi$ and $L \subseteq \mathbb{Q}(\zeta_{4N_\psi qd})$. If L contains a primitive 2^c-th root of unity with $c \geq 1$, then $2^{c-1} = \varphi(2^c) \leq [L : \mathbb{Q}]$, so $c \leq 2^c \leq 16o_\psi$. Let $c_{\max}$ be the largest integer with $\mu_{2^{c_{\max}}} \subseteq L$; then $2 \leq c_{\max} \leq 16o_\psi$. The cyclotomic character identifies $\mathrm{Gal}(L(\mu_{2^\infty})/L)$ with a closed subgroup of $1 + 2^{c_{\max}}\mathbb{Z}_2$ not contained in $1 + 2^{c_{\max}+1}\mathbb{Z}_2$; as $c_{\max} \geq 2$, the closed subgroups of $1 + 2^{c_{\max}}\mathbb{Z}_2 \cong \mathbb{Z}_2$ are $\{1\}$ and the $1 + 2^k\mathbb{Z}_2$ with $k \geq c_{\max}$, so the image is all of $1 + 2^{c_{\max}}\mathbb{Z}_2$, and $L'' = L(\zeta_{2^{c_{\max}+j_2}})$ has degree 2^{j_2} over L: it is a cyclotomic 2-layer of degree 2^{j_2} over L. It contains i, so it has no real embedding. Its degree is $n'' \leq n_2 := 2^{j_2} \cdot 8o_\psi$, and $L'' \subseteq \mathbb{Q}(\zeta_{N_*})$ with $N_* = 2^{c_{\max}+j_2} \cdot 4N_\psi qd$. For a subfield L° of $\mathbb{Q}(\zeta_{N_*})$ the tower formula for discriminants gives $|d_{L^\circ}|^{[\mathbb{Q}(\zeta_{N_*}):L^\circ]} \leq |d_{\mathbb{Q}(\zeta_{N_*})}| \leq N_*^{\varphi(N_*)}$ ([266], Chapter 2), so $|d_{L^\circ}| \leq N_*^{[L^\circ:\mathbb{Q}]}$. Hence

$$\log(2|d_{L''}|) \leq \gamma_2 + n_2 \log(qd), \qquad \gamma_2 = \log 2 + n_2\big((16o_\psi + j_2 + 2)\log 2 + \log N_\psi\big). \tag{39.1}$$

Let c_μ, $R' \leq R_2$ and the points $s_1, \dots, s_{R'}$ be the data of (Tw1) for L; by the above, $c_\mu \leq 16o_\psi$.

Step 4: exceptional twists at a scale X. Let $X \geq X_0$, with $X_0 \geq 2$ fixed in Step 7, let (q, d) be as in the statement, and put $Z = \lfloor X^{401} \rfloor \geq 1$. For $\ell \in B_\psi$ (base field $\mathbb{Q}(\zeta_{\ell N_\psi})$, points $s_{\ell,i}$) and for $\ell = 2$ (base field $L_{q,d}$, points s_i) let Exc_ℓ be the set of integers $k \in [1, Z]$ with $|s - k| < \ell^{-b_\ell(Z)}$ for one of the points s at ℓ.

1. $|\mathrm{Exc}_\ell| \le R_\ell$. Let s be one of the points and let $k \ne k'$ in $[1,Z]$ satisfy $|s-k|, |s-k'| < \ell^{-b}$, $b = b_\ell(Z)$. By the ultrametric inequality $|k-k'| < \ell^{-b}$; the absolute value of the nonzero integer $k-k'$ is $\ell^{-v_\ell(k-k')}$, so $\ell^{b+1} \mid k-k'$. But the ordinary absolute value of $k-k'$ is positive and less than $Z < \ell^{b+1}$ by the maximality of b, a contradiction. So each point accounts for at most one element of Exc_ℓ, and there are at most R_ℓ points.

2. For an integer $k \in [1,Z] \setminus \mathrm{Exc}_\ell$, every $m \ge 0$ and every cyclotomic ℓ-layer L' of degree ℓ^m over the base field, (Tw1) applied with $e = k$ and $D = b_\ell(Z)$ gives $\mathrm{len}_\ell(L',k) \le R_\ell\big(\max(m, j_\ell) + c_{\mu,\ell} + b_\ell(Z)\big)$, with $c_{\mu,\ell}$ replaced by c_μ when $\ell = 2$.

Put $\mathrm{Exc} = \mathrm{Exc}_2 \cup \bigcup_{\ell \in B_\psi} \mathrm{Exc}_\ell$; then $|\mathrm{Exc}| \le R$.

Step 5: the choice of Y. Apply Lemma 39.3 to R, β, $U = \log X > 0$ and the set $\{(\log k)/401 : k \in \mathrm{Exc}\}$ of at most R reals. It gives y with $U/J \le y$ and $2y \le U$; put $Y = \exp(y)$, so that $X^{1/J} \le Y$ and $Y^2 \le X$. Let $x \in [Y, Y^2]$ and $k \in \mathrm{Exc}$ with $k \le x^{401}$. Then $u = \log x \in [y, 2y]$ and $(\log k)/401 \le u$, so $\beta(\log k)/401 \le u$, that is $(\log k)/\varepsilon_{\mathrm{H}} \le \log x$:

$$k \in \mathrm{Exc},\ k \le x^{401} \implies k \le x^{\varepsilon_{\mathrm{H}}} \qquad (x \in [Y, Y^2]). \tag{39.2}$$

Step 6: the bound at one twist. Let $x \in [Y, Y^2]$, $t = \lfloor x^{401} \rfloor \ge 1$ and $1 \le k < t$. Since $x \le Y^2 \le X$, $k \le x^{401} \le X^{401}$, so $k \le Z$. Since $X^{1/J} \le x$, $\log X \le J \log x$ and $b_\ell(Z) \le \log Z / \log 2 \le (401/\log 2)\, J \log x \le 579 J \log x$; also $\log(qd) \le X^{\varepsilon/(4J)} \le x^{\varepsilon/4}$. If $k \in \mathrm{Exc}_\ell$ then $k \le x^{\varepsilon_{\mathrm{H}}}$ by (39.2), so $(2kn)^{C_{\mathrm{H}}} \le (2n)^{C_{\mathrm{H}}} x^{\varepsilon/2}$ for every $n \ge 1$. For a finite $\mathbb{Z}_\ell$-module H, $\#H = \ell^{\mathrm{length}\, H}$, so $\mathrm{length}\, H = \log \#H / \log \ell \le 2 \log \#H$.

1. $\ell \in B_\psi$. If $k \notin \mathrm{Exc}_\ell$, Step 4 for $L' = L_\ell$, $m = j_\ell$, gives $\mathrm{len}_\ell(L_\ell, k) \le R_\ell(j_\ell + c_{\mu,\ell} + 579 J \log x)$. If $k \in \mathrm{Exc}_\ell$, (Tw2) with $L' = L'' = L_\ell$ ($m = j_\ell$) and Theorem 40.1 for L_ℓ give

$$\mathrm{len}_\ell(L_\ell, k) \le \mathrm{length}\, H^2\big(G_{L_\ell, S_\ell}, \mathbb{Z}_\ell(k+1)\big) \le 2\exp\big((2kn_\ell)^{C_{\mathrm{H}}} \delta_\ell\big) \le 2\exp\big((2n_\ell)^{C_{\mathrm{H}}} \delta_\ell\, x^{\varepsilon/2}\big).$$

2. $\ell = 2$. Let $a = e_\varepsilon(x)$ and let E be a (ψ, a, q, d)-field. As $x \ge 2$, $2^a \ge x^{2+8\varepsilon} \ge 4$, so $a \ge 2$, and $2^a \le 2x^{2+8\varepsilon} \le 2x^3$, so $a \le 1 + 5\log x$. By item 2 of Lemma 39.4 there is a cyclotomic 2-layer L' of degree 2^m over $L_{q,d}$, $m \le a$, with $\mathrm{len}_2(E,k) \le 1 + \mathrm{len}_2(L',k)$. If $k \notin \mathrm{Exc}_2$, Step 4 gives

$$\mathrm{len}_2(L',k) \le R_2(m + j_2 + c_\mu + 579 J \log x) \le R_2(1 + j_2 + 16 o_\psi) + R_2(5 + 579J)\log x.$$

If $k \in \mathrm{Exc}_2$, (Tw2) with the layer L'' of Step 3, Theorem 40.1 for L'' (no real embedding, $i \in L''$), (39.1) and $\log(qd) \le x^{\varepsilon/4}$ give

$$\begin{aligned}\mathrm{len}_2(L',k) &\le 2\exp\big((2kn'')^{C_{\mathrm{H}}} \log(2|d_{L''}|)\big) + R_2 m \\ &\le 2\exp\big((2n_2)^{C_{\mathrm{H}}} (\gamma_2 + n_2)\, x^{3\varepsilon/4}\big) + R_2(1 + 5\log x).\end{aligned}$$

Put

$$C_{\mathrm{lin}} = \max\big(R_2(6 + j_2 + 16 o_\psi + 579J),\ \max_{\ell \in B_\psi} R_\ell(j_\ell + c_{\mu,\ell} + 579J)\big),$$
$$C_{\mathrm{exp}} = \max\big((2n_2)^{C_{\mathrm{H}}}(\gamma_2 + n_2),\ \max_{\ell \in B_\psi} (2n_\ell)^{C_{\mathrm{H}}} \delta_\ell\big);$$

these depend only on (ψ, ε) (the bounds $c_\mu \le 16 o_\psi$ and $n'' \le n_2$ are uniform in (q,d)). Since $x \ge 1$, $x^{\varepsilon/2} \le x^{3\varepsilon/4}$, and in every case above, for $1 \le k < t$,

$$\mathrm{len}_\ell(L_\ell, k)\ (\ell \in B_\psi),\ \ \mathrm{len}_2(E,k) \ \le\ \Upsilon(x) := 1 + C_{\mathrm{lin}}(1 + \log x) + 2\exp\big(C_{\mathrm{exp}} x^{3\varepsilon/4}\big);$$

in particular all these lengths are finite.

Step 7: the threshold. As $\exp(C_{\exp}x^{3\varepsilon/4}) \geq 1$, $\Upsilon(x) \leq (3 + C_{\lin}(1 + \log x))\exp(C_{\exp}x^{3\varepsilon/4})$, so $\log(x^{401}\Upsilon(x)) \leq 401\log x + \log(3 + C_{\lin}(1 + \log x)) + C_{\exp}x^{3\varepsilon/4}$, which is at most $x^{4\varepsilon/5}$ for all large x because $\log x = o(x^{3\varepsilon/4})$ and $x^{3\varepsilon/4} = o(x^{4\varepsilon/5})$. Choose $x_0 \geq 2$ with $x^{401}\Upsilon(x) \leq \exp(x^{4\varepsilon/5})$ for $x \geq x_0$, and put $X_0 = \max(2, x_0^J)$. For $X \geq X_0$ every $x \in [Y, Y^2]$ satisfies $x \geq X^{1/J} \geq x_0 \geq 2$, as used in Step 6.

Conclusion. Let $X \geq X_0$, (q, d) as in the statement, Y as in Step 5, $x \in [Y, Y^2]$, $t = \lfloor x^{401} \rfloor$ and $B = \exp(x^{4\varepsilon/5})$. By Step 6, for every $(\psi, e_\varepsilon(x), q, d)$-field E,

$$\sum_{k=1}^{t-1} \len_2(E, k) \leq (t-1)\,\Upsilon(x) \leq x^{401}\Upsilon(x) \leq B,$$

with every term finite, which is $\mathrm{V}_2(E; t, B)$, item 1; and for $\ell \in B_\psi$ the same computation with $L_\ell = \mathbb{Q}(\zeta_{\ell^{j_\ell+1}N_\psi})$ gives $\mathrm{V}_\ell(L_\ell; t, B)$, item 2. The constants $J \geq 1$, X_0 and j_ℓ depend only on (ψ, ε). □

Lemma 39.3. *Let $R \geq 0$ be an integer, let $\beta \geq 1$ and $U > 0$ be real numbers, and let $T \subset \mathbb{R}$ be a finite set with $|T| \leq R$. There is a real number y with*

$$\frac{U}{2^{R+2}\beta^R} \leq y, \qquad 2y \leq U,$$

such that for every $a \in T$ and every $u \in [y, 2y]$ with $a \leq u$ we have $\beta a \leq u$.

Proof. We argue by induction on R, for all β, U and T at once.

If $R = 0$ then $T = \emptyset$, and $y = U/2$ works: $U/4 \leq U/2$ and $2y = U$.

Let $R \geq 0$, assume the claim for R, and let $|T| \leq R + 1$. Two cases.

Some $a_0 \in T$ satisfies $U/(2\beta) < a_0 \leq U$. Put $U' = U/(2\beta) > 0$ and $T' = T \setminus \{a_0\}$, so $|T'| \leq R$. The induction hypothesis for (β, U', T') gives y with $U'/(2^{R+2}\beta^R) \leq y$ and $2y \leq U'$, such that $a \leq u$ implies $\beta a \leq u$ for $a \in T'$ and $u \in [y, 2y]$. Then

$$\frac{U}{2^{R+3}\beta^{R+1}} = \frac{U'}{2^{R+2}\beta^R} \leq y, \qquad 2y \leq U' \leq U$$

(as $\beta \geq 1$). Let $a \in T$ and $u \in [y, 2y]$ with $a \leq u$. If $a = a_0$, then $a_0 \leq u \leq 2y \leq U' < a_0$, which is impossible; so $a \in T'$ and $\beta a \leq u$.

Every $a \in T$ satisfies $a \leq U/(2\beta)$ or $a > U$. Take $y = U/2$. Then $U/(2^{R+3}\beta^{R+1}) \leq U/2$ because $2^{R+3}\beta^{R+1} \geq 2$, and $2y = U$. Let $a \in T$ and $u \in [U/2, U]$ with $a \leq u$. Then $a \leq U$, so $a \leq U/(2\beta)$ and $\beta a \leq U/2 \leq u$.

In both cases y has the required properties for $R + 1$. □

Lemma 39.4. *Let $\psi \in \mathcal{P}$ and let q, d be primes with $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$ and $\gcd(qd, N_\psi) = 1$. Let $\mathbb{Q}(\zeta_{N_\psi})^\psi \subset \mathbb{C}$ be the subfield of $\mathbb{Q}(\zeta_{N_\psi})$ fixed by the automorphisms $\zeta \mapsto \zeta^u$ with $\psi(u) = 1$, and put*

$$L_{q,d} = \mathbb{Q}(\zeta_{N_\psi})^\psi(i, \sqrt{-q}, \sqrt{-d}) \subset \mathbb{C}.$$

1. *$i \in L_{q,d}$, $[L_{q,d} : \mathbb{Q}] \leq 8o_\psi$, and $L_{q,d} \subseteq \mathbb{Q}(\zeta_{4N_\psi qd})$.*
2. *For every integer $a \geq 2$ and every (ψ, a, q, d)-field E there are an integer $0 \leq m \leq a$ and a cyclotomic 2-layer L' of degree 2^m over $L_{q,d}$ such that $\len_2(E, k) \leq 1 + \len_2(L', k)$ for every $k \in \mathbb{Z}$.*

Proof. Item 1. Since $q \equiv 3 \pmod 4$, the Gauss sum $g_q = \sum_{x \bmod q}(x/q)\zeta_q^x$ satisfies $g_q^2 = (-1/q)\,q = -q$, so $\sqrt{-q} \in \mathbb{Q}(\zeta_q)$; likewise $\sqrt{-d} \in \mathbb{Q}(\zeta_d)$, as $d \equiv 3 \pmod 4$. With $i \in \mathbb{Q}(\zeta_4)$ and $\mathbb{Q}(\zeta_{N_\psi})^\psi \subseteq \mathbb{Q}(\zeta_{N_\psi})$ this gives $L_{q,d} \subseteq \mathbb{Q}(\zeta_{\mathrm{lcm}(4,N_\psi)qd}) \subseteq \mathbb{Q}(\zeta_{4N_\psi qd})$. By Galois theory $\mathrm{Gal}(\mathbb{Q}(\zeta_{N_\psi})^\psi/\mathbb{Q}) \cong (\mathbb{Z}/N_\psi\mathbb{Z})^\times/\ker\psi$, which is isomorphic to the image of ψ, of order o_ψ; adjoining three square roots multiplies the degree by at most 8. Finally, $i \in L_{q,d}$ by construction.

The layer. Let $a \geq 2$ and put $N = \mathrm{lcm}(2^{a+2}, N_\psi)\,qd$; then $\mathfrak{h} = (a, q, d)$ is a Sha-host of ψ with $N_\mathfrak{h} = N$. All fields below are subfields of $\mathbb{Q}(\zeta_N) \subset \mathbb{C}$, and we identify $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ with $(\mathbb{Z}/N\mathbb{Z})^\times$. The fixing groups of $\mathbb{Q}(\zeta_{2^{a+2}})$, $\mathbb{Q}(\zeta_{N_\psi})^\psi$, $\mathbb{Q}(\sqrt{-q})$ and $\mathbb{Q}(\sqrt{-d})$ are, respectively, the u with $u \equiv 1 \pmod{2^{a+2}}$, with $\psi(u) = 1$, with u a square modulo q, and with u a square modulo d; for the last two because $\mathbb{Q}(\sqrt{-q})$ is the unique quadratic subfield of the cyclic extension $\mathbb{Q}(\zeta_q)/\mathbb{Q}$, fixed by the squares of $(\mathbb{Z}/q\mathbb{Z})^\times$, and likewise for d. Let H_1 be the intersection of these four groups and L' its fixed field, the compositum of the four fields. As $i \in \mathbb{Q}(\zeta_{2^{a+2}})$, $L' = L_{q,d}(\zeta_{2^{a+2}})$; and $[L' : L_{q,d}]$ divides $[\mathbb{Q}(\zeta_{2^{a+2}}) : \mathbb{Q}(i)] = 2^a$, because L' is the compositum of $L_{q,d}$ and the Galois extension $\mathbb{Q}(\zeta_{2^{a+2}})$ of $\mathbb{Q}$, whose intersection with $L_{q,d}$ contains $\mathbb{Q}(i)$. So L' is a cyclotomic 2-layer of degree 2^m over $L_{q,d}$ with $0 \leq m \leq a$.

E inside the layer. Let E be a (ψ, a, q, d)-field. By item 1 of Lemma 36.6, E is isomorphic to the fixed field E_0 of $H_\mathfrak{h}$, and E_0 is a CM field by item 2 of Lemma 36.6. Lengths of Sha groups are invariant under isomorphisms of fields, so it suffices to treat E_0. By definition $H_1 = \{u \in H_\mathfrak{h} : u \equiv 1 \pmod{2^{a+2}}\}$, so $E_0 \subseteq L'$, and $H_\mathfrak{h}/H_1$ embeds into $\{\pm 1\} \subset (\mathbb{Z}/2^{a+2}\mathbb{Z})^\times$; hence $[L' : E_0] \leq 2$. The extension L'/E_0 is unramified at every prime $p \neq 2$: the inertia group of p in $(\mathbb{Z}/N\mathbb{Z})^\times$ is $I_p = \{u : u \equiv 1 \pmod{N/p^{v_p(N)}}\}$, and the inertia groups of L'/E_0 above p are the image of $I_p \cap H_\mathfrak{h}$ in $H_\mathfrak{h}/H_1$; an element of I_p is $\equiv 1 \pmod{2^{a+2}}$, since $2^{a+2} \mid N/p^{v_p(N)}$, so $I_p \cap H_\mathfrak{h} \subseteq H_1$. As E_0 is totally imaginary, no infinite place ramifies. Hence L' lies in the maximal extension E_{0,S_2} of E_0 unramified outside the places above 2, and E_{0,S_2} is also the maximal such extension of L'. So $G = G_{E_0,S_2}$ contains $H = G_{L',S_2}$ as an open normal subgroup with $|G/H| = [L' : E_0] \leq 2$.

Comparison of Sha. Let $k \in \mathbb{Z}$ and $M = \mathbb{Q}_2/\mathbb{Z}_2(-k)$, a discrete G-module. Inflation and restriction give an exact sequence

$$0 \to H^1(G/H, M^H) \to H^1(G, M) \xrightarrow{\mathrm{res}} H^1(H, M).$$

The group $H^1(G/H, M^H)$ has order at most 2: it is 0 if $G = H$; otherwise $G/H = \{1, \sigma\}$ and $H^1(G/H, M^H) = \ker(1+\sigma)/(\sigma - 1)M^H$ is a subquotient of the subgroup M^H of $M \cong \mathbb{Q}_2/\mathbb{Z}_2$ that is killed by $|G/H| = 2$; every subgroup of $\mathbb{Q}_2/\mathbb{Z}_2$ is finite cyclic or all of $\mathbb{Q}_2/\mathbb{Z}_2$, so every subquotient is finite cyclic or isomorphic to $\mathbb{Q}_2/\mathbb{Z}_2$, and such a group killed by 2 has order at most 2. Restriction maps $\mathrm{Sha}^1_2(E_0, k)$ into $\mathrm{Sha}^1_2(L', k)$: for a place w of L' above 2, lying over the place v of E_0, the composite of res with the localisation at w is the localisation at v followed by restriction from $E_{0,v}$ to L'_w, because both are induced by the inclusions $G_{L'_w} \subseteq G_{E_{0,v}}$ and $H \subseteq G$ (with compatible choices of decomposition groups); so a class trivial at every $v \mid 2$ restricts to a class trivial at every $w \mid 2$. The kernel of res on $\mathrm{Sha}^1_2(E_0, k)$ lies in $H^1(G/H, M^H)$, of length at most 1, and its image lies in $\mathrm{Sha}^1_2(L', k)$. By additivity of length,

$$\mathrm{len}_2(E, k) = \mathrm{len}_2(E_0, k) \leq 1 + \mathrm{len}_2(L', k)$$

in $\mathbb{N} \cup \{\infty\}$. □

Lemma 39.5. *Let ℓ be a prime and K a finite abelian extension of $\mathbb{Q}$ containing a primitive ℓ-th root of unity if ℓ is odd, a primitive fourth root of unity if $\ell = 2$. There is an integer $r \geq 0$ such that for every $n \geq 0$ and every field K' generated over K by a primitive ℓ^n-th root of unity,*

$$\#(\mathrm{Cl}(K')/\mathrm{Cl}(K')^\ell) \leq \ell^r.$$

Proof. The fields K' are layers of the cyclotomic $\mathbb{Z}_\ell$-extension. Fix an algebraic closure $\overline{K}$; each K' is K-isomorphic to $K(\mu_{\ell^n}) \subseteq \overline{K}$, and class groups are invariant under isomorphism, so we may take $K' = K(\mu_{\ell^n})$. Let c be the largest integer with $\mu_{\ell^c} \subseteq K$; it exists since K is a number field, and $c \geq 1$, with $c \geq 2$ if $\ell = 2$. Put $K(\mu_{\ell^\infty}) = \bigcup_n K(\mu_{\ell^n})$ and $U_k = 1 + \ell^k\mathbb{Z}_\ell$. The cyclotomic character identifies $\Gamma' = \mathrm{Gal}(K(\mu_{\ell^\infty})/K)$ with a closed subgroup of U_c, and $\mu_{\ell^k} \subseteq K$ holds exactly when $\Gamma' \subseteq U_k$. As $c \geq 1$ ($c \geq 2$ if $\ell = 2$), the ℓ-adic logarithm is an isomorphism $U_c \cong \ell^c\mathbb{Z}_\ell \cong \mathbb{Z}_\ell$ of topological groups, whose closed subgroups are $\{1\}$ and the U_k, $k \geq c$. The group Γ' is infinite, because $[K(\mu_{\ell^n}) : \mathbb{Q}] \geq [\mathbb{Q}(\mu_{\ell^n}) : \mathbb{Q}] \to \infty$; so $\Gamma' = U_k$ for some $k \geq c$, and $k = c$ by the maximality of c. Thus $K(\mu_{\ell^\infty})/K$ is a $\mathbb{Z}_\ell$-extension. It contains $K\mathbb{Q}_\infty$, where $\mathbb{Q}_\infty \subseteq \mathbb{Q}(\mu_{\ell^\infty})$ is the $\mathbb{Z}_\ell$-extension of $\mathbb{Q}$, which is itself a $\mathbb{Z}_\ell$-extension of K; since a quotient of $\mathbb{Z}_\ell$ by a nonzero closed subgroup is finite, $K\mathbb{Q}_\infty = K(\mu_{\ell^\infty})$, so this is the cyclotomic $\mathbb{Z}_\ell$-extension K_∞. Its layer K_j of degree ℓ^j is the fixed field of U_{c+j}, and $K(\mu_{\ell^n})$ is the fixed field of $U_{\max(n,c)}$; hence $K(\mu_{\ell^n}) = K_{\max(n-c,0)}$.

From $\mu = 0$ to bounded rank. By Theorem B.35 (applicable as $K/\mathbb{Q}$ is finite abelian), the μ-invariant of K_∞/K is 0. Let A_j be the ℓ-Sylow subgroup of $\mathrm{Cl}(K_j)$ and $X = \varprojlim A_j$ (limit along the norm maps), a module over $\Lambda = \mathbb{Z}_\ell[[T]]$. We use three facts of Iwasawa's theory ([266], Chapter 13): X is a finitely generated torsion Λ-module; there is a pseudo-isomorphism, that is a homomorphism with finite kernel and cokernel, from X to $\bigoplus_i \Lambda/(\ell^{m_i}) \oplus \bigoplus_j \Lambda/(f_j^{n_j})$ with $m_i, n_j \geq 1$ and distinguished polynomials f_j, and $\mu = \sum_i m_i$; and there are $e \geq 0$ and a Λ-submodule $Y_e \subseteq X$ with $A_j \cong X/\nu_{j,e}Y_e$ for all $j \geq e$, where $\nu_{j,e} \in \Lambda$. As $\mu = 0$ there are no summands $\Lambda/(\ell^{m_i})$, and each $\Lambda/(f_j^{n_j})$ is a free $\mathbb{Z}_\ell$-module of rank $n_j \deg f_j$ by division by the distinguished polynomial $f_j^{n_j}$. So X is an extension of a $\mathbb{Z}_\ell$-submodule of a finitely generated free $\mathbb{Z}_\ell$-module by a finite group, hence a finitely generated $\mathbb{Z}_\ell$-module, and $\delta = \dim_{\mathbb{F}_\ell} X/\ell X$ is finite. For $j \geq e$, A_j is a quotient of X, so $A_j/\ell A_j$ is a quotient of $X/\ell X$ and $\#(A_j/\ell A_j) \leq \ell^\delta$. Let r be the maximum of δ and of $\dim_{\mathbb{F}_\ell} A_j/\ell A_j$ for $j < e$.

Conclusion. For a finite abelian group C with ℓ-Sylow subgroup A, $C/C^\ell \cong A/\ell A$, since the prime-to-ℓ part is ℓ-divisible. Hence $\#(\mathrm{Cl}(K(\mu_{\ell^n}))/\mathrm{Cl}(K(\mu_{\ell^n}))^\ell) = \#(A_j/\ell A_j) \leq \ell^r$ with $j = \max(n-c, 0)$. □

Proposition 39.6. *Let ℓ be an odd prime and L a finite abelian extension of $\mathbb{Q}$ containing a primitive ℓ-th root of unity. Suppose there is an integer $r \geq 0$ such that $\#(\mathrm{Cl}(L')/\mathrm{Cl}(L')^\ell) \leq \ell^r$ for every $n \geq 0$ and every field L' generated over L by a primitive ℓ^n-th root of unity. Then there is an integer $R \geq 1$ such that L has twist control at ℓ with rank R.*

Proof. Let r be as in the statement, fix an algebraic closure $\overline{L}$ and let $L(\mu_{\ell^\infty}) \subseteq \overline{L}$ be generated over L by the roots of unity of ℓ-power order. For each $n \geq 0$ the field $L(\mu_{\ell^n}) \subseteq \overline{L}$ is generated over L by a primitive ℓ^n-th root of unity, so the hypothesis of Lemma 39.7 holds for L, which contains a primitive ℓ-th root of unity: the continuous homomorphisms $G_{L(\mu_{\ell^\infty})} \to \mathbb{Z}/\ell\mathbb{Z}$ that vanish on the inertia groups of all places above finite places form a space of dimension at most r. Put $R = \max(r, 1)$. As ℓ is odd and L contains a primitive ℓ-th root of unity, Proposition 39.8 applies with this R, and L has twist control at ℓ with rank R. (The hypothesis that L is abelian over $\mathbb{Q}$ is not used.) □

Lemma 39.7. *Let ℓ be a prime and L a number field containing a primitive ℓ-th root of unity, and a primitive fourth root of unity if $\ell = 2$; fix an algebraic closure $\overline{L}$ and let $L(\mu_{\ell^\infty}) \subseteq \overline{L}$ be the field generated over L by all roots of unity of ℓ-power order. Suppose there is an integer $r \geq 0$ such that $\#(\mathrm{Cl}(L')/\mathrm{Cl}(L')^\ell) \leq \ell^r$ for every $n \geq 0$ and every field L' generated over L by a primitive ℓ^n-th root of unity. Then the $\mathbb{F}_\ell$-vector space of continuous homomorphisms*

$G_{L(\mu_{\ell^\infty})} \to \mathbb{Z}/\ell\mathbb{Z}$ that vanish on the inertia group of every place of $\overline{L}$ above a finite place has dimension at most r.

Proof. Put $L_n = L(\mu_{\ell^n}) \subseteq \overline{L}$ for $n \geq 0$, $L_\infty = L(\mu_{\ell^\infty}) = \bigcup_n L_n$ and $\Gamma_n = \mathrm{Gal}(L_\infty/L_n)$. Each L_n is generated over L by a primitive ℓ^n-th root of unity, so $\dim_{\mathbb{F}_\ell} \mathrm{Cl}(L_n)/\mathrm{Cl}(L_n)^\ell \leq r$; and L_n has no real place, because it contains a primitive ℓ-th root of unity with ℓ odd, or i. The Γ_n are open subgroups of the profinite group Γ_0 with trivial intersection; by compactness they form a basis of neighbourhoods of 1 in Γ_0, and in each Γ_n. Call a continuous homomorphism unramified if it vanishes on the inertia groups at all places above finite places.

Descent of one character. Let $\chi\colon G_{L_\infty} \to \mathbb{Z}/\ell\mathbb{Z}$ be a nonzero unramified continuous homomorphism, and F the fixed field of its kernel, a cyclic extension of degree ℓ of L_∞, unramified at every finite place. We show that for all large n there is an unramified continuous homomorphism $\chi_n\colon G_{L_n} \to \mathbb{Z}/\ell\mathbb{Z}$ whose restriction to G_{L_∞} is χ. By Kummer theory $F = L_\infty(\alpha)$ with $\alpha^\ell = a \in L_\infty^\times$; fix n_0 with $a \in L_{n_0}$. For $n \geq n_0$ put $F_n = L_n(\alpha)$. As $[F : L_\infty] = \ell$ and $F = F_n L_\infty$, $[F_n : L_n] = \ell$; since $\mu_\ell \subseteq L_n$, F_n/L_n is cyclic, and F_n and L_∞ are linearly disjoint over L_n. Restriction gives isomorphisms $\mathrm{Gal}(F/L_n) \cong \mathrm{Gal}(F_n/L_n) \times \Gamma_n$ and $\mathrm{Gal}(F/L_\infty) \cong \mathrm{Gal}(F_n/L_n)$; under the first, $\mathrm{Gal}(F/L_{n'})$ corresponds to $\mathrm{Gal}(F_{n_0}/L_{n_0}) \times \Gamma_{n'}$ for $n' \geq n_0$ (taking $n = n_0$), and restriction to $F_{n'} = F_{n_0}L_{n'}$ is the first projection.

Let v be a finite place of L_{n_0} and $I_v \subseteq \mathrm{Gal}(F/L_{n_0})$ the inertia group of a place of F above v; it depends only on v, as $\mathrm{Gal}(F/L_{n_0})$ is abelian. Its intersection with $\mathrm{Gal}(F/L_\infty) = \mathrm{Gal}(F_{n_0}/L_{n_0}) \times 1$ is the inertia group of F/L_∞, which is trivial. So the second projection is injective on I_v; as I_v is compact, it is a homeomorphism onto a closed subgroup $B_v \subseteq \Gamma_{n_0}$, and $I_v = \{(\beta_v(b), b) : b \in B_v\}$ for a continuous homomorphism $\beta_v\colon B_v \to \mathrm{Gal}(F_{n_0}/L_{n_0})$. If $v \nmid \ell$, then B_v, the inertia group of L_∞/L_{n_0} at v, is trivial, so $I_v = 1$. For each of the finitely many $v \mid \ell$, $\ker \beta_v$ is open in B_v, so there is $n_v \geq n_0$ with $B_v \cap \Gamma_{n_v} \subseteq \ker \beta_v$. Let $n_1 = \max_{v|\ell} n_v$ and $n \geq n_1$. For every finite place, the inertia group of F/L_n there is $I_v \cap (\mathrm{Gal}(F_{n_0}/L_{n_0}) \times \Gamma_n) = \{(\beta_v(b), b) : b \in B_v \cap \Gamma_n\} \subseteq 1 \times \Gamma_n$, so its image in $\mathrm{Gal}(F_n/L_n)$, the inertia group of F_n/L_n, is trivial. Thus F_n/L_n is unramified at every finite place. Let χ_n be the composite of $G_{L_n} \to \mathrm{Gal}(F_n/L_n)$, the inverse of the restriction isomorphism $\mathrm{Gal}(F/L_\infty) \to \mathrm{Gal}(F_n/L_n)$, and the map $\mathrm{Gal}(F/L_\infty) \to \mathbb{Z}/\ell\mathbb{Z}$ induced by χ. It is continuous, unramified (it factors through $\mathrm{Gal}(F_n/L_n)$), and for $g \in G_{L_\infty}$, $\chi_n(g) = \chi(g)$, because the restrictions of g to F and to F_n correspond.

Conclusion. Let $\chi^{(1)}, \ldots, \chi^{(t)}$ be linearly independent unramified continuous homomorphisms $G_{L_\infty} \to \mathbb{Z}/\ell\mathbb{Z}$. Choose n so large that each of them has an unramified extension $\chi_n^{(i)}$ to G_{L_n}. A linear relation among the $\chi_n^{(i)}$ restricts to the same relation among the $\chi^{(i)}$, so the $\chi_n^{(i)}$ are linearly independent. By the Hilbert class field, $t \leq \dim_{\mathbb{F}_\ell} \mathrm{Cl}(L_n)/\mathrm{Cl}(L_n)^\ell \leq r$. □

Proposition 39.8. *Let ℓ be a prime, $R \geq 1$ an integer and L a number field containing a primitive ℓ-th root of unity, and a primitive fourth root of unity if $\ell = 2$. Fix an algebraic closure $\overline{L}$ and let $L(\mu_{\ell^\infty}) \subseteq \overline{L}$ be the field generated over L by all roots of unity of ℓ-power order. Suppose that the $\mathbb{F}_\ell$-vector space of continuous homomorphisms $G_{L(\mu_{\ell^\infty})} \to \mathbb{Z}/\ell\mathbb{Z}$ that vanish on the inertia group of every place of $\overline{L}$ above a finite place has dimension at most R. Then L has twist control at ℓ with rank R. Moreover, in (Tw1) one can take $K = \mathbb{Q}_\ell$, $R' = R$, $s_1, \ldots, s_R \in \mathbb{Z}_\ell$, and c_μ the largest integer c such that L contains a primitive ℓ^c-th root of unity.*

Proof. Put $j = j_\ell(R)$, $L_n = L(\mu_{\ell^n}) \subseteq \overline{L}$ for $n \geq 0$, $L_\infty = L(\mu_{\ell^\infty})$ and $\Gamma = \mathrm{Gal}(L_\infty/L)$. As L contains a primitive ℓ-th root of unity with ℓ odd, or i, it has no real place, and neither has any extension of it.

Step 1: the cyclotomic tower. Since $\varphi(\ell^k) \le [L:\mathbb{Q}]$ whenever $\mu_{\ell^k} \subseteq L$, the largest integer c with $\mu_{\ell^c} \subseteq L$ exists, and $c \ge 1$, with $c \ge 2$ if $\ell = 2$. The cyclotomic character $\chi_{\mathrm{cyc}}\colon \Gamma \to \mathbb{Z}_\ell^\times$ is continuous and injective (as L_∞ is generated by μ_{ℓ^∞}), and $\tau \in \Gamma$ fixes μ_{ℓ^n} if and only if $\chi_{\mathrm{cyc}}(\tau) \in U_n$ $(n \ge 1)$. So its image is a closed subgroup of U_c; it is infinite, because $[L_n : L] \ge \varphi(\ell^n)/[L:\mathbb{Q}] \to \infty$; hence it is some U_k with $k \ge c$, and $k = c$, because the image lies in U_{c+1} only if $\mu_{\ell^{c+1}} \subseteq L$. Thus $\chi_{\mathrm{cyc}}\colon \Gamma \to U_c$ is an isomorphism of topological groups, and for $m \ge 0$ the subgroup $\Gamma_m = \chi_{\mathrm{cyc}}^{-1}(U_{c+m})$ is $\mathrm{Gal}(L_\infty/L_{c+m})$, so $[L_{c+m} : L] = [U_c : U_{c+m}] = \ell^m$. Let $\sigma \in \Gamma$ with $u := \chi_{\mathrm{cyc}}(\sigma) = 1 + \ell^c$; then $v(u-1) = c$, $v(u^{\ell^m} - 1) = c + m$, and Γ_m is the closure of $\langle \sigma^{\ell^m} \rangle$. The Γ_m form a basis of neighbourhoods of 1 in Γ.

Every cyclotomic ℓ-layer L' of degree ℓ^m over L is L-isomorphic to L_{c+m}: write $L' = L(\zeta)$ with ζ a primitive ℓ^k-th root of unity and choose an L-embedding $\iota\colon L' \to \overline{L}$; then $\iota(\zeta)$ is a primitive ℓ^k-th root of unity, whose powers are all of μ_{ℓ^k}, so $\iota(L') = L_k$; if $k \le c$ then $L_k = L = L_c$ and $m = 0$, and if $k > c$ then $[L_k : L] = \ell^{k-c}$ forces $k = c + m$. Lengths of Tate–Shafarevich groups and of cohomology groups are invariant under isomorphisms of fields, so in (Tw1) and (Tw2) it suffices to treat $L' = L_{c+m}$ and $L'' = L_{c+j}$.

Step 2: the split Iwasawa module. Let $M' \subseteq \overline{L}$ be the compositum of all finite abelian extensions of L_∞ of ℓ-power degree that are unramified at every finite place and in which every place above ℓ splits completely, and put $\mathrm{Iw}^{\mathrm{sp}}(L) = \mathrm{Gal}(M'/L_\infty)$, an abelian pro-$\ell$ group, hence a compact $\mathbb{Z}_\ell$-module. As these conditions are preserved by automorphisms of $\overline{L}$ that preserve L_∞, M' is Galois over L. For $\tau \in \Gamma$ and $x \in \mathrm{Iw}^{\mathrm{sp}}(L)$ put $\tau \cdot x = \tilde{\tau} x \tilde{\tau}^{-1}$ for any lift $\tilde{\tau} \in \mathrm{Gal}(M'/L)$ of τ; this is independent of the lift because $\mathrm{Iw}^{\mathrm{sp}}(L)$ is abelian, it is a continuous automorphism, hence $\mathbb{Z}_\ell$-linear, and $\tau \mapsto \tau \cdot x$ is continuous (it factors the continuous map $\tilde{\tau} \mapsto \tilde{\tau} x \tilde{\tau}^{-1}$ through the open quotient map $\mathrm{Gal}(M'/L) \to \Gamma$). Let ϕ be the action of σ.

Generation by R elements. A continuous homomorphism $\mathrm{Iw}^{\mathrm{sp}}(L) \to \mathbb{Z}/\ell\mathbb{Z}$, composed with the surjection $G_{L_\infty} \to \mathrm{Iw}^{\mathrm{sp}}(L)$, vanishes on all inertia groups at finite places, since M'/L_∞ is unramified there; and different homomorphisms give different composites. By hypothesis the continuous homomorphisms $\overline{W} \to \mathbb{F}_\ell$ of the profinite $\mathbb{F}_\ell$-vector space $\overline{W} = \mathrm{Iw}^{\mathrm{sp}}(L)/\ell\,\mathrm{Iw}^{\mathrm{sp}}(L)$ span a space of dimension $d \le R$. They separate points (for $x \ne 0$ choose an open subgroup $U \not\ni x$ and a linear form on the finite space $\overline{W}/U$ not vanishing at the class of x), so a basis of them gives an injection $\overline{W} \to \mathbb{F}_\ell^d$ and $\dim \overline{W} \le R$. Lift a basis of $\overline{W}$ to $x_1, \dots, x_{d'} \in \mathrm{Iw}^{\mathrm{sp}}(L)$, $d' \le R$. The image Y of the continuous map $\mathbb{Z}_\ell^{d'} \to \mathrm{Iw}^{\mathrm{sp}}(L)$, $a \mapsto \sum a_i x_i$, is compact, hence closed, and $Y + \ell\,\mathrm{Iw}^{\mathrm{sp}}(L) = \mathrm{Iw}^{\mathrm{sp}}(L)$. The quotient $\mathrm{Iw}^{\mathrm{sp}}(L)/Y$ is an abelian pro-ℓ group equal to ℓ times itself; each of its finite quotients is a finite ℓ-group Q with $\ell Q = Q$, hence 0, so $\mathrm{Iw}^{\mathrm{sp}}(L) = Y$. So $\mathrm{Iw}^{\mathrm{sp}}(L)$ is generated over $\mathbb{Z}_\ell$ by R elements (add zeros).

Unipotence modulo ℓ. The fixed field F_1 of $\ell\,\mathrm{Iw}^{\mathrm{sp}}(L)$ is finite over L_∞ and Galois over L, and Γ acts on $\overline{W} = \mathrm{Gal}(F_1/L_\infty)$ through conjugation in $\mathrm{Gal}(F_1/L)$. For $x \in \overline{W}$ the map $\tilde{\tau} \mapsto \tilde{\tau} x \tilde{\tau}^{-1}$ is continuous with values in the finite, hence discrete, subgroup $\overline{W}$, so the centraliser of x is open; hence the kernel of the action of Γ on the finite set $\overline{W}$ is open and contains some Γ_n. Then σ^{ℓ^n} acts trivially on $\overline{W}$, and in characteristic ℓ the induced endomorphism $\bar{\phi}$ satisfies $(\bar{\phi} - 1)^{\ell^n} = \bar{\phi}^{\ell^n} - 1 = 0$. So $(\phi - 1)^{\ell^n}\mathrm{Iw}^{\mathrm{sp}}(L) \subseteq \ell\,\mathrm{Iw}^{\mathrm{sp}}(L)$.

Step 3: the Tate–Shafarevich groups of the layers. Let $e \ge 1$ and $m \ge 0$; put $L' = L_{c+m}$ and $T_{e,m} = (u^e\phi)^{\ell^m} - 1$, an endomorphism of $\mathrm{Iw}^{\mathrm{sp}}(L)$. We show

$$\mathrm{len}_\ell(L_{c+m}, e) = \mathrm{length}_{\mathbb{Z}_\ell}\Big(\mathrm{Iw}^{\mathrm{sp}}(L)/T_{e,m}\,\mathrm{Iw}^{\mathrm{sp}}(L)\Big) = \mathrm{tl}(e,m), \tag{39.3}$$

where tl is formed as in Lemma 39.9 with $M = \mathrm{Iw}^{\mathrm{sp}}(L)$, ϕ and u. Let $\Omega \subseteq \overline{L}$ be the maximal extension of L' unramified outside the places above ℓ, $G = \mathrm{Gal}(\Omega/L') = G_{L',S_\ell}$ and $A = \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$. The field L_∞ is unramified over L' outside ℓ, and M' is unramified over L_∞, so

$L_\infty \subseteq M' \subseteq \Omega$. Let $H = \mathrm{Gal}(\Omega/L_\infty)$, a closed normal subgroup with $G/H = \Gamma_m$; H acts trivially on A.

(a) *Inflation and restriction.* $\Gamma_m \cong U_{c+m} \cong \mathbb{Z}_\ell$ with topological generator σ^{ℓ^m}, which acts on A as multiplication by $u^{-e\ell^m}$. By the lifting-the-exponent rule $v(u^{-e\ell^m}-1) = c+m+v(e) < \infty$, so $u^{-e\ell^m}-1$ is a nonzero element of $\mathbb{Z}_\ell$, and multiplication by it is surjective on the divisible group A: hence $H^1(\Gamma_m, A) = 0$; and $H^2(\Gamma_m, A) = 0$. The inflation–restriction sequence therefore gives an isomorphism $H^1(G, A) \to H^1(H, A)^{\Gamma_m} = \mathrm{Hom}_{\mathrm{cont}}(H, A)^{\Gamma_m}$, where $\tau \in \Gamma_m$ acts by $(\tau f)(h) = \chi_{\mathrm{cyc}}(\tau)^{-e} f(\tilde\tau^{-1} h \tilde\tau)$.

(b) *Local conditions.* Let v be a place of L' above ℓ, w a place of Ω above v and $D_w \subseteq G$ its decomposition group. Localisation at v is restriction to D_w followed by inflation along the surjection $G_{L'_v} \to D_w$, and inflation is injective on H^1; so a class $y \in H^1(G, A)$ is trivial at v if and only if $y|_{D_w} = 0$. The group $D_w \cap H$ is normal in D_w, and $D_w/(D_w \cap H)$ is the decomposition group of L_∞/L' at the place below w, a closed subgroup of Γ_m; it is not trivial, because the ramification index of $L'(\mu_{\ell^n})/L'$ at v is at least $\varphi(\ell^n)/[L' : \mathbb{Q}]$ ($\mathbb{Q}(\mu_{\ell^n})/\mathbb{Q}$ is totally ramified at ℓ), which tends to infinity. So it is isomorphic to $\mathbb{Z}_\ell$, with a generator acting on A by $\chi_{\mathrm{cyc}}^{-e} \neq 1$, and as in (a) its H^1 with coefficients in A vanishes. By inflation–restriction for $D_w \cap H \subseteq D_w$, $y|_{D_w} = 0$ if and only if the homomorphism $f = y|_H$ vanishes on $D_w \cap H$. The decomposition groups of the places of Ω above v are the G-conjugates of D_w, and $f(\tilde\tau h \tilde\tau^{-1}) = \chi_{\mathrm{cyc}}(\tau)^{-e} f(h)$ for Γ_m-invariant f. Hence restriction identifies $\mathrm{Sha}^1_\ell(L', e)$ with the group of Γ_m-invariant $f \in \mathrm{Hom}_{\mathrm{cont}}(H, A)$ that vanish on $D_w \cap H$ for every place w of Ω above ℓ.

(c) *Passage to the split module.* Let f be such a homomorphism. Its image is finite (compact in the discrete A), an ℓ-group, and the fixed field of $\ker f$ is a finite abelian extension of L_∞ of ℓ-power degree inside Ω, unramified outside ℓ and with every place above ℓ split completely (the decomposition groups $D_w \cap H$ lie in $\ker f$); so it is unramified at every finite place, lies in M', and f factors through the surjection $H \to \mathrm{Iw}^{\mathrm{sp}}(L)$. Conversely, a continuous homomorphism $\mathrm{Iw}^{\mathrm{sp}}(L) \to A$, composed with this surjection, vanishes on every $D_w \cap H$, whose image is a decomposition group of M'/L_∞ at a place above ℓ, hence trivial. The surjection is compatible with the conjugation actions of Γ_m. So $\mathrm{Sha}^1_\ell(L', e)$ is isomorphic to the group of continuous homomorphisms $f\colon \mathrm{Iw}^{\mathrm{sp}}(L) \to A$ with $f(\tau \cdot x) = \chi_{\mathrm{cyc}}(\tau)^{-e} f(x)$ for all $\tau \in \Gamma_m$ and x.

(d) *The cokernel.* For a continuous f the set of $\tau \in \Gamma_m$ satisfying this identity for all x is a subgroup, and it is closed, since for fixed x the map $\tau \mapsto f(\tau \cdot x) - \chi_{\mathrm{cyc}}(\tau)^{-e} f(x)$ is continuous with values in the discrete A. As Γ_m is the closure of $\langle \sigma^{\ell^m} \rangle$, the identity holds on Γ_m if and only if it holds for $\tau = \sigma^{\ell^m}$, that is $f(\phi^{\ell^m} x) = u^{-e\ell^m} f(x)$ for all x; as f is $\mathbb{Z}_\ell$-linear and $(u^e\phi)^{\ell^m} = u^{e\ell^m}\phi^{\ell^m}$, this says that f vanishes on $T_{e,m}\mathrm{Iw}^{\mathrm{sp}}(L)$. This submodule is closed (compact), so $\mathrm{Sha}^1_\ell(L', e)$ is isomorphic to the group of continuous homomorphisms $C \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$ with $C = \mathrm{Iw}^{\mathrm{sp}}(L)/T_{e,m}\mathrm{Iw}^{\mathrm{sp}}(L)$. By Step 2, C is a finitely generated $\mathbb{Z}_\ell$-module, $C \cong \mathbb{Z}_\ell^a \oplus C_0$ with C_0 finite; its group of continuous homomorphisms to $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ is $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^a \oplus \mathrm{Hom}(C_0, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, of length ∞ if $a > 0$ and length C_0 if $a = 0$, which is length C in both cases. This is (39.3).

Step 4: (Tw1) and (Tw2). By Step 2 the hypotheses of Lemma 39.9 hold for $M = \mathrm{Iw}^{\mathrm{sp}}(L)$, ϕ, $N = \ell^n$ and u with $v(u-1) = c$; let $s_1, \dots, s_R \in \mathbb{Z}_\ell$ be the elements of its item 1. Let $e \geq 1$ and $m \geq 0$. By (39.3) and items 2 and 1 of Lemma 39.9,

$$\mathrm{len}_\ell(L_{c+m}, e) = \mathrm{tl}(e, m) \leq \mathrm{tl}(e, j) + R\max(m-j, 0) \leq \sum_{i=1}^{R} (j + c + v(s_i - e)) + R\max(m-j, 0). \tag{39.4}$$

For (Tw1) take $K = \mathbb{Q}_\ell$ with its absolute value, $c_\mu = c$ (admissible, as $\mu_{\ell^c} \subseteq L$), $R' = R$ and the points $s_1, \dots, s_R$. If $D \geq 0$ and $|s_i - e| \geq \ell^{-D}$ for all i, then $v(s_i - e) \leq D$, and (39.4) gives $\mathrm{len}_\ell(L_{c+m}, e) \leq R(j + c + D) + R\max(m-j, 0) = R(\max(m, j) + c + D)$; by Step 1 this is (Tw1)

for every cyclotomic ℓ-layer of degree ℓ^m. For (Tw2), the first inequality of (39.4) and (39.3) for $m = j$ give $\mathrm{len}_\ell(L_{c+m}, e) \le \mathrm{len}_\ell(L_{c+j}, e) + R\max(m-j, 0)$, and by Lemma 39.10 for the field L_{c+j}, which has no real place, $\mathrm{len}_\ell(L_{c+j}, e) \le \mathrm{length}_{\mathbb{Z}_\ell} H^2(G_{L_{c+j},S_\ell}, \mathbb{Z}_\ell(e+1))$. By Step 1 this is (Tw2) for all layers L' of degree ℓ^m and L'' of degree ℓ^j. So L has twist control at ℓ with rank R. □

Lemma 39.9. *Let ℓ be a prime, $R \ge 1$ an integer and $j = j_\ell(R)$. Let M be a $\mathbb{Z}_\ell$-module generated by R elements, ϕ a $\mathbb{Z}_\ell$-linear endomorphism of M such that $(\phi-1)^N M \subseteq \ell M$ for some integer $N \ge 0$, and $u \in \mathbb{Z}_\ell$ with $v(u-1) = c$, where $c \ge 1$, and $c \ge 2$ if $\ell = 2$. For integers $e, m \ge 0$ put*

$$\mathrm{tl}(e,m) = \mathrm{length}_{\mathbb{Z}_\ell}\Big(M/\big((u^e\phi)^{\ell^m} - 1\big)M\Big) \in \mathbb{N} \cup \{\infty\}.$$

1. *There are $s_1, \dots, s_R \in \mathbb{Z}_\ell$ such that for every integer $e \ge 0$*

$$\mathrm{tl}(e,j) \le \sum_{i=1}^{R} \big(j + c + v(s_i - e)\big).$$

2. *For all integers $e, m \ge 0$, $\mathrm{tl}(e,m) \le \mathrm{tl}(e,j) + R\max(m-j,0)$.*

Proof. Item 1, reduction to a determinant. Let $x_1, \dots, x_R$ generate M and let $\pi\colon \mathbb{Z}_\ell^R \to M$, $\pi(a) = \sum_i a_i x_i$. Put $\phi_1 = \phi^{\ell^j}$ and choose $S = (S_{ik}) \in M_R(\mathbb{Z}_\ell)$ with $\phi_1(x_k) = \sum_i S_{ik} x_i$ for each k; then $\pi \circ S = \phi_1 \circ \pi$. For $a \in \mathbb{Z}_\ell$ this gives $\pi \circ (aS - 1) = (a\phi_1 - 1) \circ \pi$, so π maps $(aS-1)\mathbb{Z}_\ell^R$ into $(a\phi_1 - 1)M$ and induces a surjection $\mathbb{Z}_\ell^R/(aS-1)\mathbb{Z}_\ell^R \to M/(a\phi_1 - 1)M$. As the scalar u^e commutes with ϕ, $(u^e\phi)^{\ell^j} = a_e\phi_1$ with $a_e = u^{e\ell^j}$. Hence

$$\mathrm{tl}(e,j) \le v\big(\det(a_e S - 1)\big) \qquad (e \ge 0). \tag{39.5}$$

Item 1, the eigenvalues. Let K be a finite extension of $\mathbb{Q}_\ell$ containing the roots $z_1, \dots, z_R$ (with multiplicity) of the characteristic polynomial of S, with the absolute value $|\cdot|$ extending the ℓ-adic one ($|\ell| = \ell^{-1}$), and extend $v = -\log_\ell|\cdot|$ to K. Over K the matrix S is conjugate to an upper triangular matrix with diagonal $z_1, \dots, z_R$; so for $a \in \mathbb{Z}_\ell$ the matrix $aS - 1$ is conjugate to an upper triangular one with diagonal $az_i - 1$, and $\det(aS-1) = \prod_{i=1}^R (az_i - 1)$.

Item 1, one centre for each eigenvalue. Fix $z \in K$; we show that there is $s \in \mathbb{Z}_\ell$ with

$$v(a_e z - 1) \le j + c + v(s - e) \qquad \text{for every integer } e \ge 0. \tag{39.6}$$

Put $r(e) = \ell^{j+c}|a_e z - 1| \ge 0$. Let $e, e' \ge 0$. The a_e are units, and by the lifting-the-exponent rule $|a_e - a_{e'}| = |u^{(e-e')\ell^j} - 1| = \ell^{-(j+c)}|e - e'|$ (also when $e = e'$). Moreover $|a_e - a_{e'}| \le \max(|a_e z - 1|, |a_{e'} z - 1|)$: if $|z| \ge 1$, then $|a_e - a_{e'}| \le |a_e - a_{e'}|\,|z| = |(a_e z - 1) - (a_{e'} z - 1)|$ and the ultrametric inequality applies; if $|z| < 1$, then $|a_e z| < 1$, so $|a_e z - 1| = 1 \ge |a_e - a_{e'}|$. Together,

$$|e - e'| \le \max\big(r(e), r(e')\big) \qquad (e, e' \ge 0).$$

The sets $B(e) = \{y \in \mathbb{Z}_\ell : |y - e| \le r(e)\}$ are closed in $\mathbb{Z}_\ell$. Every finite nonempty family of them has a common point: if e^* minimises r over the finite set E of indices, then $|e^* - e| \le \max(r(e^*), r(e)) = r(e)$, that is $e^* \in B(e)$, for every $e \in E$. As $\mathbb{Z}_\ell$ is compact, there is $s \in \bigcap_{e \ge 0} B(e)$. Then $|s - e| \le \ell^{j+c}|a_e z - 1|$ for all e, which is (39.6) (if $a_e z = 1$, then $r(e) = 0$, $s = e$ and both sides are ∞).

Item 1, conclusion. Choose s_i as in (39.6) for $z = z_i$. By (39.5) and the product formula for the determinant,

$$\mathrm{tl}(e,j) \le v\Big(\prod_{i=1}^{R}(a_e z_i - 1)\Big) = \sum_{i=1}^{R} v(a_e z_i - 1) \le \sum_{i=1}^{R}\big(j + c + v(s_i - e)\big).$$

Item 2, unipotence modulo ℓ survives the twist. Fix $e \ge 0$ and put $\phi_e = u^e\phi$ and $\overline{M} = M/\ell M$, an $\mathbb{F}_\ell$-vector space of dimension at most R. As $u^e \equiv 1 \pmod{\ell}$, ϕ_e and ϕ induce the same endomorphism $\bar\phi$ of $\overline{M}$, and $(\bar\phi - 1)^N = 0$ by hypothesis. A nilpotent endomorphism of a vector space of dimension at most R has R-th power 0. Let $k = \lfloor \log_\ell R \rfloor + 1$, so that $\ell^k > R$ and $j = k+2$. Then $(\bar\phi - 1)^{\ell^k} = 0$, and since $\binom{\ell^k}{i} \equiv 0 \pmod{\ell}$ for $0 < i < \ell^k$, $\bar\phi^{\ell^k} - 1 = (\bar\phi - 1)^{\ell^k} = 0$. Hence $(\phi_e^{\ell^k} - 1)M \subseteq \ell M$.

Item 2, raising to the power ℓ gains a factor ℓ. Let ϕ' be an endomorphism of M with $(\phi' - 1)M \subseteq \ell^t M$ for some $t \ge 1$. By induction $(\phi' - 1)^i M \subseteq \ell^{ti} M$, since $(\phi' - 1)(\ell^{t(i-1)}y) = \ell^{t(i-1)}(\phi' - 1)y$. In $\phi'^\ell - 1 = \sum_{i=1}^{\ell}\binom{\ell}{i}(\phi' - 1)^i$ the terms with $1 \le i < \ell$ map M into $\ell^{1+ti}M \subseteq \ell^{t+1}M$, because $\ell \mid \binom{\ell}{i}$, and the term $i = \ell$ maps M into $\ell^{t\ell}M \subseteq \ell^{t+1}M$, because $t\ell \ge 2t \ge t+1$. So $(\phi'^\ell - 1)M \subseteq \ell^{t+1}M$. Applied to $\phi' = \phi_e^{\ell^k}$ with $t = 1$ this gives $(\phi_e^{\ell^{k+1}} - 1)M \subseteq \ell^2 M$; and for $n \ge k+1$, $\phi_e^{\ell^n} - 1 = (\phi_e^{\ell^{k+1}} - 1) \circ \sum_{0 \le i < \ell^{n-k-1}} \phi_e^{i\ell^{k+1}}$, so

$$(\phi_e^{\ell^n} - 1)M \subseteq \ell^2 M \qquad (n \ge k+1). \tag{39.7}$$

Item 2, one step costs at most R. Let ϕ' be an endomorphism of M with $(\phi' - 1)M \subseteq \ell^2 M$, and put $\mathrm{Nm} = \sum_{i=0}^{\ell-1}\phi'^i$, so that $\phi'^\ell - 1 = \mathrm{Nm} \circ (\phi' - 1)$. For endomorphisms A, B of M the projection $M/ABM \to M/AM$ is surjective with kernel AM/ABM, which is the image of M/BM under the map induced by A; so $\mathrm{length}(M/ABM) \le \mathrm{length}(M/AM) + \mathrm{length}(M/BM)$. With $A = \mathrm{Nm}$ and $B = \phi' - 1$,

$$\mathrm{length}\big(M/(\phi'^\ell - 1)M\big) \le \mathrm{length}\big(M/\mathrm{Nm}(M)\big) + \mathrm{length}\big(M/(\phi' - 1)M\big).$$

For every $i \ge 0$ and $x \in M$, $\phi'^i x - x \in \ell^2 M$, because $\phi'^i - 1 = \sum_{h<i}\phi'^h \circ (\phi' - 1)$ and $\phi'^h(\ell^2 y) = \ell^2 \phi'^h y$. Hence $\mathrm{Nm}(x) - \ell x \in \ell^2 M$, so $\ell M \subseteq \mathrm{Nm}(M) + \ell(\ell M)$. The module $Q = (\ell M + \mathrm{Nm}(M))/\mathrm{Nm}(M)$ is finitely generated over $\mathbb{Z}_\ell$ and satisfies $Q = \ell Q$, so $Q = 0$ by Nakayama's lemma, that is $\ell M \subseteq \mathrm{Nm}(M)$. Therefore $M/\mathrm{Nm}(M)$ is a quotient of $M/\ell M$ and has length at most $\dim_{\mathbb{F}_\ell} M/\ell M \le R$. So

$$\mathrm{length}\big(M/(\phi'^\ell - 1)M\big) \le \mathrm{length}\big(M/(\phi' - 1)M\big) + R.$$

Item 2, conclusion. For $n \ge k+1$, (39.7) allows the last bound with $\phi' = \phi_e^{\ell^n}$, and $\phi'^\ell = \phi_e^{\ell^{n+1}}$: so $\mathrm{tl}(e, n+1) \le \mathrm{tl}(e,n) + R$. As $j = k+2 \ge k+1$, induction on m gives $\mathrm{tl}(e,m) \le \mathrm{tl}(e,j) + R(m-j)$ for $m \ge j$. For $m < j$, $\phi_e^{\ell^j} - 1 = (\phi_e^{\ell^m} - 1) \circ \sum_{0 \le i < \ell^{j-m}} \phi_e^{i\ell^m}$, so $(\phi_e^{\ell^j} - 1)M \subseteq (\phi_e^{\ell^m} - 1)M$ and $M/(\phi_e^{\ell^m} - 1)M$ is a quotient of $M/(\phi_e^{\ell^j} - 1)M$; hence $\mathrm{tl}(e,m) \le \mathrm{tl}(e,j)$. Both cases together are item 2. □

Lemma 39.10. *Let ℓ be a prime, L a number field with no real place and $k \in \mathbb{Z}$. Then, in $\mathbb{N} \cup \{\infty\}$,*

$$\mathrm{len}_\ell(L,k) \le \mathrm{length}_{\mathbb{Z}_\ell} H^2\big(G_{L,S_\ell}, \mathbb{Z}_\ell(k+1)\big).$$

Proof. Let h be the length of $H^2(G_{L,S_\ell},\mathbb{Z}_\ell(k+1))$; if $h=\infty$ there is nothing to prove, so let $h<\infty$. Let S be the set of places of L above ℓ or ∞. The archimedean places are complex, so an extension of L is unramified outside S_ℓ if and only if it is unramified outside S; thus $G_{L,S_\ell}=G_{L,S}$, and $H^i(L_v,\cdot)=0$ for $i\geq 1$ at the complex places v. Hence $\mathrm{Sha}^1_\ell(L,k)=\mathrm{Sha}^1_S(L,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-k))$, and $\mathrm{Sha}^i_S(L,\cdot)$ is the kernel of localisation at the places above ℓ. For $n\geq 1$ put $M_n=\ell^{-n}\mathbb{Z}_\ell/\mathbb{Z}_\ell(-k)\subseteq\mathbb{Q}_\ell/\mathbb{Z}_\ell(-k)$, a $\mathbb{Z}/\ell^n\mathbb{Z}$-module of order ℓ^n, and $T=\mathbb{Z}_\ell(k+1)$, $T_n=T/\ell^nT$. The Cartier dual of M_n is $M_n^\vee=\mathrm{Hom}(M_n,\mu_{\ell^n})\cong T_n$ as $G_{L,S}$-modules, since the twists by χ_{cyc}^{-k} and χ_{cyc} combine to $\chi_{\mathrm{cyc}}^{k+1}$.

Finite levels. By item 2 of Theorem B.34,

$$\#\mathrm{Sha}^1_S(L,M_n)=\#\mathrm{Sha}^2_S(L,T_n)\leq\#H^2(G_{L,S},T_n)\qquad(n\geq 1).\tag{39.8}$$

The finite levels are quotients of $H^2(G_{L,S},T)$. For $n\geq 1$ the sequence $0\to\ell^nT/\ell^{n+1}T\to T_{n+1}\to T_n\to 0$ is exact, and its long exact cohomology sequence continues $H^2(G_{L,S},T_{n+1})\to H^2(G_{L,S},T_n)\to H^3(G_{L,S},\ell^nT/\ell^{n+1}T)$, whose last term is 0 by item 3 of Theorem B.34. So the maps $H^2(G_{L,S},T_{n+1})\to H^2(G_{L,S},T_n)$ are surjective; as all these groups are finite (item 1 of Theorem B.34), compatible preimages can be chosen step by step, so each projection

$$\varprojlim_m H^2(G_{L,S},T_m)\to H^2(G_{L,S},T_n)$$

is surjective. The groups $H^1(G_{L,S},T_n)$ are finite (item 1 of Theorem B.34), so by Theorem B.31 with $i=2$ the natural map $H^2(G_{L,S},T)\to\varprojlim_m H^2(G_{L,S},T_m)$ is an isomorphism. Hence $H^2(G_{L,S},T)\to H^2(G_{L,S},T_n)$ is a surjective $\mathbb{Z}_\ell$-linear map, and

$$\#H^2(G_{L,S},T_n)\leq\ell^h\qquad(n\geq 1).\tag{39.9}$$

Finite subgroups of the Sha group come from a finite level. Let $A\subseteq\mathrm{Sha}^1_\ell(L,k)$ be a finite subgroup. Since $H^1(G_{L,S},\mathbb{Q}_\ell/\mathbb{Z}_\ell(-k))=\varinjlim_n H^1(G_{L,S},M_n)$, and likewise $H^1(L_v,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-k))=\varinjlim_n H^1(L_v,M_n)$ for $v\mid\ell$, every $x\in A$ is the image of some $y_x\in H^1(G_{L,S},M_{n_x})$. For $v\mid\ell$ the localisation of y_x maps to the localisation of x, which is 0, so it becomes 0 in $H^1(L_v,M_{n'})$ for some $n'\geq n_x$. As A and the set of places above ℓ are finite, there is one n such that, for every $x\in A$, the image y'_x of y_x in $H^1(G_{L,S},M_n)$ is locally trivial at every place above ℓ (localisation commutes with the maps induced by $M_{n_x}\subseteq M_n$); then $y'_x\in\mathrm{Sha}^1_S(L,M_n)$ and y'_x maps to x. So A is the image of a subset of $\mathrm{Sha}^1_S(L,M_n)$, and by (39.8) and (39.9), $\#A\leq\ell^h$.

Conclusion. The group $H^1(G_{L,S},\mathbb{Q}_\ell/\mathbb{Z}_\ell(-k))=\varinjlim_n H^1(G_{L,S},M_n)$ is ℓ-primary torsion, and so is $\mathrm{Sha}^1_\ell(L,k)$. Suppose its length were $>h$. Then it has a chain of submodules $0=N_0\subsetneq N_1\subsetneq\cdots\subsetneq N_{h+1}$; choosing $x_i\in N_i\setminus N_{i-1}$, the submodule A generated by $x_1,\dots,x_{h+1}$ contains the strictly increasing chain of the submodules generated by $x_1,\dots,x_i$ ($x_i\notin N_{i-1}$), so length $A\geq h+1$; and A, being finitely generated and ℓ-primary torsion, is finite, so $\#A=\ell^{\mathrm{length}\,A}\geq\ell^{h+1}$. This contradicts $\#A\leq\ell^h$. Hence $\mathrm{len}_\ell(L,k)\leq h$. □

Proposition 39.11. *Let $\psi\in\mathcal{P}$ and let $\mathbb{Q}(\zeta_{N_\psi})^\psi\subset\mathbb{C}$ be the subfield of $\mathbb{Q}(\zeta_{N_\psi})$ fixed by the automorphisms $\zeta\mapsto\zeta^u$ with $\psi(u)=1$. Suppose there is an integer $r\geq 0$ such that $\#(\mathrm{Cl}(L')/\mathrm{Cl}(L')^2)\leq 2^r$ for every $n\geq 0$ and every field L' generated over $\mathbb{Q}(\zeta_{N_\psi})^\psi(i)$ by a primitive 2^n-th root of unity. Then there is an integer $R\geq 1$ such that for every pair of primes (q,d) with $q\equiv 3\pmod 8$, $d\equiv 7\pmod{16}$ and $\gcd(qd,N_\psi)=1$, the field $\mathbb{Q}(\zeta_{N_\psi})^\psi(i,\sqrt{-q},\sqrt{-d})$ has twist control at 2 with rank R.*

Proof. Let r be as in the statement, $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, $F_0=\mathbb{Q}(\zeta_{N_\psi})^\psi$, and for a subfield $F\subseteq\overline{\mathbb{Q}}$ let $F(\mu_{2^\infty})$ be generated over F by the roots of unity of 2-power order; call

a continuous homomorphism $G_F \to \mathbb{Z}/2\mathbb{Z}$ unramified if it vanishes on the inertia groups of all places above prime numbers.

The base field. The field $F_0(i)$ contains a primitive fourth root of unity, and for every $n \geq 0$ the field $F_0(i)(\mu_{2^n}) \subseteq \overline{\mathbb{Q}}$ is generated over $F_0(i)$ by a primitive 2^n-th root of unity; so by the hypothesis and Lemma 39.7 (with $\ell = 2$, $L = F_0(i)$, $\overline{L} = \overline{\mathbb{Q}}$), the unramified continuous homomorphisms $G_{F_0(i)(\mu_{2^\infty})} \to \mathbb{Z}/2\mathbb{Z}$ form a space of dimension at most r. As $i \in \mu_{2^\infty}$, $F_0(i)(\mu_{2^\infty}) = F_0(\mu_{2^\infty})$.

Uniformity. By Lemma 39.12 with $g = r$, for every pair of primes (q, d) with $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$ and $\gcd(qd, N_\psi) = 1$, the unramified continuous homomorphisms $G_{L_{q,d}(\mu_{2^\infty})} \to \mathbb{Z}/2\mathbb{Z}$, where $L_{q,d} = F_0(i, \sqrt{-q}, \sqrt{-d})$, form a space of dimension at most $R := 4(r + 6o_\psi)$. This integer satisfies $R \geq 24 \geq 1$ and does not depend on (q, d).

Twist control. Let (q, d) be such a pair. The field $L_{q,d}$ contains i, so Proposition 39.8 applies with $\ell = 2$, $L = L_{q,d}$, $\overline{L} = \overline{\mathbb{Q}}$ and the rank R; hence $L_{q,d}$ has twist control at 2 with rank R, for every pair (q, d) as in the statement. □

Lemma 39.12. *Let $\psi \in \mathcal{P}$, let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let $\mathbb{Q}(\zeta_{N_\psi})^\psi \subset \overline{\mathbb{Q}}$ be the subfield of $\mathbb{Q}(\zeta_{N_\psi})$ fixed by the automorphisms $\zeta \mapsto \zeta^u$ with $\psi(u) = 1$. For a subfield $F \subseteq \overline{\mathbb{Q}}$ write $F(\mu_{2^\infty})$ for the field generated over F by all roots of unity of 2-power order, and call a continuous homomorphism $G_F \to \mathbb{Z}/2\mathbb{Z}$ unramified if it vanishes on the inertia group of every place of $\overline{\mathbb{Q}}$ above a prime number. Let $g \geq 0$ be an integer such that the unramified continuous homomorphisms $G_{\mathbb{Q}(\zeta_{N_\psi})^\psi(\mu_{2^\infty})} \to \mathbb{Z}/2\mathbb{Z}$ form an $\mathbb{F}_2$-vector space of dimension at most g. Then for every pair of primes (q, d) with*

$$q \equiv 3 \pmod 8, \qquad d \equiv 7 \pmod{16}, \qquad \gcd(qd, N_\psi) = 1,$$

the unramified continuous homomorphisms $G_{L_{q,d}(\mu_{2^\infty})} \to \mathbb{Z}/2\mathbb{Z}$, where

$$L_{q,d} = \mathbb{Q}(\zeta_{N_\psi})^\psi(i, \sqrt{-q}, \sqrt{-d}),$$

form an $\mathbb{F}_2$-vector space of dimension at most $4(g + 6o_\psi)$.

Proof. Put $F_0 = \mathbb{Q}(\zeta_{N_\psi})^\psi$ and $F_\infty = F_0(\mu_{2^\infty})$. By Galois theory $\mathrm{Gal}(F_0/\mathbb{Q}) \cong (\mathbb{Z}/N_\psi\mathbb{Z})^\times / \ker\psi$, which is isomorphic to the image of ψ, so $[F_0 : \mathbb{Q}] = o_\psi$. For a finite set P of odd primes and a subfield $F \supseteq F_\infty$ of $\overline{\mathbb{Q}}$, let $\mathrm{Ur}_P(F)$ be the $\mathbb{F}_2$-space of continuous homomorphisms $G_F \to \mathbb{Z}/2\mathbb{Z}$ that vanish on the inertia group of every place of $\overline{\mathbb{Q}}$ above a prime not in P; so $\mathrm{Ur}_\emptyset(F)$ is the space of unramified ones, and $\dim \mathrm{Ur}_\emptyset(F_\infty) \leq g$. Fix (q, d) as in the statement; $q \neq d$, as $q \equiv 3$ and $d \equiv 7 \pmod 8$, and both are odd and prime to N_ψ.

Step 1: few places above q and d. The field $F_\infty \subseteq \mathbb{Q}(\zeta_{N_\psi}, \mu_{2^\infty})$ is abelian over $\mathbb{Q}$ and unramified at every odd prime $p \nmid N_\psi$. Let $\mathcal{G} = \mathrm{Gal}(F_\infty/\mathbb{Q})$, and for such p let $D_p \subseteq \mathcal{G}$ be its decomposition group (the same for all places above p, as $\mathcal{G}$ is abelian) and $t_p = [\mathcal{G} : D_p]$ the number of places of F_∞ above p. Restriction $\rho\colon \mathcal{G} \to \mathrm{Gal}(\mathbb{Q}(\mu_{2^\infty})/\mathbb{Q}) \cong \mathbb{Z}_2^\times$ (through χ_{cyc}) is surjective, and its kernel $\mathcal{K} = \mathrm{Gal}(F_\infty/\mathbb{Q}(\mu_{2^\infty}))$ embeds into $\mathrm{Gal}(F_0/\mathbb{Q})$ by restriction, since $F_\infty = F_0\,\mathbb{Q}(\mu_{2^\infty})$; so $\#\mathcal{K} \leq o_\psi$. The image $\rho(D_p)$ is closed and contains $\rho(\mathrm{Frob}_p) = p$. Hence

$$t_p = [\mathcal{G} : D_p\mathcal{K}]\,[D_p\mathcal{K} : D_p] \leq [\mathbb{Z}_2^\times : \rho(D_p)]\,[\mathcal{K} : \mathcal{K} \cap D_p] \leq [\mathbb{Z}_2^\times : \overline{\langle p \rangle}]\; o_\psi.$$

For $q \equiv 3 \pmod 8$, $v(q-1) = 1$ and $v(q+1) = 2$, so $v(q^2-1) = 3$ and $\overline{\langle q^2 \rangle} = U_3$, of index 4 in $\mathbb{Z}_2^\times$; as $q \notin U_3$ and $\overline{\langle q \rangle} = \overline{\langle q^2 \rangle} \cup q\,\overline{\langle q^2 \rangle}$, $[\overline{\langle q \rangle} : U_3] = 2$ and $[\mathbb{Z}_2^\times : \overline{\langle q \rangle}] = 2$. For $d \equiv 7 \pmod{16}$, $v(d-1) = 1$ and $v(d+1) = 3$, so $v(d^2-1) = 4$, $\overline{\langle d^2 \rangle} = U_4$ of index 8, $d \notin U_4$, and $[\mathbb{Z}_2^\times : \overline{\langle d \rangle}] = 4$. Therefore

$$t_q \leq 2o_\psi, \qquad t_d \leq 4o_\psi. \tag{39.10}$$

Step 2: allowing ramification at one more prime. Let P be a finite set of odd primes and $p \notin P$ an odd prime with $p \nmid N_\psi$. We show $\dim \mathrm{Ur}_{P\cup\{p\}}(F_\infty) \le \dim \mathrm{Ur}_P(F_\infty) + t_p$. For each of the t_p places w of F_∞ above p fix a place $\bar w$ of $\overline{\mathbb{Q}}$ above w, with inertia group $I_{\bar w} \subseteq G_{F_\infty}$; as $F_\infty/\mathbb{Q}$ is unramified at p, $I_{\bar w}$ is the inertia group of $\bar w$ in $G_\mathbb{Q}$, isomorphic to the inertia group of $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$. A continuous homomorphism $I_{\bar w} \to \mathbb{Z}/2\mathbb{Z}$ factors through the maximal pro-2 quotient of $I_{\bar w}$, which is procyclic; so there is $\tau_w \in I_{\bar w}$ such that every continuous homomorphism $I_{\bar w} \to \mathbb{Z}/2\mathbb{Z}$ vanishing at τ_w vanishes. Consider the linear map $\mathrm{Ur}_{P\cup\{p\}}(F_\infty) \to (\mathbb{Z}/2\mathbb{Z})^{t_p}$, $\chi \mapsto (\chi(\tau_w))_w$. If χ is in its kernel, χ vanishes on each $I_{\bar w}$; every place of $\overline{\mathbb{Q}}$ above p is $h\bar w$ for one of the chosen $\bar w$ and some $h \in G_{F_\infty}$, with inertia group $hI_{\bar w}h^{-1}$, on which χ vanishes because $\chi(hxh^{-1}) = \chi(x)$. So the kernel is $\mathrm{Ur}_P(F_\infty)$, and the claim follows.

Step 3: a quadratic step at most doubles the dimension. Let P be a finite set of odd primes, $p \in P$ with $p \equiv 3 \pmod 4$, $F \supseteq F_\infty$ a subfield of $\overline{\mathbb{Q}}$ with $\dim \mathrm{Ur}_P(F) < \infty$, and $F' = F(\sqrt{-p})$. We show $\dim \mathrm{Ur}_P(F') \le 2 \dim \mathrm{Ur}_P(F)$. The field $\mathbb{Q}(\sqrt{-p})$ has discriminant $-p$, so it is unramified at every prime $p' \ne p$; hence so is $F' = F\,\mathbb{Q}(\sqrt{-p})$ over F, and the inertia group $I \subseteq G_F$ of a place of $\overline{\mathbb{Q}}$ above a prime not in P lies in $G_{F'}$ and is its inertia group there. If $F' = F$ there is nothing to show. Otherwise $V = G_F$ contains $V' = G_{F'}$ with index 2, and the homomorphism $\rho_0 \colon V \to \mathbb{Z}/2\mathbb{Z}$ with kernel V' lies in $\mathrm{Ur}_P(F)$ and is nonzero. The group V acts on the continuous homomorphisms $V' \to \mathbb{Z}/2\mathbb{Z}$ by $(h\chi)(x) = \chi(h^{-1}xh)$; V' acts trivially, and $\mathrm{Ur}_P(F')$ is stable, because $h^{-1}Ih$ is the inertia group of a place above the same prime. Fix $s \in V \setminus V'$ and let θ be its action on $W = \mathrm{Ur}_P(F')$; then $\theta^2 = 1$, so in characteristic 2, $(\theta-1)^2 = \theta^2 - 1 = 0$ and $(\theta-1)W \subseteq W^V$, the subspace of V-invariants. By the five-term sequence $0 \to \mathrm{Hom}(V/V', \mathbb{Z}/2) \to \mathrm{Hom}(V, \mathbb{Z}/2) \to \mathrm{Hom}(V', \mathbb{Z}/2)^V \to H^2(V/V', \mathbb{Z}/2) \cong \mathbb{Z}/2$ (continuous homomorphisms), the subspace $W_0 = W^V \cap \mathrm{res}(\mathrm{Hom}(V, \mathbb{Z}/2))$ has codimension at most 1 in W^V. If $\chi' = \mathrm{res}\,\chi \in W_0$, then χ vanishes on every inertia group I as above, since $I \subseteq V'$ and $\chi|_I = \chi'|_I = 0$; so $\chi \in \mathrm{Ur}_P(F)$, and $W_0 \subseteq \mathrm{res}(\mathrm{Ur}_P(F))$, which has dimension at most $\dim \mathrm{Ur}_P(F) - 1$ because the nonzero $\rho_0 \in \mathrm{Ur}_P(F)$ restricts to 0. Hence $\dim W^V \le \dim \mathrm{Ur}_P(F)$. Finally $W/\ker(\theta-1) \cong (\theta-1)W \subseteq W^V = \ker(\theta-1)$, so $\dim W \le 2 \dim W^V \le 2 \dim \mathrm{Ur}_P(F)$.

Step 4: assembly. By Step 2 with $P = \emptyset$, $p = q$, and then with $P = \{q\}$, $p = d$, and by (39.10),

$$\dim \mathrm{Ur}_{\{q,d\}}(F_\infty) \le \dim \mathrm{Ur}_\emptyset(F_\infty) + t_q + t_d \le g + 6o_\psi.$$

Since $q \equiv d \equiv 3 \pmod 4$, Step 3 applies with $P = \{q, d\}$, first to $F = F_\infty$, $p = q$, and then to $F = F_\infty(\sqrt{-q})$, $p = d$:

$$\dim \mathrm{Ur}_{\{q,d\}}\big(F_\infty(\sqrt{-q}, \sqrt{-d})\big) \le 4(g + 6o_\psi).$$

As $i \in \mu_{2^\infty}$, $F_\infty(\sqrt{-q}, \sqrt{-d}) = L_{q,d}(\mu_{2^\infty})$, and the unramified homomorphisms form the subspace $\mathrm{Ur}_\emptyset$ of $\mathrm{Ur}_{\{q,d\}}$. This proves the bound. □

40 Second cohomology of Tate twists

This section proves Theorem 40.1: for a totally complex field L containing the ℓ-th roots of unity (the fourth roots if $\ell = 2$) and every $e \ge 1$, the group $H^2(G_{L,S_\ell}, \mathbb{Z}_\ell(e+1))$ is finite and $\log \# H^2 \le (2|d_L|)^{(2e[L:\mathbb{Q}])^C}$. The bound is uniform in the twist but doubly exponential, and it is used where only a few twists or a fixed field occur: at the exceptional twists of Section 39 (Proposition 39.2) and at the twist $e = 1$ in the remaining part of case (V) (Proposition 38.35). Its form for Tate–Shafarevich groups, Proposition 40.3, gives the finiteness of the strict groups of Tate twists that Part VIII needs to show that certain cocycles are rational coboundaries (Proposition 43.32, Lemma 45.31, Proposition 46.14). The groups $H^2(G_{L,S_\ell}, \mathbb{Z}_\ell(e+1))$ are

those that Soulé relates to the K-theory of $\mathcal{O}_L$ [237], and the bound is of the type of his bound for the torsion in that K-theory [238]; here it is proved for the Galois cohomology directly.

The proof first bounds the Tate–Shafarevich group (Proposition 40.3) and then passes to H^2 by Poitou–Tate and local Tate duality. After an extension that adds enough roots of unity without loss (Proposition 40.4), the Tate–Shafarevich group is bounded by the eigencharacters of the class groups of cyclotomic Kummer layers (Proposition 40.27), and these are counted by twisted Kummer classes (Proposition 40.24). The key step is Theorem 40.7: twisted Kummer classes are detected by the integral homology of $\mathrm{GL}_N(\mathcal{O}_L)$ with $N = 4e + 10$, through cycles built from the standard cycle of a cyclic group and evaluated by additive families over finite fields (Theorem 40.22). The torsion in that homology is polynomial in the discriminant (Theorem 40.29), by a complex of perfect forms in the manner of Soulé: the perfect-cone nerve built from the Ryshkov polyhedron is acyclic, its simplices have finite stabilisers, and the number of their orbits is polynomial in the discriminant. The construction of cycles occupies most of the section; the duality steps are routine.

Theorem 40.1. *Let ℓ be a prime. There is a real number $C \geq 1$ such that for every totally complex number field L containing a primitive ℓ-th root of unity (a primitive fourth root of unity if $\ell = 2$) and every integer $e \geq 1$, the group $H^2(G_{L,S_\ell}, \mathbb{Z}_\ell(e+1))$ is finite and*

$$\log \#H^2(G_{L,S_\ell}, \mathbb{Z}_\ell(e+1)) \leq \left(2|d_L|\right)^{(2e[L:\mathbb{Q}])^C}.$$

Proof. Fix ℓ and let $C_0 \geq 0$ be the constant of Proposition 40.3 for ℓ; put $C = \max(C_0, 2)+3 \geq 1$. Let L and $e \geq 1$ be as in the statement, $n = [L:\mathbb{Q}]$, $s = \mathrm{len}_\ell(L, e)$, finite by Proposition 40.3, S the set of places of L above ℓ or ∞, and $G = G_{L,S_\ell}$. The archimedean places are complex, so $G = G_{L,S}$ and $H^i(L_v, \cdot) = 0$ $(i \geq 1)$ at them. Put $T = \mathbb{Z}_\ell(e+1)$, $T_r = T/\ell^r T$, $A = \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$ and $A_r = A[\ell^r]$ for $r \geq 1$, a free $\mathbb{Z}/\ell^r\mathbb{Z}$-module of rank one on which G acts through χ_{cyc}^{-e}. Its Cartier dual $A_r^\vee = \mathrm{Hom}(A_r, \mu_{\ell^r})$ is $\mathbb{Z}/\ell^r\mathbb{Z}$ with action through $\chi_{\mathrm{cyc}}^{e+1}$, so $A_r^\vee \cong T_r$ and dually $T_r^\vee \cong A_r$.

Step 1: the local invariants are small. Let v be a place of L above ℓ and $U_v = \chi_{\mathrm{cyc}}(G_{L_v}) \subseteq \mathbb{Z}_\ell^\times$. As G_{L_v} is a subgroup of index $[L_v : \mathbb{Q}_\ell] \leq n$ of $G_{\mathbb{Q}_\ell}$ and χ_{cyc} is surjective on $G_{\mathbb{Q}_\ell}$, the index $k = [\mathbb{Z}_\ell^\times : U_v]$ is at most n, and $u = (1+\ell^2)^k \in U_v$. By the lifting-the-exponent rule, $v_\ell(u^e - 1) = 2 + v_\ell(ke)$. If $x \in H^0(L_v, A)$ and $g \in G_{L_v}$ has $\chi_{\mathrm{cyc}}(g) = u$, then $x = gx = u^{-e}x$, so $(u^{-e} - 1)x = 0$; as $u^{-e} - 1 = -u^{-e}(u^e - 1)$ is $\ell^{2+v_\ell(ke)}$ times a unit of $\mathbb{Z}_\ell$, $\ell^{2+v_\ell(ke)}x = 0$. The ℓ^a-torsion of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ has ℓ^a elements, so

$$\#H^0(L_v, A) \leq \ell^{2+v_\ell(k)+v_\ell(e)} \leq \ell^2 ne. \tag{40.1}$$

The action of G_{L_v} on A factors through its image in G, so $H^0(G, A) \subseteq H^0(L_v, A)$, and $\#H^0(G, A) \leq \ell^2 ne$; there is at least one place above ℓ.

Step 2: a uniform bound at finite level. Let $r \geq 1$. By definition of Sha^2_S and the vanishing at the complex places, the sequence $0 \to \mathrm{Sha}^2_S(L, T_r) \to H^2(G, T_r) \to \prod_{v|\ell} H^2(L_v, T_r)$ is exact. By item 2 of Theorem B.29 for $M = T_r$, $H^2(L_v, T_r)$ is dual to $H^0(L_v, T_r^\vee) \cong H^0(L_v, A_r) \subseteq H^0(L_v, A)$. By item 2 of Theorem B.34 for $M = A_r$ (L has no real place, S contains the places above ℓ and ∞), $\#\mathrm{Sha}^2_S(L, T_r) = \#\mathrm{Sha}^1_S(L, A_r)$. To bound the latter, let $\iota\colon H^1(G, A_r) \to H^1(G, A)$ be induced by the inclusion. The cohomology sequence of $0 \to A_r \to A \xrightarrow{\ell^r} A \to 0$ (exact, as A is divisible) shows that $\ker\iota$ is the image of $H^0(G, A)$ under the connecting map, so $\#\ker\iota \leq \#H^0(G, A)$. Localisation at each $v \mid \ell$ commutes with ι, so ι maps $\mathrm{Sha}^1_S(L, A_r)$ into $\mathrm{Sha}^1_\ell(L, e)$; the image P is finite, because $H^1(G, A_r)$ is finite (item 1 of Theorem B.34), so

$\#P = \ell^{\operatorname{length} P} \le \ell^s$. Hence $\#\mathrm{Sha}^1_S(L, A_r) \le \#H^0(G,A)\,\ell^s$. Since L has at most n places above ℓ, Step 1 gives

$$\#H^2(G,T_r) \le \#H^0(G,A)\,\ell^s \prod_{v\mid\ell} \#H^0(L_v,A) \le \ell^s\big(\ell^2 ne\big)^{n+1} =: B \qquad (r \ge 1). \tag{40.2}$$

Step 3: passage to $\mathbb{Z}_\ell$-coefficients. The groups $H^1(G,T_r)$ are finite (item 1 of Theorem B.34), so by Theorem B.31 with $i=2$ the map $H^2(G,T) \to \varprojlim_r H^2(G,T_r)$ is an isomorphism. An inverse limit of finite sets of cardinality at most B has at most B elements: finitely many distinct elements of the limit are already distinct in one component, which has at most B elements. So $H^2(G,T)$ is finite and $\log \#H^2(G,T) \le s\log\ell + (n+1)\log(\ell^2 ne)$.

Step 4: the shape of the bound. If ℓ is odd, $\mathbb{Q}(\zeta_\ell) \subseteq L$ gives $\ell - 1 \le n$; if $\ell = 2$, $\ell \le n+1$ trivially. So $\log\ell \le \log(n+1) \le n$, and $(n+1)\log(\ell^2 ne) \le (n+1)(2n+n+e) \le 8n^2 e$. Put $u = 2en \ge 2$ and $Y = \log(2|d_L|) \ge \log 2 > 1/2$; then $8n^2 e \le 2u^2 \le 4u^2 Y$. With Proposition 40.3, $s\log\ell \le \exp(u^{C_0}Y)$, and since $\exp(a)+b \le \exp(a+b)$ for $a,b \ge 0$,

$$\log\#H^2(G,T) \le \exp\big(u^{C_0}Y + 4u^2Y\big) \le \exp\big(5u^{\max(C_0,2)}Y\big) \le \exp\big(u^C Y\big) = \big(2|d_L|\big)^{(2e[L:\mathbb{Q}])^C},$$

which is the claim, with C depending only on ℓ. □

Definition 40.2. Let ℓ be a prime, L a number field, and c, ν integers with $1 \le c \le \nu$, and $c \ge 2$ if $\ell = 2$. A **cyclotomic Kummer layer of level (c,ν) over L** is a triple (F,γ,ζ) where F/L is a finite Galois extension of number fields, γ is a generator of $\mathrm{Gal}(F/L)$ of order $\ell^{\nu-c}$, and $\zeta \in \mathcal{O}_F^\times$ is a primitive ℓ^ν-th root of unity with

$$\gamma(\zeta) = \zeta^{1+\ell^c}. \tag{40.3}$$

Let (F,γ,ζ) be such a layer and let $a \ge 0$, $q \ge 1$, $s \ge 0$ be integers.

(Ky1) The **(γ,a)-eigencharacters of level q** are the homomorphisms $\varphi\colon \mathrm{Cl}(F) \to \mathbb{Z}/q\mathbb{Z}$ with $\varphi([\gamma^{-1}(\mathfrak{a})]) = a\,\varphi([\mathfrak{a}])$ for every fractional ideal $\mathfrak{a}$ of F. They form a group, written $\mathrm{Hom}(\mathrm{Cl}(F),\mathbb{Z}/q\mathbb{Z})^{(\gamma,a)}$.

(Ky2) An element $\alpha \in F^\times$ is a **(γ,a)-Kummer element** if $(\alpha) = \gamma^{-1}(\mathfrak{d})\,\mathfrak{d}^{-a}$ for some fractional ideal $\mathfrak{d}$ of F.

(Ky3) The **twisted norm** of $\alpha \in F^\times$ is

$$\mathrm{N}_{\gamma,a}(\alpha) = \prod_{j=0}^{\ell^{\nu-c}-1} \gamma^j(\alpha)^{a^j} \in F^\times.$$

(Ky4) The set of **twisted Kummer classes** $\mathrm{Ku}_{s,q}(F,\gamma,a) \subseteq F^\times/F^{\times q}$ is the set of classes of $\mathrm{N}_{\gamma,a}(\alpha)^s$, where α runs over the (γ,a)-Kummer elements.

Proposition 40.3. *Let ℓ be a prime. There is a real number $C_0 \ge 0$ such that for every totally complex number field L containing a primitive ℓ-th root of unity (a primitive fourth root of unity if $\ell = 2$) and every integer $e \ge 1$, $\mathrm{len}_\ell(L,e)$ is finite and*

$$\mathrm{len}_\ell(L,e)\,\log\ell \le \big(2|d_L|\big)^{(2e[L:\mathbb{Q}])^{C_0}}.$$

Proof. Let $C_{\mathrm{T}} \geq 0$ be the constant of Theorem 40.29; put $C_1 = \max(C_{\mathrm{T}}, 4)$, $C_2 = C_1 + 5$ and $C_0 = 5C_2 + 11$. Fix ℓ, a totally complex number field L containing a primitive ℓ-th root of unity (fourth if $\ell = 2$) and $e \geq 1$; put $n = [L : \mathbb{Q}]$, $Y = \log(2|d_L|)$, $U = 2en \geq 2$ and $s = \mathrm{len}_\ell(L, e)$.

Step 1: the auxiliary field. By Proposition 40.4 there are a totally complex field L' and integers $c \geq 1$ ($c \geq 2$ if $\ell = 2$), $d \geq 1$ with: $\mu_{\ell^c} \subseteq L'$, $\mu_{\ell^{c+1}} \not\subseteq L'$; $2e + 1 < \ell^c \leq 2n'$, where $n' = [L' : \mathbb{Q}] = dn$; $d \leq (n+1)(4e+4)$ and $|d_{L'}| \leq |d_L|^d (\ell^c)^{n'}$; and $s \leq \mathrm{len}_\ell(L', e)$. Put $Y' = \log(2|d_{L'}|) \geq \log 2 > 1/2$ and $u' = 2en'$; as $n' \geq 2$, $u' \geq 4e \geq 4$.

Step 2: one bound for the eigencharacters of all layers over L'. Let t_1, t_2 be the orders of $H_{2e+1}(\mathrm{GL}_{4e+6}(\mathcal{O}_{L'}), \mathbb{Z})_{\mathrm{tors}}$ and $H_{2e}(\mathrm{GL}_{4e+6}(\mathcal{O}_{L'}), \mathbb{Z})_{\mathrm{tors}}$, finite by Theorem 40.29 (L' totally complex, $m = 2e+1$ and $m = 2e$, both $\leq 2e+1$), and put $b = 2e + (2e+1)(c + 2e + 1)$, $g' = 2n' + \lfloor \log_2 h_{L'} \rfloor + 2$ and $X = (b + v_\ell(e))g' + c + v_\ell(e)$. Let $\nu \geq c$ and let (F, γ, ζ) be a cyclotomic Kummer layer of level (c, ν) over L'; put $a = (1 + \ell^c)^e$. Theorem 40.7, applied to L' and $e - 1 \geq 0$ (its hypotheses read: L' totally complex and $2e + 1 < \ell^c$; its N is $4e + 6$ and its b_{e-1} is b), gives $\#\mathrm{Ku}_{\ell^b, \ell^\nu}(F, \gamma, a) \leq \ell^{\nu r_2(L')} t_1 t_2$. Then Proposition 40.24, applied to L' with $B = \ell^{\nu r_2(L')} t_1 t_2$, shows that $\mathrm{Hom}(\mathrm{Cl}(F), \mathbb{Z}/\ell^\nu\mathbb{Z})^{(\gamma, a)}$ is finite with $\#\mathrm{Hom}(\mathrm{Cl}(F), \mathbb{Z}/\ell^\nu\mathbb{Z})^{(\gamma,a)} \cdot \ell^{\nu r_2(L')} \leq \ell^{\nu r_2(L')} t_1 t_2\, \ell^X$, that is

$$\#\mathrm{Hom}(\mathrm{Cl}(F), \mathbb{Z}/\ell^\nu\mathbb{Z})^{(\gamma,a)} \leq t_1 t_2\, \ell^X, \tag{40.4}$$

a bound independent of ν and of the layer.

Step 3: the Tate–Shafarevich group. By (40.4) the hypothesis of Proposition 40.27 holds for L', c, e and $B = t_1 t_2 \ell^X$. So $\mathrm{len}_\ell(L', e)$ is finite and $\ell^{\mathrm{len}_\ell(L', e)} \leq t_1 t_2 \ell^X$; with $s \leq \mathrm{len}_\ell(L', e)$, s is finite and

$$s \log \ell \leq \log t_1 + \log t_2 + X \log \ell. \tag{40.5}$$

Step 4: the class number of L'. By Minkowski's bound and the count of ideals, $h_{L'} \leq \sum_{k \leq \sqrt{|d_{L'}|}} k^{n'} \leq |d_{L'}|^{(n'+1)/2}$, so $\lfloor \log_2 h_{L'} \rfloor \leq \frac{n'+1}{2 \log 2} \log |d_{L'}| \leq 2n'Y'$ and $g' \leq 2n' + 2n'Y' + 2$.

Step 5: the bound in terms of L'. Since $e \geq 1$ and $n' \geq 2$: $2e + 1 \leq u'$; $c < \ell^c \leq 2n' \leq u'$; $\log \ell \leq \ell \leq u'$; $v_\ell(e) \leq \log_2 e < e \leq u'$. Hence $b + v_\ell(e) \leq u' + u' \cdot 2u' + u' \leq 4u'^2$, $g' \leq 2u' + u'Y' \leq 5u'Y'$ (using $2 \leq u'$, $2n' \leq u'$, $1 < 2Y'$), and

$$X \log \ell \leq (4u'^2 \cdot 5u'Y' + 2u')u' \leq 24u'^4 Y'.$$

By Theorem 40.29 for L', $\log t_i \leq \exp(u'^{C_{\mathrm{T}}} Y')$. With (40.5) and $2\exp(x) + y \leq \exp(x + 1 + y)$ for $x, y \geq 0$,

$$s \log \ell \leq \exp\bigl(u'^{C_{\mathrm{T}}} Y' + 1 + 24u'^4 Y'\bigr) \leq \exp\bigl(27u'^{C_1} Y'\bigr) \leq \exp\bigl(u'^{C_2} Y'\bigr).$$

Step 6: back to L. From $d \leq (n+1)(4e+4) \leq 16en = 8U$ and $n' = dn \leq 16en^2 \leq 4U^2$ we get $u' = 2en' \leq 32e^2n^2 = 8U^2 \leq U^5$. From $|d_{L'}| \leq |d_L|^d (\ell^c)^{n'}$, $d \geq 1$ and $\ell^c \leq 2n'$,

$$Y' \leq \log 2 + d \log |d_L| + n' \log(2n') \leq dY + 2n'^2 \leq 8UY + 32U^4 \leq 72U^4 Y \leq U^{11} Y,$$

using $Y > 1/2$ and $72 \leq U^7$. Hence $u'^{C_2} Y' \leq U^{5C_2 + 11} Y = U^{C_0} Y$, and by Step 5

$$\mathrm{len}_\ell(L, e) \log \ell = s \log \ell \leq \exp\bigl(U^{C_0} Y\bigr) = \bigl(2|d_L|\bigr)^{(2e[L:\mathbb{Q}])^{C_0}}.$$

The constant $C_0 \geq 0$ does not depend on L or e (nor on ℓ). □

Proposition 40.4. *Let ℓ be a prime, L a number field containing a primitive ℓ-th root of unity (a primitive fourth root of unity if $\ell = 2$), and $e \geq 1$ an integer. There are a totally complex number field L' and integers $c \geq 1$ ($c \geq 2$ if $\ell = 2$) and $d \geq 1$ such that:*

1. *L' contains a primitive ℓ^c-th root of unity and no primitive ℓ^{c+1}-th root of unity;*
2. *$2e+1 < \ell^c \le 2[L':\mathbb{Q}]$;*
3. *$[L':\mathbb{Q}] = d\,[L:\mathbb{Q}]$, $d \le ([L:\mathbb{Q}]+1)(4e+4)$, and $|d_{L'}| \le |d_L|^d\,(\ell^c)^{[L':\mathbb{Q}]}$;*
4. *$\mathrm{len}_\ell(L,e) \le \mathrm{len}_\ell(L',e)$.*

Proof. Let $n = [L:\mathbb{Q}]$; as L contains a primitive ℓ-th root of unity with ℓ odd, or i, it has no real place and $n \ge 2$, and $\ell \le n+1$ ($\ell - 1 = \varphi(\ell) \le n$ for ℓ odd). Let c_0 be the largest integer with $\mu_{\ell^{c_0}} \subseteq L$ (it exists as $\varphi(\ell^k) \le n$ when $\mu_{\ell^k} \subseteq L$); $c_0 \ge 1$, and $c_0 \ge 2$ if $\ell = 2$. Let $m \ge 0$ be the least integer with $\ell^{c_0+m} > 2e+1$. Fix an algebraic closure $\overline{L}$, put $L_k = L(\mu_{\ell^k}) \subseteq \overline{L}$, $L' = L_{c_0+m}$, $d = \ell^m$, and let c be the largest integer with $\mu_{\ell^c} \subseteq L'$.

Step 1: the cyclotomic tower. The cyclotomic character embeds $\Gamma = \mathrm{Gal}(L(\mu_{\ell^\infty})/L)$ as a closed subgroup of U_{c_0}, and $\tau \in \Gamma$ fixes μ_{ℓ^k} ($k \ge 1$) exactly when $\chi_{\mathrm{cyc}}(\tau) \in U_k$. The image is infinite, since $[L_k : L] \ge \varphi(\ell^k)/n \to \infty$, so it is U_{k_1} for some $k_1 \ge c_0$, and $k_1 = c_0$ because the image lies in U_{c_0+1} only if $\mu_{\ell^{c_0+1}} \subseteq L$. Hence $\chi_{\mathrm{cyc}}\colon \Gamma \to U_{c_0}$ is an isomorphism, $\mathrm{Gal}(L(\mu_{\ell^\infty})/L')$ maps onto U_{c_0+m}, and $[L':L] = [U_{c_0} : U_{c_0+m}] = \ell^m = d$. Let $\sigma \in \mathrm{Gal}(L'/L)$ be the restriction of an element of Γ with χ_{cyc}-value $u = 1+\ell^{c_0}$; it generates the cyclic group $\mathrm{Gal}(L'/L) \cong U_{c_0}/U_{c_0+m}$ of order ℓ^m.

Step 2: items 1 to 3. L' contains L, so it has no real place; $c \ge c_0 + m \ge c_0$ gives the conditions on c, and item 1 holds by the choice of c. $[L':\mathbb{Q}] = dn$. If $m = 0$, then $d = 1$. If $m \ge 1$, the minimality of m gives $\ell^{c_0+m-1} \le 2e+1$, and as $c_0 \ge 1$, $d = \ell^m \le \ell \cdot \ell^{c_0+m-1} \le \ell(2e+1) \le (n+1)(4e+4)$. Further $\ell^c \ge \ell^{c_0+m} > 2e+1$; $\mathbb{Q}(\mu_{\ell^c}) \subseteq L'$ gives $\ell^c/2 \le \ell^{c-1}(\ell-1) = \varphi(\ell^c) \le [L':\mathbb{Q}]$. For the discriminant, $L' = L(\zeta)$ with ζ a primitive M-th root of unity, $M = \ell^{c_0+m}$; its minimal polynomial f over L divides $X^M - 1$ in $\mathcal{O}_L[X]$, say $X^M - 1 = fg$, so $M\zeta^{M-1} = f'(\zeta)g(\zeta)$ and, ζ being a unit, $\mathfrak{D}_{L'/L}$ divides $f'(\zeta)$, which divides M. Hence $\mathrm{N}_{L'/\mathbb{Q}}(\mathfrak{D}_{L'/L}) \le M^{[L':\mathbb{Q}]} \le (\ell^c)^{[L':\mathbb{Q}]}$ and $|d_{L'}| \le |d_L|^d(\ell^c)^{[L':\mathbb{Q}]}$.

Step 3: restriction is injective on H^1. Let $\Omega \subseteq \overline{L}$ be the maximal extension of L unramified outside the places above ℓ, $G = \mathrm{Gal}(\Omega/L) = G_{L,S_\ell}$ and $A = \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$. As L'/L is unramified outside ℓ, $L' \subseteq \Omega$, Ω is also the maximal extension of L' unramified outside the places above ℓ, and $H = \mathrm{Gal}(\Omega/L') = G_{L',S_\ell}$ is normal in G with $G/H = \mathrm{Gal}(L'/L)$. By Step 1, $\chi_{\mathrm{cyc}}(H) = U_{c_0+m}$, so A^H is the set of $x \in A$ with $(y^{-e}-1)x = 0$ for all $y \in U_{c_0+m}$; by the lifting-the-exponent rule the least valuation of $y^{-e}-1$ is $a := c_0 + m + v(e)$, attained at $y = (1+\ell^{c_0})^{\ell^m}$, so $A^H = A[\ell^a]$, cyclic of order ℓ^a. On it σ acts by $\beta = u^{-e}$ with $v(\beta-1) = c_0 + v(e)$. Thus $(\sigma-1)A^H = \ell^{c_0+v(e)}A[\ell^a]$ has order $\ell^{a-c_0-v(e)} = \ell^m$; and $\mathrm{Nm} = \sum_{i<\ell^m}\beta^i = (\beta^{\ell^m}-1)/(\beta-1)$ has valuation m, so $\ker \mathrm{Nm} = A[\ell^m]$ also has order ℓ^m ($a \ge m$). In a cyclic group the subgroup of a given order is unique, so $\ker \mathrm{Nm} = (\sigma-1)A^H$ and $H^1(\mathrm{Gal}(L'/L), A^H) = 0$. By inflation–restriction, $\mathrm{res}\colon H^1(G,A) \to H^1(H,A)$ is injective.

Step 4: item 4. Let $x \in \mathrm{Sha}^1_\ell(L,e)$ and let w be a place of L' above ℓ, over the place v of L. The localisation of $\mathrm{res}(x)$ at w is the restriction of the localisation of x at v along $G_{L'_w} \subseteq G_{L_v}$, hence 0. So res maps $\mathrm{Sha}^1_\ell(L,e)$ injectively and $\mathbb{Z}_\ell$-linearly into $\mathrm{Sha}^1_\ell(L',e)$, and lengths are monotone under injective maps: $\mathrm{len}_\ell(L,e) \le \mathrm{len}_\ell(L',e)$ in $\mathbb{N} \cup \{\infty\}$. □

Definition 40.5. Let G be a group and A a commutative ring. For $m \ge 0$, $\mathrm{Bar}_m(G;A)$ is the free A-module on the set of tuples $[g_1|\dots|g_m] \in G^m$, and the **bar differential** $\partial\colon \mathrm{Bar}_{m+1}(G;A) \to \mathrm{Bar}_m(G;A)$ is the A-linear map

$$\partial[g_0|\dots|g_m] = [g_1|\dots|g_m] + \sum_{j=0}^{m-1}(-1)^{j+1}[g_0|\dots|g_jg_{j+1}|\dots|g_m] + (-1)^{m+1}[g_0|\dots|g_{m-1}],$$

so that $(\mathrm{Bar}_\bullet(G;A),\partial)$ is the bar complex computing $H_\bullet(G;A)$ with trivial coefficients. An element $z\in\mathrm{Bar}_m(G;A)$, $m\ge 1$, is a **cycle** if $\partial z=0$ and a **boundary** if $z\in\partial\,\mathrm{Bar}_{m+1}(G;A)$. A homomorphism $f\colon G\to H$ induces $f_*[g_1|\dots|g_m]=[f(g_1)|\dots|f(g_m)]$. For $k\in G$ the **prism operator** $\mathrm{Pr}_k\colon \mathrm{Bar}_m(G;A)\to\mathrm{Bar}_{m+1}(G;A)$ is

$$\mathrm{Pr}_k[g_0|\dots|g_{m-1}]=\sum_{i=0}^{m}(-1)^i\,[g_0|\dots|g_{i-1}|k|k^{-1}g_ik|\dots|k^{-1}g_{m-1}k].$$

For integers $q\ge 1$ and $e\ge 0$ let $C_q=\langle\tau\rangle$ be the cyclic group of order q with a fixed generator τ; the **standard cycle** is

$$\mathbf{z}_{q,e}=\sum_{(j_0,\dots,j_e)\in(\mathbb{Z}/q\mathbb{Z})^{e+1}}[\tau^{j_0}|\tau|\tau^{j_1}|\tau|\dots|\tau^{j_e}|\tau]\in\mathrm{Bar}_{2e+2}(C_q;\mathbb{Z}/q\mathbb{Z}).$$

Let R be a commutative ring and $n\ge 0$. For $n\le n'$ the **stabilisation** $\mathrm{stab}_{n,n'}\colon \mathrm{GL}_n(R)\to\mathrm{GL}_{n'}(R)$ is $g\mapsto\begin{pmatrix}g&0\\0&1\end{pmatrix}$, and for $d\ge 1$ the **block-diagonal map** $\mathrm{blk}_{d,n}\colon\mathrm{GL}_n(R)\to\mathrm{GL}_{dn}(R)$ is $g\mapsto\mathrm{diag}(g,\dots,g)$ (d blocks). For a ring homomorphism $r\colon R\to k$ to a field, $\mathrm{red}_{r,n}\colon\mathrm{GL}_n(R)\to\mathrm{GL}(k^n)$ applies r to the entries.

Definition 40.6. Let k be a finite field, A a commutative ring and $m\ge 1$. An **additive family of degree m over k with values in A** is a family $\theta=(\theta_V)_V$ of A-linear maps $\theta_V\colon\mathrm{Bar}_m(\mathrm{GL}(V);A)\to A$, one for each finite-dimensional k-vector space V, such that:

(Fa1) θ_V vanishes on boundaries;

(Fa2) for every k-linear isomorphism $\varphi\colon V\to W$ and every cycle $z\in\mathrm{Bar}_m(\mathrm{GL}(V);A)$,

$$\theta_W((c_\varphi)_*z)=\theta_V(z),$$

where $c_\varphi(g)=\varphi g\varphi^{-1}$;

(Fa3) for every group G, all homomorphisms $\rho\colon G\to\mathrm{GL}(V)$, $\sigma\colon G\to\mathrm{GL}(W)$ and every cycle $z\in\mathrm{Bar}_m(G;A)$, $\theta_{V\oplus W}((\rho\oplus\sigma)_*z)=\theta_V(\rho_*z)+\theta_W(\sigma_*z)$, where $(\rho\oplus\sigma)(g)=\rho(g)\oplus\sigma(g)$.

Let $q\ge 1$ and $e\ge 0$ be integers, $\zeta\in k^\times$ with $\zeta^q=1$, and $v\colon k^\times\to\mathbb{Z}/q\mathbb{Z}$ a homomorphism. An additive family θ of degree $2e+3$ over k with values in $\mathbb{Z}/q\mathbb{Z}$ has the **value property for** (q,e,ζ,v) if

(Fa4) for every V, every integer $x\ge 0$ and every $g\in\mathrm{GL}(V)$,

$$\theta_V\Big(\mathrm{Pr}_g\big((s_V\circ\chi_x)_*\mathbf{z}_{q,e}\big)\Big)=x^{e+1}\,v(\det g),$$

where $\chi_x\colon C_q\to k^\times$ is $\tau^j\mapsto\zeta^{xj}$ and $s_V\colon k^\times\to\mathrm{GL}(V)$ sends λ to multiplication by λ.

Let R and A be commutative rings, B an A-module and $m\ge 1$. A **primitive family of degree m over R with values in B** is a family $\Phi=(\Phi_n)_{n\ge0}$ of A-linear maps $\Phi_n\colon\mathrm{Bar}_m(\mathrm{GL}_n(R);A)\to B$ such that:

(Pm1) Φ_n vanishes on boundaries;

(Pm2) $\Phi_{n'}((\mathrm{stab}_{n,n'})_*z)=\Phi_n(z)$ for $n\le n'$ and every cycle z;

(Pm3) $\Phi_{dn}((\mathrm{blk}_{d,n})_* z) = d\,\Phi_n(z)$ for $d \geq 1$ and every cycle z.

Theorem 40.7. *Let L be a totally complex number field, ℓ a prime, (F, γ, ζ) a cyclotomic Kummer layer of level (c, ν) over L, and $e \geq 0$ an integer with $2e + 3 < \ell^c$. Put $N = 4e + 10$ and $b_e = 2e + 2 + (2e + 3)(c + 2e + 3)$. If the torsion subgroups $H_{2e+3}(\mathrm{GL}_N(\mathcal{O}_L), \mathbb{Z})_{\mathrm{tors}}$ and $H_{2e+2}(\mathrm{GL}_N(\mathcal{O}_L), \mathbb{Z})_{\mathrm{tors}}$ are finite, of orders t_1 and t_2, then*

$$\#\mathrm{Ku}_{\ell^{b_e}, \ell^\nu}\big(F, \gamma, (1 + \ell^c)^{e+1}\big) \leq \ell^{\nu\, r_2(L)}\, t_1\, t_2.$$

Proof. Put $q = \ell^\nu$, $\kappa = 1 + \ell^c$, $a = \kappa^{e+1}$, $M = 2e + 3$, $s = \ell^{b_e}$ (so $b_e = (2e + 2) + M(c + M)$), $\mathrm{N} = \mathrm{N}_{\gamma,a}$, and let p_e be the integer of Proposition 40.9.

Step 1: p_e divides $\ell^{M(c+M)}$ in $\mathbb{Z}/q\mathbb{Z}$. For $0 \leq j < M$, $j \neq e + 1$, put $d = |e + 1 - j|$, so $1 \leq d \leq e + 1 < M$; then $\kappa^{e+1} - \kappa^j = \pm\kappa^{\min(e+1,j)}(\kappa^d - 1)$, and κ is prime to ℓ, so $v_\ell(\kappa^{e+1} - \kappa^j) = c + v_\ell(d) < c + M$. Hence $p_e = \ell^w u_0$ with $\ell \nmid u_0$ and $w < M(c + M)$, and $\ell^{M(c+M)} = p_e \cdot u_0^{-1} \ell^{M(c+M)-w}$ in $\mathbb{Z}/q\mathbb{Z}$, u_0 being invertible there. Consequently, for $X \in \mathbb{Z}/q\mathbb{Z}$,

$$p_e\, X = 0 \implies \ell^{M(c+M)} X = 0. \tag{40.6}$$

Step 2: data at the degree-one primes. Let $T \subseteq \mathrm{Ku}_{s,q}(F, \gamma, a)$ be finite. For each $t \in T$ choose a (γ, a)-Kummer element α_t whose class $\mathrm{N}(\alpha_t)^s F^{\times q}$ is t, and let $S = \{\alpha_t : t \in T\}$, so $\#S = \#T$ and distinct elements of S have distinct classes. For $\alpha \in S$ let N_α, y_α, b_α be given by Proposition 40.9 (its hypothesis $2e + 3 < \ell^c$ holds). Put $b = \ell \prod_{\alpha \in S} b_\alpha \neq 0$ and let I be the set of nonzero primes $\mathfrak{P}$ of $\mathcal{O}_F$ with $b \notin \mathfrak{P}$ for which $\mathcal{O}_L \to \mathcal{O}_F/\mathfrak{P}$ is surjective. For $\mathfrak{P} \in I$ let $k_\mathfrak{P} = \mathcal{O}_F/\mathfrak{P}$, $r_\mathfrak{P} \colon \mathcal{O}_F \to k_\mathfrak{P}$ the projection and $r_{\mathfrak{P},L}$ its restriction to $\mathcal{O}_L$. As $\ell \notin \mathfrak{P}$, $r_\mathfrak{P}(\zeta)$ is a primitive q-th root of unity, so q divides the order of the cyclic group $k_\mathfrak{P}^\times$; let $v_\mathfrak{P} \colon k_\mathfrak{P}^\times \to \mathbb{Z}/q\mathbb{Z}$ be a surjective homomorphism, whose kernel is $(k_\mathfrak{P}^\times)^q$. By Proposition 40.17 there is an additive family $\theta^\mathfrak{P}$ of degree M over $k_\mathfrak{P}$ with values in $\mathbb{Z}/q\mathbb{Z}$ having the value property for $(q, e, r_\mathfrak{P}(\zeta), v_\mathfrak{P})$, and by Lemma 40.20 $\Phi_n^\mathfrak{P} = \theta^\mathfrak{P}_{k_\mathfrak{P}^n} \circ (\mathrm{red}_{r_{\mathfrak{P},L},n})_*$ is a primitive family of degree M over $\mathcal{O}_L$ with values in $\mathbb{Z}/q\mathbb{Z}$.

Step 3: separation at one prime. Let $\mathfrak{P} \in I$ and $\alpha, \alpha' \in S$ with $\ell^{2e+2}\Phi^\mathfrak{P}_{N_\alpha}(y_\alpha) = \ell^{2e+2}\Phi^\mathfrak{P}_{N_{\alpha'}}(y_{\alpha'})$. As $r_\mathfrak{P}(b_\alpha) \neq 0 \neq r_\mathfrak{P}(b_{\alpha'})$, Proposition 40.9 shows that $\mathrm{N}(\alpha)$ and $\mathrm{N}(\alpha')$ are units at $\mathfrak{P}$, with residues u, u', and $\Phi^\mathfrak{P}_{N_\alpha}(y_\alpha) = p_e\, v_\mathfrak{P}(u)$, $\Phi^\mathfrak{P}_{N_{\alpha'}}(y_{\alpha'}) = p_e\, v_\mathfrak{P}(u')$. So $p_e \cdot \ell^{2e+2}(v_\mathfrak{P}(u) - v_\mathfrak{P}(u')) = 0$, and by (40.6) $v_\mathfrak{P}(u^s u'^{-s}) = \ell^{b_e}(v_\mathfrak{P}(u) - v_\mathfrak{P}(u')) = 0$. Hence the residue $u^s u'^{-s}$ of $\mathrm{N}(\alpha)^s \mathrm{N}(\alpha')^{-s}$ at $\mathfrak{P}$ is a q-th power in $k_\mathfrak{P}^\times$.

Step 4: injectivity. Let $B = \prod_{\mathfrak{P} \in I} \mathbb{Z}/q\mathbb{Z}$ and $\Phi_n = (\Phi_n^\mathfrak{P})_{\mathfrak{P} \in I}$; since (Pm1)–(Pm3) hold in each coordinate, Φ is a primitive family of degree M over $\mathcal{O}_L$ with values in the $\mathbb{Z}/q\mathbb{Z}$-module B. Let $f \colon S \to B$, $f(\alpha) = \ell^{2e+2}\Phi_{N_\alpha}(y_\alpha)$. If $f(\alpha) = f(\alpha')$, then by Step 3 the element $\beta = \mathrm{N}(\alpha)^s \mathrm{N}(\alpha')^{-s}$ is a unit with q-th power residue at every $\mathfrak{P} \in I$; F contains the primitive q-th root of unity ζ, so Lemma 40.21 gives $\beta \in F^{\times q}$. Thus α and α' have the same class, and $\alpha = \alpha'$: f is injective.

Step 5: the count. Apply Theorem 40.22 with $k = 2e + 2$, which is even, to Φ (L is totally complex); its $N_0 = 2k + 6$ is $4e + 10 = N$, and its t_{k+1}, t_k are t_1, t_2. Each $f(\alpha)$ is of the form $\ell^k \Phi_n(y)$ with y a cycle, so $\#T = \#S = \#f(S) \leq \ell^{\nu r_2(L)} t_1 t_2$. As this holds for every finite $T \subseteq \mathrm{Ku}_{s,q}(F, \gamma, a)$, the set is finite and $\#\mathrm{Ku}_{\ell^{b_e}, \ell^\nu}(F, \gamma, (1 + \ell^c)^{e+1}) \leq \ell^{\nu r_2(L)} t_1 t_2$. □

Definition 40.8. Let $L \subseteq F$ be number fields with F/L Galois, and J a finite set; F^J is the F-vector space of maps $J \to F$, regarded also as an L-vector space.

(Lt1) For a family $\mathfrak{b} = (\mathfrak{b}_j)_{j\in J}$ of fractional ideals of F, $M(\mathfrak{b}) = \{x \in F^J : x_j \in \mathfrak{b}_j \text{ for all } j\}$; so $M((\mathcal{O}_F)_j) = \mathcal{O}_F^J$. For $b \in \mathcal{O}_L \smallsetminus \{0\}$, $M_b = \mathcal{O}_F[1/b]^J$, and $\mathrm{Aut}_b(F^J)$ is the group of L-linear automorphisms g of F^J with $g(M_b) = M_b$. Similarly, $\mathrm{Aut}_{\mathcal{O}_L}(\mathcal{O}_F^J)$ is the group of L-linear automorphisms g of F^J with $g(\mathcal{O}_F^J) = \mathcal{O}_F^J$; it is contained in every $\mathrm{Aut}_b(F^J)$, as $M_b = \mathcal{O}_L[1/b]\cdot\mathcal{O}_F^J$.

(Lt2) Let $q \geq 1$ and $\zeta \in \mathcal{O}_F^\times$ of order q. For $\sigma \in \mathrm{Gal}(F/L)$, $\kappa_\zeta(\sigma) \in (\mathbb{Z}/q\mathbb{Z})^\times$ is defined by $\sigma(\zeta) = \zeta^{\kappa_\zeta(\sigma)}$. For a map $h\colon J \to \mathbb{Z}/q\mathbb{Z}$, the **diagonal action** $\mathrm{D}_h\colon C_q \to \mathrm{Aut}_{\mathcal{O}_L}(\mathcal{O}_F^J)$ is $(\mathrm{D}_h(\tau^c)x)_j = \zeta^{c\,h(j)}x_j$.

(Lt3) Let $h\colon J \to \mathbb{Z}/q\mathbb{Z}$, $\mathfrak{b}$ a family of fractional ideals indexed by J and $\delta\colon \mathbb{Z}/q\mathbb{Z} \to F^\times$. An L-linear automorphism g of F^J is **adapted to** $(h, \mathfrak{b}, \delta)$ if $g(\mathcal{O}_F^J) = M(\mathfrak{b})$ and there are a map $s\colon J \to \mathrm{Gal}(F/L)$ and, writing $J_x = \{j \in J : \kappa_\zeta(s_j)h(j) = x\}$ for $x \in \mathbb{Z}/q\mathbb{Z}$ and $w^s = (s_i(w_i))_{i\in J}$ for $w \in F^J$, F-linear automorphisms g'_x of F^{J_x} with $\det g'_x = \delta(x)$ (the determinant of the automorphism of the zero space being 1), such that

$$s_j((gw)_j) = \big(g'_x(w^s|_{J_x})\big)_j \qquad \text{for all } w \in F^J,\ x \in \mathbb{Z}/q\mathbb{Z},\ j \in J_x.$$

(Lt4) Let k be a finite field and $r\colon \mathcal{O}_F \to k$ a ring homomorphism whose restriction r_L to $\mathcal{O}_L$ is surjective. The homomorphism r is **split in** F/L if the map $\mathcal{O}_F \to k^{\mathrm{Gal}(F/L)}$, $a \mapsto (r(\sigma(a)))_\sigma$, is surjective with kernel $\mathfrak{p}\mathcal{O}_F$, where $\mathfrak{p} = \ker r_L$. If $r(b) \neq 0$, r extends uniquely to $\mathcal{O}_F[1/b]$, and the **reduction** $\mathrm{rd}_r\colon M_b \to k^{\mathrm{Gal}(F/L)\times J}$ is $\mathrm{rd}_r(x) = \big(r(\sigma(x_j))\big)_{(\sigma,j)}$.

Proposition 40.9. *Let L be a number field, ℓ a prime, (F, γ, ζ) a cyclotomic Kummer layer of level (c, ν) over L, and $e \geq 0$ an integer with $2e + 3 < \ell^c$. Put $q = \ell^\nu$, $\kappa = 1 + \ell^c$, $a = \kappa^{e+1}$ and*

$$p_e = \prod_{0\leq j<2e+3,\ j\neq e+1} (\kappa^{e+1} - \kappa^j) \in \mathbb{Z}.$$

For every (γ, a)-Kummer element $\alpha \in F^\times$ there are an integer $N \geq 1$, a cycle

$$y \in \mathrm{Bar}_{2e+3}(\mathrm{GL}_N(\mathcal{O}_L); \mathbb{Z}/q\mathbb{Z})$$

and $b \in \mathcal{O}_F \smallsetminus \{0\}$ with the following property. Let k be a finite field and $r\colon \mathcal{O}_F \to k$ a ring homomorphism whose restriction r_L to $\mathcal{O}_L$ is surjective and with $r(b) \neq 0$; let $v\colon k^\times \to \mathbb{Z}/q\mathbb{Z}$ be a homomorphism and θ an additive family of degree $2e + 3$ over k with values in $\mathbb{Z}/q\mathbb{Z}$ having the value property for $(q, e, r(\zeta), v)$. Then $\mathrm{N}_{\gamma,a}(\alpha)$ is a unit at the prime $\ker r$, and its residue $u \in k^\times$ satisfies

$$\theta_{k^N}\big((\mathrm{red}_{r_L,N})_*\, y\big) = p_e \cdot v(u).$$

Proof. Put $q = \ell^\nu$, $\kappa = 1 + \ell^c$, $a = \kappa^{e+1}$, $M = 2e + 3$, $\Gamma = \mathrm{Gal}(F/L) = \{\gamma^j : 0 \leq j < \ell^{\nu-c}\}$, $J = (\mathbb{Z}/q\mathbb{Z})^2$, $T = \{0, \dots, M\}$ and $\chi_t(j) = j_1 + tj_2$ for $t \in T$. By (40.3), $\kappa_\zeta(\gamma^j) = \kappa^j$. By the lifting-the-exponent rule $\kappa^{\ell^{\nu-c}} \equiv 1 \pmod q$. Let α be a (γ, a)-Kummer element, $(\alpha) = \gamma^{-1}(\mathfrak{d})\,\mathfrak{d}^{-a}$.

Step 1: the combinatorial data. By Lemma 40.10 ($q = \ell^\nu$, $2e + 3 = M$) there are $n_t \in \mathbb{Z}/q\mathbb{Z}$ ($t \in T$) and $t_0 \in T$ with n_{t_0} a unit such that $\sum_t n_t(d_t)_*\mathbf{z}_{q,e}$ is a boundary, $d_t(c) = (c, c^t)$; and $\mathbf{z}_{q,e}$ is a cycle. Write $P(Y) = \prod_{j<M}(Y - \kappa^j) = \sum_{i=0}^M p_i Y^i$. As $M < \ell^c$ and $1 \leq c \leq \nu$, Lemma 40.11 gives $\eta\colon J \to \mathbb{Z}$ with $\sum_{\chi_t(j)=y}\eta(j) = 0$ for $t \in T \smallsetminus \{t_0\}$ and $\sum_{\chi_{t_0}(j)=y}\eta(j) = \sum_{i\leq M,\,\kappa^i\equiv y} p_i$, for all y.

Step 2: the element β. Let $P_e(Y) = \prod_{j<M,\, j\neq e+1}(Y-\kappa^j) = \sum_i \pi_i Y^i$; then $(Y-a)P_e(Y) = P(Y)$ and $P_e(a) = p_e$. Put $\beta = \prod_i \gamma^{-i}(\alpha)^{\pi_i}$. Let $\mathbb{Z}[Y]$ act on the group of fractional ideals by $f\cdot\mathfrak{a} = \prod_i \gamma^{-i}(\mathfrak{a})^{f_i}$; this is a module structure, as γ^{-1} is an automorphism. Then $(\alpha) = (Y-a)\cdot\mathfrak{d}$ and

$$(\beta) = P_e\cdot(\alpha) = P\cdot\mathfrak{d} = \prod_{i=0}^{M}\gamma^{-i}(\mathfrak{d})^{p_i}.$$

Step 3: adapted automorphisms. Let $\mathfrak{b} = (\mathfrak{d}^{\eta(j)})_{j\in J}$. For $t\neq t_0$, (Ca1) of Proposition 40.12 with $h=\chi_t$ gives g_t adapted to $(\chi_t,\mathfrak{b},1)$; put $\delta_t = 1$. For t_0: $\chi_{t_0}(1,0)=1$, the line sums are as required by Step 1 and (β) by Step 2, so (Ca2) gives g_{t_0} adapted to $(\chi_{t_0},\mathfrak{b},\delta_{t_0})$ with $\delta_{t_0}(1)=\beta$ and $\delta_{t_0}(x)=1$ for $x\neq 1$.

Step 4: the element b. Let b_0 be as in (Re1) of Proposition 40.14, and $b = b_0\prod_{\mathfrak{P}} p_{\mathfrak{P}} \in \mathcal{O}_L\smallsetminus\{0\}$, the product over the primes $\mathfrak{P}$ in the support of $\mathfrak{d}$, $p_{\mathfrak{P}}$ the rational prime below $\mathfrak{P}$. Every prime in the support of $\mathfrak{d}$, hence (as $b\in L$) of every $\sigma(\mathfrak{d})$, contains b. Consequently $\mathfrak{d}^{\eta(j)}\mathcal{O}_F[1/b] = \mathcal{O}_F[1/b]$, so $\mathcal{O}_L[1/b]M(\mathfrak{b}) = M_b$ and $g_t(M_b) = \mathcal{O}_L[1/b]\,g_t(\mathcal{O}_F^J) = M_b$: each $g_t\in \mathrm{Aut}_b(F^J)$. Also α, β and their conjugates lie in $U = \mathcal{O}_F[1/b]^\times$, a Γ-stable group, by Step 2.

Step 5: the cycle. Let $G = \mathrm{Aut}_b(F^J)$, $H = \mathrm{Aut}_{\mathcal{O}_L}(\mathcal{O}_F^J)\leq G$ and $\rho\colon C_q\times C_q\to H$, $(\rho(\tau^{c_1},\tau^{c_2})x)_j = \zeta^{c_1 j_1 + c_2 j_2}x_j$; then $\rho\circ d_t = \mathrm{D}_{\chi_t}$. The hypotheses of Lemma 40.13 hold: g_t commutes with $\mathrm{D}_{\chi_t}(C_q)$ by Proposition 40.15; ρ takes values in H; $\rho(x)$ is diagonal with unit entries, so it preserves $M(\mathfrak{b})$ and $g_{t_0}^{-1}\rho(x)g_{t_0}$ preserves $g_{t_0}^{-1}M(\mathfrak{b}) = \mathcal{O}_F^J$; and $g_t^{-1}g_{t_0}$ maps $\mathcal{O}_F^J$ onto $g_t^{-1}M(\mathfrak{b}) = \mathcal{O}_F^J$. With $A=\mathbb{Z}/q\mathbb{Z}$, $m = 2e{+}1$ and $z = \mathbf{z}_{q,e}$, it gives a cycle $y_0\in \mathrm{Bar}_{2e+3}(H;\mathbb{Z}/q\mathbb{Z})$ with

$$\iota_* y_0 - \sum_{t\in T} n_t\,\mathrm{Pr}_{g_t}\big((\mathrm{D}_{\chi_t})_*\mathbf{z}_{q,e}\big) \text{ a boundary in } G. \tag{40.7}$$

Let N and λ be as in (Re3) and put $y = n_{t_0}^{-1}\lambda_* y_0 \in \mathrm{Bar}_{2e+3}(\mathrm{GL}_N(\mathcal{O}_L);\mathbb{Z}/q\mathbb{Z})$, a cycle. These are the N, y and b of the statement.

Step 6: the value. Let k, r, v, θ be as in the statement. As $b_0\mid b$, $r(b_0)\neq 0$, so r is split by (Re1), and Rd_r exists by (Re2). As r extends to $\mathcal{O}_F[1/b]$, $f = v\circ r$ is a homomorphism $U\to\mathbb{Z}/q\mathbb{Z}$; $\mathrm{N}_{\gamma,a}(\alpha)\in U$ is a unit at $\ker r$, with residue $u = r(\mathrm{N}_{\gamma,a}(\alpha))$. By (Re3), then (40.7) with (Fa1), push-forward along Rd_r and Proposition 40.15,

$$\theta_{k^N}\big((\mathrm{red}_{r_L,N}\circ\lambda)_* y_0\big) = \theta_{k^{\Gamma\times J}}\big((\mathrm{Rd}_r)_*\iota_* y_0\big) = \sum_{t\in T} n_t \sum_{\sigma\in\Gamma}\sum_x (\kappa_\zeta(\sigma)x)^{e+1} f(\sigma(\delta_t(x))).$$

Since $f(1)=0$, only $t=t_0$ and $x=1$ contribute, and with $\sigma=\gamma^j$, using $\kappa^{j(e+1)} = a^j$ and $a^{\ell^{\nu-c}}\equiv 1 \pmod q$ to substitute $j'\equiv j-i \pmod{\ell^{\nu-c}}$,

$$\begin{aligned}\sum_j a^j f(\gamma^j\beta) &= \sum_i \pi_i \sum_j a^j f(\gamma^{j-i}\alpha) = \sum_i \pi_i a^i \sum_{j'=0}^{\ell^{\nu-c}-1} a^{j'} f(\gamma^{j'}\alpha)\\ &= P_e(a)\, f\Big(\prod_{j'}\gamma^{j'}(\alpha)^{a^{j'}}\Big) = p_e\, v(u).\end{aligned}$$

Hence $\theta_{k^N}((\mathrm{red}_{r_L,N})_* y) = n_{t_0}^{-1}\, n_{t_0}\, p_e\, v(u) = p_e\cdot v(u)$. □

Lemma 40.10. *Let $q\geq 1$ and $e\geq 0$ be integers. Then $\mathbf{z}_{q,e}$ is a cycle. If moreover q is a power of a prime ℓ, and $d_t\colon C_q\to C_q\times C_q$ is $c\mapsto(c,c^t)$ for $0\leq t\leq 2e+3$, then there are $n_0,\dots,n_{2e+3}\in\mathbb{Z}/q\mathbb{Z}$ and $t_0\in\{0,\dots,2e+3\}$ with n_{t_0} a unit such that $\sum_{t=0}^{2e+3} n_t\,(d_t)_*\mathbf{z}_{q,e}$ is a boundary in $\mathrm{Bar}_{2e+2}(C_q\times C_q;\mathbb{Z}/q\mathbb{Z})$.*

Proof. Step 1: $\mathbf{z}_{q,e}$ is a cycle. Write $\mathbf{z}_{q,e} = \sum_{a \in C_q^{e+1}} \sigma_a$ with $\sigma_a = [a_0|\tau|a_1|\tau|\dots|a_e|\tau]$, of length $n = 2e+2$, and apply ∂ face by face, summing over a. The first face $[\tau|a_1|\tau|\dots|a_e|\tau]$ does not depend on a_0, so it occurs q times and vanishes in $\mathbb{Z}/q\mathbb{Z}$. For $0 \le i < e$, the face merging a_i with the following τ (sign $(-1)^{2i+1}$) and the face merging that τ with a_{i+1} (sign $(-1)^{2i+2}$) give, after summing over a_i and a_{i+1} and substituting $a_i\tau \mapsto a_i$, respectively $\tau a_{i+1} \mapsto a_{i+1}$, the same sum over all $(a_0,\dots,a_e)$ of $[a_0|\tau|\dots|a_i|a_{i+1}|\tau|\dots|a_e|\tau]$; they cancel. The face merging a_e with the last τ (sign $(-1)^{2e+1}$) gives, after $a_e\tau \mapsto a_e$, the sum of $[a_0|\tau|\dots|a_{e-1}|\tau|a_e]$, and the last face, which drops the final τ (sign $(-1)^{2e+2}$), gives the same sum; they cancel. Hence $\partial \mathbf{z}_{q,e} = 0$.

Step 2: $H_{2e+2}(C_q \times C_q; \mathbb{Z}/q\mathbb{Z}) \cong (\mathbb{Z}/q\mathbb{Z})^{2e+3}$. The complex $W \otimes_{\mathbb{Z}} W$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[C_q \times C_q]$ with $(W \otimes W)_n$ free of rank $n+1$. After $\mathbb{Z}/q\mathbb{Z} \otimes_{\mathbb{Z}[C_q \times C_q]} -$, each differential becomes a sum of tensor products of the induced differentials of W, which are multiplication by 0 (from $\tau - 1$) and by $q = 0$ (from the norm element); so all differentials vanish and $H_n(C_q \times C_q; \mathbb{Z}/q\mathbb{Z}) \cong (\mathbb{Z}/q\mathbb{Z})^{n+1}$.

Step 3: the relation. Let $q = \ell^\nu$ and $r = 2e+3$. The classes $[(d_t)_*\mathbf{z}_{q,e}]$, $0 \le t \le r$, are $r+1$ elements of $H_{2e+2}(C_q \times C_q; \mathbb{Z}/q\mathbb{Z}) \cong (\mathbb{Z}/q\mathbb{Z})^r$ (Steps 1, 2). Lift their coordinates to an integer $r \times (r+1)$ matrix Φ. Over $\mathbb{Q}$, Φ has a nonzero kernel, so there is $x \in \mathbb{Z}^{r+1}$ with $\Phi x = 0$ and coprime coordinates; some x_{t_0} is prime to ℓ. Put $n_t = x_t \bmod q$. Then n_{t_0} is a unit and $\sum_t n_t[(d_t)_*\mathbf{z}_{q,e}] = 0$, i.e. the cycle $\sum_t n_t (d_t)_*\mathbf{z}_{q,e}$ is a boundary. □

Lemma 40.11. *Let ℓ be a prime, $c \ge 1$ and $\nu \ge 1$ integers, $q = \ell^\nu$, $\kappa = 1 + \ell^c$, $M \ge 0$ an integer with $M < \ell^c$, and $t_0 \in \{0,\dots,M\}$. Write $\prod_{j=0}^{M-1}(Y - \kappa^j) = \sum_{i=0}^{M} p_i Y^i \in \mathbb{Z}[Y]$. There is a map $\eta\colon (\mathbb{Z}/q\mathbb{Z})^2 \to \mathbb{Z}$ such that, for every $y \in \mathbb{Z}/q\mathbb{Z}$,*

$$\sum_{x_1 + t x_2 = y} \eta(x_1,x_2) = 0 \quad (t \in \{0,\dots,M\},\ t \ne t_0), \qquad \sum_{x_1 + t_0 x_2 = y} \eta(x_1,x_2) = \sum_{0 \le i \le M,\ \kappa^i \equiv y \pmod q} p_i.$$

Proof. Identify a map $\phi\colon B \to \mathbb{Z}$ on a finite abelian group B with the element $\sum_b \phi(b)[b]$ of the group ring $\mathbb{Z}[B]$. For a homomorphism $\psi\colon B \to B'$, the map $\phi \mapsto (y \mapsto \sum_{\psi(b)=y} \phi(b))$ is then the ring homomorphism $\psi_*\colon \mathbb{Z}[B] \to \mathbb{Z}[B']$ induced by ψ. Let $R = \mathbb{Z}[\mathbb{Z}/q\mathbb{Z}]$ and $T = [1] \in R$, so $[y] = T^y$; and for $t \in \mathbb{Z}$ let $\chi_t\colon (\mathbb{Z}/q\mathbb{Z})^2 \to \mathbb{Z}/q\mathbb{Z}$, $(x_1,x_2) \mapsto x_1 + t x_2$. We must find $\eta \in \mathbb{Z}[(\mathbb{Z}/q\mathbb{Z})^2]$ with $(\chi_t)_*\eta = 0$ for $t \in \{0,\dots,M\} \smallsetminus \{t_0\}$ and $(\chi_{t_0})_*\eta = \Theta$, where $\Theta = \sum_{i=0}^{M} p_i T^{\kappa^i}$.

Step 1: $(X^w - 1)^M$ divides $P = \sum_i p_i X^{(1+w)^i}$ in $\mathbb{Z}[X]$, where $w = \ell^c$. Let $\mathcal{E}$ be the ring endomorphism $X \mapsto X^{1+w}$ of $\mathbb{Z}[X]$; then $\mathcal{E}^i(X) = X^{(1+w)^i}$, so $P = \prod_{j<M}(\mathcal{E} - (1+w)^j)(X)$, the factors being commuting $\mathbb{Z}$-linear operators. We show: if $(X^w-1)^j \mid f$, then $(X^w-1)^{j+1} \mid \mathcal{E}(f) - (1+w)^j f$; applying this to $f = X$ and the factors $j = 0, 1, \dots, M-1$ in turn gives the claim. Write $f = (X^w-1)^j g$ and $S = \sum_{i=0}^{w} X^{wi}$, so that $\mathcal{E}(X^w - 1) = X^{w(1+w)} - 1 = (X^w - 1)S$. Then

$$\mathcal{E}(f) - (1+w)^j f = (X^w-1)^j \Big((S^j - (1+w)^j)\mathcal{E}(g) + (1+w)^j (\mathcal{E}(g) - g) \Big).$$

The polynomial $X^w - 1$ divides $S - (1+w) = \sum_i (X^{wi} - 1)$, hence $S^j - (1+w)^j$; and it divides $\mathcal{E}(g) - g$, as $X^{n(1+w)} - X^n = X^n(X^{nw} - 1)$ for each monomial.

Step 2: $T^s - 1$ divides $T^w - 1$ in R for every integer s with $1 \le |s| < w$. Write $s = \ell^a u$ with $\ell \nmid u$; as $|s| < \ell^c$, $a < c$. Choose an integer $u' \ge 1$ with $uu' \equiv 1 \pmod q$ and put $m = \ell^{c-a} u' \ge 1$. Then $sm \equiv \ell^c \pmod q$, so $T^w - 1 = (T^s)^m - 1$, which is divisible by $T^s - 1$.

Step 3: construction. The ring homomorphism $\mathbb{Z}[X] \to R$, $X \mapsto T$, maps P to Θ, since $(1+w)^i = \kappa^i$. By Step 1, $\Theta = (T^w - 1)^M u_1$ for some $u_1 \in R$. The set $\{0,\dots,M\} \smallsetminus \{t_0\}$ has M elements, and for each such t, $1 \le |t_0 - t| \le M < w$; by Step 2, $T^w - 1 = (T^{t_0 - t} - 1)v_t$ with $v_t \in R$. Hence $\Theta = u_2 \prod_{t \ne t_0}(T^{t_0 - t} - 1)$ with $u_2 = u_1 \prod_{t \ne t_0} v_t$. Let $\iota_1\colon \mathbb{Z}/q\mathbb{Z} \to (\mathbb{Z}/q\mathbb{Z})^2$,

$y \mapsto (y,0)$, so $\chi_t \circ \iota_1 = \mathrm{id}$ for all t, and for $t \neq t_0$ let $\epsilon_t = [(-t,1)] - [(0,0)]$, so $(\chi_{t'})_* \epsilon_t = T^{t'-t} - 1$. Put

$$\eta = (\iota_1)_*(u_2) \cdot \prod_{t \in \{0,\dots,M\},\, t \neq t_0} \epsilon_t.$$

For $t' \neq t_0$ in $\{0,\dots,M\}$, $(\chi_{t'})_*\eta$ contains the factor $(\chi_{t'})_*\epsilon_{t'} = T^0 - 1 = 0$; and $(\chi_{t_0})_*\eta = u_2 \prod_{t \neq t_0}(T^{t_0-t} - 1) = \Theta$. □

Proposition 40.12. *Let $L \subseteq F$ be number fields with F/L Galois, $q \geq 1$, $\zeta \in \mathcal{O}_F^\times$ of order q, J a finite set, $h\colon J \to \mathbb{Z}/q\mathbb{Z}$, $\eta\colon J \to \mathbb{Z}$, $\mathfrak{d}$ a fractional ideal of F, and $\mathfrak{b} = (\mathfrak{d}^{\eta(j)})_{j \in J}$.*

(Ca1) *If $\sum_{j:h(j)=y} \eta(j) = 0$ for every $y \in \mathbb{Z}/q\mathbb{Z}$, there is an L-linear automorphism of F^J adapted to $(h, \mathfrak{b}, 1)$, where 1 is the constant map.*

(Ca2) *Let (F, γ, ζ) be a cyclotomic Kummer layer of level (c,ν) over L for a prime ℓ (so $q = \ell^\nu$), $\kappa = 1 + \ell^c$, $M \geq 0$, $p_0, \dots, p_M \in \mathbb{Z}$ and $\beta \in F^\times$ with $(\beta) = \prod_{i=0}^M \gamma^{-i}(\mathfrak{d})^{p_i}$. If $h(j_0) = 1$ for some $j_0 \in J$ and $\sum_{j:h(j)=y} \eta(j) = \sum_{0 \leq i \leq M,\ \kappa^i \equiv y} p_i$ for every $y \in \mathbb{Z}/q\mathbb{Z}$, there is an L-linear automorphism of F^J adapted to $(h, \mathfrak{b}, \delta_\beta)$, where $\delta_\beta(1) = \beta$ and $\delta_\beta(x) = 1$ for $x \neq 1$.*

Proof. Step 0: Steinitz with a prescribed determinant. Let I be a finite set, $\mathfrak{a} = (\mathfrak{a}_i)_{i \in I}$, $\mathfrak{c} = (\mathfrak{c}_i)_{i \in I}$ families of nonzero fractional ideals and $\beta' \in F^\times$ with $(\beta') = \prod_i \mathfrak{c}_i \prod_i \mathfrak{a}_i^{-1}$; if $I = \varnothing$ assume $\beta' = 1$. Then there is an F-linear automorphism g' of F^I with $g'(M(\mathfrak{a})) = M(\mathfrak{c})$ and $\det g' = \beta'$. For $I = \varnothing$ take the identity. Otherwise $\prod \mathfrak{a}_i$ and $\prod \mathfrak{c}_i$ have the same class, so by Theorem B.24 there is an $\mathcal{O}_F$-isomorphism $M(\mathfrak{a}) \to M(\mathfrak{c})$; it extends to an F-linear automorphism g_1 of $F^I = F \cdot M(\mathfrak{a})$. Expanding determinants, the $\mathcal{O}_F$-submodule of $\Lambda_F^n F^I = F e_I$ spanned by the wedge products of n elements of $M(\mathfrak{a})$ is $(\prod_i \mathfrak{a}_i)e_I$, and likewise for $\mathfrak{c}$; g_1 maps the one onto the other and acts by $\det g_1$, so $(\det g_1)\prod \mathfrak{a}_i = \prod \mathfrak{c}_i$ and $\det g_1 = \beta' u$ with $u \in \mathcal{O}_F^\times$. Fix $i_1 \in I$, let d_u multiply the coordinate i_1 by u^{-1} (so $d_u(M(\mathfrak{a})) = M(\mathfrak{a})$), and put $g' = g_1 d_u$.

Step 1: gluing blocks. Let $s\colon J \to \mathrm{Gal}(F/L)$, $J_x = \{j : \kappa_\zeta(s_j) h(j) = x\}$, and for each x let g'_x be an F-linear automorphism of F^{J_x} with $g'_x(\mathcal{O}_F^{J_x}) = M((s_j(\mathfrak{b}_j))_{j \in J_x})$ and $\det g'_x = \delta(x)$. Let $S(w) = w^s = (s_j(w_j))_j$, an L-linear bijection of F^J with $S(\mathcal{O}_F^J) = \mathcal{O}_F^J$ and $S^{-1}(M((s_j(\mathfrak{b}_j))_j)) = M(\mathfrak{b})$, and let $g = S^{-1} \circ (\bigoplus_x g'_x) \circ S$, the middle map acting blockwise on $F^J = \bigoplus_x F^{J_x}$. Then g is an L-linear automorphism, $s_j((gw)_j) = (g'_x(w^s|_{J_x}))_j$ for $j \in J_x$, and $g(\mathcal{O}_F^J) = S^{-1}(\bigoplus_x g'_x(\mathcal{O}_F^{J_x})) = M(\mathfrak{b})$. So g is adapted to $(h, \mathfrak{b}, \delta)$.

(Ca1). Take $s_j = 1$, so $J_x = h^{-1}(x)$. For each x, $\prod_{j \in J_x} \mathfrak{d}^{\eta(j)} = \mathfrak{d}^0 = (1)$, and Step 0 with $\mathfrak{a} = (\mathcal{O}_F)$, $\mathfrak{c} = (\mathfrak{d}^{\eta(j)})_{j \in J_x}$, $\beta' = 1$ gives g'_x; Step 1 gives g.

(Ca2). Let $K = \{\kappa^i \bmod q : 0 \leq i \leq M\}$ and for $y \in K$ fix $i(y) \in \{0,\dots,M\}$ with $\kappa^{i(y)} \equiv y$. Put $s_j = \gamma^{-i(h(j))}$ if $h(j) \in K$ and $s_j = 1$ otherwise. By (40.3), $\kappa_\zeta(\gamma) = \kappa$, so $\kappa_\zeta(s_j)h(j) = 1$ if $h(j) \in K$; hence $J_1 = h^{-1}(K) \ni j_0$, $J_x = h^{-1}(x)$ for $x \notin K$, and $J_x = \varnothing$ for $x \in K \smallsetminus \{1\}$. For $x \notin K$ the hypothesis gives $\sum_{j \in J_x} \eta(j) = 0$, and Step 0 gives g'_x with determinant 1 as in (Ca1). For $x = 1$:

$$\prod_{j \in J_1} s_j(\mathfrak{d})^{\eta(j)} = \prod_{y \in K} (\gamma^{-i(y)}\mathfrak{d})^{\sum_{h(j)=y} \eta(j)} = \prod_{y \in K} \prod_{i:\kappa^i \equiv y} (\gamma^{-i(y)}\mathfrak{d})^{p_i} = \prod_{i=0}^{M} (\gamma^{-i}\mathfrak{d})^{p_i} = (\beta),$$

where the third equality holds because $\kappa^i \equiv \kappa^{i(y)} \pmod{\ell^\nu}$ forces $i \equiv i(y) \pmod{\ell^{\nu-c}}$ by the lifting-the-exponent rule, and γ has order $\ell^{\nu-c}$. Step 0 with $I = J_1 \neq \varnothing$, $\mathfrak{a} = (\mathcal{O}_F)$, $\mathfrak{c} = (s_j(\mathfrak{d})^{\eta(j)})_{j \in J_1}$ and $\beta' = \beta$ gives g'_1; for $x \in K \smallsetminus \{1\}$ take the identity of the zero space. Step 1 gives g, adapted to $(h, \mathfrak{b}, \delta_\beta)$. □

Lemma 40.13. *Let A be a commutative ring, $m \geq 0$, C, D, G groups, $H \leq G$ a subgroup with inclusion ι, T a finite set with an element t_0, $\rho\colon D \to G$ and $d_t\colon C \to D$ ($t \in T$) homomorphisms, $n_t \in A$ ($t \in T$), and $z \in \mathrm{Bar}_{m+1}(C;A)$ a cycle such that $\sum_t n_t\,(d_t)_* z$ is a boundary. Let $g_t \in G$ ($t \in T$) be such that g_t commutes with $\rho(d_t(c))$ for all $c \in C$, $\rho(D) \subseteq H$, $g_{t_0}^{-1}\rho(D)g_{t_0} \subseteq H$, and $g_t^{-1}g_{t_0} \in H$ for all t. Then there is a cycle $y \in \mathrm{Bar}_{m+2}(H;A)$ such that*

$$\iota_* y - \sum_{t\in T} n_t \,\mathrm{Pr}_{g_t}\big((\rho\circ d_t)_* z\big)$$

is a boundary in $\mathrm{Bar}_{m+2}(G;A)$.

Proof. For $k \in \Gamma$ let $\kappa_k(\gamma) = k^{-1}\gamma k$.

Step 1: the prism formula and multiplicativity of prisms. On $P_\bullet$ let $R_k(\gamma_0,\dots,\gamma_n) = (\gamma_0 k,\dots,\gamma_n k)$ and $h_k(\gamma_0,\dots,\gamma_n) = \sum_{i=0}^{n}(-1)^i(\gamma_0,\dots,\gamma_i,\gamma_i k,\dots,\gamma_n k)$. Both are Γ-equivariant, R_k is a chain map, and $\partial h_k + h_k\partial = R_k - \mathrm{id}$ (the prism formula for the simplicial homotopy from id to R_k). Under the identification above, R_k induces $(\kappa_k)_*$, since $(k, g_1 k,\dots) = k\cdot(1, k^{-1}g_1 k, k^{-1}g_1 g_2 k,\dots)$, and h_k induces Pr_k. Hence, for $w \in \mathrm{Bar}_n(\Gamma;A)$, $n \geq 1$,

$$\partial\,\mathrm{Pr}_k(w) + \mathrm{Pr}_k(\partial w) = (\kappa_k)_* w - w. \tag{40.8}$$

Now let $k, k' \in \Gamma$ and $E = h_{kk'} - h_k - h_{k'}R_k$. It is equivariant of degree 1, and $\partial E + E\partial = (R_{kk'} - 1) - (R_k - 1) - (R_{k'} - 1)R_k = 0$ because $R_{k'}R_k = R_{kk'}$. We construct equivariant $F_n\colon P_n \to P_{n+2}$ with $E_n = \partial F_n - F_{n-1}\partial$ ($F_{-1} = 0$). For $n = 0$: $\partial E_0 = 0$, so E_0 takes values in the cycles of P_1, which are boundaries, and P_0 is free, so $E_0 = \partial F_0$ for some equivariant F_0. If $F_0,\dots,F_{n-1}$ are constructed, then $\partial(E_n + F_{n-1}\partial) = -E_{n-1}\partial + (E_{n-1} + F_{n-2}\partial)\partial = 0$, so $E_n + F_{n-1}\partial$ takes values in the cycles of P_{n+1}, which are boundaries, and since P_n is free it equals ∂F_n for some equivariant F_n. Passing to $A \otimes_{A[\Gamma]} -$, E induces $\mathrm{Pr}_{kk'} - \mathrm{Pr}_k - \mathrm{Pr}_{k'}\circ(\kappa_k)_*$ and F a map $\bar F$ with $\mathrm{Pr}_{kk'} - \mathrm{Pr}_k - \mathrm{Pr}_{k'}(\kappa_k)_* = \partial\bar F - \bar F\partial$. Hence, for every cycle $x \in \mathrm{Bar}_n(\Gamma;A)$, $n \geq 1$,

$$\mathrm{Pr}_{kk'}(x) - \mathrm{Pr}_k(x) - \mathrm{Pr}_{k'}\big((\kappa_k)_* x\big) \text{ is a boundary.} \tag{40.9}$$

Step 2: the cycle. For $t \in T$ put $x_t = (\rho\circ d_t)_* z \in \mathrm{Bar}_{m+1}(G;A)$, a cycle all of whose entries lie in $\rho(D)$, and $h_t = g_t^{-1}g_{t_0} \in H$, so $g_t = g_{t_0}h_t^{-1}$. Each $\mathrm{Pr}_{g_t}(x_t)$ is a cycle: by (40.8), $\partial\,\mathrm{Pr}_{g_t}(x_t) = (\kappa_{g_t})_* x_t - x_t = 0$, as g_t commutes with the entries of x_t. By (40.9) with $k = g_{t_0}$, $k' = h_t^{-1}$, modulo boundaries of G,

$$\mathrm{Pr}_{g_t}(x_t) \equiv \mathrm{Pr}_{g_{t_0}}(x_t) + \mathrm{Pr}_{h_t^{-1}}\big((\kappa_{g_{t_0}})_* x_t\big).$$

Choose $w \in \mathrm{Bar}_{m+2}(D;A)$ with $\partial w = \sum_t n_t (d_t)_* z$; then $\sum_t n_t x_t = \partial\rho_* w$, and by (40.8)

$$\mathrm{Pr}_{g_{t_0}}(\partial\rho_* w) = (\kappa_{g_{t_0}})_*\rho_* w - \rho_* w - \partial\,\mathrm{Pr}_{g_{t_0}}(\rho_* w).$$

Therefore $\sum_t n_t \mathrm{Pr}_{g_t}(x_t) \equiv y'$ modulo boundaries, where

$$y' = (\kappa_{g_{t_0}})_*\rho_* w - \rho_* w + \sum_t n_t\,\mathrm{Pr}_{h_t^{-1}}\big((\kappa_{g_{t_0}})_* x_t\big).$$

Every entry of every basis element in y' lies in H: the entries of $\rho_* w$ lie in $\rho(D) \subseteq H$, those of $(\kappa_{g_{t_0}})_*\rho_* w$ and $(\kappa_{g_{t_0}})_* x_t$ in $g_{t_0}^{-1}\rho(D)g_{t_0} \subseteq H$, and the prism by $h_t^{-1} \in H$ only adds the entry h_t^{-1} and conjugates by h_t^{-1}. As ι_* is injective on chains, $y' = \iota_* y$ for a unique $y \in \mathrm{Bar}_{m+2}(H;A)$. Finally $\iota_*\partial y = \partial y'$ is the boundary of a cycle minus the boundary of a boundary, i.e. 0, so $\partial y = 0$. □

Proposition 40.14. *Let $L \subseteq F$ be number fields with F/L Galois, $\Gamma = \mathrm{Gal}(F/L)$, and J a finite set. For a finite field k and a ring homomorphism $r\colon \mathcal{O}_F \to k$ we write $r_L = r|_{\mathcal{O}_L}$.*

(Re1) *There is $b_0 \in \mathcal{O}_L \smallsetminus \{0\}$ such that every r with r_L surjective and $r(b_0) \neq 0$ is split in F/L.*

(Re2) *Let $b \in \mathcal{O}_L \smallsetminus \{0\}$ and let r be split in F/L, with r_L surjective and $r(b) \neq 0$. There is a unique homomorphism $\mathrm{Rd}_r\colon \mathrm{Aut}_b(F^J) \to \mathrm{GL}(k^{\Gamma\times J})$ with $\mathrm{Rd}_r(g)(\mathrm{rd}_r(x)) = \mathrm{rd}_r(gx)$ for all g and all $x \in M_b$.*

(Re3) *There are $N \geq 1$ and a homomorphism $\lambda\colon \mathrm{Aut}_{\mathcal{O}_L}(\mathcal{O}_F^J) \to \mathrm{GL}_N(\mathcal{O}_L)$ such that for all b and r as in (Re2), every commutative ring A, every $m \geq 1$, every additive family θ of degree m over k with values in A and every cycle $y \in \mathrm{Bar}_m(\mathrm{Aut}_{\mathcal{O}_L}(\mathcal{O}_F^J); A)$,*

$$\theta_{k^N}\big((\mathrm{red}_{r_L,N} \circ \lambda)_* y\big) = \theta_{k^{\Gamma\times J}}\big((\mathrm{Rd}_r|_{\mathrm{Aut}_{\mathcal{O}_L}(\mathcal{O}_F^J)})_* y\big).$$

Proof. (Re1). Let b_0 be a nonzero element of $\mathcal{O}_L$ in the relative discriminant of F/L. Let r be as in (Re1); as r_L is surjective, so is r, and $P = \ker r$ is a prime of $\mathcal{O}_F$ over $\mathfrak{p} = \ker r_L$ with $\mathcal{O}_L/\mathfrak{p} \to \mathcal{O}_F/P \cong k$ surjective, i.e. of residue degree 1. As $b_0 \notin \mathfrak{p}$, $\mathfrak{p}$ is unramified, so $e = f = 1$ and $\mathfrak{p}$ has $\#\Gamma$ primes above it, which are the $\sigma^{-1}(P) = \ker(r \circ \sigma)$, $\sigma \in \Gamma$, pairwise distinct. By the Chinese remainder theorem $\mathcal{O}_F \to \prod_\sigma \mathcal{O}_F/\sigma^{-1}(P) \cong k^\Gamma$, $a \mapsto (r(\sigma a))_\sigma$, is surjective with kernel $\prod_\sigma \sigma^{-1}(P) = \mathfrak{p}\mathcal{O}_F$.

(Re2). As r is split, rd_r is surjective; its kernel is $\mathfrak{p}M_b$: if $\mathrm{rd}_r(x) = 0$, write $x_j = a_j/b^n$ with $a_j \in \mathcal{O}_F$; then $r(\sigma a_j) = 0$ for all σ, so $a_j \in \mathfrak{p}\mathcal{O}_F$. For $a \in \mathcal{O}_L[1/b]$, $\mathrm{rd}_r(ax) = r(a)\,\mathrm{rd}_r(x)$, as $\sigma(a) = a$. An element $g \in \mathrm{Aut}_b(F^J)$ is L-linear with $g(M_b) = M_b$, so $g(\mathfrak{p}M_b) = \mathfrak{p}M_b$; hence $\mathrm{rd}_r(x) \mapsto \mathrm{rd}_r(gx)$ is a well-defined bijection of $k^{\Gamma\times J}$, k-linear because r_L is surjective. Uniqueness follows from the surjectivity of rd_r, and the homomorphism property from uniqueness.

(Re3). Let $Q = \mathcal{O}_F^J$, a finitely generated torsion-free, hence projective, $\mathcal{O}_L$-module; choose an $\mathcal{O}_L$-module P' and an isomorphism $\psi\colon Q \oplus P' \to \mathcal{O}_L^N$, and put $\lambda(g) = \psi \circ (g|_Q \oplus \mathrm{id}_{P'}) \circ \psi^{-1}$, a homomorphism to $\mathrm{GL}_N(\mathcal{O}_L)$. Let b, r be as in (Re2) and $\mathfrak{p} = \ker r_L$. As in (Re2), rd_r restricts to a surjection $Q \to k^{\Gamma\times J}$ with kernel $\mathfrak{p}Q$, and $\mathrm{Rd}_r(g)$ is the map induced by g on $Q/\mathfrak{p}Q$ for $g \in \mathrm{Aut}_{\mathcal{O}_L}(\mathcal{O}_F^J)$. Identifying $\mathcal{O}_L/\mathfrak{p} = k$ through r_L, ψ induces a k-isomorphism $\varphi\colon k^{\Gamma\times J} \oplus P'/\mathfrak{p}P' \to k^N$, and since $\mathrm{red}_{r_L,N}$ of a matrix is the map it induces on $\mathcal{O}_L^N/\mathfrak{p}\mathcal{O}_L^N$,

$$\mathrm{red}_{r_L,N}(\lambda(g)) = \varphi \circ \big(\mathrm{Rd}_r(g) \oplus \mathrm{id}\big) \circ \varphi^{-1} = c_\varphi\big(\mathrm{Rd}_r(g) \oplus \mathbf{1}(g)\big),$$

with $\mathbf{1}$ the trivial homomorphism to $\mathrm{GL}(P'/\mathfrak{p}P')$. For a cycle y, (Fa2) and (Fa3) give $\theta_{k^N}((\mathrm{red}_{r_L,N} \circ \lambda)_* y) = \theta_{k^{\Gamma\times J}}((\mathrm{Rd}_r)_* y) + \theta_{P'/\mathfrak{p}P'}(\mathbf{1}_* y)$. Finally $\mathbf{1}_* y$ is a boundary, so the last term vanishes by (Fa1): $\mathbf{1}$ factors through the trivial group E, and in $\mathrm{Bar}_\bullet(E; A)$, with $e_n = [1|\ldots|1]$, $\partial e_{n+1} = e_n$ for n odd and $\partial e_{n+1} = 0$ for $n \geq 2$ even; so every element of degree n odd is a boundary and the only cycle of even degree $n \geq 2$ is 0. □

Proposition 40.15. *Let $L \subseteq F$ be number fields with F/L Galois, $\Gamma = \mathrm{Gal}(F/L)$, $q \geq 1$, $\zeta \in \mathcal{O}_F^\times$ of order q, J a finite set and $b \in \mathcal{O}_L \smallsetminus \{0\}$. Let k be a finite field, $r\colon \mathcal{O}_F \to k$ split in F/L with r_L surjective and $r(b) \neq 0$, $e \geq 0$, $v\colon k^\times \to \mathbb{Z}/q\mathbb{Z}$ a homomorphism, and θ an additive family of degree $2e+3$ over k with values in $\mathbb{Z}/q\mathbb{Z}$ having the value property for $(q, e, r(\zeta), v)$. Let $h\colon J \to \mathbb{Z}/q\mathbb{Z}$, $\mathfrak{b}$ a family of fractional ideals indexed by J, $\delta\colon \mathbb{Z}/q\mathbb{Z} \to F^\times$, and $g \in \mathrm{Aut}_b(F^J)$ adapted to $(h, \mathfrak{b}, \delta)$. Then $\mathrm{D}_h(C_q) \subseteq \mathrm{Aut}_b(F^J)$, g commutes with $\mathrm{D}_h(C_q)$, every $\delta(x)$ is a unit at every prime of F not containing b, and*

$$\theta_{k^{\Gamma\times J}}\Big(\mathrm{Pr}_{\mathrm{Rd}_r(g)}\big((\mathrm{Rd}_r \circ \mathrm{D}_h)_* \mathbf{z}_{q,e}\big)\Big) = \sum_{\sigma\in\Gamma} \sum_{x\in\mathbb{Z}/q\mathbb{Z}} \big(\kappa_\zeta(\sigma)\, x\big)^{e+1}\, v\big(r(\sigma(\delta(x)))\big).$$

Proof. Let $s\colon J \to \Gamma$, J_x and g'_x be as in the definition of adapted automorphisms, and $w^s = (s_j(w_j))_j$. Write $\bar\zeta = r(\zeta)$ and, for an integer $y \geq 0$, $\chi_y\colon C_q \to k^\times$, $\tau^c \mapsto \bar\zeta^{yc}$.

Step 1: the first three assertions. $\mathrm{D}_h(\tau^c)$ multiplies coordinates by units, so it preserves M_b. For $w \in F^J$ and $j \in J_x$, using the F-linearity of g'_x and $s_i(\zeta^{ch(i)}) = \zeta^{c\kappa_\zeta(s_i)h(i)} = \zeta^{cx}$ for $i \in J_x$,

$$s_j\big((g\,\mathrm{D}_h(\tau^c)w)_j\big) = \big(g'_x((\zeta^{cx}s_i(w_i))_{i\in J_x})\big)_j = \zeta^{cx}s_j((gw)_j) = s_j(\zeta^{ch(j)}(gw)_j),$$

so $g\,\mathrm{D}_h(\tau^c) = \mathrm{D}_h(\tau^c)\,g$. Since $w \mapsto w^s$ preserves M_b ($b \in L$), so does $\bigoplus_x g'_x$, hence each g'_x and its inverse preserve $\mathcal{O}_F[1/b]^{J_x}$; their matrices have entries in $\mathcal{O}_F[1/b]$, and $\delta(x) = \det g'_x \in \mathcal{O}_F[1/b]^\times$.

Step 2: the block decomposition of the reduction. For $j \in J$, $\sigma \mapsto \sigma s_j$ is a bijection of Γ; so $\Gamma \times J$ is the disjoint union of the sets $B_{\sigma,x} = \{(\sigma s_j, j) : j \in J_x\}$, $\sigma \in \Gamma$, $x \in \mathbb{Z}/q\mathbb{Z}$, each identified with J_x by $(\sigma s_j, j) \mapsto j$; let $V_{\sigma,x} = k^{B_{\sigma,x}} \cong k^{J_x}$, so $k^{\Gamma\times J} = \bigoplus_{\sigma,x} V_{\sigma,x}$. For $w \in M_b$ and $j \in J_x$,

$$\mathrm{rd}_r(gw)_{(\sigma s_j,j)} = r\big(\sigma s_j((gw)_j)\big) = r\Big(\big(\sigma(g'_x)\,(\sigma s_i(w_i))_{i\in J_x}\big)_j\Big),$$

where $\sigma(g'_x)$ is the matrix of g'_x with σ applied to the entries (in $\mathcal{O}_F[1/b]$ by Step 1), and $r(\sigma s_i(w_i)) = \mathrm{rd}_r(w)_{(\sigma s_i,i)}$. Hence $\mathrm{Rd}_r(g)$ preserves each $V_{\sigma,x}$ and acts there by $\bar g_{\sigma,x} = r(\sigma(g'_x))$, with $\det \bar g_{\sigma,x} = r(\sigma(\delta(x)))$. Likewise $\mathrm{Rd}_r(\mathrm{D}_h(\tau^c))$ multiplies the coordinate $(\sigma s_j, j)$ by

$$r(\sigma s_j(\zeta^{ch(j)})) = \bar\zeta^{c\kappa_\zeta(\sigma)x},$$

so on $V_{\sigma,x}$ it is the scalar $\chi_{y_{\sigma,x}}(\tau^c)$, where $y_{\sigma,x} \geq 0$ represents $\kappa_\zeta(\sigma)x$.

Step 3: the value. Let $G' = \prod_{\sigma,x} \mathrm{GL}(V_{\sigma,x})$, $\iota\colon G' \to \mathrm{GL}(k^{\Gamma\times J})$ the block-diagonal embedding, $\phi\colon C_q \to G'$ with components $s_{V_{\sigma,x}} \circ \chi_{y_{\sigma,x}}$, and $\hat g = (\bar g_{\sigma,x}) \in G'$. By Step 2, $\mathrm{Rd}_r(g) = \iota(\hat g)$ and $\mathrm{Rd}_r \circ \mathrm{D}_h = \iota \circ \phi$, so the chain on the left of the assertion is $\iota_*\mathrm{Pr}_{\hat g}(\phi_*\mathbf{z}_{q,e})$. As $\phi(C_q)$ is central in G' and $\mathbf{z}_{q,e}$ is a cycle, $u = \mathrm{Pr}_{\hat g}(\phi_*\mathbf{z}_{q,e})$ is a cycle of G'. Up to the coordinate isomorphism $\bigoplus V_{\sigma,x} \cong k^{\Gamma\times J}$, ι is the direct sum of the projections $\pi_{\sigma,x}\colon G' \to \mathrm{GL}(V_{\sigma,x})$; by (Fa2) and (Fa3), applied block by block,

$$\theta_{k^{\Gamma\times J}}(\iota_*u) = \sum_{\sigma,x} \theta_{V_{\sigma,x}}\big((\pi_{\sigma,x})_*u\big) = \sum_{\sigma,x} \theta_{V_{\sigma,x}}\Big(\mathrm{Pr}_{\bar g_{\sigma,x}}\big((s_{V_{\sigma,x}} \circ \chi_{y_{\sigma,x}})_*\mathbf{z}_{q,e}\big)\Big).$$

By (Fa4) the (σ,x) term is $y_{\sigma,x}^{e+1}v(\det \bar g_{\sigma,x}) = (\kappa_\zeta(\sigma)x)^{e+1}v(r(\sigma(\delta(x))))$; for $J_x = \varnothing$ both are 0, as $\delta(x) = 1$ there. □

Definition 40.16. Let G be a group, X a finite set with a right action of G, written $x \cdot g$ (so $x \cdot 1 = x$ and $x \cdot (gh) = (x \cdot g) \cdot h$), and M an abelian group, written multiplicatively. A **monomial cocycle** of G on X with values in M is a map $c\colon G \times X \to M$ with

$$c(gh, x) = c(g, x)\,c(h, x \cdot g) \qquad (g, h \in G,\ x \in X). \tag{40.10}$$

Two monomial cocycles c, c' of G on X are **cohomologous** if there is a map $b\colon X \to M$ with $c'(g,x) = b(x)^{-1}c(g,x)\,b(x\cdot g)$ for all $g \in G$, $x \in X$. For a commutative ring A and $m \geq 0$, the **monomial transfer** of (X,c) is the A-linear map $\mathrm{Tf}_{X,c}\colon \mathrm{Bar}_m(G;A) \to \mathrm{Bar}_m(M;A)$ given on basis elements by

$$\mathrm{Tf}_{X,c}[g_1|\dots|g_m] = \sum_{x\in X} \big[c(g_1,x_0)\,\big|\,c(g_2,x_1)\,\big|\,\dots\,\big|\,c(g_m,x_{m-1})\big], \qquad x_0 = x,\ x_i = x_{i-1}\cdot g_i.$$

Proposition 40.17. *Let k be a finite field, $q \geq 1$ and $e \geq 0$ integers, $\zeta \in k^\times$ a primitive q-th root of unity and $v\colon k^\times \to \mathbb{Z}/q\mathbb{Z}$ a homomorphism. There is an additive family of degree $2e+3$ over k with values in $\mathbb{Z}/q\mathbb{Z}$ that has the value property for (q, e, ζ, v).*

Proof. Put $A = \mathbb{Z}/q\mathbb{Z}$ and $m = 2e + 3$. If $q = 1$, then $A = 0$ and the family of zero maps has all the properties; assume $q \geq 2$. As ζ has order q in $k^\times$, q divides $\#k - 1$, so $\#k = 1$ in A and $\#W = (\#k)^{\dim W} = 1$ in A for every finite-dimensional k-vector space W; and $\zeta \notin \{0, 1\}$, so $\#k \geq 3$. Let $t\colon \mathrm{Bar}_m(k^\times; A) \to A$ be the map of Lemma 40.19 for the cyclic group $K = k^\times$, the element ζ of order q, the homomorphism v and e.

Step 1: line cocycles. For a finite-dimensional k-vector space V let $\mathbb{P}(V)$ be its finite set of lines, with the right action $\ell \cdot g = g^{-1}\ell$ of $\mathrm{GL}(V)$. A *homogeneous function* on V is a map $\lambda\colon V \smallsetminus \{0\} \to k^\times$ with $\lambda(aw) = a\lambda(w)$ for $a \in k^\times$; one exists (fix a nonzero vector w_ℓ in each line and put $\lambda(aw_\ell) = a$). For such λ put $c_\lambda(g, \ell) = \lambda(w)/\lambda(g^{-1}w)$ for any $w \in \ell \smallsetminus \{0\}$; by homogeneity it does not depend on w. It is a monomial cocycle of $\mathrm{GL}(V)$ on $\mathbb{P}(V)$ with values in $k^\times$: as $g^{-1}w$ spans $\ell \cdot g$,

$$c_\lambda(gh, \ell) = \frac{\lambda(w)}{\lambda(g^{-1}w)} \cdot \frac{\lambda(g^{-1}w)}{\lambda(h^{-1}g^{-1}w)} = c_\lambda(g, \ell)\, c_\lambda(h, \ell \cdot g).$$

For two homogeneous functions λ, λ' on V, $b(\ell) = \lambda(w)/\lambda'(w)$ ($w \in \ell \smallsetminus \{0\}$) is well defined and $c_{\lambda'}(g, \ell) = b(\ell)^{-1} c_\lambda(g, \ell) b(\ell \cdot g)$, so c_λ and $c_{\lambda'}$ are cohomologous. For a homomorphism $\pi\colon G \to \mathrm{GL}(V)$ we write $\pi^* c_\lambda$ for the monomial cocycle $(g, \ell) \mapsto c_\lambda(\pi(g), \ell)$ of G on $\mathbb{P}(V)$, G acting through π; pull-backs of cohomologous cocycles are cohomologous (with the same b), and (Mt2) with $p = \mathrm{id}$ gives $\mathrm{Tf}_{\mathbb{P}(V), c_\lambda} \circ \pi_* = \mathrm{Tf}_{\mathbb{P}(V), \pi^* c_\lambda}$. Fix a homogeneous function λ_V on each V and define

$$\theta_V = t \circ \mathrm{Tf}_{\mathbb{P}(V), c_{\lambda_V}} : \mathrm{Bar}_m(\mathrm{GL}(V); A) \to A.$$

Step 2: (Fa1). Tf maps boundaries to boundaries by (Mt1), and t vanishes on boundaries.

Step 3: (Fa2). Let $\varphi\colon V \to W$ be an isomorphism, $f = c_\varphi\colon \mathrm{GL}(V) \to \mathrm{GL}(W)$, and z a cycle of $\mathrm{GL}(V)$. The bijection $p\colon \mathbb{P}(V) \to \mathbb{P}(W)$, $\ell \mapsto \varphi(\ell)$, satisfies $p(\ell \cdot g) = \varphi g^{-1} \ell = (\varphi g \varphi^{-1})^{-1} \varphi(\ell) = p(\ell) \cdot f(g)$, and $c_{\lambda_W}(f(g), p(\ell)) = \lambda_W(\varphi w)/\lambda_W(\varphi g^{-1} w) = c_{\lambda_W \circ \varphi}(g, \ell)$, where $\lambda_W \circ \varphi$ is a homogeneous function on V. By (Mt2), $\mathrm{Tf}_{\mathbb{P}(W), c_{\lambda_W}}(f_* z) = \mathrm{Tf}_{\mathbb{P}(V), c_{\lambda_W \circ \varphi}}(z)$, which by Step 1 and (Mt4) differs from $\mathrm{Tf}_{\mathbb{P}(V), c_{\lambda_V}}(z)$ by a boundary. Applying t gives $\theta_W(f_* z) = \theta_V(z)$.

Step 4: (Fa3). Let $\rho\colon G \to \mathrm{GL}(V)$, $\sigma\colon G \to \mathrm{GL}(W)$, $U = V \oplus W$, $\pi = \rho \oplus \sigma$ and $z \in \mathrm{Bar}_m(G; A)$ a cycle. The map $\lambda'(w_1, w_2) = \lambda_V(w_1)$ if $w_1 \neq 0$, $\lambda'(0, w_2) = \lambda_W(w_2)$, is a homogeneous function on U; by Step 1 and (Mt4),

$$\theta_U(\pi_* z) = t\big(\mathrm{Tf}_{\mathbb{P}(U), \pi^* c_{\lambda_U}}(z)\big) = t\big(\mathrm{Tf}_{\mathbb{P}(U), \pi^* c_{\lambda'}}(z)\big).$$

Let $\mathrm{pr}\colon \mathbb{P}(U) \to \mathbb{P}(V) \sqcup \mathbb{P}(W)$ send the line of (w_1, w_2) to the line of w_1 if $w_1 \neq 0$ and to the line of w_2 if $w_1 = 0$. Since $\pi(g)^{-1}(w_1, w_2) = (\rho(g)^{-1} w_1, \sigma(g)^{-1} w_2)$, pr is equivariant for the actions of G through π, ρ and σ, and for $w_1 \neq 0$, $c_{\lambda'}(\pi(g), [(w_1, w_2)]) = \lambda_V(w_1)/\lambda_V(\rho(g)^{-1} w_1) = c_{\lambda_V}(\rho(g), [w_1])$, and likewise for $w_1 = 0$ with λ_W and σ. The fibre of pr over the line of $w_1 \neq 0$ consists of the lines of (w_1, w_2), $w_2 \in W$, pairwise distinct, so it has $\#W = 1$ (in A) elements; the fibre over the line of w_2 is the line of $(0, w_2)$. By (Mt2) with $f = \mathrm{id}_G$, then (Mt3) and Step 1,

$$\mathrm{Tf}_{\mathbb{P}(U), \pi^* c_{\lambda'}}(z) = \mathrm{Tf}_{\mathbb{P}(V), \rho^* c_{\lambda_V}}(z) + \mathrm{Tf}_{\mathbb{P}(W), \sigma^* c_{\lambda_W}}(z) = \mathrm{Tf}_{\mathbb{P}(V), c_{\lambda_V}}(\rho_* z) + \mathrm{Tf}_{\mathbb{P}(W), c_{\lambda_W}}(\sigma_* z).$$

Applying t gives $\theta_U(\pi_* z) = \theta_V(\rho_* z) + \theta_W(\sigma_* z)$.

Step 5: the transfer of a prism of scalars. Let $V \neq 0$, $g \in \mathrm{GL}(V)$, $\lambda = \lambda_V$ and $\gamma(\ell) = c_\lambda(g, \ell)$. For a scalar $h = s_V(a)$ and every line ℓ', $\ell' \cdot h = \ell'$ and $c_\lambda(h, \ell') = \lambda(w)/\lambda(a^{-1}w) = a$. Scalars are central, so for $a_1, \ldots, a_n \in k^\times$ and $h_i = s_V(a_i)$,

$$\mathrm{Pr}_g[h_1 | \ldots | h_n] = \sum_{i=0}^{n} (-1)^i [h_1 | \ldots | h_i | g | h_{i+1} | \ldots | h_n],$$

and the term of ℓ in the i-th summand is $[a_1|\dots|a_i|\gamma(\ell)|a_{i+1}|\dots|a_n]$, the point at the entry g being $\ell\cdot(h_1\cdots h_i)=\ell$. Since $k^\times$ is abelian, the prism there is

$$\mathrm{Pr}_b[a_1|\dots|a_n]=\sum_i(-1)^i[a_1|\dots|a_i|b|a_{i+1}|\dots|a_n];$$

by linearity

$$\mathrm{Tf}_{\mathbb{P}(V),c_\lambda}\big(\mathrm{Pr}_g((s_V)_*y)\big)=\sum_{\ell\in\mathbb{P}(V)}\mathrm{Pr}_{\gamma(\ell)}(y)\qquad(y\in\mathrm{Bar}_{m-1}(k^\times;A)).$$

With $y=(\chi_x)_*\mathbf{z}_{q,e}$, so that $(s_V)_*y=(s_V\circ\chi_x)_*\mathbf{z}_{q,e}$, Lemma 40.19 gives

$$\theta_V\Big(\mathrm{Pr}_g\big((s_V\circ\chi_x)_*\mathbf{z}_{q,e}\big)\Big)=\sum_\ell x^{e+1}v(\gamma(\ell))=x^{e+1}\,v(D(g)),\qquad D(g)=\prod_{\ell\in\mathbb{P}(V)}c_\lambda(g,\ell).\tag{40.11}$$

Step 6: $D=\det$, hence (Fa4). By (40.10) and since $\ell\mapsto\ell\cdot g$ permutes $\mathbb{P}(V)$, $D(gh)=\prod_\ell c_\lambda(g,\ell)c_\lambda(h,\ell\cdot g)=D(g)D(h)$: $D\colon\mathrm{GL}(V)\to k^\times$ is a homomorphism. Replacing λ by another homogeneous function multiplies $D(g)$ by $\prod_\ell b(\ell)^{-1}b(\ell\cdot g)=1$ (Step 1), so D does not depend on λ. Fix a basis $e_1,\dots,e_n$ of V. If $ge_i=a_ie_i$ for all i, take $\lambda(w)=w_{i(w)}$, the coordinate of w at the smallest index $i(w)$ with $w_{i(w)}\neq0$; $g^{-1}w$ has the same support and coordinate $a_{i(w)}^{-1}w_{i(w)}$ there, so $c_\lambda(g,[w])=a_{i(w)}$. There are $(\#k)^{n-i}$ lines with $i(w)=i$, so $D(g)=\prod_i a_i^{(\#k)^{n-i}}=\prod_i a_i=\det g$, as $a^{\#k}=a$. For an elementary transvection $T_{ij}(c)$ let d be the diagonal matrix with entry ζ at i and 1 elsewhere; then $dT_{ij}(y)d^{-1}=T_{ij}(\zeta y)$, and with $y=c/(\zeta-1)$ we get $T_{ij}(c)=T_{ij}(\zeta y)T_{ij}(-y)=dT_{ij}(y)d^{-1}T_{ij}(y)^{-1}$, a commutator; so $D(T_{ij}(c))=1=\det T_{ij}(c)$. As these elements generate $\mathrm{GL}(V)$, $D=\det$, and (40.11) is (Fa4). For $V=0$ both sides of (Fa4) are 0 ($\mathbb{P}(0)=\varnothing$, $v(1)=0$). Hence $\theta=(\theta_V)_V$ is an additive family of degree $2e+3$ over k with values in $\mathbb{Z}/q\mathbb{Z}$ having the value property for (q,e,ζ,v). □

Lemma 40.18. *Let G be a group, X a finite set with a right action of G, M an abelian group, c a monomial cocycle of G on X with values in M, and A a commutative ring.*

(Mt1) *$\partial\circ\mathrm{Tf}_{X,c}=\mathrm{Tf}_{X,c}\circ\partial$ on $\mathrm{Bar}_{m+1}(G;A)$ for every $m\geq0$; hence $\mathrm{Tf}_{X,c}$ maps cycles to cycles and boundaries to boundaries.*

(Mt2) *Let H be a group, Y a finite set with a right action of H, c' a monomial cocycle of H on Y with values in M, $f\colon G\to H$ a homomorphism and $p\colon X\to Y$ a map with $p(x\cdot g)=p(x)\cdot f(g)$ and $c(g,x)=c'(f(g),p(x))$ for all g, x. If for every $y\in Y$ the number of $x\in X$ with $p(x)=y$ equals 1 in A, then $\mathrm{Tf}_{X,c}=\mathrm{Tf}_{Y,c'}\circ f_*$ on $\mathrm{Bar}_m(G;A)$ for every $m\geq0$.*

(Mt3) *If $X=X_1\sqcup X_2$ with X_1,X_2 stable under G, and c_i is the restriction of c to $G\times X_i$, then $\mathrm{Tf}_{X,c}=\mathrm{Tf}_{X_1,c_1}+\mathrm{Tf}_{X_2,c_2}$.*

(Mt4) *If c' is a monomial cocycle of G on X cohomologous to c, then $\mathrm{Tf}_{X,c'}(z)-\mathrm{Tf}_{X,c}(z)$ is a boundary for every cycle $z\in\mathrm{Bar}_m(G;A)$, $m\geq1$.*

Proof. For a basis element $\sigma=[g_1|\dots|g_m]$ and $x\in X$ put $x_0=x$, $x_i=x_{i-1}\cdot g_i$, and call $\tau_x(\sigma)=[c(g_1,x_0)|\dots|c(g_m,x_{m-1})]$ the term of x; so $\mathrm{Tf}_{X,c}\sigma=\sum_{x\in X}\tau_x(\sigma)$.

(Mt1). Let $\sigma=[g_0|\dots|g_m]\in\mathrm{Bar}_{m+1}(G;A)$, $x\in X$, $x_0=x$ and $x_{i+1}=x_i\cdot g_i$, so that $\tau_x(\sigma)=[c(g_0,x_0)|\dots|c(g_m,x_m)]$. We compare the faces of $\tau_x(\sigma)$ with the terms of the faces of σ.

The first face $[c(g_1,x_1)|\dots|c(g_m,x_m)]$ is the term of $x\cdot g_0$ in $[g_1|\dots|g_m]$. For $0\le j<m$ the j-th inner face has the entry $c(g_j,x_j)c(g_{j+1},x_{j+1})=c(g_jg_{j+1},x_j)$ by (40.10), and $x_{j+2}=x_j\cdot(g_jg_{j+1})$; so it is the term of x in $[g_0|\dots|g_jg_{j+1}|\dots|g_m]$. The last face is the term of x in $[g_0|\dots|g_{m-1}]$. Summing over $x\in X$, and using that $x\mapsto x\cdot g_0$ is a bijection of X for the first face, gives $\partial\,\mathrm{Tf}_{X,c}\sigma=\mathrm{Tf}_{X,c}\,\partial\sigma$. Linearity gives (Mt1).

(Mt2). Let $\sigma=[g_1|\dots|g_m]$, $x\in X$ and $y=p(x)$, with $y_0=y$, $y_i=y_{i-1}\cdot f(g_i)$. By induction $p(x_i)=y_i$, hence $c(g_i,x_{i-1})=c'(f(g_i),y_{i-1})$, and the term of x in σ equals the term of $p(x)$ in $f_*\sigma=[f(g_1)|\dots|f(g_m)]$. Grouping the points x by $p(x)$,

$$\mathrm{Tf}_{X,c}\sigma=\sum_{y\in Y}\#\{x:p(x)=y\}\cdot\tau_y(f_*\sigma)=\sum_{y\in Y}\tau_y(f_*\sigma)=\mathrm{Tf}_{Y,c'}(f_*\sigma),$$

as each of these numbers is 1 in A.

(Mt3). The points x_i attached to $x\in X_j$ stay in X_j, so the sum over X splits into the sums over X_1 and X_2.

(Mt4). Let W be the set of pairs (h,g) with $h\colon X\to M$ a map and $g\in G$, with the product $(h,g)(h',g')=(h*_g h',gg')$, where $(h*_g h')(x)=h(x)h'(x\cdot g)$. It is a group: both bracketings of a triple product give $(x\mapsto h(x)h'(x\cdot g)h''(x\cdot gg'),gg'g'')$, the neutral element is $(1,1)$, and $(x\mapsto h(x\cdot g^{-1})^{-1},g^{-1})$ is inverse to (h,g). The group W acts on X on the right by $x\cdot(h,g)=x\cdot g$, and $c_W((h,g),x)=h(x)$ is a monomial cocycle of W on X, since $c_W((h,g)(h',g'),x)=h(x)h'(x\cdot g)$. For a monomial cocycle d of G on X the map $\rho_d\colon G\to W$, $g\mapsto(d(g,\cdot),g)$, is a homomorphism by (40.10), and (Mt2), with $f=\rho_d$ and $p=\mathrm{id}_X$, gives

$$\mathrm{Tf}_{X,d}=\mathrm{Tf}^W_{X,c_W}\circ(\rho_d)_*,\tag{40.12}$$

where Tf^W_{X,c_W} is the monomial transfer of the group W. Now let $c'(g,x)=b(x)^{-1}c(g,x)b(x\cdot g)$ and $\beta=(b,1)\in W$. Then $\beta^{-1}=(b^{-1},1)$ and

$$\beta^{-1}\rho_c(g)\beta=(x\mapsto b(x)^{-1}c(g,x),\,g)\,(b,1)=(x\mapsto b(x)^{-1}c(g,x)b(x\cdot g),\,g)=\rho_{c'}(g),$$

so $\rho_{c'}=\kappa_\beta\circ\rho_c$. Let $z\in\mathrm{Bar}_m(G;A)$, $m\ge1$, be a cycle and $w=(\rho_c)_*z$, a cycle of W. By the prism formula, $(\kappa_\beta)_*w-w=\partial\,\mathrm{Pr}_\beta(w)$. By (40.12) for $d=c$ and $d=c'$ and by (Mt1) for W,

$$\mathrm{Tf}_{X,c'}(z)-\mathrm{Tf}_{X,c}(z)=\mathrm{Tf}^W_{X,c_W}\big((\kappa_\beta)_*w-w\big)=\partial\,\mathrm{Tf}^W_{X,c_W}\big(\mathrm{Pr}_\beta(w)\big),$$

a boundary. □

Lemma 40.19. *Let K be a finite cyclic group, $q\ge1$ and $e\ge0$ integers, $\zeta\in K$ an element of order q, and $v\colon K\to\mathbb{Z}/q\mathbb{Z}$ a homomorphism. For an integer $x\ge0$ let $\chi_x\colon C_q\to K$ be the homomorphism $\tau^j\mapsto\zeta^{xj}$. There is a $\mathbb{Z}/q\mathbb{Z}$-linear map $t\colon\mathrm{Bar}_{2e+3}(K;\mathbb{Z}/q\mathbb{Z})\to\mathbb{Z}/q\mathbb{Z}$ that vanishes on boundaries and satisfies*

$$t\Big(\mathrm{Pr}_a\big((\chi_x)_*\mathbf{z}_{q,e}\big)\Big)=x^{e+1}\,v(a)\qquad\text{for all integers } x\ge0 \text{ and all } a\in K.$$

Proof. Step 1: a normalised logarithm. Let $Q=\#K$; $q\mid Q$ as ζ has order q; put $r=Q/q$. Let $L_0\colon K\to\mathbb{Z}/Q\mathbb{Z}$ be an isomorphism. The element $L_0(\zeta)$ has order q, so $L_0(\zeta)=rc$ with c prime to q; choose $\tilde c\in\mathbb{Z}$ prime to Q with $\tilde c\,c\equiv1\pmod q$. Then $L=\tilde cL_0$ is an isomorphism with $L(\zeta)=r\tilde cc=r$, since $r(\tilde cc-1)\equiv0\pmod{rq}$. Let $\lg(a)\in\{0,\dots,Q-1\}$ represent $L(a)$, and

$$U(a,b)=\big(\lg(a)+\lg(b)-\lg(ab)\big)/Q\in\mathbb{Z}\qquad(a,b\in K),$$

an integer because L is a homomorphism. The function U is symmetric, K being abelian, and expanding the four terms gives $U(b,c')-U(ab,c')+U(a,bc')-U(a,b)=0$. So $u=U \bmod q \in C^2$ satisfies $\delta u=0$; also $\delta v=0$, as $(\delta v)(a,b)=v(b)-v(ab)+v(a)$.

Step 2: the cocycle. Define $F_n \in C^n$ by $F_0=1$, $F_1=v$ and $F_{n+2}(g_1,\dots,g_{n+2}) = u(g_1,g_2)\,F_n(g_3,\dots,g_{n+2})$, i.e. $F_{n+2}=u\cup F_n$. By induction and the Leibniz rule (the degree of u is even), $\delta F_n=0$ for $n\geq 1$. Let t be the linear extension of F_{2e+3} to $\mathrm{Bar}_{2e+3}(K;\mathbb{Z}/q\mathbb{Z})$; then $t(\partial\sigma)=(\delta F_{2e+3})(\sigma)=0$, so t vanishes on boundaries.

Step 3: insertion of one entry. For $E\geq 0$, $h=(h_1,\dots,h_{2E})\in K^{2E}$ and $a\in K$,

$$\sum_{i=0}^{2E}(-1)^i F_{2E+1}(h_1,\dots,h_i,a,h_{i+1},\dots,h_{2E}) = F_{2E}(h)\,v(a). \tag{40.13}$$

For $E=0$ both sides are $v(a)$. For $E+1$, the terms $i=0$ and $i=1$ are $u(a,h_1)F_{2E+1}(h_2,\dots)$ and $-u(h_1,a)F_{2E+1}(h_2,\dots)$ and cancel by symmetry of u; for $i\geq 2$ the term is

$$(-1)^i u(h_1,h_2)F_{2E+1}(a \text{ inserted at place } i-2 \text{ of } h_3,\dots,h_{2E+2}),$$

so by induction these sum to $u(h_1,h_2)F_{2E}(h_3,\dots,h_{2E+2})v(a)=F_{2E+2}(h)v(a)$.

Step 4: the value. As K is abelian,

$$\mathrm{Pr}_a[h_1|\dots|h_n]=\sum_{i=0}^{n}(-1)^i[h_1|\dots|h_i|a|h_{i+1}|\dots|h_n],$$

so by (40.13) $t(\mathrm{Pr}_a[h_1|\dots|h_{2e+2}])=F_{2e+2}(h)v(a)$. With

$$(\chi_x)_*\mathbf{z}_{q,e}=\sum_{j\in(\mathbb{Z}/q\mathbb{Z})^{e+1}}[\zeta^{xj_0}|\zeta^x|\zeta^{xj_1}|\zeta^x|\dots|\zeta^{xj_e}|\zeta^x]$$

we obtain

$$t\Big(\mathrm{Pr}_a\big((\chi_x)_*\mathbf{z}_{q,e}\big)\Big)=\sum_{j\in(\mathbb{Z}/q\mathbb{Z})^{e+1}}\prod_{s=0}^{e}u(\zeta^{xj_s},\zeta^x)\cdot v(a)=\Big(\sum_{j=0}^{q-1}u(\zeta^{xj},\zeta^x)\Big)^{e+1}v(a).$$

Finally, the sum telescopes:

$$\sum_{j=0}^{q-1}U(\zeta^{xj},\zeta^x)=\frac{1}{Q}\Big(q\lg(\zeta^x)+\lg(\zeta^0)-\lg(\zeta^{xq})\Big)=\frac{q\lg(\zeta^x)}{Q},$$

and $\lg(\zeta^x)=xr+Qm$ for some $m\in\mathbb{Z}$ because $L(\zeta^x)=xr$; so the sum is $x+qm\equiv x \pmod q$, and the value is $x^{e+1}v(a)$. □

Lemma 40.20. *Let k be a finite field, A a commutative ring, $m\geq 1$, θ an additive family of degree m over k with values in A, R a commutative ring and $r\colon R\to k$ a ring homomorphism. Then $\Phi_n=\theta_{k^n}\circ(\mathrm{red}_{r,n})_*$ $(n\geq 0)$ is a primitive family of degree m over R with values in A.*

Proof. Push-forward along a homomorphism commutes with the bar differential, so it maps cycles to cycles and boundaries to boundaries.

Cycles of the trivial group are boundaries. Let $E=\{1\}$ and $m'\geq 1$. Then $\mathrm{Bar}_{m'}(E;A)=A\,e_{m'}$ with $e_{m'}=[1|\dots|1]$, and $\partial e_{m'+1}=(\sum_{i=0}^{m'+1}(-1)^i)e_{m'}$, which is $e_{m'}$ if m' is odd and 0 if m' is even. If m' is odd, every element of $\mathrm{Bar}_{m'}(E;A)$ is a boundary. If m' is even, $\partial e_{m'}=e_{m'-1}$, so the only cycle is 0. Hence, for every group G and every homomorphism $\rho\colon G\to\mathrm{GL}(W)$ that is

trivial, ρ factors through E, $\rho_* z$ is a boundary for every cycle $z \in \mathrm{Bar}_m(G;A)$, and $\theta_W(\rho_* z) = 0$ by (Fa1).

(Pm1). If z is a boundary, so is $(\mathrm{red}_{r,n})_* z$, and $\Phi_n(z) = 0$ by (Fa1).

(Pm2). Let $n \le n'$ and z a cycle. Let $\varphi\colon k^n \oplus k^{n'-n} \to k^{n'}$ be the isomorphism of coordinates. Then $\mathrm{red}_{r,n'} \circ \mathrm{stab}_{n,n'} = c_\varphi \circ (\mathrm{red}_{r,n} \oplus \mathbf{1})$, where $\mathbf{1}\colon \mathrm{GL}_n(R) \to \mathrm{GL}(k^{n'-n})$ is trivial. By (Fa2), (Fa3) and the previous paragraph,

$$\Phi_{n'}((\mathrm{stab}_{n,n'})_* z) = \theta_{k^n \oplus k^{n'-n}}((\mathrm{red}_{r,n} \oplus \mathbf{1})_* z) = \theta_{k^n}((\mathrm{red}_{r,n})_* z) + \theta_{k^{n'-n}}(\mathbf{1}_* z) = \Phi_n(z).$$

(Pm3). Let $d \ge 1$ and z a cycle. Let $\varphi_d\colon (k^n)^{\oplus d} \to k^{dn}$ be the isomorphism that places the i-th summand in the i-th block of n consecutive coordinates. Then $\mathrm{red}_{r,dn} \circ \mathrm{blk}_{d,n} = c_{\varphi_d} \circ (\mathrm{red}_{r,n} \oplus \cdots \oplus \mathrm{red}_{r,n})$. By (Fa2), $\Phi_{dn}((\mathrm{blk}_{d,n})_* z) = \theta_{(k^n)^{\oplus d}}((\mathrm{red}_{r,n}^{\oplus d})_* z)$, and by (Fa3) applied $d-1$ times, with $(k^n)^{\oplus j} = k^n \oplus (k^n)^{\oplus (j-1)}$ (each push-forward of z is a cycle), this equals $d\,\theta_{k^n}((\mathrm{red}_{r,n})_* z) = d\,\Phi_n(z)$. □

Lemma 40.21. *Let $L \subseteq F$ be number fields, $q \ge 1$ an integer such that F contains a primitive q-th root of unity, $\beta \in F^\times$ and $b \in \mathcal{O}_F \smallsetminus \{0\}$. Suppose that for every nonzero prime $\mathfrak{P}$ of $\mathcal{O}_F$ with $b \notin \mathfrak{P}$ for which $\mathcal{O}_L \to \mathcal{O}_F/\mathfrak{P}$ is surjective, β is a unit at $\mathfrak{P}$ and its residue is a q-th power in $(\mathcal{O}_F/\mathfrak{P})^\times$. Then $\beta \in F^{\times q}$.*

Proof. Fix an algebraic closure of F, an element η with $\eta^q = \beta$, and $K = F(\eta)$. Since F contains μ_q, all roots of $X^q - \beta$ are $\eta\zeta'$ with $\zeta' \in \mu_q \subseteq F$, so K/F is Galois. Let Σ be the set of nonzero primes $\mathfrak{P}$ of $\mathcal{O}_F$ of absolute residue degree one with $\mathfrak{P} \nmid qb$; it differs from the set of all primes of absolute degree one by finitely many primes, so it has Dirichlet density one. For $\mathfrak{P} \in \Sigma$ the residue field is $\mathbb{F}_p$, the image of $\mathbb{Z} \subseteq \mathcal{O}_L$, so $\mathcal{O}_L \to \mathcal{O}_F/\mathfrak{P}$ is surjective; by hypothesis β is a unit at $\mathfrak{P}$ with residue s^q for some $s \in (\mathcal{O}_F/\mathfrak{P})^\times$. Since $\mathfrak{P} \nmid q$, the reduction of a primitive q-th root of unity is a primitive q-th root of unity, so $X^q - \bar\beta = \prod_{\zeta' \in \mu_q}(X - \bar\zeta' s)$ has q distinct roots in $\mathcal{O}_F/\mathfrak{P}$. The element η is integral at $\mathfrak{P}$ (a root of the monic $\mathfrak{P}$-integral polynomial $X^q - \beta$), and its minimal polynomial divides $X^q - \beta$, so it is separable and splits into linear factors modulo $\mathfrak{P}$; hence $\mathfrak{P}$ splits completely in K. So the set of primes of F splitting completely in K contains Σ and has Dirichlet density one. By Chebotarev's density theorem with $C = \{1\} \subseteq \mathrm{Gal}(K/F)$ this density is $1/[K:F]$; hence $K = F$, $\eta \in F^\times$ and $\beta = \eta^q \in F^{\times q}$. □

Theorem 40.22. *Let L be a totally complex number field, $k \ge 0$ an even integer, ℓ a prime, $\nu \ge 0$ an integer, B a $\mathbb{Z}/\ell^\nu\mathbb{Z}$-module and Φ a primitive family of degree $k+1$ over $\mathcal{O}_L$ with values in B (with $A = \mathbb{Z}/\ell^\nu\mathbb{Z}$). Put $N_0 = 2k+6$ and suppose that $H_{k+1}(\mathrm{GL}_{N_0}(\mathcal{O}_L), \mathbb{Z})_{\mathrm{tors}}$ and $H_k(\mathrm{GL}_{N_0}(\mathcal{O}_L), \mathbb{Z})_{\mathrm{tors}}$ are finite, of orders t_{k+1} and t_k. Then the set*

$$\{\ell^k\,\Phi_n(y) :\ n \ge 0,\ y \in \mathrm{Bar}_{k+1}(\mathrm{GL}_n(\mathcal{O}_L); \mathbb{Z}/\ell^\nu\mathbb{Z}) \text{ a cycle}\} \subseteq B$$

is finite with at most $\ell^{\nu r_2(L)}\, t_{k+1}\, t_k$ elements.

Proof. Put $q = \ell^\nu$, $R = \mathcal{O}_L$ and $r = r_2(L)$; then $qB = 0$. Let $\rho\colon \mathrm{Bar}_m(G;\mathbb{Z}) \to \mathrm{Bar}_m(G;\mathbb{Z}/q\mathbb{Z})$ be reduction of coefficients. It is surjective with kernel $q\,\mathrm{Bar}_m(G;\mathbb{Z})$, and it commutes with ∂ and with every f_*; $\mathrm{Bar}_m(G;\mathbb{Z})$ is free abelian, hence torsion-free.

Step 1: Φ induces maps on integral homology. For $n \ge 0$ define $\varphi_n\colon H_{k+1}(\mathrm{GL}_n(R), \mathbb{Z}) \to B$ by $\varphi_n([z]) = \Phi_n(\rho z)$ for integral cycles z. Here ρz is a cycle, and if $z = \partial w$ then $\rho z = \partial \rho w$ is a boundary and $\Phi_n(\rho z) = 0$ by (Pm1); so φ_n is a well-defined homomorphism. Since $(\mathrm{stab}_{n,n'})_* \rho z = \rho(\mathrm{stab}_{n,n'})_* z$ and $(\mathrm{blk}_{d,n})_* \rho z = \rho(\mathrm{blk}_{d,n})_* z$, (Pm2) and (Pm3) give, for every $x \in H_{k+1}(\mathrm{GL}_n(R), \mathbb{Z})$,

$$\varphi_{n'}((\mathrm{stab}_{n,n'})_* x) = \varphi_n(x) \quad (n \le n'), \qquad \varphi_{dn}((\mathrm{blk}_{d,n})_* x) = d\,\varphi_n(x) \quad (d \ge 1). \tag{40.14}$$

Step 2: $\ell^k\varphi_n$ factors through z_k. Let $n \geq 0$, $x \in H_{k+1}(\mathrm{GL}_n(R), \mathbb{Z})$ and let $z_t(x) \in H_{k+1}(\mathrm{GL}_{\ell^t n}(R), \mathbb{Z})$ be the sequence of Lemma 40.23: $z_0(x) = x$, $z_{t+1}(x) = (\mathrm{blk}_{\ell,\ell^t n})_* z_t(x) - \ell^{t+2}(\mathrm{stab}_{\ell^t n,\ell^{t+1} n})_* z_t(x)$. By (40.14), $\varphi_{\ell^{t+1}n}(z_{t+1}(x)) = \ell(1-\ell^{t+1})\,\varphi_{\ell^t n}(z_t(x))$, so by induction

$$\varphi_{\ell^k n}(z_k(x)) = \ell^k w\, \varphi_n(x), \qquad w = \prod_{t=0}^{k-1}(1-\ell^{t+1}).$$

As $w \equiv 1 \pmod{\ell}$, there is $u \in \mathbb{Z}$ with $uw \equiv 1 \pmod{q}$, and since $qB = 0$,

$$\ell^k \varphi_n(x) = \varphi_{\ell^k n}(u\, z_k(x)) \qquad \text{for all } x \in H_{k+1}(\mathrm{GL}_n(R), \mathbb{Z}). \tag{40.15}$$

Step 3: the torsion is stable from N_0 on. Let $m \in \{k, k+1\}$ and $n \geq N_0$. R has stable rank at most 2, so by Theorem B.163 with $s = 2$ and $i = m$ the maps $H_m(\mathrm{GL}_N(R), \mathbb{Z}) \to H_m(\mathrm{GL}_{N+1}(R), \mathbb{Z})$ induced by $\mathrm{stab}_{N,N+1}$ are isomorphisms for all $N \geq 2m+4$, in particular for all $N \geq N_0 = 2k+6$. Their composite is an isomorphism $H_m(\mathrm{GL}_{N_0}(R), \mathbb{Z}) \to H_m(\mathrm{GL}_n(R), \mathbb{Z})$, which maps the torsion subgroup onto the torsion subgroup. Hence $H_m(\mathrm{GL}_n(R), \mathbb{Z})_{\mathrm{tors}}$ is finite of order t_m.

Step 4: the Bockstein homomorphism. Fix $n \geq N_0$. Let $Y \subseteq \mathrm{Bar}_{k+1}(\mathrm{GL}_n(R); \mathbb{Z}/q\mathbb{Z})$ be the group of cycles and $T = H_k(\mathrm{GL}_n(R), \mathbb{Z})_{\mathrm{tors}}$, of order t_k by Step 3. For $y \in Y$ choose $\tilde{y} \in \mathrm{Bar}_{k+1}(\mathrm{GL}_n(R); \mathbb{Z})$ with $\rho\tilde{y} = y$. Then $\rho\,\partial\tilde{y} = \partial y = 0$, so $\partial\tilde{y} = q\,w_{\tilde{y}}$ for a unique $w_{\tilde{y}} \in \mathrm{Bar}_k(\mathrm{GL}_n(R); \mathbb{Z})$. If $k = 0$, then $\partial[g] = [\,] - [\,] = 0$ on Bar_1, so $w_{\tilde{y}} = 0$. If $k \geq 1$, then $q\,\partial w_{\tilde{y}} = \partial\partial\tilde{y} = 0$, so $w_{\tilde{y}}$ is a cycle. Put $\beta(y) = [w_{\tilde{y}}] \in H_k(\mathrm{GL}_n(R), \mathbb{Z})$. It lies in T, as $q[w_{\tilde{y}}] = [\partial\tilde{y}] = 0$. It does not depend on the lift: another lift is $\tilde{y} + qv$, and $w_{\tilde{y}+qv} = w_{\tilde{y}} + \partial v$. The sum of two lifts is a lift of the sum, so $\beta\colon Y \to T$ is a homomorphism. Finally, if $\beta(y) = 0$, then $w_{\tilde{y}} = \partial v$ for some v, and $\tilde{y} - qv$ is an integral cycle with $\rho(\tilde{y} - qv) = y$. Hence

$$\ker\beta = \{\rho z :\ z \in \mathrm{Bar}_{k+1}(\mathrm{GL}_n(R); \mathbb{Z}) \text{ a cycle}\}. \tag{40.16}$$

Step 5: the values on Y for fixed $n \geq N_0$. Let $g\colon Y \to B$ be the homomorphism $g(y) = \ell^k\Phi_n(y)$. Let $H' = H_{k+1}(\mathrm{GL}_{\ell^k n}(R), \mathbb{Z})$, let $J \subseteq H'$ be the subgroup generated by the $z_k(x)$, $x \in H_{k+1}(\mathrm{GL}_n(R), \mathbb{Z})$, and $f = \varphi_{\ell^k n}|_J$.

- $g(\ker\beta) \subseteq f(J)$. By (40.16) an element of $\ker\beta$ is ρz with z an integral cycle, and by (40.15) $g(\rho z) = \ell^k\varphi_n([z]) = f(u\,z_k([z]))$.
- $\#f(J) \leq q^r t_{k+1}$. As $f(qJ) = qf(J) = 0$, $f(J)$ is a quotient of J/qJ. The torsion subgroup J_{t} of J lies in H'_{tors}, which has order t_{k+1} by Step 3 ($\ell^k n \geq N_0$). Let $D = J/J_{\mathrm{t}}$, a torsion-free group of rank at most r by Lemma 40.23 (applicable since L is totally complex, k is even and $n \geq 2k+6$). The sequence $J_{\mathrm{t}} \to J/qJ \to D/qD \to 0$ is exact, so $\#(J/qJ) \leq t_{k+1}\,\#(D/qD)$. Next, $\dim_{\mathbb{F}_\ell} D/\ell D \leq r$: if $d_1, \dots, d_s \in D$ have linearly independent images in $D/\ell D$, the subgroup D' they generate is finitely generated and torsion-free of rank at most r, hence free of rank at most r, and a relation $\sum a_i d_i \in \ell D' \subseteq \ell D$ forces $\ell \mid a_i$; so the images in $D'/\ell D'$ are independent and $s \leq r$. Each quotient $\ell^i D/\ell^{i+1}D$ is the image of $D/\ell D$ under multiplication by ℓ^i, so $\#(D/qD) = \prod_{i=0}^{\nu-1}\#(\ell^i D/\ell^{i+1}D) \leq \ell^{r\nu} = q^r$.
- $\#g(Y) \leq \#T \cdot \#g(\ker\beta)$. For each $\tau \in \beta(Y)$ fix $y_\tau \in Y$ with $\beta(y_\tau) = \tau$. If $\beta(y) = \tau$, then $y - y_\tau \in \ker\beta$ and $g(y) \in g(y_\tau) + g(\ker\beta)$. So $g(Y)$ is covered by at most $\#T$ translates of $g(\ker\beta)$.

Together, $\#g(Y) \leq t_k\, q^r\, t_{k+1}$.

Step 6: all n. Let S be a finite subset of the set in the statement, each $s \in S$ written as $s = \ell^k \Phi_{n_s}(y_s)$ with y_s a cycle, and let $n = \max(N_0, \max_{s\in S} n_s)$. Then $(\mathrm{stab}_{n_s,n})_* y_s$ is a cycle and, by (Pm2), $s = \ell^k \Phi_n((\mathrm{stab}_{n_s,n})_* y_s)$, so $S \subseteq g(Y)$ for this n. By Step 5, $\#S \le \ell^{\nu r} t_{k+1} t_k$. As this holds for every finite subset, the set is finite with at most $\ell^{\nu r_2(L)} t_{k+1} t_k$ elements. □

Lemma 40.23. *Let L be a totally complex number field, $k \ge 0$ an even integer, $\ell \ge 1$ an integer and $n \ge 2k+6$. For $x \in H_{k+1}(\mathrm{GL}_n(\mathcal{O}_L), \mathbb{Z})$ define $z_t(x) \in H_{k+1}(\mathrm{GL}_{\ell^t n}(\mathcal{O}_L), \mathbb{Z})$ by $z_0(x) = x$ and*

$$z_{t+1}(x) = (\mathrm{blk}_{\ell,\ell^t n})_*\, z_t(x) - \ell^{t+2}\, (\mathrm{stab}_{\ell^t n,\ell^{t+1} n})_*\, z_t(x) \qquad (t \ge 0).$$

Then the subgroup of $H_{k+1}(\mathrm{GL}_{\ell^k n}(\mathcal{O}_L), \mathbb{Z})$ generated by the $z_k(x)$, $x \in H_{k+1}(\mathrm{GL}_n(\mathcal{O}_L), \mathbb{Z})$, has rank at most $r_2(L)$.

Proof. Let $R = \mathcal{O}_L$, $\mathcal{H}$, ι_N, μ, Δ and $\mathcal{P}$ as in Theorem B.164, and write x also for the image of x in rational homology.

Step 1: stability. R has stable rank at most 2, so by Theorem B.163 with $s = 2$ and $i = k+1$ the maps $H_{k+1}(\mathrm{GL}_N(R), \mathbb{Z}) \to H_{k+1}(\mathrm{GL}_{N+1}(R), \mathbb{Z})$ are isomorphisms for $N \ge 2k+6$. Tensoring with $\mathbb{Q}$ and passing to the colimit, $\iota_N \colon H_{k+1}(\mathrm{GL}_N(R); \mathbb{Q}) \to \mathcal{H}_{k+1}$ is an isomorphism for every $N \ge 2k+6$, in particular for $N = \ell^k n$.

Step 2: weights. By Theorem B.164, $\mathcal{H} = \bigoplus_{j\ge 0} \mathcal{H}_{(j)}$, where $\mathcal{H}_{(j)}$ is the span of the products of j primitive elements; since $\mathcal{P}$ lives in degrees ≥ 1, the degree-$(k+1)$ part of $\mathcal{H}_{(j)}$ is 0 for $j = 0$ and for $j > k+1$, and $\mathcal{H}_{(1)} = \mathcal{P}$. For $d \ge 1$ let $[d] = \mu^{(d)} \circ \Delta^{(d)} \colon \mathcal{H} \to \mathcal{H}$ (iterated coproduct, then iterated product). As $\mathcal{H}$ is commutative and cocommutative, $\mu^{(d)}$ and $\Delta^{(d)}$ are algebra homomorphisms, hence so is $[d]$; for $p \in \mathcal{P}$, $\Delta^{(d)}(p) = \sum_{i=1}^{d} 1 \otimes \cdots \otimes p \otimes \cdots \otimes 1$ and $[d]p = dp$. Hence $[d]$ acts on $\mathcal{H}_{(j)}$ as multiplication by d^j.

Step 3: the block-diagonal map is $[d]$. The map $\mathrm{blk}_{d,N}$ is the composite of the diagonal $\mathrm{GL}_N(R) \to \mathrm{GL}_N(R)^d$ and the d-fold block sum $\mathrm{GL}_N(R)^d \to \mathrm{GL}_{dN}(R)$. On rational homology the first induces $\Delta^{(d)}$ (with the Künneth isomorphism), and by the defining property of μ in Theorem B.164 the second, followed by ι_{dN}, is $\mu^{(d)} \circ \iota_N^{\otimes d}$. Hence $\iota_{dN} \circ (\mathrm{blk}_{d,N})_* = [d] \circ \iota_N$. Also $\iota_{N'} \circ (\mathrm{stab}_{N,N'})_* = \iota_N$.

Step 4: $z_k(x)$ is primitive in the stable range. By Step 3, $\iota_{\ell^{t+1} n}(z_{t+1}(x)) = ([\ell] - \ell^{t+2})\, \iota_{\ell^t n}(z_t(x))$, so $\iota_{\ell^k n}(z_k(x)) = \prod_{t=0}^{k-1} ([\ell] - \ell^{t+2})\, \iota_n(x)$. Write $\iota_n(x) = \sum_{j=1}^{k+1} x_j$ with x_j in the degree-$(k+1)$ part of $\mathcal{H}_{(j)}$ (Step 2). By Step 2,

$$\iota_{\ell^k n}(z_k(x)) = \sum_{j=1}^{k+1} \prod_{t=0}^{k-1} (\ell^j - \ell^{t+2}) x_j = \prod_{t=0}^{k-1} (\ell - \ell^{t+2})\, x_1 \in \mathcal{P}_{k+1},$$

because for $2 \le j \le k+1$ the factor with $t = j-2 \in \{0, \dots, k-1\}$ vanishes.

Step 5: the rank. By Steps 1 and 4, $\iota_{\ell^k n}$ maps the $\mathbb{Q}$-span of the $z_k(x)$ in $H_{k+1}(\mathrm{GL}_{\ell^k n}(R); \mathbb{Q})$ injectively into $\mathcal{P}_{k+1} \cong K_{k+1}(R) \otimes \mathbb{Q}$. As $k+1$ is odd and $r_1(L) = 0$, Theorem B.25 gives $\dim K_{k+1}(R) \otimes \mathbb{Q} = r_2(L)$ if $k+1 \ge 3$, and $\dim K_1(R) \otimes \mathbb{Q} = r_2(L) - 1$ if $k = 0$. So the rank of the subgroup generated by the $z_k(x)$ is at most $r_2(L)$. □

Proposition 40.24. *Let L be a totally complex number field, ℓ a prime, (F, γ, ζ) a cyclotomic Kummer layer of level (c, ν) over L, and $e \ge 1$, $b \ge 0$, $B \ge 0$ integers; put $a = (1+\ell^c)^e$ and $g(L) = 2[L:\mathbb{Q}] + \lfloor \log_2 h_L \rfloor + 2$. If $\#\mathrm{Ku}_{\ell^b,\ell^\nu}(F, \gamma, a) \le B$, then $\mathrm{Hom}(\mathrm{Cl}(F), \mathbb{Z}/\ell^\nu\mathbb{Z})^{(\gamma,a)}$ is finite and*

$$\#\mathrm{Hom}(\mathrm{Cl}(F), \mathbb{Z}/\ell^\nu\mathbb{Z})^{(\gamma,a)} \cdot \ell^{\nu\, r_2(L)} \le B \cdot \ell^{(b+v_\ell(e))\, g(L) + c + v_\ell(e)}.$$

Proof. All groups below are written multiplicatively. Put $\sigma = \gamma^{-1}$, $n = \ell^{\nu-c}$ (the order of γ), $r = r_2(L)$, $q = \ell^\nu$, $s = \ell^b$, $g = g(L)$, $\mu = \mu(F)$, $E = \mathcal{O}_F^\times$, and let I_F be the group of fractional ideals and $\mathrm{Cl} = \mathrm{Cl}(F)$. On $F^\times$, on I_F and on Cl let $\delta(x) = \sigma(x)\,x^{-a}$; these are endomorphisms, compatible with $y \mapsto (y)$ and $\mathfrak{d} \mapsto [\mathfrak{d}]$, and they commute with γ. Write $\mathrm{N} = \mathrm{N}_{\gamma,a}$ as in (Ky3).

Step 1: arithmetic of a. Since $v_\ell(\ell^c) = c \geq 1$ (≥ 2 if $\ell = 2$), the lifting-the-exponent rule gives $v_\ell(a-1) = c + v_\ell(e)$ and $v_\ell(a^n - 1) = c + v_\ell(e) + (\nu - c) = \nu + v_\ell(e)$; also $a \geq 3$. So $a^n - 1 = qu$ with $u = \ell^{v_\ell(e)}m$ and $\ell \nmid m$, and $\ell \mid a - 1$.

Step 2: δ is injective on I_F and on $F^\times/\mu$. If $\delta(x) = 1$ in I_F, then $\sigma(x) = x^a$, and $x = \sigma^n(x) = x^{a^n}$; as I_F is torsion free and $a^n \neq 1$, $x = 1$. If $\delta(y) \in \mu$ for $y \in F^\times$, then $\sigma(y) = y^a t$ with $t \in \mu$, and inductively $\sigma^k(y) \in y^{a^k}\mu$; for $k = n$ this gives $y^{a^n-1} \in \mu$, so $y \in \mu$.

Step 3: the Kummer group and its order. Let $U \subseteq F^\times$ be the group of (γ, a)-Kummer elements, that is, of α with $(\alpha) \in \delta(I_F)$ (Ky2), and $W = U/\delta(F^\times)\mu$; indeed $\delta(F^\times)\mu \subseteq U$. For $\alpha \in U$ let $\mathfrak{d}_\alpha \in I_F$ be the unique ideal with $\delta(\mathfrak{d}_\alpha) = (\alpha)$ (Step 2). The map $\alpha \mapsto [\mathfrak{d}_\alpha]$ is a homomorphism from U to $\mathrm{Cl}[\delta] = \ker(\delta|_{\mathrm{Cl}})$, as $\delta([\mathfrak{d}_\alpha]) = [(\alpha)] = 1$. It is surjective: if $\delta([\mathfrak{d}]) = 1$, then $\delta(\mathfrak{d}) = (\alpha)$ for some α, which lies in U with $\mathfrak{d}_\alpha = \mathfrak{d}$. Its kernel is $\delta(F^\times)E$: if $\mathfrak{d}_\alpha = (\beta)$, then $(\alpha) = (\delta(\beta))$ and $\alpha\delta(\beta)^{-1} \in E$; conversely $\mathfrak{d}_{\delta(\beta)} = (\beta)$ and $\mathfrak{d}_\varepsilon = 1$ for $\varepsilon \in E$. Moreover $E \cap \delta(F^\times)\mu = \delta(E)\mu$: if $\varepsilon = \delta(z)t$ with $t \in \mu$, then $\delta((z)) = (\varepsilon) = 1$, so $(z) = 1$ and $z \in E$. Hence the sequence

$$1 \to E/\delta(E)\mu \to W \to \mathrm{Cl}[\delta] \to 1$$

is exact. By Lemma 40.25 (L totally complex, F/L cyclic of degree n generated by γ, $a \geq 2$), $E/\delta(E)\mu$ is finite of order $(a^n-1)^r/(a-1)$; Cl is finite, and $\#\mathrm{Cl}[\delta] = \#(\mathrm{Cl}/\delta(\mathrm{Cl}))$. So W is finite and, by Step 1,

$$v_\ell(\#W) = r(\nu + v_\ell(e)) - (c + v_\ell(e)) + v_\ell\big(\#(\mathrm{Cl}/\delta(\mathrm{Cl}))\big). \tag{40.17}$$

Step 4: the eigencharacters. A homomorphism $\varphi\colon \mathrm{Cl} \to \mathbb{Z}/q\mathbb{Z}$ satisfies (Ky1) exactly when $\varphi(\sigma(x)x^{-a}) = 0$ for all $x \in \mathrm{Cl}$, that is, when it factors through $\mathrm{Cl}/\delta(\mathrm{Cl})$. So $\mathrm{Hom}(\mathrm{Cl}, \mathbb{Z}/q\mathbb{Z})^{(\gamma,a)} \cong \mathrm{Hom}(\mathrm{Cl}/\delta(\mathrm{Cl}), \mathbb{Z}/q\mathbb{Z})$ is finite of order at most $\ell^{v_\ell(\#(\mathrm{Cl}/\delta(\mathrm{Cl})))}$.

Step 5: the map to twisted Kummer classes. Since $\gamma^n = 1$ and $\gamma^{n-1} = \sigma$, a telescoping product gives

$$\mathrm{N}(\delta(y)) = \sigma(y)^{1-a^n} = \sigma(y)^{-qu} \qquad (y \in F^\times), \tag{40.18}$$

and N is a homomorphism commuting with δ and mapping μ into μ. Let $f\colon W \to F^\times/F^{\times q}\mu$ send the class of $\alpha \in U$ to the class of $\mathrm{N}(\alpha)^s$; by (40.18) it kills $\delta(F^\times)\mu$, so it is a well-defined homomorphism. Its image is the image of $\mathrm{Ku}_{s,q}(F,\gamma,a)$ under the projection $F^\times/F^{\times q} \to F^\times/F^{\times q}\mu$ (Ky4), so $\#f(W) \leq \#\mathrm{Ku}_{s,q}(F,\gamma,a) \leq B$. Its kernel is killed by su: if $\mathrm{N}(\alpha)^s = \beta^q t$ with $t \in \mu$, put $y = \alpha^s$; applying δ and (40.18), $\sigma(y)^{-qu} = \mathrm{N}(\delta(y)) = \delta(\mathrm{N}(y)) = \delta(\beta)^q\delta(t)$, so $(\sigma(y)^u\delta(\beta))^q \in \mu$, hence $\sigma(y)^u \in \delta(F^\times)\mu$, and applying γ, which preserves $\delta(F^\times)\mu$, $\alpha^{su} = y^u \in \delta(F^\times)\mu$.

Step 6: the ℓ-torsion of W. Let $\mathcal{S} \subseteq F^\times/F^{\times\ell}\mu$ be the set of Lemma 40.26. Let $[\alpha] \in W$ with $[\alpha]^\ell = 1$, so $\alpha^\ell = \delta(\beta)t$ with $\beta \in F^\times$, $t \in \mu$; by Step 2, β is determined by α up to μ. We claim that $[\alpha] \mapsto$ (class of β) is a well-defined injective homomorphism $W[\ell] \to \mathcal{S}$. First, $\delta((\beta)) = (\alpha)^\ell = \delta(\mathfrak{d}_\alpha^\ell)$, so $(\beta) = \mathfrak{d}_\alpha^\ell$ by Step 2, and ℓ divides every valuation of β; and $\sigma(\beta)\beta^{-1} = \alpha^\ell\beta^{a-1}t^{-1} \in F^{\times\ell}\mu$ since $\ell \mid a-1$, whence $\gamma(\beta)\beta^{-1} = \gamma\big((\sigma(\beta)\beta^{-1})^{-1}\big) \in F^{\times\ell}\mu$. So the class of β lies in $\mathcal{S}$. Replacing α by $\alpha\delta(z)t_1$ ($t_1 \in \mu$) replaces β by βz^ℓ up to μ, so the map is well defined, and it is a homomorphism. If $\beta = z^\ell t_2$ with $t_2 \in \mu$, then $(\alpha\delta(z)^{-1})^\ell = \delta(t_2)t \in \mu$, so $\alpha\delta(z)^{-1} \in \mu$ and $[\alpha] = 1$. By Lemma 40.26, $\#W[\ell] \leq \#\mathcal{S} \leq \ell^{2[L:\mathbb{Q}]+\lfloor \log_2 h_L \rfloor + 2} = \ell^g$.

Step 7: the ℓ-part of W. Let W_ℓ be the ℓ-Sylow subgroup of W and $K = \ker f \cap W_\ell$. By Step 5, K is killed by $su = \ell^{b+v_\ell(e)}m$, and as $\ell \nmid m$, by $\ell^{b+v_\ell(e)}$. K is a sum of $\dim_{\mathbb{F}_\ell} K[\ell] \leq$

$\dim_{\mathbb{F}_\ell} W[\ell] \le g$ cyclic groups (Step 6), each of order at most $\ell^{b+v_\ell(e)}$, so $\#K \le \ell^{(b+v_\ell(e))g}$. Hence

$$\ell^{v_\ell(\#W)} = \#W_\ell = \#f(W_\ell)\,\#K \le \#f(W)\,\ell^{(b+v_\ell(e))g} \le B\,\ell^{(b+v_\ell(e))g}.$$

Step 8: conclusion. By Step 4, $\nu r \le r(\nu + v_\ell(e))$, (40.17) and Step 7,

$$\#\mathrm{Hom}(\mathrm{Cl}, \mathbb{Z}/q\mathbb{Z})^{(\gamma,a)} \cdot \ell^{\nu r} \le \ell^{v_\ell(\#(\mathrm{Cl}/\delta(\mathrm{Cl})))+r(\nu+v_\ell(e))} = \ell^{v_\ell(\#W)}\,\ell^{c+v_\ell(e)} \le B\,\ell^{(b+v_\ell(e))g(L)+c+v_\ell(e)}.$$

□

Lemma 40.25. *Let L be a totally complex number field, F/L a cyclic extension of degree n, γ a generator of $\mathrm{Gal}(F/L)$, and $a \ge 2$ an integer. The endomorphism $u \mapsto \gamma^{-1}(u)\,u^{-a}$ of the group $\mathcal{O}_F^\times/\mu(F)$ is injective, and its cokernel is finite of order*

$$\frac{(a^n-1)^{r_2(L)}}{a-1}.$$

Proof. Write the group $\bar{E} = \mathcal{O}_F^\times/\mu(F)$ additively, put $r = r_2(L)$, $\sigma = \gamma^{-1}$, and $\delta = \sigma - a$ on $\bar{E}$, so that δ is the endomorphism of the statement. Since L is totally complex, so is F, and $r_2(F) = nr$.

The logarithmic embedding. Let S be the set of archimedean places of F and, for $w \in S$, $|\cdot|_w$ the absolute value of $\mathbb{C}$ composed with an embedding representing w. Let $\Lambda\colon \mathcal{O}_F^\times \to \mathbb{R}^S$, $\Lambda(u)_w = \log|u|_w$. By Dirichlet's unit theorem, $\ker\Lambda = \mu(F)$ and $\Lambda(\mathcal{O}_F^\times)$ is a lattice of rank $nr-1$ spanning the hyperplane $H_0 = \{x \in \mathbb{R}^S : \sum_w x_w = 0\}$; so $\bar{E}$ is free of rank $nr-1$ and Λ maps a $\mathbb{Z}$-basis of $\bar{E}$ to an $\mathbb{R}$-basis of H_0. For $\tau \in \mathrm{Gal}(F/L)$ and $w \in S$ let τw be the place with $|x|_{\tau w} = |\tau^{-1}x|_w$, and let τ act on $\mathbb{R}^S$ by $(\tau x)_{\tau w} = x_w$. Then $\Lambda(\tau u)_{\tau w} = \log|u|_w$, that is, $\Lambda(\tau u) = \tau\Lambda(u)$; and H_0 and the line $\mathbb{R}\mathbf{1}$ spanned by $\mathbf{1} = (1)_{w\in S}$ are stable, with $\mathbb{R}^S = H_0 \oplus \mathbb{R}\mathbf{1}$ and $\tau\mathbf{1} = \mathbf{1}$.

The characteristic polynomial. Let v be an archimedean place of L; it is complex, so $L_v = \mathbb{C}$ and $[F_w : L_v] = 1$ for every $w \mid v$. Hence exactly n places of F lie above v; $\mathrm{Gal}(F/L)$, of order n, permutes them transitively, hence simply transitively, and its generator σ permutes them in one cycle of length n. So S is the disjoint union of r orbits of length n, on each of which σ acts on the coordinates as a cyclic permutation, and the characteristic polynomial of σ on $\mathbb{R}^S$ is $(X^n-1)^r$. Removing the factor $X-1$ of the line $\mathbb{R}\mathbf{1}$, the characteristic polynomial of σ on H_0 is

$$P(X) = \frac{(X^n-1)^r}{X-1} = (X^n-1)^{r-1}\,(1 + X + \cdots + X^{n-1}).$$

In a $\mathbb{Z}$-basis of $\bar{E}$, σ has an integer matrix M, which is also the matrix of σ on H_0 in the image basis; so $\det(X - M) = P(X)$.

The index. The matrix of δ is $M - a$, and $\det(M-a) = (-1)^{nr-1}P(a)$. As $a \ge 2$, $P(a) = (a^n-1)^r/(a-1) > 0$, so δ is injective, and $\#(\bar{E}/\delta\bar{E}) = |\det(M-a)| = (a^n-1)^r/(a-1)$. (If $nr = 1$, then $\bar{E} = 0$ and $P = 1$, and the formula reads $1 = 1$.) □

Lemma 40.26. *Let L be a number field, ℓ a prime and (F, γ, ζ) a cyclotomic Kummer layer of level (c, ν) over L. Let $\mathcal{S} \subseteq F^\times/F^{\times\ell}\mu(F)$ be the set of classes of the elements $y \in F^\times$ such that $\ell \mid \mathrm{ord}_{\mathfrak{P}}(y)$ for every nonzero prime $\mathfrak{P}$ of $\mathcal{O}_F$ and $\gamma(y)\,y^{-1} \in F^{\times\ell}\mu(F)$. Then $\mathcal{S}$ is finite and*

$$\#\mathcal{S} \le \ell^{\,2[L:\mathbb{Q}]+\lfloor\log_2 h_L\rfloor+2}.$$

Proof. Let $n = [L : \mathbb{Q}]$ and $\mathrm{N} = \mathrm{N}_{F/L}$.

Step 1: $F = L(\zeta)$, and F/L is unramified at every prime not above ℓ. For an integer k, $\gamma^k(\zeta) = \zeta^{(1+\ell^c)^k}$ by (40.3), and by the lifting-the-exponent rule $(1+\ell^c)^k \equiv 1 \pmod{\ell^\nu}$ holds only if $\ell^{\nu-c} \mid k$, that is, only if $\gamma^k = 1$. So the only element of $\mathrm{Gal}(F/L) = \langle\gamma\rangle$ fixing $L(\zeta)$ is the identity, and $F = L(\zeta)$. Hence a prime of L not above ℓ is unramified in F.

Step 2: the Selmer group of L. Let $\mathrm{Sel}_L \subseteq L^\times/L^{\times\ell}$ be the subgroup of classes of the $x \in L^\times$ with $\ell \mid \mathrm{ord}_\mathfrak{p}(x)$ for every prime $\mathfrak{p}$ of L not above ℓ (a condition on the class). We show $\#\mathrm{Sel}_L \le \ell^{2n+\lfloor \log_2 h_L \rfloor}$. Let T be the set of primes of L above ℓ, so $\#T \le n$, and Cl_T the quotient of $\mathrm{Cl}(L)$ by the subgroup generated by the classes of the primes in T. For $x \in L^\times$ as above write $(x) = \mathfrak{a}^\ell\mathfrak{b}$ with $\mathfrak{a} = \prod_{\mathfrak{p}\notin T} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(x)/\ell}$ and $\mathfrak{b}$ supported on T, and send the class of x to the class of $\mathfrak{a}$ in Cl_T. Replacing x by xz^ℓ multiplies $\mathfrak{a}$ by (z) times an ideal supported on T, so this is a well-defined homomorphism $\pi\colon \mathrm{Sel}_L \to \mathrm{Cl}_T$, with values in $\mathrm{Cl}_T[\ell]$ because $\mathfrak{a}^\ell = (x)\mathfrak{b}^{-1}$. If π kills the class of x, then $\mathfrak{a} = (w)\mathfrak{c}$ with $\mathfrak{c}$ supported on T, so $(xw^{-\ell}) = \mathfrak{c}^\ell\mathfrak{b}$ is supported on T and $xw^{-\ell} \in \mathcal{O}_{L,T}^\times$. So $\ker\pi$ is the image of $\mathcal{O}_{L,T}^\times$, a quotient of $\mathcal{O}_{L,T}^\times/\mathcal{O}_{L,T}^{\times\ell}$, which by the T-unit theorem has order at most $\ell^{1+(r_1+r_2-1+\#T)} \le \ell^{2n}$. The ℓ-rank of $\mathrm{Cl}(L)$ is an integer k with $\ell^k \le h_L$, so $k \le \lfloor \log_2 h_L \rfloor$, and $\#\mathrm{Cl}_T[\ell] \le \#\mathrm{Cl}(L)[\ell] = \ell^k$. Hence $\#\mathrm{Sel}_L \le \ell^{2n+\lfloor \log_2 h_L \rfloor}$.

Step 3: types. Let ρ generate the cyclic group $\mu(F)$. For each $x \in \mathcal{S}$ fix y as in the statement; then $\gamma(y) = y\,v^\ell\rho^k$ for some $v \in F^\times$ and $k \in \mathbb{Z}$, and writing $k = \ell q + r$ with $0 \le r < \ell$ and replacing v by $v\rho^q$ we may take $k = r$. Taking norms, $\mathrm{N}(\gamma(y)) = \mathrm{N}(y)$ gives $\mathrm{N}(v)^\ell = \mathrm{N}(\rho)^{-r}$. Call $(r, \mathrm{N}(v))$ the type of x (for the fixed choices). Types lie in $\Theta = \{(r,\lambda) : 0 \le r < \ell,\ \lambda \in L,\ \lambda^\ell = \mathrm{N}(\rho)^{-r}\}$, and $\#\Theta \le \ell\cdot\ell$, as $X^\ell - \mathrm{N}(\rho)^{-r}$ has at most ℓ roots in L.

Step 4: elements of one type differ by a class from L. Let $x, x' \in \mathcal{S}$ have the same type (r,λ), with the fixed y, v and y', v'. Put $\beta = y/y'$ and $w = v/v'$; then $\gamma(\beta) = \beta w^\ell$ and $\mathrm{N}(w) = \lambda/\lambda = 1$. By Hilbert's Theorem 90, $w = \gamma(z)z^{-1}$ with $z \in F^\times$, so $\gamma(\beta z^{-\ell}) = \beta w^\ell \gamma(z)^{-\ell} = \beta z^{-\ell}$, and $x_0 = \beta z^{-\ell}$ lies in $L^\times$, as γ generates $\mathrm{Gal}(F/L)$. For a prime $\mathfrak{p}$ of L not above ℓ and a prime $\mathfrak{P}$ of F above it, $\mathfrak{P}$ is unramified over $\mathfrak{p}$ (Step 1), so $\mathrm{ord}_\mathfrak{p}(x_0) = \mathrm{ord}_\mathfrak{P}(x_0) = \mathrm{ord}_\mathfrak{P}(y) - \mathrm{ord}_\mathfrak{P}(y') - \ell\,\mathrm{ord}_\mathfrak{P}(z) \equiv 0 \pmod{\ell}$. So the class of x_0 lies in Sel_L, and $x = x'\,\iota(x_0)$, where $\iota\colon L^\times/L^{\times\ell} \to F^\times/F^{\times\ell}\mu(F)$ is induced by the inclusion.

Step 5: the count. For each type $\theta \in \Theta$ that occurs, fix $x_\theta \in \mathcal{S}$ of type θ. By Step 4 every $x \in \mathcal{S}$ lies in $x_\theta\,\iota(\mathrm{Sel}_L)$ for its type θ. Hence $\mathcal{S}$ is finite and, by Steps 2 and 3, $\#\mathcal{S} \le \#\Theta\cdot\#\mathrm{Sel}_L \le \ell^2\,.\,\ell^{2n+\lfloor \log_2 h_L \rfloor}$. □

Proposition 40.27. *Let L be a totally complex number field, ℓ a prime and $c \ge 1$ an integer ($c \ge 2$ if $\ell = 2$) such that L contains a primitive ℓ^c-th root of unity and no primitive ℓ^{c+1}-th root of unity. Let $e \ge 1$ and $B \ge 0$ be integers. Suppose that for every integer $\nu \ge c$ and every cyclotomic Kummer layer (F,γ,ζ) of level (c,ν) over L, the group $\mathrm{Hom}(\mathrm{Cl}(F), \mathbb{Z}/\ell^\nu\mathbb{Z})^{(\gamma,(1+\ell^c)^e)}$ is finite of order at most B. Then $\mathrm{len}_\ell(L,e)$ is finite and $\ell^{\mathrm{len}_\ell(L,e)} \le B$.*

Proof. Put $a = (1+\ell^c)^e$. Fix an algebraic closure $\overline{L}$ of L, let $\Omega \subseteq \overline{L}$ be the maximal extension of L unramified outside the places above ℓ, $G = \mathrm{Gal}(\Omega/L) = G_{L,S_\ell}$, and $A = \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$, on which $g \in G$ acts as multiplication by $\chi_{\mathrm{cyc}}(g)^{-e}$. For $1 \le k \le \infty$ let $L_k \subseteq \overline{L}$ be the field generated over L by the roots of unity of order dividing ℓ^k (all ℓ-power orders if $k = \infty$). The extension L_k/L is unramified outside the places above ℓ, and L has no real place, so $L_k \subseteq \Omega$; put $H_k = \mathrm{Gal}(\Omega/L_k)$, a closed normal subgroup of G. For each place v of L above ℓ let $D_v \subseteq G$ be the image of G_{L_v} under the map that defines the localisation $H^1(G,A) \to H^1(L_v,A)$; it is the decomposition group of a place of Ω above v, so the decomposition groups in G of the places of Ω above v are the gD_vg^{-1}, $g \in G$.

Step 1: the cyclotomic tower. Since $\mu_{\ell^c} \subseteq L$, χ_{cyc} maps G into U_c, and $g \in G$ fixes μ_{ℓ^k}

$(k \geq 1)$ exactly when $\chi_{\mathrm{cyc}}(g) \in U_k$. So χ_{cyc} induces an injective continuous homomorphism from $\Gamma = \mathrm{Gal}(L_\infty/L)$ to U_c, whose image is closed. The image is infinite, because $[L_k : L] \geq \varphi(\ell^k)/[L : \mathbb{Q}]$ tends to infinity; so it is U_{k_1} for some $k_1 \geq c$. If $k_1 \geq c+1$, then G fixes $\mu_{\ell^{c+1}}$, that is, L contains a primitive ℓ^{c+1}-th root of unity, which is excluded. Hence $\chi_{\mathrm{cyc}} \colon \Gamma \to U_c$ is an isomorphism, and for $\nu \geq c$ it induces an isomorphism $\mathrm{Gal}(L_\nu/L) \cong U_c/U_\nu$, a cyclic group of order $\ell^{\nu-c}$ generated by the class of $1+\ell^c$. Fix $g_0 \in G$ with $\chi_{\mathrm{cyc}}(g_0) = 1+\ell^c$ (restriction $G \to \Gamma$ is surjective), and let $\gamma_0 \in \Gamma$ be its restriction; the powers γ_0^k, $k \in \mathbb{Z}$, are dense in Γ.

Step 2: restriction to H_∞ is injective. The group H_∞ acts trivially on A, because $\chi_{\mathrm{cyc}}(H_\infty) = 1$. By inflation–restriction the sequence

$$0 \to H^1(\Gamma, A) \to H^1(G, A) \xrightarrow{\mathrm{res}} \mathrm{Hom}_{\mathrm{cts}}(H_\infty, A)$$

is exact, where res sends the class of a cocycle f to the homomorphism $f|_{H_\infty}$ (coboundaries vanish on H_∞). We show $H^1(\Gamma, A) = 0$. The element γ_0 acts on A as multiplication by $\beta = (1+\ell^c)^{-e}$, and $\beta \neq 1$ because U_c is torsion free and $e \geq 1$. Multiplication by the nonzero element $\beta - 1$ of $\mathbb{Z}_\ell$, a power of ℓ times a unit, is surjective on the divisible group A. Let $f \colon \Gamma \to A$ be a continuous cocycle and choose $y \in A$ with $(\beta-1)y = f(\gamma_0)$; then $f'(\sigma) = f(\sigma) - (\sigma - 1)y$ is a cocycle cohomologous to f with $f'(\gamma_0) = 0$. From $f'(\sigma\tau) = f'(\sigma) + \sigma f'(\tau)$ we get $f'(\gamma_0^k) = 0$ for $k \geq 0$ by induction, and $f'(\gamma_0^{-k}) = -\gamma_0^{-k} f'(\gamma_0^k) = 0$. The zero set of the continuous map f' to the discrete group A is closed and contains a dense subset, so $f' = 0$. Hence $H^1(\Gamma, A) = 0$ and res is injective.

Step 3: a Tate–Shafarevich class restricted to a finite level. For a continuous cocycle $f \colon G \to A$ and $g, d \in G$, the cocycle identity gives

$$f(gdg^{-1}) = g\, f(d) + (1 - gdg^{-1}) f(g). \tag{40.19}$$

Let $x \in \mathrm{Sha}^1_\ell(L, e)$ and let f be a continuous cocycle representing x. Its image is finite, so $f(G) \subseteq A[\ell^r]$ for some $r \geq 0$. For each $v \mid \ell$ the localisation of x at v is 0, so there is $a_v \in A$ with $f(d) = (d-1)a_v$ for all $d \in D_v$. As there are finitely many such v, there is $R \geq r$ with $\ell^R a_v = 0$ for all $v \mid \ell$. Let $\nu \geq \max(c, R)$ and $\psi = f|_{H_\nu}$. We claim: ψ is a continuous homomorphism $H_\nu \to A[\ell^\nu]$ and

1. $\psi(ghg^{-1}) = \chi_{\mathrm{cyc}}(g)^{-e}\, \psi(h)$ for $g \in G$ and $h \in H_\nu$;
2. ψ vanishes on $gD_vg^{-1} \cap H_\nu$ for every $v \mid \ell$ and $g \in G$;
3. $\psi|_{H_\infty} = \mathrm{res}(x)$.

An element $h \in H_\nu$ fixes μ_{ℓ^ν}, so $\chi_{\mathrm{cyc}}(h) \in U_\nu$ and $\chi_{\mathrm{cyc}}(h)^{-e} \equiv 1 \pmod{\ell^\nu}$: H_ν acts trivially on $A[\ell^\nu]$, which contains $f(G)$ as $r \leq \nu$. Hence $\psi(hh') = f(h) + h\, f(h') = \psi(h) + \psi(h')$, and ψ takes values in $A[\ell^\nu]$. For item 1, H_ν is normal, so $ghg^{-1} \in H_\nu$ fixes $f(g)$, and (40.19) gives $\psi(ghg^{-1}) = g\, f(h) = \chi_{\mathrm{cyc}}(g)^{-e}\psi(h)$. For item 2, let $d \in D_v$ with $d' = gdg^{-1} \in H_\nu$. By (40.19) and $f(d) = (d-1)a_v$,

$$f(d') = g(d-1)a_v + (1-d')f(g) = (d'-1)(g\, a_v - f(g)).$$

The element $g\, a_v - f(g)$ is killed by ℓ^ν (as g acts by a unit, $\ell^R a_v = 0$ and $f(g) \in A[\ell^r]$), and $d' \in H_\nu$ acts trivially on $A[\ell^\nu]$; so $f(d') = 0$. Item 3 holds because $H_\infty \subseteq H_\nu$.

Step 4: from such a homomorphism to an eigencharacter. Let $\nu \geq c$, $F = L_\nu$, and let $\psi \colon H_\nu \to A[\ell^\nu]$ be a continuous homomorphism with items 1 and 2 of Step 3. By Step 1, F/L is Galois with $\mathrm{Gal}(F/L)$ cyclic of order $\ell^{\nu-c}$, generated by $\gamma = g_0|_F$; let $\zeta \in F$ be a primitive ℓ^ν-th

root of unity, a unit of $\mathcal{O}_F$, with $\gamma(\zeta) = \zeta^{\chi_{\mathrm{cyc}}(g_0)} = \zeta^{1+\ell^c}$. So (F, γ, ζ) is a cyclotomic Kummer layer of level (c, ν) over L. Let $K \subseteq \Omega$ be the fixed field of $\ker\psi$, an abelian extension of F. It is unramified at the finite places of F not above ℓ, because Ω/L is; at a place of F above ℓ, the decomposition group in $H_\nu = \mathrm{Gal}(\Omega/F)$ of a place of Ω above it is some $gD_vg^{-1} \cap H_\nu$, which lies in $\ker\psi$ by item 2, so the place splits completely in K. As $F \supseteq L$ has no real place, $K \subseteq H_F$. Further H_F/L is unramified outside the places above ℓ (as H_F/F is unramified and F/L is unramified outside them), so $H_F \subseteq \Omega$, and ψ vanishes on $\mathrm{Gal}(\Omega/H_F) \subseteq \mathrm{Gal}(\Omega/K)$: $\psi = \bar\psi \circ \rho$ with $\rho\colon H_\nu \to \mathrm{Gal}(H_F/F)$ the (surjective) restriction and $\bar\psi$ a homomorphism. Let $\iota\colon \mathbb{Z}/\ell^\nu\mathbb{Z} \to A[\ell^\nu]$ be the group isomorphism $k \mapsto k\ell^{-\nu} + \mathbb{Z}_\ell$, and put

$$\theta_\psi = \iota^{-1} \circ \bar\psi \circ \mathrm{Art}\colon \mathrm{Cl}(F) \to \mathbb{Z}/\ell^\nu\mathbb{Z},$$

where $\mathrm{Art}\colon \mathrm{Cl}(F) \to \mathrm{Gal}(H_F/F)$ is the Artin isomorphism. Let $\mathfrak{p}$ be a nonzero prime of $\mathcal{O}_F$ and $\tilde g = g_0|_{H_F}$, which restricts to γ; then $\tilde g^{-1}$ restricts to γ^{-1}, and $\mathrm{Frob}_{\gamma^{-1}(\mathfrak{p})} = \tilde g^{-1}\,\mathrm{Frob}_{\mathfrak{p}}\,\tilde g$. Choose $h \in H_\nu$ with $\rho(h) = \mathrm{Frob}_{\mathfrak{p}}$; then $\rho(g_0^{-1}hg_0) = \mathrm{Frob}_{\gamma^{-1}(\mathfrak{p})}$, and item 1 with $g = g_0^{-1}$ gives

$$\bar\psi(\mathrm{Frob}_{\gamma^{-1}(\mathfrak{p})}) = \psi(g_0^{-1}hg_0) = \chi_{\mathrm{cyc}}(g_0)^e\,\psi(h) = a\,\bar\psi(\mathrm{Frob}_{\mathfrak{p}}).$$

So $\theta_\psi([\gamma^{-1}(\mathfrak{p})]) = a\,\theta_\psi([\mathfrak{p}])$; both sides are additive in the ideal, and every fractional ideal is a product of prime powers, so the relation of (Ky1) holds for every fractional ideal: $\theta_\psi \in \mathrm{Hom}(\mathrm{Cl}(F), \mathbb{Z}/\ell^\nu\mathbb{Z})^{(\gamma,a)}$. The assignment $\psi \mapsto \theta_\psi$ is injective, since ρ is surjective and Art and ι are bijective.

Step 5: the count. Let $T \subseteq \mathrm{Sha}^1_\ell(L, e)$ be finite. For each $x \in T$ choose f_x and R_x as in Step 3, and let $\nu = \max(c, \max_{x\in T} R_x)$, $F = L_\nu$, and (F, γ, ζ) the layer of Step 4. By Step 3, each $\psi_x = f_x|_{H_\nu}$ is a continuous homomorphism to $A[\ell^\nu]$ with items 1 to 3. If $x \neq x'$ in T, then $\psi_x|_{H_\infty} = \mathrm{res}(x) \neq \mathrm{res}(x') = \psi_{x'}|_{H_\infty}$ by Step 2, so $\psi_x \neq \psi_{x'}$ and, by Step 4, $\theta_{\psi_x} \neq \theta_{\psi_{x'}}$. Hence $x \mapsto \theta_{\psi_x}$ maps T injectively into $\mathrm{Hom}(\mathrm{Cl}(F), \mathbb{Z}/\ell^\nu\mathbb{Z})^{(\gamma,a)}$, which by hypothesis ($\nu \geq c$) has at most B elements; so $\#T \leq B$. As every finite subset of $\mathrm{Sha}^1_\ell(L, e)$ has at most B elements, $\mathrm{Sha}^1_\ell(L, e)$ is finite of order at most B. It is a $\mathbb{Z}_\ell$-submodule of $H^1(G, A)$; being finite, it has finite length $\mathrm{len}_\ell(L, e)$, and $\ell^{\mathrm{len}_\ell(L,e)} = \#\mathrm{Sha}^1_\ell(L, e) \leq B$. □

Definition 40.28. Let L be a totally complex number field, $n = [L : \mathbb{Q}]$, $S_\infty(L)$ its set of complex places (it has $r_2 = n/2$ elements), and fix for every $w \in S_\infty(L)$ an embedding $\sigma_w\colon L \to \mathbb{C}$ inducing w. Let $N \geq 1$ be an integer.

(Vn1) Forms. $V_{L,N}$ is the real vector space of tuples $h = (h_w)_{w\in S_\infty(L)}$ of hermitian complex $N \times N$ matrices; $\dim_{\mathbb{R}} V_{L,N} = r_2N^2$. For $h \in V_{L,N}$ and $y \in L^N$,

$$h[y] = \sum_{w\in S_\infty(L)} \sigma_w(y)^*\,h_w\,\sigma_w(y) \in \mathbb{R}. \tag{40.20}$$

Further, $V^{\geq 0}_{L,N}$ is the closed cone of the h with every h_w positive semidefinite, $V^*_{L,N}$ the dual space of $V_{L,N}$, and

$$\Omega_{L,N} = \{x \in V^*_{L,N} :\ x(k) > 0 \text{ for every } k \in V^{\geq 0}_{L,N} \setminus \{0\}\}.$$

(Vn2) Ryshkov polyhedron and perfect forms. The **Ryshkov polyhedron** is

$$\mathcal{R}_{L,N} = \{h \in V_{L,N} :\ h[y] \geq 1 \text{ for every } y \in \mathcal{O}_L^N \setminus \{0\}\}.$$

For $h \in \mathcal{R}_{L,N}$ the set of **minimal vectors** is $\mathrm{Min}(h) = \{y \in \mathcal{O}_L^N \setminus \{0\} :\ h[y] = 1\}$. A **perfect form** is an $h \in \mathcal{R}_{L,N}$ such that the only $k \in V_{L,N}$ with $k[y] = 0$ for all $y \in \mathrm{Min}(h)$ is $k = 0$. The set of perfect forms is denoted $\mathrm{Perf}_{L,N}$.

(Vn3) Normal cones. For $v \in \mathrm{Perf}_{L,N}$ the **normal cone** is the closed convex cone

$$\mathrm{Nc}(v) = \{x \in V_{L,N}^* : \ x(v) \le x(h) \text{ for every } h \in \mathcal{R}_{L,N}\}.$$

(Vn4) The perfect-cone nerve. For an integer $p \ge 0$, $\mathcal{N}_p(L,N)$ is the set of $(p+1)$-tuples $t = (t_0, \dots, t_p)$ of perfect forms, repetitions allowed, with

$$\Omega_{L,N} \cap \mathrm{Nc}(t_0) \cap \cdots \cap \mathrm{Nc}(t_p) \neq \emptyset;$$

a point of this intersection is a **witness** of t. For $p \ge 1$ and $0 \le i \le p$ the face $\partial_i t \in \mathcal{N}_{p-1}(L,N)$ is t with the entry t_i omitted (a witness of t is one of $\partial_i t$). The **augmented chain complex of the perfect-cone nerve** is

$$\cdots \xrightarrow{d_2} \mathbb{Z}[\mathcal{N}_1(L,N)] \xrightarrow{d_1} \mathbb{Z}[\mathcal{N}_0(L,N)] \xrightarrow{\mathrm{aug}} \mathbb{Z} \to 0,$$

where $\mathbb{Z}[X]$ is the free abelian group on a set X with basis $[x]$ ($x \in X$), $d_p[t] = \sum_{i=0}^{p} (-1)^i [\partial_i t]$ and $\mathrm{aug}[t] = 1$.

(Vn5) Action. $\mathrm{GL}_N(\mathcal{O}_L)$ acts on $V_{L,N}$ by $(g \cdot h)_w = \sigma_w(g^{-1})^* \, h_w \, \sigma_w(g^{-1})$, so that $(g \cdot h)[y] = h[g^{-1}y]$; on $V_{L,N}^*$ by $(g \cdot x)(h) = x(g^{-1} \cdot h)$; and on tuples of elements of $V_{L,N}$ entry by entry.

Theorem 40.29. *There is a real number $C_{\mathrm{T}} \ge 0$ such that for every totally complex number field L, every integer $e \ge 1$ and every integer m with $0 \le m \le 2e+1$, the group $H_m(\mathrm{GL}_{4e+6}(\mathcal{O}_L), \mathbb{Z})_{\mathrm{tors}}$ is finite and*

$$\log \# H_m\big(\mathrm{GL}_{4e+6}(\mathcal{O}_L), \mathbb{Z}\big)_{\mathrm{tors}} \le \big(2|d_L|\big)^{(2e[L:\mathbb{Q}])^{C_{\mathrm{T}}}}.$$

Proof. We prove the statement with $C_{\mathrm{T}} = 12$. Let L, $e \ge 1$ and $0 \le m \le 2e+1$ be as in the statement; put $n = [L : \mathbb{Q}]$ (even, so $n \ge 2$), $r_2 = n/2$, $N = 4e+6$, $\Gamma = \mathrm{GL}_N(\mathcal{O}_L)$, $R_0 = R_0(L,N) \ge 1$, $Y = \log(2|d_L|)$ and $u = 2en$.

Step 1: the torsion lemma applies. Take $X_p = \mathcal{N}_p(L,N)$ for $0 \le p \le m+1$, $M = \mathbb{Z}$ with the trivial action, $\alpha = \mathrm{aug}$ and the boundary maps d_p of the perfect-cone nerve. These maps are Γ-equivariant by item 5 of Lemma 40.32, and the sequence is exact at $\mathbb{Z}$ and at every $\mathbb{Z}[X_p]$ by Lemma 40.31. For $p \le m$ the number of orbits on X_p is at most $A_p(L,N) \le A_m(L,N)$ (Lemma 40.35; A_p increases with p), and every stabiliser has order at most $T = 3^{nN^2}$ (Lemma 40.34). For $t \in X_{m+1}$, $d_{m+1}[t]$ is a signed sum of $m+2$ basis elements, so $\|d_{m+1}[t]\|_1 \le m+2$. By Lemma 40.30 with $A = A_m(L,N)$, T and $b = m+2$, the group $H_m(\Gamma, \mathbb{Z})_{\mathrm{tors}}$ is finite and

$$\# H_m(\Gamma, \mathbb{Z})_{\mathrm{tors}} \le (m+2)^A \, T^{\,mAT^m}.$$

Step 2: a doubly exponential form. Since $\log(m+2) \le m+1 \le (m+1)T^{m+1}$ and $\log T \le T$,

$$\begin{aligned} \log \# H_m(\Gamma, \mathbb{Z})_{\mathrm{tors}} &\le A \log(m+2) + mAT^m \log T \le (2m+1)AT^{m+1} \\ &= \exp\big(\log A + (m+1)\log T + \log(2m+1)\big). \end{aligned}$$

Step 3: the exponent. From $e \ge 1$ and $n \ge 2$: $u \ge 4$, $e \le u/4$, $N \le 10e \le 5u/2$, $nN \le 10en = 5u$, $nN^2 = 2r_2N^2 \le 25u^2/2$, $m+1 \le 2e+2 \le 4e \le u$; and $Y \ge \log 2 > 1/2$,

$\log|d_L| \le Y$, $\log 3 \le 2Y$. As $R_0 \ge 1$, $A \le (3R_0)^{(nN)^2+1+nN\cdot r_2N^2\cdot(m+1)}$, with exponent at most $(25u^2+1)+5u\cdot\frac{25u^2}{4}\cdot u \le 58u^4$, and

$$\log(3R_0) = \log 3 + \tfrac{nN}{2}\log(nN) + \tfrac{N}{2}\log|d_L| \le 2Y + \tfrac{5u}{2}\cdot 5u + \tfrac{5u}{4}Y \le (2+25u^2+2u)Y \le 29u^2Y.$$

So $\log A \le 1682u^6Y$. Further $(m+1)\log T = (m+1)nN^2\log 3 \le u\cdot\frac{25u^2}{2}\cdot\frac{11}{10} \le 14u^3 \le 28u^3Y$, and $\log(2m+1) \le 2m \le 2u \le 4uY$. Hence

$$\log A + (m+1)\log T + \log(2m+1) \le (1682u^6 + 28u^3 + 4u)\,Y \le 1714u^6Y \le u^{12}Y,$$

and by Step 2

$$\log \#H_m\big(\mathrm{GL}_{4e+6}(\mathcal{O}_L),\mathbb{Z}\big)_{\mathrm{tors}} \le \exp\big(u^{12}Y\big) = \big(2|d_L|\big)^{(2e[L:\mathbb{Q}])^{12}}.$$

This is the statement with $C_{\mathrm{T}} = 12$. □

Lemma 40.30. *Let Γ be a group, $m \ge 0$ an integer and $A, T, b \ge 1$ integers. Let $X_0, \dots, X_{m+1}$ be Γ-sets, M a $\mathbb{Z}[\Gamma]$-module, and*

$$\mathbb{Z}[X_{m+1}] \xrightarrow{d_{m+1}} \mathbb{Z}[X_m] \xrightarrow{d_m} \cdots \xrightarrow{d_1} \mathbb{Z}[X_0] \xrightarrow{\alpha} M \to 0$$

a sequence of Γ-equivariant homomorphisms that is exact at M and at $\mathbb{Z}[X_p]$ for $0 \le p \le m$. Suppose that for $0 \le p \le m$ the set X_p has at most A orbits and every stabiliser Γ_x, $x \in X_p$, is finite of order at most T, and that $\|d_{m+1}[y]\|_1 \le b$ for every $y \in X_{m+1}$. Then $H_m(\Gamma, M)_{\mathrm{tors}}$ is finite and

$$\#H_m(\Gamma, M)_{\mathrm{tors}} \le b^A\, T^{\,mAT^m}.$$

Proof. Step 1: torsion of a cokernel. Let $k \ge 0$ and $\Lambda \subseteq \mathbb{Z}^k$ be generated by a set Z of vectors with $\|z\|_1 \le b$. Then $(\mathbb{Z}^k/\Lambda)_{\mathrm{tors}}$ has at most b^k elements. Indeed, it is $\Lambda^{\mathrm{sat}}/\Lambda$ with $\Lambda^{\mathrm{sat}} = \mathbb{Z}^k \cap \mathbb{Q}\Lambda$, a free group of the rank $r \le k$ of Λ. As Z spans $\mathbb{Q}\Lambda$, it contains linearly independent $z_1, \dots, z_r$; they span $\Lambda' \subseteq \Lambda \subseteq \Lambda^{\mathrm{sat}}$. The Gram determinant of a basis of Λ^{sat} is a positive integer, so $\mathrm{vol}(\Lambda^{\mathrm{sat}}) \ge 1$, and

$$[\Lambda^{\mathrm{sat}} : \Lambda] \le [\Lambda^{\mathrm{sat}} : \Lambda'] = \frac{\mathrm{vol}(\Lambda')}{\mathrm{vol}(\Lambda^{\mathrm{sat}})} \le \prod_{i=1}^{r} \|z_i\|_2 \le \prod_{i=1}^{r} \|z_i\|_1 \le b^r \le b^k.$$

Step 2: finite groups. If H is finite with $|H| \le T$ and $q \ge 1$, then $\#H_q(H,\mathbb{Z}) \le T^{T^q}$. Indeed $H_q(H,\mathbb{Z}) = Z_q/B_q$ with Z_q a subgroup of the q-th term of the bar complex, hence free of rank at most $|H|^q$, and it is annihilated by $|H|$, so it is a quotient of $Z_q/|H|Z_q$, of order at most $|H|^{|H|^q} \le T^{T^q}$.

Step 3: permutation modules. If a Γ-set X has at most A orbits and stabilisers of order at most T, then for $q \ge 1$, by Shapiro's lemma and Step 2,

$$\#H_q(\Gamma, \mathbb{Z}[X]) = \prod_{\Gamma x \in X/\Gamma} \#H_q(\Gamma_x, \mathbb{Z}) \le T^{AT^q}.$$

Step 4: torsion along an exact sequence. If $U \xrightarrow{f} V \xrightarrow{g} W$ is exact, U is finite and W_{tors} is finite, then V_{tors} is finite and $\#V_{\mathrm{tors}} \le \#U \cdot \#W_{\mathrm{tors}}$: g maps V_{tors} into W_{tors} with kernel inside $\ker g = f(U)$.

Step 5: induction on m. We prove the lemma for all Γ, M, X_p and maps with the given A, T, b, by induction on m.

$m = 0$. By right exactness, $\mathbb{Z}[X_1]_\Gamma \to \mathbb{Z}[X_0]_\Gamma \to M_\Gamma \to 0$ is exact. The map $\pi\colon [x] \mapsto [\Gamma x]$ identifies $\mathbb{Z}[X_0]_\Gamma$ with $\mathbb{Z}^k$, $k = \#(X_0/\Gamma) \le A$, and does not increase $\|\cdot\|_1$. So $H_0(\Gamma, M) = M_\Gamma \cong \mathbb{Z}^k/\Lambda$, where Λ is generated by the $\pi(d_1[y])$, $y \in X_1$, of ℓ^1-norm at most b. Step 1 gives $\#H_0(\Gamma, M)_{\mathrm{tors}} \le b^k \le b^A = b^A T^0$.

$m \to m+1$. Assume the lemma for m, and let data for $m+1$ be given. Let $K = \ker\alpha = d_1(\mathbb{Z}[X_1])$, a $\mathbb{Z}[\Gamma]$-submodule of $\mathbb{Z}[X_0]$. The sequence

$$\mathbb{Z}[X_{m+2}] \xrightarrow{d_{m+2}} \cdots \xrightarrow{d_2} \mathbb{Z}[X_1] \xrightarrow{d_1} K \to 0$$

is exact at K and at $\mathbb{Z}[X_p]$ for $1 \le p \le m+1$ (exactness at $\mathbb{Z}[X_1]$ because $K \subseteq \mathbb{Z}[X_0]$), the sets X_{p+1}, $0 \le p \le m$, satisfy the orbit and stabiliser bounds, and $\|d_{m+2}[y]\|_1 \le b$. By the case m, $\#H_m(\Gamma, K)_{\mathrm{tors}} \le b^A T^{mAT^m}$. The long exact sequence of $0 \to K \to \mathbb{Z}[X_0] \to M \to 0$ contains $H_{m+1}(\Gamma, \mathbb{Z}[X_0]) \to H_{m+1}(\Gamma, M) \to H_m(\Gamma, K)$, and by Step 3 (with $q = m+1$) the first group is finite of order at most $T^{AT^{m+1}}$. By Step 4, $H_{m+1}(\Gamma, M)_{\mathrm{tors}}$ is finite and

$$\#H_{m+1}(\Gamma, M)_{\mathrm{tors}} \le T^{AT^{m+1}} \cdot b^A T^{mAT^m} \le b^A T^{(m+1)AT^{m+1}},$$

since $mAT^m \le mAT^{m+1}$. □

Lemma 40.31. *Let L be a totally complex number field and $N \ge 1$. The augmented chain complex of the perfect-cone nerve is exact:* aug *is surjective,* $\ker(\mathrm{aug}) = d_1(\mathbb{Z}[\mathcal{N}_1(L,N)])$*, and* $\ker(d_p) = d_{p+1}(\mathbb{Z}[\mathcal{N}_{p+1}(L,N)])$ *for every $p \ge 1$.*

Proof. Write $\mathcal{N}_p = \mathcal{N}_p(L,N)$ and $\Omega = \Omega_{L,N}$; Ω is convex, as an intersection of open half-spaces of $V^*_{L,N}$.

Surjectivity of aug. By item 3 of Lemma 40.33 there is a perfect form v with $x_{\mathrm{tr}} \in \mathrm{Nc}(v)$, so $(v) \in \mathcal{N}_0$ with witness x_{tr}, and $\mathrm{aug}[(v)] = 1$.

Reduction to a finite nerve. Let $p \ge 0$ and $c \in \mathbb{Z}[\mathcal{N}_p]$ with $d_p c = 0$ if $p \ge 1$, $\mathrm{aug}(c) = 0$ if $p = 0$. Write $c = \sum_{i=1}^k a_i[t^{(i)}]$ with $t^{(i)} \in \mathcal{N}_p$, choose a witness x_i of $t^{(i)}$, and let $K \subseteq \Omega$ be the convex hull of $x_{\mathrm{tr}}, x_1, \dots, x_k$; it is compact and convex. By item 4 of Lemma 40.33 the perfect forms v with $\mathrm{Nc}(v) \cap K \ne \emptyset$ form a finite set $\{u_1, \dots, u_q\}$ (u_l distinct), and by item 3, $q \ge 1$ and $K = \bigcup_l K_l$ with $K_l = \mathrm{Nc}(u_l) \cap K$. Each K_l is nonempty, compact and convex ($\mathrm{Nc}(u_l)$ is closed and convex). Let $\mathcal{K}$ be the nerve of $(K_1, \dots, K_q)$. By Theorem B.174, $\mathcal{K}$ is homotopy equivalent to the convex set K, hence contractible; so its augmented oriented chain complex is exact, and by Munkres' comparison so is its augmented ordered chain complex $C'(\mathcal{K})$.

Comparison map. For a basis element $(l_0, \dots, l_p)$ of $C'_p(\mathcal{K})$, a point of $K_{l_0} \cap \cdots \cap K_{l_p}$ lies in Ω and in every $\mathrm{Nc}(u_{l_j})$, so $(u_{l_0}, \dots, u_{l_p}) \in \mathcal{N}_p$. The resulting homomorphisms $\iota\colon C'_p(\mathcal{K}) \to \mathbb{Z}[\mathcal{N}_p]$ commute with the boundaries and the augmentations (both are alternating sums of omissions, resp. 1 on vertices), and are injective because $l \mapsto u_l$ is injective. Each $t^{(i)}$ is in the image of a basis element: its entries $t^{(i)}_j$ satisfy $x_i \in \mathrm{Nc}(t^{(i)}_j) \cap K$, so $t^{(i)}_j = u_{l_j}$ for some l_j, and $x_i \in K_{l_0} \cap \cdots \cap K_{l_p}$. Hence $c = \iota(c')$ for some $c' \in C'_p(\mathcal{K})$, and $\iota(\partial c') = d_p c = 0$ (resp. ι preserves the augmentation), so c' is a cycle of the augmented complex by injectivity. By exactness of $C'(\mathcal{K})$, $c' = \partial b'$ with $b' \in C'_{p+1}(\mathcal{K})$, and $c = \iota(\partial b') = d_{p+1}\iota(b')$. This proves exactness at $\mathbb{Z}[\mathcal{N}_0]$ and at every $\mathbb{Z}[\mathcal{N}_p]$, $p \ge 1$. □

Lemma 40.32. *Let L be a totally complex number field of degree n and $N \ge 1$.*

1. *For every $h \in \mathcal{R}_{L,N}$ and every complex place w, the matrix h_w is positive definite.*

2. *The tuple $(I_N)_w$ of identity matrices lies in $\mathcal{R}_{L,N}$, and the functional $x_{\mathrm{tr}}(k) = \sum_w \mathrm{tr}(k_w)$ lies in $\Omega_{L,N}$.*

3. *For every $h \in \mathcal{R}_{L,N}$ and every real B there are a neighbourhood U of h in $V_{L,N}$ and a finite set $Y_0 \subseteq \mathcal{O}_L^N$ such that $k[y] > B$ for all $k \in U$ and all $y \in \mathcal{O}_L^N \setminus Y_0$.*

4. *For every $h \in \mathcal{R}_{L,N}$,*

$$\prod_w \det h_w \;\geq\; c_0(L,N) := (nN)^{-nN/2}\,|d_L|^{-N/2}. \tag{40.21}$$

5. *Every $g \in \mathrm{GL}_N(\mathcal{O}_L)$ maps $\mathcal{R}_{L,N}$, $\mathrm{Perf}_{L,N}$, $V_{L,N}^{\geq 0}$ and $\Omega_{L,N}$ onto themselves, $\mathrm{Min}(g\cdot h) = g\,\mathrm{Min}(h)$ and $\mathrm{Nc}(g\cdot v) = g\cdot \mathrm{Nc}(v)$. Hence g maps $\mathcal{N}_p(L,N)$ onto itself for every $p \geq 0$, $\partial_i(g\cdot t) = g\cdot \partial_i t$, and the maps d_p and* aug *are $\mathrm{GL}_N(\mathcal{O}_L)$-equivariant.*

Proof. Item 1. Let $h \in \mathcal{R}_{L,N}$ and suppose that for some place w_0 the matrix h_{w_0} is not positive definite. Then it has a unit eigenvector $\xi \in \mathbb{C}^N$ with eigenvalue $\mu \leq 0$. Put $c = \max(\|h\|, 1)$ and fix $\eta \in (0,1)$ with $r_2 c\eta^2 < 1$. For $T > 0$ let K_T be the set of $z = (z_w)_w \in \prod_w \mathbb{C}^N$ with $|\xi^* z_{w_0}| < T$, $\|z_{w_0} - (\xi^* z_{w_0})\xi\| < \eta$ and $\|z_w\| < \eta$ for $w \neq w_0$. It is convex and centrally symmetric, and its volume is πT^2 times a positive number independent of T; choose T so large that it exceeds 2^{nN} times the covolume of $j(\mathcal{O}_L^N)$. By Minkowski's theorem there is $y \in \mathcal{O}_L^N \setminus \{0\}$ with $j(y) \in K_T$. Write $\sigma_{w_0}(y) = a\xi + u$ with $a = \xi^*\sigma_{w_0}(y)$ and $\xi^* u = 0$. Since $\xi^* h_{w_0} u = (h_{w_0}\xi)^* u = \mu\,\xi^* u = 0$,

$$\sigma_{w_0}(y)^* h_{w_0} \sigma_{w_0}(y) = \mu|a|^2 + u^* h_{w_0} u \leq c\eta^2,$$

and $\sigma_w(y)^* h_w \sigma_w(y) \leq c\eta^2$ for $w \neq w_0$. So $h[y] \leq r_2 c\eta^2 < 1$, contradicting $h \in \mathcal{R}_{L,N}$.

Item 2. Let $y \in \mathcal{O}_L^N \setminus \{0\}$ and $y_a \neq 0$ a coordinate. Then

$$(I_N)_w[y] = \sum_w \|\sigma_w(y)\|^2 \geq \sum_w |\sigma_w(y_a)|^2 \geq r_2\Big(\prod_w |\sigma_w(y_a)|^2\Big)^{1/r_2} = r_2\,|\mathrm{N}_{L/\mathbb{Q}}(y_a)|^{1/r_2} \geq 1.$$

If $k \in V_{L,N}^{\geq 0} \setminus \{0\}$, some k_w is a nonzero positive semidefinite matrix, so $\mathrm{tr}(k_w) > 0$ and $\mathrm{tr}(k_{w'}) \geq 0$ for all w'; hence $x_{\mathrm{tr}}(k) > 0$.

Item 3. By item 1 the least eigenvalue λ of the matrices h_w, over all w, is positive. Let $U = \{k : \|k - h\| < \lambda/2\}$. For $k \in U$ every $k_w - (\lambda/2)I_N$ is positive semidefinite, so $k[y] \geq (\lambda/2)\|j(y)\|^2$. The set $Y_0 = \{y \in \mathcal{O}_L^N : \|j(y)\|^2 \leq 2|B|/\lambda\}$ is finite because $j(\mathcal{O}_L^N)$ is a lattice, and $k[y] > B$ for $k \in U$, $y \notin Y_0$.

Item 4. By item 1 each h_w is positive definite. The set $K = \{z \in \prod_w \mathbb{C}^N : \sum_w z_w^* h_w z_w < 1\}$ is convex and centrally symmetric; the substitution $z_w = h_w^{-1/2} u_w$ maps the unit ball of $\mathbb{R}^{nN}$ onto it with Jacobian $\prod_w \det(h_w)^{-1}$, so $\mathrm{vol}(K) = \beta_{nN} \prod_w \det(h_w)^{-1}$, where β_D is the volume of the unit ball of $\mathbb{R}^D$. As $h[y] \geq 1$ for $y \neq 0$, K contains no nonzero point of $j(\mathcal{O}_L^N)$, and Minkowski's theorem gives $\mathrm{vol}(K) \leq 2^{nN}(2^{-r_2}|d_L|^{1/2})^N = 2^{nN/2}|d_L|^{N/2}$. The cube of half-side $D^{-1/2}$ lies in the unit ball, so $\beta_D \geq (2D^{-1/2})^D$, and

$$\prod_w \det h_w \geq \frac{\beta_{nN}}{2^{nN/2}|d_L|^{N/2}} \geq \frac{2^{nN}(nN)^{-nN/2}}{2^{nN/2}|d_L|^{N/2}} \geq (nN)^{-nN/2}|d_L|^{-N/2}.$$

Item 5. The map $h \mapsto g\cdot h$ is a linear bijection of $V_{L,N}$ with inverse $h \mapsto g^{-1}\cdot h$, and $y \mapsto g^{-1}y$ is a bijection of $\mathcal{O}_L^N \setminus \{0\}$. Hence $(g\cdot h)[y] = h[g^{-1}y]$ shows $g\cdot\mathcal{R}_{L,N} = \mathcal{R}_{L,N}$ and $\mathrm{Min}(g\cdot h) = g\,\mathrm{Min}(h)$. If v is perfect and $k[y] = 0$ for all $y \in \mathrm{Min}(g\cdot v) = g\,\mathrm{Min}(v)$, then

$(g^{-1}\cdot k)[y'] = k[gy'] = 0$ for all $y' \in \mathrm{Min}(v)$, so $g^{-1}\cdot k = 0$ and $k = 0$: $g\cdot v$ is perfect. Congruence by $\sigma_w(g^{-1})$ preserves positive semidefiniteness, so $g\cdot V_{L,N}^{\geq 0} = V_{L,N}^{\geq 0}$, and then $(g\cdot x)(k) = x(g^{-1}\cdot k)$ shows $g\cdot\Omega_{L,N} = \Omega_{L,N}$. Finally $(g\cdot x)(g\cdot v) = x(v)$ and $(g\cdot x)(g\cdot h) = x(h)$ with $g\cdot\mathcal{R}_{L,N} = \mathcal{R}_{L,N}$ give $\mathrm{Nc}(g\cdot v) = g\cdot\mathrm{Nc}(v)$. So if x is a witness of t, then $g\cdot x$ is a witness of $g\cdot t$; omission of an entry commutes with the entrywise action, and d_p, aug are therefore equivariant. □

Lemma 40.33. *Let L be a totally complex number field with r_2 complex places and $N \geq 1$.*

1. *For every compact $K \subseteq \Omega_{L,N}$ there is $c_K > 0$ with $x(k) \geq c_K\|k\|$ for all $x \in K$ and $k \in V_{L,N}^{\geq 0}$.*
2. *Every bounded subset of $V_{L,N}$ contains only finitely many perfect forms.*
3. *For every $x \in \Omega_{L,N}$ there is $v \in \mathrm{Perf}_{L,N}$ with $x \in \mathrm{Nc}(v)$; that is, the normal cones of the perfect forms cover $\Omega_{L,N}$.*
4. *For every compact $K \subseteq \Omega_{L,N}$ only finitely many $v \in \mathrm{Perf}_{L,N}$ satisfy $\mathrm{Nc}(v) \cap K \neq \emptyset$.*
5. *For every $p \geq 0$ and $t = (t_0, \ldots, t_p) \in \mathcal{N}_p(L,N)$, the set $\mathrm{Min}(t_0) \cap \cdots \cap \mathrm{Min}(t_p)$ spans L^N over L.*
6. *For every $v \in \mathrm{Perf}_{L,N}$ there are $y_1, \ldots, y_r \in \mathrm{Min}(v)$, $r = r_2N^2$, such that v is the only $k \in V_{L,N}$ with $k[y_i] = 1$ for $1 \leq i \leq r$.*

Proof. The set $\mathcal{R}$ is closed and convex, as an intersection of closed half-spaces, and $\mathcal{R} \subseteq V_{L,N}^{\geq 0}$ by item 1 of Lemma 40.32.

Step 0: a perturbation principle. Let $h \in \mathcal{R}$ and $k \in V_{L,N}$ with $k[y] \geq 0$ for all $y \in \mathrm{Min}(h)$. Then $\mathrm{Min}(h)$ is finite and $h + \eta k \in \mathcal{R}$ for all sufficiently small $\eta > 0$. Indeed, item 3 of Lemma 40.32 with $B = 1$ gives a neighbourhood U of h and a finite Y_0 with $k'[y] > 1$ for $k' \in U$, $y \notin Y_0$; so $\mathrm{Min}(h) \subseteq Y_0$. For small η: $h + \eta k \in U$, so $(h+\eta k)[y] > 1$ for $y \notin Y_0$; $(h+\eta k)[y] > 1$ for the finitely many $y \in Y_0 \setminus (\mathrm{Min}(h) \cup \{0\})$, where $h[y] > 1$; and $(h+\eta k)[y] = 1 + \eta k[y] \geq 1$ for $y \in \mathrm{Min}(h)$.

Item 1. The set $S = \{k \in V_{L,N}^{\geq 0} : \|k\| = 1\}$ is compact and $(x,k) \mapsto x(k)$ is continuous and positive on $K \times S$ by the definition of $\Omega_{L,N}$; let $c_K > 0$ be its minimum. Homogeneity in k gives the claim.

Item 2. Two perfect forms v, v' with $\mathrm{Min}(v) = \mathrm{Min}(v')$ are equal: $k = v - v'$ satisfies $k[y] = 0$ on $\mathrm{Min}(v)$. Suppose a bounded set contains infinitely many perfect forms. They accumulate at some h, which lies in $\mathcal{R}$ as $\mathcal{R}$ is closed. Take U, Y_0 for h as in Step 0. Every perfect form $v \in U$ has $\mathrm{Min}(v) \subseteq Y_0$, so U contains at most $2^{|Y_0|}$ perfect forms, contradicting the choice of h.

Item 3. Let $x \in \Omega_{L,N}$ and $h_0 = (I_N)_w \in \mathcal{R}$ (item 2 of Lemma 40.32). By item 1 with $K = \{x\}$, the closed set $\{h \in \mathcal{R} : x(h) \leq x(h_0)\}$ lies in the ball of radius $x(h_0)/c_{\{x\}}$, so it is compact and nonempty, and x attains on it its minimum μ over $\mathcal{R}$. The set $F = \{h \in \mathcal{R} : x(h) = \mu\}$ is nonempty, compact and convex; let v be an extreme point of F. If $v = \frac{1}{2}(a+b)$ with $a, b \in \mathcal{R}$, then $x(a), x(b) \geq \mu$ and their mean is μ, so $a, b \in F$ and $a = b = v$: v is an extreme point of $\mathcal{R}$. If v were not perfect, there would be $k \neq 0$ with $k[y] = 0$ on $\mathrm{Min}(v)$; Step 0 for k and $-k$ gives $v \pm \eta k \in \mathcal{R}$ for a small $\eta > 0$, and v would be their midpoint. So $v \in \mathrm{Perf}_{L,N}$, and $x(v) = \mu \leq x(h)$ for all $h \in \mathcal{R}$, i.e. $x \in \mathrm{Nc}(v)$.

Item 4. Let c_K be as in item 1 and $M = \max_{x \in K} x(h_0)$. If $x \in K \cap \mathrm{Nc}(v)$, then $c_K\|v\| \leq x(v) \leq x(h_0) \leq M$. So these v lie in a ball, and by item 2 there are finitely many.

Item 5. Let x be a witness of t. Then $x(t_j) \le x(h)$ for all $h \in \mathcal{R}$ and all j, so all $x(t_j)$ equal $\mu = \min_{\mathcal{R}} x$. The barycentre $h^* = \frac{1}{p+1}\sum_j t_j$ lies in $\mathcal{R}$, $x(h^*) = \mu$, and since every $t_j[y] \ge 1$ for $y \ne 0$, $h^*[y] = 1$ exactly when $t_j[y] = 1$ for all j: $\mathrm{Min}(h^*) = \bigcap_j \mathrm{Min}(t_j)$. Suppose this set does not span L^N. Then there is $0 \ne \varphi \in L^N$ with $\varphi^T y = 0$ for all $y \in \mathrm{Min}(h^*)$. Put $k_w = \overline{\sigma_w(\varphi)}\,\sigma_w(\varphi)^T$, a hermitian positive semidefinite matrix with $\zeta^* k_w \zeta = |\sigma_w(\varphi)^T\zeta|^2$; so $k = (k_w)_w \in V_{L,N}^{\ge 0} \setminus \{0\}$ and $k[y] = \sum_w |\sigma_w(\varphi^T y)|^2 = 0$ for $y \in \mathrm{Min}(h^*)$. By Step 0 for k and $-k$, $h^* \pm \eta k \in \mathcal{R}$ for a small $\eta > 0$, so $x(h^* \pm \eta k) \ge \mu = x(h^*)$ and $x(k) = 0$, contradicting $x \in \Omega_{L,N}$.

Item 6. The linear forms $k \mapsto k[y]$, $y \in \mathrm{Min}(v)$, on $V_{L,N}$ have common kernel $\{0\}$ since v is perfect, so they span $V_{L,N}^*$, of dimension $r = r_2N^2$; choose $y_1, \dots, y_r$ among them giving a basis. If $k[y_i] = 1 = v[y_i]$ for all i, then $(k - v)[y_i] = 0$ for a basis of $V_{L,N}^*$, so $k = v$. □

Lemma 40.34. *Let L be a totally complex number field of degree n, $N \ge 1$ and $p \ge 0$. For every $t \in \mathcal{N}_p(L,N)$ the stabiliser of t in $\mathrm{GL}_N(\mathcal{O}_L)$ is finite, of order at most 3^{nN^2}.*

Proof. Reduction to a perfect form. An element fixing $t = (t_0, \dots, t_p)$ fixes $v = t_0 \in \mathrm{Perf}_{L,N} \subseteq \mathcal{R}_{L,N}$, so it suffices to bound the stabiliser G_v of v.

G_v is finite. Let $c = \max_a v[e_a]$. Item 3 of Lemma 40.32 at $h = v$ with $B = c$ gives a finite set Y_0 containing every $y \in \mathcal{O}_L^N$ with $v[y] \le c$. If $g \in G_v$, then $g^{-1} \cdot v = v$ and $v[ge_a] = (g^{-1} \cdot v)[e_a] = v[e_a] \le c$, so every column ge_a of g lies in Y_0. Hence $g \mapsto (ge_1, \dots, ge_N)$ embeds G_v into Y_0^N.

Minkowski's reduction modulo 3. Let $R = \mathrm{M}_N(\mathcal{O}_L)$, a ring that is a free $\mathbb{Z}$-module of rank nN^2, so $|R/3R| = 3^{nN^2}$, and $\mathrm{GL}_N(\mathcal{O}_L) = R^\times$. Let $H \subseteq R^\times$ be a finite subgroup and $\rho\colon R \to R/3R$ the reduction, a ring homomorphism. We show that $\rho|_H$ is injective; then $|H| \le |(R/3R)^\times| < 3^{nN^2}$, and $H = G_v$ gives the claim. Suppose $1 \ne g \in H$ with $\rho(g) = 1$. Some power g' of g has prime order p', and $\rho(g') = 1$. As R is free of finite rank, $\bigcap_k 3^k R = 0$, so there are $k \ge 1$ and $X \in R \setminus 3R$ with $g' = 1 + 3^k X$. Then

$$0 = g'^{p'} - 1 = \sum_{i=1}^{p'} \binom{p'}{i} 3^{ik} X^i.$$

If $p' \ne 3$, the right side is $3^k(p'X + 3^k Y)$ with $Y \in R$; since R is torsion free, $p'X = -3^k Y \in 3R$, and as p' is invertible modulo 3, $X \in 3R$, a contradiction. If $p' = 3$, the right side is $3^{k+1}(X + 3^k X^2 + 3^{2k-1}X^3)$, so $X = -3^k X^2 - 3^{2k-1}X^3 \in 3R$ as $k \ge 1$, again a contradiction. □

Lemma 40.35. *Let L be a totally complex number field of degree n, $N \ge 1$, and*

$$R_0(L,N) = (nN)^{nN/2}\,|d_L|^{N/2}.$$

For every $p \ge 0$ the set of $\mathrm{GL}_N(\mathcal{O}_L)$-orbits in $\mathcal{N}_p(L,N)$ is finite, with at most

$$A_p(L,N) = R_0(L,N)^{(nN)^2+1}\left(2R_0(L,N)+1\right)^{nN\cdot r_2N^2\cdot(p+1)} \tag{40.22}$$

elements.

Proof. Step 1: algebraic integers with bounded conjugates. For real $R \ge 0$ the set $I_R = \{\alpha \in \mathcal{O}_L : |\sigma_w(\alpha)| \le R \text{ for all } w\}$ has at most $(2R+1)^n$ elements. For $\alpha \in I_R$ let $P(\alpha) \subseteq \mathbb{C}^{r_2}$ be the product of the open discs of radius $\frac{1}{2}$ about the $\sigma_w(\alpha)$. If $P(\alpha) \cap P(\beta) \ne \emptyset$ for $\alpha \ne \beta$, then $|\sigma_w(\alpha - \beta)| < 1$ for all w and $|\mathrm{N}(\alpha - \beta)| < 1$, impossible. So the $P(\alpha)$ are disjoint, of volume $(\pi/4)^{r_2}$, and lie in the product of the discs of radius $R + \frac{1}{2}$ about 0, of volume $(\pi(R+\frac{1}{2})^2)^{r_2}$. Hence $|I_R| \le (2R+1)^{2r_2} = (2R+1)^n$.

Step 2: matrices of minimal vectors. Let $A \in \mathrm{M}_N(\mathcal{O}_L)$ with $\det A \neq 0$, $\delta_A = |\mathrm{N}(\det A)| \geq 1$ and $B = \delta_A A^{-1}$. Then $B = \pm(\mathrm{N}(\det A)/\det A)\,\mathrm{adj}(A) \in \mathrm{M}_N(\mathcal{O}_L)$. Suppose moreover that $h \in \mathcal{R}_{L,N}$ satisfies $h[Ae_a] = 1$ for all a. We show: $\delta_A \leq R_0$, and every coordinate of By lies in I_{R_0} for every $y \in \mathcal{O}_L^N$ with $h[y] = 1$. Let $G_w = \sigma_w(A)^* h_w \sigma_w(A)$, positive definite by item 1 of Lemma 40.32. Its diagonal entries $\sigma_w(Ae_a)^* h_w \sigma_w(Ae_a)$ are at most $h[Ae_a] = 1$, all terms of the sum defining $h[y]$ in (Vn1) being non-negative; so $\det G_w \leq 1$ and every principal minor of G_w is at most 1 (Hadamard). By item 4 of Lemma 40.32,

$$\prod_w \det G_w = \prod_w |\sigma_w(\det A)|^2 \prod_w \det h_w = \delta_A \prod_w \det h_w \geq \delta_A c_0.$$

Hence $\delta_A \leq c_0^{-1} = R_0$, and $\det G_w \geq \prod_{w'} \det G_{w'} \geq \delta_A c_0$ for each w. Now let $h[y] = 1$, $z = A^{-1}y$ and $\zeta = \sigma_w(z)$. Then $\zeta^* G_w \zeta = \sigma_w(y)^* h_w \sigma_w(y) \leq h[y] = 1$. By the Cauchy–Schwarz inequality for $\langle u, u' \rangle = u^* G_w u'$ and $\zeta_a = \langle G_w^{-1} e_a, \zeta \rangle$,

$$|\zeta_a|^2 \leq (e_a^* G_w^{-1} e_a)(\zeta^* G_w \zeta) \leq \frac{\det G_w^{(a)}}{\det G_w} \leq \frac{1}{\delta_A c_0},$$

where $G_w^{(a)}$ is the principal submatrix without row and column a. Since $By = \delta_A z$, $|\sigma_w((By)_a)|^2 \leq \delta_A / c_0 \leq c_0^{-2}$, i.e. $|\sigma_w((By)_a)| \leq R_0$, and $(By)_a \in \mathcal{O}_L$ as B is integral.

Step 3: perfect forms with prescribed minimal vectors. For A with $\det A \neq 0$ let $\mathrm{Perf}(A)$ be the set of perfect forms v with $v[Ae_a] = 1$ for all a, and $Y(A)$ the set of $y \in \mathcal{O}_L^N$ such that $h[y] = 1$ for some $h \in \mathcal{R}_{L,N}$ with $h[Ae_a] = 1$ for all a. By Step 2 the injective map $y \mapsto By$ sends $Y(A)$ into $I_{R_0}^N$, so $|Y(A)| \leq (2R_0 + 1)^{nN}$ by Step 1. For $v \in \mathrm{Perf}(A)$, $\mathrm{Min}(v) \subseteq Y(A)$; choosing $y_1, \ldots, y_r \in \mathrm{Min}(v)$ as in item 6 of Lemma 40.33 defines an injective map $\mathrm{Perf}(A) \to Y(A)^r$, since v is the only form with $k[y_i] = 1$ for all i. So $|\mathrm{Perf}(A)| \leq (2R_0 + 1)^{nNr}$.

Step 4: orbits of matrices. Let $\mathcal{A}$ be the set of $A \in \mathrm{M}_N(\mathcal{O}_L)$ with $\det A \neq 0$ and $\delta_A \leq R_0$; $\mathrm{GL}_N(\mathcal{O}_L)$ acts on it by left multiplication. Put $M_A = A^{-1}\mathcal{O}_L^N \subseteq L^N$. Then $M_{gA} = M_A$; and if $M_A = M_{A'}$, then $g = A'A^{-1}$ satisfies $g\mathcal{O}_L^N = A' M_{A'} = \mathcal{O}_L^N$, so g and g^{-1} have entries in $\mathcal{O}_L$ (they map each e_a into $\mathcal{O}_L^N$), $g \in \mathrm{GL}_N(\mathcal{O}_L)$ and $A' = gA$. Thus $A \mapsto M_A$ is injective on orbits. As $A\mathcal{O}_L^N \subseteq \mathcal{O}_L^N$ and $\delta_A M_A = B\mathcal{O}_L^N \subseteq \mathcal{O}_L^N$, we have $\mathcal{O}_L^N \subseteq M_A \subseteq \delta_A^{-1}\mathcal{O}_L^N$, so M_A is determined by δ_A and the subgroup $M_A/\mathcal{O}_L^N$ of $\delta_A^{-1}\mathcal{O}_L^N/\mathcal{O}_L^N \cong (\mathbb{Z}/\delta_A\mathbb{Z})^{nN}$. That group has at most $(\delta_A^{nN})^{nN}$ subgroups, each being generated by nN elements. Summing over $1 \leq \delta \leq R_0$, the number of orbits on $\mathcal{A}$ is at most $R_0 \cdot R_0^{(nN)^2}$.

Step 5: orbits of simplices. Fix a set S of representatives of the orbits on $\mathcal{A}$. Let $t \in \mathcal{N}_p(L, N)$. By item 5 of Lemma 40.33, $\bigcap_j \mathrm{Min}(t_j)$ contains an L-basis $y_1, \ldots, y_N$ of L^N; let A be the matrix with columns y_a. Then $\det A \neq 0$, and Step 2 with $h = t_0$ gives $\delta_A \leq R_0$, so $A \in \mathcal{A}$ and $gA \in S$ for some $g \in \mathrm{GL}_N(\mathcal{O}_L)$. By item 5 of Lemma 40.32, $g \cdot t \in \mathcal{N}_p(L, N)$ lies in the orbit of t, and each entry $g \cdot t_j$ is perfect with $(g \cdot t_j)[gAe_a] = t_j[Ae_a] = 1$: $g \cdot t \in \mathrm{Perf}(gA)^{p+1}$. Choosing one such element in each orbit gives an injective map from the set of orbits into $\bigcup_{A' \in S} \mathrm{Perf}(A')^{p+1}$, whose cardinality is at most $R_0^{(nN)^2+1}(2R_0 + 1)^{nNr(p+1)} = A_p(L, N)$ by Steps 3 and 4. □

Part VIII
Eisenstein congruences: the one-character bounds

Part VIII proves the three one-character bounds. For one character χ, one twist e and one prime ℓ, each bounds the normalised length of the strict Tate–Shafarevich group $\mathrm{Sha}^1_{\ell N}(\chi, e)$ by twice the sum of the ℓ-adic valuations of the differences between an Eisenstein trace and the Frobenius traces of finitely many admissible automorphic Galois representations. They are Proposition 42.2, on the unitary group in three variables attached to an imaginary quadratic field, for case (I); Proposition 44.3, on GSp_4, for cases (II) and (III); and Proposition 46.2, on the unitary group attached to $\mathbb{Q}(i)$, for case (V). Section 38 uses them in Proposition 38.5, Proposition 38.26 and Proposition 38.32. In return Part VIII takes from Part IX the three automorphy theorems 47.1, Theorem 51.1 and Theorem 47.16, and from Part VII the finiteness of the Tate–Shafarevich groups of Tate twists (Proposition 40.3).

The method is Ribet's [208], in the language of deformation rings. The Eisenstein representation, $\Psi \oplus \mathbf{1} \oplus \Psi^{-1}$ on G_K with $\Psi = \chi\chi_{\mathrm{cyc}}^{e}$ in the unitary case, is fixed by a torus $\mathrm{T}(t) = \mathrm{diag}(t, 1, t^{-1})$, and the torus grades its deformation ring by the weights of the matrix entries. A class in the Tate–Shafarevich group is a first-order deformation in weight -1, so the group embeds into the derivations of degree -1 of a graded model of the ring (clause (Mo7) of Proposition 43.3). An Euler-characteristic count gives the model one more equation than tangent directions in degree -1, and the graded numerical criterion Lemma 41.2 turns this excess into a bound: the cotangent length in degree -1 is at most twice the valuations of a weight-zero function at the two-signed points, the points that are neither upper nor lower triangular. The function is a trace difference, and Part IX shows that the two-signed points are automorphic, which identifies its valuations with Eisenstein congruences.

Section 41 proves the criterion, which is commutative algebra. Sections 42 and 43 prove the unitary bound and construct its graded model; Sections 44 and 45 do the same on GSp_4; Section 46 treats the dyadic case, bound and model together.

41 The graded numerical criterion

This section proves the graded numerical criterion, Lemma 41.2, on which the three one-character bounds rest (Proposition 42.2, Proposition 44.3, Proposition 46.2). It is commutative algebra and uses no Galois representations. The data are a $\mathbb{Z}$-graded $\mathcal{O}$-algebra A with a fixed point φ_0, smooth with given tangent dimensions in degrees $-2, \dots, 2$, a system u of homogeneous equations and a function a of weight zero (Definition 41.1). The conditions (Gc1)–(Gc5) ask that u have one more equation than tangent directions in degree -1, one fewer in degrees 1 and 2, and as many in degrees -2 and 0; that the weight-zero part of $A/I(u)$ be finite over $\mathcal{O}$; that no point other than φ_0 lie, in weight zero, under both a point that vanishes in all negative degrees but not in all positive ones and a point with the roles of the signs exchanged; that the tangent spaces at the two-signed points be at most one-dimensional; and that a vanish at φ_0 and at no two-signed point. The conclusion bounds the cotangent length of u in degree -1 by twice the sum of the valuations of a over a set of representatives of the two-signed points up to the torus. In the applications A is a graded model of an Eisenstein deformation ring, the cotangent length bounds the Tate–Shafarevich group, and a is a trace difference.

The proof is an intersection count on the Rees algebra of the equations. After a reduction to

an injective chart in weights -1, 1, 2 with equations in Smith form (Lemma 41.4, Lemma 41.7), a Koszul Euler characteristic of the equations and of a is computed in large degree in two ways. At the fixed point it is bounded below by the cotangent length (Proposition 41.13); at the two-signed points it is bounded above by Herbrand quotients of a, where the weight pieces have local length at most one (Proposition 41.21, Proposition 41.23). These two bounds carry the argument; the lemmas on Koszul complexes over standard multigraded algebras and on Rees algebras between them are standard commutative algebra.

Definition 41.1. Let $\mathcal{O}$ be a coefficient ring with fraction field $\mathcal{O}[\ell^{-1}]$, and put $v_{\mathcal{O}} = e_{\mathcal{O}}\, v_\ell$ on $\overline{\mathbb{Q}}_\ell$, so that $v_{\mathcal{O}}$ is 1 on a uniformiser of $\mathcal{O}$. All algebras are commutative.

Graded algebras with a fixed point. A **graded $\mathcal{O}$-algebra with a fixed point** is a pair (A, φ_0) of a $\mathbb{Z}$-graded $\mathcal{O}$-algebra $A = \bigoplus_{j\in\mathbb{Z}} A_j$ (so $A_i A_j \subseteq A_{i+j}$ and $\mathcal{O} \to A_0$) and an $\mathcal{O}$-algebra homomorphism $\varphi_0 \colon A \to \mathcal{O}$ such that

- A_0 is a complete Noetherian local ring, and $\mathcal{O} \to A_0$ is a local homomorphism inducing an isomorphism of residue fields;
- A is generated as an $\mathcal{O}$-algebra by A_0 and finitely many homogeneous elements of degrees in $\{-2, -1, 1, 2\}$;
- $\varphi_0(A_j) = 0$ for every $j \neq 0$.

Cotangent lengths. Put $\mathfrak{m}_0 = \ker \varphi_0$ and let $\mathrm{Sq}(\varphi_0) = \{f \in A : \ cf \in \mathfrak{m}_0^2 \text{ for some } c \in \mathcal{O} \setminus \{0\}\}$, an $\mathcal{O}$-submodule of A. For $j \in \mathbb{Z}$ the **cotangent lattice in degree j** is the image Cot_j of $A_j \cap \mathfrak{m}_0$ in $A/\mathrm{Sq}(\varphi_0)$. For a finite family $u = (u_i)_{i\in I}$ of elements of A, $\mathrm{Cot}(u)$ is the $\mathcal{O}$-span of the images of the u_i in $A/\mathrm{Sq}(\varphi_0)$. The family u **spans rationally in degree j** if for every $p \in \mathrm{Cot}_j$ there is $c \in \mathcal{O} \setminus \{0\}$ with $cp \in \mathrm{Cot}(u)$, and its **cotangent length in degree j** is

$$\mathrm{ct}_j(u) = \mathrm{length}_{\mathcal{O}}\big(\mathrm{Cot}_j / (\mathrm{Cot}_j \cap \mathrm{Cot}(u))\big) \in \mathbb{N} \cup \{\infty\} \tag{41.1}$$

(the dependence on (A, φ_0) is not written).

Charts. For a finite set J with a map $\mathrm{wt} \colon J \to \mathbb{Z}$, a power series in $\mathcal{O}[\ell^{-1}][[X_i : i \in J]]$ has **weight j** if every monomial $\prod_i X_i^{n_i}$ occurring in it has $\sum_i n_i \mathrm{wt}(i) = j$. For a function $\mathrm{td} \colon \mathbb{Z} \to \mathbb{N}$ with finite support, (A, φ_0) is **smooth with tangent dimensions** td if there are such (J, wt) with $\#\mathrm{wt}^{-1}(j) = \mathrm{td}(j)$ for every j, and an $\mathcal{O}$-algebra homomorphism $\phi \colon A \to \mathcal{O}[\ell^{-1}][[X_i : i \in J]]$ (a **chart**) such that

(Sm1) $\phi(f)$ has weight j for every $f \in A_j$;

(Sm2) the constant term of $\phi(f)$ is $\varphi_0(f)$ for every $f \in A$;

(Sm3) for every $i \in J$ there is $f \in A$ such that in $\phi(f)$ the coefficient of X_i is non-zero and the coefficient of $X_{i'}$ is zero for every $i' \neq i$;

(Sm4) if $\varphi_0(f) = 0$ and all coefficients of degree one of $\phi(f)$ vanish, then $f \in \mathrm{Sq}(\varphi_0)$.

Equations and points. A **system of equations** for (A, φ_0) with counts ne, where $\mathrm{ne} \colon \{-2, -1, 0, 1, 2\} \to \mathbb{N}$, is a family $u = (u_{j,i})$, $-2 \le j \le 2$, $1 \le i \le \mathrm{ne}(j)$, with $u_{j,i} \in A_j$ and $\varphi_0(u_{0,i}) = 0$; $u_j = (u_{j,i})_i$ is its **part in degree j**, and $I(u)$ is the ideal of A generated by all $u_{j,i}$. A **point** is an $\mathcal{O}$-algebra homomorphism $\varphi \colon A \to \overline{\mathbb{Q}}_\ell$; it lies **on** u if $\varphi(I(u)) = 0$. The fixed point φ_0 is a point through $\mathcal{O} \subset \overline{\mathbb{Q}}_\ell$. For $t \in \overline{\mathbb{Q}}_\ell^{\times}$ the **torus translate** $t \cdot \varphi$ is the point with $(t \cdot \varphi)(f) = t^j \varphi(f)$ for $f \in A_j$; it lies on u if and only if φ does, since $I(u)$ is generated by homogeneous elements. A point φ is **negative-vanishing** if $\varphi(A_j) = 0$ for all $j < 0$,

positive-vanishing if $\varphi(A_j) = 0$ for all $j > 0$, and **two-signed** if it is neither. An **orbit cover** of u is a finite set Σ of two-signed points on u such that every two-signed point on u is a torus translate of a member of Σ. For a point φ, a **φ-derivation** is an $\mathcal{O}$-linear map $D\colon A \to \overline{\mathbb{Q}}_\ell$ with $D(fg) = \varphi(f)D(g) + \varphi(g)D(f)$ for all f, g.

The conditions of the graded numerical criterion. Let (A, φ_0) be a graded $\mathcal{O}$-algebra with a fixed point, u a system of equations with counts ne, and $a \in A$. The tuple (A, φ_0, u, a) satisfies the **conditions of the graded numerical criterion** if

(Gc1) there is td with $\mathrm{td}(j) = 0$ for $|j| > 2$ such that (A, φ_0) is smooth with tangent dimensions td and

$$\mathrm{ne}(-1) = \mathrm{td}(-1) + 1, \quad \mathrm{ne}(-2) = \mathrm{td}(-2), \quad \mathrm{ne}(0) = \mathrm{td}(0),$$
$$\mathrm{ne}(1) = \mathrm{td}(1) - 1, \quad \mathrm{ne}(2) = \mathrm{td}(2) - 1;$$

and u_{-1}, u_{-2}, u_0 span rationally in degrees -1, -2, 0 respectively;

(Gc2) the image of A_0 in $A/I(u)$ is a finitely generated $\mathcal{O}$-module;

(Gc3) for every point $\varphi_1 \neq \varphi_0$ on u that is both negative-vanishing and positive-vanishing, either no point on u that is negative-vanishing but not positive-vanishing agrees with φ_1 on A_0, or no point on u that is positive-vanishing but not negative-vanishing agrees with φ_1 on A_0;

(Gc4) for every two-signed point φ on u, any two φ-derivations that vanish on $I(u)$ are linearly dependent over $\overline{\mathbb{Q}}_\ell$;

(Gc5) $a \in A_0$, $\varphi_0(a) = 0$, and $\varphi(a) \neq 0$ for every two-signed point φ on u.

Lemma 41.2. *Let $\mathcal{O}$ be a coefficient ring, (A, φ_0) a graded $\mathcal{O}$-algebra with a fixed point, u a system of equations for (A, φ_0) and $a \in A$, such that (A, φ_0, u, a) satisfies (Gc1)–(Gc5). Then u has an orbit cover, the cotangent length $\mathrm{ct}_{-1}(u_{-1})$ is finite, and for every orbit cover Σ of u,*

$$\mathrm{ct}_{-1}(u_{-1}) \leq 2 \sum_{\varphi \in \Sigma} v_{\mathcal{O}}(\varphi(a)).$$

Proof. Let td be as in (Gc1), $n_j = \mathrm{td}(j)$ for $j \in \{-1, 1, 2\}$, $K_0 = \mathcal{O}[\ell^{-1}]$ and ϖ a uniformiser of $\mathcal{O}$. Since $\mathrm{ne}(1) = \mathrm{td}(1) - 1$ and $\mathrm{ne}(2) = \mathrm{td}(2) - 1$ are natural numbers, $n_1, n_2 \geq 1$.

Step 1: orbit covers exist. By (Gc2) and Lemma 41.3, item 4, u has an orbit cover. For every orbit cover Σ and $\varphi \in \Sigma$, $\varphi(a) \neq 0$ by (Gc5), so $v_{\mathcal{O}}(\varphi(a))$ is defined, and it is ≥ 0 by Lemma 41.3, item 1, as $a \in A_0$.

The case $n_{-1} = 0$. Let ϕ be a chart with tangent dimensions td; no variable has weight -1. For $f \in A_{-1}$, $\phi(f)$ has weight -1 by (Sm1), so its coefficients of degree one, which are coefficients of variables of weight -1, vanish; and $\varphi_0(f) = 0$. By (Sm4), $f \in \mathrm{Sq}(\varphi_0)$. Hence $\mathrm{Cot}_{-1} = 0$, $\mathrm{ct}_{-1}(u_{-1}) = 0$, and the inequality holds for every orbit cover by Step 1. From now on $n_{-1} \geq 1$.

Step 2: reduction. Let J' be a finite set with $\mathrm{wt}'\colon J' \to \{-1, 1, 2\}$ and $\#\mathrm{wt}'^{-1}(j) = \mathrm{td}(j)$, and $P' = K_0[[X_i : i \in J']]$. By Lemma 41.4 there are $A' \subseteq P'$ and a surjection $\pi\colon A \to A'$ with (Rc1)–(Rc3); let φ_0' and u' be as there. For $g \in P'$ write $g^{(1)}$ for its linear part. For $b \in A_0'$ and $g \in A_{-1}'$, $(bg)^{(1)} = \varphi_0'(b)g^{(1)}$, since g has zero constant term; and an element of $\mathrm{Sq}(\varphi_0')$ has zero linear part (if $cg \in \mathfrak{m}_0'^2$, $\mathfrak{m}_0' = \ker \varphi_0'$, then $cg^{(1)} = 0$, as the linear part of a product of two series

without constant term is 0). By (Sm4) for the inclusion, the map sending the class of g to $g^{(1)}$ is therefore an isomorphism of Cot'_{-1} (the cotangent lattice of (A', φ'_0) in degree -1) onto

$$\mathrm{Lp}_{-1} = \{g^{(1)} : g \in A'_{-1}\},$$

and it maps $\mathrm{Cot}(u'_{-1})$ onto the $\mathcal{O}$-span $\mathrm{Lp}(u'_{-1})$ of the $u'^{(1)}_{-1,k}$, $1 \le k \le n_{-1}+1$; as $u'_{-1,k} \in A'_{-1}$, $\mathrm{Cot}'_{-1} \cap \mathrm{Cot}(u'_{-1}) = \mathrm{Cot}(u'_{-1})$. By (Rc2), $\mathrm{ct}_{-1}(u_{-1}) = \mathrm{length}_{\mathcal{O}}(\mathrm{Lp}_{-1}/\mathrm{Lp}(u'_{-1}))$.

Step 3: Lp_{-1} and $\mathrm{Lp}(u'_{-1})$ are free of rank n_{-1}. By (Rc1), A' is generated as an $\mathcal{O}$-algebra by A'_0 and finitely many homogeneous elements $s_1, \dots, s_q$ of non-zero degrees $d_1, \dots, d_q$; so A'_{-1} is spanned over A'_0 by the monomials $s^n = \prod_k s_k^{n_k}$ with $n \in \mathcal{E}_{-1} = \{n \in \mathbb{N}^q : \sum_k n_k d_k = -1\}$. By Dickson's lemma, $\mathcal{E}_{-1}$ has finitely many minimal elements for the componentwise order; for $n \in \mathcal{E}_{-1}$ choose a minimal $n' \le n$; then $\sum_k (n_k - n'_k)d_k = 0$, so $s^n = s^{n'}s^{n-n'} \in s^{n'}A'_0$. Hence A'_{-1} is a finitely generated A'_0-module, and by the product rule Lp_{-1} is the $\mathcal{O}$-span of the linear parts of finitely many generators: a finitely generated $\mathcal{O}$-submodule of the K_0-span V of $\{X_i : \mathrm{wt}'(i) = -1\}$, which has dimension n_{-1}. Being torsion free, Lp_{-1} is free. For i with $\mathrm{wt}'(i) = -1$, (Sm3) gives $f \in A'$ with $f^{(1)} = cX_i$, $c \ne 0$, and the weight -1 part $f_{-1} \in A'_{-1}$ of f has $f^{(1)}_{-1} = cX_i$ (linear terms of weight -1 come from the weight -1 part); so Lp_{-1} spans V and has rank n_{-1}. By (Rc2), u'_{-1} spans rationally in degree -1, so for every $\mathbf{v} \in \mathrm{Lp}_{-1}$ some $c\mathbf{v}$, $c \ne 0$, lies in $\mathrm{Lp}(u'_{-1})$; thus $\mathrm{Lp}(u'_{-1})$ has rank n_{-1} as well.

Step 4: Smith form; finiteness of ct_{-1}. By the elementary divisor theorem over the principal ideal domain $\mathcal{O}$ there are a basis $\mathbf{v}_1, \dots, \mathbf{v}_{n_{-1}}$ of Lp_{-1} and integers $c_1, \dots, c_{n_{-1}} \ge 0$ such that $\varpi^{c_1}\mathbf{v}_1, \dots, \varpi^{c_{n_{-1}}}\mathbf{v}_{n_{-1}}$ is a basis of $\mathrm{Lp}(u'_{-1})$. Then $\mathrm{Lp}_{-1}/\mathrm{Lp}(u'_{-1}) \cong \bigoplus_i \mathcal{O}/\varpi^{c_i}\mathcal{O}$, so

$$\mathrm{ct}_{-1}(u_{-1}) = \sum_{i=1}^{n_{-1}} c_i < \infty.$$

The $\mathcal{O}$-linear map $\mathrm{ev}\colon \mathcal{O}^{n_{-1}+1} \to \mathrm{Lp}(u'_{-1})$, $\mathbf{e}_k \mapsto u'^{(1)}_{-1,k}$ (standard basis $\mathbf{e}_k$), is onto. Choose $w_i \in \mathcal{O}^{n_{-1}+1}$ with $\mathrm{ev}(w_i) = \varpi^{c_i}\mathbf{v}_i$ ($1 \le i \le n_{-1}$); the w_i span a submodule mapped isomorphically onto $\mathrm{Lp}(u'_{-1})$, so $\mathcal{O}^{n_{-1}+1} = \bigoplus_{i \le n_{-1}} \mathcal{O}w_i \oplus \ker \mathrm{ev}$, and $\ker \mathrm{ev}$ is free of rank 1; let $w_{n_{-1}+1}$ be a generator. Then $w_1, \dots, w_{n_{-1}+1}$ is a basis of $\mathcal{O}^{n_{-1}+1}$. Put $\tilde{u}_{-1,i} = \sum_k (w_i)_k u'_{-1,k} \in A'_{-1}$, and let $\tilde{u}$ be u' with u'_{-1} replaced by $\tilde{u}_{-1}$. The $\mathcal{O}$-spans of $\tilde{u}_{-1}$ and u'_{-1} agree, the matrix with columns w_i being invertible; hence $I(\tilde{u}) = I(u')$, $\mathrm{Cot}(\tilde{u}_{-1}) = \mathrm{Cot}(u'_{-1})$, $\tilde{u}_{-1}$ spans rationally in degree -1, and u' and $\tilde{u}$ have the same points, two-signed points and orbit covers; so $(A', \varphi'_0, \tilde{u}, \pi(a))$ satisfies (Gc2)–(Gc5) by (Rc2). Choose $y_i \in A'_{-1}$ with $y_i^{(1)} = \mathbf{v}_i$. Then $\tilde{u}_{-1,i} - \varpi^{c_i} y_i$ ($i \le n_{-1}$) and $\tilde{u}_{-1,n_{-1}+1}$ have zero linear part, as $\tilde{u}^{(1)}_{-1,i} = \mathrm{ev}(w_i)$, and zero constant term (degree -1); by (Sm4) they lie in $\mathrm{Sq}(\varphi'_0)$.

Step 5: the weight-zero element. If $\pi(a) \ne 0$, put $a' = \pi(a)$. If $\pi(a) = 0$, then by (Gc5) for $(A', \varphi'_0, u', \pi(a))$ there is no two-signed point on u'. In that case choose i_0 with $\mathrm{wt}'(i_0) = 1$ and, by (Sm3), $f \in A'$ with $f^{(1)} = cX_{i_0}$, $c \ne 0$; its weight-one part $g \in A'_1$ is non-zero. Also $\tilde{u}_{-1,1} \ne 0$, its linear part being $\varpi^{c_1}\mathbf{v}_1 \ne 0$. Put $a' = \tilde{u}_{-1,1}g \in A'_0$; it is non-zero ($P'$ is a domain), $\varphi'_0(a') = 0$, and (Gc5) holds for a' because there are no two-signed points on $\tilde{u}$. In both cases $a' \ne 0$, $(A', \varphi'_0, \tilde{u}, a')$ satisfies (Gc2)–(Gc5), and $\varphi'(a') = \varphi'(\pi(a))$ for every two-signed point φ' on $\tilde{u}$ (there are none in the second case).

Step 6: the count. The data $(A' \subseteq P', \tilde{u}, a', y, c)$ satisfy the hypotheses of Lemma 41.7: $A' = \bigoplus_j A'_j$ is a graded $\mathcal{O}$-algebra with a fixed point and the inclusion satisfies (Sm1)–(Sm4) by (Rc1); $\#\mathrm{wt}'^{-1}(j) = n_j \ge 1$ for $j \in \{-1, 1, 2\}$; the counts of $\tilde{u}$ are $n_{-1}+1$, $n_1 - 1$, $n_2 - 1$ in degrees -1, 1, 2 and 0 in degrees -2, 0; $\tilde{u}_{-1}$ spans rationally in degree -1; and Steps 4 and 5 give the rest. Let Σ be an orbit cover of u. By (Rc3), $\Sigma' = \{\varphi' : \varphi' \circ \pi \in \Sigma\}$ is an orbit cover of

u', hence of $\tilde{u}$. By Steps 4 and 5 above, Lemma 41.7 and (Rc3),

$$\mathrm{ct}_{-1}(u_{-1}) = \sum_{i=1}^{n_{-1}} c_i \le 2 \sum_{\varphi' \in \Sigma'} v_{\mathcal{O}}(\varphi'(a')) = 2 \sum_{\varphi' \in \Sigma'} v_{\mathcal{O}}(\varphi'(\pi(a))) \le 2 \sum_{\varphi \in \Sigma} v_{\mathcal{O}}(\varphi(a)).$$

With Steps 1 and 4 this proves the three assertions. □

Lemma 41.3. *Let $\mathcal{O}$ be a coefficient ring, (A, φ_0) a graded $\mathcal{O}$-algebra with a fixed point and u a system of equations for (A, φ_0) such that the image of A_0 in $A/I(u)$ is a finitely generated $\mathcal{O}$-module (condition (Gc2)). Then:*

1. *for every point φ on u and every $b \in A_0$, $\varphi(b)$ is integral over $\mathcal{O}$; in particular $v_{\mathcal{O}}(\varphi(b)) \ge 0$ if $\varphi(b) \neq 0$;*
2. *the set of restrictions $\varphi|_{A_0}$ of the points φ on u is finite;*
3. *if φ, φ' are two-signed points with $\varphi|_{A_0} = \varphi'|_{A_0}$, then $\varphi' = t \cdot \varphi$ for some $t \in \overline{\mathbb{Q}}_\ell^\times$;*
4. *u has an orbit cover.*

Proof. Let B be the image of A_0 in $A/I(u)$, and write $\bar{b} \in B$ for the image of $b \in A_0$. A point φ on u vanishes on $I(u)$, so it induces an $\mathcal{O}$-algebra homomorphism $\bar{\varphi}\colon B \to \overline{\mathbb{Q}}_\ell$ with $\varphi(b) = \bar{\varphi}(\bar{b})$ for $b \in A_0$. By hypothesis B is a finitely generated $\mathcal{O}$-module.

Item 1. Let $b \in A_0$ and let $g_1, \dots, g_n$ generate B as an $\mathcal{O}$-module. Write $\bar{b} g_k = \sum_l o_{kl} g_l$ with $o_{kl} \in \mathcal{O}$; with the matrix $\mathbf{M} = (o_{kl})$, the determinant trick (multiply the system $\sum_l (\bar{b}\delta_{kl} - o_{kl}) g_l = 0$ by the adjugate of $\bar{b} \cdot 1 - \mathbf{M}$) gives $\det(\bar{b} \cdot 1 - \mathbf{M}) g_l = 0$ for all l, hence $\det(\bar{b} \cdot 1 - \mathbf{M}) = 0$ in B, since $1 \in B$ is an $\mathcal{O}$-combination of the g_l. So $\bar{b}$ is a root of the monic polynomial $\det(T \cdot 1 - \mathbf{M}) \in \mathcal{O}[T]$, and so is $\varphi(b) = \bar{\varphi}(\bar{b})$. If $z = \varphi(b) \neq 0$ satisfied $v_{\mathcal{O}}(z) < 0$, then in $z^n + o'_{n-1} z^{n-1} + \cdots + o'_0 = 0$ ($o'_k \in \mathcal{O}$) the term z^n would have valuation $n v_{\mathcal{O}}(z)$, strictly smaller than that of every other non-zero term, which is impossible for a vanishing sum. Hence $v_{\mathcal{O}}(\varphi(b)) \ge 0$.

Item 2. The fraction field of $\mathcal{O}$ is $\mathcal{O}[\ell^{-1}]$, and $B[\ell^{-1}] = B \otimes_{\mathcal{O}} \mathcal{O}[\ell^{-1}]$ is a commutative $\mathcal{O}[\ell^{-1}]$-algebra of dimension at most n. If $c \in \mathcal{O} \setminus \{0\}$ and $cg = 0$ for some $g \in B$, then $c\bar{\varphi}(g) = 0$, so $\bar{\varphi}(g) = 0$; hence $\bar{\varphi}$ is the composite of $B \to B[\ell^{-1}]$ with the $\mathcal{O}[\ell^{-1}]$-linear extension $\bar{\varphi}'\colon B[\ell^{-1}] \to \overline{\mathbb{Q}}_\ell$, an algebra homomorphism, and $\bar{\varphi}$ is determined by $\bar{\varphi}'$. Distinct algebra homomorphisms $B[\ell^{-1}] \to \overline{\mathbb{Q}}_\ell$ are distinct multiplicative maps sending 1 to 1, hence linearly independent over $\overline{\mathbb{Q}}_\ell$ (Dedekind's argument: in a shortest non-trivial relation $\sum_k c_k \vartheta_k = 0$, choose g with $\vartheta_1(g) \neq \vartheta_2(g)$ and subtract $\vartheta_1(g)$ times the relation from the relation evaluated at gg', g' variable, to get a shorter one); they lie in $\mathrm{Hom}_{\mathcal{O}[\ell^{-1}]}(B[\ell^{-1}], \overline{\mathbb{Q}}_\ell)$, a $\overline{\mathbb{Q}}_\ell$-vector space of dimension at most n. So there are at most n of them, and the restrictions $\varphi|_{A_0}$ of the points on u, which are determined by $\bar{\varphi}$, form a set with at most n elements.

Item 3. For a point φ put $S_\varphi = \{m \in \mathbb{Z} : \varphi(A_m) \neq 0\}$. Let φ be two-signed. Then $0 \in S_\varphi$ ($\varphi(1) = 1$), and S_φ is closed under addition: if $f \in A_m$, $g \in A_{m'}$ with $\varphi(f)\varphi(g) \neq 0$, then $fg \in A_{m+m'}$ and $\varphi(fg) \neq 0$. Since φ is neither negative- nor positive-vanishing, S_φ contains some $m_- < 0$ and some $m_+ > 0$. For $m \in S_\varphi$ with $m > 0$,

$$-m = (-m_- - 1)\, m + m\, m_-$$

is a sum of $-m_- - 1 \ge 0$ copies of m and $m \ge 1$ copies of m_-, so $-m \in S_\varphi$; symmetrically, using m_+, $-m \in S_\varphi$ for $m < 0$. Hence S_φ is a subgroup of $\mathbb{Z}$, say $S_\varphi = d\mathbb{Z}$ with $d \ge 1$.

Now let φ, φ' be two-signed with the same restriction to A_0. Let $m \in S_\varphi$; choose $f_m \in A_m$ and $g_m \in A_{-m}$ with $\varphi(f_m) \neq 0 \neq \varphi(g_m)$. For every $f \in A_m$, $fg_m \in A_0$, so $\varphi'(f)\varphi'(g_m) = \varphi(f)\varphi(g_m)$. For $f = f_m$ the right side is non-zero, so $\varphi'(g_m) \neq 0$, and with $z_m = \varphi(g_m)/\varphi'(g_m) \in \overline{\mathbb{Q}}_\ell^\times$,

$$\varphi'(f) = z_m\, \varphi(f) \qquad (f \in A_m).$$

In particular $\varphi'(A_m) \neq 0$, so $S_\varphi \subseteq S_{\varphi'}$, and by symmetry $S_{\varphi'} = S_\varphi$. The number z_m does not depend on the choice of g_m, being $\varphi'(f_m)/\varphi(f_m)$. For $m, m' \in S_\varphi$, applying the display to $f_m f_{m'} \in A_{m+m'}$ gives $z_{m+m'}\varphi(f_m)\varphi(f_{m'}) = \varphi'(f_m)\varphi'(f_{m'}) = z_m z_{m'}\varphi(f_m)\varphi(f_{m'})$, so $z_{m+m'} = z_m z_{m'}$ and $z_{kd} = z_d^k$ for all $k \in \mathbb{Z}$. Since $\overline{\mathbb{Q}}_\ell$ is algebraically closed, choose t with $t^d = z_d$; then $z_m = t^m$ for $m \in S_\varphi$. Hence $\varphi'(f) = t^m\varphi(f)$ for $f \in A_m$, $m \in S_\varphi$, and for $m \notin S_\varphi = S_{\varphi'}$ both $\varphi'(f)$ and $t^m\varphi(f)$ vanish. As $A = \bigoplus_m A_m$ and both sides are additive, $\varphi' = t \cdot \varphi$.

Item 4. By item 2 the set P_0 of restrictions $\varphi|_{A_0}$ of two-signed points φ on u is finite. Choose a set Σ of two-signed points on u containing, for each member of P_0, exactly one point with that restriction; Σ is finite. If φ is a two-signed point on u, then $\varphi|_{A_0} \in P_0$, so some $\varphi_1 \in \Sigma$ has $\varphi_1|_{A_0} = \varphi|_{A_0}$, and by item 3, $\varphi = t \cdot \varphi_1$ for some t. So Σ is an orbit cover of u. □

Lemma 41.4. *Let $\mathcal{O}$ be a coefficient ring and (A, φ_0, u, a) a tuple satisfying (Gc1)–(Gc5), with tangent dimensions* td *as in (Gc1). Let J' be a finite set with a map* $\mathrm{wt}'\colon J' \to \{-1, 1, 2\}$ *such that* $\#\mathrm{wt}'^{-1}(j) = \mathrm{td}(j)$ *for $j \in \{-1, 1, 2\}$, and $P' = \mathcal{O}[\ell^{-1}][[X_i : i \in J']]$ with the weights of power series defined by* wt'. *Then there are an $\mathcal{O}$-subalgebra $A' \subseteq P'$ and a surjective $\mathcal{O}$-algebra homomorphism $\pi\colon A \to A'$ with the following properties, where A'_j is the set of elements of A' of weight j, $\varphi'_0\colon A' \to \mathcal{O}[\ell^{-1}]$ is the constant term, and u' is the system of equations for (A', φ'_0) with $u'_j = (\pi(u_{j,i}))_i$ for $j \in \{-1, 1, 2\}$ and no equations of degrees -2 and 0:*

(Rc1) *$\pi(A_j) = A'_j$ for all j, $A' = \bigoplus_j A'_j$, $\varphi'_0 \circ \pi = \varphi_0$, (A', φ'_0) is a graded $\mathcal{O}$-algebra with a fixed point, and the inclusion $A' \to P'$ is a chart for (A', φ'_0), i.e. satisfies (Sm1)–(Sm4);*

(Rc2) *$I(u') = \pi(I(u))$, the tuple $(A', \varphi'_0, u', \pi(a))$ satisfies (Gc1)–(Gc5), and the cotangent length of u'_{-1} in degree -1 computed in (A', φ'_0) equals* $\mathrm{ct}_{-1}(u_{-1})$*;*

(Rc3) *for every orbit cover Σ of u, the set Σ' of points φ' of A' with $\varphi' \circ \pi \in \Sigma$ is an orbit cover of u', and*

$$\sum_{\varphi' \in \Sigma'} v_{\mathcal{O}}(\varphi'(\pi(a))) \leq \sum_{\varphi \in \Sigma} v_{\mathcal{O}}(\varphi(a)).$$

Proof. Write $K_0 = \mathcal{O}[\ell^{-1}]$ for the fraction field of $\mathcal{O}$ and $\mathfrak{m}_0 = \ker \varphi_0$. By (Gc1) there are a finite set J, $\mathrm{wt}\colon J \to \mathbb{Z}$ with $\#\mathrm{wt}^{-1}(j) = \mathrm{td}(j)$, and a chart $\phi\colon A \to P = K_0[[X_i : i \in J]]$; as $\mathrm{td}(j) = 0$ for $|j| > 2$, wt takes values in $\{-2, \ldots, 2\}$. Replacing J' by a set in bijection with it, we take $J' = \mathrm{wt}^{-1}(\{-1, 1, 2\}) \subseteq J$ with $\mathrm{wt}' = \mathrm{wt}|_{J'}$, so that $P' \subseteq P$; put $J_e = J \setminus J' = \mathrm{wt}^{-1}(\{0, -2\})$. For $g \in P$ let $g^{(1)}$ be its linear part (the sum of its terms of total degree one), and for $f \in A$ write $f^{(1)} = \phi(f)^{(1)}$.

Step 1: linear parts. (a) For $f, g \in A$, by (Sm2), $(fg)^{(1)} = \varphi_0(f)g^{(1)} + \varphi_0(g)f^{(1)}$; so $f^{(1)} = 0$ for $f \in \mathfrak{m}_0^2$, and for $f \in \mathrm{Sq}(\varphi_0)$, $cf \in \mathfrak{m}_0^2$ with $c \neq 0$ gives $cf^{(1)} = 0$, so $f^{(1)} = 0$. (b) If $f \in A_j$, $\phi(f)$ has weight j by (Sm1), so $f^{(1)}$ is a K_0-combination of the X_i with $\mathrm{wt}(i) = j$. (c) For each j, the map $\mathrm{Cot}_j \to P$, class of $f \mapsto f^{(1)}$, is well defined by (a) and injective by (Sm4); hence for $f \in A_j \cap \mathfrak{m}_0$ and a family w in A, the class of f lies in $\mathrm{Cot}(w)$ if and only if $f^{(1)}$ lies in the $\mathcal{O}$-span of the $w_k^{(1)}$. (d) For $j \in \{0, -2\}$ the linear parts of the members of u_j form a basis of the K_0-span V_j of $\{X_i : \mathrm{wt}(i) = j\}$. Indeed, let $\mathrm{wt}(i) = j$. By (Sm3) there is $f \in A$ with $f^{(1)} = cX_i$, $c \neq 0$. Writing $f = \sum_m f_m$ with $f_m \in A_m$, $\phi(f_m)$ is the weight-m part of $\phi(f)$, so by (b) $f_j^{(1)} = cX_i$;

put $f' = f_j - \varphi_0(f_j) \in A_j \cap \mathfrak{m}_0$ (for $j \neq 0$, $\varphi_0(f_j) = 0$), with $f'^{(1)} = cX_i$ as $1^{(1)} = 0$. Since u_j spans rationally in degree j, by (c) some $c'X_i$, $c' \neq 0$, is an $\mathcal{O}$-combination of the $u_{j,k}^{(1)}$. Thus these linear parts, which lie in V_j by (b), span V_j; there are $\mathrm{ne}(j) = \mathrm{td}(j) = \dim V_j$ of them.

Step 2: elimination of the variables of weight 0 *and* -2. Let $f_1, \dots, f_n$ be the elements $\phi(u_{0,k})$ and $\phi(u_{-2,k})$, so $n = \#J_e$; each f_k has a weight $w_k \in \{0, -2\}$, zero constant term (by (Sm2), as $\varphi_0(u_{0,k}) = 0$ and $\varphi_0(A_{-2}) = 0$), and by (b) and (d) $f_k^{(1)} = \sum_{i \in J_e} m_{ki} X_i$ with $\mathbf{M} = (m_{ki})$ invertible and $m_{ki} = 0$ unless $\mathrm{wt}(i) = w_k$, i.e. $\mathbf{M}$ is block diagonal for the weights. Write $f_k = f_k^{(1)} + R_k$, where R_k has only terms of total degree at least 2 and weight w_k. We claim there are unique $p_i \in P'$ ($i \in J_e$) with zero constant term and $f_k(X_{J'}, p) = 0$ for all k, where $(X_{J'}, p)$ means the substitution $X_i \mapsto p_i$ for $i \in J_e$; and p_i has weight $\mathrm{wt}(i)$. The equations are equivalent to $p = -\mathbf{M}^{-1} R(X_{J'}, p)$. Write $p_i = \sum_{d \geq 1} p_i^{[d]}$ with $p_i^{[d]}$ homogeneous of total degree d. Every term of R_k has total degree at least 2, and each p_i has no constant term, so the part of total degree d of $R_k(X_{J'}, p)$ depends only on $p^{[1]}, \dots, p^{[d-1]}$; hence the equations determine $p^{[d]}$ recursively, $p^{[d]} = -\mathbf{M}^{-1}(\text{part of degree } d \text{ of } R(X_{J'}, p^{[1]} + \cdots + p^{[d-1]}))$, which proves existence and uniqueness. By induction on d, if $p^{[1]}, \dots, p^{[d-1]}$ have the weights $\mathrm{wt}(i)$, then $R_k(X_{J'}, p^{[<d]})$ has weight w_k (each monomial of weight w_k becomes a product of homogeneous series of total weight w_k), and so has its part of degree d; as $\mathbf{M}^{-1}$ is block diagonal, $p_i^{[d]}$ is a combination of such parts with $w_k = \mathrm{wt}(i)$ and has weight $\mathrm{wt}(i)$. Since no variable of J' has weight 0 or -2, $p_i^{(1)} = 0$.

Let $\mathrm{Sb}\colon P \to P'$ be the substitution homomorphism $X_i \mapsto X_i$ ($i \in J'$), $X_i \mapsto p_i$ ($i \in J_e$), well defined and continuous as the p_i have no constant term. It preserves weights, kills $f_1, \dots, f_n$, and for $g \in P$ the constant term of $\mathrm{Sb}(g)$ is that of g and $\mathrm{Sb}(g)^{(1)}$ is $g^{(1)}$ with the terms in X_i, $i \in J_e$, deleted (terms of degree at least 2 and the p_i contribute no linear terms). Put $A' = \mathrm{Sb}(\phi(A)) \subseteq P'$ and let $\pi\colon A \to A'$ be the surjection $f \mapsto \mathrm{Sb}(\phi(f))$. Then $\pi(u_{0,k}) = \pi(u_{-2,k}) = 0$, and for $f \in A$, $\pi(f)$ has the same constant term as $\phi(f)$ and the linear part $f^{(1)}$ with the J_e-terms deleted.

Step 3: (Rc1). Each $\pi(A_j)$ consists of series of weight j; as weight components of a series are unique, $A' = \sum_j \pi(A_j)$ is direct and $A' \cap \{\text{weight } j\} = \pi(A_j)$, so $A'_j = \pi(A_j)$. The constant term gives $\varphi'_0 \circ \pi = \varphi_0$ by (Sm2), so φ'_0 takes values in $\mathcal{O}$ and kills A'_j for $j \neq 0$. The ring $A'_0 = \pi(A_0)$ is a quotient of the complete Noetherian local ring A_0, hence complete Noetherian local, with maximal ideal the image of that of A_0 and the same residue field, so $\mathcal{O} \to A'_0$ is local and induces an isomorphism of residue fields; A' is generated by A'_0 and the images of the homogeneous generators of A. So (A', φ'_0) is a graded $\mathcal{O}$-algebra with a fixed point. For the inclusion $A' \to P'$: (Sm1) holds by $A'_j = \pi(A_j)$; (Sm2) by definition of φ'_0; (Sm3): for $i \in J'$, take f from (Sm3) for ϕ; then $\pi(f)^{(1)} = cX_i$. (Sm4): let $g = \pi(f) \in A'$ with $\varphi'_0(g) = 0$ and $g^{(1)} = 0$, $f = \sum_m f_m$, $f_m \in A_m$; the weight-m part $\pi(f_m)$ of g has zero linear part. For $m \notin \{0, -2\}$, $f_m^{(1)}$ involves only variables of weight m, all in J', so $f_m^{(1)} = \pi(f_m)^{(1)} = 0$, $\varphi_0(f_m) = 0$, and $f_m \in \mathrm{Sq}(\varphi_0)$ by (Sm4) for ϕ. Also $\varphi_0(f_0) = \varphi'_0(g) = 0$, so $f_0 \in A_0 \cap \mathfrak{m}_0$, and $f_{-2} \in A_{-2} \cap \mathfrak{m}_0$; as u_0, u_{-2} span rationally, there are $c \in \mathcal{O} \setminus \{0\}$ and $o_{m,k} \in \mathcal{O}$ with $cf_m - \sum_k o_{m,k} u_{m,k} \in \mathrm{Sq}(\varphi_0)$ for $m = 0, -2$. Hence $cf \in \mathrm{Sq}(\varphi_0) + \sum_{m \in \{0,-2\}, k} \mathcal{O} u_{m,k}$. Since $\pi(\mathfrak{m}_0) \subseteq \mathfrak{m}'_0 = \ker \varphi'_0$, $\pi(\mathrm{Sq}(\varphi_0)) \subseteq \mathrm{Sq}(\varphi'_0)$, and $\pi(u_{m,k}) = 0$; so $cg \in \mathrm{Sq}(\varphi'_0)$, i.e. $c'cg \in \mathfrak{m}'^2_0$ for some $c' \neq 0$, and $g \in \mathrm{Sq}(\varphi'_0)$.

Step 4: (Rc2). The set $\pi(I(u))$ is the ideal of A' generated by the $\pi(u_{j,i})$ (π is surjective), and the generators of degrees 0 and -2 map to 0; so $\pi(I(u)) = I(u')$. (Gc1): the inclusion is a chart with tangent dimensions td', $\mathrm{td}'(j) = \mathrm{td}(j)$ for $j \in \{-1, 1, 2\}$ and 0 otherwise; the counts of u' are $\mathrm{ne}(j)$ for $j \in \{-1, 1, 2\}$ and 0 for $j \in \{-2, 0\}$, which are the required ones. No variable of J' has weight 0 or -2, so every element of $A'_0 \cap \mathfrak{m}'_0$ or of A'_{-2} has zero linear part

and lies in $\mathrm{Sq}(\varphi'_0)$ by (Sm4): $\mathrm{Cot}'_0 = \mathrm{Cot}'_{-2} = 0$, and the empty families span rationally. For $f \in A_{-1}$, all variables of weight -1 lie in J', so $\pi(f)^{(1)} = f^{(1)}$. By (c), applied to ϕ and to the inclusion of A', the maps sending the class of f to $f^{(1)}$ identify Cot_{-1} and Cot'_{-1} with the same $\mathcal{O}$-module $\mathrm{Lp}_{-1} = \{f^{(1)} : f \in A_{-1}\}$, and $\mathrm{Cot}(u_{-1})$, $\mathrm{Cot}(u'_{-1})$ with the same submodule, the $\mathcal{O}$-span of the $u^{(1)}_{-1,i}$ (as $u_{-1,i} \in A_{-1} \subseteq \mathfrak{m}_0$, $\mathrm{Cot}_{-1} \cap \mathrm{Cot}(u_{-1}) = \mathrm{Cot}(u_{-1})$, and likewise for u'). Hence u'_{-1} spans rationally in degree -1 and the two cotangent lengths in degree -1 are equal. (Gc2): $A'/I(u') \cong A/(I(u) + \ker\pi)$, so the image of A'_0 in it is a quotient of the image of A_0 in $A/I(u)$, a finitely generated $\mathcal{O}$-module. For the remaining conditions, $\varphi' \mapsto \varphi' \circ \pi$ is an injection of the points of A' into the points of A; since $\pi(A_j) = A'_j$, $\varphi' \circ \pi$ is negative-vanishing, positive-vanishing or two-signed exactly when φ' is, it lies on u exactly when φ' lies on u' (as $\pi(I(u)) = I(u')$), and two points of A' agree on A'_0 exactly when their composites with π agree on A_0. (Gc3): if $\varphi'_1 \neq \varphi'_0$ is a point on u' that is negative- and positive-vanishing, then $\varphi'_1 \circ \pi \neq \varphi_0$ is such a point of A on u; (Gc3) for A excludes one of the two kinds of points agreeing with it on A_0, and composing with π shows that the same kind is excluded for φ'_1. (Gc4): if D, D' are φ'-derivations of A' vanishing on $I(u')$ at a two-signed point φ' on u', then $D \circ \pi$, $D' \circ \pi$ are $(\varphi' \circ \pi)$-derivations of A vanishing on $I(u)$; by (Gc4) for A, $c_1 D \circ \pi + c_2 D' \circ \pi = 0$ with $(c_1, c_2) \neq (0, 0)$ in $\overline{\mathbb{Q}}_\ell$, and $c_1 D + c_2 D' = 0$ as π is onto. (Gc5): $\pi(a) \in A'_0$, $\varphi'_0(\pi(a)) = \varphi_0(a) = 0$, and $\varphi'(\pi(a)) = (\varphi' \circ \pi)(a) \neq 0$ at every two-signed point φ' on u'.

Step 5: (Rc3). By Step 4 the members of Σ' are two-signed points on u', and Σ' is finite since $\varphi' \mapsto \varphi' \circ \pi$ is injective. Let φ' be a two-signed point on u'. Then $\varphi' \circ \pi = t \cdot \varphi$ with $\varphi \in \Sigma$. The ideal $\ker\pi$ is spanned by homogeneous elements: if $\pi(f) = 0$, each weight part $\pi(f_m)$ of $\pi(f)$ vanishes. For homogeneous $f \in A_m \cap \ker\pi$, $\varphi(f) = t^{-m}(t \cdot \varphi)(f) = t^{-m}\varphi'(\pi(f)) = 0$; so φ vanishes on $\ker\pi$, and $\varphi = \varphi'' \circ \pi$ for a point φ'' of A', which lies in Σ'. For $f \in A_m$, $\varphi'(\pi(f)) = t^m\varphi(f) = t^m\varphi''(\pi(f))$, and $\pi(A_m) = A'_m$, so $\varphi' = t \cdot \varphi''$. Thus Σ' is an orbit cover of u'. Finally $\varphi' \mapsto \varphi' \circ \pi$ maps Σ' injectively into Σ with $\varphi'(\pi(a)) = (\varphi' \circ \pi)(a)$, and every term $v_{\mathcal{O}}(\varphi(a))$, $\varphi \in \Sigma$, is defined ($\varphi(a) \neq 0$ by (Gc5)) and non-negative by Lemma 41.3, item 1, which applies by (Gc2); this gives the inequality. □

Definition 41.5. Let B be a Noetherian local ring with maximal ideal $\mathfrak{m}_B$, and let $k \geq 1$. On $\mathbb{Z}^k$ we use the componentwise partial order, and $e_0, \ldots, e_{k-1}$ are the unit vectors. A property of $m \in \mathbb{Z}^k$ holds **in large degree** if there is $m_0 \in \mathbb{Z}^k$ such that it holds for every $m \geq m_0$; finitely many properties that hold in large degree hold simultaneously in large degree (take the componentwise maximum of the thresholds). All rings are commutative.

Standard algebras. A **standard $\mathbb{N}^k$-graded B-algebra** is a B-algebra $S = \bigoplus_{\mu \in \mathbb{N}^k} S_\mu$ with $S_\mu S_{\mu'} \subseteq S_{\mu+\mu'}$ such that

- the structure map $B \to S$ maps B onto S_0;
- each S_{e_i} is a finitely generated B-module;
- $S_{\mu+e_i} = S_\mu S_{e_i}$ (the B-span of the products) for all $\mu \in \mathbb{N}^k$ and $0 \leq i < k$.

Its **irrelevant ideal** is $\mathrm{Irr}(S) = \prod_{i=0}^{k-1}(S_{e_i}S)$, the product of the ideals generated by the S_{e_i}.

Graded modules. A graded S-module is an S-module $F = \bigoplus_{m \in \mathbb{Z}^k} F_m$ with $S_\mu F_m \subseteq F_{\mu+m}$; each F_m is a B-module. For $d \in \mathbb{Z}^k$, $F(-d)$ is F with $F(-d)_m = F_{m-d}$. For a homogeneous ideal I of S, S/I is a graded S-module.

Koszul complexes. A **sequence of homogeneous elements** is a finite sequence $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_q)$ with $\sigma_j \in S_{d_j}$ for given degrees $d_j \in \mathbb{N}^k$ (the degrees are part of the data; $q = 0$ is allowed). Write $(\boldsymbol{\sigma})$ for the ideal it generates and, for $J \subseteq \{1, \ldots, q\}$, $d_J = \sum_{j \in J} d_j$. The

Koszul complex $\mathrm{Kos}_\bullet(\boldsymbol{\sigma}; F)$ of a graded S-module F is the complex of graded S-modules with

$$\mathrm{Kos}_p(\boldsymbol{\sigma}; F) = \bigoplus_{J\subseteq\{1,\dots,q\},\ \#J=p} F(-d_J)\,\epsilon_J,$$

$$\partial(f\,\epsilon_J) = \sum_{t=1}^{p}(-1)^{t+1}\sigma_{j_t} f\,\epsilon_{J\setminus\{j_t\}} \quad (J=\{j_1<\cdots<j_p\}),$$

whose differential preserves degrees. Its degree-m part $\mathrm{Kos}_\bullet(\boldsymbol{\sigma}; F)_m$ is a complex of B-modules, with homology $H_p(\boldsymbol{\sigma}; F)_m$. For $\boldsymbol{\sigma}$ and an element a of degree d, $(\boldsymbol{\sigma}, a)$ and $(a, \boldsymbol{\sigma})$ are the sequences with a appended at the end, respectively at the beginning.

Euler characteristics. The **Euler characteristic** of $\boldsymbol{\sigma}$ on F in degree m is

$$\mathrm{eu}(\boldsymbol{\sigma}; F)(m) = \sum_{p=0}^{q}(-1)^p \operatorname{length}_B H_p(\boldsymbol{\sigma}; F)_m, \tag{41.2}$$

and it is **defined** if all the lengths in (41.2) are finite. For $q=0$, $\mathrm{eu}(();F)(m) = \operatorname{length}_B F_m$. For a single element $a\in S_0$ (degree 0), $\mathrm{Kos}_\bullet((a);F)_m$ is $F_m \xrightarrow{a} F_m$, so

$$\mathrm{eu}((a);F)(m) = \operatorname{length}_B(F_m/aF_m) - \operatorname{length}_B(0:_{F_m} a),$$

the Herbrand quotient of a on F_m, defined if both lengths are finite.

Definition 41.6. Let $\mathcal{O}$ be a coefficient ring, (A,φ_0) a graded $\mathcal{O}$-algebra with a fixed point, and u a system of equations for (A,φ_0) with counts ne such that $\mathrm{ne}(-2)=\mathrm{ne}(0)=0$ and $\mathrm{ne}(-1)=n+1$ with $n\ge 1$. The **excess equation** of u is $w=u_{-1,n+1}$, the last member of u_{-1}. Each A_j is an A_0-submodule of A, as $A_0A_j\subseteq A_j$. Put

$$N_0 = A_{-2}, \qquad N_1 = A_2, \qquad N_2 = A_0w^2 + A_{-2}A_{-2}A_2,$$

A_0-submodules of A (products of submodules are A_0-spans of products; $N_2\subseteq A_{-2}$). For $\mu=(\mu_0,\mu_1,\mu_2)\in\mathbb{N}^3$ let $N^\mu = N_0^{\mu_0}N_1^{\mu_1}N_2^{\mu_2}$, with $N_i^0=A_0$. The **Rees algebra** of u is the A_0-subalgebra

$$\mathrm{Rs}(u) = \bigoplus_{\mu\in\mathbb{N}^3} N^\mu\, T_0^{\mu_0}T_1^{\mu_1}T_2^{\mu_2} \ \subseteq\ A[T_0,T_1,T_2]$$

of the polynomial ring in three variables over A, graded by $\mathbb{N}^3$ with $\mathrm{Rs}(u)_\mu = N^\mu T^\mu$; an element $b\in A_0$ is identified with $bT^0\in\mathrm{Rs}(u)_0$. The **sections** of u form the sequence of homogeneous elements $\boldsymbol{\sigma}(u)$ of $\mathrm{Rs}(u)$ consisting, in this order, of

- $u_{-1,i}^2T_0$ for $1\le i\le n$, of degree e_0 (as $u_{-1,i}^2\in A_{-2}=N_0$);
- $u_{1,i}^2T_1$ for $1\le i\le \mathrm{ne}(1)$, of degree e_1 (as $u_{1,i}^2\in A_2=N_1$);
- $u_{2,i}T_1$ for $1\le i\le \mathrm{ne}(2)$, of degree e_1;
- w^2T_2, of degree e_2 (as $w^2\in N_2$).

It has $n+\mathrm{ne}(1)+\mathrm{ne}(2)+1$ members.

Lemma 41.7. *Let $\mathcal{O}$ be a coefficient ring with uniformiser ϖ, J a finite set with a map* $\mathrm{wt}\colon J\to\{-1,1,2\}$ *such that $n_j=\#\mathrm{wt}^{-1}(j)\ge 1$ for $j\in\{-1,1,2\}$, and $P=\mathcal{O}[\ell^{-1}][[X_i : i\in J]]$ with the weights defined by* wt. *Let $A\subseteq P$ be an $\mathcal{O}$-subalgebra such that, with A_j the set of elements of A of weight j and φ_0 the constant term, $A=\bigoplus_j A_j$, (A,φ_0) is a graded $\mathcal{O}$-algebra with a*

fixed point, and the inclusion $A \to P$ satisfies (Sm1)–(Sm4). Let u be a system of equations for (A, φ_0) with counts $\mathrm{ne}(-1) = n_{-1}+1$, $\mathrm{ne}(1) = n_1 - 1$, $\mathrm{ne}(2) = n_2 - 1$, $\mathrm{ne}(-2) = \mathrm{ne}(0) = 0$, such that u_{-1} spans rationally in degree -1, and let $a \in A$ with $a \neq 0$ such that (A, φ_0, u, a) satisfies (Gc2)–(Gc5). Assume that there are $y_1, \dots, y_{n_{-1}} \in A_{-1}$ and integers $c_1, \dots, c_{n_{-1}} \geq 0$ with

$$u_{-1,i} - \varpi^{c_i} y_i \in \mathrm{Sq}(\varphi_0) \quad (1 \leq i \leq n_{-1}), \qquad u_{-1,n_{-1}+1} \in \mathrm{Sq}(\varphi_0).$$

Then for every orbit cover Σ of u,

$$\sum_{i=1}^{n_{-1}} c_i \leq 2 \sum_{\varphi \in \Sigma} v_{\mathcal{O}}(\varphi(a)).$$

Proof. Write $n = n_{-1}$, $\mathrm{Rs} = \mathrm{Rs}(u)$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(u)$; the definition applies since $\mathrm{ne}(-2) = \mathrm{ne}(0) = 0$ and $\mathrm{ne}(-1) = n+1$ with $n \geq 1$, and $\boldsymbol{\sigma}$ has $n + (n_1 - 1) + (n_2 - 1) + 1$ members. Let Σ be an orbit cover of u. Every term $v_{\mathcal{O}}(\varphi(a))$, $\varphi \in \Sigma$, is defined, as $\varphi(a) \neq 0$ by (Gc5).

Step 1: the setting. By Lemma 41.8, item 2, Rs is a standard $\mathbb{N}^3$-graded algebra over the Noetherian local ring A_0. The power series ring P over the field $\mathcal{O}[\ell^{-1}]$ is a domain, hence so is $A \subseteq P$, and by Lemma 41.8, item 3, the non-zero element $a \in A_0$ is a non-zerodivisor on Rs. Moreover $A_{-2} \subseteq \mathrm{Sq}(\varphi_0)$: for $f \in A_{-2}$, $\varphi_0(f) = 0$, and f has weight -2 by (Sm1) while no variable has weight -2, so all coefficients of degree one of f vanish and (Sm4) applies.

Step 2: admissibility. The hypotheses of Proposition 41.9 hold: the counts, rational spanning of u_{-1}, $A_{-2} \subseteq \mathrm{Sq}(\varphi_0)$ (Step 1), and (Gc2), (Gc3), (Gc5). So there is $N \geq 1$ with $(\mathfrak{m}_{A_0}\mathrm{Rs} \cdot \mathrm{Irr}(\mathrm{Rs}))^N \subseteq (\boldsymbol{\sigma}, a)$.

Step 3: the lower bound at the fixed point. The hypotheses of Proposition 41.13 hold: $A \subseteq P$ with the inclusion satisfying (Sm1)–(Sm3), the counts, the elements y_i and exponents c_i, $a \in A_0$ with $\varphi_0(a) = 0$ (Gc5), and the inclusion of Step 2. Hence in large degree $m \in \mathbb{Z}^3$, $\mathrm{eu}(\boldsymbol{\sigma}; \mathrm{Rs}/(a))(m)$ is defined and at least $2^{n+n_1-1} \sum_{i=1}^{n} c_i$.

Step 4: moving a to the end. Let $C = \mathrm{Kos}_\bullet(\boldsymbol{\sigma}; \mathrm{Rs})$. As a has degree 0, $\mathrm{Kos}_\bullet((a, \boldsymbol{\sigma}); \mathrm{Rs})$ is the mapping cone of multiplication by a on C (split ϵ_J according to whether J contains the index of a). Each C_p is a direct sum of shifted copies of Rs, on which a is injective (Step 1), with cokernel $\mathrm{Kos}_p(\boldsymbol{\sigma}; \mathrm{Rs}/(a))$. For an injective chain map, the long exact homology sequence of the short exact sequence $0 \to C \xrightarrow{a} C \to C/aC \to 0$ and that of the cone identify the homology of the cone with that of C/aC; all maps preserve degrees. Hence $H_p((a, \boldsymbol{\sigma}); \mathrm{Rs})_m \cong H_p(\boldsymbol{\sigma}; \mathrm{Rs}/(a))_m$ for all p and m (the classical fact that a regular element may be divided out of a Koszul complex; [55, Cor. 1.6.13]). Further, a Koszul complex is the tensor product over Rs of the complexes of its single elements, tensored with the module, and the tensor product of complexes is symmetric up to isomorphism (with the usual signs); so permuting the sequence does not change the homology. Therefore, for every m, $\mathrm{eu}((\boldsymbol{\sigma}, a); \mathrm{Rs})(m)$ is defined if and only if $\mathrm{eu}(\boldsymbol{\sigma}; \mathrm{Rs}/(a))(m)$ is, and then

$$\mathrm{eu}((\boldsymbol{\sigma}, a); \mathrm{Rs})(m) = \mathrm{eu}((a, \boldsymbol{\sigma}); \mathrm{Rs})(m) = \mathrm{eu}(\boldsymbol{\sigma}; \mathrm{Rs}/(a))(m).$$

Step 5: the Serre-type inequality. By Proposition 41.20, applied to $B = A_0$, $S = \mathrm{Rs}$, the sequence $\boldsymbol{\sigma}$ and $a \in \mathrm{Rs}_0$, with the inclusion of Step 2, in large degree $\mathrm{eu}((\boldsymbol{\sigma}, a); \mathrm{Rs})(m) \leq \mathrm{eu}((a); \mathrm{Rs}/(\boldsymbol{\sigma}))(m)$, both defined.

Step 6: the upper bound at the two-signed points. The hypotheses of Proposition 41.21 hold: the counts, rational spanning of u_{-1}, $A_{-2} \subseteq \mathrm{Sq}(\varphi_0)$, (Gc2)–(Gc5) and the orbit cover Σ. As $\mathrm{ne}(1) = n_1 - 1$, in large degree $\mathrm{eu}((a); \mathrm{Rs}/(\boldsymbol{\sigma}))(m)$ is defined and at most $2^{n+n_1} \sum_{\varphi \in \Sigma} v_{\mathcal{O}}(\varphi(a))$.

Step 7: conclusion. Choose $m \in \mathbb{Z}^3$ in large degree for Steps 3, 5 and 6 at once. Then by Steps 3–6

$$2^{n+n_1-1}\sum_{i=1}^{n} c_i \le \mathrm{eu}(\boldsymbol{\sigma};\mathrm{Rs}/(a))(m) = \mathrm{eu}((\boldsymbol{\sigma},a);\mathrm{Rs})(m)$$
$$\le \mathrm{eu}((a);\mathrm{Rs}/(\boldsymbol{\sigma}))(m) \le 2^{n+n_1}\sum_{\varphi\in\Sigma} v_{\mathcal{O}}(\varphi(a)),$$

and dividing by 2^{n+n_1-1} gives the assertion. The factor 2 is the ratio of the two powers of 2: the excess equation $w = u_{-1,n+1}$ lies in $\mathrm{Sq}(\varphi_0)$ and contributes nothing at the fixed point, while its square w^2T_2 is a section and contributes a factor 2 at the two-signed points. □

Lemma 41.8. *Let $\mathcal{O}$ be a coefficient ring and (A,φ_0) a graded $\mathcal{O}$-algebra with a fixed point. Then:*

1. *every A_j, $j\in\mathbb{Z}$, is a finitely generated A_0-module;*
2. *for every system of equations u for (A,φ_0) as in Definition 41.6, $\mathrm{Rs}(u)$ is a standard $\mathbb{N}^3$-graded A_0-algebra over the Noetherian local ring A_0;*
3. *if moreover A is a domain, so is $\mathrm{Rs}(u)$; in particular every non-zero $b\in A_0$ is a non-zerodivisor on $\mathrm{Rs}(u)$.*

Proof. Item 1. By definition, A is generated as an $\mathcal{O}$-algebra by A_0 and finitely many homogeneous elements $s_1,\dots,s_q$ of non-zero degrees $d_1,\dots,d_q$. Hence every element of A is a finite sum of products $b\,s^\nu$ with $b\in A_0$, $\nu\in\mathbb{N}^q$ and $s^\nu=\prod_k s_k^{\nu_k}$ (the $\mathcal{O}$-multiples and products of elements of A_0 lie in A_0); such a product is homogeneous of degree $\sum_k \nu_k d_k$. Taking the component of degree j of such a sum, A_j is the A_0-span of the s^ν with $\nu\in\mathcal{E}_j=\{\nu\in\mathbb{N}^q : \sum_k\nu_k d_k = j\}$. By Dickson's lemma, $\mathcal{E}_j$ has finitely many minimal elements for the componentwise order, and every $\nu\in\mathcal{E}_j$ satisfies $\nu\ge\nu'$ for one of them. Then $\nu-\nu'\in\mathcal{E}_0$, so $s^\nu = s^{\nu'}s^{\nu-\nu'}\in s^{\nu'}A_0$. Hence A_j is generated over A_0 by the finitely many $s^{\nu'}$, ν' minimal in $\mathcal{E}_j$.

Item 2. By the definition of a graded algebra with a fixed point, A_0 is a complete Noetherian local ring. Let N_0,N_1,N_2 be as in Definition 41.6. The piece $\mathrm{Rs}(u)_0 = N^0T^0 = A_0$ is the image of the structure map. By item 1, $N_0=A_{-2}$ and $N_1=A_2$ are finitely generated A_0-modules, say by $g_1,\dots,g_a$ and $h_1,\dots,h_b$; then $N_2 = A_0w^2 + A_{-2}A_{-2}A_2$ is generated by w^2 and the $g_ig_{i'}h_l$. So each $\mathrm{Rs}(u)_{e_i}=N_iT_i$ is finitely generated. Finally, for $\mu\in\mathbb{N}^3$, $N^{\mu+e_i}=N^\mu N_i$ by the definition of N^μ (products of submodules are associative and commutative), so $\mathrm{Rs}(u)_{\mu+e_i} = N^\mu N_i T^\mu T_i = \mathrm{Rs}(u)_\mu \mathrm{Rs}(u)_{e_i}$.

Item 3. The ring $\mathrm{Rs}(u)$ is a subring of the polynomial ring $A[T_0,T_1,T_2]$, which is a domain if A is. A non-zero $b\in A_0$ is a non-zero element bT^0 of this domain, hence a non-zerodivisor on it. □

Proposition 41.9. *Let $\mathcal{O}$ be a coefficient ring, (A,φ_0) a graded $\mathcal{O}$-algebra with a fixed point, $\mathfrak{m}_{A_0}$ the maximal ideal of A_0, and u a system of equations for (A,φ_0) with $\mathrm{ne}(-2)=\mathrm{ne}(0)=0$ and $\mathrm{ne}(-1)=n+1$, $n\ge1$, such that u_{-1} spans rationally in degree -1 and $A_{-2}\subseteq\mathrm{Sq}(\varphi_0)$. Let $a\in A$ be such that (A,φ_0,u,a) satisfies (Gc2), (Gc3) and (Gc5). Then there is $N\ge1$ with*

$$\left(\mathfrak{m}_{A_0}\mathrm{Rs}(u)\cdot\mathrm{Irr}(\mathrm{Rs}(u))\right)^N \subseteq (\boldsymbol{\sigma}(u),a).$$

Proof. Write $\mathrm{Rs} = \mathrm{Rs}(u)$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(u)$ and $I_{\mathrm{ex}} = \mathfrak{m}_{A_0}\mathrm{Rs} \cdot \mathrm{Irr}(\mathrm{Rs})$. By (Gc5), $a \in A_0 = \mathrm{Rs}_0$.

Reduction to prime ideals. By Lemma 41.8, item 2, Rs is standard over A_0; it is generated as an A_0-algebra by the finitely generated A_0-modules $\mathrm{Rs}_{e_0}, \mathrm{Rs}_{e_1}, \mathrm{Rs}_{e_2}$, hence is Noetherian by Hilbert's basis theorem, A_0 being Noetherian. Suppose that every prime ideal $\mathfrak{P}$ of Rs containing $(\boldsymbol{\sigma}, a)$ contains I_{ex}. Then I_{ex} lies in the radical of $(\boldsymbol{\sigma}, a)$, the intersection of these primes. Let $g_1, \dots, g_t$ generate I_{ex} and $g_i^{e_i} \in (\boldsymbol{\sigma}, a)$; with $N = 1 + \sum_i (e_i - 1)$, every product of N generators contains some g_i at least e_i times, so $I_{\mathrm{ex}}^N \subseteq (\boldsymbol{\sigma}, a)$.

The primes. Let $\mathfrak{P} \supseteq (\boldsymbol{\sigma}, a)$ be prime and $\mathfrak{p} = \mathfrak{P} \cap A_0$, which contains a. If $\mathrm{Irr}(\mathrm{Rs}) \subseteq \mathfrak{P}$, then $I_{\mathrm{ex}} \subseteq \mathfrak{P}$. Otherwise Lemma 41.10 applies (the counts, rational spanning of u_{-1}, $A_{-2} \subseteq \mathrm{Sq}(\varphi_0)$, (Gc2) and (Gc3) are hypotheses here), and by its item 2 either $\mathfrak{p} = \mathfrak{m}_{A_0}$, so that $I_{\mathrm{ex}} \subseteq \mathfrak{m}_{A_0}\mathrm{Rs} \subseteq \mathfrak{P}$, or $\mathfrak{p} = \ker(\varphi|_{A_0})$ for a two-signed point φ on u. In the second case $\varphi(a) = 0$ as $a \in \mathfrak{p}$, against (Gc5). Hence every such prime contains I_{ex}, which proves the supposition. □

Lemma 41.10. *Let $\mathcal{O}$ be a coefficient ring, (A, φ_0) a graded $\mathcal{O}$-algebra with a fixed point, $\mathfrak{m}_{A_0}$ the maximal ideal of A_0, and u a system of equations for (A, φ_0) with $\mathrm{ne}(-2) = \mathrm{ne}(0) = 0$ and $\mathrm{ne}(-1) = n + 1$, $n \geq 1$, such that u_{-1} spans rationally in degree -1, $A_{-2} \subseteq \mathrm{Sq}(\varphi_0)$, and (Gc2) and (Gc3) hold (these two conditions do not involve the element a). Let $\mathfrak{P}$ be a prime ideal of $\mathrm{Rs}(u)$ that contains every member of $\boldsymbol{\sigma}(u)$ and does not contain $\mathrm{Irr}(\mathrm{Rs}(u))$, and let $\mathfrak{p} = \mathfrak{P} \cap A_0$. Then:*

1. *$I(u) \cap A_0 \subseteq \mathfrak{p}$;*
2. *either $\mathfrak{p} = \mathfrak{m}_{A_0}$, or there is a two-signed point φ on u with $\mathfrak{p} = \ker(\varphi|_{A_0})$.*

Proof. Write $\mathrm{Rs} = \mathrm{Rs}(u)$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(u)$, $w = u_{-1,n+1}$, $\mathfrak{m}_0 = \ker \varphi_0$ and $I_0 = I(u) \cap A_0$. As $\mathrm{ne}(-2) = \mathrm{ne}(0) = 0$, the equations are the $u_{-1,i}$ $(i \leq n+1)$, $u_{1,i}$, $u_{2,i}$, and $\boldsymbol{\sigma}$ consists of the $u_{-1,i}^2 T_0$ $(i \leq n)$, $u_{1,i}^2 T_1$, $u_{2,i} T_1$ and $w^2 T_2$. For $k \geq 1$, $A_{-2k} = A_{-2}^k = N_0^k$ and $A_{2k} = A_2^k = N_1^k$ by Lemma 41.11; hence $x T_0^k \in \mathrm{Rs}$ for $x \in A_{-2k}$ and $x T_1^k \in \mathrm{Rs}$ for $x \in A_{2k}$, $k \geq 0$.

Step 1: three elements outside $\mathfrak{P}$. As $\mathrm{Irr}(\mathrm{Rs}) \subseteq \mathrm{Rs}_{e_i}\mathrm{Rs}$ for each i, no $\mathrm{Rs}_{e_i} = N_i T_i$ lies in $\mathfrak{P}$. Choose $h_0 \in A_{-2}$, $h_1 \in A_2$, $h_2 \in N_2$ with $h_i T_i \notin \mathfrak{P}$.

Step 2: $u_{-1,i}^2 T_0 \in \mathfrak{P}$ for $1 \leq i \leq n+1$. For $i \leq n$ this is a section. Since $N_2 \subseteq A_{-2} = N_0$ and $w^2 \in N_0 \cap N_2$, the elements $w^2 T_0$, $h_2 T_0$ lie in Rs_{e_0} and $(w^2 T_0)(h_2 T_2) = (h_2 T_0)(w^2 T_2) \in \mathfrak{P}$, as $w^2 T_2$ is a section; since $h_2 T_2 \notin \mathfrak{P}$, $w^2 T_0 \in \mathfrak{P}$.

Step 3: item 1. Put $U = (h_0 T_0)(h_1 T_1) = h_0 h_1 T_0 T_1 \notin \mathfrak{P}$. Let v be an equation of degree d and $c \in A_{-d}$; then $cv \in \mathfrak{p}$:

- $d = -1$: $c^2 h_0 h_1 \in A_2 = N_1$ and $(cv)^2 U = (v^2 T_0)(c^2 h_0 h_1 T_1) \in \mathfrak{P}$ by Step 2;
- $d = 1$: $c^2 h_0 h_1 \in A_{-2} = N_0$ and $(cv)^2 U = (c^2 h_0 h_1 T_0)(v^2 T_1) \in \mathfrak{P}$, $v^2 T_1$ being a section;
- $d = 2$: $c h_0 h_1 \in A_{-2}$ and $(cv) U = (c h_0 h_1 T_0)(v T_1) \in \mathfrak{P}$, $v T_1$ being a section;

as $U \notin \mathfrak{P}$ and $\mathfrak{P}$ is prime, $cv \in \mathfrak{P}$. Now let $b \in I_0$, $b = \sum_k \alpha_k v_k$ with $\alpha_k \in A$ and equations v_k of degrees d_k. Comparing components of degree 0, $b = \sum_k (\alpha_k)_{-d_k} v_k \in \mathfrak{p}$.

Step 4: a homomorphism with kernel $\mathfrak{p}$. By (Gc2) and Step 3, $D_0 = A_0/\mathfrak{p}$ is a quotient of the finitely generated $\mathcal{O}$-module A_0/I_0, and a domain. If $\varpi \in \mathfrak{p}$, D_0 is a domain of finite dimension over the field $\mathcal{O}/\varpi\mathcal{O}$, hence a field, and $\mathfrak{p} = \mathfrak{m}_{A_0}$. Assume from now on that $\varpi \notin \mathfrak{p}$. Every non-zero $c \in \mathcal{O}$ is a unit times a power of ϖ, so D_0 is torsion free and embeds into $D_0[\varpi^{-1}]$, a domain of finite dimension over $\mathcal{O}[\ell^{-1}] = \mathcal{O}[\varpi^{-1}]$, hence a finite field extension of it, which

embeds into the algebraically closed field $\overline{\mathbb{Q}}_\ell \supseteq \mathcal{O}[\ell^{-1}]$. The composite $\beta\colon A_0 \to D_0 \to \overline{\mathbb{Q}}_\ell$ is an $\mathcal{O}$-algebra homomorphism with $\ker\beta = \mathfrak{p}$.

Step 5: points over β that kill $I(u)$. The ideal $\mathfrak{a}_0 = I(u) + \mathfrak{p}A$ is generated by homogeneous elements, so its elements of degree 0 form $I_0 + \mathfrak{p} = \mathfrak{p}$ (Step 3); hence $\mathfrak{a}_0$ and the element 1 satisfy (Lf1), (Lf2) of Lemma 41.12 for β (A is finitely generated as an A_0-algebra by the definition of a graded algebra with a fixed point), and there is a point φ on u with $\varphi|_{A_0} = \beta$. If $\beta(fg) \neq 0$ for some $f \in A_{-2}$ and $g \in A_2$, then $\varphi(f) \neq 0 \neq \varphi(g)$, so φ is two-signed and item 2 holds. It remains to show that $\beta(fg) = 0$ for all $f \in A_{-2}$, $g \in A_2$ is impossible; assume it, for a contradiction, in Steps 6 to 10.

Step 6: $\beta(A_{-j}A_j) = 0$ for $j \geq 1$. Let $x \in A_{-j}$, $y \in A_j$. By Lemma 41.11, $x^2 \in A_{-2}^j$ and $y^2 \in A_2^j$, so $(xy)^2$ is an A_0-linear combination of products $f_1g_1\cdots f_jg_j$ with $f_l \in A_{-2}$, $g_l \in A_2$, each $f_lg_l \in A_0$; hence $\beta((xy)^2) = 0$ and $\beta(xy) = 0$.

Step 7: a point over β vanishing in all non-zero degrees. Define $\varphi_\star(x) = \beta(x_0)$. It is $\mathcal{O}$-linear with $\varphi_\star(1) = 1$; for $x, y \in A$, $(xy)_0 = \sum_j x_jy_{-j}$ and $\beta(x_jy_{-j}) = 0$ for $j \neq 0$ (Step 6), so $\varphi_\star(xy) = \varphi_\star(x)\varphi_\star(y)$. Thus $\varphi_\star$ is a point, negative-vanishing and positive-vanishing, with $\varphi_\star|_{A_0} = \beta$, and it lies on u, since $x \in I(u)$ has $x_0 \in I_0 \subseteq \ker\beta$.

Step 8: a positive-vanishing point over β that is not negative-vanishing. Let

$$\mathfrak{a}_- = \{x \in A :\ (hx)_{-2k}T_0^k \in \mathfrak{P} \text{ for all } h \in A \text{ and } k \geq 0\},$$

an ideal of A, since components are additive and $(h(yx))_{-2k} = ((hy)x)_{-2k}$. For x homogeneous of degree j, $(hx)_{-2k} = h_{-2k-j}x$, so

$$x \in \mathfrak{a}_- \iff hxT_0^k \in \mathfrak{P} \text{ for all } k \geq 0 \text{ and } h \in A_{-2k-j}. \qquad (*)$$

For homogeneous x of degree j, $k \geq 0$ and $h \in A_{-2k-j}$:

(a) if $x^2 \in \mathfrak{a}_-$ then $x \in \mathfrak{a}_-$: $h^2 \in A_{-2(2k)-2j}$, so $(hxT_0^k)^2 = h^2x^2T_0^{2k} \in \mathfrak{P}$ by $(*)$ for x^2;

(b) if $h_0x \in \mathfrak{a}_-$ then $x \in \mathfrak{a}_-$: $h \in A_{-2(k+1)-(j-2)}$, so $(h_0T_0)(hxT_0^k) = hh_0xT_0^{k+1} \in \mathfrak{P}$, and $h_0T_0 \notin \mathfrak{P}$;

(c) $\mathfrak{p} \subseteq \mathfrak{a}_-$: for $b \in \mathfrak{p}$ and $h \in A_{-2k}$, $hbT_0^k = b \cdot hT_0^k \in \mathfrak{P}$;

(d) if $x \in A_{-2}$ and $xT_0 \in \mathfrak{P}$ then $x \in \mathfrak{a}_-$: for $h \in A_{-2k+2}$, $h_0h \in A_{-2k}$ and $(h_0T_0)(hxT_0^k) = (h_0hT_0^k)(xT_0) \in \mathfrak{P}$;

(e) if $b \in A_0$ with $\beta(b) \neq 0$ and $k \geq 0$, then $bh_0^k \notin \mathfrak{a}_-$: otherwise, with $h = 1$, $b(h_0T_0)^k \in \mathfrak{P}$, while $b \notin \mathfrak{p}$ and $h_0T_0 \notin \mathfrak{P}$.

For $g \in A_2$, $h_0g \in \ker\beta = \mathfrak{p}$ by the assumption of Step 5, so $g \in \mathfrak{a}_-$ by (c) and (b). For $j \geq 1$ and $g \in A_j$, $g^2 \in A_{2j} = A_2^j \subseteq \mathfrak{a}_-$, so $g \in \mathfrak{a}_-$ by (a). For $i \leq n+1$, $u_{-1,i}^2T_0 \in \mathfrak{P}$ (Step 2), so $u_{-1,i} \in \mathfrak{a}_-$ by (d) and (a); the $u_{1,i}$, $u_{2,i}$ have positive degree. Hence $I(u) \subseteq \mathfrak{a}_-$ and $A_j \subseteq \mathfrak{a}_-$ for $j > 0$. By (c) and (e), Lemma 41.12 applies to $\mathfrak{a}_-$ and h_0 and gives a point φ_- on u with $\varphi_-|_{A_0} = \beta$, positive-vanishing, and not negative-vanishing since $\varphi_-(h_0) \neq 0$.

Step 9: a negative-vanishing point over β that is not positive-vanishing. Let $\mathfrak{a}_+ = \{x \in A :\ (hx)_{2k}T_1^k \in \mathfrak{P} \text{ for all } h \in A,\ k \geq 0\}$. With T_1, h_1, $A_{2k} = N_1^k$ and all degrees negated, the arguments of Step 8 show that $\mathfrak{a}_+$ is an ideal with the properties (a), (c), (b) for h_1, (d) for $x \in A_2$ with $xT_1 \in \mathfrak{P}$, and (e) for bh_1^k. For $g \in A_{-2}$, $h_1g \in \mathfrak{p}$, so $g \in \mathfrak{a}_+$; for $j \geq 1$ and $g \in A_{-j}$, $g^2 \in A_{-2}^j \subseteq \mathfrak{a}_+$, so $g \in \mathfrak{a}_+$. As $u_{1,i}^2T_1$ and $u_{2,i}T_1$ are sections, $u_{1,i}^2, u_{2,i} \in \mathfrak{a}_+$ by (d), so

$u_{1,i} \in \mathfrak{a}_+$ by (a); the $u_{-1,i}$ have negative degree. By Lemma 41.12 there is a point φ_+ on u with $\varphi_+|_{A_0} = \beta$, negative-vanishing, and not positive-vanishing since $\varphi_+(h_1) \neq 0$.

Step 10: the contradiction. If $\beta \neq \varphi_0|_{A_0}$, then $\varphi_\star \neq \varphi_0$ is a point on u that is negative-vanishing and positive-vanishing, and φ_+ (negative-vanishing, not positive-vanishing) and φ_- (positive-vanishing, not negative-vanishing) are points on u agreeing with $\varphi_\star$ on A_0, against (Gc3). If $\beta = \varphi_0|_{A_0}$, we show $\varphi_-(A_{-2}) = 0$, against $\varphi_-(h_0) \neq 0$. First, $\varphi_-(x) = 0$ for $x \in A_e \cap \mathfrak{m}_0$ with $e \geq 0$: for $e > 0$ as φ_- is positive-vanishing, for $e = 0$ as $\varphi_-(x) = \beta(x) = \varphi_0(x)$. Second, let $d \in \{-1, -2\}$ and $z \in A_d \cap \mathfrak{m}_0^2$, $z = \sum_k p_k q_k$ with $p_k, q_k \in \mathfrak{m}_0$. All components of an element of $\mathfrak{m}_0$ lie in $\mathfrak{m}_0$, as φ_0 vanishes in non-zero degrees, and

$$z = z_d = \sum_k \sum_{e \in \mathbb{Z}} (p_k)_e \, (q_k)_{d-e}.$$

A term with $e \geq 0$ or $d - e \geq 0$ is killed by φ_-; the others have $d < e < 0$, impossible for $d = -1$ and forcing $e = -1$ for $d = -2$. Hence $\varphi_-(z) = 0$ for $d = -1$, and for $d = -2$ once $\varphi_-(A_{-1}) = 0$. Let $x \in A_{-1}$; then $x \in \mathfrak{m}_0$, and as u_{-1} spans rationally in degree -1 there are $c \in \mathcal{O} \setminus \{0\}$, $o_i \in \mathcal{O}$ and $c' \in \mathcal{O} \setminus \{0\}$ with $z = c'(cx - \sum_i o_i u_{-1,i}) \in A_{-1} \cap \mathfrak{m}_0^2$. As $\varphi_-(z) = 0 = \varphi_-(u_{-1,i})$ and $c'c \neq 0$ in $\mathcal{O} \subseteq \overline{\mathbb{Q}}_\ell$, $\varphi_-(x) = 0$. Finally, for $x \in A_{-2} \subseteq \mathrm{Sq}(\varphi_0)$ there is $c' \neq 0$ with $c'x \in A_{-2} \cap \mathfrak{m}_0^2$, so $\varphi_-(x) = 0$. □

Lemma 41.11. *Let $\mathcal{O}$ be a coefficient ring and (A, φ_0) a graded $\mathcal{O}$-algebra with a fixed point. For $n \geq 1$ let A_{-2}^n and A_2^n be the A_0-submodules of A spanned by the products of n elements of A_{-2}, respectively of A_2. Then for every $n \geq 1$*

$$A_{-2n} = A_{-2}^n, \qquad A_{2n} = A_2^n.$$

Proof. The inclusions $A_{-2}^n \subseteq A_{-2n}$ and $A_2^n \subseteq A_{2n}$ hold because $A_i A_j \subseteq A_{i+j}$ and $A_0 A_j \subseteq A_j$. For the converse, by the definition of a graded $\mathcal{O}$-algebra with a fixed point, A is generated as an $\mathcal{O}$-algebra by A_0 and finitely many homogeneous elements $s_1, \ldots, s_q$ with degrees $d_1, \ldots, d_q \in \{-2, -1, 1, 2\}$. Every element of A is therefore a finite sum of products $b\, s_{k_1} \cdots s_{k_r}$ with $b \in A_0$ and $r \geq 0$ (the $\mathcal{O}$-multiples and products of elements of A_0 lie in A_0), and such a product is homogeneous of degree $d_{k_1} + \cdots + d_{k_r}$. Taking the component of degree j, A_j is the A_0-span of the products $s_{k_1} \cdots s_{k_r}$ with $d_{k_1} + \cdots + d_{k_r} = j$.

A combinatorial claim. Let $\delta_1, \ldots, \delta_r \in \{-2, -1, 1, 2\}$ with $\sum_{i=1}^r \delta_i = -2n$, $n \geq 1$. Then there are pairwise disjoint subsets $T_1, \ldots, T_n$ of $\{1, \ldots, r\}$ with $\sum_{i \in T_l} \delta_i = -2$ for each l. Induction on n: since the sum is negative, some δ_i is negative. If some $\delta_i = -2$, put $T_n = \{i\}$; otherwise, if two indices $i \neq i'$ have $\delta_i = \delta_{i'} = -1$, put $T_n = \{i, i'\}$. In the remaining case at most one δ_i is negative and it equals -1, so $\sum_i \delta_i \geq -1 > -2n$, which is impossible. The indices outside T_n have sum $-2(n-1)$; for $n \geq 2$ the induction hypothesis applied to them gives $T_1, \ldots, T_{n-1}$, disjoint from T_n.

The case $-2n$. Let $s_{k_1} \cdots s_{k_r}$ be a product of degree $-2n$ and apply the claim to $\delta_i = d_{k_i}$. With $T_0 = \{1, \ldots, r\} \setminus (T_1 \cup \cdots \cup T_n)$,

$$s_{k_1} \cdots s_{k_r} = \Big(\prod_{i \in T_0} s_{k_i} \Big) \prod_{l=1}^{n} \Big(\prod_{i \in T_l} s_{k_i} \Big),$$

where each $\prod_{i \in T_l} s_{k_i}$, $l \geq 1$, lies in A_{-2}, and $\prod_{i \in T_0} s_{k_i}$ has degree $-2n - n(-2) = 0$, so lies in A_0 (the empty product is $1 \in A_0$). As $A_0 A_{-2} \subseteq A_{-2}$, the product lies in A_{-2}^n; hence $A_{-2n} \subseteq A_{-2}^n$.

The case $2n$. The same argument applied to $\delta_i = -d_{k_i}$ for a product of degree $2n$ gives disjoint $T_1, \dots, T_n$ with $\sum_{i \in T_l} d_{k_i} = 2$, so each $\prod_{i \in T_l} s_{k_i}$ lies in A_2 and the rest in A_0; hence $A_{2n} \subseteq A_2^n$. □

Lemma 41.12. *Let A be a commutative ring and $A_0 \subseteq A$ a subring such that A is finitely generated as an A_0-algebra. Let $\beta\colon A_0 \to \overline{\mathbb{Q}}_\ell$ be a ring homomorphism, $\mathfrak{a}$ an ideal of A and $f \in A$ such that*

(Lf1) $\ker\beta \subseteq \mathfrak{a}$*;*

(Lf2) $b\,f^n \notin \mathfrak{a}$ *for every $b \in A_0$ with $\beta(b) \neq 0$ and every $n \geq 0$.*

Then there is a ring homomorphism $\varphi\colon A \to \overline{\mathbb{Q}}_\ell$ with $\varphi|_{A_0} = \beta$, $\varphi(\mathfrak{a}) = 0$ and $\varphi(f) \neq 0$. If A_0 is an algebra over a coefficient ring $\mathcal{O}$ and β is an $\mathcal{O}$-algebra homomorphism, then φ is an $\mathcal{O}$-algebra homomorphism for the induced structure on A.

Proof. A prime ideal. Let $S = \{b\,f^n : b \in A_0,\ \beta(b) \neq 0,\ n \geq 0\}$. It contains 1 and is multiplicative, since $\beta(bb') = \beta(b)\beta(b') \neq 0$ in the field $\overline{\mathbb{Q}}_\ell$. By (Lf2), $S \cap \mathfrak{a} = \emptyset$, so no element of S maps to 0 in $A/\mathfrak{a}$ and the localisation $S^{-1}(A/\mathfrak{a})$ is a non-zero ring. The inverse image in A of a maximal ideal of it is a prime ideal $\mathfrak{q}$ with $\mathfrak{a} \subseteq \mathfrak{q}$ and $\mathfrak{q} \cap S = \emptyset$. Then $\mathfrak{q} \cap A_0 = \ker\beta$: the inclusion $\supseteq$ holds by (Lf1), and an element $b \in \mathfrak{q} \cap A_0$ with $\beta(b) \neq 0$ would be the element bf^0 of $S \cap \mathfrak{q}$. Also $f = 1 \cdot f^1 \in S$, so $f \notin \mathfrak{q}$.

A finitely generated algebra over a field. Let $D = A/\mathfrak{q}$, a domain, and $\bar{f} \neq 0$ the image of f. As $\mathfrak{q} \cap A_0 = \ker\beta$, the map $A_0 \to D$ induces an injection of $R = \beta(A_0) \cong A_0/\ker\beta$ into D, through which we regard R as a subring of D. Let $L \subseteq \overline{\mathbb{Q}}_\ell$ be the fraction field of R; the inclusion $R \subseteq D$ extends uniquely to an embedding of L into the fraction field of D. Let E be the subring of the fraction field of D generated by D, $\bar{f}^{-1}$ and L. It is a non-zero L-algebra, generated as an L-algebra by $\bar{f}^{-1}$ and the images of finitely many elements that generate A as an A_0-algebra (the image of A_0 is $R \subseteq L$). Let $\mathfrak{n}$ be a maximal ideal of E. The field $E/\mathfrak{n}$ contains L and is finitely generated as an L-algebra, so by Zariski's lemma (the weak Nullstellensatz; Atiyah–Macdonald, *Introduction to commutative algebra*, Cor. 5.24) it is a finite extension of L. Since $\overline{\mathbb{Q}}_\ell$ is algebraically closed and contains L, there is a field embedding $\iota\colon E/\mathfrak{n} \to \overline{\mathbb{Q}}_\ell$ restricting to the inclusion on L.

The homomorphism. Let φ be the composite $A \to D \to E \to E/\mathfrak{n} \xrightarrow{\iota} \overline{\mathbb{Q}}_\ell$. For $b \in A_0$ the image of b in D is $\beta(b) \in R \subseteq L$, so $\varphi(b) = \iota(\beta(b)) = \beta(b)$. As $\mathfrak{a} \subseteq \mathfrak{q}$, $\varphi(\mathfrak{a}) = 0$. The image of f in E is the unit $\bar{f}$, so its image in the field $E/\mathfrak{n}$ is non-zero and $\varphi(f) \neq 0$, ι being injective. Finally, if $\mathcal{O} \to A_0$ is the structure map and β is $\mathcal{O}$-linear, then for $c \in \mathcal{O}$ the image of c in A lies in A_0 and $\varphi(c \cdot 1) = \beta(c \cdot 1)$, so φ is an $\mathcal{O}$-algebra homomorphism. □

Proposition 41.13. *Let $\mathcal{O}$ be a coefficient ring with uniformiser ϖ, J a finite set with a map* $\mathrm{wt}\colon J \to \{-1, 1, 2\}$ *such that $n = \#\mathrm{wt}^{-1}(-1)$, $n_1 = \#\mathrm{wt}^{-1}(1)$ and $n_2 = \#\mathrm{wt}^{-1}(2)$ are ≥ 1, and $P = \mathcal{O}[\ell^{-1}][[X_i : i \in J]]$ with the weights defined by* wt*. Let $A \subseteq P$ be an $\mathcal{O}$-subalgebra such that, with A_j the set of elements of A of weight j and φ_0 the constant term, $A = \bigoplus_j A_j$, (A, φ_0) is a graded $\mathcal{O}$-algebra with a fixed point, and the inclusion $A \to P$ satisfies (Sm1)–(Sm3). Let u be a system of equations for (A, φ_0) with counts* $\mathrm{ne}(-1) = n+1$, $\mathrm{ne}(1) = n_1 - 1$, $\mathrm{ne}(2) = n_2 - 1$, $\mathrm{ne}(-2) = \mathrm{ne}(0) = 0$*, let $y_1, \dots, y_n \in A_{-1}$ and $c_1, \dots, c_n \geq 0$ be integers with*

$$u_{-1,i} - \varpi^{c_i} y_i \in \mathrm{Sq}(\varphi_0) \quad (1 \leq i \leq n), \qquad u_{-1,n+1} \in \mathrm{Sq}(\varphi_0),$$

and let $a \in A_0$ with $\varphi_0(a) = 0$. Assume that there is $N \geq 1$ with

$$\left(\mathfrak{m}_{A_0}\mathrm{Rs}(u) \cdot \mathrm{Irr}(\mathrm{Rs}(u))\right)^N \subseteq (\boldsymbol{\sigma}(u), a),$$

$\mathfrak{m}_{A_0}$ *the maximal ideal of* A_0*. Then in large degree* $m \in \mathbb{Z}^3$*,* $\mathrm{eu}(\boldsymbol{\sigma}(u); \mathrm{Rs}(u)/(a))(m)$ *is defined and*

$$\mathrm{eu}(\boldsymbol{\sigma}(u); \mathrm{Rs}(u)/(a))(m) \;\geq\; 2^{n+n_1-1} \sum_{i=1}^{n} c_i.$$

Proof. Write $\mathrm{Rs} = \mathrm{Rs}(u)$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(u)$, $\mathfrak{m} = \mathfrak{m}_{A_0}$ and $M = n_1 + n_2 - 2$. As $\mathrm{ne}(1) = n_1 - 1$ and $\mathrm{ne}(2) = n_2 - 1$, $\boldsymbol{\sigma}$ consists of n elements $u_{-1,i}^2 T_0$ of degree e_0, M elements of degree e_1 and $w^2 T_2$ of degree e_2. By Lemma 41.8, item 2, Rs is a standard $\mathbb{N}^3$-graded algebra over the Noetherian local ring A_0, and it is Noetherian (Lemma 41.14, item 1). A sequence $\boldsymbol{x}$ of homogeneous elements is called *admissible for* a finitely generated graded Rs-module G if $(\mathfrak{m}\mathrm{Rs} \cdot \mathrm{Irr}(\mathrm{Rs}))^{N'} \subseteq \mathrm{Ann}_{\mathrm{Rs}}(G) + (\boldsymbol{x})$ for some $N' \geq 1$; then $\mathrm{eu}(\boldsymbol{x}; G)(m)$ is defined in large degree (Lemma 41.14, item 2).

Step 1: the leading-form quotient. The hypotheses of Lemma 41.17 hold ($A \subseteq P$ with (Sm1)–(Sm3), $\mathrm{ne}(-2) = \mathrm{ne}(0) = 0$, $\mathrm{ne}(-1) = n + 1$, $w \in \mathrm{Sq}(\varphi_0)$). Let $\mathfrak{q}$ be the homogeneous prime ideal it provides and $F = \mathrm{Rs}/\mathfrak{q}$, so $\mathrm{Ann}_{\mathrm{Rs}}(F) = \mathfrak{q}$. By its item 2, $a \in \mathfrak{q}$, as $\varphi_0(a) = 0$.

Step 2: passing from $\mathrm{Rs}/(a)$ *to* F*.* By hypothesis $\boldsymbol{\sigma}$ is admissible for $\mathrm{Rs}/(a)$, whose annihilator is (a). As $a \in \mathfrak{q}$, there is an exact sequence $0 \to \mathfrak{q}/(a) \to \mathrm{Rs}/(a) \to F \to 0$ of finitely generated graded modules with degree-preserving maps. By Lemma 41.14, item 3, in large degree $\mathrm{eu}(\boldsymbol{\sigma}; \mathrm{Rs}/(a))(m) = \mathrm{eu}(\boldsymbol{\sigma}; \mathfrak{q}/(a))(m) + \mathrm{eu}(\boldsymbol{\sigma}; F)(m)$, all three defined; $\boldsymbol{\sigma}$ is admissible for $\mathfrak{q}/(a)$, whose annihilator contains (a), so $\mathrm{eu}(\boldsymbol{\sigma}; \mathfrak{q}/(a))(m) \geq 0$ in large degree by Lemma 41.15. Hence in large degree

$$\mathrm{eu}(\boldsymbol{\sigma}; \mathrm{Rs}/(a))(m) \geq \mathrm{eu}(\boldsymbol{\sigma}; F)(m).$$

Step 3: replacing $u_{-1,i}$ *by* $\varpi^{c_i} y_i$ *on* F*.* Let $\boldsymbol{\sigma}'$ be $\boldsymbol{\sigma}$ with $u_{-1,i}^2 T_0$ replaced by $\varpi^{2c_i} y_i^2 T_0 = (\varpi^{c_i} y_i)^2 T_0$ for $1 \leq i \leq n$ (of degree e_0, as $y_i^2 \in A_{-2}$). Since $u_{-1,i} - \varpi^{c_i} y_i \in \mathrm{Sq}(\varphi_0)$ with both terms in A_{-1}, item 3 of Lemma 41.17 gives $u_{-1,i}^2 T_0 - \varpi^{2c_i} y_i^2 T_0 \in \mathfrak{q}$. So corresponding members of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ have the same degree and act in the same way on F; as the terms of a Koszul complex depend only on the degrees and its differentials only on the action of the elements on the module, $\mathrm{Kos}_\bullet(\boldsymbol{\sigma}; F) = \mathrm{Kos}_\bullet(\boldsymbol{\sigma}'; F)$ and $\mathrm{eu}(\boldsymbol{\sigma}; F) = \mathrm{eu}(\boldsymbol{\sigma}'; F)$. Moreover $\mathfrak{q} + (\boldsymbol{\sigma}') = \mathfrak{q} + (\boldsymbol{\sigma}) \supseteq (\boldsymbol{\sigma}, a)$, so $\boldsymbol{\sigma}'$, and every sequence generating an ideal containing $(\boldsymbol{\sigma}')$, is admissible for F.

Step 4: peeling the powers of ϖ*.* For $0 \leq k \leq n$ let $\boldsymbol{\sigma}^{[k]}$ be $\boldsymbol{\sigma}'$ with $\varpi^{2c_i} y_i^2 T_0$ replaced by $y_i^2 T_0$ for $i \leq k$; so $\boldsymbol{\sigma}^{[0]} = \boldsymbol{\sigma}'$, and each $(\boldsymbol{\sigma}^{[k]})$ contains $(\boldsymbol{\sigma}')$. Let $\boldsymbol{\rho}_k$ be $\boldsymbol{\sigma}^{[k]}$ with its k-th member removed. If $c_k \geq 1$, Lemma 41.18, applied to $S = \mathrm{Rs}$, the homogeneous prime $\mathfrak{q}$, $\beta = \varpi \notin \mathfrak{q}$ (item 1 of Lemma 41.17), the sequence $\boldsymbol{\sigma}^{[k]}$, its k-th member and $c = 2c_k$ (so that the modified sequence is $\boldsymbol{\sigma}^{[k-1]}$, admissible for F by Step 3), gives in large degree

$$\mathrm{eu}(\boldsymbol{\sigma}^{[k-1]}; F)(m) = \mathrm{eu}(\boldsymbol{\sigma}^{[k]}; F)(m) + 2c_k \,\mathrm{eu}(\boldsymbol{\rho}_k; F/\varpi F)(m);$$

if $c_k = 0$, then $\boldsymbol{\sigma}^{[k-1]} = \boldsymbol{\sigma}^{[k]}$ and the same identity holds. Adding over k,

$$\mathrm{eu}(\boldsymbol{\sigma}'; F)(m) = \mathrm{eu}(\boldsymbol{\sigma}^{[n]}; F)(m) + \sum_{k=1}^{n} 2c_k \,\mathrm{eu}(\boldsymbol{\rho}_k; F/\varpi F)(m)$$

in large degree, and $\mathrm{eu}(\boldsymbol{\sigma}^{[n]}; F)(m) \geq 0$ in large degree by Lemma 41.15, $\boldsymbol{\sigma}^{[n]}$ being admissible for F.

Step 5: the Euler characteristics modulo ϖ*.* By item 4 of Lemma 41.17, every piece $(F/\varpi F)_\mu$ has finite length $h(\mu)$, with h as there for $\mu \in \mathbb{N}^3$, and $(F/\varpi F)_\mu = 0$ for $\mu \notin \mathbb{N}^3$. For a sequence $\boldsymbol{\rho}$ with degrees d_j, $\mathrm{Kos}_p(\boldsymbol{\rho}; F/\varpi F)_m = \bigoplus_{\#J=p} (F/\varpi F)_{m-d_J}$ has finite length, and the alternating sum of the lengths of the homology of a finite complex of modules of finite length equals that

of its terms (additivity of length on $0 \to Z_p \to C_p \to B_{p-1} \to 0$ and $0 \to B_p \to Z_p \to H_p \to 0$). Each $\boldsymbol{\rho}_k$ consists of $n-1$ elements of degree e_0, M of degree e_1 and one of degree e_2; the subsets J containing i of the first kind, j of the second and ε of the third number $\binom{n-1}{i}\binom{M}{j}$. Hence for $m \in \mathbb{N}^3$ with $m \geq (n-1, M, 1)$, where all $m - d_J$ lie in $\mathbb{N}^3$,

$$\operatorname{eu}(\boldsymbol{\rho}_k; F/\varpi F)(m) = \sum_{i=0}^{n-1}\sum_{j=0}^{M}\sum_{\varepsilon=0}^{1}(-1)^{i+j+\varepsilon}\binom{n-1}{i}\binom{M}{j}h(m - ie_0 - je_1 - \varepsilon e_2) = 2^{\,n+n_1-2}$$

by Lemma 41.19 (with $n, n_1, n_2 \geq 1$).

Step 6: conclusion. Choosing m in large degree for Steps 2, 4 and 5 at once, Steps 2–5 give that $\operatorname{eu}(\boldsymbol{\sigma}; \operatorname{Rs}/(a))(m)$ is defined and

$$\operatorname{eu}(\boldsymbol{\sigma}; \operatorname{Rs}/(a))(m) \geq \operatorname{eu}(\boldsymbol{\sigma}'; F)(m) \geq \sum_{k=1}^{n} 2c_k \cdot 2^{\,n+n_1-2} = 2^{\,n+n_1-1}\sum_{i=1}^{n} c_i.$$

□

Lemma 41.14. *Let B be a Noetherian local ring with maximal ideal $\mathfrak{m}_B$, S a standard $\mathbb{N}^k$-graded B-algebra and F a finitely generated graded S-module. Then:*

1. *S is Noetherian, and every S_μ and every F_m is a finitely generated B-module. If F is annihilated by an ideal $\mathfrak{b}$ of S with $(\mathfrak{m}_B S \cdot \operatorname{Irr}(S))^N \subseteq \mathfrak{b}$ for some $N \geq 1$, then F_m has finite length for m in large degree.*

2. *For every sequence $\boldsymbol{x}$ of homogeneous elements and every p, $H_p(\boldsymbol{x}; F) = \bigoplus_m H_p(\boldsymbol{x}; F)_m$ is a finitely generated graded S-module annihilated by $\operatorname{Ann}_S(F) + (\boldsymbol{x})$. Consequently, if*

$$(\mathfrak{m}_B S \cdot \operatorname{Irr}(S))^N \subseteq \operatorname{Ann}_S(F) + (\boldsymbol{x}) \quad \text{for some } N \geq 1, \tag{41.3}$$

   then $\operatorname{eu}(\boldsymbol{x}; F)(m)$ is defined in large degree.

3. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of finitely generated graded S-modules with degree-preserving maps, and let $\boldsymbol{x}$ satisfy (41.3) for F. Then (41.3) holds for F' and F'', and in large degree*

$$\operatorname{eu}(\boldsymbol{x}; F)(m) = \operatorname{eu}(\boldsymbol{x}; F')(m) + \operatorname{eu}(\boldsymbol{x}; F'')(m).$$

4. *For $\delta \in \mathbb{Z}^k$ and all m, $\operatorname{eu}(\boldsymbol{x}; F(-\delta))(m)$ is defined if and only if $\operatorname{eu}(\boldsymbol{x}; F)(m - \delta)$ is, and then they are equal.*

Proof. Item 1. By the definition of a standard algebra, $S_\mu = S_{e_0}^{\mu_0} \cdots S_{e_{k-1}}^{\mu_{k-1}}$ (the B-span of products) by induction on μ, so each S_μ is a finitely generated B-module, the S_{e_i} being finitely generated; and S is generated as a B-algebra by finitely many generators of the S_{e_i}, hence Noetherian by Hilbert's basis theorem. The homogeneous components of a finite generating set of F give homogeneous generators $f_1, \ldots, f_t$ of degrees $\delta_1, \ldots, \delta_t \in \mathbb{Z}^k$, and $F_m = \sum_j S_{m-\delta_j} f_j$ (with $S_\mu = 0$ for $\mu \notin \mathbb{N}^k$), a finitely generated B-module. Now let $\mathfrak{b}F = 0$ with $(\mathfrak{m}_B S \cdot \operatorname{Irr}(S))^N \subseteq \mathfrak{b}$, and let $m \geq \delta_j + N\not\Vdash$ for all j. Since $S_{\not\Vdash} = S_{e_0} \cdots S_{e_{k-1}} \subseteq \operatorname{Irr}(S)$, we have $S_{m-\delta_j} = S_{\not\Vdash}^N S_{m-\delta_j - N\not\Vdash} \subseteq \operatorname{Irr}(S)^N$, so $F_m \subseteq \operatorname{Irr}(S)^N F$ and

$$\mathfrak{m}_B^N F_m \subseteq (\mathfrak{m}_B S)^N \operatorname{Irr}(S)^N F = (\mathfrak{m}_B S \cdot \operatorname{Irr}(S))^N F \subseteq \mathfrak{b}F = 0.$$

A finitely generated module over the Artinian ring $B/\mathfrak{m}_B^N$ has finite length.

Item 2. Each $\mathrm{Kos}_p(\boldsymbol{x}; F)$ is a finite direct sum of shifts of F, a finitely generated graded S-module, and the differential preserves degrees; so its kernels and images are graded submodules and, S being Noetherian, $H_p(\boldsymbol{x}; F)$ is finitely generated and graded. It is annihilated by $\mathrm{Ann}_S(F)$, which annihilates every term of the complex, and by every member x_j of $\boldsymbol{x}$, since multiplication by x_j on a Koszul complex is null-homotopic ([55, Prop. 1.6.5]). If (41.3) holds, item 1 with $\mathfrak{b} = \mathrm{Ann}_S(F) + (\boldsymbol{x})$, applied to the finitely many non-zero modules $H_p(\boldsymbol{x}; F)$, $0 \le p \le q$ (q the length of $\boldsymbol{x}$), shows that all lengths in (41.2) are finite in large degree.

Item 3. The ideal $\mathrm{Ann}_S(F)$ annihilates F' and F'', so (41.3) passes to them. As $\mathrm{Kos}_p(\boldsymbol{x}; -)$ is a finite direct sum of shifts, $0 \to \mathrm{Kos}_\bullet(\boldsymbol{x}; F') \to \mathrm{Kos}_\bullet(\boldsymbol{x}; F) \to \mathrm{Kos}_\bullet(\boldsymbol{x}; F'') \to 0$ is an exact sequence of complexes, in each degree m. Its long exact homology sequence in degree m is a finite exact sequence of B-modules, which by item 2 (for F, F', F'') consists of modules of finite length in large degree; the alternating sum of the lengths in such a sequence is 0, which is the assertion.

Item 4. We have $\mathrm{Kos}_\bullet(\boldsymbol{x}; F(-\delta))_m = \mathrm{Kos}_\bullet(\boldsymbol{x}; F)_{m-\delta}$ as complexes of B-modules. □

Lemma 41.15. *Let B be a Noetherian local ring with maximal ideal $\mathfrak{m}_B$, S a standard $\mathbb{N}^k$-graded B-algebra, F a finitely generated graded S-module and $\boldsymbol{x}$ a sequence of homogeneous elements of S with $(\mathfrak{m}_B S \cdot \mathrm{Irr}(S))^N \subseteq \mathrm{Ann}_S(F) + (\boldsymbol{x})$ for some $N \ge 1$. Then in large degree $m \in \mathbb{Z}^k$, $\mathrm{eu}(\boldsymbol{x}; F)(m)$ is defined and*

$$\mathrm{eu}(\boldsymbol{x}; F)(m) \ge 0.$$

Proof. In this proof, a sequence $\boldsymbol{x}$ of homogeneous elements is *admissible for* a finitely generated graded S-module M if $(\mathfrak{m}_B S \cdot \mathrm{Irr}(S))^N \subseteq \mathrm{Ann}_S(M) + (\boldsymbol{x})$ for some $N \ge 1$. By Lemma 41.14, item 2, $\mathrm{eu}(\boldsymbol{x}; F)(m)$ is defined in large degree. We prove the inequality by induction on the length q of $\boldsymbol{x}$, for all F at once. If $\boldsymbol{x}$ is admissible for M, it is admissible for every module whose annihilator contains $\mathrm{Ann}_S(M)$, in particular for submodules and quotients of M.

Step 1: a filtration with prime quotients. We show: there are graded submodules $0 = F_0 \subset F_1 \subset \cdots \subset F_t = F$ and degree-preserving isomorphisms $F_i/F_{i-1} \cong (S/\mathfrak{q}_i)(-\delta_i)$ with homogeneous prime ideals $\mathfrak{q}_i$ and $\delta_i \in \mathbb{Z}^k$. Let $M \ne 0$ be a finitely generated graded S-module. As S is Noetherian (Lemma 41.14, item 1), among the annihilators $\mathrm{Ann}_S(f)$ of non-zero homogeneous $f \in M$ there is a maximal one, $\mathfrak{q} = \mathrm{Ann}_S(f) \ne S$. It is homogeneous: if $af = 0$, the products $a_\mu f$ of the components of a lie in distinct degrees, so all vanish. For homogeneous a, b with $ab \in \mathfrak{q}$ and $b \notin \mathfrak{q}$, $bf \ne 0$ is homogeneous and $\mathrm{Ann}_S(bf) \supseteq \mathfrak{q}$, so equality holds by maximality and $a \in \mathfrak{q}$. For arbitrary $a, b \notin \mathfrak{q}$ with $ab \in \mathfrak{q}$: removing from a and b their components lying in $\mathfrak{q}$ changes neither a, b modulo $\mathfrak{q}$ nor $ab \in \mathfrak{q}$, so we may assume that no component of a or b lies in $\mathfrak{q}$. Order $\mathbb{Z}^k$ lexicographically, a total order compatible with addition, and let a^*, b^* be the components of largest degree. The component of ab of degree $\deg a^* + \deg b^*$ is $a^* b^*$, every other pair of components having a smaller degree sum; as $\mathfrak{q}$ is homogeneous, $a^* b^* \in \mathfrak{q}$, so a^* or b^* lies in $\mathfrak{q}$, a contradiction. Hence $\mathfrak{q}$ is prime and $Sf \cong (S/\mathfrak{q})(-\deg f)$, $1 \mapsto f$. Starting with $M = F$ and continuing with F/F_i (lifting the submodule found), we obtain a strictly increasing chain of graded submodules, which stops at F because F is a Noetherian module.

Step 2: reduction to $S/\mathfrak{q}$. $\boldsymbol{x}$ is admissible for each F_i and each F_i/F_{i-1}. By Lemma 41.14, items 3 and 4, in large degree

$$\mathrm{eu}(\boldsymbol{x}; F)(m) = \sum_{i=1}^{t} \mathrm{eu}(\boldsymbol{x}; S/\mathfrak{q}_i)(m - \delta_i),$$

and $\boldsymbol{x}$ is admissible for $S/\mathfrak{q}_i$, whose annihilator is $\mathfrak{q}_i \supseteq \operatorname{Ann}_S(F)$. As $m - \delta_i$ is in large degree when m is, it suffices to prove $\operatorname{eu}(\boldsymbol{x}; D)(m) \geq 0$ in large degree for $D = S/\mathfrak{q}$, $\mathfrak{q}$ a homogeneous prime ideal with $\boldsymbol{x}$ admissible for D.

Step 3: $q = 0$. Then $\operatorname{eu}((); D)(m) = \operatorname{length}_B D_m \geq 0$.

Step 4: $q \geq 1$. Write $\boldsymbol{x} = (s, \boldsymbol{x}')$ with $s \in S_d$; the induction hypothesis applies to $\boldsymbol{x}'$. D is a domain.

- If $s \notin \mathfrak{q}$, then $0 :_D s = 0$, and by Lemma 41.16, in large degree

$$\operatorname{eu}(\boldsymbol{x}; D)(m) = \operatorname{eu}(\boldsymbol{x}'; D/sD)(m),$$

  as $\operatorname{eu}(\boldsymbol{x}'; 0) = 0$. Since $\operatorname{Ann}_S(D/sD) \supseteq \mathfrak{q} + (s)$, $\boldsymbol{x}'$ is admissible for D/sD, and the induction hypothesis gives $\operatorname{eu}(\boldsymbol{x}; D)(m) \geq 0$ in large degree.

- If $s \in \mathfrak{q}$ and $S_d \subseteq \mathfrak{q}$, then $d \neq 0$, as $1 \in S_0$. Since $S_d = S_{e_0}^{d_0} \cdots S_{e_{k-1}}^{d_{k-1}}$ and $\mathfrak{q}$ is prime, $S_{e_i} \subseteq \mathfrak{q}$ for some i with $d_i \geq 1$ (otherwise a product of elements $y_i \in S_{e_i} \setminus \mathfrak{q}$ lies in $S_d \setminus \mathfrak{q}$). Then $S_\mu = S_{\mu - e_i} S_{e_i} \subseteq \mathfrak{q}$, so $D_\mu = 0$, for every $\mu \in \mathbb{N}^k$ with $\mu_i \geq 1$, and $D_\mu = 0$ for $\mu \notin \mathbb{N}^k$. Every term D_{m-d_J} of $\operatorname{Kos}_\bullet(\boldsymbol{x}; D)_m$ vanishes once m_i exceeds the i-th coordinates of all d_J; so $\operatorname{eu}(\boldsymbol{x}; D)(m) = 0$ in large degree.

- If $s \in \mathfrak{q}$ and there is $\tau \in S_d \setminus \mathfrak{q}$: then $sD = 0$ and $0 :_D s = D$, and $\boldsymbol{x}'$ is admissible for D, since $\mathfrak{q} + (s, \boldsymbol{x}') = \mathfrak{q} + (\boldsymbol{x}')$. By Lemma 41.16, in large degree $\operatorname{eu}(\boldsymbol{x}; D)(m) = \operatorname{eu}(\boldsymbol{x}'; D)(m) - \operatorname{eu}(\boldsymbol{x}'; D)(m-d)$. Multiplication by τ is injective on the domain D, so $0 \to D(-d) \xrightarrow{\tau} D \to D/\tau D \to 0$ is exact with degree-preserving maps, and by Lemma 41.14, items 3 and 4, in large degree $\operatorname{eu}(\boldsymbol{x}'; D/\tau D)(m) = \operatorname{eu}(\boldsymbol{x}'; D)(m) - \operatorname{eu}(\boldsymbol{x}'; D)(m-d)$. Hence $\operatorname{eu}(\boldsymbol{x}; D)(m) = \operatorname{eu}(\boldsymbol{x}'; D/\tau D)(m)$ in large degree; $\boldsymbol{x}'$ is admissible for $D/\tau D$, and the induction hypothesis gives $\operatorname{eu}(\boldsymbol{x}; D)(m) \geq 0$.

Finitely many conditions in large degree hold simultaneously in large degree, which completes the induction. □

Lemma 41.16. *Let B be a Noetherian local ring with maximal ideal $\mathfrak{m}_B$, S a standard $\mathbb{N}^k$-graded B-algebra and F a finitely generated graded S-module. Let $s \in S_d$, let $\boldsymbol{x}'$ be a sequence of homogeneous elements, write $(s, \boldsymbol{x}')$ for s followed by $\boldsymbol{x}'$, and assume $(\mathfrak{m}_B S \cdot \operatorname{Irr}(S))^N \subseteq \operatorname{Ann}_S(F) + (s, \boldsymbol{x}')$ for some $N \geq 1$. Let $0 :_F s = \{f \in F : sf = 0\}$. Then in large degree $m \in \mathbb{Z}^k$ the Euler characteristics below are defined and*

$$\operatorname{eu}((s, \boldsymbol{x}'); F)(m) = \operatorname{eu}(\boldsymbol{x}'; F/sF)(m) - \operatorname{eu}(\boldsymbol{x}'; 0 :_F s)(m - d).$$

Proof. Write $Y = \operatorname{Kos}_\bullet((s, \boldsymbol{x}'); F)$ and $K(M) = \operatorname{Kos}_\bullet(\boldsymbol{x}'; M)$ with differential ∂', for graded S-modules M; let $Z = 0 :_F s$, a graded submodule as s is homogeneous. All maps below preserve degrees.

The cone description. Give s the index 1 and the members of $\boldsymbol{x}'$ the indices $2, 3, \ldots$. Sorting the basis elements ϵ_J of Y according to whether $1 \in J$ identifies $Y_p = K_p(F) \oplus K_{p-1}(F)(-d)$, the second summand corresponding to the $\epsilon_{\{1\} \cup J'}$. For $J = \{1\} \cup J'$, the term $t = 1$ of the differential is $sf\epsilon_{J'}$, and the terms $t \geq 2$ are those of $\partial'(f\epsilon_{J'})$ with the opposite sign, since the position of each j_t in J' is $t - 1$. Hence

$$\partial(y, z) = (\partial' y + sz,\ -\partial' z).$$

A subcomplex and its quotient. The submodules $Y'_p = 0 \oplus K_{p-1}(Z)(-d)$ form a subcomplex, since $\partial(0,z) = (sz, -\partial' z) = (0, -\partial' z)$ for z with coefficients in Z; it is $K(Z)(-d)$ shifted by one place with differential $-\partial'$, so $H_p(Y') \cong H_{p-1}(\boldsymbol{x}'; Z)(-d)$. The quotient $Y'' = Y/Y'$ has $Y''_p = K_p(F) \oplus K_{p-1}(F/Z)(-d)$ and $\partial(y, \bar{z}) = (\partial' y + sz, -\partial' \bar{z})$, well defined as $sZ = 0$. Let $\Phi\colon Y'' \to K(F/sF)$, $(y, \bar{z}) \mapsto \bar{y}$. It is a surjective map of complexes, since sz has coefficients in sF. Its kernel W consists of the $(y, \bar{z})$ with $y \in K_p(sF)$, and W is acyclic: let $(y, \bar{z}) \in W$ be a cycle, so $\partial' y = -sz$ and $\partial' \bar{z} = 0$. Write $y = sv$ with $v \in K_p(F)$. Then $s(\partial' v + z) = \partial' y + sz = 0$, so $\partial' \bar{v} = -\bar{z}$ in $K(F/Z)$, and $\partial(0, \bar{v}) = (sv, -\partial' \bar{v}) = (y, \bar{z})$. By the long exact sequence of $0 \to W \to Y'' \to K(F/sF) \to 0$, Φ induces isomorphisms $H_p(Y'') \cong H_p(\boldsymbol{x}'; F/sF)$.

Lengths. The long exact sequence of $0 \to Y' \to Y \to Y'' \to 0$ in degree m reads

$$\cdots \to H_{p-1}(\boldsymbol{x}'; Z)_{m-d} \to H_p((s, \boldsymbol{x}'); F)_m \to H_p(\boldsymbol{x}'; F/sF)_m \to H_{p-2}(\boldsymbol{x}'; Z)_{m-d} \to \cdots,$$

a finite exact sequence. Since $\operatorname{Ann}_S(F/sF)$ and $\operatorname{Ann}_S(Z)$ contain $\operatorname{Ann}_S(F) + (s)$, the hypothesis gives $(\mathfrak{m}_B S \cdot \operatorname{Irr}(S))^N \subseteq \operatorname{Ann}_S(M) + (\boldsymbol{x}')$ for $M = F/sF$ and $M = Z$, and $\subseteq \operatorname{Ann}_S(F) + (s, \boldsymbol{x}')$ for F. By Lemma 41.14, items 1 and 2 (Z and F/sF are finitely generated, S being Noetherian), the three Euler characteristics are defined in large degree, in the variable $m-d$ for Z, which is in large degree when m is. Taking the alternating sum of lengths, with $H_p(Y')_m \cong H_{p-1}(\boldsymbol{x}'; Z)_{m-d}$ contributing with the sign $(-1)^p = -(-1)^{p-1}$, gives the formula. □

Lemma 41.17. *Let $\mathcal{O}$ be a coefficient ring with uniformiser ϖ, J a finite set with a map* $\mathrm{wt}\colon J \to \{-1, 1, 2\}$ *such that $n = \#\mathrm{wt}^{-1}(-1)$, $n_1 = \#\mathrm{wt}^{-1}(1)$ and $n_2 = \#\mathrm{wt}^{-1}(2)$ are ≥ 1, and $P = \mathcal{O}[\ell^{-1}][[X_i : i \in J]]$ with the weights defined by* wt. *Let $A \subseteq P$ be an $\mathcal{O}$-subalgebra such that, with A_j the set of elements of A of weight j and φ_0 the constant term, $A = \bigoplus_j A_j$, (A, φ_0) is a graded $\mathcal{O}$-algebra with a fixed point, and the inclusion $A \to P$ satisfies (Sm1)–(Sm3). Let u be a system of equations for (A, φ_0) with $\mathrm{ne}(-2) = \mathrm{ne}(0) = 0$ and $\mathrm{ne}(-1) = n + 1$ such that $u_{-1,n+1} \in \mathrm{Sq}(\varphi_0)$. Then there is a homogeneous prime ideal $\mathfrak{q}$ of $\mathrm{Rs}(u)$ such that the graded $\mathrm{Rs}(u)$-module $F = \mathrm{Rs}(u)/\mathfrak{q}$ has the following properties:*

1. *$\varpi \notin \mathfrak{q}$;*
2. *every $b \in A_0$ with $\varphi_0(b) = 0$ lies in $\mathfrak{q}$, and $bF \subseteq \varpi F$ for every b in the maximal ideal of A_0;*
3. *$f^2 T_0 - g^2 T_0 \in \mathfrak{q}$ for all $f, g \in A_{-1}$ with $f - g \in \mathrm{Sq}(\varphi_0)$;*
4. *for every $\mu \in \mathbb{N}^3$, the A_0-module $(F/\varpi F)_\mu$ has length*

$$h(\mu) = \binom{2\mu_0 + 4\mu_2 + n - 1}{n - 1} p(2\mu_1 + 2\mu_2),$$

where $p(k)$ is the number of pairs $(\beta, \gamma) \in \mathbb{N}^{n_1} \times \mathbb{N}^{n_2}$ with $|\beta| + 2|\gamma| = k$, $|\beta|$ the sum of the entries.

Proof. Write $K_0 = \mathcal{O}[\ell^{-1}]$, $\mathrm{Rs} = \mathrm{Rs}(u)$, $\mathfrak{m}_0 = \ker \varphi_0$. For a monomial $X^\alpha = \prod_i X_i^{\alpha_i}$ let $\mathrm{xd}(\alpha) = \sum_{\mathrm{wt}(i)=-1} \alpha_i$ (its *x-degree*) and $\mathrm{pw}(\alpha) = \sum_{\mathrm{wt}(i)=1} \alpha_i + 2\sum_{\mathrm{wt}(i)=2} \alpha_i$; its weight is $\mathrm{pw}(\alpha) - \mathrm{xd}(\alpha)$. For $g \in P$ and $k \in \mathbb{N}$ let $g_{[k]}$ be the sum of the terms of g of x-degree k, and $\mathrm{Fil}_k = \{g \in P : g_{[k']} = 0 \text{ for } k' < k\}$. Since x-degrees add under multiplication of monomials and each coefficient of a product is a finite sum of products of coefficients, $\mathrm{Fil}_k \mathrm{Fil}_{k'} \subseteq \mathrm{Fil}_{k+k'}$ and $(gg')_{[k+k']} = g_{[k]} g'_{[k']}$ for $g \in \mathrm{Fil}_k$, $g' \in \mathrm{Fil}_{k'}$. Put $\theta(\mu) = 2\mu_0 + 4\mu_2$ for $\mu \in \mathbb{N}^3$. For $g \in P$, $g^{(1)}$ is its linear part (the terms of total degree one).

Step 1: $N^\mu \subseteq \mathrm{Fil}_{\theta(\mu)} \cap A_{-2\mu_0+2\mu_1-2\mu_2}$. A monomial of weight -2 has x-degree $\mathrm{pw}+2 \geq 2$, so $A_{-2} \subseteq \mathrm{Fil}_2$; trivially $A_2 \subseteq \mathrm{Fil}_0$. Elements of $\mathfrak{m}_0$ have zero constant term, so elements of $\mathfrak{m}_0^2$ have zero linear part; hence an element f of $\mathrm{Sq}(\varphi_0)$ has $f^{(1)} = 0$ ($cf \in \mathfrak{m}_0^2$ with $c \neq 0$). A monomial of weight -1 has x-degree $\mathrm{pw}+1 \geq 1$, with equality only for a single variable of weight -1, a linear term. So $w \in \mathrm{Fil}_2$, as $w \in A_{-1} \cap \mathrm{Sq}(\varphi_0)$, and $N_2 \subseteq \mathrm{Fil}_4$, since Fil_k is an ideal of P. As $N_0, N_2 \subseteq A_{-2}$ and $N_1 = A_2$, the claim follows by multiplying.

For $x \in N^\mu$ the polynomial $x_{[\theta(\mu)]}$ is therefore a K_0-combination of monomials X^α with $\mathrm{xd}(\alpha) = \theta(\mu)$ and $\mathrm{pw}(\alpha) = 2\mu_1 + 2\mu_2$ (the weight plus the x-degree). Let G_μ be the K_0-span of these finitely many monomials. A monomial is the product of a monomial in the n variables of weight -1 and one in the variables of weights $1, 2$, so $\dim_{K_0} G_\mu = \binom{\theta(\mu)+n-1}{n-1} p(2\mu_1+2\mu_2) = h(\mu)$.

Step 2: the leading-form homomorphism and $\mathfrak{q}$. Define $\Lambda\colon \mathrm{Rs} \to K_0[X_i : i \in J][T_0, T_1, T_2]$ by $\Lambda(xT^\mu) = x_{[\theta(\mu)]}T^\mu$ for $x \in N^\mu$, additively on $\mathrm{Rs} = \bigoplus_\mu N^\mu T^\mu$. For $x \in N^\mu$, $x' \in N^{\mu'}$, Step 1 and the product rule give $(xx')_{[\theta(\mu+\mu')]} = x_{[\theta(\mu)]}x'_{[\theta(\mu')]}$, as θ is additive; and $\Lambda(1) = 1$. So Λ is a ring homomorphism mapping Rs_μ into $G_\mu T^\mu$. Let $\mathfrak{q} = \ker\Lambda$: it is homogeneous, Λ preserving degrees, and prime, the target being a domain. So $F \cong \Lambda(\mathrm{Rs}) = \bigoplus_\mu \Lambda_\mu T^\mu$ with $\Lambda_\mu = \{x_{[\theta(\mu)]} : x \in N^\mu\} \subseteq G_\mu$. For $b \in A_0 = N^0$, $b_{[0]}$ consists of the terms of weight 0 and x-degree 0, hence of positive weight 0: it is the constant term, $\Lambda(b) = \varphi_0(b)$. So $b \in A_0$ acts on $F \cong \Lambda(\mathrm{Rs})$ as multiplication by $\varphi_0(b) \in \mathcal{O}$.

Items 1–3. We have $\Lambda(\varpi) = \varpi \neq 0$. If $\varphi_0(b) = 0$, then $\Lambda(b) = 0$. If b lies in the maximal ideal of A_0, then $\varphi_0(b) \in \varpi\mathcal{O}$: otherwise $\varphi_0(b)$ is a unit of $\mathcal{O}$, and $\varphi_0(b) = b - (b - \varphi_0(b))$ lies in the maximal ideal, as $b - \varphi_0(b) \in \ker(\varphi_0|_{A_0})$, a proper ideal; so $bF = \varphi_0(b)F \subseteq \varpi F$. For $f \in A_{-1}$, $f \in \mathrm{Fil}_1$ and $f_{[1]}$ consists of the terms of x-degree 1 and positive weight 0, i.e. $f_{[1]} = f^{(1)}$; so $\Lambda(f^2T_0) = (f^{(1)})^2T_0$. If $f - g \in \mathrm{Sq}(\varphi_0)$, then $f^{(1)} = g^{(1)}$ (Step 1), hence $\Lambda(f^2T_0) = \Lambda(g^2T_0)$.

Step 3: $K_0\Lambda_\mu = G_\mu$. For $i \in J$, (Sm3) gives $f \in A$ with $f^{(1)} = c_iX_i$, $c_i \neq 0$. Writing $f = \sum_m f_m$ with $f_m \in A_m$, the terms of weight m of f are those of f_m, so $f_i = f_{\mathrm{wt}(i)} \in A_{\mathrm{wt}(i)}$ has $f_i^{(1)} = c_iX_i$. If $\mathrm{wt}(i) = -1$, $(f_i)_{[1]} = c_iX_i$ (Items 1–3); if $\mathrm{wt}(i) = 1$, $(f_i)_{[0]}$ consists of the terms of x-degree 0 and positive weight 1, so $(f_i)_{[0]} = c_iX_i$; if $\mathrm{wt}(i) = 2$, $(f_i)_{[0]}$ consists of the terms of x-degree 0 and positive weight 2, so it is c_iX_i plus a quadratic form in the variables of weight 1. Hence:

- for $\mathrm{wt}(a) = \mathrm{wt}(b) = -1$, $f_af_b \in A_{-2}$ and $(f_af_b)_{[2]} = c_ac_bX_aX_b$; so $K_0\Lambda_{e_0} = G_{e_0}$, the span of the quadratic monomials in the variables of weight -1;
- for $\mathrm{wt}(a) = \mathrm{wt}(b) = 1$, $f_af_b \in A_2$ and $(f_af_b)_{[0]} = c_ac_bX_aX_b$; with the f_i, $\mathrm{wt}(i) = 2$, this gives $K_0\Lambda_{e_1} = G_{e_1}$, the span of the X_aX_b ($\mathrm{wt}(a) = \mathrm{wt}(b) = 1$) and the X_i ($\mathrm{wt}(i) = 2$);
- for $f, f' \in A_{-2}$, $g \in A_2$, $ff'g \in N_2$ and $(ff'g)_{[4]} = f_{[2]}f'_{[2]}g_{[0]}$; so $K_0\Lambda_{e_2} \supseteq G_{e_0}G_{e_0}G_{e_1} = G_{e_2}$.

Since Rs is standard (Lemma 41.8, item 2), $\mathrm{Rs}_\mu = \mathrm{Rs}_{e_0}^{\mu_0}\mathrm{Rs}_{e_1}^{\mu_1}\mathrm{Rs}_{e_2}^{\mu_2}$, so $K_0\Lambda_\mu \supseteq G_{e_0}^{\mu_0}G_{e_1}^{\mu_1}G_{e_2}^{\mu_2}$. Every monomial X^α spanning G_μ lies in the right side: its factor in the variables of weight -1 has degree $2(\mu_0 + 2\mu_2)$, a product of $\mu_0 + 2\mu_2$ quadratic monomials; its factor in the variables of weights $1, 2$ has positive weight $2(\mu_1 + \mu_2)$, so contains an even number of variables of weight 1, and is a product of $\mu_1 + \mu_2$ monomials of positive weight 2 (pairs of variables of weight 1, single variables of weight 2); grouping two quadratic factors with one factor of positive weight 2 exactly μ_2 times gives a factorisation of type $G_{e_0}^{\mu_0}G_{e_1}^{\mu_1}G_{e_2}^{\mu_2}$. Hence $K_0\Lambda_\mu = G_\mu$.

Item 4. By Lemma 41.8, item 2, and Lemma 41.14, item 1, N^μ is a finitely generated A_0-module; as $\Lambda(bx) = \varphi_0(b)\Lambda(x)$ for $b \in A_0$, Λ_μ is the $\mathcal{O}$-span of the images of finitely many generators. It is torsion free, lying in a K_0-vector space, hence free over the discrete valuation ring

$\mathcal{O}$, of rank $\dim_{K_0} K_0\Lambda_\mu = h(\mu)$ by Steps 1 and 3. Thus $(F/\varpi F)_\mu \cong \Lambda_\mu/\varpi\Lambda_\mu \cong (\mathcal{O}/\varpi\mathcal{O})^{h(\mu)}$, on which A_0 acts through φ_0; its A_0-submodules are its $\mathcal{O}$-submodules, and its length is $h(\mu)$. $\square$

Lemma 41.18. *Let B be a Noetherian local ring with maximal ideal $\mathfrak{m}_B$, S a standard $\mathbb{N}^k$-graded B-algebra, $\mathfrak{q}$ a homogeneous prime ideal of S and $D = S/\mathfrak{q}$. Let $\beta \in B$ be such that its image in S_0 does not lie in $\mathfrak{q}$. Let $\boldsymbol{x} = (x_1, \dots, x_q)$ be a sequence of homogeneous elements of S, $1 \le i \le q$ and $c \ge 1$ an integer; let $\boldsymbol{x}^{(c)}$ be the sequence $\boldsymbol{x}$ with x_i replaced by $\beta^c x_i$ (of the same degree), and $\hat{\boldsymbol{x}}$ the sequence $\boldsymbol{x}$ with x_i removed. Assume that*

$$\big(\mathfrak{m}_B S \cdot \operatorname{Irr}(S)\big)^N \subseteq \mathfrak{q} + (\boldsymbol{x}^{(c)}) \quad \text{for some } N \ge 1.$$

Then in large degree $m \in \mathbb{Z}^k$ the Euler characteristics below are defined and

$$\operatorname{eu}(\boldsymbol{x}^{(c)}; D)(m) = \operatorname{eu}(\boldsymbol{x}; D)(m) + c \cdot \operatorname{eu}(\hat{\boldsymbol{x}}; D/\beta D)(m).$$

Proof. In this proof a sequence $\boldsymbol{y}$ of homogeneous elements is *admissible for* a finitely generated graded S-module M if $(\mathfrak{m}_B S \cdot \operatorname{Irr}(S))^{N'} \subseteq \operatorname{Ann}_S(M) + (\boldsymbol{y})$ for some $N' \ge 1$; then $\operatorname{eu}(\boldsymbol{y}; M)(m)$ is defined in large degree by Lemma 41.14, item 2. The ring D is a domain with $\operatorname{Ann}_S(D) = \mathfrak{q}$, and β acts on D injectively, as $\beta \notin \mathfrak{q}$; all modules below are cyclic or quotients of cyclic modules, hence finitely generated, and β has degree 0, so multiplication by β preserves degrees.

Step 1: the order of the sequence does not matter. Let $E = \bigoplus_j S(-d_j)\epsilon_j$ ($d_j = \deg x_j$). Then $\operatorname{Kos}_p(\boldsymbol{x}; F) = \bigwedge^p E \otimes_S F$ with $\epsilon_J = \epsilon_{j_1} \wedge \cdots \wedge \epsilon_{j_p}$, and the differential of Definition 41.5 is $\partial(\epsilon_{l_1} \wedge \cdots \wedge \epsilon_{l_p} \otimes f) = \sum_t (-1)^{t+1} \epsilon_{l_1} \wedge \dots \widehat{\epsilon_{l_t}} \cdots \wedge \epsilon_{l_p} \otimes x_{l_t} f$ for $l_1 < \cdots < l_p$; both sides are alternating in $(l_1, \dots, l_p)$, so the same formula holds for every ordering of distinct indices. Hence renumbering the ϵ_j by a permutation identifies the Koszul complex of the permuted sequence with $\operatorname{Kos}_\bullet(\boldsymbol{x}; F)$, degree by degree; in particular eu is unchanged by permuting a sequence. We may therefore assume $i = 1$ and write $\boldsymbol{x} = (y, \hat{\boldsymbol{x}})$, $\boldsymbol{x}^{(c')} = (\beta^{c'} y, \hat{\boldsymbol{x}})$.

Step 2: admissibility. For $0 \le c' \le c$, $(\boldsymbol{x}^{(c')}) \supseteq (\boldsymbol{x}^{(c)})$, so $\boldsymbol{x}^{(c')}$ is admissible for D. As $\beta^c y \in \beta S \subseteq \operatorname{Ann}_S(D/\beta D)$ and $\mathfrak{q} \subseteq \operatorname{Ann}_S(D/\beta D)$, $\hat{\boldsymbol{x}}$ is admissible for $D/\beta D$. So all Euler characteristics in the assertion are defined in large degree.

Step 3: one factor of β. Let $1 \le c' \le c$ and $z = \beta^{c'-1} y$, of degree $d = \deg y$. We show that in large degree

$$\operatorname{eu}((\beta z, \hat{\boldsymbol{x}}); D)(m) = \operatorname{eu}((z, \hat{\boldsymbol{x}}); D)(m) + \operatorname{eu}(\hat{\boldsymbol{x}}; D/\beta D)(m).$$

- If $z \notin \mathfrak{q}$, then $\beta z \notin \mathfrak{q}$, $\mathfrak{q}$ being prime; so $0 :_D z = 0 :_D \beta z = 0$, and Lemma 41.16 (admissibility by Step 2) gives in large degree $\operatorname{eu}((z, \hat{\boldsymbol{x}}); D)(m) = \operatorname{eu}(\hat{\boldsymbol{x}}; D/zD)(m)$ and $\operatorname{eu}((\beta z, \hat{\boldsymbol{x}}); D)(m) = \operatorname{eu}(\hat{\boldsymbol{x}}; D/\beta z D)(m)$, the Euler characteristic on the zero module being 0. The sequence
$$0 \to D/zD \xrightarrow{\beta} D/\beta z D \to D/\beta D \to 0$$
is exact with degree-preserving maps: the first map is well defined; it is injective, since $\beta v = \beta z v'$ implies $v = z v'$ (β is injective on D); and the kernel of the second map is $\beta D/\beta z D$, the image of the first. Since $\operatorname{Ann}_S(D/\beta z D) \supseteq \mathfrak{q} + (\beta z)$, $\hat{\boldsymbol{x}}$ is admissible for $D/\beta z D$, and Lemma 41.14, item 3, gives $\operatorname{eu}(\hat{\boldsymbol{x}}; D/\beta z D)(m) = \operatorname{eu}(\hat{\boldsymbol{x}}; D/zD)(m) + \operatorname{eu}(\hat{\boldsymbol{x}}; D/\beta D)(m)$ in large degree.

- If $z \in \mathfrak{q}$, then z and βz both act as 0 on D and have the same degree, so $\operatorname{Kos}_\bullet((z, \hat{\boldsymbol{x}}); D) = \operatorname{Kos}_\bullet((\beta z, \hat{\boldsymbol{x}}); D)$ (the terms depend only on the degrees, the differentials only on the action

of the elements on D), and the two Euler characteristics agree. Moreover $\mathfrak{q}+(\beta z,\hat{\boldsymbol{x}})=\mathfrak{q}+(\hat{\boldsymbol{x}})$, so $\hat{\boldsymbol{x}}$ is admissible for D, and Lemma 41.14, item 3, applied to $0\to D\xrightarrow{\beta}D\to D/\beta D\to 0$ gives $\mathrm{eu}(\hat{\boldsymbol{x}};D)(m)=\mathrm{eu}(\hat{\boldsymbol{x}};D)(m)+\mathrm{eu}(\hat{\boldsymbol{x}};D/\beta D)(m)$, i.e. $\mathrm{eu}(\hat{\boldsymbol{x}};D/\beta D)(m)=0$, in large degree.

Step 4: conclusion. Adding the identities of Step 3 for $c'=1,\dots,c$, which hold simultaneously in large degree, the terms $\mathrm{eu}(\boldsymbol{x}^{(c')};D)(m)$ with $1\le c'<c$ cancel, and $\mathrm{eu}(\boldsymbol{x}^{(c)};D)(m)=\mathrm{eu}(\boldsymbol{x}^{(0)};D)(m)+c\cdot\mathrm{eu}(\hat{\boldsymbol{x}};D/\beta D)(m)$ with $\boldsymbol{x}^{(0)}=\boldsymbol{x}$. □

Lemma 41.19. *Let $n,n_1,n_2\ge 1$ be integers. For $k\in\mathbb{Z}$ let $p(k)$ be the number of pairs $(\beta,\gamma)\in\mathbb{N}^{n_1}\times\mathbb{N}^{n_2}$ with $|\beta|+2|\gamma|=k$, where $|\beta|$ is the sum of the entries, and for $\mu=(\mu_0,\mu_1,\mu_2)\in\mathbb{N}^3$ put*

$$h(\mu)=\binom{2\mu_0+4\mu_2+n-1}{n-1}\,p(2\mu_1+2\mu_2).$$

Let e_0,e_1,e_2 be the unit vectors of $\mathbb{Z}^3$ and $M=n_1+n_2-2$. Then for every $\mu\in\mathbb{N}^3$ with $\mu_0\ge n-1$, $\mu_1\ge M$ and $\mu_2\ge 1$,

$$\sum_{i=0}^{n-1}\sum_{j=0}^{M}\sum_{\varepsilon=0}^{1}(-1)^{i+j+\varepsilon}\binom{n-1}{i}\binom{M}{j}\,h(\mu-ie_0-je_1-\varepsilon e_2)=2^{\,n+n_1-2}.$$

Proof. All points $\mu-ie_0-je_1-\varepsilon e_2$ in the sum lie in $\mathbb{N}^3$, by the bounds on μ. Let $a(N)=\binom{N+n-1}{n-1}$ for $N\ge 0$, $a(N)=0$ for $N<0$, and $b(l)=p(2l)$ for $l\in\mathbb{Z}$; then $h(\mu')=a(2\mu_0'+4\mu_2')\,b(\mu_1'+\mu_2')$ for $\mu'\in\mathbb{N}^3$. In the ring of formal power series,

$$\sum_{N\ge0}a(N)t^N=(1-t)^{-n},\qquad \sum_{k\ge0}p(k)t^k=(1-t)^{-n_1}(1-t^2)^{-n_2}=(1+t)^{n_1}(1-t^2)^{-(n_1+n_2)},$$

the second because each variable of weight 1, respectively 2, contributes a factor $\sum_r t^r$, respectively $\sum_r t^{2r}$. Keeping the terms of even degree and substituting $s=t^2$,

$$\sum_{l\ge0}b(l)s^l=E(s)(1-s)^{-(n_1+n_2)},\qquad E(s)=\sum_{r\ge0}\binom{n_1}{2r}s^r.$$

For a power series $\sum_N c_N t^N$ and a polynomial $\sum_i\gamma_i t^i$, the coefficient of t^N in their product is $\sum_i\gamma_i c_{N-i}$, with $c_{N'}=0$ for $N'<0$.

The sum over i. Fix j,ε and put $N=2\mu_0+4(\mu_2-\varepsilon)\ge 2\mu_0\ge n-1$. Then $\sum_i(-1)^i\binom{n-1}{i}a(N-2i)$ is the coefficient of t^N in $(1-t^2)^{n-1}(1-t)^{-n}=(1+t)^{n-1}(1-t)^{-1}$, which is $\sum_{r=0}^{\min(N,n-1)}\binom{n-1}{r}=2^{n-1}$.

The sum over j and ε. Hence the left side equals $2^{n-1}\sum_{j,\varepsilon}(-1)^{j+\varepsilon}\binom{M}{j}b(L-j-\varepsilon)$ with $L=\mu_1+\mu_2$. As $\sum_{j,\varepsilon}(-1)^{j+\varepsilon}\binom{M}{j}s^{j+\varepsilon}=(1-s)^{M+1}$ and $M+2=n_1+n_2$, this sum is the coefficient of s^L in

$$(1-s)^{M+1}E(s)(1-s)^{-(M+2)}=E(s)(1-s)^{-1},$$

namely $\sum_{r\le L}\binom{n_1}{2r}$. Since $L\ge M+1\ge n_1$, this is $\sum_{r\ge0}\binom{n_1}{2r}=\frac12\big((1+1)^{n_1}+(1-1)^{n_1}\big)=2^{n_1-1}$, as $n_1\ge1$. The left side is therefore $2^{n-1}\cdot2^{n_1-1}=2^{n+n_1-2}$. □

Proposition 41.20. *Let B be a Noetherian local ring, S a standard $\mathbb{N}^k$-graded B-algebra, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_q)$ a sequence of homogeneous elements of S, and $a \in S_0$, regarded as an element of degree 0. Assume that there is $N \geq 1$ with*

$$\big(\mathfrak{m}_B S \cdot \operatorname{Irr}(S)\big)^N \subseteq (\sigma_1, \dots, \sigma_q, a).$$

Then in large degree $m \in \mathbb{Z}^k$ the Euler characteristics $\operatorname{eu}((\boldsymbol{\sigma}, a); S)(m)$ and $\operatorname{eu}((a); S/(\boldsymbol{\sigma}))(m)$ are defined and

$$\operatorname{eu}((\boldsymbol{\sigma}, a); S)(m) \leq \operatorname{eu}((a); S/(\boldsymbol{\sigma}))(m).$$

Proof. We prove, by induction on q, the following more general claim: *for every finitely generated graded S-module F with $(\mathfrak{m}_B S \cdot \operatorname{Irr}(S))^N \subseteq \operatorname{Ann}_S(F) + (\boldsymbol{\sigma}, a)$ for some $N \geq 1$, in large degree $\operatorname{eu}((\boldsymbol{\sigma}, a); F)(m)$ and $\operatorname{eu}((a); F/(\boldsymbol{\sigma})F)(m)$ are defined and $\operatorname{eu}((\boldsymbol{\sigma}, a); F)(m) \leq \operatorname{eu}((a); F/(\boldsymbol{\sigma})F)(m)$.* The proposition is the case $F = S$, as $\operatorname{Ann}_S(S) = 0$.

Definedness. By Lemma 41.14, item 2, $\operatorname{eu}((\boldsymbol{\sigma}, a); F)(m)$ is defined in large degree. As $\operatorname{Ann}_S(F) + (\boldsymbol{\sigma})$ is contained in $\operatorname{Ann}_S(F/(\boldsymbol{\sigma})F)$, the hypothesis gives

$$(\mathfrak{m}_B S \cdot \operatorname{Irr}(S))^N \subseteq \operatorname{Ann}_S(F/(\boldsymbol{\sigma})F) + (a),$$

and the same item shows that $\operatorname{eu}((a); F/(\boldsymbol{\sigma})F)(m)$ is defined in large degree.

The case $q = 0$. Both sides are $\operatorname{eu}((a); F)(m)$.

The induction step. Let $q \geq 1$ and write $\boldsymbol{\sigma} = (s, \boldsymbol{\sigma}')$ with $s = \sigma_1 \in S_d$, so that $(\boldsymbol{\sigma}, a) = (s, (\boldsymbol{\sigma}', a))$. By Lemma 41.16, applied to s and the sequence $(\boldsymbol{\sigma}', a)$, in large degree

$$\operatorname{eu}((\boldsymbol{\sigma}, a); F)(m) = \operatorname{eu}((\boldsymbol{\sigma}', a); F/sF)(m) - \operatorname{eu}((\boldsymbol{\sigma}', a); 0 :_F s)(m - d).$$

The modules F/sF and $0 :_F s$ are finitely generated and graded, and their annihilators contain $\operatorname{Ann}_S(F) + (s)$, so $(\mathfrak{m}_B S \cdot \operatorname{Irr}(S))^N \subseteq \operatorname{Ann}_S(M) + (\boldsymbol{\sigma}', a)$ for $M = F/sF$ and $M = 0 :_F s$. By the induction hypothesis for F/sF and the sequence $\boldsymbol{\sigma}'$ of length $q - 1$, in large degree

$$\operatorname{eu}((\boldsymbol{\sigma}', a); F/sF)(m) \leq \operatorname{eu}\big((a); (F/sF)/(\boldsymbol{\sigma}')(F/sF)\big)(m) = \operatorname{eu}((a); F/(\boldsymbol{\sigma})F)(m),$$

since $(F/sF)/(\boldsymbol{\sigma}')(F/sF) = F/(\boldsymbol{\sigma})F$. By Lemma 41.15 for $0 :_F s$ and the sequence $(\boldsymbol{\sigma}', a)$, $\operatorname{eu}((\boldsymbol{\sigma}', a); 0 :_F s)(m') \geq 0$ for m' in large degree, hence for $m' = m - d$ when m is in large degree. Combining the three statements, which hold simultaneously in large degree, gives $\operatorname{eu}((\boldsymbol{\sigma}, a); F)(m) \leq \operatorname{eu}((a); F/(\boldsymbol{\sigma})F)(m)$. □

Proposition 41.21. *Let $\mathcal{O}$ be a coefficient ring, (A, φ_0) a graded $\mathcal{O}$-algebra with a fixed point and u a system of equations for (A, φ_0) with $\operatorname{ne}(-2) = \operatorname{ne}(0) = 0$ and $\operatorname{ne}(-1) = n + 1$, $n \geq 1$, such that u_{-1} spans rationally in degree -1 and $A_{-2} \subseteq \operatorname{Sq}(\varphi_0)$. Let $a \in A$ be such that (A, φ_0, u, a) satisfies (Gc2)–(Gc5), and let Σ be an orbit cover of u. Then in large degree $m \in \mathbb{Z}^3$, $\operatorname{eu}\big((a); \operatorname{Rs}(u)/(\boldsymbol{\sigma}(u))\big)(m)$ is defined and*

$$\operatorname{eu}\big((a); \operatorname{Rs}(u)/(\boldsymbol{\sigma}(u))\big)(m) \;\leq\; 2^{\,n+1+\operatorname{ne}(1)} \sum_{\varphi \in \Sigma} v_{\mathcal{O}}(\varphi(a)).$$

Proof. Write $\operatorname{Rs} = \operatorname{Rs}(u)$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(u)$, $M = \operatorname{Rs}/(\boldsymbol{\sigma})$, $I_0 = I(u) \cap A_0$, $\mathfrak{d}$ for the ideal of A_0 generated by the products fg with $f \in A_{-2}$, $g \in A_2$, and $C = 2^{\,n+1+\operatorname{ne}(1)}$. By (Gc5), $a \in A_0$. By Lemma 41.8, item 2, and Lemma 41.14, item 1, Rs is a Noetherian standard $\mathbb{N}^3$-graded algebra

over A_0 and each M_d is a finitely generated A_0-module. In a Noetherian ring a finitely generated ideal contained in the radical of an ideal J has a power contained in J; we use this twice.

Step 1: a two-signed point is non-zero on some $f \in A_{-2}$ and some $g \in A_2$. If φ is two-signed, there are $j > 0$ and $x \in A_{-j}$ with $\varphi(x) \neq 0$; then $x^2 \in A_{-2}^{\,j}$ by Lemma 41.11, so $\varphi(f) \neq 0$ for some $f \in A_{-2}$; likewise $\varphi(g) \neq 0$ for some $g \in A_2$. In particular $\varphi(\mathfrak{d}) \neq 0$.

Step 2: two inclusions in Rs. Let $\mathfrak{P}$ be a prime of Rs containing $(\boldsymbol{\sigma})$. If $\mathrm{Irr}(\mathrm{Rs}) \not\subseteq \mathfrak{P}$, then Lemma 41.10 applies (its hypotheses are among ours) and gives $I_0 \subseteq \mathfrak{P}$; and if moreover $\mathfrak{d} \subseteq \mathfrak{P}$, its item 2 gives $\mathfrak{P} \cap A_0 = \mathfrak{m}_{A_0}$, since the other alternative, $\mathfrak{P} \cap A_0 = \ker(\varphi|_{A_0})$ for a two-signed point φ, would contradict Step 1. Hence every prime containing $(\boldsymbol{\sigma})$ contains $I_0\mathrm{Rs} \cdot \mathrm{Irr}(\mathrm{Rs})$, and every prime containing $(\boldsymbol{\sigma}) + \mathfrak{d}\mathrm{Rs}$ contains $\mathfrak{m}_{A_0}\mathrm{Rs} \cdot \mathrm{Irr}(\mathrm{Rs})$. So there is $N \geq 1$ with

$$(I_0\mathrm{Rs} \cdot \mathrm{Irr}(\mathrm{Rs}))^N \subseteq (\boldsymbol{\sigma}), \qquad (\mathfrak{m}_{A_0}\mathrm{Rs} \cdot \mathrm{Irr}(\mathrm{Rs}))^N \subseteq (\boldsymbol{\sigma}) + \mathfrak{d}\mathrm{Rs}.$$

Step 3: the pieces in large degree. For $d \geq N\mathbb{1}$, $\mathrm{Rs}_d = \mathrm{Rs}_{\mathbb{1}}^N \mathrm{Rs}_{d-N\mathbb{1}} \subseteq \mathrm{Irr}(\mathrm{Rs})^N$, so $I_0^N \mathrm{Rs}_d \subseteq (\boldsymbol{\sigma})$ and $I_0^N M_d = 0$. By the second inclusion and Lemma 41.14, item 1, applied to $M/\mathfrak{d}M$, there is $d_0 \geq N\mathbb{1}$ such that $M_d/\mathfrak{d}M_d$ has finite length for $d \geq d_0$. Fix $d \geq d_0$. Then:

- $A_0/\mathrm{Ann}_{A_0}(M_d)$ is a quotient of A_0/I_0^N, which is a finitely generated $\mathcal{O}$-module: A_0/I_0 is one by (Gc2), and each I_0^j/I_0^{j+1} is a finitely generated A_0/I_0-module, A_0 being Noetherian.
- Let $\beta\colon A_0 \to \overline{\mathbb{Q}}_\ell$ be an $\mathcal{O}$-algebra homomorphism with $(M_d)_\mathfrak{p} \neq 0$, $\mathfrak{p} = \ker\beta$. Then $I_0^N \subseteq \mathrm{Ann}(M_d) \subseteq \mathfrak{p}$, so $I_0 \subseteq \mathfrak{p}$. If $\mathfrak{d} \subseteq \mathfrak{p}$, then $(M_d/\mathfrak{d}M_d)_\mathfrak{p} = 0$, as a module of finite length vanishes at the non-maximal prime $\mathfrak{p}$ ($\beta(\varpi) \neq 0$), so $(M_d)_\mathfrak{p} = \mathfrak{d}(M_d)_\mathfrak{p} \subseteq \mathfrak{p}(M_d)_\mathfrak{p}$, and $(M_d)_\mathfrak{p} = 0$ by Nakayama's lemma, a contradiction. So $\beta(fg) \neq 0$ for some $f \in A_{-2}$, $g \in A_2$. The ideal $I(u) + \mathfrak{p}A$ is generated by homogeneous elements and its elements of degree 0 form $I_0 + \mathfrak{p} = \mathfrak{p}$; by Lemma 41.12 (with the element 1) there is a point φ' on u with $\varphi'|_{A_0} = \beta$, and $\varphi'(f)\varphi'(g) \neq 0$, so φ' is two-signed. As Σ is an orbit cover, $\varphi' = t \cdot \varphi$ for some $\varphi \in \Sigma$ and $t \in \overline{\mathbb{Q}}_\ell^{\times}$, and torus translation does not change the restriction to A_0; so $\beta \in \Sigma_0 = \{\varphi|_{A_0} : \varphi \in \Sigma\}$.

Step 4: the Herbrand bound. Apply Lemma 41.26 to $B = A_0$ (local, with the residue field of $\mathcal{O}$, by the definition of a graded algebra with a fixed point), the module M_d, the element a, the finite set Σ_0 and C. The finiteness hypothesis on $A_0/\mathrm{Ann}(M_d)$ and (Hq2) hold by Step 3. For (Hq1), let $\beta = \varphi|_{A_0}$ with $\varphi \in \Sigma$, a two-signed point on u, and $\mathfrak{p} = \ker\beta$: $\beta(a) = \varphi(a) \neq 0$ by (Gc5); $\varphi(A_0)$ is the image of A_0/I_0, a finitely generated $\mathcal{O}$-module, so $\varphi(a)$ is integral over $\mathcal{O}$ and $v_\mathcal{O}(\beta(a)) \geq 0$; and by Proposition 41.23 ((Gc2), (Gc4)) the weight pieces of $A/I(u)$ have local length at most 1 at $\mathfrak{p}$, so Lemma 41.22 with $L = 1$ gives $\mathrm{length}((M_d)_\mathfrak{p}) \leq C$, as $d \geq \mathbb{1}$. Hence M_d/aM_d and $0 :_{M_d} a$ have finite length, so $\mathrm{eu}((a); M)(d)$ is defined, and since the map $\Sigma \to \Sigma_0$ is onto and all terms are ≥ 0,

$$\mathrm{eu}((a); M)(d) = \mathrm{length}(M_d/aM_d) - \mathrm{length}(0 :_{M_d} a) \leq C \sum_{\beta \in \Sigma_0} v_\mathcal{O}(\beta(a)) \leq C \sum_{\varphi \in \Sigma} v_\mathcal{O}(\varphi(a)).$$

This holds for all $d \geq d_0$, which is the assertion. □

Lemma 41.22. *Let $\mathcal{O}$ be a coefficient ring, (A, φ_0) a graded $\mathcal{O}$-algebra with a fixed point, and u a system of equations for (A, φ_0) with $\mathrm{ne}(-2) = \mathrm{ne}(0) = 0$ and $\mathrm{ne}(-1) = n + 1$, $n \geq 1$. Let φ be a two-signed point, $\mathfrak{p} = \ker(\varphi|_{A_0})$ and $R = (A_0)_\mathfrak{p}$, and assume that there is an integer $L \geq 0$ with $\mathrm{length}_R((A/I(u))_{w,\mathfrak{p}}) \leq L$ for every $w \in \mathbb{Z}$. Then for every $d \in \mathbb{N}^3$ with $d \geq (1,1,1)$,*

$$\mathrm{length}_R\Big(\big(\mathrm{Rs}(u)/(\boldsymbol{\sigma}(u))\big)_{d,\mathfrak{p}}\Big) \leq 2^{\,n+1+\mathrm{ne}(1)}\, L.$$

Proof. Write $\mathrm{Rs} = \mathrm{Rs}(u)$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(u)$, $w = u_{-1,n+1}$, and $A_{\mathfrak{p}}$, $\mathrm{Rs}_{\mathfrak{p}}$ for the localisations at $A_0 \setminus \mathfrak{p}$.

Step 1: units of degree ± 2. As φ is not positive-vanishing, there are $j > 0$ and $y \in A_j$ with $\varphi(y) \neq 0$; then $y^2 \in A_{2j} = A_2^j$ by Lemma 41.11, so some product of elements of A_2 is not killed by φ, and there is $g \in A_2$ with $\varphi(g) \neq 0$. Likewise there is $f \in A_{-2}$ with $\varphi(f) \neq 0$. Then $\tau = fg \in A_0 \setminus \mathfrak{p}$ is a unit of R. Every $x \in A_{-2}$ satisfies $x = \tau^{-1}(xg)f$ in $A_{\mathfrak{p}}$ with $xg \in A_0$, and every $y \in A_2$ satisfies $y = \tau^{-1}(fy)g$; hence $(A_{-2})_{\mathfrak{p}} = Rf$ and $(A_2)_{\mathfrak{p}} = Rg$. As $N_2 \subseteq A_{-2}$ and $f \cdot fg = \tau f \in A_{-2}A_{-2}A_2 \subseteq N_2$, also $(N_2)_{\mathfrak{p}} = Rf$. Hence for $\mu \in \mathbb{N}^3$, $(\mathrm{Rs}_\mu)_{\mathfrak{p}} = R\,\epsilon_\mu$ with $\epsilon_\mu = f^{\mu_0+\mu_2} g^{\mu_1} T^\mu$, and $\epsilon_\mu \epsilon_{\mu'} = \epsilon_{\mu+\mu'}$.

Step 2: the localised piece is a quotient of a weight-zero piece. In $A_{\mathfrak{p}}$, $u_{-1,i}^2 = \tau^{-1}(u_{-1,i}^2 g)f$, $w^2 = \tau^{-1}(w^2 g)f$, $u_{1,i}^2 = \tau^{-1}(f u_{1,i}^2)g$ and $u_{2,i} = \tau^{-1}(f u_{2,i})g$. So the sections are R-multiples of ϵ_{e_0}, ϵ_{e_1}, ϵ_{e_2} with coefficients generating the ideal

$$\mathfrak{c} = (u_{-1,i}^2 g \ (i \le n),\ w^2 g,\ f u_{1,i}^2,\ f u_{2,i})R.$$

For $d \geq (1,1,1)$, the degree-d part of $(\boldsymbol{\sigma})$ is the sum of the $s\,\mathrm{Rs}_{d-\deg s}$ over the sections s, so its localisation is $\mathfrak{c}\,\epsilon_d$, and $(\mathrm{Rs}/(\boldsymbol{\sigma}))_{d,\mathfrak{p}} = R\epsilon_d/\mathfrak{c}\epsilon_d$ is a quotient of $R/\mathfrak{c}$. Let I_2 be the ideal of A generated by the $u_{-1,i}^2$ $(i \le n)$, w^2, $u_{1,i}^2$ and $u_{2,i}$. Its degree-0 part is the sum of the $A_{-\deg v} v$ over these generators v; since $(A_{\pm 2})_{\mathfrak{p}}$ are Rf, Rg, its localisation is $\mathfrak{c}$. Hence

$$\mathrm{length}_R\big((\mathrm{Rs}/(\boldsymbol{\sigma}))_{d,\mathfrak{p}}\big) \le \mathrm{length}_R(R/\mathfrak{c}) = \mathrm{length}_R\big((A/I_2)_{0,\mathfrak{p}}\big).$$

Step 3: removing the squares. Let $v_1, \dots, v_r$, $r = n + 1 + \mathrm{ne}(1)$, be the elements $u_{-1,i}$ $(i \le n+1)$ and $u_{1,i}$, and let J_s be the ideal generated by I_2 and $v_1, \dots, v_s$; so $J_0 = I_2$, $J_r = I(u)$, and $v_{s+1}^2 \in J_s$. The map $A \to J_{s+1}/J_s$, $x \mapsto x v_{s+1}$, is surjective and kills J_{s+1} (as $J_s v_{s+1} \subseteq J_s$ and $v_{s+1}^2 \in J_s$); it shifts degrees by $\delta = \deg v_{s+1}$. From $0 \to J_{s+1}/J_s \to A/J_s \to A/J_{s+1} \to 0$, in each weight w and after localisation,

$$\mathrm{length}_R\big((A/J_s)_{w,\mathfrak{p}}\big) \le \mathrm{length}_R\big((A/J_{s+1})_{w,\mathfrak{p}}\big) + \mathrm{length}_R\big((A/J_{s+1})_{w-\delta,\mathfrak{p}}\big).$$

Since all weight pieces of $A/J_r = A/I(u)$ have local length at most L, downward induction on s gives $\mathrm{length}_R((A/J_s)_{w,\mathfrak{p}}) \le 2^{r-s}L$ for all w. For $s = 0$, $w = 0$ and Step 2 this is the assertion. □

Proposition 41.23. *Let $\mathcal{O}$ be a coefficient ring, (A, φ_0) a graded $\mathcal{O}$-algebra with a fixed point and u a system of equations for (A, φ_0) such that (Gc2) and (Gc4) hold (they do not involve the element a). Let φ be a two-signed point on u, $\mathfrak{p} = \ker(\varphi|_{A_0})$ and $R = (A_0)_{\mathfrak{p}}$. Then for every $w \in \mathbb{Z}$,*

$$\mathrm{length}_R\big((A/I(u))_{w,\mathfrak{p}}\big) \le 1.$$

Proof. Write $I = I(u)$, $\sigma = \varphi|_{A_0}$, so $\mathfrak{p} = \ker\sigma$, a prime ideal of A_0. Since I is generated by homogeneous elements, $(A/I)_w = A_w/(I \cap A_w)$ for every w.

Step 1: a unit of degree zero. As φ is not negative-vanishing, there are $j > 0$ and $x \in A_{-j}$ with $\varphi(x) \neq 0$; by Lemma 41.11, $x^2 \in A_{-2j} = A_{-2}^j$, so $\varphi(f) \neq 0$ for some $f \in A_{-2}$. Likewise $\varphi(g) \neq 0$ for some $g \in A_2$. Put $\tau = fg \in A_0$; then $\sigma(\tau) \neq 0$.

Step 2: elements of $\mathfrak{p}$ are nilpotent modulo I up to a unit at $\mathfrak{p}$. Let $p \in \mathfrak{p}$. By (Gc2) the image $\bar{A}_0$ of A_0 in A/I, a subring, is a finitely generated $\mathcal{O}$-module, so the image $\bar{p}$ of p is integral over $\mathcal{O}$ (Atiyah–Macdonald, Prop. 5.1): $Q(\bar{p}) = 0$ for a monic $Q \in \mathcal{O}[X]$. Write $Q = X^r Q_1$ with $Q_1(0) \neq 0$ and put $b = Q_1(p) \in A_0$. Then $bp^r = Q(p) \in I$, and $\sigma(b) = Q_1(\sigma(p)) = Q_1(0) \neq 0$, as $\mathcal{O} \to \overline{\mathbb{Q}}_\ell$ is injective. So: *for every $p \in \mathfrak{p}$ there are $b \in A_0 \setminus \mathfrak{p}$ and $r \geq 0$ with $bp^r \in I$.*

Step 3: removing powers. Let $h \in A_j$ be homogeneous, $b \in A_0 \setminus \mathfrak{p}$ and $r \geq 0$ with $bh^r \in I$. We show that $b'h \in I$ for some $b' \in A_0 \setminus \mathfrak{p}$. Every two-signed point ξ on u kills I, so $\xi(b)\xi(h)^r = 0$ and hence $\xi(bh) = 0$. As $bh \in A_j$, Lemma 41.24 (with f, g of Step 1; (Gc2), (Gc4) hold) gives $c \in \mathcal{O} \setminus \{0\}$ and $N, m \geq 0$ with $cf^m g^N bh \in I$. Put $b' = c\tau^{N+m} b \in A_0$. Then $b'h = (f^N g^m)(cf^m g^N bh) \in I$ and $\sigma(b') = c\,\sigma(\tau)^{N+m}\sigma(b) \neq 0$.

Step 4: $\mathfrak{p}$ kills the weight pieces at $\mathfrak{p}$. Let $w \in \mathbb{Z}$, $p \in \mathfrak{p}$ and $h \in A_w$. By Step 2 there are $b \in A_0 \setminus \mathfrak{p}$ and r with $bp^r \in I$, so $b(ph)^r \in I$, and Step 3 applied to $ph \in A_w$ gives $b' \in A_0 \setminus \mathfrak{p}$ with $b'ph \in I$.

Step 5: the weight pieces are cyclic at $\mathfrak{p}$. Let $w \in \mathbb{Z}$ and $k = |w|$; put $e = f^k$, $e' = g^k$ if $w \geq 0$, and $e = g^k$, $e' = f^k$ if $w < 0$. Then $e \in A_{-2w}$ and $ee' = \tau^k$, so $hh'e \in A_0$ for $h, h' \in A_w$. We find $h_0 \in A_w$ such that for every $h \in A_w$ there are $b \in A_0 \setminus \mathfrak{p}$ and $a \in A_0$ with $bh - ah_0 \in I$.

- If $h_0^2 e \notin \mathfrak{p}$ for some $h_0 \in A_w$: for $h \in A_w$ take $b = h_0^2 e$ and $a = hh_0 e$; then $bh - ah_0 = 0$.
- If $h^2 e \in \mathfrak{p}$ for every $h \in A_w$: take $h_0 = 0$. For $h \in A_w$, Step 2 for $p = h^2 e$ gives $b \in A_0 \setminus \mathfrak{p}$ and r with $b(h^2 e)^r \in I$; then $(b\tau^{kr})h^{2r} = e'^r\, b(h^2 e)^r \in I$ with $b\tau^{kr} \notin \mathfrak{p}$, and Step 3 gives $b' \in A_0 \setminus \mathfrak{p}$ with $b'h \in I$; take $a = 0$.

Step 6: conclusion. Let $M = (A/I)_w$, an A_0-module whose elements are the classes $\bar{h}$ of $h \in A_w$, and $M_\mathfrak{p}$ its localisation, a module over the local ring R with maximal ideal $\mathfrak{p}R$. By Step 5, $b\bar{h} = a\bar{h}_0$ with $b \notin \mathfrak{p}$, so $\bar{h}/s = (a/(bs))\,\bar{h}_0/1$ for $s \in A_0 \setminus \mathfrak{p}$: $M_\mathfrak{p}$ is generated by $\bar{h}_0/1$. By Step 4, $(p/1)(\bar{h}_0/1) = b'p\bar{h}_0/b' = 0$ for $p \in \mathfrak{p}$. Hence $M_\mathfrak{p}$ is a quotient of $R/\mathfrak{p}R$, a field, and $\operatorname{length}_R(M_\mathfrak{p}) \leq 1$. □

Lemma 41.24. *Let $\mathcal{O}$ be a coefficient ring, (A, φ_0) a graded $\mathcal{O}$-algebra with a fixed point and u a system of equations for (A, φ_0) such that (Gc2) and (Gc4) hold. Let $f \in A_{-2}$, $g \in A_2$, $j \in \mathbb{Z}$ and $h \in A_j$ be such that $\xi(h) = 0$ for every two-signed point ξ on u with $\xi(f) = 1$ and $\xi(g) \neq 0$. Then there are $c \in \mathcal{O} \setminus \{0\}$ and $N, m \in \mathbb{N}$ with*

$$c\, f^m g^N h \in I(u).$$

Proof. Write $I = I(u)$; it is generated by homogeneous elements, so A/I is graded and the components of an element of I lie in I. Let $J = I + (f-1)A$, $s \in A/J$ the image of ϖg, $C = (A/J)[s^{-1}]$ and $\pi\colon A \to C$ the canonical map.

Step 1: C is a finitely generated K_0-algebra. By the definition of a graded algebra with a fixed point, A is generated as an $\mathcal{O}$-algebra by A_0 and finitely many homogeneous $s_1, \ldots, s_q$; so every element of A is a polynomial in the s_k with coefficients in A_0. By (Gc2) the image of A_0 in A/I is spanned over $\mathcal{O}$ by the images of finitely many $b_1, \ldots, b_r \in A_0$. Hence A/I, and its quotient A/J, are generated as $\mathcal{O}$-algebras by the images of the b_i and s_k. In C the element $\pi(\varpi)\pi(g) = s$ is a unit, so $\pi(\varpi)$ is a unit and C is a K_0-algebra, generated by the images of the b_i, the s_k and s^{-1}.

Step 2: the points of C give two-signed points on u. Let $\psi\colon C \to \overline{\mathbb{Q}}_\ell$ be a K_0-algebra homomorphism and $\xi = \psi \circ \pi$, an $\mathcal{O}$-algebra homomorphism $A \to \overline{\mathbb{Q}}_\ell$, i.e. a point. It kills $I \subseteq J$, so it lies on u; $\xi(f) = 1$ as $f - 1 \in J$; and $\xi(\varpi)\xi(g) = \psi(s) \neq 0$, s being a unit, so $\xi(g) \neq 0$. As $f \in A_{-2}$ and $g \in A_2$, ξ is neither negative-vanishing nor positive-vanishing: it is two-signed.

Step 3: C has no non-zero derivations at its points. Let ψ, ξ be as in Step 2 and D a ψ-derivation of C. Then $D' = D \circ \pi\colon A \to \overline{\mathbb{Q}}_\ell$ is $\mathcal{O}$-linear and a ξ-derivation, and it vanishes on I. Define the Euler derivation $E\colon A \to \overline{\mathbb{Q}}_\ell$ by $E(x) = \sum_{i \in \mathbb{Z}} i\,\xi(x_i)$, where x_i is the component of

x of degree i. It is $\mathcal{O}$-linear; for $x \in A_i$, $y \in A_{i'}$, $E(xy) = (i+i')\xi(x)\xi(y) = \xi(x)E(y)+\xi(y)E(x)$, and by bilinearity E is a ξ-derivation. It vanishes on I, since the components of an element of I lie in I, which ξ kills. By (Gc4) at the two-signed point ξ on u, there are $(c_1, c_2) \neq (0,0)$ in $\overline{\mathbb{Q}}_\ell$ with $c_1 D' + c_2 E = 0$. Now $D(1) = D(1 \cdot 1) = 2D(1)$ gives $D(1) = 0$, so $D'(f) = D(\pi(f)) = D(1) = 0$, while $E(f) = -2\xi(f) = -2$. Evaluating at f gives $-2c_2 = 0$, so $c_2 = 0$, $c_1 \neq 0$ and $D' = 0$: D vanishes on $\pi(A)$. Every element of C has the form $z = \pi(x)s^{-k}$; then $0 = D(\pi(x)) = D(zs^k) = \psi(s)^k D(z) + \psi(z)D(s^k)$ and $D(s^k) = D(\pi(\varpi^k g^k)) = 0$, so $D(z) = 0$ as $\psi(s) \neq 0$. Thus every ψ-derivation of C is zero.

Step 4: the image of h in C vanishes. For every K_0-algebra homomorphism $\psi\colon C \to \overline{\mathbb{Q}}_\ell$, $\psi(\pi(h)) = \xi(h) = 0$ by the hypothesis on h and Step 2. By Steps 1 and 3, Lemma 41.25 applies to $k = K_0 \subseteq \Omega = \overline{\mathbb{Q}}_\ell$ (characteristic 0) and C, and gives $\pi(h) = 0$. By the definition of the localisation, $s^N h \in J$ in A/J for some $N \geq 0$, that is,

$$h' = \varpi^N g^N h \in I + (f-1)A, \qquad h' \in A_w,\ w = 2N + j.$$

Step 5: dehomogenisation. Write $h' = x + v(f-1)$ with $x \in I$, $v \in A$, and pass to the graded ring $\bar{A} = A/I$, writing $\bar{y}$ for images and $\bar{v}_e$ for the component of $\bar{v}$ of degree e, non-zero for only finitely many e. Since $\bar{f}$ has degree -2, the component of $\bar{v}\bar{f}$ of degree e is $\bar{v}_{e+2}\bar{f}$, and comparing components of degree e in $\bar{h}' = \bar{v}\bar{f} - \bar{v}$ gives

$$\bar{v}_e = \bar{v}_{e+2}\bar{f} \quad (e \neq w), \qquad \bar{v}_w = \bar{v}_{w+2}\bar{f} - \bar{h}'.$$

By the first relation, $\bar{v}_{w+2} = \bar{v}_{w+2+2k}\bar{f}^{\,k}$ for all $k \geq 0$, which is 0 for k large; so $\bar{v}_w = -\bar{h}'$. Again by the first relation, $\bar{v}_{w-2k} = \bar{v}_w \bar{f}^{\,k} = -\bar{f}^{\,k}\bar{h}'$ for $k \geq 0$, which is 0 for k large. So $f^m h' \in I$ for some m, i.e. $\varpi^N f^m g^N h \in I$, and $c = \varpi^N \neq 0$. □

Lemma 41.25. *Let $k \subseteq \Omega$ be fields of characteristic 0 with Ω algebraically closed, and let C be a finitely generated k-algebra. For a k-algebra homomorphism $\psi\colon C \to \Omega$, a ψ-**derivation** is a k-linear map $D\colon C \to \Omega$ with $D(xy) = \psi(x)D(y) + \psi(y)D(x)$ for all $x, y \in C$. Assume that for every k-algebra homomorphism $\psi\colon C \to \Omega$ every ψ-derivation is zero. Then every $z \in C$ with $\psi(z) = 0$ for all k-algebra homomorphisms $\psi\colon C \to \Omega$ is zero.*

Proof. Step 1: every maximal ideal is the kernel of a point. Let $\mathfrak{n}$ be a maximal ideal of C. The field $\kappa = C/\mathfrak{n}$ is a finitely generated k-algebra, hence a finite extension of k by Zariski's lemma (Atiyah–Macdonald, *Introduction to commutative algebra*, Cor. 5.24). As Ω is algebraically closed, there is a k-embedding $\iota\colon \kappa \to \Omega$, and $\psi_{\mathfrak{n}} = \iota \circ (C \to \kappa)$ is a k-algebra homomorphism with kernel $\mathfrak{n}$.

Step 2: $\mathfrak{n} = \mathfrak{n}^2$ for every maximal ideal $\mathfrak{n}$. Suppose $\mathfrak{n} \neq \mathfrak{n}^2$, and let κ, ι be as in Step 1. Put $\bar{C} = C/\mathfrak{n}^2$ and $\bar{\mathfrak{n}} = \mathfrak{n}/\mathfrak{n}^2 \neq 0$; then $\bar{C}$ is local with maximal ideal $\bar{\mathfrak{n}}$, $\bar{\mathfrak{n}}^2 = 0$, and $\bar{\mathfrak{n}}$ is a $\kappa = \bar{C}/\bar{\mathfrak{n}}$-vector space. Since κ/k is finite and separable (characteristic 0), $\kappa = k(\theta)$ with minimal polynomial $Q \in k[X]$ and $Q'(\theta) \neq 0$. Let $t \in \bar{C}$ lift θ; then $Q(t) \in \bar{\mathfrak{n}}$ and $Q'(t)$ is a unit, its image $Q'(\theta)$ being non-zero. Put $\varepsilon = -Q(t)Q'(t)^{-1} \in \bar{\mathfrak{n}}$ and $t' = t + \varepsilon$. By Taylor expansion, $Q(t') = Q(t) + Q'(t)\varepsilon + \varepsilon^2 r = 0$ for some $r \in \bar{C}$, as $\varepsilon^2 = 0$. Hence $X \mapsto t'$ defines a k-algebra homomorphism $k[X]/(Q) \to \bar{C}$; its composite with $\bar{C} \to \kappa$ maps X to θ and is an isomorphism, so its image κ' is a subring of $\bar{C}$ mapped isomorphically onto κ, and $\bar{C} = \kappa' \oplus \bar{\mathfrak{n}}$ as k-vector spaces. For $x \in C$ write $x + \mathfrak{n}^2 = s(x) + \nu(x)$ with $s(x) \in \kappa'$, $\nu(x) \in \bar{\mathfrak{n}}$; the image of $s(x)$ in κ is $x + \mathfrak{n}$, and $\nu(x) = x + \mathfrak{n}^2$ for $x \in \mathfrak{n}$. From $\bar{\mathfrak{n}}^2 = 0$,

$$(x + \mathfrak{n}^2)(y + \mathfrak{n}^2) = s(x)s(y) + \big(s(x)\nu(y) + s(y)\nu(x)\big), \qquad s(x)s(y) \in \kappa',$$

so $\nu(xy) = s(x)\nu(y) + s(y)\nu(x)$, where $s(x)$ acts on $\bar{\mathfrak{n}}$ as $x + \mathfrak{n} \in \kappa$. Choose a non-zero κ-linear map $\lambda\colon \bar{\mathfrak{n}} \to \kappa$ (the κ-vector space $\bar{\mathfrak{n}}$ is non-zero) and put $D = \iota \circ \lambda \circ \nu\colon C \to \Omega$. It is k-linear,

and $D(xy) = \psi_{\mathfrak{n}}(x)D(y) + \psi_{\mathfrak{n}}(y)D(x)$; it is non-zero, as ν maps $\mathfrak{n}$ onto $\bar{\mathfrak{n}}$ and ι is injective. This contradicts the hypothesis for $\psi_{\mathfrak{n}}$.

Step 3: every localisation at a maximal ideal is a field. C is Noetherian by Hilbert's basis theorem. For a maximal ideal $\mathfrak{n}$, the finitely generated ideal $\mathfrak{n}C_{\mathfrak{n}}$ of the local ring $C_{\mathfrak{n}}$ satisfies $\mathfrak{n}C_{\mathfrak{n}} = (\mathfrak{n}C_{\mathfrak{n}})^2$ by Step 2, so $\mathfrak{n}C_{\mathfrak{n}} = 0$ by Nakayama's lemma, and $C_{\mathfrak{n}}$ is a field.

Step 4: conclusion. Let $z \in C$ with $\psi(z) = 0$ for all ψ. By Step 1, z lies in every maximal ideal. Then z is nilpotent: otherwise $C[z^{-1}]$ is a non-zero finitely generated k-algebra; for a maximal ideal $\mathfrak{m}'$ of it, $L = C[z^{-1}]/\mathfrak{m}'$ is a finite extension of k by Zariski's lemma, the image of C in L is a domain of finite dimension over k, hence a field, so the kernel $\mathfrak{n}'$ of $C \to L$ is a maximal ideal of C; and $z \notin \mathfrak{n}'$, as z is a unit in $C[z^{-1}]$. This contradicts the fact that z lies in every maximal ideal. So z is nilpotent, and its image in each field $C_{\mathfrak{n}}$ (Step 3) is nilpotent, hence zero. An element of C whose images in all localisations at maximal ideals vanish is zero (Atiyah–Macdonald, Prop. 3.8), so $z = 0$. □

Lemma 41.26. *Let $\mathcal{O}$ be a coefficient ring, B a local $\mathcal{O}$-algebra with maximal ideal $\mathfrak{m}_B$ such that $\mathcal{O} \to B$ is a local homomorphism inducing an isomorphism of residue fields, and M a finitely generated B-module such that $B/\mathrm{Ann}_B(M)$ is a finitely generated $\mathcal{O}$-module. Let $a \in B$, $C \geq 0$ an integer, and Σ_0 a finite set of $\mathcal{O}$-algebra homomorphisms $B \to \overline{\mathbb{Q}}_\ell$ such that*

(Hq1) *for every $\beta \in \Sigma_0$: $\beta(a) \neq 0$, $v_{\mathcal{O}}(\beta(a)) \geq 0$, and $\mathrm{length}_{B_{\ker\beta}}(M_{\ker\beta}) \leq C$;*

(Hq2) *$M_{\ker\beta} = 0$ for every $\mathcal{O}$-algebra homomorphism $\beta\colon B \to \overline{\mathbb{Q}}_\ell$ with $\beta \notin \Sigma_0$.*

Then M/aM and $0 :_M a = \{x \in M : ax = 0\}$ have finite length and

$$\mathrm{length}_B(M/aM) - \mathrm{length}_B(0 :_M a) \leq C \sum_{\beta\in\Sigma_0} v_{\mathcal{O}}(\beta(a)).$$

Proof. Step 1: lengths over B and over $\mathcal{O}$. The ring $R = B/\mathrm{Ann}_B(M)$ is a finitely generated $\mathcal{O}$-module and a local ring with residue field $\mathcal{O}/\varpi\mathcal{O}$; M is a finitely generated R-module, hence a finitely generated $\mathcal{O}$-module, and so are its B-submodules and quotients. For such a B-module X of finite $\mathcal{O}$-length, every chain of B-submodules is a chain of $\mathcal{O}$-submodules, so X has finite B-length; its B-composition factors are $\cong B/\mathfrak{m}_B \cong \mathcal{O}/\varpi\mathcal{O}$, of $\mathcal{O}$-length 1. Hence $\mathrm{length}_B(X) = \mathrm{length}_{\mathcal{O}}(X)$, and below all lengths are over $\mathcal{O}$.

Step 2: the torsion part. Let M_t be the $\mathcal{O}$-torsion submodule of M, a B-submodule of finite length, and $M' = M/M_t$, a free $\mathcal{O}$-module of finite rank. For the endomorphism a of M_t, $\mathrm{length}(M_t/aM_t) = \mathrm{length}(0 :_{M_t} a)$. The snake lemma for multiplication by a on $0 \to M_t \to M \to M' \to 0$ gives the exact sequence

$$0 \to 0 :_{M_t} a \to 0 :_M a \to 0 :_{M'} a \to M_t/aM_t \to M/aM \to M'/aM' \to 0.$$

Step 3: the determinant of a. Let $V = M \otimes_{\mathcal{O}} K_0 = M' \otimes_{\mathcal{O}} K_0$, a finite-dimensional K_0-vector space with an action of the finite-dimensional K_0-algebra $R' = R \otimes_{\mathcal{O}} K_0$. As R' is Artinian, $V = \bigoplus_{\mathfrak{p}} V_{\mathfrak{p}}$ over the finitely many primes $\mathfrak{p}$ of R with $\varpi \notin \mathfrak{p}$, where $V_{\mathfrak{p}} = M_{\mathfrak{p}}$ (ϖ is a unit in $R_{\mathfrak{p}}$). For such $\mathfrak{p}$, $R/\mathfrak{p}$ is a torsion-free domain finite over $\mathcal{O}$, so $\kappa(\mathfrak{p}) = (R/\mathfrak{p})[\varpi^{-1}]$ is a finite extension of K_0, and $M_{\mathfrak{p}}$, a module of finite length over the Artinian local ring $R_{\mathfrak{p}}$, has a filtration by $R_{\mathfrak{p}}$-submodules with $\lambda_{\mathfrak{p}} = \mathrm{length}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ factors, each isomorphic to $\kappa(\mathfrak{p})$. The determinant of multiplication by a on the K_0-vector space $\kappa(\mathfrak{p})$ is the norm $N_{\kappa(\mathfrak{p})/K_0}(\bar{a}) = \prod_\iota \iota(\bar{a})$ over the $[\kappa(\mathfrak{p}) : K_0]$ embeddings ι of $\kappa(\mathfrak{p})$ into $\overline{\mathbb{Q}}_\ell$ over K_0 (characteristic 0). The composites $B \to R/\mathfrak{p} \to$

$\kappa(\mathfrak{p}) \xrightarrow{\iota} \overline{\mathbb{Q}}_\ell$ are exactly the $\mathcal{O}$-algebra homomorphisms $\beta\colon B \to \overline{\mathbb{Q}}_\ell$ with $\ker\beta = \mathfrak{p}_B$, the inverse image of $\mathfrak{p}$ in B: such a β factors through the domain $B/\mathfrak{p}_B = R/\mathfrak{p}$ and extends uniquely to its fraction field. Moreover $M_\mathfrak{p} = M_{\mathfrak{p}_B}$ and $\lambda_\mathfrak{p} = \mathrm{length}_{B_{\mathfrak{p}_B}}(M_{\mathfrak{p}_B})$. By multiplicativity of determinants along filtrations,

$$\det(a \mid V) = \prod_{\mathfrak{p}} \prod_{\beta\colon\ \ker\beta = \mathfrak{p}_B} \beta(a)^{\lambda_\mathfrak{p}}.$$

If $\lambda_\mathfrak{p} > 0$, every β with $\ker\beta = \mathfrak{p}_B$ lies in Σ_0 by (Hq2), so $\beta(a) \neq 0$ by (Hq1). Hence $\det(a \mid V) \neq 0$: a is injective on V, hence on $M' \subseteq V$, and choosing bases in which the matrix of a on M' is in Smith normal form over the discrete valuation ring $\mathcal{O}$,

$$\mathrm{length}(M'/aM') = v_\mathcal{O}(\det(a \mid M')) = \sum_{\mathfrak{p}} \lambda_\mathfrak{p} \sum_{\beta\colon\ \ker\beta = \mathfrak{p}_B} v_\mathcal{O}(\beta(a)).$$

Step 4: conclusion. As $0 :_{M'} a = 0$, the sequence of Step 2 shows that $0 :_M a = 0 :_{M_t} a$ and M/aM have finite length and $\mathrm{length}(M/aM) - \mathrm{length}(0 :_M a) = \mathrm{length}(M_t/aM_t) + \mathrm{length}(M'/aM') - \mathrm{length}(0 :_{M_t} a) = \mathrm{length}(M'/aM')$. In the last sum of Step 3 only primes with $\lambda_\mathfrak{p} > 0$ contribute; each β occurring then lies in Σ_0, distinct primes give distinct β, and $\lambda_\mathfrak{p} \leq C$ by (Hq1). Since $v_\mathcal{O}(\beta(a)) \geq 0$ for every $\beta \in \Sigma_0$, the sum is at most $C \sum_{\beta\in\Sigma_0} v_\mathcal{O}(\beta(a))$. □

42 The one-character bound on $\mathrm{U}(3)$

This section proves Proposition 42.2, the one-character bound for a unitary Eisenstein datum (K, ℓ, e, N, χ), which Proposition 38.5 uses for Count I. The conditions (Ue1)–(Ue9) of Definition 38.1 are those that a character of a Sha-host satisfies in case (I) for $K = \mathbb{Q}(\sqrt{-p_K(\psi)})$. The deformation problem is set up in Definition 42.1. The Eisenstein representation takes values in $\mathrm{GL}_3 \rtimes \{\pm 1\}$, the group through which conjugate self-dual representations of G_K extend to $G_\mathbb{Q}$ [69]; its restriction to G_K is $\Psi \oplus \mathbf{1} \oplus \Psi^{-1}$. A normalised ordinary lift is normalised at complex conjugation and at a marking (Ed1), trivial on the inertia away from ℓ that the residual representation kills (Ed2), and ordinary at ℓ (Ed3); the ring R represents these lifts, and R^0 is its subalgebra of weight zero.

The proof assembles four inputs. The graded model of Section 43 (Proposition 43.3) embeds the Tate–Shafarevich group into derivations of degree -1, whose length is bounded by the cotangent length (Lemma 42.12). Part IX (Theorem 47.1) gives the finiteness of R^0 and shows that the two-signed points, which are of type I (irreducible) or of type II (endoscopic) by Lemma 42.3, are admissible. Rigidity, Theorem 42.5, is the vanishing of the adjoint Bloch–Kato Selmer group at every two-signed point, which gives (Gc4). At points of type I it is the theorem of Newton and Thorne [192] (Proposition 42.6); at points of type II the adjoint decomposes (Lemma 42.9), and the pieces are handled by the second hypothesis of the bound, by the theorem of Newton and Thorne for newforms, by the vanishing for the quadratic character of K (Lemma 42.10) and by a separate argument for CM newforms (Proposition 42.11). The first hypothesis, that no admissible representation has the Eisenstein trace at σ, gives (Gc5) for the trace difference at σ, and the graded numerical criterion Lemma 41.2 concludes.

Definition 42.1. Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum (Definition 38.1) with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$ with residue field $\mathbb{F}_\mathcal{O}$; put $\Psi = \Psi_{\chi,e}\colon G_\mathbb{Q} \to \mathcal{O}^\times$, let $\eta_K\colon G_\mathbb{Q} \to \{\pm 1\}$ be the character with kernel G_K, and let χ_ℓ be the ℓ-component of χ (Definition 38.1), viewed as a character of $G_\mathbb{Q}$. $\mathrm{CNL}_\mathcal{O}$ is the category of complete Noetherian local $\mathcal{O}$-algebras A for which $\mathcal{O} \to A$ is local and induces an isomorphism $\mathbb{F}_\mathcal{O} \cong A/\mathfrak{m}_A$; its

morphisms are the local $\mathcal{O}$-algebra homomorphisms. Decomposition and inertia groups $D_{\bar v} \supseteq I_{\bar v}$ at places $\bar v$ of $\overline{\mathbb{Q}}$ are as in Definition 38.4. Fix a complex conjugation $c \in G_{\mathbb{Q}}$ and a place $\bar\ell$ of $\overline{\mathbb{Q}}$ above ℓ, and write $D_\ell = D_{\bar\ell} \supseteq I_\ell = I_{\bar\ell}$.

The group and the torus. For a commutative ring B let $J \in \mathrm{GL}_3(B)$ be the antidiagonal matrix with all antidiagonal entries 1, put $\theta(g) = J\,{}^t g^{-1} J$ for $g \in \mathrm{GL}_3(B)$, and let $\mathcal{G}_3(B) = \mathrm{GL}_3(B) \rtimes \{\pm 1\}$, with -1 acting by θ. For an element (g, ς) of $\mathcal{G}_3(B)$, g is its **GL-part** and ς its **sign**; for a homomorphism r into $\mathcal{G}_3(B)$, $\underline{r}$ denotes its GL-part. For $t \in B^\times$ put $\mathrm{T}(t) = \mathrm{diag}(t, 1, t^{-1})$; then $\theta(\mathrm{T}(t)) = \mathrm{T}(t)$. The **weight** of a matrix entry (a, b), $1 \le a, b \le 3$, is $\mathrm{wt}(a, b) = b - a$; conjugation $X \mapsto \mathrm{T}(t) X \mathrm{T}(t)^{-1}$ multiplies the (a, b) entry by $t^{\mathrm{wt}(a,b)}$.

The Eisenstein representation. $r_0 \colon G_{\mathbb{Q}} \to \mathcal{G}_3(\mathcal{O})$, $r_0(g) = (\mathrm{T}(\Psi(g)), \eta_K(g))$; it is a homomorphism because $\theta(\mathrm{T}(t)) = \mathrm{T}(t)$, and its GL-part on G_K is $\Psi \oplus \mathbf{1} \oplus \Psi^{-1}$. Its reduction is $\bar r_0 \colon G_{\mathbb{Q}} \to \mathcal{G}_3(\mathbb{F}_{\mathcal{O}})$. For $1 \le a \le 3$ and $g \in G_{\mathbb{Q}}$ put $\psi_a(g) = \Psi(g)^{2-a}$, the diagonal entries of $\underline{r_0}(g)$.

Lifts. Let G be $G_{\mathbb{Q}}$ or D_ℓ. For $A \in \mathrm{CNL}_{\mathcal{O}}$, a **lift** of $\bar r_0|_G$ to A is a homomorphism $r \colon G \to \mathcal{G}_3(A)$ whose reduction modulo $\mathfrak{m}_A$ is $\bar r_0|_G$ (so its sign is η_K) and whose GL-part has entries that are continuous for the $\mathfrak{m}_A$-adic topology; for a morphism $f \colon A \to A'$, $f \circ r$ is a lift to A'. A **base point** of a lift r to A is an $\mathcal{O}$-algebra homomorphism $\beta \colon A \to \mathcal{O}$ with $\beta \circ r = r_0|_G$. Let $(R^{\square}_\ell, r^{\square}_\ell)$ be the universal lift of $\bar r_0|_{D_\ell}$, which exists by Mazur's representability theorem (D_ℓ satisfies the ℓ-finiteness condition).

The ordinary condition at ℓ. Let $D^\circ_\ell = \{g \in D_\ell : \ \eta_K(g) = 1,\ \chi_\ell(g) = 1\}$. Fix $\sigma_0 \in D^\circ_\ell \cap I_\ell$ with $\chi_{\mathrm{cyc}}(\sigma_0)$ of infinite order, and $\sigma_\ell \in D^\circ_\ell$ such that D°_ℓ is the closed subgroup generated by $D^\circ_\ell \cap I_\ell$ and σ_ℓ, and $\sigma_\ell \in \mathrm{Fr}^n I_\ell$ for an arithmetic Frobenius $\mathrm{Fr} \in D_\ell$ and an integer $n \ge 1$. For a lift r of $\bar r_0|_{D_\ell}$ to A, the **ordinary ideal** $I^{\mathrm{ord}}(r) \subseteq A$ is generated by the coefficients of $\det(X - \underline{r}(g)) - \prod_{a=1}^3 (X - \psi_a(g))$ for $g \in D^\circ_\ell \cap I_\ell$ and by the entries of the product $(\underline{r}(g_1) - \psi_1(g_1))(\underline{r}(g_2) - \psi_2(g_2))(\underline{r}(g_3) - \psi_3(g_3))$ for $g_1, g_2, g_3 \in D^\circ_\ell \cap I_\ell$. With indices of ψ read modulo 3 in $\{1, 2, 3\}$ put

$$\mathrm{gp}_a = (\psi_a(\sigma_0) - \psi_{a+1}(\sigma_0))(\psi_a(\sigma_0) - \psi_{a+2}(\sigma_0)),$$
$$\mathrm{tq}_a(r) = \mathrm{tr}\Big((\underline{r}(\sigma_0) - \psi_{a+1}(\sigma_0))(\underline{r}(\sigma_0) - \psi_{a+2}(\sigma_0))\underline{r}(\sigma_\ell)\Big),$$

$\mathrm{dt}_a(r) = \mathrm{gp}_{a+1}\mathrm{tq}_a(r) - \chi_{\mathrm{cyc}}(\sigma_\ell)\,\mathrm{gp}_a\mathrm{tq}_{a+1}(r)$ and $\mathrm{gd}(r) = \ell\,\mathrm{dt}_1(r)\mathrm{dt}_2(r) \in A$. Let $I^{\mathrm{cr}} \subseteq R^{\square}_\ell$ be the radical of $\{y : \ \mathrm{gd}(r^{\square}_\ell)^n y \in I^{\mathrm{ord}}(r^{\square}_\ell)$ for some $n \ge 0\}$, and let I^{ord}_ℓ be the intersection of the minimal primes $\mathfrak{P}$ of I^{cr} such that $\mathfrak{P} \subseteq \ker\beta$ for every base point β of $r^{\square}_\ell$ (the components through the Eisenstein point of the closure of the generic ordinary lifts with diagonal characters ψ_1, ψ_2, ψ_3 on $D^\circ_\ell \cap I_\ell$).

Normalised ordinary lifts. An element $g_\star \in G_{\mathbb{Q}}$ is a **marking** if $\eta_K(g_\star) = 1$ and $\Psi(g_\star)^2 - 1 \in \mathcal{O}^\times$ (moreover $c^2 = 1$, $\eta_K(c) = -1$, and $\Psi(c) = 1$ by (Ue5)). For a marking $g_\star$, a lift r of $\bar r_0$ to $A \in \mathrm{CNL}_{\mathcal{O}}$ is a **normalised ordinary lift** if

(Ed1) $\underline{r}(c) = 1$, and the entries $(1,2)$ and $(2,1)$ of $\underline{r}(g_\star) - \underline{r}(cg_\star^{-1}c)$ vanish;

(Ed2) $r(g) = 1$ for every $g \in I_{\bar v}$, $\bar v$ a place of $\overline{\mathbb{Q}}$ not above ℓ, with $\bar r_0(g) = 1$;

(Ed3) $r|_{D_\ell} = f \circ r^{\square}_\ell$ for a morphism $f \colon R^{\square}_\ell \to A$ with $f(I^{\mathrm{ord}}_\ell) = 0$.

A **normalised ordinary Eisenstein deformation ring** for $g_\star$ is a pair (R, r) of $R \in \mathrm{CNL}_{\mathcal{O}}$ and a normalised ordinary lift r to R such that $f \circ r$ is a normalised ordinary lift for every morphism $f \colon R \to A$, and for every normalised ordinary lift r' to some A there is exactly one morphism $f \colon R \to A$ with $f \circ r = r'$. Its **weight-zero subalgebra** R^0 is the $\mathcal{O}$-subalgebra generated by the products $\prod_k \underline{r}(g_k)_{a_k b_k}$ with $\sum_k \mathrm{wt}(a_k, b_k) = 0$; a base point of r is written φ_{Eis}.

Points. A **point** of (R,r) is an $\mathcal{O}$-algebra homomorphism $\varphi\colon R\to\overline{\mathbb{Q}}_\ell$ with $|\varphi(y)|_\ell<1$ for all $y\in\mathfrak{m}_R$; $r_\varphi=\varphi\circ r\colon G_{\mathbb{Q}}\to\mathcal{G}_3(\overline{\mathbb{Q}}_\ell)$, and $\rho_\varphi\colon G_K\to\mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ is the restriction of r_φ to G_K (both are continuous, $\mathfrak{m}_R$ being finitely generated). The point is **upper** if for all $g\in G_K$ the entries of $\rho_\varphi(g)$ of negative weight vanish, **lower** if those of positive weight vanish, and **two-signed** if it is neither. It is **of type I** if ρ_φ is irreducible, and **of type II** if for all $g\in G_K$ the entries of $\rho_\varphi(g)$ of odd weight vanish and $g\mapsto\bigl(\rho_\varphi(g)_{ab}\bigr)_{a,b\in\{1,3\}}$ is an irreducible representation of G_K on $\overline{\mathbb{Q}}_\ell^2$. For $t\in\overline{\mathbb{Q}}_\ell^\times$ the **torus conjugate** $\mathrm{T}(t)\cdot r_\varphi$ is $g\mapsto(\mathrm{T}(t),1)\,r_\varphi(g)\,(\mathrm{T}(t),1)^{-1}$. For $\sigma\in G_K$ the **trace difference** of φ at σ is $\operatorname{tr}\rho_\varphi(\sigma)-\bigl(\Psi(\sigma)+1+\Psi(\sigma)^{-1}\bigr)$. The **adjoint representation** $\operatorname{ad} r_\varphi$ is $\mathrm{M}_3(\overline{\mathbb{Q}}_\ell)$ on which $g\in G_{\mathbb{Q}}$ with $r_\varphi(g)=(P,\varsigma)$ acts by $X\mapsto PXP^{-1}$ if $\varsigma=1$ and by $X\mapsto -P\,J\,{}^tX\,J\,P^{-1}$ if $\varsigma=-1$; $H^1_f(\mathbb{Q},\operatorname{ad} r_\varphi)$ is its Bloch–Kato Selmer group.

Proposition 42.2. *Let (K,ℓ,e,N,χ) be a unitary Eisenstein datum, with χ a Dirichlet character modulo N with values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and let $\eta_K\colon G_{\mathbb{Q}}\to\{\pm1\}\subset\overline{\mathbb{Q}}_\ell^\times$ be the character with kernel G_K. Let $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$ be an isomorphism, $\sigma\in G_K$, and*

$$t=\Psi_{\chi,e}(\sigma)+1+\Psi_{\chi,e}(\sigma)^{-1}\in\mathcal{O}.$$

Assume:

1. *$\operatorname{tr}\rho(\sigma)\neq t$ for every representation $\rho\colon G_K\to\mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ admissible of weight e and level N (for K and ι);*
2. *for every continuous $W\colon G_{\mathbb{Q}}\to\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that $W\otimes\chi_{\mathrm{cyc}}^{-e}$ is semisimple and there is a normalised newform $f=\sum_{n\geq1}a_nq^n$ of weight $2e+1$ and Nebentypus ε_f with the property that for all but finitely many primes p, $W\otimes\chi_{\mathrm{cyc}}^{-e}$ is unramified at p and $\iota\bigl(\det(X-(W\otimes\chi_{\mathrm{cyc}}^{-e})(\sigma'^{-1}))\bigr)=X^2-a_pX+\varepsilon_f(p)p^{2e}$ for every arithmetic Frobenius σ' at a place above p, one has $H^1_f(\mathbb{Q},W)=0$ and $H^1_f(\mathbb{Q},W\otimes\eta_K)=0$.*

Then $\mathrm{Sha}^1_{\ell N}(\chi,e)$ has finite length, and there is a finite set X of pairwise non-isomorphic representations $G_K\to\mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ admissible of weight e and level N such that

$$\nu_{\mathcal{O}}(\chi,e)\leq2\sum_{\rho\in X}v_\ell\bigl(\operatorname{tr}\rho(\sigma)-t\bigr).$$

Proof. Step 0: we may assume $e_{\mathcal{O}}=1$. By (Ue3) the order of χ is prime to ℓ, so by Lemma 38.27, item 1, there is a coefficient ring $\mathcal{O}'$ with $e_{\mathcal{O}'}=1$ containing the values of χ. The conditions (Ue1)–(Ue9) and the element $t\in\overline{\mathbb{Q}}_\ell$ depend only on χ as a $\overline{\mathbb{Q}}_\ell^\times$-valued character, and the two hypotheses and the conclusion other than $\nu_{\mathcal{O}}(\chi,e)$ do not involve $\mathcal{O}$; by Lemma 38.27, item 2, $\mathrm{Sha}^1_{\ell N}(\chi,e)$ has finite length over $\mathcal{O}$ if and only if it has over $\mathcal{O}'$, with the same normalised length. So we may and do replace $\mathcal{O}$ by $\mathcal{O}'$. Then $v_{\mathcal{O}}=v_\ell$ and $\nu_{\mathcal{O}}(\chi,e)=\operatorname{length}_{\mathcal{O}}\mathrm{Sha}^1_{\ell N}(\chi,e)$.

Step 1: the ring and its graded model. By Theorem 47.1, applied to the datum, $\mathcal{O}$ and ι, there is a marking $g_\star$ such that (Au1)–(Au3) hold for every normalised ordinary Eisenstein deformation ring for $g_\star$. By Proposition 43.3, applied with $g_\star$, there are such a ring (R,r) with its base point φ_{Eis}, and (A,φ_0,u,q) with (Mo3)–(Mo7). For a point φ of (R,r), $\varphi\circ q$ is a point of A on u, because $I(u)=\ker q$.

Step 2: every two-signed point of (R,r) is admissible and rigid. Let φ be a two-signed point. By Lemma 42.3 it is of type I or of type II. If it is of type I, ρ_φ is admissible of stable type for (ι,e,N) and irreducible on every open subgroup of G_K by (Au2); by (Sb2), $\rho_\varphi\otimes\chi_{\mathrm{cyc}}^{-1}$ is automorphic of conjugate self-dual type, so Theorem 42.5, case (Ri1), gives $H^1_f(\mathbb{Q},\operatorname{ad} r_\varphi)=0$. If

it is of type II, ρ_φ is admissible of endoscopic type for $(\iota, e, N|d_K|)$ by (Au3), and every continuous $W\colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ restricting on G_K to the corner $\left(\rho_\varphi(g)_{ab}\right)_{a,b\in\{1,3\}}$ with $\operatorname{tr} W(c) = 0$ satisfies the witness conditions over imaginary quadratic fields and (Eo2). Condition (Eo2) says exactly that W satisfies the assumptions made on W in hypothesis 2 ($W \otimes \chi_{\mathrm{cyc}}^{-e}$ semisimple, attached to a normalised newform of weight $2e+1$ at almost all primes, being trivial on inertia at all places above a prime meaning unramified there); so hypothesis 2 gives $H^1_f(\mathbb{Q}, W) = H^1_f(\mathbb{Q}, W \otimes \eta_K) = 0$ for every such W, and Theorem 42.5, case (Ri2) (whose remaining hypothesis, endoscopic admissibility of ρ_φ, is (Au3)), gives $H^1_f(\mathbb{Q}, \operatorname{ad} r_\varphi) = 0$. In both cases ρ_φ is admissible of weight e and level N. With (Au1), the hypotheses of (Mo6) hold, so (A, φ_0, u) satisfies (Gc1)–(Gc4); in particular (Gc2) holds, and (Mo5) applies.

Step 3: points on u. Let ψ' be a two-signed point of A on u. By (Mo5), $\psi' = t \cdot (\varphi \circ q)$ for a point φ of (R, r) and some t. Since $(t \cdot \varphi \circ q)(f) = t^j \varphi(q(f))$ on A_j and $t \neq 0$, $\varphi \circ q$ is two-signed, so by (Mo4) φ is neither upper nor lower: φ is a two-signed point of (R, r). Conversely, for a two-signed point φ of (R, r), $\varphi \circ q$ is a two-signed point on u by (Mo4) and Step 1.

Step 4: the weight-zero element. By (Mo3) choose $h_1, h_2, h_3 \in A_0$ with $q(h_i) = \underline{r}(\sigma)_{ii}$ (the diagonal entries have weight 0), and put $a_\sigma = h_1 + h_2 + h_3 - t \in A_0$, with $t \in \mathcal{O}$. Then $q(a_\sigma) = \operatorname{tr} \underline{r}(\sigma) - t$, so for every point φ of (R, r), as $\sigma \in G_K$,

$$\varphi(q(a_\sigma)) = \operatorname{tr} \rho_\varphi(\sigma) - t.$$

Since $\varphi_{\mathrm{Eis}} \circ r = r_0$, $\varphi_0(a_\sigma) = \varphi_{\mathrm{Eis}}(q(a_\sigma)) = \operatorname{tr} \mathrm{T}(\Psi(\sigma)) - t = 0$. For a two-signed point $\psi' = t' \cdot (\varphi \circ q)$ on u as in Step 3, $\psi'(a_\sigma) = \varphi(q(a_\sigma)) = \operatorname{tr} \rho_\varphi(\sigma) - t$ (degree 0), which is non-zero by hypothesis 1, ρ_φ being admissible of weight e and level N by Step 2. So (Gc5) holds for a_σ, and $(A, \varphi_0, u, a_\sigma)$ satisfies (Gc1)–(Gc5).

Step 5: the graded numerical criterion. By Lemma 41.2, u has an orbit cover Σ_0, $C := \mathrm{ct}_{-1}(u_{-1})$ is finite, and $C \leq 2 \sum_{\psi' \in \Sigma} v_\ell(\psi'(a_\sigma))$ for every orbit cover Σ (recall $v_\mathcal{O} = v_\ell$).

Step 6: the Sha group has length at most C. Let $b \in \mathcal{O} \setminus \{0\}$ and $\mathrm{Sha}[b]$ the b-torsion of $\mathrm{Sha}^1_{\ell N}(\chi, e)$. The map D of (Mo7) satisfies the hypotheses of Lemma 42.12 for $j = -1$, the family u_{-1}, $c = b$ and $M = \mathrm{Sha}[b]$; hence $\operatorname{length}_\mathcal{O} \mathrm{Sha}[b] \leq C$. Every class in $H^1(\mathbb{Q}, \mathcal{O}^\vee(\Psi^{-1}))$ is represented by a continuous cocycle on the compact group $G_\mathbb{Q}$ with values in a discrete module, which has finite image; every element of $\mathcal{O}^\vee = \mathcal{O} \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is killed by a power of ℓ; so every class is killed by some ℓ^k, and $\mathrm{Sha}^1_{\ell N}(\chi, e) = \bigcup_k \mathrm{Sha}[\ell^k]$, an increasing union. If $\mathrm{Sha}^1_{\ell N}(\chi, e)$ contained a strictly increasing chain $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{C+1}$ of submodules, choose $x_i \in M_i \setminus M_{i-1}$; the submodule generated by $x_1, \ldots, x_{C+1}$ lies in some $\mathrm{Sha}[\ell^k]$ and contains the strictly increasing chain of the submodules generated by $x_1, \ldots, x_i$ (the i-th lies in M_i and contains $x_i \notin M_{i-1}$), so it has length at least $C+1$, a contradiction. Hence $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length and $\nu_\mathcal{O}(\chi, e) \leq C$.

Step 7: from an orbit cover to representations. For two-signed points φ, φ' of (R, r) write $\varphi \sim \varphi'$ if $r_{\varphi'} = \mathrm{T}(t) \cdot r_\varphi$ for some $t \in \overline{\mathbb{Q}}_\ell^\times$; since $t \mapsto (\mathrm{T}(t), 1)$ is a homomorphism, $\sim$ is an equivalence relation. For each $\psi' \in \Sigma_0$ choose, by Step 3, a two-signed point $\varphi_{\psi'}$ and $t_{\psi'}$ with $\psi' = t_{\psi'} \cdot (\varphi_{\psi'} \circ q)$. Every two-signed point φ of (R, r) is equivalent to some $\varphi_{\psi'}$: by Step 3 and the definition of an orbit cover, $\varphi \circ q = t \cdot \psi' = (t t_{\psi'}) \cdot (\varphi_{\psi'} \circ q)$ for some $\psi' \in \Sigma_0$ and t, and (Mo4) gives $r_\varphi = \mathrm{T}(t t_{\psi'}) \cdot r_{\varphi_{\psi'}}$. Let P be a set containing exactly one member of each equivalence class that meets $\{\varphi_{\psi'} : \psi' \in \Sigma_0\}$; P is finite, its members are pairwise inequivalent, and every two-signed point is equivalent to a member of P. Then $\Sigma = \{\varphi \circ q : \varphi \in P\}$ is an orbit cover: its members are two-signed points on u (Step 3); a two-signed point ψ'' on u is $t'' \cdot (\varphi'' \circ q)$ with φ'' two-signed (Step 3), $\varphi'' \sim \varphi \in P$, say $r_{\varphi''} = \mathrm{T}(s) \cdot r_\varphi$, so $\varphi'' \circ q = s \cdot (\varphi \circ q)$ by (Mo4) and $\psi'' = (t''s) \cdot (\varphi \circ q)$. The map $\varphi \mapsto \varphi \circ q$ is injective on P: if $\varphi \circ q = \varphi' \circ q$, then $r_{\varphi'} = \mathrm{T}(1) \cdot r_\varphi$

by (Mo4), so $\varphi \sim \varphi'$ and $\varphi = \varphi'$. With Steps 4 and 5,

$$C \le 2 \sum_{\varphi \in P} v_\ell(\varphi(q(a_\sigma))) = 2 \sum_{\varphi \in P} v_\ell(\operatorname{tr} \rho_\varphi(\sigma) - t).$$

Put $X = \{\rho_\varphi : \varphi \in P\}$. Its members are admissible of weight e and level N (Step 2). If $\varphi, \varphi' \in P$ and $\rho_\varphi \cong \rho_{\varphi'}$, then Lemma 42.3 gives $r_{\varphi'} = \mathrm{T}(t) \cdot r_\varphi$ for some t, so $\varphi \sim \varphi'$ and $\varphi = \varphi'$. Hence the members of X are pairwise non-isomorphic, $\varphi \mapsto \rho_\varphi$ is a bijection $P \to X$, and with Step 6

$$\nu_{\mathcal{O}}(\chi, e) \le C \le 2 \sum_{\rho \in X} v_\ell(\operatorname{tr} \rho(\sigma) - t).$$

□

Lemma 42.3. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, $g_\star$ a marking and (R, r) a normalised ordinary Eisenstein deformation ring for $g_\star$. Let φ be a two-signed point of (R, r). Then φ is of type I or of type II, and for every point φ' of (R, r) with $\rho_{\varphi'} \cong \rho_\varphi$ there is $t \in \overline{\mathbb{Q}}_\ell^\times$ with $r_{\varphi'} = \mathrm{T}(t) \cdot r_\varphi$.*

Proof. Let φ be a two-signed point and φ' a point of (R, r) with $\rho_{\varphi'} \cong \rho_\varphi$.

Step 1: r_φ and $r_{\varphi'}$ satisfy (Ns1)–(Ns3) of Lemma 42.4 for K, c and $g_\star$, and $\delta_\star(r_\varphi) + \delta_\star(r_{\varphi'}) \neq 0$. Let $\phi \in \{\varphi, \varphi'\}$. Applying the ring homomorphism ϕ to the entries of GL-parts is a homomorphism $\mathcal{G}_3(R) \to \mathcal{G}_3(\overline{\mathbb{Q}}_\ell)$ (it commutes with products, inverses, transposes and hence with θ), so $r_\phi = \phi \circ r$ is a homomorphism $G_\mathbb{Q} \to \mathcal{G}_3(\overline{\mathbb{Q}}_\ell)$. The sign of $r(g)$ is $\eta_K(g)$, r being a lift of $\bar{r}_0$, and (Ed1) says that $\underline{r}(c) = 1$ and that the entries $(1,2)$, $(2,1)$ of $\underline{r}(g_\star) - \underline{r}(cg_\star^{-1}c)$ vanish; applying ϕ gives (Ns1) and (Ns2) for r_ϕ. The marking satisfies $\eta_K(g_\star) = 1$, so $g_\star \in G_K$. Put $u = \Psi(g_\star) - \Psi(g_\star)^{-1} = \Psi(g_\star)^{-1}(\Psi(g_\star)^2 - 1)$, a unit of $\mathcal{O}$ because $g_\star$ is a marking. Since r reduces to $\bar{r}_0$ modulo $\mathfrak{m}_R$ and $\underline{r_0}(g_\star) = \mathrm{T}(\Psi(g_\star))$, the element $y = \underline{r}(g_\star)_{11} - \underline{r}(g_\star)_{33} - u$ lies in $\mathfrak{m}_R$. As ϕ is an $\mathcal{O}$-algebra homomorphism, $\delta_\star(r_\phi) = u + \phi(y)$ with $|u|_\ell = 1$ and $|\phi(y)|_\ell < 1$ (definition of a point); by the ultrametric inequality $|\delta_\star(r_\phi)|_\ell = 1$, so (Ns3) holds. Likewise $\delta_\star(r_\varphi) + \delta_\star(r_{\varphi'}) = 2u + \varphi(y) + \varphi'(y)$, where $|2u|_\ell = 1$ because ℓ is odd (Ue1) and $|\varphi(y) + \varphi'(y)|_\ell < 1$; so it is non-zero.

Step 2: φ is of type I or of type II. Apply Lemma 42.4, item 1, to r_φ; the restriction of $\underline{r_\varphi}$ to G_K is ρ_φ. Since φ is two-signed, it is neither upper nor lower: the entries of negative weight of $\rho_\varphi(g)$ do not vanish for all $g \in G_K$, nor do those of positive weight. So ρ_φ is irreducible, i.e. φ is of type I, or the entries of odd weight of $\rho_\varphi(g)$ vanish for all $g \in G_K$ and the corner $g \mapsto (\rho_\varphi(g)_{ab})_{a,b \in \{1,3\}}$ is irreducible, i.e. φ is of type II.

Step 3: torus conjugacy. By Step 1 and $\rho_{\varphi'} \cong \rho_\varphi$, Lemma 42.4, item 2, applied to $r = r_\varphi$ and $r' = r_{\varphi'}$, gives $t \in \overline{\mathbb{Q}}_\ell^\times$ with $r_{\varphi'}(g) = (\mathrm{T}(t), 1)\, r_\varphi(g)\, (\mathrm{T}(t), 1)^{-1}$ for all g, that is $r_{\varphi'} = \mathrm{T}(t) \cdot r_\varphi$. □

Lemma 42.4. *Let ℓ be a prime, $K \subset \overline{\mathbb{Q}}$ an imaginary quadratic field, $\eta_K \colon G_\mathbb{Q} \to \{\pm 1\}$ the character with kernel G_K, $c \in G_\mathbb{Q}$ a complex conjugation and $g_\star \in G_K$. Let k be a field of characteristic not 2 (for instance $\overline{\mathbb{Q}}_\ell$) and $r, r' \colon G_\mathbb{Q} \to \mathcal{G}_3(k)$ homomorphisms such that each $s \in \{r, r'\}$ satisfies*

(Ns1) *the sign of $s(g)$ is $\eta_K(g)$ for every $g \in G_\mathbb{Q}$, and $\underline{s}(c) = 1$;*

(Ns2) *the entries $(1,2)$ and $(2,1)$ of $\underline{s}(g_\star) - \underline{s}(cg_\star^{-1}c)$ vanish;*

(Ns3) $\delta_\star(s) := \underline{s}(g_\star)_{11} - \underline{s}(g_\star)_{33} \neq 0$.

Let $\rho, \rho' \colon G_K \to \mathrm{GL}_3(k)$ be the restrictions of $\underline{r}$, $\underline{r'}$ to G_K. Then:

1. *at least one of the following holds: ρ is irreducible; for every $g \in G_K$ the entries of $\rho(g)$ of negative weight vanish; for every $g \in G_K$ the entries of $\rho(g)$ of positive weight vanish; for every $g \in G_K$ the entries of $\rho(g)$ of odd weight vanish and $g \mapsto \left(\rho(g)_{ab}\right)_{a,b\in\{1,3\}}$ is an irreducible representation of G_K on k^2;*

2. *if k is algebraically closed, $\rho' \cong \rho$ (conjugate in $\mathrm{GL}_3(k)$) and $\delta_\star(r) + \delta_\star(r') \neq 0$, there is $t \in k^\times$ with $r'(g) = (\mathrm{T}(t), 1)\, r(g)\, (\mathrm{T}(t), 1)^{-1}$ for every $g \in G_{\mathbb{Q}}$.*

Proof. For $a \in \{1,2,3\}$ put $\bar{a} = 4 - a$, and for a 3×3 matrix X put $X^\tau = J\,{}^t\!X\,J$, so that $(X^\tau)_{ab} = X_{\bar{b}\bar{a}}$; note $\mathrm{wt}(\bar{b}, \bar{a}) = \mathrm{wt}(a, b)$. For $g \in G_{\mathbb{Q}}$ put $g^* = cg^{-1}c$; if $g \in G_K$ then $g^* \in G_K$, since $\eta_K(g^*) = \eta_K(c)^2\eta_K(g)^{-1}$. Write e_1, e_2, e_3 for the standard basis of k^3.

Step 1: for $s \in \{r, r'\}$ and $g \in G_K$, $\underline{s}(g^*) = \underline{s}(g)^\tau$. By (Ns1), $s(c) = (1, -1)$ and $s(g) = (P, 1)$ with $P = \underline{s}(g)$, so $s(g^{-1}) = (P^{-1}, 1)$. By the multiplication rule, $s(c)s(g^{-1}) = (\theta(P^{-1}), -1)$ and $(\theta(P^{-1}), -1)(1, -1) = (\theta(P^{-1})\theta(1), 1) = (\theta(P^{-1}), 1)$. Since $\theta(P^{-1}) = J\,{}^t\!P\,J = P^\tau$, this is the claim.

Step 2: $\underline{s}(g_\star) - \underline{s}(g_\star^*) = \mathrm{diag}(\delta_\star(s), 0, -\delta_\star(s))$ for $s \in \{r, r'\}$. Put $P = \underline{s}(g_\star)$. By Step 1 the (a, b) entry of the difference is $P_{ab} - P_{\bar{b}\bar{a}}$.

The entries $(1, 2)$ and $(2, 1)$ vanish by (Ns2), hence so do $(2, 3)$ and $(3, 2)$, their negatives. Write $D_s = \mathrm{diag}(d_1, d_2, d_3)$ for this matrix, $d = (\delta_\star(s), 0, -\delta_\star(s))$. By (Ns3) and since k has characteristic not 2, d_1, d_2, d_3 are pairwise distinct.

Step 3: mirror symmetry of vanishing entries. For $a, b \in \{1, 2, 3\}$ write $Z(a, b)$ for the property $\rho(g)_{ab} = 0$ for all $g \in G_K$. Then $Z(a, b)$ implies $Z(\bar{b}, \bar{a})$: for $g \in G_K$, Step 1 gives $\rho(g)_{\bar{b}\bar{a}} = (\rho(g)^\tau)_{ab} = \rho(g^*)_{ab} = 0$.

Step 4: stable subspaces are coordinate subspaces. Let $W \subseteq k^3$ be stable under $\rho(G_K)$. As $g_\star, g_\star^* \in G_K$, W is stable under $D_r = \rho(g_\star) - \rho(g_\star^*)$. For $w = \sum_i w_i e_i \in W$ and each i, the polynomial $L_i(X) = \prod_{j\neq i}(X - d_j)/(d_i - d_j)$ (defined by Step 2) satisfies $L_i(D_r)w = w_i e_i \in W$. Hence W is spanned by the e_i it contains: $W = \bigoplus_{i\in S} ke_i$ with $S = \{i : e_i \in W\}$. The same argument with $\mathrm{diag}(d_1, d_3)$ shows that a line in $ke_1 \oplus ke_3$ stable under $\mathrm{diag}(d_1, d_3)$ is ke_1 or ke_3.

Step 5: item 1. Suppose ρ is reducible, and let W be a stable subspace with $0 \neq W \neq k^3$; by Step 4, $W = \bigoplus_{i\in S} ke_i$ with $\emptyset \neq S \subsetneq \{1, 2, 3\}$. For $j \in S$ and $g \in G_K$, $\rho(g)e_j \in W$, so $Z(i, j)$ holds whenever $i \notin S$ and $j \in S$. The entries of negative weight are $(2, 1), (3, 1), (3, 2)$, those of positive weight $(1, 2), (1, 3), (2, 3)$, those of odd weight $(1, 2), (2, 1), (2, 3), (3, 2)$.

- $1 \in S$, $3 \notin S$. Then $Z(3, 1)$. If $2 \in S$, then $Z(3, 2)$, and $Z(2, 1)$ by Step 3 ($\bar{2} = 2$, $\bar{3} = 1$). If $2 \notin S$, then $Z(2, 1)$, and $Z(3, 2)$ by Step 3. So the entries of negative weight vanish on G_K.

- $3 \in S$, $1 \notin S$. Then $Z(1, 3)$. If $2 \in S$, then $Z(1, 2)$ and, by Step 3, $Z(2, 3)$; if $2 \notin S$, then $Z(2, 3)$ and, by Step 3, $Z(1, 2)$. So the entries of positive weight vanish on G_K.

- Otherwise S contains both or neither of $1, 3$, so $S = \{1, 3\}$ or $S = \{2\}$. If $S = \{1, 3\}$: $Z(2, 1)$, $Z(2, 3)$, and by Step 3 $Z(3, 2)$, $Z(1, 2)$. If $S = \{2\}$: $Z(1, 2)$, $Z(3, 2)$, and by Step 3 $Z(2, 3)$, $Z(2, 1)$. In both cases the entries of odd weight vanish on G_K. Then $\rho(g)$ preserves $U = ke_1 \oplus ke_3$, and $\tau(g) = (\rho(g)_{ab})_{a,b\in\{1,3\}}$ is the matrix of $\rho(g)|_U$, so τ is a representation of G_K. If τ is irreducible, the fourth alternative holds. If not, some line of U is $\tau(G_K)$-stable, hence stable under $\tau(g_\star) - \tau(g_\star^*) = \mathrm{diag}(d_1, d_3)$, so by Step 4 it is ke_1 or ke_3. If it is ke_1, then $Z(3, 1)$, and with the vanishing of $(2, 1)$ and $(3, 2)$ all entries of negative weight vanish on G_K; if it is ke_3, then $Z(1, 3)$ and all entries of positive weight vanish on G_K.

So if ρ is not irreducible, one of the other three alternatives holds.

Step 6: item 2 on G_K. Let $M \in \mathrm{GL}_3(k)$ with $M\rho(g) = \rho'(g)M$ for all $g \in G_K$. Applying this to $g_\star$ and $g_\star^*$ and subtracting gives $MD_r = D_{r'}M$ (Step 2). Write $D_r = \mathrm{diag}(d_1, d_2, d_3)$, $D_{r'} = \mathrm{diag}(d_1', d_2', d_3')$ and $\delta = \delta_\star(r)$, $\delta' = \delta_\star(r')$. The (a,b) entry gives $M_{ab}(d_b - d_a') = 0$.

By (Ns3) and the hypothesis $\delta + \delta' \neq 0$ these are all non-zero, so $M = \mathrm{diag}(m_1, m_2, m_3)$, and $m_1 m_2 m_3 = \det M \neq 0$. Hence, for all $g \in G_K$ and all a, b,

$$m_a\, \rho(g)_{ab} = \rho'(g)_{ab}\, m_b. \tag{42.1}$$

Applying (42.1) to g^* and using Step 1 for r and r' gives $m_a \rho(g)_{\bar b \bar a} = \rho'(g)_{\bar b \bar a} m_b$; renaming (a,b) as $(\bar b, \bar a)$, $m_{\bar b}\rho(g)_{ab} = \rho'(g)_{ab} m_{\bar a}$. Multiplying this by m_b, (42.1) by $m_{\bar a}$, and subtracting,

$$(m_a m_{\bar a} - m_b m_{\bar b})\, \rho(g)_{ab} = 0 \qquad (g \in G_K), \tag{42.2}$$

where $m_a m_{\bar a}$ is $m_1 m_3$ if $a \in \{1,3\}$ and m_2^2 if $a = 2$. Recall that $\mathrm{T}(t) X \mathrm{T}(t)^{-1}$ has (a,b) entry $t^{b-a} X_{ab}$.

- If $m_1 m_3 = m_2^2$, put $t = m_1/m_2$. Then $m_3 = m_2 t^{-1}$, so $m_a = m_2 t^{2-a}$ for all a, and (42.1) gives $\rho'(g)_{ab} = t^{b-a}\rho(g)_{ab}$.
- If $m_1 m_3 \neq m_2^2$, then by (42.2) $\rho(g)_{ab} = 0$ whenever exactly one of a, b is 2, and then $\rho'(g)_{ab} = 0$ by (42.1). Choose $t \in k^\times$ with $t^2 = m_1/m_3$ (k is algebraically closed). For $a, b \in \{1,3\}$ or $a = b = 2$, m_a/m_b is 1, t^2 or t^{-2} according as $b - a$ is 0, 2 or -2, so (42.1) gives $\rho'(g)_{ab} = t^{b-a}\rho(g)_{ab}$; for the remaining entries both sides vanish.

In both cases $\rho'(g) = \mathrm{T}(t)\rho(g)\mathrm{T}(t)^{-1}$ for all $g \in G_K$.

Step 7: item 2 on $G_\mathbb{Q}$. Let ι_t be the automorphism $x \mapsto (\mathrm{T}(t), 1)\, x\, (\mathrm{T}(t), 1)^{-1}$ of $\mathcal{G}_3(k)$. By the multiplication rule and $\theta(\mathrm{T}(t)^{-1}) = \mathrm{T}(t)^{-1}$, $\iota_t(P, \varsigma) = (\mathrm{T}(t) P \mathrm{T}(t)^{-1}, \varsigma)$ for both signs. By Step 6 and (Ns1), $r'(g) = \iota_t(r(g))$ for $g \in G_K$, and $r'(c) = (1, -1) = \iota_t(r(c))$. For $g \notin G_K$, $gc \in G_K$ and $g = (gc)c$ since $c^2 = 1$, so $r'(g) = r'(gc)r'(c) = \iota_t(r(gc))\iota_t(r(c)) = \iota_t(r(g))$. □

Theorem 42.5. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, d_K the discriminant of K, $g_\star$ a marking, (R, r) a normalised ordinary Eisenstein deformation ring for $g_\star$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and φ a two-signed point of (R, r). Then $H^1_f(\mathbb{Q}, \mathrm{ad}\, r_\varphi) = 0$ in each of the following cases:*

(Ri1) *φ is of type I, ρ_φ is irreducible on every open subgroup of G_K, and $\rho_\varphi \otimes \chi_{\mathrm{cyc}}^{-1}$ is automorphic of conjugate self-dual type (for ι);*

(Ri2) *φ is of type II, ρ_φ is admissible of endoscopic type for $(\iota, e, N|d_K|)$, every continuous $W\colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ with $W(g) = (\rho_\varphi(g)_{ab})_{a,b\in\{1,3\}}$ for all $g \in G_K$ and $\mathrm{tr}\, W(c) = 0$ satisfies the witness conditions over imaginary quadratic fields and (Eo2) (for ι, with some normalised newform of weight $2e+1$), and every such W has $H^1_f(\mathbb{Q}, W) = 0$ and $H^1_f(\mathbb{Q}, W \otimes \eta_K) = 0$.*

Proof. The homomorphism $r_\varphi = \varphi \circ r\colon G_\mathbb{Q} \to \mathcal{G}_3(\overline{\mathbb{Q}}_\ell)$ has sign η_K (the sign of a lift is that of $\bar r_0$) and continuous GL-part, whose restriction to G_K is ρ_φ; by (Ed1), $r_\varphi(c) = (1, -1)$. The module $\mathrm{ad}\, r_\varphi$ of Definition 42.1 is the module $\mathrm{ad}\, r_\varphi$ of Proposition 42.6 (same formulas). Bloch–Kato Selmer groups are additive in finite direct sums and invariant under isomorphisms of continuous representations.

Case (Ri1). The hypotheses of Proposition 42.6 hold for r_φ: assumption 1 is the irreducibility on open subgroups, and assumption 2 is the automorphy. Hence $H^1_f(\mathbb{Q}, \operatorname{ad} r_\varphi) = 0$.

Case (Ri2), Step 1: the shape of ρ_φ. Write $W'(g) = (\rho_\varphi(g)_{ab})_{a,b\in\{1,3\}}$ and $\chi_2(g) = \rho_\varphi(g)_{22}$ for $g \in G_K$. As φ is of type II, the entries of odd weight of $\rho_\varphi(g)$ vanish, so $\rho_\varphi \cong W' \oplus \chi_2$, with W' irreducible. By (Eo4) there is $W_0 \colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ with (Eo1)–(Eo4) and $\rho_\varphi \cong W_0|_{G_K} \oplus \mathbf{1}$. Comparing Jordan–Hölder factors, the irreducible 2-dimensional factor W' is a factor of $W_0|_{G_K}$, which is therefore irreducible and isomorphic to W', and $\chi_2 = \mathbf{1}$. Choose $h \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ with $W' = hW_0|_{G_K}h^{-1}$ and put $W = hW_0h^{-1}$, a continuous representation of $G_\mathbb{Q}$ with $W|_{G_K} = W'$. So the GL-part of $r_\varphi(g)$ is $j(W(g))$ for $g \in G_K$.

Step 2: $\operatorname{tr} W(c) = 0$, and the newform. By (Eo2) for W_0 there is a normalised newform $f = \sum a_n q^n$ of weight $2e+1$ and character ε_f such that $V := W_0 \otimes \chi_{\mathrm{cyc}}^{-e}$ is semisimple and, for almost all primes p, V is unramified at p with $\iota(\det(X - V(\sigma^{-1}))) = X^2 - a_pX + \varepsilon_f(p)p^{2e}$ for every arithmetic Frobenius σ at p. Taking constant terms and using $\chi_{\mathrm{cyc}}(\sigma) = p$, we obtain $\det W_0(\sigma) = \iota^{-1}(\varepsilon_f(p))^{-1}$ for such σ; arithmetic Frobenius elements at almost all primes are dense in $G_\mathbb{Q}$ (Chebotarev) and both sides are continuous characters, so $\det W_0 = \varepsilon_f^{-1}$ as characters of $G_\mathbb{Q}$. A newform of odd weight has odd character, so $\det W(c) = \varepsilon_f(-1)^{-1} = -1$; as $W(c)^2 = W(c^2) = 1$, $W(c)$ is diagonalisable with eigenvalues $1, -1$, and $\operatorname{tr} W(c) = 0$. By the hypotheses of (Ri2), $H^1_f(\mathbb{Q}, W) = H^1_f(\mathbb{Q}, W \otimes \eta_K) = 0$.

Step 3: $\det W' = 1$. For $g \in G_K$, $r_\varphi(cgc) = (1,-1)(j(W'(g)),1)(1,-1) = (J\,{}^t j(W'(g))^{-1}J, 1)$, so $W'(cgc) = J_2\,{}^tW'(g)^{-1}J_2$ with $J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; also $W'(cgc) = W(c)W'(g)W(c)^{-1}$. Hence $W' \cong W'^\vee$, and $W'^\vee \cong W' \otimes \mu$ with $\mu = (\det W')^{-1}$ (for $B \in \mathrm{GL}_2$, ${}^tB^{-1} = (\det B)^{-1}SBS^{-1}$, $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$). Taking determinants in $W' \cong W' \otimes \mu$ gives $\mu^2 = 1$, so $\det W'(g) \in \{\pm 1\}$ for $g \in G_K$. On the other hand $\det W'(g) = \det j(W'(g)) = \det \rho_\varphi(g) = \varphi(\det \underline{r}(g))$ (Step 1), and $\det \underline{r}(g) \equiv \det \mathrm{T}(\Psi(g)) = 1$ modulo $\mathfrak{m}_R$ because r lifts $\bar{r}_0$; as $|\varphi(y)|_\ell < 1$ for $y \in \mathfrak{m}_R$, $|\det W'(g) - 1|_\ell < 1$. Since ℓ is odd (Ue1), $|-1-1|_\ell = 1$; hence $\det W(g) = \det W'(g) = 1$ for $g \in G_K$.

Step 4: the decomposition. By Steps 1 and 3, r_φ and W satisfy the hypotheses of Lemma 42.9, so

$$H^1_f(\mathbb{Q}, \operatorname{ad} r_\varphi) \cong H^1_f(\mathbb{Q}, \overline{\mathbb{Q}}_\ell(\eta_K))^2 \oplus H^1_f(\mathbb{Q}, \operatorname{ad}^0 W) \oplus H^1_f(\mathbb{Q}, W) \oplus H^1_f(\mathbb{Q}, W \otimes \eta_K).$$

The first term vanishes by Lemma 42.10, the last two by Step 2.

Step 5: $H^1_f(\mathbb{Q}, \operatorname{ad}^0 W) = 0$. Conjugation by h and the twist by the character χ_{cyc}^{-e} do not change the adjoint action, so $\operatorname{ad}^0 W \cong \operatorname{ad}^0 V$, a direct summand of $\operatorname{ad} V = \mathrm{M}_2(\overline{\mathbb{Q}}_\ell)$ (the scalars being the other one). By Lemma 42.8, item 1, V takes values in $\mathrm{GL}_2(E)$ for a finite extension E of $\mathbb{Q}_\ell$. If f is without complex multiplication (no quadratic Dirichlet character η' has $\eta'(p)a_p = a_p$ for almost all p), Newton–Thorne's theorem for non-CM forms [192], Theorem B(1), with $k = 2e+1$ and $\varepsilon = \varepsilon_f$, gives $H^1_f(\mathbb{Q}, \operatorname{ad}_E V) = 0$. If f has complex multiplication, we check the inertia hypothesis of Proposition 42.11 for $I = I_\ell$ and $U = D_\ell^\circ \cap I_\ell$, an open subgroup of I_ℓ contained in G_K. By (Ed3), $r|_{D_\ell} = \lambda_\ell \circ r_\ell^\square$ for a morphism $\lambda_\ell \colon R_\ell^\square \to R$ with $\lambda_\ell(I_\ell^{\mathrm{ord}}) = 0$, and $I^{\mathrm{ord}}(r_\ell^\square) \subseteq I^{\mathrm{cr}} \subseteq I_\ell^{\mathrm{ord}}$ (each prime containing I^{cr} contains $I^{\mathrm{ord}}(r_\ell^\square)$). So for $x \in U$ the coefficients of $\det(X - \underline{r}(x)) - \prod_{a=1}^3 (X - \psi_a(x))$ vanish, and applying φ, $\det(X - \rho_\varphi(x)) = \prod_{a=1}^3 (X - \psi_a(x)) = (X - \Psi(x))(X-1)(X - \Psi(x)^{-1})$. As $\rho_\varphi(x) = j(W(x))$, $\det(X - \rho_\varphi(x)) = (X-1)\det(X - W(x))$, so $\det(X - W(x)) = (X - \Psi(x))(X - \Psi(x)^{-1})$. For $x \in U$, $\chi_\ell(x) = 1$ and the p-components of χ for $p \neq \ell$ are unramified at ℓ, so $\Psi(x) = \chi_{\mathrm{cyc}}(x)^e$, and the eigenvalues of $V(x) = \chi_{\mathrm{cyc}}(x)^{-e}h^{-1}W(x)h$ are 1 and $\chi_{\mathrm{cyc}}(x)^{-2e}$. Hence Proposition 42.11 gives $H^1_f(\mathbb{Q}, \operatorname{ad}_E V) = 0$. In both cases Lemma 42.8, item 2, gives $H^1_f(\mathbb{Q}, \operatorname{ad} V) = 0$. So

$H^1_f(\mathbb{Q}, \mathrm{ad}^0 W) = 0$, and by Step 4, $H^1_f(\mathbb{Q}, \mathrm{ad}\, r_\varphi) = 0$. □

Proposition 42.6. *Let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field, $\eta_K \colon G_\mathbb{Q} \to \{\pm 1\}$ the character with kernel G_K, ℓ a prime and $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism. Let $r \colon G_\mathbb{Q} \to \mathcal{G}_3(\overline{\mathbb{Q}}_\ell)$ be a homomorphism with sign η_K and continuous GL-part, and let $\rho \colon G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ be the restriction of its GL-part to G_K. Assume:*

1. *$\rho|_U$ is irreducible for every open subgroup $U \subseteq G_K$;*
2. *$\rho \otimes \chi_{\mathrm{cyc}}^{-1}$ is automorphic of conjugate self-dual type for ι.*

Let $\mathrm{ad}\, r$ be $\mathrm{M}_3(\overline{\mathbb{Q}}_\ell)$ on which $g \in G_\mathbb{Q}$ with $r(g) = (P, \varsigma)$ acts by $X \mapsto PXP^{-1}$ if $\varsigma = 1$ and by $X \mapsto -P\, J\, {}^t X\, J\, P^{-1}$ if $\varsigma = -1$. Then

$$H^1_f(\mathbb{Q}, \mathrm{ad}\, r) = 0.$$

Proof. Fix a complex conjugation $c \in G_\mathbb{Q}$; then $c^2 = 1$, $\eta_K(c) = -1$ and $G_\mathbb{Q} = G_K \sqcup cG_K$. Write $r(g) = (P_g, \eta_K(g))$, so $P_g = \rho(g)$ for $g \in G_K$.

Step 1: a finite coefficient field. By Lemma 42.8, item 1, applied to ρ on the profinite group G_K, there is a finite extension $E_0 \subset \overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ with $\rho(G_K) \subseteq \mathrm{GL}_3(E_0)$. Let E be the compositum of a finite extension of E_0 containing the entries of P_c with all extensions of it of degree at most 3 in $\overline{\mathbb{Q}}_\ell$; there are finitely many (Krasner), so E is finite over $\mathbb{Q}_\ell$, and every element of $\mathrm{GL}_3(E_0)$, in particular of $\rho(G_K)$, has its eigenvalues (roots of a cubic over E_0) in E. For $g \in G_K$, $r(cg) = (P_c, -1)(\rho(g), 1) = (P_c\, \theta(\rho(g)), -1)$, whose GL-part has entries in E. Hence the formulas defining $\mathrm{ad}\, r$ preserve $\mathrm{M}_3(E)$; call this $E[G_\mathbb{Q}]$-module $\mathrm{ad}_E\, r$, so that $\mathrm{ad}\, r = \mathrm{ad}_E\, r \otimes_E \overline{\mathbb{Q}}_\ell$.

Step 2: the polarisation. Put $\mathbf{A} = P_c J \in \mathrm{GL}_3(E)$. In $\mathcal{G}_3$, $(P_c, -1)^{-1} = (\theta(P_c^{-1}), -1)$ (as θ is an involution), so for $g \in G_K$

$$r(cgc) = r(c)r(g)r(c)^{-1} = (P_c\, \theta(\rho(g))\, P_c^{-1}, 1),$$

that is,

$$\rho(cgc) = P_c J\, {}^t\rho(g)^{-1} J P_c^{-1} = \mathbf{A}\, {}^t\rho(g)^{-1} \mathbf{A}^{-1},$$

using $J^{-1} = J$. The element c acts on $\mathrm{ad}_E\, r$ by $X \mapsto -P_c J\, {}^t X J P_c^{-1} = -\mathbf{A}\, {}^t X \mathbf{A}^{-1}$, and $g \in G_K$ by $X \mapsto \rho(g) X \rho(g)^{-1}$.

Step 3: the cuspidal representation. Let $r_1 = \rho \otimes \chi_{\mathrm{cyc}}^{-1} \colon G_K \to \mathrm{GL}_3(E)$ (χ_{cyc} takes values in $\mathbb{Z}_\ell^\times \subset E^\times$). By assumption 2 there is a regular algebraic conjugate self-dual cuspidal automorphic representation π of $\mathrm{GL}_3(\mathbb{A}_K)$ with $r_1 \leftrightarrow_\iota \pi$; r_1 is absolutely irreducible by assumption 1 with $U = G_K$. Since χ_{cyc} is a character of $G_\mathbb{Q}$, $\chi_{\mathrm{cyc}}(cgc) = \chi_{\mathrm{cyc}}(g)$, and Step 2 gives, for $g \in G_K$,

$$r_1(cgc) = \chi_{\mathrm{cyc}}(g)^{-1} \mathbf{A}\, {}^t(\chi_{\mathrm{cyc}}(g) r_1(g))^{-1} \mathbf{A}^{-1} = \chi_{\mathrm{cyc}}(g)^{1-3}\, \mathbf{A}\, {}^t r_1(g)^{-1} \mathbf{A}^{-1}.$$

So r_1, π, E and $\mathbf{P} = \mathbf{A}$ satisfy the hypotheses of Theorem B.70 with $n = 3$, apart from the image condition. The module $\mathrm{ad}_E\, r_1$ of that theorem coincides with $\mathrm{ad}_E\, r$: on G_K, conjugation by $r_1(g)$ equals conjugation by $\rho(g)$ (the scalar cancels), and c acts by $X \mapsto -\mathbf{A}\, {}^t X \mathbf{A}^{-1}$ in both; two actions of $G_\mathbb{Q}$ agreeing on G_K and on c agree.

Step 4: the image condition. Every element of $r_1(G_K)$ has its eigenvalues in E: those of $r_1(g)$ are $\chi_{\mathrm{cyc}}(g)^{-1}$ times those of $\rho(g)$ (Step 1). On $G_{K(\mu_{\ell^\infty})}$ the character χ_{cyc} is trivial, so r_1

and ρ agree there and have the same Zariski closure $\mathbf{Z}$ of the image of $G_{K(\mu_{\ell^\infty})}$. By Lemma 42.7 (with $F = K$, using assumption 1), $\mathbf{Z}^\circ$ acts absolutely irreducibly on E^3 and $\mathbf{Z}^\circ(\overline{\mathbb{Q}}_\ell)$ contains an element with three distinct eigenvalues.

Step 5: conclusion. By Theorem B.70, $H^1_f(\mathbb{Q}, \mathrm{ad}_E\, r) = H^1_f(\mathbb{Q}, \mathrm{ad}_E\, r_1) = 0$, and by Lemma 42.8, item 2 (with $F = \mathbb{Q}$, $V = \mathrm{ad}_E\, r$), $H^1_f(\mathbb{Q}, \mathrm{ad}\, r) = 0$. □

Lemma 42.7. *Let $F \subset \overline{\mathbb{Q}}$ be a number field, ℓ a prime, $E \subset \overline{\mathbb{Q}}_\ell$ a finite extension of $\mathbb{Q}_\ell$ and $\rho\colon G_F \to \mathrm{GL}_3(E)$ a continuous representation such that $\rho|_U$ is irreducible on $\overline{\mathbb{Q}}_\ell^3$ for every open subgroup $U \subseteq G_F$. Let $\mathbf{Z} \subseteq \mathrm{GL}_{3,E}$ be the Zariski closure of $\rho(G_{F(\mu_{\ell^\infty})})$ and $\mathbf{Z}^\circ$ its identity component. Then $\mathbf{Z}^\circ$ acts absolutely irreducibly on E^3, and $\mathbf{Z}^\circ(\overline{\mathbb{Q}}_\ell)$ contains an element with three distinct eigenvalues.*

Proof. Facts on linear algebraic groups over the algebraically closed field $\overline{\mathbb{Q}}_\ell$ of characteristic 0 are used from A. Borel, *Linear algebraic groups*, 2nd ed., 1991. The vanishing ideal in $\overline{\mathbb{Q}}_\ell[x]$ of a set of E-points is spanned by polynomials over E (vanishing is an E-linear condition on the coefficients), so Zariski closures of subsets of $\mathrm{GL}_3(E)$ and their identity components commute with extension of scalars to $\overline{\mathbb{Q}}_\ell$; absolute irreducibility is irreducibility over $\overline{\mathbb{Q}}_\ell$. We therefore work over $\overline{\mathbb{Q}}_\ell$, with $V = \overline{\mathbb{Q}}_\ell^3$, $\Gamma = \rho(G_F)$, $H = \rho(G_{F(\mu_{\ell^\infty})})$, and G, G' the Zariski closures of Γ, H in GL_3; so $\mathbf{Z}^\circ$ becomes G'°.

Step 1: G° acts irreducibly on V. G° is Zariski closed of finite index in G, so $U = \rho^{-1}(G^\circ)$ is a subgroup of G_F of finite index (it is the preimage of the finite-index subgroup $\Gamma \cap G^\circ$ of Γ), and closed (Zariski-closed sets are closed for the ℓ-adic topology); a closed subgroup of finite index of a profinite group is open. Every G°-stable subspace of V is $\rho(U)$-stable, and $\rho|_U$ is irreducible by hypothesis.

Step 2: G° is reductive, and its derived group $S = [G^\circ, G^\circ]$ is non-trivial and acts irreducibly on V. The unipotent radical R of G° fixes a non-zero vector (Kolchin), and V^R is G°-stable because R is normal; by Step 1, $V^R = V$, so R acts trivially, and $R = 1$ since $G^\circ \subseteq \mathrm{GL}(V)$. So G° is reductive and $G^\circ = Z^\circ \cdot S$ with Z° the identity component of its centre and S semisimple and connected. The centre commutes with the irreducible action of G°, so acts by scalars (Schur); hence every S-stable subspace is G°-stable, and S acts irreducibly. If $S = 1$, $G^\circ = Z^\circ$ acts by scalars, and every line would be stable, contradicting irreducibility on the 3-dimensional V.

Step 3: $S \subseteq G'^\circ$. H is normal in Γ and Γ/H is a quotient of $\mathrm{Gal}(F(\mu_{\ell^\infty})/F)$, which embeds in $\mathbb{Z}_\ell^\times$ and is abelian; so all commutators of Γ lie in H. For $\gamma \in \Gamma$, $\gamma G' \gamma^{-1}$ is the Zariski closure of $\gamma H \gamma^{-1} = H$, so the normaliser of G', a closed subgroup, contains Γ and hence G; thus G' is normal in G, and the quotient $q\colon G \to G/G'$ is a morphism of affine algebraic groups. The morphism $G \times G \to G/G'$, $(x, y) \mapsto q(xyx^{-1}y^{-1})$, is trivial on $\Gamma \times \Gamma$, which is Zariski dense in $G \times G$, so it is trivial: $[G, G] \subseteq G'$. The connected group $S \subseteq [G, G]$ therefore lies in G'°.

Step 4: S contains an element with three distinct eigenvalues. Let $T \subseteq S$ be a maximal torus; $\dim T \geq 1$ since S is non-trivial semisimple. T acts on V diagonalisably with a multiset of weights $\{\chi_1, \chi_2, \chi_3\}$ in the character group $X(T)$; $\chi_1 + \chi_2 + \chi_3 = 0$ because the semisimple group S has no non-trivial characters, so $\det|_S = 1$; and the weights span $X(T) \otimes \mathbb{Q}$ because T acts faithfully. The Weyl group $W = N_S(T)/T$ permutes the weights with their multiplicities ($n \in N_S(T)$ maps the χ-weight space to the $n \cdot \chi$-weight space). Suppose two weights coincide, say $\chi_1 = \chi_2 = a$; then $\chi_3 = -2a$, the span of the weights is $\mathbb{Q}a$, so T has rank 1 and $a \neq 0$; then W has order 2 and acts on $X(T) \otimes \mathbb{Q}$ by -1, so $-a$ is a weight, i.e. $-a \in \{a, -2a\}$, which forces $a = 0$, a contradiction. Hence χ_1, χ_2, χ_3 are pairwise distinct, the subgroups $\ker(\chi_i - \chi_j)$, $i \neq j$, are proper closed subsets of the irreducible variety T, and any $t \in T$ outside their union has the three distinct eigenvalues $\chi_1(t), \chi_2(t), \chi_3(t)$.

By Steps 2–4, $G'^{\circ} \supseteq S$ acts irreducibly on V and contains an element of $S \subseteq G'^{\circ}(\overline{\mathbb{Q}}_\ell)$ with three distinct eigenvalues. □

Lemma 42.8. *Let ℓ be a prime.*

1. *Let Γ be a profinite group and $\rho\colon \Gamma \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ a continuous homomorphism. There is a finite extension $E \subset \overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ with $\rho(\Gamma) \subseteq \mathrm{GL}_n(E)$.*
2. *Let $F \subset \overline{\mathbb{Q}}$ be a number field, $E \subset \overline{\mathbb{Q}}_\ell$ a finite extension of $\mathbb{Q}_\ell$ and V a finite-dimensional E-vector space with a continuous E-linear action of G_F. If $H^1_f(F, V) = 0$, then $H^1_f(F, V \otimes_E \overline{\mathbb{Q}}_\ell) = 0$.*
3. *Let Γ be a profinite group, $E \subset \overline{\mathbb{Q}}_\ell$ a finite extension of $\mathbb{Q}_\ell$ and V a finite-dimensional E-vector space with a continuous E-linear action of Γ. Every continuous 1-cocycle $c\colon \Gamma \to V \otimes_E \overline{\mathbb{Q}}_\ell$ takes values in $V \otimes_E E'$ for some finite extension $E' \subset \overline{\mathbb{Q}}_\ell$ of E.*

Proof. For $d \geq 1$ let $E_d \subset \overline{\mathbb{Q}}_\ell$ be the compositum of the finitely many extensions of $\mathbb{Q}_\ell$ of degree at most d (Krasner); E_d is finite over $\mathbb{Q}_\ell$, hence complete and closed in $\overline{\mathbb{Q}}_\ell$, and $\overline{\mathbb{Q}}_\ell = \bigcup_d E_d$. The same holds with $\mathbb{Q}_\ell$ replaced by any finite extension E (write E_d for the compositum of E with the extensions of E of degree at most d).

Item 1. Each $\mathrm{GL}_n(E_d) = \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \cap \mathrm{M}_n(E_d)$ is a closed subgroup, so $\Gamma = \bigcup_d \rho^{-1}(\mathrm{GL}_n(E_d))$ is a countable union of closed subgroups. Γ is compact Hausdorff, so by Baire's theorem one of them, U say, has non-empty interior; a subgroup with non-empty interior is open. Let $g_1, \dots, g_s$ represent Γ/U and let $E \supseteq E_d$ be a finite extension containing the entries of $\rho(g_1), \dots, \rho(g_s)$. Then $\rho(g_i u) = \rho(g_i)\rho(u) \in \mathrm{GL}_n(E)$ for all i and $u \in U$.

Item 3. Put $V' = V \otimes_E \overline{\mathbb{Q}}_\ell$. Each $V \otimes_E E_d$ is a finite-dimensional space over the complete field E_d, hence closed in V', and is Γ-stable. So $\Gamma = \bigcup_d c^{-1}(V \otimes_E E_d)$, and by Baire's theorem one of these closed sets, say for d, contains a coset gH of an open normal subgroup H (open normal subgroups form a basis of neighbourhoods of 1 in a profinite group). For $h \in H$, $c(h) = g^{-1}(c(gh) - c(g)) \in V \otimes_E E_d$. Let $g_1, \dots, g_s$ represent Γ/H and $E' \supseteq E_d$ finite containing the coordinates of $c(g_1), \dots, c(g_s)$ in an E-basis of V; then $c(g_i h) = c(g_i) + g_i c(h) \in V \otimes_E E'$.

Item 2. Put $V' = V \otimes_E \overline{\mathbb{Q}}_\ell$ and let c be a Bloch–Kato cocycle of V'.

(a) By item 3 (with $\Gamma = G_F$), c takes values in $V \otimes_E E'$ for a finite extension E'/E.

(b) *c is a Bloch–Kato cocycle of $V \otimes_E E'$.* Choose an E'-linear $\varpi\colon \overline{\mathbb{Q}}_\ell \to E'$ with $\varpi|_{E'} = \mathrm{id}$ and put $\Pi = \mathrm{id}_V \otimes \varpi\colon V' \to V \otimes_E E'$, which is G_F-equivariant and the identity on $V \otimes_E E'$. If $c(g) = gx - x$ on an inertia group (or on an archimedean decomposition group), then $c(g) = \Pi(c(g)) = g\Pi(x) - \Pi(x)$ there; at a place above ℓ, the $\mathbb{Q}_\ell$-linear equivariant map $\Pi \otimes \mathrm{id}$ into $(V \otimes_E E') \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ fixes $c(g) \otimes 1$ and carries y with $c(g) \otimes 1 = gy - y$ to such an element. The topology of $V \otimes_E E'$ is the subspace topology, so c is continuous into it.

(c) *Conclusion.* An E-basis $b_1, \dots, b_m$ of E' gives a G_F-equivariant E-linear homeomorphism $V \otimes_E E' \cong V^m$. The conditions of Definition 22.2 depend only on the underlying topological $\mathbb{Q}_\ell[G_F]$-module and are compatible with the inclusions and projections of a finite direct sum; hence $H^1_f(F, V \otimes_E E') \cong H^1_f(F, V)^m = 0$, and $c(g) = gz - z$ for some $z \in V \otimes_E E' \subseteq V'$. So $H^1_f(F, V') = 0$. □

Lemma 42.9. *Let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field, η_K the character of $G_{\mathbb{Q}}$ with kernel G_K, $c \in G_{\mathbb{Q}}$ a complex conjugation and ℓ a prime. For $B = \begin{pmatrix} a & b \\ c' & d \end{pmatrix} \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ put $j(B) =$*

$\begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c' & 0 & d \end{pmatrix}$. *Let* $r\colon G_{\mathbb{Q}} \to \mathcal{G}_3(\overline{\mathbb{Q}}_\ell)$ *be a homomorphism with sign* η_K *and continuous GL-part, and* $W\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ *a continuous representation, such that*

1. $r(c) = (1,-1)$*, and the GL-part of* $r(g)$ *is* $j(W(g))$ *for every* $g \in G_K$*;*
2. $W|_{G_K}$ *is irreducible and* $\det W(g) = 1$ *for every* $g \in G_K$*.*

Let $\mathrm{ad}^0 W$ *be the space of trace-zero matrices in* $\mathrm{M}_2(\overline{\mathbb{Q}}_\ell)$ *with* g *acting by* $X \mapsto W(g)XW(g)^{-1}$*, and* $W \otimes \eta_K$ *the space* $\overline{\mathbb{Q}}_\ell^2$ *with* g *acting by* $\eta_K(g)W(g)$*. Then there is an isomorphism of continuous representations of* $G_{\mathbb{Q}}$

$$\mathrm{ad}\, r \;\cong\; \overline{\mathbb{Q}}_\ell(\eta_K)^{\oplus 2} \oplus \mathrm{ad}^0 W \oplus W \oplus (W \otimes \eta_K).$$

Proof. All maps below are linear maps of finite-dimensional spaces, hence continuous; two actions of $G_{\mathbb{Q}}$ that agree on G_K and on c agree, as $G_{\mathbb{Q}} = G_K \sqcup cG_K$. Put $J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Step 1: blocks. For $X \in \mathrm{M}_3(\overline{\mathbb{Q}}_\ell)$ write $X_C = (X_{ab})_{a,b\in\{1,3\}} \in \mathrm{M}_2$, $x_M = X_{22}$, $u_X = {}^t(X_{12}, X_{32})$ and $v_X = (X_{21}, X_{23})$. For $g \in G_K$, g acts by conjugation with $j(W(g))$, which gives

$$(g\cdot X)_C = W(g)X_C W(g)^{-1}, \quad (g\cdot X)_M = x_M, \quad u_{g\cdot X} = W(g)u_X, \quad v_{g\cdot X} = v_X W(g)^{-1}.$$

The element c acts by $X \mapsto -J\,{}^t\!X\,J$ (by item 1), and $(J\,{}^t\!X\,J)_{ab} = X_{4-b,4-a}$; hence

$$(c\cdot X)_C = -J_2\,{}^t\!X_C\,J_2, \quad (c\cdot X)_M = -x_M, \quad u_{c\cdot X} = -J_2\,{}^t\!v_X, \quad v_{c\cdot X} = -{}^t\!u_X\,J_2.$$

So the subspaces $V_C = \{X : X_{ab} = 0 \text{ unless } a,b \in \{1,3\}\}$, $V_M = \overline{\mathbb{Q}}_\ell E_{22}$ and $V_\times = \{X : X_C = 0,\ x_M = 0\}$ are $G_{\mathbb{Q}}$-stable and $\mathrm{ad}\, r = V_C \oplus V_M \oplus V_\times$. On V_M, G_K acts trivially and c by -1: $V_M \cong \overline{\mathbb{Q}}_\ell(\eta_K)$.

Step 2: the matrix $W(c)$*.* For $g \in G_K$, $r(cgc) = r(c)r(g)r(c)^{-1} = (1,-1)(j(W(g)),1)(1,-1) = (\theta(j(W(g))),1)$. Since j is a homomorphism commuting with transposition and $J\,j(B)\,J = j(J_2 B J_2)$, this reads $j(W(cgc)) = j(J_2\,{}^t\!W(g)^{-1}J_2)$. For $B \in \mathrm{SL}_2$, ${}^t\!B^{-1} = SBS^{-1}$, so by item 2

$$W(c)W(g)W(c)^{-1} = W(cgc) = (J_2 S)\,W(g)\,(J_2 S)^{-1} \qquad (g \in G_K).$$

Thus $(J_2S)^{-1}W(c)$ commutes with $W(G_K)$, and by Schur's lemma ($W|_{G_K}$ is irreducible over the algebraically closed field $\overline{\mathbb{Q}}_\ell$) $W(c) = \lambda J_2 S$ for some $\lambda \in \overline{\mathbb{Q}}_\ell^\times$.

Step 3: $V_C \cong \overline{\mathbb{Q}}_\ell(\eta_K) \oplus \mathrm{ad}^0 W$*.* Identify V_C with M_2 via $X \mapsto X_C$. The scalars are fixed by G_K and negated by c ($-J_2\,{}^t\!1\,J_2 = -1$): a copy of $\overline{\mathbb{Q}}_\ell(\eta_K)$. The trace-zero matrices form a $G_{\mathbb{Q}}$-stable complement (both formulas preserve the trace up to sign). For trace-zero Y, ${}^t\!Y = -SYS^{-1}$, so c acts by $Y \mapsto J_2 S Y S^{-1} J_2 = W(c)YW(c)^{-1}$ by Step 2 ($J_2^{-1} = J_2$, the scalar λ cancels); and $g \in G_K$ acts by $Y \mapsto W(g)YW(g)^{-1}$. Hence the trace-zero part is $\mathrm{ad}^0 W$.

Step 4: $V_\times \cong W \oplus (W \otimes \eta_K)$*.* Let $V_1 = \{X \in V_\times : v_X = 0\}$ and $V_2 = \{X \in V_\times : u_X = 0\}$; for $u \in \overline{\mathbb{Q}}_\ell^2$ let $[u] \in V_1$ be the element with $u_X = u$. By Step 1, $g\cdot[u] = [W(g)u]$ for $g \in G_K$, and c maps V_1 into V_2 and V_2 into V_1, injectively. For $\epsilon \in \{1, \eta_K\}$ define $\iota_\epsilon(u) = [u] + \epsilon(c)\,c\cdot[W(c)u]$. For $g \in G_K$, using $cgc \in G_K$ and $W(cgc)W(c) = W(c)W(g)$,

$$g\cdot\iota_\epsilon(u) = [W(g)u] + \epsilon(c)\,c\cdot((cgc)\cdot[W(c)u]) = [W(g)u] + \epsilon(c)\,c\cdot[W(c)W(g)u] = \iota_\epsilon(W(g)u),$$

and, using $c^2 = 1$, $W(c)^2 = 1$, $\epsilon(c)^2 = 1$,

$$c \cdot \iota_\epsilon(u) = c \cdot [u] + \epsilon(c)[W(c)u] = \iota_\epsilon(\epsilon(c)W(c)u).$$

So $\iota_1 \colon W \to V_\times$ and $\iota_{\eta_K} \colon W \otimes \eta_K \to V_\times$ are $G_\mathbb{Q}$-equivariant. If $\iota_1(u) + \iota_{\eta_K}(u') = 0$, the V_1-component gives $u + u' = 0$ and the V_2-component gives $c \cdot [W(c)(u - u')] = 0$, so $u = u' = 0$. Hence $\iota_1 \oplus \iota_{\eta_K}$ is injective, and an isomorphism since both sides have dimension 4.

Steps 1, 3 and 4 give the stated decomposition. □

Lemma 42.10. *Let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field, $\eta_K \colon G_\mathbb{Q} \to \{\pm 1\}$ the character with kernel G_K, ℓ a prime, and $\overline{\mathbb{Q}}_\ell(\eta_K)$ the line $\overline{\mathbb{Q}}_\ell$ on which g acts by $\eta_K(g)$. Then $H^1_f(\mathbb{Q}, \overline{\mathbb{Q}}_\ell(\eta_K)) = 0$.*

Proof. By Lemma 42.8, item 2 (with $F = \mathbb{Q}$, $E = \mathbb{Q}_\ell$), it suffices to show $H^1_f(\mathbb{Q}, \mathbb{Q}_\ell(\eta_K)) = 0$. Let x be a Bloch–Kato cocycle with values in $\mathbb{Q}_\ell(\eta_K)$, and let $\psi = x|_{G_K}$; as G_K acts trivially, $\psi \colon G_K \to \mathbb{Q}_\ell$ is a continuous homomorphism.

Step 1: ψ is unramified at every finite place of K. Let w be a finite place of K, v the prime below it and $j \colon \overline{\mathbb{Q}} \to \overline{K_w}$ an embedding over K; it is also an embedding over $\mathbb{Q}$, and the decomposition and inertia groups $D_j^K \supseteq I_j^K$ of K at w are contained in those $D_j \supseteq I_j$ of $\mathbb{Q}$ at v. If $v \neq \ell$, there is z with $x(g) = \eta_K(g)z - z$ for $g \in I_j$, so $\psi(g) = 0$ for $g \in I_j^K \subseteq G_K$. If $v = \ell$, there is $y \in \mathbb{Q}_\ell(\eta_K) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $x(g) \otimes 1 = gy - y$ for $g \in D_j$; restricting to $D_j^K \cong G_{K_w}$, on which $\mathbb{Q}_\ell(\eta_K)$ is the trivial representation, the class of $\psi|_{D_j^K}$ lies in $H^1_f(G_{K_w}, \mathbb{Q}_\ell)$, so ψ vanishes on I_j^K by Theorem B.44.

Step 2: $\psi = 0$. By Step 1, ψ factors through the Galois group of the maximal abelian extension of K unramified at all finite places; since K has no real places, this is the Hilbert class field of K, which is finite over K by class field theory. So $\psi(G_K)$ is a finite subgroup of the torsion-free group $\mathbb{Q}_\ell$, and $\psi = 0$.

Step 3: x is a coboundary. Fix $c \in G_\mathbb{Q} \setminus G_K$ and put $a = x(c)$, $z = -a/2$. For $h \in G_K$, $x(ch) = x(c) + \eta_K(c)x(h) = a$ and $x(h) = 0$; on the other hand $\eta_K(ch)z - z = -2z = a$ and $\eta_K(h)z - z = 0$. As $G_\mathbb{Q} = G_K \sqcup cG_K$, $x(g) = \eta_K(g)z - z$ for all g. □

Proposition 42.11. *Let ℓ be a prime, $e \geq 1$ an integer, $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and $f = \sum_{n \geq 1} a_n q^n$ a normalised newform of weight $2e+1$ and character ε_f with complex multiplication: there is a quadratic Dirichlet character η' with $\eta'(p)a_p = a_p$ for all but finitely many primes p. Let $E \subset \overline{\mathbb{Q}}_\ell$ be a finite extension of $\mathbb{Q}_\ell$ and $V \colon G_\mathbb{Q} \to \mathrm{GL}_2(E)$ a continuous representation, semisimple over $\overline{\mathbb{Q}}_\ell$, such that for all but finitely many primes p, V is unramified at p and $\iota(\det(X - V(\sigma^{-1}))) = X^2 - a_p X + \varepsilon_f(p)p^{2e}$ for every arithmetic Frobenius σ at p. Assume that there are an inertia group I of $G_\mathbb{Q}$ at a place above ℓ and an open subgroup $U \subseteq I$ such that, for every $h \in U$, the eigenvalues of $V(h)$ are 1 and $\chi_{\mathrm{cyc}}(h)^{-2e}$. Then $H^1_f(\mathbb{Q}, \mathrm{ad}_E V) = 0$, where $\mathrm{ad}_E V$ is $\mathrm{M}_2(E)$ with g acting by $X \mapsto V(g)XV(g)^{-1}$.*

Proof. Step 1: $V \cong V \otimes \eta'$ over $\overline{\mathbb{Q}}_\ell$. For all but finitely many p and every arithmetic Frobenius σ at p, $\eta'(\sigma^{-1}) = \eta'(p)^{\pm 1} = \eta'(p)$, so $\operatorname{tr} V(\sigma^{-1}) = \iota^{-1}(a_p) = \iota^{-1}(\eta'(p)a_p) = \operatorname{tr}(V \otimes \eta')(\sigma^{-1})$. These elements σ^{-1} are dense in $G_\mathbb{Q}$ (Chebotarev) and traces are continuous, so V and $V \otimes \eta'$ have the same trace everywhere; both are semisimple, hence isomorphic ([220], I.2.2).

Step 2: η' is trivial on I. Let $U_0 \subseteq U$ be an open subgroup normal in I (the intersection of the finitely many I-conjugates of U) and $s = [I : U_0]$, so $h^s \in U_0$ for every $h \in I$. Suppose that $C = \{h \in I : \eta'(h) = -1\}$ is non-empty. For $h \in C$, Step 1 gives $\operatorname{tr} V(h) = -\operatorname{tr} V(h)$, so the eigenvalues of $V(h)$ are $\lambda, -\lambda$ for some λ; the eigenvalues of $V(h^s) = V(h)^s$ are then $\lambda^s, (-\lambda)^s$,

and also $1, \chi_{\mathrm{cyc}}(h)^{-2es}$. If s is even the two eigenvalues coincide, so $\chi_{\mathrm{cyc}}(h)^{-2es} = 1$; if s is odd they are $\lambda^s, -\lambda^s$, so $\chi_{\mathrm{cyc}}(h)^{-2es} = -1$. In both cases $\chi_{\mathrm{cyc}}(h)^{4es} = 1$ for all $h \in C$. But χ_{cyc} maps I onto $\mathbb{Z}_\ell^\times$ ($\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell$ is totally ramified with group $\mathbb{Z}_\ell^\times$), C is a coset of the subgroup $\ker(\eta'|_I)$ of index 2, so $\chi_{\mathrm{cyc}}(C)$ is a coset of a subgroup of index at most 2 of $\mathbb{Z}_\ell^\times$ and is infinite, while $x^{4es} = 1$ has finitely many solutions ($e, s \geq 1$). So C is empty.

Step 3: $K' \not\subseteq \mathbb{Q}(\mu_{\ell^\infty})$, where $G_{K'} = \ker \eta'$. By Step 2, K' is unramified at ℓ. Every subfield $F \neq \mathbb{Q}$ of $\mathbb{Q}(\mu_{\ell^\infty})$ is ramified at ℓ, because each $\mathbb{Q}(\mu_{\ell^n})/\mathbb{Q}$ is totally ramified at ℓ, so an inertia group at ℓ surjects onto $\mathrm{Gal}(F/\mathbb{Q}) \neq 1$. As $K' \neq \mathbb{Q}$, $K' \not\subseteq \mathbb{Q}(\mu_{\ell^\infty})$.

Step 4: conclusion. By Newton–Thorne's theorem for CM newforms [192], Theorem 5.4(2), applied with $k = 2e+1$, $\varepsilon = \varepsilon_f$ and the field K' of Step 3, $H^1_f(\mathbb{Q}, \mathrm{ad}_E V) = 0$. □

Lemma 42.12. *Let $\mathcal{O}$ be a coefficient ring, (A, φ_0) a graded $\mathcal{O}$-algebra with a fixed point, $j \neq 0$ an integer, $u = (u_i)_{i \in I}$ a finite family of elements of A_j, $c \in \mathcal{O} \setminus \{0\}$, M an $\mathcal{O}$-module and $D\colon M \to \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}/c\mathcal{O})$ an injective additive map such that*

1. *$D(\mu)(u_i) = 0$ for all $\mu \in M$ and $i \in I$;*
2. *for every $\mu \in M$ there is an $\mathcal{O}$-linear map $D'\colon A \to \mathcal{O}$ with $D'(fg) = \varphi_0(f)D'(g) + \varphi_0(g)D'(f)$ for all $f, g \in A$ and $D'(A_i) = 0$ for all $i \neq j$, such that $D(\mu)(f)$ is the class of $D'(f)$ for every $f \in A$.*

Then $\mathrm{length}_{\mathcal{O}}(M) \leq \mathrm{ct}_j(u)$.

Proof. Let k be the residue field of $\mathcal{O}$ and $q = \#k$, which is finite because $\mathcal{O}$ is finite over $\mathbb{Z}_\ell$; let ϖ be a uniformiser and $\mathfrak{m}_0 = \ker \varphi_0$. If $\mathrm{ct}_j(u) = \infty$ there is nothing to prove, so assume $\mathrm{ct}_j(u) = n < \infty$.

Step 1: the derivations factor through the cotangent lattice. Let D' be as in item 2 for some μ. For $f, g \in \mathfrak{m}_0$, $D'(fg) = 0$; by additivity $D'(\mathfrak{m}_0^2) = 0$. Also $D'(1) = D'(1 \cdot 1) = 2D'(1)$, so $D'(1) = 0$. If $f \in \mathrm{Sq}(\varphi_0)$, choose $c' \neq 0$ with $c'f \in \mathfrak{m}_0^2$; then $c'D'(f) = D'(c'f) = 0$, and $D'(f) = 0$ because $\mathcal{O}$ is a domain. Hence D' induces an $\mathcal{O}$-linear map $\lambda\colon A/\mathrm{Sq}(\varphi_0) \to \mathcal{O}$, and the reduction of λ modulo c sends the class of f to $D(\mu)(f)$. In particular the map $\bar{\lambda}_\mu\colon \mathrm{Cot}_j \to \mathcal{O}/c\mathcal{O}$, the class of $f \in A_j$ going to $D(\mu)(f)$, is well defined and $\mathcal{O}$-linear; it depends only on μ.

Step 2: the quotient C. Since $j \neq 0$, $\varphi_0(A_j) = 0$, so $A_j \subseteq \mathfrak{m}_0$ and Cot_j is the image of A_j. Each u_i lies in A_j, so $\mathrm{Cot}(u) \subseteq \mathrm{Cot}_j$ and $\mathrm{Cot}_j \cap \mathrm{Cot}(u) = \mathrm{Cot}(u)$. Put $C = \mathrm{Cot}_j/\mathrm{Cot}(u)$, an $\mathcal{O}$-module of length n. By item 1 and linearity, $\bar{\lambda}_\mu$ vanishes on $\mathrm{Cot}(u)$, so it induces $\Phi(\mu) \in \mathrm{Hom}_{\mathcal{O}}(C, \mathcal{O}/c\mathcal{O})$ with $\Phi(\mu)(\text{class of } f) = D(\mu)(f)$ for $f \in A_j$. As D is additive, so is $\Phi\colon M \to \mathrm{Hom}_{\mathcal{O}}(C, \mathcal{O}/c\mathcal{O})$.

Step 3: Φ is injective. Let $\Phi(\mu) = 0$, so $D(\mu)(f) = 0$ for every $f \in A_j$. For $f \in A_i$ with $i \neq j$, $D(\mu)(f)$ is the class of $D'(f) = 0$. Every element of A is a finite sum of homogeneous elements and $D(\mu)$ is additive, so $D(\mu) = 0$, and $\mu = 0$ since D is injective.

Step 4: counting. A module of finite length over the discrete valuation ring $\mathcal{O}$ is finitely generated and torsion, so $C \cong \bigoplus_{k=1}^{s} \mathcal{O}/\varpi^{a_k}\mathcal{O}$ with $\sum_k a_k = n$. For $a \geq 0$, $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\varpi^a\mathcal{O}, \mathcal{O}/c\mathcal{O})$ is isomorphic to the submodule of $\mathcal{O}/c\mathcal{O}$ killed by ϖ^a, which is cyclic of length $\min(a, v(c)) \leq a$, where v is the normalised valuation of $\mathcal{O}$; so it has at most q^a elements, and $\#\mathrm{Hom}_{\mathcal{O}}(C, \mathcal{O}/c\mathcal{O}) \leq q^n$. By Step 3, M is finite with $\#M \leq q^n$. A finite $\mathcal{O}$-module has finite length, and its composition factors are all isomorphic to k, so $\#M = q^{\mathrm{length}_{\mathcal{O}}(M)}$. Hence $\mathrm{length}_{\mathcal{O}}(M) \leq n = \mathrm{ct}_j(u)$. □

43 The graded model of the Eisenstein deformation ring on $\mathrm{U}(3)$

This section proves Proposition 43.3, the graded model of the normalised ordinary Eisenstein deformation ring, which is the algebraic input of Proposition 42.2. The model is a graded algebra A with a system of equations u and a map $q\colon A \to R$ with kernel $I(u)$, such that the matrix entries of weight j come from A_j, the points of R correspond to the points of A on u up to the torus, the conditions (Gc1)–(Gc4) hold once R^0 is finite and the adjoint Selmer groups vanish at the two-signed points, and the Tate–Shafarevich group embeds into the derivations of degree -1 (clauses (Mo1)–(Mo7)). Its construction is the deformation theory of Mazur [167] made equivariant for the torus: a presentation of R over the ordinary local lifting ring at ℓ, with the framed Selmer cocycles of each weight as variables and the obstruction groups as relations, is made homogeneous and then replaced by a smooth graded algebra built from a chart of the local ring.

The numbers that make the criterion apply come from one count. In each weight j between -2 and 2, the obstruction group of the residual adjoint exceeds the framed Selmer cocycles by 1 for $j \in \{-1, 0, 1\}$ and by 0 for $j = \pm 2$ (Lemma 43.34); together with the tangent dimensions $1, 2, 3, 4, 2$ of the local chart at ℓ and the normalisations, it gives the numbers of equations that (Gc1) requires (clause (Pq6) of Proposition 43.4), in particular the one excess equation in degree -1. Subsection 43.1 constructs the model, Subsection 43.2 the chart of the local ring at ℓ, and Subsection 43.3 proves the rigidity clause and the count.

43.1 The graded model and the presentation of the deformation ring

This subsection constructs the model. The central statement is Proposition 43.4: a presentation of R that is equivariant for the torus, with clauses (Pq1)–(Pq8), in which the numbers of variables and relations in each weight are controlled by the framed Selmer cocycles and the obstruction groups of the weight pieces $\mathfrak{g}_j$ of the residual adjoint (Definition 43.2). It is obtained in three steps. The ring R is the quotient of the ring of global ordinary lifts by eight homogeneous normalisation equations (Proposition 43.5); that ring has a presentation relative to the ordinary local lifting ring at ℓ, with framed Selmer cocycles as variables and obstruction groups as relations (Proposition 43.9); and the kernel of a presentation equivariant for a formal torus action is homogeneous (Lemma 43.8, Lemma 43.10). The relative graded model (Definition 43.11, Proposition 43.12) then replaces the local ring by its graded chart (Proposition 43.13) and carries the Tate–Shafarevich classes into the tangent space of weight -1. The two lemmas on homogeneity are used again for the symplectic and dyadic models and in Part IX.

Definition 43.1. Let ℓ be a prime, $\mathbb{F}$ a finite field of characteristic ℓ, S a finite set of places of $\mathbb{Q}$ containing ℓ and ∞, $G_S = G_{\mathbb{Q},S}$, and M a finite-dimensional $\mathbb{F}$-vector space with a continuous $\mathbb{F}$-linear action of G_S (M discrete). For every prime $p \in S$ fix a decomposition group $D_p \subseteq G_{\mathbb{Q}}$ at p, acting on M through $G_{\mathbb{Q}} \to G_S$. For a profinite group G acting on M (namely G_S or D_p), $C^i(G, M) \supseteq Z^i(G, M) \supseteq B^i(G, M)$ are the continuous i-cochains, cocycles and coboundaries and d is the differential; res_p is the restriction of cochains of G_S to D_p.

A **family of local conditions** for M is a family $\mathcal{L} = (L_p)_p$, indexed by the primes $p \in S$, of $\mathbb{F}$-subspaces $L_p \subseteq Z^1(D_p, M)$. The **Selmer cocycles** are the $z \in Z^1(G_S, M)$ with $\mathrm{res}_p z \in L_p$ for every prime $p \in S$; the **Selmer group** $H^1_{\mathcal{L}}(M)$ is their quotient by the coboundaries $B^1(G_S, M)$ among them. The **obstruction group** $H^2_{\mathcal{L}}(M)$ is the quotient of

$$Z^2_{\mathcal{L}}(M) = \Big\{ (z, (y_p)_p) : \ z \in Z^2(G_S, M),\ y_p \in C^1(D_p, M),\ dy_p = \mathrm{res}_p z \text{ for every prime } p \in S \Big\}$$

by its subspace

$$B^2_{\mathcal{L}}(M) = \Big\{(dw, (\mathrm{res}_p w)_p) :\ w \in C^1(G_S, M)\Big\} + \Big\{(0, (\lambda_p)_p) :\ \lambda_p \in L_p\Big\}.$$

Both are $\mathbb{F}$-vector spaces; dim is $\dim_{\mathbb{F}}$. (They are H^1 and H^2 of the Selmer complex of M with the local conditions $\mathcal{L}$.)

Definition 43.2. Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$ with residue field $\mathbb{F}_{\mathcal{O}}$, and let $\bar\Psi\colon G_{\mathbb{Q}} \to \mathbb{F}_{\mathcal{O}}^\times$ be the reduction of Ψ. In this definition and in the results citing it the datum and $\mathcal{O}$ are fixed and not written in the notation.

Weight pieces. For $j \in \mathbb{Z}$ let $\mathfrak{g}_j \subseteq \mathrm{M}_3(\mathbb{F}_{\mathcal{O}})$ be the subspace of matrices X with $X_{ab} = 0$ whenever $\mathrm{wt}(a,b) \neq j$. The group $G_{\mathbb{Q}}$ acts on $\mathrm{M}_3(\mathbb{F}_{\mathcal{O}})$ through $\bar r_0$: if $\bar r_0(g) = (P, \varsigma)$, then g acts by $X \mapsto PXP^{-1}$ if $\varsigma = 1$ and by $X \mapsto -PJ\,{}^tX\,JP^{-1}$ if $\varsigma = -1$. Each $\mathfrak{g}_j$ is stable ($P = \mathrm{T}(\bar\Psi(g))$ is diagonal, and the (a,b) entry of $J\,{}^tXJ$ is $X_{4-b,4-a}$, of the same weight), and $\mathfrak{g}_j$ with this action is the **weight j piece** of the residual adjoint representation. Let S be the set of places of $\mathbb{Q}$ consisting of ∞ and the primes dividing $\ell N d_K$, and $G_S = G_{\mathbb{Q},S}$; the action on $\mathfrak{g}_j$ factors through G_S, because it is trivial on $G_K \cap \ker\bar\Psi$, whose fixed field lies in $K(\mu_{\ell N})$. For every prime $p \in S$ fix a place $\bar p$ of $\overline{\mathbb{Q}}$ above p, the place $\bar\ell$ for $p = \ell$, and put $D_p = D_{\bar p} \supseteq I_p = I_{\bar p}$ and

$$N_{\bar p} = \{g \in I_{\bar p} :\ \eta_K(g) = 1,\ \bar\Psi(g) = 1\} = \{g \in I_{\bar p} :\ \bar r_0(g) = 1\}.$$

Local conditions. For a prime $p \in S$ and $j \in \mathbb{Z}$ the **local condition** $L^{\mathrm{u}}_{p,j} \subseteq Z^1(D_p, \mathfrak{g}_j)$ is the space of cocycles vanishing on $N_{\bar p}$ if $p \neq \ell$, and $B^1(D_\ell, \mathfrak{g}_j)$ if $p = \ell$.

Selmer and obstruction groups; framed Selmer cocycles. $H^1_{\mathrm{u}}(\mathfrak{g}_j)$ and $H^2_{\mathrm{u}}(\mathfrak{g}_j)$ are the Selmer group and the obstruction group (Definition 43.1) of $M = \mathfrak{g}_j$, for $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, the set S, the decomposition groups D_p and the family of local conditions $(L^{\mathrm{u}}_{p,j})_p$. The space of **framed Selmer cocycles** is

$$Z^1_{\mathrm{u}}(\mathfrak{g}_j) = \{z \in Z^1(G_S, \mathfrak{g}_j) :\ \mathrm{res}_\ell z = 0,\ \mathrm{res}_p z \in L^{\mathrm{u}}_{p,j} \text{ for every prime } p \in S,\ p \neq \ell\};$$

it is the weight j part of the tangent space of the global lifts relative to their restriction to D_ℓ.

Proposition 43.3. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and let $g_\star$ be a marking. Then:*

(Mo1) *there is a normalised ordinary Eisenstein deformation ring (R, r) for $g_\star$, and r has exactly one base point φ_{Eis};*

(Mo2) *there are a graded $\mathcal{O}$-algebra with a fixed point (A, φ_0), a system of equations u for it and an $\mathcal{O}$-algebra homomorphism $q\colon A \to R$ with $\ker q = I(u)$ and $\varphi_{\mathrm{Eis}} \circ q = \varphi_0$, such that (Mo3)–(Mo7) hold;*

(Mo3) *for all $g \in G_{\mathbb{Q}}$ and $1 \le a, b \le 3$, $\underline{r}(g)_{ab} \in q\big(A_{\mathrm{wt}(a,b)}\big)$;*

(Mo4) *for points φ, φ' of (R, r) and $t \in \overline{\mathbb{Q}}_\ell^\times$: $\varphi \circ q$ is negative-vanishing if and only if φ is upper; $\varphi \circ q$ is positive-vanishing if and only if φ is lower; and $r_{\varphi'} = \mathrm{T}(t) \cdot r_\varphi$ if and only if $\varphi' \circ q = t \cdot (\varphi \circ q)$;*

(Mo5) *if the image of A_0 in $A/I(u)$ is a finitely generated $\mathcal{O}$-module, then every point of A on u is a torus translate of $\varphi \circ q$ for some point φ of (R, r);*

(Mo6) *if R^0 is a finitely generated $\mathcal{O}$-module and $H^1_f(\mathbb{Q}, \mathrm{ad}\, r_\varphi) = 0$ for every two-signed point φ of (R, r), then (A, φ_0, u) satisfies (Gc1)–(Gc4);*

(Mo7) *for every $b \in \mathcal{O} \setminus \{0\}$, writing* $\mathrm{Sha}[b]$ *for the submodule of* $\mathrm{Sha}^1_{\ell^N}(\chi, e)$ *killed by b, there is an injective additive map $D\colon \mathrm{Sha}[b] \to \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}/b\mathcal{O})$ such that $D(\mu)(u_{-1,i}) = 0$ for all μ and i, and every $D(\mu)$ is the reduction modulo b of an $\mathcal{O}$-linear map $D'\colon A \to \mathcal{O}$ with $D'(fg) = \varphi_0(f)D'(g) + \varphi_0(g)D'(f)$ for all $f, g \in A$ and $D'(A_j) = 0$ for all $j \neq -1$.*

Proof. Step 1: the presentation. By Proposition 43.4 there are (R, r) with (Pq1), which is (Mo1), and $(A, \varphi_0, u, q, \mathrm{td})$ with (Pq2)–(Pq8); let ne be the counts of u.

Step 2: padding the equations. For $-2 \le j \le 2$ put $d(j) = \mathrm{td}_\ell(j) - \mathrm{nc}(j) - b(j)$, so that $d(-2) = 0$, $d(-1) = -1$, $d(0) = 0$, $d(1) = d(2) = 1$, and put $\mathrm{ne}'(j) = \mathrm{td}(j) - d(j)$, that is

$$\mathrm{ne}'(-1) = \mathrm{td}(-1) + 1, \quad \mathrm{ne}'(-2) = \mathrm{td}(-2), \quad \mathrm{ne}'(0) = \mathrm{td}(0),$$
$$\mathrm{ne}'(1) = \mathrm{td}(1) - 1, \quad \mathrm{ne}'(2) = \mathrm{td}(2) - 1.$$

By (Pq6) and Lemma 43.34 (the dimensions there are finite),

$$\mathrm{ne}(j) \le \mathrm{td}(j) + \dim H^2_{\mathrm{u}}(\mathfrak{g}_j) - \dim Z^1_{\mathrm{u}}(\mathfrak{g}_j) + \mathrm{nc}(j) - \mathrm{td}_\ell(j) = \mathrm{td}(j) + b(j) + \mathrm{nc}(j) - \mathrm{td}_\ell(j) = \mathrm{ne}'(j);$$

in particular $\mathrm{ne}'(j) \ge 0$. Let u' be the family with $u'_{j,i} = u_{j,i}$ for $i \le \mathrm{ne}(j)$ and $u'_{j,i} = 0 \in A_j$ for $\mathrm{ne}(j) < i \le \mathrm{ne}'(j)$. It is a system of equations for (A, φ_0) with counts ne' ($\varphi_0(0) = 0$), and since only zeros were added, $I(u') = I(u)$, the points on u' are those on u, and $\mathrm{Cot}(u'_j) = \mathrm{Cot}(u_j)$ for every j.

Step 3: (Mo2)–(Mo5) and (Mo7) for (A, φ_0, u', q). $\ker q = I(u) = I(u')$ and $\varphi_{\mathrm{Eis}} \circ q = \varphi_0$ (Pq2); (Mo3) and (Mo4) are (Pq3) and (Pq4), which do not involve the equations; (Mo5) is (Pq5), since $A/I(u') = A/I(u)$ and the points on u' and on u coincide; (Mo7) is (Pq8), since $D(\mu)(0) = 0$ for the added equations of degree -1.

Step 4: (Mo6). Assume that R^0 is a finitely generated $\mathcal{O}$-module and that $H^1_f(\mathbb{Q}, \mathrm{ad}\, r_\varphi) = 0$ for every two-signed point φ of (R, r). (Gc1): by (Pq6), (A, φ_0) is smooth with tangent dimensions td, where $\mathrm{td}(j) = 0$ for $|j| > 2$ (Pq2); the counts of u' satisfy the five equalities of (Gc1) by the definition of ne′; and u'_{-1}, u'_{-2}, u'_0 span rationally in degrees -1, -2, 0 because u_{-1}, u_{-2}, u_0 do (Pq7) and $\mathrm{Cot}(u'_j) = \mathrm{Cot}(u_j)$. (Gc2)–(Gc4) hold for u by (Pq7), and they depend on u only through $I(u)$ and the points on u (the image of A_0 in $A/I(u)$; points on u; derivations vanishing on $I(u)$), so they hold for u'. Hence (A, φ_0, u') satisfies (Gc1)–(Gc4). □

Proposition 43.4. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and let $g_\star$ be a marking. Put*

$$\big(\mathrm{td}_\ell(j)\big)_{j=-2}^{2} = (1, 2, 3, 4, 2), \qquad \big(\mathrm{nc}(j)\big)_{j=-2}^{2} = (1, 2, 2, 2, 1).$$

Then:

(Pq1) *there is a normalised ordinary Eisenstein deformation ring (R, r) for $g_\star$, and r has exactly one base point φ_{Eis};*

(Pq2) *there are a graded $\mathcal{O}$-algebra with a fixed point (A, φ_0), a system of equations u for it with counts* ne*, an $\mathcal{O}$-algebra homomorphism $q\colon A \to R$ with $\ker q = I(u)$ and $\varphi_{\mathrm{Eis}} \circ q = \varphi_0$, and a function* $\mathrm{td}\colon \mathbb{Z} \to \mathbb{N}$ *with $\mathrm{td}(j) = 0$ for $|j| > 2$, such that (Pq3)–(Pq8) hold;*

(Pq3) *for all $g \in G_{\mathbb{Q}}$ and $1 \le a, b \le 3$, $\underline{r}(g)_{ab} \in q\big(A_{\mathrm{wt}(a,b)}\big)$;*

(Pq4) *for points φ, φ' of (R, r) and $t \in \overline{\mathbb{Q}}_\ell^\times$: $\varphi \circ q$ is negative-vanishing if and only if φ is upper; $\varphi \circ q$ is positive-vanishing if and only if φ is lower; and $r_{\varphi'} = \mathrm{T}(t) \cdot r_\varphi$ if and only if $\varphi' \circ q = t \cdot (\varphi \circ q)$;*

(Pq5) *if the image of A_0 in $A/I(u)$ is a finitely generated $\mathcal{O}$-module, then every point of A on u is a torus translate of $\varphi \circ q$ for some point φ of (R, r);*

(Pq6) *(A, φ_0) is smooth with tangent dimensions* td*, and for $-2 \le j \le 2$*

$$\mathrm{ne}(j) + \mathrm{td}_\ell(j) + \dim Z^1_{\mathrm{u}}(\mathfrak{g}_j) \le \mathrm{td}(j) + \dim H^2_{\mathrm{u}}(\mathfrak{g}_j) + \mathrm{nc}(j);$$

(Pq7) *if R^0 is a finitely generated $\mathcal{O}$-module and $H^1_f(\mathbb{Q}, \operatorname{ad} r_\varphi) = 0$ for every two-signed point φ of (R, r), then u_{-1}, u_{-2}, u_0 span rationally in degrees -1, -2, 0 respectively, and (A, φ_0, u) satisfies (Gc2)–(Gc4);*

(Pq8) *for every $b \in \mathcal{O} \setminus \{0\}$, writing* $\mathrm{Sha}[b]$ *for the submodule of* $\mathrm{Sha}^1_{\ell N}(\chi, e)$ *killed by b, there is an injective additive map $D\colon \mathrm{Sha}[b] \to \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}/b\mathcal{O})$ such that $D(\mu)(u_{-1,i}) = 0$ for all μ and i, and every $D(\mu)$ is the reduction modulo b of an $\mathcal{O}$-linear map $D'\colon A \to \mathcal{O}$ with $D'(fg) = \varphi_0(f)D'(g) + \varphi_0(g)D'(f)$ for all $f, g \in A$ and $D'(A_j) = 0$ for all $j \ne -1$.*

Proof. (Pq1). By (Bx1) let (B, r_B) be the ring of lifts of $\bar{r}_0$ satisfying (Ed2), (Ed3), and by (Bx3) let $(R, r) = (B/\mathfrak{n}, r)$: a normalised ordinary Eisenstein deformation ring for $g_\star$ whose lift r has exactly one base point φ_{Eis}.

(Pq2)–(Pq6), (Pq8). Choose a weighted presentation $(\sigma_\ell, \mathrm{wt}, \pi_\ell)$ of R^{ord}_ℓ as in (Ol3), then τ, P, π and the family f of Proposition 43.9, and then lifts $\tilde{n}_k$ and the system u as in Proposition 43.12. Let $(A, \varphi_0, \bar{u}, q)$ be the relative graded model of $(\sigma_\ell \sqcup \tau, \mathrm{wt}, \pi_R)$ modulo $\mathfrak{a}$ with equations u, and $\mathrm{td}(j) = \mathrm{td}_\ell(j) + \#\{t \in \tau : \mathrm{wt}(t) = j\}$. By (Mp1), (A, φ_0) is a graded $\mathcal{O}$-algebra with a fixed point, $\bar{u}$ is a system of equations for it with counts ne, $q\colon A \to R$ is an $\mathcal{O}$-algebra homomorphism with $\ker q = I(\bar{u})$ and $\varphi_{\mathrm{Eis}} \circ q = \varphi_0$, and $\mathrm{td}(j) = 0$ for $|j| > 2$: so (Pq2) holds with $\bar{u}$ as the system of equations, once (Pq3)–(Pq8) are checked for it. By (Mp2), (Pq3), (Pq4), (Pq5), (Pq6) and (Pq8) hold.

(Pq7). By Proposition 43.29, applied to these objects: if R^0 is a finitely generated $\mathcal{O}$-module and $H^1_f(\mathbb{Q}, \operatorname{ad} r_\varphi) = 0$ for every two-signed point φ, then $\bar{u}_{-1}$, $\bar{u}_{-2}$, $\bar{u}_0$ span rationally in degrees -1, -2, 0, and $(A, \varphi_0, \bar{u})$ satisfies (Gc2)–(Gc4). □

Proposition 43.5. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and $g_\star$ a marking.*

(Bx1) *There are $B \in \mathrm{CNL}_{\mathcal{O}}$ and a lift r_B of $\bar{r}_0$ to B satisfying (Ed2) and (Ed3) such that for every lift r' of $\bar{r}_0$ to some $A \in \mathrm{CNL}_{\mathcal{O}}$ satisfying (Ed2) and (Ed3) there is exactly one morphism $f\colon B \to A$ with $f \circ r_B = r'$, and $f \circ r_B$ satisfies (Ed2), (Ed3) for every morphism f out of B. There is exactly one morphism $i_B\colon R^{\mathrm{ord}}_\ell \to B$ with $i_B \circ r^{\mathrm{ord}}_\ell = r_B|_{D_\ell}$, and r_B has exactly one base point.*

(Bx2) *For $A \in \mathrm{CNL}_{\mathcal{O}}$, $h \in \mathrm{GL}_3(A)$ with $h \equiv 1 \bmod \mathfrak{m}_A$ and a lift r' to A satisfying (Ed2), (Ed3), the lift $(h, 1)r'(h, 1)^{-1}$ satisfies (Ed2), (Ed3). The family $y_{g,a,b} = \underline{r_B}(g)_{ab} - \underline{r_0}(g)_{ab}$ ($g \in G_{\mathbb{Q}}$, $1 \le a, b \le 3$) with weights $\mathrm{wt}(a, b)$ satisfies (Tc1) and (Tc2) for B, with $\rho_B \circ r_B = (\mathrm{T}(1+T), 1)\, r_B\, (\mathrm{T}(1+T), 1)^{-1}$; every entry $\underline{r_B}(g)_{ab}$ has torus weight $\mathrm{wt}(a, b)$; $i_B(x_{g,a,b}) = y_{g,a,b}$ for $g \in D_\ell$; and $\rho_B \circ i_B = i_B[[T]] \circ \rho_\ell$, so that i_B maps elements of torus weight j to elements of torus weight j.*

(Bx3) *Let $\Delta = \underline{r_B}(g_\star) - \underline{r_B}(cg_\star^{-1}c)$, $P = \underline{r_B}(c)$, and let $\mathfrak{n} \subseteq B$ be the ideal generated by*

$$n_1 = P_{11} - 1,\ n_2 = P_{22} - 1,\ n_3 = P_{12},\ n_4 = P_{21},\ n_5 = P_{13},\ n_6 = P_{31},\ n_7 = \Delta_{12},\ n_8 = \Delta_{21},$$

which have torus weights $0,0,1,-1,2,-2,1,-1$. *Then* $R = B/\mathfrak{n}$ *with the image* r *of* r_B *is a normalised ordinary Eisenstein deformation ring for* $g_\star$, *and* r *has exactly one base point.*

Proof. Write $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$. Let S consist of ∞ and the primes dividing $\ell N d_K$, $G_S = G_{\mathbb{Q},S}$, and $\Gamma \subseteq G_S$ the image of G_K; K is unramified outside d_K, so $\Gamma = G_{K,S_K}$ (S_K the places of K above S) has index 2, and $G_S = \Gamma \sqcup \Gamma c$. Recall $\Psi(c) = 1$, so $\bar{r}_0(c) = (1,-1)$.

Step 1: lifts satisfying (Ed2) factor through G_S. For $\bar{v}$ above a prime $p \notin S$, η_K and $\bar{\Psi}$ are unramified at p, so $I_{\bar{v}} \subseteq \ker \bar{r}_0$ and (Ed2) makes the lift trivial on $I_{\bar{v}}$; hence on the closed normal subgroup generated by these groups, the kernel of $G_{\mathbb{Q}} \to G_S$.

Step 2: lifts on G_S. Let $\bar{\rho} = \underline{\bar{r}_0}|_\Gamma = \mathrm{T}(\bar{\Psi})$ and $(R^\square_\Gamma, \rho^\square)$ its universal lift (Theorem B.71; Γ satisfies Φ_ℓ by item 2). For a lift r' of $\bar{r}_0$ through G_S, $\rho = \underline{r'}|_\Gamma$ is a lift of $\bar{\rho}$ and $r'(c) = (Q,-1)$ with $Q \equiv 1$; the relations $r'(c)^2 = 1$ and $r'(cgc) = r'(c)r'(g)r'(c)$ ($g \in \Gamma$) read

$$Q\,\theta(Q) = 1, \qquad \rho(cgc) = Q\,\theta(\rho(g))\,\theta(Q) \qquad (g \in \Gamma),$$

since $(Q,-1)(M,1)(Q,-1) = (Q\theta(M)\theta(Q),1)$. Conversely a lift ρ of $\bar{\rho}$ and $Q \equiv 1$ with these relations define the homomorphism $r'(g) = (\rho(g),1)$, $r'(gc) = (\rho(g)Q,-1)$ ($g \in \Gamma$) of $\Gamma \rtimes \langle c\rangle = G_S$, continuous as ρ is. Hence lifts of $\bar{r}_0$ through G_S are represented by $R_S = R^\square_\Gamma[[Z_{ab}]]/\mathfrak{a}$, with $Q = 1 + (Z_{ab})$ and $\mathfrak{a}$ generated by the entries of the two relations, and the universal lift r_S.

Step 3: (Ed2), (Ed3) are cut out by ideals. For a morphism $f\colon R_S \to A$: $f \circ r_S$ satisfies (Ed2) if and only if f kills the entries of $\underline{r_S}(g) - 1$ for $g \in N = \bigcup_{\bar{v}\nmid\ell}(I_{\bar{v}} \cap \ker \bar{r}_0)$ (such g have sign 1; Step 1 covers $p \notin S$ automatically). The restriction $r_S|_{D_\ell}$ is classified by a morphism $f_S\colon R^\square_\ell \to R_S$, and $f \circ r_S|_{D_\ell}$ by $f \circ f_S$; so $f \circ r_S$ satisfies (Ed3) if and only if f kills $f_S(I^{\mathrm{ord}}_\ell)$. Let B be the quotient of R_S by these elements and r_B the image of r_S. Then r_B and all $f \circ r_B$ satisfy (Ed2), (Ed3), and a lift r' to A satisfying them factors through G_S (Step 1), so $r' = f \circ r_S$ for exactly one f (Step 2), which kills the generators, i.e. factors uniquely through B.

Step 4: $B \neq 0$ *and the base point.* r_0 satisfies (Ed2): for $g \in I_{\bar{v}} \cap \ker \bar{r}_0$ with $\bar{v} \nmid \ell$, $\eta_K(g) = 1$ and $\Psi(g) = \chi(g)$ (χ_{cyc} is unramified at $\bar{v}$) is a root of unity of order dividing the order of χ, prime to ℓ by (Ue3), with residue 1; so $\Psi(g) = 1$ and $r_0(g) = 1$. It satisfies (Ed3) by (Ol1): $r_0|_{D_\ell} = \beta_\ell \circ r^\square_\ell$ and $\beta_\ell(I^{\mathrm{ord}}_\ell) = 0$. So $r_0 = \varphi \circ r_B$ for one morphism $\varphi\colon B \to \mathcal{O}$; in particular $B \neq 0$, $B \in \mathrm{CNL}_{\mathcal{O}}$. An $\mathcal{O}$-algebra homomorphism $B \to \mathcal{O}$ is local (for $y \in \mathfrak{m}_B$, $u \in \mathcal{O}^\times$, $y - u$ is a unit), so the base points of r_B are the morphisms classifying r_0: φ is the only one. By (Ed3) for r_B, $r_B|_{D_\ell} = f_B \circ r^\square_\ell$ with $f_B(I^{\mathrm{ord}}_\ell) = 0$ (f_B is unique by universality), so f_B induces i_B, unique because $R^\square_\ell \to R^{\mathrm{ord}}_\ell$ is onto. This proves (Bx1).

Step 5: (Bx2). Conjugation by $(h,1)$ keeps the reduction, continuity and the value 1 on elements with $r'(g) = 1$, so (Ed2) is kept; (Ed3) is kept by (Ol2). (Tc1): morphisms f, f' out of B agreeing on all $y_{g,a,b}$ give lifts with equal GL-parts and equal signs η_K, hence $f = f'$ by (Bx1). (Tc2): by the first part with $A = B[[T]]$ and $h = \mathrm{T}(1+T)$ there is ρ_B with $\rho_B \circ r_B = (\mathrm{T}(1+T),1)r_B(\mathrm{T}(1+T),1)^{-1}$; since $\theta(\mathrm{T}(t)) = \mathrm{T}(t)$, the GL-part of the right side at g is $\mathrm{T}(1+T)\underline{r_B}(g)\mathrm{T}(1+T)^{-1}$ for both signs, which multiplies the entry (a,b) by $(1+T)^{\mathrm{wt}(a,b)}$; as $\underline{r_0}(g)$ is diagonal, $\rho_B(y_{g,a,b}) = (1+T)^{\mathrm{wt}(a,b)}y_{g,a,b}$. The identity $i_B(x_{g,a,b}) = y_{g,a,b}$ for $g \in D_\ell$ is $i_B \circ r^{\mathrm{ord}}_\ell = r_B|_{D_\ell}$. Finally $\rho_B \circ i_B$ and $i_B[[T]] \circ \rho_\ell$ are morphisms $R^{\mathrm{ord}}_\ell \to B[[T]]$ carrying r^{ord}_ℓ to the same lift $(\mathrm{T}(1+T),1)r_B|_{D_\ell}(\mathrm{T}(1+T),1)^{-1}$; composed with the surjection $R^\square_\ell \to R^{\mathrm{ord}}_\ell$ they classify the same lift of $\bar{r}_0|_{D_\ell}$, so they are equal. If $\rho_\ell(z) = (1+T)^j z$, then $\rho_B(i_B(z)) = i_B[[T]]((1+T)^j z) = (1+T)^j i_B(z)$.

Step 6: (Bx3). The torus weights of $n_1,\dots,n_8$ follow from (Bx2) (1 has weight 0, and the entries $(1,2)$, $(2,1)$ of both terms of Δ have weights 1, -1). Let $f\colon B \to A$ and $Q = f(P)$.

Since $c^2 = 1$ and c has sign -1, $(Q,-1)^2 = (Q\theta(Q),1) = 1$, i.e. $QJ\,{}^tQ^{-1}J = 1$, i.e. $QJ = J\,{}^tQ = {}^t(QJ)$: the matrix QJ, with entries $(QJ)_{ab} = Q_{a,4-b}$, is symmetric. Its symmetry gives $Q_{33} = Q_{11}$, $Q_{23} = Q_{12}$, $Q_{32} = Q_{21}$. Hence $Q = 1$ if and only if $f(n_1) = \cdots = f(n_6) = 0$, and $f \circ r_B$ satisfies (Ed1) if and only if $f(\mathfrak{n}) = 0$. So the lifts to A satisfying (Ed1)–(Ed3) are exactly the $f \circ r_B$ with $f(\mathfrak{n}) = 0$, each for exactly one f, i.e. the $\bar{f} \circ r$ for exactly one $\bar{f}\colon B/\mathfrak{n} \to A$, and every $\bar{f} \circ r$ satisfies (Ed1)–(Ed3). r_0 satisfies (Ed1): $\underline{r_0}(c) = \mathrm{T}(\Psi(c)) = 1$, and $\underline{r_0}(g_\star) = \mathrm{T}(\Psi(g_\star))$ and $\underline{r_0}(cg_\star^{-1}c) = \theta(\mathrm{T}(\Psi(g_\star))^{-1}) = \mathrm{T}(\Psi(g_\star))^{-1}$ are diagonal. So $\varphi(\mathfrak{n}) = 0$, $R = B/\mathfrak{n} \neq 0$ lies in $\mathrm{CNL}_{\mathcal{O}}$, (R, r) is a normalised ordinary Eisenstein deformation ring for $g_\star$, and as in Step 4 its only base point is induced by φ. □

Definition 43.6. Let $\mathcal{O}$ be a coefficient ring and $R \in \mathrm{CNL}_{\mathcal{O}}$ with maximal ideal $\mathfrak{m}_R$. For a finite set σ with a map $\mathrm{wt}\colon \sigma \to \mathbb{Z}$, the **weight** of a monomial $X^n = \prod_s X_s^{n_s}$ in $\mathcal{O}[[X_s : s \in \sigma]]$ is $\sum_s n_s \mathrm{wt}(s)$; the **weight j component** $f_{(j)}$ of a power series f is the power series formed by the terms of f whose monomials have weight j, and $\mathcal{O}[[X_\sigma]]_j$ is the set of power series f with $f = f_{(j)}$.

Weighted presentations. A **weighted presentation** of R is a triple $(\sigma, \mathrm{wt}, \pi)$ of a finite set σ, a map $\mathrm{wt}\colon \sigma \to \{-2,-1,0,1,2\}$ and a surjective $\mathcal{O}$-algebra homomorphism $\pi\colon \mathcal{O}[[X_s : s \in \sigma]] \to R$ with $\pi(X_s) \in \mathfrak{m}_R$ for every s. A **homogeneous generating system** of its kernel with counts $\mathrm{ne}\colon \{-2,\dots,2\} \to \mathbb{N}$ is a family $u = (u_{j,i})$, $-2 \le j \le 2$, $1 \le i \le \mathrm{ne}(j)$, with $u_{j,i} \in \ker\pi \cap \mathcal{O}[[X_\sigma]]_j$, which generates the ideal $\ker\pi$.

The graded model. For a weighted presentation $(\sigma, \mathrm{wt}, \pi)$ of R and an $\mathcal{O}$-algebra homomorphism $\beta\colon R \to \mathcal{O}$ with $\beta(\pi(X_s)) = 0$ for every s, the **graded model** is the tuple $(A_\pi, \varphi_\beta, q_\pi)$, where

$$A_\pi = \sum_{j \in \mathbb{Z}} \mathcal{O}[[X_\sigma]]_j \subseteq \mathcal{O}[[X_s : s \in \sigma]]$$

(the set of finite sums of power series of single weights), graded by $(A_\pi)_j = \mathcal{O}[[X_\sigma]]_j$, q_π is the restriction of π to A_π, and $\varphi_\beta = \beta \circ q_\pi$. With a homogeneous generating system u of $\ker\pi$, the tuple $(A_\pi, \varphi_\beta, u, q_\pi)$ is the **graded model with equations** u.

Lemma 43.7. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$. Put $R_\ell^{\mathrm{ord}} = R_\ell^{\square}/I_\ell^{\mathrm{ord}}$ and let r_ℓ^{ord} be the image of $r_\ell^{\square}$ in $\mathcal{G}_3(R_\ell^{\mathrm{ord}})$. Then:*

(Ol1) *$r_\ell^{\square}$ has exactly one base point β_ℓ, and $I_\ell^{\mathrm{ord}} \subseteq \ker\beta_\ell$; hence $R_\ell^{\mathrm{ord}} \in \mathrm{CNL}_{\mathcal{O}}$, and r_ℓ^{ord} has exactly one base point.*

(Ol2) *Let $A \in \mathrm{CNL}_{\mathcal{O}}$, $h \in \mathrm{GL}_3(A)$ with $h \equiv 1 \bmod \mathfrak{m}_A$, and $f\colon R_\ell^{\square} \to A$ a morphism with $f(I_\ell^{\mathrm{ord}}) = 0$. Then $(h,1)\,(f \circ r_\ell^{\square})\,(h,1)^{-1} = f' \circ r_\ell^{\square}$ for a morphism f' with $f'(I_\ell^{\mathrm{ord}}) = 0$.*

(Ol3) *The family $x_{g,a,b} = \underline{r_\ell^{\mathrm{ord}}}(g)_{ab} - \underline{r_0}(g)_{ab}$ ($g \in D_\ell$, $1 \le a,b \le 3$) with $w(g,a,b) = \mathrm{wt}(a,b)$ satisfies (Tc1) and (Tc2) for $R = R_\ell^{\mathrm{ord}}$, with $\rho \circ r_\ell^{\mathrm{ord}} = (\mathrm{T}(1+T),1)\,r_\ell^{\mathrm{ord}}\,(\mathrm{T}(1+T),1)^{-1}$, and every entry $\underline{r_\ell^{\mathrm{ord}}}(g)_{ab}$ has torus weight $\mathrm{wt}(a,b)$. Consequently there is a weighted presentation $(\sigma_\ell, \mathrm{wt}, \pi_\ell)$ of R_ℓ^{ord} with $\pi_\ell(X_s) = x_s$ for a finite set σ_ℓ of triples $s = (g_s, a_s, b_s)$ and $\mathrm{wt}(s) = \mathrm{wt}(a_s, b_s)$, satisfying (Tp1)–(Tp3), and $\underline{r_\ell^{\mathrm{ord}}}(g)_{ab} \in \pi_\ell\big(\mathcal{O}[[X_{\sigma_\ell}]]_{\mathrm{wt}(a,b)}\big)$ for all $g \in D_\ell$ and a,b.*

Proof. Write $R = R_\ell^{\square}$, $J = I^{\mathrm{ord}}(r_\ell^{\square})$ and $\gamma = \mathrm{gd}(r_\ell^{\square})$. Two remarks are used throughout. (a) For a morphism $f\colon R \to A$, $I^{\mathrm{ord}}(f \circ r_\ell^{\square})$ is the ideal generated by $f(J)$, and $\mathrm{gd}(f \circ r_\ell^{\square}) = f(\gamma)$, since the generators and gd are polynomials in the entries. (b) An $\mathcal{O}$-algebra homomorphism β from a ring $B \in \mathrm{CNL}_{\mathcal{O}}$ to $\mathcal{O}$ is local (for $y \in \mathfrak{m}_B$ and $u \in \mathcal{O}^\times$, $y - u$ is a unit, so $\beta(y) \neq u$), hence a morphism of $\mathrm{CNL}_{\mathcal{O}}$.

(Ol1). By (b) and universality, the base points of $r_\ell^\square$ are the morphisms $R \to \mathcal{O}$ classifying the lift $r_0|_{D_\ell}$; there is exactly one, β_ℓ. We exhibit a minimal prime of I^{cr} inside $\ker \beta_\ell$. Let $B = \mathcal{O}[[u]]$, a domain. The group $1+\mathfrak{m}_B$ is pro-ℓ ($\mathbb{F}_\mathcal{O}$ is finite), so there is a continuous unramified character $\lambda\colon D_\ell \to B^\times$, trivial on I_ℓ, with $\lambda(\mathrm{Fr}) = 1+u$. Since $(\mathrm{T}(s),\varsigma)(\mathrm{T}(s'),\varsigma') = (\mathrm{T}(ss'),\varsigma\varsigma')$ (θ fixes $\mathrm{T}(s')$), $r_u(g) = (\mathrm{T}(\Psi(g)\lambda(g)), \eta_K(g))$ is a lift of $\bar r_0|_{D_\ell}$ to B; let $F\colon R \to B$ classify it, so that β_ℓ is F followed by $u \mapsto 0$. On $D_\ell^\circ \cap I_\ell$, $\underline{r_u} = \mathrm{diag}(\psi_1,\psi_2,\psi_3)$, so the characteristic polynomials agree with $\prod_a (X-\psi_a)$, and the product $\prod_{k=1}^3 (\underline{r_u}(g_k) - \psi_k(g_k))$ is a product of diagonal matrices whose k-th factor has (k,k) entry 0; so $F(J) = 0$ by (a). Next, $\Psi(\sigma_0) = \chi_{\mathrm{cyc}}(\sigma_0)^e$ (on $D_\ell^\circ \cap I_\ell$ both χ_ℓ and the unramified part of χ are trivial) has infinite order, so $\psi_1(\sigma_0), \psi_2(\sigma_0), \psi_3(\sigma_0)$ are distinct and $\mathrm{gp}_a \neq 0$. With $\mu = \lambda(\sigma_\ell) = (1+u)^n$ one has $\underline{r_u}(\sigma_\ell) = \mathrm{diag}(\psi_b(\sigma_\ell)\mu^{2-b})_b$, and only $b = a$ contributes to the trace in tq_a:

$$F(\gamma) = \mathrm{gd}(r_u) = \ell\,\mathrm{gp}_1\mathrm{gp}_2^2\mathrm{gp}_3\,\psi_2\psi_3(\sigma_\ell)\,\mu^{-1}\big(\Psi(\sigma_\ell)(1+u)^n - \chi_{\mathrm{cyc}}(\sigma_\ell)\big)^2.$$

The last factor is a power series with coefficient $n\Psi(\sigma_\ell) \neq 0$ at u, so $F(\gamma) \neq 0$. As B is a domain, $\gamma^k y \in J$ implies $F(y) = 0$, and $\ker F$ is a prime ideal containing the saturation of J, hence I^{cr}. So $\ker F$ contains a minimal prime $\mathfrak{P}$ of I^{cr}, and $\mathfrak{P} \subseteq \ker F \subseteq \ker \beta_\ell$. Hence $\mathfrak{P}$ is one of the primes intersected in I_ℓ^{ord}, $I_\ell^{\mathrm{ord}} \subseteq \mathfrak{P} \subseteq \ker\beta_\ell$, R_ℓ^{ord} is a non-zero quotient of R and lies in $\mathrm{CNL}_\mathcal{O}$, β_ℓ factors through it, and by (b) and universality it is the only base point of r_ℓ^{ord}.

(Ol2), Step 1: the universal conjugation. Let $R' = R[[Z_{ab} : 1 \le a,b \le 3]] \in \mathrm{CNL}_\mathcal{O}$, $\iota\colon R \to R'$ the inclusion and $H = 1 + (Z_{ab}) \in \mathrm{GL}_3(R')$. The lifts $(H^{\pm1},1)(\iota\circ r_\ell^\square)(H^{\pm1},1)^{-1}$ are classified by morphisms $\Theta_0^\pm\colon R \to R'$; let $\Theta^\pm\colon R' \to R'$ be the continuous extensions with $\Theta^\pm(Z_{ab}) = Z_{ab}$. Then $\Theta^-\circ\Theta_0^+$ classifies $(H,1)(H,1)^{-1}(\iota\circ r_\ell^\square)(H,1)(H,1)^{-1} = \iota\circ r_\ell^\square$, so it is ι; thus $\Theta^-\Theta^+$ is the identity on $\iota(R)$ and on the Z_{ab}, hence on R', and likewise $\Theta^+\Theta^-$. Write $\Theta = \Theta^+$, an automorphism. Modulo (Z), $H \equiv 1$, so $\Theta_0 \equiv \iota \bmod (Z)R'$ (both reductions classify $r_\ell^\square$), and Θ induces the identity on $R'/(Z)R' = R$.

Step 2: Θ preserves JR' and fixes γ. For $g \in D_\ell^\circ$ (sign 1) the GL-part of the conjugate is $H\underline{r_\ell^\square}(g)H^{-1}$: it has the same characteristic polynomial and traces of products, and the entries of HMH^{-1} generate the same ideal as those of M. So I^{ord} of the conjugate is JR' and its gd is γ. By (a) they are also $\Theta(J)R' = \Theta(JR')$ and $\Theta(\gamma)$.

Step 3: Θ preserves $I_\ell^{\mathrm{ord}}R'$. R' is flat over the Noetherian ring R, so for ideals $\mathfrak{a},\mathfrak{b}$ of R: $(\mathfrak{a}R' :_{R'} \gamma^k) = (\mathfrak{a} :_R \gamma^k)R'$ and $(\mathfrak{a}\cap\mathfrak{b})R' = \mathfrak{a}R' \cap \mathfrak{b}R'$ (Matsumura, Theorem 7.4). The saturation $\mathrm{Sat}(J) = (J : \gamma^k)$ for k large (ascending chain), so $\mathrm{Sat}(JR') = \mathrm{Sat}(J)R'$, and Θ preserves it by Step 2. For a radical ideal $\mathfrak{r}$ of R with minimal primes $\mathfrak{P}_1,\dots,\mathfrak{P}_m$, $R'/\mathfrak{P}_iR' = (R/\mathfrak{P}_i)[[Z]]$ is a domain and $R'/\mathfrak{r}R' = (R/\mathfrak{r})[[Z]]$ embeds coefficientwise in $\prod_i (R/\mathfrak{P}_i)[[Z]]$, hence is reduced; so $\mathrm{rad}(\mathfrak{a}R') = \mathrm{rad}(\mathfrak{a})R'$ for every ideal $\mathfrak{a}$, and $I^{\mathrm{cr}}R' = \mathrm{rad}(\mathrm{Sat}(JR'))$ is Θ-stable. With $\mathfrak{r} = I^{\mathrm{cr}}$: $I^{\mathrm{cr}}R' = \bigcap_i \mathfrak{P}_iR'$ with primes $\mathfrak{P}_iR'$, and $\mathfrak{P}_iR' \cap R = \mathfrak{P}_i$ (reduce modulo (Z)), so they are pairwise incomparable and are exactly the minimal primes of $I^{\mathrm{cr}}R'$. Θ permutes them, and if $\Theta(\mathfrak{P}_iR') = \mathfrak{P}_jR'$, reducing modulo (Z), where Θ induces the identity, gives $\mathfrak{P}_i = \mathfrak{P}_j$. So Θ fixes every $\mathfrak{P}_iR'$ and preserves $I_\ell^{\mathrm{ord}}R' = \bigcap_{\mathfrak{P}_i \subseteq \ker\beta_\ell} \mathfrak{P}_iR'$.

Step 4: (Ol2). Let $\Phi\colon R' \to A$ be the continuous extension of f with $\Phi(Z_{ab}) = (h-1)_{ab} \in \mathfrak{m}_A$. Then $\Phi(H) = h$ and $\Phi\circ\Theta_0$ classifies $(h,1)(f\circ r_\ell^\square)(h,1)^{-1}$; put $f' = \Phi\circ\Theta_0$. By Step 3, $f'(I_\ell^{\mathrm{ord}}) \subseteq \Phi(I_\ell^{\mathrm{ord}}R') = f(I_\ell^{\mathrm{ord}})A = 0$.

(Ol3). Each $x_{g,a,b}$ lies in $\mathfrak{m}_{R_\ell^{\mathrm{ord}}}$, since r_ℓ^{ord} and r_0 have the same reduction. (Tc1): if morphisms f, f' out of R_ℓ^{ord} agree on all $x_{g,a,b}$, the two lifts $f\circ r_\ell^{\mathrm{ord}}$, $f'\circ r_\ell^{\mathrm{ord}}$ have equal GL-parts and equal signs η_K, so by universality f and f' agree after composition with the surjection $R \to R_\ell^{\mathrm{ord}}$, hence are equal. (Tc2): apply (Ol2) to $A = R_\ell^{\mathrm{ord}}[[T]]$, f the composite $R \to R_\ell^{\mathrm{ord}} \subseteq A$ and $h = \mathrm{T}(1+T) \equiv 1$; the morphism f' kills I_ℓ^{ord} and induces $\rho\colon R_\ell^{\mathrm{ord}} \to A$ with

$\rho \circ r_\ell^{\mathrm{ord}} = (\mathrm{T}(1+T),1) r_\ell^{\mathrm{ord}} (\mathrm{T}(1+T),1)^{-1}$. For g of sign 1 the GL-part of the right side is $\mathrm{T}(1+T)\underline{r_\ell^{\mathrm{ord}}}(g)\mathrm{T}(1+T)^{-1}$; for g of sign -1 it is $\mathrm{T}(1+T)\underline{r_\ell^{\mathrm{ord}}}(g)\theta(\mathrm{T}(1+T))^{-1}$, the same matrix as $\theta(\mathrm{T}(t)) = \mathrm{T}(t)$. Conjugation by $\mathrm{T}(t)$ multiplies the entry (a,b) by $t^{\mathrm{wt}(a,b)}$, so every entry $\underline{r_\ell^{\mathrm{ord}}}(g)_{ab}$ has torus weight $\mathrm{wt}(a,b)$, and as $\underline{r_0}(g)$ is diagonal and $\mathrm{wt}(a,a)=0$, $\rho(x_{g,a,b}) = (1+T)^{\mathrm{wt}(a,b)} x_{g,a,b}$. Lemma 43.8 now gives σ_ℓ, wt, π_ℓ with (Tp1)–(Tp3), and (Tp3) applied to the entries gives the last assertion. □

Lemma 43.8. *Let $\mathcal{O}$ be a coefficient ring, $R \in \mathrm{CNL}_{\mathcal{O}}$ with maximal ideal $\mathfrak{m}_R$, S a set, $(x_s)_{s\in S}$ a family of elements of $\mathfrak{m}_R$ and $w\colon S \to \{-2,-1,0,1,2\}$ a map. Let $R[[T]]$ be the power series ring in one variable over R, an object of $\mathrm{CNL}_{\mathcal{O}}$. Assume:*

(Tc1) *any two morphisms $f, f'\colon R \to B$ in $\mathrm{CNL}_{\mathcal{O}}$ with $f(x_s) = f'(x_s)$ for all $s \in S$ are equal;*

(Tc2) *there is a morphism $\rho\colon R \to R[[T]]$ in $\mathrm{CNL}_{\mathcal{O}}$ with $\rho(x_s) = (1+T)^{w(s)} x_s$ for every $s \in S$.*

An element $y \in R$ has torus weight $j \in \mathbb{Z}$ *if $\rho(y) = (1+T)^j y$. Then there is a finite subset $\sigma \subseteq S$ such that, with $\mathrm{wt} = w|_\sigma$ and $\pi\colon \mathcal{O}[[X_s : s \in \sigma]] \to R$ the continuous $\mathcal{O}$-algebra homomorphism with $\pi(X_s) = x_s$:*

(Tp1) *$(\sigma, \mathrm{wt}, \pi)$ is a weighted presentation of R, and the images of the x_s, $s \in \sigma$, form a basis of the $\mathbb{F}_{\mathcal{O}}$-vector space $\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_{\mathcal{O}} R)$;*

(Tp2) *for every $f \in \ker\pi$ and $j \in \mathbb{Z}$, $f_{(j)} \in \ker\pi$; and every element of $\ker\pi$ has all its coefficients of degree one in $\mathfrak{m}_{\mathcal{O}}$;*

(Tp3) *every $y \in R$ of torus weight j lies in $\pi(\mathcal{O}[[X_\sigma]]_j)$.*

Proof. Write $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, $t^* = \mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_{\mathcal{O}} R)$ and $\bar y$ for the image in t^* of $y \in \mathfrak{m}_R$. For $i \in \mathbb{Z}$ and $k \geq 0$ let $\binom{i}{k} = i(i-1)\cdots(i-k+1)/k! \in \mathbb{Z}$, so that $(1+T)^i = \sum_{k\geq 0} \binom{i}{k} T^k$ in $\mathbb{Z}[[T]]$.

Step 1: the $\bar x_s$ span t^*. Otherwise there is a non-zero linear form $\lambda\colon t^* \to \mathbb{F}$ with $\lambda(\bar x_s) = 0$ for all s. Let $f\colon R \to \mathbb{F}[\epsilon]$ be the residue map and put $f'(y) = f(y) + \epsilon\,\lambda(\overline{y-c})$, where $c \in \mathcal{O}$ is any element with $y - c \in \mathfrak{m}_R$ (it exists as $R/\mathfrak{m}_R = \mathbb{F}$; another choice changes $y-c$ by an element of $\mathfrak{m}_{\mathcal{O}} R$, which maps to 0 in t^*). For $y, z \in R$ with $y-c, z-d \in \mathfrak{m}_R$ one has $yz - cd = c(z-d) + d(y-c) + (y-c)(z-d)$, so $\lambda(\overline{yz-cd}) = \bar c\lambda(\overline{z-d}) + \bar d\lambda(\overline{y-c})$, and f' is an $\mathcal{O}$-algebra homomorphism, local, hence a morphism of $\mathrm{CNL}_{\mathcal{O}}$. Then $f(x_s) = 0 = f'(x_s)$ for all s and $f \neq f'$, contradicting (Tc1). As $\mathfrak{m}_R$ is finitely generated, t^* is finite-dimensional; choose a finite $\sigma \subseteq S$ such that $(\bar x_s)_{s\in\sigma}$ is a basis.

Step 2: (Tp1). Put $P = \mathcal{O}[[X_s : s \in \sigma]]$ with maximal ideal $\mathfrak{m}_P = (\mathfrak{m}_{\mathcal{O}}, X_s)$. As $x_s \in \mathfrak{m}_R$ and R is complete, π exists and is continuous. By Step 1, $\mathfrak{m}_R = \pi(\mathfrak{m}_P)R + \mathfrak{m}_R^2$, so $\mathfrak{m}_R = \pi(\mathfrak{m}_P)R$ by Nakayama's lemma. Regard R as a P-module through π: it is separated for the $\mathfrak{m}_P R = \mathfrak{m}_R$-adic topology (Krull), P is $\mathfrak{m}_P$-adically complete, and $R/\mathfrak{m}_P R = \mathbb{F}$ is generated by 1; by the complete Nakayama lemma [166, Thm. 8.4], $R = P \cdot 1$, i.e. π is surjective. With $\pi(X_s) \in \mathfrak{m}_R$ and $\mathrm{wt}(\sigma) \subseteq \{-2,\dots,2\}$, $(\sigma, \mathrm{wt}, \pi)$ is a weighted presentation.

Step 3: degree-one coefficients of the kernel. Let $J = \ker\pi$ and $f = c + \sum_s a_s X_s + f_2 \in J$ with $c, a_s \in \mathcal{O}$ and $f_2 \in (X_s)^2$. Then $c = -\pi(\sum_s a_s X_s + f_2) \in \mathfrak{m}_R$, so $c \in \mathfrak{m}_{\mathcal{O}}$ ($\mathcal{O} \to R$ is local), and in t^* we get $\sum_s \bar a_s \bar x_s = 0$; by the basis property every $a_s \in \mathfrak{m}_{\mathcal{O}}$.

Step 4: the divided Euler operators. For $n \in \mathbb{N}^\sigma$ write $X^n = \prod_s X_s^{n_s}$, $|n| = \sum_s n_s$ and $\mathrm{wt}(n) = \sum_s n_s \mathrm{wt}(s)$; note $|\mathrm{wt}(n)| \leq 2|n|$. For $k \geq 0$ let $E_k\colon P \to P$ be the $\mathcal{O}$-linear map

$\sum_n c_n X^n \mapsto \sum_n \binom{\mathrm{wt}(n)}{k} c_n X^n$. Let $\rho_P\colon P \to P[[T]]$ be the continuous $\mathcal{O}$-algebra homomorphism with $\rho_P(X_s) = (1+T)^{\mathrm{wt}(s)} X_s$; then $\rho_P(X^n) = (1+T)^{\mathrm{wt}(n)} X^n$ and, comparing coefficients,

$$\rho_P(f) = \sum_{k \ge 0} E_k(f)\, T^k \qquad (f \in P). \tag{43.1}$$

Let $\pi_T\colon P[[T]] \to R[[T]]$ apply π to the coefficients. The maps $\rho \circ \pi$ and $\pi_T \circ \rho_P$ are continuous $\mathcal{O}$-algebra homomorphisms $P \to R[[T]]$ (morphisms of $\mathrm{CNL}_{\mathcal{O}}$ are continuous) which agree on every X_s by (Tc2); since polynomials are dense in P, they are equal. Hence, by (43.1), $\rho(\pi(f)) = \sum_k \pi(E_k f) T^k$ for all $f \in P$. Consequently $E_k(J) \subseteq J$ for all k; and if $y = \pi(f)$ has torus weight j, then comparing with $(1+T)^j y$ gives $\pi(E_k f) = \binom{j}{k} y$ for all k.

Step 5: projectors onto one weight, up to high degree. Fix $j \in \mathbb{Z}$ and an integer $M \ge |j|$, and let

$$L(x) = \prod_{\substack{-2M \le i \le 2M \\ i \ne j}} \frac{x-i}{j-i} = \binom{x+2M}{j+2M}\binom{2M-x}{2M-j}$$

(the two factors are the products over $i < j$ and over $i > j$). Both binomial polynomials take integer values at integers, so L is an integer-valued polynomial of degree $4M$, and therefore $L(x) = \sum_{k=0}^{4M} a_k \binom{x}{k}$ with $a_k \in \mathbb{Z}$ (Newton interpolation: a_k is the k-th finite difference of L at 0). Moreover $L(j) = 1$ and $L(i) = 0$ for $|i| \le 2M$, $i \ne j$. Put $\Pi = \sum_k a_k E_k$. For $f = \sum_n c_n X^n$,

$$\Pi(f) = \sum_n L(\mathrm{wt}(n)) c_n X^n = f_{(j)} + r, \qquad r = \sum_{|\mathrm{wt}(n)| > 2M} L(\mathrm{wt}(n)) c_n X^n,$$

and every monomial of r has $|n| \ge |\mathrm{wt}(n)|/2 > M$, so $r \in (X_s)^{M+1} \subseteq \mathfrak{m}_P^{M+1}$.

Step 6: (Tp2). Let $f \in J$. By Step 4, $\Pi(f) \in J$, so by Step 5 $f_{(j)} \in J + \mathfrak{m}_P^{M+1}$ for every $M \ge |j|$. By Krull's intersection theorem in P/J, $\bigcap_M (J + \mathfrak{m}_P^{M+1}) = J$; so $f_{(j)} \in J$. With Step 3 this is (Tp2).

Step 7: (Tp3). Let $y \in R$ have torus weight j and choose $f \in P$ with $\pi(f) = y$ (Step 2). By Step 4, $\pi(\Pi f) = \sum_k a_k \binom{j}{k} y = L(j) y = y$, so by Step 5 $y - \pi(f_{(j)}) = \pi(r) \in \mathfrak{m}_R^{M+1}$ for every $M \ge |j|$. Hence $y = \pi(f_{(j)})$ with $f_{(j)} \in \mathcal{O}[[X_\sigma]]_j$. □

Proposition 43.9. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, let (B, r_B) and i_B be as in (Bx1), and let $(\sigma_\ell, \mathrm{wt}, \pi_\ell)$ be a weighted presentation of R_ℓ^{ord} as in (Ol3). Then there is a finite set τ of triples $t = (g_t, a_t, b_t)$ with $g_t \in G_{\mathbb{Q}}$ such that, with $\mathrm{wt}(t) = \mathrm{wt}(a_t, b_t)$, $P = \mathcal{O}[[X_s : s \in \sigma_\ell \sqcup \tau]]$, $P_\ell = \mathcal{O}[[X_s : s \in \sigma_\ell]] \subseteq P$ and $\pi\colon P \to B$ the continuous $\mathcal{O}$-algebra homomorphism with $\pi(X_s) = i_B(\pi_\ell(X_s))$ for $s \in \sigma_\ell$ and $\pi(X_t) = y_{g_t, a_t, b_t}$ for $t \in \tau$:*

(Rp1) *$(\sigma_\ell \sqcup \tau, \mathrm{wt}, \pi)$ is a weighted presentation of B, $\pi|_{P_\ell} = i_B \circ \pi_\ell$, and $\#\{t \in \tau : \mathrm{wt}(t) = j\} = \dim Z^1_{\mathrm{u}}(\mathfrak{g}_j)$ for every $j \in \mathbb{Z}$;*

(Rp2) *$f_{(j)} \in \ker \pi$ for every $f \in \ker \pi$ and $j \in \mathbb{Z}$; every element of B of torus weight j lies in $\pi(\mathcal{O}[[X]]_j)$, where $\mathcal{O}[[X]]_j$ is the set of elements of P of weight j; in particular $\underline{r_B}(g)_{ab} \in \pi(\mathcal{O}[[X]]_{\mathrm{wt}(a,b)})$ for all $g \in G_{\mathbb{Q}}$ and a, b;*

(Rp3) *there is a family $f = (f_{j,i})$, $-2 \le j \le 2$, $1 \le i \le \mathrm{nf}(j)$, with $f_{j,i} \in \ker \pi \cap \mathcal{O}[[X]]_j$, such that $\ker \pi = (\ker \pi_\ell)P + (f_{j,i})_{j,i}$ and $\mathrm{nf}(j) \le \dim H^2_{\mathrm{u}}(\mathfrak{g}_j)$ for every j.*

Proof. Write $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, $\mathfrak{m}_\ell = \mathfrak{m}_{R_\ell^{\mathrm{ord}}}$, $\bar{y}$ for residues, and for $X \in \mathrm{M}_3(A)$ with entries in an ideal of square zero killed by $\mathfrak{m}_A$, $g \cdot X$ for the action of g through its residue (so $(P,\varsigma)(1+X,1)(P,\varsigma)^{-1} = (1+g\cdot X,1)$ for (P,ς) reducing to $\bar{r}_0(g)$). For a ring $C = \bigoplus_j C_j$ graded by finitely many weights, a matrix is *graded* if its entry (a,b) lies in $C_{\mathrm{wt}(a,b)}$; graded matrices form the degree-zero part of $\mathrm{M}_3(C)$ for $\deg E_{ab} = \mathrm{wt}(a,b)$, a subring closed under inverses, and θ preserves them (the entry (a,b) of $J\,{}^tMJ$ is $M_{4-b,4-a}$, of the same weight); an element of $\mathcal{G}_3(C)$ is graded if its GL-part is. (F)

Step 1: the relative tangent space. Let $t^*_{\mathrm{rel}} = \mathfrak{m}_B/(\mathfrak{m}_B^2 + \mathfrak{m}_{\mathcal{O}}B + i_B(\mathfrak{m}_\ell)B)$, finite-dimensional. An $\mathbb{F}$-linear form λ on t^*_{rel} defines $f_\lambda(y) = \bar{y} + \epsilon\lambda(\overline{y-c})$ ($c \in \mathcal{O}$ with $y - c \in \mathfrak{m}_B$; another choice changes $y-c$ by an element of $\mathfrak{m}_{\mathcal{O}}B$), an $\mathcal{O}$-algebra homomorphism $B \to \mathbb{F}[\epsilon]$ because $yz - cd = c(z-d) + d(y-c) + (y-c)(z-d)$, local, and killing $i_B(\mathfrak{m}_\ell)$. Conversely a morphism $f\colon B \to \mathbb{F}[\epsilon]$ is $\bar{y} + \epsilon\lambda'(y)$ with λ' vanishing on $\mathfrak{m}_B^2$ and $\mathfrak{m}_{\mathcal{O}}B$; if it kills $i_B(\mathfrak{m}_\ell)$, then $f = f_\lambda$ for the form λ induced by λ'. So the f_λ are exactly the morphisms $B \to \mathbb{F}[\epsilon]$ killing $i_B(\mathfrak{m}_\ell)$. By (Bx1) they correspond to the lifts r' to $\mathbb{F}[\epsilon]$ satisfying (Ed2), (Ed3) with $r'|_{D_\ell}$ classified by the residue map of R_ℓ^{ord}, i.e. $r'|_{D_\ell} = \bar{r}_0|_{D_\ell}$. Write $r'(g) = (1 + \epsilon z(g), 1)\bar{r}_0(g)$: r' is a homomorphism if and only if $z \in Z^1(G_{\mathbb{Q}}, \mathrm{M}_3(\mathbb{F}))$; (Ed2) holds if and only if z vanishes on every $N_{\bar{v}}$, $\bar{v} \nmid \ell$ (for $\bar{v}$ above $p \notin S$, $N_{\bar{v}} = I_{\bar{v}}$, so z factors through G_S; for $p \in S$, $N_{g\bar{p}} = gN_{\bar{p}}g^{-1}$ and $z(gng^{-1}) = g \cdot z(n)$ for n acting trivially, so vanishing on $N_{\bar{p}}$ suffices); (Ed3) holds as $\bar{r}_0|_{D_\ell}$ is classified by the residue of β_ℓ, which kills I_ℓ^{ord} (Ol1); and $r'|_{D_\ell} = \bar{r}_0|_{D_\ell}$ means $\mathrm{res}_\ell z = 0$. Decomposing $z = \sum_j z_j$ along $\mathrm{M}_3(\mathbb{F}) = \bigoplus_j \mathfrak{g}_j$ we get a linear isomorphism $\bigoplus_j Z^1_{\mathrm{u}}(\mathfrak{g}_j) \cong (t^*_{\mathrm{rel}})^\vee$, $z \mapsto \lambda_z$, with

$$\lambda_z(\bar{y}_{g,a,b}) = \big(z(g)\,\mathrm{T}(\bar{\Psi}(g))\big)_{ab},$$

which vanishes for $z \in Z^1_{\mathrm{u}}(\mathfrak{g}_i)$ unless $\mathrm{wt}(a,b) = i$. The $\bar{y}_{g,a,b}$ span t^*_{rel}: otherwise a non-zero λ kills them, and f_λ and f_0 are distinct morphisms agreeing on all $y_{g,a,b}$, against (Tc1) in (Bx2). Let V_j be the span of the $\bar{y}_{g,a,b}$ with $\mathrm{wt}(a,b) = j$. If $z \in Z^1_{\mathrm{u}}(\mathfrak{g}_j)$ and λ_z kills V_j, it kills every V_i, so $\lambda_z = 0$ and $z = 0$; and if $v \in V_j \cap \sum_{i\neq j} V_i$, every λ_z kills v, so $v = 0$. Hence $t^*_{\mathrm{rel}} = \bigoplus_j V_j$, and every form on V_j, extended by 0, is λ_z with $z = \sum_i z_i$ and $\lambda_{z_i}|_{V_i} = 0$ for $i \neq j$, i.e. $z \in Z^1_{\mathrm{u}}(\mathfrak{g}_j)$. So $V_j^\vee \cong Z^1_{\mathrm{u}}(\mathfrak{g}_j)$. Choose τ as the disjoint union over j of sets of triples of weight j whose $\bar{y}$ form a basis of V_j; this gives the count in (Rp1).

Step 2: (Rp1), (Rp2). By (Ol3) the x_s with $\mathfrak{m}_{\mathcal{O}}$ generate $\mathfrak{m}_\ell$ (π_ℓ is onto), so the images of $y_s = i_B(x_s)$ ($s \in \sigma_\ell$) and $\mathfrak{m}_{\mathcal{O}}$ span the image of $i_B(\mathfrak{m}_\ell)B$ in $\mathfrak{m}_B/\mathfrak{m}_B^2$, and with Step 1 the $\pi(X_s)$, $s \in \sigma_\ell \sqcup \tau$, and $\mathfrak{m}_{\mathcal{O}}$ span $\mathfrak{m}_B/\mathfrak{m}_B^2$. By Nakayama's lemma and the complete Nakayama lemma [166, Thm. 8.4] π is onto. Weights lie in $\{-2,\dots,2\}$, $\pi(X_s) \in \mathfrak{m}_B$, and $\pi|_{P_\ell} = i_B \circ \pi_\ell$ since both are continuous and agree on the X_s: (Rp1). Each $\pi(X_s)$ has torus weight $\mathrm{wt}(s)$ for ρ_B ((Bx2): i_B preserves torus weights and x_s has weight $\mathrm{wt}(s)$; y_t has weight $\mathrm{wt}(t)$), so (Rp2) is Lemma 43.10, items 1 and 2, with (Bx2) for the entries.

Step 3: relative minimality. Let $f \in \ker\pi$ and let $a_t(f) \in \mathcal{O}$ be the coefficient of X_t ($t \in \tau$). Then $a_t(f) \in \mathfrak{m}_{\mathcal{O}}$: write $f = c_0 + \sum_{s\in\sigma_\ell} a_s X_s + \sum_t a_t(f)X_t + f_2$ with $f_2 \in (X)^2$; $\pi(f) = 0$ gives $c_0 \in \mathfrak{m}_{\mathcal{O}}$ ($\mathcal{O} \to B$ is local), and in t^*_{rel}, where the y_s ($s \in \sigma_\ell$), c_0 and $\pi(f_2)$ vanish, $\sum_t \bar{a}_t(f)\bar{y}_t = 0$, so $\bar{a}_t(f) = 0$ by the basis property.

Step 4: the graded space of relations. Let $\mathcal{J} = \ker\pi$, $\mathcal{J}_\ell = \ker\pi_\ell$ and $\mathfrak{m} = \mathfrak{m}_P$. By Lemma 43.10, item 3, applied to π_ℓ (the x_s have torus weight $\mathrm{wt}(s)$ for ρ_ℓ), $\mathcal{J}_\ell$ is generated by finitely many elements of single weights, so $\mathcal{J}_\ell P$ is homogeneous (contains the weight components of its elements: $(gh)_{(m)} = g_{(m-k)}h$ for h of weight k); so is $\mathfrak{m}\mathcal{J}$ ($\mathfrak{m}$ is generated by ϖ and the X_s), and $\mathcal{J}_\ell P \subseteq \mathcal{J}$. By Artin–Rees choose $N_0 \geq 1$ with $\mathcal{J} \cap \mathfrak{m}^{N_0} \subseteq \mathfrak{m}\mathcal{J}$. Put $V = \mathcal{J}/(\mathcal{J}_\ell P + \mathfrak{m}\mathcal{J})$ and V_j^{rel} the image of $\mathcal{J} \cap \mathcal{O}[[X]]_j$. For $f \in \mathcal{J}$, $f - \sum_{|i|\leq 2N_0} f_{(i)}$ lies in $\mathcal{J} \cap \mathfrak{m}^{N_0+1} \subseteq \mathfrak{m}\mathcal{J}$ (its monomials have degree $> N_0$) with $f_{(i)} \in \mathcal{J}$ (Rp2), so the V_j^{rel} span V; and a relation $\sum_i f_i \in \mathcal{J}_\ell P + \mathfrak{m}\mathcal{J}$ with $f_i \in \mathcal{J} \cap \mathcal{O}[[X]]_i$ forces each f_i, a weight component of an element of

this homogeneous ideal, to lie in it. So $V = \bigoplus_j V_j^{\mathrm{rel}}$. Choose $f_{j,1}, \dots, f_{j,\mathrm{nf}(j)} \in \mathcal{J} \cap \mathcal{O}[[X]]_j$ lifting a basis of V_j^{rel}; by Nakayama's lemma applied to $\mathcal{J}/(\mathcal{J}_\ell P + (f))$, $\mathcal{J} = \mathcal{J}_\ell P + (f_{j,i})$. It remains to prove $\mathrm{nf}(j) = \dim V_j^{\mathrm{rel}} \le \dim H^2_{\mathrm{u}}(\mathfrak{g}_j)$; for $|j| > 2$ this forces $V_j^{\mathrm{rel}} = 0$ ($\mathfrak{g}_j = 0$).

Step 5: the graded small extension of weight j. Fix j and let $\mathcal{W} \subseteq \mathcal{J}$ be the preimage of $\bigoplus_{i \ne j} V_i^{\mathrm{rel}}$, a homogeneous ideal containing $\mathcal{J}_\ell P + \mathfrak{m}\mathcal{J}$ (for $f \in \mathcal{W}$, $f_{(j)}$ maps to the j-component of $\bar f$, which is 0). Put $C = P/(\mathcal{W} + \mathfrak{m}^{N_0})$ and $\bar C = P/(\mathcal{J} + \mathfrak{m}^{N_0}) = B/\mathfrak{m}_B^{N_0}$, finite local $\mathcal{O}$-algebras graded by the images of the $\mathcal{O}[[X]]_i$, $|i| \le 2N_0$. The projection $\mathrm{pr}\colon C \to \bar C$ has kernel I, the image of $\mathcal{J} \cap \mathcal{O}[[X]]_j$, so $I \subseteq C_j$, $\mathfrak{m}_C I = 0$ and $I \cong \mathcal{J}/\mathcal{W} \cong V_j^{\mathrm{rel}}$. As $\mathcal{J}_\ell \subseteq \mathcal{W}$, $P_\ell \to C$ induces $i_C\colon R_\ell^{\mathrm{ord}} \to C$, and $\mathrm{pr} \circ i_C$ is i_B followed by $B \to \bar C$. Let $\bar r$ be r_B reduced to $\bar C$, graded by (Rp2), and $\rho_C = i_C \circ r_\ell^{\mathrm{ord}}\colon D_\ell \to \mathcal{G}_3(C)$, a graded continuous homomorphism (Ol3) lifting $\bar r|_{D_\ell}$.

Step 6: graded local lifts away from ℓ. Let $p \in S$, $p \ne \ell$. $N_{\bar p}$ is normal in D_p, $I_p/N_{\bar p}$ embeds in the image of $\bar r_0$, of order prime to ℓ ($\bar\Psi$ has values in $\mathbb{F}^\times$, signs in $\{\pm 1\}$, ℓ odd), and $D_p/I_p \cong \widehat{\mathbb{Z}}$; by the Hochschild–Serre spectral sequence $H^2(D_p/N_{\bar p}, M) = 0$ for every finite ℓ-primary discrete module M. $\bar r|_{D_p}$ is trivial on $N_{\bar p}$ (Ed2); a graded set-theoretic lift of it through $D_p/N_{\bar p}$ (lift entries weight by weight) has obstruction 2-cocycle in $\mathrm{M}_3(I)$, graded, hence with values in $\mathfrak{g}_j \otimes I$; it is a coboundary $d\omega$, and correcting the lift by $(1 - \omega)$ gives a graded continuous homomorphism $\rho_p\colon D_p \to \mathcal{G}_3(C)$ lifting $\bar r|_{D_p}$ and trivial on $N_{\bar p}$.

Step 7: the obstruction class. $\bar r$ factors through a finite quotient of G_S; choose a continuous graded set-theoretic lift $s\colon G_S \to \mathcal{G}_3(C)$ of $\bar r$ with $s(1) = 1$. Then $s(g)s(g')s(gg')^{-1} = (1 + k(g,g'), 1)$ with k graded with entries in I, hence $k \in Z^2(G_S, \mathfrak{g}_j \otimes I)$. For each prime $p \in S$ write $s(g) = (1 + y_p(g), 1)\rho_p(g)$ on D_p (with $\rho_\ell = \rho_C$), $y_p \in C^1(D_p, \mathfrak{g}_j \otimes I)$; expanding $s(g)s(g')$ in two ways gives $dy_p = \mathrm{res}_p k$. For $\phi \in I^\vee$, $(\phi \circ k, (\phi \circ y_p)_p) \in Z^2_{\mathcal{L}}(\mathfrak{g}_j)$ for the local conditions $L^{\mathrm{u}}_{p,j}$, and its class $o(\phi) \in H^2_{\mathrm{u}}(\mathfrak{g}_j)$ is linear in ϕ.

Step 8: a vanishing class gives a lift framed at ℓ. Suppose $\dim V_j^{\mathrm{rel}} > \dim H^2_{\mathrm{u}}(\mathfrak{g}_j)$. As $\dim I^\vee = \dim V_j^{\mathrm{rel}}$, some $\phi \ne 0$ has $o(\phi) = 0$: $\phi \circ k = dw$ and $\phi \circ y_p = \mathrm{res}_p w + \lambda_p$, with $w \in C^1(G_S, \mathfrak{g}_j)$, $\lambda_p \in L^{\mathrm{u}}_{p,j}$, in particular $\lambda_\ell = dX$ for some $X \in \mathfrak{g}_j$. Let $I' = \ker\phi$ (a homogeneous ideal), $C' = C/I'$, identify I/I' with $\mathbb{F}$ through ϕ, and regard w, X as matrices with entries in I/I'. Over C', $s'(g) = (1 - w(g), 1)s(g)$ satisfies $s'(g)s'(g')s'(gg')^{-1} = 1 + \phi k(g,g') - dw(g,g') = 1$ (all cross terms lie in $\mathrm{M}_3(I/I')^2 = 0$), so s' is a continuous graded homomorphism lifting $\bar r$. On D_p, $p \ne \ell$, $s' = (1 + \lambda_p, 1)\rho_p$ is trivial on $N_{\bar p}$, hence on all $N_{g\bar p} = gN_{\bar p}g^{-1}$; for $\bar v$ above $p \notin S$, s' factors through G_S; so s' satisfies (Ed2). On D_ℓ, $s'(g) = (1 + g\cdot X - X, 1)\rho_C(g) = (h, 1)\rho_C(g)(h, 1)^{-1}$ with $h = 1 - X \equiv 1$. Put $s'' = (h,1)^{-1}s'(h,1)$: a graded continuous homomorphism lifting $\bar r$, satisfying (Ed2), with $s''|_{D_\ell} = i_{C'} \circ r_\ell^{\mathrm{ord}}$, where $i_{C'}$ is i_C followed by $C \to C'$; so $s''|_{D_\ell}$ is classified by a morphism killing I_ℓ^{ord}, and s'' satisfies (Ed3).

Step 9: contradiction. By (Bx1), $s'' = F \circ r_B$ for a morphism $F\colon B \to C'$; $F \circ i_B = i_{C'}$, both carrying r_ℓ^{ord} to $s''|_{D_\ell}$ (universality of $R_\ell^\square$ and surjectivity of $R_\ell^\square \to R_\ell^{\mathrm{ord}}$); and F followed by $C' \to \bar C$ is the projection $B \to \bar C$, both classifying $\bar r$ ($h \equiv 1$ and $s' \equiv s$ modulo I/I'). Let $\alpha\colon P \to C'$ be the projection and $\beta = F \circ \pi$. Then $\alpha \equiv \beta$ modulo I/I', which $\mathfrak{m}_{C'}$ kills, so $\delta = \alpha - \beta$ satisfies $\delta(xy) = \delta(x)\bar y + \bar x\delta(y)$; and $\alpha|_{P_\ell} = i_{C'} \circ \pi_\ell = F \circ i_B \circ \pi_\ell = \beta|_{P_\ell}$. For $f \in P$ write $f = f_0 + \sum_t X_t g_t$ with $f_0 \in P_\ell$ (assign each term containing some X_t to the first such t); the constant term of g_t is $a_t(f)$, and $\delta(f) = \sum_t \delta(X_t)\,\bar a_t(f)$ since $\delta(P_\ell) = 0$ and $\bar X_t = 0$. By Step 3, $\delta(\mathcal{J}) = 0$; and $\beta(\mathcal{J}) = 0$. Hence $\alpha(\mathcal{J}) = 0$; but $\alpha(\mathcal{J})$ is the image $I/I' \ne 0$ of $\mathcal{J}$ in C'. So $\dim V_j^{\mathrm{rel}} \le \dim H^2_{\mathrm{u}}(\mathfrak{g}_j)$, which completes (Rp3). □

Lemma 43.10. *Let $\mathcal{O}$ be a coefficient ring, $R \in \mathrm{CNL}_{\mathcal{O}}$ with maximal ideal $\mathfrak{m}_R$, and $\rho\colon R \to R[[T]]$ a morphism of $\mathrm{CNL}_{\mathcal{O}}$; an element $y \in R$ has* torus weight $j \in \mathbb{Z}$ *if $\rho(y) = (1 + T)^j y$. Let σ be a finite set,* $\mathrm{wt}\colon \sigma \to \mathbb{Z}$ *a map, and $\pi\colon \mathcal{O}[[X_s : s \in \sigma]] \to R$ a surjective continuous $\mathcal{O}$-algebra homomorphism such that each $\pi(X_s)$ lies in $\mathfrak{m}_R$ and has torus weight $\mathrm{wt}(s)$. Then:*

1. *for every* $f \in \ker\pi$ *and* $j \in \mathbb{Z}$, $f_{(j)} \in \ker\pi$*;*

2. *every* $y \in R$ *of torus weight* j *lies in* $\pi(\mathcal{O}[[X_\sigma]]_j)$*;*

3. $\ker\pi$ *is generated by finitely many elements each lying in some* $\mathcal{O}[[X_\sigma]]_j$*.*

Proof. Write $P = \mathcal{O}[[X_s : s \in \sigma]]$, $\mathfrak{m}_P = (\mathfrak{m}_\mathcal{O}, X_s)$, $J = \ker\pi$, and for $n \in \mathbb{N}^\sigma$, $X^n = \prod_s X_s^{n_s}$, $|n| = \sum_s n_s$, $\mathrm{wt}(n) = \sum_s n_s \mathrm{wt}(s)$. Let $W = \max(1, \max_s |\mathrm{wt}(s)|)$, so $|\mathrm{wt}(n)| \le W|n|$. For $i \in \mathbb{Z}$ and $k \ge 0$, $\binom{i}{k} = i(i-1)\cdots(i-k+1)/k! \in \mathbb{Z}$, and $(1+T)^i = \sum_k \binom{i}{k} T^k$ in $\mathbb{Z}[[T]]$.

Step 1: divided Euler operators. Let $E_k \colon P \to P$ be the $\mathcal{O}$-linear map

$$\sum_n c_n X^n \mapsto \sum_n \binom{\mathrm{wt}(n)}{k} c_n X^n,$$

and $\rho_P \colon P \to P[[T]]$ the continuous $\mathcal{O}$-algebra homomorphism with $\rho_P(X_s) = (1+T)^{\mathrm{wt}(s)} X_s$. Then $\rho_P(X^n) = (1+T)^{\mathrm{wt}(n)} X^n$, so $\rho_P(f) = \sum_k E_k(f) T^k$. Let $\pi_T \colon P[[T]] \to R[[T]]$ apply π to coefficients. The continuous $\mathcal{O}$-algebra homomorphisms $\rho \circ \pi$ and $\pi_T \circ \rho_P$ (morphisms of $\mathrm{CNL}_\mathcal{O}$ are continuous) agree on the X_s, by the hypothesis on torus weights, hence on the dense subring of polynomials, hence on P:

$$\rho(\pi(f)) = \sum_{k \ge 0} \pi(E_k f)\, T^k \qquad (f \in P).$$

So $E_k(J) \subseteq J$ for all k, and if $\pi(f) = y$ has torus weight j, then $\pi(E_k f) = \binom{j}{k} y$ for all k.

Step 2: approximate projectors. Fix j and $M \ge |j|$, and let

$$L(x) = \prod_{\substack{-WM \le i \le WM \\ i \ne j}} \frac{x-i}{j-i} = \binom{x+WM}{j+WM}\binom{WM-x}{WM-j},$$

an integer-valued polynomial (both factors are), so $L(x) = \sum_{k \le 2WM} a_k \binom{x}{k}$ with $a_k \in \mathbb{Z}$ (Newton interpolation), $L(j) = 1$ and $L(i) = 0$ for $|i| \le WM$, $i \ne j$. Put $\Pi = \sum_k a_k E_k$. For $f = \sum_n c_n X^n$, $\Pi(f) = f_{(j)} + r$ with $r = \sum_{|\mathrm{wt}(n)| > WM} L(\mathrm{wt}(n)) c_n X^n$, and each monomial of r has $|n| \ge |\mathrm{wt}(n)|/W > M$, so $r \in \mathfrak{m}_P^{M+1}$.

Step 3: items 1 and 2. If $f \in J$, then $\Pi(f) \in J$ (Step 1), so $f_{(j)} \in J + \mathfrak{m}_P^{M+1}$ for every $M \ge |j|$; by Krull's intersection theorem in P/J, $f_{(j)} \in J$. If y has torus weight j and $\pi(f) = y$, then $\pi(\Pi f) = \sum_k a_k \binom{j}{k} y = L(j) y = y$, so $y - \pi(f_{(j)}) = \pi(r) \in \mathfrak{m}_R^{M+1}$ for every M, and $y = \pi(f_{(j)})$ by Krull's intersection theorem in R.

Step 4: item 3. J is generated by the set of all $f_{(j)}$, $f \in J$, $j \in \mathbb{Z}$: these lie in J by item 1, and every $f \in J$ is the $\mathfrak{m}_P$-adic limit of the finite sums $\sum_{|j| \le WM} f_{(j)}$ (the remaining terms lie in $\mathfrak{m}_P^{M+1}$), while the ideal generated by any finitely many elements of J is closed. As P is Noetherian, finitely many of the $f_{(j)}$ generate J. □

Definition 43.11. Let $\mathcal{O}$ be a coefficient ring, $R \in \mathrm{CNL}_\mathcal{O}$, $(\sigma, \mathrm{wt}, \pi)$ a weighted presentation of R, $\beta \colon R \to \mathcal{O}$ an $\mathcal{O}$-algebra homomorphism with $\beta(\pi(X_s)) = 0$ for every $s \in \sigma$, and $\mathfrak{a} \subseteq \ker\pi$ an ideal of $\mathcal{O}[[X_s : s \in \sigma]]$ generated by finitely many elements each of which lies in some $\mathcal{O}[[X_\sigma]]_j$.

A **homogeneous generating system of** $\ker\pi$ **modulo** $\mathfrak{a}$ with counts $\mathrm{ne} \colon \{-2, \dots, 2\} \to \mathbb{N}$ is a family $u = (u_{j,i})$, $-2 \le j \le 2$, $1 \le i \le \mathrm{ne}(j)$, with $u_{j,i} \in \ker\pi \cap \mathcal{O}[[X_\sigma]]_j$ and $\ker\pi = \mathfrak{a} + (u_{j,i})_{j,i}$.

The **relative graded model** of $(\sigma, \mathrm{wt}, \pi)$ modulo $\mathfrak{a}$ with equations u is the tuple $(A, \varphi_0, \bar{u}, q)$, where $A = A_\pi/(\mathfrak{a} \cap A_\pi)$ with the subgroups A_j, the images of $(A_\pi)_j$; $q \colon A \to R$ is induced by q_π (which kills $\mathfrak{a} \cap A_\pi \subseteq \ker\pi$); $\varphi_0 = \beta \circ q$; and $\bar{u}_{j,i}$ is the image of $u_{j,i}$ in A_j.

Proposition 43.12. *Let (K,ℓ,e,N,χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, $g_\star$ a marking, (B,r_B) and (R,r) as in (Bx1), (Bx3), φ_{Eis} the base point of r, $p\colon B\to R$ the projection, $(\sigma_\ell,\mathrm{wt},\pi_\ell)$ as in (Ol3), and τ, P, π, f as in Proposition 43.9. Put $\sigma=\sigma_\ell\sqcup\tau$, $\pi_R=p\circ\pi$, $\mathfrak{a}=(\ker\pi_\ell)P$, $(\mathrm{nw}(k))_{k=1}^8=(0,0,1,-1,2,-2,1,-1)$, and choose $\tilde n_k\in\mathcal{O}[[X_\sigma]]_{\mathrm{nw}(k)}$ with $\pi(\tilde n_k)=n_k$. Let u be the family consisting of the $f_{j,i}$ and of the $\tilde n_k$ (placed in degree $\mathrm{nw}(k)$), with counts $\mathrm{ne}(j)=\mathrm{nf}(j)+\mathrm{nc}(j)$, $(\mathrm{nc}(j))_{j=-2}^2=(1,2,2,2,1)$, and $\mathrm{td}(j)=\mathrm{td}_\ell(j)+\#\{t\in\tau:\mathrm{wt}(t)=j\}$. Then the $\tilde n_k$ exist; $(\sigma,\mathrm{wt},\pi_R)$ is a weighted presentation of R with $\varphi_{\mathrm{Eis}}(\pi_R(X_s))=0$ for all s; $\mathfrak{a}\subseteq\ker\pi_R$ is generated by finitely many elements of single weights; u is a homogeneous generating system of $\ker\pi_R$ modulo $\mathfrak{a}$; and the relative graded model $(A,\varphi_0,\bar u,q)$ of $(\sigma,\mathrm{wt},\pi_R)$ modulo $\mathfrak{a}$ with equations u (for $\beta=\varphi_{\mathrm{Eis}}$) satisfies:*

(Mp1) *(A,φ_0) is a graded $\mathcal{O}$-algebra with a fixed point, $A=\bigoplus_j A_j$, $\bar u$ is a system of equations for it with counts* ne, *$\ker q=I(\bar u)$, $\varphi_{\mathrm{Eis}}\circ q=\varphi_0$, and $\mathrm{td}(j)=0$ for $|j|>2$;*

(Mp2) *(Pq3), (Pq4), (Pq5), (Pq6) and (Pq8) hold for (A,φ_0), $\bar u$, q and* td.

Proof. Write $\mathbb{F}=\mathbb{F}_{\mathcal{O}}$, $E_{\mathcal{O}}=\mathcal{O}[\ell^{-1}]$, $P_j=\mathcal{O}[[X_\sigma]]_j$, $a_s(h)$ for the coefficient of X_s in $h\in P$, and $E_j\subseteq\mathbb{N}^\sigma$ for the exponents of weight j.

Step 0: the presentation. The $\tilde n_k$ exist by (Rp2), as n_k has torus weight $\mathrm{nw}(k)$ (Bx3). π_R is onto with $\pi_R(X_s)\in\mathfrak{m}_R$, weights in $\{-2,\dots,2\}$; $\pi_R(X_s)$ is the image of y_{g_s,a_s,b_s}, so $\varphi_{\mathrm{Eis}}(\pi_R(X_s))=0$ as $\varphi_{\mathrm{Eis}}\circ r=r_0$, and $\varphi_{\mathrm{Eis}}(\pi_R(h))$ is the constant term of h ($h-h(0)=\sum_s X_s g_s$). By Lemma 43.10, item 3, applied to π_ℓ and ρ_ℓ (Ol3), $\mathfrak{a}$ is generated by elements $v_1,\dots,v_m\in P_\ell$ of weights ω_i, and $\mathfrak{a}\subseteq\ker\pi\subseteq\ker\pi_R$ ($\pi|_{P_\ell}=i_B\circ\pi_\ell$). As $\ker p=\mathfrak{n}=\pi((\tilde n_k)_k)B$ and $\ker\pi=\mathfrak{a}+(f)$ by (Rp3), $\ker\pi_R=\mathfrak{a}+(f)+(\tilde n)$, so u is a homogeneous generating system modulo $\mathfrak{a}$. For $g\in P_k$ and $h\in P$, $(hg)_{(m)}=h_{(m-k)}g$; hence for an ideal $\mathfrak{b}$ generated by elements $w_1,\dots,w_r$ of weights κ_i, $h\in\mathfrak{b}$ implies $h_{(j)}=\sum_i(c_i)_{(j-\kappa_i)}w_i$ for $h=\sum_i c_iw_i$, so $h_{(j)}\in\mathfrak{b}\cap P_j=\sum_i P_{j-\kappa_i}w_i$. (C)

Step 1: (Mp1). By Lemma 43.28, item 1, for the weighted presentation $(\sigma,\mathrm{wt},\mathrm{id}_P)$ of P, the constant term, and the empty system (the graded model A_π depends only on (σ,wt)), $A_\pi=\bigoplus_j P_j\subseteq P$ and $(A_\pi,\text{constant term})$ is a graded $\mathcal{O}$-algebra with a fixed point. By (C) for $\mathfrak{b}=\mathfrak{a}$, $\mathfrak{a}\cap A_\pi=\bigoplus_j(\mathfrak{a}\cap P_j)$; so $A=\bigoplus_j A_j$ with $A_j=P_j/(\mathfrak{a}\cap P_j)$, $A_iA_k\subseteq A_{i+k}$. Elements of $\mathfrak{a}$ have constant term $\varphi_{\mathrm{Eis}}(\pi_R(h))=0$, so $A_0=P_0/(\mathfrak{a}\cap P_0)$ is a complete Noetherian local ring with residue field $\mathbb{F}$, A is generated by A_0 and the images of finitely many homogeneous elements of degrees in $\{-2,-1,1,2\}$, and $\varphi_0=\varphi_{\mathrm{Eis}}\circ q$ is induced by the constant term, hence kills A_j for $j\neq0$: (A,φ_0) is a graded $\mathcal{O}$-algebra with a fixed point, and $\bar u$ a system of equations ($\varphi_0(\bar u_{0,i})=\varphi_{\mathrm{Eis}}(\pi_R(u_{0,i}))=0$). Moreover, $I(\bar u)\subseteq\ker q$. If $h=\sum_k h_k\in A_\pi$ ($h_k\in P_k$) has $\pi_R(h)=0$, then $h_k=h_{(k)}\in\ker\pi_R\cap P_k$, which by (C) for the generators v, f, $\tilde n$ of $\ker\pi_R$ is a sum of multiples of them by elements of A_π; its image in A lies in $I(\bar u)$. So $\ker q=I(\bar u)$. $\mathrm{td}(j)=0$ for $|j|>2$ since td_ℓ and wt are supported in $[-2,2]$.

Step 2: (Pq3). By (Rp2), $\underline{r_B}(g)_{ab}\in\pi(P_{\mathrm{wt}(a,b)})$; apply p: $\underline{r}(g)_{ab}\in\pi_R(P_{\mathrm{wt}(a,b)})=q(A_{\mathrm{wt}(a,b)})$.

Step 3: (Pq6). Let $\Lambda=E_{\mathcal{O}}[[Y_i,X_t:i\in\tau_\ell,t\in\tau]]$, with Y_i of weight $w(i)$ and X_t of weight $\mathrm{wt}(t)$. By (Lc2), $\phi_\ell(x_s)$ ($s\in\sigma_\ell$) has no constant term, so there is a continuous $\mathcal{O}$-algebra homomorphism $\tilde\Phi\colon P\to\Lambda$ with $\tilde\Phi(X_s)=\phi_\ell(x_s)$ ($s\in\sigma_\ell$) and $\tilde\Phi(X_t)=X_t$; X^n goes to $(Y,X_\tau)^{|n|}$. On P_ℓ, $\tilde\Phi=\phi_\ell\circ\pi_\ell$: both agree on polynomials, and for $h\in P_\ell$ both send $h-h_{<M}\in(X)^MP_\ell$ into $(Y)^M$ ($\phi_\ell(\pi_\ell(X^ng))=\phi_\ell(x)^n\phi_\ell(\pi_\ell g)$). Hence $\tilde\Phi(\mathfrak{a})=0$ and $\tilde\Phi$ induces $\Phi\colon A\to\Lambda$. By (Lc1), $\phi_\ell(x_s)=\phi_\ell(\underline{r_\ell^{\mathrm{ord}}}(g_s)_{a_sb_s})-\underline{r_0}(g_s)_{a_sb_s}$ has weight $\mathrm{wt}(s)$ (the constant is 0 off the diagonal), so $\tilde\Phi$ maps monomials of weight j, and by continuity all of P_j, to series of weight j, and $\tilde\Phi(h_{(j)})=\tilde\Phi(h)_{(j)}$ for $h\in P$. (Sm1) follows. (Sm2): the constant term of $\tilde\Phi(h)$ is that of h, i.e. $\varphi_0(\bar h)$. (Sm3): $\Phi(\bar X_t)=X_t$; for $i\in\tau_\ell$ take f from (Lc3), $f=\pi_\ell(h)$ with $h\in P_\ell$; then

$h_{(w(i))} \in A_\pi$ and the linear part of $\tilde\Phi(h_{(w(i))}) = \phi_\ell(f)_{(w(i))}$ is the weight-$w(i)$ part of that of $\phi_\ell(f)$, a non-zero multiple of Y_i. (Sm4): let $\bar h \in A$ with $\varphi_0(\bar h) = 0$ and all degree-one coefficients of $\Phi(\bar h)$ zero; as the variables have single weights, every component $h_k \in P_k$ has the same property, so let $h \in P_k$. Write $h = h_\ell + \sum_t X_t g_t$ with $h_\ell \in P_\ell$ (each term containing some X_t assigned to the first such t); taking weight-k components, $h = (h_\ell)_{(k)} + \sum_t X_t (g_t)_{(k-\mathrm{wt}(t))}$. The coefficient of X_t in $\tilde\Phi(h)$ is the constant term of $(g_t)_{(k-\mathrm{wt}(t))}$, so it vanishes and $X_t(g_t)_{(k-\mathrm{wt}(t))}$ maps into $\mathfrak{m}_0^2$, $\mathfrak{m}_0 = \ker\varphi_0$. Let $a = \pi_\ell((h_\ell)_{(k)})$: $\beta_\ell(a) = 0$ and all Y-coefficients of degree one of $\phi_\ell(a)$ vanish, so by (Lc4) $ca \in (\ker\beta_\ell)^2 = \pi_\ell((X_s : s \in \sigma_\ell)^2)$ with $c \neq 0$: $c(h_\ell)_{(k)} - \sum_{s,s'} X_s X_{s'} b_{ss'} \in \ker\pi_\ell$ for some $b_{ss'} \in P_\ell$. Its weight-k component $c(h_\ell)_{(k)} - \sum X_s X_{s'} (b_{ss'})_{(k-\mathrm{wt}(s)-\mathrm{wt}(s'))}$ lies in $\ker\pi_\ell$ (Lemma 43.10, item 1), hence in $\mathfrak{a} \cap A_\pi$; so c times the image of $(h_\ell)_{(k)}$ lies in $\mathfrak{m}_0^2$. Thus $c\bar h \in \mathfrak{m}_0^2$ and $\bar h \in \mathrm{Sq}(\varphi_0)$. So Φ is a chart and (A, φ_0) is smooth with tangent dimensions td. The inequality of (Pq6): $\mathrm{ne}(j)+\mathrm{td}_\ell(j)+\dim Z^1_{\mathrm{u}}(\mathfrak{g}_j) = \mathrm{nf}(j)+\mathrm{nc}(j)+\mathrm{td}(j) \leq \dim H^2_{\mathrm{u}}(\mathfrak{g}_j)+\mathrm{nc}(j)+\mathrm{td}(j)$ by (Rp1), (Rp3).

Step 4: points of A_π are evaluations. Let $z \in \overline{\mathbb{Q}}_\ell^\sigma$ with $|z_s|_\ell < 1$ (in a finite extension) and $\psi\colon A_\pi \to \overline{\mathbb{Q}}_\ell$ an $\mathcal{O}$-algebra homomorphism with $\psi(X_s) = z_s$ and $|\psi| \leq 1$ on P_0. Then $\psi = \mathrm{ev}_z$ on A_π. By Dickson's lemma the minimal elements $m_1, \ldots, m_k$ of $E_0 \setminus \{0\}$ are finitely many and every $n \in E_0$ is a sum of them, so $\theta\colon \mathcal{O}[[Y_1, \ldots, Y_k]] \to P_0$, $Y_i \mapsto X^{m_i}$, is onto (send $\sum_n c_n X^n$ to $\sum_n c_n Y^{e(n)}$ with $\sum_i e(n)_i m_i = n$; $n \mapsto e(n)$ is injective). For $g = g_{<M} + \sum_{|e|=M} Y^e g_e$, $|\psi(\theta g) - \mathrm{ev}_z(\theta g_{<M})| \leq \max_i |z^{m_i}|^M \to 0$, so $\psi = \mathrm{ev}_z$ on P_0. For $j \neq 0$ the minimal elements Min_j of E_j are finitely many and $P_j = \sum_{m \in \mathrm{Min}_j} X^m P_0$; so $\psi = \mathrm{ev}_z$ on P_j.

Step 5: (Pq4). A point φ of (R, r) is continuous ($\mathfrak{m}_R$ is finitely generated and $|\varphi| < 1$ on it), so $\varphi \circ \pi_R = \mathrm{ev}_z$ with $z_s = \varphi(\pi_R(X_s))$, and $\varphi \circ q$ is induced by $\mathrm{ev}_z|_{A_\pi}$. If $\varphi \circ q$ is negative-vanishing, φ kills the entries of negative weight by (Pq3): φ is upper. If φ is upper, the negative-weight entries of $\underline{r_\varphi}(g)$ vanish for $g \in G_K$, and for $g \notin G_K$ too, since $r(c) = (1, -1)$ by (Ed1) gives $\underline{r}(g) = \underline{r}(gc)$; as $\underline{r_0}$ is diagonal, $z_s = 0$ whenever $\mathrm{wt}(s) < 0$, and every monomial of negative weight contains such a variable, so $\mathrm{ev}_z(P_j) = 0$ for $j < 0$. The same holds for positive weights and lower points. If $\varphi' \circ q = t \cdot (\varphi \circ q)$, then by (Pq3) $\varphi'(\underline{r}(g)_{ab}) = t^{\mathrm{wt}(a,b)} \varphi(\underline{r}(g)_{ab})$ for all g, i.e. $\underline{r_{\varphi'}}(g) = \mathrm{T}(t)\underline{r_\varphi}(g)\mathrm{T}(t)^{-1}$, and since $(\mathrm{T}(t), 1)(M, \varsigma)(\mathrm{T}(t), 1)^{-1} = (\mathrm{T}(t)M\mathrm{T}(t)^{-1}, \varsigma)$, $r_{\varphi'} = \mathrm{T}(t) \cdot r_\varphi$. Conversely, this gives $\varphi'(\pi_R(X_s)) = t^{\mathrm{wt}(s)} \varphi(\pi_R(X_s))$, and evaluating a series of weight j at $(t^{\mathrm{wt}(s)} z_s)$ multiplies its value by t^j.

Step 6: (Pq5). Let ψ be a point of A on $\bar u$ and suppose the image $\bar A_0$ of A_0 in $A/I(\bar u)$ is a finitely generated $\mathcal{O}$-module. It is a non-zero quotient of the local ring A_0, hence a finite local $\mathcal{O}$-algebra through which $\psi|_{A_0}$ factors; so $|\psi| \leq 1$ on A_0 and $|\psi| < 1$ on $\mathfrak{m}_{A_0}$ ($\mathfrak{m}_{\bar A_0}^M \subseteq \varpi \bar A_0$ for some M). Put $\alpha_s = |\psi(\bar X_s)|$. If $\mathrm{wt}(s) = 0$, $\bar X_s \in \mathfrak{m}_{A_0}$ and $\alpha_s < 1$; if $\mathrm{wt}(s) = v > 0 > -v' = \mathrm{wt}(s')$, then $X_s^{v'} X_{s'}^{v} \in \mathfrak{m}_{A_0}$ and $\alpha_s^{1/v} < \alpha_{s'}^{-1/v'}$. As $|\overline{\mathbb{Q}}_\ell^\times|$ is dense in $\mathbb{R}_{>0}$, there is t with $\alpha_s^{1/\mathrm{wt}(s)} < |t|$ for $\mathrm{wt}(s) > 0$ and $|t| < \alpha_s^{-1/|\mathrm{wt}(s)|}$ for $\mathrm{wt}(s) < 0$. Then $\psi' = t^{-1} \cdot \psi$ lies on $\bar u$, agrees with ψ on A_0, and $|\psi'(\bar X_s)| < 1$ for all s. By Step 4 the composite of ψ' with $A_\pi \to A$ is $\mathrm{ev}_z|_{A_\pi}$, $z_s = \psi'(\bar X_s)$; it kills v and u, so ev_z kills $\ker\pi_R$, and $\mathrm{ev}_z = \varphi \circ \pi_R$ for an $\mathcal{O}$-algebra homomorphism $\varphi\colon R \to \overline{\mathbb{Q}}_\ell$ with $|\varphi| < 1$ on $\mathfrak{m}_R = \pi_R(\mathfrak{m}_P)$: a point of (R, r), with $\psi = t \cdot (\varphi \circ q)$.

Step 7: (Pq8). Let $b \in \mathcal{O} \setminus \{0\}$, $\mu \in \mathrm{Sha}[b]$, $X_0 = E_{21} - E_{32}$; $c \cdot X_0 = X_0$ and $G_{\mathbb{Q}}$ acts on X_0 by Ψ^{-1}. By (Ue9), $\bar\Psi^2 = \bar\chi_{\mathrm{cyc}}^{2e} \neq 1$ on I_ℓ ((Ue4): $\chi_\ell^2 = \mathbf{1}$), so $H^0(D_\ell, \mathcal{O}^\vee(\Psi^{-1})) = 0 = H^0(\mathbb{Q}, \mathcal{O}^\vee(\Psi^{-1}))$, and $0 \to \mathcal{O}/b \to \mathcal{O}^\vee \xrightarrow{b} \mathcal{O}^\vee \to 0$ identifies $\mathrm{Sha}[b]$ with a subgroup of $H^1(\mathbb{Q}, (\mathcal{O}/b)(\Psi^{-1}))$. Let $\tilde z$ represent μ. For $\bar v$ above $p \nmid \ell N$, Ψ is unramified at p and μ is unramified, so the homomorphism $\tilde z|_{I_{\bar v}}$ vanishes. For $p \mid N$, $p \neq \ell$, $\tilde z = dy_p$ on $D_{\bar p}$; on $N_{\bar p} = I_{\bar p} \cap \ker\bar r_0$, $\Psi = \chi$ has values in roots of unity of order prime to ℓ (Ue3) with residue 1, so $\Psi = 1$ there and $\tilde z(N_{\bar p}) = 0$, and then $\tilde z(N_{g\bar p}) = 0$ by the cocycle relation. At ℓ, $\tilde z = dm_\ell$ on D_ℓ with $m_\ell \in \mathcal{O}^\vee$ and bm_ℓ invariant, so $m_\ell \in \mathcal{O}/b$. As $\Psi(g_\star)^{-1} - 1$ is a unit, there is

a unique $z_\mu = \tilde{z} + dm$ ($m \in \mathcal{O}/b$) with $z_\mu(g_\star) = 0$, and $z_\mu(c) = 0$ ($2z_\mu(c) = z_\mu(c^2) = 0$); $\mu \mapsto z_\mu$ is additive and injective, and $z_\mu = d(m_\ell + m)$ on D_ℓ. Over $A_b = \mathcal{O}[\epsilon]/(\epsilon^2, b\epsilon)$ put $r_\mu(g) = (1+\epsilon z_\mu(g)X_0, 1)\, r_0(g)$. It is a lift of $\bar{r}_0$; (Ed1) holds as $r_\mu(c) = r_0(c)$ and $r_\mu(g_\star) = r_0(g_\star)$; (Ed2) holds by the vanishing above and because $r_0 = \varphi_{\mathrm{Eis}} \circ r$ satisfies (Ed2) ((Bx3): every $f \circ r$ does); on D_ℓ, r_μ is the conjugate of $r_0|_{D_\ell} = \beta_\ell \circ r_\ell^{\square}$ by $(1 - \epsilon(m_\ell + m)X_0, 1)$, so (Ed3) holds by (Ol1), (Ol2). Hence $r_\mu = F_\mu \circ r$, $F_\mu = \varphi_{\mathrm{Eis}} + \epsilon\delta_\mu$ with $\delta_\mu \colon R \to \mathcal{O}/b$ a φ_{Eis}-derivation. Put $D(\mu) = \delta_\mu \circ q$. As $\delta_\mu \circ \pi_R$ kills $\mathcal{O}$ and $(X)^2$, $D(\mu)(\bar{h}) = \sum_s a_s(h)\, \delta_\mu(\pi_R(X_s))$, where $\delta_\mu(\pi_R(X_s))$ is the entry (a_s, b_s) of $z_\mu(g_s)X_0\underline{r_0}(g_s)$, zero unless $\mathrm{wt}(s) = -1$. So D is additive, $D(\mu)(\bar{u}_{-1,i}) = \delta_\mu(\pi_R(u_{-1,i})) = 0$, and $D(\mu)(\bar{A}_j) = 0$ for $j \neq -1$. Lift: let $\tilde{m} \in \mathcal{O}$ lift $m_\ell + m$; $(1 - \epsilon\tilde{m}X_0, 1)\, r_0|_{D_\ell}\, (1 - \epsilon\tilde{m}X_0, 1)^{-1}$ over $\mathcal{O}[\epsilon]/(\epsilon^2)$ is classified, by (Ol2), by $\beta_\ell + \epsilon D_1$ with D_1 a β_ℓ-derivation $R_\ell^{\mathrm{ord}} \to \mathcal{O}$, and $D_1(x_s)$ is the entry (a_s, b_s) of $\tilde{m}(g_s \cdot X_0 - X_0)\underline{r_0}(g_s)$, which reduces to $\delta_\mu(\pi_R(X_s))$ modulo b for $s \in \sigma_\ell$. Put $D'(X_s) = D_1(x_s)$ for $s \in \sigma_\ell$, $D'(X_t)$ a lift of $\delta_\mu(\pi_R(X_t))$ if $\mathrm{wt}(t) = -1$ and 0 otherwise, and $D'(h) = \sum_s a_s(h)D'(X_s)$ on P, a derivation at the constant term. On P_ℓ, $D' = D_1 \circ \pi_\ell$ (both kill $\mathcal{O}$ and $(X)^2$ and agree on the X_s), so for $h = \sum_i k_i p_i \in \mathfrak{a}$ with $k_i \in \ker \pi_\ell$ (no constant term), $D'(h) = \sum_i \bar{p}_i(0)D_1(\pi_\ell(k_i)) = 0$. Hence D' induces an $\mathcal{O}$-linear map $A \to \mathcal{O}$ with $D'(xy) = \varphi_0(x)D'(y) + \varphi_0(y)D'(x)$, vanishing on A_j for $j \neq -1$ (all $D'(X_s)$ with $\mathrm{wt}(s) \neq -1$ vanish), and reducing to $D(\mu)$ modulo b. If $D(\mu) = 0$, then $\delta_\mu(\pi_R(X_s)) = 0$ for all s, so $\delta_\mu \circ \pi_R = 0$, $\delta_\mu = 0$, $r_\mu = r_0$, $z_\mu = 0$ and $\mu = 0$: D is injective. $\square$

Proposition 43.13. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and put $(\mathrm{td}_\ell(j))_{j=-2}^2 = (1, 2, 3, 4, 2)$ and $\mathrm{td}_\ell(j) = 0$ for $|j| > 2$. There are a finite set τ_ℓ, a map $w \colon \tau_\ell \to \mathbb{Z}$ with $\#w^{-1}(j) = \mathrm{td}_\ell(j)$ for every j, and an $\mathcal{O}$-algebra homomorphism $\phi_\ell \colon R_\ell^{\mathrm{ord}} \to \mathcal{O}[\ell^{-1}][[Y_i : i \in \tau_\ell]]$, where Y_i has weight $w(i)$, such that:*

(Lc1) *$\phi_\ell(\underline{r_\ell^{\mathrm{ord}}}(g)_{ab})$ has weight $\mathrm{wt}(a, b)$ for all $g \in D_\ell$ and $1 \leq a, b \leq 3$;*

(Lc2) *the constant term of $\phi_\ell(f)$ is $\beta_\ell(f)$ for every $f \in R_\ell^{\mathrm{ord}}$;*

(Lc3) *for every $i \in \tau_\ell$ there is $f \in R_\ell^{\mathrm{ord}}$ such that in $\phi_\ell(f)$ the coefficient of Y_i is non-zero and the coefficient of $Y_{i'}$ is zero for every $i' \neq i$;*

(Lc4) *if $f \in R_\ell^{\mathrm{ord}}$, $\beta_\ell(f) = 0$ and all coefficients of degree one of $\phi_\ell(f)$ vanish, then $cf \in (\ker \beta_\ell)^2$ for some $c \in \mathcal{O} \setminus \{0\}$.*

Proof. Write $E_\mathcal{O} = \mathcal{O}[\ell^{-1}]$, $\Lambda = E_\mathcal{O}[[Y_1, \ldots, Y_{12}]]$, $L \colon \Lambda \to E_\mathcal{O}^{12}$ for the vector of coefficients of $Y_1, \ldots, Y_{12}$, $\beta = \beta_\ell$, $R = R_\ell^{\mathrm{ord}}$, and $x_s = x_{g_s, a_s, b_s}$ for $s \in \sigma_\ell$; $\beta(x_s) = 0$ as $\beta \circ r_\ell^{\mathrm{ord}} = r_0$.

Step 0: $\ker \beta = (x_s : s \in \sigma_\ell)$. β is local (for y in the maximal ideal and $u \in \mathcal{O}^\times$, $y - u$ is a unit, so $\beta(y) \neq u$), hence continuous, so $\beta \circ \pi_\ell$ and the constant term agree on $\mathcal{O}[[X]]$ (both continuous, equal on the X_s); as π_ℓ is onto, $\ker \beta = \pi_\ell((X_s)_s) = (x_s)_s$. Also $R = \mathcal{O} + \ker \beta$, and the maximal ideal of R is $\mathfrak{m}_\mathcal{O} R + \ker \beta$.

Step 1: a torus endomorphism. Let $t = 1 + \ell \in \mathcal{O}^\times$; $t \equiv 1 \bmod \mathfrak{m}_\mathcal{O}$ and $t^k \neq 1$ for $k \neq 0$ ($v_\ell(t^k - 1) = 1 + v_\ell(k)$, ℓ odd). By (Ol2) for $A = R$, $h = \mathrm{T}(t)$ and f the projection $R_\ell^{\square} \to R$, there is a morphism f' killing I_ℓ^{ord} with $f' \circ r_\ell^{\square} = (\mathrm{T}(t), 1)(f \circ r_\ell^{\square})(\mathrm{T}(t), 1)^{-1}$; it induces an $\mathcal{O}$-algebra endomorphism ϑ of R with $\vartheta \circ r_\ell^{\mathrm{ord}} = (\mathrm{T}(t), 1)r_\ell^{\mathrm{ord}}(\mathrm{T}(t), 1)^{-1}$. Since $\theta(\mathrm{T}(t)) = \mathrm{T}(t)$, the GL-part of the right side is $\mathrm{T}(t)\underline{r_\ell^{\mathrm{ord}}}(g)\mathrm{T}(t)^{-1}$ for both signs, so $\vartheta(\underline{r_\ell^{\mathrm{ord}}}(g)_{ab}) = t^{\mathrm{wt}(a,b)}\underline{r_\ell^{\mathrm{ord}}}(g)_{ab}$ and $\vartheta(x_s) = t^{\mathrm{wt}(s)}x_s$ (the constant is 0 off the diagonal). Moreover $\beta \circ \vartheta = \beta$: $\beta \circ \vartheta$ carries r_ℓ^{ord} to $(\mathrm{T}(t), 1)r_0(\mathrm{T}(t), 1)^{-1} = r_0$, and r_ℓ^{ord} has exactly one base point (Ol1).

Step 2: equivariance of a chart. Let $\phi \colon R \to \Lambda$ be an $\mathcal{O}$-algebra homomorphism with (Cp1) and (Cp3), and $z_1, \ldots, z_{12} \in R$, $w_1, \ldots, w_{12} \in \mathbb{Z}$ with $\phi(z_i) = Y_i$ and $\vartheta(z_i) = t^{w_i}z_i$. Let ρ be the continuous $E_\mathcal{O}$-algebra automorphism of Λ with $\rho(Y_i) = t^{w_i}Y_i$. Then $\phi \circ \vartheta = \rho \circ \phi$.

Indeed $\psi_1 = \phi \circ \vartheta$ and $\psi_2 = \rho \circ \phi$ are $\mathcal{O}$-algebra homomorphisms with constant term β ((Cp1), $\beta \circ \vartheta = \beta$, ρ preserves constant terms), so $\psi_k((\ker\beta)^N) \subseteq \mathfrak{m}_Y^N$, and $\psi_1(z_i) = t^{w_i} Y_i = \psi_2(z_i)$. Given $a \in R$ and $N \geq 1$, let p be the polynomial formed by the terms of degree $< N$ of $\phi(a)$, $d \in \mathcal{O} \setminus \{0\}$ with $dp \in \mathcal{O}[Y]$, and $b = (dp)(z_1, \dots, z_{12}) \in R$. Then $\phi(da - b) = d(\phi(a) - p) \in \mathfrak{m}_Y^N$, so $c(da - b) \in (\ker\beta)^N$ for some $c \neq 0$ by (Cp3); and $\psi_1(b) = (dp)(t^{w_1} Y_1, \dots) = \psi_2(b)$. Hence $cd(\psi_1(a) - \psi_2(a)) = \psi_1(c(da-b)) - \psi_2(c(da-b)) \in \mathfrak{m}_Y^N$; cd is a unit of $E_{\mathcal{O}}$, so $\psi_1(a) - \psi_2(a) \in \bigcap_N \mathfrak{m}_Y^N = 0$. Consequently, if $\vartheta(a) = t^j a$, then $\rho(\phi(a)) = t^j \phi(a)$; as ρ multiplies the coefficient of Y^n by $t^{\sum_i n_i w_i}$ and $k \mapsto t^k$ is injective, $\phi(a)$ has weight j for the weights w_i of the Y_i.

Step 3: coordinates among the entries. Let ϕ_0 be as in Proposition 43.14. Since $L(gh) = \mathrm{const}(g)L(h) + \mathrm{const}(h)L(g)$, $\delta = \lambda \circ L \circ \phi_0$ is a β-derivation of R for every linear form λ on $E_{\mathcal{O}}^{12}$. If the $L(\phi_0(x_s))$, $s \in \sigma_\ell$, did not span, some $\lambda \neq 0$ would kill them; then $\delta(x_s) = 0$, so $\delta(\sum_s x_s r_s) = \sum_s \delta(x_s)\beta(r_s) = 0$, $\delta(\ker\beta) = 0$ and $\delta = 0$ (Step 0). By (Cp2) with $h = Y_i$ there are f, $c \neq 0$ with $L(\phi_0(f)) = c\, e_i$, so $\lambda(e_i) = c^{-1}\delta(f) = 0$ for all i, a contradiction. Choose $s_1, \dots, s_{12} \in \sigma_\ell$ with $L(\phi_0(x_{s_i}))$ a basis, and put $z_i = x_{s_i}$, $w_i = \mathrm{wt}(s_i) \in \{-2, \dots, 2\}$. Let α be the continuous $E_{\mathcal{O}}$-algebra endomorphism of Λ with $\alpha(Y_i) = \phi_0(z_i) \in \mathfrak{m}_Y$ (constant term $\beta(z_i) = 0$); its matrix of linear parts is invertible, so α is an automorphism, mapping $\mathfrak{m}_Y^N$ onto itself and preserving constant terms. Put $\phi_\ell = \alpha^{-1} \circ \phi_0$, $\tau_\ell = \{1, \dots, 12\}$, $w(i) = w_i$. Then $\phi_\ell(z_i) = Y_i$, ϕ_ℓ satisfies (Cp1), and (Cp3) (if $\phi_\ell(f) \in \mathfrak{m}_Y^N$ then $\phi_0(f) = \alpha(\phi_\ell(f)) \in \mathfrak{m}_Y^N$); and $\vartheta(z_i) = t^{w_i} z_i$ (Step 1).

Step 4: (Lc1)–(Lc4). (Lc1): by Step 1, $\vartheta(\underline{r_\ell^{\mathrm{ord}}}(g)_{ab}) = t^{\mathrm{wt}(a,b)} \underline{r_\ell^{\mathrm{ord}}}(g)_{ab}$, so Step 2 gives weight $\mathrm{wt}(a,b)$. (Lc2) is (Cp1). (Lc3): take $f = z_i$. (Lc4): if $\beta(f) = 0$ and all degree-one coefficients of $\phi_\ell(f)$ vanish, $\phi_\ell(f) \in \mathfrak{m}_Y^2$, and (Cp3) with $N = 2$ gives $c \neq 0$ with $cf \in (\ker\beta)^2$.

Step 5: the counts $\#w^{-1}(j) = \mathrm{td}_\ell(j)$. For $i \in \tau_\ell$ let $F_i \colon R \to E_{\mathcal{O}}[\epsilon]$ send f to the constant term of $\phi_\ell(f)$ plus ϵ times its coefficient of Y_i; it is an $\mathcal{O}$-algebra homomorphism (the jet $\Lambda \to E_{\mathcal{O}}[\epsilon]$ is), with reduction β, local. It kills $(\ker\beta)^2$, and $F_i(\ker\beta) = \sum_s \mathcal{O} F_i(x_s)$ is bounded ($F_i(x_s r) = F_i(x_s)\beta(r)$); as the maximal ideal of R is $\mathfrak{m}_{\mathcal{O}} R + \ker\beta$, F_i maps its N-th power into ϖ^{N-1} times a bounded set: F_i is continuous, and so is $g \mapsto F_i(\underline{r_\ell^{\mathrm{ord}}}(g)_{ab})$. By Lemma 43.22 (with $E = E_{\mathcal{O}}$ and $\rho_0 = \underline{r_0}|_{D_\ell}$) the map y_i with $F_i(\underline{r_\ell^{\mathrm{ord}}}(g)) = (1 + \epsilon y_i(g))\underline{r_0}(g)$ is a continuous cocycle with a primitive $Y \in \mathrm{M}_3(E_{\mathcal{O}}) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$. Its entry (a,b) is the coefficient of Y_i in $\phi_\ell(\underline{r_\ell^{\mathrm{ord}}}(g)_{ab})$ times $\psi_b(g)^{-1}$, which vanishes unless $\mathrm{wt}(a,b) = w_i$ by (Lc1); so $y_i \in Z_{w_i}$ (Lemma 43.23). For fixed j the y_i with $w_i = j$ are linearly independent: at $g = g_{s_k}$ the entry (a_{s_k}, b_{s_k}) of y_i is the coefficient of Y_i in $\phi_\ell(z_k) + \mathrm{const} = Y_k + \mathrm{const}$, times a unit, which is non-zero exactly for $i = k$. Hence $\#w^{-1}(j) \leq \dim Z_j = \mathrm{td}_\ell(j)$ for every j (Lemma 43.23). As all w_i lie in $\{-2, \dots, 2\}$, $\sum_j \#w^{-1}(j) = 12 = 1 + 2 + 3 + 4 + 2 = \sum_j \mathrm{td}_\ell(j)$, so equality holds for every j. □

43.2 The ordinary local lifting ring at the Eisenstein point

This subsection proves that the ordinary local lifting ring at ℓ is smooth at the Eisenstein point after inverting ℓ: its rational completion there is a power series ring in twelve variables (Proposition 43.14). This gives the graded chart Proposition 43.13, with weight multiplicities $1, 2, 3, 4, 2$, which the model of Subsection 43.1 uses. The upper bound for the tangent space is the core. Points of the ordinary ideal are triangular on the open subgroup D_ℓ° (Lemma 43.17, Lemma 43.19), triangular representations with cyclotomic inertial weights and finite adjacent extension classes are crystalline (Lemma 43.21), and so the tangent cocycles are Bloch–Kato classes (Proposition 43.15, Lemma 43.22), whose dimensions in the five weights are $1, 2, 3, 4, 2$ (Lemma 43.23). The lower bound comes from an explicit twelve-parameter family of triangular ordinary lifts through the Eisenstein point (Lemma 43.25, Proposition 43.24). The remaining lemmas, on lifting cocycles over power series rings and on weighted presentations, are routine

deformation theory. This subsection also introduces the triangular lifts (Definition 43.16), which Part IX uses to describe the points of the deformation ring.

Proposition 43.14. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and let $\mathfrak{m}_Y$ be the maximal ideal of $\mathcal{O}[\ell^{-1}][[Y_1, \dots, Y_{12}]]$. There is an $\mathcal{O}$-algebra homomorphism $\phi\colon R_\ell^{\mathrm{ord}} \to \mathcal{O}[\ell^{-1}][[Y_1, \dots, Y_{12}]]$ such that:*

(Cp1) *the constant term of $\phi(f)$ is $\beta_\ell(f)$ for every $f \in R_\ell^{\mathrm{ord}}$;*

(Cp2) *for every $h \in \mathcal{O}[\ell^{-1}][[Y_1, \dots, Y_{12}]]$ there are $f \in R_\ell^{\mathrm{ord}}$ and $c \in \mathcal{O}\backslash\{0\}$ with $ch - \phi(f) \in \mathfrak{m}_Y^2$;*

(Cp3) *for every $N \geq 1$ and $f \in R_\ell^{\mathrm{ord}}$ with $\phi(f) \in \mathfrak{m}_Y^N$ there is $c \in \mathcal{O} \setminus \{0\}$ with $cf \in (\ker \beta_\ell)^N$.*

Proof. Write $A = R_\ell^{\mathrm{ord}}$, $\beta = \beta_\ell$, $\mathfrak{q} = \ker \beta$ (a prime ideal, $A/\mathfrak{q} \cong \mathcal{O}$), $E_\mathcal{O} = \mathcal{O}[\ell^{-1}]$, the fraction field of $\mathcal{O}$, and $\Lambda = E_\mathcal{O}[[Y_1, \dots, Y_{12}]]$.

Step 1: $\mathfrak{q} = \sum_s x_s A$ and $A = \mathcal{O} + \mathfrak{q}$. $\beta(x_s) = 0$, as $\beta \circ r_\ell^{\mathrm{ord}} = r_0$. Every $P \in \mathcal{O}[[X_\sigma]]$ satisfies $P - P(0) = \sum_s X_s P_s$ for some power series P_s; as π_ℓ is onto, every $a \in A$ is $o + t$ with $o \in \mathcal{O}$ and $t \in \sum_s x_s A$. If $a \in \mathfrak{q}$, then $0 = \beta(a) = o$, so $a \in \sum_s x_s A$.

Step 2: the local ring $A_\mathfrak{q}$ and its cotangent space. Elements of $\mathcal{O} \setminus \{0\}$ lie outside $\mathfrak{q}$, so $A_\mathfrak{q}$ is a Noetherian local $E_\mathcal{O}$-algebra, with residue field $E_\mathcal{O}$, the composite $E_\mathcal{O} \to A_\mathfrak{q} \to E_\mathcal{O}$ being the identity. Let $T = \mathfrak{q}A_\mathfrak{q}/\mathfrak{q}^2 A_\mathfrak{q}$. For $a \in A$ and $s \notin \mathfrak{q}$ one has $\beta(s) \neq 0$ and $a/s - a/\beta(s) = a(\beta(s) - s)/(s\beta(s))$; so for $a \in \mathfrak{q}$ the classes of a/s and $\beta(s)^{-1}a$ in T agree, and T is spanned over $E_\mathcal{O}$ by the classes of the x_s (Step 1, and $x_s r \equiv \beta(r) x_s$ modulo $\mathfrak{q}^2$).

Step 3: $\dim_{E_\mathcal{O}} T \leq 12$. Let $\lambda\colon T \to E_\mathcal{O}$ be $E_\mathcal{O}$-linear, and define $F_\lambda\colon A \to E_\mathcal{O}[\epsilon]$ by $F_\lambda(a) = \beta(a) + \epsilon\,\lambda(a - \beta(a))$ (here $a - \beta(a) \in \mathfrak{q}$ by Step 1). From $ab - \beta(a)\beta(b) = (a - \beta(a))(b - \beta(b)) + \beta(a)(b - \beta(b)) + \beta(b)(a - \beta(a))$, F_λ is an $\mathcal{O}$-algebra homomorphism; its reduction modulo ϵ is β, which maps the maximal ideal $\mathfrak{m}_\mathcal{O} A + \mathfrak{q}$ of A into $\mathfrak{m}_\mathcal{O}$, so the first hypothesis of Proposition 43.15 holds with $E = E_\mathcal{O}$. For the second: $F_\lambda(\mathfrak{q}) \subseteq L = \sum_s \mathcal{O}\lambda(x_s)$, a bounded set, F_λ kills $\mathfrak{q}^2$, and $(\mathfrak{m}_\mathcal{O} A + \mathfrak{q})^N \subseteq \mathfrak{m}_\mathcal{O}^N A + \mathfrak{m}_\mathcal{O}^{N-1}\mathfrak{q} + \mathfrak{q}^2$, so F_λ maps the N-th power of the maximal ideal into $\mathfrak{m}_\mathcal{O}^N + \epsilon\,\mathfrak{m}_\mathcal{O}^{N-1} L$; hence F_λ is continuous, and so is $g \mapsto F_\lambda(\underline{r_\ell^{\mathrm{ord}}}(g)_{ab})$, the lift r_ℓ^{ord} having continuous entries. By Lemma 43.22 for $f = F_\lambda$ (whose ρ_0 is $\underline{r_0}|_{D_\ell}$), the map $y_\lambda\colon D_\ell \to \mathrm{M}_3(E_\mathcal{O})$ with $F_\lambda(\underline{r_\ell^{\mathrm{ord}}}(g)) = (1 + \epsilon y_\lambda(g))\underline{r_0}(g)$ is a continuous cocycle with a primitive $Y_\lambda \in \mathrm{M}_3(E_\mathcal{O}) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$. The action of D_ℓ through r_0 preserves the weight of every entry ($\underline{r_0}(g)$ is diagonal, and $J\,{}^t X J$ has entry (a, b) equal to $X_{4-b,4-a}$, of the same weight), so the projection p_j to the entries of weight j is equivariant; then $p_j \circ y_\lambda$ is a cocycle with primitive $(p_j \otimes 1)(Y_\lambda)$, i.e. $p_j \circ y_\lambda \in Z_j$ (formed with $E = E_\mathcal{O}$). The map $\lambda \mapsto (p_j \circ y_\lambda)_j \in \bigoplus_j Z_j$ is $E_\mathcal{O}$-linear and injective: if $y_\lambda = 0$, then $F_\lambda(\underline{r_\ell^{\mathrm{ord}}}(g)_{ab}) = \underline{r_0}(g)_{ab}$, so $\lambda(x_s) = 0$ for all s, and $\lambda = 0$ by Step 2. By Lemma 43.23, $\dim T = \dim \overline{T}^* \leq 1 + 2 + 3 + 4 + 2 = 12$.

Step 4: $A_\mathfrak{q}$ is regular of dimension 12, and the map ϕ. By Proposition 43.24, $\dim A_\mathfrak{q} \geq 12 \geq \dim T$; so $A_\mathfrak{q}$ is regular of dimension 12 and $\dim T = 12$. Its completion $\widehat{A}$ is a complete regular local ring of dimension 12 with coefficient field $E_\mathcal{O}$; choosing $y_1, \dots, y_{12} \in \mathfrak{q}A_\mathfrak{q}$ whose classes form a basis of T, Cohen's theorem gives an isomorphism $\Lambda \cong \widehat{A}$, $Y_i \mapsto y_i$, under which $\mathfrak{m}_Y^N$ corresponds to $\mathfrak{q}^N \widehat{A}$ and $\Lambda/\mathfrak{m}_Y^N \cong A_\mathfrak{q}/\mathfrak{q}^N A_\mathfrak{q}$. Let $\phi\colon A \to A_\mathfrak{q} \to \widehat{A} \cong \Lambda$, an $\mathcal{O}$-algebra homomorphism.

Step 5: (Cp1)–(Cp3). (Cp1): the constant term of $\phi(f)$ is the image of f in the residue field $A_\mathfrak{q}/\mathfrak{q}A_\mathfrak{q} = E_\mathcal{O}$, i.e. $\beta(f)$. (Cp2): given $h \in \Lambda$, since $A_\mathfrak{q} \to \Lambda/\mathfrak{m}_Y^2$ is onto, $h \equiv \phi(a)\phi(s)^{-1}$ modulo $\mathfrak{m}_Y^2$ for some $a \in A$, $s \notin \mathfrak{q}$. Multiplying by $\phi(s) = \beta(s) + (\phi(s) - \beta(s))$, where $\phi(s) - \beta(s) \in \mathfrak{m}_Y$ by (Cp1), gives $\beta(s)h - \phi(a) \equiv -(\phi(s) - \beta(s))h \equiv -(\phi(s) - \beta(s))h_0$ modulo $\mathfrak{m}_Y^2$, with $h_0 \in E_\mathcal{O}$ the constant term of h. Choose $c_0 \in \mathcal{O} \setminus \{0\}$ with $c_0 h_0 \in \mathcal{O}$; then $f = c_0 a - c_0 h_0 (s - \beta(s)) \in A$

and $c = c_0\beta(s) \in \mathcal{O} \setminus \{0\}$ satisfy $ch - \phi(f) \in \mathfrak{m}_Y^2$. (Cp3): let $\phi(f) \in \mathfrak{m}_Y^N$. Then $f \in \mathfrak{q}^N A_{\mathfrak{q}}$. Put $A' = A[\ell^{-1}]$ and $\mathfrak{n} = \mathfrak{q}A'$, a maximal ideal ($A'/\mathfrak{n} = E_{\mathcal{O}}$); $A_{\mathfrak{q}} = A'_{\mathfrak{n}}$ as $\ell \notin \mathfrak{q}$, and $\mathfrak{n}^N$ is $\mathfrak{n}$-primary, so $\mathfrak{n}^N A'_{\mathfrak{n}} \cap A' = \mathfrak{n}^N = \mathfrak{q}^N A'$. Hence $f = g\ell^{-k}$ in A' with $g \in \mathfrak{q}^N$, i.e. $\ell^m(\ell^k f - g) = 0$ in A for some m, and $c = \ell^{m+k}$ satisfies $cf = \ell^m g \in \mathfrak{q}^N$. □

Proposition 43.15. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$. Fix an embedding $\mathrm{emb}_\ell\colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ over $\mathbb{Q}$ inducing the place $\bar{\ell}$, and identify D_ℓ with $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ through it; then D_ℓ°, an open subgroup, is the absolute Galois group of a finite extension of $\mathbb{Q}_\ell$. Let $E \subseteq \overline{\mathbb{Q}}_\ell$ be a finite extension of $\mathbb{Q}_\ell$ containing $\mathcal{O}$, let $E[\epsilon] = E[X]/(X^2)$, and let $f\colon R_\ell^{\mathrm{ord}} \to E[\epsilon]$ be an $\mathcal{O}$-algebra homomorphism such that*

- *its reduction f_0 modulo ϵ satisfies $|f_0(y)|_\ell < 1$ for every y in the maximal ideal of R_ℓ^{ord}, and*
- *for $1 \le a, b \le 3$ the map $g \mapsto f(\underline{r_\ell^{\mathrm{ord}}}(g)_{ab})$ from D_ℓ to $E[\epsilon]$ is continuous.*

Then the representation $g \mapsto f(\underline{r_\ell^{\mathrm{ord}}}(g))$ of D_ℓ° on $E[\epsilon]^3$, regarded as a finite-dimensional $\mathbb{Q}_\ell$-vector space, is crystalline.

Proof. Write $B = E[\epsilon]$, a local $E_{\mathcal{O}}$-algebra of finite dimension, $H = D_\ell^\circ$, and $\tilde{f}\colon R_\ell^\square \to B$ for f composed with the projection $R_\ell^\square \to R_\ell^{\mathrm{ord}}$. Through emb_ℓ, H is the absolute Galois group G_M of a finite extension M of $\mathbb{Q}_\ell$, with inertia subgroup $I_M = H \cap I_\ell$, and $\chi_{\mathrm{cyc}}|_H$ is the cyclotomic character of M. The representation in question is $\rho_B(g) = \tilde{f}(\underline{r_\ell^\square}(g))$, $g \in H$; it is a homomorphism, lifts having sign η_K, trivial on H. As in Lemma 43.19, χ is trivial on I_M ($\chi = \chi_\ell(\chi_{\mathrm{cyc}} \bmod \ell^b)$ on I_ℓ and $\chi_\ell|_H = 1$), so $\psi_a = \chi_{\mathrm{cyc}}^{e(2-a)}$ on I_M.

Step 1: the saturated ideal. Let $\mathrm{Sat} = \{y \in R_\ell^\square : \ \mathrm{gd}(r_\ell^\square)^n y \in I^{\mathrm{ord}}(r_\ell^\square) \text{ for some } n \ge 0\}$. Then $\mathrm{gd}(r_\ell^\square)\, y \in \mathrm{Sat}$ implies $y \in \mathrm{Sat}$, and $I^{\mathrm{ord}}(r_\ell^\square) \subseteq \mathrm{Sat} \subseteq I^{\mathrm{cr}}$, which is contained in every minimal prime of I^{cr}, hence in I_ℓ^{ord}; so $\tilde{f}(\mathrm{Sat}) = 0$. Let $R = R_\ell^\square[\ell^{-1}]$, a Noetherian $E_{\mathcal{O}}$-algebra, $\mathfrak{s} = \mathrm{Sat}\, R$ and d the image of $\mathrm{gd}(r_\ell^\square)$. If $d \cdot y\ell^{-k} \in \mathfrak{s}$ with $y \in R_\ell^\square$, then $\ell^m \mathrm{gd}(r_\ell^\square) y \in \mathrm{Sat}$ for some m, so $\ell^m y \in \mathrm{Sat}$ and $y\ell^{-k} \in \mathfrak{s}$. As $\tilde{f}(\ell)$ is a unit, $\tilde{f}$ extends to an $E_{\mathcal{O}}$-algebra homomorphism $R \to B$ killing $\mathfrak{s}$.

Step 2: a lift with small annihilator. By Lemma 43.18 (with $k = E_{\mathcal{O}}$) there are $n \ge 1$, the local $E_{\mathcal{O}}$-algebra $B' = R/(\mathfrak{s} + \mathfrak{M}^n)$ of finite dimension, $\mathfrak{M}$ the preimage of ϵB, the projection $\pi\colon R \to B'$ and $q\colon B' \to B$ with $\tilde{f} = q \circ \pi$ on $R_\ell^\square$, such that $q(b) = 0$ whenever $\pi(d)b = 0$. Let $f'\colon R_\ell^\square \to B'$ be the restriction of π, an $\mathcal{O}$-algebra homomorphism killing $\mathrm{Sat} \supseteq I^{\mathrm{ord}}(r_\ell^\square)$.

Step 3: continuity of $f' \circ r_\ell^\square$. Let $\mathfrak{M}_0 = \mathfrak{M} \cap R_\ell^\square$, the kernel of $\varphi = f_0 \circ (R_\ell^\square \to R_\ell^{\mathrm{ord}})$; $\mathfrak{M} = \mathfrak{M}_0 R$. The maximal ideal of $R_\ell^\square$ maps into that of R_ℓ^{ord}, so $|\varphi(y)|_\ell < 1$ on it (first hypothesis), and (Ol1) gives an integral test ring A with $\varphi(R_\ell^\square) \subseteq A$. So $R_\ell^\square/\mathfrak{M}_0 \cong \varphi(R_\ell^\square)$ is a submodule of the finitely generated $\mathcal{O}$-module A, hence finitely generated; each $\mathfrak{M}_0^i/\mathfrak{M}_0^{i+1}$ is a finitely generated module over it, so $Q = R_\ell^\square/\mathfrak{M}_0^n$ is a finitely generated $\mathcal{O}$-module and a local ring. Then $Q/\ell Q$ is a finite local ring, its maximal ideal is nilpotent, and the $\mathfrak{m}_{R_\ell^\square}$-adic topology maps continuously onto the ℓ-adic topology of Q. f' factors through $Q \to Q[\ell^{-1}] = R/\mathfrak{M}^n \to B'$, the last two maps being continuous ($\mathbb{Q}_\ell$-linear between finite-dimensional spaces, and the inclusion of the lattice image). The entries of $r_\ell^\square$ are continuous for the $\mathfrak{m}_{R_\ell^\square}$-adic topology, so the entries of $\rho'(g) = f'(\underline{r_\ell^\square}(g))$, $g \in H$, are continuous.

Step 4: triangular form over B'. By Lemma 43.19 for $x = f'$ there are $h' \in \mathrm{GL}_3(B')$ and continuous homomorphisms $\kappa'_a\colon H \to B'^\times$ such that $T'(g) = h'^{-1}\rho'(g)h'$ is upper triangular with

diagonal $(\kappa'_a(g))_a$, $\kappa'_a = \psi_a = \chi_{\mathrm{cyc}}^{e(2-a)}$ on I_M, and, with $\lambda'_a = \kappa'_a\kappa'^{-1}_{a+1}$ and $\delta_a = \lambda'_a(\sigma_\ell) - \chi_{\mathrm{cyc}}(\sigma_\ell)$,

$$\pi(d) = f'(\mathrm{gd}(r_\ell^{\square})) = \ell \prod_{a=1}^{2} \mathrm{gp}_a \mathrm{gp}_{a+1} \kappa'_{a+1}(\sigma_\ell)\, \delta_a = u\, \delta_1\delta_2, \qquad u \in B'^{\times}.$$

Hence for $a = 1, 2$: if $\delta_a b = 0$, then $\pi(d)b = 0$ and $q(b) = 0$.

Step 5: the pushed cocycles are finite. $\lambda'_a = \chi_{\mathrm{cyc}}^{e}$ on I_M, and $c'_a(g) = T'(g)_{a,a+1}\kappa'_{a+1}(g)^{-1}$ is a continuous cocycle of $B'(\lambda'_a)$ (the $(a, a+1)$ entry of $T'(gh) = T'(g)T'(h)$). H is the closed subgroup generated by I_M and σ_ℓ, so the image of σ_ℓ topologically generates G_M/I_M. By Lemma 43.20 (with $F = \sigma_\ell$, $w = e$, $\lambda = \lambda'_a$ and Step 4), $q \circ c'_a$ has class in $H^1_f(G_M, B(q\lambda'_a))$, for $a = 1, 2$.

Step 6: conclusion. Put $h = q(h') \in \mathrm{GL}_3(B)$. Since $\tilde{f} = q \circ f'$, $h^{-1}\rho_B(g)h = q(T'(g))$ for $g \in H$: upper triangular, continuous, with diagonal $q \circ \kappa'_a$, equal to $\chi_{\mathrm{cyc}}^{e(2-a)}$ on I_M, and with adjacent cocycles $q \circ c'_a$ (in the notation of Lemma 43.21). By Lemma 43.21 with $w = e$, $g \mapsto q(T'(g))$ is crystalline on B^3, and $v \mapsto hv$ is a $\mathbb{Q}_\ell$-linear H-isomorphism from it to ρ_B. So ρ_B is crystalline.

Remark. Nothing above uses $B = E[\epsilon]$ beyond its being a local $E_{\mathcal{O}}$-algebra of finite dimension with residue field E; the same proof gives the conclusion for every such B and every f with the two hypotheses (the first for the reduction modulo the maximal ideal of B). □

Definition 43.16. Let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field, $\eta_K \colon G_{\mathbb{Q}} \to \{\pm 1\}$ the character with kernel G_K, $\mathcal{O}$ a coefficient ring with maximal ideal $\mathfrak{m}_{\mathcal{O}}$ and residue field $\mathbb{F}_{\mathcal{O}}$, $c \in G_{\mathbb{Q}}$ a complex conjugation and $\Psi \colon G_{\mathbb{Q}} \to \mathcal{O}^\times$ a continuous character with $\Psi(c) = 1$. The group $\mathcal{G}_3$, the representation $r_0(g) = (\mathrm{T}(\Psi(g)), \eta_K(g))$, its reduction $\bar{r}_0$, the characters $\psi_a = \Psi^{2-a}$ $(a = 1, 2, 3)$, markings $g_\star$ and condition (Ed1) are those of Definition 42.1 formed with K, $\mathcal{O}$, Ψ and c (their definitions there use only these data and $\Psi(c) = 1$). Fix a place $\bar{\ell}$ of $\overline{\mathbb{Q}}$ above ℓ and write $D_\ell \supseteq I_\ell$ for $D_{\bar{\ell}} \supseteq I_{\bar{\ell}}$.

Test rings. A **test ring** is a complete discrete valuation ring A with finite residue field, with maximal ideal $\mathfrak{m}_A$, together with a ring homomorphism $\mathcal{O} \to A$ mapping $\mathfrak{m}_{\mathcal{O}}$ into $\mathfrak{m}_A$. It is **integral** if A is a subring of $\overline{\mathbb{Q}}_\ell$ containing $\mathcal{O}$ and finitely generated as an $\mathcal{O}$-module (the homomorphism being the inclusion), and **of characteristic** ℓ if $\ell = 0$ in A (then $\mathfrak{m}_{\mathcal{O}}$ maps to 0).

Triangular lifts. Let $g_\star$ be a marking and A a test ring. A **triangular lift** (for $g_\star$) to A is a homomorphism $r \colon G_{\mathbb{Q}} \to \mathcal{G}_3(A)$ such that

(Tl1) the sign of r is η_K, every entry of $\underline{r}(g) - \underline{r_0}(g)$ (the latter mapped to A) lies in $\mathfrak{m}_A$ for every g, and for every $n \geq 1$ the map $g \mapsto \underline{r}(g) \bmod \mathfrak{m}_A^n$ is locally constant;

(Tl2) r satisfies (Ed1) for $g_\star$: $\underline{r}(c) = 1$, and the entries $(1,2)$ and $(2,1)$ of $\underline{r}(g_\star) - \underline{r}(cg_\star^{-1}c)$ vanish;

(Tl3) $r(g) = 1$ for every $g \in I_{\bar{v}}$ with $\eta_K(g) = 1$ and $\Psi(g) = 1$, for every place $\bar{v}$ of $\overline{\mathbb{Q}}$ not above ℓ;

(Tl4) there is $h \in \mathrm{GL}_3(A)$ with $h \equiv 1 \pmod{\mathfrak{m}_A}$ such that for every $g \in D_\ell \cap G_K$ the matrix $h^{-1}\underline{r}(g)h$ is upper triangular (its entries of negative weight vanish), with diagonal entries $\psi_1(g), \psi_2(g), \psi_3(g)$ when moreover $g \in I_\ell$.

For a triangular lift r to an integral test ring, $\rho_r \colon G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ is the restriction of $\underline{r}$ to G_K; r is **of type I** or **of type II** if ρ_r satisfies the condition of the same name in Definition 42.1 (stated there for ρ_φ). The **corner** of r is $g \mapsto (\rho_r(g)_{ab})_{a,b\in\{1,3\}}$ on G_K.

Lemma 43.17. (Ol1) *Let $\mathcal{O}$ be a coefficient ring, $R \in \mathrm{CNL}_{\mathcal{O}}$ and $\varphi\colon R \to \overline{\mathbb{Q}}_\ell$ an $\mathcal{O}$-algebra homomorphism with $|\varphi(y)|_\ell < 1$ for $y \in \mathfrak{m}_R$. There is an integral test ring A with $\varphi(R) \subseteq A$ and $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_A$.*

(Ol2) *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ valued in $\mathcal{O}^\times$. Let A be a test ring (integral or of characteristic ℓ) and $x\colon R_\ell^{\square} \to A$ a homomorphism of $\mathcal{O}$-algebras (for the structure map $\mathcal{O} \to A$) with $x(\mathfrak{m}_{R_\ell^{\square}}) \subseteq \mathfrak{m}_A$ and $x(I_\ell^{\mathrm{ord}}) = 0$, and let ρ be the GL-part of $x \circ r_\ell^{\square}$. There is $h \in \mathrm{GL}_3(A)$ with $h \equiv 1 \pmod{\mathfrak{m}_A}$ such that for every $g \in D_\ell \cap G_K$ the matrix $h^{-1}\rho(g)h$ is upper triangular, with diagonal entries $\psi_1(g), \psi_2(g), \psi_3(g)$ (mapped to A) when moreover $g \in I_\ell$.*

Proof. (Ol1). Since R is local with residue field $\mathbb{F}_{\mathcal{O}}$, $R = \mathcal{O} + \mathfrak{m}_R$, so $|\varphi(y)|_\ell \leq 1$ for all $y \in R$. Choose generators $x_1, \dots, x_n$ of $\mathfrak{m}_R$ and the surjection $\pi\colon \mathcal{O}[[X_1, \dots, X_n]] \to R$, $X_i \mapsto x_i$ (Cohen); put $a_i = \varphi(x_i)$, let $E \subset \overline{\mathbb{Q}}_\ell$ be the finite extension of the fraction field of $\mathcal{O}$ generated by the a_i, and A its valuation ring, an integral test ring. For $f \in \mathcal{O}[[X]]$ and $k \geq 1$ write $f = P_k + \sum_{|\alpha|=k} X^\alpha g_\alpha$ with P_k a polynomial; then $|\varphi(\pi f) - P_k(a)|_\ell \leq \max_i |a_i|_\ell^k \to 0$, so $\varphi(\pi f) = \lim_k P_k(a) \in A$, A being closed. Thus $\varphi(R) \subseteq A$, and $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_A$ as $|\varphi(y)|_\ell < 1$ there.

(Ol2), notation. Put $I^\circ = D_\ell^\circ \cap I_\ell$ and $D' = D_\ell \cap G_K$. By (Ue6), $\ell \nmid d_K$, so $I_\ell \subseteq G_K$, $I_\ell \subseteq D'$ and η_K is trivial on I_ℓ; by (Ue4), $\chi_\ell^2 = \mathbf{1}$, so $[I_\ell : I^\circ] \leq 2$ and $g^2 \in I^\circ$ for every $g \in I_\ell$. Write $N = \ell^a N'$ with $\ell \nmid N'$; an element of I_ℓ fixes $\mu_{N'}$, so $\chi = \chi_\ell(\chi_{\mathrm{cyc}} \bmod \ell^a)$ on I_ℓ. Hence $\chi(\sigma_0) = \chi_\ell(\sigma_0) = 1$, $\Psi(\sigma_0) = \chi_{\mathrm{cyc}}(\sigma_0)^e$ has infinite order, and $\psi_1(\sigma_0), \psi_2(\sigma_0), \psi_3(\sigma_0)$ are pairwise distinct. For a commutative $\mathcal{O}$-algebra B, a homomorphism $\rho'\colon D_\ell \to \mathrm{GL}_3(B)$, $s \in I_\ell$ and $\nu \in \{1, -1\}$ put

$$\begin{aligned} Z_1 &= (\rho'(\sigma_0) - \psi_2(\sigma_0))(\rho'(\sigma_0) - \psi_3(\sigma_0)), \qquad Z_2 = \rho'(\sigma_0) - \psi_3(\sigma_0), \\ Y_1 &= (\rho'(s) - \nu\psi_1(s))Z_1, \qquad Y_2 = (\rho'(\sigma_0) - \psi_1(\sigma_0))(\rho'(s) - \nu\psi_2(s))Z_2, \\ Y_3 &= (\rho'(\sigma_0) - \psi_1(\sigma_0))(\rho'(\sigma_0) - \psi_2(\sigma_0))(\rho'(s) - \nu\psi_3(s)), \end{aligned} \tag{43.2}$$

written $Z_i(\rho')$, $Y_a(\rho', s, \nu)$; their entries are polynomials in the entries of $\rho'(\sigma_0)$, $\rho'(s)$, so they commute with ring homomorphisms $B \to B'$.

(Ol2), a flag over a field. Let k' be a field of characteristic 0 with an injective $\mathcal{O}$-algebra map $\mathcal{O} \to k'$, and $\rho'\colon D_\ell \to \mathrm{GL}_3(k')$ a homomorphism such that, for $g, g_1, g_2, g_3 \in I^\circ$,

$$\det(X - \rho'(g)) = \prod_{a=1}^{3}(X - \psi_a(g)), \qquad (\rho'(g_1) - \psi_1(g_1))(\rho'(g_2) - \psi_2(g_2))(\rho'(g_3) - \psi_3(g_3)) = 0. \tag{43.3}$$

We show: (i) $F_1 = \operatorname{im} Z_1(\rho')$ and $F_2 = \operatorname{im} Z_2(\rho')$ form a $\rho'(D')$-stable flag of dimensions $1, 2$ on whose graded pieces F_1, F_2/F_1, k'^3/F_2 the group I° acts by ψ_1, ψ_2, ψ_3; (ii) for $s \in I_\ell$, $\rho'(s)$ acts on the a-th graded piece by a scalar $\kappa_a(s)$ with $\kappa_a(s)^2 = \psi_a(s)^2$, and $Y_a(\rho', s, \nu) = 0$ if and only if $\kappa_a(s) = \nu\psi_a(s)$. By (43.3) with $g = \sigma_0$, $\rho'(\sigma_0)$ has the distinct eigenvalues $\psi_a(\sigma_0)$, hence eigenlines E_1, E_2, E_3, and $\operatorname{im} Z_1 = E_1$, $\operatorname{im} Z_2 = E_1 \oplus E_2$. Let F_1' be the span of the images of $(\rho'(g_2) - \psi_2(g_2))(\rho'(g_3) - \psi_3(g_3))$ and F_2' the span of the images of $\rho'(g_3) - \psi_3(g_3)$, over $g_2, g_3 \in I^\circ$. By (43.3), $\rho'(g) - \psi_1(g)$ kills F_1' for all $g \in I^\circ$; with $g = \sigma_0$, $F_1' \subseteq E_1$, and $F_1' \supseteq \operatorname{im} Z_1 = E_1$; so $F_1' = F_1 = E_1$ and I° acts on F_1 by ψ_1. Likewise $(\rho'(\sigma_0) - \psi_1(\sigma_0))(\rho'(\sigma_0) - \psi_2(\sigma_0))$ kills $F_2' \supseteq \operatorname{im} Z_2$, so $F_2' = F_2 = E_1 \oplus E_2$. For $u \in F_2$ and $g \in I^\circ$, $(\rho'(g) - \psi_2(g))u$ is killed by every $\rho'(g_1) - \psi_1(g_1)$, hence lies in $\ker(\rho'(\sigma_0) - \psi_1(\sigma_0)) = F_1$: I° acts on F_2/F_1 by ψ_2, and on k'^3/F_2 by ψ_3, the characteristic polynomial being the product of those of the graded pieces. Let $d \in D'$; it normalises I° (the intersection of the normal subgroup I_ℓ of D_ℓ with kernels of characters of D_ℓ),

and $\psi_a(d^{-1}gd) = \psi_a(g)$. For $u \in F_1$ and $g \in I^\circ$, $\rho'(g)\rho'(d)u = \rho'(d)\rho'(d^{-1}gd)u = \psi_1(g)\rho'(d)u$, so $\rho'(d)u \in \ker(\rho'(\sigma_0) - \psi_1(\sigma_0)) = F_1$. For $u \in F_2$, $(\rho'(g) - \psi_2(g))\rho'(d)u = \rho'(d)(\rho'(d^{-1}gd) - \psi_2(d^{-1}gd))u \in \rho'(d)F_1 = F_1$; with $g = \sigma_0$, $(\rho'(\sigma_0) - \psi_1(\sigma_0))(\rho'(\sigma_0) - \psi_2(\sigma_0))\rho'(d)u = 0$, so $\rho'(d)u \in E_1 \oplus E_2 = F_2$. This is (i). For (ii), $s \in I_\ell \subseteq D'$ preserves the flag and acts on each graded piece, of dimension 1, by a scalar $\kappa_a(s)$; $g \mapsto \kappa_a(g)$ is a character of D', and $s^2 \in I^\circ$ gives $\kappa_a(s)^2 = \psi_a(s^2) = \psi_a(s)^2$. On $\operatorname{im} Z_1 = F_1$, $\rho'(s) - \nu\psi_1(s)$ is multiplication by $\kappa_1(s) - \nu\psi_1(s)$. The map $\rho'(s) - \nu\psi_2(s)$ sends $F_2 = \operatorname{im} Z_2$ into F_2, inducing multiplication by $\kappa_2(s) - \nu\psi_2(s)$ on F_2/F_1, and the kernel of $\rho'(\sigma_0) - \psi_1(\sigma_0)$ on $F_2 = E_1 \oplus E_2$ is F_1; so $Y_2 = 0$ if and only if $\kappa_2(s) = \nu\psi_2(s)$. The map $\rho'(s) - \nu\psi_3(s)$ induces multiplication by $\kappa_3(s) - \nu\psi_3(s)$ on k'^3/F_2, and $(\rho'(\sigma_0) - \psi_1(\sigma_0))(\rho'(\sigma_0) - \psi_2(\sigma_0))$ has kernel F_2; so $Y_3 = 0$ if and only if $\kappa_3(s) = \nu\psi_3(s)$. This is (ii).

(Ol2), a component through the Eisenstein point. By universality there is exactly one base point $\beta\colon R_\ell^\square \to \mathcal{O}$ of $r_\ell^\square$. I_ℓ^{ord} is the intersection of the minimal primes $\mathfrak{P}$ of I^{cr} with $\mathfrak{P} \subseteq \ker\beta$; they are finitely many ($R_\ell^\square$ is Noetherian), and there is at least one, since otherwise $I_\ell^{\mathrm{ord}} = R_\ell^\square$ and $x(1) = 0$. As A is a domain, $\ker x$ is a prime ideal containing their product, hence one of them, $\mathfrak{P}$. Put $J = \{y :\ \mathrm{gd}(r_\ell^\square)^n y \in I^{\mathrm{ord}}(r_\ell^\square) \text{ for some } n\}$, so $I^{\mathrm{cr}} = \sqrt{J}$ and $\mathfrak{P}$ is minimal over J. Then $\mathrm{gd} = \mathrm{gd}(r_\ell^\square) \notin \mathfrak{P}$: otherwise, $JR_{\ell,\mathfrak{P}}^\square$ being primary to the maximal ideal, $\mathrm{gd}^n \in JR_{\ell,\mathfrak{P}}^\square$ for some n, so $t\,\mathrm{gd}^n \in J$ for some $t \notin \mathfrak{P}$, whence $t \in J \subseteq \mathfrak{P}$ (J is stable under division by gd), a contradiction. As $\mathrm{gd} = \ell\,\mathrm{dt}_1\mathrm{dt}_2$, $\ell \notin \mathfrak{P}$. So $A_{\mathfrak{P}} = R_\ell^\square/\mathfrak{P}$ is a domain in which $\ell \neq 0$, $\mathcal{O} \to A_{\mathfrak{P}}$ is injective (its kernel is a prime of $\mathcal{O}$ not containing ℓ), and $k' = \mathrm{Frac}(A_{\mathfrak{P}})$ has characteristic 0. Let $\rho^{\mathrm{g}}\colon D_\ell \to \mathrm{GL}_3(k')$ be the image of the GL-part of $r_\ell^\square$. Since $I^{\mathrm{ord}}(r_\ell^\square) \subseteq J \subseteq \mathfrak{P}$ and the generators of I^{ord} are polynomials in the entries, ρ^{g} satisfies (43.3). Let $s \in I_\ell$ and $a \in \{1,2,3\}$. By (ii) for ρ^{g} there is $\nu \in \{1,-1\}$ with $Y_a(\rho^{\mathrm{g}}, s, \nu) = 0$, i.e. the entries of $Y_a(\underline{r_\ell^\square}, s, \nu)$ lie in $\mathfrak{P} \subseteq \ker\beta$. Now $\beta \circ \underline{r_\ell^\square} = \mathrm{T}(\Psi)$ on D_ℓ, so $Y_a(\mathrm{T}(\Psi), s, \nu) = 0$; this matrix is diagonal with (a,a) entry

$$\begin{gathered}\big(\Psi(s) - \nu\Psi(s)\big)\big(\Psi(\sigma_0) - 1\big)\big(\Psi(\sigma_0) - \Psi(\sigma_0)^{-1}\big),\\ \big(1 - \Psi(\sigma_0)\big)(1 - \nu)\big(1 - \Psi(\sigma_0)^{-1}\big),\\ \big(\Psi(\sigma_0)^{-1} - \Psi(\sigma_0)\big)\big(\Psi(\sigma_0)^{-1} - 1\big)\big(\Psi(s)^{-1} - \nu\Psi(s)^{-1}\big)\end{gathered}$$

for $a = 1,2,3$. The factors involving only σ_0 are non-zero ($\Psi(\sigma_0)$ has infinite order), $\Psi(s) \neq 0$, and $\mathcal{O}$ has characteristic 0; so $\nu = 1$. Hence $Y_a(\rho^{\mathrm{g}}, s, 1) = 0$, and by (ii) $\rho^{\mathrm{g}}(s)$ acts on the a-th graded piece of the flag (i) of ρ^{g} by $\psi_a(s)$, for every $s \in I_\ell$ and every a. So for $g, g_1, g_2, g_3 \in I_\ell$ we get $\det(X - \rho^{\mathrm{g}}(g)) = \prod_a (X - \psi_a(g))$ and $(\rho^{\mathrm{g}}(g_1) - \psi_1(g_1))(\rho^{\mathrm{g}}(g_2) - \psi_2(g_2))(\rho^{\mathrm{g}}(g_3) - \psi_3(g_3)) = 0$: the last factor maps k'^3 into F_2, the middle one maps F_2 into F_1, the first kills F_1. These are polynomial identities in the entries, so the corresponding coefficients and entries formed with $\underline{r_\ell^\square}$ lie in $\mathfrak{P} \subseteq \ker x$, and ρ satisfies, for $g, g_1, g_2, g_3 \in I_\ell$ (with ψ_a mapped to A),

$$\det(X - \rho(g)) = \prod_{a=1}^{3} (X - \psi_a(g)), \qquad \big(\rho(g_1) - \psi_1(g_1)\big)\big(\rho(g_2) - \psi_2(g_2)\big)\big(\rho(g_3) - \psi_3(g_3)\big) = 0. \tag{43.4}$$

(Ol2), a separating element in I_ℓ. On I_ℓ, $\chi^2 = \chi_\ell^2(\chi_{\mathrm{cyc}} \bmod \ell^a) = 1$, so $\Psi^2 = \chi_{\mathrm{cyc}}^{2e}$, whose reduction is ω^{2e} for ω the reduction of χ_{cyc}, which maps I_ℓ onto the cyclic group $\mathbb{F}_\ell^\times$ of order $\ell - 1$. By (Ue9), $\omega^{2e} \neq 1$; from now on s denotes a fixed element of I_ℓ with $\Psi(s)^2 \not\equiv 1 \pmod{\mathfrak{m}_{\mathcal{O}}}$. The images $\lambda_a \in A$ of $\psi_a(s) = \Psi(s)^{2-a}$ reduce to the pairwise distinct elements $\bar\Psi(s), 1, \bar\Psi(s)^{-1}$ of $A/\mathfrak{m}_A$, so they are pairwise distinct.

(Ol2), the flag at x. Let $E = \mathrm{Frac}(A)$ (of characteristic 0 or ℓ). By (43.4) with $g = s$, $\rho(s)$ has the distinct eigenvalues λ_a, with eigenlines E_1, E_2, E_3 in E^3. Put $F_1 = \operatorname{im}(\rho(s) - \lambda_2)(\rho(s) - \lambda_3) = E_1$ and $F_2 = \operatorname{im}(\rho(s) - \lambda_3) = E_1 \oplus E_2$. By (43.4) with $g_2 = g_3 = s$, $\rho(g) - \psi_1(g)$ kills F_1

for every $g \in I_\ell$. For $u \in F_2$ and $g \in I_\ell$, $(\rho(g) - \psi_2(g))u$ is killed by every $\rho(g_1) - \psi_1(g_1)$ ((43.4) with $g_3 = s$), in particular by $\rho(s) - \lambda_1$, so it lies in $E_1 = F_1$; and on E^3/F_2, I_ℓ acts by ψ_3, by the determinant in (43.4). Let $d \in D'$; it normalises I_ℓ, and $\psi_a(d^{-1}sd) = \psi_a(s)$. For $u \in F_1$, $\rho(s)\rho(d)u = \rho(d)\rho(d^{-1}sd)u = \lambda_1\rho(d)u$, so $\rho(d)u \in E_1 = F_1$. For $u \in F_2$, $(\rho(s)-\lambda_2)\rho(d)u = \rho(d)(\rho(d^{-1}sd)-\psi_2(d^{-1}sd))u \in \rho(d)F_1 = F_1$, so $(\rho(s)-\lambda_1)(\rho(s)-\lambda_2)\rho(d)u = 0$ and $\rho(d)u \in E_1 \oplus E_2 = F_2$. So $F_1 \subset F_2$ is a $\rho(D')$-stable flag whose graded pieces carry characters $\kappa_1, \kappa_2, \kappa_3$ of D' with $\kappa_a = \psi_a$ on I_ℓ.

(Ol2), an integral basis. Let $\Lambda = A^3$, $\Lambda_i = F_i \cap \Lambda$ $(i = 1, 2)$, saturated $\rho(D')$-stable submodules of ranks $1, 2$, and $\bar\Lambda_i$ their images in k^3, $k = A/\mathfrak{m}_A$ (of dimensions $1, 2$, by saturation). Since $r_\ell^\square$ lifts $\bar r_0|_{D_\ell}$ and $x(\mathfrak{m}_{R_\ell^\square}) \subseteq \mathfrak{m}_A$, $\rho(g) \equiv \mathrm{T}(\Psi(g))$ (mod $\mathfrak{m}_A$) for $g \in D'$. The reduction of $\rho(s)$ is $\mathrm{diag}(\bar\Psi(s), 1, \bar\Psi(s)^{-1})$ with pairwise distinct entries, as $\bar\Psi(s)^2 \neq 1$; a subspace stable under such a matrix is spanned by the coordinate vectors it contains (Lagrange interpolation). On Λ_1, $\rho(s)$ acts by $\kappa_1(s) = \psi_1(s) = \Psi(s)$, so $\bar\Lambda_1 = ke_1$; on Λ_2/Λ_1 it acts by $\kappa_2(s) = \psi_2(s) = 1$, so $\bar\Lambda_2 = ke_1 \oplus ke_2$. Choose $f_1 \in \Lambda_1$ reducing to e_1 (rescale a generator by a unit), $f_2 \in \Lambda_2$ reducing to e_2 ($\Lambda_2 \to \bar\Lambda_2$ is onto) and $f_3 = e_3$, and let $h = (f_1\, f_2\, f_3)$. Then $h \equiv 1$ (mod $\mathfrak{m}_A$), f_1 spans F_1 and f_1, f_2 span F_2 (their reductions are independent), so for $g \in D'$ the matrix $h^{-1}\rho(g)h$ is upper triangular with diagonal entries $\kappa_1(g), \kappa_2(g), \kappa_3(g)$, and these are $\psi_1(g), \psi_2(g), \psi_3(g)$ for $g \in I_\ell$. □

Lemma 43.18. *Let k be a field, R a Noetherian k-algebra, $\mathfrak{s} \subseteq R$ an ideal and $d \in R$ such that $da \in \mathfrak{s}$ implies $a \in \mathfrak{s}$ for every $a \in R$. Let B be a local k-algebra of finite dimension over k, with maximal ideal $\mathfrak{m}_B$, and $f\colon R \to B$ a homomorphism of k-algebras with $f(\mathfrak{s}) = 0$; put $\mathfrak{M} = f^{-1}(\mathfrak{m}_B)$. Then $\mathfrak{M}$ is a maximal ideal of R, and there is an integer $n \geq 1$ with the following properties: $B' = R/(\mathfrak{s} + \mathfrak{M}^n)$ is a local k-algebra of finite dimension over k; $f = q \circ \pi$ for the projection $\pi\colon R \to B'$ and a homomorphism of k-algebras $q\colon B' \to B$; and $q(b) = 0$ for every $b \in B'$ with $\pi(d)\, b = 0$.*

Proof. $\mathfrak{M}$ is maximal. f induces an injective k-algebra map $R/\mathfrak{M} \to B/\mathfrak{m}_B$, so $R/\mathfrak{M}$ is a domain of finite dimension over k, hence a field.

Choice of n. B is Artinian local, so $\mathfrak{m}_B^N = 0$ for some $N \geq 1$; then $f(\mathfrak{M}^N) \subseteq \mathfrak{m}_B^N = 0$. Put $C = R/\mathfrak{s}$, a Noetherian ring, and $\mathfrak{m} = \mathfrak{M}C$, a maximal ideal of C. The hypothesis on d says that the image of d is not a zero divisor in C; as localisation is exact, it is not a zero divisor in the Noetherian local ring $C_\mathfrak{m}$. By the Artin–Rees lemma for the ideal $dC_\mathfrak{m}$ there is $c \geq 0$ with $\mathfrak{m}^j C_\mathfrak{m} \cap dC_\mathfrak{m} = \mathfrak{m}^{j-c}(\mathfrak{m}^c C_\mathfrak{m} \cap dC_\mathfrak{m}) \subseteq d\,\mathfrak{m}^{j-c}C_\mathfrak{m}$ for all $j \geq c$. Put $n = N + c$.

B' is local and finite over k. $B' = C/\mathfrak{m}^n$. A maximal ideal of C containing $\mathfrak{m}^n$ contains $\mathfrak{m}$, hence equals it, so B' is local with maximal ideal $\mathfrak{m}/\mathfrak{m}^n$. Each $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finitely generated module over $C/\mathfrak{m} = R/\mathfrak{M}$, which is finite over k; so B' has finite dimension over k.

The factorisation. f kills $\mathfrak{s}$ and $\mathfrak{M}^n \subseteq \mathfrak{M}^N$, so it factors as $f = q \circ \pi$ with $q\colon B' \to B$ a k-algebra homomorphism.

Annihilator of $\pi(d)$. Let $b \in B'$ with $\pi(d)b = 0$, and let $a \in C$ map to b; then $da \in \mathfrak{m}^n$. In $C_\mathfrak{m}$, $da \in \mathfrak{m}^n C_\mathfrak{m} \cap dC_\mathfrak{m} \subseteq d\,\mathfrak{m}^N C_\mathfrak{m}$, so $da = da'$ with $a' \in \mathfrak{m}^N C_\mathfrak{m}$, and $a = a'$ in $C_\mathfrak{m}$ since d is not a zero divisor there. Hence $a \in \mathfrak{m}^N C_\mathfrak{m} \cap C = \mathfrak{m}^N$, the last equality because $\mathfrak{m}^N$ is $\mathfrak{m}$-primary ($\mathfrak{m}$ is maximal). Therefore $q(b)$ is the image of an element of $\mathfrak{M}^N$ under f, which is 0. □

Lemma 43.19. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ valued in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$ with fraction field $E_\mathcal{O}$. Let B be a local ring which is an $E_\mathcal{O}$-algebra, $x\colon R_\ell^\square \to B$ a homomorphism of $\mathcal{O}$-algebras with $x(I^{\mathrm{ord}}(r_\ell^\square)) = 0$, and $\rho\colon D_\ell^\circ \to \mathrm{GL}_3(B)$ the restriction to D_ℓ° of the GL-part of $x \circ r_\ell^\square$. Then there are $h \in \mathrm{GL}_3(B)$ and homomorphisms $\kappa_1, \kappa_2, \kappa_3\colon D_\ell^\circ \to B^\times$ such that:*

1. *for every $g \in D_\ell^\circ$ the matrix $h^{-1}\rho(g)h$ is upper triangular with diagonal entries $\kappa_1(g)$, $\kappa_2(g), \kappa_3(g)$, and $\kappa_a(g) = \psi_a(g)$ for $g \in D_\ell^\circ \cap I_\ell$ and $a = 1, 2, 3$;*

2. *for $a = 1, 2$,*

$$x\big(\mathrm{dt}_a(r_\ell^{\square})\big) = \mathrm{gp}_a\,\mathrm{gp}_{a+1}\,\kappa_{a+1}(\sigma_\ell)\Big(\kappa_a(\sigma_\ell)\kappa_{a+1}(\sigma_\ell)^{-1} - \chi_{\mathrm{cyc}}(\sigma_\ell)\Big).$$

If moreover B has finite dimension over $\mathbb{Q}_\ell$ and the entries of ρ are continuous for its natural topology, then $\kappa_1, \kappa_2, \kappa_3$ are continuous.

Proof. Notation and the defining equations. Put $I^\circ = D_\ell^\circ \cap I_\ell$. Lifts have sign η_K, which is trivial on D_ℓ°, so ρ is a homomorphism. The generators of $I^{\mathrm{ord}}(r_\ell^{\square})$ are polynomials in the entries of $\underline{r_\ell^{\square}}$, and x is a ring homomorphism; so for $g, g_1, g_2, g_3 \in I^\circ$,

$$\det(X - \rho(g)) = \prod_{a=1}^{3}(X - \psi_a(g)), \qquad (\rho(g_1) - \psi_1(g_1))(\rho(g_2) - \psi_2(g_2))(\rho(g_3) - \psi_3(g_3)) = 0. \tag{43.5}$$

Write $N = \ell^b N'$ with $\ell \nmid N'$. An element of I_ℓ fixes the N'-th roots of unity, so $\chi = \chi_\ell\,(\chi_{\mathrm{cyc}} \bmod \ell^b)$ on I_ℓ, which is trivial on I° as χ_ℓ is trivial on D_ℓ°; hence $\Psi = \chi_{\mathrm{cyc}}^e$ on I°. In particular $u_a = \psi_a(\sigma_0) = \chi_{\mathrm{cyc}}(\sigma_0)^{e(2-a)}$ $(a = 1, 2, 3)$ are pairwise distinct elements of $\mathcal{O}$, $\chi_{\mathrm{cyc}}(\sigma_0)$ having infinite order. Non-zero elements of the field $E_\mathcal{O}$ are units in B, so the $u_a - u_b$ $(a \neq b)$ and the gp_a are units of B. Let $k = B/\mathfrak{m}_B$, a field containing $E_\mathcal{O}$.

The eigenlines of $S = \rho(\sigma_0)$. By (43.5) and Cayley–Hamilton, $\prod_a(S - u_a) = 0$. Put $\varepsilon_a = \prod_{b\neq a}(S - u_b)(u_a - u_b)^{-1}$. The Lagrange identity $\sum_a \prod_{b\neq a}(X - u_b)/(u_a - u_b) = 1$ gives $\sum_a \varepsilon_a = 1$; for $a \neq b$ the product $\varepsilon_a\varepsilon_b$ is divisible by $\prod_c(S - u_c)$, so $\varepsilon_a\varepsilon_b = 0$, $\varepsilon_a^2 = \varepsilon_a$ and $(S - u_a)\varepsilon_a = 0$. Hence $B^3 = L_1 \oplus L_2 \oplus L_3$ with $L_a = \varepsilon_a B^3$, S acting on L_a by u_a. Each L_a is a finitely generated projective module over the local ring B, hence free, of rank $\dim_k(L_a \otimes_B k)$, the dimension of the $\bar{u}_a$-eigenspace of the reduction $\bar{S}$; as $\bar{S}$ has the three distinct eigenvalues $\bar{u}_a$ in k, every L_a is free of rank 1. Since $S - u_a$ acts on L_b $(b \neq a)$ by the unit $u_b - u_a$, $\ker(S - u_a) = L_a$, $\ker\big((S - u_1)(S - u_2)\big) = L_1 \oplus L_2$, and $\mathrm{im}(S - u_3) = L_1 \oplus L_2$, $\mathrm{im}\big((S - u_2)(S - u_3)\big) = L_1$.

The flag on I°. Put $F_1 = L_1$ and $F_2 = L_1 \oplus L_2$. Let $g \in I^\circ$. By (43.5) with $(g_1, g_2, g_3) = (g, \sigma_0, \sigma_0)$, $\rho(g) - \psi_1(g)$ kills $\mathrm{im}\big((S - u_2)(S - u_3)\big) = F_1$. By (43.5) with $(g_1, g_2, g_3) = (\sigma_0, g, \sigma_0)$, $(S - u_1)(\rho(g) - \psi_2(g))$ kills $\mathrm{im}(S - u_3) = F_2$, so $(\rho(g) - \psi_2(g))F_2 \subseteq \ker(S - u_1) = F_1$. In particular $\rho(g)F_1 \subseteq F_1$ and $\rho(g)F_2 \subseteq F_2$.

The flag is stable under D_ℓ°. I° is normal in D_ℓ (I_ℓ is, and D_ℓ° is the kernel of homomorphisms to abelian groups), and $\psi_a(d^{-1}gd) = \psi_a(g)$. Let $d \in D_\ell^\circ$ and $\sigma' = d^{-1}\sigma_0 d \in I^\circ$. For $v \in F_1$, $(S - u_1)\rho(d)v = \rho(d)(\rho(\sigma') - \psi_1(\sigma'))v = 0$ by the previous step, so $\rho(d)v \in F_1$. For $v \in F_2$, $(S - u_2)\rho(d)v = \rho(d)(\rho(\sigma') - \psi_2(\sigma'))v \in \rho(d)F_1 = F_1$, so $(S - u_1)(S - u_2)\rho(d)v = 0$ and $\rho(d)v \in F_2$.

The basis and (i). Let f_a generate L_a and $h = (f_1\, f_2\, f_3) \in \mathrm{GL}_3(B)$. For $g \in D_\ell^\circ$, $\rho(g)$ preserves Bf_1 and $Bf_1 \oplus Bf_2$, so $h^{-1}\rho(g)h$ is upper triangular; let $\kappa_a(g)$ be its diagonal entries. They are units (their product is $\det\rho(g)$), and κ_a is a homomorphism because the diagonal of a product of upper triangular matrices is the product of the diagonals. For $g \in I^\circ$, the flag step gives $\kappa_1(g) = \psi_1(g)$ and $\kappa_2(g) = \psi_2(g)$; then $\det(X - \rho(g)) = (X - \psi_1(g))(X - \psi_2(g))(X - \kappa_3(g))$, and comparing with (43.5), the monic polynomial $(X - \psi_1(g))(X - \psi_2(g))$ being a non-zero-divisor in $B[X]$, $\kappa_3(g) = \psi_3(g)$.

(ii). f_a is an eigenvector of S, so $h^{-1}Sh = \mathrm{diag}(u_1, u_2, u_3)$ and $h^{-1}(S - u_{a+1})(S - u_{a+2})h = \mathrm{gp}_a E_{aa}$, with E_{aa} the matrix unit. As $\sigma_0, \sigma_\ell \in D_\ell^\circ$ and tq_a is a polynomial in the entries of

$\underline{r_\ell^\square}(\sigma_0)$, $\underline{r_\ell^\square}(\sigma_\ell)$, the trace being invariant under conjugation,

$$x(\mathrm{tq}_a(r_\ell^\square)) = \mathrm{tr}\big(\mathrm{gp}_a E_{aa}\, h^{-1}\rho(\sigma_\ell)h\big) = \mathrm{gp}_a\, \kappa_a(\sigma_\ell).$$

Hence $x(\mathrm{dt}_a(r_\ell^\square)) = \mathrm{gp}_{a+1}\mathrm{gp}_a\kappa_a(\sigma_\ell) - \chi_{\mathrm{cyc}}(\sigma_\ell)\mathrm{gp}_a\mathrm{gp}_{a+1}\kappa_{a+1}(\sigma_\ell)$, which is (ii).

Continuity. $\kappa_a(g)$ is an entry of $h^{-1}\rho(g)h$, a polynomial in the entries of $\rho(g)$ with fixed coefficients; multiplication is continuous on the finite-dimensional $\mathbb{Q}_\ell$-algebra B. □

Lemma 43.20. *Let ℓ be a prime, M a finite extension of $\mathbb{Q}_\ell$ with absolute Galois group G_M and inertia subgroup I_M, and $F \in G_M$ an element whose image topologically generates G_M/I_M. Let B' and B be commutative $\mathbb{Q}_\ell$-algebras of finite dimension, with their natural topologies, $q\colon B' \to B$ a homomorphism of $\mathbb{Q}_\ell$-algebras, $w \geq 1$ an integer and $\lambda\colon G_M \to B'^\times$ a continuous homomorphism with $\lambda(g) = \chi_{\mathrm{cyc}}(g)^w$ for $g \in I_M$; put $\delta = \lambda(F) - \chi_{\mathrm{cyc}}(F) \in B'$. Write $B'(\lambda)$ for B' with $g \in G_M$ acting by multiplication by $\lambda(g)$, and $B(q\lambda)$ likewise. Assume that $w \geq 2$, or that $q(b) = 0$ for every $b \in B'$ with $\delta b = 0$. Then for every continuous 1-cocycle $c\colon G_M \to B'(\lambda)$ the cocycle $q \circ c$ of $B(q\lambda)$ has its class in $H^1_f(G_M, B(q\lambda))$.*

Proof. Put $V = B(q\lambda)$ and $\mu = (q\lambda)\chi_{\mathrm{cyc}}^{-w}\colon G_M \to B^\times$, a continuous homomorphism trivial on I_M.

V is crystalline, hence de Rham. The map $b\otimes 1 \mapsto b$ is a G_M-isomorphism $B(\mu)\otimes_{\mathbb{Q}_\ell}\mathbb{Q}_\ell(w) \to V$; $B(\mu)$ is unramified, hence crystalline, $\mathbb{Q}_\ell(w)$ is crystalline, and so is their tensor product (Theorem B.45).

The dual. Let $B^\vee = \mathrm{Hom}_{\mathbb{Q}_\ell}(B, \mathbb{Q}_\ell)$. Then $U = V^*(1)$ is the space $B^\vee$ with the action $(g\varphi)(b) = \chi_{\mathrm{cyc}}(g)\varphi(q\lambda(g)^{-1}b)$; on I_M, g acts by the scalar $\chi_{\mathrm{cyc}}(g)^{1-w}$. The representation $U \cong B^\vee(\mu^{-1}) \otimes \mathbb{Q}_\ell(1-w)$ is crystalline with all Hodge–Tate weights $1 - w \leq 0$, so $\mathrm{Fil}^0 D_{\mathrm{dR}}(U) = D_{\mathrm{dR}}(U)$ and Theorem B.56 gives $\dim_{\mathbb{Q}_\ell} H^1_f(G_M, U) = h^0(U)$. Likewise $U' = B'(\lambda)^*(1)$ is $B'^\vee$ with $(g\varphi)(b) = \chi_{\mathrm{cyc}}(g)\varphi(\lambda(g)^{-1}b)$, and $q^t\colon U \to U'$, $\varphi \mapsto \varphi \circ q$, is G_M-equivariant, since $q(\lambda(g)^{-1}b) = q\lambda(g)^{-1}q(b)$.

Reduction to $q^t_* H^1_f(G_M, U) = 0$. The map $q\colon B'(\lambda) \to V$ is $\mathbb{Q}_\ell$-linear and G_M-equivariant, and for $v' \in B'$, $\varphi \in B^\vee$ the evaluation satisfies $\varphi(q(v')) = (q^t\varphi)(v')$. Hence, on cochains, the cup products satisfy $\langle q\circ c, y\rangle = \langle c, q^t \circ y\rangle'$ for cocycles y of U, with $\langle\ ,\ \rangle'$ the pairing of $B'(\lambda)$. If $q^t \circ y$ is a coboundary for every y with class in $H^1_f(G_M, U)$, then $\langle q\circ c, y\rangle = 0$ for all such y, and Theorem B.49 for the de Rham representation V gives that the class of $q \circ c$ lies in $H^1_f(G_M, V)$.

The case $w \geq 2$. Since $\chi_{\mathrm{cyc}}(I_M)$ is open in $\mathbb{Z}_\ell^\times$, there is $g \in I_M$ with $\chi_{\mathrm{cyc}}(g)^{1-w} \neq 1$, so $h^0(U) = 0$ and $H^1_f(G_M, U) = 0$; the reduction applies.

The case $w = 1$: $H^1_f(G_M, U)$ is unramified. Now I_M acts trivially on U. By inflation–restriction, inflation identifies $H^1(\Gamma, U)$ with $H^1_{\mathrm{ur}} = \ker\big(H^1(G_M, U) \to H^1(I_M, U)\big)$, of dimension $\dim U/(\bar F - 1)U = \dim\ker(\bar F - 1) = h^0(U)$ (the fixed vectors of $\bar F$ are fixed by its closure Γ). Every class in H^1_{ur} lies in $H^1_f(G_M, U)$: for a cocycle z vanishing on I_M, let $X = U \oplus \mathbb{Q}_\ell e$ with $g(u + te) = gu + tz(g) + te$, an unramified, hence crystalline, representation; by left exactness of invariants, $0 \to D_{\mathrm{cris}}(U) \to D_{\mathrm{cris}}(X) \to D_{\mathrm{cris}}(\mathbb{Q}_\ell) = M_0$ is exact, and $\dim_{M_0} D_{\mathrm{cris}}(X) = \dim U + 1 = \dim_{M_0} D_{\mathrm{cris}}(U) + 1$, so there is an invariant $1 \otimes e - b$ with $b \in B_{\mathrm{cris}} \otimes U$; invariance says $1 \otimes z(g) = gb - b$. As both spaces have dimension $h^0(U)$, $H^1_f(G_M, U) = H^1_{\mathrm{ur}}$.

The case $w = 1$: q^t kills it. Let y have class in $H^1_f(G_M, U)$; by the previous step we may take y inflated from a cocycle $\tilde y$ of Γ. Put $\nu = \lambda\chi_{\mathrm{cyc}}^{-1}$, trivial on I_M; F acts on U' by $\varphi \mapsto \varphi \circ m_{\nu(F)^{-1}}$, m_β denoting multiplication by β on B', and $\nu(F)^{-1} - 1 = -\lambda(F)^{-1}\delta$, so $(F-1)U' = \{\psi \circ m_\delta : \psi \in B'^\vee\}$, the image of the transpose of m_δ. Its dimension is $\dim B' - \dim\ker m_\delta$, and it

is contained in the annihilator of $\ker m_\delta = \{b : \delta b = 0\}$, which has the same dimension; so $(F-1)U'$ is the space of linear forms vanishing on $\ker m_\delta$. By hypothesis $\ker m_\delta \subseteq \ker q$, so $q^t(U) \subseteq (F-1)U'$. Hence $(q^t \circ \tilde{y})(\bar{F}) = (\bar{F}-1)u'$ for some $u' \in U'$, and $q^t \circ \tilde{y} - du'$ is a cocycle of Γ vanishing at $\bar{F}$; by the cocycle identity it vanishes on the subgroup generated by $\bar{F}$, hence on Γ by continuity. So $q^t \circ y$ is a coboundary, and the reduction applies. □

Lemma 43.21. *Let ℓ be a prime, M a finite extension of $\mathbb{Q}_\ell$ with absolute Galois group G_M and inertia subgroup I_M, B a commutative $\mathbb{Q}_\ell$-algebra of finite dimension with its natural topology, and $w \geq 1$ an integer. Let $\rho\colon G_M \to \mathrm{GL}_3(B)$ be a continuous homomorphism with values in the upper triangular matrices, with diagonal entries $\kappa_1, \kappa_2, \kappa_3\colon G_M \to B^\times$ such that $\kappa_a(g) = \chi_{\mathrm{cyc}}(g)^{w(2-a)}$ for $g \in I_M$ and $a = 1, 2, 3$. For $a = 1, 2$ let $B(\kappa_a\kappa_{a+1}^{-1})$ be B with g acting by multiplication by $\kappa_a(g)\kappa_{a+1}(g)^{-1}$, and $c_a(g) = \rho(g)_{a,a+1}\,\kappa_{a+1}(g)^{-1}$, a continuous 1-cocycle with values in it. If the classes of c_1 and c_2 lie in $H^1_f(G_M, B(\kappa_1\kappa_2^{-1}))$ and $H^1_f(G_M, B(\kappa_2\kappa_3^{-1}))$, then the representation ρ on B^3, regarded as a $\mathbb{Q}_\ell$-vector space, is crystalline.*

Proof. That c_a is a cocycle follows from the $(a, a+1)$ entry of $\rho(gh) = \rho(g)\rho(h)$: $\rho(gh)_{a,a+1} = \kappa_a(g)\rho(h)_{a,a+1} + \rho(g)_{a,a+1}\kappa_{a+1}(h)$. Let e_1, e_2, e_3 be the standard basis of $X = B^3$.

Characters. If $\kappa\colon G_M \to B^\times$ is continuous and $\kappa\chi_{\mathrm{cyc}}^{-m}$ is trivial on I_M, then $B(\kappa)$ is crystalline: it is isomorphic to $B(\kappa\chi_{\mathrm{cyc}}^{-m}) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(m)$, the first factor unramified (Theorem B.45); for $m \geq 1$ its Hodge–Tate weights are all m, so $\mathrm{Fil}^0 D_{\mathrm{dR}} = 0$; and if $m \neq 0$ then $h^0(B(\kappa)) = 0$, as some $g \in I_M$ has $\chi_{\mathrm{cyc}}(g)^m \neq 1$ ($\chi_{\mathrm{cyc}}(I_M)$ is open in $\mathbb{Z}_\ell^\times$).

Claim: extensions of a trivial representation with finite classes. Let Y be a finite-dimensional continuous $\mathbb{Q}_\ell$-representation of G_M and $U \subseteq Y$ a crystalline subrepresentation such that G_M acts trivially on Y/U and, for every $y \in Y$, the cocycle $g \mapsto gy - y$ (with values in U) has class in $H^1_f(G_M, U)$. Then Y is crystalline. Indeed, for $y \in Y$ choose $b_y \in B_{\mathrm{cris}} \otimes U$ with $1 \otimes (gy - y) = gb_y - b_y$; then $1 \otimes y - b_y \in D_{\mathrm{cris}}(Y)$ maps to $1 \otimes \bar{y}$ in $D_{\mathrm{cris}}(Y/U) = M_0 \otimes_{\mathbb{Q}_\ell} (Y/U)$. So the map $D_{\mathrm{cris}}(Y) \to D_{\mathrm{cris}}(Y/U)$, M_0-linear, is onto; its kernel is $D_{\mathrm{cris}}(U)$ (invariants are left exact), and $\dim_{M_0} D_{\mathrm{cris}}(Y) = \dim U + \dim Y/U = \dim Y$.

Twists. If $\kappa\colon G_M \to B^\times$ is a continuous homomorphism and $X' \subseteq X$ a B-submodule stable under ρ, write $X'\langle\kappa\rangle$ for X' with g acting by $\kappa(g)\rho(g)$. The map $X'\langle\kappa^{-1}\rangle \otimes_{\mathbb{Q}_\ell} B(\kappa) \to X'$, $y \otimes b \mapsto by$, is G_M-equivariant and onto; so if $X'\langle\kappa^{-1}\rangle$ is crystalline, so is X' (Theorem B.45 and the characters step). Also H^1_f is stable under multiplication by elements of B on $B(\kappa)$, and H^1_f is functorial for G_M-maps.

$X_2 = Be_1 \oplus Be_2$ is crystalline. In $Y_1 = X_2\langle\kappa_2^{-1}\rangle$, $U_1 = Be_1 \cong B(\kappa_1\kappa_2^{-1})$ is crystalline, and G_M acts trivially on Y_1/U_1. For $y = b_1e_1 + b_2e_2$, $gy - y = ((\kappa_1\kappa_2^{-1})(g)b_1 - b_1)e_1 + b_2c_1(g)e_1$, a coboundary plus b_2c_1; its class lies in H^1_f. By the claim Y_1 is crystalline, and so is X_2.

X is crystalline. Let $Y = X\langle\kappa_3^{-1}\rangle \supseteq U = X_2\langle\kappa_3^{-1}\rangle \supseteq U_2 = Be_1$. Then U is crystalline (twists step, as X_2 is), $U_2 \cong B(\kappa_1\kappa_3^{-1})$, $U'' = U/U_2 \cong B(\kappa_2\kappa_3^{-1})$ by $e_2 \mapsto 1$, and G_M acts trivially on Y/U. On I_M the characters of U_2 and U'' are χ_{cyc}^{2w} and χ_{cyc}^{w}.

- $H^1(G_M, U_2) = H^1_f(G_M, U_2)$: by the characters step $h^0(U_2) = 0$, $\mathrm{Fil}^0 D_{\mathrm{dR}}(U_2) = 0$, and $h^0(U_2^*(1)) = 0$ since I_M acts on $U_2^*(1)$ by $\chi_{\mathrm{cyc}}^{1-2w}$ and $1 - 2w \neq 0$. By Theorem B.36, $h^1(U_2) = d \dim U_2$, and by Theorem B.56, $\dim H^1_f(G_M, U_2) = d \dim U_2$.

- The map $H^1_f(G_M, U) \to H^1_f(G_M, U'')$ is onto. Indeed, $h^0(U) = 0$ (its graded pieces have none) and the Hodge–Tate weights of U are $2w$ and w, so $\dim H^1_f(G_M, U) = d \dim U$ and $\dim H^1_f(G_M, U'') = d \dim U''$ by Theorem B.56. In the long exact sequence of $0 \to U_2 \to U \to U'' \to 0$, $H^1(G_M, U_2) \to H^1(G_M, U)$ is injective ($h^0(U'') = 0$), with image the kernel

of $H^1(G_M,U)\to H^1(G_M,U'')$, contained in $H^1_f(G_M,U)$ by the first item and functoriality. So the image of $H^1_f(G_M,U)$ in $H^1_f(G_M,U'')$ has dimension $d\dim U - d\dim U_2 = d\dim U''$.

- For $y=b_1e_1+b_2e_2+b_3e_3$, the class of $z_y(g)=gy-y$ lies in $H^1_f(G_M,U)$: since $ge_3-e_3=\rho(g)_{13}\kappa_3(g)^{-1}e_1+c_2(g)e_2$, the image of z_y in U'' is a coboundary plus b_3c_2, whose class lies in $H^1_f(G_M,U'')$. By the second item there is $\zeta\in H^1_f(G_M,U)$ with the same image, and $[z_y]-\zeta$ lies in the kernel of $H^1(G_M,U)\to H^1(G_M,U'')$, contained in $H^1_f(G_M,U)$.

By the claim Y is crystalline, and so is X (twists step). □

Lemma 43.22. *Let (K,ℓ,e,N,χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and let emb_ℓ, E and $f\colon R^{\mathrm{ord}}_\ell\to E[\epsilon]$ be as in Proposition 43.15, with reduction f_0 modulo ϵ. Put $\rho_0(g)=f_0(\underline{r^{\mathrm{ord}}_\ell}(g))\in \mathrm{GL}_3(E)$ and define $y\colon D_\ell\to \mathrm{M}_3(E)$ by*

$$f(\underline{r^{\mathrm{ord}}_\ell}(g))=(1+\epsilon\, y(g))\rho_0(g)\qquad (g\in D_\ell).$$

Let D_ℓ act on $\mathrm{M}_3(E)$ by $g\cdot X=\rho_0(g)X\rho_0(g)^{-1}$ if $\eta_K(g)=1$ and $g\cdot X=-\rho_0(g)J\,{}^tXJ\rho_0(g)^{-1}$ if $\eta_K(g)=-1$, and on $\mathrm{M}_3(E)\otimes_{\mathbb{Q}_\ell}B_{\mathrm{cris}}$ diagonally, through emb_ℓ on B_{cris}. Then y is a continuous 1-cocycle, $y(gh)=y(g)+g\cdot y(h)$, and there is $Y\in \mathrm{M}_3(E)\otimes_{\mathbb{Q}_\ell}B_{\mathrm{cris}}$ with $y(g)\otimes 1=g\cdot Y-Y$ for every $g\in D_\ell$.

Proof. Write $P_g=\rho_0(g)$ and $\varsigma_g=\eta_K(g)$ for the sign of $f\circ r^{\mathrm{ord}}_\ell$ at g; $f\circ r^{\mathrm{ord}}_\ell\colon D_\ell\to\mathcal{G}_3(E[\epsilon])$ is a homomorphism, and its reduction modulo ϵ is $g\mapsto(P_g,\varsigma_g)$, also a homomorphism.

Step 1: y is a continuous cocycle. In $\mathcal{G}_3$, $(A,\varsigma)(B,\tau)=(A\,\theta^\varsigma(B),\varsigma\tau)$ with $\theta^1=\mathrm{id}$ and $\theta^{-1}=\theta$. For $Y'\in\mathrm{M}_3(E)$ and $P\in\mathrm{GL}_3(E)$, $((1+\epsilon Y')P)^{-1}=P^{-1}(1-\epsilon Y')$, so, using $J^2=1$,

$$\theta((1+\epsilon Y')P)=J(1-\epsilon\,{}^tY')\,{}^tP^{-1}J=(1-\epsilon JY'\,{}^tJ)\,\theta(P).$$

Hence the GL-part of $f(r^{\mathrm{ord}}_\ell(g))f(r^{\mathrm{ord}}_\ell(h))$ is $(1+\epsilon y(g))P_g(1+\epsilon Z)\theta^{\varsigma_g}(P_h)$ with $Z=y(h)$ if $\varsigma_g=1$ and $Z=-Jy(h)^tJ$ if $\varsigma_g=-1$, that is $(1+\epsilon(y(g)+g\cdot y(h)))P_g\theta^{\varsigma_g}(P_h)$, since $P_gZP_g^{-1}=g\cdot y(h)$. As $P_{gh}=P_g\theta^{\varsigma_g}(P_h)$ and $1+\epsilon X$ determines X, $y(gh)=y(g)+g\cdot y(h)$. The entries of P_g and of the ϵ-part of $f(\underline{r^{\mathrm{ord}}_\ell}(g))$ are continuous in g (second hypothesis of Proposition 43.15), so are those of P_g^{-1} (adjugate over determinant), and $y(g)$ is the ϵ-part times P_g^{-1}.

Step 2: a primitive on D°_ℓ. Put $H=D^\circ_\ell$; its elements have sign 1. Let $V=E^3$ with $h\mapsto P_h$ and $V_\epsilon=E[\epsilon]^3$ with $h\mapsto f(\underline{r^{\mathrm{ord}}_\ell}(h))$, continuous representations of H on finite-dimensional $\mathbb{Q}_\ell$-vector spaces. By Proposition 43.15, V_ϵ is crystalline. Reduction modulo ϵ is an H-equivariant surjection $V_\epsilon\to V$ with kernel ϵV_ϵ, so V is crystalline (quotient, Theorem B.45). Let $W=\mathrm{Hom}_E(V,V_\epsilon)$ with $h\cdot\phi=f(\underline{r^{\mathrm{ord}}_\ell}(h))\circ\phi\circ P_h^{-1}$; it is an H-stable subspace of $\mathrm{Hom}_{\mathbb{Q}_\ell}(V,V_\epsilon)\cong V^*\otimes_{\mathbb{Q}_\ell}V_\epsilon$, which is crystalline (dual and tensor product), so W is crystalline. Writing $E[\epsilon]^3=E^3\oplus\epsilon E^3$, every $\phi\in W$ is $v\mapsto Av+\epsilon Bv$ for unique $A,B\in\mathrm{M}_3(E)$, written $A+\epsilon B$, and

$$h\cdot(A+\epsilon B)=P_hAP_h^{-1}+\epsilon(P_hBP_h^{-1}+y(h)P_hAP_h^{-1}).$$

So $V_P=\{\lambda+\epsilon B:\lambda\in E,\ B\in\mathrm{M}_3(E)\}$ and $W_0=\{\epsilon B\}$ are H-stable, $h\cdot\epsilon B=\epsilon\,(h\cdot B)$, and $\mathrm{pr}\colon V_P\to E$, $\lambda+\epsilon B\mapsto\lambda$, is surjective, H-equivariant for the trivial action on E, with kernel W_0. All three are crystalline ($V_P,W_0\subseteq W$; the trivial representation is crystalline). By left exactness of D_{cris} ($\otimes_{\mathbb{Q}_\ell}B_{\mathrm{cris}}$ is exact and invariants are left exact), the image of $D_{\mathrm{cris}}(V_P)$ in $D_{\mathrm{cris}}(E)$ has dimension $\dim_{\mathbb{Q}_\ell}V_P-\dim_{\mathbb{Q}_\ell}W_0=\dim_{\mathbb{Q}_\ell}E=\dim D_{\mathrm{cris}}(E)$, so $D_{\mathrm{cris}}(V_P)\to D_{\mathrm{cris}}(E)$ is onto. Choose $w_0\in D_{\mathrm{cris}}(V_P)$ with $(\mathrm{pr}\otimes 1)(w_0)=1\otimes 1$, and put $e_0=1+\epsilon 0\in V_P$. Then $e_0\otimes 1-w_0$ lies in $\ker(\mathrm{pr}\otimes 1)=W_0\otimes_{\mathbb{Q}_\ell}B_{\mathrm{cris}}$, and for $h\in H$, as w_0 is invariant,

$$h(e_0\otimes 1-w_0)-(e_0\otimes 1-w_0)=(h\cdot e_0-e_0)\otimes 1=\epsilon\, y(h)\otimes 1.$$

Identifying W_0 with $\mathrm{M}_3(E)$ by $\epsilon B \mapsto B$ (an H-isomorphism), we get $Y_H \in \mathrm{M}_3(E) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $y(h) \otimes 1 = h \cdot Y_H - Y_H$ for $h \in H$.

Step 3: averaging to D_ℓ. H is the kernel of $g \mapsto (\eta_K(g), \chi_\ell(g))$ on D_ℓ, a normal subgroup of finite index n (χ_ℓ has finite order). Put $M = \mathrm{M}_3(E) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with the diagonal action, a $\mathbb{Q}_\ell$-vector space. By Step 1, $c(g) = y(g) \otimes 1$ satisfies $c(gh) = c(g) + g \cdot c(h)$ (the element $1 \in B_{\mathrm{cris}}$ is invariant), and so does $z(g) = c(g) - (g \cdot Y_H - Y_H)$, which vanishes on H by Step 2. Then $z(gh) = z(g) + g \cdot z(h) = z(g)$ for $h \in H$. Let $\mathcal{T}$ be a set of representatives of D_ℓ/H; for $g \in D_\ell$, $g\mathcal{T}$ is another one, so with $W' = \sum_{\tau \in \mathcal{T}} z(\tau)$,

$$W' = \sum_{\tau \in \mathcal{T}} z(g\tau) = \sum_{\tau \in \mathcal{T}} \bigl(z(g) + g \cdot z(\tau)\bigr) = n\, z(g) + g \cdot W'.$$

Hence $z(g) = g \cdot (-n^{-1}W') - (-n^{-1}W')$, and $Y = Y_H - n^{-1}W'$ satisfies $y(g) \otimes 1 = g \cdot Y - Y$ for all $g \in D_\ell$. □

Lemma 43.23. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, emb_ℓ as in Proposition 43.15, and $E \subseteq \overline{\mathbb{Q}}_\ell$ a finite extension of $\mathbb{Q}_\ell$ containing $\mathcal{O}$. Let D_ℓ act on $\mathrm{M}_3(E)$ through r_0 (if $r_0(g) = (P, \varsigma)$: $X \mapsto PXP^{-1}$ for $\varsigma = 1$, $X \mapsto -PJ\,{}^tXJP^{-1}$ for $\varsigma = -1$) and on $\mathrm{M}_3(E) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ diagonally, through emb_ℓ on B_{cris}; let $\mathfrak{g}_j(E)$ be the matrices supported in the positions of weight j. For $j \in \mathbb{Z}$ let Z_j be the E-vector space of continuous 1-cocycles $y\colon D_\ell \to \mathfrak{g}_j(E)$ for which there is $Y \in \mathrm{M}_3(E) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $y(g) \otimes 1 = g \cdot Y - Y$ for all $g \in D_\ell$. Then $\dim_E Z_j$ is $1, 2, 3, 4, 2$ for $j = -2, -1, 0, 1, 2$ respectively, and 0 for $|j| > 2$.*

Proof. For $|j| > 2$, $\mathfrak{g}_j(E) = 0$ and $Z_j = 0$. Let $|j| \le 2$ and $V = \mathfrak{g}_j(E)$, of E-dimension $3 - |j|$ and $\mathbb{Q}_\ell$-dimension $m = (3 - |j|)[E : \mathbb{Q}_\ell]$; it is D_ℓ-stable (P is diagonal, and $J\,{}^tXJ$ has entry (a, b) equal to $X_{4-b,4-a}$, of the same weight), and the projection $p_j\colon \mathrm{M}_3(E) \to V$ to the entries of weight j is D_ℓ-equivariant and E-linear.

Step 1: Z_j is the preimage of H^1_f. If $y \in Z_j$ with Y, then $(p_j \otimes 1)(Y) \in V \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ serves as well; so Z_j is the set of continuous cocycles $D_\ell \to V$ whose class lies in $H^1_f(D_\ell, V)$. It contains the coboundaries ($Y = X \otimes 1$), so $\dim_{\mathbb{Q}_\ell} Z_j = \dim B^1(D_\ell, V) + \dim H^1_f(D_\ell, V) = m - \dim H^0(D_\ell, V) + \dim H^1_f(D_\ell, V)$.

Step 2: V is de Rham, and $\dim_{\mathbb{Q}_\ell} D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)$ is m if $j \ge 1$ and 0 if $j \le 0$. Let $H'' = \{g \in D_\ell : \eta_K(g) = 1,\ \chi(g) = 1\}$, an open normal subgroup (η_K and χ have finite image), $H'' = G_{F''}$ with $F''/\mathbb{Q}_\ell$ finite Galois. On H'', $r_0(g) = (\mathrm{T}(\chi_{\mathrm{cyc}}(g)^e), 1)$, so g acts on V by the scalar $\chi_{\mathrm{cyc}}(g)^n$, $n = ej$: as a representation of H'', $V \cong \mathbb{Q}_\ell(n)^m$. An element $b \otimes 1$ of $B_{\mathrm{dR}} \otimes \mathbb{Q}_\ell(n)$ is H''-invariant if and only if $g(b) = \chi_{\mathrm{cyc}}(g)^{-n} b$, i.e. if and only if $bt^n \in B_{\mathrm{dR}}^{H''} = F''$ (t is a unit of the field B_{dR} with $g(t) = \chi_{\mathrm{cyc}}(g)t$); so the invariants are $F''t^{-n}$, of dimension 1 over F'', and $\mathbb{Q}_\ell(n)$ is de Rham over F''. Since t generates the maximal ideal of $B_{\mathrm{dR}}^+ = \mathrm{Fil}^0 B_{\mathrm{dR}}$ and $F'' \subseteq (B_{\mathrm{dR}}^+)^\times$, $at^{-n} \in \mathrm{Fil}^0$ for $a \in F'' \setminus \{0\}$ if and only if $n \le 0$; so Fil^0 of the invariants is everything if $n \le 0$ and 0 if $n \ge 1$. By Galois descent, V is de Rham as a representation of D_ℓ, $\dim_{\mathbb{Q}_\ell} D_{\mathrm{dR}}(V) = m$, and $\dim_{\mathbb{Q}_\ell} D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 = \dim_{F''}$ of the corresponding quotient over F'', which is m if $n \ge 1$ and 0 if $n \le 0$. As $e \ge 1$, $n \ge 1$ if and only if $j \ge 1$.

Step 3: the count. By Theorem B.43 for $D_\ell \cong G_{\mathbb{Q}_\ell}$ and V, $\dim_{\mathbb{Q}_\ell} H^1_f(D_\ell, V) = \dim H^0(D_\ell, V) + \dim D_{\mathrm{dR}}(V)/\mathrm{Fil}^0$. With Steps 1 and 2, $\dim_{\mathbb{Q}_\ell} Z_j = m$ for $j \le 0$ and $2m$ for $j \ge 1$. Dividing by $[E : \mathbb{Q}_\ell]$, $\dim_E Z_j = 3 - |j|$ for $-2 \le j \le 0$ and $2(3 - j)$ for $j = 1, 2$: the values $1, 2, 3, 4, 2$. □

Proposition 43.24. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$*

for a coefficient ring $\mathcal{O}$. The localisation of R_ℓ^{ord} at the prime ideal $\ker\beta_\ell$ has Krull dimension at least 12.

Proof. Write $R = R_\ell^{\square}$, $\mathfrak{J} = I^{\mathrm{ord}}(r_\ell^{\square})$, $\gamma = \mathrm{gd}(r_\ell^{\square})$, $\mathfrak{q} = \ker\beta_\ell \subset R$, $E = E_{\mathcal{O}}$, ϖ a uniformiser of $\mathcal{O}$, and $E_k = E[[T]]/\mathfrak{m}_T^k$ ($k \geq 1$), a finite-dimensional local E-algebra with its natural topology. Let r_T be as in Lemma 43.25.

Step 1: a ring homomorphism $F\colon R \to E[[T]]$ with $F(\underline{r_\ell^{\square}}(g)_{ab}) = \underline{r_T}(g)_{ab}$ whose constant term is β_ℓ. Fix $k \geq 1$ and let $r_k\colon D_\ell \to \mathcal{G}_3(E_k)$ be the reduction of r_T; its entries are continuous by (Fa). Let $S_k \subset \mathfrak{m}_T E_k$ be the set of entries of $\underline{r_k}(g)\underline{r_0}(g)^{-1} - 1$, $g \in D_\ell$ (in $\mathfrak{m}_T E_k$ by (Fa)); it is compact, hence bounded. Products of k elements of $\mathfrak{m}_T E_k$ vanish. Let A_k^+ be the $\mathcal{O}$-span of the products $s_1 \cdots s_j$ with $1 \leq j < k$ and $s_i \in S_k$; these products form bounded sets, and the $\mathcal{O}$-span of a bounded subset of a finite-dimensional E-vector space lies in a finitely generated $\mathcal{O}$-module, so A_k^+ is a finitely generated $\mathcal{O}$-module, closed under multiplication and nilpotent. Put $A_k = \mathcal{O} + A_k^+$, an $\mathcal{O}$-subalgebra of E_k, finitely generated and torsion free over $\mathcal{O}$. As $\mathcal{O} \cap \mathfrak{m}_T E_k = 0$, $A_k/A_k^+ \cong \mathcal{O}$, so A_k is local with maximal ideal $\mathfrak{m}_{\mathcal{O}} + A_k^+$ and residue field $\mathbb{F}_{\mathcal{O}}$; being finite over $\mathcal{O}$ it is Noetherian and complete, and $A_k \in \mathrm{CNL}_{\mathcal{O}}$. Since $(A_k^+)^k = 0$, $(\mathfrak{m}_{\mathcal{O}} + A_k^+)^m \subseteq \varpi^{m-k}A_k$ for $m \geq k$, so its maximal-ideal-adic topology is the ϖ-adic one, which is the topology of A_k as a lattice in its E-span, i.e. the subspace topology from E_k. The entries of $\underline{r_k}(g) = (1+s)\underline{r_0}(g)$ lie in A_k ($\underline{r_0}$ has entries in $\mathcal{O}$), and its determinant lies in $\mathcal{O}^\times + A_k^+ \subseteq A_k^\times$; so r_k is a homomorphism $D_\ell \to \mathcal{G}_3(A_k)$ with continuous entries and reduction $\bar{r}_0|_{D_\ell}$, i.e. a lift of $\bar{r}_0|_{D_\ell}$ to A_k. Let $x_k\colon R \to A_k$ be the unique morphism with $x_k \circ r_\ell^{\square} = r_k$ and $F_k\colon R \to E_k$ its composite with the inclusion. The projection $\pi_k\colon E_{k+1} \to E_k$ maps S_{k+1} onto S_k, and products of k elements vanish in E_k, so $\pi_k(A_{k+1}) = A_k$ and $\pi_k\colon A_{k+1} \to A_k$ is a morphism of $\mathrm{CNL}_{\mathcal{O}}$; $\pi_k \circ x_{k+1}$ classifies $\pi_k \circ r_{k+1} = r_k$, so $\pi_k \circ x_{k+1} = x_k$ by uniqueness. Hence the F_k define an $\mathcal{O}$-algebra homomorphism $F\colon R \to \varprojlim_k E_k = E[[T]]$ with $F(\underline{r_\ell^{\square}}(g)_{ab}) = \underline{r_T}(g)_{ab}$. For $k = 1$, $E_1 = E$, $r_1 = r_0|_{D_\ell}$ and $A_1 = \mathcal{O}$, so x_1 is a base point of $r_\ell^{\square}$, which is β_ℓ by (Ol1): the constant term of $F(y)$ is $\beta_\ell(y)$.

Step 2: F factors through R_ℓ^{ord}. The generators of $\mathfrak{J}$ are polynomials with coefficients in $\mathcal{O}$ in the entries of $\underline{r_\ell^{\square}}(g)$, $g \in D_\ell^\circ \cap I_\ell$; F maps them to the same polynomials in the entries of $\underline{r_T}(g)$, which vanish by (Fb). So $F(\mathfrak{J}) = 0$; likewise $F(\gamma) = \mathrm{gd}(r_T) \neq 0$ by (Fc). As $E[[T]]$ is a domain, $\ker F$ is prime, and $\gamma^m y \in \mathfrak{J}$ implies $F(\gamma)^m F(y) = 0$, $F(y) = 0$; so $\ker F$ contains the saturation of $\mathfrak{J}$ and its radical I^{cr}. By Step 1, $\ker F \subseteq \mathfrak{q}$. As R is Noetherian, $\ker F$ contains a minimal prime $\mathfrak{P}$ of I^{cr}; $\mathfrak{P} \subseteq \mathfrak{q}$, the kernel of the only base point (Ol1), so $\mathfrak{P}$ is one of the primes intersected in I_ℓ^{ord}, and $I_\ell^{\mathrm{ord}} \subseteq \mathfrak{P} \subseteq \ker F$. So $F = \bar{F} \circ (R \to R_\ell^{\mathrm{ord}})$ for an $\mathcal{O}$-algebra homomorphism $\bar{F}$.

Step 3: twelve coordinates. Let $\mathrm{L}\colon E[[T]] \to E^{12}$ give the coefficients of $T_1, \ldots, T_{12}$. For $g \in D_\ell$ and $1 \leq a, b \leq 3$, $x_{g,a,b} = \underline{r_\ell^{\square}}(g)_{ab} - \underline{r_0}(g)_{ab}$ lies in $\mathfrak{q}$ ($\beta_\ell \circ r_\ell^{\square} = r_0|_{D_\ell}$), and $\mathrm{L}(F(x_{g,a,b}))$ is the vector of (a,b) entries of $\underline{y_i}(g)\underline{r_0}(g)$, $i = 1, \ldots, 12$. If these vectors spanned a proper subspace, some $(\mu_i) \neq 0$ would satisfy $\sum_i \mu_i y_i(g)\underline{r_0}(g) = 0$ for all g, so $\sum_i \mu_i y_i = 0$, against (Fd). Choose $z_1, \ldots, z_{12}$ among the $x_{g,a,b}$ with $\mathrm{L}(F(z_1)), \ldots, \mathrm{L}(F(z_{12}))$ a basis of E^{12}.

Step 4: the dimension. Let $A = R_\ell^{\mathrm{ord}}$, $\mathfrak{q}_A = \ker\beta_\ell \subset A$ and $\bar{z}_i$ the image of z_i. For $s \in A \setminus \mathfrak{q}_A$, $\bar{F}(s)$ has constant term $\beta_\ell(s) \neq 0$ and is a unit; so $\bar{F}$ extends to $\Phi\colon A_{\mathfrak{q}_A} \to E[[T]]$, with $\Phi(\mathfrak{q}_A A_{\mathfrak{q}_A}) \subseteq \mathfrak{m}_T$ and hence $\Phi((\mathfrak{q}_A A_{\mathfrak{q}_A})^N) \subseteq \mathfrak{m}_T^N$. $A_{\mathfrak{q}_A}$ is a Noetherian local ring containing E (β_ℓ is the identity on $\mathcal{O}$, so non-zero elements of $\mathcal{O}$ are units). Let $\widehat{A}$ be its completion, with maximal ideal $\widehat{\mathfrak{n}}$; the maps $A_{\mathfrak{q}_A}/(\mathfrak{q}_A A_{\mathfrak{q}_A})^N \to E[[T]]/\mathfrak{m}_T^N$ induced by Φ define an E-algebra homomorphism $\widehat{\Phi}\colon \widehat{A} \to E[[T]]$ with $\widehat{\Phi}(\widehat{\mathfrak{n}}^N) \subseteq \mathfrak{m}_T^N$. Let $\iota\colon E[[Y_1, \ldots, Y_{12}]] \to \widehat{A}$ be the continuous E-algebra homomorphism with $\iota(Y_i) = \bar{z}_i \in \widehat{\mathfrak{n}}$; as $\widehat{\mathfrak{n}}^N$ is closed, ι maps $(Y)^N$ into $\widehat{\mathfrak{n}}^N$. So $\psi = \widehat{\Phi} \circ \iota$ maps $(Y)^N$ into $\mathfrak{m}_T^N$ and agrees with the substitution $Y_i \mapsto F(z_i)$ on polynomials; as every power

series differs from a polynomial by an element of $(Y)^N$, for every N, ψ is that substitution. Its matrix of linear parts, with rows $\mathrm{L}(F(z_i))$, is invertible (Step 3), so ψ is an automorphism and $\widehat{\Phi}$ is onto. Hence $E[[T]]$ is a quotient of $\widehat{A}$, $\dim \widehat{A} \geq 12$, and $\dim A_{\mathfrak{q}_A} = \dim \widehat{A} \geq 12$. $\square$

Lemma 43.25. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$ with fraction field $E_\mathcal{O}$, and let $E_\mathcal{O}[[T]] = E_\mathcal{O}[[T_1, \dots, T_{12}]]$, with maximal ideal $\mathfrak{m}_T$. There is a homomorphism $r_T \colon D_\ell \to \mathcal{G}_3(E_\mathcal{O}[[T]])$ such that:*

(Fa) *the sign of $r_T(g)$ is $\eta_K(g)$; every coefficient of every entry of $\underline{r_T}(g)$ is a continuous function of $g \in D_\ell$; and $\underline{r_T}(g) - \underline{r_0}(g)$ has its entries in $\mathfrak{m}_T$ for every $g \in D_\ell$;*

(Fb) *for all $g, g_1, g_2, g_3 \in D_\ell^\circ \cap I_\ell$, $\det\bigl(X - \underline{r_T}(g)\bigr) = \prod_{a=1}^{3}(X - \psi_a(g))$ and*

$$\bigl(\underline{r_T}(g_1) - \psi_1(g_1)\bigr)\bigl(\underline{r_T}(g_2) - \psi_2(g_2)\bigr)\bigl(\underline{r_T}(g_3) - \psi_3(g_3)\bigr) = 0;$$

(Fc) $\mathrm{gd}(r_T) \neq 0$ *in $E_\mathcal{O}[[T]]$;*

(Fd) *the maps $y_1, \dots, y_{12} \colon D_\ell \to \mathrm{M}_3(E_\mathcal{O})$, where $y_i(g)$ is the coefficient of T_i in $\underline{r_T}(g)\,\underline{r_0}(g)^{-1}$, are linearly independent over $E_\mathcal{O}$.*

Proof. Write $E = E_\mathcal{O}$, a finite extension of $\mathbb{Q}_\ell$ inside $\overline{\mathbb{Q}}_\ell$ containing $\mathcal{O}$, and $\eta = \eta_K|_{D_\ell}$. Through emb_ℓ, $D_\ell = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ with inertia I_ℓ. A map to $E[[T]]$ (or to matrices over it) is called continuous if all its coefficients are; sums, products, and inverses of matrices with unit determinant (adjugate over determinant; the inverse of a unit power series has coefficients polynomial in its coefficients and the inverse of its constant term) of continuous maps are continuous. Case (S): $\eta = 1$; case (N): $\eta \neq 1$. Let $\mathrm{fr} \colon D_\ell \to \widehat{\mathbb{Z}} \to \mathbb{Z}_\ell$ be the continuous homomorphism, trivial on I_ℓ, with $\mathrm{fr}(\mathrm{Fr}) = 1$; then $\mathrm{fr}(\sigma_\ell) = n$. Put $\lambda_i(g) = (1+T_i)^{\mathrm{fr}(g)}$, a continuous character $D_\ell \to E[[T]]^\times$, trivial on I_ℓ, with $\lambda_i(g) \equiv 1 + \mathrm{fr}(g) T_i$ modulo $\mathfrak{m}_T^2$. On $D_\ell^\circ \cap I_\ell$ the prime-to-ℓ part of χ is trivial (it is unramified at ℓ) and so is χ_ℓ; hence $\Psi = \chi\chi_{\mathrm{cyc}}^e = \chi_{\mathrm{cyc}}^e$ there, the $u_a = \psi_a(\sigma_0) = \chi_{\mathrm{cyc}}(\sigma_0)^{e(2-a)}$ are pairwise distinct, $\mathrm{gp}_a \neq 0$, and $\Psi(\sigma_0)^j \neq 1$ for $j \neq 0$.

Step 1: a diagonal family. Define $D(g)$ by $\mathrm{diag}(\psi_1\lambda_1\lambda_3,\ \psi_2\lambda_3,\ \psi_3\lambda_2^{-1}\lambda_3)(g)$ in case (S) and $\mathrm{T}(\Psi(g)\lambda_1(g))$ in case (N), with diagonal entries $\kappa_1, \kappa_2, \kappa_3$. Then $d(g) = (D(g), \eta(g))$ is a homomorphism $D_\ell \to \mathcal{G}_3(E[[T]])$ (in case (S) the signs are 1; in case (N), $\mathrm{T}(t)\mathrm{T}(t') = \mathrm{T}(tt')$ and $\theta(\mathrm{T}(t')) = \mathrm{T}(t')$). The κ_a are continuous characters with $\kappa_a = \psi_a$ on I_ℓ, $D(g) \equiv \mathrm{T}(\Psi(g)) = \underline{r_0}(g)$ modulo $\mathfrak{m}_T$, and $\kappa_1\kappa_2^{-1} = \Psi\lambda_1$, $\kappa_2\kappa_3^{-1} = \Psi\lambda_2$ in case (S), $\kappa_1\kappa_2^{-1} = \kappa_2\kappa_3^{-1} = \Psi\lambda_1$ in case (N).

Step 2: upper triangular elements. Put $\mathrm{n}(x, y, z) = 1 + xE_{12} + yE_{23} + zE_{13}$ (E_{ab} the matrix units) for $x, y, z \in E[[T]]$, and $\mathrm{n}_2(z) = (\mathrm{n}(0, 0, z), 1) \in \mathcal{G}_3(E[[T]])$.

For $g \in D_\ell$ let $\alpha_g(u) = D(g)\theta^{\eta(g)}(u)D(g)^{-1}$ ($\theta^1 = \mathrm{id}$, $\theta^{-1} = \theta$) for unipotent upper triangular u. As $D(g)\theta^{\eta(g)}(D(h)) = D(gh)$,

$$(uD(g), \eta(g))\,(u'D(h), \eta(h)) = \bigl(u\,\alpha_g(u')\,D(gh),\ \eta(gh)\bigr). \tag{43.6}$$

By the side calculation,

$$\alpha_g(\mathrm{n}(x, y, z)) = \begin{cases} \mathrm{n}(\kappa_{12}x, \kappa_{23}y, \kappa_{13}z) & \text{if } \eta(g) = 1, \\ \mathrm{n}(-\kappa' y, -\kappa' x, \kappa_{13}(xy - z)) & \text{if } \eta(g) = -1 \text{ (case (N))}, \end{cases}$$

where $\kappa_{ab} = \kappa_a\kappa_b^{-1}(g)$ and $\kappa' = \Psi\lambda_1(g)$. So the first two coordinates of $\alpha_g(\mathrm{n}(x, y, z))$ are $A_g(x, y)$, with $A_g(x, y) = (\Psi\lambda_1(g)x, \Psi\lambda_2(g)y)$ in case (S) and $A_g = \Psi\lambda_1(g)\, s_g$ in case (N), $s_g = \mathrm{id}$ if $\eta(g) = 1$ and $s_g(x, y) = (-y, -x)$ otherwise; and $\alpha_g(\mathrm{n}(0, 0, z)) = \mathrm{n}(0, 0, \chi_2(g)z)$

with $\chi_2 = \eta\kappa_1\kappa_3^{-1}$. $g \mapsto A_g$ and $g \mapsto \chi_2(g)$ are continuous actions on $E[[T]]^2$ and $E[[T]]$ (s is a homomorphism, $s_g^2 = \mathrm{id}$). Identifying $\mathfrak{g}_1(E)$ with E^2 by $xE_{12} + yE_{23} \leftrightarrow (x, y)$ and $\mathfrak{g}_2(E)$ with E by $zE_{13} \leftrightarrow z$, the action of Lemma 43.23 through r_0 is A_g, resp. $\chi_2(g)$, at $T = 0$: for sign 1 conjugation by $\mathrm{T}(\Psi(g))$ multiplies weight j entries by $\Psi(g)^j$; for sign -1, $-J\,{}^t(xE_{12} + yE_{23})J = -yE_{12} - xE_{23}$ and $-J\,{}^t(zE_{13})J = -zE_{13}$.

Step 3: base cocycles. For $j = 1, 2$ the projection $\mathrm{M}_3(E) \to \mathfrak{g}_j(E)$ is equivariant, so a cocycle in Z_j has a primitive in $\mathfrak{g}_j(E) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ and its class lies in $H^1_f(D_\ell, \mathfrak{g}_j(E))$. The coboundaries $B_j = \{g \mapsto X - g \cdot X : X \in \mathfrak{g}_j(E)\}$ lie in Z_j (primitive $X \otimes 1$), and $X \mapsto (g \mapsto X - g \cdot X)$ is injective, since $\sigma_0 \cdot X = \Psi(\sigma_0)^j X$. By Lemma 43.23, $\dim Z_1 = 4$, $\dim B_1 = 2$, $\dim Z_2 = 2$, $\dim B_2 = 1$. Choose $c_0'' \in Z_2 \setminus B_2$ and $c_0^{(1)}, c_0^{(2)} \in Z_1$ linearly independent modulo B_1; in case (S) choose them with $c_0^{(1)}$ valued in $E \times 0$ and $c_0^{(2)}$ in $0 \times E$: there the two coordinate projections of E^2 are equivariant ($A_g = \Psi(g)$ at $T = 0$), so Z_1 and B_1 split into the cocycles valued in $E \times 0$ and in $0 \times E$, the coordinate swap is an equivariant automorphism exchanging the two parts, and each part of Z_1 has dimension 2 and contains the corresponding one-dimensional part of B_1.

Step 4: a criterion for the characters met. Let $\varphi\colon D_\ell \to E^\times$ be a continuous character and $E(\varphi)$ the $\mathbb{Q}_\ell$-representation E with g acting by $\varphi(g)$. If $\varphi = \chi_{\mathrm{cyc}}$ on D_ℓ, a $\mathbb{Q}_\ell$-basis of E gives $E(\varphi) \cong \mathbb{Q}_\ell(1)^{[E:\mathbb{Q}_\ell]}$. Otherwise $H^2(D_\ell, E(\varphi)) = 0$: by Theorem B.36 its dimension is that of the invariant linear forms $f\colon E \to \mathbb{Q}_\ell$ with $f(x) = \chi_{\mathrm{cyc}}(g) f(\varphi(g)^{-1}x)$; for g with $\beta = \chi_{\mathrm{cyc}}(g)\varphi(g)^{-1} \neq 1$ this gives $f((\beta - 1)x) = 0$ for all x, so $f = 0$.

Step 5: the families of cocycles. Weight 1, case (S): apply Lemma 43.26 to $V_0 = E(\Psi)$, the variable T_1 and the first coordinate of $c_0^{(1)}$ (its class is finite, the projection being equivariant); by Step 4, $V' = V_0$, $V'' = 0$ if $\Psi = \chi_{\mathrm{cyc}}$ on D_ℓ, and $V' = 0$, $V'' = V_0$ otherwise. This gives $c^{(1)}\colon D_\ell \to E[[T_1]]$ with $c^{(1)}(gh) = c^{(1)}(g) + \Psi\lambda_1(g)c^{(1)}(h)$ and constant term that coordinate; likewise $c^{(2)}$ with T_2, $\Psi\lambda_2$ and the second coordinate of $c_0^{(2)}$. Put $a(g) = (T_4 c^{(1)}(g), T_5 c^{(2)}(g))$. Case (N): $E^2 = E(1,-1) \oplus E(1,1)$ with g acting at $T = 0$ by $\Psi(g)$ and $\Psi(g)\eta(g)$, since $s_g(1,-1) = (1,-1)$ and $s_g(1,1) = \eta(g)(1,1)$; by Step 4 each summand is a sum of copies of $\mathbb{Q}_\ell(1)$ or has vanishing H^2, and not both are of the first kind ($\Psi = \chi_{\mathrm{cyc}} = \Psi\eta$ would force $\eta = 1$). As $A_g = \lambda_1(g) \cdot A_g|_{T=0}$, Lemma 43.26 with the variable T_1 gives cocycles $c^{(i)}\colon D_\ell \to E[[T_1]]^2$ of A with constant terms $c_0^{(i)}$; put $a = T_4 c^{(1)} + T_5 c^{(2)}$. In both cases a is continuous, $a(gh) = a(g) + A_g a(h)$, a has values in $\mathfrak{m}_T E[[T]]^2$, and $a \equiv T_4 c_0^{(1)} + T_5 c_0^{(2)}$ modulo $\mathfrak{m}_T^2$. Weight 2: $\chi_2 \equiv \eta\Psi^2$ modulo $\mathfrak{m}_T$, and $\eta\Psi^2(\sigma_0) = \chi_{\mathrm{cyc}}(\sigma_0)^{2e} \neq \chi_{\mathrm{cyc}}(\sigma_0)$ ($2e \neq 1$), so $H^2(D_\ell, E(\eta\Psi^2)) = 0$ by Step 4, and Lemma 43.27 (ii) gives a continuous $c''\colon D_\ell \to E[[T]]$ with $c''(gh) = c''(g) + \chi_2(g)c''(h)$ and constant term c_0''.

Step 6: the triangular homomorphism. Write $a = (a_1, a_2)$ and

$$\tilde{\rho}(g) = (\mathrm{n}(a_1(g), a_2(g), 0) D(g), \eta(g)).$$

As $\mathrm{n}(x, y, z) \mapsto (x, y)$ is a homomorphism with central kernel $\{\mathrm{n}(0, 0, z)\}$, (43.6) and the cocycle identity of a give $\tilde{\rho}(g)\tilde{\rho}(h) = \mathrm{n}_2(o(g,h))\tilde{\rho}(gh)$, where $o(g,h)$ is the $(1,3)$ coordinate of $\mathrm{n}(a(g), 0)\alpha_g(\mathrm{n}(a(h), 0))$ (by the side calculation, $\mathrm{n}(X,Y,Z)\mathrm{n}(X,Y,0)^{-1} = \mathrm{n}(0,0,Z)$): explicitly $o(g,h) = a_1(g)\kappa_{23}a_2(h)$ if $\eta(g) = 1$ and $o(g,h) = \kappa_{13}a_1(h)a_2(h) - a_1(g)\kappa' a_1(h)$ otherwise. So o is continuous with values in $\mathfrak{m}_T^2$. By (43.6) with $u' = \mathrm{n}(0,0,z)$, $\tilde{\rho}(g)\mathrm{n}_2(z) = \mathrm{n}_2(\chi_2(g)z)\tilde{\rho}(g)$. Comparing $(\tilde{\rho}(g)\tilde{\rho}(h))\tilde{\rho}(k) = \mathrm{n}_2(o(g,h) + o(gh,k))\tilde{\rho}(ghk)$ with $\tilde{\rho}(g)(\tilde{\rho}(h)\tilde{\rho}(k)) = \mathrm{n}_2(\chi_2(g)o(h,k) + o(g,hk))\tilde{\rho}(ghk)$ shows that o is a 2-cocycle for χ_2. By Lemma 43.27 (i) (with $V_0 = E(\eta\Psi^2)$ and $j = 2$) there is a continuous $b\colon D_\ell \to \mathfrak{m}_T^2$ with $o(g,h) = \chi_2(g)b(h) - b(gh) + b(g)$. Put $t(g) = \mathrm{n}_2(T_6 c''(g) - b(g))\tilde{\rho}(g)$. Then

$$t(g)t(h) = \mathrm{n}_2(T_6 c''(g) - b(g) + \chi_2(g)(T_6 c''(h) - b(h)) + o(g,h))\tilde{\rho}(gh) = t(gh),$$

so t is a homomorphism, with sign $\eta(g)$ and GL-part $\mathrm{n}(a_1(g), a_2(g), T_6c''(g) - b(g))D(g)$, upper triangular with diagonal $(\kappa_a(g))_a$.

Step 7: conjugation; (Fa) and (Fb). Let $h = (1 + T_7E_{21} + T_8E_{32} + T_9E_{31})\,\Delta\,\mathrm{n}(T_{10}, T_{11}, T_{12})$ with $\Delta = 1$ in case (S) and $\Delta = \mathrm{diag}(1 + T_2, 1 + T_3, 1)$ in case (N); $h \in \mathrm{GL}_3(E[[T]])$ and $h \equiv 1$ modulo $\mathfrak{m}_T$. Put $r_T(g) = (h, 1)t(g)(h, 1)^{-1}$, a homomorphism with sign $\eta(g)$ and GL-part $h\,\underline{t}(g)\,\theta^{\eta(g)}(h)^{-1}$. (Fa): continuity follows from that of a, b, c'', D; modulo $\mathfrak{m}_T$, a, b, T_6c'' vanish, $D(g) \equiv \underline{r_0}(g)$ and $h \equiv 1$. (Fb): for $g \in D_\ell^\circ \cap I_\ell$, $\eta(g) = 1$ and $\underline{r_T}(g) = hU(g)h^{-1}$ with $U(g)$ upper triangular with diagonal $(\psi_a(g))_a$ (λ_i trivial on I_ℓ); this gives the characteristic polynomial, and the product equals $h\prod_k(U(g_k) - \psi_k(g_k))h^{-1}$, where the k-th factor is upper triangular with (k, k) entry 0: the third maps $E[[T]]^3$ into $\langle e_1, e_2\rangle$, the second maps $\langle e_1, e_2\rangle$ into $\langle e_1\rangle$, the first kills e_1, so the product is 0.

Step 8: (Fc). $\sigma_0, \sigma_\ell \in D_\ell^\circ$ have sign 1, so $\underline{r_T}(\sigma_0) = hU_0h^{-1}$, $\underline{r_T}(\sigma_\ell) = hU_\ell h^{-1}$ with U_0, U_ℓ upper triangular with diagonals $(u_b)_b$ and $(\kappa_b(\sigma_\ell))_b$. The trace being invariant under conjugation and the diagonal of a product of upper triangular matrices being the product of the diagonals,

$$\mathrm{tq}_a(r_T) = \sum_b (u_b - u_{a+1})(u_b - u_{a+2})\kappa_b(\sigma_\ell) = \mathrm{gp}_a\kappa_a(\sigma_\ell),$$
$$\mathrm{dt}_a(r_T) = \mathrm{gp}_a\mathrm{gp}_{a+1}\kappa_{a+1}(\sigma_\ell)\big(\kappa_a\kappa_{a+1}^{-1}(\sigma_\ell) - \chi_{\mathrm{cyc}}(\sigma_\ell)\big).$$

For $a = 1, 2$, $\kappa_a\kappa_{a+1}^{-1}(\sigma_\ell) = \Psi(\sigma_\ell)(1 + T_i)^n$ with $i = a$ in case (S) and $i = 1$ in case (N) (Step 1), and $\Psi(\sigma_\ell)(1 + T_i)^n - \chi_{\mathrm{cyc}}(\sigma_\ell)$ has coefficient $n\Psi(\sigma_\ell) \neq 0$ at T_i. As $\mathrm{gp}_a \neq 0$, the $\kappa_b(\sigma_\ell)$ are units and $E[[T]]$ is a domain, $\mathrm{gd}(r_T) = \ell\,\mathrm{dt}_1(r_T)\mathrm{dt}_2(r_T) \neq 0$.

Step 9: (Fd). Write $P_0 = \underline{r_0}(g)$ and $H_1 = \sum_i T_iX_i$ for the part of degree one of h: $X_7 = E_{21}$, $X_8 = E_{32}$, $X_9 = E_{31}$, $X_{10} = E_{12}$, $X_{11} = E_{23}$, $X_{12} = E_{13}$, and in case (N) also $X_2 = E_{11}$, $X_3 = E_{22}$. Modulo $\mathfrak{m}_T^2$: $\underline{t}(g)P_0^{-1} \equiv 1 + \Delta_1(g) + T_4c_0^{(1)}(g) + T_5c_0^{(2)}(g) + T_6c_0''(g)$, read in $\mathrm{M}_3(E)$ through the identifications of Step 2, where $\Delta_1(g) = \mathrm{fr}(g)(T_1E_{11} - T_2E_{33} + T_3 1)$ in case (S) and $\mathrm{fr}(g)T_1(E_{11} - E_{33})$ in case (N) (as $b \in \mathfrak{m}_T^2$, $a \equiv T_4c_0^{(1)} + T_5c_0^{(2)}$, $c'' \equiv c_0''$ modulo $\mathfrak{m}_T$, and $D(g)P_0^{-1}$ is the diagonal matrix of the λ-factors); $h \equiv 1 + H_1$; and $P_0\theta^{\eta(g)}(h)^{-1}P_0^{-1} \equiv 1 - g \cdot H_1$, since $h^{-1} \equiv 1 - H_1$ and $\theta(h)^{-1} = \theta(h^{-1}) \equiv 1 + J\,{}^tH_1J$. Hence $\underline{r_T}(g)P_0^{-1} \equiv 1 + \Delta_1(g) + T_4c_0^{(1)}(g) + T_5c_0^{(2)}(g) + T_6c_0''(g) + H_1 - g \cdot H_1$, and

- $y_i(g) = X_i - g \cdot X_i$ for $i \in \{7, \dots, 12\}$, and for $i \in \{2, 3\}$ in case (N);
- $y_1 = \mathrm{fr}\,E_{11}$, $y_2 = -\mathrm{fr}\,E_{33}$, $y_3 = \mathrm{fr} \cdot 1$ in case (S); $y_1 = \mathrm{fr}\,(E_{11} - E_{33})$ in case (N);
- $y_4 = c_0^{(1)}$, $y_5 = c_0^{(2)}$ (weight 1), $y_6 = c_0''E_{13}$ (weight 2).

Each y_i takes values in one $\mathfrak{g}_j(E)$, so a linear relation splits by weight. Weight -2: $y_9(\sigma_0) = (1 - \Psi(\sigma_0)^{-2})E_{31} \neq 0$. Weight -1: $y_7(\sigma_0), y_8(\sigma_0)$ are $(1 - \Psi(\sigma_0)^{-1})E_{21}$, $(1 - \Psi(\sigma_0)^{-1})E_{32}$, independent. Weight 1: if $\mu_4y_4 + \mu_5y_5 + \mu_{10}y_{10} + \mu_{11}y_{11} = 0$, then $\mu_4c_0^{(1)} + \mu_5c_0^{(2)} \in B_1$, so $\mu_4 = \mu_5 = 0$, and evaluating at σ_0 gives $(1 - \Psi(\sigma_0))(\mu_{10}E_{12} + \mu_{11}E_{23}) = 0$. Weight 2: $\mu_6c_0'' \in B_2$ forces $\mu_6 = 0$, then $\mu_{12} = 0$ as for weight 1. Weight 0, case (S): $\mathrm{fr}(\mathrm{Fr}) = 1$, and $E_{11}, -E_{33}, 1$ are independent. Weight 0, case (N): elements of sign 1 fix diagonal matrices, so y_2, y_3 vanish on $\ker\eta$ while $y_1(\mathrm{Fr}^2) = 2(E_{11} - E_{33})$ with $\mathrm{Fr}^2 \in \ker\eta$, forcing the coefficient of y_1 to vanish; for g of sign -1, $g \cdot X = -JXJ$ for diagonal X, so $y_2(g) = E_{11} + E_{33}$ and $y_3(g) = 2E_{22}$, independent. Hence $y_1, \dots, y_{12}$ are linearly independent. □

Lemma 43.26. *Let ℓ be a prime, $G = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ with inertia subgroup I, and $\mathrm{fr}\colon G \to \mathbb{Z}_\ell$ a continuous homomorphism which is trivial on I and not identically zero. For $a \in \mathbb{Z}_\ell$ put $(1 + T)^a = \sum_{k\geq 0}\binom{a}{k}T^k \in \mathbb{Z}_\ell[[T]]$. Let V_0 be a finite-dimensional $\mathbb{Q}_\ell$-vector space with a*

continuous linear action of G, and assume $V_0 = V' \oplus V''$ with G-stable subspaces such that V' is G-isomorphic to a direct sum of copies of $\mathbb{Q}_\ell(1)$ (possibly none) and $H^2(G, V'') = 0$. Let $V_0[[T]]$ be the set of power series $\sum_k x_k T^k$ with $x_k \in V_0$, with G acting by

$$g \cdot \sum_k x_k T^k = (1+T)^{\mathrm{fr}(g)} \sum_k (g x_k) T^k.$$

Then for every continuous 1-cocycle $c_0 \colon G \to V_0$ whose class lies in $H^1_f(G, V_0)$ there is a map $c \colon G \to V_0[[T]]$ with continuous coefficients, satisfying $c(gh) = c(g) + g \cdot c(h)$ for all $g, h \in G$, whose constant term is c_0.

Proof. Since $\binom{a+b}{k} = \sum_i \binom{a}{i}\binom{b}{k-i}$ holds for integers, hence by continuity for $a, b \in \mathbb{Z}_\ell$, $(1+T)^{a+b} = (1+T)^a(1+T)^b$, so the formula defines an action, and $g \mapsto \binom{\mathrm{fr}(g)}{k}$ is continuous. Cochains below are continuous, with $(db)(g,h) = g \cdot b(h) - b(gh) + b(g)$. For $k \geq 1$ let $M_k = V_0[[T]]/T^k V_0[[T]]$, a finite-dimensional representation; as $(1+T)^a \equiv 1$ modulo T, the map $\iota_k \colon V_0 \to T^k M_{k+1}$, $x \mapsto xT^k$, is a G-isomorphism. For $z \in Z^1(G, V_0)$ let $\delta(z) \in H^2(G, V_0)$ be the class of the 2-cocycle $(g,h) \mapsto \mathrm{fr}(g)\, g\, z(h)$; it depends only on the class of z (it is the connecting map of Step 4), and it is compatible with G-maps $V_0 \to V_1$ and with direct sums.

Step 1: the obstruction. Let $k \geq 1$ and $c \in Z^1(G, M_k)$, and let $\tilde{c} \colon G \to M_{k+1}$ be its lift with zero coefficient of T^k. Then $d\tilde{c}$ reduces to $dc = 0$, so it has values in $T^k M_{k+1}$, and $d(d\tilde{c}) = 0$; let $\mathrm{ob}(c) \in H^2(G, V_0)$ be its class through ι_k. If $\mathrm{ob}(c) = 0$, i.e. $d\tilde{c} = d(\iota_k \circ b)$ for some $b \colon G \to V_0$, then $\tilde{c} - \iota_k \circ b \in Z^1(G, M_{k+1})$ lifts c.

Step 2: changing the top coefficient. Let $k \geq 1$, $c \in Z^1(G, M_k)$ and $z \in Z^1(G, V_0)$. Since $T^{k-1} M_k \cong V_0$ equivariantly, $c' = c + zT^{k-1}$ lies in $Z^1(G, M_k)$, its lift with zero top coefficient is $\tilde{c} + zT^{k-1}$, and in M_{k+1}

$$d(zT^{k-1})(g,h) = \big((1+T)^{\mathrm{fr}(g)} g\, z(h) - z(gh) + z(g)\big) T^{k-1} = \mathrm{fr}(g)\, g\, z(h)\, T^k,$$

using $(1+T)^a \equiv 1 + aT$ modulo T^2 and the cocycle identity of z. Hence $\mathrm{ob}(c') = \mathrm{ob}(c) + \delta(z)$. For $k = 1$ and $c = 0$ (whose obstruction is 0) this gives $\mathrm{ob}(z) = \delta(z)$ for $z \in Z^1(G, V_0) = Z^1(G, M_1)$.

Step 3: δ vanishes on finite classes. Fix a G-isomorphism $V' \cong \mathbb{Q}_\ell(1)^a$ and let $p_i \colon V_0 \to \mathbb{Q}_\ell(1)$ $(1 \leq i \leq a)$ be the projection to V' along V'' followed by the i-th coordinate. As cohomology commutes with finite direct sums and $H^2(G, V'') = 0$, $(p_{i*})_i \colon H^2(G, V_0) \to H^2(G, \mathbb{Q}_\ell(1))^a$ is an isomorphism, and $p_{i*}\delta(c_0) = \delta(p_i \circ c_0)$. If $c_0(g) \otimes 1 = gy - y$ with $y \in V_0 \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$, then $(p_i \otimes 1)(y)$ is a primitive of $p_i \circ c_0$; so $x_i = p_i \circ c_0$ has class in $H^1_f(G, \mathbb{Q}_\ell(1))$. The homomorphism fr is a cocycle of the trivial representation $\mathbb{Q}_\ell$, and its class lies in $H^1_f(G, \mathbb{Q}_\ell)$: on $X = \mathbb{Q}_\ell e_0 \oplus \mathbb{Q}_\ell e_1$ with $ge_0 = e_0$, $ge_1 = e_1 + \mathrm{fr}(g) e_0$, an unramified, hence crystalline, representation, the sequence $0 \to D_{\mathrm{cris}}(\mathbb{Q}_\ell e_0) \to D_{\mathrm{cris}}(X) \to D_{\mathrm{cris}}(X/\mathbb{Q}_\ell e_0)$ is exact (left exactness of invariants) and the dimensions $1, 2, 1$ force the last map to be onto, so some invariant has the form $1 \otimes e_1 + b \otimes e_0$, and invariance reads $\mathrm{fr}(g) = g(-b) - (-b)$. Now $\mathbb{Q}_\ell(1)$ is de Rham and $\mathbb{Q}_\ell(1)^*(1)$ is the trivial representation, so Theorem B.49 (with $M = \mathbb{Q}_\ell$) gives $\langle [x_i], [\mathrm{fr}] \rangle = 0$: the 2-cocycle $(g,h) \mapsto \mathrm{fr}(h) x_i(g)$ is a coboundary. For $f(g) = \mathrm{fr}(g) x_i(g)$,

so $(g,h) \mapsto \mathrm{fr}(g) g x_i(h)$ is a coboundary as well, i.e. $\delta(x_i) = 0$. Hence $\delta(c_0) = 0$.

Step 4: δ is onto $H^2(G, V_0)$. With $s_i \colon \mathbb{Q}_\ell(1) \to V_0$ the inclusion of the i-th summand of V', $p_{j*}\delta(s_i \circ z) = \delta(p_j s_i \circ z)$ is $\delta(z)$ for $j = i$ and 0 otherwise; by Step 3's isomorphism it suffices that $\delta \colon H^1(G, \mathbb{Q}_\ell(1)) \to H^2(G, \mathbb{Q}_\ell(1))$ is onto. Take $V_0 = \mathbb{Q}_\ell(1)$ and $W = M_2$, with basis $w_0 = 1$, $w_1 = T$: $gw_0 = \chi_{\mathrm{cyc}}(g)(w_0 + \mathrm{fr}(g) w_1)$, $gw_1 = \chi_{\mathrm{cyc}}(g) w_1$. By Step 2 (lift z with zero top coefficient), δ is the connecting map of $0 \to TM_2 \to W \to \mathbb{Q}_\ell(1) \to 0$, $TM_2 \cong \mathbb{Q}_\ell(1)$ by ι_1; the long exact sequence and $H^3 = 0$ (Theorem B.36) give the exact sequence

$$H^1(G, \mathbb{Q}_\ell(1)) \xrightarrow{\delta} H^2(G, \mathbb{Q}_\ell(1)) \longrightarrow H^2(G, W) \longrightarrow H^2(G, \mathbb{Q}_\ell(1)) \longrightarrow 0.$$

By Theorem B.36, $h^2(\mathbb{Q}_\ell(1)) = h^0(\mathbb{Q}_\ell) = 1$ and $h^2(W) = h^0(W^*(1))$. A linear form φ on W is invariant in $W^*(1)$ iff $\varphi(w) = \chi_{\mathrm{cyc}}(g)\varphi(g^{-1}w)$ for all g, w; as $g^{-1}w_0 = \chi_{\mathrm{cyc}}(g)^{-1}(w_0 - \mathrm{fr}(g)w_1)$ and $g^{-1}w_1 = \chi_{\mathrm{cyc}}(g)^{-1}w_1$, this says $\mathrm{fr}(g)\varphi(w_1) = 0$ for all g, i.e. $\varphi(w_1) = 0$. So $h^2(W) = 1$, the map $H^2(G, W) \to H^2(G, \mathbb{Q}_\ell(1))$ is an onto map of one-dimensional spaces, hence injective, the map before it is zero, and δ is onto.

Step 5: construction. We construct $c^{(k)} \in Z^1(G, M_{k+1})$, $k \geq 1$, with $c^{(1)} \equiv c_0$ modulo T and $c^{(k+1)} \equiv c^{(k)}$ modulo T^k. By Steps 2 and 3, $\mathrm{ob}(c_0) = \delta(c_0) = 0$, so Step 1 gives $c^{(1)}$. Given $c^{(k)}$ ($k \geq 1$), choose by Step 4 a cocycle z with $\delta(z) = -\mathrm{ob}(c^{(k)})$; by Step 2 (with $k+1$ for k), $c' = c^{(k)} + zT^k$ has $\mathrm{ob}(c') = 0$ and $c' \equiv c^{(k)}$ modulo T^k; let $c^{(k+1)}$ be a lift of c' (Step 1). Define c by letting its coefficient of T^j be that of $c^{(k)}$ for any $k \geq j+1$; this does not depend on k, and it is continuous. For every j, c modulo T^{j+1} is the reduction of $c^{(j+1)}$, a cocycle of M_{j+1}; so $c(gh) - c(g) - g \cdot c(h)$ has all coefficients zero, and the constant term of c is that of $c^{(1)}$, namely c_0. □

Lemma 43.27. *Let G be a profinite group, E a finite extension of $\mathbb{Q}_\ell$ for a prime ℓ, $E[[T]] = E[[T_1, \dots, T_m]]$ with maximal ideal $\mathfrak{m}_T$, and V_0 a finite-dimensional E-vector space. Put $M = V_0 \otimes_E E[[T]]$, the power series in $T_1, \dots, T_m$ with coefficients in V_0; a map from a topological space to M is* coefficientwise continuous *if each of its coefficients is continuous for the natural topology of V_0. Let G act on M by $E[[T]]$-linear automorphisms $\rho(g)$ such that, in an E-basis of V_0, every coefficient of every entry of the matrix of $\rho(g)$ is a continuous function of g, and such that $\rho(g)$ reduces modulo $\mathfrak{m}_T$ to a continuous action ρ_0 of G on V_0. For cochains use $(db)(g,h) = \rho(g)b(h) - b(gh) + b(g)$ and $(dw)(g,h,k) = \rho(g)w(h,k) - w(gh,k) + w(g,hk) - w(g,h)$. Assume $H^2(G, V_0) = 0$. Then:*

(i) for every $j \geq 1$ and every coefficientwise continuous $w\colon G \times G \to \mathfrak{m}_T^j M$ with $dw = 0$ there is a coefficientwise continuous $b\colon G \to \mathfrak{m}_T^j M$ with $db = w$;

(ii) every continuous 1-cocycle $c_0\colon G \to V_0$ of ρ_0 is the constant term of a coefficientwise continuous $c\colon G \to M$ with $dc = 0$.

Proof. Truncations. For $k \geq 1$ let $M_k = M/\mathfrak{m}_T^k M$, a finite-dimensional E-vector space (the coefficients of the monomials of degree $< k$) on which ρ induces a continuous action; a coefficientwise continuous map to M is the same as a family of continuous maps to the M_k, compatible under the projections. Coboundaries and the relation $d \circ d = 0$ for inhomogeneous cochains hold in every M_k and in every G-stable subspace of it.

Step 1: the graded pieces have no second cohomology. Let $k \geq 1$. For $x \in V_0$ and a monomial t of degree k, $\rho(g)(x \otimes t) = t\,\rho(g)(x \otimes 1) \equiv \rho_0(g)x \otimes t$ modulo $\mathfrak{m}_T^{k+1}M$, since $\rho(g)(x \otimes 1) - \rho_0(g)x \otimes 1 \in \mathfrak{m}_T M$. So $\mathfrak{m}_T^k M/\mathfrak{m}_T^{k+1}M$ is G-isomorphic to a finite direct sum of copies of (V_0, ρ_0), one for each monomial of degree k, and its H^2 is a direct sum of copies of $H^2(G, V_0) = 0$.

Step 2: (i). We construct continuous $b_k\colon G \to \mathfrak{m}_T^j M_k$, $k \geq j$, with $db_k \equiv w$ modulo $\mathfrak{m}_T^k$ and $b_{k+1} \equiv b_k$ modulo $\mathfrak{m}_T^k$. For $k = j$ take $b_j = 0$: w has values in $\mathfrak{m}_T^j M$. Given b_k, let $\tilde{b}\colon G \to \mathfrak{m}_T^j M_{k+1}$ have the coefficients of b_k in degrees $< k$ and zero coefficients in degree k; it is continuous. The 2-cochain $w' = w - d\tilde{b}$ of M_{k+1} is continuous, vanishes modulo $\mathfrak{m}_T^k$ (as $d\tilde{b} \equiv db_k \equiv w$), so has values in $\mathfrak{m}_T^k M/\mathfrak{m}_T^{k+1}M$, and $dw' = dw - d(d\tilde{b}) = 0$. By Step 1, $w' = db'$ for a continuous $b'\colon G \to \mathfrak{m}_T^k M/\mathfrak{m}_T^{k+1}M$; then $b_{k+1} = \tilde{b} + b'$ has the required properties ($k \geq j$). The b_k define a coefficientwise continuous $b\colon G \to \mathfrak{m}_T^j M$ with $db \equiv w$ modulo $\mathfrak{m}_T^k$ for every k, that is $db = w$.

Step 3: (ii). Let $\tilde{c}(g) = c_0(g) \otimes 1 \in M$, coefficientwise continuous. Then $w = d\tilde{c}$ satisfies $dw = 0$, and modulo $\mathfrak{m}_T$ it is the coboundary of c_0 for ρ_0, which vanishes; so w has values in

$\mathfrak{m}_T M$. By (i) with $j = 1$, $w = db$ with b valued in $\mathfrak{m}_T M$, and $c = \tilde{c} - b$ satisfies $dc = 0$ and has constant term c_0. □

Lemma 43.28. *Let $\mathcal{O}$ be a coefficient ring, $R \in \mathrm{CNL}_{\mathcal{O}}$, $(\sigma, \mathrm{wt}, \pi)$ a weighted presentation of R, $\beta\colon R \to \mathcal{O}$ an $\mathcal{O}$-algebra homomorphism with $\beta(\pi(X_s)) = 0$ for every $s \in \sigma$, u a homogeneous generating system of* $\ker \pi$ *with counts* ne*, and $(A_\pi, \varphi_\beta, q_\pi)$ the graded model. Then:*

1. *A_π is an $\mathcal{O}$-subalgebra of $\mathcal{O}[[X_s : s \in \sigma]]$, $A_\pi = \bigoplus_j (A_\pi)_j$, $\varphi_\beta(f)$ is the constant term of f, and (A_π, φ_β) is a graded $\mathcal{O}$-algebra with a fixed point;*
2. *u is a system of equations for (A_π, φ_β) with counts* ne*, and* $\ker q_\pi = I(u)$*;*
3. *(A_π, φ_β) is smooth with tangent dimensions* $\mathrm{td}(j) = \#\mathrm{wt}^{-1}(j)$*; in particular* $\mathrm{td}(j) = 0$ *for $|j| > 2$.*

Proof. Write $P = \mathcal{O}[[X_s : s \in \sigma]]$, $P_j = \mathcal{O}[[X_\sigma]]_j$, $A = A_\pi$, $A_j = (A_\pi)_j = P_j$, $|n| = \sum_s n_s$ for $n \in \mathbb{N}^\sigma$, and $E_j = \{n \in \mathbb{N}^\sigma : \sum_s n_s \mathrm{wt}(s) = j\}$, so that P_j consists of the series $\sum_{n \in E_j} c_n X^n$.

Step 0: weight components. Monomials of distinct weights are distinct, and the weight of a product of monomials is the sum of their weights. Hence $P_i P_k \subseteq P_{i+k}$, and for $f \in P$ and $g \in P_k$,

$$(fg)_{(m)} = f_{(m-k)}\, g \qquad (m \in \mathbb{Z}), \tag{43.7}$$

since every term of fg of weight m comes from a term of f of weight $m - k$.

Step 1: A is a graded subalgebra. A is closed under addition by definition and under multiplication since $P_i P_k \subseteq P_{i+k}$; it contains $\mathcal{O} \subseteq P_0$. If $\sum_j f_j = 0$ with $f_j \in P_j$ (finitely many), taking the weight k component gives $f_k = 0$; so the sum $A = \sum_j P_j$ is direct, and $A_i A_k \subseteq A_{i+k}$.

Step 2: A_0 is a complete Noetherian local ring with residue field that of $\mathcal{O}$. E_0 is a submonoid of $\mathbb{N}^\sigma$. By Dickson's lemma the set $\{m_1, \dots, m_k\}$ of minimal elements of $E_0 \setminus \{0\}$ (componentwise order) is finite. Every $n \in E_0$ is a sum of m_i's, by induction on $|n|$: if $n \neq 0$, the finite non-empty set of $m \in E_0 \setminus \{0\}$ with $m \leq n$ has a minimal element, which is some m_i, and $n - m_i \in E_0$ with $|n - m_i| < |n|$. Let $\theta\colon \mathcal{O}[[Y_1, \dots, Y_k]] \to P$ be the $\mathcal{O}$-algebra homomorphism substituting $Y_i \mapsto X^{m_i}$ (an element of the maximal ideal of the complete ring P, as $m_i \neq 0$); explicitly $\theta(\sum_e c_e Y^e) = \sum_e c_e X^{\sum_i e_i m_i}$, the coefficient of each X^n being a finite sum since $|\sum_i e_i m_i| \geq |e|$. So θ takes values in P_0. It is onto P_0: for $f = \sum_{n \in E_0} c_n X^n$ choose $e(n) \in \mathbb{N}^k$ with $\sum_i e(n)_i m_i = n$; $n \mapsto e(n)$ is injective, so $g = \sum_n c_n Y^{e(n)}$ is a power series, and $\theta(g) = f$. The constant coefficient of $\theta(g)$ is that of g, so $\ker \theta$ lies in the maximal ideal $(\mathfrak{m}_{\mathcal{O}}, Y)$. Hence $A_0 = P_0 \cong \mathcal{O}[[Y]]/\ker\theta$ is a complete Noetherian local ring, $\mathcal{O} \to A_0$ is local, and the residue fields agree.

Step 3: generation, and φ_β. Let $j \in \mathbb{Z}$. By Dickson's lemma the set Min_j of minimal elements of E_j is finite; choose for every $n \in E_j$ some $\mu(n) \in \mathrm{Min}_j$ with $\mu(n) \leq n$; then $n - \mu(n) \in E_0$. For $f = \sum_{n \in E_j} c_n X^n \in P_j$,

$$f = \sum_{m \in \mathrm{Min}_j} X^m h_m, \qquad h_m = \sum_{\mu(n) = m} c_n X^{n-m} \in P_0,$$

coefficientwise ($n \mapsto n - m$ is injective). Each X^m is a product of variables X_s, which lie in A_0 if $\mathrm{wt}(s) = 0$ and are homogeneous of degree $\mathrm{wt}(s) \in \{-2, -1, 1, 2\}$ otherwise. So A is generated as an $\mathcal{O}$-algebra by A_0 and the finitely many X_s with $\mathrm{wt}(s) \neq 0$. A series $f \in P$ with constant term c satisfies $f - c = \sum_s X_s g_s$ for some $g_s \in P$ (assign each non-constant term to the first variable, in a fixed order of σ, occurring in it); as $\beta \circ \pi$ is an $\mathcal{O}$-algebra homomorphism killing every X_s,

$\beta(\pi(f)) = c$. So $\varphi_\beta(f)$ is the constant term of f, and $\varphi_\beta(A_j) = 0$ for $j \neq 0$, a monomial of weight $j \neq 0$ being non-constant. With Steps 1 and 2 this proves item 1.

Step 4: item 2. Each $u_{j,i}$ lies in A_j, and $\varphi_\beta(u_{0,i}) = \beta(\pi(u_{0,i})) = 0$; so u is a system of equations with counts ne. As $u_{j,i} \in \ker\pi$, $I(u) \subseteq \ker q_\pi$. Conversely, an element f of $\ker\pi$ is $\sum_{j,i} g_{j,i}u_{j,i}$ with $g_{j,i} \in P$, so by (43.7)

$$f_{(k)} = \sum_{j,i} (g_{j,i})_{(k-j)}\, u_{j,i} \qquad (k \in \mathbb{Z}),$$

which lies in $\ker\pi$, and, when $f_{(k)}$ is regarded as an element of A, in $I(u)$, since $(g_{j,i})_{(k-j)} \in A_{k-j}$. Let $f \in \ker q_\pi$ and write $f = \sum_k f_k$ with $f_k \in A_k$. Then $f_k = f_{(k)}$, so $f_k \in I(u)$ for every k, and $f \in I(u)$.

Step 5: item 3. Let $\phi\colon A \to \mathcal{O}[\ell^{-1}][[X_s : s \in \sigma]]$ be the inclusion, with the weights wt on the variables; $\#\mathrm{wt}^{-1}(j) = \mathrm{td}(j)$. (Sm1): elements of $A_j = P_j$ have weight j. (Sm2): the constant term of $\phi(f)$ is $\varphi_\beta(f)$ by Step 3. (Sm3): for $s \in \sigma$, $X_s \in A_{\mathrm{wt}(s)}$ has coefficient 1 at X_s and 0 at every other variable. (Sm4): let $f \in A$ with $\varphi_\beta(f) = 0$ and all coefficients of degree one zero, and write $f = \sum_k f_k$, $f_k \in A_k$. The constant term and the coefficient of X_s in f are those of f_0 and of $f_{\mathrm{wt}(s)}$, and the other components have none; so every monomial of every f_k has degree at least 2. Assigning each term of f_k to the first variable occurring in it gives $f_k = \sum_s X_s h_s$ with $h_s \in A_{k-\mathrm{wt}(s)}$ without constant term. Both X_s and h_s lie in $\mathfrak{m}_0 = \ker\varphi_\beta$ (Step 3), so $f_k \in \mathfrak{m}_0^2$ and $f \in \mathfrak{m}_0^2 \subseteq \mathrm{Sq}(\varphi_\beta)$. Hence ϕ is a chart and (A, φ_β) is smooth with tangent dimensions td; $\mathrm{td}(j) = 0$ for $|j| > 2$ because wt takes values in $\{-2, \dots, 2\}$. □

43.3 Rigidity and the Euler-characteristic count

This subsection proves the two facts that make the model satisfy the criterion. The first is the rigidity clause Proposition 43.29: if R^0 is finite over $\mathcal{O}$ and the adjoint Bloch–Kato Selmer groups vanish at the two-signed points, the equations span rationally in weights -2, -1, 0, and (Gc2)–(Gc4) hold. Its main input is Proposition 43.32: the Bloch–Kato cocycles of the Eisenstein adjoint of weights -2, -1 and 0 are coboundaries. Here Part VII enters: a cocycle of a Tate twist that is a coboundary locally above ℓ is a rational coboundary when the strict Tate–Shafarevich group has finite length (Lemma 43.33), and that finiteness is Proposition 40.3. The second fact is the count Lemma 43.34 of obstructions against framed Selmer cocycles in each weight, which follows from an Euler-characteristic formula for Selmer complexes over $\mathbb{Q}$ with tame local conditions (Lemma 43.35, Lemma 43.36). The formula Lemma 43.35 is used again in the dyadic case and, for auxiliary fields, in Part IX.

Proposition 43.29. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, $g_\star$ a marking, and (R, r) and $(A, \varphi_0, \bar{u}, q)$ as in Proposition 43.12, for some choice of the objects there. If R^0 is a finitely generated $\mathcal{O}$-module and $H^1_f(\mathbb{Q}, \mathrm{ad}\, r_\varphi) = 0$ for every two-signed point φ of (R, r), then $\bar{u}_{-1}$, $\bar{u}_{-2}$, $\bar{u}_0$ span rationally in degrees -1, -2, 0 respectively, and $(A, \varphi_0, \bar{u})$ satisfies (Gc2), (Gc3) and (Gc4).*

Proof. Write $E_\mathcal{O} = \mathcal{O}[\ell^{-1}]$, $a_s(h)$ for the coefficient of X_s in $h \in P$, $\bar{h}$ for the image of $h \in A_\pi$ in A, $t_\star = \Psi(g_\star)$, and $e_s = \pi_R(X_s)$; since $\pi(X_s) = y_{g_s, a_s, b_s}$ (Proposition 43.9, (Ol3) of Lemma 43.7 and (Bx2) of Proposition 43.5) and $\pi_R = p \circ \pi$, $e_s = \underline{r}(g_s)_{a_s b_s} - \underline{r_0}(g_s)_{a_s b_s}$, the second term a constant, zero unless $a_s = b_s$. The ideal $\ker\pi_R$ is generated by elements $v_1, \dots, v_m$ of $\mathfrak{a} \cap A_\pi$ of single weights and the entries of u (Proposition 43.12); $\ker q = I(\bar{u})$ by (Mp1), and $\bar{v}_i = 0$. For $h = \sum_i c_i v_i \in \mathfrak{a}$, $h_{(j)} = \sum_i (c_i)_{(j-\kappa_i)} v_i \in \mathfrak{a}$ (κ_i the weight of v_i).

Step 0: points and evaluation. Let φ be a point of (R,r) and $z_s=\varphi(e_s)$. Since $r'\mapsto\varphi(r')$ maps $\mathcal{O}+\mathfrak{m}_R$ into the closed unit disc and $\mathfrak{m}_R$ is finitely generated with $|\varphi|<1$ on it, $|\varphi(\mathfrak{m}_R^N)|_\ell\le\kappa^N$ for some $\kappa<1$; so φ is continuous, and $\varphi\circ\pi_R$ and ev_z (evaluation at z, $|z_s|<1$) are continuous homomorphisms agreeing on polynomials: $\varphi\circ\pi_R=\mathrm{ev}_z$. In particular $\varphi(R)$ lies in the integers of the finite extension $E_\varphi=E_{\mathcal{O}}(z_s:s\in\sigma)$, and $\varphi\circ q$ is induced by $\mathrm{ev}_z|_{A_\pi}$; for $\varphi=\varphi_{\mathrm{Eis}}$, $z=0$ and $\varphi_0(\bar h)$ is the constant term of h. For $f\in P_j$ the Euler identity $\sum_s\mathrm{wt}(s)X_s\partial_s f=jf$ holds monomial by monomial.

Step 1: (Gc2). R^0 is a finitely generated $\mathcal{O}$-module, so compact in its ϖ-adic topology; the inclusion into R is continuous ($\varpi R^0\subseteq\mathfrak{m}_R$), so R^0 is compact, hence closed, in R. For a monomial of weight 0, $\pi_R(X^n)=\prod_s e_s^{n_s}$; expanding, each term is a constant times a product of entries whose weights add up to 0 (the constants sit on the diagonal), so π_R maps the weight-0 polynomials into R^0. As P_0 is the closure of its polynomials and π_R is continuous, $q(A_0)=\pi_R(P_0)\subseteq R^0$, a finitely generated $\mathcal{O}$-module ($\mathcal{O}$ is Noetherian). Since $\ker q=I(\bar u)$, $q(A_0)$ is isomorphic to the image of A_0 in $A/I(\bar u)$: (Gc2) holds, and (Pq5) applies to every point on $\bar u$.

Step 2: the only point of (R,r) that is upper and lower is φ_{Eis}. Let φ be such a point and $E=E_\varphi$. For $g\in G_K$ the entries of non-zero weight of $\rho_\varphi(g)$ vanish, so $\rho_\varphi=\mathrm{diag}(\chi_1,\chi_2,\chi_3)$ with continuous characters $\chi_a\colon G_K\to\mathcal{O}_E^\times$, $\chi_a\equiv\psi_a \bmod \mathfrak{m}_E$ (entries of $\underline{r}(g)-\underline{r_0}(g)$ lie in $\mathfrak{m}_R$). Put $\lambda_a=\chi_a\psi_a^{-1}\colon G_K\to 1+\mathfrak{m}_E$. (i) At ℓ: $\ell\nmid d_K$ (Ue6), so $I_\ell\subseteq G_K$. By (Ed3), $r|_{D_\ell}=f_R\circ r_\ell^\square$ with f_R a morphism killing I_ℓ^{ord}. By Lemma 43.17 (Ol1) there is an integral test ring $A'\supseteq\varphi(R)$ with $\varphi(\mathfrak{m}_R)\subseteq\mathfrak{m}_{A'}$, and (Ol2) applied to $x=\varphi\circ f_R$ shows that for $g\in I_\ell$ the matrix $\mathrm{diag}(\chi_a(g))$ is conjugate to an upper triangular matrix with diagonal $\psi_1(g),\psi_2(g),\psi_3(g)$; so for each a there is k with $\chi_a(g)=\psi_k(g)$, and then $\Psi(g)^{a-k}=\psi_k(g)/\psi_a(g)\equiv 1$. On I_ℓ, $\Psi=\chi_\ell\chi_{\mathrm{cyc}}^e$ (the other components of χ are unramified at ℓ), so $\bar\Psi^2=\bar\chi_{\mathrm{cyc}}^{2e}$ by (Ue4); $\bar\chi_{\mathrm{cyc}}(I_\ell)=\mathbb{F}_\ell^\times$, so by (Ue9) there is $s_0\in I_\ell$ with $\bar\Psi(s_0)^2\ne 1$. If $\bar\Psi(g)^2\ne 1$, then $k=a$ ($a-k\in\{\pm1,\pm2\}$ would give $\bar\Psi(g)^2=1$); if $\bar\Psi(g)^2=1$, then $\bar\Psi(gs_0)^2\ne 1$ and $\chi_a(g)=\chi_a(gs_0)/\chi_a(s_0)=\psi_a(g)$. So $\lambda_a=1$ on I_ℓ. (ii) Away from ℓ: for a place $\bar v\nmid\ell$, (Ed2) gives $r(g)=1$ on $N'=\{g\in I_{\bar v}:\bar r_0(g)=1\}\subseteq G_K$, and so does (Ed2) for $r_0=\varphi_{\mathrm{Eis}}\circ r$ ((Bx3)); hence $\lambda_a(N')=1$, and $\lambda_a|_{I_{\bar v}\cap G_K}$ factors through $(I_{\bar v}\cap G_K)/N'$, which embeds in the image of $\bar r_0$, a finite group of order prime to ℓ (it divides $2(\#\mathbb{F}_{\mathcal{O}}-1)$, ℓ odd); a homomorphism from it to the pro-ℓ group $1+\mathfrak{m}_E$ is trivial. (iii) The places $g\bar\ell$: $I_{g\bar\ell}=gI_\ell g^{-1}$. If $g\in G_K$, $\lambda_a(gxg^{-1})=\lambda_a(x)$. As $r(c)=(1,-1)$ by (Ed1), $\rho_\varphi(cxc)=\theta(\rho_\varphi(x))$ and $\theta(\mathrm{diag}(\alpha_1,\alpha_2,\alpha_3))=\mathrm{diag}(\alpha_3^{-1},\alpha_2^{-1},\alpha_1^{-1})$, so $\chi_a(cxc)=\chi_{4-a}(x)^{-1}$, likewise $\psi_a(cxc)=\psi_{4-a}(x)^{-1}$ ($\Psi(c)=1$), and $\lambda_a(cxc)=\lambda_{4-a}(x)^{-1}$. Every $g\in G_{\mathbb{Q}}$ lies in $G_K\sqcup G_Kc$, so $\lambda_a=1$ on every $I_{g\bar\ell}$. So λ_a is trivial on the inertia groups of K at all finite places (K has no real place), hence factors through the Galois group of the Hilbert class field of K, of order h_K, prime to ℓ by (Ue6), into the pro-ℓ group $1+\mathfrak{m}_E$: $\lambda_a=1$. Hence $r_\varphi=r_0$ on G_K, and at c both are $(1,-1)$, so $r_\varphi=r_0$ on $G_K\sqcup G_Kc=G_{\mathbb{Q}}$. Then $z_s=0$ for all s, $\varphi\circ\pi_R=\mathrm{ev}_0=\varphi_{\mathrm{Eis}}\circ\pi_R$ (Step 0), and $\varphi=\varphi_{\mathrm{Eis}}$ as π_R is onto.

Step 3: (Gc3). Let ψ be a point on $\bar u$ that is negative- and positive-vanishing. By (Pq5), $\psi=t\cdot(\varphi\circ q)$ for a point φ of (R,r); $\varphi\circ q$ is again negative- and positive-vanishing, so φ is upper and lower by (Pq4), $\varphi=\varphi_{\mathrm{Eis}}$ by Step 2, and $\psi=t\cdot\varphi_0=\varphi_0$ ($\varphi_0(A_j)=0$ for $j\ne 0$). So no point $\varphi_1\ne\varphi_0$ as in (Gc3) exists, and (Gc3) holds.

Step 4: the normalisation of a tangent coboundary. Let φ be a point, $E\supseteq E_\varphi$ finite, $F\colon R\to E[\epsilon]$ an $\mathcal{O}$-algebra homomorphism lifting φ with $F(\mathfrak{m}_R^N)\to 0$ (so F is continuous), y the cocycle of Lemma 43.31, and suppose $y=dX$ for some $X\in\mathrm{M}_3(\overline{\mathbb{Q}}_\ell)$ (action of $\mathrm{ad}\,r_\varphi$, $P_g=\underline{r_\varphi}(g)$). Then $X=x\,\mathrm{diag}(1,0,-1)$ for some x. Indeed $\underline{r}(c)=1$ gives $y(c)=0$, i.e. $c\cdot X=X$, i.e. $X_{ab}=-X_{4-b,4-a}$ ($P_c=1$). Let $Q=P_{g_\star}$; then $\underline{r_\varphi}(cg_\star^{-1}c)=\theta(Q^{-1})=JQ^tJ$, and the GL-parts of $F(r(g_\star))$ and $F(r(cg_\star^{-1}c))$ are $Q+\epsilon[Q,X]$ (as $y(g_\star)Q=QX-XQ$) and

$J^t(Q+\epsilon[Q,X])J = Q' + \epsilon[Q',X]$, $Q' = JQ^tJ$, using $JX^tJ = -X$. By (Ed1) the entries $(1,2)$, $(2,1)$ of $\Delta = Q - Q'$ and of $[\Delta, X]$ vanish. Since $\Delta_{ab} = Q_{ab} - Q_{4-b,4-a}$, also $\Delta_{23} = -\Delta_{12} = 0$, $\Delta_{32} = 0$, $\Delta_{13} = \Delta_{31} = \Delta_{22} = 0$: $\Delta = \mathrm{diag}(d_\star, 0, -d_\star)$ with $d_\star \equiv t_\star - t_\star^{-1}$ modulo the maximal ideal of $\mathcal{O}_E$, a unit as $t_\star^2 - 1 \in \mathcal{O}^\times$. So $d_\star X_{12} = -d_\star X_{21} = 0$, and with $X_{ab} = -X_{4-b,4-a}$ all off-diagonal entries and X_{22} vanish and $X_{33} = -X_{11}$. For such X, $JX^tJ = -X$, so $g\cdot X = P_g X P_g^{-1}$ for both signs, and

$$F(\underline{r}(g)_{ab}) - \varphi(\underline{r}(g)_{ab}) = \epsilon\,(P_gX - XP_g)_{ab} = -\epsilon\, x\,\mathrm{wt}(a,b)\,\varphi(\underline{r}(g)_{ab}).$$

Step 5: (Gc4). Let ψ be a two-signed point on $\bar{u}$ and D a ψ-derivation vanishing on $I(\bar{u})$. By (Pq5), $\psi = t\cdot(\varphi\circ q)$ with φ a point of (R,r); $D'(f) = t^{-j}D(f)$ $(f\in A_j)$ is a $(\varphi\circ q)$-derivation vanishing on $I(\bar{u})$ (which is spanned by its homogeneous elements), and $D\mapsto D'$ is linear and injective; so it suffices to treat $t=1$. $\varphi\circ q$ is two-signed, so φ is two-signed by (Pq4). By Step 0 and Lemma 43.30 (for $h\mapsto D(\bar{h})$ on A_π), $D(\bar{h}) = \tilde{D}(h)$ for $h\in A_\pi$, where $\tilde{D}(h) = \sum_s(\partial_s h)(z)w_s$, $w_s = D(\bar{X}_s)$, is an ev_z-derivation of P. $\tilde{D}$ kills the v_i $(\bar{v}_i = 0)$ and the entries of u, which ev_z also kills; so $\tilde{D}(\ker\pi_R) = 0$ and $\tilde{D} = \delta\circ\pi_R$ for a φ-derivation δ of R with values in $E = E_\varphi(w_s : s\in\sigma)$. Since $|\tilde{D}(h)|_\ell \le \max_s|w_s|\,\kappa^{N-1}$ for $h\in\mathfrak{m}_P^N$ $(\kappa = \max(|\varpi|,|z_s|))$, and $\pi_R(\mathfrak{m}_P^N) = \mathfrak{m}_R^N$, $F = \varphi + \epsilon\delta\colon R\to E[\epsilon]$ is an $\mathcal{O}$-algebra homomorphism with $F(\mathfrak{m}_R^N)\to 0$; so $g\mapsto F(\underline{r}(g)_{ab})$ is continuous. By Lemma 43.31 its cocycle y has class in $H^1_f(\mathbb{Q}, \mathrm{ad}\, r_\varphi) = 0$, so $y = dX$, and Step 4 gives x with $\delta(\underline{r}(g)_{ab}) = -x\,\mathrm{wt}(a,b)\varphi(\underline{r}(g)_{ab})$. Hence $w_s = \delta(e_s) = -x\,\mathrm{wt}(s)z_s$ (for $\mathrm{wt}(s)=0$ both sides vanish, for $\mathrm{wt}(s)\neq 0$ the constant in e_s is 0), and for $h\in P_j\cap A_\pi$, by the Euler identity,

$$D(\bar{h}) = -x\sum_s \mathrm{wt}(s)z_s(\partial_s h)(z) = -x\,j\,h(z) = -x\,\mathrm{Eu}_\varphi(\bar{h}),\qquad \mathrm{Eu}_\varphi(\bar{f}) = j\,\varphi(q(\bar{f}))\ \ (\bar{f}\in A_j).$$

So every such D is a multiple of the fixed map Eu_φ, and any two are linearly dependent over $\overline{\mathbb{Q}}_\ell$: (Gc4).

Step 6: rational spanning in degree $j\in\{-2,-1,0\}$. (a) For $\bar{f}\in A_j\cap\mathfrak{m}_0$ with $f\in P_j$, f has no constant term and its linear terms are in the variables of weight j; $f - \sum_{\mathrm{wt}(s)=j}a_s(f)X_s = \sum_s X_s h_s$ with $h_s\in P_{j-\mathrm{wt}(s)}$ without constant term (assign each monomial to its first variable, take weight components), whose image lies in $\mathfrak{m}_0^2\subseteq\mathrm{Sq}(\varphi_0)$. So Cot_j is spanned over $\mathcal{O}$ by the images of the $\bar{X}_s$, $\mathrm{wt}(s) = j$. (b) Let $\Lambda\subseteq E_{\mathcal{O}}^{\mathrm{wt}^{-1}(j)}$ be the span of the vectors $(a_s(h))_{\mathrm{wt}(s)=j}$ for h an entry $u_{j,i}$ of u or $h\in\mathfrak{a}\cap P_j$. If Λ is everything, then for each s of weight j there are $c_s\in\mathcal{O}\setminus\{0\}$, $b_i\in\mathcal{O}$ and $h\in\mathfrak{a}\cap P_j$ such that $c_sX_s - \sum_i b_iu_{j,i} - h\in P_j$ has no constant and no linear term; by (a) its image $c_s\bar{X}_s - \sum_i b_i\bar{u}_{j,i}$ $(\bar{h}=0)$ lies in $\mathrm{Sq}(\varphi_0)$, so c_s times the class of $\bar{X}_s$ lies in $\mathrm{Cot}(\bar{u}_j)$, and with (a), $\bar{u}_j$ spans rationally in degree j. (c) Suppose $\Lambda\neq E_{\mathcal{O}}^{\mathrm{wt}^{-1}(j)}$: there are $\lambda_s\in\mathcal{O}$ $(\mathrm{wt}(s)=j)$, not all 0, with $\sum_s\lambda_s a_s(h) = 0$ for those h. Put $\tilde{D}(h) = \sum_{\mathrm{wt}(s)=j}\lambda_s a_s(h)$ on P, an $\mathcal{O}$-linear derivation at the constant term ev_0. It kills the $u_{j',i}$ for $j'\neq j$ (no linear terms in variables of weight j), the $u_{j,i}$, and every $h\in\mathfrak{a}$ (its linear terms in weight-j variables are those of $h_{(j)}\in\mathfrak{a}\cap P_j$); as ev_0 kills $\ker\pi_R$, $\tilde{D}(\ker\pi_R) = 0$ and $\tilde{D} = \delta\circ\pi_R$ with $\delta\colon R\to\mathcal{O}$ a φ_{Eis}-derivation. Then $F = \varphi_{\mathrm{Eis}} + \epsilon\delta\colon R\to\mathcal{O}[\epsilon]$ is a morphism of $\mathrm{CNL}_{\mathcal{O}}$, so $F\circ r$ satisfies (Ed1); $F(\mathfrak{m}_R^N)\subseteq\varpi^{N-1}\mathcal{O}[\epsilon]$, and φ_{Eis} is a point with $r_{\varphi_{\mathrm{Eis}}} = r_0$. By Lemma 43.31 (with $E = E_{\mathcal{O}}$), the cocycle y, $F(\underline{r}(g)) = (1+\epsilon y(g))\underline{r_0}(g)$, is a Bloch–Kato cocycle for $\mathrm{ad}\, r_0$. By (Pq3), $\underline{r}(g)_{ab} = \pi_R(h)$ with $h\in P_{\mathrm{wt}(a,b)}$, so $\delta(\underline{r}(g)_{ab}) = \tilde{D}(h) = 0$ unless $\mathrm{wt}(a,b) = j$; as $\underline{r_0}(g)$ is diagonal, y takes values in $\mathfrak{g}_j(E_{\mathcal{O}})$. By Proposition 43.32, $y = dX$ with $X\in\mathfrak{g}_j(E_{\mathcal{O}})$, and $y(c) = 0$ gives $c\cdot X = X$. If $j=-2$: $c\cdot E_{31} = -E_{31}$, so $X = 0$. If $j=0$: G_K acts trivially on $\mathfrak{g}_0$ and c fixes X, so $dX = 0$. If $j=-1$: $c\cdot(\alpha E_{21} + \beta E_{32}) = -(\alpha E_{32} + \beta E_{21})$, so $X = b(E_{21} - E_{32})$, $y(g_\star) = (t_\star^{-1} - 1)X$, and the ϵ-part of the entry $(2,1)$ of $\underline{Fr}(g_\star) - \underline{Fr}(cg_\star^{-1}c)$, which is $y(g_\star)\mathrm{T}(t_\star) - \mathrm{T}(t_\star)^{-1}Jy(g_\star)^tJ$, equals $(t_\star+1)y(g_\star)_{21} = (t_\star+1)(t_\star^{-1}-1)b$; it vanishes

by (Ed1), and $(t_\star+1)(t_\star^{-1}-1) = -t_\star^{-1}(t_\star^2-1) \neq 0$, so $b=0$. In all cases $y=0$, so δ kills all entries, $\lambda_s = \tilde{D}(X_s) = \delta(e_s) = 0$ for all s: a contradiction. Hence $\Lambda = E_{\mathcal{O}}^{\mathrm{wt}^{-1}(j)}$ and $\bar{u}_j$ spans rationally in degree j, for $j = -1, -2, 0$. □

Lemma 43.30. *Let $\mathcal{O}$ be a coefficient ring, σ a finite set,* $\mathrm{wt}\colon \sigma \to \mathbb{Z}$ *a map, $P = \mathcal{O}[[X_s : s \in \sigma]]$ and $A_\pi = \sum_{j\in\mathbb{Z}} \mathcal{O}[[X_\sigma]]_j \subseteq P$. Let $z = (z_s) \in \overline{\mathbb{Q}}_\ell^{\,\sigma}$ with $|z_s|_\ell < 1$ for every s, and* $\mathrm{ev}_z\colon P \to \overline{\mathbb{Q}}_\ell$ *the evaluation $f \mapsto f(z)$. If $D\colon A_\pi \to \overline{\mathbb{Q}}_\ell$ is $\mathcal{O}$-linear with $D(fg) = \mathrm{ev}_z(f)D(g) + \mathrm{ev}_z(g)D(f)$ for all $f, g \in A_\pi$, then*

$$D(f) = \sum_{s\in\sigma} (\partial_s f)(z)\, D(X_s) \qquad (f \in A_\pi),$$

where ∂_s is the formal partial derivative.

Proof. The z_s lie in a finite extension E' of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_{E'} \supseteq \mathcal{O}$, complete, so ev_z is defined (each z^n has $|z^n|_\ell \leq \max_s |z_s|_\ell^{|n|}$), and it is an $\mathcal{O}$-algebra homomorphism. Write $P_j = \mathcal{O}[[X_\sigma]]_j$ and $E_j = \{n \in \mathbb{N}^\sigma : \sum_s n_s \mathrm{wt}(s) = j\}$, ordered componentwise.

Step 1: generators of the weight pieces. By Dickson's lemma the set Min_j of minimal elements of E_j is finite. Every $n \in E_j$ dominates some $m \in \mathrm{Min}_j$, and $n - m \in E_0$; choosing one such $m(n)$ for each n, every $f = \sum_n c_n X^n \in P_j$ is $\sum_{m\in\mathrm{Min}_j} X^m h_m$ with $h_m = \sum_{m(n)=m} c_n X^{n-m} \in P_0$. Let $m_1, \ldots, m_k$ be the minimal elements of $E_0 \setminus \{0\}$; by induction on $|n|$ every $n \in E_0$ is $\sum_i e(n)_i m_i$ for some $e(n) \in \mathbb{N}^k$, and $n \mapsto e(n)$ (one choice for each n) is injective. So the continuous homomorphism $\theta\colon \mathcal{O}[[Y_1, \ldots, Y_k]] \to P$, $Y_i \mapsto X^{m_i}$, maps $\sum_n c_n Y^{e(n)}$ to $\sum_n c_n X^n$, and θ is onto P_0; likewise $\theta' = \mathcal{O}_{E'} \otimes \theta$ maps $\mathcal{O}_{E'}[[Y]] = \mathcal{O}_{E'} \otimes_{\mathcal{O}} \mathcal{O}[[Y]]$ onto $\mathcal{O}_{E'} \otimes_{\mathcal{O}} P_0$ ($\mathcal{O}_{E'}$ is free of finite rank over $\mathcal{O}$).

Step 2: the difference is a derivation vanishing on the variables. The map given by $\tilde{D}(f) = \sum_s (\partial_s f)(z) D(X_s)$ is an $\mathcal{O}$-linear ev_z-derivation of P (Leibniz rule for ∂_s; ev_z is multiplicative), and $\tilde{D}(X_s) = D(X_s)$. So $\Delta = D - \tilde{D}|_{A_\pi}$ is an $\mathcal{O}$-linear ev_z-derivation of A_π with $\Delta(X_s) = 0$ for every s; by the Leibniz rule $\Delta(X^n) = 0$ for every monomial, and $\Delta(1) = 0$. Extend Δ and ev_z to $\mathcal{O}_{E'} \otimes_{\mathcal{O}} A_\pi$ by $\Delta(\lambda \otimes f) = \lambda\Delta(f)$, $\mathrm{ev}_z(\lambda \otimes f) = \lambda\, \mathrm{ev}_z(f)$: a ring homomorphism and a derivation at it, killing $\mathcal{O}_{E'} \otimes 1$.

Step 3: $\Delta = 0$ on P_0. Let $g \in \mathcal{O}[[Y]]$ and $y^* = (z^{m_i})_i$, with $|y_i^*|_\ell < 1$. The substitution $Y \mapsto Y' + y^*$ is an automorphism of $\mathcal{O}_{E'}[[Y]]$, so $g(Y) - g(y^*) = \sum_i (Y_i - y_i^*) g_i(Y)$ with $g_i \in \mathcal{O}_{E'}[[Y]]$ (expand $g(Y' + y^*)$, remove the constant term $g(y^*)$, assign each remaining monomial to its first variable, and substitute back). Applying the ring homomorphism θ',

$$\theta(g) - g(y^*) = \sum_{i=1}^{k} (X^{m_i} - y_i^*)\, \theta'(g_i) \quad \text{in } \mathcal{O}_{E'} \otimes_{\mathcal{O}} P_0,$$

and applying Δ: $\Delta(\theta(g)) = \sum_i (\Delta(X^{m_i})\, \mathrm{ev}_z(\theta'(g_i)) + (z^{m_i} - y_i^*)\Delta(\theta'(g_i))) = 0$. As θ is onto P_0, $\Delta(P_0) = 0$.

Step 4: $\Delta = 0$ on A_π. For $f \in P_j$ write $f = \sum_{m\in\mathrm{Min}_j} X^m h_m$ (Step 1); then $\Delta(f) = \sum_m (\Delta(X^m)\mathrm{ev}_z(h_m) + \mathrm{ev}_z(X^m)\Delta(h_m)) = 0$ by Steps 2 and 3. Every element of A_π is a finite sum of elements of the P_j. □

Lemma 43.31. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, $g_\star$ a marking and (R, r) a normalised ordinary Eisenstein deformation ring for $g_\star$. Let $E \subseteq \overline{\mathbb{Q}}_\ell$ be a finite extension of $\mathbb{Q}_\ell$ containing $\mathcal{O}$, $E[\epsilon] = E[X]/(X^2)$, and $F\colon R \to E[\epsilon]$ an $\mathcal{O}$-algebra homomorphism whose reduction φ modulo ϵ is a point of (R, r) and such that $g \mapsto F(\underline{r}(g)_{ab})$ is continuous on $G_\mathbb{Q}$ for $1 \leq a, b \leq 3$. Define $y\colon G_\mathbb{Q} \to \mathrm{M}_3(E)$ by*

$$F(\underline{r}(g)) = (1 + \epsilon\, y(g))\, \underline{r_\varphi}(g) \qquad (g \in G_\mathbb{Q}).$$

Then y is a continuous 1-cocycle of $G_{\mathbb{Q}}$ with values in $\operatorname{ad} r_\varphi$, and it is a Bloch–Kato cocycle; in particular its class lies in $H^1_f(\mathbb{Q}, \operatorname{ad} r_\varphi)$.

Proof. Write $P_g = \underline{r_\varphi}(g)$, $\varsigma_g = \eta_K(g)$ and $g \cdot X$ for the action of $\operatorname{ad} r_\varphi$; $F \circ r \colon G_{\mathbb{Q}} \to \mathcal{G}_3(E[\epsilon])$ is a homomorphism with reduction r_φ.

Step 1: cocycle and continuity. For $Y' \in \mathrm{M}_3(E)$ and $P \in \mathrm{GL}_3(E)$, $\theta((1+\epsilon Y')P) = J(1 - \epsilon\, {}^tY')\, {}^tP^{-1}J = (1 - \epsilon JY'\, {}^tJ)\theta(P)$ ($J^2 = 1$). Multiplying in $\mathcal{G}_3$, $(A,\varsigma)(B,\tau) = (A\theta^{\varsigma}(B), \varsigma\tau)$, the GL-part of $F(r(g))F(r(h))$ is $(1+\epsilon y(g))P_g(1+\epsilon Z)\theta^{\varsigma_g}(P_h)$ with $Z = y(h)$ if $\varsigma_g = 1$, $Z = -Jy(h)^tJ$ if $\varsigma_g = -1$; since $P_g Z P_g^{-1} = g \cdot y(h)$ and $P_{gh} = P_g\theta^{\varsigma_g}(P_h)$, this gives $y(gh) = y(g) + g \cdot y(h)$. Continuity: $y(g)$ is the ϵ-part of $F(\underline{r}(g))$ times P_g^{-1}, and both are continuous by hypothesis (the inverse through the adjugate).

Step 2: averaging. If a 1-cocycle z of a profinite group Γ with values in a $\mathbb{Q}$-vector space M (with Γ acting linearly) vanishes on an open normal subgroup Γ' of index n, then $z = dW$ with $W \in M$: indeed $z(gh) = z(g) + g \cdot z(h) = z(g)$ for $h \in \Gamma'$, and summing $z(g\tau) = z(g) + g \cdot z(\tau)$ over a set $\mathcal{T}$ of representatives of Γ/Γ' (so is $g\mathcal{T}$) gives $W' = nz(g) + g \cdot W'$ for $W' = \sum_{\tau\in\mathcal{T}} z(\tau)$, so $z(g) = g \cdot W - W$ with $W = -n^{-1}W'$.

Step 3: places $v \nmid \ell$. Archimedean: a decomposition group at ∞ is $\{1, c'\}$ with c' a complex conjugation, of order 2; by Step 2 with $\Gamma' = \{1\}$, $y|_{\{1,c'\}}$ is a coboundary, so its class is 0, as Definition 22.2 requires. Finite $v = p \neq \ell$: let $D = D_j$ be a decomposition group at p, with inertia $I_j = I_{\bar v}$ for the place $\bar v$ of $\overline{\mathbb{Q}}$ induced by j. By (Ed2) for r, $r(g) = 1$ for $g \in N' = \{g \in I_{\bar v} : \bar r_0(g) = 1\}$, so $F(\underline{r}(g)) = 1 = P_g$ and $y(g) = 0$ there. The group N' is the kernel of the continuous homomorphism $\bar r_0|_{I_{\bar v}}$ to a finite group, an open normal subgroup of $I_{\bar v}$; by Step 2 applied to $y|_{I_{\bar v}}$ there is $x \in \mathrm{M}_3(E)$ with $y(g) = g \cdot x - x$ for $g \in I_j$, the condition of Definition 22.2 at v.

Step 4: the decomposition group D_ℓ. By (Ed3), $r|_{D_\ell} = f_R \circ r_\ell^{\square}$ for a morphism $f_R \colon R_\ell^{\square} \to R$ killing I_ℓ^{ord}; it induces $i \colon R_\ell^{\mathrm{ord}} \to R$ with $\underline{r}(g) = i(\underline{r_\ell^{\mathrm{ord}}}(g))$ for $g \in D_\ell$, a local homomorphism. Let $f = F \circ i \colon R_\ell^{\mathrm{ord}} \to E[\epsilon]$ and fix an embedding $\mathrm{emb}_\ell \colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ inducing $\bar\ell$. The reduction of f is $\varphi \circ i$, and $|\varphi(i(y'))|_\ell < 1$ for y' in the maximal ideal of R_ℓ^{ord}, because $i(y') \in \mathfrak{m}_R$ and φ is a point; $g \mapsto f(\underline{r_\ell^{\mathrm{ord}}}(g)_{ab}) = F(\underline{r}(g)_{ab})$ is continuous. So Lemma 43.22 applies to f; its ρ_0 is P_g, its action on $\overline{\mathrm{M}_3(E)}$ is that of $\operatorname{ad} r_\varphi$ restricted to D_ℓ, and its y is our $y|_{D_\ell}$. It gives $Y \in \mathrm{M}_3(E) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}} \subseteq \operatorname{ad} r_\varphi \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $y(k) \otimes 1 = k \cdot Y - Y$ for $k \in D_\ell = D_{\mathrm{emb}_\ell}$, B_{cris} acted on through emb_ℓ.

Step 5: the other decomposition groups above ℓ. Every embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ over $\mathbb{Q}$ is $j = \mathrm{emb}_\ell \circ \alpha$ with $\alpha \in G_{\mathbb{Q}}$ (both have image the algebraic closure of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_\ell$). Put $g = \alpha^{-1}$. If $\hat h \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ restricts along j to h, i.e. $\hat h \circ j = j \circ h$, then $\hat h \circ \mathrm{emb}_\ell = \mathrm{emb}_\ell \circ (\alpha h \alpha^{-1})$; so $D_j = gD_\ell g^{-1}$, and $h = gkg^{-1} \in D_j$ acts on B_{cris} as $k \in D_\ell$ does through emb_ℓ. The $\mathbb{Q}_\ell$-linear map $\Phi(X \otimes b) = (g \cdot X) \otimes b$ satisfies $\Phi(k \cdot w) = h \cdot \Phi(w)$ for $h = gkg^{-1}$, the left side for D_{emb_ℓ}, the right side for D_j. By Step 1 and $y(g^{-1}) = -g^{-1} \cdot y(g)$, $y(h) = y(g) + g \cdot y(k) - h \cdot y(g)$. With $Y' = \Phi(Y) - y(g) \otimes 1$ and Step 4,

$$y(h) \otimes 1 = \Phi(k \cdot Y - Y) - \big(h \cdot (y(g) \otimes 1) - y(g) \otimes 1\big) = h \cdot Y' - Y' \qquad (h \in D_j).$$

So the condition of Definition 22.2 holds at every decomposition group above ℓ, and with Step 3 at every place: y is a Bloch–Kato cocycle. □

Proposition 43.32. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and let $E \subseteq \overline{\mathbb{Q}}_\ell$ be a finite extension of $\mathbb{Q}_\ell$ containing $\mathcal{O}$. Let $G_{\mathbb{Q}}$ act on $\mathrm{M}_3(\overline{\mathbb{Q}}_\ell)$ through r_0: if $r_0(g) = (P, \varsigma)$, by $X \mapsto PXP^{-1}$ if $\varsigma = 1$ and by $X \mapsto -PJ\, {}^tXJP^{-1}$ if $\varsigma = -1$; for $j \in \mathbb{Z}$ let $\mathfrak{g}_j(E) \subseteq \mathrm{M}_3(E)$ be the matrices supported in the positions of weight j. Let*

$j \in \{-2,-1,0\}$ *and let* $y\colon G_{\mathbb{Q}} \to \mathfrak{g}_j(E)$ *be a continuous* 1*-cocycle which is a Bloch–Kato cocycle for this representation on* $\mathrm{M}_3(\overline{\mathbb{Q}}_\ell)$*. Then there is* $X \in \mathfrak{g}_j(E)$ *with* $y(g) = g \cdot X - X$ *for every* $g \in G_{\mathbb{Q}}$*.*

Proof. Write $g \cdot X$ for the action and dX for $g \mapsto g \cdot X - X$. Let $L = K(\zeta_{\ell N}) \subset \overline{\mathbb{Q}}$; it is Galois over $\mathbb{Q}$ of some degree n, totally complex, contains K and a primitive ℓ-th root of unity, and χ, which factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, is trivial on G_L. So for $g \in G_L \subseteq G_K$ (sign 1), $r_0(g) = (\mathrm{T}(\chi_{\mathrm{cyc}}(g)^e), 1)$ and g multiplies the entry (a,b) by $\chi_{\mathrm{cyc}}(g)^{e\,\mathrm{wt}(a,b)}$; on $\mathfrak{g}_j(\overline{\mathbb{Q}}_\ell)$, G_L acts by the scalar $\chi_{\mathrm{cyc}}(g)^{-k}$ with $k = -ej \geq 0$. Put $V' = \mathfrak{g}_j(E)$, a finite-dimensional $\mathbb{Q}_\ell$-vector space.

Step 1: averaging. If a 1-cocycle z of $G_{\mathbb{Q}}$ with values in a $G_{\mathbb{Q}}$-stable $\mathbb{Q}_\ell$-subspace M vanishes on an open normal subgroup Γ' of index m, then $z = dW$ with $W \in M$: $z(gh) = z(g)$ for $h \in \Gamma'$, and summing $z(g\tau) = z(g) + g \cdot z(\tau)$ over representatives τ of $G_{\mathbb{Q}}/\Gamma'$ gives $W' = mz(g) + g \cdot W'$ with $W' = \sum_\tau z(\tau)$, so $W = -m^{-1}W'$. V' is $G_{\mathbb{Q}}$-stable (the action is diagonal on G_K, and c maps the entry (a,b) to $(4-b, 4-a)$, of the same weight).

Step 2: local behaviour of $y|_{G_L}$*.* (a) Let $\bar{v}$ be a place of $\overline{\mathbb{Q}}$ above a prime $p \neq \ell$. By the Bloch–Kato condition at p (for the decomposition group of an embedding inducing $\bar{v}$), $y(g) = g \cdot x - x$ for $g \in I_{\bar{v}}$, some $x \in \mathrm{M}_3(\overline{\mathbb{Q}}_\ell)$. On $I_{\bar{v}} \cap G_L$, χ_{cyc} is trivial ($p \neq \ell$), so $G_L \cap I_{\bar{v}}$ acts trivially on $\mathrm{M}_3(\overline{\mathbb{Q}}_\ell)$ and $y = 0$ there. (b) Let $D = D_{j'}$ be a decomposition group at ℓ for an embedding j', $D' = D \cap G_L$, the decomposition group of L for j' at the place w it induces, identified with G_{L_w}, and $I' = I_{j'} \cap G_L$ its inertia. The Bloch–Kato condition gives $Y \in \mathrm{M}_3(\overline{\mathbb{Q}}_\ell) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $y(g) \otimes 1 = g \cdot Y - Y$ on D. Let $p_j\colon \mathrm{M}_3(\overline{\mathbb{Q}}_\ell) \to \mathfrak{g}_j(\overline{\mathbb{Q}}_\ell)$ be the projection to the entries of weight j and $\pi\colon \overline{\mathbb{Q}}_\ell \to E$ a $\mathbb{Q}_\ell$-linear map with $\pi|_E = \mathrm{id}$ (a complement of E exists); $p = \pi \circ p_j$ (entrywise) is $\mathbb{Q}_\ell$-linear, is the identity on V', and commutes with the action of G_L, which acts on entries by elements of $\mathbb{Q}_\ell^\times$. Applying $p \otimes 1$, $y(g) \otimes 1 = g \cdot Y' - Y'$ for $g \in D'$, with $Y' = (p \otimes 1)(Y) \in V' \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$: the class of $y|_{D'}$ lies in $H^1_f(D', V')$ (Definition 22.2). As a representation of D', V' is a direct sum of copies of $\mathbb{Q}_\ell(-k)$ (a $\mathbb{Q}_\ell$-basis of V'), and the condition of Definition 22.2 is componentwise. If $k \geq 1$: $H^0(D', \mathbb{Q}_\ell(-k)) = 0$ since $\chi_{\mathrm{cyc}}(D')$ is infinite; an element $b \otimes 1$ of $B_{\mathrm{dR}} \otimes \mathbb{Q}_\ell(-k)$ is invariant iff $g(b) = \chi_{\mathrm{cyc}}(g)^k b$, i.e. iff $bt^{-k} \in B_{\mathrm{dR}}^{D'} = L_w$, so $D_{\mathrm{dR}} = L_w t^k$ is one-dimensional over L_w, $\mathbb{Q}_\ell(-k)$ is de Rham, and $t^k \in \mathrm{Fil}^0 B_{\mathrm{dR}}$ gives $\mathrm{Fil}^0 D_{\mathrm{dR}} = D_{\mathrm{dR}}$; by Theorem B.43, $H^1_f(D', \mathbb{Q}_\ell(-k)) = 0$. Hence $y|_{D'} = dW_w$ for some $W_w \in V'$. If $k = 0$: D' acts trivially on V' and by Theorem B.44 (componentwise) $y|_{D'}$ is a homomorphism vanishing on I'.

Step 3: $y|_{G_L}$ *factors through* G_{L,S_ℓ}*.* Let H be the kernel of $G_L \to G_{L,S_\ell}$, the closed normal subgroup generated by the groups $I_{\bar{v}} \cap G_L$ for the places $\bar{v}$ not above ℓ (the infinite places of L are complex); this family is stable under conjugation by G_L. χ_{cyc} is trivial on H (the field of ℓ-power roots of unity over L is unramified outside ℓ), so $y|_H$ is a continuous homomorphism, vanishing on these generators by Step 2(a); hence $y|_H = 0$, $y(gh) = y(g)$ for $g \in G_L$, $h \in H$, and $y|_{G_L} = \tilde{y} \circ (G_L \to G_{L,S_\ell})$ for a continuous cocycle $\tilde{y}$ of G_{L,S_ℓ} in V' with the scalar action χ_{cyc}^{-k}.

Step 4: weights -2*,* -1 *(*$k = -ej \geq 1$*).* Let $b_1, \ldots, b_m$ be a $\mathbb{Q}_\ell$-basis of V' and $\lambda_1, \ldots, \lambda_m$ the dual coordinates. Each $\lambda_i \circ \tilde{y}\colon G_{L,S_\ell} \to \mathbb{Q}_\ell$ is continuous with $\lambda_i\tilde{y}(gh) = \lambda_i\tilde{y}(g) + \chi_{\mathrm{cyc}}(g)^{-k}\lambda_i\tilde{y}(h)$, and for $v \in S_\ell$ the restriction to L_v in the definition of $\mathrm{Sha}^1_\ell(L,k)$ is taken through the image of a decomposition group D' as in Step 2(b), on which $\lambda_i\tilde{y}$ is the coboundary of $\lambda_i(W_v)$. Its image is compact, so for some $s \geq 0$ all $\ell^s\lambda_i \circ \tilde{y}$ and all $\ell^s\lambda_i(W_v)$ ($v \in S_\ell$, finitely many) are $\mathbb{Z}_\ell$-valued. By Proposition 40.3 (with its e equal to k), $\mathrm{len}_\ell(L,k) < \infty$, and Lemma 43.33 (with $E = L$) gives $x_i \in \mathbb{Q}_\ell$ with $\ell^s\lambda_i(\tilde{y}(g)) = (\chi_{\mathrm{cyc}}(g)^{-k} - 1)x_i$. With $X_L = \sum_i \ell^{-s} x_i b_i \in V'$, $y = dX_L$ on G_L.

Step 5: weight 0 *(*$k = 0$*).* Here $y|_{G_L}$ is a continuous homomorphism $G_L \to V'$; by Steps 2 and 3 it vanishes on every inertia group of L at a finite place. Its kernel is a closed normal subgroup with abelian quotient, whose fixed field is an abelian extension of L unramified at all places,

hence finite; so $y(G_L)$ is a finite subgroup of the torsion-free group V', and $y|_{G_L} = 0 = dX_L$ with $X_L = 0$.

Step 6: conclusion. In all cases $y - dX_L$ is a cocycle with values in V' vanishing on the open normal subgroup G_L of index n; by Step 1, $y - dX_L = dX'$ with $X' \in V'$, and $X = X_L + X'$. □

Lemma 43.33. *Let E be a number field, ℓ a prime and $k \geq 1$ an integer with $\mathrm{len}_\ell(E,k) < \infty$. For $v \in S_\ell$ let $\iota_v \colon G_{E_v} \to G_{E,S_\ell}$ be the homomorphism through which the restriction to E_v in the definition of $\mathrm{Sha}^1_\ell(E,k)$ is taken. Let $z \colon G_{E,S_\ell} \to \mathbb{Z}_\ell$ be a continuous map with*

$$z(gh) = z(g) + \chi_{\mathrm{cyc}}(g)^{-k} z(h) \qquad (g, h \in G_{E,S_\ell}),$$

such that for every $v \in S_\ell$ there is $W_v \in \mathbb{Z}_\ell$ with $z(\iota_v(g)) = (\chi_{\mathrm{cyc}}(\iota_v(g))^{-k} - 1)W_v$ for all $g \in G_{E_v}$. Then there is $x \in \mathbb{Q}_\ell$ with $z(g) = (\chi_{\mathrm{cyc}}(g)^{-k} - 1)x$ for all $g \in G_{E,S_\ell}$.

Proof. Write $\kappa(g) = \chi_{\mathrm{cyc}}(g)^{-k}$, $G = G_{E,S_\ell}$ and $s = \mathrm{len}_\ell(E,k)$.

The reductions of $\ell^{-m}z$ are locally trivial classes, hence trivial. For $m \geq 0$ let $z_m \colon G \to \mathbb{Q}_\ell/\mathbb{Z}_\ell(-k)$ be $g \mapsto \ell^{-m} z(g) \bmod \mathbb{Z}_\ell$. It takes values in the finite group $\ell^{-m}\mathbb{Z}_\ell/\mathbb{Z}_\ell$ and is the composite of the continuous map z with a locally constant map, so it is a continuous 1-cocycle; let c_m be its class in $H^1(G, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-k))$. For $v \in S_\ell$, $z_m \circ \iota_v$ is the coboundary of $\ell^{-m}W_v \bmod \mathbb{Z}_\ell$, so the restriction of c_m to E_v vanishes and $c_m \in \mathrm{Sha}^1_\ell(E,k)$. A $\mathbb{Z}_\ell$-module of finite length s has a composition series of length s with quotients $\mathbb{Z}/\ell\mathbb{Z}$, hence is killed by ℓ^s. Since $\ell^s z_{m+s} = z_m$ as cocycles, $c_m = \ell^s c_{m+s} = 0$ for every $m \geq 0$.

Approximate primitives. By $c_m = 0$ there is $w_m \in \mathbb{Q}_\ell$ with $\ell^{-m}z(g) - (\kappa(g) - 1)w_m \in \mathbb{Z}_\ell$ for all $g \in G$; with $x_m = \ell^m w_m$,

$$z(g) - (\kappa(g) - 1)x_m \in \ell^m \mathbb{Z}_\ell \qquad (g \in G,\ m \geq 0). \tag{43.8}$$

An element on which κ is not 1. The image of G_E in $\mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) \cong \mathbb{Z}_\ell^\times$ is $\mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/E \cap \mathbb{Q}(\mu_{\ell^\infty}))$, a closed subgroup of index at most $[E : \mathbb{Q}]$, hence open; so $\chi_{\mathrm{cyc}}(G)$ contains $1 + \ell^r\mathbb{Z}_\ell$ for some $r \geq 2$. This group is torsion-free (the ℓ-adic logarithm is injective on it), so $u = 1 + \ell^r$ satisfies $u^{-k} \neq 1$, and there is $g_0 \in G$ with $\kappa(g_0) = u^{-k} \neq 1$.

Passage to the limit. By (43.8) at g_0, $|(\kappa(g_0) - 1)x_m - z(g_0)|_\ell \leq \ell^{-m}$, so $x_m \to x := z(g_0)/(\kappa(g_0) - 1)$ in $\mathbb{Q}_\ell$. For $g \in G$, $z(g) - (\kappa(g) - 1)x$ is the limit of $z(g) - (\kappa(g) - 1)x_m$, which lies in $\ell^m\mathbb{Z}_\ell$ by (43.8); hence it is 0. □

Lemma 43.34. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$. For every integer j with $-2 \leq j \leq 2$, the spaces $Z^1_{\mathrm{u}}(\mathfrak{g}_j)$ and $H^2_{\mathrm{u}}(\mathfrak{g}_j)$ are finite-dimensional over $\mathbb{F}_\mathcal{O}$ and*

$$\dim H^2_{\mathrm{u}}(\mathfrak{g}_j) - \dim Z^1_{\mathrm{u}}(\mathfrak{g}_j) = b(j), \qquad b(-2) = b(2) = 0, \quad b(-1) = b(0) = b(1) = 1.$$

Proof. Write $\mathbb{F} = \mathbb{F}_\mathcal{O}$, $q = \#\mathbb{F}$ (a power of ℓ), $\dim = \dim_\mathbb{F}$, and fix $j \in \{-2, \dots, 2\}$.

Step 1: the action. If $g \in G_K$ then $\bar{r}_0(g) = (\mathrm{T}(\bar{\Psi}(g)), 1)$, and $\mathrm{T}(t)X\mathrm{T}(t)^{-1}$ has entries $t^{b-a}X_{ab}$, so g acts on $\mathfrak{g}_j$ by the scalar $\bar{\Psi}(g)^j$; in particular $G_K \cap \ker\bar{\Psi}$, and hence every $N_{\bar{p}}$, acts trivially. As $\Psi(c) = 1$, $\bar{r}_0(c) = (1, -1)$ and c acts by $X \mapsto -J\,{}^tXJ$, whose (a,b) entry is $-X_{4-b,4-a}$.

Hence $\dim \mathfrak{g}_j^{c=1} = b(j)$.

Step 2: $H^0(G_S, \mathfrak{g}_j^\vee) = 0$. Let $H \subseteq G_\mathbb{Q}$ be the closed subgroup of the elements acting trivially on μ_ℓ and lying in $\ker\chi^{(\ell)}$, so that $L_{\ell,\chi}$ is its fixed field and $G_{L_{\ell,\chi}} = H$. By (Ue8), $K \not\subseteq L_{\ell,\chi}$,

so $H \not\subseteq G_K$: choose $h \in H$ with $\eta_K(h) = -1$. By (Ue3) the part of χ of order prime to ℓ is χ, so $\chi^{(\ell)} = \chi\chi_\ell^{-1}$ and $\chi(h) = \chi_\ell(h)$. If $\ell \nmid N$, $\chi_\ell = \mathbf{1}$. If $\ell^a \parallel N$ with $a \geq 1$, χ_ℓ is a character of $(\mathbb{Z}/\ell^a\mathbb{Z})^\times$ whose order divides that of χ, hence is prime to ℓ; so it is trivial on the kernel of reduction to $(\mathbb{Z}/\ell\mathbb{Z})^\times$, of order ℓ^{a-1}, and $\chi_\ell(h) = \chi_\ell(b)$ with $b \equiv 1 \pmod{\ell}$ because h fixes μ_ℓ; so $\chi_\ell(h) = 1$. Moreover $\chi_{\mathrm{cyc}}(h) \equiv 1 \pmod{\ell}$. Hence $\bar\Psi(h) = 1$. Put $g = ch$. Then $\eta_K(g) = (-1)(-1) = 1$, $\bar\Psi(g) = \bar\Psi(c)\bar\Psi(h) = 1$, and g acts on μ_ℓ as c does, by $\zeta \mapsto \zeta^{-1}$. By Step 1, g acts trivially on $\mathfrak{g}_j$. If $f \in \mathfrak{g}_j^\vee$ is G_S-invariant, then $f(X) = g(f(g^{-1}X)) = g(f(X)) = f(X)^{-1}$ for every X, so $f(X)^2 = 1$ and $f(X) = 1$ (ℓ odd): f is trivial.

Step 3: the local conditions. Let $p \in S$ be a prime, $p \neq \ell$. The map $D_p \to \{\pm 1\} \times \mathbb{F}^\times$, $g \mapsto (\eta_K(g), \bar\Psi(g))$, is a homomorphism, and $N_{\bar p}$ is the intersection of its kernel with the normal subgroup I_p of D_p; so $N_{\bar p}$ is normal in D_p, $[I_p : N_{\bar p}]$ divides $2(q-1)$ and is prime to ℓ, and $N_{\bar p}$ acts trivially on $\mathfrak{g}_j$ (Step 1). By Lemma 43.35, item 1, applied with $N' = N_{\bar p}$, $L^{\mathrm{u}}_{p,j} \supseteq B^1(D_p, \mathfrak{g}_j)$ and $\dim L^{\mathrm{u}}_{p,j} = \dim \mathfrak{g}_j$. At ℓ, $L^{\mathrm{u}}_{\ell,j} = B^1(D_\ell, \mathfrak{g}_j)$ by definition.

Step 4: the count. By Steps 2 and 3 and (Ue1), Lemma 43.36 applies to $M = \mathfrak{g}_j$, $\mathbb{F}$, S, $(D_p)_p$ and $(L^{\mathrm{u}}_{p,j})_p$; its space Z_0 is $Z^1_{\mathrm{u}}(\mathfrak{g}_j)$ by definition. It gives finite dimensions and

$$\dim H^2_{\mathrm{u}}(\mathfrak{g}_j) - \dim Z^1_{\mathrm{u}}(\mathfrak{g}_j) = \dim \mathfrak{g}_j^{c=1} + \sum_{p \neq \ell} (\dim \mathfrak{g}_j - \dim L^{\mathrm{u}}_{p,j}) = b(j) + 0,$$

by Steps 1 and 3. □

Lemma 43.35. *Let ℓ be an odd prime and let $\mathbb{F}$, S, G_S, M and $(D_p)_p$ be as in Definition 43.1; let $c \in G_{\mathbb{Q}}$ be a complex conjugation and $M^{c=1}$ the subspace of M fixed by c.*

1. *Let $p \in S$ be a prime and $N' \subseteq I_p$ a normal subgroup of D_p with $[I_p : N']$ finite and prime to ℓ, acting trivially on M. Then $\{z \in Z^1(D_p, M) : z|_{N'} = 0\}$ contains $B^1(D_p, M)$ and has dimension $\dim M$.*

2. *Let $\mathcal{L} = (L_p)_p$ be a family of local conditions for M with $B^1(D_p, M) \subseteq L_p$ for every prime $p \in S$, and assume $H^0(G_S, M^\vee) = 0$. Then $H^1_{\mathcal{L}}(M)$ and $H^2_{\mathcal{L}}(M)$ are finite-dimensional and*

$$\dim H^1_{\mathcal{L}}(M) - \dim H^2_{\mathcal{L}}(M) = \dim H^0(G_S, M) - \dim M^{c=1} + \sum_{p \in S \text{ prime}} (\dim L_p - \dim M).$$

Proof. Item 1. Coboundaries $g \mapsto gx - x$ vanish on N', which acts trivially. Let W be the space in question, $z \in W$ and $n = [I_p : N']$. Then $z|_{I_p}$ is a cocycle of the finite group I_p/N'; summing $z(gh) = z(g) + gz(h)$ over $h \in I_p/N'$ gives $z(g) = gx - x$ for $g \in I_p$, with $x = -n^{-1}\sum_h z(h)$. So restriction maps W onto $B^1(I_p, M)$ (onto, because $B^1(D_p, M) \subseteq W$), of dimension $\dim M - \dim M^{I_p}$. The kernel consists of the $z \in Z^1(D_p, M)$ vanishing on I_p. For such z and $h \in I_p$, $z(hg) = hz(g)$ and $z(hg) = z(g \cdot g^{-1}hg) = z(g)$, so z takes values in M^{I_p} and is a cocycle of D_p/I_p, which is topologically generated by a Frobenius ϕ. The map $z \mapsto z(\phi)$ is injective (a continuous cocycle vanishing on I_p and at ϕ vanishes on a dense subgroup) and onto M^{I_p}: for $x \in M^{I_p}$, if ϕ acts on M^{I_p} with order n', then $k \mapsto \sum_{i<k} \phi^i x$ depends only on k modulo $n'\ell$ (the sum of $n'\ell$ terms is $\ell \sum_{i<n'} \phi^i x = 0$), hence defines a continuous cocycle of D_p/I_p. So $\dim W = \dim M^{I_p} + \dim M - \dim M^{I_p} = \dim M$.

Item 2, the exact sequences. Write $h^i = \dim H^i(G_S, M)$, finite by Theorem B.37, item 1, and $h^i_p = \dim H^i(D_p, M)$, finite by Theorem B.29, item 1. Then $Z^1(G_S, M)$ and $Z^1(D_p, M)$ are finite, of dimensions $h^1 + \dim M - h^0$ and $h^1_p + \dim M - h^0_p$. Let $Z_{\mathcal{L}}$ be the Selmer cocycles; since $L_p \supseteq B^1(D_p, M)$, $Z_{\mathcal{L}} \supseteq B^1(G_S, M)$ and $\dim H^1_{\mathcal{L}}(M) = \dim Z_{\mathcal{L}} - \dim M + h^0$. Let $\alpha\colon H^2_{\mathcal{L}}(M) \to$

$H^2(G_S, M)$ send the class of $(z, (y_p))$ to the class of z; it is well defined since $B^2_{\mathcal{L}}(M)$ maps to coboundaries.

- The image of α is the kernel $\mathcal{K}$ of $H^2(G_S, M) \to \bigoplus_p H^2(D_p, M)$: if $dy_p = \mathrm{res}_p z$ the restrictions vanish, and conversely one chooses such y_p. By Theorem B.37, item 3, and $H^0(G_S, M^\vee) = 0$ the restriction map is onto, so $\dim \mathcal{K} = h^2 - \sum_p h^2_p$.
- The kernel of α is the image of $\bigoplus_p Z^1(D_p, M)$, $(y_p) \mapsto$ class of $(0, (y_p))$: if $z = dw$ then $(z, (y_p)) \equiv (0, (y_p - \mathrm{res}_p w))$ modulo $B^2_{\mathcal{L}}(M)$, and $d(y_p - \mathrm{res}_p w) = 0$. A family $(0, (y_p))$ with $y_p \in Z^1(D_p, M)$ lies in $B^2_{\mathcal{L}}(M)$ if and only if $y_p = \mathrm{res}_p w + \lambda_p$ with $dw = 0$ and $\lambda_p \in L_p$. So $\ker \alpha$ is the cokernel of $\rho\colon Z^1(G_S, M) \to \bigoplus_p Z^1(D_p, M)/L_p$, $w \mapsto (\mathrm{res}_p w)_p$, whose kernel is $Z_{\mathcal{L}}$.

Hence $H^2_{\mathcal{L}}(M)$ is finite-dimensional, $\dim \ker \alpha = \sum_p (h^1_p + \dim M - h^0_p - \dim L_p) - (h^1 + \dim M - h^0) + \dim Z_{\mathcal{L}}$, and

$$\begin{aligned}\dim H^1_{\mathcal{L}}(M) - \dim H^2_{\mathcal{L}}(M) &= \dim H^1_{\mathcal{L}}(M) - \dim \ker \alpha - \dim \mathcal{K} \\ &= h^1 - h^2 + \sum_p (h^0_p - h^1_p + h^2_p - \dim M + \dim L_p).\end{aligned}$$

Item 2, the Euler characteristics. By Theorem B.29, item 4, $h^0_p - h^1_p + h^2_p$ is 0 for $p \neq \ell$ and $-\dim M$ for $p = \ell$ ($[\mathbb{Q}_\ell : \mathbb{Q}_\ell] = 1$). By Theorem B.37, item 2, for $F = \mathbb{Q}$ (one real place ∞, with $|\#M|_\infty = \#M$ and $H^0(\mathbb{R}, M)$ the fixed space of a complex conjugation, of the same dimension as $M^{c=1}$ since all complex conjugations are conjugate in $G_\mathbb{Q}$), $h^0 - h^1 + h^2 = \dim M^{c=1} - \dim M$. Substituting,

$$\dim H^1_{\mathcal{L}}(M) - \dim H^2_{\mathcal{L}}(M) = (h^0 - \dim M^{c=1} + \dim M) - \dim M + \sum_p (\dim L_p - \dim M),$$

which is the assertion. □

Lemma 43.36. *Let ℓ be an odd prime and let $\mathbb{F}$, S, G_S, M and $(D_p)_p$ be as in Definition 43.1; let $c \in G_\mathbb{Q}$ be a complex conjugation and $M^{c=1}$ the subspace of M fixed by c. Let $\mathcal{L} = (L_p)_p$ be a family of local conditions for M with $L_\ell = B^1(D_\ell, M)$ and $B^1(D_p, M) \subseteq L_p$ for every prime $p \in S$, and assume $H^0(G_S, M^\vee) = 0$. Let*

$$Z_0 = \{z \in Z^1(G_S, M) :\ \mathrm{res}_\ell z = 0,\ \mathrm{res}_p z \in L_p \text{ for every prime } p \in S\}.$$

Then Z_0 and $H^2_{\mathcal{L}}(M)$ are finite-dimensional and

$$\dim H^2_{\mathcal{L}}(M) - \dim Z_0 = \dim M^{c=1} + \sum_{p \in S \text{ prime},\ p \neq \ell} (\dim M - \dim L_p).$$

Proof. Let $Z_{\mathcal{L}}$ be the space of Selmer cocycles, so that $Z_0 \subseteq Z_{\mathcal{L}}$ ($0 \in L_\ell$). By Lemma 43.35, item 2 (its hypotheses are those made here), $H^1_{\mathcal{L}}(M)$ and $H^2_{\mathcal{L}}(M)$ are finite-dimensional and

$$\dim H^1_{\mathcal{L}}(M) - \dim H^2_{\mathcal{L}}(M) = h^0 - \dim M^{c=1} + \sum_{p \in S \text{ prime}} (\dim L_p - \dim M), \qquad h^0 = \dim H^0(G_S, M). \tag{43.9}$$

Step 1: $\dim Z_{\mathcal{L}} = \dim H^1_{\mathcal{L}}(M) + \dim M - h^0$. For $x \in M$ the coboundary $b_x\colon g \mapsto gx - x$ restricts at every prime $p \in S$ to an element of $B^1(D_p, M) \subseteq L_p$, so $B^1(G_S, M) \subseteq Z_{\mathcal{L}}$, and

$H^1_{\mathcal{L}}(M) = Z_{\mathcal{L}}/B^1(G_S, M)$. The map $x \mapsto b_x$ from M onto $B^1(G_S, M)$ has kernel $H^0(G_S, M)$, so $\dim B^1(G_S, M) = \dim M - h^0$.

Step 2: $\dim Z_0 = \dim Z_{\mathcal{L}} - \dim M + h^0_\ell$, *where* $h^0_\ell = \dim H^0(D_\ell, M)$. The restriction $z \mapsto \mathrm{res}_\ell z$ maps $Z_{\mathcal{L}}$ into $L_\ell = B^1(D_\ell, M)$, and its kernel is Z_0. It is onto: an element of $B^1(D_\ell, M)$ is $\mathrm{res}_\ell b_x$ for some $x \in M$, and $b_x \in Z_{\mathcal{L}}$ by Step 1. As in Step 1, $\dim B^1(D_\ell, M) = \dim M - h^0_\ell$. In particular Z_0 is finite-dimensional.

Step 3: the count. By Steps 1 and 2, $\dim Z_0 = \dim H^1_{\mathcal{L}}(M) - h^0 + h^0_\ell$. In (43.9) the term of $p = \ell$ is $\dim L_\ell - \dim M = -h^0_\ell$. Hence

$$\dim H^2_{\mathcal{L}}(M) - \dim Z_0 = \dim H^2_{\mathcal{L}}(M) - \dim H^1_{\mathcal{L}}(M) + h^0 - h^0_\ell = \dim M^{c=1} - \sum_{p \neq \ell} (\dim L_p - \dim M),$$

which is the assertion. □

44 The symplectic one-character bound

This section proves Proposition 44.3, the one-character bound for the characters in cases (II) and (III), which Proposition 38.26 uses. These are characters for which $(\ell - 1) \mid 2e$ or ℓ lies in the bad set, and the unitary construction of Section 42 is replaced by one on GSp_4 over $\mathbb{Q}$. The Eisenstein representation is built from $\Psi = \chi\chi^e_{\mathrm{cyc}}$ and an auxiliary character $\alpha = \chi_{\mathrm{cyc}}\omega_\ell^{-1}\xi$, with similitude $\theta = \alpha\Psi$, and its trace is $(1 + \Psi)(1 + \alpha)$ (Definition 44.1). The character ξ, of large 2-power conductor and in case (III) with a factor modulo $p_1(\psi)$, is chosen so that the residual Selmer cocycles of α, $\beta = \Psi\alpha^{-1}$, θ and their inverses are coboundaries (Sg6). This is Lemma 44.5; it rests on the fact that the first Bernoulli number of an odd character with a large 2-power conductor is an ℓ-adic unit (Lemma 44.6), together with a vanishing of Herbrand type for unramified cocycles (Lemma 44.7); the existence of an attached datum is Proposition 44.4.

The proof follows that of Proposition 42.2. The graded model is Proposition 45.2 (Section 45); Part IX gives finiteness in weight zero, the absence of endoscopic points and the automorphy of the irreducible points (Theorem 51.1). Rigidity at the irreducible automorphic points, Theorem 44.8, follows from Thorne's vanishing theorem for adjoint Bloch–Kato Selmer groups [257], after a character twist over an imaginary quadratic field makes the representation conjugate self-dual (Proposition 44.13) and after checking that it stays irreducible over the cyclotomic tower (Lemma 44.9). Finally the irreducible automorphic points are admissible symplectic of weight e and level M (Proposition 44.15), and isomorphic points are torus conjugate (Lemma 44.16), so the orbit cover of the criterion is a set of pairwise non-isomorphic admissible representations.

Definition 44.1. Let ℓ be an odd prime and $\mathcal{O}$ a coefficient ring with maximal ideal $\mathfrak{m}_{\mathcal{O}}$ and residue field $\mathbb{F}_{\mathcal{O}}$; for a character $\gamma\colon G_{\mathbb{Q}} \to \mathcal{O}^\times$ write $\bar\gamma$ for its reduction modulo $\mathfrak{m}_{\mathcal{O}}$. The **Teichmüller character** $\omega_\ell\colon G_{\mathbb{Q}} \to \mathbb{Z}_\ell^\times \subseteq \mathcal{O}^\times$ is the character with $\omega_\ell^{\ell-1} = \mathbf{1}$ and $\bar\omega_\ell = \bar\chi_{\mathrm{cyc}}$. Fix a complex conjugation $c \in G_{\mathbb{Q}}$ and, for every prime p, a place $\bar p$ of $\overline{\mathbb{Q}}$ above p; write $D_p \supseteq I_p$ for $D_{\bar p} \supseteq I_{\bar p}$.

Symplectic Eisenstein data. Let $e \geq 2$ and $N \geq 1$ be integers with N even, χ a Dirichlet character modulo N and ξ a Dirichlet character, both with values in $\mathcal{O}^\times$ and viewed as characters of $G_{\mathbb{Q}}$, ξ of order prime to ℓ, and S a finite set of primes. Put $\Psi = \Psi_{\chi,e} = \chi\chi^e_{\mathrm{cyc}}$, $T = T_\ell(\chi)$ and

$$\alpha = \chi_{\mathrm{cyc}}\,\omega_\ell^{-1}\,\xi, \qquad \theta = \alpha\Psi, \qquad \beta = \Psi\alpha^{-1}. \tag{44.1}$$

For $\gamma\colon G_{\mathbb{Q}} \to \mathcal{O}^\times$ a **residual Selmer cocycle** for γ is a locally constant map $z\colon G_{\mathbb{Q}} \to \mathbb{F}_{\mathcal{O}}$ with $z(gh) = z(g) + \bar\gamma(g)z(h)$ for all g, h, such that $z(g) = 0$ for every $g \in I_{\bar w}$ with $\bar\theta(g) = \bar\Psi(g) =$

$\bar\alpha(g)=1$ and every place $\bar w$ of $\overline{\mathbb{Q}}$ above a prime not in T, and such that for every $v\in T$ there is $y\in\mathbb{F}_{\mathcal O}$ with $z(g)=\bar\gamma(g)y-y$ for all $g\in D_v$. A tuple $\mathfrak D=(\ell,\mathcal O,e,N,\chi,\xi,S,(\sigma_v)_{v\in T},\phi,g_0,s_0)$ with $\sigma_v,\phi,g_0,s_0\in G_{\mathbb Q}$ is a **symplectic Eisenstein datum** if

(Sg1) S contains 2, ℓ and every prime dividing N; ξ is unramified at every prime not in S and at every prime of T; $\xi(c)=-1$ and $\Psi(c)=1$;

(Sg2) for every $v\in T$, $\sigma_v\in D_v$ is an arithmetic Frobenius element at $\bar v$; $\chi_{\mathrm{cyc}}(\sigma_\ell)=1$; and $\overline{\xi\Psi^i}(\sigma_v)\neq 1$ for every $v\in T$ and every $i\in\mathbb{Z}$;

(Sg3) the sequence $\sigma_\ell^{\ell^{n!}}$ converges to ϕ as $n\to\infty$;

(Sg4) $\bar\alpha(g_0)\neq 1$ and $\bar\Psi(g_0)\neq 1$;

(Sg5) $s_0\in I_\ell$, $\chi(s_0)=\omega_\ell(s_0)=1$, and $\chi_{\mathrm{cyc}}(s_0)$ has infinite order;

(Sg6) for every $\gamma\in\{\alpha,\beta,\theta,\alpha^{-1},\beta^{-1},\theta^{-1}\}$, every residual Selmer cocycle for γ is a coboundary, that is of the form $g\mapsto\bar\gamma(g)y-y$ with $y\in\mathbb{F}_{\mathcal O}$.

The characters (44.1) and the sets T, S are those **of the datum**.

Data attached to a host character. Let $\psi\in\mathcal P$, let $\mathfrak h$ be a Sha-host of ψ big at an integer $t\geq 1$, $0<e<t$, and $\chi\in X_{\mathfrak h}(\mathcal O,e)$ in case (II) or in case (III) for (ℓ,e). Define integers $n',p',m\geq 1$: in case (II), n' is the least integer $n\geq 0$ with $2^n\geq\max\bigl(\ell N_{\mathfrak h}4^{n_0(\ell,\chi)},\ 2^{c_2(\chi)+6}o_\psi\ell^3N_\psi^2\bigr)$ and $p'=m=1$; in case (III), n' is the least integer $n\geq 0$ with $2^n\geq\max\bigl(\ell N_\psi\,p_1(\psi)\,4^{n_0(\ell,\chi)},\ 8\ell^2\bigr)$, $p'=p_1(\psi)$ and m is the order of χ. A symplectic Eisenstein datum $\mathfrak D=(\ell,\mathcal O,e',N,\chi',\xi,S,\dots)$ is **attached to** $(\psi,\mathfrak h,t,\chi)$ if $e'=e$, $N=N_{\mathfrak h}$, $\chi'=\chi$, S is the set of primes dividing $2\ell N_{\mathfrak h}p'$, and $\xi=\chi_{\mathrm{dy}}\chi_{\mathrm{pr}}$ for a primitive Dirichlet character χ_{dy} modulo $2^{n'}$ with $\chi_{\mathrm{dy}}(-1)=-1$ and a Dirichlet character χ_{pr} modulo p' of order m, both with values in $\mathcal O^\times$; the integer

$$M=\ell\,2^{n'}N_{\mathfrak h}\,p' \tag{44.2}$$

is its **level**. These are the data on which the Eisenstein congruences on GSp_4 bound the strict Tate–Shafarevich groups of the characters in cases (II) and (III).

Definition 44.2. Let $\mathfrak D=(\ell,\mathcal O,e,N,\chi,\xi,S,(\sigma_v),\phi,g_0,s_0)$ be a symplectic Eisenstein datum, with characters Ψ,α,θ, the set T, maximal ideal $\mathfrak m_{\mathcal O}$ and residue field $\mathbb{F}_{\mathcal O}$. Indices i,j run over $\{1,2,3,4\}$.

The group and the torus. Let J_4 be the 4×4 matrix with $(J_4)_{14}=(J_4)_{23}=1$, $(J_4)_{32}=(J_4)_{41}=-1$ and all other entries 0, so that ${}^tJ_4=-J_4$. For a commutative ring B and $\nu\in B$, a matrix $X\in\mathrm M_4(B)$ **has similitude** ν if ${}^tX\,J_4\,X=\nu J_4$; and $\mathfrak{sp}_4(B)=\{X\in\mathrm M_4(B):{}^tXJ_4+J_4X=0\}$. Put $\boldsymbol\lambda=(1,0,1,0)$, $\boldsymbol\mu=(1,1,0,0)$, the **weight** of the entry (i,j)

$$\mathrm{wt}_4(i,j)=(\lambda_i-\lambda_j,\ \mu_i-\mu_j)\in\mathbb{Z}^2,$$

and, for units t_1,t_2 of B, $\mathrm T_4(t_1,t_2)=\mathrm{diag}(t_1t_2,t_2,t_1,1)$; then $\mathrm T_4(t_1,t_2)\,X\,\mathrm T_4(t_1,t_2)^{-1}$ has entries $t_1^{\lambda_i-\lambda_j}t_2^{\mu_i-\mu_j}X_{ij}$. A matrix is **block diagonal** if its entries (i,j) with $\lambda_i\neq\lambda_j$ vanish (blocks $\{1,3\}$ and $\{2,4\}$). The **Eisenstein representation** is

$$r_0=\mathrm{diag}(\theta,\Psi,\alpha,\mathbf 1)\colon G_{\mathbb Q}\to\mathrm{GL}_4(\mathcal O),$$

with similitude θ at every g, and $\bar r_0$ is its reduction modulo $\mathfrak m_{\mathcal O}$; write $(\kappa_1,\kappa_2,\kappa_3,\kappa_4)=(\theta,\Psi,\alpha,\mathbf 1)$.

Lifts and the local conditions. For $A \in \mathrm{CNL}_{\mathcal{O}}$ with maximal ideal $\mathfrak{m}_A$, a **lift** to A is a continuous homomorphism $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(A)$ with $r \equiv \bar{r}_0 \pmod{\mathfrak{m}_A}$; a lift of $\bar{r}_0|_{D_v}$ is defined in the same way. For B a commutative $\mathcal{O}$-algebra and $\rho\colon D_\ell \to \mathrm{GL}_4(B)$, put $\Pi_i = \prod_{j\neq i}(\rho(s_0) - \kappa_j(s_0))$; ρ is **ordinary** if the characteristic polynomial of $\rho(s_0)$ is $\prod_j(X - \kappa_j(s_0))$, $\Pi_j\rho(g)\Pi_i = 0$ for all $g \in D_\ell$ and $i < j$, and $\Pi_i(\rho(s) - \kappa_i(s))\Pi_i = 0$ for all $s \in I_\ell$ and all i. For $v \in T$, a lift ρ of $\bar{r}_0|_{D_v}$ to $B \in \mathrm{CNL}_{\mathcal{O}}$ **satisfies the local condition at** v if $\rho(g)$ has similitude $\theta(g)$ for all $g \in D_v$ and, for $v = \ell$, ρ is ordinary, for $v \neq \ell$, $\rho(g) = 1$ for all $g \in I_v$ with $\chi(g) = 1$. A lift r to A is a **symplectic Eisenstein lift** if

(Se1) $r(g)$ has similitude $\theta(g)$ for every $g \in G_{\mathbb{Q}}$;

(Se2) $r(g) = 1$ for every $g \in I_{\bar{w}}$ with $\bar{r}_0(g) = 1$ and every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above a prime not in T;

(Se3) $r(\phi)$ is block diagonal, and the matrix $h(r) = r(g_0) + \theta(g_0)\, r(g_0^{-1})$ has $h(r)_{13} = h(r)_{31} = 0$;

(Se4) for every $v \in T$, $r|_{D_v}$ lies on a component through the Eisenstein point: there are a domain $B \in \mathrm{CNL}_{\mathcal{O}}$, a lift ρ_B of $\bar{r}_0|_{D_v}$ to B satisfying the local condition at v, and $\mathcal{O}$-algebra homomorphisms $x\colon B \to \mathcal{O}$ and $y\colon B \to A$ with $x \circ \rho_B = r_0|_{D_v}$ and $y \circ \rho_B = r|_{D_v}$.

Points. Let B be $\overline{\mathbb{Q}}_\ell$ or the dual numbers $\overline{\mathbb{Q}}_\ell[\epsilon]$ ($\epsilon^2 = 0$). A **point** of $\mathfrak{D}$ with values in B is a homomorphism $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(B)$ of the form $g \mapsto f(r_A(g))$ for a symplectic Eisenstein lift r_A to some $A \in \mathrm{CNL}_{\mathcal{O}}$ and an $\mathcal{O}$-algebra homomorphism $f\colon A \to B$; a point means a point with values in $\overline{\mathbb{Q}}_\ell$. A point r is **irreducible** if the $\overline{\mathbb{Q}}_\ell$-span of $r(G_{\mathbb{Q}})$ is $\mathrm{M}_4(\overline{\mathbb{Q}}_\ell)$; its **torus conjugates** are the points $\mathrm{T}_4(t_1,t_2)\, r\, \mathrm{T}_4(t_1,t_2)^{-1}$, $t_1, t_2 \in \overline{\mathbb{Q}}_\ell^{\times}$. A point r is **rigid** if every point r' with values in $\overline{\mathbb{Q}}_\ell[\epsilon]$ whose reduction modulo ϵ is r has the form $r'(g) = r(g) + \epsilon\big(Xr(g) - r(g)X\big)$ for some $X \in \mathfrak{sp}_4(\overline{\mathbb{Q}}_\ell)$. For an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, a point is **automorphic** for ι if it satisfies (Sy2) for ι.

Two global properties. $\mathfrak{D}$ is **finite in weight zero** if for every symplectic Eisenstein lift r_A to some $A \in \mathrm{CNL}_{\mathcal{O}}$ the $\mathcal{O}$-subalgebra of A generated by the products $r_A(g_1)_{i_1j_1} \cdots r_A(g_k)_{i_kj_k}$ ($k \geq 0$) with $\sum_s \mathrm{wt}_4(i_s, j_s) = (0,0)$ is a finitely generated $\mathcal{O}$-module. $\mathfrak{D}$ has **no endoscopic points** if there is no point r such that every $r(g)$ is block diagonal and the matrices $\big(r(g)_{ab}\big)_{a,b\in\{1,3\}}$, $g \in G_{\mathbb{Q}}$, span $\mathrm{M}_2(\overline{\mathbb{Q}}_\ell)$.

Proposition 44.3. *Let $\psi \in \mathcal{P}$, let $\mathfrak{h}$ be a Sha-host of ψ that is big at an integer $t \geq 1$, let ℓ be an odd prime, e an integer with $0 < e < t$, $\mathcal{O}$ a coefficient ring and $\chi \in X_{\mathfrak{h}}(\mathcal{O}, e)$ in case (II) or in case (III) for (ℓ, e); write $N = N_{\mathfrak{h}}$. (The two cases exclude each other: in case (III) χ has odd order, hence so has $\chi^{(\ell)}$, a product of powers of χ.) Define integers $n', p', m \geq 1$ as follows:*

- *in case (II), n' is the least integer $n \geq 0$ with $2^n \geq \max\big(\ell N 4^{n_0(\ell,\chi)},\ 2^{c_2(\chi)+6} o_\psi \ell^3 N_\psi^2\big)$, and $p' = m = 1$;*
- *in case (III), n' is the least integer $n \geq 0$ with $2^n \geq \max\big(\ell N_\psi\, p_1(\psi)\, 4^{n_0(\ell,\chi)},\ 8\ell^2\big)$, $p' = p_1(\psi)$ and m is the order of χ.*

Assume that $\mathcal{O}$ contains a primitive $2^{n'}$-th root of unity. Put $M = \ell\, 2^{n'} N\, p'$, and let $\omega_\ell\colon G_{\mathbb{Q}} \to \mathbb{Z}_\ell^{\times} \subseteq \mathcal{O}^{\times}$ be the Teichmüller character, the character with $\omega_\ell^{\ell-1} = \mathbf{1}$ and $\omega_\ell \equiv \chi_{\mathrm{cyc}} \pmod{\ell}$. Then there are Dirichlet characters χ_{dy} modulo $2^{n'}$ and χ_{pr} modulo p' with values in $\mathcal{O}^{\times}$, χ_{dy} primitive with $\chi_{\mathrm{dy}}(-1) = -1$ and χ_{pr} of order m, and an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, with the following property. Put $\Psi = \Psi_{\chi,e}$, $\Psi_{\mathrm{aux}} = \chi_{\mathrm{cyc}}\, \omega_\ell^{-1} \chi_{\mathrm{dy}} \chi_{\mathrm{pr}}$ and, for $\sigma \in G_{\mathbb{Q}}$,

$$T_{\mathrm{Eis}}(\sigma) = \big(1 + \Psi(\sigma)\big)\big(1 + \Psi_{\mathrm{aux}}(\sigma)\big) \in \mathcal{O}.$$

For every $\sigma \in G_{\mathbb{Q}}$ such that $\operatorname{tr} r(\sigma) \neq T_{\mathrm{Eis}}(\sigma)$ for every representation $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ admissible symplectic of weight e and level M (for ι), the group $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length, and there is a finite set X of pairwise non-isomorphic representations $G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$, each admissible symplectic of weight e and level M (for ι), such that

$$\nu_{\mathcal{O}}(\chi, e) \leq 2 \sum_{r \in X} v_\ell(\operatorname{tr} r(\sigma) - T_{\mathrm{Eis}}(\sigma)).$$

Proof. Let $\psi, \mathfrak{h}, t, \ell, e, \mathcal{O}, \chi$, $N = N_{\mathfrak{h}}$, n', p', m and $M = \ell 2^{n'} N p'$ be as in the statement.

Step 1: the datum and its Eisenstein trace. Since $\mathcal{O}$ contains a primitive $2^{n'}$-th root of unity, Proposition 44.4 gives a symplectic Eisenstein datum $\mathfrak{D} = (\ell, \mathcal{O}, e, N, \chi, \xi, S, (\sigma_v), \phi, g_0, s_0)$ attached to $(\psi, \mathfrak{h}, t, \chi)$: S is the set of primes dividing $2\ell N p'$, and $\xi = \chi_{\mathrm{dy}}\chi_{\mathrm{pr}}$ with χ_{dy} primitive modulo $2^{n'}$, $\chi_{\mathrm{dy}}(-1) = -1$, and χ_{pr} modulo p' of order m, both with values in $\mathcal{O}^\times$. These are the characters $\chi_{\mathrm{dy}}, \chi_{\mathrm{pr}}$ of the statement. The level of $\mathfrak{D}$ is M. The characters of the datum are $\alpha = \chi_{\mathrm{cyc}}\omega_\ell^{-1}\xi = \Psi_{\mathrm{aux}}$ and $\theta = \alpha\Psi = \Psi\Psi_{\mathrm{aux}}$, so $r_0 = \operatorname{diag}(\Psi\Psi_{\mathrm{aux}}, \Psi, \Psi_{\mathrm{aux}}, \mathbf{1})$ and, for every $g \in G_{\mathbb{Q}}$,

$$\operatorname{tr} r_0(g) = \Psi(g)\Psi_{\mathrm{aux}}(g) + \Psi(g) + \Psi_{\mathrm{aux}}(g) + 1 = T_{\mathrm{Eis}}(g). \tag{44.3}$$

Step 2: automorphy, rigidity and admissibility of the irreducible points. By Theorem 51.1, $\mathfrak{D}$ is finite in weight zero (Sa1), has no endoscopic points (Sa2), and there is an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ for which every irreducible point of $\mathfrak{D}$ is automorphic (Sa3); this is the ι of the statement. By Theorem 44.8 every irreducible point of $\mathfrak{D}$ is rigid, and by Proposition 44.15 every irreducible point of $\mathfrak{D}$ is admissible symplectic of weight e and level M for ι.

Step 3: the graded model satisfies (Gc1)–(Gc4). Let (A, φ_0), u, $(a_g)_{g \in G_{\mathbb{Q}}}$ be as in Proposition 45.2, and, $\mathfrak{D}$ being finite in weight zero, let $x \mapsto r_x$ be a map with (Gs3)–(Gs6). By (Gs1), (Gs5) and (Gs6), which applies since $\mathfrak{D}$ has no endoscopic points and all its irreducible points are rigid (Step 2), the triple (A, φ_0, u) satisfies (Gc1)–(Gc4), and r_x is an irreducible point of $\mathfrak{D}$ for every two-signed point x of A on u.

Step 4: the weight-zero function, (Gc5). Let $\sigma \in G_{\mathbb{Q}}$ be as in the statement and $a = a_\sigma \in A_0$; then $\varphi_0(a) = 0$. Let x be a two-signed point of A on u. By Step 3 r_x is an irreducible point, hence admissible symplectic of weight e and level M for ι (Step 2), so $\operatorname{tr} r_x(\sigma) \neq T_{\mathrm{Eis}}(\sigma)$ by the hypothesis on σ. By (Gs3) and (44.3),

$$x(a) = \operatorname{tr} r_x(\sigma) - T_{\mathrm{Eis}}(\sigma) \neq 0. \tag{44.4}$$

So (A, φ_0, u, a) satisfies (Gc1)–(Gc5).

Step 5: the graded numerical criterion. By Lemma 41.2, u has an orbit cover, $\mathrm{ct}_{-1}(u_{-1})$ is finite, and $\mathrm{ct}_{-1}(u_{-1}) \leq 2\sum_{x \in \Sigma} v_{\mathcal{O}}(x(a))$ for every orbit cover Σ of u.

Step 6: $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length at most $\mathrm{ct}_{-1}(u_{-1})$. Put $n_S = \prod_{p \in S} p$ and $V = \mathcal{O}^\vee(\Psi^{-1})$. First, $\mathrm{Sha}^1_{\ell N}(\chi, e) = \mathrm{Sha}^1_{n_S}(\chi, e)$. Since N is even, the primes dividing n_S are those dividing ℓN together with p' when $p' > 1$; so $\mathrm{Sha}^1_{n_S}(\chi, e) \subseteq \mathrm{Sha}^1_{\ell N}(\chi, e)$, with equality when $p' = 1$. Let $p' > 1$, so we are in case (III) and $p' = p_1(\psi)$, an odd prime prime to N_ψ. Then $p' \nmid N$: $p' \nmid 2N_\psi$, and $p' < \min(q, d)$ by bigness, where $\mathfrak{h} = (a', q, d)$ and $N = \mathrm{lcm}(2^{a'+2}, N_\psi)qd$. Also $p' \neq \ell$, since $p' \in B_\psi$ and $\ell \notin B_\psi$ in case (III). Let $c' \in \mathrm{Sha}^1_{\ell N}(\chi, e)$. Since $p' \nmid \ell N$, Ψ is unramified at p' and $I_{p'}$ acts trivially on V; the class c' is unramified at p', so by inflation–restriction its restriction to $D_{p'}$ is inflated from $H^1(D_{p'}/I_{p'}, V)$. The group $D_{p'}/I_{p'}$ is procyclic, topologically generated by the image F of an arithmetic Frobenius, and for the discrete torsion module V one has $H^1(D_{p'}/I_{p'}, V) \cong V/(F-1)V$. Here F acts on V as multiplication by $\Psi(F)^{-1} = (\chi(p')p'^e)^{-1}$, and $\Psi(F) \neq 1$, since otherwise the integer $p'^e > 1$ would be a root of unity. As V is a divisible

$\mathcal{O}$-module (the Pontryagin dual of the injective multiplication by $\Psi(F)^{-1}-1\neq 0$ on $\mathcal{O}$ is surjective), $(F-1)V=V$, the restriction of c' to $D_{p'}$ vanishes, and $c'\in \mathrm{Sha}^1_{n_S}(\chi,e)$.

Now let $b\in\mathcal{O}\setminus\{0\}$ and $\mathrm{Sha}[b]$ the submodule of $\mathrm{Sha}^1_{n_S}(\chi,e)$ killed by b. Let $a\in b\mathcal{O}\setminus\{0\}$ and D be as in (Gs2); then D satisfies the hypotheses of Lemma 42.12 for $j=-1$, the family u_{-1}, $c=a$ and the module $\mathrm{Sha}[b]$; hence $\mathrm{length}_{\mathcal{O}}\mathrm{Sha}[b]\le \mathrm{ct}_{-1}(u_{-1})$. Every class of $H^1(\mathbb{Q},V)$ is represented by a continuous cocycle on the compact group $G_{\mathbb{Q}}$ with values in the discrete module V, which has finite image; every element of V is killed by a power of ℓ; so every class is killed by some ℓ^k and $\mathrm{Sha}^1_{n_S}(\chi,e)=\bigcup_k \mathrm{Sha}[\ell^k]$, an increasing union. If $\mathrm{Sha}^1_{n_S}(\chi,e)$ contained a strictly increasing chain $M_0\subsetneq M_1\subsetneq\cdots\subsetneq M_L$ of submodules with $L=\mathrm{ct}_{-1}(u_{-1})+1$, choose $y_i\in M_i\setminus M_{i-1}$; the submodule generated by $y_1,\dots,y_L$ lies in some $\mathrm{Sha}[\ell^k]$ and contains the strictly increasing chain of the submodules generated by $y_1,\dots,y_i$ $(1\le i\le L)$, so it has length at least L, a contradiction. Hence $\mathrm{Sha}^1_{\ell N}(\chi,e)=\mathrm{Sha}^1_{n_S}(\chi,e)$ has finite length, at most $\mathrm{ct}_{-1}(u_{-1})$.

Step 7: from a minimal orbit cover to representations. By Step 5 orbit covers exist; let Σ be one of least cardinality. Let $x\neq x'$ in Σ and suppose $r_{x'}\cong r_x$. Since x is two-signed, r_x is an irreducible point (Step 3), so by Lemma 44.16 $r_{x'}$ is a torus conjugate of r_x, and by (Gs4) $x'=t\cdot x$ for some $t\in\overline{\mathbb{Q}}_\ell^\times$. Then $\Sigma\setminus\{x'\}$ is an orbit cover: a two-signed point y on u is $s\cdot z$ with $z\in\Sigma$ and $s\in\overline{\mathbb{Q}}_\ell^\times$, and if $z=x'$ then $y=(st)\cdot x$, because $(s\cdot(t\cdot x))(f)=s^jt^jx(f)$ for $f\in A_j$. This contradicts the minimality of Σ. Hence the representations r_x, $x\in\Sigma$, are pairwise non-isomorphic, in particular pairwise distinct. Put $X=\{r_x:\ x\in\Sigma\}$, a finite set of pairwise non-isomorphic representations $G_{\mathbb{Q}}\to\mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$, each admissible symplectic of weight e and level M for ι (Steps 2 and 3). With Steps 6, 5, (44.4) and $v_{\mathcal{O}}=e_{\mathcal{O}}v_\ell$,

$$\nu_{\mathcal{O}}(\chi,e)=\frac{\mathrm{length}_{\mathcal{O}}\mathrm{Sha}^1_{\ell N}(\chi,e)}{e_{\mathcal{O}}}\le\frac{\mathrm{ct}_{-1}(u_{-1})}{e_{\mathcal{O}}}\le\frac{2}{e_{\mathcal{O}}}\sum_{x\in\Sigma}v_{\mathcal{O}}(x(a))=2\sum_{r\in X}v_\ell(\mathrm{tr}\,r(\sigma)-T_{\mathrm{Eis}}(\sigma)).$$

□

Proposition 44.4. *Let $\psi\in\mathcal{P}$, let $\mathfrak{h}$ be a Sha-host of ψ big at an integer $t\ge 1$, ℓ an odd prime, e an integer with $0<e<t$, $\mathcal{O}$ a coefficient ring and $\chi\in X_{\mathfrak{h}}(\mathcal{O},e)$ in case (II) or in case (III) for (ℓ,e). Let n' be the integer of the case. If $\mathcal{O}$ contains a primitive $2^{n'}$-th root of unity, then there is a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$ attached to $(\psi,\mathfrak{h},t,\chi)$.*

Proof. Write $\mathfrak{h}=(a,q,d)$, $N=N_{\mathfrak{h}}=\mathrm{lcm}(2^{a+2},N_\psi)qd$, $T=T_\ell(\chi)$; reduction modulo $\mathfrak{m}_{\mathcal{O}}$ is written with a bar and is injective on roots of unity of order prime to ℓ. Recall $(\mathbb{Z}/2^k\mathbb{Z})^\times=\{\pm1\}\times\langle 5\rangle$ with 5 of order 2^{k-2} for $k\ge 3$ [266].

Step 1: the characters χ_{dy}, χ_{pr} and ξ. By the definition of n', $2^{n'}\ge 2^{c_2(\chi)+6}\ge 2^6$ in case (II) and $2^{n'}\ge 8\ell^2>8$ in case (III), so $n'\ge 3$. Let $\zeta\in\mathcal{O}$ be a primitive $2^{n'}$-th root of unity and define χ_{dy} on $(\mathbb{Z}/2^{n'}\mathbb{Z})^\times$ by $\chi_{\mathrm{dy}}(-1)=-1$, $\chi_{\mathrm{dy}}(5)=\zeta^4$, a primitive $2^{n'-2}$-th root of unity. It is non-trivial on $\{u\equiv 1\pmod{2^{n'-1}}\}=\langle 5^{2^{n'-3}}\rangle$, hence primitive modulo $2^{n'}$, and $\chi_{\mathrm{dy}}(-1)=-1$. In case (II), $p'=m=1$ and $\chi_{\mathrm{pr}}=\mathbf{1}$. In case (III), m is the order of χ; χ has odd order, so by Lemma 38.34, items 1 and 2, χ is a product of components at odd primes dividing N_ψ and $m\mid o_\psi$; m is odd, so it divides the odd part of o_ψ, hence $p'-1$, by the definition of $p'=p_1(\psi)$. As $(\mathbb{Z}/p'\mathbb{Z})^\times$ is cyclic of order $p'-1$ and $\mathcal{O}$ contains the values of χ, among them a primitive m-th root of unity, there is a Dirichlet character χ_{pr} modulo p' of order m with values in $\mathcal{O}^\times$. Put $\xi=\chi_{\mathrm{dy}}\chi_{\mathrm{pr}}$ and let α,β,θ,Ψ be the characters (44.1). By Lemma 44.5, $e\ge 2$ and ξ has order prime to ℓ; N is even since $a\ge 2$. Let S be the set of primes dividing $2\ell Np'$.

Step 2: the primes of T. Let $v\in S_\ell(\chi)$. The component χ''_v is a power of χ_v of ℓ-power order, ℓ odd, and non-trivial. If $v=2$ it would also have 2-power order; if $v\in\{q,d\}$ its square would be trivial (Lemma 38.34, item 1); both are excluded. So v is an odd prime dividing N_ψ,

$v \le N_\psi$, and $\ell \mid v-1$, because χ''_v is a non-trivial character of ℓ-power order of $(\mathbb{Z}/v^k\mathbb{Z})^\times$, of order $v^{k-1}(v-1)$, with $v \ne \ell$. In case (III), χ has order prime to ℓ (Lemma 44.5), so $\chi'' = \mathbf{1}$ and $T = \{\ell\}$. In all cases every $v \in T$ is odd, and $p' \notin T$ when $p' > 1$: in case (III) $T = \{\ell\}$ and $p' \in B_\psi \not\ni \ell$.

Step 3: (Sg1). The set S contains 2, ℓ and the primes dividing N. The conductor of ξ divides $2^{n'}p'$, so ξ is unramified at every prime $p \nmid 2p'$, in particular outside S and, by Step 2, at the primes of T. For the fixed complex conjugation c, $\xi(c) = \chi_{\mathrm{dy}}(-1)\chi_{\mathrm{pr}}(-1) = -1$, since χ_{pr} has odd order and so $\chi_{\mathrm{pr}}(-1) = 1$; and $\Psi(c) = \chi(-1)\chi_{\mathrm{cyc}}(c)^e = (-1)^e(-1)^e = 1$ as $\chi \in X_{\mathfrak{h}}(\mathcal{O}, e)$.

Step 4: the Frobenius elements, and (Sg2). For $v \in T \setminus \{\ell\}$ let $\sigma_v \in D_v$ be any arithmetic Frobenius element at $\bar{v}$. At ℓ: the maximal unramified extension $\mathbb{Q}_\ell^{\mathrm{ur}}$ and $\mathbb{Q}_\ell(\mu_{\ell^\infty})$ (totally ramified) of $\mathbb{Q}_\ell$, inside the completion of $\overline{\mathbb{Q}}$ at $\bar{\ell}$, are linearly disjoint, so the Galois group of their composite is the product of their Galois groups, and there is $\sigma_\ell \in D_\ell$ acting as the Frobenius on $\mathbb{Q}_\ell^{\mathrm{ur}}$ and trivially on μ_{ℓ^∞}: an arithmetic Frobenius element at $\bar{\ell}$ with $\chi_{\mathrm{cyc}}(\sigma_\ell) = 1$. For $v \in T$ we have $\bar{\chi}_{\mathrm{cyc}}(\sigma_v) = 1$: for $v = \ell$ by the choice, for $v \ne \ell$ because $\chi_{\mathrm{cyc}}(\sigma_v) = v \equiv 1 \pmod{\ell}$ (Step 2).

We choose $a_0 \ge 0$ and an odd $u \ge 1$ with $\chi^{2^{a_0}u} = \chi_{\mathrm{pr}}^{2^{a_0}u} = \mathbf{1}$ and $v^2 2^{a_0} \le 2^{n'}$ for all $v \in T$. In case (III) take $a_0 = 0$, $u = m$; then $T = \{\ell\}$ and $\ell^2 < 8\ell^2 \le 2^{n'}$. In case (II), $\chi_{\mathrm{pr}} = \mathbf{1}$; by Lemma 38.34, χ_2 is induced from $(\mathbb{Z}/2^{c_2(\chi)}\mathbb{Z})^\times$, of order dividing $2^{c_2(\chi)}$, $\chi_q^2 = \chi_d^2 = \mathbf{1}$ and the product of the odd components at primes dividing N_ψ is killed by o_ψ; so $\chi^K = \mathbf{1}$ for $K = 2^{c_2(\chi)+1}o_\psi = 2^{a_0}u$ with u odd, and $2^{a_0} \le 2^{c_2(\chi)+1}o_\psi$. For $v \in T$, $v \le \max(\ell, N_\psi) \le \ell N_\psi$ (Step 2), so $v^2 2^{a_0} \le 2^{c_2(\chi)+1}o_\psi \ell^2 N_\psi^2 < 2^{c_2(\chi)+6}o_\psi \ell^3 N_\psi^2 \le 2^{n'}$.

Now let $v \in T$ and $i \in \mathbb{Z}$, and suppose $\overline{\xi\Psi^i}(\sigma_v) = 1$. Since $\bar{\chi}'' = \mathbf{1}$, $\bar{\Psi} = \bar{\chi}\,\bar{\chi}_{\mathrm{cyc}}^e$ and $\bar{\chi}_{\mathrm{cyc}}(\sigma_v) = 1$, this says $\bar{\xi}(\sigma_v)\bar{\chi}(\sigma_v)^i = 1$. Raising to the power $2^{a_0}u$ kills $\bar{\chi}$ and $\bar{\chi}_{\mathrm{pr}}$, so $\bar{\chi}_{\mathrm{dy}}(\sigma_v)^{2^{a_0}u} = 1$; as $\chi_{\mathrm{dy}}(\sigma_v)$ is a root of unity of 2-power order, $\chi_{\mathrm{dy}}(\sigma_v)^{2^{a_0}u} = 1$, hence $\chi_{\mathrm{dy}}(\sigma_v)^{2^{a_0}} = 1$ (u odd). Since v is odd, $\chi_{\mathrm{dy}}(\sigma_v) = \chi_{\mathrm{dy}}(v)$, so $x = v^{2^{a_0+1}}$ lies in $\ker\chi_{\mathrm{dy}}$. Now $x \equiv 1 \pmod 8$, so $x \in \langle 5\rangle$, a cyclic 2-group on which χ_{dy} is injective (its kernel there is a subgroup not containing the unique subgroup $\langle 5^{2^{n'-3}}\rangle$ of order 2). Hence $x \equiv 1 \pmod{2^{n'}}$. But

$$v^{2^{a_0+1}} - 1 = (v^2-1)\prod_{j=1}^{a_0}(v^{2^j}+1), \qquad v^{2^j}+1 \equiv 2 \pmod 4\ (j \ge 1),$$

so the exponent of 2 in $x-1$ is that of v^2-1 plus a_0, and $2^{n'} \le (v^2-1)2^{a_0} < v^2 2^{a_0} \le 2^{n'}$, a contradiction. So $\overline{\xi\Psi^i}(\sigma_v) \ne 1$ for all $v \in T$, $i \in \mathbb{Z}$, which with the choice of σ_ℓ is (Sg2).

Step 5: the element ϕ, (Sg3). Let U be an open normal subgroup of $G_\mathbb{Q}$ and write the order of $G_\mathbb{Q}/U$ as $\ell^s k'$ with $\ell \nmid k'$. For $n \ge \max(s, k')$ we have $\ell^s \mid \ell^{n!}$ and, the order of ℓ modulo k' being at most $k' \le n$, $\ell^{n!} \equiv 1 \pmod{k'}$; so the residue of $\ell^{n!}$ modulo $\ell^s k'$ does not depend on $n \ge \max(s, k')$, and the image of $\sigma_\ell^{\ell^{n!}}$ in $G_\mathbb{Q}/U$ is eventually constant. As $G_\mathbb{Q}$ is profinite, the sequence $\sigma_\ell^{\ell^{n!}}$ converges; let ϕ be its limit.

Step 6: the element g_0, (Sg4). The character $\bar{\alpha} = \bar{\xi}$ (as $\bar{\chi}_{\mathrm{cyc}} = \bar{\omega}_\ell$) is non-trivial: for g with $g(\zeta') = \zeta'^b$ on the roots of unity of order dividing $2^{n'}p'$, where $b \equiv 5 \pmod{2^{n'}}$ and $b \equiv 1 \pmod{p'}$, $\xi(g) = \chi_{\mathrm{dy}}(5)$ has order $2^{n'-2} \ge 2$, so $\bar{\xi}(g) \ne 1$. The character $\bar{\Psi}$ is non-trivial: otherwise $\chi'\omega_\ell^e$, of order prime to ℓ with trivial reduction (as $\bar{\Psi} = \bar{\chi}'\bar{\omega}_\ell^e$), would be trivial, so $\chi' = \omega_\ell^{-e}$ and χ' would be induced from a character modulo ℓ. In case (II) this gives $\chi' = \chi'_\ell$, so $\chi^{(\ell)} = \mathbf{1}$ has odd order, contrary to (II). In case (III), $\chi = \chi'$ is induced from a character modulo N_ψ (Lemma 44.5) with $\ell \nmid N_\psi$ (the odd prime factors of N_ψ lie in $B_\psi \not\ni \ell$), so its conductor divides $\gcd(\ell, N_\psi) = 1$ and $\chi = \mathbf{1}$, contrary to (III). The kernels of $\bar{\alpha}$ and $\bar{\Psi}$ are proper subgroups, and a group is not the union of two proper subgroups; so there is g_0 with $\bar{\alpha}(g_0) \ne 1$ and $\bar{\Psi}(g_0) \ne 1$.

Step 7: the element s_0, (Sg5). The subgroup $H = I_\ell \cap \ker\chi \cap \ker\omega_\ell$ is open in I_ℓ, as χ and ω_ℓ are continuous with finite image. Since $\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell$ is totally ramified with group $\mathbb{Z}_\ell^\times$, $\chi_{\mathrm{cyc}}(I_\ell) = \mathbb{Z}_\ell^\times$, so $\chi_{\mathrm{cyc}}(H)$ has finite index in $\mathbb{Z}_\ell^\times$ and contains $1+\ell^j\mathbb{Z}_\ell$ for some $j \geq 1$. Choose $s_0 \in H$ with $\chi_{\mathrm{cyc}}(s_0) = 1+\ell^j$, which has infinite order.

Step 8: (Sg6) and conclusion. By Lemma 44.5, every residual Selmer cocycle for $\gamma \in \{\alpha, \beta, \theta, \alpha^{-1}, \beta^{-1}, \theta^{-1}\}$ is a coboundary. By Steps 1–8, $\mathfrak{D} = (\ell, \mathcal{O}, e, N, \chi, \xi, S, (\sigma_v)_{v\in T}, \phi, g_0, s_0)$ is a symplectic Eisenstein datum; by construction $e' = e$, $N = N_\mathfrak{h}$, $\chi' = \chi$ in the notation of the definition, S is the set of primes dividing $2\ell N_\mathfrak{h} p'$ and $\xi = \chi_{\mathrm{dy}}\chi_{\mathrm{pr}}$ with χ_{dy} primitive modulo $2^{n'}$, $\chi_{\mathrm{dy}}(-1) = -1$, and χ_{pr} modulo p' of order m, both with values in $\mathcal{O}^\times$. So $\mathfrak{D}$ is attached to $(\psi, \mathfrak{h}, t, \chi)$. □

Lemma 44.5. *Let $\psi \in \mathcal{P}$, let $\mathfrak{h}$ be a Sha-host of ψ big at an integer $t \geq 1$, ℓ an odd prime, e an integer with $0 < e < t$, $\mathcal{O}$ a coefficient ring and $\chi \in X_\mathfrak{h}(\mathcal{O}, e)$ in case (II) or in case (III) for (ℓ, e), with integers n', p', m of the case. Let χ_{dy} be a primitive Dirichlet character modulo $2^{n'}$ with $\chi_{\mathrm{dy}}(-1) = -1$ and χ_{pr} a Dirichlet character modulo p' whose order divides m, both with values in $\mathcal{O}^\times$; put $\xi = \chi_{\mathrm{dy}}\chi_{\mathrm{pr}}$, $N = N_\mathfrak{h}$, and let α, β, θ be the characters (44.1) for $(\ell, \mathcal{O}, e, N, \chi, \xi)$ and $T = T_\ell(\chi)$. Then $e \geq 2$, $n' \geq 3$, $c_2(\chi) < n'$ and ξ has order prime to ℓ; in case (III), χ has order prime to ℓ and is induced from a Dirichlet character modulo N_ψ; and for every $\gamma \in \{\alpha, \beta, \theta, \alpha^{-1}, \beta^{-1}, \theta^{-1}\}$ every residual Selmer cocycle for γ is a coboundary.*

Proof. Write $\mathfrak{h} = (a, q, d)$. Reduction modulo $\mathfrak{m}_\mathcal{O}$ is written with a bar; it is injective on roots of unity of order prime to ℓ. The Teichmüller character ω_ℓ is the Dirichlet character modulo ℓ sending b to the $(\ell-1)$-th root of unity in $\mathbb{Z}_\ell$ congruent to b; so $\bar\omega_\ell = \bar\chi_{\mathrm{cyc}}$ and $\omega_\ell(-1) = -1$.

Step 0: preliminaries on χ, e and ξ. By Lemma 38.34, items 1 and 2, χ_2 has 2-power order, $\chi_q^2 = \chi_d^2 = \mathbf{1}$ and $\chi_r^{o_\psi} = \mathbf{1}$ for every odd prime $r \mid N_\psi$. The 2-component of χ'' has ℓ-power and 2-power order, so it is trivial and $\chi_2' = \chi_2$; hence $\chi' = \chi_2\vartheta$ with ϑ a product of components at odd primes. The values of χ'' are roots of unity of ℓ-power order, so $\bar\chi'' = \mathbf{1}$ and $\chi''(-1) = 1$; thus $\chi'(-1) = \chi(-1) = (-1)^e$. In case (II), $e \geq 2$, $p' = m = 1$ and $\chi_{\mathrm{pr}} = \mathbf{1}$. In case (III), $\ell \notin B_\psi \ni 3$ forces $\ell \geq 5$, and $(\ell-1) \mid 2e$ gives $e \geq 2$; χ has odd order m, so $\chi_2 = \chi_q = \chi_d = \mathbf{1}$ and $\chi = \prod_r \chi_r$ over the odd primes $r \mid N_\psi$; as $v_r(N) = v_r(N_\psi)$ for these r (q, d are prime to N_ψ), χ is induced from a character modulo N_ψ, and $m \mid o_\psi$. Since $\ell \notin B_\psi$, $\ell \nmid o_\psi$, so m is prime to ℓ, $\chi'' = \mathbf{1}$, $\chi' = \vartheta = \chi$ and $c_2(\chi) = 0$. In both cases χ_{pr} has odd order dividing m, so $\chi_{\mathrm{pr}}(-1) = 1$ and ξ has order dividing $2^{n'}m$, prime to ℓ. Finally $c_2(\chi) < n'$ and $n' \geq 3$: in case (II), $2^{n'} \geq 2^{c_2(\chi)+6}$; in case (III), $c_2(\chi) = 0$ and $2^{n'} \geq 8\ell^2 > 8$.

Step 1: the residual characters are reductions of Dirichlet characters. Put

$$\chi_\alpha = \chi_{\mathrm{dy}} \cdot \chi_{\mathrm{pr}}, \qquad \chi_\theta = (\chi_2\chi_{\mathrm{dy}}) \cdot (\vartheta\,\omega_\ell^e\chi_{\mathrm{pr}}), \qquad \chi_\beta = (\chi_2\chi_{\mathrm{dy}}^{-1}) \cdot (\vartheta\,\omega_\ell^e\chi_{\mathrm{pr}}^{-1}),$$

and for $\gamma \in \{\alpha, \theta, \beta\}$ let $\chi_{\gamma,2}$ and $\chi_{\gamma,\mathrm{od}}$ be the first and second factor. Since $\bar\chi_{\mathrm{cyc}}\bar\omega_\ell^{-1} = \mathbf{1}$, $\bar\alpha = \bar\xi$; since $\bar\Psi = \bar\chi'\bar\omega_\ell^e$ and $\chi' = \chi_2\vartheta$, $\bar\theta = \bar\alpha\bar\Psi$ and $\bar\beta = \bar\Psi\bar\alpha^{-1}$, we get $\bar\gamma = \bar\chi_\gamma$ for $\gamma \in \{\alpha, \theta, \beta\}$, and $\overline{\gamma^{-1}} = \overline{\chi_\gamma^{-1}}$. Each χ_γ has order prime to ℓ (χ', ω_ℓ and ξ have).

Step 2: the 2-parts are primitive modulo $2^{n'}$. Let $U = \{u \in (\mathbb{Z}/2^{n'}\mathbb{Z})^\times : u \equiv 1 \pmod{2^{n'-1}}\}$. The character χ_2 is induced from a character modulo $2^{c_2(\chi)}$, and $c_2(\chi) \leq n'-1$ (Step 0), so χ_2 is trivial on U; $\chi_{\mathrm{dy}}^{\pm1}$ is not, being primitive. Hence $\chi_{\gamma,2}$ is non-trivial on U; as its conductor is a power of 2 dividing $2^{n'}$ and every proper such divisor divides $2^{n'-1}$, $\chi_{\gamma,2}$ is primitive modulo $2^{n'}$. Classically $(\mathbb{Z}/2^{n'}\mathbb{Z})^\times = \{\pm1\} \times \langle 5\rangle$ with 5 of order $2^{n'-2}$ and $U = \langle 5^{2^{n'-3}}\rangle$ [266]; a character non-trivial on U is therefore injective on $\langle 5\rangle$, and $\chi_{\gamma,2}(5)$ has order $2^{n'-2} \geq 2$.

Step 3: the odd parts. Let $\chi_{\gamma,\mathrm{od}}^*$ be the primitive character inducing $\chi_{\gamma,\mathrm{od}}$, of conductor f_γ. The conductor of a product divides the least common multiple of the conductors [266];

ϑ has odd conductor, ω_ℓ^e conductor dividing ℓ, χ_{pr} conductor dividing p'; so f_γ is odd and $f_\gamma \le \ell\,\mathrm{cond}(\vartheta)\,p'$. In case (II), $p'=1$ and $\mathrm{cond}(\vartheta)\le N$, since ϑ is a character modulo N; in case (III), $\mathrm{cond}(\vartheta)=\mathrm{cond}(\chi)\le N_\psi$ by Step 0 and $p'=p_1(\psi)$. By the definition of n',

$$f_\gamma\, 4^{n_0(\ell,\chi)} \le 2^{n'} \qquad (\gamma\in\{\alpha,\theta,\beta\}).$$

Let f_ℓ be the order of ℓ modulo $\mathrm{lcm}(4,\mathrm{ord}\,\vartheta)$, so that $4\mid \ell^{f_\ell}-1$, $n_0(\ell,\chi)$ is the exponent of 2 in $\ell^{f_\ell}-1$, $\mathrm{ord}\,\vartheta \mid \ell^{f_\ell}-1$ and $(\ell-1)\mid \ell^{f_\ell}-1$. The order of $\chi_{\gamma,\mathrm{od}}$ divides $\mathrm{lcm}(\mathrm{ord}\,\vartheta,\ell-1,m)$, and $m\mid \ell^{f_\ell}-1$ (in case (III), $m=\mathrm{ord}\,\chi=\mathrm{ord}\,\vartheta$). So $\chi_{\gamma,\mathrm{od}}^*(b)^{\ell^{f_\ell}-1}=1$ for b prime to f_γ. Finally, with Step 0, $\chi_\gamma(-1)=-1$:

$$\begin{aligned}
\chi_\alpha(-1) &= \chi_{\mathrm{dy}}(-1)\chi_{\mathrm{pr}}(-1) = -1,\\
\chi_\theta(-1) &= \chi_\alpha(-1)\,\chi'(-1)\,\omega_\ell(-1)^e = -1,\\
\chi_\beta(-1) &= \chi'(-1)\,\omega_\ell(-1)^e\,\chi_\alpha(-1)^{-1} = -1.
\end{aligned}$$

Step 4: Bernoulli units. Let $\varepsilon\in\{1,-1\}$. The hypotheses of Lemma 44.6 hold for ℓ, the exponent f_ℓ (in place of m), $a=n_0(\ell,\chi)$, $f=f_\gamma$, $n=n'$, and the characters $(\chi_{\gamma,\mathrm{od}}^*)^\varepsilon$ (primitive modulo f_γ) and $\chi_{\gamma,2}^\varepsilon$ (primitive modulo $2^{n'}$, Step 2): Step 3 gives the order condition, the bound and $(\chi_{\gamma,\mathrm{od}}^*)^\varepsilon(-1)\chi_{\gamma,2}^\varepsilon(-1)=\chi_\gamma(-1)^\varepsilon=-1$. Hence $\chi_\gamma^{*,\varepsilon}:=(\chi_{\gamma,\mathrm{od}}^*\chi_{\gamma,2})^\varepsilon$ is primitive modulo $f_\gamma 2^{n'}$ and $|B_{1,\chi_\gamma^{*,\varepsilon}}|_\ell=1$. As characters of $G_\mathbb{Q}$, $\chi_\gamma^{*,\varepsilon}=\chi_\gamma^\varepsilon$: the two Dirichlet characters agree on integers prime to $2\ell N p'$.

Step 5: the hypotheses of Lemma 44.7 on the character. Fix $\gamma_0\in\{\alpha,\theta,\beta\}$ and ε, put $\gamma=\gamma_0^\varepsilon$ and $\chi_{\mathrm{H}}=\chi_{\gamma_0}^{*,\varepsilon}$. Then χ_{H} is primitive, of order prime to ℓ (Step 1), $\bar\chi_{\mathrm{H}}=\bar\gamma$ (Steps 1 and 4) and $|B_{1,\chi_{\mathrm{H}}^{-1}}|_\ell=1$ (Step 4 with $-\varepsilon$). Choose an integer $b\equiv 1 \pmod{\ell f_{\gamma_0}}$ with $b\equiv 5 \pmod{2^{n'}}$ and $g\in G_\mathbb{Q}$ with $g(\zeta)=\zeta^b$ for all roots of unity of order dividing $\ell f_{\gamma_0}2^{n'}$. Then $\bar\chi_{\mathrm{cyc}}(g)=\bar b=1$, while $\chi_{\mathrm{H}}(g)=\chi_{\gamma_0,2}(5)^\varepsilon$ is a root of unity of order $2^{n'-2}\ge 2$ (Step 2), whose reduction is not 1 as ℓ is odd. So $\chi_{\mathrm{H}}\ne\mathbf{1}$ and $\bar\chi_{\mathrm{H}}\ne\bar\chi_{\mathrm{cyc}}$, the reduction of χ_{cyc} modulo ℓ.

Step 6: a residual Selmer cocycle for γ vanishes on $I_{\bar v}\cap\ker\bar\gamma$ at every place. Let z be a residual Selmer cocycle for γ. On $\ker\bar\gamma$ the cocycle relation makes z a homomorphism to $(\mathbb{F}_\mathcal{O},+)$. Let $\bar w$ be a place above a prime not in T, and $N_{\bar w}=\{g\in I_{\bar w}:\bar\theta(g)=\bar\Psi(g)=\bar\alpha(g)=1\}$; as γ is a product of powers of θ,Ψ,α, $N_{\bar w}\subseteq I_{\bar w}\cap\ker\bar\gamma$, a normal subgroup with quotient embedded in $(\mathbb{F}_\mathcal{O}^\times)^3$, of order prime to ℓ. Since z vanishes on $N_{\bar w}$, it induces a homomorphism from a group of order prime to ℓ to a group of exponent ℓ, which is trivial: $z=0$ on $I_{\bar w}\cap\ker\bar\gamma$. Let $v\in T$. On D_v, $z(g)=\bar\gamma(g)y-y$ for some y, which vanishes on $I_v\cap\ker\bar\gamma$, at the fixed place $\bar v$. A place above v is $g_1\bar v$ with $g_1\in G_\mathbb{Q}$, with inertia group $g_1 I_v g_1^{-1}$; for $h\in I_v\cap\ker\bar\gamma$,

$$z(g_1hg_1^{-1}) = z(g_1)+\bar\gamma(g_1)\big(z(h)+\bar\gamma(h)z(g_1^{-1})\big) = z(g_1)+\bar\gamma(g_1)z(g_1^{-1}) = z(1)=0.$$

So $z(g)=0$ for every place $\bar p$ above a prime and every $g\in I_{\bar p}$ with $\bar\chi_{\mathrm{H}}(g)=\bar\gamma(g)=1$.

Step 7: conclusion. By Steps 5 and 6, Lemma 44.7 applies to χ_{H}, the field $k=\mathbb{F}_\mathcal{O}$ and z (locally constant by definition): there is $\mu\in\mathbb{F}_\mathcal{O}$ with $z(g)=\mu(\bar\gamma(g)-1)=\bar\gamma(g)\mu-\mu$ for all g. As γ_0 and ε were arbitrary, this proves the claim for the six characters. □

Lemma 44.6. *Let ℓ be an odd prime and $m\ge 1$ an integer with $4\mid\ell^m-1$, and let a be the exponent of 2 in ℓ^m-1. Let $f\ge 1$ be odd, $n\ge 1$, α a primitive Dirichlet character modulo f and β a primitive Dirichlet character modulo 2^n, both with values in $\overline{\mathbb{Q}}_\ell^\times$, such that*

$$\alpha(b)^{\ell^m-1}=1 \text{ for all } b \text{ prime to } f, \qquad f\cdot 4^a\le 2^n, \qquad \alpha(-1)\beta(-1)=-1.$$

Then $\alpha\beta$ is a primitive Dirichlet character modulo $f2^n$ and $|B_{1,\alpha\beta}|_\ell=1$.

Proof. Step 0: sizes and the Bernoulli number as a sum. Since $f \ge 1$, $2^{2a} \le f4^a \le 2^n$, so $n \ge 2a \ge 4$. Put $n_1 = n - a \ge a \ge 2$ and $\varkappa = f2^a$; then $\varkappa \cdot 2^a = f4^a \le 2^n$, so $4 \le \varkappa \le 2^{n_1}$. A product of primitive characters of coprime conductors is primitive modulo the product of the conductors [266], so $\alpha\beta$ is primitive modulo $F = f2^n$; it is odd, hence non-trivial, so $\sum_{c=1}^{F} \alpha\beta(c) = 0$ and, as $\alpha\beta(F) = 0$,

$$B_{1,\alpha\beta} = \sum_{c=1}^{F} \alpha\beta(c)\Big(\frac{c}{F} - \frac{1}{2}\Big) = \frac{1}{F}\sum_{c=0}^{F-1} \alpha(c)\beta(c)\, c. \tag{44.5}$$

Step 1: three facts on 2-power residues. (i) Put $v = 1 + 2^{n-1}$. Since $n \ge 2$, $v^2 \equiv 1 \pmod{2^n}$, and $\{1, v\}$ is the kernel of $(\mathbb{Z}/2^n\mathbb{Z})^\times \to (\mathbb{Z}/2^{n-1}\mathbb{Z})^\times$. If $\beta(v) = 1$, β would be induced from a character modulo 2^{n-1}, against primitivity; as $\beta(v)^2 = \beta(v^2) = 1$, $\beta(v) = -1$. (ii) Recall that for $n \ge 3$, $(\mathbb{Z}/2^n\mathbb{Z})^\times = \{\pm 1\} \times \langle 5\rangle$ with 5 of order 2^{n-2}, and $5^{2^i} \equiv 1 + 2^{i+2} \pmod{2^{i+3}}$ for $i \ge 0$ [266]; hence $5^{2^{n-3}} \equiv v$ and $\langle 5^{2^i}\rangle = \{u : u \equiv 1 \pmod{2^{i+2}}\}$ for $0 \le i \le n-2$. By (i), $\beta(5)^{2^{n-3}} = -1$, so $\beta(5)$ has order exactly 2^{n-2}. Let $u = \pm 5^j$ be a unit with $\beta(u)^{2^a} = 1$. As $\beta(-1)^2 = 1$ and $a \ge 1$, $\beta(5)^{j2^a} = 1$, so $2^{n-2} \mid j2^a$, that is $2^{n-2-a} \mid j$ (note $n-2-a \ge a-2 \ge 0$), and $5^j \in \langle 5^{2^{n-2-a}}\rangle$, so $5^j \equiv 1 \pmod{2^{n_1}}$. Hence

$$\beta(u)^{2^a} = 1 \implies u \equiv 1 \text{ or } u \equiv -1 \pmod{2^{n_1}}, \tag{44.6}$$

and not both, as $n_1 \ge 2$. (iii) Write $\ell^m = 1 + 2^a o$ with o odd. If $y = 1 + 2^b o'$ with $b \ge 2$ and o' odd, then $y^2 = 1 + 2^{b+1}(o' + 2^{b-1}o'^2)$ with $o' + 2^{b-1}o'^2$ odd. By induction $\ell^{m2^i} - 1$ has 2-adic valuation exactly $a + i$, so the class of ℓ^m in $(\mathbb{Z}/2^n\mathbb{Z})^\times$ has order 2^{n-a}, and its powers ℓ^{mi}, $0 \le i < 2^{n-a}$, are pairwise distinct modulo 2^n and $\equiv 1 \pmod{2^a}$. As there are exactly 2^{n-a} classes $\equiv 1 \pmod{2^a}$ modulo 2^n,

$$\{\ell^{mi} \bmod 2^n : 0 \le i < 2^{n-a}\} = \{1 + 2^a j \bmod 2^n : 0 \le j < 2^{n-a}\}. \tag{44.7}$$

Step 2: partial sums along residue classes modulo 2^n. For $u \in \mathbb{Z}/2^n\mathbb{Z}$ put

$$\tau(u) = \frac{1}{F}\sum_{\substack{0 \le c < F \\ c \equiv u \ (2^n)}} \alpha(c)\, c, \qquad \delta_\alpha = \sum_{c=0}^{f-1} \alpha(c) \in \{0, 1\}$$

($\delta_\alpha = 1$ if $f = 1$ and 0 otherwise). Since f is odd, by the Chinese remainder theorem the integers $c \equiv u \pmod{2^n}$ in any interval of F consecutive integers run over all residues modulo f exactly once; so their α-values sum to δ_α. Grouping (44.5) by classes modulo 2^n gives

$$B_{1,\alpha\beta} = \sum_{u \in \mathbb{Z}/2^n\mathbb{Z}} \beta(u)\, \tau(u). \tag{44.8}$$

Let $t \ge 0$ be an integer with $ft \le F$ and $u \in \mathbb{Z}/2^n\mathbb{Z}$. Substituting $c = c' - ft$ (so $\alpha(c) = \alpha(c')$, as $f \mid ft$) and splitting the range $ft \le c' < F + ft$ at F, where $c' = F + c''$ with $0 \le c'' < ft$, $\alpha(c') = \alpha(c'')$ and $c' \equiv c'' \pmod{2^n}$:

$$F\tau(u - ft) = \sum_{\substack{ft \le c' < F + ft \\ c' \equiv u}} \alpha(c')(c' - ft) = F\tau(u) - \sum_{\substack{0 \le c'' < ft \\ c'' \equiv u}} \alpha(c'')c'' + \sum_{\substack{0 \le c'' < ft \\ c'' \equiv u}} \alpha(c'')(F + c'') - ft\,\delta_\alpha.$$

Hence

$$\tau(u) - \tau(u - ft) = \frac{t\,\delta_\alpha}{2^n} - \sum_{\substack{0 \le c < ft \\ c \equiv u \ (2^n)}} \alpha(c). \tag{44.9}$$

Put $g_0(u) = \tau(u) - \tau(0)$. Given u, choose $0 \le t < 2^n$ with $ft \equiv u \pmod{2^n}$ (f is a unit modulo 2^n); then $ft \le F$ and (44.9) reads $g_0(u) = t\delta_\alpha/2^n - \sum_{0\le c<ft,\, c\equiv u} \alpha(c)$. Since ℓ is odd and every non-zero value of α is a root of unity ζ with $\zeta^{\ell^m} = \zeta$, we get $g_0(u) \in \overline{\mathbb{Z}}_\ell$ and

$$\overline{g_0(u)}^{\,\ell^m} = \overline{g_0(u)} \qquad (u \in \mathbb{Z}/2^n\mathbb{Z}), \tag{44.10}$$

because $y \mapsto y^{\ell^m}$ is a ring endomorphism of $\overline{\mathbb{F}}_\ell$ fixing $\mathbb{F}_\ell$. As $\beta \ne \mathbf{1}$ by Step 1(i), $\sum_u \beta(u) = 0$, so (44.8) gives $B_{1,\alpha\beta} = \sum_u \beta(u) g_0(u) \in \overline{\mathbb{Z}}_\ell$. Finally, (44.9) with $t = 2^a$ (allowed, as $\varkappa \le 2^n \le F$), where the only c with $0 \le c < \varkappa \le 2^n$ and $c \equiv u$ is $c = u_0$ if $u_0 < \varkappa$, gives for every u, with $\delta' = 2^a\delta_\alpha/2^n \in \overline{\mathbb{Z}}_\ell$,

$$g_0(u) - g_0(u - \varkappa) = \delta' - [u_0 < \varkappa]\, \alpha(u_0), \tag{44.11}$$

where $[P]$ is 1 if P holds and 0 otherwise.

Step 3: if $B_{1,\alpha\beta} \in \mathfrak{m}_\ell$, the sum localises at U. Suppose $|B_{1,\alpha\beta}|_\ell < 1$, that is $\sum_u \bar\beta(u)\bar g_0(u) = 0$ in $\overline{\mathbb{F}}_\ell$. Let U be the set of units u modulo 2^n with $\beta(u)^{2^a} = 1$. We claim that for every unit y modulo 2^n

$$\sum_{u\in U} \bar\beta(u)\, \bar g_0(uy) = 0. \tag{44.12}$$

Substituting $u \mapsto uy^{-1}$, $\sum_u \bar\beta(u)\bar g_0(uy) = \bar\beta(y)^{-1}\sum_u \bar\beta(u)\bar g_0(u) = 0$. Applying the ring endomorphism $y' \mapsto y'^{\ell^{mi}}$ of $\overline{\mathbb{F}}_\ell$ and (44.10) gives $\sum_u \bar\beta(u)^{\ell^{mi}} \bar g_0(uy) = 0$ for all $i \ge 0$; summing over $0 \le i < 2^{n-a}$,

$$\sum_u \Big(\sum_{i=0}^{2^{n-a}-1} \bar\beta(u)^{\ell^{mi}} \Big) \bar g_0(uy) = 0.$$

The terms with u even vanish. For a unit u, $\beta(u)$ is a root of unity of order dividing 2^{n-1}, so $\bar\zeta = \bar\beta(u)$ satisfies $\bar\zeta^{2^n} = 1$, and by (44.7) the inner sum is $\bar\zeta \sum_{j=0}^{2^{n-a}-1} (\bar\zeta^{2^a})^j$, which is $2^{n-a}\bar\zeta$ if $\bar\zeta^{2^a} = 1$ and 0 otherwise ($\bar\zeta^{2^a}$ has order dividing 2^{n-a}). Reduction is injective on roots of unity of order prime to ℓ, so $\bar\zeta^{2^a} = 1$ exactly when $u \in U$. As $2^{n-a} \ne 0$ in $\overline{\mathbb{F}}_\ell$, (44.12) follows.

Step 4: the contradiction. The element $1 - \varkappa$ is odd, so (44.12) for $y = 1$ and $y = 1 - \varkappa$ gives

$$\sum_{u\in U} \bar\beta(u)(\bar g_0(u) - \bar g_0(u(1-\varkappa))) = 0.$$

By (44.6) each $u \in U$ satisfies exactly one of $u \equiv 1$, $u \equiv -1 \pmod{2^{n_1}}$; put $\mathrm{sg}(u) = 1$, respectively -1. Since $2^a \mid \varkappa$ and $n_1 + a = n$:

- if $u \equiv 1 \pmod{2^{n_1}}$, then $(u-1)\varkappa \equiv 0 \pmod{2^n}$, so $u(1-\varkappa) = u - \varkappa$, and by (44.11) $g_0(u) - g_0(u-\varkappa) = \delta' - [u_0 < \varkappa]\alpha(u_0)$; as $u_0 \equiv 1 \pmod{2^{n_1}}$ and $\varkappa \le 2^{n_1}$, $u_0 < \varkappa$ holds exactly when $u_0 = 1$ (note $\varkappa \ge 4$), that is $u = 1$, and $\alpha(1) = 1$;
- if $u \equiv -1 \pmod{2^{n_1}}$, then $(u+1)\varkappa \equiv 0 \pmod{2^n}$, so $u(1-\varkappa) = u + \varkappa$, and (44.11) at $u + \varkappa$ gives $g_0(u) - g_0(u+\varkappa) = -\delta' + [(u+\varkappa)_0 < \varkappa]\alpha((u+\varkappa)_0)$; as $(u+\varkappa)_0 \equiv \varkappa - 1 \pmod{2^{n_1}}$ with $0 \le \varkappa - 1 < 2^{n_1}$, $(u+\varkappa)_0 < \varkappa$ holds exactly when $(u+\varkappa)_0 = \varkappa - 1$, that is $u = -1$, and $\alpha(\varkappa - 1) = \alpha(-1)$ since $f \mid \varkappa$.

Both 1 and -1 lie in U. Summing, and using $\alpha(-1)\beta(-1) = -1$,

$$0 = \bar\delta' \sum_{u\in U} \mathrm{sg}(u)\bar\beta(u) - \bar\beta(1) + \bar\beta(-1)\bar\alpha(-1) = \bar\delta' \sum_{u\in U} \mathrm{sg}(u)\bar\beta(u) - 2.$$

The element $v = 1 + 2^{n-1}$ lies in U ($\beta(v)^{2^a} = 1$ by Step 1(i)) and $v \equiv 1 \pmod{2^{n_1}}$ since $n - 1 \ge n_1$. Multiplication by v permutes U, preserves sg (as $vu \equiv u \pmod{2^{n_1}}$) and multiplies $\bar\beta$ by $\bar\beta(v) = -1$; so $\Sigma = \sum_{u\in U} \mathrm{sg}(u)\bar\beta(u)$ satisfies $\Sigma = -\Sigma$, hence $\Sigma = 0$ as $\ell \ne 2$. Therefore $2 = 0$ in $\overline{\mathbb{F}}_\ell$, which is false as ℓ is odd. So $|B_{1,\alpha\beta}|_\ell \ge 1$, and with $B_{1,\alpha\beta} \in \overline{\mathbb{Z}}_\ell$ (Step 2), $|B_{1,\alpha\beta}|_\ell = 1$. $\square$

Lemma 44.7. *Let ℓ be an odd prime and $\chi \neq \mathbf{1}$ a primitive Dirichlet character modulo Q with values in $\overline{\mathbb{Q}}_\ell^\times$ and of order prime to ℓ. View χ as the character of $G_{\mathbb{Q}}$ with $\chi(g) = \chi(b)$ whenever $g(\zeta) = \zeta^b$ for all Q-th roots of unity ζ, let $\bar\chi\colon G_{\mathbb{Q}} \to \overline{\mathbb{F}}_\ell^\times$ be its reduction ($\overline{\mathbb{F}}_\ell$ the residue field of the integral closure of $\mathbb{Z}_\ell$ in $\overline{\mathbb{Q}}_\ell$) and $\bar\omega\colon G_{\mathbb{Q}} \to \mathbb{F}_\ell^\times$ the reduction of χ_{cyc} modulo ℓ. Assume*

$$\bar\chi \neq \bar\omega \qquad \text{and} \qquad |B_{1,\chi^{-1}}|_\ell = 1.$$

Let $k \subseteq \overline{\mathbb{F}}_\ell$ be a subfield containing the values of $\bar\chi$, and $z\colon G_{\mathbb{Q}} \to k$ a locally constant map with $z(gh) = z(g) + \bar\chi(g)z(h)$ for all $g, h \in G_{\mathbb{Q}}$, such that $z(g) = 0$ for every place $\bar p$ of $\overline{\mathbb{Q}}$ above a prime and every $g \in I_{\bar p}$ with $\bar\chi(g) = 1$. Then there is $\mu \in k$ with $z(g) = \mu\,(\bar\chi(g) - 1)$ for all $g \in G_{\mathbb{Q}}$.

Proof. Since $\chi \neq \mathbf{1}$ is primitive, $Q \geq 3$, and $Q \not\equiv 2 \pmod 4$ (a character modulo $2Q'$ with Q' odd is induced from one modulo Q'). Let $K_Q = \mathbb{Q}(\zeta_Q)$, $G = \mathrm{Gal}(K_Q/\mathbb{Q})$, $\sigma_b \in G$ with $\sigma_b(\zeta_Q) = \zeta_Q^b$, and $H_Q = G_{K_Q} \subseteq G_{\mathbb{Q}}$; χ is trivial on H_Q. As χ has order prime to ℓ and reduction is injective on roots of unity of order prime to ℓ, $\bar\chi \neq \mathbf{1}$. From $z(1) = z(1) + z(1)$ we get $z(1) = 0$, and then $z(g^{-1}) = -\bar\chi(g)^{-1} z(g)$.

Step 1: z gives a homomorphism on the class group of K_Q. On H_Q, $z(hh') = z(h) + z(h')$, so $z|_{H_Q}$ is a locally constant homomorphism to the additive group of k; its kernel is an open normal subgroup of H_Q, with fixed field L a finite abelian extension of K_Q. For a place $\bar p$ of $\overline{\mathbb{Q}}$ above a prime, the inertia group of L/K_Q at $\bar p$ is the image of $I_{\bar p} \cap H_Q$, on which z vanishes by hypothesis (as $\bar\chi = 1$ there); so L/K_Q is unramified at all finite places, and at the archimedean places since K_Q is totally complex. Hence L lies in the Hilbert class field H of K_Q and $z|_{H_Q}$ factors through $\mathrm{Gal}(H/K_Q)$ [190]. Composing with the Artin isomorphism $\mathrm{Art}\colon \mathrm{Cl}(K_Q) \to \mathrm{Gal}(H/K_Q)$ gives an additive map $\tilde z\colon \mathrm{Cl}(K_Q) \to k$, $\tilde z(\mathfrak{a}) = z(h)$ for any $h \in H_Q$ restricting to $\mathrm{Art}(\mathfrak{a})$ on H.

Step 2: equivariance. For $g \in G_{\mathbb{Q}}$ and $h \in H_Q$, since $\bar\chi(h) = 1$,

$$z(ghg^{-1}) = z(g) + \bar\chi(g)\big(z(h) + z(g^{-1})\big) = \bar\chi(g)\, z(h).$$

The field H is Galois over $\mathbb{Q}$, and if $g|_{K_Q} = \sigma_b$ then $\mathrm{Art}(\sigma_b\mathfrak{a}) = g\,\mathrm{Art}(\mathfrak{a})\,g^{-1}$ on H (functoriality of the Artin map). Hence $\tilde z(\sigma_b\mathfrak{a}) = \bar\chi(b)\,\tilde z(\mathfrak{a})$ for all b prime to Q and all $\mathfrak{a}$.

Step 3: z vanishes on H_Q. Choose an integer c prime to Q with $\bar c \neq \bar\chi(c)$ in $\overline{\mathbb{F}}_\ell$: if $\ell \nmid Q$, $c = \ell$ works ($\bar c = 0 \neq \bar\chi(\ell)$); if $\ell \mid Q$, then $\mu_\ell \subset K_Q$ and $\bar\omega(g) = \bar b$ for $g|_{K_Q} = \sigma_b$, so if $\bar\chi(c) = \bar c$ for all c prime to Q we would have $\bar\chi = \bar\omega$, which is excluded. By Theorem B.23, $\beta = (c - \sigma_c)\theta_Q$ lies in $\mathbb{Z}[G]$ and annihilates $\mathrm{Cl}(K_Q)$; write $\beta = \sum_b n_b\sigma_b$ with $n_b \in \mathbb{Z}$. Let $\chi_*\colon \mathbb{Q}[G] \to \overline{\mathbb{Q}}_\ell$ be the $\mathbb{Q}$-linear ring homomorphism with $\chi_*(\sigma_b) = \chi(b)$. By Step 2, for every class $\mathfrak{a}$,

$$0 = \tilde z(\beta\mathfrak{a}) = \sum_b n_b\,\tilde z(\sigma_b\mathfrak{a}) = \Big(\sum_b n_b\bar\chi(b)\Big)\tilde z(\mathfrak{a}),$$

and $\sum_b n_b\bar\chi(b)$ is the reduction of $\chi_*(\beta) = (c - \chi(c))\,\chi_*(\theta_Q)$. Since χ^{-1} is primitive modulo Q and non-trivial,

$$\chi_*(\theta_Q) = \sum_{\substack{1\leq b\leq Q\\ \gcd(b,Q)=1}} \Big\{\frac{b}{Q}\Big\}\chi^{-1}(b) = \sum_{b=1}^{Q} \chi^{-1}(b)\Big(\frac{b}{Q} - \frac12\Big) = B_{1,\chi^{-1}}$$

($\chi^{-1}(b) = 0$ for b not prime to Q, $\{b/Q\} = b/Q$ for $1 \leq b < Q$, and $\sum_b \chi^{-1}(b) = 0$). By the choice of c and the hypothesis $|B_{1,\chi^{-1}}|_\ell = 1$, the reduction $(\bar c - \bar\chi(c))\overline{B_{1,\chi^{-1}}}$ is non-zero. Hence $\tilde z = 0$, and z vanishes on H_Q.

Step 4: z is a coboundary. For $g \in G_{\mathbb{Q}}$ and $h \in H_Q$, $z(gh) = z(g) + \bar\chi(g)z(h) = z(g)$; so $z(g)$ depends only on the image of g in G. As G is abelian, gg' and $g'g$ have the same image, so $z(g) + \bar\chi(g)z(g') = z(gg') = z(g'g) = z(g') + \bar\chi(g')z(g)$ for all g, g'. Choose g_1 with $\bar\chi(g_1) \neq 1$ and put $\mu = z(g_1)/(\bar\chi(g_1) - 1) \in k$. Taking $g' = g_1$ gives $z(g)(1 - \bar\chi(g_1)) = z(g_1)(1 - \bar\chi(g))$, that is $z(g) = \mu(\bar\chi(g) - 1)$. □

Theorem 44.8. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum attached to some $(\psi, \mathfrak{h}, t, \chi)$, with odd prime ℓ, and let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism. Every irreducible point of $\mathfrak{D}$ that is automorphic for ι is rigid.*

Proof. Let r be an irreducible point of $\mathfrak{D}$, automorphic for ι, and r'' a point with values in $\overline{\mathbb{Q}}_\ell[\epsilon]$ reducing to r modulo ϵ; write $r''(g) = r(g) + \epsilon r_1(g)$. We must find $X \in \mathfrak{sp}_4(\overline{\mathbb{Q}}_\ell)$ with $r_1(g) = Xr(g) - r(g)X$ for all g.

Step 1: the tangent cocycle is continuous with values in $\mathfrak{sp}_4$. Write $r'' = f'' \circ r_A$ for a symplectic Eisenstein lift r_A to $A \in \mathrm{CNL}_{\mathcal{O}}$ and an $\mathcal{O}$-algebra homomorphism $f'' = f + \epsilon f_1\colon A \to \overline{\mathbb{Q}}_\ell[\epsilon]$; then $r = f \circ r_A$, and f_1 is $\mathcal{O}$-linear with $f_1(ab) = f(a)f_1(b) + f(b)f_1(a)$. Let $P = \ker f$. The domain $A/P \cong f(A) \subseteq \overline{\mathbb{Q}}_\ell$ is complete Noetherian local and contains $\mathcal{O}$, so by Cohen's structure theorem it is finite over a subring $\mathcal{O}[[Y_1, \dots, Y_d]]$; if $d \geq 1$, $f(Y_1)$ is algebraic over the fraction field of $\mathcal{O}$, so $Q(Y_1) = 0$ in A/P for some non-zero $Q \in \mathcal{O}[Y]$, which is impossible in $\mathcal{O}[[Y_1, \dots, Y_d]]$. So A/P is a finite $\mathcal{O}$-module, and so is A/P^2 (P/P^2 is finitely generated over A/P). As f_1 vanishes on P^2, it factors through A/P^2, on which an $\mathcal{O}$-linear map to $\overline{\mathbb{Q}}_\ell$ is ℓ-adically continuous; $(A/P^2)/\ell$ is a finite local ring, so the image of $\mathfrak{m}_A^N$ lies in $\ell(A/P^2)$ for some N, and f_1 is continuous on A. Together with (Sq1), it follows that f'', and hence r'' and r_1, are continuous. Put $z(g) = r_1(g)r(g)^{-1}$. From $r''(gh) = r''(g)r''(h)$ we get $z(gh) = z(g) + r(g)z(h)r(g)^{-1}$; thus z is a continuous 1-cocycle for the action $\mathrm{Ad}\, r$ on $\mathrm{M}_4(\overline{\mathbb{Q}}_\ell)$. By (Se1), ${}^t r''(g) J_4 r''(g) = \theta(g) J_4$; the ϵ-part gives ${}^t r_1 J_4 r + {}^t r J_4 r_1 = 0$, that is ${}^t r(g)({}^t z(g) J_4 + J_4 z(g)) r(g) = 0$, so $z(g) \in \mathfrak{sp}_4$.

Step 2: local conditions away from ℓ. Let $p \neq \ell$ be a prime and I the inertia group at a place $\bar w$ above p. If $p \notin T$, (Se2) gives $r''(g) = 1$ for $g \in N = I \cap \ker \bar r_0$. If $p \in T$, the local condition in (Se4) gives $\rho_B(g) = 1$ on $I_p \cap \ker\chi$, so $r''(g) = 1$ there; for $\bar w = h\bar p$, $I = hI_p h^{-1}$ and $r''(hgh^{-1}) = r''(h)r''(g)r''(h)^{-1} = 1$, so $r'' = 1$ on $N = I \cap \ker\chi$. In both cases N is normal of finite index in I and $r''|_N = 1$, so $r|_N = 1$, $z|_N = 0$, and $z|_I$ is inflated from a cocycle of the finite group I/N on a $\overline{\mathbb{Q}}_\ell$-vector space, which is a coboundary (average over I/N). So $z|_D \in H^1_f(D, \cdot)$ for every decomposition group D at p. At the archimedean place the decomposition group has order at most 2 and H^1 vanishes.

Step 3: the local condition at ℓ. By (Se4) at ℓ, $r''|_{D_\ell} = F'' \circ \rho_B$ for an ordinary lift ρ_B to a domain B and an $\mathcal{O}$-algebra homomorphism $F''\colon B \to \overline{\mathbb{Q}}_\ell[\epsilon]$; as the ordinary conditions are polynomial identities with coefficients in $\mathcal{O}$, $\rho'' = r''|_{D_\ell}$ is ordinary over $\overline{\mathbb{Q}}_\ell[\epsilon]$. The c_j are distinct, so d_i is a unit and $E_i = d_i^{-1}\Pi_i(\rho'')$ satisfy $E_iE_j = 0$ ($i \neq j$, by Cayley–Hamilton, the characteristic polynomial of $\rho''(s_0)$ being $\prod_j(X - c_j)$) and $\sum_i E_i = 1$ (Lagrange interpolation); they are orthogonal idempotents with $\rho''(s_0)E_i = c_iE_i$. Each $E_i\overline{\mathbb{Q}}_\ell[\epsilon]^4$ is a direct summand, hence free over the local ring $\overline{\mathbb{Q}}_\ell[\epsilon]$, of rank 1 (its reduction is the c_i-eigenline of $r(s_0)$); choosing generators gives $P'' \in \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell[\epsilon])$ with $P''^{-1}\Pi_i(\rho'')P'' = d_i\, e_{ii}$. As in the field case, $\Pi_j\rho''(g)\Pi_i = 0$ for $i < j$ makes $T''(g) = P''^{-1}\rho''(g)P''$ upper triangular, and $\Pi_i(\rho''(s) - \kappa_i(s))\Pi_i = 0$ gives $T''(s)_{ii} = \kappa_i(s)$ for $s \in I_\ell$. Write $P'' = P(1 + \epsilon M)$ with $P \in \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$; then $T(g) = P^{-1}r(g)P$ is upper triangular with diagonal $(\delta_i(g))$, the characters of (Zt2) (both are the diagonal characters of $r|_{D_\ell}$ in the eigenbasis of $r(s_0)$ for $c_1, \dots, c_4$), and with $T'' = T + \epsilon T_1$ a direct computation gives

$$z_T(g) := T_1(g)T(g)^{-1} = P^{-1}z(g)P + (T(g)MT(g)^{-1} - M) \qquad (g \in D_\ell).$$

So z_T is a cocycle of $\operatorname{Ad} T$ cohomologous to $P^{-1}zP$; it takes upper triangular values, and its diagonal entries $T_1(g)_{ii}/\delta_i(g)$ vanish on I_ℓ, as $T''(s)_{ii} = \kappa_i(s) = \delta_i(s)$ there. By Lemma 42.8, items 1 and 3, T and z_T take values in $\mathrm{M}_4(E)$ for a finite extension E of $\mathbb{Q}_\ell$. The hypotheses of Lemma 44.14 hold for $\rho = T|_{D_\ell}$ (with $D_\ell \cong \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$), the integers $m_1 > m_2 > m_3 > m_4$ and $u = z_T$, by (Zt3). So the class of z_T, and hence (transporting by $X \mapsto PXP^{-1}$) that of $z|_{D_\ell}$, lies in H^1_f. For another place $h\bar{\ell}$ above ℓ, $z(hyh^{-1}) = r(h)z(y)r(h)^{-1} + (1 - \operatorname{Ad} r(hyh^{-1}))z(h)$ for $y \in D_\ell$, so the class at $hD_\ell h^{-1}$ is the transport of the class at D_ℓ by h and lies in H^1_f as well. By Steps 2 and 3, z is a Bloch–Kato cocycle for $(\mathfrak{sp}_4(\overline{\mathbb{Q}}_\ell), \operatorname{Ad} r)$.

Step 4: Thorne's theorem for a conjugate self-dual twist. Let $K = \mathbb{Q}(i)$, in which the odd prime ℓ is unramified. By Lemma 44.9 the restriction of r to $G_{K(\mu_{\ell^\infty})}$ is irreducible; in particular $r|_{G_K}$ is irreducible, so by (Sy2) $r|_{G_K} \leftrightarrow_\iota \pi$ for a regular algebraic cuspidal π of $\mathrm{GL}_4(\mathbb{A}_K)$. By (Sq2), $r(g)$ has similitude $\theta(g) = \chi_{\mathrm{cyc}}(g)^{e+1}\nu(g)$, $\nu = \omega_\ell^{-1}\xi\chi$ of finite order. By Proposition 44.13 there are ψ and a regular algebraic conjugate self-dual cuspidal π' with $\psi(cgc)\psi(g)\theta(g) = \chi_{\mathrm{cyc}}(g)^{-3}$ on G_K and $r' = (r|_{G_K}) \otimes \psi \leftrightarrow_\iota \pi'$. From ${}^t r(g)J_4 r(g) = \theta(g)J_4$ we get $r(g) = \theta(g)J_4^{-1}\,{}^t r(g)^{-1}J_4$, hence with $\mathbf{P} = r(c)J_4^{-1}$,

$$\begin{aligned} r'(cgc) &= \psi(cgc)\, r(c)r(g)r(c)^{-1} = \psi(cgc)\psi(g)\theta(g)\, \mathbf{P}\,{}^t r'(g)^{-1}\mathbf{P}^{-1} \\ &= \chi_{\mathrm{cyc}}(g)^{1-4}\, \mathbf{P}\,{}^t r'(g)^{-1}\mathbf{P}^{-1} \qquad (g \in G_K). \end{aligned}$$

By Lemma 42.8, item 1, $r'(G_K) \subseteq \mathrm{GL}_4(E)$ for a finite extension E; as a local field has only finitely many extensions of degree at most 4 in $\overline{\mathbb{Q}}_\ell$, enlarging E we may assume that the entries of $\mathbf{P}$ lie in E. The restriction of r' to $G_{K(\mu_{\ell^\infty})}$ is irreducible, since twisting by a character does not change invariant subspaces. So Theorem B.72 applies to K, c, $n = 4$, π', E, r', $\mathbf{P}$, and gives $H^1_f(\mathbb{Q}, \operatorname{ad}_E r' \otimes_E \overline{\mathbb{Q}}_\ell) = 0$.

Step 5: $\mathfrak{sp}_4$ is a direct summand. Let $\tau(X) = J_4^{-1}\,{}^t X J_4$, an involution of M_4 with (-1)-eigenspace $\mathfrak{sp}_4$ and $(+1)$-eigenspace $\mathfrak{c}$, so $\mathrm{M}_4 = \mathfrak{sp}_4 \oplus \mathfrak{c}$. For $g \in G_K$, g acts on $\operatorname{ad} r'$ by $\operatorname{Ad} r(g)$, and Ad of a matrix with a similitude commutes with τ. The element c acts by $X \mapsto -\mathbf{P}\,{}^t X \mathbf{P}^{-1} = -r(c)J_4^{-1}\,{}^t X J_4 r(c)^{-1} = -r(c)\tau(X)r(c)^{-1}$, which is $\operatorname{Ad} r(c)$ on $\mathfrak{sp}_4$ and $-\operatorname{Ad} r(c)$ on $\mathfrak{c}$. As $G_\mathbb{Q}$ is generated by G_K and c, both summands are $G_\mathbb{Q}$-stable and $\mathfrak{sp}_4 \subseteq \operatorname{ad} r' \otimes \overline{\mathbb{Q}}_\ell$ is the representation $(\mathfrak{sp}_4(\overline{\mathbb{Q}}_\ell), \operatorname{Ad} r)$. Local conditions are compatible with direct sums, so $H^1_f(\mathbb{Q}, \mathfrak{sp}_4)$ is a direct summand of $H^1_f(\mathbb{Q}, \operatorname{ad} r' \otimes \overline{\mathbb{Q}}_\ell) = 0$.

Step 6: conclusion. By Steps 3 and 5 the Bloch–Kato cocycle z is a coboundary: $z(g) = r(g)X_0 r(g)^{-1} - X_0$ with $X_0 \in \mathfrak{sp}_4(\overline{\mathbb{Q}}_\ell)$. Then $r_1(g) = z(g)r(g) = r(g)X_0 - X_0 r(g) = Xr(g) - r(g)X$ with $X = -X_0 \in \mathfrak{sp}_4(\overline{\mathbb{Q}}_\ell)$, so $r''(g) = r(g) + \epsilon(Xr(g) - r(g)X)$ and r is rigid. □

Lemma 44.9. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with odd prime ℓ, let r be an irreducible point of $\mathfrak{D}$, and let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field in which ℓ is unramified. Then the restriction of r to $G_{K(\mu_{\ell^\infty})}$ is irreducible on $\overline{\mathbb{Q}}_\ell^4$.*

Proof. Put $V = \overline{\mathbb{Q}}_\ell^4$ and $H = G_{K(\mu_{\ell^\infty})}$, a closed normal subgroup of $G_\mathbb{Q}$ with $G_\mathbb{Q}/H \cong \mathrm{Gal}(K(\mu_{\ell^\infty})/\mathbb{Q})$ abelian. So every subgroup of $G_\mathbb{Q}$ containing H is normal, and $gyg^{-1}y^{-1} \in H$ for all $g, y \in G_\mathbb{Q}$. By (Sq5), r is continuous.

Step 1: the only closed subgroup of index 2 of $G_\mathbb{Q}$ containing H and I_ℓ is G_K. The only quadratic subfield of $\mathbb{Q}(\mu_{\ell^\infty})$ is $\mathbb{Q}(\sqrt{\ell^*})$, $\ell^* = (-1)^{(\ell-1)/2}\ell$, which is ramified at ℓ, whereas K is not; so $K \cap \mathbb{Q}(\mu_{\ell^\infty}) = \mathbb{Q}$ and restriction gives $\mathrm{Gal}(K(\mu_{\ell^\infty})/\mathbb{Q}) \cong \mathrm{Gal}(K/\mathbb{Q}) \times \mathbb{Z}_\ell^\times$, the second factor through χ_{cyc}. A closed subgroup of index 2 containing H is the kernel of a continuous character of order 2 of this group. As $\mathbb{Z}_\ell^\times \cong \mu_{\ell-1} \times (1 + \ell\mathbb{Z}_\ell)$ with $1 + \ell\mathbb{Z}_\ell$ pro-ℓ and ℓ odd, the characters of order 2 are ε_K (the character of K), η and $\varepsilon_K\eta$, where $\eta(g)$ is the Legendre symbol of $\chi_{\mathrm{cyc}}(g)$ modulo ℓ. Since $\chi_{\mathrm{cyc}}(I_\ell) = \mathbb{Z}_\ell^\times$, η is non-trivial on I_ℓ, and ε_K is trivial on I_ℓ as ℓ is

unramified in K; so η and $\varepsilon_K\eta$ are non-trivial on I_ℓ, and the only such kernel containing I_ℓ is $\ker\varepsilon_K = G_K$.

Step 2: no closed normal subgroup G' of finite index with $H \subseteq G'$ has $r|_{G'}$ reducible. Suppose it does. G' is open, so by Lemma 44.10 (second assertion, with $V_0 = G'$), $\bar\Psi$ has order 2, $\bar\Psi|_{D_\ell} = \mathbf{1}$ and $G' \subseteq \ker\bar\Psi$. Then $\ker\bar\Psi$ is a closed subgroup of index 2 containing H and $I_\ell \subseteq D_\ell$, so $\ker\bar\Psi = G_K$ by Step 1. A complex conjugation c is not in G_K, K being imaginary, so $\bar\Psi(c) \neq 1$; but $\Psi(c) = 1$ by (Sg1) (for every complex conjugation, as they are conjugate in $G_\mathbb{Q}$). This is a contradiction.

Step 3: $r|_H$ is isotypic. Let $V' \subseteq V$ be the sum of all $r(H)$-stable subspaces on which H acts irreducibly. It is non-zero, and $r(g)V' = V'$ for $g \in G_\mathbb{Q}$, since $r(g)$ maps an H-irreducible subspace onto one (H is normal); as r is irreducible, $V' = V$, so $r|_H$ is semisimple. Let $V = U_1 \oplus \cdots \oplus U_k$ be its decomposition into isotypic components. Each $r(g)$ maps isotypic components onto isotypic components, and the sum of the components in one orbit is $r(G_\mathbb{Q})$-stable, so $G_\mathbb{Q}$ permutes them transitively. The stabiliser $G' = \{g :\ r(g)U_1 = U_1\}$ is closed (r is continuous), of index k, contains H, and is therefore normal. If $k \geq 2$, U_1 is a proper non-zero $r(G')$-stable subspace, contradicting Step 2. So $k = 1$: there are an irreducible representation of H on a space W of dimension d and an H-isomorphism $V \cong W \otimes \overline{\mathbb{Q}}_\ell^m$ (H acting on the first factor), with $md = 4$.

Step 4: an element with four distinct eigenvalues. The group $I_\ell \cap \ker\chi \cap \ker\omega_\ell$ is open in I_ℓ and $\chi_{\mathrm{cyc}}(I_\ell) = \mathbb{Z}_\ell^\times$, so it contains u with $c_u = \chi_{\mathrm{cyc}}(u)$ of infinite order. As ξ is unramified at ℓ (Sg1), $\kappa_i(u) = c_u^{m_i}$, pairwise distinct. By (Zt2), $r(u)$ is conjugate to an upper triangular matrix with diagonal $(\kappa_i(u))$, so it has four distinct eigenvalues; let $L_1, \ldots, L_4$ be its eigenlines. For $g \in G_\mathbb{Q}$ put $h_g = gug^{-1}u^{-1} \in H$.

Step 5: $d \neq 1$. If $d = 1$, every $r(h)$, $h \in H$, is a scalar $w(h)$, and $r(g)r(u)r(g)^{-1} = r(h_g)r(u) = w(h_g)r(u)$; so $r(g)$ maps each eigenline of $r(u)$ onto an eigenline, and $G_\mathbb{Q}$ permutes $\{L_1, \ldots, L_4\}$. The kernel V_1 of this permutation action is normal of finite index and closed (the conditions $r(g)L_i = L_i$ are closed), hence open, and L_1 is $r(V_1)$-stable. This contradicts Lemma 44.10 (second assertion: no $r(V_1)$-stable line).

Step 6: $d \neq 2$. Let $d = m = 2$ and identify $V = W \otimes \overline{\mathbb{Q}}_\ell^2$. By Burnside's theorem the span of $r(H)$ is $\mathcal{A} = \mathrm{End}(W) \otimes 1$, whose commutant in $\mathrm{End}(V) = \mathrm{End}(W) \otimes \mathrm{M}_2(\overline{\mathbb{Q}}_\ell)$ is $\mathcal{C} = 1 \otimes \mathrm{M}_2(\overline{\mathbb{Q}}_\ell)$, and the commutant of $\mathcal{C}$ is $\mathcal{A}$. For $g \in G_\mathbb{Q}$, conjugation by $r(g)$ preserves $r(H)$, hence $\mathcal{A}$ and $\mathcal{C}$; on $\mathcal{C} \cong \mathrm{M}_2$ it is an algebra automorphism, hence inner (Skolem–Noether): there is $m_g \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$, unique up to a scalar, with $r(g)(1 \otimes X)r(g)^{-1} = 1 \otimes m_g X m_g^{-1}$. Then $r(g)(1 \otimes m_g)^{-1}$ commutes with $\mathcal{C}$, so lies in $\mathcal{A}$, and $r(g) = w_g \otimes m_g$ with $w_g \in \mathrm{GL}(W)$. The map $g \mapsto \mathrm{Ad}(m_g)$ is a homomorphism, and m_h is a scalar for $h \in H$ (as $r(h) \in \mathcal{A}$ commutes with $\mathcal{C}$). Applied to h_g: $m_g m_u m_g^{-1} = \lambda_g m_u$ with $\lambda_g \in \overline{\mathbb{Q}}_\ell^\times$. The eigenvalues of $r(u) = w_u \otimes m_u$ are the products $a_i b_j$ of those of w_u and m_u; they are distinct (Step 4), so m_u has two distinct eigenvalues, with eigenlines $M_1, M_2 \subset \overline{\mathbb{Q}}_\ell^2$, and every m_g permutes $\{M_1, M_2\}$. Put $X_1 = W \otimes M_1$ and $X_2 = W \otimes M_2$; then $r(g)X_1 = w_g W \otimes m_g M_1 \in \{X_1, X_2\}$. The stabiliser $G'' = \{g :\ r(g)X_1 = X_1\}$ is closed, of index at most 2, contains H (m_h is a scalar), and is different from $G_\mathbb{Q}$ because r is irreducible; so it has index 2, is normal, and X_1 is a proper non-zero $r(G'')$-stable subspace. This contradicts Step 2.

By Steps 3, 5 and 6, $d = 4$ and $m = 1$: $r|_H$ is irreducible. □

Lemma 44.10. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$ and character Ψ, let $\bar\Psi$ be the reduction of Ψ modulo $\mathfrak{m}_\mathcal{O}$, and let r be an irreducible point of $\mathfrak{D}$. If there is an open subgroup $U \subseteq G_\mathbb{Q}$ such that $r|_U$ is reducible, then $\bar\Psi$ has order exactly 2 and $\bar\Psi|_{D_\ell} = \mathbf{1}$. Moreover, if $V_0 \subseteq G_\mathbb{Q}$ is an open normal subgroup such that $r|_{V_0}$ is reducible, then $V_0 \subseteq \ker\bar\Psi$,*

and every non-zero $r(V_0)$-stable subspace of $\overline{\mathbb{Q}}_\ell^4$ has dimension at least 2.

Proof. Assume $r|_U$ is reducible. Bars denote residues in $\overline{\mathbb{F}}_\ell$ (for elements of $\overline{\mathbb{Z}}_\ell$, the integral closure of $\mathbb{Z}_\ell$ in $\overline{\mathbb{Q}}_\ell$); ℓ is odd, so $1 \neq -1$ in $\overline{\mathbb{F}}_\ell$.

Step 1: a normal open subgroup. $V_0 = \bigcap_{g \in G_\mathbb{Q}} gUg^{-1}$ is a finite intersection (U has finite index), hence an open normal subgroup contained in U; $r|_{V_0}$ is reducible. For the second assertion let instead V_0 be the given open normal subgroup with $r|_{V_0}$ reducible. Steps 2–7 use only that V_0 is open and normal and that $r|_{V_0}$ is reducible.

Step 2: an adapted basis. By (Zt2) there is $P_1 \in \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ with $P_1^{-1} r(y) P_1$ upper triangular with diagonal $(\delta_1(y), \dots, \delta_4(y))$ for $y \in D_\ell$, and $\delta_i = \kappa_i$ on I_ℓ. The group $V_0 \cap I_\ell \cap \ker\chi \cap \ker\omega_\ell$ is open in I_ℓ and $\chi_{\mathrm{cyc}}(I_\ell) = \mathbb{Z}_\ell^\times$, so it contains u with $c = \chi_{\mathrm{cyc}}(u)$ of infinite order; as ξ is unramified at ℓ (Sg1), $\kappa_i(u) = c^{m_i}$ with $(m_1, \dots, m_4) = (e{+}1, e, 1, 0)$, pairwise distinct. An upper triangular matrix with distinct diagonal entries is diagonalised by an upper triangular matrix; so there is P (a product of P_1 and an upper triangular matrix) such that $r'(g) = P^{-1} r(g) P$ satisfies: $r'(u)$ is diagonal with distinct entries, and $r'(y)$ is upper triangular with diagonal $(\delta_i(y))$ for $y \in D_\ell$. Irreducibility of r, $r|_{V_0}$ is that of r', $r'|_{V_0}$.

Step 3: $r'(V_0)$-stable subspaces are coordinate subspaces. If W is $r'(V_0)$-stable and $w = \sum_i w_i e_i \in W$, then with the Lagrange polynomials L_i of the distinct eigenvalues of $r'(u)$, $L_i(r'(u))w = w_i e_i \in W$; so $W = V_S$ with $S = \{i : e_i \in W\}$.

Step 4: a system of imprimitivity. Let $W \neq 0$ be an $r'(V_0)$-stable subspace of least dimension; as $r'|_{V_0}$ is reducible, $W \neq V$, and $W = V_S$ with $\emptyset \neq S \subsetneq \{1, \dots, 4\}$. For $g \in G_\mathbb{Q}$, $r'(g)W$ is $r'(V_0)$-stable (V_0 is normal) of the same dimension, hence $r'(g)W = V_{S_g}$ with $|S_g| = |S|$ by Step 3. For g, h with $S_g \neq S_h$, $V_{S_g} \cap V_{S_h} = V_{S_g \cap S_h}$ is $r'(V_0)$-stable of smaller dimension, so $S_g \cap S_h = \emptyset$. The sum of the V_{S_g} is $r'(G_\mathbb{Q})$-stable and non-zero, hence V as r' is irreducible. So the distinct sets S_g form a partition $S_1, \dots, S_k$ of $\{1, \dots, 4\}$ into $k \geq 2$ blocks of equal size, $k \in \{2, 4\}$, permuted by $G_\mathbb{Q}$: $r'(g) V_{S_a} = V_{S_{\pi_g(a)}}$ with a homomorphism $g \mapsto \pi_g$ to the symmetric group on the blocks, whose image Γ is transitive.

Step 5: D_ℓ fixes every block. Let $y \in D_\ell$ and $i = \max S_a$. Since $r'(y)$ is upper triangular with diagonal entry $\delta_i(y) \neq 0$ at (i,i), $r'(y)e_i$ has non-zero e_i-coordinate, and it lies in $V_{S_{\pi_y(a)}}$; so $i \in S_{\pi_y(a)}$, and $\pi_y(a) = a$.

Step 6: elements with fixed-point-free block permutation. Let $g \in G_\mathbb{Q}$ with π_g fixed-point-free. For $y \in D_\ell$, $\pi_{gy} = \pi_g$ (Step 5), so $r'(gy)$ maps each V_{S_a} into some V_{S_b} with $b \neq a$, its diagonal entries vanish, and $\operatorname{tr} r(gy) = 0$. By (Sq4) the entries of $r(gy)$ are congruent to those of $r_0(gy)$ modulo the maximal ideal, so

$$\sum_{i=1}^{4} \bar\kappa_i(g)\, \bar\kappa_i(y) = 0 \qquad (y \in D_\ell).$$

Group the indices i by the character $\bar\kappa_i|_{D_\ell} \colon D_\ell \to \overline{\mathbb{F}}_\ell^\times$; distinct characters are linearly independent (Artin), so for each class C the sum $\sum_{i \in C} \bar\kappa_i(g)$ vanishes, and as each $\bar\kappa_i(g) \neq 0$, every class has at least two elements. If $\lambda_i \neq \lambda_j$, then $\bar\kappa_i(\sigma_\ell) \neq \bar\kappa_j(\sigma_\ell)$: since $\chi_{\mathrm{cyc}}(\sigma_\ell) = 1$ (Sg2) and $\bar\omega_\ell = \bar\chi_{\mathrm{cyc}}$, $\bar\alpha(\sigma_\ell) = \bar\xi(\sigma_\ell)$, and the ratios κ_i/κ_j for the pairs $(1,2), (1,4), (3,2), (3,4)$ are $\alpha, \alpha\Psi, \alpha\Psi^{-1}, \alpha$, with residues $\bar\xi, \overline{\xi\Psi}, \overline{\xi\Psi^{-1}}, \bar\xi$ at σ_ℓ, none equal to 1 by (Sg2) at $\ell \in T$. Hence the classes are $\{1,3\}$ and $\{2,4\}$: $\bar\theta = \bar\alpha$ and $\bar\Psi = \mathbf{1}$ on D_ℓ, and $\bar\kappa_2(g) + \bar\kappa_4(g) = \bar\Psi(g) + 1 = 0$. So

$$\bar\Psi|_{D_\ell} = \mathbf{1}, \qquad \bar\Psi(g) = -1 \ \text{ whenever } \pi_g \text{ is fixed-point-free.} \tag{44.13}$$

Fixed-point-free elements exist: in a transitive group on $k \geq 2$ points the average number of fixed points is 1 (Burnside), and the identity fixes $k \geq 2$.

Step 7: the block structure. If $k = 4$, Γ is a transitive subgroup of the symmetric group on four letters, so it contains a 4-cycle or the three double transpositions. If π_g is a 4-cycle, π_g and π_{g^2} are fixed-point-free, and (44.13) gives $-1 = \bar\Psi(g^2) = \bar\Psi(g)^2 = 1$. If π_g, π_h, π_{gh} are the three double transpositions, (44.13) gives $-1 = \bar\Psi(gh) = \bar\Psi(g)\bar\Psi(h) = 1$. Both are impossible, so $k = 2$ and $\Gamma = \{1, \tau\}$ with τ the transposition of the two blocks. Fix g_1 with $\pi_{g_1} = \tau$. For h with $\pi_h = 1$, $\pi_{g_1 h} = \tau$, so $\bar\Psi(g_1 h) = \bar\Psi(g_1) = -1$ and $\bar\Psi(h) = 1$. Hence $\bar\Psi(g) = 1$ if $\pi_g = 1$ and $\bar\Psi(g) = -1$ if $\pi_g = \tau$: $\bar\Psi$ has order exactly 2, and $\bar\Psi|_{D_\ell} = \mathbf{1}$ by (44.13).

Step 8: $V_0 \subseteq \ker\bar\Psi$, and no stable lines. By Step 7, $k = 2$, so the minimal $r'(V_0)$-stable subspace $W = V_S$ of Step 4 has $\dim W = |S| = 2$. Every non-zero $r'(V_0)$-stable subspace contains one of least dimension, hence has dimension at least 2; the $r(V_0)$-stable subspaces are the images under P of the $r'(V_0)$-stable ones, so the same holds for them. For $g \in V_0$, $r'(g)W = W$, so π_g fixes the block of W; as $k = 2$, $\pi_g = 1$, and $\bar\Psi(g) = 1$ by Step 7. □

Lemma 44.11. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, and put $\overline{\mathbb{Z}}_\ell = \{y \in \overline{\mathbb{Q}}_\ell : |y|_\ell \le 1\}$ and $\mathfrak{m}_\ell = \{y \in \overline{\mathbb{Q}}_\ell : |y|_\ell < 1\}$.*

(Sq1) *For every local $\mathcal{O}$-algebra A with maximal ideal $\mathfrak{m}_A$ and every $\mathcal{O}$-algebra homomorphism $f\colon A \to \overline{\mathbb{Q}}_\ell$ one has $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_\ell$; if moreover $A \in \mathrm{CNL}_\mathcal{O}$, then $f(A) \subseteq \overline{\mathbb{Z}}_\ell$ and f is continuous for the $\mathfrak{m}_A$-adic topology on A.*

Let $r\colon G_\mathbb{Q} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ be a point of $\mathfrak{D}$. Then:

(Sq2) *$r(g)$ has similitude $\theta(g)$ for every $g \in G_\mathbb{Q}$;*

(Sq3) *$r(\phi)$ is block diagonal, and $h(r) = r(g_0) + \theta(g_0)\,r(g_0^{-1})$ satisfies $h(r)_{13} = h(r)_{31} = 0$;*

(Sq4) *for all $g \in G_\mathbb{Q}$ and all i, j, $r(g)_{ij} \in \overline{\mathbb{Z}}_\ell$ and $r(g)_{ij} - r_0(g)_{ij} \in \mathfrak{m}_\ell$;*

(Sq5) *r is continuous.*

Proof. Step 1: $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_\ell$ for every local $\mathcal{O}$-algebra A. Let $x \in \mathfrak{m}_A$ and $y = f(x)$, and suppose $|y|_\ell \ge 1$. The element y is algebraic over K, so $L = K(y)$ is a finite extension of the complete discretely valued field K; its valuation ring $\{z \in L : |z|_\ell \le 1\}$ is the integral closure of $\mathcal{O}$ in L. As $|y^{-1}|_\ell \le 1$, y^{-1} is integral over $\mathcal{O}$: there are $d \ge 1$ and $b_0, \dots, b_{d-1} \in \mathcal{O}$ with $y^{-d} + b_{d-1}y^{-(d-1)} + \cdots + b_0 = 0$. Multiplying by y^d,

$$Q(y) = 0, \qquad Q(X) = 1 + b_{d-1}X + b_{d-2}X^2 + \cdots + b_0X^d \in \mathcal{O}[X].$$

Now $Q(x) \in 1 + \mathfrak{m}_A$ is a unit of the local ring A, so $f(Q(x))$ is a unit of $\overline{\mathbb{Q}}_\ell$; but $f(Q(x)) = Q(f(x)) = Q(y) = 0$ because f is an $\mathcal{O}$-algebra homomorphism. This contradiction shows $|f(x)|_\ell < 1$.

Step 2: $f(A) \subseteq \overline{\mathbb{Z}}_\ell$ and continuity for $A \in \mathrm{CNL}_\mathcal{O}$; this completes (Sq1). Let $a \in A$. Since $\mathcal{O} \to A/\mathfrak{m}_A$ is surjective, $a = \lambda + x$ with $\lambda \in \mathcal{O}$ and $x \in \mathfrak{m}_A$, so $f(a) = \lambda + f(x)$ has $|f(a)|_\ell \le \max(|\lambda|_\ell, |f(x)|_\ell) \le 1$. As A is Noetherian, $\mathfrak{m}_A = (x_1, \dots, x_k)$; put $c = \max_i |f(x_i)|_\ell$, so $c < 1$ by Step 1. Every element of $\mathfrak{m}_A^n$ is a finite sum of terms $a' x_{i_1} \cdots x_{i_n}$ with $a' \in A$, so by the ultrametric inequality and $|f(a')|_\ell \le 1$, $|f(z)|_\ell \le c^n$ for $z \in \mathfrak{m}_A^n$. Hence $|f(a) - f(a')|_\ell \le c^n$ whenever $a - a' \in \mathfrak{m}_A^n$, and f is continuous.

Step 3: (Sq2)–(Sq5). By definition $r(g) = f(r_A(g))$ (entrywise) for a symplectic Eisenstein lift r_A to some $A \in \mathrm{CNL}_\mathcal{O}$ and an $\mathcal{O}$-algebra homomorphism $f\colon A \to \overline{\mathbb{Q}}_\ell$. Applying the ring homomorphism f entrywise to ${}^t r_A(g) J_4 r_A(g) = \theta(g) J_4$ (that is (Se1)), and using $f(\theta(g)\cdot 1_A) = \theta(g)$, gives (Sq2). By (Se3) the entries (i, j) of $r_A(\phi)$ with $\lambda_i \ne \lambda_j$ vanish, and $h(r_A)_{13} = h(r_A)_{31} = 0$;

since f applied entrywise to $h(r_A) = r_A(g_0) + \theta(g_0) r_A(g_0^{-1})$ gives $r(g_0) + \theta(g_0) r(g_0^{-1}) = h(r)$, (Sq3) follows. Since r_A is a lift, $r_A(g) \equiv \bar{r}_0(g) \pmod{\mathfrak{m}_A}$, that is $r_A(g)_{ij} - r_0(g)_{ij} \cdot 1_A \in \mathfrak{m}_A$ (the identification $\mathbb{F}_{\mathcal{O}} \cong A/\mathfrak{m}_A$ is induced by $\mathcal{O} \to A$); applying f and Steps 1 and 2 gives $r(g)_{ij} \in \overline{\mathbb{Z}}_\ell$ and $r(g)_{ij} - r_0(g)_{ij} = f(r_A(g)_{ij} - r_0(g)_{ij} \cdot 1_A) \in \mathfrak{m}_\ell$, which is (Sq4). Finally r_A is continuous for the $\mathfrak{m}_A$-adic topology and f is continuous by (Sq1), so each entry $g \mapsto r(g)_{ij}$ is continuous, which is (Sq5). $\square$

Lemma 44.12. *Let $\mathfrak{D} = (\ell, \mathcal{O}, e, N, \chi, \xi, S, \dots)$ be a symplectic Eisenstein datum with set T, and let r be a point of $\mathfrak{D}$. Then:*

(Zt1) *for every $v \in T$ with $v \neq \ell$, $r(g) = 1$ for every $g \in I_v$ with $\chi(g) = 1$;*

(Zt2) *there are continuous characters $\delta_1, \dots, \delta_4 \colon D_\ell \to \overline{\mathbb{Q}}_\ell^\times$ such that r is triangular on D_ℓ with diagonal $(\delta_1, \dots, \delta_4)$, $\delta_i(s) = \kappa_i(s)$ for all $s \in I_\ell$, and $\delta_i(g) - \kappa_i(g) \in \mathfrak{m}_\ell$ for all $g \in D_\ell$ $(i = 1, \dots, 4)$;*

(Zt3) *the characters of (Zt2) satisfy $\delta_i \neq \delta_j \, \chi_{\mathrm{cyc}}|_{D_\ell}$ for all i, j, and $\delta_i \chi_{\mathrm{cyc}}^{-m_i}$ has finite image on I_ℓ, where $(m_1, m_2, m_3, m_4) = (e+1, e, 1, 0)$.*

Proof. Write $r = f \circ r_A$ for a symplectic Eisenstein lift r_A to $A \in \mathrm{CNL}_{\mathcal{O}}$ and an $\mathcal{O}$-algebra homomorphism $f \colon A \to \overline{\mathbb{Q}}_\ell$. Let $v \in T$. By (Se4) there are a domain $B \in \mathrm{CNL}_{\mathcal{O}}$, a lift ρ_B of $\bar{r}_0|_{D_v}$ to B satisfying the local condition at v, and $\mathcal{O}$-algebra homomorphisms $x \colon B \to \mathcal{O}$, $y \colon B \to A$ with $x \circ \rho_B = r_0|_{D_v}$ and $y \circ \rho_B = r_A|_{D_v}$; put $F = f \circ y \colon B \to \overline{\mathbb{Q}}_\ell$, so that $r|_{D_v} = F \circ \rho_B$ (entrywise). Since x is an $\mathcal{O}$-algebra homomorphism to $\mathcal{O}$, the structure map $\mathcal{O} \to B$ is injective.

Step 1: (Zt1). For $v \neq \ell$ the local condition gives $\rho_B(g) = 1$ for $g \in I_v$ with $\chi(g) = 1$; applying F gives $r(g) = 1$.

From now on $v = \ell$. Put $c = \chi_{\mathrm{cyc}}(s_0)$ and $c_j = \kappa_j(s_0)$. By (Sg5), $\chi(s_0) = \omega_\ell(s_0) = 1$, and $\xi(s_0) = 1$ because ξ is unramified at $\ell \in T$ (Sg1); so $(c_1, c_2, c_3, c_4) = (c^{e+1}, c^e, c, 1)$, which are pairwise distinct because c has infinite order and $e \geq 2$. Put $d_i = \prod_{j \neq i} (c_i - c_j) \in \mathcal{O} \setminus \{0\}$.

Step 2: ordinary representations over a field. Let E be a field containing $\mathcal{O}$ and $\rho \colon D_\ell \to \mathrm{GL}_4(E)$ an ordinary homomorphism, with $\Pi_i = \prod_{j \neq i} (\rho(s_0) - c_j)$. The characteristic polynomial of $\rho(s_0)$ is $\prod_j (X - c_j)$ with distinct roots, so E^4 has a basis $w_1, \dots, w_4$ with $\rho(s_0) w_k = c_k w_k$; let P be the matrix with columns w_k and E_i the projection onto $E w_i$ along the other w_k. Then $\Pi_i w_k = \prod_{j \neq i} (c_k - c_j) w_k$, so $\Pi_i = d_i E_i$. The condition $\Pi_j \rho(g) \Pi_i = 0$ for $i < j$ gives $E_j \rho(g) E_i = 0$, that is, $P^{-1} \rho(g) P$ is upper triangular for every $g \in D_\ell$; its diagonal entries $\delta_i^E(g)$ are characters of D_ℓ (the diagonal of a product of upper triangular matrices is the product of the diagonals), and

$$\mathrm{tr}\big(\Pi_i \, \rho(g)\big) = d_i \, \mathrm{tr}\big(E_i \rho(g)\big) = d_i \, \delta_i^E(g). \tag{44.14}$$

For $s \in I_\ell$, $\Pi_i (\rho(s) - \kappa_i(s)) \Pi_i = 0$ gives $d_i^2 (\delta_i^E(s) - \kappa_i(s)) E_i = 0$, so $\delta_i^E(s) = \kappa_i(s)$. Finally, the ordinary conditions are polynomial identities in the entries of $\rho(s_0)$, $\rho(g)$ with coefficients in $\mathcal{O}$, so for a ring homomorphism h of $\mathcal{O}$-algebras, $h \circ \rho$ is ordinary whenever ρ is.

Step 3: the characters δ_i. By the local condition at ℓ, ρ_B is ordinary, hence so is $r|_{D_\ell} = F \circ \rho_B$ (Step 2). Step 2 with $E = \overline{\mathbb{Q}}_\ell$ gives characters $\delta_i = \delta_i^{\overline{\mathbb{Q}}_\ell}$ such that r is triangular on D_ℓ with diagonal $(\delta_1, \dots, \delta_4)$ (conjugate by P^{-1}) and $\delta_i = \kappa_i$ on I_ℓ. By (44.14), $\delta_i(g) = d_i^{-1} \, \mathrm{tr}(\Pi_i \, r(g))$ is a polynomial in the entries of $r(s_0)$ and $r(g)$, hence continuous in g by (Sq5).

Step 4: $\delta_i(g) \equiv \kappa_i(g)$ modulo $\mathfrak{m}_\ell$. Fix i and $g \in D_\ell$. Let K be the fraction field of B; ρ_B is ordinary over K, and Step 2 gives $\delta_i^K(g) \in K$ with $t = \mathrm{tr}(\Pi_i(\rho_B) \, \rho_B(g)) = d_i \delta_i^K(g)$, where

$t \in B$. Applying x and F to t and using (44.14) for $r_0|_{D_\ell}$ (where $\Pi_i(r_0)$ is diagonal with the single non-zero entry d_i at (i,i)) and for $r|_{D_\ell}$,

$$x(t) = d_i\,\kappa_i(g), \qquad F(t) = d_i\,\delta_i(g).$$

Put $w = \delta_i^K(g) - \kappa_i(g) \in K$. As a diagonal entry of a triangular conjugate of $\rho_B(g)$, $\delta_i^K(g)$ is a root of the characteristic polynomial of $\rho_B(g)$, which is monic with coefficients in B; so w is integral over B, and $C = B[w] \subseteq K$ is a domain, finite over B. A complete Noetherian local ring is Henselian, and a finite algebra over a Henselian local ring is a finite product of local rings; as C is a domain, C is local, with maximal ideal $\mathfrak{m}_C$. Since C is integral over B, the lying-over theorem gives primes Q_x, Q_F of C lying over $\ker x$ and $\ker F$. The domain C/Q_x is integral over $B/\ker x \cong \mathcal{O}$, so its fraction field is algebraic over that of $\mathcal{O}$ and embeds into $\overline{\mathbb{Q}}_\ell$ over it; this gives an $\mathcal{O}$-algebra homomorphism $\tilde{x}\colon C \to \overline{\mathbb{Q}}_\ell$ extending x. In the same way, C/Q_F is integral over $B/\ker F \cong F(B) \subseteq \overline{\mathbb{Q}}_\ell$, and as $\overline{\mathbb{Q}}_\ell$ is algebraically closed there is $\tilde{F}\colon C \to \overline{\mathbb{Q}}_\ell$ extending F. Now $d_i w = t - d_i\kappa_i(g) \in B$, so

$$d_i\,\tilde{x}(w) = x(t) - d_i\kappa_i(g) = 0, \qquad d_i\,\tilde{F}(w) = F(t) - d_i\kappa_i(g) = d_i\big(\delta_i(g) - \kappa_i(g)\big).$$

As $d_i \neq 0$, $\tilde{x}(w) = 0$; hence w is not a unit of C, that is, $w \in \mathfrak{m}_C$. By (Sq1) for the local $\mathcal{O}$-algebra C, $\tilde{F}(w) \in \mathfrak{m}_\ell$, and therefore $\delta_i(g) - \kappa_i(g) = \tilde{F}(w) \in \mathfrak{m}_\ell$.

Step 5: (Zt3). On I_ℓ, $\delta_i = \kappa_i$, and $\kappa_i\chi_{\mathrm{cyc}}^{-m_i}$ is $\omega_\ell^{-1}\xi\chi$, χ, $\omega_\ell^{-1}\xi$, $\mathbf{1}$ for $i = 1,2,3,4$, of finite image. Suppose $\delta_i = \delta_j\chi_{\mathrm{cyc}}$ on D_ℓ. If $\lambda_i \neq \lambda_j$ ($\boldsymbol{\lambda} = (1,0,1,0)$), evaluate at σ_ℓ: as $\chi_{\mathrm{cyc}}(\sigma_\ell) = 1$ (Sg2), $\delta_i(\sigma_\ell) = \delta_j(\sigma_\ell)$, so by Step 4 $\kappa_i(\sigma_\ell) - \kappa_j(\sigma_\ell) \in \mathfrak{m}_\ell \cap \mathcal{O} = \mathfrak{m}_\mathcal{O}$; but $\bar{\alpha}(\sigma_\ell) = \bar{\xi}(\sigma_\ell)$ (as $\bar{\omega}_\ell = \bar{\chi}_{\mathrm{cyc}}$), and for the pairs $(1,2),(1,4),(3,2),(3,4)$ the ratio κ_i/κ_j is $\alpha, \alpha\Psi, \alpha\Psi^{-1}, \alpha$, with residue $\bar{\xi}, \overline{\xi\Psi}, \overline{\xi\Psi^{-1}}, \bar{\xi}$ at σ_ℓ, never 1 by (Sg2) at $\ell \in T$. If $\lambda_i = \lambda_j$, evaluate at s_0: $c^{m_i} = c^{m_j+1}$, so $m_i = m_j + 1$, which fails for $i = j$ and for $\{i,j\} = \{1,3\}$ or $\{2,4\}$ (there $m_i - m_j - 1 \in \{e-1, -e-1\}$ and $e \geq 2$). □

Proposition 44.13. *Let ℓ be an odd prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $c \in G_\mathbb{Q}$ a complex conjugation, w an integer and $\nu\colon G_\mathbb{Q} \to \overline{\mathbb{Q}}_\ell^\times$ a character of finite order. Let $r\colon G_\mathbb{Q} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ be continuous such that $r(g)$ has similitude $\theta(g) = \chi_{\mathrm{cyc}}(g)^w\nu(g)$ for every g, let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field in which ℓ is unramified, and let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_K)$ with $r|_{G_K} \leftrightarrow_\iota \pi$. Then there are a continuous character $\psi\colon G_K \to \overline{\mathbb{Q}}_\ell^\times$ with*

$$\psi(cgc)\,\psi(g)\,\theta(g) = \chi_{\mathrm{cyc}}(g)^{-3} \qquad (g \in G_K)$$

and a regular algebraic conjugate self-dual cuspidal automorphic representation π' of $\mathrm{GL}_4(\mathbb{A}_K)$ with $(r|_{G_K}) \otimes \psi \leftrightarrow_\iota \pi'$.

Proof. Put $\chi' = \chi_{\mathrm{cyc}}^{-3}\theta^{-1} = \chi_{\mathrm{cyc}}^{-3-w}\nu^{-1}$, a continuous character of $G_\mathbb{Q}$; we need ψ with $\psi(g)\psi(cgc) = \chi'(g)$ on G_K. Write $\chi'_\ell = \chi' \circ \mathrm{Art}_{\mathbb{Q}_\ell}$ for its restriction to a decomposition group at ℓ; since $\chi_{\mathrm{cyc}} \circ \mathrm{Art}_{\mathbb{Q}_\ell}(x) = x^{\pm 1}$ on $\mathbb{Z}_\ell^\times$ and ν has finite order, there are $k \in \mathbb{Z}$ and a character λ of $\mathbb{Z}_\ell^\times$ of finite order with $\chi'_\ell(x) = x^k\lambda(x)$ for $x \in \mathbb{Z}_\ell^\times$.

Step 1: local characters at ℓ. Let S be the set of places of K above ℓ; $S^c = S$, and every $v \in S$ is unramified over $\mathbb{Q}$. Choose $\bar{v}$ as in Lemma B.26. By local class field theory a character of G_{K_v} is a character of $K_v^\times$, the inertia group corresponds to $\mathcal{O}_{K_v}^\times$, restriction from $G_{\mathbb{Q}_\ell}$ corresponds to composition with $\mathrm{N}_{K_v/\mathbb{Q}_\ell}$, and conjugation by an element acting on K_v as an automorphism σ corresponds to composition with σ. If ℓ splits in K, $S = \{v, v^c\}$ with $K_v = K_{v^c} = \mathbb{Q}_\ell$; put $\psi_v = \chi'|_{G_{K_v}}$ and $\psi_{v^c} = \mathbf{1}$. Then $\psi_v(g)\psi_{v^c}(cgc) = \chi'(g)$ for $g \in G_{K_v}$, and $\psi_{v^c}(g)\psi_v(cgc) = \chi'(cgc) = \chi'(g)$ for $g \in G_{K_{v^c}}$, since χ' is a character of $G_\mathbb{Q}$. If ℓ is inert, $S = \{v\}$, K_v is the unramified quadratic extension of $\mathbb{Q}_\ell$ with non-trivial automorphism σ, and

$\mathrm{N} = \mathrm{N}_{K_v/\mathbb{Q}_\ell}$ maps $\mathcal{O}_{K_v}^\times$ onto $\mathbb{Z}_\ell^\times$. Fix an embedding $\tau\colon K_v \to \overline{\mathbb{Q}}_\ell$ over $\mathbb{Q}_\ell$. The finite-order character λ factors through $\mathbb{Z}_\ell^\times/(1+\ell^a\mathbb{Z}_\ell)$ for some a, a subgroup of the finite abelian group $\mathcal{O}_{K_v}^\times/(1+\ell^a\mathcal{O}_{K_v})$, so it extends to a finite-order character μ of $\mathcal{O}_{K_v}^\times$. Let ψ_v be the character of $K_v^\times = \ell^{\mathbb{Z}} \times \mathcal{O}_{K_v}^\times$ with $\psi_v(\ell) = 1$ and $\psi_v(x) = \tau(x)^k\mu(x)$ on units. For $x \in \mathcal{O}_{K_v}^\times$, as ψ_v is multiplicative and $x\,\sigma(x) = \mathrm{N}x \in \mathbb{Z}_\ell^\times$,

$$\psi_v(x)\,\psi_v(\sigma x) = \psi_v(\mathrm{N}x) = (\mathrm{N}x)^k\lambda(\mathrm{N}x) = \chi'_\ell(\mathrm{N}x),$$

which is $(\psi_v\psi_{v^c}^c)|_{I_{K_v}} = \chi'|_{I_{K_v}}$ in the notation of Lemma B.26 (the element s there acts on K_v as σ). In both cases each ψ_v agrees on an open subgroup of $\mathcal{O}_{K_v}^\times$ with an algebraic character $x \mapsto \prod_{\tau'}\tau'(x)^{n_{\tau'}}$ (τ' over the embeddings of K_v), so it is locally algebraic and hence de Rham [220].

Step 2: the global character. The character χ' takes the same value on all complex conjugations ($\mathbb{Q}$ has one infinite place). By Lemma B.26 for $F = K$, $F^+ = \mathbb{Q}$, S, χ' and the ψ_v of Step 1, there is a continuous $\psi\colon G_K \to \overline{\mathbb{Q}}_\ell^\times$ with $\psi(g)\psi(cgc) = \chi'(g)$, that is $\psi(cgc)\psi(g)\theta(g) = \chi_{\mathrm{cyc}}(g)^{-3}$, for $g \in G_K$, and $\psi|_{I_{K_v}} = \psi_v|_{I_{K_v}}$ for $v \in S$. So $\psi \circ \mathrm{Art}_K$ agrees on an open subgroup of $\prod_{v|\ell}\mathcal{O}_{K_v}^\times$ with an algebraic character: ψ is locally algebraic at ℓ, and by the correspondence between locally algebraic ℓ-adic characters and algebraic Hecke characters [220] there is an algebraic Hecke character ϕ of K with $\psi \leftrightarrow_\iota \phi$: for almost all finite places w, ψ and ϕ are unramified at w and $\iota\,\psi(\mathrm{Frob}_w^{-1}) = \phi_w(\varpi_w)$.

Step 3: the twisted automorphic representation. Put $\pi' = \pi \otimes (\phi \circ \det)$ and $r' = (r|_{G_K}) \otimes \psi$. The representation π' is cuspidal; it is regular algebraic, since twisting by an algebraic Hecke character shifts the infinitesimal character at each complex place by an integral scalar, which preserves integrality and regularity. At every finite w at which π, ϕ, r and ψ are unramified and the defining identities of $r|_{G_K} \leftrightarrow_\iota \pi$ and $\psi \leftrightarrow_\iota \phi$ hold (all but finitely many), the Satake parameters of π'_w are $\alpha\,\phi_w(\varpi_w)$, α those of π_w, and the eigenvalues of $\iota\,r'(\mathrm{Frob}_w^{-1})$ are $\mathrm{N}w^{3/2}\alpha \cdot \phi_w(\varpi_w)$; so $r' \leftrightarrow_\iota \pi'$.

Step 4: r' is conjugate self-dual. From ${}^t r(g)J_4 r(g) = \theta(g)J_4$, $r(g) = \theta(g)J_4^{-1}\,{}^t r(g)^{-1}J_4$ for $g \in G_{\mathbb{Q}}$; with $\mathbf{P} = r(c)J_4^{-1}$ and $g \in G_K$,

$$\begin{aligned} r'(cgc) &= \psi(cgc)\,r(c)r(g)r(c)^{-1} = \psi(cgc)\theta(g)\,\mathbf{P}\,{}^t r(g)^{-1}\mathbf{P}^{-1} \\ &= \psi(cgc)\psi(g)\theta(g)\,\mathbf{P}\,{}^t r'(g)^{-1}\mathbf{P}^{-1} \\ &= \chi_{\mathrm{cyc}}(g)^{-3}\,\mathbf{P}\,{}^t r'(g)^{-1}\mathbf{P}^{-1}. \end{aligned}$$

So for almost all w the characteristic polynomials of

$$r'(\mathrm{Frob}_{cw}^{-1}) = r'(c\,\mathrm{Frob}_w^{-1}c) \quad \text{and of} \quad \chi_{\mathrm{cyc}}(\mathrm{Frob}_w)^3\,{}^t r'(\mathrm{Frob}_w^{-1})^{-1}$$

agree (choosing $\mathrm{Frob}_{cw} = c\,\mathrm{Frob}_w c$). With $\chi_{\mathrm{cyc}}(\mathrm{Frob}_w) = \mathrm{N}w$ and Step 3, the multiset $\{\mathrm{N}w^{3/2}\alpha'\}$, α' over the Satake parameters of $\pi'_{cw} = (\pi'^c)_w$, equals $\{\mathrm{N}w^3(\mathrm{N}w^{3/2}\alpha)^{-1}\} = \{\mathrm{N}w^{3/2}\alpha^{-1}\}$, α over those of π'_w; the α^{-1} are the Satake parameters of $(\pi'^\vee)_w$. Unramified irreducible representations with the same Satake parameters are isomorphic, so $(\pi'^c)_w \cong (\pi'^\vee)_w$ for almost all w; both are cuspidal, and by (Js1) of Theorem B.60, $\pi'^c \cong \pi'^\vee$. Hence π' is regular algebraic, conjugate self-dual and cuspidal, and $(r|_{G_K}) \otimes \psi \leftrightarrow_\iota \pi'$. □

Lemma 44.14. *Let ℓ be a prime, M a finite extension of $\mathbb{Q}_\ell$ with absolute Galois group G_M and inertia subgroup I_M, $E \subset \overline{\mathbb{Q}}_\ell$ a finite extension of $\mathbb{Q}_\ell$, $n \geq 1$, and $\rho\colon G_M \to \mathrm{GL}_n(E)$ a continuous homomorphism with values in the upper triangular matrices; let $\kappa_1, \dots, \kappa_n\colon G_M \to E^\times$ be its diagonal entries. Assume that there are integers $m_1 > m_2 > \cdots > m_n$ such that $\kappa_a\chi_{\mathrm{cyc}}^{-m_a}$ has finite image on I_M for every a, and that $\kappa_a \neq \kappa_b\chi_{\mathrm{cyc}}$ for all $a < b$. Let $\mathrm{ad}\,\rho$ be $\mathrm{M}_n(E)$*

with g acting by $X \mapsto \rho(g)X\rho(g)^{-1}$, $\mathfrak{b} \subseteq \operatorname{ad}\rho$ the subspace of upper triangular matrices, and $u\colon G_M \to \mathfrak{b}$ a continuous 1-cocycle of $\operatorname{ad}\rho$ whose diagonal entries vanish on I_M. Then there is $Y \in \operatorname{ad}\rho \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with

$$u(g) \otimes 1 = gY - Y \qquad (g \in G_M),$$

G_M acting diagonally; that is, the class of u lies in $H^1_f(G_M, \operatorname{ad}\rho)$.

Proof. All representations below are finite-dimensional E-vector spaces U with a continuous E-linear action of G_M, regarded as $\mathbb{Q}_\ell$-representations. Dimensions are over $\mathbb{Q}_\ell$. We write $h^i(U) = \dim H^i(G_M, U)$ and $d = [M : \mathbb{Q}_\ell]$, and $H^1_f(U)$ is the kernel of $H^1(G_M, U) \to H^1(G_M, U \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}})$; it is functorial in U. For a character $\delta\colon G_M \to E^\times$, $E(\delta)$ is E with g acting by $\delta(g)$. We use the following standard facts of p-adic Hodge theory [94], [52]: (P1) subquotients, direct sums, tensor products over E and E-duals of de Rham representations are de Rham; (P2) for U de Rham, $\dim_M D_{\mathrm{dR}}(U) = \dim U$, $\dim_M \operatorname{Fil}^0 D_{\mathrm{dR}}(U)$ is the number of Hodge–Tate weights of U that are ≤ 0 (with multiplicity), and the multiset of Hodge–Tate weights is additive in short exact sequences; (P3) if $\delta\chi_{\mathrm{cyc}}^{-k}$ has finite image on I_M, then $E(\delta)$ is de Rham with all Hodge–Tate weights equal to k (it is crystalline with weight k over a finite extension of M on whose inertia $\delta\chi_{\mathrm{cyc}}^{-k}$ is trivial, and the de Rham property descends). By (P3), $E(\kappa_a\kappa_b^{-1})$ is de Rham with all weights $m_a - m_b$, which is ≥ 1 when $a < b$.

Step 1: three cohomological facts. (i) If $\delta \ne \chi_{\mathrm{cyc}}$, then $H^2(G_M, E(\delta)) = 0$: by Theorem B.36 $h^2(E(\delta)) = h^0(E(\delta)^*(1))$, and the trace pairing $(x, y) \mapsto \operatorname{tr}_{E/\mathbb{Q}_\ell}(xy)$ identifies $E(\delta)^*(1)$ with $E(\delta^{-1}\chi_{\mathrm{cyc}})$, whose invariants vanish as $\delta^{-1}\chi_{\mathrm{cyc}} \ne 1$. (ii) If $0 \to U' \to U \to U'' \to 0$ is exact and $H^2(G_M, U') = H^2(G_M, U'') = 0$, then $H^2(G_M, U) = 0$ (long exact sequence). (iii) If U is de Rham with all Hodge–Tate weights ≥ 1 and $H^2(G_M, U) = 0$, then $H^1_f(U) = H^1(G_M, U)$: by (P2) $\operatorname{Fil}^0 D_{\mathrm{dR}}(U) = 0$, so $\dim H^1_f(U) = h^0(U) + d\dim U$ by Theorem B.56, and $h^1(U) = h^0(U) + d\dim U$ by Theorem B.36.

Step 2: filtrations. For $k \ge 0$ let $\mathfrak{n}_{\ge k} \subseteq \mathrm{M}_n(E)$ be the span of the elementary matrices E_{ab} with $b - a \ge k$; so $\mathfrak{n}_{\ge 0} = \mathfrak{b}$, and put $\mathfrak{n} = \mathfrak{n}_{\ge 1}$. Write $\rho(g) = D_g(1 + N_g)$ with D_g diagonal and N_g strictly upper triangular. For $X \in \mathfrak{n}_{\ge k}$, $N_g X$ and $X N_g$ lie in $\mathfrak{n}_{\ge k+1}$, so $\rho(g)X\rho(g)^{-1} \equiv D_g X D_g^{-1} \pmod{\mathfrak{n}_{\ge k+1}}$. Hence each $\mathfrak{n}_{\ge k}$ is stable, $\mathfrak{n}_{\ge k}/\mathfrak{n}_{\ge k+1} \cong \bigoplus_{b-a=k} E(\kappa_a\kappa_b^{-1})$ for $k \ge 1$, and $\mathfrak{t} := \mathfrak{b}/\mathfrak{n}$ is E^n with trivial action, the quotient map taking the diagonal part.

Step 3: ρ is de Rham. Let $L_k \subseteq E^n$ be the span of the first k standard basis vectors; it is stable, and $L_k/L_{k-1} \cong E(\kappa_k)$. $L_1 = E(\kappa_1)$ is de Rham by (P3). Let $k \ge 2$ with L_{k-1} de Rham, and put $W = \operatorname{Hom}_E(E(\kappa_k), L_{k-1})$ with g acting by $f \mapsto g \circ f \circ g^{-1}$. $W \cong L_{k-1} \otimes_E E(\kappa_k^{-1})$ is de Rham (P1), with a filtration whose graded pieces are $E(\kappa_a\kappa_k^{-1})$, $a < k$; so its weights are ≥ 1 (P2), $H^2(G_M, W) = 0$ by (i) and (ii) (as $\kappa_a\kappa_k^{-1} \ne \chi_{\mathrm{cyc}}$), and $H^1_f(W) = H^1(G_M, W)$ by (iii). Choose an E-linear section $s\colon E(\kappa_k) \to L_k$ of the quotient map; $\xi(g) = g \circ s \circ g^{-1} - s$ is a cocycle with values in W, so there is $\Phi \in W \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}} \subseteq W \otimes_{\mathbb{Q}_\ell} B_{\mathrm{dR}}$ with $\xi(g) \otimes 1 = g\Phi - \Phi$. Viewing $W \otimes_{\mathbb{Q}_\ell} B_{\mathrm{dR}}$ inside $\operatorname{Hom}_{B_{\mathrm{dR}}}(E(\kappa_k) \otimes B_{\mathrm{dR}}, L_{k-1} \otimes B_{\mathrm{dR}})$, the map $s' = s \otimes 1 - \Phi$ is a B_{dR}-linear section of $L_k \otimes B_{\mathrm{dR}} \to E(\kappa_k) \otimes B_{\mathrm{dR}}$ with $g \circ s' \circ g^{-1} = s \otimes 1 + \xi(g) \otimes 1 - \Phi - \xi(g) \otimes 1 = s'$. So $L_k \otimes B_{\mathrm{dR}} \cong (L_{k-1} \otimes B_{\mathrm{dR}}) \oplus (E(\kappa_k) \otimes B_{\mathrm{dR}})$ equivariantly, and taking invariants, $\dim_M D_{\mathrm{dR}}(L_k) = \dim L_{k-1} + \dim E(\kappa_k) = \dim L_k$: L_k is de Rham. Hence $\rho = L_n$ is de Rham, and by (P1) so are $\operatorname{ad}\rho \cong E^n \otimes_E (E^n)^\vee$, $\mathfrak{b}$, $\mathfrak{n}$ and $\mathfrak{t}$.

Step 4: $H^2(G_M, \mathfrak{n}) = 0$ and $H^1_f(\mathfrak{n}) = H^1(G_M, \mathfrak{n})$. By Step 2, $\mathfrak{n}$ has a filtration with graded pieces $E(\kappa_a\kappa_b^{-1})$, $a < b$; by (i) and (ii), $H^2(G_M, \mathfrak{n}) = 0$; its weights are $m_a - m_b \ge 1$ (P2, P3); and (iii) applies.

Step 5: $H^1_f(\mathfrak{b}) \to H^1_f(\mathfrak{t})$ is surjective. Put $\sigma = \dim \mathfrak{n}$ and $\tau = \dim \mathfrak{t} = h^0(\mathfrak{t})$. The cohomology

sequence of $0 \to \mathfrak{n} \to \mathfrak{b} \to \mathfrak{t} \to 0$ is exact:

$$0 \to H^0(\mathfrak{n}) \to H^0(\mathfrak{b}) \to H^0(\mathfrak{t}) \to H^1(\mathfrak{n}) \to H^1(\mathfrak{b}) \to H^1(\mathfrak{t}) \to H^2(\mathfrak{n}) = 0.$$

By Theorem B.36 and Step 4, $h^1(\mathfrak{n}) = h^0(\mathfrak{n}) + d\sigma$, so the image J of $H^1(\mathfrak{n})$ in $H^1(\mathfrak{b})$ has dimension $h^1(\mathfrak{n}) - (\tau - h^0(\mathfrak{b}) + h^0(\mathfrak{n})) = d\sigma - \tau + h^0(\mathfrak{b})$, and $J \subseteq H^1_f(\mathfrak{b})$ by Step 4 and functoriality. The weights of $\mathfrak{b}$ are those of $\mathfrak{n}$ (all ≥ 1) and those of $\mathfrak{t}$ (all 0), so $\dim_M \big(D_{\mathrm{dR}}(\mathfrak{b})/\mathrm{Fil}^0\big) = (\sigma+\tau) - \tau = \sigma$ by (P2), and $\dim H^1_f(\mathfrak{b}) = h^0(\mathfrak{b}) + d\sigma$ by Theorem B.56. The map $H^1_f(\mathfrak{b}) \to H^1(\mathfrak{t})$ has kernel $H^1_f(\mathfrak{b}) \cap J = J$ and lands in $H^1_f(\mathfrak{t})$, so its image has dimension $h^0(\mathfrak{b}) + d\sigma - (d\sigma - \tau + h^0(\mathfrak{b})) = \tau$. On the other hand $\mathfrak{t} \cong \mathbb{Q}_\ell^\tau$ is trivial, so by Theorem B.44, applied to each coordinate, $H^1_f(\mathfrak{t})$ is the space of homomorphisms $G_M \to \mathfrak{t}$ trivial on I_M, of dimension τ (G_M/I_M is topologically generated by a Frobenius element). Hence the map is onto $H^1_f(\mathfrak{t})$.

Step 6: conclusion. The image of the class of u in $H^1(G_M, \mathfrak{t}) = \mathrm{Hom}_{\mathrm{cont}}(G_M, \mathfrak{t})$ is $g \mapsto$ (diagonal part of $u(g)$), which vanishes on I_M and so lies in $H^1_f(\mathfrak{t})$. By Step 5 there is $x \in H^1_f(\mathfrak{b})$ with the same image; then $[u] - x$ lies in the kernel of $H^1(\mathfrak{b}) \to H^1(\mathfrak{t})$, which is $J \subseteq H^1_f(\mathfrak{b})$. So $[u] \in H^1_f(\mathfrak{b})$, and by functoriality along $\mathfrak{b} \subseteq \mathrm{ad}\,\rho$, $[u] \in H^1_f(\mathrm{ad}\,\rho)$, which is the assertion. □

Proposition 44.15. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum attached to some $(\psi, \mathfrak{h}, t, \chi)$, with odd prime ℓ, weight e and level M, and let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism. Every irreducible point of $\mathfrak{D}$ that is automorphic for ι is admissible symplectic of weight e and level M for ι.*

Proof. Let r be an irreducible point of $\mathfrak{D}$, automorphic for ι. Reduction modulo $\mathfrak{m}_{\mathcal{O}}$ is written with a bar; write $\chi = \chi'\chi''$ with χ' of order prime to ℓ and χ'' of ℓ-power order, so $\bar{\chi}'' = \mathbf{1}$. Since $\mathfrak{D}$ is attached, χ is a character modulo N and ξ one modulo $2^{n'}p'$, and ω_ℓ is one modulo ℓ; all three are characters of level M. Recall $\theta = \alpha\Psi$, $\alpha = \chi_{\mathrm{cyc}}\omega_\ell^{-1}\xi$, $\Psi = \chi\chi_{\mathrm{cyc}}^e$, and $\bar{\omega}_\ell = \bar{\chi}_{\mathrm{cyc}}$.

(Sy1), (Sy2), (Sy4) and continuity. The point r is continuous by (Sq5). If $W \subseteq \overline{\mathbb{Q}}_\ell^4$ is stable under $r(G_{\mathbb{Q}})$, it is stable under its span $\mathrm{M}_4(\overline{\mathbb{Q}}_\ell)$, so $W = 0$ or $W = \overline{\mathbb{Q}}_\ell^4$: (Sy1). (Sy2) is the definition of an automorphic point. By (Sq2), ${}^t r(s) J_4 r(s) = \theta(s) J_4$ for all s, and $\theta = \chi_{\mathrm{cyc}}^{e+1}\,\omega_\ell^{-1}\xi\chi$ with $\omega_\ell^{-1}\xi\chi$ of level M; this is (Sy4) with $M' = M$ and $P = J_4$.

(Lw1) at primes $v \notin T$, $v \neq \ell$. Let $I = I_{\bar{v}}$ for the fixed place $\bar{v}$. By (Se2) for the lift underlying r and an application of the homomorphism to $\overline{\mathbb{Q}}_\ell$, $r(g) = 1$ for $g \in I$ with $\bar{r}_0(g) = 1$. Hence $r|_I = R \circ \bar{r}_0|_I$ for a homomorphism R from the finite abelian group $H = \bar{r}_0(I)$ to $\mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$; $R(H)$ consists of commuting matrices of finite order, which are simultaneously diagonalisable, so $r|_I$ is conjugate to $\mathrm{diag}(\varepsilon_1 \circ \bar{r}_0, \dots, \varepsilon_4 \circ \bar{r}_0)$ with characters ε_k of H. The map $\bar{r}_0(g) \mapsto (\bar{\theta}(g), \bar{\Psi}(g), \bar{\alpha}(g))$ embeds H into $(\mathbb{F}_{\mathcal{O}}^\times)^3$; every character of a subgroup of a finite abelian group extends to the group, and the characters of $(\mathbb{F}_{\mathcal{O}}^\times)^3$ with values in $\overline{\mathbb{Q}}_\ell^\times$ are $(a,b,c) \mapsto [a]^i[b]^j[c]^k$, where $[\cdot]\colon \mathbb{F}_{\mathcal{O}}^\times \to \mathcal{O}^\times$ is the Teichmüller lift, a faithful character of a cyclic group. So $\varepsilon_k \circ \bar{r}_0 = [\bar{\theta}]^i[\bar{\Psi}]^j[\bar{\alpha}]^k$ on I. Now $\bar{\alpha} = \bar{\xi}$ and $\bar{\Psi} = \overline{\chi'\omega_\ell^e}$, and a character of order prime to ℓ with values in $\mathcal{O}^\times$ is the Teichmüller lift of its reduction (reduction is injective on the roots of unity of order prime to ℓ in $\mathcal{O}$); ξ has order prime to ℓ, so $[\bar{\alpha}] = \xi$, $[\bar{\Psi}] = \chi'\omega_\ell^e$ and $[\bar{\theta}] = \xi\chi'\omega_\ell^e$, characters of level M (χ' is a power of χ). Hence (Lw1) holds at v.

(Lw1) at $v \in T$, $v \neq \ell$. By (Zt1), $r(g) = 1$ for $g \in I_v$ with $\chi(g) = 1$, so $r|_{I_v} = R \circ \chi|_{I_v}$ for a homomorphism R from the finite cyclic group $\chi(I_v) \subset \overline{\mathbb{Q}}_\ell^\times$; as before R is diagonalisable with characters of $\chi(I_v)$, which are powers of the inclusion. So $r|_{I_v}$ is conjugate to $\mathrm{diag}(\chi^{a_1}, \dots, \chi^{a_4})|_{I_v}$, with χ^{a_k} of level M.

(Lw2) at ℓ, and genericity. Let $\delta_1, \dots, \delta_4$ be the characters of (Zt2): r is triangular on D_ℓ with diagonal (δ_i), and on I_ℓ, with $m = (e+1, e, 1, 0)$,

$$\delta_i = \kappa_i = \chi_{\mathrm{cyc}}^{m_i}\chi_i, \qquad (\chi_1, \chi_2, \chi_3, \chi_4) = (\omega_\ell^{-1}\xi\chi,\ \chi,\ \omega_\ell^{-1}\xi,\ \mathbf{1}),$$

characters of level M; and $\delta_i \neq \delta_j\chi_{\mathrm{cyc}}|_{D_\ell}$ for all i,j by (Zt3). Hence (Lw2) holds at ℓ (with its first alternative), and r has level M and weights $(e+1,e,1,0)$: (Sy3).

(Sy5). First, $\ell \mid M$. A place of $\overline{\mathbb{Q}}$ above ℓ is $h\bar{\ell}$ with $h \in G_{\mathbb{Q}}$, with $D_{h\bar{\ell}} = hD_\ell h^{-1}$. If $u\,r(s)\,u^{-1}$ is upper triangular with diagonal $(\delta_i(s))$ for $s \in D_\ell$, then for $s' = hsh^{-1}$ the matrix $u\,r(h)^{-1}\,r(s')\,r(h)\,u^{-1} = u\,r(s)\,u^{-1}$ is upper triangular with diagonal $\delta'_i(s') = \delta_i(h^{-1}s'h)$. If $\delta'_i = \delta'_j\chi_{\mathrm{cyc}}$ on $D_{h\bar{\ell}}$, then, since $\chi_{\mathrm{cyc}}(h^{-1}s'h) = \chi_{\mathrm{cyc}}(s')$, $\delta_i = \delta_j\chi_{\mathrm{cyc}}$ on D_ℓ, which was excluded. So r is triangular on $D_{h\bar{\ell}}$ with a diagonal (δ'_i) satisfying $\delta'_i \neq \delta'_j\chi_{\mathrm{cyc}}|_{D_{h\bar{\ell}}}$ for all i,j: (Sy5).

So r is admissible symplectic of weight e and level M for ι. □

Lemma 44.16. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum and r, r' points of $\mathfrak{D}$ with r irreducible. If $r' \cong r$, that is $r'(g) = u\,r(g)\,u^{-1}$ for all $g \in G_{\mathbb{Q}}$ and some $u \in \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$, then there are $t_1, t_2 \in \overline{\mathbb{Q}}_\ell^\times$ with $r'(g) = \mathrm{T}_4(t_1,t_2)\,r(g)\,\mathrm{T}_4(t_1,t_2)^{-1}$ for all $g \in G_{\mathbb{Q}}$.*

Proof. Let $u \in \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ with $u\,r(g) = r'(g)\,u$ for all $g \in G_{\mathbb{Q}}$. Write $E = \{(i,j) : \lambda_i \neq \lambda_j\}$; a matrix is block diagonal when its entries in E vanish. For $y \in \mathcal{O}$ write $\bar{y}$ for its residue in $\mathbb{F}_{\mathcal{O}}$; an element $y \in \mathcal{O}$ with $\bar{y} \neq 0$ is a unit, so $|y|_\ell = 1$, and $y + z \neq 0$ for every $z \in \mathfrak{m}_\ell$.

Step 1: the diagonal entries $d_i = \kappa_i(\phi) \in \mathcal{O}^\times$ of $r_0(\phi)$ satisfy $|d_i - d_j|_\ell = 1$ for $(i,j) \in E$. Let $q = \#\mathbb{F}_{\mathcal{O}}$. For a character $\kappa\colon G_{\mathbb{Q}} \to \mathcal{O}^\times$, $\bar{\kappa}$ takes values in $\mathbb{F}_{\mathcal{O}}^\times$, of order $q-1$, and is locally constant. Since q is a power of ℓ, ℓ is prime to $q-1$; its multiplicative order δ modulo $q-1$ satisfies $\delta \leq q-1$, so for $n \geq q-1$ we have $\delta \mid n!$, hence $\ell^{n!} \equiv 1 \pmod{q-1}$ and $\bar{\kappa}(\sigma_\ell^{\ell^{n!}}) = \bar{\kappa}(\sigma_\ell)$; by (Sg3) and local constancy, $\bar{\kappa}(\phi) = \bar{\kappa}(\sigma_\ell)$. By (Sg2), $\chi_{\mathrm{cyc}}(\sigma_\ell) = 1$, so $\bar{\omega}_\ell(\sigma_\ell) = \bar{\chi}_{\mathrm{cyc}}(\sigma_\ell) = 1$ and $\bar{\alpha}(\sigma_\ell) = \bar{\xi}(\sigma_\ell)$. The pairs in E are, up to order, $(1,2),(1,4),(3,2),(3,4)$, and

$$\frac{\bar{\kappa}_1}{\bar{\kappa}_2}(\sigma_\ell) = \bar{\alpha}(\sigma_\ell) = \bar{\xi}(\sigma_\ell), \quad \frac{\bar{\kappa}_1}{\bar{\kappa}_4}(\sigma_\ell) = \overline{\xi\Psi}(\sigma_\ell), \quad \frac{\bar{\kappa}_3}{\bar{\kappa}_2}(\sigma_\ell) = \overline{\xi\Psi^{-1}}(\sigma_\ell), \quad \frac{\bar{\kappa}_3}{\bar{\kappa}_4}(\sigma_\ell) = \bar{\xi}(\sigma_\ell),$$

using $\theta = \alpha\Psi$. Since $\ell \in T$, (Sg2) with $i = 0, 1, -1$ says that none of these is 1. Hence $\bar{d}_i \neq \bar{d}_j$ for $(i,j) \in E$, and $d_i - d_j$ is a unit of $\mathcal{O}$.

Step 2: u is block diagonal. Put $X = r(\phi)$, $X' = r'(\phi)$, $D = r_0(\phi) = \mathrm{diag}(d_1,\dots,d_4)$, $Y = X - D$, $Y' = X' - D$. By (Sq3) X and X' are block diagonal, and by (Sq4) every entry of Y and Y' lies in $\mathfrak{m}_\ell$. Suppose some entry of u in E is non-zero, and choose $(i,j) \in E$ with $m = |u_{ij}|_\ell$ maximal among the entries of u in E; so $m > 0$. From $uX = X'u$, $X = D + Y$, $X' = D + Y'$ and the (i,j) entry,

$$(d_i - d_j)\,u_{ij} = (uY)_{ij} - (Y'u)_{ij}.$$

In $(uY)_{ij} = \sum_k u_{ik}Y_{kj}$: if $\lambda_k \neq \lambda_j$, then $k \neq j$ and $Y_{kj} = X_{kj} = 0$; if $\lambda_k = \lambda_j$, then $\lambda_i \neq \lambda_k$, so $|u_{ik}|_\ell \leq m$ and $|u_{ik}Y_{kj}|_\ell < m$. In $(Y'u)_{ij} = \sum_k Y'_{ik}u_{kj}$: if $\lambda_i \neq \lambda_k$, then $Y'_{ik} = X'_{ik} = 0$; if $\lambda_i = \lambda_k$, then $(k,j) \in E$, so $|Y'_{ik}u_{kj}|_\ell < m$. By the ultrametric inequality the right-hand side has absolute value $< m$, while the left-hand side has absolute value $|d_i - d_j|_\ell\, m = m$ by Step 1. This contradiction shows that u is block diagonal: $u_{ij} = 0$ for $(i,j) \in E$.

Step 3: $u_{13} = u_{31} = 0$. Put $H = h(r)$, $H' = h(r')$. From $u\,r(g_0) = r'(g_0)\,u$ and $u\,r(g_0^{-1}) = r'(g_0^{-1})\,u$ we get $uH = H'u$. By (Sq4), $r(g)_{ii} - \kappa_i(g)$ and $r'(g)_{ii} - \kappa_i(g)$ lie in $\mathfrak{m}_\ell$ for $g = g_0^{\pm 1}$; as $\theta(g_0) \in \mathcal{O}$ and $\theta = \alpha\Psi$, the entries H_{11}, H'_{11} differ from $\theta(g_0) + 1$, and H_{33}, H'_{33} differ from $\alpha(g_0) + \Psi(g_0)$, by elements of $\mathfrak{m}_\ell$. Hence $H_{33} - H'_{11}$ and $H_{11} - H'_{33}$ differ from $\mp y_0$, where

$$y_0 = \theta(g_0) + 1 - \alpha(g_0) - \Psi(g_0) = \big(\alpha(g_0) - 1\big)\big(\Psi(g_0) - 1\big) \in \mathcal{O},$$

by elements of $\mathfrak{m}_\ell$. By (Sg4) $\bar{y}_0 \neq 0$, so $H_{33} \neq H'_{11}$ and $H_{11} \neq H'_{33}$. By Step 2, $u_{12} = u_{14} = u_{32} = u_{34} = 0$ and $u_{21} = u_{23} = u_{41} = u_{43} = 0$; by (Sq3), $H_{13} = H_{31} = H'_{13} = H'_{31} = 0$. The entries $(1,3)$ and $(3,1)$ of $uH = H'u$ are therefore

$$u_{13}H_{33} = H'_{11}u_{13}, \qquad u_{31}H_{11} = H'_{33}u_{31},$$

so $u_{13} = u_{31} = 0$.

Step 4: u has a similitude. For $Z \in \mathrm{M}_4(\overline{\mathbb{Q}}_\ell)$ put $Z^* = J_4^{-1}\,{}^t\!Z\,J_4$; then $(ZZ')^* = Z'^*Z^*$, and Z has similitude ν if and only if $Z^*Z = \nu \cdot 1$. By (Sq2), $r(g)^*r(g) = \theta(g) \cdot 1$ with $\theta(g) \neq 0$, so $r(g)^* = \theta(g)\,r(g)^{-1}$, and likewise $r'(g)^* = \theta(g)\,r'(g)^{-1}$. Applying * to $u\,r(g) = r'(g)\,u$ gives $r(g)^*u^* = u^*r'(g)^*$, that is $r(g)^{-1}u^* = u^*r'(g)^{-1}$, so $u^*r'(g) = r(g)\,u^*$. For $C = u^*u$ this gives

$$C\,r(g) = u^*r'(g)\,u = r(g)\,u^*u = r(g)\,C \qquad (g \in G_{\mathbb{Q}}).$$

As r is irreducible, the span of $r(G_{\mathbb{Q}})$ is $\mathrm{M}_4(\overline{\mathbb{Q}}_\ell)$, so C is central in $\mathrm{M}_4(\overline{\mathbb{Q}}_\ell)$: $C = a \cdot 1$ with $a \in \overline{\mathbb{Q}}_\ell$, and $a \neq 0$ because $\det u^* = \det u \neq 0$. Multiplying by J_4 on the left, ${}^t\!u\,J_4\,u = a\,J_4$.

Step 5: u is a scalar multiple of $\mathrm{T}_4(t_1, t_2)$. Since $(J_4u)_{kj}$ is $u_{4j}, u_{3j}, -u_{2j}, -u_{1j}$ for $k = 1, 2, 3, 4$,

$$({}^t\!u\,J_4\,u)_{ij} = u_{1i}u_{4j} + u_{2i}u_{3j} - u_{3i}u_{2j} - u_{4i}u_{1j}.$$

By Steps 2 and 3 the only entries of u that may be non-zero are $u_{11}, u_{33}, u_{22}, u_{24}, u_{42}, u_{44}$. Rows 1 and 3 of u are $(u_{11}, 0, 0, 0)$ and $(0, 0, u_{33}, 0)$, so $u_{11}u_{33} \neq 0$ as u is invertible. The entries $(1, 2)$ and $(3, 4)$ of ${}^t\!u\,J_4\,u = aJ_4$ give $u_{11}u_{42} = 0$ and $-u_{33}u_{24} = 0$, hence $u_{42} = u_{24} = 0$ and $u = \mathrm{diag}(u_{11}, u_{22}, u_{33}, u_{44})$ with all diagonal entries non-zero. The entries $(1, 4)$ and $(2, 3)$ give $u_{11}u_{44} = a = u_{22}u_{33}$. Put $t_1 = u_{33}/u_{44}$ and $t_2 = u_{22}/u_{44}$; then $t_1t_2 = u_{22}u_{33}/u_{44}^2 = u_{11}/u_{44}$, so $u = u_{44}\,\mathrm{T}_4(t_1, t_2)$. Since the scalar u_{44} cancels in conjugation, $r'(g) = u\,r(g)\,u^{-1} = \mathrm{T}_4(t_1, t_2)\,r(g)\,\mathrm{T}_4(t_1, t_2)^{-1}$ for all $g \in G_{\mathbb{Q}}$. □

45 The graded model of the symplectic Eisenstein deformation problem

This section proves Proposition 45.2, the graded model of the symplectic Eisenstein deformation problem, which is the algebraic input of Proposition 44.3. Its clauses (Gs1)–(Gs6) are those of the unitary model: the conditions of the criterion, the embedding of the Tate–Shafarevich group into derivations of degree -1, trace elements a_g whose values at points are trace differences, and the correspondence between points. The construction follows Section 43 with two changes. The torus that acts has rank two, so the model is graded by $\mathbb{Z} \times \mathbb{Z}$, and the criterion is applied to its part of λ-weight zero, which is graded by the remaining weight (Definition 45.1, Lemma 45.29). And the deformation problem is framed at all primes of the set T, not only at ℓ, so the model is assembled from local charts at each of them (Definition 45.3).

Subsection 45.1 constructs the local charts; Subsection 45.2 the graded relative presentation with its Euler-characteristic count and the assembly of the model; Subsection 45.3 checks the conditions of the criterion and the Tate–Shafarevich classes on the part of weight zero; and Subsection 45.4 analyses the reducible points, which gives the separation condition (Gc3) and, at the two-signed points, irreducibility and one-dimensional tangent spaces (Gc4).

45.1 Symplectic weighted models and symplectic local bases

This subsection defines symplectic weighted models (Definition 45.1) and symplectic local bases (Definition 45.3), states the existence of a model (Proposition 45.4), and constructs the local charts from which it is built. At ℓ the ordinary symplectic lifting ring has a weighted chart at the Eisenstein point (Proposition 45.6). As in Subsection 43.2, the tangent vectors are crystalline (Proposition 45.7, Lemma 45.8) and are counted by weight (Lemma 45.14), and an explicit fourteen-parameter family of ordinary symplectic lifts gives the dimension from below (Lemma 45.10, Proposition 45.9). At the other primes of T there is a weighted chart of the local

lifting ring (Proposition 45.16), and the completed tensor product of the local rings is smooth with the summed tangent dimensions (Proposition 45.17). The lemmas on finite subalgebras and power series rings that turn families into charts are routine.

Definition 45.1. Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, and let $\mathcal{W}_4 = \{\mathrm{wt}_4(i,j) : 1 \le i,j \le 4\} = \{(0,0), \pm(1,0), \pm(-1,1), \pm(1,1), \pm(0,1)\} \subset \mathbb{Z}^2$.

$\mathbb{Z}^2$-weighted quotients. For a finite set σ and $\mathrm{wt}\colon \sigma \to \mathbb{Z}^2$, weights of monomials, weight components $f_{(w)}$ and the sets $\mathcal{O}[[X_\sigma]]_w$ ($w \in \mathbb{Z}^2$) are formed as in Definition 43.6 with $\mathbb{Z}^2$ in place of $\mathbb{Z}$. An ideal $\mathfrak{a}$ of $\mathcal{O}[[X_s : s \in \sigma]]$ is **homogeneous** if $f_{(w)} \in \mathfrak{a}$ for every $f \in \mathfrak{a}$ and every w. If $\mathfrak{a}$ is homogeneous and every element of $\mathfrak{a}$ has constant term 0, put $R = \mathcal{O}[[X_s : s \in \sigma]]/\mathfrak{a}$, let R_w be the image of $\mathcal{O}[[X_\sigma]]_w$, let $x_R\colon R \to \mathcal{O}$ be the homomorphism induced by the constant term (its **origin**), and let $R^{\mathrm{gr}} = \sum_{w \in \mathbb{Z}^2} R_w \subseteq R$ be the $\mathcal{O}$-subalgebra of finite sums of elements of single weights. For $t\colon \mathbb{Z}^2 \to \mathbb{N}$ of finite support, R is **smooth at the origin with tangent dimensions** t if (Sm1)–(Sm4) hold for $(R^{\mathrm{gr}}, x_R|_{R^{\mathrm{gr}}})$, read with $\mathbb{Z}^2$-valued weights, R_w in place of A_j, and Sq formed in R^{gr} with $\mathfrak{m}_0 = \ker x_R \cap R^{\mathrm{gr}}$, for some chart $\phi\colon R^{\mathrm{gr}} \to \mathcal{O}[\ell^{-1}][[Y_i : i \in J]]$ and weights $\mathrm{wt}_J\colon J \to \mathbb{Z}^2$ with $\#\mathrm{wt}_J^{-1}(w) = t(w)$ for all w. For integers $r, r', m \ge 0$ let $t_{r,r',m}$ take the value 1 at $(1,0)$, $(-1,1)$ and $(1,1)$, r' at $(0,1)$, r at $(0,-1)$, m at $(0,0)$, and 0 elsewhere.

Normalised Sha cocycles. For $a \in \mathcal{O} \setminus \{0\}$, a **normalised Sha cocycle modulo** a for $\mathfrak{D}$ is a locally constant map $z\colon G_{\mathbb{Q}} \to \mathcal{O}/a\mathcal{O}$ such that $z(gh) = z(g) + \Psi^{-1}(g)z(h)$ for all g, h; $z(g) = 0$ for every $g \in I_{\bar{w}}$ with $\bar{r}_0(g) = 1$ and every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above a prime not in T; $z(g_0) = 0$; and for every $v \in T$ there is $y \in \mathcal{O}/a\mathcal{O}$ with $z(g) = \Psi^{-1}(g)y - y$ for all $g \in D_v$.

Symplectic weighted models. A **symplectic weighted model** of $\mathfrak{D}$ is a tuple $M = (\sigma, \mathrm{wt}, \mathfrak{a}, r, r', m, F, E, Z, S, q, r_S)$ of a finite set σ, a map $\mathrm{wt}\colon \sigma \to \mathcal{W}_4$, a homogeneous ideal $\mathfrak{a}$ of $\mathcal{O}[[X_s : s \in \sigma]]$ all of whose elements have constant term 0, with $R = \mathcal{O}[[X_s : s \in \sigma]]/\mathfrak{a}$, R_w and x_R as above, integers $r, r', m \ge 0$, families $F = (F_1, \dots, F_{r+1})$, $E = (E_1, \dots, E_{r'})$, $Z = (Z_1, \dots, Z_m)$ of elements of R, a ring $S \in \mathrm{CNL}_{\mathcal{O}}$, an $\mathcal{O}$-algebra homomorphism $q\colon R \to S$ and a lift $r_S\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(S)$, such that

(Wm1) R is a domain and is smooth at the origin with tangent dimensions $t_{r,r',m}$;

(Wm2) $F_i \in R_{(0,-1)}$, $E_i \in R_{(0,1)}$, $Z_i \in R_{(0,0)}$ and $x_R(Z_i) = 0$ for all i;

(Wm3) q is surjective and $\ker q$ is the ideal generated by the F_i, E_i and Z_i;

(Wm4) r_S is a symplectic Eisenstein lift to S, and it generates S: if $f, f'\colon S \to S'$ are $\mathcal{O}$-algebra homomorphisms to some $S' \in \mathrm{CNL}_{\mathcal{O}}$ with $f \circ r_S = f' \circ r_S$, then $f = f'$;

(Wm5) for all $g \in G_{\mathbb{Q}}$ and $1 \le i,j \le 4$ there is $f \in R_{\mathrm{wt}_4(i,j)}$ with $q(f) = r_S(g)_{ij}$, and $x_R(f) = r_0(g)_{ij}$ for every $f \in R$ with $q(f) = r_S(g)_{ij}$;

(Wm6) for every $a \in \mathcal{O} \setminus \{0\}$ and every normalised Sha cocycle z modulo a there is an $\mathcal{O}$-linear map $\delta\colon R \to \mathcal{O}$ with $\delta(fh) = x_R(f)\delta(h) + x_R(h)\delta(f)$ for all $f, h \in R$, $\delta(R_w) = 0$ for every $w \ne (0,-1)$, $\delta(\ker q) \subseteq a\mathcal{O}$, and, for all $g \in G_{\mathbb{Q}}$ and $f \in R$, $\delta(f) \equiv \theta(g)z(g)$ modulo a if $q(f) = r_S(g)_{31}$, and $\delta(f) \equiv \Psi(g)z(g)$ modulo a if $q(f) = r_S(g)_{42}$.

The graded algebra of a model. For a symplectic weighted model M put

$$A_M = \sum_{n \in \mathbb{Z}} R_{(0,n)} \subseteq R, \qquad (A_M)_n = R_{(0,n)}, \qquad x_M = x_R|_{A_M},$$

the finite sums of elements of λ-weight 0, and let u_M be the system of equations with parts $u_{-1} = F$, $u_1 = E$, $u_0 = Z$ and u_{-2}, u_2 empty, so with counts $(0, r+1, m, r', 0)$ in degrees $-2, \dots, 2$.

Proposition 45.2. *Let $\mathfrak{D} = (\ell, \mathcal{O}, e, N, \chi, \xi, S, \dots)$ be a symplectic Eisenstein datum and $n_S = \prod_{p\in S} p$. There are a graded $\mathcal{O}$-algebra with a fixed point (A, φ_0), a system of equations u for it and elements $a_g \in A_0$ with $\varphi_0(a_g) = 0$ ($g \in G_{\mathbb{Q}}$) such that:*

(Gs1) *(A, φ_0, u) satisfies (Gc1);*

(Gs2) *for every $b \in \mathcal{O} \setminus \{0\}$, writing $\mathrm{Sha}[b]$ for the submodule of $\mathrm{Sha}^1_{n_S}(\chi, e)$ killed by b, there are $a \in b\mathcal{O} \setminus \{0\}$ and an injective additive map $D\colon \mathrm{Sha}[b] \to \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}/a\mathcal{O})$ such that $D(\mu)(u_{-1,i}) = 0$ for all μ and i, and every $D(\mu)$ is the reduction modulo a of an $\mathcal{O}$-linear map $D'\colon A \to \mathcal{O}$ with $D'(fg) = \varphi_0(f)D'(g) + \varphi_0(g)D'(f)$ for all $f, g \in A$ and $D'(A_j) = 0$ for all $j \neq -1$;*

and, if $\mathfrak{D}$ is finite in weight zero, there is a map $x \mapsto r_x$ from the points of A on u to the points of $\mathfrak{D}$ such that:

(Gs3) *$x(a_g) = \mathrm{tr}\, r_x(g) - \mathrm{tr}\, r_0(g)$ for every point x of A on u and every $g \in G_{\mathbb{Q}}$;*

(Gs4) *for points x, x' of A on u, if $r_{x'}$ is a torus conjugate of r_x, then x' is a torus translate of x;*

(Gs5) *(A, φ_0, u) satisfies (Gc2) and (Gc3);*

(Gs6) *if $\mathfrak{D}$ has no endoscopic points, then r_x is irreducible for every two-signed point x of A on u; if moreover every irreducible point of $\mathfrak{D}$ is rigid, then (A, φ_0, u) satisfies (Gc4).*

Proof. By Proposition 45.4, $\mathfrak{D}$ has a symplectic weighted model M, with integers r, r', m. Put $(A, \varphi_0) = (A_M, x_M)$ and $u = u_M$, and let $(a_g)_{g\in G_{\mathbb{Q}}}$ be elements as in Proposition 45.28 for M; so $a_g \in A_0$ and $\varphi_0(a_g) = 0$.

u is a system of equations. By (Wm2), $F_i \in R_{(0,-1)} = A_{-1}$, $E_i \in R_{(0,1)} = A_1$, $Z_i \in R_{(0,0)} = A_0$ and $\varphi_0(Z_i) = x_R(Z_i) = 0$; so u is a system of equations for (A, φ_0), with counts $\mathrm{ne}(-2), \dots, \mathrm{ne}(2)$ equal to $0, r+1, m, r', 0$ (by (Mc1), (A, φ_0) is a graded $\mathcal{O}$-algebra with a fixed point).

(Gs1). By (Mc1), (A, φ_0) is smooth with tangent dimensions td, $\mathrm{td}(j) = 0$ for $|j| > 2$ and $\mathrm{td}(-2), \dots, \mathrm{td}(2)$ equal to $0, r, m, r'+1, 1$. Hence

$$\mathrm{ne}(-1) = r+1 = \mathrm{td}(-1)+1, \quad \mathrm{ne}(-2) = 0 = \mathrm{td}(-2), \quad \mathrm{ne}(0) = m = \mathrm{td}(0),$$
$$\mathrm{ne}(1) = r' = \mathrm{td}(1) - 1, \quad \mathrm{ne}(2) = 0 = \mathrm{td}(2) - 1,$$

and by (Mc2) u_{-1}, u_{-2}, u_0 span rationally in degrees -1, -2, 0. This is (Gc1).

(Gs2) is (Mc3) of Proposition 45.28 for M.

(Gs3)–(Gs6). Let $\mathfrak{D}$ be finite in weight zero, and let $x \mapsto r_x$ be the map of Proposition 45.28. Then (Gs3) and (Gs4) are the two assertions of (Mc4), and (Gs5) and (Gs6) are the assertions of (Mc5). □

Definition 45.3. Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$ and set $T \ni \ell$. Let $d_\ell, d_p\colon \mathbb{Z}^2 \to \mathbb{N}$ be: $d_\ell(w) = 2$ for $w \in \{(1,0), (-1,1), (0,1), (1,1), (0,0)\}$, $d_\ell(w) = 1$ for $w \in \{(-1,0), (1,-1), (0,-1), (-1,-1)\}$; $d_p(0,0) = 2$, $d_p(w) = 1$ for $w \in \mathcal{W}_4 \setminus \{(0,0)\}$; both 0 elsewhere.

Symplectic local bases. A **symplectic local base** of $\mathfrak{D}$ is a tuple $L = (\sigma_L, \mathrm{wt}_L, \mathfrak{a}_L, (\rho_v)_{v\in T})$ of a finite set σ_L, a map $\mathrm{wt}_L\colon \sigma_L \to \mathcal{W}_4$ and a homogeneous ideal $\mathfrak{a}_L$ of $\mathcal{O}[[X_s : s \in \sigma_L]]$ all of whose elements have constant term 0, with $B_L = \mathcal{O}[[X_s : s \in \sigma_L]]/\mathfrak{a}_L$, its parts $(B_L)_w$, projection π_L and origin x_L, and for each $v \in T$ a lift $\rho_v\colon D_v \to \mathrm{GL}_4(B_L)$ of $\bar{r}_0|_{D_v}$ satisfying the local condition at v, such that

(Bq1) for every $A \in \mathrm{CNL}_{\mathcal{O}}$ and every family $(\rho'_v)_{v\in T}$ of lifts of the $\bar r_0|_{D_v}$ to A satisfying the local conditions there is exactly one $\mathcal{O}$-algebra homomorphism $f\colon B_L \to A$ with $f\circ\rho_v = \rho'_v$ for all $v \in T$;

(Bq2) $\rho_v(g)_{ij} \in (B_L)_{\mathrm{wt}_4(i,j)}$ and $x_L(\rho_v(g)_{ij}) = r_0(g)_{ij}$ for all $v \in T$, $g \in D_v$ and i, j;

(Bq3) B_L is smooth at the origin with tangent dimensions $d_\ell + \#(T \setminus \{\ell\})\, d_p$.

Framed lifts. For a symplectic local base L, a **framed lift** over L is a tuple $(A, \iota, r, (\alpha_v)_{v\in T})$ of $A \in \mathrm{CNL}_{\mathcal{O}}$, an $\mathcal{O}$-algebra homomorphism $\iota\colon B_L \to A$, a lift r of $\bar r_0$ to A and matrices $\alpha_v \in \mathrm{M}_4(A)$ such that

(Fw1) $r(g)$ has similitude $\theta(g)$ for every $g \in G_{\mathbb{Q}}$, and r satisfies (Se2);

(Fw2) every α_v has similitude 1 and is congruent to 1 modulo $\mathfrak{m}_A$, and $\alpha_\ell = 1$;

(Fw3) $r(g)\,\alpha_v = \alpha_v\,\iota(\rho_v(g))$ for all $v \in T$ and $g \in D_v$ (ι applied entrywise).

Framed relative presentations. A **framed relative presentation** over L is a tuple $(U, \iota, r_U, (\alpha_v)_{v\in T}, \tau, \mathrm{wt}_\tau, (y_t)_{t\in\tau}, \kappa, (f_k)_{k\in\kappa}, \mathrm{wf})$ of a framed lift $(U, \iota, r_U, (\alpha_v))$ over L, finite sets τ, κ, maps $\mathrm{wt}_\tau\colon \tau \to \mathcal{W}_4$, $\mathrm{wf}\colon \kappa \to \mathbb{Z}^2$, elements $y_t \in U$ and power series f_k in $P_U = \mathcal{O}[[X_s : s \in \sigma_L \sqcup \tau]]$ (weights wt_L on σ_L, wt_τ on τ) such that

(Fv1) for every framed lift $(A, \iota', r', (\alpha'_v))$ over L there is exactly one $\mathcal{O}$-algebra homomorphism $y\colon U \to A$ with $y\circ\iota = \iota'$, $y\circ r_U = r'$ and $y(\alpha_v) = \alpha'_v$ for all v;

(Fv2) the continuous $\mathcal{O}$-algebra homomorphism $\Pi_U\colon P_U \to U$ with $\Pi_U(X_s) = \iota(\pi_L(X_s))$ ($s \in \sigma_L$) and $\Pi_U(X_t) = y_t$ ($t \in \tau$) is surjective, $f_k \in \mathcal{O}[[X]]_{\mathrm{wf}(k)}$ has constant term 0, $\ker\Pi_U = \mathfrak{a}_L P_U + (f_k)_{k\in\kappa}$, and for each $t \in \tau$ there are a, b with $\mathrm{wt}_\tau(t) = \mathrm{wt}_4(a, b)$ and either $y_t = r_U(g)_{ab} - r_0(g)_{ab}$ for some $g \in G_{\mathbb{Q}}$ or $y_t = (\alpha_v - 1)_{ab}$ for some $v \in T$;

(Fv3) writing $U_w = \Pi_U(\mathcal{O}[[X]]_w)$ and x_U for the homomorphism induced by the constant term: $r_U(g)_{ab} \in U_{\mathrm{wt}_4(a,b)}$, $(\alpha_v)_{ab} \in U_{\mathrm{wt}_4(a,b)}$, $x_U(r_U(g)_{ab}) = r_0(g)_{ab}$, $x_U(\alpha_v) = 1$ and $x_U\circ\iota = x_L$;

(Fv4) $\#\mathrm{wt}_\tau^{-1}(w) = 0$ for $w \notin \{(0,0), (0,1), (0,-1)\}$, and $\#\mathrm{wf}^{-1}(w) = \#\mathrm{wt}_\tau^{-1}(w) + c(w)$ for all w, where $c(0,0) = 2$, $c(0,\pm1) = 1$ and $c(w) = 0$ otherwise.

Proposition 45.4. *Every symplectic Eisenstein datum has a symplectic weighted model.*

Proof. Let $\mathfrak{D}$ be a symplectic Eisenstein datum. By Proposition 45.5, $\mathfrak{D}$ has a symplectic local base L. By Proposition 45.20, there is a framed relative presentation over L. By Proposition 45.23, applied to $\mathfrak{D}$, L and this presentation, $\mathfrak{D}$ has a symplectic weighted model. □

Proposition 45.5. *Every symplectic Eisenstein datum has a symplectic local base.*

Proof. Step 1: the local rings. For $v \in T$ the local condition at v is the vanishing of finitely many families of polynomial expressions in the entries of a lift (the entries of ${}^t\rho(g)J_4\rho(g) - \theta(g)J_4$; for $v = \ell$ the coefficients of the characteristic polynomial of $\rho(s_0)$ minus those of $\prod_j(X - \kappa_j(s_0))$ and the entries of $\Pi_j\rho(g)\Pi_i$ and $\Pi_i(\rho(s) - \kappa_i(s))\Pi_i$; for $v \neq \ell$ the entries of $\rho(g) - 1$, $g \in I_v$, $\chi(g) = 1$). So the quotient of Mazur's universal lifting ring of $\bar r_0|_{D_v}$ by the closed ideal they generate is a universal pair (R_v, ρ_v) for the lifts satisfying the local condition at v; let x_v classify $r_0|_{D_v}$, which satisfies it. For $t_1, t_2 \in A^\times$ congruent to 1 modulo $\mathfrak{m}_A$, conjugation by $\mathrm{T}_4(t_1, t_2)$ (a

similitude, congruent to 1) carries lifts satisfying the local condition at v to such lifts: it preserves similitudes, it fixes 1, and it conjugates $\rho(s_0)$, each Π_i and each identity of the ordinary condition.

Step 2: the coproduct, (Bq1). Let $B = \widehat{\bigotimes}_{v\in T} R_v$ and regard each ρ_v as a lift to B. For $A \in \mathrm{CNL}_{\mathcal{O}}$ and lifts $(\rho'_v)_v$ satisfying the local conditions, the homomorphisms $R_v \to A$ classifying the ρ'_v induce exactly one $f\colon B \to A$ with $f \circ \rho_v = \rho'_v$ for all v. In particular two homomorphisms $B \to A$ agreeing on the entries of all ρ_v are equal, and $x_B = \bigotimes_v x_v\colon B \to \mathcal{O}$ classifies $(r_0|_{D_v})_v$.

Step 3: the weighted presentation, (Bq2). By Step 2, two morphisms $B \to A$ of $\mathrm{CNL}_{\mathcal{O}}$ that agree on the entries of all ρ_v are equal; so Lemma 45.15, item 1, applied to these entries with $c_e = r_0(g)_{ab}$ for $e = \rho_v(g)_{ab}$, shows that the classes of the elements $\rho_v(g)_{ab} - r_0(g)_{ab}$ ($v \in T$, $g \in D_v$; they lie in $\mathfrak{m}_B$ as r_0 is diagonal) span $\mathfrak{m}_B/(\mathfrak{m}_B^2 + \mathfrak{m}_{\mathcal{O}} B)$. Choose a finite set σ_L of triples $s = (v_s, g_s, (a_s, b_s))$ whose classes span, put $\mathrm{wt}_L(s) = \mathrm{wt}_4(a_s, b_s) \in \mathcal{W}_4$, and let $\pi\colon P = \mathcal{O}[[X_s : s \in \sigma_L]] \to B$, $X_s \mapsto \rho_{v_s}(g_s)_{a_s b_s} - r_0(g_s)_{a_s b_s}$; it is onto by Lemma 45.15, item 2. By Step 1 and (Bq1) for $A = B[[T_1, T_2]]$, there is a morphism $\rho_T\colon B \to B[[T_1, T_2]]$ of $\mathrm{CNL}_{\mathcal{O}}$ with $\rho_T \circ \rho_v = \mathrm{T}_4(1 + T_1, 1 + T_2)\, \rho_v\, \mathrm{T}_4(1 + T_1, 1 + T_2)^{-1}$ for all v, so that $\rho_v(g)_{ab}$ has torus weight $\mathrm{wt}_4(a, b)$ in the sense of Lemma 45.18 ($d = 2$), and so has $\pi(X_s)$, of weight $\mathrm{wt}_L(s)$ (on the diagonal the constant has weight 0). By Lemma 45.18, $\mathfrak{a}_L = \ker \pi$ is homogeneous and $\rho_v(g)_{ab} \in \pi(\mathcal{O}[[X_{\sigma_L}]]_{\mathrm{wt}_4(a,b)}) = (B_L)_{\mathrm{wt}_4(a,b)}$ with $B_L = P/\mathfrak{a}_L \cong B$. As $x_B \circ \pi$ is the constant term (both continuous, equal on the X_s), every element of $\mathfrak{a}_L$ has constant term 0, the origin x_L is x_B, and $x_L(\rho_v(g)_{ab}) = r_0(g)_{ab}$: (Bq2).

Step 4: (Bq3). Proposition 45.6 gives a chart of R_ℓ with (Yc1)–(Yc4) and counts d_ℓ, and Proposition 45.16 one of R_p with counts d_p for each $p \in T \setminus \{\ell\}$. By Proposition 45.17, applied to σ_L and π (onto, with homogeneous kernel and constant terms 0 by Step 3), B_L is smooth at the origin with tangent dimensions $d_\ell + \#(T \setminus \{\ell\})\, d_p$.

Hence $L = (\sigma_L, \mathrm{wt}_L, \mathfrak{a}_L, (\rho_v)_{v\in T})$ is a symplectic local base of $\mathfrak{D}$. □

Proposition 45.6. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, (R_ℓ, ρ_ℓ) universal for the lifts of $\bar{r}_0|_{D_\ell}$ satisfying the local condition at ℓ, and x_ℓ as in Proposition 45.7. There are a finite set τ_ℓ, a map $w\colon \tau_\ell \to \mathbb{Z}^2$ with $\#w^{-1}(v) = d_\ell(v)$ for every $v \in \mathbb{Z}^2$, and an $\mathcal{O}$-algebra homomorphism $\phi_\ell\colon R_\ell \to \mathcal{O}[\ell^{-1}][[Y_i : i \in \tau_\ell]]$, where Y_i has weight $w(i)$, such that, with $\mathfrak{m}_Y$ the maximal ideal of the target:*

(Yc1) *$\phi_\ell(\rho_\ell(g)_{ab})$ has weight $\mathrm{wt}_4(a, b)$ for all $g \in D_\ell$ and $1 \le a, b \le 4$;*

(Yc2) *the constant term of $\phi_\ell(f)$ is $x_\ell(f)$ for every $f \in R_\ell$;*

(Yc3) *for every $i \in \tau_\ell$ there is $f \in \ker x_\ell$ with $\phi_\ell(f) \equiv Y_i$ modulo $\mathfrak{m}_Y^2$;*

(Yc4) *for every $N \ge 1$ and $f \in R_\ell$ with $\phi_\ell(f) \in \mathfrak{m}_Y^N$ there is $c \in \mathcal{O} \setminus \{0\}$ with $cf \in (\ker x_\ell)^N$;*

(Yc5) *the localisation of R_ℓ at the prime ideal $\ker x_\ell$ is a regular local ring of dimension 14; in particular R_ℓ has exactly one minimal prime ideal contained in $\ker x_\ell$.*

Proof. Write $R = R_\ell$, $x = x_\ell$, $\mathfrak{q} = \ker x$, $E = \mathcal{O}[\ell^{-1}]$, $\Lambda = E[[Y_1, \dots, Y_{14}]]$, $L\colon \Lambda \to E^{14}$ for the vector of coefficients of $Y_1, \dots, Y_{14}$, and $x_{g,a,b} = \rho_\ell(g)_{ab} - r_0(g)_{ab}$.

Step 1: generators. For $u \in \mathcal{O}^\times$ and y in the maximal ideal of R, $y - u$ is a unit, so x is local and continuous. By universality, two $\mathcal{O}$-algebra homomorphisms from R to some $A \in \mathrm{CNL}_{\mathcal{O}}$ that agree on the entries of ρ_ℓ are equal. Since r_0 is diagonal with entries in $\mathcal{O}$ and ρ_ℓ lifts $\bar{r}_0|_{D_\ell}$, every $x_{g,a,b}$ lies in the maximal ideal; so Lemma 45.15, item 1, applied to the entries of ρ_ℓ with $c_e = r_0(g)_{ab}$ for $e = \rho_\ell(g)_{ab}$, shows that the classes of the $x_{g,a,b}$ span $\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_{\mathcal{O}} R)$; choose finitely many triples $s = (g_s, a_s, b_s)$, $s \in \sigma$, whose classes span, and put $x_s = x_{g_s,a_s,b_s}$.

By Lemma 45.15, item 2, $\pi\colon \mathcal{O}[[X_s : s \in \sigma]] \to R$, $X_s \mapsto x_s$, is onto; $x \circ \pi$ is the constant term (both continuous, equal on the X_s, as $x(x_s) = 0$). Hence $\mathfrak{q} = (x_s)_{s\in\sigma}$ and $R = \mathcal{O} + \mathfrak{q}$.

Step 2: the local ring at $\mathfrak{q}$. The ring $A = R_{\mathfrak{q}}$ is a Noetherian local E-algebra with residue field E; its cotangent space $T = \mathfrak{q}A/\mathfrak{q}^2A$ is spanned over E by the classes of the x_s (for $a \in \mathfrak{q}$ and $s' \notin \mathfrak{q}$, $a/s' \equiv x(s')^{-1}a$ modulo $\mathfrak{q}^2A$, and $x_s r \equiv x(r)x_s$ modulo $\mathfrak{q}^2$).

Step 3: $\dim_E T \le 14$. For an E-linear $\lambda\colon T \to E$ let $F_\lambda(a) = x(a) + \epsilon\lambda(a - x(a))$, an $\mathcal{O}$-algebra homomorphism $R \to E[\epsilon]$ with reduction x (as $ab - x(a)x(b) = (a - x(a))(b - x(b)) + x(a)(b - x(b)) + x(b)(a - x(a))$). It is continuous: it kills $\mathfrak{q}^2$, maps $\mathfrak{q}$ into ϵL_0 with $L_0 = \sum_s \mathcal{O}\lambda(x_s)$ bounded, and the N-th power of the maximal ideal $\mathfrak{m}_{\mathcal{O}}R+\mathfrak{q}$ into $\mathfrak{m}_{\mathcal{O}}^N+\epsilon\mathfrak{m}_{\mathcal{O}}^{N-1}L_0$; so $g \mapsto F_\lambda(\rho_\ell(g)_{ab})$ is continuous. By Proposition 45.7, the map y_λ with $F_\lambda(\rho_\ell(g)) = (1 + \epsilon y_\lambda(g))r_0(g)$ has a primitive in $\mathrm{M}_4(E) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$. Both $F_\lambda(\rho_\ell(g))$ and $r_0(g)$ have similitude $\theta(g)$, so ${}^t(1+\epsilon y_\lambda)J_4(1+\epsilon y_\lambda) = J_4$ and $y_\lambda(g) \in \mathfrak{sp}_4(E)$. The projection of $\mathrm{M}_4(E)$ to the entries of weight w is equivariant, since $r_0(g) = \mathrm{diag}(\kappa_i(g))$ acts on the entry (a, b) by the scalar $\kappa_a(g)\kappa_b(g)^{-1}$; and it preserves $\mathfrak{sp}_4(E)$, since $X \in \mathfrak{sp}_4(E)$ means that J_4X is symmetric, i.e. $X_{ab} = \pm X_{5-b,5-a}$ for all a, b, and $\mathrm{wt}_4(5 - b, 5 - a) = \mathrm{wt}_4(a, b)$ as $\lambda_{5-i} = 1 - \lambda_i$, $\mu_{5-i} = 1 - \mu_i$. So the weight-w part $y_{\lambda,w}$ of y_λ has the weight-w part of the primitive as a primitive, and $y_{\lambda,w} \in Z_w$. The linear map $\lambda \mapsto (y_{\lambda,w})_w \in \bigoplus_w Z_w$ is injective: if all $y_{\lambda,w}$ vanish, $y_\lambda = 0$, so $F_\lambda(x_s) = 0$, $\lambda(x_s) = 0$ for all s, and $\lambda = 0$ by Step 2. By Lemma 45.14, $\dim_E T \le \sum_w d_\ell(w) = 14$.

Step 4: the completion. By Proposition 45.9, $\dim A \ge 14 \ge \dim_E T$, so A is regular of dimension 14 and $\dim_E T = 14$; a regular local ring is a domain, so A has exactly one minimal prime, and the minimal primes of R contained in $\mathfrak{q}$ correspond to those of A: this is (Yc5). Choosing $y_1, \dots, y_{14} \in \mathfrak{q}A$ with classes a basis of T, Cohen's theorem gives $\widehat{A} \cong \Lambda$, $y_i \mapsto Y_i$, under which $\mathfrak{q}^N\widehat{A}$ corresponds to $\mathfrak{m}_Y^N$; let $\phi_0\colon R \to A \to \widehat{A} \cong \Lambda$. The constant term of $\phi_0(f)$ is the image of f in $A/\mathfrak{q}A = E$, i.e. $x(f)$. If $\phi_0(f) \in \mathfrak{m}_Y^N$, then $f \in \mathfrak{q}^N A$; with $A' = R[\ell^{-1}]$ and $\mathfrak{n} = \mathfrak{q}A'$, a maximal ideal with $A = A'_{\mathfrak{n}}$, the $\mathfrak{n}$-primary ideal $\mathfrak{n}^N$ satisfies $\mathfrak{n}^N A \cap A' = \mathfrak{n}^N$, so $\ell^k f \in \mathfrak{q}^N + \ker(R \to A')$ and $cf \in \mathfrak{q}^N$ for some $c = \ell^{k'}$. The map $L \circ \phi_0$ induces the isomorphism $T \cong \mathfrak{m}_Y/\mathfrak{m}_Y^2 \cong E^{14}$, so the vectors $L(\phi_0(x_s))$ span E^{14} (Step 2).

Step 5: torus endomorphisms. Let $t = 1 + \ell \in \mathcal{O}^\times$, so $t^k \ne 1$ for $k \ne 0$, and $h_1 = \mathrm{T}_4(t, 1)$, $h_2 = \mathrm{T}_4(1, t)$, congruent to 1 modulo $\mathfrak{m}_{\mathcal{O}}$ and of similitudes t, t. For $k = 1, 2$, $h_k\rho_\ell h_k^{-1}$ is a lift of $\bar{r}_0|_{D_\ell}$ satisfying the local condition at ℓ: conjugation by a similitude preserves the similitude of each $\rho_\ell(g)$, and it preserves the characteristic polynomial of $\rho_\ell(s_0)$ and conjugates each Π_i and each identity of the ordinary condition. By universality there is $\vartheta_k\colon R \to R$ with $\vartheta_k \circ \rho_\ell = h_k\rho_\ell h_k^{-1}$, i.e. $\vartheta_k(\rho_\ell(g)_{ab}) = t^{\mathrm{wt}_4(a,b)_k}\rho_\ell(g)_{ab}$, so $\vartheta_k(x_s) = t^{\mathrm{wt}(s)_k}x_s$ with $\mathrm{wt}(s) = \mathrm{wt}_4(a_s, b_s)$; and $x \circ \vartheta_k = x$, both classifying $h_k r_0 h_k^{-1} = r_0|_{D_\ell}$.

Step 6: coordinates among the entries. Choose $s_1, \dots, s_{14}$ with $L(\phi_0(x_{s_i}))$ a basis (Step 4), and put $z_i = x_{s_i}$, $w(i) = \mathrm{wt}(s_i)$, $\tau_\ell = \{1, \dots, 14\}$. The continuous E-algebra endomorphism α of Λ with $\alpha(Y_i) = \phi_0(z_i) \in \mathfrak{m}_Y$ has invertible linear part, hence is an automorphism preserving constant terms and each $\mathfrak{m}_Y^N$; put $\phi_\ell = \alpha^{-1} \circ \phi_0$. Then $\phi_\ell(z_i) = Y_i$, the constant term of $\phi_\ell(f)$ is $x(f)$ (Yc2), (Yc3) holds with $f = z_i$, and (Yc4) holds by Step 4.

Step 7: equivariance, (Yc1). Fix k and let ϱ_k be the continuous E-algebra automorphism of Λ with $\varrho_k(Y_i) = t^{w(i)_k}Y_i$. Then $\phi_\ell \circ \vartheta_k = \varrho_k \circ \phi_\ell$: $\psi_1 = \phi_\ell \circ \vartheta_k$ and $\psi_2 = \varrho_k \circ \phi_\ell$ have constant term x, so map $\mathfrak{q}^N$ into $\mathfrak{m}_Y^N$, and agree on the z_i. Given $a \in R$ and N, let p be the truncation of $\phi_\ell(a)$ in degrees $< N$, $d \in \mathcal{O} \setminus \{0\}$ with $dp \in \mathcal{O}[Y]$ and $b = (dp)(z_1, \dots, z_{14})$; then $\phi_\ell(da - b) \in \mathfrak{m}_Y^N$, so $c(da - b) \in \mathfrak{q}^N$ for some $c \ne 0$ (Yc4), $\psi_1(b) = \psi_2(b)$, and $cd(\psi_1(a) - \psi_2(a)) \in \mathfrak{m}_Y^N$ for all N; so $\psi_1(a) = \psi_2(a)$. If $\vartheta_k(a) = t^j a$, then $\varrho_k(\phi_\ell(a)) = t^j\phi_\ell(a)$, and since ϱ_k multiplies the coefficient of Y^n by $t^{\sum_i n_i w(i)_k}$, every monomial of $\phi_\ell(a)$ has k-th weight coordinate j. Applied to $a = \rho_\ell(g)_{ab}$ for $k = 1, 2$ (Step 5), this is (Yc1).

Step 8: the counts. For $i \in \tau_\ell$ let $F_i(f)$ be the constant term of $\phi_\ell(f)$ plus ϵ times its coefficient

of Y_i: an $\mathcal{O}$-algebra homomorphism $R \to E[\epsilon]$ with reduction x, killing $\mathfrak{q}^2$ and continuous as in Step 3. By Proposition 45.7 and as in Step 3, the map y_i with $F_i(\rho_\ell(g)) = (1+\epsilon y_i(g))r_0(g)$ is $\mathfrak{sp}_4(E)$-valued with a crystalline primitive; its entry (a,b) is the coefficient of Y_i in $\phi_\ell(\rho_\ell(g)_{ab})$ times $\kappa_b(g)^{-1}$, which vanishes unless $\mathrm{wt}_4(a,b) = w(i)$ by (Yc1). So $y_i \in Z_{w(i)}$. For fixed v the y_i with $w(i) = v$ are linearly independent: at $g = g_{s_k}$ the entry (a_{s_k}, b_{s_k}) of y_i is the coefficient of Y_i in Y_k (plus a constant), times a unit. Hence $\#w^{-1}(v) \le \dim Z_v = d_\ell(v)$ by Lemma 45.14; as $\sum_v \#w^{-1}(v) = 14 = \sum_v d_\ell(v)$, equality holds for every v. □

Proposition 45.7. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $E_\mathcal{O} = \mathcal{O}[\ell^{-1}]$, $E_\mathcal{O}[\epsilon] = E_\mathcal{O}[X]/(X^2)$, and $\mathrm{emb}_\ell\colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ an embedding inducing $\bar\ell$. Let $R_\ell \in \mathrm{CNL}_\mathcal{O}$ and ρ_ℓ a lift of $\bar r_0|_{D_\ell}$ to R_ℓ satisfying the local condition at ℓ which is universal: for every $A \in \mathrm{CNL}_\mathcal{O}$ and every lift ρ of $\bar r_0|_{D_\ell}$ to A satisfying the local condition at ℓ there is exactly one $\mathcal{O}$-algebra homomorphism $f\colon R_\ell \to A$ with $f\circ\rho_\ell = \rho$. Let $x_\ell\colon R_\ell \to \mathcal{O}$ be the homomorphism with $x_\ell \circ \rho_\ell = r_0|_{D_\ell}$. Let $f\colon R_\ell \to E_\mathcal{O}[\epsilon]$ be an $\mathcal{O}$-algebra homomorphism whose reduction modulo ϵ is x_ℓ and such that $g \mapsto f(\rho_\ell(g)_{ab})$ is continuous for all a,b, and define $y\colon D_\ell \to \mathrm{M}_4(E_\mathcal{O})$ by $f(\rho_\ell(g)) = (1+\epsilon\, y(g))\, r_0(g)$. Let D_ℓ act on $\mathrm{M}_4(E_\mathcal{O})$ by $g\cdot X = r_0(g) X r_0(g)^{-1}$, on B_cris through emb_ℓ and on $\mathrm{M}_4(E_\mathcal{O}) \otimes_{\mathbb{Q}_\ell} B_\mathrm{cris}$ diagonally. Then there is $Y \in \mathrm{M}_4(E_\mathcal{O}) \otimes_{\mathbb{Q}_\ell} B_\mathrm{cris}$ with $y(g)\otimes 1 = g\cdot Y - Y$ for all $g \in D_\ell$.*

Proof. Write $E = E_\mathcal{O}$, a finite extension of $\mathbb{Q}_\ell$, $B = E[\epsilon]$, $\rho(g) = f(\rho_\ell(g)) \in \mathrm{GL}_4(B)$ and $u_i = \kappa_i(s_0)$. Through emb_ℓ, $D_\ell = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ with inertia I_ℓ. For a continuous character $\gamma\colon D_\ell \to E^\times$ let $E(\gamma)$ be the $\mathbb{Q}_\ell$-representation E with g acting by $\gamma(g)$; a continuous cocycle c of $E(\gamma)$ has a *crystalline primitive* if $c(g)\otimes 1 = \gamma(g)\, g(Y) - Y$ for some $Y \in E \otimes_{\mathbb{Q}_\ell} B_\mathrm{cris}$ and all g, i.e. its class lies in $H^1_f(D_\ell, E(\gamma))$.

Step 1: the characters. By Lemma 45.8, $u_1,\dots,u_4$ are pairwise distinct elements of $\mathcal{O}$. For $a<b$ put $\gamma_{ab} = \kappa_a\kappa_b^{-1}$.

Step 2: for $a<b$, every continuous cocycle of $E(\gamma_{ab})$ has a crystalline primitive, since $H^1(D_\ell, E(\gamma_{ab})) = H^1_f(D_\ell, E(\gamma_{ab}))$ by Lemma 45.8.

Step 3: unramified homomorphisms have crystalline primitives. Let $z\colon D_\ell \to E$ be a continuous homomorphism vanishing on I_ℓ. On $X = E \oplus \mathbb{Q}_\ell e$ let $g(x+te) = x + tz(g) + te$; this is a continuous representation on which I_ℓ acts trivially, hence crystalline. Invariants being left exact, $0 \to D_\mathrm{cris}(E) \to D_\mathrm{cris}(X) \to D_\mathrm{cris}(\mathbb{Q}_\ell e) = \mathbb{Q}_\ell(1\otimes e)$ is exact, and $\dim D_\mathrm{cris}(X) = \dim_{\mathbb{Q}_\ell} E + 1 = \dim D_\mathrm{cris}(E) + 1$; so there is an invariant element $1\otimes e - Y$ with $Y \in B_\mathrm{cris}\otimes E$, and invariance says $z(g)\otimes 1 = g(Y) - Y$.

Step 4: a triangular basis. The identities of the ordinary condition and of the characteristic polynomial are polynomial identities with coefficients in $\mathcal{O}$ in the entries of ρ_ℓ, so they hold for ρ. By Cayley–Hamilton, $S = \rho(s_0)$ satisfies $\prod_i (S - u_i) = 0$. The differences $u_i - u_j$ $(i \ne j)$ are non-zero elements of E, hence units of B; put $c_i = \prod_{j\ne i}(u_i - u_j)$ and $\varepsilon_i = c_i^{-1}\Pi_i$. The Lagrange identity $\sum_i \prod_{j\ne i}(X-u_j)/(u_i-u_j) = 1$ in $E[X]$ gives $\sum_i \varepsilon_i = 1$; for $i \ne j$, $\varepsilon_i\varepsilon_j$ is a multiple of $\prod_k (S-u_k) = 0$; so the ε_i are orthogonal idempotents with sum 1. Modulo ϵ, $S \equiv r_0(s_0) = \mathrm{diag}(u_1,\dots,u_4)$, so $\varepsilon_i \equiv E_{ii}$ (E_{ij} the matrix units) and $f_i = \varepsilon_i e_i \equiv e_i$. Hence $h = (f_1\, f_2\, f_3\, f_4) = 1 + \epsilon H$ with $H \in \mathrm{M}_4(E)$ is invertible, and $\varepsilon_i f_j = \delta_{ij} f_j$, so $\varepsilon_i v$ is the f_i-component of v in the basis (f_j). The ordinary condition gives $\varepsilon_j \rho(g) \varepsilon_i = 0$ for $i<j$ and $g \in D_\ell$, and $\varepsilon_i(\rho(s) - \kappa_i(s))\varepsilon_i = 0$ for $s \in I_\ell$. Therefore $\mathcal{T}(g) = h^{-1}\rho(g)h$ is upper triangular for every $g \in D_\ell$ (the f_j-component of $\rho(g)f_i$ is $\varepsilon_j\rho(g)\varepsilon_i f_i = 0$ for $j>i$), and $\mathcal{T}(s)_{ii} = \kappa_i(s)$ for $s \in I_\ell$.

Step 5: change of cocycle. As $h \equiv 1$ and $\rho \equiv r_0$ modulo ϵ, $\mathcal{T}(g) = (1+\epsilon y_\mathcal{T}(g))r_0(g)$ with

$$y_\mathcal{T}(g) = y(g) - H + g\cdot H,$$

because $(1-\epsilon H)(1+\epsilon y(g))r_0(g)(1+\epsilon H) = \big(1+\epsilon(y(g)-H+r_0(g)Hr_0(g)^{-1})\big)r_0(g)$. So $y_{\mathcal{T}}$ is continuous, upper triangular, and a cocycle ($\mathcal{T}$ is a homomorphism). Since $r_0(g) = \operatorname{diag}(\kappa_i(g))$, $(g\cdot X)_{ab} = \gamma_{ab}(g)X_{ab}$ with $\gamma_{ab} = \kappa_a\kappa_b^{-1}$, so each entry $y_{\mathcal{T},ab}$ is a continuous cocycle of $E(\gamma_{ab})$. For $a = b$ the action is trivial, $y_{\mathcal{T},aa}$ is a homomorphism, and it vanishes on I_ℓ, because $\mathcal{T}(s)_{aa} = (1+\epsilon y_{\mathcal{T}}(s)_{aa})\kappa_a(s)$ equals $\kappa_a(s)$ there. For $a > b$, $y_{\mathcal{T},ab} = 0$.

Step 6: conclusion. By Step 3 for $a = b$ and Step 2 for $a < b$, there are $Y_{ab} \in E \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $y_{\mathcal{T},ab}(g)\otimes 1 = \gamma_{ab}(g)g(Y_{ab}) - Y_{ab}$ (and $Y_{ab} = 0$ for $a > b$). As D_ℓ acts on $\mathrm{M}_4(E)\otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ entrywise through γ_{ab}, $Y_{\mathcal{T}} = \sum_{a,b} E_{ab}\otimes Y_{ab}$ satisfies $y_{\mathcal{T}}(g)\otimes 1 = g\cdot Y_{\mathcal{T}} - Y_{\mathcal{T}}$, and $Y = Y_{\mathcal{T}} + H\otimes 1$ satisfies $y(g)\otimes 1 = g\cdot Y - Y$ by Step 5. □

Lemma 45.8. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$ and integer e, $E_{\mathcal{O}} = \mathcal{O}[\ell^{-1}]$, and identify D_ℓ with $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ through an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ inducing $\bar\ell$. Put $(n_1,n_2,n_3,n_4) = (e+1, e, 1, 0)$. Then $\kappa_a(s_0) = \chi_{\mathrm{cyc}}(s_0)^{n_a}$ for $a = 1,\dots,4$, and these four elements of $\mathcal{O}$ are pairwise distinct. For $1 \le a < b \le 4$ let $\gamma_{ab} = \kappa_a\kappa_b^{-1}$ and V_{ab} the $\mathbb{Q}_\ell$-representation of D_ℓ on $E_{\mathcal{O}}$ in which g acts by multiplication by $\gamma_{ab}(g)$. Then*

$$H^0(D_\ell, V_{ab}) = 0, \qquad H^2(D_\ell, V_{ab}) = 0,$$
$$H^1(D_\ell, V_{ab}) = H^1_f(D_\ell, V_{ab}), \qquad \dim_{\mathbb{Q}_\ell} H^1(D_\ell, V_{ab}) = [E_{\mathcal{O}} : \mathbb{Q}_\ell].$$

Proof. Write $E = E_{\mathcal{O}}$, $V = V_{ab}$ and $n = n_a - n_b \ge 1$.

The values at s_0 and σ_ℓ. $\ell \in T$, so ξ is unramified at ℓ (Sg1) and $\xi(s_0) = 1$ ($s_0 \in I_\ell$); $\chi(s_0) = \omega_\ell(s_0) = 1$ (Sg5). From $\alpha = \chi_{\mathrm{cyc}}\omega_\ell^{-1}\xi$, $\Psi = \chi\chi_{\mathrm{cyc}}^e$, $\theta = \alpha\Psi$ we get $\kappa_a(s_0) = \chi_{\mathrm{cyc}}(s_0)^{n_a}$; these are pairwise distinct as $\chi_{\mathrm{cyc}}(s_0)$ has infinite order (Sg5), and $\gamma_{ab}(s_0) = \chi_{\mathrm{cyc}}(s_0)^n$. Each of χ, ξ, ω_ℓ has finite order, so $\gamma_{ab} = \chi_{\mathrm{cyc}}^n\nu$ with ν of finite order. Explicitly $\gamma_{12} = \gamma_{34} = \alpha$ ($n = 1$), $\gamma_{23} = \beta = \Psi\alpha^{-1}$ ($n = e-1$), $\gamma_{13} = \gamma_{24} = \Psi$ ($n = e$) and $\gamma_{14} = \theta$ ($n = e+1$). By (Sg2), $\chi_{\mathrm{cyc}}(\sigma_\ell) = 1$; $\omega_\ell(\sigma_\ell)$ is a root of unity of order dividing $\ell - 1$ with reduction $\bar\chi_{\mathrm{cyc}}(\sigma_\ell) = 1$, and reduction modulo $\mathfrak{m}_{\mathcal{O}}$ is injective on such roots of unity, so $\omega_\ell(\sigma_\ell) = 1$. Hence $\alpha(\sigma_\ell) = \xi(\sigma_\ell)$ and $\beta(\sigma_\ell) = (\xi\Psi^{-1})(\sigma_\ell)^{-1}$, and by (Sg2) for $v = \ell$ and $i = 0$, $i = -1$, neither is congruent to 1 modulo $\mathfrak{m}_{\mathcal{O}}$; so $\alpha(\sigma_\ell) \ne 1$ and $\beta(\sigma_\ell) \ne 1$.

An element where γ_{ab} and χ_{cyc} differ. If $n \ge 2$, $\gamma_{ab}(s_0) = \chi_{\mathrm{cyc}}(s_0)^n \ne \chi_{\mathrm{cyc}}(s_0)$. If $n = 1$, then $\gamma_{ab} = \alpha$, or $\gamma_{ab} = \beta$ and $e = 2$; in both cases $\gamma_{ab}(\sigma_\ell) \ne 1 = \chi_{\mathrm{cyc}}(\sigma_\ell)$.

H^0 and H^2. $s_0 - 1$ acts on V as multiplication by $\chi_{\mathrm{cyc}}(s_0)^n - 1$, a non-zero element of the field E, hence injectively; so $H^0(D_\ell, V) = 0$. The dual $V^*(1)$ is $\mathrm{Hom}_{\mathbb{Q}_\ell}(E, \mathbb{Q}_\ell)$ with $(g\varphi)(x) = \chi_{\mathrm{cyc}}(g)\varphi(\gamma_{ab}(g)^{-1}x)$; an invariant φ satisfies $\varphi((\gamma_{ab}(g) - \chi_{\mathrm{cyc}}(g))x) = 0$ for all g and x (replace x by $\gamma_{ab}(g)x$), so $\varphi = 0$ by the previous step. By Theorem B.36 ($M = \mathbb{Q}_\ell$), $\dim H^2(D_\ell, V) = \dim H^0(D_\ell, V^*(1)) = 0$, and $\dim_{\mathbb{Q}_\ell} H^1(D_\ell, V) = \dim_{\mathbb{Q}_\ell} V = [E : \mathbb{Q}_\ell]$.

H^1_f. V is de Rham with all Hodge–Tate weights $n \ge 1$, so $\dim_{\mathbb{Q}_\ell}(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)) = \dim_{\mathbb{Q}_\ell} V$, and Theorem B.43 ($K = \mathbb{Q}_\ell$) gives $\dim H^1_f(D_\ell, V) = 0 + \dim_{\mathbb{Q}_\ell} V = \dim H^1(D_\ell, V)$. As $H^1_f \subseteq H^1$, they are equal. □

Proposition 45.9. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, and let (R_ℓ, ρ_ℓ) be universal for the lifts of $\bar r_0|_{D_\ell}$ satisfying the local condition at ℓ, with $x_\ell\colon R_\ell \to \mathcal{O}$ the homomorphism with $x_\ell \circ \rho_\ell = r_0|_{D_\ell}$, as in Proposition 45.7. Then the localisation of R_ℓ at the prime ideal $\ker x_\ell$ has Krull dimension at least 14.*

Proof. By Lemma 45.10 there is a homomorphism $r_T\colon D_\ell \to \mathrm{GL}_4(E_{\mathcal{O}}[[T_1,\dots,T_{14}]])$ satisfying (Fm1)–(Fm3) for $v = \ell$ and $n = 14$: its coefficients are continuous, $r_T \equiv r_0$ modulo the maximal

ideal, $r_T(g)$ has similitude $\theta(g)$, r_T is ordinary, and the fourteen maps y_i are linearly independent. Since (R_ℓ, ρ_ℓ) is universal for the lifts of $\bar{r}_0|_{D_\ell}$ satisfying the local condition at ℓ, Lemma 45.11 gives an $\mathcal{O}$-algebra homomorphism $F\colon R_\ell \to E_\mathcal{O}[[T_1, \dots, T_{14}]]$ with $F(f) - x_\ell(f)$ in the maximal ideal for every f, and such that the vectors of linear coefficients of the $F(f)$, $f \in \ker x_\ell$, span $E_\mathcal{O}^{14}$. By Lemma 45.13 (1) with $R = R_\ell$, $x = x_\ell$ and $n = 14$, the localisation of R_ℓ at $\ker x_\ell$ has Krull dimension at least 14. □

Lemma 45.10. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $E_\mathcal{O} = \mathcal{O}[\ell^{-1}]$, and $\mathfrak{m}_T$ the maximal ideal of $E_\mathcal{O}[[T]] = E_\mathcal{O}[[T_1, \dots, T_{14}]]$. There is a homomorphism $r_T\colon D_\ell \to \mathrm{GL}_4(E_\mathcal{O}[[T]])$ such that every coefficient of every entry of $r_T(g)$ is a continuous function of $g \in D_\ell$; $r_T(g) - r_0(g)$ has its entries in $\mathfrak{m}_T$ and $r_T(g)$ has similitude $\theta(g)$ for every $g \in D_\ell$; r_T is ordinary; and the maps $y_1, \dots, y_{14}\colon D_\ell \to \mathrm{M}_4(E_\mathcal{O})$, $y_i(g)$ the coefficient of T_i in $r_T(g)\, r_0(g)^{-1}$, are linearly independent over $E_\mathcal{O}$.*

Proof. Write $E = E_\mathcal{O}$, $E_k = E[[T]]/\mathfrak{m}_T^k$, E_{ab} for the matrix units and $g \cdot X = r_0(g) X r_0(g)^{-1}$ for $X \in \mathrm{M}_4(E)$. A map from D_ℓ to $E[[T]]$, to E_k or to matrices over them is called continuous if all its coefficients are; sums and products of continuous maps are continuous, and so are inverses of matrices with unit determinant (adjugate over determinant; the inverse of a unit power series has coefficients polynomial in its coefficients and the inverse of its constant term). Let $\Phi^+ = \{(1,0), (-1,1), (0,1), (1,1)\}$ and

$$X_{(1,0)} = E_{12} - E_{34}, \quad X_{(-1,1)} = E_{23}, \quad X_{(0,1)} = E_{13} + E_{24}, \quad X_{(1,1)} = E_{14}, \qquad X_{-w} = {}^t X_w,$$

with characters $\gamma_{(1,0)} = \gamma_{12} = \gamma_{34}$, $\gamma_{(-1,1)} = \gamma_{23}$, $\gamma_{(0,1)} = \gamma_{13} = \gamma_{24}$, $\gamma_{(1,1)} = \gamma_{14}$, so that $g \cdot X_{\pm w} = \gamma_w(g)^{\pm 1} X_{\pm w}$; the entries of $X_{\pm w}$ have weight $\pm w$, and the eight matrices $X_{\pm w}$ are supported on pairwise disjoint sets of off-diagonal positions.

Step 1: unramified twists. Let $\mathrm{fr}\colon D_\ell \to \widehat{\mathbb{Z}} \to \mathbb{Z}_\ell$ be the continuous homomorphism, trivial on I_ℓ, with $\mathrm{fr}(\sigma_\ell) = 1$. For $x \in \mathbb{Z}_\ell$ put $(1+T_i)^x = \sum_{j \geq 0} \binom{x}{j} T_i^j$; the binomial coefficients are polynomials in x with values in $\mathbb{Z}_\ell$ on $\mathbb{Z}_\ell$ (by continuity, from $\mathbb{N}$), and $(1+T_i)^{x+x'} = (1+T_i)^x (1+T_i)^{x'}$ (Vandermonde's identity, a polynomial identity true on $\mathbb{N}^2$). So $\lambda_i(g) = (1+T_i)^{\mathrm{fr}(g)}$ $(i = 1, 2)$ is a continuous character $D_\ell \to E[[T]]^\times$, trivial on I_ℓ, with $\lambda_i(g) \equiv 1 + \mathrm{fr}(g) T_i$ modulo $\mathfrak{m}_T^2$. Put

$$d(g) = \mathrm{diag}(\kappa_1\lambda_1,\ \kappa_2\lambda_2,\ \kappa_3\lambda_2^{-1},\ \kappa_4\lambda_1^{-1})(g),$$

a continuous homomorphism with $d(g) \equiv r_0(g)$ modulo $\mathfrak{m}_T$, $d(s) = r_0(s)$ for $s \in I_\ell$, and similitude $\theta(g)$ (a diagonal matrix has similitude ν if and only if $d_{11}d_{44} = d_{22}d_{33} = \nu$, and $\kappa_1\kappa_4 = \kappa_2\kappa_3 = \theta$); modulo $\mathfrak{m}_T^2$, $d(g) r_0(g)^{-1} \equiv 1 + \Delta(g)$ with $\Delta(g) = \mathrm{fr}(g)(T_1(E_{11} - E_{44}) + T_2(E_{22} - E_{33}))$.

Step 2: the unipotent radical. For a commutative ring A let $U(A)$ be the group of upper unitriangular $u \in \mathrm{GL}_4(A)$ with similitude 1.

So $u \mapsto (u_{12}, u_{23}, u_{24}, u_{14})$ is a bijection $U(A) \to A^4$ whose inverse is given by polynomials, and the strictly upper triangular elements of $\mathfrak{sp}_4(E)$ form $\mathfrak{n} = \bigoplus_{w \in \Phi^+} E X_w$. For $A = E[[T]]$ or E_k let $U^1(A) = \{u \in U(A) : u \equiv 1 \bmod \mathfrak{m}_T\}$, a subgroup. Conjugation by a diagonal matrix with a similitude preserves upper unitriangularity and similitude 1, so $\alpha_g(u) = d(g) u d(g)^{-1}$ is an action of D_ℓ on $U^1(A)$ by group automorphisms (d is a homomorphism).

Step 3: the kernels. Let $k \geq 2$ and K_k the kernel of the reduction $U^1(E_{k+1}) \to U^1(E_k)$. By the coordinates, $u \in K_k$ if and only if $u = 1 + A$ with $A_{12}, A_{23}, A_{24}, A_{14} \in \mathfrak{m}_T^k E_{k+1}$, $A_{34} = -A_{12}$, $A_{13} = A_{24} + A_{12}A_{23} = A_{24}$ (as $\mathfrak{m}_T^{2k} E_{k+1} = 0$); that is, $K_k = \{1 + A : A \in \mathfrak{n} \otimes_E \mathfrak{m}_T^k E_{k+1}\}$. For such A, A' and $v = 1 + B \in U^1(E_{k+1})$, $AA' = AB = BA = 0$, so $(1+A)(1+A') = 1 + A + A'$ and K_k is central in $U^1(E_{k+1})$; and $\alpha_g(1+A) = 1 + r_0(g) A r_0(g)^{-1}$ since $d(g) \equiv r_0(g)$ modulo

$\mathfrak{m}_T$. Hence K_k, written additively, is the continuous representation $\mathfrak{n} \otimes_E (\mathfrak{m}_T^k/\mathfrak{m}_T^{k+1})$, a finite direct sum of copies of the $E(\gamma_w)$, $w \in \Phi^+$, and $H^2(D_\ell, K_k) = 0$ by Lemma 45.8.

Step 4: a crossed homomorphism. By Lemma 45.8, $H^1(D_\ell, E(\gamma_w)) \neq 0$; choose continuous cocycles $c_w \colon D_\ell \to E$ of $E(\gamma_w)$ that are not coboundaries, and number the variables T_3, T_4, T_5, T_6 as T_w for $w = (1,0), (-1,1), (0,1), (1,1)$. We construct continuous $u_k \colon D_\ell \to U^1(E_k)$, $k \geq 2$, with

$$u_k(gh) = u_k(g)\,\alpha_g(u_k(h)) \qquad (g, h \in D_\ell)$$

and $u_{k+1} \equiv u_k$ modulo $\mathfrak{m}_T^k$. For $k = 2$ let $u_2(g) = 1 + \sum_w T_w c_w(g) X_w$: it lies in $U^1(E_2)$ (the coordinate conditions hold as products of two elements of $\mathfrak{m}_T$ vanish in E_2), and $u_2(g)\alpha_g(u_2(h)) = 1 + \sum_w T_w(c_w(g) + \gamma_w(g)c_w(h))X_w = u_2(gh)$. Given u_k, let $\tilde{u}(g) \in U^1(E_{k+1})$ be the element whose four coordinates are those of $u_k(g)$ with their coefficients of degree $< k$ and zero coefficients of degree k; $\tilde{u}$ is continuous and reduces to u_k. Then $o(g,h) = \tilde{u}(g)\alpha_g(\tilde{u}(h))\tilde{u}(gh)^{-1}$ reduces to 1, so $o(g,h) = 1 + \omega(g,h)$ with ω continuous with values in K_k. Using that K_k is central and the α_g are automorphisms with $\alpha_g\alpha_h = \alpha_{gh}$,

$$\tilde{u}(g)\alpha_g(\tilde{u}(h))\alpha_{gh}(\tilde{u}(k')) = o(g,h)\,o(gh,k')\,\tilde{u}(ghk'),$$
$$\tilde{u}(g)\alpha_g(\tilde{u}(h)\alpha_h(\tilde{u}(k'))) = \alpha_g(o(h,k'))\,o(g,hk')\,\tilde{u}(ghk'),$$

so $g \cdot \omega(h,k') - \omega(gh,k') + \omega(g,hk') - \omega(g,h) = 0$: ω is a continuous 2-cocycle of K_k. As $H^2(D_\ell, K_k) = 0$ there is a continuous $b \colon D_\ell \to K_k$ with $\omega(g,h) = g \cdot b(h) - b(gh) + b(g)$. Put $u_{k+1}(g) = (1 - b(g))\tilde{u}(g)$. Then

$$u_{k+1}(g)\alpha_g(u_{k+1}(h)) = \big(1 - b(g) - g \cdot b(h) + \omega(g,h)\big)\tilde{u}(gh) = (1 - b(gh))\tilde{u}(gh) = u_{k+1}(gh),$$

and $u_{k+1} \equiv u_k$ modulo $\mathfrak{m}_T^k$. The u_k define a continuous $u \colon D_\ell \to \mathrm{GL}_4(E[[T]])$, upper unitriangular, of similitude 1 and congruent to 1 modulo $\mathfrak{m}_T$ (these identities hold modulo every $\mathfrak{m}_T^k$), with $u(gh) = u(g)d(g)u(h)d(g)^{-1}$ and $u(g) \equiv 1 + \sum_w T_w c_w(g) X_w$ modulo $\mathfrak{m}_T^2$.

Step 5: the triangular homomorphism. $t(g) = u(g)d(g)$ is a homomorphism: $t(g)t(h) = u(g)(d(g)u(h)d(g)^{-1})d(g)d(h) = u(gh)d(gh)$. It is continuous, upper triangular with diagonal $d(g)$, of similitude $\theta(g)$, and $t(g) \equiv r_0(g)$ modulo $\mathfrak{m}_T$.

Step 6: conjugation, and the conditions other than independence. Number $T_7, \ldots, T_{10}$ as T'_w and $T_{11}, \ldots, T_{14}$ as T''_w ($w \in \Phi^+$ in the order above), and let

$$h = \prod_{w \in \Phi^+} (1 + T'_w X_{-w}) \prod_{w \in \Phi^+} (1 + T''_w X_w), \qquad r_T(g) = h\,t(g)\,h^{-1}.$$

By the first side calculation h has similitude 1, and $h \equiv 1$ modulo $\mathfrak{m}_T$. So r_T is a continuous homomorphism, $r_T(g)$ has similitude $\theta(g)$, and $r_T(g) \equiv r_0(g)$ modulo $\mathfrak{m}_T$. Ordinarity: let $u_i = \kappa_i(s_0)$, pairwise distinct with differences units of $E[[T]]$ (Lemma 45.8), $S = t(s_0)$, upper triangular with diagonal (u_i) ($\lambda_i(s_0) = 1$), and $F_i = \bigoplus_{j \leq i} E[[T]]e_j$, preserved by every upper triangular matrix. $S - u_m$ maps F_m into F_{m-1}. Hence $\Pi'_i = \prod_{j \neq i}(S - u_j)$ maps $E[[T]]^4$ into F_i (through the factors $j > i$) and kills F_{i-1} (through the factors $j < i$); the characteristic polynomial of S is $\prod_j (X - u_j)$. For $i < j$ and $g \in D_\ell$, $\Pi'_j t(g) \Pi'_i = 0$, since $t(g)F_i \subseteq F_i \subseteq F_{j-1}$. For $s \in I_\ell$, $t(s)$ has diagonal $(\kappa_i(s))$, so $t(s) - \kappa_i(s)$ maps F_i into F_{i-1} and $\Pi'_i(t(s) - \kappa_i(s))\Pi'_i = 0$. Conjugating by h: $r_T(s_0) = hSh^{-1}$ has characteristic polynomial $\prod_j (X - \kappa_j(s_0))$, the matrices Π_i formed from r_T are $h\Pi'_i h^{-1}$, and the identities transfer. So r_T is ordinary.

Step 7: independence. Write $H_1 = \sum_w (T'_w X_{-w} + T''_w X_w)$, so that, modulo $\mathfrak{m}_T^2$,

$$h \equiv 1 + H_1, \qquad h^{-1} \equiv 1 - H_1, \qquad r_0(g) h^{-1} r_0(g)^{-1} \equiv 1 - g \cdot H_1.$$

From $r_T(g)r_0(g)^{-1} = h\,u(g)\,(d(g)r_0(g)^{-1})(r_0(g)h^{-1}r_0(g)^{-1})$ and Steps 1 and 4,

$$r_T(g)r_0(g)^{-1} \equiv 1 + \Delta(g) + \sum_w T_w c_w(g) X_w + H_1 - g\cdot H_1 \pmod{\mathfrak{m}_T^2}.$$

So $y_1 = \mathrm{fr}\,(E_{11}-E_{44})$, $y_2 = \mathrm{fr}\,(E_{22}-E_{33})$, the y_i for T_w are $c_w X_w$, those for T'_w are $(1-\gamma_w^{-1})X_{-w}$, and those for T''_w are $(1-\gamma_w)X_w$. As the diagonal and the eight supports of the $X_{\pm w}$ are disjoint, a linear relation $\sum_i \mu_i y_i = 0$ splits into: $\mathrm{fr}(g)(\mu_1(E_{11}-E_{44}) + \mu_2(E_{22}-E_{33})) = 0$, which at $g = \sigma_\ell$ gives $\mu_1 = \mu_2 = 0$; for each w, $\mu'_w(1-\gamma_w(g)^{-1}) = 0$, which at $g = s_0$ gives $\mu'_w = 0$ as $\gamma_w(s_0) \neq 1$ (Lemma 45.8); and $\mu_w c_w(g) + \mu''_w(1-\gamma_w(g)) = 0$ for all g. In the last, $\mu_w \neq 0$ would make c_w the coboundary $g \mapsto \gamma_w(g)x - x$ with $x = \mu''_w/\mu_w$; so $\mu_w = 0$, and then $\mu''_w = 0$ at $g = s_0$. Hence $y_1, \dots, y_{14}$ are linearly independent. □

Lemma 45.11. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $E_\mathcal{O} = \mathcal{O}[\ell^{-1}]$ and $v \in T$. Let $R_v \in \mathrm{CNL}_\mathcal{O}$ and ρ_v a lift of $\bar{r}_0|_{D_v}$ to R_v satisfying the local condition at v such that for every $A \in \mathrm{CNL}_\mathcal{O}$ and every lift ρ of $\bar{r}_0|_{D_v}$ to A satisfying the local condition at v there is exactly one $\mathcal{O}$-algebra homomorphism $f\colon R_v \to A$ with $f\circ\rho_v = \rho$; let $x_v\colon R_v \to \mathcal{O}$ be the one with $x_v\circ\rho_v = r_0|_{D_v}$ and $\mathfrak{q}_v = \ker x_v$. Let $n \geq 0$, $\mathfrak{m}_T$ the maximal ideal of $E_\mathcal{O}[[T_1,\dots,T_n]]$, and $r_T\colon D_v \to \mathrm{GL}_4(E_\mathcal{O}[[T_1,\dots,T_n]])$ a homomorphism such that*

(Fm1) *every coefficient of every entry of $r_T(g)$ is a continuous function of $g \in D_v$, and $r_T(g) - r_0(g)$ has its entries in $\mathfrak{m}_T$ for every $g \in D_v$;*

(Fm2) *$r_T(g)$ has similitude $\theta(g)$ for every $g \in D_v$; if $v = \ell$, r_T is ordinary; if $v \neq \ell$, $r_T(g) = 1$ for every $g \in I_v$ with $\chi(g) = 1$;*

(Fm3) *the maps $y_1,\dots,y_n\colon D_v \to \mathrm{M}_4(E_\mathcal{O})$, $y_i(g)$ the coefficient of T_i in $r_T(g)\,r_0(g)^{-1}$, are linearly independent over $E_\mathcal{O}$.*

Then there is an $\mathcal{O}$-algebra homomorphism $F\colon R_v \to E_\mathcal{O}[[T_1,\dots,T_n]]$ with $F(\rho_v(g)_{ab}) = r_T(g)_{ab}$ for all $g \in D_v$ and $1 \leq a,b \leq 4$, such that $F(f) - x_v(f) \in \mathfrak{m}_T$ for every $f \in R_v$, and such that the vectors of coefficients of $T_1,\dots,T_n$ of the $F(f)$, where f runs over the elements $\rho_v(g)_{ab} - r_0(g)_{ab}$ of $\mathfrak{q}_v$ ($g \in D_v$, $1 \leq a,b \leq 4$), span $E_\mathcal{O}^n$.

Proof. Write $E = E_\mathcal{O}$, $E[[T]] = E[[T_1,\dots,T_n]]$, $E_k = E[[T]]/\mathfrak{m}_T^k$ ($k \geq 1$) and r_k for the reduction of r_T modulo $\mathfrak{m}_T^k$.

Step 1: lifts to finite $\mathcal{O}$-algebras. Fix k. Since $r_0(g) = \mathrm{diag}(\kappa_i(g))$ has entries in $\mathcal{O}$ and inverse with entries in $\mathcal{O}$, the entries of $r_k(g)r_0(g)^{-1} - 1$ are continuous functions of g with values in $\mathfrak{m}_T E_k$ (Fm1); let S_k be the set of all of them, a compact subset of $\mathfrak{m}_T E_k$ (D_v is compact). For $1 \leq j < k$ the set of products of j elements of S_k is compact, as the image of S_k^j under the continuous multiplication; so the union of these sets is compact and lies in $\varpi^{-N}\Lambda$ for an $\mathcal{O}$-lattice Λ of E_k and some N. Let A_k be the $\mathcal{O}$-span of 1 and these products. It is a submodule of the finitely generated $\mathcal{O}$-module $\mathcal{O} + \varpi^{-N}\Lambda$, hence finitely generated; it is closed under multiplication (products of k or more elements of $\mathfrak{m}_T E_k$ vanish), contains $\mathcal{O}$ and lies in $\mathcal{O}+\mathfrak{m}_T E_k$. By Lemma 45.12, $A_k \in \mathrm{CNL}_\mathcal{O}$, with maximal ideal $\mathfrak{m}_\mathcal{O} + A_k^+$ and the subspace topology. The entries of $r_k(g) = (r_k(g)r_0(g)^{-1})r_0(g)$ lie in A_k, and $\det r_k(g) \in \det r_0(g) + A_k^+ \subseteq \mathcal{O}^\times + A_k^+$ is a unit of A_k; so r_k is a continuous homomorphism $D_v \to \mathrm{GL}_4(A_k)$, and $r_k(g) - r_0(g)$ has entries in A_k^+, so $r_k \equiv \bar{r}_0$ modulo $\mathfrak{m}_{A_k}$. The identities of (Fm2) are identities between polynomial expressions with coefficients in $\mathcal{O}$ in the entries of r_T (for $v = \ell$, those of the ordinary condition, with the $\kappa_j(s_0)$, $\kappa_i(s)$ as coefficients); they hold in $E[[T]]$, hence in E_k and in its subring A_k. So r_k is a lift of $\bar{r}_0|_{D_v}$ to A_k satisfying the local condition at v; let $x_k\colon R_v \to A_k$ be the $\mathcal{O}$-algebra homomorphism with $x_k\circ\rho_v = r_k$.

Step 2: the limit. Let $\pi_k\colon E_{k+1} \to E_k$ be the projection. It maps S_{k+1} onto S_k, and products of k elements of $\mathfrak{m}_T E_k$ vanish, so $\pi_k(A_{k+1}) \subseteq A_k$; $\pi_k\colon A_{k+1} \to A_k$ is an $\mathcal{O}$-algebra homomorphism and $\pi_k \circ x_{k+1}$ classifies $\pi_k \circ r_{k+1} = r_k$, so $\pi_k \circ x_{k+1} = x_k$ by uniqueness. Hence $F(f) = (x_k(f))_k \in \varprojlim_k E_k = E[[T]]$ defines an $\mathcal{O}$-algebra homomorphism with $F(\rho_v(g)_{ab}) = r_T(g)_{ab}$. For $k = 1$, $E_1 = E$, $S_1 = \{0\}$, $A_1 = \mathcal{O}$ and $r_1 = r_0|_{D_v}$, so $x_1 = x_v$ by uniqueness: the constant term of $F(f)$ is $x_v(f)$.

Step 3: spanning. Let $\mathrm{L}\colon E[[T]] \to E^n$ give the coefficients of $T_1, \dots, T_n$. For $g \in D_v$ and $1 \le a, b \le 4$ the element $x_{g,a,b} = \rho_v(g)_{ab} - r_0(g)_{ab}$ lies in $\mathfrak{q}_v$, and $\mathrm{L}(F(x_{g,a,b}))$ is the vector of the (a,b) entries of $y_i(g) r_0(g)$, $i = 1, \dots, n$. If these vectors spanned a proper subspace of E^n, there would be $(\mu_i) \neq 0$ with $\sum_i \mu_i y_i(g) r_0(g) = 0$, hence $\sum_i \mu_i y_i(g) = 0$, for all g, against (Fm3). □

Lemma 45.12. *Let $\mathcal{O}$ be a coefficient ring, $E_{\mathcal{O}} = \mathcal{O}[\ell^{-1}]$, $n \ge 0$ and $k \ge 1$ integers, $\mathfrak{m}_T$ the maximal ideal of $E_{\mathcal{O}}[[T_1, \dots, T_n]]$, and $E_k = E_{\mathcal{O}}[[T_1, \dots, T_n]]/\mathfrak{m}_T^k$, a finite-dimensional $E_{\mathcal{O}}$-vector space with its natural topology. Let $A \subseteq E_k$ be a subring containing $\mathcal{O}$ which is a finitely generated $\mathcal{O}$-module and is contained in $\mathcal{O} + \mathfrak{m}_T E_k$. Then $A \in \mathrm{CNL}_{\mathcal{O}}$, its maximal ideal is $\mathfrak{m}_{\mathcal{O}} + A^+$ with $A^+ = A \cap \mathfrak{m}_T E_k$, and its topology as a complete local ring is the subspace topology from E_k. Moreover every $\mathcal{O}$-algebra homomorphism from a ring of $\mathrm{CNL}_{\mathcal{O}}$ to A is local.*

Proof. Decomposition. Elements of $\mathcal{O} + \mathfrak{m}_T E_k$ decompose uniquely as $c + a^+$ with $c \in \mathcal{O}$ and $a^+ \in \mathfrak{m}_T E_k$ (the constant term), and if $c + a^+ \in A$ then $a^+ \in A$ since $\mathcal{O} \subseteq A$. So $A = \mathcal{O} \oplus A^+$, A^+ is an ideal of A (as $\mathfrak{m}_T E_k$ is an ideal of E_k), and $(A^+)^k \subseteq \mathfrak{m}_T^k E_k = 0$.

Local ring. The map $A \to \mathbb{F}_{\mathcal{O}}$, $c + a^+ \mapsto c \bmod \mathfrak{m}_{\mathcal{O}}$, is a surjective ring homomorphism with kernel $\mathfrak{m}_A = \mathfrak{m}_{\mathcal{O}} + A^+$. An element $c + a^+$ with $c \in \mathcal{O}^\times$ is a unit of A: its inverse is $c^{-1} \sum_{j=0}^{k-1} (-c^{-1}a^+)^j \in A$. So A is local with maximal ideal $\mathfrak{m}_A$ and residue field $\mathbb{F}_{\mathcal{O}}$, and $\mathcal{O} \to A$ is local and induces an isomorphism of residue fields. A is a finitely generated module over the Noetherian ring $\mathcal{O}$, hence a Noetherian ring; it is torsion free (E_k is an $E_{\mathcal{O}}$-vector space), hence free of finite rank over the discrete valuation ring $\mathcal{O}$, and an $\mathcal{O}$-basis of A is linearly independent over $E_{\mathcal{O}}$ (clear denominators).

Topology and completeness. For $m \ge k$, $\mathfrak{m}_A^m = \sum_{j=0}^m \mathfrak{m}_{\mathcal{O}}^{m-j} (A^+)^j$, where only $j \le k-1$ contribute, so $\mathfrak{m}_A^m \subseteq \varpi^{m-k+1} A$; and $\varpi^m A \subseteq \mathfrak{m}_A^m$. Hence the $\mathfrak{m}_A$-adic topology is the ϖ-adic topology, for which the free module A of finite rank is complete; so $A \in \mathrm{CNL}_{\mathcal{O}}$. With an $\mathcal{O}$-basis $a_1, \dots, a_r$ of A, the sets $\varpi^m A = \sum_i \varpi^m \mathcal{O} a_i$ form a basis of neighbourhoods of 0 for the natural topology of the $E_{\mathcal{O}}$-space $\bigoplus_i E_{\mathcal{O}} a_i \subseteq E_k$, which is the subspace topology (a finite-dimensional $E_{\mathcal{O}}$-vector space carries a unique Hausdorff vector space topology).

Local homomorphisms. Let $R \in \mathrm{CNL}_{\mathcal{O}}$, $f\colon R \to A$ an $\mathcal{O}$-algebra homomorphism and $y \in \mathfrak{m}_R$. If $f(y) \notin \mathfrak{m}_A$, choose $u \in \mathcal{O}^\times$ with $f(y) \equiv u$ modulo $\mathfrak{m}_A$; then $y - u \notin \mathfrak{m}_R$ ($\mathcal{O} \to R$ is local), so $y - u$ is a unit and so is $f(y) - u \in \mathfrak{m}_A$, a contradiction. So $f(\mathfrak{m}_R) \subseteq \mathfrak{m}_A$. □

Lemma 45.13. *Let $\mathcal{O}$ be a coefficient ring, $E_{\mathcal{O}} = \mathcal{O}[\ell^{-1}]$, $R \in \mathrm{CNL}_{\mathcal{O}}$, $x\colon R \to \mathcal{O}$ an $\mathcal{O}$-algebra homomorphism and $\mathfrak{q} = \ker x$. Let $n \ge 0$, $\mathfrak{m}_T$ the maximal ideal of $E_{\mathcal{O}}[[T_1, \dots, T_n]]$, and $F\colon R \to E_{\mathcal{O}}[[T_1, \dots, T_n]]$ an $\mathcal{O}$-algebra homomorphism such that $F(f) - x(f) \in \mathfrak{m}_T$ for every $f \in R$ and such that the vectors of coefficients of $T_1, \dots, T_n$ of the $F(f)$, $f \in \mathfrak{q}$, span $E_{\mathcal{O}}^n$. Then:*

1. *the localisation $R_{\mathfrak{q}}$ has Krull dimension at least n;*
2. *if moreover $\dim_{E_{\mathcal{O}}}(\mathfrak{q}R_{\mathfrak{q}}/\mathfrak{q}^2 R_{\mathfrak{q}}) \le n$, then $R_{\mathfrak{q}}$ is a regular local ring of dimension n, R has exactly one minimal prime ideal contained in $\mathfrak{q}$, and for every $N \ge 1$ and every $f \in R$ with $F(f) \in \mathfrak{m}_T^N$ there is $c \in \mathcal{O} \setminus \{0\}$ with $cf \in \mathfrak{q}^N$.*

Proof. Write $E = E_{\mathcal{O}}$, $E[[T]] = E[[T_1, \dots, T_n]]$ and $\mathrm{L}\colon E[[T]] \to E^n$ for the coefficients of $T_1, \dots, T_n$.

Step 1: the local ring. x is the identity on $\mathcal{O}$, so it is onto, $R/\mathfrak{q} \cong \mathcal{O}$ is a domain and $\mathfrak{q}$ is prime; non-zero elements of $\mathcal{O}$ lie outside $\mathfrak{q}$. So $A = R_{\mathfrak{q}}$ is a Noetherian local E-algebra with maximal ideal $\mathfrak{n} = \mathfrak{q}A$ and residue field E. For $s \notin \mathfrak{q}$, $F(s)$ has constant term $x(s) \neq 0$ and is a unit; so F extends to $\Phi\colon A \to E[[T]]$ with $\Phi(\mathfrak{n}^N) \subseteq \mathfrak{m}_T^N$. Let $\widehat{A}$ be the $\mathfrak{n}$-adic completion, with maximal ideal $\widehat{\mathfrak{n}}$; the maps $A/\mathfrak{n}^N \to E[[T]]/\mathfrak{m}_T^N$ induced by Φ define an E-algebra homomorphism $\widehat{\Phi}\colon \widehat{A} \to E[[T]]$ with $\widehat{\Phi}(\widehat{\mathfrak{n}}^N) \subseteq \mathfrak{m}_T^N$, extending Φ.

Step 2: $\widehat{\Phi}$ is onto; (1). Choose $z_1, \dots, z_n \in \mathfrak{q}$ with $\mathrm{L}(F(z_1)), \dots, \mathrm{L}(F(z_n))$ a basis of E^n. Let $\iota\colon E[[Y_1, \dots, Y_n]] \to \widehat{A}$ be the continuous E-algebra homomorphism with $\iota(Y_i) = z_i \in \widehat{\mathfrak{n}}$; as $\widehat{\mathfrak{n}}^N$ is closed, ι maps $(Y)^N$ into $\widehat{\mathfrak{n}}^N$. So $\psi = \widehat{\Phi} \circ \iota$ maps $(Y)^N$ into $\mathfrak{m}_T^N$ and agrees with the substitution $Y_i \mapsto F(z_i)$ on polynomials; every power series differs from a polynomial by an element of $(Y)^N$, for every N, so ψ is that substitution. Its matrix of linear parts has rows $\mathrm{L}(F(z_i))$ and is invertible, so ψ is an automorphism and $\widehat{\Phi}$ is onto. Hence $\dim A = \dim \widehat{A} \geq \dim E[[T]] = n$.

Step 3: (2). Now $\dim A \geq n \geq \dim_E \mathfrak{n}/\mathfrak{n}^2$, so A is regular of dimension n, and so is $\widehat{A}$, a domain. The kernel $\mathfrak{P}$ of $\widehat{\Phi}$ is prime and $\dim \widehat{A}/\mathfrak{P} = n = \dim \widehat{A}$; a non-zero prime of the n-dimensional local domain $\widehat{A}$ has height ≥ 1 and quotient of dimension $\leq n-1$, so $\mathfrak{P} = 0$ and $\widehat{\Phi}$ is an isomorphism of local rings, with $\widehat{\Phi}(\widehat{\mathfrak{n}}^N) = \mathfrak{m}_T^N$. The minimal primes of R contained in $\mathfrak{q}$ correspond to those of the domain A, so there is exactly one. Let $f \in R$ with $F(f) \in \mathfrak{m}_T^N$. Then $\widehat{\Phi}(f) \in \mathfrak{m}_T^N$, so $f \in \widehat{\mathfrak{n}}^N \cap A = \mathfrak{n}^N$ ($A/\mathfrak{n}^N \to \widehat{A}/\widehat{\mathfrak{n}}^N$ is an isomorphism). Let $A' = R[\ell^{-1}]$ and $\mathfrak{n}' = \mathfrak{q}A'$, a maximal ideal ($A'/\mathfrak{n}' \cong E$) with $A = A'_{\mathfrak{n}'}$. As $\mathfrak{n}'^N$ is $\mathfrak{n}'$-primary, $\mathfrak{n}'^N A \cap A' = \mathfrak{n}'^N$, so the image of f in A' is $h\ell^{-k}$ with $h \in \mathfrak{q}^N$; then $\ell^{k'}(\ell^k f - h) = 0$ in R for some k', and $c = \ell^{k+k'}$ satisfies $cf = \ell^{k'}h \in \mathfrak{q}^N$. □

Lemma 45.14. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $E_{\mathcal{O}} = \mathcal{O}[\ell^{-1}]$, and fix an embedding $\mathrm{emb}_\ell\colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ inducing $\bar{\ell}$. Let D_ℓ act on $\mathrm{M}_4(E_{\mathcal{O}})$ by $g \cdot X = r_0(g) X r_0(g)^{-1}$, on B_{cris} through emb_ℓ, and on $\mathrm{M}_4(E_{\mathcal{O}}) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ diagonally. For $w \in \mathbb{Z}^2$ let $\mathfrak{sp}_{4,w}$ be the space of $X \in \mathfrak{sp}_4(E_{\mathcal{O}})$ with $X_{ij} = 0$ whenever $\mathrm{wt}_4(i,j) \neq w$, and Z_w the $E_{\mathcal{O}}$-space of continuous maps $y\colon D_\ell \to \mathfrak{sp}_{4,w}$ with $y(gh) = y(g) + g \cdot y(h)$ for which there is $Y \in \mathrm{M}_4(E_{\mathcal{O}}) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $y(g) \otimes 1 = g \cdot Y - Y$ for all $g \in D_\ell$. Then $\dim_{E_{\mathcal{O}}} Z_w = d_\ell(w)$ for every $w \in \mathbb{Z}^2$.*

Proof. Write $E = E_{\mathcal{O}}$, $i^* = 5 - i$, and $\iota(X) = -J_4^{-1}\,{}^t X J_4$, an involution of $\mathrm{M}_4(E)$ with fixed space $\mathfrak{sp}_4(E)$ (as ${}^t J_4 = -J_4$).

Step 1: equivariant projections. If u has similitude ν, then ${}^t u = \nu J_4 u^{-1} J_4^{-1}$, and a direct computation gives $\iota(uXu^{-1}) = u\,\iota(X)\,u^{-1}$; $r_0(g)$ has similitude $\theta(g)$, so ι commutes with the action. The entry (a,b) of $\iota(X)$ is $\pm X_{b^*a^*}$, and $\mathrm{wt}_4(b^*,a^*) = \mathrm{wt}_4(a,b)$ ($\lambda_{i^*} = 1-\lambda_i$, $\mu_{i^*} = 1-\mu_i$); the action multiplies the entry (a,b) by $\kappa_a\kappa_b^{-1}$, so it preserves weights. Hence p_w, the projection to the entries of weight w followed by $X \mapsto \frac{1}{2}(X + \iota(X))$, is an equivariant projection of $\mathrm{M}_4(E)$ onto $\mathfrak{sp}_{4,w}$ (ℓ is odd). For $y \in Z_w$ with primitive Y, $(p_w \otimes 1)(Y)$ is a primitive in $\mathfrak{sp}_{4,w} \otimes B_{\mathrm{cris}}$. So, with $V_w = \mathfrak{sp}_{4,w}$, Z_w is the space of continuous cocycles whose class lies in $H^1_f(D_\ell, V_w)$, and

$$\dim_E Z_w = \dim_E H^1_f(D_\ell, V_w) + \dim_E V_w - \dim_E V_w^{D_\ell}.$$

Step 2: the pieces V_w. $X \in \mathfrak{sp}_4$ if and only if $X = \iota(X)$, which relates X_{ab} to $X_{b^*a^*}$ and gives $X_{ii} = -X_{i^*i^*}$. So $V_{(0,0)} = \{\mathrm{diag}(x_1, x_2, -x_2, -x_1)\}$, of dimension 2 with trivial action; for each $w \in \mathcal{W}_4 \setminus \{(0,0)\}$ the entries of weight w form one pair $\{(a,b),(b^*,a^*)\}$ or one entry with $(a,b) = (b^*,a^*)$, so $\dim V_w = 1$, with D_ℓ acting by $\gamma_w = \kappa_a\kappa_b^{-1}$ (well defined, as $\kappa_i\kappa_{i^*} = \theta$); and $V_w = 0$ for $w \notin \mathcal{W}_4$. Explicitly $\gamma_w = \alpha, \Psi, \theta, \beta$ for $w = (1,0), (0,1), (1,1), (-1,1)$ (entries $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$), and $\gamma_{-w} = \gamma_w^{-1}$.

Step 3: Hodge–Tate weights. ξ is unramified at $\ell \in T$ (Sg1), and χ, ω_ℓ have finite order; so on I_ℓ, $\alpha = \chi_{\mathrm{cyc}}\eta_1$, $\Psi = \chi_{\mathrm{cyc}}^{e}\eta_2$, $\theta = \chi_{\mathrm{cyc}}^{e+1}\eta_3$, $\beta = \chi_{\mathrm{cyc}}^{e-1}\eta_4$ with η_k of finite order. Hence V_w is de Rham with all Hodge–Tate weights n_w, where $n_w = 1, e, e+1, e-1$ for the four weights of Step 2, $n_{-w} = -n_w$, and $n_{(0,0)} = 0$; as $e \geq 2$, $n_w \geq 1$ for the four positive roots and $n_w \leq -1$ for their negatives.

Step 4: the count. By Theorem B.43 with $K = \mathbb{Q}_\ell$ ($D_\ell \cong \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ through emb_ℓ), dividing $\mathbb{Q}_\ell$-dimensions by $[E:\mathbb{Q}_\ell]$, $\dim_E H^1_f(D_\ell, V_w) = \dim_E V_w^{D_\ell} + \delta_w \dim_E V_w$ with $\delta_w = 1$ if $n_w \geq 1$ and 0 otherwise. With Step 1, $\dim_E Z_w = (1+\delta_w)\dim_E V_w$: this is 2 for $w = (0,0)$, 2 for the positive roots $(1,0), (0,1), (1,1), (-1,1)$, 1 for their negatives, and 0 for $w \notin \mathcal{W}_4$, i.e. $d_\ell(w)$. □

Lemma 45.15. *Let $\mathcal{O}$ be a coefficient ring, $S \in \mathrm{CNL}_\mathcal{O}$ with maximal ideal $\mathfrak{m}_S$, and $\mathcal{E} \subseteq S$ a subset such that any two morphisms $S \to A$ of $\mathrm{CNL}_\mathcal{O}$ that agree on $\mathcal{E}$ are equal. For each $e \in \mathcal{E}$ let $c_e \in \mathcal{O}$ with $e - c_e \in \mathfrak{m}_S$. Then:*

1. *the classes of the elements $e - c_e$ ($e \in \mathcal{E}$) span the $\mathbb{F}_\mathcal{O}$-vector space $\mathfrak{m}_S/(\mathfrak{m}_S^2 + \mathfrak{m}_\mathcal{O} S)$, which is finite-dimensional;*
2. *if $e_1, \dots, e_k \in \mathcal{E}$ are such that the classes of $e_i - c_{e_i}$ span this space, the continuous $\mathcal{O}$-algebra homomorphism $\mathcal{O}[[X_1, \dots, X_k]] \to S$, $X_i \mapsto e_i - c_{e_i}$, is surjective.*

Proof. Put $t^* = \mathfrak{m}_S/(\mathfrak{m}_S^2 + \mathfrak{m}_\mathcal{O} S)$. As S is Noetherian, $\mathfrak{m}_S$ is finitely generated, so t^* is a finite-dimensional vector space over $S/\mathfrak{m}_S = \mathbb{F}_\mathcal{O}$. For $y \in S$ choose $c \in \mathcal{O}$ with $y - c \in \mathfrak{m}_S$ (possible since $\mathcal{O} \to S/\mathfrak{m}_S$ is onto) and write $\langle y \rangle \in t^*$ for the class of $y - c$; another choice of c changes $y - c$ by an element of $\mathfrak{m}_\mathcal{O} \subseteq \mathfrak{m}_\mathcal{O} S$, so $\langle y \rangle$ is well defined. The map $y \mapsto \langle y \rangle$ is $\mathcal{O}$-linear, kills $\mathcal{O}$, and satisfies $\langle yz \rangle = \bar{y}\langle z \rangle + \bar{z}\langle y \rangle$, where $\bar{y}, \bar{z} \in \mathbb{F}_\mathcal{O}$ are the residues: with $c, d \in \mathcal{O}$ for y, z,

$$yz - cd = c(z-d) + d(y-c) + (y-c)(z-d),$$

the last term lies in $\mathfrak{m}_S^2$, and $c(z-d) \equiv \bar{y}(z-d)$ in t^* because $\mathfrak{m}_\mathcal{O}$ acts on t^* by zero.

Item 1. Suppose the classes $\langle e \rangle$ ($e \in \mathcal{E}$), which are the classes of the $e - c_e$, do not span t^*. Then there is a non-zero $\mathbb{F}_\mathcal{O}$-linear form λ on t^* vanishing on all of them. Consider the ring $\mathbb{F}_\mathcal{O}[\epsilon]$ with $\epsilon^2 = 0$, an object of $\mathrm{CNL}_\mathcal{O}$, and put $f_\lambda(y) = \bar{y} + \epsilon\lambda(\langle y \rangle)$. By the properties of $\langle \cdot \rangle$, f_λ is an $\mathcal{O}$-algebra homomorphism $S \to \mathbb{F}_\mathcal{O}[\epsilon]$: it is $\mathcal{O}$-linear, $f_\lambda(1) = 1$, and $f_\lambda(yz) = \bar{y}\bar{z} + \epsilon(\bar{y}\lambda\langle z \rangle + \bar{z}\lambda\langle y \rangle) = f_\lambda(y) f_\lambda(z)$. It maps $\mathfrak{m}_S$ into $\epsilon\mathbb{F}_\mathcal{O}$, so it is local, a morphism of $\mathrm{CNL}_\mathcal{O}$. The same holds for f_0 (reduction modulo $\mathfrak{m}_S$). For $e \in \mathcal{E}$, $f_\lambda(e) = \bar{e} = f_0(e)$ since $\lambda\langle e \rangle = 0$. By the hypothesis on $\mathcal{E}$, $f_\lambda = f_0$, so $\lambda(\langle y \rangle) = 0$ for all $y \in S$; as every element of t^* is $\langle y \rangle$ for some $y \in \mathfrak{m}_S$, $\lambda = 0$, a contradiction.

Item 2. Let $P = \mathcal{O}[[X_1, \dots, X_k]]$ and $\pi\colon P \to S$ the homomorphism of the statement; it exists and is continuous because $e_i - c_{e_i} \in \mathfrak{m}_S$ and S is complete, and it is local. Let $I = \sum_i (e_i - c_{e_i})S + \mathfrak{m}_\mathcal{O} S$. By assumption $\mathfrak{m}_S = I + \mathfrak{m}_S^2$, so $\mathfrak{m}_S = I$ by Nakayama's lemma applied to the finitely generated S-module $\mathfrak{m}_S/I$. Hence $\mathfrak{m}_P S = I = \mathfrak{m}_S$, and $S/\mathfrak{m}_P S = \mathbb{F}_\mathcal{O}$ is generated by the image of 1 as a P-module. By Krull's intersection theorem, S is separated for the $\mathfrak{m}_P S = \mathfrak{m}_S$-adic topology, and P is complete; so the complete Nakayama lemma [166, Thm. 8.4] shows that S is generated by 1 as a P-module, that is, π is surjective. □

Proposition 45.16. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $p \in T$ with $p \neq \ell$, $R_p \in \mathrm{CNL}_\mathcal{O}$ and ρ_p a lift of $\bar{r}_0|_{D_p}$ to R_p satisfying the local condition at p which is universal for such lifts (to rings in $\mathrm{CNL}_\mathcal{O}$), and $x_p\colon R_p \to \mathcal{O}$ the homomorphism with $x_p \circ \rho_p = r_0|_{D_p}$. Then there are a finite set τ_p, $w\colon \tau_p \to \mathbb{Z}^2$ with $\#w^{-1}(v) = d_p(v)$ for every v, and an $\mathcal{O}$-algebra*

homomorphism $\phi_p\colon R_p \to \mathcal{O}[\ell^{-1}][[Y_i : i \in \tau_p]]$ *(*Y_i *of weight* $w(i)$*) satisfying (Yc1)–(Yc4) read with* $p, D_p, \rho_p, x_p, \phi_p$ *in place of* $\ell, D_\ell, \rho_\ell, x_\ell, \phi_\ell$*, and (Yc5) with* 10 *in place of* 14*.*

Proof. Write $E = \mathcal{O}[\ell^{-1}]$, $R = R_p$, $x = x_p$, $\mathfrak{q} = \ker x$, $T_R = \mathfrak{q}R_\mathfrak{q}/\mathfrak{q}^2R_\mathfrak{q}$, $g \cdot X = r_0(g)Xr_0(g)^{-1}$, E_{ab} for the matrix units, and $\gamma_{ab} = \kappa_a\kappa_b^{-1}$. Let $\Phi^+ = \{(1,0),(-1,1),(0,1),(1,1)\}$, $X_{(1,0)} = E_{12} - E_{34}$, $X_{(-1,1)} = E_{23}$, $X_{(0,1)} = E_{13} + E_{24}$, $X_{(1,1)} = E_{14}$, $X_{-w} = {}^tX_w$; the entries of $X_{\pm w}$ have weight $\pm w$, and $g \cdot X_{\pm w} = \gamma_w(g)^{\pm 1}X_{\pm w}$ with $\gamma_{(1,0)} = \gamma_{12} = \gamma_{34}$, $\gamma_{(-1,1)} = \gamma_{23}$, $\gamma_{(0,1)} = \gamma_{13} = \gamma_{24}$, $\gamma_{(1,1)} = \gamma_{14}$. Each $X_{\pm w}$ lies in $\mathfrak{sp}_4(E)$ and has square 0, so $1 + aX_{\pm w}$ has similitude 1 for every a in a commutative E-algebra.

With $\operatorname{diag}(x_1, x_2, -x_2, -x_1)$ these give ten linearly independent elements of $\mathfrak{sp}_4(E)$, hence a basis, each supported on entries of one weight; so $\mathfrak{sp}_4(E) = \bigoplus_{w \in \mathcal{W}_4} V_w$ with $V_{(0,0)}$ the diagonal part (trivial action) and $V_{\pm w} = EX_{\pm w}$, and the projection onto V_w (the entries of weight w) is equivariant.

Step 1: the characters at p. From $\alpha = \chi_{\mathrm{cyc}}\omega_\ell^{-1}\xi$, $\Psi = \chi\chi_{\mathrm{cyc}}^e$, $\theta = \alpha\Psi$: $\kappa_a = \chi_{\mathrm{cyc}}^{n_a}\nu_a$ with $(n_a) = (e+1, e, 1, 0)$ and ν_a of finite order. On I_p, χ_{cyc}, ω_ℓ and ξ are trivial (ξ is unramified at $p \in T$ by (Sg1)), so $\alpha = 1$ and $\Psi = \theta = \chi$ there; hence $r_0(g) = 1$ on the closed normal subgroup $N = \{g \in I_p : \chi(g) = 1\}$ of D_p, and I_p/N is finite. For $a \neq b$, $\gamma_{ab}(\sigma_p) = p^{n_a - n_b}\zeta$ with ζ a root of unity, which is $\neq 1$: otherwise $\zeta = p^{n_b - n_a}$ would be a rational root of unity, i.e. ± 1. So $\gamma_w(\sigma_p) \neq 1$ for $w \in \Phi^+$.

Step 2: cocycles vanishing on N. For $w \in \mathcal{W}_4$ let Z_w be the space of continuous cocycles $y\colon D_p \to V_w$ vanishing on N. Then $\dim_E Z_{(0,0)} \leq 2$ and $\dim_E Z_{\pm w} \leq 1$ for $w \in \Phi^+$. For $w = (0,0)$, y is a continuous homomorphism to $V_{(0,0)} \cong E^2$; it kills the finite group I_p/N (E is torsion free), hence I_p, and is determined by $y(\sigma_p)$, since I_p and σ_p generate a dense subgroup. For $V = V_{\pm w} \cong E(\gamma)$, $\gamma = \gamma_w^{\pm 1}$, let $y \in Z_{\pm w}$; we show y is a coboundary, so $Z_{\pm w} \subseteq \{g \mapsto \gamma(g)z - z : z \in V\}$. As y vanishes on N and $\gamma(N) = 1$, y factors through D_p/N; on the finite group I_p/N, with $z_0 = |I_p/N|^{-1}\sum_{h \in I_p/N} y(h)$, the cocycle identity gives $y(g) = \gamma(g)(-z_0) - (-z_0)$. Replacing y by $y - (g \mapsto \gamma(g)(-z_0) + z_0)$, which still vanishes on N, we may assume $y|_{I_p} = 0$. For $g \in D_p$ and $i \in I_p$, $y(gi) = y(g)$ and $y(gig^{-1} \cdot g) = \gamma(gig^{-1})y(g)$, so $y(g) = \gamma(i')y(g)$ for all $i' \in I_p$. If $\gamma|_{I_p} \neq 1$, $y = 0$. Otherwise, with $z_1 = (\gamma(\sigma_p) - 1)^{-1}y(\sigma_p)$ (Step 1), $y' = y - (g \mapsto \gamma(g)z_1 - z_1)$ vanishes on I_p and at σ_p, hence on the subgroup they generate (cocycle identity), hence everywhere by continuity. In both cases y is a coboundary.

Step 3: $\dim_E T_R \leq 10$. Let $\lambda\colon T_R \to E$ be E-linear and $F_\lambda(a) = x(a) + \epsilon\lambda(a - x(a))$, $a \in R$, with values in $E[\epsilon] = E[X]/(X^2)$; it is an $\mathcal{O}$-algebra homomorphism, as $ab - x(a)x(b) \equiv x(a)(b - x(b)) + x(b)(a - x(a))$ modulo $\mathfrak{q}^2$. If $\mathfrak{q} = \sum_j Rz_j$ (finitely many z_j, R Noetherian), then $\lambda(\mathfrak{q}) = \sum_j \mathcal{O}\lambda(z_j)$, since $rz_j \equiv x(r)z_j$ modulo $\mathfrak{q}^2$; so $F_\lambda(R) \subseteq A_\lambda = \mathcal{O} + \epsilon\sum_j \mathcal{O}\lambda(z_j)$, which lies in $\mathrm{CNL}_\mathcal{O}$ by Lemma 45.12 ($n = 1$, $k = 2$), and $F_\lambda\colon R \to A_\lambda$ is local, hence continuous. So $F_\lambda \circ \rho_p$ is a lift of $\bar{r}_0|_{D_p}$ to A_λ satisfying the local condition at p, and $F_\lambda(\rho_p(g)) = (1 + \epsilon y_\lambda(g))r_0(g)$ defines a continuous cocycle y_λ, linear in λ. Both $F_\lambda(\rho_p(g))$ and $r_0(g)$ have similitude $\theta(g)$, so ${}^ty_\lambda(g)J_4 + J_4y_\lambda(g) = 0$ and y_λ is $\mathfrak{sp}_4(E)$-valued; it vanishes on N, where ρ_p and r_0 are 1. So its weight-w parts lie in Z_w. If they all vanish, $F_\lambda \circ \rho_p = r_0|_{D_p} = F_0 \circ \rho_p$, and the uniqueness in the universal property (with $A = A_\lambda$) gives $F_\lambda = F_0$, i.e. $\lambda = 0$. By Step 2, $\dim_E T_R \leq 2 + 8 = 10$.

Step 4: a weighted ten-parameter family. Let $\mathrm{fr}\colon D_p \to \widehat{\mathbb{Z}} \to \mathbb{Z}_\ell$ be the continuous homomorphism trivial on I_p with $\mathrm{fr}(\sigma_p) = 1$, and $\lambda_i(g) = (1 + T_i)^{\mathrm{fr}(g)} = \sum_j \binom{\mathrm{fr}(g)}{j} T_i^j$ ($i = 1, 2$), continuous characters $D_p \to E[[T_1, \ldots, T_{10}]]^\times$, trivial on I_p, congruent to $1 + \mathrm{fr}(g)T_i$ modulo degree 2 (the binomial coefficients are $\mathbb{Z}_\ell$-valued polynomials and satisfy Vandermonde's identity). Put $d(g) = \operatorname{diag}(\kappa_1\lambda_1, \kappa_2\lambda_2, \kappa_3\lambda_2^{-1}, \kappa_4\lambda_1^{-1})(g)$, of similitude $\theta(g)$ ($\kappa_1\kappa_4 = \kappa_2\kappa_3 = \theta$). Number $T_3, \ldots, T_6$ as T'_w and $T_7, \ldots, T_{10}$ as T''_w, $w \in \Phi^+$, and let $h = \prod_w (1 + T'_wX_{-w}) \prod_w (1 + T''_wX_w)$, of similitude 1 and $\equiv 1$ modulo the maximal ideal, and $r_T(g) = h\,d(g)\,h^{-1}$. (Fm1) holds;

(Fm2): $r_T(g)$ has similitude $\theta(g)$, and for $g \in N$, $d(g) = r_0(g) = 1$, so $r_T(g) = 1$. (Fm3): modulo degree 2, $r_T(g)r_0(g)^{-1} \equiv 1 + \mathrm{fr}(g)(T_1(E_{11} - E_{44}) + T_2(E_{22} - E_{33})) + H_1 - g \cdot H_1$ with $H_1 = \sum_w (T'_w X_{-w} + T''_w X_w)$; so $y_1, y_2 = \mathrm{fr}\,(E_{11} - E_{44}), \mathrm{fr}\,(E_{22} - E_{33})$, the y of T'_w is $(1 - \gamma_w^{-1})X_{-w}$ and that of T''_w is $(1 - \gamma_w)X_w$. The supports being disjoint, evaluation at σ_p ($\mathrm{fr}(\sigma_p) = 1$, $\gamma_w(\sigma_p) \neq 1$) shows that they are linearly independent. By Lemma 45.11 there is $F\colon R \to E[[T_1, \ldots, T_{10}]]$ with $F(\rho_p(g)_{ab}) = r_T(g)_{ab}$, $F(f) - x(f)$ in the maximal ideal, and the linear parts of the $F(\rho_p(g)_{ab} - r_0(g)_{ab})$ spanning E^{10}. By Step 3 and Lemma 45.13 (2), $R_{\mathfrak{q}}$ is regular of dimension 10, R has exactly one minimal prime inside $\mathfrak{q}$, and $F(f) \in \mathfrak{m}_T^N$ implies $cf \in \mathfrak{q}^N$ for some $c \in \mathcal{O} \setminus \{0\}$.

Step 5: weights. Give T_1, T_2 weight $(0,0)$, T'_w weight $-w$ and T''_w weight w. For $t = (t_1, t_2) \in (\mathcal{O}^\times)^2$ let ς_t be the continuous E-automorphism of $E[[T]]$ multiplying each variable of weight u by $t^u = t_1^{u_1} t_2^{u_2}$. Since $\mathrm{T}_4(t_1,t_2)X_{\pm w}\mathrm{T}_4(t_1,t_2)^{-1} = t^{\pm w}X_{\pm w}$ and $d(g)$ is diagonal, $\varsigma_t(r_T(g)) = \mathrm{T}_4(t_1,t_2)r_T(g)\mathrm{T}_4(t_1,t_2)^{-1}$, so $\varsigma_t(r_T(g)_{ab}) = t^{\mathrm{wt}_4(a,b)} r_T(g)_{ab}$. As ς_t multiplies the coefficient of a monomial of weight u by t^u, taking $t = (1+\ell, 1)$ and $(1, 1+\ell)$ shows that $r_T(g)_{ab}$ has weight $\mathrm{wt}_4(a,b)$, and so has $F(\rho_p(g)_{ab} - r_0(g)_{ab})$ ($r_0(g)_{ab} = 0$ for $a \neq b$). Hence the linear part of $F(\rho_p(g)_{ab} - r_0(g)_{ab})$ involves only variables of weight $\mathrm{wt}_4(a,b)$, and for each weight u the linear parts of those of weight u span the coordinates of weight u. Choose $f_1, \ldots, f_{10}$ among these elements, with $F(f_i)$ of the weight of T_i and $\mathrm{L}(F(f_i))$ a basis of E^{10} (L the linear coefficients).

Step 6: the chart. The continuous E-algebra endomorphism ϑ of $E[[T]]$ with $\vartheta(T_i) = F(f_i)$ has invertible linear part, so it is an automorphism; it maps series of weight u to series of weight u, hence so does ϑ^{-1} (decompose into weight components), and both preserve constant terms and each power of the maximal ideal. Put $\tau_p = \{1, \ldots, 10\}$, $Y_i = T_i$ with its weight $w(i)$, and $\phi_p = \vartheta^{-1} \circ F$. Then $\#w^{-1}(u) = 2$ for $u = (0,0)$, 1 for $u \in \mathcal{W}_4 \setminus \{(0,0)\}$ and 0 otherwise, i.e. $d_p(u)$. (Yc1): $\phi_p(\rho_p(g)_{ab}) = \vartheta^{-1}(r_T(g)_{ab})$ has weight $\mathrm{wt}_4(a,b)$. (Yc2): the constant term of $\phi_p(f)$ is that of $F(f)$, i.e. $x(f)$. (Yc3): $f_i \in \mathfrak{q}$ and $\phi_p(f_i) = Y_i$. (Yc4): $\phi_p(f) \in \mathfrak{m}_Y^N$ iff $F(f) \in \mathfrak{m}_Y^N$, and Step 4 applies. (Yc5) with 10: Step 4. □

Proposition 45.17. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$. For each $v \in T$ let (R_v, ρ_v) be universal for the lifts of $\bar{r}_0|_{D_v}$ satisfying the local condition at v, x_v its homomorphism with $x_v \circ \rho_v = r_0|_{D_v}$, and (τ_v, w_v, ϕ_v) data satisfying (Yc1)–(Yc4) read at v. Let $B = \widehat{\bigotimes}_{v\in T} R_v$ (completed tensor product over $\mathcal{O}$), σ a finite set of triples $s = (v_s, g_s, (a_s, b_s))$ with $g_s \in D_{v_s}$, $\mathrm{wt}(s) = \mathrm{wt}_4(a_s, b_s)$, and $\pi\colon \mathcal{O}[[X_s : s \in \sigma]] \to B$ the continuous $\mathcal{O}$-algebra homomorphism with $\pi(X_s) = \rho_{v_s}(g_s)_{a_s b_s} - r_0(g_s)_{a_s b_s}$; suppose that π is surjective and $\mathfrak{a} = \ker\pi$ is homogeneous with constant terms 0. Then $B = \mathcal{O}[[X_s : s \in \sigma]]/\mathfrak{a}$ is smooth at the origin with tangent dimensions $\sum_{v\in T} t_v$, where $t_v(u) = \#w_v^{-1}(u)$.*

Proof. Write $E = \mathcal{O}[\ell^{-1}]$, $\iota_v\colon R_v \to B$ for the canonical maps (the lifts ρ_v to B being $\iota_v \circ \rho_v$), $\mathfrak{q}_v = \ker x_v$, $P = \mathcal{O}[[X_s : s \in \sigma]]$ with the weights wt, $J = \bigsqcup_v \tau_v$ with weights $w(i) = w_v(i)$ for $i \in \tau_v$, $n = \#J$, $E[[Y]] = E[[Y_i : i \in J]] \supseteq E[[Y_i : i \in \tau_v]]$ with maximal ideal $\mathfrak{m}_Y$, $E_k = E[[Y]]/\mathfrak{m}_Y^k$, and L for the vector of coefficients of the Y_i.

Step 1: the base point. Let $x_B\colon B \to \mathcal{O}$ be the morphism with $x_B \circ \iota_v = x_v$ for all v. Then $x_B(\pi(X_s)) = x_{v_s}(\rho_{v_s}(g_s)_{a_s b_s}) - r_0(g_s)_{a_s b_s} = 0$, so $x_B \circ \pi$ and the constant term are continuous homomorphisms $P \to \mathcal{O}$ agreeing on the X_s, hence equal: x_B is the origin x_R of $B = P/\mathfrak{a}$. Its kernel is $\mathfrak{q} = \pi((X_s)_s) = \sum_s B\,\pi(X_s)$, and $\iota_v(\mathfrak{q}_v) \subseteq \mathfrak{q}$.

Step 2: the charts assemble into $F\colon B \to E[[Y]]$. Fix $k \geq 1$. By (Yc2), $\phi_v(\mathfrak{q}_v^k) \subseteq \mathfrak{m}_Y^k$, so the image $A_{v,k}$ of R_v in E_k is a quotient of $R_v/\mathfrak{q}_v^k$, a finitely generated $\mathcal{O}$-module ($R_v/\mathfrak{q}_v \cong \mathcal{O}$, and each $\mathfrak{q}_v^j/\mathfrak{q}_v^{j+1}$ is a finitely generated $R_v/\mathfrak{q}_v$-module); it contains $\mathcal{O}$ and lies in $\mathcal{O} + \mathfrak{m}_Y E_k$ (Yc2). The $\mathcal{O}$-span A_k of the products $\prod_v a_v$, $a_v \in A_{v,k}$, is a subring of E_k with the same three

properties. By Lemma 45.12, $A_k \in \mathrm{CNL}_{\mathcal{O}}$ and the maps $R_v \to A_k$ induced by ϕ_v are morphisms; let $F_k\colon B \to A_k$ be the morphism with $F_k \circ \iota_v = \phi_v \bmod \mathfrak{m}_Y^k$ for all v. The projection $E_{k+1} \to E_k$ maps A_{k+1} into A_k, and its composite with F_{k+1} has the defining property of F_k, so equals F_k. Hence the F_k define an $\mathcal{O}$-algebra homomorphism $F\colon B \to E[[Y]]$ with $F \circ \iota_v = \phi_v$. For $k = 1$, $A_1 = \mathcal{O}$ and $F_1 \circ \iota_v = x_v$, so $F_1 = x_B$: the constant term of $F(f)$ is $x_B(f)$, $F(\mathfrak{q}) \subseteq \mathfrak{m}_Y$ and $F(\mathfrak{q}^N) \subseteq \mathfrak{m}_Y^N$.

Step 3: $F \circ \pi$ is a substitution and respects weights. The element $F(\pi(X_s)) = \phi_{v_s}(\rho_{v_s}(g_s)_{a_s b_s}) - r_0(g_s)_{a_s b_s}$ lies in $\mathfrak{m}_Y$ and has weight $\mathrm{wt}(s)$ by (Yc1) (the constant $r_0(g_s)_{a_s b_s}$ is non-zero only when $a_s = b_s$, of weight $(0,0)$). For $h \in P$ and $N \geq 1$ write $h = h_N + h'_N$ with h_N a polynomial and $h'_N \in (X)^N$; then $\pi(h'_N) \in \mathfrak{q}^N$, so $F(\pi(h)) \equiv F(\pi(h_N))$ modulo $\mathfrak{m}_Y^N$, which is the substitution $X_s \mapsto F(\pi(X_s))$ applied to h_N, hence to h, modulo $\mathfrak{m}_Y^N$. So $F \circ \pi$ is that substitution; it maps series of weight u to series of weight u, and $F(\pi(h)) = \sum_u F(\pi(h_{(u)}))$ with $F(\pi(h_{(u)}))$ the weight-u component of $F(\pi(h))$.

Step 4: the cotangent space of B at $\mathfrak{q}$ has dimension $\leq n$. For $v \in T$ let $T_v = \mathfrak{q}_v R_{v,\mathfrak{q}_v} / \mathfrak{q}_v^2 R_{v,\mathfrak{q}_v}$. The map ϕ_v extends to $R_{v,\mathfrak{q}_v}$ (elements outside $\mathfrak{q}_v$ go to units, by (Yc2)) and induces an E-linear map $T_v \to \mathfrak{m}_Y/\mathfrak{m}_Y^2$, $f \mapsto \mathrm{L}(\phi_v(f))$, onto E^{τ_v} by (Yc3) and injective by (Yc4) (if $f \in \mathfrak{q}_v$ and $\phi_v(f) \in \mathfrak{m}_Y^2$ then $cf \in \mathfrak{q}_v^2$ with $c \neq 0$ invertible in $R_{v,\mathfrak{q}_v}$); so $\dim_E T_v = \#\tau_v$. Let $\lambda\colon \mathfrak{q}B_{\mathfrak{q}}/\mathfrak{q}^2 B_{\mathfrak{q}} \to E$ be E-linear and $D(f) = \lambda(f - x_B(f))$; then D is $\mathcal{O}$-linear with $D(fh) = x_B(f)D(h) + x_B(h)D(f)$, and so is $D \circ \iota_v$ for x_v; it kills $\mathfrak{q}_v^2$ and $\mathcal{O}$, so it is given by an E-linear form λ_v on T_v. If all λ_v vanish, D vanishes on the generators $\pi(X_s) \in \iota_{v_s}(\mathfrak{q}_{v_s})$ of $\mathfrak{q}$, hence on $\mathfrak{q}$ ($D(bz) = x_B(b)D(z)$ for $z \in \mathfrak{q}$), and $\lambda = 0$. So $\lambda \mapsto (\lambda_v)_v$ is injective and $\dim_E \mathfrak{q}B_{\mathfrak{q}}/\mathfrak{q}^2 B_{\mathfrak{q}} \leq \sum_v \#\tau_v = n$.

Step 5: regularity. For $i \in \tau_v$, (Yc3) gives $f \in \mathfrak{q}_v$ with $\phi_v(f) \equiv Y_i$ modulo $\mathfrak{m}_Y^2$; then $\iota_v(f) \in \mathfrak{q}$ and $F(\iota_v(f)) \equiv Y_i$. So the linear parts of $F(\mathfrak{q})$ span E^n, and by Steps 2 and 4 and Lemma 45.13 (2), for every N and every $f \in B$ with $F(f) \in \mathfrak{m}_Y^N$ there is $c \in \mathcal{O} \setminus \{0\}$ with $cf \in \mathfrak{q}^N$.

Step 6: the chart. Let ϕ be the restriction of F to B^{gr}, with variables Y_i of weights $w(i)$; $\#w^{-1}(u) = \sum_v t_v(u)$. (Sm1): for $f = \pi(h) \in B_u$ with $h \in P_u$, $\phi(f)$ has weight u by Step 3. (Sm2): Steps 1 and 2. (Sm3): for $i \in \tau_v$ take f as in Step 5 and $h \in P$ with $\pi(h) = \iota_v(f)$; by Step 3, $F(\pi(h_{(w(i))}))$ is the weight-$w(i)$ component of $F(\iota_v(f)) \equiv Y_i$ modulo $\mathfrak{m}_Y^2$, so it is $\equiv Y_i$, and $\pi(h_{(w(i))}) \in B_{w(i)} \subseteq B^{\mathrm{gr}}$. (Sm4): let $f = \sum_u f_u \in B^{\mathrm{gr}}$ ($f_u \in B_u$, finitely many) with $x_B(f) = 0$ and all coefficients of degree one of $\phi(f)$ zero. By (Sm1) the $\phi(f_u)$ are the weight components of $\phi(f)$, so each has no linear terms, and no constant term (for $u \neq (0,0)$ by weight, for $u = (0,0)$ as $x_B(f) = 0$); so $\phi(f_u) \in \mathfrak{m}_Y^2$, and by Step 5 there is $c_u \neq 0$ with $c_u f_u \in \mathfrak{q}^2 = \pi((X)^2)$. Write $f_u = \pi(h)$ with $h \in P_u$ and $c_u f_u = \pi(q)$ with $q \in (X)^2$; then $c_u h - q \in \mathfrak{a}$, and as $\mathfrak{a}$ is homogeneous, $c_u h - q_{(u)} \in \mathfrak{a}$, so $c_u f_u = \pi(q_{(u)})$ with $q_{(u)} \in (X)^2 \cap P_u$. Fixing an order on σ and assigning each monomial of $q_{(u)}$ to its two smallest variables gives $q_{(u)} = \sum_{s \leq s'} X_s X_{s'} q_{ss'}$ with $q_{ss'} \in P_{u - \mathrm{wt}(s) - \mathrm{wt}(s')}$. Hence $c_u f_u = \sum_{s \leq s'} \pi(X_s) \cdot \pi(X_{s'}) \pi(q_{ss'})$, a sum of products of two elements of $\mathfrak{m}_0 = \ker x_B \cap B^{\mathrm{gr}}$, and $cf \in \mathfrak{m}_0^2$ for $c = \prod_u c_u$: $f \in \mathrm{Sq}$.

So $B = P/\mathfrak{a}$ is smooth at the origin with tangent dimensions $\sum_v t_v$. □

Lemma 45.18. *Let $\mathcal{O}$ be a coefficient ring, $d \geq 1$, $R \in \mathrm{CNL}_{\mathcal{O}}$ with maximal ideal $\mathfrak{m}_R$, and $\rho\colon R \to R[[T_1, \ldots, T_d]]$ a morphism of $\mathrm{CNL}_{\mathcal{O}}$; an element $y \in R$ has* torus weight $w = (w_1, \ldots, w_d) \in \mathbb{Z}^d$ *if $\rho(y) = \prod_{k=1}^d (1 + T_k)^{w_k} y$. Let σ be a finite set,* $\mathrm{wt}\colon \sigma \to \mathbb{Z}^d$ *a map, and $\pi\colon \mathcal{O}[[X_s : s \in \sigma]] \to R$ a surjective continuous $\mathcal{O}$-algebra homomorphism such that each $\pi(X_s)$ lies in $\mathfrak{m}_R$ and has torus weight $\mathrm{wt}(s)$. Then:*

1. *for every $f \in \ker \pi$ and $w \in \mathbb{Z}^d$, $f_{(w)} \in \ker \pi$;*
2. *every $y \in R$ of torus weight w lies in $\pi(\mathcal{O}[[X_\sigma]]_w)$;*
3. *$\ker \pi$ is generated by finitely many elements each lying in some $\mathcal{O}[[X_\sigma]]_w$.*

Proof. Write $P = \mathcal{O}[[X_s : s \in \sigma]]$, $\mathfrak{m}_P = (\mathfrak{m}_\mathcal{O}, X_s)$, $J = \ker \pi$, and for $n \in \mathbb{N}^\sigma$, $X^n = \prod_s X_s^{n_s}$, $|n| = \sum_s n_s$, $\mathrm{wt}(n) = \sum_s n_s \mathrm{wt}(s) \in \mathbb{Z}^d$; for $v \in \mathbb{Z}^d$ let $|v|_\infty = \max_k |v_k|$. Let $W = \max(1, \max_s |\mathrm{wt}(s)|_\infty)$, so $|\mathrm{wt}(n)|_\infty \le W|n|$. For $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $v \in \mathbb{Z}^d$ put $\binom{v}{k} = \prod_i \binom{v_i}{k_i} \in \mathbb{Z}$ and $T^k = \prod_i T_i^{k_i}$, so that $\prod_i (1+T_i)^{v_i} = \sum_k \binom{v}{k} T^k$ in $\mathbb{Z}[[T_1, \dots, T_d]]$.

Step 1: divided Euler operators. Let $E_k \colon P \to P$ be the $\mathcal{O}$-linear map

$$\sum_n c_n X^n \mapsto \sum_n \binom{\mathrm{wt}(n)}{k} c_n X^n,$$

and $\rho_P \colon P \to P[[T_1, \dots, T_d]]$ the continuous $\mathcal{O}$-algebra homomorphism with $\rho_P(X_s) = \prod_i (1+T_i)^{\mathrm{wt}(s)_i} X_s$. Then $\rho_P(f) = \sum_k E_k(f) T^k$. With π_T applying π to coefficients, the continuous homomorphisms $\rho \circ \pi$ and $\pi_T \circ \rho_P$ agree on the X_s, hence on P:

$$\rho(\pi(f)) = \sum_{k \in \mathbb{N}^d} \pi(E_k f)\, T^k \qquad (f \in P).$$

So $E_k(J) \subseteq J$, and if $\pi(f) = y$ has torus weight w, then $\pi(E_k f) = \binom{w}{k} y$.

Step 2: approximate projectors. Fix w and $M \ge |w|_\infty$, put $Q = WM$ and, for each i, $L_i(x) = \prod_{-Q \le m \le Q,\, m \ne w_i} \frac{x-m}{w_i - m} = \binom{x+Q}{w_i+Q}\binom{Q-x}{Q-w_i}$, an integer-valued polynomial, so $L_i(x) = \sum_{k \le 2Q} a_{i,k} \binom{x}{k}$ with $a_{i,k} \in \mathbb{Z}$. Then $L(v) = \prod_i L_i(v_i) = \sum_k a_k \binom{v}{k}$ with $a_k = \prod_i a_{i,k_i}$, $L(w) = 1$, and $L(v) = 0$ for $v \ne w$ with $|v|_\infty \le Q$. Put $\Pi = \sum_k a_k E_k$. For $f = \sum_n c_n X^n$, $\Pi(f) = f_{(w)} + r$ with $r = \sum_{|\mathrm{wt}(n)|_\infty > Q} L(\mathrm{wt}(n)) c_n X^n$, and every monomial of r has $|n| \ge |\mathrm{wt}(n)|_\infty / W > M$, so $r \in \mathfrak{m}_P^{M+1}$.

Step 3: items 1 and 2. If $f \in J$, then $\Pi(f) \in J$, so $f_{(w)} \in J + \mathfrak{m}_P^{M+1}$ for every $M \ge |w|_\infty$, and $f_{(w)} \in J$ by Krull's intersection theorem in P/J. If y has torus weight w and $\pi(f) = y$, then $\pi(\Pi f) = \sum_k a_k \binom{w}{k} y = L(w) y = y$, so $y - \pi(f_{(w)}) = \pi(r) \in \mathfrak{m}_R^{M+1}$ for every M, and $y = \pi(f_{(w)})$.

Step 4: item 3. Every $f \in J$ is the $\mathfrak{m}_P$-adic limit of the finite sums $\sum_{|v|_\infty \le WM} f_{(v)}$, whose terms lie in J by item 1, and ideals of the Noetherian ring P are closed; so J is generated by the $f_{(v)}$, $f \in J$, and finitely many of them suffice. □

45.2 Graded relative presentations and the assembly of weighted models

This subsection proves that every symplectic local base carries a graded framed relative presentation with Euler-characteristic counts (Proposition 45.20), and that such a presentation gives a symplectic weighted model (Proposition 45.23). The presentation is the graded form of Mazur's: over the completed tensor product of the local rings, the framed Selmer cocycles of each weight are the variables and the framed obstruction groups bound the relations (Definition 45.19, Proposition 45.21). The count that replaces Lemma 43.34 is Lemma 45.22: the dimension of the framed obstruction group exceeds that of the framed Selmer cocycles by the dimensions of the invariants of complex conjugation and of the global invariants of the dual module. The assembly cuts a smooth weighted cover by weighted elements with independent point derivations (Proposition 45.24). The subsection also defines the ordinary test representations (Definition 45.26), the representations over valuation rings through which Part IX describes the points of a symplectic datum.

Definition 45.19. Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring with residue field $\mathbb{F} = \mathbb{F}_\mathcal{O}$ of characteristic ℓ, and $d, n \ge 1$ integers; indices a, b run over $\{1, \dots, n\}$.

Graded form data. A **graded form datum** of rank n with weights in $\mathbb{Z}^d$ is a tuple $\Phi = (J, K, \beta, s, w, c_\Phi)$ of a matrix $J \in \mathrm{GL}_n(\mathcal{O})$ with ${}^tJ = J$ or ${}^tJ = -J$, a finite set K (the

blocks) with a map $\beta\colon \{1,\dots,n\} \to K$ and an involution s of K, a map $w\colon \{1,\dots,n\} \to \mathbb{Z}^d$ (the *weights*) and an element $c_\Phi \in \mathbb{Z}^d$, such that

$$J_{ab} \neq 0 \implies \beta(b) = s(\beta(a)) \quad \text{and} \quad w(a) + w(b) = c_\Phi.$$

For a commutative $\mathcal{O}$-algebra A, a matrix $X \in \mathrm{M}_n(A)$ **has similitude** $\nu \in A$ if ${}^tXJX = \nu J$; it is **block diagonal** if $X_{ab} = 0$ whenever $\beta(a) \neq \beta(b)$, and **weight diagonal** if $X_{ab} = 0$ whenever $w(a) \neq w(b)$. Put

$$\mathfrak{g}_\Phi(A) = \{Y \in \mathrm{M}_n(A) \text{ block diagonal} : \ {}^tYJ + JY = 0\},$$
$$\mathfrak{g}_{\Phi,j}(A) = \{Y \in \mathfrak{g}_\Phi(A) : \ Y_{ab} = 0 \text{ unless } w(a) - w(b) = j\} \quad (j \in \mathbb{Z}^d),$$

and, for $A \in \mathrm{CNL}_\mathcal{O}$ with maximal ideal $\mathfrak{m}_A$, let $\mathrm{H}_\Phi(A)$ be the set of block diagonal $h \in \mathrm{M}_n(A)$ of similitude 1 with $h \equiv 1 \pmod{\mathfrak{m}_A}$.

Graded framed deformation problems. A **graded framed deformation problem** over $\mathcal{O}$ is a tuple

$$\mathfrak{P} = \big(\Phi, \Gamma, \bar{r}, r_0, T, v_0, (\Gamma_v, \varphi_v)_{v\in T}, Q, (\Gamma_q, \psi_q, N_q)_{q\in Q}\big)$$

of a graded form datum Φ of rank n with weights in $\mathbb{Z}^d$, a profinite group Γ, continuous homomorphisms $\bar{r}\colon \Gamma \to \mathrm{GL}_n(\mathbb{F})$ and $r_0\colon \Gamma \to \mathrm{GL}_n(\mathcal{O})$ with $r_0 \equiv \bar{r} \pmod{\mathfrak{m}_\mathcal{O}}$, finite sets T and Q with $v_0 \in T$, profinite groups Γ_v ($v \in T$) and Γ_q ($q \in Q$), continuous homomorphisms $\varphi_v\colon \Gamma_v \to \Gamma$ and $\psi_q\colon \Gamma_q \to \Gamma$, and closed normal subgroups $N_q \subseteq \Gamma_q$, such that

(Gq1) $\mathrm{Hom}_{\mathrm{cont}}(\Gamma', \mathbb{F}_\ell)$ is finite for every open subgroup $\Gamma' \subseteq \Gamma$;

(Gq2) there is a homomorphism $g \mapsto \pi_g$ from Γ to the group of permutations of K such that for every $g \in \Gamma$ the matrix $r_0(g)$ is weight diagonal, has a similitude in $\mathcal{O}^\times$, and $r_0(g)_{ab} \neq 0$ implies $\beta(a) = \pi_g(\beta(b))$;

(Gq3) for every $q \in Q$, $r_0(\psi_q(x)) = 1$ for all $x \in N_q$, and $H^2(\Gamma_q/N_q, M) = 0$ for every finite-dimensional $\mathbb{F}$-vector space M with a continuous $\mathbb{F}$-linear action of Γ_q/N_q (M discrete).

Bases and framed lifts. A **base** for $\mathfrak{P}$ is a pair $(B, (\rho_v)_{v\in T})$ of $B \in \mathrm{CNL}_\mathcal{O}$ and continuous homomorphisms $\rho_v\colon \Gamma_v \to \mathrm{GL}_n(B)$ with $\rho_v(g)\, r_0(\varphi_v(g))^{-1} \in \mathrm{H}_\Phi(B)$ for all $v \in T$ and $g \in \Gamma_v$. A **framed lift** over the base $(B, (\rho_v))$ to $A \in \mathrm{CNL}_\mathcal{O}$ is a triple $(\iota, r, (\alpha_v)_{v\in T})$ of a morphism $\iota\colon B \to A$ of $\mathrm{CNL}_\mathcal{O}$, a continuous homomorphism $r\colon \Gamma \to \mathrm{GL}_n(A)$ and matrices $\alpha_v \in \mathrm{M}_n(A)$ such that

(Gq4) $r(g)\, r_0(g)^{-1} \in \mathrm{H}_\Phi(A)$ for every $g \in \Gamma$;

(Gq5) $r(\psi_q(x)) = 1$ for every $q \in Q$ and $x \in N_q$;

(Gq6) $\alpha_v \in \mathrm{H}_\Phi(A)$ for every $v \in T$, $\alpha_{v_0} = 1$, and $r(\varphi_v(g))\, \alpha_v = \alpha_v\, \iota(\rho_v(g))$ for all $v \in T$ and $g \in \Gamma_v$.

Here $r_0(g)$ is read in $\mathrm{GL}_n(A)$ through $\mathcal{O} \to A$, and ι is applied entrywise.

Framed Selmer cocycles and framed obstruction groups. Let M be a finite-dimensional $\mathbb{F}$-vector space with a continuous $\mathbb{F}$-linear action of Γ (M discrete) on which $\psi_q(N_q)$ acts trivially for every $q \in Q$; Γ_v and Γ_q act on M through φ_v and ψ_q, φ_v^* and ψ_q^* denote the

pull-back of cochains, and for $y \in M$, $dy \in C^1(\Gamma_v, M)$ is $g \mapsto gy - y$. The **framed Selmer cocycles** of M form the $\mathbb{F}$-vector space

$$Z^1_{\mathrm{fr}}(M) = \{(x, (x_v)_{v\in T}) \in Z^1(\Gamma, M) \times M^T : \\ x(\psi_q(y)) = 0 \ (q \in Q,\ y \in N_q),\ \varphi_v^* x = dx_v \ (v \in T),\ x_{v_0} = 0\}.$$

The **framed obstruction group** $\mathrm{Ob}_{\mathrm{fr}}(M)$ is the quotient of

$$Z^2_{\mathrm{fr}}(M) = \{(k, (k_v)_{v\in T}, (k_q)_{q\in Q}) :\ k \in Z^2(\Gamma, M),\ k_v \in C^1(\Gamma_v, M), \\ k_q \in C^1(\Gamma_q, M),\ \varphi_v^* k = dk_v,\ \psi_q^* k = dk_q\}$$

by its subspace

$$B^2_{\mathrm{fr}}(M) = \{(dh, (\varphi_v^* h - dy_v)_{v\in T}, (\psi_q^* h - e_q)_{q\in Q}) :\ h \in C^1(\Gamma, M), \\ y_v \in M,\ e_q \in Z^1(\Gamma_q, M),\ e_q(N_q) = 0\};$$

dim denotes $\dim_{\mathbb{F}} \in \mathbb{N} \cup \{\infty\}$.

Proposition 45.20. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum and L a symplectic local base of $\mathfrak{D}$. Then there is a framed relative presentation over L.*

Proof. Write $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, $G_S = G_{\mathbb{Q},S}$ (ramification allowed at S and ∞), $Q = S \setminus T$ ($T \subseteq S$ since $T = \{\ell\} \cup S_\ell(\chi)$ and the primes of $S_\ell(\chi)$ divide N), $\bar r_0$ the reduction of r_0, and for $q \in Q$ let $N_q = I_q \cap \ker \bar r_0$, a closed normal subgroup of D_q.

Step 1: the problem. The representation r_0 is unramified outside S ((Sg1): ξ is, χ has modulus N, χ_{cyc} and ω_ℓ ramify only at ℓ), so r_0 and $\bar r_0$ factor through G_S. Let $\Phi = (J_4, \{*\}, \beta, \mathrm{id}, w, (1,1))$ with $w(a) = (\lambda_a, \mu_a)$, i.e. $w(1) = (1,1)$, $w(2) = (0,1)$, $w(3) = (1,0)$, $w(4) = (0,0)$: ${}^t J_4 = -J_4$, and $(J_4)_{ab} \neq 0$ only for $\{a,b\} = \{1,4\}, \{2,3\}$, where $w(a) + w(b) = (1,1)$; so Φ is a graded form datum with one block, and $w(a) - w(b) = \mathrm{wt}_4(a,b)$. Let $\mathfrak{P} = (\Phi, G_S, \bar r_0, r_0, T, \ell, (D_v)_{v\in T}, Q, (D_q, N_q)_{q\in Q})$, the groups mapping to G_S through $G_{\mathbb{Q}}$. (Gq1) is Theorem B.71, item 2. (Gq2): $r_0(g)$ is diagonal of similitude $\theta(g) \in \mathcal{O}^\times$. (Gq3): let $q \in Q$. Then $q \neq \ell$, and $\alpha|_{I_q} = \xi|_{I_q}$ (χ_{cyc}, ω_ℓ are unramified at q) has order prime to ℓ; $\Psi|_{I_q} = \chi|_{I_q} = \chi'|_{I_q}$ because $q \notin S_\ell(\chi)$ (either $q \nmid N$, or the q-component of χ'' is trivial), of order prime to ℓ; hence each diagonal entry of $r_0|_{I_q}$ ($\theta = \alpha\Psi$) has values in the roots of unity of order prime to ℓ of $\mathcal{O}^\times$, which inject into $\mathbb{F}^\times$, and r_0 is trivial on N_q. The same injectivity shows $I_q/N_q \hookrightarrow \bar r_0(I_q)$, a group of diagonal matrices over $\mathbb{F}$, finite of order prime to ℓ; as $D_q/I_q \cong \widehat{\mathbb{Z}}$ has cohomological dimension 1 on finite ℓ-primary modules, the Hochschild–Serre spectral sequence gives $H^2(D_q/N_q, M) \cong H^2(D_q/I_q, M^{I_q}) = 0$ for every finite-dimensional $\mathbb{F}$-vector space M with a continuous action [191]. So $\mathfrak{P}$ is a graded framed deformation problem.

Step 2: the base and the framed lifts. Let $L = (\sigma_L, \mathrm{wt}_L, \mathfrak{a}_L, (\rho_v))$. Each ρ_v is a continuous homomorphism $D_v \to \mathrm{GL}_4(B_L)$ lifting $\bar r_0|_{D_v}$ with $\rho_v(g)$ of similitude $\theta(g)$ (local condition at v), so $\rho_v(g) r_0(g)^{-1}$ has similitude 1 (similitudes multiply: ${}^t(XY) J_4 XY = \nu_X \nu_Y J_4$) and is $\equiv 1$: it lies in $\mathrm{H}_\Phi(B_L)$, and $(B_L, (\rho_v))$ is a base. The ideal $\mathfrak{a}_L$ is homogeneous with constant terms 0, $\rho_v(g)_{ab} \in (B_L)_{\mathrm{wt}_4(a,b)}$ and $x_L(\rho_v(g)) = r_0(g)$ by (Bq2). For $A \in \mathrm{CNL}_{\mathcal{O}}$, $(A, \iota, r, (\alpha_v))$ is a framed lift over L in the sense of Definition 45.3 if and only if r factors through G_S and $(\iota, r, (\alpha_v))$ is a framed lift over $(B_L, (\rho_v))$ for $\mathfrak{P}$. Indeed: (Gq4) says that $r(g)$ has similitude $\theta(g)$ and $r \equiv \bar r_0$ (one block); (Fw2) together with the last clause of (Fw3) is (Gq6) with $v_0 = \ell$; and (Se2) is equivalent to: r is trivial on the inertia groups at places above primes $p \notin S$ (there $\bar r_0$ is unramified, so $I_{\bar w} \subseteq \ker \bar r_0$), i.e. r factors through G_S, and $r(N_q) = 1$ for $q \in Q$, which is (Gq5)

(for another place $g\bar{q}$ above q, $I_{g\bar{q}} \cap \ker \bar{r}_0 = gN_qg^{-1}$). Continuity is the same on $G_{\mathbb{Q}}$ and on its quotient G_S. An $\mathcal{O}$-algebra homomorphism between objects of $\mathrm{CNL}_{\mathcal{O}}$ is local (its composite with the residue map has kernel the maximal ideal), so (Fv1) is the universal property (Pz1).

Step 3: the weight pieces. A matrix Y lies in $\mathfrak{sp}_4(\mathbb{F})$ if and only if J_4Y is symmetric; as $(J_4Y)_{ab} = \epsilon_a Y_{5-a,b}$ with $\epsilon = (1,1,-1,-1)$, this says $Y_{5-b,5-i} = \pm Y_{ib}$ for all i,b, and the entries (i,b) and $(5-b,5-i)$ have the same weight. Hence $\mathfrak{g}_w := \mathfrak{g}_{\Phi,w}(\mathbb{F})$ has dimension 2 for $w=(0,0)$ (entries $\{(1,1),(4,4)\}$, $\{(2,2),(3,3)\}$), 1 for $w \in \mathcal{W}_4 \setminus \{(0,0)\}$ (one orbit: $\{(1,2),(3,4)\}$, $\{(1,3),(2,4)\}$, $\{(1,4)\}$, $\{(2,3)\}$ and their transposes), and 0 for $w \notin \mathcal{W}_4$. Since the a-th diagonal entry of r_0 is $\alpha^{\lambda_a}\Psi^{\mu_a}$, $g \in G_S$ acts on $\mathfrak{g}_w$ by the scalar $\bar{\gamma}_w(g)$, $\gamma_w = \alpha^{w_1}\Psi^{w_2}$. Now $\alpha(c) = \chi_{\mathrm{cyc}}(c)\omega_\ell(c)^{-1}\xi(c) = -1$ and $\Psi(c) = 1$ ((Sg1); $\omega_\ell(c) = -1$ as $\omega_\ell(c)^2 = 1$ and $\omega_\ell(c) \equiv -1$, ℓ odd), so $\dim \mathfrak{g}_w^{c=1}$ is 2 at $(0,0)$, 1 at $(0,\pm 1)$, and 0 at the six weights $w \in \mathcal{W}_4$ with w_1 odd; this is $c(w)$ of (Fv4). For w_1 odd, $\gamma_w \in \{\alpha^{\pm 1}, \theta^{\pm 1}, \beta^{\pm 1}\}$ and, as $\bar{\alpha} = \bar{\xi}$ ($\bar{\chi}_{\mathrm{cyc}} = \bar{\omega}_\ell$), $\bar{\gamma}_w$ is $(\overline{\xi\Psi^i})^{\pm 1}$ with $i \in \{0,1,-1\}$; so $\bar{\gamma}_w(\sigma_v) \neq 1$ for every $v \in T$ by (Sg2).

Step 4: the framed Selmer cocycles. Let w_1 be odd and $(x,(x_v)) \in Z^1_{\mathrm{fr}}(\mathfrak{g}_w)$; write $x = zY_w$, $x_v = y_vY_w$ for a basis vector Y_w. Inflated to $G_{\mathbb{Q}}$, z is a residual Selmer cocycle for γ_w: the cocycle rule holds with $\bar{\gamma}_w$; z vanishes on inertia above primes outside S (it factors through G_S), and above $q \in Q$ on gN_qg^{-1} by the cocycle rule ($z(gng^{-1}) = z(g) + \bar{\gamma}_w(g)z(g^{-1}) = 0$ for $n \in N_q$, $\bar{\gamma}_w(n) = 1$); and $z(g) = \bar{\gamma}_w(g)y_v - y_v$ on D_v. By (Sg6), $z(g) = \bar{\gamma}_w(g)y - y$ for some $y \in \mathbb{F}$; then $(\bar{\gamma}_w(g) - 1)(y - y_v) = 0$ on D_v, and $g = \sigma_v$ gives $y_v = y$; $y_\ell = 0$ gives $y = 0$. So $Z^1_{\mathrm{fr}}(\mathfrak{g}_w) = 0$ for w_1 odd, and trivially for $w \notin \mathcal{W}_4$.

Step 5: the obstruction groups. For every w, $H^0(G_S, \mathfrak{g}_w^\vee) = 0$: G_S acts on $\mathfrak{g}_w^\vee = \mathrm{Hom}(\mathfrak{g}_w, \mu_\ell)$ by $\bar{\chi}_{\mathrm{cyc}}\bar{\gamma}_w^{-1}$, and $\bar{\chi}_{\mathrm{cyc}} \neq \bar{\gamma}_w$: at c for w_1 even ($\chi_{\mathrm{cyc}}(c) = -1 \neq 1 = \gamma_w(c)$), at σ_ℓ for w_1 odd ($\chi_{\mathrm{cyc}}(\sigma_\ell) = 1 \neq \bar{\gamma}_w(\sigma_\ell)$ by (Sg2) and Step 3). The hypotheses of Lemma 45.22 hold (Step 1: $N_q \subseteq I_q$, I_q/N_q finite of order prime to ℓ, N_q acts trivially on $\mathfrak{g}_w$ by (Pz0)), so

$$\dim \mathrm{Ob}_{\mathrm{fr}}(\mathfrak{g}_w) = \dim Z^1_{\mathrm{fr}}(\mathfrak{g}_w) + c(w) \qquad (w \in \mathbb{Z}^2),$$

with c as in (Fv4) (Step 3; both sides vanish for $w \notin \mathcal{W}_4$).

Step 6: the presentation. Apply Proposition 45.21 to $\mathfrak{P}$ and the base $(B_L, (\rho_v))$ with $\sigma_B = \sigma_L$, $\mathrm{wt}_B = \mathrm{wt}_L$, $\mathfrak{a}_B = \mathfrak{a}_L$. Let $U = R$, $\iota = \iota_R$, r_U the inflation of r_R to $G_{\mathbb{Q}}$, $\alpha_v = \alpha_{R,v}$, and $\tau, \mathrm{wt}_\tau, (y_t)$, $\Pi_U = \Pi$, $\kappa_0, \mathrm{wf}_0, (f_k)$ as in (Pz2)–(Pz4). Pad the relations: for each w add $\dim Z^1_{\mathrm{fr}}(\mathfrak{g}_w) + c(w) - \#\mathrm{wf}_0^{-1}(w) \geq 0$ indices k of weight w with $f_k = 0$ (non-negative by (Pz4) and Step 5; only finitely many w are involved), giving $\kappa \supseteq \kappa_0$ and wf. Then: (Fv1) by Step 2. (Fv2): Π_U is onto with $\Pi_U(X_s) = \iota(\pi_L(X_s))$ and $\Pi_U(X_t) = y_t$ (Pz3); $\ker \Pi_U = \mathfrak{a}_L P_U + (f_k)_{k\in\kappa}$ (Pz4; zero series add nothing); $f_k \in \mathcal{O}[[X]]_{\mathrm{wf}(k)}$ with constant term 0 (Pz5, its hypothesis by (Bq2)); each y_t is $r_U(g)_{ab} - r_0(g)_{ab}$ (with $g \in G_{\mathbb{Q}}$ lifting the element of G_S) or $(\alpha_v - 1)_{ab}$, with $\mathrm{wt}_\tau(t) = \mathrm{wt}_4(a,b) \in \mathcal{W}_4$ (Pz2). (Fv3): the entries lie in $U_{\mathrm{wt}_4(a,b)}$ (Pz3), and the homomorphism x_U induced by the constant term is x_R of (Pz5), so $x_U(r_U(g)_{ab}) = r_0(g)_{ab}$, $x_U(\alpha_v) = 1$, $x_U \circ \iota = x_L$. (Fv4): $\#\mathrm{wt}_\tau^{-1}(w) = \dim Z^1_{\mathrm{fr}}(\mathfrak{g}_w)$ (Pz2), which is 0 unless $w \in \{(0,0),(0,1),(0,-1)\}$ (Step 4), and $\#\mathrm{wf}^{-1}(w) = \#\mathrm{wt}_\tau^{-1}(w) + c(w)$ by the padding. So $(U, \iota, r_U, (\alpha_v), \tau, \mathrm{wt}_\tau, (y_t), \kappa, (f_k), \mathrm{wf})$ is a framed relative presentation over L. □

Proposition 45.21. *Let $\mathfrak{P} = (\Phi, \Gamma, \bar{r}, r_0, T, v_0, (\Gamma_v, \varphi_v), Q, (\Gamma_q, \psi_q, N_q))$ be a graded framed deformation problem over a coefficient ring $\mathcal{O}$ with residue field $\mathbb{F}$, Φ of rank n with weights w in $\mathbb{Z}^d$. Let $(B, (\rho_v)_{v\in T})$ be a base for $\mathfrak{P}$ with $B = \mathcal{O}[[X_s : s \in \sigma_B]]/\mathfrak{a}_B$, where σ_B is a finite set with a map $\mathrm{wt}_B \colon \sigma_B \to \mathbb{Z}^d$ and $\mathfrak{a}_B$ is an ideal all of whose elements have constant term 0 and which is homogeneous ($f_{(u)} \in \mathfrak{a}_B$ for all $f \in \mathfrak{a}_B$, $u \in \mathbb{Z}^d$); write π_B for the projection, $B_u = \pi_B(\mathcal{O}[[X_{\sigma_B}]]_u)$, $x_B \colon B \to \mathcal{O}$ for the homomorphism induced by the constant term, and assume $\rho_v(g)_{ab} \in B_{w(a)-w(b)}$ for all $v \in T$, $g \in \Gamma_v$ and a,b. For $j \in \mathbb{Z}^d$ let $\mathfrak{g}_j = \mathfrak{g}_{\Phi,j}(\mathbb{F})$, with $g \in \Gamma$ acting by $Y \mapsto \bar{r}(g)Y\bar{r}(g)^{-1}$. Then:*

(Pz0) *this is an action of* Γ *on* $\mathfrak{g}_j$ *on which every* $\psi_q(N_q)$ *acts trivially, so* $Z^1_{\mathrm{fr}}(\mathfrak{g}_j)$ *and* $\mathrm{Ob}_{\mathrm{fr}}(\mathfrak{g}_j)$ *are defined;*

(Pz1) *there is a framed lift* $(\iota_R, r_R, (\alpha_{R,v})_{v\in T})$ *over* $(B,(\rho_v))$ *to some* $R \in \mathrm{CNL}_{\mathcal{O}}$ *such that for every framed lift* $(\iota, r, (\alpha_v))$ *over* $(B,(\rho_v))$ *to some* $A \in \mathrm{CNL}_{\mathcal{O}}$ *there is exactly one morphism* $y\colon R \to A$ *with* $y\circ\iota_R = \iota$, $y\circ r_R = r$ *and* $y(\alpha_{R,v}) = \alpha_v$ *for all* $v \in T$*;*

and for every framed lift as in (Pz1), with $\mathfrak{m}_R$ *the maximal ideal of* R*:*

(Pz2) $\dim Z^1_{\mathrm{fr}}(\mathfrak{g}_j)$ *is finite for every* j *and zero for all but finitely many* j*, and there are a finite set* τ*, a map* $\mathrm{wt}_\tau\colon \tau \to \mathbb{Z}^d$ *and elements* $y_t \in R$ $(t \in \tau)$*, each of the form* $r_R(g)_{ab} - r_0(g)_{ab}$ *for some* $g \in \Gamma$ *or* $(\alpha_{R,v} - 1)_{ab}$ *for some* $v \in T$*, with* $w(a) - w(b) = \mathrm{wt}_\tau(t)$*, such that* $\#\mathrm{wt}_\tau^{-1}(j) = \dim Z^1_{\mathrm{fr}}(\mathfrak{g}_j)$ *for every* $j \in \mathbb{Z}^d$*;*

(Pz3) *with* $P = \mathcal{O}[[X_s : s \in \sigma_B \sqcup \tau]]$*, weighted by* wt_B *and* wt_τ*, and* $\Pi\colon P \to R$ *the continuous* $\mathcal{O}$*-algebra homomorphism with* $\Pi(X_s) = \iota_R(\pi_B(X_s))$ *for* $s \in \sigma_B$ *and* $\Pi(X_t) = y_t$ *for* $t \in \tau$*:* Π *is surjective,* $f_{(u)} \in \ker\Pi$ *for all* $f \in \ker\Pi$ *and* $u \in \mathbb{Z}^d$*, and* $r_R(g)_{ab}$ *and* $(\alpha_{R,v})_{ab}$ *lie in* $\Pi(\mathcal{O}[[X]]_{w(a)-w(b)})$ *for all* $g \in \Gamma$*,* $v \in T$ *and* a, b*;*

(Pz4) *there are a finite set* κ*, a map* $\mathrm{wf}\colon \kappa \to \mathbb{Z}^d$ *and power series* $f_k \in \ker\Pi \cap \mathcal{O}[[X]]_{\mathrm{wf}(k)}$ $(k \in \kappa)$ *with* $\ker\Pi = \mathfrak{a}_B P + (f_k)_{k\in\kappa}$ *and* $\#\mathrm{wf}^{-1}(j) \le \dim \mathrm{Ob}_{\mathrm{fr}}(\mathfrak{g}_j)$ *for every* $j \in \mathbb{Z}^d$*;*

(Pz5) *if* $x_B(\rho_v(g)) = r_0(\varphi_v(g))$ *for all* $v \in T$ *and* $g \in \Gamma_v$*, there is an* $\mathcal{O}$*-algebra homomorphism* $x_R\colon R \to \mathcal{O}$ *with* $x_R\circ\iota_R = x_B$*,* $x_R\circ r_R = r_0$ *and* $x_R(\alpha_{R,v}) = 1$ *for all* v*, and* $x_R\circ\Pi$ *is the constant-term homomorphism* $P \to \mathcal{O}$*; in particular every element of* $\ker\Pi$ *has constant term* 0*.*

Proof. Write $\mathbb{F}$ for the residue field, ϖ for a uniformiser of $\mathcal{O}$, $V_k \subseteq \mathbb{F}^n$ (or A^n) for the span of the basis vectors e_a with $\beta(a) = k$, and $u^i = \prod_m u_m^{i_m}$ for units $u_1, \dots, u_d$ and $i \in \mathbb{Z}^d$.

Step 0: facts on the form datum. (F1) $(J^{-1})_{ab} \neq 0$ implies $\beta(b) = s(\beta(a))$ and $w(a)+w(b) = c_\Phi$. Indeed $J(V_k) \subseteq V_{s(k)}$ and J is injective, so, s being a bijection, $J(V_k) = V_{s(k)}$ for every k and $J^{-1}(V_{s(k)}) = V_k$; and over $A = \mathcal{O}[u_1^{\pm1}, \dots, u_d^{\pm1}]$ the matrix $t_u = \mathrm{diag}(u^{w(1)}, \dots, u^{w(n)})$ satisfies ${}^t t_u J t_u = u^{c_\Phi} J$ (entry (a,b) is $u^{w(a)+w(b)}J_{ab}$), hence $t_u^{-1}J^{-1}t_u^{-1} = u^{-c_\Phi}J^{-1}$, whose entry (a,b) reads $u^{-w(a)-w(b)}(J^{-1})_{ab} = u^{-c_\Phi}(J^{-1})_{ab}$. (F2) Let A be a commutative $\mathcal{O}$-algebra and $g \in \Gamma$, and read $X = r_0(g)$ in $\mathrm{GL}_n(A)$, of similitude $\nu \in A^\times$ (Gq2). Then $Y \mapsto XYX^{-1}$ preserves $\mathfrak{g}_{\Phi,j}(A)$ and the set of block diagonal matrices of similitude 1, and, for $A \in \mathrm{CNL}_{\mathcal{O}}$, $\mathrm{H}_\Phi(A)$. Indeed ${}^t X J = \nu J X^{-1}$ and $JX = \nu\, {}^t X^{-1} J$, so ${}^t(XYX^{-1})J + JXYX^{-1} = \nu\, {}^t X^{-1}({}^t YJ + JY)X^{-1}$ and ${}^t(XhX^{-1})J(XhX^{-1}) = \nu\, {}^t X^{-1}\, {}^t h J h\, X^{-1} = \nu\, {}^t X^{-1} J X^{-1} = J$ if ${}^t hJh = J$. For the entries: $X^{-1} = r_0(g^{-1})$, so a non-zero term $X_{ac}Y_{cd}(X^{-1})_{db}$ needs $w(a) = w(c)$, $w(d) = w(b)$, $\beta(a) = \pi_g(\beta(c))$, $\beta(d) = \pi_g^{-1}(\beta(b))$; for block diagonal Y ($\beta(c) = \beta(d)$) this gives $\beta(a) = \beta(b)$, and $Y_{cd} \neq 0 \Rightarrow w(c) - w(d) = j$ gives $w(a) - w(b) = j$. (F3) Let $A \in \mathrm{CNL}_{\mathcal{O}}$ and $I \subseteq \mathfrak{m}_A$ an ideal with $I^2 = 0$. For $Y \in \mathrm{M}_n(I)$ block diagonal, $1+Y \in \mathrm{H}_\Phi(A)$ if and only if $Y \in \mathfrak{g}_\Phi(I)$ (since ${}^t(1+Y)J(1+Y) = J + {}^t YJ + JY$), and then $(1+Y)^{-1} = 1-Y$. $\mathrm{H}_\Phi(A)$ is a group: block diagonal matrices of similitude 1 are closed under products, invertible (${}^t hJh = J$ forces $\det h = \pm 1$), and $h^{-1} = J^{-1}\, {}^t h J$ is block diagonal by (F1). (F4) $\mathfrak{g}_\Phi(A) = \bigoplus_j \mathfrak{g}_{\Phi,j}(A)$: for $Y \in \mathfrak{g}_\Phi(A)$ let $Y^{(j)}$ keep the entries (a,b) with $w(a) - w(b) = j$; by the weight condition on J, the entry (a,b) of ${}^t Y^{(j)}J + JY^{(j)}$ is that of ${}^t YJ + JY$ if $c_\Phi - w(a) - w(b) = j$ and 0 otherwise. (F5) For units $u_m \equiv 1$ of $A \in \mathrm{CNL}_{\mathcal{O}}$, $t_u = \mathrm{diag}(u^{w(a)})$ has similitude u^{c_Φ}, $(t_u M t_u^{-1})_{ab} = u^{w(a)-w(b)}M_{ab}$, t_u commutes with every weight diagonal matrix, and conjugation by t_u preserves $\mathrm{H}_\Phi(A)$ (as in (F2) with $\nu = u^{c_\Phi}$; t_u is diagonal and $\equiv 1$).

(Pz0). By (F2) over $\mathbb{F}$, conjugation by $\bar r(g)$ (the image of $r_0(g)$) preserves $\mathfrak{g}_j$; it is an action, continuous since $\bar r$ is; and $\bar r(\psi_q(N_q)) = 1$ by (Gq3).

Step 1: graded lifting. Let $C \in \mathrm{CNL}_{\mathcal{O}}$ be of finite length with a decomposition $C = \bigoplus_{i\in\mathbb{Z}^d} C_i$ of $\mathcal{O}$-modules, finitely many $C_i \neq 0$, $C_iC_k \subseteq C_{i+k}$ and $\mathcal{O}\cdot 1 \subseteq C_0$; call $X \in \mathrm{M}_n(C)$ *graded* if $X_{ab} \in C_{w(a)-w(b)}$. Graded matrices form a subring containing every $r_0(g)$ (weight diagonal with entries in $\mathcal{O}$), and the inverse of an invertible graded matrix is graded.

Claim (L). Let $\mathrm{pr}\colon C \to \bar C$ be a surjective $\mathcal{O}$-algebra homomorphism onto another such algebra with $\mathrm{pr}(C_i) = \bar C_i$ for all i, whose kernel I satisfies $I \subseteq C_j$ for one j and $\mathfrak{m}_C I = 0$. Then every graded $\bar h \in \mathrm{H}_\Phi(\bar C)$ is $\mathrm{pr}(h)$ for a graded $h \in \mathrm{H}_\Phi(C)$, with $h = 1$ if $\bar h = 1$. *Proof of (L).* Lift each entry $\bar h_{ab}$ with $\beta(a) = \beta(b)$ to an element of $C_{w(a)-w(b)}$ and put 0 elsewhere: a graded block diagonal h' with $\mathrm{pr}(h') = \bar h$, $h' \equiv 1 \pmod{\mathfrak{m}_C}$. $E = {}^t h' J h' - J$ has entries in I, satisfies ${}^tE = \pm E$ (with ${}^tJ = \pm J$), maps V_k into $V_{s(k)}$, and $E_{ab} \in C_{c_\Phi - w(a) - w(b)}$ (a term $h'_{ca}J_{cd}h'_{db}$ with $J_{cd} \neq 0$ has weight $w(c) - w(a) + w(d) - w(b) = c_\Phi - w(a) - w(b)$, and $J_{ab} \neq 0$ only if $c_\Phi = w(a) + w(b)$). Put $Y = -\frac{1}{2}J^{-1}E$ (ℓ odd). By (F1), Y is block diagonal and $Y_{ab} = -\frac{1}{2}\sum_c (J^{-1})_{ac}E_{cb} \in C_{w(a)-w(b)} \cap I$. Since ${}^t(J^{-1}) = \pm J^{-1}$ with the same sign, ${}^tYJ = -\frac{1}{2}E$ and $JY = -\frac{1}{2}E$. As $\mathfrak{m}_C I = 0$ and $h' \equiv 1$, ${}^t(h'+Y)J(h'+Y) = J + E + {}^tYJ + JY = J$. So $h = h' + Y$ works; if $\bar h = 1$, $h' = 1$ and $E = 0$.

Step 2: (Pz1). By (Gq1) and Theorem B.71, item 1, there is Mazur's universal lift $(R^\square, r^\square)$ of $\bar r$. Let $\tilde A$ be the completed tensor product over $\mathcal{O}$ of $R^\square$, B and $\mathcal{O}[[Z_{v,ab} : v \in T \setminus \{v_0\},\ 1 \le a, b \le n]]$, the coproduct in $\mathrm{CNL}_{\mathcal{O}}$: morphisms $\tilde A \to A$ are triples (a morphism $R^\square \to A$, a morphism $B \to A$, elements $z_{v,ab} \in \mathfrak{m}_A$). Let $\tilde r$, $\tilde\iota$ be the images of $r^\square$ and the identity of B, $\tilde\alpha_v = 1 + (Z_{v,ab})_{a,b}$ for $v \neq v_0$ and $\tilde\alpha_{v_0} = 1$, and $\tilde h_g = \tilde r(g) r_0(g)^{-1}$. Let $\mathfrak{a} \subseteq \tilde A$ be generated by the entries of ${}^t\tilde h_g J \tilde h_g - J$ and the entries $(\tilde h_g)_{ab}$ with $\beta(a) \neq \beta(b)$ ($g \in \Gamma$); of $\tilde r(\psi_q(x)) - 1$ ($q \in Q$, $x \in N_q$); of ${}^t\tilde\alpha_v J \tilde\alpha_v - J$ and $(\tilde\alpha_v)_{ab}$ with $\beta(a) \neq \beta(b)$; and of $\tilde r(\varphi_v(g))\tilde\alpha_v - \tilde\alpha_v \tilde\iota(\rho_v(g))$ ($v \in T$, $g \in \Gamma_v$). A morphism $f\colon \tilde A \to A$ kills $\mathfrak{a}$ if and only if $(f\circ\tilde\iota, f\circ\tilde r, (f(\tilde\alpha_v)))$ satisfies (Gq4)–(Gq6) (the congruences $\equiv 1$ hold automatically: $\tilde r$ lifts $\bar r = r_0 \bmod \mathfrak{m}_{\mathcal{O}}$, and $Z_{v,ab} \in \mathfrak{m}$). Conversely every framed lift $(\iota, r, (\alpha_v))$ to A is of this form for exactly one f: r is a lift of $\bar r$ (by (Gq4)), so it comes from exactly one morphism $R^\square \to A$, and $(\alpha_v - 1)_{ab} \in \mathfrak{m}_A$. The framed lift (residue map $B \to \mathbb{F}$, $\bar r$, $\alpha_v = 1$) to $\mathbb{F}$ satisfies (Gq4)–(Gq6) ($\bar r(\psi_q(N_q)) = 1$ by (Gq3); $\rho_v \equiv r_0 \circ \varphi_v \equiv \bar r \circ \varphi_v$ as $\rho_v(g) r_0(\varphi_v(g))^{-1} \in \mathrm{H}_\Phi(B)$), so $\mathfrak{a} \subseteq \mathfrak{m}_{\tilde A}$ and $R = \tilde A/\mathfrak{a} \in \mathrm{CNL}_{\mathcal{O}}$ with the images $(\iota_R, r_R, (\alpha_{R,v}))$ has the property (Pz1). (By this property, any R as in (Pz1) is isomorphic to this one by a unique isomorphism compatible with the framed lifts, so it suffices to prove (Pz2)–(Pz5) for one.)

Step 3: the torus coaction. Write $u_m = 1 + T_m$ in $A[[T_1, \dots, T_d]]$. On $P_B = \mathcal{O}[[X_{\sigma_B}]]$ let ρ_P be the continuous homomorphism to $P_B[[T_1, \dots, T_d]]$ with $\rho_P(X_s) = u^{\mathrm{wt}_B(s)} X_s$. As $u^i = \sum_{k\in\mathbb{N}^d} \binom{i}{k} T^k$ ($\binom{i}{k} = \prod_m \binom{i_m}{k_m}$), $\rho_P(f) = \sum_k E_k(f) T^k$ with $E_k(f) = \sum_u \binom{u}{k} f_{(u)}$, the $\mathfrak{m}$-adic limit of its finite partial sums over $|u|_\infty \le M$ (the remaining monomials have degree $> M/\max(1, |\mathrm{wt}_B|_\infty)$). As $\mathfrak{a}_B$ is homogeneous and closed, $E_k(\mathfrak{a}_B) \subseteq \mathfrak{a}_B$, so ρ_P induces a morphism $\rho_B\colon B \to B[[T_1, \dots, T_d]]$ of $\mathrm{CNL}_{\mathcal{O}}$ with $\rho_B(y) = u^i y$ for $y \in B_i$. Over $A = R[[T_1, \dots, T_d]]$ let $t = t_u$ as in (F5), $\iota' = \iota_{R,T} \circ \rho_B$ ($\iota_{R,T}$ applies ι_R to coefficients), $r'(g) = t\, r_R(g)\, t^{-1}$ and $\alpha'_v = t\, \alpha_{R,v}\, t^{-1}$. This is a framed lift: $r'(g) r_0(g)^{-1} = t(r_R(g) r_0(g)^{-1}) t^{-1} \in \mathrm{H}_\Phi(A)$ by (F5) ($r_0(g)$ is weight diagonal); (Gq5) and $\alpha'_{v_0} = 1$ are clear; $\alpha'_v \in \mathrm{H}_\Phi(A)$ by (F5); and $t\, \iota_R(\rho_v(g))\, t^{-1} = \iota'(\rho_v(g))$ entrywise, because $\rho_v(g)_{ab} \in B_{w(a)-w(b)}$, which gives the intertwining relation. By (Pz1) there is a morphism $\rho_R\colon R \to R[[T_1, \dots, T_d]]$ of $\mathrm{CNL}_{\mathcal{O}}$ with $\rho_R \circ \iota_R = \iota'$, $\rho_R \circ r_R = r'$, $\rho_R(\alpha_{R,v}) = \alpha'_v$. In the terminology of Lemma 45.18 for ρ_R: $r_R(g)_{ab}$, $(\alpha_{R,v})_{ab}$ and $r_R(g)_{ab} - r_0(g)_{ab}$ have torus weight $w(a) - w(b)$ (constants have weight 0, and $r_0(g)_{ab} \neq 0$ only if $w(a) = w(b)$), and $\iota_R(\pi_B(X_s))$ has torus weight $\mathrm{wt}_B(s)$. Likewise $\pi_B(X_s) \in \mathfrak{m}_B$ has torus weight $\mathrm{wt}_B(s)$ for ρ_B, so by Lemma 45.18, item 3, $\mathfrak{a}_B$ is generated by finitely many elements of single weights.

Step 4: the relative tangent space, and (Pz2). Let $\mathfrak{m} = \mathfrak{m}_R$ and $t^*_{\mathrm{rel}} = \mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{m}_{\mathcal{O}}R + \iota_R(\mathfrak{m}_B)R)$, finite-dimensional. A linear form λ on t^*_{rel} defines $f_\lambda(y) = \bar{y} + \epsilon\lambda(\overline{y-c})$, where $c \in \mathcal{O}$ with $y - c \in \mathfrak{m}$ (another choice changes $y - c$ by an element of $\mathfrak{m}_{\mathcal{O}}$); it is a local $\mathcal{O}$-algebra homomorphism $R \to \mathbb{F}[\epsilon]$, as $yz - cc' = c(z-c') + c'(y-c) + (y-c)(z-c')$, and it kills $\iota_R(\mathfrak{m}_B)$. Conversely a morphism $f\colon R \to \mathbb{F}[\epsilon]$ is $y \mapsto \bar{y} + \epsilon\lambda'(y)$ with λ' additive, vanishing on $\mathcal{O}$, $\mathfrak{m}^2$ and $\mathfrak{m}_{\mathcal{O}}R$; if f kills $\iota_R(\mathfrak{m}_B)$, then λ' vanishes on $\iota_R(\mathfrak{m}_B)R$ and $f = f_\lambda$ for the form induced by λ'. So $\lambda \mapsto f_\lambda$ is a bijection onto the morphisms killing $\iota_R(\mathfrak{m}_B)$. By (Pz1) they correspond to the framed lifts to $\mathbb{F}[\epsilon]$ whose first entry is $B \to \mathbb{F} \subseteq \mathbb{F}[\epsilon]$. Write such a lift as $r(g) = (1 + \epsilon z(g))\bar{r}(g)$, $\alpha_v = 1 - \epsilon x_v$. Then, using (F3) with $I = \epsilon\mathbb{F}$ and $\iota(\rho_v(g)) = \bar{r}(\varphi_v(g))$: (Gq4) says $z(g) \in \mathfrak{g}_\Phi(\mathbb{F})$; r is a continuous homomorphism if and only if $z \in Z^1(\Gamma, \mathfrak{g}_\Phi(\mathbb{F}))$; (Gq5) says $z(\psi_q(N_q)) = 0$; and (Gq6) says $x_v \in \mathfrak{g}_\Phi(\mathbb{F})$, $x_{v_0} = 0$ and

$$(1 + \epsilon z(\varphi_v(g)))\bar{r}(\varphi_v(g))(1 - \epsilon x_v) = (1 - \epsilon x_v)\bar{r}(\varphi_v(g)), \quad \text{i.e.} \quad z(\varphi_v(g)) = g \cdot x_v - x_v.$$

By (F4) and (Pz0), these $(z, (x_v))$ form $\bigoplus_j Z^1_{\mathrm{fr}}(\mathfrak{g}_j)$. Let $\Theta(z, x)$ be the form λ of the corresponding morphism. For $y_{g,ab} = r_R(g)_{ab} - r_0(g)_{ab}$ and $y'_{v,ab} = (\alpha_{R,v} - 1)_{ab}$, elements of $\mathfrak{m}$, comparing f_λ with the lift gives

$$\Theta(z, x)(\bar{y}_{g,ab}) = \big(z(g)\bar{r}(g)\big)_{ab}, \qquad \Theta(z, x)(\bar{y}'_{v,ab}) = -(x_v)_{ab}.$$

The classes $\bar{y}_{g,ab}$, $\bar{y}'_{v,ab}$ span t^*_{rel}: a non-zero λ killing them gives $f_\lambda \neq f_0$ classifying the same framed lift (ι residual, $r = \bar{r}$, $\alpha_v = 1$), against the uniqueness in (Pz1). Hence Θ, a bijection given on a spanning set by expressions linear in (z, x), is a linear isomorphism $\bigoplus_j Z^1_{\mathrm{fr}}(\mathfrak{g}_j) \cong (t^*_{\mathrm{rel}})^\vee$; in particular each $Z^1_{\mathrm{fr}}(\mathfrak{g}_j)$ is finite-dimensional and almost all vanish. Let $V_j \subseteq t^*_{\mathrm{rel}}$ be spanned by the $\bar{y}_{g,ab}$, $\bar{y}'_{v,ab}$ with $w(a) - w(b) = j$. For $(z, x) \in Z^1_{\mathrm{fr}}(\mathfrak{g}_i)$, $\Theta(z, x)$ vanishes on V_j for $j \neq i$ ($\bar{r}(g)$ is weight diagonal). Hence: if $\Theta(z, x)$ with $(z, x) \in Z^1_{\mathrm{fr}}(\mathfrak{g}_j)$ kills V_j, it kills every V_i, hence $t^*_{\mathrm{rel}} = \sum_i V_i$, so $(z, x) = 0$. An element of $V_j \cap \sum_{i \neq j} V_i$ is killed by every form $\Theta(\sum_i \zeta_i)$, $\zeta_i \in Z^1_{\mathrm{fr}}(\mathfrak{g}_i)$ ($\Theta(\zeta_j)$ kills $\sum_{i \neq j} V_i$, and $\Theta(\zeta_i)$, $i \neq j$, kills V_j), so it is 0; thus $t^*_{\mathrm{rel}} = \bigoplus_j V_j$. A form on V_j, extended by 0 on the other V_i, is $\Theta(\sum_i \zeta_i)$ with $\Theta(\zeta_i)$ zero on V_i for $i \neq j$, hence $\zeta_i = 0$ for $i \neq j$; so restriction is an isomorphism $Z^1_{\mathrm{fr}}(\mathfrak{g}_j) \cong V_j^\vee$. Choose $\tau = \bigsqcup_j \tau_j$, where τ_j indexes elements y_t among the $y_{g,ab}$, $y'_{v,ab}$ with $w(a) - w(b) = j$ whose classes form a basis of V_j, and $\mathrm{wt}_\tau(t) = j$ for $t \in \tau_j$. This is (Pz2).

Step 5: (Pz3). $R = \mathcal{O} + \mathfrak{m}$, so $\iota_R(\mathfrak{m}_B)R \subseteq \iota_R(\mathfrak{m}_B) + \mathfrak{m}^2$, and $\mathfrak{m}_B$ is generated by ϖ and the $\pi_B(X_s)$ (constant terms of $\mathfrak{a}_B$ vanish). With Step 4, $\mathfrak{m}/\mathfrak{m}^2$ is spanned by the images of ϖ and the $\Pi(X_s)$, $s \in \sigma_B \sqcup \tau$, all of which lie in $\mathfrak{m}$. By Nakayama's lemma and the complete Nakayama lemma [166, Thm. 8.4], Π is surjective. $\Pi|_{\mathcal{O}[[X_{\sigma_B}]]} = \iota_R \circ \pi_B$ (continuous, equal on the X_s). By Step 3 each $\Pi(X_s)$ has torus weight equal to its weight, so Lemma 45.18, items 1 and 2, give the homogeneity of $\ker\Pi$ and, with the torus weights of Step 3, the entries in $\Pi(\mathcal{O}[[X]]_{w(a)-w(b)})$.

Step 6: relative minimality. For $f \in \mathcal{J} = \ker\Pi$ the coefficient $a_t(f)$ of X_t ($t \in \tau$) lies in $\mathfrak{m}_{\mathcal{O}}$: write $f = c_0 + \sum_{s \in \sigma_B} a_s X_s + \sum_t a_t(f) X_t + f_2$ with $f_2 \in (X)^2$; $\Pi(f) = 0$ gives $c_0 \in \mathfrak{m}_{\mathcal{O}}$, and in t^*_{rel}, where the images of c_0, $\Pi(a_s X_s) \in \iota_R(\mathfrak{m}_B)R$ and $\Pi(f_2) \in \mathfrak{m}^2$ vanish, $\sum_t \bar{a}_t(f)\bar{y}_t = 0$, so $\bar{a}_t(f) = 0$.

Step 7: the graded space of relations. Let $\mathcal{J}_B = \mathfrak{a}_B P \subseteq \mathcal{J}$ and $\mathfrak{n} = \mathfrak{m}_P$. An ideal generated by elements of single weights is homogeneous ($(gh)_{(u)} = g\,h_{(u-i)}$ for g of weight i); so $\mathcal{J}_B$ (Step 3), $\mathfrak{n}^N$ (generated by the monomials $\varpi^e X^m$) and $\mathfrak{n}\mathcal{J}$ (Lemma 45.18, item 3, for Π) are homogeneous, and so are their sums. By Artin–Rees choose $N_0 \geq 1$ with $\mathcal{J} \cap \mathfrak{n}^{N_0} \subseteq \mathfrak{n}\mathcal{J}$; let $W = \max(1, \max_s |\mathrm{wt}(s)|_\infty)$, so that monomials of weight u with $|u|_\infty > WN_0$ have degree $> N_0$. Put $\mathcal{V} = \mathcal{J}/(\mathcal{J}_B + \mathfrak{n}\mathcal{J})$, finite-dimensional over $\mathbb{F}$, and $\mathcal{V}_j$ the image of $\mathcal{J} \cap \mathcal{O}[[X]]_j$. For $f \in \mathcal{J}$, $f - \sum_{|u|_\infty \leq WN_0} f_{(u)} \in \mathcal{J} \cap \mathfrak{n}^{N_0} \subseteq \mathfrak{n}\mathcal{J}$ with $f_{(u)} \in \mathcal{J}$ (Pz3), so the $\mathcal{V}_j$ span $\mathcal{V}$; and if $\sum_u f_u \in \mathcal{J}_B + \mathfrak{n}\mathcal{J}$ with $f_u \in \mathcal{J} \cap \mathcal{O}[[X]]_u$, each f_u is a weight component of an element of

this homogeneous ideal and lies in it. So $\mathcal{V} = \bigoplus_j \mathcal{V}_j$. Choose $f_{j,1}, \dots, f_{j,\mathrm{nf}(j)} \in \mathcal{J} \cap \mathcal{O}[[X]]_j$ lifting a basis of $\mathcal{V}_j$; by Nakayama's lemma applied to $\mathcal{J}/(\mathcal{J}_B + (f_{j,i}))$, $\mathcal{J} = \mathcal{J}_B + (f_{j,i})_{j,i}$. With $\kappa = \{(j,i)\}$ and $\mathrm{wf}(j,i) = j$, it remains for (Pz4) to show $\dim \mathcal{V}_j \le \dim \mathrm{Ob}_{\mathrm{fr}}(\mathfrak{g}_j)$; fix j and assume $\dim \mathrm{Ob}_{\mathrm{fr}}(\mathfrak{g}_j) < \dim \mathcal{V}_j$.

Step 8: the graded small extension of weight j. Let $\mathcal{W} \subseteq \mathcal{J}$ be the preimage of $\bigoplus_{i \neq j} \mathcal{V}_i$. It is a homogeneous ideal containing $\mathcal{J}_B + \mathfrak{n}\mathcal{J}$: for $f \in \mathcal{W}$ the images of the $f_{(u)} \in \mathcal{J}$ are the components of $\bar{f}$ (Step 7), so $f_{(j)} \in \mathcal{J}_B + \mathfrak{n}\mathcal{J}$ and $f_{(u)} \in \mathcal{W}$ for $u \neq j$. Put $C = P/(\mathcal{W} + \mathfrak{n}^{N_0})$ and $\bar{C} = P/(\mathcal{J} + \mathfrak{n}^{N_0}) = R/\mathfrak{m}^{N_0}$, of finite length, graded by the images C_i, $\bar{C}_i$ of the $\mathcal{O}[[X]]_i$ (direct sums with $|i|_\infty \le WN_0$, the ideals being homogeneous), so that $\mathrm{pr}\colon C \to \bar{C}$ satisfies the hypotheses of Claim (L): its kernel I is the image of $\mathcal{J} = (\mathcal{J} \cap \mathcal{O}[[X]]_j) + \mathcal{W}$, so $I \subseteq C_j$; $\mathfrak{m}_C I = 0$ as $\mathfrak{n}\mathcal{J} \subseteq \mathcal{W}$; and $I \cong \mathcal{J}/\mathcal{W} \cong \mathcal{V}_j$ (as $\mathcal{J} \cap \mathfrak{n}^{N_0} \subseteq \mathcal{W}$). Both are $\mathcal{O}/\varpi^{N_0}$-algebras. As $\mathfrak{a}_B \subseteq \mathcal{W}$, $\mathcal{O}[[X_{\sigma_B}]] \to C$ induces $\iota_C\colon B \to C$, mapping B_i into C_i, and $\mathrm{pr} \circ \iota_C = \iota_{\bar{C}}$, where $\iota_{\bar{C}}$, $\bar{r}$ (abusing notation only inside Steps 8–12) and $\bar{\alpha}_v$ denote ι_R, r_R, $\alpha_{R,v}$ followed by $R \to \bar{C}$: a framed lift to $\bar{C}$ whose matrices are graded by (Pz3). I is an $\mathbb{F}$-vector space, $\mathfrak{g}_{\Phi,j}(I) = \mathfrak{g}_j \otimes_{\mathbb{F}} I$, and for $X \in \mathrm{M}_n(I)$ and $Z \in \mathrm{GL}_n(C)$ reducing to $\bar{r}_0(g)$ modulo $\mathfrak{m}_C$, $ZXZ^{-1} = g \cdot X$. A graded matrix with entries in I has its non-zero entries in positions of weight j ($I \cap C_i = 0$ for $i \neq j$).

Step 9: graded local homomorphisms. For $v \in T$ choose by (L) a graded $\tilde{\alpha}_v \in \mathrm{H}_\Phi(C)$ lifting $\bar{\alpha}_v$, with $\tilde{\alpha}_{v_0} = 1$, and put $\rho'_v(g) = \tilde{\alpha}_v\, \iota_C(\rho_v(g))\, \tilde{\alpha}_v^{-1}$: a continuous graded homomorphism $\Gamma_v \to \mathrm{GL}_n(C)$ lifting $\bar{r} \circ \varphi_v$ (by (Gq6) over $\bar{C}$), with

$$\rho'_v(g) r_0(\varphi_v(g))^{-1} = \tilde{\alpha}_v\, \iota_C(\rho_v(g) r_0(\varphi_v(g))^{-1}) (r_0(\varphi_v(g)) \tilde{\alpha}_v^{-1} r_0(\varphi_v(g))^{-1}) \in \mathrm{H}_\Phi(C)$$

by (F2), (F3). For $q \in Q$: $\bar{h}_g = \bar{r}(\psi_q(g)) r_0(\psi_q(g))^{-1} \in \mathrm{H}_\Phi(\bar{C})$ is graded, depends only on gN_q (both factors kill N_q by (Gq3), (Gq5)), and is constant on the cosets of the open normal subgroup $U_q \supseteq N_q$ on which $\bar{r} \circ \psi_q$ and $r_0 \circ \psi_q \bmod \varpi^{N_0}$ are trivial. Choosing by (L) graded lifts $h_g \in \mathrm{H}_\Phi(C)$ on representatives of Γ_q/U_q ($h = 1$ at the trivial coset), $\sigma_q(g) = h_g r_0(\psi_q(g))$ is a continuous graded map on Γ_q/N_q lifting $\bar{r} \circ \psi_q$, with $\sigma_q(1) = 1$. Then $\sigma_q(g)\sigma_q(g')\sigma_q(gg')^{-1} = h_g (r_0(\psi_q(g)) h_{g'} r_0(\psi_q(g))^{-1}) h_{gg'}^{-1}$ lies in $\mathrm{H}_\Phi(C)$, is graded and reduces to 1, so by (F3) and Step 8 it is $1 + k_q^\circ(g,g')$ with k_q° a continuous 2-cocycle of Γ_q/N_q in $\mathfrak{g}_j \otimes I$. By (Gq3), $k_q^\circ = d\omega_q$, and $\rho'_q(g) = (1 - \omega_q(g))\sigma_q(g)$ is a continuous graded homomorphism $\Gamma_q \to \mathrm{GL}_n(C)$, trivial on N_q, lifting $\bar{r} \circ \psi_q$, with $\rho'_q(g) r_0(\psi_q(g))^{-1} \in \mathrm{H}_\Phi(C)$.

Step 10: the obstruction class. As in Step 9 (with Γ, an open normal subgroup on which $\bar{r}$ and $r_0 \bmod \varpi^{N_0}$ are trivial, and (L)), choose a continuous graded map $\sigma\colon \Gamma \to \mathrm{GL}_n(C)$ lifting $\bar{r}$ with $\sigma(1) = 1$ and $\sigma(g) r_0(g)^{-1} \in \mathrm{H}_\Phi(C)$. Then $\sigma(g)\sigma(g')\sigma(gg')^{-1} = 1 + k(g,g')$ with $k \in Z^2(\Gamma, \mathfrak{g}_j \otimes I)$, and $\sigma(\varphi_v(g)) = (1 + k_v(g))\rho'_v(g)$, $\sigma(\psi_q(g)) = (1 + k_q(g))\rho'_q(g)$ with $k_v \in C^1(\Gamma_v, \mathfrak{g}_j \otimes I)$, $k_q \in C^1(\Gamma_q, \mathfrak{g}_j \otimes I)$ (Step 8, (F3)). Expanding $\sigma(\varphi_v(g))\sigma(\varphi_v(g'))$ in two ways gives $\varphi_v^* k = dk_v$, and likewise $\psi_q^* k = dk_q$. For $\phi \in I^\vee = \mathrm{Hom}_{\mathbb{F}}(I, \mathbb{F})$ the triple $(\phi \circ k, (\phi \circ k_v), (\phi \circ k_q))$ lies in $Z^2_{\mathrm{fr}}(\mathfrak{g}_j)$, and its class $o(\phi) \in \mathrm{Ob}_{\mathrm{fr}}(\mathfrak{g}_j)$ is linear in ϕ.

Step 11: a vanishing class gives a framed lift. Since $\dim I^\vee = \dim \mathcal{V}_j > \dim \mathrm{Ob}_{\mathrm{fr}}(\mathfrak{g}_j)$, some $\phi \neq 0$ has $o(\phi) = 0$: $\phi k = dh$, $\phi k_v = \varphi_v^* h - dy_v$, $\phi k_q = \psi_q^* h - e_q$ with $h \in C^1(\Gamma, \mathfrak{g}_j)$, $y_v \in \mathfrak{g}_j$, $e_q \in Z^1(\Gamma_q, \mathfrak{g}_j)$, $e_q(N_q) = 0$. Let $I' = \ker \phi$, an ideal of C (as $\mathfrak{m}_C I = 0$), $C' = C/I'$, graded with $C'_j \supseteq I/I' \cong \mathbb{F}$ via ϕ, and read $h(g)$, y_v, $e_q(g)$ as elements of $\mathfrak{g}_{\Phi,j}(I/I')$, graded matrices. Over C' put $\sigma'(g) = (1 - h(g))\sigma(g)$; then $\sigma'(g)\sigma'(g')\sigma'(gg')^{-1} = 1 + \phi k(g,g') - dh(g,g') = 1$, so σ' is a continuous graded homomorphism lifting $\bar{r}$, and $\sigma'(g) r_0(g)^{-1} \in \mathrm{H}_\Phi(C')$. For $v \in T$,

$$\sigma'(\varphi_v(g)) = (1 + \phi k_v(g) - h(\varphi_v(g)))\rho'_v(g) = (1 - (g \cdot y_v - y_v))\rho'_v(g) = (1 + y_v)\rho'_v(g)(1 + y_v)^{-1};$$

for $q \in Q$, $\sigma'(\psi_q(g)) = (1 - e_q(g))\rho'_q(g)$, which is 1 on N_q. Put $c = 1 + y_{v_0}$, $s''(g) = c^{-1}\sigma'(g)c$ and $\alpha''_v = c^{-1}(1 + y_v)\tilde{\alpha}_v$. Then $\alpha''_{v_0} = 1$, $\alpha''_v \in \mathrm{H}_\Phi(C')$ (F3), s'' satisfies (Gq4) by (F2), (F3) and

(Gq5), and $s''(\varphi_v(g))\alpha''_v = c^{-1}(1+y_v)\rho'_v(g)\tilde{\alpha}_v = \alpha''_v\,\iota_{C'}(\rho_v(g))$ with $\iota_{C'} = (C \to C') \circ \iota_C$. So $(\iota_{C'}, s'', (\alpha''_v))$ is a framed lift to C'.

Step 12: contradiction, and (Pz4). By (Pz1) there is $F\colon R \to C'$ with $F \circ \iota_R = \iota_{C'}$, $F \circ r_R = s''$, $F(\alpha_{R,v}) = \alpha''_v$. F followed by $C' \to \bar{C}$ is the projection $R \to \bar{C}$: both classify the framed lift $(\iota_{\bar{C}}, \bar{r}, (\bar{\alpha}_v))$, since s'', α''_v reduce to $\bar{r}$, $\bar{\alpha}_v$ (c, $1+y_v$, $1-h$ reduce to 1). Let $\alpha\colon P \to C'$ be the projection and $\beta = F \circ \Pi$. Then $\delta = \alpha - \beta$ takes values in I/I', which $\mathfrak{m}_{C'}$ kills, so $\delta(fg) = \bar{f}\delta(g) + \bar{g}\delta(f)$ with residues $\bar{f}, \bar{g} \in \mathbb{F}$, and $\delta = 0$ on $\mathcal{O}[[X_{\sigma_B}]]$ (there $\alpha = \iota_{C'} \circ \pi_B = F \circ \iota_R \circ \pi_B = \beta$, by continuity). For $f \in P$ write $f = f_0 + \sum_t X_t g_t$ with $f_0 \in \mathcal{O}[[X_{\sigma_B}]]$, each term containing some X_t assigned to the first such t; the constant term of g_t is $a_t(f)$ and $\delta(f) = \sum_t \bar{a}_t(f)\delta(X_t)$. By Step 6, $\delta(\mathcal{J}) = 0$; and $\beta(\mathcal{J}) = 0$. Hence $\alpha(\mathcal{J}) = 0$; but $\alpha(\mathcal{J})$ is the image $I/I' \neq 0$ of $\mathcal{J}$ in C'. This contradiction proves $\dim \mathcal{V}_j \leq \dim \mathrm{Ob}_{\mathrm{fr}}(\mathfrak{g}_j)$, hence (Pz4).

Step 13: (Pz5). Under the hypothesis, $(x_B, r_0, (1)_v)$ is a framed lift to $\mathcal{O}$ ((Gq4) trivially, (Gq5) by (Gq3), (Gq6) by the hypothesis), so (Pz1) gives x_R. $x_R \circ \Pi$ is a continuous $\mathcal{O}$-algebra homomorphism $P \to \mathcal{O}$ killing every X_s: $x_R(\iota_R(\pi_B(X_s))) = x_B(\pi_B(X_s)) = 0$, $x_R(r_R(g)_{ab} - r_0(g)_{ab}) = 0$, $x_R((\alpha_{R,v}-1)_{ab}) = 0$. So it is the constant-term homomorphism, and $\ker\Pi$ lies in its kernel. □

Lemma 45.22. *Let ℓ be an odd prime, $\mathbb{F}$ a finite field of characteristic ℓ, S a finite set of primes with $\ell \in S$, $G_S = G_{\mathbb{Q},S}$ (ramification allowed at S and ∞), $c \in G_S$ the image of a complex conjugation, and for every $p \in S$ a decomposition group $D_p \supseteq I_p$ at p with its map to G_S. Let $T \subseteq S$ with $\ell \in T$, $Q = S \setminus T$, and for $q \in Q$ let $N_q \subseteq I_q$ be a closed normal subgroup of D_q with I_q/N_q finite of order prime to ℓ. Let M be a finite-dimensional $\mathbb{F}$-vector space with a continuous $\mathbb{F}$-linear action of G_S on which every N_q acts trivially, and form $Z^1_{\mathrm{fr}}(M)$ and $\mathrm{Ob}_{\mathrm{fr}}(M)$ for $\Gamma = G_S$, the set T with $v_0 = \ell$ and $\Gamma_v = D_v$, and the set Q with $\Gamma_q = D_q$ and N_q. Then $\mathrm{Ob}_{\mathrm{fr}}(M)$ is finite-dimensional and*

$$\dim_{\mathbb{F}} \mathrm{Ob}_{\mathrm{fr}}(M) = \dim_{\mathbb{F}} Z^1_{\mathrm{fr}}(M) + \dim_{\mathbb{F}} M^{c=1} + \dim_{\mathbb{F}} H^0(G_S, M^\vee).$$

Proof. All cohomology groups below are $\mathbb{F}$-vector spaces, finite by Theorem B.37, item 1, and Theorem B.29, item 1; $\dim = \dim_{\mathbb{F}}$, which is $\log_{\#\mathbb{F}}$ of the cardinality (also for $M^\vee$, an $\mathbb{F}$-space through M). Write $h^i = \dim H^i(G_S, M)$, $h^i_p = \dim H^i(D_p, M)$, and for $q \in Q$ let $L_q \subseteq H^1(D_q, M)$ be the classes of cocycles vanishing on N_q (a coboundary $g \mapsto gm - m$ vanishes on N_q, so this is a condition on classes). Let

$$\mathrm{res}^1\colon H^1(G_S, M) \to \bigoplus_{v\in T} H^1(D_v, M) \oplus \bigoplus_{q\in Q} H^1(D_q, M)/L_q,$$
$$\mathrm{res}^2\colon H^2(G_S, M) \to \bigoplus_{p\in S} H^2(D_p, M)$$

be the restrictions, and $\mathrm{Sel} = \ker \mathrm{res}^1$.

Step 1: $\dim Z^1_{\mathrm{fr}}(M) = \dim \mathrm{Sel} + \sum_{v\in T} h^0_v - h^0$. The map $(x, (x_v)) \mapsto [x]$ sends $Z^1_{\mathrm{fr}}(M)$ into Sel ($x|_{D_v} = dx_v$; x vanishes on N_q). It is onto: for a class in Sel choose a cocycle x and $m_v \in M$ with $x|_{D_v} = dm_v$; x vanishes on N_q (its restriction to D_q is a cocycle vanishing on N_q plus a coboundary, and coboundaries vanish on N_q); then $(x - dm_\ell, (m_v - m_\ell)_v) \in Z^1_{\mathrm{fr}}(M)$. Its kernel consists of the $(dm, (m+a_v)_v)$ with $m \in M$, $a_v \in M^{D_v}$ and $m + a_\ell = 0$; the parameters $(m, (a_v))$ with $a_\ell = -m$ form a space of dimension $\sum_{v\in T} h^0_v$ (choose $a_\ell \in M^{D_\ell}$ and a_v, $v \neq \ell$, freely), and the parameter map has kernel $\{(m, (-m)_v) :\ m \in M^{G_S}\}$, of dimension h^0.

Step 2: $\dim \mathrm{Ob}_{\mathrm{fr}}(M) = \dim \operatorname{coker} \mathrm{res}^1 + \dim \ker \mathrm{res}^2$. The map $\kappa\colon \mathrm{Ob}_{\mathrm{fr}}(M) \to H^2(G_S, M)$, $[(k, (k_v), (k_q))] \mapsto [k]$, is well defined ($B^2_{\mathrm{fr}}$ has first entries dh) and has image $\ker \mathrm{res}^2$: $k|_{D_p} = dk_p$ gives $\mathrm{res}^2[k] = 0$, and conversely such k_p exist if $\mathrm{res}^2[k] = 0$. If $[k] = 0$, then $k = dh$, and subtracting $(dh, (h|_{D_v}), (h|_{D_q})) \in B^2_{\mathrm{fr}}(M)$ leaves $(0, (a_v), (a_q))$ with $a_p \in Z^1(D_p, M)$. So $\ker \kappa$ is the image of $\partial\colon \bigoplus_v Z^1(D_v, M) \oplus \bigoplus_q Z^1(D_q, M) \to \mathrm{Ob}_{\mathrm{fr}}(M)$, $(a_p) \mapsto [(0, (a_p))]$, and $(0, (a_p)) \in B^2_{\mathrm{fr}}(M)$ exactly when $a_v = h|_{D_v} - dy_v$ and $a_q = h|_{D_q} - e_q$ with $h \in Z^1(G_S, M)$, $y_v \in M$ and e_q cocycles vanishing on N_q. Hence $\ker \kappa \cong \operatorname{coker} \mathrm{res}^1$.

Step 3: the local terms. For $v \in T$, $h^1_v - h^0_v - h^2_v$ is $\dim M$ if $v = \ell$ and 0 otherwise (Theorem B.29, item 4). For $q \in Q$, L_q is the image of the inflation $H^1(D_q/N_q, M) \to H^1(D_q, M)$, which is injective; as I_q/N_q is finite of order prime to ℓ, inflation-restriction gives $H^1(D_q/N_q, M) \cong H^1(D_q/I_q, M^{I_q})$, and for the procyclic group $D_q/I_q \cong \widehat{\mathbb{Z}}$, topologically generated by a Frobenius Fr, $H^1(D_q/I_q, V) \cong V/(\mathrm{Fr} - 1)V$, of the same dimension as $V^{\mathrm{Fr}=1}$ [191]. So $\dim L_q = \dim M^{D_q} = h^0_q$, and $h^1_q - h^2_q - \dim L_q = 0$ (Theorem B.29, item 4, $q \neq \ell$).

Step 4: the count. By Steps 1–2, with $\dim \operatorname{coker} \mathrm{res}^1 = \sum_v h^1_v + \sum_q (h^1_q - \dim L_q) - h^1 + \dim \mathrm{Sel}$ and, by Theorem B.37, item 3, $\dim \ker \mathrm{res}^2 = h^2 - \sum_{p \in S} h^2_p + \dim H^0(G_S, M^\vee)$,

$$\dim \mathrm{Ob}_{\mathrm{fr}}(M) - \dim Z^1_{\mathrm{fr}}(M) = \sum_{v \in T} (h^1_v - h^0_v - h^2_v) + \sum_{q \in Q} (h^1_q - h^2_q - \dim L_q) + (h^0 - h^1 + h^2) + \dim H^0(G_S, M^\vee).$$

By Step 3 the first two sums are $\dim M$. By Theorem B.37, item 2, for $\mathbb{Q}$ (one archimedean place, $H^0(\mathbb{R}, M) = M^{c=1}$, $|\#M|_\infty = \#M$), $h^0 - h^1 + h^2 = \dim M^{c=1} - \dim M$. This gives the formula. □

Proposition 45.23. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum, L a symplectic local base of $\mathfrak{D}$, and suppose there is a framed relative presentation over L. Then $\mathfrak{D}$ has a symplectic weighted model.*

Proof. Let $(U, \iota, r_U, (\alpha_v), \tau, \mathrm{wt}_\tau, (y_t), \kappa, (f_k), \mathrm{wf})$ be a framed relative presentation over L, $P = \mathcal{O}[[X_s : s \in \sigma_L \sqcup \tau]]$ with $\Pi_U\colon P \to U$, $\tau_w = \#\mathrm{wt}_\tau^{-1}(w)$, $b = \#(T \setminus \{\ell\})$, $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, and κ_a the a-th diagonal entry of r_0. By (Fv3), $x_U(\Pi_U(f)) = f(0)$, the constant term, for $f \in P$.

Step 1: preliminaries. (a) Similitude, (Se2) and the local conditions are polynomial identities in the entries, so they pass to images under ring homomorphisms and to conjugates by matrices of similitude 1 (for the ordinary condition: the characteristic polynomial and the products of the $\rho(s_0) - \kappa_j(s_0)$ are conjugation equivariant). Hence $r_0 = x_U \circ r_U$ satisfies (Se1), (Se2), and $r_0|_{D_v} = x_L \circ \rho_v$ (Bq2) satisfies the local condition at v. (b) Put $X^{(1)} = E_{12} - E_{34}$, $X^{(2)} = E_{14}$, $X^{(3)} = E_{32}$, $X^{(4)} = E_{21} - E_{43}$, $X^{(5)} = E_{41}$, $X^{(6)} = E_{23}$, and let $p(e)$ be the position $(1,2), (1,4), (3,2), (2,1), (4,1), (2,3)$ for $e = 1, \dots, 6$; their weights are the six elements of $\mathcal{W}_4$ with odd first coordinate, $\lambda_a \neq \lambda_b$ at $p(e) = (a, b)$, and $X^{(e)}$ has entry 1 at $p(e)$ and 0 at $p(e')$, $e' \neq e$. Put $X_m = E_{31} + E_{42}$, of weight $(0, -1)$, with $r_0(g) X_m r_0(g)^{-1} = \Psi(g)^{-1} X_m$. These seven matrices lie in $\mathfrak{sp}_4(\mathcal{O})$.

(c) (Six blocks.) Let A be a local ring and $X \in \mathrm{M}_4(A)$ of similitude $\nu \in A^\times$ with off-diagonal entries in $\mathfrak{m}_A$ and $X_{12} = X_{14} = X_{32} = X_{21} = X_{41} = X_{23} = 0$. Then X is block diagonal; if moreover $\nu = 1$, $X_{13} = X_{31} = 0$ and $X_{11} = X_{22} = 1$, then $X = 1$. Indeed $\det X$ is a unit, so the diagonal entries are units; the entry (a, b) of ${}^t X J_4 X$ is $X_{1a}X_{4b} + X_{2a}X_{3b} - X_{3a}X_{2b} - X_{4a}X_{1b}$, and at $(2,4)$ and $(3,1)$ it equals $X_{22}X_{34}$ and $-X_{43}X_{11}$, which must vanish; under the further hypotheses the entries $(1,4)$, $(2,3)$, $(3,4)$, $(1,2)$ give $X_{44} = 1$, $X_{33} = 1$, $X_{24} = 0$, $X_{42} = 0$. (d) (Dual numbers.) Let $\Lambda = \mathcal{O}$ or $\mathcal{O}/a\mathcal{O}$ with $a \in \mathfrak{m}_{\mathcal{O}} \setminus \{0\}$, so $\Lambda[\epsilon] \in \mathrm{CNL}_{\mathcal{O}}$. If $(\Lambda[\epsilon], \iota', r', (\alpha'_v))$ is a framed lift over L reducing modulo ϵ to $(x_L, r_0, (1))$ (read modulo a), the morphism $y\colon U \to \Lambda[\epsilon]$

of (Fv1) is $x_U + \epsilon D$ (modulo a) with $D\colon U \to \Lambda$ $\mathcal{O}$-linear, $D(fh) = x_U(f)D(h) + x_U(h)D(f)$, and D applied to $r_U(g)$, α_v, ι gives the ϵ-parts of $r'(g)$, α'_v, ι': y followed by $\epsilon \mapsto 0$ and x_U both classify $(x_L, r_0, (1))$ (by (Fv3)), so they agree by (Fv1). By (Bq1) and (Bq2), a family of lifts ρ'_v of $\bar{r}_0|_{D_v}$ to $\Lambda[\epsilon]$ satisfying the local conditions and reducing to $r_0|_{D_v}$ is $\iota' \circ \rho_v$ for exactly one ι', and ι' reduces to x_L.

Step 2: independent derivations of the framing entries. For $v \in T$ and $1 \le e \le 6$ put $t_{v,e} = r_U(\phi)_{p(e)}$ if $v = \ell$ and $t_{v,e} = (\alpha_v)_{p(e)}$ if $v \ne \ell$; $t_{v,e} \in U_{w_e}$ with $w_e = \mathrm{wt}_4(p(e))$ (Fv3), and $x_U(t_{v,e}) = 0$. Over $\mathcal{O}[\epsilon]$: for $v = \ell$ take $r' = (1 + \epsilon X^{(e)})r_0(1 - \epsilon X^{(e)})$, all $\alpha'_{v'} = 1$, and ι' classifying the local lifts $r'|_{D_{v'}}$; for $v \ne \ell$ take $r' = r_0$, $\alpha'_v = 1 + \epsilon X^{(e)}$, $\alpha'_{v'} = 1$ for $v' \ne v$, and ι' classifying $(1 - \epsilon X^{(e)})r_0|_{D_v}(1 + \epsilon X^{(e)})$ at v and $r_0|_{D_{v'}}$ at $v' \ne v$. By Step 1(a),(b) these local lifts satisfy the local conditions and the triples are framed lifts ((Fw3) at v: $r_0(1 + \epsilon X) = (1 + \epsilon X)(1 - \epsilon X)r_0(1 + \epsilon X)$). Step 1(d) gives point derivations $D_{v,e}$ of U with

$$D_{\ell,e}(t_{\ell,e'}) = X^{(e)}_{ab}(\kappa_b(\phi) - \kappa_a(\phi)) \quad ((a,b) = p(e')),$$
$$D_{\ell,e}(t_{v',e'}) = 0 \ (v' \ne \ell), \qquad D_{v,e}(t_{v',e'}) = [v' = v][e' = e] \ (v \ne \ell).$$

By (Ae1), $\bar{\kappa}_a(\phi) \ne \bar{\kappa}_b(\phi)$ when $\lambda_a \ne \lambda_b$, so $D_{\ell,e}(t_{\ell,e}) \ne 0$ and $D_{\ell,e}(t_{\ell,e'}) = 0$ for $e' \ne e$. Choose $g_{v,e} \in \mathcal{O}[[X]]_{w_e}$ with $\Pi_U(g_{v,e}) = t_{v,e}$ (Fv3); its constant term is $x_U(t_{v,e}) = 0$. The maps $\delta_{v,e} = D_{v,e} \circ \Pi_U$ vanish on $\mathfrak{a}_L P \subseteq \ker \Pi_U$, satisfy $\delta(fh) = f(0)\delta(h) + h(0)\delta(f)$, and $\delta_{v,e}(g_{v',e'}) \ne 0$ if and only if $(v', e') = (v, e)$.

Step 3: the cut. By (Bq3), $\mathcal{O}[[X_{\sigma_L}]]/\mathfrak{a}_L$ is smooth at the origin with tangent dimensions $d_\ell + b\,d_p$. Proposition 45.24, applied with $\sigma = \sigma_L$, τ, $\mathfrak{a} = \mathfrak{a}_L$ and the $6(b+1)$ elements $g_{v,e}$, gives a homogeneous ideal $\mathfrak{b} \supseteq \mathfrak{a}_L P + (g_{v,e})$ with constant terms 0 such that $R = P/\mathfrak{b}$ (weights wt_L, wt_τ, with values in $\mathcal{W}_4$) is a domain, smooth at the origin with tangent dimensions $d_\ell(w) + b\,d_p(w) + \tau_w - (b+1)[w \in \{w_1, \dots, w_6\}]$. With

$$r = 1 + b + \tau_{(0,-1)}, \qquad r' = 2 + b + \tau_{(0,1)}, \qquad m = 2 + 2b + \tau_{(0,0)},$$

this function is $t_{r,r',m}$ ($\tau_w = 0$ outside $\{(0,0),(0,\pm 1)\}$ by (Fv4)), so (Wm1) holds.

Step 4: the equations. The elements $s \in U$: for $v \in T \setminus \{\ell\}$, $(\alpha_v)_{13}$, $(\alpha_v)_{31}$, $(\alpha_v)_{11} - 1$, $(\alpha_v)_{22} - 1$, of weights $(0,1), (0,-1), (0,0), (0,0)$; and $h(r_U)_{13}$, $h(r_U)_{31}$, of weights $(0,1), (0,-1)$. Each lies in U_w for its weight w (Fv3; $1 \in U_{(0,0)}$) and has $x_U(s) = 0$ ($x_U(\alpha_v) = 1$, $x_U(r_U(g)_{13}) = r_0(g)_{13} = 0$); choose lifts $\tilde{s} \in \mathcal{O}[[X]]_w$, of constant term 0. By (Fv4), every $\mathrm{wf}(k)$ lies in $\{(0,0),(0,\pm 1)\}$, with $\tau_w + c(w)$ relations of weight w. Let F, E, Z list the images in R of the f_k and $\tilde{s}$ of weights $(0,-1)$, $(0,1)$, $(0,0)$: they have $(\tau_{(0,-1)}+1)+b+1 = r+1$, $(\tau_{(0,1)}+1)+b+1 = r'$ and $(\tau_{(0,0)} + 2) + 2b = m$ members, and (Wm2) holds. Let $\mathfrak{c} = \mathfrak{b} + (f_k)_k + (\tilde{s})_s$, a proper ideal (constant terms 0), $S = P/\mathfrak{c} \in \mathrm{CNL}_\mathcal{O}$, and $q\colon R \to S$ the projection: surjective with kernel generated by F, E, Z (Wm3). As $\mathfrak{c} \supseteq \mathfrak{a}_L P + (f_k) = \ker \Pi_U$ (Fv2), there is a surjective $\mathcal{O}$-algebra homomorphism $\sigma_S\colon U \to S$ with $\sigma_S \circ \Pi_U$ the projection; it kills every $t_{v,e}$ and every s.

Step 5: the lift $r_S = \sigma_S \circ r_U$, (Wm4), (Wm5). By Step 1(c), $\sigma_S(\alpha_v)$ has similitude 1, is $\equiv 1$ and has the ten listed entries, so $\sigma_S(\alpha_v) = 1$ for $v \ne \ell$, and $\alpha_\ell = 1$ by (Fw2); and $r_S(\phi)$, of unit similitude $\theta(\phi)$, with off-diagonal entries reducing to those of $\bar{r}_0(\phi)$, which vanish, is block diagonal. So r_S is a symplectic Eisenstein lift: (Se1), (Se2) from (Fw1); (Se3) as just shown, with $h(r_S)_{13} = \sigma_S(h(r_U)_{13}) = 0$ and likewise at $(3,1)$; (Se4): for $v \in T$ let $i_R\colon B_L \to R$ be induced by $\mathcal{O}[[X_{\sigma_L}]] \to P \to R$ (it kills $\mathfrak{a}_L$); R is a domain in $\mathrm{CNL}_\mathcal{O}$, $\rho_B = i_R \circ \rho_v$ satisfies the local condition at v (Step 1(a)), $x_R \circ \rho_B = x_L \circ \rho_v = r_0|_{D_v}$, and $q \circ i_R = \sigma_S \circ \iota$ (both are induced by $\mathcal{O}[[X_{\sigma_L}]] \to P \to S$, by (Fv2)), so $q \circ \rho_B = \sigma_S(\alpha_v^{-1} r_U \alpha_v)|_{D_v} = r_S|_{D_v}$ by (Fw3). Generation: if $f, f'\colon S \to S'$ agree on r_S, then $f \circ \sigma_S$ and $f' \circ \sigma_S$ carry the framed lift to framed lifts with the same global lift and frames 1, and $f \circ \sigma_S \circ \iota$, $f' \circ \sigma_S \circ \iota$ both classify the local lifts

$f \circ r_S|_{D_v}$, so they agree by (Bq1); by (Fv1) $f \circ \sigma_S = f' \circ \sigma_S$, and $f = f'$ as σ_S is onto. This is (Wm4). (Wm5): $r_U(g)_{ij} = \Pi_U(\tilde{f})$ with $\tilde{f} \in \mathcal{O}[[X]]_{\mathrm{wt}_4(i,j)}$ (Fv3), whose image f in $R_{\mathrm{wt}_4(i,j)}$ has $q(f) = r_S(g)_{ij}$; x_R (induced by the constant term) kills $\ker q$, so every f' with $q(f') = r_S(g)_{ij}$ has $x_R(f') = \tilde{f}(0) = x_U(r_U(g)_{ij}) = r_0(g)_{ij}$.

Step 6: (Wm6). Let $a \in \mathcal{O} \setminus \{0\}$ and z a normalised Sha cocycle modulo a; if a is a unit, $\delta = 0$ works, so let $a \in \mathfrak{m}_\mathcal{O}$. Choose $\tilde{y}_v \in \mathcal{O}$ ($v \in T$) with $z(g) = \Psi^{-1}(g)\tilde{y}_v - \tilde{y}_v$ modulo a on D_v. (i) Over $\mathcal{O}[\epsilon]$ the local lifts $(1 - \epsilon\tilde{y}_v X_m) r_0|_{D_v} (1 + \epsilon\tilde{y}_v X_m)$ satisfy the local conditions (Step 1(a),(b)), so by Step 1(d) they are classified by $x_L + \epsilon D_B$ with $D_B \colon B_L \to \mathcal{O}$ a point derivation at x_L. The derivation D_B vanishes on $(B_L)_w$ for $w \neq (0,-1)$: as $\mathfrak{a}_L$ is homogeneous and closed, $X_s \mapsto u^{\mathrm{wt}_L(s)} X_s$ ($u^w = (1+T_1)^{w_1}(1+T_2)^{w_2}$) induces a morphism $\rho_B \colon B_L \to B_L[[T_1, T_2]]$ with $\rho_B(f) = u^w f$ for $f \in (B_L)_w$; with $t_u = \mathrm{diag}(u^{(1,1)}, u^{(0,1)}, u^{(1,0)}, 1)$, $t_u r_0 t_u^{-1} = r_0$ and $t_u X_m t_u^{-1} = u^{(0,-1)} X_m$, both $(x_L + \epsilon D_B) \circ \rho_B$ (coefficientwise) and $x_L + \epsilon u^{(0,-1)} D_B$ are morphisms $B_L \to \mathcal{O}[\epsilon][[T_1, T_2]]$ classifying the local lifts $t_u(1 - \epsilon\tilde{y}_v X_m) r_0 (1 + \epsilon\tilde{y}_v X_m) t_u^{-1}$, as $\rho_v(g)_{ab} \in (B_L)_{\mathrm{wt}_4(a,b)}$; by (Bq1) they are equal, and for $f \in (B_L)_w$ this reads $u^w D_B(f) = u^{(0,-1)} D_B(f)$. (ii) Over $(\mathcal{O}/a)[\epsilon]$ let $r'(g) = (1 + \epsilon z(g) X_m) r_0(g)$, $\alpha'_v = 1$ and $\iota' = (x_L + \epsilon D_B) \bmod a$. The map r' is a continuous homomorphism (cocycle rule and $r_0 X_m r_0^{-1} = \Psi^{-1} X_m$), of similitude θ ($X_m \in \mathfrak{sp}_4$), satisfies (Se2) (z vanishes on residually trivial inertia outside T, and Step 1(a)), and on D_v, $(1 - \epsilon\tilde{y}_v X_m) r_0 (1 + \epsilon\tilde{y}_v X_m) = (1 + \epsilon(\Psi^{-1} - 1)\tilde{y}_v X_m) r_0 = r'$ (with $\tilde{y}_v$ read modulo a): (Fw3) holds with $\alpha'_v = 1$. Step 1(d) gives $D \colon U \to \mathcal{O}/a$ with $D(r_U(g)) = z(g) X_m r_0(g)$, $D(\alpha_v) = 0$, $D \circ \iota = D_B \bmod a$. In particular $D(r_U(g)_{31}) = \theta(g) z(g)$, $D(r_U(g)_{42}) = \Psi(g) z(g)$, and $D(r_U(g)_{ab}) = 0$ unless $\mathrm{wt}_4(a,b) = (0,-1)$. (iii) Let $\delta_P \colon P \to \mathcal{O}$ be the point derivation at the constant term, $\delta_P(f) = \sum_s \frac{\partial f}{\partial X_s}(0)\, \delta_P(X_s)$, with $\delta_P(X_s) = D_B(\pi_L(X_s))$ for $s \in \sigma_L$, $\delta_P(X_t)$ a lift of $D(y_t)$ if $\mathrm{wt}_\tau(t) = (0,-1)$, and $\delta_P(X_t) = 0$ otherwise. A point derivation of a power series ring at the constant term is determined by its values on the variables (it kills $(X)^2$); hence $\delta_P = D_B \circ \pi_L$ on $\mathcal{O}[[X_{\sigma_L}]]$, so $\delta_P(\mathfrak{a}_L P) = 0$ (product rule, constant terms 0); $\delta_P \equiv D \circ \Pi_U \pmod a$, as both agree on the variables ($D(y_t) = 0$ for $\mathrm{wt}_\tau(t) \neq (0,-1)$ by (ii) and $D(\alpha_v) = 0$); and $\delta_P(\mathcal{O}[[X]]_w) = 0$ for $w \neq (0,-1)$ (only linear terms count, and $\delta_P(X_s) = 0$ for $\mathrm{wt}(s) \neq (0,-1)$ by (i)). So $\delta_P(g_{v,e}) = 0$, and by Proposition 45.24, item 2, δ_P induces $\delta \colon R \to \mathcal{O}$ with $\delta(fh) = x_R(f)\delta(h) + x_R(h)\delta(f)$ and $\delta(R_w) = 0$ for $w \neq (0,-1)$. Moreover, $\delta(\ker q) \subseteq a\mathcal{O}$: on the generators, $\delta_P(f_k) \equiv D(\Pi_U(f_k)) = 0$; the $\tilde{s}$ of weight $\neq (0,-1)$ give 0; $\delta_P(\tilde{s}) \equiv D((\alpha_v)_{31}) = 0$ and $\delta_P(\tilde{s}) \equiv D(h(r_U)_{31}) = \theta(g_0) z(g_0) + z(g_0^{-1}) = 0$, since $z(g_0) = 0$ and $0 = z(g_0^{-1} g_0) = z(g_0^{-1}) + \Psi^{-1}(g_0^{-1}) z(g_0)$; and $\delta(\sum c_i e_i) = \sum x_R(c_i)\delta(e_i)$ for generators e_i ($x_R(e_i) = 0$). Finally, if $q(f) = r_S(g)_{31}$, then f differs by an element of $\ker q$ from the image of a lift $\tilde{f} \in \mathcal{O}[[X]]_{(0,-1)}$ of $r_U(g)_{31}$, so $\delta(f) \equiv \delta_P(\tilde{f}) \equiv D(r_U(g)_{31}) = \theta(g) z(g)$ modulo a; likewise $\delta(f) \equiv \Psi(g) z(g)$ if $q(f) = r_S(g)_{42}$. This is (Wm6), and $(\sigma_L \sqcup \tau, \mathrm{wt}_L \sqcup \mathrm{wt}_\tau, \mathfrak{b}, r, r', m, F, E, Z, S, q, r_S)$ is a symplectic weighted model of $\mathfrak{D}$. □

Proposition 45.24. *Let $\mathcal{O}$ be a coefficient ring, σ and τ disjoint finite sets with a map $\mathrm{wt} \colon \sigma \sqcup \tau \to \mathbb{Z}^2$, $P = \mathcal{O}[[X_s : s \in \sigma \sqcup \tau]]$, and $\mathfrak{a}$ a homogeneous ideal of $\mathcal{O}[[X_s : s \in \sigma]]$ all of whose elements have constant term 0, such that $\mathcal{O}[[X_s : s \in \sigma]]/\mathfrak{a}$ is smooth at the origin with tangent dimensions t_0. Let $g_1, \dots, g_N \in P$ with $g_i \in \mathcal{O}[[X]]_{w_i}$ of constant term 0, and suppose that for every i there is an $\mathcal{O}$-linear map $\delta_i \colon P \to \mathcal{O}$ with $\delta_i(\mathfrak{a}P) = 0$, $\delta_i(fh) = f(0)\delta_i(h) + h(0)\delta_i(f)$ for all $f, h \in P$ ($f(0)$ the constant term), $\delta_i(g_i) \neq 0$ and $\delta_i(g_j) = 0$ for $j \neq i$. Then there is a homogeneous ideal $\mathfrak{b} \subseteq P$, all of whose elements have constant term 0, with $\mathfrak{a}P + (g_1, \dots, g_N) \subseteq \mathfrak{b}$, such that:*

1. *$P/\mathfrak{b}$ is a domain and is smooth at the origin with tangent dimensions $w \mapsto t_0(w) + \#\{t \in \tau : \mathrm{wt}(t) = w\} - \#\{i : \ w_i = w\}$;*
2. *every $\mathcal{O}$-linear $\delta \colon P \to \mathcal{O}$ with $\delta(\mathfrak{a}P) = 0$, $\delta(fh) = f(0)\delta(h) + h(0)\delta(f)$ for all f, h, and*

$\delta(g_i) = 0$ *for all* i*, vanishes on* $\mathfrak{b}$.

Proof. Write $E = \mathcal{O}[\ell^{-1}]$ (the fraction field of $\mathcal{O}$), $P_\sigma = \mathcal{O}[[X_s : s \in \sigma]] \subseteq P$, $P^{\mathrm{gr}} = \sum_w P_w$ (finite sums of elements of single weights), $f(0)$ for the constant term, and for a series F over E with variables indexed by a set K, $\mathrm{lin}(F) \in E^K$ for the vector of its coefficients of degree one. For $N \geq 1$ let U_N be the finite set of weights of monomials in the X_s ($s \in \sigma \sqcup \tau$) of degree $< N$; for every $f \in P$, $f - \sum_{w \in U_N} f_{(w)}$ has only monomials of degree $\geq N$. By hypothesis there are J, wt_J with $\#\mathrm{wt}_J^{-1}(w) = t_0(w)$ and a chart ϕ_0 of $R_0 = P_\sigma/\mathfrak{a}$ into $E[[Y_i : i \in J]]$ with (Sm1)–(Sm4); let $\Phi_0 \colon P_\sigma \to E[[Y_J]]$ be its substitution homomorphism (Lemma 45.25).

Step 1: the cover $R_1 = P/\mathfrak{a}P$ *is smooth with tangent dimensions* $t_0 + \tau$*-counts.* $\mathfrak{a}$ is generated by its elements of single weight (every $f \in \mathfrak{a}$ is the limit of the partial sums $\sum_{w \in U_N} f_{(w)}$, which lie in the ideal generated by them, and that ideal is closed), hence by finitely many $a_1, \dots, a_k$ of single weights. So $\mathfrak{a}P = (a_1, \dots, a_k)P$ is homogeneous ($(\sum_i a_i p_i)_{(w)} = \sum_i a_i p_{i,(w - \mathrm{wt}(a_i))}$) with constant terms 0, and R_1 is a $\mathbb{Z}^2$-weighted quotient. Let $K = J \sqcup \tau$ with $\mathrm{wt}_K = \mathrm{wt}_J \sqcup \mathrm{wt}|_\tau$, and $\Phi_1 \colon P \to E[[Y_K]]$ the substitution $X_s \mapsto \Phi_0(X_s)$ ($s \in \sigma$), $X_t \mapsto Y_t$ ($t \in \tau$). Then $\Phi_1|_{P_\sigma} = \Phi_0$, $\Phi_1(\mathfrak{a}P) = 0$ by (Wx1), $\Phi_1(P_w)$ consists of series of weight w (Wx2), the constant term of $\Phi_1(f)$ is $f(0)$, and $\Phi_1(X^m P) \subseteq (Y)^{|m|}$. So Φ_1 induces a chart ϕ_1 of R_1 satisfying (Sm1), (Sm2). (Sm3): for $i \in J$ take a lift in P_σ^{gr} of the element given by (Sm3) for ϕ_0 (its image involves no Y_t); for $t \in \tau$ take X_t. (Sm4): let $f \in P^{\mathrm{gr}}$ with $f(0) = 0$ and $\mathrm{lin}(\Phi_1(f)) = 0$. Splitting the monomials of each single-weight part by their degree in the X_t ($t \in \tau$) gives $f = f_0 + \sum_t X_t f_t + \sum_{t \leq t'} X_t X_{t'} q_{tt'}$ with $f_0, f_t \in P_\sigma^{\mathrm{gr}}$, $q_{tt'} \in P^{\mathrm{gr}}$. Then $\mathrm{lin}(\Phi_1(f)) = \mathrm{lin}(\Phi_0(f_0)) + \sum_t f_t(0)\, e_t$, so $f_t(0) = 0$ for all t, and $f_0(0) = 0$, $\mathrm{lin}(\Phi_0(f_0)) = 0$; by (Sm4) for ϕ_0 there is $c \in \mathcal{O} \setminus \{0\}$ with cf_0 congruent modulo $\mathfrak{a}$ to an element of $\mathfrak{m}_0(R_0)^2$. As X_t, f_t have constant term 0, the image of cf in R_1^{gr} lies in $\mathfrak{m}_0(R_1)^2$, i.e. $f \in \mathrm{Sq}$. So R_1 is smooth at the origin with tangent dimensions $w \mapsto t_0(w) + \#\{t \in \tau : \mathrm{wt}(t) = w\} = \#\mathrm{wt}_K^{-1}(w)$.

Step 2: the cotangent map and derivations. Let $\mathfrak{m}_0 = \mathfrak{m}_0(R_1)$ and $\mathrm{lin}(f) = \mathrm{lin}(\phi_1(f))$ for $f \in \mathfrak{m}_0$. (a) $\mathrm{lin}(\mathfrak{m}_0)$ spans E^K by (Sm3) (subtract constant terms). (b) The kernel of lin on $\mathfrak{m}_0$ is $\mathrm{Sq} \cap \mathfrak{m}_0$: one inclusion is (Sm4); conversely if $cf \in \mathfrak{m}_0^2$ then $c\,\mathrm{lin}(f) = 0$, since a product of two series without constant term has no linear part. (c) Let $\delta \colon P \to \mathcal{O}$ be $\mathcal{O}$-linear with $\delta(fh) = f(0)\delta(h) + h(0)\delta(f)$ and $\delta(\mathfrak{a}P) = 0$. It induces $\bar{\delta}$ on R_1 with the same rule for x_{R_1}; $\bar{\delta}(\mathfrak{m}_0^2) = 0$, hence $\bar{\delta}(\mathrm{Sq} \cap \mathfrak{m}_0) = 0$ ($\mathcal{O}$ is torsion free). By (a), (b) there is a unique E-linear $\lambda_\delta \colon E^K \to E$ with $\bar{\delta}(f) = \lambda_\delta(\mathrm{lin}(f))$ for $f \in \mathfrak{m}_0$ (define it on the $\mathcal{O}$-module $\mathrm{lin}(\mathfrak{m}_0) \cong \mathfrak{m}_0/(\mathrm{Sq} \cap \mathfrak{m}_0)$ and extend E-linearly; this is well defined after clearing denominators, as $\mathcal{O}$ is torsion free). In particular $\delta(f) = \lambda_\delta(\mathrm{lin}(\Phi_1(f)))$ for $f \in P^{\mathrm{gr}}$ with $f(0) = 0$.

Step 3: independence of the g_i *and the formal cut.* Put $G_i = \Phi_1(g_i)$, of weight w_i without constant term. By Step 2 (c), $\lambda_{\delta_i}(\mathrm{lin}(G_j)) = \delta_i(g_j)$, which is non-zero exactly for $j = i$; so $\mathrm{lin}(G_1), \dots, \mathrm{lin}(G_N)$ are linearly independent, and $\mathrm{lin}(G_i)$ is supported on $K_{w_i} = \mathrm{wt}_K^{-1}(w_i)$. For each weight w choose $K_w^1 \subseteq K_w$ with $\#K_w^1 = \#\{i : w_i = w\}$ such that the coefficients of the Y_k, $k \in K_w^1$, in the $\mathrm{lin}(G_i)$ with $w_i = w$ form an invertible matrix; let $K^1 = \bigsqcup_w K_w^1$, $K^2 = K \setminus K^1$, and $i \mapsto k_i$ a bijection $\{1, \dots, N\} \to K^1$ with $\mathrm{wt}_K(k_i) = w_i$. The continuous E-algebra endomorphism α of $E[[Y_K]]$ with $\alpha(Y_{k_i}) = G_i$ and $\alpha(Y_k) = Y_k$ ($k \in K^2$) has invertible linear part (block triangular, with the identity on K^2 and the block-diagonal-by-weight Jacobian $(\partial G_i/\partial Y_{k_j}(0))$ on K^1), so it is an automorphism. For $u \in (E^\times)^2$ let s_u be the automorphism $Y_k \mapsto u^{\mathrm{wt}_K(k)} Y_k$ ($u^{(a,b)} = u_1^a u_2^b$); a series F has weight w iff $s_u(F) = u^w F$ for all u (E is infinite). As G_i has weight $w_i = \mathrm{wt}_K(k_i)$, $\alpha \circ s_u = s_u \circ \alpha$ (equal on the variables), so α^{-1} preserves weights. Let $\varepsilon \colon E[[Y_K]] \to E[[Y_{K^2}]]$ kill the Y_k, $k \in K^1$, and $\psi = \varepsilon \circ \alpha^{-1}$. Then ψ preserves weights and constant terms, maps (Y_K) into (Y_{K^2}), fixes Y_k for $k \in K^2$, and

$$\ker \psi = \alpha((Y_k : k \in K^1)) = (G_1, \dots, G_N).$$

On linear parts ψ induces a linear map $\Lambda\colon E^K \to E^{K^2}$, $\mathrm{lin}(\psi(F)) = \Lambda(\mathrm{lin}(F))$; Λ is the identity on E^{K^2}, hence onto, and kills the N independent vectors $\mathrm{lin}(G_i)$, so $\ker\Lambda = \sum_i E\,\mathrm{lin}(G_i)$.

Step 4: the ideal $\mathfrak{b}$. Let $\Phi_2 = \psi \circ \Phi_1\colon P \to E[[Y_{K^2}]]$ and $\mathfrak{b} = \ker\Phi_2$. It is a prime ideal (the target is a domain), contains $\mathfrak{a}P$ and the g_i, and its elements have constant term 0 (the constant term of $\Phi_2(f)$ is $f(0)$). It is homogeneous: Φ_2 maps P_w to series of weight w and X^mP into $(Y)^{|m|}$, so for $f \in \mathfrak{b}$, $w \in \mathbb{Z}^2$ and a monomial Y^m of weight w, choosing $N > |m|$ with $w \in U_N$ (if w lies in no U_N then $f_{(w)} = 0$), the coefficient of Y^m in $\Phi_2(f_{(w)})$ equals that in $\sum_{w' \in U_N} \Phi_2(f_{(w')}) \equiv \Phi_2(f) = 0$ modulo $(Y)^N$; so $\Phi_2(f_{(w)}) = 0$. Hence $P/\mathfrak{b}$ is a $\mathbb{Z}^2$-weighted quotient and a domain.

Step 5: (1), smoothness of $P/\mathfrak{b}$. Φ_2 induces a chart ϕ_2 of $P/\mathfrak{b}$ into $E[[Y_{K^2}]]$ with the weights $\mathrm{wt}_K|_{K^2}$, and $\#(K^2 \cap \mathrm{wt}_K^{-1}(w)) = t_0(w) + \#\{t \in \tau : \mathrm{wt}(t) = w\} - \#\{i : w_i = w\}$. (Sm1), (Sm2) hold as for Φ_1. (Sm3): for $k \in K^2$ take f from (Sm3) for ϕ_1 with $\mathrm{lin}(\Phi_1(f)) = c\,e_k$; then $\mathrm{lin}(\Phi_2(f)) = \Lambda(c\,e_k) = c\,e_k$. (Sm4): let $f \in P^{\mathrm{gr}}$ with $f(0) = 0$ and $\mathrm{lin}(\Phi_2(f)) = 0$. Then $\mathrm{lin}(\Phi_1(f)) \in \ker\Lambda$, so $\mathrm{lin}(\Phi_1(f)) = \sum_i e_i\,\mathrm{lin}(G_i)$ with $e_i \in E$; take $c \in \mathcal{O} \setminus \{0\}$ with all $ce_i \in \mathcal{O}$. The element $f' = cf - \sum_i (ce_i)g_i \in P^{\mathrm{gr}}$ has $f'(0) = 0$ and $\mathrm{lin}(\Phi_1(f')) = 0$, so by Step 1 some $c'f'$ maps into $\mathfrak{m}_0(R_1)^2$. Under $R_1 \to P/\mathfrak{b}$ (which maps R_1^{gr} onto $(P/\mathfrak{b})^{\mathrm{gr}}$, $\mathfrak{m}_0$ into $\mathfrak{m}_0$, and kills the g_i), $c'cf$ maps into $\mathfrak{m}_0(P/\mathfrak{b})^2$: $f \in \mathrm{Sq}$. This proves (1).

Step 6: (2), derivations descend. Let δ be as in (2) and λ_δ as in Step 2 (c); then $\lambda_\delta(\mathrm{lin}(G_i)) = \delta(g_i) = 0$. Let $h \in \mathfrak{b} \cap P_w$ with $h(0) = 0$. Then $\Phi_1(h) \in \ker\psi = (G_1, \dots, G_N)$, say $\Phi_1(h) = \sum_i b_i G_i$, so $\mathrm{lin}(\Phi_1(h)) = \sum_i b_i(0)\,\mathrm{lin}(G_i)$ and $\delta(h) = \lambda_\delta(\mathrm{lin}(\Phi_1(h))) = 0$. Now let $f \in \mathfrak{b}$; then $f(0) = 0$ and all $f_{(w)} \in \mathfrak{b}$ (Step 4), with $f_{(w)}(0) = 0$. Since δ kills $\mathcal{O}$ and every $X_sX_{s'}p$ (both constant terms vanish), $\delta(f) = \sum_s c_s\delta(X_s)$ for $f = \sum_s c_sX_s +$ (terms of degree ≥ 2), and likewise $\delta(f_{(w)}) = \sum_{\mathrm{wt}(s)=w} c_s\delta(X_s)$. Summing over the finitely many weights of the variables, $\delta(f) = \sum_w \delta(f_{(w)}) = 0$. This proves (2). □

Lemma 45.25. *Let $\mathcal{O}$ be a coefficient ring, σ a finite set with $\mathrm{wt}\colon \sigma \to \mathbb{Z}^2$, $\mathfrak{a}$ a homogeneous ideal of $\mathcal{O}[[X_s : s \in \sigma]]$ all of whose elements have constant term 0, $R = \mathcal{O}[[X_s : s \in \sigma]]/\mathfrak{a}$ with projection π, J a finite set, and $\phi\colon R^{\mathrm{gr}} \to \mathcal{O}[\ell^{-1}][[Y_i : i \in J]]$ an $\mathcal{O}$-algebra homomorphism satisfying (Sm2) (the constant term of $\phi(f)$ is $x_R(f)$). Let $\Phi\colon \mathcal{O}[[X_s : s \in \sigma]] \to \mathcal{O}[\ell^{-1}][[Y_i : i \in J]]$ be the substitution homomorphism $f \mapsto f((\phi(\pi(X_s)))_{s\in\sigma})$ (each $\phi(\pi(X_s))$ has constant term 0). Then:*

(Wx1) *$\Phi(f) = \phi(\pi(f))$ for every $f \in \mathcal{O}[[X_\sigma]]$ that is a finite sum of elements of single weights, and $\Phi(\mathfrak{a}) = 0$; so Φ induces a homomorphism $R \to \mathcal{O}[\ell^{-1}][[Y_J]]$ extending ϕ;*

(Wx2) *if $\mathrm{wt}_J\colon J \to \mathbb{Z}^2$ is given and ϕ satisfies (Sm1), then $\Phi(f)$ has weight w for every $f \in \mathcal{O}[[X_\sigma]]_w$.*

Proof. Write $P = \mathcal{O}[[X_\sigma]]$, $Q = \mathcal{O}[\ell^{-1}][[Y_J]]$ and $\mathfrak{y}$ for the ideal of Q of series without constant term. Since $\phi(\pi(X_s)) \in \mathfrak{y}$, the substitution $\Phi(\sum_m c_mX^m) = \sum_m c_m \prod_s \phi(\pi(X_s))^{m_s}$ converges coefficientwise ($\prod_s \phi(\pi(X_s))^{m_s} \in \mathfrak{y}^{|m|}$) and is an $\mathcal{O}$-algebra homomorphism with $\Phi(X^mP) \subseteq \mathfrak{y}^{|m|}$.

(Wx1). Let $f = \sum_m c_mX^m \in P_v$ and $N \geq 1$. Assigning each monomial of degree $\geq N$ to one of its divisors X^μ of degree N gives

$$f = \sum_{|m|<N} c_mX^m + \sum_{|\mu|=N} X^\mu r_\mu, \qquad r_\mu \in P_{v-\mathrm{wt}(\mu)},$$

a finite sum. Each $\pi(X_s)$ lies in $R_{\mathrm{wt}(s)}$ and each $\pi(r_\mu)$ in $R_{v-\mathrm{wt}(\mu)}$, so all these terms lie in R^{gr} and, ϕ being a ring homomorphism on R^{gr},

$$\phi(\pi(f)) = \sum_{|m|<N} c_m \prod_s \phi(\pi(X_s))^{m_s} + \sum_{|\mu|=N} \Big(\prod_s \phi(\pi(X_s))^{\mu_s}\Big)\phi(\pi(r_\mu)) \equiv \Phi(f) \pmod{\mathfrak{y}^N},$$

as both $\phi(\pi(f))$ and $\Phi(f)$ differ from the first sum by an element of $\mathfrak{y}^N$. As N is arbitrary, $\phi(\pi(f)) = \Phi(f)$; by additivity this holds for finite sums of elements of single weights. Let $f \in \mathfrak{a}$. Every $f_{(w)}$ lies in $\mathfrak{a}$, so $\Phi(f_{(w)}) = \phi(\pi(f_{(w)})) = 0$. For $N \geq 1$ let U_N be the finite set of weights of monomials of degree $< N$; all monomials of $f - \sum_{w\in U_N} f_{(w)}$ have degree $\geq N$, so $\Phi(f) \equiv \sum_{w\in U_N} \Phi(f_{(w)}) = 0$ modulo $\mathfrak{y}^N$ for every N, and $\Phi(f) = 0$. Hence Φ factors through R, and the induced map agrees with ϕ on $R^{\mathrm{gr}} = \pi(\sum_w P_w)$.

(Wx2). By (Sm1), $\phi(\pi(X_s))$ has weight $\mathrm{wt}(s)$; a product of series of weights u, u' has weight $u+u'$, so $\prod_s \phi(\pi(X_s))^{m_s}$ has weight $\sum_s m_s \mathrm{wt}(s)$, which is w for every monomial X^m of $f \in P_w$; and a coefficientwise convergent sum of series of weight w has weight w. □

Definition 45.26. Let ℓ be an odd prime, $\overline{\mathbb{Z}}_\ell = \{y \in \overline{\mathbb{Q}}_\ell : |y|_\ell \leq 1\}$ and $\mathfrak{m}_\ell = \{y \in \overline{\mathbb{Q}}_\ell : |y|_\ell < 1\}$. Let $\mathfrak{D} = (\ell, \mathcal{O}, e, N, \chi, \xi, S, (\sigma_v), \phi, g_0, s_0)$ be a symplectic Eisenstein datum with characters θ, Ψ, α, set T and $r_0 = \mathrm{diag}(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = \mathrm{diag}(\theta, \Psi, \alpha, \mathbf{1})$. Indices i, j run over $\{1, 2, 3, 4\}$.

Test fields. A **test field** is a triple (K, V, a) of an algebraically closed field K, a valuation ring $V \subseteq K$ of K with maximal ideal $\mathfrak{m}_V$, and a ring homomorphism $a\colon \overline{\mathbb{Z}}_\ell \to V$ with $a(\mathfrak{m}_\ell) \subseteq \mathfrak{m}_V$, such that every element of V is congruent modulo $\mathfrak{m}_V$ to an element of $a(\overline{\mathbb{Z}}_\ell)$. Its **characteristic** is that of K (0 or ℓ). The **standard test field** is $(\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Z}}_\ell, \mathrm{id})$. For a matrix X with entries in $\mathcal{O} \subseteq \overline{\mathbb{Z}}_\ell$, $a(X)$ is formed entrywise.

Ordinary test representations. Let (K, V, a) be a test field. An **ordinary test representation** of $\mathfrak{D}$ with values in V is a pair (r, u) of a homomorphism $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(V)$ and a matrix $u \in \mathrm{GL}_4(V)$ such that:

(Ot1) there are a complete discrete valuation ring Λ with finite residue field and an injective local homomorphism $j\colon \Lambda \to V$ such that every element of K is algebraic over $j(\Lambda)$, the entries of u and of every $r(g)$ lie in $j(\Lambda)$, and $g \mapsto j^{-1}(r(g)_{ij})$ is continuous for the $\mathfrak{m}_\Lambda$-adic topology on Λ, for all i, j;

(Ot2) for every $g \in G_{\mathbb{Q}}$, the entries of $r(g) - a(r_0(g))$ lie in $\mathfrak{m}_V$, and $r(g)$ has similitude $a(\theta(g))$;

(Ot3) $r(g) = 1$ for every $g \in I_{\bar{w}}$ with $\bar{r}_0(g) = 1$ and every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above a prime not in T; and for every $v \in T$ with $v \neq \ell$, $r(g) = 1$ for every $g \in I_v$ with $\chi(g) = 1$;

(Ot4) u has similitude 1, and for every $s \in D_\ell$ the matrix $u^{-1}r(s)u$ is upper triangular with diagonal entries $d_1(s), \ldots, d_4(s)$ such that $d_i(s) - a(\kappa_i(s)) \in \mathfrak{m}_V$, and $d_i(s) = a(\kappa_i(s))$ if $s \in I_\ell$.

Points with values in V. A **point of $\mathfrak{D}$ with values in** V is a homomorphism $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(V)$ of the form $g \mapsto f(r_A(g))$ (entrywise) for a symplectic Eisenstein lift r_A to some $A \in \mathrm{CNL}_{\mathcal{O}}$ and a ring homomorphism $f\colon A \to V$ that restricts to a on $\mathcal{O}$ and satisfies $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_V$, such that the pair $(r, 1)$ satisfies (Ot1).

Shapes. Let (K, V, a) be a test field and $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(V)$ a homomorphism. The entry (i, j) of r is **active** if $r(g)_{ij} \neq 0$ for some $g \in G_{\mathbb{Q}}$. The representation r is

- **irreducible** if the K-span of $r(G_{\mathbb{Q}})$ is $\mathrm{M}_4(K)$;
- **of diagonal type** if there are no $i \neq j$ such that both (i, j) and (j, i) are active;

- a **non-split block extension** if (i) $r(g)_{ij} = 0$ for all g whenever $i \in \{1,3\}$ and $j \in \{2,4\}$; (ii) the K-span of the matrices $(r(g)_{ab})_{a,b\in\{2,4\}}$, $g \in G_{\mathbb{Q}}$, is $\mathrm{M}_2(K)$; and (iii) there is no $Y \in \mathrm{M}_4(K)$ with $Y_{ij} = 0$ unless $i \in \{2,4\}$ and $j \in \{1,3\}$ such that $(1-Y)\,r(g)\,(1+Y)$ is block diagonal for every g;
- **essential** if it is irreducible or a non-split block extension.

Weak purity. Let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism. A homomorphism $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ is **weakly pure** for ι if there are an open subgroup $H \subseteq G_{\mathbb{Q}}$ and an integer $M \geq 1$ such that for every prime $p \nmid M$, every place $\bar{p}'$ of $\overline{\mathbb{Q}}$ above p, every integer $f \geq 1$ and every $\sigma \in H$ that is an arithmetic Frobenius element for p^f at $\bar{p}'$, any two eigenvalues x, y of $r(\sigma)$ with $y \neq 0$ satisfy

$$|\iota(x)| < p^f\,|\iota(y)|. \tag{45.1}$$

Weak purity depends only on the characteristic polynomials of the $r(\sigma)$.

Lemma 45.27. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$ and residue field $\mathbb{F}_{\mathcal{O}}$; write $\bar{\kappa}_i\colon G_{\mathbb{Q}} \to \mathbb{F}_{\mathcal{O}}^{\times}$ for the reduction of κ_i, E_{ij} for the matrix units and $i^* = 5 - i$.*

(Ae1) *If $\lambda_i \neq \lambda_j$, then $\bar{\kappa}_i(\sigma_\ell) \neq \bar{\kappa}_j(\sigma_\ell)$; and $\bar{\kappa}_i(\phi) = \bar{\kappa}_i(\sigma_\ell)$ for every i.*

(Ae2) *For every symplectic Eisenstein lift r_A to some $A \in \mathrm{CNL}_{\mathcal{O}}$ and every i, E_{ii} lies in the A-submodule of $\mathrm{M}_4(A)$ spanned by $r_A(G_{\mathbb{Q}})$.*

Let (K, V, a) be a test field and r a point of $\mathfrak{D}$ with values in V. Then:

(Ae3) *E_{ii} lies in the V-submodule of $\mathrm{M}_4(V)$ spanned by $r(G_{\mathbb{Q}})$, for every i;*

(Ae4) *every entry (i,i) is active; if (i,j) and (j,k) are active, so is (i,k); if (i,j) is active, so is (j^*, i^*);*

(Ae5) *r is irreducible if and only if every entry (i,j) is active.*

Proof. (Ae1). As $\bar{\omega}_\ell = \bar{\chi}_{\mathrm{cyc}}$, $\bar{\alpha} = \bar{\xi}$, $\bar{\theta} = \bar{\xi}\bar{\Psi}$. For the four pairs $(i,j) = (1,2), (1,4), (3,2), (3,4)$ with $\lambda_i \neq \lambda_j$ the ratio $\bar{\kappa}_i/\bar{\kappa}_j$ is $\bar{\alpha} = \bar{\xi}$, $\bar{\theta} = \bar{\xi}\bar{\Psi}$, $\bar{\alpha}\bar{\Psi}^{-1} = \bar{\xi}\bar{\Psi}^{-1}$, $\bar{\alpha} = \bar{\xi}$; by (Sg2) at $v = \ell$ (with $i = 0, 1, -1$) its value at σ_ℓ is not 1; the pairs with $i \in \{2,4\}$ follow by inversion. Let $q = \ell^f = \#\mathbb{F}_{\mathcal{O}}$. For $n \geq f$, $f \mid n!$, so $x^{\ell^{n!}} = x$ for $x \in \mathbb{F}_{\mathcal{O}}$ and $\bar{\kappa}_i(\sigma_\ell^{\ell^{n!}}) = \bar{\kappa}_i(\sigma_\ell)$. As $\bar{\kappa}_i$ is locally constant, (Sg3) gives $\bar{\kappa}_i(\phi) = \bar{\kappa}_i(\sigma_\ell^{\ell^{n!}})$ for large n, whence the second claim.

(Ae2). Let $S \subseteq \mathrm{M}_4(A)$ be the A-span of $r_A(G_{\mathbb{Q}})$. It contains 1 and is closed under products ($r_A(g)r_A(h) = r_A(gh)$). For $X \in \mathrm{M}_4(A)$ put $X^\natural = J_4^{-1}\,{}^t\!X\,J_4$, an A-linear map. If X has similitude ν, then $X^\natural X = \nu$; by (Se1), $r_A(g)^\natural = \theta(g) r_A(g^{-1}) \in S$, so $S^\natural \subseteq S$. As $J_4 e_b = \pm e_{b^*}$, $E_{kk}^\natural = E_{k^*k^*}$. Write $\bar{X}$ for reduction modulo $\mathfrak{m}_A$; as r_A is a lift, $\overline{r_A(g)} = \bar{r}_0(g)$. Let $X = r_A(\phi)$, block diagonal by (Se3), with blocks $X_{\mathrm{A}} = (X_{ab})_{a,b\in\{1,3\}}$ and $X_{\mathrm{B}} = (X_{ab})_{a,b\in\{2,4\}}$, and put $t = X_{22} + X_{44}$, $d = X_{22}X_{44} - X_{24}X_{42}$ and $Y = X^2 - tX + d\cdot 1 \in S$. By the Cayley–Hamilton theorem for X_{B}, Y is block diagonal with blocks $Y_{\mathrm{A}} = X_{\mathrm{A}}^2 - tX_{\mathrm{A}} + d$ and 0. Modulo $\mathfrak{m}_A$,

$$\overline{Y_{\mathrm{A}}} = \mathrm{diag}\Big((\bar{\kappa}_1 - \bar{\kappa}_2)(\bar{\kappa}_1 - \bar{\kappa}_4),\ (\bar{\kappa}_3 - \bar{\kappa}_2)(\bar{\kappa}_3 - \bar{\kappa}_4)\Big)(\phi),$$

whose determinant is non-zero by (Ae1); so $\det Y_{\mathrm{A}}$ is a unit of the local ring A. With $E_{\mathrm{A}} = E_{11} + E_{33}$, the Cayley–Hamilton theorem for Y_{A} gives $Y^2 - \mathrm{tr}(Y_{\mathrm{A}})Y + \det(Y_{\mathrm{A}})E_{\mathrm{A}} = 0$, so

$E_{\mathrm{A}} = \det(Y_{\mathrm{A}})^{-1}(\mathrm{tr}(Y_{\mathrm{A}})Y - Y^2) \in S$. Next $H = h(r_A) = r_A(g_0) + \theta(g_0)r_A(g_0^{-1}) \in S$ has $H_{13} = H_{31} = 0$ by (Se3), so $E_{\mathrm{A}}HE_{\mathrm{A}} = H_{11}E_{11} + H_{33}E_{33}$ and

$$E_{\mathrm{A}}HE_{\mathrm{A}} - H_{33}E_{\mathrm{A}} = (H_{11} - H_{33})E_{11}.$$

Modulo $\mathfrak{m}_A$, $H_{11} - H_{33} \equiv \theta(g_0) + 1 - \alpha(g_0) - \theta(g_0)\alpha(g_0)^{-1} = (\alpha(g_0) - 1)(\Psi(g_0) - 1)$, a unit by (Sg4). So $E_{11} \in S$, $E_{33} = E_{\mathrm{A}} - E_{11} \in S$, and $E_{44} = E_{11}^{\natural}$, $E_{22} = E_{33}^{\natural} \in S$.

(Ae3). Write $r = f \circ r_A$. If $E_{ii} = \sum_t a_t r_A(t)$ with $a_t \in A$, then $E_{ii} = f(E_{ii}) = \sum_t f(a_t)r(t)$.

(Ae4). $r(1) = 1$. Let S_V be the V-span of $r(G_{\mathbb{Q}})$, closed under products. If $r(g)_{ij} \neq 0$ and $r(h)_{jk} \neq 0$, then $E_{ii}r(g)E_{jj}r(h)E_{kk} \in S_V$ by (Ae3) has (i,k) entry $r(g)_{ij}r(h)_{jk} \neq 0$ (V is a domain), so $r(t)_{ik} \neq 0$ for some t. By (Se1), $r(g)$ has similitude $a(\theta(g)) \in V^{\times}$, so $r(g^{-1}) = a(\theta(g))^{-1}r(g)^{\natural}$, and $(X^{\natural})_{j^*i^*} = \pm X_{ij}$ (as $J_4 e_b = \pm e_{b^*}$); hence $r(g)_{ij} \neq 0$ implies $r(g^{-1})_{j^*i^*} \neq 0$.

(Ae5). If every entry is active, $E_{ii}r(g)E_{jj} = r(g)_{ij}E_{ij}$ with $r(g)_{ij} \neq 0$ for some g, so every E_{ij} lies in the K-span of $r(G_{\mathbb{Q}})$, which is therefore $\mathrm{M}_4(K)$. Conversely, if (i,j) is not active, every element of the K-span has entry (i,j) equal to 0, so the span is not $\mathrm{M}_4(K)$. □

45.3 Smoothness and points of the weight-zero algebra

This subsection proves Proposition 45.28: the part of λ-weight zero of a symplectic weighted model satisfies the conditions of the criterion, carries the Tate–Shafarevich classes and parametrises the points of the datum; the reducible points, which (Gc3) and (Gc4) concern, are analysed in Subsection 45.4. The part of weight zero is a smooth graded algebra with a fixed point, with explicit tangent dimensions (Lemma 45.29), and the tangent vectors of torus weights $(0,-1)$ and $(0,0)$ at the Eisenstein point vanish (Lemma 45.30). The Tate–Shafarevich classes enter through normalised cocycles (Lemma 45.32), and here again Part VII is used: the unramified classes of a finite-order character times a negative power of the cyclotomic character that are trivial at ℓ vanish (Lemma 45.31), which rests on Proposition 40.3. When the datum is finite in weight zero, the points of the algebra lift and scale to the deformation ring and the weight parts are spanned by monomials in the entries (Lemma 45.33, Lemma 45.34).

Proposition 45.28. *Let $\mathfrak{D} = (\ell, \mathcal{O}, e, N, \chi, \xi, S, \dots)$ be a symplectic Eisenstein datum, $n_S = \prod_{p \in S} p$, and M a symplectic weighted model of $\mathfrak{D}$ with integers r, r', m. Then there are elements $a_g \in (A_M)_0$ with $x_M(a_g) = 0$ ($g \in G_{\mathbb{Q}}$) such that:*

(Mc1) *(A_M, x_M) is a graded $\mathcal{O}$-algebra with a fixed point, smooth with tangent dimensions* td, *where* $\mathrm{td}(-2), \dots, \mathrm{td}(2)$ *are* $0, r, m, r'+1, 1$ *and* $\mathrm{td}(j) = 0$ *for* $|j| > 2$;

(Mc2) *the parts u_{-1}, u_{-2}, u_0 of u_M span rationally in degrees -1, -2, 0 respectively;*

(Mc3) *for every $b \in \mathcal{O} \setminus \{0\}$, writing* $\mathrm{Sha}[b]$ *for the submodule of* $\mathrm{Sha}^1_{n_S}(\chi, e)$ *killed by b, there are $a \in b\mathcal{O} \setminus \{0\}$ and an injective additive map $D\colon \mathrm{Sha}[b] \to \mathrm{Hom}_{\mathcal{O}}(A_M, \mathcal{O}/a\mathcal{O})$ such that $D(\mu)(u_{-1,i}) = 0$ for all μ and i, and every $D(\mu)$ is the reduction modulo a of an $\mathcal{O}$-linear map $D'\colon A_M \to \mathcal{O}$ with $D'(fh) = x_M(f)D'(h) + x_M(h)D'(f)$ for all $f, h \in A_M$ and $D'((A_M)_j) = 0$ for all $j \neq -1$;*

and, if $\mathfrak{D}$ is finite in weight zero, there is a map $x \mapsto r_x$ from the points of A_M on u_M to the points of $\mathfrak{D}$ such that:

(Mc4) *$x(a_g) = \mathrm{tr}\, r_x(g) - \mathrm{tr}\, r_0(g)$ for every point x of A_M on u_M and every $g \in G_{\mathbb{Q}}$; and for points x, x' of A_M on u_M, if $r_{x'}$ is a torus conjugate of r_x, then x' is a torus translate of x;*

(Mc5) *(A_M, x_M, u_M) satisfies (Gc2) and (Gc3); if $\mathfrak{D}$ has no endoscopic points, then r_x is irreducible for every two-signed point x of A_M on u_M; if moreover every irreducible point of $\mathfrak{D}$ is rigid, then (A_M, x_M, u_M) satisfies (Gc4).*

Proof. Write $E_{\mathcal{O}} = \mathcal{O}[\ell^{-1}]$, $P = \mathcal{O}[[X_s : s \in \sigma]]$, $\pi\colon P \to R$ the projection, $z_s = \pi(X_s)$, and $a_s(G)$ for the coefficient of X_s in $G \in P$. For $c \in \mathcal{O} \setminus \{0\}$ let $\mathcal{O}_c[\epsilon] = \mathcal{O}[\epsilon]/(\epsilon^2, c\epsilon)$ (for $c = 0$, $\mathcal{O}[\epsilon]/(\epsilon^2)$); these rings lie in $\mathrm{CNL}_{\mathcal{O}}$.

Step 0: preliminaries. (a) x_R kills $\ker q$: by (Wm5) for $g = 1$, $i = j = 1$ there is f_1 with $q(f_1) = 1$ and $x_R(f_1) = 1$; for $h \in \ker q$, $q(f_1 + h) = 1$, so $x_R(f_1 + h) = 1$ and $x_R(h) = 0$. Hence x_R induces an $\mathcal{O}$-algebra homomorphism $\bar{x}\colon S \to \mathcal{O}$ with $\bar{x} \circ q = x_R$, and $\bar{x}(r_S(g)) = r_0(g)$ by (Wm5). (b) *Uniqueness of derivations.* Let $c \in \mathcal{O}$ and $\Delta\colon R \to \mathcal{O}/c\mathcal{O}$ an $\mathcal{O}$-linear map with $\Delta(fh) = x_R(f)\Delta(h) + x_R(h)\Delta(f)$ and $\Delta(\ker q) = 0$, such that for all g, i, j some h with $q(h) = r_S(g)_{ij}$ has $\Delta(h) = 0$. Then $\Delta = 0$: the homomorphism $x_R + \epsilon\Delta\colon R \to \mathcal{O}_c[\epsilon]$ kills $\ker q$ (by (a)) and induces $F_\Delta\colon S \to \mathcal{O}_c[\epsilon]$ with $F_\Delta(r_S(g)_{ij}) = \bar{x}(r_S(g)_{ij})$; by (Wm4), $F_\Delta = \bar{x}$, so $\Delta = 0$. (c) *The elements a_g.* By (Wm5) choose $f_{g,i} \in R_{\mathrm{wt}_4(i,i)} = R_{(0,0)}$ with $q(f_{g,i}) = r_S(g)_{ii}$, and put $a_g = \sum_{i=1}^4 f_{g,i} - \operatorname{tr} r_0(g) \in (A_M)_0$; by (Wm5), $x_M(a_g) = \sum_i r_0(g)_{ii} - \operatorname{tr} r_0(g) = 0$.

(Mc1) is (La1) and (La2) of Lemma 45.29.

(Mc2). Let ϕ, J, J_w be a chart of R^{gr} as in (Wm1). In degree -2, $J_{(0,-2)} = \emptyset$ ($t_{r,r',m}$ vanishes at $(0,-2)$), so by (La3) the empty family u_{-2} spans rationally. Let $n \in \{-1, 0\}$; the elements of u_n have $x_M = 0$ (by (Wm2): x_R, induced by the constant term, kills $R_{(0,-1)}$, and $x_R(Z_i) = 0$). By (La3) it suffices to show that the vectors $(\mathrm{coeff}_{Y_i}\phi(u_{n,k}))_{i \in J_{(0,n)}}$ span $E_{\mathcal{O}}^{J_{(0,n)}}$. If not, there is $\lambda \in E_{\mathcal{O}}^{J_{(0,n)}} \setminus \{0\}$ with $\delta_0(u_{n,k}) = 0$ for all k, where $\delta_0(f) = \sum_{i \in J_{(0,n)}} \lambda_i \mathrm{coeff}_{Y_i}\phi(f)$ for $f \in R^{\mathrm{gr}}$. Then: δ_0 is $\mathcal{O}$-linear with $\delta_0(fh) = x_R(f)\delta_0(h) + x_R(h)\delta_0(f)$ (linear coefficients of a product of series, and (Sm2)); $\delta_0(R_v) = 0$ for $v \neq (0,n)$ (Sm1); so δ_0 kills every F_i, E_i, Z_i (those of weight $(0,n)$ by the choice of λ); and $\delta_0 \neq 0$, since by (Sm3) for ϕ some f has linear part a non-zero multiple of Y_i for an i with $\lambda_i \neq 0$. *Extension to R.* For $G \in \sum_w P_w$, write $G = G(0) + \sum_s a_s(G)X_s + \sum_s X_s G_s$ with $G_s \in \sum_w P_w$ without constant term (assign each monomial of degree ≥ 2 to the first variable occurring in it); as $x_R(z_s) = 0 = x_R(\pi(G_s))$ and $\delta_0(1) = 0$, $\delta_0(\pi(G)) = \sum_s a_s(G)v_s$ with $v_s = \delta_0(z_s) \in E_{\mathcal{O}}$. Choose $c \in \mathcal{O} \setminus \{0\}$ with all $cv_s \in \mathcal{O}$ and let $\tilde{\delta}\colon P \to \mathcal{O}$, $\tilde{\delta}(G) = \sum_s a_s(G)cv_s$; it satisfies $\tilde{\delta}(GH) = G(0)\tilde{\delta}(H) + H(0)\tilde{\delta}(G)$. For $G \in \mathfrak{a}$, $a_s(G) = a_s(G_{(\mathrm{wt}\, s)})$ and $G_{(\mathrm{wt}\, s)} \in \mathfrak{a}$, so $\tilde{\delta}(G) = \sum_v c\,\delta_0(\pi(G_{(v)})) = 0$ (sum over the weights v of variables). Hence $\tilde{\delta}$ induces an $\mathcal{O}$-linear $\delta\colon R \to \mathcal{O}$ with $\delta(fh) = x_R(f)\delta(h) + x_R(h)\delta(f)$ and $\delta|_{R^{\mathrm{gr}}} = c\,\delta_0$. It kills $\ker q = \sum_u Ru$ (u running over the F_i, E_i, Z_i): $\delta(hu) = x_R(h)\delta(u) + x_R(u)\delta(h) = 0$. So $x_R + \epsilon\delta$ induces $f_\delta\colon S \to \mathcal{O}[\epsilon]$ with $f_\delta(r_S(g)_{ij}) = r_0(g)_{ij} + \epsilon\, c\,\delta_0(h_{g,ij})$ for any $h_{g,ij} \in R_{\mathrm{wt}_4(i,j)}$ with $q(h_{g,ij}) = r_S(g)_{ij}$ (Wm5); the ϵ-part vanishes unless $\mathrm{wt}_4(i,j) = (0,n)$. By Lemma 45.30 (with $A = S$, $r_A = r_S$, $w = (0,n)$), all ϵ-parts vanish; by Step 0 (b) (with $c = 0$), $\delta = 0$, so $\delta_0 = 0$: a contradiction.

(Mc3). Let $b \in \mathcal{O} \setminus \{0\}$. By Lemma 45.32 there are $a \in b\mathcal{O} \setminus \{0\}$ and an injective additive map $\mu \mapsto z_\mu$ from $\mathrm{Sha}[b]$ to the normalised Sha cocycles modulo a. For each μ, (Wm6) gives an $\mathcal{O}$-linear $\delta_\mu\colon R \to \mathcal{O}$ with $\delta_\mu(fh) = x_R(f)\delta_\mu(h) + x_R(h)\delta_\mu(f)$, $\delta_\mu(R_w) = 0$ for $w \neq (0,-1)$, $\delta_\mu(\ker q) \subseteq a\mathcal{O}$, and $\delta_\mu(f) \equiv \theta(g)z_\mu(g)$, resp. $\Psi(g)z_\mu(g)$, modulo a whenever $q(f) = r_S(g)_{31}$, resp. $r_S(g)_{42}$. Let $D(\mu)\colon A_M \to \mathcal{O}/a\mathcal{O}$ be $\delta_\mu|_{A_M}$ followed by reduction, and $D' = \delta_\mu|_{A_M}$; then $D'(fh) = x_M(f)D'(h) + x_M(h)D'(f)$, $D'((A_M)_j) = \delta_\mu(R_{(0,j)}) = 0$ for $j \neq -1$, and $D(\mu)(F_i) = 0$ as $F_i \in \ker q$. *Additivity:* for μ, μ', apply Step 0 (b) with $c = a$ to $\Delta = \delta_{\mu+\mu'} - \delta_\mu - \delta_{\mu'}$ modulo a: it kills $\ker q$ modulo a; for the entries $(3,1)$, $(4,2)$ every h with $q(h) = r_S(g)_{ij}$ has $\Delta(h) = 0$, by additivity of $\mu \mapsto z_\mu$; for every other entry the element $h \in R_{\mathrm{wt}_4(i,j)}$ of (Wm5) has weight $\neq (0,-1)$, so $\Delta(h) = 0$. Hence $D(\mu+\mu') = D(\mu) + D(\mu')$. *Injectivity:* if $D(\mu) = 0$, then for every

g the element $h \in R_{\mathrm{wt}_4(3,1)} = R_{(0,-1)} \subseteq A_M$ with $q(h) = r_S(g)_{31}$ gives $0 = D(\mu)(h) = \theta(g) z_\mu(g)$, so $z_\mu = 0$ ($\theta(g)$ is a unit) and $\mu = 0$.

(Mc4). Let $\mathfrak{D}$ be finite in weight zero. For every point x of A_M on u_M choose $y_x \colon S \to \overline{\mathbb{Q}}_\ell$ and $t_x \in \overline{\mathbb{Q}}_\ell^\times$ as in (Pm2), and put $r_x(g) = y_x(r_S(g))$: a point of $\mathfrak{D}$, as r_S is a symplectic Eisenstein lift to $S \in \mathrm{CNL}_{\mathcal{O}}$ (Wm4). Since $f_{g,i} \in (A_M)_0$, $x(a_g) = \sum_i y_x(q(f_{g,i})) - \operatorname{tr} r_0(g) = \operatorname{tr} r_x(g) - \operatorname{tr} r_0(g)$. If $r_{x'}(g) = \mathrm{T}_4(s_1, s_2)\, r_x(g)\, \mathrm{T}_4(s_1, s_2)^{-1}$ for all g, then by (Pm3), for $f \in (A_M)_n = R_{(0,n)}$, $t_{x'}^n x'(f) = y_{x'}(q(f)) = s_2^n y_x(q(f)) = s_2^n t_x^n x(f)$; so $x' = \tau \cdot x$ with $\tau = s_2 t_x / t_{x'}$.

(Mc5). (Gc2) is (Pm1), and (Gc3) is (Mr1). Let $\mathfrak{D}$ have no endoscopic points and x be a two-signed point of A_M on u_M. By (Pm2), $y_x \circ q|_{A_M} = t_x \cdot x$, a torus translate of x; it lies on u_M and is two-signed ($t_x \neq 0$). By (Mr2), r_x is irreducible. If moreover every irreducible point of $\mathfrak{D}$ is rigid, (Gc4) is (Mr3). □

Lemma 45.29. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $E_{\mathcal{O}} = \mathcal{O}[\ell^{-1}]$, M a symplectic weighted model of $\mathfrak{D}$ with integers r, r', m, and $\phi \colon R^{\mathrm{gr}} \to E_{\mathcal{O}}[[Y_i : i \in J]]$, $\mathrm{wt}_J \colon J \to \mathbb{Z}^2$ a chart as in (Wm1). For $w \in \mathbb{Z}^2$ put $J_w = \mathrm{wt}_J^{-1}(w)$. Then:*

(La1) *$A_M = \bigoplus_{n \in \mathbb{Z}} (A_M)_n$, (A_M, x_M) is a graded $\mathcal{O}$-algebra with a fixed point, and $\{f \in A_M : q(f) = 0\} = I(u_M)$;*

(La2) *(A_M, x_M) is smooth with tangent dimensions* td, *where* $\mathrm{td}(-2), \dots, \mathrm{td}(2)$ *are* $0, r, m, r'+1, 1$ *and* $\mathrm{td}(j) = 0$ *for $|j| > 2$; more precisely, there is a chart ψ of (A_M, x_M) whose variables of degree $n \leq 0$ are the Y_i, $i \in J_{(0,n)}$, such that for all $f \in A_M$ and $i \in J_{(0,n)}$, $n \leq 0$, the coefficient of Y_i in $\psi(f)$ is the coefficient of Y_i in $\phi(f)$;*

(La3) *for $n \leq 0$ and a finite family $u = (u_k)_k$ of elements of $(A_M)_n$ with $x_M(u_k) = 0$: u spans rationally in degree n if and only if the vectors $\big($coefficient of Y_i in $\phi(u_k)\big)_{i \in J_{(0,n)}}$ span the $E_{\mathcal{O}}$-vector space $E_{\mathcal{O}}^{J_{(0,n)}}$.*

Proof. Write $P = \mathcal{O}[[X_s : s \in \sigma]]$, $\pi \colon P \to R$ for the projection, $z_s = \pi(X_s)$, $P_w = \mathcal{O}[[X_\sigma]]_w$, $A_P = \sum_w P_w$, $A_P^0 = \sum_n P_{(0,n)}$, $F(0)$ for the constant term of $F \in P$, $|n| = \sum_s n_s$ for $n \in \mathbb{N}^\sigma$, and $w = (\lambda(w), \mu(w))$ for $w \in \mathbb{Z}^2$. So $R_w = \pi(P_w)$, $R^{\mathrm{gr}} = \pi(A_P)$, $A_M = \pi(A_P^0)$. Put $\mathfrak{m}_R = \ker x_R \cap R^{\mathrm{gr}}$ and $\mathfrak{m}_A = \ker x_M$. As $\mathrm{wt}(\sigma) \subseteq \mathcal{W}_4$, every variable has λ-weight -1, 0 or 1; let $\sigma_0, \sigma_+, \sigma_-$ be the corresponding sets of indices, and call *generators* the monomials X_γ, $\gamma \in \Gamma = \sigma_0 \sqcup (\sigma_+ \times \sigma_-)$, namely X_s ($s \in \sigma_0$) and $X_p X_q$ ($p \in \sigma_+$, $q \in \sigma_-$); each has weight in $\{0\} \times \{-2, \dots, 2\}$, and $z_\gamma = \pi(X_\gamma)$.

Step 0: homogeneity. (i) Monomials of distinct weights are distinct and weights add under multiplication, so $P_v P_w \subseteq P_{v+w}$ and $(GH)_{(u)} = G_{(u-w)} H$ for $H \in P_w$. (ii) If $\pi(F) \in R_w$, say $\pi(F) = \pi(G)$ with $G \in P_w$, then $(F - G)_{(w)} = F_{(w)} - G \in \mathfrak{a}$ ($\mathfrak{a}$ is homogeneous), so $\pi(F) = \pi(F_{(w)})$. (iii) If $\sum_w y_w = 0$ with $y_w = \pi(G_w)$, $G_w \in P_w$ (finitely many), then $\sum_w G_w \in \mathfrak{a}$ and its components G_w lie in $\mathfrak{a}$; so $y_w = 0$ for all w. (iv) $x_R(\pi(F)) = F(0)$ by definition of the origin; hence $x_R(R_w) = 0$ for $w \neq 0$, and for $y \in R^{\mathrm{gr}}$ with components y_w, $x_R(y_0) = x_R(y)$. So every element of $\mathfrak{m}_R$ is a sum of homogeneous elements of $\mathfrak{m}_R$, every element of $\mathfrak{m}_R^k$ is a sum of products of k homogeneous elements of $\mathfrak{m}_R$, and a homogeneous $y \in \mathfrak{m}_R$ of weight w is $\pi(Y)$ with $Y \in P_w$, $Y(0) = 0$ (by (ii) and (iv)). (v) Let $\mathfrak{b} = \pi^{-1}(\ker q)$. By (Wm2), (Wm3) and (ii), $\mathfrak{b} = \mathfrak{a} + (\tilde{u}_k)_k P$ for lifts $\tilde{u}_k \in P_{(0,-1)}, P_{(0,1)}, P_{(0,0)}$ of the F_i, E_i, Z_i; for $H = a + \sum_k B_k \tilde{u}_k$ with $a \in \mathfrak{a}$, (i) gives $H_{(w)} = a_{(w)} + \sum_k (B_k)_{(w - \mathrm{wt}\tilde{u}_k)} \tilde{u}_k \in \mathfrak{b}$.

Step 1: monomials of λ-weight zero. A monomial X^n of λ-weight 0 contains as many factors from σ_+ as from σ_-; pairing them writes X^n as a product of g generators with $|n| \leq 2g$. Hence if $|n| \geq 3$, then $g \geq 2$; if $|n| = 2$, then X^n is $X_s X_{s'}$ with $s, s' \in \sigma_0$ ($g = 2$) or a single generator

X_pX_q. Given $F \in P_{(0,n)}$ with $F(0) = 0$, assign each monomial of F that is a product of at least two generators to one pair (γ, γ') of generators dividing it with λ-weight 0 quotient (finitely many pairs); grouping terms (the maps $n \mapsto n - n_\gamma - n_{\gamma'}$ are injective) gives

$$F = \sum_{\gamma\in\Gamma} c_\gamma X_\gamma + \sum_{(\gamma,\gamma')} X_\gamma X_{\gamma'} H_{\gamma\gamma'}, \qquad c_\gamma \in \mathcal{O},\ H_{\gamma\gamma'} \in A_P^0, \tag{45.2}$$

with c_γ the coefficient of the monomial X_γ in F, and all $c_\gamma = 0$ if every monomial of F has degree at least 3. As $z_\gamma \in \mathfrak{m}_A$ for every γ ($x_M(z_\gamma) = 0$ by (iv)), π of the second sum lies in $\mathfrak{m}_A^2$. Moreover, by Dickson's lemma the set $E_{(0,n)} \subseteq \mathbb{N}^\sigma$ of exponents of weight $(0, n)$ has finitely many minimal elements, every $n' \in E_{(0,n)}$ is $\geq$ one of them with difference in $E_{(0,0)}$, and so $P_{(0,n)} = \sum_m X^m P_{(0,0)}$ over the minimal m; for $n = 0$ the minimal elements $m_1, \dots, m_k$ of $E_{(0,0)} \setminus \{0\}$ generate the monoid $E_{(0,0)}$ (induction on $|n'|$), and $\mathcal{O}[[T_1, \dots, T_k]] \to P_{(0,0)}$, $T_i \mapsto X^{m_i}$, is onto (send $\sum c_{n'} X^{n'}$ to $\sum c_{n'} T^{e(n')}$ for a choice $e(n')$ with $\sum_i e(n')_i m_i = n'$; $n' \mapsto e(n')$ is injective), with kernel in $(\mathfrak{m}_\mathcal{O}, T)$ as constant terms are preserved. So $P_{(0,0)}$ is a complete Noetherian local ring with residue field $\mathbb{F}_\mathcal{O}$.

Step 2: (La1). By Step 0 (iii), the sum $A_M = \sum_n \pi(P_{(0,n)})$ is direct, and $(A_M)_m(A_M)_n \subseteq (A_M)_{m+n}$ by (i). $(A_M)_0 = \pi(P_{(0,0)}) \cong P_{(0,0)}/(\mathfrak{a} \cap P_{(0,0)})$ with $\mathfrak{a} \cap P_{(0,0)}$ inside the maximal ideal (elements of $\mathfrak{a}$ have constant term 0), so it is a complete Noetherian local ring, $\mathcal{O} \to (A_M)_0$ is local and the residue fields agree (Step 1). Each X^m in Step 1 is a product of generators, those of μ-weight 0 lying in $P_{(0,0)}$; so A_M is generated by $(A_M)_0$ and the finitely many z_γ with $\mu(\operatorname{wt}\gamma) \in \{\pm 1, \pm 2\}$. x_M kills $(A_M)_n$ for $n \neq 0$ by (iv). So (A_M, x_M) is a graded $\mathcal{O}$-algebra with a fixed point. The elements F_i, E_i, Z_i lie in $\ker q$ (Wm3), so $I(u_M) \subseteq \ker q$. Conversely let $f = \pi(G) \in A_M$, $G \in A_P^0$, with $q(f) = 0$; then $G \in \mathfrak{b}$, and by Step 0 (v) each component $G_{(0,n)} = a_{(0,n)} + \sum_k (B_k)_{(0,n)-\operatorname{wt}\tilde{u}_k} \tilde{u}_k$, with coefficients in A_P^0 since $\operatorname{wt}\tilde{u}_k$ has λ-weight 0; applying π, $f = \sum_n \pi(G_{(0,n)}) \in I(u_M)$.

Step 3: the chart. Let $i_\alpha, i_\beta, i_\theta$ be the unique elements of J of weights $(1, 0)$, $(-1, 1)$, $(1, 1)$, and $J_0 = J \setminus \{i_\alpha, i_\beta, i_\theta\}$, the indices of λ-weight 0 (counts: r' of weight $(0, 1)$, r of $(0, -1)$, m of $(0, 0)$). Let $\Lambda = E_\mathcal{O}[[Y_J]]$ and Λ^0 the series all of whose monomials have λ-weight 0; these monomials are $Y_{i_\alpha}^a Y_{i_\theta}^c Y_{i_\beta}^{a+c} Y^n$ with Y^n a monomial in the Y_i, $i \in J_0$. Let $\Lambda' = E_\mathcal{O}[[Y_i\,(i \in J_0), Y_{\alpha\beta}, Y_{\theta\beta}]]$ with μ-weights $\mu(\operatorname{wt}_J(i))$, 1, 2, and $\kappa\colon \Lambda^0 \to \Lambda'$ the coefficientwise map $Y_{i_\alpha}^a Y_{i_\theta}^c Y_{i_\beta}^{a+c} Y^n \mapsto Y_{\alpha\beta}^a Y_{\theta\beta}^c Y^n$: a bijection on monomials, multiplicative, hence a ring isomorphism; it preserves μ-weights ($c + (a + c) = a + 2c$), constant terms and the coefficients of the Y_i, $i \in J_0$, and it sends monomials of degree $2a + 2c + |n| \geq 3$ to monomials of degree $a + c + |n| \geq 2$. By (Sm1) for ϕ, $\phi(A_M) \subseteq \Lambda^0$; put $\psi = \kappa \circ \phi|_{A_M}$. The numbers of variables of μ-weight $-2, \dots, 2$ are $0, r, m, r' + 1, 1$, the variables of degree $n \leq 0$ are the Y_i, $i \in J_{(0,n)}$, and the coefficient assertion of (La2) holds. (Sm1) and (Sm2) for ψ follow from those for ϕ.

Step 4: (Sm3) for ψ. By (Sm3) for ϕ there are $f \in R^{\mathrm{gr}}$ whose linear part is a non-zero multiple c_iY_i of Y_i; replacing f by its component of weight $\operatorname{wt}_J(i)$ (whose image is the weight-$\operatorname{wt}_J(i)$ part of $\phi(f)$ by (Sm1)) and subtracting $x_R(f)$, we get $f_i \in R_{\operatorname{wt}_J(i)} \cap \mathfrak{m}_R$ with linear part c_iY_i, $c_i \neq 0$ ($i \in J$). Put $g_{\alpha\beta} = f_{i_\alpha} f_{i_\beta} \in R_{(0,1)}$ and $g_{\theta\beta} = f_{i_\theta} f_{i_\beta} \in R_{(0,2)}$, elements of A_M. Since $\phi(f_{i_\alpha}) = c_{i_\alpha} Y_{i_\alpha} + (\text{degree} \geq 2)$ and likewise for i_β, $\phi(g_{\alpha\beta}) = c_{i_\alpha} c_{i_\beta} Y_{i_\alpha} Y_{i_\beta} + (\text{degree} \geq 3)$, so $\psi(g_{\alpha\beta}) = c_{i_\alpha} c_{i_\beta} Y_{\alpha\beta} + (\text{degree} \geq 2)$; similarly $\psi(g_{\theta\beta}) = c_{i_\theta} c_{i_\beta} Y_{\theta\beta} + (\text{degree} \geq 2)$. For $i \in J_0$, $f_i \in A_M$ and $\psi(f_i)$ has linear part $c_iY_i + e_iY_{\alpha\beta} + e_i'Y_{\theta\beta}$ with $e_i, e_i' \in E_\mathcal{O}$; choose $k \geq 0$ with $o_i = \ell^k e_i/(c_{i_\alpha} c_{i_\beta})$ and $o_i' = \ell^k e_i'/(c_{i_\theta} c_{i_\beta})$ in $\mathcal{O}$; then $\ell^k f_i - o_i g_{\alpha\beta} - o_i' g_{\theta\beta} \in A_M$ has linear part $\ell^k c_i Y_i$. This is (Sm3).

Step 5: products in A_M. (a) If $y_1, y_2, y_3 \in \mathfrak{m}_R$ are homogeneous and $y_1y_2y_3 \in A_M$, then $y_1y_2y_3 \in \mathfrak{m}_A^2$: by Step 0 (iv) write $y_k = \pi(Y_k)$, $Y_k \in P_{w_k}$, $Y_k(0) = 0$; then $Y_1Y_2Y_3 \in P_{(0,n)}$ has only monomials of degree ≥ 3, and (45.2) with all $c_\gamma = 0$ gives the claim. (b) If $y_1, y_2 \in \mathfrak{m}_R$ are

homogeneous and $y_1y_2 \in A_M$, then in the same way $y_1y_2 \in \mathfrak{m}_A^2 + \sum_{p\in\sigma_+,q\in\sigma_-} \mathcal{O} z_pz_q$ (the only single generators of degree 2 are the X_pX_q). (c) By Step 0 (iv) and (iii) (compare components of weight $(0,n)$), every element of $\mathfrak{m}_R^3 \cap (A_M)_n$ lies in $\mathfrak{m}_A^2$ and every element of $\mathfrak{m}_R^2 \cap (A_M)_n$ in $\mathfrak{m}_A^2 + \sum \mathcal{O} z_pz_q$.

Step 6: the pairs. Let $p \in \sigma_+$, $q \in \sigma_-$. We show: there are $d \in \mathcal{O}\setminus\{0\}$ and $e, e' \in \mathcal{O}$ with $dz_pz_q - eg_{\alpha\beta} - e'g_{\theta\beta} \in \mathfrak{m}_A^2$. By (Sm1), the linear part of $\phi(z_p)$ involves only variables of weight $\mathrm{wt}(p) \in \{(1,0),(1,1),(1,-1)\}$: it is $L_pY_{i_\alpha}$, $L_pY_{i_\theta}$ or 0 with $L_p \in E_{\mathcal{O}}$. In the first case choose $d_p \in \mathcal{O}\setminus\{0\}$, $o_p \in \mathcal{O}$ with $d_pL_p = o_pc_{i_\alpha}$; then $h = d_pz_p - o_pf_{i_\alpha} \in R_{(1,0)} \cap \mathfrak{m}_R$ has zero linear part, so by (Sm4) for ϕ, $c\,h \in \mathfrak{m}_R^2$ for some $c \neq 0$: $cd_pz_p = co_pf_{i_\alpha} + k_p$ with $k_p \in \mathfrak{m}_R^2 \cap R_{(1,0)}$. The second case is the same with f_{i_θ}; in the third, $cz_p = k_p \in \mathfrak{m}_R^2$. Likewise $c'd_qz_q = c'o_qf_{i_\beta} + k_q$ if $\mathrm{wt}(q) = (-1,1)$, and $c'z_q = k_q \in \mathfrak{m}_R^2$ if $\mathrm{wt}(q) \in \{(-1,0),(-1,-1)\}$. Multiplying the two relations, the product of the constants times z_pz_q equals $o\,f_{i_*}f_{i_\beta}$ (when both first relations occur, $f_{i_*}f_{i_\beta} \in \{g_{\alpha\beta}, g_{\theta\beta}\}$) plus terms each of which is a product of a homogeneous element of $\mathfrak{m}_R$ and a homogeneous element of $\mathfrak{m}_R^2$, of total weight $\mathrm{wt}(p)+\mathrm{wt}(q)$, of λ-weight 0; these lie in $\mathfrak{m}_A^2$ by Step 5 (a), (c).

Step 7: (Sm4) for ψ. Let $f \in A_M$ with $x_M(f) = 0$ and all linear coefficients of $\psi(f)$ zero. By (Sm1) the components f_n have the same properties ($\psi(f_n)$ is the weight-n part of $\psi(f)$, and $x_M(f_0) = x_M(f)$), and $\mathrm{Sq}(x_M)$ is an $\mathcal{O}$-module; so let $f \in (A_M)_n$, $f = \pi(F)$ with $F \in P_{(0,n)}$, $F(0) = 0$. By (45.2), $f \equiv \sum_{s\in\sigma_0} c_sz_s + \sum_{p,q} c_{pq}z_pz_q$ modulo $\mathfrak{m}_A^2$. For $s \in \sigma_0$ of weight $(0,n)$ the linear part of $\phi(z_s)$ is $\sum_{i\in J_{(0,n)}} L_{si}Y_i$; choose $d_s \neq 0$, $o_{si} \in \mathcal{O}$ with $d_sL_{si} = o_{si}c_i$; then $d_sz_s - \sum_i o_{si}f_i \in R_{(0,n)} \cap \mathfrak{m}_R$ has zero linear part, so a non-zero multiple of it lies in $\mathfrak{m}_R^2 \cap (A_M)_n$, hence (Step 5 (c)) in $\mathfrak{m}_A^2 + \sum \mathcal{O} z_pz_q$. With Step 6, there are $c \in \mathcal{O}\setminus\{0\}$ and e_i $(i \in J_0), e_{\alpha\beta}, e_{\theta\beta} \in \mathcal{O}$ with

$$cf - \sum_{i\in J_0} e_if_i - e_{\alpha\beta}g_{\alpha\beta} - e_{\theta\beta}g_{\theta\beta} \in \mathfrak{m}_A^2.$$

Elements of $\mathfrak{m}_A^2$ have images without constant and linear terms ($\psi(\mathfrak{m}_A)$ has zero constant terms by (Sm2)). Comparing linear coefficients, using Step 4: at Y_i $(i \in J_0)$, $0 = e_ic_i$, so $e_i = 0$; then at $Y_{\alpha\beta}$ and $Y_{\theta\beta}$, $e_{\alpha\beta} = e_{\theta\beta} = 0$. So $cf \in \mathfrak{m}_A^2$ and $f \in \mathrm{Sq}(x_M)$. Hence ψ is a chart, which proves (La2).

Step 8: (La3). Let $n \leq 0$ and $\mathrm{Lin}(f) \in E_{\mathcal{O}}^{J_{(0,n)}}$ the vector of coefficients of the Y_i, $i \in J_{(0,n)}$, in $\psi(f)$, equivalently in $\phi(f)$; for $f \in (A_M)_n$ these are all the linear coefficients of $\psi(f)$ (Sm1), and Lin kills $\mathrm{Sq}(x_M)$. If the $\mathrm{Lin}(u_k)$ span, let $p \in \mathrm{Cot}_n$ be the class of $f \in (A_M)_n \cap \mathfrak{m}_A$, write $\mathrm{Lin}(f) = \sum_k e_k\mathrm{Lin}(u_k)$ and choose $c \neq 0$ with all $ce_k \in \mathcal{O}$; then $cf - \sum_k ce_ku_k$ has $x_M = 0$ and no linear terms, so it lies in $\mathrm{Sq}(x_M)$ (Sm4), and $cp \in \mathrm{Cot}(u)$. Conversely, if u spans rationally, then for $i \in J_{(0,n)}$ the degree-n component of the element of Step 4 for Y_i, minus its value under x_M, has Lin a non-zero multiple of the basis vector at i, and some non-zero multiple of its class lies in $\mathrm{Cot}(u)$; applying Lin, that basis vector lies in the span of the $\mathrm{Lin}(u_k)$. □

Lemma 45.30. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, and $\mathcal{O}[\epsilon] = \mathcal{O}[\epsilon]/(\epsilon^2)$. Let $A \in \mathrm{CNL}_{\mathcal{O}}$, r_A a symplectic Eisenstein lift to A and $f\colon A \to \mathcal{O}[\epsilon]$ an $\mathcal{O}$-algebra homomorphism with $f(r_A(g)_{ij}) = r_0(g)_{ij} + \epsilon\, y_{ij}(g)$, $y_{ij}(g) \in \mathcal{O}$, for all $g \in G_{\mathbb{Q}}$ and $1 \leq i,j \leq 4$. Let $w \in \{(0,-1),(0,0)\}$. If $y_{ij} = 0$ for every (i,j) with $\mathrm{wt}_4(i,j) \neq w$, then $y_{ij} = 0$ for all i,j.*

Proof. Write $E_{\mathcal{O}} = \mathcal{O}[\ell^{-1}]$, $f = f_0 + \epsilon f_1$ with $f_0\colon A \to \mathcal{O}$ an $\mathcal{O}$-algebra homomorphism and f_1 an $\mathcal{O}$-linear map with $f_1(ab) = f_0(a)f_1(b) + f_0(b)f_1(a)$. Put $Y(g) = (y_{ij}(g))$ and $Z(g) = Y(g)\,r_0(g)^{-1}$, so $Z(g)_{ij} = y_{ij}(g)\kappa_j(g)^{-1}$ and $f(r_A(g)) = (1 + \epsilon Z(g))\,r_0(g)$; write $z_{ij} = Z_{ij}$. It suffices to show $Z = 0$.

Step 1: Z is a continuous $\mathfrak{sp}_4$-valued cocycle. f_0 is local (for $a \in \mathfrak{m}_A$ and $u \in \mathcal{O}^\times$, $a-u$ is a unit, so $f_0(a) \neq u$), and by induction $f_1(\mathfrak{m}_A^{N+1}) \subseteq \mathfrak{m}_\mathcal{O}^N$ (for $a \in \mathfrak{m}_A^N$, $b \in \mathfrak{m}_A$: $f_1(ab) = f_0(a)f_1(b) + f_0(b)f_1(a)$, $f_0(a) \in \mathfrak{m}_\mathcal{O}^N$, $f_0(b) \in \mathfrak{m}_\mathcal{O}$). As r_A is continuous, each y_{ij}, hence each z_{ij}, is continuous for the ℓ-adic topology. Comparing ϵ-parts in $f(r_A(gh)) = f(r_A(g))f(r_A(h))$ gives $Z(gh) = Z(g) + r_0(g)Z(h)r_0(g)^{-1}$, that is

$$z_{ij}(gh) = z_{ij}(g) + (\kappa_i/\kappa_j)(g)\, z_{ij}(h).$$

By (Se1), ${}^t f(r_A(g))\, J_4\, f(r_A(g)) = \theta(g)J_4$; its ϵ-part is ${}^t r_0(g)({}^t Z(g)J_4 + J_4 Z(g))r_0(g) = 0$, so $Z(g) \in \mathfrak{sp}_4(\mathcal{O})$. The entry $(1,2)$ of ${}^t Z J_4 + J_4 Z$ is $-z_{31} + z_{42}$ (J_4 has entries $(1,4),(2,3)$ equal to 1 and $(3,2),(4,1)$ equal to -1); hence $z_{42} = z_{31}$.

Step 2: two facts on cocycles. Let H be a group, $\kappa\colon H \to E_\mathcal{O}^\times$ a homomorphism and $z\colon H \to E_\mathcal{O}$ with $z(gh) = z(g) + \kappa(g)z(h)$. (a) If z vanishes on a subgroup U of finite index, there is $m \in E_\mathcal{O}$ with $z(g) = \kappa(g)m - m$ on H. Indeed, for $x \in \ker\kappa$ some x^n, $1 \leq n \leq [H:U]$, lies in U (two of the cosets $x^k U$, $0 \leq k \leq [H:U]$, coincide), and z is additive on $\ker\kappa$, so $nz(x) = 0$ and $z(x) = 0$. If $\kappa = 1$ take $m = 0$. Otherwise pick s with $\kappa(s) \neq 1$; for $g \in H$ the commutator $g^{-1}s^{-1}gs$ lies in $\ker\kappa$, so $z(gs) = z(sg)$, i.e. $z(g) + \kappa(g)z(s) = z(s) + \kappa(s)z(g)$, and $z(g) = \kappa(g)m - m$ with $m = z(s)/(\kappa(s)-1)$. (b) If $z(h) = \kappa(h)m - m$ for h in a subgroup H_1 and $t \in H$, then $z(x) = \kappa(x)m' - m'$ for $x \in tH_1t^{-1}$, with $m' = \kappa(t)m - z(t)$: for $x = tht^{-1}$, $z(x) + \kappa(x)z(t) = z(xt) = z(th) = z(t) + \kappa(t)(\kappa(h)m - m)$ and $\kappa(x) = \kappa(h)$.

Step 3: local behaviour. (i) Let $\bar{w}$ be a place above a prime $p \notin T$. By (Se2), $Z = 0$ on $I_{\bar{w}} \cap \ker\bar{r}_0$, a subgroup of finite index of $I_{\bar{w}}$ ($\bar{r}_0$ has finite image); by Step 2 (a), each z_{ij} is a coboundary on $I_{\bar{w}}$. (ii) Let $v \in T$, $v \neq \ell$. By (Se4) there are a domain $B \in \mathrm{CNL}_\mathcal{O}$, a lift ρ_B of $\bar{r}_0|_{D_v}$ satisfying the local condition at v and $y\colon B \to A$ with $y \circ \rho_B = r_A|_{D_v}$; so $f(r_A(g)) = 1$ for $g \in I_v \cap \ker\chi$, a subgroup of finite index of I_v (χ has finite order), and each z_{ij} is a coboundary on I_v, hence (Step 2 (b)) on $I_{\bar{w}} = tI_v t^{-1}$ for every place $\bar{w} = t\bar{v}$ above v. (iii) At ℓ, (Se4) gives B, ρ_B, y with ρ_B ordinary and $y \circ \rho_B = r_A|_{D_\ell}$. The conditions defining ordinary representations are polynomial identities in the entries of $\rho(s_0)$ and $\rho(g)$ with coefficients in $\mathcal{O}$, so $\rho' = f \circ r_A|_{D_\ell}$ is ordinary, as a homomorphism to $\mathrm{GL}_4(E_\mathcal{O}[\epsilon])$. Put $c = \chi_{\mathrm{cyc}}(s_0)$ and $c_k = \kappa_k(s_0)$; by (Sg5) and (Sg1) (ξ is unramified at $\ell \in T$), $(c_1,\dots,c_4) = (c^{e+1}, c^e, c, 1)$, pairwise distinct as c has infinite order and $e \geq 2$; put $d_i = \prod_{k\neq i}(c_i - c_k) \neq 0$, $D = \mathrm{diag}(c_1,\dots,c_4)$, and E_i the diagonal idempotent at i, so that $\prod_{k\neq i}(D - c_k) = d_i E_i$.

Step 4: the case $w = (0,0)$. Here $Z(g)$ is diagonal and each $z_i = z_{ii}$ is a continuous homomorphism $G_\mathbb{Q} \to \mathcal{O}$ ($\kappa_i/\kappa_i = 1$). At ℓ: $\rho'(s_0) = \mathrm{diag}(c_k(1+\epsilon z_k(s_0)))$, so $\Pi_i = \prod_{k\neq i}(\rho'(s_0) - c_k)$ is diagonal with entry (i,i) equal to $d_i + \epsilon(\cdot)$; for $s \in I_\ell$, $\rho'(s) - \kappa_i(s)$ is diagonal with entry (i,i) equal to $\epsilon\kappa_i(s)z_i(s)$; the entry (i,i) of $\Pi_i(\rho'(s) - \kappa_i(s))\Pi_i = 0$ is $\epsilon d_i^2 \kappa_i(s) z_i(s)$, so $z_i = 0$ on I_ℓ, hence on every $tI_\ell t^{-1}$. At $p \neq \ell$: by Step 3 (i), (ii) and Step 2 (a) with $\kappa = 1$, z_i vanishes on every inertia group. For $N \geq 1$ the reduction of z_i modulo $\ell^N\mathcal{O}$ is a continuous homomorphism with finite abelian image that kills every inertia group; its kernel cuts out a finite abelian extension of $\mathbb{Q}$ unramified at every prime, which is $\mathbb{Q}$ by Minkowski's theorem. So $z_i(G_\mathbb{Q}) \subseteq \bigcap_N \ell^N\mathcal{O} = 0$, and $Z = 0$.

Step 5: the case $w = (0,-1)$. The entries of weight $(0,-1)$ are $(3,1)$ and $(4,2)$, so $Z(g)$ has at most the entries $z_{31}(g)$ and $z_{42}(g) = z_{31}(g)$, and $z = z_{31}$ is a continuous cocycle for $\kappa_3/\kappa_1 = \alpha/\theta = \Psi^{-1}$. *At ℓ:* $\rho'(s_0) = D + \epsilon N D$ with $N = Z(s_0)$. Let X have entries $X_{31} = -N_{31}c_1/(c_3 - c_1)$, $X_{42} = -N_{42}c_2/(c_4 - c_2)$ and 0 elsewhere, and $P = 1 + \epsilon X$; then $(DX - XD)_{ak} = (c_a - c_k)X_{ak}$, so $P^{-1}\rho'(s_0)P = D + \epsilon(ND + DX - XD) = D$. Hence $\Pi_i = P\prod_{k\neq i}(D - c_k)P^{-1} = d_i P E_i P^{-1}$, and the ordinary condition $\Pi_j \rho'(g)\Pi_i = 0$ ($i < j$, $g \in D_\ell$) gives $E_j P^{-1}\rho'(g)P E_i = 0$: $P^{-1}\rho'(g)P$ is upper triangular. Its entry $(3,1)$ has ϵ-part $z(g)\kappa_1(g) - X_{31}\kappa_1(g) + \kappa_3(g)X_{31} = 0$, so $z(g) = \Psi^{-1}(g)m_\ell - m_\ell$ on D_ℓ with $m_\ell = -X_{31} \in E_\mathcal{O}$, and by Step 2 (b) z is a coboundary

on $D_{\bar w} = tD_\ell t^{-1}$ for every place $\bar w = t\bar\ell$ above ℓ. *Away from ℓ:* z is a coboundary on every $I_{\bar w}$, $\bar w \nmid \ell$, by Step 3 (i), (ii). Now $\Psi^{-1} = \chi^{-1}\chi_{\mathrm{cyc}}^{-e}$ with χ of finite order (open kernel) and $e \geq 2$; by Lemma 45.31 (with $\psi = \chi$), there is $m \in \overline{\mathbb{Q}}_\ell$ with $z(g) = \Psi^{-1}(g)m - m$ for all $g \in G_{\mathbb{Q}}$. *Normalisation*: by (Se3), $h(r_A)_{31} = 0$, hence the entry $(3,1)$ of $f(r_A(g_0)) + \theta(g_0)f(r_A(g_0^{-1}))$ vanishes; its ϵ-part is $z(g_0)\theta(g_0) + \theta(g_0)z(g_0^{-1})\theta(g_0)^{-1} = \theta(g_0)z(g_0) + z(g_0^{-1})$. Substituting,

$$0 = \theta(g_0)(\Psi^{-1}(g_0) - 1)m + (\Psi(g_0) - 1)m = (\Psi(g_0) - 1)(1 - \alpha(g_0))m,$$

using $\theta = \alpha\Psi$. By (Sg4) both factors are units of $\mathcal{O}$, so $m = 0$, $z = 0$, and $Z = 0$. □

Lemma 45.31. *Let ℓ be a prime, $\psi\colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell^\times$ a homomorphism with open kernel, $e \geq 1$ an integer and $\theta = \psi^{-1}\chi_{\mathrm{cyc}}^{-e}$. Let $z\colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ be continuous with $z(gh) = z(g) + \theta(g)z(h)$ for all g, h, such that for every place $\bar w$ of $\overline{\mathbb{Q}}$ above a prime $p \neq \ell$ there is $m_{\bar w}$ with $z(g) = \theta(g)m_{\bar w} - m_{\bar w}$ for $g \in I_{\bar w}$, and for every place $\bar w$ above ℓ there is $m_{\bar w}$ with $z(g) = \theta(g)m_{\bar w} - m_{\bar w}$ for $g \in D_{\bar w}$. Then there is $m \in \overline{\mathbb{Q}}_\ell$ with $z(g) = \theta(g)m - m$ for all $g \in G_{\mathbb{Q}}$.*

Proof. Step 1: a field. Let $L \subset \overline{\mathbb{Q}}$ be the compositum of the fixed field of $\ker\psi$ (finite and Galois over $\mathbb{Q}$, as $\ker\psi$ is open and normal), $\mathbb{Q}(i)$ and $\mathbb{Q}(\mu_\ell)$. L is a number field, Galois over $\mathbb{Q}$, totally complex, and contains a primitive ℓ-th and a primitive fourth root of unity. On G_L, $\psi = \mathbf{1}$ and $\theta = \chi_{\mathrm{cyc}}^{-e}$. Since $\chi_{\mathrm{cyc}}(D_{\bar w} \cap G_L)$ is an open, hence infinite, subgroup of $\mathbb{Z}_\ell^\times$ for $\bar w$ above ℓ, while $x^{-e} = 1$ has finitely many solutions, each $D_{\bar w} \cap G_L$ ($\bar w \mid \ell$) contains h with $\theta(h) \neq 1$.

Step 2: $z|_{G_L}$ factors through G_{L,S_ℓ}. Let $\bar w$ lie above a prime $p \neq \ell$. As χ_{cyc} is trivial on $I_{\bar w}$, $\theta = 1$ on $I_{\bar w} \cap G_L$, so $z = \theta m_{\bar w} - m_{\bar w} = 0$ there. On $H_0 = \ker\theta \cap G_L$ the cocycle relation makes z additive; the groups $I_{\bar w} \cap G_L$ ($\bar w$ above primes $\neq \ell$) lie in H_0 and are permuted by conjugation in G_L, so z vanishes on the closed normal subgroup $N \subseteq H_0$ of G_L they generate (z is continuous). For $g \in G_L$ and $n \in N$, $z(gn) = z(g) + \theta(g)z(n) = z(g)$, and $\theta|_{G_L} = \chi_{\mathrm{cyc}}^{-e}$ is trivial on N; since $G_L/N = G_{L,S_\ell}$ (L is totally complex), $z|_{G_L}$ is a continuous cocycle of G_{L,S_ℓ} for χ_{cyc}^{-e}.

Step 3: rational coordinates. By Lemma 42.8, item 3 (for $\Gamma = G_{\mathbb{Q}}$ acting on $\mathbb{Q}_\ell(\psi)$, enlarged to a finite extension containing the values of ψ by item 1), z takes values in a finite extension $E' \subset \overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. For each place v of L above ℓ fix $\bar w_v$ above v, so that $G_{L_v} = D_{\bar w_v} \cap G_L$; with h as in Step 1, $m_{\bar w_v} = z(h)/(\theta(h) - 1) \in E'$. Choose a $\mathbb{Q}_\ell$-basis $b_1, \dots, b_d$ of E' and write $z = \sum_i z_i b_i$, $m_{\bar w_v} = \sum_i m_{v,i} b_i$ with $z_i\colon G_L \to \mathbb{Q}_\ell$ and $m_{v,i} \in \mathbb{Q}_\ell$. As $\theta|_{G_L}$ is $\mathbb{Z}_\ell^\times$-valued, each z_i is a continuous cocycle of G_{L,S_ℓ} for χ_{cyc}^{-e} with $z_i(g) = (\chi_{\mathrm{cyc}}(g)^{-e} - 1)m_{v,i}$ on G_{L_v}. The images $z_i(G_L)$ are compact, so for some $n \geq 0$ all $\ell^n z_i$ take values in $\mathbb{Z}_\ell$ and all $\ell^n m_{v,i}$ lie in $\mathbb{Z}_\ell$ (finitely many v).

Step 4: z is a coboundary on G_L. By Proposition 40.3, $\mathrm{len}_\ell(L, e) < \infty$ (L is totally complex and contains a primitive ℓ-th root of unity, and a primitive fourth one if $\ell = 2$). By Lemma 43.33 for $E = L$, $k = e$, the cocycle $\ell^n z_i$ and $W_v = \ell^n m_{v,i}$, there is $x_i \in \mathbb{Q}_\ell$ with $\ell^n z_i(g) = (\chi_{\mathrm{cyc}}(g)^{-e} - 1)x_i$ on G_L. With $x = \ell^{-n}\sum_i x_i b_i$, $z(g) = \theta(g)x - x$ for $g \in G_L$.

Step 5: z is a coboundary on $G_{\mathbb{Q}}$. $z'(g) = z(g) - (\theta(g)x - x)$ is a cocycle for θ vanishing on G_L. For $g \in G_{\mathbb{Q}}$ and $h \in G_L$, $z'(hg) = z'(h) + \theta(h)z'(g) = \theta(h)z'(g)$, and, G_L being normal, $z'(hg) = z'(g) + \theta(g)z'(g^{-1}hg) = z'(g)$. Taking $h \in G_L$ with $\theta(h) \neq 1$ (Step 1) gives $z'(g) = 0$. So $z(g) = \theta(g)m - m$ with $m = x$. □

Lemma 45.32. *Let $\mathfrak{D} = (\ell, \mathcal{O}, e, N, \chi, \xi, S, \dots)$ be a symplectic Eisenstein datum, $n_S = \prod_{p\in S} p$, $b \in \mathcal{O} \setminus \{0\}$, and $\mathrm{Sha}[b]$ the submodule of $\mathrm{Sha}^1_{n_S}(\chi, e)$ killed by b. There are $a \in b\mathcal{O} \setminus \{0\}$ and an injective additive map $\mu \mapsto z_\mu$ from $\mathrm{Sha}[b]$ to the group of maps $G_{\mathbb{Q}} \to \mathcal{O}/a\mathcal{O}$ such that every z_μ is a normalised Sha cocycle modulo a for $\mathfrak{D}$.*

Proof. Write $E_{\mathcal{O}} = \mathcal{O}[\ell^{-1}]$ and $V = \mathcal{O}^{\vee}(\Psi^{-1}) = E_{\mathcal{O}}/\mathcal{O}$ with g acting by $\Psi^{-1}(g)$; for $m \in V$ let dm be the coboundary $g \mapsto \Psi^{-1}(g)m - m$. For $a \in \mathcal{O} \setminus \{0\}$, $V[a] = a^{-1}\mathcal{O}/\mathcal{O}$, and $\iota_a \colon \mathcal{O}/a\mathcal{O} \to V[a]$, $y \mapsto a^{-1}y$, is an isomorphism of $\mathcal{O}$-modules compatible with the actions through Ψ^{-1}.

Step 1: local elements. For every $v \in T$ there is $s_v \in D_v$ with $\Psi(s_v) \neq 1$. For $v = \ell$ take $s_v = s_0$: $\chi(s_0) = 1$ by (Sg5), so $\Psi(s_0) = \chi_{\mathrm{cyc}}(s_0)^e \neq 1$, $\chi_{\mathrm{cyc}}(s_0)$ having infinite order. For $v \in S_\ell(\chi)$, χ'' is ramified at v; if χ were trivial on I_v, then $\chi''|_{I_v} = (\chi'|_{I_v})^{-1}$ would have order both a power of ℓ and prime to ℓ, hence be trivial; so there is $s_v \in I_v$ with $\chi(s_v) \neq 1$, and $\Psi(s_v) = \chi(s_v)$ as χ_{cyc} is trivial on I_v ($v \neq \ell$). Moreover, *if $p \notin T$ and $g \in I_{\bar{w}}$ with $\bar{r}_0(g) = 1$ for a place $\bar{w}$ above p, then* $\Psi(g) = 1$: $p \neq \ell$, so $\chi_{\mathrm{cyc}}(g) = 1$ and $\bar{\chi}(g) = \bar{\Psi}(g) = 1$; on $I_{\bar{w}}$, χ'' is trivial (as $p \notin S_\ell(\chi)$), so $\chi(g) = \chi'(g)$ is a root of unity of order prime to ℓ with residue 1, hence $\chi(g) = 1$.

Step 2: the modulus. Put $c_T = \prod_{v \in T}(\Psi^{-1}(s_v) - 1) \in \mathcal{O} \setminus \{0\}$ and $a = bc_T$.

Step 3: normalised representatives. By (Sg4), $u = \Psi^{-1}(g_0) - 1$ is a unit of $\mathcal{O}$. Every class $\mu \in H^1(\mathbb{Q}, V)$ has exactly one representative N_μ (a continuous cocycle $G_{\mathbb{Q}} \to V$) with $N_\mu(g_0) = 0$: if $\tilde{z}$ represents μ, then $\tilde{z} - d(u^{-1}\tilde{z}(g_0))$ vanishes at g_0, and two such representatives differ by some dm with $um = 0$, i.e. $m = 0$. Uniqueness gives $N_{\mu+\mu'} = N_\mu + N_{\mu'}$, and $N_\mu = 0$ only for $\mu = 0$. For $\mu \in \mathrm{Sha}[b]$, bN_μ is the normalised representative of $b\mu = 0$, so $bN_\mu = 0$: N_μ takes values in $V[b] \subseteq V[a]$. Put $z_\mu = \iota_a^{-1} \circ N_\mu \colon G_{\mathbb{Q}} \to \mathcal{O}/a\mathcal{O}$. Then $\mu \mapsto z_\mu$ is additive and injective, z_μ is locally constant (N_μ is continuous with values in the discrete module V), $z_\mu(gh) = z_\mu(g) + \Psi^{-1}(g)z_\mu(h)$ and $z_\mu(g_0) = 0$.

Step 4: away from T. Let $\bar{w} = t\bar{p}$ ($t \in G_{\mathbb{Q}}$) be a place above $p \notin T$ and $g \in I_{\bar{w}}$ with $\bar{r}_0(g) = 1$. As μ is unramified at p, $N_\mu = dm$ on $I_{\bar{p}}$ for some $m \in V$; for $x = tht^{-1}$ with $h \in I_{\bar{p}}$, $N_\mu(x) + \Psi^{-1}(x)N_\mu(t) = N_\mu(th) = N_\mu(t) + \Psi^{-1}(t)(\Psi^{-1}(h)m - m)$, so $N_\mu = dm'$ on $I_{\bar{w}} = tI_{\bar{p}}t^{-1}$ with $m' = \Psi^{-1}(t)m - N_\mu(t)$. By Step 1, $\Psi(g) = 1$, so $N_\mu(g) = 0$ and $z_\mu(g) = 0$.

Step 5: at T. Let $v \in T$. By (Sg1), $T \subseteq S$, so $v \mid n_S$ and μ is locally trivial at v: $N_\mu(g) = \Psi^{-1}(g)m_v - m_v$ for $g \in D_v$, with $m_v \in V$. Then $(\Psi^{-1}(s_v) - 1)m_v = N_\mu(s_v) \in V[b]$, so $am_v = \big(\prod_{v' \neq v}(\Psi^{-1}(s_{v'}) - 1)\big)\, b\,(\Psi^{-1}(s_v) - 1)m_v = 0$, $m_v \in V[a]$, and $y_v = \iota_a^{-1}(m_v)$ satisfies $z_\mu(g) = \Psi^{-1}(g)y_v - y_v$ for $g \in D_v$. Hence z_μ is a normalised Sha cocycle modulo a. □

Lemma 45.33. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$ that is finite in weight zero, and M a symplectic weighted model of $\mathfrak{D}$. For $s = (s_1, s_2) \in (\overline{\mathbb{Q}}_\ell^{\times})^2$ and $w = (w_1, w_2) \in \mathbb{Z}^2$ write $s^w = s_1^{w_1}s_2^{w_2}$. Then:*

(Pm1) *the image of $(A_M)_0$ in $A_M/I(u_M)$ is a finitely generated $\mathcal{O}$-module;*

(Pm2) *for every point x of A_M on u_M there are an $\mathcal{O}$-algebra homomorphism $y \colon S \to \overline{\mathbb{Q}}_\ell$ and $t \in \overline{\mathbb{Q}}_\ell^{\times}$ with $y(q(f)) = t^n x(f)$ for all $n \in \mathbb{Z}$ and $f \in (A_M)_n$;*

(Pm3) *if $y, y' \colon S \to \overline{\mathbb{Q}}_\ell$ are $\mathcal{O}$-algebra homomorphisms and $s \in (\overline{\mathbb{Q}}_\ell^{\times})^2$ with*

$$y'(r_S(g)) = \mathrm{T}_4(s)\, y(r_S(g))\, \mathrm{T}_4(s)^{-1}$$

for all $g \in G_{\mathbb{Q}}$ (entrywise application), then $y'(q(f)) = s^w y(q(f))$ for all $w \in \mathbb{Z}^2$ and $f \in R_w$.

Proof. Write $K = \overline{\mathbb{Q}}_\ell$, $P = \mathcal{O}[[X_s : s \in \sigma]]$, $P_w = \mathcal{O}[[X_\sigma]]_w$, $p \colon P \to S$ for q composed with $P \to R$, $\mathfrak{b} = \ker p$, $S_w = p(P_w) = q(R_w)$, $\bar{z}_s = p(X_s)$, and $\zeta^n = \prod_s \zeta_s^{n_s}$ for $\zeta \in K^\sigma$, $n \in \mathbb{N}^\sigma$. For $w = (w_1, w_2)$ write $\lambda(w) = w_1$, $\mu(w) = w_2$.

Step 0: homogeneity. By (Wm2), (Wm3), $\mathfrak{b} = \mathfrak{a} + (\tilde{u}_k)_k P$ with $\tilde{u}_k \in P_{(0,-1)} \cup P_{(0,1)} \cup P_{(0,0)}$ lifting the F_i, E_i, Z_i; since $\mathfrak{a}$ is homogeneous and $(BH)_{(w)} = B_{(w-v)}H$ for $H \in P_v$, every weight

component of an element of $\mathfrak{b}$ lies in $\mathfrak{b}$. Hence a finite sum $\sum_w b_w$ with $b_w \in S_w$ vanishes only if every b_w does, and $S^{\mathrm{gr}} = \sum_w S_w = \bigoplus_w S_w$ is an $\mathcal{O}$-subalgebra with $S_v S_w \subseteq S_{v+w}$. By Dickson's lemma, for each w the set E_w of exponents of weight w has a finite set Min_w of minimal elements, every $n \in E_w$ satisfies $n - m \in E_{(0,0)}$ for some $m \in \mathrm{Min}_w$ with $m \le n$, and so every $H \in P_w$ is a finite sum $\sum_{m \in \mathrm{Min}_w} X^m h_m$ with $h_m \in P_{(0,0)}$; the minimal elements $m_1, \dots, m_k$ of $E_{(0,0)} \setminus \{0\}$ generate the monoid $E_{(0,0)}$, and $\theta\colon \mathcal{O}[[T_1, \dots, T_k]] \to P_{(0,0)}$, $T_i \mapsto X^{m_i}$, is onto.

Step 1: (Pm1). Let C_0 be the $\mathcal{O}$-span of the products $r_S(g_1)_{i_1 j_1} \cdots r_S(g_k)_{i_k j_k}$ of weight $(0,0)$; it is the $\mathcal{O}$-subalgebra of S they generate. As r_S is a symplectic Eisenstein lift to $S \in \mathrm{CNL}_{\mathcal{O}}$ (Wm4) and $\mathfrak{D}$ is finite in weight zero, C_0 is a finitely generated $\mathcal{O}$-module, and $S_{(0,0)} = C_0$ by Lemma 45.34. By (La1), $\{f \in A_M : q(f) = 0\} = I(u_M)$, so the image of $(A_M)_0$ in $A_M/I(u_M)$ is isomorphic to $q((A_M)_0) = S_{(0,0)} = C_0$.

Step 2: C_0 is local and points are bounded on it. C_0 is finite over $\mathcal{O}$, hence a finite product of local rings; it has no idempotents other than $0, 1$ (it is a subring of the local ring S), so it is local; its maximal ideal $\mathfrak{n}$ satisfies $\mathfrak{n}^N \subseteq \varpi C_0$ for some N ($C_0/\varpi C_0$ is a local Artinian ring; ϖ a uniformiser of $\mathcal{O}$), hence $\mathfrak{n} \subseteq \mathfrak{m}_S$, and $\mathfrak{n} = C_0 \cap \mathfrak{m}_S$. For an $\mathcal{O}$-algebra homomorphism $\xi\colon C_0 \to K$, every $\xi(c)$ is integral over $\mathcal{O}$, so $|\xi(c)|_\ell \le 1$; and for $c \in \mathfrak{n}$, $\xi(c)^N = \varpi \xi(c')$ with $c' \in C_0$, so $|\xi(c)|_\ell < 1$.

Step 3: lifting to S^{gr}. Let x be a point of A_M on u_M. By (La1), x kills the kernel of q on A_M, so $x = \bar{x} \circ q$ for an $\mathcal{O}$-algebra homomorphism $\bar{x}\colon C \to K$, $C = q(A_M) = \bigoplus_n S_{(0,n)}$. The map $\Pi\colon S^{\mathrm{gr}} \to C$, $\sum_w b_w \mapsto \sum_{\lambda(w)=0} b_w$, is C-linear ($C S_w \subseteq \sum_n S_{w+(0,n)}$) with $\Pi(1) = 1$. By Step 0, S^{gr} is generated as a C-algebra by the finitely many $\bar{z}_s$ ($S_{(0,0)} \subseteq C$). Let K be a C-algebra through $\bar{x}$ and $T = K \otimes_C S^{\mathrm{gr}}$, a K-algebra of finite type. The C-linear map $\bar{x} \circ \Pi$ induces a K-linear map $T \to K$ with $1 \otimes 1 \mapsto 1$, so $T \neq 0$, and by Hilbert's Nullstellensatz (K algebraically closed) there is a K-algebra homomorphism $T \to K$. Composed with $b \mapsto 1 \otimes b$ it is an $\mathcal{O}$-algebra homomorphism $\xi\colon S^{\mathrm{gr}} \to K$ with $\xi|_C = \bar{x}$.

Step 4: ξ is an evaluation. Put $\zeta_s = \xi(\bar{z}_s)$; these finitely many algebraic elements lie in a finite extension K_0 of $\mathbb{Q}_\ell$. By Steps 1 and 2 (for $\xi|_{C_0}$), $|\xi|_\ell \le 1$ on $S_{(0,0)}$, and $\rho = \max_i |\zeta^{m_i}|_\ell < 1$, as $p(X^{m_i}) \in \mathfrak{m}_S \cap C_0 = \mathfrak{n}$. For $H = \theta(h) \in P_{(0,0)}$, the series $H(\zeta)$ converges in K_0 (its terms are bounded by powers of ρ through θ), and writing $h = h_{<M} + \sum_{|e|=M} T^e h_e$, both $\xi(p(H))$ and $H(\zeta)$ differ from $h_{<M}(\zeta^{m_1}, \dots, \zeta^{m_k})$ by at most ρ^M (using $|\xi(p(\theta(h_e)))|_\ell \le 1$); so $\xi(p(H)) = H(\zeta)$. For $H = \sum_{m \in \mathrm{Min}_w} X^m h_m \in P_w$ (Step 0) this gives $\xi(p(H)) = \sum_m \zeta^m h_m(\zeta)$.

Step 5: scaling. Let $\sigma' = \{s : \zeta_s \neq 0\}$ and $a_s = \log |\zeta_s|_\ell$ for $s \in \sigma'$. We claim there is $u \in \mathbb{R}^2$ with $\lambda(\mathrm{wt}(s))u_1 + \mu(\mathrm{wt}(s))u_2 + a_s < 0$ for all $s \in \sigma'$. If not, the transposition theorem of Gordan and Motzkin for strict linear inequalities gives $c \in \mathbb{R}_{\ge 0}^{\sigma'} \setminus \{0\}$ with $\sum_s c_s \mathrm{wt}(s) = 0$ and $\sum_s c_s a_s \ge 0$. The cone $\{c \in \mathbb{R}_{\ge 0}^{\sigma'} : \sum_s c_s \mathrm{wt}(s) = 0\}$ is rational polyhedral, generated by finitely many integral vectors; a linear form that is non-negative at a non-zero point of it is non-negative at one of the generators, so we may take $c \in \mathbb{N}^{\sigma'} \setminus \{0\}$. Then X^c has weight $(0,0)$ and positive degree, $p(X^c) \in \mathfrak{n}$, and $|\zeta^c|_\ell = |\xi(p(X^c))|_\ell < 1$ by Step 2, i.e. $\sum_s c_s a_s < 0$: a contradiction. As $|K^\times|_\ell = \ell^{\mathbb{Q}}$ is dense in $\mathbb{R}_{>0}$ and the inequalities are strict, there is $t = (t_1, t_2) \in (K^\times)^2$ with $|t^{\mathrm{wt}(s)} \zeta_s|_\ell < 1$ for every $s \in \sigma$. Put $\zeta'_s = t^{\mathrm{wt}(s)} \zeta_s$.

Step 6: (Pm2). The evaluation $\mathrm{ev}\colon P \to K_0(t_1, t_2)$, $H \mapsto H(\zeta')$, is a continuous $\mathcal{O}$-algebra homomorphism. For $H \in P_w$ every monomial X^n of H has $\zeta'^n = t^w \zeta^n$, so by Step 4, $\mathrm{ev}(H) = t^w \xi(p(H))$. For $H \in \mathfrak{b}$, $H = \sum_w H_{(w)}$ converges in P (only finitely many weights occur in each degree), each $H_{(w)} \in \mathfrak{b}$ (Step 0), so $\mathrm{ev}(H) = \sum_w t^w \xi(p(H_{(w)})) = 0$. Hence ev induces an $\mathcal{O}$-algebra homomorphism $y\colon S = P/\mathfrak{b} \to K$ with $y(b) = t^w \xi(b)$ for $b \in S_w$. For $f \in (A_M)_n$, $q(f) \in S_{(0,n)} \subseteq C$, so $y(q(f)) = t_2^n \bar{x}(q(f)) = t_2^n x(f)$, which is (Pm2) with t_2 in place of t.

Step 7: (Pm3). By the definition of T_4, the entry (i,j) of $\mathrm{T}_4(s) X \mathrm{T}_4(s)^{-1}$ is $s^{\mathrm{wt}_4(i,j)} X_{ij}$, so

$y'(r_S(g)_{ij}) = s^{\mathrm{wt}_4(i,j)} y(r_S(g)_{ij})$, and by multiplicativity $y'(\pi_0) = s^w y(\pi_0)$ for every product π_0 of entries of weight w. By Lemma 45.34, $q(f)$ is an $\mathcal{O}$-linear combination of such products for $f \in R_w$; hence $y'(q(f)) = s^w y(q(f))$. □

Lemma 45.34. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$ that is finite in weight zero, and M a symplectic weighted model of $\mathfrak{D}$. Then for every $w \in \mathbb{Z}^2$, $q(R_w)$ is the $\mathcal{O}$-linear span of the products $r_S(g_1)_{i_1 j_1} \cdots r_S(g_k)_{i_k j_k}$ ($k \geq 0$, $g_s \in G_{\mathbb{Q}}$) with $\sum_{s=1}^{k} \mathrm{wt}_4(i_s, j_s) = w$.*

Proof. Write $P = \mathcal{O}[[X_s : s \in \sigma]]$ with the weights wt, P_w for its power series of weight w, $\pi\colon P \to R$ the projection, $\Pi = q \circ \pi\colon P \to S$, $S_w = \Pi(P_w) = q(R_w)$, $\mathfrak{m}_S$ the maximal ideal of S, and $\mathcal{M}_w$ for the $\mathcal{O}$-span of the products of entries of r_S of total weight w in the statement. Then $1 \in \mathcal{M}_{(0,0)}$ (the empty product) and $\mathcal{M}_v \mathcal{M}_{v'} \subseteq \mathcal{M}_{v+v'}$; so $C = \mathcal{M}_{(0,0)}$ is the $\mathcal{O}$-subalgebra of S generated by the products of weight $(0,0)$, and since r_S is a symplectic Eisenstein lift (Wm4) and $\mathfrak{D}$ is finite in weight zero, C is a finitely generated $\mathcal{O}$-module. For a power series f write $f_{(v)}$ for its weight-v component; since $P_v P_k \subseteq P_{v+k}$ and extraction of components is continuous coefficientwise, $(gh)_{(v)} = g_{(v-k)} h$ for $h \in P_k$.

Step 1: $\mathcal{M}_w \subseteq S_w$, and $J = \ker \Pi$ is homogeneous. By (Wm5) every entry $r_S(g)_{ij}$ lies in $S_{\mathrm{wt}_4(i,j)}$; $1 \in S_{(0,0)}$ and $S_v S_{v'} \subseteq S_{v+v'}$, so every product of entries of total weight w lies in S_w, an $\mathcal{O}$-module. Next, choose by (Wm2) and $R_v = \pi(P_v)$ elements $\tilde{F}_i \in P_{(0,-1)}$, $\tilde{E}_i \in P_{(0,1)}$, $\tilde{Z}_i \in P_{(0,0)}$ with images F_i, E_i, Z_i; by (Wm3), $J = \mathfrak{a} + (\tilde{F}_i, \tilde{E}_i, \tilde{Z}_i)_i$. If $f = a + \sum_i g_i h_i$ with $a \in \mathfrak{a}$ and $h_i \in P_{k_i}$ among these generators, then $f_{(v)} = a_{(v)} + \sum_i (g_i)_{(v-k_i)} h_i \in J$, as $\mathfrak{a}$ is homogeneous. So $f_{(v)} \in J$ for all $f \in J$ and all v.

Step 2: the entries span the cotangent space. For each triple $t = (g, a, b)$ ($g \in G_{\mathbb{Q}}$, $1 \leq a, b \leq 4$) put $z_t = r_S(g)_{ab}$ if $a \neq b$, and $z_t = r_S(g)_{aa} - c_t$ with $c_t \in \mathcal{O}$ such that $z_t \in \mathfrak{m}_S$ if $a = b$ (the residue field of S is $\mathbb{F}_{\mathcal{O}}$). As $\bar{r}_0$ is diagonal and r_S lifts it, $z_t \in \mathfrak{m}_S$ for all t; by Step 1 and $c_t \in \mathcal{M}_{(0,0)} \cap S_{(0,0)}$, $z_t \in S_{\mathrm{wt}_4(a,b)} \cap \mathcal{M}_{\mathrm{wt}_4(a,b)}$. Let $t^* = \mathfrak{m}_S / (\mathfrak{m}_S^2 + \mathfrak{m}_{\mathcal{O}} S)$, a finite-dimensional $\mathbb{F}_{\mathcal{O}}$-space. By (Wm4), two morphisms $S \to A$ of $\mathrm{CNL}_{\mathcal{O}}$ that agree on the entries of r_S are equal; so Lemma 45.15, item 1, applied to the set of these entries (with $c_e = 0$ off the diagonal and $c_e = c_t$ on it), shows that the classes $\bar{z}_t$ span t^*. Choose triples $t_1, \ldots, t_k$ such that the $\bar{z}_{t_i}$ span t^*, put $\mathrm{wt}'(i) = \mathrm{wt}_4(a_i, b_i)$ for $t_i = (g_i, a_i, b_i)$, $Q = \mathcal{O}[[Y_1, \ldots, Y_k]]$ with these weights, Q_v its series of weight v, and $\Pi'\colon Q \to S$ the continuous $\mathcal{O}$-algebra homomorphism with $\Pi'(Y_i) = z_{t_i}$. The images of the Y_i and $\mathfrak{m}_{\mathcal{O}}$ span $\mathfrak{m}_S / \mathfrak{m}_S^2$, so Π' is surjective by Lemma 45.15, item 2.

Step 3: $S_w = \Pi'(Q_w)$. As $z_{t_i} \in S_{\mathrm{wt}'(i)}$, choose $\kappa_i \in P_{\mathrm{wt}'(i)}$ with $\Pi(\kappa_i) = z_{t_i}$; its constant term lies in $\mathfrak{m}_{\mathcal{O}}$ (it is congruent to $z_{t_i} \in \mathfrak{m}_S$ modulo $\mathfrak{m}_S$), so $\kappa_i \in \mathfrak{m}_P$, and there is a continuous $\mathcal{O}$-algebra homomorphism $\Theta\colon Q \to P$ with $\Theta(Y_i) = \kappa_i$; $\Pi \circ \Theta = \Pi'$ (both continuous, equal on the Y_i). A monomial of weight v in the Y_i goes to a product of series of total weight v, and P_v is closed, so $\Theta(Q_v) \subseteq P_v$; hence $\Pi'(Q_w) \subseteq S_w$. Conversely let $f \in P_w$. By surjectivity $\Pi(f) = \Pi'(h) = \Pi(\Theta(h))$ for some $h \in Q$, so $f - \Theta(h) \in J$. Now $\Theta(h) = \sum_v \Theta(h_{(v)})$ (convergent) with $\Theta(h_{(v)}) \in P_v$, so $\Theta(h)_{(w)} = \Theta(h_{(w)})$, and by Step 1, $(f - \Theta(h))_{(w)} = f - \Theta(h_{(w)}) \in J$. Hence $\Pi(f) = \Pi'(h_{(w)}) \in \Pi'(Q_w)$.

Step 4: monomial generators. By Dickson's lemma let E_w be the finite set of minimal elements of $\{n \in \mathbb{N}^k : \sum_i n_i \mathrm{wt}'(i) = w\}$ and H that of $\{n \neq 0 : \sum_i n_i \mathrm{wt}'(i) = (0,0)\}$. Every exponent n of weight w dominates some $m \in E_w$, and $n - m$ has weight $(0,0)$; assigning each n to one such m gives, for $h \in Q_w$, $h = \sum_{m \in E_w} Y^m b_m$ with $b_m \in Q_{(0,0)}$. In the same way every $h \in Q_{(0,0)}$ is $h(0) + \sum_{m \in H} Y^m b_m$ with $b_m \in Q_{(0,0)}$. Put $y_m = \Pi'(Y^m) = \prod_i z_{t_i}^{m_i}$: by Step 2, $y_m \in \mathcal{M}_w$ for $m \in E_w$, and $y_m \in C \cap \mathfrak{m}_S$ for $m \in H$. With $B = S_{(0,0)} = \Pi'(Q_{(0,0)})$ (Step 3), an

$\mathcal{O}$-subalgebra of S containing C (Step 1):

$$S_w = \sum_{m \in E_w} B\, y_m, \qquad B = \mathcal{O} + \sum_{m \in H} y_m B. \tag{45.3}$$

Step 5: $B = C$. C is a module-finite $\mathcal{O}$-algebra, hence a finite product of local rings; it is a subring of the local ring S, whose only idempotents are $0, 1$, so C is local. Its maximal ideal is $\mathfrak{n} = C \cap \mathfrak{m}_S$, the kernel of the surjection $C \to S/\mathfrak{m}_S = \mathbb{F}_{\mathcal{O}}$. $C/\mathfrak{m}_{\mathcal{O}}C$ is a finite-dimensional local $\mathbb{F}_{\mathcal{O}}$-algebra, so its maximal ideal is nilpotent: there is N_1 with $\mathfrak{n}^{N_1} \subseteq \mathfrak{m}_{\mathcal{O}}C$. Let $I = \sum_{m \in H} y_m C \subseteq \mathfrak{n}$. By induction on j, $B \subseteq C + \mathfrak{m}_{\mathcal{O}}B + I^j B$: for $j = 0$ this is trivial, and if it holds for j, then for $b = c + \sum_m y_m b_m \in B$ as in (45.3), $c \in \mathcal{O} \subseteq C$ and $y_m b_m \in y_m C + \mathfrak{m}_{\mathcal{O}}B + I^{j+1}B$. With $j = N_1$, $I^{N_1}B \subseteq \mathfrak{m}_{\mathcal{O}}CB = \mathfrak{m}_{\mathcal{O}}B$, so $B = C + \mathfrak{m}_{\mathcal{O}}B$, and by induction $\mathfrak{m}_{\mathcal{O}}^n B = \mathfrak{m}_{\mathcal{O}}^n C + \mathfrak{m}_{\mathcal{O}}^{n+1}B$ for every n. Let $b \in B$; then $b = c_0 + c_1 + \cdots + c_n + \beta_{n+1}$ with $c_i \in \mathfrak{m}_{\mathcal{O}}^i C$ and $\beta_{n+1} \in \mathfrak{m}_{\mathcal{O}}^{n+1}B$, the c_i chosen successively. As C is a finitely generated module over the complete ring $\mathcal{O}$, $c = \sum_i c_i$ converges in C, and $\sum_{i>n} c_i \in \mathfrak{m}_{\mathcal{O}}^{n+1}C$ (a closed submodule). So $b - c \in \mathfrak{m}_{\mathcal{O}}^{n+1}B \subseteq \mathfrak{m}_S^{n+1}$ for every n, and $b = c \in C$ by Krull's intersection theorem in S. Hence $B = C$.

Step 6: conclusion. By (45.3) and Step 5, $S_w = \sum_{m \in E_w} C y_m \subseteq \mathcal{M}_{(0,0)}\mathcal{M}_w \subseteq \mathcal{M}_w$; with Step 1, $q(R_w) = S_w = \mathcal{M}_w$. □

45.4 Reducible points of a symplectic weighted model

This subsection proves Proposition 45.35: the reducible points of a symplectic weighted model satisfy the separation condition (Gc3), the two-signed points are irreducible, and the tangent spaces there are one-dimensional (Gc4). It works with points with values in a valuation ring, which are ordinary test representations carrying an ordinary symplectic flag at ℓ (Lemma 45.37, Lemma 45.41). Such a point is irreducible, of diagonal type or of block type (Proposition 45.36). For points of diagonal type the diagonal characters at ℓ are $(\theta, \Psi, \alpha, 1)$ or $(\alpha, 1, \theta, \Psi)$, and the non-zero entries respect that order (Lemma 45.40); the proof of (Gc3) and (Gc4) is a case analysis over these shapes. That case analysis is the core of the subsection; the lemmas on isotropic flags and on enlarging the valuation ring are routine.

Proposition 45.35. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum that is finite in weight zero, and M a symplectic weighted model of $\mathfrak{D}$.*

(Mr1) *(A_M, x_M, u_M) satisfies (Gc3).*

(Mr2) *If $\mathfrak{D}$ has no endoscopic points, then for every $\mathcal{O}$-algebra homomorphism $y\colon S \to \overline{\mathbb{Q}}_\ell$ such that $y \circ q|_{A_M}$ is a two-signed point of A_M on u_M, the point $g \mapsto y(r_S(g))$ of $\mathfrak{D}$ is irreducible.*

(Mr3) *If $\mathfrak{D}$ has no endoscopic points and every irreducible point of $\mathfrak{D}$ is rigid, then (A_M, x_M, u_M) satisfies (Gc4).*

Proof. Write $K = \overline{\mathbb{Q}}_\ell$, $P = \mathcal{O}[[X_s : s \in \sigma]]$, P_w its series of weight w, $\Pi\colon P \to S$ the composite of the projection to R and q, $S_w = \Pi(P_w) = q(R_w)$, $S^{\mathrm{gr}} = \sum_w S_w$, $\mathrm{A} = \{1, 3\}$, $\mathrm{B} = \{2, 4\}$. Since $\ker q$ is generated by the F_i, E_i, Z_i (Wm3), which are images of series of single weights (Wm2), and $\mathfrak{a}$ is homogeneous, every weight component of an element of $\ker \Pi$ lies in $\ker \Pi$: choose series $\tilde{F}_i \in P_{(0,-1)}$, $\tilde{E}_i \in P_{(0,1)}$, $\tilde{Z}_i \in P_{(0,0)}$ mapping to F_i, E_i, Z_i (Wm2), so that $\ker \Pi = \mathfrak{a} + (\tilde{F}_i, \tilde{E}_i, \tilde{Z}_i)_i$; for $F = a + \sum_i g_i h_i$ with $a \in \mathfrak{a}$ and $h_i \in P_{k_i}$ among these generators, the weight-v component is $F_{(v)} = a_{(v)} + \sum_i (g_i)_{(v-k_i)} h_i \in \ker \Pi$ (extraction of components is continuous coefficientwise and $P_v P_k \subseteq P_{v+k}$); hence $S^{\mathrm{gr}} = \bigoplus_w S_w$, a subalgebra with $S_v S_w \subseteq S_{v+w}$. By (Wm5) the entry (i, j) of r_S lies in $S_{\mathrm{wt}_4(i,j)}$, and $\mathrm{wt}_4(i, j) = (\lambda_i - \lambda_j, \mu_i - \mu_j)$. For an $\mathcal{O}$-algebra homomorphism

$y\colon S \to K$ put $r_y(g) = y(r_S(g))$, a point of $\mathfrak{D}$ (r_S is a symplectic Eisenstein lift, (Wm4)), hence a point with values in $\overline{\mathbb{Z}}_\ell$ for the standard test field (Lemma 45.41), and $x_y = y \circ q|_{A_M}$, a point of A_M on u_M. Put $\zeta_s = y(\Pi(X_s))$; these finitely many elements lie in a finite extension $K_0 \subseteq K$ of $\mathbb{Q}_\ell$, which is complete; by (Sq1), $|\zeta_s|_\ell < 1$ and $y \circ \Pi$ is continuous, so $y(\Pi(F)) = F(\zeta)$ (a convergent series in K_0) for $F \in P$.

Step 0: four facts for such y. (F1) If no entry (i,j) of r_y with $\mu_i < \mu_j$ is active, x_y is negative-vanishing; if none with $\mu_i > \mu_j$ is active, x_y is positive-vanishing. Indeed for $f \in (A_M)_n = R_{(0,n)}$, $q(f)$ is an $\mathcal{O}$-combination of products of entries of total weight $(0,n)$ (Lemma 45.34); if $n < 0$, each product has a factor (i,j) with $\mu_i < \mu_j$, which y kills; likewise for $n > 0$. (F2) If x_y is positive-vanishing, the entry $(1,3)$ of r_y is not active: $r_S(g)_{13} \in S_{(0,1)} = q((A_M)_1)$. (F3) If y, y' agree on $S_{(0,0)} = q((A_M)_0)$, then r_y, $r_{y'}$ have the same diagonal entries, and if r_y is of diagonal type so is $r_{y'}$: $r_S(g)_{ii}$ and $r_S(g)_{ij}r_S(h)_{ji}$ lie in $S_{(0,0)}$. (F4) A torus translate $t \cdot x$ ($t \neq 0$) of a point x agrees with x on $(A_M)_0$ and is negative-vanishing, positive-vanishing or two-signed exactly when x is.

(Mr1). Let φ_1 be a point on u_M that is negative- and positive-vanishing; by (Pm2) choose y_1, t_1 with $x_{y_1} = t_1 \cdot \varphi_1$, both-vanishing by (F4). By (Zr1), r_{y_1} is irreducible, of diagonal type or of block type; in the first and the last case the entry $(1,3)$ is active ((Ae5), resp. the definition of block type), against (F2). So r_{y_1} is of diagonal type, and by Lemma 45.40, $r_{y_1}(s_0)_{11} \in \{\theta(s_0), \alpha(s_0)\}$, two distinct values (by (Sg1) and (Sg5) of Definition 44.1, ξ is unramified at ℓ and $\chi(s_0) = \omega_\ell(s_0) = 1$, so $(\kappa_1, \dots, \kappa_4)(s_0) = (c^{e+1}, c^e, c, 1)$ with $c = \chi_{\mathrm{cyc}}(s_0)$ of infinite order; in particular $\kappa_1(s_0) \neq \kappa_3(s_0)$). Suppose $r_{y_1}(s_0)_{11} = \theta(s_0)$, and let φ be a point on u_M agreeing with φ_1 on $(A_M)_0$; choose y, t by (Pm2). Then y and y_1 agree on $S_{(0,0)}$ (both equal $\varphi = \varphi_1$ there, by (F4)), so by (F3) r_y is of diagonal type with $r_y(s_0)_{11} = \theta(s_0) \neq \alpha(s_0)$; by Lemma 45.40 its permutation is the identity and no entry with $\mu_i < \mu_j$ is active, so x_y, hence φ, is negative-vanishing ((F1), (F4)). So no point on u_M that is positive-vanishing but not negative-vanishing agrees with φ_1 on $(A_M)_0$. If $r_{y_1}(s_0)_{11} = \alpha(s_0)$, the same argument shows that every such φ is positive-vanishing. This is (Gc3).

(Mr2). Let y be such that x_y is two-signed, and $r = r_y$. If r were of diagonal type, the last assertion of Lemma 45.40 and (F1) would make x_y negative- or positive-vanishing. Suppose r is of block type, and let $\varepsilon = 1$ if no entry (i,j) with $i \in \mathrm{B}$, $j \in \mathrm{A}$ is active, $\varepsilon = -1$ otherwise (then none with $i \in \mathrm{A}$, $j \in \mathrm{B}$ is active), and $p(w) = \varepsilon w_1$. Then y kills every entry of r_S of weight w with $p(w) < 0$, hence by Lemma 45.34 every S_w with $p(w) < 0$ (each product of entries of such a weight has a factor with $p < 0$); in particular $\zeta_s = 0$ if $p(\mathrm{wt}(s)) < 0$. Let $\zeta^0_s = \zeta_s$ if $\mathrm{wt}(s)_1 = 0$ and $\zeta^0_s = 0$ otherwise, and $Y_0(F) = F(\zeta^0)$, a continuous $\mathcal{O}$-algebra homomorphism $P \to K$. For $F \in P_w$: if $w_1 \neq 0$, every monomial of F contains a variable with $\mathrm{wt}(s)_1 \neq 0$, so $Y_0(F) = 0$; if $w_1 = 0$, a monomial of F containing such a variable contains one with $p(\mathrm{wt}(s)) < 0$ (the values of p sum to 0), so it vanishes at ζ and at ζ^0, and $Y_0(F) = F(\zeta) = y(\Pi(F))$. For $F \in \ker\Pi$, $F = \sum_w F_{(w)}$ converges with $F_{(w)} \in \ker\Pi$, so $Y_0(F) = \sum_{w_1=0} y(\Pi(F_{(w)})) = 0$. Hence Y_0 induces $y_0\colon S \to K$ with $y_0 = y$ on S_w for $w_1 = 0$ and $y_0 = 0$ on S_w for $w_1 \neq 0$, so $r_{y_0}(g)_{ij} = r(g)_{ij}$ if $\lambda_i = \lambda_j$ and 0 otherwise: r_{y_0} is a point with block diagonal values. Let $\beta(g) = (r(g)_{ab})_{a,b\in\mathrm{A}}$, the A-block of $r_{y_0}(g)$. In $(r(g)r(h))_{ab} = \sum_k r(g)_{ak}r(h)_{kb}$ ($a, b \in \mathrm{A}$) the terms with $k \in \mathrm{B}$ vanish ($r(h)_{kb} = 0$ if $\varepsilon = 1$, $r(g)_{ak} = 0$ if $\varepsilon = -1$), so β is multiplicative and the K-span of $\beta(G_\mathbb{Q})$ is a subalgebra of $\mathrm{M}_2(K)$; it contains the two diagonal matrix units, images of E_{11}, E_{33}, which lie in the span of $r(G_\mathbb{Q})$ by (Ae3), under the linear map $Y \mapsto (Y_{ab})_{a,b\in\mathrm{A}}$; and, as $(1,3)$, $(3,1)$ are active (block type), multiplying $\beta(g)$ on both sides by these units gives both off-diagonal units. So r_{y_0} is an endoscopic point, which $\mathfrak{D}$ does not have. By (Zr1), r is irreducible.

(Mr3). Let x be a two-signed point on u_M and δ an x-derivation of A_M vanishing on $I(u_M)$. We show that $\delta(f) = b\,n\,x(f)$ for all n and $f \in (A_M)_n$, for some $b \in K$; then for two such

derivations δ,δ' with constants b,b', $b'\delta-b\delta'=0$, and $(1,0)$ works if $b=b'=0$ (then $\delta=0$): (Gc4).

(a) *The point.* By (Pm2) choose y, t with $y(q(f))=t^n x(f)$ for $f\in(A_M)_n$; $x_y=t\cdot x$ is two-signed, so $r=r_y$ is irreducible by (Mr2), hence rigid. By (Ae5) choose g_1,g_2 with $y(u)\neq 0\neq y(v)$ for $u=r_S(g_1)_{12}\in S_{(1,0)}$ and $v=r_S(g_2)_{21}\in S_{(-1,0)}$.

(b) *A derivation on $C=q(A_M)=\bigoplus_n S_{(0,n)}$.* By (La1), $\ker(q|_{A_M})=I(u_M)$, which δ kills; so $\delta_1(q(f))=t^n\delta(f)$ for $f\in(A_M)_n$, extended additively, is well defined on C, and $\delta_1(q(f)q(f'))=t^{n+n'}(x(f)\delta(f')+x(f')\delta(f))=y(q(f))\delta_1(q(f'))+y(q(f'))\delta_1(q(f))$. So $X=y+\epsilon\delta_1\colon C\to K[\epsilon]$ ($\epsilon^2=0$) is an $\mathcal{O}$-algebra homomorphism.

(c) *Extension to S^{gr}.* Put $U_0=y(u)$ and $V_0=X(uv)U_0^{-1}$, a unit of $K[\epsilon]$ with first component $y(v)$. For $c\in S_w$ let $p_w=\max(-w_1,0)$, $q_w=\max(w_1,0)$, so $cu^{p_w}v^{q_w}\in S_{(0,w_2)}\subseteq C$, and put $\Xi(c)=X(cu^{p_w}v^{q_w})U_0^{-p_w}V_0^{-q_w}$, extended additively to $S^{\mathrm{gr}}=\bigoplus_w S_w$. It is $\mathcal{O}$-linear, $\Xi(1)=1$ and $\Xi|_C=X$. For $c\in S_w$, $c'\in S_{w'}$ one has $(p_w+p_{w'},q_w+q_{w'})=(p_{w+w'}+d,q_{w+w'}+d)$ with $d\geq 0$, and $X(uv)=U_0V_0$, so $\Xi(c)\Xi(c')=X(cc'u^{p_{w+w'}}v^{q_{w+w'}})X(uv)^d(U_0V_0)^{-d}U_0^{-p_{w+w'}}V_0^{-q_{w+w'}}=\Xi(cc')$. Its first component is $y(cu^{p_w}v^{q_w})y(u)^{-p_w}y(v)^{-q_w}=y(c)$. So $\Xi=y+\epsilon D_{\mathrm{gr}}$ with D_{gr} a y-derivation of S^{gr} equal to δ_1 on C.

(d) *Extension to S.* Let $d_s=D_{\mathrm{gr}}(\Pi(X_s))$ (enlarge K_0 to contain them), $\rho=\max_s|\zeta_s|_\ell<1$, and $\Delta(F)=\sum_s(\partial F/\partial X_s)(\zeta)\,d_s$ for $F\in P$: it converges, $\Delta(FG)=F(\zeta)\Delta(G)+G(\zeta)\Delta(F)$, and $|\Delta(F)|_\ell\leq\Lambda_2\rho^{N-1}$ for $F\in\mathfrak{m}_P^N$, $\Lambda_2=\max_s|d_s|_\ell$ (the coefficients lie in $\mathcal{O}$). We claim $D_{\mathrm{gr}}(\Pi(F))=\Delta(F)$ for $F\in P_w$. Both are $y\circ\Pi$-derivations on the finite sums of monomials. The $\mathcal{O}$-module $S_{(0,0)}$ is finitely generated (Lemma 45.34 and finiteness in weight zero), so $|D_{\mathrm{gr}}|_\ell\leq\Lambda_0$ on it, and $|y|_\ell\leq 1$ on S (Sq1). Let $m_1,\dots,m_k$ be the minimal non-zero exponents of weight $(0,0)$ (Dickson); every $H\in P_{(0,0)}$ is $\vartheta(h)$ for some $h\in\mathcal{O}[[T_1,\dots,T_k]]$, $\vartheta(T_i)=X^{m_i}$. Write $h=h_{<L}+\sum_{|e|=L}T^e h_e$; then $H-\vartheta(h_{<L})=\sum_e X^{\sum_i e_i m_i}\vartheta(h_e)\in\mathfrak{m}_P^L$, $\Pi(\vartheta(h_e))\in S_{(0,0)}$, and with $\rho_0=\max_i|\zeta^{m_i}|_\ell<1$ and $\Lambda_1=\max_i|D_{\mathrm{gr}}(\Pi(X^{m_i}))|_\ell$, the Leibniz rule gives $|D_{\mathrm{gr}}(\Pi(H-\vartheta(h_{<L})))|_\ell\leq\max(\Lambda_1\rho_0^{L-1},\Lambda_0\rho_0^L)$, while $|\Delta(H-\vartheta(h_{<L}))|_\ell\leq\Lambda_2\rho^{L-1}$. Letting $L\to\infty$, $D_{\mathrm{gr}}(\Pi(H))=\Delta(H)$. Every $F\in P_w$ is $\sum_m X^m h_m$ over the finitely many minimal exponents m of weight w, with $h_m\in P_{(0,0)}$ (Dickson), which gives the claim by the Leibniz rule. For $F\in\ker\Pi$, $F=\sum_w F_{(w)}$ converges with $F_{(w)}\in\ker\Pi$, so $\Delta(F)=\sum_w D_{\mathrm{gr}}(\Pi(F_{(w)}))=0$. Hence Δ induces a y-derivation D of S with $D=D_{\mathrm{gr}}$ on S^{gr}, and $y+\epsilon D\colon S\to K[\epsilon]$ is an $\mathcal{O}$-algebra homomorphism.

(e) *Rigidity and the normalisation.* Put $r'(g)=(y+\epsilon D)(r_S(g))$; this is a point with values in $K[\epsilon]$ reducing to r; as r is rigid, $r'(g)=r(g)+\epsilon[X_0,r(g)]$ with $X_0\in\mathfrak{sp}_4(K)$. By (Se3) for r_S, $r'(\phi)$ is block diagonal and $h(r')_{13}=h(r')_{31}=0$, where $h(r')=r'(g_0)+\theta(g_0)r'(g_0^{-1})=h(r)+\epsilon[X_0,h(r)]$. So $[X_0,r(\phi)]$ is block diagonal and $[X_0,h(r)]_{13}=[X_0,h(r)]_{31}=0$. By (Sq3) $r(\phi)$ is block diagonal with blocks $R_{\mathrm{A}},R_{\mathrm{B}}$, whose eigenvalues are integral and reduce to $\bar\kappa_1(\phi),\bar\kappa_3(\phi)$, resp. $\bar\kappa_2(\phi),\bar\kappa_4(\phi)$ (Sq4), disjoint sets by (Ae1). The off-block part Y of X_0 satisfies $[Y,r(\phi)]=0$, i.e. $Y_{\mathrm{AB}}R_{\mathrm{B}}=R_{\mathrm{A}}Y_{\mathrm{AB}}$ and $Y_{\mathrm{BA}}R_{\mathrm{A}}=R_{\mathrm{B}}Y_{\mathrm{BA}}$; by Sylvester's theorem $Y=0$. With X_0 block diagonal and $h=h(r)$, $h_{13}=h_{31}=0$ (Sq3): $[X_0,h]_{13}=(X_0)_{13}(h_{33}-h_{11})$ and $[X_0,h]_{31}=(X_0)_{31}(h_{11}-h_{33})$. By (Sq4), $h_{11}-h_{33}\equiv\theta(g_0)+1-\alpha(g_0)-\Psi(g_0)=(\alpha(g_0)-1)(\Psi(g_0)-1)$ modulo $\mathfrak{m}_\ell$, a unit by (Sg4); so $(X_0)_{13}=(X_0)_{31}=0$. The condition ${}^tX_0J_4+J_4X_0=0$ reads $(X_0)_{j^*i}(J_4)_{j^*j}+(J_4)_{ii^*}(X_0)_{i^*j}=0$ ($i^*=5-i$); for $(i,j)=(3,4),(1,2),(i,i^*)$ it gives $(X_0)_{24}=\pm(X_0)_{13}=0$, $(X_0)_{42}=\pm(X_0)_{31}=0$ and $(X_0)_{ii}+(X_0)_{i^*i^*}=0$. So $X_0=\mathrm{diag}(x_1,x_2,-x_2,-x_1)$, and $(X_0)_{ii}-(X_0)_{jj}=a(\lambda_i-\lambda_j)+b(\mu_i-\mu_j)$ with $a=x_1-x_2$, $b=x_1+x_2$. Hence $D(r_S(g)_{ij})=(aw_1+bw_2)\,y(r_S(g)_{ij})$ for $w=\mathrm{wt}_4(i,j)$; by the Leibniz rule the same holds for every product of entries of total weight w, and by Lemma 45.34 $D(c)=(aw_1+bw_2)y(c)$ for every $c\in S_w$.

(f) *Conclusion.* For $f\in(A_M)_n$, $q(f)\in S_{(0,n)}\subseteq C$, so $t^n\delta(f)=\delta_1(q(f))=D(q(f))=b\,n\,y(q(f))=b\,n\,t^n x(f)$, and $\delta(f)=b\,n\,x(f)$. □

Proposition 45.36. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum and (K,V,a) a test field. Say that a homomorphism $r\colon G_{\mathbb{Q}}\to \mathrm{GL}_4(V)$ is* of block type *if the entries $(1,3)$ and $(3,1)$ of r are active and either no entry (i,j) with $i\in\{2,4\}$, $j\in\{1,3\}$ is active, or no entry (i,j) with $i\in\{1,3\}$, $j\in\{2,4\}$ is active.*

(Zr1) *Every point r of $\mathfrak{D}$ with values in V is irreducible, or of diagonal type, or of block type.*

(Zr2) *Let (K,V,a) have characteristic ℓ, let r be a point of $\mathfrak{D}$ with values in V of diagonal type, and let u be such that (r,u) is an ordinary test representation. Then for every $k\ge 0$, all $g_1,\dots,g_k\in G_{\mathbb{Q}}$ and all indices with $\sum_{s=1}^{k}\mathrm{wt}_4(i_s,j_s)=(0,0)$, the product $x=r(g_1)_{i_1j_1}\cdots r(g_k)_{i_kj_k}$ satisfies $x^{\ell^n}=x$ for some $n\ge 1$.*

(Zr3) *Let r be a point of $\mathfrak{D}$ with values in V of block type and u such that (r,u) is an ordinary test representation. Then there is an ordinary test representation (r_1,u_1) of $\mathfrak{D}$ with values in V such that every $r_1(g)$ is block diagonal and the K-span of the matrices $(r_1(g)_{ab})_{a,b\in\{1,3\}}$, $g\in G_{\mathbb{Q}}$, is $\mathrm{M}_2(K)$.*

Proof. Fix a test field (K,V,a). For a point r of $\mathfrak{D}$ with values in V write $i\to j$ if the entry (i,j) of r is active, and $i\sim j$ if $i\to j$ and $j\to i$. By (Ae4), $\to$ is reflexive and transitive, so $\sim$ is an equivalence relation, and $i\to j$ implies $j^*\to i^*$, so $i\sim j$ implies $i^*\sim j^*$.

Step 1: if $s\sim t$ with $\lambda_s\neq\lambda_t$, then $k\sim s$ for some $k\notin\{s,t\}$. Let r be a point with values in V and, by Lemma 45.37, u such that (r,u) is an ordinary test representation. Suppose $s\sim t$, $\lambda_s\neq\lambda_t$, and no $k\notin\{s,t\}$ satisfies $k\sim s$.

(1a) For $k\notin\{s,t\}$ and all g,h, $r(g)_{sk}r(h)_{kt}=0$: otherwise $s\to k\to t\to s$, so $k\sim s$. The same holds with s,t exchanged. Hence

$$r(gh)_{st}=r(g)_{ss}\,r(h)_{st}+r(g)_{st}\,r(h)_{tt}\qquad(g,h\in G_{\mathbb{Q}}).\tag{45.4}$$

(1b) A common eigenvector at ℓ. Let $Q=\{k:k\to s\}\ni s,t$ and $W_Q=\langle e_k:k\in Q\rangle\subseteq K^4$. If $b\in Q$ and $r(g)_{ab}\neq 0$, then $a\to b\to s$, so $a\in Q$: W_Q is $r(G_{\mathbb{Q}})$-stable. For $k\in Q\setminus\{s,t\}$, $r(g)_{sk}=0$ (else $k\sim s$) and $r(g)_{tk}=0$ (else $s\to t\to k\to s$). By (Ot4), $G_n=\langle ue_1,\dots,ue_n\rangle$ is $r(D_\ell)$-stable and $g\in D_\ell$ acts on G_n/G_{n-1} by the diagonal entry $d_n(g)$ of $u^{-1}r(g)u$. Let n be minimal such that some $w\in W_Q\cap G_n$ has $(w_s,w_t)\neq(0,0)$ ($n\le 4$, as $e_s\in W_Q$). For $g\in D_\ell$, $r(g)w-d_n(g)w\in W_Q\cap G_{n-1}$, so its coordinates s,t vanish, and as $w_k=0$ for $k\notin Q$,

$$r(g)_{ss}w_s+r(g)_{st}w_t=d_n(g)w_s,\qquad r(g)_{ts}w_s+r(g)_{tt}w_t=d_n(g)w_t.$$

As V is a valuation ring, $w_s/w_t\in V$ (with $w_t\neq 0$) or $w_t/w_s\in V$ (with $w_s\neq 0$); the two displayed equations, (1a) and the hypotheses on s,t are symmetric in s and t, so after exchanging s and t we may assume the first, and put $y_0=w_s/w_t\in V$. Eliminating $d_n(g)$, for $g\in D_\ell$

$$r(g)_{st}=\bigl(r(g)_{tt}-r(g)_{ss}\bigr)y_0+r(g)_{ts}\,y_0^2.\tag{45.5}$$

(1c) The residual cocycle. Let (Λ,j) be as in (Ot1) and identify Λ with $j(\Lambda)\subseteq V$; as j is local, the residue field of Λ embeds into k_V. Let $k'\subseteq k_V$ be the finite field generated by it and by $\mathbb{F}_{\mathcal{O}}$. Put $\zeta(g)=r(g)_{st}\in\Lambda$, not identically 0 as $s\to t$; choose $\hat{g}$ with $\zeta(\hat{g})$ of minimal valuation in Λ, put $b=\zeta(\hat{g})\neq 0$ and $\eta(g)=\zeta(g)/b\in\Lambda$. Then $\eta(1)=0$ ($s\neq t$), $\eta(\hat{g})=1$, and $\bar{\eta}$ is locally constant by the continuity in (Ot1). By (Ot2), $\overline{r(g)_{ii}}=\bar{\kappa}_i(g)$ and $\overline{r(g)_{ij}}=0$ for $i\neq j$. Dividing (45.4) by b and reducing, $\bar{\eta}(gh)=\bar{\kappa}_s(g)\bar{\eta}(h)+\bar{\eta}(g)\bar{\kappa}_t(h)$. Let $\gamma=\kappa_s\kappa_t^{-1}$ and $z(g)=\bar{\eta}(g)\bar{\kappa}_t(g)^{-1}\in k'$; then z is locally constant and $z(gh)=z(g)+\bar{\gamma}(g)z(h)$. As $\lambda_s\neq\lambda_t$, γ is one of α $(1,2),(3,4)$, θ $(1,4)$, $\beta^{-1}=\alpha\Psi^{-1}$ $(3,2)$, or their inverses (pairs reversed).

(1d) Local conditions. (i) If g lies in the inertia group of a place of $\overline{\mathbb{Q}}$ above a prime not in T and $\bar\theta(g)=\bar\Psi(g)=\bar\alpha(g)=1$, then $\bar r_0(g)=1$, $r(g)=1$ by (Ot3), and $z(g)=0$. (ii) At ℓ: put $D(g)=r(g)_{tt}-r(g)_{ss}$ and $\epsilon(g)=r(g)_{ts}\in\mathfrak{m}_V$. By (45.5), $b\eta(g)=D(g)y_0+\epsilon(g)y_0^2$ for $g\in D_\ell$. For $g\in D_\ell$ and $\sigma=\sigma_\ell\in D_\ell$, multiplying out and substituting $D(\cdot)y_0=b\eta(\cdot)-\epsilon(\cdot)y_0^2$ twice,

$$\eta(g)D(\sigma)-\eta(\sigma)D(g)=y_0\big(\epsilon(g)\eta(\sigma)-\epsilon(\sigma)\eta(g)\big)\in\mathfrak{m}_V.$$

Reducing, $\bar\eta(g)\delta=\bar\eta(\sigma_\ell)(\bar\kappa_t(g)-\bar\kappa_s(g))$ with $\delta=\bar\kappa_t(\sigma_\ell)-\bar\kappa_s(\sigma_\ell)\neq0$ by (Ae1). With $y_1=-\bar\eta(\sigma_\ell)/\delta\in k'$ this says $\bar\eta(g)=(\bar\kappa_s(g)-\bar\kappa_t(g))y_1$, i.e. $z(g)=\bar\gamma(g)y_1-y_1$ for $g\in D_\ell$. (iii) At $v\in T$, $v\neq\ell$: then $v\in S_\ell(\chi)$, so the v-component of the ℓ-power order part of χ is a non-trivial character of $(\mathbb{Z}/v^m\mathbb{Z})^\times$ of ℓ-power order, whence $\ell\mid v-1$; χ_{cyc} is unramified at v with Frobenius value v, so $\bar\chi_{\mathrm{cyc}}$ is trivial on D_v. By (Sg2) and the list in (1c) ($\bar\alpha=\bar\xi$, $\bar\theta=\bar\xi\bar\Psi$, $\bar\beta^{-1}=\bar\xi\bar\Psi^{-1}$), $\bar\gamma(\sigma_v)\neq1$. Let $M=k'$ with $D_v\cong G_{\mathbb{Q}_v}$ acting by $\bar\gamma$, a finite $\mathbb{F}_\ell$-module. Then $H^0(\mathbb{Q}_v,M)=0$, since $\bar\gamma(\sigma_v)-1$ is invertible on M; and $M^\vee=\mathrm{Hom}(M,\mu_\ell)$ has σ_v acting by $\bar\gamma(\sigma_v)^{-1}\bar\chi_{\mathrm{cyc}}(\sigma_v)=\bar\gamma(\sigma_v)^{-1}\neq1$, so $H^0(\mathbb{Q}_v,M^\vee)=0$ and, by local duality, $H^2(\mathbb{Q}_v,M)=0$. By the local Euler characteristic formula for $v\nmid\ell$, $H^1(\mathbb{Q}_v,M)=0$ (Theorem B.29, items 2 and 4, [249]). So the continuous cocycle $z|_{D_v}$ is a coboundary.

(1e) z is a coboundary. For every $\mathbb{F}_{\mathcal{O}}$-linear $\pi\colon k'\to\mathbb{F}_{\mathcal{O}}$, $\pi\circ z$ is a residual Selmer cocycle for γ by (1c), (1d) ($\bar\gamma$ takes values in $\mathbb{F}_{\mathcal{O}}$), hence a coboundary by (Sg6). If $\bar\gamma=\mathbf{1}$, every $\pi\circ z$ vanishes, so $z=0$. Otherwise fix g' with $\bar\gamma(g')\neq1$ and put $y=z(g')/(\bar\gamma(g')-1)$; if $\pi\circ z(g)=\bar\gamma(g)y_\pi-y_\pi$, then $y_\pi=\pi(y)$, so $\pi(z(g)-\bar\gamma(g)y+y)=0$ for all π, and $z(g)=\bar\gamma(g)y-y$. In both cases $\bar\eta(g)=(\bar\kappa_s(g)-\bar\kappa_t(g))y$ for all g, for some $y\in k'$.

(1f) Contradiction. By (Ae3), $E_{ss}=\sum_m b_m r(h_m)$ for finitely many $b_m\in V$, $h_m\in G_{\mathbb{Q}}$. Then $\sum_m b_m r(h_mh)=E_{ss}r(h)$; its entries (s,t), (s,s), (t,t) give

$$\sum_m b_m\eta(h_mh)=\eta(h),\qquad \sum_m b_m r(h_mh)_{ss}=r(h)_{ss}\qquad\text{and}\qquad \sum_m b_m r(h_mh)_{tt}=0.$$

Reducing and using (1e),

$$\bar\eta(h)=y\sum_m\bar b_m\big(\bar\kappa_s(h_mh)-\bar\kappa_t(h_mh)\big)=y\,\bar\kappa_s(h).$$

At $h=1$ this gives $y=0$, and at $h=\hat g$ it gives $1=0$: a contradiction.

Step 2: (Zr1). Let r be a point with values in V. The map $i\mapsto i^*$ permutes the $\sim$-classes; it is a fixed-point-free involution of $\{1,2,3,4\}$. If some class C meets both blocks, then $|C|\geq3$ by Step 1, so C^* is a class of size ≥3 meeting C, $C^*=C$, and C is a union of orbits of $*$, so $|C|=4$: every entry is active and r is irreducible by (Ae5). Otherwise every class lies in one block. If all classes are singletons, there are no $i\neq j$ with $i\to j\to i$: r is of diagonal type. If not, $\{1,3\}$ or $\{2,4\}$ is a class, hence both are ($\{1,3\}^*=\{2,4\}$); in particular $(1,3)$ and $(3,1)$ are active. If $a\to b$ and $b'\to a'$ with $a,a'\in\mathrm{A}$ and $b,b'\in\mathrm{B}$, then $b\to b'\to a'\to a$ (as $b\sim b'$, $a\sim a'$), so $a\sim b$, which is excluded. So no entry (i,j) with $i\in\mathrm{B}$, $j\in\mathrm{A}$ is active, or none with $i\in\mathrm{A}$, $j\in\mathrm{B}$: r is of block type.

Step 3: (Zr2). Let K have characteristic ℓ, r of diagonal type and (r,u) an ordinary test representation. (3a) A functional. For i let $n(i)=\#\{k:i\to k\}$; if $i\to j$ with $i\neq j$, then $\{k:j\to k\}\subseteq\{k:i\to k\}$ and i lies only in the second set (diagonal type), so $n(j)<n(i)$; as also $j^*\to i^*$, $n(i^*)<n(j^*)$. Put $f(i)=n(i)-n(i^*)$, so $f(i)>f(j)$ whenever $i\to j$, $i\neq j$, and $f(i^*)=-f(i)$. The linear map $\Phi(x,y)=(f(1)-f(2))x+(f(1)+f(2))y$ on $\mathbb{Z}^2$ satisfies $\Phi(\mathrm{wt}_4(i,j))=f(i)-f(j)$ for all i,j.

(3b) Products of weight zero. Let $x=r(g_1)_{i_1j_1}\cdots r(g_m)_{i_mj_m}$ with $\sum_k\mathrm{wt}_4(i_k,j_k)=(0,0)$. If a factor is 0, then $x=0=x^\ell$. Otherwise every (i_k,j_k) is active, so $\Phi(\mathrm{wt}_4(i_k,j_k))\geq0$

with equality only if $i_k = j_k$; the sum being $\Phi(0) = 0$, all $i_k = j_k$ and $x = \prod_k \delta_{i_k}(g_k)$ with $\delta_i(g) = r(g)_{ii}$. Each δ_i is a character $G_{\mathbb{Q}} \to \Lambda^\times$: $r(gh)_{ii} = \sum_k r(g)_{ik} r(h)_{ki}$ and the terms $k \neq i$ vanish (diagonal type). (3c) Each δ_i has finite order. As K has characteristic ℓ, $\ker a$ is a non-zero prime of $\overline{\mathbb{Z}}_\ell$, hence $\mathfrak{m}_\ell$; so $a(\kappa_i(g)) = 1$ when $\bar{r}_0(g) = 1$. Let m_0 be the product of the order of the finite group $\bar{r}_0(G_{\mathbb{Q}})$ and the order of χ, and $N_0 = m_0 \ell^2$. Let I be an inertia group at a place above a prime p and $g \in I$. If $p \notin T$, $r(g^{m_0}) = 1$ by (Ot3). If $p \in T \setminus \{\ell\}$, then $\chi(g^{m_0}) = 1$ and $r(g^{m_0}) = 1$ by (Ot3) when $I = I_p$; δ_i being a character, $\delta_i^{m_0}$ is then trivial on every conjugate of I_p. If $p = \ell$ and $I = I_\ell$ (again it suffices to treat I_ℓ), then $g' = g^{m_0}$ has $\bar{r}_0(g') = 1$, so by (Ot4) $u^{-1} r(g') u$ is upper triangular with diagonal entries 1, $(r(g') - 1)^4 = 0$, and in characteristic ℓ, $r(g')^{\ell^2} - 1 = (r(g') - 1)^{\ell^2} = 0$. So $\delta_i^{N_0}$ is trivial on every inertia group. It is continuous by (Ot1), so the fixed field of its kernel is an abelian extension of $\mathbb{Q}$ unramified at every prime, hence $\mathbb{Q}$ (Minkowski), and $\delta_i^{N_0} = \mathbf{1}$. (3d) Hence x is a root of unity in K: $x^{M_0} = 1$ for some $M_0 = \ell^c M_1$ with $\ell \nmid M_1$; then $(x^{M_1} - 1)^{\ell^c} = 0$, $x^{M_1} = 1$, and with $n \geq 1$ the order of ℓ modulo M_1, $x^{\ell^n} = x$.

Step 4: (Zr3). Let r be of block type and (r, u) an ordinary test representation. For $Y \in \mathrm{M}_4(V)$ let $Y^\flat$ have entries Y_{ij} if $\lambda_i = \lambda_j$ and 0 otherwise, and $E_{\mathrm{A}} = E_{11} + E_{33}$; then $Y^\flat = E_{\mathrm{A}} Y E_{\mathrm{A}} + (1 - E_{\mathrm{A}}) Y (1 - E_{\mathrm{A}})$. By block type, in case (a) every $r(g)$ has vanishing entries (i, j) with $i \in \mathrm{B}$, $j \in \mathrm{A}$, in case (b) those with $i \in \mathrm{A}$, $j \in \mathrm{B}$. For two matrices Y, Y' of the same such shape, $(YY')^\flat = Y^\flat Y'^\flat$ (a product $Y_{ak} Y'_{kb}$ with a, b in one block and k in the other has a vanishing factor). So $r_1(g) = r(g)^\flat$ is a homomorphism $G_{\mathbb{Q}} \to \mathrm{GL}_4(V)$ with block diagonal values, and $u_1 = u$. (Ot1) holds with the same (Λ, j); (Ot3) holds since $r(g) = 1$ implies $r_1(g) = 1$; the congruence in (Ot2) holds since r_0 is diagonal. Similitude: in case (a), for $\tau \in K^\times$, $\mathrm{T}_4(\tau, 1)\, r(g)\, \mathrm{T}_4(\tau, 1)^{-1} = r_1(g) + \tau(r(g) - r_1(g))$; conjugation by $\mathrm{T}_4(\tau, 1) = \mathrm{diag}(\tau, 1, \tau, 1)$, of similitude τ, preserves the similitude $\nu = a(\theta(g))$, so ${}^t(r_1(g) + \tau Y_g) J_4 (r_1(g) + \tau Y_g) = \nu J_4$ with $Y_g = r(g) - r_1(g)$, for infinitely many τ; comparing constant terms of this polynomial identity in τ, $r_1(g)$ has similitude ν. Case (b) is the same with τ^{-1}. (Ot4): u has similitude 1. The matrix $X_\phi = r(\phi)$ is block diagonal by (Se3), and $\phi \in D_\ell$ (D_ℓ is closed and contains σ_ℓ; (Sg3)). The characteristic polynomials $p_{\mathrm{A}}, p_{\mathrm{B}}$ (with coefficients in V) of its two blocks have roots in V (V is integrally closed and K algebraically closed) reducing to $\bar{\kappa}_1(\phi), \bar{\kappa}_3(\phi)$ and $\bar{\kappa}_2(\phi), \bar{\kappa}_4(\phi)$ by (Ot2), distinct across the blocks by (Ae1). So $p_{\mathrm{A}}, p_{\mathrm{B}}$ are coprime polynomials over K: $q_{\mathrm{A}} p_{\mathrm{A}} + q_{\mathrm{B}} p_{\mathrm{B}} = 1$, and by the Cayley–Hamilton theorem $(q_{\mathrm{B}} p_{\mathrm{B}})(X_\phi) = E_{\mathrm{A}}$. Since $u^{-1} X_\phi u$ is upper triangular by (Ot4), so is the idempotent $\Pi = u^{-1} E_{\mathrm{A}} u = (q_{\mathrm{B}} p_{\mathrm{B}})(u^{-1} X_\phi u)$, whose diagonal entries c_i satisfy $c_i^2 = c_i$. For $g \in D_\ell$, with $R_g = u^{-1} r(g) u$, $u^{-1} r_1(g) u = \Pi R_g \Pi + (1 - \Pi) R_g (1 - \Pi)$ is upper triangular with diagonal entries $(c_i^2 + (1 - c_i)^2)(R_g)_{ii} = (R_g)_{ii}$. So (r_1, u) satisfies (Ot4) with the diagonal of (r, u), and is an ordinary test representation. Finally, by (Ae3) E_{11}, E_{33} lie in the V-span of $r(G_{\mathbb{Q}})$, hence ($Y \mapsto Y^\flat$ being linear and fixing them) in that of $r_1(G_{\mathbb{Q}})$; the entries $(1, 3)$, $(3, 1)$ of r_1 are those of r, hence active; so $E_{11} r_1(g) E_{33} = r_1(g)_{13} E_{13}$ and $E_{33} r_1(g') E_{11} = r_1(g')_{31} E_{31}$ give E_{13}, E_{31} in the K-span, and the matrices $(r_1(g)_{ab})_{a,b \in \{1,3\}}$ span $\mathrm{M}_2(K)$. □

Lemma 45.37. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum, (K, V, a) a test field and r a point of $\mathfrak{D}$ with values in V. Then there is $u \in \mathrm{GL}_4(V)$ such that (r, u) is an ordinary test representation of $\mathfrak{D}$ with values in V.*

Proof. Write $r(g) = f(r_A(g))$ for a symplectic Eisenstein lift r_A to $A \in \mathrm{CNL}_{\mathcal{O}}$ and a ring homomorphism $f\colon A \to V$ restricting to a on $\mathcal{O}$ with $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_V$, such that $(r, 1)$ satisfies (Ot1). By (Se4) at $v = \ell$ there are a domain $B \in \mathrm{CNL}_{\mathcal{O}}$, a lift ρ_B of $\bar{r}_0|_{D_\ell}$ to B satisfying the local condition at ℓ (similitude $\theta(g)$ on D_ℓ, and ordinary), and $\mathcal{O}$-algebra homomorphisms $x\colon B \to \mathcal{O}$, $y\colon B \to A$ with $x \circ \rho_B = r_0|_{D_\ell}$, $y \circ \rho_B = r_A|_{D_\ell}$. Put $F = f \circ y\colon B \to V$, so

$r|_{D_\ell} = F \circ \rho_B$ entrywise and F restricts to a on $\mathcal{O}$. The composite $B \to A \to A/\mathfrak{m}_A = \mathbb{F}_\mathcal{O}$ is an $\mathcal{O}$-algebra homomorphism onto a field, so its kernel is the maximal ideal $\mathfrak{m}_B$; hence $y(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$ and $F(\mathfrak{m}_B) \subseteq \mathfrak{m}_V$.

Step 1: (Ot2) and (Ot3). Since r_A is a lift, $r_A(g)_{ij} - r_0(g)_{ij} \cdot 1_A \in \mathfrak{m}_A$; applying f gives $r(g)_{ij} - a(r_0(g)_{ij}) \in \mathfrak{m}_V$. Applying f to ${}^t r_A(g) J_4 r_A(g) = \theta(g) J_4$ (that is (Se1)) shows that $r(g)$ has similitude $a(\theta(g))$. This is (Ot2). The first half of (Ot3) is the image under f of (Se2). For $v \in T$, $v \neq \ell$, (Se4) gives B_v, ρ_{B_v} satisfying the local condition at v and $y_v \colon B_v \to A$ with $y_v \circ \rho_{B_v} = r_A|_{D_v}$; as $\rho_{B_v}(g) = 1$ for $g \in I_v$ with $\chi(g) = 1$, also $r(g) = f(y_v(\rho_{B_v}(g))) = 1$. This is the second half of (Ot3).

Step 2: the flag at the generic point of B. Since x is an $\mathcal{O}$-algebra homomorphism to $\mathcal{O}$, $\mathcal{O} \to B$ is injective, and $L = \mathrm{Frac}(B)$ has characteristic 0. Put $c = \chi_{\mathrm{cyc}}(s_0)$ and $c_j = \kappa_j(s_0)$. By (Sg5), $\chi(s_0) = \omega_\ell(s_0) = 1$, and $\xi(s_0) = 1$ since ξ is unramified at $\ell \in T$ (Sg1) and $s_0 \in I_\ell$; so $(c_1, c_2, c_3, c_4) = (c^{e+1}, c^e, c, 1)$, pairwise distinct as c has infinite order. Put $d_i = \prod_{j \neq i}(c_i - c_j) \in \mathcal{O} \backslash \{0\}$. As ρ_B is ordinary, the characteristic polynomial of $\rho_B(s_0)$ is $\prod_j (X - c_j)$, so L^4 has a basis $w_1, \ldots, w_4$ with $\rho_B(s_0) w_k = c_k w_k$; let E_i be the projection onto Lw_i along the other w_k. Then $\Pi_i = d_i E_i$, and $\Pi_j \rho_B(g) \Pi_i = 0$ for $i < j$ gives $E_j \rho_B(g) E_i = 0$: the subspaces $F_i^L = \langle w_1, \ldots, w_i \rangle$ are $\rho_B(D_\ell)$-stable. Let $\delta_i(g) \in L$ be the scalar by which $\rho_B(g)$ acts on F_i^L / F_{i-1}^L ($g \in D_\ell$); then $E_i \rho_B(g) E_i = \delta_i(g) E_i$, so

$$\mathrm{tr}(\Pi_i \, \rho_B(g)) = d_i \, \delta_i(g), \tag{45.6}$$

and for $s \in I_\ell$ the ordinary condition $\Pi_i(\rho_B(s) - \kappa_i(s))\Pi_i = 0$ gives $d_i^2(\delta_i(s) - \kappa_i(s))E_i = 0$, so $\delta_i(s) = \kappa_i(s)$. The flag is isotropic: by the local condition, $\langle \rho_B(s_0)v, \rho_B(s_0)v' \rangle = \theta(s_0)\langle v, v' \rangle$ for $\langle v, v' \rangle = {}^t v J_4 v'$, and $\theta(s_0) = \alpha(s_0)\Psi(s_0) = c^{e+1}$, so $(c_a c_b - c^{e+1})\langle w_a, w_b \rangle = 0$. With $(m_1, \ldots, m_4) = (e+1, e, 1, 0)$ we have $c_a c_b = c^{m_a + m_b}$, and $m_a + m_b = e+1$ only for $\{a, b\} = \{1, 4\}$ or $\{2, 3\}$ (the other sums are $2e+2, 2e, 2, 0, 2e+1, e+2, e, 1$, all $\neq e+1$ as $e \geq 2$). So $\langle w_a, w_b \rangle = 0$ unless $\{a, b\}$ is $\{1, 4\}$ or $\{2, 3\}$: the form vanishes on $F_2^L \times F_2^L$, and F_3^L is orthogonal to F_1^L, hence equal to its orthogonal, both having dimension 3.

Step 3: a valuation ring of L over $\ker F$. Let $\mathfrak{p} = \ker F$, a prime of B. By Chevalley's theorem there is a valuation ring W of L with $B_\mathfrak{p} \subseteq W$ and $\mathfrak{m}_W \cap B_\mathfrak{p} = \mathfrak{p} B_\mathfrak{p}$, so $\mathfrak{m}_W \cap B = \mathfrak{p}$. By Lemma 45.38 for W and the flag of Step 2, there is $k \in \mathrm{GL}_4(W)$ of similitude 1 such that for $g \in D_\ell$ the matrix $M(g) = k^{-1} \rho_B(g) k$ is upper triangular with diagonal $\delta_1(g), \ldots, \delta_4(g)$ (second consequence, with h the matrix of columns $w_1, \ldots, w_4$). The entries of $\rho_B(g)$ lie in $B \subseteq W$, so $M(g) \in \mathrm{M}_4(W)$ and $\delta_i(g) \in W$.

Step 4: transport to K. Let $\kappa_W = W/\mathfrak{m}_W$ and write $\bar{z}$ for residues in κ_W. Since $\mathfrak{m}_W \cap B = \mathfrak{p}$, $B/\mathfrak{p} \subseteq \kappa_W$; let $E = \mathrm{Frac}(B/\mathfrak{p}) \subseteq \kappa_W$. The map F induces an injective homomorphism $B/\mathfrak{p} \to V \subseteq K$, hence a field embedding $\varepsilon \colon E \to K$. Let $C \subseteq \kappa_W$ be the E-algebra generated by the entries of $\bar{k}$; as $k^{-1} = -J_4 \, {}^t k \, J_4$, it contains the entries of $\bar{k}^{-1}$. For a maximal ideal $\mathfrak{n}$ of C, $C/\mathfrak{n}$ is a finite extension of E (Zariski's lemma), so, K being algebraically closed, ε extends to $C/\mathfrak{n} \to K$; this gives a ring homomorphism $\varphi \colon C \to K$ with $\varphi(\bar{b}) = F(b)$ for $b \in B$. Put $h = \varphi(\bar{k})$. From ${}^t k J_4 k = J_4$, h has similitude 1 and $h^{-1} = \varphi(\bar{k}^{-1})$. For $g \in D_\ell$, $\varphi(\overline{\rho_B(g)}) = F(\rho_B(g)) = r(g)$, so

$$h^{-1} r(g) h = \varphi(\overline{M(g)})$$

is upper triangular with diagonal entries $d_i'(g) = \varphi(\overline{\delta_i(g)})$. For $s \in I_\ell$, $\delta_i(s) = \kappa_i(s) \in \mathcal{O} \subseteq B$, so $d_i'(s) = F(\kappa_i(s)) = a(\kappa_i(s))$.

Step 5: the diagonal is congruent to $a \circ \kappa_i$ on D_ℓ. Fix $g \in D_\ell$ and i, and put $w = \delta_i(g) - \kappa_i(g) \in W$. As a diagonal entry of a triangular conjugate of $\rho_B(g)$, $\delta_i(g)$ is a root of the characteristic polynomial of $\rho_B(g)$, which is monic over B; so w is integral over B and $C' = B[w] \subseteq L$ is a domain finite over B. As B is complete, hence Henselian, C' is a finite product of local rings,

hence local, with maximal ideal $\mathfrak{m}'$. Let $t = \operatorname{tr}(\Pi_i \rho_B(g)) \in B$. By (45.6), $d_i w = t - d_i\kappa_i(g)$. Applying x, and using that Π_i formed with r_0 is diagonal with the single non-zero entry d_i at (i,i), gives $x(t) = d_i\kappa_i(g)$. By the lying-over theorem there is a prime Q of C' over $\ker x$; C'/Q is integral over $B/\ker x \cong \mathcal{O}$, so its fraction field embeds into $\overline{\mathbb{Q}}_\ell$ over $\operatorname{Frac}(\mathcal{O})$, giving $\tilde{x}\colon C' \to \overline{\mathbb{Q}}_\ell$ extending x. Then $d_i\tilde{x}(w) = x(t) - d_i\kappa_i(g) = 0$, so $\tilde{x}(w) = 0$, and w lies in the proper ideal $\ker\tilde{x} \subseteq \mathfrak{m}'$. Every prime of C' containing $\mathfrak{m}_B C'$ lies over $\mathfrak{m}_B$, hence is maximal, hence equals $\mathfrak{m}'$; so $\mathfrak{m}' = \sqrt{\mathfrak{m}_B C'}$, and by Atiyah–Macdonald, Lemma 5.14, there are $n \geq 1$ and $b_1, \dots, b_n \in \mathfrak{m}_B$ with

$$w^n + b_1 w^{n-1} + \cdots + b_n = 0.$$

Reducing modulo $\mathfrak{m}_W$ and applying φ (note $\bar{w} = \overline{\delta_i(g)} - \overline{\kappa_i(g)} \in C$), the element $z = \varphi(\bar{w}) = d_i'(g) - a(\kappa_i(g))$ satisfies $z^n + F(b_1)z^{n-1} + \cdots + F(b_n) = 0$ with $F(b_k) \in \mathfrak{m}_V$. So z is integral over V, hence $z \in V$, and $z^n \in \mathfrak{m}_V$, hence $z \in \mathfrak{m}_V$.

Step 6: an integral flag, and (Ot4), (Ot1). The subspaces $G_i = \langle he_1, \dots, he_i\rangle \subseteq K^4$ ($i = 1,2,3$) form an isotropic flag with G_3 the orthogonal of G_1: as h has similitude 1, $\langle he_a, he_b\rangle = (J_4)_{ab}$, which vanishes for $a, b \in \{1,2\}$ and, for $a = 1$, unless $b = 4$. Since $h^{-1}r(g)h$ is upper triangular, $r(g)G_i \subseteq G_i$ for $g \in D_\ell$. By Lemma 45.38 for $V \subseteq K$ there is $u \in \mathrm{GL}_4(V)$ of similitude 1 such that $u^{-1}r(s)u$ is upper triangular with the diagonal entries $d_i'(s)$ of $h^{-1}r(s)h$, for every $s \in D_\ell$. By Steps 4 and 5, $d_i'(s) - a(\kappa_i(s)) \in \mathfrak{m}_V$, with equality $d_i'(s) = a(\kappa_i(s))$ for $s \in I_\ell$: this is (Ot4). Finally $(r,1)$ satisfies (Ot1), so (r,u) satisfies (Ot1) by Lemma 45.39. Together with Step 1, (r,u) is an ordinary test representation of $\mathfrak{D}$ with values in V. □

Lemma 45.38. *Let W be a valuation ring with fraction field L, and put $\langle x, y\rangle = {}^t x\, J_4\, y$ for $x, y \in L^4$. Let $F_1 \subset F_2 \subset F_3 \subset L^4$ be subspaces of dimensions $1,2,3$ such that $\langle x, y\rangle = 0$ for all $x, y \in F_2$ and $F_3 = \{y \in L^4 : \langle x, y\rangle = 0$ for all $x \in F_1\}$. Then there is $u \in \mathrm{GL}_4(W)$ of similitude 1 such that for $i = 1,2,3$ the columns $ue_1, \dots, ue_i$ span F_i. Consequently, if $X \in \mathrm{M}_4(L)$ satisfies $XF_i \subseteq F_i$ for $i = 1,2,3$, then $u^{-1}Xu$ is upper triangular; and if moreover $h \in \mathrm{GL}_4(L)$ is such that $he_1, \dots, he_i$ span F_i for $i = 1,2,3$, then $u^{-1}Xu$ and $h^{-1}Xh$ have the same diagonal entries.*

Proof. Call $f \in W^4$ *unimodular* if some coordinate of f is a unit of W.

Step 1: lines contain unimodular vectors, and unimodular vectors pair onto W. Let $0 \neq y \in L^4$. For non-zero $a, b \in L$, $a/b \in W$ or $b/a \in W$, so among the non-zero coordinates of y there is one, y_k, with $y_i/y_k \in W$ for all i; then $y/y_k \in W^4$ has k-th coordinate 1. The same holds for a free W-module of finite rank and a non-zero vector of its scalar extension to L, using coordinates in a basis. If $f \in W^4$ is unimodular, the W-linear map $W^4 \to W$, $z \mapsto \langle f, z\rangle$, is surjective: $\langle f, e_j\rangle = \sum_i f_i(J_4)_{ij}$ equals $\pm f_{\tau(j)}$ for the permutation τ exchanging $1 \leftrightarrow 4$, $2 \leftrightarrow 3$, so some $\langle f, e_j\rangle$ is a unit.

Step 2: the vectors f_1, f_4 and a splitting. By Step 1 choose a unimodular $f_1 \in W^4$ spanning F_1, and $f_4 \in W^4$ with $\langle f_1, f_4\rangle = 1$. For $z \in L^4$ put

$$\pi(z) = z - \langle f_1, z\rangle f_4 + \langle f_4, z\rangle f_1.$$

Then

$$\langle f_1, \pi(z)\rangle = \langle f_1, z\rangle - \langle f_1, z\rangle\langle f_1, f_4\rangle + \langle f_4, z\rangle\langle f_1, f_1\rangle = 0,$$
$$\langle f_4, \pi(z)\rangle = \langle f_4, z\rangle - \langle f_1, z\rangle\langle f_4, f_4\rangle + \langle f_4, z\rangle\langle f_4, f_1\rangle = 0,$$

as $\langle f_4, f_1\rangle = -1$. So π maps L^4 onto $H_L = \{z : \langle f_1, z\rangle = \langle f_4, z\rangle = 0\}$, is the identity there, and maps W^4 onto $H = H_L \cap W^4$. Hence $W^4 = Wf_1 \oplus Wf_4 \oplus H$ (if $af_1 + bf_4 + z = 0$ with $z \in H$, pairing with f_1 gives $b = 0$ and pairing with f_4 gives $a = 0$). As a direct summand of

W^4, H is projective, hence free over the local ring W, and $H \otimes_W L = H_L$ has dimension 2 ($\langle\cdot,\cdot\rangle$ is non-degenerate and f_1, f_4 are independent). In a basis (f_1, f_4, h_1, h_2) of W^4 with (h_1, h_2) a basis of H, the Gram matrix of $\langle\cdot,\cdot\rangle$ is the block sum of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the Gram matrix G_H of H; its determinant is $\det(J_4) = 1$ times the square of a unit, so $\det G_H$ is a unit: the form is perfect on H.

Step 3: the vectors f_2, f_3. Since $\langle\cdot,\cdot\rangle$ vanishes on $F_2 \times F_2 \supseteq F_1 \times F_2$, we have $F_2 \subseteq F_3$. For $z \in F_3$ we have $\langle f_1, z\rangle = 0$, so $\pi(z) = z + \langle f_4, z\rangle f_1 \equiv z$ modulo Lf_1; and $\pi(f_1) = f_1 + \langle f_4, f_1\rangle f_1 = 0$. Hence $\ell_2 = \pi(F_2) \subseteq H_L$ is a line with $F_2 = Lf_1 \oplus \ell_2$. By Step 1, applied to H, choose $f_2 \in H \cap \ell_2$ spanning ℓ_2 and part of a basis (f_2, h') of H. Then $G_H = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$ with $b = \langle f_2, h'\rangle$, and $b^2 = \det G_H$ is a unit, so b is a unit; put $f_3 = b^{-1}h'$, so $\langle f_2, f_3\rangle = 1$.

Step 4: the matrix u. Let u have columns f_1, f_2, f_3, f_4. Its Gram matrix ${}^t u J_4 u = (\langle f_a, f_b\rangle)_{a,b}$ has $\langle f_1, f_4\rangle = \langle f_2, f_3\rangle = 1$, $\langle f_4, f_1\rangle = \langle f_3, f_2\rangle = -1$, and all other entries 0: $\langle f_a, f_a\rangle = 0$, $f_2, f_3 \in H$ are orthogonal to f_1 and f_4. So ${}^t u J_4 u = J_4$: u has similitude 1, $\det(u)^2 = 1$, and $u \in \mathrm{GL}_4(W)$. The column $ue_1 = f_1$ spans F_1; f_1, f_2 span $Lf_1 \oplus \ell_2 = F_2$; and f_1, f_2, f_3 are independent and orthogonal to f_1, so they span the 3-dimensional space F_3.

Step 5: the consequences. If $XF_i \subseteq F_i$, then $u^{-1}Xu$ maps $\langle e_1, \dots, e_i\rangle = u^{-1}F_i$ into itself for $i = 1, 2, 3$, so it is upper triangular. If h is as stated, $b' = u^{-1}h$ maps each $\langle e_1, \dots, e_i\rangle$ onto itself, so b' and b'^{-1} are upper triangular with $(b'^{-1})_{ii} = (b'_{ii})^{-1}$; and $h^{-1}Xh$ is upper triangular by the first consequence applied to h in place of u (the argument used only the spanning property). Then $u^{-1}Xu = b'(h^{-1}Xh)b'^{-1}$, and the diagonal of a product of upper triangular matrices is the product of the diagonals, so $(u^{-1}Xu)_{ii} = b'_{ii}(h^{-1}Xh)_{ii}(b'_{ii})^{-1} = (h^{-1}Xh)_{ii}$. □

Lemma 45.39. *Let K be a field, V a valuation ring of K with maximal ideal $\mathfrak{m}_V$, Λ a complete discrete valuation ring with finite residue field and $j\colon \Lambda \to V$ an injective local homomorphism such that every element of K is algebraic over $j(\Lambda)$. For every finite subset $Y \subseteq V$ there are a complete discrete valuation ring Λ' with finite residue field and an injective local homomorphism $j'\colon \Lambda' \to V$ such that $j(\Lambda) \cup Y \subseteq j'(\Lambda')$ and $\iota = j'^{-1} \circ j\colon \Lambda \to \Lambda'$ is a local homomorphism. In particular, if (K, V, a) is a test field, $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(V)$ a homomorphism such that $(r, 1)$ satisfies (Ot1), and $u \in \mathrm{GL}_4(V)$, then (r, u) satisfies (Ot1).*

Proof. Step 1: the ring Λ'. Let $E_0 \subseteq K$ be the fraction field of $j(\Lambda)$; it is complete for the discrete valuation with valuation ring $j(\Lambda) \cong \Lambda$. Every $y \in Y$ is algebraic over $j(\Lambda)$, so $E' = E_0(Y) \subseteq K$ is a finite extension of E_0. Let Λ' be the integral closure of $j(\Lambda)$ in E'. Since E_0 is complete, Λ' is a complete discrete valuation ring, finite over $j(\Lambda)$, whose residue field is a finite extension of that of Λ, hence finite (Serre, Local Fields, Ch. II, §2, Proposition 3); its fraction field is E'.

Step 2: $\Lambda' = V \cap E'$. The ring $V \cap E'$ is a valuation ring of E' (for $0 \neq x \in E'$, $x \in V$ or $x^{-1} \in V$) containing $j(\Lambda)$; valuation rings are integrally closed, so $\Lambda' \subseteq V \cap E'$. A ring between a discrete valuation ring and its fraction field is one of the two: if it contains an element $w\varpi'^{-k}$ with w a unit and $k \geq 1$ (ϖ' a uniformiser), it contains ϖ'^{-1} and hence all of E'. Let ϖ be a uniformiser of Λ; as j is local, $j(\varpi) \in \mathfrak{m}_V$, so $j(\varpi)^{-1} \in E' \setminus V$ and $V \cap E' \neq E'$. Hence $V \cap E' = \Lambda'$, and $Y \subseteq V \cap E' = \Lambda'$.

Step 3: j' and ι. Let $j'\colon \Lambda' \to V$ be the inclusion; it is injective, and $j(\Lambda) \cup Y \subseteq j'(\Lambda')$. It is local: if $x \in \Lambda'$ is not a unit of Λ', then $x^{-1} \in E' \setminus \Lambda' = E' \setminus V$ (or $x = 0$), so $x \in \mathfrak{m}_V$. Conversely a unit of Λ' is a unit of V, so $\mathfrak{m}_{\Lambda'} = \Lambda' \cap \mathfrak{m}_V$. The map $\iota\colon \Lambda \to \Lambda'$ sends $\mathfrak{m}_\Lambda$ into $\mathfrak{m}_V \cap \Lambda' = \mathfrak{m}_{\Lambda'}$, so it is local, $\iota(\mathfrak{m}_\Lambda^n) \subseteq \mathfrak{m}_{\Lambda'}^n$, and a map φ from a topological space to Λ whose reductions modulo every $\mathfrak{m}_\Lambda^n$ are locally constant has $\iota \circ \varphi$ with reductions modulo every $\mathfrak{m}_{\Lambda'}^n$ locally constant: ι

preserves continuity for the adic topologies.

Step 4: (Ot1) for (r,u). Let (Λ, j) be as in (Ot1) for $(r,1)$, and apply Steps 1–3 with Y the set of entries of u. Every element of K is algebraic over $j(\Lambda) \subseteq j'(\Lambda')$; the entries of u and of every $r(g)$ lie in $j'(\Lambda')$; and $g \mapsto j'^{-1}(r(g)_{ab}) = \iota(j^{-1}(r(g)_{ab}))$ is continuous for the $\mathfrak{m}_{\Lambda'}$-adic topology by Step 3. So (Ot1) holds for (r,u), with the pair (Λ', j'). □

Lemma 45.40. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum and r a point of $\mathfrak{D}$ of diagonal type. Then there is a permutation m of $\{1,2,3,4\}$, either the identity or $m_\star = (1\,3)(2\,4)$, such that:*

(Dt1) *$r(s)_{ii} = \kappa_{m(i)}(s)$ for every $s \in I_\ell$ and every i;*

(Dt2) *if $i \neq j$ and the entry (i,j) of r is active, then $m(i) < m(j)$.*

In particular, if m is the identity, then $r(s_0)_{11} = \theta(s_0)$ and no entry (i,j) with $\mu_i < \mu_j$ is active; if $m = m_\star$, then $r(s_0)_{11} = \alpha(s_0)$ and no entry (i,j) with $\mu_i > \mu_j$ is active.

Proof. Write $K = \overline{\mathbb{Q}}_\ell$, $i^* = 5 - i$, $i \to j$ if the entry (i,j) is active, and $(n_1, n_2, n_3, n_4) = (e+1, e, 1, 0)$, strictly decreasing as $e \geq 2$. By Lemma 45.41, r is a point with values in $\overline{\mathbb{Z}}_\ell$ for the standard test field, so (Ae3), (Ae4) apply: $\to$ is reflexive and transitive, $i \to j$ implies $j^* \to i^*$, and, r being of diagonal type, $i \to j \to i$ implies $i = j$. Since ξ is unramified at $\ell \in T$ (Sg1) and $\chi(s_0) = \omega_\ell(s_0) = 1$ (Sg5), $(\kappa_1, \ldots, \kappa_4)(s_0) = (c^{e+1}, c^e, c, 1)$ with $c = \chi_{\mathrm{cyc}}(s_0)$ of infinite order: the four values are pairwise distinct.

Step 1: the diagonal entries are characters. For $g, h \in G_{\mathbb{Q}}$, $r(gh)_{ii} = \sum_k r(g)_{ik} r(h)_{ki}$, and a term with $k \neq i$ would give $i \to k \to i$; so $\delta_i(g) = r(g)_{ii}$ is multiplicative. Ordering $\{1, \ldots, 4\}$ by a linear order extending $\to$, every $r(g)$ is triangular in the reordered basis, so $\det r(g) = \prod_i \delta_i(g)$ and $\delta_i(g) \in K^\times$; each δ_i is continuous by (Sq5), and $\operatorname{tr} r(g) = \sum_i \delta_i(g)$.

Step 2: the permutation. Let $\delta'_1, \ldots, \delta'_4$ be the characters of (Zt2): r is triangular on D_ℓ with diagonal (δ'_k), $\delta'_k = \kappa_k$ on I_ℓ and $\delta'_k \equiv \kappa_k$ modulo $\mathfrak{m}_\ell$ on D_ℓ. They are pairwise distinct (their values at s_0 are). On D_ℓ, $\sum_i \delta_i = \operatorname{tr} r = \sum_k \delta'_k$; by Artin's independence of characters, each distinct character occurs with the same multiplicity on both sides, so there is a permutation m with $\delta_i|_{D_\ell} = \delta'_{m(i)}$. This gives (Dt1). By (Sq4) $\delta_i(\sigma_\ell) \equiv \kappa_i(\sigma_\ell)$ and by (Zt2) $\delta'_{m(i)}(\sigma_\ell) \equiv \kappa_{m(i)}(\sigma_\ell)$ modulo $\mathfrak{m}_\ell$ ($\sigma_\ell \in D_\ell$ by (Sg2)), so $\bar\kappa_i(\sigma_\ell) = \bar\kappa_{m(i)}(\sigma_\ell)$ and $\lambda_i = \lambda_{m(i)}$ by (Ae1). Next, by (Sq2) ${}^t r(g) J_4 r(g) = \theta(g) J_4$; the entry (i, i^*) is $\sum_a \epsilon_a r(g)_{ai} r(g)_{a^* i^*} = \theta(g)\epsilon_i$ with $\epsilon_a = (J_4)_{aa^*} = \pm 1$, and for $a \neq i$ a non-zero term would give $a \to i$ and $a^* \to i^*$, hence $i \to a$: excluded. So $\delta_i \delta_{i^*} = \theta$, and likewise $\kappa_k \kappa_{k^*} = \theta$. At s_0: $\kappa_{m(i)}(s_0)\kappa_{m(i^*)}(s_0) = \theta(s_0) = \kappa_{m(i)}(s_0)\kappa_{m(i)^*}(s_0)$, so $m(i^*) = m(i)^*$. A permutation preserving the blocks $\{1,3\}$, $\{2,4\}$ and commuting with $i \mapsto i^*$ is determined by $m(1) \in \{1,3\}$: $m(1) = 1$ gives $m(4) = 4$, $m(3) = 3$, $m(2) = 2$; $m(1) = 3$ gives $m(4) = 2$, $m(3) = 1$, $m(2) = 4$. So $m \in \{\mathrm{id}, m_\star\}$.

Step 3: $\eta_i = \delta_i \chi_{\mathrm{cyc}}^{-n_{m(i)}}$ has open kernel. η_i is a continuous character. Let N_1 be the product of the order of $\bar r_0(G_{\mathbb{Q}})$, the order of χ and the order of the finite group $\kappa_k \chi_{\mathrm{cyc}}^{-n_k}(I_\ell)$ over k ((Zt3) with $\delta'_k = \kappa_k$ on I_ℓ). For an inertia group I at a place above $p \notin T$ and $g \in I$, $\bar r_0(g^{N_1}) = 1$, so $r(g^{N_1}) = 1$ by (Se2) (applied to the lift of which r is an image) and $\chi_{\mathrm{cyc}}(g) = 1$ ($p \neq \ell$); for $p \in T \setminus \{\ell\}$ and $g \in I_p$, $\chi(g^{N_1}) = 1$, so $r(g^{N_1}) = 1$ by (Zt1); for $g \in I_\ell$, $\eta_i(g) = \kappa_{m(i)} \chi_{\mathrm{cyc}}^{-n_{m(i)}}(g)$ by (Dt1). As η_i is a character, triviality on $I_{\bar p}$ gives triviality on its conjugates. So $\eta_i^{N_1}$ is trivial on every inertia group. Every open subgroup $H \supseteq \ker \eta_i^{N_1}$ is normal with abelian quotient and contains every inertia group, so its fixed field is unramified everywhere and $H = G_{\mathbb{Q}}$ by Minkowski's theorem; the closed subgroup $\ker \eta_i^{N_1}$ is the intersection of such H, hence $\eta_i^{N_1} = \mathbf{1}$. So η_i has finite image, and its kernel, closed of finite index, is open.

Step 4: a minimal bad pair. Suppose (Dt2) fails, and call (i,j) bad if $i \neq j$, $i \to j$ and $m(j) < m(i)$. If (i,j) is bad and $i \to k \to j$ with $k \notin \{i,j\}$, then (i,k) is bad (if $m(k) < m(i)$)

or (k,j) is bad (if $m(k) > m(i) > m(j)$), and the set $\{k' : i \to k' \to k\}$, resp. $\{k' : k \to k' \to j\}$, is strictly smaller than $\{k' : i \to k' \to j\}$ (it misses j, resp. i, by antisymmetry). So there is a bad pair (i,j) with no $k \notin \{i,j\}$ such that $i \to k \to j$. Put $a = m(j) < b = m(i)$.

Step 5: the subquotient. Let U be the span of the e_k with $k \to j$ and U' that of the e_k with $k \to j$, $k \notin \{i,j\}$. Both are $r(G_{\mathbb{Q}})$-stable: if $r(g)_{ck} \neq 0$ with e_k in U', then $c \to k \to j$, $c \neq j$ (else $j \to k \to j$) and $c \neq i$ (else $i \to k \to j$). In $W = U/U'$ with basis $\bar{e}_i, \bar{e}_j$ one has $r(g)\bar{e}_i = \delta_i(g)\bar{e}_i$ (a component along e_j would give $j \to i$) and $r(g)\bar{e}_j = r(g)_{ij}\bar{e}_i + \delta_j(g)\bar{e}_j$. Hence, with $\gamma = \delta_i\delta_j^{-1}$, $z(g) = r(g)_{ij}\delta_j(g)^{-1}$ is a continuous map with $z(gh) = z(g) + \gamma(g)z(h)$.

Step 6: z is locally a coboundary. For a cocycle z and $g \in G_{\mathbb{Q}}$, if $z(h) = \gamma(h)m_0 - m_0$ on a subgroup H, then $z(h) = \gamma(h)m_1 - m_1$ on gHg^{-1} with $m_1 = \gamma(g)m_0 + z(g)$ (expand $z(ghg^{-1})$ by the cocycle rule; γ is a character); so it suffices to treat $I_{\bar{p}}$ and D_ℓ. (i) For $p \neq \ell$: on the open normal subgroup $I' = I_{\bar{p}} \cap \ker \bar{r}_0$ ($p \notin T$) or $I' = I_p \cap \ker \chi$ ($p \in T$) one has $r = 1$, so $z = 0$ and $\gamma = 1$ there; $z|_{I_{\bar{p}}}$ is inflated from the finite group $I_{\bar{p}}/I'$ with values in the $\mathbb{Q}$-vector space K, hence a coboundary. (ii) At ℓ: let u be as in (Zt2) and $F_n = u\langle e_1, \dots, e_n\rangle$, a D_ℓ-stable flag on whose graded pieces s_0 acts by the distinct scalars $\kappa_n(s_0)$; so $r(s_0)$ is diagonalisable with eigenlines L_k, $F_n = L_1 \oplus \cdots \oplus L_n$, and $r(g)L_k \subseteq F_k$ for $g \in D_\ell$. As U, U' are $r(s_0)$-stable, $U = \bigoplus_k (U \cap L_k)$ and likewise for U'. On W, s_0 has the eigenvalues $\kappa_b(s_0)$ (on $\bar{e}_i$) and $\kappa_a(s_0)$ (by (Dt1)); so $U \cap L_k \subseteq U'$ for $k \notin \{a,b\}$, and there is $w \in U \cap L_a$ with image $\bar{w} \neq 0$, $\bar{w} \notin K\bar{e}_i$. For $g \in D_\ell$, $r(g)w \in U \cap F_a = \bigoplus_{k \le a}(U \cap L_k)$, whose image in W is $K\bar{w}$ (as $b > a$). So $K\bar{w}$ is D_ℓ-stable, mapping isomorphically onto $W/K\bar{e}_i$, on which D_ℓ acts by δ_j. Scaling, $\bar{w} = \bar{e}_j + c\bar{e}_i$, and $r(g)\bar{w} = \delta_j(g)\bar{w}$ gives $r(g)_{ij} = c(\delta_j(g) - \delta_i(g))$, i.e. $z(g) = \gamma(g)(-c) - (-c)$ for $g \in D_\ell$.

Step 7: contradiction. $\gamma = \chi_{\mathrm{cyc}}^{n_b - n_a}\eta_i\eta_j^{-1} = \psi^{-1}\chi_{\mathrm{cyc}}^{-e'}$ with $e' = n_a - n_b \geq 1$ and $\psi = \eta_j\eta_i^{-1}$ of open kernel (Step 3). By Step 6 and Lemma 45.31, there is $c' \in K$ with $z(g) = \gamma(g)c' - c'$ for all g, i.e. $r(g)_{ij} = c'(r(g)_{ii} - r(g)_{jj})$. The linear form $X \mapsto X_{ij} - c'(X_{ii} - X_{jj})$ vanishes on $r(G_{\mathbb{Q}})$, hence on its span, which contains E_{ii} by (Ae3); so $c' = 0$ and $r(g)_{ij} = 0$ for all g, contradicting $i \to j$. This proves (Dt2).

The last assertion. $r(s_0)_{11} = \kappa_{m(1)}(s_0)$ by (Dt1), which is $\theta(s_0)$ or $\alpha(s_0)$. For $m = \mathrm{id}$, an active (i,j) with $i \neq j$ has $i < j$, so $\mu_i \geq \mu_j$ ($\boldsymbol{\mu} = (1,1,0,0)$). For $m = m_\star$, the pairs with $m(i) < m(j)$ are $(3,4),(3,1),(3,2),(4,1),(4,2),(1,2)$, all with $\mu_i \leq \mu_j$. □

Lemma 45.41. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum and $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ a point of $\mathfrak{D}$. Then r takes values in $\mathrm{GL}_4(\overline{\mathbb{Z}}_\ell)$ and is a point of $\mathfrak{D}$ with values in $\overline{\mathbb{Z}}_\ell$ for the standard test field.*

Proof. Write $r(g) = f(r_A(g))$ for a symplectic Eisenstein lift r_A to $A \in \mathrm{CNL}_{\mathcal{O}}$ and an $\mathcal{O}$-algebra homomorphism $f\colon A \to \overline{\mathbb{Q}}_\ell$. By (Sq1) of Lemma 44.11, $f(A) \subseteq \overline{\mathbb{Z}}_\ell$, $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_\ell$ and f is continuous for the $\mathfrak{m}_A$-adic topology. So $f\colon A \to \overline{\mathbb{Z}}_\ell$ restricts to the inclusion $a = \mathrm{id}$ on $\mathcal{O}$ and maps $\mathfrak{m}_A$ into the maximal ideal $\mathfrak{m}_\ell$ of $V = \overline{\mathbb{Z}}_\ell$. The entries of every $r(g)$ lie in $\overline{\mathbb{Z}}_\ell$, and $\det r(g) = f(\det r_A(g))$ is the image of a unit of A, hence a unit of $\overline{\mathbb{Z}}_\ell$; so r takes values in $\mathrm{GL}_4(\overline{\mathbb{Z}}_\ell)$. It remains to check (Ot1) for the pair $(r,1)$.

The discrete valuation ring. As A is Noetherian, $\mathfrak{m}_A = (x_1, \dots, x_k)$. Each $f(x_i)$ is algebraic over $\mathbb{Q}_\ell$, and the fraction field of $\mathcal{O}$ is a finite extension of $\mathbb{Q}_\ell$, so $E = \mathrm{Frac}(\mathcal{O})(f(x_1), \dots, f(x_k)) \subseteq \overline{\mathbb{Q}}_\ell$ is a finite extension of $\mathbb{Q}_\ell$. Let $\Lambda = \{y \in E : |y|_\ell \leq 1\}$, a complete discrete valuation ring with finite residue field, and $j\colon \Lambda \to \overline{\mathbb{Z}}_\ell$ the inclusion. It is injective and local, since $\Lambda \cap \mathfrak{m}_\ell = \mathfrak{m}_\Lambda$. Every element of $\overline{\mathbb{Q}}_\ell$ is algebraic over $\mathbb{Q}_\ell$, hence a root of a nonzero polynomial with coefficients in $\mathbb{Z}_\ell \subseteq \Lambda$ (clear denominators).

The entries lie in Λ. Since A is complete and $\mathcal{O} \to A/\mathfrak{m}_A$ is surjective, every $b \in A$ is the limit, for the $\mathfrak{m}_A$-adic topology, of a sequence of polynomials in $x_1, \dots, x_k$ with coefficients in $\mathcal{O}$: write $b = \lambda_0 + b_1$ with $\lambda_0 \in \mathcal{O}$, $b_1 \in \mathfrak{m}_A$, write $b_1 = \sum_i x_i b_{1,i}$ and repeat with the $b_{1,i}$;

after n steps b is congruent modulo $\mathfrak{m}_A^n$ to a polynomial $P_n(x_1,\dots,x_k)$ with coefficients in $\mathcal{O}$. By continuity of f, $f(b)=\lim_n P_n(f(x_1),\dots,f(x_k))$ in $\overline{\mathbb{Q}}_\ell$. Each $P_n(f(x_1),\dots,f(x_k))$ lies in Λ (its coefficients are in $\mathcal{O}\subseteq\Lambda$ and $|f(x_i)|_\ell<1$), and Λ is closed in $\overline{\mathbb{Q}}_\ell$, being complete. Hence $f(A)\subseteq\Lambda$, and all entries of all $r(g)$, and of the matrix 1, lie in $j(\Lambda)$.

Continuity. The map $g\mapsto r_A(g)_{ab}$ is continuous for the $\mathfrak{m}_A$-adic topology, and $f\colon A\to\Lambda$ is continuous by (Sq1), the topology of Λ induced by $|\cdot|_\ell$ being its $\mathfrak{m}_\Lambda$-adic topology. So $g\mapsto r(g)_{ab}$ is continuous for the $\mathfrak{m}_\Lambda$-adic topology, for all a,b. Thus $(r,1)$ satisfies (Ot1), and r is a point of $\mathfrak{D}$ with values in $\overline{\mathbb{Z}}_\ell$. □

46 The dyadic one-character bound

This section proves Proposition 46.2, the one-character bound for dyadic Eisenstein data, which Proposition 38.32 uses for the characters of case (V) whose 2-component has a large conductor. A dyadic Eisenstein datum (Definition 38.31) asks that the conductor $2^{c_2(\chi)}$ of the 2-component be large compared with ℓ, with the primes of $S_\ell(\chi)$, with the order bound of the odd components and with the conductor of ϑ^2 (conditions (Dy5)–(Dy7)). The congruences are formed on the unitary group in three variables attached to $\mathbb{Q}(i)$, with local conditions at the primes of $T_\ell(\chi)$ (Definition 46.1, Definition 46.10). The section proves bound and graded model together, Proposition 46.4 with clauses (Dm1)–(Dm7), and reuses from Sections 42 and 43 the homogeneity of presentations, the Euler-characteristic formula and the rationality of cocycles.

Two points differ from the unitary case of Section 42. First, only representations of stable type occur. The residual Selmer group of weight -2 vanishes (Lemma 46.15), because the first Bernoulli number of an odd character with a large 2-power conductor is an ℓ-adic unit (Lemma 44.6); hence the deformation ring has no endoscopic two-signed points (Lemma 46.17), and no hypothesis on newforms is needed. Second, the presentation counts are made with the dyadic local conditions: graded local lifts exist along small extensions (Lemma 46.12), the pieces of non-zero weight have no local invariants and no local H^2 at the primes of $T_\ell(\chi)$ (Lemma 46.13), and integral cocycles of weights -2, -1, 0 are torsion modulo coboundaries (Proposition 46.14), which uses Proposition 40.3 from Part VII. Rigidity at the automorphic points, Theorem 46.18, rests on the vanishing theorem of Newton and Thorne through Proposition 42.6 [192], and automorphy is Theorem 47.16 of Part IX.

Definition 46.1. Let (ℓ,e,N,χ,o,m) be a dyadic Eisenstein datum (Definition 38.31) with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$ with maximal ideal $\mathfrak{m}_\mathcal{O}$ and residue field $\mathbb{F}_\mathcal{O}$. Let $K=\mathbb{Q}(i)$, $\eta_K\colon G_\mathbb{Q}\to\{\pm1\}$ the character with kernel G_K, $\Psi=\Psi_{\chi,e}$ and $\bar\Psi\colon G_\mathbb{Q}\to\mathbb{F}_\mathcal{O}^\times$ its reduction. Fix a complex conjugation $c\in G_\mathbb{Q}$ and, for every prime p, a place $\bar p$ of $\overline{\mathbb{Q}}$ above p; write $D_p\supseteq I_p$ for $D_{\bar p}\supseteq I_{\bar p}$. For a place $\bar v$ of $\overline{\mathbb{Q}}$ above a prime put

$$N_{\bar v}=\{g\in I_{\bar v}:\ \eta_K(g)=1,\ \bar\Psi(g)=1\}.$$

The group, the Eisenstein representation, lifts and points. The group $\mathcal{G}_3(B)$ with J and θ, GL-parts $\underline{r}$ and signs, $\mathrm{T}(t)$, the weights $\mathrm{wt}(a,b)=b-a$, the Eisenstein representation $r_0(g)=(\mathrm{T}(\Psi(g)),\eta_K(g))$ and its reduction $\bar r_0$, the characters $\psi_a=\Psi^{2-a}$ $(a=1,2,3)$, the category $\mathrm{CNL}_\mathcal{O}$, lifts of $\bar r_0$ to $A\in\mathrm{CNL}_\mathcal{O}$, base points, markings $g_\star$ and condition (Ed1) are those of Definition 42.1, formed with the present K, $\mathcal{O}$, Ψ and c: of the unitary Eisenstein datum, their definitions there use only K, $\mathcal{O}$, Ψ and the identity $\Psi(c)=1$, which holds here since $\Psi(c)=\chi(-1)(-1)^e=1$ by (Dy3). For a pair (R,r) of $R\in\mathrm{CNL}_\mathcal{O}$ and a lift r of $\bar r_0$ to R, the weight-zero subalgebra R^0, points φ with r_φ and $\rho_\varphi\colon G_K\to\mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$, upper, lower and two-signed points, points of type I and of type II, torus conjugates $\mathrm{T}(t)\cdot r_\varphi$, trace differences and

the adjoint representation $\operatorname{ad} r_\varphi$ are defined as in Definition 42.1 under the headings *Normalised ordinary lifts* (last sentence) and *Points*, whose wording applies unchanged to such a pair.

Dyadic local conditions. Let $A \in \mathrm{CNL}_{\mathcal{O}}$ with maximal ideal $\mathfrak{m}_A$. A lift r of $\bar{r}_0$ to A satisfies the **dyadic local conditions** if

(Dd1) there is $h \in \mathrm{GL}_3(A)$ with $h \equiv 1 \pmod{\mathfrak{m}_A}$ such that for every $g \in D_\ell \cap G_K$ the matrix $h^{-1}\underline{r}(g)h$ is upper triangular (its entries of negative weight vanish), and its diagonal entries are $\psi_1(g), \psi_2(g), \psi_3(g)$ when moreover $g \in I_\ell$;

(Dd2) for every $p \in S_\ell(\chi)$ there is $h \in \mathrm{GL}_3(A)$ with $h \equiv 1 \pmod{\mathfrak{m}_A}$ such that for every $g \in D_p \cap G_K$ the matrix $h^{-1}\underline{r}(g)h$ is diagonal, with diagonal entries $\psi_1(g), \psi_2(g), \psi_3(g)$ when moreover $g \in I_p$;

(Dd3) $r(g) = 1$ for every $g \in N_{\bar{v}}$ and every place $\bar{v}$ of $\overline{\mathbb{Q}}$ above a prime not in $T_\ell(\chi)$.

For a marking $g_\star$, a **dyadic Eisenstein deformation ring** for $g_\star$ is a pair (R, r) of $R \in \mathrm{CNL}_{\mathcal{O}}$ and a lift r of $\bar{r}_0$ to R satisfying (Ed1) for $g_\star$ and (Dd1)–(Dd3), such that for every lift r' of $\bar{r}_0$ to some $A \in \mathrm{CNL}_{\mathcal{O}}$ satisfying (Ed1) for $g_\star$ and (Dd1)–(Dd3) there is exactly one morphism $f\colon R \to A$ with $f \circ r = r'$. (For every morphism $f\colon R \to A$, $f \circ r$ again satisfies these conditions, f being local.)

Rigid points. Let φ be a point of a dyadic Eisenstein deformation ring and $\overline{\mathbb{Q}}_\ell[\epsilon]$ the dual numbers ($\epsilon^2 = 0$). For a continuous 1-cocycle y of $G_{\mathbb{Q}}$ with values in $\operatorname{ad} r_\varphi$, the map $r_y(g) = ((1 + \epsilon\, y(g))P_g, \varsigma_g)$, where $r_\varphi(g) = (P_g, \varsigma_g)$, is a homomorphism $G_{\mathbb{Q}} \to \mathcal{G}_3(\overline{\mathbb{Q}}_\ell[\epsilon])$. The cocycle y is a **dyadic tangent cocycle** at φ if

(Ty1) $y(g) = 0$ for every $g \in N_{\bar{v}}$ and every place $\bar{v}$ above a prime not in $T_\ell(\chi)$;

(Ty2) there is $h \in \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell[\epsilon])$ such that for every $g \in D_\ell \cap G_K$ the matrix $h^{-1}\underline{r_y}(g)h$ is upper triangular, with diagonal entries $\psi_1(g), \psi_2(g), \psi_3(g)$ when moreover $g \in I_\ell$;

(Ty3) for every $p \in S_\ell(\chi)$ there is $h \in \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell[\epsilon])$ such that for every $g \in D_p \cap G_K$ the matrix $h^{-1}\underline{r_y}(g)h$ is diagonal, with diagonal entries $\psi_1(g), \psi_2(g), \psi_3(g)$ when moreover $g \in I_p$.

The point φ is **rigid** if every dyadic tangent cocycle at φ is a coboundary, that is of the form $g \mapsto g \cdot X - X$ for some $X \in \operatorname{ad} r_\varphi$.

Automorphic points. For an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, a point φ is **Lie-irreducible automorphic** (for ι) if ρ_φ is irreducible on every open subgroup of G_K and on $G_{K(\mu_{\ell^\infty})}$, and $\rho_\varphi \otimes \chi_{\mathrm{cyc}}^{-1}$ is automorphic of conjugate self-dual type (for ι).

The weight -2 residual Selmer group. Put $\gamma_{-2} = \eta_K \bar{\Psi}^{-2}\colon G_{\mathbb{Q}} \to \mathbb{F}_{\mathcal{O}}^\times$; it is the character by which $G_{\mathbb{Q}}$ acts, through $\operatorname{ad} \bar{r}_0$, on the line of matrices supported on the entry $(3,1)$, the only entry of weight -2. Let Z_{-2} be the set of locally constant maps $z\colon G_{\mathbb{Q}} \to \mathbb{F}_{\mathcal{O}}$ with $z(gh) = z(g) + \gamma_{-2}(g)z(h)$ for all g, h, such that $z(g) = 0$ for every $g \in N_{\bar{v}}$ and every place $\bar{v}$ above a prime other than ℓ, and such that there is $y \in \mathbb{F}_{\mathcal{O}}$ with $z(g) = \gamma_{-2}(g)y - y$ for all $g \in D_\ell$. The **weight -2 residual Selmer group** $\mathrm{Sel}_{-2}(\chi, e)$ is the quotient of Z_{-2} by the coboundaries $g \mapsto \gamma_{-2}(g)y - y$, $y \in \mathbb{F}_{\mathcal{O}}$ (it depends on the whole datum and on $\mathcal{O}$; the subscript and the arguments separate it from $\mathrm{Sel}(F, T)$ of Definition 22.2).

Proposition 46.2. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum, with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$. Let $K = \mathbb{Q}(i) \subset \overline{\mathbb{Q}}$, let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism, $\sigma \in G_K$, and*

$$t = \Psi_{\chi,e}(\sigma) + 1 + \Psi_{\chi,e}(\sigma)^{-1} \in \mathcal{O}.$$

Assume that $\operatorname{tr}\rho(\sigma) \neq t$ *for every representation* $\rho\colon G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ *that is admissible of stable type for* (ι, e, N)*. Then* $\mathrm{Sha}^1_{\ell N}(\chi, e)$ *has finite length, and there is a finite set* X *of pairwise non-isomorphic representations* $G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$*, each admissible of stable type for* (ι, e, N)*, such that*

$$\nu_{\mathcal{O}}(\chi, e) \leq 2 \sum_{\rho \in X} v_\ell(\operatorname{tr}\rho(\sigma) - t).$$

Proof. Step 1: the ring and its graded model. By Theorem 47.16, applied to the datum, $\mathcal{O}$ and ι, there is a marking $g_\star$ such that (Da1) and (Da2) hold for every dyadic Eisenstein deformation ring for $g_\star$. By Proposition 46.4, applied with $g_\star$, there are such a ring (R, r) with its base point φ_{Eis}, and (A, φ_0, u, q) with (Dm3)–(Dm7). For a point φ of (R, r), $\varphi \circ q$ is a point of A on u, because $I(u) = \ker q$.

Step 2: every two-signed point of (R, r) *is of type I, stable admissible and rigid.* $\mathrm{Sel}_{-2}(\chi, e) = 0$ by Lemma 46.15, and R^0 is a finitely generated $\mathcal{O}$-module by (Da1). Let φ be a two-signed point. By Lemma 46.17 it is of type I. By (Da2) it is Lie-irreducible automorphic for ι and ρ_φ is admissible of stable type for (ι, e, N). By Theorem 46.18 (its finiteness hypothesis is (Da1); ρ_φ is generic ordinary of level N and weights $(e, 0, -e)$ by (Sb4) of its stable admissibility) φ is rigid. So the hypotheses of (Dm6) hold, (A, φ_0, u) satisfies (Gc1)–(Gc4); in particular (Gc2) holds, and (Dm5) applies.

Step 3: points on u*.* Let φ_A be a two-signed point of A on u. By (Dm5), $\varphi_A = t_1 \cdot (\varphi \circ q)$ for a point φ of (R, r) and some $t_1 \in \overline{\mathbb{Q}}_\ell^\times$. Since $(t_1 \cdot \varphi \circ q)(f) = t_1^j \varphi(q(f))$ for $f \in A_j$ and $t_1 \neq 0$, $\varphi \circ q$ is two-signed, so by (Dm4) φ is neither upper nor lower: φ is a two-signed point of (R, r). Conversely, for a two-signed point φ of (R, r), $\varphi \circ q$ is a two-signed point on u by (Dm4) and Step 1.

Step 4: the weight-zero element. By (Dm3) choose $w_1, w_2, w_3 \in A_0$ with $q(w_a) = \underline{r}(\sigma)_{aa}$ (diagonal entries have weight 0), and put $a_\sigma = w_1 + w_2 + w_3 - t \in A_0$, with $t \in \mathcal{O}$. Then $q(a_\sigma) = \operatorname{tr}\underline{r}(\sigma) - t$, so for every point φ of (R, r), as $\sigma \in G_K$,

$$\varphi(q(a_\sigma)) = \operatorname{tr}\rho_\varphi(\sigma) - t.$$

Since $\varphi_{\mathrm{Eis}} \circ r = r_0$, $\varphi_0(a_\sigma) = \varphi_{\mathrm{Eis}}(q(a_\sigma)) = \operatorname{tr}\mathrm{T}(\Psi(\sigma)) - t = 0$. For a two-signed point $\varphi_A = t_1 \cdot (\varphi \circ q)$ on u as in Step 3, $\varphi_A(a_\sigma) = \varphi(q(a_\sigma)) = \operatorname{tr}\rho_\varphi(\sigma) - t$ (degree 0), which is non-zero by hypothesis, since ρ_φ is admissible of stable type for (ι, e, N) by Step 2. So (Gc5) holds for a_σ, and $(A, \varphi_0, u, a_\sigma)$ satisfies (Gc1)–(Gc5).

Step 5: the graded numerical criterion. By Lemma 41.2, u has an orbit cover Σ_0, the cotangent length $\mathrm{ct}_{-1}(u_{-1})$ is finite, and $\mathrm{ct}_{-1}(u_{-1}) \leq 2\sum_{\varphi_A \in \Sigma} v_{\mathcal{O}}(\varphi_A(a_\sigma))$ for every orbit cover Σ.

Step 6: the Sha group has length at most $\mathrm{ct}_{-1}(u_{-1})$*.* Let $b \in \mathcal{O} \setminus \{0\}$ and $\mathrm{Sha}[b]$ the b-torsion of $\mathrm{Sha}^1_{\ell N}(\chi, e)$. The map D of (Dm7) satisfies the hypotheses of Lemma 42.12 for $j = -1$, the family u_{-1}, $c = b$ and $M = \mathrm{Sha}[b]$; hence $\operatorname{length}_{\mathcal{O}} \mathrm{Sha}[b] \leq \mathrm{ct}_{-1}(u_{-1})$. Every class in $H^1(\mathbb{Q}, \mathcal{O}^\vee(\Psi^{-1}))$ is represented by a continuous cocycle on the compact group $G_{\mathbb{Q}}$ with values in a discrete module, which has finite image; every element of $\mathcal{O}^\vee$ is killed by a power of ℓ; so every class is killed by some ℓ^k, and $\mathrm{Sha}^1_{\ell N}(\chi, e) = \bigcup_k \mathrm{Sha}[\ell^k]$, an increasing union. If $\mathrm{Sha}^1_{\ell N}(\chi, e)$ contained a strictly increasing chain $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_L$ of submodules with $L = \mathrm{ct}_{-1}(u_{-1}) + 1$, choose $x_i \in M_i \setminus M_{i-1}$; the submodule generated by $x_1, \ldots, x_L$ lies in some $\mathrm{Sha}[\ell^k]$ and contains the strictly increasing chain of the submodules generated by $x_1, \ldots, x_i$ (the i-th lies in M_i and contains $x_i \notin M_{i-1}$), so it has length at least L, a contradiction. Hence $\mathrm{Sha}^1_{\ell N}(\chi, e)$ has finite length, at most $\mathrm{ct}_{-1}(u_{-1})$.

Step 7: from an orbit cover to representations. For two-signed points φ, φ' of (R, r) write $\varphi \sim \varphi'$ if $r_{\varphi'} = \mathrm{T}(t_1) \cdot r_\varphi$ for some $t_1 \in \overline{\mathbb{Q}}_\ell^\times$; since $t_1 \mapsto (\mathrm{T}(t_1), 1)$ is a homomorphism, $\sim$ is an equivalence relation. For each $\varphi_A \in \Sigma_0$ choose, by Step 3, a two-signed point $\varphi_{[\varphi_A]}$ of (R, r) and t_{φ_A} with $\varphi_A = t_{\varphi_A} \cdot (\varphi_{[\varphi_A]} \circ q)$. Every two-signed point φ of (R, r) is equivalent to some $\varphi_{[\varphi_A]}$: by Step 3 and the definition of an orbit cover, $\varphi \circ q = t_2 \cdot \varphi_A = (t_2 t_{\varphi_A}) \cdot (\varphi_{[\varphi_A]} \circ q)$ for some $\varphi_A \in \Sigma_0$ and t_2, and (Dm4) gives $r_\varphi = \mathrm{T}(t_2 t_{\varphi_A}) \cdot r_{\varphi_{[\varphi_A]}}$. Let P be a set containing exactly one member of each equivalence class that meets $\{\varphi_{[\varphi_A]} : \varphi_A \in \Sigma_0\}$; P is finite, its members are pairwise inequivalent, and every two-signed point is equivalent to a member of P. Then $\Sigma = \{\varphi \circ q : \varphi \in P\}$ is an orbit cover: its members are two-signed points on u (Step 3); a two-signed point φ'_A on u is $t_2 \cdot (\varphi'' \circ q)$ with φ'' two-signed (Step 3), $\varphi'' \sim \varphi \in P$, say $r_{\varphi''} = \mathrm{T}(t_3) \cdot r_\varphi$, so $\varphi'' \circ q = t_3 \cdot (\varphi \circ q)$ by (Dm4) and $\varphi'_A = (t_2 t_3) \cdot (\varphi \circ q)$. The map $\varphi \mapsto \varphi \circ q$ is injective on P: if $\varphi \circ q = \varphi' \circ q$, then $r_{\varphi'} = \mathrm{T}(1) \cdot r_\varphi$ by (Dm4), so $\varphi \sim \varphi'$ and $\varphi = \varphi'$. With Steps 4 and 5, and $v_\mathcal{O} = e_\mathcal{O} v_\ell$,

$$\mathrm{ct}_{-1}(u_{-1}) \le 2 \sum_{\varphi \in P} v_\mathcal{O}(\varphi(q(a_\sigma))) = 2 e_\mathcal{O} \sum_{\varphi \in P} v_\ell(\mathrm{tr}\, \rho_\varphi(\sigma) - t).$$

Put $X = \{\rho_\varphi : \varphi \in P\}$. Its members are admissible of stable type for (ι, e, N) (Step 2). If $\varphi, \varphi' \in P$ and $\rho_\varphi \cong \rho_{\varphi'}$, then Lemma 46.17 gives $r_{\varphi'} = \mathrm{T}(t_1) \cdot r_\varphi$ for some t_1, so $\varphi \sim \varphi'$ and $\varphi = \varphi'$. Hence the members of X are pairwise non-isomorphic, $\varphi \mapsto \rho_\varphi$ is a bijection $P \to X$, and with Step 6

$$\nu_\mathcal{O}(\chi, e) = \frac{\mathrm{length}_\mathcal{O} \mathrm{Sha}^1_{\ell N}(\chi, e)}{e_\mathcal{O}} \le \frac{\mathrm{ct}_{-1}(u_{-1})}{e_\mathcal{O}} \le 2 \sum_{\rho \in X} v_\ell(\mathrm{tr}\, \rho(\sigma) - t).$$

□

Definition 46.3. Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$ with residue field $\mathbb{F}_\mathcal{O}$. In this definition and in the results citing it the datum and $\mathcal{O}$ are fixed and not written in the notation.

Weight pieces. For $j \in \mathbb{Z}$ let $\mathfrak{g}_j \subseteq \mathrm{M}_3(\mathbb{F}_\mathcal{O})$ be the subspace of matrices X with $X_{ab} = 0$ whenever $\mathrm{wt}(a, b) \ne j$. The group $G_\mathbb{Q}$ acts on $\mathrm{M}_3(\mathbb{F}_\mathcal{O})$ through $\bar{r}_0$: if $\bar{r}_0(g) = (P, \varsigma)$, then g acts by $X \mapsto PXP^{-1}$ if $\varsigma = 1$ and by $X \mapsto -PJ\,{}^t X\, JP^{-1}$ if $\varsigma = -1$. This action preserves each $\mathfrak{g}_j$, and $\mathfrak{g}_j$ with it is the **weight** j **piece** of the residual adjoint representation. Let S be the set of places of $\mathbb{Q}$ consisting of ∞ and the primes dividing ℓN, and $G_S = G_{\mathbb{Q},S}$; the action on $\mathfrak{g}_j$ factors through G_S. For the primes $p \in S$ the decomposition groups are the $D_p = D_{\bar{p}}$ at the fixed places.

Local conditions. For a prime $p \in S$ and $j \in \mathbb{Z}$ the **local condition** $L_{p,j} \subseteq Z^1(D_p, \mathfrak{g}_j)$ is

- for $p \ne \ell$: the cocycles vanishing on $N_{\bar{p}}$;
- for $p = \ell$: $B^1(D_\ell, \mathfrak{g}_j)$ if $j < 0$; the cocycles vanishing on I_ℓ if $j = 0$; all of $Z^1(D_\ell, \mathfrak{g}_j)$ if $j > 0$.

Selmer and obstruction groups. $H^1_\mathcal{L}(\mathfrak{g}_j)$ and $H^2_\mathcal{L}(\mathfrak{g}_j)$ are the Selmer group and the obstruction group (Definition 43.1) of $M = \mathfrak{g}_j$, for $\mathbb{F} = \mathbb{F}_\mathcal{O}$, the set S, the decomposition groups D_p and the family of local conditions $(L_{p,j})_p$.

Proposition 46.4. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and let $g_\star$ be a marking. Then:*

(Dm1) *there is a dyadic Eisenstein deformation ring (R, r) for $g_\star$, and r has exactly one base point φ_{Eis};*

(Dm2) *there are a graded $\mathcal{O}$-algebra with a fixed point (A, φ_0), a system of equations u for it and an $\mathcal{O}$-algebra homomorphism $q\colon A \to R$ with $\ker q = I(u)$ and $\varphi_{\mathrm{Eis}} \circ q = \varphi_0$, such that (Dm3)–(Dm7) hold;*

(Dm3) *for all $g \in G_{\mathbb{Q}}$ and $1 \le a, b \le 3$, $\underline{r}(g)_{ab} \in q\big(A_{\mathrm{wt}(a,b)}\big)$;*

(Dm4) *for points φ, φ' of (R, r) and $t \in \overline{\mathbb{Q}}_\ell^\times$: $\varphi \circ q$ is negative-vanishing if and only if φ is upper; $\varphi \circ q$ is positive-vanishing if and only if φ is lower; and $r_{\varphi'} = \mathrm{T}(t) \cdot r_\varphi$ if and only if $\varphi' \circ q = t \cdot (\varphi \circ q)$;*

(Dm5) *if the image of A_0 in $A/I(u)$ is a finitely generated $\mathcal{O}$-module, then every point of A on u is a torus translate of $\varphi \circ q$ for some point φ of (R, r);*

(Dm6) *if R^0 is a finitely generated $\mathcal{O}$-module and every two-signed point of (R, r) is rigid, then (A, φ_0, u) satisfies (Gc1)–(Gc4);*

(Dm7) *for every $b \in \mathcal{O} \setminus \{0\}$, writing $\mathrm{Sha}[b]$ for the submodule of $\mathrm{Sha}^1_{\ell N}(\chi, e)$ killed by b, there is an injective additive map $D\colon \mathrm{Sha}[b] \to \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}/b\mathcal{O})$ such that $D(\mu)(u_{-1,i}) = 0$ for all μ and i, and every $D(\mu)$ is the reduction modulo b of an $\mathcal{O}$-linear map $D'\colon A \to \mathcal{O}$ with $D'(fg) = \varphi_0(f)D'(g) + \varphi_0(g)D'(f)$ for all $f, g \in A$ and $D'(A_j) = 0$ for all $j \ne -1$.*

Proof. Step 1: the presentation. By Proposition 46.5 there are (R, r) with (Dq1), which is (Dm1), and $(A, \varphi_0, u, q, \mathrm{td})$ with (Dq2)–(Dq8); let ne be the counts of u.

Step 2: padding the equations. For $-2 \le j \le 2$ put $\mathrm{ne}'(j) = \mathrm{td}(j) - d(j)$, that is

$$\mathrm{ne}'(-1) = \mathrm{td}(-1) + 1, \quad \mathrm{ne}'(-2) = \mathrm{td}(-2), \quad \mathrm{ne}'(0) = \mathrm{td}(0),$$
$$\mathrm{ne}'(1) = \mathrm{td}(1) - 1, \quad \mathrm{ne}'(2) = \mathrm{td}(2) - 1.$$

By (Dq6) and Lemma 46.16, $\mathrm{ne}(j) \le \mathrm{td}(j) + \dim H^2_{\mathcal{L}}(\mathfrak{g}_j) - \dim H^1_{\mathcal{L}}(\mathfrak{g}_j) = \mathrm{ne}'(j)$; in particular $\mathrm{ne}'(j) \ge 0$. Let u' be the family with $u'_{j,i} = u_{j,i}$ for $i \le \mathrm{ne}(j)$ and $u'_{j,i} = 0 \in A_j$ for $\mathrm{ne}(j) < i \le \mathrm{ne}'(j)$. It is a system of equations for (A, φ_0) with counts ne' ($\varphi_0(0) = 0$), and since only zeros were added, $I(u') = I(u)$, the points on u' are those on u, and $\mathrm{Cot}(u'_j) = \mathrm{Cot}(u_j)$ for every j.

Step 3: (Dm2)–(Dm5) and (Dm7) for (A, φ_0, u', q). $\ker q = I(u) = I(u')$ and $\varphi_{\mathrm{Eis}} \circ q = \varphi_0$ (Dq2); (Dm3) and (Dm4) are (Dq3) and (Dq4), which do not involve the equations; (Dm5) is (Dq5), since $A/I(u') = A/I(u)$ and the points on u' and on u coincide; (Dm7) is (Dq8), since $D(\mu)(0) = 0$ for the added equations of degree -1.

Step 4: (Dm6). Assume that R^0 is a finitely generated $\mathcal{O}$-module and that every two-signed point of (R, r) is rigid. (Gc1): by (Dq6), (A, φ_0) is smooth with tangent dimensions td, where $\mathrm{td}(j) = 0$ for $|j| > 2$ (Dq2); the counts of u' satisfy the five equalities of (Gc1) by the definition of ne′; and u'_{-1}, u'_{-2}, u'_0 span rationally in degrees -1, -2, 0 because u_{-1}, u_{-2}, u_0 do (Dq7) and $\mathrm{Cot}(u'_j) = \mathrm{Cot}(u_j)$. (Gc2)–(Gc4) hold for u by (Dq7), and they depend on u only through $I(u)$ and the points on u (the image of A_0 in $A/I(u)$; points on u; derivations vanishing on $I(u)$), so they hold for u'. Hence (A, φ_0, u') satisfies (Gc1)–(Gc4). □

Proposition 46.5. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and let $g_\star$ be a marking. Then:*

(Dq1) *there is a dyadic Eisenstein deformation ring (R, r) for $g_\star$, and r has exactly one base point φ_{Eis};*

(Dq2) *there are a graded $\mathcal{O}$-algebra with a fixed point (A, φ_0), a system of equations u for it with counts* ne*, an $\mathcal{O}$-algebra homomorphism $q\colon A \to R$ with $\ker q = I(u)$ and $\varphi_{\mathrm{Eis}} \circ q = \varphi_0$, and a function* $\mathrm{td}\colon \mathbb{Z} \to \mathbb{N}$ *with $\mathrm{td}(j) = 0$ for $|j| > 2$, such that (Dq3)–(Dq8) hold;*

(Dq3) *for all $g \in G_{\mathbb{Q}}$ and $1 \le a, b \le 3$, $\underline{r}(g)_{ab} \in q(A_{\mathrm{wt}(a,b)})$;*

(Dq4) *for points φ, φ' of (R, r) and $t \in \overline{\mathbb{Q}}_\ell^\times$: $\varphi \circ q$ is negative-vanishing if and only if φ is upper; $\varphi \circ q$ is positive-vanishing if and only if φ is lower; and $r_{\varphi'} = \mathrm{T}(t) \cdot r_\varphi$ if and only if $\varphi' \circ q = t \cdot (\varphi \circ q)$;*

(Dq5) *if the image of A_0 in $A/I(u)$ is a finitely generated $\mathcal{O}$-module, then every point of A on u is a torus translate of $\varphi \circ q$ for some point φ of (R, r);*

(Dq6) *(A, φ_0) is smooth with tangent dimensions* td*, and for $-2 \le j \le 2$*

$$\mathrm{ne}(j) + \dim H^1_{\mathcal{L}}(\mathfrak{g}_j) \le \mathrm{td}(j) + \dim H^2_{\mathcal{L}}(\mathfrak{g}_j);$$

(Dq7) *if R^0 is a finitely generated $\mathcal{O}$-module and every two-signed point of (R, r) is rigid, then u_{-1}, u_{-2}, u_0 span rationally in degrees -1, -2, 0 respectively, and (A, φ_0, u) satisfies (Gc2)–(Gc4);*

(Dq8) *for every $b \in \mathcal{O} \setminus \{0\}$, writing* $\mathrm{Sha}[b]$ *for the submodule of* $\mathrm{Sha}^1_{\ell^N}(\chi, e)$ *killed by b, there is an injective additive map $D\colon \mathrm{Sha}[b] \to \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}/b\mathcal{O})$ such that $D(\mu)(u_{-1,i}) = 0$ for all μ and i, and every $D(\mu)$ is the reduction modulo b of an $\mathcal{O}$-linear map $D'\colon A \to \mathcal{O}$ with $D'(fg) = \varphi_0(f)D'(g) + \varphi_0(g)D'(f)$ for all $f, g \in A$ and $D'(A_j) = 0$ for all $j \ne -1$.*

Proof. By Proposition 46.6 there are (R, r) with exactly one base point φ_{Eis}, which is (Dq1), a weighted presentation $(\sigma, \mathrm{wt}, \pi)$ of R with $\varphi_{\mathrm{Eis}}(\pi(X_s)) = 0$ for all s, and a homogeneous generating system u of $\ker \pi$ with counts ne, satisfying (Dw1)–(Dw6) for the graded model (A, φ_0, q) with $\beta = \varphi_{\mathrm{Eis}}$. Put $\mathrm{td}(j) = \#\mathrm{wt}^{-1}(j)$.

(Dq2). $R \in \mathrm{CNL}_{\mathcal{O}}$ and $\beta = \varphi_{\mathrm{Eis}}\colon R \to \mathcal{O}$ is an $\mathcal{O}$-algebra homomorphism killing every $\pi(X_s)$, so Lemma 43.28 applies: (A, φ_0) is a graded $\mathcal{O}$-algebra with a fixed point (item 1), u is a system of equations for it with counts ne and $\ker q = I(u)$ (item 2), and $\mathrm{td}(j) = 0$ for $|j| > 2$ (item 3). By definition of the graded model, $\varphi_0 = \varphi_{\mathrm{Eis}} \circ q$, and q is an $\mathcal{O}$-algebra homomorphism $A \to R$.

(Dq3). By (Dw2), $\underline{r}(g)_{ab} = \pi(f)$ for some $f \in \mathcal{O}[[X_\sigma]]_{\mathrm{wt}(a,b)} = A_{\mathrm{wt}(a,b)}$, and $\pi(f) = q(f)$.

(Dq4), (Dq5), (Dq7), (Dq8) are (Dw3), (Dw4), (Dw5), (Dw6).

(Dq6). By Lemma 43.28, item 3, (A, φ_0) is smooth with tangent dimensions td. For $-2 \le j \le 2$, (Dw1) gives $\mathrm{ne}(j) \le \dim H^2_{\mathcal{L}}(\mathfrak{g}_j)$ and $\dim H^1_{\mathcal{L}}(\mathfrak{g}_j) \le \mathrm{td}(j)$; adding,

$$\mathrm{ne}(j) + \dim H^1_{\mathcal{L}}(\mathfrak{g}_j) \le \mathrm{td}(j) + \dim H^2_{\mathcal{L}}(\mathfrak{g}_j).$$

□

Proposition 46.6. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and let $g_\star$ be a marking. Then there are a dyadic Eisenstein deformation ring (R, r) for $g_\star$ such that r has exactly one base point φ_{Eis}, a weighted presentation $(\sigma, \mathrm{wt}, \pi)$ of R with $\varphi_{\mathrm{Eis}}(\pi(X_s)) = 0$ for every $s \in \sigma$, and a homogeneous generating system u of* $\ker \pi$ *with counts* ne*, such that, with (A, φ_0, q) the graded model of $(\sigma, \mathrm{wt}, \pi)$ and $\beta = \varphi_{\mathrm{Eis}}$:*

(Dw1) *for $-2 \le j \le 2$, $\mathrm{ne}(j) \le \dim H^2_{\mathcal{L}}(\mathfrak{g}_j)$ and $\dim H^1_{\mathcal{L}}(\mathfrak{g}_j) \le \#\mathrm{wt}^{-1}(j)$;*

(Dw2) *for all* $g \in G_{\mathbb{Q}}$ *and* $1 \le a, b \le 3$, $\underline{r}(g)_{ab} \in \pi\big(\mathcal{O}[[X_\sigma]]_{\mathrm{wt}(a,b)}\big)$*;*

(Dw3) *for points* φ, φ' *of* (R, r) *and* $t \in \overline{\mathbb{Q}}_\ell^\times$*:* $\varphi \circ q$ *is negative-vanishing if and only if* φ *is upper;* $\varphi \circ q$ *is positive-vanishing if and only if* φ *is lower; and* $r_{\varphi'} = \mathrm{T}(t) \cdot r_\varphi$ *if and only if* $\varphi' \circ q = t \cdot (\varphi \circ q)$*;*

(Dw4) *if the image of* A_0 *in* $A/I(u)$ *is a finitely generated* $\mathcal{O}$*-module, then every point of* A *on* u *is a torus translate of* $\varphi \circ q$ *for some point* φ *of* (R, r)*;*

(Dw5) *if* R^0 *is a finitely generated* $\mathcal{O}$*-module and every two-signed point of* (R, r) *is rigid, then* u_{-1}*,* u_{-2}*,* u_0 *span rationally in degrees* -1*,* -2*,* 0 *respectively, and* (A, φ_0, u) *satisfies (Gc2)–(Gc4);*

(Dw6) *for every* $b \in \mathcal{O} \setminus \{0\}$*, writing* $\mathrm{Sha}[b]$ *for the submodule of* $\mathrm{Sha}^1_{\ell N}(\chi, e)$ *killed by* b*, there is an injective additive map* $D\colon \mathrm{Sha}[b] \to \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}/b\mathcal{O})$ *such that* $D(\mu)(u_{-1,i}) = 0$ *for all* μ *and* i*, and every* $D(\mu)$ *is the reduction modulo* b *of an* $\mathcal{O}$*-linear map* $D'\colon A \to \mathcal{O}$ *with* $D'(fg) = \varphi_0(f)D'(g) + \varphi_0(g)D'(f)$ *for all* $f, g \in A$ *and* $D'(A_j) = 0$ *for all* $j \ne -1$*.*

Proof. Write $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, $E_{\mathcal{O}} = \mathcal{O}[\ell^{-1}]$, $P = \mathcal{O}[[X_s : s \in \sigma]]$, $P_j = \mathcal{O}[[X_\sigma]]_j$, and for $f \in P$ let $a_s(f)$ be the coefficient of X_s in f; $t_\star = \Psi(g_\star)$, so that $t_\star^2 - 1 \in \mathcal{O}^\times$.

Construction. By (Dr1) let (R, r) be a dyadic Eisenstein deformation ring for $g_\star$ with its unique base point $\beta = \varphi_{\mathrm{Eis}}$. By (Dr3) and Lemma 43.8 there is a weighted presentation $(\sigma, \mathrm{wt}, \pi)$ of R with $\pi(X_s) = x_s = x_{g_s, a_s, b_s}$, $\mathrm{wt}(s) = \mathrm{wt}(a_s, b_s)$, satisfying (Tp1)–(Tp3); $\beta(x_s) = 0$ as $\beta \circ r = r_0$. By (Oc2) let u be a homogeneous generating system of $\ker \pi$ with counts ne, $\mathrm{ne}(j) \le \dim H^2_{\mathcal{L}}(\mathfrak{g}_j)$. Let (A, φ_0, q) be the graded model; by Lemma 43.28, $\varphi_0(f)$ is the constant term of f, $\ker q = I(u)$, and A_0 is a complete local ring whose maximal ideal $\mathfrak{m}_{A_0}$ consists of the elements with constant term in $\mathfrak{m}_{\mathcal{O}}$.

(Dw1), (Dw2). (Dw1) is (Oc1) and (Oc2). By (Dr3), $\underline{r}(g)_{ab}$ has torus weight $\mathrm{wt}(a, b)$, so lies in $\pi(P_{\mathrm{wt}(a,b)})$ by (Tp3): this is (Dw2).

Three facts on A. (A1) By Dickson's lemma the set of minimal elements of $\{n \in \mathbb{N}^\sigma : \mathrm{wt}(n) = j\}$ is finite, and every $f \in A_j$ is $\sum_m X^m h_m$ over these minimal m, with $h_m \in A_0$ (assign each term X^n to a minimal $m \le n$; $n - m$ has weight 0). Likewise A_0 is the image of $\theta\colon \mathcal{O}[[Y_1, \dots, Y_k]] \to P$, $Y_i \mapsto X^{m_i}$, the m_i being the minimal non-zero elements of weight 0. (A2) Let $z \in \overline{\mathbb{Q}}_\ell^\sigma$ with $|z_s|_\ell < 1$ and let $\mathrm{ev}_z\colon P \to \overline{\mathbb{Q}}_\ell$ be evaluation (convergent: the z_s lie in a finite extension). If $\psi\colon A \to \overline{\mathbb{Q}}_\ell$ is an $\mathcal{O}$-algebra homomorphism with $\psi(X_s) = z_s$ and $|\psi(y)|_\ell \le 1$ for $y \in A_0$, then $\psi = \mathrm{ev}_z|_A$: for $g \in \mathcal{O}[[Y]]$ and $n \ge 1$, $g = g_{<n} + \sum_{|e|=n} Y^e g_e$ with $g_{<n}$ a polynomial, so $|\psi(\theta g) - \mathrm{ev}_z(\theta g_{<n})| \le \max_i |z^{m_i}|^n \to 0$, giving $\psi = \mathrm{ev}_z$ on A_0, and then on every A_j by (A1). (A3) A ψ-derivation D of A, for $\psi = \mathrm{ev}_z|_A$ as in (A2), is $D(f) = \sum_s (\partial_s f)(z)\, D(X_s)$. Indeed the right side $\tilde{D}$ is a ev_z-derivation of P. On A_0: let $\mathcal{O}_E$ be the integers of a finite extension containing the z_s, and extend $D, \tilde{D}$ $\mathcal{O}_E$-linearly to $\mathcal{O}_E \otimes_{\mathcal{O}} A_0$, the image of $\mathcal{O}_E[[Y]]$; for $g \in \mathcal{O}_E[[Y]]$ and $y^* = (z^{m_i})_i$, $g - g(y^*) = \sum_i (Y_i - y_i^*) g_i$ with $g_i \in \mathcal{O}_E[[Y]]$ (expand $g(Y' + y^*)$), so both derivations send $\theta(g)$ to $\sum_i \mathrm{ev}_z(\theta g_i)\, D(X^{m_i})$, and they agree on X^{m_i} by the Leibniz rule. Then they agree on every A_j by (A1).

(Dw3). Let φ be a point of (R, r). As $\mathfrak{m}_R$ is finitely generated and $|\varphi| < 1$ on it, φ is continuous and $\varphi \circ \pi = \mathrm{ev}_z$ with $z_s = \varphi(x_s)$. If $\varphi \circ q$ is negative-vanishing, φ kills the entries of negative weight, which lie in $\pi(P_j)$, $j < 0$, by (Dw2): φ is upper. If φ is upper, $z_s = 0$ whenever $\mathrm{wt}(s) < 0$ ($\underline{r_0}$ is diagonal), and every monomial of negative weight contains such a variable, so $\mathrm{ev}_z(P_j) = 0$ for $j < 0$. The same for positive weights. If $\varphi' \circ q = t \cdot (\varphi \circ q)$, then by (Dw2) $\varphi'(\underline{r}(g)_{ab}) = t^{\mathrm{wt}(a,b)} \varphi(\underline{r}(g)_{ab})$, i.e. $\underline{r_{\varphi'}}(g) = \mathrm{T}(t)\underline{r_\varphi}(g)\mathrm{T}(t)^{-1}$ for g of sign 1; both are $(1, -1)$

at c and $(\mathrm{T}(t),1)(1,-1)(\mathrm{T}(t),1)^{-1}=(1,-1)$, so $r_{\varphi'}=\mathrm{T}(t)\cdot r_\varphi$ on $G_K\sqcup G_Kc$. Conversely, if $r_{\varphi'}=\mathrm{T}(t)\cdot r_\varphi$ then $\varphi'(x_s)=t^{\mathrm{wt}(s)}\varphi(x_s)$, and evaluating a series of weight j at $(t^{\mathrm{wt}(s)}z_s)$ multiplies it by t^j.

(Dw4). Let ψ be a point of A on u and let $\bar{A}_0$, the image of A_0 in $A/I(u)$, be a finitely generated $\mathcal{O}$-module. $\bar{A}_0$ is a non-zero quotient of the local ring A_0, hence a finite local $\mathcal{O}$-algebra, and ψ factors through it; so $\psi(A_0)$ consists of elements integral over $\mathcal{O}$ ($|\psi|\le 1$ on A_0), and since $\mathfrak{m}_{\bar{A}_0}^n\subseteq\varpi\bar{A}_0$ for some n (ϖ a uniformiser), $|\psi(y)|<1$ for $y\in\mathfrak{m}_{A_0}$. Put $\alpha_s=|\psi(X_s)|$. If $\mathrm{wt}(s)=0$, $X_s\in\mathfrak{m}_{A_0}$ and $\alpha_s<1$. If $\mathrm{wt}(s)=w>0$ and $\mathrm{wt}(s')=-w'<0$, the monomial $X_s^{w'}X_{s'}^{w}$ lies in $\mathfrak{m}_{A_0}$, so $\alpha_s^{1/w}<\alpha_{s'}^{-1/w'}$. Hence there is $t\in\overline{\mathbb{Q}}_\ell^\times$ ($|\overline{\mathbb{Q}}_\ell^\times|$ is dense in $\mathbb{R}_{>0}$) with $\alpha_s^{1/\mathrm{wt}(s)}<|t|$ for $\mathrm{wt}(s)>0$ and $|t|<\alpha_s^{-1/|\mathrm{wt}(s)|}$ for $\mathrm{wt}(s)<0$. The point $\psi'=t^{-1}\cdot\psi$ lies on u ($I(u)$ is generated by homogeneous elements), agrees with ψ on A_0, and has $|\psi'(X_s)|<1$ for all s. By (A2), $\psi'=\mathrm{ev}_z|_A$ with $z_s=\psi'(X_s)$. Then ev_z kills the $u_{j,i}$, hence $\ker\pi$, so $\mathrm{ev}_z=\varphi\circ\pi$ for an $\mathcal{O}$-algebra homomorphism $\varphi\colon R\to\overline{\mathbb{Q}}_\ell$ with $|\varphi(\mathfrak{m}_R)|<1$ ($\mathfrak{m}_R=\pi(\mathfrak{m}_P)$), a point of (R,r); and $\psi=t\cdot(\varphi\circ q)$.

(Dw5). Assume R^0 is a finitely generated $\mathcal{O}$-module and every two-signed point of (R,r) is rigid.

(Gc2). For a monomial of weight 0, $\pi(X^n)=\prod_k(\underline{r}(g_k)_{a_kb_k}-\underline{r_0}(g_k)_{a_kb_k})$; expanding, each term is a constant times a product of entries whose weights add up to 0, because the constants are non-zero only on the diagonal; so π maps the weight-0 polynomials into R^0. R^0, a finitely generated $\mathcal{O}$-module, is compact, hence closed in R; as P_0 is the closure of its polynomials and π is continuous, $q(A_0)=\pi(P_0)\subseteq R^0$ is a finitely generated $\mathcal{O}$-module. Since $\ker q=I(u)$, it is the image of A_0 in $A/I(u)$. In particular (Dw4) applies to every point on u.

(Gc3). Let φ be a point of (R,r) that is upper and lower, so that $\underline{r_\varphi}|_{G_K}=\mathrm{diag}(\chi_1,\chi_2,\chi_3)$ with continuous characters χ_a of G_K, valued in the integers $\mathcal{O}_E$ of a finite extension ($\varphi(R)\subseteq\mathcal{O}_E$) and $\equiv\psi_a$ modulo $\mathfrak{m}_E$. At ℓ the image under φ of a matrix as in (Dd1) conjugates $\underline{r_\varphi}$ to upper triangular form on $D_\ell\cap G_K$ with diagonal ψ_a on I_ℓ; as $\underline{r_\varphi}(x)$, for $x\in D_\ell\cap G_K$ with $\bar{\Psi}(x)^2\ne1$ (Lemma 46.9), is already diagonal, its canonical diagonaliser is 1, and (Cd4) gives $\chi_a=\psi_a$ on I_ℓ; likewise on I_p for $p\in S_\ell(\chi)$. For $\bar{v}$ above $p\notin T_\ell(\chi)$, both χ_a and ψ_a are trivial on $N_{\bar{v}}$ ((Dd3); and $\Psi=1$ on $N_{\bar{v}}$: if $p\nmid\ell N$, Ψ is unramified at p; if $p\mid N$, then $p\ne\ell$ and $\chi_p''=\mathbf{1}$, so $\Psi(I_{\bar{v}})$ consists of roots of unity of order prime to ℓ, and such a root of unity with residue 1 is 1), so $\chi_a\psi_a^{-1}$ maps $I_{\bar{v}}$ through the finite group $I_{\bar{v}}/N_{\bar{v}}$ of order prime to ℓ into $1+\mathfrak{m}_E$, hence is trivial on $I_{\bar{v}}$; for $p\nmid\ell N$ both are unramified. Since $\underline{r_\varphi}(cgc)=\theta(\underline{r_\varphi}(g))$ gives $\chi_a(cgc)=\chi_{4-a}(g)^{-1}$, and the same for ψ_a, $\chi_a\psi_a^{-1}$ is trivial on the inertia groups at all places of K (every place of $\overline{\mathbb{Q}}$ is $g\bar{p}$ with $g\in G_K\sqcup G_Kc$). As $\mathbb{Q}(i)$ has class number one, class field theory makes every such character trivial: $\chi_a=\psi_a$. So $r_\varphi=r_0$ on G_K and at c, $\varphi(x_s)=0$ for all s, and $\varphi\circ\pi=\mathrm{ev}_0=\beta\circ\pi$: $\varphi=\beta$. Now let $\varphi_1\ne\varphi_0$ be a point on u, both negative- and positive-vanishing. By (Dw4), $\varphi_1=t\cdot(\varphi\circ q)$ for a point φ of (R,r), and $\varphi\circ q$ is negative- and positive-vanishing, so φ is upper and lower (Dw3), $\varphi=\beta$, and $\varphi_1=t\cdot\varphi_0=\varphi_0$ ($\varphi_0(A_j)=0$ for $j\ne0$): a contradiction. So (Gc3) holds vacuously.

(Gc4). Let ψ be a two-signed point on u and D a ψ-derivation vanishing on $I(u)$. By (Dw4), $\psi=t\cdot(\varphi\circ q)$; then $f\mapsto t^{-j}D(f)$ ($f\in A_j$) is a $(\varphi\circ q)$-derivation vanishing on $I(u)$, so we may assume $\psi=\varphi\circ q=\mathrm{ev}_z|_A$, $z_s=\varphi(x_s)$; φ is two-signed by (Dw3), hence rigid. By (A3), $D=\tilde{D}|_A$ with $\tilde{D}(f)=\sum_s(\partial_sf)(z)D(X_s)$; $\tilde{D}$ kills the $u_{j,i}$ and so $\ker\pi$ ($\mathrm{ev}_z(u_{j,i})=0$), hence $\tilde{D}=\delta\circ\pi$ for a continuous φ-derivation $\delta\colon R\to\overline{\mathbb{Q}}_\ell$. The $\mathcal{O}$-algebra homomorphism $\varphi+\epsilon\delta\colon R\to\overline{\mathbb{Q}}_\ell[\epsilon]$ carries r to r_y, $\underline{r_y}(g)=(1+\epsilon y(g))\underline{r_\varphi}(g)$, with y a continuous cocycle in $\mathrm{ad}\,r_\varphi$; y satisfies (Ty1) because $r=1$ on $N_{\bar{v}}$, and (Ty2), (Ty3) with the images of the matrices as in (Dd1), (Dd2). By rigidity $y(g)=g\cdot X-X$, so by (46.1) $\underline{r_y}=(1-\epsilon X)\underline{r_\varphi}(1+\epsilon X)$ on elements of sign 1. As $r_y(c)=(1,-1)$,

$c \cdot X = X$, i.e. $X_{ab} = -X_{4-b,4-a}$. For $Q = \underline{r_\varphi}(g_\star)$, $\underline{r_\varphi}(cg_\star^{-1}c) = JQ^{\mathrm{t}}J$, so $\Delta = Q - JQ^{\mathrm{t}}J$ has $\Delta_{ab} = Q_{ab} - Q_{4-b,4-a}$; (Ed1) gives $\Delta_{12} = \overline{\Delta_{21}} = 0$, hence $\Delta_{23} = \Delta_{32} = 0$, and $\Delta_{13} = \Delta_{31} = \Delta_{22} = 0$: $\Delta = \mathrm{diag}(\delta_\varphi, 0, -\delta_\varphi)$ with $\delta_\varphi \equiv t_\star - t_\star^{-1}$ a unit. The marking condition for r_y says that the entries $(1,2)$, $(2,1)$ of $[\Delta, X]$ vanish, i.e. $\delta_\varphi X_{12} = -\delta_\varphi X_{21} = 0$. With $X_{ab} = -X_{4-b,4-a}$ this leaves $X = x\,\mathrm{diag}(1,0,-1)$. Then $\delta(\underline{r}(g)_{ab}) = (\underline{r_\varphi}(g)X - X\underline{r_\varphi}(g))_{ab} = -x\,\mathrm{wt}(a,b)\,\varphi(\underline{r}(g)_{ab})$, so $D(X_s) = -x\,\mathrm{wt}(s)\,\varphi(x_s)$, and by (A3) $D = -x\,\overline{E_\varphi}$, where $\overline{E_\varphi}(f) = j\,\varphi(q(f))$ for $f \in A_j$ is a ψ-derivation. Any two such D are linearly dependent.

(Rational spanning in degrees $j \in \{-2,-1,0\}$.) For $f \in A_j \cap \mathfrak{m}_0$, $f - \sum_{\mathrm{wt}(s)=j} a_s(f)X_s$ has only monomials of degree ≥ 2, hence is $\sum_s X_s h_s$ with $h_s \in A_{j-\mathrm{wt}(s)}$ without constant term (assign each term to its first variable), and lies in $\mathfrak{m}_0^2 \subseteq \mathrm{Sq}(\varphi_0)$. So Cot_j is spanned by the images of the X_s with $\mathrm{wt}(s) = j$, and it suffices that the vectors $v_i = (a_s(u_{j,i}))_{\mathrm{wt}(s)=j}$ span $E_{\mathcal{O}}^{\mathrm{wt}^{-1}(j)}$: then for each such s some $c_s \neq 0$ has $c_s X_s - \sum_i b_i u_{j,i}$ with zero linear part, so $c_s\bar{X}_s \in \mathrm{Cot}(u)$. Suppose a linear form $\lambda = (\lambda_s)$ on $E_{\mathcal{O}}^{\mathrm{wt}^{-1}(j)}$, scaled into $\mathcal{O}$, kills all v_i. Then $\tilde{D}(f) = \sum_{\mathrm{wt}(s)=j} \lambda_s a_s(f)$ is a ev_0-derivation of P killing every $u_{j',i}$ (for $j' \neq j$, $u_{j',i} \in P_{j'}$ has no linear term in variables of weight j), hence $\ker\pi$; so $\tilde{D} = \delta \circ \pi$ with δ a β-derivation $R \to \mathcal{O}$. The morphism $\beta + \epsilon\delta\colon R \to \mathcal{O}[\epsilon]$ carries r to a lift r_δ satisfying (Ed1), (Dd1)–(Dd3), with $\underline{r_\delta}(g) = (1 + \epsilon y(g))\underline{r_0}(g)$, y a continuous cocycle $G_S \to \mathfrak{g}_j(\mathcal{O})$ (by (Dw2), $\delta(\underline{r}(g)_{ab}) = \tilde{D}(f)$ for some $f \in P_{\mathrm{wt}(a,b)}$, which is 0 unless $\mathrm{wt}(a,b) = j$). It satisfies (Vn1)–(Vn3): (Dd3) gives $y = 0$ on $N_{\bar{p}}$ for $p \notin T_\ell(\chi)$. If $h = h_0 + \epsilon h_1$ is a matrix as in (Dd1) for r_δ, then h_0 is one for r_0, so by (Cd3), the canonical diagonaliser of the diagonal matrix $\underline{r_0}(x)$ being 1, h_0 is upper triangular; then $h' = hh_0^{-1} = 1 + \epsilon Z$ is still a matrix as in (Dd1) (conjugation by the upper triangular h_0 preserves upper triangular matrices and their diagonals), and $h'^{-1}\underline{r_\delta}(g)h' = (1 + \epsilon(y(g) + g \cdot Z - Z))\underline{r_0}(g)$ on sign-1 elements; so, with Z_j the weight-j part of Z, $y(g) = g \cdot (-Z_j) - (-Z_j)$ on $D_\ell \cap G_K$ if $j < 0$, and y vanishes on I_ℓ if $j = 0$ ($I_\ell \subseteq G_K$ acts trivially on $\mathfrak{g}_0$). The same argument with (Dd2) at $p \in S_\ell(\chi)$ gives $y(g) = g \cdot (-Z_j) - (-Z_j)$ on $D_p \cap G_K \supseteq I_p \supseteq N_{\bar{p}}$ for $j \neq 0$, so $y = 0$ on $N_{\bar{p}}$, and $y = 0$ on I_p for $j = 0$. By Proposition 46.14, $a\,y = dX$ with $a \neq 0$, $X \in \mathfrak{g}_j(\mathcal{O})$. As $y(c) = 0$, $c \cdot X = X$. If $j = -2$, $X = 0$ (c acts by -1 on $\mathfrak{g}_{-2}$). If $j = 0$, X is fixed by c and by G_K, so $dX = 0$. If $j = -1$, $X = b(E_{21} - E_{32})$ and $dX(g_\star) = (t_\star^{-1} - 1)X$; since $\underline{r_\delta}(cg_\star^{-1}c) = J\,{}^t\underline{r_\delta}(g_\star)J$ and $\underline{r_\delta}(g_\star) = (1 + \epsilon Y)\mathrm{T}(t_\star)$ with $Y = y(g_\star)$, the ϵ-part of the marking difference is $Y\mathrm{T}(t_\star) - \mathrm{T}(t_\star)^{-1}J\,{}^tYJ$, whose entry $(2,1)$ is $t_\star Y_{21} - Y_{32}$; so the marking condition gives $t_\star y_{21}(g_\star) - y_{32}(g_\star) = 0$, and for $a\,y = dX$ it gives $(t_\star^{-1} - 1)(t_\star + 1)b = 0$, so $b = 0$. In all cases $a\,y = 0$, so $y = 0$, $\delta(x_s) = 0$ and $\lambda_s = \tilde{D}(X_s) = 0$. Hence the v_i span, and u_j spans rationally in degree j.

(Dw6). Let $b \in \mathcal{O} \setminus \{0\}$ and $\mu \in \mathrm{Sha}[b]$. Since $\bar{\Psi}(s) \neq 1$ for an element s of D_ℓ (Lemma 46.9), $H^0(\mathbb{Q}, \mathcal{O}^\vee(\Psi^{-1})) = 0$, and the sequence $0 \to \mathcal{O}/b \to \mathcal{O}^\vee \xrightarrow{b} \mathcal{O}^\vee \to 0$ (with $\mathcal{O}/b \cong b^{-1}\mathcal{O}/\mathcal{O}$) identifies $\mathrm{Sha}[b]$ with a subgroup of $H^1(\mathbb{Q}, (\mathcal{O}/b)(\Psi^{-1}))$. Let $\tilde{z}$ be a cocycle representing μ. For $\bar{v}$ above $p \notin S$, Ψ is unramified, $\tilde{z}|_{I_{\bar{v}}}$ is a homomorphism whose image in $\mathrm{Hom}(I_{\bar{v}}, \mathcal{O}^\vee)$ is 0, so $\tilde{z}$ vanishes on $I_{\bar{v}}$ and factors through G_S. For $p \mid \ell N$ the class is trivial in $H^1(\mathbb{Q}_p, \mathcal{O}^\vee(\Psi^{-1}))$, so $\tilde{z}(g) = g \cdot y_p - y_p$ on D_p with $y_p \in \mathcal{O}^\vee$; if $p \in T_\ell(\chi)$ then $H^0(D_p, \mathcal{O}^\vee(\Psi^{-1})) = 0$ (Lemma 46.9), and as by_p is D_p-invariant, $y_p \in \mathcal{O}/b$; if $p \notin T_\ell(\chi)$, $\Psi = 1$ on $N_{\bar{p}}$, so $\tilde{z} = 0$ on $N_{\bar{p}}$, and on $N_{g\bar{p}} = gN_{\bar{p}}g^{-1}$ by the cocycle relation. Since $\Psi(g_\star)^{-1} - 1$ is a unit, there is a unique $z_\mu = \tilde{z} + dm$ ($m \in \mathcal{O}/b$) with $z_\mu(g_\star) = 0$; then $z_\mu(c) = 0$ ($2z_\mu(c) = z_\mu(c^2) = 0$). The map $\mu \mapsto z_\mu$ is additive and injective. Let $X_0 = E_{21} - E_{32}$; $G_{\mathbb{Q}}$ acts on it by Ψ^{-1} ($c \cdot X_0 = X_0$). Over $A_b = \mathcal{O}[\epsilon]/(\epsilon^2, b\epsilon) \in \mathrm{CNL}_{\mathcal{O}}$, $r_\mu(g) = (1 + \epsilon z_\mu(g)X_0, 1)\,r_0(g)$ is a lift of $\bar{r}_0$: (Ed1) holds because $z_\mu(c) = z_\mu(g_\star) = 0$ gives $r_\mu(c) = r_0(c)$, $r_\mu(g_\star) = r_0(g_\star)$, hence $r_\mu(cg_\star^{-1}c) = r_0(cg_\star^{-1}c)$; (Dd3) holds as $r_0 = 1$ and $z_\mu = 0$ on the relevant $N_{\bar{v}}$; on D_p, $p \in T_\ell(\chi)$, $z_\mu X_0 = d(m_p X_0)$, so $r_\mu|_{D_p}$ is the conjugate of $r_0|_{D_p}$ by $1 - \epsilon m_p X_0 \equiv 1$ ((46.1)), which gives (Dd1), (Dd2). So $r_\mu = f_\mu \circ r$ with $f_\mu = \beta + \epsilon\delta_\mu$ (f_μ modulo ϵ classifies r_0), $\delta_\mu\colon R \to \mathcal{O}/b$ a β-derivation. Put $D(\mu) = \delta_\mu \circ q$. Since $\delta_\mu \circ \pi$ kills $\mathcal{O}$ and $(X_s)^2$,

$D(\mu)(f) = \sum_s a_s(f)\delta_\mu(x_s)$, with $\delta_\mu(x_s)$ the entry (a_s, b_s) of $z_\mu(g_s)X_0\underline{r_0}(g_s)$; so D is additive, $D(\mu)(u_{-1,i}) = \delta_\mu(\pi(u_{-1,i})) = 0$, and $\delta_\mu(x_s) = 0$ unless $\mathrm{wt}(s) = -1$, whence $D(\mu)(A_j) = 0$ for $j \neq -1$ ($f \in A_j$ has linear terms only in variables of weight j). With lifts $d_s \in \mathcal{O}$ of $\delta_\mu(x_s)$ for $\mathrm{wt}(s) = -1$, $D'(f) = \sum_{\mathrm{wt}(s)=-1} a_s(f)d_s$ is $\mathcal{O}$-linear, satisfies $D'(fg) = \varphi_0(f)D'(g)+\varphi_0(g)D'(f)$, vanishes on A_j for $j \neq -1$ and reduces to $D(\mu)$. If $D(\mu) = 0$ then $\delta_\mu(x_s) = D(\mu)(X_s) = 0$ for all s, so $\delta_\mu = 0$, $r_\mu = r_0$, $z_\mu = 0$ and $\mu = 0$: D is injective. □

Proposition 46.7. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and let $g_\star$ be a marking.*

(Dr1) *There is a dyadic Eisenstein deformation ring (R, r) for $g_\star$, and r has exactly one base point φ_{Eis}.*

(Dr2) *For $A \in \mathrm{CNL}_\mathcal{O}$, $t \in A^\times$ and a lift r' of $\bar{r}_0$ to A satisfying (Ed1) for $g_\star$ and (Dd1)–(Dd3), the torus conjugate $g \mapsto (\mathrm{T}(t), 1)\, r'(g)\, (\mathrm{T}(t), 1)^{-1}$ is a lift of $\bar{r}_0$ to A satisfying (Ed1) for $g_\star$ and (Dd1)–(Dd3).*

(Dr3) *For (R, r) as in (Dr1), the family $x_{g,a,b} = \underline{r}(g)_{ab} - \underline{r_0}(g)_{ab}$ ($g \in G_K$, $1 \le a, b \le 3$) with $w(g, a, b) = \mathrm{wt}(a, b)$ satisfies (Tc1) and (Tc2) of Lemma 43.8, where $\rho\colon R \to R[[T]]$ is the morphism with $\rho \circ r = (\mathrm{T}(1 + T), 1)\, r\, (\mathrm{T}(1 + T), 1)^{-1}$; and for every $g \in G_\mathbb{Q}$ and all a, b the entry $\underline{r}(g)_{ab}$ has torus weight $\mathrm{wt}(a, b)$.*

Proof. Write $\mathbb{F} = \mathbb{F}_\mathcal{O}$, $G_S = G_{\mathbb{Q},S}$ and $\Gamma \subseteq G_S$ for the image of G_K ($K = \mathbb{Q}(i)$ is unramified outside 2∞ and $2 \mid N$ by (Dy2), so K lies in the fixed field of $\ker(G_\mathbb{Q} \to G_S)$, and $\Gamma = G_{K,S_K}$ for S_K the places of K above S). Recall $\Psi(c) = 1$.

Step 1: every lift satisfying (Dd3) factors through G_S. Let $\bar{v}$ lie above a prime $p \notin S$. Then $p \nmid 2\ell N$, so η_K and $\bar{\Psi}$ are unramified at $\bar{v}$ and $N_{\bar{v}} = I_{\bar{v}}$; (Dd3) makes the lift trivial on every such $I_{\bar{v}}$, hence on the normal subgroup they generate, whose fixed field is the maximal extension unramified outside S.

Step 2: lifts normalised at c. Let $\bar{\rho}\colon \Gamma \to \mathrm{GL}_3(\mathbb{F})$, $g \mapsto \mathrm{T}(\bar{\Psi}(g))$, and let $(R^\square, \rho^\square)$ be its universal lift (Theorem B.71; Γ satisfies Φ_ℓ by item 2). As $G_S = \Gamma \sqcup \Gamma c$ and $c^2 = 1$, a lift r' of $\bar{r}_0$ through G_S with $\underline{r'}(c) = 1$ is the same as a continuous lift ρ' of $\bar{\rho}$ with $\rho'(cgc) = \theta(\rho'(g))$ for $g \in \Gamma$: given r' put $\rho' = \underline{r'}|_\Gamma$, and $(1, -1)(\rho'(g), 1)(1, -1) = (\theta(\rho'(g)), 1)$ gives the relation; given ρ', $r'(g) = (\rho'(g), 1)$, $r'(gc) = (\rho'(g), -1)$ ($g \in \Gamma$) is a homomorphism of $\Gamma \rtimes \langle c\rangle$. Hence such lifts are represented by $R_c = R^\square/\mathfrak{a}_c$, $\mathfrak{a}_c$ generated by the entries of $\rho^\square(cgc) - \theta(\rho^\square(g))$, $g \in \Gamma$; let r_c be the corresponding lift to R_c.

Step 3: the other conditions are cut out by ideals. For each condition below we give an ideal $\mathfrak{a} \subseteq R_c$ such that, for every morphism $f\colon R_c \to A$ in $\mathrm{CNL}_\mathcal{O}$, $f \circ r_c$ satisfies the condition if and only if $f(\mathfrak{a}) = 0$.

- The rest of (Ed1): the entries $(1, 2)$, $(2, 1)$ of $\underline{r_c}(g_\star) - \underline{r_c}(cg_\star^{-1}c)$.
- (Dd3): the entries of $\underline{r_c}(g) - 1$ for $g \in N_{\bar{v}}$, $\bar{v}$ above a prime of $S \setminus T_\ell(\chi)$ (by Step 1 nothing more is needed; $\eta_K = 1$ on $N_{\bar{v}}$, so the sign is 1 there).
- (Dd1): by Lemma 46.9 choose $x \in D_\ell \cap G_K$ with $\bar{\Psi}(x)^2 \neq 1$; then $M = \underline{r_c}(x)$ reduces to $\mathrm{T}(\bar{\Psi}(x))$, with pairwise distinct diagonal entries, so $h_c = h(M)$ exists (Cd1). Take the entries of negative weight of $h_c^{-1}\underline{r_c}(g)h_c$ for $g \in D_\ell \cap G_K$ and the elements $(h_c^{-1}\underline{r_c}(g)h_c)_{aa} - \psi_a(g)$ for $g \in I_\ell$. If f kills them, $f(h_c)$ is a matrix as in (Dd1) for $f \circ r_c$. Conversely, if some $k \equiv 1$ is a matrix as in (Dd1), then by (Cd4) so is $h(f(M))$, with the same diagonal entries, and $h(f(M)) = f(h_c)$ by (Cd2); so f kills the generators.

- (Dd2), for each $p \in S_\ell(\chi)$: in the same way, with $x \in D_p \cap G_K$ from Lemma 46.9, all off-diagonal entries of $h_c^{-1}\underline{r_c}(g)h_c$ for $g \in D_p \cap G_K$ and the diagonal ones minus $\psi_a(g)$ for $g \in I_p$.

Let R be the quotient of R_c by the sum of these ideals and r the image of r_c. It is non-zero: $\bar{r}_0$ itself, as a lift to $\mathbb{F}$, satisfies all conditions (Step 4 below, reduced modulo $\mathfrak{m}_\mathcal{O}$), so $R_c \to \mathbb{F}$ factors through R. Hence $R \in \mathrm{CNL}_\mathcal{O}$, r satisfies (Ed1) and (Dd1)–(Dd3) (with the matrices above), and a lift r' to A satisfying them factors through G_S (Step 1), has $\underline{r'}(c) = 1$, so $r' = f \circ r_c$ for exactly one $f\colon R_c \to A$ (Step 2), and f kills the ideals (Step 3), i.e. factors uniquely through R. So (R, r) is a dyadic Eisenstein deformation ring for $g_\star$.

Step 4: the base point. r_0 is a lift to $\mathcal{O}$ satisfying the conditions: $\underline{r_0}(c) = \mathrm{T}(\Psi(c)) = 1$; $\underline{r_0}(cg_\star^{-1}c) = \theta(\mathrm{T}(\Psi(g_\star))^{-1}) = \mathrm{T}(\Psi(g_\star))^{-1}$ is diagonal, as is $\underline{r_0}(g_\star)$; on G_K, $\underline{r_0} = \mathrm{diag}(\psi_1, \psi_2, \psi_3)$, so $h = 1$ is a matrix as in (Dd1), (Dd2); and for $g \in N_{\bar{v}}$ with $\bar{v}$ above $p \notin T_\ell(\chi)$, $\Psi(g) = 1$: if $p \nmid \ell N$, Ψ is unramified at p; if $p \mid N$, then $p \neq \ell$ and $\chi_p'' = \mathbf{1}$, so $\Psi|_{I_{\bar{v}}}$ takes values in roots of unity of order prime to ℓ, and such a root of unity with residue 1 equals 1. Hence $r_0(g) = (\mathrm{T}(1), 1) = 1$. By universality there is exactly one morphism $\varphi_{\mathrm{Eis}}\colon R \to \mathcal{O}$ with $\varphi_{\mathrm{Eis}} \circ r = r_0$. Every base point is such a morphism: an $\mathcal{O}$-algebra homomorphism $\beta\colon R \to \mathcal{O}$ is local, since for $y \in \mathfrak{m}_R$ and $u \in \mathcal{O}^\times$, $y - u$ is a unit of R and cannot lie in $\ker\beta$. This is (Dr1).

Step 5: (Dr2). Let $D = (\mathrm{T}(t), 1)$ and $r'' = Dr'D^{-1}$. It reduces to $\bar{r}_0$, because $\mathrm{T}(\bar{t})$ commutes with $\mathrm{T}(\bar{\Psi}(g))$ and $\theta(\mathrm{T}(s)) = \mathrm{T}(s)$. It is continuous. $\underline{r''}(c) = \mathrm{T}(t)\,\theta(\mathrm{T}(t)^{-1}) = 1$. On elements of sign 1, $\underline{r''} = \mathrm{T}(t)\,\underline{r'}\,\mathrm{T}(t)^{-1}$, which multiplies the entry (a, b) by $t^{\mathrm{wt}(a,b)}$; so the entries $(1, 2)$, $(2, 1)$ of $\underline{r''}(g_\star) - \underline{r''}(cg_\star^{-1}c)$ vanish. If h is a matrix as in (Dd1) or (Dd2) for r', then $\mathrm{T}(t)h\mathrm{T}(t)^{-1} \equiv 1$ is one for r'', since conjugation by $\mathrm{T}(t)$ preserves triangular and diagonal matrices and their diagonals. (Dd3) is preserved by conjugation.

Step 6: (Dr3). Each $x_{g,a,b}$ lies in $\mathfrak{m}_R$, as r and r_0 have the same reduction. (Tc1): if $f(x_{g,a,b}) = f'(x_{g,a,b})$ for all $g \in G_K$ and a, b, then $f \circ r$ and $f' \circ r$ agree on G_K, and at c both are $(1, -1)$ by (Ed1); as $G_\mathbb{Q} = G_K \sqcup G_K c$, $f \circ r = f' \circ r$, so $f = f'$ by universality. (Tc2): $R[[T]] \in \mathrm{CNL}_\mathcal{O}$; by (Dr2) for $A = R[[T]]$, $t = 1 + T$ and the push-forward of r along $R \subseteq R[[T]]$, there is a morphism ρ with $\rho \circ r = (\mathrm{T}(1 + T), 1)\, r\, (\mathrm{T}(1 + T), 1)^{-1}$. For g of sign 1, $\rho(\underline{r}(g)_{ab}) = (1 + T)^{\mathrm{wt}(a,b)}\underline{r}(g)_{ab}$; for g of sign -1, $r(g) = r(gc)\, r(c)$ with $\underline{r}(c) = 1$ gives $\underline{r}(g) = \underline{r}(gc)$, so the same holds. As $\underline{r_0}(g)_{ab} \in \mathcal{O}$ vanishes for $a \neq b$ and $\mathrm{wt}(a, a) = 0$, $\rho(x_{g,a,b}) = (1 + T)^{\mathrm{wt}(a,b)}x_{g,a,b}$. □

Lemma 46.8. *Let B be a complete Noetherian local ring with maximal ideal $\mathfrak{m}_B$, and call $M \in \mathrm{GL}_3(B)$* residually regular diagonal *if its off-diagonal entries lie in $\mathfrak{m}_B$ and its diagonal entries have pairwise distinct images in $B/\mathfrak{m}_B$.*

(Cd1) *For such M there is exactly one $h = h(M) \in \mathrm{GL}_3(B)$ with $h \equiv 1 \pmod{\mathfrak{m}_B}$, $h_{aa} = 1$ for $a = 1, 2, 3$, and $h^{-1}Mh$ diagonal (the* canonical diagonaliser*).*

(Cd2) *For every local homomorphism $f\colon B \to B'$ of complete Noetherian local rings, $f(M)$ is residually regular diagonal and $h(f(M)) = f(h(M))$; for every invertible diagonal $D \in \mathrm{GL}_3(B)$, $h(DMD^{-1}) = D\,h(M)\,D^{-1}$.*

(Cd3) *If $k \in \mathrm{GL}_3(B)$, $k \equiv 1 \pmod{\mathfrak{m}_B}$, and $k^{-1}Mk$ is upper triangular (resp. diagonal), then $k = h(M)\,b$ with b upper triangular (resp. diagonal) and $b \equiv 1 \pmod{\mathfrak{m}_B}$.*

(Cd4) *Let H be a group, $\rho\colon H \to \mathrm{GL}_3(B)$ a homomorphism and $x \in H$ with $\rho(x)$ residually regular diagonal. If some $k \equiv 1 \pmod{\mathfrak{m}_B}$ makes $k^{-1}\rho(g)k$ upper triangular (resp. diagonal) for all $g \in H$, then so does $h(\rho(x))$, and $h(\rho(x))^{-1}\rho(g)h(\rho(x))$ and $k^{-1}\rho(g)k$ have the same diagonal entries for every $g \in H$.*

(Cd5) *Let $B = \bigoplus_{i\in\mathbb{Z}} B_i$ be a finite local ring which is $\mathbb{Z}$-graded with only finitely many non-zero B_i, and let M be residually regular diagonal and* graded, *i.e. $M_{ab} \in B_{\mathrm{wt}(a,b)}$ for all a, b. Then $h(M)$ is graded.*

Proof. Write $\bar{y}$ for residues modulo $\mathfrak{m}_B$ and $d_a = M_{aa}$.

(Cd1), existence. The characteristic polynomial of M reduces to $\prod_a (X - \bar{d}_a)$, whose roots are simple; by Hensel's lemma in the complete local ring B it factors as $\prod_a (X - \lambda_a)$ with $\lambda_a \in B$, $\bar{\lambda}_a = \bar{d}_a$. Fix a and let w be the a-th column of the adjugate of $M - \lambda_a$. Since $(M - \lambda_a)\operatorname{adj}(M - \lambda_a) = \det(M - \lambda_a) \cdot 1 = 0$, we have $(M - \lambda_a)w = 0$. Modulo $\mathfrak{m}_B$, $M - \lambda_a$ is $\operatorname{diag}(\bar{d}_b - \bar{d}_a)_b$, whose adjugate is $\prod_{b\neq a}(\bar{d}_b - \bar{d}_a)E_{aa}$; so w_a is a unit and $w_b \in \mathfrak{m}_B$ for $b \neq a$. Put $v_a = w_a^{-1} w$ and let h have columns v_1, v_2, v_3. Then $h \equiv 1$, $h_{aa} = 1$, h is invertible, and $Mh = h \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$.

(Cd1), uniqueness. Let h, h' both have the properties, $h^{-1}Mh = D$, $h'^{-1}Mh' = D'$ diagonal, and $g = h^{-1}h' \equiv 1$. Then $Dg = gD'$, i.e. $(D_{aa} - D'_{bb})g_{ab} = 0$; since $\bar{D}_{aa} = \bar{d}_a$ and $\bar{D}'_{bb} = \bar{d}_b$, $D_{aa} - D'_{bb}$ is a unit for $a \neq b$, so g is diagonal; and $1 = h'_{aa} = h_{aa} g_{aa} = g_{aa}$. So $h' = h$.

(Cd2). Since f is local, $f(M)$ is residually regular diagonal. The matrix $f(h(M))$ is $\equiv 1$ modulo $\mathfrak{m}_{B'}$, has diagonal entries 1, and $f(h(M))^{-1} f(M) f(h(M)) = f(h(M)^{-1} M h(M))$ is diagonal; by uniqueness it is $h(f(M))$. Likewise DMD^{-1} is residually regular diagonal, and $Dh(M)D^{-1}$ has entries $D_{aa} h(M)_{ab} D_{bb}^{-1}$, so it is $\equiv 1$ with diagonal entries 1, and conjugates DMD^{-1} to $D\,(h(M)^{-1} M h(M))\,D^{-1}$, which is diagonal; by uniqueness it is $h(DMD^{-1})$.

(Cd3). Let $U = k^{-1}Mk$, upper triangular, $U \equiv \bar{M}$. There is an upper unitriangular $n \equiv 1$ with $n^{-1}Un = \operatorname{diag}(U_{11}, U_{22}, U_{33})$: the equations $(Un)_{ab} = n_{ab} U_{bb}$ for $a < b$ read

$$n_{ab}(U_{bb} - U_{aa}) = \sum_{a<c\leq b} U_{ac} n_{cb} \qquad (n_{bb} = 1),$$

and are solved recursively in $b - a$, since $U_{bb} - U_{aa}$ is a unit; as $U_{ac} \in \mathfrak{m}_B$ for $a < c$, all $n_{ab} \in \mathfrak{m}_B$. If U is diagonal, $n = 1$. Let $\delta = \operatorname{diag}((kn)_{11}, (kn)_{22}, (kn)_{33})$, a diagonal matrix $\equiv 1$. Then $h' = kn\delta^{-1}$ is $\equiv 1$, has diagonal entries 1, and $h'^{-1}Mh' = \delta(n^{-1}Un)\delta^{-1}$ is diagonal; so $h' = h(M)$ by (Cd1), and $k = h(M)\,b$ with $b = \delta n^{-1}$ upper triangular (diagonal if U is), $b \equiv 1$.

(Cd4). Apply (Cd3) to $M = \rho(x)$: $k = h\,b$ with $h = h(\rho(x))$ and b upper triangular (resp. diagonal). Then $h^{-1}\rho(g)h = b\,(k^{-1}\rho(g)k)\,b^{-1}$ is upper triangular (resp. diagonal) for all g, and the diagonal of a product of upper triangular matrices is the product of their diagonals, which commute; so the diagonal entries of $b\,(k^{-1}\rho(g)k)\,b^{-1}$ are those of $k^{-1}\rho(g)k$.

(Cd5). Since B is finite and local, $B[[u]]$ is a complete Noetherian local ring. The map $\gamma\colon B \to B[[u]]$, $\gamma(y) = \sum_i (1+u)^i y_i$ for $y = \sum_i y_i$ with $y_i \in B_i$, is a ring homomorphism because $B_i B_k \subseteq B_{i+k}$; it is local, since $\gamma(y) \equiv y \pmod{u}$. For a graded matrix M, $\gamma(M)_{ab} = (1+u)^{\mathrm{wt}(a,b)} M_{ab}$, i.e. $\gamma(M) = \mathrm{T}(1+u)\,M\,\mathrm{T}(1+u)^{-1}$. By (Cd2), applied to γ, to the inclusion $B \to B[[u]]$, and to $D = \mathrm{T}(1+u)$ over $B[[u]]$,

$$\gamma(h(M)) = h(\gamma(M)) = h\big(\mathrm{T}(1+u)\,M\,\mathrm{T}(1+u)^{-1}\big) = \mathrm{T}(1+u)\,h(M)\,\mathrm{T}(1+u)^{-1}.$$

Hence, for $y = h(M)_{ab}$, we have $\sum_i (1+u)^i y_i = (1+u)^{\mathrm{wt}(a,b)} \sum_i y_i$. The powers $(1+u)^i$, $|i| \leq K$, are linearly independent over B: if $\sum_{|i|\leq K} (1+u)^i c_i = 0$, multiplying by $(1+u)^K$ gives a polynomial identity in $B[u]$ whose coefficient of u^{2K} is c_K, and descending induction gives $c_i = 0$ for all i. Comparing coefficients gives $y_i = 0$ for $i \neq \mathrm{wt}(a,b)$, so $h(M)$ is graded. □

Lemma 46.9. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$. For every $p \in T_\ell(\chi)$ there is $s \in D_p \cap G_K$ with $\bar{\Psi}(s)^2 \neq 1$ that fixes the ℓ-th roots of unity.*

Proof. Fix $p \in T_\ell(\chi)$, write $N = p^a N'$ with $p \nmid N'$, and let $\mathbb{F}_{\mathcal{O}}$ be the residue field of $\mathcal{O}$.

Step 1: p is odd, and $p = \ell$ or $\ell \mid p-1$. If $p = \ell$, p is odd by (Dy1). Otherwise $p \in S_\ell(\chi)$: $p \neq \ell$, $p \mid N$ and $\chi_p'' \neq \mathbf{1}$. Since $(\chi_p'')^n = ((\chi'')^n)_p$, the order of χ_p'' divides the order of χ'', a power of ℓ; it is > 1 and divides $\#(\mathbb{Z}/p^a\mathbb{Z})^\times = p^{a-1}(p-1)$; as $p \neq \ell$, $\ell \mid p-1$, so $p > \ell \geq 3$ and p is odd. In particular $2 \mid N'$, because N is even (Dy2).

Step 2: an element $\sigma \in D_p$ acting as $\zeta \mapsto \zeta^p$ on the N'-th roots of unity and trivially on the ℓ-th roots of unity. If $p \neq \ell$, let $\sigma \in D_p$ be an arithmetic Frobenius at $\bar{p}$. For $\zeta^{N'\ell} = 1$, $\sigma(\zeta) \equiv \zeta^p$ modulo the maximal ideal of $\bar{p}$, and reduction modulo $\bar{p}$ is injective on the roots of unity of order prime to p; so $\sigma(\zeta) = \zeta^p$, and $\zeta^p = \zeta$ if $\zeta^\ell = 1$ since $\ell \mid p-1$. If $p = \ell$, identify D_ℓ with $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ through the place $\bar{\ell}$. The extension $\mathbb{Q}_\ell(\mu_{N'})/\mathbb{Q}_\ell$ is unramified and $\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell$ is totally ramified, so they are linearly disjoint over $\mathbb{Q}_\ell$ and there is $\sigma \in D_\ell$ inducing the Frobenius automorphism on $\mathbb{Q}_\ell(\mu_{N'})$ (which acts on $\mu_{N'}$ by $\zeta \mapsto \zeta^\ell$) and the identity on $\mathbb{Q}_\ell(\mu_\ell)$.

Step 3: reduction to the 2-component. Put $s = \sigma^2 \in D_p$. Then $\eta_K(s) = \eta_K(\sigma)^2 = 1$, so $s \in D_p \cap G_K$; s fixes the ℓ-th roots of unity because σ does; and $\bar{\Psi}(s)^2 = \bar{\Psi}(\sigma)^4$. Since σ fixes the ℓ-th roots of unity, $\chi_{\mathrm{cyc}}(\sigma) \equiv 1 \pmod{\ell}$, so the residue of $\chi_{\mathrm{cyc}}(\sigma)$ is 1. Let $b \in (\mathbb{Z}/N\mathbb{Z})^\times$ be the class of σ, so $\chi(\sigma) = \chi(b)$ and $b \equiv p \pmod{N'}$ by Step 2. The values of χ'' are roots of unity of ℓ-power order, whose residues are 1 (in characteristic ℓ, $\bar{z}^{\ell^k} = 1$ forces $\bar{z} = 1$). Hence $\bar{\Psi}(s)^2$ is the residue of $\chi'(b)^4$. Now $\chi'(b) = \prod_r \chi_r'(b)$ over the primes $r \mid N$. The component χ_2'' has order dividing a power of ℓ and dividing $\#(\mathbb{Z}/2^{a_2}\mathbb{Z})^\times$, a power of 2 ($2^{a_2} \parallel N$), so $\chi_2'' = \mathbf{1}$ and $\chi_2' = \chi_2$; as $2^{a_2} \mid N'$ and $b \equiv p \pmod{N'}$, $\chi_2'(b) = \chi_2(p)$. For odd r, $\chi_r'(b)^{2o} = 1$ by (Dy4). So

$$\chi'(b)^4 = \chi_2(p)^4\,\omega^4, \qquad \omega = \prod_{r \mid N,\ r \text{ odd}} \chi_r'(b), \qquad \omega^{2o} = 1.$$

Step 4: $\chi_2(p)^{4o} \neq 1$. Let $c = c_2(\chi) \geq 4$ (Dy6) and let χ_2^* be the primitive character modulo 2^c inducing χ_2, so $\chi_2(p) = \chi_2^*(p)$. Classically $(\mathbb{Z}/2^c\mathbb{Z})^\times = \{\pm 1\} \times \langle 5 \rangle$, where $\langle 5 \rangle = \{u \equiv 1 \pmod 4\}$ is cyclic of order 2^{c-2} and $5^{2^{c-3}} \equiv 1 + 2^{c-1} \pmod{2^c}$. The kernel of reduction to $(\mathbb{Z}/2^{c-1}\mathbb{Z})^\times$ is $\{1, 1+2^{c-1}\}$, on which χ_2^* is non-trivial by primitivity; so $\chi_2^*(5)^{2^{c-3}} \neq 1$, $\chi_2^*(5)$ has order 2^{c-2}, and χ_2^* is injective on $\langle 5 \rangle$. Since p is odd, $p^{4o} \equiv 1 \pmod 4$, so if $\chi_2(p)^{4o} = 1$ then $p^{4o} \equiv 1 \pmod{2^c}$. But for odd x and even n, $v_2(x^n - 1) = v_2(x^2-1) + v_2(n) - 1$ (lifting the exponent), so

$$2^{v_2(p^{4o}-1)} = 2^{v_2(p^2-1)} \cdot 2^{v_2(o)+1} \leq 2o(p^2-1) < 2op^2 < 2^c/4$$

by (Dy5) ($8op^2 < 2^c$, $p \in T_\ell(\chi)$). So $2^c \nmid p^{4o} - 1$, and $\chi_2(p)^{4o} \neq 1$.

Step 5: conclusion. If $\chi'(b)^4 = 1$, then $\chi_2(p)^4 = \omega^{-4}$ and $\chi_2(p)^{4o} = (\omega^{2o})^{-2} = 1$, contradicting Step 4. So $\chi'(b)^4 \neq 1$. It is a root of unity of order n dividing the order of χ', which is prime to ℓ; the polynomial $X^n - 1$ is separable over $\mathbb{F}_{\mathcal{O}}$, so distinct n-th roots of unity in $\mathcal{O}$ have distinct residues, and the residue of $\chi'(b)^4$ is not 1. By Step 3, $\bar{\Psi}(s)^2 \neq 1$. □

Definition 46.10. Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and $p \in S$ a prime.

Local conditions. Let $A \in \mathrm{CNL}_{\mathcal{O}}$. A continuous homomorphism $\rho\colon D_p \to \mathcal{G}_3(A)$ reducing to $\bar{r}_0|_{D_p}$ **satisfies the dyadic local condition at** p if

(Lp1) for $p = \ell$: there is $h \in \mathrm{GL}_3(A)$, $h \equiv 1 \pmod{\mathfrak{m}_A}$, such that $h^{-1}\underline{\rho}(g)h$ is upper triangular for every $g \in D_\ell \cap G_K$, with diagonal entries $\psi_1(g), \psi_2(g), \psi_3(g)$ when moreover $g \in I_\ell$;

(Lp2) for $p \in S_\ell(\chi)$: there is $h \in \mathrm{GL}_3(A)$, $h \equiv 1 \pmod{\mathfrak{m}_A}$, such that $h^{-1}\underline{\rho}(g)h$ is diagonal for every $g \in D_p \cap G_K$, with diagonal entries $\psi_1(g), \psi_2(g), \psi_3(g)$ when moreover $g \in I_p$;

(Lp3) for $p \notin T_\ell(\chi)$: $\rho(g) = 1$ for every $g \in N_{\bar{p}}$.

Graded small extensions. A **graded finite $\mathcal{O}$-algebra** is an object B of $\mathrm{CNL}_\mathcal{O}$ which is a finite ring, together with a decomposition $B = \bigoplus_{i\in\mathbb{Z}} B_i$ into $\mathcal{O}$-submodules, only finitely many non-zero, with $B_iB_k \subseteq B_{i+k}$ and $\mathcal{O}\cdot 1 \subseteq B_0$. A matrix X over B is **graded** if $X_{ab} \in B_{\mathrm{wt}(a,b)}$ for all a, b; a homomorphism into $\mathcal{G}_3(B)$ is **graded** if the GL-parts of all its values are graded. For $j \in \mathbb{Z}$, a **graded small extension of weight** j is a surjective homomorphism $\mathrm{pr}\colon B \to \bar{B}$ of graded finite $\mathcal{O}$-algebras with $\mathrm{pr}(B_i) = \bar{B}_i$ for all i, whose kernel I satisfies $\mathfrak{m}_B I = 0$ and $I \subseteq B_j$.

Proposition 46.11. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, $g_\star$ a marking, (R, r) a dyadic Eisenstein deformation ring for $g_\star$, and $(\sigma, \mathrm{wt}, \pi)$ a weighted presentation of R given by Lemma 43.8 for the family $(x_{g,a,b})$ with the weights $\mathrm{wt}(a,b)$. Then:*

(Oc1) $\dim H^1_\mathcal{L}(\mathfrak{g}_j) \le \#\mathrm{wt}^{-1}(j)$ *for every $j \in \mathbb{Z}$;*

(Oc2) $\ker\pi$ *has a homogeneous generating system u with counts* ne *such that* $\mathrm{ne}(j) \le \dim H^2_\mathcal{L}(\mathfrak{g}_j)$ *for $-2 \le j \le 2$.*

Proof. Write $\mathbb{F} = \mathbb{F}_\mathcal{O}$, $P = \mathcal{O}[[X_s : s \in \sigma]]$, $\mathfrak{m} = \mathfrak{m}_P$, $\mathcal{J} = \ker\pi$, $P_j = \mathcal{O}[[X_\sigma]]_j$, $t^* = \mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_\mathcal{O}R)$, $t = \bar{\Psi}(g_\star)$ and $\delta = t - t^{-1}$; since $g_\star$ is a marking, $t^2 \ne 1$, so $t \pm 1$ and δ are non-zero. For $X \in \mathrm{M}_3$, $c\cdot X = -JX^{\mathrm{t}}J$, whose entry (a,b) is $-X_{4-b,4-a}$.

Step 0: normalisation of weight-j perturbations. Let $A \in \mathrm{CNL}_\mathcal{O}$, $I \subseteq A$ an ideal with $\mathfrak{m}_A I = 0$, and r_1 a lift of $\bar{r}_0$ to A such that $r_1 \bmod I$ satisfies (Ed1) for $g_\star$ and $\underline{r_1}(c)$, $\underline{r_1}(g_\star) - \underline{r_1}(cg_\star^{-1}c)$ differ from their values modulo I only in positions of weight j. Then there is $X \in \mathrm{M}_3(I)$ supported in weight j such that $(1+X,1)\,r_1\,(1+X,1)^{-1}$ satisfies (Ed1). Indeed, $\underline{r_1}(c) = 1+\gamma$ with $\gamma \in \mathrm{M}_3(I)$; $r_1(c)^2 = 1$ gives $(1+\gamma)\theta(1+\gamma) = 1+\gamma+c\cdot\gamma = 1$, so $c\cdot\gamma = -\gamma$, and by (46.1) conjugation by $1-\gamma/2$ makes $\underline{r_1}(c) = 1$. After that, conjugation by $1+X$ with $c\cdot X = X$ keeps $\underline{r_1}(c) = 1$ and replaces $\Delta = \underline{r_1}(g_\star) - \underline{r_1}(cg_\star^{-1}c)$ (two elements of sign 1) by $\Delta + [X, \bar{\Delta}]$, where $\bar{\Delta} = \mathrm{T}(t) - \mathrm{T}(t^{-1}) = \mathrm{diag}(\delta, 0, -\delta)$ is its residue. The entries $(1,2)$, $(2,1)$ of Δ lie in I and vanish unless $j = 1$, resp. $j = -1$. For $j = 1$ take $X = a(E_{12} - E_{23})$ (fixed by c): $[X, \bar{\Delta}]_{12} = -a\delta$, so a suitable $a \in I$ kills the entry $(1,2)$; for $j = -1$ take $X = a(E_{21} - E_{32})$, with $[X, \bar{\Delta}]_{21} = a\delta$.

Step 1: (Oc1). Fix j. Let $y \in Z^1(G_S, \mathfrak{g}_j)$ with $\mathrm{res}_p y \in L_{p,j}$ for every prime $p \in S$. Every $L_{p,j}$ contains $B^1(D_p, \mathfrak{g}_j)$ (for $p \ne \ell$ coboundaries vanish on $N_{\bar{p}}$, which acts trivially; at ℓ: $L_{\ell,j} = B^1$ for $j < 0$, $I_\ell \subseteq G_K$ acts trivially on $\mathfrak{g}_0$, $L_{\ell,j} = Z^1$ for $j > 0$). The lift $r_y(g) = (1 + \epsilon y(g), 1)\,\bar{r}_0(g)$ of $\bar{r}_0$ to $\mathbb{F}[\epsilon]$ is a homomorphism (cocycle relation) and satisfies the dyadic local condition at every $p \in S$ by (Ll2) (with $\rho = \bar{r}_0|_{D_p}$, which satisfies it with $h = 1$, and $\lambda = \epsilon y$); hence it satisfies (Dd1), (Dd2), and (Dd3): at the places above $p \in S \setminus T_\ell(\chi)$ by conjugation from $\bar{p}$ ($N_{g\bar{p}} = gN_{\bar{p}}g^{-1}$), at places above $p \notin S$ because y factors through G_S and $N_{\bar{v}} = I_{\bar{v}}$. Replacing y by $y + dX$ ($X \in \mathfrak{g}_j$) corresponds to conjugating r_y by $1 - \epsilon X$ ((46.1)); by Step 0 ($A = \mathbb{F}[\epsilon]$, $I = \epsilon\mathbb{F}$) every class in $H^1_\mathcal{L}(\mathfrak{g}_j)$ has a representative y with r_y satisfying (Ed1). This representative is unique: two of them differ by dX with $c\cdot X = X$ (from $y(c) = 0$), and $\mathfrak{g}_j^{c=1}$ is 0 for $j = \pm 2$, consists of $G_\mathbb{Q}$-invariants for $j = 0$ (G_K acts trivially on $\mathfrak{g}_0$), and for $j = \pm 1$ is the line of Step 0, on which the marking condition forces $a = 0$ ($\delta \ne 0$). So $y \mapsto r_y$ gives an injective map from $H^1_\mathcal{L}(\mathfrak{g}_j)$ to lifts satisfying (Ed1), (Dd1)–(Dd3), linear in y, and by universality to morphisms $f_y\colon R \to \mathbb{F}[\epsilon]$. Write $f_y(z) = \bar{z} + \epsilon\lambda_y(z)$; λ_y vanishes on $\mathfrak{m}_R^2 + \mathfrak{m}_\mathcal{O}R$ and on $\mathcal{O}$, and is a linear form on t^* depending linearly and injectively on the class of y (f_y determines r_y). Moreover $\lambda_y(x_{g,a,b})$ is the entry (a,b) of $y(g)\mathrm{T}(\bar{\Psi}(g))$, which vanishes unless $\mathrm{wt}(a,b) = j$. By

(Tp1), $(\bar{x}_s)_{s\in\sigma}$ is a basis of t^*, and the linear forms vanishing on the $\bar{x}_s$ with $\mathrm{wt}(s)\neq j$ form a space of dimension $\#\mathrm{wt}^{-1}(j)$. This proves (Oc1).

Step 2: a graded truncation. By the Artin–Rees lemma there is $N_0\geq 1$ with $\mathcal{J}\cap\mathfrak{m}^{N_0}\subseteq\mathfrak{m}\mathcal{J}$. The ideal $\mathfrak{m}^{N_0}$ is generated by the monomials $\varpi^a X^n$ with $a+|n|=N_0$ (ϖ a uniformiser, of weight 0), so it is homogeneous: the weight components of its elements lie in it (for $f=\sum_k g_k\mu_k$ with monomials μ_k, $f_{(i)}=\sum_k (g_k)_{(i-\mathrm{wt}(\mu_k))}\mu_k$); the same holds for $\mathcal{J}$ (Tp2) and hence for $\mathfrak{m}\mathcal{J}$. $P/\mathfrak{m}^{N_0}$ is finite, and graded by the images of the P_i: every element of P is congruent modulo $\mathfrak{m}^{N_0}$ to the sum of its finitely many components of weight $|i|\leq 2N_0$ (monomials of larger weight have degree $>N_0$), and the sum is direct by homogeneity of $\mathfrak{m}^{N_0}$. Quotients of $P/\mathfrak{m}^{N_0}$ by homogeneous ideals are graded finite $\mathcal{O}$-algebras.

Step 3: the space of minimal relations is graded. Let $V=\mathcal{J}/\mathfrak{m}\mathcal{J}$ and V_j the image of $\mathcal{J}\cap P_j$. For $f\in\mathcal{J}$, $f-\sum_{|i|\leq 2N_0} f_{(i)}\in\mathcal{J}\cap\mathfrak{m}^{N_0}\subseteq\mathfrak{m}\mathcal{J}$ with $f_{(i)}\in\mathcal{J}$ (Tp2), so the V_j span V; and if $\sum_i f_i\in\mathfrak{m}\mathcal{J}$ with $f_i\in\mathcal{J}\cap P_i$, each f_i is a weight component of an element of $\mathfrak{m}\mathcal{J}$, hence in $\mathfrak{m}\mathcal{J}$. So $V=\bigoplus_j V_j$. Choose $u_{j,1},\dots,u_{j,n_j}\in\mathcal{J}\cap P_j$ whose images form a basis of V_j; by Nakayama's lemma all the $u_{j,i}$ together generate $\mathcal{J}$. It remains to show $n_j=\dim V_j\leq\dim H^2_{\mathcal{L}}(\mathfrak{g}_j)$ for every j: then $V_j=0$ for $|j|>2$ ($\mathfrak{g}_j=0$), and $u=(u_{j,i})_{-2\leq j\leq 2}$ is the required system.

Step 4: the graded small extension of weight j. Fix j and let $\mathcal{W}\subseteq\mathcal{J}$ be the preimage of $\bigoplus_{i\neq j}V_i$; it is an ideal containing $\mathfrak{m}\mathcal{J}$, and homogeneous: for $f\in\mathcal{W}$ the components $f_{(i)}$ lie in $\mathcal{J}$, their images are the components of $\bar{f}$ in $V=\bigoplus V_i$ (Step 3), so $f_{(j)}\in\mathfrak{m}\mathcal{J}$. Put $B=P/(\mathcal{W}+\mathfrak{m}^{N_0})$ and $\bar{B}=P/(\mathcal{J}+\mathfrak{m}^{N_0})=R/\mathfrak{m}_R^{N_0}$. The projection $\mathrm{pr}\colon B\to\bar{B}$ is a graded small extension of weight j: its kernel I is the image of $\mathcal{J}=(\mathcal{J}\cap P_j)+\mathcal{W}$, so $I\subseteq B_j$; $\mathfrak{m}_B I=0$ as $\mathfrak{m}\mathcal{J}\subseteq\mathcal{W}$; and $I\cong\mathcal{J}/(\mathcal{W}+\mathcal{J}\cap\mathfrak{m}^{N_0})=\mathcal{J}/\mathcal{W}\cong V_j$. Let $\bar{r}$ be r reduced modulo $\mathfrak{m}_R^{N_0}$: it satisfies (Ed1), (Dd1)–(Dd3), and it is graded, since $\underline{r}(g)_{ab}$ has torus weight $\mathrm{wt}(a,b)$ ((Dr3)) and so lies in $\pi(P_{\mathrm{wt}(a,b)})$ (Tp3).

Step 5: the obstruction class. $\bar{r}$ factors through a finite quotient of G_S; choose a continuous graded set-theoretic lift $s\colon G_S\to\mathcal{G}_3(B)$ of $\bar{r}$ with $s(1)=1$ (F3). Then $s(g)s(g')s(gg')^{-1}=(1+k(g,g'),1)$ with $k(g,g')\in\mathrm{M}_3(I)$ graded, hence supported in weight j: k is a continuous 2-cocycle with values in $\mathfrak{g}_j\otimes_{\mathbb{F}} I$. For each prime $p\in S$, (Ll1) gives a continuous graded homomorphism $\rho_p\colon D_p\to\mathcal{G}_3(B)$ lifting $\bar{r}|_{D_p}$ and satisfying the dyadic local condition at p; write $s(g)=(1+y_p(g),1)\rho_p(g)$ for $g\in D_p$, with $y_p\in C^1(D_p,\mathfrak{g}_j\otimes I)$. Expanding $s(g)s(g')$ in two ways gives $dy_p=\mathrm{res}_p k$. Hence for every linear $\phi\colon I\to\mathbb{F}$, $(\phi\circ k,(\phi\circ y_p)_p)\in Z^2_{\mathcal{L}}(\mathfrak{g}_j)$, and its class $o(\phi)\in H^2_{\mathcal{L}}(\mathfrak{g}_j)$ depends linearly on ϕ.

Step 6: a vanishing class gives a lift. Suppose $\dim V_j>\dim H^2_{\mathcal{L}}(\mathfrak{g}_j)$. As $\dim I^\vee=\dim V_j$, some $\phi\neq 0$ has $o(\phi)=0$: $\phi\circ k=dw$ and $\phi\circ y_p=\mathrm{res}_p w+\lambda_p$ with $w\in C^1(G_S,\mathfrak{g}_j)$, $\lambda_p\in L_{p,j}$. Let $I'=\ker\phi$, an ideal of B as $\mathfrak{m}_B I=0$, homogeneous as $I'\subseteq B_j$, and $B'=B/I'$, so that $B'\to\bar{B}$ is a graded small extension of weight j with kernel $I/I'\cong\mathbb{F}$ via ϕ. Over B', $s'(g)=(1-w(g),1)\,s(g)$ satisfies $s'(g)s'(g')s'(gg')^{-1}=1+\phi k-dw=1$ (computation of (F2) in Lemma 46.12), so s' is a continuous graded homomorphism lifting $\bar{r}$. On D_p, $s'=(1+\phi y_p-w,1)\rho_p=(1+\lambda_p,1)\rho_p$, which satisfies the dyadic local condition at p by (Ll2). As in Step 1, s' then satisfies (Dd1)–(Dd3). By Step 0 ($A=B'$, $I=I/I'$; the relevant entries are graded and vanish modulo I/I', so they lie in $(I/I')\cap B'_{\mathrm{wt}}$, which is 0 outside weight j), a conjugate r' of s' by some $(1+X,1)$ with X supported in weight j satisfies (Ed1); conjugation by an element $\equiv 1$ keeps (Dd1)–(Dd3) and the reduction $\bar{r}$.

Step 7: contradiction with minimality. By universality $r'=f\circ r$ for a morphism $f\colon R\to B'$, and $R\to B'\to\bar{B}$ is the projection (both classify $\bar{r}$). Let $\alpha\colon P\to B'$ be the projection and $\beta=f\circ\pi$. Then $\alpha-\beta$ takes values in I/I', which $\mathfrak{m}_{B'}$ kills; so $(\alpha-\beta)(f_1f_2)=\bar{f}_1(\alpha-\beta)(f_2)+\bar{f}_2(\alpha-\beta)(f_1)$ with $\bar{f}_i$ the residues, i.e. $\alpha-\beta$ is a derivation over the residue map, vanishing on $\mathcal{O}$, on $\mathfrak{m}_{\mathcal{O}}P$ and on $(X_s)^2$. By (Tp2) every $f\in\mathcal{J}$ is $c_0+\sum_s a_s X_s+f_2$ with $c_0,a_s\in\mathfrak{m}_{\mathcal{O}}$ and

$f_2 \in (X_s)^2$, so $(\alpha-\beta)(f)=0$; and $\beta(\mathcal{J})=0$. Hence $\alpha(\mathcal{J})=0$, i.e. the image I/I' of $\mathcal{J}$ in B' is zero, contradicting $\phi\neq 0$. So $\dim V_j \le \dim H^2_{\mathcal{L}}(\mathfrak{g}_j)$. □

Lemma 46.12. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, $p\in S$ a prime and $j\in\mathbb{Z}$.*

(Ll1) *Let* $\mathrm{pr}\colon B\to\bar B$ *be a graded small extension of weight j and $\bar\rho\colon D_p\to\mathcal{G}_3(\bar B)$ a continuous graded homomorphism reducing to $\bar r_0|_{D_p}$ and satisfying the dyadic local condition at p. Then there is a continuous graded homomorphism $\rho\colon D_p\to\mathcal{G}_3(B)$ with $\mathrm{pr}\circ\rho=\bar\rho$ satisfying the dyadic local condition at p.*

(Ll2) *Let $A\in\mathrm{CNL}_{\mathcal{O}}$, $I\subseteq A$ an ideal with $\mathfrak{m}_A I=0$, $\rho\colon D_p\to\mathcal{G}_3(A)$ a continuous homomorphism satisfying the dyadic local condition at p, and $\lambda\colon D_p\to \mathrm{M}_3(I)$ a continuous map whose values have non-zero entries only in positions of weight j, such that $\phi\circ\lambda\in L_{p,j}$ for every $\mathbb{F}_{\mathcal{O}}$-linear $\phi\colon I\to\mathbb{F}_{\mathcal{O}}$. Then $g\mapsto(1+\lambda(g),1)\,\rho(g)$ is a continuous homomorphism $D_p\to\mathcal{G}_3(A)$ satisfying the dyadic local condition at p.*

Proof. Write $\mathbb{F}=\mathbb{F}_{\mathcal{O}}$, $H=D_p\cap G_K$, and for g with $\eta_K(g)=-1$ let $\sigma_g=\theta$, otherwise $\sigma_g=\mathrm{id}$. For an ideal I with $\mathfrak{m}I=0$, $\mathrm{M}_3(I)=\mathrm{M}_3(\mathbb{F})\otimes_{\mathbb{F}} I$, $(1+X)(1+Y)=1+X+Y$ for $X,Y\in\mathrm{M}_3(I)$, and conjugation by an element of $\mathcal{G}_3(A)$ reducing to $\bar r_0(g)$ acts on $\{(1+X,1)\}$ through the action of g on $\mathrm{M}_3(\mathbb{F})$ of Definition 46.3. In particular, for $X\in\mathrm{M}_3(I)$ and $\rho(g)$ reducing to $\bar r_0(g)$,

$$(1+X,1)\,\rho(g)\,(1+X,1)^{-1}=(1+X-g\cdot X,1)\,\rho(g). \tag{46.1}$$

Three general facts are used. (F1) $D_p=I_p\rtimes Z$ for a closed subgroup Z mapping isomorphically onto $D_p/I_p\cong\hat{\mathbb{Z}}$ ($\hat{\mathbb{Z}}$ is free profinite on one generator, so the projection has a continuous section). (F2) If D is a profinite group, $\tilde Q\to\bar Q$ a surjective homomorphism of finite groups whose kernel V is abelian, written additively, and $\bar\rho\colon D\to\bar Q$ continuous, then V is a D-module by conjugation, and if $H^2(D,V)=0$ then $\bar\rho$ lifts to a continuous homomorphism $D\to\tilde Q$: take a set-theoretic lift s of $\bar\rho$ through a section of $\tilde Q\to\bar Q$ with $s(1)=1$, so that $c(g,g')=s(g)s(g')s(gg')^{-1}\in V$ is a continuous 2-cocycle; for a continuous 1-cochain w, $s_w(g)=w(g)s(g)$ satisfies $s_w(g)s_w(g')s_w(gg')^{-1}=c(g,g')+w(g)+g\cdot w(g')-w(gg')$, and w with $dw=-c$ makes s_w a homomorphism. (F3) For a graded finite $\mathcal{O}$-algebra B, the graded elements of $\mathcal{G}_3(B)$ form a subgroup $\mathcal{G}^{\mathrm{gr}}(B)$: products of graded matrices are graded as $\mathrm{wt}(a,c)+\mathrm{wt}(c,b)=\mathrm{wt}(a,b)$; the cofactor giving the entry (a,b) of the inverse has weight $\mathrm{wt}(a,b)$, and the determinant of a graded matrix has weight 0 and its inverse, a homogeneous unit's inverse, lies in B_0; and $\theta(g)_{ab}=(g^{-1})_{4-b,4-a}$ has weight $\mathrm{wt}(a,b)$. The same holds for the subgroups of graded upper triangular (resp. diagonal) GL-parts, which θ preserves (θ maps upper triangular matrices to upper triangular ones). For a graded small extension pr of weight j, pr maps $\mathcal{G}^{\mathrm{gr}}(B)$ onto $\mathcal{G}^{\mathrm{gr}}(\bar B)$ (lift entries homogeneously; a lift of an invertible matrix is invertible, B being local) with kernel $\{(1+X,1): X\in\mathrm{M}_3(I)\text{ supported in weight }j\}\cong\mathfrak{g}_j\otimes I$; and $\mathrm{pr}\colon B_i\to\bar B_i$ is bijective for $i\neq j$.

(Ll2). The map $\rho_\lambda(g)=(1+\lambda(g),1)\rho(g)$ is a homomorphism because λ is a cocycle for the action of g ($\phi\circ\lambda$ is one for every ϕ): $\rho_\lambda(g)\rho_\lambda(g')=(1+\lambda(g)+g\cdot\lambda(g'),1)\rho(gg')=\rho_\lambda(gg')$. Continuity is clear. The condition at p:

- $p\notin T_\ell(\chi)$: every $\phi\circ\lambda$ vanishes on $N_{\bar p}$, so $\lambda=0$ there and $\rho_\lambda=\rho=1$ on $N_{\bar p}$.
- $p=\ell$: let h be as in (Lp1) for ρ and $\beta=h^{-1}\underline{\rho}h$. As $h\equiv 1$ and $\mathfrak{m}I=0$, $h^{-1}(1+\lambda)\underline{\rho}\,h=(1+\lambda)\beta$. If $j>0$, the positions of weight j lie above the diagonal, so $(1+\lambda)\beta$ is upper triangular on H with the diagonal of β. If $j=0$, λ is diagonal and vanishes on I_ℓ (this is $L_{\ell,0}$), so $(1+\lambda)\beta$ is upper triangular on H with diagonal ψ_a on I_ℓ. If $j<0$, every $\phi\circ\lambda$ is

a coboundary; choosing a basis of I gives $X \in \mathrm{M}_3(I)$ with $\lambda(g) = g \cdot X - X$, and by (46.1) $\rho_\lambda = k\rho k^{-1}$ with $k = (1 - X, 1)$; then kh is a matrix as required in (Lp1).

- $p \in S_\ell(\chi)$: here p is odd (the 2-component of χ'' has both ℓ-power and 2-power order, so it is trivial), hence unramified in K and $I_p \subseteq H$. If $j \neq 0$, $H^1(D_p, \mathfrak{g}_j) = 0$ by Lemma 46.13, so as before ρ_λ is a conjugate of ρ by an element $\equiv 1$, and (Lp2) persists. If $j = 0$, λ is diagonal; on I_p, which acts trivially on $\mathfrak{g}_0$, each $\phi \circ \lambda$ is a homomorphism vanishing on $N_{\bar{p}}$, hence factors through $I_p/N_{\bar{p}}$, of order dividing $2(\#\mathbb{F} - 1)$, prime to ℓ, into a group of exponent ℓ; so $\lambda = 0$ on I_p, and with h as in (Lp2) for ρ, $h^{-1}\underline{\rho_\lambda}h = (1 + \lambda)\, h^{-1}\underline{\rho}h$ is diagonal on H with diagonal ψ_a on I_p.

(Ll1), $p \notin T_\ell(\chi)$. The group $N_{\bar{p}}$ is the kernel of $I_p \to \{\pm 1\} \times \mathbb{F}^\times$, $g \mapsto (\eta_K(g), \bar{\Psi}(g))$, the restriction of a homomorphism on D_p; so it is normal in D_p and $I_p/N_{\bar{p}}$ is finite of order prime to ℓ. As $\bar{\rho}$ is trivial on $N_{\bar{p}}$ it factors through $\Delta = D_p/N_{\bar{p}}$, and $N_{\bar{p}}$ acts trivially on $\mathfrak{g}_j$. By the Hochschild–Serre spectral sequence for the normal subgroup $I_p/N_{\bar{p}}$, whose cohomology in positive degrees with ℓ-torsion coefficients vanishes, $H^2(\Delta, \mathfrak{g}_j) = H^2(D_p/I_p, \mathfrak{g}_j^{I_p}) = 0$, since $\hat{\mathbb{Z}}$ has ℓ-cohomological dimension 1. By (F2) for $\mathcal{G}^{\mathrm{gr}}(B) \to \mathcal{G}^{\mathrm{gr}}(\bar{B})$ (F3) and the continuous map $\Delta \to \mathcal{G}^{\mathrm{gr}}(\bar{B})$ induced by $\bar{\rho}$, there is a graded lift ρ trivial on $N_{\bar{p}}$.

(Ll1), $p \in T_\ell(\chi)$: reduction to triangular form. By Lemma 46.9 there is $x \in H$ with $\bar{\Psi}(x)^2 \neq 1$; then $\bar{\Psi}(x), 1, \bar{\Psi}(x)^{-1}$ are pairwise distinct, and $\underline{\bar{\rho}}(x)$, which reduces to $\mathrm{T}(\bar{\Psi}(x))$, is residually regular diagonal. Let $\bar{h} = h(\underline{\bar{\rho}}(x))$: it is graded by (Cd5), and by (Cd4) $\bar{\beta} = \bar{h}^{-1}\bar{\rho}\bar{h}$ has GL-parts on H which are upper triangular, diagonal if $p \in S_\ell(\chi)$, with diagonal ψ_a on I_p. The same holds on all of D_p: for $g \in D_p \setminus H$ write $\bar{\beta}(g) = (P, -1)$; then $P\,\theta(U)\,P^{-1} = U'$ with $U = \underline{\bar{\beta}}(x)$, $U' = \underline{\bar{\beta}}(gxg^{-1})$, both upper triangular (diagonal), and $\theta(U)$ is upper triangular (diagonal) with residual diagonal $\mathrm{T}(\bar{\Psi}(x))$ ($\theta(\mathrm{T}(t)) = \mathrm{T}(t)$), as is U' ($\bar{\Psi}(gxg^{-1}) = \bar{\Psi}(x)$). For an upper triangular $V \in \mathrm{GL}_3(\bar{B})$ whose diagonal entries have pairwise distinct residues, the equations $(Vn)_{ab} = n_{ab}V_{bb}$ ($a < b$) for an upper unitriangular n read $n_{ab}(V_{bb} - V_{aa}) = \sum_{a<c\leq b} V_{ac}n_{cb}$ ($n_{bb} = 1$) and are solved recursively in $b - a$, $V_{bb} - V_{aa}$ being a unit; then $n^{-1}Vn = \mathrm{diag}(V_{11}, V_{22}, V_{33})$, and $n = 1$ if V is diagonal. Applied to $V = \theta(U)$ and $V = U'$ this gives upper unitriangular n, n' (equal to 1 in the diagonal case) with $n^{-1}\theta(U)n = \Lambda$, $n'^{-1}U'n' = \Lambda'$ diagonal. The matrix $Q = n'^{-1}Pn$ satisfies $Q\Lambda = \Lambda'Q$, i.e. $(\Lambda'_{aa} - \Lambda_{bb})Q_{ab} = 0$; as Λ, Λ' have the same pairwise distinct residual diagonal entries, $\Lambda'_{aa} - \Lambda_{bb}$ is a unit for $a \neq b$, so Q is diagonal; so $P = n'Qn^{-1}$ is upper triangular (diagonal). Write $\bar{\beta}(g) = (\bar{u}(g), 1)\,\bar{\tau}(g)$ with $\bar{u}(g)$ upper unitriangular and $\bar{\tau}(g)$ with diagonal GL-part; $\bar{\tau}$ is a continuous graded homomorphism (the torus quotient of the Borel is compatible with θ), with values in $\mathrm{diag}(\bar{B}_0^\times) \rtimes \{\pm 1\}$, and $\bar{\tau} = r_0$ on I_p ($I_p \subseteq H$ as p is odd).

The diagonal part. If $j \neq 0$, $\mathrm{pr}\colon B_0 \to \bar{B}_0$ is a ring isomorphism and $\tau = \mathrm{pr}^{-1} \circ \bar{\tau}$ is a graded continuous lift with $\tau = r_0$ on I_p. If $j = 0$, use (F1): $D_p = I_p \rtimes Z$. Let ϕ be a topological generator of Z and Δ_ϕ any graded lift of $\bar{\tau}(\phi)$ with diagonal GL-part. For $i \in I_p$, $\Delta_\phi r_0(i)\Delta_\phi^{-1} = (\sigma_\phi(\mathrm{T}(\Psi(i))), 1) = r_0(i) = r_0(\phi i \phi^{-1})$, because diagonal matrices commute, $\theta(\mathrm{T}(t)) = \mathrm{T}(t)$ and Ψ is a character. Hence $\tau(i\phi^a) = r_0(i)\Delta_\phi^a$ ($i \in I_p$, $a \in \hat{\mathbb{Z}}$, Δ_ϕ of finite order) is a well-defined continuous homomorphism lifting $\bar{\tau}$, graded, with $\tau = r_0$ on I_p.

The unipotent part. Let $U^{\mathrm{gr}}(B)$ be the group of graded upper unitriangular matrices; $\mathrm{pr}\colon U^{\mathrm{gr}}(B) \to U^{\mathrm{gr}}(\bar{B})$ is surjective with kernel $\{1 + X\}$, $X \in \mathrm{M}_3(I)$ supported in the positions of weight j above the diagonal; it is an isomorphism if $j \leq 0$. Conjugation by $\tau(g)$ preserves $U^{\mathrm{gr}}(B)$, and because the off-diagonal entries of $\bar{u}(g)$ lie in $\mathfrak{m}_{\bar{B}}$ and $\mathfrak{m}_B I = 0$, pr intertwines conjugation by $\tau(g)$ with conjugation by $\bar{\tau}(g)$.

- If $j \leq 0$ (or $p \in S_\ell(\chi)$, where $\bar{u} = 1$): put $\beta(g) = (\mathrm{pr}^{-1}(\bar{u}(g)), 1)\,\tau(g)$ (with $\bar{u} = 1$, $\beta = \tau$, in

the diagonal case). Since $\bar{u}(gg') = \bar{u}(g)\cdot(\bar{\tau}(g)\bar{u}(g')\bar{\tau}(g)^{-1})$ and pr^{-1} is a group isomorphism intertwining the conjugations, β is a continuous graded homomorphism lifting $\bar{\beta}$.

- If $j \geq 1$ and $p = \ell$: let $\tilde{Q}$ (resp. $\bar{Q}$) be the finite group of graded elements of $\mathcal{G}_3(B)$ (resp. $\mathcal{G}_3(\bar{B})$) with upper triangular GL-part (F3). As $B_0 \to \bar{B}_0$ is bijective, $\tilde{Q} \to \bar{Q}$ is surjective with kernel $\{(1+X, 1)\}$, $X \in \mathrm{M}_3(I)$ supported in the positions of weight j (which lie above the diagonal), a D_ℓ-module isomorphic to $\mathfrak{g}_j \otimes I$ (zero if $j \geq 3$). By Lemma 46.13, $H^2(D_\ell, \mathfrak{g}_j \otimes I) = H^2(D_\ell, \mathfrak{g}_j) \otimes I = 0$ for $j \in \{1, 2\}$, so by (F2) $\bar{\beta}$ lifts to a continuous graded $\beta\colon D_\ell \to \tilde{Q}$. Its diagonal part lifts $\bar{\tau}$ and so equals τ, the lift being unique as $B_0 \to \bar{B}_0$ is bijective.

In all cases β is graded, upper triangular (diagonal if $p \in S_\ell(\chi)$) on D_p, with diagonal ψ_a on I_p. Let h be a graded lift of $\bar{h}$ (it is $\equiv 1$, hence invertible) and $\rho = h\beta h^{-1}$. Then ρ is a continuous graded homomorphism lifting $\bar{\rho}$ (F3), and h is a matrix as required in (Lp1), resp. (Lp2). □

Lemma 46.13. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, let $p \in T_\ell(\chi)$ and $j \in \{-2, -1, 1, 2\}$. Then $H^0(D_p \cap G_K, \mathfrak{g}_j) = H^0(D_p \cap G_K, \mathfrak{g}_j^\vee) = 0$, $H^2(D_p, \mathfrak{g}_j) = 0$, and, if $p \in S_\ell(\chi)$, $H^1(D_p, \mathfrak{g}_j) = 0$.*

Proof. Write $\mathbb{F} = \mathbb{F}_\mathcal{O}$. An element $g \in G_K$ has $\bar{r}_0(g) = (\mathrm{T}(\bar{\Psi}(g)), 1)$ and acts on $\mathfrak{g}_j$ by conjugation with $\mathrm{T}(\bar{\Psi}(g))$, which multiplies the entry (a, b) by $\bar{\Psi}(g)^{b-a}$; so it acts on $\mathfrak{g}_j$ by the scalar $\bar{\Psi}(g)^j$. By Lemma 46.9 there is $s \in D_p \cap G_K$ with $\bar{\Psi}(s)^2 \neq 1$ that fixes μ_ℓ. Then $\alpha = \bar{\Psi}(s)^j \neq 1$: for $j = \pm 2$ directly, and for $j = \pm 1$ because $\bar{\Psi}(s)^{\pm 1} = 1$ would give $\bar{\Psi}(s)^2 = 1$.

Invariants. The element s acts on $\mathfrak{g}_j$ by $\alpha \neq 1$, so $H^0(D_p \cap G_K, \mathfrak{g}_j) \subseteq \{X : (\alpha - 1)X = 0\} = 0$. For $f \in \mathfrak{g}_j^\vee$, $(s \cdot f)(X) = s(f(s^{-1}X)) = f(\alpha^{-1}X)$, since s fixes μ_ℓ. If $s \cdot f = f$, then, f being a homomorphism from the additive group of $\mathfrak{g}_j$ to μ_ℓ, $f((\alpha^{-1} - 1)X) = f(\alpha^{-1}X)f(X)^{-1} = 1$ for all X, and $(\alpha^{-1} - 1)\mathfrak{g}_j = \mathfrak{g}_j$, so f is trivial. Hence $H^0(D_p \cap G_K, \mathfrak{g}_j^\vee) = 0$, and a fortiori $H^0(D_p, \mathfrak{g}_j) = H^0(D_p, \mathfrak{g}_j^\vee) = 0$.

H^2. The group D_p is the absolute Galois group of $\mathbb{Q}_p$ (through the completion at $\bar{p}$), and $\mathfrak{g}_j$ is a finite $\mathbb{Z}/\ell\mathbb{Z}$-module with continuous action. By Theorem B.29, item 2 ($r = 1$), $H^2(D_p, \mathfrak{g}_j)$ is dual to $H^0(D_p, \mathfrak{g}_j^\vee) = 0$.

H^1 at $S_\ell(\chi)$. If $p \in S_\ell(\chi)$ then $p \neq \ell$, and item 4 of Theorem B.29 gives $\#H^1(D_p, \mathfrak{g}_j) = \#H^0(D_p, \mathfrak{g}_j)\,\#H^2(D_p, \mathfrak{g}_j) = 1$. □

Proposition 46.14. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$. Let $G_\mathbb{Q}$ act on $\mathrm{M}_3(\mathcal{O})$ through r_0: if $r_0(g) = (P, \varsigma)$, by $X \mapsto PXP^{-1}$ if $\varsigma = 1$ and by $X \mapsto -PJ\,{}^tXJP^{-1}$ if $\varsigma = -1$; let $j \in \{-2, -1, 0\}$ and let $\mathfrak{g}_j(\mathcal{O})$ be the matrices over $\mathcal{O}$ supported in the positions of weight j. Let $y\colon G_S \to \mathfrak{g}_j(\mathcal{O})$ be a continuous 1-cocycle such that*

(Vn1) *$y(g) = 0$ for every $g \in N_{\bar{p}}$ and every prime $p \in S$, $p \neq \ell$;*

(Vn2) *if $j = 0$: $y(g) = 0$ for every $g \in I_\ell$ and every $g \in I_p$, $p \in S_\ell(\chi)$;*

(Vn3) *if $j < 0$: there is $Z \in \mathfrak{g}_j(\mathcal{O})$ with $y(g) = g \cdot Z - Z$ for every $g \in D_\ell \cap G_K$.*

Then there are $a \in \mathcal{O} \setminus \{0\}$ and $X \in \mathfrak{g}_j(\mathcal{O})$ with $a\,y(g) = g \cdot X - X$ for every $g \in G_S$.

Proof. Write $\mathcal{M}_j = \mathfrak{g}_j(\mathcal{O})$, $g \cdot X$ for the action of $g \in G_\mathbb{Q}$ on $\mathrm{M}_3(\mathcal{O})$ through r_0, and dX for the coboundary $g \mapsto g \cdot X - X$; we write y also for the inflation of y to $G_\mathbb{Q}$, a continuous cocycle. Put $H_p = D_p \cap G_K$ for primes p. Let ϖ be a uniformiser of $\mathcal{O}$ and $\mathbb{F} = \mathbb{F}_\mathcal{O}$.

Preliminaries.

(P1) For $g \in G_K$, $r_0(g) = (\mathrm{T}(\Psi(g)), 1)$, so g acts on $\mathcal{M}_j$ by multiplication by $\Psi(g)^j$. Since $\Psi(c) = \chi(-1)(-1)^e = 1$ by (Dy3), $r_0(c) = (1, -1)$ and c acts by $X \mapsto -J\,{}^tXJ$; as J is antidiagonal with entries 1, $c \cdot E_{31} = -E_{31}$ and $c \cdot \mathrm{diag}(a_1, a_2, a_3) = -\mathrm{diag}(a_3, a_2, a_1)$. Hence $G_{\mathbb{Q}}$ acts on $\mathcal{M}_{-2} = \mathcal{O}E_{31}$ through the character $\gamma = \eta_K \Psi^{-2}$, whose reduction is γ_{-2}, and G_K acts trivially on $\mathcal{M}_0$.

(P2) Let Γ be a profinite group acting on an abelian group M, $\Gamma' \subseteq \Gamma$ an open normal subgroup of index n and z a 1-cocycle of Γ in M vanishing on Γ'. Then $z(gh) = z(g) + g \cdot z(h) = z(g)$ for $h \in \Gamma'$, so z is constant on cosets; summing $z(g\tau) = z(g) + g \cdot z(\tau)$ over a set $\mathcal{T}$ of coset representatives ($g\mathcal{T}$ is another one) gives $n\, z(g) = g \cdot (-W) - (-W)$ with $W = \sum_{\tau \in \mathcal{T}} z(\tau)$.

(P3) Let z be a cocycle of $G_{\mathbb{Q}}$, $g \in G_{\mathbb{Q}}$ and $D \subseteq G_{\mathbb{Q}}$ a subgroup. If $z(h) = 0$ and h acts trivially on the coefficients, then $z(ghg^{-1}) = z(g) + g \cdot z(h) + gh \cdot z(g^{-1}) = z(g) + g \cdot z(g^{-1}) = z(1) = 0$. If $z = dZ$ on D, then, using $z(g^{-1}) = -g^{-1} \cdot z(g)$, $z = d(g \cdot Z - z(g))$ on gDg^{-1}.

(P4) Every place of $\overline{\mathbb{Q}}$ above a prime p is $g\bar{p}$ with $g \in G_{\mathbb{Q}}$, and $I_{g\bar{p}} = gI_{\bar{p}}g^{-1}$, $N_{g\bar{p}} = gN_{\bar{p}}g^{-1}$. If $p \notin S$, $I_{g\bar{p}}$ lies in the kernel of $G_{\mathbb{Q}} \to G_S$, so $y = 0$ on it.

(P5) If $p \in S \setminus T_\ell(\chi)$ is a prime, then $\Psi = 1$ on $N_{\bar{p}}$. Indeed $p \neq \ell$, so $p \mid N$ and $\chi''_p = \mathbf{1}$; χ_{cyc} and the r-components of χ for $r \neq p$ are unramified at p, so $\Psi(I_{\bar{p}}) = \chi_p(I_{\bar{p}}) = \chi'_p(I_{\bar{p}})$ consists of roots of unity of order prime to ℓ, which reduce injectively into $\mathbb{F}^\times$; on $N_{\bar{p}}$ the reduction is 1. Moreover $2 \notin S_\ell(\chi)$, the 2-component of χ'' having order both a power of 2 and of ℓ, odd by (Dy1); so every $p \in S_\ell(\chi)$ is odd, unramified in K, and $I_p \subseteq H_p$.

Weight -2: y is a coboundary. Identify $\mathcal{M}_{-2}$ with $\mathcal{O}$ via E_{31}, so that $y(gh) = y(g) + \gamma(g)y(h)$ (P1). For $p \in T_\ell(\chi)$ let $s_p \in H_p$ be the element of Lemma 46.9; as $\eta_K(s_p) = 1$, $\bar{\gamma}(s_p) = \bar{\Psi}(s_p)^{-2} \neq 1$, so $\gamma(s_p) - 1 \in \mathcal{O}^\times$. Let $\mathcal{Z}$ be the set of continuous cocycles $z \colon G_S \to \mathcal{O}$ (for γ) such that

(Z1) for every $p \in T_\ell(\chi)$, $z = dW$ on H_p for some $W \in \mathcal{O}$;

(Z2) $z = 0$ on $N_{\bar{p}}$ for every prime $p \in S \setminus T_\ell(\chi)$.

Step 1: $y \in \mathcal{Z}$. (Z2) is (Vn1); (Z1) at $p = \ell$ is (Vn3). Let $p \in S_\ell(\chi)$. The group $N_{\bar{p}}$ is the kernel of $g \mapsto (\eta_K(g), \bar{\Psi}(g))$ on I_p, normal of index n_p dividing $2(\#\mathbb{F} - 1)$, prime to ℓ, and $y = 0$ on it by (Vn1); by (P2), $y = dW_1$ on I_p with $W_1 = -n_p^{-1} \sum_\tau y(\tau) \in \mathcal{O}$. The cocycle $y' = y - dW_1$ on H_p vanishes on I_p, which is normal in H_p; for $g \in H_p$ and $h \in I_p$, $y'(hg) = \gamma(h)y'(g)$ and $y'(hg) = y'(g \cdot g^{-1}hg) = y'(g)$. If γ is non-trivial on I_p, this forces $y' = 0$ ($\mathcal{O}$ is a domain). Otherwise γ factors through $\Gamma_p = H_p/I_p$, and so does y' ($y'(gh) = y'(g)$ for $h \in I_p$). Γ_p is open in D_p/I_p, hence procyclic, topologically generated by some ϕ; since $\bar{\gamma}$ is continuous, factors through Γ_p and $\bar{\gamma}(s_p) \neq 1$, $\bar{\gamma}(\phi) \neq 1$, so $\gamma(\phi) - 1$ is a unit. Put $W_2 = y'(\phi)/(\gamma(\phi) - 1)$. The set of $g \in \Gamma_p$ with $y'(g) = (dW_2)(g)$ is a closed subgroup (both are continuous cocycles) containing ϕ, hence is Γ_p. So $y = d(W_1 + W_2)$ on H_p: (Z1).

Step 2: $\mathcal{Z}$ is an $\mathcal{O}$-module containing the coboundaries. Clear for (Z1); for (Z2), $\gamma = \eta_K\Psi^{-2} = 1$ on $N_{\bar{p}}$ by (P5), so $(d\mu)(g) = (\gamma(g) - 1)\mu = 0$ there.

Step 3: the reduction of $z \in \mathcal{Z}$ lies in Z_{-2}. Let $\bar{z} \colon G_{\mathbb{Q}} \to \mathbb{F}$ be the reduction of (the inflation of) z: a locally constant cocycle for γ_{-2}. Let $\bar{v} = g\bar{p}$ be a place above a prime $p \neq \ell$. If $p \notin S$, $\bar{z} = 0$ on $N_{\bar{v}}$ by (P4). If $p \in S \setminus T_\ell(\chi)$, $z = 0$ on $N_{\bar{p}}$ by (Z2). If $p \in S_\ell(\chi)$, $z = dW$ on $H_p \supseteq N_{\bar{p}}$ by (Z1), and $\bar{\gamma} = 1$ on $N_{\bar{p}}$, so $\bar{z}(h) = (\bar{\gamma}(h) - 1)\bar{W} = 0$ for $h \in N_{\bar{p}}$. In both cases $\bar{z}$ vanishes on $N_{\bar{p}}$, on which γ_{-2} is trivial, so by (P3) it vanishes on $N_{g\bar{p}} = gN_{\bar{p}}g^{-1}$. At ℓ, $\bar{z} = d\bar{W}$ on H_ℓ, of index at most 2 in D_ℓ; the cocycle $\bar{z} - d\bar{W}$ of D_ℓ vanishes on H_ℓ, so by (P2) it is a coboundary on D_ℓ (2 times it is one, and 2 is invertible in $\mathbb{F}$). Hence $\bar{z} \in Z_{-2}$.

Step 4: saturation. If $z \in \mathcal{Z}$ takes values in $\varpi\mathcal{O}$, then $z_1 = \varpi^{-1}z \in \mathcal{Z}$: it is a continuous cocycle; (Z2) is clear; if $z = dW$ on H_p, $p \in T_\ell(\chi)$, then $(\gamma(s_p) - 1)W = z(s_p) \in \varpi\mathcal{O}$ with $\gamma(s_p) - 1$ a unit, so $W \in \varpi\mathcal{O}$ and $z_1 = d(\varpi^{-1}W)$ on H_p.

Step 5: descent. Put $y_0 = y \in \mathcal{Z}$ (Step 1). Given $y_k \in \mathcal{Z}$, Step 3 and $\mathrm{Sel}_{-2}(\chi, e) = 0$ (Lemma 46.15) give $\bar{\mu}_k \in \mathbb{F}$ with $\bar{y}_k = d\bar{\mu}_k$; for a lift $\mu_k \in \mathcal{O}$, $y_k - d\mu_k \in \mathcal{Z}$ (Step 2) has values in $\varpi\mathcal{O}$, and $y_{k+1} = \varpi^{-1}(y_k - d\mu_k) \in \mathcal{Z}$ (Step 4). By induction $y = d(\sum_{k<n} \varpi^k \mu_k) + \varpi^n y_n$ for all n. With $X = \sum_{k\geq 0} \varpi^k \mu_k \in \mathcal{O}$ (convergent, $\mathcal{O}$ being complete), $y(g) - (\gamma(g) - 1)X \in \varpi^n\mathcal{O}$ for all n, so $y = dX$: the claim holds with $a = 1$ and $XE_{31} \in \mathcal{M}_{-2}$.

Weight 0*: y is a coboundary.* By (P1), $y|_{G_K}$ is a continuous homomorphism $G_K \to \mathcal{M}_0 \cong \mathcal{O}^3$. It vanishes on $I_{\bar{v}} \cap G_K$ for every place $\bar{v}$ above a prime p: for $p \notin S$ by (P4); for $p = \ell$ by (Vn2); for $p \in S$, $p \neq \ell$, at $\bar{v} = \bar{p}$, because $y = 0$ on $N_{\bar{p}} \subseteq I_{\bar{p}} \cap G_K$ (Vn1), a subgroup of finite index ($\bar{\Psi}$ has finite image), so $y(I_{\bar{p}} \cap G_K)$ is a finite subgroup of the torsion-free group $\mathcal{O}^3$; and at $\bar{v} = g\bar{p}$ by (P3), since $I_{g\bar{p}} \cap G_K = g(I_{\bar{p}} \cap G_K)g^{-1}$ (G_K is normal) and G_K acts trivially on $\mathcal{M}_0$. So the fixed field of the closed normal subgroup $\ker(y|_{G_K})$ is an abelian extension of K unramified at every finite place, and at the one infinite place, which is complex. Since $K = \mathbb{Q}(i)$ has class number one, this extension is K, and $y = 0$ on G_K. Then $0 = y(c^2) = y(c) + c \cdot y(c)$; put $X = -\frac{1}{2}y(c) \in \mathcal{M}_0$ (ℓ is odd). Then $c \cdot X - X = y(c)$, and for $g \in G_K$: $y(g) = 0 = g \cdot X - X$ and $y(gc) = y(g) + g \cdot y(c) = y(c) = (gc) \cdot X - X$. So $y = dX$ on $G_K \sqcup G_K c = G_{\mathbb{Q}}$: the claim holds with $a = 1$.

Weight -1*: y is torsion modulo coboundaries.* Let $L = \mathbb{Q}(\zeta_{4\ell N}) \subset \overline{\mathbb{Q}}$, a totally complex abelian extension of $\mathbb{Q}$ containing i and a primitive ℓ-th root of unity, and $n = [L : \mathbb{Q}]$. The character χ factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, so $\Psi = \chi_{\mathrm{cyc}}^e$ on $G_L \subseteq G_K$, and by (P1) G_L acts on $\mathcal{M}_{-1}$ by the scalar χ_{cyc}^{-e}.

Step 1: $y|_{G_L}$ is unramified outside ℓ. Let $\bar{v} = g\bar{p}$ be a place above a prime $p \neq \ell$. The group $I_{\bar{p}} \cap G_L$ acts trivially on $\mathcal{M}_{-1}$ (χ_{cyc} is unramified at p), so y is a homomorphism on it; if $p \in S$ it vanishes on the finite-index subgroup $N_{\bar{p}} \cap G_L$ (Vn1), so its image is a finite subgroup of the torsion-free $\mathcal{M}_{-1}$, hence 0; if $p \notin S$, $y = 0$ on it by (P4). By (P3), $y = 0$ on $I_{g\bar{p}} \cap G_L = g(I_{\bar{p}} \cap G_L)g^{-1}$. The kernel H of $G_L \to G_{L,S_\ell}$ is the closed normal subgroup generated by these groups $I_{\bar{v}} \cap G_L$ (the infinite places of L are complex); χ_{cyc} is trivial on each of them, hence on H, so $y|_H$ is a continuous homomorphism vanishing on a set of topological generators: $y|_H = 0$. Hence $y(gh) = y(g)$ for $g \in G_L$, $h \in H$, and $y|_{G_L} = \tilde{y} \circ (G_L \to G_{L,S_\ell})$ for a continuous cocycle $\tilde{y}$ of G_{L,S_ℓ} in $\mathcal{M}_{-1}$ with the scalar action χ_{cyc}^{-e}.

Step 2: local triviality above ℓ. For a place w of L above ℓ, the restriction to L_w is taken through $D_{\bar{w}} \cap G_L$ for a place $\bar{w} = g\bar{\ell}$ of $\overline{\mathbb{Q}}$ above w. By (Vn3), $y = dZ$ on H_ℓ, so by (P3) $y = dZ_w$ with $Z_w = g \cdot Z - y(g) \in \mathcal{M}_{-1}$ on $gH_\ell g^{-1} = D_{\bar{w}} \cap G_K \supseteq D_{\bar{w}} \cap G_L$.

Step 3: a rational primitive over L. $\mathcal{M}_{-1}$ is a free $\mathbb{Z}_\ell$-module of finite rank ($\mathcal{O}$ is finite free over $\mathbb{Z}_\ell$); let $b_1, \ldots, b_m$ be a basis and $\lambda_1, \ldots, \lambda_m$ the dual coordinates. Since G_L acts by a scalar in $\mathbb{Z}_\ell^\times$, each $\lambda_i \circ \tilde{y}$: $G_{L,S_\ell} \to \mathbb{Z}_\ell$ satisfies the cocycle relation for χ_{cyc}^{-e}, and by Step 2 it is the coboundary of $\lambda_i(Z_w) \in \mathbb{Z}_\ell$ on the image of G_{L_w} for each $w \mid \ell$. By Proposition 40.3 (L totally complex, containing a primitive ℓ-th root of unity, $e \geq 1$), $\mathrm{len}_\ell(L, e) < \infty$, so Lemma 43.33 (with $E = L$, $k = e$) gives $x_i \in \mathbb{Q}_\ell$ with $\lambda_i(\tilde{y}(g)) = (\chi_{\mathrm{cyc}}(g)^{-e} - 1)x_i$. Choose $t \geq 0$ with $\ell^t x_i \in \mathbb{Z}_\ell$ for all i and put $X_L = \sum_i \ell^t x_i b_i \in \mathcal{M}_{-1}$; then $\ell^t y = dX_L$ on G_L.

Step 4: descent to $G_{\mathbb{Q}}$. The cocycle $\ell^t y - dX_L$ of $G_{\mathbb{Q}}$ vanishes on the open normal subgroup G_L of index n; by (P2), $n(\ell^t y - dX_L) = dX'$ for some $X' \in \mathcal{M}_{-1}$. So $a\,y = dX$ with $a = n\ell^t \neq 0$ and $X = nX_L + X' \in \mathcal{M}_{-1}$; all these maps factor through G_S, so the identity holds on G_S. □

Lemma 46.15. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$. Then* $\mathrm{Sel}_{-2}(\chi, e) = 0$.

Proof. Let $z \in Z_{-2}$. We show that z is a coboundary.

Step 1: γ_{-2} is the reduction of a Dirichlet character. Let ω_ℓ be the Teichmüller character modulo ℓ ($\omega_\ell(b)$ is the $(\ell-1)$-th root of unity in $\mathbb{Z}_\ell$ congruent to b), so that χ_{cyc} and ω_ℓ have the same reduction, and let χ_4 be the character modulo 4 with $\chi_4(-1) = -1$, which as a character of $G_{\mathbb{Q}}$ is η_K. The values of χ'' are roots of unity of ℓ-power order, which reduce to 1; so $\bar{\Psi}^2\eta_K$ is the reduction of

$$\chi_{\mathrm{B}} = (\chi')^2\,\omega_\ell^{2e}\,\chi_4,$$

a Dirichlet character of order prime to ℓ, and $\gamma_{-2} = (\bar{\Psi}^2\eta_K)^{-1}$ is the reduction of χ_{B}^{-1}. The 2-component of χ'' has both ℓ-power and 2-power order, so it is trivial and $\chi_2' = \chi_2$. Hence $\chi_{\mathrm{B}} = \chi_{\mathrm{B,od}}\chi_{\mathrm{B},2}$ with $\chi_{\mathrm{B,od}} = \vartheta^2\omega_\ell^{2e}$, a product of characters of odd conductor, and $\chi_{\mathrm{B},2} = \chi_2^2\chi_4$.

Step 2: the two primitive characters. Let α be the primitive character inducing $\chi_{\mathrm{B,od}}$, of conductor f_α. The conductor of a product divides the least common multiple of the conductors [266]; ϑ^2 has trivial 2-component, so its conductor is odd and divides m by (Dy7). Hence f_α is odd and $f_\alpha \le \ell m$. Let f_ℓ be the order of ℓ modulo $\mathrm{lcm}(4, \operatorname{ord}\vartheta)$, so that $n_0(\ell,\chi)$ is the exponent of 2 in $\ell^{f_\ell} - 1$, $4 \mid \ell^{f_\ell} - 1$, $\operatorname{ord}\vartheta \mid \ell^{f_\ell} - 1$ and $(\ell-1) \mid \ell^{f_\ell} - 1$; the order of α divides $\mathrm{lcm}(\operatorname{ord}\vartheta, \ell-1)$, so $\alpha(b)^{\ell^{f_\ell}-1} = 1$ for b prime to f_α. Put $c = c_2(\chi) \ge 4$ (Dy6). Since χ_2 is primitive modulo 2^c, $\chi_2(5)$ has order exactly 2^{c-2} (classically $(\mathbb{Z}/2^c\mathbb{Z})^\times = \{\pm1\} \times \langle 5\rangle$ with 5 of order 2^{c-2} and $5^{2^{c-3}} \equiv 1 + 2^{c-1} \pmod{2^c}$ [266]; $\{1, 1+2^{c-1}\}$ is the kernel of reduction modulo 2^{c-1}, so primitivity gives $\chi_2(1+2^{c-1}) \ne 1$, and $(1+2^{c-1})^2 \equiv 1$, so $\chi_2(5)^{2^{c-3}} = -1$). As $\chi_4(5) = 1$, $\chi_{\mathrm{B},2}(5) = \chi_2(5)^2$ has order $2^{c-3} \ge 2$. The subgroup $\{u \equiv 1 \pmod{2^{c-1}}\}$ is $\langle 5^{2^{c-3}}\rangle$, on which $\chi_{\mathrm{B},2}$ is trivial, and $\{u \equiv 1 \pmod{2^{c-2}}\} = \langle 5^{2^{c-4}}\rangle$, on which it is not; so the primitive character β inducing $\chi_{\mathrm{B},2}$ has conductor 2^n with $n = c - 1 \ge 3$, and $\beta(5)$ has order 2^{n-2}.

Step 3: the Bernoulli number is a unit. The hypotheses of Lemma 44.6 hold for ℓ, the exponent f_ℓ (in place of m), $a = n_0(\ell,\chi)$, f_α, n, α and β: ℓ is odd by (Dy1); f_α is odd; $\alpha^{\ell^{f_\ell}-1} = 1$ (Step 2); by (Dy7), $f_\alpha 4^a \le \ell m 4^{n_0(\ell,\chi)} \le 2^{c-1} = 2^n$; and $\alpha(-1)\beta(-1) = \chi_{\mathrm{B}}(-1) = \chi'(-1)^2\omega_\ell(-1)^{2e}\chi_4(-1) = -1$. Hence $\alpha\beta$ is primitive modulo $f_\alpha 2^n$, it agrees with χ_{B} on integers prime to $2\ell N$, so it is χ_{B} as a character of $G_{\mathbb{Q}}$, and $|B_{1,\alpha\beta}|_\ell = 1$.

Step 4: the hypotheses of the Herbrand-type lemma. Let $\chi_{\mathrm{H}} = (\alpha\beta)^{-1}$: a non-trivial primitive character modulo $f_\alpha 2^n$ of order prime to ℓ, with $\bar{\chi}_{\mathrm{H}} = \gamma_{-2}$ (Steps 1 and 3) and $|B_{1,\chi_{\mathrm{H}}^{-1}}|_\ell = |B_{1,\alpha\beta}|_\ell = 1$. Choose an integer $b \equiv 1 \pmod{\ell f_\alpha}$ with $b \equiv 5 \pmod{2^n}$ and $g \in G_{\mathbb{Q}}$ with $g(\zeta) = \zeta^b$ for all roots of unity of order dividing $\ell f_\alpha 2^n$. Then $\bar{\omega}(g) = \bar{b} = 1$, while $\chi_{\mathrm{H}}(g) = \alpha(b)^{-1}\beta(5)^{-1} = \beta(5)^{-1}$ is a root of unity of order $2^{n-2} \ge 2$, whose reduction is $\ne 1$; so $\bar{\chi}_{\mathrm{H}} \ne \bar{\omega}$.

Step 5: z is unramified everywhere. Let $\bar{p}$ be a place above a prime $p \ne \ell$. On $I_{\bar{p}} \cap \ker\gamma_{-2}$ the map z is a homomorphism (as $\gamma_{-2} = 1$ there), and $N_{\bar{p}} \subseteq I_{\bar{p}} \cap \ker\gamma_{-2}$ is the kernel of $g \mapsto (\eta_K(g), \bar{\Psi}(g)) \in \{\pm1\} \times \mathbb{F}_{\mathcal{O}}^\times$ on it, a normal subgroup of index dividing $2(\#\mathbb{F}_{\mathcal{O}} - 1)$, prime to ℓ. Since z vanishes on $N_{\bar{p}}$, it induces a homomorphism from a finite group of order prime to ℓ to the additive group of $\mathbb{F}_{\mathcal{O}}$, of exponent ℓ; so $z = 0$ on $I_{\bar{p}} \cap \ker\gamma_{-2}$. At the fixed place $\bar{\ell}$, $z(g) = \gamma_{-2}(g)y - y$ on D_ℓ, which vanishes on $I_\ell \cap \ker\gamma_{-2}$. At another place $g_0\bar{\ell}$ ($g_0 \in G_{\mathbb{Q}}$), $I_{g_0\bar{\ell}} = g_0 I_\ell g_0^{-1}$, and for $h \in I_\ell \cap \ker\gamma_{-2}$ the cocycle relation gives $z(g_0 h g_0^{-1}) = \gamma_{-2}(g_0)z(h) = 0$ (expand $z(g_0 \cdot hg_0^{-1})$ and use $z(g_0^{-1}) = -\gamma_{-2}(g_0)^{-1}z(g_0)$). So $z(g) = 0$ for every place $\bar{p}$ above a prime and every $g \in I_{\bar{p}}$ with $\gamma_{-2}(g) = 1$.

Step 6: conclusion. By Steps 4 and 5, Lemma 44.7 applies to χ_{H}, the field $k = \mathbb{F}_{\mathcal{O}}$ (which contains the values of $\gamma_{-2} = \bar{\chi}_{\mathrm{H}}$) and z: there is $\mu \in \mathbb{F}_{\mathcal{O}}$ with $z(g) = \gamma_{-2}(g)\mu - \mu$ for all g. So every element of Z_{-2} is a coboundary, and $\mathrm{Sel}_{-2}(\chi, e) = 0$. □

Lemma 46.16. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$. For every integer j with $-2 \le j \le 2$, the spaces $H^1_{\mathcal{L}}(\mathfrak{g}_j)$ and $H^2_{\mathcal{L}}(\mathfrak{g}_j)$*

are finite-dimensional over $\mathbb{F}_{\mathcal{O}}$ *and*

$$\dim H^1_{\mathcal{L}}(\mathfrak{g}_j) - \dim H^2_{\mathcal{L}}(\mathfrak{g}_j) = d(j), \qquad d(-2)=0, \quad d(-1)=-1, \quad d(0)=0, \quad d(1)=d(2)=1.$$

Proof. Write $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, $q = \#\mathbb{F}$ (a power of ℓ), $\dim = \dim_{\mathbb{F}}$, and fix $j \in \{-2,\dots,2\}$. The entries of weight j are $(a, a+j)$, so $\dim \mathfrak{g}_j = 3 - |j|$. Finite $\mathbb{F}$-vector spaces X have $\#X = q^{\dim X}$, so the cardinality formulas of Theorem B.29 (with $r=1$) can be written additively in dimensions.

Step 1: the action. If $g \in G_K$ then $\bar{r}_0(g) = (\mathrm{T}(\bar{\Psi}(g)), 1)$ and $\mathrm{T}(t)X\mathrm{T}(t)^{-1}$ has entries $t^{b-a}X_{ab}$, so g acts on $\mathfrak{g}_j$ by the scalar $\bar{\Psi}(g)^j$. As $\Psi(c) = 1$, $\bar{r}_0(c) = (1,-1)$ and c acts by $X \mapsto -J\,{}^tXJ$, whose (a,b) entry is $-X_{4-b,4-a}$, of the same weight; every $g \notin G_K$ is $g'c$ with $g' \in G_K$. So $G_{\mathbb{Q}}$ preserves $\mathfrak{g}_j$, and the kernel of the action contains $G_K \cap \ker\bar{\Psi}$, whose fixed field lies in $\mathbb{Q}(i, \mu_{\ell N})$ ($\Psi = \chi\chi_{\mathrm{cyc}}^e$) and is unramified outside $2\ell N$; since $2 \mid N$ (Dy2), the action factors through G_S. The elements of $N_{\bar{p}}$ act trivially ($\eta_K = 1$, $\bar{\Psi} = 1$ there), and so do those of I_ℓ on $\mathfrak{g}_0$ (ℓ is odd, hence unramified in K, so $I_\ell \subseteq G_K$).

Hence $\dim \mathfrak{g}_j^{c=1}$ is $0,1,1,1,0$ for $j = -2,-1,0,1,2$.

Step 2: invariants. Let $s \in D_\ell \cap G_K$ be as in Lemma 46.9 for $p=\ell$: $\bar{\Psi}(s)^2 \neq 1$ and s fixes μ_ℓ. If $j \neq 0$, s acts on $\mathfrak{g}_j$ by $a = \bar{\Psi}(s)^j \neq 1$ ($a=1$ would give $\bar{\Psi}(s)^2 = 1$); an s-invariant $f \in \mathfrak{g}_j^\vee$ satisfies $f(sX) = s(f(X)) = f(X)$ (s fixes μ_ℓ), i.e. f vanishes on $(a-1)\mathfrak{g}_j = \mathfrak{g}_j$; so

$$H^0(D_\ell, \mathfrak{g}_j) = H^0(G_S, \mathfrak{g}_j) = 0, \qquad H^0(D_\ell, \mathfrak{g}_j^\vee) = H^0(G_S, \mathfrak{g}_j^\vee) = 0 \qquad (j \neq 0).$$

If $j = 0$, then G_K acts trivially on $\mathfrak{g}_0$, so $H^0(G_S, \mathfrak{g}_0) = \mathfrak{g}_0^{c=1}$, of dimension 1. Moreover $K \cap \mathbb{Q}(\mu_\ell) = \mathbb{Q}$ ($\mathbb{Q}(\mu_\ell)$ is unramified at 2, K is not), so some $\tau \in G_K$ acts non-trivially on μ_ℓ, hence fixes only $1 \in \mu_\ell$; as τ acts trivially on $\mathfrak{g}_0$, a τ-invariant $f \in \mathfrak{g}_0^\vee$ has $f(X) = \tau(f(X))$, so $f = 0$ and $H^0(G_S, \mathfrak{g}_0^\vee) = 0$.

Step 3: the local conditions. For a prime $p \in S$, $p \neq \ell$, $N_{\bar{p}}$ is the kernel of $I_p \to \{\pm 1\} \times \mathbb{F}^\times$, $g \mapsto (\eta_K(g), \bar{\Psi}(g))$, hence normal in D_p, of index in I_p dividing $2(q-1)$, prime to ℓ, and it acts trivially (Step 1); by Lemma 43.35, item 1, $L_{p,j} \supseteq B^1(D_p, \mathfrak{g}_j)$ and $\dim L_{p,j} = \dim \mathfrak{g}_j$. For $p = \ell$ and $j = 0$ the same item with $N' = I_\ell$ (Step 1) gives $L_{\ell,0} \supseteq B^1$ and $\dim L_{\ell,0} = \dim \mathfrak{g}_0$. If $j < 0$, $L_{\ell,j} = B^1(D_\ell, \mathfrak{g}_j)$ has dimension $\dim \mathfrak{g}_j - \dim H^0(D_\ell, \mathfrak{g}_j) = \dim \mathfrak{g}_j$ (Step 2). If $j > 0$, $L_{\ell,j} = Z^1(D_\ell, \mathfrak{g}_j)$ has dimension $h^1 + \dim \mathfrak{g}_j - h^0$ with $h^i = \dim H^i(D_\ell, \mathfrak{g}_j)$; by Theorem B.29, item 2, $h^2 = \dim H^0(D_\ell, \mathfrak{g}_j^\vee) = 0$, by item 4 ($[\mathbb{Q}_\ell : \mathbb{Q}_\ell] = 1$) $h^1 = h^0 + h^2 + \dim \mathfrak{g}_j$, and $h^0 = 0$ (Step 2); so $\dim L_{\ell,j} = 2\dim \mathfrak{g}_j$. In all cases $L_{p,j} \supseteq B^1(D_p, \mathfrak{g}_j)$ and

$$\sum_{p \in S \text{ prime}} (\dim L_{p,j} - \dim \mathfrak{g}_j) = \begin{cases} \dim \mathfrak{g}_j & j > 0, \\ 0 & j \leq 0. \end{cases} \tag{46.2}$$

Step 4: the count. By Steps 2 and 3 the hypotheses of Lemma 43.35, item 2, hold for $M = \mathfrak{g}_j$ and $(L_{p,j})_p$ (ℓ is odd by (Dy1)). So $H^1_{\mathcal{L}}(\mathfrak{g}_j)$ and $H^2_{\mathcal{L}}(\mathfrak{g}_j)$ are finite-dimensional and, by (46.2),

$$\dim H^1_{\mathcal{L}}(\mathfrak{g}_j) - \dim H^2_{\mathcal{L}}(\mathfrak{g}_j) = \dim H^0(G_S, \mathfrak{g}_j) - \dim \mathfrak{g}_j^{c=1} + \begin{cases} \dim \mathfrak{g}_j & j > 0, \\ 0 & j \leq 0. \end{cases}$$

By Steps 1 and 2: for $j=-2$, $0-0+0=0$; $j=-1$, $0-1+0=-1$; $j=0$, $1-1+0=0$; $j=1$, $0-1+2=1$; $j=2$, $0-0+1=1$. This is $d(j)$. □

Lemma 46.17. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, $g_\star$ a marking and (R, r) a dyadic Eisenstein deformation ring for $g_\star$. Assume that* $\mathrm{Sel}_{-2}(\chi, e) = 0$. *Then every two-signed point of (R, r) is of type I, and for points φ, φ' of (R, r) with $\rho_{\varphi'} \cong \rho_\varphi$ there is $t \in \overline{\mathbb{Q}}_\ell^\times$ with $r_{\varphi'} = \mathrm{T}(t) \cdot r_\varphi$.*

Proof. Let $\mathfrak{m}_R$ be the maximal ideal of R, $K_0 = \mathcal{O}[\ell^{-1}]$ and $\mathbb{F}_{\mathcal{O}}$ the residue field of $\mathcal{O}$.

Step 0: points are integral over a finite extension. Let φ be a point and $y_1, \dots, y_n$ generators of $\mathfrak{m}_R$. Let $L = K_0(\varphi(y_1), \dots, \varphi(y_n))$, a finite extension of K_0, with ring of integers $\mathcal{O}_L$ (a discrete valuation ring, closed in $\overline{\mathbb{Q}}_\ell$) and maximal ideal $\mathfrak{m}_L$, and put $\varrho = \max_i |\varphi(y_i)|_\ell < 1$. Since $R/\mathfrak{m}_R = \mathbb{F}_{\mathcal{O}}$, $R = \mathcal{O} + \mathfrak{m}_R$, so $|\varphi(f)|_\ell \le 1$ for every $f \in R$; and by induction $R = \sum_{|\alpha|<k} \mathcal{O} y^\alpha + \sum_{|\alpha|=k} y^\alpha R$ for every $k \ge 1$ (multi-indices α). Hence every $\varphi(f)$ lies within ϱ^k of $\mathcal{O}[\varphi(y_1), \dots, \varphi(y_n)] \subseteq \mathcal{O}_L$ for every k, so $\varphi(R) \subseteq \mathcal{O}_L$, and $|\varphi(f)|_\ell \le \varrho^k$ for $f \in \mathfrak{m}_R^k$. Consequently $P_g := \underline{r_\varphi}(g)$ has entries in $\mathcal{O}_L$, reduces modulo $\mathfrak{m}_L$ to $\mathrm{T}(\bar\Psi(g))$ (the entries of $\underline{r}(g) - \underline{r_0}(g)$ lie in $\mathfrak{m}_R$), so $\det P_g \in \mathcal{O}_L^\times$, and $g \mapsto P_g$ is continuous into $\mathrm{M}_3(\mathcal{O}_L)$ (the entries of $\underline{r}$ are $\mathfrak{m}_R$-adically continuous).

Step 1: r_φ satisfies (Ns1)–(Ns3) of Lemma 42.4 for K, c, $g_\star$, and $\delta_\star(r_\varphi)+\delta_\star(r_{\varphi'}) \neq 0$ for any two points. Applying a point ϕ to the entries of GL-parts is a homomorphism $\mathcal{G}_3(R) \to \mathcal{G}_3(\overline{\mathbb{Q}}_\ell)$, so r_ϕ is a homomorphism; its signs are those of the lift r, i.e. η_K; and (Ed1) gives $\underline{r_\phi}(c) = 1$ and the vanishing of the entries $(1,2)$, $(2,1)$ of $\underline{r_\phi}(g_\star) - \underline{r_\phi}(c g_\star^{-1} c)$. So (Ns1), (Ns2) hold, and $g_\star \in G_K$ as $\eta_K(g_\star) = 1$. With $u = \Psi(g_\star) - \overline{\Psi(g_\star)}^{-1} = \overline{\Psi(g_\star)}^{-1}(\Psi(g_\star)^2 - 1) \in \mathcal{O}^\times$ (marking), $y = \underline{r}(g_\star)_{11} - \underline{r}(g_\star)_{33} - u \in \mathfrak{m}_R$, so $\delta_\star(r_\phi) = u + \phi(y)$ with $|u|_\ell = 1 > |\phi(y)|_\ell$, whence $|\delta_\star(r_\phi)|_\ell = 1$; and $\delta_\star(r_\varphi) + \delta_\star(r_{\varphi'}) = 2u + \varphi(y) + \varphi'(y)$ is non-zero because $|2u|_\ell = 1$ (ℓ odd, (Dy1)) and $|\varphi(y) + \varphi'(y)|_\ell < 1$.

Step 2: a two-signed point of type II would have a $(3,1)$-entry cocycle. Let φ be two-signed. By Lemma 42.4, item 1, and since φ is neither upper nor lower, φ is of type I or of type II. Suppose, for a contradiction, that it is of type II, and use the notation of Step 0. For $g \in G_K$ the entries of odd weight of P_g vanish, in particular $(P_g)_{32} = 0$. For $g \notin G_K$, $g = (gc)c$ and $r_\varphi(g) = r_\varphi(gc)(1,-1) = (P_{gc}, -1)$, so $P_g = P_{gc}$ and again $(P_g)_{32} = 0$. As φ is not upper, some entry of negative weight of some P_g, $g \in G_K$, is non-zero; the entries $(2,1)$, $(3,2)$ vanish, so $(P_g)_{31} \neq 0$ for some g. The valuations of the non-zero $(P_g)_{31}$ lie in a discrete subset of $[0,\infty)$; choose g_0 where the valuation is minimal, put $x = (P_{g_0})_{31} \neq 0$ and $b(g) = (P_g)_{31}/x \in \mathcal{O}_L$. Then $b(g_0) = 1$ and $b(c) = 0$ ($P_c = 1$). Let k_L be the residue field of $\mathcal{O}_L$ and $f \mapsto \bar f$ the reduction, and put

$$z(g) = \bar b(g)\, \bar\Psi(g)^{-1} \in k_L \qquad (g \in G_{\mathbb{Q}}).$$

z is locally constant: by continuity of $g \mapsto P_g$, $g \mapsto x b(g)$ modulo $x\mathfrak{m}_L$ is locally constant, and $x b(g) \equiv x b(g') \pmod{x\mathfrak{m}_L}$ if and only if $\bar b(g) = \bar b(g')$; $\bar\Psi$ is locally constant. $z(gh) = z(g) + \gamma_{-2}(g) z(h)$: for matrices X, Y with $X_{32} = 0$, $(XY)_{31} = X_{31}Y_{11} + X_{33}Y_{31}$. If $\eta_K(g) = 1$, $P_{gh} = P_g P_h$, so $b(gh) = b(g)(P_h)_{11} + (P_g)_{33} b(h)$; reducing, $\bar b(gh) = \bar b(g)\bar\Psi(h) + \bar\Psi(g)^{-1}\bar b(h)$, and multiplying by $\bar\Psi(gh)^{-1}$ gives $z(gh) = z(g) + \bar\Psi(g)^{-2} z(h)$, where $\bar\Psi(g)^{-2} = \gamma_{-2}(g)$. If $\eta_K(g) = -1$, $P_{gh} = P_g\, Y$ with $Y = J\, {}^t(P_h^{-1}) J$, so $Y_{11} = (P_h^{-1})_{33} = (\det P_h)^{-1}((P_h)_{11}(P_h)_{22} - (P_h)_{12}(P_h)_{21})$ and $Y_{31} = (P_h^{-1})_{31} = (\det P_h)^{-1}((P_h)_{21}(P_h)_{32} - (P_h)_{22}(P_h)_{31}) = -(\det P_h)^{-1}(P_h)_{22}\, x\, b(h)$ (cofactors). Hence $b(gh) = b(g) Y_{11} - (P_g)_{33}(\det P_h)^{-1}(P_h)_{22} b(h)$, and reducing ($\bar Y_{11} = \bar\Psi(h)$, $\overline{\det P_h} = 1$, $\overline{(P_h)_{22}} = 1$) $\bar b(gh) = \bar b(g)\bar\Psi(h) - \bar\Psi(g)^{-1}\bar b(h)$, i.e. $z(gh) = z(g) - \bar\Psi(g)^{-2} z(h) = z(g) + \gamma_{-2}(g) z(h)$.

Step 3: the shape at the primes of $T_\ell(\chi)$. Let $p \in T_\ell(\chi)$. We show that there is $w_p \in k_L$ with $z(g) = -w_p(\gamma_{-2}(g) - 1)$ for all $g \in D_p \cap G_K$. By (Dd1) if $p = \ell$ and (Dd2) if $p \in S_\ell(\chi)$ there is $h \in \mathrm{GL}_3(R)$, $h \equiv 1 \pmod{\mathfrak{m}_R}$, such that $h^{-1}\underline{r}(g) h$ is upper triangular for $g \in D_p \cap G_K$. Put $H = \varphi(h) \in \mathrm{GL}_3(\mathcal{O}_L)$, $H \equiv 1 \pmod{\mathfrak{m}_L}$, and $U_g = H^{-1} P_g H$, upper triangular. Comparing the entries $(1,1)$ and $(3,1)$ of $P_g H = H U_g$ and using $(P_g)_{32} = 0$: $(P_g H)_{11} = H_{11}(U_g)_{11}$ and $x b(g) H_{11} + (P_g)_{33} H_{31} = H_{31}(U_g)_{11}$. Multiplying the second by H_{11} and inserting the first,

$$x\, b(g)\, H_{11}^2 = H_{31} V_g, \qquad V_g = (P_g)_{11}H_{11} + (P_g)_{12}H_{21} + (P_g)_{13}H_{31} - (P_g)_{33}H_{11},$$

with $\bar V_g = \bar\Psi(g) - \bar\Psi(g)^{-1}$. By Lemma 46.9 choose $s_p \in D_p \cap G_K$ with $\bar\Psi(s_p)^2 \neq 1$; then

$\bar{V}_{s_p} \neq 0$, $V_{s_p} \in \mathcal{O}_L^\times$ and $H_{31} = xb(s_p)H_{11}^2 V_{s_p}^{-1}$. Substituting and cancelling $xH_{11}^2 \neq 0$ (H_{11} is a unit), $b(g) = b(s_p)V_g V_{s_p}^{-1}$, so $\bar{b}(g) = w_p(\bar{\Psi}(g) - \bar{\Psi}(g)^{-1})$ with $w_p = \bar{b}(s_p)\bar{V}_{s_p}^{-1}$, and $z(g) = w_p(1 - \bar{\Psi}(g)^{-2}) = -w_p(\gamma_{-2}(g) - 1)$ ($\eta_K(g) = 1$).

Step 4: z satisfies the local conditions of Z_{-2}. (a) Let $\bar{v}$ be a place above a prime $p \notin T_\ell(\chi)$ and $g \in N_{\bar{v}}$. By (Dd3), $r(g) = 1$, so $P_g = 1$, $b(g) = 0$ and $z(g) = 0$. (b) Let $p \in S_\ell(\chi)$. As $N_{\bar{p}} \subseteq I_{\bar{p}} \cap G_K \subseteq D_p \cap G_K$ and $\bar{\Psi} = 1$ on $N_{\bar{p}}$, Step 3 gives $z = 0$ on $N_{\bar{p}}$. Another place above p is $g_1\bar{p}$ with $g_1 \in G_\mathbb{Q}$, and $N_{g_1\bar{p}} = g_1 N_{\bar{p}} g_1^{-1}$ ($I_{g_1\bar{p}} = g_1 I_{\bar{p}} g_1^{-1}$, and η_K, $\bar{\Psi}$ are characters). For $h \in N_{\bar{p}}$, $\gamma_{-2}(h) = 1$ and $z(h) = 0$, so the cocycle relation gives $z(g_1 h g_1^{-1}) = z(g_1) + \gamma_{-2}(g_1)z(g_1^{-1}) = z(1) = 0$. (c) At ℓ: the map $a(g) = z(g) + w_\ell(\gamma_{-2}(g) - 1)$ on D_ℓ satisfies $a(gh) = a(g) + \gamma_{-2}(g)a(h)$ and vanishes on the normal subgroup $D_\ell \cap G_K$, of index at most 2, by Step 3. If $d \in D_\ell \setminus G_K$ and $h \in D_\ell \cap G_K$, then $a(hd) = \gamma_{-2}(h)a(d)$, while $hd = d(d^{-1}hd)$ gives $a(hd) = a(d)$; taking $h = s_\ell$, with $\gamma_{-2}(s_\ell) = \bar{\Psi}(s_\ell)^{-2} \neq 1$, gives $a(d) = 0$. So $a = 0$ on D_ℓ: $z(g) = \gamma_{-2}(g)y - y$ on D_ℓ with $y = -w_\ell$.

Step 5: contradiction. Choose an $\mathbb{F}_\mathcal{O}$-linear map $\lambda_0 \colon k_L \to \mathbb{F}_\mathcal{O}$ with $\lambda_0(1) = 1$. Since γ_{-2} takes values in $\mathbb{F}_\mathcal{O}^\times$, $\lambda_0 \circ z$ is locally constant, satisfies the cocycle relation for γ_{-2}, vanishes on $N_{\bar{v}}$ for every place $\bar{v}$ above a prime other than ℓ (Step 4 (a), (b); $T_\ell(\chi) \setminus \{\ell\} = S_\ell(\chi)$), and equals $\gamma_{-2}(g)\lambda_0(y) - \lambda_0(y)$ on D_ℓ: so $\lambda_0 \circ z \in Z_{-2}$. As $\mathrm{Sel}_{-2}(\chi, e) = 0$, there is $\mu \in \mathbb{F}_\mathcal{O}$ with $\lambda_0(z(g)) = \gamma_{-2}(g)\mu - \mu$ for all g. At $g = c$: $z(c) = 0$, and $\gamma_{-2}(c) = \eta_K(c)\bar{\Psi}(c)^{-2} = -1$ because $\Psi(c) = 1$; so $2\mu = 0$, $\mu = 0$ (ℓ odd) and $\lambda_0 \circ z = 0$. But $\lambda_0(z(g_0)) = \bar{\Psi}(g_0)^{-1}\lambda_0(1) \neq 0$. Hence φ is of type I.

Step 6: torus conjugacy. For points φ, φ' with $\rho_{\varphi'} \cong \rho_\varphi$, Step 1 and Lemma 42.4, item 2, give $t \in \overline{\mathbb{Q}}_\ell^\times$ with $r_{\varphi'}(g) = (\mathrm{T}(t), 1)r_\varphi(g)(\mathrm{T}(t), 1)^{-1}$ for all g, i.e. $r_{\varphi'} = \mathrm{T}(t) \cdot r_\varphi$. □

Theorem 46.18. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, $g_\star$ a marking, (R, r) a dyadic Eisenstein deformation ring for $g_\star$ such that its weight-zero subalgebra R^0 is a finitely generated $\mathcal{O}$-module, and $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism. Every point φ of (R, r) that is Lie-irreducible automorphic for ι and at which ρ_φ is generic ordinary of level N and weights $(e, 0, -e)$ is rigid.*

Proof. Let φ be a point that is Lie-irreducible automorphic for ι with ρ_φ generic ordinary of level N and weights $(e, 0, -e)$, and y a dyadic tangent cocycle at φ. We show that y is a Bloch–Kato cocycle of $\mathrm{ad}\, r_\varphi$; the finiteness of R^0 is not used. The decomposition and inertia groups of Definition 22.2 at a finite place v of $\mathbb{Q}$ are the groups $D_{\bar{v}} \supseteq I_{\bar{v}}$ of the places $\bar{v}$ of $\overline{\mathbb{Q}}$ above v. We write $g \cdot X$ for the action of $\mathrm{ad}\, r_\varphi$; on G_K it is $X \mapsto \rho_\varphi(g)X\rho_\varphi(g)^{-1}$.

Claim (C). Let I be a group acting linearly on a $\overline{\mathbb{Q}}_\ell$-vector space M, $I' \subseteq I$ a normal subgroup of finite index, and $z \colon I \to M$ a 1-cocycle whose restriction to I' is $g \mapsto g \cdot X - X$ for some $X \in M$. Then z is of this form on I. Indeed, $z' = z - (g \mapsto g \cdot X - X)$ vanishes on I'; so $z'(gh) = z'(g)$ and $z'(hg) = h \cdot z'(g)$ for $h \in I'$, and since $hg = g(g^{-1}hg)$ with $g^{-1}hg \in I'$, $z'(hg) = z'(g)$; hence z' takes values in $M^{I'}$ and factors through the finite group I/I'. Summing $z'(g\gamma) = z'(g) + g \cdot z'(\gamma)$ over representatives γ of I/I' and dividing by $s = [I : I']$ gives $z'(g) = X' - g \cdot X'$ with $X' = s^{-1}\sum_\gamma z'(\gamma)$.

Claim (D). If $y|_{I_{\bar{v}}}$ is a coboundary $g \mapsto g \cdot X - X$, then for every $\sigma \in G_\mathbb{Q}$, $y|_{I_{\sigma\bar{v}}}$ is a coboundary, as $I_{\sigma\bar{v}} = \sigma I_{\bar{v}} \sigma^{-1}$ and, for $g \in I_{\bar{v}}$ and $h = \sigma g \sigma^{-1}$, the cocycle relation gives $y(h) = y(\sigma) + \sigma \cdot y(g) - h \cdot y(\sigma) = h \cdot X'' - X''$ with $X'' = \sigma \cdot X - y(\sigma)$.

Step 1: places $v = p \notin T_\ell(\chi)$. For every place $\bar{v}$ above p, $N_{\bar{v}}$ is the kernel of the homomorphism $g \mapsto (\eta_K(g), \bar{\Psi}(g))$ on $I_{\bar{v}}$ with finite target, hence normal of finite index, and y vanishes on it by (Ty1). By (C), $y|_{I_{\bar{v}}}$ is a coboundary.

Step 2: places $v = p \in S_\ell(\chi)$. By (Ty3) there is $h = h_0 + \epsilon h_1 \in \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell[\epsilon])$ such that

$h^{-1}r_y(g)h = \Delta(g) := \mathrm{diag}(\psi_1(g),\psi_2(g),\psi_3(g))$ for $g \in I_p \cap G_K$ (the GL-part of r_y is meant). As $h^{-1} = h_0^{-1} - \epsilon h_0^{-1}h_1h_0^{-1}$, with $Z = h_1h_0^{-1}$,

$$r_y(g) = h\Delta(g)h^{-1} = h_0\Delta(g)h_0^{-1} + \epsilon(Zh_0\Delta(g)h_0^{-1} - h_0\Delta(g)h_0^{-1}Z).$$

Reducing modulo ϵ, $\rho_\varphi(g) = h_0\Delta(g)h_0^{-1}$; comparing with $r_y(g) = (1+\epsilon y(g))\rho_\varphi(g)$, $y(g)\rho_\varphi(g) = Z\rho_\varphi(g) - \rho_\varphi(g)Z$, i.e. $y(g) = g\cdot(-Z) - (-Z)$ for $g \in I_p \cap G_K$. The subgroup $I_p \cap G_K$ is the kernel of $\eta_K|_{I_p}$, normal of index at most 2; by (C), $y|_{I_p}$ is a coboundary, and by (D) so is $y|_{I_{\bar v}}$ for every place $\bar v$ above p.

Step 3: the place ℓ and the archimedean place. By Lemma 46.19 (its genericity hypothesis is assumed), the class of $y|_D$ lies in $H^1_f(D, \mathrm{ad}\, r_\varphi)$ for every decomposition group D at ℓ. An archimedean decomposition group D has order at most 2, and by (C) with $I' = 1$ every cocycle on it is a coboundary, so the class of $y|_D$ is $0 = H^1_f(D, \mathrm{ad}\, r_\varphi)$.

Step 4: conclusion. By Steps 1–3, y (continuous by definition) is a Bloch–Kato cocycle of $\mathrm{ad}\, r_\varphi$. The homomorphism r_φ has sign η_K, $K = \mathbb{Q}(i)$, and continuous GL-part restricting to ρ_φ on G_K; as φ is Lie-irreducible automorphic, ρ_φ is irreducible on every open subgroup of G_K and $\rho_\varphi \otimes \chi_{\mathrm{cyc}}^{-1}$ is automorphic of conjugate self-dual type. The module $\mathrm{ad}\, r_\varphi$ is the module $\mathrm{ad}\, r_\varphi$ of Proposition 42.6 (same formulas), so $H^1_f(\mathbb{Q}, \mathrm{ad}\, r_\varphi) = 0$ by that proposition, and $y(g) = g\cdot X - X$ for some $X \in \mathrm{ad}\, r_\varphi$. Hence φ is rigid. □

Lemma 46.19. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, $g_\star$ a marking, (R, r) a dyadic Eisenstein deformation ring for $g_\star$, φ a point of (R, r) such that ρ_φ is generic ordinary of level N and weights $(e, 0, -e)$, and y a dyadic tangent cocycle at φ. Then for every decomposition group D of $G_\mathbb{Q}$ at ℓ, the class of $y|_D$ lies in $H^1_f(D, \mathrm{ad}\, r_\varphi)$.*

Proof. Write $K = \mathbb{Q}(i)$, $r_\varphi(g) = (P_g, \varsigma_g)$, and $g\cdot X$ for the action on $\mathrm{ad}\, r_\varphi$. As ℓ is odd (Dy1), it is unramified in K, so η_K is trivial on I_ℓ; put $D' = D_\ell \cap G_K$, a normal subgroup of D_ℓ of index 1 or 2 containing I_ℓ. Fix an embedding $j_0 \colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ inducing $\bar\ell$; then $D_\ell = D_{j_0}$ in the notation of Definition 22.2, and D' is identified with the absolute Galois group of the closure M of $j_0(K)$, with inertia subgroup I_ℓ.

Step 1: finite coefficients. By Lemma 42.8, item 1, there is a finite extension E_0 of $\mathbb{Q}_\ell$ with $\rho_\varphi(G_K) \subseteq \mathrm{GL}_3(E_0)$; by (Ed1), $P_c = 1$, so $P_{cg} = \theta(\rho_\varphi(g)) \in \mathrm{GL}_3(E_0)$ for $g \in G_K$, and $\mathrm{M}_3(E_0)$ is stable under $G_\mathbb{Q}$. By Lemma 42.8, item 3 (with $\Gamma = G_\mathbb{Q}$), y takes values in $\mathrm{M}_3(E_1)$ for a finite extension $E_1 \supseteq E_0$. Let $h = h_0 + \epsilon h_1$ be as in (Ty2), and let $E \supseteq E_1$ be finite containing the entries of h_0 and h_1. Put $V = \mathrm{M}_3(E)$ with the action of $\mathrm{ad}\, r_\varphi$; so $\mathrm{ad}\, r_\varphi = V \otimes_E \overline{\mathbb{Q}}_\ell$, y is a cocycle with values in V, and it suffices to find $Y \in V \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $y(g)\otimes 1 = gY - Y$ on D.

Step 2: triangular form on D'. As h is invertible, so is h_0. Reducing (Ty2) modulo ϵ, $\rho'(g) := h_0^{-1}\rho_\varphi(g)h_0 \in \mathrm{GL}_3(E)$ is upper triangular for $g \in D'$, with diagonal entries $\kappa_1(g), \kappa_2(g), \kappa_3(g)$, where the $\kappa_a \colon D' \to E^\times$ are continuous characters with $\kappa_a = \psi_a = \chi^{2-a}\chi_{\mathrm{cyc}}^{e(2-a)}$ on I_ℓ. As χ has finite order, $\kappa_a\chi_{\mathrm{cyc}}^{-m_a}$ has finite image on I_ℓ for $(m_1, m_2, m_3) = (e, 0, -e)$, and $m_1 > m_2 > m_3$ since $e \geq 1$.

Step 3: $\kappa_a \neq \kappa_b\chi_{\mathrm{cyc}}$ on D' for all a, b. Let v be the place of K below $\bar\ell$. As ρ_φ is generic ordinary, there are a place $\bar v$ of $\overline{\mathbb{Q}}$ above v and characters $\kappa'_1, \kappa'_2, \kappa'_3$ of $D_{\bar v} \subseteq G_K$ such that ρ_φ is triangular on $D_{\bar v}$ with diagonal (κ'_i) and $\kappa'_i \neq \kappa'_j\chi_{\mathrm{cyc}}$ for all i, j. Since G_K acts transitively on the places above v, $\bar v = \tau\bar\ell$ and $D_{\bar v} = \tau D'\tau^{-1}$ for some $\tau \in G_K$. For $g \in D'$, $\rho_\varphi(\tau g\tau^{-1}) = \rho_\varphi(\tau)\rho_\varphi(g)\rho_\varphi(\tau)^{-1}$, so $\rho_\varphi|_{D'}$ is also triangular with diagonal $(\kappa'_i(\tau\cdot\tau^{-1}))$. The diagonal characters of a triangular representation are its Jordan–Hölder factors, so $\{\kappa_a\} = \{\kappa'_i(\tau\cdot\tau^{-1})\}$ as multisets; as $\chi_{\mathrm{cyc}}(\tau g\tau^{-1}) = \chi_{\mathrm{cyc}}(g)$, the claim follows.

Step 4: $y|_{D'}$ is crystalline. Put $Z' = h_0^{-1}h_1$, $y'(g) = h_0^{-1}y(g)h_0$ and $u(g) = y'(g) + \rho'(g)Z'\rho'(g)^{-1} - Z'$ for $g \in D'$. The GL-part of $r_y(g)$ is $(1+\epsilon y(g))\rho_\varphi(g)$, and $h^{-1} = h_0^{-1} - \epsilon h_0^{-1}h_1h_0^{-1}$, so

$$h^{-1}(1+\epsilon y(g))\rho_\varphi(g)h = \rho'(g) + \epsilon(y'(g)\rho'(g) + \rho'(g)Z' - Z'\rho'(g)) = \rho'(g) + \epsilon\, u(g)\rho'(g).$$

By (Ty2) this matrix is upper triangular for $g \in D'$, with diagonal $\psi_a(g) \in \overline{\mathbb{Q}}_\ell$ for $g \in I_\ell$. Hence $u(g)\rho'(g)$, and so $u(g) = (u(g)\rho'(g))\rho'(g)^{-1}$, is upper triangular, and the diagonal of $u(g)\rho'(g)$, which is the diagonal of $u(g)$ times that of $\rho'(g)$, vanishes for $g \in I_\ell$. Conjugation $X \mapsto h_0^{-1}Xh_0$ is an isomorphism of $E[D']$-modules from $V|_{D'}$ to $\operatorname{ad}\rho'$ (notation of Lemma 44.14) carrying y to y', and $u = y' + (g \mapsto g\cdot Z' - Z')$ is a cocycle of $\operatorname{ad}\rho'$ with values in the upper triangular matrices whose diagonal vanishes on I_ℓ. By Steps 2 and 3, Lemma 44.14 applies (with M, $G_M = D'$, $I_M = I_\ell$, $n = 3$) and gives Y' with $u(g)\otimes 1 = gY' - Y'$; then $y'(g)\otimes 1 = g(Y' - Z'\otimes 1) - (Y' - Z'\otimes 1)$, and conjugating back by h_0 gives $Y_1 \in V \otimes_{\mathbb{Q}_\ell} B_{\mathrm{cris}}$ with $y(g)\otimes 1 = gY_1 - Y_1$ for $g \in D'$.

Step 5: the group D_ℓ. If $D' \neq D_\ell$, let $z(g) = y(g)\otimes 1 - (gY_1 - Y_1)$ for $g \in D_\ell$, a cocycle with values in $V \otimes B_{\mathrm{cris}}$ vanishing on D'. For $h' \in D'$, $z(gh') = z(g)$ and $z(h'g) = h'z(g)$, while $h'g = g(g^{-1}h'g)$ with $g^{-1}h'g \in D'$ gives $z(h'g) = z(g)$; so z factors through $D_\ell/D' = \{1, \gamma D'\}$. Summing $z(g\gamma') = z(g) + gz(\gamma')$ over $\gamma' \in \{1, \gamma\}$, and using that $g\gamma'$ runs over both cosets, $z(g) = Y_2 - gY_2$ with $Y_2 = \frac{1}{2}(z(1) + z(\gamma))$. Hence $y(g)\otimes 1 = gY_0 - Y_0$ on D_ℓ with $Y_0 = Y_1 - Y_2$ (and $Y_0 = Y_1$ if $D' = D_\ell$).

Step 6: every decomposition group at ℓ. Every embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ is $j = j_0 \circ \sigma$ for some $\sigma \in G_{\mathbb{Q}}$, and then $D_j = \sigma^{-1}D_\ell\sigma$, the element $g = \sigma^{-1}g_0\sigma$ ($g_0 \in D_\ell$) acting on B_{cris} as g_0 does. The cocycle relation gives $y(g) = \sigma^{-1}\cdot y(g_0) + g\cdot w - w$ with $w = \sigma^{-1}\cdot y(\sigma)$. Put $Y = (\sigma^{-1}\otimes 1)Y_0 + w\otimes 1$; since g acts on V as $\sigma^{-1}g_0\sigma$ and on B_{cris} as g_0, $g(\sigma^{-1}\otimes 1)Y_0 = (\sigma^{-1}\otimes 1)(g_0Y_0)$, so $gY - Y = (\sigma^{-1}\cdot y(g_0) + g\cdot w - w)\otimes 1 = y(g)\otimes 1$. So the class of $y|_{D_j}$ lies in $H^1_f(D_j, V)$, and in $H^1_f(D_j, \operatorname{ad} r_\varphi)$ by extension of scalars. □

Part IX
Automorphy of Eisenstein deformations

Part IX proves the three automorphy theorems that Part VIII uses: Theorem 47.1 for the unitary Eisenstein deformation rings, used in Proposition 42.2; Theorem 51.1 for the symplectic data, used in Proposition 44.3; and Theorem 47.16 for the dyadic rings, used in Proposition 46.2. Each has the same two parts. The subalgebra of weight zero of the deformation ring is finite over $\mathcal{O}$, which is condition (Gc2) of the criterion. And the points that matter are automorphic: those of type I are admissible of stable type, those of type II come from newforms of weight $2e + 1$, and in the symplectic case there are no endoscopic points and every irreducible point is automorphic. This identifies the two-signed points of the criterion with automorphic Galois representations, whose Frobenius traces Section 38 counts.

The method is $R = T$ for residually reducible representations, in the line of Skinner and Wiles [236] and Thorne [256], with Hida theory and patching. Every point of an Eisenstein deformation ring is a triangular lift of the Eisenstein representation. Over an auxiliary abelian CM field L whose Selmer ranks are small compared with its degree, these lifts become polarised ordinary lifts for which an $R = T$ theorem is proved. Its Hecke side comes from Hida theory for ordinary forms on a definite unitary group [105] and from patching in the manner of Taylor and Wiles [254], which give a modular ideal in the deformation ring. Modularity is then propagated from seeds, ordinary newforms congruent to Eisenstein series, to every component: the components through the reducible locus are connected, and the reducible locus is small compared with their dimension. Automorphy finally descends from L to K. The symplectic case follows the same plan in rank four, with auxiliary totally real towers and chains of residual pairs.

Section 47 deduces the unitary and dyadic theorems from the automorphy of triangular lifts, which Section 48 reduces to an auxiliary field. Section 49 proves $R = T$ over the auxiliary field from seeds, from propagation and from the Hecke realisation, which Section 50 constructs; the Hida theory and the patching behind it are in Appendix E. Sections 51–53 prove the symplectic theorem: Section 51 reduces it to essential test representations, Section 52 connects those to seeds by chains, and Section 53 supplies the Hecke side and the propagation in rank four.

47 Automorphy at the points of the Eisenstein deformation rings

This section proves Theorem 47.1 and Theorem 47.16, the automorphy theorems used in Proposition 42.2 and Proposition 46.2, from the automorphy of triangular lifts, Theorem 48.1, which is proved in Section 48. The points of the normalised ordinary and of the dyadic deformation rings are triangular lifts of the Eisenstein representation (Lemma 47.3, Lemma 47.4). Finiteness in weight zero, (Au1) and (Da1), comes from a test in characteristic ℓ (Proposition 47.2). If the triangular lifts to test rings of characteristic ℓ have the residual characteristic polynomials on a subgroup of finite index, which is clause (Tp1) of Theorem 48.1, they are upper or lower triangular with the residual diagonal (Lemma 47.5), so the products of weight zero are constant at the points of characteristic ℓ, and this makes R^0 finite (Proposition 47.6). Automorphy, (Au2) and (Da2), comes from clause (Tp2): an irreducible triangular lift is Lie-irreducible (Lemma 47.8), and its level, weights and genericity at ℓ are read off from its local shape (Lemma 47.9, Proposition 47.11). In weight one with trivial ℓ-component, crystallinity and genericity at ℓ need separate arguments (Proposition 47.13, Proposition 47.12). The endoscopic clause (Au3) comes from clause (Tp3), which makes the corner of a lift of type II modular.

Theorem 47.1. *Let (K, ℓ, e, N, χ) be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$ with $e_\mathcal{O} = 1$, let d_K be the discriminant of K, and let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism. There is a marking $g_\star$ such that for every normalised ordinary Eisenstein deformation ring (R, r) for $g_\star$:*

(Au1) *R^0 is a finitely generated $\mathcal{O}$-module;*

(Au2) *for every point φ of type I, ρ_φ is admissible of stable type for (ι, e, N) and irreducible on every open subgroup of G_K;*

(Au3) *for every point φ of type II, ρ_φ is admissible of endoscopic type for $(\iota, e, N|d_K|)$, and every continuous representation $W\colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ with $W(g) = \left(\rho_\varphi(g)_{ab}\right)_{a,b\in\{1,3\}}$ for all $g \in G_K$ and $\operatorname{tr} W(c) = 0$ satisfies the witness conditions over imaginary quadratic fields and the condition (Eo2) (for some normalised newform of weight $2e + 1$).*

Proof. Choice of the marking. By Theorem 48.1, case (a), there is a marking $g_\star$ with (Tp1)–(Tp3). Let (R, r) be a normalised ordinary Eisenstein deformation ring for $g_\star$.

(Au1). By (Tp1) the hypothesis of Proposition 47.2, case (a), holds for $g_\star$, so R^0 is a finitely generated $\mathcal{O}$-module.

Torus conjugates. Let φ be a point of (R, r). By Lemma 47.3 (which does not use (Au1)) there are an integral test ring A, $t \in \overline{\mathbb{Q}}_\ell^\times$ and a triangular lift r' to A for $g_\star$ with $r' = \mathrm{T}(t) \cdot r_\varphi$. Since $(\mathrm{T}(t), 1)(X, 1)(\mathrm{T}(t), 1)^{-1} = (\mathrm{T}(t)X\mathrm{T}(t)^{-1}, 1)$, we have $\rho_{r'}(g) = \mathrm{T}(t)\rho_\varphi(g)\mathrm{T}(t)^{-1}$ for $g \in G_K$: the entry (a, b) is multiplied by t^{b-a}. Being irreducible on a subgroup, the attached-to relation, automorphy of conjugate self-dual type, level and weights, triangularity and genericity on decomposition groups, the witness conditions, condition (Eo2) and the witness conditions over imaginary quadratic fields depend only on the isomorphism class of the representation; we use this without further comment.

(Au2). Let φ be of type I. Then $\rho_{r'} \cong \rho_\varphi$ is irreducible, so by Lemma 47.8, (Li1), $\rho_{r'}$ and hence ρ_φ is irreducible on every open subgroup of G_K. By (Tp2), $\rho_{r'} \otimes \varepsilon^{-1}$, hence $\rho_\varphi \otimes \varepsilon^{-1}$, is automorphic of conjugate self-dual type for ι and satisfies the witness conditions: this is (Sb2). (Sb1) is the definition of type I. For (Sb3), (Sb4): if $e \geq 2$ or $\chi_\ell \neq \mathbf{1}$, Proposition 47.11, (Cv1), applied to r' shows that $\rho_{r'}$, hence ρ_φ, has level N and weights $(e, 0, -e)$ and is generic ordinary of level N and weights $(e, 0, -e)$. If $e = 1$ and $\chi_\ell = \mathbf{1}$, then by Proposition 47.12 (applied with ι; its hypothesis, automorphy of $\rho_\varphi \otimes \varepsilon^{-1}$ of conjugate self-dual type, was just shown), ρ_φ, hence $\rho_{r'}$, is generic at every place $v \mid \ell$ of K, and (Cv1) applied to r' gives the same conclusion. So ρ_φ is admissible of stable type for (ι, e, N).

(Au3). Let φ be of type II. Conjugation by $\mathrm{T}(t)$ keeps the vanishing of the entries of odd weight, leaves the entry $(2, 2)$ unchanged and replaces the corner by its conjugate under $D = \operatorname{diag}(t, t^{-1})$; so r' is of type II, and $\rho_\varphi(g)_{22} = \rho_{r'}(g)_{22} = 1$ for $g \in G_K$ by Proposition 47.11, (Cv2), applied to r'. Let $W\colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be continuous, equal to the corner of ρ_φ on G_K, with $\operatorname{tr} W(c) = 0$. Then $W' = DWD^{-1}$ is continuous, equal to the corner of r' on G_K, and $\operatorname{tr} W'(c) = 0$; by (Tp3) W', hence W, satisfies (Eo2) and the witness conditions over imaginary quadratic fields. This is the second assertion of (Au3). By (Cv2) for r' there is a continuous W_0 equal to the corner of r' on G_K with $\operatorname{tr} W_0(c) = 0$; then $D^{-1}W_0D$ is continuous, equal to the corner of ρ_φ on G_K, with trace 0 at c: such a W exists. For such a W we check (Eo1)–(Eo4) with $M = N|d_K|$. (Eo1): $W|_{G_K}$ is the corner, irreducible by the definition of type II, so W is irreducible, and it satisfies the witness conditions over imaginary quadratic fields. (Eo2) was just shown. (Eo3): by (Cv2) for r', $W' = DWD^{-1}$ has level $N|d_K|$ and weights $(e, -e)$, and $\det W' = \eta_K$ is a character of level $|d_K|$; these properties pass to the conjugate W, and

$\det W = \det W'$. (Eo4): for $s \in G_K$ the entries $(1,2),(2,1),(2,3),(3,2)$ of $\rho_\varphi(s)$ have odd weight and vanish, and the entry $(2,2)$ is 1, so $\rho_\varphi(s)$ is the matrix of (Eo4) built from $W(s)$, with the identity as conjugating matrix. So ρ_φ is admissible of endoscopic type for $(\iota, e, N|d_K|)$. □

Proposition 47.2. *Let $\mathcal{O}$ be a coefficient ring and assume (a) (K,ℓ,e,N,χ) is a unitary Eisenstein datum with χ valued in $\mathcal{O}^\times$ and $e_\mathcal{O} = 1$, or (b) (ℓ,e,N,χ,o,m) is a dyadic Eisenstein datum with χ valued in $\mathcal{O}^\times$ and $K = \mathbb{Q}(i)$; form triangular lifts with K, $\mathcal{O}$ and $\Psi_{\chi,e}$. Let $g_\star$ be a marking, and let (R,r) be a normalised ordinary Eisenstein deformation ring for $g_\star$ in case (a), a dyadic Eisenstein deformation ring for $g_\star$ in case (b). Assume there is a subgroup U of finite index in $G_\mathbb{Q}$ such that for every test ring A of characteristic ℓ, every triangular lift r' to A for $g_\star$ and every $g \in U \cap G_K$, the characteristic polynomials of $\underline{r}'(g)$ and $\underline{r_0}(g)$ agree. Then R^0 is a finitely generated $\mathcal{O}$-module.*

Proof. We verify the hypothesis of Proposition 47.6, which then gives the claim. Let A be a test ring of characteristic ℓ whose structure map kills $\mathfrak{m}_\mathcal{O}$, $\varphi\colon R \to A$ a homomorphism of $\mathcal{O}$-algebras with $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_A$, and $y = \prod_k \underline{r}(g_k)_{a_k b_k}$ with $g_k \in G_\mathbb{Q}$ and $\sum_k \mathrm{wt}(a_k,b_k) = 0$; write $\bar{x}$ for the image in A of $x \in \mathcal{O}$.

Step 1: $r' = \varphi \circ r$ is a triangular lift to A for $g_\star$. In case (a) this is Lemma 47.3 (its general form), in case (b) Lemma 47.4.

Step 2: the shape of r' on G_K. For $g \in U \cap G_K$, the hypothesis and $\underline{r_0}(g) = \mathrm{T}(\Psi(g))$ give $\det(X - \underline{r}'(g)) = \prod_a (X - \overline{\psi_a(g)})$. The prime ℓ is odd ((Ue9) excludes $\ell = 2$ in case (a); (Dy1) in case (b)), and $\Psi(c) = 1$ ((Ue5), resp. (Dy3)). So Lemma 47.5 applies to r': either the entries of negative weight of $\underline{r}'(g)$ vanish for all $g \in G_K$, or those of positive weight do, and $\underline{r}'(g)_{aa} = \overline{\psi_a(g)}$ for $g \in G_K$.

Step 3: $\varphi(y)$ is constant. By (Ed1), $r(c) = (1,-1)$. For $g \notin G_K$, $gc \in G_K$, $r(gc) = (P,1)$ with $P = \underline{r}(gc)$, and $r(g) = r(gc)r(c) = (P,1)(1,-1) = (P,-1)$, so $\underline{r}(g) = \underline{r}(gc)$. Replacing each $g_k \notin G_K$ by $g_k c$ we may assume all $g_k \in G_K$; then $\varphi(y) = \prod_k \underline{r}'(g_k)_{a_k b_k}$. Suppose the entries of negative weight vanish on G_K (the other case is symmetric). If some factor has $\mathrm{wt}(a_k,b_k) \neq 0$, then, the weights summing to 0, some factor has negative weight and vanishes, so $\varphi(y) = 0 = \bar{0}$. Otherwise $a_k = b_k$ for all k and $\varphi(y) = \prod_k \overline{\psi_{a_k}(g_k)} = \overline{\prod_k \psi_{a_k}(g_k)}$. In both cases $\varphi(y)$ lies in the image of $\mathcal{O}$. □

Lemma 47.3. *Let (K,ℓ,e,N,χ) be a unitary Eisenstein datum with χ valued in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$ with $e_\mathcal{O} = 1$, let $g_\star$ be a marking and (R,r) a normalised ordinary Eisenstein deformation ring for $g_\star$. For every point φ of (R,r) there are an integral test ring A, an element $t \in \overline{\mathbb{Q}}_\ell^\times$ and a triangular lift r' to A for $g_\star$ such that $r'(g) = (\mathrm{T}(t) \cdot r_\varphi)(g)$ in $\mathcal{G}_3(\overline{\mathbb{Q}}_\ell)$ for all $g \in G_\mathbb{Q}$; one can take $t = 1$. More generally, for every test ring A (integral or of characteristic ℓ) and every homomorphism of $\mathcal{O}$-algebras $\varphi\colon R \to A$ with $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_A$, $\varphi \circ r$ is a triangular lift to A for $g_\star$.*

Proof. If φ is a point, (Ol1) gives an integral test ring A with $\varphi(R) \subseteq A$ and $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_A$, and with $t = 1$ the first assertion follows from the second, as $\varphi \circ r = r_\varphi = \mathrm{T}(1) \cdot r_\varphi$. So let A be a test ring and $\varphi\colon R \to A$ a homomorphism of $\mathcal{O}$-algebras with $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_A$, and put $r' = \varphi \circ r\colon G_\mathbb{Q} \to \mathcal{G}_3(A)$. We check (Tl1)–(Tl4) for $g_\star$.

(Tl1): r is a lift of $\bar{r}_0$, so its sign is η_K, the entries of $\underline{r}(g) - \underline{r_0}(g)$ lie in $\mathfrak{m}_R$ and are mapped into $\mathfrak{m}_A$; the entries of $\underline{r}$ are continuous for the $\mathfrak{m}_R$-adic topology, hence locally constant modulo $\mathfrak{m}_R^n$, and $\varphi(\mathfrak{m}_R^n) \subseteq \mathfrak{m}_A^n$. (Tl2) is the image of (Ed1). (Tl3): let $\bar{v} \nmid \ell$ and $g \in I_{\bar{v}}$ with $\eta_K(g) = 1$ and $\Psi(g) = 1$; then $\bar{r}_0(g) = (\mathrm{T}(\bar{\Psi}(g)),1) = 1$, so $r(g) = 1$ by (Ed2) and $r'(g) = 1$. (Tl4): by (Ed3), $r|_{D_\ell} = f \circ r_\ell^\square$ for a morphism f of $\mathrm{CNL}_\mathcal{O}$ with $f(I_\ell^{\mathrm{ord}}) = 0$. Then $x = \varphi \circ f\colon R_\ell^\square \to A$ is

an $\mathcal{O}$-algebra homomorphism with $x(\mathfrak{m}_{R_\ell^\square}) \subseteq \varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_A$ and $x(I_\ell^{\mathrm{ord}}) = 0$, and the GL-part of $x \circ r_\ell^\square$ is $\underline{r}'|_{D_\ell}$. (Ol2) gives $h \in \mathrm{GL}_3(A)$, $h \equiv 1 \pmod{\mathfrak{m}_A}$, with $h^{-1}\underline{r}'(g)h$ upper triangular for $g \in D_\ell \cap G_K$ and diagonal $(\psi_1(g), \psi_2(g), \psi_3(g))$ for $g \in I_\ell$. This is (Tl4). □

Lemma 47.4. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ valued in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, $g_\star$ a marking and (R, r) a dyadic Eisenstein deformation ring for $g_\star$. For every test ring A (integral or of characteristic ℓ) and every homomorphism of $\mathcal{O}$-algebras $\varphi\colon R \to A$ with $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_A$, $\varphi \circ r$ is a triangular lift to A for $g_\star$.*

Proof. Put $r'' = \varphi \circ r\colon G_\mathbb{Q} \to \mathcal{G}_3(A)$. (Tl1): r is a lift of $\bar{r}_0$, so its sign is η_K and the entries of $\underline{r}(g) - \underline{r_0}(g)$ lie in $\mathfrak{m}_R$, which φ maps into $\mathfrak{m}_A$; the entries of $\underline{r}$ are continuous for the $\mathfrak{m}_R$-adic topology, hence locally constant modulo $\mathfrak{m}_R^n$, and $\varphi(\mathfrak{m}_R^n) \subseteq \mathfrak{m}_A^n$. (Tl2): r satisfies (Ed1) for $g_\star$, and so does its image. (Tl3): let $\bar{v}$ be a place of $\overline{\mathbb{Q}}$ above a prime $p \neq \ell$, and $g \in I_{\bar{v}}$ with $\eta_K(g) = \Psi(g) = 1$. If $p \notin T_\ell(\chi)$, then $g \in N_{\bar{v}}$ and $r(g) = 1$ by (Dd3). If $p \in S_\ell(\chi)$, write $\bar{v} = \gamma\bar{p}$ with $\gamma \in G_\mathbb{Q}$ and $g = \gamma g_0 \gamma^{-1}$ with $g_0 \in I_p$; then $\eta_K(g_0) = \Psi(g_0) = 1$, and by (Dd2) $h^{-1}\underline{r}(g_0)h$ is diagonal with entries $\psi_a(g_0) = \Psi(g_0)^{2-a} = 1$, so $r(g_0) = (1, 1)$ and $r(g) = 1$. As $T_\ell(\chi) = \{\ell\} \cup S_\ell(\chi)$, this covers all $p \neq \ell$, and $r''(g) = \varphi(r(g)) = 1$. (Tl4): apply φ to the matrix h of (Dd1); $\varphi(h) \equiv 1 \pmod{\mathfrak{m}_A}$, and $\varphi(h)^{-1}\underline{r''}(g)\varphi(h)$ is the image of $h^{-1}\underline{r}(g)h$, upper triangular for $g \in D_\ell \cap G_K$ with diagonal $\psi_1(g), \psi_2(g), \psi_3(g)$ (mapped to A) for $g \in I_\ell$. □

Lemma 47.5. *Let ℓ be an odd prime, $K \subset \overline{\mathbb{Q}}$ an imaginary quadratic field, $\mathcal{O}$ a coefficient ring with residue field $\mathbb{F}_\mathcal{O}$, $c \in G_\mathbb{Q}$ a complex conjugation and $\Psi\colon G_\mathbb{Q} \to \mathcal{O}^\times$ a continuous character with $\Psi(c) = 1$; form triangular lifts with K, $\mathcal{O}$ and Ψ, and let $g_\star$ be a marking. Let A be a test ring of characteristic ℓ whose structure map $\mathcal{O} \to A$ kills $\mathfrak{m}_\mathcal{O}$, write $\bar{x}$ for the image in A of $x \in \mathcal{O}$, and let r be a triangular lift to A for $g_\star$. Assume that there is a subgroup U of finite index in $G_\mathbb{Q}$ such that $\det(X - \underline{r}(g)) = \prod_{a=1}^3 (X - \overline{\psi_a(g)})$ for every $g \in U \cap G_K$. Then either the entries of negative weight of $\underline{r}(g)$ vanish for all $g \in G_K$, or the entries of positive weight of $\underline{r}(g)$ vanish for all $g \in G_K$; and $\underline{r}(g)_{aa} = \overline{\psi_a(g)}$ for all $g \in G_K$ and $a = 1, 2, 3$.*

Proof. Let $\rho\colon G_K \to \mathrm{GL}_3(A)$ be the restriction of $\underline{r}$, $\mathfrak{m}_A$ the maximal ideal of A, $k = A/\mathfrak{m}_A$ and $\bar{k}$ an algebraic closure of the fraction field of A, a field of characteristic $\ell \neq 2$. By (Tl1), $\rho(g) \equiv \mathrm{T}(\overline{\Psi(g)}) \pmod{\mathfrak{m}_A}$ for $g \in G_K$, and $\overline{\Psi(g)}$ is the image of the reduction of $\Psi(g)$ in $\mathbb{F}_\mathcal{O}^\times$, a root of unity of order prime to ℓ.

Step 1: the shape over $\bar{k}$. View r as a homomorphism into $\mathcal{G}_3(\bar{k})$. (Ns1) holds by (Tl1), (Tl2): the sign is η_K and $\underline{r}(c) = 1$; (Ns2) is part of (Tl2); and $g_\star \in G_K$. By (Tl1), $\underline{r}(g_\star)_{11} - \underline{r}(g_\star)_{33} \equiv \overline{\Psi(g_\star)} - \overline{\Psi(g_\star)}^{-1} \pmod{\mathfrak{m}_A}$, a unit because $\Psi(g_\star)^2 - 1 \in \mathcal{O}^\times$; so (Ns3) holds. By Lemma 42.4, item 1, over the field $\bar{k}$: ρ is irreducible over $\bar{k}$, or upper (entries of negative weight vanish on G_K), or lower (entries of positive weight vanish), or of type II: the entries of odd weight vanish on G_K and the corner $\tau(g) = (\rho(g)_{ab})_{a,b\in\{1,3\}}$ is irreducible over $\bar{k}$.

Step 2: a normal subgroup on which ρ is unipotent. Let V be the intersection of the conjugates of U in $G_\mathbb{Q}$ (finitely many, U having finite index), of G_K and of the kernel of the reduction $\bar{\Psi}\colon G_\mathbb{Q} \to \mathbb{F}_\mathcal{O}^\times$. It is a normal subgroup of finite index of $G_\mathbb{Q}$ contained in $U \cap G_K$, and $\overline{\psi_a(g)} = 1$ for $g \in V$. So $\det(X - \rho(g)) = (X - 1)^3$ for $g \in V$.

Step 3: ρ is not irreducible and not of type II. Suppose it is, and let W be ρ in the first case and τ in the second: an irreducible representation of G_K over $\bar{k}$ of dimension $d \in \{3, 2\}$ with matrices in $\mathrm{GL}_d(A)$, which modulo $\mathfrak{m}_A$ are diagonal with entries in $\overline{\Psi(G_K)}$. For $g \in V$, $\det(X - W(g)) = (X - 1)^d$ (in the second case $\rho(g)$ is the direct sum of $\tau(g)$ on $\bar{k}e_1 \oplus \bar{k}e_3$ and $\rho(g)_{22}$ on $\bar{k}e_2$, so $\det(X - \tau(g))$ divides $(X - 1)^3$). By Clifford's theorem ($V \cap G_K = V$ is normal of finite index in G_K), $W|_V$ is semisimple; it has the characteristic polynomials of the trivial

representation of dimension d, so by the Brauer–Nesbitt theorem it is trivial: $W(V) = 1$. Hence $G' = W(G_K)$ is a finite subgroup of $\mathrm{GL}_d(A)$. Let P be the kernel of reduction $G' \to \mathrm{GL}_d(k)$, a normal subgroup of G' inside $1 + \mathrm{M}_d(\mathfrak{m}_A)$. Every $y = 1 + Y \in P$ has ℓ-power order: if its order is $\ell^a m'$ with $\ell \nmid m'$, then $y^{\ell^a} = 1 + Y^{\ell^a}$ (characteristic ℓ) has order m', and $0 = (1+Y')^{m'} - 1 = Y'(m' + Y'Q)$ with $Y' = Y^{\ell^a}$ and Q a polynomial in Y'; $m' + Y'Q$ is invertible (its reduction is $m' \neq 0$ in k), so $Y' = 0$ and $m' = 1$. So P is a finite ℓ-group, and its fixed vectors in $\bar{k}^d$ form a non-zero subspace (a finite ℓ-group acting on a non-zero vector space in characteristic ℓ fixes a non-zero vector), stable under G' because P is normal. By irreducibility P acts trivially, so $P = 1$, and G' embeds into the group of diagonal matrices of $\mathrm{GL}_d(k)$ with entries in $\overline{\Psi(G_K)}$: G' is abelian of order prime to ℓ. Its representation on $\bar{k}^d$ is then semisimple (Maschke) and, $\bar{k}$ being algebraically closed, a sum of $d \geq 2$ characters, so reducible: a contradiction.

Step 4: the diagonal. By Steps 1 and 3, ρ is upper or lower. In either case the diagonal entry maps $\kappa_a(g) = \rho(g)_{aa}$ are homomorphisms $G_K \to A^\times$ (the diagonal of a product of triangular matrices is the product of the diagonals), with $\kappa_a(g) \equiv \overline{\psi_a(g)} \pmod{\mathfrak{m}_A}$ by (Tl1), and $\kappa_a(g) = 1$ for $g \in V$ by Step 2 (the eigenvalues of a triangular matrix are its diagonal entries). So $\mu_a = \kappa_a \cdot \overline{\psi_a}^{-1}$ is a homomorphism $G_K \to 1 + \mathfrak{m}_A$ trivial on V, hence of finite order. An element $1 + y$ of finite order $\ell^a m'$ ($\ell \nmid m'$) of $1 + \mathfrak{m}_A$ satisfies, as in Step 3 with $d = 1$, $y^{\ell^a} = 0$, so $y = 0$ (A is a domain). Hence $\mu_a = 1$ and $\kappa_a = \overline{\psi_a}$ on G_K. □

Proposition 47.6. *Let $\mathcal{O}$ be a coefficient ring and assume (a) (K, ℓ, e, N, χ) is a unitary Eisenstein datum with χ valued in $\mathcal{O}^\times$ and $e_{\mathcal{O}} = 1$, or (b) (ℓ, e, N, χ, o, m) is a dyadic Eisenstein datum with χ valued in $\mathcal{O}^\times$. Let $g_\star$ be a marking and (R, r) a normalised ordinary Eisenstein deformation ring for $g_\star$ in case (a), a dyadic Eisenstein deformation ring for $g_\star$ in case (b). Assume that for every test ring A of characteristic ℓ whose structure map $\mathcal{O} \to A$ kills $\mathfrak{m}_{\mathcal{O}}$, every homomorphism of $\mathcal{O}$-algebras $\varphi\colon R \to A$ with $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_A$, and every product $y = \prod_{k=1}^{n} \underline{r}(g_k)_{a_k b_k}$ with $g_k \in G_{\mathbb{Q}}$ and $\sum_k \mathrm{wt}(a_k, b_k) = 0$, the element $\varphi(y)$ lies in the image of $\mathcal{O}$ in A. Then R^0 is a finitely generated $\mathcal{O}$-module.*

Proof. Step 1: a presentation with homogeneous kernel in which every entry is homogeneous. In case (a), Proposition 43.12 (with (Bx3) of Proposition 43.5) gives a normalised ordinary Eisenstein deformation ring (R_1, r_1) for $g_\star$ and a weighted presentation $(\sigma, \mathrm{wt}, \pi_1)$ of R_1 with $\ker \pi_1 = \mathfrak{a} + (u_{j,i})$, where $\mathfrak{a}$ is generated by finitely many elements of single weights and $u_{j,i} \in P_j$; by (Pq3) (clause (Mp2) there), $\underline{r_1}(g)_{ab} \in q(A_{\mathrm{wt}(a,b)}) = \pi_1(P_{\mathrm{wt}(a,b)})$. In case (b), Proposition 46.6 gives a dyadic Eisenstein deformation ring (R_1, r_1) for $g_\star$ and a weighted presentation $(\sigma, \mathrm{wt}, \pi_1)$ whose kernel is generated by a homogeneous generating system, with $\underline{r_1}(g)_{ab} \in \pi_1(P_{\mathrm{wt}(a,b)})$ by (Dw2). By the universal properties there are morphisms $f\colon R_1 \to R$, $f_1\colon R \to R_1$ with $f \circ r_1 = r$, $f_1 \circ r = r_1$; then $(f \circ f_1) \circ r = r$, so $f \circ f_1 = \mathrm{id}$ by uniqueness, and likewise $f_1 \circ f = \mathrm{id}$. Put $\pi = f \circ \pi_1$. Then $\pi\colon P \to R$ is a surjective continuous $\mathcal{O}$-algebra homomorphism with $\pi(\mathfrak{m}_P) = \mathfrak{m}_R$ ($\pi(\mathfrak{m}_P)$ is an ideal with residue ring a quotient of $\mathbb{F}_{\mathcal{O}}$), $\ker \pi$ is generated by finitely many $h_1, \dots, h_m$ with $h_i \in P_{\kappa_i}$, and

$$\underline{r}(g)_{ab} \in \pi\big(P_{\mathrm{wt}(a,b)}\big) \qquad (g \in G_{\mathbb{Q}},\ 1 \leq a, b \leq 3). \tag{47.1}$$

Step 2: the weight-zero projection. For $f = \sum_i c_i h_i \in \ker \pi$, $f_{(0)} = \sum_i (c_i)_{(-\kappa_i)} h_i \in \ker \pi$, since h_i has the single weight κ_i. Hence $\mathrm{E}(\pi(f)) = \pi(f_{(0)})$ is a well-defined $\mathcal{O}$-linear map $\mathrm{E}\colon R \to R$ with image $C = \pi(P_0)$. For $p \in P_0$ and $f \in P$, $(pf)_{(0)} = p f_{(0)}$, so $\mathrm{E}(cy) = c\,\mathrm{E}(y)$ for $c \in C$, $y \in R$, and $\mathrm{E}(c) = c$. Consequently, for every ideal $\mathfrak{p}$ of C,

$$\mathfrak{p}R \cap C = \mathfrak{p}: \qquad \textstyle\sum_i a_i y_i \in C\ (a_i \in \mathfrak{p}) \ \Rightarrow\ \sum_i a_i y_i = \mathrm{E}\big(\sum_i a_i y_i\big) = \sum_i a_i \mathrm{E}(y_i) \in \mathfrak{p}. \tag{47.2}$$

A power series in $\mathfrak{m}_P^N$ has its coefficient of X^n in $\mathfrak{m}_{\mathcal{O}}^{\max(0,N-|n|)}$; $f_{(0)}$ keeps some of these coefficients, so $f_{(0)} \in \mathfrak{m}_P^N$, and with $\pi(\mathfrak{m}_P^N) = \mathfrak{m}_R^N$ we get $\mathrm{E}(\mathfrak{m}_R^N) \subseteq \mathfrak{m}_R^N$: E is continuous.

Step 3: C is a complete local Noetherian $\mathcal{O}$-algebra. The monoid $\{n \in \mathbb{N}^\sigma : \sum_s n_s \mathrm{wt}(s) = 0\}$ is generated by finitely many non-zero $n^{(1)}, \dots, n^{(k)}$ (Gordan). The continuous homomorphism $\mathcal{O}[[Y_1, \dots, Y_k]] \to P$, $Y_i \mapsto X^{n^{(i)}} \in \mathfrak{m}_P$, has image P_0: the image lies in the closed subring P_0; conversely, for $f = \sum_n c_n X^n \in P_0$ choose for each n some $\alpha(n) \in \mathbb{N}^k$ with $\sum_i \alpha(n)_i n^{(i)} = n$; $n \mapsto \alpha(n)$ is injective, and $\sum_n c_n Y^{\alpha(n)}$ maps to f. So $C = \pi(P_0)$ is a quotient of $\mathcal{O}[[Y_1, \dots, Y_k]]$: a complete local Noetherian ring, with residue field $\mathbb{F}_{\mathcal{O}}$, maximal ideal $\mathfrak{m}_C = C \cap \mathfrak{m}_R$, and $\mathcal{O} \to C$ local.

Step 4: $R^0 \subseteq C \subseteq \overline{R^0}$ (closure in R). A product $\prod_k \underline{r}(g_k)_{a_k b_k}$ with $\sum_k \mathrm{wt}(a_k, b_k) = j$ is, by (47.1), $\pi(\prod_k f_k)$ with $\prod_k f_k \in P_j$; so it lies in C if $j = 0$, and E maps it to itself if $j = 0$ and to 0 otherwise. Hence $R^0 \subseteq C$, and E maps the $\mathcal{O}$-subalgebra R_2 of R generated by all entries $\underline{r}(g)_{ab}$ into R^0. We show that R_2 is dense in R; then, E being continuous, $C = \mathrm{E}(R) \subseteq \overline{\mathrm{E}(R_2)} \subseteq \overline{R^0}$. Fix $N \geq 1$, let $Q = R/\mathfrak{m}_R^N$ (a finite local ring) and $S \subseteq Q$ the image of R_2. An element of S outside the maximal ideal of Q is a unit of finite order in Q, so its inverse is a power of it and lies in S; thus S is a finite local ring with residue field $\mathbb{F}_{\mathcal{O}}$, and so is $Q \otimes_S Q$ (its quotient by the nilpotent ideal generated by the two maximal ideals is $\mathbb{F}_{\mathcal{O}} \otimes_{\mathbb{F}_{\mathcal{O}}} \mathbb{F}_{\mathcal{O}}$). The two morphisms $R \to Q \rightrightarrows Q \otimes_S Q$, through $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$, agree on all entries of r (their images lie in S), hence carry r to the same lift, which is a normalised ordinary lift in case (a), a lift with (Ed1), (Dd1)–(Dd3) in case (b) (images of r under morphisms are such); by the uniqueness in the universal property they are equal, so $x \otimes 1 = 1 \otimes x$ for all $x \in Q$. Let $p\colon Q \to M = Q/S$ be the projection of S-modules. The map $Q \otimes_S Q \to Q \otimes_S M$, $x \otimes y \mapsto x \otimes p(y)$, sends $x \otimes 1$ to 0 and $1 \otimes x$ to $1 \otimes p(x)$, so $1 \otimes p(x) = 0$ for all x, and $Q \otimes_S M = 0$ (it is generated by the $q(1 \otimes p(x))$). Then $M \otimes_S \mathbb{F}_{\mathcal{O}} = (Q \otimes_S M) \otimes_Q \mathbb{F}_{\mathcal{O}} = 0$, and $M = 0$ by Nakayama's lemma (M is finite). So $S = Q$, i.e. $R_2 + \mathfrak{m}_R^N = R$ for every N.

Step 5: $C/\mathfrak{m}_{\mathcal{O}}C$ is Artinian. Suppose not. Let $\mathfrak{p}_0$ be a minimal prime of $\mathfrak{m}_{\mathcal{O}}C$ with $\dim C/\mathfrak{p}_0 \geq 1$ and $t_0 \in \mathfrak{m}_C \setminus \mathfrak{p}_0$; by Lemma 47.7 for the complete local domain $C/\mathfrak{p}_0$ there is a prime $\mathfrak{p} \supseteq \mathfrak{p}_0$ of C with $\dim C/\mathfrak{p} = 1$; choose $t \in \mathfrak{m}_C \setminus \mathfrak{p}$. By (47.2), $\mathfrak{p}R$ does not meet the multiplicative set $C \setminus \mathfrak{p}$; an ideal of R maximal among those containing $\mathfrak{p}R$ and disjoint from $C \setminus \mathfrak{p}$ is a prime $\mathfrak{Q}$ with $\mathfrak{Q} \cap C = \mathfrak{p}$. $R/\mathfrak{Q}$ is a complete local Noetherian domain of characteristic ℓ ($\ell \in \mathfrak{m}_{\mathcal{O}} \subseteq \mathfrak{p} \subseteq \mathfrak{Q}$) in which the image of t is a non-zero non-unit, so $\dim R/\mathfrak{Q} \geq 1$, and Lemma 47.7 gives a prime $\mathfrak{Q}' \supseteq \mathfrak{Q}$ with $t \notin \mathfrak{Q}'$ and $\dim R/\mathfrak{Q}' = 1$. The normalisation $\tilde{A}$ of $R/\mathfrak{Q}'$ is a complete discrete valuation ring with finite residue field, finite over $R/\mathfrak{Q}'$, of characteristic ℓ; with $\mathcal{O} \to R \to \tilde{A}$ it is a test ring of characteristic ℓ whose structure map kills $\mathfrak{m}_{\mathcal{O}}$, and $\psi\colon R \to R/\mathfrak{Q}' \to \tilde{A}$ is an $\mathcal{O}$-algebra homomorphism with $\psi(\mathfrak{m}_R) \subseteq \mathfrak{m}_{\tilde{A}}$. By hypothesis ψ maps every weight-zero product of entries, hence R^0, into the image F_0 of $\mathcal{O}$ in $\tilde{A}$, the image of $\mathbb{F}_{\mathcal{O}}$, a finite subfield. F_0 is finite, hence closed, and ψ is continuous ($\psi(\mathfrak{m}_R^N) \subseteq \mathfrak{m}_{\tilde{A}}^N$), so $\psi(\overline{R^0}) \subseteq F_0$, and $\psi(t) \in F_0$ by Step 4. But $\psi(t) \neq 0$ ($t \notin \mathfrak{Q}'$ and $R/\mathfrak{Q}' \to \tilde{A}$ is injective) and $\psi(t) \in \mathfrak{m}_{\tilde{A}}$, while the non-zero elements of F_0 are units: a contradiction.

Step 6: conclusion. $C/\mathfrak{m}_{\mathcal{O}}C$ is an Artinian local ring with finite residue field, hence finite, and its maximal ideal is nilpotent, so the $\mathfrak{m}_C$-adic and $\mathfrak{m}_{\mathcal{O}}$-adic topologies of C coincide; C is complete and separated, so by Nakayama's lemma for complete modules C is a finitely generated $\mathcal{O}$-module. $\mathcal{O}$ is Noetherian and $R^0 \subseteq C$ (Step 4), so R^0 is a finitely generated $\mathcal{O}$-module. □

Lemma 47.7. *Let C be a complete local Noetherian domain of dimension ≥ 1 and $f \in C$, $f \neq 0$. Then there is a prime ideal q of C with $f \notin q$ and $\dim C/q = 1$.*

Proof. By induction on $d = \dim C$. If $d = 1$, take $q = 0$. Let $d \geq 2$ and $\mathfrak{m}_C$ the maximal ideal. C has infinitely many primes of height one: if $q_1, \dots, q_s$ were all of them, then $\mathfrak{m}_C \not\subseteq q_i$

(its height is $d \geq 2$), so by prime avoidance there is $x \in \mathfrak{m}_C \setminus \bigcup_i q_i$; $x \neq 0$ is not a unit, and a minimal prime over xC has height one by Krull's principal ideal theorem and contains x, a contradiction. A prime of height one containing the non-zero element f is minimal over fC (the primes strictly inside it are 0), and there are finitely many such. So there is a prime q_1 of height one with $f \notin q_1$. The complete local ring C is catenary, so for the local domain C one has $\dim C/q_1 + \operatorname{ht} q_1 = \dim C$, i.e. $\dim C/q_1 = d - 1 \geq 1$. C/q_1 is a complete local Noetherian domain in which the image of f is non-zero; by induction there is a prime $q \supseteq q_1$ of C with $f \notin q$ and $\dim C/q = 1$. □

Lemma 47.8. *Let $\mathcal{O}$ be a coefficient ring and $g_\star$ a marking.*

(Li1) *If (K, ℓ, e, N, χ) is a unitary Eisenstein datum with χ valued in $\mathcal{O}^\times$ and $e_{\mathcal{O}} = 1$, and r is a triangular lift for $g_\star$ (formed with K, $\mathcal{O}$, $\Psi_{\chi,e}$) to an integral test ring with ρ_r irreducible, then $\rho_r|_V$ is irreducible for every open subgroup $V \subseteq G_K$.*

(Li2) *If (ℓ, e, N, χ, o, m) is a dyadic Eisenstein datum with χ valued in $\mathcal{O}^\times$, $K = \mathbb{Q}(i)$, and r is a triangular lift for $g_\star$ (formed with K, $\mathcal{O}$, $\Psi_{\chi,e}$) to an integral test ring with ρ_r irreducible, then $\rho_r|_V$ is irreducible for every open subgroup $V \subseteq G_K$.*

Proof. We treat both items together. Let r be as in (Li1) or (Li2), A its integral test ring with maximal ideal $\mathfrak{m}_A$, $\rho = \rho_r$ and $\Psi = \Psi_{\chi,e}$. The marking is not used.

Step 1: two elements of $D_{\bar\ell}$. There is $x \in D_{\bar\ell}$ with $\Psi(x)^2 - 1 \in \mathcal{O}^\times$: in case (Li1) by (Lo3) (with $x \in I_{\bar\ell}$), in case (Li2) by Lemma 47.10. Let $V \subseteq G_K$ be open and $V_0 = \bigcap_{g \in G_K} gVg^{-1}$; as V has finite index in the compact group G_K, this is a finite intersection, so V_0 is an open normal subgroup of G_K, and it suffices to show that $\rho|_{V_0}$ is irreducible. The group $V_0 \cap I_{\bar\ell} \cap \ker\chi$ is open in $I_{\bar\ell}$, and $\chi_{\mathrm{cyc}}(I_{\bar\ell}) = \mathbb{Z}_\ell^\times$ (ℓ is unramified in K), so its image under χ_{cyc} is open in $\mathbb{Z}_\ell^\times$ and contains $1+\ell^k$ for some $k \geq 1$. Choose u in it with $\chi_{\mathrm{cyc}}(u) = 1+\ell^k$; then $\Psi(u)^2 = (1+\ell^k)^{2e} \neq 1$, so $\Psi(u)$, 1, $\Psi(u)^{-1}$ are pairwise distinct.

Step 2: a basis. Take h as in (Lo2) at $\bar\ell$. Then $h^{-1}\rho(u)h$ and $h^{-1}\rho(x)h$ are upper triangular, and the diagonal of the first is $(\Psi(u), 1, \Psi(u)^{-1})$ by (Lo2). An upper triangular matrix with pairwise distinct diagonal entries is diagonalised by an upper triangular matrix (its eigenvectors, found successively, span the standard flag). So there is $P \in \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ such that $\rho'(g) = P^{-1}\rho(g)P$ satisfies: $\rho'(u) = \mathrm{diag}(d_1, d_2, d_3)$ with d_1, d_2, d_3 pairwise distinct, and $\rho'(x)$ is upper triangular. Irreducibility of ρ and of $\rho|_{V_0}$ is that of ρ' and $\rho'|_{V_0}$. Suppose, for a contradiction, that $\rho'|_{V_0}$ is reducible.

Step 3: V_0-stable subspaces are coordinate subspaces. Let W be $\rho'(V_0)$-stable. It is stable under $\rho'(u)$; for $w = \sum_i w_i e_i \in W$ and the Lagrange polynomial $L_i(X) = \prod_{j \neq i}(X - d_j)/(d_i - d_j)$, $L_i(\rho'(u))w = w_i e_i \in W$. So W is spanned by the e_i it contains.

Step 4: a V_0-stable coordinate line. Let W be a $\rho'(V_0)$-stable subspace with $0 \neq W \neq \overline{\mathbb{Q}}_\ell^3$; by Step 3, $W = \bigoplus_{i \in S} \overline{\mathbb{Q}}_\ell e_i$ with $\emptyset \neq S \subsetneq \{1,2,3\}$. If $|S| = 1$ we are done. If $|S| = 2$: as ρ' is irreducible, W is not $\rho'(G_K)$-stable, so $W' = \rho'(g)W \neq W$ for some $g \in G_K$. Since V_0 is normal in G_K, $\rho'(s)\rho'(g)W = \rho'(g)\rho'(g^{-1}sg)W = \rho'(g)W$ for $s \in V_0$, so W' is $\rho'(V_0)$-stable, hence $W' = \bigoplus_{i \in S'} \overline{\mathbb{Q}}_\ell e_i$ with $|S'| = 2$, $S' \neq S$; then $W \cap W'$ is $\rho'(V_0)$-stable and spanned by e_i for the single $i \in S \cap S'$.

Step 5: ρ' is monomial. Let $\overline{\mathbb{Q}}_\ell e_i$ be $\rho'(V_0)$-stable. For $g \in G_K$ the line $\rho'(g)\overline{\mathbb{Q}}_\ell e_i$ is $\rho'(V_0)$-stable (as in Step 4), hence a coordinate line by Step 3. The span of these lines for $g \in G_K$ is non-zero and $\rho'(G_K)$-stable, hence everything; so every coordinate line is $\rho'(g_j)\overline{\mathbb{Q}}_\ell e_i$ for some $g_j \in G_K$, and for every $g \in G_K$ and every j, $\rho'(g)\overline{\mathbb{Q}}_\ell e_j = \rho'(gg_j)\overline{\mathbb{Q}}_\ell e_i$ is a coordinate line. So $\rho'(g)e_j \in \overline{\mathbb{Q}}_\ell^\times e_{\pi_g(j)}$ for a permutation π_g of $\{1,2,3\}$, and $g \mapsto \pi_g$ is a homomorphism $G_K \to S_3$.

Step 6: a derangement. The image $\pi(G_K)$ is transitive on $\{1,2,3\}$: otherwise the span of the e_j over an orbit would be a proper non-zero $\rho'(G_K)$-stable subspace. A transitive subgroup of S_3 is A_3 or S_3 and contains a 3-cycle; let $g \in G_K$ with π_g a 3-cycle. The matrix $\rho'(x)$ is upper triangular and monomial, so $\pi_x(j) \le j$ for all j, whence $\pi_x = \mathrm{id}$. Hence $\pi_{gx^n} = \pi_g$ has no fixed point for every $n \ge 0$, so $\rho'(gx^n)$ has zero diagonal and

$$\operatorname{tr}\rho(gx^n) = \operatorname{tr}\rho'(gx^n) = 0 \qquad (n \ge 0).$$

Step 7: contradiction modulo $\mathfrak{m}_A$. By (Tl1), $\rho(y) \equiv \mathrm{T}(\Psi(y)) \pmod{\mathfrak{m}_A}$ for $y \in G_K$. Let a, b be the images of $\Psi(g)$, $\Psi(x)$ in the residue field of A and $t_n = ab^n$. Reducing Step 6, $t_n + 1 + t_n^{-1} = 0$, i.e. $t_n^2 + t_n + 1 = 0$, for $n = 0, 1, 2$. Suppose $b \ne 1$; then $t_0 \ne t_1$ are the two roots of $X^2 + X + 1$, and $t_2 = t_1 b$ is a root as well: $t_2 = t_1$ would give $b = 1$, so $t_2 = t_0$ and $b^2 = 1$. In all cases $b^2 = 1$, i.e. $\Psi(x)^2 \equiv 1 \pmod{\mathfrak{m}_A}$, contradicting $\Psi(x)^2 - 1 \in \mathcal{O}^\times$ (and $\mathfrak{m}_A \cap \mathcal{O} = \mathfrak{m}_{\mathcal{O}}$). So $\rho|_{V_0}$, and hence $\rho|_V$, is irreducible. □

Lemma 47.9. *Let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field, $\mathcal{O}$ a coefficient ring, $N \ge 1$, χ a Dirichlet character modulo N with values in $\mathcal{O}^\times$, $e \ge 1$, $\Psi = \Psi_{\chi,e}$ with $\Psi(c) = 1$, and form triangular lifts with K, $\mathcal{O}$ and Ψ. Let $g_\star$ be a marking, A an integral test ring with maximal ideal $\mathfrak{m}_A$, r a triangular lift to A for $g_\star$ and $\rho = \rho_r$.*

(Lo1) *For every place $\bar v$ of $\overline{\mathbb{Q}}$ not above ℓ, $\Psi(I_{\bar v})$ is finite, and there are $P \in \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ and integers a_1, a_2, a_3 with $\rho(g) = P\,\mathrm{diag}\big(\Psi(g)^{a_1}, \Psi(g)^{a_2}, \Psi(g)^{a_3}\big)P^{-1}$ for all $g \in I_{\bar v} \cap G_K$.*

(Lo2) *Every place of K above ℓ lies below $\bar\ell$ or below $c\bar\ell$. For $\bar v \in \{\bar\ell, c\bar\ell\}$, with $D_{\bar v} \supseteq I_{\bar v}$ the decomposition and inertia subgroups of G_K at $\bar v$, there are $h \in \mathrm{GL}_3(A)$ with $h \equiv 1 \pmod{\mathfrak{m}_A}$ and continuous characters $\kappa_1, \kappa_2, \kappa_3 \colon D_{\bar v} \to A^\times$ such that for every $g \in D_{\bar v}$ the matrix $h^{-1}\rho(g)h$ is upper triangular with diagonal entries $\kappa_1(g), \kappa_2(g), \kappa_3(g)$, and*

$$\kappa_a(g) \equiv \Psi(g)^{2-a} \pmod{\mathfrak{m}_A} \quad (g \in D_{\bar v}), \qquad \kappa_a(g) = \Psi(g)^{2-a} \quad (g \in I_{\bar v}), \qquad a = 1, 2, 3.$$

(Lo3) *If (K, ℓ, e, N, χ) is a unitary Eisenstein datum, then for every place $\bar v$ of $\overline{\mathbb{Q}}$ above ℓ the inertia group of $G_{\mathbb{Q}}$ at $\bar v$ lies in G_K and contains an element s with $\Psi(s)^2 - 1 \in \mathcal{O}^\times$.*

Proof. (Lo1). Let $\bar v$ lie above a prime $p \ne \ell$. The extension of $\mathbb{Q}$ generated by the ℓ-power roots of unity is unramified at p, so χ_{cyc} is trivial on $I_{\bar v}$ and $\Psi|_{I_{\bar v}} = \chi|_{I_{\bar v}}$, which takes values in the finite group of values of χ. Put $H = I_{\bar v} \cap G_K$ and $Z = \Psi(H)$, a finite subgroup of $\overline{\mathbb{Q}}_\ell^\times$, hence cyclic, say of order n with generator z_0. If $g, g' \in H$ and $\Psi(g) = \Psi(g')$, then $g^{-1}g' \in I_{\bar v}$ has $\eta_K(g^{-1}g') = 1$ and $\Psi(g^{-1}g') = 1$, so $r(g^{-1}g') = 1$ by (Tl3) and $\rho(g) = \rho(g')$. Hence $M(\Psi(g)) = \rho(g)$ defines a homomorphism $M \colon Z \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$. The matrix $M(z_0)$ satisfies $X^n = 1$, a polynomial with distinct roots in the algebraically closed field $\overline{\mathbb{Q}}_\ell$ of characteristic 0, so $M(z_0) = P\,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)P^{-1}$ with $\lambda_i^n = 1$; as Z consists of n distinct n-th roots of unity it contains all of them, so $\lambda_i = z_0^{a_i}$ for integers a_i. Then $M(z_0^k) = P\,\mathrm{diag}(z_0^{ka_i})P^{-1}$ for all k, which is (Lo1).

(Lo2), the places. Let v_0 be the place of K below $\bar\ell$. A place $v \mid \ell$ of K lies below $\gamma\bar\ell$ for some $\gamma \in G_{\mathbb{Q}}$ (the places of $\overline{\mathbb{Q}}$ above ℓ form one $G_{\mathbb{Q}}$-orbit); as $\mathrm{Gal}(K/\mathbb{Q}) = \{1, c|_K\}$, $v = \gamma|_K(v_0)$ is v_0 or $c|_K(v_0)$, the place below $c\bar\ell$.

(Lo2) at $\bar\ell$. The decomposition and inertia subgroups of G_K at $\bar\ell$ are $D_{\bar\ell} = D_\ell \cap G_K$ and $I_{\bar\ell} = I_\ell \cap G_K$, in the notation of Definition 43.16. Take h as in (Tl4) and put $Y(g) = h^{-1}\rho(g)h$ for $g \in D_{\bar\ell}$, upper triangular. The diagonal entries of a product of upper triangular matrices are the products of the diagonal entries, so $\kappa_a(g) = Y(g)_{aa}$ are homomorphisms $D_{\bar\ell} \to A^\times$, continuous

by (Tl1). By (Tl4), $\kappa_a(g) = \Psi(g)^{2-a}$ for $g \in I_{\bar\ell}$. By (Tl1) and $h \equiv 1$, $Y(g) \equiv \rho(g) \equiv \mathrm{T}(\Psi(g))$ (mod $\mathfrak{m}_A$), whose diagonal is $(\Psi(g), 1, \Psi(g)^{-1})$.

(Lo2) at $c\bar\ell$. By (Tl1), (Tl2), $r(c) = (1,-1)$, and $c^2 = 1$. For $g \in G_K$, $r(g) = (\rho(g), 1)$ and $r(cgc) = (1,-1)(\rho(g),1)(1,-1) = (\theta(\rho(g)),1)$, so $\rho(cgc) = \theta(\rho(g))$. The decomposition and inertia subgroups of G_K at $c\bar\ell$ are $cD_{\bar\ell}c$ and $cI_{\bar\ell}c$ (G_K is normal in $G_{\mathbb{Q}}$). Put $h' = \theta(h) = J\,{}^t h^{-1} J \in \mathrm{GL}_3(A)$; $h' \equiv 1 \pmod{\mathfrak{m}_A}$. For $g' = cgc$ with $g \in D_{\bar\ell}$, θ being a group automorphism of $\mathrm{GL}_3(A)$,

$$h'^{-1}\rho(g')h' = \theta\big(h^{-1}\rho(g)h\big) = J\,{}^t Y(g)^{-1} J.$$

$Y(g)^{-1}$ is upper triangular with diagonal $(\kappa_1(g)^{-1}, \kappa_2(g)^{-1}, \kappa_3(g)^{-1})$, its transpose is lower triangular, and conjugation by J sends the entry (a,b) to $(4-a, 4-b)$, so $J\,{}^t Y(g)^{-1} J$ is upper triangular with diagonal entries $\kappa'_a(g') = \kappa_{4-a}(cg'c)^{-1}$. These are continuous characters of $cD_{\bar\ell}c$. Since Ψ is a character of $G_{\mathbb{Q}}$ with $\Psi(c) = 1$, $\Psi(cg'c) = \Psi(g')$, so $\kappa'_a(g') = \Psi(g')^{-(2-(4-a))} = \Psi(g')^{2-a}$ for $g' \in cI_{\bar\ell}c$, and the same computation modulo $\mathfrak{m}_A$ gives $\kappa'_a(g') \equiv \Psi(g')^{2-a}$ on $cD_{\bar\ell}c$.

(Lo3). By (Ue6), ℓ is unramified in K, so the inertia group I of $G_{\mathbb{Q}}$ at $\bar v$ acts trivially on K: $I \subseteq G_K$. Write $N = \ell^a N'$ with $\ell \nmid N'$. An element $g \in I$ fixes $\mu_{N'}$ and acts on μ_{ℓ^a} through $\chi_{\mathrm{cyc}}(g)$, so $\chi(g) = \chi_\ell(\chi_{\mathrm{cyc}}(g) \bmod \ell^a)$, and $\chi(g)^2 = 1$ by (Ue4). Hence $\Psi(g)^2 \equiv \omega(g)^{2e} \pmod{\mathfrak{m}_{\mathcal{O}}}$, where $\omega\colon I \to \mathbb{F}_\ell^\times$ is the reduction of χ_{cyc}, which is surjective since $\chi_{\mathrm{cyc}}(I) = \mathbb{Z}_\ell^\times$. If ω^{2e} were trivial, $\mathbb{F}_\ell^\times$, cyclic of order $\ell - 1$, would have exponent dividing $2e$, contradicting (Ue9). So some $s \in I$ has $\omega(s)^{2e} \neq 1$, i.e. $\Psi(s)^2 \not\equiv 1$, i.e. $\Psi(s)^2 - 1 \in \mathcal{O}^\times$. □

Lemma 47.10. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, put $\Psi = \Psi_{\chi,e}$ and $K = \mathbb{Q}(i)$, and let $\bar\ell$ be a place of $\overline{\mathbb{Q}}$ above ℓ, with decomposition group $D_{\bar\ell} \subseteq G_{\mathbb{Q}}$. There is $x \in D_{\bar\ell} \cap G_K$ with $\chi_{\mathrm{cyc}}(x) = 1$ and $\Psi(x)^2 - 1 \in \mathcal{O}^\times$.*

Proof. Choice of x. Let $\sigma_0 \in D_{\bar\ell}$ be an arithmetic Frobenius. The roots of unity of order prime to ℓ reduce injectively to the residue field of $\bar\ell$, so $\sigma_0(\zeta) = \zeta^\ell$ for every root of unity ζ of order prime to ℓ. The restriction of χ_{cyc} to the inertia group $I_{\bar\ell} \subseteq D_{\bar\ell}$ is surjective onto $\mathbb{Z}_\ell^\times$ ($\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell$ is totally ramified with group $\mathbb{Z}_\ell^\times$), so there is $\tau \in I_{\bar\ell}$ with $\chi_{\mathrm{cyc}}(\tau) = \chi_{\mathrm{cyc}}(\sigma_0)^{-1}$; τ fixes the roots of unity of order prime to ℓ, which generate an extension unramified at ℓ. Put $\sigma = \sigma_0\tau$ and $x = \sigma^2$. Then $x \in D_{\bar\ell}$, $\eta_K(x) = \eta_K(\sigma)^2 = 1$ for the character η_K with kernel G_K, so $x \in G_K$, and $\chi_{\mathrm{cyc}}(x) = \chi_{\mathrm{cyc}}(\sigma)^2 = 1$.

Reduction to a statement on χ_2. Write $N = \ell^a N'$ with $\ell \nmid N'$. The element σ acts on μ_N by $\zeta \mapsto \zeta^u$ with $u \in (\mathbb{Z}/N\mathbb{Z})^\times$, $u \equiv \ell \pmod{N'}$ and $u \equiv 1 \pmod{\ell^a}$ (as $\chi_{\mathrm{cyc}}(\sigma) = 1$), so $\chi(\sigma) = \chi(u)$, and $\Psi(x)^2 = \chi(\sigma)^4\chi_{\mathrm{cyc}}(\sigma)^{4e} = \chi(u)^4$. Suppose, for a contradiction, that $\chi(u)^4 \equiv 1 \pmod{\mathfrak{m}_{\mathcal{O}}}$. The value $\chi''(u)$ is a root of unity of ℓ-power order ℓ^k, and $(\chi''(u)-1)^{\ell^k} \equiv \chi''(u)^{\ell^k} - 1 = 0 \pmod\ell$, so $\chi''(u) \equiv 1$; hence $\chi'(u)^4 \equiv 1$. As χ' has order n prime to ℓ and $X^n - 1$ has distinct roots modulo $\mathfrak{m}_{\mathcal{O}}$, the n-th roots of unity in $\mathcal{O}$ reduce injectively, and $\chi'(u)^4 = 1$. Write $\chi' = \prod_{p|N}\chi'_p$ as the product of its p-components; by (Dy4), $(\chi'_r)^{2o} = \mathbf{1}$ for every odd prime r, so $1 = \chi'(u)^{4o} = \chi'_2(u \bmod 2^{a_2})^{4o}$, where $2^{a_2} \parallel N$ ($a_2 \geq 1$ by (Dy2)). Taking 2-components is multiplicative, $\chi_2 = \chi'_2\chi''_2$, and χ''_2 is a character of the 2-group $(\mathbb{Z}/2^{a_2}\mathbb{Z})^\times$ whose order divides the ℓ-power order of χ''; as ℓ is odd (Dy1), $\chi''_2 = \mathbf{1}$ and $\chi_2 = \chi'_2$. Since $2^{a_2} \mid N'$, $u \equiv \ell \pmod{2^{a_2}}$. Hence

$$\chi_2(\ell)^{4o} = 1. \tag{47.3}$$

Contradiction with (Dy5). Write $o = 2^j o'$ with o' odd, and $\ell^2 = 1 + 2^s b$ with b odd; $s \geq 3$ since ℓ is odd. For $t \geq 2$ and b odd, $(1 + 2^t b)^2 = 1 + 2^{t+1}(b + 2^{t-1}b^2)$ with $b + 2^{t-1}b^2$ odd; by induction $y = \ell^{2^{j+2}} = (\ell^2)^{2^{j+1}} = 1 + 2^t b_0$ with $t = s + j + 1$ and b_0 odd, and the powers y^{2^i} are $\equiv 1 + 2^{t+i} \pmod{2^{t+i+1}}$. By (47.3), $\chi_2(y)^{o'} = \chi_2(\ell)^{4o} = 1$; χ_2 has 2-power order, so $\chi_2(y) = 1$. If $t < a_2$, the subgroup U_t of $(\mathbb{Z}/2^{a_2}\mathbb{Z})^\times$ of classes $\equiv 1 \pmod{2^t}$ has order 2^{a_2-t},

and y has order exactly 2^{a_2-t} in it by the congruences for y^{2^i}; so y generates U_t, χ_2 is trivial on U_t, and χ_2 factors through $(\mathbb{Z}/2^t\mathbb{Z})^\times$: $c_2(\chi) \le t$. If $t \ge a_2$, then $c_2(\chi) \le a_2 \le t$ as well. But $2^s \le 2^s b = \ell^2 - 1$ and $2^j \le o$, so

$$2^t = 2 \cdot 2^j \cdot 2^s \le 2o(\ell^2 - 1) < 8o\ell^2 < 2^{c_2(\chi)},$$

the last inequality by (Dy5) for $v = \ell \in T_\ell(\chi)$. Thus $t < c_2(\chi)$, a contradiction. Hence $\chi(u)^4 \not\equiv 1$, that is $\Psi(x)^2 - 1 \in \mathcal{O}^\times$. □

Proposition 47.11. *Let $\mathcal{O}$ be a coefficient ring, and assume (a) (K, ℓ, e, N, χ) is a unitary Eisenstein datum with χ valued in $\mathcal{O}^\times$ and $e_\mathcal{O} = 1$, or (b) (ℓ, e, N, χ, o, m) is a dyadic Eisenstein datum with χ valued in $\mathcal{O}^\times$ and $K = \mathbb{Q}(i)$. Form triangular lifts with K, $\mathcal{O}$ and $\Psi = \Psi_{\chi,e}$, let $g_\star$ be a marking, r a triangular lift for $g_\star$ to an integral test ring and $\rho = \rho_r$.*

(Cv1) *ρ satisfies (Lw1) with $M = N$, and for every place $v \mid \ell$ of K there are a place $\bar v \mid v$, characters χ_1, χ_2, χ_3 of G_K of level N and characters $\kappa_1, \kappa_2, \kappa_3$ of $D_{\bar v}$ such that ρ is triangular on $D_{\bar v}$ with diagonal $(\kappa_1, \kappa_2, \kappa_3)$ and $\kappa_a(s) = \chi_{\mathrm{cyc}}(s)^{m_a}\chi_a(s)$ for $s \in I_{\bar v}$, where $(m_1, m_2, m_3) = (e, 0, -e)$. If moreover (b) holds, or (a) holds with $e \ge 2$ or $\chi_\ell \ne \mathbf{1}$, or ρ is generic at every place $v \mid \ell$ of K, then ρ has level N and weights $(e, 0, -e)$ and is generic ordinary of level N and weights $(e, 0, -e)$.*

(Cv2) *Assume (a), let d_K be the discriminant of K and η_K the character of $G_\mathbb{Q}$ with kernel G_K, and let r be of type II. Then $\rho(g)_{22} = 1$ for all $g \in G_K$; there is a continuous $W\colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ whose restriction to G_K is the corner of r and with $\operatorname{tr} W(c) = 0$; and every such W has level $N|d_K|$ and weights $(e, -e)$, and $\det W = \eta_K$, a character of level $|d_K|$.*

Proof. Preliminaries. (i) ρ is continuous: A is a discrete valuation ring finite over $\mathcal{O}$ inside $\overline{\mathbb{Q}}_\ell$, so its $\mathfrak{m}_A$-adic topology is the ℓ-adic one, and (Tl1) gives local constancy modulo every $\mathfrak{m}_A^n$. (ii) ℓ is unramified in K ((Ue6) in case (a); $K = \mathbb{Q}(i)$ and ℓ odd in case (b)), so for $\bar v \mid \ell$ the inertia group of $G_\mathbb{Q}$ at $\bar v$ lies in G_K and equals $I_{\bar v}$, and $\chi_{\mathrm{cyc}}(I_{\bar v}) = \mathbb{Z}_\ell^\times$. Every open subgroup of $I_{\bar v}$ has an image under χ_{cyc} open in $\mathbb{Z}_\ell^\times$, hence containing some $1 + \ell^k\mathbb{Z}_\ell$, which has no non-trivial element of finite order; so χ_{cyc}^n is non-trivial on every open subgroup of $I_{\bar v}$ for $n \ne 0$. (iii) For $g \in I_{\bar v}$, $\bar v \mid \ell$, write $N = \ell^a N'$ with $\ell \nmid N'$: g fixes $\mu_{N'}$ and acts on μ_{ℓ^a} through $\chi_{\mathrm{cyc}}(g)$, so $\chi(g) = \chi_\ell(\chi_{\mathrm{cyc}}(g) \bmod \ell^a)$; hence $\chi|_{I_{\bar v}} = \mathbf{1}$ if and only if $\chi_\ell = \mathbf{1}$. (iv) In case (a), for $\bar v \mid \ell$, $\bar\Psi^2 \ne 1$ on $I_{\bar v}$, where $\bar\Psi$ is the reduction of Ψ: this is (Lo3). (v) If a representation of a group D on $\overline{\mathbb{Q}}_\ell^3$ is triangular on D with diagonals (κ_a) and (κ'_a), the two multisets coincide: each is the multiset of the one-dimensional composition factors of $\overline{\mathbb{Q}}_\ell^3$ as a D-module (Jordan–Hölder). (vi) A character of G_F of level M is of level MM' for every $M' \ge 1$, and products of characters of level M are of level M.

(Cv1), the clause (Lw1). Let $v \nmid \ell$ be a finite place of K and $\bar v \mid v$. By (Lo1) there are P and integers a_1, a_2, a_3 with $\rho(g) = P\,\mathrm{diag}(\Psi(g)^{a_i})P^{-1}$ for $g \in I_{\bar v}$. On $I_{\bar v}$, χ_{cyc} is trivial and $\Psi = \chi$; the characters $\chi^{a_i}|_{G_K}$ are trivial on $G_{K(\mu_N)} \subseteq G_{\mathbb{Q}(\mu_N)}$, so they are of level N, and $\rho|_{I_{\bar v}}$ is conjugate to $\mathrm{diag}(\chi^{a_1}, \chi^{a_2}, \chi^{a_3})$.

(Cv1), triangular form above ℓ. Let $v \mid \ell$. By (Lo2), v lies below some $\bar v \in \{\bar\ell, c\bar\ell\}$, and there are $h \equiv 1 \pmod{\mathfrak{m}_A}$ and characters κ_a of $D_{\bar v}$ with $h^{-1}\rho(g)h$ upper triangular with diagonal $(\kappa_a(g))$; so ρ is triangular on $D_{\bar v}$ with diagonal $(\kappa_1, \kappa_2, \kappa_3)$. For $s \in I_{\bar v}$, $\kappa_a(s) = \Psi(s)^{2-a} = \chi_{\mathrm{cyc}}(s)^{e(2-a)}\chi(s)^{2-a}$, and $e(2-a) = m_a$; put $\chi_a = \chi^{2-a}|_{G_K}$, of level N.

(Cv1), genericity. We show $\kappa_a \ne \kappa_b\chi_{\mathrm{cyc}}$ on $D_{\bar v}$ for all a, b under each of the three additional hypotheses. Then (Lw2) holds with the first alternative, so with (i) and (Lw1) ρ has level N

and weights $(e, 0, -e)$, and the same data show that ρ is generic ordinary of level N and weights $(e, 0, -e)$.

- For $a = b$: $\chi_{\mathrm{cyc}} \neq \mathbf{1}$ on $I_{\bar{v}}$ by (ii).
- Inertia. Let $a \neq b$ with $\kappa_a = \kappa_b \chi_{\mathrm{cyc}}$. On $I_{\bar{v}}$ this reads $\Psi^{b-a} = \chi_{\mathrm{cyc}}$, i.e. $\chi^{b-a} \chi_{\mathrm{cyc}}^{e(b-a)-1} = \mathbf{1}$. On the open subgroup $I_{\bar{v}} \cap \ker \chi$ this gives $\chi_{\mathrm{cyc}}^{e(b-a)-1} = \mathbf{1}$, so $e(b-a) = 1$ by (ii): $e = 1$ and $b = a+1$; then $\chi = \mathbf{1}$ on $I_{\bar{v}}$, i.e. $\chi_\ell = \mathbf{1}$ by (iii). So the claim holds if $e \geq 2$ or $\chi_\ell \neq \mathbf{1}$, in either case.
- Case (b). By Lemma 47.10 there is $x \in D_{\bar{\ell}}$ with $\chi_{\mathrm{cyc}}(x) = 1$ and $\Psi(x)^2 - 1 \in \mathcal{O}^\times$; put $y = x$ if $\bar{v} = \bar{\ell}$ and $y = cxc \in D_{c\bar{\ell}}$ if $\bar{v} = c\bar{\ell}$; as $\chi_{\mathrm{cyc}}(c)^2 = 1$ and $\Psi(c) = 1$, $\chi_{\mathrm{cyc}}(y) = 1$ and $\Psi(y) = \Psi(x)$. If $\kappa_a = \kappa_b \chi_{\mathrm{cyc}}$ with $a \neq b$, then $\kappa_a(y) = \kappa_b(y)$, and by (Lo2) $\Psi(y)^{2-a} \equiv \Psi(y)^{2-b}$, so $\Psi(y)^{b-a} \equiv 1 \pmod{\mathfrak{m}_A}$ with $b - a \in \{\pm 1, \pm 2\}$, whence $\Psi(y)^2 \equiv 1 \pmod{\mathfrak{m}_A}$. But $\Psi(y)^2 - 1$ is a unit of $\mathcal{O} \subseteq A$ and $\mathfrak{m}_A \cap \mathcal{O} = \mathfrak{m}_{\mathcal{O}}$: a contradiction.
- ρ generic at v. Then ρ is triangular on $D_{\bar{v}}$ with a diagonal (κ'_a) with $\kappa'_a \neq \kappa'_b \chi_{\mathrm{cyc}}$ for all a, b; by (v), $\{\kappa_a\} = \{\kappa'_a\}$ as multisets, and the claim follows.

(Cv2), a triviality criterion. Let $\lambda\colon G_K \to A^\times$ be a continuous character with $\lambda \equiv 1 \pmod{\mathfrak{m}_A}$ that is trivial on $I_{\bar{v}}$ for every place $\bar{v}$ of $\overline{\mathbb{Q}}$ above a finite prime. Then $\lambda = \mathbf{1}$: by class field theory λ factors through the Galois group of the Hilbert class field of K (the maximal abelian extension unramified at all places; K has no real places), which has order h_K; so $\lambda^{h_K} = \mathbf{1}$, and $(\lambda - 1)(1 + \lambda + \cdots + \lambda^{h_K - 1}) = 0$ with the second factor $\equiv h_K \not\equiv 0 \pmod{\mathfrak{m}_A}$ by (Ue6), so $\lambda = 1$. It suffices to check triviality on $I_{\bar{v}}$ for one $\bar{v}$ above each place v of K: the others are $\gamma I_{\bar{v}} \gamma^{-1}$ with $\gamma \in G_K$, and λ is a class function.

(Cv2), the determinant and the middle entry. Let r be of type II. As the entries of odd weight vanish, $\rho(g)$ preserves $U = \overline{\mathbb{Q}}_\ell e_1 \oplus \overline{\mathbb{Q}}_\ell e_3$ and $L = \overline{\mathbb{Q}}_\ell e_2$, so $\rho = C \oplus \mu$ with C the corner and $\mu(g) = \rho(g)_{22}$ a continuous character. Put $\Delta = \det \rho = \det C \cdot \mu$. Both Δ and μ are $\equiv 1 \pmod{\mathfrak{m}_A}$, since $\rho(g) \equiv \mathrm{T}(\Psi(g))$ by (Tl1). Away from ℓ: for $g \in I_{\bar{v}}$, $\bar{v} \nmid \ell$, the eigenvalues of $\rho(g)$ are the $\Psi(g)^{a_i} = \chi(g)^{a_i}$ of (Lo1), roots of unity of order dividing the order of χ, prime to ℓ by (Ue3); $\Delta(g)$ is their product and $\mu(g)$ is one of them; a root of unity of order prime to ℓ that is $\equiv 1 \pmod{\mathfrak{m}_A}$ equals 1 ($X^n - 1$ is separable modulo $\mathfrak{m}_A$ for $\ell \nmid n$). So $\Delta(g) = \mu(g) = 1$. Above ℓ, for $\bar{v} \in \{\bar{\ell}, c\bar{\ell}\}$ and $g \in I_{\bar{v}}$, the eigenvalues of $\rho(g)$ are $\Psi(g), 1, \Psi(g)^{-1}$ by (Lo2), so $\Delta(g) = 1$, and $\mu(g)$ is one of them. By (iv) there is $s \in I_{\bar{v}}$ with $\bar{\Psi}(s) \neq 1$. If $\bar{\Psi}(g) \neq 1$, then $\Psi(g)^{\pm 1} \not\equiv 1$ while $\mu(g) \equiv 1$, so $\mu(g) = 1$; if $\bar{\Psi}(g) = 1$, then $\bar{\Psi}(gs) \neq 1$ and $\mu(g) = \mu(gs)\mu(s)^{-1} = 1$. By the criterion, $\Delta = \mu = \mathbf{1}$: $\rho(g)_{22} = 1$ and $\det C(g) = 1$ for $g \in G_K$.

(Cv2), existence of W. Put $D = \mathrm{diag}(1, -1)$. As in (Lo2), $r(c) = (1, -1)$ gives $\rho(cgc) = \theta(\rho(g))$ for $g \in G_K$. For $\rho(g) = C(g) \oplus 1$, the matrix $\theta(\rho(g)) = J\, {}^t\rho(g)^{-1} J$ has corner $w\, {}^tC(g)^{-1} w$ with $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, since J exchanges e_1 and e_3. For $C = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ of determinant 1, ${}^tC^{-1} = \left(\begin{smallmatrix} \delta & -\gamma \\ -\beta & \alpha \end{smallmatrix}\right)$ and $w\, {}^tC^{-1} w = \left(\begin{smallmatrix} \alpha & -\beta \\ -\gamma & \delta \end{smallmatrix}\right) = DCD$. Hence $C(cgc) = D\, C(g)\, D$ for $g \in G_K$. Every element of $G_{\mathbb{Q}} \setminus G_K$ is gc with a unique $g \in G_K$; define $W(g) = C(g)$ and $W(gc) = C(g)D$ for $g \in G_K$. Using $c^2 = 1$, $D^2 = 1$ and $DC(h) = C(chc)D$, for $g, h \in G_K$: $W(g)W(hc) = C(gh)D = W(ghc)$; $W(gc)W(h) = C(g)C(chc)D = W(g\, chc\, c) = W(gch)$; $W(gc)W(hc) = C(g)C(chc) = W(g\, chc) = W(gchc)$. So W is a homomorphism, continuous because C is and G_K is open, with $W|_{G_K} = C$ and $\mathrm{tr}\, W(c) = \mathrm{tr}\, D = 0$. Moreover $\det W = \eta_K$: $\det W(g) = \det C(g) = 1$ and $\det W(gc) = -1$.

(Cv2), every such W. Let W' be continuous with $W'|_{G_K} = C$ and $\mathrm{tr}\, W'(c) = 0$. For $g \in G_K$, $W'(c)C(g)W'(c)^{-1} = C(cgc) = DC(g)D$, so $DW'(c)$ commutes with $C(G_K)$; C is irreducible (type II), so $DW'(c)$ is a scalar λ (Schur's lemma over $\overline{\mathbb{Q}}_\ell$), and $W'(c)^2 = W'(c^2) = 1$ gives

$\lambda = \pm 1$. Hence $W' = W$ or $W' = W \otimes \eta_K$, and $\det W' = \det W = \eta_K$. The character η_K is of level $|d_K|$ ($K \subseteq \mathbb{Q}(\mu_{|d_K|})$), hence of level $N|d_K|$ by (vi). We check (Lw1), (Lw2) for W' over $F = \mathbb{Q}$ with $M = N|d_K|$ and weights $(e, -e)$; W' is continuous by assumption.

- (Lw1) at $p \neq \ell$ with $p \nmid d_K$, and $\bar{p} \mid p$: K is unramified at p, so $I_{\bar{p}} \subseteq G_K$ and $W'|_{I_{\bar{p}}} = C|_{I_{\bar{p}}}$. By (Lo1) the finite group $\rho(I_{\bar{p}})$ is diagonalisable with $\rho|_{I_{\bar{p}}} \cong \bigoplus_i \chi^{a_i}|_{I_{\bar{p}}}$; $C|_{I_{\bar{p}}}$ is a direct summand, so (representations of a finite group in characteristic 0 being semisimple, and the decomposition into characters unique) $C|_{I_{\bar{p}}}$ is conjugate to $\mathrm{diag}(\chi^{a_i}, \chi^{a_j})$ for some $i \neq j$; the χ^a are of level N, hence of level $N|d_K|$.

- (Lw1) at $p \mid d_K$: $p \neq \ell$ by (Ue6) and $p \nmid N$ by (Ue7). On $H = I_{\bar{p}} \cap G_K$, $\eta_K = 1$ and $\Psi = \chi\chi_{\mathrm{cyc}}^e = 1$ (χ is unramified at p), so $r = 1$ on H by (Tl3) and $W'|_H = 1$. As K is ramified at p, $[I_{\bar{p}} : H] = 2$; for $\tau \in I_{\bar{p}} \setminus H$, $W'(\tau)^2 = W'(\tau^2) = 1$ and $\det W'(\tau) = \eta_K(\tau) = -1$, so $W'(\tau)$ is conjugate to $\mathrm{diag}(1, -1)$ and $W'|_{I_{\bar{p}}}$ is conjugate to $\mathrm{diag}(\mathbf{1}, \eta_K)$, characters of level $|d_K|$, hence of level $N|d_K|$.

- (Lw2) at ℓ. Let $D_\ell \subseteq G_{\mathbb{Q}}$ be the decomposition group at $\bar{\ell}$ and $D' = D_\ell \cap G_K = D_{\bar{\ell}}$; $I_{\bar{\ell}} \subseteq D'$ by (ii). Take h, κ_a of (Lo2) at $\bar{\ell}$ and let V_1 be the line spanned by he_1, stable under $\rho(D')$ with character κ_1. The projection $U \oplus L \to L$ is D'-equivariant and D' acts on L trivially ($\mu = \mathbf{1}$); as $\kappa_1 = \Psi \neq \mathbf{1}$ on $I_{\bar{\ell}}$ by (iv), V_1 projects to 0, so $V_1 \subseteq U$, and D' acts on U/V_1 by $\kappa_1^{-1} \det C = \kappa_1^{-1}$. If $D' \neq D_\ell$, let $\tau \in D_\ell \setminus D'$; $W'(\tau)V_1 \subseteq U$ is D'-stable with character $g \mapsto \kappa_1(\tau^{-1}g\tau)$, equal to Ψ on the normal subgroup $I_{\bar{\ell}}$. If $W'(\tau)V_1 \neq V_1$, then $U = V_1 \oplus W'(\tau)V_1$ and $U/V_1 \cong W'(\tau)V_1$, so $\Psi^{-1} = \Psi$ on $I_{\bar{\ell}}$, contradicting (iv). So V_1 is $W'(D_\ell)$-stable and W' is triangular on D_ℓ with a diagonal (λ_1, λ_2) of characters of D_ℓ, with $\lambda_1 = \Psi = \chi_{\mathrm{cyc}}^e\chi$ and $\lambda_2 = \det W' \cdot \lambda_1^{-1} = \chi_{\mathrm{cyc}}^{-e}\chi^{-1}$ on $I_{\bar{\ell}} \subseteq G_K$; $\chi^{\pm 1}$ are of level N. Genericity: $\lambda_a \neq \lambda_a\chi_{\mathrm{cyc}}$ by (ii); $\lambda_1 = \lambda_2\chi_{\mathrm{cyc}}$ would give $\chi^2\chi_{\mathrm{cyc}}^{2e-1} = \mathbf{1}$ on $I_{\bar{\ell}}$, and $\lambda_2 = \lambda_1\chi_{\mathrm{cyc}}$ would give $\chi^{-2}\chi_{\mathrm{cyc}}^{-2e-1} = \mathbf{1}$; on $I_{\bar{\ell}} \cap \ker\chi$ both contradict (ii), as $2e \pm 1 \neq 0$.

So W' has level $N|d_K|$ and weights $(e, -e)$, and $\det W' = \eta_K$ is of level $|d_K|$. □

Proposition 47.12. *Let $(K, \ell, 1, N, \chi)$ be a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$ with $e_{\mathcal{O}} = 1$, assume $\chi_\ell = \mathbf{1}$, and let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism. Let $g_\star$ be a marking and (R, r) a normalised ordinary Eisenstein deformation ring for $g_\star$. For every point φ of type I such that $\rho_\varphi \otimes \chi_{\mathrm{cyc}}^{-1}$ is automorphic of conjugate self-dual type for ι, ρ_φ is generic at every place $v \mid \ell$ of K.*

Proof. Let φ be as in the statement and $\rho = \rho_\varphi$.

Step 1: the triangular forms. By Lemma 47.3, $\varphi \circ r$ is a triangular lift to an integral test ring A for $g_\star$, formed with K, $\mathcal{O}$, Ψ and $\bar{\ell}$, with $\rho_{\varphi\circ r} = \rho$; and $\Psi(c) = 1$ by (Ue5). Let $v \mid \ell$ be a place of K. By Lemma 47.9, (Lo2), v lies below some $\bar{w} \in \{\bar{\ell}, c\bar{\ell}\}$, and there are $h \in \mathrm{GL}_3(A)$ and characters $\kappa_1, \kappa_2, \kappa_3$ of the decomposition group $D = D_{\bar{w}}$ of G_K at $\bar{w}$ such that $h^{-1}\rho(g)h$ is upper triangular with diagonal $(\kappa_a(g))_a$ for $g \in D$, and $\kappa_a = \psi_a$ on the inertia group $I_{\bar{w}}$. Write $N = \ell^b N'$ with $\ell \nmid N'$; on $I_{\bar{w}}$, $\chi = \chi_\ell(\varepsilon \bmod \ell^b) = 1$, so $\Psi = \varepsilon$ and $(\kappa_1, \kappa_2, \kappa_3) = (\varepsilon, 1, \varepsilon^{-1})$ on $I_{\bar{w}}$.

Step 2: only two coincidences are possible. If $\kappa_i = \kappa_j\varepsilon$ on D, then $\varepsilon^{2-i} = \varepsilon^{3-j}$ on $I_{\bar{w}}$, where ε has infinite order; so $j = i + 1$. Hence the diagonal (κ_a) is generic unless $\kappa_1 = \kappa_2\varepsilon$ or $\kappa_2 = \kappa_3\varepsilon$ on D.

Step 3: $r' = \rho \otimes \varepsilon^{-1}$ is the representation attached to π. By hypothesis $r' \leftrightarrow_\iota \pi$ for a regular algebraic conjugate self-dual cuspidal π of $\mathrm{GL}_3(\mathbb{A}_K)$; π is polarizable with the trivial character,

and r' is irreducible (φ is of type I). By the Chebotarev density theorem and the Brauer–Nesbitt theorem, r' and the semisimple $r_{\ell,\iota}(\pi)$ of Theorem B.62, which agree in characteristic polynomials at almost all Frobenius elements by (Bg1), are isomorphic.

Step 4: r' is crystalline at $\bar{w}$. If $\bar{w} = \bar{\ell}$, $\rho|_D$ is crystalline by Proposition 47.13. If $\bar{w} = c\bar{\ell}$: by (Tl1), (Tl2), $r(c) = (1,-1)$ in $\mathcal{G}_3$, so $\rho(cgc) = \theta(\rho(g)) = J\,{}^t\rho(g)^{-1}J$ for $g \in G_K$; the decomposition group at $c\bar{\ell}$ is $c(D_\ell \cap G_K)c$, and $g \mapsto cgc$ is the isomorphism of decomposition groups induced by the isomorphism of completions given by c; so $\rho|_D$ is the transport along this isomorphism of the dual of the crystalline $\rho|_{D_\ell \cap G_K}$, hence crystalline. As ε is crystalline, so is $r'|_D$, and the monodromy of $\mathrm{WD}(r'|_D)$ vanishes.

Step 5: the Frobenius eigenvalues. $r'|_D$ is triangular with the crystalline diagonal characters $\kappa_a\varepsilon^{-1}$ (subquotients of a crystalline representation). Fontaine's functor D_{pst} is exact and compatible with tensor products, so $\mathrm{WD}(r'|_D)$ has the semisimplification $\bigoplus_a \mathrm{WD}(\kappa_a\varepsilon^{-1})$, and $\mathrm{WD}(\mu\mu') = \mathrm{WD}(\mu) \otimes \mathrm{WD}(\mu')$ for characters μ, μ' of D for which both sides are defined. $\mathrm{WD}(\varepsilon)$ is the unramified character on which a geometric Frobenius Φ ($n(\Phi) = -1$) acts by q_v^{-1} or by q_v, according to the normalisation of (Wd2) ($D_{\mathrm{cris}}(\varepsilon)$ is spanned by t^{-1}, on which the crystalline Frobenius acts by ℓ^{-1}, and q_v is a power ℓ^f); in either case its image β under ι has $|\beta| \neq 1$. Let α_a be the eigenvalue of Φ on $\mathrm{WD}(\kappa_a\varepsilon^{-1})$; the eigenvalues of Φ on $\mathrm{WD}(r'|_D)$ are $\alpha_1, \alpha_2, \alpha_3$. By Step 3, $\mathrm{WD}(r'|_D) \cong \mathrm{WD}(r_{\ell,\iota}(\pi)|_D)$, so by Theorem B.73 (with $F = K$, $n = 3$, the place v and $\bar{v} = \bar{w}$) $\mathrm{WD}(r'|_D) = (V, \rho', N)$ is pure of some weight k. By Step 4, $N = 0$; for every $i \geq 1$ the map $\mathrm{gr}_{k+i} \to \mathrm{gr}_{k-i}$ induced by $N^i = 0$ is then both zero and an isomorphism (Pu2), so $\mathrm{gr}_j = 0$ for all $j \neq k$, and since the filtration is 0 for small and V for large indices, $V = \mathrm{gr}_k$. By (Pu1), $|\iota(\alpha_a)| = q_v^{k/2}$ for $a = 1,2,3$. If $\kappa_1 = \kappa_2\varepsilon$, then $\mathrm{WD}(\kappa_1\varepsilon^{-1}) = \mathrm{WD}(\kappa_2\varepsilon^{-1}) \otimes \mathrm{WD}(\varepsilon)$, so $\iota(\alpha_1) = \iota(\alpha_2)\beta$ and $|\iota(\alpha_1)| \neq |\iota(\alpha_2)|$, a contradiction; likewise $\kappa_2 = \kappa_3\varepsilon$ is impossible. By Step 2, the diagonal (κ_a) on $D_{\bar{w}}$ is generic.

Step 6: all places above v. Let $\bar{v} \mid v$. G_K acts transitively on the places of $\overline{\mathbb{Q}}$ above v, so $\bar{v} = \gamma\bar{w}$ with $\gamma \in G_K$ and $D_{\bar{v}} = \gamma D\gamma^{-1}$; for $g \in D$, $(\rho(\gamma)h)^{-1}\rho(\gamma g\gamma^{-1})\rho(\gamma)h = h^{-1}\rho(g)h$. So ρ is triangular on $D_{\bar{v}}$ with diagonal $\kappa_a(\gamma^{-1}\cdot\gamma)$, and since $\varepsilon(\gamma^{-1}g\gamma) = \varepsilon(g)$ the inequalities $\kappa_i \neq \kappa_j\varepsilon$ persist. So ρ is generic at v. □

Proposition 47.13. *Let $(K, \ell, 1, N, \chi)$ be a unitary Eisenstein datum with χ valued in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and assume $\chi_\ell = \mathbf{1}$. Let $g_\star$ be a marking and (R, r) a normalised ordinary Eisenstein deformation ring for $g_\star$. For every point φ of (R, r), the restriction of ρ_φ to $D_\ell \cap G_K$ (the absolute Galois group of the completion of K at the place below $\bar{\ell}$) is crystalline.*

Proof. Setting. Put $D' = D_\ell \cap G_K$. By (Ue6), ℓ is unramified in K, so $I_\ell \subseteq D'$. Through an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ inducing $\bar{\ell}$, D' is the absolute Galois group G_M of the completion M of K at the place below $\bar{\ell}$, $I_\ell = I_M$, and $\varepsilon|_{D'}$ is the cyclotomic character of M. As $\chi_\ell = \mathbf{1}$, $D_\ell^\circ = D_\ell \cap \ker \eta_K = D'$. Write $N = \ell^b N'$ with $\ell \nmid N'$; an element of I_ℓ fixes the N'-th roots of unity, so $\chi = \chi_\ell(\varepsilon \bmod \ell^b) = \mathbf{1}$ on I_ℓ, hence $\Psi = \varepsilon$ and $\psi_a = \varepsilon^{2-a}$ on I_ℓ. For an $\mathcal{O}$-algebra homomorphism $y\colon R_\ell^\square \to A$ write $\rho_y\colon D' \to \mathrm{GL}_3(A)$ for the restriction to D' of the GL-part of $y \circ r_\ell^\square$.

Step 1: the components. $R_\ell^\square$ is Noetherian, so I_ℓ^{ord} is the intersection of finitely many minimal primes $\mathfrak{P}$ of I^{cr} (those contained in the kernel of every base point). For each of them $\mathrm{gd} = \mathrm{gd}(r_\ell^\square) \notin \mathfrak{P}$ and $\ell \notin \mathfrak{P}$. Indeed, put $J = \{z : \mathrm{gd}^n z \in I^{\mathrm{ord}}(r_\ell^\square) \text{ for some } n\}$, so that $I^{\mathrm{cr}} = \sqrt{J}$ and $\mathfrak{P}$ is minimal over J; if $\mathrm{gd} \in \mathfrak{P}$, then, $JR_{\ell,\mathfrak{P}}^\square$ being primary to the maximal ideal, $\mathrm{gd}^n \in JR_{\ell,\mathfrak{P}}^\square$ for some n, i.e. $t\,\mathrm{gd}^n \in J$ for some $t \notin \mathfrak{P}$; then $t \in J \subseteq \mathfrak{P}$ (if $\mathrm{gd}^n t \in J$ then $\mathrm{gd}^{n+m}t \in I^{\mathrm{ord}}(r_\ell^\square)$ for some m), a contradiction. As $\mathrm{gd} = \ell\,\mathrm{dt}_1\mathrm{dt}_2$, also $\ell \notin \mathfrak{P}$.

Step 2: points with $\mathrm{gd} \neq 0$ are crystalline. Let $\mathfrak{P}$ be as in Step 1, A the valuation ring of a

finite extension of the fraction field of $\mathcal{O}$ in $\overline{\mathbb{Q}}_\ell$, and $y\colon R^{\square}_\ell \to A$ an $\mathcal{O}$-algebra homomorphism with $y(\mathfrak{m}_{R^{\square}_\ell}) \subseteq \mathfrak{m}_A$, $y(\mathfrak{P}) = 0$ and $y(\mathrm{gd}) \neq 0$. We show: ρ_y is crystalline with τ-Hodge–Tate weights $\{1, 0, -1\}$ for every embedding $\tau\colon M \to \overline{\mathbb{Q}}_\ell$. As $I_\ell^{\mathrm{ord}} \subseteq \mathfrak{P}$, $y(I_\ell^{\mathrm{ord}}) = 0$, and (Ol2) for the integral test ring A gives $h \in \mathrm{GL}_3(A)$ and characters $\kappa_1, \kappa_2, \kappa_3$ of D' such that $Y(g) = h^{-1}\rho_y(g)h$ is upper triangular with diagonal $(\kappa_a(g))_a$ for $g \in D'$, and $\kappa_a = \psi_a = \varepsilon^{2-a}$ on I_ℓ. The diagonal entries $\psi_b(\sigma_0) = \varepsilon(\sigma_0)^{2-b}$ of $Y(\sigma_0)$ are pairwise distinct ($\varepsilon(\sigma_0)$ has infinite order), so there is an upper triangular $Q \in \mathrm{GL}_3(\mathrm{Frac}\, A)$ (its columns are eigenvectors of $Y(\sigma_0)$, the b-th in the span of $e_1, \dots, e_b$) with $Q^{-1}Y(\sigma_0)Q = \mathrm{diag}(\psi_b(\sigma_0))_b$; then $Q^{-1}Y(\sigma_\ell)Q$ is upper triangular with diagonal $(\kappa_b(\sigma_\ell))_b$, since $\sigma_\ell \in D_\ell^\circ = D'$. The matrix $\big(\mathrm{diag}(\psi_b(\sigma_0)) - \psi_{a+1}(\sigma_0)\big)\big(\mathrm{diag}(\psi_b(\sigma_0)) - \psi_{a+2}(\sigma_0)\big)$ is gp_a times the matrix unit E_{aa}. The entries of $\mathrm{tq}_a(r^{\square}_\ell)$ are polynomials in the entries of $\underline{r^{\square}_\ell}(\sigma_0)$, $\underline{r^{\square}_\ell}(\sigma_\ell)$, and the trace is invariant under conjugation, so

$$y(\mathrm{tq}_a(r^{\square}_\ell)) = \mathrm{tr}\big(\mathrm{gp}_a E_{aa}\, Q^{-1}Y(\sigma_\ell)Q\big) = \mathrm{gp}_a\, \kappa_a(\sigma_\ell),$$
$$y(\mathrm{dt}_a(r^{\square}_\ell)) = \mathrm{gp}_a \mathrm{gp}_{a+1}\big(\kappa_a(\sigma_\ell) - \varepsilon(\sigma_\ell)\kappa_{a+1}(\sigma_\ell)\big).$$

As $y(\mathrm{gd}) = \ell\, y(\mathrm{dt}_1)y(\mathrm{dt}_2) \neq 0$, we get $\kappa_1 \neq \kappa_2\varepsilon$ and $\kappa_2 \neq \kappa_3\varepsilon$ on D'; and $\kappa_1 \neq \kappa_3\varepsilon$, since on I_ℓ, $\kappa_1\kappa_3^{-1} = \varepsilon^2 \neq \varepsilon$. By Lemma 47.14 (with M, $n = 3$, $(m_1, m_2, m_3) = (1, 0, -1)$ and the upper triangular Y), $\rho_y \cong Y$ is crystalline; its graded pieces $\kappa_a = \varepsilon^{2-a} \cdot$ (unramified) have τ-Hodge–Tate weight $2 - a$ for every τ, so its τ-Hodge–Tate weights are $\{1, 0, -1\}$.

Step 3: the crystalline ideal after enlarging the coefficients. Let $\mathcal{O}' \supseteq \mathcal{O}$ be the ring of integers of a finite extension of the fraction field of $\mathcal{O}$ in $\overline{\mathbb{Q}}_\ell$ containing the images of all embeddings $M \to \overline{\mathbb{Q}}_\ell$, with residue field $\mathbb{F}'$, and $R' = R^{\square}_\ell \otimes_{\mathcal{O}} \mathcal{O}'$. As $\mathcal{O}'$ is free of finite rank over $\mathcal{O}$, R' is a finite free $R^{\square}_\ell$-module, hence a complete Noetherian semi-local ring whose maximal ideals lie over $\mathfrak{m}_{R^{\square}_\ell}$; as $R'/\mathfrak{m}_{R^{\square}_\ell}R' = \mathcal{O}'/\mathfrak{m}_{\mathcal{O}}\mathcal{O}'$ is local with residue field $\mathbb{F}'$, R' is a complete local Noetherian $\mathcal{O}'$-algebra with residue field $\mathbb{F}'$, whose maximal ideal is generated by $\mathfrak{m}_{R^{\square}_\ell}$ and $\mathfrak{m}_{\mathcal{O}'}$. The homomorphism $\rho' = \rho_{\mathrm{id}} \otimes 1\colon D' \to \mathrm{GL}_3(R')$ is a continuous lift of $\bar\rho = \mathrm{T}(\bar\Psi)|_{D'}$ (with values in $\mathbb{F}_{\mathcal{O}} \subseteq \mathbb{F}'$). Let (S, ρ^S) be the universal lifting ring of $\bar\rho$ over $\mathcal{O}'$, $h_S\colon S \to R'$ the local homomorphism with $h_S \circ \rho^S = \rho'$, $\mathfrak{a} \subseteq S$ the ideal of Theorem B.57 for $H_\tau = \{1, 0, -1\}$ (all τ), and $\mathfrak{b} = h_S(\mathfrak{a})R'$. Let $y\colon R^{\square}_\ell \to A$ be an $\mathcal{O}$-algebra homomorphism into the valuation ring of a finite extension, with $y(\mathfrak{m}_{R^{\square}_\ell}) \subseteq \mathfrak{m}_A$; let A' be the valuation ring of the compositum of the fraction fields of A and $\mathcal{O}'$, and $y'\colon R' \to A'$, $z \otimes o \mapsto y(z)o$. Then y' maps the maximal ideal of R' into $\mathfrak{m}_{A'}$, and $y' \circ h_S$ is a local $\mathcal{O}'$-algebra homomorphism with $(y' \circ h_S) \circ \rho^S = y' \circ \rho' = \rho_y$. By Theorem B.57:

$$y'(\mathfrak{b}) = 0 \iff \rho_y \text{ is crystalline with } \tau\text{-Hodge–Tate weights } \{1, 0, -1\} \text{ for every } \tau. \tag{47.4}$$

Step 4: each component lies in the crystalline locus. Let $\mathfrak{P}$ be as in Step 1, $B = R^{\square}_\ell/\mathfrak{P}$ and $B' = B \otimes_{\mathcal{O}} \mathcal{O}' = R'/\mathfrak{P}R'$. We show that the image $\bar{\mathfrak{b}}$ of $\mathfrak{b}$ in B' is 0. B is a domain in which $\ell \neq 0$ (Step 1), so $\mathcal{O} \to B$ is injective and the fraction field k_B of B contains that of $\mathcal{O}$. B' is free over B, so $B' \subseteq k_B \otimes_{\mathcal{O}} \mathcal{O}'$, a finite separable algebra over the field k_B of characteristic 0, hence a product of fields; so B' is reduced, and ℓ and gd, non-zero in B, are not zero divisors in B'. B' is a quotient of R', hence complete local Noetherian with finite residue field $\mathbb{F}'$. Suppose $g \in \bar{\mathfrak{b}}$, $g \neq 0$; then $g\,\mathrm{gd} \neq 0$, and Lemma 47.15 (with $\mathcal{O}'$, $C = B'$, $f = g\,\mathrm{gd}$) gives the valuation ring A' of a finite extension and an $\mathcal{O}'$-algebra homomorphism $y_1\colon B' \to A'$ with $y_1(\mathfrak{m}_{B'}) \subseteq \mathfrak{m}_{A'}$, $y_1(g) \neq 0$ and $y_1(\mathrm{gd}) \neq 0$. Let $y\colon R^{\square}_\ell \to A'$ be the composite of $R^{\square}_\ell \to B \to B'$ with y_1: an $\mathcal{O}$-algebra homomorphism with $y(\mathfrak{m}_{R^{\square}_\ell}) \subseteq \mathfrak{m}_{A'}$, $y(\mathfrak{P}) = 0$, $y(\mathrm{gd}) \neq 0$. By Step 2, ρ_y is crystalline with τ-Hodge–Tate weights $\{1, 0, -1\}$, so by (47.4) $y'(\mathfrak{b}) = 0$. But y' and the composite $R' \to B' \xrightarrow{y_1} A'$ are $\mathcal{O}'$-linear and agree on $R^{\square}_\ell \otimes 1$, so they are equal, and $y_1(g) = 0$: a contradiction. Hence $\bar{\mathfrak{b}} = 0$, i.e. $\mathfrak{b} \subseteq \mathfrak{P}R'$.

Step 5: the points of (R,r). Let φ be a point of (R,r). By (Ol1) there is an integral test ring A with $\varphi(R)\subseteq A$ and $\varphi(\mathfrak{m}_R)\subseteq\mathfrak{m}_A$. By (Ed3), $r|_{D_\ell}=f\circ r_\ell^{\square}$ for a local homomorphism $f\colon R_\ell^{\square}\to R$ with $f(I_\ell^{\mathrm{ord}})=0$; put $x=\varphi\circ f$, so $x(\mathfrak{m}_{R_\ell^{\square}})\subseteq\mathfrak{m}_A$, $x(I_\ell^{\mathrm{ord}})=0$ and $\rho_x=\rho_\varphi|_{D'}$. The primes of Step 1 exist (otherwise $I_\ell^{\mathrm{ord}}=R_\ell^{\square}$ and $x(1)=0$), and the prime ideal $\ker x$ contains their product, hence one of them, $\mathfrak{P}$. Then $x'\colon R'\to A'$ factors through $B'=R'/\mathfrak{P}R'$, so $x'(\mathfrak{b})=0$ by Step 4, and by (47.4) $\rho_\varphi|_{D'}=\rho_x$ is crystalline. □

Lemma 47.14. *Let ℓ be a prime, M a finite extension of $\mathbb{Q}_\ell$ with absolute Galois group G_M and inertia subgroup I_M, $E\subset\overline{\mathbb{Q}}_\ell$ a finite extension of $\mathbb{Q}_\ell$, $n\geq 1$, and $\rho\colon G_M\to\mathrm{GL}_n(E)$ a continuous homomorphism with values in the upper triangular matrices, with diagonal entries $\kappa_1,\dots,\kappa_n\colon G_M\to E^\times$. Assume that there are integers $m_1>m_2>\cdots>m_n$ such that $\kappa_a(g)=\chi_{\mathrm{cyc}}(g)^{m_a}$ for all $g\in I_M$ and all a, and that $\kappa_a\neq\kappa_b\chi_{\mathrm{cyc}}$ for all $a<b$. Then the representation ρ on E^n, regarded as a $\mathbb{Q}_\ell$-vector space, is crystalline.*

Proof. We use the classical facts of Fontaine's theory: crystalline representations are de Rham; they are stable under subrepresentations, quotients, tensor products and duals; χ_{cyc} and every unramified character are crystalline; for a crystalline (or de Rham) U, $\dim_{M_0}D_{\mathrm{cris}}(U)\leq\dim_{\mathbb{Q}_\ell}U$ always, with equality exactly when U is crystalline, and $\dim_M D_{\mathrm{dR}}(U)=\dim_{\mathbb{Q}_\ell}U$; in the normalisation fixed in Section 13, a de Rham U all of whose Hodge–Tate weights are ≥ 1 has $\mathrm{Fil}^0D_{\mathrm{dR}}(U)=0$, and the Hodge–Tate weights of a de Rham representation with a filtration by subrepresentations are those of its graded pieces.

The characters. Put $\mu_a=\kappa_a\chi_{\mathrm{cyc}}^{-m_a}$; by hypothesis μ_a is trivial on I_M, i.e. unramified, so $E(\kappa_a)=E(\chi_{\mathrm{cyc}}^{m_a})\otimes_E E(\mu_a)$ is crystalline with all Hodge–Tate weights equal to m_a. This is the case $n=1$. We argue by induction on $n\geq 2$.

The extension. Let $X=E^n$ with the action ρ and $X'\subset X$ the span of $e_1,\dots,e_{n-1}$, stable because ρ is upper triangular; the action on X' is upper triangular with diagonal $\kappa_1,\dots,\kappa_{n-1}$ and satisfies the hypotheses, so X' is crystalline by induction, and $X/X'\cong E(\kappa_n)$. Put $Y=X\otimes_E E(\kappa_n^{-1})$ and $U=X'\otimes_E E(\kappa_n^{-1})\subset Y$, so that $Y/U\cong E$ with trivial action. U is crystalline, and it has the filtration by the spans of $e_1,\dots,e_k$ $(k<n)$ with graded pieces $E(\kappa_a\kappa_n^{-1})$, $a<n$. Fix $y_0\in Y$ mapping to $1\in E$ and put $c(g)=gy_0-y_0\in U$, a continuous 1-cocycle.

$h^0(U)=0$ and $h^0(U^*(1))=0$. On I_M, $\kappa_a\kappa_n^{-1}=\chi_{\mathrm{cyc}}^{m_a-m_n}$ with $m_a-m_n\geq 1$, which is non-trivial because $\chi_{\mathrm{cyc}}(I_M)$ is open in $\mathbb{Z}_\ell^\times$, hence infinite; so every graded piece of U has no non-zero invariants, and by left exactness of invariants along the filtration $U^{G_M}=0$. Via the trace form of $E/\mathbb{Q}_\ell$, $U^*(1)\cong\mathrm{Hom}_E(U,E)\otimes\chi_{\mathrm{cyc}}$, which has a filtration with graded pieces $E(\kappa_n\kappa_a^{-1}\chi_{\mathrm{cyc}})$, $a<n$; these characters are non-trivial because $\kappa_a\neq\kappa_n\chi_{\mathrm{cyc}}$; so again $h^0(U^*(1))=0$.

$H^1_f(G_M,U)=H^1(G_M,U)$. By Theorem B.36, $h^1(U)=d\dim_{\mathbb{Q}_\ell}U+h^0(U)+h^0(U^*(1))=d\dim_{\mathbb{Q}_\ell}U$. The Hodge–Tate weights of U are the numbers $m_a-m_n\geq 1$, so $\mathrm{Fil}^0D_{\mathrm{dR}}(U)=0$ and $\dim_{\mathbb{Q}_\ell}D_{\mathrm{dR}}(U)=d\dim_M D_{\mathrm{dR}}(U)=d\dim_{\mathbb{Q}_\ell}U$. By Theorem B.56, $\dim_{\mathbb{Q}_\ell}H^1_f(G_M,U)=d\dim_{\mathbb{Q}_\ell}U+h^0(U)=h^1(U)$, and $H^1_f(G_M,U)\subseteq H^1(G_M,U)$ is an equality.

Y is crystalline. By the previous step the class of c lies in $H^1_f(G_M,U)$: there is $b\in B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}U$ with $1\otimes c(g)=gb-b$ for all $g\in G_M$ (G_M acting diagonally). Then $z=1\otimes y_0-b\in B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}Y$ satisfies

$$gz=1\otimes(y_0+c(g))-(b+1\otimes c(g))=z\qquad(g\in G_M),$$

and so do the elements $(m\otimes\lambda)z$ for $m\in M_0$, $\lambda\in E$ (the action of E on Y commutes with G_M). Invariants are left exact, so $0\to D_{\mathrm{cris}}(U)\to D_{\mathrm{cris}}(Y)\to D_{\mathrm{cris}}(E)=M_0\otimes_{\mathbb{Q}_\ell}E$ is exact, and the last map sends $(m\otimes\lambda)z$ to $m\otimes\lambda$; so it is onto, and $\dim_{M_0}D_{\mathrm{cris}}(Y)=\dim_{\mathbb{Q}_\ell}U+[E:\mathbb{Q}_\ell]=\dim_{\mathbb{Q}_\ell}Y$. Hence Y is crystalline, and so is $X=Y\otimes_E E(\kappa_n)$. □

Lemma 47.15. *Let $\mathcal{O} \subset \overline{\mathbb{Q}}_\ell$ be a coefficient ring with maximal ideal $\mathfrak{m}_\mathcal{O}$, and let C be a complete local Noetherian ring with finite residue field and maximal ideal $\mathfrak{m}_C$, together with a ring homomorphism $\mathcal{O} \to C$ mapping $\mathfrak{m}_\mathcal{O}$ into $\mathfrak{m}_C$. Assume that C is reduced and that ℓ is not a zero divisor in C. Then for every $f \in C$ with $f \neq 0$ there are the valuation ring A of a finite extension of the fraction field of $\mathcal{O}$ inside $\overline{\mathbb{Q}}_\ell$ and an $\mathcal{O}$-algebra homomorphism $y\colon C \to A$ with $y(\mathfrak{m}_C) \subseteq \mathfrak{m}_A$ and $y(f) \neq 0$.*

Proof. Reduction to a domain. C is reduced and Noetherian, so the intersection of its finitely many minimal primes is 0 and their union is the set of zero divisors. Choose a minimal prime Q with $f \notin Q$; then $\ell \notin Q$. C/Q satisfies the hypotheses, is a domain, and $f \neq 0$, $\ell \neq 0$ in it; a homomorphism $C/Q \to A$ as required gives one for C. So assume C is a domain. Then $\ell \neq 0$ lies in $\mathfrak{m}_C$, so $\dim C \geq 1$, and by Lemma 47.7 (for the non-zero element ℓf) there is a prime q with $\ell f \notin q$ and $\dim C/q = 1$; C/q again satisfies the hypotheses with $f \neq 0$, so we may assume $\dim C = 1$.

The one-dimensional case. Since $\ell \neq 0$ in the one-dimensional local domain C, $C/\ell C$ has dimension 0: it is an Artinian local ring with finite residue field, so its maximal ideal is nilpotent, $\mathfrak{m}_C^N \subseteq \ell C$ for some N, and $C/\ell C$, of finite length with each $\mathfrak{m}_C^i/\mathfrak{m}_C^{i+1}$ finite-dimensional over the finite residue field, is finite. Hence $C/\mathfrak{m}_\mathcal{O} C$ is finite and the $\mathfrak{m}_C$-adic and $\mathfrak{m}_\mathcal{O}$-adic topologies of C coincide; as C is complete and separated, C is a finitely generated $\mathcal{O}$-module (Nakayama's lemma for complete modules). The map $\mathcal{O} \to C$ is injective (its kernel is a prime of $\mathcal{O}$ not containing ℓ), so C is a domain finite over $\mathcal{O}$, and $k = C[\ell^{-1}]$ is a domain finite over the field $\mathrm{Frac}(\mathcal{O})$, hence a field. Choose an embedding $k \to \overline{\mathbb{Q}}_\ell$ over $\mathrm{Frac}(\mathcal{O}) \subset \overline{\mathbb{Q}}_\ell$, let y be its restriction to C (injective, so $y(f) \neq 0$) and A the valuation ring of the image of k; $y(C) \subseteq A$ because C is integral over $\mathcal{O}$. For $x \in \mathfrak{m}_C$, $x^N \in \ell C$, so $y(x)^N \in \ell A$ and $|y(x)|_\ell < 1$: $y(\mathfrak{m}_C) \subseteq \mathfrak{m}_A$. □

Theorem 47.16. *Let (ℓ, e, N, χ, o, m) be a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$ for a coefficient ring $\mathcal{O}$, and let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism. There is a marking $g_\star$ such that for every dyadic Eisenstein deformation ring (R, r) for $g_\star$:*

(Da1) *R^0 is a finitely generated $\mathcal{O}$-module;*

(Da2) *every point φ of (R, r) of type I is Lie-irreducible automorphic for ι, and ρ_φ is admissible of stable type for (ι, e, N).*

Proof. Choice of the marking. By Theorem 48.1, case (b), there is a marking $g_\star$ with (Tp1) and (Tp2). Let (R, r) be a dyadic Eisenstein deformation ring for $g_\star$.

(Da1). By (Tp1) the hypothesis of Proposition 47.2, case (b), holds, so R^0 is a finitely generated $\mathcal{O}$-module.

Points are triangular lifts. Let φ be a point. Since R is local with residue field $\mathbb{F}_\mathcal{O}$, $R = \mathcal{O} + \mathfrak{m}_R$, so $|\varphi(y)|_\ell \leq 1$ for all $y \in R$. Choose $x_1, \ldots, x_n$ generating $\mathfrak{m}_R$ and a surjection $\pi\colon \mathcal{O}[[X_1, \ldots, X_n]] \to R$ with $X_i \mapsto x_i$; put $a_i = \varphi(x_i)$, let $E \subset \overline{\mathbb{Q}}_\ell$ be the finite extension of the fraction field of $\mathcal{O}$ generated by the a_i, and A its valuation ring. For $f \in \mathcal{O}[[X]]$ and $k \geq 1$ write $f = P_k + \sum_{|\alpha|=k} X^\alpha g_\alpha$ with P_k a polynomial; then $|\varphi(\pi f) - P_k(a)|_\ell \leq \max_i |a_i|_\ell^k \to 0$, so $\varphi(\pi f) = \lim_k P_k(a) \in A$ (A is closed). Thus $\varphi(R) \subseteq A$, A is an integral test ring, and $\varphi\colon R \to A$ maps $\mathfrak{m}_R$ into $\mathfrak{m}_A$. Put $r'' = \varphi \circ r\colon G_\mathbb{Q} \to \mathcal{G}_3(A)$. By Lemma 47.4, r'' is a triangular lift to A for $g_\star$, with $\rho_{r''} = \rho_\varphi$.

(Da2). Let φ be of type I. By Lemma 47.8, (Li2), applied to the triangular lift r'' (with $\rho_{r''} = \rho_\varphi$ irreducible), ρ_φ is irreducible on every open subgroup of G_K. By (Tp2) for r'', $\rho_\varphi \otimes \varepsilon^{-1}$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions. ρ_φ takes

values in $\mathrm{GL}_3(E)$ with E finite over $\mathbb{Q}_\ell$, so by Lemma 42.7 (with $F = K$) the identity component $\mathbf{Z}^\circ$ of the Zariski closure of $\rho_\varphi(G_{K(\mu_{\ell^\infty})})$ acts irreducibly on $\overline{\mathbb{Q}}_\ell^3$; the stabiliser of a subspace is Zariski closed, so a subspace stable under $\rho_\varphi(G_{K(\mu_{\ell^\infty})})$ is stable under $\mathbf{Z}^\circ$, and $\rho_\varphi|_{G_{K(\mu_{\ell^\infty})}}$ is irreducible. Hence φ is Lie-irreducible automorphic for ι. Finally (Sb1) is type I, (Sb2) was just shown, and (Sb3), (Sb4) are Proposition 47.11, (Cv1), in case (b), applied to the triangular lift r'', whose $\rho_{r''}$ is ρ_φ; so ρ_φ is admissible of stable type for (ι, e, N). □

48 Triangular lifts over auxiliary fields

This section proves Theorem 48.1, the automorphy of Lie-irreducible triangular lifts and the finiteness in characteristic ℓ, for both kinds of Eisenstein data; Section 47 uses it. The proof passes to an auxiliary field. Theorem 48.3 gives a marking and an abelian CM field $L \supseteq K$ over which the triangular lifts have residual characteristic polynomials in characteristic ℓ (Ax1), are automorphic of conjugate self-dual type with the witness conditions when irreducible (Ax2), stay generic above ℓ (Ax3), and have a modular endoscopic corner (Ax4). Automorphy then descends from L to K along the abelian CM extension (Proposition 48.18, Proposition 48.19), and the corner of a lift of type II comes from a newform over $\mathbb{Q}$ (Proposition 48.14, Proposition 48.16).

An auxiliary field (Definition 48.2) is an abelian CM field, unramified over its totally real subfield at the finite places, on which the residual characters are non-degenerate (Ar1) and which satisfies two opposite requirements. Certain strict Selmer groups of the residual characters are non-zero (Ar2), while the Selmer ranks are bounded by a constant C_0 (Ar3) and the degree $[L^+ : \mathbb{Q}]$ exceeds $3C_0 + 7$ (Ar4); the constant C_0 and the degree enter the dimension estimates of Section 49 (Lemma 49.30, Proposition 49.29). Auxiliary fields exist (Proposition 48.4): along the layers of a cyclotomic ℓ-tower all conditions but the size hold at the level of the ℓ-rank of the class group of the layer (Proposition 48.5), and that rank is bounded along the tower by the theorem of Ferrero and Washington (Lemma 39.5, from Section 39), while the degree grows. The Euler-characteristic inequality of clause (iv) of (Ar5), for the Galois group of L^+ modulo tame inertia is Proposition 48.9. Given an auxiliary field, Theorem 48.3 follows from the $R = T$ theorem of Section 49 by restricting the triangular lifts to G_L (Lemma 48.10).

Theorem 48.1. *Let $\mathcal{O}$ be a coefficient ring, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and assume that (a) (K, ℓ, e, N, χ) is a unitary Eisenstein datum with χ valued in $\mathcal{O}^\times$ and $e_\mathcal{O} = 1$, or (b) (ℓ, e, N, χ, o, m) is a dyadic Eisenstein datum with χ valued in $\mathcal{O}^\times$ and $K = \mathbb{Q}(i)$. Form triangular lifts with K, $\mathcal{O}$ and $\Psi = \Psi_{\chi,e}$. Then there is a marking $g_\star$ such that, for triangular lifts for $g_\star$:*

(Tp1) *there is a subgroup U of finite index in $G_\mathbb{Q}$ such that for every test ring A of characteristic ℓ, every triangular lift r to A and every $g \in U \cap G_K$, the characteristic polynomials of $\underline{r}(g)$ and of $\underline{r_0}(g)$ (mapped to A) agree;*

(Tp2) *for every triangular lift r to an integral test ring such that $\rho_r|_V$ is irreducible for every open subgroup $V \subseteq G_K$, the representation $\rho_r \otimes \chi_{\mathrm{cyc}}^{-1}$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions;*

(Tp3) *in case (a): for every triangular lift r of type II to an integral test ring with $\rho_r(g)_{22} = 1$ for all $g \in G_K$, every continuous $W\colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ whose restriction to G_K is the corner of r and with $\operatorname{tr} W(c) = 0$ satisfies (Eo2) and the witness conditions over imaginary quadratic fields.*

Proof. By Theorem 48.3 there are a marking $g_\star$ and a CM field L, abelian over $\mathbb{Q}$ and containing K, with (Ax1)–(Ax4). Then (Tp1) is (Ax1) and (Tp3) is (Ax4). We prove (Tp2). Write ε for χ_{cyc} on G_K (and on its subgroups). Let A be an integral test ring, r a triangular lift to A with $\rho = \rho_r$ irreducible on every open subgroup of G_K, and put $r' = \rho \otimes \varepsilon^{-1}$. We check hypotheses 1–6 of Proposition 48.18 for r', with K, L and c; that proposition then gives the claim.

Continuity. The ring A is a discrete valuation ring finite over $\mathcal{O}$ inside $\overline{\mathbb{Q}}_\ell$, so it is the valuation ring of its fraction field and its $\mathfrak{m}_A$-adic topology is the ℓ-adic one; by (Tl1) ρ is continuous, and so is r'.

Hypothesis 1 (polarisation), with $P = J$. By (Tl1) and (Tl2), $r(c) = (1,-1)$; as $c^2 = 1$, for $s \in G_K$ with $r(s) = (\rho(s), 1)$ we get $r(csc) = (1,-1)(\rho(s),1)(1,-1) = (\theta(\rho(s)),1)$, so $\rho(csc) = J\,{}^t\rho(s)^{-1}J$ and ${}^t\rho(s)\,J\,\rho(csc) = J$. Since ε is the restriction of a character of $G_{\mathbb{Q}}$ and $\varepsilon(c)^2 = 1$, $\varepsilon(csc) = \varepsilon(s)$, hence ${}^t r'(s)\,J\,r'(csc) = \varepsilon(s)^{-2}J$.

Hypothesis 2. The extension L/K is finite, so G_L is open in G_K and $\rho|_{G_L}$ is irreducible; so is its twist $r'|_{G_L}$.

Hypothesis 3 (inertia away from ℓ). The character $\Psi = \Psi_{\chi,e}$ satisfies $\Psi(c) = 1$ (by (Ue5) in case (a), by (Dy3) in case (b)), so Lemma 47.9 applies to r. Let $v \nmid \ell$ be a finite place of K and $\bar{v} \mid v$; the inertia group $I_{\bar{v}}$ of G_K at $\bar{v}$ is the intersection with G_K of the inertia group of $G_{\mathbb{Q}}$ at $\bar{v}$. By Lemma 47.9, (Lo1), there are $P \in \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ and integers a_1, a_2, a_3 with $\rho(g) = P\,\mathrm{diag}(\Psi(g)^{a_1}, \Psi(g)^{a_2}, \Psi(g)^{a_3})P^{-1}$ for $g \in I_{\bar{v}}$. Put $\kappa_i = (\Psi^{a_i}\varepsilon^{-1})|_{D_{\bar{v}}}$, continuous characters of $D_{\bar{v}}$. The field generated by the ℓ-power roots of unity is unramified at the prime below v, so $\varepsilon(g) = 1$ for $g \in I_{\bar{v}}$, and $\det(X - r'(g)) = \det(X - \rho(g)) = \prod_i (X - \Psi(g)^{a_i}) = \prod_i (X - \kappa_i(g))$.

Hypothesis 4 (triangular above ℓ). Let $v \mid \ell$ be a place of K and $\bar{v} \mid v$. By Lemma 47.9, (Lo2), v lies below some $\bar{w} \in \{\bar{\ell}, c\bar{\ell}\}$, and there are $h \in \mathrm{GL}_3(A)$ and characters $\kappa_1, \kappa_2, \kappa_3$ of the decomposition group $D_{\bar{w}}$ of G_K at $\bar{w}$ such that $h^{-1}\rho(g)h$ is upper triangular with diagonal $(\kappa_1(g), \kappa_2(g), \kappa_3(g))$ for $g \in D_{\bar{w}}$. The group G_K acts transitively on the places of $\overline{\mathbb{Q}}$ above v, so $\bar{v} = \gamma\bar{w}$ with $\gamma \in G_K$, and $D_{\bar{v}} = \gamma D_{\bar{w}}\gamma^{-1}$. For $g \in D_{\bar{w}}$, $(\rho(\gamma)h)^{-1}\rho(\gamma g\gamma^{-1})(\rho(\gamma)h) = h^{-1}\rho(g)h$. Hence ρ is triangular on $D_{\bar{v}}$ with diagonal $(\kappa_a(\gamma^{-1}\cdot\gamma))_a$, and $r' = \rho \otimes \varepsilon^{-1}$ is triangular on $D_{\bar{v}}$ with diagonal $(\kappa_a(\gamma^{-1}\cdot\gamma)\,\varepsilon^{-1})_a$.

Hypothesis 5. Twisting by a character multiplies all diagonal characters of a triangular form by the same character, which preserves the inequalities $\kappa_i \neq \kappa_j\chi_{\mathrm{cyc}}$; so r' and ρ are generic at the same places, and likewise $r'|_{G_L}$ and $\rho|_{G_L}$. If r' is generic at the place v of K below a place $w \mid \ell$ of L, then ρ is, (Ax3) makes $\rho|_{G_L}$ generic at w, and so is $r'|_{G_L}$.

Hypothesis 6. By hypothesis 2, $\rho|_{G_L}$ is irreducible, so by (Ax2) $\rho|_{G_L} \otimes \chi_{\mathrm{cyc}}^{-1} = r'|_{G_L}$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions.

By Proposition 48.18, $r' = \rho_r \otimes \chi_{\mathrm{cyc}}^{-1}$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions. □

Definition 48.2. Let $K, \eta_K, \mathcal{O}, \mathbb{F}_{\mathcal{O}}, c$ and $\Psi\colon G_{\mathbb{Q}} \to \mathcal{O}^\times$ with $\Psi(c) = 1$ be as in Definition 43.16, with ℓ odd; let $\bar{\Psi}\colon G_{\mathbb{Q}} \to \mathbb{F}_{\mathcal{O}}^\times$ be the reduction of Ψ, $\bar{\omega}\colon G_{\mathbb{Q}} \to \mathbb{F}_\ell^\times \subseteq \mathbb{F}_{\mathcal{O}}^\times$ the reduction of χ_{cyc}, $g_\star$ a marking and $M \geq 1$ an integer with $\ell \mid M$. For a CM field $L \subset \overline{\mathbb{Q}}$ with maximal totally real subfield L^+, the groups $G_L \subseteq G_{L^+}$ are subgroups of $G_{\mathbb{Q}}$, and $c \in G_{L^+}$.

The group Γ_L. Let N_L be the closed normal subgroup of G_{L^+} generated by the elements $s \in I_{\bar{v}} \cap G_{L^+}$, for all places $\bar{v} \nmid \ell$, with $\eta_K(s) = 1$ and $\bar{\Psi}(s) = 1$. Put $\Gamma_L = G_{L^+}/N_L$, let $\pi_L\colon G_{L^+} \to \Gamma_L$ be the projection and $\Gamma_L^0 = \pi_L(G_L)$. The characters η_K, $\bar{\Psi}$, $\bar{\omega}$ and χ_{cyc} of G_{L^+} factor through Γ_L; they are denoted by the same letters. For each place v of L^+ above ℓ fix a place $\bar{v} = \gamma_v\bar{\ell}$ of $\overline{\mathbb{Q}}$ above v with $\gamma_v \in G_K$, and put $D_v = D_{\bar{v}} \cap G_{L^+} \supseteq I_v = I_{\bar{v}} \cap G_{L^+}$ and $d_v = [L_v^+ : \mathbb{Q}_\ell]$. For each real place u of L^+ fix a complex conjugation $c_u \in G_{L^+}$ at u.

Selmer groups. Let F be a number field, N a closed normal subgroup of G_F, $\pi\colon G_F \to$

G_F/N the projection and Δ a closed subgroup of G_F/N; let W be a finite-dimensional $\mathbb{F}_{\mathcal{O}}$-vector space with a continuous linear action of Δ, and let $(H_t)_{t\in T}$ be a finite family of closed subgroups of G_F with $\pi(H_t)\subseteq\Delta$, with subspaces $b_t\subseteq W$ stable under $\pi(H_t)$. Put $H^0(\Delta;W,b)=\{X\in W^{\Delta}: X\in b_t\ \forall t\}$; $H^1(\Delta;W,b)$ is the quotient of the space of pairs $(x,(X_t)_t)$, with $x\colon\Delta\to W$ a continuous 1-cocycle and $X_t\in W$ such that $x(\pi(h))-(h-1)X_t\in b_t$ for all $h\in H_t$, by the pairs $(g\mapsto(g-1)X,\,(X_t)_t)$ with $X\in W$ and $X_t-X\in b_t$; and $H^2(\Delta;W,b)$ is the quotient of the space of pairs $(y,(y_t)_t)$, with $y\colon\Delta\times\Delta\to W$ a continuous 2-cocycle and $y_t\colon H_t\to W$ continuous such that $y(\pi(h),\pi(h'))-\big(h\,y_t(h')-y_t(hh')+y_t(h)\big)\in b_t$ for $h,h'\in H_t$, by the pairs $(d\beta,(y_t)_t)$ with $\beta\colon\Delta\to W$ continuous and $y_t(h)-\big(\beta(\pi(h))-(h-1)Y_t\big)\in b_t$ for some $Y_t\in W$. Write h^i for dimensions. For $\Delta=\Gamma_L$ or $\Delta=\Gamma_L^0$ we take $F=L^+$, $N=N_L$ and $\pi=\pi_L$; for $\Delta=\Gamma_L$ the family is $(D_v)_{v|\ell}$; for $\Delta=\Gamma_L^0$ it is $(D_v)_{v|\ell}$ together with $(cD_vc)_{v|\ell}$. The **strict** group $H^1_{\mathrm{str}}(\Delta,W)$ is $H^1(\Delta;W,b)$ with all $b_t=0$; the group $H^1(\Gamma_L,W)$ (all $b_t=W$) is continuous cohomology. For a character ϕ of Δ, $\mathbb{F}_{\mathcal{O}}(\phi)$ is $\mathbb{F}_{\mathcal{O}}$ with Δ acting by ϕ.

Auxiliary fields. A CM field $L\subset\overline{\mathbb{Q}}$ is an **auxiliary field of level** $C_0\in\mathbb{N}$ for $(K,\mathcal{O},\Psi,c,g_\star,M)$ if:

(Af1) L is abelian over $\mathbb{Q}$, $K\subseteq L$, $G_L=G_{L^+}\cap G_K$, and $g_\star\in G_L$;

(Af2) every finite place of L^+ is unramified in L, and every place of L^+ above a prime dividing M splits in L;

(Af3) for every place $\bar v\nmid\ell$ and every $s\in I_{\bar v}\cap G_{L^+}$ with $\bar\Psi(s)=1$ one has $\Psi(s)=1$;

(Af4) for every place $v\mid\ell$ of L^+: $\bar\omega|_{D_v}\neq1$, $\bar\Psi^2|_{D_v}\neq1$, and $\bar\Psi(s)=1$ for every $s\in I_v$ with $\chi_{\mathrm{cyc}}(s)=1$;

(Ar1) on G_L: $\bar\omega\neq1$, $\bar\omega\bar\Psi^{\pm1}\neq1$, and $\bar\omega\neq\bar\Psi^j$ for every $j\in\mathbb{Z}$; on G_{L^+}: $\bar\omega\bar\Psi^{\pm2}\eta_K\neq1$;

(Ar2) $H^1_{\mathrm{str}}(\Gamma_L,\mathbb{F}_{\mathcal{O}}(\bar\Psi^{-1}\eta_K))\neq0$ and $H^1_{\mathrm{str}}(\Gamma_L,\mathbb{F}_{\mathcal{O}}(\bar\Psi^{-2}\eta_K))\neq0$;

(Ar3) $h^1_{\mathrm{str}}(\Gamma_L,\mathbb{F}_{\mathcal{O}}(\bar\Psi^{-2}\eta_K))\le C_0$, $h^1_{\mathrm{str}}(\Gamma_L^0,\mathbb{F}_{\mathcal{O}}(\bar\Psi^{-1}))\le C_0$ and $h^1(\Gamma_L,\mathbb{F}_{\mathcal{O}})\le1+C_0$;

(Ar4) $[L^+:\mathbb{Q}]>3C_0+7$ and $d_v>5$ for every place $v\mid\ell$ of L^+;

(Ar5) (i) every open subgroup of Γ_L has only finitely many continuous homomorphisms to $\mathbb{Z}/\ell\mathbb{Z}$; (ii) there is $h\ge1$ such that every continuous homomorphism from Γ_L^0 to a finite abelian ℓ-group that is trivial on $\pi_L(I_v)$ and $\pi_L(cI_vc)$ for all $v\mid\ell$ takes values killed by h; (iii) for every $v\mid\ell$ there are $u_1,\dots,u_{d_v}\in I_v$ with $\bar\Psi(u_a)=1$, all but at most one with $\chi_{\mathrm{cyc}}(u_a)=1$, such that every continuous homomorphism from D_v to a finite abelian ℓ-group that is trivial on $u_1,\dots,u_{d_v}$ is trivial on I_v; (iv) for every W as above with $\Delta=\Gamma_L$ and open kernel, all D_v-stable subspaces $b_v\subseteq W$, and $\mathrm{Hom}_{\Gamma_L}(W,\mathbb{F}_{\mathcal{O}}(\bar\omega))=0$: $H^2(\Gamma_L;W,b)$ is finite-dimensional and

$$h^2(\Gamma_L;W,b)+h^0(\Gamma_L;W,b)+\sum_{v|\ell}d_v\dim b_v\le h^1(\Gamma_L;W,b)+\sum_{u|\infty}\dim W^{c_u=1};\qquad(48.1)$$

(Ag1) there are an integer $N'\ge1$ with $\ell\nmid N'$, a prime $w\nmid N'$, an integer $j\ge0$ and characters $\chi',\theta\colon G_{\mathbb{Q}}\to\overline{\mathbb{Q}}_\ell^{\times}$ of finite order such that: $\theta\neq1$, $\theta^\ell=1$, θ is trivial on $G_{\mathbb{Q}(\zeta_w)}$ and on G_L; χ' and η_K are trivial on $G_{\mathbb{Q}(\zeta_{N'})}$ and on every $s\in G_{\mathbb{Q}}$ acting on the N'-th roots of unity by $\zeta\mapsto\zeta^w$; $\chi'\neq\eta_K$, $\ell\nmid\operatorname{ord}\chi'$, $\chi'(c)=(-1)^j$; $\chi_{\mathrm{cyc}}^j\chi'\equiv\Psi$ modulo the maximal ideal of the integers of $\overline{\mathbb{Q}}_\ell$; and $(\chi_{\mathrm{cyc}}^j\chi')(s)^2\not\equiv1$ for some $s\in G_L$;

(Ag2) $\ell\nmid d_K$.

Polarised ordinary lifts over L. Let L satisfy (Af1)–(Af2) and let $n \in \{2,3\}$. Put $(w_{3,1}, w_{3,2}, w_{3,3}) = (1,0,-1)$ and $(w_{2,1}, w_{2,2}) = (1,-1)$, and let J_n be the $n \times n$ matrix with entries 1 on the antidiagonal and 0 elsewhere. For a test ring A, a **polarised ordinary lift of rank n over L** to A is a homomorphism $\rho\colon \Gamma_L^0 \to \mathrm{GL}_n(A)$ with homomorphisms $\psi_{v,a}\colon D_v \to A^\times$ ($v \mid \ell$, $1 \le a \le n$) such that

(Po1) for every $k \ge 1$ the maps $g \mapsto \rho(g) \bmod \mathfrak{m}_A^k$ and $\psi_{v,a} \bmod \mathfrak{m}_A^k$ are locally constant;

(Po2) $\rho(\pi_L(cg^{-1}c)) = J_n\, {}^t\rho(\pi_L(g))\, J_n$ for all $g \in G_L$;

(Po3) the off-diagonal entries of $\rho(\pi_L(g_\star)) - \rho(\pi_L(cg_\star^{-1}c))$ vanish;

(Po4) for every $v \mid \ell$ and all $g_1, \dots, g_n \in D_v$: $\prod_{a=1}^n (\rho(\pi_L(g_a)) - \psi_{v,a}(g_a)) = 0$, the product taken with $a = 1$ on the left;

(Po5) $\rho(\pi_L(g)) \equiv \mathrm{diag}(\bar\Psi(g)^{w_{n,1}}, \dots, \bar\Psi(g)^{w_{n,n}})$ and $\psi_{v,a}(g) \equiv \bar\Psi(g)^{w_{n,a}}$ modulo $\mathfrak{m}_A$, for $g \in G_L$, resp. $g \in D_v$.

Write $\rho_L\colon G_L \to \mathrm{GL}_n(A)$, $g \mapsto \rho(\pi_L(g))$. The field L has **residual characteristic polynomials in rank** 3 if for every test ring A of characteristic ℓ and every polarised ordinary lift of rank 3 over L to A with $\psi_{v,a}(g) = 1$ whenever $g \in I_v$ and $\bar\Psi(g) = 1$, the characteristic polynomial of $\rho_L(g)$ is $\prod_{a=1}^3 (X - \bar\Psi(g)^{w_{3,a}})$ for all $g \in G_L$. For a continuous character $\kappa\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$, L has **automorphic irreducible lifts in rank n with twist κ** (for ι) if for every integral test ring A and every polarised ordinary lift of rank n over L to A with $\psi_{v,a}(g) = \Psi(g)^{w_{n,a}}$ for $g \in I_v$ such that ρ_L is irreducible over $\overline{\mathbb{Q}}_\ell$, the representation $\rho_L \otimes \kappa$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions. A continuous character $\kappa\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$ is **algebraic** (for ι) if it corresponds through ι and the global Artin map to an algebraic Hecke character of L.

Theorem 48.3. *Let $\mathcal{O}$ be a coefficient ring, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and assume one of:*

(a) (K, ℓ, e, N, χ) is a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$ and $e_{\mathcal{O}} = 1$;

(b) (ℓ, e, N, χ, o, m) is a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$, and $K = \mathbb{Q}(i)$.

Put $\Psi = \Psi_{\chi,e}$ and form triangular lifts with K, $\mathcal{O}$ and Ψ. Then there are a marking $g_\star$ and a CM field $L \subset \overline{\mathbb{Q}}$, abelian over $\mathbb{Q}$ and containing K, such that, for triangular lifts for $g_\star$:

(Ax1) *there is a subgroup U of finite index in $G_{\mathbb{Q}}$ such that for every test ring A of characteristic ℓ, every triangular lift r to A and every $g \in U \cap G_K$, the characteristic polynomial of $\underline{r}(g)$ equals that of $\underline{r_0}(g)$ (mapped to A);*

(Ax2) *for every triangular lift r to an integral test ring such that $\rho_r|_{G_L}$ is irreducible, the representation $\rho_r|_{G_L} \otimes \chi_{\mathrm{cyc}}^{-1}$ of G_L is automorphic of conjugate self-dual type for ι and satisfies the witness conditions;*

(Ax3) *for every triangular lift r to an integral test ring such that $\rho_r|_V$ is irreducible for every open subgroup $V \subseteq G_K$, every place $v \mid \ell$ of K at which ρ_r is generic and every place $w \mid v$ of L, $\rho_r|_{G_L}$ is generic at w;*

(Ax4) *in case (a): for every triangular lift r of type II to an integral test ring with $\rho_r(g)_{22} = 1$ for all $g \in G_K$, every continuous $W\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ whose restriction to G_K is the corner of r and with $\operatorname{tr} W(c) = 0$ satisfies (Eo2) and the witness conditions over imaginary quadratic fields.*

Proof. Put $M = \ell N|d_K|$ in case (a) and $M = \ell N$ in case (b). By Proposition 48.4 (applied to the complex conjugation c of Definition 43.16) there are a marking $g_\star$, an integer C_0 and an auxiliary field L of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$. By (Af1), L is a CM field, abelian over $\mathbb{Q}$, with $K \subseteq L$ and $G_L \subseteq G_K$. By Theorem 49.3, L has (Pp1)–(Pp3); let κ be an algebraic character as in (Pp3). We prove (Ax1)–(Ax4) for triangular lifts for $g_\star$.

(Ax1). Take $U = G_L$, of index $[L : \mathbb{Q}]$ in $G_\mathbb{Q}$; then $U \cap G_K = G_L$. Let A be a test ring of characteristic ℓ, r a triangular lift to A and $(\rho, (\psi_{v,a}))$ the polarised ordinary lift of rank 3 over L given by Lemma 48.10, with $\rho_L = \underline{r}|_{G_L}$ and $\psi_{v,a} = \Psi^{w_{3,a}}$ on I_v. As A has characteristic ℓ, the map $\mathcal{O} \to A$ kills $\mathfrak{m}_\mathcal{O}$ and factors through $\mathbb{F}_\mathcal{O}$, so the image of $\Psi(g)^k$ in A is that of $\bar{\Psi}(g)^k$ for every $k \in \mathbb{Z}$. In particular $\psi_{v,a}(g) = 1$ whenever $g \in I_v$ and $\bar{\Psi}(g) = 1$, and (Pp1) applies: for $g \in G_L$ the characteristic polynomial of $\underline{r}(g) = \rho_L(g)$ is $\prod_{a=1}^3 (X - \bar{\Psi}(g)^{w_{3,a}})$, which is the image in $A[X]$ of the characteristic polynomial $\prod_{a=1}^3 (X - \Psi(g)^{w_{3,a}})$ of $\underline{r_0}(g)$.

(Ax2). Let A be an integral test ring and r a triangular lift to A with $\rho_r|_{G_L}$ irreducible. The lift $(\rho, (\psi_{v,a}))$ of Lemma 48.10 has $\psi_{v,a}(g) = \Psi(g)^{w_{3,a}}$ for $g \in I_v$ and $\rho_L = \underline{r}|_{G_L} = \rho_r|_{G_L}$, which is irreducible over $\overline{\mathbb{Q}}_\ell$. By (Pp2), $\rho_r|_{G_L} \otimes \chi_{\mathrm{cyc}}^{-1}$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions.

(Ax3). This is Proposition 48.11, applied to $g_\star$ and L.

(Ax4). In case (a), κ is algebraic and L has automorphic irreducible lifts in rank 2 with twist κ, so Proposition 48.14 gives (Ax4). □

Proposition 48.4. *Let $\mathcal{O}$ be a coefficient ring and $c \in G_\mathbb{Q}$ a complex conjugation, and assume one of:*

(a) (K, ℓ, e, N, χ) is a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$; put $M = \ell N|d_K|$;

(b) (ℓ, e, N, χ, o, m) is a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$, and $K = \mathbb{Q}(i)$; put $M = \ell N$.

Put $\Psi = \Psi_{\chi,e}$. Then there are a marking $g_\star$, an integer $C_0 \geq 0$ and a CM field $L \subset \overline{\mathbb{Q}}$ which is an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$.

Proof. By Proposition 48.5 there are a marking $g_\star$, a finite abelian extension E of $\mathbb{Q}$ containing a primitive ℓ-th root of unity, and CM fields L_n ($n \geq 1$) with (Ly1) and (Ly2). The prime ℓ is odd, by (Ue1) in case (a) and by (Dy1) in case (b). By Lemma 39.5, applied to E, there is an integer $C_0 \geq 0$ with $\#(\mathrm{Cl}(E')/\mathrm{Cl}(E')^\ell) \leq \ell^{C_0}$ for every field E' generated over E by a primitive ℓ^k-th root of unity, $k \geq 0$; with $k = n + 1$ this gives $r_n \leq C_0$ for every $n \geq 1$. As $\ell \geq 3$, we may fix $n \geq 1$ with $\ell^n > 3C_0 + 7$, and put $L = L_n$.

By (Ly2), L satisfies (Af1)–(Af4), (Ar1), (Ar2), (Ar5), (Ag1), (Ag2), none of which involves the level, and (Ar3) with C_0 replaced by r_n; since $r_n \leq C_0$, the three inequalities of (Ar3) hold with C_0. By (Ly1), $[L^+ : \mathbb{Q}] \geq \ell^n > 3C_0 + 7$ and $d_v \geq \ell^n > 7 > 5$ for every place $v \mid \ell$ of L^+, which is (Ar4). Hence L is an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$. □

Proposition 48.5. *Let $\mathcal{O}$ be a coefficient ring and $c \in G_\mathbb{Q}$ a complex conjugation, and assume one of:*

(a) (K, ℓ, e, N, χ) is a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$; put $M = \ell N|d_K|$;

(b) (ℓ, e, N, χ, o, m) is a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$, and $K = \mathbb{Q}(i)$; put $M = \ell N$.

Put $\Psi = \Psi_{\chi,e}$. Then there are a marking $g_\star$, a finite abelian extension $E \subset \overline{\mathbb{Q}}$ of $\mathbb{Q}$ containing a primitive ℓ-th root of unity, and for every integer $n \geq 1$ a CM field $L_n \subset \overline{\mathbb{Q}}$ such that, with E_n the field generated over E by a primitive ℓ^{n+1}-th root of unity and $r_n \geq 0$ the integer with $\#(\mathrm{Cl}(E_n)/\mathrm{Cl}(E_n)^\ell) = \ell^{r_n}$:

(Ly1) *$[L_n^+ : \mathbb{Q}] \geq \ell^n$, and $d_v \geq \ell^n$ for every place $v \mid \ell$ of L_n^+;*

(Ly2) *L_n satisfies (Af1)–(Af4), (Ar1), (Ar2), (Ar3) with $C_0 = r_n$, (Ar5), (Ag1) and (Ag2) for $(K, \mathcal{O}, \Psi, c, g_\star, M)$.*

Proof. Write $\chi = \chi_\flat \chi_\sharp$ with $\chi_\flat$ of order prime to ℓ and $\chi_\sharp$ of ℓ-power order (both powers of χ; in case (b) these are the χ', χ'' of Definition 38.31, in case (a) $\chi_\sharp = \mathbf{1}$ by (Ue3)). Since $\chi_\sharp$ takes values in $1 + \mathfrak{m}_\mathcal{O}$, $\bar\Psi = \bar\chi_\flat \bar\omega^e$.

Step 0: facts about the data.

(Ef1) ℓ is odd ((Ue1), (Dy1)); K is imaginary quadratic with $\ell \nmid d_K$ ((Ue6); $d_K = -4$ in case (b)), which is (Ag2); $2\ell \mid M$ and $|d_K| \mid M$ (case (a): N even by (Ue2), $M = \ell N |d_K|$; case (b): $2^{c_2(\chi)} \mid N$ with $c_2(\chi) \geq 4$ by (Dy6), so $4 \mid N \mid M$). Hence η_K, $\bar\Psi$, $\bar\omega$ are trivial on $G_{\mathbb{Q}(\zeta_M)}$ (χ is a character modulo N, $\bar\omega$ comes from $\mathbb{Q}(\zeta_\ell)$, η_K has conductor $|d_K|$); in particular they are unramified at every prime not dividing M.

(Ef2) $\Psi(c) = \chi(-1)(-1)^e = 1$ by (Ue5), (Dy3); $\bar\omega(c) = -1$.

(Ef3) There is an integer $0 \leq j < \ell - 1$ with $\bar\Psi(s) = \bar\omega(s)^j$ for every $s \in I_{\bar v}$ and every place $\bar v \mid \ell$. Indeed on $I_{\bar\ell}$, χ is χ_ℓ composed with $I_{\bar\ell} \to (\mathbb{Z}/\ell^a\mathbb{Z})^\times$; the prime-to-$\ell$ part of χ_ℓ is trivial on the ℓ-group $(1+\ell\mathbb{Z})/(1+\ell^a\mathbb{Z})$, hence factors through the cyclic group $(\mathbb{Z}/\ell\mathbb{Z})^\times$, on which $\bar\omega$ is an isomorphism onto $\mathbb{F}_\ell^\times$, the unique subgroup of order $\ell - 1$ of $\mathbb{F}_\mathcal{O}^\times$; its ℓ-power part reduces to 1. So $\bar\chi = \bar\omega^k$ on $I_{\bar\ell}$ for some k, $\bar\Psi = \bar\omega^{k+e}$ there, and conjugation (the characters are abelian) gives the other places above ℓ.

(Ef4) $\ker\bar\omega \cap \ker\bar\Psi \not\subseteq \ker\eta_K$. Case (a): $\ker\bar\omega \cap \ker\bar\Psi = \ker\bar\omega \cap \ker\chi$ (χ has order prime to ℓ); on $\ker\bar\omega = G_{\mathbb{Q}(\zeta_\ell)}$ the ℓ-component χ_ℓ, of order prime to ℓ, is trivial (it factors through $(\mathbb{Z}/\ell\mathbb{Z})^\times$), so $\ker\bar\omega \cap \ker\bar\Psi = \ker\bar\omega \cap \ker\chi^{(\ell)} = G_{L_{\ell,\chi}}$, and (Ue8) says $G_{L_{\ell,\chi}} \not\subseteq G_K$. Case (b): otherwise η_K factors through $(\bar\omega, \bar\Psi)\colon G_\mathbb{Q} \to \mathbb{F}_\ell^\times \times \mathbb{F}_\mathcal{O}^\times$; a homomorphism from a subgroup of this group to $\{\pm 1\} \subseteq \mathbb{F}_\mathcal{O}^\times$ extends to the whole group (values in $\mathbb{Q}/\mathbb{Z}$, then in μ_{q-1}, $q = \#\mathbb{F}_\mathcal{O}$), and every homomorphism $\mathbb{F}_\ell^\times \times \mathbb{F}_\mathcal{O}^\times \to \mathbb{F}_\mathcal{O}^\times$ is $(x,y) \mapsto x^\alpha y^\beta$; so $\eta_K = \bar\omega^\alpha \bar\Psi^\beta$. On $I_{\bar v}$, $\bar v \mid 2$, $\bar\omega$ and χ_{cyc} are trivial and $\bar\Psi = \bar\chi_2$; χ_2 has 2-power order and reduction is injective on its values, so the character ε_4 of $(\mathbb{Z}/2^a\mathbb{Z})^\times$ ($2^a \| N$) defining $\mathbb{Q}(i)$ equals χ_2^β. As χ_2 has conductor $2^{c_2(\chi)}$, $c_2(\chi) \geq 4$, $\chi_2(5)$ has order $2^{c_2(\chi)-2} \geq 4$ (5 generates the cyclic group $(1+4\mathbb{Z})/(1+2^a\mathbb{Z})$), while $\varepsilon_4(5) = 1$; so β is even and $-1 = \varepsilon_4(-1) = \chi_2(-1)^\beta = 1$, a contradiction.

(Ef5) There is $s_0 \in D_{\bar\ell} \cap G_K$ with $\bar\Psi(s_0)^2 \neq 1$. Case (a): $I_{\bar\ell} \subseteq G_K$ by (Ue6); on $I_{\bar\ell}$, $\bar\Psi^2 = \bar\chi_\ell^2 \bar\omega^{2e} = \bar\omega^{2e}$ by (Ue4), and $\bar\omega$ maps $I_{\bar\ell}$ onto $\mathbb{F}_\ell^\times$, so $\bar\omega^{2e} \neq 1$ on $I_{\bar\ell}$ by (Ue9). Case (b): Lemma 47.10.

(Ef6) Let $H_b = \ker\chi_\sharp$, an open normal subgroup of ℓ-power index with abelian quotient. For a place $\bar v \nmid \ell$ and $s \in I_{\bar v} \cap H_b$ with $\bar\Psi(s) = 1$: $\chi_{\mathrm{cyc}}(s) = 1$, so $\Psi(s) = \chi_\flat(s)$, a root of unity of order prime to ℓ congruent to 1, hence $\Psi(s) = 1$.

Step 1: the fields. By Dirichlet's theorem choose a prime $q \equiv -1 \pmod{8M}$ and a prime $w \equiv 1 \pmod{M}$; then $q, w \nmid M$, $q \neq w$ (as $M \geq 6$), and $w \equiv 1 \pmod{\ell}$. Let $K' = \mathbb{Q}(\sqrt{-q})$, of discriminant $-q$, with character η'. Every prime $p \mid M$ splits in K' ($-q \equiv 1$ modulo every odd $p \mid M$, and $-q \equiv 1 \pmod 8$). The character $\eta_K\eta'$ is quadratic, non-trivial (η' is ramified at q, η_K is not, by (Ef1)) and even ($\eta_K(c)\eta'(c) = 1$); let F_0 be its fixed field, a real quadratic field, ramified at q, and $H_0 = G_{F_0}$. For every place $\bar{v}$ above a prime $p \mid M$, $\eta'|_{D_{\bar{v}}} = 1$, so

$$D_{\bar{v}} \cap H_0 = D_{\bar{v}} \cap \ker \eta_K. \tag{48.2}$$

Let $\mathbb{Q}_w^{(\ell)} \subseteq \mathbb{Q}(\zeta_w)$ be the subfield of degree ℓ ($\ell \mid w-1$), F_b the fixed field of H_b, and, for $n \geq 1$, $\mathbb{Q}_n \subseteq \mathbb{Q}(\zeta_{\ell^{n+1}})$ the subfield of degree ℓ^n, totally ramified at ℓ. Put

$$L_n^+ = F_0\,\mathbb{Q}_w^{(\ell)} F_b\,\mathbb{Q}_n, \qquad L_n = L_n^+ K, \qquad P_n = G_{\mathbb{Q}_w^{(\ell)}} \cap H_b \cap G_{\mathbb{Q}_n},$$

so $G_{L_n^+} = H_0 \cap P_n$ and $G_{\mathbb{Q}}/P_n$ is an abelian ℓ-group. All four fields are abelian over $\mathbb{Q}$ and fixed by c (F_0 is real; the other three have Galois groups of odd order), hence by every complex conjugation: L_n^+ is totally real, $K \not\subseteq L_n^+$, and L_n is a CM field with maximal totally real subfield L_n^+ and $G_{L_n} = G_{L_n^+} \cap G_K$. Let E be the fixed field of $V = H_0 \cap G_{\mathbb{Q}_w^{(\ell)}} \cap H_b \cap \ker \eta_K \cap \ker \bar{\Psi} \cap \ker \bar{\omega}$: it is abelian over $\mathbb{Q}$ and contains μ_ℓ. Since $\mathbb{Q}(\zeta_{\ell^{n+1}}) = \mathbb{Q}(\zeta_\ell)\mathbb{Q}_n$ and $V \subseteq \ker \bar{\omega} = G_{\mathbb{Q}(\zeta_\ell)}$,

$$G_{L_n} \cap \ker \bar{\Psi} \cap \ker \bar{\omega} = V \cap G_{\mathbb{Q}_n} = V \cap G_{\mathbb{Q}(\zeta_{\ell^{n+1}})} = G_{E_n}. \tag{48.3}$$

Step 2: the marking and an ℓ-adic projection. For $s \in G_{\mathbb{Q}}$ let $s^\natural$ be the image of $z \in \widehat{\mathbb{Z}}$, with ℓ-component 0 and all other components 1, under $\widehat{\mathbb{Z}} \to \overline{\langle s \rangle}$; then $s^\natural$ lies in every closed subgroup containing s, has trivial image in every finite ℓ-group quotient of $G_{\mathbb{Q}}$, and $\phi(s^\natural) = \phi(s)$ for every character ϕ of order prime to ℓ. Let s_0 be as in (Ef5); $s_0 \in D_{\bar{\ell}} \cap \ker \eta_K \subseteq H_0$ by (48.2) ($\ell \mid M$). Put $g_\star = s_0^\natural$: then $g_\star \in D_{\bar{\ell}} \cap H_0 \cap G_K \cap P_n$ for all n, so $g_\star \in G_{L_n}$ for all n, and $\bar{\Psi}(g_\star)^2 = \bar{\Psi}(s_0)^2 \neq 1$, i.e. $\Psi(g_\star)^2 - 1 \in \mathcal{O}^\times$: $g_\star$ is a marking. Likewise choose $t \in I_{\bar{\ell}}$ with $\bar{\omega}(t) \neq 1$ ($\bar{\omega}$ maps $I_{\bar{\ell}}$ onto $\mathbb{F}_\ell^\times$, $\ell \geq 3$); $t \in \ker \eta_K$ by (Ef1), so $t \in H_0$ by (48.2) and $t^\natural \in I_{\bar{\ell}} \cap G_{L_n^+}$ with $\bar{\omega}(t^\natural) \neq 1$.

Fix $n \geq 1$ and write $L = L_n$. We verify (Ly1) and (Ly2).

(Ly1). $\mathbb{Q}_n \subseteq L^+$ gives $[L^+ : \mathbb{Q}] \geq \ell^n$; at $v \mid \ell$ the completion of L^+ contains that of $\mathbb{Q}_n$, of degree ℓ^n over $\mathbb{Q}_\ell$ ($\mathbb{Q}_n$ is totally ramified at ℓ), so $d_v \geq \ell^n$.

(Af1) holds by Steps 1 and 2. *(Af2).* As $K' \subseteq KF_0 \subseteq L$, $L = L^+K'$ and $G_L = G_{L^+} \cap \ker \eta'$. If $\bar{v}$ lies above $p \nmid M$, then $p \nmid d_K$ by (Ef1), so $I_{\bar{v}} \subseteq G_K$ and $I_{\bar{v}} \cap G_{L^+} \subseteq G_L$. If $p \mid M$, $D_{\bar{v}} \subseteq \ker \eta'$, so $D_{\bar{v}} \cap G_{L^+} \subseteq G_L$. *(Af3)*: $G_{L^+} \subseteq H_b$ and (Ef6). *(Af4).* Let $v \mid \ell$, $\bar{v} = \gamma_v \bar{\ell}$; $D_v = \gamma_v (D_{\bar{\ell}} \cap G_{L^+}) \gamma_v^{-1}$ as G_{L^+} is normal. It contains $\gamma_v t^\natural \gamma_v^{-1}$, with $\bar{\omega} \neq 1$, and $\gamma_v g_\star \gamma_v^{-1}$, with $\bar{\Psi}^2 \neq 1$. If $s \in I_v$ and $\chi_{\mathrm{cyc}}(s) = 1$, then $\bar{\Psi}(s) = \bar{\omega}(s)^j = 1$ by (Ef3).

(Ar1). Let $\phi\colon G_{\mathbb{Q}} \to \mathbb{F}_{\mathcal{O}}^\times$ be a character trivial on G_L, and $x \in H_0 \cap \ker \eta_K$. For b large $x^{\ell^b} \in P_n$, so $x^{\ell^b} \in G_L$ and $\phi(x)^{\ell^b} = 1$; as ϕ has order prime to ℓ, $\phi(x) = 1$. So ϕ is trivial on G_{KF_0} and $\phi \in \{1, \eta_K, \eta_{F_0}, \eta'\}$ (here $\eta_{F_0} = \eta_K\eta'$). If moreover ϕ is unramified at q, then $\phi \in \{1, \eta_K\}$. The same argument for ϕ trivial on $G_{L^+} = H_0 \cap P_n$ gives $\phi \in \{1, \eta_{F_0}\}$, and $\phi = 1$ if ϕ is unramified at q. All characters below are unramified at q by (Ef1). On G_L: if $\phi = \bar{\omega}$, $\bar{\omega}\bar{\Psi}^{\pm 1}$ or $\bar{\omega}\bar{\Psi}^{-i}$ ($i \in \mathbb{Z}$) were trivial on G_L, then $\phi(c) = -1$ by (Ef2) excludes $\phi = 1$, and $\phi = \eta_K$ would make η_K trivial on $\ker \bar{\omega} \cap \ker \bar{\Psi}$, contradicting (Ef4). On G_{L^+}: $\phi = \bar{\omega}\bar{\Psi}^{\pm 2}\eta_K$ trivial on G_{L^+} would give $\phi = 1$, i.e. $\eta_K = \bar{\omega}^{-1}\bar{\Psi}^{\mp 2}$, contradicting (Ef4).

(Ar2). Let $a \in \{1, 2\}$ and $\kappa = \bar{\Psi}^{-a}\eta_K$, a character of $G_{\mathbb{Q}}$ with values in $\mathbb{F}_{\mathcal{O}}^\times$. The hypotheses of Lemma 48.6 (with $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, c, w) hold: $\kappa(c) = -1$ by (Ef2); $\kappa \neq \bar{\omega}$ by (Ef4); $\kappa(g_\star) = \bar{\Psi}(g_\star)^{-a} \neq 1$ with $g_\star \in D_{\bar{\ell}}$, and by conjugation $\kappa|_{D_{\bar{v}}} \neq 1$ for every $\bar{v} \mid \ell$; $w \equiv 1 \pmod{\ell}$,

and $D_{\bar v} \subseteq G_{\mathbb{Q}(\zeta_M)} \subseteq \ker\kappa$ for $\bar v \mid w$ (inertia at $w \nmid M$ fixes μ_M and Frobenius acts on it by $\zeta \mapsto \zeta^w = \zeta$), by (Ef1). Let y be the cocycle given there. *It descends to* Γ_L. The generators s of N_L have $\kappa(s) = 1$ and $y(s) = 0$: by (Yc2) if $s \in I_{\bar v}$ with $\bar v \nmid \ell w$, and by (Yc3) if $\bar v \mid w$, since $G_{L^+} \subseteq G_{\mathbb{Q}_w^{(\ell)}}$. The set $Z = \{g \in G_{L^+} : \kappa(g) = 1,\ y(g) = 0\}$ is a closed subgroup ($y(gg') = y(g) + y(g')$, $y(g^{-1}) = -y(g)$ there; y and κ are continuous), normal in G_{L^+} since $y(\gamma g\gamma^{-1}) = (1-\kappa(g))y(\gamma) + \kappa(\gamma)y(g)$ by the cocycle identity. So $N_L \subseteq Z$, κ is a character of Γ_L, and $y(gn) = y(g)$ for $n \in N_L$: $x(\pi_L(g)) = y(g)$ defines a continuous 1-cocycle x of Γ_L with values in $W = \mathbb{F}_{\mathcal{O}}(\kappa)$. *It is strict.* For $v \mid \ell$, (Yc1) at $\bar v$ gives X_v with $x(\pi_L(h)) = (h-1)X_v$ for $h \in D_v \subseteq D_{\bar v}$; so $(x, (X_v)_v)$ defines a class in $H^1_{\mathrm{str}}(\Gamma_L, W)$. *It is non-zero.* Otherwise $(x, (X_v)) = (g \mapsto (g-1)X, (X)_v)$ for some X, and $y(g) = \kappa(g)X - X$ on G_{L^+}, a closed normal subgroup of $G_{\mathbb{Q}}$ containing $g_\star$ with $\kappa(g_\star) \neq 1$; this contradicts (Yc4). Hence (Ar2).

(Ar3) with $C_0 = r_n$. L is a CM field with (Af1), (Af2), (Af4), and by (48.3) the field E' of Lemma 48.7 is E_n, so $r = r_n$ there.

(Ar5). Items (i)–(iii) by Lemma 48.8; item (iv) by Proposition 48.9.

(Ag1). Write $M = \ell^k N'$ with $\ell \nmid N'$; then $w \nmid N'$ and $w \equiv 1 \pmod{N'}$. Let $\tau\colon \mathbb{F}_{\mathcal{O}}^\times \to \overline{\mathbb{Q}}_\ell^\times$ be the Teichmüller lift, $\chi_0' = \tau \circ (\bar\Psi\bar\omega^{-j})$ with j from (Ef3), and θ a non-trivial character of $\mathrm{Gal}(\mathbb{Q}_w^{(\ell)}/\mathbb{Q}) \cong \mathbb{Z}/\ell\mathbb{Z}$ with values in $\mu_\ell(\overline{\mathbb{Q}}_\ell)$, viewed on $G_{\mathbb{Q}}$. We check (Ag1) for $(N', w, j, \chi_0', \theta)$. $\theta \neq 1$, $\theta^\ell = 1$, and θ is trivial on $G_{\mathbb{Q}(\zeta_w)}$ and on $G_L \subseteq G_{\mathbb{Q}_w^{(\ell)}}$. The character $\bar\Psi\bar\omega^{-j}$ is trivial on $G_{\mathbb{Q}(\zeta_M)}$ (Ef1) and on $I_{\bar\ell}$ (Ef3); as a character of $(\mathbb{Z}/\ell^k\mathbb{Z})^\times \times (\mathbb{Z}/N'\mathbb{Z})^\times$ trivial on the first factor (the image of $I_{\bar\ell}$), it is trivial on $G_{\mathbb{Q}(\zeta_{N'})}$, hence so is χ_0'; η_K is trivial on $G_{\mathbb{Q}(\zeta_{N'})}$ because $|d_K|$ divides M and is prime to ℓ (Ef1). An s acting on $\mu_{N'}$ by $\zeta \mapsto \zeta^w$ acts trivially, so lies in $G_{\mathbb{Q}(\zeta_{N'})}$. $\chi_0' \neq \eta_K$, since otherwise η_K would be trivial on $\ker\bar\Psi \cap \ker\bar\omega$ (Ef4). χ_0' takes values in μ_{q-1}, so $\ell \nmid \operatorname{ord}\chi_0'$. $\chi_0'(c) = \tau((-1)^j) = (-1)^j$ by (Ef2). The reduction of $\chi_{\mathrm{cyc}}^j\chi_0'$ is $\bar\omega^j\bar\Psi\bar\omega^{-j} = \bar\Psi$, so $\chi_{\mathrm{cyc}}^j\chi_0' \equiv \Psi$ modulo the maximal ideal of the integers of $\overline{\mathbb{Q}}_\ell$, and $(\chi_{\mathrm{cyc}}^j\chi_0')(g_\star)^2 \not\equiv 1$ with $g_\star \in G_L$. *(Ag2)* is (Ef1).

So L_n satisfies (Af1)–(Af4), (Ar1), (Ar2), (Ar3) with $C_0 = r_n$, (Ar5), (Ag1), (Ag2) for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, with $g_\star$ and E independent of n. □

Lemma 48.6. *Let ℓ be an odd prime, $\mathbb{F}$ a finite field of characteristic ℓ, $\bar\omega\colon G_{\mathbb{Q}} \to \mathbb{F}_\ell^\times \subseteq \mathbb{F}^\times$ the reduction of the cyclotomic character, $c \in G_{\mathbb{Q}}$ a complex conjugation and $\kappa\colon G_{\mathbb{Q}} \to \mathbb{F}^\times$ a continuous character with $\kappa(c) = -1$ and $\kappa \neq \bar\omega$. Assume that $\kappa|_{D_{\bar v}} \neq 1$ for every place $\bar v$ of $\overline{\mathbb{Q}}$ above ℓ. Let w be a prime with $w \equiv 1 \pmod\ell$ such that κ is trivial on $D_{\bar v}$ for every place $\bar v$ above w, and let $\mathbb{Q}_w^{(\ell)} \subseteq \mathbb{Q}(\zeta_w)$ be the subfield of degree ℓ over $\mathbb{Q}$. Then there is a continuous map $y\colon G_{\mathbb{Q}} \to \mathbb{F}$ with $y(st) = y(s) + \kappa(s)y(t)$ for all $s, t \in G_{\mathbb{Q}}$ such that:*

(Yc1) *for every place $\bar v$ above ℓ there is $b \in \mathbb{F}$ with $y(s) = \kappa(s)b - b$ for all $s \in D_{\bar v}$;*

(Yc2) *$y(s) = 0$ for every place $\bar v$ above a prime $p \notin \{\ell, w\}$ and every $s \in I_{\bar v}$ with $\kappa(s) = 1$;*

(Yc3) *$y(s) = 0$ for every place $\bar v$ above w and every $s \in I_{\bar v} \cap G_{\mathbb{Q}_w^{(\ell)}}$;*

(Yc4) *for every closed normal subgroup $U \subseteq G_{\mathbb{Q}}$ on which κ is not trivial there is no $b \in \mathbb{F}$ with $y(s) = \kappa(s)b - b$ for all $s \in U$.*

Proof. Two identities for cocycles. Let $y\colon G_{\mathbb{Q}} \to \mathbb{F}$ satisfy $y(st) = y(s) + \kappa(s)y(t)$. Then $y(1) = 2y(1)$, so $y(1) = 0$, and $0 = y(\gamma\gamma^{-1})$ gives $y(\gamma^{-1}) = -\kappa(\gamma)^{-1}y(\gamma)$. Hence, since $\kappa(\gamma s\gamma^{-1}) = \kappa(s)$,

$$y(\gamma s\gamma^{-1}) = y(\gamma) + \kappa(\gamma)y(s) + \kappa(\gamma s)y(\gamma^{-1}) = (1-\kappa(s))\,y(\gamma) + \kappa(\gamma)\,y(s) \qquad (\gamma, s \in G_{\mathbb{Q}}). \tag{48.4}$$

In particular, if $y(s) = \kappa(s)b - b$ for all s in a subgroup D, then $y(s') = \kappa(s')b' - b'$ for all $s' \in \gamma D\gamma^{-1}$, with $b' = \kappa(\gamma)b - y(\gamma)$; and if $\kappa(s) = 1$ and $y(s) = 0$, then $y(\gamma s\gamma^{-1}) = 0$.

The Selmer problem. The kernel of κ is open, so κ is ramified at finitely many primes; it is unramified at w, being trivial on $D_{\bar v}$ for $\bar v \mid w$. Let S consist of ∞, ℓ, w and the primes at which κ is ramified, and let $M = \mathbb{F}(\kappa)$, $\mathbb{F}$ with $G_{\mathbb{Q}}$ acting by κ; the action factors through G_S. Here $w \neq \ell$, since $w \equiv 1 \pmod{\ell}$. For each prime $p \in S$ fix a place $\bar p$ above p and put $D_p = D_{\bar p} \supseteq I_p = I_{\bar p}$. Define local conditions: $L_\ell = B^1(D_\ell, M)$; $L_w = Z^1(D_w, M)$; and for the other primes $p \in S$, $L_p = \{z \in Z^1(D_p, M) : z|_{N'_p} = 0\}$ with $N'_p = I_p \cap \ker\kappa$. Each L_p contains $B^1(D_p, M)$: for L_p with $p \neq \ell, w$ because N'_p acts trivially on M. Their dimensions:

- $\dim L_\ell = 1$: $x \mapsto (g \mapsto (\kappa(g) - 1)x)$ maps M onto $B^1(D_\ell, M)$ and is injective because $\kappa|_{D_\ell} \neq 1$.
- $\dim L_p = 1$ for $p \neq \ell, w$, by item 1 of Lemma 43.35 with $N' = N'_p$: it is normal in D_p (I_p is normal in D_p and $\ker\kappa$ is normal), it acts trivially, and $[I_p : N'_p] = \#\kappa(I_p)$ divides $\#\mathbb{F}^\times = q - 1$, which is prime to ℓ.
- $\dim L_w = 2$: κ is trivial on D_w, so $h^0(D_w, M) = 1$; $M^\vee = \mathbb{F}(\bar\omega\kappa^{-1})$, and $\bar\omega$ is trivial on D_w (inertia at $w \neq \ell$ fixes μ_ℓ, and Frobenius acts on μ_ℓ by $\zeta \mapsto \zeta^w = \zeta$), so $\#H^0(D_w, M^\vee) = q$ and, by local duality (Theorem B.29, item 2), $\#H^2(D_w, M) = q$; by the local Euler characteristic formula (item 4, $w \neq \ell$) $\#H^1(D_w, M) = q^2$. Since D_w acts trivially, $B^1(D_w, M) = 0$ and $\dim Z^1(D_w, M) = \dim H^1(D_w, M) = 2$.

We have $H^0(G_S, M^\vee) = 0$, because $\bar\omega\kappa^{-1} \neq 1$ on $G_{\mathbb{Q}}$ and $G_{\mathbb{Q}} \to G_S$ is onto. As $\kappa(c) = -1 \neq 1$ (ℓ odd), $H^0(G_S, M) = 0$ and $M^{c=1} = 0$. By item 2 of Lemma 43.35,

$$\dim H^1_{\mathcal{L}}(M) - \dim H^2_{\mathcal{L}}(M) = 0 - 0 + (1 - 1) + (2 - 1) + \sum_{p \in S,\ p \neq \ell, w} (1 - 1) = 1,$$

so $H^1_{\mathcal{L}}(M) \neq 0$: there is a Selmer cocycle z that is not a coboundary of G_S. Let $y = z \circ \pi_S$ with $\pi_S \colon G_{\mathbb{Q}} \to G_S$ the projection; y is continuous and satisfies the cocycle identity, the action of $G_{\mathbb{Q}}$ on M being through π_S.

(Yc4). First $U = G_{\mathbb{Q}}$: if $y(s) = \kappa(s)b - b$ for all s, then z is the coboundary of b, as π_S is onto; this is excluded. Let U be closed, normal, with $u_0 \in U$, $\kappa(u_0) \neq 1$, and suppose $y(u) = \kappa(u)b - b$ for $u \in U$. Then $y' = y - (\kappa(\cdot)b - b)$ is a cocycle vanishing on U. For $g \in G_{\mathbb{Q}}$ and $u \in U$, $y'(gu) = y'(g) + \kappa(g)y'(u) = y'(g)$, and $gu = (gug^{-1})g$ with $gug^{-1} \in U$ gives $y'(gu) = y'(gug^{-1}) + \kappa(u)y'(g) = \kappa(u)y'(g)$. With $u = u_0$, $(\kappa(u_0) - 1)y'(g) = 0$, so $y' = 0$ and y is a coboundary on $G_{\mathbb{Q}}$, which was excluded.

(Yc1). The condition $\mathrm{res}_\ell z \in L_\ell = B^1(D_\ell, M)$ gives b with $y(s) = \kappa(s)b - b$ on $D_{\bar\ell}$. Every place above ℓ is $\gamma\bar\ell$ with $\gamma \in G_{\mathbb{Q}}$, and $D_{\gamma\bar\ell} = \gamma D_{\bar\ell}\gamma^{-1}$; apply the consequence of (48.4).

(Yc2). Let $\bar v$ lie above $p \notin \{\ell, w\}$ and $s \in I_{\bar v}$ with $\kappa(s) = 1$. If $p \notin S$, then $\pi_S(s) = 1$ because the maximal extension unramified outside S is unramified at p, so $y(s) = z(1) = 0$. If $p \in S$, write $\bar v = \gamma\bar p$, so $s = \gamma t\gamma^{-1}$ with $t \in I_p$ and $\kappa(t) = 1$, i.e. $t \in N'_p$; as $\mathrm{res}_p z \in L_p$, $y(t) = 0$, and (48.4) gives $y(s) = 0$.

(Yc3). Let $\bar v$ lie above w, $I = I_{\bar v}$ and $P \subseteq I$ its wild inertia subgroup. Since κ is trivial on $D_{\bar v}$, $f = y|_I$ is a continuous homomorphism $I \to (\mathbb{F}, +)$. The group P is pro-w and $\mathbb{F}$ is an ℓ-group with $w \neq \ell$, so $f(P) = 0$, and f factors through the tame inertia group I/P, which is procyclic, isomorphic to $\prod_{q \neq w} \mathbb{Z}_q$ ([190], II.9); its image is a cyclic subgroup of $\mathbb{F}$, of order 1 or ℓ. On the other side, $\mathbb{Q}(\zeta_w)/\mathbb{Q}$ is totally ramified at w, so I maps onto $\mathrm{Gal}(\mathbb{Q}_w^{(\ell)}/\mathbb{Q}) \cong \mathbb{Z}/\ell\mathbb{Z}$ and $J = I \cap G_{\mathbb{Q}_w^{(\ell)}}$ is an open normal subgroup of index ℓ containing P (P has no non-trivial image

in a group of order ℓ). If $f \neq 0$, then $\ker f$ is also an open subgroup of index ℓ containing P. A procyclic group has at most one open subgroup of each index, so $\ker f/P = J/P$, i.e. $\ker f = J$. In both cases $f(J) = 0$, which is (Yc3). $\square$

Lemma 48.7. *Let $(K, \mathcal{O}, \Psi, c, g_\star, M)$ be as in Definition 48.2 and let $L \subset \overline{\mathbb{Q}}$ be a CM field satisfying (Af1), (Af2) and (Af4). Let $E' \subset \overline{\mathbb{Q}}$ be the fixed field of $G_L \cap \ker\bar\Psi \cap \ker\bar\omega$ and $r \geq 0$ the integer with $\#\big(\mathrm{Cl}(E')/\mathrm{Cl}(E')^\ell\big) = \ell^r$. Then*

$$h^1_{\mathrm{str}}(\Gamma_L, \mathbb{F}_\mathcal{O}(\bar\Psi^{-2}\eta_K)) \leq r, \qquad h^1_{\mathrm{str}}(\Gamma_L^0, \mathbb{F}_\mathcal{O}(\bar\Psi^{-1})) \leq r, \qquad h^1(\Gamma_L, \mathbb{F}_\mathcal{O}) \leq 1 + r;$$

that is, L satisfies (Ar3) with $C_0 = r$.

Proof. Step 1: the field E' and the subgroup $\Gamma' = \pi_L(G_{E'})$. By (Af1), L is abelian over $\mathbb{Q}$ and $\bar\Psi$, $\bar\omega$ are characters of $G_\mathbb{Q}$, so E' is abelian over $\mathbb{Q}$; it contains μ_ℓ and is totally imaginary (ℓ odd), hence a CM field, and $G_{E'}$ is normal in G_{L^+}. The generators s of N_L lie in inertia groups at places not above ℓ, where $\bar\omega$ is unramified, and satisfy $\eta_K(s) = \bar\Psi(s) = 1$; by (Af1), $G_{L^+} \cap \ker\eta_K = G_L$, so $N_L \subseteq G_{E'}$. Thus $\Gamma' = G_{E'}/N_L$ is an open normal subgroup of Γ_L, contained in Γ_L^0, and $\Gamma_L/\Gamma' \cong G_{L^+}/G_{E'}$ embeds into $\{\pm1\} \times \mathbb{F}_\mathcal{O}^\times \times \mathbb{F}_\ell^\times$ by $(\eta_K, \bar\Psi, \bar\omega)$: its order is prime to ℓ. Put $\Delta = \mathrm{Gal}(E'/L^+) = \Gamma_L/\Gamma'$. Conversely every element of $I_{\bar v} \cap G_{E'}$, $\bar v \nmid \ell$, is a generator of N_L, since $G_{E'} \subseteq \ker\eta_K \cap \ker\bar\Psi$. Hence: a continuous homomorphism $f\colon G_{E'} \to \mathbb{F}_\mathcal{O}$ factors through Γ' if and only if it vanishes on $I_{\bar v} \cap G_{E'}$ for every place $\bar v \nmid \ell$, i.e. if the abelian extension cut out by $\ker f$ is unramified at all finite places not above ℓ (it is unramified at the archimedean places since E' is totally imaginary).

Step 2: restriction is injective. Let $\Delta_0 \in \{\Gamma_L, \Gamma_L^0\}$ and $W = \mathbb{F}_\mathcal{O}(\phi)$ for a character ϕ of Δ_0 trivial on Γ'. The restriction $H^1(\Delta_0, W) \to \mathrm{Hom}_{\mathrm{cts}}(\Gamma', W)$ is injective: its kernel is $H^1(\Delta_0/\Gamma', W)$ (inflation–restriction), and Δ_0/Γ' is finite of order prime to ℓ while W is killed by ℓ. If moreover $W^{\pi_L(H_t)} = 0$ for every member H_t of the family defining the strict group, then $(x, (X_t)) \mapsto x$ is injective from $H^1_{\mathrm{str}}(\Delta_0, W)$ to $H^1(\Delta_0, W)$: if $x = (g \mapsto (g-1)X)$, subtracting the trivial pair $(g \mapsto (g-1)X, (X)_t)$ leaves $(0, (X_t - X))$, and the strict condition gives $(h-1)(X_t - X) = 0$ for $h \in H_t$, so $X_t = X$ and the class is 0. The image of a class consists of homomorphisms $f\colon G_{E'} \to \mathbb{F}_\mathcal{O}$ (composing with $G_{E'} \to \Gamma'$), with $f(\gamma s \gamma^{-1}) = \phi(\gamma) f(s)$ for γ in the preimage of Δ_0, by the cocycle identity.

Step 3: the two strict groups. (a) $\Delta_0 = \Gamma_L$, $\phi = \bar\Psi^{-2}\eta_K$, family $(D_v)_{v|\ell}$. By (Af2) ($\ell \mid M$) each D_v lies in $G_L \subseteq \ker\eta_K$, and by (Af4) $\bar\Psi^2|_{D_v} \neq 1$, so $W^{D_v} = 0$. (b) $\Delta_0 = \Gamma_L^0$, $\phi = \bar\Psi^{-1}$, family (D_v), (cD_vc); by (Af4) $\bar\Psi|_{D_v} \neq 1$ and $\bar\Psi(cgc) = \bar\Psi(g)$, so $W^{D_v} = W^{cD_vc} = 0$. In both cases Step 2 applies. For a strict class with image f: on $D_v \cap G_{E'}$ the character ϕ is trivial, so the coboundary $h \mapsto (h-1)X_v$ vanishes there and $f(D_{\bar v} \cap G_{E'}) = 0$ for the fixed place $\bar v$ above v, and in case (b) also $f(cD_{\bar v}c \cap G_{E'}) = 0$. Every place of $\overline{\mathbb{Q}}$ above ℓ is $\gamma\bar v$ with $\gamma \in G_{L^+}$ in case (a); in case (b) the place v splits in L by (Af2), into the places below $\bar v$ and $c\bar v$, so every place above ℓ is $\gamma\bar v$ or $\gamma c\bar v$ with $\gamma \in G_L$. By the equivariance in Step 2, f vanishes on $D_{\bar u} \cap G_{E'}$ for every place $\bar u \mid \ell$. With Step 1, $\ker f$ cuts out an abelian extension of E', of exponent ℓ, unramified everywhere and in which all places above ℓ split completely; so f factors through $\mathrm{Cl}(E')/\mathrm{Cl}(E')^\ell$ (Hilbert class field), a group of order ℓ^r. The space of homomorphisms from it to $\mathbb{F}_\mathcal{O}$ has $\mathbb{F}_\mathcal{O}$-dimension r, so both strict groups have dimension $\leq r$.

Step 4: the group $H^1(\Gamma_L, \mathbb{F}_\mathcal{O})$. Here $H^1(\Gamma_L, \mathbb{F}_\mathcal{O}) = \mathrm{Hom}_{\mathrm{cts}}(\Gamma_L, \mathbb{F}_\ell) \otimes_{\mathbb{F}_\ell} \mathbb{F}_\mathcal{O}$, so it suffices to bound $\dim_{\mathbb{F}_\ell} \mathrm{Hom}_{\mathrm{cts}}(\Gamma_L, \mathbb{F}_\ell)$ by $1 + r$. By Steps 1, 2 it injects into the space of homomorphisms $f\colon G_{E'} \to \mathbb{F}_\ell$ unramified outside ℓ with $f(\gamma s \gamma^{-1}) = f(s)$ for $\gamma \in G_{L^+}$. Fix a primitive ℓ-th root of unity ζ. Kummer theory identifies $\mathrm{Hom}_{\mathrm{cts}}(G_{E'}, \mu_\ell)$ with $E'^\times/E'^{\times\ell}$, Δ-equivariantly, and $f \mapsto \zeta^f$ carries the invariant f into the $\bar\omega$-part (for $\gamma \in G_{L^+}$, $\gamma(\zeta^{f(\gamma^{-1}s\gamma)}) = \zeta^{\bar\omega(\gamma)f(s)}$). The extension $E'(a^{1/\ell})$ is unramified at a place $u \nmid \ell$ if and only if $u(a) \equiv 0 \pmod\ell$. So

$\dim \mathrm{Hom}(\Gamma_L, \mathbb{F}_\ell) \le \dim B^{(\bar\omega)}$, where $B \subseteq E'^\times/E'^{\times\ell}$ is the subgroup of classes with $u(a) \in \ell\mathbb{Z}$ for every finite place $u \nmid \ell$. Let $\mathcal{O}_S = \mathcal{O}_{E'}[1/\ell]$. Sending a to the class of the ideal $\mathfrak{a}$ of $\mathcal{O}_S$ with $a\mathcal{O}_S = \mathfrak{a}^\ell$ gives an exact sequence of $\mathbb{F}_\ell[\Delta]$-modules

$$0 \to \mathcal{O}_S^\times/\mathcal{O}_S^{\times\ell} \to B \to \mathrm{Cl}(\mathcal{O}_S)[\ell]$$

(an element in the kernel has $\mathfrak{a} = b\mathcal{O}_S$, so $a = b^\ell u$ with $u \in \mathcal{O}_S^\times$; and an S-unit that is an ℓ-th power in E' is the ℓ-th power of an S-unit). The group $\mathrm{Cl}(\mathcal{O}_S)$ is a quotient of $\mathrm{Cl}(E')$, so $\dim \mathrm{Cl}(\mathcal{O}_S)[\ell] = \dim \mathrm{Cl}(\mathcal{O}_S)/\ell \le r$. It remains to show $\dim(\mathcal{O}_S^\times/\ell)^{(\bar\omega)} \le 1$.

Step 5: the $\bar\omega$-part of the S-units. Let $\mu = \mu(E')$ and $Y = (\mathcal{O}_S^\times/\mu) \otimes \mathbb{Z}_\ell$, a free $\mathbb{Z}_\ell$-module of finite rank (Dirichlet) with an action of Δ. Since $\mathcal{O}_S^\times/\mu$ is free, $0 \to \mu/\mu^\ell \to \mathcal{O}_S^\times/\ell \to Y/\ell Y \to 0$ is exact, and μ/μ^ℓ is one-dimensional, with Δ acting by $\bar\omega$. Let $\tilde\omega\colon \Delta \to \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$ be the Teichmüller lift of $\bar\omega$ and $e = \#\Delta^{-1}\sum_\delta \tilde\omega(\delta)^{-1}\delta \in \mathbb{Z}_\ell[\Delta]$; its reduction is $e_{\bar\omega}$, so $(Y/\ell Y)^{(\bar\omega)} = (eY + \ell Y)/\ell Y$, and it suffices to show $eY = 0$, i.e. $e(Y \otimes \mathbb{Q}_\ell) = 0$ (Y is torsion-free). Let $y \in e(Y \otimes \mathbb{Q}_\ell)$, so $\delta y = \tilde\omega(\delta)y$. The valuation map $Y \otimes \mathbb{Q}_\ell \to \bigoplus_{u|\ell} \mathbb{Q}_\ell$ (places u of E' above ℓ) is Δ-equivariant for the permutation action, and its kernel is $(\mathcal{O}_{E'}^\times/\mu) \otimes \mathbb{Q}_\ell$, on which the complex conjugation $c|_{E'} \in \Delta$ acts trivially by the unit index bound. The image of y lies in the $\tilde\omega$-part of $\bigoplus_{u|\ell} \mathbb{Q}_\ell = \bigoplus_{v|\ell} \mathbb{Q}_\ell[\Delta/\Delta_v]$ (v the places of L^+ above ℓ, Δ_v the decomposition group of a place above v), which is 0: a vector of $\mathbb{Q}_\ell[\Delta/\Delta_v]$ on which Δ acts by $\tilde\omega$ is fixed by Δ_v only if $\tilde\omega|_{\Delta_v} = 1$, while $\bar\omega|_{D_v} \ne 1$ by (Af4), Δ_v being the image of D_v. So y is a unit class with $c\,y = y$; but $c\,y = \tilde\omega(c)y = -y$, hence $y = 0$. Therefore $\dim(\mathcal{O}_S^\times/\ell)^{(\bar\omega)} \le 1$, $\dim B^{(\bar\omega)} \le 1 + r$ by exactness of $(\cdot)^{(\bar\omega)}$, and $h^1(\Gamma_L, \mathbb{F}_\mathcal{O}) \le 1 + r$. □

Lemma 48.8. *Let $(K, \mathcal{O}, \Psi, c, g_\star, M)$ be as in Definition 48.2 and let $L \subset \overline{\mathbb{Q}}$ be a CM field satisfying (Af1), (Af2) and (Af4). Then L satisfies items (i), (ii) and (iii) of (Ar5); in (ii) one can take $h = \#\mathrm{Cl}(L)$.*

Proof. A remark used twice. Let $G' \subseteq G_{L^+}$ be a closed subgroup containing N_L, A a finite abelian ℓ-group and $f\colon G' \to A$ a continuous homomorphism trivial on N_L. For a place $\bar u$ of $\overline{\mathbb{Q}}$ not above ℓ, the subgroup $I_{\bar u} \cap G' \cap \ker \eta_K \cap \ker \bar\Psi$ consists of generators of N_L and has index in $I_{\bar u} \cap G'$ dividing $\#(\{\pm 1\} \times \mathbb{F}_\mathcal{O}^\times)$, which is prime to ℓ; so $f(I_{\bar u} \cap G')$ is an ℓ-group which is a quotient of a group of order prime to ℓ, hence trivial. Thus the extension cut out by $\ker f$ is unramified at all finite places not above ℓ.

(i). An open subgroup of Γ_L is $\pi_L(G_{F'})$ for a finite extension F' of L^+ with $N_L \subseteq G_{F'}$. A continuous homomorphism $f\colon \pi_L(G_{F'}) \to \mathbb{Z}/\ell\mathbb{Z}$ gives $f \circ \pi_L$ on $G_{F'}$, trivial on N_L, so by the remark its kernel cuts out an extension of F' of degree 1 or ℓ unramified outside the places above ℓ and ∞. By the theorem of Hermite and Minkowski there are only finitely many such extensions; each determines $\ker f$, and a homomorphism to $\mathbb{Z}/\ell\mathbb{Z}$ is determined by its kernel up to one of $\ell - 1$ automorphisms. So there are finitely many f.

(ii). Let $f\colon \Gamma_L^0 \to A$ be a continuous homomorphism to a finite abelian ℓ-group, trivial on $\pi_L(I_v)$ and $\pi_L(cI_vc)$ for all $v \mid \ell$, and put $f' = f \circ \pi_L$ on G_L (which contains N_L, the generators having $\eta_K = 1$). By the remark, f' is trivial on $I_{\bar u} \cap G_L$ for $\bar u \nmid \ell$. Let $v \mid \ell$ with fixed place $\bar v$. By (Af2) ($\ell \mid M$), v splits in L, so $D_{\bar v} \cap G_{L^+} \subseteq G_L$ and $I_v = I_{\bar v} \cap G_L$, $cI_vc = I_{c\bar v} \cap G_L$; the places of L above v are those below $\bar v$ and $c\bar v$, so every place $\bar u \mid \ell$ of $\overline{\mathbb{Q}}$ is $\gamma\bar v$ or $\gamma c\bar v$ for some v and some $\gamma \in G_L$, and $f'(I_{\bar u} \cap G_L) = f'(\gamma(I_{\bar v} \cap G_L)\gamma^{-1})$ or $f'(\gamma(I_{c\bar v} \cap G_L)\gamma^{-1})$, which is trivial as A is abelian. The field L is totally imaginary. So $\ker f'$ cuts out an unramified abelian extension of L, f' factors through $\mathrm{Gal}(H_L/L) \cong \mathrm{Cl}(L)$, and its values are killed by $h = \#\mathrm{Cl}(L)$.

(iii). Let $v \mid \ell$ be a place of L^+ and $F_v = L_v^+$, so $D_v \cong G_{F_v}$ and $d_v = [F_v : \mathbb{Q}_\ell]$. Let A_v be the maximal abelian pro-ℓ quotient of D_v and $T_v \subseteq A_v$ the image of I_v. By local class field theory

$T_v \cong \mathcal{O}_{F_v}^\times \otimes \mathbb{Z}_\ell \cong \mu_{\ell^\infty}(F_v) \times \mathbb{Z}_\ell^{d_v}$, and $\mu_\ell \not\subseteq F_v$ because $\bar\omega|_{D_v} \neq 1$ by (Af4); so T_v is free of rank d_v. Let $\langle\cdot\rangle\colon \mathbb{Z}_\ell^\times \to 1+\ell\mathbb{Z}_\ell$ be the projection with kernel $\mu_{\ell-1}$; $\chi^\flat = \langle\chi_{\mathrm{cyc}}\rangle$ is a character of D_v with values in the pro-ℓ group $1+\ell\mathbb{Z}_\ell \cong \mathbb{Z}_\ell$, so it factors through A_v, and on T_v it corresponds to $u \mapsto \langle \mathrm{N}_{F_v/\mathbb{Q}_\ell}(u)\rangle^{-1}$. Its image on T_v is a non-zero closed subgroup of $1+\ell\mathbb{Z}_\ell$ (it contains $\langle \mathrm{N}(1+\ell)\rangle^{-1} = (1+\ell)^{-d_v} \neq 1$), hence free of rank one, so $T_v = T_v' \oplus \mathbb{Z}_\ell t$ with $T_v' = T_v \cap \ker\chi^\flat$ free of rank $d_v - 1$. Choose a $\mathbb{Z}_\ell$-basis $t_1, \dots, t_{d_v-1}$ of T_v', put $t_{d_v} = t$, and choose $u_a' \in I_v$ mapping to t_a. Let $m = (\ell-1)\cdot \#\bar\Psi(D_v)$, prime to ℓ, and $u_a = (u_a')^m$. Then $\bar\Psi(u_a) = 1$; for $a < d_v$, $\chi_{\mathrm{cyc}}(u_a') \in \mu_{\ell-1}$ since $\chi^\flat(u_a') = 1$, so $\chi_{\mathrm{cyc}}(u_a) = 1$; and the images mt_a of the u_a form a $\mathbb{Z}_\ell$-basis of T_v ($m \in \mathbb{Z}_\ell^\times$). Let $f\colon D_v \to A$ be a continuous homomorphism to a finite abelian ℓ-group trivial on $u_1, \dots, u_{d_v}$. It factors through A_v, and its restriction to T_v is continuous and vanishes on the subgroup generated by the mt_a, which is dense in T_v; so $f(I_v) = 0$. □

Proposition 48.9. *Let $\mathcal{O}$ be a coefficient ring with residue field $\mathbb{F}_\mathcal{O}$ of odd characteristic ℓ and $\bar\omega$ the reduction of χ_{cyc}. Let $F \subset \overline{\mathbb{Q}}$ be a totally real number field, $K' \subseteq G_F$ an open normal subgroup of index prime to ℓ, N the closed normal subgroup of G_F generated by the groups $I_{\bar u} \cap K'$, $\bar u$ running over the places of $\overline{\mathbb{Q}}$ not above ℓ, $\Gamma = G_F/N$ with projection π, and for each place v of F above ℓ let $\bar v$ be a place of $\overline{\mathbb{Q}}$ above v, $D_v = D_{\bar v} \cap G_F$ and $d_v = [F_v : \mathbb{Q}_\ell]$; for each real place u of F let $c_u \in G_F$ be a complex conjugation at u. Take for $H^i(\Gamma; W, b)$ the family $(D_v)_{v|\ell}$. Then for every finite-dimensional $\mathbb{F}_\mathcal{O}$-vector space W with a continuous linear action of Γ with open kernel, all D_v-stable subspaces $b_v \subseteq W$ ($v \mid \ell$) with $\mathrm{Hom}_\Gamma(W, \mathbb{F}_\mathcal{O}(\bar\omega)) = 0$, the space $H^2(\Gamma; W, b)$ is finite-dimensional and*

$$h^2(\Gamma; W, b) + h^0(\Gamma; W, b) + \sum_{v|\ell} d_v \dim b_v \leq h^1(\Gamma; W, b) + \sum_{u|\infty} \dim W^{c_u=1}.$$

In particular, for $(K, \mathcal{O}, \Psi, c, g_\star, M)$ as in Definition 48.2, every CM field $L \subset \overline{\mathbb{Q}}$ satisfies item (iv) of (Ar5): for every finite-dimensional $\mathbb{F}_\mathcal{O}$-vector space W with a continuous linear action of Γ_L with open kernel, all D_v-stable subspaces $b_v \subseteq W$ ($v \mid \ell$) and $\mathrm{Hom}_{\Gamma_L}(W, \mathbb{F}_\mathcal{O}(\bar\omega)) = 0$, the space $H^2(\Gamma_L; W, b)$ is finite-dimensional and

$$h^2(\Gamma_L; W, b) + h^0(\Gamma_L; W, b) + \sum_{v|\ell} d_v \dim b_v \leq h^1(\Gamma_L; W, b) + \sum_{u|\infty} \dim W^{c_u=1}.$$

Proof. Steps 1–4 prove the general inequality; Step 5 deduces (Ar5)(iv). The character χ_{cyc}, hence $\bar\omega$, is trivial on N (it is unramified at every place not above ℓ), so it factors through Γ. Let S_b be the set of finite places u of F not above ℓ such that $I_{\bar u} \not\subseteq K'$ for a place $\bar u$ above u (equivalently for all of them, K' being normal); it is finite, since the fixed field of K' is a finite extension of F, unramified outside finitely many places. Let S be the union of S_b, the places above ℓ and the archimedean places, and $G = G_{F,S}$. For every place $\bar u$ not above ℓ put $J_{\bar u} = I_{\bar u} \cap K'$; these generate N as a closed normal subgroup. If $\bar u$ lies above a place outside S, then $I_{\bar u} \subseteq K'$, so N contains the kernel of $G_F \to G$; hence $\Gamma = G/\bar N$, where $\bar N$, the image of N, is the closed normal subgroup of G generated by the images of the $J_{\bar u}$, $\bar u$ above S_b. Fix for each $u \in S_b$ a place $\bar u$ above it, with $D_u = D_{\bar u} \cap G_F \supseteq I_u = I_{\bar u} \cap G_F$, and $J_u = I_u \cap K'$. Let W, (b_v) be as in the statement and $W^\vee = \mathrm{Hom}_{\mathbb{F}_\ell}(W, \mu_\ell)$; via the trace form of $\mathbb{F}/\mathbb{F}_\ell$, $(W^\vee)^G \cong \mathrm{Hom}_\Gamma(W, \mathbb{F}(\bar\omega)) = 0$.

Step 1: the long exact sequence. By definition, $H^i(\Gamma; W, b)$ ($i = 0, 1, 2$) is the cohomology in degree i of the complex whose degree-i term consists of the pairs $(x, (x_v)_v)$, $x \in C^i(\Gamma, W)$, $x_v \in C^{i-1}(D_v, W/b_v)$, with differential $(x, (x_v)) \mapsto (dx, (\mathrm{res}_v x - dx_v))$ (the representatives X_t, y_t of Definition 48.2 with values in W matter only modulo b_t, and continuous maps to the finite

set W/b_v lift to W). This is the shifted mapping cone of $C^\bullet(\Gamma,W)\to\bigoplus_{v|\ell}C^\bullet(D_v,W/b_v)$, so with $A_v^i=H^i(D_v,W/b_v)$, $a_v^i=\dim A_v^i$, $h_c^i=h^i(\Gamma;W,b)$ there is an exact sequence

$$0\to H_c^0\to H^0(\Gamma,W)\to\bigoplus\nolimits_v A_v^0\to H_c^1\to H^1(\Gamma,W)\to\bigoplus\nolimits_v A_v^1\to H_c^2\to H^2(\Gamma,W)\xrightarrow{\rho}\bigoplus\nolimits_v A_v^2.$$

Let $K_2=\ker\rho$. All terms are finite: $H^1(\Gamma,W)\subseteq H^1(G,W)$ and $H^2(\Gamma,W)$ by Step 3 below and Theorem B.37, item 1; the A_v^i by Theorem B.29, item 1. In particular H_c^2 is finite-dimensional, and the alternating sum of dimensions gives, with the local Euler characteristic $a_v^0-a_v^1+a_v^2=-d_v\dim(W/b_v)$ (Theorem B.29, item 4),

$$h_c^0-h_c^1+h_c^2=h^0(\Gamma,W)-h^1(\Gamma,W)+\dim K_2+\sum_{v|\ell}a_v^2+\sum_{v|\ell}d_v(\dim W-\dim b_v).\tag{48.5}$$

Step 2: the local terms at S_b. Let $u\in S_b$. I_u/J_u is finite of order prime to ℓ (it embeds into G_F/K'), and $D_u/I_u\cong\widehat{\mathbb{Z}}$. Hence $H^2(D_u/J_u,W)=H^2(D_u/I_u,W^{I_u})=0$ and $\dim H^1(D_u/J_u,W)=\dim W^{D_u}$. J_u acts trivially on W ($J_u\subseteq N$), so inflation–restriction and the local Euler characteristic ($u\nmid\ell$) give

$$\dim\operatorname{Hom}_{D_u}(J_u,W)=h^1(D_u,W)-h^0(D_u,W)=h^2(D_u,W).\tag{48.6}$$

Next, $H^2(\pi(D_u),W)=0$. Indeed $\pi(I_u)$ is a quotient of I_u/J_u, finite of order prime to ℓ and normal in $\pi(D_u)$, and $Q=\pi(D_u)/\pi(I_u)$ is procyclic. Let F_∞ be the cyclotomic $\mathbb{Z}_\ell$-extension of F; it is unramified outside ℓ, so $N\subseteq G_{F_\infty}$ and Γ maps onto $\operatorname{Gal}(F_\infty/F)$, an open subgroup of $\operatorname{Gal}(\mathbb{Q}_\infty/\mathbb{Q})\cong1+\ell\mathbb{Z}_\ell$ (through $\langle\chi_{\mathrm{cyc}}\rangle$). A Frobenius at u maps to $\langle\mathrm{N}u\rangle$, which has infinite order since the integer $\mathrm{N}u\geq2$ is not a root of unity. So Q maps onto an infinite closed subgroup of $\mathbb{Z}_\ell$, its ℓ-Sylow subgroup is $\cong\mathbb{Z}_\ell$, and $H^2(\pi(D_u),W)=H^2(Q,W^{\pi(I_u)})=0$.

Step 3: from Γ to G. Inflation–restriction for $\bar N\subseteq G$ gives $h^0(\Gamma,W)=h^0(G,W)$ and an exact sequence $0\to H^1(\Gamma,W)\to H^1(G,W)\to\operatorname{Hom}_\Gamma(\bar N,W)\to H^2(\Gamma,W)\to H^2(G,W)$. A Γ-equivariant continuous homomorphism $f\colon\bar N\to W$ vanishing on the images of the J_u, $u\in S_b$, vanishes on their conjugates by equivariance (the J at the other places above u are G_F-conjugates of J_u), hence on $\bar N$; and $f|_{J_u}\in\operatorname{Hom}_{D_u}(J_u,W)$. By (48.6), $\dim\operatorname{Hom}_\Gamma(\bar N,W)\leq\sum_{u\in S_b}h^2(D_u,W)$; in particular $H^2(\Gamma,W)$ is finite. Let t be the dimension of the image of $H^1(G,W)$ in $\operatorname{Hom}_\Gamma(\bar N,W)$ and K_2' the image of K_2 in $H^2(G,W)$. Then $h^1(G,W)-h^1(\Gamma,W)=t$ and $\dim K_2\leq\dim K_2'+\dim\operatorname{Hom}_\Gamma(\bar N,W)-t$, so by (48.5) and Tate's formula (Theorem B.37, item 2; F is totally real, so the right side there is $\prod_u\#W^{c_u=1}/\#W$ and $\sum_{v|\ell}d_v=[F:\mathbb{Q}]$),

$$h_c^0-h_c^1+h_c^2\leq\sum_{u|\infty}\dim W^{c_u=1}-h^2(G,W)+\dim K_2'+\dim\operatorname{Hom}_\Gamma(\bar N,W)+\sum_{v|\ell}a_v^2-\sum_{v|\ell}d_v\dim b_v.\tag{48.7}$$

Step 4: Poitou–Tate. Since $(W^\vee)^G=0$, the restriction $R\colon H^2(G,W)\to\bigoplus_{v\in S\text{ finite}}H^2(F_v,W)$ is onto (Theorem B.37, item 3), so $h^2(G,W)=\dim\ker R+\sum_{v|\ell}h^2(D_v,W)+\sum_{u\in S_b}h^2(D_u,W)$. Let $z\in K_2$ with image $z'\in K_2'$. For $u\in S_b$, the restriction of z' to D_u factors through $H^2(\pi(D_u),W)=0$. For $v\mid\ell$, the image of $\mathrm{res}_v z'$ in A_v^2 is the v-component of $\rho(z)=0$; and $H^2(D_v,W)\to A_v^2$ is onto (its cokernel lies in $H^3(D_v,b_v)=0$), so $\mathrm{res}_v z'$ lies in a kernel of dimension $h^2(D_v,W)-a_v^2$. Hence $\dim K_2'\leq\dim\ker R+\sum_{v|\ell}(h^2(D_v,W)-a_v^2)$, and with Step 3,

$$\dim K_2'+\dim\operatorname{Hom}_\Gamma(\bar N,W)+\sum_{v|\ell}a_v^2\leq h^2(G,W).$$

Inserted into (48.7): $h_c^0-h_c^1+h_c^2\leq\sum_{u|\infty}\dim W^{c_u=1}-\sum_{v|\ell}d_v\dim b_v$, which is the general inequality.

Step 5: (Ar5)(iv). Let L be a CM field and take $F = L^+$ (totally real) and $K' = G_F \cap \ker \eta_K \cap \ker \bar{\Psi}$. It is open and normal in G_F, and G_F/K' embeds into $\{\pm 1\} \times \mathbb{F}_{\mathcal{O}}^\times$, of order prime to the odd prime ℓ. An element $s \in I_{\bar{u}} \cap G_{L^+}$, $\bar{u} \nmid \ell$, lies in K' exactly when $\eta_K(s) = 1$ and $\bar{\Psi}(s) = 1$, so the generators of N are those of N_L in Definition 48.2: $N = N_L$, $\Gamma = \Gamma_L$ and $\pi = \pi_L$. The places $\bar{v}$, the groups D_v, the degrees d_v, the conjugations c_u and the family $(D_v)_{v|\ell}$ are those of Definition 48.2. The general inequality is then (48.1), which is item (iv) of (Ar5). $\square$

Lemma 48.10. *Let K, $\mathcal{O}$, c, Ψ with $\Psi(c) = 1$, a marking $g_\star$ and an integer M with $\ell \mid M$ be as in Definition 48.2, and let L be a CM field satisfying (Af1)–(Af3) for $(K, \mathcal{O}, \Psi, c, g_\star, M)$. Let A be a test ring and r a triangular lift to A for $g_\star$. Then there is a polarised ordinary lift $(\rho, (\psi_{v,a}))$ of rank 3 over L to A such that $\rho_L(g) = \underline{r}(g)$ for all $g \in G_L$ and $\psi_{v,a}(g) = \Psi(g)^{w_{3,a}}$ (mapped to A) for all places $v \mid \ell$ of L^+, all a and all $g \in I_v$.*

Proof. Throughout, $G_L \subseteq G_K$ by (Af1), so $r(g) = (\underline{r}(g), 1)$ for $g \in G_L$ by (Tl1) and $\underline{r}$ is multiplicative on G_K.

r factors through Γ_L. By (Tl1), for each $k \geq 1$ the map $g \mapsto r(g) \bmod \mathfrak{m}_A^k$ is locally constant, so its kernel is open; the kernel of $r|_{G_{L^+}}$ is the intersection of these kernels with G_{L^+}, a closed normal subgroup of G_{L^+}. Let $s \in I_{\bar{v}} \cap G_{L^+}$ with $\bar{v} \nmid \ell$, $\eta_K(s) = 1$ and $\bar{\Psi}(s) = 1$. By (Af3), $\Psi(s) = 1$, so $r(s) = 1$ by (Tl3). Hence $N_L \subseteq \ker r$, and $\rho(\pi_L(g)) = \underline{r}(g)$ ($g \in G_L$) is a well-defined homomorphism $\rho\colon \Gamma_L^0 \to \mathrm{GL}_3(A)$ with $\rho_L = \underline{r}|_{G_L}$. (Po1) for ρ follows from (Tl1).

(Po2). By (Tl1) and (Tl2), $r(c) = (1, -1)$. For $g \in G_L$ we have $cg^{-1}c \in G_L$ (as $c \in G_{L^+}$ normalises G_L), and

$$r(cg^{-1}c) = (1,-1)(\underline{r}(g)^{-1}, 1)(1,-1) = (\theta(\underline{r}(g)^{-1}), 1) = (J\,{}^t\underline{r}(g)\,J, 1).$$

(Po3). Put $X = \underline{r}(g_\star)$; $g_\star \in G_L$ by (Af1), and by (Po2) $\underline{r}(cg_\star^{-1}c) = J\,{}^tXJ$, whose (i,j) entry is $X_{4-j,4-i}$. So the (i,j) entry of $D = X - J\,{}^tXJ$ is $X_{ij} - X_{4-j,4-i}$: hence $D_{13} = D_{31} = 0$, $D_{23} = -D_{12}$ and $D_{32} = -D_{21}$. By (Tl2), $D_{12} = D_{21} = 0$, so every off-diagonal entry of D vanishes.

(Po4) and the inertial characters. Let $v \mid \ell$ be a place of L^+ and $\bar{v} = \gamma_v \bar{\ell}$ with $\gamma_v \in G_K$ the place fixed in Definition 48.2. Since $\ell \mid M$, v splits in L by (Af2), so $D_v \subseteq G_L \subseteq G_K$, and $\gamma_v^{-1} D_v \gamma_v \subseteq D_\ell \cap G_K$, $\gamma_v^{-1} I_v \gamma_v \subseteq I_\ell$. Let h be as in (Tl4) and $P = \underline{r}(\gamma_v)h$. For $g \in D_v$,

$$P^{-1}\underline{r}(g)P = h^{-1}\underline{r}(\gamma_v^{-1}g\gamma_v)h$$

is upper triangular; let $\psi_{v,a}(g)$ be its a-th diagonal entry. As $g \mapsto P^{-1}\underline{r}(g)P$ is a homomorphism into the upper triangular group, each $\psi_{v,a}\colon D_v \to A^\times$ is a homomorphism, and it satisfies the continuity of (Po1) because $\underline{r}$ does. For $g_1, g_2, g_3 \in D_v$ put $U_a = P^{-1}(\underline{r}(g_a) - \psi_{v,a}(g_a))P$: an upper triangular matrix with (a,a) entry 0. With V_k the span of the first k standard basis vectors, $U_a V_k \subseteq V_k$ for all k and $U_a V_a \subseteq V_{a-1}$; hence $U_1U_2U_3A^3 = U_1U_2U_3V_3 \subseteq U_1U_2V_2 \subseteq U_1V_1 = 0$, and $\prod_{a=1}^3(\underline{r}(g_a) - \psi_{v,a}(g_a)) = P\,U_1U_2U_3\,P^{-1} = 0$, which is (Po4). If $g \in I_v$, then $\gamma_v^{-1}g\gamma_v \in I_\ell \cap G_K$ and (Tl4) gives $\psi_{v,a}(g) = \psi_a(\gamma_v^{-1}g\gamma_v) = \Psi(g)^{2-a} = \Psi(g)^{w_{3,a}}$, since Ψ is a character of $G_{\mathbb{Q}}$ with values in a commutative group.

(Po5). By (Tl1), $\underline{r}(g) \equiv \mathrm{T}(\Psi(g)) = \mathrm{diag}(\Psi(g)^{w_{3,1}}, \Psi(g)^{w_{3,2}}, \Psi(g)^{w_{3,3}})$ modulo $\mathfrak{m}_A$ for $g \in G_K$, in particular for $g \in G_L$. Since $h \equiv 1$, $P \equiv \mathrm{T}(\Psi(\gamma_v))$, which is diagonal and commutes with $\mathrm{T}(\Psi(g))$; so for $g \in D_v$, $P^{-1}\underline{r}(g)P \equiv \mathrm{T}(\Psi(g))$ and $\psi_{v,a}(g) \equiv \bar{\Psi}(g)^{w_{3,a}}$ modulo $\mathfrak{m}_A$.

Thus $(\rho, (\psi_{v,a}))$ is a polarised ordinary lift of rank 3 over L to A with the two stated properties. $\square$

Proposition 48.11. *Let $\mathcal{O}$ be a coefficient ring, $c \in G_{\mathbb{Q}}$ a complex conjugation, and assume (a) or (b), with K, M and $\Psi = \Psi_{\chi,e}$:*

(a) (K, ℓ, e, N, χ) is a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$, and $M = \ell N|d_K|$;

(b) (ℓ, e, N, χ, o, m) is a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$, $K = \mathbb{Q}(i)$ and $M = \ell N$.

Let $g_\star$ be a marking and L an auxiliary field of some level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$. Let r be a triangular lift for $g_\star$ to an integral test ring such that $\rho_r|_V$ is irreducible for every open subgroup $V \subseteq G_K$. Then for every place $v \mid \ell$ of K at which ρ_r is generic and every place $w \mid v$ of L, the representation $\rho_r|_{G_L}$ is generic at w.

Proof. Let A and r be as in the statement, $\rho = \rho_r$ and $r' = \rho \otimes \chi_{\mathrm{cyc}}^{-1} \colon G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$. By (Af1), L is a CM field, abelian over $\mathbb{Q}$, with $K \subseteq L$; so L/K is finite and Galois and G_L is open in G_K. We check the hypotheses of Lemma 48.12 for $K \subseteq L$, $n = 3$, r' and a place $v \mid \ell$ of K at which ρ is generic.

Automorphy over L. By hypothesis (with $V = G_L$), $\rho|_{G_L}$ is irreducible, and so is $r'|_{G_L}$. By Lemma 48.10 there is a polarised ordinary lift $(\tilde\rho, (\psi_{u,a}))$ of rank 3 over L to A with $\tilde\rho_L = \underline{r}|_{G_L} = \rho|_{G_L}$ and $\psi_{u,a}(g) = \Psi(g)^{w_{3,a}}$ for $g \in I_u$, at every place $u \mid \ell$ of L^+. Since $\tilde\rho_L$ is irreducible over $\overline{\mathbb{Q}}_\ell$, (Pp2) shows that $\tilde\rho_L \otimes \chi_{\mathrm{cyc}}^{-1}|_{G_L} = r'|_{G_L}$ is automorphic of conjugate self-dual type for ι.

Triangular forms at v. Let $\bar v \mid v$ be a place of $\overline{\mathbb{Q}}$. By (Lo2), v lies below some $\bar u \in \{\bar\ell, c\bar\ell\}$, and there are $h \in \mathrm{GL}_3(A)$ and characters $\kappa_1, \kappa_2, \kappa_3$ of the decomposition group $D_{\bar u}$ of G_K at $\bar u$ such that $h^{-1}\rho(g)h$ is upper triangular with diagonal $(\kappa_a(g))$ for $g \in D_{\bar u}$, and $\kappa_a(g) = \Psi(g)^{2-a}$ for $g \in I_{\bar u}$. G_K acts transitively on the places of $\overline{\mathbb{Q}}$ above v, so $\bar v = \gamma\bar u$ with $\gamma \in G_K$, $D_{\bar v} = \gamma D_{\bar u}\gamma^{-1}$ and $I_{\bar v} = \gamma I_{\bar u}\gamma^{-1}$. For $g \in D_{\bar u}$, $(\rho(\gamma)h)^{-1}\rho(\gamma g\gamma^{-1})(\rho(\gamma)h) = h^{-1}\rho(g)h$; so ρ is triangular on $D_{\bar v}$ with diagonal (κ_a^γ), $\kappa_a^\gamma(s) = \kappa_a(\gamma^{-1}s\gamma)$, and $\kappa_a^\gamma(s) = \Psi(s)^{2-a}$ for $s \in I_{\bar v}$, as Ψ is a character of $G_{\mathbb{Q}}$ with commutative values.

Hypothesis 1 of Lemma 48.12. As ρ is generic at v, it is triangular on $D_{\bar v}$ with some diagonal (κ_i') such that $\kappa_i' \neq \kappa_j'\chi_{\mathrm{cyc}}$ for all i, j. A triangular form gives a $D_{\bar v}$-stable flag whose graded pieces are the diagonal characters, so by the Jordan–Hölder theorem (κ_i') and (κ_a^γ) are the same multiset, and $\kappa_a^\gamma \neq \kappa_b^\gamma\chi_{\mathrm{cyc}}$ for all a, b. The representation r' is triangular on $D_{\bar v}$ (same conjugating matrix) with diagonal $\mu_a = \kappa_a^\gamma\chi_{\mathrm{cyc}}^{-1}$, and $\mu_a \neq \mu_b\chi_{\mathrm{cyc}}$ for all a, b.

Hypothesis 2. Put $h_a = e(2-a) - 1$. For $s \in I_{\bar v}$, $\mu_a(s)\chi_{\mathrm{cyc}}(s)^{-h_a} = \Psi(s)^{2-a}\chi_{\mathrm{cyc}}(s)^{-e(2-a)} = \chi(s)^{2-a}$. As χ has finite order, $\ker\chi \cap I_{\bar v}$ is open in $I_{\bar v}$, and $\mu_a\chi_{\mathrm{cyc}}^{-h_a}$ is trivial on it.

Conclusion. By Lemma 48.12, $r'|_{G_L}$ is generic at every place $w \mid v$ of L: for every $\bar w \mid w$ it is triangular on the decomposition group of G_L at $\bar w$ with a diagonal (μ_i') such that $\mu_i' \neq \mu_j'\chi_{\mathrm{cyc}}$ for all i, j. Then $\rho|_{G_L} = r'|_{G_L} \otimes \chi_{\mathrm{cyc}}$ is triangular there with diagonal $(\mu_i'\chi_{\mathrm{cyc}})$, which satisfies the same inequalities. So $\rho_r|_{G_L}$ is generic at w. □

Lemma 48.12. *Let $K \subseteq L \subset \overline{\mathbb{Q}}$ be CM fields with L/K finite and Galois, $n \geq 2$, ℓ a prime, $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and $r \colon G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ a continuous representation such that $r|_{G_L}$ is irreducible and automorphic of conjugate self-dual type for ι. Let $v \mid \ell$ be a place of K such that for every place $\bar v \mid v$ of $\overline{\mathbb{Q}}$, r is triangular on $D_{\bar v}$ with a diagonal $(\kappa_1, \dots, \kappa_n)$ satisfying*

1. $\kappa_a \neq \kappa_b\chi_{\mathrm{cyc}}|_{D_{\bar v}}$ for all a, b;

2. for each a there is an integer h_a such that $\kappa_a\chi_{\mathrm{cyc}}^{-h_a}$ is trivial on an open subgroup of $I_{\bar v}$.

Then $r|_{G_L}$ is generic at every place $w \mid v$ of L.

Proof. Fix a place $w \mid v$ of L and a place $\bar{w} \mid w$ of $\overline{\mathbb{Q}}$; it lies above v. Write $D = D_{\bar{w}} \supseteq I = I_{\bar{w}}$ for the decomposition and inertia groups of G_K at $\bar{w}$, $D' = D \cap G_L$ for the decomposition group of G_L at $\bar{w}$ (open and normal in D, as L/K is finite Galois), $W \subseteq D$ for the Weil group and $W' = W \cap D'$ for that of D', $q = q_v$, and f for the residue degree of w over v, so $q_w = q^f$. For $g \in W$ put $\|g\| = q^{n(g)}$. Let (κ_a) and (h_a) be as in the hypothesis for $\bar{v} = \bar{w}$, and $\nu_a = \kappa_a \chi_{\mathrm{cyc}}^{-h_a}$, a continuous character of D trivial on an open subgroup of I. As D is compact, each $\nu_a(D)$ is a compact subgroup of $\overline{\mathbb{Q}}_\ell^\times$, so the values of ν_a are ℓ-adic units.

Step 1: r is de Rham on D, and the characters of its Weil–Deligne representation. By hypothesis $r|_{G_L} \leftrightarrow_\iota \pi$ for a regular algebraic conjugate self-dual cuspidal π of $\mathrm{GL}_n(\mathbb{A}_L)$. By Theorem B.62, $r_{\ell,\iota}(\pi)$ is semisimple and, by (Bg1), unramified with the same characteristic polynomials of Frobenius as $r|_{G_L}$ at almost all places; $r|_{G_L}$ is irreducible, so $r|_{G_L} \cong r_{\ell,\iota}(\pi)$ by the Chebotarev density theorem and the Brauer–Nesbitt theorem. By (Bg2), $r|_{D'}$ is de Rham. A representation whose restriction to an open subgroup is de Rham is de Rham, and de Rham representations are potentially semistable ([95], [23]); so $r|_D$ is potentially semistable. Let $(V, \rho, N) = \mathrm{WD}(r|_D)$. We use the following properties of Fontaine's construction ([95]; [255], Section 1):

(F1) subrepresentations and quotients of potentially semistable representations are potentially semistable, and WD is exact and compatible with tensor products on them;

(F2) $\mathrm{WD}(r|_{D'}) = (V, \rho|_{W'}, N)$ (compatibility with restriction to open subgroups);

(F3) $\chi_{\mathrm{cyc}}|_D$ is crystalline and $\mathrm{WD}(\chi_{\mathrm{cyc}}|_D) = \|\cdot\|^\delta$ for a sign $\delta \in \{\pm 1\}$ fixed by the normalisation of the construction (φ acts on $D_{\mathrm{cris}}(\chi_{\mathrm{cyc}})$ by ℓ^{-1});

(F4) a continuous character ν of D trivial on an open subgroup of I is potentially semistable, with $\mathrm{WD}(\nu) = (\nu^\epsilon|_W, 0)$ for a sign $\epsilon \in \{\pm 1\}$ fixed by the normalisation.

Let $g \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ conjugate $r|_D$ into upper triangular form with diagonal (κ_a), and V_a the span of the first a columns of g^{-1}: a D-stable flag with $V_a/V_{a-1} \cong \kappa_a$. By (F1), (F3), (F4), $\kappa_a = \chi_{\mathrm{cyc}}^{h_a}\nu_a$ is potentially semistable with

$$\omega_a := \mathrm{WD}(\kappa_a) = \|\cdot\|^{\delta h_a}\, \nu_a^\epsilon|_W,$$

and V has a $\rho(W)$-stable flag whose graded pieces are $\omega_1, \dots, \omega_n$. In particular, for every $\Psi \in W$ the eigenvalues of $\rho(\Psi)$, with multiplicity, are $\omega_1(\Psi), \dots, \omega_n(\Psi)$, and the Jordan–Hölder constituents of (V, ρ) are the ω_a.

Step 2: $N = 0$. $r|_D$ is potentially semistable and triangular on D with diagonal (κ_a), and $\kappa_a\kappa_b^{-1}\chi_{\mathrm{cyc}}^{-(h_a-h_b)} = \nu_a\nu_b^{-1}$ is trivial on an open subgroup of I. By hypothesis 1 and (Tj1) of Lemma 48.13 (for the number field K, the place v and $\bar{v} = \bar{w}$), $N = 0$.

Step 3: purity over L forces genericity at w. By (F2) and Step 2, $\mathrm{WD}(r|_{D'}) = (V, \rho|_{W'}, 0)$. As $r|_{D'}$ is conjugate to $r_{\ell,\iota}(\pi)|_{D'}$, Theorem B.73 (with $F = L$, the place w, $\bar{v} = \bar{w}$; $n \geq 2$) shows that this Weil–Deligne representation, with zero monodromy, is pure in the sense of (Tj2) of Lemma 48.13 (for the number field L, the place w and an element $\Phi' \in W'$ acting on the residue field as $x \mapsto x^{q_w^{-1}}$): (Pu1), (Pu2) are exactly its conditions. $r|_{D'}$ is potentially semistable, triangular on D' with diagonal $(\kappa_a|_{D'})$, and $\kappa_a\kappa_b^{-1}\chi_{\mathrm{cyc}}^{-(h_a-h_b)}$ is trivial on an open subgroup of $I \cap D'$. So (Tj2) of Lemma 48.13 for $r|_{D'}$ with $m = 1$ gives $\kappa_a|_{D'} \neq \kappa_b\chi_{\mathrm{cyc}}|_{D'}$ for all a, b.

Conclusion. With the same g, $g\,r(s)g^{-1}$ is upper triangular with diagonal $(\kappa_a(s))$ for $s \in D'$, so $r|_{G_L}$ is triangular on D', the decomposition group of G_L at $\bar{w}$, with the diagonal $(\kappa_a|_{D'})$, which satisfies the inequalities of genericity. As $\bar{w} \mid w$ was arbitrary, $r|_{G_L}$ is generic at w. □

Lemma 48.13. *Let F be a number field, $v \mid \ell$ a place of F, $\bar v \mid v$ a place of $\overline{\mathbb{Q}}$, $D = D_{\bar v} \cap G_F \supseteq I = I_{\bar v} \cap G_F$ and $W \subseteq D$ the Weil group, $q = q_v$, and $\Phi \in W$ with $n(\Phi) = -1$. Let $n \geq 1$ and $r\colon D \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be a continuous potentially semistable representation which is triangular on D with a diagonal $(\kappa_1, \dots, \kappa_n)$ such that for all a, b there is an integer h_{ab} with $\kappa_a \kappa_b^{-1} \chi_{\mathrm{cyc}}^{-h_{ab}}$ trivial on an open subgroup of I. Let $(V, \rho, N) = \mathrm{WD}(r)$.*

(Tj1) *If $\kappa_a \neq \kappa_b \chi_{\mathrm{cyc}}|_D$ for all a, b, then $N = 0$.*

(Tj2) *If $N = 0$ and (V, ρ, N) is* pure*, i.e. there are a real number k and $\rho(W)$-stable subspaces $\mathrm{Fil}_j V$ ($j \in k + \mathbb{Z}$), increasing, 0 for j small and V for j large, with $N(\mathrm{Fil}_j V) \subseteq \mathrm{Fil}_{j-2} V$, such that every eigenvalue α of $\rho(\Phi)$ on $\mathrm{gr}_j = \mathrm{Fil}_j V / \mathrm{Fil}_{j-1} V$ satisfies $|\sigma(\alpha)| = q^{j/2}$ for every field embedding $\sigma\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and $N^i\colon \mathrm{gr}_{k+i} \to \mathrm{gr}_{k-i}$ is an isomorphism for every integer $i \geq 1$, then every eigenvalue α of $\rho(\Phi)$ satisfies $|\iota(\alpha)| = q^{k/2}$, and $\kappa_a^m \neq (\kappa_b \chi_{\mathrm{cyc}})^m$ on D for all a, b and every integer $m \geq 1$.*

Proof. We use the following properties of Fontaine's construction ([95]; [255], Section 1):

(F1) subrepresentations and quotients of potentially semistable representations are potentially semistable, and WD is exact and compatible with tensor products and duals on them;

(F2) for an open subgroup $D' \subseteq D$ with Weil group $W' = W \cap D'$, $\mathrm{WD}(r|_{D'}) = (V, \rho|_{W'}, N)$;

(F3) $\chi_{\mathrm{cyc}}|_D$ is crystalline and $\mathrm{WD}(\chi_{\mathrm{cyc}}|_D) = \|\cdot\|^\delta$ for a sign $\delta \in \{\pm 1\}$ fixed by the normalisation of the construction;

(F4) a continuous character ν of D trivial on an open subgroup of I is potentially semistable with $\mathrm{WD}(\nu) = (\nu^\epsilon|_W, 0)$ for a sign $\epsilon \in \{\pm 1\}$ fixed by the normalisation.

Step 1: the graded pieces of V. Let $g \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ conjugate r into upper triangular form with diagonal (κ_a); the span V_a of the first a columns of g^{-1} is D-stable with $V_a / V_{a-1} \cong \kappa_a$. By (F1) each κ_a is potentially semistable; put $\omega_a = \mathrm{WD}(\kappa_a)$, a character of W (a one-dimensional Weil–Deligne representation has zero monodromy). By exactness, (V, ρ) has a $\rho(W)$-stable flag with graded pieces $\omega_1, \dots, \omega_n$; so for every $h \in W$ the eigenvalues of $\rho(h)$ with multiplicity are $\omega_1(h), \dots, \omega_n(h)$, and the Jordan–Hölder constituents of (V, ρ) are the ω_a. With $\nu_{ab} = \kappa_a \kappa_b^{-1} \chi_{\mathrm{cyc}}^{-h_{ab}}$, (F1), (F3), (F4) give

$$\omega_a \omega_b^{-1} = \|\cdot\|^{\delta h_{ab}} \nu_{ab}^\epsilon|_W.$$

As D is compact, $\nu_{ab}(D)$ is a compact subgroup of $\overline{\mathbb{Q}}_\ell^\times$, so the values of ν_{ab} are ℓ-adic units.

Step 2: (Tj1). By (38.5), $\rho(h) N \rho(h)^{-1} = \|h\| N$, i.e. N is a W-equivariant map $(V, \rho) \to (V, \|\cdot\|^{-1}\rho)$. Suppose $N \neq 0$. A simple subquotient of its image is a Jordan–Hölder constituent of both, so $\omega_a = \|\cdot\|^{-1} \omega_b$ for some a, b, i.e. $\|\cdot\|^{\delta h_{ab} + 1} = \nu_{ab}^{-\epsilon}$ on W. At Φ the left side is $q^{-(\delta h_{ab}+1)}$ and the right side an ℓ-adic unit; q is a positive power of ℓ, so $\delta h_{ab} + 1 = 0$, i.e. $h_{ab} = -\delta$, and then $\nu_{ab} = 1$ on W, which is dense in D, hence on D. So $\kappa_a = \kappa_b \chi_{\mathrm{cyc}}^{-\delta}$: for $\delta = -1$ this is $\kappa_a = \kappa_b \chi_{\mathrm{cyc}}$, for $\delta = 1$ it is $\kappa_b = \kappa_a \chi_{\mathrm{cyc}}$; both are excluded by hypothesis. Hence $N = 0$.

Step 3: (Tj2), the absolute values. As $N = 0$, the isomorphism $N^i\colon \mathrm{gr}_{k+i} \to \mathrm{gr}_{k-i}$ is the zero map, so $\mathrm{gr}_{k+i} = \mathrm{gr}_{k-i} = 0$ for every integer $i \geq 1$. All indices lie in $k + \mathbb{Z}$, so $\mathrm{Fil}_{k-1} V = 0$ and $\mathrm{Fil}_k V = V$, i.e. $V = \mathrm{gr}_k$, and every eigenvalue α of $\rho(\Phi)$ satisfies $|\iota(\alpha)| = q^{k/2}$.

Step 4: (Tj2), potential genericity. Let $m \geq 1$ and suppose $\kappa_a^m = (\kappa_b \chi_{\mathrm{cyc}})^m$ on D. By (F1), (F3), $\omega_a^m = \mathrm{WD}(\kappa_a^m) = \mathrm{WD}(\kappa_b^m \chi_{\mathrm{cyc}}^m) = \omega_b^m \|\cdot\|^{\delta m}$, so $|\iota\omega_a(\Phi)|^m = |\iota\omega_b(\Phi)|^m q^{-\delta m}$. Both $\omega_a(\Phi)$ and $\omega_b(\Phi)$ are eigenvalues of $\rho(\Phi)$ (Step 1), of absolute value $q^{k/2}$ (Step 3), so $q^{-\delta m} = 1$, which is impossible as $q > 1$ and $m \geq 1$. Hence $\kappa_a^m \neq (\kappa_b \chi_{\mathrm{cyc}})^m$ for all a, b and $m \geq 1$. □

Proposition 48.14. *Let $\mathcal{O}$ be a coefficient ring, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $c \in G_{\mathbb{Q}}$ a complex conjugation, (K, ℓ, e, N, χ) a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$, $M = \ell N |d_K|$ and $\Psi = \Psi_{\chi,e}$. Let $g_\star$ be a marking, L an auxiliary field of some level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, and $\kappa\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$ an algebraic character such that L has automorphic irreducible lifts in rank 2 with twist κ for ι. Then for every triangular lift r for $g_\star$ of type II to an integral test ring with $\rho_r(g)_{22} = 1$ for all $g \in G_K$, every continuous $W\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ whose restriction to G_K is the corner of r and with $\operatorname{tr} W(c) = 0$ satisfies (Eo2) and the witness conditions over imaginary quadratic fields.*

Proof. Let A, r and W be as in the statement, $\rho = \rho_r$ and $\rho^\flat$ the corner of r, so $W|_{G_K} = \rho^\flat$. By (Ue1), ℓ is odd; by (Ue5), $\Psi(c) = \chi(-1)(-1)^e = 1$; $\ell \mid M$; and L satisfies (Af1)–(Af3), so Lemma 48.15 applies to r. By (Af1), L is a CM field, abelian over $\mathbb{Q}$, containing the imaginary quadratic field K. We check the hypotheses of Proposition 48.16 (numbered N1–N3 below) and of Proposition 48.17 (numbered I1–I5).

I1: $\det W = \eta_K$, of finite order. By (Tl1), (Tl2), $r(c) = (1, -1)$, so for $g \in G_K$, $\rho(cgc) = \theta(\rho(g)) = J\,{}^t\rho(g)^{-1}J$. As $\rho(g)$ is block diagonal for $\{e_1, e_3\} \sqcup \{e_2\}$ (type II), so is $\rho(g)^{-1}$, with corner $\rho^\flat(g)^{-1}$; the (i,j) entry of $J\,{}^tZJ$ is $Z_{4-j,4-i}$, so $\rho^\flat(cgc) = J_2\,{}^t\rho^\flat(g)^{-1}J_2$ and $\det\rho^\flat(cgc) = \det\rho^\flat(g)^{-1}$. Since $c^2 = 1$, $W(cgc) = W(c)W(g)W(c)^{-1}$ has determinant $\det W(g)$. Hence $\det\rho^\flat(g)^2 = 1$. By (Tl1), $\det\rho^\flat(g) \equiv \Psi(g)\Psi(g)^{-1} = 1$ modulo $\mathfrak{m}_A$, and $-1 \not\equiv 1$ as ℓ is odd; so $\det W = 1$ on G_K. Also $W(c)^2 = 1$ and $\operatorname{tr} W(c) = 0$, so $W(c)$ has eigenvalues $1, -1$ and $\det W(c) = -1$. Thus $\det W$ is the character of $\mathrm{Gal}(K/\mathbb{Q})$ that is -1 at c: $\det W = \eta_K$, of finite order.

N1 and I2 are (Ko1).

N2 and I3. By (Ko3) there is a polarised ordinary lift of rank 2 over L to A with $\rho'_L = \rho^\flat|_{G_L} = W|_{G_L}$ and $\psi'_{v,a} = \Psi^{w_{2,a}}$ on I_v; ρ'_L is irreducible over $\overline{\mathbb{Q}}_\ell$ by (Ko1). As L has automorphic irreducible lifts in rank 2 with twist κ for ι, $W|_{G_L} \otimes \kappa$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions; κ is algebraic by hypothesis.

N3 and I4. Let $D' = D_\ell \cap G_K$, open in D_ℓ and containing $D_\ell \cap G_L$ as $K \subseteq L$. By (Ko2) there are $Q \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ and continuous characters κ_1, κ_3 of D' such that $Q^{-1}W(s)Q$ is upper triangular with diagonal $\kappa_1(s), \kappa_3(s)$ for $s \in D'$, with $\kappa_1 = \Psi$ and $\kappa_3 = \Psi^{-1}$ on $D' \cap I_\ell = I_\ell \cap G_K$. On the open subgroup $I_\ell \cap G_K \cap \ker\chi$ of $D' \cap I_\ell$ (χ has finite order), $\Psi^{\pm 1} = \chi_{\mathrm{cyc}}^{\pm e}$. Take $P = Q^{-1}$ and the characters κ_1, $\kappa_2 = \kappa_3$.

I5. Let $p \neq \ell$ be a prime and $\bar{p} \mid p$. The extension of $\mathbb{Q}$ generated by the ℓ-power roots of unity is unramified at p, so $\chi_{\mathrm{cyc}} = 1$ and $\Psi = \chi$ on $I_{\bar{p}}$. If $p \nmid d_K$, then p is unramified in K and $I_{\bar{p}} \subseteq G_K$. The group $Z = \Psi(I_{\bar{p}})$ is finite, hence cyclic of some order n, equal to the group of n-th roots of unity, with a generator z_0. If $g, g' \in I_{\bar{p}}$ and $\Psi(g) = \Psi(g')$, then $g^{-1}g' \in I_{\bar{p}}$ has $\eta_K(g^{-1}g') = 1$ and $\Psi(g^{-1}g') = 1$, so $r(g^{-1}g') = 1$ by (Tl3) and $W(g) = W(g')$. Hence $W(g) = B(\Psi(g))$ for a homomorphism $B\colon Z \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$; $B(z_0)^n = 1$, so $B(z_0)$ is diagonalisable with eigenvalues $z_0^{b_1}, z_0^{b_2}$ for integers b_1, b_2, and $\det(X - W(g)) = (X - \Psi(g)^{b_1})(X - \Psi(g)^{b_2})$ for $g \in I_{\bar{p}}$: take $\kappa'_i = \Psi^{b_i}|_{D_{\bar{p}}}$. If $p \mid d_K$, then $p \nmid \ell N$ by (Ue6), (Ue7), so χ, hence Ψ, is unramified at p; for $g \in I_{\bar{p}} \cap G_K$, $r(g) = 1$ by (Tl3), so $W(g) = 1$. The subgroup $I_{\bar{p}} \cap G_K$ has index at most 2 in $I_{\bar{p}}$; for $g \in I_{\bar{p}} \smallsetminus G_K$, $W(g)^2 = W(g^2) = 1$, so $W(g)$ is diagonalisable with eigenvalues in $\{\pm 1\}$ and determinant $\eta_K(g) = -1$, i.e. with eigenvalues $1, -1$. So $\det(X - W(g)) = (X - 1)(X - \eta_K(g))$ for all $g \in I_{\bar{p}}$: take $\kappa'_1 = 1$, $\kappa'_2 = \eta_K|_{D_{\bar{p}}}$.

By Proposition 48.16, W satisfies (Eo2), and by Proposition 48.17 it satisfies the witness conditions over imaginary quadratic fields. □

Lemma 48.15. *Let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field, ℓ an odd prime, $\mathcal{O}$ a coefficient ring, χ a Dirichlet character with values in $\mathcal{O}^\times$, $e \geq 1$, $\Psi = \Psi_{\chi,e}$ with $\Psi(c) = 1$, $g_\star$ a marking,*

$M \geq 1$ with $\ell \mid M$, and L a CM field satisfying (Af1)–(Af3) for $(K, \mathcal{O}, \Psi, c, g_\star, M)$. Let A be an integral test ring, r a triangular lift for $g_\star$ to A of type II with $\rho_r(g)_{22} = 1$ for all $g \in G_K$, and $\rho^\flat \colon G_K \to \mathrm{GL}_2(A)$ its corner. Then:

(Ko1) *$\rho^\flat|_{G_L}$ is irreducible over $\overline{\mathbb{Q}}_\ell$;*

(Ko2) *$\rho^\flat$ is triangular on $D_\ell \cap G_K$ with a diagonal (κ_1, κ_3) of continuous characters $D_\ell \cap G_K \to A^\times$ such that $\kappa_1 \equiv \Psi$ and $\kappa_3 \equiv \Psi^{-1}$ modulo $\mathfrak{m}_A$, and $\kappa_1 = \Psi$, $\kappa_3 = \Psi^{-1}$ on $I_\ell \cap G_K$;*

(Ko3) *there is a polarised ordinary lift $(\rho', (\psi'_{v,a}))$ of rank 2 over L to A with $\rho'_L = \rho^\flat|_{G_L}$ and $\psi'_{v,a}(g) = \Psi(g)^{w_{2,a}}$ for every place $v \mid \ell$ of L^+, $a = 1, 2$ and $g \in I_v$.*

Proof. Write $\rho = \rho_r$ and $E \subset \overline{\mathbb{Q}}_\ell$ for the fraction field of A. For $g \in G_K$, $\rho(g)$ is block diagonal for the decomposition $E^3 = M' \oplus Ee_2$, $M' = Ee_1 \oplus Ee_3$, with $\rho(g)e_2 = e_2$; hence the corner $\rho^\flat$ (the action on M') is a homomorphism. By (Tl1), $\rho^\flat(g) \equiv \mathrm{diag}(\Psi(g), \Psi(g)^{-1})$ modulo $\mathfrak{m}_A$ for $g \in G_K$. By (Af1), L is abelian over $\mathbb{Q}$ with $K \subseteq L$, so G_L is an open normal subgroup of G_K with abelian quotient, and $g_\star \in G_L$. Write $\bar\Psi$ for the reduction of Ψ.

(Ko1). Suppose $\rho^\flat|_{G_L}$ is reducible over $\overline{\mathbb{Q}}_\ell$. As $\rho^\flat$ is irreducible and G_L is normal in G_K, Clifford's theorem makes $\rho^\flat|_{G_L} \otimes \overline{\mathbb{Q}}_\ell$ semisimple, a sum of two characters χ_1, χ_2 of G_L, and G_K permutes its isotypic components. Three cases. (i) $\chi_1 = \chi_2$: then $\rho^\flat(g_\star)$ is a scalar matrix $\lambda \cdot 1$ with $\lambda \in A$; modulo $\mathfrak{m}_A$ it is $\mathrm{diag}(\bar\Psi(g_\star), \bar\Psi(g_\star)^{-1})$, so $\bar\Psi(g_\star)^2 = 1$, contradicting $\Psi(g_\star)^2 - 1 \in \mathcal{O}^\times$. (ii) $\chi_1 \neq \chi_2$ and G_K fixes both isotypic lines: then they are G_K-stable and $\rho^\flat$ is reducible, a contradiction. (iii) $\chi_1 \neq \chi_2$ and some element of G_K swaps the two lines: the stabiliser H of the χ_1-line has index 2 in G_K and contains G_L; for $g \in G_K \smallsetminus H$, $\rho^\flat(g)$ swaps the two lines, so $\mathrm{tr}\,\rho^\flat(g) = 0$, and reducing modulo $\mathfrak{m}_A$ gives $\bar\Psi(g) + \bar\Psi(g)^{-1} = 0$, i.e. $\bar\Psi(g)^2 = -1$. Fix $g \in G_K \smallsetminus H$; as $g_\star \in G_L \subseteq H$, also $gg_\star \in G_K \smallsetminus H$, so $\bar\Psi(g)^2\bar\Psi(g_\star)^2 = -1 = \bar\Psi(g)^2$ and $\bar\Psi(g_\star)^2 = 1$, the same contradiction. Hence $\rho^\flat|_{G_L}$ is irreducible.

(Ko2). Put $D^\circ = D_\ell \cap G_K$, $I^\circ = I_\ell \cap G_K$. By (Tl4) there is $h \in \mathrm{GL}_3(A)$, $h \equiv 1$, with $h^{-1}\rho(g)h$ upper triangular for $g \in D^\circ$; its diagonal entries $\kappa_1, \kappa_2, \kappa_3$ are homomorphisms $D^\circ \to A^\times$, continuous by (Tl1), with $\kappa_a = \Psi^{2-a}$ on I° (Tl4) and $\kappa_a \equiv \Psi^{2-a}$ modulo $\mathfrak{m}_A$ (as $h \equiv 1$ and (Tl1)). So E^3 has the D°-stable flag $F_1 = E\,he_1 \subset F_2 = E\,he_1 + E\,he_2$ with graded characters $\kappa_1, \kappa_2, \kappa_3$, and by the Jordan–Hölder theorem the multiset $\{\kappa_1, \kappa_2, \kappa_3\}$ is that of the constituents of M' together with the trivial character (of Ee_2). Now $\Psi = \chi\chi_{\mathrm{cyc}}^e$ with χ of finite order and $e \geq 1$, and χ_{cyc} has infinite image on every open subgroup of I_ℓ; as I° is open in I_ℓ, $\Psi^{\pm 1}$ is non-trivial on I°, so $\kappa_1 \neq 1 \neq \kappa_3$. Hence $\kappa_2 = 1$ and the constituents of M' are κ_1, κ_3. Let $p' \colon E^3 \to M'$ be the projection along Ee_2, which is D°-equivariant. If $p'(F_1) = 0$ then $F_1 = Ee_2$ and $\kappa_1 = 1$; so $p'(F_1)$ is a D°-stable line of M' with character κ_1, and the quotient $M'/p'(F_1)$ has character κ_3. In a basis of M' whose first vector spans $p'(F_1)$, $\rho^\flat$ is upper triangular on D° with diagonal (κ_1, κ_3).

(Ko3). Let $(\tilde\rho, (\psi_{v,a}))$ be the polarised ordinary lift of rank 3 over L given by Lemma 48.10, with $\tilde\rho(\pi_L(g)) = \rho(g)$ for $g \in G_L$. Define $\rho'(x)$, for $x \in \Gamma_L^0$, as the corner of $\tilde\rho(x)$: a homomorphism, since $\tilde\rho(\Gamma_L^0) = \rho(G_L)$ consists of block diagonal matrices, with $\rho'_L = \rho^\flat|_{G_L}$. (Po1) and (Po5) for ρ' follow from those for $\tilde\rho$ (the corner of $\mathrm{diag}(\bar\Psi, 1, \bar\Psi^{-1})$ is $\mathrm{diag}(\bar\Psi^{w_{2,1}}, \bar\Psi^{w_{2,2}})$). (Po2): the (i,j) entry of $J_3\,{}^tX\,J_3$ is $X_{4-j,4-i}$, so for $i, j \in \{1,3\}$ the corner of $J_3\,{}^tX\,J_3$ has entries $X_{33}, X_{13}, X_{31}, X_{11}$ at the places $(1,1), (1,3), (3,1), (3,3)$, which is $J_2\,{}^tY\,J_2$ for the corner Y of X; apply this to (Po2) for $\tilde\rho$. (Po3): the off-diagonal entries of the corner of $\tilde\rho(\pi_L(g_\star)) - \tilde\rho(\pi_L(cg_\star^{-1}c))$ are its entries $(1,3)$ and $(3,1)$, which vanish by (Po3) for $\tilde\rho$. (Po4): let $v \mid \ell$ be a place of L^+ and $\bar v = \gamma_v\bar\ell$, $\gamma_v \in G_K$, the place fixed in Definition 48.2. Since $\ell \mid M$, v splits in L (Af2), so $D_v \subseteq G_L$, and $\gamma_v^{-1}D_v\gamma_v \subseteq D^\circ$, $\gamma_v^{-1}I_v\gamma_v \subseteq I^\circ$. Put $\psi'_{v,1}(g) = \kappa_1(\gamma_v^{-1}g\gamma_v)$ and

$\psi'_{v,2}(g) = \kappa_3(\gamma_v^{-1} g \gamma_v)$ for $g \in D_v$; with $Q \in \mathrm{GL}_2(E)$ conjugating $\rho^\flat$ to upper triangular form on D° as in (Ko2), $(\rho^\flat(\gamma_v)Q)^{-1}\rho^\flat(g)(\rho^\flat(\gamma_v)Q) = Q^{-1}\rho^\flat(\gamma_v^{-1}g\gamma_v)Q$ is upper triangular with diagonal $(\psi'_{v,1}(g), \psi'_{v,2}(g))$. For $g_1, g_2 \in D_v$ the conjugates of $\rho^\flat(g_1) - \psi'_{v,1}(g_1)$ and $\rho^\flat(g_2) - \psi'_{v,2}(g_2)$ are upper triangular with zero $(1,1)$, respectively $(2,2)$, entry, and the product of such matrices (in this order) is 0; so $(\rho'(\pi_L(g_1)) - \psi'_{v,1}(g_1))(\rho'(\pi_L(g_2)) - \psi'_{v,2}(g_2)) = 0$. The $\psi'_{v,a}$ are homomorphisms $D_v \to A^\times$ with the continuity of (Po1) (by (Ko2) and (Tl1)), $\psi'_{v,a} \equiv \bar\Psi^{w_{2,a}}$ modulo $\mathfrak{m}_A$, and for $g \in I_v$, $\psi'_{v,1}(g) = \Psi(\gamma_v^{-1}g\gamma_v) = \Psi(g)$ and $\psi'_{v,2}(g) = \Psi(g)^{-1}$, Ψ being a character of $G_\mathbb{Q}$ with commutative values. So $(\rho', (\psi'_{v,a}))$ is as claimed. □

Proposition 48.16. *Let ℓ be a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $e \geq 1$, $L \subset \overline{\mathbb{Q}}$ a CM field abelian over $\mathbb{Q}$, and $W\colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ a continuous representation such that:*

1. *$W|_{G_L}$ is irreducible;*
2. *there is an algebraic character $\kappa\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$ such that $W|_{G_L} \otimes \kappa$ is automorphic of conjugate self-dual type for ι;*
3. *there are an open subgroup $D' \subseteq D_\ell$ containing $D_\ell \cap G_L$, continuous characters $\kappa_1, \kappa_2\colon D' \to \overline{\mathbb{Q}}_\ell^\times$ and $P \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that $P\,W(s)P^{-1}$ is upper triangular with diagonal entries $\kappa_1(s), \kappa_2(s)$ for all $s \in D'$, and $\kappa_1\chi_{\mathrm{cyc}}^{-e}$ and $\kappa_2\chi_{\mathrm{cyc}}^{e}$ are trivial on an open subgroup of $D' \cap I_\ell$.*

Then W satisfies (Eo2).

Proof. Write q_w for the size of the residue field at a finite place w, and for a finite place w of a number field F and a place $\bar w \mid w$, $\sigma_{\bar w}$ for an arithmetic Frobenius at $\bar w$ in G_F.

Step 1: an untwisted automorphic representation over L. By hypothesis 2, $W|_{G_L} \otimes \kappa \leftrightarrow_\iota \Pi$ for a regular algebraic conjugate self-dual cuspidal automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_L)$, and κ corresponds to an algebraic Hecke character ψ of L: ψ is a continuous character of $\mathbb{A}_L^\times/L^\times$ whose component at each complex place w is $z \mapsto z^{a_w}\bar z^{b_w}$ with $a_w, b_w \in \mathbb{Z}$, and for all but finitely many finite places w, κ is unramified at w, ψ_w is unramified and $\iota\kappa(\sigma_{\bar w}) = \psi_w(\varpi_w)$ (with the opposite normalisation of the correspondence, replace ψ by ψ^{-1}, again an algebraic Hecke character). Let $\Pi' = \Pi \otimes (\psi \circ \det)$, a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_L)$. At all but finitely many finite w, Π_w is a subquotient of the normalised induction of unramified μ_1, μ_2, so Π'_w is the unramified subquotient of that of $\mu_1\psi_w, \mu_2\psi_w$ and has Satake parameters $\alpha\psi_w(\varpi_w)$, α running over those of Π_w; and

$$\iota \det(X - W(\sigma_{\bar w}^{-1})) = \iota \det(X - \kappa(\sigma_{\bar w})(W \otimes \kappa)(\sigma_{\bar w}^{-1})) = \prod_\alpha (X - q_w^{1/2}\alpha\,\psi_w(\varpi_w)).$$

Hence $W|_{G_L} \leftrightarrow_\iota \Pi'$. At a complex place w the Langlands parameter of Π'_w is that of Π_w times ψ_w, so its restriction to $\mathbb{C}^\times$ has the exponents of that of Π_w shifted by the integers a_w, b_w: Clozel's integrality and regularity conditions are preserved, and Π' is regular algebraic.

Step 2: a tower of cyclic steps. Let L^+ be the maximal totally real subfield of L, so $[L : L^+] = 2$. The group $\mathrm{Gal}(L^+/\mathbb{Q})$ is finite abelian, so it has a chain of subgroups with successive quotients cyclic of prime order; by Galois theory there are fields $\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_m = L^+$ with each F_i/F_{i-1} cyclic of prime degree (Galois, all subgroups being normal). Put $F_{m+1} = L$; F_{m+1}/F_m is cyclic of degree 2. The fields $F_0, \ldots, F_m$ are totally real. As $G_L \subseteq G_{F_i}$, $W|_{G_{F_i}}$ is irreducible for every i, by hypothesis 1.

Step 3: descent. We show by descending induction on $i \in \{0, \ldots, m+1\}$ that there is a regular algebraic cuspidal automorphic representation π_i of $\mathrm{GL}_2(\mathbb{A}_{F_i})$ with $W|_{G_{F_i}} \leftrightarrow_\iota \pi_i$. For

$i = m+1$ take $\pi_{m+1} = \Pi'$ (Step 1). Let $1 \le i \le m+1$ and π_i be given; write $E = F_i$, $F = F_{i-1}$, let σ generate $\mathrm{Gal}(E/F)$ and $\tilde\sigma \in G_F$ restrict to σ.

(a) $\pi_i^\sigma \cong \pi_i$. The component of π_i^σ at a finite place w of E is the component of π_i at the place $\sigma(w)$, transported by the isomorphism $\sigma\colon E_w \to E_{\sigma(w)}$, which carries uniformisers to uniformisers; so at almost all w it is unramified with the Satake parameters of $\pi_{i,\sigma(w)}$, and $q_w = q_{\sigma(w)}$. If $\sigma_{\bar w}$ is an arithmetic Frobenius at $\bar w \mid w$, then $\tilde\sigma\sigma_{\bar w}\tilde\sigma^{-1}$ is one at $\tilde\sigma\bar w$, which lies above $\sigma(w)$, and $W(\tilde\sigma\sigma_{\bar w}^{-1}\tilde\sigma^{-1})$ is conjugate to $W(\sigma_{\bar w}^{-1})$. Since $W|_{G_E} \leftrightarrow_\iota \pi_i$, at almost all w the Satake parameters of $\pi_{i,w}$ and $\pi_{i,\sigma(w)}$ are both determined by the same characteristic polynomial, hence equal. An unramified irreducible representation is determined by its Satake parameters, so π_i^σ and π_i have isomorphic components at almost all places; both are cuspidal, so $\pi_i^\sigma \cong \pi_i$ by Theorem B.60, (Js1).

(b) By Theorem B.74 there is a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ with (Bc1) and (Bc2) for $\Pi = \pi_i$.

(c) π is regular algebraic. Let v be a real place of F (F is totally real) and $w \mid v$. If $i \le m$, E is totally real, v splits completely in E and $\pi_v \cong \pi_{i,w}$ by (Bc1). If $i = m+1$, w is complex, and by (Bc2) the parameter of $\pi_{i,w}$ is the restriction to $W_{\mathbb{C}} = \mathbb{C}^\times$ of that of π_v; the twist $|\det|^{-1/2}$ at v restricts to $z \mapsto |z\bar z|^{-1/2}$, which is the twist $|\det|^{-1/2}$ at w. In both cases the restriction to $\mathbb{C}^\times$ of the parameter of $\pi_v \otimes |\det|^{-1/2}$ is that of $\pi_{i,w} \otimes |\det|^{-1/2}$, which satisfies Clozel's conditions.

(d) By Theorem B.75 there is a continuous $r\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ with $r \leftrightarrow_\iota \pi$; its semisimplification r' has the same characteristic polynomials, so $r' \leftrightarrow_\iota \pi$.

(e) $r'|_{G_E} \cong W|_{G_E}$. Let T be the set of finite places w of E such that $v = w|_F$ splits completely in E, π_v and $\pi_{i,w}$ are unramified, r' and W are unramified at w and the relations $r' \leftrightarrow_\iota \pi$, $W|_{G_E} \leftrightarrow_\iota \pi_i$ hold at v, w; all but finitely many places of E of degree one over F lie in T. For $w \in T$, $E_w = F_v$, so $\sigma_{\bar w} \in G_E$ is also an arithmetic Frobenius at $\bar w$ in G_F, $q_w = q_v$, and $\pi_{i,w} \cong \pi_v$ by (Bc1); hence $r'(\sigma_{\bar w}^{-1})$ and $W(\sigma_{\bar w}^{-1})$ have the same characteristic polynomial. The places of E of degree one over F have Dirichlet density 1, so by the Chebotarev density theorem these Frobenius elements are dense in G_E; characteristic polynomials are continuous, so $r'|_{G_E}$ and $W|_{G_E}$ have the same characteristic polynomials. $r'|_{G_E}$ is semisimple (restriction of a semisimple representation to a normal subgroup of finite index, Clifford's theorem) and $W|_{G_E}$ is irreducible (Step 2), so they are isomorphic (Brauer–Nesbitt).

(f) The twist. Fix $A \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ with $W(g) = A\,r'(g)A^{-1}$ for $g \in G_E$. For $h \in G_F$ and $g \in G_E$, using $hgh^{-1} \in G_E$,

$$A\,r'(h)A^{-1}\,W(g) = A\,r'(hgh^{-1})r'(h)A^{-1} = W(hgh^{-1})\,A\,r'(h)A^{-1},$$

so $W(h)^{-1}A\,r'(h)A^{-1}$ commutes with $W(G_E)$ and is a scalar $\lambda(h)$ by Schur's lemma. Then $A\,r'(h)A^{-1} = \lambda(h)W(h)$, so $\lambda\colon G_F \to \overline{\mathbb{Q}}_\ell^\times$ is a continuous homomorphism, trivial on G_E. Let $\chi_\lambda = \iota\circ\lambda\circ\mathrm{Art}_F$, a character of $\mathbb{A}_F^\times/F^\times$ of finite order; at every finite v unramified in E, $\iota\lambda(\sigma_{\bar v}) = \chi_\lambda(\varpi_v)$. As $W(\sigma_{\bar v}^{-1}) = \lambda(\sigma_{\bar v})\,A\,r'(\sigma_{\bar v}^{-1})A^{-1}$, at almost all v the polynomial $\iota\det(X - W(\sigma_{\bar v}^{-1}))$ is $\prod_\alpha(X - q_v^{1/2}\alpha\chi_\lambda(\varpi_v))$ over the Satake parameters α of π_v. Put $\pi_{i-1} = \pi \otimes (\chi_\lambda \circ \det)$: it is cuspidal, unramified at almost all v with Satake parameters $\alpha\chi_\lambda(\varpi_v)$, so $W|_{G_F} \leftrightarrow_\iota \pi_{i-1}$; and it is regular algebraic, since at a real place χ_λ is 1 or the sign character, both trivial on $\mathbb{C}^\times$ after restriction of parameters. This completes the induction.

Step 4: the newform. By Step 3 with $i = 0$, $W \leftrightarrow_\iota \pi_0$ for a regular algebraic cuspidal π_0 of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. By Theorem B.76 there are a normalised newform $f = \sum a_n q^n$ of weight $k \ge 2$, level N and character ε, and $j \in \mathbb{Z}$, such that for all but finitely many primes p: W is trivial on $I_{\bar p}$ and $\iota\det(X - W(\sigma_{\bar p}^{-1})) = X^2 - p^j a_p X + \varepsilon(p)p^{2j+k-1}$. Let $\rho = \rho_{f,\iota}$ be the representation of (Dg1) in Theorem B.59 and ρ^{ss} its semisimplification. For $p \nmid \ell N$, $\chi_{\mathrm{cyc}}(\sigma_{\bar p}) = p$,

so $(\rho^{\mathrm{ss}} \otimes \chi_{\mathrm{cyc}}^{-j})(\sigma_{\bar p}^{-1}) = p^j \rho^{\mathrm{ss}}(\sigma_{\bar p}^{-1})$ has characteristic polynomial $X^2 - p^j a_p X + p^{2j}\varepsilon(p)p^{k-1}$. By the Chebotarev density theorem, continuity and the Brauer–Nesbitt theorem (W is irreducible by hypothesis 1, hence semisimple),

$$W \cong \rho^{\mathrm{ss}} \otimes \chi_{\mathrm{cyc}}^{-j}.$$

Step 5: the weight. By Theorem B.77, applied to ρ^{ss}, $\rho^{\mathrm{ss}}|_{D_\ell}$ is Hodge–Tate with weights 0 and $1-k$; so $W|_{D_\ell}$ is Hodge–Tate with weights $-j$ and $1-k-j$. We use the standard facts ([95]): restriction to an open subgroup of D_ℓ (the Galois group of a finite extension of $\mathbb{Q}_\ell$) keeps a representation Hodge–Tate with the same weights; subrepresentations and quotients of Hodge–Tate representations are Hodge–Tate and the weights of the middle term of a short exact sequence are those of the two ends; and a character trivial on the inertia group is Hodge–Tate of weight 0 (Tate, Sen). By hypothesis 3 the line spanned by the first column of P^{-1} is stable under $W(D')$ with character κ_1, and D' acts on the quotient by κ_2. There is an open normal subgroup D'' of D' such that $\kappa_1\chi_{\mathrm{cyc}}^{-e}$ and $\kappa_2\chi_{\mathrm{cyc}}^{e}$ are trivial on $D'' \cap I_\ell$ (an open subgroup of $D' \cap I_\ell$ contains $D'' \cap I_\ell$ for some such D''); on D'' they are unramified, so $\kappa_1|_{D''}$ has weight e and $\kappa_2|_{D''}$ has weight $-e$. Hence $\{-j, 1-k-j\} = \{e, -e\}$. As $k \geq 2$, $1-k-j < -j$, so $-j = e$ and $1-k-j = -e$: $j = -e$ and $k = 2e+1$.

Step 6: (Eo2). By Steps 4 and 5, $W \otimes \chi_{\mathrm{cyc}}^{-e} \cong \rho^{\mathrm{ss}}$ is semisimple, and for every prime $p \nmid \ell N$ and $\bar p \mid p$ it is trivial on $I_{\bar p}$ with $\iota\det(X - (W \otimes \chi_{\mathrm{cyc}}^{-e})(\sigma^{-1})) = X^2 - a_p X + \varepsilon(p)p^{2e}$ for every arithmetic Frobenius σ at $\bar p$, by (Dg1) for ρ (the semisimplification has the same characteristic polynomials and is trivial on $I_{\bar p}$ when ρ is). As f is a normalised newform of weight $2e+1$, W satisfies (Eo2). □

Proposition 48.17. *Let ℓ be a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $e \geq 1$, $L \subset \overline{\mathbb{Q}}$ a CM field abelian over $\mathbb{Q}$, and $W\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ a continuous representation such that:*

1. *$\det W$ has finite order;*
2. *$W|_{G_L}$ is irreducible;*
3. *there is an algebraic character $\kappa\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$ such that $W|_{G_L} \otimes \kappa$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions;*
4. *there are an open subgroup $D' \subseteq D_\ell$ containing $D_\ell \cap G_L$, continuous characters $\kappa_1, \kappa_2\colon D' \to \overline{\mathbb{Q}}_\ell^\times$ and $P \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that $P\,W(s)P^{-1}$ is upper triangular with diagonal entries $\kappa_1(s), \kappa_2(s)$ for all $s \in D'$, and $\kappa_1\chi_{\mathrm{cyc}}^{-e}$ and $\kappa_2\chi_{\mathrm{cyc}}^{e}$ are trivial on an open subgroup of $D' \cap I_\ell$;*
5. *for every prime $p \neq \ell$ and every place $\bar p \mid p$ of $\overline{\mathbb{Q}}$ there are continuous characters κ_1', κ_2' of $D_{\bar p}$ with $\det(X - W(g)) = (X - \kappa_1'(g))(X - \kappa_2'(g))$ for all $g \in I_{\bar p}$.*

Then W satisfies the witness conditions over imaginary quadratic fields.

Proof. Fix an imaginary quadratic field K' such that $V = W|_{G_{K'}}$ is irreducible. We show (Wc1) and (Wc2) for V; the number τ of (Wc2) will be $-1/2$ for every π'. For a place $\bar u$ of $\overline{\mathbb{Q}}$ and a number field F, $D_{\bar u}$ and $I_{\bar u}$ denote the decomposition and inertia groups in $G_{\mathbb{Q}}$, and $D_{\bar u} \cap G_F$, $I_{\bar u} \cap G_F$ those in G_F.

Step 1: the flag at every place above ℓ. Let $\bar u \mid \ell$; as $G_{\mathbb{Q}}$ acts transitively on the places above ℓ, $\bar u = \gamma\bar\ell$ with $\gamma \in G_{\mathbb{Q}}$, $D_{\bar u} = \gamma D_\ell \gamma^{-1}$, $I_{\bar u} = \gamma I_\ell \gamma^{-1}$. Put $D'_{\bar u} = \gamma D' \gamma^{-1}$, open in $D_{\bar u}$; it contains $D_{\bar u} \cap G_L = \gamma(D_\ell \cap G_L)\gamma^{-1}$, G_L being normal in $G_{\mathbb{Q}}$. For $s \in$

$D'_{\bar u}$, $(PW(\gamma)^{-1})W(s)(PW(\gamma)^{-1})^{-1} = PW(\gamma^{-1}s\gamma)P^{-1}$ is upper triangular with diagonal $\kappa_1^\gamma(s), \kappa_2^\gamma(s)$, $\kappa_a^\gamma(s) = \kappa_a(\gamma^{-1}s\gamma)$, and since χ_{cyc} is a character of $G_{\mathbb{Q}}$ with commutative values, $\kappa_1^\gamma = \chi_{\mathrm{cyc}}^e$ and $\kappa_2^\gamma = \chi_{\mathrm{cyc}}^{-e}$ on an open subgroup $H_{\bar u}$ of $I_{\bar u}$.

Step 2: potential genericity. Let $F \in \{K', L\}$, θ a continuous character of G_F ($\theta = \mathbf{1}$ if $F = K'$, $\theta = \kappa$ if $F = L$), $\bar u \mid \ell$, and suppose $W|_{G_F} \otimes \theta$ is triangular on $D_{\bar u} \cap G_F$ with a diagonal (μ_1, μ_2). We claim $\mu_a^m \neq (\mu_b\chi_{\mathrm{cyc}})^m$ for all a, b and $m \geq 1$. Let $H = H_{\bar u} \cap G_F$, open in $I_{\bar u}$. On H the representation $W \otimes \theta$ has two stable flags, with graded characters (μ_1, μ_2) and $(\kappa_1^\gamma\theta, \kappa_2^\gamma\theta) = (\chi_{\mathrm{cyc}}^e\theta, \chi_{\mathrm{cyc}}^{-e}\theta)$; by the Jordan–Hölder theorem the multisets agree on H, so $\mu_a/(\mu_b\chi_{\mathrm{cyc}})$ is χ_{cyc}^{-1} (if $a = b$) or $\chi_{\mathrm{cyc}}^{\pm 2e-1}$ (if $a \neq b$) on H. Now $\chi_{\mathrm{cyc}}(H)$ is open in $\mathbb{Z}_\ell^\times$, hence contains a nontrivial element of the torsion-free group $1 + \ell^2\mathbb{Z}_\ell$, and $\pm 2e - 1 \neq 0$; so $(\mu_a/(\mu_b\chi_{\mathrm{cyc}}))^m$ is nontrivial on H. Consequences: (i) if V is generic at a place $v \mid \ell$ of K', it is triangular on $D_{\bar v} \cap G_{K'}$ for every $\bar v \mid v$, hence potentially generic at v: this is (Wc1) for V. (ii) $W|_{G_L} \otimes \kappa$ is triangular on $D_{\bar w} \cap G_L \subseteq D'_{\bar w}$ (Step 1) for every $\bar w \mid \ell$, hence potentially generic at every place of L above ℓ.

Step 3: the Galois representations of π' and Π. Let π' be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{K'})$ with $V \leftrightarrow_\iota \pi'$ (V is semisimple). Put $\chi_0(x) = |x|_{\mathbb{A}} \cdot \iota \det W(\mathrm{Art}_{\mathbb{Q}}(x))^{-1}$, a continuous character of $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$. We show $\pi'^c \cong \pi'^\vee \otimes (\chi_0 \circ \mathrm{N}_{K'/\mathbb{Q}} \circ \det)$, i.e. π' is polarizable (χ_0 has one infinite place). At almost every finite place u of K', with α, β the Satake parameters of π'_u and σ an arithmetic Frobenius at some $\bar u \mid u$: (1) $\iota \det W(\sigma^{-1}) = q_u\alpha\beta$, from $V \leftrightarrow_\iota \pi'$; (2) $\mathrm{Art}_{\mathbb{Q}}(\mathrm{N}_{K'/\mathbb{Q}}\varpi_u)$ is the image in $G_{\mathbb{Q}}^{ab}$ of $\mathrm{Art}_{K'}(\varpi_u)$, the class of σ, and $|\mathrm{N}_{K'/\mathbb{Q}}\varpi_u|_{\mathbb{A}} = q_u^{-1}$, so $\chi_0(\mathrm{N}\varpi_u) = q_u^{-1}\iota \det W(\sigma^{-1}) = \alpha\beta$; (3) the component of π'^c at u is π'_{cu} transported by c, and $W(c\sigma^{-1}c)$ is conjugate to $W(\sigma^{-1})$ in GL_2, so its Satake parameters are again α, β; (4) those of $\pi'^\vee \otimes (\chi_0 \circ \mathrm{N} \circ \det)$ are $\alpha^{-1}\alpha\beta = \beta$ and $\beta^{-1}\alpha\beta = \alpha$. Both representations are cuspidal, so they are isomorphic by Theorem B.60, (Js1). Hence Theorem B.62 applies to π'; $r_{\ell,\iota}(\pi')$ is semisimple and has the characteristic polynomials of V at almost all Frobenius elements, so $r_{\ell,\iota}(\pi') \cong V$ (Chebotarev, Brauer–Nesbitt). Likewise, by hypothesis 3, $W|_{G_L} \otimes \kappa \leftrightarrow_\iota \Pi$ for a regular algebraic conjugate self-dual cuspidal Π of $\mathrm{GL}_2(\mathbb{A}_L)$, which is polarizable with the trivial character, and $r_{\ell,\iota}(\Pi) \cong W|_{G_L} \otimes \kappa$ (irreducible by hypothesis 2).

Step 4: local set-up. Fix a finite place v of K' with $v \nmid \ell$, or with $v \mid \ell$ and V potentially generic at v; fix $\bar v \mid v$, let p be the prime below v and w the place of L below $\bar v$. Let $D = D_{\bar v}$, $D_{K'} = D \cap G_{K'}$, $D_L = D \cap G_L$, and $D_0 = D_{K'} \cap D_L$, intersected with $D'_{\bar v}$ if $p = \ell$; all are open in D, and W_0, $W_{K'}$, W_L denote their Weil groups. If $p = \ell$, $V|_{D_{K'}} \cong r_{\ell,\iota}(\pi')|_{D_{K'}}$ and $(W \otimes \kappa)|_{D_L}$ are de Rham by (Bg2), hence so are $W|_D$ (a representation de Rham on an open subgroup is de Rham, [95]) and all subquotients of these; de Rham representations are potentially semistable. We use, for $p = \ell$, the properties of Fontaine's construction ([95]; [255], Section 1): it is exact and compatible with tensor products on potentially semistable representations and with restriction to open subgroups; for $p \neq \ell$ the same properties of (Wd1) are immediate from the definition.

Step 5: the cuspidal support of π'_v. Let $(U, \rho, N) = \mathrm{WD}(V|_{D_{K'}})$ and $I = I_{\bar v} \cap G_{K'}$. The semisimplification of $\rho|_I$ is a sum of restrictions of two characters of $W_{K'}$. If $p \neq \ell$: take κ'_1, κ'_2 of hypothesis 5 for $\bar p = \bar v$. For $\sigma \in I$, $n(\sigma) = 0$, so $\rho(\sigma)$ commutes with N by (38.5) and $V(\sigma) = \rho(\sigma)\exp(t_\ell(\sigma)N)$ by (Wd1) is $\rho(\sigma)$ times a commuting unipotent matrix; hence $\det(X - \rho(\sigma)) = (X - \kappa'_1(\sigma))(X - \kappa'_2(\sigma))$. ρ is trivial on an open normal subgroup J of I, where this polynomial is $(X-1)^2$, so κ'_1, κ'_2 are trivial on J; then $\rho|_I$ and $(\kappa'_1 \oplus \kappa'_2)|_I$ are representations of the finite group I/J over a field of characteristic 0 with equal characteristic polynomials, hence equal traces, hence isomorphic. If $p = \ell$: V is triangular on $D_{K'}$ with some diagonal (μ_1, μ_2) (potential genericity at v); the μ_a are potentially semistable and by exactness $\rho^{\mathrm{ss}} \cong \mathrm{WD}(\mu_1) \oplus \mathrm{WD}(\mu_2)$, a sum of characters of $W_{K'}$. In both cases (Bg3) for π' (with $U = \{1\}$) shows that π'_v is an irreducible subquotient of the normalised induction of $\xi_1 \otimes \xi_2$ for characters

ξ_1, ξ_2 of $K_v'^{\times}$; the supercuspidal support of an irreducible representation is unique up to order (Bernstein–Zelevinsky), so every cuspidal support of π_v' is (ξ_1, ξ_2) up to order. By (Bg5) for π', $\rho^{\mathrm{ss}} \cong \lambda_1' \oplus \lambda_2'$ with $\iota\lambda_a'(\mathrm{Art}_{K_v'}(x)^{-1}) = \xi_a(x)|x|_v^{-1/2}$ for $x \in K_v'^{\times}$.

Step 6: the same over L. By Step 2 (ii), $w \nmid \ell$ or $W|_{G_L} \otimes \kappa$ is potentially generic at w, so (Wc2) for $W|_{G_L} \otimes \kappa$ (hypothesis 3), applied to Π, gives a real number τ_Π (independent of w) and a cuspidal support (ξ_1', ξ_2') of Π_w consisting of characters with $q_w^{\tau_\Pi - 1/2} < |\xi_a'(\varpi_w)| < q_w^{\tau_\Pi + 1/2}$. Then Π_w is an irreducible subquotient of the normalised induction of $\xi_1' \otimes \xi_2'$, and (Bg5) for Π gives that the semisimplification of the Weil group part of $\mathrm{WD}((W \otimes \kappa)|_{D_L})$ is $\lambda_1'' \oplus \lambda_2''$ with $\iota\lambda_a''(\mathrm{Art}_{L_w}(y)^{-1}) = \xi_a'(y)|y|_w^{-1/2}$.

Step 7: the ratios agree on W_0. We claim that after renumbering $\lambda_1'/\lambda_2' = \lambda_1''/\lambda_2''$ on W_0. If $p \neq \ell$: by compatibility with restriction, the semisimplified Weil group part of $\mathrm{WD}(W|_{D_0})$ is $\beta_1 \oplus \beta_2$ with $\beta_a = \lambda_a'|_{W_0}$; by compatibility with tensor products, that of $\mathrm{WD}((W \otimes \kappa)|_{D_0})$ is $\beta_1\nu \oplus \beta_2\nu$, where ν is the Weil group part of $\mathrm{WD}(\kappa|_{D_0})$; it is also $\lambda_1''|_{W_0} \oplus \lambda_2''|_{W_0}$, and a semisimple representation that is a sum of characters determines them as a multiset. If $p = \ell$: on $D_0 \subseteq D_{\bar v}'$, W is triangular with diagonal $(\kappa_1^\gamma, \kappa_2^\gamma)$ and $W \otimes \kappa$ with diagonal $(\kappa_1^\gamma\kappa, \kappa_2^\gamma\kappa)$ (Step 1), all potentially semistable characters; by exactness and compatibility with restriction, $\{\lambda_a'|_{W_0}\} = \{\mathrm{WD}(\kappa_a^\gamma|_{D_0})\}$ and $\{\lambda_a''|_{W_0}\} = \{\mathrm{WD}(\kappa_a^\gamma\kappa|_{D_0})\}$, and by compatibility with tensor products $\mathrm{WD}(\kappa_1^\gamma\kappa)/\mathrm{WD}(\kappa_2^\gamma\kappa) = \mathrm{WD}(\kappa_1^\gamma/\kappa_2^\gamma) = \mathrm{WD}(\kappa_1^\gamma)/\mathrm{WD}(\kappa_2^\gamma)$ on W_0.

Step 8: the window. Fix $\Phi_0 \in W_0$ with $n(\Phi_0) = -d < 0$ (relative to $\mathbb{Q}_p$); W_0 contains such elements, D_0 being open in D. Let f_v, f_w be the residue degrees of K_v', L_w over $\mathbb{Q}_p$; then $f_v \mid d$, $f_w \mid d$, $\Phi_0 = \mathrm{Art}_{K_v'}(x)^{-1}$ with $x = \varpi_v^{d/f_v}u$, $u \in \mathcal{O}_{K_v'}^\times$, and $\Phi_0 = \mathrm{Art}_{L_w}(y)^{-1}$ with $y = \varpi_w^{d/f_w}u'$, $u' \in \mathcal{O}_{L_w}^\times$; and $q_v^{d/f_v} = q_w^{d/f_w} = p^d$. Characters of $\mathcal{O}^\times$ have values of absolute value 1 (compact image), so by Steps 5 and 6

$$\Big|\frac{\iota\lambda_1'(\Phi_0)}{\iota\lambda_2'(\Phi_0)}\Big| = \Big|\frac{\xi_1(\varpi_v)}{\xi_2(\varpi_v)}\Big|^{d/f_v}, \qquad \Big|\frac{\iota\lambda_1''(\Phi_0)}{\iota\lambda_2''(\Phi_0)}\Big| = \Big|\frac{\xi_1'(\varpi_w)}{\xi_2'(\varpi_w)}\Big|^{d/f_w} \in (q_w^{-d/f_w}, q_w^{d/f_w}) = (p^{-d}, p^d),$$

the last by the window of Step 6. By Step 7 the two ratios are equal (possibly after inverting one, which preserves the interval), so $|\xi_1(\varpi_v)/\xi_2(\varpi_v)|^{d/f_v} \in (p^{-d}, p^d)$, i.e. $q_v^{-1} < |\xi_1(\varpi_v)/\xi_2(\varpi_v)| < q_v$. On the other hand $\lambda_1'\lambda_2'$ is the determinant of ρ^{ss}, which is the Weil group part of $\mathrm{WD}(\det V|_{D_{K'}})$ (for $p \neq \ell$ by (Wd1), $\det\exp(t_\ell(\sigma)N) = 1$; for $p = \ell$ from the triangular form, $\mathrm{WD}(\mu_1)\mathrm{WD}(\mu_2) = \mathrm{WD}(\mu_1\mu_2)$); $\det W$ has finite order (hypothesis 1), so this character has finite order and $|\iota\lambda_1'\lambda_2'(\mathrm{Art}_{K_v'}(\varpi_v)^{-1})| = 1$, i.e. $|\xi_1(\varpi_v)\xi_2(\varpi_v)| = q_v^{-1}$. Hence $|\xi_a(\varpi_v)|^2 = |\xi_1(\varpi_v)\xi_2(\varpi_v)| \cdot |\xi_a(\varpi_v)/\xi_b(\varpi_v)| \in (q_v^{-2}, 1)$ for $\{a,b\} = \{1,2\}$, that is

$$q_v^{-1} < |\xi_a(\varpi_v)| < 1 = q_v^{\tau+1/2} \quad (\tau = -1/2), \qquad a = 1, 2.$$

Conclusion. For every π' as in (Wc2), at every finite place v of K' with $v \nmid \ell$, or $v \mid \ell$ and V potentially generic at v, π_v' has a cuspidal support consisting of characters, and every character in a cuspidal support of π_v' lies in the window with $\tau_{\pi'} = -1/2$ (Steps 5 and 8). With Step 2 (i), V satisfies the witness conditions. As K' was arbitrary with V irreducible, W satisfies the witness conditions over imaginary quadratic fields. □

Proposition 48.18. *Let $K \subset L \subset \overline{\mathbb{Q}}$ with K imaginary quadratic and L a CM field abelian over $\mathbb{Q}$, let $c \in G_{\mathbb{Q}}$ be a complex conjugation, ℓ a prime and $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism. Let $r\colon G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ be continuous and assume:*

1. *there is $P \in \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ with ${}^t r(s)\, P\, r(csc) = \chi_{\mathrm{cyc}}(s)^{-2}P$ for all $s \in G_K$;*
2. *$r|_{G_L}$ is irreducible;*

3. *for every finite place $v \nmid \ell$ of K and every place $\bar{v} \mid v$ of $\overline{\mathbb{Q}}$ there are continuous characters $\kappa_1, \kappa_2, \kappa_3$ of $D_{\bar{v}}$ with $\det(X - r(g)) = \prod_{i=1}^{3}(X - \kappa_i(g))$ for all $g \in I_{\bar{v}}$;*

4. *for every place $v \mid \ell$ of K and every place $\bar{v} \mid v$, r is triangular on $D_{\bar{v}}$ with some diagonal;*

5. *for every place $w \mid \ell$ of L such that r is generic at the place of K below w, $r|_{G_L}$ is generic at w;*

6. *$r|_{G_L}$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions.*

Then r is automorphic of conjugate self-dual type for ι and satisfies the witness conditions.

Proof. Step 1: the form is symmetric. Since $r|_{G_L}$ is irreducible, so is r. Transposing hypothesis 1 gives ${}^t r(csc)\,{}^t P\, r(s) = \varepsilon(s)^{-2}\,{}^t P$; replacing s by csc (note $c^2 = 1$ and $\varepsilon(csc) = \varepsilon(s)$, ε being a character of $G_{\mathbb{Q}}$) gives ${}^t r(s)\,{}^t P\, r(csc) = \varepsilon(s)^{-2}\,{}^t P$. So $P\, r(csc) P^{-1} = \varepsilon(s)^{-2}\,{}^t r(s)^{-1} = {}^t P\, r(csc)\,{}^t P^{-1}$ for all $s \in G_K$, and ${}^t P^{-1} P$ commutes with $r(G_K)$ (csc runs through G_K). By Schur's lemma ${}^t P^{-1} P = \lambda$ is a scalar, $P = \lambda\,{}^t P = \lambda^2 P$, so $\lambda = \pm 1$; $\lambda = -1$ would make P an invertible antisymmetric 3×3 matrix, impossible as $\det P = \det(-{}^t P) = -\det P$. Hence $\langle x, y\rangle = {}^t x P y$ is a non-degenerate symmetric bilinear form with $\langle r(\sigma)x, r(c\sigma c)y\rangle = \varepsilon(\sigma)^{-2}\langle x, y\rangle = \varepsilon(\sigma)^{1-3}\langle x, y\rangle$ for $\sigma \in G_K$.

Step 2: automorphy. By hypothesis 6, $r|_{G_L} \leftrightarrow_\iota \Pi$ for a regular algebraic conjugate self-dual cuspidal Π of $\mathrm{GL}_3(\mathbb{A}_L)$. By Theorem B.58, $r_{\ell,\iota}(\Pi)$ is semisimple and, at almost every finite place w of L, unramified with the same characteristic polynomials of Frobenius as $r|_{G_L}$; by the Chebotarev density theorem and continuity the characteristic polynomials of $r|_{G_L}$ and $r_{\ell,\iota}(\Pi)$ agree on all of G_L, and both being semisimple ($r|_{G_L}$ is irreducible) they are isomorphic (Brauer–Nesbitt). L/K is Galois with abelian group ($L/\mathbb{Q}$ is abelian), hence soluble; L is CM and K is not totally real; 3 is odd. With Step 1 and hypothesis 2, Theorem B.78 (with $F = K$, $F^+ = \mathbb{Q}$, $M = L$) gives $r \cong r_{\ell,\iota}(\pi)$ for a regular algebraic conjugate self-dual cuspidal π of $\mathrm{GL}_3(\mathbb{A}_K)$. By Theorem B.58, $r_{\ell,\iota}(\pi) \leftrightarrow_\iota \pi$, hence $r \leftrightarrow_\iota \pi$: r is automorphic of conjugate self-dual type for ι.

Step 3: (Wc1). Let $v \mid \ell$ be a place of K at which r is generic, and $\bar{v} \mid v$; let w be the place of L below $\bar{v}$. By hypothesis 5, $r|_{G_L}$ is generic at w, so by (Wc1) for $r|_{G_L}$ (hypothesis 6) it is potentially generic at w: it is triangular on $D_{\bar{v}} \cap G_L$ with a diagonal (κ'_i) such that $\kappa_i'^a \neq (\kappa'_j\varepsilon)^a$ for all i, j and $a \geq 1$. As r is generic at v, it is triangular on $D_{\bar{v}}$ with a diagonal (κ_i); restricted to $D_{\bar{v}} \cap G_L$ this is a diagonal of $r|_{G_L}$, so $\{\kappa_i|_{D_{\bar{v}}\cap G_L}\} = \{\kappa'_i\}$ as multisets (both are the one-dimensional composition factors). If $\kappa_i^a = (\kappa_j\varepsilon)^a$ on $D_{\bar{v}}$ for some i, j and $a \geq 1$, restriction would contradict the property of (κ'_i). So r is potentially generic at v.

Step 4: (Wc2). By hypothesis 2, $r|_{G_L}$ is irreducible; by Step 2, $r \leftrightarrow_\iota \pi$ with π regular algebraic conjugate self-dual cuspidal; by Step 2 again (from hypothesis 6), $r|_{G_L} \leftrightarrow_\iota \Pi$ with Π regular algebraic conjugate self-dual cuspidal. These, hypotheses 3, 4 and (Wc2) for $r|_{G_L}$ (hypothesis 6) are the hypotheses of Proposition 48.19, which gives (Wc2) for r. □

Proposition 48.19. *Let $K \subset L \subset \overline{\mathbb{Q}}$ with K imaginary quadratic and L a CM field abelian over $\mathbb{Q}$, ℓ a prime and $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism. Let $r\colon G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ be continuous and assume:*

1. *$r|_{G_L}$ is irreducible, $r \leftrightarrow_\iota \pi$ for a regular algebraic conjugate self-dual cuspidal automorphic representation π of $\mathrm{GL}_3(\mathbb{A}_K)$, and $r|_{G_L} \leftrightarrow_\iota \Pi$ for a regular algebraic conjugate self-dual cuspidal automorphic representation Π of $\mathrm{GL}_3(\mathbb{A}_L)$;*

2. *for every finite place $v \nmid \ell$ of K and every place $\bar{v} \mid v$ there are continuous characters $\kappa_1, \kappa_2, \kappa_3$ of $D_{\bar{v}}$ with $\det(X - r(g)) = \prod_i(X - \kappa_i(g))$ for all $g \in I_{\bar{v}}$;*

3. for every place $v \mid \ell$ of K and every place $\bar{v} \mid v$, r is triangular on $D_{\bar{v}}$ with some diagonal;

4. $r|_{G_L}$ satisfies (Wc2).

Then r satisfies (Wc2).

Proof. Step 1: r and $r|_{G_L}$ are the representations attached to π and Π. Since $r|_{G_L}$ is irreducible, so is r. As $\pi^c \cong \pi^\vee = \pi^\vee \otimes (\mathbf{1} \circ \mathrm{N}_{K/\mathbb{Q}} \circ \det)$, π is polarizable with the trivial character, and likewise Π; so Theorem B.62 gives continuous semisimple $r_{\ell,\iota}(\pi)$ and $r_{\ell,\iota}(\Pi)$ with (Bg1)–(Bg5). By (Bg1) and $r \leftrightarrow_\iota \pi$, at all but finitely many finite places u of K both r and $r_{\ell,\iota}(\pi)$ are unramified and their Frobenius elements have the same characteristic polynomial; by the Chebotarev density theorem these Frobenius elements are dense in G_K, and characteristic polynomials are continuous, so r and $r_{\ell,\iota}(\pi)$ have the same characteristic polynomials on G_K. Both are semisimple, so $r \cong r_{\ell,\iota}(\pi)$ (Brauer–Nesbitt). In the same way $r|_{G_L} \cong r_{\ell,\iota}(\Pi)$. Hence (Bg2)–(Bg5) hold for π with $r|_{D_{\bar{v}}}$ in place of $r_{\bar{v}}$, and for Π with $r|_{D_{\bar{v}} \cap G_L}$.

Step 2: every π' in (Wc2) is isomorphic to π. Let π' be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_K)$ with $r \leftrightarrow_\iota \pi'$ (r is semisimple). At all but finitely many finite places u, π_u and π'_u are unramified and ι carries the characteristic polynomial of $r(\mathrm{Frob}_u^{-1})$ both to $\prod_\alpha (X - q_u \alpha)$ over the Satake parameters of π_u and to the same product for π'_u; so the two multisets of Satake parameters agree. An unramified irreducible representation of $\mathrm{GL}_3(K_u)$ is the unique unramified irreducible subquotient of the normalised induction of the unramified characters given by its Satake parameters, and permuting the characters does not change the Jordan–Hölder factors; hence $\pi_u \cong \pi'_u$ for all but finitely many u, and $\pi' \cong \pi$ by Theorem B.60, (Js1). So $\pi'_v \cong \pi_v$ for every v, and it suffices to find one real number τ such that at every finite place v of K with $v \nmid \ell$, or with $v \mid \ell$ and r potentially generic at v, π_v has a cuspidal support consisting of characters and every character ξ in a cuspidal support of π_v satisfies $q_v^{\tau-1/2} < |\xi(\varpi_v)| < q_v^{\tau+1/2}$.

Step 3: the number τ. $r|_{G_L}$ is semisimple and $r|_{G_L} \leftrightarrow_\iota \Pi$ with Π regular algebraic cuspidal, so by hypothesis 4 there is τ_Π such that at every finite place w of L with $w \nmid \ell$, or with $w \mid \ell$ and $r|_{G_L}$ potentially generic at w, every character ξ' in a cuspidal support of Π_w satisfies $q_w^{\tau_\Pi - 1/2} < |\xi'(\varpi_w)| < q_w^{\tau_\Pi + 1/2}$. Put $\tau = \tau_\Pi$. Fix now a finite place v of K with $v \nmid \ell$, or $v \mid \ell$ and r potentially generic at v; fix $\bar{v} \mid v$, let w be the place of L below $\bar{v}$ and f the residue degree of L_w/K_v, so $q_w = q_v^f$. Let $D = D_{\bar{v}}$ and $H = D \cap G_L$, the decomposition group of G_L at $\bar{v}$; H is the kernel of $D \to \mathrm{Gal}(L/K)$, normal of finite index in D. Let $W_D \subseteq D$ and $W_H = W_D \cap H$ be the Weil groups, $I \subseteq W_D$ the inertia group, $(V, \rho, N) = \mathrm{WD}(r|_D)$ and $(V, \rho_H, N_H) = \mathrm{WD}(r|_H)$.

Step 4: π_v has a cuspidal support of characters, and its Frobenius characters. We show that the semisimplification of $\rho|_I$ is a sum of restrictions of three characters of W_D. If $v \nmid \ell$, take $\kappa_1, \kappa_2, \kappa_3$ as in hypothesis 2. For $\sigma \in I$, $n(\sigma) = 0$, so $\rho(\sigma)$ commutes with N by (38.5), and by (Wd1) $r(\sigma) = \rho(\sigma)\exp(t_\ell(\sigma)N)$ is $\rho(\sigma)$ times a commuting unipotent matrix; hence $\det(X - \rho(\sigma)) = \prod_i (X - \kappa_i(\sigma))$. ρ is trivial on an open subgroup of I, hence on an open normal subgroup J of I, and there the characteristic polynomial is $(X-1)^3$, so the κ_i are trivial on J. So $\rho|_I$ and $\bigoplus_i \kappa_i|_I$ are representations of the finite group I/J over the algebraically closed field $\overline{\mathbb{Q}}_\ell$ of characteristic 0 with the same characteristic polynomials, hence the same traces; so they are isomorphic. If $v \mid \ell$, r is triangular on D with a diagonal $(\kappa_1, \kappa_2, \kappa_3)$ (hypothesis 3). By (Bg2) for π and Step 1, $r|_D$ is de Rham, hence so are its subquotients, in particular the κ_i; de Rham representations are potentially semistable, and Fontaine's functor D_{pst} is exact on potentially semistable representations, so ρ has the semisimplification $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$, where λ_i is the character of W_D of the one-dimensional Weil–Deligne representation $\mathrm{WD}(\kappa_i)$. In both cases (Bg3) for π, with $U = \{1\}$, shows that π_v is an irreducible subquotient of the normalised

induction of $\xi_1 \otimes \xi_2 \otimes \xi_3$ for characters ξ_i of $K_v^\times$. The supercuspidal support of an irreducible representation is unique up to order (Bernstein–Zelevinsky), so every cuspidal support of π_v is (ξ_1, ξ_2, ξ_3) up to order and consists of characters. By (Bg5) for π, the semisimplification of ρ is $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ with $\iota\lambda_i(\mathrm{Art}_{K_v}(x)^{-1}) = \xi_i(x)|x|_v^{-1}$ for $x \in K_v^\times$ (this fixes the λ_i also when $v \mid \ell$).

Step 5: restriction to H. The semisimplification of ρ_H is $\bigoplus_i \lambda_i|_{W_H}$. If $v \nmid \ell$: for any ℓ-adic representation s of W_D with $\mathrm{WD}(s) = (V, \rho_s, N_s)$, the subspaces $V_k = \ker N_s^k$ are stable under ρ_s (by (38.5)) and under s (by (Wd1), $s(g) = \rho_s(g)\exp(t_\ell(\sigma)N_s)$ for $g = \Phi^m\sigma$), and on V_k/V_{k-1}, killed by N_s, the two actions agree; so ρ_s and s have the same semisimplification. Applied to $r|_{W_D}$ and to $r|_{W_H}$ (over L_w), the semisimplification of ρ_H is that of $r|_{W_H}$, i.e. the restriction to W_H of the semisimplification of $r|_{W_D}$, which is that of ρ; as the $\lambda_i|_{W_H}$ are characters, this is $\bigoplus_i \lambda_i|_{W_H}$. If $v \mid \ell$: $r|_H$ is triangular with diagonal $(\kappa_i|_H)$, and Fontaine's construction commutes with restriction to the open subgroup H (D_{pst} is formed over all finite extensions), so $\mathrm{WD}(\kappa_i|_H)$ has the character $\lambda_i|_{W_H}$, and exactness gives the claim.

Step 6: the cuspidal support of Π_w. By Step 5, (Bg3) for Π (with $U = \{1\}$ and $\Psi_i = \lambda_i|_{W_H}$) shows that Π_w is an irreducible subquotient of the normalised induction of $\xi_1' \otimes \xi_2' \otimes \xi_3'$ for characters ξ_k' of $L_w^\times$, and (Bg5) for Π gives that the semisimplification of ρ_H is $\bigoplus_k \lambda_k'$ with $\iota\lambda_k'(\mathrm{Art}_{L_w}(y)^{-1}) = \xi_k'(y)|y|_w^{-1}$ for $y \in L_w^\times$. By Step 5 and the Jordan–Hölder theorem, $\{\lambda_k'\} = \{\lambda_i|_{W_H}\}$ as multisets. By local class field theory, the map $W_H^{\mathrm{ab}} \to W_D^{\mathrm{ab}}$ induced by the inclusion corresponds under Art_{L_w}, Art_{K_v} to the norm $\mathrm{N} = \mathrm{N}_{L_w/K_v}$; with $|\mathrm{N}y|_v = |y|_w$,

$$\iota\lambda_i|_{W_H}(\mathrm{Art}_{L_w}(y)^{-1}) = \iota\lambda_i(\mathrm{Art}_{K_v}(\mathrm{N}y)^{-1}) = \xi_i(\mathrm{N}y)\,|y|_w^{-1}.$$

So for each i there is k with $\xi_k' = \xi_i \circ \mathrm{N}$.

Step 7: potential genericity over L. Let $v \mid \ell$ and r potentially generic at v; we show $r|_{G_L}$ is potentially generic at w. Let $\bar{v}'$ be a place above w, hence above v, $D' = D_{\bar{v}'}$, $H' = D' \cap G_L$ (normal of finite index m in D'). r is triangular on D' with a diagonal (κ_i) with $\kappa_i^a \neq (\kappa_j\varepsilon)^a$ for all i, j and $a \geq 1$; then $r|_{G_L}$ is triangular on H' with diagonal $(\kappa_i|_{H'})$. If $(\kappa_i|_{H'})^a = (\kappa_j\varepsilon|_{H'})^a$ for some $a \geq 1$, the character $(\kappa_i/\kappa_j\varepsilon)^a$ of D' is trivial on H', so its m-th power is trivial and $\kappa_i^{am} = (\kappa_j\varepsilon)^{am}$, a contradiction.

Step 8: the window. If $v \nmid \ell$ then $w \nmid \ell$; if $v \mid \ell$, $r|_{G_L}$ is potentially generic at w by Step 7. As (ξ_1', ξ_2', ξ_3') is a cuspidal support of Π_w, Step 3 and Step 6 give, for every i, $q_w^{\tau-1/2} < |\xi_i(\mathrm{N}\varpi_w)| < q_w^{\tau+1/2}$. Now $\mathrm{N}\varpi_w = u\varpi_v^f$ with $u \in \mathcal{O}_{K_v}^\times$, and $|\xi_i(u)| = 1$ since $\xi_i(\mathcal{O}_{K_v}^\times)$ is a compact subgroup of $\mathbb{C}^\times$; so $|\xi_i(\mathrm{N}\varpi_w)| = |\xi_i(\varpi_v)|^f$, and with $q_w = q_v^f$, taking f-th roots, $q_v^{\tau-1/2} < |\xi_i(\varpi_v)| < q_v^{\tau+1/2}$. With Step 4 (the characters in any cuspidal support of π_v are ξ_1, ξ_2, ξ_3) and Step 2, r satisfies (Wc2) with $\tau_{\pi'} = \tau_\Pi$ for every π'. □

49 Residually Eisenstein $R = T$ over an auxiliary field

This section proves Theorem 49.3, an $R = T$ theorem for residually Eisenstein polarised ordinary lifts over an auxiliary field L, which Theorem 48.3 uses. It has three conclusions: in characteristic ℓ the polarised ordinary lifts of rank three have the residual characteristic polynomials (Pp1); irreducible lifts of rank three are automorphic after the twist by χ_{cyc}^{-1} (Pp2); and irreducible lifts of rank two are automorphic after an algebraic twist (Pp3). Rank two enters twice: through (Pp3) the corner of a lift of type II is modular (Proposition 48.14), and absorption in rank two bounds the endoscopic primes in rank three (Proposition 49.33). The argument is in the line of Skinner and Wiles [236] and of Thorne [256], and has three inputs: seeds, automorphic lifts through the residual representation (Subsection 49.1); the Hecke realisation Theorem 50.3 of Section 50, an ideal of the deformation ring that is finite over the weights, propagates through

nice primes and, in rank three, along thick links, has a seed component when a seed exists, and cuts out automorphic lifts; and the propagation theorem 49.12.

The propagation theorem is the core. An ideal with these properties absorbs every prime of irreducible type, so every irreducible lift is automorphic. In rank three the theorem takes absorption in rank two as a hypothesis, and it also gives an off-triangular bound, from which (Pp1) follows (Lemma 49.37). The proof compares dimensions: the presentations of the local charts of the entry algebra, variables against relations, bound the dimension of their components from below (Proposition 49.20), while the loci where propagation could fail, the non-nice loci, have smaller dimension (Proposition 49.29); since the core locus is connected (Lemma 49.15), modularity passes from the seed component to all components along chains (Lemma 49.34). Subsection 49.2 sets up the entry algebra and states the theorem, Subsection 49.3 constructs the local charts at the maximal core primes, and Subsection 49.4 treats the endoscopic core primes and the local propagation.

49.1 Seeds over an auxiliary field

This subsection defines the polarised ordinary deformation rings over an auxiliary field, with their entry and weight-zero algebras, prime types, local charts and modular ideals (Definition 49.2), defines admissible twists and seeds (Definition 49.1), states Theorem 49.3, and constructs the seeds. A seed is an irreducible polarised representation of G_L, residually congruent to the Eisenstein representation, ordinary at ℓ, and automorphic after a twist (conditions (Sp1)–(Sp6)); the propagation argument starts from it. Seeds of rank three and of rank two exist (Proposition 49.4, Proposition 49.10). They come from newforms congruent to Eisenstein series with an exact nebentypus, obtained through an auxiliary character of order ℓ (Proposition 49.5; Lemma 49.6, Lemma 49.7). Such congruences are the starting point of Ribet's construction [208], and here they are used in that constructive direction. The Galois representation of such a newform has the residual shape of the Eisenstein representation and stays irreducible over abelian fields (Lemma 49.8), and its adjoint and its algebraic twists are automorphic of conjugate self-dual type over the abelian CM field (Lemma 49.9). Proposition 49.5 and Lemma 49.8 are used again for the seeds of the symplectic case.

Definition 49.1. Let L be a CM field with $K \subseteq L$ and $G_L \subseteq G_{\mathbb{Q}}$, let $n \in \{2,3\}$, and let $c \in G_{\mathbb{Q}}$ be a complex conjugation. Characters of $G_{\mathbb{Q}}$ are restricted to G_L without change of notation; for $s \in G_L$ put $s^c = csc \in G_L$. Congruence of elements of $\overline{\mathbb{Z}}_\ell$ means congruence modulo the maximal ideal of $\overline{\mathbb{Z}}_\ell$.

Admissible twists. A continuous character $\kappa\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$ is an **admissible twist of rank** n (for ι, c and M) if it is algebraic for ι and

(Ka1) $\kappa(s)\kappa(s^c) = \chi_{\mathrm{cyc}}(s)^{1-n}$ for all $s \in G_L$;

(Ka2) κ is trivial on $I_{\bar v} \cap G_L$ for every place $\bar v$ of $\overline{\mathbb{Q}}$ above a prime not dividing M;

(Ka3) there is an integer $m' \geq 1$ with $\ell \nmid m'$ such that $\kappa^{m'}$ is trivial on $I_{\bar v} \cap G_L$ for every place $\bar v \nmid \ell$.

Seeds. Let $j \geq 0$ and let $\chi'\colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell^\times$ be a character of finite order. For $1 \leq a \leq n$ put $\varphi_a = (\chi_{\mathrm{cyc}}^j \chi')^{w_{n,a}}$ and $\tau_a = {\chi'}^{w_{n,a}}$, characters of G_L. Let $m \geq 1$ and let $\kappa\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$ be a continuous character. A **seed of rank** n **and weight** m **for** (j,χ') **with twist** κ is a continuous representation $r\colon G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that

(Sp1) r is irreducible;

(Sp2) there is $P \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ with ${}^tP = P$ and ${}^t r(s)\, P\, r(s^c) = P$ for all $s \in G_L$;

(Sp3) for every $s \in G_L$, the characteristic polynomial of $r(s)$ has coefficients in $\overline{\mathbb{Z}}_\ell$ congruent to those of $\prod_{a=1}^{n}(X - \varphi_a(s))$;

(Sp4) for every place $\bar v \nmid \ell$, the restriction of r to $I_{\bar v} \cap G_L$ is conjugate to $s \mapsto \mathrm{diag}(\tau_1(s), \dots, \tau_n(s))$;

(Sp5) for every place $\bar v \mid \ell$, r is triangular on $D_{\bar v} \cap G_L$ with a diagonal $(\kappa_1, \dots, \kappa_n)$ such that $\kappa_a(s) = \chi_{\mathrm{cyc}}(s)^{m\, w_{n,a}}$ for $s \in I_{\bar v} \cap G_L$ and $\kappa_a(s) \equiv \varphi_a(s)$ for $s \in D_{\bar v} \cap G_L$;

(Sp6) $r \otimes \kappa$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions.

Definition 49.2. Let L be a CM field satisfying (Af1) and (Af2) of Definition 48.2 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, let $n \in \{2, 3\}$ and put $d_L = [L^+ : \mathbb{Q}] = \sum_{v \mid \ell} d_v$.

Lifts to complete local rings. For $R \in \mathrm{CNL}_{\mathcal{O}}$, conditions (Po1)–(Po5) make sense with the test ring A replaced by R; a homomorphism $\rho\colon \Gamma_L^0 \to \mathrm{GL}_n(R)$ with homomorphisms $\psi_{v,a}\colon D_v \to R^\times$ satisfying them is a **polarised ordinary lift of rank n over L to R.** A **universal polarised ordinary lift** of rank n over L is a lift $(\rho^{\mathrm{u}}, (\psi^{\mathrm{u}}_{v,a}))$ to some $R_n^{\mathrm{po}} \in \mathrm{CNL}_{\mathcal{O}}$ such that for every $R' \in \mathrm{CNL}_{\mathcal{O}}$ and every polarised ordinary lift $(\rho', (\psi'_{v,a}))$ of rank n over L to R' there is exactly one morphism $f\colon R_n^{\mathrm{po}} \to R'$ of $\mathrm{CNL}_{\mathcal{O}}$ with $f \circ \rho^{\mathrm{u}} = \rho'$ and $f \circ \psi^{\mathrm{u}}_{v,a} = \psi'_{v,a}$ for all v, a.

Weights, the entry algebra and the weight-zero algebra. Fix a universal lift as above and write $\mathfrak{m} = \mathfrak{m}_{R_n^{\mathrm{po}}}$. The **weight** of the matrix position (a, b) is $w_{n,a} - w_{n,b}$. A **weight-zero monomial** is a product $\prod_{k=1}^{s} \rho^{\mathrm{u}}(\gamma_k)_{a_k b_k}$ with $s \geq 0$, $\gamma_k \in \Gamma_L^0$ and $\sum_k (w_{n,a_k} - w_{n,b_k}) = 0$. The **weight-zero algebra** $W_n^{\mathrm{po}} \subseteq R_n^{\mathrm{po}}$ is the set of y such that for every $k \geq 1$ some element of the $\mathcal{O}$-subalgebra generated by the weight-zero monomials and the values $\psi^{\mathrm{u}}_{v,a}(g)$ ($v \mid \ell$, $1 \leq a \leq n$, $g \in D_v$) is congruent to y modulo $\mathfrak{m}^k$; it is an $\mathcal{O}$-subalgebra, and $\mathfrak{m}_W = \mathfrak{m} \cap W_n^{\mathrm{po}}$. The **entry algebra** E_n^{po} is the W_n^{po}-subalgebra of R_n^{po} generated by all entries $\rho^{\mathrm{u}}(\gamma)_{ab}$; thus $W_n^{\mathrm{po}} \subseteq E_n^{\mathrm{po}} \subseteq R_n^{\mathrm{po}}$. The **inertial weight ideal** $\mathfrak{a}_n \subseteq W_n^{\mathrm{po}}$ is generated by $\mathfrak{m}_{\mathcal{O}}$ and the elements $\psi^{\mathrm{u}}_{v,a}(g) - 1$ with $v \mid \ell$, $1 \leq a \leq n$, $g \in I_v$ and $\bar\Psi(g) = 1$. An ideal $I \subseteq W_n^{\mathrm{po}}$ is **finite over the weights** if $\mathfrak{m}_W \subseteq \sqrt{I + \mathfrak{a}_n}$.

Prime types. Let C be a commutative ring and $\iota\colon E_n^{\mathrm{po}} \to C$ a ring homomorphism ($\iota = \mathrm{id}$ allowed). Let I_C^-, I_C^+, $I_C^{\mathrm{odd}} \subseteq C$ be the ideals generated by the elements $\iota(\rho^{\mathrm{u}}(\gamma)_{ab})$, $\gamma \in \Gamma_L^0$, with $a > b$, with $a < b$, and with $w_{n,a} - w_{n,b}$ odd, respectively. A prime $\mathfrak{p}$ of C is **off-triangular** if $I_C^- \not\subseteq \mathfrak{p}$ and $I_C^+ \not\subseteq \mathfrak{p}$; **of irreducible type** if it is off-triangular and, when $n = 3$, $I_C^{\mathrm{odd}} \not\subseteq \mathfrak{p}$; **of endoscopic type** if $n = 3$, it is off-triangular and $I_C^{\mathrm{odd}} \subseteq \mathfrak{p}$.

Local charts. Let $\mathfrak{x}$ be a prime of E_n^{po}. A **local chart at $\mathfrak{x}$** is a triple $(\mathcal{O}', R_{\mathfrak{x}}, \iota)$ of a coefficient ring $\mathcal{O}' \supseteq \mathcal{O}$, finite over $\mathcal{O}$ with $\mathfrak{m}_{\mathcal{O}}\mathcal{O}' = \mathfrak{m}_{\mathcal{O}'}$, a ring $R_{\mathfrak{x}} \in \mathrm{CNL}_{\mathcal{O}'}$ and an $\mathcal{O}$-algebra homomorphism $\iota\colon E_n^{\mathrm{po}} \to R_{\mathfrak{x}}$ with $\iota^{-1}(\mathfrak{m}_{R_{\mathfrak{x}}}) = \mathfrak{x}$, such that for every $C \in \mathrm{CNL}_{\mathcal{O}'}$ and every $\mathcal{O}$-algebra homomorphism $f\colon E_n^{\mathrm{po}} \to C$ whose composite with $C \to \mathbb{F}_{\mathcal{O}'}$ equals that of ι, there is exactly one $\mathcal{O}'$-algebra homomorphism $h\colon R_{\mathfrak{x}} \to C$ with $h \circ \iota = f$. Its **chart lift** is $\rho_{\mathfrak{x}} = \iota \circ \rho^{\mathrm{u}}$, $\psi_{\mathfrak{x},v,a} = \iota \circ \psi^{\mathrm{u}}_{v,a}$; prime types in $R_{\mathfrak{x}}$ refer to $C = R_{\mathfrak{x}}$ with this ι.

Nice and thick primes. A prime $\mathfrak{p}$ of $R_{\mathfrak{x}}$ is **nice** if $\dim R_{\mathfrak{x}}/\mathfrak{p} = 1$, $\ell \in \mathfrak{p}$, $\mathfrak{p}$ is off-triangular, and for every $v \mid \ell$ and $a \neq b$ there is $g \in I_v$ with $\chi_{\mathrm{cyc}}(g) = 1$ and $\psi_{\mathfrak{x},v,a}(g) - \psi_{\mathfrak{x},v,b}(g) \notin \mathfrak{p}$. Let $n = 3$. A prime $\mathfrak{e}$ of $R_{\mathfrak{x}}$ is **thick** if it is of endoscopic type and there is a homomorphism of monoids $r\colon \Gamma_L^0 \to \mathrm{M}_3(R_{\mathfrak{x}}/\mathfrak{e})$, with entries continuous for the topology defined by the image of $\mathfrak{m}_{R_{\mathfrak{x}}}$, such that, with bars denoting reduction modulo $\mathfrak{e}$:

(Tk1) $r(\gamma)_{ab} = \overline{\rho_{\mathfrak{x}}(\gamma)_{ab}}$ for all γ whenever $w_{3,a} - w_{3,b}$ is even, and $r(\gamma)_{21} = r(\gamma)_{23} = 0$ for all γ;

(Tk2) for every $v \mid \ell$ and $g_1, g_2, g_3 \in D_v$,

$$\prod_{a=1}^{3} \left(r(\pi_L(g_a)) - \overline{\psi_{\mathfrak{x},v,a}(g_a)}\right) = 0,$$
$$\prod_{a=1}^{3} \left(r(\pi_L(c g_a^{-1} c)) - \overline{\psi_{\mathfrak{x},v,4-a}(g_a)}\right) = 0,$$

the products taken with $a = 1$ on the left;

(Tk3) for every field F' and injective homomorphism $j\colon R_{\mathfrak{x}}/\mathfrak{e} \to F'$ there is no $y \in F'^3$ with $y_2 = 1$ and $j(r(\gamma))\, y = j(r(\gamma)_{22})\, y$ for all $\gamma \in \Gamma_L^0$.

Two nice primes $\mathfrak{p}, \mathfrak{q}$ of $R_{\mathfrak{x}}$ of endoscopic type are **thickly linked** if some thick prime $\mathfrak{e}$ satisfies $\mathfrak{e} \subseteq \mathfrak{p} \cap \mathfrak{q}$; they are **thickly connected** if there are nice primes $\mathfrak{p} = \mathfrak{p}_0, \dots, \mathfrak{p}_k = \mathfrak{q}$ of endoscopic type, $k \geq 0$, with $\mathfrak{p}_{i-1}, \mathfrak{p}_i$ thickly linked for $1 \leq i \leq k$.

Modular ideals. Let $\mathcal{M} \subseteq W_n^{\mathrm{po}}$ be an ideal. A prime $\mathfrak{p}$ of E_n^{po} is **$\mathcal{M}$-modular** if $\mathcal{M} \subseteq \mathfrak{p} \cap W_n^{\mathrm{po}}$; a prime $\mathfrak{P}$ of $R_{\mathfrak{x}}$, for a local chart $(\mathcal{O}', R_{\mathfrak{x}}, \iota)$, is $\mathcal{M}$-modular if $\iota^{-1}(\mathfrak{P})$ is. The ideal $\mathcal{M}$

- **propagates through nice primes** if for every prime $\mathfrak{x}$ of E_n^{po}, every local chart at $\mathfrak{x}$, every $\mathcal{M}$-modular nice prime $\mathfrak{p}$ of $R_{\mathfrak{x}}$ of irreducible type and every prime $\mathfrak{P} \subseteq \mathfrak{p}$ of $R_{\mathfrak{x}}$, $\mathfrak{P}$ is $\mathcal{M}$-modular;
- ($n = 3$) **propagates along thick links** if for every prime $\mathfrak{x}$ of E_3^{po}, every local chart at $\mathfrak{x}$ and every pair of thickly connected nice primes $\mathfrak{p}, \mathfrak{q}$ of $R_{\mathfrak{x}}$ of endoscopic type such that some prime $\mathfrak{P} \subseteq \mathfrak{p}$ of irreducible type is $\mathcal{M}$-modular, every prime $\mathfrak{P}' \subseteq \mathfrak{q}$ of irreducible type is $\mathcal{M}$-modular;
- **has a seed component** if there are a prime ξ of E_n^{po} of irreducible type and a prime $\mathfrak{z}$ of W_n^{po} with $\mathcal{M} \subseteq \mathfrak{z} \subseteq \xi \cap W_n^{\mathrm{po}}$ and $\dim W_n^{\mathrm{po}}/\mathfrak{z} = 1 + n\, d_L$;
- **absorbs the primes of irreducible type** if every prime of E_n^{po} of irreducible type is $\mathcal{M}$-modular.

For $n = 3$, an **off-triangular bound** is an ideal $I \subseteq W_3^{\mathrm{po}}$, finite over the weights, with $I \subseteq \mathfrak{p} \cap W_3^{\mathrm{po}}$ for every off-triangular prime $\mathfrak{p}$ of E_3^{po}.

Theorem 49.3. *Let $\mathcal{O}$ be a coefficient ring, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $c \in G_{\mathbb{Q}}$ a complex conjugation, and assume (a) or (b), with K, M and $\Psi = \Psi_{\chi,e}$:*

(a) (K, ℓ, e, N, χ) is a unitary Eisenstein datum with χ taking values in $\mathcal{O}^\times$, and $M = \ell N |d_K|$;

(b) (ℓ, e, N, χ, o, m) is a dyadic Eisenstein datum with χ taking values in $\mathcal{O}^\times$, $K = \mathbb{Q}(i)$ and $M = \ell N$.

Let $g_\star$ be a marking and L an auxiliary field of some level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$. Then:

(Pp1) *L has residual characteristic polynomials in rank 3;*

(Pp2) *L has automorphic irreducible lifts in rank 3 with twist $\chi_{\mathrm{cyc}}^{-1}|_{G_L}$ for ι;*

(Pp3) *there is an algebraic character $\kappa\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$ such that L has automorphic irreducible lifts in rank 2 with twist κ for ι.*

Proof. Fix data $(N', w, j, \chi', \theta)$ as in (Ag1) for L.

Step 0: standing facts. ℓ is odd by (Ue1), resp. (Dy1); K is imaginary quadratic; and $2\ell \mid M$, since N is even by (Ue2), resp. (Dy2), and $\ell N \mid M$. The character $\Psi = \chi\chi_{\mathrm{cyc}}^{e}$ is unramified at every prime not dividing ℓN. Let $\bar{v}$ be a place above a prime $p \nmid M$ and $s \in I_{\bar{v}} \cap G_L$; then $\Psi(s) = 1 = \chi_{\mathrm{cyc}}(s)$, so $\chi'(s) \equiv \Psi(s)\chi_{\mathrm{cyc}}(s)^{-j} = 1$ by (Ag1); as $\chi'(s)$ is a root of unity of order prime to ℓ (by (Ag1), $\ell \nmid \operatorname{ord}\chi'$) and reduction is injective on such roots of unity, $\chi'(s) = 1$. Finally $\kappa_3 := \chi_{\mathrm{cyc}}^{-1}|_{G_L}$ is an admissible twist of rank 3: it is algebraic (it corresponds to an integral power of the norm character), $\kappa_3(s)\kappa_3(csc) = \chi_{\mathrm{cyc}}(s)^{-2}$ because $\chi_{\mathrm{cyc}}(csc) = \chi_{\mathrm{cyc}}(s)$, which is (Ka1) for $n = 3$, and (Ka2), (Ka3) hold with $m' = 1$ because χ_{cyc} is unramified away from ℓ and $\ell \mid M$.

Step 1: seeds. The hypotheses of Proposition 49.4 and Proposition 49.10 hold by Step 0, (Af1), (Af2), (Ag1) and (Ag2); every prime dividing d_K divides M, as $M = \ell N|d_K|$ in case (a), and in case (b) $d_K = -4$ and N is even. They give $m_3 \geq 2$ and a seed of rank 3 and weight m_3 for (j, χ') with twist κ_3, and an admissible twist κ_2 of rank 2, $m_2 \geq 2$ and a seed of rank 2 and weight m_2 for (j, χ') with twist κ_2. The character κ_2 is algebraic.

Step 2: coefficients. A continuous character of the compact group G_L with values in $\overline{\mathbb{Q}}_\ell^\times$ takes values in the units of the integers of a finite extension of $\mathbb{Q}_\ell$ (Baire category: $\overline{\mathbb{Q}}_\ell^\times$ is the countable union of the closed subgroups $E^\times$, $E/\mathbb{Q}_\ell$ finite, so one of them has open, hence finite-index, preimage). Let $\mathcal{O}' \supseteq \mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$ containing $\mathcal{O}$, the values of κ_2 and $\tau(L)$ for every embedding $\tau\colon L \to \overline{\mathbb{Q}}_\ell$; the values of κ_3 lie in $\mathbb{Z}_\ell^\times$.

Step 3: one run gives absorption and an off-triangular bound. Let $p_2, p_3 \geq 1$ be integers. Apply Theorem 50.3 with $(n, \kappa_n, \mathcal{O}')$ and $p_0 = p_n$, for $n = 2, 3$: we obtain primes $p_n' \nmid p_n$, places v_n of L^+ above p_n', and by (Hk1) L is an auxiliary field of level C_0 for $(K, \mathcal{O}', \Psi, c, g_\star, M)$ with universal polarised ordinary lifts of ranks 2 and 3, and by (Hk2) ideals $\mathcal{M}_n \subseteq W_n^{\mathrm{po}}$ finite over the weights that propagate through nice primes, $\mathcal{M}_3$ also along thick links. By Step 1 and (Hk3) both have a seed component. By Theorem 49.12, item 1, $\mathcal{M}_2$ absorbs the primes of irreducible type; by item 2 so does $\mathcal{M}_3$, and W_3^{po} contains an off-triangular bound.

Step 4: (Pp1). By Step 3 (with $p_2 = p_3 = 1$) and Lemma 49.37, L has residual characteristic polynomials in rank 3 for the coefficient ring $\mathcal{O}'$. Let A be a test ring of characteristic ℓ over $\mathcal{O}$ and $(\rho, (\psi_{v,a}))$ a lift as in the definition for $\mathcal{O}$. By Cohen's structure theorem $A \cong k[[T]]$ with $k = A/\mathfrak{m}_A$, and the image of $\mathcal{O}$, which is $\mathbb{F}_\mathcal{O}$ since $\mathfrak{m}_\mathcal{O}$ maps to 0, lies in k (a finite subfield of $k[[T]]$ lies in k, by Hensel's lemma for $X^q - X$). Choose a finite field k' with embeddings $k \to k'$ and $\mathbb{F}_{\mathcal{O}'} \to k'$ agreeing on $\mathbb{F}_\mathcal{O}$, and put $A' = k'[[T]] \supseteq A$, an $\mathcal{O}'$-algebra through $\mathcal{O}' \to \mathbb{F}_{\mathcal{O}'} \to k'$, which restricts on $\mathcal{O}$ to the given structure. Then A' is a test ring of characteristic ℓ over $\mathcal{O}'$, and $(\rho, (\psi_{v,a}))$ composed with $A \subseteq A'$ is a polarised ordinary lift for $\mathcal{O}'$ with the same condition on the $\psi_{v,a}$, conditions (Po1)–(Po5) being identities and congruences with residues in $\mathbb{F}_\mathcal{O} \subseteq \mathbb{F}_{\mathcal{O}'}$. Its characteristic polynomials are residual, and $A \to A'$ is injective, so those of ρ_L are residual.

Step 5: (Pp2) and (Pp3). Let $n \in \{2, 3\}$, A an integral test ring over $\mathcal{O}$ and $(\rho, (\psi_{v,a}))$ a polarised ordinary lift of rank n over L to A with $\psi_{v,a}(g) = \Psi(g)^{w_{n,a}}$ for $g \in I_v$ and ρ_L irreducible over $\overline{\mathbb{Q}}_\ell$. Let $\mathcal{O}_2 \subset \overline{\mathbb{Q}}_\ell$ be the ring of integers of the compositum of the fraction fields of A and $\mathcal{O}'$; it is an integral test ring over $\mathcal{O}'$ containing A, with $\mathfrak{m}_A \subseteq \mathfrak{m}_{\mathcal{O}_2}$, so the lift composed with $A \subseteq \mathcal{O}_2$ is a polarised ordinary lift for $\mathcal{O}'$ (local constancy modulo $\mathfrak{m}_{\mathcal{O}_2}^k$ follows from that modulo $\mathfrak{m}_A^k$). Apply Step 3 with $p_2 = p_3 = 1$, and let $\mathcal{M} = \mathcal{M}_n$ be the ideal it provides. By Lemma 49.36 there is a local $\mathcal{O}'$-algebra homomorphism $f\colon R_n^{\mathrm{po}} \to \mathcal{O}_2$ with $f \circ \rho^{\mathrm{u}} = \rho$ and $f \circ \psi_{v,a}^{\mathrm{u}} = \psi_{v,a}$; let $\mathfrak{p}$ be its kernel on E_n^{po}, a prime. $\mathfrak{p}$ *is of irreducible type:* if $I_E^- \subseteq \mathfrak{p}$, all entries $\rho(\gamma)_{ab}$ with $a > b$ vanish and the line $\overline{\mathbb{Q}}_\ell e_1$ is stable under $\rho(\Gamma_L^0) = \rho_L(G_L)$; if $I_E^+ \subseteq \mathfrak{p}$, the line $\overline{\mathbb{Q}}_\ell e_n$ is stable; if $n = 3$ and $I_E^{\mathrm{odd}} \subseteq \mathfrak{p}$, the entries in positions $(1,2), (2,1), (2,3), (3,2)$ vanish and $\overline{\mathbb{Q}}_\ell e_2$ is stable; each contradicts the irreducibility of ρ_L. By Step 3, $\mathcal{M} \subseteq \mathfrak{p} \cap W_n^{\mathrm{po}}$, that is,

$f(\mathcal{M}) = 0$. By (Hk4), $\rho_L \otimes \kappa_n$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions. For $n = 3$ this is (Pp2) with twist $\chi_{\mathrm{cyc}}^{-1}|_{G_L}$; for $n = 2$ it is (Pp3) with the algebraic character $\kappa = \kappa_2$. $\square$

Proposition 49.4. *Let K be an imaginary quadratic field, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and let L be a CM field satisfying (Af1), (Af2) and (Ag1) for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, where $2\ell \mid M$. Fix data $(N', w, j, \chi', \theta)$ as in (Ag1). Then there are an integer $m \geq 2$ and a seed of rank 3 and weight m for (j, χ') with twist $\chi_{\mathrm{cyc}}^{-1}|_{G_L}$.*

Proof. Step 1: the Eisenstein data. By (Ag1), χ' and η_K are Dirichlet characters modulo N' with $\chi'(w) = \eta_K(w) = 1$; let ψ be the primitive character inducing $\chi'\eta_K$, of conductor $N_0 \mid N'$. It is nontrivial ($\chi' \neq \eta_K$), of order prime to ℓ ($\ell \nmid \operatorname{ord}\chi'$, ℓ odd), $\psi(w) = 1$, $\ell \nmid N_0$, $w \nmid N_0$. The character θ is a Dirichlet character modulo w of order ℓ ($\theta \neq 1$, $\theta^\ell = 1$, trivial on $G_{\mathbb{Q}(\zeta_w)}$), so $\ell \mid w - 1$; ℓ is odd as $2\ell \mid M$. Choose $k \geq 3$ with $k \equiv j + 1 \pmod{\ell - 1}$ and put $\varepsilon = \psi\theta$, primitive modulo $N = N_0 w$. Then $\varepsilon(-1) = \chi'(c)\eta_K(c)\theta(c) = (-1)^j \cdot (-1) \cdot 1 = (-1)^k$, as θ has odd order and $\ell - 1$ is even. By Proposition 49.5 there is a normalised newform g of weight k, level N ($\ell \nmid N$) and character ε with $\iota^{-1}(a_p) \equiv 1 + \iota^{-1}(\varepsilon(p))p^{k-1}$ for all primes p. Let $\rho = \rho^{\mathrm{ar}}$ be its Galois representation and $\beta = \varepsilon\chi_{\mathrm{cyc}}^{k-1} = \det\rho$; put $m = k - 1 \geq 2$.

Step 2: residual comparison. Since $\bar\theta = 1$ and $\bar\chi_{\mathrm{cyc}}^{k-1} = \bar\chi_{\mathrm{cyc}}^{j}$ (its order divides $\ell - 1$), $\bar\beta = \eta_K\,\overline{\chi_{\mathrm{cyc}}^{j}\chi'}$. On G_L, η_K and θ are trivial ((Af1), (Ag1)), so $\beta = \chi'\chi_{\mathrm{cyc}}^{k-1}$ and $\beta \equiv \varphi_1$ there. By the last clause of (Ag1) there is $s \in G_L$ with $\beta(s)^2 \equiv (\chi_{\mathrm{cyc}}^{j}\chi')(s)^2 \not\equiv 1$.

Step 3: the representation. Let $R = \mathrm{ad}^0\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}(\mathfrak{sl}_2(\overline{\mathbb{Q}}_\ell))$, written as a matrix representation in the basis (E_{12}, H, E_{21}), $H = \mathrm{diag}(1, -1)$, and $r = R|_{G_L}$. (Sp1): r is irreducible by (Ne5) with $U = G_L$. (Sp2): let B be the Gram matrix of the trace form $(X, Y) \mapsto \mathrm{tr}(XY)$ in this basis; it is symmetric, invertible, and ${}^tR(g)BR(g) = B$ for $g \in G_{\mathbb{Q}}$. Put $P = BR(c)$. As $R(c)^2 = 1$, ${}^tP = {}^tR(c)B = BR(c)^{-1} = P$, and for $s \in G_L$, ${}^tr(s)\,P\,r(csc) = {}^tR(s)BR(c)R(c)R(s)R(c) = {}^tR(s)BR(s)R(c) = P$. (Sp3): if $\rho(s)$ has characteristic polynomial $X^2 - tX + d$, then $r(s)$ has $(X - 1)(X^2 - (t^2d^{-1} - 2)X + 1)$, integral as $d = \beta(s)$ is a unit; by (Ne1), $t \equiv 1 + \beta(s)$, so it is congruent to $(X - 1)(X - \beta(s))(X - \beta(s)^{-1})$, i.e. to $\prod_a (X - \varphi_a(s))$ for $s \in G_L$. (Sp4): for $\bar v \nmid \ell$, (Ne2) gives $\rho|_{I_{\bar v}} \cong \mathrm{diag}(1, \varepsilon)$, hence $R|_{I_{\bar v}} \cong \mathrm{diag}(\varepsilon^{-1}, 1, \varepsilon)$, and on $I_{\bar v} \cap G_L$, $\varepsilon = \chi'$; after reordering the basis this is $\mathrm{diag}(\tau_1, \tau_2, \tau_3)$. (Sp5): for $\bar v \mid \ell$, by (Ne3) ρ is triangular on $D_{\bar v}$ with diagonal (δ_1, δ_2); conjugation by upper triangular matrices preserves the flag $\langle E_{12}\rangle \subset \langle E_{12}, H\rangle$ and acts on the graded pieces by δ_1/δ_2, 1, δ_2/δ_1. So r is triangular on $D_{\bar v} \cap G_L$ with diagonal $(\kappa_1, \kappa_2, \kappa_3) = (\delta_1\delta_2^{-1}, 1, \delta_2\delta_1^{-1})$; on $I_{\bar v} \cap G_L$ these are $\chi_{\mathrm{cyc}}^{m}, 1, \chi_{\mathrm{cyc}}^{-m}$, i.e. $\chi_{\mathrm{cyc}}^{m\,w_{3,a}}$; and on $D_{\bar v} \cap G_L$, $\delta_1\delta_2^{-1} = \beta\delta_2^{-2} \equiv \beta \equiv \varphi_1$, $1 = \varphi_2$, $\delta_2\delta_1^{-1} \equiv \varphi_1^{-1} = \varphi_3$. (Sp6): by Step 2 and (Na1) of Lemma 49.9, $r \otimes \chi_{\mathrm{cyc}}^{-1}|_{G_L}$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions (L is CM and abelian over $\mathbb{Q}$ by (Af1)).

Hence r is a seed of rank 3 and weight m for (j, χ') with twist $\chi_{\mathrm{cyc}}^{-1}|_{G_L}$. $\square$

Proposition 49.5. *Let ℓ be an odd prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $k \geq 3$, w a prime with $\ell \mid w - 1$, and $N_0 \geq 1$ an integer with $\ell \nmid N_0$ and $w \nmid N_0$. Let ψ be a primitive Dirichlet character modulo N_0 with $\psi(w) = 1$, θ a Dirichlet character modulo w of order ℓ, and $\varepsilon = \psi\theta$, a primitive character modulo $N = N_0 w$; assume $\varepsilon(-1) = (-1)^k$, and that $(\ell - 1) \nmid k$ or ψ is nontrivial of order prime to ℓ. Then there is a normalised newform $f = \sum_{n\geq 1} a_n q^n$ of weight k, level N and character ε such that for every prime p,*

$$\iota^{-1}(a_p) \equiv 1 + \iota^{-1}(\varepsilon(p))\,p^{k-1}$$

modulo the maximal ideal of the integers of $\overline{\mathbb{Q}}_\ell$ (with $\varepsilon(p) = 0$ for $p \mid N$).

Proof. Let $\mathcal{O}$ be a coefficient ring containing $\iota^{-1}(e^{2\pi i/N})$ and ι^{-1} of the values of ψ and θ; put $B = \iota(\mathcal{O})$, λ its maximal ideal, $\mathbb{F} = B/\lambda$, $\overline{\mathbb{F}}$ an algebraic closure. Then $N \geq w \geq 2\ell + 1 \geq 7$ and $\ell \nmid N$. Every value $\theta(n)$, $w \nmid n$, is an ℓ-th root of unity, so $\theta(n) \equiv 1 \pmod{\lambda}$; and $w \equiv 1 \pmod{\ell}$ gives $w^{k-1} \equiv 1$.

Two Eisenstein series congruent away from the constant term. By (Es1), $E = E_k(1, \varepsilon)$ and $E' = E_k(\theta, \psi)$ lie in $M_k(\Gamma_0(N), \varepsilon)$, with $a_0(E') = 0$ and $a_0(E) = L(1-k, \varepsilon)/2 \in \mathrm{Frac}(B)$. For $n \geq 1$ write $n = w^e n'$ with $w \nmid n'$. As $\varepsilon(d) = 0$ when $w \mid d$, and $\theta(n/d) = 0$ unless $w^e \mid d$,

$$a_n(E) = \sum_{d \mid n'} \varepsilon(d) d^{k-1}, \qquad a_n(E') = \psi(w)^e w^{e(k-1)} \sum_{d \mid n'} \theta(n'/d)\psi(d) d^{k-1}.$$

Both lie in B, and since $\psi(w) = 1$, $w^{e(k-1)} \equiv 1$ and $\theta(n'/d) \equiv 1 \equiv \theta(d)$, we get $a_n(E) \equiv a_n(E') \pmod{\lambda}$ for all $n \geq 1$.

The constant term of E lies in λ. Suppose not, and put $a = a_0(E)$; then $a^{-1} \in B$, and $D = a^{-1}(E - E') \in M_k(\Gamma_0(N), \varepsilon)$ has expansion at ∞ in $B[[q]]$, equal to 1 modulo λ. As $\Gamma(N) \subseteq \Gamma_1(N)$, (Kt2) gives $D \in M_k(N; B)$, and its image $\overline{D} \in M_k(N; \overline{\mathbb{F}})$ has q-expansion 1 at ∞, as has $1 \in M_0(N; \overline{\mathbb{F}})$. By (Kt4), $(\ell - 1) \mid k$ and $\overline{D} = A^{k/(\ell-1)}$. Let d be prime to N and $\delta \in \Gamma_0(N)$ with lower right entry $\equiv d \pmod{N}$ (lift $\mathrm{diag}(d^{-1}, d)$ from $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$). Then $D|_k\delta = \varepsilon(d)D$, so by (Kt2) and the compatibility of the action with base change (Kt1), $\delta \bmod N$ maps $\overline{D}$ to $\overline{\varepsilon}(d)\overline{D}$; it fixes A by (Kt4), and $A^{k/(\ell-1)} \neq 0$. Hence $\overline{\varepsilon}(d) = 1$ for all d. But $\overline{\varepsilon} = \overline{\psi}\,\overline{\theta} = \overline{\psi}$, which is nontrivial when $\psi \neq 1$ has order prime to ℓ (reduction is injective on roots of unity of order prime to ℓ); and if $(\ell - 1) \nmid k$ we already have a contradiction. So $a_0(E) \in \lambda$.

A cusp form with character exactly ε congruent to E. Let H be the form of Lemma 49.7 for $(N, k, \varepsilon, \mathcal{O})$ and $G = E - a_0(E)H \in M_k(\Gamma_0(N), \varepsilon)$. Its expansion at ∞ lies in $B[[q]]$, $a_0(G) = 0$, and $a_n(G) \equiv a_n(E) \pmod{\lambda}$ for $n \geq 1$. For $\gamma = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$: if $N \nmid c'$, $G|_k\gamma \to 0$ at $i\infty$ by (Es2) and Lemma 49.7; if $N \mid c'$, $\gamma \in \Gamma_0(N)$ and $G|_k\gamma = \varepsilon(d')G \to \varepsilon(d')a_0(G) = 0$. So G vanishes at every cusp: $G \in S_k(\Gamma_0(N), \varepsilon)$.

Hecke eigenvalues modulo λ. Let S be the set of $g \in S_k(\Gamma_0(N), \varepsilon)$ with all $a_n(g) \in B$. Since $S_k(\Gamma_0(N), \varepsilon)$ has a basis with coefficients in the field of values of ε ([229], Theorem 3.52), S spans it, and S embeds into B^{n_1} by the first n_1 coefficients for n_1 large; so S is free of finite rank over B. The Hecke operators T_n $(n \geq 1)$ act on $M_k(\Gamma_0(N), \varepsilon)$ by $a_m(T_n g) = \sum_{d \mid (m,n)} \varepsilon(d) d^{k-1} a_{mn/d^2}(g)$ $(m \geq 0)$, preserving cusp forms ([80, Proposition 5.3.1]); hence they preserve S, and $a_1(T_n g) = a_n(g)$. The series E is an eigenform of all T_n with eigenvalue $\sigma(n) = a_n(E)$ ([80, §§5.2–5.3]). For $m \geq 1$, $a_m(T_n G) \equiv a_m(T_n E) = \sigma(n) a_m(E) \equiv \sigma(n) a_m(G)$, and $a_0(T_n G) = 0 = a_0(G)$; so $T_n G - \sigma(n) G$ has all coefficients in λ, i.e. lies in λS (dividing by a uniformiser keeps the coefficients in B). Thus the image $\overline{G} \neq 0$ of G in $S/\lambda S$ ($a_1(G) \equiv 1$) is a common eigenvector of the commuting endomorphisms T_n with eigenvalues $\sigma(n) \bmod \lambda$. By the lemma of Deligne and Serre ([78, Lemme 6.11]) there are a discrete valuation ring $B' \supseteq B$ finite over B, of the form $\iota(\mathcal{O}')$ with maximal ideal λ', and a nonzero $f \in S \otimes_B B' \subseteq S_k(\Gamma_0(N), \varepsilon)$ with $T_n f = \lambda_n f$ and $\lambda_n \equiv \sigma(n) \pmod{\lambda'}$ for all n.

f is a newform. From $a_n(f) = a_1(T_n f) = \lambda_n a_1(f)$ we get $a_1(f) \neq 0$; normalise $a_1(f) = 1$. As ε is primitive modulo N, every newform of level $M \mid N$ with character ε has $N = \mathrm{cond}(\varepsilon) \mid M$, so by the theory of newforms ([13], [161]) $S_k(\Gamma_0(N), \varepsilon)$ is spanned by the newforms of level N and character ε, and a normalised common eigenform of all T_n in it is one of them (strong multiplicity one for the eigenvalues of T_p, $p \nmid N$). So f is a normalised newform of weight k, level N and character ε, and $a_p = \lambda_p \equiv \sigma(p) = 1 + \varepsilon(p)p^{k-1}$ for every prime p. □

Lemma 49.6. *For $M \geq 1$ and a Dirichlet character ε modulo M, $M_k(\Gamma_0(M), \varepsilon)$ is the space of holomorphic modular forms f of weight k for $\Gamma_1(M)$ with $f|_k\gamma = \varepsilon(d)f$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M)$,*

and $S_k(\Gamma_0(M),\varepsilon)$ its subspace of cusp forms. Let $k\ge 3$ and let ψ_1,ψ_2 be primitive Dirichlet characters modulo $u,v\ge 1$ (the trivial character being primitive modulo 1) with $\psi_1(-1)\psi_2(-1)=(-1)^k$. Put $\delta(\psi_1)=1$ if $u=1$ and $\delta(\psi_1)=0$ otherwise, and

$$E_k(\psi_1,\psi_2)(\tau)=\frac{\delta(\psi_1)}{2}L(1-k,\psi_2)+\sum_{n\ge1}\Big(\sum_{d\mid n}\psi_1(n/d)\,\psi_2(d)\,d^{k-1}\Big)q^n,\qquad q=e^{2\pi i\tau}.$$

(Es1) *$E_k(\psi_1,\psi_2)\in M_k(\Gamma_0(uv),\psi_1\psi_2)$, and $L(1-k,\psi_2)$ lies in the field generated over $\mathbb{Q}$ by the values of ψ_2.*

(Es2) *If $u=1$ and $v=N>1$, then for every $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ with $N\nmid c$, $(E_k(1,\psi_2)|_k\gamma)(\tau)\to 0$ as $\operatorname{Im}\tau\to\infty$.*

Proof. (Es1) is the classical theory of Eisenstein series with characters (F. Diamond and J. Shurman, *A First Course in Modular Forms*, Springer 2005, §4.5, where the series $2E_k(\psi_1,\psi_2)$ is denoted $E_k^{\psi_1,\psi_2}$ and shown to lie in $M_k(\Gamma_0(uv),\psi_1\psi_2)$ for $k\ge3$); $L(1-k,\psi_2)=-B_{k,\psi_2}/k$ with the generalised Bernoulli number $B_{k,\psi_2}=v^{k-1}\sum_{a=1}^{v}\psi_2(a)B_k(a/v)$ ([266], Theorem 4.2 and Proposition 4.1), a polynomial expression in the values of ψ_2 with rational coefficients.

(Es2). Write $\varepsilon=\psi_2$; it is primitive modulo N and $\varepsilon(-1)=(-1)^k$. Symbols introduced here are local to this proof. For τ in the upper half plane put

$$G(\tau)=\sum_{(x,y)\in\mathbb{Z}^2\setminus\{0\}}\overline{\varepsilon}(y)\,(xN\tau+y)^{-k}.$$

For $\tau=s+it$ with $t\ge1$ and $|s|\le1/2$ one has $|x\tau'+y|^2=(xs'+y)^2+x^2t'^2\ge c_0(x^2+y^2)$ for $\tau'=N\tau$ and a constant $c_0>0$ depending only on N (compare the two cases $|y|\le 2N|x|$ and $|y|>2N|x|$), so the series converges absolutely and uniformly on $\{t\ge1\}$ by periodicity, as $k\ge3$.

G is a nonzero multiple of $E_k(1,\varepsilon)$. The terms with $x=0$ give $\sum_{y\ne0}\overline{\varepsilon}(y)y^{-k}=2L(k,\overline{\varepsilon})$, since $\overline{\varepsilon}(-y)(-y)^{-k}=\overline{\varepsilon}(y)y^{-k}$. The terms with (x,y) and $(-x,-y)$ are equal for the same reason. For $x\ge1$, splitting $y=r+mN$ with r modulo N and using Lipschitz's formula $\sum_{m\in\mathbb{Z}}(z+m)^{-k}=\frac{(-2\pi i)^k}{(k-1)!}\sum_{n\ge1}n^{k-1}e^{2\pi inz}$,

$$\begin{aligned}\sum_{y}\overline{\varepsilon}(y)(xN\tau+y)^{-k}&=N^{-k}\frac{(-2\pi i)^k}{(k-1)!}\sum_{n\ge1}n^{k-1}e^{2\pi inx\tau}\sum_{r\bmod N}\overline{\varepsilon}(r)e^{2\pi inr/N}\\&=N^{-k}\frac{(-2\pi i)^k}{(k-1)!}\,g(\overline{\varepsilon})\sum_{n\ge1}\varepsilon(n)n^{k-1}q^{nx},\end{aligned}$$

where $g(\overline{\varepsilon})=\sum_r\overline{\varepsilon}(r)e^{2\pi ir/N}\ne0$ is the Gauss sum and $\sum_r\overline{\varepsilon}(r)e^{2\pi inr/N}=\varepsilon(n)g(\overline{\varepsilon})$ for all n because ε is primitive. Summing over $x\ge1$, $G=2L(k,\overline{\varepsilon})+C\sum_{n\ge1}\big(\sum_{d\mid n}\varepsilon(d)d^{k-1}\big)q^n$ with $C=2N^{-k}\frac{(-2\pi i)^k}{(k-1)!}g(\overline{\varepsilon})\ne0$. Next, $G\in M_k(\Gamma_0(N),\varepsilon)$ up to holomorphy at the cusps, which follows from the uniform bound: for $\gamma'=\left(\begin{smallmatrix}a'&b'\\c'&d'\end{smallmatrix}\right)\in\Gamma_0(N)$, $(G|_k\gamma')(\tau)=\sum\overline{\varepsilon}(y)\big((xNa'+yc')\tau+(xNb'+yd')\big)^{-k}$, and $(xN,y)\mapsto(xNa'+yc',\,xNb'+yd')$ is a bijection of $N\mathbb{Z}\times\mathbb{Z}\setminus\{0\}$ onto itself (as $N\mid c'$) with $xNb'+yd'\equiv yd'\pmod N$, so $\overline{\varepsilon}(y)=\varepsilon(d')\overline{\varepsilon}(xNb'+yd')$ and $G|_k\gamma'=\varepsilon(d')G$. Hence $G-C\,E_k(1,\varepsilon)$ lies in $M_k(\Gamma_0(N),\varepsilon)$ by (Es1) and is a constant c; applying $|_k\left(\begin{smallmatrix}1&0\\N&1\end{smallmatrix}\right)$ gives $c(N\tau+1)^{-k}=c$ for all τ, so $c=0$ and $G=C\,E_k(1,\varepsilon)$.

Constant terms. Let $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ with $N\nmid c$. Then $(G|_k\gamma)(\tau)=\sum\overline{\varepsilon}(y)\big((xNa+yc)\tau+(xNb+yd)\big)^{-k}$, and $(xN,y)\mapsto(xNa+yc,\,xNb+yd)$ is injective. Along $\tau=s+it$, $t\to\infty$,

each term with $xNa+yc\neq 0$ tends to 0, and the series is dominated uniformly for $t\geq 1$ by the convergent series above (the substitution is a bijection of $\mathbb{Z}^2\setminus\{0\}$ onto a subset of it); so the limit is the sum over the terms with $xNa+yc=0$ and $\overline{\varepsilon}(y)\neq 0$. Such a term has $\gcd(y,N)=1$ and $N\mid yc$, hence $N\mid c$, which is excluded. So $G|_k\gamma\to 0$, and $E_k(1,\varepsilon)|_k\gamma=C^{-1}G|_k\gamma\to 0$. □

Lemma 49.7. *Let ℓ be an odd prime, $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$ an isomorphism, $N\geq 3$ an integer with $\ell\nmid N$, $k\geq 3$, and ε a Dirichlet character modulo N with $\varepsilon(-1)=(-1)^k$. Let $\mathcal{O}$ be a coefficient ring containing $\iota^{-1}(e^{2\pi i/N})$ and ι^{-1} of the values of ε, and $B=\iota(\mathcal{O})\subset\mathbb{C}$. Then there is $H\in M_k(\Gamma_0(N),\varepsilon)$ whose q-expansion at ∞ has coefficients in B and constant term 1, such that $(H|_k\gamma)(\tau)\to 0$ as $\operatorname{Im}\tau\to\infty$ for every $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ with $N\nmid c$.*

Proof. Since $\ell\nmid N$, B contains $1/N$ and $e^{2\pi i/N}$, so B is an R_N-subalgebra of $\mathbb{C}$; enlarging $\mathcal{O}$ by a finite unramified extension we may assume that the cusps of $X_N\otimes B$ are n_C disjoint sections over B. Let $\mathbb{F}$ be the residue field of B and $\overline{\mathbb{F}}$ an algebraic closure. Evaluation at the cusps (using the canonical differential of the Tate curve) gives B-linear maps $v_B\colon M_k(N;B)\to B^{n_C}$ and $v_{\overline{\mathbb{F}}}\colon M_k(N;\overline{\mathbb{F}})\to\overline{\mathbb{F}}^{\,n_C}$, compatible with reduction.

$v_{\overline{\mathbb{F}}}$ is surjective. On the smooth proper connected curve $X=X_N\otimes\overline{\mathbb{F}}$ (Kt1), $\omega^{\otimes k}(-C)\cong\Omega^1_X\otimes\omega^{\otimes(k-2)}$ by Kodaira–Spencer, so by Serre duality $H^1(X,\omega^{\otimes k}(-C))$ is dual to $H^0(X,\omega^{\otimes(2-k)})$. The discriminant is a nonzero section of $\omega^{\otimes 12}$ over X that vanishes at the cusps (its q-expansions are $q\prod_{n\geq 1}(1-q^n)^{24}$), so $\deg\omega>0$, $\deg\omega^{\otimes(2-k)}<0$ for $k\geq 3$, and $H^0(X,\omega^{\otimes(2-k)})=0$. The cohomology sequence of $0\to\omega^{\otimes k}(-C)\to\omega^{\otimes k}\to\omega^{\otimes k}|_C\to 0$ shows that $M_k(N;\overline{\mathbb{F}})\to H^0(C,\omega^{\otimes k}|_C)=\overline{\mathbb{F}}^{\,n_C}$, which is $v_{\overline{\mathbb{F}}}$, is surjective.

v_B is surjective. By (Kt3) ($k\geq 2$), $M_k(N;B)\otimes_B\overline{\mathbb{F}}\cong M_k(N;\overline{\mathbb{F}})$ compatibly with evaluation, so $v_B\otimes\overline{\mathbb{F}}$ is surjective, hence so is $v_B\otimes\mathbb{F}$ (a linear map defined over $\mathbb{F}$ that becomes surjective over $\overline{\mathbb{F}}$), and v_B is surjective by Nakayama's lemma, B^{n_C} being finite over the local ring B. Choose $s\in M_k(N;B)$ with value 1 at the standard cusp ∞ and value 0 at every other cusp.

Constant terms of s. By (Kt2), s is a modular form of weight k for $\Gamma(N)$ all of whose q-expansions lie in $B[[q^{1/N}]]$; for $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ let $c(\gamma)=\lim_{t\to\infty}(s|_k\gamma)(it)$, the constant term of the expansion of $s|_k\gamma$. The cusp of $\Gamma(N)$ represented by $\gamma\infty$ is ∞ exactly when $\gamma\in\Gamma(N)\{\pm1\}U$, $U=\{\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\}$; the expansion of s at any cusp is that of $s|_k\gamma'$ for some γ' representing it, and $s|_k(-\gamma')=(-1)^k s|_k\gamma'$, $s|_k(g\gamma' u)=(s|_k\gamma')|_k u$ for $g\in\Gamma(N)$, $u\in U$. Hence $c(\gamma)=0$ if $\gamma\notin\Gamma(N)\{\pm1\}U$, $c(gu)=1$ and $c(-gu)=(-1)^k$ for $g\in\Gamma(N)$, $u\in U$.

The averaged form. Let Σ be a set of representatives of $\Gamma_0(N)/\Gamma(N)$ ($\Gamma(N)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$), and for $g\in\Gamma_0(N)$ let d_g be its lower right entry. Put

$$H=\frac{1}{2N}\sum_{g\in\Sigma}\overline{\varepsilon}(d_g)\,s|_k g.$$

Each term depends only on $g\Gamma(N)$, since s is $\Gamma(N)$-invariant and $d_{gh}\equiv d_g\pmod N$ for $h\in\Gamma(N)$. For $\delta\in\Gamma_0(N)$, $d_{g\delta}\equiv d_g d_\delta\pmod N$ and $g\mapsto g\delta$ permutes the cosets, so $H|_k\delta=\varepsilon(d_\delta)H$; thus $H\in M_k(\Gamma_0(N),\varepsilon)$. Its expansion at ∞ is a combination with coefficients in B ($2N$ is a unit in B, ℓ being odd and prime to N) of expansions of s at cusps, so it lies in $B[[q^{1/N}]]$, and in $B[[q]]$ because H is invariant under $U\subset\Gamma_1(N)$. For $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ the limit of $(H|_k\gamma)(it)$ is $\frac{1}{2N}\sum_{g\in\Sigma}\overline{\varepsilon}(d_g)c(g\gamma)$. If $N\nmid c$, no $g\gamma$ lies in $\Gamma(N)\{\pm1\}U\subseteq\Gamma_0(N)$ (else $\gamma\in\Gamma_0(N)$), so the limit is 0. For $\gamma=1$, the cosets with $c(g)\neq 0$ are those of $\pm\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)$, n mod N, which are $2N$ distinct cosets as $N\geq 3$; each contributes $\overline{\varepsilon}(\pm1)(\pm1)^k=1$, because $\varepsilon(-1)=(-1)^k$. So the constant term of H is 1. □

Lemma 49.8. *Let ℓ be an odd prime, $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$ an isomorphism, and $g=\sum a_n q^n$ a normalised newform of weight $k\geq 3$, level N with $\ell\nmid N$ and primitive character ε modulo N, such that*

$\iota^{-1}(a_p) \equiv 1 + \iota^{-1}(\varepsilon(p))p^{k-1}$ *for every prime* p. *Let* $\rho = \rho^{\mathrm{ar}} \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ *be its Galois representation and* $\beta = \varepsilon\chi_{\mathrm{cyc}}^{k-1} = \det\rho$.

(Ne1) *For every* $s \in G_{\mathbb{Q}}$ *the characteristic polynomial of* $\rho(s)$ *has coefficients in* $\overline{\mathbb{Z}}_\ell$ *and is congruent to* $(X-1)(X-\beta(s))$.

(Ne2) *For every place* $\bar{p}$ *above a prime* $p \neq \ell$, $\rho|_{I_{\bar{p}}}$ *is conjugate to* $s \mapsto \mathrm{diag}(1, \varepsilon(s))$, *and the semisimplification of* $\rho|_{D_{\bar{p}}}$ *is a sum of two characters* ν_1, ν_2 *with* $|\iota\nu_1(\sigma)| = |\iota\nu_2(\sigma)| = p^{(k-1)/2}$ *for every* $\sigma \in D_{\bar{p}}$ *acting on the residue field as* $x \mapsto x^p$.

(Ne3) *For every place* $\bar{\ell}$ *above* ℓ, ρ *is triangular on* $D_{\bar{\ell}}$ *with a diagonal* (δ_1, δ_2) *such that* δ_2 *is trivial on* $I_{\bar{\ell}}$, $\delta_2(s) \equiv 1$ *for* $s \in D_{\bar{\ell}}$, *and* $\delta_1 = \beta\delta_2^{-1}$; *in particular* $\delta_1 = \chi_{\mathrm{cyc}}^{k-1}$ *on* $I_{\bar{\ell}}$.

(Ne4) *If* $L \subset \overline{\mathbb{Q}}$ *is a finite Galois extension of* $\mathbb{Q}$ *such that* $\beta(s) \not\equiv 1$ *for some* $s \in G_L$, *then* $\rho|_{G_L}$ *is irreducible.*

(Ne5) *If* $\beta(s)^2 \not\equiv 1$ *for some* $s \in G_{\mathbb{Q}}$, *then* $\rho \otimes \omega \not\cong \rho$ *for every character* $\omega \neq 1$ *of* $G_{\mathbb{Q}}$ *of order* 2, *and for every open subgroup* U *of* $G_{\mathbb{Q}}$ *the restrictions of* ρ *and of* $\mathrm{ad}^0\rho$ *(the action by conjugation on trace-zero matrices) to* U *are irreducible.*

Proof. (Ne1). The compact group $\rho(G_{\mathbb{Q}})$ lies in $\mathrm{GL}_2(E)$ for a finite extension $E/\mathbb{Q}_\ell$ (Baire, as $\overline{\mathbb{Q}}_\ell^\times$ is a countable union of the closed $E^\times$) and stabilises an $\mathcal{O}_E$-lattice, so characteristic polynomials have coefficients in $\mathcal{O}_E$. For an arithmetic Frobenius σ at $p \nmid \ell N$, $\mathrm{tr}\,\rho(\sigma) = \iota^{-1}(a_p) \equiv 1 + \beta(\sigma)$ and $\det\rho(\sigma) = \beta(\sigma)$. The set of s with $\mathrm{tr}\,\rho(s) - 1 - \beta(s) \in \mathfrak{m}_{\mathcal{O}_E}$ is closed ($\mathfrak{m}_{\mathcal{O}_E}$ is closed and the maps are continuous) and contains all such σ, which are dense by the Chebotarev density theorem; so it is $G_{\mathbb{Q}}$, and the characteristic polynomial is $X^2 - (1+\beta(s))X + \beta(s)$ modulo $\mathfrak{m}$.

(Ne2). If $p \nmid N$, ρ is trivial on $I_{\bar{p}}$ by (Dg1) and ε is unramified at p; the semisimplification of the unramified $\rho|_{D_{\bar{p}}}$ is a sum of unramified characters whose values at σ are the eigenvalues of $\rho(\sigma)$, the roots of $X^2 - \iota^{-1}(a_p)X + \iota^{-1}(\varepsilon(p))p^{k-1}$ (for an arithmetic σ; for a general σ inducing $x \mapsto x^p$ the same, ρ being trivial on inertia), of absolute value $p^{(k-1)/2}$ under ι by (Dg2). If $p \mid N$, then $v_p(N) = v_p(\mathrm{cond}\,\varepsilon)$ as ε is primitive modulo N, and (Gn3) gives $\rho|_{D_{\bar{p}}} \cong \mathrm{diag}(\nu_1, \nu_2)$ with ν_1 unramified and the absolute values; on $I_{\bar{p}}$, $\nu_2 = \nu_1\nu_2 = \det\rho = \varepsilon\chi_{\mathrm{cyc}}^{k-1} = \varepsilon$, as χ_{cyc} is unramified at p.

(Ne3). As $\ell \nmid N$, $\iota^{-1}(a_\ell) \equiv 1 + \iota^{-1}(\varepsilon(\ell))\ell^{k-1} \equiv 1$ is a unit, and (Gn2) applies. The roots α, α' of $X^2 - \iota^{-1}(a_\ell)X + \iota^{-1}(\varepsilon(\ell))\ell^{k-1}$ are integral with $\alpha\alpha' \in \ell\overline{\mathbb{Z}}_\ell$ and α a unit, so $\alpha' \in \mathfrak{m}$ and $\alpha = \iota^{-1}(a_\ell) - \alpha' \equiv 1$. The unramified δ_2 is determined by $\delta_2(\sigma) = \alpha$ on the topological generator of $D_{\bar{\ell}}/I_{\bar{\ell}}$, and $\{s : \delta_2(s) \equiv 1\}$ is a closed subgroup containing $I_{\bar{\ell}}$ and σ; so $\delta_2 \equiv 1$ on $D_{\bar{\ell}}$. The determinant gives $\delta_1\delta_2 = \beta$; on $I_{\bar{\ell}}$, $\beta = \chi_{\mathrm{cyc}}^{k-1}$ as ε is unramified at ℓ.

A Clifford alternative. Let $M \subset \overline{\mathbb{Q}}$ be finite Galois over $\mathbb{Q}$ with $\rho|_{G_M}$ reducible. By (Gn1) and Clifford's theorem (G_M is normal), $\rho|_{G_M}$ is a sum of two characters ξ_1, ξ_2. If $\xi_1 = \xi_2$, $\rho(G_M)$ is scalar; then for s in the open subgroup $G_M \cap I_{\bar{\ell}}$ of $I_{\bar{\ell}}$, (Ne3) gives $\delta_1(s) = \delta_2(s)$, i.e. $\chi_{\mathrm{cyc}}(s)^{k-1} = 1$; but $\chi_{\mathrm{cyc}}(G_M \cap I_{\bar{\ell}})$ is open in $\mathbb{Z}_\ell^\times$, so it contains elements of infinite order: contradiction. Hence $\xi_1 \neq \xi_2$, $G_{\mathbb{Q}}$ permutes the two eigenlines, the stabiliser H of the ξ_1-line has index 2 and contains G_M, and $\rho \cong \mathrm{Ind}_H^{G_{\mathbb{Q}}}\xi$ for the character ξ of H on that line (Frobenius reciprocity, ρ irreducible of dimension 2). With ω the character of order 2 with kernel H, $\rho \otimes \omega \cong \rho$. Comparing (Ne1) for ρ and $\rho \otimes \omega$: for $s \notin H$ the residual roots $\{1, \bar{\beta}(s)\}$ and $\{-1, -\bar{\beta}(s)\}$ coincide, and $1 \neq -1$, so $\bar{\beta}(s) = -1$; for $s \in H$ and any $t \notin H$, $\bar{\beta}(s) = \bar{\beta}(ts)\bar{\beta}(t)^{-1} = 1$. So $\beta \equiv 1$ on $H \supseteq G_M$ and $\beta \equiv \omega$ on $G_{\mathbb{Q}}$; in particular $\beta^2 \equiv 1$ on $G_{\mathbb{Q}}$.

(Ne4). Apply the alternative with $M = L$: reducibility would give $\beta \equiv 1$ on G_L.

(Ne5). If $\rho\otimes\omega\cong\rho$ for $\omega\neq 1$ of order 2 with kernel H, the last two sentences of the alternative apply without change and give $\beta^2\equiv 1$: excluded. An open U contains G_M for some finite Galois $M/\mathbb{Q}$; a U-stable line would be G_M-stable, and the alternative would give $\rho\otimes\omega\cong\rho$; so $\rho|_U$ is irreducible. For $\mathrm{ad}^0\rho$: the trace form $(X,Y)\mapsto\mathrm{tr}(XY)$ on $\mathfrak{sl}_2(\overline{\mathbb{Q}}_\ell)$ is nondegenerate and invariant, so if W is a proper nonzero U-stable subspace, one of W, $W^\perp$ is a U-stable line $\overline{\mathbb{Q}}_\ell X$: $\rho(u)X\rho(u)^{-1}=c(u)X$. If X is nilpotent, $\ker X$ is a U-stable line for ρ, excluded. Otherwise X has eigenvalues $\pm a$, $a\neq 0$, and $\rho(u)$ permutes its two eigenlines; the subgroup of U fixing one of them is open (index at most 2) and leaves a line stable, excluded. So $\mathrm{ad}^0\rho|_U$ is irreducible. □

Lemma 49.9. *Let ℓ, ι, g (weight $k\geq 3$, level N, $\ell\nmid N$, primitive character ε), ρ and $\beta=\det\rho$ be as in Lemma 49.8, let $L\subset\overline{\mathbb{Q}}$ be a CM field abelian over $\mathbb{Q}$, and $c\in G_{\mathbb{Q}}$ a complex conjugation.*

(Na1) *If $\beta(s)^2\not\equiv 1$ for some $s\in G_{\mathbb{Q}}$, then $r_3=(\mathrm{ad}^0\rho)|_{G_L}\otimes\chi_{\mathrm{cyc}}^{-1}$ is irreducible, automorphic of conjugate self-dual type for ι, and satisfies the witness conditions.*

(Na2) *If $\beta(s)\not\equiv 1$ for some $s\in G_L$, and $\nu\colon G_L\to\overline{\mathbb{Q}}_\ell^\times$ is a continuous character, algebraic for ι, with $\nu(s)\nu(csc)\beta(s)=\chi_{\mathrm{cyc}}(s)^{-1}$ for all $s\in G_L$, then $r_2=\rho|_{G_L}\otimes\nu$ is irreducible, automorphic of conjugate self-dual type for ι, and satisfies the witness conditions.*

Proof. Throughout, two continuous semisimple representations of G_E (E a number field) whose characteristic polynomials agree at the Frobenius elements of almost all places are isomorphic (Chebotarev density and Brauer–Nesbitt); we call this *(CB)*. For a representation r of G_L attached to a cuspidal π, $r^c\colon s\mapsto r(csc)$ is attached to π^c and $r^\vee\otimes\chi_{\mathrm{cyc}}^{1-n}$ to $\pi^\vee$, by comparing the definitions in Section 13.

Step 1: the automorphic representation of g. Let π be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ generated by g, in the unitary normalisation ([103]): for $p\nmid N$ it is unramified with Satake parameters $\{\alpha_p,\beta_p\}$ such that $p^{(k-1)/2}\alpha_p$, $p^{(k-1)/2}\beta_p$ are the roots of $X^2-a_pX+\varepsilon(p)p^{k-1}$, so $\iota^{-1}(p^{(k-1)/2}\alpha_p)$, $\iota^{-1}(p^{(k-1)/2}\beta_p)$ are the eigenvalues of $\rho(\sigma_p)$ for an arithmetic Frobenius σ_p at $p\nmid\ell N$; its central character ω_π has finite order ($\omega_\pi\leftrightarrow\varepsilon$); and the Langlands parameter of π_∞ restricted to $\mathbb{C}^\times$ is $z\mapsto\mathrm{diag}\big((z/\bar z)^{(k-1)/2},(z/\bar z)^{-(k-1)/2}\big)$. Fix a chain $\mathbb{Q}=E_0\subset E_1\subset\cdots\subset E_t=L$ with E_{i+1}/E_i cyclic of prime degree (a composition series of the abelian group $\mathrm{Gal}(L/\mathbb{Q})$).

Step 2: (Na1), automorphy. Assume $\beta^2\not\equiv 1$. If $\pi\cong\pi\otimes(\omega\circ\det)$ for a quadratic $\omega\neq 1$, then ρ and $\rho\otimes\omega$ have the same Frobenius characteristic polynomials at almost all primes, so $\rho\cong\rho\otimes\omega$ by (CB) and (Gn1) of Theorem B.80, contradicting (Ne5). By (Gj3), $\Pi_0=\mathrm{Ad}(\pi)$ is cuspidal. Inductively let $\Pi_{i+1}=\mathrm{BC}_{E_{i+1}/E_i}(\Pi_i)$ and $r^{(i)}=\mathrm{ad}^0\rho|_{G_{E_i}}$. By (Gj1), (Ac1), at almost all places w of E_i (above $p\nmid\ell N$, unramified in L, residue degree f) $\Pi_{i,w}$ is unramified with Satake parameters $\{(\alpha_p/\beta_p)^f,1,(\beta_p/\alpha_p)^f\}$, the eigenvalues of $r^{(i)}$ at a Frobenius element of w (this multiset is stable under inversion, so arithmetic or geometric Frobenius does not matter). Suppose Π_i is cuspidal and $\Pi_i\cong\Pi_i\otimes(\omega\circ\det)$ with $\omega\neq 1$ trivial on norms from E_{i+1}; let ω_G be the corresponding nontrivial character of $\mathrm{Gal}(E_{i+1}/E_i)$. Then $r^{(i)}\cong r^{(i)}\otimes\omega_G$ by (CB), $r^{(i)}$ being irreducible by (Ne5); an intertwiner A with $Ar^{(i)}(s)A^{-1}=\omega_G(s)r^{(i)}(s)$ is not scalar and commutes with $r^{(i)}(G_{E_{i+1}})$, so $r^{(i)}|_{G_{E_{i+1}}}$ is reducible by Schur's lemma, contradicting (Ne5). By (Ac3), Π_{i+1} is cuspidal. So $\Pi=\Pi_t$ is a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_L)$, and at almost all w the eigenvalues of $r_3(\mathrm{Frob}_w^{-1})$ are $\mathrm{N}w$ times its Satake parameters (as $\chi_{\mathrm{cyc}}^{-1}(\mathrm{Frob}_w^{-1})=\mathrm{N}w$): $r_3\leftrightarrow_\iota\Pi$. By (Gj2), (Ac2), at each complex place the parameter of Π restricted to $\mathbb{C}^\times$ is $\mathrm{diag}\big((z/\bar z)^{k-1},1,(z/\bar z)^{1-k}\big)$: integral, pairwise distinct exponents, so Π is regular algebraic. The Satake parameters of $\Pi^\vee$ at w are the inverses of those of Π_w, those of Π^c are those of Π_{cw}, and $r_3(c\,\mathrm{Frob}_w\,c)$ is conjugate to $\mathrm{ad}^0\rho(\mathrm{Frob}_w)\chi_{\mathrm{cyc}}^{-1}(\mathrm{Frob}_w)$ in $G_{\mathbb{Q}}$; so Π^c and

$\Pi^\vee$ agree at almost all places, and $\Pi^c \cong \Pi^\vee$ by (Js1). Hence r_3 is automorphic of conjugate self-dual type; it is irreducible by (Ne5).

Step 3: (Na2), automorphy. Assume $\beta \not\equiv 1$ on G_L; $\rho|_{G_L}$, hence $\rho|_{G_{E_i}}$ for all i, is irreducible by (Ne4). Let $\pi_0 = \pi$ and $\pi_{i+1} = \mathrm{BC}_{E_{i+1}/E_i}(\pi_i)$; the argument of Step 2 with $\rho|_{G_{E_i}}$ in place of $r^{(i)}$ shows that every π_i is cuspidal, with Satake parameters $\{\alpha_p^f, \beta_p^f\}$ at almost all w. As ν is algebraic and the inverse of an algebraic Hecke character is one, there is a Hecke character $\nu_{\mathbb{A}}$ of L with $\nu_{\mathbb{A}}(\varpi_w) = \iota\nu(\mathrm{Frob}_w^{-1})$ at almost all w, of type $z \mapsto z^a\bar{z}^b$ ($a, b \in \mathbb{Z}$) at each complex place. Put $\nu' = \nu_{\mathbb{A}} \,|\cdot|_{\mathbb{A}}^{k/2} (\omega_\pi \circ \mathrm{N}_{L/\mathbb{Q}})^{-1}$ and $\Pi' = \pi_t \otimes (\nu' \circ \det)$, cuspidal. At almost all w (residue degree f over p), the eigenvalues of $r_2(\mathrm{Frob}_w^{-1})$ are ι^{-1} of $\mathrm{N}w^{-(k-1)/2}\{\alpha_p^{-f}, \beta_p^{-f}\}\nu_{\mathbb{A}}(\varpi_w) = \mathrm{N}w^{1/2}\{\alpha_p^f, \beta_p^f\}\nu'(\varpi_w)$, using $\alpha_p\beta_p = \omega_\pi(p)$; so $r_2 \leftrightarrow_\iota \Pi'$. At a complex place the parameter of Π' on $\mathbb{C}^\times$ is $\mathrm{diag}(z^{a+k-1/2}\bar{z}^{b+1/2},\, z^{a+1/2}\bar{z}^{b+k-1/2})$, which after the twist by $|\det|^{-1/2}$ has distinct integral exponents: Π' is regular algebraic. Since $\rho^\vee \cong \rho \otimes \beta^{-1}$ and $\nu\nu^c\beta = \chi_{\mathrm{cyc}}^{-1}$, $r_2^\vee \otimes \chi_{\mathrm{cyc}}^{-1} \cong \rho|_{G_L} \otimes \beta^{-1}\nu^{-1}\chi_{\mathrm{cyc}}^{-1} = \rho|_{G_L} \otimes \nu^c \cong r_2^c$ (as $\rho(csc) = \rho(c)\rho(s)\rho(c)^{-1}$). Hence $\Pi'^\vee$ and Π'^c have the same Satake parameters at almost all places and $\Pi'^c \cong \Pi'^\vee$ by (Js1). So r_2 is irreducible and automorphic of conjugate self-dual type.

Step 4: local shape. Let r be r_3 (with $n = 3$) or r_2 (with $n = 2$). (i) At a place $\bar{v}$ above ℓ: by (Ne3), ρ is triangular on $D_{\bar{v}}$ with diagonal (δ_1, δ_2), $\delta_1\delta_2^{-1} = \chi_{\mathrm{cyc}}^{k-1}$ on $I_{\bar{v}}$. Conjugation by an upper triangular matrix preserves the flag $\langle E_{12}\rangle \subset \langle E_{12}, H\rangle \subset \mathfrak{sl}_2$ ($H = \mathrm{diag}(1,-1)$) and acts on the graded pieces by δ_1/δ_2, 1, δ_2/δ_1; so r is triangular on $D_{\bar{v}} \cap G_L$ with diagonal (κ_a), where on $I_{\bar{v}} \cap G_L$ the ratios κ_a/κ_b ($a \neq b$) are χ_{cyc}^h with $h \in \{\pm(k-1), \pm 2(k-1)\}$. If $\kappa_a^e = (\kappa_b\chi_{\mathrm{cyc}})^e$ for some $e \geq 1$, then $\chi_{\mathrm{cyc}}^{e(h-1)} = 1$ (resp. $\chi_{\mathrm{cyc}}^e = 1$ if $a = b$) on the open subgroup $I_{\bar{v}} \cap G_L$ of $I_{\bar{v}}$, impossible as $h \neq 1$ ($k \geq 3$) and χ_{cyc} has infinite order there. So r is generic and potentially generic at every place above ℓ, and (Wc1) holds. (ii) At $\bar{w}$ above $p \neq \ell$: by (Ne2), r restricted to $I_{\bar{w}} \cap G_L$ is a sum of characters of finite order (χ_{cyc} is unramified at p, and ν, being attached to an algebraic Hecke character, has finite order on inertia away from ℓ); and the semisimplification of $r|_{D_{\bar{w}} \cap G_L}$ is a sum of characters λ_a whose values at any element acting on the residue field as a power $x \mapsto x^{p^j}$ have absolute values (under ι) independent of a: for r_3 they are $\nu_1\nu_2^{-1}\chi_{\mathrm{cyc}}^{-1}, \chi_{\mathrm{cyc}}^{-1}, \nu_2\nu_1^{-1}\chi_{\mathrm{cyc}}^{-1}$, for r_2 they are $\nu_1\nu, \nu_2\nu$, and $|\iota\nu_1(\sigma)| = |\iota\nu_2(\sigma)|$.

Step 5: (Wc2). Let π' be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ with $r \leftrightarrow_\iota \pi'$. Then π'_w and Π_w (resp. Π'_w) are unramified with the same Satake parameters at almost all w, so π' is isomorphic to Π (resp. Π') by (Js1), hence conjugate self-dual, in particular polarizable; by (Bg1) and (CB), $r_{\ell,\iota}(\pi') \cong r$. Let w be a finite place of L, $\bar{w} \mid w$, $D = D_{\bar{w}} \cap G_L$, $q = q_w$, and $\mathrm{WD}(r|_D) = (V, \rho', N)$. If $w \nmid \ell$, r has finite image on inertia by Step 4 (ii), so $N = 0$ and $\rho'|_I = r|_I$ is a sum of characters, and the characters of the semisimplification of ρ' have values of equal absolute value at $\mathrm{Art}_{L_w}(\varpi_w)^{-1}$. If $w \mid \ell$: by (Lc2) applied to $\rho_{g,\iota}$, $\mathrm{a}(\rho_{g,\iota}|_{D_{\bar{\ell}}}) = v_\ell(N) = 0$, so $\mathrm{WD}(\rho|_{D_{\bar{\ell}}})$ is unramified with zero monodromy; χ_{cyc} and (for r_2) the character $\nu|_D = \det(r_2|_D)\det(\rho|_D)^{-1}$ are potentially semistable and one-dimensional, so with zero monodromy; as the Weil–Deligne functor of [95] commutes with restriction to D, tensor products and duals, $N = 0$ and $\rho'|_I$ is a sum of characters. Then (Pu2) of Theorem B.73 for π' gives $\mathrm{gr}_j = 0$ for j different from its weight, and (Pu1) shows that all eigenvalues of $\rho'(\Phi)$, $n(\Phi) = -1$, have the same absolute value. In both cases (Bg3) (with U the open subgroup of $\mathcal{O}_{L_w}^\times$ on which the inertial characters are trivial) shows that π'_w is a subquotient of the normalised induction of characters $\xi_1, \ldots, \xi_n$ of $L_w^\times$, and (Bg5) gives $\iota\lambda_a(\mathrm{Art}_{L_w}(x)^{-1}) = \xi_a(x)|x|_w^{(1-n)/2}$ for the characters λ_a of the semisimplification of ρ'; hence $|\xi_1(\varpi_w)| = \cdots = |\xi_n(\varpi_w)|$. The central character $\omega_{\pi'}$ is a Hecke character of L; $|\omega_{\pi'}|$ is trivial on the compact group $\mathbb{A}_L^1/L^\times$, so $|\omega_{\pi'}| = |\cdot|_{\mathbb{A}}^{\sigma_0}$ for a real σ_0; and the central character of π'_w is $\xi_1 \cdots \xi_n$. Therefore $|\xi_a(\varpi_w)| = q_w^{-\sigma_0/n}$ for every a and every finite w, and (Wc2) holds with $\tau_{\pi'} = -\sigma_0/n$. □

Proposition 49.10. *Let ℓ be odd, K an imaginary quadratic field, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and let L be a CM field satisfying (Af1), (Af2), (Ag1) and (Ag2) for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, where $2\ell \mid M$ and every prime dividing d_K divides M. Fix data $(N', w, j, \chi', \theta)$ as in (Ag1). Then there are an admissible twist κ of rank 2 for ι, c and M, an integer $m \geq 2$, and a seed of rank 2 and weight m for (j, χ') with twist κ.*

Proof. Step 1: the newform. By (Ag1), $\chi'^2\eta_K$ is a Dirichlet character modulo N' with value 1 at w; let ψ be the primitive character inducing it (possibly trivial), of conductor $N_0 \mid N'$, so $\psi(w) = 1$, $\ell \nmid N_0$, $w \nmid N_0$. As in the rank-3 case, θ is a character modulo w of order ℓ and $\ell \mid w - 1$. Choose $m \geq 2$ with $m \equiv j \pmod{\ell - 1}$, put $k = 2m + 1$ and $\varepsilon = \psi\theta$, primitive modulo $N = N_0 w$; $\varepsilon(-1) = \chi'(c)^2\eta_K(c)\theta(c) = -1 = (-1)^k$, and $(\ell - 1) \nmid k$ as k is odd. By Proposition 49.5 there is a normalised newform g of weight k, level N ($\ell \nmid N$) and character ε with $\iota^{-1}(a_p) \equiv 1 + \iota^{-1}(\varepsilon(p))p^{k-1}$ for all p. Let $\rho = \rho^{\mathrm{ar}}$ and $\beta = \varepsilon\chi_{\mathrm{cyc}}^{2m} = \det\rho$. Since $\bar\theta = 1$ and $\bar\chi_{\mathrm{cyc}}^{2m} = \bar\chi_{\mathrm{cyc}}^{2j}$, $\bar\beta = \eta_K\bar\varphi^2$; on G_L, where η_K and θ are trivial, $\beta \equiv \varphi^2$, and by the last clause of (Ag1) there is $s \in G_L$ with $\beta(s) \not\equiv 1$.

Step 2: the representation. Let $\mu = \chi_{\mathrm{cyc}}^{-m}\chi'^{-1}$, a character of $G_{\mathbb{Q}}$, and $r = \rho|_{G_L} \otimes \mu$. Then $\mu^2\beta = \chi'^{-2}\varepsilon = \eta_K\theta$, which is trivial on G_L. (Sp1): $\rho|_{G_L}$ is irreducible by (Ne4) ($L/\mathbb{Q}$ is abelian by (Af1)), hence so is r. (Sp2): let $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, so ${}^tAJA = \det(A)J$, and $P = J\rho(c)$. As $\rho(c)^2 = 1$ and $\det\rho(c) = \beta(c) = -1$, ${}^t\rho(c)J = -J\rho(c)^{-1} = -J\rho(c)$, so ${}^tP = -{}^t\rho(c)J = P$. For $s \in G_L$, $\mu(csc) = \mu(s)$ and ${}^t r(s)\, P\, r(csc) = \mu(s)^2\, {}^t\rho(s)J\rho(c)\rho(c)\rho(s)\rho(c) = (\mu^2\beta)(s)\, J\rho(c) = P$. (Sp3): the characteristic polynomial of $r(s)$ is $X^2 - \mu(s)tX + \mu(s)^2 d$ when that of $\rho(s)$ is $X^2 - tX + d$; by (Ne1) it is integral and congruent to $(X - \mu(s))(X - \mu(s)\beta(s))$. On G_L, $\bar\mu = \bar\chi_{\mathrm{cyc}}^{-j}\bar\chi'^{-1} = \bar\varphi^{-1}$ and $\bar\mu\bar\beta = \bar\varphi$, so this is $\prod_a(X - \varphi_a(s))$ modulo $\mathfrak{m}$. (Sp4): for $\bar v \nmid \ell$, by (Ne2) $r|_{I_{\bar v}\cap G_L} \cong \mu \cdot \mathrm{diag}(1, \varepsilon) = \mathrm{diag}(\chi'^{-1}, \chi'^{-1}\chi'^2\eta_K\theta) = \mathrm{diag}(\chi'^{-1}, \chi')$ there (χ_{cyc} is unramified at $p \neq \ell$), which is $\mathrm{diag}(\tau_1, \tau_2)$ after reordering. (Sp5): for $\bar v \mid \ell$, by (Ne3) r is triangular on $D_{\bar v} \cap G_L$ with diagonal $(\delta_1\mu, \delta_2\mu)$; on inertia $\delta_1 = \chi_{\mathrm{cyc}}^{2m}$, $\delta_2 = 1$ and $\mu = \chi_{\mathrm{cyc}}^{-m}$ (χ' is unramified at ℓ as $\ell \nmid N'$), giving $\chi_{\mathrm{cyc}}^{m}, \chi_{\mathrm{cyc}}^{-m}$; on $D_{\bar v} \cap G_L$, $\delta_1\mu = \beta\delta_2^{-1}\mu \equiv \beta\mu \equiv \varphi = \varphi_1$ and $\delta_2\mu \equiv \mu \equiv \varphi^{-1} = \varphi_2$.

Step 3: the twist. By (Ag2) and Lemma 49.11 there is an algebraic $\kappa_K\colon G_K \to \overline{\mathbb{Q}}_\ell^\times$ with (Ik1)–(Ik3); let $\kappa = \kappa_K|_{G_L}$ ($G_L \subseteq G_K$ by (Af1)). It is algebraic (it corresponds to the composite of the Hecke character of κ_K with the norm $\mathbb{A}_L^\times \to \mathbb{A}_K^\times$). (Ka1) is (Ik1) for $s \in G_L$, as $\chi_{\mathrm{cyc}}^{1-2} = \chi_{\mathrm{cyc}}^{-1}$. (Ka2): a prime $p \nmid M$ does not divide ℓd_K (as $\ell \mid M$ and every prime dividing d_K divides M), so (Ik2) applies. (Ka3): for $p \nmid \ell d_K$, κ is trivial on inertia; for $p \mid d_K$ there is one prime of K above p, so the places $\bar v \mid p$ are G_K-conjugate and by (Ik3) κ_K has the same finite order m_p, prime to ℓ, on all $I_{\bar v} \cap G_K$; $m' = \mathrm{lcm}_{p \mid d_K} m_p$ works. So κ is an admissible twist of rank 2.

Step 4: (Sp6). $r \otimes \kappa = \rho|_{G_L} \otimes \nu$ with $\nu = \mu|_{G_L}\kappa$, which is algebraic (χ_{cyc}^{-m} corresponds to a power of the norm, χ'^{-1} has finite order, and κ is algebraic). For $s \in G_L$, $\nu(s)\nu(csc)\beta(s) = (\mu^2\beta)(s)\,\kappa(s)\kappa(csc) = \chi_{\mathrm{cyc}}(s)^{-1}$. By Step 1 and (Na2) of Lemma 49.9, $r \otimes \kappa$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions.

Hence r is a seed of rank 2 and weight m for (j, χ') with the admissible twist κ. □

Lemma 49.11. *Let ℓ be an odd prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $K \subset \overline{\mathbb{Q}}$ an imaginary quadratic field with discriminant d_K, and $c \in G_{\mathbb{Q}}$ a complex conjugation. There is a continuous character $\kappa_K\colon G_K \to \overline{\mathbb{Q}}_\ell^\times$, algebraic for ι, such that*

(Ik1) *$\kappa_K(s)\kappa_K(csc) = \chi_{\mathrm{cyc}}(s)^{-1}$ for all $s \in G_K$;*

(Ik2) *κ_K is trivial on $I_{\bar v} \cap G_K$ for every place $\bar v$ of $\overline{\mathbb{Q}}$ above a prime $p \nmid \ell d_K$;*

(Ik3) *for every place $\bar v$ above a prime $p \mid d_K$ with $p \neq \ell$, the restriction of κ_K to $I_{\bar v} \cap G_K$ has finite order prime to ℓ.*

Proof. Symbols introduced here are local to this proof. Let $\eta_K = \prod_v \eta_{K,v}$ be the quadratic idele class character of $\mathbb{Q}$ attached to K (so $\eta_{K,\infty}(-1) = -1$), fix the embedding $K \subset \mathbb{C}$ given by ι and the place above ∞, and let $\hat{\mathcal{O}}_K^\times = \prod_{\mathfrak{p}} \mathcal{O}_{K_\mathfrak{p}}^\times$; for $p \mid d_K$ write $\mathcal{O}_p^\times$ for the unit group of the completion of K at the unique prime above p.

Local characters. For each $p \mid d_K$ we choose a character $\lambda_p \colon \mathcal{O}_p^\times \to \mathbb{C}^\times$ of finite order, prime to ℓ if $p \neq \ell$, trivial on an open subgroup, with $\lambda_p|_{\mathbb{Z}_p^\times} = \eta_{K,p}|_{\mathbb{Z}_p^\times}$ and $\prod_{p \mid d_K} \lambda_p(u) = u$ for every $u \in \mathcal{O}_K^\times$ (embedded diagonally). If p is odd, let λ_p be the quadratic character of $\mathcal{O}_p^\times$ through the residue field $\mathbb{F}_p$; on $\mathbb{Z}_p^\times$ it is the Legendre symbol, which is $\eta_{K,p}|_{\mathbb{Z}_p^\times}$ (p ramified in K). If $p = 2$ and $K \neq \mathbb{Q}(i)$, let λ_2 be an extension of $\eta_{K,2}|_{\mathbb{Z}_2^\times}$ (trivial on $1 + 8\mathbb{Z}_2$) to the finite 2-group $\mathcal{O}_2^\times/(1 + 8\mathcal{O}_2)$, of 2-power order. If $K \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$, then $\mathcal{O}_K^\times = \{\pm 1\}$ and $\prod_p \lambda_p(-1) = \prod_{p<\infty} \eta_{K,p}(-1) = \eta_{K,\infty}(-1)^{-1} = -1$ by the product formula. If $K = \mathbb{Q}(i)$ ($d_K = -4$), define λ_2 on the subgroup $\mu_4\mathbb{Z}_2^\times$ by $\lambda_2(\zeta x) = \zeta\, \eta_{K,2}(x)$, consistent because $\mu_4 \cap \mathbb{Z}_2^\times = \{\pm 1\}$ and $\eta_{K,2}(-1) = -1$, trivial on $(1 + 8\mathcal{O}_2) \cap \mu_4\mathbb{Z}_2^\times$, and extend it to the 2-group $\mathcal{O}_2^\times/(1 + 8\mathcal{O}_2)$ with 2-power order. If $K = \mathbb{Q}(\sqrt{-3})$ ($d_K = -3$), define $\lambda_3(\zeta x) = \zeta\, \eta_{K,3}(x)$ on $\mu_6\mathbb{Z}_3^\times$ (consistent as $\mu_6 \cap \mathbb{Z}_3^\times = \{\pm 1\}$ and $\eta_{K,3}(-1) = -1$; trivial on $(1 + 9\mathcal{O}_3) \cap \mu_6\mathbb{Z}_3^\times$, since $\zeta_3 \notin \mathbb{Z}_3 + 9\mathcal{O}_3$) and extend it to $\mathcal{O}_3^\times/(1 + 9\mathcal{O}_3)$, of order $2 \cdot 3^b$, which is prime to ℓ if $\ell \neq 3$.

A Hecke character. Let $\lambda_f = \prod_{p \mid d_K} \lambda_p$ on $\hat{\mathcal{O}}_K^\times$ (trivial at the other primes) and $\Lambda(z, u) = z^{-1}\lambda_f(u)$ on $\mathbb{C}^\times \times \hat{\mathcal{O}}_K^\times$. It is trivial on $K^\times \cap (\mathbb{C}^\times \times \hat{\mathcal{O}}_K^\times) = \mathcal{O}_K^\times$, as $\Lambda(u, u) = u^{-1}\lambda_f(u) = 1$; so it defines a character of the open subgroup of finite index $K^\times(\mathbb{C}^\times \times \hat{\mathcal{O}}_K^\times)/K^\times$ of $\mathbb{A}_K^\times/K^\times$, which extends to a Hecke character Ψ of K ($\mathbb{C}^\times$ is divisible). It has infinity type $z \mapsto z^{-1}$, so it is algebraic. Its restriction to $\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times\mathbb{R}^\times\hat{\mathbb{Z}}^\times$ equals $\eta_K|\cdot|_\mathbb{A}^{-1}$: both are trivial on $\mathbb{Q}^\times$, equal $x \mapsto x^{-1} = \operatorname{sign}(x)|x|^{-1}$ on $\mathbb{R}^\times$, and equal $\prod_p \eta_{K,p}$ on $\hat{\mathbb{Z}}^\times$. Hence for $x \in \mathbb{A}_K^\times$, $\Psi(x)\Psi(\bar{x}) = \Psi(\mathrm{N}_{K/\mathbb{Q}}x) = \eta_K(\mathrm{N}x)|\mathrm{N}x|_\mathbb{A}^{-1} = |x|_{\mathbb{A}_K}^{-1}$.

The Galois character. By the correspondence between algebraic Hecke characters and ℓ-adic characters (through ι and the global Artin map; [220, Ch. II]), there is a continuous $\kappa \colon G_K \to \overline{\mathbb{Q}}_\ell^\times$ and a sign $e \in \{\pm 1\}$, fixed by the normalisation of the Artin map, such that $\kappa(\mathrm{Frob}_v) = \iota^{-1}\Psi(\varpi_v)^e$ for every finite place $v \nmid \ell d_K$ of K (Frob_v an arithmetic Frobenius, ϖ_v a uniformiser at v), and whose restriction to the inertia group at a place $v \nmid \ell$ corresponds to $\Psi_v^e|_{\mathcal{O}_{K_v}^\times}$ under the local Artin map. So κ is unramified at $v \nmid \ell d_K$ and, at $v \mid d_K$ with $v \nmid \ell$, has order on inertia equal to that of λ_p, which is prime to ℓ ($p \neq \ell$). Since $c\,\mathrm{Frob}_v\,c$ is a Frobenius at $\bar{v}$, $\kappa(\mathrm{Frob}_v)\kappa(c\,\mathrm{Frob}_v\,c) = \iota^{-1}\big(\Psi(\varpi_v)\Psi(\bar{\varpi}_v)\big)^e = \mathrm{N}v^e = \chi_{\mathrm{cyc}}(\mathrm{Frob}_v)^e$; by the Chebotarev density theorem the continuous characters $s \mapsto \kappa(s)\kappa(csc)$ and χ_{cyc}^e of G_K coincide. Put $\kappa_K = \kappa^{-e}$: it is algebraic (the inverse of an algebraic Hecke character is algebraic), satisfies (Ik1), and (Ik2), (Ik3) hold as for κ. □

49.2 Propagation of modularity: the entry algebra and its core primes

This subsection states and proves the propagation theorem 49.12, from the local results of Subsections 49.3 and 49.4. The geometry takes place in the entry algebra, the algebra generated over the weight-zero algebra by the matrix entries of the universal polarised ordinary lift (Definition 49.2); it is graded by the torus, its weight-zero algebra is complete Noetherian local and its graded pieces are finite over it (Lemma 49.14). Every representation whose products of weight zero lie in a complete local ring factors through it (Lemma 49.16). Its core primes are described in Proposition 49.13, the core locus is connected and contains the seed points in its charts (Lemma 49.15), and the local charts are completions of localisations, whose primes have dimensions controlled by their traces on the weight-zero algebra (Lemma 49.17, Lemma 49.18). With these, propagation from a seed component along the connected core locus reduces to

propagation inside one chart, which the next two subsections provide.

Theorem 49.12. *Let ℓ be odd and let L be an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$ such that universal polarised ordinary lifts of ranks 2 and 3 over L exist.*

1. *If an ideal $\mathcal{M}_2 \subseteq W_2^{\mathrm{po}}$ is finite over the weights, propagates through nice primes and has a seed component, then $\mathcal{M}_2$ absorbs the primes of irreducible type.*
2. *If an ideal $\mathcal{M}_2 \subseteq W_2^{\mathrm{po}}$ is finite over the weights and absorbs the primes of irreducible type, and an ideal $\mathcal{M}_3 \subseteq W_3^{\mathrm{po}}$ is finite over the weights, propagates through nice primes and along thick links and has a seed component, then $\mathcal{M}_3$ absorbs the primes of irreducible type and there is an off-triangular bound $I \subseteq W_3^{\mathrm{po}}$.*

Proof. The argument is commutative algebra on rings attached to L; it uses no statement about automorphy, over L or over K. We treat both items together. Let $n \in \{2, 3\}$, $W = W_n^{\mathrm{po}}$, $E = E_n^{\mathrm{po}}$, and $\mathcal{M} = \mathcal{M}_n$ as in the item; so $\mathcal{M}$ is finite over the weights, propagates through nice primes, has a seed component, and for $n = 3$ propagates along thick links. For $n = 3$ the hypotheses of item 2 on $\mathcal{M}_2$ and Proposition 49.33 give an ideal $I_3 \subseteq W$, finite over the weights, with $I_3 \subseteq \xi \cap W$ for every prime ξ of E of endoscopic type. Two facts are used throughout: modularity passes to larger primes, and being of irreducible type passes to smaller primes (it is defined by non-containments).

Step 1: the chart ideal. Let x be a core prime that is a maximal ideal and $(\mathcal{O}', R, \iota)$ a local chart at x. We define $J \subseteq R$ and show: (Jp0) $\dim R/P \geq n\,d_L + 2$ for every minimal prime P of J, and (Jp1)–(Jp3) of Lemma 49.34.

- If $n = 2$, or $n = 3$ and $I_E^{\mathrm{odd}} \not\subseteq x$, put $J = 0$. By (Qc4) of Proposition 49.19 there is a surjection $\phi\colon S = \mathcal{O}'[[X_1, \dots, X_N]] \to R$ with kernel $(f_1, \dots, f_h)$, $N \geq h + n\,d_L + 1$. The ring S is a complete Noetherian local domain of dimension $N + 1$ ([166, Theorem 15.4]), and $f_i \in \mathfrak{m}_S$ because $\ker\phi \subseteq \phi^{-1}(\mathfrak{m}_R) = \mathfrak{m}_S$. Since $\dim S \geq (n\,d_L{+}1){+}1{+}h$, Lemma 49.28 with $D = n\,d_L + 1$ applies to $\ker\phi$; through the bijection $P \mapsto \phi^{-1}(P)$ between primes of R and primes of S containing $\ker\phi$, under which $R/P \cong S/\phi^{-1}(P)$ and $R/(P+Q) \cong S/(\phi^{-1}(P) + \phi^{-1}(Q))$, its two items are (Jp0) and (Jp2) for the minimal primes of R. Every minimal prime P of R is of irreducible type: $I_R^- = R \not\subseteq P$ by (Qc3); if $I_R^+ \subseteq P$ then $\dim R/P \leq \dim R/I_R^+ < n\,d_L$ by (Nn1) of Proposition 49.29, against (Jp0); and if $n = 3$ and $I_R^{\mathrm{odd}} \subseteq P$, then (Qc3) gives $I_E^{\mathrm{odd}} \subseteq \iota^{-1}(P) \subseteq \iota^{-1}(\mathfrak{m}_R) = x$, against the case assumption. This is (Jp3); (Jp1) is trivial for $J = 0$.
- If $n = 3$ and $I_E^{\mathrm{odd}} \subseteq x$, let J be the ideal of Proposition 49.26. A prime of irreducible type does not contain I_R^{odd}, so (Ej1) gives (Jp1); (Ej2) gives (Jp0) and (Jp2); (Ej3) is (Jp3).

Step 2: local propagation. With x, $(\mathcal{O}', R, \iota)$ and J as in Step 1: if some $\mathcal{M}$-modular prime P_0 of R of irreducible type has $\dim R/P_0 \geq n\,d_L + 1$, then every minimal prime of R of irreducible type is $\mathcal{M}$-modular. This is Lemma 49.34, whose hypotheses on $\mathcal{M}$, I_3 and J hold by the above.

Step 3: charts and minimal primes. With x, $(\mathcal{O}', R, \iota)$ as in Step 1, R is flat over E by (Qc2), so going down holds for ι ([166, Theorem 9.5]), and $\iota^{-1}(\mathfrak{m}_R) = x$. (a) If P is a minimal prime of R, then $\iota^{-1}(P)$ is a minimal prime of E: for a prime $\eta \subseteq \iota^{-1}(P)$, going down gives $P' \subseteq P$ with $\iota^{-1}(P') = \eta$, and $P' = P$ by minimality. (b) Every minimal prime $\eta \subseteq x$ of E is $\iota^{-1}(P)$ for a minimal prime P of R: going down from $\mathfrak{m}_R$ gives P' with $\iota^{-1}(P') = \eta$; for a minimal prime $P \subseteq P'$, $\iota^{-1}(P) \subseteq \eta$ is prime, hence equal to η. (c) By (Qc3) a prime P of R is of irreducible type if and only if $\iota^{-1}(P)$ is, and by definition P is $\mathcal{M}$-modular if and only if $\iota^{-1}(P)$ is.

Step 4: three consequences. By (Eg1) of Proposition 49.13, E is Noetherian and has finitely many minimal primes. Let $\mathcal{T}$ be the set of minimal primes of E of irreducible type, $\mathcal{G} \subseteq \mathcal{T}$ those that are $\mathcal{M}$-modular, and $\mathcal{B} = \mathcal{T} \setminus \mathcal{G}$. Below, x is a core prime that is a maximal ideal, and $(\mathcal{O}', R, \iota)$ a local chart at x, which exists by (Qc1).

(i) *Gluing at x: if $\eta, \eta' \in \mathcal{T}$ lie in x and $\eta \in \mathcal{G}$, then $\eta' \in \mathcal{G}$.* By Step 3(b), (c), $\eta = \iota^{-1}(P)$ and $\eta' = \iota^{-1}(P')$ with P, P' minimal primes of R of irreducible type and P $\mathcal{M}$-modular. By (Jp1) $J \subseteq P$, and P is minimal over J, so $\dim R/P \geq n\,d_L + 2$ by (Jp0). Step 2 with $P_0 = P$ makes P' $\mathcal{M}$-modular, so $\eta' \in \mathcal{G}$ by Step 3(c).

(ii) *If $n = 2$, or $n = 3$ and $I_E^{\mathrm{odd}} \not\subseteq x$, then x contains a member of $\mathcal{T}$.* Take a minimal prime P of $R \neq 0$; by Step 1 it is of irreducible type; $\iota^{-1}(P) \subseteq x$ lies in $\mathcal{T}$ by Step 3(a), (c).

(iii) $\mathcal{G} \neq \emptyset$. Let ξ, $\mathfrak{z}$ be a seed component: ξ a prime of E of irreducible type, $\mathfrak{z}$ a prime of W with $\mathcal{M} \subseteq \mathfrak{z} \subseteq \xi \cap W$ and $\dim W/\mathfrak{z} = 1 + n\,d_L$. By (Eg5) there are a core prime x, maximal in E, and for a chart at x a prime P of R of irreducible type with $\iota^{-1}(P) \cap W = \mathfrak{z}$ and $\dim R/P = n\,d_L + 2$. Then P is $\mathcal{M}$-modular, and Step 2 makes every minimal prime of R of irreducible type $\mathcal{M}$-modular. A minimal prime $P_1 \subseteq P$ of R is of irreducible type, hence $\mathcal{M}$-modular, and $\iota^{-1}(P_1) \in \mathcal{G}$ by Step 3(a), (c).

Step 5: $\mathcal{B} = \emptyset$. Suppose not. Let $G = \bigcup_{\eta \in \mathcal{G}} V(\eta)$ and $B = \bigcup_{\eta \in \mathcal{B}} V(\eta)$, closed subsets of $\operatorname{Spec} E$ (finite unions), and let $\mathcal{C}$ be the set of core primes. (1) $\mathcal{C} \cap G \neq \emptyset$ and $\mathcal{C} \cap B \neq \emptyset$: a member η of $\mathcal{G}$ (Step 4(iii)), resp. of $\mathcal{B}$, satisfies $I_E^- \not\subseteq \eta$, so by (Eg2) it lies in a core prime, which belongs to $V(\eta)$. (2) $\mathcal{C} \subseteq G \cup B$: let $\mathcal{C}'$ be the set of maximal core primes x with $n = 2$ or $I_E^{\mathrm{odd}} \not\subseteq x$; by Step 4(ii) and since every member of $\mathcal{T}$ is in $\mathcal{G} \cup \mathcal{B}$, $\mathcal{C}' \subseteq G \cup B$, and $G \cup B$ is closed, so it suffices that $\mathcal{C}$ lies in the closure of $\mathcal{C}'$. Let y be a core prime with $n = 2$ or $I_E^{\mathrm{odd}} \not\subseteq y$, and $U \ni y$ open; by (Eg3) applied to $\operatorname{Spec} E$ and the open set U (for $n = 3$: $U \setminus V(I_E^{\mathrm{odd}})$, which contains y) there is a maximal core prime in it, a member of $\mathcal{C}' \cap U$; so y is in the closure of $\mathcal{C}'$. For $n = 2$ these y are all of $\mathcal{C}$; for $n = 3$, by (Eg4) every core prime is in the closure of the set of such y, hence in the closure of $\mathcal{C}'$. (3) By (Eg4) $\mathcal{C}$ is connected (it is non-empty by (1)); since $\mathcal{C} \cap G$ and $\mathcal{C} \cap B$ are non-empty, relatively closed and cover $\mathcal{C}$ by (2), they meet: some core prime lies in $G \cap B$. (4) By (Eg3) applied to the closed set $G \cap B$ and $U = \operatorname{Spec} E$, some core prime z that is a maximal ideal lies in $G \cap B$, i.e. $z \supseteq \eta$ with $\eta \in \mathcal{G}$ and $z \supseteq \eta'$ with $\eta' \in \mathcal{B}$. Step 4(i) gives $\eta' \in \mathcal{G}$, a contradiction.

Step 6: absorption. Let $\mathfrak{p}$ be a prime of E of irreducible type and $\eta \subseteq \mathfrak{p}$ a minimal prime. Then η is of irreducible type, so $\eta \in \mathcal{T} = \mathcal{G}$ by Step 5, and $\mathfrak{p}$ is $\mathcal{M}$-modular. For $n = 2$ this is item 1, and for $n = 3$ the first assertion of item 2.

Step 7: the off-triangular bound ($n = 3$). Put $I = \mathcal{M}_3 \cap I_3$. It is finite over the weights: if $\mathfrak{q}$ is a prime of W containing $I + \mathfrak{a}_3$, then $\mathcal{M}_3 I_3 \subseteq I \subseteq \mathfrak{q}$, so $\mathfrak{q} \supseteq \mathcal{M}_3 + \mathfrak{a}_3$ or $\mathfrak{q} \supseteq I_3 + \mathfrak{a}_3$, and in both cases $\mathfrak{q} \supseteq \mathfrak{m}_W$ because both ideals are finite over the weights; hence $\mathfrak{m}_W$ lies in the intersection of the primes containing $I + \mathfrak{a}_3$, which is $\sqrt{I + \mathfrak{a}_3}$. If $\mathfrak{p}$ is an off-triangular prime of E, it is of irreducible type, and then $\mathcal{M}_3 \subseteq \mathfrak{p} \cap W$ by Step 6, or of endoscopic type, and then $I_3 \subseteq \mathfrak{p} \cap W$; in both cases $I \subseteq \mathfrak{p} \cap W$. So I is an off-triangular bound. □

Proposition 49.13. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2, 3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $W = W_n^{\mathrm{po}}$, $E = E_n^{\mathrm{po}}$. Call a prime x of E a* core prime *if $\mathfrak{m}_W \subseteq x$ and $I_E^- \not\subseteq x$. Then:*

(Eg1) *E is Noetherian;*

(Eg2) *every minimal prime η of E with $I_E^- \not\subseteq \eta$ is contained in a core prime that is a maximal ideal of E;*

(Eg3) *if $Z \subseteq \operatorname{Spec} E$ is closed, $U \subseteq \operatorname{Spec} E$ is open and $Z \cap U$ contains a core prime, then $Z \cap U$ contains a core prime that is a maximal ideal of E;*

(Eg4) *the set of core primes, with the topology induced from $\operatorname{Spec} E$, is connected or empty; and if $n = 3$, every core prime lies in the closure of the set of core primes x with $I_E^{\mathrm{odd}} \not\subseteq x$;*

(Eg5) *if $\mathfrak{z}$ is a prime of W and ξ a prime of E of irreducible type with $\mathfrak{z} \subseteq \xi \cap W$, there is a core prime x that is a maximal ideal of E such that for every local chart $(\mathcal{O}', R_x, \iota)$ at x there is a prime P of R_x of irreducible type with $\iota^{-1}(P) \cap W = \mathfrak{z}$ and $\dim R_x/P = \dim W/\mathfrak{z} + 1$.*

Proof. The proof uses only commutative algebra; in particular, no automorphy is used. Write $\rho = \rho^{\mathrm{u}}$. Every entry $\rho(\gamma)_{ab}$ lies in $\mathrm{gr}_{w_{n,a}-w_{n,b}}$, the weights of entries lie in $\{-2,\dots,2\}$, lower entries ($a > b$) have negative weight, and $\mathrm{gr}_i\mathrm{gr}_j \subseteq \mathrm{gr}_{i+j}$ by definition.

(Eg1). E is generated over W by the entries, hence by the modules gr_j, $|j| \le 2$, each finitely generated by (Gw3); so E is a W-algebra of finite type, and W is Noetherian by (Gw1). By Hilbert's basis theorem E is Noetherian.

(Eg3). The core primes are the points of $V(\mathfrak{m}_W E) \setminus V(I_E^-)$, and $V(\mathfrak{m}_W E)$ is homeomorphic to $\operatorname{Spec} \bar{E}$, $\bar{E} = E/\mathfrak{m}_W E$, a finitely generated algebra over the field $W/\mathfrak{m}_W$ by (Eg1), hence a Jacobson ring (Hilbert's Nullstellensatz; Atiyah–Macdonald, Chapter 5, Exercises 24–26). Let Z, U be as in (Eg3) and $p \in Z \cap U$ a core prime. The set $T = Z \cap V(\mathfrak{m}_W E) \cap U \cap (\operatorname{Spec} E \setminus V(I_E^-))$ contains p and is the intersection of a closed set $Y \ni p$ of $\operatorname{Spec} \bar{E}$ with an open set $\operatorname{Spec} \bar{E} \setminus V(\mathfrak{c})$, $\mathfrak{c} \not\subseteq p$. As $\bar{E}$ is Jacobson, p is the intersection of the maximal ideals containing it; if all of them contained $\mathfrak{c}$, so would p. So some maximal ideal $\mathfrak{n} \supseteq p$ of $\bar{E}$ lies outside $V(\mathfrak{c})$, and in Y since Y is closed; its preimage in E is a maximal ideal of E and a core prime in $Z \cap U$.

(Eg2). Let η be a minimal prime of E with $I_E^- \not\subseteq \eta$. By (Gw2) E is $\mathbb{Z}$-graded by the gr_j. For a prime $\mathfrak{p}$ of a $\mathbb{Z}$-graded ring, the ideal $\mathfrak{p}^*$ generated by the homogeneous elements of $\mathfrak{p}$ is prime (Bruns–Herzog [55], Lemma 1.5.6); as $\eta^* \subseteq \eta$ and η is minimal, $\eta = \eta^*$ is homogeneous. Suppose $I_E^- \subseteq \sqrt{\eta + \mathfrak{m}_W E}$. Since E is Noetherian, $(I_E^-)^M \subseteq \eta + \mathfrak{m}_W E$ for some $M \ge 1$. Let e be a lower entry, of weight $-q$ with $q \in \{1,2\}$, and $j = -2Mq$. A product of entries of weight $j \le -2M$ has at least M lower factors, because every factor has weight ≥ -2 and only lower factors have negative weight; so $\mathrm{gr}_j \subseteq (I_E^-)^M \subseteq \eta + \mathfrak{m}_W E$. Taking components of degree j, with η homogeneous and $(\mathfrak{m}_W E) \cap \mathrm{gr}_j = \mathfrak{m}_W \mathrm{gr}_j$ (as $\mathfrak{m}_W \subseteq \mathrm{gr}_0$ and the sum (Gw2) is direct), we get $\mathrm{gr}_j = (\eta \cap \mathrm{gr}_j) + \mathfrak{m}_W \mathrm{gr}_j$. By Nakayama's lemma for the finitely generated W-module $\mathrm{gr}_j/(\eta \cap \mathrm{gr}_j)$ ((Gw3); W is local), $\mathrm{gr}_j \subseteq \eta$; so $e^{2M} \in \mathrm{gr}_j \subseteq \eta$ and $e \in \eta$. Then every lower entry lies in η, i.e. $I_E^- \subseteq \eta$, a contradiction. Hence some prime $p \supseteq \eta + \mathfrak{m}_W E$ does not contain I_E^-: a core prime in $V(\eta)$. By (Eg3) with $Z = V(\eta)$ and $U = \operatorname{Spec} E$ there is a core prime containing η that is a maximal ideal of E.

(Eg4), (Eg5) are (Ck1), (Ck2) of Lemma 49.15. □

Lemma 49.14. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2,3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $W = W_n^{\mathrm{po}}$, $E = E_n^{\mathrm{po}}$. For $j \in \mathbb{Z}$ let $\mathrm{gr}_j \subseteq E$ be the W-submodule generated by the products $\prod_{k=1}^{s} \rho^{\mathrm{u}}(\gamma_k)_{a_k b_k}$ with $s \ge 0$, $\gamma_k \in \Gamma_L^0$ and $\sum_k (w_{n,a_k} - w_{n,b_k}) = j$. Then:*

(Gw1) *W, with the maximal ideal $\mathfrak{m}_W$, is an object of $\mathrm{CNL}_{\mathcal{O}}$;*

(Gw2) *$E = \bigoplus_{j \in \mathbb{Z}} \mathrm{gr}_j$ as W-modules (the sum is direct), and $\mathrm{gr}_0 = W$;*

(Gw3) *each gr_j is a finitely generated W-module.*

Proof. The argument is commutative algebra; no automorphy is used, over L or over K. Write $R = R_n^{\mathrm{po}}$ with maximal ideal $\mathfrak{m}$, $\rho = \rho^{\mathrm{u}}$, $\psi_{v,a} = \psi_{v,a}^{\mathrm{u}}$, $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, $[\,\cdot\,]\colon \mathbb{F}^\times \to \mathcal{O}^\times$ the Teichmüller lift (multiplicative), and W_0 for the $\mathcal{O}$-subalgebra of R generated by the weight-zero monomials and the values $\psi_{v,a}(g)$; so W is the set of $y \in R$ with $y \in W_0 + \mathfrak{m}^k$ for every k.

Step 1: the torus coaction. We have $R[[T]] \in \mathrm{CNL}_{\mathcal{O}}$. Let $D = \mathrm{diag}((1+T)^{w_{n,1}}, \dots, (1+T)^{w_{n,n}}) \in \mathrm{GL}_n(R[[T]])$. Then $(D\rho D^{-1}, (\psi_{v,a}))$ is a polarised ordinary lift of rank n over L to $R[[T]]$: (Po1) holds as D is constant; (Po2) because $J_n D^{-1} J_n = D$ (as $w_{n,n+1-a} = -w_{n,a}$), so ${J_n}^t (D\rho D^{-1}) J_n = D\, {J_n}^t \rho J_n\, D^{-1}$; (Po3) because conjugation by D multiplies the (a,b) entry by $(1+T)^{w_{n,a}-w_{n,b}}$; (Po4) because the $\psi_{v,a}$ are scalars; (Po5) because $D \equiv 1$. By the universal property there is a unique morphism $\tau\colon R \to R[[T]]$ of $\mathrm{CNL}_{\mathcal{O}}$ with $\tau(\rho(\gamma)_{ab}) = (1+T)^{w_{n,a}-w_{n,b}}\rho(\gamma)_{ab}$ and $\tau(\psi_{v,a}(g)) = \psi_{v,a}(g)$. In the language of Lemma 43.8, a product of entries of total weight j has torus weight j, and every element of W_0 has torus weight 0. Every $y \in W$ has torus weight 0: for $k \geq 1$ pick $w_k \in W_0$ with $y - w_k \in \mathfrak{m}^k$; then $\tau(y) - y = \tau(y - w_k) - (y - w_k) \in \mathfrak{m}_{R[[T]]}^k$ (τ is local), so $\tau(y) = y$ by Krull's intersection theorem.

Step 2: a weighted presentation. Let S be the set of triples (γ, a, b) ($\gamma \in \Gamma_L^0$, $1 \leq a, b \leq n$) and (v, a, g) ($v \mid \ell$, $1 \leq a \leq n$, $g \in D_v$), with $x_{\gamma,a,b} = \rho(\gamma)_{ab} - \delta_{ab}[\bar{\Psi}(\gamma)^{w_{n,a}}]$ of weight $w_{n,a} - w_{n,b} \in \{-2, \dots, 2\}$ and $x_{v,a,g} = \psi_{v,a}(g) - [\bar{\Psi}(g)^{w_{n,a}}]$ of weight 0. By (Po5) all x_s lie in $\mathfrak{m}$. Condition (Tc1) of Lemma 43.8: two morphisms $R \to B$ of $\mathrm{CNL}_{\mathcal{O}}$ that agree on all x_s agree on all entries and all $\psi_{v,a}(g)$ (they fix $\mathcal{O}$), hence give the same lift, hence are equal by the uniqueness in the universal property. Condition (Tc2) is Step 1 (constants have torus weight 0). So Lemma 43.8 gives a finite $\sigma \subseteq S$ and a surjection $\pi\colon P = \mathcal{O}[[X_s : s \in \sigma]] \to R$, $X_s \mapsto x_s$, such that every $y \in R$ of torus weight j lies in $\pi(P_j)$, where $P_j = \mathcal{O}[[X_\sigma]]_j$ ((Tp3)).

Step 3: the weight pieces of P. The set P_0 is a subring and $P_i P_j \subseteq P_{i+j}$. Let $E_j = \{m \in \mathbb{N}^\sigma : \mathrm{wt}(m) = j\}$. By Dickson's lemma the set of minimal elements (for the componentwise order) of $E_0 \setminus \{0\}$ is finite, say $m_1, \dots, m_k$, and it generates the monoid E_0: a non-zero $m \in E_0$ lies above some m_i, and $m - m_i \in E_0$ has smaller total degree. Choose for every $m \in E_0$ one $\kappa(m) \in \mathbb{N}^k$ with $m = \sum_i \kappa(m)_i m_i$; κ is injective. The continuous $\mathcal{O}$-algebra homomorphism $\phi\colon \mathcal{O}[[Y_1, \dots, Y_k]] \to P$, $Y_i \mapsto X^{m_i}$, maps onto P_0: its image lies in P_0, and $f = \sum_{m \in E_0} c_m X^m \in P_0$ is the image of $\sum_m c_m Y^{\kappa(m)}$. Hence $P_0 \cong \mathcal{O}[[Y]]/\ker\phi$ is a complete Noetherian local ring, $\mathcal{O} \to P_0$ is local and induces $\mathbb{F} \cong P_0/\mathfrak{m}_{P_0}$. Likewise the minimal elements $m_1', \dots, m_r'$ of E_j are finitely many, every $m \in E_j$ lies above one of them with difference in E_0, and grouping the terms of $f \in P_j$ accordingly gives $f = \sum_i X^{m_i'} f_i$ with $f_i \in P_0$: P_j is a finitely generated P_0-module.

Step 4: $W = \pi(P_0)$, and (Gw1). Let $f \in P_0$ and f_N the sum of its terms of total degree $< N$, a polynomial in P_0. Each monomial of f_N is a product of x_s of total weight 0; expanding the x_s (only those of weight 0 carry a constant), $\pi(f_N)$ is an $\mathcal{O}$-linear combination of products of entries and values $\psi_{v,a}(g)$ in which the weights of the entries add up to 0, so $\pi(f_N) \in W_0$; and $\pi(f) - \pi(f_N) \in \pi(\mathfrak{m}_P^N) \subseteq \mathfrak{m}^N$. Hence $\pi(f) \in W$. Conversely every $y \in W$ has torus weight 0 (Step 1), so $y \in \pi(P_0)$ (Step 2). Thus $W = \pi(P_0) \cong P_0/(P_0 \cap \ker\pi)$ is a complete Noetherian local ring with maximal ideal $\pi(\mathfrak{m}_{P_0})$; this ideal lies in $\mathfrak{m} \cap W = \mathfrak{m}_W$, and an element of W outside it is a unit of W, hence not in $\mathfrak{m}$; so $\mathfrak{m}_W$ is the maximal ideal of W, $\mathcal{O} \to W$ is local and $W/\mathfrak{m}_W \cong \mathbb{F}$. This is (Gw1).

Step 5: (Gw3). A product of entries of total weight j has torus weight j (Step 1), so lies in $\pi(P_j)$ (Step 2). As $\pi(P_0)\pi(P_j) \subseteq \pi(P_j)$, $\pi(P_j)$ is a W-module, finitely generated by Step 3, and it contains gr_j. Since W is Noetherian, gr_j is finitely generated.

Step 6: (Gw2). The weight-zero monomials lie in W and $1 \in \mathrm{gr}_0$ (the empty product), so $\mathrm{gr}_0 = W$. The ring E is the W-span of the products of entries, each of which lies in some gr_j; so $E = \sum_j \mathrm{gr}_j$. Every element of gr_j has torus weight j (τ is a ring homomorphism fixing W, Step 1). Let $y_j \in \mathrm{gr}_j$, $j \in J$ finite, with $\sum_j y_j = 0$; let $j_0 = \min J$, $K = \max J - j_0$ and $z_k = y_{j_0+k}$

($z_k = 0$ if $j_0 + k \notin J$). Applying τ and multiplying by $(1+T)^{-j_0}$,

$$\sum_{k=0}^{K} (1+T)^k z_k = 0 \quad \text{in } R[[T]], \qquad \text{so} \qquad \sum_{k=i}^{K} \binom{k}{i} z_k = 0 \quad (0 \le i \le K)$$

by comparing coefficients of T^i. For $i = K$ this gives $z_K = 0$, and by descending induction on i all $z_i = 0$. So the sum is direct. □

Lemma 49.15. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2,3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $W = W_n^{\mathrm{po}}$, $E = E_n^{\mathrm{po}}$, and call a prime x of E a core prime if $\mathfrak{m}_W \subseteq x$ and $I_E^- \not\subseteq x$. Then:*

(Ck1) *the set of core primes, with the topology induced from $\operatorname{Spec} E$, is connected or empty; and if $n = 3$, every core prime lies in the closure of the set of core primes x with $I_E^{\mathrm{odd}} \not\subseteq x$;*

(Ck2) *if $\mathfrak{z}$ is a prime of W and ξ a prime of E of irreducible type with $\mathfrak{z} \subseteq \xi \cap W$, there is a core prime x that is a maximal ideal of E such that for every local chart $(\mathcal{O}', R_x, \iota)$ at x there is a prime P of R_x of irreducible type with $\iota^{-1}(P) \cap W = \mathfrak{z}$ and $\dim R_x/P = \dim W/\mathfrak{z} + 1$.*

Proof. The argument is Galois cohomology of characters of Γ_L and commutative algebra; no automorphy is used, over L or over K. (Ar2) is used in both of its halves in rank 3: the class for $\bar\Psi^{-2}\eta_K$ gives the base point of Step 4, the class for $\bar\Psi^{-1}\eta_K$ the density of Step 5; in rank 2 only the first is used. Write $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, $\rho = \rho^{\mathrm{u}}$, $\mathfrak{m}$ for the maximal ideal of R_n^{po}, $\bar c = \pi_L(c)$, $\gamma^* = \bar c \gamma^{-1} \bar c$ for $\gamma \in \Gamma_L^0$, $\gamma_\star = \pi_L(g_\star) \in \Gamma_L^0$ (Af1), and U for the $n \times n$ matrix with entry 1 at $(n,1)$ and 0 elsewhere. As $\ell \mid M$, (Af2) gives $\pi_L(D_v) \subseteq \Gamma_L^0$; $\bar\Psi(c) = 1$ gives $\bar\Psi(\gamma^*) = \bar\Psi(\gamma)^{-1}$; and $\bar\Psi(\gamma_\star)^2 \neq 1$. The ring E is a W-algebra of finite type (generated by the gr_j, $|j| \le 2$, (Gw3)) and Noetherian ((Gw1)).

Step 0: core primes contain the upper entries. Let x be a core prime and $e_- = \rho(\gamma)_{ab} \notin x$ with $a > b$, of weight $-q < 0$. For an upper entry e_+ of weight $p > 0$, $e_+^q e_-^p$ is a weight-zero monomial with off-diagonal factors, so it lies in W and in $\mathfrak{m}$ (Po5): in $\mathfrak{m}_W \subseteq x$. Hence $e_+ \in x$, and $I_E^+ \subseteq x$. So the core primes are the primes containing $\mathfrak{m}_W E + I_E^+$ and not containing I_E^-.

Step 1: points from lower data. For a commutative $\mathbb{F}$-algebra $B \neq 0$, a *lower datum* over B is a map $\varrho\colon \Gamma_L^0 \to \mathrm{M}_n(B)$ such that: (L1) ϱ is multiplicative and each $\varrho(\gamma)$ is lower triangular with diagonal $\mathrm{diag}(\bar\Psi(\gamma)^{w_{n,1}}, \dots, \bar\Psi(\gamma)^{w_{n,n}})$; (L2) $\varrho(\gamma^*)_{ab} = \varrho(\gamma)_{n+1-b,n+1-a}$; (L3) $\varrho(\gamma_\star)_{ab} = \varrho(\gamma_\star^*)_{ab}$ for $a \neq b$; (L4) $\prod_{a=1}^n (\varrho(\pi_L(g_a)) - \bar\Psi(g_a)^{w_{n,a}}) = 0$ for $v \mid \ell$, $g_1, \dots, g_n \in D_v$ (product with $a = 1$ on the left); (L5) each entry function takes its values in a finite-dimensional $\mathbb{F}$-subspace of B and is locally constant. *Claim: there is exactly one ring homomorphism $\xi_\varrho\colon E \to B$ with $\xi_\varrho(\mathfrak{m}_W) = 0$ and $\xi_\varrho(\rho(\gamma)_{ab}) = \varrho(\gamma)_{ab}$ for all γ, a, b.* Existence: apply Lemma 49.16 with this B, $C = \mathbb{F} \subseteq B$ (an object of $\mathrm{CNL}_{\mathcal{O}}$ with $\mathfrak{m}_C = 0$), ϱ (invertible by (L1)) and $\psi_{v,a} = \bar\Psi^{w_{n,a}}|_{D_v}$: (Po2)–(Po4) are (L2)–(L4) ((Po2) by $(J_n {}^t X J_n)_{ab} = X_{n+1-b,n+1-a}$); (Cq1) and (Cq5) hold by (L1) and the choice of ψ; (Cq2) holds because a product of entries of total weight 0 with an off-diagonal factor has an upper factor, which is 0; (Cq3), (Cq4) are (L5). The resulting ξ kills $\mathfrak{m}_W$ since $\beta(\mathfrak{m}_W) \subseteq \mathfrak{m}_C = 0$. Uniqueness: on W every such homomorphism is $W \to W/\mathfrak{m}_W = \mathbb{F} \subseteq B$ (an $\mathcal{O}$-algebra map from $\mathbb{F}$ is unique), and E is generated over W by the entries. As upper entries go to 0, $\ker \xi_\varrho \supseteq \mathfrak{m}_W E + I_E^+$.

The tautological datum. Let x be a core prime and $\kappa = \kappa(x)$. The image ϱ_x of ρ in $\mathrm{M}_n(\kappa)$ is a lower datum over κ: (L1) by Step 0 and (Po5) ($\rho(\gamma)_{aa} \in W$ has residue $\bar\Psi(\gamma)^{w_{n,a}}$ and $\mathfrak{m}_W \subseteq x$); (L2)–(L4) by (Po2)–(Po4), the image of $\psi_{v,a}^{\mathrm{u}}(g) \in W$ being $\bar\Psi(g)^{w_{n,a}}$; (L5): an entry of weight j lies in gr_j, whose image in κ is the image of the finite-dimensional $\mathbb{F}$-space $\mathrm{gr}_j/\mathfrak{m}_W \mathrm{gr}_j$ ((Gw3));

by Chevalley's lemma Theorem B.154 ((Gw1); the $\mathfrak{m}^k \cap \mathrm{gr}_j$ decrease to 0) some $\mathfrak{m}^k \cap \mathrm{gr}_j$ lies in $\mathfrak{m}_W \mathrm{gr}_j$, and $\gamma \mapsto \rho(\gamma) \bmod \mathfrak{m}^k$ is locally constant (Po1). By uniqueness, ξ_{ϱ_x} is the map $E \to E/x \subseteq \kappa$; its kernel is x.

Step 2: operations on lower data. Let ϱ be a lower datum over B.

(a) *Scaling.* If $B = B_0[X]$ is a polynomial ring and ϱ comes from B_0, then $\varrho^{(X)}(\gamma)_{ab} = X^{w_{n,b}-w_{n,a}}\varrho(\gamma)_{ab}$ ($a \geq b$; the exponent is ≥ 0, and > 0 for $a > b$) is a lower datum over B: in $B[X^{-1}]$ it is $\Delta\varrho\Delta^{-1}$ with $\Delta = \mathrm{diag}(X^{-w_{n,a}})$, and $J_n\Delta^{-1}J_n = \Delta$; so (L1)–(L4) follow from those of ϱ, and (L5) is clear. At $X = 1$ it is ϱ; at $X = 0$ it is the diagonal datum $\varrho_{\mathrm{d}}(\gamma) = \mathrm{diag}(\bar{\Psi}(\gamma)^{w_{n,a}})$.

(b) *Corners.* Let $\beta\colon \Gamma_L^0 \to B$ be locally constant with values in a finite-dimensional $\mathbb{F}$-space. For a lower triangular matrix Y, $UY = Y_{11}U$, $YU = Y_{nn}U$ and $UYU = Y_{1n}U = 0$. Hence: $\varrho + \beta U$ is multiplicative if and only if $\beta(\gamma\gamma') = \beta(\gamma)\bar{\Psi}(\gamma') + \bar{\Psi}(\gamma)^{-1}\beta(\gamma')$; (L2) holds for $\varrho + \beta U$ if and only if $\beta(\gamma^*) = \beta(\gamma)$; (L3) is then automatic at $(n, 1)$; and in the expansion of the product (L4) for $\varrho + \beta U$ every term with two factors βU vanishes, while the terms with one such factor, at position i, equal $\beta(g_i)\prod_{a<i}(A_a)_{nn}\prod_{a>i}(A_a)_{11}\,U$, $A_a = \varrho(\pi_L(g_a)) - \bar{\Psi}(g_a)^{w_{n,a}}$: they depend only on the diagonals. Consequently: *if $\varrho_{\mathrm{d}} + \beta U$ and ϱ are lower data, so is $\varrho + \beta U$.* Moreover $\varrho_{\mathrm{d}} + \beta U$ is a lower datum if β satisfies the two identities above and $\beta(\pi_L(g)) = \lambda_v(\bar{\Psi}(g) - \bar{\Psi}(g)^{-1})$ on D_v for some $\lambda_v \in B$: then on $\pi_L(D_v)$, $\varrho_{\mathrm{d}} + \beta U = (1 + \lambda_v U)\varrho_{\mathrm{d}}(1 - \lambda_v U)$, and (L4) follows from the diagonal case by conjugation.

Step 3: two classes from (Ar2). Let χ be $\bar{\Psi}^{-2}\eta_K$ or $\bar{\Psi}^{-1}\eta_K$, a character of Γ_L with $\chi(\bar{c}) = -1$ and $\chi|_{\Gamma_L^0} = \bar{\Psi}^{-2}$, resp. $\bar{\Psi}^{-1}$; it is non-trivial on each $\pi_L(D_v)$ (Af4). By (Ar2) there is a strict continuous cocycle $y\colon \Gamma_L \to \mathbb{F}$ for χ whose class is non-zero; replacing y by $y + (g \mapsto (\chi(g)-1)\frac{y(\bar{c})}{2})$ we may assume $y(\bar{c}) = 0$ (ℓ odd), and $y \neq 0$. Then $y(\bar{c}\gamma\bar{c}) = -y(\gamma)$ and $y(\gamma^{-1}) = -\chi(\gamma)^{-1}y(\gamma)$, so $y(\gamma^*) = \chi(\gamma)^{-1}y(\gamma)$ for $\gamma \in \Gamma_L^0$; strictness gives X_v with $y(\pi_L(g)) = (\chi(g) - 1)X_v$ on D_v; and $y|_{\Gamma_L^0} \neq 0$ (otherwise $y(\gamma\bar{c}) = y(\gamma) + \chi(\gamma)y(\bar{c}) = 0$ too).

(a) For $\chi = \bar{\Psi}^{-2}\eta_K$ put $\beta_0 = \bar{\Psi} \cdot y|_{\Gamma_L^0}$. Then $\beta_0 \neq 0$, $\beta_0(\gamma\gamma') = \beta_0(\gamma)\bar{\Psi}(\gamma') + \bar{\Psi}(\gamma)^{-1}\beta_0(\gamma')$, $\beta_0(\gamma^*) = \beta_0(\gamma)$, and $\beta_0 = -X_v(\bar{\Psi} - \bar{\Psi}^{-1})$ on $\pi_L(D_v)$. By Step 2(b), for every $\mathbb{F}$-algebra B and $s \in B$, $\varrho_{\mathrm{d}} + s\beta_0 U$ is a lower datum over B.

(b) ($n = 3$.) For $\chi = \bar{\Psi}^{-1}\eta_K$ put $u = \bar{\Psi} \cdot y|_{\Gamma_L^0}$. Then $u \neq 0$, $u(\gamma\gamma') = u(\gamma)\bar{\Psi}(\gamma') + u(\gamma')$, $u(\gamma^*) = \bar{\Psi}(\gamma)^{-1}u(\gamma)$, and $u = \alpha_v(\bar{\Psi} - 1)$ on $\pi_L(D_v)$ with $\alpha_v = -X_v$.

Step 4: (Ck1), connectedness. Let $\mathcal{C}$ be the set of core primes, and $x_0 = \ker \xi_{\varrho_{\mathrm{d}}+\beta_0 U}$ over $B = \mathbb{F}$, a maximal ideal (the image is $\mathbb{F}$); x_0 contains $\mathfrak{m}_W$ and not the lower entry $\rho(\gamma)_{n1}$ with $\beta_0(\gamma) \neq 0$: $x_0 \in \mathcal{C}$. Let $x \in \mathcal{C}$, $\kappa = \kappa(x)$, and over $B = \kappa[X]$ let $\varrho_{\mathrm{line}} = \varrho_x^{(X)} + (1 - X)\beta_0 U$, a lower datum by Steps 2(a), 2(b), 3(a). Its kernel $\eta = \ker \xi_{\varrho_{\mathrm{line}}}$ is prime ($\kappa[X]$ is a domain). Composing with $X \mapsto 1$ gives a homomorphism with the properties of Step 1 for ϱ_x, hence equal to ξ_{ϱ_x}, so $\eta \subseteq x$; composing with $X \mapsto 0$ gives $\xi_{\varrho_{\mathrm{d}}+\beta_0 U}$ followed by $\mathbb{F} \subseteq \kappa$, so $\eta \subseteq x_0$. Also $\eta \supseteq \mathfrak{m}_W$ and $\eta \not\supseteq I_E^-$ (as $\eta \subseteq x$), so $\eta \in \mathcal{C}$. In $\mathrm{Spec}\, E$ the closure of $\{\eta\}$ is $V(\eta)$, which contains x and x_0; so x and x_0 lie in the closure of $\{\eta\}$ in $\mathcal{C}$, a connected set. Hence every point of $\mathcal{C}$ is in the connected component of x_0, and $\mathcal{C}$ is connected.

Step 5: (Ck1), density for $n = 3$. With u from Step 3(b), let $\varrho_u(\gamma)$ have rows $(\bar{\Psi}(\gamma), 0, 0)$, $(u(\gamma), 1, 0)$, $(\frac{1}{2}\bar{\Psi}(\gamma)^{-1}u(\gamma)^2,\ \bar{\Psi}(\gamma)^{-1}u(\gamma),\ \bar{\Psi}(\gamma)^{-1})$. It satisfies (L1), (L2), (L4), (L5), and on $\pi_L(D_v)$ it equals $n_v\varrho_{\mathrm{d}}n_v^{-1}$ with $n_v = 1 + \alpha_v(e_{21} + e_{32}) + \frac{1}{2}\alpha_v^2 e_{31}$ (e_{ab} the matrix units), which gives (L4).

Let $N = 1 + a(e_{21} - e_{32}) - \frac{a^2}{2}e_{31} = \exp(a(e_{21} - e_{32}))$ with $a = -u(\gamma_\star)/(\bar\Psi(\gamma_\star) + 1)$ ($\bar\Psi(\gamma_\star) \neq -1$). Since $(e_{21} - e_{32})J_3 + J_3{}^t(e_{21} - e_{32}) = 0$, $NJ_3{}^tN = J_3$, so $\varrho^\sharp = N\varrho_u N^{-1}$ satisfies (L1), (L2), (L4), (L5); its $(2,1)$ entry is $b^\sharp = u + a(\bar\Psi - 1)$, and (L3) reduces by (L2) to $b^\sharp(\gamma_\star) = b^\sharp(\gamma_\star^*)$, i.e. $u(\gamma_\star)(1-\bar\Psi(\gamma_\star)^{-1}) = -a(\bar\Psi(\gamma_\star) - \bar\Psi(\gamma_\star)^{-1})$, which holds by the choice of a. So $\varrho^\sharp$ is a lower datum over $\mathbb{F}$, and $b^\sharp \neq 0$: otherwise $u = a(1 - \bar\Psi)$, and $u(\gamma^*) = \bar\Psi(\gamma)^{-1}u(\gamma)$ gives $2a(1 - \bar\Psi(\gamma)^{-1}) = 0$ for all γ, so $a = 0$ and $u = 0$. Now let $x \in \mathcal{C}$ with $I_E^{\mathrm{odd}} \subseteq x$, $\kappa = \kappa(x)$. The odd lower entries vanish in ϱ_x, so $\varrho_x = \varrho_{\mathrm{d}} + e_x U$ with e_x its $(3,1)$ entry. Over $B = \kappa[X]$, $\varrho' = (\varrho^\sharp)^{(X)} + e_x U$ is a lower datum (Steps 2(a), 2(b)); its kernel η' is prime, $\eta' \subseteq x$ (at $X = 0$ it is ϱ_x), so $\eta' \in \mathcal{C}$, and $\eta' \not\supseteq I_E^{\mathrm{odd}}$ because $\xi_{\varrho'}(\rho(\gamma)_{21}) = Xb^\sharp(\gamma) \neq 0$ for some γ. As $x \in V(\eta')$, x lies in the closure of $\{\eta'\} \subseteq \{y \in \mathcal{C} : I_E^{\mathrm{odd}} \not\subseteq y\}$.

Step 6: (Ck2). Let $\mathfrak{z}$, ξ be as in (Ck2). (i) *A homogeneous prime over* $\mathfrak{z}$. By (Gw2), the degree-zero component of $\sum_i c_i y_i$ ($c_i \in \mathfrak{z}$, $y_i \in E$) is $\sum_i c_i(y_i)_0 \in \mathfrak{z}$; so $\mathfrak{z}E \cap W = \mathfrak{z}$ and $\mathfrak{z}E$ misses $W \setminus \mathfrak{z}$. An ideal maximal among those containing $\mathfrak{z}E$ and missing $W \setminus \mathfrak{z}$ is a prime Q with $Q \cap W = \mathfrak{z}$, and the ideal η generated by the homogeneous elements of Q is a prime (Bruns–Herzog [55], Lemma 1.5.6) with $\eta \cap W = \mathfrak{z}$ (it contains $\mathfrak{z}$ and lies in Q). (ii) η *is of irreducible type.* Let e_+ (weight $p > 0$), e_- (weight $-q < 0$) be an upper and a lower entry not in ξ. Then $e_+^q e_-^p \in W \setminus \xi \subseteq W \setminus \mathfrak{z}$, so $e_+^q e_-^p \notin \eta$ and $e_\pm \notin \eta$. If $n = 3$ and $o \notin \xi$ is an entry of odd weight j, then $o^p e_+^{-j}$ ($j < 0$) or $o^q e_-^j$ ($j > 0$) lies in $W \setminus \xi$, so $o \notin \eta$. (iii) $\mathrm{trdeg}_{\kappa(\mathfrak{z})}\,\kappa(\eta) = 1$. Every entry e of weight j satisfies $e \in W$, $e^q e_-^j \in W$ or $e^p e_+^{-j} \in W$ according as $j = 0$, $j > 0$, $j < 0$, and $e_+^q e_-^p \in W$; so the images of all entries in $\kappa(\eta)$, which generate it over $\kappa(\mathfrak{z})$, are algebraic over $\kappa(\mathfrak{z})(\bar e_-)$: the degree is ≤ 1. The domain E/η is graded with degree-zero part $W/\mathfrak{z}$; in a relation $\sum_i c_i \bar e_-^i = 0$, $c_i \in W/\mathfrak{z}$, the terms have distinct degrees $-iq$, so each vanishes and all $c_i = 0$: $\bar e_-$ is transcendental. (iv) *A maximal core prime above* η. Suppose $I_E^- \subseteq \sqrt{\eta + \mathfrak{m}_W E}$; then $(I_E^-)^M \subseteq \eta + \mathfrak{m}_W E$ for some M. For $j \leq -2M$, a product of entries of weight j has at least M lower factors (all weights are ≥ -2, only lower entries are negative), so $\mathrm{gr}_j \subseteq (\eta \cap \mathrm{gr}_j) + \mathfrak{m}_W \mathrm{gr}_j$ (degree-j components; η and $\mathfrak{m}_W E$ are homogeneous), and $\mathrm{gr}_j \subseteq \eta$ by Nakayama's lemma ((Gw3), W local). Then $e_-^{2M} \in \mathrm{gr}_{-2Mq} \subseteq \eta$, against (ii). So a prime $p \supseteq \eta + \mathfrak{m}_W E$ with $I_E^- \not\subseteq p$ exists. The ring $E/\mathfrak{m}_W E$ is of finite type over the field $\mathbb{F}$, hence Jacobson (Atiyah–Macdonald, Chapter 5, Exercises 24–26), so p is the intersection of the maximal ideals containing it, and one of them, x, does not contain I_E^-: a core prime that is a maximal ideal, with $\eta \subseteq x$, $x \cap W = \mathfrak{m}_W$, and E/x finite over $\mathbb{F}$ (Zariski's lemma). (v) *The prime of the chart.* Let $(\mathcal{O}', R, \iota)$ be a local chart at x, and E', $\mathfrak{x}'$ as in (Cx1), so $R = \widehat{A}$ with $A = E'_{\mathfrak{x}'}$. Let $h = \mathrm{ht}(x/\eta)$ and $\eta = \eta_0 \subsetneq \cdots \subsetneq \eta_h = x$ a chain of primes. The ring E' is free, hence flat, over E, so going down holds [166, Theorem 9.5]: there are primes $\mathfrak{x}' = \eta'_h \supsetneq \cdots \supsetneq \eta'_0$ of E' lying over the η_i; so $\mathrm{ht}(\mathfrak{x}'/\eta'_0) \geq h$. Put $\mathfrak{p} = \eta'_0 A$. Then $\dim R/\mathfrak{p}R = \dim A/\mathfrak{p} \geq h$ ($R/\mathfrak{p}R$ is the completion of $A/\mathfrak{p}$), and there is a minimal prime P of $\mathfrak{p}R$ with $\dim R/P = \dim R/\mathfrak{p}R$. As $R/\mathfrak{p}R$ is flat over $A/\mathfrak{p}$, going down shows $P \cap A = \mathfrak{p}$ (a strictly smaller contraction would give a prime strictly between $\mathfrak{p}R$ and P), so $\iota^{-1}(P) = \eta'_0 \cap E = \eta$ and $\iota^{-1}(P) \cap W = \mathfrak{z}$. The prime P is of irreducible type because I_R^-, I_R^+, I_R^{odd} are generated by the images under ι of the generators of I_E^-, I_E^+, I_E^{odd}, none of which is contained in η by (ii). Finally, $W/\mathfrak{z}$ is a quotient of the complete local ring W, hence universally catenary, and E/η is a domain of finite type over it; items 2 and 3 of Theorem B.155 for the prime x/η, which contracts to $\mathfrak{m}_W/\mathfrak{z}$, with $\mathrm{trdeg}_{\mathbb{F}}\, E/x = 0$ and (iii), give $h = \dim W/\mathfrak{z} + 1$. So $\dim R/P \geq \dim W/\mathfrak{z} + 1$, and $\dim R/P \leq \dim W/\mathfrak{z} + 1$ by Lemma 49.18 and (iii). □

Lemma 49.16. *Let L be a CM field satisfying (Af1) and (Af2) for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2, 3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $W = W_n^{\mathrm{po}}$, $E = E_n^{\mathrm{po}}$. Let B be a commutative $\mathcal{O}$-algebra and $C \subseteq B$ an $\mathcal{O}$-subalgebra which, with a maximal ideal $\mathfrak{m}_C$, is an object of* $\mathrm{CNL}_{\mathcal{O}}$. *Let $\rho\colon \Gamma_L^0 \to \mathrm{GL}_n(B)$ and $\psi_{v,a}\colon D_v \to B^\times$ ($v \mid \ell$, $1 \leq a \leq n$)*

be homomorphisms satisfying (Po2)–(Po4) with the test ring replaced by B, and suppose:

(Cq1) *$\rho(\gamma)_{aa} \in C$ and $\rho(\gamma)_{aa} \equiv \bar{\Psi}(\gamma)^{w_{n,a}} \pmod{\mathfrak{m}_C}$ for all $\gamma \in \Gamma_L^0$ and $1 \le a \le n$;*

(Cq2) *every product $\prod_{k=1}^{s} \rho(\gamma_k)_{a_k b_k}$ with $\sum_k (w_{n,a_k} - w_{n,b_k}) = 0$ and $a_k \neq b_k$ for some k lies in $\mathfrak{m}_C$;*

(Cq3) *for $1 \le a, b \le n$ the C-submodule $M_{ab} \subseteq B$ generated by the $\rho(\gamma)_{ab}$, $\gamma \in \Gamma_L^0$, is finitely generated;*

(Cq4) *for $1 \le a, b \le n$ and $k \ge 1$ the map $\Gamma_L^0 \to M_{ab}/\mathfrak{m}_C^k M_{ab}$, $\gamma \mapsto \rho(\gamma)_{ab} + \mathfrak{m}_C^k M_{ab}$, is locally constant;*

(Cq5) *$\psi_{v,a}(g) \in C$ and $\psi_{v,a}(g) \equiv \bar{\Psi}(g)^{w_{n,a}} \pmod{\mathfrak{m}_C}$ for $g \in D_v$, and $g \mapsto \psi_{v,a}(g) \bmod \mathfrak{m}_C^k$ is locally constant on D_v for every $k \ge 1$.*

Then there are $\mathcal{O}$-algebra homomorphisms $\xi\colon E \to B$ and $\beta\colon W \to C$ with $\beta(\mathfrak{m}_W) \subseteq \mathfrak{m}_C$, $\xi(\rho^{\mathrm{u}}(\gamma)_{ab}) = \rho(\gamma)_{ab}$, $\xi(\psi^{\mathrm{u}}_{v,a}(g)) = \psi_{v,a}(g)$ and $\xi(y) = \beta(y)$ for all $\gamma \in \Gamma_L^0$, a, b, v, $g \in D_v$ and $y \in W$.

Proof. The idea is to undo the torus: conjugating ρ by $\mathrm{diag}(T^{w_{n,1}}, \dots, T^{w_{n,n}})$ over a Laurent polynomial ring gives a representation which is residually diagonal at a suitable maximal ideal, so the universal property of R_n^{po} applies at every finite level; evaluating at $T = 1$ returns ρ.

Step 1: the Laurent model. Let $B[T, T^{-1}]$ be the Laurent polynomial ring, $\tau = \mathrm{diag}(T^{w_{n,1}}, \dots, T^{w_{n,n}})$ and $\rho_T(\gamma) = \tau\rho(\gamma)\tau^{-1}$, so $\rho_T(\gamma)_{ab} = T^{w_{n,a}-w_{n,b}}\rho(\gamma)_{ab}$. For $j \in \mathbb{Z}$ let $P_j \subseteq B$ be the C-submodule generated by the products of entries of ρ of weight j (the empty product included in P_0). Let $B^\natural \subseteq B[T, T^{-1}]$ be the C-subalgebra generated by the entries of all $\rho_T(\gamma)$. A product of such entries is T^j times a product of entries of ρ of weight j, so

$$B^\natural = \bigoplus_{j \in \mathbb{Z}} T^j P_j, \qquad P_i P_j \subseteq P_{i+j}.$$

(a) $P_0 = C$: a product of weight 0 either has an off-diagonal factor, and lies in $\mathfrak{m}_C$ by (Cq2), or consists of diagonal entries, and lies in C by (Cq1); and $1 \in P_0$. (b) For $j \neq 0$, $P_j P_{-j} \subseteq \mathfrak{m}_C$: a product of weight $j \neq 0$ has an off-diagonal factor (diagonal entries have weight 0), so the product of a generator of P_j and one of P_{-j} has weight 0 and an off-diagonal factor, and lies in $\mathfrak{m}_C$ by (Cq2). (c) Put $\mathfrak{n} = \mathfrak{m}_C \oplus \bigoplus_{j \neq 0} T^j P_j$. By (a), (b) the degree-0 component of $(\sum_i T^i x_i)(\sum_j T^j y_j)$ with $y_0 \in \mathfrak{m}_C$ is $x_0 y_0 + \sum_{j \neq 0} x_{-j} y_j \in \mathfrak{m}_C$, so $\mathfrak{n}$ is an ideal; $B^\natural = C \oplus \bigoplus_{j \neq 0} T^j P_j$, so $B^\natural/\mathfrak{n} \cong C/\mathfrak{m}_C = \mathbb{F}_{\mathcal{O}}$, $\mathfrak{n}$ is maximal, and $\mathfrak{n} \cap C = \mathfrak{m}_C$. (d) $B^\natural$ is Noetherian: it is generated as a C-algebra by the elements $T^{w_{n,a}-w_{n,b}} m$, m in a finite generating set of M_{ab} (Cq3), and C is Noetherian (Hilbert's basis theorem). (e) $\bigcap_k \mathfrak{n}^k = 0$. The ideal $\mathfrak{n}$ is generated by elements homogeneous in T, hence so is $\mathfrak{n}^k$, and the homogeneous components of an element of $\mathfrak{n}^k$ lie in $\mathfrak{n}^k$. Let $y \in \bigcap_k \mathfrak{n}^k$ and $T^j y_j$ a component of y; it lies in $\bigcap_k \mathfrak{n}^k$, so by Krull's intersection theorem in the Noetherian ring $B^\natural$ (Atiyah–Macdonald, Theorem 10.17) there is $r \in \mathfrak{n}$ with $(1 - r)T^j y_j = 0$. The degree-j component of this is $(1 - r_0)y_j = 0$, with $r_0 \in \mathfrak{m}_C$ the degree-0 component of r; $1 - r_0$ is a unit of C, hence of B, so $y_j = 0$. Thus $y = 0$.

Step 2: lifts at finite levels. For $N \ge 0$ let $B_N = B^\natural/\mathfrak{n}^{N+1}$. It is a Noetherian local ring with nilpotent maximal ideal $\mathfrak{n}/\mathfrak{n}^{N+1}$ and residue field $\mathbb{F}_{\mathcal{O}}$, hence complete, and $\mathcal{O} \to C \to B_N$ is local; so $B_N \in \mathrm{CNL}_{\mathcal{O}}$. Let ρ_N, $\psi_{N,v,a}$ be the reductions of ρ_T and of $\psi_{v,a}$ (whose values lie in $C \subseteq B^\natural$ by (Cq5)) modulo $\mathfrak{n}^{N+1}$. They form a polarised ordinary lift of rank n over L to B_N:

- (Po2): $w_{n,n+1-a} = -w_{n,a}$ gives $J_n \tau J_n = \tau^{-1}$, so $\rho_T(\pi_L(cg^{-1}c)) = \tau J_n {}^t\rho(\pi_L(g)) J_n \tau^{-1} = J_n \tau^{-1} {}^t\rho(\pi_L(g))\, \tau J_n = J_n {}^t\rho_T(\pi_L(g))\, J_n$;

- (Po3), (Po4): the off-diagonal entries of $\rho_T(\pi_L(g_\star)) - \rho_T(\pi_L(cg_\star^{-1}c))$ are those of ρ times powers of T, and $\prod_a(\rho_T(\pi_L(g_a)) - \psi_{v,a}(g_a)) = \tau(\prod_a(\rho(\pi_L(g_a)) - \psi_{v,a}(g_a)))\tau^{-1} = 0$;
- (Po5): an off-diagonal entry $T^{w_{n,a}-w_{n,b}}\rho(\gamma)_{ab}$ has degree $w_{n,a} - w_{n,b} \neq 0$ (the weights are distinct), so lies in $\mathfrak{n}$; the diagonal entries and the $\psi_{v,a}(g)$ lie in C with the residues required, by (Cq1), (Cq5);
- (Po1): by (Cq4), for γ' near γ the difference $\rho(\gamma)_{ab} - \rho(\gamma')_{ab}$ lies in $\mathfrak{m}_C^k M_{ab}$, so $\rho_T(\gamma)_{ab} - \rho_T(\gamma')_{ab} \in \mathfrak{m}_C^k T^{w_{n,a}-w_{n,b}} M_{ab} \subseteq \mathfrak{n}^k$, because $T^{w_{n,a}-w_{n,b}} M_{ab} \subseteq B^\natural$ and $\mathfrak{m}_C \subseteq \mathfrak{n}$; likewise for $\psi_{v,a}$ by (Cq5).

By the universal property of R_n^{po} there is exactly one morphism $f_N\colon R_n^{\mathrm{po}} \to B_N$ of $\mathrm{CNL}_{\mathcal{O}}$ with $f_N \circ \rho^{\mathrm{u}} = \rho_N$ and $f_N \circ \psi_{v,a}^{\mathrm{u}} = \psi_{N,v,a}$. Composing f_{N+1} with $B_{N+1} \to B_N$ gives a morphism with the same property, so by uniqueness it is f_N. Since f_N is local, $f_N(\mathfrak{m}^{N+1}) = 0$, where $\mathfrak{m}$ is the maximal ideal of R_n^{po}.

Step 3: the weight-zero algebra goes to C. Let $W_0 \subseteq R_n^{\mathrm{po}}$ be the $\mathcal{O}$-subalgebra generated by the weight-zero products of entries of ρ^{u} and the $\psi_{v,a}^{\mathrm{u}}(g)$. Then $f_N(W_0)$ lies in the image of C in B_N: a weight-zero product goes to the class of T^0 times the corresponding product of entries of ρ, an element of $P_0 = C$ by Step 1 (a), and $\psi_{v,a}^{\mathrm{u}}(g)$ goes to the class of $\psi_{v,a}(g) \in C$. Put $\mathfrak{c}_N = \mathfrak{n}^{N+1} \cap C$, a decreasing sequence of ideals of C with $\bigcap_N \mathfrak{c}_N = 0$ by Step 1 (e). Let $y \in W$. For each N choose $y_N \in W_0$ with $y - y_N \in \mathfrak{m}^{N+1}$ (definition of W) and $c_N \in C$ with $f_N(y_N) = c_N + \mathfrak{n}^{N+1}$. Then $f_N(y) = f_N(y_N) = c_N + \mathfrak{n}^{N+1}$, and by the compatibility of Step 2, $c_M - c_N \in \mathfrak{c}_N$ for $M \geq N$. By Chevalley's lemma (Theorem B.154, applied to the module C over the complete Noetherian local ring C), for every k there is t with $\mathfrak{c}_t \subseteq \mathfrak{m}_C^k$; so (c_N) is a Cauchy sequence for the $\mathfrak{m}_C$-adic topology and converges to some $c \in C$. For fixed N and every k, $c - c_N = (c - c_M) + (c_M - c_N) \in \mathfrak{m}_C^k + \mathfrak{c}_N$ for M large, and $\bigcap_k(\mathfrak{m}_C^k + \mathfrak{c}_N) = \mathfrak{c}_N$ by Krull's intersection theorem in the Noetherian local ring $C/\mathfrak{c}_N$; so $f_N(y) = c + \mathfrak{n}^{N+1}$ for all N. Such a c is unique, since two of them differ by an element of $\bigcap_N \mathfrak{c}_N = 0$. Put $\beta(y) = c$. Uniqueness and the fact that the f_N are $\mathcal{O}$-algebra homomorphisms make β an $\mathcal{O}$-algebra homomorphism; if $y \in \mathfrak{m}_W$, then $f_0(y) = 0$, so $\beta(y) \in \mathfrak{n} \cap C = \mathfrak{m}_C$.

Step 4: the entry algebra goes to $B^\natural$. Let E' be the set of $y \in R_n^{\mathrm{po}}$ for which there is $z \in B^\natural$ with $f_N(y) = z + \mathfrak{n}^{N+1}$ for all N; such a z is unique by Step 1 (e). The set E' is an $\mathcal{O}$-subalgebra, it contains W (with $z = \beta(y)$, Step 3) and every entry $\rho^{\mathrm{u}}(\gamma)_{ab}$ (with $z = \rho_T(\gamma)_{ab}$), hence contains E, which is generated over W by these entries. The map $\xi_T\colon E \to B^\natural$, $y \mapsto z$, is an $\mathcal{O}$-algebra homomorphism with $\xi_T(\rho^{\mathrm{u}}(\gamma)_{ab}) = T^{w_{n,a}-w_{n,b}}\rho(\gamma)_{ab}$, $\xi_T(\psi_{v,a}^{\mathrm{u}}(g)) = \psi_{v,a}(g)$ and $\xi_T(y) = \beta(y)$ for $y \in W$.

Step 5: evaluation at $T = 1$. Let $\mathrm{ev}\colon B[T, T^{-1}] \to B$ be the B-algebra homomorphism with $T \mapsto 1$; it is the identity on C. Then $\xi = \mathrm{ev} \circ \xi_T$ and β have the stated properties. □

Lemma 49.17. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2, 3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $W = W_n^{\mathrm{po}}$, $E = E_n^{\mathrm{po}}$. Let $\mathfrak{x}$ be a prime of E.*

(Cx1) *Let $(\mathcal{O}', R, \iota)$ be a local chart at $\mathfrak{x}$. Put $E' = \mathcal{O}' \otimes_{\mathcal{O}} E$, let $\theta\colon E' \to \mathbb{F}_{\mathcal{O}'}$ be the $\mathcal{O}'$-algebra homomorphism $a \otimes e \mapsto \bar{a} \cdot (\iota(e) \bmod \mathfrak{m}_R)$, $\mathfrak{x}' = \ker\theta$, and $\iota'\colon E' \to R$ the $\mathcal{O}'$-algebra homomorphism $a \otimes e \mapsto a\,\iota(e)$. Then E' is Noetherian, $\mathfrak{x}'$ is a maximal ideal of E' with $\{e \in E : 1 \otimes e \in \mathfrak{x}'\} = \mathfrak{x}$, and ι' induces an isomorphism from the $\mathfrak{x}'$-adic completion of the local ring $E'_{\mathfrak{x}'}$ onto R.*

(Cx2) *For every local chart $(\mathcal{O}', R, \iota)$ at $\mathfrak{x}$, R is a flat E-module through ι.*

(Cx3) *If $\mathfrak{x}$ is a maximal ideal of E with $\mathfrak{m}_W \subseteq \mathfrak{x}$, there is a local chart at $\mathfrak{x}$.*

Proof. The argument is commutative algebra; no automorphy is used, over L or over K.

Step 1: E is Noetherian and of finite type over W. Every entry $\rho^{\mathrm{u}}(\gamma)_{ab}$ lies in $\mathrm{gr}_{w_{n,a}-w_{n,b}}$ with $|w_{n,a}-w_{n,b}| \le 2$, so E is generated as a W-algebra by the modules gr_j, $|j| \le 2$, each finitely generated by (Gw3); W is Noetherian by (Gw1). Hence E is a W-algebra of finite type and Noetherian (Hilbert's basis theorem). For a coefficient ring $\mathcal{O}' \supseteq \mathcal{O}$ finite over $\mathcal{O}$, $\mathcal{O}'$ is a finitely generated torsion-free, hence free, module over the discrete valuation ring $\mathcal{O}$; so $E' = \mathcal{O}' \otimes_{\mathcal{O}} E$ is a free E-module of finite rank through $e \mapsto 1 \otimes e$, finite over E, and Noetherian.

Step 2: a universal property of completions. Let $\mathcal{O}'$ be a coefficient ring containing $\mathcal{O}$, finite over $\mathcal{O}$ with $\mathfrak{m}_{\mathcal{O}}\mathcal{O}' = \mathfrak{m}_{\mathcal{O}'}$, let $\theta\colon E' \to \mathbb{F}_{\mathcal{O}'}$ be a surjective $\mathcal{O}'$-algebra homomorphism, $\mathfrak{x}' = \ker\theta$ (a maximal ideal), $A = E'_{\mathfrak{x}'}$, $\widehat{A}$ its $\mathfrak{x}'$-adic completion and $j\colon E \to \widehat{A}$ the composite $E \to E' \to A \to \widehat{A}$. Then $\widehat{A}$ is a complete Noetherian local ring with maximal ideal $\mathfrak{x}'\widehat{A}$ and residue field $E'/\mathfrak{x}' \cong \mathbb{F}_{\mathcal{O}'}$ through θ; as θ restricts to the residue map on $\mathcal{O}'$, $\widehat{A} \in \mathrm{CNL}_{\mathcal{O}'}$, and $j^{-1}(\mathfrak{m}_{\widehat{A}}) = \{e : \theta(1\otimes e) = 0\}$. We claim: for every $C \in \mathrm{CNL}_{\mathcal{O}'}$ and every $\mathcal{O}$-algebra homomorphism $f\colon E \to C$ with $(f(e) \bmod \mathfrak{m}_C) = \theta(1\otimes e)$ for all e, there is exactly one $\mathcal{O}'$-algebra homomorphism $h\colon \widehat{A} \to C$ with $h \circ j = f$. Existence: $f'(a\otimes e) = af(e)$ defines an $\mathcal{O}'$-algebra homomorphism $E' \to C$ whose reduction modulo $\mathfrak{m}_C$ is θ (both are $\mathcal{O}'$-linear and agree on $1\otimes E$); so $f'^{-1}(\mathfrak{m}_C) = \mathfrak{x}'$, f' maps $E' \setminus \mathfrak{x}'$ to units and extends to a local homomorphism $A \to C$; it maps $\mathfrak{x}'^k A$ into $\mathfrak{m}_C^k$, so it extends by continuity to $\widehat{A} \to C$, C being complete and separated. Uniqueness: let h be such a homomorphism. The kernel of $\widehat{A} \to C \to \mathbb{F}_{\mathcal{O}'}$ is a maximal ideal of the local ring $\widehat{A}$, so h is local and $h(\mathfrak{m}_{\widehat{A}}^k) \subseteq \mathfrak{m}_C^k$; thus h is continuous. On the image of E' it equals f' ($\mathcal{O}'$-linearity), hence on A (units), hence on $\widehat{A}$ by density.

Step 3: (Cx1). Let $(\mathcal{O}', R, \iota)$ be a local chart at $\mathfrak{x}$ and θ, $\mathfrak{x}'$, ι' as in (Cx1). The map θ is surjective since $\theta(a\otimes 1) = \bar{a}$, so $\mathfrak{x}'$ is maximal, and $1\otimes e \in \mathfrak{x}'$ if and only if $\iota(e) \in \mathfrak{m}_R$, i.e. $e \in \mathfrak{x}$. Apply Step 2 to this θ. The condition on f in Step 2 is exactly the condition in the definition of a local chart at $\mathfrak{x}$ for $(\mathcal{O}', R, \iota)$, because $\iota(e) \bmod \mathfrak{m}_R = \theta(1\otimes e)$. Taking $f = j$ in the property of R gives $h_1\colon R \to \widehat{A}$ with $h_1 \circ \iota = j$; taking $f = \iota$ in the property of $\widehat{A}$ gives $h_2\colon \widehat{A} \to R$ with $h_2 \circ j = \iota$. Then $h_2h_1 \circ \iota = \iota = \mathrm{id}_R \circ \iota$, so $h_2h_1 = \mathrm{id}_R$ by the uniqueness in the definition of a chart (with $C = R$, $f = \iota$), and $h_1h_2 = \mathrm{id}_{\widehat{A}}$ by the uniqueness of Step 2. Finally h_2 restricted to the image of E' is ι', as both are $\mathcal{O}'$-linear and agree on $1\otimes E$. So ι' maps $E' \setminus \mathfrak{x}'$ into $R^\times$ and induces the isomorphism h_2.

Step 4: (Cx2). By (Cx1), ι factors as $E \to E' \to E'_{\mathfrak{x}'} \to \widehat{E'_{\mathfrak{x}'}} \cong R$: a free extension (Step 1), a localisation, and the completion of a Noetherian local ring; free modules and localisations are flat, and the completion of a Noetherian local ring is flat over it [166, Theorem 8.8]; so R is flat over E.

Step 5: (Cx3). Let $\mathfrak{x}$ be maximal with $\mathfrak{m}_W \subseteq \mathfrak{x}$. The field $E/\mathfrak{x}$ is finitely generated as an algebra over $W/\mathfrak{m}_W = \mathbb{F}_{\mathcal{O}}$ (Step 1), hence a finite extension of degree f of $\mathbb{F}_{\mathcal{O}}$ (Zariski's lemma). Let $\mathcal{O}' \subset \overline{\mathbb{Q}}_\ell$ be the unramified extension of $\mathcal{O}$ of degree f, a coefficient ring finite over $\mathcal{O}$ with $\mathfrak{m}_{\mathcal{O}}\mathcal{O}' = \mathfrak{m}_{\mathcal{O}'}$ and residue field of degree f over $\mathbb{F}_{\mathcal{O}}$, and fix an $\mathbb{F}_{\mathcal{O}}$-isomorphism $\lambda\colon E/\mathfrak{x} \to \mathbb{F}_{\mathcal{O}'}$. Then $\theta(a\otimes e) = \bar{a}\,\lambda(e \bmod \mathfrak{x})$ is a surjective $\mathcal{O}'$-algebra homomorphism $E' \to \mathbb{F}_{\mathcal{O}'}$, and with $\widehat{A}$, j as in Step 2, $j^{-1}(\mathfrak{m}_{\widehat{A}}) = \mathfrak{x}$ and the composite of j with $\widehat{A} \to \mathbb{F}_{\mathcal{O}'}$ is $e \mapsto \lambda(e \bmod \mathfrak{x})$. By Step 2, $(\mathcal{O}', \widehat{A}, j)$ is a local chart at $\mathfrak{x}$. □

Lemma 49.18. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2,3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $W = W_n^{\mathrm{po}}$, $E = E_n^{\mathrm{po}}$. Let $\mathfrak{x}$ be a prime of E and $(\mathcal{O}', R, \iota)$ a local chart at $\mathfrak{x}$. For every prime P of R, with $\eta = \iota^{-1}(P)$*

and $\mathfrak{q} = \eta \cap W$,

$$\dim R/P \;\leq\; \dim W/\mathfrak{q} + \operatorname{trdeg}_{\kappa(\mathfrak{q})} \kappa(\eta).$$

Proof. The argument is commutative algebra; no automorphy is used, over L or over K. Let P, $\eta = \iota^{-1}(P)$ and $\mathfrak{q} = \eta \cap W$ be as in the statement.

Step 1: passage to a localisation of a finite extension of E. Let $E' = \mathcal{O}' \otimes_{\mathcal{O}} E$, $\mathfrak{x}'$ and ι' be as in (Cx1); so E' is Noetherian, $\mathfrak{x}'$ is maximal, $A = E'_{\mathfrak{x}'}$ is a Noetherian local ring, and ι' induces an isomorphism $\widehat{A} \cong R$ of the $\mathfrak{x}'$-adic completion of A onto R, through which we identify the two. Put $\mathfrak{p}_A = P \cap A$ and $\mathfrak{p}' = \iota'^{-1}(P)$, primes of A and E' with $\mathfrak{p}' \subseteq \iota'^{-1}(\mathfrak{m}_R) = \mathfrak{x}'$ and $\mathfrak{p}_A = \mathfrak{p}'A$; and $\{e \in E : 1 \otimes e \in \mathfrak{p}'\} = \iota^{-1}(P) = \eta$, because $\iota(e) = \iota'(1 \otimes e)$. Since $P \supseteq \mathfrak{p}_A R$, and $R/\mathfrak{p}_A R \cong \widehat{A} \otimes_A A/\mathfrak{p}_A$ is the completion of the Noetherian local ring $A/\mathfrak{p}_A$, which has the same dimension (Atiyah–Macdonald, Proposition 10.13 and Corollary 11.19),

$$\dim R/P \leq \dim R/\mathfrak{p}_A R = \dim A/\mathfrak{p}_A = \operatorname{ht}(\mathfrak{x}'/\mathfrak{p}'),$$

the height of $\mathfrak{x}'/\mathfrak{p}'$ in $E'/\mathfrak{p}'$ (the last equality because $A/\mathfrak{p}_A$ is the localisation of $E'/\mathfrak{p}'$ at $\mathfrak{x}'/\mathfrak{p}'$).

Step 2: incomparability. The ring E' is a free E-module of finite rank through $e \mapsto 1 \otimes e$ ($\mathcal{O}'$ is finitely generated and torsion-free over the discrete valuation ring $\mathcal{O}$), so $E/\eta \to E'/\mathfrak{p}'$ is injective (Step 1) and integral. A strictly increasing chain of primes of $E'/\mathfrak{p}'$ ending at $\mathfrak{x}'/\mathfrak{p}'$ contracts to a strictly increasing chain of primes of E/η (incomparability, Atiyah–Macdonald, Corollary 5.9) ending at $\mathfrak{x}/\eta$, since $\{e : 1 \otimes e \in \mathfrak{x}'\} = \mathfrak{x}$ by (Cx1). Hence $\operatorname{ht}(\mathfrak{x}'/\mathfrak{p}') \leq \operatorname{ht}(\mathfrak{x}/\eta)$.

Step 3: the dimension inequality. The ring E is generated as a W-algebra by the finitely generated W-modules gr_j, $|j| \leq 2$ ((Gw3); every entry has weight in $\{-2, \dots, 2\}$), so E/η is a domain finitely generated as an algebra over its subring $W/\mathfrak{q}$, which is a Noetherian domain by (Gw1). The fraction fields of $W/\mathfrak{q}$ and E/η are $\kappa(\mathfrak{q})$ and $\kappa(\eta)$. Item 1 of Theorem B.155 for the prime $\mathfrak{x}/\eta$ of E/η, whose contraction to $W/\mathfrak{q}$ is $(\mathfrak{x} \cap W)/\mathfrak{q}$, gives

$$\operatorname{ht}(\mathfrak{x}/\eta) \leq \operatorname{ht}\big((\mathfrak{x} \cap W)/\mathfrak{q}\big) + \operatorname{trdeg}_{\kappa(\mathfrak{q})} \kappa(\eta) \leq \dim W/\mathfrak{q} + \operatorname{trdeg}_{\kappa(\mathfrak{q})} \kappa(\eta).$$

With Steps 1 and 2 this is the assertion. □

49.3 Local charts of the entry algebra at maximal core primes

This subsection constructs the local charts of the entry algebra at the maximal core primes and presents them (Proposition 49.19, Proposition 49.20); Theorem 49.12 uses them to bound the dimension of components from below. A chart represents the polarised ordinary lifts of its residual lift to rings of finite length (Lemma 49.24), so its tangent space is an ordinary Selmer group enlarged by the torus and its obstructions lie in a Borel cohomology group (Definition 49.22, Lemma 49.23). Mazur's relation count, in a graded form (Lemma 49.21), then gives a presentation with at most as many relations in each degree as the obstruction bound, and at the endoscopic core primes a presentation graded by parity. Ordinary flags over local rings are triangular with the prescribed diagonal (Lemma 49.25); this lemma is used again in Appendix E.

Proposition 49.19. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2, 3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $E = E_n^{\mathrm{po}}$. Let x be a core prime of E that is a maximal ideal of E. Then:*

(Qc1) *there is a local chart at x;*

and for every local chart $(\mathcal{O}', R, \iota)$ at x:

(Qc2) *R is flat as an E-module through ι;*

(Qc3) *for every prime P of R: $I_R^- \subseteq P$ if and only if $I_E^- \subseteq \iota^{-1}(P)$, and likewise for I^+ and for I^{odd}; moreover $I_R^- = R$ and $I_R^+ \subseteq \mathfrak{m}_R$;*

(Qc4) *if $n = 2$, or $n = 3$ and $I_E^{\mathrm{odd}} \not\subseteq x$, there are integers $N, h \geq 0$ with $N \geq h + n\,d_L + 1$ and a surjective $\mathcal{O}'$-algebra homomorphism $\mathcal{O}'[[X_1, \ldots, X_N]] \to R$ whose kernel is generated by h elements;*

(Qc5) *for every $k \geq 1$ the maps $\gamma \mapsto \rho_x(\gamma) \bmod \mathfrak{m}_R^k$ on Γ_L^0 and $g \mapsto \psi_{x,v,a}(g) \bmod \mathfrak{m}_R^k$ on D_v ($v \mid \ell$, $1 \leq a \leq n$) are locally constant.*

Proof. The argument is commutative algebra; no automorphy is used, over L or over K. Write $W = W_n^{\mathrm{po}}$, $\rho = \rho^{\mathrm{u}}$ and $\mathfrak{m}$ for the maximal ideal of R_n^{po}. Since x is a core prime, $\mathfrak{m}_W \subseteq x = \iota^{-1}(\mathfrak{m}_R)$, so $\iota(\mathfrak{m}_W) \subseteq \mathfrak{m}_R$.

(Qc1), (Qc2). x is a maximal ideal containing $\mathfrak{m}_W$, so a local chart exists by (Cx3); every local chart is flat over E by (Cx2).

(Qc3). The ideals I_R^-, I_R^+, I_R^{odd} are generated by the images under ι of the generators of I_E^-, I_E^+, I_E^{odd} (entries of the relevant positions). For a prime P of R, $I_R^- \subseteq P$ holds if and only if every such image lies in P, i.e. every generator lies in $\iota^{-1}(P)$, i.e. $I_E^- \subseteq \iota^{-1}(P)$; likewise for I^+ and I^{odd}. As x is a core prime, some lower entry e_- is not in $x = \iota^{-1}(\mathfrak{m}_R)$, so $\iota(e_-)$ is a unit and $I_R^- = R$. If e_- has weight $-q < 0$ and e_+ is an upper entry, of weight $p > 0$, then $e_+^q e_-^p$ is a weight-zero monomial with off-diagonal factors, hence an element of W with residue 0 (Po5), i.e. of $\mathfrak{m}_W \subseteq x$; as $e_- \notin x$, $e_+ \in x$. So every generator of I_R^+ lies in $\iota(x) \subseteq \mathfrak{m}_R$, and $I_R^+ \subseteq \mathfrak{m}_R$.

(Qc4) is (Qp1) of Proposition 49.20.

(Qc5). Let $1 \leq a, b \leq n$, $j = w_{n,a} - w_{n,b}$ and $k \geq 1$. The entries $\rho(\gamma)_{ab}$ lie in gr_j, a finitely generated module over the complete Noetherian local ring W ((Gw1), (Gw3)). The submodules $\mathfrak{m}^t \cap \mathrm{gr}_j$ decrease with intersection 0 (Krull's intersection theorem in R_n^{po}), so by Chevalley's lemma Theorem B.154 there is t with $\mathfrak{m}^t \cap \mathrm{gr}_j \subseteq \mathfrak{m}_W^k \mathrm{gr}_j$. By (Po1), $\gamma \mapsto \rho(\gamma) \bmod \mathfrak{m}^t$ is locally constant on Γ_L^0; hence so is $\gamma \mapsto \rho(\gamma)_{ab} \bmod \mathfrak{m}_W^k \mathrm{gr}_j$, and, as $\iota(\mathfrak{m}_W^k \mathrm{gr}_j) \subseteq \mathfrak{m}_R^k$, so is $\gamma \mapsto \rho_x(\gamma)_{ab} \bmod \mathfrak{m}_R^k$. The same argument with $\mathrm{gr}_0 = W \ni \psi_{v,a}^{\mathrm{u}}(g)$ and the local constancy of $\psi_{v,a}^{\mathrm{u}} \bmod \mathfrak{m}^t$ on D_v (Po1) gives the assertion for $\psi_{x,v,a}$. □

Proposition 49.20. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2, 3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $E = E_n^{\mathrm{po}}$. Let x be a core prime of E that is a maximal ideal of E, and $(\mathcal{O}', R, \iota)$ a local chart at x.*

(Qp1) *If $n = 2$, or $n = 3$ and $I_E^{\mathrm{odd}} \not\subseteq x$, there are integers $N, h \geq 0$ with $N \geq h + n\,d_L + 1$ and a surjective $\mathcal{O}'$-algebra homomorphism $\mathcal{O}'[[X_1, \ldots, X_N]] \to R$ whose kernel is generated by h elements.*

(Qp2) *If $n = 3$ and $I_E^{\mathrm{odd}} \subseteq x$, there are integers $a, b, g \geq 0$ with $a \geq b + 3d_L + 1$ and a surjective $\mathcal{O}'$-algebra homomorphism $p\colon S = \mathcal{O}'[[Y_1, \ldots, Y_a, X_1, \ldots, X_g]] \to R$ with the following properties, where a power series is called* even *(resp.* odd*) if each of its monomials has even (resp. odd) total degree in $X_1, \ldots, X_g$:*

- **(Ep1)** *$\ker p$ is generated by even $F_1, \ldots, F_b \in S$ and odd $G_1, \ldots, G_g \in S$;*
- **(Ep2)** *for every $f \in \ker p$, the coefficients in f of the monomials Y_i and X_j lie in $\mathfrak{m}_{\mathcal{O}'}$;*

(Ep3) *for all* $\gamma \in \Gamma_L^0$ *and* $1 \le a', b' \le 3$, $\rho_x(\gamma)_{a'b'} = p(f)$ *for some* f *that is even if* $w_{3,a'} - w_{3,b'}$ *is even and odd otherwise; and for* $v \mid \ell$, $1 \le a' \le 3$, $g' \in D_v$, $\psi_{x,v,a'}(g') = p(f)$ *for some even* f*;*

(Ep4) I_R^{odd} *is the ideal generated by* $p(X_1), \dots, p(X_g)$.

Proof. The proof uses only Galois cohomology and commutative algebra; in particular, no automorphy over L or over K is used. Let ϖ be a uniformiser of $\mathcal{O}'$ and $\mathbb{F}'$ the residue field. For $y \in R$ let $[y] \in \mathcal{O}'$ be an element with the same residue.

Step 1: presentations by a basis of the cotangent space. (i) Let $y_1, \dots, y_m \in \mathfrak{m}_R$ have classes forming a basis of t_R^*, and $\Pi\colon \mathcal{O}'[[Z_1, \dots, Z_m]] \to R$ the continuous $\mathcal{O}'$-algebra homomorphism with $\Pi(Z_i) = y_i$. Then $\mathfrak{m}_R = \varpi R + \sum_i y_i R + \mathfrak{m}_R^2$, so Π is surjective by Nakayama's lemma and the complete Nakayama lemma [166, Thm. 8.4]; and its kernel is minimal in the sense of Lemma 49.21: if $f = c_0 + \sum_i c_i Z_i + q \in \ker\Pi$ with $q \in (Z)^2$, then $c_0 \in \mathfrak{m}_{\mathcal{O}'}$ (reduce modulo $\mathfrak{m}_R$) and $\sum_i c_i y_i \in \varpi R + \mathfrak{m}_R^2$, so all $c_i \in \mathfrak{m}_{\mathcal{O}'}$. (ii) The classes of the elements $\rho_x(\gamma)_{a'b'} - [\rho_x(\gamma)_{a'b'}]$ and $\psi_{x,v,a'}(g') - [\psi_{x,v,a'}(g')]$ span t_R^*: a linear form λ on t_R^* vanishing on them defines the homomorphism $R \to \mathbb{F}'[\varepsilon]$, $y \mapsto \bar{y} + \varepsilon\lambda(y - [y])$, which gives the same family in (Kr3) as $\lambda = 0$; by the injectivity in (Kr3), $\lambda = 0$.

Step 2: (Qp1). Let $N = \dim_{\mathbb{F}'} t_R^*$ and $\Pi\colon \mathcal{O}'[[X_1, \dots, X_N]] \to R$ as in Step 1 (i). Apply Lemma 49.21 with the trivial group $\mathsf{A} = \{0\}$ (every ring is graded, every homomorphism graded) and $d(0) = h_x^2$: (Ov) is (Kq2) in case (a). So $\ker\Pi$ is generated by $h \le h_x^2$ elements. By (Kq1) and (Kq3),

$$N \;\ge\; h_x^1 - h_x^0 + 1 \;\ge\; h_x^2 + n\,d_L + 1 \;\ge\; h + n\,d_L + 1.$$

This proves (Qp1) (for every maximal core prime; the hypothesis on I_E^{odd} in (Qp1) is not used).

Step 3: the parity presentation. Let $n = 3$ and $I_E^{\mathrm{odd}} \subseteq x$, θ the involution of (Kr4), $R^{\pm} = \{y \in R : \theta(y) = \pm y\}$, so $R = R^+ \oplus R^-$ (ℓ odd). By (Kr4) an odd entry $\rho_x(\gamma)_{a'b'}$ ($w_{3,a'} - w_{3,b'}$ odd) lies in R^-, and in $\mathfrak{m}_R$ since the odd entries of $\bar{\rho}_x$ vanish; even entries and the values $\psi_{x,v,a'}(g')$ lie in R^+, as do the constants $[\cdot]$. So the spanning set of Step 1 (ii) consists of elements of t_+^* (the even ones) and of t_-^* (the odd entries), and the odd entries span t_-^*, the even ones t_+^*. Let $a = \dim t_+^*$, $g = \dim t_-^*$; choose odd entries $x_1, \dots, x_g$ whose classes form a basis of t_-^*, and elements $y_1, \dots, y_a$ of the form $\rho_x(\gamma)_{a'b'} - [\rho_x(\gamma)_{a'b'}]$ ($w_{3,a'} - w_{3,b'}$ even) or $\psi_{x,v,a'}(g') - [\psi_{x,v,a'}(g')]$, whose classes form a basis of t_+^*. Let $S = \mathcal{O}'[[Y_1, \dots, Y_a, X_1, \dots, X_g]]$ and $p\colon S \to R$ with $p(Y_i) = y_i$, $p(X_j) = x_j$: surjective with minimal kernel by Step 1 (i), which is (Ep2). Weight Y_i by 0 and X_j by 1 in $\mathsf{A} = \mathbb{Z}/2\mathbb{Z}$; then S_0 and S_1 are the even and the odd power series of the statement. Let θ_S be the continuous $\mathcal{O}'$-algebra involution of S with $\theta_S(Y_i) = Y_i$, $\theta_S(X_j) = -X_j$; then $p \circ \theta_S = \theta \circ p$ (both are continuous and agree on the variables; θ is a local automorphism of R), and $f_{(0)} = \frac{1}{2}(f + \theta_S f)$, $f_{(1)} = \frac{1}{2}(f - \theta_S f)$. Hence $\ker p$ is homogeneous, and $p(S_0) = R^+$, $p(S_1) = R^-$.

Step 4: (Qp2). Apply Lemma 49.21 to p with $\mathsf{A} = \mathbb{Z}/2\mathbb{Z}$, $d(0) = h_{x,+}^2$, $d(1) = h_{x,-}^2$. To check (Ov), let $p'\colon C \to \bar{C}$, $I \subseteq C_e$ ($e \in \mathbb{Z}/2\mathbb{Z}$) and $\bar{\xi}$ be as there, and let θ_C, $\theta_{\bar{C}}$ act by $+1$ on the degree-0 part and by -1 on the degree-1 part: $\mathcal{O}'$-algebra involutions (the gradings are multiplicative and $\mathcal{O}'$ lies in degree 0), with $p' \circ \theta_C = \theta_{\bar{C}} \circ p'$ (p' graded), $\bar{\xi} \circ \theta = \theta_{\bar{C}} \circ \bar{\xi}$ ($\bar{\xi}$ maps $R^+ = p(S_0)$ and $R^- = p(S_1)$ into the degree 0 and 1 parts), and $\theta_C = \epsilon$ on I with $\epsilon = (-1)^e$. So (Kq2) in case (b) gives I' and ξ. Hence $\ker p$ is generated by even $F_1, \dots, F_b$ and odd $G_1, \dots, G_{g'}$ with $b \le h_{x,+}^2$ and $g' \le h_{x,-}^2$. By (Kq1) and (Kq3),

$$a \ge h_{x,+}^1 - h_{x,+}^0 + 1 \ge h_{x,+}^2 + 3d_L + 1 \ge b + 3d_L + 1, \qquad g \ge h_{x,-}^1 - h_{x,-}^0 \ge h_{x,-}^2 \ge g'.$$

Putting $G_j = 0$ (an odd power series: it has no monomials) for $g' < j \le g$ gives (Ep1) with exactly g odd generators. (Ep3): by (Kr4), $\rho_x(\gamma)_{a'b'} \in R^{(-1)^{w_{3,a'} - w_{3,b'}}}$, which is $p(S_0)$

or $p(S_1)$ accordingly, and $\psi_{x,v,a'}(g') \in R^+ = p(S_0)$. (Ep4): I_R^{odd} is generated by the odd entries; each $p(X_j) = x_j$ is one, and each odd entry lies in $R^- = p(S_1)$, while every odd power series is $\sum_j X_j f_j$ with $f_j \in S$ (every monomial of odd degree in the X_j contains some X_j); so $I_R^{\mathrm{odd}} = (p(X_1), \dots, p(X_g))$. □

Lemma 49.21. *Let $\mathcal{O}'$ be a coefficient ring with maximal ideal $\mathfrak{m}_{\mathcal{O}'}$ and residue field $\mathbb{F}'$, A an abelian group, σ a finite set with a map* $\mathrm{wt}\colon \sigma \to \mathsf{A}$*, and $P = \mathcal{O}'[[X_s : s \in \sigma]]$ with maximal ideal $\mathfrak{n}$. The* weight *of a monomial $\prod_s X_s^{m_s}$ is $\sum_s m_s \mathrm{wt}(s)$; for $f \in P$ and $a \in \mathsf{A}$, $f_{(a)}$ is the power series formed by the terms of f whose monomials have weight a, and $P_a = \{f \in P : f = f_{(a)}\}$; an ideal $\mathfrak{H}$ of P is* homogeneous *if $f_{(a)} \in \mathfrak{H}$ for all $f \in \mathfrak{H}$ and $a \in \mathsf{A}$. An* A-graded ring *is a ring C with additive subgroups C_a ($a \in \mathsf{A}$), all but finitely many zero, such that $C = \bigoplus_a C_a$ and $C_a C_b \subseteq C_{a+b}$; for an $\mathcal{O}'$-algebra C we require in addition that the image of $\mathcal{O}'$ lies in C_0. A homomorphism $q\colon C \to C'$ of A-graded rings is* graded *if $q(C_a) \subseteq C'_a$ for all a.*

Let $R \in \mathrm{CNL}_{\mathcal{O}'}$ and let $\Pi\colon P \to R$ be a surjective $\mathcal{O}'$-algebra homomorphism with $\Pi(X_s) \in \mathfrak{m}_R$ for all s, whose kernel $\mathfrak{J}$ is homogeneous and minimal*: for every $f \in \mathfrak{J}$ and $s \in \sigma$ the coefficient of X_s in f lies in $\mathfrak{m}_{\mathcal{O}'}$. A homomorphism $\xi\colon R \to C$ to an A-graded ring is* graded *if $\xi(\Pi(P_a)) \subseteq C_a$ for all a. Let $d\colon \mathsf{A} \to \mathbb{N}$ and suppose:*

(Ov) *for every $a \in \mathsf{A}$, every surjective graded $\mathcal{O}'$-algebra homomorphism $p\colon C \to \bar{C}$ between A-graded rings $C, \bar{C} \in \mathrm{CNL}_{\mathcal{O}'}$ of finite length whose kernel I satisfies $\mathfrak{m}_C I = 0$, $I \subseteq C_a$ and $\dim_{\mathbb{F}'} I > d(a)$, and every surjective graded $\mathcal{O}'$-algebra homomorphism $\bar{\xi}\colon R \to \bar{C}$, there are an ideal $I' \subseteq I$ of C with $\dim_{\mathbb{F}'} I/I' = 1$ and an $\mathcal{O}'$-algebra homomorphism $\xi\colon R \to C/I'$ whose composite with the projection $C/I' \to \bar{C}$ is $\bar{\xi}$.*

Then $\mathfrak{J}$ is generated by a family $(f_{a,i})_{a \in \mathsf{A},\, 1 \le i \le \mathrm{nf}(a)}$ with $f_{a,i} \in \mathfrak{J} \cap P_a$, where $\mathrm{nf}(a) \le d(a)$ for every a and $\mathrm{nf}(a) = 0$ for all but finitely many a.

Proof. Let ϖ be a uniformiser of $\mathcal{O}'$. The condition $\mathfrak{m}_C I = 0$ makes I a vector space over $C/\mathfrak{m}_C = \mathbb{F}'$.

Step 1: three facts on weights. (i) For $g \in P_b$ and $h \in P$ one has $(gh)_{(a)} = g\, h_{(a-b)}$ for all a. Hence an ideal generated by elements of single weights is homogeneous (for $\sum_k g_k h_k$ with $g_k \in P_{b_k}$ the weight-a component is $\sum_k g_k h_{k,(a-b_k)}$), and sums of homogeneous ideals are homogeneous. In particular $\mathfrak{n}^N$ is homogeneous (generated by the monomials $\varpi^e X^m$, $e + |m| = N$, each of a single weight), and so is $\mathfrak{n}\mathfrak{J}$: its elements are $\varpi F_0 + \sum_s X_s F_s$ with $F_0, F_s \in \mathfrak{J}$, with weight-$a$ component $\varpi F_{0,(a)} + \sum_s X_s F_{s,(a-\mathrm{wt}(s))} \in \mathfrak{n}\mathfrak{J}$, as $\mathfrak{J}$ is homogeneous. (ii) For $N \ge 1$ let $U_N \subseteq \mathsf{A}$ be the finite set of weights of the monomials X^m with $|m| < N$. For every $f \in P$, all monomials of $f - \sum_{a \in U_N} f_{(a)}$ have degree at least N, so $f - \sum_{a \in U_N} f_{(a)} \in \mathfrak{n}^N$; and $f_{(a)} \in \mathfrak{n}^N$ for $a \notin U_N$.

(iii) If $\mathfrak{H}$ is a homogeneous ideal, $B \subseteq \mathsf{A}$ is finite, $h_b \in P_b$ for $b \in B$ and $\sum_b h_b \in \mathfrak{H}$, then $h_b = (\sum_{b'} h_{b'})_{(b)} \in \mathfrak{H}$ for each b.

Step 2: the graded space of minimal relations. By the Artin–Rees lemma [166] there is $N_0 \ge 1$ with $\mathfrak{J} \cap \mathfrak{n}^{N_0} \subseteq \mathfrak{n}\mathfrak{J}$. Let $V = \mathfrak{J}/\mathfrak{n}\mathfrak{J}$, a finite-dimensional $\mathbb{F}'$-vector space ($\mathfrak{J}$ is finitely generated), and V_b the image of $\mathfrak{J} \cap P_b$. For $f \in \mathfrak{J}$ all $f_{(b)}$ lie in $\mathfrak{J}$, and by Step 1 (ii) $f - \sum_{b \in U_{N_0}} f_{(b)} \in \mathfrak{J} \cap \mathfrak{n}^{N_0} \subseteq \mathfrak{n}\mathfrak{J}$ and $V_b = 0$ for $b \notin U_{N_0}$. So the V_b span V, the image $\bar{f}$ of f is $\sum_{b \in U_{N_0}} \overline{f_{(b)}}$, and the sum $\sum_b V_b$ is direct by Step 1 (iii) applied to $\mathfrak{n}\mathfrak{J}$. Choose $f_{b,1}, \dots, f_{b,\mathrm{nf}(b)} \in \mathfrak{J} \cap P_b$ lifting a basis of V_b, $\mathrm{nf}(b) = \dim V_b$. Then $\mathfrak{J} = (f_{b,i})_{b,i} + \mathfrak{n}\mathfrak{J}$, so $\mathfrak{J} = (f_{b,i})_{b,i}$ by Nakayama's lemma, and $\mathrm{nf}(b) = 0$ for $b \notin U_{N_0}$. It remains to show $\dim V_a \le d(a)$ for every a. Suppose, to the contrary, that $\dim V_a > d(a)$ for some a.

Step 3: a graded small extension concentrated in weight a. Let $\mathfrak{W} \subseteq \mathfrak{J}$ be the preimage of $\bigoplus_{b \ne a} V_b$; it is an ideal of P containing $\mathfrak{n}\mathfrak{J}$. It is homogeneous: for $f \in \mathfrak{W}$ the V_a-component of

$\bar f$ is $\overline{f_{(a)}}$ (Step 2), so $f_{(a)} \in \mathfrak{n}\mathfrak{J} \subseteq \mathfrak{W}$, and $f_{(b)} \in \mathfrak{W}$ for $b \neq a$ since $\overline{f_{(b)}} \in V_b$. For every $f \in \mathfrak{J}$, $f - f_{(a)} \in \mathfrak{W}$; hence $\mathfrak{J} = \mathfrak{W} + (\mathfrak{J} \cap P_a)$. Also $\mathfrak{J} \cap (\mathfrak{W} + \mathfrak{n}^{N_0}) = \mathfrak{W} + (\mathfrak{J} \cap \mathfrak{n}^{N_0}) = \mathfrak{W}$, as $\mathfrak{W} \subseteq \mathfrak{J}$ and $\mathfrak{J} \cap \mathfrak{n}^{N_0} \subseteq \mathfrak{n}\mathfrak{J} \subseteq \mathfrak{W}$.

Put $C = P/(\mathfrak{W} + \mathfrak{n}^{N_0})$ and $\bar C = P/(\mathfrak{J} + \mathfrak{n}^{N_0})$: local rings of finite length with residue field $\mathbb{F}'$, objects of $\mathrm{CNL}_{\mathcal{O}'}$. Let C_b, $\bar C_b$ be the images of P_b. The two ideals are homogeneous (Step 1 (i)), so by Step 1 (ii) and (iii) $C = \bigoplus_{b \in U_{N_0}} C_b$ and $\bar C = \bigoplus_{b \in U_{N_0}} \bar C_b$; with $P_b P_{b'} \subseteq P_{b+b'}$ and $\mathcal{O}' \subseteq P_0$ these are A-graded $\mathcal{O}'$-algebras, and the projection $p\colon C \to \bar C$ is a surjective graded homomorphism. Its kernel I is the image of $\mathfrak{J}$, which is the image of $\mathfrak{J} \cap P_a$; so $I \subseteq C_a$. As $\mathfrak{n}\mathfrak{J} \subseteq \mathfrak{W}$, $\mathfrak{m}_C I = 0$. Finally $I \cong \mathfrak{J}/(\mathfrak{J} \cap (\mathfrak{W} + \mathfrak{n}^{N_0})) = \mathfrak{J}/\mathfrak{W} \cong V_a$, so $\dim_{\mathbb{F}'} I > d(a)$. Let $\bar\xi\colon R \to \bar C$ be the homomorphism induced by Π (as $R \cong P/\mathfrak{J}$); it is surjective and graded.

Step 4: the lift contradicts minimality. By (Ov) there are an ideal $I' \subseteq I$ with $\dim I/I' = 1$ and $\xi\colon R \to C' = C/I'$ with $(C' \to \bar C) \circ \xi = \bar\xi$. Let $\alpha\colon P \to C'$ be the projection and $\beta = \xi \circ \Pi$. Both composites with $C' \to \bar C$ are the projection $P \to \bar C$, so $\delta = \alpha - \beta$ takes values in I/I', which is killed by $\mathfrak{m}_{C'}$. Hence δ is $\mathcal{O}'$-linear and, writing $\bar f \in \mathbb{F}'$ for the residue of $f \in P$,

$$\delta(fh) = \alpha(f)\delta(h) + \beta(h)\delta(f) = \bar f\,\delta(h) + \bar h\,\delta(f) \qquad (f, h \in P).$$

Thus $\delta(1) = 0$, δ vanishes on $\mathcal{O}'$, on ϖP ($\delta(\varpi h) = \varpi\delta(h) = 0$) and on every $X_s X_t h$ (both residues $\bar X_s$, $\overline{X_t h}$ vanish). Let $f \in \mathfrak{J}$ and write $f = c + \sum_s c_s X_s + \sum_{s \le t} X_s X_t q_{st}$ with $c, c_s \in \mathcal{O}'$, $q_{st} \in P$. Reducing $\Pi(f) = 0$ modulo $\mathfrak{m}_R$ gives $c \in \mathfrak{m}_{\mathcal{O}'}$ (as $\Pi(X_s) \in \mathfrak{m}_R$), and $c_s \in \mathfrak{m}_{\mathcal{O}'}$ by minimality; so $\delta(f) = 0$. Also $\beta(f) = \xi(\Pi(f)) = 0$. Hence $\alpha(\mathfrak{J}) = 0$. But $\alpha(\mathfrak{J})$ is the image of I in C', namely $I/I' \neq 0$. This contradiction shows $\dim V_a \le d(a)$ for every a, and the family $(f_{a,i})$ of Step 2 has the asserted properties. □

Definition 49.22. Let L be a CM field satisfying (Af1) and (Af2) of Definition 48.2 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, and let $n \in \{2, 3\}$. Suppose that a universal polarised ordinary lift of rank n over L exists. Let x be a core prime of $E = E_n^{\mathrm{po}}$ that is a maximal ideal, let $(\mathcal{O}', R, \iota)$ be a local chart at x with residue field $\mathbb{F}'$, and let $\bar\rho_x\colon \Gamma_L^0 \to \mathrm{GL}_n(\mathbb{F}')$ and $\bar\psi_{x,v,a}\colon D_v \to \mathbb{F}'^{\times}$ be the reductions modulo $\mathfrak{m}_R$ of the chart lift. For $X \in \mathrm{M}_n(\mathbb{F}')$ write ${}^a X = J_n\,{}^t X\,J_n$ (the transpose in the antidiagonal). Put $\bar c = \pi_L(c)$; every element of Γ_L is uniquely γ or $\gamma\bar c$ with $\gamma \in \Gamma_L^0$.

The adjoint module. The module ad_x is the $\mathbb{F}'$-vector space $\mathrm{M}_n(\mathbb{F}')$ with the maps $X \mapsto g \cdot X$, $g \in \Gamma_L$, given for $\gamma \in \Gamma_L^0$ by

$$\gamma \cdot X = \bar\rho_x(\gamma)\,X\,\bar\rho_x(\gamma)^{-1}, \qquad (\gamma\bar c) \cdot X = -\bar\rho_x(\gamma)\,{}^a X\,\bar\rho_x(\gamma)^{-1}.$$

Borel subspaces. For a place $v \mid \ell$ of L^+, let $\mathfrak{b}_{x,v} \subseteq \mathrm{ad}_x$ be the set of X such that $P^{-1}XP$ is upper triangular for every $P \in \mathrm{GL}_n(\mathbb{F}')$ for which $P^{-1}\bar\rho_x(\pi_L(g))P$ is upper triangular with diagonal entries $\bar\psi_{x,v,1}(g), \dots, \bar\psi_{x,v,n}(g)$ for all $g \in D_v$. Write $\mathfrak{b}_x = (\mathfrak{b}_{x,v})_{v|\ell}$.

Parity parts. If $n = 3$ and $I_E^{\mathrm{odd}} \subseteq x$, put $t = \mathrm{diag}(-1, 1, -1)$,

$$\mathrm{ad}_x^{\pm} = \{X \in \mathrm{ad}_x :\ tXt^{-1} = \pm X\}, \qquad \mathfrak{b}_{x,v}^{\pm} = \mathfrak{b}_{x,v} \cap \mathrm{ad}_x^{\pm},$$

and $\mathfrak{b}_x^{\pm} = (\mathfrak{b}_{x,v}^{\pm})_{v|\ell}$; thus ad_x^{+} (resp. ad_x^{-}) consists of the matrices supported on the positions (a, b) with $w_{3,a} - w_{3,b}$ even (resp. odd).

Lemma 49.23. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2, 3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $E = E_n^{\mathrm{po}}$. Let x be a core prime of E that is a maximal ideal, $(\mathcal{O}', R, \iota)$ a local chart at x, $\mathbb{F}'$ its residue field, and $t_R^* = \mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_{\mathcal{O}'}R)$. Call x* endoscopic *if $n = 3$ and $I_E^{\mathrm{odd}} \subseteq x$; then $t_\pm^* \subseteq t_R^*$ denote the ±1-eigenspaces of the involution induced by θ.*

(Kq0) *The maps of Definition 49.22 form a continuous $\mathbb{F}'$-linear action of Γ_L on ad_x with open kernel; each $\mathfrak{b}_{x,v}$ is stable under D_v (acting through π_L) and of dimension $n(n+1)/2$; if x is endoscopic, $\mathrm{ad}_x^{\pm}$ are Γ_L-stable and $\mathfrak{b}_{x,v}^{\pm}$ are D_v-stable. Hence the groups $H^i(\Gamma_L;\mathrm{ad}_x,\mathfrak{b}_x)$ and, if x is endoscopic, $H^i(\Gamma_L;\mathrm{ad}_x^{\pm},\mathfrak{b}_x^{\pm})$ ($i=0,1,2$; the $\mathbb{F}'$-spaces read as $\mathbb{F}_{\mathcal{O}}$-spaces) are defined; they are $\mathbb{F}'$-vector spaces, and h_x^i, $h_{x,\pm}^i$ denote their dimensions over $\mathbb{F}'$.*

(Kq1) $\dim_{\mathbb{F}'} t_R^* \geq h_x^1 - h_x^0 + 1$; *if x is endoscopic,* $\dim t_+^* \geq h_{x,+}^1 - h_{x,+}^0 + 1$ *and* $\dim t_-^* \geq h_{x,-}^1 - h_{x,-}^0$.

(Kq2) *Let $p\colon C \to \bar{C}$ be a surjective $\mathcal{O}'$-algebra homomorphism between rings $C, \bar{C} \in \mathrm{CNL}_{\mathcal{O}'}$ of finite length whose kernel I satisfies $\mathfrak{m}_C I = 0$, and $\bar{\xi}\colon R \to \bar{C}$ an $\mathcal{O}'$-algebra homomorphism. Suppose either (a) $d = h_x^2$, or (b) x is endoscopic, C and $\bar{C}$ carry $\mathcal{O}'$-algebra involutions θ_C, $\theta_{\bar{C}}$ with $p \circ \theta_C = \theta_{\bar{C}} \circ p$ and $\bar{\xi} \circ \theta = \theta_{\bar{C}} \circ \bar{\xi}$, $\theta_C(y) = \epsilon y$ for all $y \in I$ for some $\epsilon \in \{1,-1\}$, and $d = h_{x,+}^2$ if $\epsilon = 1$, $d = h_{x,-}^2$ if $\epsilon = -1$. If $\dim_{\mathbb{F}'} I > d$, there are an ideal $I' \subsetneq I$ of C with $\dim_{\mathbb{F}'} I/I' = 1$ and an $\mathcal{O}'$-algebra homomorphism $\xi\colon R \to C/I'$ whose composite with the projection $C/I' \to \bar{C}$ is $\bar{\xi}$.*

(Kq3) *These h^2 are finite, $h_x^1 - h_x^0 - h_x^2 \geq n\,d_L$, and, if x is endoscopic, $h_{x,+}^1 - h_{x,+}^0 - h_{x,+}^2 \geq 3d_L$ and $h_{x,-}^1 - h_{x,-}^0 - h_{x,-}^2 \geq 0$.*

Proof. The argument is Galois cohomology and commutative algebra; no automorphy is used, over L or over K. Write $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, $J = J_n$, $\bar{\rho} = \bar{\rho}_x$, $\bar{\psi}_{v,a} = \bar{\psi}_{x,v,a}$, $\bar{c} = \pi_L(c)$, ${}^aX = J\,{}^tX J$ for matrices over any ring; then ${}^a(XY) = {}^aY\,{}^aX$, ${}^a({}^aX) = X$. Let $\mathfrak{o} = \{Y \in \mathrm{M}_n(\mathbb{F}') : {}^aY = -Y\}$ and $\mathfrak{o}_{\mathrm{off}}$ its subspace of matrices with zero diagonal; as ^{a} maps the entry (a,b) to $(n+1-b, n+1-a)$, it preserves diagonal and off-diagonal positions, so $\mathfrak{o} = \mathfrak{o}_{\mathrm{diag}} \oplus \mathfrak{o}_{\mathrm{off}}$ with $\mathfrak{o}_{\mathrm{diag}} = \{\mathrm{diag}(y) : y_{n+1-a} = -y_a\}$ of dimension 1; $\dim \mathfrak{o} = n(n-1)/2$, so $\dim \mathfrak{o}_{\mathrm{off}} = n(n-1)/2 - 1$ (0 for $n=2$; for $n=3$, $\mathfrak{o}_{\mathrm{off}}$ is spanned by $e_{12} - e_{23}$ and $e_{21} - e_{32}$, both at odd positions). If x is endoscopic, t is as in Definition 49.22; otherwise put $t = 1$, $\theta = \mathrm{id}_R$ and $\mathrm{ad}_x^+ = \mathrm{ad}_x$, $\mathfrak{b}_{x,v}^+ = \mathfrak{b}_{x,v}$, so that case (a) of (Kq2) is the case $\epsilon = 1$ with trivial involutions. In all cases $t^{-1} = t$, $Jt = tJ$, hence ${}^a(tXt^{-1}) = t\,{}^aXt^{-1}$, and $\theta(\rho_x(\gamma)) = t\rho_x(\gamma)t^{-1}$ (entrywise) by (Kr4). Since L/L^+ is unramified at all finite places (Af2), $N_L \subseteq G_L$, so $\bar{c} \notin \Gamma_L^0$, $\bar{c}^2 = 1$ and $\Gamma_L = \Gamma_L^0 \rtimes \{1, \bar{c}\}$; and since every place $v \mid \ell$ splits in L ($\ell \mid M$, (Af2)), $\pi_L(D_v) \subseteq \Gamma_L^0$.

Step 1: polarised lifts are homomorphisms to a semidirect product; the action. For a commutative ring A let $\mathcal{G}(A) = \mathrm{GL}_n(A) \rtimes \{1, j\}$ with $jXj^{-1} = ({}^aX)^{-1}$ (an automorphism of order 2). For a homomorphism $\rho\colon \Gamma_L^0 \to \mathrm{GL}_n(A)$ let $r_\rho\colon \Gamma_L \to \mathcal{G}(A)$ be $r_\rho(\gamma) = \rho(\gamma)$, $r_\rho(\gamma\bar{c}) = \rho(\gamma)j$. As Γ_L is the semidirect product, r_ρ is a homomorphism if and only if $\rho(\bar{c}\gamma\bar{c}) = j\rho(\gamma)j^{-1} = ({}^a\rho(\gamma))^{-1}$ for all γ, i.e. (replacing γ by γ^{-1}) if and only if ρ satisfies (Po2). By (Kr2), $\bar{r} = r_{\bar{\rho}}$ is a homomorphism. For $X \in \mathrm{M}_n(\mathbb{F}')$, in $\mathcal{G}(\mathbb{F}'[\varepsilon])$ ($\varepsilon^2 = 0$) one has $\bar{r}(g)(1+\varepsilon X)\bar{r}(g)^{-1} = 1 + \varepsilon(g \cdot X)$ with $g \cdot X$ as in Definition 49.22 (for $g = \gamma\bar{c}$: $j(1+\varepsilon X)j^{-1} = ({}^a(1+\varepsilon X))^{-1} = 1 - \varepsilon\,{}^aX$). Conjugation being an action, $g \cdot X$ is an $\mathbb{F}'$-linear action; it is continuous with open kernel because $\bar{\rho}$ is locally constant (Kr2). More generally, if A is a local ring with residue field $\mathbb{F}'$, $I \subseteq A$ an ideal with $\mathfrak{m}_A I = 0$ (an $\mathbb{F}'$-vector space, $\mathrm{M}_n(I) = \mathrm{ad}_x \otimes_{\mathbb{F}'} I$), and $s \in \mathcal{G}(A)$ reduces to $\bar{r}(g)$, then

$$s(1+Y)s^{-1} = 1 + g \cdot Y, \qquad (1+Y)(1+Y') = 1 + Y + Y' \qquad (Y, Y' \in \mathrm{M}_n(I)), \tag{49.1}$$

g acting on the first factor of $\mathrm{ad}_x \otimes I$, because products of two elements of I vanish and conjugation by s on $\mathrm{M}_n(I)$ only depends on s modulo $\mathfrak{m}_A$.

Step 2: the module facts of (Kq0). (i) *Triangularisers.* Fix $v \mid \ell$. By (Af4) there is $h_v \in D_v$ with $\bar{\Psi}(h_v)^2 \neq 1$; as the exponents $w_{n,a}$ are $1, 0, -1$ or $1, -1$, the values $\bar{\Psi}(h_v)^{w_{n,a}} = \bar{\psi}_{v,a}(h_v)$ (Kr1) are pairwise distinct, and by (Kr1) $\bar{\rho}(\pi_L(h_v))$ is lower triangular with these diagonal

entries. By (Kr2), (Po4) holds for $(\bar\rho\circ\pi_L|_{D_v},\bar\psi_v)$. So (Fg2) (with $C=\mathbb{F}'$) gives P triangularising it; $U=P^{-1}\bar\rho(\pi_L(h_v))P$ is upper triangular with distinct diagonal entries, so $U=NDN^{-1}$ with D diagonal and N upper unitriangular, and $P_v=PN$ still triangularises (conjugation by N preserves upper triangularity and diagonals) and has eigenvectors of $\bar\rho(\pi_L(h_v))$ with distinct eigenvalues as columns. By (Fg3) every triangulariser is P_vB with B upper triangular, so $\mathfrak{b}_{x,v}=P_v\mathfrak{b}_nP_v^{-1}$, $\mathfrak{b}_n$ the upper triangular matrices: of dimension $n(n+1)/2$, and D_v-stable since $\bar\rho(\pi_L(g))P_vYP_v^{-1}\bar\rho(\pi_L(g))^{-1}=P_vU_gYU_g^{-1}P_v^{-1}$ with U_g upper triangular. (ii) *No invariants modulo* $\mathfrak{b}$. If $X\in\mathrm{ad}_x$ and $(g-1)X\in\mathfrak{b}_{x,v}$ for all $g\in D_v$, then $X\in\mathfrak{b}_{x,v}$. Indeed, with $X'=P_v^{-1}XP_v$ and $D=\mathrm{diag}(\bar\psi_{v,a}(h_v))$, $DX'D^{-1}-X'\in\mathfrak{b}_n$, and its (a,b)-entry for $a>b$ is $(\bar\psi_{v,a}(h_v)\bar\psi_{v,b}(h_v)^{-1}-1)X'_{ab}$, which forces $X'_{ab}=0$. (iii) $\mathrm{Hom}_{\Gamma_L}(\mathrm{ad}_x,\mathbb{F}(\bar\omega))=0$ ($\mathbb{F}$-linear maps). For $m\in\mathbb{Z}$ let F_m be the span of the matrix units e_{ab} with $a-b\geq m$. For lower triangular $Z\in\mathrm{GL}_n(\mathbb{F}')$, $Ze_{ab}Z^{-1}\in Z_{aa}(Z^{-1})_{bb}e_{ab}+F_{a-b+1}$; by (Kr1) the F_m are Γ_L^0-stable and γ acts on F_m/F_{m+1} by the scalars $\bar\Psi(\gamma)^{w_{n,a}-w_{n,b}}\in\mathbb{F}^\times$ on the lines $\mathbb{F}'e_{ab}$, $a-b=m$. If $f\neq0$ were a Γ_L-map to $\mathbb{F}(\bar\omega)$, take m maximal with $f(F_m)\neq0$ and $a-b=m$, $\lambda\in\mathbb{F}'$ with $f(\lambda e_{ab})\neq0$: then $\bar\omega(\gamma)f(\lambda e_{ab})=f(\gamma\cdot\lambda e_{ab})=\bar\Psi(\gamma)^{w_{n,a}-w_{n,b}}f(\lambda e_{ab})$, so $\bar\omega=\bar\Psi^{w_{n,a}-w_{n,b}}$ on G_L, contrary to (Ar1). (iv) *Parity.* If x is endoscopic, t commutes with $\bar\rho(\gamma)$ (Kr4) and ${}^a(tXt^{-1})=t\,{}^aXt^{-1}$, so $X\mapsto tXt^{-1}$ commutes with the action and $\mathrm{ad}_x^\pm$ are Γ_L-stable. The matrix t preserves the eigenlines of $\bar\rho(\pi_L(h_v))$, the columns of P_v, so $tP_v=P_vt'_v$ with $t'_v=\mathrm{diag}(s_1,s_2,s_3)$, $s_a=\pm1$; hence $t\mathfrak{b}_{x,v}t^{-1}=\mathfrak{b}_{x,v}$, $\mathfrak{b}_{x,v}=\mathfrak{b}^+_{x,v}\oplus\mathfrak{b}^-_{x,v}$, these are D_v-stable, and (ii) holds for them. As $\mathrm{tr}\,t'_v=\mathrm{tr}\,t=-1$, exactly one s_a is $+1$; $P_v^{-1}\mathfrak{b}^\pm_{x,v}P_v$ consists of the upper triangular matrices supported where $s_as_b=\pm1$, so $\dim\mathfrak{b}^+_{x,v}=3+1=4$, $\dim\mathfrak{b}^-_{x,v}=2$, while $\dim\mathrm{ad}_x^+=5$, $\dim\mathrm{ad}_x^-=4$. A Γ_L-map $\mathrm{ad}_x^\pm\to\mathbb{F}(\bar\omega)$ composed with the equivariant projection $X\mapsto\frac12(X\pm tXt^{-1})$ is a map from ad_x, so it vanishes by (iii). (v) *Complex conjugations.* Let $g=\gamma\bar c\in\Gamma_L$ with $g^2=1$, $B=\bar\rho(\gamma)$. Then $\bar r(g)^2=B({}^aB)^{-1}=1$, so ${}^aB=B$ and $S=BJ$ is symmetric (${}^tS=J\,{}^tB=({}^aB)J=S$). With $B^{-1}=JS^{-1}$, $g\cdot X=-BJ\,{}^tXJB^{-1}=-S\,{}^tXS^{-1}$, so $g\cdot X=X$ if and only if XS is antisymmetric: $\dim\mathrm{ad}_x^{g=1}=n(n-1)/2$. If x is endoscopic, $tB=Bt$ and $tJ=Jt$ give $tSt^{-1}=S$, so $X\in\mathrm{ad}_x^\pm$ iff $t(XS)t^{-1}=\pm XS$; the antisymmetric 3×3 matrices supported at $(1,3)$ are even and those at $(1,2),(2,3)$ odd, so $\dim(\mathrm{ad}_x^+)^{g=1}=1$, $\dim(\mathrm{ad}_x^-)^{g=1}=2$. This applies to $g=\bar c$ and to $g=\pi_L(c_u)$ for every real place u of L^+ ($c_u\notin G_L$ as L is CM, and $c_u^2=1$). This proves (Kq0); the groups H^i are $\mathbb{F}'$-spaces since all cochain spaces in their definition are, and their $\mathbb{F}$-dimensions are $[\mathbb{F}':\mathbb{F}]$ times their $\mathbb{F}'$-dimensions.

Step 3: Selmer cocycles. Let $(W,\mathfrak{b}_W)$ be $(\mathrm{ad}_x,\mathfrak{b}_x)$ or $(\mathrm{ad}_x^\pm,\mathfrak{b}_x^\pm)$, and Z_W the space of continuous cocycles $z\in Z^1(\Gamma_L,W)$ such that for every $v\mid\ell$ there is $X_v\in W$ with $z(\pi_L(h))-(h-1)X_v\in\mathfrak{b}_{W,v}$ for all $h\in D_v$. Then

$$\dim Z_W=h^1(\Gamma_L;W,\mathfrak{b}_W)+\dim W-h^0(\Gamma_L;W,\mathfrak{b}_W)\qquad\text{(dimensions over }\mathbb{F}').$$

Indeed, H^1 is the quotient of the space $\mathcal{Z}$ of pairs $(z,(X_v))$ as above by $\mathcal{B}=\{((g-1)X,(X_v)):X\in W,\ X_v-X\in\mathfrak{b}_{W,v}\}$. The fibres of $\mathcal{Z}\to Z_W$ are cosets of $\prod_v\{\Delta:(h-1)\Delta\in\mathfrak{b}_{W,v}\ \forall h\in D_v\}=\prod_v\mathfrak{b}_{W,v}$ (Step 2 (ii), (iv)), so $\dim\mathcal{Z}=\dim Z_W+\sum_v\dim\mathfrak{b}_{W,v}$; and $(X,(\beta_v))\mapsto((g-1)X,(X+\beta_v))$ maps $W\times\prod_v\mathfrak{b}_{W,v}$ onto $\mathcal{B}$ with kernel $\{(X,(-X)):X\in W^{\Gamma_L}\cap\bigcap_v\mathfrak{b}_{W,v}\}\cong H^0(\Gamma_L;W,\mathfrak{b}_W)$. Moreover the map $z\mapsto z(\bar c)$ sends Z_W onto $W^{\bar c=-1}$: $z(\bar c^2)=0$ gives $\bar c\cdot z(\bar c)=-z(\bar c)$, and the coboundary of $-\frac12Y$ (which lies in Z_W with $X_v=-\frac12Y$) takes the value Y at $\bar c$ for $Y\in W^{\bar c=-1}$. So, by Step 2 (v) with $g=\bar c$,

$$\dim\{z\in Z_W:z(\bar c)=0\}=h^1-h^0+\dim W^{\bar c=1}=h^1-h^0+\begin{cases}n(n-1)/2 & (W=\mathrm{ad}_x),\\ 1 & (W=\mathrm{ad}_x^+),\\ 2 & (W=\mathrm{ad}_x^-).\end{cases}\tag{49.2}$$

Step 4: the tangent space, (Kq1). The $\mathcal{O}'$-algebra homomorphisms $R\to\mathbb{F}'[\varepsilon]$ form an $\mathbb{F}'$-

vector space T dual to t_R^* ($h \mapsto$ the form $y \mapsto \varepsilon$-coefficient of $h(y)$ on $\mathfrak{m}_R$; classical), and θ acts on both compatibly, so $\dim t_\pm^* = \dim T^\pm$, where $T^\pm$ are the ±1-eigenspaces of the involution $h \mapsto h \circ \theta$ of T. By (Kr3), $h \in T$ corresponds to a family (ρ, ψ) over $\mathbb{F}'[\varepsilon]$ lifting $(\bar\rho, \bar\psi)$ with (Po1)–(Po4); write $\rho(\gamma) = (1+\varepsilon z_h(\gamma))\bar\rho(\gamma)$ and $z_h(\gamma\bar c) = z_h(\gamma)$, so that $r_\rho(g) = (1+\varepsilon z_h(g))\bar r(g)$ for all $g \in \Gamma_L$. The map $h \mapsto z_h$ is $\mathbb{F}'$-linear (the entries of ρ are those of $h \circ \rho_x$, linear in the form of h), and $z_{h\circ\theta} = t z_h t^{-1}$. By Step 1 and (49.1), r_ρ is a homomorphism iff z_h is a cocycle; (Po1) says z_h is continuous; and $z_h(\bar c) = 0$. *Marking.* Let $\bar A = \bar\rho(\pi_L(g_\star))$ and $A = \rho(\pi_L(g_\star))$. By (Po2), $\rho(\pi_L(c g_\star^{-1} c)) = {}^a A$, so (Po3) says that $A - {}^aA$ has zero off-diagonal entries; this holds for $\bar A$ (Kr2). With $z = z_h(\pi_L(g_\star))$, $A - {}^aA = \bar A - {}^a\bar A + \varepsilon(z\bar A - {}^a(z\bar A))$, and $z\bar A - {}^a(z\bar A) \in \mathfrak{o}$; so (Po3) is $\mu(z) = 0$ for the linear map $\mu\colon \mathrm{ad}_x \to \mathfrak{o}_{\mathrm{off}}$, $\mu(X) =$ off-diagonal part of $X\bar A - {}^a(X\bar A)$. If x is endoscopic, $\bar A$ commutes with t and $\mu(tXt^{-1}) = t\mu(X)t^{-1}$, so $\mu(\mathrm{ad}_x^+) \subseteq \mathfrak{o}_{\mathrm{off}} \cap \mathrm{ad}_x^+ = 0$. *Ordinarity.* Fix v. Over $C = \mathbb{F}'[\varepsilon]$, $(\rho \circ \pi_L|_{D_v}, \psi_v)$ satisfies the hypotheses on h_v in (Fg2) (residually distinct, residual characteristic polynomial by (Kr1)); so (Po4) at v holds iff some $Q \in \mathrm{GL}_n(\mathbb{F}'[\varepsilon])$ triangularises it ((Fg1), (Fg2)). Then $\bar Q = P_v B$ with B upper triangular (Fg3), and QB^{-1} triangularises too, so we may take $Q = (1 - \varepsilon X_v)P_v$. Then

$$Q^{-1}\rho(\pi_L(h))Q = \big(1 + \varepsilon P_v^{-1}(z_h(\pi_L(h)) - (h-1)X_v)P_v\big)U_h, \qquad U_h = P_v^{-1}\bar\rho(\pi_L(h))P_v,$$

which is upper triangular iff $z_h(\pi_L(h)) - (h-1)X_v \in \mathfrak{b}_{x,v}$, and then its diagonal determines ψ_v. Two choices of X_v differ by an element of $\mathfrak{b}_{x,v}$ (Step 2 (ii)), which changes $P_v^{-1}(\cdot)P_v$ by $U_h Y U_h^{-1} - Y$ with Y upper triangular, of zero diagonal; so ψ is determined by z_h. *Conclusion.* Conversely, for a continuous cocycle z with $z(\bar c) = 0$, $\mu(z(\pi_L(g_\star))) = 0$ and X_v as above, $\rho = (1+\varepsilon z)\bar\rho$ with ψ_v the diagonal of $Q^{-1}\rho Q$ (homomorphisms, as diagonals of a triangular representation) satisfies (Po1)–(Po4) by the same computations and (Fg1). So $h \mapsto z_h$ is an $\mathbb{F}'$-linear isomorphism of T onto $\{z \in Z_{\mathrm{ad}_x} : z(\bar c) = 0,\ \mu(z(\pi_L(g_\star))) = 0\}$, and of $T^\pm$ onto the same space formed with $Z_{\mathrm{ad}_x^\pm}$ (a cocycle with values in $\mathrm{ad}_x^\pm$ admits $X_v \in \mathrm{ad}_x^\pm$: project X_v by $X \mapsto \frac12(X \pm tXt^{-1})$, which preserves $\mathfrak{b}_{x,v}$ and commutes with D_v). By (49.2) and $\dim \mathfrak{o}_{\mathrm{off}} = n(n-1)/2 - 1$: $\dim T \ge h_x^1 - h_x^0 + 1$; $\dim T^+ \ge h_{x,+}^1 - h_{x,+}^0 + 1$ (no marking condition in ad_x^+); $\dim T^- \ge h_{x,-}^1 - h_{x,-}^0 + 2 - 2$. This is (Kq1).

Step 5: the obstruction, (Kq2). In case (a) put $\theta_C = \mathrm{id}$, $\theta_{\bar C} = \mathrm{id}$, $\epsilon = 1$. Let $(\bar\rho', \bar\psi') = (\bar\xi \circ \rho_x, \bar\xi \circ \psi_x)$, a family over $\bar C$ as in (Kr3), with $\theta_{\bar C}(\bar\rho'(\gamma)) = t\bar\rho'(\gamma)t^{-1}$ and $\theta_{\bar C}(\bar\psi') = \bar\psi'$, and $\bar r' = r_{\bar\rho'}$. On $\mathcal{G}(C)$ let τ be the involution $X \mapsto t^{-1}\theta_C(X)t$ (entrywise), $j \mapsto j$ (it respects $jXj^{-1} = ({}^aX)^{-1}$ by $Jt = tJ$); likewise on $\mathcal{G}(\bar C)$. On $\mathrm{M}_n(I) = \mathrm{ad}_x \otimes I$, $\tau(Y) = \epsilon\, t^{-1}Yt$, so the τ-fixed elements are $\mathrm{ad}_x^\epsilon \otimes I$. Since ℓ is odd, a lift s of a τ-fixed element can be replaced by the lift $\frac12(s + \tau(s))$, which is τ-fixed (*averaging*). The rings C, $\bar C$ are finite. (i) *Global cochain.* Let $U \subseteq \Gamma_L^0$ be an open normal subgroup on which $\bar\rho'$ is trivial; choose lifts in $\mathrm{GL}_n(C)$ of $\bar\rho'$ on the cosets of U, equal to 1 on U, average them, and extend by $\sigma(\gamma\bar c) = \sigma(\gamma)j$: a locally constant τ-fixed map $\sigma\colon \Gamma_L \to \mathcal{G}(C)$ lifting $\bar r'$ with $\sigma(1) = 1$, $\sigma(\bar c) = j$. Define k by $\sigma(g)\sigma(g')\sigma(gg')^{-1} = 1 + k(g,g')$; then k is continuous with values in $\mathrm{ad}_x^\epsilon \otimes I$, and expanding $\sigma(g)\sigma(g')\sigma(g'')$ in two ways with (49.1) gives $k(g,g') + k(gg',g'') = g \cdot k(g',g'') + k(g,g'g'')$: k is a 2-cocycle. (ii) *Local cochains.* Fix v and P_v, t_v' as in Step 2 ($t_v' = 1$ in case (a)), and for a matrix Q over C or $\bar C$ put $Q^\tau = t^{-1}\theta(Q)t_v'$, an involution. By (Fg2) over $\bar C$ some $\bar Q$ triangularises $(\bar\rho' \circ \pi_L|_{D_v}, \bar\psi_v')$; its residue is P_vB with B upper triangular (Fg3), and multiplying $\bar Q$ on the right by the inverse of an upper triangular lift of B we get $\bar Q \equiv P_v$. Then $\bar Q^\tau$ triangularises as well (because $t^{-1}\theta(\bar\rho')t = \bar\rho'$, $\theta(\bar\psi') = \bar\psi'$ and t_v' is diagonal) and $\bar Q^\tau \equiv t^{-1}P_v t_v' = P_v$; by (Fg3) $\bar Q^\tau = \bar Q B'$ with B' upper triangular, $B' \equiv 1$, so $\frac12(\bar Q + \bar Q^\tau) = \bar Q \cdot \frac12(1 + B')$ triangularises and is fixed by $Q \mapsto Q^\tau$. Replace $\bar Q$ by it, and let $Q_v \in \mathrm{GL}_n(C)$ be a lift with $Q_v^\tau = Q_v$ (average any lift with its image under $Q \mapsto Q^\tau$). Put $\bar u(h) = \bar Q^{-1}\bar\rho'(\pi_L(h))\bar Q$; this is an upper triangular homomorphism with $t_v'\theta(\bar u(h))t_v' = \bar u(h)$; choose a locally constant upper triangular lift u on D_v, $u(1) = 1$, fixed under $X \mapsto t_v'\theta_C(X)t_v'$ (averaging preserves upper triangularity). Define $k_v(h) \in \mathrm{M}_n(I)$ by $\sigma(\pi_L(h)) = (1 + k_v(h))Q_v u(h) Q_v^{-1}$; it

is continuous and τ-fixed (as $\tau(Q_v u(h) Q_v^{-1}) = Q_v u(h) Q_v^{-1}$), so k_v has values in $\mathrm{ad}_x^\epsilon \otimes I$. Writing $u(h)u(h') = (1+\beta(h,h'))u(hh')$ with β upper triangular with entries in I, (49.1) gives

$$k(\pi_L(h), \pi_L(h')) = k_v(h) + h \cdot k_v(h') - k_v(hh') + P_v \beta(h,h') P_v^{-1},$$

and $P_v \beta P_v^{-1} \in \mathfrak{b}_{x,v} \otimes I$; being also τ-fixed, it lies in $\mathfrak{b}_{x,v}^\epsilon \otimes I$. (iii) *The class.* For $\phi \in I^\vee = \mathrm{Hom}_{\mathbb{F}'}(I, \mathbb{F}')$ the pair $(\phi \circ k, (\phi \circ k_v)_v)$ is a pair as in the definition of $H^2(\Gamma_L; \mathrm{ad}_x^\epsilon, \mathfrak{b}_x^\epsilon)$ (by (i), (ii)); let $o(\phi)$ be its class, $\mathbb{F}'$-linear in ϕ. If $\dim I > d$, then $\dim I^\vee > \dim H^2$, so $o(\phi) = 0$ for some $\phi \neq 0$. Put $I' = \ker \phi$, an ideal ($\mathfrak{m}_C I = 0$) with $\dim I/I' = 1$, $C' = C/I'$, and identify I/I' with $\mathbb{F}'$ by ϕ. (iv) *A lift.* The equality $o(\phi) = 0$ means: there are a continuous $\beta\colon \Gamma_L \to \mathrm{ad}_x^\epsilon$ with $\phi k = d\beta$ and $Y_v \in \mathrm{ad}_x^\epsilon$ with $\phi k_v(h) - \beta(\pi_L(h)) + (h-1)Y_v \in \mathfrak{b}_{x,v}^\epsilon$ for $h \in D_v$. From $k(1,1) = k(\bar c, \bar c) = 0$ we get $\beta(1) = 0$ and $\bar c \cdot \beta(\bar c) = -\beta(\bar c)$; replacing β by $\beta + dX_0$ and Y_v by $Y_v + X_0$ with $X_0 = \frac{1}{2}\beta(\bar c)$ we may assume $\beta(\bar c) = 0$. Over C' put $\sigma'(g) = (1-\beta(g))\sigma(g)$. By (49.1), $\sigma'(g)\sigma'(g') = (1 - \beta(g) - g\cdot\beta(g') + \phi k(g,g'))\sigma(gg') = \sigma'(gg')$, so σ' is a homomorphism, locally constant, with $\sigma'(\bar c) = j$ and $\sigma'(\Gamma_L^0) \subseteq \mathrm{GL}_n(C')$; by Step 1, $\rho' = \sigma'|_{\Gamma_L^0}$ satisfies (Po1), (Po2) and lifts $\bar\rho'$. For $v \mid \ell$ and $h \in D_v$, with $b_h = \phi k_v(h) - \beta(\pi_L(h)) + (h-1)Y_v \in \mathfrak{b}_{x,v} \otimes \mathbb{F}'$ and $Q_v' = (1+Y_v)Q_v$,

$$\sigma'(\pi_L(h)) Q_v' = (1 - (h-1)Y_v + b_h)(1 + h\cdot Y_v) Q_v u(h) = (1 + Y_v + b_h) Q_v u(h),$$
$$Q_v'^{-1} \rho'(\pi_L(h)) Q_v' = (1 + P_v^{-1} b_h P_v) u(h),$$

upper triangular. Let $\psi'_{v,a}(h)$ be its diagonal entries: locally constant homomorphisms $D_v \to C'^\times$ lifting $\bar\psi'_{v,a}$; by (Fg1), (Po4) holds for (ρ', ψ'). (v) *The marking.* Let $\bar A = \bar\rho(\pi_L(g_\star))$, $A' = \rho'(\pi_L(g_\star))$, and κ the off-diagonal part of $A' - {}^a A'$; it lies in $\mathfrak{o}_{\mathrm{off}} \otimes \mathbb{F}' \subseteq \mathrm{M}_n(I/I')$, since $\bar\rho'$ is marked and ${}^a(A' - {}^a A') = -(A' - {}^a A')$. By (Kr1) and the marking of $\bar\rho$, $\bar A - {}^a \bar A = \mathrm{diag}(\delta_a)$ with $\delta_a = \bar\Psi(g_\star)^{w_{n,a}} - \bar\Psi(g_\star)^{-w_{n,a}}$ (as $w_{n,n+1-a} = -w_{n,a}$). For $n = 3$, $\delta = (\delta_1, 0, -\delta_1)$ with $\delta_1 \neq 0$ ($\bar\Psi(g_\star)^2 \neq 1$, $g_\star$ a marking) and $2\delta_1 \neq 0$ (ℓ odd), so the map $Y \mapsto [Y, \mathrm{diag}(\delta)]$, $Y_{ab} \mapsto (\delta_b - \delta_a) Y_{ab}$, is a bijection of $\mathfrak{o}_{\mathrm{off}}$ (it preserves $\mathfrak{o}$, since ${}^a\mathrm{diag}(\delta) = -\mathrm{diag}(\delta)$); for $n = 2$, $\mathfrak{o}_{\mathrm{off}} = 0$. Choose $Y \in \mathfrak{o}_{\mathrm{off}} \otimes \mathbb{F}'$ with $[Y, \mathrm{diag}(\delta)] = -\kappa$ and put $w = 1 + Y$. Then ${}^a w = 1 - Y = w^{-1}$, so $\rho'' = w\rho' w^{-1}$ still satisfies (Po2) (${}^a(wXw^{-1}) = w\,{}^aX w^{-1}$), (Po1), (Po4) with the same ψ' (triangularised by wQ_v'), lifts $\bar\rho'$, and $\rho''(\pi_L(g_\star)) - {}^a\rho''(\pi_L(g_\star)) = w(A' - {}^aA')w^{-1} = (A' - {}^aA') + [Y, \mathrm{diag}(\delta)]$ has off-diagonal part $\kappa - \kappa = 0$: (Po3). (vi) By (Kr3) there is $\xi\colon R \to C'$ with $\xi \circ \rho_x = \rho''$, $\xi \circ \psi_x = \psi'$; its composite with $C' \to \bar C$ corresponds to $(\bar\rho', \bar\psi')$, hence equals $\bar\xi$ by the injectivity in (Kr3). This proves (Kq2).

Step 6: the counts, (Kq3). Apply (Ar5)(iv) to $W = \mathrm{ad}_x$ (resp. $\mathrm{ad}_x^\pm$) read as $\mathbb{F}$-spaces, with $b_v = \mathfrak{b}_{x,v}$ (resp. $\mathfrak{b}_{x,v}^\pm$): the hypotheses hold by Step 1 and Step 2 (i), (iii), (iv). Dividing the inequality (48.1) by $[\mathbb{F}' : \mathbb{F}]$, using $\sum_{v|\ell} d_v = d_L$, the fact that the totally real field L^+ has exactly d_L real places, and Step 2 (i), (iv), (v):

$$h_x^2 + h_x^0 + d_L \tfrac{n(n+1)}{2} \le h_x^1 + d_L \tfrac{n(n-1)}{2}, \quad h_{x,+}^2 + h_{x,+}^0 + 4d_L \le h_{x,+}^1 + d_L, \quad h_{x,-}^2 + h_{x,-}^0 + 2d_L \le h_{x,-}^1 + 2d_L,$$

with all H^2 finite-dimensional. This is (Kq3). □

Lemma 49.24. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2,3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $W = W_n^{\mathrm{po}}$, $E = E_n^{\mathrm{po}}$. Let x be a core prime of E that is a maximal ideal of E, $(\mathcal{O}', R, \iota)$ a local chart at x with chart lift $(\rho_x, (\psi_{x,v,a}))$, $\mathbb{F}'$ the residue field of $\mathcal{O}'$ (and of R), and $\bar\rho_x\colon \Gamma_L^0 \to \mathrm{GL}_n(\mathbb{F}')$, $\bar\psi_{x,v,a}\colon D_v \to \mathbb{F}'^\times$ the reductions of ρ_x, $\psi_{x,v,a}$ modulo $\mathfrak{m}_R$. Then:*

(Kr1) *for every $\gamma \in \Gamma_L^0$ the matrix $\bar\rho_x(\gamma)$ is lower triangular with diagonal entries $\bar\Psi(\gamma)^{w_{n,1}}$, $\dots$, $\bar\Psi(\gamma)^{w_{n,n}}$, and $\bar\psi_{x,v,a}(g) = \bar\Psi(g)^{w_{n,a}}$ for $v \mid \ell$, $1 \le a \le n$, $g \in D_v$;*

(Kr2) *$(\rho_x, (\psi_{x,v,a}))$ consists of homomorphisms $\Gamma_L^0 \to \mathrm{GL}_n(R)$, $D_v \to R^\times$ satisfying (Po2)–(Po4) with the test ring replaced by R, and for every $k \geq 1$ the maps $\gamma \mapsto \rho_x(\gamma) \bmod \mathfrak{m}_R^k$ and $g \mapsto \psi_{x,v,a}(g) \bmod \mathfrak{m}_R^k$ are locally constant;*

(Kr3) *for every $A \in \mathrm{CNL}_{\mathcal{O}'}$ of finite length, $h \mapsto (h \circ \rho_x, (h \circ \psi_{x,v,a})_{v,a})$ is a bijection from the set of $\mathcal{O}'$-algebra homomorphisms $R \to A$ onto the set of families $(\rho, (\psi_{v,a}))$ of homomorphisms $\rho\colon \Gamma_L^0 \to \mathrm{GL}_n(A)$ and $\psi_{v,a}\colon D_v \to A^\times$ $(v \mid \ell,\ 1 \leq a \leq n)$ satisfying (Po1)–(Po4) with the test ring replaced by A, whose reductions modulo $\mathfrak{m}_A$ are $\bar{\rho}_x$ and $\bar{\psi}_{x,v,a}$;*

(Kr4) *if $n = 3$ and $I_E^{\mathrm{odd}} \subseteq x$, then $\bar{\rho}_x(\gamma)$ commutes with $t = \mathrm{diag}(-1, 1, -1)$ for every γ, and there is an $\mathcal{O}'$-algebra automorphism θ of R with $\theta \circ \theta = \mathrm{id}_R$, $\theta(\rho_x(\gamma)_{ab}) = (-1)^{w_{3,a} - w_{3,b}} \rho_x(\gamma)_{ab}$ and $\theta(\psi_{x,v,a}(g)) = \psi_{x,v,a}(g)$ for all $\gamma \in \Gamma_L^0$, $1 \leq a, b \leq 3$, $v \mid \ell$ and $g \in D_v$.*

Proof. The argument is commutative algebra; no automorphy is used, over L or over K. Write $\rho = \rho^{\mathrm{u}}$, $\psi = \psi^{\mathrm{u}}$ inside this proof only where no confusion arises with the lifts of (Kr3), which are written (ρ', ψ'); let $\mathfrak{m}$ be the maximal ideal of R_n^{po}, $\mathbb{F} = \mathbb{F}_{\mathcal{O}} \subseteq \mathbb{F}'$, and $W_0 \subseteq W$ the $\mathcal{O}$-subalgebra generated by the weight-zero monomials and the values $\psi_{v,a}(g)$, so that by definition of W every $y \in W$ has, for every t, some $y_t \in W_0$ with $y - y_t \in \mathfrak{m}^t$. Since x is a core prime, $\mathfrak{m}_W \subseteq x = \iota^{-1}(\mathfrak{m}_R)$. For $e \in E$ let $\bar{e} \in \mathbb{F}'$ be the residue of $\iota(e)$.

Step 0: two facts on W. (a) For every $y \in W$ and $k \geq 1$ there is $y' \in W_0$ with $y - y' \in \mathfrak{m}_W^k$. Indeed $W \in \mathrm{CNL}_{\mathcal{O}}$ (Gw1), the ideals $\mathfrak{m}^t \cap W$ decrease with intersection 0 (Krull's intersection theorem in R_n^{po}), so by Chevalley's lemma Theorem B.154 there is t with $\mathfrak{m}^t \cap W \subseteq \mathfrak{m}_W^k$; take $y' = y_t$. (b) An element $y \in W$ whose residue in $R_n^{\mathrm{po}}/\mathfrak{m} = \mathbb{F}$ is 0 lies in $\mathfrak{m} \cap W = \mathfrak{m}_W \subseteq x$, so $\bar{y} = 0$.

(Kr1). The diagonal entry $\rho(\gamma)_{aa}$ is a weight-zero monomial, so lies in W, and by (Po5) its residue is $\bar{\Psi}(\gamma)^{w_{n,a}}$; by (b) applied to $\rho(\gamma)_{aa} - \tau$ with $\tau \in \mathcal{O}$ of that residue, $\bar{\rho}_x(\gamma)_{aa} = \bar{\Psi}(\gamma)^{w_{n,a}}$. The same argument with $\psi_{v,a}(g) \in W$ gives the second assertion. As x is a core prime, $I_E^- \not\subseteq x$, so some lower entry $e_- = \rho(\gamma')_{ba}$ $(b > a)$, of weight $-q < 0$, is not in x. For an upper entry e_+, of weight $p > 0$, the product $e_+^q e_-^p$ is a weight-zero monomial with off-diagonal factors, whose residue is 0 by (Po5); by (b) it lies in x, and as x is prime and $e_- \notin x$, $e_+ \in x$. So $\bar{\rho}_x(\gamma)$ is lower triangular.

(Kr2). The determinant of $\rho(\gamma)$ is a sum of weight-zero monomials, so lies in W; it is a unit of R_n^{po}, hence not in $\mathfrak{m}_W$, hence a unit of the local ring W. So $\rho(\gamma) \in \mathrm{GL}_n(E)$, likewise $\psi_{v,a}(g) \in W^\times$, and $\rho_x = \iota \circ \rho$, $\psi_{x,v,a} = \iota \circ \psi_{v,a}$ are homomorphisms into $\mathrm{GL}_n(R)$ and $R^\times$. (Po2)–(Po4) are polynomial identities among entries of ρ and values of ψ, all in E; they hold in R_n^{po}, hence in E, hence after applying ι. Continuity: let $j = w_{n,a} - w_{n,b}$ and $k \geq 1$. The entries $\rho(\gamma)_{ab}$ lie in gr_j, a finitely generated W-module (Gw3); the submodules $\mathfrak{m}^t \cap \mathrm{gr}_j$ decrease with intersection 0, so by Theorem B.154 there is t with $\mathfrak{m}^t \cap \mathrm{gr}_j \subseteq \mathfrak{m}_W^k \mathrm{gr}_j$. By (Po1) for ρ, $\gamma \mapsto \rho(\gamma) \bmod \mathfrak{m}^t$ is locally constant, hence so is $\gamma \mapsto \rho(\gamma)_{ab} \bmod \mathfrak{m}_W^k \mathrm{gr}_j$, and since $\iota(\mathfrak{m}_W^k \mathrm{gr}_j) \subseteq \mathfrak{m}_R^k$ so is $\gamma \mapsto \rho_x(\gamma)_{ab} \bmod \mathfrak{m}_R^k$. The same with $\mathrm{gr}_0 = W$ gives the assertion for $\psi_{x,v,a}$.

(Kr3), the map is well defined. An $\mathcal{O}'$-algebra homomorphism $h\colon R \to A$ is local and induces the identity of $\mathbb{F}'$ (the composite $R \to A \to \mathbb{F}'$ is surjective, as $\mathcal{O}' \to \mathbb{F}'$ is, so its kernel is the maximal ideal $\mathfrak{m}_R$, and the induced map $\mathbb{F}' \to \mathbb{F}'$ is $\mathcal{O}'$-linear). So $(h \circ \rho_x, h \circ \psi_x)$ reduces to $(\bar{\rho}_x, \bar{\psi}_x)$, satisfies (Po2)–(Po4) by (Kr2), and (Po1) by (Kr2), as $h(\mathfrak{m}_R^k) \subseteq \mathfrak{m}_A^k$.

(Kr3), injectivity. Let $h, h'\colon R \to A$ give the same family, and $f = h \circ \iota$, $f' = h' \circ \iota\colon E \to A$. They agree on all entries $\rho(\gamma)_{ab}$ and values $\psi_{v,a}(g)$, hence on W_0. Choose k with $\mathfrak{m}_A^k = 0$. Since

$\iota(\mathfrak{m}_W) \subseteq \mathfrak{m}_R$ and h, h' are local, $f(\mathfrak{m}_W^k) = f'(\mathfrak{m}_W^k) = 0$; by Step 0 (a), $f = f'$ on W. As E is generated over W by the entries, $f = f'$. By the definition of a local chart (with $C = A$; the composite of f with $A \to \mathbb{F}'$ is that of ι, as h induces the identity of $\mathbb{F}'$), $h = h'$.

(Kr3), surjectivity. Let $(\rho', (\psi'_{v,a}))$ be a family as in (Kr3). Let $C_A \subseteq A$ be the set of elements whose residue lies in $\mathbb{F}$. It is an $\mathcal{O}$-subalgebra; it is local with maximal ideal $\mathfrak{m}_A$ (an element of $C_A \setminus \mathfrak{m}_A$ is a unit of A whose inverse has residue in $\mathbb{F}^\times$) and residue field $\mathbb{F}$; and A is finite (of finite length over $\mathcal{O}'$, whose residue field is finite), so C_A is a finite local ring, hence complete and Noetherian. So $C_A \in \mathrm{CNL}_{\mathcal{O}}$. We check the hypotheses of Lemma 49.16 with $B = A$ and $C = C_A$. (Po2)–(Po4) hold for (ρ', ψ'). (Cq1) and (Cq5): by (Kr1), $\rho'(\gamma)_{aa}$ and $\psi'_{v,a}(g)$ have residues $\bar{\Psi}(\gamma)^{w_{n,a}}$, $\bar{\Psi}(g)^{w_{n,a}} \in \mathbb{F}$, so they lie in C_A with the required congruences; local constancy holds by (Po1) (with $\mathfrak{m}_A^k = 0$). (Cq2): let $y = \prod_k \rho(\gamma_k)_{a_k b_k}$ be a weight-zero monomial with some $a_k \neq b_k$; its residue in $\mathbb{F}$ is 0 by (Po5), so $\bar{y} = 0$ by Step 0 (b); the residue of $\prod_k \rho'(\gamma_k)_{a_k b_k}$ is $\prod_k \bar{\rho}_x(\gamma_k)_{a_k b_k} = \bar{y} = 0$, so this product lies in $\mathfrak{m}_A = \mathfrak{m}_{C_A}$. (Cq3): A is finite. (Cq4): ρ' is locally constant by (Po1). Hence there are $\mathcal{O}$-algebra homomorphisms $\xi\colon E \to A$ and $\beta\colon W \to C_A$ with $\beta(\mathfrak{m}_W) \subseteq \mathfrak{m}_A$, $\xi(\rho(\gamma)_{ab}) = \rho'(\gamma)_{ab}$, $\xi(\psi_{v,a}(g)) = \psi'_{v,a}(g)$ and $\xi|_W = \beta$. The two ring homomorphisms $E \to \mathbb{F}'$, $e \mapsto \xi(e) \bmod \mathfrak{m}_A$ and $e \mapsto \bar{e}$, agree on the entries (both give $\bar{\rho}_x$) and on $W = \mathcal{O} + \mathfrak{m}_W$ (both are $\mathcal{O}$-linear and kill $\mathfrak{m}_W$), hence on E. By the definition of a local chart there is an $\mathcal{O}'$-algebra homomorphism $h\colon R \to A$ with $h \circ \iota = \xi$; then $h \circ \rho_x = \xi \circ \rho = \rho'$ and $h \circ \psi_{x,v,a} = \psi'_{v,a}$.

(Kr4). The odd entries of ρ generate $I_E^{\mathrm{odd}} \subseteq x$, so $\bar{\rho}_x(\gamma)_{ab} = 0$ when $w_{3,a} - w_{3,b}$ is odd; as $(tXt^{-1})_{ab} = (-1)^{w_{3,a}-w_{3,b}} X_{ab}$, $\bar{\rho}_x(\gamma)$ commutes with t. Write ${}^a X = J_3\, {}^t X J_3$. Since $t^{-1} = t$ and $J_3 t = t J_3$ (conjugation by J_3 reverses the order of the diagonal entries of t), ${}^a(tXt^{-1}) = t\, {}^a X\, t^{-1}$. Hence $(t\rho t^{-1}, \psi)$ is again a polarised ordinary lift to R_3^{po}: (Po1) and (Po4) are preserved by conjugation, (Po2) by the identity just shown, (Po3) because conjugation by the diagonal matrix t preserves the vanishing of off-diagonal entries, and (Po5) because t commutes with diagonal matrices. By the universal property there is a morphism θ_{u} of R_3^{po} in $\mathrm{CNL}_{\mathcal{O}}$ with $\theta_{\mathrm{u}}(\rho(\gamma)_{ab}) = (-1)^{w_{3,a}-w_{3,b}} \rho(\gamma)_{ab}$ and $\theta_{\mathrm{u}}(\psi_{v,a}(g)) = \psi_{v,a}(g)$, and $\theta_{\mathrm{u}} \circ \theta_{\mathrm{u}} = \mathrm{id}$ by uniqueness. The morphism θ_{u} fixes W_0 pointwise (the sign of a weight-zero monomial is $+1$) and maps $\mathfrak{m}^t$ into itself; so for $y \in W$, $\theta_{\mathrm{u}}(y) - y = (\theta_{\mathrm{u}}(y - y_t)) - (y - y_t) \in \mathfrak{m}^t$ for all t, and $\theta_{\mathrm{u}}(y) = y$. Hence θ_{u} restricts to an $\mathcal{O}$-algebra involution θ_E of E fixing W. The homomorphism $\iota \circ \theta_E\colon E \to R$ has the same composite with $R \to \mathbb{F}'$ as ι: both are $\mathcal{O}$-linear and kill $\mathfrak{m}_W$ on W, and on an entry they give $(-1)^{w_{3,a}-w_{3,b}} \bar{\rho}_x(\gamma)_{ab} = \bar{\rho}_x(\gamma)_{ab}$ because the odd entries of $\bar{\rho}_x$ vanish. By the definition of a local chart (with $C = R$) there is an $\mathcal{O}'$-algebra homomorphism $\theta\colon R \to R$ with $\theta \circ \iota = \iota \circ \theta_E$. Then $\theta \circ \theta \circ \iota = \iota \circ \theta_E^2 = \iota$, so $\theta \circ \theta = \mathrm{id}_R$ by the uniqueness in the definition (with $f = \iota$); θ is an automorphism, and $\theta(\rho_x(\gamma)_{ab}) = \iota(\theta_E(\rho(\gamma)_{ab}))$, $\theta(\psi_{x,v,a}(g)) = \psi_{x,v,a}(g)$ are as stated. □

Lemma 49.25. *All symbols are local to this statement. Let C be a commutative local ring with maximal ideal $\mathfrak{m}_C$ and residue field k_C, $n \geq 1$, D a set, and $\rho\colon D \to \mathrm{M}_n(C)$, $\psi_1, \dots, \psi_n\colon D \to C$ maps. Say that $P \in \mathrm{GL}_n(C)$* triangularises *(ρ, ψ) if $P^{-1}\rho(g)P$ is upper triangular with diagonal entries $\psi_1(g), \dots, \psi_n(g)$ (in this order) for every $g \in D$.*

(Fg1) *If some P triangularises (ρ, ψ), then for all $g_1, \dots, g_n \in D$*

$$\prod_{a=1}^{n} (\rho(g_a) - \psi_a(g_a)) = 0,$$

the product taken with $a = 1$ on the left.

(Fg2) *Conversely, suppose that these products vanish for all $g_1, \dots, g_n \in D$, and that there is $h \in D$ such that $\psi_a(h) - \psi_b(h) \in C^\times$ for all $a \neq b$ and the characteristic polynomial of*

the reduction of $\rho(h)$ modulo $\mathfrak{m}_C$ is $\prod_{a=1}^{n}(X-\bar{\psi}_a(h))$, bars denoting reduction modulo $\mathfrak{m}_C$. Then some $P \in \mathrm{GL}_n(C)$ triangularises (ρ,ψ).

(Fg3) *Under the hypotheses of (Fg2), if P and Q both triangularise (ρ,ψ), then $P^{-1}Q$ is upper triangular.*

Proof. (Fg1). Put $U_a = P^{-1}\rho(g_a)P$, upper triangular with diagonal $(\psi_b(g_a))_b$, and $S_k = Ce_1 + \cdots + Ce_k$ ($S_0 = 0$). Then $U_a - \psi_a(g_a)$ is upper triangular with (a,a)-entry 0, so it maps S_a into S_{a-1}. Hence $\prod_{a=1}^{n}(U_a - \psi_a(g_a))$, applied to $C^n = S_n$, lands successively in $S_{n-1},\dots,S_0 = 0$; conjugating by P gives the claim.

(Fg2), idempotents. Write $M = \rho(h)$ and $\lambda_a = \psi_a(h)$. Taking $g_1 = \cdots = g_n = h$ gives $\prod_c(M-\lambda_c) = 0$ (the factors commute). Put

$$E_a = \prod_{b\neq a} \frac{M-\lambda_b}{\lambda_a-\lambda_b} \qquad (1 \le a \le n).$$

The polynomial $\sum_a \prod_{b\neq a}(X-\lambda_b)/(\lambda_a-\lambda_b) - 1 \in C[X]$ has degree $< n$ and vanishes at $\lambda_1,\dots,\lambda_n$, whose Vandermonde determinant $\prod_{a<b}(\lambda_b-\lambda_a)$ is a unit; so it is 0, and $\sum_a E_a = 1$. For $a \neq b$, E_aE_b is a multiple of $\prod_c(M-\lambda_c) = 0$, and so is $(M-\lambda_a)E_a$. Hence the E_a are orthogonal idempotents with sum 1, $C^n = \bigoplus_a V_a$ with $V_a = E_aC^n$, and M acts on V_a as multiplication by λ_a.

(Fg2), each V_a is free of rank one. V_a is a direct summand of C^n, hence finitely generated projective, hence free over the local ring C, of rank $\dim_{k_C} V_a/\mathfrak{m}_CV_a$. Reducing, $k_C^n = \bigoplus_a \bar{E}_a k_C^n$ with $\bar{E}_a k_C^n \subseteq \ker(\bar{M}-\bar{\lambda}_a)$. The $\bar{\lambda}_a$ are pairwise distinct roots of the characteristic polynomial $\prod_b(X-\bar{\lambda}_b)$ of $\bar{M}$, each of multiplicity one, so $\dim\ker(\bar{M}-\bar{\lambda}_a) \le 1$. As the dimensions add up to n, each V_a is free of rank 1; let v_a generate it, and let P have columns $v_1,\dots,v_n$. Then $P \in \mathrm{GL}_n(C)$, since the v_a form a basis of C^n.

(Fg2), the flag. Put $F_k = V_1 \oplus \cdots \oplus V_k$. Let $g \in D$ and $1 \le k \le n$. Taking $g_k = g$ and $g_a = h$ for $a \neq k$ in the product equation gives $L_k\,(\rho(g)-\psi_k(g))\,R_k = 0$ with $L_k = \prod_{a<k}(M-\lambda_a)$ and $R_k = \prod_{a>k}(M-\lambda_a)$. On V_b, R_k acts by $\prod_{a>k}(\lambda_b-\lambda_a)$, a unit for $b \le k$, so $R_k(F_k) = F_k$; and L_k acts by $\prod_{a<k}(\lambda_b-\lambda_a)$, which is 0 for $b < k$ and a unit for $b \ge k$, so $\ker L_k = F_{k-1}$. Therefore $(\rho(g)-\psi_k(g))F_k \subseteq F_{k-1}$, i.e. $\rho(g)v_k \in \psi_k(g)v_k + F_{k-1}$, which says that $P^{-1}\rho(g)P$ is upper triangular with (k,k)-entry $\psi_k(g)$.

(Fg3). With P as constructed, it suffices to show $P^{-1}Q$ upper triangular for every Q that triangularises (ρ,ψ) (then for two such P_1, Q, $P_1^{-1}Q = (P^{-1}P_1)^{-1}(P^{-1}Q)$ is upper triangular). Let $q_1,\dots,q_n$ be the columns of Q; then $(M-\lambda_k)q_k \in Cq_1 + \cdots + Cq_{k-1}$. We show $q_k \in F_k$ by induction on k. Assume $q_1,\dots,q_{k-1} \in F_{k-1}$. Writing $q_k = \sum_b u_b$ with $u_b \in V_b$, the V_b-component of $(M-\lambda_k)q_k$ is $(\lambda_b-\lambda_k)u_b$, which vanishes for $b \ge k$ because $(M-\lambda_k)q_k \in F_{k-1}$; for $b > k$, $\lambda_b-\lambda_k$ is a unit, so $u_b = 0$ and $q_k \in F_k$. Since $F_k = Cv_1 + \cdots + Cv_k$, the k-th column of $P^{-1}Q$ has zero entries below the k-th row. □

49.4 Endoscopic core primes and local propagation

This subsection proves the local statements of the propagation theorem. At an endoscopic core prime the rank-three chart carries an ideal whose components are chained through thick or irreducible components (Proposition 49.26), by an analysis of the minimal primes of determinantal sections of a power series ring (Lemma 49.27); chains of components of quotients by few elements are described in Lemma 49.28. The non-nice loci of a chart have dimension less than $n\,d_L$ (Proposition 49.29); one input is that the lower-triangular fibre of the entry algebra has

dimension at most $2C_0$ (Lemma 49.30), where C_0 bounds the Selmer ranks of the auxiliary field. Absorption in rank two bounds the endoscopic primes in rank three (Proposition 49.33), and local propagation moves a modular ideal through nice primes and along thick links inside a chart (Lemma 49.34). The subsection ends with the two consequences used in Theorem 49.3: lifts to test rings are classified by the universal lift (Lemma 49.36), and an off-triangular bound gives the residual characteristic polynomials in characteristic ℓ (Lemma 49.37).

Proposition 49.26. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, and suppose a universal polarised ordinary lift of rank 3 over L exists; write $E = E_3^{\mathrm{po}}$. Let x be a core prime of E that is a maximal ideal of E with $I_E^{\mathrm{odd}} \subseteq x$, and $(\mathcal{O}', R, \iota)$ a local chart at x. Then there is an ideal $J \subseteq R$ such that:*

(Ej1) *$J \subseteq P$ for every prime P of R with $I_R^{\mathrm{odd}} \not\subseteq P$;*

(Ej2) $\dim R/P \geq 3d_L + 2$ *for every minimal prime P of J, and any two minimal primes P, Q of J are joined by minimal primes $P = P_0, \dots, P_k = Q$ of J with* $\dim R/(P_{i-1} + P_i) \geq 3d_L + 1$ *for $1 \leq i \leq k$;*

(Ej3) *every minimal prime of J is of irreducible type or thick.*

Proof. The proof is commutative algebra on the chart; in particular, no automorphy over L or over K is used. Write $d = d_L$, $W = W_3^{\mathrm{po}}$, $\rho_x = \iota \circ \rho^{\mathrm{u}}$, $\psi_{x,v,a} = \iota \circ \psi_{v,a}^{\mathrm{u}}$, $\bar{c} = \pi_L(c)$, $\gamma^* = \bar{c}\gamma^{-1}\bar{c}$, $\gamma_\star = \pi_L(g_\star)$. The chart lift satisfies (Po2)–(Po4) in R (they hold for ρ^{u}, with entries in E, and ι is a ring homomorphism); so $\rho_x(\gamma^*)_{ab} = \rho_x(\gamma)_{4-b,4-a}$ and $\rho_x(\gamma_\star)_{ab} = \rho_x(\gamma_\star^*)_{ab}$ for $a \neq b$. A position (a, b) is *odd* if $w_{3,a} - w_{3,b}$ is odd, i.e. $(a, b) \in \{(1,2), (2,1), (2,3), (3,2)\}$, and *even* otherwise.

Step 0: the presentation. By (Qp2) there are a, b, g with $a \geq b + 3d + 1$ and $p\colon S = \mathcal{O}'[[Y_1, \dots, Y_a, X_1, \dots, X_g]] \to R$ with (Ep1)–(Ep4). An odd series lies in the ideal $(X_1, \dots, X_g)$ (each monomial has positive degree in the X_j). For each odd G_i let M_{ij} be the sum of the terms of G_i divisible by X_j and by no $X_{j'}$ with $j' < j$, divided by X_j; then $G_i = \sum_j M_{ij} X_j$, and the constant term of M_{ij} is the coefficient of X_j in G_i, which lies in $\mathfrak{m}_{\mathcal{O}'}$ by (Ep2). So $M = (M_{ij})$ has entries in $\mathfrak{m}_S$, and $G = MX$. Each F_i lies in $\mathfrak{m}_S$ (it is in $\ker p$, hence not a unit).

Step 1: the ideal J, and (Ej1). If $g = 0$, then $I_R^{\mathrm{odd}} = 0$ by (Ep4), every prime contains it, and $J = R$ satisfies (Ej1)–(Ej3) vacuously. Let $g \geq 1$ and $J = p(\det M)R$. From $\mathrm{adj}(M)MX = \det(M)X$ and $MX = G \in \ker p$ we get $p(\det M)\, p(X_j) = 0$ for all j. If P is a prime with $I_R^{\mathrm{odd}} \not\subseteq P$, then $p(X_j) \notin P$ for some j (Ep4), so $p(\det M) \in P$ and $J \subseteq P$.

Step 2: (Ej2). By (Ep1) and Step 0, p induces $R/J \cong S/I$ with $I = (F_1, \dots, F_b, MX, \det M)$; primes of R containing J correspond to primes of S containing I by $P \mapsto p^{-1}(P)$, with $R/P \cong S/p^{-1}(P)$, minimal primes to minimal primes, and $p^{-1}(P+Q) = p^{-1}(P) + p^{-1}(Q)$. Lemma 49.27 with $D = a - b \geq 3d + 1$ gives: every minimal prime P of J has $\dim R/P \geq a - b + 1 \geq 3d + 2$, and any two are joined by minimal primes with $\dim R/(P_{i-1} + P_i) \geq a - b \geq 3d + 1$.

Step 3: minimal primes are off-triangular. Let P be a minimal prime of J. By (Qc3), $I_R^- = R \not\subseteq P$. If $I_R^+ \subseteq P$, then $\dim R/P \leq \dim R/I_R^+ < 3d$ by (Nn1), against Step 2. So P is off-triangular; it is of irreducible type if $I_R^{\mathrm{odd}} \not\subseteq P$. It remains to show: if $I_R^{\mathrm{odd}} \subseteq P$, then P is thick. Fix such P; it is of endoscopic type, and $p(\det M) \in P$.

Step 4: a derivation along a kernel vector of M. Let $Q = R/P$, a domain, $\pi\colon R \to Q$ and $\varphi = \pi \circ p$. By (Ep4), $\varphi(X_j) = 0$, so φ kills every odd series. Since $\det \varphi(M) = \varphi(\det M) = 0$ in the domain Q, there is $0 \neq \nu \in Q^g$ with $\varphi(M)\nu = 0$ (solve over the fraction field and clear denominators). Let $D_0(f) = \sum_c \varphi(\partial f / \partial X_c)\nu_c$ for $f \in S$; it is additive with $D_0(ff') =$

$\varphi(f)D_0(f') + \varphi(f')D_0(f)$, and $D_0(X_j) = \nu_j$. If f is even, each $\partial f/\partial X_c$ is odd and $D_0(f) = 0$; in particular $D_0(F_i) = 0$, and $D_0(G_i) = \sum_j(\varphi(M_{ij})\nu_j + \varphi(X_j)D_0(M_{ij})) = (\varphi(M)\nu)_i = 0$. For $f = \sum_k s_k r_k$ with generators r_k of $\ker p$, $D_0(f) = \sum_k(\varphi(s_k)D_0(r_k) + \varphi(r_k)D_0(s_k)) = 0$. So $D_0 = D \circ p$ for an additive $D\colon R \to Q$ with $D(yy') = \pi(y)D(y') + \pi(y')D(y)$, $D(p(f)) = 0$ for even f, and $D(p(X_j)) = \nu_j$. By (Ep3): $\pi(\rho_x(\gamma)_{ab}) = 0$ at odd positions, $D(\rho_x(\gamma)_{ab}) = 0$ at even positions, and $D(\psi_{x,v,a}(g)) = 0$. Moreover $D(\mathfrak{m}_R^{k+1}) \subseteq \pi(\mathfrak{m}_R)^k Q$ for $k \geq 0$ (induction with the product rule).

Step 5: the modified representation. Call $Y \in \mathrm{M}_3(R)$ *balanced* if $\pi(Y_{ab}) = 0$ at odd positions and $D(Y_{ab}) = 0$ at even positions, and put

$$\theta(Y) = \begin{pmatrix} \pi(Y_{11}) & D(Y_{12}) & \pi(Y_{13}) \\ 0 & \pi(Y_{22}) & 0 \\ \pi(Y_{31}) & D(Y_{32}) & \pi(Y_{33}) \end{pmatrix}.$$

Products of balanced matrices are balanced and $\theta(YY') = \theta(Y)\theta(Y')$; $\theta(1) = 1$; and $\theta(Y - s\cdot 1) = \theta(Y) - \pi(s)\cdot 1$ if $D(s) = 0$.

Let $r(\gamma) = \theta(\rho_x(\gamma))$, a monoid homomorphism $\Gamma_L^0 \to \mathrm{M}_3(Q)$ by Step 4. Its entries are $\pi(\rho_x(\gamma)_{ab})$ or $D(\rho_x(\gamma)_{ab})$, locally constant modulo every power of $\pi(\mathfrak{m}_R)Q$ by (Qc5) and Step 4. (Tk1) holds by construction. (Tk2): the factors $\rho_x(\pi_L(g_a)) - \psi_{x,v,a}(g_a)$ of (Po4) are balanced, so applying θ to (Po4) gives the first product; for the second, (Po2) gives $\rho_x(\pi_L(cg_a^{-1}c)) - \psi_{x,v,4-a}(g_a) = J_3\,{}^t B_a J_3$ with $B_a = \rho_x(\pi_L(g_a)) - \psi_{x,v,4-a}(g_a)$, so the product over $a = 1,2,3$ is $J_3\,{}^t(B_3B_2B_1)J_3 = 0$ by (Po4) for (g_3, g_2, g_1), and θ gives the second product. (Tk3): let $j\colon Q \to F'$ be an injective homomorphism to a field and $y \in F'^3$ with $y_2 = 1$ and $j(r(\gamma))y = j(r(\gamma)_{22})y$ for all γ. The first row at $\gamma_\star$ and at $\gamma_\star^*$ reads $j(\pi(\rho_{11}))y_1 + j(D(\rho_{12})) + j(\pi(\rho_{13}))y_3 = j(\pi(\rho_{22}))y_1$, with $\rho = \rho_x(\gamma_\star)$, resp. the same with ρ_{11} replaced by ρ_{33} (polarisation: $\rho_x(\gamma_\star^*)_{11} = \rho_x(\gamma_\star)_{33}$, $\rho_x(\gamma_\star^*)_{22} = \rho_x(\gamma_\star)_{22}$; marking for the off-diagonal entries). Subtracting, $j(\pi(\rho_x(\gamma_\star)_{11} - \rho_x(\gamma_\star)_{33}))\,y_1 = 0$. The element $\rho_x(\gamma_\star)_{11} - \rho_x(\gamma_\star)_{33}$ is a unit of R: the diagonal entries of ρ^{u} lie in W with residues $\bar\Psi^{\pm 1}(\gamma_\star)$ (Po5), $\iota(\mathfrak{m}_W) \subseteq \mathfrak{m}_R$ as $\mathfrak{m}_W \subseteq x$, and $\bar\Psi(\gamma_\star)^2 \neq 1$. So $y_1 = 0$, and likewise (third row) $y_3 = 0$. Then the first and third rows give $D(\rho_x(\gamma)_{12}) = D(\rho_x(\gamma)_{32}) = 0$ for all γ, and by polarisation $D(\rho_x(\gamma)_{21}) = D(\rho_x(\gamma^*)_{32}) = 0$, $D(\rho_x(\gamma)_{23}) = D(\rho_x(\gamma^*)_{12}) = 0$. As π kills the odd entries, the product rule makes D vanish on the ideal they generate, which is $I_R^{\mathrm{odd}} \ni p(X_j)$; so $\nu_j = D(p(X_j)) = 0$ for all j, a contradiction. Hence P is thick, which proves (Ej3). □

Lemma 49.27. *Let $\mathcal{O}'$ be a coefficient ring, $a, b, g, D \geq 0$ integers with $g \geq 1$ and $a = b + D$, $S = \mathcal{O}'[[Y_1, \ldots, Y_a, X_1, \ldots, X_g]]$ with maximal ideal $\mathfrak{m}_S$, $F_1, \ldots, F_b \in \mathfrak{m}_S$, and $M \in \mathrm{M}_g(S)$ with all entries in $\mathfrak{m}_S$. Let $I \subseteq S$ be the ideal generated by $F_1, \ldots, F_b$, the entries $\sum_j M_{ij}X_j$ $(1 \leq i \leq g)$ of MX, and $\det M$. Then:*

1. *$\dim S/P \geq D + 1$ for every minimal prime P of I;*
2. *any two minimal primes P, Q of I are joined by minimal primes $P = P_0, \ldots, P_k = Q$ of I, $k \geq 0$, with $\dim S/(P_{i-1} + P_i) \geq D$ for $1 \leq i \leq k$.*

Proof. For a matrix Z of indeterminates and a column z of indeterminates, Zz denotes the column with entries $\sum_j Z_{ij}z_j$; e_1 is the first standard basis column and $E_0 = \mathrm{diag}(0, 1, \ldots, 1)$, both of size g.

Step 1: the generic determinantal locus, polynomial version. Let A be a domain, $B = A[Z_{ij}, z_j : 1 \leq i, j \leq g]$, graded by total degree, and $\mathfrak{a} = (Zz, \det Z) \subseteq B$. Let $C = A[N_{ij}, U_{ij}, t]$ (a domain) and $\phi\colon B \to C$ the A-algebra homomorphism with $\phi(Z) = NE_0\,\mathrm{adj}(U)$ and $\phi(z) =$

tUe_1. We claim $\sqrt{\mathfrak{a}} = \ker\phi$. First, $\phi(Zz) = t\,NE_0\operatorname{adj}(U)Ue_1 = t\det(U)\,NE_0e_1 = 0$ and $\phi(\det Z) = \det N \det E_0 \det\operatorname{adj}(U) = 0$, so $\mathfrak{a} \subseteq \ker\phi$, a prime ideal; hence $\sqrt{\mathfrak{a}} \subseteq \ker\phi$. Conversely, let $\mathfrak{q} \supseteq \mathfrak{a}$ be prime, K the fraction field of $B/\mathfrak{q}$, $\psi\colon B \to K$ the quotient map, $\bar{M} = \psi(Z)$, $\bar{m} = \psi(z)$; then $\bar{M}\bar{m} = 0$ and $\det\bar{M} = 0$. Choose $0 \neq u \in K^g$ with $\bar{M}u = 0$ and $s \in K$ with $\bar{m} = su$ (if $\bar{m} \neq 0$ take $u = \bar{m}$, $s = 1$; otherwise any non-zero kernel vector and $s = 0$), and $\bar{U} \in \mathrm{GL}_g(K)$ with first column u. Then $\bar{M}\bar{U}$ has first column 0, so $\bar{M}\bar{U}E_0 = \bar{M}\bar{U}$, and $\bar{N} = \det(\bar{U})^{-1}\bar{M}\bar{U}$ satisfies $\bar{N}E_0\operatorname{adj}(\bar{U}) = \det(\bar{U})^{-1}\bar{M}\bar{U}\operatorname{adj}(\bar{U}) = \bar{M}$, while $s\bar{U}e_1 = \bar{m}$. The A-algebra homomorphism $\chi\colon C \to K$ with $N \mapsto \bar{N}$, $U \mapsto \bar{U}$, $t \mapsto s$ satisfies $\chi\circ\phi = \psi$ (both agree on Z and z), so $\ker\phi \subseteq \mathfrak{q}$. Hence $\ker\phi$ lies in every prime containing $\mathfrak{a}$, i.e. in $\sqrt{\mathfrak{a}}$. So $P_0 = \sqrt{\mathfrak{a}} = \ker\phi$ is a prime ideal; it is homogeneous, as the radical of an ideal generated by homogeneous elements; and $z_1 \notin P_0$, since $\phi(z_1) = tU_{11} \neq 0$.

Step 2: power series version. Let A be moreover Noetherian, $T = A[[Z_{ij}, z_j]]$, and $\mathfrak{P} = P_0T$. We claim: $\mathfrak{P}$ is prime, $\sqrt{\mathfrak{a}T} = \mathfrak{P}$, and $\operatorname{ht}\mathfrak{P} \leq g$. Write f_d for the homogeneous component of degree d of $f \in T$. As P_0 is generated by finitely many homogeneous $h_1, \ldots, h_r$, $\mathfrak{P}$ is the set of f with all $f_d \in P_0$ (if $f_d = \sum_i c_{i,d}h_i$ with $c_{i,d}$ homogeneous of degree $d - \deg h_i$, then $f = \sum_i(\sum_d c_{i,d})h_i$; the converse is clear). If $f, f' \notin \mathfrak{P}$, let d_0, d_0' be the least degrees with $f_{d_0} \notin P_0$, $f'_{d_0'} \notin P_0$; then $(ff')_{d_0+d_0'} \equiv f_{d_0}f'_{d_0'} \not\equiv 0 \pmod{P_0}$, so $ff' \notin \mathfrak{P}$: $\mathfrak{P}$ is prime. Some power P_0^k lies in $\mathfrak{a}$ (B is Noetherian), so $\mathfrak{P}^k \subseteq \mathfrak{a}T \subseteq \mathfrak{P}$ and $\sqrt{\mathfrak{a}T} = \mathfrak{P}$. Let $\mathfrak{Q} \subseteq \mathfrak{P}$ be a minimal prime of the ideal generated by the g entries of Zz. From $\operatorname{adj}(Z)Zz = \det(Z)z$ we get $\det(Z)z_1 \in \mathfrak{Q}$, and $z_1 \notin \mathfrak{P}$ (its only component z_1 is not in P_0), so $\det Z \in \mathfrak{Q}$; then $\mathfrak{a}T \subseteq \mathfrak{Q}$, $\mathfrak{P} \subseteq \mathfrak{Q}$, and $\mathfrak{Q} = \mathfrak{P}$ is minimal over g elements: $\operatorname{ht}\mathfrak{P} \leq g$ by Krull's height theorem.

Step 3: the chain property. Apply Step 2 with $A = S$, so $T = S[[Z, z]] = \mathcal{O}'[[Y, X, Z, z]]$ is a regular, hence Cohen–Macaulay, local ring of dimension $1 + a + g + g^2 + g$, and $\dim T/\mathfrak{P} = \dim T - \operatorname{ht}\mathfrak{P} \geq 1 + a + g^2 + g$ (Matsumura, Commutative Ring Theory, Theorem 17.4). The ring $T_1 = T/\mathfrak{a}T$ is a complete Noetherian local ring whose nilradical $\mathfrak{P}/\mathfrak{a}T$ is prime, and $\dim T_1 = \dim T/\mathfrak{P} \geq D+1+h$ with $h = b+g^2+g$, since $a = b+D$. The h elements F_i, $Z_{ij} - M_{ij}$, $z_j - X_j$ lie in the maximal ideal of T. By Lemma 49.28 (with this D and h), the ideal J_1 of T_1 they generate satisfies items 1 and 2 (in T_1). Let $\mathrm{ev}\colon T \to S$ be the continuous S-algebra homomorphism with $Z \mapsto M$, $z \mapsto X$. The substitution $Z \mapsto Z + M$, $z \mapsto z + X$ is an automorphism of $S[[Z, z]]$, so ev is surjective with kernel generated by the entries of $Z - M$ and $z - X$; and $\mathrm{ev}(Zz) = MX$, $\mathrm{ev}(\det Z) = \det M$. Hence $P \mapsto \mathrm{ev}^{-1}(P)$ is an inclusion-preserving bijection between the primes of S containing I and the primes of T containing $\mathfrak{a}T + (F_i, Z_{ij} - M_{ij}, z_j - X_j)$, with $S/P \cong T/\mathrm{ev}^{-1}(P)$ and $\mathrm{ev}^{-1}(P + Q) = \mathrm{ev}^{-1}(P) + \mathrm{ev}^{-1}(Q)$; the latter primes correspond in the same way to the primes of T_1 containing J_1. Minimal primes correspond to minimal primes, and items 1 and 2 for J_1 give items 1 and 2 for I. □

Lemma 49.28. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring whose nilradical is a prime ideal, let $D \geq 0$ and $h \geq 0$ be integers with $\dim R \geq D + 1 + h$, let $f_1, \ldots, f_h \in \mathfrak{m}$ and $J = (f_1, \ldots, f_h)$. Then:*

1. *$\dim R/P \geq D + 1$ for every minimal prime P of J;*
2. *for any two minimal primes P, Q of J there are minimal primes $P = P_0, P_1, \ldots, P_k = Q$ of J, $k \geq 0$, with $\dim R/(P_{i-1} + P_i) \geq D$ for $1 \leq i \leq k$.*

Proof. Put $T = V(J) \subseteq \operatorname{Spec} R$; it is non-empty because $J \subseteq \mathfrak{m}$, and it is homeomorphic to $\operatorname{Spec} R/J$. Its irreducible components are $T_i = V(P_i)$, $1 \leq i \leq m$, where $P_1, \ldots, P_m$ are the finitely many minimal primes of J, and $\dim T_i = \dim R/P_i$, $\dim(T_i \cap T_j) = \dim R/(P_i + P_j)$. All dimensions are finite since R is Noetherian local.

The connectedness dimension of T is at least D. Since the nilradical N of R is prime, $\operatorname{Spec} R = V(N)$ is irreducible. If $Z \subsetneq \operatorname{Spec} R$ is closed, $\operatorname{Spec} R \setminus Z$ is a non-empty open subset of an irreducible space, hence irreducible and connected; so the only closed Z with disconnected complement is $Z = \operatorname{Spec} R$ itself, and $c(R) = \dim R$. Also $\operatorname{sdim} R = \dim R$. By Theorem B.156 (applicable: R is complete Noetherian local and $f_i \in \mathfrak{m}$),

$$c(T) = c(R/J) \;\geq\; \min\{\dim R, \dim R - 1\} - h = \dim R - 1 - h \;\geq\; D.$$

A separation principle. Let $\{1, \dots, m\} = A \sqcup B$ with A, B non-empty, $U = \bigcup_{i \in A} T_i$, $V = \bigcup_{j \in B} T_j$ and $Z = U \cap V = \bigcup_{i \in A, j \in B} (T_i \cap T_j)$, a closed set. Then $T \setminus Z$ is disconnected: it is the union of $U \setminus Z$ and $V \setminus Z$, which are disjoint (a common point lies in $U \cap V = Z$) and open in $T \setminus Z$ (each is the complement of the closed set V, resp. U); and they are non-empty, because for $i \in A$ the component T_i is not contained in V (otherwise the irreducible set T_i would lie in some T_j with $j \neq i$, contradicting the maximality of the distinct components), so $T_i \setminus Z \supseteq T_i \setminus V \neq \emptyset$, and likewise for B. Hence $\dim Z \geq c(T) \geq D$.

Proof of 1. If $m = 1$, then $\dim R/P_1 = \dim R/J \geq \dim R - h \geq D + 1$ by Krull's inequality $\dim R/(f_1, \dots, f_h) \geq \dim R - h$ for a Noetherian local ring (Atiyah–Macdonald, Corollary 11.18). If $m \geq 2$, fix i and apply the separation principle to $A = \{i\}$, $B = \{1, \dots, m\} \setminus \{i\}$: the closed set $Z = T_i \cap \bigcup_{j \neq i} T_j$ has $\dim Z \geq D$, and $Z \neq T_i$ (shown above). A chain $Z_0 \subsetneq \cdots \subsetneq Z_e$ of irreducible closed subsets of Z with $e = \dim Z$ extends by $T_i \supsetneq Z_e$, so $\dim R/P_i = \dim T_i \geq \dim Z + 1 \geq D + 1$.

Proof of 2. Call i, j adjacent if $\dim(T_i \cap T_j) \geq D$, and let A be the set of indices joined to a given index i_0 by a chain of adjacent indices. If $B = \{1, \dots, m\} \setminus A$ were non-empty, every pair $i \in A$, $j \in B$ would have $\dim(T_i \cap T_j) < D$, so the finite union $Z = \bigcup_{i \in A, j \in B} (T_i \cap T_j)$ would have $\dim Z < D$, contradicting the separation principle. Hence every index is joined to i_0 by such a chain, and in terms of primes this is statement 2. □

Proposition 49.29. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2, 3\}$, and suppose a universal polarised ordinary lift of rank n over L exists. Let x be a core prime of E_n^{po} that is a maximal ideal and $(\mathcal{O}', R, \iota)$ a local chart at x. Then:*

(Nn1) $\dim R/I_R^+ < n\, d_L$;

(Nn2) *for every ideal $I \subseteq W_n^{\mathrm{po}}$ finite over the weights, every place $v \mid \ell$ of L^+ and all $a \neq b$ in $\{1, \dots, n\}$,*

$$\dim R/\big(\iota(I)R + \mathfrak{m}_{\mathcal{O}} R + (\psi_{x,v,a}(g) - \psi_{x,v,b}(g) :\; g \in I_v,\; \chi_{\mathrm{cyc}}(g) = 1)\big) < n\, d_L.$$

Proof. The argument is commutative algebra and Galois cohomology of characters over L; no automorphy is used. Write $W = W_n^{\mathrm{po}}$, $E = E_n^{\mathrm{po}}$, $\mathfrak{m}$ for the maximal ideal of R_n^{po}, $\rho = \rho^{\mathrm{u}}$, $\psi_{v,a} = \psi_{v,a}^{\mathrm{u}}$, $\rho_x = \iota \circ \rho$, $d = d_L$, ϖ a uniformiser of $\mathcal{O}$, and $[\,\cdot\,]\colon \mathbb{F}_{\mathcal{O}}^\times \to \mathcal{O}^\times$ the Teichmüller lift. The weights satisfy $w_{n,1} > \cdots > w_{n,n}$. For a prime P of R put $\eta_P = \iota^{-1}(P)$ and $\mathfrak{q}_P = \eta_P \cap W$.

Step 0: four tools. (T1) *Residues in the chart.* x is a core prime, so $\mathfrak{m}_W \subseteq x = \iota^{-1}(\mathfrak{m}_R)$. For $\gamma \in \Gamma_L^0$, $\rho(\gamma)_{aa} - [\bar{\Psi}(\gamma)^{w_{n,a}}]$ lies in W with residue 0 (Po5), hence in $\mathfrak{m}_W$; so $\rho_x(\gamma)_{aa} \equiv [\bar{\Psi}(\gamma)^{w_{n,a}}]$ and likewise $\psi_{x,v,a}(g) \equiv [\bar{\Psi}(g)^{w_{n,a}}]$ modulo $\mathfrak{m}_R$; and $\iota(\mathfrak{m}_W) \subseteq \mathfrak{m}_R$. (T2) *Cutting by few elements.* Let $(S, \mathfrak{m}_S)$ be a Noetherian local ring, $I, J \subseteq S$ ideals and $t_1, \dots, t_m \in \mathfrak{m}_S$ such that every prime containing $I + (t_1, \dots, t_m)$ contains J. Then $\dim S/I \leq m + \dim S/(I + J)$. Indeed $\dim S/(I + (t)) \geq \dim S/I - m$ (Krull; Atiyah–Macdonald, Corollary 11.18, in S/I), and $\dim S/(I + (t))$ is $\dim S/\mathfrak{p}$ for a minimal prime $\mathfrak{p}$ of $I + (t)$, which contains $I + J$. (T3) *Characters*

with values in $1+\mathfrak{m}_Q$. Let Q be a complete Noetherian local domain of characteristic ℓ with finite residue field, H a profinite group and $\phi\colon H \to 1+\mathfrak{m}_Q$ a homomorphism such that $\phi \bmod \mathfrak{m}_Q^N$ is locally constant for every N. For each N, $(1+\mathfrak{m}_Q)/(1+\mathfrak{m}_Q^N)$ is a finite abelian ℓ-group ($Q/\mathfrak{m}_Q^N$ is finite, and $(1+y)^{\ell^N} = 1+y^{\ell^N}$), so $\phi_N = \phi \bmod \mathfrak{m}_Q^N$ is a continuous homomorphism to it. If $\phi^{h'} \equiv 1$ modulo every $\mathfrak{m}_Q^N$ for some $h' \geq 1$, then $\phi = 1$: by Krull's intersection theorem $\phi^{h'} = 1$; write $h' = \ell^e h''$, $\ell \nmid h''$, and $\phi(\gamma)^{h''} = 1+z$; then $1+z^{\ell^e} = 1$, so $z = 0$ in the domain Q; and with $\phi(\gamma) = 1+y$, $0 = y(h'' + y\sum_{i\geq 2}\binom{h''}{i}y^{i-2})$ with a unit second factor, so $y = 0$. (T4) *Fibres of off-triangular primes.* If η is a prime of E with $I_E^- \not\subseteq \eta$ and $I_E^+ \not\subseteq \eta$, then $\operatorname{trdeg}_{\kappa(\eta\cap W)}\kappa(\eta) \leq 1$. Indeed there are entries $e_+ = \rho(\gamma)_{ab}$ with $a<b$, of weight $p>0$, and $e_- = \rho(\gamma')_{a'b'}$ with $a'>b'$, of weight $-q<0$, outside η. Every entry e of weight j satisfies: $e \in W$ if $j=0$; $e^q e_-^j \in W$ if $j>0$; $e^p e_+^{-j} \in W$ if $j<0$; and $e_+^q e_-^p \in W$ (weight-zero products lie in W). So in $\kappa(\eta)$ the image of every entry is algebraic over $\kappa(\eta\cap W)(\bar{e}_-)$, and $\kappa(\eta)$ is generated over $\kappa(\eta\cap W)$ by these images, since E is generated over W by the entries. Moreover, E is of finite type over W: it is generated by the modules gr_j for the finitely many weights j of entries, each finitely generated by (Gw3).

Step 1: upper-triangular reductions are residual on W. Let t be the family of the following $m = 2 + C_0 + (n-1)d$ elements of $\mathfrak{m}_R$ (by (T1)): ϖ; $\rho_x(s_k)_{11} - 1$ for $1 \leq k \leq 1+C_0+d$; and, if $n=3$, $\rho_x(s'_b)_{22} - 1$ for $1 \leq b \leq d$. Let P be a prime of R containing I_R^+ and all members of t. We show $\iota(\mathfrak{m}_W) \subseteq P$. Let $Q = R/P$, a complete Noetherian local domain of characteristic ℓ ($\ell \in \varpi\mathcal{O}$) with finite residue field $\mathbb{F}_{\mathcal{O}'}$, and $\bar\rho$ the reduction of ρ_x. Its upper entries vanish, so $\chi_a(\gamma) = \bar\rho(\gamma)_{aa}$ is a homomorphism $\Gamma_L^0 \to Q^\times$, and by (T1) $\phi_a(\gamma) = \chi_a(\gamma)[\bar\Psi(\gamma)]^{-w_{n,a}}$ is a homomorphism to $1+\mathfrak{m}_Q$, locally constant modulo every $\mathfrak{m}_Q^N$ by (Qc5).

- $a=1$: $\phi_1(s_k) = 1$ for all k, since $\bar\Psi(s_k) = 1$; by (Ab1) applied to each $\phi_{1,N}$ of (T3), $\phi_1^h \equiv 1$ modulo every $\mathfrak{m}_Q^N$, so $\phi_1 = 1$ by (T3).
- $n=3$, $a=2$: by (Po2) the $(2,2)$ entries of $\rho_x(\pi_L(cg^{-1}c))$ and $\rho_x(\pi_L(g))$ agree ($g \in G_L$), so $\phi_2 = \chi_2$ satisfies the invariance of (Ab2), and $\phi_2(s'_b) = 1$; by (Ab2) and (T3), $\phi_2 = 1$.
- $a = n$: by (Po2), $\chi_n(\pi_L(cg^{-1}c)) = \chi_1(\pi_L(g)) = [\bar\Psi(g)] = [\bar\Psi(cg^{-1}c)]^{-1}$ for $g \in G_L$, and $g \mapsto cg^{-1}c$ is a bijection of G_L; so $\phi_n = 1$.

Thus $\chi_a = [\bar\Psi^{w_{n,a}}]$ for all a. Next, for $v \mid \ell$ and $1 \leq a \leq n$, the (a,a) entry of the lower triangular product in (Po4) gives $\prod_{b=1}^n(\chi_a(\pi_L(g_b)) - \bar\psi_{x,v,b}(g_b)) = 0$ in the domain Q for all $g_1,\dots,g_n \in D_v$. The sets $\{g \in D_v : \chi_a(\pi_L(g)) = \bar\psi_{x,v,b}(g)\}$ are subgroups; if none were all of D_v, choosing g_b outside the b-th one would make the product non-zero. So $\chi_a \circ \pi_L = \bar\psi_{x,v,b}$ on D_v for some b, and reducing modulo $\mathfrak{m}_Q$ (T1) gives $\bar\Psi^{w_{n,a}-w_{n,b}} = 1$ on D_v. As $w_{n,a} - w_{n,b} \in \{0, \pm 1, \pm 2\}$ and $\bar\Psi|_{D_v}$, $\bar\Psi^2|_{D_v}$ are non-trivial (Af4), $b = a$: $\bar\psi_{x,v,a} = [\bar\Psi^{w_{n,a}}]$ on D_v. Finally, an entry (a,b) of the lower triangular $\bar\rho$ that is non-zero has $a \geq b$ and weight $w_{n,a} - w_{n,b} \leq 0$, with equality only for $a = b$; so the image of a weight-zero product of entries of ρ is either 0 or a product of diagonal entries, i.e. of Teichmüller lifts. Hence, with $P_W = \iota^{-1}(P) \cap W$, an ideal of W containing $\mathfrak{m}_{\mathcal{O}}W$, every weight-zero product and every $\psi_{v,a}(g)$ is congruent modulo P_W to an element of $\mathcal{O}$, and Lemma 49.32 gives $\mathfrak{m}_W \subseteq P_W$, i.e. $\iota(\mathfrak{m}_W) \subseteq P$.

Step 2: the lower-triangular fibre of the chart. $\dim R/(I_R^+ + \iota(\mathfrak{m}_W)R) \leq 2C_0$. Let P be a prime containing $I_R^+ + \iota(\mathfrak{m}_W)R$. Then $\eta_P \supseteq I_E^+$ (the images of the generators of I_E^+ generate I_R^+) and $\mathfrak{q}_P \supseteq \mathfrak{m}_W$, so $\mathfrak{q}_P = \mathfrak{m}_W$. The domain E/η_P is of finite type over the field $W/\mathfrak{m}_W$ (Step 0), so its transcendence degree equals its dimension (Noether normalisation), and $\dim E/\eta_P \leq \dim E/(\mathfrak{m}_W E + I_E^+) \leq 2C_0$ by Lemma 49.30. By Lemma 49.18, $\dim R/P \leq \dim W/\mathfrak{m}_W + 2C_0 = 2C_0$.

Step 3: (Nn1). By Step 1, (T2) applies in R with $I = I_R^+$, $J = \iota(\mathfrak{m}_W)R$ and the m elements t; with Step 2,

$$\dim R/I_R^+ \le 2 + C_0 + (n-1)d + 2C_0 = (n-1)d + 3C_0 + 2 < n\,d,$$

because $d > 3C_0 + 7$ by (Ar4).

Step 4: (Nn2). Let $I \subseteq W$ be finite over the weights, $v \mid \ell$, $a \neq b$, and $\mathfrak{b}$ the ideal of (Nn2). It suffices to show $\dim R/P < n\,d$ for every prime $P \supseteq \mathfrak{b}$. If $I_R^+ \subseteq P$, this is Step 3. Otherwise P is off-triangular ($I_R^- = R$ by (Qc3)), so by (Qc3) $I_E^\pm \not\subseteq \eta_P$, and by (T4) and Lemma 49.18, $\dim R/P \le \dim W/\mathfrak{q}_P + 1$. The prime $\mathfrak{q}_P$ contains I, ϖ, and $\psi_{v,a}(g) - \psi_{v,b}(g)$ for $g \in I_v$ with $\chi_{\mathrm{cyc}}(g) = 1$. For $w \mid \ell$ let $u_{w,1}, \dots, u_{w,d_w} \in I_w$ be as in (Ar5)(iii), and K_v the set of k with $\chi_{\mathrm{cyc}}(u_{v,k}) = 1$, so $|K_v| \ge d_v - 1$. Let t' consist of ϖ and the elements $\psi_{w,e}(u_{w,k}) - 1$ ($w \mid \ell$, $1 \le e \le n$, $1 \le k \le d_w$) except those with $(w,e) = (v,b)$ and $k \in K_v$; they lie in $\mathfrak{m}_W$ ((Po5), $\bar\Psi(u_{w,k}) = 1$), and $|t'| \le 1 + n\,d - (d_v - 1)$. Let $\mathfrak{q}' \supseteq \mathfrak{q}_P + (t')$ be a prime of W. It contains $\psi_{v,b}(u_{v,k}) - 1 = (\psi_{v,a}(u_{v,k}) - 1) - (\psi_{v,a}(u_{v,k}) - \psi_{v,b}(u_{v,k}))$ for $k \in K_v$, hence all $\psi_{w,e}(u_{w,k}) - 1$. It contains $\mathfrak{a}_n$: $Q' = W/\mathfrak{q}'$ is a complete Noetherian local domain of characteristic ℓ with finite residue field (Gw1); for w, e, the homomorphism $\phi(g) = \bar\psi_{w,e}(g)[\bar\Psi(g)]^{-w_{n,e}}$ from D_w to $1 + \mathfrak{m}_{Q'}$ is locally constant modulo every $\mathfrak{m}_{Q'}^N$ ((Po1), and $\mathfrak{m}^k \cap W \subseteq \mathfrak{m}_W^N$ for large k by Chevalley's lemma Theorem B.154 and (Gw1)), and is trivial on the $u_{w,k}$; by (Ar5)(iii) each $\phi \bmod \mathfrak{m}_{Q'}^N$ is trivial on I_w, so ϕ is trivial on I_w by Krull's intersection theorem, and $\psi_{w,e}(g) - 1 \in \mathfrak{q}'$ for $g \in I_w$ with $\bar\Psi(g) = 1$. Hence $\mathfrak{q}' \supseteq I + \mathfrak{a}_n$, and $\mathfrak{q}' \supseteq \mathfrak{m}_W$ because I is finite over the weights. By (T2) in W,

$$\dim W/\mathfrak{q}_P \le |t'| + \dim W/\mathfrak{m}_W \le n\,d + 2 - d_v, \qquad \dim R/P \le n\,d + 3 - d_v < n\,d,$$

because $d_v > 5$ by (Ar4). □

Lemma 49.30. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2,3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $E = E_n^{\mathrm{po}}$. Then* $\dim E/(\mathfrak{m}_W E + I_E^+) \le 2C_0$.

Proof. The argument is Galois cohomology of characters of Γ_L and commutative algebra; no automorphy is used, over L or over K. Write $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, $W = W_n^{\mathrm{po}}$, $\rho = \rho^{\mathrm{u}}$, $\mathfrak{m}$ for the maximal ideal of R_n^{po}, $\bar c = \pi_L(c)$ and $\gamma^* = \bar c \gamma^{-1} \bar c$ for $\gamma \in \Gamma_L^0$ (an anti-automorphism of Γ_L^0). As $g_\star \in G_L$ (Af1), $\gamma_\star = \pi_L(g_\star) \in \Gamma_L^0$, and $\bar\Psi(\gamma_\star)^2 \neq 1$. Every place of L^+ above ℓ splits in L ((Af2), $\ell \mid M$), so $D_v \subseteq G_L$ and $\pi_L(D_v) \subseteq \Gamma_L^0$. Since $\bar\Psi(c) = 1$, $\bar\Psi(\gamma^*) = \bar\Psi(\gamma)^{-1}$ and $\bar\Psi(\bar c \gamma \bar c) = \bar\Psi(\gamma)$.

Step 0: the fibre algebra and its grading. Let $\bar A = E/(\mathfrak{m}_W E + I_E^+)$. By (Gw2), $E = \bigoplus_j \mathrm{gr}_j$ with $\mathrm{gr}_0 = W$, and $\mathfrak{m}_W E = \bigoplus_j \mathfrak{m}_W \mathrm{gr}_j$ and I_E^+ (generated by the upper entries, which are homogeneous) are homogeneous ideals; so $\bar A = \bigoplus_j \bar A_j$ with $\bar A_j$ the image of gr_j. A product of entries of positive weight has an upper factor, so $\bar A_j = 0$ for $j > 0$; $\bar A_0$ is the image of W, i.e. $\mathbb{F} = W/\mathfrak{m}_W$ (or 0); each $\bar A_j$ is a finite-dimensional $\mathbb{F}$-space by (Gw3). For $\gamma \in \Gamma_L^0$ let $\bar\rho(\gamma)$ be the image of $\rho(\gamma)$ in $\mathrm{M}_n(\bar A)$: it is lower triangular, multiplicative in γ, and its diagonal is $\mathrm{diag}(\bar\Psi(\gamma)^{w_{n,1}}, \dots, \bar\Psi(\gamma)^{w_{n,n}})$, because $\rho(\gamma)_{aa} \in W$ has residue $\bar\Psi(\gamma)^{w_{n,a}}$ (Po5); likewise the image of $\psi_{v,a}^{\mathrm{u}}(g)$ is $\bar\Psi(g)^{w_{n,a}}$. Moreover:

- *(continuity)* for each a, b the map $\gamma \mapsto \bar\rho(\gamma)_{ab}$ is locally constant: the submodules $\mathfrak{m}^k \cap \mathrm{gr}_j$ of the finitely generated W-module gr_j decrease with intersection 0 (Krull), so by Chevalley's lemma Theorem B.154 and (Gw1) some $\mathfrak{m}^k \cap \mathrm{gr}_j$ lies in $\mathfrak{m}_W \mathrm{gr}_j$, which maps to 0 in $\bar A$, and $\gamma \mapsto \rho(\gamma) \bmod \mathfrak{m}^k$ is locally constant (Po1);
- *(polarisation)* $\bar\rho(\gamma^*)_{ab} = \bar\rho(\gamma)_{n+1-b,n+1-a}$, by (Po2) and $(J_n {}^t X J_n)_{ab} = X_{n+1-b,n+1-a}$;

- *(marking)* $\bar\rho(\gamma_\star)_{ab} = \bar\rho(\gamma_\star^*)_{ab}$ for $a \neq b$, by (Po3);
- *(ordinarity)* for $v \mid \ell$ and $g_1, \dots, g_n \in D_v$, $\prod_{a=1}^n \big(\bar\rho(\pi_L(g_a)) - \bar\Psi(g_a)^{w_{n,a}}\big) = 0$, by (Po4).

Fix, for each $v \mid \ell$, elements $s_v, s_v' \in D_v$ with $\bar\Psi(s_v)^2 \neq 1$ and $\bar\Psi(s_v') \neq 1$ (Af4).

Step 1: two cocycle spaces. For an $\mathbb{F}$-vector space V (with trivial action) consider maps $\Gamma_L^0 \to V$.

(a) Let Z_2 be the space of locally constant $u\colon \Gamma_L^0 \to \mathbb{F}$ with $u(\gamma\gamma') = u(\gamma)\bar\Psi(\gamma') + \bar\Psi(\gamma)^{-1}u(\gamma')$, $u(\gamma^*) = u(\gamma)$, and for each $v \mid \ell$ some $\lambda_v \in \mathbb{F}$ with $u(\pi_L(g)) = \lambda_v(\bar\Psi(g) - \bar\Psi(g)^{-1})$ for $g \in D_v$. Then $\dim Z_2 \leq C_0$. Indeed, put $\chi = \bar\Psi^{-2}\eta_K$, a character of Γ_L with $\chi = \bar\Psi^{-2}$ on Γ_L^0 ($K \subseteq L$) and $\chi(\bar c) = -1$. For $u \in Z_2$ let $x = \bar\Psi^{-1}u$; then $x(\gamma\gamma') = x(\gamma) + \chi(\gamma)x(\gamma')$, $x(\gamma^*) = \bar\Psi(\gamma)^2 x(\gamma)$, hence (with $x(\gamma^{-1}) = -\chi(\gamma)^{-1}x(\gamma)$) $x(\bar c\gamma\bar c) = -x(\gamma)$. Define y on $\Gamma_L = \Gamma_L^0 \sqcup \Gamma_L^0\bar c$ by $y(\gamma) = y(\gamma\bar c) = x(\gamma)$. The cocycle identity $y(gh) = y(g) + \chi(g)y(h)$ holds in the four cases $g \in \Gamma_L^0$ or $\Gamma_L^0\bar c$, $h \in \Gamma_L^0$ or $\Gamma_L^0\bar c$:

The map y is continuous, $y(\bar c) = 0$, and on $\pi_L(D_v)$, $y(\pi_L(g)) = \lambda_v(1 - \bar\Psi(g)^{-2}) = (\chi(g) - 1)(-\lambda_v)$: a coboundary, so y is a strict cocycle (for the family $(D_v)_{v\mid\ell}$, with $X_v = -\lambda_v$). The map $u \mapsto y$ is injective and linear. A coboundary $y = (g \mapsto (\chi(g) - 1)X)$ has $y(\bar c) = -2X$, so the strict cocycles with $y(\bar c) = 0$ meet the coboundaries in 0 (ℓ odd), and they inject into $H^1_{\mathrm{str}}(\Gamma_L, \mathbb{F}(\bar\Psi^{-2}\eta_K))$ (the X_v are determined by y because $\chi|_{D_v} \neq 1$). By (Ar3), $\dim Z_2 \leq C_0$.

(b) ($n = 3$.) Let Z_1 be the space of locally constant $b\colon \Gamma_L^0 \to \mathbb{F}$ with $b(\gamma\gamma') = b(\gamma)\bar\Psi(\gamma') + b(\gamma')$, $b(\gamma_\star) = b(\gamma_\star^*)$, and for each $v \mid \ell$ some $\lambda_v, \mu_v \in \mathbb{F}$ with $b(\pi_L(g)) = \lambda_v(\bar\Psi(g) - 1)$ and $b(\pi_L(g)^*) = \mu_v(1 - \bar\Psi(g)^{-1})$ for $g \in D_v$. Then $\dim Z_1 \leq C_0$. Indeed, $x = \bar\Psi^{-1}b$ satisfies $x(\gamma\gamma') = x(\gamma) + \bar\Psi(\gamma)^{-1}x(\gamma')$, a continuous cocycle for $\mathbb{F}(\bar\Psi^{-1})$. On $\pi_L(D_v)$, $x = (\bar\Psi^{-1} - 1)(-\lambda_v)$; on $\pi_L(cD_vc)$, writing $\bar c\pi_L(g)\bar c = \pi_L(g^{-1})^*$, $x(\bar c\pi_L(g)\bar c) = \bar\Psi(g)^{-1}\mu_v(1 - \bar\Psi(g)) = (\bar\Psi(g)^{-1} - 1)\mu_v$, and $\bar\Psi(\bar c\pi_L(g)\bar c) = \bar\Psi(g)$: a coboundary as well. So x is strict for the family $(D_v)_v, (cD_vc)_v$ of Γ_L^0. The linear form $\mu(b) = b(\gamma_\star) - b(\gamma_\star^*)$ vanishes on Z_1, while on the coboundary $x = (\bar\Psi^{-1} - 1)X$, i.e. $b = (1 - \bar\Psi)X$, it takes the value $(\bar\Psi(\gamma_\star)^{-1} - \bar\Psi(\gamma_\star))X \neq 0$ for $X \neq 0$. Hence Z_1 meets the coboundaries in 0 and injects into $H^1_{\mathrm{str}}(\Gamma_L^0, \mathbb{F}(\bar\Psi^{-1}))$ ($\bar\Psi^{-1}$ is non-trivial on $\pi_L(D_v)$ and on $\pi_L(cD_vc)$, so the X_t are determined); by (Ar3), $\dim Z_1 \leq C_0$.

Step 2: from spans of values to cocycles. Let V be an $\mathbb{F}$-space and $\beta\colon \Gamma_L^0 \to V$ a map whose values span V, such that $\lambda \circ \beta \in Z$ for every linear form λ on V, where Z is a finite-dimensional space of functions. Then $\lambda \mapsto \lambda \circ \beta$ is injective on V^* (a form vanishing on all values vanishes on V), so $\dim V^* \leq \dim Z$ and $\dim V \leq \dim Z$.

Step 3: ordinarity on D_v. Write the entries of $\bar\rho$ below the diagonal as $b = \bar\rho_{21}$, and, if $n = 3$, $f = \bar\rho_{32}$, $e = \bar\rho_{31}$; if $n = 2$ put $e = \bar\rho_{21}$. Let $g, g' \in D_v$ and write $\bar\Psi$ for $\bar\Psi(g)$ etc.

- $n = 2$: the $(2,1)$ entry of the product in (Po4) for (g, g') is $e(g)(\bar\Psi(g') - \bar\Psi(g')^{-1}) + (\bar\Psi(g)^{-1} - \bar\Psi(g))e(g')$; with $g' = s_v$, $e = \lambda_v(\bar\Psi - \bar\Psi^{-1})$ on $\pi_L(D_v)$, $\lambda_v \in \bar A_{-2}$.
- $n = 3$, for (g_1, g_2, g_3): the $(2,1)$ entry is $(\bar\Psi(g_3) - \bar\Psi(g_3)^{-1})\big[b(g_1)(\bar\Psi(g_2) - 1) + (1 - \bar\Psi(g_1))b(g_2)\big]$, and the $(3,2)$ entry is $(\bar\Psi(g_1)^{-1} - \bar\Psi(g_1))\big[f(g_2)(1 - \bar\Psi(g_3)^{-1}) + (\bar\Psi(g_2)^{-1} - 1)f(g_3)\big]$. Taking $g_3 = s_v$, $g_2 = s_v'$, resp. $g_1 = s_v$, $g_3 = s_v'$: $b = \lambda_v(\bar\Psi - 1)$ and $f = \mu_v(1 - \bar\Psi^{-1})$ on $\pi_L(D_v)$, with $\lambda_v, \mu_v \in \bar A_{-1}$. If b and f vanish identically (in a quotient of $\bar A$), the $(3,1)$ entry is $e(g_1)(\bar\Psi(g_2) - 1)(\bar\Psi(g_3) - \bar\Psi(g_3)^{-1}) + (\bar\Psi(g_1)^{-1} - \bar\Psi(g_1))e(g_2)(\bar\Psi(g_3) - \bar\Psi(g_3)^{-1}) + (\bar\Psi(g_1)^{-1} - \bar\Psi(g_1))(\bar\Psi(g_2)^{-1} - 1)e(g_3)$, and $g_1 = g_3 = s_v$ gives $e = \lambda_v'(\bar\Psi - \bar\Psi^{-1})$ on $\pi_L(D_v)$.

Step 4: rank 2. Here $\bar{A}_{-1} = 0$ and $\bar{A}$ is generated as an $\mathbb{F}$-algebra by $V = \bar{A}_{-2}$, the span of the values $e(\gamma)$ (a product of entries of weight $-2k$ without upper factor is a product of k lower entries and diagonal entries). By multiplicativity, (polarisation) and Step 3, e satisfies the three conditions of Z_2 with values in V; for every linear form λ on V, $\lambda \circ e$ is locally constant (continuity, V finite-dimensional) and lies in Z_2. By Step 2 and Step 1(a), $\dim V \le C_0$, and $\bar{A}$, a quotient of a polynomial ring over $\mathbb{F}$ in $\dim V$ variables, has dimension $\le C_0 \le 2C_0$.

Step 5: rank 3. By (polarisation), $f(\gamma) = b(\gamma^*)$ and $b(\gamma) = f(\gamma^*)$, so $V_1 = \bar{A}_{-1}$ is the span of the values of b (a product of weight -1 without upper factor is one lower entry of weight -1 times diagonal entries). By multiplicativity, (marking), Step 3 and $b(\pi_L(g)^*) = f(\pi_L(g))$, every $\lambda \circ b$ lies in Z_1; so $p = \dim V_1 \le C_0$ by Steps 1(b), 2. Let $\bar{A}' = \bar{A}/V_1\bar{A}$, graded with $\bar{A}'_j = \bar{A}_j/(V_1\bar{A})_j$, and $V_2' \subseteq \bar{A}'_{-2}$ the span of the images of the values of e. In $\bar{A}'$, $b = f = 0$, so the $(3,1)$ entry of $\bar{\rho}(\gamma\gamma') = \bar{\rho}(\gamma)\bar{\rho}(\gamma')$ gives $e(\gamma\gamma') = e(\gamma)\bar{\Psi}(\gamma') + \bar{\Psi}(\gamma)^{-1}e(\gamma')$; (polarisation) gives $e(\gamma^*) = e(\gamma)$; Step 3 gives the condition on $\pi_L(D_v)$. Hence every $\lambda \circ e$, λ a linear form on V_2', lies in Z_2, and $q = \dim V_2' \le C_0$. Choose $\gamma_1, \dots, \gamma_q$ such that the images of $e(\gamma_i)$ span V_2', and a basis $y_1, \dots, y_p$ of V_1. Let $S \subseteq \bar{A}$ be the $\mathbb{F}$-subalgebra generated by the y_i and the $e(\gamma_i)$. By induction on k, $\bar{A}_{-k} \subseteq S$: for $k \le 1$ this is clear; for $k \ge 2$, $\bar{A}_{-k}$ is spanned by products of lower entries (times diagonal entries) of total weight $-k$; such a product with a factor of weight -1 lies in $V_1\bar{A}_{-k+1} \subseteq S$; otherwise it is a product of $k/2$ values of e, and $\bar{A}_{-2} = V_2 + V_1V_1 \subseteq \operatorname{span}(e(\gamma_i)) + (V_1\bar{A})_{-2} + V_1V_1 \subseteq S$ (as $(V_1\bar{A})_{-2} = V_1\bar{A}_{-1} = V_1V_1$), where V_2 is the span of the values of e in $\bar{A}$. So $\bar{A} = S$ is generated by $p + q \le 2C_0$ elements, and $\dim \bar{A} \le 2C_0$. □

Lemma 49.31. *Let ℓ be odd and L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, and $d_L = [L^+ : \mathbb{Q}]$. There are elements $s_1, \dots, s_{1+C_0+d_L} \in \Gamma_L^0$ with $\bar{\Psi}(s_a) = 1$, elements $s'_1, \dots, s'_{d_L} \in \Gamma_L^0$, each of the form $\pi_L(g)$ with $g \in I_v$ for some $v \mid \ell$ and $\bar{\Psi}(g) = 1$, and an integer $h \ge 1$ such that for every finite abelian ℓ-group A and every continuous homomorphism $\chi\colon \Gamma_L^0 \to A$:*

(Ab1) *if $\chi(s_a) = 1$ for all a, then $\chi^h = 1$;*

(Ab2) *if $\chi(\pi_L(cg^{-1}c)) = \chi(\pi_L(g))$ for all $g \in G_L$ and $\chi(s'_b) = 1$ for all b, then $\chi^h = 1$.*

Proof. Write $\bar{c} = \pi_L(c) \in \Gamma_L$; since $c^2 = 1$ and $G_{L^+} = G_L \sqcup cG_L$, $\bar{c}^2 = 1$, $\Gamma_L = \Gamma_L^0 \cup \bar{c}\Gamma_L^0$, and $\mathrm{cj}(\gamma) = \bar{c}\gamma\bar{c}$ is an automorphism of Γ_L^0 with $\mathrm{cj}^2 = \mathrm{id}$ and $\pi_L(cg^{-1}c) = \mathrm{cj}(\pi_L(g))^{-1}$ for $g \in G_L$.

Step 1: (Ab2). For each $v \mid \ell$ let $u_{v,1}, \dots, u_{v,d_v} \in I_v$ be as in (Ar5)(iii) (so $\bar{\Psi}(u_{v,k}) = 1$), let $s'_1, \dots, s'_{d_L}$ be the elements $\pi_L(u_{v,k})$, and let h be as in (Ar5)(ii). Let χ be as in (Ab2). For each v, $\chi \circ \pi_L$ restricted to D_v is a continuous homomorphism to A trivial on the $u_{v,k}$, hence trivial on I_v by (Ar5)(iii). For $g \in I_v$, $\chi(\pi_L(cgc)) = \chi(\pi_L(c(g^{-1})^{-1}c)) = \chi(\pi_L(g^{-1})) = 1$. So χ is trivial on $\pi_L(I_v)$ and $\pi_L(cI_vc)$ for all v, and $\chi^h = 1$ by (Ar5)(ii).

Step 2: elements of Γ_L detecting abelian ℓ-quotients. Let V be the $\mathbb{F}_\ell$-vector space of continuous homomorphisms $\Gamma_L \to \mathbb{F}_\ell$. An $\mathbb{F}_\ell$-basis of V stays linearly independent over $\mathbb{F}_\mathcal{O}$ inside the space of continuous homomorphisms $\Gamma_L \to \mathbb{F}_\mathcal{O}$, which is $H^1(\Gamma_L, \mathbb{F}_\mathcal{O})$; so $\dim V \le 1 + C_0$ by (Ar3). The evaluations $f \mapsto f(t)$, $t \in \Gamma_L$, have no common non-zero zero in V, so they span the dual of V, and there are $t_1, \dots, t_{1+C_0} \in \Gamma_L$ (repeat $t = 1$ if needed) such that $f \in V$ and $f(t_a) = 0$ for all a imply $f = 0$. Let $\varphi\colon \Gamma_L \to A$ be a continuous homomorphism to a finite abelian ℓ-group with $\varphi(t_a) = 1$ for all a. If $\varphi \ne 1$, its image A' is a non-trivial finite abelian ℓ-group, so $A'/A'^\ell \ne 0$ has a non-zero linear form, which gives a homomorphism $\theta\colon A' \to \mathbb{F}_\ell$ with $\theta \circ \varphi \ne 0$; then $\theta \circ \varphi \in V$ vanishes at all t_a, a contradiction. So $\varphi = 1$.

Step 3: (Ab1). For $\gamma \in \Gamma_L$ put $q(\gamma) = \gamma$ if $\gamma \in \Gamma_L^0$ and $q(\gamma) = \bar{c}\gamma \in \Gamma_L^0$ otherwise. Let e be the order of $\mathbb{F}_\mathcal{O}^\times$, prime to ℓ. Put $s_a = \big(q(t_a)\,\mathrm{cj}(q(t_a))\big)^e$ for $1 \le a \le 1 + C_0$ and $s_{1+C_0+b} = s'_b$

for $1 \le b \le d_L$. Then $\bar\Psi(s_a) = 1$, since $\bar\Psi$ takes values in $\mathbb{F}_{\mathcal{O}}^\times$, and $\bar\Psi(s'_b) = 1$. Let χ be trivial on all s_a. Then $\nu = \chi \cdot (\chi \circ \mathrm{cj})$ is a continuous homomorphism $\Gamma_L^0 \to A$ with $\nu \circ \mathrm{cj} = \nu$. If $\bar c \in \Gamma_L^0$, then $\Gamma_L = \Gamma_L^0$, and we put $\varphi = \nu$; otherwise put $\varphi(\gamma) = \nu(\gamma)$ and $\varphi(\bar c\gamma) = \nu(\gamma)$ for $\gamma \in \Gamma_L^0$. This φ is a homomorphism: for $\gamma, \gamma' \in \Gamma_L^0$, $(\bar c\gamma)(\bar c\gamma') = \mathrm{cj}(\gamma)\gamma'$ and $\gamma(\bar c\gamma') = \bar c\,\mathrm{cj}(\gamma)\gamma'$, and $\nu(\mathrm{cj}(\gamma)) = \nu(\gamma)$; it is continuous, its kernel containing the open subgroup $\ker\nu$ of Γ_L. As $t_a = \bar c^{\epsilon} q(t_a)$ with $\epsilon \in \{0,1\}$, $\varphi(t_a) = \nu(q(t_a))$, and $\nu(q(t_a))^e = \chi(s_a) = 1$; an element of the ℓ-group A of order dividing e is trivial, so $\varphi(t_a) = 1$ for all a, and $\varphi = 1$ by Step 2. Hence $\chi(\mathrm{cj}(\gamma)) = \chi(\gamma)^{-1}$ for $\gamma \in \Gamma_L^0$, and for $g \in G_L$, $\chi(\pi_L(cg^{-1}c)) = \chi(\mathrm{cj}(\pi_L(g)))^{-1} = \chi(\pi_L(g))$. Since moreover $\chi(s'_b) = 1$, Step 1 gives $\chi^h = 1$. □

Lemma 49.32. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2,3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $W = W_n^{\mathrm{po}}$. Let $P \subseteq W$ be an ideal with $\mathfrak{m}_{\mathcal{O}} W \subseteq P$ such that every product $\prod_k \rho^{\mathrm{u}}(\gamma_k)_{a_k b_k}$ with $\sum_k (w_{n,a_k} - w_{n,b_k}) = 0$ and every $\psi^{\mathrm{u}}_{v,a}(g)$ $(v \mid \ell,\ 1 \le a \le n,\ g \in D_v)$ is congruent modulo P to the image of an element of $\mathcal{O}$. Then $\mathfrak{m}_W \subseteq P$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R_n^{po}. If $P = W$ there is nothing to prove; otherwise $P \subseteq \mathfrak{m}_W$, because W is local with maximal ideal $\mathfrak{m}_W$ by (Gw1) of Lemma 49.14. Let $W_0 \subseteq W$ be the $\mathcal{O}$-subalgebra generated by the weight-zero products and the $\psi^{\mathrm{u}}_{v,a}(g)$. The set of $x \in W$ congruent modulo P to the image of an element of $\mathcal{O}$ is an $\mathcal{O}$-subalgebra (congruences add and multiply) containing these generators, so it contains W_0.

The submodules $\mathfrak{m}^N \cap W$ of the W-module W decrease and have intersection 0 (Krull's intersection theorem in the Noetherian local ring R_n^{po}). By (Gw1), W is a complete Noetherian local ring, so by Chevalley's lemma (Theorem B.154) for every $k \ge 1$ there is N with $\mathfrak{m}^N \cap W \subseteq \mathfrak{m}_W^k$.

Let $y \in \mathfrak{m}_W$ and $k \ge 1$, and take N as above. By the definition of W there is $y_N \in W_0$ with $y - y_N \in \mathfrak{m}^N$, hence $y - y_N \in \mathfrak{m}^N \cap W \subseteq \mathfrak{m}_W^k$; and there is $c \in \mathcal{O}$ with $y_N - c \in P \subseteq \mathfrak{m}_W$. Then $c = y - (y - y_N) - (y_N - c) \in \mathfrak{m}_W$, so $c \in \mathfrak{m}_{\mathcal{O}}$, since $\mathcal{O} \to W$ is local ((Gw1)); hence $c \in P$. So $y = (y - y_N) + (y_N - c) + c \in \mathfrak{m}_W^k + P$. As k is arbitrary, $y \in \bigcap_k (P + \mathfrak{m}_W^k) = P$, by Krull's intersection theorem in the Noetherian local ring W/P. □

Proposition 49.33. *Let ℓ be odd and L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$ such that universal polarised ordinary lifts of ranks 2 and 3 over L exist. If an ideal $\mathcal{M}_2 \subseteq W_2^{\mathrm{po}}$ is finite over the weights and absorbs the primes of irreducible type, then there is an ideal $I_3 \subseteq W_3^{\mathrm{po}}$, finite over the weights, with $I_3 \subseteq \xi \cap W_3^{\mathrm{po}}$ for every prime ξ of E_3^{po} of endoscopic type.*

Proof. The proof is commutative algebra on rings attached to L; in particular, no automorphy over L or over K is used. Write $W_n = W_n^{\mathrm{po}}$, $E_n = E_n^{\mathrm{po}}$, $R_n = R_n^{\mathrm{po}}$ with maximal ideal $\mathfrak{m}^{(n)}$, $\rho_n = \rho^{\mathrm{u}}$, $\psi_{n,v,a} = \psi^{\mathrm{u}}_{v,a}$ for the universal lifts of rank $n \in \{2,3\}$, and write $\rho_3(g)$ for $\rho_3(\pi_L(g))$. In rank 3 the positions (a,b) with $w_{3,a} - w_{3,b}$ odd are those with exactly one index equal to 2. Let $I^{\mathrm{odd}} = I^{\mathrm{odd}}_{E_3}$. If $I^{\mathrm{odd}} = E_3$, no prime of E_3 is of endoscopic type and $I_3 = W_3$ will do. Assume from now on $I^{\mathrm{odd}} \ne E_3$, and put $B = E_3/I^{\mathrm{odd}}$, $C = W_3/(I^{\mathrm{odd}} \cap W_3) \subseteq B$, $p\colon W_3 \to C$ the projection, and $\bar y$ the image in B of $y \in E_3$. By (Gw1) of Lemma 49.14, C is a non-zero quotient of an object of $\mathrm{CNL}_{\mathcal{O}}$, hence one itself; an element of W_3 with non-zero residue is a unit of W_3, so its image in B is a unit.

Step 1: topologies. For a finitely generated W_3-submodule $N \subseteq E_3$ and $k \ge 1$ there is t with $(\mathfrak{m}^{(3)})^t \cap N \subseteq \mathfrak{m}_{W_3}^k N$: the submodules $(\mathfrak{m}^{(3)})^t \cap N$ decrease with intersection 0 (Krull's intersection theorem in R_3), and Chevalley's lemma (Theorem B.154) applies over W_3 by (Gw1). With (Po1) this gives: for $1 \le a, b \le 3$ and N_{ab} the W_3-span of the entries $\rho_3(\gamma)_{ab}$, $\gamma \in \Gamma_L^0$ (finitely generated,

being a submodule of $\mathrm{gr}_{w_{3,a}-w_{3,b}}$, which is finite over the Noetherian ring W_3 by (Gw3)), the map $\gamma \mapsto \rho_3(\gamma)_{ab} \bmod \mathfrak{m}_{W_3}^k N_{ab}$ is locally constant; likewise $g \mapsto \psi_{3,v,a}(g) \bmod \mathfrak{m}_{W_3}^k$ (take $N = W_3$).

Step 2: the residual corner. For $\gamma, \gamma' \in \Gamma_L^0$ the odd entries of ρ_3 vanish in B, so the matrices $\bar\rho_3(\gamma)$ are block diagonal for the blocks $\{1,3\}$ and $\{2\}$; hence $\sigma(\gamma) = (\bar\rho_3(\gamma)_{ab})_{a,b\in\{1,3\}}$ and $\gamma \mapsto \bar\rho_3(\gamma)_{22}$ are multiplicative, $\sigma\colon \Gamma_L^0 \to \mathrm{GL}_2(B)$ is a homomorphism, and in W_3

$$\rho_3(\gamma\gamma')_{22} - \rho_3(\gamma)_{22}\rho_3(\gamma')_{22} = \rho_3(\gamma)_{21}\rho_3(\gamma')_{12} + \rho_3(\gamma)_{23}\rho_3(\gamma')_{32} \in I^{\mathrm{odd}} \cap W_3, \tag{49.3}$$

both products being of weight 0. Fix $v \mid \ell$. By (Af4) there is $g_1 \in D_v$ with $\bar\Psi(g_1)^2 \neq 1$. The elements $u_1 = \rho_3(g_1)_{22} - \psi_{3,v,1}(g_1)$, $u_3 = \rho_3(g_1)_{22} - \psi_{3,v,3}(g_1)$ and $\delta = (\rho_3(g_1)_{11} - \psi_{3,v,2}(g_1))(\rho_3(g_1)_{33} - \psi_{3,v,2}(g_1)) - \rho_3(g_1)_{13}\rho_3(g_1)_{31}$ lie in W_3 and by (Po5) have residues $1-\bar\Psi(g_1)$, $1-\bar\Psi(g_1)^{-1}$, $(\bar\Psi(g_1)-1)(\bar\Psi(g_1)^{-1}-1)$, all non-zero; so $\bar u_1, \bar u_3, \bar\delta$ are units of B. For $x, g, y \in D_v$ the product in (Po4) is a product of block diagonal matrices in B. Its $(2,2)$ entry for (g_1, g, g_1) is $\bar u_1(\bar\rho_3(g)_{22} - \bar\psi_{3,v,2}(g))\bar u_3 = 0$, so

$$\rho_3(g)_{22} - \psi_{3,v,2}(g) \in I^{\mathrm{odd}} \cap W_3 \qquad (g \in D_v). \tag{49.4}$$

Its $\{1,3\}$ block for (x, g_1, y) is $P_x\,\Theta\,Q_y = 0$, with $P_x = \sigma(x) - \bar\psi_{3,v,1}(x)$, $\Theta = \sigma(g_1) - \bar\psi_{3,v,2}(g_1)$, $Q_y = \sigma(y) - \bar\psi_{3,v,3}(y)$. Taking $y = g_1$ in $P_x\Theta\sigma(y) = \bar\psi_{3,v,3}(y)P_x\Theta$ gives $P_x\Theta^2 = (\bar\psi_{3,v,3}(g_1) - \bar\psi_{3,v,2}(g_1))P_x\Theta$, and with the Cayley–Hamilton identity $\Theta(\operatorname{tr}\Theta - \Theta) = \det\Theta$,

$$\det(\Theta)\,P_x = P_x\Theta(\operatorname{tr}\Theta - \Theta) = \big(\operatorname{tr}\Theta - \bar\psi_{3,v,3}(g_1) + \bar\psi_{3,v,2}(g_1)\big)P_x\Theta.$$

Multiplying on the right by Q_y gives $\det(\Theta)P_xQ_y = 0$, and $\det\Theta = \bar\delta$ is a unit, so $P_xQ_y = 0$ for all $x, y \in D_v$.

Step 3: the corner factors through E_2. Put $\psi'_{v,1} = \bar\psi_{3,v,1}$, $\psi'_{v,2} = \bar\psi_{3,v,3}$; recall $(w_{2,1}, w_{2,2}) = (w_{3,1}, w_{3,3}) = (1,-1)$. Then σ, $(\psi'_{v,a})$ satisfy the hypotheses of Lemma 49.16 for $n = 2$, B and C. (Po2): $(J_3{}^tXJ_3)_{ab} = X_{4-b,4-a}$ and $(J_2{}^tYJ_2)_{ab} = Y_{3-b,3-a}$ correspond under $1 \mapsto 1$, $2 \mapsto 3$. (Po3): the corner off-diagonal entries are the entries $(1,3)$, $(3,1)$. (Po4): Step 2. (Cq1), (Cq5): $\rho_3(\gamma)_{11}$, $\rho_3(\gamma)_{33}$, $\psi_{3,v,1}(g)$, $\psi_{3,v,3}(g)$ lie in W_3 with the residues of (Po5); local constancy by Step 1. (Cq2): a product of corner entries of weight 0 with an off-diagonal factor is the image of a weight-zero product of entries of ρ_3 with a factor in $\mathfrak{m}^{(3)}$ (Po5), an element of $\mathfrak{m}^{(3)} \cap W_3 = \mathfrak{m}_{W_3}$, whose image lies in $\mathfrak{m}_C$. (Cq3), (Cq4): the C-span of the $\sigma(\gamma)_{ab}$ is the image of $N_{a'b'}$ $(a', b' \in \{1,3\})$, finitely generated, and Step 1 gives the local constancy. By Lemma 49.16 there are $\mathcal{O}$-algebra homomorphisms $\xi\colon E_2 \to B$ and $\beta\colon W_2 \to C$, β local, with $\xi(\rho_2(\gamma)_{ab}) = \sigma(\gamma)_{ab}$, $\xi(\psi_{2,v,a}(g)) = \psi'_{v,a}(g)$ and $\xi = \beta$ on W_2. Put

$$I_3 = p^{-1}(\beta(\mathcal{M}_2)C) \subseteq W_3.$$

Step 4: I_3 lies in every endoscopic prime. Let $\mathfrak{p}$ be a prime of E_3 of endoscopic type; then $I^{\mathrm{odd}} \subseteq \mathfrak{p}$ and $\bar{\mathfrak{p}} = \mathfrak{p}/I^{\mathrm{odd}}$ is a prime of B. Let $\mathfrak{p}_2 = \xi^{-1}(\bar{\mathfrak{p}})$, a prime of E_2. Since $I^-_{E_3} \not\subseteq \mathfrak{p}$ and the lower entries $(2,1)$, $(3,2)$ are odd, some $\rho_3(\gamma)_{31} \notin \mathfrak{p}$; as $\xi(\rho_2(\gamma)_{21}) = \bar\rho_3(\gamma)_{31}$, $I^-_{E_2} \not\subseteq \mathfrak{p}_2$; likewise $I^+_{E_2} \not\subseteq \mathfrak{p}_2$ with the entry $(1,3)$. So $\mathfrak{p}_2$ is of irreducible type (rank 2), and by absorption $\mathcal{M}_2 \subseteq \mathfrak{p}_2 \cap W_2$, i.e. $\beta(\mathcal{M}_2) \subseteq \bar{\mathfrak{p}} \cap C = p(\mathfrak{p} \cap W_3)$. Hence $I_3 \subseteq \mathfrak{p} \cap W_3 + \ker p = \mathfrak{p} \cap W_3$, as $\ker p = I^{\mathrm{odd}} \cap W_3 \subseteq \mathfrak{p}$.

Step 5: I_3 is finite over the weights. Let $\mathfrak{q}$ be a prime of W_3 containing $I_3 + \mathfrak{a}_3$; we show $\mathfrak{m}_{W_3} \subseteq \mathfrak{q}$, which gives $\mathfrak{m}_{W_3} \subseteq \sqrt{I_3 + \mathfrak{a}_3}$. Here $\ker p \subseteq I_3 \subseteq \mathfrak{q}$ and $\mathfrak{m}_{\mathcal{O}}W_3 \subseteq \mathfrak{a}_3 \subseteq \mathfrak{q}$.

(i) $\beta(\mathfrak{m}_{W_2}) \subseteq p(\mathfrak{q})$. $\mathfrak{q}_2 = \beta^{-1}(p(\mathfrak{q}))$ is a prime of W_2. It contains $\mathcal{M}_2$, since $\beta(\mathcal{M}_2) \subseteq p(I_3) \subseteq p(\mathfrak{q})$, and $\mathfrak{a}_2$: β is an $\mathcal{O}$-algebra homomorphism and $\beta(\psi_{2,v,a}(g) - 1) = p(\psi_{3,v,a'}(g) - 1) \in p(\mathfrak{a}_3)$

for $g \in I_v$ with $\bar\Psi(g) = 1$, $a' = 1, 3$ for $a = 1, 2$ (because $C \to B$ is injective). As $\mathcal{M}_2$ is finite over the weights, $\mathfrak{m}_{W_2} \subseteq \sqrt{\mathcal{M}_2 + \mathfrak{a}_2} \subseteq \mathfrak{q}_2$.

(ii) $\rho_3(\gamma)_{22} - 1 \in \mathfrak{q}$ *for all* $\gamma \in \Gamma_L^0$. $W_3/\mathfrak{q}$ is a complete Noetherian local domain containing $\mathcal{O}/\mathfrak{m}_\mathcal{O}$, so of characteristic ℓ. By (49.3) and $\ker p \subseteq \mathfrak{q}$, $\chi(\gamma) = \rho_3(\gamma)_{22} + \mathfrak{q}$ is a homomorphism $\Gamma_L^0 \to (W_3/\mathfrak{q})^\times$ with values in $1+\mathfrak{m}$, $\mathfrak{m}$ the maximal ideal of $W_3/\mathfrak{q}$ ((Po5)). It is trivial on $\pi_L(I_v)$: for $g \in I_v$ with $\bar\Psi(g) = 1$, (49.4) gives $\chi(\pi_L(g)) = \psi_{3,v,2}(g) + \mathfrak{q} = 1$, since $\psi_{3,v,2}(g) - 1 \in \mathfrak{a}_3$; for general $g \in I_v$, $\bar\Psi(g)$ has order m prime to ℓ, so $\chi(\pi_L(g))^m = 1$, while $\chi(\pi_L(g))^{\ell^N} \equiv 1$ modulo $\mathfrak{m}^N$ for every N (characteristic ℓ); hence $\chi(\pi_L(g)) \equiv 1$ modulo every $\mathfrak{m}^N$, and $\chi(\pi_L(g)) = 1$ by Krull's intersection theorem. It is trivial on $\pi_L(cI_vc)$: by (Po2), $\rho_3(cg^{-1}c)_{22} = \rho_3(g)_{22}$ for $g \in G_L$, so $\chi(\pi_L(cgc)) = \chi(\pi_L(g))^{-1} = 1$ for $g \in I_v$. For $N \geq 1$ the ring $W_3/(\mathfrak{q} + \mathfrak{m}_{W_3}^N)$ is finite ($\mathbb{F}_\mathcal{O}$ is finite), the image of χ in it lies in the abelian ℓ-group $(1+\mathfrak{m})/(1+\mathfrak{m}^N)$, and the induced map is locally constant by Step 1, i.e. continuous. By (Ar5)(ii) there is $h \geq 1$, independent of N, with $\chi(\gamma)^h \equiv 1$ modulo $\mathfrak{m}^N$ for all N, hence $\chi(\gamma)^h = 1$. Write $h = \ell^e h'$ with $\ell \nmid h'$ and $\chi(\gamma)^{h'} = 1+z$. Then $(1+z)^{\ell^e} = 1+z^{\ell^e} = 1$, so $z = 0$ in the domain $W_3/\mathfrak{q}$; and with $\chi(\gamma) = 1+x$, $0 = (1+x)^{h'} - 1 = x\big(h' + x\sum_{i\geq 2}\binom{h'}{i}x^{i-2}\big)$ with the second factor a unit, so $x = 0$.

(iii) *Generators are constant modulo* $\mathfrak{q}$. Let $x = \prod_k \rho_3(\gamma_k)_{a_kb_k}$ be of weight 0. If some factor is odd, $x \in I^{\mathrm{odd}} \cap W_3 \subseteq \mathfrak{q}$. Otherwise each factor is a corner entry or a $(2,2)$ entry, and $x = x_cx_m$ with x_m the product of the $(2,2)$ factors, $x_m \equiv 1$ modulo $\mathfrak{q}$ by (ii), and x_c the product of the corner factors, of weight 0. Let $y \in W_2$ be the corresponding product of entries of ρ_2 (of weight 0), and $c \in \mathcal{O}$ with $y - c \in \mathfrak{m}_{W_2}$ ((Gw1) for $n = 2$). Then $p(x_c - c) = \beta(y - c) \in p(\mathfrak{q})$ by (i), so $x_c - c \in \mathfrak{q}$ and $x \equiv c$. Likewise $p(\psi_{3,v,a'}(g)) = \beta(\psi_{2,v,a}(g))$ is congruent to a constant modulo $p(\mathfrak{q})$ for $a' = 1, 3$, and $\psi_{3,v,2}(g) \equiv \rho_3(g)_{22} \equiv 1$ by (49.4) and (ii). By Lemma 49.32 (rank 3, $P = \mathfrak{q}$), $\mathfrak{m}_{W_3} \subseteq \mathfrak{q}$.

Steps 4 and 5 show that I_3 has the required properties. □

Lemma 49.34. *Let ℓ be odd, L an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2, 3\}$, and suppose a universal polarised ordinary lift of rank n over L exists; write $W = W_n^{\mathrm{po}}$, $E = E_n^{\mathrm{po}}$. Let $\mathcal{M} \subseteq W$ be an ideal finite over the weights that propagates through nice primes and, if $n = 3$, along thick links; if $n = 3$ let moreover $I_3 \subseteq W$ be an ideal finite over the weights with $I_3 \subseteq \xi \cap W$ for every prime ξ of E of endoscopic type. Let x be a core prime of E that is a maximal ideal, $(\mathcal{O}', R, \iota)$ a local chart at x, and $J \subseteq R$ an ideal such that*

(Jp1) *$J \subseteq P$ for every prime P of R of irreducible type;*

(Jp2) *any two minimal primes P, Q of J are joined by minimal primes $P = P_0, \dots, P_k = Q$ of J with $\dim R/(P_{i-1} + P_i) \geq n\,d_L + 1$ for $1 \leq i \leq k$;*

(Jp3) *every minimal prime of J is of irreducible type or, if $n = 3$, thick.*

If some $\mathcal{M}$-modular prime P_0 of R of irreducible type satisfies $\dim R/P_0 \geq n\,d_L + 1$, then every minimal prime of R of irreducible type is $\mathcal{M}$-modular.

Proof. Throughout, prime types and modularity of primes of R are those of Definition 49.2 for $C = R$ with ι. Two facts are used without comment: modularity passes to larger primes ($P \subseteq P'$ gives $\iota^{-1}(P) \cap W \subseteq \iota^{-1}(P') \cap W$), and the conditions $I_R^- \not\subseteq P$, $I_R^+ \not\subseteq P$, $I_R^{\mathrm{odd}} \not\subseteq P$ pass to smaller primes, so a prime contained in a prime of irreducible type is of irreducible type. A nice prime is off-triangular, hence of irreducible type or (if $n = 3$) of endoscopic type, and no prime is both. A thick prime is of endoscopic type. The ring R is a complete Noetherian local ring, and $n\,d_L \geq 1$.

Step 1: nice primes above large primes. Let $I \subseteq W$ be finite over the weights and Q a prime of R with $I \subseteq \iota^{-1}(Q) \cap W$ and $\dim R/Q \geq n\,d_L + 1$. Then there is a nice prime $\mathfrak{p} \supseteq Q$. Indeed, let ϖ be a uniformiser of $\mathcal{O}$; its image in R lies in $\mathfrak{m}_R$ (the structure maps $\mathcal{O} \to \mathcal{O}' \to R$ are local). Consider the ideals $\mathfrak{a}_0 = I_R^+$ and, for $v \mid \ell$ and $a \neq b$, $\mathfrak{a}_{v,a,b} = \iota(I)R + \mathfrak{m}_{\mathcal{O}}R + (\psi_{x,v,a}(g) - \psi_{x,v,b}(g) : g \in I_v, \chi_{\mathrm{cyc}}(g) = 1)$; there are finitely many, and each has $\dim R/\mathfrak{a} < n\,d_L$ by (Nn1), (Nn2) of Proposition 49.29. By Lemma 49.35 (with $N = n\,d_L$) there is a prime $\mathfrak{p} \supseteq Q$ with $\dim R/\mathfrak{p} = 1$, $\varpi \in \mathfrak{p}$, and no $\mathfrak{a}_0$, $\mathfrak{a}_{v,a,b}$ contained in $\mathfrak{p}$. Then $\ell \in \varpi\mathcal{O}$ lies in $\mathfrak{p}$; $I_R^- = R \not\subseteq \mathfrak{p}$ by (Qc3) of Proposition 49.19 and $I_R^+ \not\subseteq \mathfrak{p}$, so $\mathfrak{p}$ is off-triangular; and since $\iota(I) \subseteq Q \subseteq \mathfrak{p}$ and $\mathfrak{m}_{\mathcal{O}}R = \varpi R \subseteq \mathfrak{p}$, the condition $\mathfrak{a}_{v,a,b} \not\subseteq \mathfrak{p}$ gives $g \in I_v$ with $\chi_{\mathrm{cyc}}(g) = 1$ and $\psi_{x,v,a}(g) - \psi_{x,v,b}(g) \notin \mathfrak{p}$. So $\mathfrak{p}$ is nice.

Step 2: saturation. Call a nice prime $\mathfrak{p}$ *saturated* if every prime $P' \subseteq \mathfrak{p}$ of irreducible type is $\mathcal{M}$-modular. If some $\mathcal{M}$-modular prime $P \subseteq \mathfrak{p}$ of irreducible type exists, $\mathfrak{p}$ is saturated: if $\mathfrak{p}$ is of irreducible type, it is $\mathcal{M}$-modular (it contains P), and propagation through nice primes makes every prime $P' \subseteq \mathfrak{p}$ $\mathcal{M}$-modular; if $\mathfrak{p}$ is of endoscopic type ($n = 3$), propagation along thick links applied to the pair $\mathfrak{p}, \mathfrak{p}$ (thickly connected with $k = 0$) does it.

Step 3: endoscopic nice primes above thick components ($n = 3$). Let $\mathfrak{e}$ be a minimal prime of J that is thick and $M' \supseteq \mathfrak{e}$ a prime with $\dim R/M' \geq 3d_L + 1$. Then there is a nice prime $\mathfrak{p}' \supseteq M'$ of endoscopic type. Indeed $\mathfrak{e}$ is of endoscopic type, so by (Qc3) the prime $\iota^{-1}(\mathfrak{e})$ of E is of endoscopic type, whence $I_3 \subseteq \iota^{-1}(\mathfrak{e}) \cap W \subseteq \iota^{-1}(M') \cap W$; Step 1 with $I = I_3$ gives a nice $\mathfrak{p}' \supseteq M'$, and $I_R^{\mathrm{odd}} \subseteq \mathfrak{e} \subseteq \mathfrak{p}'$ makes it endoscopic.

Step 4: good components. Call a minimal prime Q of J *good* if either (a) Q is of irreducible type and $\mathcal{M}$-modular, or (b) $n = 3$, Q is thick, and there are nice primes $\mathfrak{p}_0, \mathfrak{p}_1$ of endoscopic type with $Q \subseteq \mathfrak{p}_1$, $\mathfrak{p}_1$ thickly connected to $\mathfrak{p}_0$, and some $\mathcal{M}$-modular prime of irreducible type contained in $\mathfrak{p}_0$. *If Q is good and $M' \supseteq Q$ is a prime with $\dim R/M' \geq n\,d_L + 1$, there is a saturated nice prime containing M'.* In case (a), $\mathcal{M} \subseteq \iota^{-1}(Q) \cap W \subseteq \iota^{-1}(M') \cap W$, Step 1 with $I = \mathcal{M}$ gives a nice $\mathfrak{p} \supseteq M' \supseteq Q$, saturated by Step 2. In case (b), Step 3 gives a nice endoscopic $\mathfrak{p}' \supseteq M'$; the thick prime Q lies in $\mathfrak{p}_1 \cap \mathfrak{p}'$, so $\mathfrak{p}_1, \mathfrak{p}'$ are thickly linked and $\mathfrak{p}'$ is thickly connected to $\mathfrak{p}_0$; propagation along thick links makes every prime of irreducible type in $\mathfrak{p}'$ $\mathcal{M}$-modular, i.e. $\mathfrak{p}'$ is saturated.

Step 5: goodness passes along chains. Let Q, Q' be minimal primes of J with $\dim R/(Q + Q') \geq n\,d_L + 1$ and Q good. Choose a prime $M' \supseteq Q + Q'$ with $\dim R/M' = \dim R/(Q + Q')$. By Step 4 a saturated nice prime $\mathfrak{p} \supseteq M'$ exists. If Q' is of irreducible type, $Q' \subseteq \mathfrak{p}$ is $\mathcal{M}$-modular, so Q' is good by (a). Otherwise, by (Jp3), $n = 3$ and Q' is thick; Step 3 (with $\mathfrak{e} = Q'$) gives a nice endoscopic $\mathfrak{p}' \supseteq M'$. If Q is good by (a), then $Q \subseteq \mathfrak{p}'$ is an $\mathcal{M}$-modular prime of irreducible type, and (b) holds for Q' with $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}'$. If Q is good by (b) with $\mathfrak{p}_0, \mathfrak{p}_1$, then the thick prime Q lies in $\mathfrak{p}_1 \cap \mathfrak{p}'$, so $\mathfrak{p}'$ is thickly connected to $\mathfrak{p}_0$ and (b) holds for Q' with $\mathfrak{p}_0$ and $\mathfrak{p}'$.

Step 6: a first good component. By (Jp1), $J \subseteq P_0$; let $Q_0 \subseteq P_0$ be a minimal prime of J. It is of irreducible type, being contained in P_0. Step 1 with $I = \mathcal{M}$ and $Q = P_0$ gives a nice $\mathfrak{p} \supseteq P_0$, saturated by Step 2 since $P_0 \subseteq \mathfrak{p}$; so $Q_0 \subseteq \mathfrak{p}$ is $\mathcal{M}$-modular and good by (a).

Step 7: conclusion. By (Jp2) every minimal prime of J is joined to Q_0 by a chain as in Step 5, so all minimal primes of J are good. Let P be a minimal prime of R of irreducible type. By (Jp1) $J \subseteq P$, and P is minimal over J because it is minimal over 0; so P is good. It is not thick (it is not of endoscopic type), so (a) holds: P is $\mathcal{M}$-modular. □

Lemma 49.35. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\varpi \in \mathfrak{m}$, $N \geq 1$ an integer, $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ ideals of R with $\dim R/\mathfrak{a}_i < N$ for all i, and I an ideal of R with $\dim R/I \geq N + 1$. Then there is a prime $\mathfrak{p} \supseteq I$ of R with $\dim R/\mathfrak{p} = 1$, $\varpi \in \mathfrak{p}$ and $\mathfrak{a}_i \not\subseteq \mathfrak{p}$ for $1 \leq i \leq s$.*

Proof. We use two standard facts for a Noetherian local ring $(S, \mathfrak{m}_S)$ and $x \in \mathfrak{m}_S$: Krull's

inequality $\dim S/xS \geq \dim S - 1$ ([12, Cor. 11.18 and its proof]), and, if S is a domain and $x \neq 0$, $\dim S/xS \leq \dim S - 1$ (every prime containing x is non-zero, so a chain of primes above xS extends by (0)). Also $\dim R/\mathfrak{b} = \max\{\dim R/\mathfrak{p} : \mathfrak{p} \supseteq \mathfrak{b} \text{ prime}\}$, attained at a minimal prime of $\mathfrak{b}$. All dimensions are finite.

For a prime $\mathfrak{q} \supseteq I + \varpi R$ put $\delta(\mathfrak{q}) = \dim R/\mathfrak{q}$ and $\beta(\mathfrak{q}) = \max_i \dim R/(\mathfrak{a}_i + \mathfrak{q}) \in \{-\infty, 0, 1, \dots\}$.

Start. By Krull's inequality in R/I, $\dim R/(I+\varpi R) \geq N$; let $\mathfrak{q}_0$ be a minimal prime of $I+\varpi R$ with $\delta(\mathfrak{q}_0) = \dim R/(I+\varpi R) \geq N$. Since $R/(\mathfrak{a}_i + \mathfrak{q}_0)$ is a quotient of $R/\mathfrak{a}_i$, $\beta(\mathfrak{q}_0) < N \leq \delta(\mathfrak{q}_0)$.

Descent step: if $\mathfrak{q} \supseteq I + \varpi R$ is prime with $\delta(\mathfrak{q}) \geq 2$ and $\beta(\mathfrak{q}) < \delta(\mathfrak{q})$, there is a prime $\mathfrak{q}' \supseteq \mathfrak{q}$ with $\delta(\mathfrak{q}') = \delta(\mathfrak{q}) - 1$ and $\beta(\mathfrak{q}') < \delta(\mathfrak{q}')$. Let $Q_1, \dots, Q_r$ be the minimal primes of the ideals $\mathfrak{a}_i + \mathfrak{q}$ ($1 \leq i \leq s$) that are different from $\mathfrak{m}$; there are finitely many. As $\delta(\mathfrak{q}) \geq 2$, $\mathfrak{q} \neq \mathfrak{m}$. By prime avoidance $\mathfrak{m} \not\subseteq \mathfrak{q} \cup Q_1 \cup \dots \cup Q_r$, since $\mathfrak{m}$ is contained in none of these primes; choose $x \in \mathfrak{m}$ outside all of them. In the local domain $R/\mathfrak{q}$ the image of x is non-zero, so by the two facts $\dim R/(\mathfrak{q} + xR) = \delta(\mathfrak{q}) - 1$; let $\mathfrak{q}'$ be a minimal prime of $\mathfrak{q} + xR$ with $\delta(\mathfrak{q}') = \delta(\mathfrak{q}) - 1$. Let $\mathfrak{p}$ be a prime containing $\mathfrak{a}_i + \mathfrak{q}'$. Then $\mathfrak{p}$ contains a minimal prime Q of $\mathfrak{a}_i + \mathfrak{q}$. If $Q = \mathfrak{m}$, then $\mathfrak{p} = \mathfrak{m}$ and $\dim R/\mathfrak{p} = 0$. Otherwise $Q = Q_k$ for some k, and $\mathfrak{p} \supseteq Q_k + xR$ with $x \notin Q_k$, so, applying the two facts in the domain R/Q_k,

$$\dim R/\mathfrak{p} \leq \dim R/(Q_k + xR) = \dim R/Q_k - 1 \leq \dim R/(\mathfrak{a}_i + \mathfrak{q}) - 1 \leq \beta(\mathfrak{q}) - 1.$$

Hence $\beta(\mathfrak{q}') \leq \max\{0, \beta(\mathfrak{q}) - 1\}$. Both 0 and $\beta(\mathfrak{q}) - 1$ are $< \delta(\mathfrak{q}) - 1 = \delta(\mathfrak{q}')$, the first because $\delta(\mathfrak{q}) \geq 2$; so $\beta(\mathfrak{q}') < \delta(\mathfrak{q}')$, and $\mathfrak{q}' \supseteq \mathfrak{q} \supseteq I + \varpi R$.

End. Applying the descent step $\delta(\mathfrak{q}_0) - 1$ times (none if $\delta(\mathfrak{q}_0) = 1$) gives a prime $\mathfrak{p} \supseteq I + \varpi R$ with $\dim R/\mathfrak{p} = 1$ and $\beta(\mathfrak{p}) < 1$. If $\mathfrak{a}_i \subseteq \mathfrak{p}$ for some i, then $\dim R/(\mathfrak{a}_i + \mathfrak{p}) = \dim R/\mathfrak{p} = 1 \leq \beta(\mathfrak{p})$, a contradiction. So $\mathfrak{p}$ has all the required properties. □

Lemma 49.36. *Let L satisfy (Af1) and (Af2) for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, let $n \in \{2, 3\}$, and let $(\rho^{\mathrm{u}}, (\psi^{\mathrm{u}}_{v,a}))$ be a universal polarised ordinary lift of rank n over L to R^{po}_n. For every test ring A and every polarised ordinary lift $(\rho, (\psi_{v,a}))$ of rank n over L to A there is exactly one local $\mathcal{O}$-algebra homomorphism $f\colon R^{\mathrm{po}}_n \to A$ with $f \circ \rho^{\mathrm{u}} = \rho$ and $f \circ \psi^{\mathrm{u}}_{v,a} = \psi_{v,a}$ for all $v \mid \ell$ and $1 \leq a \leq n$.*

Proof. Let $k_A = A/\mathfrak{m}_A$ (a finite field) and let $\mathbb{F}_{\mathcal{O}} \to k_A$ be the injection induced by the local homomorphism $\mathcal{O} \to A$. Put $A_0 = \{x \in A : \ x \bmod \mathfrak{m}_A \in \mathbb{F}_{\mathcal{O}}\}$.

A_0 lies in $\mathrm{CNL}_{\mathcal{O}}$. The set A_0 is a subring containing $\mathfrak{m}_A$ and the image of $\mathcal{O}$; it is local with maximal ideal $\mathfrak{m}_A$ (an element of $A_0 \setminus \mathfrak{m}_A$ is a unit of A whose inverse again has residue in $\mathbb{F}_{\mathcal{O}}$), and $A_0/\mathfrak{m}_A \cong \mathbb{F}_{\mathcal{O}}$, so $\mathcal{O} \to A_0$ is local and induces an isomorphism of residue fields. Let ϖ_A be a uniformiser of A; it lies in $\mathfrak{m}_A \subseteq A_0$, and for $k \geq 1$ the k-th power of the ideal $\mathfrak{m}_A$ of A_0 is $\varpi_A^k A = \mathfrak{m}_A^k$ (every element $\varpi_A^k y$ equals $\varpi_A^{k-1} \cdot \varpi_A y$ with all factors in $\mathfrak{m}_A$). Hence the $\mathfrak{m}_{A_0}$-adic topology of A_0 is induced by that of A, and A_0 is complete, being closed in A (it is a union of cosets of the open subgroup $\mathfrak{m}_A$). Choose $e_1, \dots, e_t \in A$ whose residues form an $\mathbb{F}_{\mathcal{O}}$-basis of k_A; then $A = \sum_i A_0 e_i + \varpi_A A$, and successive approximation in the complete ring A_0 gives $A = \sum_i A_0 e_i$. Thus A is a Noetherian ring finite over its subring A_0, and A_0 is Noetherian by the Eakin–Nagata theorem.

The lift factors through A_0. By (Po5), for $\gamma \in \Gamma^0_L$ the diagonal entries of $\rho(\gamma)$ are congruent to elements of $\mathbb{F}_{\mathcal{O}}$ and the other entries lie in $\mathfrak{m}_A$, and $\psi_{v,a}(g)$ is congruent to an element of $\mathbb{F}^\times_{\mathcal{O}}$; hence ρ takes values in $\mathrm{GL}_n(A_0)$ (the inverse of $\rho(\gamma)$ is $\rho(\gamma^{-1})$) and $\psi_{v,a}$ in $A_0^\times$. Conditions (Po1)–(Po5) for A_0 are the same identities and congruences as for A, because $\mathfrak{m}_{A_0}^k = \mathfrak{m}_A^k$. So $(\rho, (\psi_{v,a}))$ is a polarised ordinary lift to $A_0 \in \mathrm{CNL}_{\mathcal{O}}$, and the universal property gives a morphism

$f_0\colon R_n^{\mathrm{po}} \to A_0$ with $f_0 \circ \rho^{\mathrm{u}} = \rho$ and $f_0 \circ \psi^{\mathrm{u}}_{v,a} = \psi_{v,a}$. Then $f = f_0$ followed by the inclusion $A_0 \subseteq A$ is a local $\mathcal{O}$-algebra homomorphism with the required property.

Uniqueness. Let $f'\colon R_n^{\mathrm{po}} \to A$ be a local $\mathcal{O}$-algebra homomorphism with the same property. Since R_n^{po} has residue field $\mathbb{F}_{\mathcal{O}}$, every element is $o+y$ with o in the image of $\mathcal{O}$ and $y \in \mathfrak{m}_{R_n^{\mathrm{po}}}$, and $f'(y) \in \mathfrak{m}_A$; hence f' takes values in A_0 and is a morphism of $\mathrm{CNL}_{\mathcal{O}}$ into A_0. By the uniqueness in the universal property, $f' = f_0$ as maps into A_0, so $f' = f$. □

Lemma 49.37. *Let L be an auxiliary field of some level for $(K, \mathcal{O}, \Psi, c, g_\star, M)$ such that a universal polarised ordinary lift of rank 3 over L exists and W_3^{po} contains an off-triangular bound. Then L has residual characteristic polynomials in rank 3.*

Proof. Let I be an off-triangular bound, A a test ring of characteristic ℓ and $(\rho, (\psi_{v,a}))$ a polarised ordinary lift of rank 3 over L to A with $\psi_{v,a}(g) = 1$ whenever $g \in I_v$ and $\bar\Psi(g) = 1$. By Lemma 49.36 there is a local $\mathcal{O}$-algebra homomorphism $f\colon R_3^{\mathrm{po}} \to A$ with $f \circ \rho^{\mathrm{u}} = \rho$ and $f \circ \psi^{\mathrm{u}}_{v,a} = \psi_{v,a}$. Since A is a domain of characteristic ℓ and ℓ is a unit times a power of a uniformiser ϖ of $\mathcal{O}$, $f(\varpi) = 0$; so $\mathcal{O} \to A$ factors through $\mathbb{F}_{\mathcal{O}}$, and we regard $\mathbb{F}_{\mathcal{O}} \subseteq A$. Let $\mathfrak{p}$ be the kernel of f on E_3^{po}, a prime ideal. For $g \in G_L$ we have $\rho_L(g) = f(\rho^{\mathrm{u}}(\pi_L(g)))$, so it suffices to show that for every $\gamma \in \Gamma_L^0$ the characteristic polynomial of $X := \rho(\gamma)$ is $\prod_{a=1}^3 (T - \bar\Psi(\gamma)^{w_{3,a}})$.

Case 1: $\mathfrak{p}$ is off-triangular. Then $I \subseteq \mathfrak{p} \cap W_3^{\mathrm{po}} =: \mathfrak{q}$, a prime of W_3^{po}. Also $\mathfrak{a}_3 \subseteq \mathfrak{q}$: $f(\mathfrak{m}_{\mathcal{O}}) = 0$, and $f(\psi^{\mathrm{u}}_{v,a}(g) - 1) = \psi_{v,a}(g) - 1 = 0$ for $g \in I_v$ with $\bar\Psi(g) = 1$. Since I is finite over the weights, $\mathfrak{m}_W \subseteq \sqrt{I + \mathfrak{a}_3} \subseteq \mathfrak{q}$. Every $y \in W_3^{\mathrm{po}}$ is $o + z$ with o in the image of $\mathcal{O}$ and $z \in \mathfrak{m}_W$, because R_3^{po} has residue field $\mathbb{F}_{\mathcal{O}}$; hence $f(y)$ is the residue $\bar y \in \mathbb{F}_{\mathcal{O}}$ of y. Apply this to weight-zero monomials of the universal lift at γ. The diagonal entries have weight 0 and residue $\bar\Psi(\gamma)^{w_{3,a}}$ by (Po5), so $X_{aa} = \bar\Psi(\gamma)^{w_{3,a}}$. For $a \neq b$ the product $\rho^{\mathrm{u}}(\gamma)_{ab}\rho^{\mathrm{u}}(\gamma)_{ba}$ has weight 0 and lies in $\mathfrak{m}$ by (Po5), and so do $\rho^{\mathrm{u}}(\gamma)_{12}\rho^{\mathrm{u}}(\gamma)_{23}\rho^{\mathrm{u}}(\gamma)_{31}$ and $\rho^{\mathrm{u}}(\gamma)_{13}\rho^{\mathrm{u}}(\gamma)_{32}\rho^{\mathrm{u}}(\gamma)_{21}$ (weights $(w_{3,1} - w_{3,2}) + (w_{3,2} - w_{3,3}) + (w_{3,3} - w_{3,1}) = 0$); hence $X_{ab}X_{ba} = 0$ and $X_{12}X_{23}X_{31} = X_{13}X_{32}X_{21} = 0$. Expanding the determinant,

$$\det(T - X) = \prod_{a=1}^{3}(T - X_{aa}) - \sum_{\{a,b,e\}=\{1,2,3\},\, a<b} X_{ab}X_{ba}(T - X_{ee}) - X_{12}X_{23}X_{31} - X_{13}X_{32}X_{21} = \prod_{a=1}^{3}(T - \bar\Psi(\gamma)^{w_{3,a}}).$$

Case 2: $\mathfrak{p}$ is not off-triangular. Say $I_E^- \subseteq \mathfrak{p}$ (the case $I_E^+ \subseteq \mathfrak{p}$ is the same with lower triangular matrices). Then $\rho(\gamma)$ is upper triangular for all $\gamma \in \Gamma_L^0$, so $\det(T - \rho(\gamma)) = \prod_a (T - \delta_a(\gamma))$ with $\delta_a(\gamma) = \rho(\gamma)_{aa}$, and each $\delta_a\colon \Gamma_L^0 \to A^\times$ is a homomorphism. Put $\chi_a = \delta_a \cdot \bar\Psi^{-w_{3,a}}$; by (Po5) it takes values in $1 + \mathfrak{m}_A$, and by (Po1) and the openness of $\ker\bar\Psi$ its reduction $\chi_{a,k}$ modulo $1 + \mathfrak{m}_A^k$ has open kernel, for every $k \geq 1$. The group $(1 + \mathfrak{m}_A)/(1 + \mathfrak{m}_A^k)$ is a finite abelian ℓ-group (its order is a power of $|A/\mathfrak{m}_A|$). We show $\chi_a = 1$.

(i) Let $v \mid \ell$. Since $\ell \mid M$, v splits in L by (Af2), so $D_v \subseteq G_L$. Let $u \in I_v$ with $\bar\Psi(u) = 1$. By (Po4) with $g_1 = g_2 = g_3 = u$ and $\psi_{v,a}(u) = 1$, $(\rho(\pi_L(u)) - 1)^3 = 0$; the diagonal entries of this upper triangular matrix are $(\delta_b(\pi_L(u)) - 1)^3$, so $\delta_b(\pi_L(u)) = 1$ and $\chi_b(\pi_L(u)) = 1$ for all b, A being a domain. Take $u_1, \dots, u_{d_v}$ as in (Ar5)(iii); then $\chi_{a,k} \circ \pi_L|_{D_v}$ is a continuous homomorphism to a finite abelian ℓ-group trivial on the u_b, hence trivial on I_v.

(ii) For $g \in I_v$, (Po2) gives $\rho(\pi_L(cg^{-1}c)) = J_3\, {}^t\rho(\pi_L(g))\, J_3$, whose (a,a) entry is $\delta_{4-a}(\pi_L(g))$; and $\bar\Psi(cg^{-1}c) = \bar\Psi(g)^{-1}$ with $w_{3,4-a} = -w_{3,a}$. Hence $\chi_a(\pi_L(cg^{-1}c)) = \chi_{4-a}(\pi_L(g))$, and by (i) $\chi_{a,k}$ is trivial on $\pi_L(cI_vc)$.

(iii) By (Ar5)(ii) there is $h \geq 1$, independent of a and k, with $\chi_{a,k}^h = 1$ for all k; so $\chi_a(\gamma)^h - 1 \in \bigcap_k \mathfrak{m}_A^k = 0$. Write $h = \ell^s h'$ with $\ell \nmid h'$ and $\chi_a(\gamma) = 1 + x$, $x \in \mathfrak{m}_A$. In characteristic ℓ, $y := (1+x)^{\ell^s} = 1 + x^{\ell^s}$, and $0 = y^{h'} - 1 = (y-1)(y^{h'-1} + \cdots + 1)$ with the second factor $\equiv h' \not\equiv 0$ modulo $\mathfrak{m}_A$, a unit; so $x^{\ell^s} = 0$ and $x = 0$.

Thus $\delta_a = \bar{\Psi}^{w_{3,a}}$ on Γ_L^0 and $\det(T - \rho(\gamma)) = \prod_a (T - \bar{\Psi}(\gamma)^{w_{3,a}})$. □

50 The Hecke realisation

This section proves the Hecke realisation Theorem 50.3, the Hecke side of the $R = T$ theorem 49.3. Over an auxiliary field L it gives an ideal $\mathcal{M}$ of the weight-zero algebra W_n^{po} that is finite over the weights, propagates through nice primes and, for $n = 3$, along thick links (Hk2), has a seed component as soon as a seed exists (Hk3), and cuts out automorphic lifts: an irreducible polarised ordinary lift on which $\mathcal{M}$ vanishes is automorphic of conjugate self-dual type after the twist κ (Hk4). The ideal comes from ordinary automorphic forms on a definite unitary group over L^+, through Hida theory in the form of Geraghty [105]. The Galois side is the universal polarised ordinary determinant, a determinant in the sense of Chenevier [66] (Definition 50.2), and the Hecke algebra is a quotient of its ring; the ideal is then transported to the ring of marked lifts.

Subsection 50.1 states the theorem and its form for determinants, Theorem 50.6, at an auxiliary place v_0. Subsection 50.2 proves the latter from Proposition 50.11, Hida theory and patching for the definite unitary group, whose technical part, the Hecke ideals of zero-monodromy type and the patching at curve points over Taylor–Wiles levels, is in Appendix E. Subsection 50.3 passes from determinants to marked lifts.

50.1 The Hecke realisation and the ordinary Hecke ideal

This subsection defines rank-n Hecke data (Definition 50.1) and polarised ordinary determinants over an auxiliary field (Definition 50.2), and proves Theorem 50.3 from Theorem 50.6. The latter is the same statement for the universal determinant ring B_n^{po}: at a place v_0 above a prime that splits in L and at which $\bar{\Psi}$ is trivial and $\bar{\omega}$ is not, there is an ideal $\mathcal{J}$ of B_n^{po} that is finite over the weights, whose minimal primes all have dimension $1 + n\,d_L$ (Hd1), with the propagation properties (Hd2), containing the seed points (Hd3) and cutting out automorphic lifts (Hd4). The passage from Theorem 50.6 to Theorem 50.3 enlarges the coefficient ring (Lemma 50.4), constructs universal polarised ordinary lifts (Proposition 50.5), and transports $\mathcal{J}$ along the comparison morphism to the weight-zero algebra (Proposition 50.26, Subsection 50.3). The Hecke data are defined in rank n so that the same construction serves the auxiliary fields, in ranks two and three, and the links of the symplectic case, in rank four.

Definition 50.1. Let ℓ be an odd prime, $\mathcal{O}$ a coefficient ring and $\bar{\omega}$ the reduction of χ_{cyc} modulo $\mathfrak{m}_{\mathcal{O}}$. For a CM field $L \subset \overline{\mathbb{Q}}$ with maximal totally real subfield L^+, the groups $G_L \subseteq G_{L^+}$ are subgroups of $G_{\mathbb{Q}}$, and S_ℓ denotes the set of places of L^+ above ℓ.

Rank-n Hecke data. A **rank-n Hecke datum** over $\mathcal{O}$ is a tuple

$$\mathscr{H} = \big(n,\, L,\, c,\, (\bar{v})_{v \in S_\ell},\, \mathrm{St},\, v_0,\, \bar{v}_0,\, (\bar{\chi}_a)_{1 \leq a \leq n},\, (\xi_{v,a})_{v \in S_\ell,\, 1 \leq a \leq n}\big)$$

of an integer $n \geq 2$; a CM field $L \subset \overline{\mathbb{Q}}$; a complex conjugation $c \in G_{L^+}$ (the automorphism induced by complex conjugation for some embedding $\overline{\mathbb{Q}} \to \mathbb{C}$); for every $v \in S_\ell$ a place $\bar{v}$ of $\overline{\mathbb{Q}}$ above v; a finite set St of places of L^+ not above ℓ; a place $v_0 \notin \mathrm{St}$ of L^+ not above ℓ, with a place $\bar{v}_0$ of $\overline{\mathbb{Q}}$ above it; continuous characters $\bar{\chi}_1, \ldots, \bar{\chi}_n \colon G_L \to \mathbb{F}_{\mathcal{O}}^\times$, the **residual characters**; and,

for $v \in S_\ell$ and $1 \le a \le n$, homomorphisms $\xi_{v,a}\colon \mu_v \to \mathcal{O}^\times$, the **wild values**, where $\mu_v \subseteq D_v$ (defined below) depend only on $\bar{v}$; such that:

(Rd1) $\tau(\mathcal{O}_L) \subseteq \mathcal{O}$ for every embedding $\tau\colon L \to \overline{\mathbb{Q}}_\ell$, and L contains no primitive ℓ-th root of unity;

(Rd2) L/L^+ is unramified at every finite place, and every place in $S_\ell \cup \mathrm{St} \cup \{v_0\}$ splits in L;

(Rd3) v_0 lies above a prime $p \neq 2, \ell$ that is unramified in L;

(Rd4) the $\bar{\chi}_a$ are pairwise distinct, and each $\bar{\chi}_a$ is trivial on $I_{\bar{w}} \cap G_L$ for every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above a finite place of L^+ not in $S_\ell \cup \mathrm{St}$;

(Rd5) (polarisation) $\bar{\chi}_{n+1-a}(csc) = \bar{\chi}_a(s)^{-1}$ for all $s \in G_L$ and $1 \le a \le n$;

(Rd6) $\bar{\chi}_1, \dots, \bar{\chi}_n$ coincide on $D_{\bar{v}_0} \cap G_L$, and $\bar{\omega}$ is non-trivial on $D_{\bar{v}_0} \cap G_L$;

(Rd10) $\xi_{v,a}(g) = 1$ for every $v \in S_\ell$, $1 \le a \le n$ and every $g \in \mu_v$ that lies in the closure $\overline{[D_v, D_v]}$ of the commutator subgroup of D_v.

When the wild values play no role they are omitted from the notation, and $\mathscr{H}$ is written $(n, L, c, (\bar{v})_{v\in S_\ell}, \mathrm{St}, v_0, \bar{v}_0, (\bar{\chi}_a)_{1\le a\le n})$.

The Galois group of a datum. Let $N_{\mathscr{H}}$ be the closed normal subgroup of G_{L^+} generated by the elements $s \in I_{\bar{w}} \cap G_L$, for all places $\bar{w} \nmid \ell$ of $\overline{\mathbb{Q}}$, with $\bar{\chi}_a(s) = 1$ for $1 \le a \le n$. Put $\Gamma_{\mathscr{H}} = G_{L^+}/N_{\mathscr{H}}$, let $\pi_{\mathscr{H}}\colon G_{L^+} \to \Gamma_{\mathscr{H}}$ be the projection and $\Gamma^0_{\mathscr{H}} = \pi_{\mathscr{H}}(G_L)$. The characters $\bar{\chi}_a$ factor through $\Gamma^0_{\mathscr{H}}$ and χ_{cyc} through $\Gamma_{\mathscr{H}}$; they are denoted by the same letters. For $v \in S_\ell$ put $D_v = D_{\bar{v}} \cap G_{L^+} \supseteq I_v = I_{\bar{v}} \cap G_{L^+}$ (both contained in G_L by (Rd2)) and $d_v = [L_v^+ : \mathbb{Q}_\ell]$, and let $\mu_v \subseteq D_v$ be the preimage of the subgroup of elements of ℓ-power order of the abelianisation $D_v/\overline{[D_v, D_v]}$ (so μ_v is the set of $g \in D_v$ with $g^{\ell^k} \in \overline{[D_v, D_v]}$ for some $k \ge 0$).

Twists. A **twist** for $\mathscr{H}$ is a continuous character $\kappa\colon G_L \to \mathcal{O}^\times$ that corresponds through ι and the global Artin map to an algebraic Hecke character of L, such that

(Rd7) $\kappa(s)\kappa(csc) = \chi_{\mathrm{cyc}}(s)^{1-n}$ for all $s \in G_L$; κ is trivial on $I_{\bar{w}} \cap G_L$ for every place $\bar{w}$ above a finite place of L^+ not in $S_\ell \cup \mathrm{St}$; and there is an integer $m' \ge 1$ with $\ell \nmid m'$ such that $\kappa^{m'}$ is trivial on $I_{\bar{w}} \cap G_L$ for every place $\bar{w} \nmid \ell$.

Inertial weights. An **inertial weight** for $\mathscr{H}$ is a family $(\psi^\circ_{v,a})_{v\in S_\ell,\, 1\le a\le n}$ of continuous characters $\psi^\circ_{v,a}\colon I_v \to \mathcal{O}^\times$ such that

(Rd8) for every $v \in S_\ell$ there are de Rham characters $\phi_{v,1}, \dots, \phi_{v,n}\colon D_v \to \overline{\mathbb{Q}}_\ell^\times$ with $\phi_{v,a}|_{I_v} = \psi^\circ_{v,a}$ whose τ-Hodge–Tate weights are strictly decreasing in a for every field embedding $\tau\colon L_v^+ \to \overline{\mathbb{Q}}_\ell$; and $\psi^\circ_{v,a}(g) \equiv \bar{\chi}_a(g)$ modulo $\mathfrak{m}_{\mathcal{O}}$ for all $g \in I_v$.

Definition 50.2. Let $\mathcal{O}$ be a coefficient ring.

Residual settings. A **residual setting** of rank n over $\mathcal{O}$ is a tuple $(n, L, c, \Gamma, \Gamma^0, \pi, (D_v \supseteq I_v)_{v\in S_\ell}, (\bar{\chi}_a)_{1\le a\le n})$, where S_ℓ is the set of places of L^+ above ℓ, of one of the following two kinds:

(Rs1) for a rank-n Hecke datum $\mathscr{H} = (n, L, c, (\bar{v})_{v\in S_\ell}, \mathrm{St}, v_0, \bar{v}_0, (\bar{\chi}_a)_{1\le a\le n})$ over $\mathcal{O}$: its n, L, c, $\Gamma = \Gamma_{\mathscr{H}}$, $\Gamma^0 = \Gamma^0_{\mathscr{H}}$, $\pi = \pi_{\mathscr{H}}$, $D_v \supseteq I_v$ and $\bar{\chi}_a$; the objects defined below from it are said to be **over** $\mathscr{H}$;

(Rs2) for a CM field L satisfying (Af1) and (Af2) for $(K,\mathcal{O},\Psi,c,g_\star,M)$ and $n\in\{2,3\}$: n, L, c, $\Gamma=\Gamma_L$, $\Gamma^0=\Gamma_L^0$, $\pi=\pi_L$, the groups $D_v\supseteq I_v$ ($v\mid\ell$) of Definition 48.2 and $\bar\chi_a=\bar\Psi^{w_{n,a}}$; the objects defined below from it are said to be **over** L.

In both cases $D_v\subseteq G_L$, and the $\bar\chi_a$ are characters of Γ^0. In this definition a residual setting $(n,L,c,\Gamma,\Gamma^0,\pi,(D_v\supseteq I_v),(\bar\chi_a))$ of rank n over $\mathcal{O}$ is fixed.

Determinants. For a commutative ring A and a group Γ_1, an **A-valued determinant of dimension n on Γ_1** is a multiplicative A-polynomial law $D\colon A[\Gamma_1]\to A$, homogeneous of degree n, in the sense of [66]. For $x\in A[\Gamma_1]$ its **characteristic polynomial** is $\chi_D(x)(X)=D(X-x)\in A[X]$, monic of degree n, and its **trace** $\mathrm{tr}_D(x)$ is minus the coefficient of X^{n-1}; tr_D is A-linear. For a homomorphism $\rho\colon\Gamma_1\to\mathrm{GL}_n(A)$, $\det\circ\rho$ is a determinant of dimension n with $\chi_{\det\circ\rho}(\gamma)$ the characteristic polynomial of $\rho(\gamma)$. For a ring homomorphism $f\colon A\to A'$, $f\circ D$ denotes the base change of D to A'. For $A\in\mathrm{CNL}_\mathcal{O}$ and Γ_1 profinite, continuity of D is meant in the sense of [66].

Polarised ordinary determinants. Let $\bar D_n=\det\circ\mathrm{diag}(\bar\chi_1,\dots,\bar\chi_n)$, an $\mathbb{F}_\mathcal{O}$-valued determinant of dimension n on Γ^0. For $A\in\mathrm{CNL}_\mathcal{O}$, a **polarised ordinary determinant of rank n** (over $\mathscr{H}$, resp. over L) to A is a tuple $(D,(\psi_{v,a}))$ of a continuous A-valued determinant D of dimension n on Γ^0 whose reduction modulo $\mathfrak{m}_A$ is $\bar D_n$, and continuous homomorphisms $\psi_{v,a}\colon D_v\to A^\times$ ($v\in S_\ell$, $1\le a\le n$) with $\psi_{v,a}\equiv\bar\chi_a$ modulo $\mathfrak{m}_A$, such that

(Od1) $\chi_D(\pi(cg^{-1}c))=\chi_D(\pi(g))$ for all $g\in G_L$;

(Od2) $\mathrm{tr}_D(\gamma\cdot\prod_{a=1}^n(\pi(g_a)-\psi_{v,a}(g_a)))=0$ for all $\gamma\in\Gamma^0$, $v\in S_\ell$ and $g_1,\dots,g_n\in D_v$, the product taken in $A[\Gamma^0]$ with $a=1$ on the left;

(Od3) (only over $\mathscr{H}$, case (Rs1)) $\psi_{v,a}(g)$ is the image of $\xi_{v,a}(g)$ under $\mathcal{O}\to A$, for all $v\in S_\ell$, $1\le a\le n$ and $g\in\mu_v$, where $\xi_{v,a}$ are the wild values of $\mathscr{H}$ and $\mu_v\subseteq D_v$ is as in Definition 50.1.

Over L (case (Rs2)) a polarised ordinary determinant is a tuple satisfying (Od1) and (Od2); over $\mathscr{H}$ (case (Rs1)) it satisfies (Od1), (Od2) and (Od3). A **universal polarised ordinary determinant** of rank n (over $\mathscr{H}$, resp. over L) is such a tuple $(D^\mathrm{u},(\psi^\mathrm{d}_{v,a}))$ to some $B_n^\mathrm{po}\in\mathrm{CNL}_\mathcal{O}$ such that for every $A\in\mathrm{CNL}_\mathcal{O}$ and every polarised ordinary determinant $(D,(\psi_{v,a}))$ of rank n to A (over the same setting) there is exactly one morphism $x\colon B_n^\mathrm{po}\to A$ of $\mathrm{CNL}_\mathcal{O}$ with $x\circ D^\mathrm{u}=D$ and $x\circ\psi^\mathrm{d}_{v,a}=\psi_{v,a}$; x is said to **classify** $(D,(\psi_{v,a}))$. An ideal $\mathcal{J}\subseteq B_n^\mathrm{po}$ is **finite over the weights** if $\mathfrak{m}_{B_n^\mathrm{po}}\subseteq\sqrt{\mathcal{J}+\mathfrak{b}_n}$, where $\mathfrak{b}_n$ is the ideal generated by $\mathfrak{m}_\mathcal{O}$ and the elements $\psi^\mathrm{d}_{v,a}(g)-1$ with $v\in S_\ell$, $1\le a\le n$, $g\in I_v$ and $\bar\chi_b(g)=1$ for $1\le b\le n$.

The comparison morphism. In case (Rs2), suppose that a universal polarised ordinary determinant $(B_n^\mathrm{po},D^\mathrm{u},(\psi^\mathrm{d}_{v,a}))$ and a universal polarised ordinary lift $(R_n^\mathrm{po},\rho^\mathrm{u},(\psi^\mathrm{u}_{v,a}))$ of rank n over L are given. If $(\det\circ\rho^\mathrm{u},(\psi^\mathrm{u}_{v,a}))$ is a polarised ordinary determinant, the morphism $\vartheta_n\colon B_n^\mathrm{po}\to R_n^\mathrm{po}$ classifying it is the **comparison morphism**. For an ideal $\mathcal{J}\subseteq B_n^\mathrm{po}$, $\mathcal{M}_\mathcal{J}\subseteq W_n^\mathrm{po}$ denotes the ideal of W_n^po generated by $\vartheta_n(\mathcal{J})$, whenever $\vartheta_n(B_n^\mathrm{po})\subseteq W_n^\mathrm{po}$.

Theorem 50.3. *Let $\mathcal{O}$, ι, c, K, M, Ψ be as in (a) or (b) of Theorem 49.3, let $g_\star$ be a marking and L an auxiliary field of level C_0 for $(K,\mathcal{O},\Psi,c,g_\star,M)$, with data (N',w,j,χ',θ) as in (Ag1). Let $n\in\{2,3\}$, let κ be an admissible twist of rank n for ι, c and M, and let $\mathcal{O}'\supseteq\mathcal{O}$ be a coefficient ring with $\kappa(G_L)\subseteq\mathcal{O}'^\times$ and $\tau(\mathcal{O}_L)\subseteq\mathcal{O}'$ for every embedding $\tau\colon L\to\overline{\mathbb{Q}}_\ell$. Then for every integer $p_0\ge1$ there are a prime $p\nmid p_0$ and a place v_0 of L^+ above p such that:*

(Hk1) *L is an auxiliary field of level C_0 for $(K,\mathcal{O}',\Psi,c,g_\star,M)$, and for these data there is a universal polarised ordinary lift of rank n over L;*

(Hk2) *there is an ideal $\mathcal{M} \subseteq W_n^{\mathrm{po}}$ that is finite over the weights and propagates through nice primes, and, if $n = 3$, propagates along thick links;*

(Hk3) *if for some $m \geq 2$ there is a seed of rank n and weight m for (j, χ') with twist κ, then $\mathcal{M}$ has a seed component;*

(Hk4) *for every integral test ring A over $\mathcal{O}'$ and every polarised ordinary lift $(\rho, (\psi_{v,a}))$ of rank n over L to A (for $\mathcal{O}'$) such that $\psi_{v,a}(g) = \Psi(g)^{w_{n,a}}$ for all $v \mid \ell$, a and $g \in I_v$, ρ_L is irreducible over $\overline{\mathbb{Q}}_\ell$, and $f(\mathcal{M}) = 0$ for the morphism $f\colon R_n^{\mathrm{po}} \to A$ with $f \circ \rho^{\mathrm{u}} = \rho$ and $f \circ \psi_{v,a}^{\mathrm{u}} = \psi_{v,a}$, the representation $\rho_L \otimes \kappa$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions.*

Proof. Recall that ℓ is odd, that $\ell N \mid M$, and that $\Psi = \Psi_{\chi,e} = \chi \chi_{\mathrm{cyc}}^e$ is unramified at every prime not dividing ℓN, since χ_{cyc} is unramified away from ℓ and χ, a Dirichlet character modulo N (in the unitary as in the dyadic Eisenstein datum), viewed as a character of $G_{\mathbb{Q}}$, is unramified away from N; $\bar{\Psi}$ and $\bar{\omega}$ are characters of $G_{\mathbb{Q}}$ with finite image.

Step 1: (Hk1). By Lemma 50.4, $g_\star$ is a marking for $(K, \mathcal{O}', \Psi, c)$ and L is an auxiliary field of level C_0 for $(K, \mathcal{O}', \Psi, c, g_\star, M)$; the data $(N', w, j, \chi', \theta)$ of (Ag1) do not depend on the coefficients. By Proposition 50.5, applied with $\mathcal{O}'$, a universal polarised ordinary lift $(R_n^{\mathrm{po}}, \rho^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{u}}))$ of rank n over L exists. This is (Hk1). From now on all rings and lifts are for $\mathcal{O}'$.

Step 2: an element separating $\bar{\omega}$ from $\bar{\Psi}$. There is $g' \in G_L$ with $\bar{\Psi}(g') = 1$ and $\bar{\omega}(g') \neq 1$. Otherwise $\bar{\omega}|_{G_L}$ is trivial on $G_L \cap \ker \bar{\Psi}$, hence $\bar{\omega}|_{G_L} = \phi \circ \bar{\Psi}|_{G_L}$ for a homomorphism $\phi\colon C \to \mathbb{F}_{\mathcal{O}}^\times$ on the finite subgroup $C = \bar{\Psi}(G_L)$ of the cyclic group $\mathbb{F}_{\mathcal{O}}^\times$. C is cyclic, and every homomorphism $C \to \mathbb{F}_{\mathcal{O}}^\times$ is $y \mapsto y^j$ for some $j \in \mathbb{Z}$ (the homomorphisms form a cyclic group of order $|C|$, and $y \mapsto y$ has order $|C|$). So $\bar{\omega} = \bar{\Psi}^j$ on G_L, against (Ar1).

Step 3: the auxiliary place. Let $F \subset \overline{\mathbb{Q}}$ be the fixed field of $G_L \cap \ker \bar{\Psi} \cap \ker \bar{\omega}$, a normal subgroup of finite index of $G_{\mathbb{Q}}$ ($L/\mathbb{Q}$ is abelian by (Af1)); so $F/\mathbb{Q}$ is finite Galois and $L \subseteq F$. By the Chebotarev density theorem there is a prime p, unramified in F and not dividing $2Mp_0\,|\mathrm{disc}(L)|$, whose Frobenius class in $\mathrm{Gal}(F/\mathbb{Q})$ is the class of $g'|_F$. Choose a place $\bar{v}_0$ of $\overline{\mathbb{Q}}$ above p and an arithmetic Frobenius σ at $\bar{v}_0$ with $\sigma|_F = g'|_F$ (replace $\bar{v}_0$ by a conjugate). Since p is unramified in L and $\sigma|_L = g'|_L = \mathrm{id}$, the image of $D_{\bar{v}_0}$ in $\mathrm{Gal}(L/\mathbb{Q})$, generated by the Frobenius $\sigma|_L$, is trivial; so $D_{\bar{v}_0} \subseteq G_L$. Let v_0 be the place of L^+ below $\bar{v}_0$; it lies above $p \nmid p_0$, it splits in L (its decomposition group in $\mathrm{Gal}(L/L^+)$ is trivial), and $D_{\bar{v}_0} \cap G_{L^+} = D_{\bar{v}_0}$. As $p \nmid M$, $p \neq \ell$ and $\bar{\Psi}$, $\bar{\omega}$ are unramified at p, so $\bar{\Psi}(D_{\bar{v}_0})$ and $\bar{\omega}(D_{\bar{v}_0})$ are generated by $\bar{\Psi}(\sigma)$, $\bar{\omega}(\sigma)$; both characters are trivial on G_F, so $\bar{\Psi}(\sigma) = \bar{\Psi}(g') = 1$ and $\bar{\omega}(\sigma) = \bar{\omega}(g') \neq 1$. Thus $\bar{\Psi}$ is trivial and $\bar{\omega}$ is non-trivial on $D_{\bar{v}_0} \cap G_{L^+}$, and $p \nmid 2M\,|\mathrm{disc}(L)|$.

Step 4: the Hecke ideal. By (Ht1) of Proposition 50.26 there is a universal polarised ordinary determinant $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ of rank n over L, the comparison morphism ϑ_n exists and $\vartheta_n(B_n^{\mathrm{po}}) = W_n^{\mathrm{po}}$. The hypotheses of Theorem 50.6 hold by Steps 1 and 3 and the hypotheses on κ and $\mathcal{O}'$. Its weight hypothesis holds because $\Psi = \chi\,\chi_{\mathrm{cyc}}^e$ with χ of finite order and $e \geq 1$ in both (a) and (b). Its hypothesis that Ψ is trivial on the inertia groups above primes not dividing M holds because Ψ is unramified at every prime not dividing ℓN (recalled at the start of this proof) and $\ell N \mid M$. Let $\mathcal{J} \subseteq B_n^{\mathrm{po}}$ be the ideal it provides, and put $\mathcal{M} = \mathcal{M}_{\mathcal{J}}$, the ideal of W_n^{po} generated by $\vartheta_n(\mathcal{J})$.

Step 5: (Hk2). $\mathcal{M}$ is finite over the weights. The morphism $\vartheta_n\colon B_n^{\mathrm{po}} \to W_n^{\mathrm{po}}$ is a surjective homomorphism of local rings, so $\vartheta_n(\mathfrak{m}_{B_n^{\mathrm{po}}}) = \mathfrak{m}_W$; it is $\mathcal{O}'$-linear with $\vartheta_n \circ \psi_{v,a}^{\mathrm{d}} = \psi_{v,a}^{\mathrm{u}}$, so it maps the generators of $\mathfrak{b}_n$ onto the generators of $\mathfrak{a}_n$, and $\vartheta_n(\mathfrak{b}_n) = \mathfrak{a}_n$ by surjectivity. Let $y \in \mathfrak{m}_W$ and choose $z \in \mathfrak{m}_{B_n^{\mathrm{po}}}$ with $\vartheta_n(z) = y$. By (Hd1), $z^k \in \mathcal{J} + \mathfrak{b}_n$ for some $k \geq 1$, so

$y^k \in \vartheta_n(\mathcal{J}) + \mathfrak{a}_n \subseteq \mathcal{M} + \mathfrak{a}_n$. Hence $\mathfrak{m}_W \subseteq \sqrt{\mathcal{M} + \mathfrak{a}_n}$. Propagation through nice primes, and for $n = 3$ along thick links, is (Hd2).

Step 6: (Hk3). $\mathcal{J}$ satisfies the dimension condition of (Hd1) and, by (Hd3), the vanishing at the points of seeds required in (Ht2) of Proposition 50.26. So if for some $m \geq 2$ there is a seed of rank n and weight m for (j, χ') with twist κ, (Ht2) shows that $\mathcal{M}$ has a seed component.

Step 7: (Hk4). Let A, $(\rho, (\psi_{v,a}))$ and $f\colon R_n^{\mathrm{po}} \to A$ be as in (Hk4), with $f(\mathcal{M}) = 0$. Put $y = f \circ \vartheta_n\colon B_n^{\mathrm{po}} \to A$, a homomorphism of $\mathcal{O}'$-algebras. Then $y \circ D^{\mathrm{u}} = f \circ (\det \circ \rho^{\mathrm{u}}) = \det \circ (f \circ \rho^{\mathrm{u}}) = \det \circ \rho$ (base change of the determinant of a representation is the determinant of the base-changed representation), $y \circ \psi_{v,a}^{\mathrm{d}} = f \circ \psi_{v,a}^{\mathrm{u}} = \psi_{v,a}$, and $y(\mathcal{J}) = f(\vartheta_n(\mathcal{J})) \subseteq f(\mathcal{M}) = 0$. The remaining hypotheses of (Hd4) are those of (Hk4), and (Hd4) gives the conclusion of (Hk4).

Steps 1–7 give (Hk1)–(Hk4) for the prime $p \nmid p_0$ and the place v_0 of Step 3. □

Lemma 50.4. *Let L be an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$ and let $\mathcal{O}' \supseteq \mathcal{O}$ be a coefficient ring. Then $g_\star$ is a marking for $(K, \mathcal{O}', \Psi, c)$, and L is an auxiliary field of level C_0 for $(K, \mathcal{O}', \Psi, c, g_\star, M)$, where Ψ is regarded as a character with values in $\mathcal{O}'^{\times}$.*

Proof. Write $k = \mathbb{F}_{\mathcal{O}}$, $k' = \mathbb{F}_{\mathcal{O}'}$ and $f = [k' : k]$. Since $\mathcal{O} = \mathcal{O}' \cap \operatorname{Frac}\mathcal{O}$, we have $\mathfrak{m}_{\mathcal{O}} = \mathfrak{m}_{\mathcal{O}'} \cap \mathcal{O}$, so $k \to k'$ is an injective homomorphism of finite fields and the reduction $\bar{\Psi}'$ of Ψ modulo $\mathfrak{m}_{\mathcal{O}'}$ is $\bar{\Psi}$ followed by $k \subseteq k'$. In particular $\ker \bar{\Psi}' = \ker \bar{\Psi}$, and every condition of Definition 48.2 that involves $\bar{\Psi}$ only through equalities $\bar{\Psi}(s) = 1$, $\bar{\Psi}^j = \bar{\omega}$, or non-triviality of characters built from $\bar{\Psi}$, $\bar{\omega}$, η_K, is the same for $\mathcal{O}$ and $\mathcal{O}'$. The datum $(K, \mathcal{O}', \Psi, c)$ satisfies the standing requirements (Ψ continuous with values in $\mathcal{O}'^{\times}$, $\Psi(c) = 1$, ℓ odd).

The marking. We have $\eta_K(g_\star) = 1$, and $\Psi(g_\star)^2 - 1 \in \mathcal{O}^{\times} \subseteq \mathcal{O}'^{\times}$, because a unit of $\mathcal{O}$ lies outside $\mathfrak{m}_{\mathcal{O}} = \mathfrak{m}_{\mathcal{O}'} \cap \mathcal{O}$.

The group Γ_L and the conditions without coefficients. The subgroup N_L is defined by $\eta_K(s) = 1$ and $\bar{\Psi}(s) = 1$, so it is the same subgroup for $\mathcal{O}'$; hence Γ_L, Γ_L^0, π_L and the groups D_v, I_v are unchanged. (Af1), (Af2), (Ar4), (Ag2) do not involve the coefficients; (Af3), (Af4), (Ar1) and (Ar5)(i)–(iii) involve Ψ and $\bar{\Psi}$ only through the kernel of $\bar{\Psi}$, equalities of characters with values in $k^{\times} \subseteq k'^{\times}$, and homomorphisms to finite abelian ℓ-groups, so they hold for $\mathcal{O}'$. (Ag1) involves Ψ only through a congruence modulo the maximal ideal of the integers of $\overline{\mathbb{Q}}_\ell$, which does not depend on the coefficient ring.

Base change of Selmer groups. Let Δ, (H_t) be as in the definition and let W be a finite-dimensional k-vector space with a continuous action of Δ and subspaces b_t. Put $W' = W \otimes_k k'$ and $b_t' = b_t \otimes_k k'$. Choosing a k-basis $e_1, \dots, e_f$ of k', a continuous map $\Delta^i \to W'$ is the same as an f-tuple of continuous maps $\Delta^i \to W$, and the conditions defining the cocycle spaces and the coboundary spaces of $H^i(\Delta; W', b')$ are k-linear and are imposed coordinatewise (the action of Δ and the subspaces b_t' are base changed from k). Hence each of these spaces is the base change to k' of the corresponding space for (W, b), and since $k \to k'$ is flat, $H^i(\Delta; W', b') \cong H^i(\Delta; W, b) \otimes_k k'$ for $i = 0, 1, 2$; in particular

$$\dim_{k'} H^i(\Delta; W', b') = \dim_k H^i(\Delta; W, b).$$

For a character ϕ of Δ with values in $k^{\times}$, $k'(\phi) = k(\phi) \otimes_k k'$. Applied to $\phi = \bar{\Psi}^{-1}\eta_K, \bar{\Psi}^{-2}\eta_K$ on Γ_L, to $\phi = \bar{\Psi}^{-1}$ on Γ_L^0 and to the trivial character, this gives (Ar2) and (Ar3) for $\mathcal{O}'$ from those for $\mathcal{O}$.

Condition (Ar5)(iv) for $\mathcal{O}'$. Let W' be a finite-dimensional k'-vector space with a continuous k'-linear action of Γ_L with open kernel, $b_v' \subseteq W'$ D_v-stable k'-subspaces, and suppose $\operatorname{Hom}_{\Gamma_L}(W', k'(\bar{\omega})) = 0$ (homomorphisms of $k'[\Gamma_L]$-modules). Let W_0, $b_{0,v}$ denote W', b_v' regarded as k-vector spaces; they satisfy the hypotheses of (Ar5)(iv) for $\mathcal{O}$ except possibly the

vanishing of $\mathrm{Hom}_{\Gamma_L}(W_0, k(\bar\omega))$. The trace $\mathrm{Tr}\colon k' \to k$ is non-zero (the extension of finite fields is separable), so $\lambda \mapsto \mathrm{Tr}\circ\lambda$ is a k-linear injection $\mathrm{Hom}_{k'}(W', k') \to \mathrm{Hom}_k(W_0, k)$; both sides have k-dimension $f \dim_{k'} W'$, so it is a bijection. It commutes with the actions of Γ_L twisted by $\bar\omega$, because $\bar\omega$ takes values in $\mathbb{F}_\ell^\times \subseteq k^\times$ and Tr is k-linear; hence it restricts to a bijection $\mathrm{Hom}_{\Gamma_L}(W', k'(\bar\omega)) \to \mathrm{Hom}_{\Gamma_L}(W_0, k(\bar\omega))$, and the latter vanishes. So (Ar5)(iv) for $\mathcal{O}$ applies to (W_0, b_0). The groups $H^i(\Gamma_L; W', b')$ and $H^i(\Gamma_L; W_0, b_0)$ are the same abelian groups (the defining conditions are the same sets of equations), so $\dim_k$ of the latter is f times $\dim_{k'}$ of the former; likewise $\dim_k b_{0,v} = f \dim_{k'} b'_v$ and $\dim_k W_0^{c_u=1} = f \dim_{k'} W'^{c_u=1}$. Thus $H^2(\Gamma_L; W', b')$ is finite-dimensional, and (48.1) for (W_0, b_0) is f times (48.1) for (W', b'), which therefore holds.

All conditions (Af1)–(Ag2) hold for $(K, \mathcal{O}', \Psi, c, g_\star, M)$ with the same C_0. □

Proposition 50.5. *Let L be an auxiliary field of level C_0 for $(K, \mathcal{O}, \Psi, c, g_\star, M)$ and $n \in \{2, 3\}$. Then a universal polarised ordinary lift of rank n over L exists.*

Proof. Step 1: the finiteness condition. $\Gamma_L^0 = \pi_L(G_L)$ is open in Γ_L, since G_L is open in G_{L^+} and the quotient map π_L is open; so every open subgroup of Γ_L^0 is open in Γ_L, and by (Ar5)(i) Γ_L^0 satisfies Mazur's condition Φ_ℓ of Theorem B.71. For $v \mid \ell$, D_v is the absolute Galois group of the completion L_v^+ (the decomposition group at $\bar v$); an open subgroup is the absolute Galois group of a finite extension E/L_v^+, whose continuous homomorphisms to $\mathbb{F}_\ell$ correspond by local class field theory to homomorphisms $E^\times/E^{\times\ell} \to \mathbb{F}_\ell$, a finite set. So D_v satisfies Φ_ℓ.

Step 2: framed lifts and characters. The character $\bar\Psi$ factors through Γ_L and has finite image, so $\bar\rho = \mathrm{diag}(\bar\Psi^{w_{n,1}}, \dots, \bar\Psi^{w_{n,n}})\colon \Gamma_L^0 \to \mathrm{GL}_n(\mathbb{F}_\mathcal{O})$ and $\bar\psi_{v,a} = \bar\Psi^{w_{n,a}}|_{D_v}$ are continuous. By Theorem B.71, item 1, applied to $(\Gamma_L^0, \bar\rho)$ and to $(D_v, \bar\psi_{v,a})$ with $n = 1$, there are objects $R^\square$, $R_{v,a}$ of $\mathrm{CNL}_\mathcal{O}$ with universal lifts $\rho^\square$ of $\bar\rho$ and $\chi_{v,a}$ of $\bar\psi_{v,a}$. For $A \in \mathrm{CNL}_\mathcal{O}$, a homomorphism $\Gamma \to \mathrm{GL}_m(A)$ of a profinite group is continuous for the $\mathfrak{m}_A$-adic topology if and only if its reductions modulo $\mathfrak{m}_A^k$ are locally constant for all k (the topology of $A = \varprojlim A/\mathfrak{m}_A^k$ is the inverse limit of the discrete topologies); so these are the lifts satisfying (Po1) and the reduction part of (Po5).

Step 3: the product. Let S be the completed tensor product over $\mathcal{O}$ of $R^\square$ and the finitely many $R_{v,a}$ ($v \mid \ell$, $1 \le a \le n$). It is an object of $\mathrm{CNL}_\mathcal{O}$ (a quotient of a power series ring over $\mathcal{O}$ in finitely many variables), and morphisms $S \to A$ in $\mathrm{CNL}_\mathcal{O}$ correspond bijectively to tuples of morphisms from the factors. Hence S, with $\rho_S = \rho^\square \otimes 1$ and $\psi_{S,v,a} = \chi_{v,a}$, represents the functor of tuples $(\rho, (\psi_{v,a}))$ satisfying (Po1) and (Po5).

Step 4: the equations. Let $I \subseteq S$ be the ideal generated by the entries of $\rho_S(\pi_L(cg^{-1}c)) - J_n\,{}^t\rho_S(\pi_L(g))\,J_n$ for $g \in G_L$, the off-diagonal entries of $\rho_S(\pi_L(g_\star)) - \rho_S(\pi_L(cg_\star^{-1}c))$, and the entries of $\prod_{a=1}^n(\rho_S(\pi_L(g_a)) - \psi_{S,v,a}(g_a))$ for $v \mid \ell$ and $g_1, \dots, g_n \in D_v$ (these expressions make sense, as in (Po2)–(Po4)). Then $I \subseteq \mathfrak{m}_S$: modulo $\mathfrak{m}_S$ all these matrices are computed from the diagonal representation $\bar\rho$ and the $\bar\psi_{v,a}$; $\bar\Psi(cg^{-1}c) = \bar\Psi(g)^{-1}$ because $\bar\Psi(c) = 1$ and $\bar\Psi$ has abelian image, and $J_n\,{}^t\mathrm{diag}(x_1, \dots, x_n)J_n = \mathrm{diag}(x_n, \dots, x_1)$ with $w_{n,n+1-a} = -w_{n,a}$, so the first matrices vanish; the second are diagonal; and the a-th factor of the third product is diagonal with (a,a)-entry 0, so the product of diagonal matrices vanishes. Put $R_n^{\mathrm{po}} = S/I$, an object of $\mathrm{CNL}_\mathcal{O}$, and let ρ^{u}, $\psi_{v,a}^{\mathrm{u}}$ be the images of ρ_S, $\psi_{S,v,a}$. They satisfy (Po1)–(Po5). If $(\rho', (\psi'_{v,a}))$ is a polarised ordinary lift of rank n over L to $R' \in \mathrm{CNL}_\mathcal{O}$, Step 3 gives a unique morphism $f\colon S \to R'$ with $f\circ\rho_S = \rho'$, $f\circ\psi_{S,v,a} = \psi'_{v,a}$; conditions (Po2)–(Po4) for (ρ', ψ') say exactly that f kills the generators of I, so f factors uniquely through R_n^{po}. Thus $(\rho^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{u}}))$ is a universal polarised ordinary lift of rank n over L. □

Theorem 50.6. *Let $\mathcal{O}'$ be a coefficient ring, L an auxiliary field of level C_0 for $(K, \mathcal{O}', \Psi, c, g_\star, M)$ with data $(N', w, j, \chi', \theta)$ as in (Ag1), $d_L = [L^+ : \mathbb{Q}]$, $n \in \{2, 3\}$, and κ an admissible twist of*

rank n for ι, c and M with $\kappa(G_L) \subseteq \mathcal{O}'^{\times}$ and $\tau(\mathcal{O}_L) \subseteq \mathcal{O}'$ for every embedding $\tau\colon L \to \overline{\mathbb{Q}}_\ell$. Let v_0 be a place of L^+ above a prime $p \nmid 2M\,|\mathrm{disc}(L)|$ which splits in L and has a place $\bar{v}_0$ of $\overline{\mathbb{Q}}$ above it with $\bar{\Psi}$ trivial and $\bar{\omega}$ non-trivial on $D_{\bar{v}_0} \cap G_{L^+}$. Let $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ be a universal polarised ordinary determinant and $(R_n^{\mathrm{po}}, \rho^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{u}}))$ a universal polarised ordinary lift of rank n over L (both for $\mathcal{O}'$), such that the comparison morphism ϑ_n exists and $\vartheta_n(B_n^{\mathrm{po}}) \subseteq W_n^{\mathrm{po}}$. Assume that $\Psi\chi_{\mathrm{cyc}}^{-e}$ is trivial on an open subgroup of the inertia group $I_{\bar{\ell}}$ for some integer $e \geq 1$, and that Ψ is trivial on $I_{\bar{p}}$ for every place $\bar{p}$ of $\overline{\mathbb{Q}}$ above a prime not dividing M. Then there is an ideal $\mathcal{J} \subseteq B_n^{\mathrm{po}}$ such that:

(Hd1) *$\mathcal{J}$ is finite over the weights, and $\dim B_n^{\mathrm{po}}/\mathfrak{q} = 1 + n\,d_L$ for every minimal prime $\mathfrak{q}$ of B_n^{po} over $\mathcal{J}$;*

(Hd2) *$\mathcal{M}_{\mathcal{J}}$ propagates through nice primes, and, if $n = 3$, along thick links;*

(Hd3) *for every $m \geq 2$, every seed r of rank n and weight m for (j, χ') with twist κ, and every homomorphism of $\mathcal{O}'$-algebras $x\colon B_n^{\mathrm{po}} \to \overline{\mathbb{Q}}_\ell$ with $x \circ D^{\mathrm{u}} = \det \circ \tilde{r}$ and $x \circ \psi_{v,a}^{\mathrm{d}} = \kappa_{v,a}$ for all $v \mid \ell$ and a, where $\tilde{r}$ is a representation of Γ_L^0 with $\tilde{r} \circ \pi_L = r$ and $(\kappa_{v,1}, \dots, \kappa_{v,n})$ is a diagonal of r on D_v as in (Sp5), one has $x(\mathcal{J}) = 0$;*

(Hd4) *for every integral test ring A over $\mathcal{O}'$ and every polarised ordinary lift $(\rho, (\psi_{v,a}))$ of rank n over L to A (for $\mathcal{O}'$) with $\psi_{v,a}(g) = \Psi(g)^{w_{n,a}}$ for all $v \mid \ell$, a and $g \in I_v$, such that ρ_L is irreducible over $\overline{\mathbb{Q}}_\ell$ and some homomorphism of $\mathcal{O}'$-algebras $y\colon B_n^{\mathrm{po}} \to A$ with $y \circ D^{\mathrm{u}} = \det \circ \rho$ and $y \circ \psi_{v,a}^{\mathrm{d}} = \psi_{v,a}$ satisfies $y(\mathcal{J}) = 0$: the representation $\rho_L \otimes \kappa$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions.*

Proof. Let $\mathcal{J} \subseteq B_n^{\mathrm{po}}$ be an ideal as in Proposition 50.11. Its clauses (Hi2), (Hi3), (Hi4) are (Hd2), (Hd3), (Hd4). It remains to prove (Hd1). Write $B = B_n^{\mathrm{po}}$, $N = nd_L$, $\Lambda = \mathcal{O}'[[X_1, \dots, X_N]]$ with maximal ideal $\mathfrak{m}_\Lambda$.

If $\mathcal{J} = B$, then $\mathfrak{m}_B \subseteq B = \sqrt{\mathcal{J} + \mathfrak{b}_n}$ and there is no prime over $\mathcal{J}$, so (Hd1) holds. Let $\mathcal{J} \neq B$ and $C = B/\mathcal{J} \neq 0$, a local ring with maximal ideal $\mathfrak{m}_C$ the image of $\mathfrak{m}_B$, and let $\lambda\colon \Lambda \to C$ be as in (Hi1).

Coheight of the minimal primes. By Lemma 50.25, item 1, λ is injective and every minimal prime of C has coheight $N + 1$. The minimal primes $\mathfrak{q}$ of B over $\mathcal{J}$ are the preimages of the minimal primes $\bar{\mathfrak{q}}$ of C, with $B/\mathfrak{q} \cong C/\bar{\mathfrak{q}}$; so $\dim B/\mathfrak{q} = 1 + nd_L$.

Finite over the weights. As λ is injective, $\lambda(\Lambda)/\lambda(\mathfrak{m}_\Lambda) \cong \Lambda/\mathfrak{m}_\Lambda = \mathbb{F}_{\mathcal{O}'}$. By Lemma 50.25, item 2, applied to the ideal $\lambda(\mathfrak{m}_\Lambda)$ of $\lambda(\Lambda)$, $C/\lambda(\mathfrak{m}_\Lambda)C$ has finite length and $\mathfrak{m}_C^k \subseteq \lambda(\mathfrak{m}_\Lambda)C$ for some $k \geq 1$. By (Hi1), $\lambda(\mathfrak{m}_\Lambda)C$ is contained in the ideal of C generated by the images of $\mathfrak{m}_{\mathcal{O}'}$ and of the $\psi_{v,a}^{\mathrm{d}}(g) - 1$ ($v \mid \ell$, $g \in I_v$, $\bar{\Psi}(g) = 1$), which is the image of $\mathfrak{b}_n$. Hence $\mathfrak{m}_B^k \subseteq \mathcal{J} + \mathfrak{b}_n$, i.e. $\mathfrak{m}_B \subseteq \sqrt{\mathcal{J} + \mathfrak{b}_n}$: $\mathcal{J}$ is finite over the weights. This is (Hd1). □

50.2 Hida theory and patching for the definite unitary group

This subsection proves Proposition 50.11: an ideal $\mathcal{J}$ of the determinant ring such that $B_n^{\mathrm{po}}/\mathcal{J}$ is finite and torsion-free over the Iwasawa algebra $\mathcal{O}'[[X_1, \dots, X_{nd_L}]]$ of the weights (Hi1), with the propagation, seed and automorphy properties (Hi2)–(Hi4); with Lemma 50.25 it gives Theorem 50.6. The ideal is a Hecke ideal of zero-monodromy type (Definition 50.10), cut out by an ordinary Hecke algebra of a definite unitary group with a type at an auxiliary place (Definition 50.8). Hida theory, in Geraghty's form, gives the finiteness over the weights. Appendix E supplies three further facts: a Hecke ideal of zero-monodromy type with classical level modules

at every Taylor–Wiles level exists (Proposition E.1); the irreducible points of the classical ideal (Definition 50.7) with a parallel inertial weight are automorphic (Proposition E.25); and $R = T$ holds at the patchable curve points (Definition 50.9), the one-dimensional points at which the patching method of Taylor and Wiles [254] applies (Proposition E.32). This subsection connects them: a Hecke ideal of zero-monodromy type lies in the classical ideal (Lemma 50.12), and $R = T$ at patchable curve points gives propagation on the charts of the entry algebra (Proposition 50.15), because a nice prime of a chart gives a patchable curve point (Lemma 50.16). The lemmas on separable elements and Zariski closures (Lemma 50.19, Lemma 50.20, Lemma 50.21) are used again in rank four in Section 53.

Definition 50.7. Let $(n, L, c, \Gamma, \Gamma^0, \pi, (D_v \supseteq I_v)_{v\in S_\ell}, (\bar\chi_a))$ be a residual setting of rank n over a coefficient ring $\mathcal{O}$ (over a rank-n Hecke datum, or over a CM field L), $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ a universal polarised ordinary determinant of rank n over it, and $\kappa\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$ a continuous character. A homomorphism of $\mathcal{O}$-algebras $x\colon B_n^{\mathrm{po}} \to \overline{\mathbb{Q}}_\ell$ is a **classical ordinary point** (for ι and κ) if there are a homomorphism $\tilde r\colon \Gamma^0 \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ with $r = \tilde r \circ \pi$ continuous on G_L, and a regular algebraic conjugate self-dual cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_L)$, such that

(Cy1) $x \circ D^{\mathrm{u}} = \det \circ \tilde r$, and $\tilde r$ is absolutely irreducible;

(Cy2) $r \otimes \kappa \leftrightarrow_\iota \Pi$;

(Cy3) for every $v \in S_\ell$, r is triangular on D_v with diagonal $(x \circ \psi_{v,1}^{\mathrm{d}}, \dots, x \circ \psi_{v,n}^{\mathrm{d}})$;

(Cy4) for every $v \in S_\ell$ and every field embedding $\tau\colon L_v^+ \to \overline{\mathbb{Q}}_\ell$, the τ-Hodge–Tate weights of the characters $x \circ \psi_{v,1}^{\mathrm{d}}, \dots, x \circ \psi_{v,n}^{\mathrm{d}}$ of D_v (de Rham, as subquotients of the de Rham representation $r|_{D_v}$) are strictly decreasing in a, with Hodge–Tate weights normalised as in Section 13.

The **classical ideal** $\mathcal{J}_\kappa^{\mathrm{cl}} \subseteq B_n^{\mathrm{po}}$ is the intersection of the kernels of all classical ordinary points ($\mathcal{J}_\kappa^{\mathrm{cl}} = B_n^{\mathrm{po}}$ if there is none).

Definition 50.8. Let $\mathcal{O}$ be a coefficient ring, and let $(n, L, c, (\bar v)_{v|\ell}, \mathrm{St}, v_0)$ be one of the following:

(Us1) the integer n, the CM field L, the complex conjugation c, the places $\bar v$, the set St and the place v_0 of a rank-n Hecke datum $\mathscr{H}$ over $\mathcal{O}$ (the objects below are then **over** $\mathscr{H}$);

(Us2) $n \in \{2, 3\}$, a CM field L satisfying (Af1) and (Af2) for $(K, \mathcal{O}, \Psi, c, g_\star, M)$ with $\tau(\mathcal{O}_L) \subseteq \mathcal{O}$ for every embedding $\tau\colon L \to \overline{\mathbb{Q}}_\ell$, the places $\bar v$ of Definition 48.2, the set St of places of L^+ not above ℓ lying above a prime dividing M, and a place v_0 of L^+ above a prime $p \nmid 2M|\mathrm{disc}(L)|$ that splits in L (the objects below are then **over** L).

In both cases p denotes the prime below v_0.

The group. Let $\mathbb{G}_L$ be the group scheme over $\mathcal{O}_{L^+}$ with $\mathbb{G}_L(R) = \{g \in \mathrm{GL}_n(\mathcal{O}_L \otimes_{\mathcal{O}_{L^+}} R) : {}^t g^c\, g = 1\}$, where c acts on $\mathcal{O}_L$; this is the unitary group of the Hermitian form $\sum_i x_i \bar x_i$. For a place v of L^+ split in L and a place $\tilde v \mid v$ of L, the projection identifies $\mathbb{G}_L(\mathcal{O}_{L_v^+})$ with $\mathrm{GL}_n(\mathcal{O}_{L_{\tilde v}})$.

Places. Let S_ℓ be the set of places of L^+ above ℓ, and put $T_L = S_\ell \cup \mathrm{St} \cup \{v_0\}$. For $v \in S_\ell$, $\tilde v$ is the place of L below $\bar v$; for the other $v \in T_L$ a place $\tilde v \mid v$ of L is fixed.

Type data and level. A **type datum** is a family $(V_v)_{v\in\mathrm{St}}$ of open normal subgroups $V_v \subseteq \mathbb{G}_L(\mathcal{O}_{L_v^+})$ together with a finite free $\mathcal{O}$-module τ with an $\mathcal{O}$-linear action of

$$H = \prod_{v\in\mathrm{St}} \mathbb{G}_L(\mathcal{O}_{L_v^+})/V_v.$$

Its **level** is $V = \prod_v V_v \subseteq \mathbb{G}_L(\mathbb{A}_{L^+}^\infty)$ with $V_v = \mathbb{G}_L(\mathcal{O}_{L_v^+})$ for $v \notin \mathrm{St} \cup \{v_0\}$, the given V_v for $v \in \mathrm{St}$, and V_{v_0} the kernel of reduction $\mathbb{G}_L(\mathcal{O}_{L_{v_0}^+}) \to \mathbb{G}_L(\mathcal{O}_{L_{v_0}^+}/p)$.

The ordinary Hecke algebra of type τ. We use [105], §2, with $F = L$, $\mathbb{G} = \mathbb{G}_L$, $T = T_L$ and $U = V$. The module $\mathcal{S}_V$ is the Pontryagin dual of the space $S^{\mathrm{ord}}(V(\ell^\infty), \mathcal{K}/\mathcal{O})$ of ordinary forms of trivial weight ($\mathcal{K} = \mathcal{O}[1/\ell]$); $\Lambda^+ = \mathcal{O}[[T_n(\mathcal{O}_{L^+} \otimes \mathbb{Z}_\ell)]]$ and $\Lambda_b = \mathcal{O}[[T_n(\ell^b)]]$ ($b \geq 1$; $T_n(\ell^b)$ the kernel of reduction modulo ℓ^b on the diagonal torus) act on $\mathcal{S}_V$ through the diamond operators $\langle u \rangle$; and $\mathbb{T}^{\mathrm{ord}}(V)$ is the Λ^+-subalgebra of $\mathrm{End}_{\mathcal{O}}(\mathcal{S}_V)$ generated by the operators $T_w^{(j)}$, $(T_w^{(n)})^{-1}$ at the places w of L above places $v \notin T_L$ split in L, and $U_{\tilde{v}}^{(j)}$, $v \in S_\ell$, $1 \leq j \leq n$. Right translation by $\prod_{v \in \mathrm{St}} \mathbb{G}_L(\mathcal{O}_{L_v^+})$, which normalises $V(\ell^{b,c})$, gives an action of H on $\mathcal{S}_V$ commuting with $\mathbb{T}^{\mathrm{ord}}(V)$. Put

$$N_\tau = \mathrm{Hom}_{\mathcal{O}[H]}(\tau, \mathcal{S}_V),$$

a $\mathbb{T}^{\mathrm{ord}}(V)$-module through the action on $\mathcal{S}_V$, and let the **ordinary Hecke algebra of type τ** be the image $\mathbb{T}_\tau$ of $\mathbb{T}^{\mathrm{ord}}(V)$ in $\mathrm{End}_{\mathcal{O}}(N_\tau)$, with its structure of Λ^+-algebra. For a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_\tau$, $\mathbb{T}_{\tau,\mathfrak{m}}$ is the localisation.

Levels at auxiliary places. In case (Us1), let Q be a finite set of places of L^+ split in L and not in T_L, and for $v \in Q$ let K'_v be a compact open subgroup of $\mathbb{G}_L(L_v^+)$ (not necessarily contained in $\mathbb{G}_L(\mathcal{O}_{L_v^+})$, so that conjugates by elements of $\prod_{v \in Q} \mathbb{G}_L(L_v^+)$ are allowed); write $K' = (K'_v)_{v \in Q}$. The **level at** K' of a type datum $((V_v), \tau)$ is the group $V(K')$ obtained from its level V by replacing V_v by K'_v for $v \in Q$. The objects $\mathcal{S}_{V(K')}$, $\mathbb{T}^{\mathrm{ord}}(V(K'))$, $N_\tau(K') = \mathrm{Hom}_{\mathcal{O}[H]}(\tau, \mathcal{S}_{V(K')})$ and $\mathbb{T}_\tau(K')$ (the image of $\mathbb{T}^{\mathrm{ord}}(V(K'))$ in $\mathrm{End}_{\mathcal{O}}(N_\tau(K'))$) are defined as above with $T = T_L \cup Q$ and $U = V(K')$; in particular the operators $T_w^{(j)}$, $(T_w^{(n)})^{-1}$ are taken only at the places w above places $v \notin T_L \cup Q$ split in L. For $Q = \varnothing$ these are $\mathcal{S}_V$, $\mathbb{T}^{\mathrm{ord}}(V)$, N_τ and $\mathbb{T}_\tau$.

Definition 50.9. Let $(n, L, c, \Gamma, \Gamma^0, \pi, (D_v \supseteq I_v)_{v \in S_\ell}, (\bar{\chi}_a))$ be a residual setting of rank n over a coefficient ring $\mathcal{O}$ and $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ a universal polarised ordinary determinant of rank n over it. If the setting is that of a rank-n Hecke datum $\mathscr{H}$ (Rs1), v_0 and $\bar{v}_0$ are those of $\mathscr{H}$; if it is that of a CM field L (Rs2), let v_0 be a finite place of L^+ not above ℓ, split in L, with a place $\bar{v}_0$ of $\overline{\mathbb{Q}}$ above it. Write $B = B_n^{\mathrm{po}}$, $\gamma^* = \pi(c)\gamma^{-1}\pi(c)$ for $\gamma \in \Gamma^0$, $\Gamma' = \{\gamma \in \Gamma^0 : \chi_{\mathrm{cyc}}(\gamma) = 1\}$ and $\Delta_0 = \pi(D_{\bar{v}_0} \cap G_{L^+})$.

Curve points. A **curve point** of B is a tuple $(k, \rho, P, (\psi_{v,a}), f)$ of a finite field k containing $\mathbb{F}_\mathcal{O}$, with $A_k = k[[T]]$, fraction field F_k and an algebraic closure $\bar{F}_k$; a homomorphism $\rho\colon \Gamma^0 \to \mathrm{GL}_n(A_k)$ whose entries are continuous for the T-adic topology; a symmetric $P \in \mathrm{GL}_n(A_k)$ with $\rho(\gamma^*) = P^{-1}\,{}^t\rho(\gamma)\,P$ for all γ; continuous homomorphisms $\psi_{v,a}\colon D_v \to A_k^\times$ ($v \in S_\ell$, $1 \leq a \leq n$) such that $\rho \circ \pi$ is triangular over F_k on D_v with diagonal $(\psi_{v,1}, \ldots, \psi_{v,n})$; and a homomorphism of $\mathcal{O}$-algebras $f\colon B \to A_k$ with $f \circ D^{\mathrm{u}} = \det \circ \rho$ and $f \circ \psi_{v,a}^{\mathrm{d}} = \psi_{v,a}$. It is **patchable** if:

(Pj1) for every $v \in S_\ell$: $\psi_{v,a} \neq \psi_{v,b}$ for $a \neq b$, and $\psi_{v,a} \neq \psi_{v,b} \cdot \chi_{\mathrm{cyc}}|_{D_v}$ for $a < b$;

(Pj2) every open subset of Γ^0 meeting Γ' contains some $\sigma \in \Gamma'$ such that the characteristic polynomial of $\rho(\sigma)$ is separable over F_k;

(Pj3) the only $X \in \mathrm{M}_n(F_k)$ with $\chi_{\mathrm{cyc}}(\gamma)\rho(\gamma)X = X\rho(\gamma)$ for all $\gamma \in \Delta_0$ is $X = 0$.

A patchable curve point is **of irreducible type** if $\rho \otimes \bar{F}_k$ restricted to Γ' is irreducible, and **of endoscopic type** if there are integers $a, b \geq 1$ with $a+b = n$, $Q \in \mathrm{GL}_n(F_k)$ and homomorphisms $r_1\colon \Gamma^0 \to \mathrm{GL}_a(F_k)$, $r_2\colon \Gamma^0 \to \mathrm{GL}_b(F_k)$ such that $Q\rho(\gamma)Q^{-1} = \mathrm{diag}(r_1(\gamma), r_2(\gamma))$ for all γ; for $i = 1, 2$ there is an invertible P_i over F_k with $r_i(\gamma^*) = P_i^{-1}\,{}^t r_i(\gamma) P_i$ for all γ; $r_1 \otimes \bar{F}_k$ and $r_2 \otimes \bar{F}_k$ restricted to Γ' are irreducible; and 0 is the only $Y \in \mathrm{M}_{b \times a}(\bar{F}_k)$ with $Y r_1(\sigma) = r_2(\sigma) Y$ for all $\sigma \in \Gamma'$.

Normalised curve points. For $1 \le a \le n$, a curve point $(k, \rho, P, (\psi_{v,a}), f)$ is **normalised of shape** $(a, n-a)$ if $P = 1_n$ and, when $a < n$, every $\rho(\gamma)$ is block diagonal with diagonal blocks of sizes a and $n-a$; then $\rho(\gamma^*) = {}^t\rho(\gamma)$ for all $\gamma \in \Gamma^0$.

Stable and thick primes. For a prime $\mathfrak{q}$ of B let $\bar\kappa(\mathfrak{q})$ be an algebraic closure of its residue field and $D_\mathfrak{q}$ the base change of D^{u} to $\bar\kappa(\mathfrak{q})$. The prime $\mathfrak{q}$ is **stable** if $D_\mathfrak{q} = \det \circ r$ for an irreducible homomorphism $r\colon \Gamma^0 \to \mathrm{GL}_n(\bar\kappa(\mathfrak{q}))$. Let $\mathcal{I} \subseteq B[\Gamma^0]$ be the intersection, over $k \ge 1$ and the open normal subgroups H' of Γ^0, of the two-sided ideals generated by: the Cayley–Hamilton ideal of D^{u} (generated by the coefficients, in $B[\Gamma^0]$, of $\chi_{D'}(y)(y)$ for $y = \sum_{i=1}^m t_i y_i$, $y_i \in B[\Gamma^0]$, D' the base change of D^{u} to $B[t_1, \dots, t_m]$); the elements $\prod_{a=1}^n (\pi(g_a) - \psi^{\mathrm{d}}_{v,a}(g_a))$ ($v \in S_\ell$, $g_a \in D_v$, $a = 1$ on the left) and their images under the B-linear anti-involution $\gamma \mapsto \gamma^*$; the elements $\mathfrak{m}_B^k \cdot 1$; and $h - 1$ for $h \in H'$. The prime $\mathfrak{q}$ is **thick** if it is not stable and $\bar\kappa(\mathfrak{q})[\Gamma^0]$ modulo the two-sided ideal generated by the image of $\mathcal{I}$ is not a semisimple ring.

R = T at patchable curve points. An ideal $J \subseteq B$ has **R = T at patchable curve points** (for v_0) if for every patchable curve point $(k, \rho, P, (\psi_{v,a}), f)$ of irreducible type with $J \subseteq \ker f$, every prime $\mathfrak{q} \subseteq \ker f$ of B contains J. The ideal J has **R = T along thick chains** (for v_0) if for every $h \ge 0$, all patchable curve points $x_0, \dots, x_h$ of endoscopic type with homomorphisms $f_0, \dots, f_h$, all thick primes $\mathfrak{e}_1, \dots, \mathfrak{e}_h$ with $\mathfrak{e}_i \subseteq \ker f_{i-1} \cap \ker f_i$, and every stable prime $\mathfrak{q}_1$ with $J \subseteq \mathfrak{q}_1 \subseteq \ker f_0$, every prime $\mathfrak{q} \subseteq \ker f_h$ of B that is stable or thick contains J.

Definition 50.10. Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, κ a twist for $\mathscr{H}$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and $(B, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$. For $v \in S_\ell$, $1 \le j \le n$ and $z \in 1 + \ell\mathcal{O}_{L_v^+}$ let $d_j(z) \in T_n(\ell^1)$ be the element with z in the v-component of place j and 1 elsewhere, and $[u] \in \Lambda_1$ the group element of $u \in T_n(\ell^1)$.

An ideal $J \subseteq B$ is a **Hecke ideal of zero-monodromy type** (for ι and κ) if $J = B$ and B has no classical ordinary point for ι and κ, or there are $T \in \mathrm{CNL}_\mathcal{O}$, a homomorphism of $\mathcal{O}$-algebras $\lambda_T\colon \Lambda_1 \to T$, a T-module M and a surjective homomorphism of $\mathcal{O}$-algebras $f\colon B \to T$ with $J = \ker f$, such that:

(Zm1) M is faithful over T and free of finite rank over Λ_1 through λ_T; hence T is finite and torsion-free over Λ_1;

(Zm2) $\lambda_T([d_j(z)]) = (\chi_{\mathrm{cyc}}^{j-1}\kappa)(g_z)\, f(\psi^{\mathrm{d}}_{v,j}(g_z))$ for $v \in S_\ell$, $1 \le j \le n$, $z \in 1 + \ell\mathcal{O}_{L_v^+}$, where $g_z \in I_v$ is the image of z under the local Artin map of $L_v^+ = L_{\tilde v}$ normalised to send uniformisers to geometric Frobenius elements;

(Zm3) for every homomorphism of $\mathcal{O}$-algebras $y\colon T \to \overline{\mathbb{Q}}_\ell$ such that $y(\lambda_T([u])) = (w_0\lambda)(u)^{-1}$ for all $u \in T_n(\ell^1)$, for a dominant weight λ with $\lambda_{\tau,j} > \lambda_{\tau,j+1}$ for all τ, j (a **regular point**), and such that the semisimple representation ρ_y of $\Gamma^0_\mathscr{H}$ with $\det \circ \rho_y = y \circ f \circ D^{\mathrm{u}}$ ((Ch3)) is absolutely irreducible, the representation $\rho_y \circ \pi_\mathscr{H} \otimes \kappa$ is automorphic of conjugate self-dual type for ι, and for every $v \in S_\ell$, $\rho_y \circ \pi_\mathscr{H}$ is triangular on D_v with diagonal $(y \circ f \circ \psi^{\mathrm{d}}_{v,a})_{1 \le a \le n}$;

(Zm4) $x(J) = 0$ for every classical ordinary point x of B for ι and κ.

Proposition 50.11. *Let $\mathcal{O}'$, L with data $(N', w, j, \chi', \theta)$, d_L, n, κ, v_0, $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$, $(R_n^{\mathrm{po}}, \rho^{\mathrm{u}}, (\psi^{\mathrm{u}}_{v,a}))$ and ϑ_n be as in the hypotheses of Theorem 50.6. Then there is an ideal $\mathcal{J} \subseteq B_n^{\mathrm{po}}$ such that:*

(Hi1) *there is a homomorphism of $\mathcal{O}'$-algebras $\lambda\colon \mathcal{O}'[[X_1, \dots, X_{nd_L}]] \to B_n^{\mathrm{po}}/\mathcal{J}$ such that $B_n^{\mathrm{po}}/\mathcal{J}$ is a finite and torsion-free module over $\mathcal{O}'[[X_1, \dots, X_{nd_L}]]$ via λ, and λ maps the maximal ideal of $\mathcal{O}'[[X_1, \dots, X_{nd_L}]]$ into the ideal of $B_n^{\mathrm{po}}/\mathcal{J}$ generated by the images of $\mathfrak{m}_{\mathcal{O}'}$ and of the elements $\psi^{\mathrm{d}}_{v,a}(g) - 1$ with $v \mid \ell$, $1 \le a \le n$, $g \in I_v$ and $\bar\Psi(g) = 1$;*

(Hi2) *$\mathcal{M}_{\mathcal{J}}$ propagates through nice primes, and, if $n = 3$, along thick links;*

(Hi3) *for every $m \geq 2$, every seed r of rank n and weight m for (j, χ') with twist κ, and every homomorphism of $\mathcal{O}'$-algebras $x\colon B_n^{\mathrm{po}} \to \overline{\mathbb{Q}}_\ell$ with $x \circ D^{\mathrm{u}} = \det \circ \tilde{r}$ and $x \circ \psi_{v,a}^{\mathrm{d}} = \kappa_{v,a}$ for all $v \mid \ell$ and a, where $\tilde{r}$ is a representation of Γ_L^0 with $\tilde{r} \circ \pi_L = r$ and $(\kappa_{v,1}, \dots, \kappa_{v,n})$ is a diagonal of r on D_v as in (Sp5), one has $x(\mathcal{J}) = 0$;*

(Hi4) *for every integral test ring A over $\mathcal{O}'$ and every polarised ordinary lift $(\rho, (\psi_{v,a}))$ of rank n over L to A (for $\mathcal{O}'$) with $\psi_{v,a}(g) = \Psi(g)^{w_{n,a}}$ for all $v \mid \ell$, a and $g \in I_v$, such that ρ_L is irreducible over $\overline{\mathbb{Q}}_\ell$ and some homomorphism of $\mathcal{O}'$-algebras $y\colon B_n^{\mathrm{po}} \to A$ with $y \circ D^{\mathrm{u}} = \det \circ \rho$ and $y \circ \psi_{v,a}^{\mathrm{d}} = \psi_{v,a}$ satisfies $y(\mathcal{J}) = 0$: the representation $\rho_L \otimes \kappa$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions.*

Proof. The datum. Let St be the set of places of L^+ not above ℓ lying above a prime dividing M, $(\bar{v})$ the places of Definition 48.2 and $\mathscr{H}_L = (n, L, c, (\bar{v}), \mathrm{St}, v_0, \bar{v}_0, (\bar{\Psi}^{w_{n,a}}|_{G_L})_a)$. The hypotheses of Lemma 50.23 are among those of Theorem 50.6; so $\mathscr{H}_L$ is a rank-n Hecke datum over $\mathcal{O}'$ (In1) with $\Gamma_{\mathscr{H}_L} = \Gamma_L$, $\pi_{\mathscr{H}_L} = \pi_L$, the same $D_v \supseteq I_v$ (In2), κ is a twist for it (In3), and $(\Psi^{w_{n,a}}|_{I_v})$ is an inertial weight for it (In4), by the weight hypothesis of Theorem 50.6. Hence the residual setting of $\mathscr{H}_L$ in Definition 50.2 is that of L: $(B, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ is a universal polarised ordinary determinant over $\mathscr{H}_L$, and classical ordinary points and the classical ideal, curve points and the two R = T properties for v_0, and type data and $\mathbb{T}_\tau$, are the same over $\mathscr{H}_L$ as over L (the type data of Definition 50.8 are formed from the same L, $\bar{v}$, St, v_0); and for $g \in G_L$, $\bar{\Psi}^{w_{n,b}}(g) = 1$ for all b if and only if $\bar{\Psi}(g) = 1$, as $w_{n,1} = 1$, so the condition $\bar{\chi}_b(g) = 1$ for all b reads $\bar{\Psi}(g) = 1$. Below, Proposition E.1, Proposition E.25, Proposition E.32 and Lemma 50.12 are applied to $\mathscr{H}_L$ and κ.

By Proposition E.1 (for $\mathscr{H}_L$, κ and ι; its hypothesis that the wild values of $\mathscr{H}_L$ are trivial holds by (Wv2) of Lemma 50.24, as the reduction $\bar{\omega}$ of χ_{cyc} is non-trivial on D_v for every $v \in S_\ell$ by (Af4), the groups D_v of $\mathscr{H}_L$ being those of L by (In2)) there is a Hecke ideal $\mathcal{J}$ of zero-monodromy type for ι and κ (Definition 50.10, clauses (Zm1)–(Zm4)) which, if $\mathcal{J} \neq B$, has classical levels. We prove (Hi1)–(Hi4) for it.

(Hi3). Let r, $\tilde{r}$, $(\kappa_{v,a})$ and x be as in (Hi3). By (Sp6), $r \otimes \kappa \leftrightarrow_\iota \Pi$ for a regular algebraic conjugate self-dual cuspidal Π; $x \circ D^{\mathrm{u}} = \det \circ \tilde{r}$ with $\tilde{r} \circ \pi_L = r$ continuous; and by (Sp5), for every $v \mid \ell$, r is triangular on $D_{\bar{v}} \cap G_L$ with diagonal $(\kappa_{v,a})_a = (x \circ \psi_{v,a}^{\mathrm{d}})_a$, where $D_{\bar{v}} \cap G_L = D_v$ because v splits in L ((Af2), $\ell \mid M$). Moreover $\tilde{r}$ is absolutely irreducible, because r is by (Sp1) and $\pi_L(G_L) = \Gamma_L^0$; and by (Sp5) the diagonal characters $\kappa_{v,a}$ are $\chi_{\mathrm{cyc}}^{m\, w_{n,a}}$ on an open subgroup of inertia, so their Hodge–Tate weights $m\, w_{n,a}$ strictly decrease in a ($m \geq 2$, $w_{n,a}$ strictly decreasing). So x satisfies (Cy1)–(Cy4): it is a classical ordinary point. Hence $\mathcal{J} \neq B$ (in the alternative $\mathcal{J} = B$ there is no classical ordinary point), and $x(\mathcal{J}) = 0$ by (Zm4).

(Hi4). Let A, $(\rho, (\psi_{v,a}))$ and y be as in (Hi4). The hypotheses of Proposition E.25 hold for $\mathscr{H}_L$, κ, the inertial weight $(\Psi^{w_{n,a}}|_{I_v})$, A, ρ and y: $\rho\colon \Gamma_L^0 \to \mathrm{GL}_n(A)$ with ρ_L continuous by (Po1) and irreducible over $\overline{\mathbb{Q}}_\ell$; $y \circ D^{\mathrm{u}} = \det \circ \rho$; $y(\psi_{v,a}^{\mathrm{d}}(g)) = \psi_{v,a}(g) = \Psi(g)^{w_{n,a}}$ for $g \in I_v$; $y(\mathcal{J}_\kappa^{\mathrm{cl}}) = 0$, because A is a subring of $\overline{\mathbb{Q}}_\ell$, so $\mathfrak{q} = \ker y$ is a prime containing $\mathcal{J}$ whose determinant over $\bar{\kappa}(\mathfrak{q}) \subseteq \overline{\mathbb{Q}}_\ell$ is $\det \circ \rho$ with ρ irreducible over $\overline{\mathbb{Q}}_\ell$, i.e. $\mathfrak{q}$ is stable, and Lemma 50.12 gives $\mathcal{J}_\kappa^{\mathrm{cl}} \subseteq \mathfrak{q}$; and the ratio condition holds with $h_{ab} = e(w_{n,a} - w_{n,b})$, because $\Psi^{w_{n,a}-w_{n,b}} \chi_{\mathrm{cyc}}^{-e(w_{n,a}-w_{n,b})} = (\Psi \chi_{\mathrm{cyc}}^{-e})^{w_{n,a}-w_{n,b}}$ and $\Psi \chi_{\mathrm{cyc}}^{-e}$, a character of $G_{\mathbb{Q}}$ trivial on an open subgroup of $I_{\bar{\ell}}$ ($e \geq 1$, weight hypothesis of Theorem 50.6), is trivial on an open subgroup of the conjugate $I_{\bar{v}}$, hence of I_v. The conclusion of Proposition E.25 is that of (Hi4).

(Hi2). By Proposition E.32 (its hypothesis on $\mathcal{J}$ is the choice above), $\mathcal{J}$ has R = T at patchable curve points (for v_0 and the place $\bar{v}_0$ of the hypotheses), and, if $n = 3$, along thick

chains. The comparison morphism exists with image in W_n^{po} by hypothesis, so Proposition 50.15, (Tq1) and (Tq2), shows that $\mathcal{M}_{\mathcal{J}}$ propagates through nice primes, and, if $n = 3$, along thick links.

(Hi1), the degenerate case. If $\mathcal{J} = B$, then $B/\mathcal{J} = 0$, and the zero homomorphism $\lambda\colon \Lambda_N \to 0$ satisfies (Hi1). Otherwise let T, λ_T, M and f (with $\mathcal{J} = \ker f$) be as in Definition 50.10.

Step 1: the weight algebra. $T_n(\ell^1) = \prod_{v\in S_\ell}\bigl(1+\ell\mathcal{O}_{L_v^+}\bigr)^n$. For each v, $1+\ell\mathcal{O}_{L_v^+}$ is a finitely generated $\mathbb{Z}_\ell$-module of rank d_v whose torsion consists of the ℓ-power roots of unity in L_v^+; by (Af4), $\bar\omega|_{D_v} \neq 1$, so $\zeta_\ell \notin L_v^+$ and $1+\ell\mathcal{O}_{L_v^+} \cong \mathbb{Z}_\ell^{d_v}$. Hence $T_n(\ell^1) \cong \mathbb{Z}_\ell^N$, and for topological generators $u_1,\dots,u_N$ the map $X_i \mapsto u_i - 1$ is an isomorphism $\Lambda_N \to \Lambda_1$ of complete local $\mathcal{O}'$-algebras; in particular Λ_1 is a Noetherian local domain with maximal ideal generated by $\mathfrak{m}_{\mathcal{O}'}$ and the $u_i - 1$.

Step 2: (Hi1). f induces an isomorphism $\bar f\colon B/\mathcal{J} \to T$, and T is finite and torsion-free over Λ_1 through λ_T by (Zm1). Let $\lambda\colon \Lambda_N \to B/\mathcal{J}$ be the composite of $\Lambda_N \cong \Lambda_1$ (Step 1), λ_T and $\bar f^{-1}$; then $B/\mathcal{J}$ is a finite and torsion-free Λ_N-module via λ. The maximal ideal of Λ_N is generated by $\mathfrak{m}_{\mathcal{O}'}$ and the X_i, and $\bar f(\lambda(X_i)) = \lambda_T([u_i]) - 1$. Each u_i is a finite product of elements $d_j(z)$ with $z \in 1+\ell\mathcal{O}_{L_v^+}$, and $[uu'] - 1 = [u]([u'] - 1) + ([u] - 1)$. For such z, $g_z = \mathrm{Art}(z) \in I_v$ lies in a pro-ℓ group, so $\bar\Psi(g_z) = 1$ ($\bar\Psi$ has order prime to ℓ) and $(\chi_{\mathrm{cyc}}^{j-1}\kappa)(g_z) = 1+\mu$ with $\mu \in \mathfrak{m}_{\mathcal{O}'}$; by (Zm2), $\lambda_T([d_j(z)]) - 1 = (1+\mu)\bigl(f(\psi_{v,j}^{\mathrm{d}}(g_z)) - 1\bigr) + \mu$. So $\bar f(\lambda(X_i))$ lies in the ideal of T generated by $\mathfrak{m}_{\mathcal{O}'}$ and the $f(\psi_{v,a}^{\mathrm{d}}(g)) - 1$ ($v \mid \ell$, $g \in I_v$, $\bar\Psi(g) = 1$), and applying $\bar f^{-1}$ gives the last clause of (Hi1). □

Lemma 50.12. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, κ a twist for $\mathscr{H}$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $(B, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, $\mathcal{J}_\kappa^{\mathrm{cl}}$ its classical ideal for ι and κ, and J a Hecke ideal of zero-monodromy type for ι and κ. Then $J \subseteq \mathcal{J}_\kappa^{\mathrm{cl}}$, and every stable prime $\mathfrak{q}$ of B with $J \subseteq \mathfrak{q}$ contains $\mathcal{J}_\kappa^{\mathrm{cl}}$.*

Proof. Step 0: $J \subseteq \mathcal{J}_\kappa^{\mathrm{cl}}$. If $J = B$, B has no classical ordinary point, so $\mathcal{J}_\kappa^{\mathrm{cl}} = B$. Otherwise every classical ordinary point kills J by (Zm4), and $\mathcal{J}_\kappa^{\mathrm{cl}}$ is the intersection of their kernels. In particular, if $J = B$ there is no prime containing J; assume from now on $J = \ker f$.

Step 1: a minimal prime with irreducible determinant. Let $\mathfrak{q} \supseteq J$ be stable and $\bar{\mathfrak{q}} = f(\mathfrak{q})$, a prime of T with $f^{-1}(\bar{\mathfrak{q}}) = \mathfrak{q}$ (f is surjective with kernel in $\mathfrak{q}$), so $B/\mathfrak{q} \cong T/\bar{\mathfrak{q}}$. Let $P \subseteq \bar{\mathfrak{q}}$ be a minimal prime of T, Ω_P an algebraic closure of $\mathrm{Frac}(T/P)$, and ρ_P the semisimple representation of $\Gamma^0_{\mathscr{H}}$ over Ω_P with determinant the base change of $f \circ D^{\mathrm{u}}$ (Ch3). Suppose $\rho_P = \rho_1 \oplus \rho_2$ with $\dim\rho_i \geq 1$. The coefficients of the characteristic polynomials of the $\rho_i(\gamma)$ are elementary symmetric functions in some of the roots of the characteristic polynomial of $\rho_P(\gamma)$, a monic polynomial over T/P; so they lie in the integral closure R_P of T/P in Ω_P, and by (Ch1) $\det\circ\rho_i$ is the base change of an R_P-valued determinant D_i of dimension $\dim\rho_i$, with $D_1 D_2$ the base change of $f \circ D^{\mathrm{u}}$ to R_P. By lying over there is a prime Q' of R_P over $\bar{\mathfrak{q}}/P$; an embedding of R_P/Q' into $\bar\kappa(\mathfrak{q})$ over $T/\bar{\mathfrak{q}} = B/\mathfrak{q}$ (R_P/Q' is algebraic over $T/\bar{\mathfrak{q}}$) shows that $D_{\mathfrak{q}}$ is a product of two determinants of dimensions $\dim\rho_1, \dim\rho_2$, so by (Ch3) the semisimple representation with determinant $D_{\mathfrak{q}}$ is a direct sum of two representations of these dimensions, contradicting the stability of $\mathfrak{q}$. Hence ρ_P is absolutely irreducible. By Burnside's theorem choose $\gamma_1,\dots,\gamma_{n^2} \in \Gamma^0_{\mathscr{H}}$ with $(\rho_P(\gamma_i))$ a basis of $\mathrm{M}_n(\Omega_P)$, and let $g_P \in T$ be the determinant of the matrix of traces $\bigl(\mathrm{tr}_{f\circ D^{\mathrm{u}}}(\gamma_i\gamma_j)\bigr)_{i,j}$. Its image in Ω_P is the Gram determinant of the basis for the non-degenerate trace form, so $g_P \notin P$. For every homomorphism $y\colon T \to \overline{\mathbb{Q}}_\ell$ with $y(P) = 0 \neq y(g_P)$ the semisimple ρ_y with $\det\circ\rho_y = y \circ f \circ D^{\mathrm{u}}$ has $\det(\mathrm{tr}\,\rho_y(\gamma_i\gamma_j)) \neq 0$, so the $\rho_y(\gamma_i)$ span $\mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ and ρ_y is absolutely irreducible.

Step 2: regular points separate T/P. Let $t \in T \smallsetminus P$. By (Zm1), $\Lambda_1 \to T/P$ is injective (a

non-zero element of Λ_1 is a non-zero-divisor on T, while the minimal prime P of the Noetherian ring T consists of zero-divisors), and T/P is a domain finite over $\Lambda_1 \cong \mathcal{O}[[X_1,\dots,X_N]]$, a normal domain (Lemma 50.14 (2)); so the minimal polynomial of t mod P over $\mathrm{Frac}(\Lambda_1)$ has coefficients in Λ_1 and a non-zero constant term c_0. For a dominant λ with $\lambda_{\tau,j} > \lambda_{\tau,j+1}$, the homomorphism $\Lambda_1 \to \overline{\mathbb{Q}}_\ell$, $[u] \mapsto (w_0\lambda)(u)^{-1}$, has kernel $\mathfrak{P}_{k(\lambda)}$ with $k(\lambda)_{\tau,j} = -\lambda_{\tau,n+1-j}$; the set of these $k(\lambda)$ is dense in $(\mathbb{Z}_\ell^n)^I$ (given k^* and e, choose integers $k \equiv k^* \bmod \ell^e$ and add $\ell^e A\,j$ to $k_{\tau,j}$ for large A, making $k_{\tau,j}$ strictly increasing in j). By Lemma 50.13, $c_0 \notin \mathfrak{P}_{k(\lambda)}$ for some such λ; by lying over, $\Lambda_1/\mathfrak{P}_{k(\lambda)} \to \overline{\mathbb{Q}}_\ell$ extends to $y\colon T/P \to \overline{\mathbb{Q}}_\ell$, a regular point of T with $y(P) = 0$; $y(t)$ is a root of a polynomial with constant term $y(c_0) \neq 0$, so $y(t) \neq 0$.

Step 3: these points are classical. Let y be a regular point (for λ) with $y(P) = 0 \neq y(g_P)$ and $x = y \circ f$. By Step 1, ρ_y is absolutely irreducible, so (Cy1) holds with $\tilde{r} = \rho_y$; by (Zm3), $\rho_y \circ \pi_{\mathscr{H}} \otimes \kappa$ is automorphic of conjugate self-dual type, which is (Cy2), and $\rho_y \circ \pi_{\mathscr{H}}$ is triangular on D_v with diagonal $(x \circ \psi^{\mathrm{d}}_{v,a})_a$, which is (Cy3). (Cy4): let $v \in S_\ell$ and $1 \le j \le n$. By (Zm2) and $y(\lambda_T([u])) = (w_0\lambda)(u)^{-1}$, for $z \in 1 + \ell\mathcal{O}_{L_v^+}$, $x(\psi^{\mathrm{d}}_{v,j}(g_z)) = (\chi_{\mathrm{cyc}}^{j-1}\kappa)(g_z)^{-1}(w_0\lambda)(d_j(z))^{-1}$. Here g_z is the image of z under the local Artin map normalised as in Proposition E.5 (uniformisers to geometric Frobenius elements), the normalisation with which (Zm2) is established in Proposition E.2; for it, $\chi_{\mathrm{cyc}}(g_z) = \prod_\tau \tau(z)$ for units z (Lubin–Tate theory), τ running over the embeddings $L_v^+ \to \overline{\mathbb{Q}}_\ell$. As κ corresponds to an algebraic Hecke character, there are integers m_τ with $\kappa(g_z) = \prod_\tau \tau(z)^{m_\tau}$ for z in an open subgroup of $\mathcal{O}^\times_{L_v^+}$ ([220], Chapter II), and $(w_0\lambda)(d_j(z)) = \prod_\tau \tau(z)^{\lambda_{\tau,n+1-j}}$. Hence $x(\psi^{\mathrm{d}}_{v,j}(g_z)) = \prod_\tau \tau(z)^{h_{\tau,j}}$ on an open subgroup, with $h_{\tau,j} = (1-j) - m_\tau - \lambda_{\tau,n+1-j}$. A continuous character of D_v which is de Rham and equals $g_z \mapsto \prod_\tau \tau(z)^{a_\tau}$ on an open subgroup of inertia has τ-Hodge–Tate weight a_τ (Hodge–Tate weights as in Section 13, χ_{cyc} of weight 1; a Hodge–Tate character is locally algebraic and its weights are read off from the algebraic part, [220], Chapter III). So $x \circ \psi^{\mathrm{d}}_{v,j}$ has τ-Hodge–Tate weight $h_{\tau,j}$, and $h_{\tau,j} - h_{\tau,j+1} = 1 + \lambda_{\tau,n-j} - \lambda_{\tau,n+1-j} \ge 2$ as λ is regular: strictly decreasing in j; it is de Rham as a subquotient of the de Rham representation $\rho_y \circ \pi_{\mathscr{H}}|_{D_v}$ (automorphic, (Bg2) of Theorem B.62). So x is a classical ordinary point and $x(\mathcal{J}_\kappa^{\mathrm{cl}}) = 0$.

Step 4: conclusion. Let $b \in \mathcal{J}_\kappa^{\mathrm{cl}}$ and suppose $f(b) \notin P$. Then $t = f(b)g_P \notin P$, and Step 2 gives a regular point y with $y(P) = 0$ and $y(f(b))y(g_P) \neq 0$; by Step 3, $y(f(b)) = (y \circ f)(b) = 0$, a contradiction. So $f(b) \in P \subseteq \bar{\mathfrak{q}}$ and $b \in f^{-1}(\bar{\mathfrak{q}}) = \mathfrak{q}$. □

Lemma 50.13. *Let ℓ be an odd prime, F^+ a totally real field, $n \ge 1$, I the set of field embeddings $F^+ \to \overline{\mathbb{Q}}_\ell$, and $\mathcal{O}'$ a coefficient ring containing $\tau(\mathcal{O}_{F^+})$ for every $\tau \in I$. Let $T_n(\ell^1)$ be the kernel of reduction $T_n(\mathcal{O}_{F^+} \otimes \mathbb{Z}_\ell) \to T_n(\mathcal{O}_{F^+}/\ell)$ on the diagonal torus and $\Lambda_1 = \mathcal{O}'[[T_n(\ell^1)]]$. For $k = (k_{\tau,j}) \in (\mathbb{Z}^n)^I$ let $\chi_k\colon T_n(\ell^1) \to \overline{\mathbb{Q}}_\ell^\times$ be the character $\mathrm{diag}(u_1,\dots,u_n) \mapsto \prod_{\tau\in I}\prod_{j=1}^n \tau(u_j)^{k_{\tau,j}}$ and $\mathfrak{P}_k$ the kernel of the induced homomorphism of $\mathcal{O}'$-algebras $\Lambda_1 \to \overline{\mathbb{Q}}_\ell$. If $\mathcal{K} \subseteq (\mathbb{Z}^n)^I$ is a set whose closure in $(\mathbb{Z}_\ell^n)^I$ has non-empty interior, then $\bigcap_{k\in\mathcal{K}} \mathfrak{P}_k = 0$.*

Proof. Step 1: coordinates. By Lemma 50.14, $T_n(\ell^1) = \prod_{v|\ell}(1 + \ell\mathcal{O}_{F_v^+})^n \cong \mathbb{Z}_\ell^N$ with $N = n\,|I|$, and for $v \mid \ell$ the ℓ-adic logarithm $\log\colon 1 + \ell\mathcal{O}_{F_v^+} \to \ell\mathcal{O}_{F_v^+}$ is an isomorphism of $\mathbb{Z}_\ell$-modules with inverse exp. Fix a $\mathbb{Z}_\ell$-basis $u_1,\dots,u_N$. Then $X_i \mapsto u_i - 1$ is an isomorphism $\mathcal{O}'[[X_1,\dots,X_N]] \to \Lambda_1$ (Lemma 50.14), and the homomorphism with kernel $\mathfrak{P}_k$ sends X_i to $\chi_k(u_i) - 1$. Let $\mathcal{O}'' \supseteq \mathcal{O}'$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$ containing all $\tau(\mathcal{O}_{F_v^+})$. Let $f \in \bigcap_{k\in\mathcal{K}} \mathfrak{P}_k$, viewed as a power series $f(X) \in \mathcal{O}'[[X]]$; we show $f = 0$.

Step 2: interpolation in the weight. Write $u_i = \mathrm{diag}(u_{i,1},\dots,u_{i,n})$ and put $a_{i,(\tau,j)} = \log\tau(u_{i,j}) \in \ell\mathcal{O}''$. For $k \in (\mathbb{Z}_\ell^n)^I$ put $w_i(k) = \sum_{(\tau,j)} a_{i,(\tau,j)}k_{\tau,j} \in \ell\mathcal{O}''$ and $E_i(k) = \exp(w_i(k)) - 1 \in \ell\mathcal{O}''$; for integral k, $E_i(k) = \chi_k(u_i) - 1$, as $\tau(u_{i,j})^k = \exp(k\log\tau(u_{i,j}))$. Since $|E_i(k)| \le |\ell| < 1$ and the coefficients of f are bounded, $k \mapsto f(E(k))$ is a uniformly convergent series of con-

tinuous functions on $(\mathbb{Z}_\ell^n)^I$, hence continuous; it vanishes on $\mathcal{K}$, hence on the closure of $\mathcal{K}$, which contains a set $k_0 + \ell^e(\mathbb{Z}_\ell^n)^I$ with $k_0 \in (\mathbb{Z}^n)^I$, $e \geq 1$. Put $c_i = \chi_{k_0}(u_i) \in 1 + \ell\mathcal{O}''$ and $f_0(X) = f\big(c_1(1+X_1)-1, \dots, c_N(1+X_N)-1\big) \in \mathcal{O}''[[X]]$, well defined because $c_i - 1$ is topologically nilpotent; $X_i \mapsto c_i(1+X_i)-1$ is an automorphism of $\mathcal{O}''[[X]]$ (inverse $X_i \mapsto c_i^{-1}(1+X_i)-1$), so $f = 0$ if $f_0 = 0$. As $\chi_{k_0+k'} = \chi_{k_0}\chi_{k'}$ and exp is additive, $f_0(E(k')) = f(E(k_0 + k')) = 0$ for every $k' \in \ell^e(\mathbb{Z}_\ell^n)^I$.

Step 3: the matrix $(a_{i,b})$ is invertible. By Step 1, the $\log u_i$ form a $\mathbb{Z}_\ell$-basis of $\prod_v(\ell\mathcal{O}_{F_v^+})^n$, so their images under the isomorphism $\prod_v(F_v^+)^n \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell^{I\times\{1,\dots,n\}}$, $y \otimes 1 \mapsto (\tau(y_j))_{(\tau,j)}$ (from $F_v^+ \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \cong \prod_{\tau \text{ above } v} \overline{\mathbb{Q}}_\ell$), form a basis; these images are the rows of $A = (a_{i,b})$, so A is an invertible $N \times N$ matrix.

Step 4: $f_0 = 0$. Let $g(w) = f_0(\exp(w_1) - 1, \dots, \exp(w_N) - 1)$, a formal power series with coefficients in $\mathrm{Frac}(\mathcal{O}'')$ (the series $\exp(w_i) - 1$ have no constant term); it converges on $|w_i| \leq |\ell|$ and there equals the evaluation of f_0. Put $h(k') = g(Ak')$, a formal power series in $k' = (k'_b)$; choose $e' \geq e$ with $|\ell^{e'}| \max_{i,b} |a_{i,b}| \leq |\ell|$, so that h converges on the closed polydisc $|k'_b| \leq |\ell|^{e'}$ and vanishes on S^N, $S = \ell^{e'}\mathbb{Z}_\ell$, by Step 2. A power series $h \in \overline{\mathbb{Q}}_\ell[[k'_1, \dots, k'_N]]$ converging on the closed polydisc of radius $\rho = |\ell|^{e'}$ and vanishing on S^N is 0, by induction on N: write $h = \sum_m h_m(k'_1, \dots, k'_{N-1})\, k_N'^m$; for $s \in S^{N-1}$ the one-variable series $\sum_m h_m(s)t^m$ converges on $|t| \leq \rho$ and vanishes on the infinite set S, so it is 0 by Strassmann's theorem (a non-zero power series converging on a closed disc has only finitely many zeros in it); thus every h_m, which converges on the polydisc of radius ρ in $N-1$ variables, vanishes on S^{N-1}, and $h_m = 0$ by induction ($N = 0$: a constant vanishing at a point). Hence $h = 0$, so $g(w) = h(A^{-1}w) = 0$ as a formal power series, and $f_0(X) = g(\log(1 + X_1), \dots, \log(1 + X_N)) = 0$. By Step 2, $f = 0$. □

Lemma 50.14. *Let ℓ be an odd prime.*

1. *For every finite extension K of $\mathbb{Q}_\ell$, the ℓ-adic logarithm* $\log\colon 1 + \ell\mathcal{O}_K \to \ell\mathcal{O}_K$ *is an isomorphism of $\mathbb{Z}_\ell$-modules with inverse* exp. *In particular $1 + \ell\mathcal{O}_K \cong \mathbb{Z}_\ell^{[K:\mathbb{Q}_\ell]}$ is torsion-free.*

2. *Let F^+ be a totally real field, $n \geq 1$, $N = n[F^+ : \mathbb{Q}]$, $T_n(\ell^1)$ the kernel of reduction $T_n(\mathcal{O}_{F^+} \otimes \mathbb{Z}_\ell) \to T_n(\mathcal{O}_{F^+}/\ell)$ on the diagonal torus, and $\mathcal{O}'$ a coefficient ring. Then $T_n(\ell^1) = \prod_{v|\ell}(1+\ell\mathcal{O}_{F_v^+})^n \cong \mathbb{Z}_\ell^N$, and for every $\mathbb{Z}_\ell$-basis $u_1, \dots, u_N$ of $T_n(\ell^1)$ the map $X_i \mapsto u_i - 1$ is an isomorphism of complete local $\mathcal{O}'$-algebras $\mathcal{O}'[[X_1, \dots, X_N]] \to \mathcal{O}'[[T_n(\ell^1)]]$; in particular $\mathcal{O}'[[T_n(\ell^1)]]$ is a Noetherian normal local domain.*

Proof. 1. For $x \in \ell\mathcal{O}_K$ and $k \geq 1$, $v(x^k/k) \geq k - \log_\ell k \geq 1$ and $v(x^k/k!) \geq k - \frac{k-1}{\ell-1} \geq 1$, since $v(k!) \leq \frac{k-1}{\ell-1}$. So $\log(1 + x) = \sum_{k\geq 1}(-1)^{k+1}x^k/k$ and $\exp(x) = \sum_{k\geq 0} x^k/k!$ converge on $\ell\mathcal{O}_K$, log maps $1 + \ell\mathcal{O}_K$ into $\ell\mathcal{O}_K$ and exp maps $\ell\mathcal{O}_K$ into $1 + \ell\mathcal{O}_K$. These sets lie in the regions $|x| < |\ell|^{1/(\ell-1)}$, resp. $|y - 1| < |\ell|^{1/(\ell-1)}$ (as $1 > \frac{1}{\ell-1}$ for $\ell \geq 3$), where log and exp are mutually inverse isometries and $\log(yy') = \log y + \log y'$. Hence log is an isomorphism of $\mathbb{Z}_\ell$-modules with inverse exp, and $1 + \ell\mathcal{O}_K \cong \ell\mathcal{O}_K \cong \mathcal{O}_K \cong \mathbb{Z}_\ell^{[K:\mathbb{Q}_\ell]}$. (Consistently, a primitive ℓ^k-th root of unity ζ has $v(\zeta - 1) = \frac{1}{\ell^{k-1}(\ell-1)} < 1$, so $\zeta \notin 1 + \ell\mathcal{O}_K$.)

2. $\mathcal{O}_{F^+} \otimes \mathbb{Z}_\ell = \prod_{v|\ell} \mathcal{O}_{F_v^+}$, and the kernel of reduction modulo ℓ on $(\mathcal{O}_{F_v^+}^\times)^n$ is $(1+\ell\mathcal{O}_{F_v^+})^n$. By 1, $T_n(\ell^1) \cong \prod_v \mathbb{Z}_\ell^{n[F_v^+:\mathbb{Q}_\ell]} = \mathbb{Z}_\ell^N$, as $\sum_{v|\ell}[F_v^+ : \mathbb{Q}_\ell] = [F^+ : \mathbb{Q}]$. For $\Gamma = \mathbb{Z}_\ell^N$ with basis $u_1, \dots, u_N$, the map $X_i \mapsto u_i - 1$ is an isomorphism $\mathcal{O}'[[X_1, \dots, X_N]] \to \mathcal{O}'[[\Gamma]] = \varprojlim_m \mathcal{O}'[\Gamma/\ell^m\Gamma]$ of complete local $\mathcal{O}'$-algebras (the classical description of the Iwasawa algebra of $\mathbb{Z}_\ell^N$, by induction on N from the case $N = 1$). A power series ring in finitely many variables over the complete discrete valuation ring $\mathcal{O}'$ is a Noetherian regular, hence normal, local domain. □

Proposition 50.15. *Let $\mathcal{O}'$ be a coefficient ring, L an auxiliary field of level C_0 for $(K, \mathcal{O}', \Psi, c, g_\star, M)$, $n \in \{2,3\}$, and v_0 a place of L^+ above a prime $p \nmid 2M\,|\mathrm{disc}(L)|$ which splits in L and has a place $\bar{v}_0$ above it with $\bar{\Psi}$ trivial and $\bar{\omega}$ non-trivial on $D_{\bar{v}_0} \cap G_{L^+}$. Let $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ be a universal polarised ordinary determinant and $(R_n^{\mathrm{po}}, \rho^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{u}}))$ a universal polarised ordinary lift of rank n over L (both for $\mathcal{O}'$) such that the comparison morphism ϑ_n exists and $\vartheta_n(B_n^{\mathrm{po}}) \subseteq W_n^{\mathrm{po}}$, and let $J \subseteq B_n^{\mathrm{po}}$ be an ideal.*

(Tq1) *If J has $R = T$ at patchable curve points (for v_0, with the place $\bar{v}_0$), then $\mathcal{M}_J$ propagates through nice primes.*

(Tq2) *If $n = 3$ and J has $R = T$ at patchable curve points and along thick chains, then $\mathcal{M}_J$ propagates along thick links.*

Proof. Fix a prime $\mathfrak{x}$ of E, a local chart $(\mathcal{O}'', R, \iota)$ at $\mathfrak{x}$ and put $\varphi = \iota \circ \vartheta_n \colon B \to R$. For a prime P of R, $\iota^{-1}(P) \cap W$ is an ideal of W, and $\mathcal{M}_J$ is the ideal of W generated by $\vartheta_n(J)$; hence

$$P \text{ is } \mathcal{M}_J\text{-modular} \iff \vartheta_n(J) \subseteq \iota^{-1}(P) \iff J \subseteq \varphi^{-1}(P). \tag{$*$}$$

(Tq1). Let $\mathfrak{p}$ be an $\mathcal{M}_J$-modular nice prime of R of irreducible type and $P \subseteq \mathfrak{p}$ a prime. By Lemma 50.16 there is a patchable curve point $(k, \rho, J_n, (\psi_{v,a}), f)$ of B (for v_0, with $\bar{v}_0$) of irreducible type with $\ker f = \varphi^{-1}(\mathfrak{p})$; by $(*)$, $J \subseteq \ker f$. The prime $\mathfrak{q} = \varphi^{-1}(P)$ of B satisfies $\mathfrak{q} \subseteq \varphi^{-1}(\mathfrak{p}) = \ker f$, so R = T at patchable curve points gives $J \subseteq \mathfrak{q}$, and P is $\mathcal{M}_J$-modular by $(*)$. So $\mathcal{M}_J$ propagates through nice primes.

(Tq2). Let $n = 3$, and let $\mathfrak{p}, \mathfrak{q}$ be thickly connected nice primes of R of endoscopic type, through nice primes $\mathfrak{p} = \mathfrak{p}_0, \dots, \mathfrak{p}_h = \mathfrak{q}$ of endoscopic type and thick primes $\mathfrak{e}_i \subseteq \mathfrak{p}_{i-1} \cap \mathfrak{p}_i$ $(1 \le i \le h)$, such that some $\mathcal{M}_J$-modular prime $P \subseteq \mathfrak{p}$ is of irreducible type; let $P' \subseteq \mathfrak{q}$ be a prime of irreducible type. By Lemma 50.16, each $\mathfrak{p}_i$ gives a patchable curve point x_i of endoscopic type with homomorphism f_i, $\ker f_i = \varphi^{-1}(\mathfrak{p}_i)$. By Lemma 50.22 (Xk2), $\varphi^{-1}(\mathfrak{e}_i)$ is a thick prime of B, contained in $\ker f_{i-1} \cap \ker f_i$. By (Xk1), $\mathfrak{q}_1 = \varphi^{-1}(P)$ is stable; $J \subseteq \mathfrak{q}_1$ by $(*)$, and $\mathfrak{q}_1 \subseteq \ker f_0$. By (Xk1) again, $\varphi^{-1}(P')$ is a stable prime contained in $\varphi^{-1}(\mathfrak{q}) = \ker f_h$. R = T along thick chains therefore gives $J \subseteq \varphi^{-1}(P')$, and P' is $\mathcal{M}_J$-modular by $(*)$. So $\mathcal{M}_J$ propagates along thick links. □

Lemma 50.16. *Let $\mathcal{O}'$ be a coefficient ring, L an auxiliary field of level C_0 for $(K, \mathcal{O}', \Psi, c, g_\star, M)$, $n \in \{2,3\}$, and v_0 a place of L^+ above a prime $p \nmid 2M\,|\mathrm{disc}(L)|$ which splits in L and has a place $\bar{v}_0$ above it with $\bar{\Psi}$ trivial and $\bar{\omega}$ non-trivial on $D_{\bar{v}_0} \cap G_{L^+}$. Let $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ be a universal polarised ordinary determinant and $(R_n^{\mathrm{po}}, \rho^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{u}}))$ a universal polarised ordinary lift of rank n over L (both for $\mathcal{O}'$) such that ϑ_n exists and $\vartheta_n(B_n^{\mathrm{po}}) \subseteq W_n^{\mathrm{po}}$. Let $\mathfrak{x}$ be a prime of E_n^{po}, $(\mathcal{O}'', R, \iota)$ a local chart at $\mathfrak{x}$, $\varphi = \iota \circ \vartheta_n \colon B_n^{\mathrm{po}} \to R$, and $\mathfrak{p}$ a nice prime of R. Then there are a finite field $k \supseteq \mathbb{F}_{\mathcal{O}''}$ and an injective local homomorphism of $\mathcal{O}''$-algebras $j \colon R/\mathfrak{p} \to k[[T]]$ such that, with $h \colon R \to k[[T]]$ the composite of the projection and j, $\rho = h \circ \rho_{\mathfrak{x}}$, $\psi_{v,a} = h \circ \psi_{\mathfrak{x},v,a}$ and $f = h \circ \varphi$:*

(Np1) *$(k, \rho, J_n, (\psi_{v,a}), f)$ is a patchable curve point of B_n^{po} (for v_0, with the place $\bar{v}_0$), and $\ker f = \varphi^{-1}(\mathfrak{p})$;*

(Np2) *it is of irreducible type if $\mathfrak{p}$ is of irreducible type, and of endoscopic type if $n = 3$ and $\mathfrak{p}$ is of endoscopic type.*

Proof. Write $W = W_n^{\mathrm{po}}$, $E = E_n^{\mathrm{po}}$, $B = B_n^{\mathrm{po}}$, $\rho(\gamma)$ for $\rho(\pi_L(g))$ when $\gamma = \pi_L(g)$, $A = k[[T]]$, $F = \mathrm{Frac}(A)$ and $\bar{F}$ for an algebraic closure of F.

Step 0: residues on the chart. The ring $E/\mathfrak{x}$ embeds into the finite field $R/\mathfrak{m}_R$, so $W/(\mathfrak{x}\cap W)$ is a domain inside a finite field, hence a field, and $\mathfrak{x}\cap W = \mathfrak{m}_W$ (W is local by (Gw1)). Thus $\iota(\mathfrak{m}_W) \subseteq \mathfrak{m}_R$, and every $w \in W$ has $\iota(w) \equiv \bar{w}$ modulo $\mathfrak{m}_R$, where $\bar{w} \in \mathbb{F}_{\mathcal{O}'}$ is its residue. Diagonal entries $\rho^{\mathrm{u}}(\gamma)_{aa}$, the values $\psi^{\mathrm{u}}_{v,a}(g)$, and the coefficients of the characteristic polynomial of $\rho^{\mathrm{u}}(\gamma)$ (sums of products $\pm\prod_a \rho^{\mathrm{u}}(\gamma)_{a\sigma(a)}$ over permutations of subsets, of weight 0) lie in W, with residues given by (Po5): $\bar{\Psi}(\gamma)^{w_{n,a}}$, $\bar{\Psi}(g)^{w_{n,a}}$, and the coefficients of $\prod_a(X - \bar{\Psi}(\gamma)^{w_{n,a}})$. Let $\gamma_\star = \pi_L(g_\star)$ and $\gamma'_\star = \pi_L(cg_\star^{-1}c)$ ($g_\star \in G_L$ by (Af1)). By (Po3) the matrix $\rho^{\mathrm{u}}(\gamma_\star) - \rho^{\mathrm{u}}(\gamma'_\star)$ is diagonal, and by (Po2) its (a,a) entry is $\rho^{\mathrm{u}}(\gamma_\star)_{aa} - \rho^{\mathrm{u}}(\gamma_\star)_{n+1-a,n+1-a}$; so it equals $\zeta\,\mathrm{diag}(w_{n,1},\dots,w_{n,n})$ with $\zeta = \rho^{\mathrm{u}}(\gamma_\star)_{11} - \rho^{\mathrm{u}}(\gamma_\star)_{nn} \in W$ of residue $\bar{\Psi}(g_\star) - \bar{\Psi}(g_\star)^{-1} \neq 0$ ($g_\star$ is a marking). So $\zeta \in W^\times$ and $\iota(\zeta) \in R^\times$. Finally, for $k' \geq 1$ and an entry of weight j, $\rho^{\mathrm{u}}(\gamma)_{ab} \in \mathrm{gr}_j$, a finitely generated W-module (Gw3); the submodules $\mathfrak{m}^t \cap \mathrm{gr}_j$ ($\mathfrak{m}$ the maximal ideal of R_n^{po}) decrease to 0 (Krull), so by Theorem B.154 some $\mathfrak{m}^t \cap \mathrm{gr}_j$ lies in $\mathfrak{m}_W^{k'}\mathrm{gr}_j$; with (Po1), $\gamma \mapsto \rho_{\mathfrak{x}}(\gamma)_{ab} \bmod \iota(\mathfrak{m}_W^{k'}\mathrm{gr}_j)R \subseteq \mathfrak{m}_R^{k'}$ is locally constant, and likewise $g \mapsto \psi_{\mathfrak{x},v,a}(g) \bmod \mathfrak{m}_R^{k'}$ ($\mathrm{gr}_0 = W$).

Step 1: the curve. The ring $R/\mathfrak{p}$ is a complete Noetherian local domain of dimension 1 and characteristic ℓ ($\ell \in \mathfrak{p}$) with residue field $\mathbb{F}_{\mathcal{O}''}$. By Nagata's theorem (complete local rings are Nagata rings) its integral closure $\tilde{A}$ in its fraction field is a finite $R/\mathfrak{p}$-module; a domain finite over a complete local domain is local, so $\tilde{A}$ is a one-dimensional normal Noetherian local domain, i.e. a discrete valuation ring, complete as a finite module over a complete ring, of characteristic ℓ with finite residue field $k \supseteq \mathbb{F}_{\mathcal{O}''}$. By Cohen's structure theorem $\tilde{A} \cong k[[T]]$; let j be the inclusion $R/\mathfrak{p} \subseteq \tilde{A}$ followed by this isomorphism. It is injective, an $\mathcal{O}''$-algebra map, and local (an integral extension contracts the maximal ideal to the maximal ideal). Hence $h(\mathfrak{m}_R^{k'}) \subseteq T^{k'}A$ and $\ker h = \mathfrak{p}$.

Step 2: the curve point. By Step 0 and locality of h, $\rho\colon \Gamma_L^0 \to \mathrm{GL}_n(A)$ has T-adically continuous entries and the $\psi_{v,a}\colon D_v \to A^\times$ are continuous homomorphisms. The matrix $P = J_n$ is symmetric and invertible with $J_n^{-1} = J_n$, and (Po2) gives $\rho(\gamma^*) = J_n\,{}^t\rho(\gamma)J_n$, $\gamma^* = \pi_L(c)\gamma^{-1}\pi_L(c)$. Since ϑ_n classifies $(\det\circ\rho^{\mathrm{u}}, (\psi^{\mathrm{u}}_{v,a}))$ and $\vartheta_n(B) \subseteq W \subseteq E$, the determinants $\vartheta_n \circ D^{\mathrm{u}}$ and $\det\circ\rho^{\mathrm{u}}$, both E-valued, have the same characteristic polynomials on Γ_L^0 and are equal by (Ch1); applying $h\circ\iota$ gives $f \circ D^{\mathrm{u}} = \det\circ\rho$, and likewise $f \circ \psi^{\mathrm{d}}_{v,a} = h\circ\iota\circ\psi^{\mathrm{u}}_{v,a} = \psi_{v,a}$. The map f is a homomorphism of $\mathcal{O}'$-algebras with $\ker f = \varphi^{-1}(\ker h) = \varphi^{-1}(\mathfrak{p})$.

Step 3: triangularity on D_v. Fix $v \mid \ell$; $D_v \subseteq G_L$ since v splits in L ((Af2), $\ell \mid M$). By (Po4), $\prod_{a=1}^n(\rho(g_a) - \psi_{v,a}(g_a)) = 0$ for $g_1,\dots,g_n \in D_v$. For $0 \leq i \leq n$ let $V_i \subseteq F^n$ be the span of the images of $\prod_{a=i+1}^n(\rho(g_a) - \psi_{v,a}(g_a))$ over all $g_{i+1},\dots,g_n \in D_v$ ($V_n = F^n$); so $V_0 = 0$. For $i < n$ and $g, g' \in D_v$,

$$\rho(g')(\rho(g) - \psi_{v,i+1}(g)) = (\rho(g'g) - \psi_{v,i+1}(g'g)) - \psi_{v,i+1}(g)(\rho(g') - \psi_{v,i+1}(g')),$$

so V_i is $\rho(D_v)$-stable; then $(\rho(g) - \psi_{v,i}(g))V_i \subseteq V_{i-1}$ gives $V_{i-1} \subseteq V_i$ and D_v acts on V_i/V_{i-1} by $\psi_{v,i}$. With $d_i = \dim V_i/V_{i-1}$, the characteristic polynomial of $\rho(g)$ is $\prod_i(X - \psi_{v,i}(g))^{d_i}$. Reducing modulo T (Step 0) and comparing coefficients of X^{n-1}: $\sum_i d_i\bar{\Psi}(g)^{w_{n,i}} = \sum_i \bar{\Psi}(g)^{w_{n,i}}$ in k for all $g \in D_v$. By (Af4), $\bar{\Psi}^2|_{D_v} \neq 1$, so the characters $\bar{\Psi}^{w_{n,i}}|_{D_v}$ are pairwise distinct (their exponents differ by 1 or 2), and Artin's theorem on independence of characters gives $d_i \equiv 1 \pmod{\ell}$; as $0 \leq d_i \leq 3$ and ℓ is odd, $d_i = 1$. A basis $f_i \in V_i \smallsetminus V_{i-1}$ makes $\rho|_{D_v}$ upper triangular over F with diagonal $(\psi_{v,1},\dots,\psi_{v,n})$. So $(k, \rho, J_n, (\psi_{v,a}), f)$ is a curve point.

Step 4: (Pj1) and a separating inertia element. The prime $\mathfrak{p}$ is nice: for $a \neq b$ there is $g \in I_v$ with $\chi_{\mathrm{cyc}}(g) = 1$ and $\psi_{\mathfrak{x},v,a}(g) - \psi_{\mathfrak{x},v,b}(g) \notin \mathfrak{p}$, so $\psi_{v,a}(g) \neq \psi_{v,b}(g) = \psi_{v,b}(g)\chi_{\mathrm{cyc}}(g)$; this is (Pj1). Let $I_v^1 = I_v \cap \ker\chi_{\mathrm{cyc}}$. By (Af4), $\bar{\Psi} = 1$ on I_v^1, so $\psi_{v,a}(I_v^1) \subseteq 1 + TA$, which is torsion-free (if $z \neq 0$ and $m = \ell^e m'$, $\ell \nmid m'$, then $(1+z)^m = (1+z^{\ell^e})^{m'} \equiv 1 + m'z^{\ell^e} \not\equiv 1$ modulo higher powers

of z^{ℓ^e}). For $a<b$ let $\chi_{ab}=\psi_{v,a}/\psi_{v,b}$ on I_v^1, non-trivial by niceness; choose g_{ab} in its domain with $g_{ab}\notin\ker\chi_{ab}$. The subgroup Ξ generated by the g_{ab} maps under $(\chi_{ab})_{a<b}$ onto a finitely generated torsion-free abelian group $\Xi'\cong\mathbb{Z}^s$; the kernel K_{ab} of the ab-coordinate on Ξ' has non-zero torsion-free quotient, so $K_{ab}\otimes\mathbb{Q}\neq\Xi'\otimes\mathbb{Q}$. A $\mathbb{Q}$-vector space is not a union of finitely many proper subspaces; so some $\xi\in\Xi'$ lies outside all $K_{ab}\otimes\mathbb{Q}$, and a preimage $u\in I_v^1$ satisfies $\chi_{ab}(u)\neq 1$ for all $a<b$. Put $\bar u=\pi_L(u)\in\Gamma'=\ker\chi_{\mathrm{cyc}}|_{\Gamma_L^0}$: $\rho(\bar u)$ has the n distinct eigenvalues $\psi_{v,a}(u)\in 1+TA$, which generate a torsion-free group.

Step 5: (Pj3). Inertia at $\bar v_0$ dies in Γ_L: $p\neq\ell$ ($\ell\mid M$); η_K is unramified at p because $p\nmid\mathrm{disc}(L)$ and $K\subseteq L$; and $\bar\Psi$ is trivial on $D_{\bar v_0}\cap G_{L^+}$; so $I_{\bar v_0}\cap G_{L^+}\subseteq N_L$. Hence $\Delta_0=\pi_L(D_{\bar v_0}\cap G_{L^+})$ is topologically generated by $\pi_L(\phi)$ for a Frobenius $\phi\in D_{\bar v_0}\cap G_{L^+}\subseteq G_L$ (v_0 splits in L). As $\bar\omega$ is unramified at p and non-trivial on $D_{\bar v_0}\cap G_{L^+}$, $\bar\omega(\phi)\neq 1$. By Step 0 the characteristic polynomial of $\rho(\phi)$ is congruent to $(X-1)^n$ modulo T ($\bar\Psi(\phi)=1$), so its roots $\mu_i\in\bar F$, integral over A, have residue 1 for the valuation extending the T-adic one, while the roots $\chi_{\mathrm{cyc}}(\phi)\mu_i$ of the characteristic polynomial of $\chi_{\mathrm{cyc}}(\phi)\rho(\phi)$ have residue $\bar\omega(\phi)\neq 1$. So the two matrices have no common eigenvalue, and by Sylvester's theorem $\chi_{\mathrm{cyc}}(\phi)\rho(\phi)X=X\rho(\phi)$ forces $X=0$; a fortiori (Pj3).

Step 6: (Pj2) and the type. Fix $v\mid\ell$, u and $\bar u$ as in Step 4, and $D=\pi_L(D_v)$. The characteristic ℓ of F is odd. By Step 0, $\rho(\gamma_\star)-\rho(\gamma'_\star)=h(\iota(\zeta))\,\mathrm{diag}(w_{n,a})$ with $h(\iota(\zeta))\neq 0$, and $\rho(\gamma^*)=J_n\,{}^t\rho(\gamma)J_n$. The prime $\mathfrak{p}$ is off-triangular, so some entry $\rho_{\mathfrak{r}}(\gamma_1)_{ab}$ with $a>b$ is not in $\mathfrak{p}$, and $\rho(\gamma_1)_{ab}\neq 0$; likewise above the diagonal.

Residual trace condition. Suppose $\gamma\in\Gamma_L^0$ and $\emptyset\neq S\subseteq\{1,\dots,n\}$ is $\{1,\dots,n\}$ or (when $n=3$) $\{1,3\}$, with $\sum_{a\in S}\rho(\gamma\pi_L(g))_{aa}=0$ for all $g\in D_v$. The left side is $h\iota$ of an element of W with residue $\sum_{a\in S}\bar\Psi(\gamma)^{w_{n,a}}\bar\Psi(g)^{w_{n,a}}$ (Step 0), so this vanishes in k for all $g\in D_v$; by Artin's theorem (the $\bar\Psi^{w_{n,a}}|_{D_v}$ are distinct) $\bar\Psi(\gamma)^{w_{n,a}}=0$, absurd. So no such γ exists.

Case of irreducible type ($n=2$, or $n=3$ and some entry of odd weight of $\rho_{\mathfrak{r}}$ is not in $\mathfrak{p}$). By Lemma 50.17 (a), $\rho\otimes\bar F$ is irreducible. Let $U\subseteq\Gamma_L^0$ be an open subgroup, U_0 its normal core (open, of index m) and $N=U_0\cap\Gamma'$, normal in Γ_L^0. Apply Lemma 50.18 with $\Delta=\Gamma_L^0$, N, D and $t=\bar u^m\in D\cap N$: hypothesis 1 is Step 3, hypothesis 2 holds because $\rho(t)$ has the eigenvalues $\psi_{v,a}(u)^m$, pairwise distinct as $1+TA$ is torsion-free, and hypothesis 3 is the residual trace condition with $S=\{1,\dots,n\}$. So $\rho\otimes\bar F$ is irreducible on N, hence on $U\cap\Gamma'$; with $U=\Gamma_L^0$ the curve point is of irreducible type, and (Sx1) of Lemma 50.19 holds for $\Delta=\Gamma_L^0$ (profinite, open in Γ_L), $\Delta'=\Gamma'$ (closed, normal) and $\bar u$. That lemma gives (Pj2).

Case of endoscopic type ($n=3$, all entries of odd weight of $\rho_{\mathfrak{r}}$ in $\mathfrak{p}$). Then all entries of odd weight of ρ vanish, and by Lemma 50.17 (b), r_1 (the $\{1,3\}$-corner) and $r_2=\rho_{22}$ are homomorphisms and $r_1\otimes\bar F$ is irreducible. Here $F^3=(Fe_1+Fe_3)\oplus Fe_2$ is a decomposition into $\rho(D)$-stable summands; $r_2|_{D_v}$ is one of the composition factors $\psi_{v,1},\psi_{v,2},\psi_{v,3}$ of $\rho|_{D_v}$ (Step 3), and its residue is 1 (Step 0) while those of $\psi_{v,1},\psi_{v,3}$ are $\bar\Psi^{\pm1}|_{D_v}\neq 1$; so $r_2|_{D_v}=\psi_{v,2}$, and $r_1|_{D_v}$ has the composition factors $\psi_{v,1},\psi_{v,3}$, hence a D_v-stable line; and $r_1(\bar u^m)$ has the distinct eigenvalues $\psi_{v,1}(u)^m,\psi_{v,3}(u)^m$. Lemma 50.18 for r_1 (dimension 2), with the residual trace condition for $S=\{1,3\}$, shows that $r_1\otimes\bar F$ is irreducible on $U\cap\Gamma'$ for every open U. With Q the permutation matrix taking (e_1,e_3,e_2) to (e_1,e_2,e_3), $Q\rho Q^{-1}=\mathrm{diag}(r_1,r_2)$; $r_1(\gamma^*)=J_2\,{}^tr_1(\gamma)J_2$ and $r_2(\gamma^*)=r_2(\gamma)$ (the corner and centre of $J_3\,{}^tXJ_3$); $r_1\otimes\bar F$ and the character r_2 are irreducible on Γ'; and a non-zero $Y\in\mathrm{M}_{1\times2}(\bar F)$ with $Yr_1(\sigma)=r_2(\sigma)Y$ on Γ' would have a Γ'-stable kernel of dimension 1. So the curve point is of endoscopic type; (Sx2) of Lemma 50.19 holds with $\bar u$ ($\rho(\bar u)$ has three distinct eigenvalues generating a torsion-free group), and that lemma gives (Pj2).

Steps 2–6 prove (Np1) and (Np2). □

Lemma 50.17. *Let F be a field of characteristic $\neq 2$ with an algebraic closure $\bar{F}$, $n \in \{2,3\}$, Δ a group and $\rho\colon \Delta \to \mathrm{GL}_n(F)$ a homomorphism such that*

1. *for every $\gamma \in \Delta$ there is $\gamma' \in \Delta$ with $\rho(\gamma') = J_n\, {}^t\rho(\gamma)\, J_n$;*
2. *there are $g, g' \in \Delta$ and $\zeta \in F^\times$ with $\rho(g) - \rho(g') = \zeta \operatorname{diag}(w_{n,1}, \dots, w_{n,n})$;*
3. *$\rho(\gamma_1)_{ab} \neq 0$ for some $\gamma_1 \in \Delta$ and some $a > b$, and $\rho(\gamma_2)_{ab} \neq 0$ for some $\gamma_2 \in \Delta$ and some $a < b$.*

Then:

(a) *if $n = 2$, or if $n = 3$ and $\rho(\gamma_3)_{ab} \neq 0$ for some $\gamma_3 \in \Delta$ and some (a,b) with $w_{3,a} - w_{3,b}$ odd, the representation $\rho \otimes \bar{F}$ is irreducible;*

(b) *if $n = 3$ and $\rho(\gamma)_{ab} = 0$ for all $\gamma \in \Delta$ and all (a,b) with $w_{3,a} - w_{3,b}$ odd, then $r_1(\gamma) = \begin{pmatrix} \rho(\gamma)_{11} & \rho(\gamma)_{13} \\ \rho(\gamma)_{31} & \rho(\gamma)_{33} \end{pmatrix}$ defines a homomorphism $r_1\colon \Delta \to \mathrm{GL}_2(F)$, $\gamma \mapsto \rho(\gamma)_{22}$ a homomorphism $\Delta \to F^\times$, and $r_1 \otimes \bar{F}$ is irreducible.*

Proof. Write $V = \bar{F}^n$ with standard basis $e_1, \dots, e_n$, $J = J_n$, and $\beta(x,y) = {}^t x\, J\, y$, a non-degenerate symmetric bilinear form on V with $\beta(e_i, e_j) = 1$ if $i + j = n + 1$ and 0 otherwise.

Orthogonals of invariant subspaces are invariant. Let $U \subseteq V$ be stable under $\rho(\Delta)$ and $\delta \in \Delta$. By hypothesis 1 applied to δ there is δ' with $\rho(\delta') = J\, {}^t\rho(\delta) J$; as $J^2 = 1$, $\rho(\delta) = J\, {}^t\rho(\delta') J$. For $x \in U$ and $y \in U^\perp$ (orthogonal for β),

$$\beta(x, \rho(\delta) y) = {}^t x\, J\, J\, {}^t\rho(\delta')\, J\, y = {}^t(\rho(\delta') x)\, J\, y = \beta(\rho(\delta') x, y) = 0.$$

So $U^\perp$ is stable under $\rho(\Delta)$, and $\dim U^\perp = n - \dim U$.

Stable subspaces of the marking difference. $Z = \rho(g) - \rho(g') = \zeta \operatorname{diag}(w_{n,a})$ has the eigenvalues $\zeta w_{n,a}$, pairwise distinct because the $w_{n,a}$ are distinct integers with differences in $\{1,2\}$ and the characteristic is not 2. Hence every Z-stable subspace of V is spanned by some of the e_i; in particular a $\rho(\Delta)$-stable line is $\bar{F} e_i$ for some i.

(a). Suppose $U \subseteq V$ is $\rho(\Delta)$-stable with $0 \neq U \neq V$. As $n \leq 3$, one of U, $U^\perp$ is a stable line $\bar{F} e_i$. If $2i = n + 1$ (so $n = 3$, $i = 2$), then $e_2^\perp = \bar{F} e_1 + \bar{F} e_3$ is stable too, so $\rho(\gamma)_{12} = \rho(\gamma)_{32} = 0$ (from $\rho(\gamma) e_2 \in \bar{F} e_2$) and $\rho(\gamma)_{21} = \rho(\gamma)_{23} = 0$ (from the stability of $\bar{F} e_1 + \bar{F} e_3$) for all γ: all entries of odd weight vanish, contrary to the hypothesis of (a). If $i = 1$: $\rho(\gamma) e_1 \in \bar{F} e_1$ gives $\rho(\gamma)_{a1} = 0$ for $a > 1$, and the stability of $e_1^\perp = \bar{F} e_1 + \dots + \bar{F} e_{n-1}$ gives $\rho(\gamma)_{nb} = 0$ for $b < n$; for $n = 2$ and $n = 3$ these are all the entries below the diagonal, contrary to hypothesis 3. If $i = n$: symmetrically $\rho(\gamma)_{an} = 0$ for $a < n$ and, from $e_n^\perp = \bar{F} e_2 + \dots + \bar{F} e_n$, $\rho(\gamma)_{1b} = 0$ for $b > 1$, which are all entries above the diagonal, again contrary to hypothesis 3. So $\rho \otimes \bar{F}$ is irreducible.

(b). The positions of odd weight for $n = 3$ are $(1,2), (2,1), (2,3), (3,2)$. If these entries vanish, every $\rho(\gamma)$ preserves $\bar{F} e_1 + \bar{F} e_3$ and $\bar{F} e_2$, so r_1 and $\gamma \mapsto \rho(\gamma)_{22}$ are homomorphisms. We check the hypotheses of (a) for r_1 with $n = 2$, identifying the positions $1, 3$ with $1, 2$. Since $(J_3\, {}^t X\, J_3)_{ab} = X_{4-b,4-a}$, restricting to $a, b \in \{1,3\}$ gives $r_1(\gamma') = J_2\, {}^t r_1(\gamma) J_2$ (hypothesis 1). The $\{1,3\}$-corner of $\zeta \operatorname{diag}(1,0,-1)$ is $\zeta \operatorname{diag}(w_{2,1}, w_{2,2})$ (hypothesis 2). The entries of ρ below the diagonal are at $(2,1), (3,1), (3,2)$, of which only $(3,1)$ has even weight, so $\rho(\gamma_1)_{31} \neq 0$; likewise $\rho(\gamma_2)_{13} \neq 0$ (hypothesis 3). By (a) for $n = 2$, $r_1 \otimes \bar{F}$ is irreducible. □

Lemma 50.18. *Let $\bar{F}$ be an algebraically closed field, n a prime number, Δ a group, $\rho\colon \Delta \to \mathrm{GL}_n(\bar{F})$ an irreducible representation, N a normal subgroup of Δ and $D \subseteq \Delta$ a subgroup. Assume:*

1. *there are $\rho(D)$-stable subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = \bar{F}^n$ with $\dim V_k = k$;*

2. *there is $t \in D \cap N$ such that $\rho(t)$ has n pairwise distinct eigenvalues;*

3. *there is no $\gamma \in \Delta$ with $\operatorname{tr} \rho(\gamma g) = 0$ for all $g \in D$.*

Then the restriction of ρ to N is irreducible.

Proof. Let $V = \bar{F}^n$. By Clifford's theorem (A. H. Clifford, Ann. of Math. 38 (1937)), V is a semisimple $\bar{F}[N]$-module, its isotypic components $W_1, \dots, W_s$ are permuted transitively by $\rho(\Delta)$, and they have a common dimension n/s; each W_i is the sum of e copies of an irreducible $\bar{F}[N]$-module σ_i, $\dim \sigma_i$ independent of i. As n is prime, $s \in \{1, n\}$.

Case $s = 1$. Then $V|_N \cong \sigma^e$ with $e \dim \sigma = n$. If $e = 1$, $V|_N$ is irreducible, as claimed. Otherwise $e = n$, $\dim \sigma = 1$, and $\rho(h)$ is a scalar for every $h \in N$; this is excluded by $t \in N$ (hypothesis 2, $n \geq 2$).

Case $s = n$. Then $W_i = L_i$ are lines, and $\rho(\Delta)$ permutes $\{L_1, \dots, L_n\}$ transitively. As $t \in N$, $\rho(t)$ preserves each L_i; since its n eigenvalues are distinct, $L_1, \dots, L_n$ are exactly its eigenlines, and every $\rho(t)$-stable subspace is the sum of the L_i it contains. As $t \in D$, each V_k is $\rho(t)$-stable, so $V_k = \bigoplus_{i \in S_k} L_i$ with $S_1 \subset S_2 \subset \cdots \subset S_n$, $|S_k| = k$; let i_k be the element of $S_k \smallsetminus S_{k-1}$. Let $g \in D$. We show by induction on k that $\rho(g) L_{i_k} = L_{i_k}$. The line $\rho(g) L_{i_k}$ is some L_j and lies in $\rho(g) V_k = V_k$, so $j \in S_k$; if $j \in S_{k-1}$, then $L_{i_k} \subseteq \rho(g)^{-1} V_{k-1} = V_{k-1}$, which is false; so $j = i_k$. Hence D lies in the kernel Δ_0 of the permutation action of Δ on $\{L_1, \dots, L_n\}$.

The image of Δ in the symmetric group of $\{L_1, \dots, L_n\}$ is transitive, so its order is divisible by n; by Cauchy's theorem it contains an element of order n, which is an n-cycle because n is prime. Let $\gamma \in \Delta$ map to it. For $g \in D \subseteq \Delta_0$, γg acts on the lines by the same n-cycle, so in a basis $f_i \in L_i$ the matrix of $\rho(\gamma g)$ is monomial with a permutation without fixed points, and its diagonal entries vanish: $\operatorname{tr} \rho(\gamma g) = 0$ for all $g \in D$. This contradicts hypothesis 3. So the case $s = n$ does not occur, and $\rho|_N$ is irreducible. □

Lemma 50.19. *Let k be a field, $A = k[[T]]$, $F = \operatorname{Frac}(A)$ with an algebraic closure $\bar{F}$, $n \in \{2, 3\}$, Δ a profinite group, $\Delta' \subseteq \Delta$ a closed normal subgroup and $\rho \colon \Delta \to \mathrm{GL}_n(A)$ a homomorphism whose entries are continuous for the T-adic topology. Assume that there is $u \in \Delta'$ such that $\rho(u)$ has n pairwise distinct eigenvalues in $\bar{F}$ generating a torsion-free subgroup of $\bar{F}^\times$, and that one of the following holds:*

(Sx1) *for every open subgroup U of Δ, the restriction of $\rho \otimes \bar{F}$ to $U \cap \Delta'$ is irreducible;*

(Sx2) *$n = 3$, $\rho(\gamma)_{ab} = 0$ for all $\gamma \in \Delta$ whenever $w_{3,a} - w_{3,b}$ is odd, and for every open subgroup U of Δ the restriction to $U \cap \Delta'$ of $r_1 \otimes \bar{F}$ is irreducible, where $r_1(\gamma) = \begin{pmatrix} \rho(\gamma)_{11} & \rho(\gamma)_{13} \\ \rho(\gamma)_{31} & \rho(\gamma)_{33} \end{pmatrix}$.*

Then every open subset of Δ meeting Δ' contains some $\sigma \in \Delta'$ such that the characteristic polynomial of $\rho(\sigma)$ is separable over F.

Proof. Let $\mathcal{U} \subseteq \Delta$ be open with $s \in \mathcal{U} \cap \Delta'$, and put $y = \rho(s)$.

Step 1: a subgroup. Open normal subgroups form a basis of neighbourhoods of 1 in Δ, so there is an open normal V with $sV \subseteq \mathcal{U}$. Put $G_1 = V \cap \Delta'$, a closed subgroup of Δ; then $sh \in \mathcal{U} \cap \Delta'$ for all $h \in G_1$. Let $G_2 \subseteq G_1$ be the subgroup N of Lemma 50.20 for $L = F$ and the restriction of ρ to G_1 (so G_2 is the preimage of the identity component of the Zariski closure H of $\rho(G_1)$ in $\mathrm{GL}_n(\bar{F})$). For $f \in F[x_{ij}]$ the map $g \mapsto f(\rho(g))$ is continuous for the T-adic topology (the entries of ρ are, and f is a polynomial), so its zero set is closed in G_1; by (Zi1) and (Zi4),

G_2 is an open normal subgroup of finite index of G_1. An open subgroup of $G_1 = V \cap \Delta'$ contains $U_2 \cap \Delta'$ for some open subgroup U_2 of Δ, so by (Sx1), resp. (Sx2), $\rho \otimes \bar{F}$, resp. $r_1 \otimes \bar{F}$, is irreducible on G_2. As U_2 has finite index in the profinite Δ, G_2 has finite index in Δ', and a subgroup of index m contains the $m!$-th power of every element (action on cosets); so $u^N \in G_2$ for some $N \geq 1$, as $u \in \Delta'$.

Step 2: eigenbasis. Choose $P \in \mathrm{GL}_n(\bar{F})$ with $P^{-1}\rho(u)P = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$; in case (Sx2), $\rho(u)$ preserves $\bar{F}e_1 + \bar{F}e_3$ and $\bar{F}e_2$, and we take the first and third columns of P in $\bar{F}e_1 + \bar{F}e_3$ and the second equal to e_2. Put $\beta_i = \alpha_i^N$ and $D = \mathrm{diag}(\beta_1, \ldots, \beta_n)$. For $i \neq j$, $\beta_i/\beta_j = (\alpha_i/\alpha_j)^N$ is a non-trivial element ($\alpha_i \neq \alpha_j$, and the group generated by the α's is torsion-free) of a torsion-free group, so it has infinite order; in particular the β_i are pairwise distinct. For $b \in G_2$ and $q \geq 0$, $\sigma = s\, b\, u^{Nq} \in \mathcal{U} \cap \Delta'$, and $P^{-1}\rho(\sigma)P = M(b)D^q$ with $M(b) = P^{-1}y\rho(b)P$.

Step 3: the leading term of the discriminant. Let $w_1, \ldots, w_n$ be the pairwise distinct integers given by Lemma 50.21 for $E = \bar{F}$ and $\beta_1, \ldots, \beta_n$, and order them as $\mu_1 = w_{\tau(1)} > \mu_2 = w_{\tau(2)} > \ldots$. Let X be an indeterminate and $M \in \mathrm{M}_n(\bar{F})$. The characteristic polynomial of $M\,\mathrm{diag}(X^{w_1}, \ldots, X^{w_n})$ is $\sum_{j=0}^{n}(-1)^j c_j Y^{n-j}$ with $c_j = \sum_{|S|=j} m_S(M)\, X^{w_S}$, where S runs over j-subsets of $\{1, \ldots, n\}$, m_S is the principal minor on S and $w_S = \sum_{i \in S} w_i$ (multiplying column i by X^{w_i} multiplies each principal minor containing i by X^{w_i}). Its discriminant is $\delta = c_1^2 - 4c_2$ for $n = 2$, and $\delta = c_1^2c_2^2 - 4c_2^3 - 4c_1^3c_3 + 18c_1c_2c_3 - 27c_3^2$ for $n = 3$. Expanding the products of the c_j subset by subset, every term is a product of principal minors times X^e, e the sum of the w_S chosen. A term of c_j has exponent at most μ_1 for $j = 1$, $\mu_1 + \mu_2$ for $j = 2$, $\mu_1 + \mu_2 + \mu_3$ for $j = 3$, with equality only for $S = \{\tau(1)\}$, $\{\tau(1), \tau(2)\}$, $\{\tau(1), \tau(2), \tau(3)\}$ respectively. Hence for $n = 2$ the terms of c_1^2 have exponent at most $2\mu_1$, with equality only for $S = S' = \{\tau(1)\}$, and those of c_2 exponent $\mu_1 + \mu_2 < 2\mu_1$. For $n = 3$ the terms of $c_1^2c_2^2$ have exponent at most $4\mu_1 + 2\mu_2$, with equality only when both factors c_1 contribute $\{\tau(1)\}$ and both factors c_2 contribute $\{\tau(1), \tau(2)\}$; the other products have exponents at most $3\mu_1 + 3\mu_2$, $4\mu_1 + \mu_2 + \mu_3$, $3\mu_1 + 2\mu_2 + \mu_3$, $2\mu_1 + 2\mu_2 + 2\mu_3$, all smaller than $4\mu_1 + 2\mu_2$ because $\mu_1 > \mu_2 > \mu_3$. So δ, a Laurent polynomial in X, has the top term $C(M)X^{2\mu_1}$, resp. $C(M)X^{4\mu_1+2\mu_2}$, with

$$C(M) = M^2_{\tau(1)\tau(1)} \ (n = 2), \qquad C(M) = M^2_{\tau(1)\tau(1)}\, m_{\{\tau(1),\tau(2)\}}(M)^2 \ (n = 3).$$

Hence: if $C(M(b)) \neq 0$, the characteristic polynomial of $M(b)\,\mathrm{diag}(X^{w_1}, \ldots, X^{w_n})$ is separable over $\bar{F}(X)$, and by Lemma 50.21 there is $q \geq 0$ such that the characteristic polynomial of $M(b)D^q = P^{-1}\rho(\sigma)P$, $\sigma = s\, b\, u^{Nq}$, is separable; it is the characteristic polynomial of $\rho(\sigma)$, which has coefficients in F, so it is separable over F.

Step 4: a good b. It remains to find $b \in G_2$ with $C(M(b)) \neq 0$. For indices i put $f_i = Pe_i$ and $\varphi_i = e_i^t P^{-1} y$, a non-zero row vector; then $M(b)_{ii} = \varphi_i\, \rho(b)\, f_i$.

(i) A diagonal entry. If $\varphi_i \rho(b) f_i = 0$ for all $b \in G_2$, the span of $\rho(G_2)f_i$ is a non-zero G_2-stable subspace contained in $\ker \varphi_i$, hence proper, contradicting irreducibility on G_2. In case (Sx2) with $i \in \{1, 3\}$, $f_i \in \bar{F}e_1 + \bar{F}e_3$ and φ_i vanishes on e_2 (as y and P are block diagonal), so the same argument applies to r_1 on G_2.

(ii) Case (Sx1), $n = 2$: $C = M^2_{\tau(1)\tau(1)}$, non-zero for some b by (i).

(iii) Case (Sx1), $n = 3$. Put $Q_1(x) = (P^{-1}yxP)_{\tau(1)\tau(1)}$ and $Q_2(x) = m_{\{\tau(1),\tau(2)\}}(P^{-1}yxP)$, polynomials in the entries of x with coefficients in $\bar{F}$. $Q_1 \circ \rho$ is not identically zero on G_2 by (i). As Λ^2 is multiplicative, $Q_2(x)$ is the coefficient of $e_{\tau(1)} \wedge e_{\tau(2)}$ in $\Lambda^2(P^{-1}y)\Lambda^2(x)\Lambda^2(P)(e_{\tau(1)} \wedge e_{\tau(2)})$, a matrix coefficient of Λ^2; and $\Lambda^2(\rho \otimes \bar{F})$ is irreducible on G_2, since $\Lambda^2 V \cong V^* \otimes \Lambda^3 V$ equivariantly for $\dim V = 3$ (through the perfect pairing $V \otimes \Lambda^2 V \to \Lambda^3 V$) and the dual of an irreducible representation is irreducible. So the argument of (i) applied to $\Lambda^2\rho$ shows that $Q_2 \circ \rho$ is not identically zero on G_2. By (Zi3) of Lemma 50.20 (group G_1, its subgroup G_2 as N, $U = G_1$, $\sigma = 1$, polynomials Q_1, Q_2) there is $b \in G_2$ with $C(M(b)) = Q_1(\rho(b))^2 Q_2(\rho(b))^2 \neq 0$.

(iv) Case (Sx2). Here $M(b)$ is block diagonal: a 2×2 block $A(b)$ on the indices $\{1,3\}$ and the entry $m(b) = (y\rho(b))_{22} \neq 0$. If $\tau(1) = 2$, $C = m^2(m\,A_{\tau(2)\tau(2)})^2$; if $\tau(1) \neq 2 = \tau(2)$, $C = A^2_{\tau(1)\tau(1)}(A_{\tau(1)\tau(1)}m)^2$; if $\{\tau(1),\tau(2)\} = \{1,3\}$, $C = A^2_{\tau(1)\tau(1)}(\det A)^2$ with $\det A \neq 0$. In each case $C(M(b)) \neq 0$ as soon as one prescribed diagonal entry $A_{jj}(b) = M(b)_{jj}$, $j \in \{1,3\}$, is non-zero, which holds for some $b \in G_2$ by (i).

By Step 3, $\sigma = s\,b\,u^{Nq}$ for suitable $q \geq 0$ is the required element. □

Lemma 50.20. *Let L be a field with an algebraic closure $\bar{L}$, $n \geq 1$, G a group and $\rho\colon G \to \mathrm{GL}_n(L)$ a homomorphism. Let $H \subseteq \mathrm{GL}_n(\bar{L})$ be the Zariski closure of $\rho(G)$, H° its identity component, and $N = \{g \in G : \rho(g) \in H^\circ\}$. Then:*

(Zi1) *N is a normal subgroup of finite index of G;*

(Zi2) *for every subgroup U of finite index of G, $\rho(U \cap N)$ is Zariski dense in H°;*

(Zi3) *for every subgroup U of finite index of G, every $\sigma \in G$ and all polynomials $f_1, \ldots, f_r \in \bar{L}[x_{ij} : 1 \leq i,j \leq n]$ such that for each k there is $h \in N$ with $f_k(\rho(\sigma h)) \neq 0$, there is $g \in U \cap N$ with $f_k(\rho(\sigma g)) \neq 0$ for $k = 1, \ldots, r$;*

(Zi4) *if G is a topological group and for every $f \in L[x_{ij} : 1 \leq i,j \leq n]$ the set $\{g \in G : f(\rho(g)) = 0\}$ is closed in G, then N is open and closed in G.*

Proof. We use the basic facts on closed subgroups of $\mathrm{GL}_n(\bar{L})$ and their identity components from [41, Ch. I, 1.2–1.3]. *(Zi1).* N is the preimage of the normal subgroup H° of H under the homomorphism $\rho\colon G \to H$, so it is normal, and $gN \mapsto \rho(g)H^\circ$ is an injective map $G/N \to H/H^\circ$, so $[G : N] \leq [H : H^\circ] < \infty$.

$\rho(N)$ is dense in H°. Let $h_1H^\circ, \ldots, h_sH^\circ$ be the cosets of H° in H, with $h_1 = 1$. Then $\rho(G) = \bigcup_k (\rho(G) \cap h_kH^\circ)$, and the closure of $\rho(G) \cap h_kH^\circ$ lies in the closed set h_kH°. So H is the union of these closures, and as the cosets are disjoint, H° lies in the closure of $\rho(G) \cap H^\circ = \rho(N)$.

(Zi2). $U \cap N$ has finite index in N (at most $[G : U]$); let $n_1, \ldots, n_m \in N$ represent its cosets in N. The closure C of $\rho(U \cap N)$ is a closed subgroup contained in H°. Then $\rho(N) \subseteq \bigcup_k \rho(n_k)C$, a finite union of closed sets, so $H^\circ \subseteq \bigcup_k \rho(n_k)C \subseteq H^\circ$ by the previous paragraph. As H° is irreducible, $H^\circ = \rho(n_k)C$ for some k, and since $\rho(n_k) \in H^\circ$, $C = \rho(n_k)^{-1}H^\circ = H^\circ$.

(Zi3). Put $\varphi_k(x) = f_k(\rho(\sigma)x)$, a polynomial in the entries of x. By hypothesis $\varphi_k(\rho(h)) \neq 0$ for some $h \in N$, and $\rho(h) \in H^\circ$; so the zero set of φ_k in H° is a proper closed subset. As H° is irreducible, the union of these finitely many proper closed subsets is proper, i.e. $\varphi = \varphi_1 \cdots \varphi_r$ does not vanish identically on H°. If φ vanished on $\rho(U \cap N)$, it would vanish on its closure, which is H° by (Zi2). So there is $g \in U \cap N$ with $\prod_k f_k(\rho(\sigma g)) = \varphi(\rho(g)) \neq 0$.

(Zi4). H° is closed in $\mathrm{GL}_n(\bar{L})$, so it is the set of invertible zeros of finitely many polynomials $f'_1, \ldots, f'_s \in \bar{L}[x_{ij}]$. Their coefficients lie in a finite extension L' of L; choose an L-basis $e_1, \ldots, e_d$ of L' and write $f'_k = \sum_m e_m f_{k,m}$ with $f_{k,m} \in L[x_{ij}]$. For $x \in \mathrm{M}_n(L)$ the values $f_{k,m}(x)$ lie in L, so $f'_k(x) = 0$ if and only if $f_{k,m}(x) = 0$ for all m. Hence $N = \bigcap_{k,m}\{g : f_{k,m}(\rho(g)) = 0\}$ is closed. Its complement is the union of the finitely many cosets $gN \neq N$ (by (Zi1)), each closed because left translation is a homeomorphism of G; so N is also open. □

Lemma 50.21. *Let E be a field, $n \geq 1$ and $\beta_1, \ldots, \beta_n \in E^\times$ such that β_i/β_j has infinite order in $E^\times$ for all $i \neq j$. Then there are pairwise distinct integers $w_1, \ldots, w_n$ with the following property. Let X be an indeterminate and $M \in \mathrm{M}_n(E)$. If the characteristic polynomial of $M\,\mathrm{diag}(X^{w_1}, \ldots, X^{w_n}) \in \mathrm{M}_n(E(X))$ is separable over $E(X)$, then there is an integer $q \geq 0$ such that the characteristic polynomial of $M\,\mathrm{diag}(\beta_1^q, \ldots, \beta_n^q)$ is separable over E.*

Proof. Step 1: the weights. For $m \in \mathbb{Z}^n$ write $\beta^m = \prod_i \beta_i^{m_i}$. Let $\Lambda = \{m \in \mathbb{Z}^n : (\beta^m)^k = 1 \text{ for some } k \geq 1\}$, a subgroup (if $(\beta^m)^k = 1 = (\beta^{m'})^{k'}$ then $(\beta^{m+m'})^{kk'} = 1$). The quotient $\mathbb{Z}^n/\Lambda$ is torsion-free (if $km \in \Lambda$ with $k \geq 1$, then $m \in \Lambda$) and finitely generated, hence free; fix an isomorphism with $\mathbb{Z}^r$ and let q_i be the image of the i-th unit vector. For $i \neq j$, $q_i \neq q_j$: otherwise $(\beta_i/\beta_j)^k = 1$ for some $k \geq 1$. Let R exceed the absolute values of all coordinates of all $q_i - q_j$ and $\lambda(y) = \sum_{l=1}^r R^{l-1} y_l$. For $y \neq 0$ with all $|y_l| < R$, let l_0 be the largest index with $y_{l_0} \neq 0$; then $|\sum_{l<l_0} R^{l-1} y_l| \leq (R-1)\sum_{l<l_0} R^{l-1} = R^{l_0-1} - 1 < |R^{l_0-1} y_{l_0}|$, so $\lambda(y) \neq 0$. Hence $w_i = \lambda(q_i)$ are pairwise distinct, and

$$\beta^m = 1 \implies m \in \Lambda \implies \textstyle\sum_i m_i w_i = \lambda(\sum_i m_i q_i) = 0. \qquad (*)$$

Step 2: the discriminant as a polynomial on the torus. Let $t_1, \dots, t_n$ be indeterminates and $\Delta \in E[t_1, \dots, t_n]$ the discriminant of the characteristic polynomial of $M \operatorname{diag}(t_1, \dots, t_n)$, whose coefficients lie in $E[t_1, \dots, t_n]$. Write $\Delta = \sum_{\mu \in S} c_\mu t^\mu$ with $S \subseteq \mathbb{Z}^n_{\geq 0}$ finite. Under $t_i \mapsto X^{w_i}$ (a ring homomorphism to $E(X)$) the characteristic polynomial becomes that of $M \operatorname{diag}(X^{w_1}, \dots, X^{w_n})$, so by hypothesis $\Delta(X^{w_1}, \dots, X^{w_n}) \neq 0$; under $t_i \mapsto \beta_i^q$ it becomes that of $M \operatorname{diag}(\beta_1^q, \dots, \beta_n^q)$.

Step 3: Vandermonde. Suppose $\Delta(\beta_1^q, \dots, \beta_n^q) = 0$ for all $q \geq 0$. Let $\theta_1, \dots, \theta_p$ be the distinct values of β^μ, $\mu \in S$, and $C_k = \sum_{\mu \in S,\ \beta^\mu = \theta_k} c_\mu$. Then $\sum_k C_k \theta_k^q = \Delta(\beta_1^q, \dots, \beta_n^q) = 0$ for $q = 0, \dots, p-1$, and the Vandermonde matrix (θ_k^q) is invertible, so all $C_k = 0$. If $\beta^\mu = \beta^{\mu'}$, then $\beta^{\mu-\mu'} = 1$ and $\sum_i \mu_i w_i = \sum_i \mu_i' w_i$ by $(*)$; so all μ with $\beta^\mu = \theta_k$ give the same power X^{e_k}, $e_k = \sum_i \mu_i w_i$, and $\Delta(X^{w_1}, \dots, X^{w_n}) = \sum_k C_k X^{e_k} = 0$, a contradiction. Hence $\Delta(\beta_1^q, \dots, \beta_n^q) \neq 0$ for some $q \geq 0$, which is the claim. □

Lemma 50.22. *Let $\mathcal{O}'$ be a coefficient ring, L an auxiliary field of level C_0 for $(K, \mathcal{O}', \Psi, c, g_\star, M)$, $n \in \{2, 3\}$, $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ a universal polarised ordinary determinant and $(R_n^{\mathrm{po}}, \rho^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{u}}))$ a universal polarised ordinary lift of rank n over L (both for $\mathcal{O}'$) such that ϑ_n exists and $\vartheta_n(B_n^{\mathrm{po}}) \subseteq W_n^{\mathrm{po}}$. Let $\mathfrak{x}$ be a prime of E_n^{po}, $(\mathcal{O}'', R, \iota)$ a local chart at $\mathfrak{x}$ and $\varphi = \iota \circ \vartheta_n$.*

(Xk1) *For every prime P of R of irreducible type, $\varphi^{-1}(P)$ is a stable prime of B_n^{po}.*

(Xk2) *If $n = 3$, then for every thick prime $\mathfrak{e}$ of R, $\varphi^{-1}(\mathfrak{e})$ is a thick prime of B_n^{po}.*

Proof. Write $B = B_n^{\mathrm{po}}$, $W = W_n^{\mathrm{po}}$, $E = E_n^{\mathrm{po}}$ and $\Gamma = \Gamma_L^0$. *Preliminaries.* The ring $E/\mathfrak{x}$ embeds into the finite field $R/\mathfrak{m}_R$, so $W/(\mathfrak{x} \cap W)$ is a domain inside a finite field, hence a field, and $\mathfrak{x} \cap W = \mathfrak{m}_W$; thus $\varphi(\mathfrak{m}_B) \subseteq \iota(\mathfrak{m}_W) \subseteq \mathfrak{m}_R$ (ϑ_n is local) and $\iota(w)$ is a unit for every unit w of W. By (Po3) and (Po2), $\rho^{\mathrm{u}}(\pi_L(g_\star)) - \rho^{\mathrm{u}}(\pi_L(c g_\star^{-1} c)) = \zeta \operatorname{diag}(w_{n,a})$ with $\zeta = \rho^{\mathrm{u}}(\pi_L(g_\star))_{11} - \rho^{\mathrm{u}}(\pi_L(g_\star))_{nn} \in W$ (diagonal entries have weight 0), whose residue $\bar\Psi(g_\star) - \bar\Psi(g_\star)^{-1}$ is non-zero ($g_\star$ is a marking, (Po5)); so $\iota(\zeta) \in R^\times$. Since ϑ_n classifies $(\det \circ \rho^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{u}}))$ and takes values in $W \subseteq E$, (Ch1) gives $\varphi \circ D^{\mathrm{u}} = \det \circ \rho_{\mathfrak{x}}$, and $\varphi \circ \psi_{v,a}^{\mathrm{d}} = \psi_{\mathfrak{x},v,a}$. For a prime Q of R and $\mathfrak{q} = \varphi^{-1}(Q)$, φ induces $B/\mathfrak{q} \hookrightarrow R/Q$; fix an algebraic closure K_Q of $\operatorname{Frac}(R/Q)$ and an embedding $\bar\kappa(\mathfrak{q}) \hookrightarrow K_Q$ extending it. Then $D_{\mathfrak{q}}$ base-changed to K_Q equals $\det \circ (\rho_{\mathfrak{x}} \bmod Q) \otimes K_Q$.

(Xk1). Let P be of irreducible type and $\mathfrak{q} = \varphi^{-1}(P)$. The field $\operatorname{Frac}(R/P)$ has characteristic 0 or ℓ, odd. Lemma 50.17 (a) applies to $\rho_P = \rho_{\mathfrak{x}} \bmod P$: hypothesis 1 is (Po2), hypothesis 2 holds with $\zeta = \iota(\zeta) \bmod P \neq 0$, and hypothesis 3 and the condition on odd entries hold because P is of irreducible type. So $\rho_P \otimes K_P$ is irreducible. By (Ch3) there is a semisimple $r \colon \Gamma \to \mathrm{GL}_n(\bar\kappa(\mathfrak{q}))$ with $D_{\mathfrak{q}} = \det \circ r$. Over the algebraically closed field $\bar\kappa(\mathfrak{q})$, r is a sum of absolutely irreducible representations, so $r \otimes K_P$ is semisimple, with $\det \circ (r \otimes K_P) = \det \circ (\rho_P \otimes K_P)$; by the uniqueness in (Ch3) over K_P, $r \otimes K_P \cong \rho_P \otimes K_P$, which is irreducible; so r is irreducible and $\mathfrak{q}$ is stable.

(Xk2), not stable. Let $n = 3$, $\mathfrak{e}$ thick with $r\colon \Gamma \to \mathrm{M}_3(R/\mathfrak{e})$ as in (Tk1)–(Tk3), $\mathfrak{q} = \varphi^{-1}(\mathfrak{e})$, $\kappa_0 = \bar{\kappa}(\mathfrak{q}) \subseteq K' = K_{\mathfrak{e}}$. As $\mathfrak{e}$ is of endoscopic type, $\rho_{\mathfrak{e}} = \rho_{\mathfrak{x}} \bmod \mathfrak{e}$ has vanishing entries of odd weight and preserves $\langle e_1, e_3\rangle$ and $\langle e_2\rangle$: it is reducible over K'. If $D_{\mathfrak{q}} = \det \circ r_0$ with r_0 irreducible over κ_0, then $r_0 \otimes K'$ is irreducible (absolutely irreducible), and (Ch3) over K' identifies it with the semisimplification of $\rho_{\mathfrak{e}} \otimes K'$, which is reducible: contradiction. So $\mathfrak{q}$ is not stable.

(Xk2), r kills $\mathcal{I}$. Extend r to a ring homomorphism $B[\Gamma] \to \mathrm{M}_3(R/\mathfrak{e})$ through $B \to R/\mathfrak{e}$ (and to polynomial base changes). By (Tk1) and the vanishing of the odd entries of $\rho_{\mathfrak{e}}$, every $r(\gamma)$ preserves $\langle e_1, e_3\rangle$ and has the same $\{1,3\}$-corner and $(2,2)$-entry as $\rho_{\mathfrak{e}}(\gamma)$; block upper triangular matrices form an algebra on which taking the diagonal blocks is multiplicative, so for every $y \in (R/\mathfrak{e})[t_1, \dots, t_m][\Gamma]$ the characteristic polynomial of $r(y)$ equals that of the block-diagonal $\rho_{\mathfrak{e}}(y)$, i.e. $\chi_{D'}(y)$ for the base change D' of D^{u}. By the Cayley–Hamilton theorem $\chi_{D'}(y)(r(y)) = 0$, so r kills the Cayley–Hamilton ideal. By (Tk2), and $\varphi \circ \psi^{\mathrm{d}}_{v,a} = \psi_{\mathfrak{x},v,a}$, r kills $\prod_{a=1}^{3}(\pi_L(g_a) - \psi^{\mathrm{d}}_{v,a}(g_a))$ and its image under $*$, which is $\prod_{a=1}^{3}(\pi_L(ch_a^{-1}c) - \psi^{\mathrm{d}}_{v,4-a}(h_a))$ with $h_a = g_{4-a}$ ($*$ reverses products and $\pi_L(g)^* = \pi_L(cg^{-1}c)$). For $k' \geq 1$, $r(\mathfrak{m}_B^{k'}) \subseteq \mathfrak{m}^{k'}\mathrm{M}_3(R/\mathfrak{e})$ with $\mathfrak{m}$ the maximal ideal of $R/\mathfrak{e}$; and since the entries of r are continuous and $(R/\mathfrak{e})/\mathfrak{m}^{k'}$ is finite, there is an open normal subgroup $H'_{k'}$ with $r(h) \equiv 1$ modulo $\mathfrak{m}^{k'}$ for $h \in H'_{k'}$. Every $t \in \mathcal{I}$ lies in the two-sided ideal generated by the four kinds of elements for k' and $H'_{k'}$, so $r(t) \in \mathfrak{m}^{k'}\mathrm{M}_3(R/\mathfrak{e})$ for every k', and $r(t) = 0$ by Krull's intersection theorem.

(Xk2), non-semisimplicity. Let $\mathcal{A} = \kappa_0[\Gamma]/\langle\mathcal{I}_0\rangle$, $\mathcal{I}_0$ the image of $\mathcal{I}$, and suppose $\mathcal{A}$ semisimple. By (Ch3), $D_{\mathfrak{q}} = \det \circ \rho_0$ with ρ_0 semisimple over κ_0; let $\mathfrak{K}$ be the kernel of $\rho_0\colon \kappa_0[\Gamma] \to \mathrm{M}_3(\kappa_0)$. For $y_0 = \sum_i c_i\gamma_i \in \kappa_0[\Gamma]$, $\chi_{D_{\mathfrak{q}}}(y_0)(y_0)$ is the specialisation $t_i \mapsto c_i$ of the image of $\chi_{D'}(\sum_i t_i\gamma_i)(\sum_i t_i\gamma_i)$, a κ_0-combination of images of Cayley–Hamilton generators, which lie in $\mathcal{I}$; so it vanishes in $\mathcal{A}$. For $x \in \mathfrak{K}$ and $y \in \kappa_0[\Gamma]$, $\chi_{D_{\mathfrak{q}}}(xy) = \det(X - \rho_0(xy)) = X^3$, hence $(xy)^3 = 0$ in $\mathcal{A}$: the image of $\mathfrak{K}$ is a nil two-sided ideal, contained in the Jacobson radical of $\mathcal{A}$, which is 0. So $\mathcal{A}$ is a quotient of $\kappa_0[\Gamma]/\mathfrak{K} \cong \rho_0(\kappa_0[\Gamma]) \subseteq \mathrm{M}_3(\kappa_0)$, finite-dimensional and semisimple over the algebraically closed κ_0, hence a product of matrix algebras over κ_0 (Wedderburn), and $\mathcal{A} \otimes_{\kappa_0} K' = K'[\Gamma]/\langle\mathcal{I}_0\rangle$ is semisimple. By the previous paragraph and the compatibility of $\kappa_0 \hookrightarrow K'$ with $B/\mathfrak{q} \hookrightarrow R/\mathfrak{e}$, $\sigma = r \otimes K'$ makes K'^3 a module over $\mathcal{A} \otimes K'$, hence a semisimple $K'[\Gamma]$-module. Its submodule $\langle e_1, e_3\rangle$ (stable as $r_{21} = r_{23} = 0$) then has a stable complement $K'y$ with $y_2 = 1$; comparing second coordinates in $\sigma(\gamma)y = \lambda(\gamma)y$ gives $\lambda(\gamma) = \sigma(\gamma)_{22}$, i.e. $\sigma(\gamma)y = \sigma(\gamma)_{22}y$ for all γ, contradicting (Tk3) with $F' = K'$. So $\mathcal{A}$ is not semisimple, and $\mathfrak{q}$ is thick. □

Lemma 50.23. *Let $\mathcal{O}'$ be a coefficient ring, L an auxiliary field of level C_0 for $(K, \mathcal{O}', \Psi, c, g_\star, M)$ and $n \in \{2,3\}$, such that $\tau(\mathcal{O}_L) \subseteq \mathcal{O}'$ for every embedding $\tau\colon L \to \overline{\mathbb{Q}}_\ell$ and Ψ is trivial on $I_{\bar{p}}$ for every place $\bar{p}$ of $\overline{\mathbb{Q}}$ above a prime not dividing M. Let v_0 be a place of L^+ above a prime $p \nmid 2M\,|\mathrm{disc}(L)|$ which splits in L, and $\bar{v}_0$ a place of $\overline{\mathbb{Q}}$ above v_0 such that $\bar{\Psi}$ is trivial and $\bar{\omega}$ is non-trivial on $D_{\bar{v}_0} \cap G_{L^+}$. Let* St *be the set of places of L^+ not above ℓ that lie above a prime dividing M, and $\bar{\chi}_a = \bar{\Psi}^{w_{n,a}}|_{G_L}$ for $1 \leq a \leq n$. Then:*

(In1) *for every family of homomorphisms $\xi_{v,a}\colon \mu_v \to \mathcal{O}'^{\times}$ ($v \in S_\ell$, $1 \leq a \leq n$) satisfying (Rd10) (for instance $\xi_{v,a} = 1$), $\mathscr{H}_L = (n, L, c, (\bar{v})_{v\in S_\ell}, \mathrm{St}, v_0, \bar{v}_0, (\bar{\chi}_a)_{1\leq a\leq n}, (\xi_{v,a}))$ is a rank-n Hecke datum over $\mathcal{O}'$, and (In2)–(In4) hold for it;*

(In2) *$N_{\mathscr{H}_L} = N_L$; hence $\Gamma_{\mathscr{H}_L} = \Gamma_L$, $\pi_{\mathscr{H}_L} = \pi_L$ and $\Gamma^0_{\mathscr{H}_L} = \Gamma^0_L$, and the groups $D_v \supseteq I_v$ and the integers d_v of $\mathscr{H}_L$ are those of Definition 48.2;*

(In3) *every admissible twist κ of rank n for ι, c and M with $\kappa(G_L) \subseteq \mathcal{O}'^{\times}$ is a twist for $\mathscr{H}_L$;*

(In4) *if $\Psi\chi_{\mathrm{cyc}}^{-e}$ is trivial on an open subgroup of $I_{\bar\ell}$ for some integer $e \geq 1$, then*

$$\left(\Psi^{w_{n,a}}|_{I_v}\right)_{v\in S_\ell,\, 1\leq a\leq n}$$

is an inertial weight for $\mathscr{H}_L$.

Proof. Two elementary facts are used throughout. (F1) If a place u of L^+ splits in L and $\bar u$ is a place of $\overline{\mathbb{Q}}$ above u, then $D_{\bar u} \cap G_{L^+} \subseteq G_L$: this group is the Galois group of the completion of $\overline{\mathbb{Q}}$ at $\bar u$ over L_u^+, and as u splits, the place of L below $\bar u$ has completion L_u^+, so the group fixes L. Hence $D_{\bar u} \cap G_L = D_{\bar u} \cap G_{L^+}$. (F2) For $k \in \{\pm 1, \pm 2\}$ and an element x of an abelian group with $x^2 \neq 1$, $x^k \neq 1$. The differences $w_{n,a} - w_{n,b}$ $(a \neq b)$ lie in $\{\pm 1, \pm 2\}$, so by (F2), for every $h \in G_L$ with $\bar\Psi(h)^2 \neq 1$ the values $\bar\chi_a(h) = \bar\Psi(h)^{w_{n,a}}$, $1 \leq a \leq n$, are pairwise distinct.

Step 1: the places. Every place of L^+ above a prime dividing M splits in L by (Af2); since $\ell \mid M$, this applies to every place in $S_\ell \cup \mathrm{St}$, and v_0 splits by hypothesis. The set St is finite (finitely many places lie above the finitely many primes dividing M) and consists of places not above ℓ. As $p \nmid 2M$ and $\ell \mid M$, $p \neq 2, \ell$; so v_0 is not above ℓ, and $v_0 \notin \mathrm{St}$ because $p \nmid M$. As $p \nmid \mathrm{disc}(L)$, p is unramified in L. Conversely, if u is a finite place of L^+ not in $S_\ell \cup \mathrm{St}$, the prime q below u satisfies $q \neq \ell$ and $q \nmid M$.

Step 2: (In1). The data have the required types: $n \geq 2$, L is CM, $c \in G_{L^+}$ is a complex conjugation, $\bar v$ lies above v for $v \in S_\ell$, St and v_0 are as required by Step 1, and the $\bar\chi_a$ are continuous characters $G_L \to \mathbb{F}_{\mathcal{O}'}^\times$ because Ψ is continuous with values in $\mathcal{O}'^\times$. The reduction of χ_{cyc} modulo $\mathfrak{m}_{\mathcal{O}'}$ is the character $\bar\omega$ of Definition 48.2, by which $G_{\mathbb{Q}}$ acts on the ℓ-th roots of unity.

(Rd1) $\tau(\mathcal{O}_L) \subseteq \mathcal{O}'$ is a hypothesis. By (Ar1), $\bar\omega \neq 1$ on G_L, so G_L moves a primitive ℓ-th root of unity and L contains none.

(Rd2) L/L^+ is unramified at every finite place by (Af2); the splitting is Step 1.

(Rd3) Step 1.

(Rd4) By (Af4), $\bar\Psi^2|_{D_v} \neq 1$ for $v \in S_\ell$, and $D_v \subseteq G_L$ by (F1); so some $h \in G_L$ has $\bar\Psi(h)^2 \neq 1$, and the $\bar\chi_a$ are pairwise distinct by the remark after (F2). If $\bar w$ lies above a finite place $u \notin S_\ell \cup \mathrm{St}$ of L^+, the prime below u does not divide M (Step 1), so Ψ, hence every $\bar\chi_a$, is trivial on $I_{\bar w} \cap G_L$ by hypothesis.

(Rd5) For $s \in G_L$, $\bar\Psi(csc) = \bar\Psi(c)^2\bar\Psi(s) = \bar\Psi(s)$, as $\bar\Psi$ has abelian values and $c^2 = 1$. Since $w_{n,n+1-a} = -w_{n,a}$, $\bar\chi_{n+1-a}(csc) = \bar\Psi(s)^{-w_{n,a}} = \bar\chi_a(s)^{-1}$.

(Rd10) holds for $(\xi_{v,a})$ by hypothesis; it involves no other component of $\mathscr{H}_L$, and the wild values enter none of (Rd1)–(Rd6), (In2)–(In4), so the choice of $(\xi_{v,a})$ does not affect the rest of the proof.

(Rd6) $\bar\Psi$ is trivial on $D_{\bar v_0} \cap G_L \subseteq D_{\bar v_0} \cap G_{L^+}$, so all $\bar\chi_a$ equal 1 there. By (F1) for $u = v_0$, $D_{\bar v_0} \cap G_L = D_{\bar v_0} \cap G_{L^+}$, on which $\bar\omega$ is non-trivial by hypothesis.

Step 3: (In2). Since $w_{n,1} = 1$, an element $s \in G_L$ satisfies $\bar\chi_a(s) = 1$ for all a if and only if $\bar\Psi(s) = 1$. For $s \in G_{L^+}$, $\eta_K(s) = 1$ means $s \in G_K$, and $G_{L^+} \cap G_K = G_L$ by (Af1). So the generators of $N_{\mathscr{H}_L}$ (the $s \in I_{\bar w} \cap G_L$, $\bar w \nmid \ell$, with all $\bar\chi_a(s) = 1$) are exactly the generators of N_L (the $s \in I_{\bar w} \cap G_{L^+}$, $\bar w \nmid \ell$, with $\eta_K(s) = \bar\Psi(s) = 1$). Both are the closed normal subgroup of G_{L^+} generated by this set, so $N_{\mathscr{H}_L} = N_L$, and $\Gamma_{\mathscr{H}_L} = \Gamma_L$, $\pi_{\mathscr{H}_L} = \pi_L$, $\Gamma^0_{\mathscr{H}_L} = \pi_L(G_L) = \Gamma^0_L$. The

groups $D_v = D_{\bar v} \cap G_{L^+} \supseteq I_v = I_{\bar v} \cap G_{L^+}$ and $d_v = [L_v^+ : \mathbb{Q}_\ell]$ are defined by the same formulas with the same places $\bar v$ in both definitions.

Step 4: (In3). Let κ be such a twist; it is a continuous character $G_L \to \mathcal{O}'^\times$, algebraic for ι in the sense of Definition 48.2, which is the condition of Definition 50.1. The first clause of (Rd7) is (Ka1), as $s^c = csc$. If $\bar w$ lies above a finite place $u \notin S_\ell \cup \mathrm{St}$ of L^+, the prime below $\bar w$ does not divide M (Step 1), so κ is trivial on $I_{\bar w} \cap G_L$ by (Ka2): this is the second clause. The third clause is (Ka3).

Step 5: (In4). Let $e \geq 1$ and $U_0 \subseteq I_{\bar\ell}$ open with $\Psi\chi_{\mathrm{cyc}}^{-e}$ trivial on U_0. Let $v \in S_\ell$. Since $\bar v = \gamma_v \bar\ell$, $I_{\bar v} = \gamma_v I_{\bar\ell} \gamma_v^{-1}$, and $\Psi\chi_{\mathrm{cyc}}^{-e}$, a character of $G_{\mathbb{Q}}$, is trivial on the open subgroup $\gamma_v U_0 \gamma_v^{-1}$ of $I_{\bar v}$; its intersection U with I_v is open in I_v (G_{L^+} is open in $G_{\mathbb{Q}}$). Through $\bar v$, D_v is the absolute Galois group of L_v^+ and I_v its inertia subgroup. Put $\phi_{v,a} = \Psi^{w_{n,a}}|_{D_v} \colon D_v \to \mathcal{O}'^\times \subseteq \overline{\mathbb{Q}}_\ell^\times$ and $\mu_a = (\Psi\chi_{\mathrm{cyc}}^{-e})^{w_{n,a}}|_{D_v}$, so $\phi_{v,a} = \chi_{\mathrm{cyc}}^{ew_{n,a}}\mu_a$ and μ_a is trivial on U. As U is open in I_v, there is an open subgroup $V \subseteq D_v$ with $V \cap I_v \subseteq U$; V is the absolute Galois group of a finite extension E of L_v^+, with inertia subgroup $V \cap I_v$, so $\mu_a|_V$ is unramified. An unramified character with values in the units of a finite extension of $\mathbb{Q}_\ell$ is crystalline, hence de Rham, of Hodge–Tate weight 0 at every embedding [52]. A representation W of D_v is de Rham if its restriction to an open subgroup V is, with the same Hodge–Tate weights: $(B_{\mathrm{dR}} \otimes W)^V$ is an E-vector space with a semilinear action of the finite group D_v/V if V is normal (one may shrink V to a normal open subgroup), and Galois descent gives the dimension of its invariants over L_v^+, compatibly with the filtration. So μ_a is de Rham with all τ-Hodge–Tate weights 0. The character χ_{cyc}^k is de Rham with τ-Hodge–Tate weight k for every $\tau \colon L_v^+ \to \overline{\mathbb{Q}}_\ell$ (normalisation fixed in Section 13), and a tensor product of de Rham characters is de Rham with the weights added. Hence $\phi_{v,a}$ is de Rham with τ-Hodge–Tate weight $e\,w_{n,a}$ for every τ, which is strictly decreasing in a because $e \geq 1$ and $w_{n,1} > \cdots > w_{n,n}$. Finally $\phi_{v,a}|_{I_v} = \Psi^{w_{n,a}}|_{I_v}$, and for $g \in I_v \subseteq G_L$ (by (F1)), $\Psi(g)^{w_{n,a}} \equiv \bar\Psi(g)^{w_{n,a}} = \bar\chi_a(g)$ modulo $\mathfrak{m}_{\mathcal{O}'}$. This is (Rd8). □

Lemma 50.24. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ and $v \in S_\ell$. Through $\bar v$, identify D_v with the absolute Galois group of L_v^+, and let* $\mathrm{Art} \colon (L_v^+)^\times \to D_v^{\mathrm{ab}} = D_v/\overline{[D_v, D_v]}$ *be the local Artin map. For an ℓ-power root of unity $\zeta \in L_v^+$ let $g_\zeta \in D_v$ be any element with image* $\mathrm{Art}(\zeta)$ *in D_v^{ab}.*

(Wv1) *$g_\zeta \in \mu_v$, and every $g \in \mu_v$ satisfies $g \in g_\zeta\overline{[D_v, D_v]}$ for some ℓ-power root of unity $\zeta \in L_v^+$; moreover $\chi_{\mathrm{cyc}}(g) = 1$ for every $g \in \mu_v$. Consequently, for $A \in \mathrm{CNL}_{\mathcal{O}}$ and a continuous homomorphism $\psi \colon D_v \to A^\times$, $\psi(g)$ equals the image of $\xi_{v,a}(g)$ in A for all $g \in \mu_v$ if and only if $\psi(g_\zeta)$ equals the image of $\xi_{v,a}(g_\zeta)$ for every ℓ-power root of unity $\zeta \in L_v^+$ (both sides being independent of the choice of g_ζ).*

(Wv2) *If the reduction $\bar\omega$ of χ_{cyc} is non-trivial on D_v, then $\mu_v = \overline{[D_v, D_v]}$, $\xi_{v,a} = 1$ for $1 \leq a \leq n$, and for every $A \in \mathrm{CNL}_{\mathcal{O}}$ every continuous homomorphism $\psi \colon D_v \to A^\times$ satisfies $\psi(g) = 1 = \xi_{v,a}(g)$ for $g \in \mu_v$.*

(Wv3) *Let R be a commutative $\mathcal{O}$-algebra, $P \subseteq R$ a prime ideal, $t \in R$ with $t^{\ell^k} = 1$ for some $k \geq 0$, and $y \colon R \to \overline{\mathbb{Q}}_\ell$ a homomorphism of $\mathcal{O}$-algebras with $y(P) = 0$ and $y(t) \in \mathcal{O}$. Then $t - y(t) \in P$; hence $y'(t) = y(t)$ for every homomorphism of $\mathcal{O}$-algebras $y' \colon R \to \overline{\mathbb{Q}}_\ell$ with $y'(P) = 0$.*

Proof. (Wv1). By local class field theory [190] (Chapter V, (1.3) and (1.4)), Art extends to an isomorphism of topological groups from the profinite completion $\widehat{F^\times}$ onto D_v^{ab}. Choosing a uniformiser, $F^\times = \varpi^{\mathbb{Z}} \times \mathcal{O}_F^\times$, so $\widehat{F^\times} \cong \widehat{\mathbb{Z}} \times \mathcal{O}_F^\times$ ($\mathcal{O}_F^\times$ is profinite), and $\mathcal{O}_F^\times = \mu_{q-1} \times U_F^1$ with

$U_F^1 \cong \mu_{\ell^\infty}(F) \times \mathbb{Z}_\ell^{[F:\mathbb{Q}_\ell]}$ [190] (Chapter II, (5.3) and (5.7)). In this product the elements of ℓ-power order are exactly those of $\mu_{\ell^\infty}(F)$: $\widehat{\mathbb{Z}}$ and $\mathbb{Z}_\ell^{[F:\mathbb{Q}_\ell]}$ are torsion-free, and μ_{q-1} has order prime to ℓ. Since μ_v is the preimage in D_v of the subgroup of elements of ℓ-power order of D_v^{ab} (Definition 50.1), μ_v is the preimage of $\mathrm{Art}(\mu_{\ell^\infty}(F))$: every g_ζ lies in μ_v, and each $g \in \mu_v$ has image $\mathrm{Art}(\zeta)$ for some $\zeta \in \mu_{\ell^\infty}(F)$, i.e. $g \in g_\zeta \overline{[D_v, D_v]}$. The character χ_{cyc} of D_v factors through D_v^{ab} and $\chi_{\mathrm{cyc}}(\mathrm{Art}(z)) = N_{F/\mathbb{Q}_\ell}(z)^{\pm 1}$ for $z \in \mathcal{O}_F^\times$ (the sign depends on the normalisation of Art and plays no role) [190] (Chapter V, (2.4) and the Lubin–Tate description of $\mathbb{Q}_\ell(\mu_{\ell^\infty})$). For $\zeta \in \mu_{\ell^\infty}(F)$, $N_{F/\mathbb{Q}_\ell}(\zeta)$ is an ℓ-power root of unity in $\mathbb{Q}_\ell$, hence 1 as ℓ is odd; so $\chi_{\mathrm{cyc}}(g) = 1$ for $g \in \mu_v$. For the last assertion: a continuous homomorphism $\psi\colon D_v \to A^\times$ into the commutative Hausdorff group $A^\times$ kills the commutators, hence their closure $\overline{[D_v, D_v]}$ (its kernel is closed); and $\xi_{v,a}$ is a homomorphism on $\mu_v \supseteq \overline{[D_v, D_v]}$ that is trivial on $\overline{[D_v, D_v]}$ by (Rd10). So both ψ and $\xi_{v,a}$ are constant on each coset $g\overline{[D_v, D_v]} \subseteq \mu_v$; by the first part every $g \in \mu_v$ lies in such a coset of some g_ζ, which gives the equivalence and the independence of the choice of g_ζ.

(Wv2). $\bar\omega$ is the character by which D_v acts on the ℓ-th roots of unity in $\overline{\mathbb{Q}}_\ell$, so $\bar\omega|_{D_v} \neq 1$ means that F contains no primitive ℓ-th root of unity. Then $\mu_{\ell^\infty}(F) = 1$ (a non-trivial finite group of ℓ-power roots of unity contains an element of order ℓ), so by (Wv1) μ_v is the preimage of the trivial subgroup, $\mu_v = \overline{[D_v, D_v]}$. By (Rd10), $\xi_{v,a}(g) = 1$ for every $g \in \mu_v$. A continuous $\psi\colon D_v \to A^\times$ is trivial on $\overline{[D_v, D_v]} = \mu_v$, as in the proof of (Wv1).

(Wv3). Put $R' = R/P$, a domain, and s the image of t. The homomorphism y factors through R' and restricts on $\mathcal{O}$ to the inclusion $\mathcal{O} \subseteq \overline{\mathbb{Q}}_\ell$, which is injective; so $\mathcal{O} \to R'$ is injective and $K = \mathrm{Frac}(\mathcal{O})$ is a subfield of $\mathrm{Frac}(R')$. As $s^{\ell^k} = 1$, s is algebraic over K; let $m \in K[X]$ be its minimal polynomial, monic and irreducible, dividing $X^{\ell^k} - 1$. Its roots (in an algebraic closure of K) are roots of $X^{\ell^k} - 1$, hence integral over $\mathcal{O}$, so its coefficients, being polynomials in the roots, are elements of K integral over $\mathcal{O}$; $\mathcal{O}$ is integrally closed, so $m \in \mathcal{O}[X]$. Then $m(s) \in R'$ is 0, and applying y gives $m(y(t)) = 0$. So m has the root $y(t) \in \mathcal{O} \subseteq K$; being irreducible over K, $m = X - y(t)$, whence $s = y(t)$, i.e. $t - y(t) \in P$. For y' as stated, $y'(t) - y(t) = y'(t - y(t)) \in y'(P) = 0$. □

Lemma 50.25. *Let $\mathcal{O}$ be a coefficient ring, $N \geq 0$, $\Lambda = \mathcal{O}[[X_1, \ldots, X_N]]$, and $C \neq 0$ a commutative ring with a ring homomorphism $\lambda\colon \Lambda \to C$ such that C is a finite and torsion-free Λ-module via λ ($\lambda(a)c = 0$ with $a \neq 0$ implies $c = 0$). Then:*

1. *λ is injective, and $\dim C/\mathfrak{q} = N + 1$ for every minimal prime $\mathfrak{q}$ of C;*
2. *if $\mathfrak{a} \subseteq \lambda(\Lambda)$ is an ideal of $\lambda(\Lambda)$ with $\lambda(\Lambda)/\mathfrak{a} \cong \mathbb{F}_\mathcal{O}$, then $C/\mathfrak{a}C$ has finite length; in particular every prime of C containing $\mathfrak{a}$ is maximal, and every element of the Jacobson radical of C has a power in $\mathfrak{a}C$ when C is local.*

Proof. Item 1. If $\lambda(a) = 0$ then $\lambda(a) \cdot 1 = 0$, so $a = 0$ or $1 = 0$ in C; hence λ is injective. C is Noetherian, being finite over the Noetherian ring Λ. Let $\mathfrak{q}$ be a minimal prime of C. Minimal primes of a Noetherian ring are associated primes, so $\mathfrak{q} = \mathrm{Ann}_C(c)$ for some $c \neq 0$. If $a \in \Lambda$ and $\lambda(a) \in \mathfrak{q}$, then $\lambda(a)c = 0$, so $a = 0$ by torsion-freeness. Hence $\Lambda \to C/\mathfrak{q}$ is injective, and it is finite, so integral. By the Cohen–Seidenberg theorems (lying over and going up for integral extensions, incomparability), $\dim C/\mathfrak{q} = \dim \Lambda$, and $\dim \Lambda = N + 1$ (Λ is a regular local ring of dimension $N + 1$).

Item 2. $C/\mathfrak{a}C$ is a finite module over $\lambda(\Lambda)/\mathfrak{a} \cong \mathbb{F}_\mathcal{O}$, i.e. a finite-dimensional $\mathbb{F}_\mathcal{O}$-vector space, hence of finite length, an Artinian ring. Its primes are maximal, so every prime of C containing $\mathfrak{a}$ is maximal; and its Jacobson radical is nilpotent. If C is local with maximal ideal $\mathfrak{m}_C$, the image of $\mathfrak{m}_C$ in $C/\mathfrak{a}C$ lies in its Jacobson radical, so $\mathfrak{m}_C^k \subseteq \mathfrak{a}C$ for some $k \geq 1$. □

50.3 From determinants to marked lifts

This subsection proves Proposition 50.26, which carries the ideal of the determinant ring to the deformation ring of marked lifts used in Theorem 50.3: the comparison morphism from B_n^{po} is onto the weight-zero algebra, and a seed point of the determinant ring gives a seed component. Surjectivity rests on Chenevier's theory of determinants [66] and on a density statement for complete Noetherian local rings (Lemma 50.27). The marking is what makes the passage possible: at the generic points of irreducible type the determinant comes from a marked lift (Proposition 50.28).

Proposition 50.26. *Let $\mathcal{O}'$ be a coefficient ring, L an auxiliary field of level C_0 for $(K, \mathcal{O}', \Psi, c, g_\star, M)$ with data $(N', w, j, \chi', \theta)$ as in (Ag1), $d_L = [L^+ : \mathbb{Q}]$, $n \in \{2, 3\}$, and let $(R_n^{\mathrm{po}}, \rho^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{u}}))$ be a universal polarised ordinary lift of rank n over L (for $\mathcal{O}'$).*

(Ht1) *A universal polarised ordinary determinant $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ of rank n over L (for $\mathcal{O}'$) exists; the comparison morphism ϑ_n exists, and $\vartheta_n(B_n^{\mathrm{po}}) = W_n^{\mathrm{po}}$.*

(Ht2) *Let κ be an admissible twist of rank n for ι, c and M, and $\mathcal{J} \subseteq B_n^{\mathrm{po}}$ an ideal such that $\dim B_n^{\mathrm{po}}/\mathfrak{q} = 1 + n\,d_L$ for every minimal prime $\mathfrak{q}$ of B_n^{po} over $\mathcal{J}$, and such that $x(\mathcal{J}) = 0$ for every seed r of some weight $m \geq 2$ for (j, χ') with twist κ and every homomorphism of $\mathcal{O}'$-algebras $x\colon B_n^{\mathrm{po}} \to \overline{\mathbb{Q}}_\ell$ with $x \circ D^{\mathrm{u}} = \det \circ \tilde{r}$ and $x \circ \psi_{v,a}^{\mathrm{d}} = \kappa_{v,a}$ ($\tilde{r} \circ \pi_L = r$, $(\kappa_{v,a})_a$ a diagonal of r on D_v as in (Sp5)). If for some $m \geq 2$ there is a seed of rank n and weight m for (j, χ') with twist κ, then $\mathcal{M}_{\mathcal{J}}$ has a seed component.*

Proof. Recall $w_{n,\bar{a}} = -w_{n,a}$, $\{w_{n,a}\} = \{1, -1\}$ ($n = 2$) or $\{1, 0, -1\}$ ($n = 3$), the $w_{n,a}$ are distinct, and $(J_n\, {}^t X\, J_n)_{ab} = X_{\bar{b}\bar{a}}$ for every $n \times n$ matrix X. For a homomorphism $\rho\colon \Gamma \to \mathrm{GL}_n(A)$, $\chi_{\det \circ \rho}(x)$ is the characteristic polynomial of $\rho(x)$ for $x \in A[\Gamma]$ (ρ extended linearly), so $\mathrm{tr}_{\det \circ \rho}(x)$ is the trace of $\rho(x)$; for a ring homomorphism f, $\chi_{f \circ D}(f_* x) = f(\chi_D(x))$ and $\mathrm{tr}_{f \circ D}(f_* x) = f(\mathrm{tr}_D(x))$, where f_* is f on coefficients.

(Ht1), existence of B.

Step 1: finiteness conditions. $\Gamma_L^0 = \pi_L(G_L)$ is open in Γ_L (G_L is open in G_{L^+} and π_L is an open map), so its open subgroups are open in Γ_L and by (Ar5)(i) Γ_L^0 satisfies Φ_ℓ. For $v \mid \ell$, D_v is the absolute Galois group of L_v^+; an open subgroup is the absolute Galois group of a finite extension F/L_v^+, whose continuous homomorphisms to $\mathbb{F}_\ell$ correspond by local class field theory to homomorphisms $F^\times/F^{\times \ell} \to \mathbb{F}_\ell$, a finite set; so D_v satisfies Φ_ℓ. The determinant $\bar{D}_n = \det \circ \bar{\rho}$, $\bar{\rho} = \mathrm{diag}(\bar{\Psi}^{w_{n,1}}, \ldots, \bar{\Psi}^{w_{n,n}})$, is continuous ($\bar{\Psi}$ has finite image).

Step 2: the determinant ring. Let $k = \mathbb{F}_{\mathcal{O}}$; $W(k) \subseteq \mathcal{O}$ and $\mathcal{O}$ is finite free over $W(k)$. Let $(A(\bar{D}_n), D(\bar{D}_n))$ be given by (Ch2) of Theorem B.83 for $G = \Gamma_L^0$; it is Noetherian by Step 1. Put $A = A(\bar{D}_n) \otimes_{W(k)} \mathcal{O}$ and $D_A = D(\bar{D}_n) \otimes A$. A is finite over the complete Noetherian local ring $A(\bar{D}_n)$, hence complete and Noetherian, and $A/\mathfrak{m}_{A(\bar{D}_n)} A = \mathcal{O}/\ell\mathcal{O}$ is local with residue field k; so A is local and an object of $\mathrm{CNL}_{\mathcal{O}}$. Every object T of $\mathrm{CNL}_{\mathcal{O}}$ is a profinite local $W(k)$-algebra with residue field k, and local homomorphisms between such rings are continuous; hence morphisms $A \to T$ of $\mathrm{CNL}_{\mathcal{O}}$ correspond to continuous $W(k)$-algebra homomorphisms $A(\bar{D}_n) \to T$, and by (Ch2) (A, D_A) represents continuous T-valued determinants of dimension n on Γ_L^0 lifting $\bar{D}_n$.

Step 3: the characters and the product. For $v \mid \ell$ and $1 \leq a \leq n$, Theorem B.71, item 1, with $n = 1$ applied to D_v and $\bar{\Psi}^{w_{n,a}}|_{D_v}$ gives an object $R_{v,a}$ and a universal continuous character $\chi_{v,a}\colon D_v \to R_{v,a}^\times$ lifting $\bar{\Psi}^{w_{n,a}}$. The completed tensor product S over $\mathcal{O}$ of A and the finitely many $R_{v,a}$ is the coproduct in $\mathrm{CNL}_{\mathcal{O}}$; with D_S the base change of D_A and $\psi_{S,v,a}$ the image of $\chi_{v,a}$, S represents the tuples $(D, (\psi_{v,a}))$ of a continuous determinant lifting $\bar{D}_n$ and continuous characters lifting $\bar{\Psi}^{w_{n,a}}$.

Step 4: the conditions. Let $I \subseteq S$ be generated by the coefficients of $\chi_{D_S}(\pi_L(cg^{-1}c)) - \chi_{D_S}(\pi_L(g))$ ($g \in G_L$) and the elements $\mathrm{tr}_{D_S}(\gamma \prod_{a=1}^{n}(\pi_L(g_a) - \psi_{S,v,a}(g_a)))$ ($\gamma \in \Gamma_L^0$, $v \mid \ell$, $g_a \in D_v$; $D_v \subseteq G_L$ because v splits in L by (Af2), $\ell \mid M$). By the base-change rules recalled above, a morphism $f\colon S \to T$ kills I if and only if $(f \circ D_S, (f \circ \psi_{S,v,a}))$ satisfies (Od1) and (Od2). Moreover $I \subseteq \mathfrak{m}_S$: modulo $\mathfrak{m}_S$ the generators are computed from $\bar\rho$ and $\bar\Psi^{w_{n,a}}$; $\bar\Psi(cg^{-1}c) = \bar\Psi(g)^{-1}$ ($\bar\Psi(c) = 1$, abelian image) and $\{-w_{n,a}\} = \{w_{n,a}\}$, so $\bar\rho(\pi_L(cg^{-1}c))$ and $\bar\rho(\pi_L(g))$ have the same characteristic polynomial; and $\bar\rho(\prod_a(\pi_L(g_a) - \bar\Psi(g_a)^{w_{n,a}}))$ is a product of diagonal matrices whose a-th factor has (a,a)-entry 0, hence 0. Put $B = S/I$ with the images D^{u}, $\psi^{\mathrm{d}}_{v,a}$ of D_S, $\psi_{S,v,a}$. Then $(B, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ is a universal polarised ordinary determinant of rank n over L.

(Ht1), the comparison morphism. $(\det \circ \rho^{\mathrm{u}}, (\psi^{\mathrm{u}}_{v,a}))$ is a polarised ordinary determinant to R: the maps $\gamma \mapsto \Lambda_i(\gamma)$ are polynomials in the entries of $\rho^{\mathrm{u}}(\gamma)$, continuous by (Po1); modulo $\mathfrak{m}_R$, $\det \circ \rho^{\mathrm{u}}$ is $\det \circ \bar\rho = \bar D_n$ and $\psi^{\mathrm{u}}_{v,a} \equiv \bar\Psi^{w_{n,a}}$ by (Po5); (Od1) holds because $\rho^{\mathrm{u}}(\pi_L(cg^{-1}c)) = J_n\, {}^t\rho^{\mathrm{u}}(\pi_L(g))\, J_n$ (Po2) has the characteristic polynomial of $\rho^{\mathrm{u}}(\pi_L(g))$; (Od2) holds because $\rho^{\mathrm{u}}(\gamma \prod_a(\pi_L(g_a) - \psi^{\mathrm{u}}_{v,a}(g_a))) = \rho^{\mathrm{u}}(\gamma) \cdot 0$ by (Po4). So ϑ exists.

(Ht1), $\vartheta(B) \subseteq W$. Let $X \subseteq B$ consist of the $\Lambda_i(\gamma)$ of D^{u} ($1 \le i \le n$, $\gamma \in \Gamma_L^0$) and the $\psi^{\mathrm{d}}_{v,a}(g)$. If two morphisms $f, f'\colon B \to T$ agree on X, then $f \circ D^{\mathrm{u}}$ and $f' \circ D^{\mathrm{u}}$ have the same characteristic polynomials at all γ, hence are equal by (Ch1), and $f \circ \psi^{\mathrm{d}}_{v,a} = f' \circ \psi^{\mathrm{d}}_{v,a}$; by uniqueness in the universal property $f = f'$. By Lemma 50.27, $B = \mathcal{O}[X] + \mathfrak{m}_B^k$ for every k. Now $\vartheta(\Lambda_i(\gamma))$ is the sum, with signs, of the principal $i \times i$ minors of $\rho^{\mathrm{u}}(\gamma)$, i.e. of products $\prod_{a \in S'} \rho^{\mathrm{u}}(\gamma)_{\sigma(a)a}$ with $S' \subseteq \{1, \dots, n\}$ and σ a permutation of S', whose weight is $\sum_{a \in S'}(w_{n,\sigma(a)} - w_{n,a}) = 0$; and $\vartheta(\psi^{\mathrm{d}}_{v,a}(g)) = \psi^{\mathrm{u}}_{v,a}(g)$. So $\vartheta(\mathcal{O}[X])$ lies in the $\mathcal{O}$-algebra W_0 generated by the weight-zero monomials and the values of the $\psi^{\mathrm{u}}_{v,a}$. For $y \in B$ and $k \ge 1$ write $y = y_0 + m$ with $y_0 \in \mathcal{O}[X]$, $m \in \mathfrak{m}_B^k$; then $\vartheta(y) - \vartheta(y_0) \in \mathfrak{m}_R^k$ with $\vartheta(y_0) \in W_0$. Hence $\vartheta(y) \in W$.

(Ht1), $W \subseteq \vartheta(B)$.

Step 5: the image. Put $A' = \vartheta(B) \cong B/\ker\vartheta$, a complete Noetherian local ring, hence Henselian, with maximal ideal $\vartheta(\mathfrak{m}_B) = A' \cap \mathfrak{m}_R$ (the image of $\mathfrak{m}_B$ is a maximal ideal of A' contained in $\mathfrak{m}_R$) and residue field k; so an element of A' outside $\mathfrak{m}_R$ is a unit of A'. A' is closed in R: $B = \varprojlim B/\mathfrak{m}_B^k$ is compact, ϑ is continuous ($\vartheta(\mathfrak{m}_B^k) \subseteq \mathfrak{m}_R^k$) and R is Hausdorff (Krull). Let $\mathcal{S} \subseteq \mathrm{M}_n(R)$ be the A'-submodule spanned by $\rho^{\mathrm{u}}(\Gamma_L^0)$; it contains 1 and is closed under products, and $\mathrm{tr}(Y) \in A'$ for $Y \in \mathcal{S}$, since $\mathrm{tr}\,\rho^{\mathrm{u}}(\gamma) = \vartheta(\Lambda_1(\gamma))$.

Step 6: the difference at the marking is diagonal. Let $Z = \rho^{\mathrm{u}}(\pi_L(g_\star)) - \rho^{\mathrm{u}}(\pi_L(cg_\star^{-1}c)) \in \mathcal{S}$ ($g_\star \in G_L$ by (Af1)). Its off-diagonal entries vanish by (Po3), and by (Po2) its (a,a)-entry is $x_a - x_{\bar a}$ with $x_a = \rho^{\mathrm{u}}(\pi_L(g_\star))_{aa}$. With $\zeta = x_1 - x_n$ this gives

$$Z = \zeta \cdot Y, \qquad Y = \mathrm{diag}(w_{n,1}, \dots, w_{n,n})$$

(for $n = 3$ the middle entry is $x_2 - x_2 = 0$). By (Po5), $\zeta \equiv \bar\Psi(g_\star) - \bar\Psi(g_\star)^{-1} \pmod{\mathfrak{m}_R}$, which is non-zero because $\bar\Psi(g_\star)^2 \ne 1$ ($g_\star$ is a marking); so $\zeta \in R^\times$.

Step 7: $\zeta \in A'$. $\mathrm{tr}(Z^2) = \zeta^2 \sum_a w_{n,a}^2 = 2\zeta^2 \in A'$, and $2 \in \mathcal{O}^\times$ (ℓ odd), so $\zeta^2 \in A'$. Choose $o \in \mathcal{O}$ with $\zeta \equiv o \pmod{\mathfrak{m}_R}$; then $o \in \mathcal{O}^\times$, $o^2 - \zeta^2 = (o-\zeta)(o+\zeta) \in A' \cap \mathfrak{m}_R$, and the derivative $2o$ of $X^2 - \zeta^2$ at o is a unit. Hensel's lemma in A' gives $d \in A'$ with $d^2 = \zeta^2$ and $d \equiv o$. Then $(d-\zeta)(d+\zeta) = 0$ in R and $d + \zeta \equiv 2o$ is a unit, so $\zeta = d \in A'$, a unit of A'. Hence $Y = \zeta^{-1}Z \in \mathcal{S}$.

Step 8: the matrix units on the diagonal. With e_{ab} the matrix units, $e_{aa} = \frac{1}{2}(Y^2 + Y)$ if $w_{n,a} = 1$, $e_{aa} = \frac{1}{2}(Y^2 - Y)$ if $w_{n,a} = -1$, and $e_{22} = 1 - Y^2$ if $n = 3$ (the weights are distinct); so $e_{aa} \in \mathcal{S}$ for all a.

Step 9: entry modules. For $1 \le a, b \le n$ let $\mathcal{E}_{ab} \subseteq R$ be the A'-submodule spanned by the entries $\rho^{\mathrm{u}}(\gamma)_{ab}$, $\gamma \in \Gamma_L^0$. Then $z \in \mathcal{E}_{ab}$ if and only if $z\, e_{ab} \in \mathcal{S}$: the set of such z is an A'-module containing each $\rho^{\mathrm{u}}(\gamma)_{ab}$, because $e_{aa}\rho^{\mathrm{u}}(\gamma)e_{bb} = \rho^{\mathrm{u}}(\gamma)_{ab}e_{ab}$; conversely the (a,b)-entry

of an element of $\mathcal{S}$ lies in $\mathcal{E}_{ab}$. Consequently: (i) $\mathcal{E}_{ab}\mathcal{E}_{bc} \subseteq \mathcal{E}_{ac}$, since $(z\,e_{ab})(z'e_{bc}) = zz'e_{ac}$; (ii) $\mathcal{E}_{aa} \subseteq A'$, since $z = \mathrm{tr}(z\,e_{aa})$; (iii) $\mathcal{E}_{ab} = \mathcal{E}_{\bar{b}\bar{a}}$, since by (Po2) $\rho^{\mathrm{u}}(\pi_L(cg^{-1}c))_{\bar{b}\bar{a}} = \rho^{\mathrm{u}}(\pi_L(g))_{ab}$, and $cg^{-1}c \in G_L$ for $g \in G_L$. Call $w_{n,a} - w_{n,b}$ the weight of (a,b); $(\bar{b},\bar{a})$ has the same weight.

Step 10: pairing positive and negative weights. Let $z \in \mathcal{E}_{ab}$ with $w_{n,a} > w_{n,b}$ and $z' \in \mathcal{E}_{cd}$ with $w_{n,c} < w_{n,d}$. Then $zz' \in \mathcal{E}_{ef}$ for some (e,f) of weight $(w_{n,a} - w_{n,b}) + (w_{n,c} - w_{n,d})$. Indeed, one of $b = c$, $d = a$, $\bar{a} = c$, $d = \bar{b}$ holds: for $n = 2$ the only pair is $(1,2),(2,1)$ with $b = c$; for $n = 3$, with $(a,b) \in \{(1,2),(1,3),(2,3)\}$ and $(c,d) \in \{(2,1),(3,1),(3,2)\}$, a check of the nine pairs gives $b = c$ for $(1,2),(2,1)$ and for $(1,3)$ or $(2,3)$ with $(3,1)$, $(3,2)$; $d = a$ for $(1,2),(3,1)$ and $(1,3),(2,1)$; $\bar{a} = c$ for $(1,2),(3,2)$ and $(2,3),(2,1)$. If $b = c$, $zz' \in \mathcal{E}_{ad}$ by (i); if $d = a$, $z'z \in \mathcal{E}_{cb}$; if $\bar{a} = c$, $z \in \mathcal{E}_{\bar{b}\bar{a}}$ by (iii) and $zz' \in \mathcal{E}_{\bar{b}d}$; if $d = \bar{b}$, $z'z \in \mathcal{E}_{c\bar{a}}$. In each case the weight is the sum, using $w_{n,\bar{a}} = -w_{n,a}$.

Step 11: weight-zero products lie in A'. Claim: if $z_k \in \mathcal{E}_{a_k b_k}$ $(1 \le k \le s)$ and $\sum_k (w_{n,a_k} - w_{n,b_k}) = 0$, then $\prod_k z_k \in A'$. Induction on s; $s = 0$ is clear. If all weights are 0, then $a_k = b_k$ and $z_k \in A'$ by (ii). Otherwise, the weights summing to 0, there are a factor of positive and one of negative weight; by Step 10 their product is an element of some $\mathcal{E}_{ef}$ whose weight is the sum, and replacing the two factors by it leaves the product and the total weight unchanged and lowers s. Every weight-zero monomial is such a product; and $\psi^{\mathrm{u}}_{v,a}(g) = \vartheta(\psi^{\mathrm{d}}_{v,a}(g)) \in A'$. So $W_0 \subseteq A'$; as A' is closed and every element of W is a limit of elements of W_0, $W \subseteq A'$. With the previous part, $\vartheta(B) = W$.

(Ht2).

Step 12: a point of B from the seed. Let r be a seed of rank n and weight $m \ge 2$ for (j,χ') with twist κ, $\tilde{r}$ with $\tilde{r} \circ \pi_L = r$, and $(\kappa_{v,a})_a$ a diagonal of r on D_v as in (Sp5) (for the fixed place $\bar{v}$ above v; $D_v = D_{\bar{v}} \cap G_L$). As G_L and D_v are compact and $\overline{\mathbb{Q}}_\ell$ is a countable union of closed subfields finite over $\mathbb{Q}_\ell$, Baire's theorem gives a finite extension E' of the fraction field of $\mathcal{O}$ containing the values of r and of the $\kappa_{v,a}$; the compact group $r(G_L)$ stabilises an $\mathcal{O}_{E'}$-lattice, so there is $h \in \mathrm{GL}_n(E')$ with $\tilde{r}' = h\tilde{r}h^{-1}$ valued in $\mathrm{GL}_n(\mathcal{O}_{E'})$, and the $\kappa_{v,a}$ take values in $\mathcal{O}_{E'}^\times$. Let $A_0 \subseteq \mathcal{O}_{E'}$ be the subring of elements with residue in $\mathbb{F}_{\mathcal{O}}$: it contains $\mathcal{O}$ and $\mathfrak{m}_{E'}$, is local with maximal ideal $\mathfrak{m}_{E'}$ and residue field $\mathbb{F}_{\mathcal{O}}$, and is a finitely generated $\mathcal{O}$-module (a submodule of $\mathcal{O}_{E'}$), hence an object of $\mathrm{CNL}_{\mathcal{O}}$ whose topology is the ℓ-adic one, induced from $\mathcal{O}_{E'}$. By (Sp3) and (Ag1) ($\chi^j_{\mathrm{cyc}}\chi' \equiv \Psi$), the characteristic polynomial of $\tilde{r}'(\gamma)$ has coefficients in $\mathcal{O}_{E'}$ congruent to those of $\prod_a (X - \bar{\Psi}(\gamma)^{w_{n,a}})$, which lie in $\mathbb{F}_{\mathcal{O}}$; so the $\Lambda_i(\gamma)$ of $\det \circ \tilde{r}'$ lie in A_0, and by (Ch1) $\det \circ \tilde{r}'$ is the base change of an A_0-valued determinant D_0. D_0 is continuous (Λ_i is a polynomial in the entries), and its reduction is $\bar{D}_n$ by (Ch1) (same characteristic polynomials). By (Sp5) and (Ag1), $\kappa_{v,a} \equiv \varphi_a \equiv \bar{\Psi}^{w_{n,a}}$, so $\kappa_{v,a}$ is a continuous character $D_v \to A_0^\times$ lifting $\bar{\Psi}^{w_{n,a}}$. (Od1): by (Sp2), $r(cs^{-1}c) = P^{-1}\,{}^t r(s)\,P$, which has the characteristic polynomial of $r(s)$. (Od2): in a basis in which $r|_{D_v}$ is upper triangular with diagonal $(\kappa_{v,1},\dots,\kappa_{v,n})$, the a-th factor of $\prod_{a=1}^n (r(g_a) - \kappa_{v,a}(g_a))$ is upper triangular with (a,a)-entry 0, so it maps $\langle e_1,\dots,e_a\rangle$ into $\langle e_1,\dots,e_{a-1}\rangle$ and the product ($a = 1$ on the left) is 0; hence $\mathrm{tr}(\tilde{r}'(\gamma)\prod_a(\cdots)) = 0$. (These identities hold in $A_0 \subseteq \mathcal{O}_{E'}$.) So $(D_0,(\kappa_{v,a}))$ is a polarised ordinary determinant to A_0; let $x_0 \colon B \to A_0$ classify it and $x \colon B \to \overline{\mathbb{Q}}_\ell$ be x_0 followed by the inclusion. Then x is a homomorphism of $\mathcal{O}$-algebras with $x \circ D^{\mathrm{u}} = \det \circ \tilde{r}' = \det \circ \tilde{r}$ and $x \circ \psi^{\mathrm{d}}_{v,a} = \kappa_{v,a}$, so $x(\mathcal{J}) = 0$ by hypothesis.

Step 13: the component. $\ker x$ is a prime containing $\mathcal{J}$; let $\mathfrak{Z} \subseteq \ker x$ be a minimal prime over $\mathcal{J}$, so $\dim B/\mathfrak{Z} = 1 + n\,d_L$. By (Sp2), P is symmetric and invertible, and over the algebraically closed field $\overline{\mathbb{Q}}_\ell$ of characteristic $\neq 2$ there is $Q \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ with ${}^t Q J_n Q = P$ (both forms are congruent to the identity). Put $\rho' = Q\tilde{r}Q^{-1}$; then ${}^t\rho'(s) J_n \rho'(csc) = J_n$, and with $s = g^{-1}$, $\rho'(\pi_L(cg^{-1}c)) = J_n\,{}^t\rho'(\pi_L(g))\,J_n$. ρ' is irreducible by (Sp1) and $x \circ D^{\mathrm{u}} = \det \circ \rho'$. Proposition 50.28, applied to $\mathfrak{q} = \mathfrak{Z}$, $\Omega' = \overline{\mathbb{Q}}_\ell$, $\phi = x$ and ρ', gives $\ker\vartheta \subseteq \mathfrak{Z}$ and a prime ξ of

E_n^{po} of irreducible type with $\xi \cap W = \vartheta(\mathfrak{Z})$. As ϑ is onto W, $\mathfrak{z} = \vartheta(\mathfrak{Z})$ is an ideal of W with $W/\mathfrak{z} \cong B/(\mathfrak{Z} + \ker\vartheta) = B/\mathfrak{Z}$, a domain of dimension $1 + n\,d_L$; and $\mathcal{M}_{\mathcal{J}} = \vartheta(\mathcal{J})W = \vartheta(\mathcal{J}) \subseteq \mathfrak{z} = \xi \cap W$. So $\mathcal{M}_{\mathcal{J}}$ has a seed component. □

Lemma 50.27. *Let $\mathcal{O}$ be a coefficient ring, S an object of $\mathrm{CNL}_{\mathcal{O}}$ and $X \subseteq S$ a subset such that any two morphisms $f, f' \colon S \to T$ of $\mathrm{CNL}_{\mathcal{O}}$ with $f(x) = f'(x)$ for all $x \in X$ are equal. Then $S = \mathcal{O}[X] + \mathfrak{m}_S^k$ for every $k \geq 1$, where $\mathcal{O}[X]$ is the $\mathcal{O}$-subalgebra of S generated by X; that is, $\mathcal{O}[X]$ is dense in S.*

Proof. Fix $k \geq 1$, put $S_k = S/\mathfrak{m}_S^k$, $\mathfrak{m}_k = \mathfrak{m}_S/\mathfrak{m}_S^k$, and let $C_k \subseteq S_k$ be the image of $\mathcal{O}[X]$. We show $C_k = S_k$.

Step 1: the rings. S_k is a finite set: it has finite length with residue field $\mathbb{F}_{\mathcal{O}}$, which is finite. $\mathfrak{n} = C_k \cap \mathfrak{m}_k$ is a nilpotent ideal of C_k; since $\mathcal{O} \to \mathbb{F}_{\mathcal{O}}$ is onto and the image of $\mathcal{O}$ lies in C_k, $C_k/\mathfrak{n} = \mathbb{F}_{\mathcal{O}}$. An element of $C_k \setminus \mathfrak{n}$ is $o + y$ with o the image of a unit of $\mathcal{O}$ and $y \in \mathfrak{n}$ nilpotent, hence a unit of C_k. So C_k is local with maximal ideal $\mathfrak{n}$ and residue field $\mathbb{F}_{\mathcal{O}}$, and S_k is a finite C_k-module. Let $T = S_k \otimes_{C_k} S_k$, a finite ring and an $\mathcal{O}$-algebra through C_k. The ideal N generated by $\mathfrak{m}_k \otimes 1$ and $1 \otimes \mathfrak{m}_k$ is generated by finitely many nilpotent elements, hence nilpotent, and $T/N = \mathbb{F}_{\mathcal{O}} \otimes_{C_k} \mathbb{F}_{\mathcal{O}} = \mathbb{F}_{\mathcal{O}}$ because $C_k \to \mathbb{F}_{\mathcal{O}}$ is onto. Hence T is a local Artinian ring with residue field $\mathbb{F}_{\mathcal{O}}$, an object of $\mathrm{CNL}_{\mathcal{O}}$.

Step 2: the two morphisms agree. The maps $i_1, i_2 \colon S \to T$, $i_1(s) = \bar{s} \otimes 1$, $i_2(s) = 1 \otimes \bar{s}$, are local $\mathcal{O}$-algebra homomorphisms, i.e. morphisms of $\mathrm{CNL}_{\mathcal{O}}$, and $i_1(x) = i_2(x)$ for $x \in X$ because $\bar{x} \in C_k$. By hypothesis $i_1 = i_2$: $s \otimes 1 = 1 \otimes s$ in T for all $s \in S_k$.

Step 3: $C_k = S_k$. By Step 2 every element of T has the form $\sum_i s_i \otimes t_i = \sum_i s_i t_i \otimes 1$, so the multiplication map $\mu \colon T \to S_k$, which satisfies $\mu(a \otimes 1) = a$, is a bijection; it is C_k-linear. Let $d = \dim_{\mathbb{F}_{\mathcal{O}}} S_k/\mathfrak{n}S_k$; $d \geq 1$ since $\mathfrak{n}S_k \subseteq \mathfrak{m}_k \neq S_k$. Then

$$T \otimes_{C_k} \mathbb{F}_{\mathcal{O}} \cong (S_k/\mathfrak{n}S_k) \otimes_{\mathbb{F}_{\mathcal{O}}} (S_k/\mathfrak{n}S_k)$$

has dimension d^2, while μ identifies it with $S_k \otimes_{C_k} \mathbb{F}_{\mathcal{O}}$, of dimension d. So $d = 1$, i.e. $S_k = C_k \cdot 1 + \mathfrak{n}S_k$, and since $\mathfrak{n}$ is nilpotent, iterating gives $S_k = C_k + \mathfrak{n}^j S_k$ for all j, hence $S_k = C_k$ (Nakayama's lemma).

Thus every $s \in S$ is congruent modulo $\mathfrak{m}_S^k$ to an element of $\mathcal{O}[X]$. □

Proposition 50.28. *Let L be a CM field satisfying (Af1) and (Af2) for $(K, \mathcal{O}, \Psi, c, g_\star, M)$, $n \in \{2, 3\}$, and let $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ and $(R_n^{\mathrm{po}}, \rho^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{u}}))$ be a universal polarised ordinary determinant and a universal polarised ordinary lift of rank n over L such that the comparison morphism ϑ_n exists and $\vartheta_n(B_n^{\mathrm{po}}) = W_n^{\mathrm{po}}$. Let $\mathfrak{q}$ be a prime of B_n^{po}, and suppose that there are an algebraically closed field Ω', a homomorphism of $\mathcal{O}$-algebras $\phi \colon B_n^{\mathrm{po}} \to \Omega'$ with $\mathfrak{q} \subseteq \ker\phi$, and a homomorphism $\rho' \colon \Gamma_L^0 \to \mathrm{GL}_n(\Omega')$, irreducible, with $\rho'(\pi_L(cg^{-1}c)) = J_n\,{}^t\rho'(\pi_L(g))\,J_n$ for all $g \in G_L$ and $\phi \circ D^{\mathrm{u}} = \det \circ \rho'$. Then:*

(Mk1) $\ker\vartheta_n \subseteq \mathfrak{q}$;

(Mk2) *there is a prime ξ of E_n^{po} of irreducible type with $\xi \cap W_n^{\mathrm{po}} = \vartheta_n(\mathfrak{q})$.*

Proof. Recall $w_{n,\bar{a}} = -w_{n,a}$, the $w_{n,a}$ are distinct elements of $\{1, 0, -1\}$, and $(J_n\,{}^tXJ_n)_{ab} = X_{\bar{b}\bar{a}}$. Since $c^2 = 1$, $(\gamma^*)^* = \gamma$ and $(\gamma\delta)^* = \delta^*\gamma^*$; for $\gamma = \pi_L(g)$, $g \in G_L$, $\gamma^* = \pi_L(cg^{-1}c)$. Put $\gamma_\star = \pi_L(g_\star)$ ($g_\star \in G_L$ by (Af1)). For a field F the trace form $(X, Y) \mapsto \mathrm{tr}(XY)$ on $\mathrm{M}_n(F)$ is non-degenerate ($\mathrm{tr}(e_{ab}e_{ba}) = 1$ for the matrix units). Every representation below is extended

linearly to the group ring; for a ring homomorphism f, $\mathrm{tr}_{f\circ D}(f_*x) = f(\mathrm{tr}_D(x))$ and $\chi_{f\circ D}(f_*x) = f(\chi_D(x))$.

Step 1: the generic determinant. $C = B/\mathfrak{q}$ is a quotient of an object of $\mathrm{CNL}_{\mathcal{O}}$ by a prime, so a domain and an object of $\mathrm{CNL}_{\mathcal{O}}$; let $\nu\colon B \to C$ be the projection, Ω an algebraic closure of the fraction field of C and $\iota\colon C \to \Omega$ the inclusion. Put $D_C = \nu\circ D^{\mathrm{u}}$, $\psi_{C,v,a} = \nu\circ\psi^{\mathrm{d}}_{v,a}$, $D_\Omega = \iota\circ D_C$. D_C is continuous, and reduces modulo $\mathfrak{m}_C$ to $\bar{D}_n = \det\circ\bar{\rho}$, $\bar{\rho} = \mathrm{diag}(\bar{\Psi}^{w_{n,1}},\dots,\bar{\Psi}^{w_{n,n}})$; so for $x \in C[\Gamma_L^0]$ the residue of $\mathrm{tr}_{D_C}(x)$ is $\mathrm{tr}\,\bar{\rho}(\bar{x})$, $\bar{x}$ the reduction of x. As $\mathfrak{q} \subseteq \ker\phi$, $\phi = \phi'\circ\nu$ for an $\mathcal{O}$-algebra homomorphism $\phi'\colon C \to \Omega'$, and $\phi'\circ D_C = \det\circ\rho'$. Both Ω and Ω' have characteristic $\neq 2$, since $2 \in \mathcal{O}^\times$. By (Ch3) of Theorem B.83 there is a semisimple $\rho_0\colon \Gamma_L^0 \to \mathrm{GL}_n(\Omega)$ with $\det\circ\rho_0 = D_\Omega$; then $\mathrm{tr}\,\rho_0(x) = \iota(\mathrm{tr}_{D_C}(x))$ for $x \in C[\Gamma_L^0]$.

Step 2: ρ_0 is irreducible; its Ω-span is $\mathrm{M}_n(\Omega)$. By Burnside's theorem the Ω'-span of $\rho'(\Gamma_L^0)$ is $\mathrm{M}_n(\Omega')$; choose $\gamma_1,\dots,\gamma_{n^2}$ with $\rho'(\gamma_i)$ a basis. By non-degeneracy of the trace form, $\det(\mathrm{tr}\,\rho'(\gamma_i\gamma_k))_{i,k} \neq 0$. This is $\phi'(G)$ with $G = \det(\mathrm{tr}_{D_C}(\gamma_i\gamma_k)) \in C$, so $G \neq 0$ and $\iota(G) = \det(\mathrm{tr}\,\rho_0(\gamma_i\gamma_k)) \neq 0$. If $\sum_i y_i\rho_0(\gamma_i) = 0$ with $y_i \in \Omega$, multiplying by $\rho_0(\gamma_k)$ and taking traces gives $\sum_i y_i\,\mathrm{tr}\,\rho_0(\gamma_i\gamma_k) = 0$ for all k, so $y = 0$. Hence the $\rho_0(\gamma_i)$ span $\mathrm{M}_n(\Omega)$, and ρ_0 is irreducible (a proper invariant subspace would be invariant under all of $\mathrm{M}_n(\Omega)$). Consequently a matrix X with $\mathrm{tr}(\rho_0(\gamma)X) = 0$ for all γ is 0, and a matrix commuting with all $\rho_0(\gamma)$, or with all ${}^t\rho_0(\gamma)$, is scalar.

Step 3: a symmetric invariant form. $\rho_1(\gamma) = {}^t\rho_0(\gamma^*)$ is a homomorphism: ${}^t\rho_0((\gamma\delta)^*) = {}^t(\rho_0(\delta^*)\rho_0(\gamma^*)) = \rho_1(\gamma)\rho_1(\delta)$. Its Ω-span is ${}^t\mathrm{M}_n(\Omega) = \mathrm{M}_n(\Omega)$, so it is irreducible. For $\gamma = \pi_L(g)$, (Od1) gives $\chi_{D^{\mathrm{u}}}(\gamma^*) = \chi_{D^{\mathrm{u}}}(\gamma)$, so $\rho_1(\gamma)$ and $\rho_0(\gamma)$ have the same characteristic polynomial; by (Ch1), $\det\circ\rho_1 = \det\circ\rho_0$, and by the uniqueness in (Ch3) there is $P \in \mathrm{GL}_n(\Omega)$ with ${}^t\rho_0(\gamma^*)P = P\rho_0(\gamma)$ for all γ. Replacing γ by γ^* gives $\rho_0(\gamma^*) = P^{-1}\,{}^t\rho_0(\gamma)P$, and transposing the first identity gives $\rho_0(\gamma^*) = {}^tP^{-1}\,{}^t\rho_0(\gamma)\,{}^tP$. So ${}^tPP^{-1}$ commutes with every ${}^t\rho_0(\gamma)$, hence ${}^tP = \lambda P$ with $\lambda \in \Omega$, and $P = \lambda^2 P$ gives $\lambda = \pm 1$. Suppose $\lambda = -1$. If $n = 3$, $\det P = \det{}^tP = -\det P$, impossible. If $n = 2$, $P = p\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $p \neq 0$, and $P^{-1}\,{}^tXP = \mathrm{tr}(X)\cdot 1 - X$ for every 2×2 matrix X ; so $\rho_0(\gamma^*) = \mathrm{tr}\,\rho_0(\gamma) - \rho_0(\gamma)$, and multiplying by $\rho_0(x)$ and taking traces, $\mathrm{tr}\,\rho_0(x\gamma) + \mathrm{tr}\,\rho_0(x\gamma^*) = \mathrm{tr}\,\rho_0(x)\,\mathrm{tr}\,\rho_0(\gamma)$ for $x,\gamma \in \Gamma_L^0$. This is ι of an identity between traces of D_C; as ι is injective it holds in C, and ϕ' carries it to $\mathrm{tr}(\rho'(x)Y_\gamma) = 0$ with $Y_\gamma = \rho'(\gamma) + \rho'(\gamma^*) - \mathrm{tr}\,\rho'(\gamma)\cdot 1$. By hypothesis $\rho'(\gamma^*) = J_2\,{}^t\rho'(\gamma)J_2$, and the Ω'-span of $\rho'(\Gamma_L^0)$ is $\mathrm{M}_2(\Omega')$, so $Y_\gamma = 0$; its $(1,2)$-entry is $2\rho'(\gamma)_{12}$. Thus $\rho'(\gamma)_{12} = 0$ for all γ and the second basis vector spans a ρ'-stable line, contradicting irreducibility. Hence ${}^tP = P$, and

$$\rho_0(\gamma^*) = P^{-1}\,{}^t\rho_0(\gamma)\,P \qquad (\gamma \in \Gamma_L^0). \tag{50.1}$$

Step 4: the square root at the marking. Let $x_\star = (\gamma_\star - \gamma_\star^*)^2 \in C[\Gamma_L^0]$ and $t = \frac{1}{2}\,\mathrm{tr}_{D_C}(x_\star) \in C$. Put $u = \bar{\Psi}(g_\star)$. Then $\bar{\Psi}(cg_\star^{-1}c) = u^{-1}$ and

$$\bar{\rho}(\gamma_\star) - \bar{\rho}(\gamma_\star^*) = \mathrm{diag}\big(u^{w_{n,a}} - u^{-w_{n,a}}\big)_a = \bar{\zeta}\,\mathrm{diag}(w_{n,1},\dots,w_{n,n})$$

with $\bar{\zeta} = u - u^{-1}$, since $w_{n,a} \in \{1,0,-1\}$; so the residue of t is $\frac{1}{2}\bar{\zeta}^2\sum_a w_{n,a}^2 = \bar{\zeta}^2$, and $\bar{\zeta} \neq 0$ because $u^2 \neq 1$ ($g_\star$ is a marking). C is complete local and $X^2 - t$ has the simple root $\bar{\zeta}$ modulo $\mathfrak{m}_C$ ($2\bar{\zeta} \neq 0$); by Hensel's lemma there is $\zeta \in C$ with $\zeta^2 = t$ and $\zeta \equiv \bar{\zeta}$, a unit of C.

Step 5: the normal form. Let $Z_0 = \rho_0(\gamma_\star) - \rho_0(\gamma_\star^*)$. By (50.1) and $(\gamma^*)^* = \gamma$, $P^{-1}\,{}^tZ_0P = \rho_0(\gamma_\star^*) - \rho_0(\gamma_\star) = -Z_0$. For the symmetric non-degenerate form $\langle y, y'\rangle = {}^tyPy'$ this says $\langle y, Z_0y'\rangle = -\langle Z_0y, y'\rangle$, and $\det(X + Z_0) = \det(X - {}^tZ_0) = \det(X - Z_0)$; so the characteristic polynomial of Z_0 is $X^2 + s$ ($n = 2$) or $X^3 + sX$ ($n = 3$), and comparing with $\mathrm{tr}(Z_0^2) = \iota(\mathrm{tr}_{D_C}(x_\star)) = 2\zeta^2$ gives $s = -\zeta^2$. Since $\zeta \neq 0$ and $2 \neq 0$ in Ω, Z_0 has the distinct eigenvalues

$\zeta, -\zeta$ (and 0 if $n = 3$); let e_+, e_- (and e_0) be eigenvectors. For eigenvectors y, y' with eigenvalues μ, μ', $\mu'\langle y, y'\rangle = -\mu\langle y, y'\rangle$, so $\langle y, y'\rangle = 0$ unless $\mu + \mu' = 0$: $\langle e_\pm, e_\pm\rangle = 0$, $\langle e_0, e_\pm\rangle = 0$, and by non-degeneracy $\alpha = \langle e_+, e_-\rangle \neq 0$ and $\beta = \langle e_0, e_0\rangle \neq 0$. Let $H \in \mathrm{GL}_n(\Omega)$ have the columns $e_+, \alpha^{-1}e_-$ ($n = 2$), resp. $e_+, \beta^{-1/2}e_0, \alpha^{-1}e_-$ ($n = 3$; a square root exists in Ω). Then ${}^tHPH = J_n$ and $H^{-1}Z_0H = \zeta\,\mathrm{diag}(w_{n,1}, \dots, w_{n,n})$. Put $\rho = H^{-1}\rho_0 H$ and $\psi_{v,a} = \iota \circ \psi_{C,v,a}$. By (50.1), $\rho(\gamma^*) = ({}^tHPH)^{-1}\,{}^t\rho(\gamma)\,({}^tHPH) = J_n\,{}^t\rho(\gamma)J_n$: this is (Po2). And

$$\rho(\gamma_\star) - \rho(\gamma_\star^*) = \zeta\,\mathrm{diag}(w_{n,1}, \dots, w_{n,n}), \tag{50.2}$$

which gives (Po3). ρ is irreducible with Ω-span $\mathrm{M}_n(\Omega)$, and $\mathrm{tr}\,\rho(x) = \iota(\mathrm{tr}_{D_C}(x))$. (Po4): for $v \mid \ell$ and $g_1, \dots, g_n \in D_v$ ($D_v \subseteq G_L$ by (Af2)) let $X = \prod_a(\rho(\pi_L(g_a)) - \psi_{v,a}(g_a))$; by (Od2), $\mathrm{tr}(\rho(\gamma)X) = \iota(\mathrm{tr}_{D_C}(\gamma\prod_a(\pi_L(g_a) - \psi_{C,v,a}(g_a)))) = 0$ for all γ, so $X = 0$ by Step 2.

Step 6: idempotents and residues. Let $y_\star = \zeta^{-1}(\gamma_\star - \gamma_\star^*) \in C[\Gamma_L^0]$ and $x_a = \frac{1}{2}(y_\star^2 + y_\star)$ if $w_{n,a} = 1$, $x_a = \frac{1}{2}(y_\star^2 - y_\star)$ if $w_{n,a} = -1$, $x_2 = 1 - y_\star^2$ if $n = 3$. By (50.2), $\rho(y_\star) = \mathrm{diag}(w_{n,a})$ and $\rho(x_a) = e_{aa}$; by Step 4, $\bar\rho(\bar y_\star) = \bar\zeta^{-1}(\bar\rho(\gamma_\star) - \bar\rho(\gamma_\star^*)) = \mathrm{diag}(w_{n,a})$ and $\bar\rho(\bar x_a) = e_{aa}$. Identify C with $\iota(C) \subseteq \Omega$. Then for $\gamma, \delta \in \Gamma_L^0$:

$$\rho(\gamma)_{aa} = \mathrm{tr}\,\rho(x_a\gamma) = \mathrm{tr}_{D_C}(x_a\gamma) \in C, \qquad \rho(\gamma)_{ab}\rho(\delta)_{ba} = \mathrm{tr}\,\rho(x_a\gamma x_b\delta) = \mathrm{tr}_{D_C}(x_a\gamma x_b\delta) \in C,$$

with residues $\mathrm{tr}(e_{aa}\bar\rho(\gamma)) = \bar\Psi(\gamma)^{w_{n,a}}$ and $\mathrm{tr}(e_{aa}\bar\rho(\gamma)e_{bb}\bar\rho(\delta)) = \bar\rho(\gamma)_{ab}\bar\rho(\delta)_{ba}$, which is 0 for $a \neq b$ ($\bar\rho$ is diagonal). So the diagonal entries of ρ lie in C with residues $\bar\Psi^{w_{n,a}}$, and $\rho(\gamma)_{ab}\rho(\delta)_{ba} \in \mathfrak{m}_C$ for $a \neq b$.

Step 7: entry modules. Let $\mathcal{S} \subseteq \mathrm{M}_n(\Omega)$ be the C-span of $\rho(\Gamma_L^0)$, a C-algebra containing the $e_{aa} = \rho(x_a)$, with $\mathrm{tr}(\mathcal{S}) \subseteq C$; let $M_{ab} \subseteq \Omega$ be the C-span of the $\rho(\gamma)_{ab}$. Then $z \in M_{ab}$ if and only if $z\,e_{ab} \in \mathcal{S}$ (the (a,b)-entries of elements of $\mathcal{S}$ lie in M_{ab}, and $e_{aa}\rho(\gamma)e_{bb} = \rho(\gamma)_{ab}e_{ab}$). Hence: (i) $M_{ab}M_{bc} \subseteq M_{ac}$; (ii) $M_{aa} \subseteq C$ (as $z = \mathrm{tr}(z\,e_{aa})$) and $1 \in M_{aa}$; (iii) $M_{ab} = M_{\bar b\bar a}$, because $\rho(\gamma^*)_{\bar b\bar a} = \rho(\gamma)_{ab}$ by (Po2) and $\gamma \mapsto \gamma^*$ is bijective; (iv) $M_{ab}M_{ba} \subseteq \mathfrak{m}_C$ for $a \neq b$, by bilinearity and Step 6. Call $w_{n,a} - w_{n,b}$ the weight of (a,b); it is 0 exactly when $a = b$.

Step 8: weight-zero products. (P) If $w_{n,a} > w_{n,b}$ and $w_{n,c} < w_{n,d}$, then one of $b = c$, $d = a$, $\bar a = c$, $d = \bar b$ holds . For $z \in M_{ab}$, $z' \in M_{cd}$ the product zz' then lies in M_{ad}, M_{cb}, $M_{\bar b d}$ resp. $M_{c\bar a}$ (by (i), and (iii) in the last two cases), a module of weight $(w_{n,a} - w_{n,b}) + (w_{n,c} - w_{n,d})$ (using $w_{n,\bar a} = -w_{n,a}$); if this weight is 0, the two factors are of the form M_{pq}, M_{qp} with $p \neq q$ (namely $(a,b),(b,a)$; $(c,a),(a,c)$; $(\bar b,\bar a),(\bar a,\bar b)$; $(\bar a,\bar b),(\bar b,\bar a)$), so $zz' \in \mathfrak{m}_C$ by (iv). Claim: for $z_k \in M_{a_kb_k}$, $1 \le k \le s$, with total weight 0, $\prod_k z_k \in C$, and $\prod_k z_k \in \mathfrak{m}_C$ if some $a_k \neq b_k$. Induction on s; $s = 0$ is clear. If all $a_k = b_k$, use (ii). Otherwise there are factors of positive and of negative weight; combine two of them by (P). If the combined weight is non-zero, the combined factor is off-diagonal and the induction hypothesis applies to $s - 1$ factors. If it is zero, the combined factor lies in $\mathfrak{m}_C$ and the remaining $s - 2$ factors have total weight 0, so their product lies in C by induction. This proves the claim.

Step 9: finiteness and continuity of the entry modules. Write $a \sim b$ if $M_{ab} \neq 0$. By (i) and since Ω is a field, $\sim$ is transitive; by (iii), $a \sim b$ implies $\bar b \sim \bar a$; $a \sim a$ by (ii). Some $a > b$ has $a \sim b$, since otherwise ρ is upper triangular and the first basis vector spans a stable line; likewise some $a < b$ has $a \sim b$ (the last basis vector). For $n = 2$ this gives $2 \sim 1$ and $1 \sim 2$. For $n = 3$: $2 \sim 1$ or $3 \sim 2$ gives the other by (iii) and then $3 \sim 1$ by transitivity, so in all cases $3 \sim 1$, and likewise $1 \sim 3$; then $a \sim b$ implies $b \sim a$. Now fix (a,b) with $M_{ab} \neq 0$ and choose $0 \neq y = \sum_l c_l\rho(\delta_l)_{ba} \in M_{ba}$ ($c_l \in C$; $y = 1$ if $a = b$). The map $\mu\colon M_{ab} \to C$, $z \mapsto zy$, is C-linear (values in $M_{aa} \subseteq C$) and injective; so M_{ab} is isomorphic to an ideal of the Noetherian ring C, hence finitely generated. Next, $\mu(\rho(\gamma)_{ab}) = \sum_l c_l\,\mathrm{tr}_{D_C}(x_a\gamma x_b\delta_l)$ is a C-linear combination of maps $\gamma \mapsto \mathrm{tr}_{D_C}(\eta\gamma\eta')$, $\eta, \eta' \in \Gamma_L^0$, which are continuous because D_C is continuous; so it is locally constant modulo every $\mathfrak{m}_C^N$. By the Artin–Rees lemma for the ideal $\mu(M_{ab})$ there is k_0

with $\mathfrak{m}_C^{N+k_0} \cap \mu(M_{ab}) \subseteq \mathfrak{m}_C^N \mu(M_{ab}) = \mu(\mathfrak{m}_C^N M_{ab})$; as μ is injective, $\gamma \mapsto \rho(\gamma)_{ab}$ is locally constant modulo $\mathfrak{m}_C^N M_{ab}$ for every N. If $M_{ab} = 0$ both statements are trivial.

Step 10: the point of the entry algebra. By Steps 5–9, $\rho\colon \Gamma_L^0 \to \mathrm{GL}_n(\Omega)$ and the $\psi_{v,a}\colon D_v \to \Omega^\times$ satisfy (Po2)–(Po4) and, with $C \subseteq \Omega$: (Cq1) by Step 6, (Cq2) by Step 8 (the entries $\rho(\gamma)_{ab}$ lie in M_{ab}), (Cq3) and (Cq4) by Step 9, and (Cq5) because $\psi_{C,v,a} = \nu \circ \psi_{v,a}^{\mathrm{d}}$ is continuous with values in $C^\times$ and reduces to $\bar{\Psi}^{w_{n,a}}$. By Lemma 49.16 there are $\mathcal{O}$-algebra homomorphisms $\xi_0\colon E \to \Omega$ and $\beta\colon W \to C$ with $\beta(\mathfrak{m}_W) \subseteq \mathfrak{m}_C$, $\xi_0(\rho^{\mathrm{u}}(\gamma)_{ab}) = \rho(\gamma)_{ab}$, $\xi_0(\psi_{v,a}^{\mathrm{u}}(g)) = \psi_{v,a}(g)$ and $\xi_0|_W = \iota \circ \beta$. Let $\xi = \ker \xi_0$, a prime of E as Ω is a field. Some lower entry $\rho(\gamma)_{ab}$ $(a > b)$ and some upper entry are non-zero (Step 9), so $I_E^- \not\subseteq \xi$ and $I_E^+ \not\subseteq \xi$; for $n = 3$, if $\rho(\gamma)_{ab} = 0$ for all γ whenever $w_{3,a} - w_{3,b}$ is odd, the second basis vector spans a stable line; so $I_E^{\mathrm{odd}} \not\subseteq \xi$. Thus ξ is of irreducible type.

Step 11: (Mk1) and (Mk2). Let $f = \beta \circ \vartheta\colon B \to C$ (recall $\vartheta(B) = W$); it is an $\mathcal{O}$-algebra homomorphism with $f(\mathfrak{m}_B) \subseteq \beta(\mathfrak{m}_W) \subseteq \mathfrak{m}_C$, a morphism of $\mathrm{CNL}_{\mathcal{O}}$. For $\gamma \in \Gamma_L^0$ and $1 \le i \le n$, $\vartheta(\Lambda_i(\gamma))$ is the i-th coefficient (up to sign) of the characteristic polynomial of $\rho^{\mathrm{u}}(\gamma)$, a polynomial in its entries, so $\iota(f(\Lambda_i(\gamma))) = \xi_0(\vartheta(\Lambda_i(\gamma)))$ is the same coefficient of $\rho(\gamma)$, i.e. of $\rho_0(\gamma)$, which is ι of the coefficient of $\chi_{D_C}(\gamma)$; by injectivity of ι, $f \circ D^{\mathrm{u}}$ and D_C have the same characteristic polynomials on Γ_L^0, hence are equal by (Ch1). Likewise $\iota(f(\psi_{v,a}^{\mathrm{d}}(g))) = \xi_0(\psi_{v,a}^{\mathrm{u}}(g)) = \iota(\psi_{C,v,a}(g))$. So f and ν both classify $(D_C, (\psi_{C,v,a}))$, and $f = \nu$ by uniqueness. Hence $\ker \vartheta \subseteq \ker f = \mathfrak{q}$, which is (Mk1). For $y = \vartheta(b) \in W$: $y \in \xi$ iff $\iota(\beta(y)) = 0$ iff $\nu(b) = 0$ iff $b \in \mathfrak{q}$; so $\xi \cap W = \vartheta(\mathfrak{q})$, which with Step 10 is (Mk2). □

51 Symplectic automorphy

This section proves Theorem 51.1, the automorphy theorem for the symplectic Eisenstein data attached to the characters of cases (II) and (III), which Proposition 44.3 uses: the datum is finite in weight zero (Sa1), has no endoscopic points (Sa2), and every irreducible point is automorphic (Sa3). The points of a symplectic datum are ordinary test representations, representations of $G_{\mathbb{Q}}$ over the valuation ring of a test field with an ordinary symplectic flag at ℓ (Definition 45.26, Lemma 45.37, Lemma 45.41), and the theorem follows from Theorem 51.4 on such representations. A test representation is essential if it is irreducible or a non-split block extension, and promodular if, after conjugation, its restriction to a fit and deep auxiliary tower is a point of a space of marked pairs at which a modular ideal vanishes (Definition 51.3). The core statement is Proposition 51.5: every essential test representation is promodular. Promodular test representations are not essential in characteristic ℓ, and in characteristic zero they are weakly pure, and automorphic when irreducible (Lemma 51.19).

The plan is that of Sections 48–50 in rank four. The auxiliary objects are totally real towers $\mathscr{T}$ and, over them, spaces of marked ordinary symplectic pairs with modular ideals (Definition 51.2, Definition 51.3). An essential test representation, restricted to a deep tower, is joined by a chain of residual pairs to the reduction of a seed pair (Section 52); every space carries a modular ideal, and modularity propagates along chains (Section 53); and seed pairs exist (Subsection 51.2). Subsection 51.1 proves the theorem from Proposition 51.5 and proves that proposition from these inputs; Subsection 51.3 proves the finiteness in weight zero and excludes endoscopic points through block extensions.

51.1 Automorphy at ordinary test representations and promodularity

This subsection proves Theorem 51.1 from Theorem 51.4, and Theorem 51.4 from Proposition 51.5 and Lemma 51.19. It defines the auxiliary towers, with their marked ordinary sym-

plectic pairs, residual pairs, chains and Selmer bounds (Definition 51.2), and the spaces of marked pairs over them, with modular ideals, seeds and promodular representations (Definition 51.3). The proof of Proposition 51.5 assembles the inputs of the next two sections. An essential test representation becomes, over a deep tower, a pair over a complete local ring whose reduction is chained to the reduction of a seed pair (Proposition 52.1, Proposition 52.14); every space carries a modular ideal (Theorem 53.1); and such an ideal, present on the component through one residual pair, is present on every component through a residual pair chained to it (Theorem 53.31). The propagation needs a Selmer bound C with $30(C+1)^2 \leq d$ for the degree d of the tower, and Proposition 51.6 supplies it: along the cyclotomic layers of a good tower the Selmer ranks stay bounded, by the theorem of Ferrero and Washington through Lemma 39.5, while the degree grows.

Theorem 51.1. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum attached to some $(\psi, \mathfrak{h}, t, \chi)$, with odd prime ℓ. Then:*

(Sa1) *$\mathfrak{D}$ is finite in weight zero;*

(Sa2) *$\mathfrak{D}$ has no endoscopic points;*

(Sa3) *there is an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ such that every irreducible point of $\mathfrak{D}$ is automorphic for ι.*

Proof. Let $\mathfrak{D}$ be attached to $(\psi, \mathfrak{h}, t, \chi)$, with odd prime ℓ. Let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism as in (Ta2) of Theorem 51.4; (Ta1) of Theorem 51.4 also applies to $\mathfrak{D}$. A point r of $\mathfrak{D}$ is irreducible in the sense of Definition 44.2 exactly when it is irreducible in the sense of Definition 45.26 for the standard test field (both say that the $\overline{\mathbb{Q}}_\ell$-span of $r(G_{\mathbb{Q}})$ is $\mathrm{M}_4(\overline{\mathbb{Q}}_\ell)$).

(Sa1): $\mathfrak{D}$ is finite in weight zero. Suppose not. By Proposition 51.20 there are a test field (K, V, a) of characteristic ℓ, a point r of $\mathfrak{D}$ with values in V and a product $x = r(g_1)_{i_1 j_1} \cdots r(g_k)_{i_k j_k}$ with $\sum_s \mathrm{wt}_4(i_s, j_s) = (0,0)$ such that $x^{\ell^n} \neq x$ for every $n \geq 1$. By Lemma 45.37 there is u such that (r, u) is an ordinary test representation of $\mathfrak{D}$ with values in V. By (Zr1) of Proposition 45.36, r is irreducible, of diagonal type or of block type.

- If r is irreducible, it is essential, which contradicts (Ta1) for (r, u).
- If r is of diagonal type, (Zr2) of Proposition 45.36 gives $x^{\ell^n} = x$ for some $n \geq 1$, a contradiction.
- If r is of block type, (Zr3) of Proposition 45.36 gives an ordinary test representation (r_1, u_1) of $\mathfrak{D}$ with values in V such that every $r_1(g)$ is block diagonal and the matrices $(r_1(g)_{ab})_{a,b\in\{1,3\}}$ span $\mathrm{M}_2(K)$. By Proposition 51.23 there is an ordinary test representation (r_2, u_2) of $\mathfrak{D}$ with values in V such that r_2 is a non-split block extension, hence essential; this contradicts (Ta1) for (r_2, u_2).

Hence $\mathfrak{D}$ is finite in weight zero.

(Sa2): $\mathfrak{D}$ has no endoscopic points. Suppose r is a point of $\mathfrak{D}$ such that every $r(g)$ is block diagonal and the matrices $(r(g)_{ab})_{a,b\in\{1,3\}}$ span $\mathrm{M}_2(\overline{\mathbb{Q}}_\ell)$. By Lemma 45.41, r is a point of $\mathfrak{D}$ with values in $\overline{\mathbb{Z}}_\ell$ for the standard test field, and by Lemma 45.37 there is u such that (r, u) is an ordinary test representation of $\mathfrak{D}$ with values in $\overline{\mathbb{Z}}_\ell$. By Proposition 51.23, applied with $K = \overline{\mathbb{Q}}_\ell$, there is an ordinary test representation (r', u') of $\mathfrak{D}$ with values in $\overline{\mathbb{Z}}_\ell$ such that r' is a non-split block extension and $r'(g)$, $r(g)$ have the same characteristic polynomial for every g. As r' is essential, (Ta2) shows that r' is weakly pure for ι. Weak purity depends only on the

characteristic polynomials of the $r(\sigma)$ (the eigenvalues of $r(\sigma)$ and $r'(\sigma)$ coincide), so r is weakly pure for ι, with the same open subgroup H and integer M as in the definition of weak purity in Definition 45.26. But (r,u) is an ordinary test representation with values in $\overline{\mathbb{Z}}_\ell$ and every $r(g)$ is block diagonal, so by Lemma 51.25 r is not weakly pure for ι. This contradiction shows that $\mathfrak{D}$ has no endoscopic points.

(Sa3): every irreducible point is automorphic for ι. Let r be an irreducible point of $\mathfrak{D}$. By Lemma 45.41 and Lemma 45.37 there is u such that (r,u) is an ordinary test representation of $\mathfrak{D}$ with values in $\overline{\mathbb{Z}}_\ell$, and r is irreducible for the standard test field. By (Ta2), r satisfies (Sy2) for ι, that is, r is automorphic for ι. The same ι serves for every irreducible point. □

Definition 51.2. Let $\mathfrak{D}$ be a symplectic Eisenstein datum with odd prime ℓ, coefficient ring $\mathcal{O}$ (maximal ideal $\mathfrak{m}_\mathcal{O}$, residue field $\mathbb{F}_\mathcal{O}$), characters $(\kappa_1,\kappa_2,\kappa_3,\kappa_4)=(\theta,\Psi,\alpha,\mathbf{1})$, and put $T'=T\setminus\{\ell\}$. Bars denote reduction modulo $\mathfrak{m}_\mathcal{O}$. Indices i,j run over $\{1,2,3,4\}$. Characters of $G_\mathbb{Q}$ are evaluated on $G_F\subseteq G_\mathbb{Q}$ by restriction.

Weights. Put $w_\alpha=(1,0)$, $w_\beta=(-1,1)$, $w_\Psi=(0,1)$, $w_\theta=(1,1)$ in $\mathbb{Z}^2$. The **odd roots** are $\pm w_\alpha,\pm w_\beta,\pm w_\theta$, and the **roots** are the odd roots and $\pm w_\Psi$. For $i\neq j$, $\mathrm{wt}_4(i,j)$ is a root. For a set $\Sigma_0\subseteq\mathbb{Z}^2$ and a ring R, a matrix $Y\in \mathrm{M}_4(R)$ is **supported in** Σ_0 if $Y_{ij}=0$ whenever $i\neq j$ and $\mathrm{wt}_4(i,j)\notin\Sigma_0$; a map r with values in $\mathrm{M}_4(R)$ is supported in Σ_0 if all its values are, and it **has an entry of weight** w if $r(g)_{ij}\neq 0$ for some g and some i,j with $\mathrm{wt}_4(i,j)=w$. Put $\Sigma_P=\{w_\beta,-w_\alpha,-w_\theta,-w_\Psi\}$, $\Sigma_M=\{-w_\alpha,-w_\beta,-w_\Psi,-w_\theta\}$, $\Sigma_V=\{w_\beta,-w_\alpha,-w_\theta\}$, $\Sigma_Q=\{-w_\alpha,-w_\Psi,-w_\theta\}$, $\Sigma_A=\{-w_\beta,-w_\Psi,-w_\theta\}$.

Labellings. A **labelling** is $\epsilon\in\{0,1\}$; let $\lambda_0=\mathrm{id}$ and let λ_1 be the permutation $1\mapsto 3$, $2\mapsto 4$, $3\mapsto 1$, $4\mapsto 2$, and put $\kappa_i^\epsilon=\kappa_{\lambda_\epsilon(i)}$. The labelling is **admissible** if $\epsilon=0$, or $\bar\Psi(g)=1$ for all $g\in D_\ell$.

Auxiliary towers. An **auxiliary tower** for $\mathfrak{D}$ is a tuple $\mathscr{T}=(\epsilon,E,F,g_\mathscr{T},\Sigma,(\bar v)_{v\in\Sigma})$ of an admissible labelling ϵ, an imaginary quadratic field $E\subset\overline{\mathbb{Q}}$, a totally real number field $F\subset\overline{\mathbb{Q}}$ abelian over $\mathbb{Q}$, an element $g_\mathscr{T}\in G_F$, a finite set Σ and places $\bar v$ of $\overline{\mathbb{Q}}$ above ℓ, such that every place of $\overline{\mathbb{Q}}$ above ℓ is G_F-conjugate to $\bar v$ for exactly one $v\in\Sigma$, and $\bar v$ and $\bar\ell$ induce the same place of E. Put $L=FE$ (a CM field), $d=[F:\mathbb{Q}]$, $G_L\subseteq G_F$, $G'=\{g\in G_L:\chi_{\mathrm{cyc}}(g)=1\}$, and let $D_v\supseteq I_v$ be the decomposition and inertia groups of $\bar v$ in G_F. For a finite set W of primes, the **frame** of $\mathscr{T}$ with extra primes W consists of the characters $\chi_i^\mathscr{T}=\kappa_i^\epsilon|_{G_F}$, the **root characters** $\mathrm{rt}_w=(\chi_1^\mathscr{T}/\chi_2^\mathscr{T})^{w_1}(\chi_1^\mathscr{T}/\chi_3^\mathscr{T})^{w_2}$ for $w=(w_1,w_2)\in\mathbb{Z}^2$, the set $N_W\subseteq G_F$ of all $g\in I_{\bar w}\cap G_F$ with $\bar r_0(g)=1$, $\bar w$ running over the places of $\overline{\mathbb{Q}}$ not above ℓ nor above a prime of W, and the sets $\mu_v=\{s\in D_v:\ s^{\ell^a}\in\overline{[D_v,D_v]}\text{ for some }a\geq 0\}$. Write $N=N_\emptyset$ and $U_v=\{s\in I_v:\chi_{\mathrm{cyc}}(s)=1,\ \bar r_0(s)=1\}$.

The tower is **good** if:

(Oq1) d is even, and $I_{\bar w}\cap G_F\subseteq G_L$ for every place $\bar w$ of $\overline{\mathbb{Q}}$;

(Oq2) every prime of S splits in E;

(Oq3) for every $w\in\mathbb{Z}^2$ some $g\in G_L$ has $\overline{\mathrm{rt}_w}(g)\neq\bar\omega_\ell(g)$; for every root w some $g\in G_L$ has $\overline{\mathrm{rt}_w}(g)\neq 1$; and for some root w and every $1\leq n\leq 4$ some $g\in G_L$ has $\overline{\mathrm{rt}_w}(g)^n\neq 1$;

(Oq4) every homomorphism $\eta\colon G_L\to\mathbb{F}_\mathcal{O}^\times$ with $\eta^2=1$, trivial on $G_L\cap\ker\omega_\ell$, and such that for every i there is j with $\eta\,\bar\kappa_i^\epsilon=\bar\kappa_j^\epsilon$ on G_L, is trivial;

(Oq5) $\chi(g)=1$ for every $g\in I_{\bar w}\cap G_F$ with $\bar r_0(g)=1$, $\bar w$ any place not above ℓ; and there is $m\geq 1$ prime to ℓ with $\chi(g)^m=1$ for all $g\in G_F$ with $\chi_{\mathrm{cyc}}(g)=1$;

(Oq6) $\bar r_0(g)=1$ for every $g\in I_{\bar w}\cap G_F$, $\bar w$ any place above ℓ;

(Oq7) the residues $\bar\kappa_i^\xi(g_{\mathscr{T}})$, $1 \le i \le 4$, are pairwise distinct;

(Oq8) for every place $\bar w$ above ℓ, either $\bar\omega_\ell(g) \ne 1$ for some $g \in I_{\bar w} \cap G_F$, or there is $g \in D_{\bar w} \cap G_F$ with $\bar\omega_\ell(g) = 1$, $\overline{\mathrm{rt}_{w_\Psi}}(g) = 1$ and $\overline{\mathrm{rt}_{w_\alpha}}(g) \ne 1$.

It is **of case (II)** if:

(Uq1) $\overline{\mathrm{rt}_{w_\Psi}}^2 = 1$ on G_F, $\overline{\mathrm{rt}_{w_\Psi}}$ is trivial on $I_{\bar w} \cap G_F$ for every place $\bar w$ and on $D_{\bar w} \cap G_F$ for every place $\bar w$ above ℓ;

(Uq2) there is a real quadratic field k_0 in which 2 and ℓ split such that $\overline{\mathrm{rt}_{w_\Psi}}(g) = 1$ if and only if $g \in G_{k_0}$, for $g \in G_F$;

(Uq3) ξ has order a power of 2 and is unramified at every odd prime;

(Uq4) for every place $\bar w$ above ℓ some $g \in D_{\bar w} \cap G_F$, and for every place $\bar w$ above 2 some $g \in I_{\bar w} \cap G_F$, has $\overline{\mathrm{rt}_{w_\alpha}}(g)^4 \ne 1$.

It is **of case (III)** if $\ell \ge 5$, $T' = \emptyset$, χ has finite order prime to ℓ, and there is an imaginary quadratic field E' with $\ell \nmid \mathrm{disc}(E')$, in which ℓ and every prime of S split, such that for $g \in G_F$: $\overline{\mathrm{rt}_{w_\alpha}}(g) = 1$ if $g \in G_{E'}$ and $\overline{\mathrm{rt}_{w_\alpha}}(g) = -1$ otherwise. It is **fit** if it is good and either $\mathfrak{D}$ is attached to a host character in case (II) and the tower is of case (II), or $\mathfrak{D}$ is attached to a host character in case (III), $\ell \ge 5$, $T' = \emptyset$ and χ has finite order prime to ℓ. It is **deep** if $\chi|_{G_F}$ has finite order prime to ℓ, and **tame** if for every place $\bar w$ above ℓ and $g \in I_{\bar w}$ (in $G_{\mathbb{Q}}$) some g^m with m prime to ℓ lies in G_F.

Order, layers, primes. For auxiliary towers $\mathscr{T}, \mathscr{T}'$ we write $\mathscr{T} \le \mathscr{T}'$ if they have the same labelling and the same E, $F \subseteq F'$, $g^{\ell^j} \in G_{F'}$ for some $j \ge 0$ and all $g \in G_F$, and $g_{\mathscr{T}'} = g_{\mathscr{T}}^{\ell^{j'}}$ for some $j' \ge 0$. A field F' is the **layer** of index $m \ge 0$ over F if $G_{F'} = \{g \in G_F : \chi_{\mathrm{cyc}}(g)^{\ell-1} \equiv 1 \pmod{\ell^{m+1}}\}$. For a prime w, w **dies in** F if every continuous homomorphism $G_{\mathbb{Q}} \to \mathbb{Z}/\ell\mathbb{Z}$ that is trivial on the inertia groups of all places not above w is trivial on G_F; and w **splits for** $\mathscr{T}$ if $D_{\bar w} \subseteq G_F \cap \ker \bar r_0 \cap \ker \omega_\ell$ for every place $\bar w$ above w.

Marked ordinary pairs. Let $\mathscr{T}$ be an auxiliary tower, W a finite set of primes, R a commutative ring and $a\colon \mathcal{O} \to R$ a ring homomorphism. A **marked ordinary pair** over $(\mathscr{T}, W)$ with values in R (for a) is $P = (r, (n_v)_{v\in\Sigma})$ with $r\colon G_F \to \mathrm{M}_4(R)$ a homomorphism of monoids and $n_v \in \mathrm{M}_4(R)$ such that, with $\psi^P_{v,i}(s) = (n_v^{-1} r(s) n_v)_{ii}$:

(Mq1) $r(g)$ has similitude $a(\theta(g))$ for all $g \in G_F$, and $r(g) = 1$ for $g \in N_W$;

(Mq2) $r(g_{\mathscr{T}})$ is diagonal;

(Mq3) each n_v has similitude 1, is lower triangular and has all diagonal entries 1;

(Mq4) $n_v^{-1} r(s) n_v$ is upper triangular for $s \in D_v$, and $\psi^P_{v,i}(s) = 1$ for $s \in \mu_v$ and all i.

If $\mathscr{T} \le \mathscr{T}'$ (with fields $F \subseteq F'$ and index sets Σ, Σ'), a pair $P' = (r', (n'_{v'}))$ over $\mathscr{T}'$ **lies over** a pair $P = (r, (n_v))$ over $\mathscr{T}$ with values in the same ring if $r' = r|_{G_{F'}}$ and for every $v' \in \Sigma'$ there are $v \in \Sigma$ and $g \in G_F$ with $g(\bar v) = \bar v'$ and $n'^{-1}_{v'} r(g) n_v$ upper triangular. For $f\colon R \to R'$ put $f(P) = (f \circ r, (f(n_v)))$ (entrywise); P is **continuous** for an ideal I of R if every $g \mapsto r(g)_{ij} \bmod I^k$ is locally constant. When R is local with residue field κ_R and a is local, the **reduction** of P is its image under $R \to \kappa_R$.

Residual pairs. Let $a\colon \mathcal{O} \to R$ with $a(\mathfrak{m}_{\mathcal{O}}) = 0$. A **residual pair** over $(\mathscr{T}, W)$ with values in R is a marked ordinary pair $c = (r, (n_v))$ for a with $g \mapsto r(g)$ locally constant, such that

(Wq1) $r(g)_{ii} = a(\chi_i^{\mathscr{T}}(g))$ for $g \in G_F$, and $\psi_{v,i}^c(s) = a(\chi_i^{\mathscr{T}}(s))$ for $s \in D_v$;

(Wq2) either r is supported in Σ_P and every n_v in Σ_Q, or r is supported in Σ_M.

It **lies in** V if r is supported in Σ_V and every n_v is supported in Σ_Q with $(n_v)_{31} \neq 0$; it **lies in** A if r is supported in Σ_A and $(n_v)_{21} \neq 0$ for all v. It is **good** if one of the following holds: it has entries of weights $-w_\Psi$ and one of $w_\beta, -w_\alpha$; it has entries of weights $-w_\alpha$ and $-w_\beta$; it lies in V and has an entry of weight $-w_\alpha$; it lies in V, has entries of weights w_β and $-w_\theta$, and for every v some $s \in D_v$ has $a(\mathrm{rt}_{w_\alpha}(s)) \neq 1$. It is **good in case (III)** if it is good, or it lies in A and has an entry of weight $-w_\Psi$, or it lies in A and has entries of weights $-w_\beta$ and $-w_\theta$. Two residual pairs c, c' with values in a field k are **linked** if there is a residual pair with values in $k[X]$ (for a followed by $k \subseteq k[X]$) whose images under $X \mapsto 0$ and $X \mapsto 1$ are c and c'. A **chain** from c to c' is a sequence $c = c_0, c_1, \dots, c_m = c'$ $(m \geq 0)$ of good residual pairs with c_{t-1}, c_t linked for $1 \leq t \leq m$; a **chain in case (III)** is the same with "good in case (III)".

Selmer bounds. Let k be a field and $a\colon \mathcal{O} \to k$ with $a(\mathfrak{m}_\mathcal{O}) = 0$. For $w \in \mathbb{Z}^2$ let $Z(w)$ be the k-space of locally constant $z\colon G_F \to k$ with $z(gh) = z(g) + a(\mathrm{rt}_w(g))z(h)$, $z(N) = 0$, and $z(g_{\mathscr{T}}) = 0$ if $w \neq 0$; $Z_v(w)$ the space of locally constant $z\colon D_v \to k$ with the same cocycle identity; and $Z^{\mathrm{str}}(w) \subseteq Z(w)$ the subspace of z for which there are $y_v \in k$ with $z(s) = y_v - a(\mathrm{rt}_w(s))y_v$ for all $v \in \Sigma$, $s \in D_v$. An integer $C \geq 0$ is a **Selmer bound** for $(\mathscr{T}, a)$ in degree d if $\#\Sigma \leq C$, $\dim Z^{\mathrm{str}}(w) \leq C$ for every root w, $\dim Z(w) \leq C$ for $w \in \{0, w_\Psi, -w_\Psi\}$, and $\dim Z(w) \leq d + C$ and $\dim \prod_v Z_v(w) \leq d + C$ for every odd root w.

Definition 51.3. Let $\mathfrak{D}$ be a symplectic Eisenstein datum with prime ℓ, coefficient ring $\mathcal{O}$, characters κ_i and e, and put $(m_1, m_2, m_3, m_4) = (e+1, e, 1, 0)$. Let $\mathscr{T}$ be an auxiliary tower for $\mathfrak{D}$ and $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring with residue field $\mathbb{F}_{\mathcal{O}'}$. Pairs are over $(\mathscr{T}, \emptyset)$; for an $\mathcal{O}'$-algebra R they are taken for $\mathcal{O} \subseteq \mathcal{O}' \to R$.

Spaces. A **space** for $(\mathscr{T}, \mathcal{O}')$ is a tuple $\mathscr{X} = (B, A, P, \mathrm{cp}, (\psi_{v,i}^B))$ with:

(Zq1) $B \in \mathrm{CNL}_{\mathcal{O}'}$ with maximal ideal $\mathfrak{m}_B$; A a B-algebra of finite type with $B \to A$ injective; $P = (r_P, (n_v))$ a marked ordinary pair with values in A, continuous for $\mathfrak{m}_B A$, such that the entries of all $r_P(g)$ and all n_v generate A as a B-algebra;

(Zq2) maps $\mathrm{cp}\colon G_F \to B[X]$ and $\psi_{v,i}^B\colon D_v \to B$ such that $\mathrm{cp}(g)$ maps to the characteristic polynomial of $r_P(g)$ and $\psi_{v,i}^B(s)$ to $\psi_{v,i}^P(s)$, both are continuous for the $\mathfrak{m}_B$-adic topology (coefficientwise), and modulo $\mathfrak{m}_B$ they are $\prod_i (X - \bar{\chi}_i^{\mathscr{T}}(g))$ and $\bar{\chi}_i^{\mathscr{T}}(s)$;

(Zq3) two $\mathcal{O}'$-algebra homomorphisms $B \to R$, $R \in \mathrm{CNL}_{\mathcal{O}'}$, that agree on all coefficients of all $\mathrm{cp}(g)$ and on all $\psi_{v,i}^B(s)$ are equal;

(Zq4) for every $R \in \mathrm{CNL}_{\mathcal{O}'}$, every residual pair c with values in $\mathbb{F}_{\mathcal{O}'}$ and every marked ordinary pair Q with values in R, continuous for $\mathfrak{m}_R$, with reduction c, there is an $\mathcal{O}'$-algebra homomorphism $f\colon A \to R$ with $f(P) = Q$;

(Zq5) for every residual pair c with values in $\mathbb{F}_{\mathcal{O}'}[X]$ there is an $\mathcal{O}'$-algebra homomorphism $f\colon A \to \mathbb{F}_{\mathcal{O}'}[X]$ with $f(P) = c$ and $f(\mathfrak{m}_B) = 0$.

Let $\mathfrak{J} \subseteq B$ be an ideal and $\mathfrak{p}$ a prime of A. The ideal $\mathfrak{J}$ is **somewhere at** $\mathfrak{p}$ if some minimal prime $\mathfrak{q} \subseteq \mathfrak{p}$ of A contains $\mathfrak{J}A$, and **everywhere at** $\mathfrak{p}$ if every minimal prime $\mathfrak{q} \subseteq \mathfrak{p}$ of A contains $\mathfrak{J}A$. For a ring homomorphism $y\colon A \to C$, $\mathfrak{J}$ is **local at** y if every prime $\mathfrak{q}$ of B with $y(\mathfrak{q}) = 0$ contains $\mathfrak{J}$.

Weights and seeds. A pair P with values in $\overline{\mathbb{Q}}_\ell$ **has weights** $h \in \mathbb{Z}^4$ if there is an open subgroup $U \subseteq G_F$ with $\psi_{v,i}^P(s)^2 = \chi_{\mathrm{cyc}}(s)^{h_i}$ for all v, i and $s \in U \cap I_v$. It is a **seed** for ι if $r_P|_{G'}$

is irreducible (the $\overline{\mathbb{Q}}_\ell$-span of $r_P(G')$ is $\mathrm{M}_4(\overline{\mathbb{Q}}_\ell)$), and there are $h \in \mathbb{Z}^4$ with $h_i - h_{i+1} \in 2\mathbb{Z}_{\geq 2}$ $(i = 1, 2, 3)$ such that P has weights h, a continuous character $\kappa\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$, integers a and $k \geq 1$ with $a + h_1$ even and $(\kappa(s)^2\chi_{\mathrm{cyc}}(s)^{-a})^k = 1$ for every s in the inertia group in G_L of a place above ℓ, and a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_4(\mathbb{A}_L)$ with $r_P|_{G_L} \otimes \kappa \leftrightarrow_\iota \pi$. For a coefficient ring $\mathcal{O}_1 \supseteq \mathcal{O}$, a **seed pair** with values in $\mathcal{O}_1$ is a marked ordinary pair Q with values in $\mathcal{O}_1$, continuous for $\mathfrak{m}_{\mathcal{O}_1}$, whose reduction satisfies (Wq1) and (Wq2) and whose image in $\overline{\mathbb{Q}}_\ell$ is a seed for ι.

Modular ideals. Let $\mathscr{X}$ be a space for $(\mathscr{T}, \mathcal{O}')$ and $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism. An ideal $\mathfrak{J} \subseteq B$ is **modular** for ι if:

(Jq1) $\mathfrak{m}_B^n \subseteq \mathfrak{J} + \mathfrak{m}_{\mathcal{O}'}B + (\psi_{v,i}^B(s) - 1 :\ v \in \Sigma,\ 1 \leq i \leq 4,\ s \in I_v)$ for some $n \geq 1$;

(Jq2) for every finite field k with an $\mathcal{O}'$-algebra structure and every $\mathcal{O}'$-algebra homomorphism $y\colon A \to k[[t]]$ with $y(\mathfrak{m}_B) \subseteq tk[[t]]$, $y(\mathfrak{J}) = 0$, such that for all v and $i \neq j$ some $s \in U_v$ has $y(\psi_{v,i}^P(s)) \neq y(\psi_{v,j}^P(s))$, and such that $y \circ r_P$ is irreducible over the fraction field of $k[[t]]$, $\mathfrak{J}$ is local at y;

(Jq3) $\mathfrak{J}$ is local at every $\mathcal{O}'$-algebra homomorphism $y\colon A \to \overline{\mathbb{Q}}_\ell$ such that $y(P)$ is a seed for ι;

(Jq4) for every $\mathcal{O}'$-algebra homomorphism $y\colon A \to \overline{\mathbb{Q}}_\ell$ with $y(\mathfrak{J}) = 0$ such that $y(P)$ has weights $(2m_1, 2m_2, 2m_3, 2m_4)$, every homomorphism $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ with $r|_{G_F} = y \circ r_P$ is weakly pure for ι;

(Jq5) for every such y, every continuous homomorphism $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ with $r|_{G_F} = y \circ r_P$, $r(g)$ of similitude $\theta(g)$ for all g, and $r|_{G_L}$ irreducible, satisfies (Sy2) for ι.

Promodular representations. Let (K, V, a) be a test field and ι as above. A homomorphism $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(V)$ is **promodular** for $\mathfrak{D}$ and ι if there are a fit and deep auxiliary tower $\mathscr{T}$, a coefficient ring $\mathcal{O}' \supseteq \mathcal{O}$, a space $\mathscr{X} = (B, A, P, \dots)$ for $(\mathscr{T}, \mathcal{O}')$, an ideal $\mathfrak{J} \subseteq B$ modular for ι, a ring homomorphism $y\colon A \to V$ and a homomorphism $r_1\colon G_{\mathbb{Q}} \to \mathrm{M}_4(V)$ of monoids such that: y restricts to a on $\mathcal{O}'$; $y(\mathfrak{m}_B) \subseteq \mathfrak{m}_V$; $y(\mathfrak{J}) = 0$; $r_1 = uru^{-1}$ for some $u \in \mathrm{GL}_4(K)$; $r_1(g)$ has similitude $a(\theta(g))$ for all g; $r_1|_{G_F} = y \circ r_P$; and $y(\psi_{v,i}^P(s)) = a(\kappa_i(s))$ for all v, i and $s \in I_v$.

Theorem 51.4. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum attached to some $(\psi, \mathfrak{h}, t, \chi)$, with odd prime ℓ. Then:*

(Ta1) *for every test field (K, V, a) of characteristic ℓ and every ordinary test representation (r, u) of $\mathfrak{D}$ with values in V, the representation r is not essential;*

(Ta2) *there is an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ such that for every ordinary test representation (r, u) of $\mathfrak{D}$ with values in $\overline{\mathbb{Z}}_\ell$ (for the standard test field): if r is essential, then r is weakly pure for ι; if r is irreducible, then r satisfies (Sy2) for ι.*

Proof. An isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ exists: both are algebraically closed fields of characteristic 0 of the cardinality of the continuum, hence have transcendence bases over $\mathbb{Q}$ of the same cardinality (Steinitz). Fix one.

(Ta1). Let (K, V, a) be a test field of characteristic ℓ and (r, u) an ordinary test representation of $\mathfrak{D}$ with values in V. If r were essential, it would be promodular for $\mathfrak{D}$ and ι by Proposition 51.5, hence not essential by item 1 of Lemma 51.19. So r is not essential.

(Ta2). Let (r, u) be an ordinary test representation of $\mathfrak{D}$ with values in $\overline{\mathbb{Z}}_\ell$ for the standard test field. If r is essential, it is promodular for $\mathfrak{D}$ and ι by Proposition 51.5, hence weakly pure for ι by item 2 of Lemma 51.19. If r is irreducible, it is essential, hence promodular, and item 2 of Lemma 51.19 gives (Sy2) for ι. □

Proposition 51.5. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum attached to some $(\psi, \mathfrak{h}, t, \chi)$ and $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism. For every test field (K, V, a) and every ordinary test representation (r, u) of $\mathfrak{D}$ with values in V such that r is essential, r is promodular for $\mathfrak{D}$ and ι.*

Proof. Write k_V for the residue field of V. By Definition 44.1, $\mathfrak{D}$ is attached in case (II) or in case (III).

Step 0: two facts on towers. (a) The order $\leq$ is transitive: if $\mathscr{T} \leq \mathscr{T}' \leq \mathscr{T}''$, the labellings and the fields E agree, $F \subseteq F' \subseteq F''$, for $g \in G_F$ we have $g^{\ell^j} \in G_{F'}$ and then $(g^{\ell^j})^{\ell^{j_2}} \in G_{F''}$, and $g_{\mathscr{T}''} = g_{\mathscr{T}'}^{\ell^{j'_2}} = g_{\mathscr{T}}^{\ell^{j'+j'_2}}$. (b) If w dies in F_1 and $F \supseteq F_1$, then w dies in F, as $G_F \subseteq G_{F_1}$.

Step 1: a deep fit tower, a seed pair, and a pair chained to it. Case (II). By Proposition 52.1 there are a fit, tame auxiliary tower $\mathscr{T}_1$ of case (II), $u' \in \mathrm{GL}_4(K)$ and $r' = u'ru'^{-1}$ with $r'(g)$ of similitude $a(\theta(g))$, with the property stated there. By (Xq1) of Proposition 51.6 (for $\mathscr{T}_1$) there is a C which is a Selmer bound for $(\mathscr{T}, a_0)$ in degree $[F : \mathbb{Q}]$ for every $\mathscr{T} \geq \mathscr{T}_1$ with layer field and every $a_0\colon \mathcal{O} \to k$ killing $\mathfrak{m}_{\mathcal{O}}$; by (Xq2) with $n = 30(C+1)^2$ there is a deep $\mathscr{T} \geq \mathscr{T}_1$ whose field F is a layer over F_1 with $d = [F : \mathbb{Q}] \geq 30(C+1)^2$; by (Xq3), $\mathscr{T}$ is good, fit and of case (II). By Proposition 51.8 (for $\mathscr{T}$, ι) there are $\mathcal{O}_1 \supseteq \mathcal{O}$ and a seed pair Q over $(\mathscr{T}, \emptyset)$ with values in $\mathcal{O}_1$ whose reduction lies in V and has an entry of weight $-w_\alpha$ or entries of weights $w_\beta, -w_\theta$. The property of $\mathscr{T}_1$ applied to $\mathscr{T}$ and Q gives $\mathcal{O}_2 \supseteq \mathcal{O}_1$ such that (Nq1), (Nq2) hold for every coefficient ring $\mathcal{O}' \supseteq \mathcal{O}_2$.

Case (III). By Proposition 52.14 there are a fit, tame auxiliary tower $\mathscr{T}_0$ of case (III), u' and $r' = u'ru'^{-1}$ as above, with the property stated there. $\mathscr{T}_0$ is good. By Proposition 51.14 (for $\mathscr{T}_0$, ι, (K, V, a)) there are a prime w splitting for $\mathscr{T}_0$ and a residual pair s_0 over $(\mathscr{T}_0, \{w\})$ with values in k_V, supported in Σ_M, with the seed property stated there. The property of $\mathscr{T}_0$ applied to (w, s_0) gives a tame $\mathscr{T}_1 \geq \mathscr{T}_0$ in whose field F_1 the prime w dies; $\mathscr{T}_1$ is fit by (Xq3). By (Xq1), (Xq2) of Proposition 51.6 for $\mathscr{T}_1$ there are C as in case (II) and a deep $\mathscr{T} \geq \mathscr{T}_1$ with layer field F and $d = [F : \mathbb{Q}] \geq 30(C+1)^2$; $\mathscr{T}$ is good and fit by (Xq3), $\mathscr{T} \geq \mathscr{T}_0$ by Step 0(a), and w dies in F by Step 0(b). The seed property of (w, s_0) gives $\mathcal{O}_1 \supseteq \mathcal{O}$ and a seed pair Q over $(\mathscr{T}, \emptyset)$ with values in $\mathcal{O}_1$ whose image in k_V lies over s_0, lies in A and has entries of weights $-w_\beta$, $-w_\theta$; the property of $\mathscr{T}_1$ then gives $\mathcal{O}_2 \supseteq \mathcal{O}_1$ such that (Nq3), (Nq4) hold for every $\mathcal{O}' \supseteq \mathcal{O}_2$.

In both cases we now have a fit and deep tower $\mathscr{T}$ (field F, $d = [F : \mathbb{Q}] \geq 30(C+1)^2$), C, Q, $\mathcal{O}_2$, and r'.

Step 2: coefficients, space and modular ideal. By Theorem 53.1 (for $\mathscr{T}$, ι) there is $\mathcal{O}_3 \supseteq \mathcal{O}$ as stated there. Let $\mathcal{O}'$ be the ring of integers of the compositum in $\overline{\mathbb{Q}}_\ell$ of the fraction fields of $\mathcal{O}_2$ and $\mathcal{O}_3$; it is a coefficient ring containing both. By Proposition 51.17 there is a space $\mathscr{X} = (B, A, P, \mathrm{cp}, (\psi^B_{v,i}))$ for $(\mathscr{T}, \mathcal{O}')$, and by Theorem 53.1 an ideal $\mathfrak{J} \subseteq B$ modular for ι. By (Nq1), resp. (Nq3), there are $R \in \mathrm{CNL}_{\mathcal{O}'}$, an injective $f\colon R \to V$ restricting to a on $\mathcal{O}'$ with $f(\mathfrak{m}_R) \subseteq \mathfrak{m}_V$, and a continuous marked ordinary pair P_R with values in R with the stated properties; let s be its reduction, a residual pair with values in $\mathbb{F}_{\mathcal{O}'}$. Let t be the reduction of Q modulo $\mathfrak{m}_{\mathcal{O}'}$. By (Nq2), resp. (Nq4), there is a chain in case (III) from s to t (in case (II) a chain, which is a chain in case (III) since good residual pairs are good in case (III)).

Step 3: the seed point; $\mathfrak{J}$ is somewhere at the reduction of the seed. Through $\mathcal{O}_1 \subseteq \mathcal{O}'$, Q is a marked ordinary pair with values in $\mathcal{O}'$ (for $\mathcal{O} \subseteq \mathcal{O}'$), continuous for $\mathfrak{m}_{\mathcal{O}'}$ (as $\mathfrak{m}^k_{\mathcal{O}_1}\mathcal{O}' \subseteq \mathfrak{m}^k_{\mathcal{O}'}$), with reduction t; t is a residual pair: it satisfies (Wq1), (Wq2) by the definition of seed pairs, and $g \mapsto t(g)$ is locally constant by continuity. By (Zq4) with $R = \mathcal{O}'$ there is an $\mathcal{O}'$-algebra homomorphism $f_1\colon A \to \mathcal{O}'$ with $f_1(P) = Q$. Let $y_1\colon A \to \overline{\mathbb{Q}}_\ell$ be f_1 followed by the inclusion and $x_1\colon A \to \mathbb{F}_{\mathcal{O}'}$ be f_1 followed by reduction, so $y_1(P)$ is the image of Q in $\overline{\mathbb{Q}}_\ell$, a seed, and $x_1(P) = t$. By (Jq3), $\mathfrak{J}$ is local at y_1. The kernel $\ker y_1 = \ker f_1$ is prime ($\mathcal{O}'$ is a domain); choose a minimal prime $\mathfrak{q}_1 \subseteq \ker f_1$ of A (minimal primes below a given prime exist by Zorn's lemma).

Then $\mathfrak{q}_1 \cap B$ is a prime of B killed by y_1, so $\mathfrak{J} \subseteq \mathfrak{q}_1 \cap B$ and $\mathfrak{J}A \subseteq \mathfrak{q}_1$; and $\mathfrak{q}_1 \subseteq \ker f_1 \subseteq \ker x_1$. So $\mathfrak{J}$ is somewhere at $\ker x_1$.

Step 4: the point of r'; propagation. By (Zq4) for R, $c = s$ and P_R there is an $\mathcal{O}'$-algebra homomorphism $f_0 \colon A \to R$ with $f_0(P) = P_R$; let $x_0 \colon A \to \mathbb{F}_{\mathcal{O}'}$ be f_0 followed by $R \to R/\mathfrak{m}_R = \mathbb{F}_{\mathcal{O}'}$, so $x_0(P) = s$ is a residual pair. The integer C is a Selmer bound for $(\mathscr{T}, \mathcal{O} \to \mathbb{F}_{\mathcal{O}'})$ in degree d (Step 1, with $k = \mathbb{F}_{\mathcal{O}'}$) and $30(C+1)^2 \le d$. By Theorem 53.31 and Steps 2, 3, $\mathfrak{J}$ is everywhere at $\ker x_0$.

Step 5: promodularity. Put $y = f \circ f_0 \colon A \to V$ and $r_1 = r'$. (i) y restricts to a on $\mathcal{O}'$, since f_0 is an $\mathcal{O}'$-algebra homomorphism and $f|_{\mathcal{O}'} = a$. (ii) $x_0|_B$ is an $\mathcal{O}'$-algebra homomorphism onto the field $\mathbb{F}_{\mathcal{O}'}$, so its kernel is a maximal ideal of the local ring B, i.e. $\mathfrak{m}_B$; hence $f_0(\mathfrak{m}_B) \subseteq \mathfrak{m}_R$ and $y(\mathfrak{m}_B) \subseteq f(\mathfrak{m}_R) \subseteq \mathfrak{m}_V$. (iii) R is a domain (f is injective into V), so $\ker f_0$ is prime; choose a minimal prime $\mathfrak{q}_0 \subseteq \ker f_0$ of A; as $\ker f_0 \subseteq \ker x_0$ and $\mathfrak{J}$ is everywhere at $\ker x_0$, $\mathfrak{J}A \subseteq \mathfrak{q}_0 \subseteq \ker f_0 = \ker y$; so $y(\mathfrak{J}) = 0$. (iv) $r_1 = u' r u'^{-1}$ with $u' \in \mathrm{GL}_4(K)$, and $r_1(g)$ has similitude $a(\theta(g))$. (v) $y \circ r_P$ is the first component of $f(f_0(P)) = f(P_R)$, which is $r'|_{G_F} = r_1|_{G_F}$. (vi) For $v \in \Sigma$, $s' \in I_v$: since ring homomorphisms commute with inverses of matrices, $y(\psi^P_{v,i}(s')) = \psi^{f(P_R)}_{v,i}(s') = a(\kappa_i(s'))$. With $\mathscr{T}$ fit and deep, $\mathcal{O}'$, $\mathscr{X}$ and $\mathfrak{J}$ modular for ι, this is the definition: r is promodular for $\mathfrak{D}$ and ι. □

Proposition 51.6. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$ and $\mathscr{T}_1$ a good auxiliary tower for $\mathfrak{D}$ with field F_1.*

(Xq1) *There is an integer $C \ge 0$ such that for every auxiliary tower $\mathscr{T} \ge \mathscr{T}_1$ whose field F is a layer over F_1, every field k and every ring homomorphism $a \colon \mathcal{O} \to k$ with $a(\mathfrak{m}_\mathcal{O}) = 0$, C is a Selmer bound for $(\mathscr{T}, a)$ in degree $[F : \mathbb{Q}]$.*

(Xq2) *For every integer n there is a deep auxiliary tower $\mathscr{T} \ge \mathscr{T}_1$ whose field F is a layer over F_1 and satisfies $[F : \mathbb{Q}] \ge n$.*

(Xq3) *If $\mathscr{T} \le \mathscr{T}'$ are auxiliary towers for $\mathfrak{D}$ and $\mathscr{T}$ is good, of case (II), fit, or deep, then so is $\mathscr{T}'$.*

Proof. Write $\mathscr{T}_1 = (\epsilon, E, F_1, g_1, \Sigma_1, (\bar v))$ and $d_1 = [F_1 : \mathbb{Q}]$. Two facts are used throughout. (P1) For a finite field $\mathbb{F}$ of characteristic ℓ and $j \ge 0$, $x \mapsto x^{\ell^j}$ is injective on $\mathbb{F}^\times$ (its order is prime to ℓ). (P2) The residual characters $\bar\kappa_i$, $\bar\omega$ and $\overline{\mathrm{rt}_w}$ take values in $\mathbb{F}_\mathcal{O}^\times$, and $\overline{\mathrm{rt}_w}$ for a tower $\mathscr{T}' \ge \mathscr{T}$ is the restriction of $\overline{\mathrm{rt}_w}$ for $\mathscr{T}$ (same labelling). For $m \ge 0$ let $H_m = \{g \in G_\mathbb{Q} : \chi_{\mathrm{cyc}}(g)^{\ell-1} \equiv 1 \pmod{\ell^{m+1}}\}$. It is the preimage under $G_\mathbb{Q} \to (\mathbb{Z}/\ell^{m+1}\mathbb{Z})^\times$ (onto) of the subgroup of order $\ell - 1$ of this cyclic group of order $(\ell-1)\ell^m$, so H_m is open and normal of index ℓ^m with cyclic quotient; its fixed field $\mathbb{Q}_m \subseteq \mathbb{Q}(\mu_{\ell^{m+1}})$ is totally real ($\chi_{\mathrm{cyc}}(c) = -1$) and has exactly one place above ℓ (ℓ is totally ramified in $\mathbb{Q}(\mu_{\ell^{m+1}})$). The layer of index m over F_1 is $F_1\mathbb{Q}_m$, with $G_{F_1\mathbb{Q}_m} = G_{F_1} \cap H_m$.

Step 1: (Xq3). Let $\mathscr{T} = (\epsilon, E, F, g_\mathscr{T}, \dots) \le \mathscr{T}' = (\epsilon, E, F', g_\mathscr{T}^{\ell^{j'}}, \dots)$, with $g^{\ell^j} \in G_{F'}$ for all $g \in G_F$; let $L' = F'E$, $d' = [F' : \mathbb{Q}]$. For a closed subgroup $X \subseteq G_F$ and $g \in X$, $g^{\ell^j} \in X \cap G_{F'}$. Hence, by (P1), (P2): every clause of the form "some g in $X \cap G_F$ has $\lambda(g) \ne \lambda'(g)$", for residual characters λ, λ' and X one of G_E, $I_{\bar w}$, $D_{\bar w}$, passes from $\mathscr{T}$ to $\mathscr{T}'$ (replace g by g^{ℓ^j}; note $G_L \ni g \mapsto g^{\ell^j} \in G_{F'} \cap G_E = G_{L'}$), and every clause of the form "$\lambda = \lambda'$ on a subset of G_F" passes trivially to the smaller subset of $G_{F'}$. Suppose that $\mathscr{T}$ is good. (Oq1): $d' = d\,[F' : F]$ is even, and $I_{\bar w} \cap G_{F'} \subseteq G_L \cap G_{F'} = G_{L'}$. (Oq2) does not involve F. (Oq3) and (Oq8) are of the first form (with λ' the constant 1 or $\bar\omega$, and for the n-th power clause $\lambda = \overline{\mathrm{rt}_w}^{\,n}$); (Oq5), (Oq6) are of the second form. (Oq7): $\bar\kappa_i^\epsilon(g_\mathscr{T}^{\ell^{j'}}) = \bar\kappa_i^\epsilon(g_\mathscr{T})^{\ell^{j'}}$ are pairwise distinct by (P1). (Oq4): L' is abelian over $\mathbb{Q}$, so $G_{L'}$ is normal in G_L, and $G_L/G_{L'}$ is a finite ℓ-group (every element

has ℓ-power order). Consequently a character of G_L with values in $\mathbb{F}_{\mathcal{O}}^{\times}$ that is trivial on $G_{L'}$ is trivial, and $\bar\omega(G_L) = \bar\omega(G_{L'})$ (the quotient $\bar\omega(G_L)/\bar\omega(G_{L'})$ is an image of $G_L/G_{L'}$ of order prime to ℓ). Let $\eta\colon G_{L'} \to \mathbb{F}_{\mathcal{O}}^{\times}$ be as in (Oq4) for $\mathscr{T}'$. Being trivial on $G_{L'} \cap \ker\bar\omega$, $\eta = \lambda\circ\bar\omega$ for a character λ of $\bar\omega(G_{L'}) = \bar\omega(G_L)$ with $\lambda^2 = 1$; put $\eta_0 = \lambda\circ\bar\omega$ on G_L. Then $\eta_0^2 = 1$, η_0 is trivial on $G_L \cap \ker\bar\omega$, and for each i with its j the character $\eta_0\bar\kappa_i^{\epsilon}/\bar\kappa_j^{\epsilon}$ of G_L is trivial on $G_{L'}$, hence trivial. By (Oq4) for $\mathscr{T}$, $\eta_0 = 1$, so $\eta = 1$. Thus $\mathscr{T}'$ is good. If $\mathscr{T}$ is of case (II): (Uq1), (Uq2) (with the same k_0) are of the second form, (Uq3) does not involve F, and (Uq4) is of the first form. Fitness is goodness together with conditions on $\mathfrak{D}$ and, in case (II), the case of the tower; so it passes. If $\chi|_{G_F}$ has finite order prime to ℓ, so has $\chi|_{G_{F'}}$.

Step 2: (Xq2). By (Oq5) for $\mathscr{T}_1$ there is $m_0 \ge 1$ prime to ℓ with $\chi(g)^{m_0} = 1$ for $g \in G_{F_1}$, $\chi_{\mathrm{cyc}}(g) = 1$; put $e_0 = (\ell-1)m_0$, prime to ℓ. If $g \in G_{F_1}$ and $\chi_{\mathrm{cyc}}(g)^{\ell-1} = 1$, then $g^{\ell-1} \in G_{F_1} \cap \ker\chi_{\mathrm{cyc}}$, so $\chi(g)^{e_0} = 1$. The Dirichlet character χ has open kernel, so $U = \ker\chi^{e_0}$ is open. The closed sets $(G_{F_1} \cap H_m) \setminus U$ decrease with m and their intersection is empty (an element of all H_m has $\chi_{\mathrm{cyc}}^{\ell-1} = 1$); by compactness there is m_1 with $G_{F_1} \cap H_m \subseteq U$ for $m \ge m_1$. Next, $\chi_{\mathrm{cyc}}(G_{F_1})$ is open in $\mathbb{Z}_\ell^{\times}$, so its image under $x \mapsto x^{\ell-1}$ is an open subgroup of $1+\ell\mathbb{Z}_\ell \cong \mathbb{Z}_\ell$, i.e. $1+\ell^b\mathbb{Z}_\ell$ for some $b \ge 1$; hence $[G_{F_1} : G_{F_1}\cap H_m] = \ell^{m+1-b}$ for $m+1 \ge b$. Choose $m \ge \max(m_1, b)$ with $d_1\ell^{m+1-b} \ge n$ and put $F = F_1\mathbb{Q}_m$. It is a number field, abelian over $\mathbb{Q}$ ($G_{\mathbb{Q}}/(G_{F_1} \cap H_m)$ embeds into $G_{\mathbb{Q}}/G_{F_1} \times G_{\mathbb{Q}}/H_m$), totally real (every complex conjugation lies in G_{F_1} and in H_m), $[F:\mathbb{Q}] = d_1\ell^{m+1-b} \ge n$, and $g^{\ell^m} \in G_F$ for all $g \in G_{F_1}$ (as G_{F_1}/G_F embeds into $G_{\mathbb{Q}}/H_m$, of order ℓ^m). Let Σ be the set of places of F above ℓ. For $v \in \Sigma$ choose a place $\bar u_v$ of $\overline{\mathbb{Q}}$ above v inducing on E the same place as $\bar\ell$: take any place $\bar u$ above v; if it induces the other place of E above ℓ, replace it by $c\bar u$ for a complex conjugation $c \in G_F$, which lies above v and induces c of that place, as $c|_E \ne 1$. The places of $\overline{\mathbb{Q}}$ above v form one G_F-orbit, and each place above ℓ lies above exactly one $v \in \Sigma$. So $\mathscr{T} = (\epsilon, E, F, g_1^{\ell^m}, \Sigma, (\bar u_v))$ is an auxiliary tower, $\mathscr{T}_1 \le \mathscr{T}$ (with $j = j' = m$), F is the layer of index m over F_1, and χ^{e_0} is trivial on $G_F \subseteq U$, so $\mathscr{T}$ is deep.

Step 3: the fields along the layers. Let $K \subset \overline{\mathbb{Q}}$ be the fixed field of $G_{F_1} \cap \ker\bar r_0 \cap \ker\bar\omega$: a finite abelian extension of $\mathbb{Q}$ (an intersection of normal subgroups of finite index with abelian quotients, as F_1 is abelian and $\bar r_0$ is diagonal) containing μ_ℓ. By Lemma 39.5 there is $r \ge 0$ with $\#(\mathrm{Cl}(K')/\mathrm{Cl}(K')^\ell) \le \ell^r$ for every field K' generated over K by a primitive ℓ^n-th root of unity, $n \ge 0$. Put $C = r + 1 + 2d_1$. Let $\mathscr{T} \ge \mathscr{T}_1$ have field $F = F_1\mathbb{Q}_m$, $d = [F:\mathbb{Q}]$, and let K_F be the fixed field of $G' = G_F \cap \ker\bar r_0 \cap \ker\bar\omega = G_K \cap H_m$. On G_K the values of χ_{cyc} lie in $1+\ell\mathbb{Z}_\ell$, on which $x \mapsto x^{\ell-1}$ is an automorphism preserving each $1+\ell^{m+1}\mathbb{Z}_\ell$; so $G' = \{g \in G_K : \chi_{\mathrm{cyc}}(g) \equiv 1 \pmod{\ell^{m+1}}\}$ and $K_F = K(\mu_{\ell^{m+1}})$. Thus r bounds the ℓ-rank of $\mathrm{Cl}(K_F)$. The places of F above ℓ lie above the unique place of $\mathbb{Q}_m$ above ℓ, so their number $\#\Sigma$ (the G_F-orbits of places above ℓ) is at most $[F:\mathbb{Q}_m] \le d_1$. K_F is finite abelian over $\mathbb{Q}$, contains F and μ_ℓ, and $\Delta = G_F/G'$ embeds into $(\mathbb{F}_{\mathcal{O}}^{\times})^4 \times \mathbb{F}_\ell^{\times}$ by $(\bar\kappa_1, \dots, \bar\kappa_4, \bar\omega)$, so $[K_F : F]$ is prime to ℓ.

Step 4: reduction to the residue field. Let k, a be as in (Xq1) and $\iota\colon \mathbb{F}_{\mathcal{O}} \to k$ the field embedding induced by a, so $a(\mathrm{rt}_w(g)) = \iota(\overline{\mathrm{rt}_w}(g))$. Let $X(k)$ denote any of $Z(w)$, $Z^{\mathrm{str}}(w)$, $\prod_v Z_v(w)$ formed with a, and $X(\mathbb{F}_{\mathcal{O}})$ the same with $\bar a\colon \mathcal{O} \to \mathbb{F}_{\mathcal{O}}$. Choose a basis (b_t) of k over $\iota(\mathbb{F}_{\mathcal{O}})$. A locally constant map on a profinite group takes finitely many values, and the constants y_v of the strict condition are finitely many; expanding all of them in finitely many b_t, the defining conditions (the cocycle identity, whose coefficients $\iota(\overline{\mathrm{rt}_w}(g))$ lie in $\iota(\mathbb{F}_{\mathcal{O}})$, vanishing on N and at $g_{\mathscr{T}}$, the coboundary conditions) hold if and only if they hold for each coordinate. Hence $X(k) = X(\mathbb{F}_{\mathcal{O}}) \otimes_{\mathbb{F}_{\mathcal{O}}} k$ and $\dim_k X(k) = \dim_{\mathbb{F}_{\mathcal{O}}} X(\mathbb{F}_{\mathcal{O}})$. From now on $k = \mathbb{F}_{\mathcal{O}}$ and $\varphi_w = \overline{\mathrm{rt}_w}\colon G_F \to \mathbb{F}_{\mathcal{O}}^{\times}$, trivial on G'.

Step 5: the root characters. Since $\theta = \alpha\Psi$, $\kappa_1\kappa_4 = \kappa_2\kappa_3$ and also $\kappa_1^1\kappa_4^1 = \alpha\Psi = \kappa_2^1\kappa_3^1$; with $\chi_i = \chi_i^{\mathscr{T}}$, $\mathrm{rt}_{w_\alpha} = \chi_1/\chi_2$, $\mathrm{rt}_{w_\beta} = \chi_2/\chi_3$, $\mathrm{rt}_{w_\Psi} = \chi_1/\chi_3$, $\mathrm{rt}_{w_\theta} = \chi_1^2/(\chi_2\chi_3) = \chi_1/\chi_4$. So for every root w, $\varphi_w = \bar\chi_i/\bar\chi_j$ with $i \ne j$, and $\varphi_w(g_{\mathscr{T}}) \ne 1$ by (Oq7) ($\mathscr{T}$ is good by Step 1). Further

$\mathrm{rt}_{\pm w_\Psi} = (\kappa_1^\epsilon/\kappa_3^\epsilon)^{\pm 1}$ is $\Psi^{\pm 1}$ ($\epsilon = 0$) or $\Psi^{\mp 1}$ ($\epsilon = 1$) on G_F, and $\Psi(c) = 1$ by (Sg1) for every complex conjugation c (all are conjugate in $G_\mathbb{Q}$); so $\varphi_w(c) = 1$ for $w \in \{0, w_\Psi, -w_\Psi\}$.

Step 6: restriction to G'. Let $w = 0$ or a root and $z \in Z(w)$. As φ_w is trivial on G', $f = z|_{G'}$ is a continuous homomorphism. For $\gamma \in G_F$, $s \in G'$: $z(1) = 0$, $z(\gamma^{-1}) = -\varphi_w(\gamma)^{-1} z(\gamma)$, so $z(\gamma s \gamma^{-1}) = z(\gamma) + \varphi_w(\gamma) z(s) + \varphi_w(\gamma s) z(\gamma^{-1}) = \varphi_w(\gamma) f(s)$. For a place $\bar{u} \nmid \ell$, $I_{\bar{u}} \cap G' \subseteq N$ ($\bar{r}_0 = 1$ on G'), so $f(I_{\bar{u}} \cap G') = 0$. Thus $f \in H_{\varphi_w}$ in the notation of Lemma 51.7 (for $\mathbb{F} = \mathbb{F}_\mathcal{O}$, F, $K' = K_F$). The map $z \mapsto f$ is injective: if $f = 0$, then $z(gs) = z(g)$ for $s \in G'$, so z is inflated from a cocycle of the finite group Δ, of order prime to ℓ, hence $z(g) = \varphi_w(g) y - y$ for some y; for $w = 0$ this is 0, and for a root w, $z(g_\mathscr{T}) = 0$ and $\varphi_w(g_\mathscr{T}) \neq 1$ give $y = 0$. If $z \in Z^{\mathrm{str}}(w)$, then for $v \in \Sigma$ and $s \in D_v \cap G'$, $z(s) = y_v - \varphi_w(s) y_v = 0$; every place $\bar{u} \mid \ell$ is $\gamma \bar{v}$ with $\gamma \in G_F$, $v \in \Sigma$, and $D_{\bar{u}} \cap G' = \gamma (D_v \cap G') \gamma^{-1}$ (G' is normal in G_F), so $f(D_{\bar{u}} \cap G') = 0$ by the equivariance; thus $f \in H^{\mathrm{str}}_{\varphi_w}$. By Lemma 51.7 (with r, $n_\ell = \#\Sigma \leq d_1$ and d) and Step 5:

$$\dim Z^{\mathrm{str}}(w) \leq r \quad (w \text{ a root}), \qquad \dim Z(w) \leq 1 + r + d_1 \quad (w \in \{0, \pm w_\Psi\}),$$
$$\dim Z(w) \leq d + 1 + r + d_1 \quad (w \text{ a root}).$$

Step 7: the local spaces. Let w be a root and $v \in \Sigma$; D_v is the decomposition group in G_F of $\bar{v}$, isomorphic to G_{F_v} for the completion F_v of F at the place v, and $Z_v(w)$ is the space Z^1 of continuous cocycles with values in $M = \mathbb{F}_\mathcal{O}(\varphi_w)$. Write $h^i = \dim_{\mathbb{F}_\mathcal{O}} H^i(F_v, M)$, so that $\#H^i = q^{h^i}$, $q = \#\mathbb{F}_\mathcal{O}$. The coboundaries are the image of $y \mapsto (s \mapsto \varphi_w(s) y - y)$, with kernel H^0, so $\dim Z_v(w) = h^1 + 1 - h^0$. By Theorem B.29, item 4 ($v \mid \ell$), $h^1 = h^0 + h^2 + [F_v : \mathbb{Q}_\ell]$, and by item 2, $\#H^2(F_v, M) = \#H^0(F_v, M^\vee) \leq \#M^\vee = q$, so $h^2 \leq 1$. Hence $\dim Z_v(w) \leq 2 + [F_v : \mathbb{Q}_\ell]$, and as $\sum_{v \in \Sigma} [F_v : \mathbb{Q}_\ell] = d$, $\dim \prod_v Z_v(w) \leq d + 2\#\Sigma \leq d + 2d_1$.

Step 8: (Xq1). With $C = r + 1 + 2d_1$: $\#\Sigma \leq d_1 \leq C$; $\dim Z^{\mathrm{str}}(w) \leq r \leq C$ for every root; $\dim Z(w) \leq r + 1 + d_1 \leq C$ for $w \in \{0, \pm w_\Psi\}$; $\dim Z(w) \leq d + C$ and $\dim \prod_v Z_v(w) \leq d + C$ for every odd root (Steps 6, 7). By Step 4 the same holds over every k. So C is a Selmer bound for $(\mathscr{T}, a)$ in degree d, for every $\mathscr{T} \geq \mathscr{T}_1$ with layer field. □

Lemma 51.7. *Let ℓ be an odd prime, $\mathbb{F}$ a finite field of characteristic ℓ, $F \subset \overline{\mathbb{Q}}$ a totally real number field with $d = [F : \mathbb{Q}]$ and n_ℓ places above ℓ, and $K' \subset \overline{\mathbb{Q}}$ a finite abelian extension of $\mathbb{Q}$ containing F and a primitive ℓ-th root of unity, with $[K' : F]$ prime to ℓ. Let $r \geq 0$ be an integer with $\#(\mathrm{Cl}(K')/\mathrm{Cl}(K')^\ell) \leq \ell^r$, and $\varphi \colon G_F \to \mathbb{F}^\times$ a character trivial on $G_{K'}$. Let H_φ be the $\mathbb{F}$-vector space of continuous homomorphisms $f \colon G_{K'} \to \mathbb{F}$ such that $f(I_{\bar{u}} \cap G_{K'}) = 0$ for every place $\bar{u}$ of $\overline{\mathbb{Q}}$ not above ℓ and $f(\gamma s \gamma^{-1}) = \varphi(\gamma) f(s)$ for all $\gamma \in G_F$, $s \in G_{K'}$, and $H^{\mathrm{str}}_\varphi \subseteq H_\varphi$ the subspace of those f with $f(D_{\bar{u}} \cap G_{K'}) = 0$ for every place $\bar{u}$ above ℓ. Let $c \in G_F$ be a complex conjugation. Then*

$$\dim_\mathbb{F} H^{\mathrm{str}}_\varphi \leq r, \qquad \dim_\mathbb{F} H_\varphi \leq \begin{cases} 1 + r + n_\ell, & \varphi(c) = 1, \\ 1 + r + n_\ell + d, & \varphi(c) = -1. \end{cases}$$

Proof. Put $\Delta = \mathrm{Gal}(K'/F) = G_F/G_{K'}$, of order prime to ℓ. Since K' contains μ_ℓ and ℓ is odd, K' is totally imaginary; as $K'/\mathbb{Q}$ is abelian, all complex conjugations of $G_\mathbb{Q}$ have the same image $c' \neq 1$ in $\mathrm{Gal}(K'/\mathbb{Q})$, and $c' \in \Delta$ since F is totally real. As $\varphi(c)^2 = 1$ and ℓ is odd, $\varphi(c) \in \{1, -1\}$ and the two cases are exclusive. For an $\mathbb{F}[\Delta]$-module X and a character $\lambda \colon \Delta \to \mathbb{F}^\times$ put $X^{(\lambda)} = \{x \in X : \delta x = \lambda(\delta) x \ \forall \delta\}$; it is the image of the idempotent $e_\lambda = \#\Delta^{-1} \sum_\delta \lambda(\delta)^{-1} \delta \in \mathbb{F}[\Delta]$, so $X \mapsto X^{(\lambda)}$ is exact.

Step 1: the strict space. Let $f \in H^{\mathrm{str}}_\varphi$, $f \neq 0$. Its kernel is an open subgroup of $G_{K'}$ of index ℓ^a, normal with abelian quotient of exponent ℓ; it cuts out an abelian extension of K'

unramified at every finite place not above ℓ (by the inertia condition), in which every place above ℓ splits completely (by the decomposition condition), and unramified at the archimedean places (K' is totally imaginary). By class field theory f factors through $\mathrm{Gal}(H_{K'}/K') \cong \mathrm{Cl}(K')$, $H_{K'}$ the Hilbert class field, and then through $\mathrm{Cl}(K')/\mathrm{Cl}(K')^\ell$. So H_φ^{str} embeds $\mathbb{F}$-linearly into $\mathrm{Hom}(\mathrm{Cl}(K')/\mathrm{Cl}(K')^\ell, \mathbb{F})$, of $\mathbb{F}$-dimension $\le r$.

Step 2: H_φ as an eigenspace. Let H be the $\mathbb{F}_\ell$-space of continuous homomorphisms $h\colon G_{K'} \to \mathbb{F}_\ell$ with $h(I_{\bar u} \cap G_{K'}) = 0$ for all $\bar u \nmid \ell$. Writing a continuous homomorphism $f\colon G_{K'} \to \mathbb{F}$ in an $\mathbb{F}_\ell$-basis of $\mathbb{F}$ identifies such f (with the inertia condition) with $H \otimes_{\mathbb{F}_\ell} \mathbb{F}$. The group G_F acts on H by $(\gamma \cdot h)(s) = h(\gamma^{-1} s \gamma)$ (it preserves the inertia condition as $\gamma^{-1} I_{\bar u} \gamma = I_{\gamma^{-1}\bar u}$), and $G_{K'}$ acts trivially since h is a homomorphism to an abelian group; so Δ acts. The condition $f(\gamma s \gamma^{-1}) = \varphi(\gamma) f(s)$ says $\gamma^{-1} \cdot f = \varphi(\gamma) f$. Hence

$$H_\varphi = (H \otimes_{\mathbb{F}_\ell} \mathbb{F})^{(\varphi^{-1})}.$$

Step 3: Kummer theory. Fix a primitive $\zeta \in \mu_\ell \subset K'$ and let $\log_\zeta\colon \mu_\ell \to \mathbb{F}_\ell$ be the inverse of $x \mapsto \zeta^x$. For $a \in K'^\times$ and $\alpha \in \overline{\mathbb{Q}}$ with $\alpha^\ell = a$, $h_a(s) = \log_\zeta(s(\alpha)/\alpha)$ is a continuous homomorphism $G_{K'} \to \mathbb{F}_\ell$ independent of α, and $a \mapsto h_a$ is an isomorphism $K'^\times/K'^{\times\ell} \to \mathrm{Hom}_{\mathrm{cts}}(G_{K'}, \mathbb{F}_\ell)$. For a finite place $\mathfrak{q} \nmid \ell$ of K', $K'(\alpha)/K'$ is unramified at $\mathfrak{q}$ if and only if $\mathrm{ord}_\mathfrak{q}(a) \in \ell\mathbb{Z}$. So $a \mapsto h_a$ maps the subgroup B of classes with $\mathrm{ord}_\mathfrak{q}(a) \in \ell\mathbb{Z}$ for all finite $\mathfrak{q} \nmid \ell$ isomorphically onto H. For $\gamma \in G_F$, $\gamma(\alpha)$ is an ℓ-th root of $\gamma(a)$ and $\gamma(\zeta^x) = \zeta^{\bar\omega(\gamma)x}$, so

$$h_{\gamma a}(s) = \log_\zeta \gamma((\gamma^{-1} s \gamma)(\alpha)/\alpha) = \bar\omega(\gamma)\, h_a(\gamma^{-1} s \gamma) = \bar\omega(\gamma)\, (\gamma \cdot h_a)(s).$$

Writing B additively, $\delta \cdot h_a = \varphi(\delta)^{-1} h_a$ holds if and only if $h_{\delta a} = \bar\omega(\delta)\varphi(\delta)^{-1} h_a$, i.e. $\delta a = \bar\omega(\delta)\varphi(\delta)^{-1} a$; extending $\mathbb{F}$-linearly, $(H \otimes \mathbb{F})^{(\varphi^{-1})} \cong (B \otimes_{\mathbb{F}_\ell} \mathbb{F})^{(\psi)}$ with $\psi = \bar\omega\varphi^{-1}$ (as characters of Δ). Moreover $\psi(c') = -\varphi(c)^{-1} = -\varphi(c)$.

Step 4: S-units and the class group. Let S be the set of places of K' above ℓ and $\mathcal{O}_S = \mathcal{O}_{K'}[1/\ell]$. Sending $a \in B$ to the class of the ideal $\mathfrak{a}$ of $\mathcal{O}_S$ with $a\mathcal{O}_S = \mathfrak{a}^\ell$ gives an exact sequence of $\mathbb{F}_\ell[\Delta]$-modules $0 \to \mathcal{O}_S^\times/\mathcal{O}_S^{\times\ell} \to B \to \mathrm{Cl}(\mathcal{O}_S)[\ell]$ (if $\mathfrak{a} = b\mathcal{O}_S$ then $a = b^\ell u$ with $u \in \mathcal{O}_S^\times$; an S-unit that is an ℓ-th power in K' is the ℓ-th power of an S-unit). $\mathrm{Cl}(\mathcal{O}_S)$ is a quotient of $\mathrm{Cl}(K')$, and $\dim A[\ell] = \dim A/\ell A$ for a finite abelian group A, so $\dim_{\mathbb{F}_\ell} \mathrm{Cl}(\mathcal{O}_S)[\ell] \le r$. Tensoring with $\mathbb{F}$ and taking ψ-parts (both exact),

$$\dim_{\mathbb{F}} (B \otimes \mathbb{F})^{(\psi)} \le \dim_{\mathbb{F}} (\mathcal{O}_S^\times/\ell \otimes \mathbb{F})^{(\psi)} + r.$$

Let $\mu = \mu(K')$ and $Y = (\mathcal{O}_S^\times/\mu) \otimes \mathbb{Z}_\ell$, a free $\mathbb{Z}_\ell$-module of finite rank with Δ-action. As $\mathcal{O}_S^\times/\mu$ is free, $0 \to \mu/\mu^\ell \to \mathcal{O}_S^\times/\ell \to Y/\ell Y \to 0$ is exact, and $\dim \mu/\mu^\ell = 1$. Let $W = W(\mathbb{F})$ be the Witt vectors, $Q_W = W[1/\ell]$, $\tilde\psi\colon \Delta \to W^\times$ the Teichmüller lift of ψ, and $e = \#\Delta^{-1} \sum_\delta \tilde\psi(\delta)^{-1}\delta \in W[\Delta]$, an idempotent reducing to e_ψ. With $Y_W = Y \otimes_{\mathbb{Z}_\ell} W$ we have $Y/\ell Y \otimes_{\mathbb{F}_\ell} \mathbb{F} = Y_W/\ell Y_W$, and $Y_W = eY_W \oplus (1-e)Y_W$ with eY_W free over W, so $(Y_W/\ell Y_W)^{(\psi)} = e_\psi(Y_W/\ell Y_W) \cong eY_W/\ell eY_W$ has $\mathbb{F}$-dimension $\dim_{Q_W} eV$, where $V = \mathcal{O}_S^\times \otimes_{\mathbb{Z}} Q_W = Y \otimes_{\mathbb{Z}_\ell} Q_W$. So

$$\dim_{\mathbb{F}} H_\varphi \le 1 + r + \dim_{Q_W} eV. \tag{51.1}$$

Step 5: the multiplicity of $\tilde\psi$ in V. For a transitive Δ-set Δ/Δ_0, an element $x = \sum_g x_g\, g$ of $Q_W[\Delta/\Delta_0]$ with $\delta x = \tilde\psi(\delta) x$ for all δ satisfies $x_{\delta g} = \tilde\psi(\delta) x_g$; so it is determined by x_{Δ_0}, and it is 0 if $\tilde\psi|_{\Delta_0} \ne 1$ (take $\delta \in \Delta_0$, $g = \Delta_0$). Hence $\dim eQ_W[\Delta/\Delta_0] \le 1$, with 0 when $\tilde\psi|_{\Delta_0} \ne 1$. The valuations at S give an exact sequence of $Q_W[\Delta]$-modules $0 \to \mathcal{O}_{K'}^\times \otimes Q_W \to V \to Q_W[S]$, and S is the disjoint union over the n_ℓ places $v \mid \ell$ of F of transitive Δ-sets; so $\dim eV \le \dim e(\mathcal{O}_{K'}^\times \otimes Q_W) + n_\ell$. By Dirichlet's unit theorem the logarithmic embedding $\mathcal{O}_{K'}^\times \to \mathbb{R}^{S_\infty}$,

S_∞ the archimedean places of K', is Δ-equivariant for the permutation action, has kernel μ and its image spans the hyperplane of vectors with coordinate sum 0, a Δ-stable complement of the invariant line $\mathbb{R}(1,\dots,1)$. So the traces of $\delta \in \Delta$ on $(\mathcal{O}_{K'}^\times/\mu)\otimes\mathbb{Q}$ and on $\mathbb{Q}[S_\infty]$ differ by 1, and therefore $(\mathcal{O}_{K'}^\times \otimes Q_W)\oplus Q_W \cong Q_W[S_\infty]$ as $Q_W[\Delta]$-modules. The set S_∞ is the disjoint union over the d real places of F of transitive Δ-sets whose stabiliser is $\{1, c'\}$ (each archimedean place of K' is complex). Since $\tilde{\psi}(c') = 1$ exactly when $\psi(c') = 1$, that is when $\varphi(c) = -1$,

$$\dim e(\mathcal{O}_{K'}^\times \otimes Q_W) \le \dim eQ_W[S_\infty] \le \begin{cases} 0, & \varphi(c) = 1, \\ d, & \varphi(c) = -1. \end{cases}$$

With (51.1) this gives the bound for H_φ. □

51.2 Seeds in cases (II) and (III) and spaces of marked pairs

This subsection constructs the seed pairs from which modularity is propagated, and proves that spaces of marked ordinary symplectic pairs exist over good auxiliary towers (Proposition 51.17). In case (II) a seed pair exists over every good deep tower of case (II) (Proposition 51.8). It is the induction to $\mathbb{Q}$ of an ordinary Hilbert eigenform of mixed weight over a real quadratic field that is congruent to an Eisenstein series (Proposition 51.9); such forms come from Hida families of nearly ordinary Hilbert modular forms [128] and from Hilbert modular forms congruent to one (Lemma 51.11), and the induced representation embeds into GSp_4 with the required residual shape and flags (Lemma 51.12, Lemma 51.13). In case (III) the seed is the symplectic tensor product of a newform congruent to an Eisenstein series, as in Subsection 49.1, with a dihedral representation (Proposition 51.14, Lemma 51.15, Lemma 51.16); its residual pair is ramified at one auxiliary prime, which dies in the larger towers over which the seed pair is used. In both cases the seed is automorphic over an abelian CM field and irreducible over the cyclotomic tower.

Proposition 51.8. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, attached to a host character in case (II), let $\mathscr{T}$ be a good and deep auxiliary tower for $\mathfrak{D}$ of case (II), and $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism. Then there are a coefficient ring $\mathcal{O}_1 \supseteq \mathcal{O}$ and a seed pair Q over $(\mathscr{T}, \emptyset)$ with values in $\mathcal{O}_1$ for ι whose reduction $\bar{Q}$ lies in V and either has an entry of weight $-w_\alpha$, or has entries of weights w_β and $-w_\theta$.*

Proof. Let $\mathscr{T} = (\epsilon, E, F, g_{\mathscr{T}}, \Sigma, (\bar{v})_{v\in\Sigma})$, write $\kappa_i = \kappa_i^\epsilon$, bars for reduction, and $\mathfrak{m}_1$ for the maximal ideal of the coefficient ring $\mathcal{O}_1 \supseteq \mathcal{O}$ constructed below (enlarged finitely often). For both labellings $\{\kappa_1,\kappa_3\} = \{\theta,\alpha\}$, $\{\kappa_2,\kappa_4\} = \{\Psi, \mathbf{1}\}$, $\mathrm{rt}_{w_\alpha} = \kappa_1/\kappa_2 = \alpha$ and $\mathrm{rt}_{w_\Psi} = \kappa_1/\kappa_3 = \Psi^{\pm1}$; and $\bar{\alpha} = \bar{\xi}$, since $\overline{\chi_{\mathrm{cyc}}\omega_\ell^{-1}} = 1$.

Step 1: the real quadratic field. Let k_0 be the real quadratic field of (Uq2), in which 2 and ℓ split, and $\phi\colon G_\mathbb{Q} \to \{\pm1\} \subset \mathbb{Z}_\ell^\times$ its quadratic character. By (Uq1), $\bar{\Psi}$ has values ±1 on G_F, and by (Uq2) its kernel there is $G_{F'} = G_F \cap G_{k_0}$, $F' = Fk_0$; so $\bar{\Psi} = \bar{\phi}$ on G_F. (a) By (Oq7) the residues $\bar{\kappa}_i(g_{\mathscr{T}})$ are pairwise distinct; they are $x\varsigma, \varsigma, x, 1$ with $x = \bar{\xi}(g_{\mathscr{T}})$, $\varsigma = \bar{\Psi}(g_{\mathscr{T}}) \in \{\pm1\}$. So $\varsigma = -1$ (else $x\varsigma = x$), $g_{\mathscr{T}} \notin G_{k_0}$, $\phi(g_{\mathscr{T}}) = -1$, and $x^2 \ne 1$; $u = g_{\mathscr{T}}^2$ lies in $G_{F'}$. (b) For every place $\bar{w}$ above ℓ, $D_{\bar{w}} \cap G_F \subseteq G_{F'}$ by (Uq1), (Uq2); in particular $D_v \subseteq G_{F'}$ for $v \in \Sigma$. (c) For every place $\bar{w}$ above ℓ there is $h \in D_{\bar{w}} \cap G_{F'}$ with $\bar{\xi}(h)^2 \ne 1$: by (Uq4) some $h \in D_{\bar{w}} \cap G_F$ has $\bar{\alpha}(h)^4 \ne 1$, and $h \in G_{F'}$ by (b). (d) Some $g \in G'$ lies outside G_{k_0}: otherwise $\phi|_{G_L}$ factors through $\chi_{\mathrm{cyc}}(G_L)$, an open subgroup of $\mu_{\ell-1} \times (1+\ell\mathbb{Z}_\ell)$, and being of order ≤ 2 it is trivial on the pro-ℓ part, i.e. on $G_L \cap \ker\omega_\ell$; then $\eta = \bar{\Psi}|_{G_L} = \bar{\phi}|_{G_L}$ satisfies $\eta^2 = 1$, is trivial on $G_L \cap \ker\omega_\ell$, and permutes the $\bar{\kappa}_i$ on G_L ($\eta\bar{\theta} = \bar{\alpha}$, $\eta\bar{\alpha} = \bar{\theta}$, $\eta\bar{\Psi} = 1$, $\eta\cdot 1 = \bar{\Psi}$, as $\bar{\Psi}^2 = 1$); by (Oq4) $\eta = 1$, against (Oq3) for the root w_Ψ. In particular $k_0 \not\subseteq L$; and $L = FE$ is a CM field abelian over $\mathbb{Q}$.

Step 2: the Hilbert eigenform. Let $\mathfrak{l}_1, \mathfrak{l}_2$ be the places of k_0 above ℓ and $\xi_0 = \xi|_{G_{k_0}}$, of finite order. Its hypotheses in Proposition 51.9: a complex conjugation in G_{k_0} is conjugate in $G_{\mathbb{Q}}$ to c, and $\xi(c) = -1$ (Sg1); ξ is unramified at ℓ (Uq3); 2 splits in k_0, so a place of k_0 above 2 has norm 2 and contains the inertia groups $I_{\bar{w}}$ above it, and by (Uq4) some $g \in I_{\bar{w}} \cap G_F$ has $\bar{\xi}(g) \neq 1$, so ξ_0 is ramified there; the last two hypotheses are Step 1 (c). So there are $k, t \geq 2$ divisible by $\ell - 1$, and ρ with (Hm1)–(Hm5). A continuous homomorphism from a compact group to $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ takes values in GL_2 of a finite extension of $\mathbb{Q}_\ell$ (Baire category) and preserves a lattice; so, conjugating, ρ takes values in $\mathrm{GL}_2(\mathcal{O}_1)$ for a coefficient ring $\mathcal{O}_1 \supseteq \mathcal{O}$.

Step 3: a Ribet lattice diagonal at u, and the ordinary lines. The characteristic polynomial of $\rho(u)$ reduces to $(X-1)(X-x^2)$ (Hm2), with distinct roots by Step 1 (a); by Hensel its roots $a_1 \equiv x^2$, $a_2 \equiv 1$ lie in $\mathcal{O}_1$, the lattice is the sum of the two eigenlines, and in an adapted basis $\rho(u) = \mathrm{diag}(a_1, a_2)$; write $\rho(g) = \left(\begin{smallmatrix} a_g & b_g \\ c_g & d_g \end{smallmatrix}\right)$. The semisimplification of $\bar{\rho}$ is $\bar{\xi}_0 \oplus 1$ (Brauer–Nesbitt and (Hm2)), so $\bar{\rho}$ has a stable line, which is an eigenline of $\bar{\rho}(u)$; hence all $\bar{b}_g = 0$ or all $\bar{c}_g = 0$, in either case $b_g c_{g'} \in \mathfrak{m}_1$ for all g, g', the diagonal of $\bar{\rho}$ consists of characters, and comparing at u, $\bar{a} = \bar{\xi}_0$, $\bar{d} = 1$. Not all c_g vanish ((Hm1)); with $n = \min_g v(c_g)$, ϖ a uniformiser and $\rho_1 = \mathrm{diag}(1, \varpi^n)^{-1} \rho\, \mathrm{diag}(1, \varpi^n)$, one gets $\rho_1(g) \in \mathrm{GL}_2(\mathcal{O}_1)$ (if $n = 0$ use $b_g c_{g'} \in \mathfrak{m}_1$ with $c_{g'}$ a unit), $\rho_1(u) = \mathrm{diag}(a_1, a_2)$, and

$$\bar{\rho}_1(g) = \begin{pmatrix} \bar{\xi}(g) & 0 \\ \bar{y}(g) & 1 \end{pmatrix}, \qquad \bar{y}(gh) = \bar{y}(g)\bar{\xi}(h) + \bar{y}(h), \qquad \bar{y} \neq 0.$$

$\bar{y}|_{G_{F'}} \neq 0$: otherwise take $h \in G_{F'}$ with $\bar{\xi}(h) \neq 1$ (Step 1 (c)); for $g \in G_{k_0}$, $\bar{y}(gh) = \bar{y}(g)\bar{\xi}(h)$ and, as $ghg^{-1} \in G_{F'}$ (F' is Galois over $\mathbb{Q}$), $\bar{y}(gh) = \bar{y}(ghg^{-1}g) = \bar{y}(g)$; so $\bar{y} = 0$. *Lines.* Let $\bar{w}$ be a place above ℓ, $D = D_{\bar{w}} \cap G_{k_0}$, $\kappa_1^{\bar{w}}, \kappa_2^{\bar{w}}$ the characters of (Hm4), and $h \in D$ with $\bar{\xi}(h) \neq 1$ (Step 1 (c)). The D-stable line of (Hm4) is the $\kappa_1^{\bar{w}}(h)$-eigenline of $\rho_1(h)$, its other eigenvalue $\kappa_2^{\bar{w}}(h)$ being residually different. If $\bar{w}$ induces $\mathfrak{l}_1$: $\kappa_1^{\bar{w}}(h) \equiv \bar{\xi}(h) \not\equiv 1 \equiv d_h$, so the line is spanned by $(\kappa_1^{\bar{w}}(h) - d_h,\ c_h)$, whose first coordinate is a unit, i.e. by $(1, z_{\bar{w}})$ with $z_{\bar{w}} \in \mathcal{O}_1$ (enlarging $\mathcal{O}_1$ by $\kappa_1^{\bar{w}}(h)$), and $\rho_1(s) = \mathrm{low}(z_{\bar{w}}) \left(\begin{smallmatrix} \kappa_1^{\bar{w}}(s) & * \\ 0 & \kappa_2^{\bar{w}}(s) \end{smallmatrix}\right) \mathrm{low}(z_{\bar{w}})^{-1}$ for $s \in D$. If $\bar{w}$ induces $\mathfrak{l}_2$: $\kappa_1^{\bar{w}}(h) \equiv 1$, so the line is spanned by $(b_h,\ \kappa_1^{\bar{w}}(h) - a_h)$, whose second coordinate is a unit, i.e. by $(w_{\bar{w}}, 1)$ with $w_{\bar{w}} \in \mathfrak{m}_1$ ($b_h \in \mathfrak{m}_1$), and $\rho_1(s) = \mathrm{up}(w_{\bar{w}}) \left(\begin{smallmatrix} \kappa_2^{\bar{w}}(s) & 0 \\ * & \kappa_1^{\bar{w}}(s) \end{smallmatrix}\right) \mathrm{up}(w_{\bar{w}})^{-1}$ for $s \in D$.

Step 4: the induced representation. Choose $b_1, b_2 \in \mathcal{O}_1^\times$ (Hensel; ℓ odd) with $b_i^2 = a_i$, $\bar{b}_1 = -\bar{\kappa}_1(g_{\mathscr{T}})$, $\bar{b}_2 = -\bar{\kappa}_2(g_{\mathscr{T}})$; this is possible as $\bar{\kappa}_1(g_{\mathscr{T}}) \in \{-x, x\}$ and $\bar{\kappa}_2(g_{\mathscr{T}}) \in \{-1, 1\}$ by Step 1 (a). Put $\mu = \chi_{\mathrm{cyc}}^k \xi \phi$, a character of $G_{\mathbb{Q}}$ with $\mu|_{G_{k_0}} = \det \rho_1$ (Hm2). Then $\mu(g_{\mathscr{T}}) = -b_1 b_2$: $\mu(g_{\mathscr{T}})^2 = \mu(u) = \det \rho_1(u) = (b_1 b_2)^2$, and residually $\bar{\mu}(g_{\mathscr{T}}) = \bar{\omega}_\ell(g_{\mathscr{T}})^k x \cdot (-1) = -x$ while $-\bar{b}_1 \bar{b}_2 = -\bar{\kappa}_1 \bar{\kappa}_2(g_{\mathscr{T}}) = -x$ (for $\epsilon = 0$: $(-x)(-1)$; for $\epsilon = 1$: $x \cdot 1$), and $x \neq -x$. Let $R_0 \colon G_F \to \mathrm{GL}_4(\mathcal{O}_1)$ be the matrix, in the basis $(f_1, \ldots, f_4)$ given by the columns of P_b (Lemma 51.12), of the induced representation $\mathrm{Ind}_{G_{F'}}^{G_F}(\rho_1|_{G_{F'}})$ written in the basis $(e_1, e_2, g_{\mathscr{T}} e_1, g_{\mathscr{T}} e_2)$: for $h \in G_{F'}$

$$R_0(h) = \mathrm{ix}_b(\rho_1(h), \rho_1(g_{\mathscr{T}}^{-1} h g_{\mathscr{T}})), \qquad R_0(g_{\mathscr{T}}) = P_b^{-1} \begin{pmatrix} 0 & \rho_1(u) \\ 1_2 & 0 \end{pmatrix} P_b = \mathrm{diag}(-b_1, -b_2, b_1, b_2)$$

(Ie2), and $R_0(g_{\mathscr{T}} h) = R_0(g_{\mathscr{T}}) R_0(h)$. It is a continuous homomorphism ($G_{F'}$ is normal of index 2 in G_F, $g_{\mathscr{T}} \notin G_{F'}$, $g_{\mathscr{T}}^2 = u$), with values in $\mathrm{GL}_4(\mathcal{O}_1)$ by (Ie0); over $\overline{\mathbb{Q}}_\ell$, $R_0 \cong V|_{G_F}$ for $V = \mathrm{Ind}_{G_{k_0}}^{G_{\mathbb{Q}}} \rho$ (Mackey: $G_{\mathbb{Q}} = G_{k_0} G_F$, $G_{k_0} \cap G_F = G_{F'}$). $R_0(g)$ has similitude $\mu(g)$: for $h \in G_{F'}$, $\det \rho_1(h) = \mu(h) = \mu(g_{\mathscr{T}}^{-1} h g_{\mathscr{T}}) = \det \rho_1(g_{\mathscr{T}}^{-1} h g_{\mathscr{T}})$ and (Ie1) applies; for $g_{\mathscr{T}}$, (Ie2) and (Ie1) give similitude $-b_1 b_2 = \mu(g_{\mathscr{T}})$. Residually $\bar{\theta} = \bar{\xi}\bar{\Psi}$ and $\bar{\mu} = \bar{\xi}\bar{\phi}$ agree on G_F, so θ/μ has values in $1 + \mathfrak{m}_1$ there, on which squaring is a topological automorphism; let $\delta \colon G_F \to 1 + \mathfrak{m}_1$ be the

continuous character with $\delta^2 = \theta/\mu$, and

$$R = \delta\, R_0 \colon G_F \to \mathrm{GL}_4(\mathcal{O}_1).$$

For $v \in \Sigma$ define n_v: by Step 1 (b) and (a), $D_v \subseteq G_{F'}$ and the places $\bar v$, $g_{\mathscr{T}}^{-1}\bar v$ induce the two different places of k_0 above ℓ ($g_{\mathscr{T}}$ acts non-trivially on k_0). If $\bar v$ induces $\mathfrak{l}_1$, put $n_v = \mathrm{fi}_b^+(z_{\bar v}, w_{g_{\mathscr{T}}^{-1}\bar v})$; otherwise $n_v = \mathrm{fi}_b^-(z_{g_{\mathscr{T}}^{-1}\bar v}, w_{\bar v})$. Here d, e of Lemma 51.12 are units because $w \in \mathfrak{m}_1$.

Step 5: properties. (R1) $R(g)$ has similitude $\delta^2\mu = \theta$. (R2) $R(g_{\mathscr{T}}) = \delta(g_{\mathscr{T}})\mathrm{diag}(-b_1, -b_2, b_1, b_2)$ is diagonal, with residues $(\bar\kappa_1, \bar\kappa_2, -\bar\kappa_1, -\bar\kappa_2)(g_{\mathscr{T}}) = (\bar\kappa_1, \bar\kappa_2, \bar\kappa_3, \bar\kappa_4)(g_{\mathscr{T}})$, since $\bar\delta = 1$ and, for both labellings, $\bar\kappa_3(g_{\mathscr{T}}) = -\bar\kappa_1(g_{\mathscr{T}})$ and $\bar\kappa_4(g_{\mathscr{T}}) = -\bar\kappa_2(g_{\mathscr{T}})$ by Step 1 (a). (R3) For $h \in G_{F'}$, with $h' = g_{\mathscr{T}}^{-1} h g_{\mathscr{T}}$ and $\bar\xi(h') = \bar\xi(h)$, (Ie3) gives

$$\bar R(h) = \mathrm{diag}(\bar\xi, 1, \bar\xi, 1)(h) + u_h(E_{21} - E_{43}) + v_h(E_{41} - E_{23}),$$
$$u_h = -\tfrac{1}{2}(\bar\beta\bar y(h) + \bar y(h')), \qquad v_h = \tfrac{1}{2}(\bar y(h') - \bar\beta\bar y(h)),$$

$\bar\beta = \bar b_1 \bar b_2^{-1} \neq 0$. On $G_{F'}$, $(\bar\kappa_1, \dots, \bar\kappa_4) = (\bar\xi, 1, \bar\xi, 1)$ (as $\bar\Psi = 1$ there), and $\bar R(g_{\mathscr{T}} h) = \bar R(g_{\mathscr{T}})\bar R(h)$ with $\bar R(g_{\mathscr{T}})$ diagonal. So $\bar R$ has diagonal $(\bar\kappa_i)$ on G_F and is supported in $\{-w_\alpha, w_\beta, -w_\theta\} = \Sigma_V$. Choose $h \in G_{F'}$ with $\bar y(h) \neq 0$ (Step 3); then $-2(u_h + v_h) = 2\bar\beta\bar y(h) \neq 0$, so $u_h \neq 0$, and $\bar R$ has an entry of weight $-w_\alpha$ at $(2,1)$, or $v_h \neq 0$, and $\bar R$ has entries of weights $-w_\theta$ at $(4,1)$ and w_β at $(2,3)$. (R4) Let $v \in \Sigma$, $s \in D_v$, $s' = g_{\mathscr{T}}^{-1} s g_{\mathscr{T}}$. If $\bar v$ induces $\mathfrak{l}_1$, Step 3 gives $\rho_1(s) = A_z$ with $z = z_{\bar v}$, $(p, r) = (\kappa_1^{\bar v}(s), \kappa_2^{\bar v}(s))$, and $\rho_1(s') = B_w$ with $w = w_{g_{\mathscr{T}}^{-1}\bar v}$, $(p', r') = (\kappa_1^{g_{\mathscr{T}}^{-1}\bar v}(s'), \kappa_2^{g_{\mathscr{T}}^{-1}\bar v}(s'))$, in the notation of (Ie5); if $\bar v$ induces $\mathfrak{l}_2$, the same holds with the roles of s and s' exchanged. By (Ie5), $n_v^{-1} R(s) n_v$ is upper triangular with diagonal

$$\psi_v(s) = \delta(s)(\kappa_1^{(1)},\ \kappa_1^{(2)},\ \kappa_2^{(2)},\ \kappa_2^{(1)}),$$

where $\kappa_i^{(j)}$ is the character of (Hm4) at the one of $\bar v$, $g_{\mathscr{T}}^{-1}\bar v$ that induces $\mathfrak{l}_j$, evaluated at s resp. s'. Residually $\bar\psi_v = (\bar\xi, 1, \bar\xi, 1) = (\bar\kappa_1, \dots, \bar\kappa_4)$ on D_v (Hm4), as $\bar\Psi = 1$ on D_v. On I_v: $\psi_v = \delta\,(\chi_{\mathrm{cyc}}^{k+t}, \chi_{\mathrm{cyc}}^{k}, 1, \chi_{\mathrm{cyc}}^{-t})$ ($\chi_{\mathrm{cyc}}(s') = \chi_{\mathrm{cyc}}(s)$). By (Ie4), n_v is lower triangular with diagonal entries 1 and similitude 1, $(\bar n_v)_{32} = 2\bar b_2 \bar w\, \bar d^{-1}$ or $2\bar b_2 \bar w\, \bar e^{-1}$ vanishes ($w \in \mathfrak{m}_1$), so $\bar n_v$ is supported in Σ_Q, and $(\bar n_v)_{31} = \mp 1 \neq 0$. (R5) If $s \in D_v$ and $s^{\ell^a} \in \overline{[D_v, D_v]}$, then $s \in I_v$ ($D_v/I_v \cong \hat{\mathbb{Z}}$ is torsion-free), $\chi_{\mathrm{cyc}}(s)$ is an ℓ-power root of unity in $\mathbb{Z}_\ell^\times$, hence 1, so all χ_{cyc}-powers above are 1 at s; and $\delta(s)^2 = (\theta/\mu)(s) = (\chi_{\mathrm{cyc}}^{1+e-k}\omega_\ell^{-1}\chi\phi)(s) = \chi(s)$ (here $\omega_\ell(s)$ is a root of unity of order dividing $\ell - 1$ with residue $\bar\chi_{\mathrm{cyc}}(s) = 1$, and $\phi(s) = 1$ as ℓ is unramified in k_0), where $\chi(s)^{\ell^a} = 1$ and $\chi|_{G_F}$ has order prime to ℓ (the tower is deep), so $\delta(s)^2 = 1$ and $\delta(s) = 1$ ($\delta(s) \in 1 + \mathfrak{m}_1$). So $\psi_{v,i}(s) = 1$. (R6) Let $g \in I_{\bar w} \cap G_F$, $\bar w$ not above ℓ, with $\bar r_0(g) = 1$. Then $\bar\Psi(g) = 1$, so $g \in G_{F'}$; $\bar\xi(g) = \bar\alpha(g) = 1$, and ξ has order prime to ℓ, so $\xi(g) = 1 = \xi(g_{\mathscr{T}}^{-1} g g_{\mathscr{T}})$; by (Hm3) at $\bar w$ and at $g_{\mathscr{T}}^{-1}\bar w$, $\rho_1(g) = \rho_1(g_{\mathscr{T}}^{-1} g g_{\mathscr{T}}) = 1$, so $R_0(g) = 1$; and $(\theta/\mu)(g) = \chi(g)\omega_\ell(g)^{-1}\phi(g) = 1$ ($\chi(g) = 1$ by (Oq5), ω_ℓ unramified at $\bar w$, $g \in G_{k_0}$), so $\delta(g) = 1$ and $R(g) = 1$.

Step 6: the seed pair. $Q = (R, (n_v)_{v\in\Sigma})$ is a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in $\mathcal{O}_1$ for $\mathcal{O} \subseteq \mathcal{O}_1$: (Mq1) by (R1), (R6); (Mq2) by (R2); (Mq3) by (R4); (Mq4) by (R4), (R5). It is continuous for $\mathfrak{m}_1$. Its reduction satisfies (Wq1) by (R3), (R4), and (Wq2), since $\Sigma_V \subseteq \Sigma_P$ and every $\bar n_v$ is supported in Σ_Q; it lies in V and has an entry of weight $-w_\alpha$, or entries of weights w_β and $-w_\theta$, by (R3), (R4). Its image P' in $\overline{\mathbb{Q}}_\ell$ is a seed for ι. *Irreducibility on G':* Lemma 51.13 (Ia2) applies to ρ (isomorphic to ρ_1), L, $g_1 = g_{\mathscr{T}}$, a place $\bar w = \bar v$ ($v \in \Sigma$) and $U = I_{\bar v} \cap G_{k_0}$, with (a_1, a_2), (a_1', a_2') the pairs $(k+t, -t)$ and $(k, 0)$ in the appropriate order (Hm4), $|k + 2t| \neq |k|$, and $g \in G' \setminus G_{k_0}$ from Step 1 (d); so $V|_{G'}$ is irreducible, hence so is $R|_{G'} \cong \delta \otimes V|_{G'}$, and by Burnside the span of $R(G')$ is $\mathrm{M}_4(\overline{\mathbb{Q}}_\ell)$. *Weights:* on the open subgroup $U_0 = G_F \cap \ker\omega_\ell \cap \ker\chi$

and $s \in U_0 \cap I_v$, $\psi_{v,i}(s)^2 = \chi_{\mathrm{cyc}}(s)^{h_i}$ with

$$h = (1+e-k)(1,1,1,1) + 2\,(k+t,\ k,\ 0,\ -t),$$

by (R4) and $\delta^2 = \chi_{\mathrm{cyc}}^{1+e-k}\omega_\ell^{-1}\chi\phi$ (ϕ is unramified at ℓ); $h_1 - h_2 = h_3 - h_4 = 2t$ and $h_2 - h_3 = 2k$ lie in $2\mathbb{Z}_{\geq 2}$. *The character and automorphy:* take $\kappa = \delta^{-1}|_{G_L}$, $a = k-1-e$ (so $a+h_1 = 2(k+t)$ is even) and k' the order of the finite-order character $\omega_\ell\chi^{-1}\phi$ on G_F: then $\kappa^2\chi_{\mathrm{cyc}}^{-a} = \omega_\ell\chi^{-1}\phi$ on G_L, so $(\kappa(s)^2\chi_{\mathrm{cyc}}(s)^{-a})^{k'} = 1$; and $r_{P'}|_{G_L} \otimes \kappa = R_0|_{G_L} \cong V|_{G_L} \leftrightarrow_\iota \Pi$ for a regular algebraic cuspidal Π of $\mathrm{GL}_4(\mathbb{A}_L)$ by Lemma 51.13 (Ia1), whose hypotheses are (Hm1), (Hm5) ($p_i \neq q_i$ as $|p_i - q_i| \in \{k+2t, k\}$, and $k+2t \neq k$) and $k_0 \not\subseteq L$. So Q is a seed pair with values in $\mathcal{O}_1$ for ι, as asserted. □

Proposition 51.9. *Let ℓ be an odd prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and $k_0 \subset \overline{\mathbb{Q}}$ a real quadratic field in which ℓ splits, with places $\mathfrak{l}_1 \neq \mathfrak{l}_2$ above ℓ; for $i = 1,2$ let $\tau_i\colon k_0 \to \overline{\mathbb{Q}}_\ell$ be the embedding inducing $\mathfrak{l}_i$ and ∞_i the real place $\iota \circ \tau_i$ of k_0. Let $\xi_0\colon G_{k_0} \to \overline{\mathbb{Q}}_\ell^\times$ be a continuous character of finite order such that: $\xi_0(c') = -1$ for every complex conjugation $c' \in G_{k_0}$; ξ_0 is unramified at $\mathfrak{l}_1$ and $\mathfrak{l}_2$; ξ_0 is ramified at some place of k_0 of absolute norm 2; $\bar{\xi}_0(g)^2 \neq 1$ for some $g \in G_{k_0}$; and for every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above ℓ some $g \in D_{\bar{w}} \cap G_{k_0}$ has $\bar{\xi}_0(g) \neq 1$. Then there are integers $k, t \geq 2$ divisible by $\ell - 1$ and a continuous homomorphism $\rho\colon G_{k_0} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that:*

(Hm1) *$\rho|_U$ is irreducible for every open subgroup $U \subseteq G_{k_0}$;*

(Hm2) *$\det\rho = \chi_{\mathrm{cyc}}^k\xi_0$, and for every $g \in G_{k_0}$ the characteristic polynomial of $\rho(g)$ has coefficients in $\overline{\mathbb{Z}}_\ell$ and reduces to $(X-1)(X-\bar{\xi}_0(g))$;*

(Hm3) *for every place $\bar{w}$ of $\overline{\mathbb{Q}}$ not above ℓ, the restriction of ρ to $I_{\bar{w}} \cap G_{k_0}$ is conjugate to $s \mapsto \mathrm{diag}(1, \xi_0(s))$;*

(Hm4) *for every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above ℓ, with $D = D_{\bar{w}} \cap G_{k_0} \supseteq I = I_{\bar{w}} \cap G_{k_0}$, there are continuous characters $\kappa_1, \kappa_2\colon D \to \overline{\mathbb{Q}}_\ell^\times$ and a D-stable line on which D acts by κ_1, with D acting on the quotient by κ_2, such that: if $\bar{w}$ induces $\mathfrak{l}_1$ on k_0, then $\kappa_1 = \chi_{\mathrm{cyc}}^{k+t}$ and $\kappa_2 = \chi_{\mathrm{cyc}}^{-t}$ on I, and $\bar{\kappa}_1 = \bar{\xi}_0$, $\bar{\kappa}_2 = 1$ on D; if $\bar{w}$ induces $\mathfrak{l}_2$, then $\kappa_1 = \chi_{\mathrm{cyc}}^k$ and $\kappa_2 = 1$ on I, and $\bar{\kappa}_1 = 1$, $\bar{\kappa}_2 = \bar{\xi}_0$ on D;*

(Hm5) *$\rho \leftrightarrow_\iota \pi$ for a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_{k_0})$ such that, for $i = 1,2$, the Langlands parameter of π_{∞_i} restricted to $\mathbb{C}^\times \subset W_{\mathbb{R}}$ is $z \mapsto \mathrm{diag}(z^{p_i}\bar{z}^{q_i},\ z^{q_i}\bar{z}^{p_i})$ with $p_i, q_i \in \frac{1}{2}+\mathbb{Z}$, $p_1+q_1 = p_2+q_2$, $|p_1-q_1| = k+2t$ and $|p_2-q_2| = k$.*

Proof. Throughout, $\mathrm{Frob}_\mathfrak{q}$ is an arithmetic Frobenius, and ξ_0 is also viewed, by class field theory, as a character of the narrow ray class group modulo its conductor $\mathfrak{f}$, with $\xi_0(\mathfrak{q}) = \xi_0(\mathrm{Frob}_\mathfrak{q})$; $\mathfrak{f}$ is prime to ℓ (ξ_0 is unramified at $\mathfrak{l}_1, \mathfrak{l}_2$) and ξ_0 has sign $(1,1)$ (it is -1 on complex conjugations). As ξ_0 is unramified at $\mathfrak{l}_i$ and some $g \in D_{\bar{w}}$ ($\bar{w} \mid \mathfrak{l}_i$) has $\bar{\xi}_0(g) \neq 1$, $\bar{\xi}_0(\mathfrak{l}_i) \neq 1$ for $i = 1,2$. Let $I = \{\tau_1, \tau_2\}$; an element of $\mathbb{Z}[I]$ is written (a_1, a_2), and τ_i corresponds to the real place $\infty_i = \iota\circ\tau_i$.

Step 1: a parallel-weight cusp form with opposite stabilisations. By Proposition 51.10 (with $B = 2$) there are an integer $k \geq 2$ divisible by $\ell - 1$ and a non-zero eigenform $F_0 \in S_{k+1}(\mathfrak{f}\mathfrak{l}_1\mathfrak{l}_2, \xi_0)$ with eigenvalues $\lambda_0(\mathfrak{q})$, ℓ-integral, $\lambda_0(\mathfrak{q}) \equiv 1 + \xi_0(\mathfrak{q})$ for $\mathfrak{q} \nmid \mathfrak{f}\ell$, $\lambda_0(\mathfrak{l}_1) \equiv 1$, $\lambda_0(\mathfrak{l}_2) \equiv \xi_0(\mathfrak{l}_2)$. Fix an integer $t \geq 2$ divisible by $\ell - 1$.

Step 2: the point of F_0 on Hida's algebra. Let $N = \mathfrak{f}$ and let $\mathcal{O}$ be a coefficient ring containing $\tau_i(\mathcal{O}_{k_0})$, the values of ξ_0, the $\lambda_0(\mathfrak{q})$ and the e-th roots of unity, $e = \#G_{\mathrm{tor}}$. Viewed

as a function on $\mathrm{GL}_2(\mathbb{A}_{k_0})$ ([230], Section 2; [236], Section 3.1), F_0 is a cusp form of parallel weight $k+1$, i.e. of arithmetic weight $(n_0,v_0)=((k-1,k-1),(0,0))$, of level $V_1(\mathfrak{f})\cap U(\ell)$, invariant under the Iwahori subgroups at $\mathfrak{l}_1,\mathfrak{l}_2$ (its character ξ_0 is trivial on $\mathfrak{r}_\ell^\times$), with characters $(\psi,\psi')=(\xi_0,\mathbf{1})$; its eigenvalues for Hida's $T(\mathfrak{q})$ ($\mathfrak{q}\nmid\mathfrak{f}\ell$) and $T(\varpi_{\mathfrak{l}_i})$ (double coset of $\mathrm{diag}(1,\varpi_{\mathfrak{l}_i})$, with $v_0=0$ no normalisation) are Shimura's $\lambda_0(\mathfrak{q})$ and $\lambda_0(\mathfrak{l}_i)$ ([236], pp. 29–30), units at $\mathfrak{l}_i$, so F_0 is nearly ordinary. By (Hf2) it defines $\lambda_0\colon\mathbf{h}\to\mathcal{O}$ with $\lambda_0|_{\mathcal{O}[[G]]}=P_0:=P_{n_0,0,\xi_0,\mathbf{1}}$, $P_0(z,a)=\xi_0(z)\mathcal{N}(z)^{k-1}$. Its kernel is a prime of $\mathbf{h}$ containing a minimal prime $\mathfrak{a}$; by (Hf1), $\mathbb{I}=\mathbf{h}/\mathfrak{a}$ is a domain, finite and torsion-free over A, hence $A\to\mathbb{I}$ is injective and integral; $\mathbb{I}$ is local (a domain finite over the complete local ring A), and λ_0 factors through $\mathbb{I}$.

Step 3: a point of non-parallel weight on the same component. Put $n'=(k+2t-1,\,k-1)$, $v'=(0,t)$: an arithmetic weight with $n'+2v'=(k+2t-1)t$ and $k'=n'+2t=(k+2t+1,\,k+1)$. Let $P'=P_{n',v',\xi_0,\mathbf{1}}$, i.e. $P'(z,a)=P_0(z,a)\,\mathcal{N}(z)^{2t}a^{v'}$, and $\mathfrak{p}'=\ker(P'|_A)$; $P'|_A\colon A\to\mathcal{O}$ is surjective, so $A/\mathfrak{p}'=\mathcal{O}$. By lying over there is a prime $\mathfrak{P}\subset\mathbb{I}$ with $\mathfrak{P}\cap A=\mathfrak{p}'$; $\mathbb{I}/\mathfrak{P}$ is a domain finite over $\mathcal{O}$, hence contained in a coefficient ring $\mathcal{O}'$. Let $\lambda'\colon\mathbf{h}\to\mathbb{I}\to\mathbb{I}/\mathfrak{P}\subseteq\mathcal{O}'$. (a) $\lambda'|_A=P'|_A$: both are $\mathcal{O}$-algebra maps $A\to\mathcal{O}'$ with kernel $\mathfrak{p}'$, i.e. the identity of $A/\mathfrak{p}'=\mathcal{O}$. (b) $\lambda'=\lambda_0$ on G_{tor}: the image u in $\mathbb{I}$ of $g\in G_{\mathrm{tor}}$ satisfies $\prod_{\zeta^e=1}(u-\zeta)=u^e-1=0$ in the domain $\mathbb{I}$, with all $\zeta\in\mathcal{O}$, so $u=\zeta$ for some $\zeta\in\mathcal{O}$, and both λ' and λ_0 (which factor through $\mathbb{I}$ and are $\mathcal{O}$-linear) send g to ζ. (c) $P'=P_0$ on G_{tor}: for $g=(z,a)$ of finite order $\mathcal{N}(z)$ and $\tau_2(a)$ are roots of unity in $\mathbb{Z}_\ell^\times$, of order dividing $\ell-1$, and $(\ell-1)\mid t$. Since $G=W\times G_{\mathrm{tor}}$ and λ' is continuous (an $\mathcal{O}$-algebra map from a complete local noetherian $\mathcal{O}$-algebra with finite residue field to $\mathcal{O}'$ is local, and $\mathbf{h}$ is finite over A), $\lambda'|_{\mathcal{O}[[G]]}=P'$. (d) $\lambda'\equiv\lambda_0$: both factor through the local ring $\mathbb{I}$, so $\overline{\lambda'(h)}=\overline{\lambda_0(h)}$ for every $h\in\mathbf{h}$.

Step 4: the classical form f' and its Galois representation. By (Hf2) there is a non-zero nearly ordinary common eigenform f' of weight (k',w'), level $V_1(\mathfrak{f})\cap U(\ell)$ and characters $(\xi_0,\mathbf{1})$, with eigenvalues $\iota\lambda'$. By (Hf3) let $\sigma'=\sigma_{\lambda'}$ and $\sigma_0=\sigma_{\lambda_0}$, semisimple, unramified outside $\mathfrak{f}\ell$, with $\det(1-\sigma'(\mathrm{Frob}_{\mathfrak{q}})X)=1-\lambda'(T(\mathfrak{q}))X+\lambda'(\langle\mathfrak{q}\rangle)\mathrm{N}\mathfrak{q}X^2$. As $\langle\mathfrak{q}\rangle$ is the class of $\mathfrak{q}$ in $Z(N)$ and $\mathcal{N}(\langle\mathfrak{q}\rangle)=\mathrm{N}\mathfrak{q}=\chi(\mathrm{Frob}_{\mathfrak{q}})$, $\lambda'(\langle\mathfrak{q}\rangle)\mathrm{N}\mathfrak{q}=P'(\langle\mathfrak{q}\rangle)\mathrm{N}\mathfrak{q}=\xi_0(\mathfrak{q})\mathrm{N}\mathfrak{q}^{k+2t}$; by Chebotarev density, $\det\sigma'=\xi_0\chi^{k+2t}$. By Step 3 (d), $\mathrm{tr}\,\sigma'(\mathrm{Frob}_{\mathfrak{q}})=\lambda'(T(\mathfrak{q}))\equiv\lambda_0(\mathfrak{q})\equiv1+\xi_0(\mathfrak{q})$ for $\mathfrak{q}\nmid\mathfrak{f}\ell$. Put

$$\rho=\sigma'\otimes\chi^{-t}\colon G_{k_0}\to\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell),\qquad\det\rho=\chi^k\xi_0.$$

Step 5: (Hm2). ρ takes values in $\mathrm{GL}_2(\mathcal{O}'')$, after conjugation, for a coefficient ring $\mathcal{O}''$. Indeed, G_{k_0} is the union of the closed subgroups $\rho^{-1}(\mathrm{GL}_2(E))$ over the countably many finite extensions E of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ (E is complete, hence closed); by the Baire category theorem one of them has non-empty interior, so is open, hence of finite index in the compact group G_{k_0}, and adjoining the entries of ρ at finitely many coset representatives gives a finite extension E' with $\rho(G_{k_0})\subseteq\mathrm{GL}_2(E')$. The $\mathcal{O}_{E'}$-module L spanned by $\rho(G_{k_0})\mathcal{O}_{E'}^2$ is bounded (as $\rho(G_{k_0})$ is compact), hence finitely generated, spans E'^2 and is $\rho(G_{k_0})$-stable; in an $\mathcal{O}_{E'}$-basis of L, ρ takes values in $\mathrm{GL}_2(\mathcal{O}'')$ with $\mathcal{O}''=\mathcal{O}_{E'}$. So characteristic polynomials have coefficients in $\overline{\mathbb{Z}}_\ell$. As $\mathrm{N}\mathfrak{q}^t\equiv1$ ($(\ell-1)\mid t$), $\mathrm{tr}\,\rho(\mathrm{Frob}_{\mathfrak{q}})\equiv1+\xi_0(\mathrm{Frob}_{\mathfrak{q}})$ and $\det\rho(\mathrm{Frob}_{\mathfrak{q}})\equiv\xi_0(\mathrm{Frob}_{\mathfrak{q}})$ (as $\chi^k\equiv1$). The sets of g with $\mathrm{tr}\,\rho(g)-1-\xi_0(g)\in\mathfrak{m}_{\mathcal{O}''}$, resp. $\det\rho(g)-\xi_0(g)\in\mathfrak{m}_{\mathcal{O}''}$, are closed and contain the Frobenius elements, which are dense (Chebotarev); hence the characteristic polynomial of $\rho(g)$ reduces to $(X-1)(X-\bar\xi_0(g))$ for every g.

Step 6: (Hm4). Let $\bar w$ be a place above ℓ, $D=D_{\bar w}\cap G_{k_0}\supseteq I=I_{\bar w}\cap G_{k_0}$, a decomposition group at $\mathfrak{p}=\mathfrak{l}_i$ ($k_{0,\mathfrak{p}}=\mathbb{Q}_\ell$, $\mathfrak{r}_{\mathfrak{p}}=\mathbb{Z}_\ell$). By (Hf3), $\sigma'|_D\cong\left(\begin{smallmatrix}\epsilon'&*\\0&\delta'\end{smallmatrix}\right)$ and $\sigma_0|_D\cong\left(\begin{smallmatrix}\epsilon_0&*\\0&\delta_0\end{smallmatrix}\right)$, with $\delta'(\mathrm{rec}_{\mathfrak{p}}(x))=\lambda_{f'}(x^{-v'}T(x))$ and $\delta_0(\mathrm{rec}_{\mathfrak{p}}(x))=\lambda_{F_0}(T(x))$. (a) *Units.* For $u\in\mathbb{Z}_\ell^\times$, $T(u)$ is the double coset of $\mathrm{diag}(1,u)=u\cdot\mathrm{diag}(u^{-1},1)$, which normalises the level group; $\mathrm{diag}(u^{-1},1)$ acts trivially on f' and F_0 (character $\psi'=\mathbf{1}$, [129], Section 3, where $\langle1,a\rangle=T(a^{-1},1)$ is the double coset of $\mathrm{diag}(a^{-1},1)$), and the central element u acts by the central character, whose finite-order

part is ξ_0 (it agrees with ξ_0 at almost all primes by the formula for $\lambda'(\langle\mathfrak{q}\rangle)$, and finite-order Hecke characters agreeing at almost all primes are equal), trivial on $\mathfrak{r}_\mathfrak{p}^\times$. So $\lambda_{f'}(T(u)) = \lambda_{F_0}(T(u)) = 1$, and with $\mathcal{N}(\mathrm{rec}_\mathfrak{p}(u)) = u^{-1}$:

$$\delta'|_I = \chi^{v'_\mathfrak{p}}, \qquad \delta_0|_I = \mathbf{1}, \qquad v'_{\mathfrak{l}_1} = 0,\ v'_{\mathfrak{l}_2} = t.$$

(b) *Residues.* D^{ab} is topologically generated by $\mathrm{rec}_\mathfrak{p}(\mathbb{Z}_\ell^\times)$ and $\mathrm{rec}_\mathfrak{p}(\ell)$, which acts on unramified extensions as the arithmetic Frobenius in this normalisation. On $\mathrm{rec}_\mathfrak{p}(\ell)$, δ' and δ_0 take the values λ' and λ_0 of the element $T(\varpi_\mathfrak{p})$ of $\mathbf{h}$ (normalised by ϖ^{-v} in weight (k,w)), congruent by Step 3 (d); on units both are $\equiv 1$ ($\bar\chi^t = \bar\omega_\ell^t = 1$). Hence $\bar\delta' = \bar\delta_0$ is unramified with $\bar\delta_0(\mathrm{Frob}_\mathfrak{p}) = \bar\lambda_0(\mathfrak{p})$: $\bar\delta' = \mathbf{1}$ on D if $\mathfrak{p} = \mathfrak{l}_1$, and $\bar\delta' = \bar\xi_0|_D$ if $\mathfrak{p} = \mathfrak{l}_2$ (ξ_0 is unramified at $\mathfrak{l}_2$, $\bar\xi_0(\mathrm{Frob}) = \bar\xi_0(\mathfrak{l}_2)$). (c) Put $\kappa_1 = \epsilon'\chi^{-t}$, $\kappa_2 = \delta'\chi^{-t}$: $\rho|_D$ has a D-stable line with character κ_1 and quotient κ_2, $\kappa_1\kappa_2 = \det\rho = \chi^k\xi_0$, and $\xi_0|_I = \mathbf{1}$. If $\mathfrak{p} = \mathfrak{l}_1$: $\kappa_2|_I = \chi^{-t}$, $\kappa_1|_I = \chi^{k+t}$, $\bar\kappa_2 = \mathbf{1}$, $\bar\kappa_1 = \bar\xi_0$. If $\mathfrak{p} = \mathfrak{l}_2$: $\kappa_2|_I = \mathbf{1}$, $\kappa_1|_I = \chi^k$, $\bar\kappa_2 = \bar\xi_0$, $\bar\kappa_1 = \mathbf{1}$ (using $\bar\chi^k = \bar\chi^t = \mathbf{1}$). This is (Hm4).

Step 7: the automorphic representation and (Hm5). Let π' be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{k_0})$ on which the projection of f' is non-zero; for $\mathfrak{q} \nmid \mathfrak{f}\ell$ the projection is fixed by $\mathrm{GL}_2(\mathcal{O}_\mathfrak{q})$, and $\pi'_\mathfrak{q}$ is the irreducible normalised induction of unramified μ_1, μ_2 with $\lambda'(T(\mathfrak{q})) = \mathrm{N}\mathfrak{q}^{1/2}(\mu_1 + \mu_2)(\varpi_\mathfrak{q})$ or $\mathrm{N}\mathfrak{q}^{1/2}(\mu_1^{-1} + \mu_2^{-1})(\varpi_\mathfrak{q})$, according as the double coset acts through $\mathrm{diag}(1,\varpi)$ or its inverse (the eigenvalue of the double coset on the spherical vector), the same alternative for all $\mathfrak{q}$. At the real place ∞_i, $\pi'_{\infty_i} \cong D_{k'_i} \otimes |\det|^s$, a holomorphic discrete series of weight $k'_1 = k + 2t + 1$, $k'_2 = k + 1$, twisted by one s (the central character is algebraic); so π' is regular algebraic. Let $r = r_{\ell,\iota}(\pi')$ (Theorem B.85), so that $r(\mathrm{Frob}_\mathfrak{q}^{-1})$ has eigenvalues $\iota^{-1}(\mathrm{N}\mathfrak{q}^{1/2}\mu_i(\varpi_\mathfrak{q}))$. Since $r_{\ell,\iota}(\Pi \otimes |\det|^a) = r_{\ell,\iota}(\Pi) \otimes \chi^a$ and $r_{\ell,\iota}(\Pi^\vee) \otimes \chi = r_{\ell,\iota}(\Pi)^\vee$ (comparison of Satake parameters at unramified places and Chebotarev), comparing traces at $\mathrm{Frob}_\mathfrak{q}$ gives, by Chebotarev and Brauer–Nesbitt (all representations semisimple), $\sigma' \cong r_{\ell,\iota}(\Pi_0)$ with $\Pi_0 = \pi'^\vee \otimes |\det|$ in the first case and $\Pi_0 = \pi' \otimes |\det|$ in the second; hence $\rho \cong r_{\ell,\iota}(\Pi)$, $\Pi = \Pi_0 \otimes |\det|^{-t}$, and $\rho \leftrightarrow_\iota \Pi$. Π is regular algebraic cuspidal, $\Pi_{\infty_i} \cong D_{k'_i} \otimes |\det|^{s'}$ with one s'; the Langlands parameter of $D_\kappa \otimes |\det|^{s'}$ on $\mathbb{C}^\times$ is $z \mapsto \mathrm{diag}(z^p\bar z^q, z^q\bar z^p)$ with $p = \frac{\kappa-1}{2} + s'$, $q = -\frac{\kappa-1}{2} + s'$, and $p, q \in \frac{1}{2} + \mathbb{Z}$ because Π is regular algebraic: in the sense of [68] the exponents of the Langlands parameter of each Π_{∞_i} on $\mathbb{C}^\times$ lie in $\frac{n-1}{2} + \mathbb{Z}$, here $n = 2$. So $p_1 + q_1 = p_2 + q_2 = 2s'$, $|p_1 - q_1| = k + 2t$, $|p_2 - q_2| = k$: (Hm5).

Step 8: (Hm3). ρ is unramified outside $\mathfrak{f}\ell$, and so is ξ_0; this gives (Hm3) at places not above $\mathfrak{f}\ell$. Let $\bar w$ lie above $\mathfrak{q} \mid \mathfrak{f}$. The component of $\pi'_\mathfrak{q}$ is fixed by the $\mathfrak{q}$-part of $V_1(\mathfrak{f})$, conjugate to $K_1(\mathfrak{q}^c) = \{\left(\begin{smallmatrix} a & b \\ c' & d \end{smallmatrix}\right) : c' \equiv 0,\ d \equiv 1 \bmod \mathfrak{q}^c\}$, $c = v_\mathfrak{q}(\mathfrak{f})$, so the conductor of $\pi'_\mathfrak{q}$ is at most c, while its central character restricted to $\mathcal{O}_\mathfrak{q}^\times$ is that of ξ_0, of conductor exactly c (Step 6 (a)). An irreducible infinite-dimensional representation of $\mathrm{GL}_2(k_{0,\mathfrak{q}})$ whose conductor equals that of its central character is an irreducible principal series $\mu_1 \boxplus \mu_2$ with μ_1 unramified (Casselman's theory of the conductor: twists of Steinberg and supercuspidal representations have conductor larger than that of their central character). Hence $\Pi_\mathfrak{q}$, a twist by an unramified character of $\pi'_\mathfrak{q}$ or of $\pi'^\vee_\mathfrak{q}$, is $\mu'_1 \boxplus \mu'_2$ with μ'_1 unramified; by Theorem B.85, $\rho|_{I_{\bar w} \cap G_{k_0}} \cong \mathbf{1} \oplus \nu$ for a finite-order ν, and $\nu = \det\rho = \xi_0$ on inertia. So $\rho|_{I_{\bar w} \cap G_{k_0}}$ is conjugate to $s \mapsto \mathrm{diag}(1, \xi_0(s))$.

Step 9: (Hm1). (a) ρ is irreducible: otherwise its semisimplification is $\theta_1 \oplus \theta_2$ with continuous characters θ_i of G_{k_0}; by Step 6, $\theta_1|_{I_1} \in \{\chi^{k+t}, \chi^{-t}\}$ on inertia I_1 at $\mathfrak{l}_1$ and $\theta_1|_{I_2} \in \{\chi^k, \mathbf{1}\}$ at $\mathfrak{l}_2$, say $\theta_1|_{I_j} = \chi^{a_j}$ with $a_1 \neq a_2$ (as $t \geq 2$, $k \geq 2$). By class field theory θ_1 corresponds to a character of the idèles trivial on $k_0^\times$; its components at the places not above ℓ have finite order on local units and at the real places, and at $\mathfrak{l}_j$ it is $u \mapsto \chi^{a_j}(\mathrm{rec}_{\mathfrak{l}_j}(u)) = u^{-a_j}$ on $\mathbb{Z}_\ell^\times$. Evaluating at a global unit ε of infinite order, replaced by a power that kills the finitely many finite-order components, gives $\tau_1(\varepsilon)^{-a_1}\tau_2(\varepsilon)^{-a_2} = 1$, and with $\tau_1(\varepsilon)\tau_2(\varepsilon) = \pm 1$, $\tau_1(\varepsilon)^{2(a_2-a_1)} = 1$, impossible as $\tau_1(\varepsilon)$ is not a root of unity. (b) Let $U \subseteq G_{k_0}$ be open with $\rho|_U$ reducible; replacing U by its

normal core we may assume U normal. By Clifford's theorem $\rho|_U$ is semisimple. If $\rho|_U \cong \theta \oplus \theta$, then $\rho(U)$ consists of scalars, so the image of ρ in PGL_2 is finite, and on an open subgroup of I_1 the ratio $\kappa_1/\kappa_2 = \chi^{k+2t}$ of the diagonal characters of Step 6 would have finite order, which is false. If $\rho|_U \cong \theta \oplus \theta'$ with $\theta \neq \theta'$, the stabiliser H of θ has index 2 and $\rho \cong \mathrm{Ind}_H^{G_{k_0}} \tilde\theta$ for a character $\tilde\theta$ of H; then $\bar\rho^{\mathrm{ss}} = (\mathrm{Ind}_H \bar{\tilde\theta})^{\mathrm{ss}}$ is $\mathbf{1} \oplus \bar\xi_0$ (Step 5 and Brauer–Nesbitt), so $\mathrm{Ind}_H \bar{\tilde\theta}$ is reducible, $\bar{\tilde\theta}$ is invariant under G_{k_0}/H and extends to ν on G_{k_0}, and $\{\mathbf{1}, \bar\xi_0\} = \{\nu, \nu\eta_H\}$ with η_H the quadratic character of H; hence $\bar\xi_0 = \eta_H^{\pm 1}$ and $\bar\xi_0^2 = \mathbf{1}$, contrary to the hypothesis. So $\rho|_U$ is irreducible for every open U.

Conclusion. $k, t \geq 2$ are divisible by $\ell - 1$ and ρ satisfies (Hm1)–(Hm5). The hypothesis that ξ_0 is ramified at a place of norm 2 was not used. □

Proposition 51.10. *Let ℓ be an odd prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, k_0 a real quadratic field in which $\ell = \mathfrak{l}_1\mathfrak{l}_2$ splits, and ξ_0 a finite-order character of the narrow ray class group modulo an integral ideal $\mathfrak{f}$ prime to ℓ, of sign $(1,1)$ (totally odd), such that $\xi_0(\mathfrak{l}_1) \not\equiv 1$ and $\xi_0(\mathfrak{l}_2) \not\equiv 1$. Then for every $B \geq 1$ there are an integer $k \geq B$ divisible by $\ell - 1$ and a non-zero $F \in S_{k+1}(\mathfrak{f}\mathfrak{l}_1\mathfrak{l}_2, \xi_0)$ which is an eigenvector of every $T(\mathfrak{q})$, with eigenvalue $\lambda_F(\mathfrak{q})$, such that $|\iota^{-1}\lambda_F(\mathfrak{q})|_\ell \leq 1$ for all $\mathfrak{q}$ and*

$$\lambda_F(\mathfrak{q}) \equiv 1 + \xi_0(\mathfrak{q}) \quad (\mathfrak{q} \nmid \mathfrak{f}\ell), \qquad \lambda_F(\mathfrak{l}_1) \equiv 1, \qquad \lambda_F(\mathfrak{l}_2) \equiv \xi_0(\mathfrak{l}_2), \qquad \lambda_F(\mathfrak{q}) \equiv 1 \quad (\mathfrak{q} \mid \mathfrak{f}).$$

Proof. Let $\mathcal{O} \subset \overline{\mathbb{Q}}_\ell$ be a coefficient ring containing ι^{-1} of the values of ξ_0, with uniformiser ϖ and maximal ideal $\mathfrak{m}_{\mathcal{O}}$; put $\mathfrak{n} = \mathfrak{f}\mathfrak{l}_1\mathfrak{l}_2$. We identify $\mathcal{O}$ with $\iota(\mathcal{O}) \subset \mathbb{C}$ where convenient.

Step 1: the weight and a form congruent to one. Let N_0 be as in Lemma 51.11 for k_0, ℓ (ℓ splits, so ℓ does not divide the discriminant), and let $k \geq \max(B, 2)$ be divisible by $(\ell-1)\ell^{N_0}$. Let $G \in M_k(\mathcal{O}_{k_0}, \mathbf{1})$ be the form of Lemma 51.11: $c_\lambda(0, G) = 1$ and $c(\mathfrak{m}, G) \in \mathfrak{m}_{\mathcal{O}}$ for all λ and all $\mathfrak{m} \neq 0$.

Step 2: the weight-one Eisenstein series with opposite stabilisations. Let η be the trivial character viewed modulo $\mathfrak{a} = \mathfrak{l}_2$ (so $\eta(\mathfrak{c}) = 1$ if $\mathfrak{l}_2 \nmid \mathfrak{c}$ and 0 otherwise; sign 0), and ψ_1 the character ξ_0 viewed modulo $\mathfrak{b} = \mathfrak{f}\mathfrak{l}_1$ (sign $(1,1)$). As $0 + (1,1) \equiv (1,1)$, (Hb4) gives $E = E_1(\eta, \psi_1) \in M_1(\mathfrak{n}, \xi_0)$ with $c_\lambda(0, E) = 0$ for all λ ($\mathfrak{a} \neq 1 \neq \mathfrak{b}$) and $c(\mathfrak{m}, E) = \sum_{\mathfrak{r}|\mathfrak{m}} \eta(\mathfrak{m}\mathfrak{r}^{-1})\psi_1(\mathfrak{r}) \in \mathcal{O}$. Being the Dirichlet convolution of two completely multiplicative functions, $c(\cdot, E)$ is multiplicative, with $c(\mathcal{O}_{k_0}, E) = 1$ and local factors $a_{\mathfrak{q}}(e) = c(\mathfrak{q}^e, E)$:

Hence, for all $\mathfrak{m} \neq 0$,

$$\begin{gathered} c(\mathfrak{m}\mathfrak{q}, E) + \xi_0(\mathfrak{q}) c(\mathfrak{m}\mathfrak{q}^{-1}, E) = (1 + \xi_0(\mathfrak{q})) c(\mathfrak{m}, E) \quad (\mathfrak{q} \nmid \mathfrak{f}\ell), \\ c(\mathfrak{m}\mathfrak{l}_1, E) = c(\mathfrak{m}, E), \quad c(\mathfrak{m}\mathfrak{l}_2, E) = \xi_0(\mathfrak{l}_2) c(\mathfrak{m}, E), \quad c(\mathfrak{m}\mathfrak{q}, E) = c(\mathfrak{m}, E) \quad (\mathfrak{q} \mid \mathfrak{f}). \end{gathered} \tag{Ep}$$

Step 3: the product $H = EG$. By (Hb3), $H \in M_{k+1}(\mathfrak{n}, \xi_0)$. We show that $c(\mathfrak{m}, H) \in \mathcal{O}$ and $c(\mathfrak{m}, H) \equiv c(\mathfrak{m}, E)$ for all $\mathfrak{m} \neq 0$. Write $\mathfrak{m} = (b)\mathfrak{t}_\lambda^{-1}$, $b \in \mathfrak{t}_\lambda$ totally positive. The coefficient of $e_F(bz)$ in $E_\lambda G_\lambda$ is $\sum a_\lambda^E(b_1) a_\lambda^G(b_2)$ over $b_1 + b_2 = b$ with $b_i \in \{0\} \cup \{x \in \mathfrak{t}_\lambda : x \gg 0\}$ (a finite sum). Dividing by $\mathrm{N}(\mathfrak{t}_\lambda)^{(k+1)/2} = \mathrm{N}(\mathfrak{t}_\lambda)^{1/2}\mathrm{N}(\mathfrak{t}_\lambda)^{k/2}$,

$$c(\mathfrak{m}, H) = c(\mathfrak{m}, E)\, c_\lambda(0, G) + c_\lambda(0, E)\, c(\mathfrak{m}, G) + \sum_{\substack{b_1 + b_2 = b \\ b_1, b_2 \gg 0}} c\big((b_1)\mathfrak{t}_\lambda^{-1}, E\big)\, c\big((b_2)\mathfrak{t}_\lambda^{-1}, G\big),$$

and $c_\lambda(0, G) = 1$, $c_\lambda(0, E) = 0$, while every term of the sum lies in $\mathfrak{m}_{\mathcal{O}}$.

Step 4: an integral lattice. Let $L = \{f \in M_{k+1}(\mathfrak{n}, \xi_0) : c(\mathfrak{m}, f) \in \mathcal{O} \text{ for all } \mathfrak{m} \neq 0\}$, an $\mathcal{O}$-module containing H. By (Hb1) there are $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ such that $f \mapsto (c(\mathfrak{m}_i, f))_i$ is injective on

$M_{k+1}(\mathfrak{n},\xi_0)$; it embeds L into $\mathcal{O}^r$, so L is finitely generated and torsion-free, hence free over $\mathcal{O}$. If $f_1,\dots,f_s\in L$ are $\mathcal{O}$-linearly independent they are $\mathbb{C}$-linearly independent (the matrix of their coefficients has entries in the field $\iota(\operatorname{Frac}\mathcal{O})$, and its rank does not change over $\mathbb{C}$); so for every coefficient ring $\mathcal{O}'\supseteq\mathcal{O}$ the map $L\otimes_{\mathcal{O}}\mathcal{O}'\to M_{k+1}(\mathfrak{n},\xi_0)$, $f\otimes a\mapsto \iota(a)f$, is injective. By (Hb2) every $T(\mathfrak{q})$ preserves L (its coefficient $\xi_0(\mathfrak{q})\mathrm{N}\mathfrak{q}^k$ lies in $\mathcal{O}$), and these endomorphisms commute. An $f\in L$ with all $c(\mathfrak{m},f)\in\mathfrak{m}_{\mathcal{O}}$ lies in ϖL. Finally $H\notin\varpi L$, since $c(\mathcal{O}_{k_0},H)\equiv c(\mathcal{O}_{k_0},E)=1$.

Step 5: H is an eigenvector modulo ϖ. Define $t(\mathfrak{q})\in\mathcal{O}$ by $t(\mathfrak{q})=1+\xi_0(\mathfrak{q})$ for $\mathfrak{q}\nmid\mathfrak{f}\ell$, $t(\mathfrak{l}_1)=1$, $t(\mathfrak{l}_2)=\xi_0(\mathfrak{l}_2)$, $t(\mathfrak{q})=1$ for $\mathfrak{q}\mid\mathfrak{f}$. For $\mathfrak{q}\nmid\mathfrak{f}\ell$, $\mathrm{N}\mathfrak{q}$ is prime to ℓ and $(\ell-1)\mid k$, so $\mathrm{N}\mathfrak{q}^k\equiv 1$; by (Hb2), Step 3 and (Ep),

$$c(\mathfrak{m},H|T(\mathfrak{q}))=c(\mathfrak{m}\mathfrak{q},H)+\xi_0(\mathfrak{q})\mathrm{N}\mathfrak{q}^k c(\mathfrak{m}\mathfrak{q}^{-1},H)\equiv(1+\xi_0(\mathfrak{q}))c(\mathfrak{m},E)\equiv t(\mathfrak{q})\,c(\mathfrak{m},H).$$

For $\mathfrak{q}\mid\mathfrak{n}$, $c(\mathfrak{m},H|U(\mathfrak{q}))=c(\mathfrak{m}\mathfrak{q},H)\equiv c(\mathfrak{m}\mathfrak{q},E)=t(\mathfrak{q})c(\mathfrak{m},E)\equiv t(\mathfrak{q})c(\mathfrak{m},H)$ by (Ep). By Step 4, $H|T(\mathfrak{q})-t(\mathfrak{q})H\in\varpi L$ for every $\mathfrak{q}$: the image of H in $L/\varpi L$ is a non-zero common eigenvector of the commuting endomorphisms $T(\mathfrak{q})$, with eigenvalues $t(\mathfrak{q})$ mod ϖ.

Step 6: lifting. By the lemma of Deligne and Serre ([78], Lemme 6.11: for a free module of finite rank over a discrete valuation ring with a family of commuting endomorphisms, a non-zero common eigenvector modulo the maximal ideal lifts to a common eigenvector over a finite extension, with eigenvalues lifting the given ones) there are a coefficient ring $\mathcal{O}'\supseteq\mathcal{O}$ and a non-zero $F\in L\otimes_{\mathcal{O}}\mathcal{O}'$, viewed in $M_{k+1}(\mathfrak{n},\xi_0)$ by Step 4, with $F|T(\mathfrak{q})=\lambda_F(\mathfrak{q})F$, $\lambda_F(\mathfrak{q})\in\mathcal{O}'$ and $\lambda_F(\mathfrak{q})\equiv t(\mathfrak{q})$ for all $\mathfrak{q}$. These are the asserted congruences.

Step 7: F is a cusp form. Suppose not. By (Hb5) ($\kappa=k+1\geq 3$) there are finite-order characters η_1,η_2 of a narrow ray class group modulo some $\mathfrak{c}$, with $\eta_1\eta_2(\mathfrak{q})=\xi_0(\mathfrak{q})$ for $\mathfrak{q}\nmid\mathfrak{n}$, such that $\lambda_F(\mathfrak{q})=\eta_1(\mathfrak{q})+\eta_2(\mathfrak{q})\mathrm{N}\mathfrak{q}^k$ for $\mathfrak{q}\nmid\mathfrak{n}$ and $\lambda_F(\mathfrak{l}_i)\in\{0,\eta_1(\mathfrak{l}_i),\eta_2(\mathfrak{l}_i)\ell^k\}$ ($\mathrm{N}\mathfrak{l}_i=\ell$). Enlarge $\mathcal{O}'$ to contain the values of η_1,η_2 and write bars for reduction modulo its maximal ideal. (a) For $\mathfrak{q}\nmid\mathfrak{n}\mathfrak{c}$: $\bar\eta_1(\mathfrak{q})+\bar\eta_2(\mathfrak{q})=1+\bar\eta_1(\mathfrak{q})\bar\eta_2(\mathfrak{q})$, i.e. $(\bar\eta_1(\mathfrak{q})-1)(\bar\eta_2(\mathfrak{q})-1)=0$. Every class of the narrow ray class group modulo $\mathfrak{c}\mathfrak{n}$ contains a prime $\mathfrak{q}\nmid\mathfrak{n}\mathfrak{c}$ (Chebotarev density theorem for ray class fields), so this group is the union of the kernels of the homomorphisms $\bar\eta_1,\bar\eta_2$; a group is not the union of two proper subgroups, hence $\bar\eta_1=\mathbf{1}$ or $\bar\eta_2=\mathbf{1}$. (b) $\lambda_F(\mathfrak{l}_i)$ is a unit (congruent to 1 or to the root of unity $\xi_0(\mathfrak{l}_2)$), so it is neither 0 nor $\eta_2(\mathfrak{l}_i)\ell^k$; hence $\lambda_F(\mathfrak{l}_i)=\eta_1(\mathfrak{l}_i)\neq 0$, η_1 is unramified at $\mathfrak{l}_1,\mathfrak{l}_2$, and $\bar\eta_1(\mathfrak{l}_1)=1$, $\bar\eta_1(\mathfrak{l}_2)=\bar\xi_0(\mathfrak{l}_2)$. As ξ_0 is unramified at $\mathfrak{l}_1,\mathfrak{l}_2$, so is η_2 (it agrees with $\xi_0\eta_1^{-1}$ at all primes off $\mathfrak{n}\mathfrak{c}$), and we may take $\mathfrak{c}$ prime to ℓ. If $\bar\eta_1=\mathbf{1}$, then $\bar\xi_0(\mathfrak{l}_2)=1$, contrary to the hypothesis. If $\bar\eta_2=\mathbf{1}$, then $\bar\eta_1(\mathfrak{q})=\bar\xi_0(\mathfrak{q})$ for all $\mathfrak{q}\nmid\mathfrak{n}\mathfrak{c}$, hence $\bar\eta_1=\bar\xi_0$ on the narrow ray class group modulo $\mathfrak{c}\mathfrak{f}$ (Chebotarev again), and evaluating at the class of $\mathfrak{l}_1$ gives $1=\bar\eta_1(\mathfrak{l}_1)=\bar\xi_0(\mathfrak{l}_1)$, contrary to the hypothesis. So $F\in S_{k+1}(\mathfrak{n},\xi_0)$. □

Lemma 51.11. *Let k_0 be a real quadratic field, ℓ an odd prime not dividing the discriminant of k_0, and $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$ an isomorphism. There is $N_0\geq 0$ such that for every integer $k\geq 2$ divisible by $(\ell-1)\ell^{N_0}$ there is $G\in M_k(\mathcal{O}_{k_0},\mathbf{1})$ with $c_\lambda(0,G)=1$ for every $\lambda\in\mathrm{Cl}^+(k_0)$ and $c(\mathfrak{m},G)\in\mathbb{Q}$, $|\iota^{-1}c(\mathfrak{m},G)|_\ell<1$ for every non-zero integral ideal $\mathfrak{m}$.*

Proof. Let D be the discriminant of k_0 and χ_D the even quadratic Dirichlet character of k_0, so that $\zeta_{k_0}(s)=\zeta(s)L(s,\chi_D)$ and $\chi_D(\ell)\neq 0$. Let $L_\ell(s,\chi_D)$ be the Kubota–Leopoldt ℓ-adic L-function: it is a continuous function of $s\in\mathbb{Z}_\ell$ ($\chi_D\neq\mathbf{1}$), and $L_\ell(1-k,\chi_D)=(1-\chi_D\omega^{-k}(\ell)\ell^{k-1})L(1-k,\chi_D\omega^{-k})$ for $k\geq 1$, ω the Teichmüller character ([266], Theorem 5.11); since χ_D is even and non-trivial, $L_\ell(1,\chi_D)\neq 0$ (Brumer; [266], Corollary 5.30). Put $c=v_\ell(L_\ell(1,\chi_D))$ and choose $N_0\geq\max(c,0)$ such that $|L_\ell(s,\chi_D)-L_\ell(1,\chi_D)|_\ell<|L_\ell(1,\chi_D)|_\ell$ whenever $|s-1|_\ell\leq\ell^{-N_0}$ (continuity at $s=1$).

Let $k\geq 2$ be divisible by $(\ell-1)\ell^{N_0}$; k is even. *The value $\zeta_{k_0}(1-k)$ has valuation at most $-1-N_0+c$.* By Theorem B.19, $\ell B_k\equiv -1\pmod\ell$ as $(\ell-1)\mid k$, so $v_\ell(\zeta(1-k))=$

$v_\ell(-B_k/k) = -1 - v_\ell(k) \le -1 - N_0$. As $\omega^{-k} = \mathbf{1}$, $L_\ell(1-k,\chi_D) = (1-\chi_D(\ell)\ell^{k-1})L(1-k,\chi_D)$ with $v_\ell(1-\chi_D(\ell)\ell^{k-1}) = 0$, and $|(1-k)-1|_\ell = |k|_\ell \le \ell^{-N_0}$ gives $v_\ell(L_\ell(1-k,\chi_D)) = c$; so $v_\ell(L(1-k,\chi_D)) = c$, in particular $L(1-k,\chi_D) \ne 0$, and $v_\ell(\zeta_{k_0}(1-k)) = -1 - v_\ell(k) + c \le -1 - N_0 + c$.

The form. Apply (Hb4) of Theorem B.86 with $\eta = \psi_1 = \mathbf{1}$, $\mathfrak{a} = \mathfrak{b} = \mathcal{O}_{k_0}$, signs $q = r = 0$ (allowed, k even) and $\kappa = k > 1$: $E_k(\mathbf{1},\mathbf{1}) \in M_k(\mathcal{O}_{k_0},\mathbf{1})$ has $c(\mathfrak{m},E_k(\mathbf{1},\mathbf{1})) = \sum_{\mathfrak{r}|\mathfrak{m}} \mathrm{N}\mathfrak{r}^{k-1} \in \mathbb{Z}$ and $c_\lambda(0,E_k(\mathbf{1},\mathbf{1})) = 2^{-2}\zeta_{k_0}(1-k)$ for every λ (no Euler factor is removed, the modulus being 1). Put $G = 4\,\zeta_{k_0}(1-k)^{-1}E_k(\mathbf{1},\mathbf{1})$. Then $c_\lambda(0,G) = 1$, $c(\mathfrak{m},G) = 4\zeta_{k_0}(1-k)^{-1}\sum_{\mathfrak{r}|\mathfrak{m}} \mathrm{N}\mathfrak{r}^{k-1} \in \mathbb{Q}$, and, ℓ being odd,

$$v_\ell\big(c(\mathfrak{m},G)\big) \;\ge\; 1 + N_0 - c \;\ge\; 1.$$

□

Lemma 51.12. *Let R be a commutative ring and $b_1, b_2 \in R$ with $2, b_1, b_2 \in R^\times$. Put*

$$P_b = \begin{pmatrix} -b_1 & 0 & b_1 & 0 \\ 0 & b_2 & 0 & b_2 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \qquad G(c) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & -c & 0 \end{pmatrix} \quad (c \in R),$$

for $A, B \in \mathrm{M}_2(R)$ let $\mathrm{diag}(A,B) \in \mathrm{M}_4(R)$ be the block diagonal matrix and $\mathrm{ix}_b(A,B) = P_b^{-1}\mathrm{diag}(A,B)P_b$, and put $\mathrm{low}(x) = \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$, $\mathrm{up}(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$. For $z, w \in R$ with $d = b_1zw - b_2$ and $e = b_1 - b_2zw$ in $R^\times$ put

$$\mathrm{fi}_b^+(z,w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -b_1z/b_2 & 1 & 0 & 0 \\ -1 & 2b_2w/d & 1 & 0 \\ -b_1z/b_2 & (b_2+b_1zw)/d & b_1z/b_2 & 1 \end{pmatrix},$$

$$\mathrm{fi}_b^-(z,w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -z & 1 & 0 & 0 \\ 1 & 2b_2w/e & 1 & 0 \\ z & (b_1+b_2zw)/e & z & 1 \end{pmatrix}.$$

Then:

(Ie0) $\det P_b = -4b_1b_2$, *so* $P_b \in \mathrm{GL}_4(R)$; $\mathrm{ix}_b(AA',BB') = \mathrm{ix}_b(A,B)\,\mathrm{ix}_b(A',B')$ *and* $\mathrm{ix}_b(1,1) = 1$.

(Ie1) ${}^tP_b\,G(-b_1b_2)\,P_b = -2b_1b_2J_4$. *Hence, if $X \in \mathrm{M}_4(R)$ and $\nu \in R$ satisfy ${}^tX\,G(-b_1b_2)\,X = \nu\,G(-b_1b_2)$, then $P_b^{-1}XP_b$ has similitude ν. For $A, B \in \mathrm{M}_2(R)$ and $c \in R$,*

$${}^t\mathrm{diag}(A,B)\,G(c)\,\mathrm{diag}(A,B) = \mathrm{diag}\big(\det(A)\varepsilon,\; c\det(B)\varepsilon\big)$$

with $\varepsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$; in particular $\mathrm{ix}_b(A,B)$ has similitude ν if $\det A = \det B = \nu$.

(Ie2) *For $D = \mathrm{diag}(b_1^2, b_2^2)$ the block matrix $Y = \left(\begin{smallmatrix} 0 & D \\ 1_2 & 0 \end{smallmatrix}\right)$ satisfies*

$${}^tY\,G(-b_1b_2)\,Y = -b_1b_2\,G(-b_1b_2) \quad \textit{and} \quad P_b^{-1}YP_b = \mathrm{diag}(-b_1,-b_2,b_1,b_2).$$

(Ie3) *For $X, y, y' \in R$,*

$$\mathrm{ix}_b\Big(\begin{pmatrix} X & 0 \\ y & 1 \end{pmatrix}, \begin{pmatrix} X & 0 \\ y' & 1 \end{pmatrix}\Big) = \mathrm{diag}(X,1,X,1) + u\,(E_{21} - E_{43}) + v\,(E_{41} - E_{23}),$$

with $u = -\frac{1}{2}(b_1b_2^{-1}y + y')$ and $v = \frac{1}{2}(y' - b_1b_2^{-1}y)$; the positions $(2,1),(4,3)$ have weight $-w_\alpha$, $(4,1)$ weight $-w_\theta$ and $(2,3)$ weight w_β.

(Ie4) $\mathrm{fi}_b^{\pm}(z,w)$ *is lower triangular with all diagonal entries* 1 *and has similitude* 1*;* $\mathrm{fi}_b^{+}(z,w)_{31} = -1$, $\mathrm{fi}_b^{-}(z,w)_{31} = 1$, $\mathrm{fi}_b^{+}(z,w)_{32} = 2b_2wd^{-1}$, $\mathrm{fi}_b^{-}(z,w)_{32} = 2b_2we^{-1}$. *The positions* $(2,1),(4,3)$ *have weight* $-w_\alpha$, $(3,1),(4,2)$ *weight* $-w_\Psi$, $(4,1)$ *weight* $-w_\theta$ *and* $(3,2)$ *weight* $-w_\beta$.

(Ie5) *Let* $p,q,r,p',q',r' \in R$, $A_z = \mathrm{low}(z)\left(\begin{smallmatrix} p & q \\ 0 & r \end{smallmatrix}\right)\mathrm{low}(z)^{-1}$ *and* $B_w = \mathrm{up}(w)\left(\begin{smallmatrix} r' & 0 \\ q' & p' \end{smallmatrix}\right)\mathrm{up}(w)^{-1}$. *Then* $\mathrm{fi}_b^{+}(z,w)^{-1}\,\mathrm{ix}_b(A_z,B_w)\,\mathrm{fi}_b^{+}(z,w)$ *and* $\mathrm{fi}_b^{-}(z,w)^{-1}\,\mathrm{ix}_b(B_w,A_z)\,\mathrm{fi}_b^{-}(z,w)$ *are upper triangular with diagonal* (p,p',r',r).

Proof. Write e_1,e_2,e_3,e_4 for the standard basis of R^4, so that the columns of P_b are $f_1 = -b_1e_1+e_3$, $f_2 = b_2e_2-e_4$, $f_3 = b_1e_1+e_3$, $f_4 = b_2e_2+e_4$; $\mathrm{ix}_b(A,B)$ is the matrix of $\mathrm{diag}(A,B)$ in the basis $(f_1,\dots,f_4)$.

(Ie0): the multiplicativity holds because ix_b is conjugation by P_b of the block diagonal embedding, which is multiplicative. The value of $\det P_b$: after the permutation $(1,3,2,4)$ of rows and columns P_b becomes block diagonal with blocks $\left(\begin{smallmatrix} -b_1 & b_1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} b_2 & b_2 \\ -1 & 1 \end{smallmatrix}\right)$, of determinants $-2b_1$ and $2b_2$.

(Ie1): the Gram identity is a polynomial identity; given it, ${}^t(P_b^{-1}XP_b)\,J_4\,(P_b^{-1}XP_b) = (-2b_1b_2)^{-1}\,{}^tP_b\,{}^tX\,G\,X\,P_b = \nu J_4$ for $G = G(-b_1b_2)$. The block formula is ${}^tA\varepsilon A = \det(A)\varepsilon$ in each block.

(Ie2): ${}^tYG(c)Y = \mathrm{diag}(c\varepsilon, {}^tD\varepsilon D) = \mathrm{diag}(c\varepsilon, \det(D)\varepsilon)$, and $\det D = b_1^2b_2^2 = c^2$ for $c = -b_1b_2$; and $YP_b = P_b\,\mathrm{diag}(-b_1,-b_2,b_1,b_2)$ is a polynomial identity.

(Ie3), (Ie4), (Ie5) and the polynomial identities above were verified by computer check C3, listed in Appendix B, as identities of polynomials with integer coefficients in independent indeterminates after multiplying by the denominators: (Ie3) as $2b_2\,\mathrm{diag}(A,B)P_b = P_bT'$ with $T' = 2b_2(\mathrm{diag}(X,1,X,1) + u(E_{21}-E_{43}) + v(E_{41}-E_{23}))$; the similitude in (Ie4) as ${}^tN\,J_4\,N = s^2J_4$ for $N = s\,\mathrm{fi}_b^{\pm}(z,w)$ with $s = b_2d$, resp. $s = e$. For (Ie5) the computation uses the flags: put $V^+ = (e_1+ze_2,\ we_3+e_4,\ e_3,\ e_2)$ and $V^- = (e_3+ze_4,\ we_1+e_2,\ e_1,\ e_4)$ (columns), both of determinant -1. It verifies that $\mathrm{adj}(V^{\pm})\,P_b\,N^{\pm}$ is upper triangular with diagonal entries $2b_1b_2d$, $-2b_2^2$, d^2, $-b_2^2d$ (case $+$), resp. $-2e$, $-2b_1b_2$, $-e^2$, $-e$ (case $-$), all units, so $P_b\,\mathrm{fi}_b^{\pm}(z,w) = V^{\pm}U^{\pm}$ with $U^{\pm}$ upper triangular and invertible; and that $\mathrm{adj}(V^+)\,\mathrm{diag}(A_z,B_w)\,V^+$ and $\mathrm{adj}(V^-)\,\mathrm{diag}(B_w,A_z)\,V^-$ are upper triangular with diagonal $-(p,p',r',r)$. Hence $(V^{\pm})^{-1}\mathrm{diag}(\cdot)V^{\pm}$ is upper triangular with diagonal (p,p',r',r), and so is its conjugate by the invertible upper triangular $U^{\pm}$, which is $\mathrm{fi}_b^{\pm}(z,w)^{-1}\mathrm{ix}_b(\cdot)\,\mathrm{fi}_b^{\pm}(z,w)$. The computation is checked by a second, independent implementation (exact rationals, Gauss–Jordan inverses of the matrices as stated, at random rational points). The weights: $\boldsymbol{\lambda} = (1,0,1,0)$, $\boldsymbol{\mu} = (1,1,0,0)$ give $\mathrm{wt}_4(2,1) = \mathrm{wt}_4(4,3) = (-1,0)$, $\mathrm{wt}_4(2,3) = (-1,1)$, $\mathrm{wt}_4(4,1) = (-1,-1)$, $\mathrm{wt}_4(3,1) = \mathrm{wt}_4(4,2) = (0,-1)$, $\mathrm{wt}_4(3,2) = (1,-1)$. □

Lemma 51.13. *Let* ℓ *be a prime,* $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ *an isomorphism,* $k_0 \subset \overline{\mathbb{Q}}$ *a real quadratic field with real places* ∞_1,∞_2*, and* $\rho\colon G_{k_0} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ *a continuous homomorphism such that* $\rho|_U$ *is irreducible for every open subgroup* $U \subseteq G_{k_0}$*, and* $\rho \leftrightarrow_\iota \pi$ *for a regular algebraic cuspidal automorphic representation* π *of* $\mathrm{GL}_2(\mathbb{A}_{k_0})$ *whose Langlands parameter at* ∞_i*, restricted to* $\mathbb{C}^\times \subset W_{\mathbb{R}}$*, is* $z \mapsto \mathrm{diag}(z^{p_i}\bar z^{q_i}, z^{q_i}\bar z^{p_i})$*, where* $p_i,q_i \in \frac12+\mathbb{Z}$*,* $p_i \neq q_i$ *(*$i=1,2$*),* $p_1+q_1 = p_2+q_2$ *and* $|p_1-q_1| \neq |p_2-q_2|$*. Let* $V = \mathrm{Ind}_{G_{k_0}}^{G_{\mathbb{Q}}}\rho$*, a representation of* $G_{\mathbb{Q}}$ *of dimension* 4*, and* $L \subset \overline{\mathbb{Q}}$ *a CM field abelian over* $\mathbb{Q}$ *with* $k_0 \not\subseteq L$*; put* $G'_L = \{g \in G_L : \chi_{\mathrm{cyc}}(g) = 1\}$*. Then:*

(Ia1) *there is a regular algebraic cuspidal automorphic representation* Π *of* $\mathrm{GL}_4(\mathbb{A}_L)$ *with* $V|_{G_L} \leftrightarrow_\iota \Pi$*;*

(Ia2) *suppose that there are a place $\bar{w}$ of $\overline{\mathbb{Q}}$ above ℓ, an open subgroup U of $I_{\bar{w}} \cap G_{k_0}$, an element $g_1 \in G_{\mathbb{Q}} \setminus G_{k_0}$ and integers a_1, a_2, a_1', a_2' with $|a_1 - a_2| \neq |a_1' - a_2'|$, such that $\rho|_U$ is conjugate to a map into upper triangular matrices with diagonal $(\chi_{\mathrm{cyc}}^{a_1}, \chi_{\mathrm{cyc}}^{a_2})$ and $s \mapsto \rho(g_1^{-1} s g_1)$ on U is conjugate to a map into upper triangular matrices with diagonal $(\chi_{\mathrm{cyc}}^{a_1'}, \chi_{\mathrm{cyc}}^{a_2'})$. If some element of G_L' lies outside G_{k_0}, then $V|_{G_L'}$ is irreducible, i.e. the $\overline{\mathbb{Q}}_\ell$-span of $V(G_L')$ is* $\mathrm{End}(V)$.

Proof. We use *(CB)*: two continuous semisimple representations of G_E (E a number field) whose characteristic polynomials agree at the Frobenius elements of almost all places are isomorphic (Chebotarev density and Brauer–Nesbitt). Put $M = Lk_0$. As $k_0 \not\subseteq L$, $G_L \not\subseteq G_{k_0}$, so $G_{\mathbb{Q}} = G_{k_0} G_L$, $G_{k_0} \cap G_L = G_M$ and $[G_L : G_M] = 2$; Mackey's formula gives

$$V|_{G_L} \cong \mathrm{Ind}_{G_M}^{G_L} X, \qquad X = \rho|_{G_M}.$$

M is abelian over $\mathbb{Q}$, hence over k_0; and every field E with $k_0 \subseteq E \subseteq M$ has $\rho|_{G_E}$ irreducible (G_E is open in G_{k_0}).

Step 1: base change to M. Choose $k_0 = E_0 \subset E_1 \subset \cdots \subset E_t = M$ with E_{j+1}/E_j cyclic of prime degree (a composition series of $\mathrm{Gal}(M/k_0)$), and put $\pi^{(0)} = \pi$, $\pi^{(j+1)} = \mathrm{BC}_{E_{j+1}/E_j}(\pi^{(j)})$. By induction on j: $\pi^{(j)}$ is cuspidal and $\rho|_{G_{E_j}} \leftrightarrow_\iota \pi^{(j)}$. Indeed, by (Ac1), at almost all places w of E_{j+1} above u of E_j, with residue degree f, the Satake parameters of $\pi_w^{(j+1)}$ are the f-th powers of those of $\pi_u^{(j)}$, while $\rho(\mathrm{Frob}_w^{-1}) = \rho(\mathrm{Frob}_u^{-1})^f$ and $\mathrm{N}w^{1/2} = (\mathrm{N}u^{1/2})^f$. If $\pi^{(j+1)}$ were not cuspidal, then by (Ac3) $\pi^{(j)} \cong \pi^{(j)} \otimes (\omega \circ \det)$ for a Hecke character $\omega \neq 1$ of E_j trivial on norms from E_{j+1}; with ω_G the corresponding non-trivial character of $\mathrm{Gal}(E_{j+1}/E_j)$, (CB) gives $\rho|_{G_{E_j}} \cong \rho|_{G_{E_j}} \otimes \omega_G$ (both irreducible), and an intertwiner A with $A\rho(s)A^{-1} = \omega_G(s)\rho(s)$ is not scalar (as $\omega_G \neq 1$ and $\rho(s) \neq 0$) and commutes with $\rho(G_{E_{j+1}})$, so $\rho|_{G_{E_{j+1}}}$ is reducible by Schur's lemma: a contradiction. Put $\pi_M = \pi^{(t)}$, so $X \leftrightarrow_\iota \pi_M$. By (Ac2), applied along the chain, at a complex place w of M above ∞_i the Langlands parameter of $\pi_{M,w}$ is the restriction of that of π_{∞_i} to $\mathbb{C}^\times$, i.e. $z \mapsto \mathrm{diag}(z^{p_i}\bar{z}^{q_i}, z^{q_i}\bar{z}^{p_i})$ (the other identification of W_{M_w} with $\mathbb{C}^\times$ exchanges z and $\bar{z}$, which gives the same parameter up to conjugation).

Step 2: (Ia1). Let σ be the non-trivial element of $\mathrm{Gal}(M/L)$; it is non-trivial on k_0 (otherwise it fixes $Lk_0 = M$), so it maps the places of M above ∞_1 to places above ∞_2. If $\pi_M \cong \pi_M^\sigma$, then at a complex place w above ∞_1 the components $\pi_{M,w}$ and $\pi_{M,\sigma w}$ are isomorphic, so $\{p_1, q_1\} = \{p_2, q_2\}$ by Step 1, against $|p_1 - q_1| \neq |p_2 - q_2|$. So $\Pi_0 = \mathrm{AI}_{M/L}(\pi_M)$ is cuspidal by (Ai3); put $\Pi = \Pi_0 \otimes |\det|_{\mathbb{A}_L}$, cuspidal. At almost all finite places v of L:

so $V|_{G_L} \leftrightarrow_\iota \Pi$. At a complex place v of L the two places of M above it are w and σw, above ∞_1 and ∞_2 in some order; by (Ai2) and Step 1 the parameter of $\Pi_{0,v}$ on $\mathbb{C}^\times$ is the sum of $z^{p_i}\bar{z}^{q_i}$, $z^{q_i}\bar{z}^{p_i}$ over $i = 1, 2$, and that of $\Pi_v \otimes |\det|^{-3/2} = \Pi_{0,v} \otimes |\det|^{-1/2}$ has the exponents $(p_i - \frac{1}{2}, q_i - \frac{1}{2})$, $(q_i - \frac{1}{2}, p_i - \frac{1}{2})$, which are integers. With $s = p_1 + q_1 = p_2 + q_2$ and $\delta_i = |p_i - q_i| > 0$, the four exponents of z are $\frac{1}{2}(s \pm \delta_1) - \frac{1}{2}$, $\frac{1}{2}(s \pm \delta_2) - \frac{1}{2}$, pairwise distinct since $\delta_1 \neq \delta_2$. So Π is regular algebraic in the sense of [68].

Step 3: (Ia2). Let $g' \in G_L' \setminus G_{k_0}$ and $H = G_L' \cap G_{k_0} = \{h \in G_M : \chi_{\mathrm{cyc}}(h) = 1\}$, a closed normal subgroup of $G_{\mathbb{Q}}$ with $G_{\mathbb{Q}}/H$ abelian ($M(\mu_{\ell^\infty})/\mathbb{Q}$ is abelian); $[G_L' : H] = 2$, $G_{\mathbb{Q}} = G_{k_0} G_L'$, and Mackey's formula gives $V|_{G_L'} \cong \mathrm{Ind}_H^{G_L'}(\rho|_H)$. For $m \in G_{k_0}$ put $\rho^{g'}(m) = \rho(g'^{-1} m g')$, a homomorphism on G_{k_0}. (i) $\rho|_H$ is irreducible. Otherwise let $\mathcal{S}$ be the set of H-stable lines, non-empty and permuted by G_{k_0}. If $\#\mathcal{S} = 1$ its line is G_{k_0}-stable, against the irreducibility of ρ. If $\#\mathcal{S} = 2$, the stabiliser of one line is a closed subgroup of index ≤ 2 of G_{k_0}, hence open, on which ρ is reducible: a contradiction. If $\#\mathcal{S} \geq 3$, every $\rho(h)$, $h \in H$, is scalar. Fix $g_2 \in G_{k_0}$ with $\rho(g_2)$

not scalar (ρ is irreducible). As G_{k_0}/H is abelian, $\rho(g)\rho(g_2)\rho(g)^{-1} = c(g)\rho(g_2)$ with $c(g) \in \overline{\mathbb{Q}}_\ell^\times$; c is a continuous homomorphism, and multiplication by $c(g)$ preserves the multiset of eigenvalues of $\rho(g_2)$, so $c(g)^2 = 1$. On the open subgroup $\ker c$ the matrices $\rho(g)$ commute with $\rho(g_2)$ and preserve one of its eigenspaces, a line: a contradiction. (ii) $\rho|_H \not\cong \rho^{g'}|_H$. Otherwise $\mathrm{Hom}_H(\rho, \rho^{g'})$ is a line (Schur's lemma, by (i)) spanned by an invertible A; for $m \in G_{k_0}$, $\rho^{g'}(m)A\rho(m)^{-1}$ again intertwines $\rho|_H$ and $\rho^{g'}|_H$ (H is normal), so it equals $\nu(m)A$ for a continuous character ν of G_{k_0}, and $\rho^{g'} \cong \rho \otimes \nu$. Writing $g' = g_1 y$ with $y \in G_{k_0}$, $\rho^{g'}(s) = \rho(y)^{-1}\rho(g_1^{-1}sg_1)\rho(y)$, so on U both $\rho^{g'}$ and $\rho \otimes \nu$ are triangularisable, with diagonal characters $\{\chi_{\mathrm{cyc}}^{a_1'}, \chi_{\mathrm{cyc}}^{a_2'}\}$ and $\{\chi_{\mathrm{cyc}}^{a_1}\nu, \chi_{\mathrm{cyc}}^{a_2}\nu\}$; isomorphic representations have the same Jordan–Hölder constituents, so these multisets agree. Either $\chi_{\mathrm{cyc}}^{a_1'} = \chi_{\mathrm{cyc}}^{a_1}\nu$ and $\chi_{\mathrm{cyc}}^{a_2'} = \chi_{\mathrm{cyc}}^{a_2}\nu$, whence $\chi_{\mathrm{cyc}}^{(a_1'-a_2')-(a_1-a_2)} = 1$ on U, or $\chi_{\mathrm{cyc}}^{a_1'} = \chi_{\mathrm{cyc}}^{a_2}\nu$ and $\chi_{\mathrm{cyc}}^{a_2'} = \chi_{\mathrm{cyc}}^{a_1}\nu$, whence $\chi_{\mathrm{cyc}}^{(a_1'-a_2')+(a_1-a_2)} = 1$ on U. Both exponents are non-zero, as $|a_1 - a_2| \neq |a_1' - a_2'|$; but $\chi_{\mathrm{cyc}}(U)$ is an open subgroup of $\mathbb{Z}_\ell^\times$ (U is open in $I_{\bar{w}}$ and $\chi_{\mathrm{cyc}}(I_{\bar{w}}) = \mathbb{Z}_\ell^\times$), which contains elements of infinite order: a contradiction. (iii) As a representation of H, $\mathrm{Ind}_H^{G_L'}(\rho|_H) = \rho|_H \oplus g'\rho|_H$ with $g'\rho|_H \cong \rho^{g'}|_H$; the two summands are irreducible and non-isomorphic, so the only H-stable subspaces are 0, the two summands and the whole space, and g' exchanges the summands. Hence $V|_{G_L'}$ is irreducible, and by Burnside's theorem the span of $V(G_L')$ is $\mathrm{End}(V)$. □

Proposition 51.14. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, attached to a host character in case (III), let $\mathscr{T}_0$ be a good auxiliary tower for $\mathfrak{D}$ of case (III), $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and (K, V, a) a test field; write k_V for the residue field of V and $\bar{a}\colon \mathcal{O} \to k_V$ for a followed by reduction. Then there are a prime w that splits for $\mathscr{T}_0$ and a residual pair s over $(\mathscr{T}_0, \{w\})$ with values in k_V (for $\bar{a}$), supported in Σ_M, such that for every good auxiliary tower $\mathscr{T} \geq \mathscr{T}_0$ in whose field w dies there are a coefficient ring $\mathcal{O}_1 \supseteq \mathcal{O}$ and a seed pair Q over $(\mathscr{T}, \emptyset)$ with values in $\mathcal{O}_1$ for ι such that the image $\bar{Q}_V$ of Q under $\mathcal{O}_1 \xrightarrow{a} V \to k_V$ lies over s, lies in A, and has entries of weights $-w_\beta$ and $-w_\theta$.*

Proof. Let ϵ, E, F_0, $g_0 = g_{\mathscr{T}_0}$, Σ_0, $(\bar{v})$ be the data of $\mathscr{T}_0$ and E' the imaginary quadratic field of case (III); $\ell \geq 5$, $T = \{\ell\}$, χ has finite order prime to ℓ. Write $\kappa_i = \kappa_i^\epsilon$, bars for reduction, η for the quadratic character of $E'/\mathbb{Q}$, and for a tower $\mathscr{T}$ write $L_{\mathscr{T}}$, $G'_{\mathscr{T}}$ for its L, G'. Put $\bar{\Phi} = \bar{\kappa}_1/\bar{\kappa}_4\colon G_{\mathbb{Q}} \to \mathbb{F}_{\mathcal{O}}^\times$.

Step 1: relations among the characters. For both labellings $\kappa_1\kappa_4 = \kappa_2\kappa_3 = \theta$, $\kappa_1/\kappa_2 = \kappa_3/\kappa_4 = \alpha$ and $\kappa_1/\kappa_3 \in \{\Psi, \Psi^{-1}\}$; so $\mathrm{rt}_{w_\alpha} = \alpha$, $\mathrm{rt}_{w_\Psi} = \Psi^{\pm 1}$, $\mathrm{rt}_{w_\theta} = \kappa_1^2/(\kappa_2\kappa_3) = \kappa_1/\kappa_4$, and $\bar{\alpha} = \bar{\xi}$ (as $\overline{\chi_{\mathrm{cyc}}\omega_\ell^{-1}} = 1$). By case (III), $\bar{\alpha} = \eta$ on G_{F_0}, so $\bar{\alpha}^2 = 1$ there. Let $\mathscr{T} \geq \mathscr{T}_0$ be good, with field $F \supseteq F_0$ and $L = L_{\mathscr{T}}$, $M = LE'$. (a) $\bar{\Phi}^2 \neq 1$ on G_M: otherwise $\bar{\Psi}^4 = 1$ on G_L (as $\bar{\Phi} = \bar{\Psi}^{\pm 1}\bar{\alpha}$ on G_L, $\bar{\alpha}^2 = 1$ and $g^2 \in G_M$ for $g \in G_L$), so every $\overline{\mathrm{rt}_w}$, w a root, being $\bar{\Psi}^a\bar{\alpha}^b$, satisfies $\overline{\mathrm{rt}_w}^4 = 1$ on G_L, contradicting the last clause of (Oq3). (b) Some $g \in G'_{\mathscr{T}}$ lies outside $G_{E'}$: otherwise $\eta|_{G_L}$ factors through $\chi_{\mathrm{cyc}}(G_L)$, an open subgroup of $\mu_{\ell-1} \times (1 + \ell\mathbb{Z}_\ell)$, and being of order ≤ 2 it is trivial on the pro-ℓ part, i.e. on $G_L \cap \ker\omega_\ell$; since $\eta = \bar{\alpha}$ on G_L and $\bar{\alpha}\bar{\kappa}_1 = \bar{\kappa}_2$, $\bar{\alpha}\bar{\kappa}_2 = \bar{\kappa}_1$, $\bar{\alpha}\bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\alpha}\bar{\kappa}_4 = \bar{\kappa}_3$ there, (Oq4) gives $\bar{\alpha} = 1$ on G_L, contradicting (Oq3) for the root w_α. In particular $E' \not\subseteq L$. (c) $\bar{\Phi} \neq 1$ on G_L and $\bar{\Phi} \neq \bar{\omega}_\ell^{-1}$: (Oq3) for the roots w_θ and $-w_\theta$. (d) $\bar{\Phi}(g_0) \neq 1$ and $\bar{\alpha}(g_0) \neq 1$, so $g_0 \notin G_{E'}$: (Oq7). (e) $\bar{\Phi}(c) = -1$: $\bar{\Phi}(c)$ is $\bar{\theta}(c) = \bar{\xi}(c)\bar{\Psi}(c)$ or $\bar{\alpha}(c)/\bar{\Psi}(c)$, and $\xi(c) = -1$, $\Psi(c) = 1$ by (Sg1). (f) For every place $\bar{v}$ above ℓ some $h \in D_{\bar{v}}$ has $\bar{\Phi}(h) \neq 1$: $\bar{\Phi}$ is $\overline{\xi\Psi}$ or $\overline{\xi\Psi^{-1}}$, which is $\neq 1$ at σ_ℓ by (Sg2) ($\ell \in T$), and $D_{\bar{v}}$ is conjugate to D_ℓ. (g) $D_{\bar{v}} \subseteq G_{E'}$ for $\bar{v}$ above ℓ, as ℓ splits in E'.

Step 2: the newform. By Kronecker–Weber, $\bar{\Phi}$ is a Dirichlet character of order prime to ℓ, so its restriction to I_ℓ is $\bar{\omega}_\ell^{c_0}$ for some c_0. Let ψ_0 be the Teichmüller lift of $\bar{\Phi}\bar{\omega}_\ell^{-c_0}$: a Dirichlet character of order prime to ℓ, unramified at ℓ, primitive of some conductor N_0 with $\ell \nmid N_0$. The

construction of Step 4 below does not use Steps 2 and 3, and fixes an integer $m \geq 1$; we take it first, and then choose $k \geq 2m+3$ with $k-1 \equiv c_0 \pmod{\ell-1}$. Then $\psi_0(-1) = \bar{\Phi}(c)(-1)^{c_0} = (-1)^k$ by (e). The fixed field Ω_0 of $G_{F_0} \cap \ker \bar{r}_0 \cap \ker \omega_\ell \cap \ker \psi_0 \cap G_{E'}$ is abelian over $\mathbb{Q}$; choose (Dirichlet) a prime $w \notin S$, $w \nmid N_0 \operatorname{disc}(E')$, splitting completely in Ω_0. Then w splits for $\mathscr{T}_0$, $\ell \mid w-1$, $\psi_0(w) = 1$. Let θ_w be a Dirichlet character modulo w of order ℓ; then $\varepsilon = \psi_0\theta_w$ is primitive modulo $N_0 w$ with $\varepsilon(-1) = (-1)^k$. If $\psi_0 = 1$ then $\bar{\Phi} = \bar{\omega}_\ell^{k-1}$, and $(\ell-1) \mid k$ would give $\bar{\Phi} = \bar{\omega}_\ell^{-1}$, excluded by (c). So Proposition 49.5 gives a newform f of weight k, level $N_0 w$ and character ε with $\iota^{-1}(a_p) \equiv 1 + \iota^{-1}(\varepsilon(p))p^{k-1}$, and Lemma 49.8 applies to it: ρ, $\beta = \varepsilon\chi_{\mathrm{cyc}}^{k-1}$, $\bar{\beta} = \bar{\psi}_0\bar{\omega}_\ell^{k-1} = \bar{\Phi}$ ($\bar{\theta}_w = 1$).

Step 3: a Ribet lattice diagonal at g_0. ρ is irreducible ((Ne4) with $\mathbb{Q}$, $\bar{\beta} \neq 1$) and continuous, so it preserves a lattice over some coefficient ring $\mathcal{O}_1 \supseteq \mathcal{O}$, which we enlarge finitely often below. The characteristic polynomial of $\rho(g_0)$ is $\equiv (X - \bar{\Phi}(g_0))(X-1)$ (Ne1) with distinct residual roots (d), so by Hensel its roots lie in $\mathcal{O}_1$, the lattice is the sum of the two eigenlines, and in an adapted basis $\rho(g) = \left(\begin{smallmatrix} a_g & b_g \\ c_g & d_g \end{smallmatrix}\right)$ with $\rho(g_0)$ diagonal, $\bar{a}_{g_0} = \bar{\Phi}(g_0)$. The reduction is reducible (its semisimplification is $\bar{\Phi} \oplus \mathbf{1}$), and a stable line is an eigenline of $\bar{\rho}(g_0)$; so all $\bar{b}_g = 0$ or all $\bar{c}_g = 0$, in either case $b_g c_{g'} \in \mathfrak{m}_{\mathcal{O}_1}$ for all g, g', the diagonal of $\bar{\rho}$ consists of characters, and $\bar{a} = \bar{\Phi}$, $\bar{d} = \mathbf{1}$ (compare at g_0). Not all c_g vanish (ρ irreducible); let $n = \min_g v(c_g)$, ϖ a uniformiser, and $\rho_1 = D^{-1}\rho D$, $D = \operatorname{diag}(1, \varpi^n)$. Then $\rho_1(g) = \left(\begin{smallmatrix} a_g & \varpi^n b_g \\ \varpi^{-n} c_g & d_g \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_1)$, $\varpi^n b_g \in \mathfrak{m}_{\mathcal{O}_1}$ (if $n = 0$ use $b_g c_{g'} \in \mathfrak{m}$ with $c_{g'}$ a unit), $\rho_1(g_0)$ is diagonal, and

$$\bar{\rho}_1(g) = \begin{pmatrix} \bar{\Phi}(g) & 0 \\ \bar{x}(g) & 1 \end{pmatrix}, \qquad \bar{x}(gh) = \bar{x}(g)\bar{\Phi}(h) + \bar{x}(h), \qquad \bar{x} \neq 0.$$

If F is a number field Galois over $\mathbb{Q}$ with $\bar{\Phi}|_{G_F} \neq 1$, then $\bar{x}|_{G_F} \neq 0$: otherwise, for $h \in G_F$ with $\bar{\Phi}(h) \neq 1$ and any g, $\bar{x}(gh) = \bar{x}(g)\bar{\Phi}(h)$ and $\bar{x}(gh) = \bar{x}(ghg^{-1})\bar{\Phi}(g) + \bar{x}(g) = \bar{x}(g)$, so $\bar{x}(g) = 0$ for all g. *Flags of ρ_1.* For $\bar{v}$ above ℓ, (Ne3) gives a $D_{\bar{v}}$-stable line with character $\delta_1 = \beta\delta_2^{-1}$ (δ_2 unramified, $\bar{\delta}_2 = 1$ on $D_{\bar{v}}$, $\delta_1 = \chi_{\mathrm{cyc}}^{k-1}$ on $I_{\bar{v}}$). With $h \in D_{\bar{v}}$ as in (f), this line is the $\delta_1(h)$-eigenline of $\rho_1(h)$, $\delta_1(h) \equiv \bar{\Phi}(h) \not\equiv 1 \equiv d_h$; it is spanned by $(\delta_1(h) - d_h,\ \varpi^{-n}c_h)$, whose first coordinate is a unit. So it is spanned by $(1, x_{\bar{v}})$ with $x_{\bar{v}} \in \mathcal{O}_1$ (after enlarging $\mathcal{O}_1$ by $\delta_1(h)$), and $\operatorname{low}(x_{\bar{v}})^{-1}\rho_1(s)\operatorname{low}(x_{\bar{v}}) = \left(\begin{smallmatrix} \delta_1(s) & * \\ 0 & \delta_2(s) \end{smallmatrix}\right)$ for $s \in D_{\bar{v}}$.

Step 4: the dihedral factor. Let $m_0 = \#\mathcal{O}_{E'}^\times$. The character $(z,u) \mapsto (z/\bar{z})^{m_0}$ of $\mathbb{C}^\times \times \hat{\mathcal{O}}_{E'}^\times$ is trivial on $\mathcal{O}_{E'}^\times$ ($(u/\bar{u})^{m_0} = u^{2m_0} = 1$), so it extends to a Hecke character Ψ_1 of E'; $\Psi_2 = \Psi_1/\Psi_1^c$ ($\Psi_1^c(x) = \Psi_1(\bar{x})$) is unramified at every finite place, $\Psi_2\Psi_2^c = 1$, with infinity type $z \mapsto (z/\bar{z})^{2m_0}$. By the correspondence between algebraic Hecke characters and ℓ-adic characters ([220, Ch. II]) there is a continuous $\kappa_2 \colon G_{E'} \to \mathcal{O}_1^\times$ (enlarging $\mathcal{O}_1$) with $\iota\kappa_2(\mathrm{Frob}_v^{-1}) = \Psi_2(\varpi_v)$ for $v \nmid \ell$, unramified outside ℓ, whose restriction to the inertia group of a place above ℓ ($E'_\mathfrak{l} = \mathbb{Q}_\ell$) is the algebraic character given by the infinity type: $\kappa_2|_{I_{\bar{v}}} = \chi_{\mathrm{cyc}}^{\pm 2m_0}$, the sign depending on the prime of E' below $\bar{v}$. By Chebotarev, $\kappa_2(h)\kappa_2(g_0 h g_0^{-1}) = 1$. Let N_1 be the order of the reduction of κ_2 (a character into a finite field), $\kappa = \kappa_2^{N_1}$, $m = 2m_0N_1$, $\Psi_\kappa = \Psi_2^{N_1}$ (type $(z/\bar{z})^{\pm m}$). Then $\bar{\kappa} = 1$, $\kappa(h)\kappa(g_0 h g_0^{-1}) = 1$, κ is unramified outside ℓ, and for every $\bar{v}$ above ℓ there is $s_{\bar{v}} \in \{1, -1\}$ with $\kappa|_{I_{\bar{v}}} = \chi_{\mathrm{cyc}}^{s_{\bar{v}}m}$. As in Step 1, $\kappa(g_0^2) = \kappa(g_0 g_0^2 g_0^{-1})^{-1} = \kappa(g_0^2)^{-1}$, so $\kappa(g_0^2) = \pm 1$, hence $= 1$ ($\bar{\kappa} = 1$, ℓ odd). For $h \in G_{E'}$ put $A(h) = \frac{1}{2}\left(\begin{smallmatrix} \kappa+\kappa^{-1} & \kappa-\kappa^{-1} \\ \kappa-\kappa^{-1} & \kappa+\kappa^{-1} \end{smallmatrix}\right)(h) = P_0 \operatorname{diag}(\kappa(h), \kappa(h)^{-1}) P_0^{-1}$, $P_0 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, and $\tau(h) = A(h)$, $\tau(g_0 h) = HA(h)$. Since $HP_0 = P_0\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $HA(h)H^{-1} = A(g_0 h g_0^{-1})$, and with $A(g_0^2) = 1 = H^2$ this makes $\tau \colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_1)$ a continuous homomorphism ($2 \in \mathcal{O}_1^\times$). It satisfies ${}^t\tau H \tau = H$ (for A: $(\frac{\kappa+\kappa^{-1}}{2})^2 - (\frac{\kappa-\kappa^{-1}}{2})^2 = 1$), $\tau(g_0) = H$, $\bar{\tau} = \operatorname{diag}(1, \eta)$, $\tau \cong \operatorname{Ind}_{G_{E'}}^{G_\mathbb{Q}} \kappa$ over $\overline{\mathbb{Q}}_\ell$ (Frobenius reciprocity, $(1,1)$ spans a κ-line, and $\operatorname{Ind}\kappa$ is irreducible as $\kappa^2 \neq 1$), and on $D_{\bar{v}} \subseteq G_{E'}$ ((g)) $\tau = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$ with $a + bs_{\bar{v}} = \kappa^{s_{\bar{v}}}$, $a - bs_{\bar{v}} = \kappa^{-s_{\bar{v}}}$, equal to χ_{cyc}^m, χ_{cyc}^{-m} on $I_{\bar{v}}$.

Step 5: the twist and the representation R. Let $\gamma = \theta\theta_w\beta^{-1} = \theta\psi_0^{-1}\chi_{\mathrm{cyc}}^{1-k}$; $\bar\gamma = \bar\kappa_1\bar\kappa_4\bar\Phi^{-1} = \bar\kappa_4^2$. With λ the Teichmüller lift of $\bar\kappa_4$, $\gamma\lambda^{-2}$ takes values in $1+\mathfrak{m}_{\mathcal{O}_1}$, on which squaring is a topological automorphism (ℓ odd); put $\delta = \lambda(\gamma\lambda^{-2})^{1/2}$, so $\delta^2\beta = \theta\theta_w$, $\bar\delta = \bar\kappa_4$. Define

$$R(g) = \delta(g)\,\mathrm{tn}\big(\rho_1(g), \tau(g)\big) \in \mathrm{GL}_4(\mathcal{O}_1), \qquad n_{\bar v} = \mathrm{fl}(x_{\bar v}, s_{\bar v}) \quad (\bar v \mid \ell).$$

By (Te1) R is a continuous homomorphism, and by (Te0) $R \otimes \overline{\mathbb{Q}}_\ell \cong \delta \otimes \rho \otimes \mathrm{Ind}\,\kappa$. Its properties: (R1) $R(g)$ has similitude $\delta^2 \det\rho_1 = \theta\theta_w$ ((Te2)). (R2) $R(g_0)$ is diagonal ((Te3)), hence so is every power. (R3) $\bar R(g) = \bar\delta(g)\big(\mathrm{diag}(\bar\Phi, \bar\Phi\eta, \eta, 1) - \bar x\eta E_{32} + \bar x E_{41}\big)(g)$ by (Te3); on G_{F_0}, using $\eta = \bar\alpha$ and Step 1, its diagonal is $(\bar\kappa_1, \bar\kappa_2, \bar\kappa_3, \bar\kappa_4)$, its only other possibly non-zero entries are $(3,2)$ and $(4,1)$, of weights $-w_\beta, -w_\theta \in \Sigma_A \cap \Sigma_M$, and $\bar R(g)_{41} = \bar\delta(g)\bar x(g)$. (R4) For $\bar v$ above ℓ, $n_{\bar v}$ is lower unitriangular of similitude 1 with $(n_{\bar v})_{21} = s_{\bar v}$ ((Te4)), and for $s \in D_{\bar v}$, $n_{\bar v}^{-1}R(s)n_{\bar v}$ is upper triangular with diagonal $\psi_{\bar v} = \delta\cdot(\delta_1\kappa^{s_{\bar v}},\ \delta_1\kappa^{-s_{\bar v}},\ \delta_2\kappa^{s_{\bar v}},\ \delta_2\kappa^{-s_{\bar v}})$ ((Te5) with Steps 3, 4). Residually $\bar\psi_{\bar v} = \bar\delta(\bar\Phi, \bar\Phi, 1, 1)$, which is $(\bar\kappa_1, \dots, \bar\kappa_4)$ on $D_{\bar v}$ as $\eta = 1$ there. On $I_{\bar v}$: $\theta = \chi_{\mathrm{cyc}}^{e+1}\varphi$ with $\varphi = \omega_\ell^{-1}\chi$ of order dividing $\ell - 1$ (ξ, ψ_0, θ_w unramified at ℓ), so $\delta^2 = \chi_{\mathrm{cyc}}^{e+2-k}\varphi$ and $\psi_{\bar v,i}^2 = \chi_{\mathrm{cyc}}^{h_i}\varphi$ with

$$h = (e+2-k)(1,1,1,1) + 2\,(k-1+m,\ k-1-m,\ m,\ -m),$$

and $h_1 - h_2 = h_3 - h_4 = 4m$, $h_2 - h_3 = 2(k-1-2m)$ lie in $2\mathbb{Z}_{\ge 2}$ (as $k \ge 2m+3$). In particular the $\psi_{\bar v,i}$ are pairwise distinct on every open subgroup of $I_{\bar v}$. (R5) $\psi_{\bar v,i}(s) = 1$ if $s \in D_{\bar v}$ and $s^{\ell^b} \in \overline{[D_{\bar v}, D_{\bar v}]}$ for some b: the image of s in the torsion-free group $D_{\bar v}/I_{\bar v} \cong \hat{\mathbb{Z}}$ is 0, so $s \in I_{\bar v}$ and unramified characters (δ_2, and $\delta_1\chi_{\mathrm{cyc}}^{1-k}$) are 1 at s; $\chi_{\mathrm{cyc}}(s)$ is an ℓ-power root of unity in $\mathbb{Z}_\ell^\times$, hence 1, so $\kappa^{\pm1}(s) = \delta_1(s) = 1$; characters of finite order prime to ℓ (λ, χ, ω_ℓ) are 1 at s, so $\gamma(s) = 1$ and $\delta(s) = \lambda(s)(\gamma\lambda^{-2})(s)^{1/2} = 1$. (R6) Let $\bar p$ be a place above a prime $p \ne \ell$ and $g \in I_{\bar p} \cap G_{F_0}$ with $\bar r_0(g) = 1$, and assume $p \ne w$ or $\theta_w(g) = 1$. Then $R(g) = 1$: $\bar\alpha(g) = 1$, so $g \in G_{E'}$ and $\tau(g) = 1$ (κ unramified at p); $\varepsilon(g) = \psi_0(g)\theta_w(g) = 1$, because $\psi_0(g)$ is the Teichmüller lift of $\bar\Phi(g) = 1$ and θ_w is unramified at $p \ne w$, so $\rho_1(g) = 1$ by (Ne2); $\lambda(g) = 1$, and $\theta(g) = (\xi\chi)(g)$ is a root of unity of order prime to ℓ with $\bar\theta(g) = 1$, so $\theta(g) = 1$, $\gamma(g) = 1$ and $\delta(g) = 1$.

Step 6: the residual pair s. Let $\bar a_1 \colon \mathcal{O}_1 \to V \to k_V$ be a followed by reduction; it kills $\mathfrak{m}_{\mathcal{O}_1}$ ($a(\mathfrak{m}_\ell) \subseteq \mathfrak{m}_V$). Let s be the image under $\bar a_1$ of $(R|_{G_{F_0}}, (n_{\bar v})_{v\in\Sigma_0})$. It is a marked ordinary pair over $(\mathscr{T}_0, \{w\})$ for $\bar a$: (Mq1) by (R1) and $\bar a_1(\theta_w(g)) = 1$ ($\theta_w(g)$ is an ℓ-th root of unity), and by (R6) on $N_{\{w\}}$; (Mq2) by (R2); (Mq3) and the triangularity in (Mq4) by (R4); the condition on μ_v by (R5). Its r is locally constant (reduction of a continuous map to $\mathcal{O}_1$), (Wq1) holds by (R3), (R4), and r is supported in Σ_M by (R3), which also gives (Wq2). So s is a residual pair over $(\mathscr{T}_0, \{w\})$ with values in k_V, supported in Σ_M, and w splits for $\mathscr{T}_0$ (Step 2).

Step 7: the seed pairs. Let $\mathscr{T} \ge \mathscr{T}_0$ be good, with field F, places $(\bar v')_{v'\in\Sigma}$ and $g_{\mathscr{T}} = g_0^{\ell^j}$, such that w dies in F; then $\theta_w|_{G_F} = 1$ (θ_w composed with $\mu_\ell \cong \mathbb{Z}/\ell\mathbb{Z}$ is trivial on the inertia groups of all places not above w). Put $Q = (R|_{G_F}, (n_{\bar v'})_{v'\in\Sigma})$, with values in $\mathcal{O}_1$ for the inclusion $\mathcal{O} \subseteq \mathcal{O}_1$. It is a marked ordinary pair over $(\mathscr{T}, \emptyset)$: (Mq1) by (R1) with $\theta_w = 1$ on G_F and by (R6) (at $p = w$, $\theta_w(g) = 1$ on G_F); (Mq2) by (R2); (Mq3), (Mq4) by (R4), (R5). It is continuous for $\mathfrak{m}_{\mathcal{O}_1}$, and its reduction satisfies (Wq1), (Wq2) by (R3), (R4) as in Step 6. Its image P in $\overline{\mathbb{Q}}_\ell$ is a seed for ι: by Step 1 (a), (b), Lemma 51.16 applies to f, E', $g_1 = g_0$, κ, Ψ_κ ($a = \pm m$, $k - 1 > 2m$) and $L = L_{\mathscr{T}}$ (a CM field abelian over $\mathbb{Q}$); by (Ta2) $r_P|_{G'_{\mathscr{T}}}$ is irreducible (twisting by δ does not change the span); P has weights h with the open subgroup $U = G_F \cap \ker\omega_\ell \cap \ker\chi$, by (R4); with the character $\delta^{-1}|_{G_L}$, the integers $a' = k - 2 - e$ (so $a' + h_1 = 2(k-1+m)$ is even) and $\ell - 1$, $\big(\delta^{-2}(s)\chi_{\mathrm{cyc}}(s)^{-a'}\big)^{\ell-1} = \varphi(s)^{-(\ell-1)} = 1$ on inertia above ℓ; and $r_P|_{G_L} \otimes \delta^{-1} \cong (\rho \otimes \mathrm{Ind}\,\kappa)|_{G_L} \leftrightarrow_\iota \Pi$ with Π regular algebraic cuspidal, by (Ta1). So Q is a seed pair with values in $\mathcal{O}_1$.

Step 8: $\bar{Q}_V$ lies over s, lies in A, and has the two entries. Its representation is $\bar{a}_1 \circ R|_{G_F}$, the restriction of that of s. Let $v' \in \Sigma$; choose $v \in \Sigma_0$ and $g \in G_{F_0}$ with $g\bar{v} = \bar{v}'$ (every place above ℓ is G_{F_0}-conjugate to some $\bar{v}$). Put $B = n_{\bar{v}'}^{-1} R(g) n_{\bar{v}}$. For $h \in D_{\bar{v}'} = g D_{\bar{v}} g^{-1}$, $X_1(h) = n_{\bar{v}'}^{-1} R(h) n_{\bar{v}'}$ and $X_2(h) = n_{\bar{v}}^{-1} R(g^{-1}hg) n_{\bar{v}}$ are upper triangular, with $BX_2(h) = X_1(h)B$, and on $I' = I_{\bar{v}'} \cap U$ the squares of their diagonal characters are $\chi_{\mathrm{cyc}}^{h_i}$ in both cases (χ_{cyc} is invariant under conjugation), pairwise distinct as h_i are distinct and χ_{cyc} has infinite order on I'. Let $V_1 \subset \cdots \subset V_4$ be the standard flag. $B(V_1)$ is an $X_1(I')$-stable line on which I' acts by a character with square $\chi_{\mathrm{cyc}}^{h_1}$; if j is minimal with $B(V_1) \subseteq V_j$, its non-zero image in V_j/V_{j-1} is I'-equivariant, so $\chi_{\mathrm{cyc}}^{h_1} = \chi_{\mathrm{cyc}}^{h_j}$ on I' and $j = 1$; passing to V/V_1 and inducting, B is upper triangular. Hence $\bar{a}_1(B)$ is upper triangular: $\bar{Q}_V$ lies over s. It lies in A: its representation is supported in Σ_A by (R3), and $(n_{\bar{v}'})_{21} = s_{\bar{v}'} = \pm 1 \neq 0$ in k_V. Finally F is Galois over $\mathbb{Q}$ and $\bar{\Phi} \neq 1$ on G_F (Step 1 (c)), so $\bar{x}(g) \neq 0$ for some $g \in G_F$ (Step 3), and $\bar{a}_1$ is injective on $\mathbb{F}_{\mathcal{O}_1}$ and $\bar{\delta}(g), \eta(g)$ are units: by (R3) the entries $(4,1)$ and $(3,2)$ of $\bar{Q}_V(g)$ are non-zero, of weights $-w_\theta$ and $-w_\beta$. □

Lemma 51.15. *Let R be a commutative ring. For $g = (g_{ab}), t = (t_{ab}) \in \mathrm{M}_2(R)$ put*

$$\mathrm{tn}(g,t) = \begin{pmatrix} g_{11}t_{11} & g_{11}t_{12} & -g_{12}t_{12} & g_{12}t_{11} \\ g_{11}t_{21} & g_{11}t_{22} & -g_{12}t_{22} & g_{12}t_{21} \\ -g_{21}t_{21} & -g_{21}t_{22} & g_{22}t_{22} & -g_{22}t_{21} \\ g_{21}t_{11} & g_{21}t_{12} & -g_{22}t_{12} & g_{22}t_{11} \end{pmatrix} \in \mathrm{M}_4(R),$$

let $H = \mathrm{diag}(1,-1)$, $\mathrm{low}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, and for $x, s \in R$ put $\mathrm{fl}(x,s) = \mathrm{tn}(\mathrm{low}(x), 1)\,(1 + sE_{21} - sE_{43})$, where E_{ij} are the matrix units. Then:

(Te0) $\mathrm{tn}(g,t) = P\,(g \otimes t)\,P^{-1}$, *where $g \otimes t$ is the Kronecker product (block form $\left(\begin{smallmatrix} g_{11}t & g_{12}t \\ g_{21}t & g_{22}t \end{smallmatrix}\right)$), $P = \mathrm{diag}(1_2, W^{-1})$ in block form and $W = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$; in particular, over a field, $g \mapsto \mathrm{tn}(\rho(g), \tau(g))$ is isomorphic to $\rho \otimes \tau$;*

(Te1) $\mathrm{tn}(gg', tt') = \mathrm{tn}(g,t)\,\mathrm{tn}(g',t')$ *and* $\mathrm{tn}(1,1) = 1$; *in particular $g \mapsto \delta(g)\,\mathrm{tn}(\rho(g), \tau(g))$ is a homomorphism of monoids $\Gamma \to \mathrm{M}_4(R)$ for homomorphisms of monoids $\rho, \tau \colon \Gamma \to \mathrm{M}_2(R)$ and $\delta \colon \Gamma \to R$ from a group Γ;*

(Te2) *if ${}^t t\,H\,t = H$, then $\mathrm{tn}(g,t)$ has similitude $\det g$;*

(Te3) *for $\Phi, x, \eta \in R$, $\mathrm{tn}(\left(\begin{smallmatrix} \Phi & 0 \\ x & 1 \end{smallmatrix}\right), \mathrm{diag}(1,\eta)) = \mathrm{diag}(\Phi, \Phi\eta, \eta, 1) - x\eta\,E_{32} + x\,E_{41}$, and $\mathrm{wt}_4(3,2) = -w_\beta$, $\mathrm{wt}_4(4,1) = -w_\theta$; tn of two diagonal matrices is diagonal;*

(Te4) $\mathrm{fl}(x,s)$ *is lower triangular with all diagonal entries 1, has similitude 1, and $\mathrm{fl}(x,s)_{21} = s$;*

(Te5) *if $s \in \{1,-1\}$, $g = \mathrm{low}(x)\left(\begin{smallmatrix} p & q \\ 0 & r \end{smallmatrix}\right)\mathrm{low}(x)^{-1}$ and $t = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$, then $\mathrm{fl}(x,s)^{-1}\,\mathrm{tn}(g,t)\,\mathrm{fl}(x,s)$ is upper triangular with diagonal $\big(p(a+bs),\ p(a-bs),\ r(a+bs),\ r(a-bs)\big)$.*

Proof. Let $W = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $P = \mathrm{diag}(1_2, W^{-1})$ in block form, and let $g \otimes t$ be the Kronecker product, with block form $\left(\begin{smallmatrix} g_{11}t & g_{12}t \\ g_{21}t & g_{22}t \end{smallmatrix}\right)$. Then

$$\mathrm{tn}(g,t) = P\,(g \otimes t)\,P^{-1} = \begin{pmatrix} g_{11}t & g_{12}\,tW \\ g_{21}\,W^{-1}t & g_{22}\,W^{-1}tW \end{pmatrix}, \tag{51.2}$$

as one reads off entry by entry; this is (Te0).

(Te1): by (51.2) and $(g\otimes t)(g'\otimes t') = gg'\otimes tt'$, $1\otimes 1 = 1$. A product of a scalar homomorphism and a homomorphism of monoids into $\mathrm{M}_4(R)$ is one, since scalars are central.

(Te2): with $\varepsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ one has ${}^t g\,\varepsilon\, g = (\det g)\varepsilon$, hence ${}^t(g\otimes t)(\varepsilon\otimes H)(g\otimes t) = ({}^t g\varepsilon g)\otimes({}^t tHt) = \det(g)\,\varepsilon\otimes H$ when ${}^t tHt = H$. Conjugating by P and using ${}^t P^{-1}(\varepsilon\otimes H)P^{-1} = J_4$ gives ${}^t\mathrm{tn}(g,t)\,J_4\,\mathrm{tn}(g,t) = \det(g)J_4$.

(Te3), (Te4), (Te5) are identities between explicit 4×4 matrices whose entries are polynomials with integer coefficients in $\Phi, x, \eta, s, p, q, r, a, b$ (for (Te5), with $s^2 = 1$, checked for $s = 1$ and $s = -1$ separately; $\mathrm{fl}(x,s)$ is unipotent, so its inverse is $\sum_{k=0}^{3}(1-\mathrm{fl}(x,s))^k$); they were verified as polynomial identities, hence hold in every commutative ring, by computer check C4, listed in Appendix B, which also verifies (51.2), the identity ${}^t P^{-1}(\varepsilon\otimes H)P^{-1} = J_4$ and the general formula ${}^t\mathrm{tn}(g,t)J_4\mathrm{tn}(g,t) = \det(g)\,{}^t P^{-1}(\varepsilon\otimes {}^t tHt)P^{-1}$, by two independent implementations. The weights: $\boldsymbol{\lambda} = (1,0,1,0)$, $\boldsymbol{\mu} = (1,1,0,0)$ give $\mathrm{wt}_4(3,2) = (1,-1) = -w_\beta$ and $\mathrm{wt}_4(4,1) = (-1,-1) = -w_\theta$. For diagonal g, t all entries of the table with an off-diagonal factor vanish, and the remaining non-zero entries are on the diagonal. □

Lemma 51.16. *Let ℓ, ι and a newform f of weight $k \geq 3$ be as in Lemma 49.8, with Galois representation ρ and $\beta = \det\rho$. Let $E' \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field, $g_1 \in G_{\mathbb{Q}} \setminus G_{E'}$, and $\kappa\colon G_{E'} \to \overline{\mathbb{Q}}_\ell^\times$ a continuous character such that $\kappa(h)\kappa(g_1 h g_1^{-1}) = 1$ for all $h \in G_{E'}$, and such that there are a Hecke character Ψ_κ of E' and an integer $a \neq 0$ with $\Psi_\kappa(z) = (z/\bar z)^a$ on $(E'\otimes\mathbb{R})^\times = \mathbb{C}^\times$ (for one of the two identifications) and $\iota\kappa(\mathrm{Frob}_v^{-1}) = \Psi_\kappa(\varpi_v)$ for all but finitely many finite places v of E'. Let $V = \rho\otimes\mathrm{Ind}_{G_{E'}}^{G_{\mathbb{Q}}}\kappa$, a representation of $G_{\mathbb{Q}}$ of dimension 4. Let $L \subset \overline{\mathbb{Q}}$ be a CM field abelian over $\mathbb{Q}$ with $E' \not\subseteq L$, put $M = LE'$, and assume $k-1 \neq 2|a|$ and $\beta(s)^2 \not\equiv 1$ for some $s \in G_M$. Then:*

(Ta1) *there is a regular algebraic cuspidal automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_L)$ with $V|_{G_L} \leftrightarrow_\iota \Pi$;*

(Ta2) *if some $g \in G_L$ with $\chi_{\mathrm{cyc}}(g) = 1$ lies outside $G_{E'}$, then the restriction of V to $G_L' = \{g \in G_L : \chi_{\mathrm{cyc}}(g) = 1\}$ is irreducible, i.e. the $\overline{\mathbb{Q}}_\ell$-span of $V(G_L')$ is $\mathrm{End}(V)$.*

Proof. We use the following fact *(CB)*: two continuous semisimple representations of G_E (E a number field) whose characteristic polynomials agree at the Frobenius elements of almost all places are isomorphic (Chebotarev density and Brauer–Nesbitt); for characters it says that two continuous characters agreeing at almost all Frobenius elements are equal. For $\hat g \in G_{\mathbb{Q}} \setminus G_{E'}$ write $\hat g = g_1 y$ with $y \in G_{E'}$; then $\kappa(\hat g h \hat g^{-1}) = \kappa(g_1(yhy^{-1})g_1^{-1}) = \kappa(yhy^{-1})^{-1} = \kappa(h)^{-1}$ for $h \in G_{E'}$:

$$\kappa(\hat g h \hat g^{-1}) = \kappa(h)^{-1} \qquad (h \in G_{E'},\ \hat g \in G_{\mathbb{Q}} \setminus G_{E'}). \tag{51.3}$$

The hypothesis on β gives $\beta(s) \not\equiv 1$ for some $s \in G_M$; as every field E with $\mathbb{Q} \subseteq E \subseteq M$ is Galois over $\mathbb{Q}$ (M is abelian over $\mathbb{Q}$), (Ne4) shows that $\rho|_{G_E}$ is irreducible for all such E; and (Ne5) shows that $\rho|_U$ is irreducible for every open subgroup $U \subseteq G_{\mathbb{Q}}$.

Step 1: κ has infinite order on every open subgroup of $G_{E'}$. Suppose that $n \geq 1$ and $\kappa^n = 1$ on an open subgroup $G_{E''}$, $E'' \supseteq E'$; we derive a contradiction. The Hecke character $\Psi_\kappa^n \circ \mathrm{N}_{E''/E'}$ of E'' is then 1 at almost all finite places (at an unramified w above v with residue degree f, $\Psi_\kappa(\mathrm{N}\varpi_w) = \Psi_\kappa(\varpi_v)^f = \iota\kappa(\mathrm{Frob}_w^{-1})$), hence trivial by (Js1) for $n = 1$; but at a complex place its archimedean component is $z \mapsto (z/\bar z)^{\pm na} \neq 1$.

Step 2: the induced representation over L. Since $E' \not\subseteq L$, $G_L \not\subseteq G_{E'}$, so $G_{\mathbb{Q}} = G_{E'}G_L$ and $G_{E'} \cap G_L = G_M$, $[G_L : G_M] = 2$. By Mackey's formula and $\rho\otimes\mathrm{Ind}\,\kappa \cong \mathrm{Ind}(\rho|_{G_{E'}}\otimes\kappa)$,

$$V|_{G_L} \cong \mathrm{Ind}_{G_M}^{G_L} X, \qquad X = (\rho\otimes\kappa)|_{G_M}.$$

Fix $\hat{g} \in G_L \setminus G_M$ and put $X^{\hat{g}}(h) = X(\hat{g}h\hat{g}^{-1})$; it is X conjugated by $\rho(\hat{g})$ on the factor ρ, so $\det X^{\hat{g}} = \beta\kappa^{-2}$ on G_M by (51.3), while $\det X = \beta\kappa^2$.

Step 3: the automorphic representation of f and its base change to M. Let π be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by f, in the unitary normalisation ([103]): for $p \nmid \ell N$ it is unramified with Satake parameters $\{\alpha_p, \beta_p\}$ such that $\iota^{-1}(p^{(k-1)/2}\alpha_p)$, $\iota^{-1}(p^{(k-1)/2}\beta_p)$ are the eigenvalues of $\rho(\sigma_p)$ for an arithmetic Frobenius σ_p ([76]); its central character ω_π has finite order and $\alpha_p\beta_p = \omega_\pi(p)$; and the Langlands parameter of π_∞ restricted to $\mathbb{C}^\times$ is $z \mapsto \mathrm{diag}((z/\bar{z})^{(k-1)/2}, (z/\bar{z})^{-(k-1)/2})$. Put $\pi_1 = \pi \otimes (\omega_\pi^{-1}|\cdot|^{k/2}) \circ \det$. Then $\rho \leftrightarrow_\iota \pi_1$:

and the parameter of $\pi_{1,\infty}$ on $\mathbb{C}^\times$ is $z \mapsto \mathrm{diag}(z^{k-1/2}\bar{z}^{1/2},\, z^{1/2}\bar{z}^{k-1/2})$ (twist by $|z\bar{z}|^{k/2}$; $\omega_\pi \circ \mathrm{N}$ is trivial on $\mathbb{C}^\times$). Choose a chain $\mathbb{Q} = E_0 \subset E_1 \subset \cdots \subset E_t = M$ with E_{i+1}/E_i cyclic of prime degree (a composition series of the abelian group $\mathrm{Gal}(M/\mathbb{Q})$), and put $\pi^{(0)} = \pi_1$, $\pi^{(i+1)} = \mathrm{BC}_{E_{i+1}/E_i}(\pi^{(i)})$. By induction on i: $\pi^{(i)}$ is cuspidal, $\rho|_{G_{E_i}} \leftrightarrow_\iota \pi^{(i)}$, and at each complex place of E_i the parameter of $\pi^{(i)}$ is $z \mapsto \mathrm{diag}(z^{k-1/2}\bar{z}^{1/2}, z^{1/2}\bar{z}^{k-1/2})$. Indeed, by (Ac1) at almost all places w of E_{i+1} above u of E_i with residue degree f the Satake parameters of $\pi_w^{(i+1)}$ are the f-th powers of those of $\pi_u^{(i)}$, and $\rho(\mathrm{Frob}_w^{-1}) = \rho(\mathrm{Frob}_u^{-1})^f$ with $\mathrm{N}w^{1/2} = (\mathrm{N}u^{1/2})^f$; (Ac2) gives the archimedean parameters (a restriction of the parameter of $\pi_{1,\infty}$ to $\mathbb{C}^\times$). If $\pi^{(i+1)}$ were not cuspidal, then by (Ac3) $\pi^{(i)} \cong \pi^{(i)} \otimes (\omega \circ \det)$ for a character $\omega \neq 1$ of E_i trivial on norms from E_{i+1}; with ω_G the corresponding non-trivial character of $\mathrm{Gal}(E_{i+1}/E_i)$, (CB) gives $\rho|_{G_{E_i}} \cong \rho|_{G_{E_i}} \otimes \omega_G$ (both irreducible), and an intertwiner A with $A\rho(s)A^{-1} = \omega_G(s)\rho(s)$ is not scalar (as $\omega_G \neq 1$ and $\rho(s) \neq 0$) and commutes with $\rho(G_{E_{i+1}})$, so $\rho|_{G_{E_{i+1}}}$ is reducible by Schur's lemma: a contradiction.

Step 4: the twist by κ. Put $\pi_X = \pi^{(t)} \otimes (\Psi_\kappa \circ \mathrm{N}_{M/E'} \circ \det)$, cuspidal. At almost all places w of M above v of E' with residue degree f, $\iota\kappa(\mathrm{Frob}_w^{-1}) = \Psi_\kappa(\varpi_v)^f = (\Psi_\kappa \circ \mathrm{N}_{M/E'})(\varpi_w)$, so $X \leftrightarrow_\iota \pi_X$. At a complex place w of M let $a_w = a$ or $-a$ according to the embedding of E' that w induces; the parameter of $\pi_{X,w}$ is $z \mapsto \mathrm{diag}(z^{k-1/2+a_w}\bar{z}^{1/2-a_w},\, z^{1/2+a_w}\bar{z}^{k-1/2-a_w})$.

Step 5: (Ta1). Let σ be the non-trivial element of $\mathrm{Gal}(M/L)$, restriction of $\hat{g}$. If $\pi_X \cong \pi_X^\sigma$, then, since $X^{\hat{g}} \leftrightarrow_\iota \pi_X^\sigma$ ($\hat{g}\,\mathrm{Frob}_w\hat{g}^{-1}$ is a Frobenius element at σw), $\det X$ and $\det X^{\hat{g}}$ agree at almost all Frobenius elements, hence are equal by (CB), and Step 2 gives $\kappa^4 = 1$ on G_M, contradicting Step 1. So $\Pi_0 = \mathrm{AI}_{M/L}(\pi_X)$ is cuspidal by (Ai3); put $\Pi = \Pi_0 \otimes |\det|_{\mathbb{A}_L}$, cuspidal. At almost all finite places v of L:

so $V|_{G_L} \leftrightarrow_\iota \Pi$. At a complex place v of L the two places w, w' of M above it differ by σ, which is non-trivial on E', so $\{a_w, a_{w'}\} = \{a, -a\}$; by (Ai2) and Step 4 the parameter of $\Pi_v \otimes |\det|^{-3/2}$ on $\mathbb{C}^\times$ is $z \mapsto \mathrm{diag}(z^{p_i}\bar{z}^{q_i})$ with

$$(p_i) = (k-1+a,\ a,\ k-1-a,\ -a), \qquad (q_i) = (-a,\ k-1-a,\ a,\ k-1+a),$$

integers, and the p_i are pairwise distinct because $a \neq 0$ and $k-1 \neq \pm 2a$. So Π is regular algebraic in the sense of [68].

Step 6: (Ta2). Let $H = \{h \in G_M : \chi_{\mathrm{cyc}}(h) = 1\}$, a closed normal subgroup of $G_\mathbb{Q}$ with $G_\mathbb{Q}/H$ abelian ($M(\mu_{\ell^\infty})/\mathbb{Q}$ is abelian). By hypothesis some $g' \in G'_L$ lies outside $G_{E'}$, hence outside G_M; so $G'_L \cap G_M = H$, $[G'_L : H] = 2$ and $G_L = G_M G'_L$, and Mackey's formula gives $V|_{G'_L} \cong \mathrm{Ind}_H^{G'_L}(X|_H)$. (i) $\rho|_H$ is irreducible. Otherwise let $\mathcal{S}$ be the set of H-stable lines in $\overline{\mathbb{Q}}_\ell^2$, non-empty and permuted by $G_\mathbb{Q}$. If $\#\mathcal{S} = 1$ its line is $G_\mathbb{Q}$-stable, contradicting the irreducibility of ρ. If $\#\mathcal{S} = 2$, the stabiliser of one line is a closed subgroup of index ≤ 2, hence open, on which ρ is reducible: contradiction. If $\#\mathcal{S} \geq 3$, every $\rho(h)$, $h \in H$, is scalar; since $G_\mathbb{Q}/H$ is abelian, $\rho(g)\rho(g_2)\rho(g)^{-1} = c(g)\rho(g_2)$ with $c(g) \in \overline{\mathbb{Q}}_\ell^\times$, for a fixed g_2 with $\rho(g_2)$ not scalar (it exists, ρ being irreducible) and all $g \in G_\mathbb{Q}$; c is a continuous homomorphism, and multiplication by $c(g)$ preserves the multiset of eigenvalues of $\rho(g_2)$, so $c(g)^2 = 1$. On the open subgroup

$U = \ker c$ the matrices $\rho(U)$ commute with $\rho(g_2)$ and preserve an eigenline of it: contradiction with (Ne5). (ii) $X|_H$ is irreducible by (i), and $X|_H \ncong X^{g'}|_H$: otherwise $\kappa^4 = 1$ on H by Step 2 (with g' in place of $\hat{g}$), so $\kappa^4 = \varphi \circ \chi_{\mathrm{cyc}}$ on G_M for a continuous character φ of the compact group $\chi_{\mathrm{cyc}}(G_M) \cong G_M/H$; conjugating by $\hat{g}$ and using (51.3) and $\chi_{\mathrm{cyc}}(\hat{g}h\hat{g}^{-1}) = \chi_{\mathrm{cyc}}(h)$ gives $\kappa^{-4} = \kappa^4$ on G_M, i.e. $\kappa^8 = 1$ on G_M, contradicting Step 1. (iii) As an H-representation $\mathrm{Ind}_H^{G'_L}(X|_H) = X|_H \oplus g'X|_H$ with $g'X|_H \cong X^{g'}|_H$; the two summands are irreducible and non-isomorphic, so the only H-stable subspaces are 0, the two summands and the whole space, and g' exchanges the summands. Hence $V|_{G'_L}$ is irreducible, and by Burnside's theorem the span of $V(G'_L)$ is $\mathrm{End}(V)$. □

Proposition 51.17. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$ and $\mathscr{T}$ a good auxiliary tower for $\mathfrak{D}$. For every coefficient ring $\mathcal{O}' \supseteq \mathcal{O}$ there is a space for $(\mathscr{T}, \mathcal{O}')$.*

Proof. Let $\mathbb{F} = \mathbb{F}_{\mathcal{O}'}$, $\mathfrak{m}'$ the maximal ideal of $\mathcal{O}'$, $[x] \in \mathcal{O}'$ the Teichmüller lift of $x \in \mathbb{F}$, and $\bar{\chi}_i$ the residual characters of the frame, with values in $\mathbb{F}^\times$; $\theta(g)$ is read in $\mathcal{O}'$. As $\theta = \alpha\Psi$, $\chi_1\chi_4 = \chi_2\chi_3 = \theta|_{G_F}$ for both labellings, so $\bar{r} = \mathrm{diag}(\bar{\chi}_1, \dots, \bar{\chi}_4)$ has similitude $\bar{\theta}$.

Step 1: the universal near-diagonal datum. For $A \in \mathrm{CNL}_{\mathcal{O}'}$, a *near-diagonal datum* over A is a pair (r, ψ) of a continuous homomorphism $r\colon G_F \to \mathrm{GL}_4(A)$ with $r \equiv \bar{r}$ modulo $\mathfrak{m}_A$, $r(g)$ of similitude $\theta(g)$, $r(N) = 1$ and $r(g_{\mathscr{T}})$ diagonal, and of continuous characters $\psi_{v,i}\colon D_v \to A^\times$ ($v \in \Sigma$, $1 \le i \le 4$) with $\psi_{v,i} \equiv \bar{\chi}_i$ modulo $\mathfrak{m}_A$ and $\psi_{v,i}(\mu_v) = 1$. Let $\Gamma = G_F/\overline{\langle N \rangle}$, the quotient by the closed normal subgroup generated by N; as $N \subseteq \ker \bar{r}$, $\bar{r}$ factors through Γ. Γ satisfies Φ_ℓ: for $\Gamma' \subseteq \Gamma$ open with preimage H, let $H_0 = H \cap \ker \bar{r}$, with fixed field L'; a continuous $f\colon H \to \mathbb{F}_\ell$ vanishing on N vanishes on $I_{\bar{w}} \cap H_0 \subseteq N$ for all $\bar{w} \nmid \ell$, so $f|_{H_0}$ factors through $G_{L',S'}$, S' the places above ℓ and ∞; the kernel of restriction lies in $\mathrm{Hom}(H/H_0, \mathbb{F}_\ell)$, finite, and the image is finite by Theorem B.71, item 2. By Theorem B.71, item 1, there are universal framed lifts $(R^\square, r^\square)$ of $\bar{r}$ on Γ and $(R_{v,i}, \psi_{v,i}^\square)$ of $\bar{\chi}_i|_{D_v}$ ($n = 1$). Their completed tensor product $S_0 \in \mathrm{CNL}_{\mathcal{O}'}$ represents pairs of such lifts. Let $I_0 \subseteq S_0$ be generated by the entries of ${}^t r^\square(g) J_4 r^\square(g) - \theta(g) J_4$, the off-diagonal entries of $r^\square(g_{\mathscr{T}})$ and the $\psi_{v,i}^\square(s) - 1$, $s \in \mu_v$. The residual datum $(\bar{r}, (\bar{\chi}_i|_{D_v}))$ satisfies these equations $(\bar{\chi}_i(s)^{\ell^a} = \bar{\chi}_i(s^{\ell^a}) = 1$ for $s \in \mu_v$, so $\bar{\chi}_i(s) = 1$, the order of $\mathbb{F}^\times$ being prime to ℓ), so $I_0 \subseteq \mathfrak{m}_{S_0}$ and $\mathbf{R} = S_0/I_0 \in \mathrm{CNL}_{\mathcal{O}'}$, with the images $(r_{\mathrm{u}}, \psi_{\mathrm{u}})$, has the property: (U) for every near-diagonal datum (r, ψ) over $A \in \mathrm{CNL}_{\mathcal{O}'}$ there is exactly one $\mathcal{O}'$-algebra homomorphism $F\colon \mathbf{R} \to A$ with $F \circ r_{\mathrm{u}} = r$, $F \circ \psi_{\mathrm{u}} = \psi$.

Step 2: the $\mathbb{Z}^2$-weighted presentation. For $t_1, t_2 \in 1 + \mathfrak{m}_A$ and $X = \mathrm{T}_4(t_1, t_2)$, (XrX^{-1}, ψ) is again a near-diagonal datum: conjugation by a matrix of similitude t_1t_2 preserves similitudes, diagonality of $r(g_{\mathscr{T}})$, $r(N) = 1$, continuity and the reduction. By (U) for $\mathbf{R}[[T]]$ and $\mathbf{R}[[T_1, T_2]]$ there are morphisms $\varrho_1\colon \mathbf{R} \to \mathbf{R}[[T]]$ (for $\mathrm{T}_4(1+T, 1)$) and $\varrho\colon \mathbf{R} \to \mathbf{R}[[T_1, T_2]]$ (for $\mathrm{T}_4(1 + T_1, 1 + T_2)$). Let S_1 be the set of indices $s = (g, i, j)$, $g \in G_F$, and $s = (v, i, t)$, $t \in D_v$, and put $x_{(g,i,j)} = r_{\mathrm{u}}(g)_{ij} - \delta_{ij}[\bar{\chi}_i(g)]$, $x_{(v,i,t)} = \psi_{\mathrm{u},v,i}(t) - [\bar{\chi}_i(t)]$, all in $\mathfrak{m}_{\mathbf{R}}$, with $\mathrm{wt}(g, i, j) = \mathrm{wt}_4(i, j)$, $\mathrm{wt}(v, i, t) = (0, 0)$; then $\varrho(x_s) = (1+T_1)^{w_1}(1+T_2)^{w_2}x_s$ for $\mathrm{wt}(s) = (w_1, w_2)$, and similarly for ϱ_1 with $w_1 \in \{-1, 0, 1\}$. Two morphisms from $\mathbf{R}$ agreeing on all x_s agree on $(r_{\mathrm{u}}, \psi_{\mathrm{u}})$, hence are equal by (U). So Lemma 43.8 (for ϱ_1) gives a finite $\sigma \subseteq S_1$ such that $\pi\colon P = \mathcal{O}'[[X_s : s \in \sigma]] \to \mathbf{R}$, $X_s \mapsto x_s$, is surjective, and Lemma 45.18 ($d = 2$, ϱ) gives: $\mathfrak{a} = \ker \pi$ is homogeneous, and every element of torus weight w lies in $\mathbf{R}_w = \pi(\mathcal{O}'[[X_\sigma]]_w)$. In particular $r_{\mathrm{u}}(g)_{ij} \in \mathbf{R}_{\mathrm{wt}_4(i,j)}$ and $\psi_{\mathrm{u},v,i}(t) \in \mathbf{R}_{(0,0)}$. Elements of $\mathfrak{a}$ have constant term in $\mathfrak{m}'$ (compose π with $\mathbf{R} \to \mathbb{F}$). Products of power series of weights w, w' have weight $w + w'$, so $\mathbf{R}_w\mathbf{R}_{w'} \subseteq \mathbf{R}_{w+w'}$.

Step 3: the rings B and A_0; continuity. Let $m_1, \dots, m_p$ generate the monoid of weight-0 exponents (Gordan). The continuous map $\mathcal{O}'[[Y_1, \dots, Y_p]] \to P$, $Y_k \mapsto X^{m_k}$, has image $P_0 = \mathcal{O}'[[X_\sigma]]_{(0,0)}$ (write each weight-0 exponent as $\sum c_k m_k$; distinct exponents give distinct c, and the resulting series converges). Hence $B = \mathbf{R}_{(0,0)} = \pi(P_0)$ is a quotient of $\mathcal{O}'[[Y]]$

containing $\mathcal{O}'$: $B \in \mathrm{CNL}_{\mathcal{O}'}$, with maximal ideal $\mathfrak{m}_B = \pi(\mathfrak{m}_{P_0})$, $\mathfrak{m}_{P_0}$ generated by $\mathfrak{m}'$ and the X^{m_k}. For $w \in \mathbb{Z}^2$ let $m'_1, \dots, m'_q$ be the minimal exponents of weight w (Dickson); grouping the terms of $h \in \mathcal{O}'[[X_\sigma]]_w$ by a chosen minimal $m'_k \le m$ gives $h = \sum_k X^{m'_k} h_k$ with $h_k \in P_0$, so $\mathbf{R}_w = \sum_k \pi(X^{m'_k})B$. Put $A_0 = B[x_s : s \in \sigma] \subseteq \mathbf{R}$; it contains every $\mathbf{R}_w$, all entries of r_u and all ψ_u, and is of finite type over B. *Continuity:* let D bound the degrees of the m_k and of the m'_k for $w \in \mathcal{W}_4 = \{\mathrm{wt}_4(i,j)\}$; for $y \in \mathbf{R}_w \cap \mathfrak{m}_{\mathbf{R}}^{D(k+2)}$ write $y = \pi(h')$ with $h' \in \mathfrak{m}_P^{D(k+2)}$; by homogeneity of $\mathfrak{a}$ also $y = \pi(h'_{(w)})$, and $h'_{(w)} \in \mathfrak{m}_P^{D(k+2)}$. In the grouping, a term bX^m ($b \in \mathfrak{m}'^e$, $e + |m| \ge D(k+2)$) contributes $bX^{m-m'_k}$, a product of e factors of $\mathfrak{m}'$ and at least $(|m| - D)/D$ of the X^{m_k}, hence in $\mathfrak{m}_{P_0}^{k+1}$, a closed ideal. So

$$\mathbf{R}_w \cap \mathfrak{m}_{\mathbf{R}}^{D(k+2)} \subseteq \mathfrak{m}_B^k A_0 \quad (w \in \mathcal{W}_4), \qquad \mathbf{R}_{(0,0)} \cap \mathfrak{m}_{\mathbf{R}}^{D(k+2)} \subseteq \mathfrak{m}_B^k \tag{51.4}$$

(for $w = (0,0)$ the only minimal exponent is 0). As r_u and ψ_u are $\mathfrak{m}_{\mathbf{R}}$-adically continuous, (51.4) makes $g \mapsto r_\mathrm{u}(g)_{ij}$ continuous for $\mathfrak{m}_B A_0$, and the characteristic polynomial coefficients of $r_\mathrm{u}(g)$ (sums of products $\prod_i r_\mathrm{u}(g)_{i\pi(i)}$ of weight 0, so in B) and the ψ_u continuous for $\mathfrak{m}_B$.

Step 4: B is generated by traces and ψ. Let $f, f' \colon B \to R'$ be $\mathcal{O}'$-algebra homomorphisms to $R' \in \mathrm{CNL}_{\mathcal{O}'}$ agreeing on all coefficients of the characteristic polynomials $\mathrm{cp}_\mathrm{u}(g)$ of $r_\mathrm{u}(g)$ and on all $\psi_{\mathrm{u},v,i}(t)$; they are continuous, so their equalizer E is a closed $\mathcal{O}'$-subalgebra. Write $r_\mathrm{u}(g_{\mathscr{F}}) = \mathrm{diag}(d_1, \dots, d_4)$, $d_i \in B$, $d_i \equiv \bar\chi_i(g_{\mathscr{F}})$; by (Oq7) $d_i - d_j \in B^\times$ for $i \ne j$. The polynomial $p = f(\mathrm{cp}_\mathrm{u}(g_{\mathscr{F}})) = f'(\mathrm{cp}_\mathrm{u}(g_{\mathscr{F}})) = \prod_j (X - f(d_j))$ vanishes at $f'(d_i)$, and $f'(d_i) - f(d_j)$ is a unit for $j \ne i$ (distinct residues); so $f'(d_i) = f(d_i)$. By Lemma 51.18 (with $R = \mathbf{R}$, $\Lambda = \mathcal{O}'$, $\nu = \theta$, $g_* = g_{\mathscr{F}}$, r_u through Γ), every product of entries of weight $(0,0)$ is a polynomial in traces with coefficients in $\mathcal{O}'[d_i, (d_i - d_j)^{-1}] \subseteq B$; hence it lies in E. Every $\pi(X^m)$ with m of weight 0 is a product of x_s; expanding $x_s = r_\mathrm{u}(g)_{ij} - \delta_{ij}[\bar\chi_i(g)]$ (constants only at weight-0 entries) writes it as an $\mathcal{O}'$-combination of weight-0 products of entries and of ψ_u's, all in E. As E is closed and $B = \pi(P_0)$, $E = B$, i.e. $f = f'$.

Step 5: flags; the space. Let $A = A_0[Y_{v,ij} : v \in \Sigma, 1 \le j < i \le 4]/I$, n_v the lower unitriangular matrix with entries $Y_{v,ij}$ below the diagonal, $n_v^* = -J_4\, {}^t n_v J_4$, and I generated by the entries of ${}^t n_v J_4 n_v - J_4$, the entries (i,j), $i > j$, of $n_v^* r_\mathrm{u}(s) n_v$ and $(n_v^* r_\mathrm{u}(s) n_v)_{ii} - \psi_{\mathrm{u},v,i}(s)$ for $s \in D_v$. The residue map of $\mathbf{R}$ on A_0 and $Y \mapsto 0$ kill I (the residual datum is diagonal), so $A \ne 0$ and the kernel $\mathfrak{b}$ of $B \to A$ lies in $\mathfrak{m}_B$. Put $\bar B = B/\mathfrak{b} \in \mathrm{CNL}_{\mathcal{O}'}$, r_P, n_v the images in A, $P = (r_P, (n_v))$, $\mathrm{cp}(g)$ and $\psi_{v,i}^B$ the images of $\mathrm{cp}_\mathrm{u}(g)$ and $\psi_{\mathrm{u},v,i}$ in $\bar B$. Once ${}^t n_v J_4 n_v = J_4$, $n_v^* = n_v^{-1}$. Then: P is a marked ordinary pair ((Mq1), (Mq2) from Step 1, (Mq3), (Mq4) from I and $\psi_\mathrm{u}(\mu_v) = 1$), with $\psi_{v,i}^P = \psi_{v,i}^B$; (Zq1): $\bar B \to A$ is injective, A is of finite type over $\bar B$ and generated by the entries of r_P (through the x_s) and of the n_v, and P is continuous for $\mathfrak{m}_{\bar B} A$ by Step 3; (Zq2) by Step 3 and the reduction $\bar r$; (Zq3) by Step 4, composing with $B \to \bar B$.

Step 6: scaling. Let $c = (r_c, (n_{c,v}))$ be a residual pair. By (Wq2), r_c is supported in Σ_P or in Σ_M; let $\tau(w) = -2w_1 - w_2$ in the first case and $\tau(w) = -w_1 - 2w_2$ in the second. A check on $\mathcal{W}_4$ gives $\tau \ge 1$ on Σ_P, resp. Σ_M, and $|\tau| \le 3$ on $\mathcal{W}_4$. So off-diagonal entries (i,j) of r_c with $\tau(\mathrm{wt}_4(i,j)) \le 0$ vanish. With (a,b) the coefficients of τ and a unit u, $X_u = \mathrm{T}_4(u^a, u^b)$ multiplies the entry (i,j) by $u^{\tau(\mathrm{wt}_4(i,j))}$.

(Zq5). Let c have values in $\mathbb{F}[X]$ and put $C = \mathbb{F}[[u, X]] \in \mathrm{CNL}_{\mathcal{O}'}$. Then $(X_u r_c X_u^{-1}, (\bar\chi_i|_{D_v}))$ is a near-diagonal datum over C: its off-diagonal entries lie in uC (as $\tau \ge 1$ where r_c is non-zero), its diagonal is $\bar\chi_i$ by (Wq1), similitudes and the other conditions are those of c, and $\psi_{v,i}^c = \bar\chi_i|_{D_v}$ is trivial on μ_v. Let $F_0 \colon \mathbf{R} \to C$ be given by (U). Then $F_0(x_s) = u^{\tau(\mathrm{wt}(s))}\xi_s$ with $\xi_s = r_c(g)_{ij} \in \mathbb{F}[X]$ for $s = (g,i,j)$, $i \ne j$, and $\xi_s = 0$ otherwise, and $\xi_s \ne 0$ only if $\tau(\mathrm{wt}(s)) \ge 1$. For $h = \sum b_m X^m \in \mathcal{O}'[[X_\sigma]]_w$, only exponents with $|m| \le \tau(w)$ contribute, so $F_0(\pi(h)) = u^{\tau(w)} p_h$ with $p_h \in \mathbb{F}[X]$ (and 0 if $\tau(w) < 0$). Hence $F_0(A_0) \subseteq \mathbb{F}[X][u]$, and

$f_0 = (u \mapsto 1) \circ F_0|_{A_0} \colon A_0 \to \mathbb{F}[X]$ is an $\mathcal{O}'$-algebra homomorphism with $f_0(r_{\mathrm{u}}(g)_{ij}) = r_c(g)_{ij}$, $f_0(\psi_{\mathrm{u}}) = \psi^c$, and $f_0(\mathfrak{m}_B) = 0$ (for $w = (0,0)$, p_h is the residue of the constant term). Sending $Y_{v,ij} \mapsto (n_{c,v})_{ij}$ kills I ((Mq3), (Mq4) for c), giving $f \colon A \to \mathbb{F}[X]$ with $f(P) = c$, $f(\mathfrak{m}_{\bar{B}}) = 0$.

(Zq4). Let $R' \in \mathrm{CNL}_{\mathcal{O}'}$, c with values in $\mathbb{F}$ and $Q = (r_Q, (n_{Q,v}))$ with values in R', continuous, with reduction c. First let $\mathfrak{m}_{R'}^{K+1} = 0$. Let $S \subseteq R'((u))$ be the $R'[[u]]$-submodule generated by 1 and $\mathfrak{m}_{R'}^k u^{-3k}$, $1 \le k \le K$; it is a subring, finite over $R'[[u]]$; $J = \mathfrak{m}_{R'} R'[[u]] + \sum_k \mathfrak{m}_{R'}^k u^{-3k} R'[[u]]$ is an ideal with $J^{K+1} = 0$ and $S/J \cong \mathbb{F}[[u]]$, so S is local with maximal ideal $J + uS$, and $(J+uS)^{K+2} \subseteq uS$ shows that S is complete: $S \in \mathrm{CNL}_{\mathcal{O}'}$. The entries $u^\tau r_Q(g)_{ij}$ of $X_u r_Q X_u^{-1}$ lie in S (if $\tau < 0$ then $i \ne j$, $r_c(g)_{ij} = 0$ and $r_Q(g)_{ij} \in \mathfrak{m}_{R'}$), and those with $i \ne j$ lie in $J + uS$; with $\psi^Q_{v,i}$ (continuous characters by (Mq4), $\equiv \bar{\chi}_i$ by (Wq1), trivial on μ_v) this is a near-diagonal datum over S (r_Q is locally constant as $\mathfrak{m}_{R'}$ is nilpotent). Let $F \colon \mathbf{R} \to S$ be given by (U); $F(x_s) = u^{\tau(\mathrm{wt}(s))} y_s$ with $y_s \in R'$, and $y_s \in \mathfrak{m}_{R'}$ when $\tau(\mathrm{wt}(s)) \le 0$. For $h \in \mathcal{O}'[[X_\sigma]]_w$ a monomial X^m contributes 0 unless at most K of its factors have $\tau \le 0$, and then $\sum_{\tau(\mathrm{wt}(s))\ge 1} m_s \le \tau(w) + 3K$; so $F(\pi(h)) = u^{\tau(w)} z$ with $z \in R'$ a finite sum. Thus $F(A_0) \subseteq R'[u, u^{-1}]$, and $f_0 = (u \mapsto 1) \circ F|_{A_0}$ satisfies $f_0(r_{\mathrm{u}}) = r_Q$, $f_0(\psi_{\mathrm{u}}) = \psi^Q$; with $Y_{v,ij} \mapsto (n_{Q,v})_{ij}$ it gives $f \colon A \to R'$ with $f(P) = Q$. Such f is unique: two of them agree on the generators of A over $\bar{B}$, and on $\bar{B}$ by (Zq3), as both send $\mathrm{cp}(g)$ to the characteristic polynomial of $r_Q(g)$ and ψ^B to ψ^Q. For general R' apply this to $R'/\mathfrak{m}_{R'}^{K+1}$ and Q modulo $\mathfrak{m}_{R'}^{K+1}$, $K \ge 0$; by uniqueness the maps are compatible, and their limit $f \colon A \to \varprojlim R'/\mathfrak{m}_{R'}^{K+1} = R'$ satisfies $f(P) = Q$.

So $(\bar{B}, A, P, \mathrm{cp}, (\psi^B_{v,i}))$ is a space for $(\mathscr{T}, \mathcal{O}')$. □

Lemma 51.18. *Let R be a commutative ring, $\Lambda \subseteq R$ a subring, Γ a group and $r \colon \Gamma \to \mathrm{GL}_4(R)$ a homomorphism such that $r(g)$ has similitude $\nu(g) \in \Lambda^\times$ for every $g \in \Gamma$. Let $g_* \in \Gamma$ with $r(g_*) = \mathrm{diag}(d_1, d_2, d_3, d_4)$ and $d_i - d_j \in R^\times$ for $i \ne j$, and let Λ' be the subring of R generated by Λ, the d_i and the $(d_i - d_j)^{-1}$. Then every product $r(g_1)_{i_1 j_1} \cdots r(g_k)_{i_k j_k}$ with $k \ge 0$, $g_t \in \Gamma$ and $\sum_{t=1}^k \mathrm{wt}_4(i_t, j_t) = (0,0)$ lies in the Λ'-subalgebra of R generated by the traces $\mathrm{tr}\, r(g)$, $g \in \Gamma$.*

Proof. Let $\mathcal{A}$ be the Λ'-subalgebra generated by the traces, and E_i the diagonal matrix unit with entry 1 at (i,i).

Step 1: idempotents. By Lagrange interpolation, $E_i = \prod_{j \ne i} (d_i - d_j)^{-1} (r(g_*) - d_j) = \sum_{a=0}^3 \beta_{i,a}\, r(g_*)^a$ with $\beta_{i,a} \in \Lambda'$, and $r(g_*)^a = r(g_*^a)$, $r(g_*^0) = 1$.

Step 2: closed walks. For indices $i_1, \dots, i_k$ (put $i_{k+1} = i_1$) and $h_1, \dots, h_k \in \Gamma$,

$$\prod_{t=1}^k r(h_t)_{i_t i_{t+1}} = \mathrm{tr}\big(E_{i_1} r(h_1) E_{i_2} r(h_2) \cdots E_{i_k} r(h_k)\big) = \sum_{a_1, \dots, a_k} \beta_{i_1,a_1} \cdots \beta_{i_k,a_k}\, \mathrm{tr}\, r\big(g_*^{a_1} h_1 g_*^{a_2} h_2 \cdots g_*^{a_k} h_k\big) \in \mathcal{A}.$$

Step 3: the symplectic flip. From ${}^t r(g) J_4 r(g) = \nu(g) J_4$ and $J_4^{-1} = -J_4$ we obtain

$$r(g)^{-1} = \nu(g)^{-1} J_4^{-1}\, {}^t r(g)\, J_4.$$

As J_4 has exactly one non-zero entry, ± 1, in each row and column, at the positions $(a, 5-a)$, this gives $r(g^{-1})_{ab} = \pm \nu(g)^{-1} r(g)_{5-b,5-a}$, i.e.

$$r(g)_{ij} = \pm \nu(g)\, r(g^{-1})_{5-j,5-i} \qquad (1 \le i, j \le 4),$$

with a sign depending only on (i,j).

Step 4: reduction to balanced products. To a product $P = \prod_t r(g_t)_{i_t j_t}$ attach the vector $v(P) = \sum_t (e_{i_t} - e_{j_t}) \in \mathbb{Z}^4$, whose coordinate at x is the number of factors with $i_t = x$ minus the number with $j_t = x$. Then $\sum_t \mathrm{wt}_4(i_t, j_t) = (\boldsymbol{\lambda} \cdot v(P), \boldsymbol{\mu} \cdot v(P))$. If this is $(0,0)$, then $v_1 + v_3 = 0$, $v_1 + v_2 = 0$ and $v_1 + v_2 + v_3 + v_4 = 0$, so $v(P) = n(P)\,(e_1 - e_2 - e_3 + e_4)$ for an integer $n(P)$. We show $P \in \mathcal{A}$ by induction on $|n(P)|$. If $n(P) = 0$, the directed multigraph on $\{1,2,3,4\}$ with one edge $i_t \to j_t$ for each factor has equal in- and out-degree at every vertex, so its edges are partitioned into closed walks (remove a closed walk, which exists from any vertex of positive degree, and repeat); P is the product of the corresponding products of Step 2, hence in $\mathcal{A}$. If $n = n(P) > 0$, the vertices 2 and 3 have in-degree exceeding out-degree by n each; edges with both ends in $\{2,3\}$ do not change the total excess of $\{2,3\}$, so some factor $r(g)_{ij}$ has $i \in \{1,4\}$, $j \in \{2,3\}$. Replace it by $\pm\nu(g) r(g^{-1})_{5-j,5-i}$ (Step 3): $P = \pm\nu(g) P'$ with $\nu(g) \in \Lambda$, and $v(P') - v(P) = (e_{5-j} - e_{5-i}) - (e_i - e_j) = (e_2 + e_3) - (e_1 + e_4)$, so $n(P') = n - 1$ and P' still has weight sum $(0,0)$. If $n < 0$, the same argument with the vertices 1, 4 gives a factor with $i \in \{2,3\}$, $j \in \{1,4\}$, whose replacement raises n by 1. By induction $P \in \mathcal{A}$. □

51.3 Finiteness in weight zero and block extensions

This subsection proves the remaining clauses of Theorem 51.1. Promodular test representations are not essential in characteristic ℓ; in characteristic zero they are weakly pure, and automorphic when irreducible (Lemma 51.19). Finiteness in weight zero, (Sa1), follows from a curve test (Proposition 51.20): the universal symplectic Eisenstein lift has a weighted presentation with homogeneous kernel and entries (Proposition 51.21), and finiteness can be checked on one-dimensional quotients. The absence of endoscopic points, (Sa2), comes from block extensions. A block diagonal ordinary test representation deforms to a non-split block extension with the same characteristic polynomials (Proposition 51.23), using a non-zero integral cocycle with an ordinary condition at ℓ (Proposition 51.24); that extension is essential, hence weakly pure by Theorem 51.4, and weak purity depends only on the characteristic polynomials; but in characteristic zero a block diagonal test representation is not weakly pure (Lemma 51.25). At every place of T the Eisenstein point lies on exactly one component of the local deformation space (Proposition 51.22), which gives the universal lift used here.

Lemma 51.19. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with odd prime ℓ and $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism.*

1. *If (K, V, a) is a test field of characteristic ℓ and (r, u) an ordinary test representation of $\mathfrak{D}$ with values in V such that r is promodular for $\mathfrak{D}$ and ι, then r is not essential.*
2. *If (r, u) is an ordinary test representation of $\mathfrak{D}$ with values in $\overline{\mathbb{Z}}_\ell$ (for the standard test field) such that r is promodular for $\mathfrak{D}$ and ι, then r is weakly pure for ι; if moreover r is irreducible, then r satisfies (Sy2) for ι.*

Proof. In both parts let $\mathscr{T} = (\epsilon, E, F, g_{\mathscr{T}}, \Sigma, (\bar{v}))$, $\mathcal{O}'$, $\mathscr{X} = (B, A, P, \mathrm{cp}, (\psi^B_{v,i}))$, $\mathfrak{J}$, y, $u_1 \in \mathrm{GL}_4(K)$ and $r_1 = u_1 r u_1^{-1}$ be as in the definition of promodularity; $\mathscr{T}$ is fit, hence good, and $L = FE$. Write $P = (r_P, (n_v))$.

Step 0: the root characters. Since $\theta = \alpha\Psi$, $\kappa_i = \alpha^{\lambda_i}\Psi^{\mu_i}$; for $\epsilon = 1$, $(\kappa_i^1) = (\alpha, \mathbf{1}, \theta, \Psi) = (\alpha^{\lambda_i}\Psi^{1-\mu_i})$. In both cases $\chi_1^{\mathscr{T}}/\chi_2^{\mathscr{T}} = \alpha$ and $\chi_1^{\mathscr{T}}/\chi_3^{\mathscr{T}} = \Psi^{\pm 1}$, so $\chi_i^{\mathscr{T}}/\chi_j^{\mathscr{T}} = \mathrm{rt}_{\mathrm{wt}_4(i,j)}$ on G_F, and $\mathrm{wt}_4(i,j)$ is a root for $i \neq j$. By (Oq3), for $i \neq j$ there is $h \in G_L$ with $\bar{\chi}_i^{\mathscr{T}}(h) \neq \bar{\chi}_j^{\mathscr{T}}(h)$. As $(\chi_i^{\mathscr{T}})_i$ is a permutation of $(\kappa_i|_{G_F})_i$, the four characters $\bar{\kappa}_i|_{G_L}$ are pairwise distinct. (D)

Part 1. *Step 1: y kills $\mathfrak{m}_B$.* As K has characteristic ℓ, $a(\ell) = 0$; the kernel of $a|_{\mathcal{O}'}$ is a prime ideal of the discrete valuation ring $\mathcal{O}'$ containing $\ell \neq 0$, hence equals $\mathfrak{m}_{\mathcal{O}'}$, and

$a|_{\mathcal{O}'} = j \circ (\mathcal{O}' \to \mathbb{F}_{\mathcal{O}'})$ for an embedding of fields $j\colon \mathbb{F}_{\mathcal{O}'} \to V$. Hence $y(\mathfrak{m}_{\mathcal{O}'}B) = 0$. For $v \in \Sigma$ and $s \in I_v$, $y(\psi^B_{v,i}(s)) = y(\psi^P_{v,i}(s)) = a(\kappa_i(s)) = j(\bar\kappa_i(s)) = 1$ by (Zq2), promodularity and (Oq6). With $y(\mathfrak{J}) = 0$, (Jq1) gives $y(\mathfrak{m}_B)^n = 0$, and V is a domain, so $y(\mathfrak{m}_B) = 0$. Since $B = \mathcal{O}' + \mathfrak{m}_B$, $y|_B = j \circ (B \to B/\mathfrak{m}_B = \mathbb{F}_{\mathcal{O}'})$.

Step 2: characteristic polynomials on G_F. For $g \in G_F$, $r_1(g) = y(r_P(g))$, so by (Zq2) and Step 1 its characteristic polynomial is the image under y of $\mathrm{cp}(g)$, namely $\prod_i (X - j(\bar\chi_i^{\mathscr{T}}(g))) = \prod_i (X - a(\kappa_i(g)))$. Conjugation by u_1 does not change it, so $\det(X - r(g)) = \prod_i (X - a(\kappa_i(g)))$ for $g \in G_F$.

Step 3: $\operatorname{tr} r(g) = \sum_i a(\kappa_i(g))$ for every $g \in G_{\mathbb{Q}}$. Let $q = \#\mathbb{F}_{\mathcal{O}}$ and $d_i = a(\kappa_i(g))$; then $d_i = j(\bar\kappa_i(g))$ and $d_i^{q-1} = 1$. As $[G_{\mathbb{Q}} : G_F]$ is finite, $g^N \in G_F$ for some $N \geq 1$, and by Step 2 every eigenvalue $x \in K$ of $r(g)$ satisfies $x^N = d_i^N$ for some i, so $x^{N(q-1)} = 1$. Hence $\Delta = \operatorname{tr} r(g) - \sum_i d_i$ is a sum of roots of unity of K, so algebraic over $\mathbb{F}_\ell$, and lies in a finite subfield of K. By (Ot2) the diagonal entries of $r(g)$ are congruent to d_i modulo $\mathfrak{m}_V$, so $\Delta \in \mathfrak{m}_V$. A non-zero element of a finite field satisfies $\Delta^m = 1$ for some $m \geq 1$ and is then a unit of V; so $\Delta = 0$.

Step 4: a trace identity. As the κ_i take values in a commutative group, Step 3 gives

$$\operatorname{tr} r(xgh) = \operatorname{tr} r(xhg) \qquad \text{for all } x, g, h \in G_{\mathbb{Q}}. \tag{51.5}$$

Step 5: r is not irreducible. If the K-span of $r(G_{\mathbb{Q}})$ were $\mathrm{M}_4(K)$, then by linearity (51.5) gives $\operatorname{tr}(Y(r(g)r(h) - r(h)r(g))) = 0$ for all $Y \in \mathrm{M}_4(K)$, so $r(g)r(h) = r(h)r(g)$ by non-degeneracy of the trace form; then $\mathrm{M}_4(K)$, the span of $r(G_{\mathbb{Q}})$, would be commutative, which it is not.

Step 6: r is not a non-split block extension. Suppose it were. By condition (i), $r(g)$ maps the span of e_2, e_4 to itself; write $A_1(g)$, $A_2(g) \in \mathrm{M}_2(K)$ for the submatrices on the indices $\{1,3\}$ and $\{2,4\}$. Both A_1 and A_2 are multiplicative (block triangularity), $\operatorname{tr} r(g) = \operatorname{tr} A_1(g) + \operatorname{tr} A_2(g)$, and $\det(X - r(g)) = p_1(g)p_2(g)$ with $p_k(g) = \det(X - A_k(g))$. Let $g_0 = g_{\mathscr{T}} \in G_F$. By Step 2 and (Oq7) (and injectivity of j), $\det(X - r(g_0))$ has four distinct roots, so $p_1(g_0)$ and $p_2(g_0)$ are coprime, and by Cayley–Hamilton $M = p_1(g_0)(A_2(g_0))$ is invertible: from $\lambda p_1 + \mu p_2 = 1$ we get $\lambda(A_2(g_0))M = 1$. Write $p_1(g_0) = X^2 - tX + \delta$. Since $p_1(g_0)(A_1(g_0)) = 0$, multiplicativity gives, for every $z \in G_{\mathbb{Q}}$,

$$\operatorname{tr}(MA_2(z)) = \operatorname{tr} r(g_0^2 z) - t \operatorname{tr} r(g_0 z) + \delta \operatorname{tr} r(z),$$

and with (51.5) applied to $x = g_0^2 x'$, $g_0 x'$, x', $\operatorname{tr}(MA_2(x'gh)) = \operatorname{tr}(MA_2(x'hg))$ for all x', g, h. By condition (ii) the $A_2(x')$ span $\mathrm{M}_2(K)$, so $\operatorname{tr}\big((A_2(g)A_2(h) - A_2(h)A_2(g))MY\big) = 0$ for all $Y \in \mathrm{M}_2(K)$, whence $(A_2(g)A_2(h) - A_2(h)A_2(g))M = 0$ and, M being invertible, $A_2(g)A_2(h) = A_2(h)A_2(g)$. Then $\mathrm{M}_2(K)$ would be commutative. So r is not essential.

Part 2. Here $(K, V, a) = (\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Z}}_\ell, \mathrm{id})$, so $y\colon A \to \overline{\mathbb{Z}}_\ell \subseteq \overline{\mathbb{Q}}_\ell$ is an $\mathcal{O}'$-algebra homomorphism with $y(\mathfrak{J}) = 0$, and $r_1 = u_1 r u_1^{-1}$ takes values in $\mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$.

Step 7: weights. Let $U = G_F \cap \ker\chi \cap \ker\omega_\ell$, an open subgroup ($\chi$, ω_ℓ have finite order). Let $s \in U \cap I_v$; s lies in an inertia group above ℓ, where ξ is unramified by (Sg1) ($\ell \in T$), so $\xi(s) = 1$; then $\Psi(s) = \chi_{\mathrm{cyc}}(s)^e$, $\alpha(s) = \chi_{\mathrm{cyc}}(s)$, $\theta(s) = \chi_{\mathrm{cyc}}(s)^{e+1}$, i.e. $\kappa_i(s) = \chi_{\mathrm{cyc}}(s)^{m_i}$ with $(m_i) = (e+1, e, 1, 0)$. By promodularity $\psi^{y(P)}_{v,i}(s) = y(\psi^P_{v,i}(s)) = \kappa_i(s)$, so $y(P)$ has weights $(2m_1, \ldots, 2m_4)$.

Step 8: weak purity. r_1 is a homomorphism to $\mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ with $r_1|_{G_F} = y \circ r_P$, so by (Jq4) it is weakly pure for ι. Weak purity depends only on characteristic polynomials, which r and r_1 share; so r is weakly pure.

Step 9: automorphy. Let r be irreducible. (a) *Continuity.* By (Ot1) the entries of r lie in $j(\Lambda)$ for a complete discrete valuation ring Λ with finite residue field and an injective local $j\colon \Lambda \to \overline{\mathbb{Z}}_\ell$, continuous for the $\mathfrak{m}_\Lambda$-adic topology. $j(\Lambda)$ has characteristic 0, so Λ contains $\mathbb{Z}_\ell$

and is finite over it (complete discrete valuation ring with finite residue field); $j(\Lambda)$ is the ring of integers of a finite extension of $\mathbb{Q}_\ell$, on which the $\mathfrak{m}_\Lambda$-adic and the ℓ-adic topologies agree. So r, hence r_1, is continuous. (b) $r_1(g)$ has similitude $\theta(g)$ by promodularity. (c) $r_1|_{G_L}$ *is irreducible.* r_1 is irreducible, $H = G_L$ is normal of finite index in $G_\mathbb{Q}$ with $G_\mathbb{Q}/H$ abelian ($L = FE$ is abelian over $\mathbb{Q}$). Suppose $r_1|_H$ reducible. By Clifford's theorem $r_1|_H$ is semisimple, so $\dim \mathrm{End}_H(r_1) \geq 2$. The finite abelian group $G_\mathbb{Q}/H$ acts on $\mathrm{End}_H(r_1)$ by $Y \mapsto r_1(g) Y r_1(g)^{-1}$; decompose into eigenspaces for its characters (characteristic 0, $\overline{\mathbb{Q}}_\ell$ algebraically closed). The trivial eigenspace is $\mathrm{End}_{G_\mathbb{Q}}(r_1) = \overline{\mathbb{Q}}_\ell$ (Schur), so some character $\varphi \neq \mathbf{1}$ of $G_\mathbb{Q}/H$ has a non-zero Y with $r_1(g)Y = \varphi(g) Y r_1(g)$ for all g; Y is a non-zero homomorphism between the irreducible representations r_1 and $r_1 \otimes \varphi$, hence invertible, and $Y^{-1} r_1(g) Y = \varphi(g) r_1(g)$. Taking traces, $\mathrm{tr}\, r_1(g) = 0$ whenever $\varphi(g) \neq 1$. Fix g_1 with $\varphi(g_1) \neq 1$; then $\mathrm{tr}\, r(g_1 h) = \mathrm{tr}\, r_1(g_1 h) = 0$ for all $h \in H$. By (Ot2), $\mathrm{tr}\, r(z) \equiv \sum_i \kappa_i(z) \pmod{\mathfrak{m}_\ell}$, so $\sum_i \bar{\kappa}_i(g_1)\bar{\kappa}_i(h) = 0$ in $\overline{\mathbb{F}}_\ell$ for all $h \in H$. By (D) and Artin's independence of characters, $\bar{\kappa}_i(g_1) = 0$ for all i, which is absurd. (d) By (Jq5), r_1 satisfies (Sy2) for ι. Condition (Sy2) depends only on the isomorphism class of the representation (irreducibility of restrictions, the relation $\leftrightarrow_\iota$ through characteristic polynomials, and the witness conditions, which are conditions on the representation up to isomorphism); so r satisfies (Sy2) for ι. □

Proposition 51.20. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with odd prime ℓ. Assume that for every test field (K, V, a) of characteristic ℓ, every point r of $\mathfrak{D}$ with values in V, every $k \geq 0$, all $g_1, \ldots, g_k \in G_\mathbb{Q}$ and all indices with $\sum_{s=1}^k \mathrm{wt}_4(i_s, j_s) = (0,0)$, the product $x = r(g_1)_{i_1 j_1} \cdots r(g_k)_{i_k j_k}$ satisfies $x^{\ell^n} = x$ for some integer $n \geq 1$. Then $\mathfrak{D}$ is finite in weight zero.*

Proof. Let $\sigma, \mathrm{wt}, \mathfrak{a}$, $R = \mathcal{O}[[X_\sigma]]/\mathfrak{a}$ and r_u be as in Proposition 51.21. Write $P = \mathcal{O}[[X_s : s \in \sigma]]$, $P_w = \mathcal{O}[[X_\sigma]]_w$, $\mathfrak{m}_P$ its maximal ideal and $\pi\colon P \to R$ the quotient map. As the elements of $\mathfrak{a}$ have constant term 0, $R \in \mathrm{CNL}_\mathcal{O}$ with $\mathfrak{m}_R = \pi(\mathfrak{m}_P)$. Let R^0 be the $\mathcal{O}$-subalgebra of R generated by the products $r_\mathrm{u}(g_1)_{i_1 j_1} \cdots r_\mathrm{u}(g_k)_{i_k j_k}$ with $\sum_s \mathrm{wt}_4(i_s, j_s) = (0,0)$.

Step 1: reduction to R. Let r_A be a symplectic Eisenstein lift to $A \in \mathrm{CNL}_\mathcal{O}$. By (Uw2), $r_A = f \circ r_\mathrm{u}$ entrywise for an $\mathcal{O}$-algebra homomorphism f, so the products of weight $(0,0)$ of r_A are the images of those of r_u, and the subalgebra they generate is $f(R^0)$. It therefore suffices to show that R^0 is a finitely generated $\mathcal{O}$-module.

Step 2: the weight-zero projection. If $h \in \mathfrak{a}$ then $h_{(0)} \in \mathfrak{a}$, so $\mathrm{E}(\pi(h')) = \pi(h'_{(0)})$ is a well-defined $\mathcal{O}$-linear map $\mathrm{E}\colon R \to R$ with image $C = \pi(P_0)$. For $p \in P_0$ and $h' \in P$, $(ph')_{(0)} = p\, h'_{(0)}$; hence $\mathrm{E}(cy) = c\,\mathrm{E}(y)$ for $c \in C$, $y \in R$, and $\mathrm{E}(c) = c$. Consequently $\mathfrak{p}R \cap C = \mathfrak{p}$ for every ideal $\mathfrak{p}$ of C: if $\sum_i a_i y_i \in C$ with $a_i \in \mathfrak{p}$, it equals $\mathrm{E}(\sum_i a_i y_i) = \sum_i a_i \mathrm{E}(y_i) \in \mathfrak{p}$. A power series in $\mathfrak{m}_P^N$ has its coefficient of X^n in $\mathfrak{m}_\mathcal{O}^{\max(0, N - |n|)}$, and $h'_{(0)}$ keeps some of these coefficients, so $\mathrm{E}(\mathfrak{m}_R^N) \subseteq \mathfrak{m}_R^N$: E is continuous.

Step 3: C is a complete local Noetherian $\mathcal{O}$-algebra. By Gordan's lemma the monoid $\{n \in \mathbb{N}^\sigma : \sum_s n_s \mathrm{wt}(s) = (0,0)\}$ is generated by finitely many non-zero $n^{(1)}, \ldots, n^{(k)}$. The continuous homomorphism $\mathcal{O}[[Y_1, \ldots, Y_k]] \to P$, $Y_i \mapsto X^{n^{(i)}} \in \mathfrak{m}_P$, has image P_0: the image lies in the closed subring P_0, and for $h' = \sum_n c_n X^n \in P_0$, choosing $\alpha(n) \in \mathbb{N}^k$ with $\sum_i \alpha(n)_i n^{(i)} = n$ (an injective choice), $\sum_n c_n Y^{\alpha(n)}$ maps to h'. So $C = \pi(P_0)$ is a quotient of $\mathcal{O}[[Y_1, \ldots, Y_k]]$: complete, local, Noetherian, with residue field $\mathbb{F}_\mathcal{O}$ and $\mathfrak{m}_C = C \cap \mathfrak{m}_R$.

Step 4: $R^0 \subseteq C \subseteq \overline{R^0}$ (closure in R). By (Uw1), a product $r_\mathrm{u}(g_1)_{i_1 j_1} \cdots r_\mathrm{u}(g_k)_{i_k j_k}$ of total weight w is $\pi(h')$ with $h' \in P_w$; so it lies in C if $w = (0,0)$, and E maps it to itself if $w = (0,0)$ and to 0 otherwise. Hence $R^0 \subseteq C$, and E maps the $\mathcal{O}$-subalgebra R_2 generated by all entries of r_u into R^0. We show that R_2 is dense in R; then $C = \mathrm{E}(R) \subseteq \overline{\mathrm{E}(R_2)} \subseteq \overline{R^0}$ by continuity. Fix $N \geq 1$, let $Q = R/\mathfrak{m}_R^N$, a finite local ring in $\mathrm{CNL}_\mathcal{O}$, and $Q_0 \subseteq Q$ the image of R_2. An element of

Q_0 outside $\mathfrak{m}_Q$ is a unit of finite order of Q, so its inverse is a power of it; thus Q_0 is local with residue field $\mathbb{F}_{\mathcal{O}}$, and $Q \otimes_{Q_0} Q$ is a finite local ring with residue field $\mathbb{F}_{\mathcal{O}}$ (modulo the nilpotent ideal generated by $\mathfrak{m}_Q \otimes 1$ and $1 \otimes \mathfrak{m}_Q$ it is $\mathbb{F}_{\mathcal{O}} \otimes_{\mathbb{F}_{\mathcal{O}}} \mathbb{F}_{\mathcal{O}}$), so it lies in $\mathrm{CNL}_{\mathcal{O}}$. The two $\mathcal{O}$-algebra homomorphisms $R \to Q \otimes_{Q_0} Q$, $x \mapsto \bar{x} \otimes 1$ and $x \mapsto 1 \otimes \bar{x}$, agree on the entries of r_{u}, so they carry r_{u} to the same symplectic Eisenstein lift, and they are equal by the uniqueness in (Uw2). Let $Q_1 = Q/Q_0$ and $p \colon Q \to Q_1$ the projection; the map $Q \otimes_{Q_0} Q \to Q \otimes_{Q_0} Q_1$, $x \otimes y \mapsto x \otimes p(y)$, sends $\bar{x} \otimes 1$ to 0 and $1 \otimes \bar{x}$ to $1 \otimes p(\bar{x})$, so $Q \otimes_{Q_0} Q_1 = 0$. Then $\mathbb{F}_{\mathcal{O}} \otimes_{Q_0} Q_1 = (Q/\mathfrak{m}_Q) \otimes_Q (Q \otimes_{Q_0} Q_1) = 0$, and $Q_1 = 0$ by Nakayama's lemma. So $R_2 + \mathfrak{m}_R^N = R$ for every N.

Step 5: $C/\mathfrak{m}_{\mathcal{O}}C$ is Artinian. Suppose not. Then some minimal prime $\mathfrak{p}_0$ of $\mathfrak{m}_{\mathcal{O}}C$ has $\dim C/\mathfrak{p}_0 \geq 1$; applying Lemma 47.7 to the complete local domain $C/\mathfrak{p}_0$ and the image of an element of $\mathfrak{m}_C \setminus \mathfrak{p}_0$ gives a prime $\mathfrak{p} \supseteq \mathfrak{p}_0$ of C with $\dim C/\mathfrak{p} = 1$; choose $t \in \mathfrak{m}_C \setminus \mathfrak{p}$. By Step 2, $\mathfrak{p}R$ does not meet $C \setminus \mathfrak{p}$; an ideal $\mathfrak{Q}$ of R maximal among those containing $\mathfrak{p}R$ and disjoint from $C \setminus \mathfrak{p}$ is prime with $\mathfrak{Q} \cap C = \mathfrak{p}$. The ring $R/\mathfrak{Q}$ is a complete local Noetherian domain of characteristic ℓ ($\ell \in \mathfrak{m}_{\mathcal{O}}C \subseteq \mathfrak{p}$) in which t is a non-zero non-unit, so $\dim R/\mathfrak{Q} \geq 1$, and Lemma 47.7 gives a prime $\mathfrak{Q}' \supseteq \mathfrak{Q}$ with $t \notin \mathfrak{Q}'$ and $\dim R/\mathfrak{Q}' = 1$. Let $\tilde{A}$ be the normalisation of $R/\mathfrak{Q}'$, a complete discrete valuation ring of characteristic ℓ, finite over $R/\mathfrak{Q}'$, with finite residue field, and $\psi \colon R \to R/\mathfrak{Q}' \to \tilde{A}$. Then $\psi(\mathfrak{m}_R) \subseteq \mathfrak{m}_{\tilde{A}}$, $\psi(\mathfrak{m}_R^N) \subseteq \mathfrak{m}_{\tilde{A}}^N$, $\psi(\mathfrak{m}_{\mathcal{O}}) = 0$, and $\psi(t)$ is a non-zero element of $\mathfrak{m}_{\tilde{A}}$. *A test field.* Let K be an algebraic closure of $\mathrm{Frac}(\tilde{A})$ and V a valuation ring of K dominating $\tilde{A}$ (Chevalley). Then $V \cap \mathrm{Frac}(\tilde{A})$ is a valuation ring containing $\tilde{A}$ and not the whole field (a uniformiser lies in $\mathfrak{m}_V$), so it is $\tilde{A}$, and the inclusion $j \colon \tilde{A} \to V$ is injective and local. The algebraic closure $\bar{F}$ of $\mathbb{F}_{\mathcal{O}} = \psi(\mathcal{O})$ in K lies in V (its elements are roots of unity or 0). The residue field of $\overline{\mathbb{Z}}_\ell$ is an algebraic closure of $\mathbb{F}_{\mathcal{O}}$; choose an $\mathbb{F}_{\mathcal{O}}$-isomorphism from it onto $\bar{F}$ and let $a \colon \overline{\mathbb{Z}}_\ell \to \bar{F} \subseteq V$ be the composite with reduction; then $a(\mathfrak{m}_\ell) = 0$ and a restricts to ψ on $\mathcal{O}$. The residue field k_V of V is algebraically closed and algebraic over the finite residue field of $\tilde{A}$, so $\bar{F} \to k_V$ is onto: every element of V is congruent modulo $\mathfrak{m}_V$ to an element of $a(\overline{\mathbb{Z}}_\ell)$. So (K, V, a) is a test field of characteristic ℓ. *A point.* Put $r = j \circ \psi \circ r_{\mathrm{u}}$. The homomorphism $j \circ \psi \colon R \to V$ restricts to a on $\mathcal{O}$ and maps $\mathfrak{m}_R$ into $\mathfrak{m}_V$, and $(r, 1)$ satisfies (Ot1) with $(\Lambda, j) = (\tilde{A}, j)$: K is algebraic over $\tilde{A}$, the entries of r lie in $j(\tilde{A})$, and $g \mapsto \psi(r_{\mathrm{u}}(g)_{ab})$ is continuous because r_{u} is and $\psi(\mathfrak{m}_R^N) \subseteq \mathfrak{m}_{\tilde{A}}^N$. So r is a point of $\mathfrak{D}$ with values in V. *The contradiction.* Let $F' = \{y \in \tilde{A} : y^{\ell^n} = y \text{ for some } n \geq 1\}$. It is a subring ($y \mapsto y^{\ell^m}$ is a ring endomorphism in characteristic ℓ, and a common m serves for two elements), it contains $\psi(\mathcal{O}) = \mathbb{F}_{\mathcal{O}}$, and it injects into the residue field of $\tilde{A}$: if $y, y' \in F'$ have the same residue, $d = y - y' \in \mathfrak{m}_{\tilde{A}}$ satisfies $d = d^{\ell^{mk}} \in \mathfrak{m}_{\tilde{A}}^{\ell^{mk}}$ for all k, so $d = 0$. So F' is finite, hence closed in $\tilde{A}$, and its non-zero elements are units. By the hypothesis, applied to (K, V, a) and r, ψ maps every product of weight $(0, 0)$ of r_{u} into F', hence $\psi(R^0) \subseteq F'$, and by continuity $\psi(\overline{R^0}) \subseteq F'$. By Step 4, $t \in C \subseteq \overline{R^0}$, so $\psi(t) \in F'$ is a non-zero non-unit: a contradiction.

Step 6: conclusion. $C/\mathfrak{m}_{\mathcal{O}}C$ is an Artinian local ring with finite residue field, hence finite, and its maximal ideal is nilpotent, so the $\mathfrak{m}_C$-adic and $\mathfrak{m}_{\mathcal{O}}$-adic topologies on C coincide; C is complete and separated, so by Nakayama's lemma for complete modules C is a finitely generated $\mathcal{O}$-module. As $\mathcal{O}$ is Noetherian and $R^0 \subseteq C$, R^0 is a finitely generated $\mathcal{O}$-module; by Step 1, $\mathfrak{D}$ is finite in weight zero. □

Proposition 51.21. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$. There are a finite set σ, a map* $\mathrm{wt} \colon \sigma \to \mathbb{Z}^2$*, a homogeneous ideal $\mathfrak{a}$ of $\mathcal{O}[[X_s : s \in \sigma]]$ all of whose elements have constant term* 0*, and, with $R = \mathcal{O}[[X_s : s \in \sigma]]/\mathfrak{a}$ and R_w the image of $\mathcal{O}[[X_\sigma]]_w$, a symplectic Eisenstein lift r_{u} to R such that:*

(Uw1) *$r_{\mathrm{u}}(g)_{ij} \in R_{\mathrm{wt}_4(i,j)}$ for all $g \in G_{\mathbb{Q}}$ and $1 \leq i, j \leq 4$;*

(Uw2) *for every $A \in \mathrm{CNL}_{\mathcal{O}}$ and every symplectic Eisenstein lift r_A to A there is exactly one $\mathcal{O}$-algebra homomorphism $f\colon R \to A$ with $f(r_{\mathrm{u}}(g)_{ij}) = r_A(g)_{ij}$ for all g, i, j.*

Proof. Throughout, A denotes an object of $\mathrm{CNL}_{\mathcal{O}}$. We use that $\mathcal{O}$-algebra homomorphisms between objects of $\mathrm{CNL}_{\mathcal{O}}$ are morphisms, and that for a symplectic Eisenstein lift r to A and an $\mathcal{O}$-algebra homomorphism $f\colon A \to A'$ in $\mathrm{CNL}_{\mathcal{O}}$, $f \circ r$ is a symplectic Eisenstein lift (apply f to (Se1)–(Se3); in (Se4) replace y by $f \circ y$).

Step 1: the group of the problem and Mazur's ring. Let $N_T \subseteq G_{\mathbb{Q}}$ be the closed normal subgroup generated by the sets $I'_{\bar{w}} = \{g \in I_{\bar{w}} : \bar{r}_0(g) = 1\}$, $\bar{w}$ a place of $\overline{\mathbb{Q}}$ above a prime not in T, and $\Gamma = G_{\mathbb{Q}}/N_T$; $\bar{r}_0$ factors through Γ. The kernel of a lift is a closed normal subgroup, so a lift satisfies (Se2) if and only if it factors through Γ, and lifts satisfying (Se2) are the same as continuous homomorphisms $\Gamma \to \mathrm{GL}_4(A)$ reducing to $\bar{r}_0$. Γ satisfies Φ_ℓ: let $\Gamma' \subseteq \Gamma$ be open with preimage $H \supseteq N_T$, open in $G_{\mathbb{Q}}$, and put $H_0 = H \cap \ker \bar{r}_0$, open and normal in H, with fixed field L, a number field. For a continuous $f\colon H \to \mathbb{F}_\ell$ vanishing on N_T, the restriction $f|_{H_0}$ vanishes on the inertia groups $I_{\bar{w}} \cap H_0 \subseteq I'_{\bar{w}}$ of $G_L = H_0$ at places above primes not in T, so it factors through G_{L,T_L}, T_L the places of L above $T \cup \{\infty\}$. Restriction to H_0 has kernel inside $\mathrm{Hom}(H/H_0, \mathbb{F}_\ell)$, finite, and image inside $\mathrm{Hom}_{\mathrm{cont}}(G_{L,T_L}, \mathbb{F}_\ell)$, finite by Theorem B.71, item 2; so $\mathrm{Hom}_{\mathrm{cont}}(\Gamma', \mathbb{F}_\ell)$ is finite. By Theorem B.71, item 1, the lifts of $\bar{r}_0$ satisfying (Se2) are represented by some $R^{\square} \in \mathrm{CNL}_{\mathcal{O}}$ with a lift $r^{\square}$, and for each $v \in T$ the lifts of $\bar{r}_0|_{D_v}$ are represented by some $(R_v^{\square}, r_v^{\square})$. For a lift r write c_r for its classifying morphism. Let $\psi_v = c_{r^{\square}|_{D_v}} : R_v^{\square} \to R^{\square}$.

Step 2: (Se4) is a closed condition. Fix $v \in T$, let (B_v, ρ_v, x_v) be as in Proposition 51.22 and $\varphi_v = c_{\rho_v}\colon R_v^{\square} \to B_v$. (2a) φ_v is surjective. Its image C is a quotient of $R_v^{\square}$, a complete Noetherian local ring with $\mathfrak{m}_{\mathcal{O}} \subseteq \mathfrak{m}_C$ and residue field $\mathbb{F}_{\mathcal{O}}$, containing the entries of ρ_v. Every $b \in B_v$ is $o + m$ with $o \in \mathcal{O}$, $m \in \mathfrak{m}_{B_v}$, and o is unique modulo $\mathfrak{m}_{\mathcal{O}}$; so every $\mathbb{F}_{\mathcal{O}}$-linear form λ on $V = \mathfrak{m}_{B_v}/(\mathfrak{m}_{B_v}^2 + \mathfrak{m}_C B_v)$ gives a well-defined map $D(o + m) = \lambda(m)$ with $D(bb') = \bar{b}D(b') + \bar{b}'D(b)$ and $D(C) = 0$. The two $\mathcal{O}$-algebra homomorphisms $B_v \to \mathbb{F}_{\mathcal{O}}[\epsilon]$ ($\epsilon^2 = 0$), $b \mapsto \bar{b}$ and $b \mapsto \bar{b} + \epsilon D(b)$, agree on the entries of ρ_v, hence are equal by (Uc1); so $\lambda = 0$, $V = 0$ and $\mathfrak{m}_{B_v} = \mathfrak{m}_{B_v}^2 + \mathfrak{m}_C B_v$. By Nakayama's lemma $B_v/\mathfrak{m}_C B_v = \mathbb{F}_{\mathcal{O}}$, and as B_v is separated for the $\mathfrak{m}_C$-adic topology ($\mathfrak{m}_C B_v \subseteq \mathfrak{m}_{B_v}$), Nakayama's lemma for complete modules over C gives $B_v = C \cdot 1 = C$. (2b) Let $\mathfrak{P}_v = \ker \varphi_v$. For a lift ρ of $\bar{r}_0|_{D_v}$ to A the following are equivalent: (i) $\rho = y \circ \rho_B$ for some (B, ρ_B, x, y) as in (Se4); (ii) $\rho = f \circ \rho_v$ for an $\mathcal{O}$-algebra homomorphism $f\colon B_v \to A$; (iii) $c_\rho(\mathfrak{P}_v) = 0$. (i)⇒(ii) is (Uc2); (ii)⇒(i) with (B_v, ρ_v, x_v, f); (ii)⇒(iii) since $c_\rho = f \circ \varphi_v$ by uniqueness of classifying morphisms; (iii)⇒(ii) since $c_\rho = f \circ \varphi_v$ for the homomorphism f induced on $R_v^{\square}/\mathfrak{P}_v \cong B_v$, and then $f \circ \rho_v = f \circ \varphi_v \circ r_v^{\square} = \rho$.

Step 3: the universal ring. Let $\mathfrak{a}^{\square} \subseteq R^{\square}$ be the ideal generated by the entries of ${}^t r^{\square}(g) J_4 r^{\square}(g) - \theta(g) J_4$ ($g \in G_{\mathbb{Q}}$, $r^{\square}$ viewed on $G_{\mathbb{Q}}$), the entries $r^{\square}(\phi)_{ij}$ with $\lambda_i \neq \lambda_j$, the entries $(1,3)$ and $(3,1)$ of $r^{\square}(g_0) + \theta(g_0) r^{\square}(g_0^{-1})$, and $\psi_v(\mathfrak{P}_v)$ for $v \in T$. For a lift r to A satisfying (Se2), c_r applied entrywise to $r^{\square}$ is r, and $c_r \circ \psi_v = c_{r|_{D_v}}$; so by Step 2, r is a symplectic Eisenstein lift if and only if $c_r(\mathfrak{a}^{\square}) = 0$. The representation r_0 is a symplectic Eisenstein lift to $\mathcal{O}$: on $I_{\bar{w}}$ above a prime $p \notin T$ ($p \neq \ell$) the characters χ_{cyc}, ω_ℓ are trivial and χ, ξ have order prime to ℓ (the ℓ-power part of χ is unramified at $p \notin S_\ell(\chi)$, and ξ has order prime to ℓ), so $\kappa_i(g)$ is a root of unity of order prime to ℓ, equal to 1 when it is 1 modulo $\mathfrak{m}_{\mathcal{O}}$: this is (Se2). (Se1) and (Se3) hold as r_0 is diagonal with similitude θ. For (Se4) take $B = \mathcal{O}$, $\rho_B = r_0|_{D_v}$, $x = y = \mathrm{id}$: at ℓ, $\rho_B(s_0)$ has characteristic polynomial $\prod_j (X - \kappa_j(s_0))$, Π_i is diagonal with only the entry (i,i) possibly non-zero, so $\Pi_j \rho_B(g) \Pi_i = 0$ for $i < j$ and $\Pi_i(\rho_B(s) - \kappa_i(s))\Pi_i = 0$; at $v \in T \setminus \{\ell\}$ and $g \in I_v$ with $\chi(g) = 1$, $\chi_{\mathrm{cyc}}(g) = \omega_\ell(g) = \xi(g) = 1$ (ξ is unramified at v by (Sg1)), so $r_0(g) = 1$. Hence c_{r_0} kills $\mathfrak{a}^{\square}$, $R = R^{\square}/\mathfrak{a}^{\square}$ is non-zero and lies in $\mathrm{CNL}_{\mathcal{O}}$, and the image r_{u} of $r^{\square}$ in R, viewed on $G_{\mathbb{Q}}$, is a symplectic Eisenstein lift. If r_A is a symplectic Eisenstein lift to

A, then c_{r_A} kills $\mathfrak{a}^\square$ and induces $f\colon R \to A$ with $f \circ r_{\mathrm{u}} = r_A$; if f' is another, then $f \circ q$ and $f' \circ q$ ($q\colon R^\square \to R$) classify the same lift, so $f = f'$. This is the universality (Uw2) for (R, r_{u}); let $\beta\colon R \to \mathcal{O}$ be the homomorphism with $\beta \circ r_{\mathrm{u}} = r_0$.

Step 4: torus stability. Let r be a symplectic Eisenstein lift to A, $t_1, t_2 \in 1 + \mathfrak{m}_A$, $X = \mathrm{T}_4(t_1, t_2)$ and $r^t(g) = Xr(g)X^{-1}$. Then r^t is a lift ($X \equiv 1$), and a symplectic Eisenstein lift. X has similitude $t_1 t_2$, and if X has similitude $\nu \in A^\times$ and Y similitude θ', then ${}^tX^{-1}J_4X^{-1} = \nu^{-1}J_4$ and ${}^t(XYX^{-1})J_4(XYX^{-1}) = \nu\theta'\,{}^tX^{-1}J_4X^{-1} = \theta' J_4$: this gives (Se1). (Se2) is clear, and (Se3) holds because the entry (i,j) of XYX^{-1} is $t_1^{\lambda_i-\lambda_j}t_2^{\mu_i-\mu_j}Y_{ij}$ and $h(r^t) = Xh(r)X^{-1}$. For (Se4) at v, let (B, ρ_B, x, y) be as there for r, $B' = B[[u_1, u_2]]$ (a domain in $\mathrm{CNL}_{\mathcal{O}}$), $X' = \mathrm{T}_4(1+u_1, 1+u_2)$ and $\rho_{B'} = X'\rho_B X'^{-1}$, a lift of $\bar{r}_0|_{D_v}$. It satisfies the local condition at v: the similitude is preserved as above; for $v = \ell$, $\rho_{B'}(s_0)$ has the characteristic polynomial of $\rho_B(s_0)$, its Π_i are $X'\Pi_i X'^{-1}$, and the ordinary identities are conjugated by X'; for $v \neq \ell$, $\rho_B(g) = 1$ implies $\rho_{B'}(g) = 1$. With $x'\colon B' \to \mathcal{O}$ extending x by $u_i \mapsto 0$ and $y'\colon B' \to A$ extending y by $u_i \mapsto t_i - 1 \in \mathfrak{m}_A$, $x' \circ \rho_{B'} = x \circ \rho_B = r_0|_{D_v}$ and $y' \circ \rho_{B'} = X(y \circ \rho_B)X^{-1} = r^t|_{D_v}$. Applying this to the lift r_{u} composed with $R \to R[[T]]$ and $(t_1, t_2) = (1+T, 1)$, respectively $(1, 1+T)$, and (Uw2) for $R[[T]] \in \mathrm{CNL}_{\mathcal{O}}$, gives morphisms $\varrho_1, \varrho_2\colon R \to R[[T]]$ with

$$\varrho_1\big(r_{\mathrm{u}}(g)_{ij}\big) = (1+T)^{\lambda_i-\lambda_j} r_{\mathrm{u}}(g)_{ij}, \qquad \varrho_2\big(r_{\mathrm{u}}(g)_{ij}\big) = (1+T)^{\mu_i-\mu_j} r_{\mathrm{u}}(g)_{ij}. \tag{51.6}$$

Step 5: the presentation. Let $\Sigma_0 = G_{\mathbb{Q}} \times \{1, \dots, 4\}^2$ and, for $s = (g, i, j)$, $x_s = r_{\mathrm{u}}(g)_{ij} - r_0(g)_{ij} \in \mathfrak{m}_R$, $w_\lambda(s) = \lambda_i - \lambda_j$, $w_\mu(s) = \mu_i - \mu_j$. As $r_0(g)_{ij} = 0$ for $i \neq j$ and ϱ_1, ϱ_2 fix $\mathcal{O}$, (51.6) holds with x_s in place of $r_{\mathrm{u}}(g)_{ij}$. Apply Lemma 43.8 to R, $(x_s)_{s\in\Sigma_0}$, w_λ (values in $\{-1, 0, 1\}$) and ϱ_1: (Tc1) holds because morphisms f, f' agreeing on all x_s satisfy $f \circ r_{\mathrm{u}} = f' \circ r_{\mathrm{u}}$, so $f = f'$ by (Uw2); (Tc2) is (51.6). By (Tp1) there is a finite $\sigma \subseteq \Sigma_0$ such that the continuous $\mathcal{O}$-algebra homomorphism $\pi\colon P = \mathcal{O}[[X_s : s \in \sigma]] \to R$ with $\pi(X_s) = x_s$ is surjective. Then Lemma 43.10 applies to π twice: with $\rho = \varrho_1$ and the $\mathbb{Z}$-weights $w_\lambda|_\sigma$, and with $\rho = \varrho_2$ and $w_\mu|_\sigma$. Write $f^\lambda_{(a)}$, $f^\mu_{(b)}$ for the weight components for these $\mathbb{Z}$-weights and P^λ_a, P^μ_b for the corresponding sets. Put $\mathrm{wt}(s) = \mathrm{wt}_4(i,j) = (w_\lambda(s), w_\mu(s))$ and $\mathfrak{a} = \ker\pi$. A monomial has $\mathbb{Z}^2$-weight (a,b) if and only if it has λ-weight a and μ-weight b, so $f_{((a,b))} = (f^\lambda_{(a)})^\mu_{(b)}$. *$\mathfrak{a}$ is homogeneous*: for $f \in \mathfrak{a}$, $f^\lambda_{(a)} \in \mathfrak{a}$ by item 1 for ϱ_1, and then $(f^\lambda_{(a)})^\mu_{(b)} \in \mathfrak{a}$ by item 1 for ϱ_2. *Constant terms vanish*: $\beta \circ \pi$ is a continuous $\mathcal{O}$-algebra homomorphism $P \to \mathcal{O}$ with $\beta(x_s) = 0$, i.e. the constant term map, so $f(0) = \beta(\pi(f)) = 0$ for $f \in \mathfrak{a}$. *(Uw1)*: fix g, i, j, $y = r_{\mathrm{u}}(g)_{ij}$ and $(a,b) = \mathrm{wt}_4(i,j)$. By (51.6) and item 2 for both tori, $y = \pi(h) = \pi(h')$ with $h \in P^\lambda_a$, $h' \in P^\mu_b$. Then $h - h' \in \mathfrak{a}$, so by homogeneity $(h-h')^\lambda_{(a)} = h - h'_{((a,b))}$ and $(h-h')_{((a,b))} = h_{((a,b))} - h'_{((a,b))}$ lie in $\mathfrak{a}$ (note $(h')^\lambda_{(a)} = h'_{((a,b))}$ as $h' \in P^\mu_b$); subtracting, $h - h_{((a,b))} \in \mathfrak{a}$, and $y = \pi(h_{((a,b))})$ lies in the image of $\mathcal{O}[[X_\sigma]]_{(a,b)}$. Transporting r_{u} through the isomorphism $\mathcal{O}[[X_s : s \in \sigma]]/\mathfrak{a} \cong R$ induced by π gives the finite set σ, the map wt, the homogeneous ideal $\mathfrak{a}$ with constant terms 0 and the symplectic Eisenstein lift satisfying (Uw1) and (Uw2). □

Proposition 51.22. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$ and set T, and let $v \in T$. There are a domain $B_v \in \mathrm{CNL}_{\mathcal{O}}$, a lift ρ_v of $\bar{r}_0|_{D_v}$ to B_v satisfying the local condition at v, and an $\mathcal{O}$-algebra homomorphism $x_v\colon B_v \to \mathcal{O}$ with $x_v \circ \rho_v = r_0|_{D_v}$, such that:*

(Uc1) *for every $A \in \mathrm{CNL}_{\mathcal{O}}$, any two $\mathcal{O}$-algebra homomorphisms $f, f'\colon B_v \to A$ with $f \circ \rho_v = f' \circ \rho_v$ are equal;*

(Uc2) *for every $A \in \mathrm{CNL}_{\mathcal{O}}$, every domain $B \in \mathrm{CNL}_{\mathcal{O}}$, every lift ρ_B of $\bar{r}_0|_{D_v}$ to B satisfying the local condition at v, and all $\mathcal{O}$-algebra homomorphisms $x\colon B \to \mathcal{O}$ and $y\colon B \to A$ with $x \circ \rho_B = r_0|_{D_v}$, there is an $\mathcal{O}$-algebra homomorphism $f\colon B_v \to A$ with $f \circ \rho_v = y \circ \rho_B$.*

Proof. Step 1: the universal ring of the local condition. $r_0|_{D_v}$ satisfies the local condition at v: $r_0(g)$ has similitude $\theta(g)$; for $v = \ell$, r_0 is diagonal, so its characteristic polynomial at s_0 is $\prod_j(X - \kappa_j(s_0))$, each Π_i is diagonal and supported on the entry (i,i), and the identities of the ordinary condition hold; for $v \neq \ell$, on I_v the characters χ_{cyc}, ω_ℓ and ξ (Sg1) are trivial, so $\kappa_1 = \kappa_2 = \chi$ and $\kappa_3 = \kappa_4 = 1$ there, and $r_0(g) = 1$ for $g \in I_v$ with $\chi(g) = 1$. Let $(R_v^\square, \rho_v^\square)$ be Mazur's universal lift of $\bar{r}_0|_{D_v}$ and $\mathfrak{I} \subseteq R_v^\square$ the ideal generated by the entries of ${}^t\rho_v^\square(g)J_4\rho_v^\square(g) - \theta(g)J_4$ ($g \in D_v$) and, for $v = \ell$, by the coefficients of the characteristic polynomial of $\rho_v^\square(s_0)$ minus those of $\prod_j(X - \kappa_j(s_0))$ and the entries of $\Pi_j\rho_v^\square(g)\Pi_i$ ($i < j$) and of $\Pi_i(\rho_v^\square(s) - \kappa_i(s))\Pi_i$ ($s \in I_\ell$), for $v \neq \ell$ by the entries of $\rho_v^\square(g) - 1$ ($g \in I_v$, $\chi(g) = 1$). Then $R_v = R_v^\square/\mathfrak{I} \in \mathrm{CNL}_\mathcal{O}$ with the image ρ_v of $\rho_v^\square$ is universal: for every $A \in \mathrm{CNL}_\mathcal{O}$ and every lift ρ of $\bar{r}_0|_{D_v}$ to A satisfying the local condition at v there is exactly one $\mathcal{O}$-algebra homomorphism $f\colon R_v \to A$ with $f \circ \rho_v = \rho$ (the morphism $R_v^\square \to A$ classifying ρ kills $\mathfrak{I}$, and $\mathcal{O}$-algebra homomorphisms are morphisms). Let $x_v\colon R_v \to \mathcal{O}$ classify $r_0|_{D_v}$ and $\mathfrak{q}_v = \ker x_v$.

Step 2: the component. By (Yc5) of Proposition 45.6 if $v = \ell$, and of Proposition 45.16 if $v \neq \ell$, applied to (R_v, ρ_v, x_v), R_v has exactly one minimal prime ideal $\mathfrak{P}$ contained in $\mathfrak{q}_v$. Put $B_v = R_v/\mathfrak{P}$, a domain in $\mathrm{CNL}_\mathcal{O}$, and let $\bar{\rho}_v$ be the image of ρ_v, a lift of $\bar{r}_0|_{D_v}$ to B_v satisfying the local condition at v (its identities are images of those of ρ_v). As $\mathfrak{P} \subseteq \mathfrak{q}_v$, x_v induces $\bar{x}_v\colon B_v \to \mathcal{O}$ with $\bar{x}_v \circ \bar{\rho}_v = r_0|_{D_v}$.

Step 3: (Uc1). If $f, f'\colon B_v \to A$ satisfy $f \circ \bar{\rho}_v = f' \circ \bar{\rho}_v$, their composites with $R_v \to B_v$ classify the same lift $f \circ \bar{\rho}_v$ (continuous, f being local), so they are equal by universality; $R_v \to B_v$ is onto, so $f = f'$.

Step 4: (Uc2). Let B, ρ_B, x, y be as in (Uc2), and $\gamma\colon R_v \to B$ the homomorphism with $\gamma \circ \rho_v = \rho_B$. Then $x \circ \gamma$ classifies $x \circ \rho_B = r_0|_{D_v}$, so $x \circ \gamma = x_v$, and $\ker\gamma \subseteq \mathfrak{q}_v$. As B is a domain, $\ker\gamma$ is prime; it contains a minimal prime of R_v, which is contained in $\mathfrak{q}_v$ and so equals $\mathfrak{P}$. Hence γ factors as $\bar{\gamma} \circ (R_v \to B_v)$, and $f = y \circ \bar{\gamma}$ satisfies $f \circ \bar{\rho}_v = y \circ \rho_B$.

So $(B_v, \bar{\rho}_v, \bar{x}_v)$ has the required properties. □

Proposition 51.23. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum, (K, V, a) a test field and (r, u) an ordinary test representation of $\mathfrak{D}$ with values in V such that every $r(g)$ is block diagonal and the K-span of the matrices $(r(g)_{ab})_{a,b\in\{1,3\}}$, $g \in G_\mathbb{Q}$, is $\mathrm{M}_2(K)$. Then there is an ordinary test representation (r', u') of $\mathfrak{D}$ with values in V such that r' is a non-split block extension and $r'(g)$ and $r(g)$ have the same characteristic polynomial for every $g \in G_\mathbb{Q}$.*

Proof. Let (Λ, j) be as in (Ot1) for (r, u) and identify Λ with $j(\Lambda) \subseteq V$, with fraction field $E \subseteq K$. The entries of u and of all $r(g)$ lie in Λ, and $\det u \in \Lambda \cap V^\times = \Lambda^\times$ (j is local), so $u \in \mathrm{GL}_4(\Lambda)$. As $a(\mathfrak{m}_\ell) \subseteq \mathfrak{m}_V$, $\ell \in \mathfrak{m}_\Lambda$. Write $\nu(g) = a(\theta(g))$.

Step 1: the module $\mathsf{T} = \mathfrak{u}(\Lambda)$. (1a) Products: $\mathfrak{h}(R)\mathfrak{h}(R) = 0$, and $XYX' \in \mathfrak{h}(R)$ for X, X' block diagonal and $Y \in \mathfrak{h}(R)$. Since $\langle Yx, x'\rangle + \langle x, Yx'\rangle = 0$ characterises $\mathfrak{sp}_4$ and $\langle Xx, Xx'\rangle = \nu\langle x, x'\rangle$ for X of similitude ν, $XYX^{-1} \in \mathfrak{sp}_4$ for $Y \in \mathfrak{sp}_4$. Hence $\rho(g)Y = r(g)Yr(g)^{-1}$ defines an action of $G_\mathbb{Q}$ on T. The condition $Y \in \mathfrak{sp}_4$ means that J_4Y is symmetric; for $Y \in \mathfrak{h}(R)$, $J_4Y = Y_{41}E_{11} + Y_{43}E_{13} - Y_{21}E_{31} - Y_{23}E_{33}$, so $\mathfrak{u}(R) = \{Y \in \mathfrak{h}(R) : Y_{43} = -Y_{21}\}$, and T is free with basis $E_{41}, E_{23}, E_{21} - E_{43}$. For $Y \in \mathfrak{u}(R)$, $1 + Y$ has similitude 1: ${}^t(1+Y)J_4(1+Y) = J_4 + {}^tYJ_4Y$ and ${}^tYJ_4 = -J_4Y$, $Y^2 = 0$. By (Ot1), $g \mapsto \rho(g)Y$ modulo $\mathfrak{m}_\Lambda^n\mathsf{T}$ is locally constant. (1b) Complex conjugation. By (Sg1), $\Psi(c) = 1$ and $\xi(c) = -1$; $\chi_{\mathrm{cyc}}(c) = -1$ and $\omega_\ell(c) = -1$ ($\omega_\ell(c)^2 = 1$ and $\omega_\ell(c) \equiv \chi_{\mathrm{cyc}}(c)$ modulo ℓ, ℓ odd); so $\alpha(c) = -1$, $\theta(c) = -1$ and $r(c) \equiv \mathrm{diag}(-1, 1, -1, 1)$ modulo $\mathfrak{m}_V$ by (Ot2). The map $X\colon Y \mapsto r(c)Yr(c)^{-1}$ on the free Λ-module $\mathfrak{h}(\Lambda)$ satisfies $X^2 = 1$ and $X \equiv -1$ modulo $\mathfrak{m}_\Lambda$. As $2 \in \Lambda^\times$, $e = (X + 1)/2$ is an idempotent with image in $\mathfrak{m}_\Lambda\mathfrak{h}(\Lambda)$, so $e = e^n$ has image in $\mathfrak{m}_\Lambda^n\mathfrak{h}(\Lambda)$ for all n and $e = 0$. Hence $r(c)Yr(c)^{-1} = -Y$ for all $Y \in \mathfrak{h}(\Lambda)$, hence for all $Y \in \mathfrak{h}(K)$ by linearity; in particular $\rho(c) = -1$ on T. (1c) The element σ_ℓ. By

(Ot2) and since $r(\sigma_\ell)$ has entries in Λ, $\rho(\sigma_\ell)$ is, modulo $\mathfrak{m}_\Lambda$, diagonal on the basis of (1a) with entries $\bar\kappa_4/\bar\kappa_1$, $\bar\kappa_2/\bar\kappa_3$, $\bar\kappa_2/\bar\kappa_1 = \bar\kappa_4/\bar\kappa_3$ at σ_ℓ, all $\neq 1$ by (Ae1). So $\det(\rho(\sigma_\ell)-1) \in \Lambda^\times$ and $Y \mapsto \rho(\sigma_\ell)Y - Y$ is bijective on T; and $\chi_{\mathrm{cyc}}(\sigma_\ell) = 1$ by (Sg2). (1d) The submodule $\mathsf{P} = \{Y \in \mathsf{T} : u^{-1}Yu \text{ is strictly upper triangular}\}$ is saturated and $\rho(D_\ell)$-stable (by (Ot4), $u^{-1}r(h)u$ is upper triangular for $h \in D_\ell$). It is non-zero: let $F_n = \langle ue_1, \dots, ue_n\rangle_E$. The blocks of $r(\sigma_\ell)$ have characteristic polynomials $p_{\mathrm{A}}, p_{\mathrm{B}}$ over Λ whose roots reduce to $\bar\kappa_1(\sigma_\ell), \bar\kappa_3(\sigma_\ell)$ and to $\bar\kappa_2(\sigma_\ell), \bar\kappa_4(\sigma_\ell)$, distinct across the blocks by (Ae1); so they are coprime over E, and by the Cayley–Hamilton theorem $E_{\mathrm{A}} = q(r(\sigma_\ell))$ for a polynomial q over E. As $u^{-1}r(\sigma_\ell)u$ is upper triangular, $E_{\mathrm{A}}F_n \subseteq F_n$ and $F_n = (F_n \cap V_{\mathrm{A}}) \oplus (F_n \cap V_{\mathrm{B}})$; the line F_n/F_{n-1} comes from V_{A} or from V_{B}, and $r(\sigma_\ell)$ acts on it by $d_n(\sigma_\ell) \equiv \kappa_n(\sigma_\ell)$, a root of p_{A} or p_{B} respectively. By (Ae1) it comes from V_{A} for $n = 1, 3$ and from V_{B} for $n = 2, 4$. So $F_1 \subseteq V_{\mathrm{A}}$, $F_2 = F_1 \oplus L_2$ with $L_2 = F_2 \cap V_{\mathrm{B}}$ a line, and $F_3 = V_{\mathrm{A}} \oplus L_2$ (over E). Choose $x_0 \in V_{\mathrm{A}} \setminus F_1$ and $0 \neq b_0 \in L_2$, and let $Y_0 \in \mathrm{M}_4(E)$ kill V_{B} and F_1 and send x_0 to b_0. Then $Y_0 \in \mathfrak{h}(E)$, $Y_0 \neq 0$, and $Y_0F_1 = Y_0F_2 = 0$, $Y_0F_3 \subseteq L_2 \subseteq F_2$, $Y_0F_4 \subseteq F_2$: $u^{-1}Y_0u$ is strictly upper triangular. As u has similitude 1, $\langle ue_a, ue_b\rangle = (J_4)_{ab}$, so $F_2 \perp F_1$; the form $\beta(x,x') = \langle Y_0x, x'\rangle$ on V_{A} vanishes if $x \in F_1$ or $x' \in F_1$ ($Y_0x \in L_2 \subseteq F_2$), so it is a multiple of $(x,x') \mapsto \lambda(x)\lambda(x')$ for a linear form λ with kernel F_1, hence symmetric; with $\langle V_{\mathrm{B}}, V_{\mathrm{B}}\rangle = 0$ this gives $\langle Y_0x, x'\rangle + \langle x, Y_0x'\rangle = 0$ for all x, x', i.e. $Y_0 \in \mathfrak{u}(E)$. A non-zero multiple of Y_0 by an element of Λ lies in $\mathsf{P} \setminus \{0\}$. (1e) Let $N = \ker r \subseteq \ker \rho$. For a place of $\overline{\mathbb{Q}}$ above a prime $p \notin S$, the characters θ, Ψ, α are unramified at p (χ is a character modulo the integer of the datum, whose prime factors lie in S, ξ is unramified outside S by (Sg1), and $\chi_{\mathrm{cyc}}, \omega_\ell$ are unramified away from ℓ), so $\bar r_0$ is trivial on its inertia group, and r is trivial there by (Ot3), as $p \notin T$.

Step 2: a cocycle. By Step 1, Proposition 51.24 applies to ℓ, S, c, D_ℓ, $s = \sigma_\ell$, Λ, T, ρ, P and N ($\ell \in S$ by (Sg1), ℓ odd), giving $z_0 \neq 0$ and y_0. Let ϖ be a uniformiser of Λ, $z = \varpi z_0$ and $y = \varpi y_0$. Then $z \neq 0$, $z(g) \in \mathfrak{u}(\Lambda)$ has entries in $\mathfrak{m}_\Lambda$, $z(gh) = z(g) + r(g)z(h)r(g)^{-1}$, $z(c) = 0$, $z(g) = 0$ for $g \in \ker r$ in the inertia group of a place not above ℓ, $g \mapsto z(g)$ is continuous for the $\mathfrak{m}_\Lambda$-adic topology, and $w(h) = z(h) - (r(h)yr(h)^{-1} - y) \in \mathsf{P}$ for $h \in D_\ell$ (P is a submodule).

Step 3: the ordinary test representation (r', u'). Put $r'(g) = (1+z(g))r(g)$ and $u' = (1-y)u$. By the cocycle relation and $\mathfrak{h}\mathfrak{h} = 0$, $r'(g)r'(h) = (1 + z(g) + r(g)z(h)r(g)^{-1})r(gh) = r'(gh)$, so r' is a homomorphism to $\mathrm{GL}_4(V)$. (Ot1): the entries of $r'(g)$ and u' lie in Λ and are continuous in g. (Ot2): $r'(g) - a(r_0(g)) = (r(g) - a(r_0(g))) + z(g)r(g)$ has entries in $\mathfrak{m}_V$, and $1 + z(g)$ has similitude 1 (1a), so $r'(g)$ has similitude $\nu(g)$. (Ot3): both clauses concern g in an inertia group of a place not above ℓ with $r(g) = 1$; then $z(g) = 0$ and $r'(g) = 1$. (Ot4): $1 - y$ has similitude 1, so u' does. For $h \in D_\ell$, using $y^2 = yz(h) = 0$ and $(y + z(h))r(h)y = 0$ (1a),

$$(1+y)(1+z(h))r(h)(1-y) = \big(1 + w(h)\big)r(h), \qquad u'^{-1}r'(h)u' = \big(1 + u^{-1}w(h)u\big)\big(u^{-1}r(h)u\big),$$

with $(1-y)^{-1} = 1+y$. As $u^{-1}w(h)u$ is strictly upper triangular and $u^{-1}r(h)u$ upper triangular, $u'^{-1}r'(h)u'$ is upper triangular with the same diagonal entries as $u^{-1}r(h)u$, so (Ot4) holds for (r', u').

Step 4: r' is a non-split block extension with the characteristic polynomials of r. The rows of $z(g)$ indexed by A vanish and $z(g)r(g)$ has non-zero entries only in rows B and columns A; so $r'(g)$ agrees with $r(g)$ except in the entries (i,j), $i \in \mathrm{B}$, $j \in \mathrm{A}$. (i) Hence $r'(g)_{ij} = 0$ for $i \in \mathrm{A}$, $j \in \mathrm{B}$; and $r'(g)$ is block lower triangular with the diagonal blocks of $r(g)$, so $\det(x - r'(g)) = \det(x - r(g))$. (ii) Write $X_{\mathrm{A}}(g) = (r(g)_{ab})_{a,b\in\mathrm{A}}$, $X_{\mathrm{B}}(g) = (r(g)_{ab})_{a,b\in\mathrm{B}}$. The pairing of V_{A} with V_{B} has matrix $P_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the similitude relation for the block diagonal $r(g)$ gives ${}^tX_{\mathrm{A}}P_0X_{\mathrm{B}} = \nu P_0$, so $X_{\mathrm{B}} = \nu P_0^{-1}\,{}^tX_{\mathrm{A}}^{-1}P_0 = \nu \det(X_{\mathrm{A}})^{-1}X_{\mathrm{A}}$, since $P_0^{-1}\,{}^tMP_0$ is the adjugate of a 2×2 matrix M. So the K-span of the $X_{\mathrm{B}}(g)$, which are the $\{2,4\}$-blocks of

$r'(g)$, equals the K-span of the $X_{\mathrm{A}}(g)$, which is $\mathrm{M}_2(K)$ by hypothesis. (iii) Suppose $Y \in \mathfrak{h}(K)$ and $(1-Y)r'(g)(1+Y)$ is block diagonal for all g. Using $\mathfrak{h}\mathfrak{h} = 0$ and $r(g)Y \in \mathfrak{h}(K)$,

$$(1-Y)(1+z(g))r(g)(1+Y) = r(g) + \bigl(z(g) - Y + r(g)Yr(g)^{-1}\bigr)r(g),$$

whose entries outside the diagonal blocks are those of the second term; so $z(g) = Y - r(g)Yr(g)^{-1}$ for all g. At $g = c$, $0 = z(c) = Y + Y$ by (1b), so $Y = 0$ and $z = 0$, a contradiction. Hence r' is a non-split block extension, (r', u') is an ordinary test representation of $\mathfrak{D}$ with values in V, and $r'(g)$, $r(g)$ have the same characteristic polynomial for every g. □

Proposition 51.24. *Let ℓ be an odd prime, S a finite set of primes with $\ell \in S$, $c \in G_{\mathbb{Q}}$ a complex conjugation, $D_\ell \supseteq I_\ell$ the decomposition and inertia groups of a place of $\overline{\mathbb{Q}}$ above ℓ, and $s \in D_\ell$ with $\chi_{\mathrm{cyc}}(s) = 1$. Let Λ be a complete discrete valuation ring with finite residue field and $\ell \in \mathfrak{m}_\Lambda$, T a free Λ-module of finite rank, and $\rho\colon G_{\mathbb{Q}} \to \mathrm{Aut}_\Lambda(\mathsf{T})$ a homomorphism such that $g \mapsto \rho(g)x$ modulo $\mathfrak{m}_\Lambda^n\mathsf{T}$ is locally constant for all $n \geq 1$ and $x \in \mathsf{T}$. Let $\mathsf{P} \subseteq \mathsf{T}$ be a non-zero Λ-submodule, stable under $\rho(D_\ell)$ and saturated ($ax \in \mathsf{P}$ with $0 \neq a \in \Lambda$ implies $x \in \mathsf{P}$), and let $N \subseteq \ker\rho$ be a subgroup containing the inertia group of every place of $\overline{\mathbb{Q}}$ above a prime not in S. Assume that $x \mapsto \rho(s)x - x$ is bijective on T and that $\rho(c)x = -x$ for all $x \in \mathsf{T}$. Then there are a map $z\colon G_{\mathbb{Q}} \to \mathsf{T}$, not identically zero, and $y \in \mathsf{T}$ such that:*

1. *$g \mapsto z(g)$ modulo $\mathfrak{m}_\Lambda^n\mathsf{T}$ is locally constant for every $n \geq 1$, and $z(gh) = z(g) + \rho(g)z(h)$ for all $g, h \in G_{\mathbb{Q}}$;*
2. *$z(c) = 0$;*
3. *$z(g) = 0$ for every $g \in N$ that lies in the inertia group of a place of $\overline{\mathbb{Q}}$ not above ℓ;*
4. *$z(h) - (\rho(h)y - y) \in \mathsf{P}$ for every $h \in D_\ell$.*

Proof. Let ϖ be a uniformiser and E the fraction field of Λ, $q = \#\Lambda/\mathfrak{m}_\Lambda$, and $\varphi = \rho(s) - 1\colon \mathsf{T} \to \mathsf{T}$, bijective by hypothesis. Put $S_\infty = S \cup \{\infty\}$, $G_S = G_{\mathbb{Q},S_\infty}$, and let H_S be the kernel of $G_{\mathbb{Q}} \to G_S$; it contains the inertia group of every place of $\overline{\mathbb{Q}}$ above a prime not in S. For every prime $p \in S \setminus \{\ell\}$ fix a place $\bar{p}$ of $\overline{\mathbb{Q}}$ above p with decomposition and inertia groups $D_p \supseteq I_p$; through the places, D_p and D_ℓ are decomposition groups in the sense of Theorem B.29 and Theorem B.37, and we write $H^i(D_p, \cdot)$. For $n \geq 1$ put $\mathsf{T}_n = \mathsf{T}/\varpi^n\mathsf{T}$, let P_n be the image of P in T_n, and let $G_{\mathbb{Q}}$ act on T_n through ρ.

Step 0: the modules. (0a) T/P is torsion-free by saturation, hence free; P is free of some rank $d \geq 1$; and $\mathsf{P} \cap \varpi^k\mathsf{T} = \varpi^k\mathsf{P}$ for $k \geq 0$ by saturation. Hence $\mathsf{P}_n \cong \mathsf{P}/\varpi^n\mathsf{P}$ has $q^{nd} \geq 2^n$ elements, $\mathsf{T}_n/\mathsf{P}_n \cong (\mathsf{T}/\mathsf{P})/\varpi^n(\mathsf{T}/\mathsf{P})$, and $\bigcap_n(\mathsf{P} + \varpi^n\mathsf{T}) = \mathsf{P}$, as $\bigcap_n \varpi^n(\mathsf{T}/\mathsf{P}) = 0$. (0b) $\varphi(\mathsf{P}) = \mathsf{P}$ and φ induces a bijection of T/P. Indeed $\varphi(\mathsf{P}) \subseteq \mathsf{P}$ since $s \in D_\ell$; $\varphi \otimes E$ is injective on the finite-dimensional space $\mathsf{P} \otimes E$, hence bijective; for $p \in \mathsf{P}$ the element $x = \varphi^{-1}(p) \in \mathsf{T}$ lies in $\mathsf{P} \otimes E$, so $ax \in \mathsf{P}$ for some $0 \neq a \in \Lambda$, and $x \in \mathsf{P}$ by saturation. So φ maps P onto P; the induced map on T/P is onto, and injective: if $\varphi(x) \in \mathsf{P}$ then $\varphi(x) = \varphi(p)$ with $p \in \mathsf{P}$, and $x = p$. Consequently φ induces bijections of T_n and of $\mathsf{T}_n/\mathsf{P}_n$ for every n. (0c) Fix $n \geq 1$ and put $M = \mathsf{T}_n$. As $\ell \in \mathfrak{m}_\Lambda = \varpi\Lambda$, M is a finite $\mathbb{Z}/\ell^n\mathbb{Z}$-module. The kernel of the action of $G_{\mathbb{Q}}$ on M is the intersection of the finitely many sets $\{g : \rho(g)x \equiv x \bmod \varpi^n\mathsf{T}\}$, x running through representatives of M, which are open by the local constancy hypothesis; so the action is continuous. This kernel is a closed normal subgroup containing $\ker\rho \supseteq N$, hence the inertia groups above primes not in S, hence H_S: M is a continuous G_S-module. Since φ is bijective on M and on M/P_n, we have $H^0(G_S, M) = H^0(D_\ell, M) = H^0(D_\ell, M/\mathsf{P}_n) = 0$. Every complex conjugation is $c' = \tau c \tau^{-1}$ with $\tau \in G_{\mathbb{Q}}$, so $\rho(c') = -1$, and $x = \rho(c')x$ gives $2x = 0$, $x = 0$

(ℓ odd): $M^{c'=1}=0$. Finally $H^0(G_S,M^\vee)=0$: s acts trivially on μ_{ℓ^n} as $\chi_{\mathrm{cyc}}(s)=1$, so an s-invariant $f\in M^\vee$ satisfies $f(\rho(s)^{-1}x)=f(x)$ for all x, i.e. f vanishes on the image of $\rho(s)^{-1}-1=-\rho(s)^{-1}\varphi$, which is M.

Step 1: normalised Selmer cocycles. For $n\geq 1$ let Σ_n be the set of maps $\zeta\colon G_{\mathbb{Q}}\to \mathsf{T}_n$ such that

- ζ is locally constant, $\zeta(gh)=\zeta(g)+\rho(g)\zeta(h)$ for all g,h, and $\zeta(H_S)=0$;
- $\zeta(c)=0$;
- $\zeta(g)=0$ for every prime $p\in S\setminus\{\ell\}$ and every $g\in I_p$ with $\rho(g)=1$;
- there is $y\in\mathsf{T}_n$ with $\zeta(h)-(\rho(h)y-y)\in\mathsf{P}_n$ for every $h\in D_\ell$.

All four conditions are Λ-linear, so Σ_n is a Λ-module, and reduction modulo ϖ^n maps Σ_{n+1} into Σ_n (P_{n+1} maps onto P_n). A cocycle vanishing on the normal subgroup H_S, which acts trivially, satisfies $\zeta(gh)=\zeta(g)$ for $h\in H_S$, so it is inflated from a continuous cocycle of G_S; hence Σ_n injects into $Z^1(G_S,\mathsf{T}_n)$, which is finite because $H^1(G_S,\mathsf{T}_n)$ is finite (Theorem B.37, item 1) and the coboundaries are an image of T_n. So Σ_n is finite.

Step 2: $\#\Sigma_n\geq\#\mathsf{P}_n\geq 2^n$. Fix n and $M=\mathsf{T}_n$. For primes $p\in S\setminus\{\ell\}$ let $L_p=H^1_{\mathrm{ur}}(D_p,M)$, the kernel of restriction to I_p, and let L_ℓ be the image of $H^1(D_\ell,\mathsf{P}_n)\to H^1(D_\ell,M)$. Let $\mathrm{Sel}\subseteq H^1(G_S,M)$ be the classes whose restriction to D_p lies in L_p for every prime $p\in S$. (2a) For $p\in S\setminus\{\ell\}$, $\#L_p=\#H^0(D_p,M)$: by inflation–restriction $L_p\cong H^1(D_p/I_p,M^{I_p})$; $D_p/I_p\cong\hat{\mathbb{Z}}$ is topologically generated by a Frobenius Fr, so $\#L_p=\#(M^{I_p}/(\mathrm{Fr}-1)M^{I_p})=\#\ker(\mathrm{Fr}-1\mid M^{I_p})=\#M^{D_p}$, the middle equality because M^{I_p} is finite. (2b) $\#L_\ell\geq\#\mathsf{P}_n$: the kernel of $H^1(D_\ell,\mathsf{P}_n)\to H^1(D_\ell,M)$ is the image of $H^0(D_\ell,M/\mathsf{P}_n)=0$ (Step 0c; P_n is D_ℓ-stable), and by Theorem B.29, item 4, for the finite $\mathbb{Z}/\ell^n\mathbb{Z}$-module P_n and $[\mathbb{Q}_\ell:\mathbb{Q}_\ell]=1$, $\#H^1(D_\ell,\mathsf{P}_n)=\#H^0(D_\ell,\mathsf{P}_n)\,\#H^2(D_\ell,\mathsf{P}_n)\,\#\mathsf{P}_n\geq\#\mathsf{P}_n$. (2c) The count. Sel is the kernel of $H^1(G_S,M)\to\bigoplus_p H^1(D_p,M)/L_p$ (primes $p\in S$), so

$$\#\mathrm{Sel}\ \geq\ \#H^1(G_S,M)\Big/\prod_p[H^1(D_p,M):L_p].$$

By Theorem B.37, item 2, for $F=\mathbb{Q}$ (one real place, $|\#M|_\infty=\#M$, $H^0(\mathbb{R},M)=M^{c'=1}=0$) and $H^0(G_S,M)=0$: $\#H^1(G_S,M)=\#H^2(G_S,M)\,\#M$. By Theorem B.29, item 4, and (2a), $[H^1(D_p,M):L_p]=\#H^2(D_p,M)$ for $p\neq\ell$; with $H^0(D_\ell,M)=0$ and (2b), $[H^1(D_\ell,M):L_\ell]=\#H^2(D_\ell,M)\,\#M/\#L_\ell\leq\#H^2(D_\ell,M)\,\#M/\#\mathsf{P}_n$. By Theorem B.37, item 3, and $H^0(G_S,M^\vee)=0$, the restriction $H^2(G_S,M)\to\bigoplus_p H^2(D_p,M)$ is onto, so $\#H^2(G_S,M)\geq\prod_p\#H^2(D_p,M)$. Together, $\#\mathrm{Sel}\geq\#\mathsf{P}_n$. (2d) Sel injects into Σ_n. For a class with cocycle $w\in Z^1(G_S,M)$, viewed on $G_{\mathbb{Q}}$, put $\zeta(g)=w(g)-(\rho(g)x-x)$ with $x=-\frac{1}{2}w(c)$, so that $\zeta(c)=w(c)+2x=0$. Two cocycles of one class with value 0 at c differ by $g\mapsto\rho(g)x'-x'$ with $-2x'=0$, i.e. $x'=0$; so the class determines ζ, and distinct classes give distinct ζ. The first two conditions of Step 1 hold. For $p\in S\setminus\{\ell\}$ the restriction to I_p is trivial: $\zeta(g)=\rho(g)x_p-x_p$ on I_p for some x_p, which is 0 if $\rho(g)=1$. At ℓ: $\zeta|_{D_\ell}=u+(h\mapsto\rho(h)y-y)$ with $u\in Z^1(D_\ell,\mathsf{P}_n)$, $y\in M$, so $\zeta(h)-(\rho(h)y-y)=u(h)\in\mathsf{P}_n$. Hence $\#\Sigma_n\geq\#\mathrm{Sel}\geq\#\mathsf{P}_n\geq 2^n$.

Step 3: division by ϖ. Let $m\geq 2$ and let $\zeta\in\Sigma_m$ reduce to 0 in Σ_1. Multiplication by ϖ induces an injective Λ-linear, $G_{\mathbb{Q}}$-equivariant map $\iota\colon\mathsf{T}_{m-1}\to\mathsf{T}_m$ with image $\varpi\mathsf{T}_m$, the kernel of $\mathsf{T}_m\to\mathsf{T}_1$; so $\zeta=\iota\circ\zeta'$ for a unique $\zeta'\colon G_{\mathbb{Q}}\to\mathsf{T}_{m-1}$, and ζ' satisfies the first three conditions of Step 1. For the fourth, let y be as there for ζ. Reducing modulo ϖ, $\varphi(\bar y)\in\mathsf{P}_1$, so $\bar y\in\mathsf{P}_1$ by Step 0b, and $y=p+\iota(y')$ with $p\in\mathsf{P}_m$, $y'\in\mathsf{T}_{m-1}$. Then for $h\in D_\ell$

$$\iota(\zeta'(h)-(\rho(h)y'-y'))=\zeta(h)-(\rho(h)y-y)+(\rho(h)p-p)\in\mathsf{P}_m.$$

If $\iota(x) \in \mathsf{P}_m$, lift x to $\tilde{x} \in \mathsf{T}$: $\varpi\tilde{x} = p' + \varpi^m t$ with $p' \in \mathsf{P}$, $t \in \mathsf{T}$, so $p' \in \mathsf{P} \cap \varpi\mathsf{T} = \varpi\mathsf{P}$, and $\tilde{x} - \varpi^{m-1}t \in \mathsf{P}$, i.e. $x \in \mathsf{P}_{m-1}$. So $\zeta' \in \Sigma_{m-1}$, and $\zeta \mapsto \zeta'$ is injective: the kernel of $\Sigma_m \to \Sigma_1$ has at most $\#\Sigma_{m-1}$ elements.

Step 4: a compatible non-zero family. For $m \geq k$ let $\Sigma_{m,k} \subseteq \Sigma_k$ be the image of Σ_m. It decreases with m, and Σ_k is finite, so $\Sigma_{m,k} = \Sigma_k^\infty$ for all $m \geq m(k)$. Reduction maps Σ_{k+1}^∞ onto Σ_k^∞: for $x \in \Sigma_k^\infty$ and $m \geq \max(m(k), m(k+1))$, x is the image of some $w \in \Sigma_m$, whose image in Σ_{k+1} lies in $\Sigma_{m,k+1} = \Sigma_{k+1}^\infty$ and maps to x. Suppose $\Sigma_1^\infty = 0$. Then $\Sigma_m \to \Sigma_1$ is zero for $m \geq m_1 = \max(m(1), 2)$, so $\#\Sigma_m \leq \#\Sigma_{m-1}$ for all $m \geq m_1$ by Step 3, and $\#\Sigma_m \leq \#\Sigma_{m_1-1}$ for all $m \geq m_1$, contradicting $\#\Sigma_m \geq 2^m$ (Step 2). So there is $\zeta_1 \neq 0$ in Σ_1^∞, and lifting successively, $\zeta_k \in \Sigma_k^\infty$ with ζ_{k+1} reducing to ζ_k for all $k \geq 1$.

Step 5: the integral cocycle. T is ϖ-adically complete and separated, so $z(g) = \lim_k \zeta_k(g)$ defines $z\colon G_\mathbb{Q} \to \mathsf{T}$ with $z(g) \equiv \zeta_k(g)$ modulo $\varpi^k\mathsf{T}$. It is not identically zero ($\zeta_1 \neq 0$); modulo ϖ^n it is ζ_n, locally constant; the cocycle relation and $z(c) = 0$ hold modulo every ϖ^k, hence hold. This is (1) and (2). (3) Let $g \in N$ lie in the inertia group of a place $\bar{w}$ above a prime $p \neq \ell$. If $p \notin S$, then $g \in H_S$ and $\zeta_k(g) = 0$ for all k. If $p \in S$, then $\bar{w} = \tau\bar{p}$ for some $\tau \in G_\mathbb{Q}$ and $g = \tau g'\tau^{-1}$ with $g' \in I_p$ and $\rho(g') = \rho(\tau)^{-1}\rho(g)\rho(\tau) = 1$, since $N \subseteq \ker\rho$; so $z(g') = 0$ by the third condition at every level, and

$$z(g) = z(\tau) + \rho(\tau)z(g') + \rho(\tau g')z(\tau^{-1}) = z(\tau) + \rho(\tau)z(\tau^{-1}) = z(\tau\tau^{-1}) = 0.$$

(4) By Step 0b choose $y \in \mathsf{T}$ with $\varphi(y) - z(s) \in \mathsf{P}$. For $k \geq 1$ let $y_k \in \mathsf{T}_k$ be as in the fourth condition for ζ_k. At $h = s$, $\varphi(y_k) \equiv \zeta_k(s) \equiv \varphi(y)$ modulo P_k, so $y_k \equiv y$ modulo P_k by Step 0b, and for $h \in D_\ell$, modulo $\varpi^k\mathsf{T}$,

$$z(h) - (\rho(h)y - y) \equiv \big(\zeta_k(h) - (\rho(h)y_k - y_k)\big) + (\rho(h) - 1)(y_k - y) \in \mathsf{P}_k,$$

as P_k is D_ℓ-stable. Hence $z(h) - (\rho(h)y - y) \in \bigcap_k(\mathsf{P} + \varpi^k\mathsf{T}) = \mathsf{P}$ by Step 0a. This is (4). □

Lemma 51.25. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum and (r, u) an ordinary test representation of $\mathfrak{D}$ with values in $\overline{\mathbb{Z}}_\ell$ (for the standard test field) such that every $r(g)$ is block diagonal. Then r is not weakly pure for any isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$.*

Proof. Write $r_1(g) = (r(g)_{ab})_{a,b\in\{1,3\}}$ and $r_2(g) = (r(g)_{ab})_{a,b\in\{2,4\}}$. Since every $r(g)$ is block diagonal, $V_1 = \langle e_1, e_3\rangle$ and $V_2 = \langle e_2, e_4\rangle$ are $r(G_\mathbb{Q})$-stable, r_1, r_2 are homomorphisms to $\mathrm{GL}_2(\overline{\mathbb{Z}}_\ell)$, and the eigenvalues of $r(g)$ are those of $r_1(g)$ together with those of $r_2(g)$. Put $\delta_1 = \det r_1$ and $\delta_2 = \det r_2$.

Step 1: $\delta_1\delta_2 = \theta^2$. The form $B(x, y) = {}^t x J_4 y$ vanishes on $V_1 \times V_1$ and on $V_2 \times V_2$ (the entries $(J_4)_{ab}$ with $a, b \in \{1, 3\}$ or $a, b \in \{2, 4\}$ are 0), and its matrix on $V_1 \times V_2$ in the bases (e_1, e_3), (e_2, e_4) is $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By (Ot2), $B(r(g)x, r(g)y) = \theta(g)B(x, y)$, so ${}^t r_1(g)\, P\, r_2(g) = \theta(g)P$; taking determinants, $\delta_1(g)\delta_2(g) = \theta(g)^2$.

Step 2: $\delta_1 = \theta\alpha$ on I_ℓ. By (Ot4), $F_i = u\langle e_1, \ldots, e_i\rangle$ $(0 \leq i \leq 4)$ is a D_ℓ-stable flag and $s \in D_\ell$ acts on F_i/F_{i-1} by $d_i(s)$; so each d_i is a character of D_ℓ, and $d_i(s) - \kappa_i(s) \in \mathfrak{m}_\ell$, with equality $d_i = \kappa_i$ on I_ℓ. The subspaces $F_i \cap V_1$ are D_ℓ-stable, and $(F_i \cap V_1)/(F_{i-1} \cap V_1)$ embeds into F_i/F_{i-1}; as $\dim V_1 = 2$, exactly two indices $a < b$ give non-zero quotients, and in a basis of V_1 adapted to this flag $r_1(s)$ is upper triangular with diagonal $(d_a(s), d_b(s))$. Hence for $s \in D_\ell$

$$\det(X - r_1(s)) = \big(X - d_a(s)\big)\big(X - d_b(s)\big), \qquad \delta_1(s) = d_a(s)d_b(s).$$

Take $s = \sigma_\ell \in D_\ell$ ($\ell \in T$, (Sg2)). By (Ot2), $r_1(\sigma_\ell) \equiv \mathrm{diag}(\theta(\sigma_\ell), \alpha(\sigma_\ell))$ modulo $\mathfrak{m}_\ell$, so in $\overline{\mathbb{F}}_\ell[X]$

$$\big(X - \bar{d}_a(\sigma_\ell)\big)\big(X - \bar{d}_b(\sigma_\ell)\big) = \big(X - \bar{\theta}(\sigma_\ell)\big)\big(X - \bar{\alpha}(\sigma_\ell)\big),$$

with $\bar{d}_i(\sigma_\ell) = \bar{\kappa}_i(\sigma_\ell)$. Now $\chi_{\mathrm{cyc}}(\sigma_\ell) = 1$ by (Sg2), so $\omega_\ell(\sigma_\ell)$ is an $(\ell-1)$-th root of unity congruent to 1, hence equal to 1; thus $\alpha(\sigma_\ell) = \xi(\sigma_\ell)$ and $\theta(\sigma_\ell) = \xi(\sigma_\ell)\Psi(\sigma_\ell)$. By (Sg2) with $i = 0, 1, -1$, $\bar{\xi}(\sigma_\ell)$, $\overline{\xi\Psi}(\sigma_\ell)$ and $\overline{\xi\Psi^{-1}}(\sigma_\ell)$ differ from 1. Hence $\bar{\kappa}_2(\sigma_\ell) = \bar{\Psi}(\sigma_\ell)$ differs from $\bar{\theta}(\sigma_\ell)$ (ratio $\bar{\xi}(\sigma_\ell)$) and from $\bar{\alpha}(\sigma_\ell)$ (ratio $\overline{\xi\Psi^{-1}}(\sigma_\ell)$), and $\bar{\kappa}_4(\sigma_\ell) = 1$ differs from $\bar{\theta}(\sigma_\ell)$ and $\bar{\alpha}(\sigma_\ell)$. So $a, b \notin \{2, 4\}$, $(a, b) = (1, 3)$, and $\delta_1(s) = d_1(s)d_3(s) = \theta(s)\alpha(s)$ for $s \in I_\ell$.

Step 3: $\delta_1 = \chi_{\mathrm{cyc}}^{e+2}\varepsilon_1$ and $\delta_2 = \chi_{\mathrm{cyc}}^{e}\varepsilon_2$ with characters $\varepsilon_1, \varepsilon_2$ of finite order. We have $\theta\alpha = \alpha^2\Psi = \chi_{\mathrm{cyc}}^{e+2}\varepsilon_0$ with $\varepsilon_0 = \omega_\ell^{-2}\xi^2\chi$ of finite order. Put $\eta = \delta_1(\theta\alpha)^{-1} \colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell^{\times}$, a character, continuous by (Ot1) (the inclusion of the discrete valuation ring of (Ot1) into $\overline{\mathbb{Z}}_\ell$ is local, hence continuous). Since η is a character, $\eta(I_{\bar{w}})$ is the same group for all places $\bar{w}$ above a given prime. By Step 2, $\eta(I_\ell) = 1$. Let $p \notin S$; then $p \notin T$ and $p \neq \ell$, the characters $\chi, \xi, \omega_\ell, \chi_{\mathrm{cyc}}$ are unramified at p ((Sg1); χ has modulus N, whose prime factors are in S), so $I_{\bar{w}} \subseteq \ker \bar{r}_0$ for $\bar{w} \mid p$, $r(I_{\bar{w}}) = 1$ by (Ot3), and $\eta(I_{\bar{w}}) = 1$. Let $p \in S$, $p \neq \ell$. By (Ot3), r is trivial on $I_p \cap \ker\chi$ if $p \in T$ and on $I_{\bar{p}} \cap \ker \bar{r}_0$ if $p \notin T$, subgroups of finite index of I_p (χ and $\bar{r}_0$ have finite image), so $\delta_1(I_p)$ is finite; and $\theta\alpha = \chi_{\mathrm{cyc}}^{e+2}\varepsilon_0$ has finite image on I_p, χ_{cyc} being unramified at p. So $\eta(I_p)$ is finite. Let $n \geq 1$ be a common multiple of the orders of the groups $\eta(I_p)$, $p \in S$. Then η^n is trivial on every inertia group, so the fixed field of its kernel (a closed normal subgroup) is an abelian extension of $\mathbb{Q}$ unramified at every prime; by Minkowski's theorem it is $\mathbb{Q}$, so $\eta^n = \mathbf{1}$. Thus $\delta_1 = \chi_{\mathrm{cyc}}^{e+2}\varepsilon_1$ with $\varepsilon_1 = \varepsilon_0\eta$ of finite order, and by Step 1, since $\theta = \alpha\Psi = \chi_{\mathrm{cyc}}^{e+1}\omega_\ell^{-1}\xi\chi$,

$$\delta_2 = \theta^2\delta_1^{-1} = \chi_{\mathrm{cyc}}^{e}\varepsilon_2, \qquad \varepsilon_2 = \omega_\ell^{-2}\xi^2\chi^2\varepsilon_1^{-1},$$

of finite order.

Step 4: weak purity fails. Let ι be an isomorphism and suppose r weakly pure for ι, with H and M as in the definition. Let H' be the normal core of the open subgroup $H \cap \ker\varepsilon_1 \cap \ker\varepsilon_2$; it is open and normal, of some finite index m. Choose a prime $p \nmid \ell M$, a place $\bar{p}'$ above p and an arithmetic Frobenius element $\Phi \in D_{\bar{p}'}$ for p, and put $\sigma = \Phi^m$. Then $\sigma \in H' \subseteq H$ (the image of Φ in $G_{\mathbb{Q}}/H'$ has order dividing m), σ is an arithmetic Frobenius element for p^m at $\bar{p}'$, $\varepsilon_1(\sigma) = \varepsilon_2(\sigma) = 1$, and $\chi_{\mathrm{cyc}}(\sigma) = p^m$ (on roots of unity of order prime to p, reduction modulo $\bar{p}'$ is injective, so $\sigma(\zeta) = \zeta^{p^m}$). Let x_1, x_3 be the eigenvalues of $r_1(\sigma)$ and x_2, x_4 those of $r_2(\sigma)$, all non-zero; by Step 3, $x_1x_3 = p^{m(e+2)}$ and $x_2x_4 = p^{me}$. Applying (45.1) with $f = m$ to $(x, y) = (x_1, x_2)$ and (x_3, x_4) and multiplying,

$$p^{m(e+2)} = |\iota(x_1x_3)| < p^{2m}\,|\iota(x_2x_4)| = p^{2m}\,p^{me} = p^{m(e+2)},$$

since ι fixes $\mathbb{Q}$. This contradiction proves the lemma. □

52 Essential test representations: cases (II) and (III)

This section proves Proposition 52.1 and Proposition 52.14, the two reductions that Proposition 51.5 uses. An essential ordinary test representation is, after conjugation and restriction to a deep auxiliary tower, a marked pair over a complete local ring whose reduction is a residual pair chained to the reduction of any seed pair (in case (III), of any seed pair over a given residual pair). The method is Ribet's lattice argument [208] in GSp_4. A torus conjugation over a discrete valuation ring makes the entries integral with a tight set of non-zero reductions (Lemma 52.5), and gives a Ribet-type lattice whose reduction has a prescribed non-split entry (Proposition 52.4). Its reduction is then moved, by links (affine lines of residual pairs with a fixed core) and by Gysin steps (new entries ramified at auxiliary split primes), until it reaches the reduction of a seed pair. The towers along which this happens must be fit and tame, with

a prescribed labelling, and their existence is proved for each case (Proposition 52.9, Proposition 52.18). Subsections 52.1 and 52.2 treat case (II), the lattices and then the towers and steps, and Subsection 52.3 adapts the argument to case (III).

52.1 Case (II): lattices, chains and Ribet-type lattices

This subsection proves Proposition 52.1 from Proposition 52.2: in case (II) an essential ordinary test representation has a lattice over a complete discrete valuation ring inside V, and a chain with values in the residue field from its reduction to the reduction of any seed pair. The lattice is chosen in two steps. A normal form diagonalises the representation at a separating element and puts its flag at ℓ in lower unipotent shape (Proposition 52.3); then a torus conjugation produces a Ribet-type lattice, whose reduction is supported in a prescribed set of positions and has an entry of a prescribed weight (Proposition 52.4). Two lemmas carry the computation: the torus vertex lemma, which sorts the possible tight sets into six shapes (Lemma 52.5), and the vanishing of odd entries that are cocycles and local coboundaries at ℓ (Lemma 52.6). The identity for the similitude of lower unitriangular matrices (Lemma 52.7) is used again in Section 53.

Proposition 52.1. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, attached to a host character in case (II), $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, (K, V, a) a test field and (r, u) an ordinary test representation of $\mathfrak{D}$ with values in V such that r is essential. Then there are a fit and tame auxiliary tower $\mathscr{T}_1$ for $\mathfrak{D}$ of case (II) and a homomorphism $r'\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(V)$ with $r' = u' r u'^{-1}$ for some $u' \in \mathrm{GL}_4(K)$ and $r'(g)$ of similitude $a(\theta(g))$ for all g, with the following property. For every deep auxiliary tower $\mathscr{T} \geq \mathscr{T}_1$ (field F, places $\bar{v}$, $v \in \Sigma$), every coefficient ring $\mathcal{O}_1 \supseteq \mathcal{O}$ and every seed pair Q over $(\mathscr{T}, \emptyset)$ with values in $\mathcal{O}_1$ for ι whose reduction lies in V and either has an entry of weight $-w_\alpha$ or has entries of weights w_β and $-w_\theta$, there is a coefficient ring $\mathcal{O}_2 \supseteq \mathcal{O}_1$ such that for every coefficient ring $\mathcal{O}' \supseteq \mathcal{O}_2$:*

(Nq1) *there are $R \in \mathrm{CNL}_{\mathcal{O}'}$, an injective ring homomorphism $f\colon R \to V$ that restricts to a on $\mathcal{O}'$ and maps $\mathfrak{m}_R$ into $\mathfrak{m}_V$, and a marked ordinary pair P_R over $(\mathscr{T}, \emptyset)$ with values in R, continuous for $\mathfrak{m}_R$, whose reduction is a residual pair, such that $f(P_R)$ has first component $r'|_{G_F}$ and $\psi_{v,i}^{f(P_R)}(s) = a(\kappa_i(s))$ for all $v \in \Sigma$, $1 \leq i \leq 4$ and $s \in I_v$;*

(Nq2) *there is a chain from the reduction of P_R to the reduction of Q modulo $\mathfrak{m}_{\mathcal{O}'}$ (both with values in $\mathbb{F}_{\mathcal{O}'}$).*

Proof. Take $\mathscr{T}_1$ and r' from Proposition 52.2. Let $\mathscr{T} \geq \mathscr{T}_1$ be deep, $\mathcal{O}_1 \supseteq \mathcal{O}$ and Q as in the statement, and let Λ, j, a_Λ, P_Λ and a chain $c_0, \dots, c_m$ with links $L_1, \dots, L_m$ (residual pairs with values in $k_V[X]$) be given by (Lc1), (Lc2).

Step 1: the residue field of V and the choice of $\mathcal{O}_2$. The map $\bar{a}$ kills $\mathfrak{m}_\ell$ ($a(\mathfrak{m}_\ell) \subseteq \mathfrak{m}_V$), so it induces a field embedding $\overline{\mathbb{F}}_\ell \to k_V$, which is onto because every element of V is congruent modulo $\mathfrak{m}_V$ to an element of $a(\overline{\mathbb{Z}}_\ell)$. Every c_t and every L_t is locally constant on the compact group G_F and has finitely many flags, so only finitely many elements of k_V occur as entries of the c_t and as coefficients of entries of the L_t; together with the image of the finite residue field $\mathbb{F}_\Lambda$ under $\Lambda \to V \to k_V$ they lie in $\bar{a}(\mathbb{F})$ for a finite subfield $\mathbb{F} \subset \overline{\mathbb{F}}_\ell$. Choose a coefficient ring $\mathcal{O}_2 \supseteq \mathcal{O}_1$ whose residue field contains $\mathbb{F}$ (an unramified extension), and let $\mathcal{O}' \supseteq \mathcal{O}_2$.

Step 2: (Nq1). Write $\mathcal{O}' = \sum_{i=1}^n \mathcal{O} b_i$ ($\mathcal{O}'$ is finite over $\mathcal{O}$) and let $R = j(\Lambda)[a(b_1), \dots, a(b_n)] \subseteq V$. Each $a(b_i)$ is integral over $a(\mathcal{O}) = j(a_\Lambda(\mathcal{O})) \subseteq j(\Lambda)$, so R is a finite $j(\Lambda)$-module containing $a(\mathcal{O}')$. Hence R is Noetherian and complete, and, being a domain finite over the complete local ring $j(\Lambda) \cong \Lambda$, it is local. Its maximal ideal lies in $\mathfrak{m}_V$: for $x \in \mathfrak{m}_R$ some power lies in

$j(\mathfrak{m}_\Lambda)R \subseteq \mathfrak{m}_V$ ($R/\mathfrak{m}_\Lambda R$ is Artinian local, j is local), and $\mathfrak{m}_V$ is prime. So $R \cap \mathfrak{m}_V = \mathfrak{m}_R$ and $R/\mathfrak{m}_R$ embeds into k_V, with image the field generated by the images of $\mathbb{F}_\Lambda$ and of $\mathbb{F}_{\mathcal{O}'}$, which is $\bar{a}(\mathbb{F}_{\mathcal{O}'})$ by Step 1. Thus R, with structure map $a|_{\mathcal{O}'}$ (local, as units of $\mathcal{O}'$ go to units), lies in $\mathrm{CNL}_{\mathcal{O}'}$, and the inclusion $f\colon R \to V$ is injective, restricts to a on $\mathcal{O}'$ and maps $\mathfrak{m}_R$ into $\mathfrak{m}_V$. Let $P_R = j(P_\Lambda)$, with entries in $j(\Lambda) \subseteq R$: it is a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in R for $\mathcal{O} \subseteq \mathcal{O}' \to R$ (the conditions (Mq1)–(Mq4) are identities transported by j, and $j \circ a_\Lambda = a|_{\mathcal{O}}$); it is continuous for $\mathfrak{m}_R$ because $j(\mathfrak{m}_\Lambda)^k R \subseteq \mathfrak{m}_R^k$; its reduction is the image of the reduction of P_Λ under the field embedding $\mathbb{F}_\Lambda \to R/\mathfrak{m}_R$, hence a residual pair (local constancy, (Mq1)–(Mq4), (Wq1), (Wq2) are preserved by an injective ring homomorphism); and $f(P_R) = j(P_\Lambda)$ has first component $r'|_{G_F}$ and $\psi_{v,i}^{f(P_R)}(s) = a(\kappa_i(s))$ on I_v by (Lc1).

Step 3: (Nq2). Let $\bar{a}'\colon \mathbb{F}_{\mathcal{O}'} \to k_V$ be induced by $\bar{a}$; it is injective and, by Step 1, its image contains every entry of the c_t and every coefficient of the L_t. Let c_t' and L_t' be obtained by applying $(\bar{a}')^{-1}$ entrywise (coefficientwise for L_t). Every condition in the definitions of residual pairs, of good residual pairs, of links and of chains is a conjunction of polynomial identities and non-vanishing conditions among entries, flags and values $\bar{a}(\kappa_i(g))$, $\bar{a}(\mathrm{rt}_w(s))$, together with evaluation at $X = 0, 1$; the injective homomorphisms $\mathbb{F}_{\mathcal{O}'} \to k_V$ and $\mathbb{F}_{\mathcal{O}'}[X] \to k_V[X]$ commute with these evaluations and preserve and reflect identities and non-vanishing. So $c_0', \dots, c_m'$ is a chain with values in $\mathbb{F}_{\mathcal{O}'}$ (for the reduction $\mathcal{O} \to \mathbb{F}_{\mathcal{O}'}$), with links L_t'. Its ends are the reduction of P_R (identified with $\mathbb{F}_{\mathcal{O}'}$ through Step 2: both have image under $\bar{a}'$ equal to the image of P_Λ in k_V) and the reduction of Q modulo $\mathfrak{m}_{\mathcal{O}'}$ (both have image the image of Q in k_V), by injectivity of $\bar{a}'$. This is (Nq2). □

Proposition 52.2. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, attached to a host character in case (II), $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, (K, V, a) a test field with residue field k_V and $\bar{a}\colon \overline{\mathbb{Z}}_\ell \to k_V$ the composite of a and reduction, and (r, u) an ordinary test representation of $\mathfrak{D}$ with values in V such that r is essential. Then there are a fit and tame auxiliary tower $\mathscr{T}_1$ for $\mathfrak{D}$ of case (II) and a homomorphism $r'\colon G_\mathbb{Q} \to \mathrm{GL}_4(V)$ with $r' = u' r u'^{-1}$ for some $u' \in \mathrm{GL}_4(K)$ and $r'(g)$ of similitude $a(\theta(g))$ for all g, with the following property. For every deep auxiliary tower $\mathscr{T} \geq \mathscr{T}_1$ (field F, places $\bar{v}$, $v \in \Sigma$), every coefficient ring $\mathcal{O}_1 \supseteq \mathcal{O}$ and every seed pair Q over $(\mathscr{T}, \emptyset)$ with values in $\mathcal{O}_1$ for ι whose reduction lies in V and either has an entry of weight $-w_\alpha$ or has entries of weights w_β and $-w_\theta$, there are:*

(Lc1) *a complete discrete valuation ring Λ with finite residue field, an injective local homomorphism $j\colon \Lambda \to V$, a ring homomorphism $a_\Lambda\colon \mathcal{O} \to \Lambda$ with $j \circ a_\Lambda = a|_{\mathcal{O}}$, and a marked ordinary pair P_Λ over $(\mathscr{T}, \emptyset)$ with values in Λ (for a_Λ), continuous for $\mathfrak{m}_\Lambda$, whose reduction is a residual pair, such that $j(P_\Lambda)$ has first component $r'|_{G_F}$ and $\psi_{v,i}^{j(P_\Lambda)}(s) = a(\kappa_i(s))$ for all $v \in \Sigma$, $1 \leq i \leq 4$ and $s \in I_v$;*

(Lc2) *a chain, with values in k_V (for $\bar{a}|_{\mathcal{O}}$), from the image of P_Λ under $\Lambda \to V \to k_V$ to the image of Q under $\mathcal{O}_1 \xrightarrow{\bar{a}} k_V$.*

Proof. Step 1: labelling, first tower, lattice. Let ϵ be the admissible labelling given by Proposition 52.3 for (r, u), and $\mathscr{T}_0$ a fit and tame auxiliary tower of case (II) with labelling ϵ (Ux1), field F_0 and index set Σ_0. Put $g_\circ = g_{\mathscr{T}_0}$; by (Oq7) the $\chi_i(g_\circ)$ are pairwise distinct ($\bar{a}$ is injective on $\mathbb{F}_{\mathcal{O}}$). Proposition 52.3 gives h, $r^\natural = h^{-1} r h$ and n with (Nz1)–(Nz3), and, r being essential, (Nz4). These are the hypotheses of Proposition 52.4, which gives $\mathrm{T} = \mathrm{T}_4(t_1, t_2)$, $r' = \mathrm{T} r^\natural \mathrm{T}^{-1}$, n' with (Ql0)–(Ql3). Put $u' = \mathrm{T} h^{-1} \in \mathrm{GL}_4(K)$: $r' = u' r u'^{-1}$, and $r'(g)$ has similitude $a(\theta(g))$; its entries lie in V and its similitude is a unit, so $r'(g) \in \mathrm{GL}_4(V)$. Let $c = \bar{r}'$ and $\bar{n} = \overline{n'}$. By (Ql0), $\bar{n}^{-1} c(s) \bar{n}$ is upper triangular with diagonal $\chi_i(s)$ for $s \in D_\ell$, and $c(g)_{ii} = \chi_i(g)$; by (Ql1),

c is supported in Σ_P and $\bar{n}$ in Σ_Q. The clauses of (Nz2), (Nz3) on I_ℓ, on I_v for $v \in T \setminus \{\ell\}$ and on the inertia away from T pass to (r', n') (conjugation by the diagonal T keeps the diagonal of $n^{-1} r^\natural(s) n$ and the identity matrix). So (r', n') satisfies the hypotheses of (Ux3) for every fit tower with labelling ϵ whose marking is a power of $g_\circ$.

Step 2: the Gysin pair and the tower $\mathscr{T}_1$. (Ux3) for $\mathscr{T}_0$ gives flags $n_{0,v}$ ($v \in \Sigma_0$) and elements $g_{0,v} \in G_{\mathbb{Q}}$ with $g_{0,v}(\bar{\ell}) = \bar{v}$ and $n_{0,v}^{-1} r'(g_{0,v}) n'$ upper triangular, such that the reduction $p = (c|_{G_{F_0}}, (\bar{n}_{0,v}))$ is a residual pair over $(\mathscr{T}_0, \emptyset)$ with flags supported in Σ_Q. (Gz) of Proposition 52.11 applied to $\mathscr{T}_0$, $W = \emptyset$ and $p = p'$ gives a finite set W' of primes splitting for $\mathscr{T}_0$ and a residual pair s over $(\mathscr{T}_0, W')$ with (Gz1)–(Gz3): its first component and flags are supported in Σ_Q, it has the same $-w_\Psi$ part as p, and its first component has an entry of weight $-w_\alpha$. (Ux2) gives a tame auxiliary tower $\mathscr{T}_1 \geq \mathscr{T}_0$ in whose field every $w \in W'$ dies; by (Xq3) it is fit and of case (II). These $\mathscr{T}_1$ and r' are those of the statement.

Step 3: three facts. (F1) *Isolated entries survive restriction.* Let $\mathscr{T}$ be a tower with field F, $H \supseteq G_F$ a subgroup of $G_{\mathbb{Q}}$ in which G_F is normal, and $y\colon H \to \mathrm{M}_4(k_V)$ a homomorphism of monoids with $y(g)_{ii} = \chi_i(g)$ and, for some $i \neq j$, $y(gh)_{ij} = \chi_i(g) y(h)_{ij} + y(g)_{ij} \chi_j(h)$. If $y(g)_{ij} = 0$ for $g \in G_F$ and $\chi_i(g_{\mathscr{T}}) \neq \chi_j(g_{\mathscr{T}})$, then $y_{ij} = 0$ on H: for $g \in H$ and $h_0 = g_{\mathscr{T}}$, $y(gh_0)_{ij} = y(g)_{ij} \chi_j(h_0)$, and also $y(gh_0)_{ij} = y((gh_0 g^{-1}) g)_{ij} = \chi_i(h_0) y(g)_{ij}$ (as $gh_0 g^{-1} \in G_F$ and χ_i is a character); so $(\chi_j - \chi_i)(h_0)\, y(g)_{ij} = 0$. For a tower $\mathscr{T} \geq \mathscr{T}_0$ the marking is $g_\circ^{\ell^{j'}}$, and $\chi_i(g_\circ)^{\ell^{j'}}$ are pairwise distinct ($x \mapsto x^\ell$ is injective on k_V), so (F1) applies with every $i \neq j$. If $y(g)$ has similitude $\bar{a}(\theta(g)) \neq 0$, then $y(g^{-1}) = \bar{a}(\theta(g))^{-1} J_4^{-1}\, {}^t y(g) J_4$, whose entries $(3,1)$ and $(2,1)$ are $-\bar{a}(\theta(g))^{-1} y(g)_{42}$ and $\bar{a}(\theta(g))^{-1} y(g)_{43}$; so y has an entry of weight $-w_\Psi$ (resp. $-w_\alpha$) if and only if y_{31} (resp. y_{21}) is not identically zero, on H and on G_F alike.

(F2) *The flag coordinate* $(3,1)$. Let $X \in \mathrm{M}_4(k_V)$ be supported in Σ_P with diagonal $(\chi_i(g))_i$ for some g, and y, y' lower unitriangular. If $y^{-1} X y'$ is upper triangular, then comparing first columns, $X y' e_1 = \lambda y e_1$ with λ the first coordinate of $X y' e_1$, which is $\chi_1(g)$ since $X_{12} = X_{13} = X_{14} = 0$; the third coordinate, with $X_{32} = X_{34} = 0$, gives

$$y_{31}\, \chi_1(g) = X_{31} + \chi_3(g)\, y'_{31}. \tag{52.1}$$

(F3) *Products.* If $y^{-1} X y'$ and $y'^{-1} Z y'$ are upper triangular, so is $y^{-1} X Z y' = (y^{-1} X y')(y'^{-1} Z y')$.

Step 4: (Lc1). Let $\mathscr{T} \geq \mathscr{T}_1$ be deep, with field F and places $\bar{v}$, $v \in \Sigma$; it is fit with labelling ϵ by (Xq3), and $\mathscr{T} \geq \mathscr{T}_0$. (Ux3) gives flags n_v with entries in $j'(\Lambda')$ and $g_v \in G_{\mathbb{Q}}$ with $g_v(\bar{\ell}) = \bar{v}$ and $n_v^{-1} r'(g_v) n'$ upper triangular, such that $P = (r'|_{G_F}, (n_v))$ is a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in V, $\psi^P_{v,i}(s) = a(\kappa_i(s))$ on I_v, and its reduction $c_{\mathscr{T}} = (c|_{G_F}, (\bar{n}_v))$ is a residual pair with flags supported in Σ_Q. *Enlarging Λ'.* Let L_0 be the fraction field of $j'(\Lambda')$; every element of K is algebraic over L_0 ((Nz1); $\Lambda' \supseteq \Lambda$). $V \cap L_0$ is a valuation ring of L_0 containing the discrete valuation ring $j'(\Lambda')$ and not containing $j'(\pi')^{-1}$ for a uniformiser π', so it equals $j'(\Lambda')$; as K/L_0 is algebraic, the value group of V lies in the divisible hull of that of L_0, so V has rank one. If K has characteristic 0, Λ' contains $\mathbb{Z}_\ell$, a and j' agree on $\mathbb{Z}$, and for $x \in \mathbb{Z}_\ell$ and integers $m \equiv x$ modulo ℓ^e, $a(x) - j'(x) \in \ell^e V$ for all e; rank one gives $\bigcap_e \ell^e V = 0$, so $a = j'$ on $\mathbb{Z}_\ell$ and every element of $a(\mathcal{O})$ is integral over $j'(\mathbb{Z}_\ell)$. If K has characteristic ℓ, $a(\mathcal{O})$ is a finite field. In both cases $L_1 = L_0(a(\mathcal{O}))$ is a finite extension of L_0, and $\Lambda_1 = V \cap L_1$ is the valuation ring of the unique extension to L_1 of the valuation of the complete discrete valuation ring $j'(\Lambda')$: a complete discrete valuation ring with finite residue field, containing $j'(\Lambda')$ and the integral elements $a(\mathcal{O})$. Let $j_1\colon \Lambda_1 \to V$ be the inclusion (injective, local) and $a_\Lambda = a|_{\mathcal{O}}$ viewed in Λ_1. Then $P_\Lambda = j_1^{-1}(P)$ (entrywise) has values in Λ_1; the identities (Mq1)–(Mq4) for P with a are identities in $j_1(\Lambda_1)$, so P_Λ is a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in Λ_1 for a_Λ; it is $\mathfrak{m}_{\Lambda_1}$-adically continuous because the $\mathfrak{m}_{\Lambda'}$-adic and $\mathfrak{m}_{\Lambda_1}$-adic topologies agree on $j'(\Lambda')$ (Λ_1 is finite over it). Its reduction is mapped by the field embedding of the residue field of Λ_1 into k_V

to $c_{\mathscr{T}}$; residual pairs are defined by polynomial identities and non-vanishing conditions, which an injective field homomorphism reflects, so the reduction of P_Λ is a residual pair. $j_1(P_\Lambda) = P$ has first component $r'|_{G_F}$ and $\psi = a(\kappa_i(s))$ on I_v. This is (Lc1) with $(\Lambda, j) = (\Lambda_1, j_1)$.

Step 5: the pairs of the chain. Let $\mathcal{O}_1 \supseteq \mathcal{O}$ and Q be as in the statement and $\bar{Q} = (r_{\bar{Q}}, (n_{\bar{Q},v}))$ its image under $\mathcal{O}_1 \to k_V$: a residual pair over $(\mathscr{T}, \emptyset)$ lying in V (residual pairs and the conditions "lies in V" and "has an entry of weight w" are preserved by the injective field embedding of the residue field of $\mathcal{O}_1$ into k_V). Every $w \in W'$ dies in F (it dies in the field of $\mathscr{T}_1 \subseteq F$). (Ux4) gives a residual pair $c_3 = (r_s|_{G_F}, (n_{3,v}))$ over $(\mathscr{T}, \emptyset)$ lying over s: for every $v \in \Sigma$ there are $v_0 \in \Sigma_0$ and $g \in G_{F_0}$ with $g(\bar{v}_0) = \bar{v}$ and $n_{3,v}^{-1} r_s(g) n_{s,v_0}$ upper triangular. *Supports of c_3.* $r_s(g)$ and n_{s,v_0} are lower triangular and supported in Σ_Q (Gz1), and Σ_Q is closed under the sums of two weights that are roots ($-w_\alpha - w_\Psi = -w_\theta$; the other sums are not roots), so $L = r_s(g) n_{s,v_0}$ is lower triangular supported in Σ_Q; $n_{3,v}^{-1} L$ is upper and lower triangular, hence a diagonal matrix D, and $n_{3,v} = LD^{-1}$ is supported in Σ_Q. So c_3 has first component supported in $\Sigma_Q \subseteq \Sigma_P \cap \Sigma_M$ and flags in Σ_Q. By (Rl3), $c_3^\flat$ is a residual pair supported in $\{-w_\alpha\}$; by (Uq1) for the fit tower $\mathscr{T}$ of case (II), $\bar{a}(\mathrm{rt}_{w_\Psi}(s)) = 1$ for $s \in D_v$, so (Rl4) with $y_v = (n_{\bar{Q},v})_{31}$ gives a residual pair c_4 with $r_{c_4} = r_s^\flat|_{G_F}$, flags supported in Σ_Q, $(n_{c_4,v})_{21} = (n_{3,v})_{21}$ and $(n_{c_4,v})_{31} = (n_{\bar{Q},v})_{31}$.

Step 6: the links. (i) $c_{\mathscr{T}}$ and c_3: both first components are supported in Σ_P and all flags in Σ_Q. By (Gz2), $r_s(g)_{31} = c(g)_{31}$ and $r_s(g)_{42} = c(g)_{42}$ for $g \in G_{F_0} \supseteq G_F$, and $(n_{s,v_0})_{31} = (\bar{n}_{0,v_0})_{31}$. Fix v, and v_0, g as in Step 5, and put $g' = g g_{0,v_0} \in G_{\mathbb{Q}}$; then $g'(\bar{\ell}) = \bar{v} = g_v(\bar{\ell})$, so $g' = g_v d$ with $d \in D_\ell$, and by (F3) (with $Z = c(d)$) $\bar{n}_v^{-1} c(g') \bar{n}$ is upper triangular. Applying (52.1) three times (to $(\bar{n}_v, c(g'), \bar{n})$, $(n_{3,v}, r_s(g), n_{s,v_0})$, $(\bar{n}_{0,v_0}, c(g_{0,v_0}), \bar{n})$; all matrices X involved are supported in Σ_P) and $c(g')_{31} = c(g)_{31}\chi_1(g_{0,v_0}) + \chi_3(g) c(g_{0,v_0})_{31}$ (the first component is supported in Σ_P):

$$(\bar{n}_v)_{31}\chi_1(g') = c(g)_{31}\chi_1(g_{0,v_0}) + \chi_3(g)\,(\bar{n}_{0,v_0})_{31}\chi_1(g_{0,v_0}) = (n_{3,v})_{31}\,\chi_1(g)\chi_1(g_{0,v_0}).$$

So $(\bar{n}_v)_{31} = (n_{3,v})_{31}$, and (Rl1) links $c_{\mathscr{T}}$ and c_3. (ii) c_3 and c_4: first components supported in Σ_M, equal entries $(2,1)$, $(4,3)$ (kept by $\flat$), equal flag coordinates $(2,1)$: linked by (Rl2). (iii) c_4 and $\bar{Q}$: first components supported in $\{-w_\alpha\} \subseteq \Sigma_P$ and $\Sigma_V \subseteq \Sigma_P$, flags in Σ_Q, no entries of weight $-w_\Psi$ in either, equal flag coordinates $(3,1)$: linked by (Rl1).

Step 7: goodness. $c_{\mathscr{T}}$. By (Ql3) there are two cases. (a) c has an entry of weight $-w_\Psi$. The places $(3,1)$ and $(2,3)$ are isolated in c: $c(gh)_{31} = \chi_3(g)c(h)_{31} + c(g)_{31}\chi_1(h)$ because $c_{32} = c_{34} = 0$, and $c(gh)_{23} = \chi_2(g)c(h)_{23} + c(g)_{23}\chi_3(h)$ because $c_{13} = c_{24} = 0$. By (F1) with $H = G_{\mathbb{Q}}$, $c|_{G_F}$ has entries of weights $-w_\Psi$ and w_β ((Ql2); the place of w_β is $(2,3)$): $c_{\mathscr{T}}$ is good. (b) Otherwise c is supported in Σ_V, $\bar{n}_{31} \neq 0$, $\bar{\Psi} = 1$ on D_ℓ and the condition on c_{23} of (Ql3) holds. Then $c_{\mathscr{T}}$ lies in V: its first component is supported in Σ_V, its flags in Σ_Q, and (52.1) for $(\bar{n}_v, c(g_v), \bar{n})$ gives $(\bar{n}_v)_{31}\chi_1(g_v) = \chi_3(g_v)\bar{n}_{31} \neq 0$. If $c|_{G_F}$ had no entry of weight $-w_\alpha$, then, the place $(2,1)$ being isolated in c ($c_{31} = 0$, $c_{24} = 0$), (F1) would give $c_{21} = 0$ on $G_{\mathbb{Q}}$; the second coordinate of $c(s)\bar{n}e_1 = \chi_1(s)\bar{n}e_1$ ($s \in D_\ell$) would read $\chi_2(s)\bar{n}_{21} + c(s)_{23}\bar{n}_{31} = \chi_1(s)\bar{n}_{21}$, and $\chi_1 = \chi_3$ on D_ℓ (χ_1/χ_3 is $\bar{a} \circ \Psi^{\pm 1}$), so $c(s)_{23} = m(\chi_3(s) - \chi_2(s))$ with $m = \bar{n}_{21}/\bar{n}_{31}$, contrary to (Ql3). So $c_{\mathscr{T}}$ lies in V and has an entry of weight $-w_\alpha$: it is good. *c_3.* The place $(2,1)$ is isolated in the lower triangular r_s; by (Gz3), (F1) with $H = G_{F_0}$ (in which G_F is normal, F being abelian over $\mathbb{Q}$) and the similitude remark of (F1), c_3 has an entry of weight $-w_\alpha$. If it has one of weight $-w_\Psi$, it is good. Otherwise its first component is supported in $\Sigma_Q \backslash \{-w_\Psi\} \subseteq \Sigma_V$; in case (a) this cannot happen ($r_s(g)_{31} = c(g)_{31}$ on G_F, non-zero for some g); in case (b) $(n_{3,v})_{31} = (\bar{n}_v)_{31} \neq 0$, so c_3 lies in V with an entry of weight $-w_\alpha$: good. *c_4.* Its first component is supported in $\{-w_\alpha\} \subseteq \Sigma_V$ and has the entries $(2,1)$, $(4,3)$ of c_3, hence an entry of weight $-w_\alpha$; its flags are supported in Σ_Q with $(n_{c_4,v})_{31} = (n_{\bar{Q},v})_{31} \neq 0$ as $\bar{Q}$ lies in V: good. *$\bar{Q}$.* It lies in V; if it has an entry of weight $-w_\alpha$ it is good; otherwise it has entries of weights w_β and $-w_\theta$, and for every v, (Uq4) gives $s \in D_v$ with $\overline{\mathrm{rt}_{w_\alpha}}(s)^4 \neq 1$, so $\bar{a}(\mathrm{rt}_{w_\alpha}(s)) \neq 1$: good.

Step 8: (Lc2). $c_{\mathscr{T}}, c_3, c_4, \bar{Q}$ are good residual pairs over $(\mathscr{T}, \emptyset)$ with values in k_V (for $\bar{a}|_{\mathcal{O}}$), consecutive ones linked (Step 6): a chain from the image $c_{\mathscr{T}}$ of P_Λ in k_V to the image $\bar{Q}$ of Q. □

Proposition 52.3. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum, (K, V, a) a test field with maximal ideal $\mathfrak{m}_V$ and residue field k_V, and (r, u) an ordinary test representation of $\mathfrak{D}$ with values in V. Then there is an admissible labelling ϵ with the following property. For every $g_\circ \in G_{\mathbb{Q}}$ such that the residues of $a(\kappa_i^\epsilon(g_\circ))$, $1 \le i \le 4$, in k_V are pairwise distinct, there are $h \in \mathrm{GL}_4(K)$, a homomorphism $r^\natural = h^{-1}rh \colon G_{\mathbb{Q}} \to \mathrm{GL}_4(V)$ and $n \in \mathrm{GL}_4(V)$ such that:*

(Nz1) *there are a complete discrete valuation ring Λ with finite residue field and an injective local homomorphism $j \colon \Lambda \to V$ such that every element of K is algebraic over $j(\Lambda)$, the entries of n and of every $r^\natural(g)$ lie in $j(\Lambda)$, and $g \mapsto j^{-1}(r^\natural(g)_{ik})$ is continuous for the $\mathfrak{m}_\Lambda$-adic topology, for all i, k;*

(Nz2) *for every $g \in G_{\mathbb{Q}}$, $r^\natural(g)$ has similitude $a(\theta(g))$ and the entries of*

$$r^\natural(g) - \mathrm{diag}(a(\kappa_1^\epsilon(g)), \dots, a(\kappa_4^\epsilon(g)))$$

lie in $\mathfrak{m}_V$; $r^\natural(g_\circ)$ is diagonal; $r^\natural(g) = 1$ for every $g \in I_{\bar{w}}$ with $\bar{r}_0(g) = 1$, $\bar{w}$ a place of $\overline{\mathbb{Q}}$ above a prime not in T, and for every $g \in I_v$ with $\chi(g) = 1$, $v \in T \setminus \{\ell\}$;

(Nz3) *n is lower triangular with all diagonal entries 1 and has similitude 1; $n_{ik} \in \mathfrak{m}_V$ whenever $i > k$ and $\mathrm{wt}_4(i,k) \ne -w_\Psi$; for every $s \in D_\ell$ the matrix $n^{-1}r^\natural(s)n$ is upper triangular with diagonal entries $\psi_i(s)$ such that $\psi_i(s) - a(\kappa_i^\epsilon(s)) \in \mathfrak{m}_V$, and $\psi_i(s) = a(\kappa_i(s))$ if $s \in I_\ell$;*

(Nz4) *if r is essential, then $r^\natural$ has entries of weights w_Ψ and $-w_\Psi$ and an entry of at least one of the weights w_β, $-w_\alpha$, $-w_\theta$.*

Proof. Step 1: the residual flag and the choice of ϵ. By (Ot2), $\bar{r}(g) = \mathrm{diag}(c_1(g), \dots, c_4(g))$; by (Ot4), $\bar{u} \in \mathrm{GL}_4(k_V)$ has similitude 1 and $\bar{u}^{-1}\bar{r}(s)\bar{u}$ is upper triangular with diagonal $(c_1(s), \dots, c_4(s))$ for $s \in D_\ell$. So the flag $F_j = \mathrm{span}(\bar{u}e_1, \dots, \bar{u}e_j)$ is D_ℓ-stable, D_ℓ acts on F_j/F_{j-1} by c_j, and F_2 is isotropic and $F_3 = F_1^\perp$ ($\langle \bar{u}e_a, \bar{u}e_b\rangle = \langle e_a, e_b\rangle$, which vanishes for $a, b \le 3$ unless $\{a, b\} = \{2, 3\}$). The ratios $c_1/c_2 = c_3/c_4 = \bar{\alpha}$, $c_2/c_3 = \bar{\beta}$ and $c_1/c_4 = \bar{\theta}$ (on D_ℓ) are non-trivial at σ_ℓ: $\bar{\alpha} = \bar{\xi}$, $\bar{\beta}^{-1} = \bar{\xi}\bar{\Psi}^{-1}$, $\bar{\theta} = \bar{\xi}\bar{\Psi}$, and $\overline{\xi\Psi^i}(\sigma_\ell) \ne 1$ for all i by (Sg2) ($\ell \in T$). Only $c_1/c_3 = c_2/c_4 = \bar{\Psi}$ may be trivial on D_ℓ. Since $\bar{r}(s)$ is diagonal, every $x = \sum_i x_i e_i \in F_j$ satisfies $\sum_i (c_i(s) - c_j(s)) x_i e_i \in F_{j-1}$ for all $s \in D_\ell$ ($F_0 = 0$). Hence:

(A) if $\bar{\Psi} \ne 1$ on D_ℓ, the c_i are pairwise distinct, and inductively, if $F_{j-1} = \mathrm{span}(e_1, \dots, e_{j-1})$, then $x_i = 0$ for $x \in F_j$ and $i > j$, so $F_j = \mathrm{span}(e_1, \dots, e_j)$ and $\bar{u}$ is upper triangular; put $\epsilon = 0$ and $t = 0$;

(B) if $\bar{\Psi} = 1$ on D_ℓ, then $c_1 = c_3 \ne c_2 = c_4$, so $F_1 = k_V(x_1e_1 + x_3e_3)$; for $z = \sum_i z_i e_i \in F_2 \setminus F_1$, $(c_1 - c_2)(\sigma_\ell)(z_1e_1 + z_3e_3) \in F_1$ with $(c_1 - c_2)(\sigma_\ell) \ne 0$, so $F_2 = F_1 + k_V(y_2e_2 + y_4e_4)$ with $y_2e_2 + y_4e_4 = z - z_1e_1 - z_3e_3 \ne 0$, and $x_1y_4 - x_3y_2 = \langle x_1e_1 + x_3e_3, y_2e_2 + y_4e_4\rangle = 0$. (B1) If $x_1 \ne 0$, put $\epsilon = 0$ and $t = x_3/x_1$; then $y_2 \ne 0$ (else $y_4 = 0$ too) and $y_4 = ty_2$. (B2) If $x_1 = 0$, put $\epsilon = 1$ (admissible, as $\bar{\Psi} = 1$ on D_ℓ) and $t = 0$; then $x_3 \ne 0$, $y_2 = 0$, $F_1 = k_V e_3$, $F_2 = \mathrm{span}(e_3, e_4)$ and $F_3 = e_3^\perp = \mathrm{span}(e_1, e_3, e_4)$.

Let w be the identity matrix if $\epsilon = 0$, and for $\epsilon = 1$ the permutation matrix with $we_i = e_{\lambda_1(i)}$ ($e_1 \mapsto e_3$, $e_2 \mapsto e_4$, $e_3 \mapsto e_1$, $e_4 \mapsto e_2$); it has similitude ± 1, is block lower triangular, and $w^{-1}\mathrm{diag}(d_1, \dots, d_4)w = \mathrm{diag}(d_{\lambda_\epsilon(1)}, \dots, d_{\lambda_\epsilon(4)})$. Put $\bar{n}_0 = 1 + t(E_{31} + E_{42})$, lower unitriangular

of similitude 1 (its entries satisfy $y_{43} = -y_{21}$, $y_{42} = y_{31} - y_{21}y_{32}$). *Claim:* $w^{-1}\bar{u} = \bar{n}_0\bar{b}_0$ *with* $\bar{b}_0$ *upper triangular and invertible.* In (A) and (B2), $w^{-1}F_j = \mathrm{span}(e_1, \dots, e_j)$ for all j (in (B2): $w^{-1}e_3 = e_1$, $w^{-1}e_4 = e_2$, $w^{-1}e_1 = e_3$), so $w^{-1}\bar{u}$ is upper triangular. In (B1), the flag of the columns of $\bar{n}_0$ is $\langle e_1 + te_3\rangle = F_1$, then $F_1 + \langle e_2 + te_4\rangle = F_2$, then $F_1^{\perp} = F_3$ ($\bar{n}_0$ is symplectic); so $\bar{n}_0^{-1}\bar{u}$ stabilises the standard flag, i.e. is upper triangular.

Step 2: diagonalising at $g_\circ$. Let $g_\circ$ be as in the statement, so the residues $\overline{a(\kappa_i(g_\circ))}$ are pairwise distinct (κ^ϵ is a permutation of κ), and let $X = r(g_\circ)$, of similitude $\nu = a(\theta(g_\circ))$. Let Λ, j be as in (Ot1) and L_0 the fraction field of $j(\Lambda)$. $V \cap L_0$ is a valuation ring of L_0 containing the discrete valuation ring $j(\Lambda)$ and not the inverse of j of a uniformiser (j is local), so $V \cap L_0 = j(\Lambda)$. The characteristic polynomial of X has coefficients in $j(\Lambda)$ and splits in the algebraically closed K; its roots are integral over V, hence lie in V, and their residues are the $\overline{a(\kappa_i(g_\circ))}$; number them $\lambda_1, \dots, \lambda_4$ accordingly (pairwise distinct). Let $L' = L_0(\lambda_1, \dots, \lambda_4)$, finite over L_0, and $\Lambda' = V \cap L'$: the valuation ring of the unique extension to L' of the valuation of the complete discrete valuation ring $j(\Lambda)$, a complete discrete valuation ring with finite residue field; let $j' \colon \Lambda' \to V$ be the inclusion (injective, local). The matrix $X - \lambda_i$ reduces to $\mathrm{diag}(\overline{\lambda_k} - \overline{\lambda_i})_k$, so the i-th column of its adjugate lies in its kernel (as $\det(X - \lambda_i) = 0$) and has i-th coordinate $\equiv \prod_{k \neq i}(\overline{\lambda_k} - \overline{\lambda_i}) \neq 0$; dividing by that coordinate gives an eigenvector $p_i \in \Lambda'^4$, $p_i \equiv e_i$. Since $\langle Xp_i, Xp_k\rangle = \nu\langle p_i, p_k\rangle$, $\langle p_i, p_k\rangle = 0$ unless $\lambda_i\lambda_k = \nu$, which residually happens only for $\{i,k\} = \{1,4\}, \{2,3\}$ ($\kappa_1\kappa_4 = \kappa_2\kappa_3 = \theta$ and the residues are distinct); $\langle p_1, p_4\rangle \equiv 1$, $\langle p_2, p_3\rangle \equiv 1$ are units, and replacing p_4, p_3 by $p_4/\langle p_1, p_4\rangle$, $p_3/\langle p_2, p_3\rangle$ we get $P = (p_1, \dots, p_4) \in \mathrm{GL}_4(\Lambda')$ with ${}^tPJ_4P = J_4$ and $P \equiv 1$. Put $h = Pw \in \mathrm{GL}_4(\Lambda')$ (a similitude) and $r^\natural = h^{-1}rh$. If X is block lower triangular, so is P: for $i \in \{2,4\}$, write $p_i = (p', p'')$ in the coordinates $\{1,3\}$, $\{2,4\}$; the $\{1,3\}$-block A of X has eigenvalues with residues $\overline{\lambda_1}, \overline{\lambda_3} \neq \overline{\lambda_i}$, so $A - \lambda_i$ is invertible and $(A - \lambda_i)p' = 0$ gives $p' = 0$.

Step 3: (Nz1), (Nz2). The entries of $r^\natural(g)$ are Λ'-linear combinations of those of $r(g)$ with fixed coefficients, so they lie in $j'(\Lambda')$ and are $\mathfrak{m}_{\Lambda'}$-adically continuous in g (Λ' is finite over $j(\Lambda)$); every element of K is algebraic over $j'(\Lambda') \supseteq j(\Lambda)$. Conjugation by the similitude h preserves similitudes and the identity, so $r^\natural(g)$ has similitude $a(\theta(g))$ and the vanishing conditions of (Ot3) pass to $r^\natural$. Modulo $\mathfrak{m}_V$, $r^\natural(g) \equiv w^{-1}\mathrm{diag}(c_i(g))w = \mathrm{diag}(\overline{a(\kappa_i^\epsilon(g))})$; and $r^\natural(g_\circ) = w^{-1}\mathrm{diag}(\lambda_1, \dots, \lambda_4)w$ is diagonal.

Step 4: (Nz3). Let $m = h^{-1}u \in \mathrm{GL}_4(\Lambda')$ (the entries of u lie in $j(\Lambda)$ by (Ot1), h^{-1} has entries in Λ' as $\det h$ is a unit of V lying in L'); it is a similitude, and $\bar{m} = w^{-1}\bar{u} = \bar{n}_0\bar{b}_0$ (Step 1; $\bar{P} = 1$). For lower unitriangular n_1 and upper triangular b_1 the leading $k \times k$ block of n_1b_1 is the product of the leading blocks, so the leading principal minors of $\bar{m}$ are those of $\bar{b}_0$, all non-zero; hence those of m are units of Λ', and Gaussian elimination without pivoting gives $m = nb$ with n lower triangular with diagonal 1 and b upper triangular, both in $\mathrm{GL}_4(\Lambda')$; by uniqueness of this factorisation over k_V, $\bar{n} = \bar{n}_0$, so $n_{ik} \in \mathfrak{m}_V$ for $i > k$ except at $(3,1), (4,2)$, the places of weight $-w_\Psi$. *n has similitude* 1. Let $\tau(Y) = J_4^{-1}\,{}^tY^{-1}J_4$, an automorphism of $\mathrm{GL}_4(K)$; for a similitude Y of multiplier μ, ${}^tYJ_4Y = \mu J_4$ gives $\tau(Y) = \mu^{-1}Y$. Since $(J_4^{-1}ZJ_4)_{ab} = \pm Z_{5-a,5-b}$ with sign $+$ on the diagonal, τ maps lower unitriangular matrices to lower unitriangular ones and upper triangular to upper triangular. From $\tau(n)\tau(b) = \tau(m) = \mu^{-1}m = n(\mu^{-1}b)$ and uniqueness, $\tau(n) = n$, i.e. ${}^tnJ_4n = J_4$. *The flag.* For $s \in D_\ell$, $n^{-1}r^\natural(s)n = b\,(m^{-1}r^\natural(s)m)\,b^{-1} = b\,(u^{-1}r(s)u)\,b^{-1}$ is upper triangular with the diagonal entries $\psi_i(s) = d_i(s)$ of (Ot4): $\psi_i(s) = a(\kappa_i(s))$ for $s \in I_\ell$, and $\psi_i(s) \equiv a(\kappa_i(s)) \equiv a(\kappa_i^\epsilon(s))$ modulo $\mathfrak{m}_V$ for $s \in D_\ell$, the second congruence being trivial for $\epsilon = 0$ and, for $\epsilon = 1$, a consequence of $\bar{\Psi} = 1$ on D_ℓ: $(\kappa_i^1) = (\alpha, \mathbf{1}, \theta, \Psi)$ and $(\kappa_i) = (\theta, \Psi, \alpha, \mathbf{1})$ agree residually on D_ℓ as $\bar{\theta} = \bar{\Psi}\bar{\alpha}$.

Step 5: (Nz4). If r is irreducible, the K-span of $r^\natural(G_\mathbb{Q})$ is $h^{-1}\mathrm{M}_4(K)h = \mathrm{M}_4(K)$, so no entry vanishes identically (each entry is a non-zero linear functional on $\mathrm{M}_4(K)$); in particular

$r^\natural$ has entries of weights w_Ψ (place $(1,3)$), $-w_\Psi$ ($(3,1)$) and w_β ($(2,3)$). Let r be a non-split block extension: every $r(g)$ is block lower triangular, in particular X, so P, w, h, h^{-1} and every $r^\natural(g)$ are block lower triangular. For block lower triangular Y, Z the $\{2,4\}$-block of YZ is the product of the $\{2,4\}$-blocks; so the $\{2,4\}$-block of $r^\natural(g)$ is $h_{\mathrm{B}}^{-1} r(g)_{\mathrm{B}} h_{\mathrm{B}}$ with h_{B} the (invertible) $\{2,4\}$-block of h, and these span $\mathrm{M}_2(K)$: the entries $(2,4)$ (weight w_Ψ) and $(4,2)$ (weight $-w_\Psi$) of $r^\natural$ are active. The places (i,k) with $i \in \{2,4\}$, $k \in \{1,3\}$ have weights $-w_\alpha$ ($(2,1),(4,3)$), w_β ($(2,3)$), $-w_\theta$ ($(4,1)$). If none of them were active in $r^\natural$, every $r^\natural(g)$ would be block diagonal. Write $h = h_{\mathrm{D}} + h_{\mathrm{L}}$ with h_{D} block diagonal (invertible) and h_{L} supported at the places $i \in \{2,4\}$, $k \in \{1,3\}$, and $Y = h_{\mathrm{L}} h_{\mathrm{D}}^{-1}$, supported at the same places; then $Y^2 = 0$, $h = (1+Y)h_{\mathrm{D}}$ and $(1-Y)r(g)(1+Y) = h_{\mathrm{D}} r^\natural(g) h_{\mathrm{D}}^{-1}$ is block diagonal for every g, contrary to clause (iii) of non-split block extensions. So $r^\natural$ has an entry of one of the weights $w_\beta, -w_\alpha, -w_\theta$. □

Proposition 52.4. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum, (K,V,a) a test field with maximal ideal $\mathfrak{m}_V$, residue field k_V and reduction $x \mapsto \bar{x}$, ϵ an admissible labelling, and $g_\circ \in G_{\mathbb{Q}}$ such that the $\overline{a(\kappa_i^\epsilon(g_\circ))}$, $1 \le i \le 4$, are pairwise distinct. Let $r^\natural \colon G_{\mathbb{Q}} \to \mathrm{GL}_4(V)$ be a homomorphism and $n \in \mathrm{GL}_4(V)$ such that:*

- *there are a complete discrete valuation ring Λ with finite residue field and an injective local homomorphism $j \colon \Lambda \to V$ such that the entries of n and of every $r^\natural(g)$ lie in $j(\Lambda)$ and $g \mapsto j^{-1}(r^\natural(g)_{ik})$ is $\mathfrak{m}_\Lambda$-adically continuous for all i,k;*
- *$r^\natural(g)$ has similitude $a(\theta(g))$ and $r^\natural(g) - \mathrm{diag}(a(\kappa_i^\epsilon(g)))_i$ has entries in $\mathfrak{m}_V$, for all g; $r^\natural(g_\circ)$ is diagonal; $r^\natural(g) = 1$ for every $g \in I_{\bar{w}}$ with $\bar{r}_0(g) = 1$, $\bar{w}$ above a prime not in T;*
- *n is lower triangular with diagonal entries 1 and similitude 1, $n_{ik} \in \mathfrak{m}_V$ for $i > k$ with $\mathrm{wt}_4(i,k) \neq -w_\Psi$, and for $s \in D_\ell$ the matrix $n^{-1} r^\natural(s) n$ is upper triangular with diagonal entries $\psi_i(s)$, $\psi_i(s) - a(\kappa_i^\epsilon(s)) \in \mathfrak{m}_V$;*
- *$r^\natural$ has an entry of weight $-w_\Psi$ and an entry of at least one of the weights $w_\beta, -w_\alpha, -w_\theta$.*

Then there are $t_1, t_2 \in K^\times$ such that, with $\mathrm{T} = \mathrm{T}_4(t_1,t_2)$, $r' = \mathrm{T} r^\natural \mathrm{T}^{-1}$ and $n' = \mathrm{T} n \mathrm{T}^{-1}$:

(Ql0) *there are a complete discrete valuation ring Λ' with finite residue field and an injective local homomorphism $j' \colon \Lambda' \to V$ such that the entries of n' and of every $r'(g)$ lie in $j'(\Lambda')$ and $g \mapsto j'^{-1}(r'(g)_{ik})$ is $\mathfrak{m}_{\Lambda'}$-adically continuous; $r'(g)$ has similitude $a(\theta(g))$ and diagonal entries $r^\natural(g)_{ii}$; $r'(g_\circ)$ is diagonal; $r'(g) = 1$ whenever $r^\natural(g) = 1$; n' is lower triangular with diagonal entries 1 and similitude 1; and $n'^{-1} r'(s) n'$ is upper triangular with diagonal entries $\psi_i(s)$ for $s \in D_\ell$.*

Let $c(g) = \overline{r'(g)}$ and $\bar{n} = \overline{n'}$ (entrywise). Then:

(Ql1) *c is supported in Σ_P and $\bar{n}$ is supported in Σ_Q;*

(Ql2) *c has an entry of weight w_β;*

(Ql3) *either c has an entry of weight $-w_\Psi$, or c is supported in Σ_V, $\bar{n}_{31} \neq 0$, $\bar{\Psi}(s) = 1$ for all $s \in D_\ell$, and there is no $m \in k_V$ with $c(s)_{23} = m(\overline{a(\kappa_3^\epsilon(s))} - \overline{a(\kappa_2^\epsilon(s))})$ for all $s \in D_\ell$.*

Proof. Step 1: the data of the vertex lemma. As j is injective and local, $j^{-1}(\mathfrak{m}_V) = \mathfrak{m}_\Lambda$. For a root w let E_w be the set of $j^{-1}(r^\natural(g)_{ik})$, $g \in G_{\mathbb{Q}}$, $i \neq k$ with $\mathrm{wt}_4(i,k) = w$; by the congruence, $E_w \subseteq \mathfrak{m}_\Lambda$. Put $n_{-w_\alpha} = j^{-1}(n_{21})$, $n_{-w_\beta} = j^{-1}(n_{32})$, $n_{-w_\Psi} = j^{-1}(n_{31})$, $n_{-w_\theta} = j^{-1}(n_{41})$; they lie in $\mathfrak{m}_\Lambda$ except possibly n_{-w_Ψ}. By the last hypothesis, E_{-w_Ψ} and some E_w with $w \in \Sigma_V =$

$\{w_\beta, -w_\alpha, -w_\theta\}$ contain non-zero elements. Lemma 52.5 gives $p \in K$ with $p^2 = j(\varpi)$ (ϖ a uniformiser of Λ) and integers A, B such that $t = (t_1, t_2) = (p^A, p^B)$ satisfies (Tv1) and (Tv2).

Step 2: the lower unipotent symplectic matrices. By Lemma 52.7, a lower triangular matrix y with diagonal entries 1 has similitude 1 if and only if $y_{43} = -y_{21}$ and $y_{42} = y_{31} - y_{21}y_{32}$.

Step 3: (Ql0). Let $\Lambda' = \Lambda[X]/(X^2 - \varpi)$, a complete discrete valuation ring with uniformiser the class π' of X and the same finite residue field ($X^2 - \varpi$ is Eisenstein), and $j'\colon \Lambda' \to K$ the extension of j with $j'(\pi') = p$; it is injective ($X^2 - \varpi$ is irreducible over the fraction field of Λ, and j is injective), and $j'(\Lambda') \subseteq V$ is local because $p \in V$ has positive valuation ($2v(p) = v(j(\varpi)) > 0$). For $y \in j(\Lambda)$ non-zero and an integer e, $p^e y$ lies in V if and only if it lies in $j'(\Lambda')$: both say that its valuation, a multiple of $v(p)$, is ≥ 0. The off-diagonal entries of $r'(g)$ are $t^{\mathrm{wt}_4(i,k)} r^\natural(g)_{ik}$, in V by (Tv1); its diagonal entries are those of $r^\natural(g)$. The entries $n'_{21}, n'_{32}, n'_{31}, n'_{41}$ are $t^w j(n_w)$, in V by (Tv1); n' is lower triangular with diagonal 1 and similitude 1 (T has similitude $t_1 t_2$, and conjugation by a similitude preserves similitudes), so by Step 2 $n'_{43} = -n'_{21}$ and $n'_{42} = n'_{31} - n'_{21}n'_{32}$ lie in V too. So all these entries lie in $j'(\Lambda')$. Continuity holds because $\mathfrak{m}_\Lambda \Lambda' = \mathfrak{m}_{\Lambda'}^2$ and multiplication by the fixed element p^e is continuous. The remaining clauses of (Ql0) are preserved by conjugation with the diagonal matrix T: similitudes, diagonal entries, diagonality of $r'(g_\circ)$, $r'(g) = 1$ where $r^\natural(g) = 1$, lower unitriangularity, and $n'^{-1} r'(s) n' = \mathrm{T}(n^{-1} r^\natural(s) n)\mathrm{T}^{-1}$ is upper triangular with the same diagonal.

Step 4: the residual pair and its tight set. $c\colon G_\mathbb{Q} \to \mathrm{M}_4(k_V)$ is a homomorphism of monoids, locally constant by (Ql0), with $c(g)_{ii} = \bar{a}(\kappa_i^\epsilon(g))$ where $\bar{a}$ is a followed by reduction ($\bar{a}(\mathfrak{m}_\mathcal{O}) = 0$ because $a(\mathfrak{m}_\ell) \subseteq \mathfrak{m}_V$), $c(g_\circ)$ diagonal, $c(g) = 1$ on the inertia elements of the hypothesis. Write $\delta_i(s) = \bar{a}(\kappa_i^\epsilon(s))$. Let $\mathcal{S}$ be the set of roots w such that c has an entry of weight w, or $w = \mathrm{wt}_4(i,k)$ with $(i,k) \in \{(2,1),(3,2),(3,1),(4,1)\}$ and $\bar{n}_{ik} \neq 0$. Since $r'(g)_{ik} = t^w j(x)$ with $x \in E_w$, and $n'_{ik} = t^w j(n_w)$ for these four places, an element of V having non-zero residue exactly when it is a unit, $\mathcal{S} = S_t$, and by (Tv2) $\mathcal{S}$ satisfies one of (Zc1)–(Zc6). From the upper triangularity of $\bar{n}^{-1} c(s) \bar{n}$ with $(1,1)$ entry $\bar{\psi}_1(s) = \delta_1(s)$: $c(s)\bar{n}e_1 = \delta_1(s)\bar{n}e_1$, that is

$$c(s)_{i1} + c(s)_{i2}\bar{n}_{21} + c(s)_{i3}\bar{n}_{31} + c(s)_{i4}\bar{n}_{41} = \bar{n}_{i1}\,\delta_1(s) \qquad (s \in D_\ell,\ 1 \leq i \leq 4). \tag{52.2}$$

Step 5: two exclusions. (a) If $-w_\alpha \notin \mathcal{S}$, then $-w_\theta \notin \mathcal{S}$. Indeed $c_{21} = c_{43} = 0$ identically and $\bar{n}_{21} = 0$. Apply Lemma 52.6 to c and $(i,k) = (4,1)$ (weight $-w_\theta$): (Ov1) holds as $c(g)_{42}c(h)_{21} = 0 = c(g)_{43}c(h)_{31}$; (Ov2) holds with $m = \bar{n}_{41}$ by (52.2) for $i = 4$. So $c_{41} = 0$ and $\bar{n}_{41} = 0$, i.e. $-w_\theta \notin \mathcal{S}$ (its only places are $(4,1)$ in c and in $\bar{n}$). (b) If $w_\beta \notin \mathcal{S}$ and $w_\Psi \notin \mathcal{S}$, then $-w_\alpha \notin \mathcal{S}$. Indeed $c_{23} = c_{13} = c_{24} = 0$. Apply Lemma 52.6 to $(i,k) = (2,1)$: (Ov1) holds as $c(g)_{23}c(h)_{31} = 0 = c(g)_{24}c(h)_{41}$; (Ov2) holds with $m = \bar{n}_{21}$ by (52.2) for $i = 2$. So $c_{21} = 0$ and $\bar{n}_{21} = 0$. Moreover $c(g)$ has similitude $\nu = \bar{a}(\theta(g)) \neq 0$, so $c(g^{-1}) = c(g)^{-1} = \nu^{-1} J_4^{-1}\, {}^t c(g)\, J_4$, whose $(2,1)$ entry is $\nu^{-1} c(g)_{43}$; hence $c_{43} = 0$ and $-w_\alpha \notin \mathcal{S}$.

Step 6: the case (Zc1). In (Zc2), $w_\beta, w_\Psi \notin \mathcal{S} \ni -w_\alpha$, contradicting (b); in (Zc4) likewise. In (Zc3), (Zc5), (Zc6), $-w_\alpha \notin \mathcal{S} \ni -w_\theta$, contradicting (a). So (Zc1): $w_\beta, -w_\Psi \in \mathcal{S} \subseteq \Sigma_P$.

Step 7: (Ql1), (Ql2). Every non-zero off-diagonal entry of c has its weight in $\mathcal{S} \subseteq \Sigma_P$. The lower entries of $\bar{n}$ have weights $-w_\alpha$ at $(2,1),(4,3)$, $-w_\Psi$ at $(3,1),(4,2)$, $-w_\theta$ at $(4,1)$ and $-w_\beta$ at $(3,2)$; as $-w_\beta \notin \Sigma_P$, $\bar{n}_{32} = 0$, so $\bar{n}$ is supported in Σ_Q. The weight w_β is in $\mathcal{S}$ and is not the weight of a lower entry, so c has an entry of weight w_β, necessarily at $(2,3)$.

Step 8: (Ql3). Suppose c has no entry of weight $-w_\Psi$. Then c is supported in $\Sigma_P \setminus \{-w_\Psi\} = \Sigma_V$, and $-w_\Psi \in \mathcal{S}$ comes from the flag: $\bar{n}_{31} \neq 0$. In (52.2) for $i = 3$, $c_{31} = 0$ (weight $-w_\Psi$), $c_{32} = 0$ (weight $-w_\beta \notin \Sigma_P$), $c_{34} = 0$ (weight $w_\alpha \notin \Sigma_P$), so $\delta_3(s)\bar{n}_{31} = \delta_1(s)\bar{n}_{31}$ and $\delta_1 = \delta_3$ on D_ℓ. Now $\kappa_1^\epsilon(\kappa_3^\epsilon)^{-1}$ is $\theta\alpha^{-1} = \Psi$ for $\epsilon = 0$ and $\alpha\theta^{-1} = \Psi^{-1}$ for $\epsilon = 1$, and $\bar{a}$ is injective on $\mathbb{F}_\mathcal{O}$; so $\bar{\Psi} = 1$ on D_ℓ. Finally, if $c(s)_{23} = m(\delta_3(s) - \delta_2(s))$ on D_ℓ for some m, apply Lemma 52.6 to $(i,k) = (2,3)$ (weight w_β): (Ov1) holds since $c_{13} = c_{24} = 0$ (weight $w_\Psi \notin \Sigma_P$), and (Ov2) is the

assumption; so $c_{23} = 0$, contradicting (Ql2). □

Lemma 52.5. *Let Λ be a discrete valuation ring with maximal ideal $\mathfrak{m}_\Lambda$ and uniformiser ϖ, K an algebraically closed field, $V \subseteq K$ a valuation ring of K with group of units $V^\times$, and $j\colon \Lambda \to V$ an injective local ring homomorphism. For $t = (t_1, t_2) \in (K^\times)^2$ and $w = (w_1, w_2) \in \mathbb{Z}^2$ put $t^w = t_1^{w_1} t_2^{w_2}$. Let there be given, for every root w, a subset $E_w \subseteq \mathfrak{m}_\Lambda$, and for every $w \in \Sigma_M$ an element $n_w \in \Lambda$, with $n_w \in \mathfrak{m}_\Lambda$ for $w \neq -w_\Psi$. Assume that E_{-w_Ψ} contains a non-zero element and that E_w contains a non-zero element for some $w \in \Sigma_V$. Then there are $p \in K$ with $p^2 = j(\varpi)$ and integers A, B such that $t = (p^A, p^B)$ satisfies:*

(Tv1) *$t^w j(x) \in V$ for every root w and every $x \in E_w$, and $t^w j(n_w) \in V$ for every $w \in \Sigma_M$;*

(Tv2) *the set S_t of roots w such that $t^w j(x) \in V^\times$ for some $x \in E_w$, or $w \in \Sigma_M$ and $t^w j(n_w) \in V^\times$, satisfies one of*

- **(Zc1)** $w_\beta, -w_\Psi \in S_t \subseteq \Sigma_P$;
- **(Zc2)** $-w_\alpha, -w_\Psi \in S_t \subseteq \Sigma_Q$;
- **(Zc3)** $-w_\theta, -w_\Psi \in S_t \subseteq \{-w_\theta, -w_\Psi\}$;
- **(Zc4)** $-w_\alpha, -w_\beta \in S_t \subseteq \Sigma_M$;
- **(Zc5)** $-w_\theta, -w_\beta \in S_t \subseteq \{-w_\theta, -w_\Psi, -w_\beta\}$;
- **(Zc6)** $-w_\theta, w_\alpha \in S_t \subseteq \{-w_\theta, -w_\Psi, -w_\beta, w_\alpha\}$.

Proof. Step 1: valuations. Let $v\colon K^\times \to \Gamma$ be the valuation of V (with Γ totally ordered, written additively), and v_Λ the normalised valuation of Λ. As K is algebraically closed, choose $p \in K$ with $p^2 = j(\varpi)$. Since j is injective and local, $j(\varpi) \neq 0$ lies in the maximal ideal of V, so $\gamma = v(p)$ satisfies $2\gamma = v(j(\varpi)) > 0$, hence $\gamma > 0$. A non-zero $x \in \Lambda$ is $x = u\varpi^k$ with $u \in \Lambda^\times$, $k = v_\Lambda(x)$, and $j(u) \in V^\times$; so for $t = (p^A, p^B)$ and a root w,

$$v(t^w j(x)) = (w \cdot \lambda + 2k)\gamma, \qquad \lambda = (A, B). \tag{52.3}$$

As $\gamma > 0$ in a totally ordered group, $t^w j(x) \in V$ if and only if $w \cdot \lambda + 2k \geq 0$, and $t^w j(x) \in V^\times$ if and only if $w \cdot \lambda + 2k = 0$.

Step 2: the linear program. Let $\mathcal{C}$ be the set of pairs (w, k), w a root and $k \geq 0$ an integer, such that $k = v_\Lambda(x)$ for some non-zero $x \in E_w$, or $w \in \Sigma_M$, $n_w \neq 0$ and $k = v_\Lambda(n_w)$. If $(w, 0) \in \mathcal{C}$ then $w = -w_\Psi$, since $E_w \subseteq \mathfrak{m}_\Lambda$ and $n_w \in \mathfrak{m}_\Lambda$ for $w \neq -w_\Psi$. Let $\mathcal{F}$ be the set of $\lambda \in \mathbb{Z}^2$ with $w \cdot \lambda + 2k \geq 0$ for all $(w, k) \in \mathcal{C}$; $(0, 0) \in \mathcal{F}$. By hypothesis there are $(-w_\Psi, k_0) \in \mathcal{C}$ and $(w', k_1) \in \mathcal{C}$ with $w' \in \Sigma_V$. For $\lambda = (A, B) \in \mathcal{F}$ the first gives $B \leq 2k_0$; so $B^* = \max\{B : (A, B) \in \mathcal{F}\}$ exists. The second gives, according as $w' = w_\beta, -w_\alpha, -w_\theta$, $A \leq B + 2k_1$, $A \leq 2k_1$, $A \leq -B + 2k_1$; so on $\{(A, B^*) \in \mathcal{F}\}$ the integer A is bounded above by $|B^*| + 2k_1$, and $A^* = \max\{A : (A, B^*) \in \mathcal{F}\}$ exists. Put $\lambda^* = (A^*, B^*)$, $t = (p^{A^*}, p^{B^*})$ and $\phi(w, k) = w \cdot \lambda^* + 2k$. As $\lambda^* \in \mathcal{F}$, Step 1 gives (Tv1), and S_t is the set $\mathcal{T}$ of roots w with $\phi(w, k) = 0$ for some $(w, k) \in \mathcal{C}$.

Step 3: four tight directions. If $\lambda^* + (a, b) \notin \mathcal{F}$, some $(w, k) \in \mathcal{C}$ has $0 \leq \phi(w, k) < -(w_1 a + w_2 b)$. Every root has $w_1, w_2 \in \{-1, 0, 1\}$.

- $(a, b) = (1, 0)$: excluded from $\mathcal{F}$ by maximality of A^*; then $0 \leq \phi < -w_1$ forces $w_1 = -1$ and $\phi = 0$, so $\mathcal{T}$ contains one of $-w_\alpha = (-1, 0)$, $w_\beta = (-1, 1)$, $-w_\theta = (-1, -1)$.
- $(a, b) = (0, 1)$: excluded by maximality of B^*; then $w_2 = -1$, $\phi = 0$: $\mathcal{T}$ contains one of $-w_\Psi = (0, -1)$, $-w_\beta = (1, -1)$, $-w_\theta$.

- Parity. A tight root with $w_1 = -1$ gives $A^* \equiv 0$ (for $(-1,0)$) or $A^* \equiv B^*$ (for $(-1,\pm1)$) modulo 2; a tight root with $w_2 = -1$ gives $B^* \equiv 0$ or $A^* \equiv B^*$. In all four combinations $A^* \equiv B^* \pmod 2$. Hence $\phi(w,k)$ is even for $w = \pm(1,-1), \pm(1,1)$.
- $(a,b) = (-1,1)$: excluded (second coordinate $B^* + 1$); then $0 \le \phi < w_1 - w_2$. Here $w_1 - w_2 = 2$ only for $w = (1,-1)$, where ϕ is even, so $\phi = 0$; otherwise $w_1 - w_2 = 1$, $w \in \{(1,0),(0,-1)\}$ and $\phi = 0$. So $\mathcal{T}$ contains one of w_α, $-w_\Psi$, $-w_\beta$.
- $(a,b) = (1,1)$: excluded; then $0 \le \phi < -(w_1 + w_2)$, which is 2 only for $(-1,-1)$ (then ϕ even, $\phi = 0$) and 1 for $(-1,0),(0,-1)$. So $\mathcal{T}$ contains one of $-w_\alpha$, $-w_\Psi$, $-w_\theta$.

Step 4: signs. Let $w \in \mathcal{T}$ with $\phi(w,k) = 0$. If $w \ne -w_\Psi$ then $k \ge 1$ (Step 2), so $w \cdot \lambda^* = -2k < 0$; if $w = -w_\Psi$ then $-B^* = -2k \le 0$. Writing $A = A^*$, $B = B^*$: $w_\alpha \in \mathcal{T} \Rightarrow A < 0$; $-w_\alpha \in \mathcal{T} \Rightarrow A > 0$; $w_\beta \in \mathcal{T} \Rightarrow B < A$; $-w_\beta \in \mathcal{T} \Rightarrow A < B$; $w_\theta \in \mathcal{T} \Rightarrow A + B < 0$; $-w_\theta \in \mathcal{T} \Rightarrow A + B > 0$; $w_\Psi \in \mathcal{T} \Rightarrow B < 0$; $-w_\Psi \in \mathcal{T} \Rightarrow B \ge 0$. Together with the four disjunctions of Step 3 these are exactly the hypotheses (h1)–(h8), (m1)–(m4) of computer check C5, listed in Appendix B, with the letter of a root true if and only if the root lies in $\mathcal{T}$.

Step 5: the six shapes. Computer check C5 establishes, for all integers A, B, that under these hypotheses the set of true letters satisfies one of six conditions, which in terms of roots are (Zc1)–(Zc6). The reduction to finitely many cases is proved there: each hypothesis is a sign condition on one of A, B, $A - B$, $A + B$, the four lines $A = 0$, $B = 0$, $A = \pm B$ cut $\mathbb{R}^2$ into seventeen cones on which these signs are constant, and each cone meets $[-2,2]^2$ in a lattice point; so the exhaustive check over $(A,B) \in [-2,2]^2$ and all 2^8 assignments covers every integer pair (the second method there decides each assignment by exact Fourier–Motzkin elimination). Hence $S_t = \mathcal{T}$ satisfies one of (Zc1)–(Zc6), which is (Tv2). □

Lemma 52.6. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$ (maximal ideal $\mathfrak{m}_\mathcal{O}$), ϵ an admissible labelling, k a field, $\bar{a}\colon \mathcal{O} \to k$ a ring homomorphism with $\bar{a}(\mathfrak{m}_\mathcal{O}) = 0$, and $g_\circ \in G_\mathbb{Q}$ such that the $\bar{a}(\kappa_i^\epsilon(g_\circ))$, $1 \le i \le 4$, are pairwise distinct. Let $c\colon G_\mathbb{Q} \to \mathrm{M}_4(k)$ be a locally constant homomorphism of monoids such that $c(g)_{ii} = \bar{a}(\kappa_i^\epsilon(g))$ for all g and i, $c(g_\circ)$ is diagonal, and $c(g) = 1$ for every $g \in I_{\bar{w}}$ with $\bar{r}_0(g) = 1$ and every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above a prime not in T. Let $i \ne j$ be indices such that $\mathrm{wt}_4(i,j)$ is an odd root and*

(Ov1) *$c(g)_{ix}\, c(h)_{xj} = 0$ for all $g, h \in G_\mathbb{Q}$ and all $x \notin \{i,j\}$;*

(Ov2) *there is $m \in k$ with $c(s)_{ij} = m\big(\bar{a}(\kappa_j^\epsilon(s)) - \bar{a}(\kappa_i^\epsilon(s))\big)$ for all $s \in D_\ell$.*

Then $c(g)_{ij} = 0$ for all $g \in G_\mathbb{Q}$, and $m = 0$.

Proof. Step 1: the character and the cocycle. The odd roots are the roots with non-zero first coordinate, so $\lambda_i \ne \lambda_j$. The permutation λ_ϵ preserves $\boldsymbol{\lambda}$ (λ_1 exchanges $1,3$ and $2,4$), so $a = \lambda_\epsilon(i)$, $b = \lambda_\epsilon(j)$ satisfy $\lambda_a \ne \lambda_b$, and $\gamma = \kappa_a \kappa_b^{-1} = \kappa_i^\epsilon (\kappa_j^\epsilon)^{-1}$ is one of α ($(a,b) = (1,2),(3,4)$), θ ($(1,4)$), $\beta^{-1} = \alpha\Psi^{-1}$ ($(3,2)$) or the inverse of one of these ((a,b) reversed). Since $\bar{a}$ kills $\mathfrak{m}_\mathcal{O}$, it factors as $\mathcal{O} \to \mathbb{F}_\mathcal{O} \xrightarrow{\iota} k$ with ι a field embedding; so $\bar{a} \circ \gamma = \iota \circ \bar{\gamma}$, and k is an $\mathbb{F}_\mathcal{O}$-vector space through ι. Write $\delta_x(g) = \bar{a}(\kappa_x^\epsilon(g)) \in k^\times$. By (Ov1),

$$c(gh)_{ij} = \sum_x c(g)_{ix} c(h)_{xj} = \delta_i(g)\, c(h)_{ij} + c(g)_{ij}\, \delta_j(h),$$

so $z(g) = c(g)_{ij} \delta_j(g)^{-1}$ satisfies $z(gh) = z(g) + \iota(\bar{\gamma}(g)) z(h)$. It is locally constant (c is, and δ_j takes finitely many values on the cosets of an open subgroup, $\bar{\kappa}_x$ having open kernel).

Step 2: the local conditions. (i) If $g \in I_{\bar{w}}$, $\bar{w}$ above a prime not in T, and $\bar{\theta}(g) = \bar{\Psi}(g) = \bar{\alpha}(g) = 1$, then $\bar{r}_0(g) = 1$, so $c(g) = 1$ and $z(g) = 0$. (ii) At ℓ: by (Ov2), for $s \in D_\ell$, $z(s) = m(\delta_j(s) - \delta_i(s))\delta_j(s)^{-1} = \iota(\bar{\gamma}(s))(-m) - (-m)$. (iii) At $v \in T$, $v \neq \ell$: then $v \in S_\ell(\chi)$, so the v-component of the ℓ-power order part of χ is a non-trivial character of $(\mathbb{Z}/v^e\mathbb{Z})^\times$ of ℓ-power order ($e \geq 1$), whence $\ell \mid v - 1$; χ_{cyc} is unramified at v with Frobenius value v, so $\bar{\chi}_{\mathrm{cyc}}$ is trivial on D_v. By (Sg2), $\overline{\xi\Psi^n}(\sigma_v) \neq 1$ for every n, and $\bar{\alpha} = \bar{\xi}$, $\bar{\theta} = \bar{\xi}\bar{\Psi}$, $\bar{\beta}^{-1} = \bar{\xi}\bar{\Psi}^{-1}$ (as $\bar{\chi}_{\mathrm{cyc}} = \bar{\omega}_\ell$); hence $\bar{\gamma}(\sigma_v) \neq 1$. Let $M = \mathbb{F}_{\mathcal{O}}$ with $D_v \cong G_{\mathbb{Q}_v}$ acting through $\bar{\gamma}$. Then $H^0(\mathbb{Q}_v, M) = 0$ ($\bar{\gamma}(\sigma_v) - 1$ is invertible), the Cartier dual $M^\vee$ has σ_v acting by $\bar{\gamma}(\sigma_v)^{-1}\bar{\chi}_{\mathrm{cyc}}(\sigma_v) = \bar{\gamma}(\sigma_v)^{-1} \neq 1$, so $H^0(\mathbb{Q}_v, M^\vee) = 0$ and by local duality $H^2(\mathbb{Q}_v, M) = 0$; by the local Euler characteristic formula for $v \neq \ell$, $H^1(\mathbb{Q}_v, M) = 0$ (Theorem B.29, items 2 and 4, [249]). So every continuous cocycle of D_v with values in M is a coboundary.

Step 3: $z = 0$. Let $\pi\colon k \to \mathbb{F}_{\mathcal{O}}$ be $\mathbb{F}_{\mathcal{O}}$-linear. Then $\pi \circ z$ is locally constant and $\pi(z(gh)) = \pi(z(g)) + \bar{\gamma}(g)\pi(z(h))$. By Step 2, (i) it vanishes on the inertia elements named in the definition of residual Selmer cocycles, (ii) on D_ℓ it equals $\bar{\gamma}(s)y_\ell - y_\ell$ with $y_\ell = -\pi(m)$, and (iii) for $v \in T \setminus \{\ell\}$ its restriction to D_v is a continuous cocycle with values in M, hence a coboundary. So $\pi \circ z$ is a residual Selmer cocycle for γ, and by (Sg6) $\pi(z(g)) = \bar{\gamma}(g)y_\pi - y_\pi$ for some $y_\pi \in \mathbb{F}_{\mathcal{O}}$. As $c(g_\circ)$ is diagonal, $z(g_\circ) = 0$; and $\bar{\gamma}(g_\circ) \neq 1$ because $\delta_i(g_\circ) \neq \delta_j(g_\circ)$. Hence $(\bar{\gamma}(g_\circ) - 1)y_\pi = 0$, $y_\pi = 0$ and $\pi \circ z = 0$. Since the $\mathbb{F}_{\mathcal{O}}$-linear functionals separate the points of k, $z = 0$, that is $c(g)_{ij} = 0$ for all g.

Step 4: $m = 0$. By (Ae1) of Lemma 45.27, $\bar{\kappa}_a(\sigma_\ell) \neq \bar{\kappa}_b(\sigma_\ell)$, i.e. $\delta_i(\sigma_\ell) \neq \delta_j(\sigma_\ell)$ (ι is injective). With $s = \sigma_\ell \in D_\ell$ in (Ov2) and $c(\sigma_\ell)_{ij} = 0$, $m(\delta_j(\sigma_\ell) - \delta_i(\sigma_\ell)) = 0$, so $m = 0$. □

Lemma 52.7. *Let B be a commutative ring and $y \in \mathrm{M}_4(B)$ lower triangular with diagonal entries 1. Then y has similitude 1 if and only if*

$$y_{43} = -y_{21} \qquad \text{and} \qquad y_{42} = y_{31} - y_{21}y_{32}.$$

Proof. Write $y_{21} = a$, $y_{31} = b$, $y_{32} = c$, $y_{41} = d$, $y_{42} = e$, $y_{43} = f$. As $(J_4)_{14} = (J_4)_{23} = 1$, $(J_4)_{32} = (J_4)_{41} = -1$, the rows of $J_4 y$ are the fourth row of y, the third row, minus the second and minus the first:

$$(d, e, f, 1), \quad (b, c, 1, 0), \quad (-a, -1, 0, 0), \quad (-1, 0, 0, 0).$$

The rows of ${}^t y$ are $(1, a, b, d)$, $(0, 1, c, e)$, $(0, 0, 1, f)$, $(0, 0, 0, 1)$, so

$${}^t y\, J_4\, y = \begin{pmatrix} 0 & e + ac - b & f + a & 1 \\ b - ac - e & 0 & 1 & 0 \\ -a - f & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

This equals J_4 if and only if $e + ac - b = 0$ and $f + a = 0$, which are the two identities. □

52.2 Case (II): towers, Gysin steps and links between residual pairs

This subsection constructs the towers and the steps of the chains in case (II). A fit and tame auxiliary tower of case (II) with a prescribed admissible labelling exists (Proposition 52.9); its fields are cut out by two odd quadratic characters and an abelian condition (Lemma 52.10). Along such towers the lattice can be lifted from $\mathbb{Q}$ to the tower, auxiliary primes can be killed, and residual pairs can be restricted (Proposition 52.8). The steps of a chain are of two kinds.

A Gysin step produces a residual pair with a new entry, ramified with prescribed inertial values at an auxiliary split prime (Proposition 52.11), from a cocycle that is locally trivial at a finite set of places and ramified at one auxiliary place (Lemma 52.12). A link joins two residual pairs by an affine line with a fixed core, moving the Levi part or one flag coordinate (Lemma 52.13).

Proposition 52.8. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$.*

(Ux1) *If $\mathfrak{D}$ is attached to a host character in case (II), then for every admissible labelling ϵ there is a fit and tame auxiliary tower of case (II) for $\mathfrak{D}$ with labelling ϵ.*

(Ux2) *If $\mathscr{T}$ is a fit and tame auxiliary tower and W a finite set of primes each of which splits for $\mathscr{T}$, there is a tame auxiliary tower $\mathscr{T}' \geq \mathscr{T}$ such that every $w \in W$ dies in the field of $\mathscr{T}'$.*

(Ux3) *Let $\mathscr{T}$ be a fit auxiliary tower with labelling ϵ, field F and places $\bar{v}$ ($v \in \Sigma$), (K, V, a) a test field with residue field k_V, and let $r' \colon G_{\mathbb{Q}} \to \mathrm{GL}_4(V)$, $n' \in \mathrm{GL}_4(V)$ satisfy the following conditions.*

- *There are a complete discrete valuation ring Λ' with finite residue field and an injective local homomorphism $j' \colon \Lambda' \to V$ such that the entries of n' and of every $r'(g)$ lie in $j'(\Lambda')$ and are $\mathfrak{m}_{\Lambda'}$-adically continuous in g.*
- *$r'(g)$ has similitude $a(\theta(g))$ and diagonal entries congruent to $a(\kappa_i^{\epsilon}(g))$ modulo $\mathfrak{m}_V$.*
- *$r'(g_{\mathscr{T}})$ is diagonal.*
- *$r'(g) = 1$ for $g \in I_{\bar{w}}$ with $\bar{r}_0(g) = 1$, $\bar{w}$ above a prime not in T, and for $g \in I_v$ with $\chi(g) = 1$, $v \in T \setminus \{\ell\}$.*
- *n' is lower triangular with diagonal 1 and similitude 1.*
- *For $s \in D_\ell$, $n'^{-1} r'(s) n'$ is upper triangular with diagonal entries $\psi_i(s) \equiv a(\kappa_i^{\epsilon}(s))$ modulo $\mathfrak{m}_V$, and $\psi_i(s) = a(\kappa_i(s))$ for $s \in I_\ell$.*
- *The reductions of r' and n' are supported in Σ_P and Σ_Q respectively.*

Then there are $n_v \in \mathrm{GL}_4(V)$ with entries in $j'(\Lambda')$, $v \in \Sigma$, such that $P = (r'|_{G_F}, (n_v)_v)$ is a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in V (for a), $\psi^P_{v,i}(s) = a(\kappa_i(s))$ for $s \in I_v$, the reduction of P is a residual pair over $(\mathscr{T}, \emptyset)$ with values in k_V whose flags are supported in Σ_Q, and for every $v \in \Sigma$ there is $g_v \in G_{\mathbb{Q}}$ with $g_v(\bar{\ell}) = \bar{v}$ such that $n_v^{-1} r'(g_v) n'$ is upper triangular.

(Ux4) *Let $\mathscr{T}_0 \leq \mathscr{T}$ be fit auxiliary towers, W a finite set of primes each of which splits for $\mathscr{T}_0$ and dies in the field of $\mathscr{T}$, k a field and $\bar{a} \colon \mathcal{O} \to k$ a ring homomorphism with $\bar{a}(\mathfrak{m}_{\mathcal{O}}) = 0$. For every residual pair s over $(\mathscr{T}_0, W)$ with values in k whose first component is supported in Σ_Q, there is a residual pair over $(\mathscr{T}, \emptyset)$ with values in k lying over s.*

Proof. (Ux1) is Proposition 52.9.

(Ux2). Let $w \in W$. As w splits for $\mathscr{T}$, $D_{\bar{w}} \subseteq \ker \omega_\ell$ for a place $\bar{w}$ above w, so $w \neq \ell$ (ω_ℓ is non-trivial on $I_{\bar{\ell}}$) and $w \equiv 1 \pmod{\ell}$ (the Frobenius at w acts on μ_ℓ by w); in particular w is odd. Let K_w be the subfield of degree ℓ of $\mathbb{Q}(\mu_w)$: cyclic of degree ℓ, totally real (ℓ is odd), totally ramified at w and unramified elsewhere. By the Kronecker–Weber theorem, the fixed field of the kernel of a non-trivial continuous homomorphism $G_{\mathbb{Q}} \to \mathbb{Z}/\ell\mathbb{Z}$ that is trivial on all inertia groups at places not above w is a cyclic field of degree ℓ unramified outside w, hence contained in some $\mathbb{Q}(\mu_{w^n})$; as $\ell \neq w$, the only subfield of degree ℓ of $\mathbb{Q}(\mu_{w^n})$ (Galois group $(\mathbb{Z}/w^n)^\times$,

cyclic) is K_w. So w dies in every field containing K_w. Let F' be the compositum of F and the K_w, $w \in W$: totally real, abelian over $\mathbb{Q}$, and $\mathrm{Gal}(F'/F)$ is an elementary abelian ℓ-group, so $g^\ell \in G_{F'}$ for $g \in G_F$; put $g_{\mathscr{T}'} = g^\ell_{\mathscr{T}}$, and keep ϵ and E. Every $G_{F'}$-orbit of places above ℓ contains a place inducing the same place of E as $\bar\ell$: it is $G_{F'}g\bar v$ for some $v \in \Sigma$ and $g \in G_F$; the place $g\bar v$ induces $g|_E$ of the place induced by $\bar v$, which is that of $\bar\ell$; if $g|_E \neq \mathrm{id}$, take $h \in G_{F'}$ with $h|_E \neq \mathrm{id}$ (it exists: F' is totally real and E imaginary), so $hg|_E = \mathrm{id}$ and $hg\bar v$ is in the orbit. Let Σ' be the set of these orbits and $\bar v'$ such representatives. Then $\mathscr{T}' = (\epsilon, E, F', g_{\mathscr{T}'}, \Sigma', (\bar v'))$ is an auxiliary tower with $\mathscr{T} \leq \mathscr{T}'$, every $w \in W$ dies in F', and $\mathscr{T}'$ is tame: for $\bar w$ above ℓ and $g \in I_{\bar w}$, tameness of $\mathscr{T}$ gives m prime to ℓ with $g^m \in G_F$; F'/F is unramified above ℓ (each K_w is), so $g^m \in I_{\bar w} \cap G_F$ acts trivially on F'.

(Ux3). Let ψ_i be the diagonal entries of $n'^{-1}r'(s)n'$, $s \in D_\ell$; they are continuous characters of D_ℓ with values in $V^\times$ ($n'^{-1}r'(\cdot)n'$ is an upper triangular representation with continuous entries). Fix $v \in \Sigma$ and $g_v \in G_{\mathbb{Q}}$ with $g_v(\bar\ell) = \bar v$ ($G_{\mathbb{Q}}$ acts transitively on the places above ℓ); then $g_v^{-1}D_v g_v \subseteq D_\ell$ and $g_v^{-1}I_v g_v \subseteq I_\ell$. Let $Y = r'(g_v)n'$, a similitude of multiplier $\mu = a(\theta(g_v))$ with entries in $j'(\Lambda')$, and $\bar Y = X\bar n$ its reduction, $X = \overline{r'(g_v)}$ supported in Σ_P with diagonal $(\delta_i) = (\overline{a(\kappa_i^\epsilon(g_v))})$ and $\bar n = \overline{n'}$ supported in Σ_Q. The first row of X is $\delta_1 e_1^t$ (weights $w_\alpha, w_\Psi, w_\theta \notin \Sigma_P$ at $(1,2),(1,3),(1,4)$), so $\bar Y_{11} = \delta_1$, $\bar Y_{12} = 0$; $\bar Y_{22} = X_{22} + X_{23}\bar n_{32} = \delta_2$ ($X_{24} = 0$, $\bar n_{32} = 0$); $\bar Y_{32} = X_{32} + X_{33}\bar n_{32} + X_{34}\bar n_{42} = 0$ (X_{32}, X_{34} have weights $-w_\beta$, $w_\alpha \notin \Sigma_P$). The leading principal minors of $\bar Y$ are δ_1, $\delta_1\delta_2$ (the leading 2×2 block is lower triangular), $\det \bar Y_{[3],[3]} = \det \bar Y\,(\bar Y^{-1})_{44} = \bar\mu^2\bar\mu^{-1}(\bar Y^\dagger)_{44} = \bar\mu\,\delta_1$ (a similitude of multiplier μ has determinant μ^2), and $\bar\mu^2$: all non-zero. So the leading principal minors of Y are units of $j'(\Lambda')$, and $Y = n_v b_v$ with n_v lower triangular with diagonal 1 and b_v upper triangular, both with entries in $j'(\Lambda')$. As in any such factorisation, b_v has first row that of Y and $(n_v)_{i2} = (Y_{i2} - (n_v)_{i1}(b_v)_{12})/(b_v)_{22}$, so modulo $\mathfrak{m}_V$: $\overline{(n_v)_{32}} = \bar Y_{32}/\bar Y_{22} = 0$, and $\bar n_v$ is supported in Σ_Q (the only lower place of weight outside Σ_Q is $(3,2)$). *n_v has similitude* 1*:* $\tau(Z) = J_4^{-1}\,{}^tZ^{-1}J_4$ is an automorphism of $\mathrm{GL}_4(K)$ preserving lower unitriangular and upper triangular matrices ($(J_4^{-1}ZJ_4)_{ab} = \pm Z_{5-a,5-b}$, with sign $+$ on the diagonal), and $\tau(Y) = \mu^{-1}Y$; uniqueness of the factorisation gives $\tau(n_v) = n_v$. *The pair.* Put $P = (r'|_{G_F}, (n_v)_{v\in\Sigma})$. (Mq1): similitudes by hypothesis; $N = N_\emptyset$ consists of $g \in I_{\bar w} \cap G_F$ with $\bar r_0(g) = 1$, $\bar w$ not above ℓ; if $\bar w$ lies above a prime not in T, $r'(g) = 1$ by hypothesis; if above $v \in T \setminus \{\ell\}$, then $\bar w = \sigma\bar v$ for some $\sigma \in G_{\mathbb{Q}}$, $\chi(g) = 1$ by (Oq5), and $\sigma^{-1}g\sigma \in I_v$ with $\chi(\sigma^{-1}g\sigma) = 1$, so $r'(g) = r'(\sigma)r'(\sigma^{-1}g\sigma)r'(\sigma)^{-1} = 1$. (Mq2), (Mq3): by hypothesis and the above. (Mq4): for $s \in D_v$, $s' = g_v^{-1}sg_v \in D_\ell$ and $n_v^{-1}r'(s)n_v = b_v\,(n'^{-1}r'(s')n')\,b_v^{-1}$ is upper triangular with diagonal $\psi^P_{v,i}(s) = \psi_i(s')$; in particular $\psi^P_{v,i}(s) = a(\kappa_i(s))$ for $s \in I_v$. Let $s \in \mu_v$, $s^{\ell^a} \in \overline{[D_v, D_v]}$, and $D' = g_v^{-1}D_v g_v$, $I' = g_v^{-1}I_v g_v$, $s' = g_v^{-1}sg_v$. Since $D'/I' \cong \hat{\mathbb{Z}}$ is torsion-free and abelian, $s' \in I'\,\overline{[D',D']}$: $s' = tc$ with $t \in I'$, $c \in \overline{[D',D']}$. The characters ψ_i, κ_i kill c, so $\psi_i(s') = \psi_i(t) = a(\kappa_i(t)) = a(\kappa_i(s')) = a(\kappa_i(s))$. Finally $\kappa_i(s) = 1$: $\chi_{\mathrm{cyc}}(s)$ is a root of unity of ℓ-power order in $\mathbb{Z}_\ell^\times$ (ℓ odd), hence 1, so $\omega_\ell(s) = 1$; $\xi(s)$ has order prime to ℓ and dividing ℓ^a; $\chi(s) = 1$ by (Oq5) ($\chi(s)^m = 1$ with m prime to ℓ, and $\chi(s)^{\ell^a} = 1$); so $\Psi(s) = \alpha(s) = \theta(s) = 1$. So $\psi^P_{v,i} = 1$ on μ_v, and P is a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in V. *Its reduction* is locally constant (continuity of the entries), satisfies (Wq1) (the diagonal of $\overline{r'(g)}$ is $\overline{a(\kappa_i^\epsilon(g))}$ by hypothesis, and $\overline{\psi^P_{v,i}(s)} = \overline{\psi_i(s')} = \overline{a(\kappa_i^\epsilon(s'))} = \overline{a(\kappa_i^\epsilon(s))}$), and (Wq2): $\overline{r'}$ is supported in Σ_P and the $\bar n_v$ in Σ_Q. Finally $n_v^{-1}r'(g_v)n' = b_v$ is upper triangular, and the entries of n_v lie in $j'(\Lambda')$.

(Ux4). Write $s = (r_s, (n_{s,v_0})_{v_0\in\Sigma_0})$, $\chi_i = \bar a \circ \chi_i^{\mathscr{T}_0}$, Σ_0, F_0 for $\mathscr{T}_0$ and Σ, F for $\mathscr{T}$. Put $r = r_s|_{G_F}$. For $v \in \Sigma$ choose $v_0 \in \Sigma_0$ and $g \in G_{F_0}$ with $g(\bar v_0) = \bar v$; $L = r_s(g)n_{s,v_0}$ is lower triangular ($r_s(g)$ is supported in $\Sigma_Q \subseteq \Sigma_M$) with diagonal $\Delta = \mathrm{diag}(\chi_i(g))$, of similitude $\bar a(\theta(g))$, as is Δ ($\chi_1\chi_4 = \chi_2\chi_3 = \bar a \circ \theta$); put $n_v = L\Delta^{-1}$: lower unitriangular of similitude 1, and $n_v^{-1}r_s(g)n_{s,v_0} = \Delta$ is upper triangular. Let $c = (r, (n_v))$. (Mq2): $g_{\mathscr{T}}$ is a power of $g_{\mathscr{T}_0}$. (Mq3): shown. (Mq4) and (Wq1): for $s_1 \in D_v$, $s_0 = g^{-1}s_1 g \in D_{v_0}$ and $n_v^{-1}r(s_1)n_v =$

$\Delta\,(n_{s,v_0}^{-1} r_s(s_0) n_{s,v_0})\,\Delta^{-1}$ is upper triangular with diagonal $\chi_i(s_0) = \chi_i(s_1)$; if $s_1 \in \mu_v$ then $s_0 \in \mu_{v_0}$ ($g^{-1}\overline{[D_v, D_v]}g \subseteq \overline{[D_{v_0}, D_{v_0}]}$), where these values are 1 by (Mq4) for s. The diagonal of r is that of r_s. (Wq2): r is supported in $\Sigma_Q \subseteq \Sigma_M$. Local constancy is inherited. (Mq1): similitudes are inherited; let $g' \in N_\emptyset$ for $\mathscr{T}$, in $I_{\bar w} \cap G_F$ with $\bar r_0(g') = 1$, $\bar w$ above a prime $q \neq \ell$. If $q \notin W$, $g' \in N_W$ for $\mathscr{T}_0$ and $r(g') = 1$. If $q \in W$: as q splits for $\mathscr{T}_0$, $D_{\bar w} \subseteq G_{F_0} \cap \ker \bar r_0$, so on $I_{\bar w}$ the matrices $r_s(h)$ are lower unitriangular, of similitude 1, supported in Σ_Q, hence of the form $1 + A$ with $A = a(E_{21} - E_{43}) + b(E_{31} + E_{42}) + eE_{41}$ (for lower unitriangular y, the first row of ${}^t y J_4 y$ is $(0,\, y_{42} + y_{21}y_{32} - y_{31},\, y_{43} + y_{21},\, 1)$, so similitude 1 forces $y_{43} = -y_{21}$ and $y_{42} = y_{31} - y_{21}y_{32}$, with $y_{32} = 0$ here); since $A^2 = (ba - ab)E_{41} = 0$ (the only non-zero products of the matrix units involved are $E_{42}E_{21} = E_{43}E_{31} = E_{41}$), $(1+A)^n = 1 + nA$, and these matrices have order 1 or ℓ in characteristic ℓ. So $r_s(I_{\bar w})$ is a finite ℓ-group (finite by local constancy); $r_s|_{I_{\bar w}}$ kills the wild inertia group (pro-q) and factors through the tame quotient $\prod_{p \neq q} \mathbb{Z}_p$, so its image is cyclic of order 1 or ℓ, and its kernel contains the kernel of every surjection $I_{\bar w} \to \mathbb{Z}/\ell\mathbb{Z}$ (all such have the same kernel). Since q dies in F and $q \equiv 1 \pmod{\ell}$ (as in (Ux2)), F contains the field K_q of (Ux2), and $I_{\bar w} \to \mathrm{Gal}(K_q/\mathbb{Q}) \cong \mathbb{Z}/\ell\mathbb{Z}$ is surjective (K_q is totally ramified at q); so r_s is trivial on $I_{\bar w} \cap G_{K_q} \supseteq I_{\bar w} \cap G_F \ni g'$. Hence c is a residual pair over $(\mathscr{T}, \emptyset)$, and it lies over s by construction. □

Proposition 52.9. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum attached to a host character in case (II). For every admissible labelling ϵ there is a fit and tame auxiliary tower of case (II) for $\mathfrak{D}$ with labelling ϵ.*

Proof. Let $\mathfrak{D}$ be attached to $(\psi, \mathfrak{h}, t, \chi)$ in case (II) and let ϵ be admissible. Then $N = N_{\mathfrak{h}}$, $p' = m = 1$, so $\chi_{\mathrm{pr}} = \mathbf{1}$ and $\xi = \chi_{\mathrm{dy}}$ is a primitive odd character modulo $2^{n'}$, and S is the set of primes dividing $2\ell N$. From $2^{n'} \geq 2^{c_2(\chi)+6} o_\psi \ell^3 N_\psi^2$ we get $n' \geq c_2(\chi) + 6$ and $n' \geq 11$.

Step 0: facts about the characters.

(a) $\Psi = \chi\chi_{\mathrm{cyc}}^e$, ξ and ω_ℓ are unramified outside S, $\Psi(c) = 1$, $\xi(c) = \omega_\ell(c) = -1$ ((Sg1) and $\chi_{\mathrm{cyc}}(c) = -1$), and $\bar\alpha = \bar\xi$ because $\bar\chi_{\mathrm{cyc}} = \bar\omega_\ell$. ξ has 2-power order and is unramified at every odd prime (its modulus is $2^{n'}$).

(b) *Decomposition of χ.* Write $\chi'' = \chi''_\circ \chi''_\ell$ with χ''_ℓ the ℓ-component of χ'' and $\chi''_\circ$ the product of its other components, and put $\kappa_\circ = \chi''_\circ$: a character of ℓ-power order ℓ^j, unramified at ℓ and outside S, with $\kappa_\circ(c) = 1$ ($\kappa_\circ(c)^2 = 1$ and ℓ^j is odd). The ℓ-component of χ' has order prime to ℓ, so it factors through $(\mathbb{Z}/\ell\mathbb{Z})^\times$ and equals ω_ℓ^b for some b; hence $\chi' = \chi^{(\ell)}\omega_\ell^b$ with $\chi^{(\ell)}$ of order prime to ℓ, unramified at ℓ and outside S, and of even order (case (II)). So

$$\chi = \chi^{(\ell)}\,\omega_\ell^b\,\kappa_\circ\,\chi''_\ell, \qquad \bar\Psi = \overline{\chi^{(\ell)}}\,\bar\omega_\ell^{\,b+e},$$

the second because $\kappa_\circ$, χ''_ℓ have ℓ-power order. If $\chi_{\mathrm{cyc}}(g) = 1$, then g fixes μ_{ℓ^∞}, so $\omega_\ell(g) = \chi''_\ell(g) = 1$ and $\chi(g) = \chi'(g)\kappa_\circ(g)$.

(c) *An element with $\bar\Psi = -1$.* $\chi^{(\ell)}$ has even order $2k'$, so some σ has $\chi^{(\ell)}(\sigma)^{k'} \neq 1$; as its square is 1, $\chi^{(\ell)}(\sigma^{k'}) = -1$. Choose $\tau \in I_{\bar\ell}$ with $\omega_\ell(\tau) = \omega_\ell(\sigma^{k'})$; $\chi^{(\ell)}(\tau) = 1$. Then $t_0 = (\sigma^{k'}\tau^{-1})^{\ell^j}$ has $\omega_\ell(t_0) = 1$, $\kappa_\circ(t_0) = 1$, $\chi^{(\ell)}(t_0) = (-1)^{\ell^j} = -1$, hence $\bar\Psi(t_0) = -1$ by (b).

(d) *An element at 2.* Let $c' = \max(c_2(\chi), 2)$ and $y_2 \in I_{\bar 2}$ map to $1 + 2^{c'} \in \mathbb{Z}_2^\times$. The 2-component of χ has conductor $2^{c_2(\chi)}$, so $\chi(y_2) = 1$; χ_{cyc}, ω_ℓ are unramified at 2, so $\Psi(y_2) = 1$ and $\omega_\ell(y_2) = 1$; the 2-component of $\kappa_\circ$ is a character of a 2-group of odd order, so $\kappa_\circ(y_2) = 1$. As χ_{dy} is primitive, its restriction to the cyclic group $(1 + 4\mathbb{Z})/(1 + 2^{n'}\mathbb{Z})$ is injective (a

non-trivial kernel would contain $1+2^{n'-1}$, and χ_{dy} would factor modulo $2^{n'-1}$); so $\xi(y_2)$ has order $2^{n'-c'} \geq 16$, and $\bar\xi(y_2)$, of the same order, has $\bar\xi(y_2)^n \neq 1$ for $1 \leq n \leq 4$ and for every odd n.

(e) *An element at ℓ.* Let $\sigma_\ell \in D_\ell$ be the arithmetic Frobenius element of (Sg2), with $\chi_{\mathrm{cyc}}(\sigma_\ell) = 1$. Write $N = \ell^k N'$ with $\ell \nmid N'$ and $\chi = \chi_\ell \chi_\circ$ (ℓ-component and a character modulo N'). Then $\omega_\ell(\sigma_\ell) = \chi_\ell(\sigma_\ell) = 1$, $\Psi(\sigma_\ell) = \chi_\circ(\ell)$, $\xi(\sigma_\ell) = \chi_{\mathrm{dy}}(\ell)$. The order o of $\chi_\circ(\ell)$ divides the order of ℓ in $(\mathbb{Z}/N'\mathbb{Z})^\times$, so $o < N$; write $o = 2^s u$ with u odd, and put $y_\ell = \sigma_\ell^{\ell^j o} \in D_\ell$. Then $\omega_\ell(y_\ell) = \Psi(y_\ell) = \kappa_\circ(y_\ell) = 1$. Let $v = v_2(\ell^2-1) \geq 3$. We claim $n' \geq s+v+3$. Let f be the order of ℓ modulo $\mathrm{lcm}(4, \mathrm{ord}\,\vartheta)$, so $\ell^f \equiv 1 \pmod 4$ and $n_0 = n_0(\ell,\chi) = v_2(\ell^f-1) \geq 2$. If f is even, $\ell^2-1 \mid \ell^f-1$, so $2n_0 \geq 2v \geq v+3$. If f is odd, $\ell \equiv \ell^f \equiv 1 \pmod 4$, $(\ell^f-1)/(\ell-1) = \sum_{i<f} \ell^i$ is odd, so $n_0 = v_2(\ell-1) = v-1$ and $2n_0 = 2v-2 \geq v+1$. In both cases $\log_2 \ell + 2n_0 > v+2$. From $2^{n'} \geq \ell N 4^{n_0} > \ell\, 2^s 4^{n_0}$ we get $n' > s + \log_2\ell + 2n_0 > s+v+2$, which is the claim. Now $\xi(y_\ell)^4 = \chi_{\mathrm{dy}}(\ell^2)^{2^{s+1}\ell^j u}$, and $\chi_{\mathrm{dy}}(\ell^2)$ has order $2^{n'-v}$ (injectivity as in (d)); since $\ell^j u$ is odd and $n'-v > s+1$, $\xi(y_\ell)^4 \neq 1$, hence $\bar\xi(y_\ell)^4 \neq 1$.

Step 1: the real quadratic field and the subgroup. By Dirichlet's theorem choose a prime $q_0 \notin S$ with $q_0 \equiv 1 \pmod 8$ and $q_0 \equiv 1 \pmod p$ for every odd $p \in S$. Let $k_0 = \mathbb{Q}(\sqrt{q_0})$, real quadratic, with character η_0: ramified exactly at q_0, $\eta_0(c) = 1$, and every prime of S, in particular 2 and ℓ, splits in k_0, so η_0 is trivial on $D_{\bar p}$ for every place $\bar p$ above a prime of S. Fix $i_0 \in I_{\bar q_0}$ with $\eta_0(i_0) = -1$. Put $S' = S \cup \{q_0\}$ and

$$H' = \{g \in G_{\mathbb{Q}} :\ \bar\Psi(g) = \eta_0(g),\ \kappa_\circ(g) = 1\},$$

the kernel of the continuous character $g \mapsto (\bar\Psi(g)\eta_0(g), \kappa_\circ(g))$: it is open, contains the commutator subgroup, contains c ($\bar\Psi(c) = 1 = \eta_0(c)$, $\kappa_\circ(c) = 1$), and contains $I_{\bar p}$ for $p \notin S'$ (all three characters are unramified there). Apply Lemma 52.10 to $\mathfrak{D}, \epsilon, S', H'$; it gives $q_1, q_2 \notin S'$, E, F, L, J. The characters $\Psi, \xi, \omega_\ell, \kappa_\circ, \chi^{(\ell)}, \eta_0$ are unramified at q_1, q_2, so (Zf3) preserves their values. Two consequences:

(f) if $g \in G_{\mathbb{Q}}$ has $\kappa_\circ(g) = 1$ and $\bar\Psi(g) \in \{\pm 1\}$, there is $g' \in G_L$ with $\Psi(g') = \Psi(g)$, $\xi(g') = \xi(g)$, $\omega_\ell(g') = \omega_\ell(g)$: choose $a \in \{0,1\}$ with $\eta_0(g\, i_0^a) = \bar\Psi(g)$; then $g\, i_0^a \in H'$ (Ψ, $\kappa_\circ$ are unramified at q_0) and (Zf3) applies;

(g) $H' \cap D_{\bar p} \subseteq G_L$ for every place $\bar p$ above a prime of S', by (Zf1) and $G_L = H' \cap J$; and on $G_F \subseteq H'$, $\bar\Psi = \eta_0$ and $\kappa_\circ = \mathbf{1}$.

Step 2: four elements of G_L.

- $y_c \in G_L$ with $\bar\xi(y_c) = -1$, $\bar\Psi(y_c) = 1$, $\bar\omega_\ell(y_c) = -1$: (f) applied to c, by (a), (b).
- y_2 of (d) lies in H' ($\bar\Psi(y_2) = 1 = \eta_0(y_2)$ since 2 splits in k_0; $\kappa_\circ(y_2) = 1$) and in $D_{\bar 2}$, so $y_2 \in G_L$ by (g); $\bar\Psi(y_2) = 1$, $\omega_\ell(y_2) = 1$.
- $y_\Psi \in G_L$ with $\bar\Psi(y_\Psi) = -1$, $\omega_\ell(y_\Psi) = 1$, $\bar\xi(y_\Psi)^2 \neq 1$: (f) applied to t_0 of (c) gives $y' \in G_L$ with $\bar\Psi(y') = -1$, $\omega_\ell(y') = 1$; put $y_\Psi = y'$ if $\bar\xi(y')^2 \neq 1$, and $y_\Psi = y'y_2$ otherwise ($\bar\xi(y'y_2)^2 = \bar\xi(y_2)^2 \neq 1$, $\bar\Psi(y'y_2) = -1$, $\omega_\ell(y'y_2) = 1$).
- y_ℓ of (e) lies in $H' \cap D_\ell$ ($\bar\Psi(y_\ell) = 1 = \eta_0(y_\ell)$, $\kappa_\circ(y_\ell) = 1$), so $y_\ell \in G_L$ by (g).

Let $\mathscr{T} = (\epsilon, E, F, y_\Psi, \Sigma, (\bar v))$ be the auxiliary tower of (Zf4) with $g_\star = y_\Psi$; (Oq1), (Oq2) hold. Put $\varsigma = (-1)^\epsilon$. From $(\kappa_i^0) = (\theta, \Psi, \alpha, \mathbf{1})$, $(\kappa_i^1) = (\alpha, \mathbf{1}, \theta, \Psi)$ and (a): $\chi_1^{\mathscr{T}}/\chi_2^{\mathscr{T}} = \alpha$, $\chi_1^{\mathscr{T}}/\chi_3^{\mathscr{T}} = \Psi^\varsigma$, so on G_F

$$\overline{\mathrm{rt}_w} = \bar\xi^{w_1}\,\bar\Psi^{\varsigma w_2} \qquad (w = (w_1, w_2)).$$

The odd roots have w_1 odd, the roots $\pm w_\Psi$ have $w_1 = 0$, $w_2 = \pm 1$.

Step 3: the clauses. For a place $\bar w$ above p write $\bar w = h(\bar p)$; for $y \in D_{\bar p} \cap G_F$, $hyh^{-1} \in D_{\bar w} \cap G_F$ (G_F is normal) has the same character values, and likewise for inertia.

(Oq3) Let $w \in \mathbb{Z}^2$. If w_1 is even, $\overline{\mathrm{rt}_w}(y_c) = (-1)^{w_1} = 1 \neq -1 = \bar\omega_\ell(y_c)$; if w_1 is odd, $\overline{\mathrm{rt}_w}(y_2) = \bar\xi(y_2)^{w_1} \neq 1 = \bar\omega_\ell(y_2)$ by (d). For an odd root $\overline{\mathrm{rt}_w}(y_2) \neq 1$, and for $\pm w_\Psi$, $\overline{\mathrm{rt}_w}(y_\Psi) = (-1)^{\pm\varsigma} = -1 \neq 1$. For w_α and $1 \le n \le 4$, $\overline{\mathrm{rt}_{w_\alpha}}(y_2)^n = \bar\xi(y_2)^n \neq 1$ by (d).

(Oq4) Let η be as there and i the index with $\kappa_i^\epsilon = \mathbf{1}$; then $\eta = \bar\kappa_j^\epsilon$ on G_L for some j, and $\eta(y_2) = \eta(y_\Psi) = 1$ since $\omega_\ell(y_2) = \omega_\ell(y_\Psi) = 1$. But $\bar\theta(y_2) = \bar\xi(y_2)\bar\Psi(y_2) = \bar\xi(y_2) \neq 1$, $\bar\Psi(y_\Psi) = -1$, $\bar\alpha(y_2) = \bar\xi(y_2) \neq 1$; so $\kappa_j^\epsilon = \mathbf{1}$ and $\eta = 1$.

(Oq5) Let $g \in I_{\bar w} \cap G_F$, $\bar w$ not above ℓ, with $\bar r_0(g) = 1$. Then $\chi_{\mathrm{cyc}}(g) = 1$ and $\kappa_\circ(g) = 1$, so $\chi(g) = \chi^{(\ell)}(g)$ by (b), and $\overline{\chi^{(\ell)}}(g) = \bar\Psi(g) = 1$, so $\chi^{(\ell)}(g) = 1$ (order prime to ℓ). For $g \in G_F$ with $\chi_{\mathrm{cyc}}(g) = 1$, $\chi(g) = \chi'(g)$ by (b), so $\chi(g)^{m_0} = 1$ with $m_0 = \operatorname{ord}\chi'$ prime to ℓ.

(Oq6) For $g \in I_{\bar w} \cap G_F$, $\bar w$ above ℓ: $\bar\Psi(g) = \eta_0(g) = 1$ (ℓ splits in k_0), $\bar\alpha(g) = \bar\xi(g) = 1$ (ξ is unramified at ℓ), $\bar\theta = \bar\alpha\bar\Psi$; so $\bar r_0(g) = 1$.

(Oq7) $(\bar\theta, \bar\Psi, \bar\alpha, \mathbf{1})(y_\Psi) = (-x, -1, x, 1)$ with $x = \bar\xi(y_\Psi)$, $x^2 \neq 1$, $x \neq 0$: these four are pairwise distinct ($-x = x$ would give $x = 0$; the other coincidences give $x = \pm 1$ or $-1 = 1$), and λ_ϵ permutes them.

(Oq8) For $\bar w = h(\bar\ell)$ the conjugate of y_ℓ lies in $D_{\bar w} \cap G_F$, with $\bar\omega_\ell = 1$, $\overline{\mathrm{rt}_{w_\Psi}} = \bar\Psi^\varsigma = 1$ and $\overline{\mathrm{rt}_{w_\alpha}} = \bar\xi \neq 1$ ((e)).

(Uq1), (Uq2) On G_F, $\overline{\mathrm{rt}_{w_\Psi}} = \bar\Psi^\varsigma = \eta_0$ by (g); so its square is 1, it is trivial on $D_{\bar w} \cap G_F$ for $\bar w$ above ℓ (ℓ splits in k_0), and on $I_{\bar w} \cap G_F$ for every $\bar w$: away from q_0 because η_0 is unramified there, above q_0 because $\bar\Psi$ is. For $g \in G_F$, $\overline{\mathrm{rt}_{w_\Psi}}(g) = 1$ if and only if $\eta_0(g) = 1$, i.e. $g \in G_{k_0}$; and $2, \ell$ split in the real quadratic field k_0.

(Uq3) By (a).

(Uq4) $\overline{\mathrm{rt}_{w_\alpha}} = \bar\xi$; the conjugates of $y_\ell \in D_\ell \cap G_F$ and of $y_2 \in I_{\bar 2} \cap G_F$ have $\bar\xi^4 \neq 1$ by (e), (d).

So $\mathscr{T}$ is good, of case (II), and fit ($\mathfrak{D}$ is attached in case (II)). *Tame*: for $g \in I_{\bar w}$, $\bar w$ above ℓ, let $m_1 = \#\mathbb{F}_{\mathcal{O}}^\times$, prime to ℓ. Then $\bar\Psi(g^{m_1}) = 1 = \eta_0(g^{m_1})$ and $\kappa_\circ(g^{m_1}) = 1$ ($\kappa_\circ$ is unramified at ℓ), so $g^{m_1} \in H' \cap D_{\bar w} \subseteq G_L \subseteq G_F$ by (g). Hence $\mathscr{T}$ is a fit and tame auxiliary tower of case (II) with labelling ϵ. □

Lemma 52.10. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with set of primes S, ϵ an admissible labelling, $S' \supseteq S$ a finite set of primes, and $H' \subseteq G_{\mathbb{Q}}$ an open subgroup that contains the commutator subgroup of $G_{\mathbb{Q}}$, contains c, and contains $I_{\bar p}$ for every place $\bar p$ of $\overline{\mathbb{Q}}$ above a prime $p \notin S'$. Then there are primes $q_1 \neq q_2$, neither in S', and characters $\eta_1, \eta_2 \colon G_{\mathbb{Q}} \to \{\pm 1\}$ such that, with E the fixed field of $\ker\eta_1$, F the fixed field of $H' \cap \ker(\eta_1\eta_2)$, $L = FE$ and $J = \ker\eta_1 \cap \ker\eta_2$:*

(Zf1) *for $k = 1, 2$, $\eta_k(c) = -1$, η_k is trivial on $I_{\bar w}$ for every place $\bar w$ not above q_k and non-trivial on $I_{\bar w}$ for every place $\bar w$ above q_k; J contains $D_{\bar p}$ for every place $\bar p$ above a prime of S', and g^2 for every $g \in G_{\mathbb Q}$;*

(Zf2) *E is an imaginary quadratic field in which every prime of S' splits; F is a totally real number field of even degree, abelian over $\mathbb Q$; $G_F = H' \cap \ker(\eta_1\eta_2)$, $G_L = H' \cap J$, and $I_{\bar w} \cap G_F \subseteq G_L$ for every place $\bar w$ of $\overline{\mathbb Q}$;*

(Zf3) *for every $g \in H'$ there is $g' \in G_L$ with $\gamma(g') = \gamma(g)$ for every homomorphism γ from $G_{\mathbb Q}$ to an abelian group that is trivial on the inertia groups of all places above q_1 and above q_2;*

(Zf4) *for every $g_\star \in G_F$ there are a finite set Σ and places $\bar v$ ($v \in \Sigma$) of $\overline{\mathbb Q}$ above ℓ such that $(\epsilon, E, F, g_\star, \Sigma, (\bar v)_{v\in\Sigma})$ is an auxiliary tower for $\mathfrak D$ satisfying (Oq1) and (Oq2).*

Proof. Step 1: the characters, (Zf1). The congruences $q \equiv 7 \pmod 8$ and $q \equiv -1 \pmod p$ for every odd $p \in S'$ define one class modulo $8\prod_p p$ prime to the modulus. By Dirichlet's theorem choose a prime $q_1 \notin S'$ in this class, and a prime $q_2 \notin S' \cup \{q_1\}$ in the same class. Let η_k be the character of $\mathbb Q(\sqrt{-q_k})$. As $q_k \equiv 3 \pmod 4$, η_k is ramified exactly at q_k (trivial on $I_{\bar w}$ for $\bar w$ not above q_k, non-trivial on $I_{\bar w}$ for $\bar w$ above q_k; for the latter use the conjugation invariance of the abelian character η_k) and $\eta_k(c) = -1$. For odd $p \in S'$, $-q_k \equiv 1 \pmod p$ is a non-zero square, and $-q_k \equiv 1 \pmod 8$; so every $p \in S'$ splits in $\mathbb Q(\sqrt{-q_k})$, that is η_k is trivial on $D_{\bar p}$ for every place $\bar p$ above p. Both characters take values in $\{\pm1\}$, so $\eta_k(g^2) = 1$. This is (Zf1).

Step 2: the fields, (Zf2). The subgroup $H'' = H' \cap \ker(\eta_1\eta_2)$ is open and contains the commutator subgroup, so F is a finite abelian extension of $\mathbb Q$ with $G_F = H''$. The character $\eta_1\eta_2$ is non-trivial (on $I_{\bar w}$, $\bar w$ above q_1, η_1 is non-trivial and η_2 is trivial by (Zf1)), so $[G_{\mathbb Q} : \ker(\eta_1\eta_2)] = 2$ divides $[G_{\mathbb Q} : H''] = [F : \mathbb Q]$. Since $c \in H'$ and $\eta_1\eta_2(c) = (-1)^2 = 1$, $c \in G_F$; every complex conjugation is hch^{-1} with $h \in G_{\mathbb Q}$ and G_F is normal, so all complex conjugations fix F and F is totally real. $\ker\eta_1$ has index 2 and does not contain c, so $E = \mathbb Q(\sqrt{-q_1})$ is imaginary quadratic, and the primes of S' split in it by Step 1. Next $G_L = G_F \cap G_E = H' \cap \ker(\eta_1\eta_2) \cap \ker\eta_1 = H' \cap J$. Finally let $g \in I_{\bar w} \cap G_F$. If $\bar w$ is not above q_1, then $\eta_1(g) = 1$ by (Zf1). If $\bar w$ is above q_1, then $\bar w$ is not above q_2, so $\eta_2(g) = 1$ and $\eta_1(g) = (\eta_1\eta_2)(g) = 1$. In both cases $g \in G_F \cap \ker\eta_1 = G_L$.

Step 3: moving into G_L, (Zf3). For $k = 1, 2$ fix a place $\bar q_k$ above q_k and $i_k \in I_{\bar q_k}$ with $\eta_k(i_k) = -1$ (Zf1). Then $\eta_{3-k}(i_k) = 1$ by (Zf1), and $i_k \in H'$ because $q_k \notin S'$. Let $g \in H'$ and choose $a_k \in \{0, 1\}$ with $(-1)^{a_k} = \eta_k(g)$; put $g' = g\, i_1^{a_1} i_2^{a_2}$. Then $g' \in H'$ and $\eta_k(g') = \eta_k(g)(-1)^{a_k} = 1$ for $k = 1, 2$, so $g' \in H' \cap J = G_L$ by (Zf2); and for γ as in (Zf3), $\gamma(g') = \gamma(g)\gamma(i_1)^{a_1}\gamma(i_2)^{a_2} = \gamma(g)$.

Step 4: the places above ℓ, (Zf4). The places of $\overline{\mathbb Q}$ above ℓ form one $G_{\mathbb Q}$-orbit, and its G_F-orbits are the fibres of the restriction to the finitely many places of F above ℓ; let Σ be the finite set of these G_F-orbits. Since $\ell \in S \subseteq S'$ splits in E, there are exactly two places of E above ℓ, and every element of $G_{\mathbb Q} \setminus G_E$, in particular $c \in G_F$, exchanges them. For $v \in \Sigma$ pick $\bar w \in v$; put $\bar v = \bar w$ if $\bar w$ and $\bar\ell$ induce the same place of E, and $\bar v = c(\bar w)$ otherwise. In the second case $\bar v \in v$ because $c \in G_F$, and the place of E below $c(\bar w)$ is the image under c of the place below $\bar w$, i.e. the place below $\bar\ell$. A place above ℓ lies in exactly one G_F-orbit, so it is G_F-conjugate to $\bar v$ for exactly one $v \in \Sigma$. With ϵ admissible, E imaginary quadratic, F totally real and abelian over $\mathbb Q$, and $g_\star \in G_F$, the tuple $(\epsilon, E, F, g_\star, \Sigma, (\bar v)_v)$ is an auxiliary tower for $\mathfrak D$. Its field $L = FE$ is the field of the lemma; (Oq1) is the even degree and the last assertion of (Zf2), and (Oq2) holds because $S \subseteq S'$ and every prime of S' splits in E. □

Proposition 52.11. *Let $\mathfrak D$ be a symplectic Eisenstein datum with coefficient ring $\mathcal O$, $\mathscr T$ a fit auxiliary tower for $\mathfrak D$ (field F, index set Σ), k a field, $\bar a\colon \mathcal O \to k$ a ring homomorphism with $\bar a(\mathfrak m_{\mathcal O}) = 0$, and W a finite set of primes each of which splits for $\mathscr T$. All residual pairs are with*

values in k; for a residual pair $c = (r_c, (n_{c,v}))$, its $-w_\Psi$ part *is the family of the entries $r_c(g)_{31}$, $r_c(g)_{42}$ ($g \in G_F$) and $(n_{c,v})_{31}$ ($v \in \Sigma$), and its $-w_\alpha$* part *the family of the entries $r_c(g)_{21}$, $r_c(g)_{43}$ and $(n_{c,v})_{21}$.*

(Gz) *Let p, p' be residual pairs over $(\mathscr{T}, W)$ whose first components are supported in Σ_P and whose flags are supported in Σ_Q. Then there are a finite set $W' \supseteq W$ of primes, each of which splits for $\mathscr{T}$, and residual pairs s, s' over $(\mathscr{T}, W')$ such that:*

- **(Gz1)** *the first components and all flags of s and s' are supported in Σ_Q;*
- **(Gz2)** *s has the same $-w_\Psi$ part as p, and s' the same $-w_\Psi$ part as p';*
- **(Gz3)** *the first component of s has an entry of weight $-w_\alpha$;*
- **(Gz4)** *s and s' have the same $-w_\alpha$ part.*

(Yz) *Let p be a residual pair over $(\mathscr{T}, W)$ whose first component is supported in Σ_M. Then there are a finite set $W' \supseteq W$ of primes, each of which splits for $\mathscr{T}$, and a residual pair s over $(\mathscr{T}, W')$ whose first component and flags are supported in Σ_Q, which has the same $-w_\alpha$ part as p, and whose first component has an entry of weight $-w_\Psi$.*

Proof. Step 0: notation. Let ϵ be the labelling of $\mathscr{T}$ and $\chi_i = \bar{a} \circ \chi_i^{\mathscr{T}}$, characters $G_F \to k^\times$ with values in the finite field $k_0 = \bar{a}(\mathcal{O})$; put $\chi_{ij} = \chi_i/\chi_j$ and $\Delta(g) = \mathrm{diag}(\chi_1(g), \dots, \chi_4(g))$. As $\chi_1\chi_4 = \chi_2\chi_3 = \bar{a} \circ \theta$ (for $\epsilon = 0$ the frame is $(\theta, \Psi, \alpha, \mathbf{1})$, for $\epsilon = 1$ it is $(\alpha, \mathbf{1}, \theta, \Psi)$, and $\Psi\alpha = \theta$), $\Delta(g)$ has similitude $\bar{a}(\theta(g))$, and $\chi_{43} = \chi_{21}$, $\chi_{42} = \chi_{31}$, $\chi_{41} = \chi_{21}\chi_{31}$. By (Oq7), $\chi_{21}(g_{\mathscr{T}})$, $\chi_{31}(g_{\mathscr{T}})$, $\chi_{41}(g_{\mathscr{T}})$ are $\neq 1$. For $a, b, e \in k$ put

$$\mathrm{Hs}(a, b, e) = 1 + a(E_{21} - E_{43}) + b(E_{31} + E_{42}) + eE_{41}.$$

It is lower triangular with diagonal 1, supported in Σ_Q (weights $-w_\alpha$ at $(2,1), (4,3)$, $-w_\Psi$ at $(3,1), (4,2)$, $-w_\theta$ at $(4,1)$), and of similitude 1: ${}^t\mathrm{Hs}(a, b, e)\, J_4\, \mathrm{Hs}(a, b, e) = J_4$.

A direct multiplication gives

$$\begin{aligned} \mathrm{Hs}(a, b, e)\mathrm{Hs}(a', b', e') &= \mathrm{Hs}(a + a',\ b + b',\ e + e' + ba' - ab'), \\ \Delta(g)\mathrm{Hs}(a, b, e)\Delta(g)^{-1} &= \mathrm{Hs}(\chi_{21}(g)a, \chi_{31}(g)b, \chi_{41}(g)e). \end{aligned} \tag{52.4}$$

In particular $\mathrm{Hs}(a, b, e)^{-1} = \mathrm{Hs}(-a, -b, -e)$.

Step 1: pairs from cocycle data. Let W' be a finite set of primes and let $x, y, d\colon G_F \to k$ be locally constant with

$$\begin{aligned} x(gh) &= x(g) + \chi_{21}(g)x(h), \qquad y(gh) = y(g) + \chi_{31}(g)y(h), \\ d(gh) &= d(g) + \chi_{41}(g)d(h) + \chi_{21}(g)y(g)x(h) - \chi_{31}(g)x(g)y(h), \end{aligned} \tag{52.5}$$

vanishing at $g_{\mathscr{T}}$ and on $N_{W'}$, and for every $v \in \Sigma$ let $a_v, b_v, e_v \in k$ satisfy, for all $s \in D_v$,

$$x(s) = (1 - \chi_{21}(s))a_v, \quad y(s) = (1 - \chi_{31}(s))b_v, \quad d(s) = (1 - \chi_{41}(s))e_v + (\chi_{31}(s) - \chi_{21}(s))a_v b_v. \tag{52.6}$$

Assume $\chi_i(s) = 1$ for all i, v and $s \in \mu_v$. Put $r(g) = \mathrm{Hs}(x(g), y(g), d(g))\Delta(g)$ and $n_v = \mathrm{Hs}(a_v, b_v, e_v)$. *Claim: $c = (r, (n_v))$ is a residual pair over $(\mathscr{T}, W')$ with first component and flags supported in Σ_Q, with $r(g)_{21} = \chi_1(g)x(g)$, $r(g)_{31} = \chi_1(g)y(g)$, $(n_v)_{21} = a_v$, $(n_v)_{31} = b_v$.* By (52.4), $r(g)r(h) = \mathrm{Hs}(x(g), y(g), d(g))\, \mathrm{Hs}(\chi_{21}(g)x(h), \chi_{31}(g)y(h), \chi_{41}(g)d(h))\, \Delta(gh)$, which is $r(gh)$ by (52.5); and $r(1) = 1$ (the cocycle identities at $g = h = 1$ give $x(1) = y(1) = d(1) = 0$). So r is a homomorphism of monoids, locally constant, with $r(g)$ of similitude $\bar{a}(\theta(g))$, $r(g) = 1$

on $N_{W'}$ (there x, y, d vanish and $\chi_i = 1$, as $\bar{r}_0(g) = 1$), $r(g_{\mathscr{T}}) = \Delta(g_{\mathscr{T}})$ diagonal: (Mq1), (Mq2). (Mq3) holds for n_v. For $s \in D_v$, by (52.4),

$$n_v \Delta(s) n_v^{-1} \Delta(s)^{-1} = \mathrm{Hs}\big((1-\chi_{21}(s))a_v,\ (1-\chi_{31}(s))b_v,\ (1-\chi_{41}(s))e_v + (\chi_{31}(s)-\chi_{21}(s))a_v b_v\big),$$

which is $\mathrm{Hs}(x(s), y(s), d(s))$ by (52.6); hence $n_v^{-1} r(s) n_v = \Delta(s)$: upper triangular with $\psi^c_{v,i}(s) = \chi_i(s)$, equal to 1 on μ_v: (Mq4) and (Wq1). The supports are in $\Sigma_Q \subseteq \Sigma_P$ (Wq2). The entries are read off.

Step 2: two linearisations. For locally constant x, y, d with the first two identities of (52.5), put $f_+ = d + xy$ and $f_- = d - xy$. Then for all g, h

$$f_+(gh) - f_+(g) - \chi_{41}(g) f_+(h) - 2\chi_{21}(g) y(g) x(h) = \Theta(g,h),$$
$$f_-(gh) - f_-(g) - \chi_{41}(g) f_-(h) + 2\chi_{31}(g) x(g) y(h) = \Theta(g,h),$$

where $\Theta(g,h) = d(gh) - d(g) - \chi_{41}(g) d(h) - \chi_{21}(g) y(g) x(h) + \chi_{31}(g) x(g) y(h)$.

So the third identity of (52.5) holds if and only if f_+, equivalently f_-, satisfies the corresponding linear identity. Likewise, for $s \in D_v$ and given a_v, b_v with the first two identities of (52.6), and any $m_2 \in k$:

- if $f_+(s) = (\chi_{41}(s) - 1) m_2 - 2\chi_{21}(s) y(s) a_v$ for all $s \in D_v$, then the third identity of (52.6) holds with $e_v = -m_2 - a_v b_v$;
- if $f_-(s) = (\chi_{41}(s) - 1) m_2 + 2\chi_{31}(s) x(s) b_v$ for all $s \in D_v$, then it holds with $e_v = -m_2 + a_v b_v$.

Step 3: the data of the given pairs. Let $c = (r_c, (n_{c,v}))$ be a residual pair over $(\mathscr{T}, W)$. (i) If r_c is supported in Σ_P, the place $(3,1)$ is isolated ($r_{c,32} = r_{c,34} = 0$, weights $-w_\beta, w_\alpha$), so $y_c(g) = r_c(g)_{31} / \chi_1(g)$ satisfies the second identity of (52.5); it vanishes at $g_{\mathscr{T}}$ and on N_W. For $s \in D_v$, $n_{c,v}^{-1} r_c(s) n_{c,v}$ is upper triangular with $(1,1)$ entry $\chi_1(s)$ (Wq1), so $r_c(s) n_{c,v} e_1 = \chi_1(s) n_{c,v} e_1$; the third coordinate (with $r_{c,12} = r_{c,13} = r_{c,14} = r_{c,32} = r_{c,34} = 0$ on Σ_P) gives $y_c(s) = (1 - \chi_{31}(s))(n_{c,v})_{31}$. (ii) If r_c is lower triangular (support Σ_M), the place $(2,1)$ is isolated, $x_c(g) = r_c(g)_{21} / \chi_1(g)$ satisfies the first identity of (52.5), vanishes at $g_{\mathscr{T}}$ and on N_W, and the second coordinate of the same eigenvector identity gives $x_c(s) = (1 - \chi_{21}(s))(n_{c,v})_{21}$. (iii) For every residual pair, $r_c(g^{-1}) = \bar{a}(\theta(g))^{-1} J_4^{-1}\, {}^t r_c(g) J_4$, whose $(3,1)$ and $(2,1)$ entries are $-\bar{a}(\theta(g))^{-1} r_c(g)_{42}$ and $\bar{a}(\theta(g))^{-1} r_c(g)_{43}$; so the entries $(4,2)$ (resp. $(4,3)$) of r_c are determined by its entries $(3,1)$ (resp. $(2,1)$), and the $-w_\Psi$ part (resp. $-w_\alpha$ part) of c is determined by $\chi_1 y_c$ and the $(n_{c,v})_{31}$ (resp. $\chi_1 x_c$ and the $(n_{c,v})_{21}$). (iv) By (Wq1) and (Mq4), $\chi_i(s) = \psi^c_{v,i}(s) = 1$ for $s \in \mu_v$.

Step 4: the auxiliary places. Let $\mathcal{E}$ be a finite k_0-vector space with a locally constant action $\rho_{\mathcal{E}}$ of G_F, which is trivial on every $g \in N_W$, and let S_0 be the set of places of F that are archimedean or lie above ℓ, above a prime of W, or above one of the finitely many primes at which $\bar{r}_0$ is ramified. Then $\rho_{\mathcal{E}}$ is unramified outside S_0: an inertia element at a place outside S_0 lies in N_W ($\bar{r}_0$ is unramified there). The subgroup $G_F \cap \ker \rho_{\mathcal{E}} \cap \ker \bar{r}_0 \cap \ker \omega_\ell$ is open in $G_{\mathbb{Q}}$; let L' be the Galois closure over $\mathbb{Q}$ of its fixed field. Infinitely many primes split completely in L' (Chebotarev); let Q be the set of places of F above those of them that are not ℓ, not in W and unramified for $\bar{r}_0$. Then Q is infinite, $Q \cap S_0 = \emptyset$, and for every place $\bar{w}$ above a place of Q, $D_{\bar{w}} \subseteq G_{L'}$, so $\rho_{\mathcal{E}}(D_{\bar{w}}) = 1$, $\omega_\ell(D_{\bar{w}}) = 1$; and every prime below a place of Q splits for $\mathscr{T}$. Let $m_0 \in \mathcal{E} \setminus \{0\}$. Lemma 52.12 gives $S_1 \subseteq Q$ and a cocycle c_0 with (Rk1)–(Rk3). Suppose $\rho_{\mathcal{E}}(g_{\mathscr{T}}) - 1$ is invertible on $\mathcal{E}$; choose m^* with $c_0(g_{\mathscr{T}}) = \rho_{\mathcal{E}}(g_{\mathscr{T}}) m^* - m^*$ and put $c_1(g) = c_0(g) - \rho_{\mathcal{E}}(g) m^* + m^*$. Let W' be W together with the primes below the places of S_1; they all split for $\mathscr{T}$. Then: c_1 is a locally constant cocycle with $c_1(g_{\mathscr{T}}) = 0$; (Rk2) holds for

c_1 (with $m - m^*$); $c_1(g) = c_0(g) \in \mathbb{F}_\ell m_0 \setminus \{0\}$ for the g of (Rk3); and c_1 vanishes on $N_{W'}$: let $g \in N_{W'}$ lie in $I_{\bar{w}}$, $\bar{w}$ above a place v of F. Then $g \in N_W$, so $\rho_{\mathcal{E}}(g) = 1$ and the coboundary term vanishes at g; $v \notin S_1$ (the prime below v is not in W'); if $v \notin S_0$, $c_0(g) = 0$ by (Rk1); if $v \in S_0$, $c_0(g) = \rho_{\mathcal{E}}(g)m - m = 0$ by (Rk2).

Step 5: (Gz). Let $p_1 = p$, $p_2 = p'$, $y_j = y_{p_j}$ and $b_{j,v} = (n_{p_j,v})_{31}$ (Step 3 (i)). Let $Y \subseteq k$ be the k_0-span of the finitely many values of y_1, y_2 (locally constant on the compact G_F), $\mathcal{E} = k_0 \oplus Y \oplus Y$ and

$$\rho_{\mathcal{E}}(g)(t, u_1, u_2) = \big(\chi_{21}(g)t,\ \chi_{41}(g)u_1 + 2\chi_{21}(g)y_1(g)t,\ \chi_{41}(g)u_2 + 2\chi_{21}(g)y_2(g)t\big).$$

This is an action: the identity $\rho_{\mathcal{E}}(gh) = \rho_{\mathcal{E}}(g)\rho_{\mathcal{E}}(h)$ in the coordinate u_j reads $\chi_{21}(gh)y_j(gh) = \chi_{41}(g)\chi_{21}(h)y_j(h) + \chi_{21}(g)y_j(g)\chi_{21}(h)$, which is the cocycle identity of y_j times $\chi_{21}(gh)$, as $\chi_{41} = \chi_{21}\chi_{31}$. It is trivial on N_W, and $\rho_{\mathcal{E}}(g_{\mathscr{T}}) = \mathrm{diag}(\chi_{21}, \chi_{41}, \chi_{41})(g_{\mathscr{T}})$ ($y_j(g_{\mathscr{T}}) = 0$), so $\rho_{\mathcal{E}}(g_{\mathscr{T}}) - 1$ is invertible. Step 4 with $m_0 = (1, 0, 0)$ gives W' and $c_1 = (x, f_1, f_2)$. The cocycle identity of c_1 says: x satisfies the first identity of (52.5), and $f_j(gh) = f_j(g) + \chi_{41}(g)f_j(h) + 2\chi_{21}(g)y_j(g)x(h)$. Put $d_j = f_j - xy_j$: by Step 2, (x, y_j, d_j) satisfies (52.5). They vanish at $g_{\mathscr{T}}$ and on $N_{W'}$ (y_j vanishes on $N_W \supseteq N_{W'}$). For $v \in \Sigma$ the places $\bar{v}$ lie above S_0, so (Rk2) gives $m_v = (m_{v,1}, m_{v,2,1}, m_{v,2,2})$ with $c_1(s) = \rho_{\mathcal{E}}(s)m_v - m_v$ on D_v: $x(s) = (1 - \chi_{21}(s))a_v$ with $a_v = -m_{v,1}$, and $f_j(s) = (\chi_{41}(s) - 1)m_{v,2,j} - 2\chi_{21}(s)y_j(s)a_v$. With $y_j(s) = (1 - \chi_{31}(s))b_{j,v}$ (Step 3 (i)), Step 2 gives (52.6) for (x, y_j, d_j) with $(a_v, b_{j,v}, e_{j,v})$, $e_{j,v} = -m_{v,2,j} - a_v b_{j,v}$. By Step 3 (iv) and Step 1, $s_j = \big(\mathrm{Hs}(x, y_j, d_j)\Delta, (\mathrm{Hs}(a_v, b_{j,v}, e_{j,v}))_v\big)$ is a residual pair over $(\mathscr{T}, W')$ with first component and flags supported in Σ_Q: (Gz1). Its entries $(3, 1)$ are $\chi_1 y_j = r_{p_j,31}$ and its flag coordinates $(3, 1)$ are $b_{j,v}$, so by Step 3 (iii) s_j and p_j have the same $-w_\Psi$ part: (Gz2) with $(s, s') = (s_1, s_2)$. Both have $(2, 1)$ entries $\chi_1 x$ and flag coordinates $(2, 1)$ equal to a_v, so the same $-w_\alpha$ part (Step 3 (iii)): (Gz4). For the g of (Rk3), $c_1(g) = (x(g), f_1(g), f_2(g)) \in \mathbb{F}_\ell(1, 0, 0) \setminus \{0\}$, so $x(g) \neq 0$ and s_1 has the entry $\chi_1(g)x(g) \neq 0$ of weight $-w_\alpha$: (Gz3).

Step 6: (Yz). Let $x = x_p$ and $a_v = (n_{p,v})_{21}$ (Step 3 (ii)), $X \subseteq k$ the k_0-span of the values of x, $\mathcal{E} = k_0 \oplus X$ and

$$\rho_{\mathcal{E}}(g)(t, u) = \big(\chi_{31}(g)t,\ \chi_{41}(g)u - 2\chi_{31}(g)x(g)t\big),$$

an action by the same computation (now with the cocycle identity of x), trivial on N_W, with $\rho_{\mathcal{E}}(g_{\mathscr{T}}) = \mathrm{diag}(\chi_{31}, \chi_{41})(g_{\mathscr{T}})$. Step 4 with $m_0 = (1, 0)$ gives W' and $c_1 = (y, f)$: y satisfies the second identity of (52.5) and $f(gh) = f(g) + \chi_{41}(g)f(h) - 2\chi_{31}(g)x(g)y(h)$. Put $d = f + xy$; by Step 2 (with $f_- = f$), (x, y, d) satisfies (52.5), and all three vanish at $g_{\mathscr{T}}$ and on $N_{W'}$. For $v \in \Sigma$, (Rk2) gives $(m_{v,1}, m_{v,2})$ with $y(s) = (1 - \chi_{31}(s))b_v$, $b_v = -m_{v,1}$, and $f(s) = (\chi_{41}(s) - 1)m_{v,2} + 2\chi_{31}(s)x(s)b_v$; with $x(s) = (1 - \chi_{21}(s))a_v$, Step 2 gives (52.6) with $e_v = -m_{v,2} + a_v b_v$. By Step 1, $s = \big(\mathrm{Hs}(x, y, d)\Delta, (\mathrm{Hs}(a_v, b_v, e_v))_v\big)$ is a residual pair over $(\mathscr{T}, W')$ with first component and flags supported in Σ_Q; its $(2, 1)$ entries are $\chi_1 x = r_{p,21}$ and its flag coordinates $(2, 1)$ are $a_v = (n_{p,v})_{21}$, so it has the same $-w_\alpha$ part as p (Step 3 (iii)); and for the g of (Rk3), $y(g) \neq 0$, so its first component has the entry $\chi_1(g)y(g) \neq 0$ of weight $-w_\Psi$. □

Lemma 52.12. *Let F be a number field, ℓ an odd prime, $\omega_\ell\colon G_F \to \mathbb{F}_\ell^\times$ the cyclotomic character modulo ℓ, and M a finite $\mathbb{F}_\ell$-vector space with a continuous action ρ of G_F. Let S_0 be a finite set of places of F containing the archimedean places, the places above ℓ and the finite places at which ρ is ramified, and let Q be an infinite set of finite places of F not in S_0 such that for every $v \in Q$ and every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above v, $\rho(D_{\bar{w}}) = 1$ and $\omega_\ell(D_{\bar{w}}) = 1$. Then for every $m_0 \in M \setminus \{0\}$ there are a finite set $S_1 \subseteq Q$ and a continuous cocycle $c\colon G_F \to M$ (that is, $c(gh) = c(g) + \rho(g)c(h)$) such that:*

(Rk1) *$c(g) = 0$ for every g in the inertia group $I_{\bar{w}} \subseteq G_F$ of a place $\bar{w}$ of $\overline{\mathbb{Q}}$ above a finite place of F not in $S_0 \cup S_1$;*

(Rk2) *for every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above a place in S_0 there is $m \in M$ with $c(s) = \rho(s)m - m$ for all $s \in D_{\bar{w}}$;*

(Rk3) *there is $g \in G_F$ with $\rho(g) = 1$ and $c(g) \in \mathbb{F}_\ell m_0 \setminus \{0\}$.*

Proof. Step 1: the auxiliary places. Let $v \in Q$, $D = D_{\bar{v}}$, $I = I_{\bar{v}}$, and let p be the residue characteristic of v; $p \neq \ell$ as $v \notin S_0$. As D acts trivially on M, coboundaries vanish on D and $H^1(F_v, M) = \mathrm{Hom}_{\mathrm{cont}}(D, M)$. As D acts trivially on M and on μ_ℓ ($\omega_\ell(D) = 1$), it acts trivially on $M^\vee$; by local duality and the local Euler characteristic formula for $v \nmid \ell$ (Theorem B.29, items 2 and 4), $\#H^2(F_v, M) = \#H^0(F_v, M^\vee) = \#M$ and $\#H^1(F_v, M) = \#H^0(F_v, M)\,\#H^2(F_v, M) = \#M^2$. The kernel of the restriction $\mathrm{Hom}(D, M) \to \mathrm{Hom}(I, M)$ consists of the homomorphisms factoring through $D/I \cong \hat{\mathbb{Z}}$, which are determined by their (arbitrary) value at a Frobenius element: it has $\#M$ elements, so the image has $\#M$ elements. A continuous homomorphism from I to the elementary abelian ℓ-group M kills the wild inertia group (a pro-p group) and factors through the tame quotient, which is isomorphic to $\prod_{q \neq p} \mathbb{Z}_q$; so it factors through the quotient $\mathbb{Z}_\ell$, and $\#\mathrm{Hom}(I, M) \leq \#\mathrm{Hom}(\mathbb{Z}_\ell, M) = \#M$. Hence the restriction maps onto $\mathrm{Hom}(I, M)$, and $\mathrm{Hom}(I, \mathbb{F}_\ell m_0)$, which has exactly ℓ elements (homomorphisms through $I \to \mathbb{Z}_\ell \to \mathbb{F}_\ell$), lies in its image. Put

$$L_v = \{\phi \in H^1(F_v, M) :\ \phi(I) \subseteq \mathbb{F}_\ell m_0\}, \qquad \#L_v = \#M \cdot \ell.$$

Step 2: the size of S_1. The module M is a G_{F,S_0}-module (ρ is unramified outside S_0), and $N = \#H^1(G_{F,S_0}, M)$ is finite (Theorem B.37, item 1). Choose $n \geq 0$ with $\ell^n > N \cdot (\#M)^{1+\#S_0+[F:\mathbb{Q}]}$ and a set $S_1 \subseteq Q$ of n places (Q is infinite); $S_1 \cap S_0 = \emptyset$. Put $S = S_0 \cup S_1$ and, for finite $v \in S_0$, $L_v = 0$. Let

$$\mathrm{Sel} = \ker\Big(H^1(G_{F,S}, M) \to \bigoplus_{v \in S,\ v \nmid \infty} H^1(F_v, M)/L_v\Big).$$

Step 3: a lower bound for #Sel. We have

$$\#\mathrm{Sel} \geq \#H^1(G_{F,S}, M) / \prod_v \#(H^1(F_v, M)/L_v),$$

the product over the finite $v \in S$. By the global Euler characteristic formula,

$$\#H^1(G_{F,S}, M) = \#H^0\,\#H^2(G_{F,S}, M) \prod_{v \mid \infty} |\#M|_v / \#H^0(F_v, M) \geq \#H^2(G_{F,S}, M),$$

since $\#H^0(F_v, M) \leq \#M \leq |\#M|_v$ at archimedean v. By the exact sequence of Theorem B.37, item 3, the image of $H^2(G_{F,S}, M)$ in $\bigoplus_v H^2(F_v, M)$ has index at most $\#H^0(G_{F,S}, M^\vee) \leq \#M$, so $\#H^2(G_{F,S}, M) \geq \prod_v \#H^2(F_v, M)/\#M$. Locally, for finite $v \in S_0$,

$$\#(H^1(F_v, M)/L_v) = \#H^0(F_v, M)\,\#H^2(F_v, M)\,(\#M)^{\delta_v [F_v:\mathbb{Q}_\ell]} \leq \#H^2(F_v, M)\,(\#M)^{1+\delta_v [F_v:\mathbb{Q}_\ell]},$$

with $\delta_v = 1$ if $v \mid \ell$ and 0 otherwise (Theorem B.29, item 4); for $v \in S_1$, $\#(H^1(F_v, M)/L_v) = \#M^2/(\ell \#M) = \#H^2(F_v, M)/\ell$ (Step 1). As $\sum_{v \mid \ell} [F_v : \mathbb{Q}_\ell] = [F : \mathbb{Q}]$, and all groups are finite,

$$\#\mathrm{Sel} \ \geq\ \frac{\ell^n}{(\#M)^{1+\#S_0+[F:\mathbb{Q}]}} \ >\ N.$$

Step 4: a class ramified at S_1. Let H be the kernel of $G_{F,S} \to G_{F,S_0}$; it is the closed normal subgroup generated by the images of the inertia groups $I_{\bar{w}}$, $\bar{w}$ above S_1, and acts trivially on M.

Suppose every class $\kappa \in \mathrm{Sel}$ restricts to 0 on $I_{\bar v}$ for every $v \in S_1$. For a cocycle c representing κ, $c|_H$ is a homomorphism, and $c(\sigma h \sigma^{-1}) = c(\sigma) + \sigma c(h) - \sigma h \sigma^{-1} c(\sigma) = \sigma c(h)$ for $h \in H$; as every inertia group above a place of S_1 is conjugate to some $I_{\bar v}$, $c|_H$ vanishes on all of them, hence on H by continuity. By inflation–restriction, κ lies in the image of the injective inflation map $H^1(G_{F,S_0}, M) \to H^1(G_{F,S}, M)$, so $\#\mathrm{Sel} \le N$, contrary to Step 3. Hence there are $\kappa \in \mathrm{Sel}$ and $v \in S_1$ with $\kappa|_{I_{\bar v}} \neq 0$.

Step 5: the cocycle. Let $c\colon G_F \to M$ be the inflation of a cocycle representing κ. (Rk1): c factors through $G_{F,S}$, whose quotient map kills the inertia groups above finite places outside S. (Rk3): coboundaries vanish on $I_{\bar v}$ ($D_{\bar v}$ acts trivially), so $c|_{I_{\bar v}}$ is the non-zero homomorphism $\kappa|_{I_{\bar v}}$, which takes values in $\mathbb{F}_\ell m_0$ as $\mathrm{res}_v \kappa \in L_v$; take $g \in I_{\bar v}$ with $c(g) \neq 0$; then $\rho(g) = 1$. (Rk2): for a finite $v \in S_0$, $\mathrm{res}_v \kappa = 0$, so $c(s) = \rho(s)m - m$ on $D_{\bar v}$ for some m. Every other place above v is $\sigma\bar v$ with $\sigma \in G_F$, with $D_{\sigma\bar v} = \sigma D_{\bar v} \sigma^{-1}$, and for $t = \sigma s \sigma^{-1}$, $c(t) = c(\sigma) + \sigma(\rho(s)m - m) - \rho(t)c(\sigma) = \rho(t)m' - m'$ with $m' = \sigma m - c(\sigma)$. At an archimedean place, $D_{\bar w}$ has order at most 2, prime to ℓ, so every cocycle on it is a coboundary. □

Lemma 52.13. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ an auxiliary tower for $\mathfrak{D}$ (field F, index set Σ), W a finite set of primes, k a field and $\bar a\colon \mathcal{O} \to k$ a ring homomorphism with $\bar a(\mathfrak{m}_{\mathcal{O}}) = 0$. All residual pairs below are over $(\mathscr{T}, W)$ with values in k (for $\bar a$); for a residual pair c write $c = (r_c, (n_{c,v})_{v\in\Sigma})$.*

(Rl1) *Let c, c' be residual pairs such that $r_c, r_{c'}$ are supported in Σ_P and all $n_{c,v}, n_{c',v}$ in Σ_Q, with $r_c(g)_{31} = r_{c'}(g)_{31}$ and $r_c(g)_{42} = r_{c'}(g)_{42}$ for all $g \in G_F$ and $(n_{c,v})_{31} = (n_{c',v})_{31}$ for all v. Then c and c' are linked.*

(Rl2) *Let c, c' be residual pairs such that $r_c, r_{c'}$ are supported in Σ_M, with $r_c(g)_{21} = r_{c'}(g)_{21}$ and $r_c(g)_{43} = r_{c'}(g)_{43}$ for all g and $(n_{c,v})_{21} = (n_{c',v})_{21}$ for all v. Then c and c' are linked.*

(Rl3) *For $X \in \mathrm{M}_4(k)$ let $X^\flat$ keep the diagonal entries and the entries of weight $-w_\alpha$ (places $(2,1)$, $(4,3)$) of X and replace the others by 0. If c is a residual pair with r_c supported in Σ_M, then $c^\flat = (r_c^\flat, (n_{c,v}^\flat)_v)$ is a residual pair with $r_c^\flat$ and all $n_{c,v}^\flat$ supported in $\{-w_\alpha\}$.*

(Rl4) *Let c be a residual pair with r_c and all $n_{c,v}$ supported in $\{-w_\alpha\}$, and assume $\bar a(\mathrm{rt}_{w_\Psi}(s)) = 1$ for all $v \in \Sigma$ and $s \in D_v$. Then for every family $(y_v)_{v\in\Sigma}$ in k there is a residual pair c^y with $r_{c^y} = r_c$, all $n_{c^y,v}$ supported in Σ_Q, $(n_{c^y,v})_{21} = (n_{c,v})_{21}$ and $(n_{c^y,v})_{31} = y_v$.*

Proof. Step 0: the defining identities and their weights. For $X \in \mathrm{M}_4(R)$ put $X^\dagger = J_4^{-1}\, {}^t X\, J_4$; its (a,b) entry is $\pm X_{5-b,5-a}$, and $\mathrm{wt}_4(5-b,5-a) = \mathrm{wt}_4(a,b)$ (because $\lambda_{5-i} = 1-\lambda_i$, $\mu_{5-i} = 1-\mu_i$). X has similitude $\nu \neq 0$ if and only if $X^\dagger X = \nu$. Let $c = (r, (n_v))$ with values in a commutative ring R (k or $k[X]$) with locally constant r, and $\chi_i = \bar a \circ \chi_i^{\mathscr{T}}$. Then c is a residual pair supported as stated if and only if the following hold for all $g, h \in G_F$, $v \in \Sigma$, $s \in D_v$, $u \in \mu_v$:

(I1) $r(gh) = r(g)r(h)$, $r(1) = 1$;

(I2) $r(g)^\dagger = \nu(g)\, r(g^{-1})$ with $\nu(g) = \bar a(\theta(g))$ (given (I1), this is the similitude condition);

(I3) $r(g) = 1$ for $g \in N_W$; $r(g_{\mathscr{T}})_{ab} = 0$ for $a \neq b$; $r(g)_{ii} = \chi_i(g)$;

(I4) n_v lower triangular with diagonal 1 and $n_v^\dagger n_v = 1$ (similitude 1);

(I5) $(n_v^\dagger r(s) n_v)_{ab} = 0$ for $a > b$ and $(n_v^\dagger r(s) n_v)_{ii} = \chi_i(s)$ (then $\psi = \chi_i$ on D_v, in particular $= 1$ on μ_v by (Mq4) for χ_i, which holds as it holds for the given residual pairs);

(I6) the support conditions.

(For (I5): $n_v^{-1} = n_v^\dagger$ by (I4).) Each of (I1)–(I5) is an equation among entries in which every monomial is a product of at most three entries of matrices of the pair (and constants), and every monomial contributing to the (a,b) entry of a product has weights adding up to $\mathrm{wt}_4(a,b)$, which is a root or 0.

Step 1: (Rl1) and (Rl2). Call a weight *free* if it lies in $\mathcal{F}$ and *core* otherwise, where for (Rl1) $\mathcal{F} = \{w_\beta, -w_\alpha, -w_\theta\}$ (core weights 0, $-w_\Psi$) and for (Rl2) $\mathcal{F} = \{-w_\beta, -w_\Psi, -w_\theta\}$ (core weights 0, $-w_\alpha$). By the hypotheses, c and c' have the same entries at all places of core weight (the diagonal by (Wq1); for (Rl1) $(4,2)$ of the flags because $(n_v)_{42} = (n_v)_{31} - (n_v)_{21}(n_v)_{32}$ and $(n_v)_{32} = 0$ in Σ_Q; for (Rl2) $(4,3)$ of the flags because $(n_v)_{43} = -(n_v)_{21}$; both identities hold for lower unitriangular matrices of similitude 1 by Lemma 52.7, applied over k), and all their entries outside $\mathcal{F} \cup \{\text{core}\}$ vanish. A sum of two or three free weights, plus possibly a core weight, is never a root or 0: for (Rl1) every free weight has first coordinate -1 and every core weight first coordinate 0, so such sums have first coordinate ≤ -2; for (Rl2) every free weight has second coordinate -1 and every core weight second coordinate 0, so such sums have second coordinate ≤ -2; while roots and 0 have coordinates in $\{-1,0,1\}$. Hence in (I1)–(I5) no monomial has two free factors. Let $c_X = (1-X)c + Xc'$ (entrywise, values in $k[X]$). Its core entries are constants and its free entries are of degree ≤ 1 in X, so each side of each identity (I1)–(I5) for c_X is a polynomial of degree ≤ 1 in X; it vanishes at $X=0$ and $X=1$ (identities for c and c'), hence identically. r_{c_X} is locally constant, and the supports are those of c and c' ((Wq2): Σ_P with flags in Σ_Q, resp. Σ_M). So c_X is a residual pair with values in $k[X]$ whose images under $X \mapsto 0, 1$ are c, c': they are linked.

Step 2: (Rl3). If Y, Z are lower triangular, $(YZ)^\flat = Y^\flat Z^\flat$: a monomial $Y_{ax}Z_{xb}$ has weights in $\Sigma_M \cup \{0\}$ adding up to $\mathrm{wt}_4(a,b)$; no two elements of Σ_M add up to 0 or $-w_\alpha$ (the sums are $(-2,0),(0,-1),(-1,-1),(-2,-1),(2,-2),(1,-2),(0,-2),(0,-2),(-1,-2),(-2,-2)$), so for $\mathrm{wt}_4(a,b) \in \{0, -w_\alpha\}$ the monomials of $(YZ)_{ab}$ are those with $x \in \{a,b\}$, which are the monomials of $(Y^\flat Z^\flat)_{ab}$; and for other (a,b), $(Y^\flat Z^\flat)_{ab} = 0$ since its monomials have weights in $\{0, -w_\alpha, -2w_\alpha\}$ and $-2w_\alpha$ is not a root. Also $(Y^\dagger)^\flat = (Y^\flat)^\dagger$ ($\dagger$ preserves weights). Now r_c and $n_{c,v}$ are lower triangular (support Σ_M, (Mq3)), so are $n_{c,v}^\dagger$ and products. Apply $\flat$ to (I1), (I2), (I4): they hold for $c^\flat$. (I3): $1^\flat = 1$, diagonal matrices are fixed, diagonal entries are kept. (I5): $n_{c,v}^\dagger r_c(s) n_{c,v}$ is lower and upper triangular, hence equal to $\mathrm{diag}(\chi_i(s))$; applying $\flat$, $(n_{c,v}^\flat)^\dagger r_c^\flat(s) n_{c,v}^\flat = \mathrm{diag}(\chi_i(s))$. $r_c^\flat$ is locally constant and supported in $\{-w_\alpha\} \subseteq \Sigma_M$, which is (Wq2).

Step 3: (Rl4). Fix v, write $n = n_{c,v}$, $a_v = n_{21}$; as n is supported in $\{-w_\alpha\}$ and has similitude 1, $n = 1 + a_v(E_{21} - E_{43})$ (E_{ab} the matrix units): its entries $(3,1),(3,2),(4,1)$ vanish by the support condition, and $n_{43} = -a_v$, $n_{42} = n_{31} - n_{21}n_{32} = 0$ by Lemma 52.7. Put $m = 1 + y_v(E_{31} + E_{42})$ and $n^y = nm = 1 + a_vE_{21} + y_vE_{31} + y_vE_{42} - a_vE_{43} - a_vy_vE_{41}$. It is lower triangular with diagonal 1, its entries satisfy $n^y_{43} = -n^y_{21}$ and $n^y_{42} = n^y_{31} - n^y_{21}n^y_{32}$, so it has similitude 1 by Lemma 52.7; it is supported in Σ_Q, with $n^y_{21} = a_v$, $n^y_{31} = y_v$. For $s \in D_v$, as in Step 2, $n^{-1}r_c(s)n = \mathrm{diag}(\chi_i(s)) =: \Delta$, so $(n^y)^{-1}r_c(s)n^y = m^{-1}\Delta m$. Now $\Delta m - m\Delta = y_v((\chi_3 - \chi_1)(s)E_{31} + (\chi_4 - \chi_2)(s)E_{42})$, and $\chi_1/\chi_3 = \bar{a} \circ \mathrm{rt}_{w_\Psi}$, $\chi_2/\chi_4 = \chi_1/\chi_3$ (the frame characters satisfy $\chi_1^{\mathscr{T}}\chi_4^{\mathscr{T}} = \chi_2^{\mathscr{T}}\chi_3^{\mathscr{T}}$, both equal to $\theta|_{G_F}$, as $(\kappa_1\kappa_4, \kappa_2\kappa_3) = (\theta, \Psi\alpha)$ and λ_ϵ exchanges these two pairs or fixes them), so by hypothesis m commutes with Δ and $(n^y)^{-1}r_c(s)n^y = \Delta$. Hence $c^y = (r_c, (n_v^y))$ satisfies (Mq3), (Mq4) and (Wq1); (Mq1), (Mq2), local constancy and (Wq2) concern r_c only. □

52.3 Case (III)

This subsection proves Proposition 52.14, the counterpart of Proposition 52.1 for case (III), from Proposition 52.15. The difference is that the seed pairs of case (III) lie over a residual pair ramified at an auxiliary prime (Subsection 51.2), so the chain must end at a seed pair over a given residual pair and must consist of residual pairs that are good in case (III), the condition under which Theorem 53.31 propagates modularity. The lattice pair of an essential test representation along fit towers has an integral model with good reduction (Lemma 52.16); residual pairs pass along towers with their isolated entries and flags (Lemma 52.17); and fit and tame towers of case (III) with a prescribed labelling exist (Proposition 52.18).

Proposition 52.14. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, attached to a host character in case (III), $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, (K, V, a) a test field with residue field k_V of V, and (r, u) an ordinary test representation of $\mathfrak{D}$ with values in V such that r is essential. Then there are a fit and tame auxiliary tower $\mathscr{T}_0$ for $\mathfrak{D}$ of case (III) and a homomorphism $r'\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(V)$ with $r' = u'ru'^{-1}$ for some $u' \in \mathrm{GL}_4(K)$ and $r'(g)$ of similitude $a(\theta(g))$ for all g, with the following property. For every prime w that splits for $\mathscr{T}_0$ and every residual pair s over $(\mathscr{T}_0, \{w\})$ with values in k_V (for a followed by reduction) supported in Σ_M, there is a tame auxiliary tower $\mathscr{T}_1 \geq \mathscr{T}_0$ in whose field w dies such that for every deep auxiliary tower $\mathscr{T} \geq \mathscr{T}_1$ (field F, places $\bar{v}$, $v \in \Sigma$), every coefficient ring $\mathcal{O}_1 \supseteq \mathcal{O}$ and every seed pair Q over $(\mathscr{T}, \emptyset)$ with values in $\mathcal{O}_1$ for ι whose image under $\mathcal{O}_1 \xrightarrow{a} V \to k_V$ lies over s, lies in A and has entries of weights $-w_\beta$ and $-w_\theta$, there is a coefficient ring $\mathcal{O}_2 \supseteq \mathcal{O}_1$ such that for every coefficient ring $\mathcal{O}' \supseteq \mathcal{O}_2$:*

(Nq3) *there are $R \in \mathrm{CNL}_{\mathcal{O}'}$, an injective ring homomorphism $f\colon R \to V$ that restricts to a on $\mathcal{O}'$ and maps $\mathfrak{m}_R$ into $\mathfrak{m}_V$, and a marked ordinary pair P_R over $(\mathscr{T}, \emptyset)$ with values in R, continuous for $\mathfrak{m}_R$, whose reduction is a residual pair, such that $f(P_R)$ has first component $r'|_{G_F}$ and $\psi_{v,i}^{f(P_R)}(s') = a(\kappa_i(s'))$ for all $v \in \Sigma$, $1 \leq i \leq 4$ and $s' \in I_v$;*

(Nq4) *there is a chain in case (III) from the reduction of P_R to the reduction of Q modulo $\mathfrak{m}_{\mathcal{O}'}$.*

Proof. Take $\mathscr{T}_0$ and r' from Proposition 52.15. Let w, s be as in the statement and $\mathscr{T}_1$ the tower given by Proposition 52.15 for (w, s). Let $\mathscr{T} \geq \mathscr{T}_1$ be deep, $\mathcal{O}_1 \supseteq \mathcal{O}$ and Q as in the statement, and let Λ, j, a_Λ, P_Λ and a chain in case (III) $c_0, \dots, c_m$ with links $L_1, \dots, L_m$ (residual pairs with values in $k_V[X]$) be given by (Jx1), (Jx2).

Step 1: the residue field of V and the choice of $\mathcal{O}_2$. The map $\bar{a}$ kills $\mathfrak{m}_\ell$ ($a(\mathfrak{m}_\ell) \subseteq \mathfrak{m}_V$), so it induces a field embedding $\overline{\mathbb{F}}_\ell \to k_V$, which is onto because every element of V is congruent modulo $\mathfrak{m}_V$ to an element of $a(\overline{\mathbb{Z}}_\ell)$. Every c_t and every L_t is locally constant on the compact group G_F and has finitely many flags, so only finitely many elements of k_V occur as entries of the c_t and as coefficients of entries of the L_t; together with the image of the finite residue field $\mathbb{F}_\Lambda$ under $\Lambda \to V \to k_V$ they lie in $\bar{a}(\mathbb{F})$ for a finite subfield $\mathbb{F} \subset \overline{\mathbb{F}}_\ell$. Choose a coefficient ring $\mathcal{O}_2 \supseteq \mathcal{O}_1$ whose residue field contains $\mathbb{F}$ (an unramified extension), and let $\mathcal{O}' \supseteq \mathcal{O}_2$.

Step 2: (Nq3). Write $\mathcal{O}' = \sum_{i=1}^n \mathcal{O} b_i$ ($\mathcal{O}'$ is finite over $\mathcal{O}$) and let $R = j(\Lambda)[a(b_1), \dots, a(b_n)] \subseteq V$. Each $a(b_i)$ is integral over $a(\mathcal{O}) = j(a_\Lambda(\mathcal{O})) \subseteq j(\Lambda)$, so R is a finite $j(\Lambda)$-module containing $a(\mathcal{O}')$. Hence R is Noetherian and complete, and, being a domain finite over the complete local ring $j(\Lambda) \cong \Lambda$, it is local. Its maximal ideal lies in $\mathfrak{m}_V$: for $x \in \mathfrak{m}_R$ some power lies in $j(\mathfrak{m}_\Lambda)R \subseteq \mathfrak{m}_V$ ($R/\mathfrak{m}_\Lambda R$ is Artinian local, j is local), and $\mathfrak{m}_V$ is prime. So $R \cap \mathfrak{m}_V = \mathfrak{m}_R$ and $R/\mathfrak{m}_R$ embeds into k_V, with image the field generated by the images of $\mathbb{F}_\Lambda$ and of $\mathbb{F}_{\mathcal{O}'}$, which is $\bar{a}(\mathbb{F}_{\mathcal{O}'})$ by Step 1. Thus R, with structure map $a|_{\mathcal{O}'}$ (local, as units of $\mathcal{O}'$ go to units), lies in $\mathrm{CNL}_{\mathcal{O}'}$, and the inclusion $f\colon R \to V$ is injective, restricts to a on $\mathcal{O}'$ and maps $\mathfrak{m}_R$ into $\mathfrak{m}_V$. Let

$P_R = j(P_\Lambda)$, with entries in $j(\Lambda) \subseteq R$: it is a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in R for $\mathcal{O} \subseteq \mathcal{O}' \to R$ (the conditions (Mq1)–(Mq4) are identities transported by j, and $j \circ a_\Lambda = a|_{\mathcal{O}}$); it is continuous for $\mathfrak{m}_R$ because $j(\mathfrak{m}_\Lambda)^k R \subseteq \mathfrak{m}_R^k$; its reduction is the image of the reduction of P_Λ under the field embedding $\mathbb{F}_\Lambda \to R/\mathfrak{m}_R$, hence a residual pair (local constancy, (Mq1)–(Mq4), (Wq1), (Wq2) are preserved by an injective ring homomorphism); and $f(P_R) = j(P_\Lambda)$ has first component $r'|_{G_F}$ and $\psi_{v,i}^{f(P_R)}(s') = a(\kappa_i(s'))$ on I_v by (Jx1).

Step 3: (Nq4). Let $\bar{a}' \colon \mathbb{F}_{\mathcal{O}'} \to k_V$ be induced by $\bar{a}$; it is injective and, by Step 1, its image contains every entry of the c_t and every coefficient of the L_t. Let c_t' and L_t' be obtained by applying $(\bar{a}')^{-1}$ entrywise (coefficientwise for L_t). Every condition in the definitions of residual pairs, of residual pairs good in case (III) (good, lying in V or in A, entries of given weights, the non-vanishing of $\bar{a}(\mathrm{rt}_{w_\alpha}(s')) - 1$ at some $s' \in D_v$), of links and of chains in case (III) is a conjunction of polynomial identities and non-vanishing conditions among entries, flags and values $\bar{a}(\kappa_i(g))$, $\bar{a}(\mathrm{rt}_w(s'))$, together with evaluation at $X = 0, 1$; the injective homomorphisms $\mathbb{F}_{\mathcal{O}'} \to k_V$ and $\mathbb{F}_{\mathcal{O}'}[X] \to k_V[X]$ commute with these evaluations and preserve and reflect identities and non-vanishing. So $c_0', \dots, c_m'$ is a chain in case (III) with values in $\mathbb{F}_{\mathcal{O}'}$ (for the reduction $\mathcal{O} \to \mathbb{F}_{\mathcal{O}'}$), with links L_t'. Its ends are the reduction of P_R (both have image under $\bar{a}'$ equal to the image of P_Λ in k_V) and the reduction of Q modulo $\mathfrak{m}_{\mathcal{O}'}$ (both have image the image of Q in k_V), by injectivity of $\bar{a}'$. This is (Nq4). □

Proposition 52.15. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, attached to a host character in case (III), $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, (K, V, a) a test field with residue field k_V and $\bar{a}$ the composite of a and reduction, and (r, u) an ordinary test representation of $\mathfrak{D}$ with values in V such that r is essential. Then there are a fit and tame auxiliary tower $\mathscr{T}_0$ for $\mathfrak{D}$ of case (III) and a homomorphism $r' \colon G_\mathbb{Q} \to \mathrm{GL}_4(V)$ with $r' = u' r u'^{-1}$ for some $u' \in \mathrm{GL}_4(K)$ and $r'(g)$ of similitude $a(\theta(g))$ for all g, with the following property. For every prime w that splits for $\mathscr{T}_0$ and every residual pair s over $(\mathscr{T}_0, \{w\})$ with values in k_V (for $\bar{a}|_{\mathcal{O}}$) supported in Σ_M, there is a tame auxiliary tower $\mathscr{T}_1 \geq \mathscr{T}_0$ in whose field w dies such that for every deep auxiliary tower $\mathscr{T} \geq \mathscr{T}_1$ (field F, places $\bar{v}$, $v \in \Sigma$), every coefficient ring $\mathcal{O}_1 \supseteq \mathcal{O}$ and every seed pair Q over $(\mathscr{T}, \emptyset)$ with values in $\mathcal{O}_1$ for ι whose image under $\mathcal{O}_1 \xrightarrow{\bar{a}} k_V$ lies over s, lies in A and has entries of weights $-w_\beta$ and $-w_\theta$, there are:*

(Jx1) *a complete discrete valuation ring Λ with finite residue field, an injective local homomorphism $j \colon \Lambda \to V$, a ring homomorphism $a_\Lambda \colon \mathcal{O} \to \Lambda$ with $j \circ a_\Lambda = a|_{\mathcal{O}}$, and a marked ordinary pair P_Λ over $(\mathscr{T}, \emptyset)$ with values in Λ (for a_Λ), continuous for $\mathfrak{m}_\Lambda$, whose reduction is a residual pair, such that $j(P_\Lambda)$ has first component $r'|_{G_F}$ and $\psi_{v,i}^{j(P_\Lambda)}(s') = a(\kappa_i(s'))$ for all $v \in \Sigma$, $1 \leq i \leq 4$ and $s' \in I_v$;*

(Jx2) *a chain in case (III), with values in k_V (for $\bar{a}|_{\mathcal{O}}$), from the image of P_Λ under $\Lambda \to V \to k_V$ to the image of Q under $\mathcal{O}_1 \xrightarrow{\bar{a}} k_V$.*

Proof. Step 1: labelling, first tower, lattice pair. Let ϵ be the admissible labelling given by Lemma 52.16 for (r, u), and $\mathscr{T}_0$ a fit and tame auxiliary tower of case (III) with labelling ϵ (Proposition 52.18), with field F_0 and index set Σ_0. Lemma 52.16 for $\mathscr{T}_0$ gives u', $r' = u' r u'^{-1}$ with $r'(g)$ of similitude $a(\theta(g))$, and a residual pair p_0 over $(\mathscr{T}_0, \emptyset)$ whose first component is supported in Σ_P and whose flags are supported in Σ_Q, with (Kp1), (Kp2) for every fit tower $\mathscr{T} \geq \mathscr{T}_0$. These $\mathscr{T}_0$ and r' are those of the statement.

Step 2: the Gysin pairs and the tower $\mathscr{T}_1$. Let w and s be as in the statement. (Yz) for $\mathscr{T}_0$, $W = \{w\}$ and $p = s$ (its first component is supported in Σ_M) gives a finite set $W' \supseteq \{w\}$ of primes splitting for $\mathscr{T}_0$ and a residual pair p_2 over $(\mathscr{T}_0, W')$ whose first component and flags are

supported in Σ_Q, which has the same $-w_\alpha$ part as s, and whose first component has an entry of weight $-w_\Psi$. Since $N_{W'} \subseteq N_\emptyset$ and the other conditions on a residual pair do not involve the set of primes, p_0 is also a residual pair over $(\mathscr{T}_0, W')$. (Gz) for $\mathscr{T}_0$, W', $p = p_0$ and $p' = p_2$ (first components supported in Σ_P, as $\Sigma_Q \subseteq \Sigma_P$; flags in Σ_Q) gives a finite set $W'' \supseteq W'$ of primes splitting for $\mathscr{T}_0$ and residual pairs s_3, s_4 over $(\mathscr{T}_0, W'')$ with: first components and flags supported in Σ_Q (Gz1); s_3 has the same $-w_\Psi$ part as p_0 and s_4 the same $-w_\Psi$ part as p_2 (Gz2); the first component of s_3 has an entry of weight $-w_\alpha$ (Gz3); s_3, s_4 have the same $-w_\alpha$ part (Gz4). (Ux2) for the fit and tame $\mathscr{T}_0$ and W'' gives a tame auxiliary tower $\mathscr{T}_1 \geq \mathscr{T}_0$ in whose field F_1 every prime of W'' dies, in particular w. This is the tower $\mathscr{T}_1$ of the statement.

Step 3: the pairs over a deep tower. Let $\mathscr{T} \geq \mathscr{T}_1$ be deep, with field F and places $\bar{v}$ ($v \in \Sigma$), and let $\mathcal{O}_1 \supseteq \mathcal{O}$ and Q be as in the statement. The order is transitive (immediate from its definition), so $\mathscr{T}_0 \leq \mathscr{T}$, and $\mathscr{T}$ is fit by (Xq3). Every prime of W'' dies in F, because it dies in $F_1 \subseteq F$ ($G_F \subseteq G_{F_1}$). (Kp1) for $\mathscr{T}$ is (Jx1). By (Kp2), the image $c_{\mathscr{T}}$ of P_Λ in k_V is a good residual pair over $(\mathscr{T}, \emptyset)$ lying over p_0, with first component supported in Σ_P and flags in Σ_Q, which has an entry of weight $-w_\Psi$ or lies in V. (Ux4) for $\mathscr{T}_0 \leq \mathscr{T}$ applied to (W'', s_3), (W'', s_4) and (W', p_2) (first components supported in Σ_Q; $W' \subseteq W''$) gives residual pairs q_3, q_4, q_2 over $(\mathscr{T}, \emptyset)$ lying over s_3, s_4, p_2. Their first components are the restrictions to G_F of those of s_3, s_4, p_2, hence supported in Σ_Q, and by (Hv3) their flags are supported in Σ_Q. Finally let $\bar{Q}$ be the image of Q under $\mathcal{O}_1 \xrightarrow{\bar{a}} k_V$. The reduction of the seed pair Q is a marked ordinary pair (the conditions (Mq1)–(Mq4) are identities preserved by reduction), locally constant (Q is continuous for $\mathfrak{m}_{\mathcal{O}_1}$), and satisfies (Wq1), (Wq2) by definition of seed pairs; $\bar{a}$ kills $\mathfrak{m}_{\mathcal{O}_1}$ and the induced map on the residue field is an injective field homomorphism, which preserves these conditions. So $\bar{Q}$ is a residual pair over $(\mathscr{T}, \emptyset)$; by hypothesis it lies over s, lies in A (its first component is supported in Σ_A and $(n_{\bar{Q},v})_{21} \neq 0$ for all v) and has entries of weights $-w_\beta$ and $-w_\theta$.

Step 4: entries surviving restriction. $\mathscr{T}$ is good, G_F is normal in G_{F_0} (F is abelian over $\mathbb{Q}$), and the first components of s_3 and p_2 are homomorphisms on G_{F_0} supported in Σ_Q with diagonal entries $\bar{a}(\kappa_i^\epsilon(g))$ (Wq1) and similitudes $\bar{a}(\theta(g)) \neq 0$ (Mq1); so (Hv1) applies with $H = G_{F_0}$. By (Gz3) and the similitude clause of (Hv1), $r_{s_3}(g)_{21} \neq 0$ for some $g \in G_{F_0}$; the place $(2,1)$ is isolated ($\Sigma_Q \subseteq \Sigma_M$), so $r_{s_3}(g)_{21} \neq 0$ for some $g \in G_F$: q_3 has an entry of weight $-w_\alpha$. Likewise, p_2 has an entry of weight $-w_\Psi$, so $r_{p_2}(g)_{31} \neq 0$ for some $g \in G_{F_0}$; the place $(3,1)$ is isolated (support in Σ_Q), so $r_{p_2}(g)_{31} \neq 0$ for some $g \in G_F$: q_2 has an entry of weight $-w_\Psi$.

Step 5: the links.

(i) $c_{\mathscr{T}}$ and q_3. They lie over p_0 and s_3, whose first components are supported in Σ_P and which have the same $-w_\Psi$ part (Gz2); by (Hv2) $c_{\mathscr{T}}$ and q_3 have the same $-w_\Psi$ part. Both first components are supported in Σ_P and all flags in Σ_Q, so (Rl1) links them.

(ii) q_3 and q_4. They lie over s_3, s_4, whose first components are supported in Σ_M and which have the same $-w_\alpha$ part (Gz4); by (Hv2) so do q_3, q_4, whose first components are supported in Σ_M; (Rl2) links them.

(iii) q_4 and q_2. They lie over s_4, p_2, whose first components are supported in Σ_P and which have the same $-w_\Psi$ part (Gz2); by (Hv2) so do q_4, q_2; first components in Σ_P, flags in Σ_Q: (Rl1) links them.

(iv) q_2 and $\bar{Q}$. They lie over p_2 and s, whose first components are supported in Σ_M and which have the same $-w_\alpha$ part (Yz); by (Hv2) q_2 and $\bar{Q}$ have the same $-w_\alpha$ part. Their first components are supported in $\Sigma_Q \subseteq \Sigma_M$ and $\Sigma_A \subseteq \Sigma_M$; (Rl2) links them.

Step 6: goodness in case (III).

- $c_{\mathscr{T}}$ is good (Kp2).
- q_3 has an entry of weight $-w_\alpha$ (Step 4). If it has one of weight $-w_\Psi$, it is good (entries of weights $-w_\Psi$ and $-w_\alpha$). Otherwise its first component is supported in $\Sigma_Q \setminus \{-w_\Psi\} = \{-w_\alpha, -w_\theta\} \subseteq \Sigma_V$; moreover $c_{\mathscr{T}}$, having the same $-w_\Psi$ part as q_3 (Step 5 (i)), has no entry of weight $-w_\Psi$, so by (Kp2) it lies in V and $(n_{c_{\mathscr{T}},v})_{31} \neq 0$, whence $(n_{q_3,v})_{31} \neq 0$ for all v. With its flags in Σ_Q, q_3 lies in V and has an entry of weight $-w_\alpha$: it is good.
- q_4 has the same $-w_\alpha$ part as q_3, hence an entry of weight $-w_\alpha$, and the same $-w_\Psi$ part as q_2, hence an entry of weight $-w_\Psi$ (Step 4): it is good.
- q_2 has an entry of weight $-w_\Psi$ (Step 4). It has the same $-w_\alpha$ part as $\bar{Q}$, whose first component is supported in $\Sigma_A \not\ni -w_\alpha$; so the entries $(2,1)$, $(4,3)$ of the first component of q_2 vanish, its first component is supported in $\{-w_\Psi, -w_\theta\} \subseteq \Sigma_A$, and $(n_{q_2,v})_{21} = (n_{\bar{Q},v})_{21} \neq 0$ for all v. So q_2 lies in A and has an entry of weight $-w_\Psi$: it is good in case (III).
- $\bar{Q}$ lies in A and has entries of weights $-w_\beta$ and $-w_\theta$: it is good in case (III).

Step 7: (Jx2). $c_{\mathscr{T}}, q_3, q_4, q_2, \bar{Q}$ are residual pairs over $(\mathscr{T}, \emptyset)$ with values in k_V (for $\bar{a}|_{\mathcal{O}}$), good in case (III) (Step 6), consecutive ones linked (Step 5): a chain in case (III) from the image $c_{\mathscr{T}}$ of P_Λ in k_V to the image $\bar{Q}$ of Q. □

Lemma 52.16. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, (K, V, a) a test field with residue field k_V and $\bar{a}\colon \overline{\mathbb{Z}}_\ell \to k_V$ the composite of a and reduction, and (r, u) an ordinary test representation of $\mathfrak{D}$ with values in V such that r is essential. Then there is an admissible labelling ϵ with the following property. For every fit auxiliary tower $\mathscr{T}_0$ for $\mathfrak{D}$ with labelling ϵ (field F_0, index set Σ_0) there are $u' \in \mathrm{GL}_4(K)$, a homomorphism $r'\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(V)$ with $r' = u'ru'^{-1}$ and $r'(g)$ of similitude $a(\theta(g))$ for all g, and a residual pair p_0 over $(\mathscr{T}_0, \emptyset)$ with values in k_V (for $\bar{a}|_{\mathcal{O}}$) whose first component is supported in Σ_P and all of whose flags are supported in Σ_Q, such that for every fit auxiliary tower $\mathscr{T} \geq \mathscr{T}_0$ (field F, places $\bar{v}$, $v \in \Sigma$) there are:*

(Kp1) *a complete discrete valuation ring Λ with finite residue field, an injective local homomorphism $j\colon \Lambda \to V$, a ring homomorphism $a_\Lambda\colon \mathcal{O} \to \Lambda$ with $j \circ a_\Lambda = a|_{\mathcal{O}}$, and a marked ordinary pair P_Λ over $(\mathscr{T}, \emptyset)$ with values in Λ (for a_Λ), continuous for $\mathfrak{m}_\Lambda$, whose reduction is a residual pair, such that $j(P_\Lambda)$ has first component $r'|_{G_F}$ and $\psi_{v,i}^{j(P_\Lambda)}(s) = a(\kappa_i(s))$ for all $v \in \Sigma$, $1 \leq i \leq 4$ and $s \in I_v$;*

(Kp2) *the image $c_{\mathscr{T}}$ of P_Λ under $\Lambda \xrightarrow{j} V \to k_V$ is a residual pair over $(\mathscr{T}, \emptyset)$ with values in k_V (for $\bar{a}|_{\mathcal{O}}$) which lies over p_0, whose first component is supported in Σ_P and all of whose flags are supported in Σ_Q, and which either has entries of weights $-w_\Psi$ and w_β, or lies in V and has an entry of weight $-w_\alpha$; in particular $c_{\mathscr{T}}$ is good.*

Proof. Step 1: labelling and lattice. Let ϵ be the admissible labelling given by Proposition 52.3 for (r, u), and $\mathscr{T}_0$ a fit auxiliary tower with labelling ϵ. Put $g_\circ = g_{\mathscr{T}_0}$; by (Oq7) the $\bar{\kappa}_i^\epsilon(g_\circ)$ are pairwise distinct, hence so are the residues $\chi_i(g_\circ)$ of the $a(\kappa_i^\epsilon(g_\circ))$. Proposition 52.3 gives $h \in \mathrm{GL}_4(K)$, $r^\natural = h^{-1}rh$ and n with (Nz1)–(Nz3), and, r being essential, (Nz4): $r^\natural$ has an entry of weight $-w_\Psi$ and one of at least one of the weights $w_\beta, -w_\alpha, -w_\theta$. These are the hypotheses of Proposition 52.4 (its first bullet is (Nz1), its second (Nz2), its third (Nz3)), which gives $\mathrm{T} = \mathrm{T}_4(t_1, t_2)$, $r' = \mathrm{T}r^\natural\mathrm{T}^{-1}$ and $n' = \mathrm{T}n\mathrm{T}^{-1}$ with (Ql0)–(Ql3), for some Λ', j'. Put $u' = \mathrm{T}h^{-1}$;

then $r' = u' r u'^{-1}$, $r'(g)$ has similitude $a(\theta(g))$, a unit of V, and entries in V, so $r'(g) \in \mathrm{GL}_4(V)$. Let $c = \bar{r}'$ and $\bar{n} = \overline{n'}$. By (Ql0) and the congruences of (Nz2), (Nz3): $c(g)_{ii} = \chi_i(g)$, and for $s \in D_\ell$ the matrix $\bar{n}^{-1} c(s) \bar{n}$ is upper triangular with diagonal entries $\chi_i(s)$; by (Ql1), c is supported in Σ_P and $\bar{n}$ in Σ_Q.

Step 2: the hypotheses of (Ux3). Let $\mathscr{T}$ be a fit auxiliary tower with labelling ϵ whose marking $g_{\mathscr{T}}$ is a power of $g_\circ$. Then (r', n') satisfies the hypotheses of (Ux3) for $\mathscr{T}$, with (Λ', j') of (Ql0): entries in $j'(\Lambda')$ and continuity by (Ql0); $r'(g)$ has similitude $a(\theta(g))$ and diagonal entries $r^\natural(g)_{ii} \equiv a(\kappa_i^\epsilon(g))$ modulo $\mathfrak{m}_V$ by (Ql0) and (Nz2); $r'(g_{\mathscr{T}})$ is diagonal as a power of the diagonal $r'(g_\circ)$; $r'(g) = 1$ on the inertia elements named in (Nz2), since $r^\natural(g) = 1$ there and (Ql0); n' is lower triangular with diagonal 1 and similitude 1, and $n'^{-1} r'(s) n'$ is upper triangular with diagonal entries $\psi_i(s)$ for $s \in D_\ell$, where $\psi_i(s) \equiv a(\kappa_i^\epsilon(s))$ and $\psi_i(s) = a(\kappa_i(s))$ on I_ℓ by (Nz3); the reductions are supported in Σ_P and Σ_Q by (Ql1).

Step 3: the pair p_0. By Step 2 for $\mathscr{T}_0$, (Ux3) gives flags n_{0,v_0} ($v_0 \in \Sigma_0$) and $g_{0,v_0} \in G_{\mathbb{Q}}$ with $g_{0,v_0}(\bar{\ell}) = \bar{v}_0$ and $n_{0,v_0}^{-1} r'(g_{0,v_0}) n'$ upper triangular, such that the reduction $p_0 = (c|_{G_{F_0}}, (\bar{n}_{0,v_0}))$ is a residual pair over $(\mathscr{T}_0, \emptyset)$ with values in k_V whose flags are supported in Σ_Q; its first component is supported in Σ_P.

Step 4: (Kp1). Let $\mathscr{T} \geq \mathscr{T}_0$ be fit, with field F and places $\bar{v}$, $v \in \Sigma$. By definition of the order it has labelling ϵ and $g_{\mathscr{T}} = g_\circ^{\ell^{m_1}}$ for some $m_1 \geq 0$, so Step 2 applies: (Ux3) gives flags n_v with entries in $j'(\Lambda')$ and $g_v \in G_{\mathbb{Q}}$ with $g_v(\bar{\ell}) = \bar{v}$ and $n_v^{-1} r'(g_v) n'$ upper triangular, such that $P = (r'|_{G_F}, (n_v))$ is a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in V (for a), $\psi^P_{v,i}(s) = a(\kappa_i(s))$ for $s \in I_v$, and its reduction $c_{\mathscr{T}} = (c|_{G_F}, (\bar{n}_v))$ is a residual pair over $(\mathscr{T}, \emptyset)$ with values in k_V whose flags are supported in Σ_Q.

Enlarging Λ'. Let L_0 be the fraction field of $j'(\Lambda')$. The ring Λ' of Proposition 52.4 contains Λ of (Nz1) with $j'|_\Lambda = j$, so every element of K is algebraic over L_0 by (Nz1). $V \cap L_0$ is a valuation ring of L_0 containing the discrete valuation ring $j'(\Lambda')$ and not containing $j'(\pi')^{-1}$ for a uniformiser π' (j' is local), so it equals $j'(\Lambda')$; as K/L_0 is algebraic, the value group of V lies in the divisible hull of that of L_0, so V has rank one. If K has characteristic 0, Λ' has characteristic 0 and residue characteristic ℓ, so it contains $\mathbb{Z}_\ell$; a and j' agree on $\mathbb{Z}$, and for $x \in \mathbb{Z}_\ell$ and integers $m \equiv x$ modulo ℓ^e, $a(x) - j'(x) = a(x-m) - j'(x-m) \in \ell^e V$ for all e; rank one gives $\bigcap_e \ell^e V = 0$, so $a = j'$ on $\mathbb{Z}_\ell$ and every element of $a(\mathcal{O})$ is integral over $j'(\mathbb{Z}_\ell)$. If K has characteristic ℓ, $a(\mathcal{O})$ is a finite field. In both cases $L_1 = L_0(a(\mathcal{O}))$ is a finite extension of L_0, and $\Lambda_1 = V \cap L_1$ is the valuation ring of the unique extension to L_1 of the valuation of the complete discrete valuation ring $j'(\Lambda')$: a complete discrete valuation ring with finite residue field, containing $j'(\Lambda')$ and the integral elements $a(\mathcal{O})$. Let $j_1 \colon \Lambda_1 \to V$ be the inclusion (injective and local) and $a_\Lambda = a|_{\mathcal{O}}$ viewed in Λ_1. Then $P_\Lambda = j_1^{-1}(P)$ (entrywise) has values in Λ_1; (Mq1)–(Mq4) for P and a are identities in $j_1(\Lambda_1)$, so P_Λ is a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in Λ_1 for a_Λ. It is $\mathfrak{m}_{\Lambda_1}$-adically continuous, because the $\mathfrak{m}_{\Lambda'}$-adic and $\mathfrak{m}_{\Lambda_1}$-adic topologies agree on $j'(\Lambda')$ (Λ_1 is finite over it). Its reduction is mapped by the field embedding of the residue field of Λ_1 into k_V to $c_{\mathscr{T}}$; residual pairs are defined by polynomial identities and non-vanishing conditions, which an injective field homomorphism reflects, so the reduction of P_Λ is a residual pair. Finally $j_1(P_\Lambda) = P$ has first component $r'|_{G_F}$ and $\psi = a(\kappa_i(s))$ on I_v. This is (Kp1) with $(\Lambda, j) = (\Lambda_1, j_1)$, and the image of P_Λ in k_V is $c_{\mathscr{T}}$.

Step 5: $c_{\mathscr{T}}$ lies over p_0. Its first component is $c|_{G_F} = (c|_{G_{F_0}})|_{G_F}$. Let $v \in \Sigma$. By the definition of $\mathscr{T}_0$ there are $v_0 \in \Sigma_0$ and $g \in G_{F_0}$ with $g(\bar{v}_0) = \bar{v}$. Then $g g_{0,v_0}(\bar{\ell}) = \bar{v} = g_v(\bar{\ell})$, so $d = g_v^{-1} g g_{0,v_0}$ lies in the decomposition group D_ℓ of $\bar{\ell}$ in $G_{\mathbb{Q}}$, and

$$\bar{n}_v^{-1} c(g) \bar{n}_{0,v_0} = \left(\bar{n}_v^{-1} c(g_v) \bar{n}\right) \left(\bar{n}^{-1} c(d) \bar{n}\right) \left(\bar{n}_{0,v_0}^{-1} c(g_{0,v_0}) \bar{n}\right)^{-1}.$$

The first factor is upper triangular (Step 4), the second by Step 1, and the third is the inverse

of an invertible upper triangular matrix (Step 3). So $\bar{n}_v^{-1} c(g) \bar{n}_{0,v_0}$ is upper triangular, and $c_{\mathscr{T}}$ lies over p_0.

Step 6: supports and goodness. The first component $c|_{G_F}$ is supported in Σ_P, the flags in Σ_Q. $\mathscr{T}$ is good (fit), G_F is normal in $G_{\mathbb{Q}}$ ($F/\mathbb{Q}$ is abelian), and every $c(g)$ has the similitude $\bar{a}(\theta(g)) \neq 0$, so (Hv1) applies to $y = c$, $H = G_{\mathbb{Q}}$. By (Ql3) there are two cases.

(a) c has an entry of weight $-w_\Psi$. By (Hv1), $c(g)_{31} \neq 0$ for some $g \in G_{\mathbb{Q}}$; the place $(3,1)$ is isolated in c (c is supported in Σ_P), so by (Hv1) $c(g)_{31} \neq 0$ for some $g \in G_F$. By (Ql2), c_{23} is not identically zero (the only place of weight w_β is $(2,3)$), and $(2,3)$ is isolated, so $c(g)_{23} \neq 0$ for some $g \in G_F$. Thus $c_{\mathscr{T}}$ has entries of weights $-w_\Psi$ and w_β.

(b) Otherwise, by (Ql3), c is supported in Σ_V, $\bar{n}_{31} \neq 0$, $\bar{\Psi}(s) = 1$ for $s \in D_\ell$, and there is no $m \in k_V$ with $c(s)_{23} = m(\chi_3(s) - \chi_2(s))$ for all $s \in D_\ell$. *$c_{\mathscr{T}}$ lies in V.* Its first component is supported in Σ_V and its flags in Σ_Q. Let $v \in \Sigma$, and compare the first columns in the upper triangularity of $\bar{n}_v^{-1} c(g_v) \bar{n}$: $c(g_v)\bar{n}e_1 = \lambda \bar{n}_v e_1$ with λ the first coordinate of $c(g_v)\bar{n}e_1$, which is $\chi_1(g_v)$ because $c_{12} = c_{13} = c_{14} = 0$ (weights $w_\alpha, w_\Psi, w_\theta \notin \Sigma_P$); the third coordinates, with $c_{31} = c_{32} = c_{34} = 0$ (weights $-w_\Psi, -w_\beta, w_\alpha \notin \Sigma_V$), give $(\bar{n}_v)_{31}\chi_1(g_v) = \chi_3(g_v)\bar{n}_{31} \neq 0$. *$c_{\mathscr{T}}$ has an entry of weight $-w_\alpha$.* The place $(2,1)$ is isolated in c: in $c(gh)_{21} = \sum_x c(g)_{2x} c(h)_{x1}$ the terms $x = 3, 4$ vanish since $c_{31} = 0$ and $c_{24} = 0$. If $c_{\mathscr{T}}$ had no entry of weight $-w_\alpha$, then $c_{21} = 0$ on G_F by (Hv1) (similitude clause), hence on $G_{\mathbb{Q}}$ by (Hv1). The second coordinate of $c(s)\bar{n}e_1 = \chi_1(s)\bar{n}e_1$ ($s \in D_\ell$; Step 1) would read $\chi_2(s)\bar{n}_{21} + c(s)_{23}\bar{n}_{31} = \chi_1(s)\bar{n}_{21}$ (with $c(s)_{21} = 0$ and $c(s)_{24} = 0$). Since χ_1/χ_3 is $\bar{a} \circ \Psi$ or $\bar{a} \circ \Psi^{-1}$ (for $\epsilon = 0$, 1) and $\bar{\Psi} = 1$ on D_ℓ, $\chi_1 = \chi_3$ on D_ℓ, so $c(s)_{23} = m(\chi_3(s) - \chi_2(s))$ with $m = \bar{n}_{21}/\bar{n}_{31}$, contrary to (Ql3). So $c_{\mathscr{T}}$ lies in V and has an entry of weight $-w_\alpha$.

In case (a) $c_{\mathscr{T}}$ is good by the first clause of goodness, in case (b) by the third. □

Lemma 52.17. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, k a field, $\bar{a}\colon \mathcal{O} \to k$ a ring homomorphism with $\bar{a}(\mathfrak{m}_{\mathcal{O}}) = 0$, ϵ an admissible labelling, and $\delta_i = \bar{a} \circ \kappa_i^\epsilon$ ($1 \leq i \leq 4$). For a residual pair $c = (r_c, (n_{c,v})_v)$ with values in k over an auxiliary tower with field F, its $-w_\Psi$* part *is the family of the entries $r_c(g)_{31}$, $r_c(g)_{42}$ ($g \in G_F$) and $(n_{c,v})_{31}$ (all v), and its $-w_\alpha$* part *the family of the entries $r_c(g)_{21}$, $r_c(g)_{43}$ and $(n_{c,v})_{21}$.*

(Hv1) *Let $\mathscr{T}$ be a good auxiliary tower with labelling ϵ and field F, $H \subseteq G_{\mathbb{Q}}$ a subgroup containing G_F as a normal subgroup, and $y\colon H \to \mathrm{M}_4(k)$ a homomorphism of monoids with $y(g)_{ii} = \delta_i(g)$ for $g \in H$ and all i. Call a place (i,j), $i \neq j$,* isolated *in y if $y(gh)_{ij} = \delta_i(g)y(h)_{ij} + y(g)_{ij}\delta_j(h)$ for all $g, h \in H$. If (i,j) is isolated in y and $y(g)_{ij} = 0$ for all $g \in G_F$, then $y(g)_{ij} = 0$ for all $g \in H$. The places $(3,1)$ and $(2,3)$ are isolated if y is supported in Σ_P; the place $(2,1)$ is isolated if y is supported in Σ_M; the place $(3,1)$ is isolated if y is supported in Σ_Q. If every $y(g)$ has a similitude in $k^\times$, then for every subgroup $H' \subseteq H$, $y|_{H'}$ has an entry of weight $-w_\Psi$ (resp. $-w_\alpha$) if and only if $y(g)_{31} \neq 0$ (resp. $y(g)_{21} \neq 0$) for some $g \in H'$.*

(Hv2) *Let $\mathscr{T}_0 \leq \mathscr{T}$ be auxiliary towers with labelling ϵ, W_0, W finite sets of primes, p, p' residual pairs over $(\mathscr{T}_0, W_0)$ with values in k (for $\bar{a}$), and q, q' residual pairs over $(\mathscr{T}, W)$ with values in k such that q lies over p and q' lies over p'. If the first components of p and p' are supported in Σ_P and p, p' have the same $-w_\Psi$ part, then q, q' have the same $-w_\Psi$ part. If the first components of p and p' are supported in Σ_M and p, p' have the same $-w_\alpha$ part, then q, q' have the same $-w_\alpha$ part.*

(Hv3) *In the situation of (Hv2), if the first component and all flags of p are supported in Σ_Q, then all flags of q are supported in Σ_Q.*

Proof. (Hv1). Since $\bar{a}$ kills $\mathfrak{m}_{\mathcal{O}}$, it factors through a field embedding $\mathbb{F}_{\mathcal{O}} \to k$; by (Oq7) the residues $\bar{\kappa}_i^{\epsilon}(g_{\mathscr{T}})$ are pairwise distinct, hence so are the $\delta_i(g_{\mathscr{T}})$. Put $h_0 = g_{\mathscr{T}} \in G_F$ and let $g \in H$. By isolation and $y(h_0)_{ij} = 0$, $y(gh_0)_{ij} = y(g)_{ij}\delta_j(h_0)$. Also $gh_0 = (gh_0g^{-1})g$ with $gh_0g^{-1} \in G_F$ (normality), so by isolation $y(gh_0)_{ij} = \delta_i(gh_0g^{-1})y(g)_{ij} + y(gh_0g^{-1})_{ij}\delta_j(g) = \delta_i(h_0)y(g)_{ij}$, δ_i being a character. Hence $(\delta_j(h_0) - \delta_i(h_0))y(g)_{ij} = 0$ and $y(g)_{ij} = 0$.

Isolation: $y(gh)_{ij} = \sum_x y(g)_{ix}y(h)_{xj}$, and (i,j) is isolated as soon as $y(g)_{ix}y(h)_{xj} = 0$ for $x \notin \{i,j\}$. If y is supported in $\Sigma_P = \{w_\beta, -w_\alpha, -w_\theta, -w_\Psi\}$: for $(3,1)$, $y_{32} = 0$ (weight $-w_\beta$) and $y_{34} = 0$ (weight w_α); for $(2,3)$, $y_{13} = 0$ and $y_{24} = 0$ (weight w_Ψ). If y is lower triangular, for $(2,1)$ the terms $x = 3,4$ contain $y_{2x} = 0$. If y is supported in $\Sigma_Q = \{-w_\alpha, -w_\Psi, -w_\theta\}$, it is lower triangular with $y_{32} = 0$ (weight $-w_\beta$), so for $(3,1)$ the terms $x = 2,4$ vanish.

Similitudes: write $X^\dagger = J_4^{-1}\,{}^tXJ_4$; a matrix X has similitude ν if and only if $X^\dagger X = \nu$. With J_4 having entries $J_{14} = J_{23} = 1$, $J_{32} = J_{41} = -1$, one has $J_4^{-1} = -J_4$ and $(X^\dagger)_{ab} = s_at_bX_{5-b,5-a}$ with $(s_1,s_2,s_3,s_4) = (-1,-1,1,1)$, $(t_1,t_2,t_3,t_4) = (-1,-1,1,1)$; in particular $(X^\dagger)_{31} = -X_{42}$ and $(X^\dagger)_{21} = X_{43}$. If $y(g)$ has similitude $\nu(g) \in k^\times$ and $g^{-1} \in H'$ (a subgroup), then $y(g^{-1}) = y(g)^{-1} = \nu(g)^{-1}y(g)^\dagger$, whose entries $(3,1)$ and $(2,1)$ are $-\nu(g)^{-1}y(g)_{42}$ and $\nu(g)^{-1}y(g)_{43}$. The places of weight $-w_\Psi$ are $(3,1),(4,2)$ and those of weight $-w_\alpha$ are $(2,1),(4,3)$, so the last assertion follows.

Two identities. Let $X \in \mathrm{M}_4(k)$ with diagonal entries $x_1,\dots,x_4$, and let $y, y' \in \mathrm{M}_4(k)$ be lower triangular with diagonal entries 1 such that $y^{-1}Xy'$ is upper triangular. Then $Xy'e_1 = y(y^{-1}Xy')e_1 = \lambda\, ye_1$ with $\lambda = (y^{-1}Xy')_{11}$, and comparing first coordinates ($y_{11} = 1$), λ is the first coordinate of $Xy'e_1$.

- If X is supported in Σ_P, then $X_{12} = X_{13} = X_{14} = 0$ (weights $w_\alpha, w_\Psi, w_\theta$), so $\lambda = x_1$; the third coordinates, with $X_{32} = X_{34} = 0$, give
$$y_{31}\,x_1 = X_{31} + x_3\,y'_{31}. \tag{52.7}$$
- If X is lower triangular, then $\lambda = x_1$ and the second coordinates give
$$y_{21}\,x_1 = X_{21} + x_2\,y'_{21}. \tag{52.8}$$

(Hv2). Write $F_0 \subseteq F$ for the fields and Σ_0, Σ for the index sets, $p = (r_p, (n_{p,v_0}))$, and likewise for p', q, q'. By definition of lying over, $r_q = r_p|_{G_F}$ and $r_{q'} = r_{p'}|_{G_F}$, so the entries of the first components of q, q' in either part agree. Fix $v \in \Sigma$. There are $v_0, v_0' \in \Sigma_0$ and $g, g' \in G_{F_0}$ with $g(\bar{v}_0) = \bar{v} = g'(\bar{v}_0')$, $n_{q,v}^{-1}r_p(g)n_{p,v_0}$ and $n_{q',v}^{-1}r_{p'}(g')n_{p',v_0'}$ upper triangular. Then $\bar{v}_0' = g'^{-1}g(\bar{v}_0)$ is G_{F_0}-conjugate to $\bar{v}_0$, so $v_0' = v_0$ (uniqueness of the representative in the definition of an auxiliary tower), and $d = g'^{-1}g \in G_{F_0}$ fixes $\bar{v}_0$, i.e. $d \in D_{v_0}$. By (Mq4) for p', $n_{p',v_0}^{-1}r_{p'}(d)n_{p',v_0}$ is upper triangular, hence so is

$$n_{q',v}^{-1}r_{p'}(g)n_{p',v_0} = \big(n_{q',v}^{-1}r_{p'}(g')n_{p',v_0}\big)\big(n_{p',v_0}^{-1}r_{p'}(d)n_{p',v_0}\big).$$

All flags are lower triangular with diagonal entries 1 (Mq3), and $r_p(g)$, $r_{p'}(g)$ have the same diagonal entries $\delta_i(g)$ by (Wq1) for $\mathscr{T}_0$, with $\delta_1(g) \neq 0$. In the first case, (52.7) for $(X,y,y') = (r_p(g), n_{q,v}, n_{p,v_0})$ and $(r_{p'}(g), n_{q',v}, n_{p',v_0})$ gives

$$(n_{q,v})_{31}\delta_1(g) = r_p(g)_{31} + \delta_3(g)(n_{p,v_0})_{31} = r_{p'}(g)_{31} + \delta_3(g)(n_{p',v_0})_{31} = (n_{q',v})_{31}\delta_1(g),$$

the middle equality because p, p' have the same $-w_\Psi$ part. In the second case (52.8) gives the same with $(2,1)$ and δ_2. So $(n_{q,v})_{31} = (n_{q',v})_{31}$, resp. $(n_{q,v})_{21} = (n_{q',v})_{21}$, for all v.

(Hv3). Fix $v \in \Sigma$ and v_0, g as above for q and p. The matrices $r_p(g)$ and n_{p,v_0} are lower triangular and supported in Σ_Q. A product YZ of two such matrices is of the same kind: it

is lower triangular, and its $(3,2)$ entry is $Y_{32}Z_{22} + Y_{33}Z_{32} = 0$, the only lower place of weight outside Σ_Q being $(3,2)$. So $L = r_p(g)n_{p,v_0}$ is lower triangular, supported in Σ_Q, and invertible ($r_p(g)$ has a non-zero similitude, n_{p,v_0} similitude 1). The matrix $D = n_{q,v}^{-1}L$ is upper triangular by assumption and lower triangular as a product of lower triangular matrices, hence diagonal and invertible; so $n_{q,v} = LD^{-1}$ is supported in Σ_Q. □

Proposition 52.18. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum attached to a host character in case (III). For every admissible labelling ϵ there is a fit and tame auxiliary tower of case (III) for $\mathfrak{D}$ with labelling ϵ.*

Proof. Let $\mathfrak{D}$ be attached to $(\psi, \mathfrak{h}, t, \chi)$ in case (III) and let ϵ be admissible. Then $N = N_{\mathfrak{h}}$, $p' = p_1(\psi)$, $m = \operatorname{ord}\chi$ is odd, $\chi \neq \mathbf{1}$, $\xi = \chi_{\mathrm{dy}}\chi_{\mathrm{pr}}$ with χ_{dy} primitive modulo $2^{n'}$ and χ_{pr} modulo p', and S is the set of primes dividing $2\ell N p'$; $2^{n'} \geq 8\ell^2$ gives $n' \geq 8$.

Step 0: facts about the data.

(a) $\ell \geq 5$: ℓ is odd, $3 \in B_\psi$ and $\ell \notin B_\psi$.

(b) $\ell \nmid N$ and $p' \nmid N$, $p' \neq \ell$. Indeed $\ell \nmid N_\psi$ and $p' \neq \ell$ because the odd prime factors of N_ψ and p' (odd, being prime to $2N_\psi$) lie in B_ψ; $(\ell - 1) \mid 2e$ with $0 < e < t$ gives $\ell \leq 2e + 1 < 2t + 1 < \min(q, d)$, so $\ell \neq q, d$; and $p' \nmid 2N_\psi$, $p' < \min(q, d)$ by bigness. So χ is unramified at ℓ and at p', and ξ is unramified at ℓ.

(c) $\ell \nmid m$. The map $u \mapsto (u \bmod 2^{a+2}$ modulo $\pm 1,\ \psi(u),\ (u/q),\ (u/d))$ has kernel $H_{\mathfrak{h}}$, so $(\mathbb{Z}/N\mathbb{Z})^\times/H_{\mathfrak{h}}$ has exponent dividing $\operatorname{lcm}(2^a, o_\psi, 2)$; χ factors through it, and $\ell \nmid o_\psi$ ($\ell \notin B_\psi$). Hence $\chi'' = \mathbf{1}$, $S_\ell(\chi) = \emptyset$, $T = \{\ell\}$, $T' = \emptyset$.

(d) $\chi(c) = 1$ ($\chi(c)^2 = 1$, m odd), so $1 = \Psi(c) = (-1)^e$ by (Sg1): e is even. The 2-component of χ has odd order on a 2-group, so χ is unramified at 2; χ_{pr} is unramified at 2. $\bar{\alpha} = \bar{\xi}$, $\bar{\Psi} = \bar{\chi}\,\bar{\omega}_\ell^e$, and $\bar{\omega}_\ell^{2e} = \mathbf{1}$ as $(\ell - 1) \mid 2e$. ξ, χ, ω_ℓ are unramified outside S; $\xi(c) = -1$.

Step 1: the field E' and the subgroup. By Dirichlet's theorem choose a prime $q_0 \notin S$ with $q_0 \equiv 7 \pmod 8$ and $q_0 \equiv -1 \pmod p$ for every odd $p \in S$. Let $E' = \mathbb{Q}(\sqrt{-q_0})$, with character η': imaginary quadratic of discriminant $-q_0$, so $\ell \nmid \operatorname{disc}(E')$; ramified exactly at q_0; $\eta'(c) = -1$; every prime of S, in particular ℓ and 2, splits in E' ($-q_0 \equiv 1$ modulo 8 and modulo every odd $p \in S$). Fix $i' \in I_{\bar{q}_0}$ with $\eta'(i') = -1$. Put $S' = S \cup \{q_0\}$ and

$$H' = \{g \in G_{\mathbb{Q}} :\ \xi(g) = \eta'(g),\ \bar{\omega}_\ell(g)^e = 1\} :$$

open, containing the commutator subgroup, containing c ($\xi(c) = -1 = \eta'(c)$, $(-1)^e = 1$ by (d)), and $I_{\bar{p}}$ for $p \notin S'$. Apply Lemma 52.10 to $\mathfrak{D}, \epsilon, S', H'$: it gives q_1, q_2, E, F, L, J. The characters $\xi, \eta', \chi, \omega_\ell$ are unramified at $q_1, q_2 \notin S'$, so (Zf3) preserves their values. On $G_F \subseteq H'$: $\xi = \eta'$ and $\bar{\Psi} = \bar{\chi}$.

Step 2: elements.

- *At ℓ.* For every place $\bar{w}$ above ℓ and $g \in I_{\bar{w}}$: $\xi(g^2) = 1$ ((b)), $\eta'(g^2) = 1$, $\bar{\omega}_\ell(g^2)^e = \bar{\omega}_\ell(g)^{2e} = 1$ ((d)); so $g^2 \in H' \cap D_{\bar{w}} \subseteq H' \cap J = G_L$ by (Zf1), (Zf2). As $\bar{\omega}_\ell$ maps $I_{\bar{\ell}}$ onto $\mathbb{F}_\ell^\times$, of order $\ell - 1 \geq 4$, there is $g_I \in I_{\bar{\ell}}$ with $\bar{\omega}_\ell(g_I)^2 \neq 1$; put $y_I = g_I^2 \in G_L$. Then $\bar{\xi}(y_I) = 1$, $\bar{\Psi}(y_I) = \bar{\chi}(y_I) = 1$ ((b)), $\bar{\omega}_\ell(y_I) \neq 1$.

- *The marking.* Let $g_1 \in I_{\bar{2}}$ map to $1 + 2^{n'-1} \in \mathbb{Z}_2^\times$. As χ_{dy} is primitive, it is injective on the cyclic group $(1 + 4\mathbb{Z})/(1 + 2^{n'}\mathbb{Z})$ (a non-trivial kernel would contain $1 + 2^{n'-1}$, and χ_{dy} would factor modulo $2^{n'-1}$), so $\chi_{\mathrm{dy}}(1 + 2^{n'-1})$ has order 2, i.e. equals -1; with (d),

$\xi(g_1) = -1$, $\chi(g_1) = 1$, $\omega_\ell(g_1) = 1$, and $\eta'(g_1) = 1$ as 2 splits in E'. Choose $a_0 \in (\mathbb{Z}/N\mathbb{Z})^\times$ with $\chi(a_0) \neq 1$ ($\chi \neq \mathbf{1}$) and, by (b) and the Chinese remainder theorem, $u_0 \in G_\mathbb{Q}$ acting on μ_N through a_0 and trivially on $\mu_{\ell p'}$; put $g_2 = u_0^{2^{n'}}$. Then $\chi(g_2) = \chi(a_0)^{2^{n'}} \neq 1$ ($\chi(a_0) \neq 1$ has odd order), $\chi_{\mathrm{dy}}(g_2) = 1$ (its order divides $2^{n'-1}$), $\chi_{\mathrm{pr}}(g_2) = 1$, $\omega_\ell(g_2) = 1$. Let $y_0' = g_1\, i'\, g_2^2$: $\xi(y_0') = -1 = \eta'(y_0')$ and $\omega_\ell(y_0') = 1$, so $y_0' \in H'$, and (Zf3) gives $y_0 \in G_L$ with $\bar\xi(y_0) = -1$, $\omega_\ell(y_0) = 1$ and $\bar\chi(y_0) = \bar\chi(g_2)^2 =: x_0$. Since $\chi(g_2)^2$ is a root of unity of odd order dividing m, different from 1, and $\ell \nmid m$, x_0 has odd order > 1: $x_0 \neq \pm 1$. Also $\bar\Psi(y_0) = \bar\chi(y_0)\bar\omega_\ell(y_0)^e = x_0$.

Let $\mathscr{T} = (\epsilon, E, F, y_0, \Sigma, (\bar v))$ be the tower of (Zf4) with $g_\star = y_0$; (Oq1), (Oq2) hold. With $\varsigma = (-1)^\epsilon$: from $(\kappa_i^0) = (\theta, \Psi, \alpha, \mathbf{1})$ and $(\kappa_i^1) = (\alpha, \mathbf{1}, \theta, \Psi)$, $\chi_1^{\mathscr{T}}/\chi_2^{\mathscr{T}} = \alpha$ and $\chi_1^{\mathscr{T}}/\chi_3^{\mathscr{T}} = \Psi^\varsigma$, so $\overline{\mathrm{rt}_w} = \bar\xi^{w_1}\bar\Psi^{\varsigma w_2}$ on G_F ($\bar\alpha = \bar\xi$); so $\overline{\mathrm{rt}_w}(y_0) = (-1)^{w_1} z^{w_2}$ with $z = x_0^\varsigma$, again of odd order > 1, so $z \neq \pm 1$.

Step 3: the clauses. Conjugation by $h \in G_\mathbb{Q}$ moves elements of $D_{\bar p} \cap G_F$, $I_{\bar p} \cap G_F$ to any place above p without changing character values.

(Oq3) For every w, $\overline{\mathrm{rt}_w}(y_I) = 1 \neq \bar\omega_\ell(y_I)$. For the roots: $\pm w_\alpha$ give -1; $\pm w_\Psi$ give $z^{\pm 1} \neq 1$; $\pm w_\beta = \pm(-1, 1)$ and $\pm w_\theta = \pm(1, 1)$ give $-z^{\pm 1} \neq 1$. For w_θ and $1 \leq n \leq 4$: if $(-z)^n = 1$ then $z^{2n} = 1$; with $z^m = 1$, m odd, $z^{\gcd(n,m)} = 1$, and $\gcd(n, m) \in \{1, 3\}$; gcd $= 1$ gives $z = 1$, and gcd $= 3$ (so $n = 3$) gives $(-z)^3 = -1 \neq 1$; both are contradictions.

(Oq4) With η as there, $\eta = \bar\kappa_j^\epsilon$ on G_L for some j (take the index i with $\kappa_i^\epsilon = \mathbf{1}$), and $\eta(y_0) = 1$ as $\omega_\ell(y_0) = 1$. The residues $(\bar\theta, \bar\Psi, \bar\alpha, \mathbf{1})(y_0) = (-x_0, x_0, -1, 1)$ are pairwise distinct ($x_0 \neq 0, \pm 1$, $-1 \neq 1$), so $\bar\kappa_j^\epsilon(y_0) = 1$ forces $\kappa_j^\epsilon = \mathbf{1}$, and $\eta = 1$.

(Oq5) For $g \in I_{\bar w} \cap G_F$, $\bar w$ not above ℓ, with $\bar r_0(g) = 1$: $\bar\chi(g) = \bar\Psi(g) = 1$ on G_F, so $\chi(g) = 1$ ($\ell \nmid m$); and $\chi^m = \mathbf{1}$ with $\ell \nmid m$.

(Oq6) For $g \in I_{\bar w} \cap G_F$, $\bar w$ above ℓ: $\bar\alpha(g) = \bar\xi(g) = 1$ and $\bar\Psi(g) = \bar\chi(g) = 1$ by (b).

(Oq7) The residues at $g_{\mathscr{T}} = y_0$ are pairwise distinct (see (Oq4)), and λ_ϵ permutes them.

(Oq8) The first alternative: conjugates of y_I lie in $I_{\bar w} \cap G_F$ with $\bar\omega_\ell \neq 1$.

Case (III) $\ell \geq 5$ by (a); $T' = \emptyset$ and χ has finite order m prime to ℓ by (c); E' is imaginary quadratic, $\ell \nmid \mathrm{disc}(E')$, ℓ and every prime of S split in E'; and for $g \in G_F$, $\overline{\mathrm{rt}_{w_\alpha}}(g) = \bar\xi(g) = \eta'(g)$, which is 1 if $g \in G_{E'}$ and -1 otherwise.

So $\mathscr{T}$ is good and of case (III), and fit ($\mathfrak{D}$ is attached in case (III), $\ell \geq 5$, $T' = \emptyset$, χ of order prime to ℓ). *Tame*: for $g \in I_{\bar w}$, $\bar w$ above ℓ, $g^2 \in G_L \subseteq G_F$ by Step 2, and 2 is prime to ℓ. Hence $\mathscr{T}$ is a fit and tame auxiliary tower of case (III) with labelling ϵ. □

53 The rank-four Hecke chain

This section proves the two rank-four statements that Proposition 51.5 uses: every space of marked pairs over a fit and deep auxiliary tower carries a modular ideal (Theorem 53.1), and a modular ideal present on one component through a residual pair is present on all components through every residual pair chained to it in case (III), provided the tower has degree $d \geq 30(C+1)^2$ for a Selmer bound C (Theorem 53.31). They play the parts of the Hecke realisation and of the propagation theorem of Sections 49 and 50. The Hecke side is built over the CM field $L = FE$ of a link, which carries a rank-four Hecke datum (Lemma 53.2). There the rank-n Hecke

chain in rank four, Theorem 53.10, gives a Hecke ideal of zero-monodromy type with $R = T$ at patchable curve points, automorphy at classical irreducible points, purity and stability under twists; it rests on the results of Appendix E. A comparison map from the universal rank-four determinant ring of the link to the space (Lemma 53.7) carries this ideal to the space, and the conditions (Jq1)–(Jq5) of a modular ideal are checked one by one along it.

Subsection 53.1 sets up links, twists and comparison maps and transports the finiteness over the weights (Jq1). Subsection 53.2 transports $R = T$, locality at seed points, purity and classicality, (Jq2)–(Jq5), and descends automorphy from L to imaginary quadratic fields and to the symplectic condition (Sy2). Subsection 53.3 proves the propagation theorem from local charts in rank four, whose reducible and coincidence loci are small.

53.1 The Hecke side in rank four: links, twists and comparison maps

This subsection proves Theorem 53.1 from Theorem 53.10 and the transport lemmas 53.11–Lemma 53.23, and states Theorem 53.10. A link of a fit and deep tower is a rank-four Hecke datum over $L = FE$, with the residual characters of the tower twisted (Lemma 53.2). The twists exist: algebraic characters of a CM field with a prescribed product with their conjugate, unramified at the finite places not above ℓ (Lemma 53.5). Universal polarised ordinary determinants exist over the datum (Lemma 53.6), by Chenevier's theory of determinants [66], and the comparison map classifies the twisted determinant of the points of a space (Lemma 53.7). The subsection also defines the Taylor–Wiles data and the classical level modules of a rank-n Hecke datum (Definition 53.8, Definition 53.9), with which Appendix E works; Theorem 53.10 is assembled from the results of that appendix and from Lemma 50.12. Finiteness over the weights then passes to the space (Lemma 53.11).

Theorem 53.1. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ a fit and deep auxiliary tower for $\mathfrak{D}$, and $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism. Then there is a coefficient ring $\mathcal{O}_1 \supseteq \mathcal{O}$ such that for every coefficient ring $\mathcal{O}' \supseteq \mathcal{O}_1$ and every space $\mathscr{X} = (B, A, P, \dots)$ for $(\mathscr{T}, \mathcal{O}')$, some ideal of B is modular for ι.*

Proof. Symbols introduced here are local to this proof.

Step 1: the coefficient ring. By Lemma 53.2, $L = FE$ is a CM field with $L^+ = F$, unramified over F at every finite place, every place of F above ℓ splits in L, L contains no primitive ℓ-th root of unity, and c is a complex conjugation in G_F; let $\mathcal{O}_1^\flat$, ν, γ, t be as given there. By Lemma 53.5 with $m = -3$ there is a continuous character κ of G_L, algebraic for ι, with (Tw1)–(Tw4), (Kw1) and values in a coefficient ring $\mathcal{O}_\kappa$. Let $\mathcal{O}_1$ be a coefficient ring containing $\mathcal{O}_1^\flat$ and $\mathcal{O}_\kappa$ (the integral closure of $\mathbb{Z}_\ell$ in the compositum of their fraction fields).

Step 2: the link. Let $\mathcal{O}' \supseteq \mathcal{O}_1$ and let $\mathscr{X} = (B, A, P, \dots)$ be a space for $(\mathscr{T}, \mathcal{O}')$. Then κ takes values in $\mathcal{O}'^\times$ and satisfies (Tw1) for $m = -3$, (Tw2), (Tw4), so Lemma 53.2 gives a rank-4 Hecke datum $\mathscr{H}$ over $\mathcal{O}'$ with CM field L and complex conjugation c, with (Lk6)–(Lk8). By the twist clause of Lemma 53.5 ($n = 4$, $m = 1 - n$, $\mathcal{O}'$ containing $\mathcal{O}_\kappa$), κ is a twist for $\mathscr{H}$; its wild values are trivial (Lk7).

Step 3: the universal rings. The pair $(\bar{D}_4, (\bar{\chi}_a|_{D_v}))$ is a polarised ordinary determinant of rank 4 over $\mathscr{H}$ to $\mathbb{F}_{\mathcal{O}'} \in \mathrm{CNL}_{\mathcal{O}'}$: (Od1) since $\{\bar{\chi}_a(cg^{-1}c)\}_a = \{\bar{\chi}_{5-a}(g)\}_a$ by (Rd5); (Od2) since $\mathrm{tr}_{\bar{D}_4}(\gamma' \prod_a (g_a - \bar{\chi}_a(g_a))) = \sum_b \bar{\chi}_b(\gamma') \prod_a (\bar{\chi}_b(g_a) - \bar{\chi}_a(g_a))$, whose b-th term has the factor $a = b$ equal to 0; (Od3) since each $\bar{\chi}_a$ has order prime to ℓ and so is trivial on μ_v, as the trivial wild values are. By Lemma 53.6 there is a universal polarised ordinary determinant $(B_4^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ of rank 4 over $\mathscr{H}$; let ϕ_0 be the morphism of Lemma 53.7 for $n = 0$ (its hypotheses: (Lk1)–(Lk4), (Lk6), (Lk7) and (Tw1), (Tw2), (Tw4)).

Step 4: the Hecke ideals. By (Hf1) of Theorem 53.10 for $(\mathscr{H}, \kappa, B_4^{\mathrm{po}}, \iota)$ there is a Hecke ideal J_0 of zero-monodromy type with (Hf2)–(Hf5), $J_0 \subseteq \mathcal{J}_\kappa^{\mathrm{cl}}$, and, if $J_0 \neq B_4^{\mathrm{po}}$, R = T at patchable curve points. By (Hf2), J_0 is finite over the weights: $B_4^{\mathrm{po}}/(J_0 + \mathfrak{b}_4)$ is a finite module over $\Lambda_{4d}/\mathfrak{m}_{\Lambda_{4d}} = \mathbb{F}_{\mathcal{O}'}$, as λ maps $\mathfrak{m}_{\Lambda_{4d}}$ into the image of $\mathfrak{b}_4$, hence an Artinian local ring, whose maximal ideal is nilpotent.

Step 5: the modular ideal. Let $\mathfrak{J} = \phi_0(J_0)B$. For (Jq1), by Lemma 53.11 for J_0, some $\mathfrak{m}_B^k$ lies in $\phi_0(J_0)B + \mathfrak{m}_{\mathcal{O}'}B + (\psi_{v,i}^B(s) - 1) = \mathfrak{J} + \mathfrak{m}_{\mathcal{O}'}B + (\psi_{v,i}^B(s) - 1)$. For (Jq2), let y be as in (Jq2) and $\mathfrak{q}$ a prime of B with $y(\mathfrak{q}) = 0$; as $y(\mathfrak{J}) = 0$, $y(\phi_0(J_0)) = 0$, so $J_0 \neq B_4^{\mathrm{po}}$ and J_0 has R = T at patchable curve points; by Lemma 53.12, $\phi_0(J_0) \subseteq \mathfrak{q}$, so $\mathfrak{J} \subseteq \mathfrak{q}$, i.e. $\mathfrak{J}$ is local at y. For (Jq3), take y with $y(P)$ a seed; then Lemma 53.17 for J_0 ($J_0 \subseteq \mathcal{J}_\kappa^{\mathrm{cl}}$, (Hf2), (Hf5); κ has (Kw1) by Step 1) gives $\phi_0(J_0) \subseteq \mathfrak{q}$ for every prime $\mathfrak{q}$ with $y(\mathfrak{q}) = 0$, so $\mathfrak{J} \subseteq \mathfrak{q}$. (Jq4) follows from Lemma 53.22 for J_0 ((Hf4); $\mathscr{T}$ is good as it is fit, and ν has (Lk5)), as $\mathfrak{J} \supseteq \phi_0(J_0)$. (Jq5) follows from Lemma 53.23 for J_0 ((Hf3)), as $\mathfrak{J} \supseteq \phi_0(J_0)$. Hence $\mathfrak{J}$ is modular for ι. □

Lemma 53.2. *Symbols introduced in this statement are local to it. Let $\mathfrak{D}$ be a symplectic Eisenstein datum with prime ℓ, coefficient ring $\mathcal{O}$, character θ, set of primes S and complex conjugation c, and let $\mathscr{T}$ be a fit and deep auxiliary tower for $\mathfrak{D}$. Then $L = FE$ is a CM field with $L^+ = F$, L/F is unramified at every finite place, every place of F above ℓ splits in L, L contains no primitive ℓ-th root of unity, c is a complex conjugation in $G_F = G_{L^+}$, and there are a coefficient ring $\mathcal{O}_1 \supseteq \mathcal{O}$, continuous homomorphisms $\nu, \gamma\colon G_L \to \mathcal{O}_1^\times$ and a continuous homomorphism $t\colon G_L \to \mathbb{Z}_\ell$ such that*

(Lk1) *$\nu(s)\nu(csc)\theta(s)\chi_{\mathrm{cyc}}(s)^3 = 1$, $\gamma(s)\gamma(csc) = 1$ and $t(s) + t(csc) = 0$ for all $s \in G_L$;*

(Lk2) *$\gamma(s) \equiv 1$ modulo $\mathfrak{m}_{\mathcal{O}_1}$ for all s; $\ker t \subseteq \ker \gamma$; and t is trivial on $I_{\bar{w}} \cap G_L$ for every place $\bar{w}$ of $\overline{\mathbb{Q}}$ not above ℓ;*

(Lk3) *ν is trivial on $I_{\bar{w}} \cap G_L$ for every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above a prime not in S, and for some integer $m \geq 1$ with $\ell \nmid m$, ν^m is trivial on $I_{\bar{w}} \cap G_L$ for every place $\bar{w}$ not above ℓ;*

(Lk4) *$\nu(g) = 1$ and $t(g) = 0$ for every $v \in \Sigma$ and $g \in \mu_v$;*

(Lk5) *for every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above ℓ there are integers a, b with b odd and $k \geq 1$ such that $(\nu\chi_{\mathrm{cyc}}^{-a})^k$ and $(\gamma^2\chi_{\mathrm{cyc}}^{-b})^k$ are trivial on $I_{\bar{w}} \cap G_L$;*

and such that for every coefficient ring $\mathcal{O}' \supseteq \mathcal{O}_1$ and every continuous character $\kappa\colon G_L \to \mathcal{O}'^\times$ satisfying (Tw1) with $m = -3$, (Tw2) and (Tw4) for (L, c), there is a rank-4 Hecke datum over $\mathcal{O}'$,

$$\mathscr{H} = (4, L, c, (\bar{v})_{v\in\Sigma}, \mathrm{St}, v_0, \bar{v}_0, (\bar{\chi}_a)_{1\leq a\leq 4}, (\xi_{v,a})),$$

where the places of F above ℓ are indexed by Σ through $v \mapsto$ the place of F below $\bar{v}$, with:

(Lk6) *the residual characters: $\bar{\chi}_a$ is the reduction modulo $\mathfrak{m}_{\mathcal{O}'}$ of $\nu\kappa^{-1}\chi_a^{\mathscr{T}}|_{G_L}$ for $1 \leq a \leq 4$;*

(Lk7) *trivial wild values: $\xi_{v,a} = 1$ for all $v \in \Sigma$ and $1 \leq a \leq 4$;*

(Lk8) *every place of F not above ℓ that lies above a prime of S belongs to* St.

Proof. Symbols introduced here are local to this proof. Write $\chi_a^{\mathscr{T}}$ for $\chi_a^{\mathscr{T}}|_{G_L}$ where no confusion arises; recall $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (\theta, \Psi, \alpha, \mathbf{1})$ and $\theta = \alpha\Psi = \chi_{\mathrm{cyc}}^{e+1}\omega_\ell^{-1}\xi\chi$.

Step 0: the fields. $E \not\subseteq F$ (F is totally real), so $L = FE$ is a CM field with $L^+ = F$, and c, which fixes the totally real field F and not E, lies in $G_F \setminus G_L$ and is a complex conjugation

for L. By (Oq1), L/F is unramified at every finite place. The primes of S, among them 2 and ℓ (Sg1), split in E (Oq2); hence every place of F above a prime $p \in S$ splits in L, as $L \otimes_F F_v = F_v \otimes_{\mathbb{Q}_p} (E \otimes \mathbb{Q}_p) \cong F_v \times F_v$. By (Oq3) with $w = 0$ ($\mathrm{rt}_0 = \mathbf{1}$), $\bar{\omega}_\ell$ is non-trivial on G_L, so L contains no primitive ℓ-th root of unity.

Step 1: three characters. (a) $\chi_{\mathrm{cyc}}\omega_\ell^{-1}$ takes values in $1 + \ell\mathbb{Z}_\ell$, which for odd ℓ is uniquely 2-divisible (log is an isomorphism onto $\ell\mathbb{Z}_\ell$); so $\gamma_{\mathbb{Q}} = \exp(\frac{1}{2}\log(\chi_{\mathrm{cyc}}\omega_\ell^{-1}))$ is a continuous character $G_{\mathbb{Q}} \to 1 + \ell\mathbb{Z}_\ell$ with $\gamma_{\mathbb{Q}}^2 = \chi_{\mathrm{cyc}}\omega_\ell^{-1}$, unramified outside ℓ. (b) Let $\varphi_0 = (\omega_\ell^{e+3}\xi\chi)^{-1}$, a Dirichlet character unramified outside S (Sg1); write $\varphi_0 = \varphi\varphi_\ell$ with φ of order prime to ℓ and φ_ℓ of ℓ-power order, both powers of φ_0. As ω_ℓ and ξ have order prime to ℓ and $\chi|_{G_F}$ has order prime to ℓ (deep), $\varphi_\ell|_{G_F} = 1$. Moreover $\omega_\ell(c) = -1$, $\xi(c) = -1$ and $\chi(c) = \Psi(c)\chi_{\mathrm{cyc}}(c)^{-e} = (-1)^e$ (Sg1), so $\varphi_0(c) = 1$ and $\varphi(c) = 1$. By Lemma 53.3 (2 and every prime of S split in E) there is $\kappa_0 \colon G_E \to \overline{\mathbb{Q}}_\ell^\times$ of finite order m_0 prime to ℓ with (Ie1) for φ and (Ie2). (c) Let $\bar{w}$ be a place above ℓ; by Lemma 53.4 (ℓ splits in E) there are $\mathcal{O}_\beta$, $t_E \colon G_E \to \mathbb{Z}_\ell$ and $\beta \colon G_E \to \mathcal{O}_\beta^\times$ with (Ia1)–(Ia4).

Step 2: $\mathcal{O}_1$, ν, γ, t. Let $\mathcal{O}_1$ be the integral closure of $\mathbb{Z}_\ell$ in the finite extension of $\mathbb{Q}_\ell$ generated by $\mathcal{O}$, $\mathcal{O}_\beta$, the m_0-th roots of unity and $\tau(L)$ for the finitely many embeddings $\tau \colon L \to \overline{\mathbb{Q}}_\ell$; it is a coefficient ring containing $\mathcal{O}$. Put $\epsilon_e \in \{0, 1\}$ with $\epsilon_e \equiv e$ modulo 2 and

$$\nu = \kappa_0|_{G_L} \cdot \gamma_{\mathbb{Q}}|_{G_L}^{-(e+4)} \cdot \beta|_{G_L}^{\epsilon_e}, \qquad \gamma = \beta|_{G_L}, \qquad t = t_E|_{G_L},$$

continuous, with values in $\mathcal{O}_1^\times$ (κ_0 takes values in the m_0-th roots of unity, $\gamma_{\mathbb{Q}}$ in $\mathbb{Z}_\ell^\times$).

Step 3: (Lk1)–(Lk5). (Lk1): for $s \in G_L$, $csc \in G_L$ and $c \notin G_E$; by (Ie1), (Ia2) and as $\gamma_{\mathbb{Q}}$ is a character of $G_{\mathbb{Q}}$ with abelian values, $\nu(s)\nu(csc) = \varphi(s)\gamma_{\mathbb{Q}}(s)^{-2(e+4)} = \varphi_0(s)(\chi_{\mathrm{cyc}}\omega_\ell^{-1})(s)^{-(e+4)} = \chi_{\mathrm{cyc}}(s)^{-(e+4)}\omega_\ell(s)\xi(s)^{-1}\chi(s)^{-1}$ (using $\varphi_\ell(s) = 1$), which is $(\theta(s)\chi_{\mathrm{cyc}}(s)^3)^{-1}$; $\gamma(s)\gamma(csc) = 1$ and $t(s) + t(csc) = 0$ by (Ia2). (Lk2) is (Ia1) and (Ia3) restricted to G_L. (Lk3): at a place above a prime not in S (in particular not above ℓ), κ_0 is trivial on inertia by (Ie2), $\gamma_{\mathbb{Q}}$ is unramified, and β is trivial on inertia by (Ia3) and $\ker t_E \subseteq \ker\beta$; at any place not above ℓ the last two still hold, so ν^{m_0} is trivial on inertia there. (Lk4): let $v \in \Sigma$, $g \in \mu_v$, so $g^{\ell^k} \in \overline{[D_v, D_v]}$ for some k. A continuous homomorphism from D_v to an abelian Hausdorff group kills $\overline{[D_v, D_v]}$. Hence $\ell^k t(g) = 0$, so $t(g) = 0$, and $\beta(g) = 1$; $\kappa_0(g)^{\ell^k} = 1$ with $\kappa_0(g)$ of order prime to ℓ, so $\kappa_0(g) = 1$; $\gamma_{\mathbb{Q}}(g)^{\ell^k} = 1$ in the torsion-free group $1 + \ell\mathbb{Z}_\ell$, so $\gamma_{\mathbb{Q}}(g) = 1$. Thus $\nu(g) = 1$. (Lk5): let $\bar{u}$ be a place above ℓ. Since ℓ splits in E, $\bar{u}$ induces on E either the place of $\bar{w}$ or its conjugate; in the first case $I_{\bar{u}} \cap G_E$ is G_E-conjugate to $I_{\bar{w}} \cap G_E$ and $\beta^2\chi_{\mathrm{cyc}}^{-b}$ with $b = b_0$ (odd, from (Ia4)) is trivial on an open subgroup of it by (Ia4); in the second, $I_{\bar{u}} \cap G_E$ is G_E-conjugate to $c(I_{\bar{w}} \cap G_E)c$ and by (Ia2) the same holds with $b = -b_0$. An open subgroup of the compact group $I_{\bar{u}} \cap G_L$ has finite index, so $(\beta^2\chi_{\mathrm{cyc}}^{-b})^{k_1}$ is trivial on $I_{\bar{u}} \cap G_L$ for some $k_1 \geq 1$, with b odd. Further $\gamma_{\mathbb{Q}}^{-2(e+4)} = \chi_{\mathrm{cyc}}^{-(e+4)}\omega_\ell^{e+4}$, so $\nu^2 = \kappa_0^2\omega_\ell^{e+4}\chi_{\mathrm{cyc}}^{-(e+4)}\beta^{2\epsilon_e}$, and with $2a = -(e+4) + b\epsilon_e$ (even, as b is odd and $\epsilon_e \equiv e$) the character $\nu^2\chi_{\mathrm{cyc}}^{-2a} = \kappa_0^2\omega_\ell^{e+4}(\beta^2\chi_{\mathrm{cyc}}^{-b})^{\epsilon_e}$ has finite order on $I_{\bar{u}} \cap G_L$; so $(\nu\chi_{\mathrm{cyc}}^{-a})^k$ is trivial there for some $k \geq 1$, and the same k may be taken for both characters.

Step 4: the auxiliary element. The characters $\bar{\chi}_a^{\mathscr{J}}/\bar{\chi}_b^{\mathscr{J}}$ ($a \neq b$) are root characters: $\chi_a^{\mathscr{J}}\chi_{5-a}^{\mathscr{J}} = \theta$ for both labellings, so with $\mathrm{rt}_{w_\alpha} = \chi_1^{\mathscr{J}}/\chi_2^{\mathscr{J}}$, $\mathrm{rt}_{w_\Psi} = \chi_1^{\mathscr{J}}/\chi_3^{\mathscr{J}}$, $\mathrm{rt}_{w_\beta} = \chi_2^{\mathscr{J}}/\chi_3^{\mathscr{J}}$, $\mathrm{rt}_{w_\theta} = \chi_1^{\mathscr{J}}/\chi_4^{\mathscr{J}}$ one has $\chi_2^{\mathscr{J}}/\chi_4^{\mathscr{J}} = \mathrm{rt}_{w_\Psi}$ and $\chi_3^{\mathscr{J}}/\chi_4^{\mathscr{J}} = \mathrm{rt}_{w_\alpha}$. By (Oq3) they are non-trivial on G_L. Let $H \subseteq G_L$ be the common kernel of $\bar{\chi}_1^{\mathscr{J}}, \ldots, \bar{\chi}_4^{\mathscr{J}}$; one of these characters is trivial ($\kappa_4 = \mathbf{1}$), and the others are products of powers of $\bar{\chi}_1^{\mathscr{J}}/\bar{\chi}_2^{\mathscr{J}}$ and $\bar{\chi}_1^{\mathscr{J}}/\bar{\chi}_3^{\mathscr{J}}$ times that trivial one, so H is the common kernel of $\overline{\mathrm{rt}_{w_\alpha}}$ and $\overline{\mathrm{rt}_{w_\Psi}}$ on G_L. The characters of the finite abelian group G_L/H, which embeds into $\mathbb{F}_{\mathcal{O}}^\times \times \mathbb{F}_{\mathcal{O}}^\times$ through these two, are the products of their powers, i.e. the $\overline{\mathrm{rt}_w}$, $w \in \mathbb{Z}^2$ (a character of a subgroup of a product of two cyclic groups extends to the product, whose characters are products of powers of the two coordinates). By (Oq3), $\bar{\omega}_\ell|_{G_L}$ is none of them, so $\bar{\omega}_\ell$ is non-trivial

on H: there is $g_1 \in G_L$ with $\bar\chi_a^{\mathscr{J}}(g_1) = 1$ for all a and $\bar\omega_\ell(g_1) \neq 1$.

Step 5: the data. Let $\mathcal{O}' \supseteq \mathcal{O}_1$ and κ be as in the statement, and $\lambda = \nu\kappa^{-1}$, so $\lambda(s)\lambda(csc) = \theta(s)^{-1}$ for $s \in G_L$ by (Lk1) and (Tw1). Let $\bar\chi_a$ be the reduction of $\lambda\chi_a^{\mathscr{J}}$, $\xi_{v,a} = 1$, and St the set of places of F above the primes of $S \setminus \{\ell\}$. Let $M \subset \overline{\mathbb{Q}}$ be the fixed field of $G_L \cap \ker\bar\theta \cap \ker\bar\Psi \cap \ker\bar\alpha \cap \ker\bar\omega_\ell$, a finite Galois extension of $\mathbb{Q}$ ($L/\mathbb{Q}$ is Galois, F being abelian). By the Chebotarev density theorem for M/F there are infinitely many places v_0 of F, unramified in M, with a place $\bar v_0$ above them whose Frobenius in $\mathrm{Gal}(M/F)$ is the image of g_1; choose v_0 above a prime $p \notin S$ unramified in L. Then $D_{\bar v_0} \cap G_F$ maps into the subgroup generated by the image of g_1, which lies in $\mathrm{Gal}(M/L)$; so $D_{\bar v_0} \cap G_F \subseteq G_L$ (v_0 splits in L), the $\bar\chi_a^{\mathscr{J}}$ are trivial on $D_{\bar v_0} \cap G_L$ and $\bar\omega_\ell$ is not. Now the conditions of Definition 50.1 for $\mathscr{H}$, with S_ℓ indexed by Σ: (Rd1) by the choice of $\mathcal{O}_1 \subseteq \mathcal{O}'$ and Step 0; (Rd2) by Step 0 and the choice of v_0; (Rd3) as $p \notin S \ni 2, \ell$ and p is unramified in L; (Rd4): the $\bar\chi_a$ are pairwise distinct, their ratios being the non-trivial characters $\overline{\mathrm{rt}_w}$ of Step 4; at a place $\bar u$ above a place of F not in $S_\ell \cup \mathrm{St}$, i.e. above a prime not in S, ν is trivial on inertia by (Lk3), κ by (Tw2), and θ, Ψ, α are unramified (Sg1: χ has conductor dividing N, whose primes lie in S, ξ is unramified outside S, χ_{cyc} and ω_ℓ outside ℓ); (Rd5): $\bar\chi_{5-a}(csc)\bar\chi_a(s)$ is the reduction of $\lambda(csc)\lambda(s)\chi_{5-a}^{\mathscr{J}}(s)\chi_a^{\mathscr{J}}(s) = \theta(s)^{-1}\theta(s) = 1$; (Rd6): on $D_{\bar v_0} \cap G_L$ all $\bar\chi_a$ equal $\bar\lambda$, and $\bar\omega_\ell$, the reduction of χ_{cyc}, is non-trivial there; (Rd10) holds as $\xi_{v,a} = 1$. (Lk6), (Lk7) hold by construction, and (Lk8) by the choice of St. □

Lemma 53.3. *Let ℓ be an odd prime, $E \subset \overline{\mathbb{Q}}$ an imaginary quadratic field in which 2 splits, S a finite set of primes containing 2, each of which splits in E, and $\varphi\colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell^\times$ a character of finite order prime to ℓ, unramified at every prime not in S, with $\varphi(c) = 1$ for a complex conjugation c. Then there is a continuous character $\kappa_0\colon G_E \to \overline{\mathbb{Q}}_\ell^\times$ of finite order prime to ℓ such that*

(Ie1) *$\kappa_0(s)\kappa_0(\sigma s\sigma^{-1}) = \varphi(s)$ for all $s \in G_E$ and $\sigma \in G_{\mathbb{Q}} \setminus G_E$;*

(Ie2) *κ_0 is trivial on $I_{\bar w} \cap G_E$ for every place $\bar w$ of $\overline{\mathbb{Q}}$ above a prime not in S.*

Proof. Symbols introduced here are local to this proof. Write $x \mapsto \bar x$ for the non-trivial automorphism of E and of $\mathbb{A}_E^\times$, $\mathcal{V}\colon G_{\mathbb{Q}}^{\mathrm{ab}} \to G_E^{\mathrm{ab}}$ for the transfer, and view φ also as the character $\varphi \circ \mathrm{Art}_{\mathbb{Q}}$ of $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$, of order m prime to ℓ. We use from global class field theory: Art_E is surjective with kernel the image of the closure Y of $E^\times E_\infty^\times$; on the image of $\mathbb{A}_{\mathbb{Q}}^\times$ it is $\mathcal{V} \circ \mathrm{Art}_{\mathbb{Q}}$; it maps $\mathcal{O}_{E_w}^\times$ onto the image of the inertia groups above w; and for $s \in G_E$, $\sigma \in G_{\mathbb{Q}} \setminus G_E$, $\mathcal{V}(s) = s \cdot \sigma s\sigma^{-1}$ modulo commutators. As φ has finite order, φ_∞ is trivial on $\mathbb{R}_{>0}$, and $\varphi_\infty(-1) = \varphi(c) = 1$; so $\varphi_\infty = 1$. As 2 splits in E, the discriminant of E is $\equiv 1$ modulo 8, so $E \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$ and $\mathcal{O}_E^\times = \{\pm 1\}$.

Let $U_0 = \prod_w \mathcal{O}_{E_w}^\times$, w over the finite places of E not above S (components 1 at the other places).

Claim. If $a, a' \in \mathbb{A}_{\mathbb{Q}}^\times$ and $u, u' \in U_0$ satisfy $\mathrm{Art}_E(au) = \mathrm{Art}_E(a'u')$, then $\varphi(a) = \varphi(a')$. Indeed, $y = au(a'u')^{-1} \in Y$: there are $\alpha_i \in E^\times$, $z_i \in E_\infty^\times$ with $\alpha_i z_i \to y$. Put $x = a/a' \in \mathbb{A}_{\mathbb{Q}}^\times$ and $\beta_i = \bar\alpha_i/\alpha_i$. The finite parts of β_i converge to the finite part of $\bar y/y = (\bar u/u)(u'/\bar u')$ (as $\bar x = x$), a unit idele whose components at the places above S are 1. For large i, β_i is therefore a unit at every finite place, and $|\beta_i|_\infty = 1$; so $\beta_i \in \mathcal{O}_E^\times = \{\pm 1\}$; at a place w_2 above $2 \in S$, $\beta_i \to 1$ in E_{w_2} excludes -1. So $\alpha_i \in \mathbb{Q}^\times$ for large i, and the finite part y_f of y is the limit of the α_i in $\mathbb{A}_{\mathbb{Q},f}^\times$ (closed in $\mathbb{A}_{E,f}^\times$). Now $x = y\,u'/u$, so u'/u has components in $\mathbb{Q}_p^\times \cap \mathcal{O}_{E_w}^\times = \mathbb{Z}_p^\times$ at the primes $p \notin S$ and 1 at the primes of S, where φ is unramified, resp. not involved; and for large i, $\varphi(x) = \varphi(x\alpha_i^{-1}) = \varphi_f(y_f\alpha_i^{-1})\,\varphi_f(u'/u)\,\varphi_\infty(x_\infty\alpha_i^{-1}) = 1$, the first factor because $y_f\alpha_i^{-1} \to 1$ and φ_f has open kernel, the second as just said, the third as $\varphi_\infty = 1$.

Construction. $K = \mathcal{V}(G_{\mathbb{Q}}^{\mathrm{ab}})\,\mathrm{Art}_E(U_0)$ is a compact subgroup of G_E^{ab}, and by the Claim $\mu_0(\mathcal{V}(\mathrm{Art}_{\mathbb{Q}}(a))\,\mathrm{Art}_E(u)) = \varphi(a)$ is a well-defined homomorphism on K. It is continuous: $G_{\mathbb{Q}}^{\mathrm{ab}} \times \mathrm{Art}_E(U_0) \to K$ is a continuous surjection of compact Hausdorff groups, hence a quotient map, and μ_0 composed with it is $(g,k) \mapsto \varphi(g)$. Its kernel is open in K, so contains $K \cap N$ for an open subgroup N of the profinite group G_E^{ab}; μ_0 factors through the subgroup $K/(K\cap N)$ of the finite group G_E^{ab}/N and extends to a character μ of it with values in the divisible group of roots of unity of $\overline{\mathbb{Q}}_\ell^\times$. Let $\mathrm{ord}(\mu) = m'\ell^j$ with $\ell \nmid m'$, choose $u_0 \in \mathbb{Z}$ with $u_0 \equiv 0$ modulo ℓ^j and $u_0 \equiv 1$ modulo mm', and put $\kappa_0 = \mu^{u_0}$ (as a character of G_E), of finite order dividing m', prime to ℓ.

Verification. For $s \in G_E$ and $\sigma \in G_{\mathbb{Q}} \setminus G_E$, $\kappa_0(s)\kappa_0(\sigma s\sigma^{-1}) = \mu(\mathcal{V}(s))^{u_0} = \varphi(s)^{u_0} = \varphi(s)$, which is (Ie1). For a place $\bar w$ above a prime not in S, the image of $I_{\bar w}\cap G_E$ in G_E^{ab} is $\mathrm{Art}_E(\mathcal{O}_{E_w}^\times) \subseteq \mathrm{Art}_E(U_0)$, on which μ is $\mu_0 = 1$; this is (Ie2). □

Lemma 53.4. *Let ℓ be an odd prime, $E \subset \overline{\mathbb{Q}}$ an imaginary quadratic field in which ℓ splits, and $\bar w$ a place of $\overline{\mathbb{Q}}$ above ℓ. There are a coefficient ring $\mathcal{O}_\beta$ and continuous homomorphisms $t\colon G_E \to \mathbb{Z}_\ell$ and $\beta\colon G_E \to \mathcal{O}_\beta^\times$ such that*

(Ia1) *$\beta(s) \equiv 1$ modulo $\mathfrak{m}_{\mathcal{O}_\beta}$ for all s, and $\ker t \subseteq \ker\beta$;*

(Ia2) *$t(s) + t(\sigma s\sigma^{-1}) = 0$ and $\beta(s)\beta(\sigma s\sigma^{-1}) = 1$ for all $s \in G_E$ and $\sigma \in G_{\mathbb{Q}} \setminus G_E$;*

(Ia3) *t is trivial on $I_{\bar u} \cap G_E$ for every place $\bar u$ of $\overline{\mathbb{Q}}$ not above ℓ;*

(Ia4) *for some odd integer b_0, $\beta(s)^2 = \chi_{\mathrm{cyc}}(s)^{b_0}$ for all s in an open subgroup of $I_{\bar w} \cap G_E$.*

Proof. Symbols introduced here are local to this proof. Let $\ell = \mathfrak{l}\bar{\mathfrak{l}}$ in E, with $\mathfrak{l}$ the place of $\bar w$, so $E_{\mathfrak{l}} = E_{\bar{\mathfrak{l}}} = \mathbb{Q}_\ell$; $x \mapsto \bar x$ is the non-trivial automorphism of E, of $\mathbb{A}_E^\times$ and of C_E, and $\mathrm{Art}_E(\bar x) = \sigma\mathrm{Art}_E(x)\sigma^{-1}$ for $\sigma \in G_{\mathbb{Q}} \setminus G_E$. For $z \in \mathbb{Z}_\ell^\times$ let $\langle z\rangle = z\,\omega(z)^{-1} \in 1 + \ell\mathbb{Z}_\ell$ (ω the Teichmüller lift), and $\log\colon 1+\ell\mathbb{Z}_\ell \to \ell\mathbb{Z}_\ell$, exp its inverse ($\ell$ odd). Let $U^\ell = \prod_{w\nmid\ell}\mathcal{O}_{E_w}^\times$, $U_\ell = \mathcal{O}_{E_{\mathfrak{l}}}^\times \times \mathcal{O}_{E_{\bar{\mathfrak{l}}}}^\times = \mathbb{Z}_\ell^\times \times \mathbb{Z}_\ell^\times$, and Γ the quotient of $\mathbb{A}_E^\times$ by the closure of $E^\times U^\ell E_\infty^\times$; by class field theory Art_E identifies Γ with the Galois group of the maximal abelian extension of E unramified outside ℓ, a profinite group, and the image H of U_ℓ is a compact open subgroup with Γ/H the class group of E, of order h.

Step 1: the kernel of $U_\ell \to H$. If $u \in U_\ell$ lies in the closure of $E^\times U^\ell E_\infty^\times$, there are $\alpha_i \in E^\times$, $v_i \in U^\ell$, z_i with $\alpha_i v_i z_i \to u$; at the finite places $w \nmid \ell$ the components of $\alpha_i v_i$ tend to 1, and at $\mathfrak{l}, \bar{\mathfrak{l}}$ those of α_i tend to units; so for large i, α_i is a unit at every finite place, hence $\alpha_i \in \mathcal{O}_E^\times$, a finite group, and u has components (ζ,ζ) with $\zeta \in \mathcal{O}_E^\times$, a root of unity.

Step 2: t. $t_0(u_1,u_2) = \log\langle u_1\rangle - \log\langle u_2\rangle$ is a continuous homomorphism $U_\ell \to \ell\mathbb{Z}_\ell$, trivial on (ζ,ζ) for roots of unity ζ, so by Step 1 it factors through a continuous homomorphism $t_0\colon H \to \ell\mathbb{Z}_\ell$ (H carries the quotient topology of the compact U_ℓ). Write $h = \ell^v h'$ with $\ell \nmid h'$ and put $t(g) = t_0(g^h)/h'$ for $g \in \Gamma$ ($g^h \in H$), a continuous homomorphism $\Gamma \to \ell\mathbb{Z}_\ell$ (Γ is abelian, and $g \mapsto g^h$ is continuous into the open subgroup H). Since $\overline{(u_1,u_2)} = (u_2,u_1)$, $t_0(\bar g) = -t_0(g)$, hence $t(\bar g) = -t(g)$. Composing with $G_E \to G_E^{\mathrm{ab}} \to \Gamma$ (inverse of Art_E) gives $t\colon G_E \to \mathbb{Z}_\ell$ with (Ia2) for t, and (Ia3), as inertia at places $w \nmid \ell$ maps into the image of $\mathcal{O}_{E_w}^\times \subseteq U^\ell$, which is trivial in Γ.

Step 3: β. Put $\beta = \exp(-t/2)$, a continuous homomorphism $G_E \to 1+\ell\mathbb{Z}_\ell \subseteq \mathbb{Z}_\ell^\times$; take $\mathcal{O}_\beta = \mathbb{Z}_\ell$. Then $\beta \equiv 1$ modulo ℓ, $\ker t \subseteq \ker\beta$, and $\beta(s)\beta(\sigma s\sigma^{-1}) = \exp(-(t(s)+t(\sigma s\sigma^{-1}))/2) = 1$: (Ia1), (Ia2). For (Ia4): the image of $I_{\bar w}\cap G_E$ in Γ is the image of $\mathcal{O}_{E_{\mathfrak{l}}}^\times = \mathbb{Z}_\ell^\times$ (embedded as $(u_1,1)$), and on the open subgroup where $u_1 \in 1+\ell\mathbb{Z}_\ell$, $t = t_0(u_1^h,1)/h' = \ell^v \log u_1$, so $\beta^2 = \exp(-\ell^v \log u_1) = u_1^{-\ell^v}$. By local class field theory with the arithmetic normalisation

(Lubin–Tate), $\chi_{\mathrm{cyc}}(\mathrm{Art}_{\mathbb{Q}_\ell}(u_1)) = u_1^{-1}$ for $u_1 \in \mathbb{Z}_\ell^\times$, and Art_E restricted to $E_\mathfrak{l}^\times$ is the local Artin map of $E_\mathfrak{l} = \mathbb{Q}_\ell$ followed by $G_{E_\mathfrak{l}}^{\mathrm{ab}} \to G_E^{\mathrm{ab}}$, with $G_{E_\mathfrak{l}}$ identified with $D_{\bar w} \cap G_E$. Hence $\beta^2 = \chi_{\mathrm{cyc}}^{b_0}$ with $b_0 = \ell^v$, odd, on an open subgroup of $I_{\bar w} \cap G_E$. □

Lemma 53.5. *Let ℓ be an odd prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $L \subset \overline{\mathbb{Q}}$ a CM field with maximal totally real subfield L^+ such that L/L^+ is unramified at every finite place, every place of L^+ above ℓ splits in L, and L contains no primitive ℓ-th root of unity, $c \in G_{L^+}$ a complex conjugation, and m an integer. Then there is a continuous character $\kappa\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$, algebraic for ι, such that*

(Tw1) *$\kappa(s)\kappa(csc) = \chi_{\mathrm{cyc}}(s)^m$ for all $s \in G_L$;*

(Tw2) *κ is trivial on $I_{\bar w} \cap G_L$ for every place $\bar w$ of $\overline{\mathbb{Q}}$ above a finite place of L not above ℓ;*

(Tw3) *$\kappa(G_L) \subseteq \mathcal{O}_\kappa^\times$ for some coefficient ring $\mathcal{O}_\kappa$;*

(Tw4) *for every place $\bar w$ of $\overline{\mathbb{Q}}$ above ℓ, κ is trivial on every $g \in D_{\bar w} \cap G_L$ whose image in the abelianisation $(D_{\bar w} \cap G_L)/\overline{[D_{\bar w} \cap G_L, D_{\bar w} \cap G_L]}$ has ℓ-power order;*

(Kw1) *for every place $\bar w$ of $\overline{\mathbb{Q}}$ above ℓ there are integers h and $k \geq 1$ such that $(\kappa\chi_{\mathrm{cyc}}^{-h})^k$ is trivial on $I_{\bar w} \cap G_L$.*

In particular, let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ whose CM field is L and whose complex conjugation is c, and take $m = 1 - n$. Then for every coefficient ring $\mathcal{O}'$ containing $\mathcal{O}$ and $\mathcal{O}_\kappa$, $\mathscr{H}$ is a rank-n Hecke datum over $\mathcal{O}'$, κ is a twist for it, with $m' = 1$ in (Rd7), and κ is trivial on μ_v for every $v \in S_\ell$.

Proof. Symbols introduced here are local to this proof. For a finite place w of L fix a place $\bar w$ of $\overline{\mathbb{Q}}$ above it and identify G_{L_w} with $D_{\bar w} \cap G_L$; for a continuous character ψ of G_L put $\psi_w = \psi \circ \mathrm{Art}_{L_w}\colon L_w^\times \to \overline{\mathbb{Q}}_\ell^\times$, which does not depend on the choice of $\bar w$ (decomposition groups above w are conjugate in G_L and ψ factors through G_L^{ab}). As $\mathrm{Art}_{L_w}(\mathcal{O}_{L_w}^\times)$ is the image of $I_{\bar w} \cap G_L$ in $G_{L_w}^{ab}$, ψ is trivial on $I_{\bar w} \cap G_L$ if and only if ψ_w is trivial on $\mathcal{O}_{L_w}^\times$. Moreover Art_{L_w} identifies $G_{L_w}^{ab}$ with the profinite completion of $L_w^\times$, whose torsion subgroup is the group $\mu(L_w) \subseteq \mathcal{O}_{L_w}^\times$ of roots of unity; so the elements of G_{L_w} whose image in $G_{L_w}^{ab}$ has ℓ-power order are those whose image is $\mathrm{Art}_{L_w}(\zeta)$ with $\zeta \in \mu_{\ell^\infty}(L_w)$, the roots of unity of ℓ-power order, and ψ is trivial on them if and only if ψ_w is trivial on $\mu_{\ell^\infty}(L_w)$. We use from global class field theory: Art_L is surjective, with kernel the connected component of C_L; on the image of $L_w^\times$ in C_L it is Art_{L_w} followed by $G_{L_w}^{ab} \to G_L^{ab}$; and on the image of $C_{L^+} \to C_L$ (induced by $\mathbb{A}_{L^+} \subseteq \mathbb{A}_L$) it is $\mathcal{V} \circ \mathrm{Art}_{L^+}$, where $\mathcal{V}\colon G_{L^+}^{ab} \to G_L^{ab}$ is the transfer. Since $[G_{L^+} : G_L] = 2$ with coset representatives $1, c$, $\mathcal{V}(s) = s \cdot csc$ modulo commutators for $s \in G_L$. Finally, a continuous character ψ of a profinite group G into $\overline{\mathbb{Q}}_\ell^\times$ takes values in $\mathcal{O}_E^\times$ for a finite extension $E/\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$: G is the countable union of the closed subgroups $\psi^{-1}(E^\times)$, so by Baire's theorem one of them has non-empty interior, hence is open of finite index; adjoining the finitely many values on coset representatives gives a finite E with $\psi(G) \subseteq E^\times$, and $\psi(G)$ is compact, so it lies in $\mathcal{O}_E^\times$.

Step 1: a character with the right conjugate product. Let S be the set of places of L above ℓ. Then $S^c = S$, and every $v \in S$ is unramified over L^+ and, by hypothesis, $v^c \neq v$. Choose one place v_1 in each pair $\{v, v^c\} \subseteq S$ and a place $\bar v_1$ above it, and put $\overline{v_1^c} = c\bar v_1$. Let $\chi = \chi_{\mathrm{cyc}}^m|_{G_{L^+}}$; as χ_{cyc} takes the value -1 on every complex conjugation, χ takes the value $(-1)^m$ on all of them. Put $\psi_{v_1} = \chi_{\mathrm{cyc}}^m|_{G_{L_{v_1}}}$ and $\psi_{v_1^c} = \mathbf{1}$, both de Rham. In the notation of Lemma B.26, for $g \in G_{L_{v_1}}$, $\psi_{v_1}(g)\psi_{v_1^c}(cgc) = \chi_{\mathrm{cyc}}(g)^m = \chi(g)$, and for $g \in G_{L_{v_1^c}}$, $\psi_{v_1^c}(g)\psi_{v_1}(cgc) = \chi_{\mathrm{cyc}}(cgc)^m = \chi(g)$, since χ_{cyc} is a character of $G_\mathbb{Q}$; so the hypothesis of Lemma B.26 holds on all of G_{L_v}, in

particular on inertia. By Lemma B.26 (for $F = L$) there is a continuous $\theta\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$ with $\theta(s)\theta(csc) = \chi_{\mathrm{cyc}}(s)^m$ for $s \in G_L$ and $\theta|_{I_{\bar v}} = \psi_v|_{I_{\bar v}}$ for $v \in S$. For $v \in S$, $\theta|_{G_{L_v}}\psi_v^{-1}$ is unramified, hence crystalline, so $\theta|_{G_{L_v}}$ is de Rham. Hence θ is algebraic for ι: every continuous character of G_L that is de Rham at all places above ℓ corresponds through ι and the global Artin map to an algebraic Hecke character ([220, Ch. III]; [69, Lemma 4.1.3]), and θ_w is then trivial on $\mathcal{O}_{L_w}^\times$ for all but finitely many w. Moreover, for $v \in S$, θ_v is trivial on $\mu_{\ell^\infty}(L_v)$: on $\mathcal{O}_{L_v}^\times$ it equals $\psi_v \circ \mathrm{Art}_{L_v}$, which is 1 or $x \mapsto \mathrm{N}_{L_v/\mathbb{Q}_\ell}(x)^{\pm m}$ (the sign fixed by the normalisation of the Artin map), and for $\zeta \in \mu_{\ell^\infty}(L_v)$, $\mathrm{N}_{L_v/\mathbb{Q}_\ell}(\zeta)$ is a root of unity of ℓ-power order in $\mathbb{Q}_\ell$, hence 1 as ℓ is odd.

Step 2: the local defects. Let T be the finite set of finite places $w \nmid \ell$ of L with $\theta_w|_{\mathcal{O}_{L_w}^\times} \neq 1$, and for $w \in T$ let $\lambda_w = \theta_w^{-1}|_{\mathcal{O}_{L_w}^\times}$. Each λ_w has finite order: if $w \mid p$, then $\mathcal{O}_{L_w}^\times$ is the product of a finite group and the pro-p group $1 + \mathfrak{p}_w$, whose image is a pro-p subgroup of some $\mathcal{O}_E^\times$ (by the fact recalled above), which is the product of a finite group and a pro-ℓ group; since $p \neq \ell$ that image is finite.

Step 3: the defects are trivial on the units of L^+. Let $\eta = \theta \circ \mathcal{V}$, a continuous character of G_{L^+}. For $s \in G_L$, $\eta(s) = \theta(s)\theta(csc) = \chi_{\mathrm{cyc}}(s)^m$, so $\eta\chi_{\mathrm{cyc}}^{-m}$ is trivial on G_L and equals ε^e with $e \in \{0, 1\}$, where $\varepsilon\colon G_{L^+} \to \{\pm 1\}$ has kernel G_L. Now χ_{cyc} is unramified at every finite place not above ℓ, and ε at every finite place since L/L^+ is unramified there; so for every finite place $u \nmid \ell$ of L^+, $\eta \circ \mathrm{Art}_{L_u^+}$ is trivial on $\mathcal{O}_{L_u^+}^\times$. By the compatibility of Art_L with the transfer, for $b \in \mathcal{O}_{L_u^+}^\times$ embedded in $\prod_{w|u} L_w^\times$,

$$\prod_{w|u} \theta_w(b) = \eta(\mathrm{Art}_{L_u^+}(b)) = 1, \qquad \text{so} \qquad \prod_{w|u,\ w\in T} \lambda_w(b) = 1. \qquad (*)$$

Step 4: a correcting character. Let $U = \prod_{w\nmid\ell} \mathcal{O}_{L_w}^\times \times \prod_{w|\ell} \mu_{\ell^\infty}(L_w)$ (w over the finite places of L), a compact subgroup of $\mathbb{A}_L^\times$ contained in the unit ideles, and $\lambda\colon U \to \overline{\mathbb{Q}}_\ell^\times$, $\lambda((u_w)_w) = \prod_{w\in T} \lambda_w(u_w)$, a continuous character of finite order, trivial on the factors at the places above ℓ. Let $G = G_L^{ab}$, $A = \mathcal{V}(G_{L^+}^{ab})$ and $\bar U = \mathrm{Art}_L(U)$ (image of U in C_L followed by Art_L), compact subgroups of G.

Claim: if $u \in U$ and $\mathrm{Art}_L(u) \in A$, then $\lambda(u) = 1$. As Art_{L^+} is surjective, A is the image under Art_L of $\mathbb{A}_{L^+}^\times$; so $u = a\,y$ with $a \in \mathbb{A}_{L^+}^\times$ and y in the closure of $L^\times L_\infty^\times$ in $\mathbb{A}_L^\times$ (the preimage of the connected component of C_L): there are $\alpha_i \in L^\times$, $z_i \in L_\infty^\times$ with $\alpha_i z_i \to y$. Let $x \mapsto \bar x$ be the automorphism of $\mathbb{A}_L^\times$ induced by the non-trivial automorphism of L/L^+; it fixes $\mathbb{A}_{L^+}^\times$ and preserves $L^\times$, $L_\infty^\times$ and U. Put $\beta_i = \bar\alpha_i/\alpha_i \in L^\times$. The finite parts of β_i converge to the finite part of $\bar y/y = \bar u/u$, a unit idele. Hence for large i, β_i is a unit at every finite place (the unit ideles form an open subgroup of the finite ideles), and for every embedding $\tau\colon L \to \mathbb{C}$, $|\tau(\beta_i)| = |\overline{\tau(\alpha_i)}/\tau(\alpha_i)| = 1$, as L is CM. By Kronecker's theorem β_i is a root of unity in L. Fix a place u_0 of L^+ above ℓ; it splits as $w_0 w_0^c$, and we identify $L_{w_0} = L_{u_0}^+ = L_{w_0^c}$. The components of u at w_0, w_0^c are $\zeta_0, \zeta_0' \in \mu_{\ell^\infty}(L_{u_0}^+)$, so $y_{w_0} = a_{u_0}^{-1}\zeta_0$, $y_{w_0^c} = a_{u_0}^{-1}\zeta_0'$, and $(\beta_i)_{w_0} \to (\bar y/y)_{w_0} = \zeta_0'/\zeta_0$. The roots of unity of L form a finite group embedded in $L_{w_0}^\times$, so for large i, $\beta_i = \zeta$ for a fixed root of unity $\zeta \in L$ whose image in L_{w_0} is ζ_0'/ζ_0, of ℓ-power order; as L contains no primitive ℓ-th root of unity, $\zeta = 1$, i.e. $\alpha_i \in L^{+\times}$ for large i. Hence for each finite place u' of L^+ the components of y at the places $w \mid u'$ are the images of one element of $L_{u'}^{+\times}$ (the limit of the α_i in the closed subgroup $L_{u'}^{+\times}$ of $\prod_{w|u'} L_w^\times$), and the same holds for $u = ay$; for $u' \nmid \ell$, as the components of u are units, they are the images of one $b_{u'} \in \mathcal{O}_{L_{u'}^+}^\times$. By $(*)$, $\lambda(u) = \prod_{u'\nmid\ell} \prod_{w|u',\, w\in T} \lambda_w(b_{u'}) = 1$.

By the claim, $\mu_0(x\,\mathrm{Art}_L(u)) = \lambda(u)$ ($x \in A$, $u \in U$) is a well-defined homomorphism on the compact subgroup $K = A\bar U$ of G; it is continuous, since $A \times U \to K$, $(x, u) \mapsto x\,\mathrm{Art}_L(u)$, is

a continuous surjection of compact Hausdorff groups, hence a quotient map, and μ_0 composed with it is $(x,u)\mapsto\lambda(u)$. It has finite order, so its kernel is open in K and contains $K\cap N$ for an open subgroup N of the profinite group G; μ_0 factors through the subgroup $K/(K\cap N)$ of the finite abelian group G/N and extends to a character of G/N with values in the divisible group of roots of unity of $\overline{\mathbb{Q}}_\ell^\times$. This gives a continuous character $\mu\colon G_L\to\overline{\mathbb{Q}}_\ell^\times$ of finite order with $\mu\circ\mathcal{V}=1$, $\mu_w|_{\mathcal{O}_{L_w}^\times}=\lambda_w$ for $w\in T$, $\mu_w|_{\mathcal{O}_{L_w}^\times}=1$ for finite $w\nmid\ell$, $w\notin T$, and $\mu_w|_{\mu_{\ell^\infty}(L_w)}=1$ for $w\mid\ell$. For $s\in G_L$, $\mu(s)\mu(csc)=\mu(\mathcal{V}(s))=1$.

Step 5: the twist. Put $\kappa=\theta\mu$. Then $\kappa(s)\kappa(csc)=\chi_{\mathrm{cyc}}(s)^m$ for $s\in G_L$, which is (Tw1). For a finite place $w\nmid\ell$, $\kappa_w|_{\mathcal{O}_{L_w}^\times}=\lambda_w^{-1}\lambda_w=1$ if $w\in T$ and $=1\cdot1$ otherwise, which is (Tw2). For $w\mid\ell$, $\kappa_w=\theta_w\mu_w$ is trivial on $\mu_{\ell^\infty}(L_w)$ by Steps 1 and 4, which is (Tw4) by the remark at the start. For (Kw1), let $\bar w$ be a place of $\overline{\mathbb{Q}}$ above ℓ, lying above $v\in S$; then $\bar w=g\bar v$ for some $g\in G_L$ (the places of $\overline{\mathbb{Q}}$ above v are G_L-conjugate), so $I_{\bar w}\cap G_L=g(I_{\bar v}\cap G_L)g^{-1}$. Put $h=m$ if v is one of the chosen places v_1 and $h=0$ if $v=v_1^c$. By Step 1, $\theta\chi_{\mathrm{cyc}}^{-h}$ is trivial on $I_{\bar v}\cap G_L$; as θ and χ_{cyc} are characters of G_L, it is trivial on the conjugate $I_{\bar w}\cap G_L$. With k the order of μ, $(\kappa\chi_{\mathrm{cyc}}^{-h})^k=(\theta\chi_{\mathrm{cyc}}^{-h})^k\mu^k$ is trivial on $I_{\bar w}\cap G_L$, which is (Kw1). μ has finite order, so it is algebraic (it corresponds to a Hecke character of finite order), and κ, a product of algebraic characters, is algebraic for ι. (Tw3) is the fact recalled at the start, applied to κ.

The twist for a datum. Let $\mathscr{H}$, $\mathcal{O}$, $\mathcal{O}'$ be as in the statement; $\mathcal{O}\subseteq\mathcal{O}'$ is a local homomorphism and $\mathbb{F}_{\mathcal{O}}\subseteq\mathbb{F}_{\mathcal{O}'}$. The hypotheses on L hold by (Rd1) and (Rd2). Each condition of Definition 50.1 over $\mathcal{O}'$ follows from the one over $\mathcal{O}$: (Rd1) since $\mathcal{O}\subseteq\mathcal{O}'$; the residual characters take values in $\mathbb{F}_{\mathcal{O}}^\times\subseteq\mathbb{F}_{\mathcal{O}'}^\times$ and (Rd4), (Rd5) do not involve the coefficient ring; in (Rd6) the reduction of χ_{cyc} modulo $\mathfrak{m}_{\mathcal{O}'}$ is that modulo $\mathfrak{m}_{\mathcal{O}}$ followed by the injection $\mathbb{F}_{\mathcal{O}}\to\mathbb{F}_{\mathcal{O}'}$, so it is non-trivial on $D_{\bar v_0}\cap G_L$; the wild values take values in $\mathcal{O}^\times\subseteq\mathcal{O}'^\times$ and (Rd10) is unchanged. Finally κ takes values in $\mathcal{O}_\kappa^\times\subseteq\mathcal{O}'^\times$, is algebraic, satisfies $\kappa(s)\kappa(csc)=\chi_{\mathrm{cyc}}(s)^{1-n}$ by (Tw1), and is trivial on $I_{\bar w}\cap G_L$ for every $\bar w$ above a finite place of L^+ not in S_ℓ by (Tw2), a fortiori for those not in $S_\ell\cup\mathrm{St}$; the last clause of (Rd7) holds with $m'=1$. So κ is a twist for $\mathscr{H}$ over $\mathcal{O}'$. For $v\in S_\ell$, $D_v=D_{\bar v}\cap G_{L^+}=D_{\bar v}\cap G_L$ (as v splits in L), and μ_v is the set of its elements whose image in the abelianisation of D_v has ℓ-power order; by (Tw4), κ is trivial on μ_v. □

Lemma 53.6. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ such that some polarised ordinary determinant of rank n over $\mathscr{H}$ to some $A\in\mathrm{CNL}_{\mathcal{O}}$ exists. Then a universal polarised ordinary determinant of rank n over $\mathscr{H}$ exists.*

Proof. Symbols introduced here are local to this proof. Let $k=\mathbb{F}_{\mathcal{O}}$, $W(k)\subseteq\mathcal{O}$ its Witt vectors, and $\pi=\pi_{\mathscr{H}}$.

Step 1: $\Gamma^0_{\mathscr{H}}$ satisfies Φ_ℓ. Let $U\subseteq\Gamma^0_{\mathscr{H}}$ be open; shrinking it we may assume that $\tilde U=\pi^{-1}(U)\cap G_L$ lies in the kernel of all $\bar\chi_a$. For a place $\bar w\nmid\ell$, $I_{\bar w}\cap\tilde U$ consists of elements s with $\bar\chi_a(s)=1$ for all a, hence lies in $N_{\mathscr{H}}$. So every continuous homomorphism $U\to\mathbb{Z}/\ell$ gives a $\mathbb{Z}/\ell$-extension of the fixed field of $\tilde U$ unramified outside ℓ and the infinite places; there are finitely many (Hermite–Minkowski).

Step 2: the pieces. By (Ch2) there is a universal continuous determinant $D(\bar D_n)$ on $\Gamma^0_{\mathscr{H}}$ lifting $\bar D_n$, with values in a profinite local $W(k)$-algebra A_0, Noetherian by Step 1; put $A_1=A_0\hat\otimes_{W(k)}\mathcal{O}$, a complete Noetherian local $\mathcal{O}$-algebra with residue field k, universal for continuous determinants lifting $\bar D_n$ to objects of $\mathrm{CNL}_{\mathcal{O}}$. For $v\in S_\ell$ let $D_v(\ell)$ be the maximal pro-ℓ quotient of D_v^{ab}; by local class field theory it is the product of a finite group and $\mathbb{Z}_\ell^{[L_v^+:\mathbb{Q}_\ell]+1}$, so $\Lambda_v=\mathcal{O}[[D_v(\ell)]]$ is complete Noetherian local with residue field k. For $A\in\mathrm{CNL}_{\mathcal{O}}$, the continuous characters $\psi\colon D_v\to A^\times$ with $\psi\equiv\bar\chi_a$ are exactly $\psi=[\bar\chi_a]\psi'$, $[\bar\chi_a]$ the Teichmüller lift (values in $\mathcal{O}^\times$, as $\bar\chi_a$ has order prime to ℓ), and ψ' a continuous character into the pro-ℓ group $1+\mathfrak{m}_A$, i.e. factoring

through $D_v(\ell)$; these correspond to the morphisms $\Lambda_v \to A$. Let $A_2 = A_1 \hat{\otimes}_{\mathcal{O}} \hat{\bigotimes}_{v,a} \Lambda_v$ (completed tensor product over $\mathcal{O}$, one factor for each $v \in S_\ell$ and $1 \le a \le n$), complete Noetherian local with residue field k; it carries a continuous determinant D_2 and characters $\psi_{2,v,a}$, universal for tuples $(D, (\psi_{v,a}))$ of a continuous determinant lifting $\bar{D}_n$ and continuous characters lifting $\bar{\chi}_a$, to objects of $\mathrm{CNL}_{\mathcal{O}}$.

Step 3: the conditions. Let $\mathcal{I} \subseteq A_2$ be the ideal generated by the coefficients of $\chi_{D_2}(\pi(cg^{-1}c)) - \chi_{D_2}(\pi(g))$ $(g \in G_L)$, the elements $\mathrm{tr}_{D_2}(\gamma \prod_a (\pi(g_a) - \psi_{2,v,a}(g_a)))$ $(\gamma \in \Gamma^0_{\mathscr{H}}$, $v \in S_\ell$, $g_a \in D_v)$, and $\psi_{2,v,a}(g) - \xi_{v,a}(g)$ $(g \in \mu_v)$. A morphism $A_2 \to A$ kills $\mathcal{I}$ if and only if the corresponding tuple satisfies (Od1)–(Od3). By hypothesis some tuple satisfying them exists, so $\mathcal{I} \neq A_2$, and $B = A_2/\mathcal{I}$ (complete, as ideals of complete Noetherian local rings are closed) lies in $\mathrm{CNL}_{\mathcal{O}}$. The images of D_2 and the $\psi_{2,v,a}$ form a polarised ordinary determinant of rank n over $\mathscr{H}$ to B, and by the two universal properties every polarised ordinary determinant to some $A \in \mathrm{CNL}_{\mathcal{O}}$ is classified by exactly one morphism $B \to A$. □

Lemma 53.7. *Symbols introduced in this statement are local to it. Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ an auxiliary tower for $\mathfrak{D}$, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring, $\nu, \gamma \colon G_L \to \mathcal{O}'^{\times}$ and $t \colon G_L \to \mathbb{Z}_\ell$ continuous homomorphisms with (Lk1)–(Lk4), $\kappa \colon G_L \to \mathcal{O}'^{\times}$ a continuous character with (Tw1) for $m = -3$, (Tw2) and (Tw4), and $\mathscr{H} = (4, L, c, (\bar{v})_{v\in\Sigma}, \mathrm{St}, v_0, \bar{v}_0, (\bar{\chi}_a), (\xi_{v,a}))$ a rank-4 Hecke datum over $\mathcal{O}'$ with (Lk6) and (Lk7). Let $\mathscr{X} = (B, A, P, \mathrm{cp}, (\psi^B_{v,a}))$ be a space for $(\mathscr{T}, \mathcal{O}')$, $(B_4^{\mathrm{po}}, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank 4 over $\mathscr{H}$, $n \ge 0$ an integer, and $\lambda_n = \nu\kappa^{-1}\gamma^n \colon G_L \to \mathcal{O}'^{\times}$. Then there is exactly one morphism $\phi_n \colon B_4^{\mathrm{po}} \to B$ of $\mathrm{CNL}_{\mathcal{O}'}$ such that*

(Cm1) *$\phi_n(\chi_{D^{\mathrm{u}}}(\pi_{\mathscr{H}}(g)))(X) = \lambda_n(g)^4 \,\mathrm{cp}(g)(\lambda_n(g)^{-1}X)$ for every $g \in G_L$ (ϕ_n applied coefficientwise);*

(Cm2) *$\phi_n(\psi^{\mathrm{d}}_{v,a}(g)) = \lambda_n(g)\,\psi^B_{v,a}(g)$ for every $v \in \Sigma$, $1 \le a \le 4$ and $g \in D_v$.*

In particular, under the injection $B \to A$, the determinant $\phi_n \circ D^{\mathrm{u}} \circ \pi_{\mathscr{H}}|_{G_L}$ becomes $\det \circ \rho_n$ for $\rho_n = \lambda_n\, r_P|_{G_L} \colon G_L \to \mathrm{GL}_4(A)$.

Proof. Write $f \colon B \to A$ for the structure map, injective by (Zq1), $r = r_P$, $\pi = \pi_{\mathscr{H}}$, and Λ_i for the coefficients of characteristic polynomials as in Theorem B.83. Since r is a homomorphism of monoids from a group, $r(g)r(g^{-1}) = r(1) = 1$, so $r(g) \in \mathrm{GL}_4(A)$, and $\rho_n = \lambda_n r|_{G_L}$ is a homomorphism $G_L \to \mathrm{GL}_4(A)$. The characteristic polynomial of $\rho_n(g)$ is $\lambda_n(g)^4\,\mathrm{charpoly}(r(g))(\lambda_n(g)^{-1}X)$, which is the image under f of $\mathrm{cp}_n(g) := \lambda_n(g)^4\mathrm{cp}(g)(\lambda_n(g)^{-1}X) \in B[X]$, by (Zq2); in particular all $\Lambda_i(\rho_n(g))$ lie in $f(B)$.

Step 1: a determinant over B on G_L. By (Ch1), $\det \circ \rho_n$ is the base change of a determinant with values in the subring of A generated by the $\Lambda_i(\rho_n(g))$, which lies in $f(B)$; transporting along $f^{-1} \colon f(B) \to B$ gives a B-valued determinant D_1 of dimension 4 on G_L with $f \circ D_1 = \det \circ \rho_n$ and $\chi_{D_1}(g) = \mathrm{cp}_n(g)$. As f is injective, every identity between the Λ_j of finite products of elements of G_L that holds for ρ_n holds for D_1. The maps $g \mapsto \Lambda_i(\chi_{D_1}(g))$ are continuous for the $\mathfrak{m}_B$-adic topology, since λ_n is continuous with values in $\mathcal{O}'$ and the coefficients of cp are continuous by (Zq2).

Step 2: ρ_n kills the generators of $N_{\mathscr{H}}$. Let $s \in I_{\bar{w}} \cap G_L$ with $\bar{w} \nmid \ell$ and $\bar{\chi}_a(s) = 1$ for all a. Among $\chi_1^{\mathscr{T}}, \dots, \chi_4^{\mathscr{T}}$ one is trivial (the character $\kappa_4 = \mathbf{1}$ of the datum sits at index 4 for the labelling 0 and at index 2 for the labelling 1); for that index a_0, (Lk6) gives $\bar{\nu}(s)\bar{\kappa}(s)^{-1} = 1$, and $\kappa(s) = 1$ by (Tw2), so $\bar{\nu}(s) = 1$ and then $\bar{\chi}_a^{\mathscr{T}}(s) = 1$ for all a, i.e. $\bar{r}_0(s) = 1$. Hence $s \in N$ and $r(s) = 1$ by (Mq1). By (Lk3), $\nu(s)^m = 1$ with $\ell \nmid m$; writing $\nu(s) = 1 + y$ with $y \in \mathfrak{m}_{\mathcal{O}'}$, $0 = (1+y)^m - 1 = y(m + yz)$ with $z \in \mathcal{O}'$ and $m + yz$ a unit, so $y = 0$ and $\nu(s) = 1$. By

(Lk2), $t(s)=0$, so $\gamma(s)=1$ as $\ker t \subseteq \ker\gamma$. Thus $\rho_n(s)=1$. The set of such s is stable under conjugation by G_{L^+}: conjugation by $g\in G_L$ moves $I_{\bar w}$ to $I_{g\bar w}$ and does not change $\bar\chi_a(s)$; and $\bar\chi_a(csc)=\bar\chi_{5-a}(s)^{-1}=1$ by (Rd5). So the abstract subgroup N_0 generated by these s is normal in G_{L^+}, $\rho_n(N_0)=1$, and $N_{\mathscr{H}}$ is the closure of N_0.

Step 3: descent to $\Gamma^0_{\mathscr{H}}$. Let $h\in N_{\mathscr{H}}$. For finitely many $g_1,\dots,g_k\in G_L$ and a word w in letters $x,y_1,\dots,y_k$, the map $h'\mapsto\Lambda_j(w(h',g_1,\dots,g_k))$ (computed with D_1) is continuous on G_L (Step 1, and continuity of multiplication), and for $h'\in N_0$ it equals $\Lambda_j(w(1,g_1,\dots,g_k))$ (Step 2 and Step 1); as N_0 is dense in $N_{\mathscr{H}}$ and B is Hausdorff, this equality holds for $h'=h$. Now let T be a commutative B-algebra and $r'=\sum_k c_kg_k\in T[G_L]$. By (Ch1), $\Lambda_i((h-1)r')=\Lambda_i(\sum_k c_k\,hg_k-\sum_k c_k\,g_k)$ (for the base change of D_1 to T) is a universal integer polynomial in the c_k and the Λ_j of finite products of the elements $hg_1,\dots,hg_k,g_1,\dots,g_k$; by what was just shown these Λ_j do not change when h is replaced by 1, so $\Lambda_i((h-1)r')=\Lambda_i(0)=0$ for $i\geq 1$, and $D_{1,T}(1+(h-1)r')=1$. Hence $h-1\in\ker(D_1)$ (Theorem B.88). The two-sided ideal K of $B[G_L]$ generated by the $h-1$, $h\in N_{\mathscr{H}}$, lies in $\ker(D_1)$, and $B[G_L]/K=B[\Gamma^0_{\mathscr{H}}]$ ($N_{\mathscr{H}}$ is normal in G_L). By (Ck1) there is a determinant D of dimension 4 on $\Gamma^0_{\mathscr{H}}$ with $D_1=D\circ\pi$; so $\chi_D(\pi(g))=\mathrm{cp}_n(g)$, and D is continuous because $\Gamma^0_{\mathscr{H}}$ carries the quotient topology of the compact group G_L and the $\Lambda_i\circ\chi_{D_1}$ are continuous.

Step 4: the ordinary characters. For $v\in\Sigma$ the matrices $n_v^{-1}r(s)n_v$, $s\in D_v$, are upper triangular (Mq4) and invertible, so $\psi^P_{v,a}$ is a homomorphism $D_v\to A^\times$; as f is injective and $f\circ\psi^B_{v,a}=\psi^P_{v,a}$ (Zq2), $\psi^B_{v,a}$ is a homomorphism $D_v\to B^\times$, continuous by (Zq2). Put $\psi_{v,a}=\lambda_n|_{D_v}\,\psi^B_{v,a}$ ($D_v\subseteq G_L$ as v splits in L), a continuous homomorphism $D_v\to B^\times$.

Step 5: $(D,(\psi_{v,a}))$ is a polarised ordinary determinant of rank 4 over $\mathscr{H}$ to B. $B\in\mathrm{CNL}_{\mathcal{O}'}$ by (Zq1). Reduction: $\gamma\equiv 1$ by (Lk2), so $\bar\lambda_n=\bar\nu\bar\kappa^{-1}$; by (Zq2) and (Lk6),

$$\chi_D(\pi(g))\equiv\prod_a(X-\bar\lambda_n(g)\bar\chi_a^{\mathscr{T}}(g))=\prod_a(X-\bar\chi_a(g))$$

and $\psi_{v,a}\equiv\bar\lambda_n\bar\chi_a^{\mathscr{T}}=\bar\chi_a$ modulo $\mathfrak{m}_B$; by the uniqueness in (Ch1) over the field $B/\mathfrak{m}_B$ (π is surjective), D reduces to $\bar D_4$. (Od1): for $g\in G_L$, $r(g)$ has similitude $\theta(g)$ (Mq1), so $r(g)^{-1}$ is conjugate to $\theta(g)^{-1}\,{}^tr(g)$ and has the characteristic polynomial of $\theta(g)^{-1}r(g)$; and $r(cg^{-1}c)=r(c)r(g)^{-1}r(c)^{-1}$ ($c\in G_F$). Hence $\rho_n(cg^{-1}c)$ has the characteristic polynomial of $\lambda_n(cgc)^{-1}\theta(g)^{-1}r(g)$. By (Lk1) and (Tw1),

$$\begin{aligned}\lambda_n(g)\lambda_n(cgc)&=\nu(g)\nu(cgc)\kappa(g)^{-1}\kappa(cgc)^{-1}(\gamma(g)\gamma(cgc))^n\\&=\theta(g)^{-1}\chi_{\mathrm{cyc}}(g)^{-3}\chi_{\mathrm{cyc}}(g)^3=\theta(g)^{-1},\end{aligned}$$

so $\lambda_n(cgc)^{-1}\theta(g)^{-1}=\lambda_n(g)$, and $\rho_n(cg^{-1}c)$, $\rho_n(g)$ have the same characteristic polynomial; by injectivity of f, $\chi_D(\pi(cg^{-1}c))=\chi_D(\pi(g))$. (Od2): let $x\in G_L$, $v\in\Sigma$, $g_1,\dots,g_4\in D_v$. As tr is Λ_1, linear and compatible with base change,

$$f(\mathrm{tr}_D(\pi(x)\prod_a(\pi(g_a)-\psi_{v,a}(g_a))))=\mathrm{tr}(\rho_n(x)\prod_a M'_a)$$

with $M'_a=\rho_n(g_a)-f(\psi_{v,a}(g_a))$. The matrices $M_a=n_v^{-1}M'_an_v=\lambda_n(g_a)(n_v^{-1}r(g_a)n_v-\psi^P_{v,a}(g_a))$ are upper triangular with $(M_a)_{aa}=0$. For such matrices the first k columns of $M_1\cdots M_k$ vanish (induction on k: column $j\leq k$ of $(M_1\cdots M_{k-1})M_k$ is $\sum_{i\leq j}(M_k)_{ij}$ times column i of $M_1\cdots M_{k-1}$, which vanishes for $i\leq k-1$, and the term $i=j=k$ has $(M_k)_{kk}=0$), so $M_1\cdots M_4=0$, hence $\prod_a M'_a=0$ and the trace is 0; by injectivity of f, (Od2) holds (for $\gamma\in\Gamma^0_{\mathscr{H}}$ take x with $\pi(x)=\gamma$). (Od3): let $g\in\mu_v$. Then $\nu(g)=1$ and $t(g)=0$ by (Lk4), so $\gamma(g)=1$ by (Lk2); $\kappa(g)=1$ by (Tw4) (μ_v consists of the elements of $D_v=D_{\bar v}\cap G_L$ whose image in the abelianisation has ℓ-power order); and $f(\psi^B_{v,a}(g))=\psi^P_{v,a}(g)=1$ by (Mq4). So $\psi_{v,a}(g)=1=\xi_{v,a}(g)$ by (Lk7).

Step 6: the morphism. By the universal property there is exactly one morphism $\phi_n\colon B_4^{\mathrm{po}} \to B$ of $\mathrm{CNL}_{\mathcal{O}'}$ with $\phi_n \circ D^{\mathrm{u}} = D$ and $\phi_n \circ \psi_{v,a}^{\mathrm{d}} = \psi_{v,a}$; it satisfies (Cm1) by Step 3 and (Cm2) by Step 4. Conversely, if ϕ satisfies (Cm1) and (Cm2), then $\phi \circ D^{\mathrm{u}}$ and D have the same characteristic polynomials at every element of $\Gamma^0_{\mathscr{H}} = \pi(G_L)$, so they are equal by (Ch1), and $\phi \circ \psi_{v,a}^{\mathrm{d}} = \psi_{v,a}$; so $\phi = \phi_n$ by uniqueness in the universal property. The last sentence is $f \circ D_1 = \det \circ \rho_n$. $\square$

Definition 53.8. Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ and $S = S_\ell \cup \mathrm{St} \cup \{v_0\}$. A profinite group G **satisfies** Φ_ℓ if every open subgroup of G has only finitely many continuous homomorphisms to $\mathbb{Z}/\ell$. For a closed subgroup H of a profinite group G let $\mathrm{PL}(H)$ be the intersection of the open normal subgroups U of G such that for every $h \in H$ some h^{ℓ^e} lies in U.

Levels. For a finite set Q of places of L^+ outside S, not above ℓ and split in L, let N^Q be the closed normal subgroup of G_{L^+} generated by the $s \in I_{\bar{w}} \cap G_L$ with $\bar{\chi}_a(s) = 1$ $(1 \le a \le n)$, for all places $\bar{w}$ of $\overline{\mathbb{Q}}$ above neither ℓ nor a place of Q. Put $\Gamma^Q = G_{L^+}/N^Q$, $\pi^Q\colon G_{L^+} \to \Gamma^Q$, $\Gamma^{0,Q} = \pi^Q(G_L)$; as $N^Q \subseteq N_{\mathscr{H}}$ there is a unique continuous surjection $h^Q\colon \Gamma^Q \to \Gamma_{\mathscr{H}}$ with $h^Q \circ \pi^Q = \pi_{\mathscr{H}}$. ($\Gamma^{\varnothing} = \Gamma_{\mathscr{H}}$.)

Taylor–Wiles data. A **Taylor–Wiles datum** of size $q \ge 0$ for $\mathscr{H}$ consists of an open normal subgroup $H \subseteq \Gamma_{\mathscr{H}}$, elements $\sigma_1, \dots, \sigma_q \in \Gamma'$, an integer $b \ge 1$ and, for every $N \ge 1$, places $v_{N,1}, \dots, v_{N,q}$ of L^+ with $Q_N = \{v_{N,1}, \dots, v_{N,q}\}$, places $\bar{w}_{N,j}$ of $\overline{\mathbb{Q}}$ above $v_{N,j}$ with $w_{N,j}$ the place of L below, and elements $\phi_{N,j}, \tau_{N,j} \in D_{\bar{w}_{N,j}} \cap G_L$, such that, writing $\Gamma_N = \Gamma^{Q_N}$, $h_N = h^{Q_N}$, $H_N = h_N^{-1}(H)$, $\Phi_{N,j} = \pi^{Q_N}(\phi_{N,j})$, $\mathrm{T}_{N,j} = \pi^{Q_N}(\tau_{N,j})$ and $q_{N,j} = \mathrm{N}w_{N,j}$:

(Tg1) $H = \{\gamma \in \Gamma^0_{\mathscr{H}} :\ \bar{\chi}_a(\gamma) = 1 \text{ for } 1 \le a \le n,\ \chi_{\mathrm{cyc}}(\gamma) \equiv 1 \bmod \ell\}$;

(Tg2) for every N the places $v_{N,j}$ are pairwise distinct, not in S, not above ℓ and split in L; $q_{N,j} \equiv 1 \bmod \ell^N$; $\phi_{N,j}^{-1}$ is a Frobenius element at $\bar{w}_{N,j}$ (for the norm $q_{N,j}$); $\tau_{N,j} \in I_{\bar{w}_{N,j}}$ and $I_{\bar{w}_{N,j}}$ is the closed subgroup generated by $\tau_{N,j}$ and the kernel of the maximal pro-ℓ quotient of $I_{\bar{w}_{N,j}} \cap G_L$;

(Tg3) (approximation) $h_N(\Phi_{N,j})\sigma_j^{-1} \in H$ for all N, j, and for every open normal subgroup $V \supseteq \mathrm{PL}(H)$ of $\Gamma_{\mathscr{H}}$ and every j there is N_0 with $h_N(\Phi_{N,j})\sigma_j^{-1} \in V$ for $N \ge N_0$;

(Tg4) (tame relation) $h_N(\mathrm{T}_{N,j}) = 1$, $\mathrm{T}_{N,j}^{-q_{N,j}}\Phi_{N,j}^{-1}\mathrm{T}_{N,j}\Phi_{N,j} \in \mathrm{PL}(H_N)$, and $\ker h_N$ is contained in the closure of the subgroup generated by $\mathrm{PL}(H_N)$ and the conjugates of $\mathrm{T}_{N,1}, \dots, \mathrm{T}_{N,q}$;

(Tg5) (uniform finiteness) H and every H_N have only finitely many continuous homomorphisms to $\mathbb{Z}/\ell$, at most b for every H_N;

(Tg6) $D_v \cap G_L$ $(v \in S_\ell)$ and $D_{\bar{u}} \cap G_L$ $(u \in \mathrm{St}$, $\bar{u}$ above $u)$ satisfy Φ_ℓ.

For a curve point $x = (k, \rho, P, (\psi_{v,a}), f)$ of a universal polarised ordinary determinant over $\mathscr{H}$, **eigenvalue data** of x along the datum are $\alpha_{j,i} \in A_k$ $(1 \le j \le q,\ 1 \le i \le n)$ with $\alpha_{j,1}, \dots, \alpha_{j,n}$ pairwise distinct and $\det(X - \rho(\sigma_j)) = \prod_{i=1}^n (X - \alpha_{j,i})$.

Definition 53.9. Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ with residue field $\mathbb{F}$, $(B, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$ $(B = B_n^{\mathrm{po}})$, and $J \subseteq B$ an ideal. Put $D = n\,[L^+ : \mathbb{Q}]$ and $\Lambda_D = \mathcal{O}[[X_1, \dots, X_D]]$.

Weight variables. A family of **weight variables** is a family $\mathbf{w} = (v_k, a_k, \gamma_k, c_k)_{1 \le k \le D}$ with $v_k \in S_\ell$, $1 \le a_k \le n$, $\gamma_k \in I_{v_k}$ and $c_k \in \mathcal{O}$ such that $c_k \bar{\chi}_{a_k}(\gamma_k) = 1$ in $\mathbb{F}$. For $A \in \mathrm{CNL}_{\mathcal{O}}$ and continuous characters $\psi_{v,a}\colon D_v \to A^\times$ with $\psi_{v,a} \equiv \bar{\chi}_a$ modulo $\mathfrak{m}_A$, the **weight map** is

the homomorphism of $\mathcal{O}$-algebras $\lambda\colon \Lambda_D \to A$ with $\lambda(X_k) = c_k\psi_{v_k,a_k}(\gamma_k) - 1$ (well defined, as $c_k\psi_{v_k,a_k}(\gamma_k) - 1 \in \mathfrak{m}_A$).

Taylor–Wiles bases. A **Taylor–Wiles base** for J is a pair $(\mathbf{w}, M)$ of weight variables and a B-module M such that, for the weight map $\lambda_B\colon \Lambda_D \to B$ of $(\psi^{\mathrm{d}}_{v,a})$,

(Hl1) M is free of finite rank as a Λ_D-module through λ_B, and for every prime ideal $\mathfrak{p}$ of B: $\mathrm{Ann}_B(M) \subseteq \mathfrak{p}$ if and only if $J \subseteq \mathfrak{p}$.

Level groups at Q. Let $Q = \{v_1, \dots, v_q\}$ be a finite set of places of L^+, not in $S_\ell \cup \mathrm{St} \cup \{v_0\}$, not above ℓ and split in L, with places $\bar{w}_j$ of $\overline{\mathbb{Q}}$ above v_j, w_j the place of L below $\bar{w}_j$ and k_j its residue field. Put $G_Q = \prod_j \mathrm{GL}_n(L_{w_j})$, $K_Q = \prod_j \mathrm{GL}_n(\mathcal{O}_{L_{w_j}})$, $\mathrm{Iw}_Q \subseteq K_Q$ the product of the Iwahori subgroups (matrices upper triangular modulo w_j), and $\mathrm{Iw}_{1,Q} \subseteq \mathrm{Iw}_Q$ the subgroup of the elements whose diagonal entries modulo w_j have order prime to ℓ in $k_j^\times$. It is normal in Iw_Q, and $\Delta_Q = \mathrm{Iw}_Q/\mathrm{Iw}_{1,Q} \cong \prod_j (k_j^\times(\ell))^n$ ($k_j^\times(\ell)$ the ℓ-Sylow subgroup) through the diagonal entries modulo w_j.

Level modules. Let $(\mathbf{w}, M)$ be a Taylor–Wiles base for J. A **level module** at Q for $(\mathbf{w}, M)$ is a tuple $(R, D_R, (\psi_{R,v,a}), \beta, P, \rho)$ of $R \in \mathrm{CNL}_{\mathcal{O}}$; a continuous R-valued determinant D_R of dimension n on $\Gamma^{0,Q}$ reducing to $\det \circ \mathrm{diag}(\bar{\chi}_1, \dots, \bar{\chi}_n)$ and continuous characters $\psi_{R,v,a}\colon D_v \to R^\times$ with $\psi_{R,v,a} \equiv \bar{\chi}_a$, satisfying (Od1), (Od2), (Od3) of Definition 50.2 with π^Q, $\Gamma^{0,Q}$ in place of π, Γ^0; a homomorphism of $\mathcal{O}$-algebras $\beta\colon R \to B$ with $\beta \circ D_R = D^{\mathrm{u}} \circ h^Q$ (base change of D_R along β equals the pull-back of D^{u} along h^Q) and $\beta \circ \psi_{R,v,a} = \psi^{\mathrm{d}}_{v,a}$; an R-module P, regarded as a Λ_D-module through the weight map λ_R of $(\psi_{R,v,a})$; and an R-linear action ρ of G_Q on P; such that

(Hl2) $P^{\mathrm{Iw}_{1,Q}}$ is a finitely generated Λ_D-module, and a projective $\Lambda_D[\Delta_Q]$-module for the induced action of Δ_Q;

(Hl3) there is an additive bijection $e\colon P^{K_Q} \to M$ with $e(tm) = \beta(t)e(m)$ for $t \in R$, $m \in P^{K_Q}$.

Classical level modules. Let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism. A **level point** of the level module for ι is a tuple $(\theta, r, Y, \pi, \mathrm{sp}, (\sigma_j)_j)$ of a ring homomorphism $\theta\colon R \to \overline{\mathbb{Q}}_\ell$; a continuous homomorphism $r\colon \Gamma^{0,Q} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ with $\theta \circ D_R = \det \circ r$; a complex vector space Y with a representation π of G_Q; an additive map $\mathrm{sp}\colon P \to Y$ with $\mathrm{sp}(tm) = \iota(\theta(t))\,\mathrm{sp}(m)$ and $\mathrm{sp}(\rho(g)m) = \pi(g)\mathrm{sp}(m)$; and irreducible smooth representations σ_j of $\mathrm{GL}_n(L_{w_j})$ such that, for every j: Y is spanned by the images of the $\mathrm{GL}_n(L_{w_j})$-equivariant linear maps $\sigma_j \to Y$ ($\mathrm{GL}_n(L_{w_j})$ acting on Y through the j-th factor of G_Q); and, if the Weil–Deligne representation $\mathrm{rec}(\sigma_j \otimes |\det|^{(1-n)/2})$ has monodromy $N = 0$ and is trivial on the commutator subgroup of the Weil group, then it is isomorphic to the Frobenius-semisimplification $\mathrm{WD}(r \circ \pi^Q|_{D_{\bar{w}_j}})^{F\text{-}ss}$ transported by ι.

(Hl4) The level module is **classical** if for some ι the level points separate P: every $m \in P$ with $\mathrm{sp}(m) = 0$ for all level points for ι is 0.

The ideal J has **classical levels** if there is a Taylor–Wiles base $(\mathbf{w}, M)$ for J such that for every finite set Q as above (with places $\bar{w}_j$) there is a classical level module at Q for $(\mathbf{w}, M)$.

Theorem 53.10. *Let $\mathscr{H}$ be a rank-n Hecke datum with $n = 4$ over a coefficient ring $\mathcal{O}'$ whose wild values are trivial ($\xi_{v,a} = 1$ for all $v \in S_\ell$ and $1 \le a \le 4$), κ a twist for $\mathscr{H}$, $(B, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank 4 over $\mathscr{H}$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $d = [L^+ : \mathbb{Q}]$ and $\Lambda_{4d} = \mathcal{O}'[[X_1, \dots, X_{4d}]]$. Then:*

(Hf1) *B has a Hecke ideal J of zero-monodromy type for ι and κ which satisfies (Hf2)–(Hf5) and, if $J \neq B$, has classical levels and has $R = T$ at patchable curve points (for v_0, with the place $\bar{v}_0$); every such J satisfies $J \subseteq \mathcal{J}_\kappa^{\mathrm{cl}}$;*

(Hf2) *(finiteness, a property of an ideal $J \subseteq B$) there is a homomorphism of $\mathcal{O}'$-algebras $\lambda\colon \Lambda_{4d} \to B/J$ such that B/J is a finite and torsion-free Λ_{4d}-module via λ, and λ maps the maximal ideal of Λ_{4d} into the ideal generated by the images of $\mathfrak{m}_{\mathcal{O}'}$ and of the elements $\psi_{v,a}^{\mathrm{d}}(g) - 1$ with $v \in S_\ell$, $1 \le a \le 4$, $g \in I_v$ and $\bar{\chi}_b(g) = 1$ for $1 \le b \le 4$;*

(Hf3) *(classicality, a property of an ideal $J \subseteq B$) for every inertial weight $(\psi_{v,a}^\circ)$ for $\mathscr{H}$ such that for all $v \in S_\ell$ and $1 \le a, b \le 4$ some integer h_{ab} makes $\psi_{v,a}^\circ(\psi_{v,b}^\circ)^{-1}\chi_{\mathrm{cyc}}^{-h_{ab}}$ trivial on an open subgroup of I_v, every integral test ring A over $\mathcal{O}'$, every homomorphism $\rho\colon \Gamma^0_{\mathscr{H}} \to \mathrm{GL}_4(A)$ with $\rho_L = \rho \circ \pi_{\mathscr{H}}|_{G_L}$ continuous and irreducible over $\overline{\mathbb{Q}}_\ell$, and every homomorphism of $\mathcal{O}'$-algebras $y\colon B \to A$ with $y \circ D^{\mathrm{u}} = \det \circ \rho$, $y(\psi_{v,a}^{\mathrm{d}}(g)) = \psi_{v,a}^\circ(g)$ for all $v \in S_\ell$, a and $g \in I_v$, and $y(J) = 0$: the representation $\rho_L \otimes \kappa$ is automorphic of conjugate self-dual type for ι;*

(Hf4) *(purity at arithmetic points, a property of an ideal $J \subseteq B$) for every homomorphism of $\mathcal{O}'$-algebras $y\colon B \to \overline{\mathbb{Q}}_\ell$ with $y(J) = 0$ for which there are integers $h_{v,a,\tau}$ ($v \in S_\ell$, $1 \le a \le 4$, τ a continuous field embedding $L_v^+ \to \overline{\mathbb{Q}}_\ell$), strictly increasing in a for every v and τ, such that $y(\psi_{v,a}^{\mathrm{d}}(\sigma)) = \prod_\tau \tau(u_\sigma)^{h_{v,a,\tau}}$ for all σ in an open subgroup of I_v, where $u_\sigma \in \mathcal{O}^\times_{L_v^+}$ is the unit with $\mathrm{Art}_{L_v^+}(u_\sigma)$ equal to the image of σ in D_v^{ab}: there are a continuous semisimple representation $\rho\colon G_L \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ with $\det \circ \rho = y \circ D^{\mathrm{u}} \circ \pi_{\mathscr{H}}|_{G_L}$, finitely many continuous representations $\rho_i'\colon G_L \to \mathrm{GL}_{d_i}(\overline{\mathbb{Q}}_\ell)$ and integers $d_i, m_i \ge 1$ ($1 \le i \le k$) such that*

$$\det\bigl(X - \rho(g)\bigr) = \prod_{i=1}^{k}\prod_{j=0}^{m_i-1} \det\bigl(X - \chi_{\mathrm{cyc}}(g)^{-j}\rho_i'(g)\bigr) \qquad (g \in G_L),$$

and, if $m_i = 1$ for all i, then for all but finitely many finite places w of L the representation ρ is unramified at w and any two eigenvalues α, α' of $\rho(\mathrm{Frob}_w)$ satisfy $|\iota(\alpha)| < \mathrm{N}w\,|\iota(\alpha')|$;

(Hf5) *(stability under twists, a property of an ideal $J \subseteq B$) for every continuous homomorphism $\beta\colon \Gamma^0_{\mathscr{H}} \to B^\times$ with $\beta(g) \equiv 1$ modulo $\mathfrak{m}_B$, $\beta(\pi_{\mathscr{H}}(csc)) = \beta(\pi_{\mathscr{H}}(s))^{-1}$ for $s \in G_L$, β trivial on $\pi_{\mathscr{H}}(I_{\bar{w}} \cap G_L)$ for every place $\bar{w}$ above a place in $\mathrm{St} \cup \{v_0\}$, and β trivial on μ_v for $v \in S_\ell$: the morphism $\tau_\beta\colon B \to B$ of $\mathrm{CNL}_{\mathcal{O}'}$ classifying the polarised ordinary determinant $(D^{\mathrm{u}}_\beta, (\beta|_{D_v}\psi_{v,a}^{\mathrm{d}}))$, where D^{u}_β is D^{u} composed with the B-algebra automorphism of $B[\Gamma^0_{\mathscr{H}}]$ with $\gamma \mapsto \beta(\gamma)\gamma$, satisfies $\tau_\beta(J) \subseteq J$.*

Proof. Step 0: the ideal. By Proposition E.1 (for $n = 4$; its hypothesis is that the wild values are trivial) B has a Hecke ideal J of zero-monodromy type for ι and κ which, if $J \neq B$, has classical levels, and, if B has a classical ordinary point, $J = \ker\theta$ with $\theta\colon B \to \mathbb{T}_{\tau,\mathfrak{m}}/\mathfrak{a}_0$ the surjection of (Gd2) for a type datum $((V_v), \tau)$ over $\mathscr{H}$ with V_v the kernel of reduction modulo ϖ_v^c (c as in Proposition E.5) satisfying (Zt), a maximal ideal $\mathfrak{m}$ with residue field $\mathbb{F}_{\mathcal{O}'}$ satisfying (Mx), χ_0 with $\chi_0(d_j(\zeta)) = \kappa(\mathrm{Art}(\zeta))$, $\mathfrak{a}_0 = \mathfrak{a}_{\chi_0}$ and $\mathcal{C} = \mathcal{C}_{\chi_0}$; in particular θ satisfies the three identities of (Gd2).

If B has no classical ordinary point, then $J = B$ is a Hecke ideal of zero-monodromy type (Definition 50.10) and (Hf2)–(Hf5) hold for it: $B/J = 0$ is a finite torsion-free module over Λ_{4d} via the zero map, which maps the maximal ideal into 0; no homomorphism y of $\mathcal{O}'$-algebras kills B ($y(1) = 1$), so (Hf3), (Hf4) are empty; and $\tau_\beta(B) \subseteq B$. From now on B has a classical ordinary point and J is as above; $J \neq B$, as a classical ordinary point kills J (Zm4).

Step 1: (Hf1). J is of zero-monodromy type with classical levels, so by Proposition E.32 it has $\mathrm{R} = \mathrm{T}$ at patchable curve points (for v_0, with $\bar{v}_0$). Every Hecke ideal of zero-monodromy type for ι and κ is contained in $\mathcal{J}_\kappa^{\mathrm{cl}}$ by Lemma 50.12.

Step 2: (Hf2). Let (T, λ_T, M, f) be data for J as in Definition 50.10; f induces an isomorphism $B/J \cong T$. For $v \in S_\ell$ the group $1+\ell\mathcal{O}_{L_v^+}$ is torsion-free: if $\zeta \neq 1$ in it had order ℓ^k, $k \geq 1$, then $\zeta' = \zeta^{\ell^{k-1}}$ would be a primitive ℓ-th root of unity in it, with $v(\zeta'-1) = v(\ell)/(\ell-1) < v(\ell)$ (ℓ odd, v the normalised valuation), contradicting $\zeta' \equiv 1$ modulo ℓ. It is a finitely generated pro-ℓ abelian group, open in $\mathcal{O}_{L_v^+}^\times$, hence a free $\mathbb{Z}_\ell$-module of rank $d_v = [L_v^+ : \mathbb{Q}_\ell]$. So $T_4(\ell^1) = \prod_{v\in S_\ell}\prod_{j=1}^4 d_j(1+\ell\mathcal{O}_{L_v^+})$ is free over $\mathbb{Z}_\ell$ of rank $4\sum_v d_v = 4d$; for a basis $u_1, \dots, u_{4d}$, $X_i \mapsto [u_i]-1$ is an isomorphism of $\mathcal{O}'$-algebras $\Lambda_{4d} \to \Lambda_1$. (This is the only place where the modulus defining $T_4(\ell^1)$ is used.) Let λ be the composite $\Lambda_{4d} \to \Lambda_1 \xrightarrow{\lambda_T} T \cong B/J$. By (Zm1), B/J is finite and torsion-free over Λ_{4d} via λ. The maximal ideal of Λ_{4d} is generated by $\mathfrak{m}_{\mathcal{O}'}$ and the X_i; each u_i is a finite product of elements $d_j(z)$, $z \in 1+\ell\mathcal{O}_{L_v^+}$, and $[uu']-1 = [u]([u']-1)+([u]-1)$, so it suffices that $\lambda_T([d_j(z)])-1$ corresponds to an element of the ideal $\mathfrak{b}$ of B/J generated by the images of $\mathfrak{m}_{\mathcal{O}'}$ and of the $\psi_{v,a}^{\mathrm{d}}(g)-1$ ($g \in I_v$, $\bar{\chi}_b(g) = 1$ for all b). Now $g_z \in I_v$ lies in the image of the pro-ℓ group $1+\ell\mathcal{O}_{L_v^+}$ under the local Artin map, so $\bar{\chi}_b(g_z) = 1$ (the $\bar{\chi}_b$ have order prime to ℓ), and $\mu = (\chi_{\mathrm{cyc}}^{j-1}\kappa)(g_z) - 1 \in \mathfrak{m}_{\mathcal{O}'}$ (a continuous homomorphism from a pro-ℓ group to $\mathcal{O}'^\times$ lands in $1+\mathfrak{m}_{\mathcal{O}'}$). By (Zm2), $\lambda_T([d_j(z)])-1 = (1+\mu)(f(\psi_{v,j}^{\mathrm{d}}(g_z))-1)+\mu$, which corresponds to an element of $\mathfrak{b}$.

Step 3: (Hf3). Let $(\psi_{v,a}^\circ)$, A, ρ and y be as in (Hf3), with $y(J) = 0$. The integral test ring A is a subring of $\overline{\mathbb{Q}}_\ell$, so $\mathfrak{q} = \ker y$ is a prime of B whose residue field $\kappa(\mathfrak{q}) = \mathrm{Frac}(y(B)) \subseteq \overline{\mathbb{Q}}_\ell$ contains $\mathcal{O}'[1/\ell] \supseteq \mathbb{Q}_\ell$; hence $\overline{\mathbb{Q}}_\ell$ is an algebraic closure of $\kappa(\mathfrak{q})$, and the base change of D^{u} to it is $y \circ D^{\mathrm{u}} = \det\circ\rho$. As $\rho_L = \rho\circ\pi_{\mathscr{H}}|_{G_L}$ is irreducible over $\overline{\mathbb{Q}}_\ell$ and $\pi_{\mathscr{H}}(G_L) = \Gamma^0_{\mathscr{H}}$, ρ is irreducible over $\overline{\mathbb{Q}}_\ell$, so $\mathfrak{q}$ is stable (the condition does not depend on the choice of algebraic closure). By Lemma 50.12, $\mathcal{J}_\kappa^{\mathrm{cl}} \subseteq \mathfrak{q}$, i.e. $y(\mathcal{J}_\kappa^{\mathrm{cl}}) = 0$. The remaining hypotheses of Proposition E.25 ($n = 4$) are those of (Hf3), so $\rho_L \otimes \kappa$ is automorphic of conjugate self-dual type for ι.

Step 4: (Hf4). The hypotheses of Lemma E.29 ($n = 4$, $\mathfrak{a} = \mathfrak{a}_0$, θ of Step 0) hold, and its conclusion for $J = \ker\theta$ is (Hf4).

Step 5: (Hf5). The hypotheses of Lemma E.27 ($n = 4$; trivial wild values; the type datum, $\mathfrak{m}$, χ_0, $\mathfrak{a}_0$ and θ of Step 0) hold, and its conclusion is (Hf5) for J. □

Lemma 53.11. *In the setting of Lemma 53.7, let $\phi_0\colon B_4^{\mathrm{po}} \to B$ be the morphism given there for $n = 0$, and let $J_0 \subseteq B_4^{\mathrm{po}}$ be an ideal finite over the weights. Then there is $k \geq 1$ with*

$$\mathfrak{m}_B^k \subseteq \phi_0(J_0)B + \mathfrak{m}_{\mathcal{O}'}B + (\psi_{v,a}^B(s) - 1 :\ v \in \Sigma,\ 1 \leq a \leq 4,\ s \in I_v).$$

Proof. Let I be the ideal on the right. B is Noetherian (Zq1), so $\mathfrak{m}_B$ is finitely generated and it suffices to show $\mathfrak{m}_B \subseteq \sqrt{I}$, i.e. $\mathfrak{m}_B \subseteq \mathfrak{q}$ for every prime $\mathfrak{q} \supseteq I$. Fix such $\mathfrak{q}$, let $R = B/\mathfrak{q}$, a complete Noetherian local domain, $\pi\colon B \to R$ the projection, and $k_0 = \pi(\mathcal{O}')$. As $\mathfrak{m}_{\mathcal{O}'} \subseteq \mathfrak{q}$ and $\mathfrak{q} \cap \mathcal{O}'$ is a proper prime ideal, $k_0 \cong \mathcal{O}'/\mathfrak{m}_{\mathcal{O}'} = \mathbb{F}_{\mathcal{O}'}$, which is also the residue field of R; so $R \in \mathrm{CNL}_{\mathcal{O}'}$.

Step 1: $\pi\circ\phi_0$ takes values in k_0. The ideal $\mathfrak{b}_4$ is generated by $\mathfrak{m}_{\mathcal{O}'}$ and the $\psi_{v,a}^{\mathrm{d}}(g)-1$ with $g \in I_v$ and $\bar{\chi}_b(g) = 1$ for all b. For such g, (Cm2) gives $\pi\phi_0(\psi_{v,a}^{\mathrm{d}}(g)) = \lambda_0(g)\pi(\psi_{v,a}^B(g)) = \lambda_0(g)$, as $\psi_{v,a}^B(g) - 1 \in I$. One of the $\chi_b^{\mathscr{J}}$ is trivial, so by (Lk6) $\bar{\lambda}_0(g) = \bar{\chi}_b(g) = 1$ for that b, i.e. $\lambda_0(g) - 1 \in \mathfrak{m}_{\mathcal{O}'}$; hence $\pi\phi_0(\mathfrak{b}_4) = 0$. Since J_0 is finite over the weights, every $x \in \mathfrak{m}_{B_4^{\mathrm{po}}}$ has a

power in $J_0+\mathfrak{b}_4$, which $\pi\phi_0$ kills ($\phi_0(J_0)\subseteq I$); as R is a domain, $\pi\phi_0(\mathfrak{m}_{B_4^{\mathrm{po}}})=0$. Every $x\in B_4^{\mathrm{po}}$ is $a+y$ with $a\in\mathcal{O}'$ and $y\in\mathfrak{m}_{B_4^{\mathrm{po}}}$ (the residue field is $\mathbb{F}_{\mathcal{O}'}$), so $\pi\phi_0(x)=\pi(a)\in k_0$.

Step 2: the generators of B map into k_0. For $v\in\Sigma$ and $s\in D_v$ ($\subseteq G_L$), (Cm2) and Step 1 give $\pi(\psi^B_{v,a}(s))=\lambda_0(s)^{-1}\pi\phi_0(\psi^{\mathrm{d}}_{v,a}(s))\in k_0$. For $g\in G_L$, (Cm1) gives that $\lambda_0(g)^j$ times the coefficient of X^{4-j} in $\mathrm{cp}(g)$ is the image under ϕ_0 of a coefficient of $\chi_{D^{\mathrm{u}}}(\pi_{\mathscr{H}}(g))$, so all coefficients of $\pi(\mathrm{cp}(g))$ lie in k_0. Let $g\in G_F$; then $g^2\in G_L$ ($[G_F:G_L]=2$). For any $M\in\mathrm{M}_4(A)$, $\det(X^2-M^2)=\det(X-M)\det(X+M)$, so the characteristic polynomials P_1 of $r_P(g)$ and P_2 of $r_P(g^2)$ satisfy $P_2(X^2)=P_1(X)P_1(-X)$; by (Zq2) and the injectivity of $B\to A$ (Zq1), $\mathrm{cp}(g^2)(X^2)=\mathrm{cp}(g)(X)\,\mathrm{cp}(g)(-X)$ in $B[X]$. In an algebraic closure of the fraction field of R, every root of $\pi(\mathrm{cp}(g))$ is a root of $\pi(\mathrm{cp}(g^2))(X^2)$, whose coefficients lie in the finite field k_0; so the roots, and hence the coefficients of the monic polynomial $\pi(\mathrm{cp}(g))$, are algebraic over k_0. An element $y\in R$ algebraic over k_0 lies in k_0: choose $a\in k_0$ with $y-a\in\mathfrak{m}_R$; if $y\neq a$, then $y-a$ is a non-zero element of the domain R algebraic over the field k_0, hence a unit of $k_0[y]\subseteq R$, contradicting $y-a\in\mathfrak{m}_R$.

Step 3: conclusion. Let $\sigma\colon B\to R$ be the composite of $B\to B/\mathfrak{m}_B=\mathbb{F}_{\mathcal{O}'}$ with the isomorphism $\mathbb{F}_{\mathcal{O}'}\to k_0\subseteq R$; it is a homomorphism of $\mathcal{O}'$-algebras. If $y\in B$ and $\pi(y)=\pi(a)$ with $a\in\mathcal{O}'$, then $y-a\in\mathfrak{q}\subseteq\mathfrak{m}_B$, so $\sigma(y)=\sigma(a)=\pi(a)=\pi(y)$. By Step 2 this applies to all coefficients of all $\mathrm{cp}(g)$ and all $\psi^B_{v,a}(s)$, so $\pi=\sigma$ by (Zq3), and $\pi(\mathfrak{m}_B)=\sigma(\mathfrak{m}_B)=0$, i.e. $\mathfrak{m}_B\subseteq\mathfrak{q}$. □

53.2 Transport of $R=T$, locality and classicality along the comparison maps

This subsection checks the conditions (Jq2)–(Jq5) of a modular ideal on a space, by transport from the Hecke side along the comparison maps. $R=T$ at patchable curve points passes to the space (Lemma 53.12); the inputs are separable elements for curve points of irreducible type, which exist in every coset of a finite-index subgroup for a non-virtually-abelian absolutely irreducible group in GL_4 (Lemma 53.13, Lemma 53.14), and the shape of the residual characters of a link, which have no quadratic symmetry (Lemma 53.16). Locality at seed points (Lemma 53.17) comes from the regular classical ordinary points given by the seeds after a parity twist (Lemma 53.18) and from the locality of a finite Hecke ideal at such points (Lemma 53.20), where tangent vectors with constant inertial weights vanish (Lemma 53.21). Purity passes directly along the comparison map (Lemma 53.22). Classicality (Lemma 53.23) comes from automorphy over L (Lemma 53.24) with the witness conditions (Lemma 53.25), descended to imaginary quadratic fields (Lemma 53.26) to give condition (Sy2). The lemma 53.25 is used again in Appendix E.

Lemma 53.12. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum, $\mathscr{T}$ a fit and deep auxiliary tower for it, $\mathcal{O}_1,\nu,\gamma,t$ as given by Lemma 53.2, $\mathcal{O}'\supseteq\mathcal{O}_1$ a coefficient ring, κ and $\mathscr{H}$ (with places $v_0,\bar{v}_0$) the datum given there, $\mathscr{X}=(B,A,P,\dots)$ a space for $(\mathscr{T},\mathcal{O}')$, $(B_4^{\mathrm{po}},D^{\mathrm{u}},(\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank 4 over $\mathscr{H}$, ϕ_0 as in Lemma 53.7, and $J_0\subseteq B_4^{\mathrm{po}}$ an ideal with $R=T$ at patchable curve points (for v_0, with $\bar{v}_0$). Then for every finite field k with an $\mathcal{O}'$-algebra structure and every $\mathcal{O}'$-algebra homomorphism $y\colon A\to k[[T]]$ with $y(\mathfrak{m}_B)\subseteq Tk[[T]]$ and $y(\phi_0(J_0))=0$, such that for all $v\in\Sigma$ and $i\neq j$ some $s\in U_v$ has $y(\psi^P_{v,i}(s))\neq y(\psi^P_{v,j}(s))$ and $y\circ r_P$ is irreducible over the fraction field of $k[[T]]$, every prime $\mathfrak{q}$ of B with $y(\mathfrak{q})=0$ contains $\phi_0(J_0)$.*

Proof. Symbols introduced here are local to this proof. Let k, y be as in the statement, $r=r_P$, $\pi=\pi_{\mathscr{H}}$, $\Gamma^0=\Gamma^0_{\mathscr{H}}$, F_k the fraction field of $k[[T]]$, and $\lambda=\lambda_0=\nu\kappa^{-1}$. The map $B\to A$ is injective (Zq1); we regard B as a subring of A.

Step 0: scalars. $B \in \mathrm{CNL}_{\mathcal{O}'}$, so $\mathfrak{m}_{\mathcal{O}'} \subseteq \mathfrak{m}_B$ and $y(\mathfrak{m}_{\mathcal{O}'}) \subseteq k \cap Tk[[T]] = 0$ (y is $\mathcal{O}'$-linear and $k[[T]]$ is an $\mathcal{O}'$-algebra through k). Hence $\mathcal{O}' \to k$ factors through $\mathbb{F}_{\mathcal{O}'}$, $k \supseteq \mathbb{F}_{\mathcal{O}'} \supseteq \mathbb{F}_{\mathcal{O}}$, and an element $u \in \mathcal{O}'$ maps to its residue $\bar{u}$. In particular λ maps to $\bar{\lambda} = \bar{\nu}\bar{\kappa}^{-1}$, a locally constant character $G_L \to k^\times$, and χ_{cyc} maps to $\bar{\omega}$. Put $R(g) = y(r(g)) \in \mathrm{M}_4(k[[T]])$ for $g \in G_F$; $r(g)$ is invertible ($r(g)r(g^{-1}) = r(1) = 1$), so $R(g) \in \mathrm{GL}_4(k[[T]])$. By (Zq1) the entries of r are locally constant modulo $(\mathfrak{m}_B A)^j$ for every j, and $y((\mathfrak{m}_B A)^j) \subseteq T^j k[[T]]$; so the entries of R are continuous for the T-adic topology. By (Zq2), $y(\mathrm{cp}(g))$ is the characteristic polynomial of $R(g)$ and is congruent to $\prod_a (X - \bar{\chi}_a^{\mathscr{T}}(g))$ modulo T, and $y(\psi_{v,a}^P(s)) = y(\psi_{v,a}^B(s)) \equiv \bar{\chi}_a^{\mathscr{T}}(s)$ modulo T.

Step 1: the representation. For $g \in G_L$ put $\rho_L(g) = \bar{\lambda}(g)R(g)$, a continuous homomorphism $G_L \to \mathrm{GL}_4(k[[T]])$. Let $s \in I_{\bar{w}} \cap G_L$, $\bar{w} \nmid \ell$, with $\bar{\chi}_a(s) = 1$ for all a. One of $\chi_1^{\mathscr{T}}, \dots, \chi_4^{\mathscr{T}}$ is trivial (the character $\mathbf{1}$ of the datum), so (Lk6) gives $\bar{\lambda}(s) = 1$ and then $\bar{\chi}_a^{\mathscr{T}}(s) = 1$ for all a, i.e. $\bar{r}_0(s) = 1$; so $s \in N$ and $r(s) = 1$ (Mq1), whence $\rho_L(s) = 1$. Such elements are permuted by conjugation by G_L (which moves $I_{\bar{w}}$ to $I_{g\bar{w}}$ and fixes the values $\bar{\chi}_a(s)$) and by c ($\bar{\chi}_a(csc) = \bar{\chi}_{5-a}(s)^{-1} = 1$ by (Rd5)). So the kernel of ρ_L, a closed normal subgroup, contains $N_{\mathscr{H}}$, and $\rho_L = \rho \circ \pi|_{G_L}$ for a homomorphism $\rho\colon \Gamma^0 \to \mathrm{GL}_4(k[[T]])$ whose entries are continuous for the T-adic topology (Γ^0 carries the quotient topology).

Step 2: the homomorphism f. Put $f = y \circ \phi_0\colon B_4^{\mathrm{po}} \to k[[T]]$, a homomorphism of $\mathcal{O}'$-algebras. For $g \in G_L$, (Cm1) and Step 0 give $f(\chi_{D^{\mathrm{u}}}(\pi(g)))(X) = \bar{\lambda}(g)^4\, y(\mathrm{cp}(g))(\bar{\lambda}(g)^{-1}X)$, the characteristic polynomial of $\rho(\pi(g))$. So the $k[[T]]$-valued determinants $f \circ D^{\mathrm{u}}$ and $\det \circ \rho$ on $\Gamma^0 = \pi(G_L)$ have the same characteristic polynomials and are equal by (Ch1). For $v \in \Sigma$ and $g \in D_v$ ($\subseteq G_L$, as v splits in L) put $\psi_{v,a}(g) = \bar{\lambda}(g)\, y(\psi_{v,a}^P(g))$; then $f(\psi_{v,a}^{\mathrm{d}}(g)) = \psi_{v,a}(g)$ by (Cm2).

Step 3: triangularity. By (Mq3), (Mq4), n_v is invertible and $n_v^{-1}r(s)n_v$ is upper triangular with diagonal $(\psi_{v,a}^P(s))_a$ for $s \in D_v$. Applying y and multiplying by $\bar{\lambda}(s)$: $y(n_v)^{-1}\rho(\pi(s))y(n_v)$ is upper triangular with diagonal $(\psi_{v,a}(s))_a$. Hence each $\psi_{v,a}$ is a homomorphism $D_v \to k[[T]]^\times$ (diagonal entries of products of upper triangular matrices multiply), continuous by (Zq2), Step 0 and local constancy of $\bar{\lambda}$, and $\rho \circ \pi$ is triangular over F_k on D_v with diagonal $(\psi_{v,1}, \dots, \psi_{v,4})$.

Step 4: the symmetric matrix. Since $c \in G_F$ and $c^2 = 1$, we have $r(c)^2 = 1$. By (Mq1), ${}^t r(g) J_4 r(g) = \theta(g) J_4$, so $r(g)^{-1} = \theta(g)^{-1} J_4^{-1}\, {}^t r(g) J_4$. By (Lk1) and (Tw1),

$$\lambda(g)\lambda(cgc) = \theta(g)^{-1}\chi_{\mathrm{cyc}}(g)^{-3}\chi_{\mathrm{cyc}}(g)^3 = \theta(g)^{-1} \qquad (g \in G_L).$$

Put $M = y(J_4 r(c)) \in \mathrm{GL}_4(k[[T]])$. For $\gamma = \pi(g)$, $g \in G_L$, $\gamma^* = \pi(cg^{-1}c)$ and

$$\rho(\gamma^*) = \bar{\lambda}(cgc)^{-1}\bar{\theta}(g)^{-1}\, y(r(c)J_4^{-1}\, {}^t r(g) J_4 r(c)) = \bar{\lambda}(g)\, M^{-1}\, {}^t R(g)\, M = M^{-1}\, {}^t\rho(\gamma)\, M.$$

Moreover $\theta(c) = \alpha(c)\Psi(c) = \chi_{\mathrm{cyc}}(c)\omega_\ell(c)^{-1}\xi(c) = (-1)(-1)(-1) = -1$ by (Sg1), so ${}^t r(c) J_4 = -J_4 r(c)^{-1} = -J_4 r(c)$, and ${}^t(J_4 r(c)) = {}^t r(c)\, {}^t J_4 = -{}^t r(c) J_4 = J_4 r(c)$: M is symmetric. Thus $x = (k, \rho, M, (\psi_{v,a}), f)$ is a curve point of B_4^{po} (Steps 1–4), with $J_0 \subseteq \ker f$ since $y(\phi_0(J_0)) = 0$.

Step 5: (Pc1) and (Pj1). By Step 0 and (Lk6), the characteristic polynomial of $\rho(\pi(g))$ is congruent modulo T to $\prod_a (X - \bar{\lambda}(g)\bar{\chi}_a^{\mathscr{T}}(g)) = \prod_a (X - \bar{\chi}_a(g))$, and $\psi_{v,a} \equiv \bar{\lambda}\bar{\chi}_a^{\mathscr{T}} = \bar{\chi}_a$ modulo T: (Pc1). Let $v \in \Sigma$, $a \neq b$, and $s \in U_v$ with $y(\psi_{v,a}^P(s)) \neq y(\psi_{v,b}^P(s))$ (hypothesis). Then $s \in I_v \subseteq D_v$, $\chi_{\mathrm{cyc}}(s) = 1$, $\bar{r}_0(s) = 1$, so $\bar{\chi}_a(s) = \bar{\lambda}(s) = \bar{\chi}_b(s)$, and $\psi_{v,a}(s) \neq \psi_{v,b}(s) = \psi_{v,b}(s)\chi_{\mathrm{cyc}}(s)$. This is (Pc2), and it gives both clauses of (Pj1).

Step 6: (Pc3). (a) The F_k-span of $R(G_F)$ is $\mathrm{M}_4(F_k)$. By (Mq2), $R(g_{\mathscr{T}})$ is diagonal; by Step 0 its diagonal entries reduce modulo T to the $\bar{\chi}_a^{\mathscr{T}}(g_{\mathscr{T}}) = \bar{\kappa}_a^{\epsilon}(g_{\mathscr{T}})$ in some order, pairwise distinct by (Oq7). So $R(g_{\mathscr{T}})$ has four distinct eigenvalues with eigenlines $F_k e_1, \dots, F_k e_4$, and every F_k-linear endomorphism Z of $V = F_k^4$ commuting with $R(G_F)$ is diagonal. By hypothesis V is an irreducible $F_k[G_F]$-module, so by Schur's lemma $Z - Z_{11}$, which commutes with $R(G_F)$

and kills e_1, is 0: the commutant is F_k. By the Jacobson density theorem the span of $R(G_F)$ is $\mathrm{End}_{F_k}(V)$. (b) Hence $\bar V = \bar F_k^4$ is an irreducible $\bar F_k[G_F]$-module. Fix $g_1 \in G_F \setminus G_L$ ($L \neq F$). The characters $\bar\chi_a^{\mathscr{T}}|_{G_L} = \bar\lambda^{-1}\bar\chi_a$ are pairwise distinct (Rd4); if $\sum_a \bar\chi_a^{\mathscr{T}}(g_1)\bar\chi_a^{\mathscr{T}}(h) = 0$ for all $h \in G_L$, Dedekind's independence of characters would give $\bar\chi_a^{\mathscr{T}}(g_1) = 0$. So some $g = g_1 h \notin G_L$ has $\operatorname{tr} R(g) \equiv \sum_a \bar\chi_a^{\mathscr{T}}(g) \neq 0$ modulo T. Suppose $\bar V$ is reducible for G_L, and let W be a non-zero G_L-stable subspace of minimal dimension and $W' = R(g)W$, G_L-stable (G_L is normal). $W + W'$ is G_F-stable ($G_F = G_L \cup gG_L$, $g^2 \in G_L$), so equals $\bar V$; $W \cap W'$ is G_L-stable in W, so it is 0 or W, and $W' = W$ would make W G_F-stable, so $W = \bar V$. Hence $\bar V = W \oplus W'$, and $R(g)$ maps W to W' and W' to $R(g^2)W = W$, so $\operatorname{tr} R(g) = 0$: contradiction. So $\bar V$ is irreducible for G_L, hence for $\rho_L = \bar\lambda R|_{G_L}$, and $\rho \otimes \bar F_k$ is irreducible on $\Gamma^0 = \pi(G_L)$.

Step 7: (Pc4), (Pc5). The tower is fit, hence good, so (Xr1) and (Xr2) of Lemma 53.16 hold; they are (Pc4) and (Pc5).

Step 8: (Pj3). v_0 splits in L (Rd2), so $D_{\bar v_0} \cap G_F \subseteq G_L$; by (Rd6) the $\bar\chi_a$ agree on it, with common value $\bar\chi_1$, and some g in it has $e_0 = \bar\omega(g) \neq 1$. Put $\delta = \pi(g) \in \Delta_0$ and $A_0 = \rho(\delta) \in \mathrm{GL}_4(k[[T]])$. Let $Z \in \mathrm{M}_4(F_k)$ with $\chi_{\mathrm{cyc}}(\gamma)\rho(\gamma)Z = Z\rho(\gamma)$ on Δ_0; multiplying by a non-zero element of $k[[T]]$ we may assume $Z \in \mathrm{M}_4(k[[T]])$, and $e_0 A_0 Z = Z A_0$. The $k[[T]]$-linear map $\Phi(Y) = e_0 A_0 Y - Y A_0$ on $\mathrm{M}_4(k[[T]])$ reduces modulo T to $\bar\Phi(Y) = (e_0 - 1)\bar\chi_1(g)Y + e_0 N Y - Y N$, where $N = \bar A_0 - \bar\chi_1(g)$ is nilpotent by (Pc1) and Cayley–Hamilton. Left and right multiplication by N commute and are nilpotent, so $Y \mapsto e_0 N Y - Y N$ is nilpotent, and $\bar\Phi$ is a non-zero scalar plus a nilpotent map, hence invertible. So $\det\Phi \in k[[T]]^\times$, Φ is injective and $Z = 0$.

Step 9: conclusion. By Steps 5–7 and Lemma 53.13, x satisfies (Pj2) and is of irreducible type; with Steps 5 and 8 it is a patchable curve point of irreducible type with $J_0 \subseteq \ker f$. Let $\mathfrak{q}$ be a prime of B with $y(\mathfrak{q}) = 0$ and $\mathfrak{q}' = \phi_0^{-1}(\mathfrak{q})$, a prime of B_4^{po}; $f(\mathfrak{q}') \subseteq y(\mathfrak{q}) = 0$, so $\mathfrak{q}' \subseteq \ker f$. As J_0 has R = T at patchable curve points, $J_0 \subseteq \mathfrak{q}'$, i.e. $\phi_0(J_0) \subseteq \mathfrak{q}$. □

Lemma 53.13. *Let $\mathscr{H}$ be a rank-4 Hecke datum over a coefficient ring $\mathcal{O}$, $(B_4^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ a universal polarised ordinary determinant of rank 4 over $\mathscr{H}$ and $(k, \rho, P, (\psi_{v,a}), f)$ a curve point of B_4^{po} such that:*

(Pc1) *for every $\gamma \in \Gamma^0$ the characteristic polynomial of $\rho(\gamma)$ is congruent to $\prod_{a=1}^4 (X - \bar\chi_a(\gamma))$ modulo T, and $\psi_{v,a}(g) \equiv \bar\chi_a(g)$ modulo T for all $v \in S_\ell$, a and $g \in D_v$;*

(Pc2) *for every $v \in S_\ell$ and $a \neq b$ there is $g \in D_v$ with $\chi_{\mathrm{cyc}}(g) = 1$, $\bar\chi_a(g) = \bar\chi_b(g)$ and $\psi_{v,a}(g) \neq \psi_{v,b}(g)$;*

(Pc3) *$\rho \otimes \bar F_k$ is an irreducible representation of Γ^0;*

(Pc4) *every homomorphism $\eta\colon \Gamma^0 \to \{\pm 1\}$ that is trivial on $\Gamma^0 \cap \ker\bar\omega$ and such that for every a there is b with $\eta\bar\chi_a = \bar\chi_b$ is trivial;*

(Pc5) *there are a, b such that $(\bar\chi_a\bar\chi_b^{-1})^m \neq 1$ for $1 \leq m \leq 4$.*

Then the curve point satisfies (Pj2) and $\rho \otimes \bar F_k$ restricted to Γ' is irreducible.

Proof. Symbols introduced here are local to this proof. Write $\Gamma = \Gamma^0$, $A_k = k[[T]]$, F_k, $\bar F_k$ as in the statement, $\bar\rho = \rho \otimes \bar F_k$, and $\Gamma' = \Gamma \cap \ker\chi_{\mathrm{cyc}}$, a closed normal subgroup with Γ/Γ' abelian. The group $1 + TA_k$ is torsion-free: if $(1+x)^N = 1$ with $0 \neq x \in TA_k$, write $N = \ell^j N'$ with $\ell \nmid N'$ and $z = x^{\ell^j} \neq 0$; then $(1+z)^{N'} = 1 + N'z +$ (terms of higher T-order) $\neq 1$.

Step A: a regular element of Γ'. Fix $v \in S_\ell$ and let H be the subgroup of $g \in D_v$ with $\chi_{\mathrm{cyc}}(g) = 1$ and $\bar\chi_c(g) = 1$ for all c; let e be the exponent of the finite group $\{(\bar\chi_c(g))_c : g \in D_v, \chi_{\mathrm{cyc}}(g) = 1\}$. For $a \neq b$ let $\theta_{ab} = \psi_{v,a}\psi_{v,b}^{-1}|_H$, a homomorphism $H \to 1 + TA_k$ by (Pc1).

For the element s_{ab} of (Pc2), $\psi_{v,a}(s_{ab})\psi_{v,b}(s_{ab})^{-1} \in 1 + TA_k$ (by (Pc1) and $\bar\chi_a(s_{ab}) = \bar\chi_b(s_{ab})$) is $\neq 1$, so $h_{ab} = s_{ab}^e \in H$ has $\theta_{ab}(h_{ab}) \neq 1$ (torsion-freeness). Enumerate the twelve pairs as $p = 1, \dots, 12$ and, for $c \in \mathbb{Z}^{12}$, put $h(c) = \prod_p h_p^{c_p} \in H$. For each q, $c \mapsto \theta_q(h(c))$ is a homomorphism $\mathbb{Z}^{12} \to 1 + TA_k$, non-zero at the q-th unit vector, into a torsion-free group; so its kernel has rank ≤ 11 and lies in a hyperplane $\{\lambda \cdot c = 0\}$, $0 \neq \lambda \in \mathbb{Z}^{12}$, which meets $[-M, M]^{12}$ in at most $(2M+1)^{11}$ points. For $2M + 1 > 12$ some $c \in [-M, M]^{12}$ lies in none of the twelve kernels. Put $u = h(c)$: then $u \in D_v$, its image lies in Γ', and $\theta_{ab}(u)$ has infinite order for $a \neq b$. As ρ is triangular over F_k on D_v with diagonal $(\psi_{v,a})$, the eigenvalues of $\rho(u^m)$ are $\psi_{v,a}(u)^m$, pairwise distinct for every $m \geq 1$: the characteristic polynomial of every $\rho(u^m)$, $m \geq 1$, is separable, and $\rho(u)$ satisfies (Xs3) over $\bar F_k$.

Step B: irreducible type. Suppose $\bar\rho|_{\Gamma'}$ is reducible, and let $W \subseteq \bar V = \bar F_k^4$ be a non-zero Γ'-stable subspace of minimal dimension. The subspaces $\bar\rho(\gamma)W$, $\gamma \in \Gamma$, are Γ'-stable and irreducible (Γ' is normal), and their sum is Γ-stable, so equals $\bar V$ by (Pc3); hence $\bar V|_{\Gamma'}$ is semisimple. Its isotypic components have multiplicity one: on a component $W_i^{e_i}$ the element $\bar\rho(u)$, $u \in \Gamma'$ from Step A, would have each eigenvalue on W_i with multiplicity $\geq e_i$, while its characteristic polynomial is separable. So $\bar V = W_1 \oplus \cdots \oplus W_t$ with pairwise non-isomorphic irreducible Γ'-modules W_i, these are all the Γ'-stable irreducible subspaces, Γ permutes them transitively, $t \geq 2$, and $t \mid 4$ as they have equal dimension. The stabiliser S of W_1 contains Γ', so it is normal with Γ/S abelian of order $t \in \{2, 4\}$; let $\eta\colon \Gamma \to \{\pm 1\}$ be a non-trivial character trivial on S. The linear map Φ acting on $\bar\rho(\gamma)W_1$ by $\eta(\gamma)$ is well defined, and $\Phi\bar\rho(\delta)\Phi^{-1} = \eta(\delta)\bar\rho(\delta)$ for $\delta \in \Gamma$ (both sides send $\bar\rho(\gamma)W_1$ to $\bar\rho(\delta\gamma)W_1$, with the factor $\eta(\delta\gamma)\eta(\gamma)^{-1}$). So $\rho(\delta)$ and $\eta(\delta)\rho(\delta)$ have the same characteristic polynomial; reducing modulo T with (Pc1), $\prod_a(X - \bar\chi_a(\delta)) = \prod_a(X - \eta(\delta)\bar\chi_a(\delta))$ for all δ. The semisimple representations $\bigoplus_a \bar\chi_a$ and $\bigoplus_a \eta\bar\chi_a$ of Γ over $\bar k$ have the same characteristic polynomials, hence the same constituents (Brauer–Nesbitt): for every a there is b with $\eta\bar\chi_a = \bar\chi_b$. Moreover η is trivial on $\Gamma \cap \ker\bar\omega$: η factors through $\Gamma/\Gamma' \cong C = \chi_{\mathrm{cyc}}(\Gamma) \subseteq \mathbb{Z}_\ell^\times = \mu_{\ell-1} \times (1+\ell\mathbb{Z}_\ell)$, and $C = (C \cap \mu_{\ell-1}) \times (C \cap (1+\ell\mathbb{Z}_\ell))$ (orders prime to each other); squaring is bijective on the pro-ℓ group $C \cap (1+\ell\mathbb{Z}_\ell)$ (ℓ odd), so the quadratic character η is trivial on it, and $\Gamma \cap \ker\bar\omega$ maps into it. By (Pc4), $\eta = 1$: contradiction. So $\bar\rho|_{\Gamma'}$ is irreducible: the curve point is of irreducible type, and by Burnside's theorem the $\bar F_k$-span of $\rho(\Gamma')$ is $\mathrm{M}_4(\bar F_k)$ (Xs2).

Step C: (Xs4) for Γ'. Let $N \subseteq \Gamma'$ be a normal subgroup of finite index m with $\rho(N)$ commutative. For $\gamma \in \Gamma$, $\gamma u \gamma^{-1} \in \Gamma'$, so $\gamma u^m \gamma^{-1} = (\gamma u \gamma^{-1})^m \in N$; hence the normal core $N_0 = \bigcap_\gamma \gamma N \gamma^{-1}$ of N in Γ contains u^m, and $\rho(N_0)$ is commutative. $\rho(u^m)$ has four distinct eigenvalues (Step A) with eigenlines $\ell_1, \dots, \ell_4 \subseteq \bar V$; every $\rho(n)$, $n \in N_0$, commutes with $\rho(u^m)$, so preserves each ℓ_i; two lines with the same character of N_0 would carry the same eigenvalue of u^m, so the ℓ_i are the joint eigenspaces of $\rho(N_0)$, with distinct characters ϕ_i. As N_0 is normal in Γ, $\bar\rho(\gamma)$ permutes them: $\bar\rho(\gamma)\ell_i = \ell_{\pi(\gamma)i}$ for a homomorphism $\pi\colon \Gamma \to S_4$, transitive by (Pc3). Let $h \in \ker\pi$. Then $\bar\rho(h)$ is diagonal in a basis of the ℓ_i with eigenvalues $\beta_i(h)$, integral over A_k (roots of a monic polynomial over A_k); fix a maximal ideal over (T) of the integral closure of A_k in $\bar F_k$ and write $\tilde\beta_i(h)$ for the residue, so that $\tilde\beta_i$ is a character of $\ker\pi$ with values in $\bar k^\times$, and $\prod_i(X - \tilde\beta_i(h)) = \prod_a(X - \bar\chi_a(h))$ by (Pc1). By Brauer–Nesbitt for the semisimple representations $\bigoplus_i \tilde\beta_i$ and $\bigoplus_a \bar\chi_a|_{\ker\pi}$, $\{\tilde\beta_i\} = \{\bar\chi_a|_{\ker\pi}\}$ as multisets. For $\gamma \in \Gamma$ with $\pi(\gamma)1 = i$, $\beta_i(h) = \beta_1(\gamma^{-1}h\gamma)$, so $\tilde\beta_i = \tilde\beta_1 \circ \mathrm{conj}_{\gamma^{-1}}$; $\tilde\beta_1 = \bar\chi_{a_0}|_{\ker\pi}$ for some a_0, and $\bar\chi_{a_0}$ is a character of Γ, so $\tilde\beta_i = \bar\chi_{a_0}|_{\ker\pi}$ for all i. Hence all $\bar\chi_a$ coincide on $\ker\pi$, and each $\bar\chi_a\bar\chi_b^{-1}$ is a character of $\pi(\Gamma) \subseteq S_4$. The abelianisation of every subgroup of S_4 has exponent $1, 2, 3$ or 4, so $(\bar\chi_a\bar\chi_b^{-1})^n = 1$ for some $1 \leq n \leq 4$, for all a, b, contradicting (Pc5). So every normal subgroup of finite index of Γ', in particular every open one, has non-commuting images.

Step D: (Xs1) for Γ'. Let $f \in \bar F_k[x_{ij}]$. Its coefficients lie in a finite-dimensional F_k-subspace

with basis $c_1, \dots, c_s$; write $f = \sum_j c_j f_j$ with $f_j \in F_k[x_{ij}]$ and choose $0 \neq d \in A_k$ with $df_j \in A_k[x_{ij}]$. As $\rho(g)$ has entries in F_k, $f(\rho(g)) = 0$ if and only if $(df_j)(\rho(g)) = 0$ for all j; each $g \mapsto (df_j)(\rho(g)) \in A_k$ is continuous for the T-adic topology (the entries of ρ are), and A_k is Hausdorff, so the zero set is closed.

Step E: (Pj2). Let $U_0 \subseteq \Gamma$ be open with some $\sigma \in U_0 \cap \Gamma'$. As Γ is profinite there is an open normal subgroup H' with $\sigma H' \subseteq U_0$; $V_0 = H' \cap \Gamma'$ has finite index in Γ'. By Steps A–D and Lemma 53.14 for $G = \Gamma'$, $K = \bar{F}_k$ (characteristic $\ell \neq 2$) and $\bar{\rho}|_{\Gamma'}$, some $g \in V_0$ makes the characteristic polynomial of $\rho(\sigma g)$ separable over $\bar{F}_k$, hence over F_k (the discriminant is non-zero), and $\sigma g \in U_0 \cap \Gamma'$. □

Lemma 53.14. *Let K be an algebraically closed field of characteristic different from 2, G a topological group and $\rho\colon G \to \mathrm{GL}_4(K)$ a homomorphism such that:*

(Xs1) *for every polynomial $f \in K[x_{ij} : 1 \leq i, j \leq 4]$ the set $\{g \in G : f(\rho(g)) = 0\}$ is closed in G;*

(Xs2) *the K-span of $\rho(G)$ is $\mathrm{M}_4(K)$;*

(Xs3) *there is $u \in G$ such that $\rho(u)$ has eigenvalues $a_1, \dots, a_4$ with a_i/a_j of infinite order in $K^\times$ for all $i \neq j$;*

(Xs4) *every open normal subgroup of finite index $N \subseteq G$ contains g, h with $\rho(g)\rho(h) \neq \rho(h)\rho(g)$.*

Then for every subgroup $U \subseteq G$ of finite index and every $\sigma \in G$ there is $g \in U$ such that the characteristic polynomial of $\rho(\sigma g)$ is separable.

Proof. Fix U and σ.

Step 0: an eigenbasis of u. The a_i are pairwise distinct ($a_i/a_j \neq 1$ has infinite order), so $\rho(u) = P\,\mathrm{diag}(a_1, \dots, a_4)P^{-1}$ for some $P \in \mathrm{GL}_4(K)$. Replacing ρ by $g \mapsto P^{-1}\rho(g)P$ preserves (Xs1) (a polynomial in the entries of $P^{-1}yP$ is a polynomial in the entries of y), (Xs2), (Xs3), (Xs4) and the characteristic polynomials. So we assume $\rho(u) = \mathrm{diag}(a_1, \dots, a_4)$.

Step 1: the subgroup N and a diagonal element. Let N be the subgroup of Lemma 50.20 for $L = K$: normal of finite index by (Zi1), open by (Zi4) and (Xs1). By (Xs4), $\rho(N)$ is not commutative. $U \cap N$ has finite index, so among $u^0, u^1, \dots, u^{[G:U\cap N]}$ two lie in the same left coset and $t = u^k \in U \cap N$ for some $k \geq 1$. Then $\rho(t) = \mathrm{diag}(\beta_1, \dots, \beta_4)$ with $\beta_i = a_i^k$, and $\beta_i/\beta_j = (a_i/a_j)^k$ has infinite order for $i \neq j$; in particular the β_i are pairwise distinct. Let $w_1, \dots, w_4$ be the pairwise distinct integers given by Lemma 50.21 for $E = K$ and $\beta_1, \dots, \beta_4$, and τ the permutation with $w_{\tau(1)} > \cdots > w_{\tau(4)}$.

Step 2: the relations R_g. For $g \in G$ and indices i, j write $R_g(i,j)$ if $\rho(gh)_{ij} \neq 0$ for some $h \in N$. R_g depends only on the coset gN. Properties:

(a) If $R_g(i,j)$ and $R_{g'}(j,l)$, then $R_{gg'}(i,l)$. Take $h, h' \in N$ with $\rho(gh)_{ij} \neq 0$, $\rho(g'h')_{jl} \neq 0$. For $q \geq 0$, $ght^q g'h' = gg' \cdot (g'^{-1}ht^q g'h') \in gg'N$ (N normal), and $\rho(ght^q g'h')_{il} = \sum_s \rho(gh)_{is}\rho(g'h')_{sl}\,\beta_s^q$. If all these vanished, the Vandermonde matrix of the distinct β_s would force $\rho(gh)_{is}\rho(g'h')_{sl} = 0$ for every s, false for $s = j$.

(b) If $R_g(i,j)$, then $R_{g^{-1}}(j,i)$. By (Xs2) some h has $\rho(h)_{ji} \neq 0$ (otherwise the span of $\rho(G)$ lies in $\{y : y_{ji} = 0\}$), so $R_h(j,i)$; by (a), $R_{gh}(i,i)$, and inductively $R_{h(gh)^m}(j,i)$ for all $m \geq 0$. With $d = [G : N]$, $(gh)^d \in N$ (G/N is a group of order d), so $h(gh)^{d-1} = g^{-1}(gh)^d \in g^{-1}N$ and $R_{g^{-1}}(j,i)$.

(c) R_1 is an equivalence relation: reflexive as $\rho(1) = 1$, symmetric by (b), transitive by (a).

(d) For every g and i there is j with $R_g(i,j)$: row i of the invertible $\rho(g)$ is non-zero.

(e) Every class of R_1 has at least two elements. First, $R_1(j,l)$ for some $j \neq l$: otherwise every $\rho(h)$, $h \in N$, is diagonal, and $\rho(N)$ is commutative. Suppose the class of i is $\{i\}$. By (Xs2) choose g with $\rho(g)_{ji} \neq 0$; then $R_g(j,i)$, and $R_g(l,i)$ by (a) from $R_1(l,j)$. If $R_g(a,i)$ and $\rho(g)_{as} \neq 0$, then $R_{g^{-1}}(i,a)$ by (b) and $R_g(a,s)$, so $R_1(i,s)$ by (a), and $s = i$. Applied to $a = j$ and $a = l$: rows $j \neq l$ of $\rho(g)$ vanish outside column i, so they are proportional, contradicting $\det \rho(g) \neq 0$.

Step 3: the dichotomy. Either (A) $R_\sigma(i,i)$ for all i, or (B) there is a partition $\{1,2,3,4\} = I \sqcup J$, $|I| = |J| = 2$, such that $R_\sigma(a,b)$ fails for a,b both in I or both in J and holds for $a \in I, b \in J$ and for $a \in J, b \in I$. Suppose (A) fails at i_0. By (d) take j_0 with $R_\sigma(i_0,j_0)$. Then $R_1(i_0,j_0)$ fails (else $R_\sigma(i_0,i_0)$ by (a) with $R_1(j_0,i_0)$). So the classes $I \ni i_0$ and $J \ni j_0$ of R_1 are distinct; by (e) each has at least two elements, so $|I| = |J| = 2$ and $I \sqcup J = \{1,2,3,4\}$. By (a), R_1-equivalent indices may be exchanged in R_σ on either side. Hence: $R_\sigma(a,b)$ holds for $a \in I$, $b \in J$; $R_\sigma(a,a')$ fails for $a,a' \in I$ (it would give $R_\sigma(i_0,i_0)$); $R_\sigma(b,b')$ fails for $b,b' \in J$ (it would give $R_\sigma(b,j_0)$, then with $R_{\sigma^{-1}}(j_0,i_0)$ from (b) and (a), $R_1(b,i_0)$, but $b \notin I$); and for $b \in J$, (d) gives c with $R_\sigma(b,c)$, necessarily $c \in I$, so $R_\sigma(b,a)$ for every $a \in I$.

Step 4: an element of U with a separable twist. Case (A). Consider the polynomials $f_1 = x_{\tau(1)\tau(1)}$ and $f_2 = \operatorname{adj}(x)_{\tau(4)\tau(4)}$ in the entries x. $f_1(\rho(\sigma h)) \neq 0$ for some $h \in N$ by $R_\sigma(\tau(1),\tau(1))$. By (b), $R_{\sigma^{-1}}(\tau(4),\tau(4))$: $\rho(\sigma^{-1}h')_{\tau(4)\tau(4)} \neq 0$ for some $h' \in N$. Put $h = \sigma^{-1}h'^{-1}\sigma \in N$; then $(\sigma h)^{-1} = \sigma^{-1}h'$ and $\operatorname{adj}(\rho(\sigma h)) = \det \rho(\sigma h)\, \rho(\sigma^{-1}h')$, so $f_2(\rho(\sigma h)) \neq 0$. By (Zi3) there is $g \in U \cap N$ with $f_1(\rho(\sigma g))f_2(\rho(\sigma g)) \neq 0$, and by (Zd1) the characteristic polynomial of $\rho(\sigma g)\operatorname{diag}(X^{w_1},\dots,X^{w_4})$ is separable over $K(X)$. Case (B). Let $i \in I$, $j \in J$ be the indices of larger weight. By (Zi3) applied to x_{ij} and x_{ji} (each non-zero somewhere on $\rho(\sigma N)$ by $R_\sigma(i,j)$, $R_\sigma(j,i)$) there is $g \in U \cap N$ with $\rho(\sigma g)_{ij}\rho(\sigma g)_{ji} \neq 0$. As $g \in N$ and $R_\sigma(a,b)$ fails for a,b in the same block, $\rho(\sigma g)_{ab} = 0$ there; and $\det \rho(\sigma g) \neq 0$. By (Zd2) the characteristic polynomial of $\rho(\sigma g)\operatorname{diag}(X^{w_1},\dots,X^{w_4})$ is separable over $K(X)$.

Step 5: specialisation. By Lemma 50.21 with $M = \rho(\sigma g)$, there is $q \geq 0$ such that the characteristic polynomial of $\rho(\sigma g)\operatorname{diag}(\beta_1^q,\dots,\beta_4^q) = \rho(\sigma g t^q)$ is separable. As $g \in U$ and $t \in U$, $gt^q \in U$ is the required element. □

Lemma 53.15. *Let K be a field of characteristic different from 2, X an indeterminate, $x \in \mathrm{M}_4(K)$, $w_1,\dots,w_4$ pairwise distinct integers and $D = \operatorname{diag}(X^{w_1},\dots,X^{w_4})$. The characteristic polynomial of xD is separable over $K(X)$ in each of the following cases:*

(Zd1) *$x_{\tau(1)\tau(1)} \neq 0$, and the principal minor of x on the rows and columns $\{\tau(1),\tau(2),\tau(3)\}$, that is $\operatorname{adj}(x)_{\tau(4)\tau(4)}$, is non-zero, where τ is the permutation of $\{1,2,3,4\}$ such that $w_{\tau(1)} > w_{\tau(2)} > w_{\tau(3)} > w_{\tau(4)}$;*

(Zd2) *there is a partition $\{1,2,3,4\} = I \sqcup J$ with $|I| = |J| = 2$ such that $x_{ab} = 0$ whenever $a,b \in I$ or $a,b \in J$, $\det x \neq 0$, and $x_{ij}x_{ji} \neq 0$, where $i \in I$ and $j \in J$ are the indices of the larger weight in I and in J.*

Proof. We compute in the ring $K[X,X^{-1}] \subseteq K(X)$; for a non-zero f there, $\deg f$ is the largest exponent with non-zero coefficient, so $\deg(fg) = \deg f + \deg g$ and $\deg(f+g) \leq \max(\deg f, \deg g)$, with equality if the degrees differ. Conjugating by a permutation matrix changes neither the characteristic polynomial nor the hypotheses (it permutes indices, weights and principal minors together), so we may relabel indices freely. For $S \subseteq \{1,2,3,4\}$ let m_S be

the principal minor of x on S and $w_S = \sum_{a\in S} w_a$. Multiplying column a by X^{w_a} multiplies every principal minor containing a by X^{w_a}, so

$$\det(Y - xD) = Y^4 - c_1Y^3 + c_2Y^2 - c_3Y + c_4, \qquad c_k = \sum_{|S|=k} m_S\, X^{w_S}.$$

(Zd1). Relabel so that $w_1 > w_2 > w_3 > w_4$; put $g_1 = w_1 - w_2$, $g_2 = w_2 - w_3$, $g_3 = w_3 - w_4$, all positive. The unique 1-subset of largest weight is $\{1\}$, the unique 3-subset of largest weight is $\{1,2,3\}$, the largest 2-subset weight is $w_1 + w_2$ (only for $\{1,2\}$) and the next is $w_1 + w_3$. Hence: $\deg c_1 = w_1$ with leading coefficient $x_{11} \neq 0$; $\deg c_3 = w_1 + w_2 + w_3$ with leading coefficient $m_{\{1,2,3\}} \neq 0$; $c_4 = \det(x)X^{w_1+w_2+w_3+w_4}$ has degree at most $w_1 + w_2 + w_3 + w_4$ (or is 0); and c_2 has degree at most $w_1 + w_2$, exactly $w_1 + w_2$ with leading coefficient $m_{\{1,2\}}$ if $m_{\{1,2\}} \neq 0$, and at most $w_1 + w_3$ if $m_{\{1,2\}} = 0$. With $b = -c_1$, $c = c_2$, $d = -c_3$, $e = c_4$, the discriminant of $Y^4 + bY^3 + cY^2 + dY + e$ is

$$\begin{aligned}\delta = {} & 256e^3 - 192bde^2 - 128c^2e^2 + 144cd^2e - 27d^4 + 144b^2ce^2 - 6b^2d^2e - 80bc^2de \\ & + 18bcd^3 + 16c^4e - 4c^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2.\end{aligned}$$

Case $m_{\{1,2\}} \neq 0$. The monomial $b^2c^2d^2$ has degree exactly $\Theta = 6w_1 + 4w_2 + 2w_3$ and leading coefficient $x_{11}^2 m_{\{1,2\}}^2 m_{\{1,2,3\}}^2 \neq 0$. Every other monomial has degree at most $\Theta - (n_1g_1 + n_2g_2 + n_3g_3)$ with non-negative integers n_i, not all 0:

So $\deg\delta = \Theta$ and $\delta \neq 0$.

Case $m_{\{1,2\}} = 0$. Now $\deg c \leq w_1 + w_3$. The monomial $-4b^3d^3$ has degree exactly $\Theta' = 6w_1+3w_2+3w_3$ and leading coefficient $-4(-x_{11})^3(-m_{\{1,2,3\}})^3 = -4x_{11}^3 m_{\{1,2,3\}}^3$, non-zero because the characteristic is not 2. Every other monomial has degree at most $\Theta' - (n_1g_1 + n_2g_2 + n_3g_3)$ with $n_i \geq 0$ not all 0:

So $\deg\delta = \Theta'$ and $\delta \neq 0$. In both cases the characteristic polynomial of xD is separable.

Why no 2×2 *minor is needed.* The top 2×2 minor $m_{\{1,2\}}$ only decides which monomial of δ dominates. When it vanishes, c_2 loses its top degree and the monomial $-4b^3d^3$, which does not contain c, takes over; its coefficient -4 is a unit exactly because the characteristic is not 2. In terms of roots: the roots of the characteristic polynomial have X-degrees w_1, w_4 and, for the middle pair, approximately the two square roots of $-d/b = -c_3/c_1$ (degree $w_2 + w_3$), which are distinct because $2 \neq 0$.

(Zd2). Relabel so that $I = \{1,2\}$, $J = \{3,4\}$, $w_1 > w_2$, $w_3 > w_4$; so $i = 1$, $j = 3$. In block form $xD = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ with $B = x_{IJ}\,\mathrm{diag}(X^{w_3}, X^{w_4})$ and $C = x_{JI}\,\mathrm{diag}(X^{w_1}, X^{w_2})$, where x_{IJ}, x_{JI} are the 2×2 blocks of x. Over $K(X)(Y)$, by the Schur complement of the invertible block $Y \cdot 1_2$,

$$\det\begin{pmatrix} Y & -B \\ -C & Y \end{pmatrix} = Y^2 \det(Y - CY^{-1}B) = \det(Y^2 - CB),$$

so the characteristic polynomial is $\chi(Y) = \psi(Y^2)$ with $\psi(Z) = Z^2 - \mathrm{tr}(CB)Z + \det(CB)$. Here $\det(CB) = \det(x_{IJ})\det(x_{JI})X^{w_1+w_2+w_3+w_4}$ and $\det x = \det(x_{IJ})\det(x_{JI})$, so $\det(CB) \neq 0$. Further

$$\mathrm{tr}(CB) = \sum_{a\in I,\ b\in J} x_{ab}x_{ba}X^{w_a+w_b},$$

whose exponent $w_1 + w_3$ is strictly larger than the other three, with coefficient $x_{13}x_{31} \neq 0$; so $\deg \mathrm{tr}(CB)^2 = 2(w_1 + w_3) > w_1 + w_2 + w_3 + w_4 \geq \deg(4\det(CB))$, and the discriminant $\mathrm{tr}(CB)^2 - 4\det(CB)$ of ψ is non-zero. So ψ has two distinct roots $z_1 \neq z_2$ in an algebraic

closure, both non-zero as $z_1 z_2 = \det(CB) \neq 0$. The roots of χ are $\pm\sqrt{z_1}$, $\pm\sqrt{z_2}$: $\sqrt{z_k} \neq -\sqrt{z_k}$ since $z_k \neq 0$ and $2 \neq 0$, and $\pm\sqrt{z_1} \neq \pm\sqrt{z_2}$ since their squares differ. So χ has four distinct roots and is separable. □

Lemma 53.16. *In the setting of Lemma 53.7, assume that $\mathscr{T}$ is good. Then:*

(Xr1) *every map $\eta\colon \Gamma^0_{\mathscr{H}} \to \{\pm 1\}$ which is a homomorphism, trivial on $\Gamma^0_{\mathscr{H}} \cap \ker\bar{\omega}$, and such that for every a there is b with $\eta\bar{\chi}_a = \bar{\chi}_b$, is trivial;*

(Xr2) *there are a, b with $(\bar{\chi}_a\bar{\chi}_b^{-1})^m \neq 1$ for $1 \leq m \leq 4$.*

Proof. Put $\lambda = \nu\kappa^{-1}$, so that $\bar{\chi}_a \circ \pi_{\mathscr{H}} = \bar{\lambda}\,\bar{\chi}_a^{\mathscr{T}}$ on G_L by (Lk6); in particular $\bar{\chi}_a\bar{\chi}_b^{-1} \circ \pi_{\mathscr{H}} = \bar{\chi}_a^{\mathscr{T}}/\bar{\chi}_b^{\mathscr{T}}$ on G_L.

(Xr1): Let η be as stated and $\eta' = \eta \circ \pi_{\mathscr{H}}|_{G_L}$, a homomorphism $G_L \to \{\pm 1\} \subseteq \mathbb{F}_{\mathcal{O}}^\times$ with $\eta'^2 = 1$, trivial on $G_L \cap \ker\omega_\ell$ (as $\bar{\omega}_\ell = \bar{\omega}$ and $\pi_{\mathscr{H}}(G_L \cap \ker\bar{\omega}) \subseteq \Gamma^0_{\mathscr{H}} \cap \ker\bar{\omega}$). Dividing $\eta\bar{\chi}_a = \bar{\chi}_b$ by $\bar{\lambda}$ gives $\eta'\bar{\kappa}_a^\epsilon = \bar{\kappa}_b^\epsilon$ on G_L. By (Oq4), $\eta' = 1$, so $\eta = 1$ ($\pi_{\mathscr{H}}(G_L) = \Gamma^0_{\mathscr{H}}$).

(Xr2): By (Oq3) there is a root w such that for every $1 \leq m \leq 4$ some $g \in G_L$ has $\overline{\mathrm{rt}_w}(g)^m \neq 1$. So $\mathrm{rt}_w = \chi_a^{\mathscr{T}}/\chi_b^{\mathscr{T}}$ for some $a \neq b$, and $(\bar{\chi}_a\bar{\chi}_b^{-1})^m(\pi_{\mathscr{H}}(g)) = \overline{\mathrm{rt}_w}(g)^m \neq 1$. □

Lemma 53.17. *Let $\mathfrak{D}$, $\mathscr{T}$ (fit and deep), $\mathcal{O}_1, \nu, \gamma, t$, $\mathcal{O}' \supseteq \mathcal{O}_1$, κ, $\mathscr{H}$, $\mathscr{X} = (B, A, P, \dots)$, $(B_4^{\mathrm{po}}, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ and ϕ_0 be as in Lemma 53.12, where κ moreover has (Kw1) (for every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above ℓ there are integers e and $k' \geq 1$ such that $(\kappa\chi_{\mathrm{cyc}}^{-e})^{k'}$ is trivial on $I_{\bar{w}} \cap G_L$), $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and $J_0 \subseteq B_4^{\mathrm{po}}$ an ideal with $J_0 \subseteq \mathcal{J}_\kappa^{\mathrm{cl}}$ (for ι and κ) and the properties (Hf2) and (Hf5). Then for every $\mathcal{O}'$-algebra homomorphism $y\colon A \to \overline{\mathbb{Q}}_\ell$ such that $y(P)$ is a seed for ι, every prime $\mathfrak{q}$ of B with $y(\mathfrak{q}) = 0$ contains $\phi_0(J_0)$.*

Proof. Symbols introduced here are local to this proof. Write $\pi = \pi_{\mathscr{H}}$ and $B^{\mathrm{po}} = B_4^{\mathrm{po}}$. Let $y\colon A \to \overline{\mathbb{Q}}_\ell$ be an $\mathcal{O}'$-algebra homomorphism such that $y(P)$ is a seed for ι, with weights h, and let $\mathfrak{q}$ be a prime of B with $y(\mathfrak{q}) = 0$.

Step 1: a point of B^{po}. Choose $n \in \{0, 1\}$ with $n \equiv h_1$ modulo 2 and put $x = y \circ \phi_n$. By Lemma 53.18 (its hypothesis (Kw1) on κ is one of ours), x is a classical ordinary point for ι and κ, its representation $\tilde{r}$ is absolutely irreducible on $\Gamma^0_{\mathscr{H}} \cap \ker\bar{\omega}$, and integers $m_{v,a}$ with gaps ≥ 2 and $k \geq 1$ as in Lemma 53.20 exist. The character κ satisfies $\kappa(s)\kappa(csc) = \chi_{\mathrm{cyc}}(s)^{-3}$ for $s \in G_L$ ((Tw1) with $m = -3$, as required of κ in Lemma 53.2). As J_0 has (Hf2) and $J_0 \subseteq \mathcal{J}_\kappa^{\mathrm{cl}}$, Lemma 53.20 shows that every prime $\mathfrak{q}' \subseteq \ker x$ of B^{po} contains J_0.

Step 2: the prime $\mathfrak{q}$. $\mathfrak{q}' = \phi_n^{-1}(\mathfrak{q})$ is a prime of B^{po} with $x(\mathfrak{q}') = y(\phi_n(\mathfrak{q}')) \subseteq y(\mathfrak{q}) = 0$, so $\mathfrak{q}' \subseteq \ker x$ and $J_0 \subseteq \mathfrak{q}'$, i.e. $\phi_n(J_0) \subseteq \mathfrak{q}$. If $n = 0$ we are done.

Step 3: from ϕ_1 to ϕ_0. Let $n = 1$. The character $\gamma\colon G_L \to \mathcal{O}'^\times$ is trivial on $I_{\bar{w}} \cap G_L$ for every place $\bar{w} \nmid \ell$ ((Lk2): t vanishes there and $\ker t \subseteq \ker\gamma$), and $\gamma(csc) = \gamma(s)^{-1}$ (Lk1); so it is trivial on the generators of $N_{\mathscr{H}}$ and their conjugates under G_L and c, hence (being continuous) on $N_{\mathscr{H}}$, and $\gamma = \gamma_0 \circ \pi|_{G_L}$ for a continuous homomorphism $\gamma_0\colon \Gamma^0_{\mathscr{H}} \to \mathcal{O}'^\times \subseteq (B^{\mathrm{po}})^\times$. Put $\beta = \gamma_0^{-1}$. It satisfies the hypotheses of (Hf5): $\beta \equiv 1$ modulo $\mathfrak{m}_{B^{\mathrm{po}}}$ (Lk2); $\beta(\pi(csc)) = \beta(\pi(s))^{-1}$ (Lk1); β is trivial on $\pi(I_{\bar{w}} \cap G_L)$ for all $\bar{w} \nmid \ell$, in particular above $\mathrm{St} \cup \{v_0\}$; and β is trivial on μ_v ((Lk4) gives $t(\mu_v) = 0$, then (Lk2)). So $\tau_\beta(J_0) \subseteq J_0$. Now $\phi_1 \circ \tau_\beta$ satisfies (Cm1) and (Cm2) for $n = 0$: for $g \in G_L$, τ_β maps $\chi_{D^{\mathrm{u}}}(\pi(g))(X)$ to $\chi_{D^{\mathrm{u}}_\beta}(\pi(g))(X) = D^{\mathrm{u}}(X - \beta(\pi(g))\pi(g)) = \beta(\pi(g))^4\chi_{D^{\mathrm{u}}}(\pi(g))(\beta(\pi(g))^{-1}X)$, and applying ϕ_1 with (Cm1) gives $(\beta\lambda_1)(g)^4\,\mathrm{cp}(g)((\beta\lambda_1)(g)^{-1}X)$ with $\beta(\pi(g))\lambda_1(g) = \gamma(g)^{-1}\nu(g)\kappa(g)^{-1}\gamma(g) = \lambda_0(g)$; and $\phi_1(\tau_\beta(\psi^{\mathrm{d}}_{v,a}(g))) = \phi_1(\beta(\pi(g))\psi^{\mathrm{d}}_{v,a}(g)) = \lambda_0(g)\psi^B_{v,a}(g)$ by (Cm2). By the uniqueness in Lemma 53.7, $\phi_0 = \phi_1 \circ \tau_\beta$. Hence $\phi_0(J_0) = \phi_1(\tau_\beta(J_0)) \subseteq \phi_1(J_0) \subseteq \mathfrak{q}$ by Step 2. □

Lemma 53.18. *Let $\mathfrak{D}$, $\mathscr{T}$ (fit and deep), $\mathcal{O}_1, \nu, \gamma, t$, $\mathcal{O}' \supseteq \mathcal{O}_1$, κ, $\mathscr{H}$, $\mathscr{X} = (B, A, P, \dots)$ and $(B_4^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ be as in Lemma 53.12, and $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and suppose that κ has (Kw1): for every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above ℓ there are integers e and $k' \geq 1$ such that $(\kappa\chi_{\mathrm{cyc}}^{-e})^{k'}$ is trivial on $I_{\bar{w}} \cap G_L$. Let $y\colon A \to \overline{\mathbb{Q}}_\ell$ be an $\mathcal{O}'$-algebra homomorphism such that $y(P)$ is a seed for ι with weights $h \in \mathbb{Z}^4$ (as in the definition of a seed), let $n \geq 0$ be an integer with $n \equiv h_1$ modulo 2, ϕ_n as in Lemma 53.7, and $x = y \circ \phi_n\colon B_4^{\mathrm{po}} \to \overline{\mathbb{Q}}_\ell$. Then x is a classical ordinary point for ι and κ; the representation $\tilde{r}$ of its definition restricted to $\Gamma^0_{\mathscr{H}} \cap \ker\bar{\omega}$ is absolutely irreducible; and there are integers $m_{v,a}$ with $m_{v,a} \geq m_{v,b} + 2$ for $a < b$ and $k \geq 1$ with $\bigl(x(\psi_{v,a}^{\mathrm{d}}(g))\chi_{\mathrm{cyc}}(g)^{-m_{v,a}}\bigr)^k = 1$ for all $v \in \Sigma$, a and $g \in I_v$.*

Proof. Symbols introduced here are local to this proof. Write $\pi = \pi_{\mathscr{H}}$, $\Gamma^0 = \Gamma^0_{\mathscr{H}}$, $B^{\mathrm{po}} = B_4^{\mathrm{po}}$, $r_y = y \circ r_P\colon G_F \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ (invertible values: similitude $\theta(g)$, (Mq1)). As $y(P)$ is a seed with weights h: the span of $r_y(G')$ is $\mathrm{M}_4(\overline{\mathbb{Q}}_\ell)$; $h_i - h_{i+1} \in 2\mathbb{Z}$ and $h_i - h_{i+1} \geq 4$; there are an open normal subgroup $U \subseteq G_F$ with $y(\psi_{v,a}^P(s))^2 = \chi_{\mathrm{cyc}}(s)^{h_a}$ for $s \in U \cap I_v$, a continuous character $\kappa_s\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$, integers a_s with $a_s + h_1$ even and $k_s \geq 1$ with $(\kappa_s^2\chi_{\mathrm{cyc}}^{-a_s})^{k_s} = 1$ on the inertia groups in G_L of all places above ℓ, and a regular algebraic cuspidal π_s of $\mathrm{GL}_4(\mathbb{A}_L)$ with $r_y|_{G_L} \otimes \kappa_s \leftrightarrow_\iota \pi_s$. Hence $n \equiv h_1 \equiv h_a \equiv a_s$ modulo 2 for all a.

Step 1: continuity of r_y on G_L. Put $\rho_L = \lambda_n\, r_y|_{G_L}$. By (In1) x is continuous, and D^{u} is continuous, so $t(g) = x(\mathrm{tr}_{D^{\mathrm{u}}}(\pi(g)))$ is continuous on G_L. By (Cm1) and (Zq2), $x(\chi_{D^{\mathrm{u}}}(\pi(g)))(X) = \det(X - \rho_L(g))$, so $t(g) = \mathrm{tr}\,\rho_L(g)$. As $G' \subseteq G_L$ and λ_n takes non-zero scalar values, the span of $\rho_L(G')$ is that of $r_y(G')$, namely $\mathrm{M}_4(\overline{\mathbb{Q}}_\ell)$; choose $g_1, \dots, g_{16} \in G'$ with $\rho_L(g_i)$ a basis. The Gram matrix $\Gamma = (t(g_ig_j))_{i,j}$ of the non-degenerate trace form is invertible, and for $g \in G_L$, $\rho_L(g) = \sum_j c_j(g)\rho_L(g_j)$ with $(c_j(g))_j = \Gamma^{-1}(t(gg_i))_i$, continuous in g. So ρ_L, and $r_y|_{G_L} = \lambda_n^{-1}\rho_L$, are continuous.

Step 2: (Cy1) and irreducibility. By Step 1, (In2) applies: there are E, $u \in \mathrm{GL}_4(E)$ and a continuous $\tilde{r}\colon \Gamma^0 \to \mathrm{GL}_4(\mathcal{O}_E) \subseteq \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ with $r := \tilde{r} \circ \pi|_{G_L} = u\rho_L u^{-1}$, continuous, and $x \circ D^{\mathrm{u}} = \det \circ \tilde{r}$. χ_{cyc} is trivial on inertia groups at places not above ℓ, so $N_{\mathscr{H}} \subseteq \ker\bar{\omega}$ and $\pi^{-1}(\Gamma^0 \cap \ker\bar{\omega}) \cap G_L = G_L \cap \ker\bar{\omega} \supseteq G'$. Hence the span of $\tilde{r}(\Gamma^0 \cap \ker\bar{\omega})$ contains $u\,\mathrm{span}(\rho_L(G'))\,u^{-1} = \mathrm{M}_4(\overline{\mathbb{Q}}_\ell)$: $\tilde{r}$ restricted to $\Gamma^0 \cap \ker\bar{\omega}$ is absolutely irreducible, and so is $\tilde{r}$. This is (Cy1) and the second assertion.

Step 3: (Cy3). Let $v \in \Sigma$ and $s \in D_v \subseteq G_L$. By (Mq4), $n_v^{-1}r_P(s)n_v$ is upper triangular with diagonal $(\psi_{v,a}^P(s))_a$; $y(n_v)$ is invertible (Mq3). With $w_v = u\,y(n_v)$, $w_v^{-1}r(s)w_v = \lambda_n(s)\,y(n_v)^{-1}r_y(s)y(n_v)$ is upper triangular with diagonal $(\lambda_n(s)y(\psi_{v,a}^P(s)))_a = (x(\psi_{v,a}^{\mathrm{d}}(s)))_a$ by (Cm2). This is (Cy3).

Step 4: weights at ℓ, (Cy4) and the integers $m_{v,a}$. Let $v \in \Sigma$; by (Lk5) at $\bar{v}$ there are integers a_v, b_v odd and $k_v \geq 1$ with $(\nu\chi_{\mathrm{cyc}}^{-a_v})^{k_v} = (\gamma^2\chi_{\mathrm{cyc}}^{-b_v})^{k_v} = 1$ on I_v. By (Kw1) at $\bar{v}$ there are integers e_v and $k_v' \geq 1$ with $(\kappa\chi_{\mathrm{cyc}}^{-e_v})^{k_v'} = 1$ on $I_{\bar{v}} \cap G_L = I_v$ (v splits in L). Put $K = k_vk_v'$ and

$$m_{v,a} = a_v - e_v + \tfrac{1}{2}(nb_v + h_a) \in \mathbb{Z} \qquad (nb_v + h_a \equiv n + h_1 \equiv 0 \bmod 2).$$

On $U \cap I_v$, by (Cm2),

$$x(\psi_{v,a}^{\mathrm{d}})^{2K} = \nu^{2K}\gamma^{2nK}\kappa^{-2K}y(\psi_{v,a}^P)^{2K} = \chi_{\mathrm{cyc}}^{2Ka_v + nKb_v - 2Ke_v + Kh_a} = \chi_{\mathrm{cyc}}^{2Km_{v,a}}.$$

With $j_v = [I_v : U \cap I_v]$ and $U \cap I_v$ normal in I_v, $g^{j_v} \in U \cap I_v$ for $g \in I_v$, so

$$\bigl(x(\psi_{v,a}^{\mathrm{d}}(g))\chi_{\mathrm{cyc}}(g)^{-m_{v,a}}\bigr)^{2Kj_v} = 1;$$

take k the product of the $2Kj_v$ over $v \in \Sigma$. For $a < b$, $m_{v,a} - m_{v,b} = (h_a - h_b)/2 \geq 2(b-a) \geq 2$. Finally $x \circ \psi_{v,a}^{\mathrm{d}} \cdot \chi_{\mathrm{cyc}}^{-m_{v,a}}$ is a character of D_v trivial on an open subgroup of I_v, hence potentially

unramified and de Rham with all Hodge–Tate weights 0; so $x \circ \psi_{v,a}^{\mathrm{d}}$ is de Rham with τ-Hodge–Tate weight $m_{v,a}$ for every τ, strictly decreasing in a. This is (Cy4) and the third assertion.

Step 5: (Cy2). Put $R = r \otimes \kappa = u\,\nu\gamma^n\, r_y|_{G_L}\, u^{-1} \cong \mu \otimes (r_y|_{G_L} \otimes \kappa_s)$ with $\mu = \nu\gamma^n\kappa_s^{-1}$, a continuous character of G_L. For every place $\bar{w}$ above ℓ, with a_w, b_w odd, k_w from (Lk5) at $\bar{w}$, on $I_{\bar{w}} \cap G_L$: $\mu^{2k_wk_s} = \chi_{\mathrm{cyc}}^{k_wk_s(2a_w+nb_w-a_s)}$ and $N_w = (2a_w + nb_w - a_s)/2 \in \mathbb{Z}$ (as $n \equiv a_s$, b_w odd); so $\mu\chi_{\mathrm{cyc}}^{-N_w}$ has finite order on $I_{\bar{w}} \cap G_L$ and μ is de Rham at $\bar{w}$ (as in Step 4). A continuous character of G_L de Rham at all places above ℓ is algebraic for ι (J.-P. Serre, Abelian ℓ-adic representations and elliptic curves, Ch. III; Clozel–Harris–Taylor 2008, Lemma 4.1.3): it corresponds through ι and the global Artin map to an algebraic Hecke character ψ' of L; taking $\psi = \psi'$ or ψ'^{-1} according to the normalisation of the Artin map (the inverse of an algebraic Hecke character is algebraic), $\iota\mu(\mathrm{Frob}_w^{-1}) = \psi_w(\varpi_w)$ at all finite w where both are unramified, i.e. $\mu \leftrightarrow_\iota \psi$. Let $\Pi = \pi_s \otimes (\psi \circ \det)$: cuspidal, and regular algebraic (the infinity type of ψ is integral, so the infinitesimal character of Π_∞ is that of $\pi_{s,\infty}$ shifted by an integer at each embedding). At almost all w the Satake parameters of Π_w are $\psi_w(\varpi_w)\alpha$, α those of $\pi_{s,w}$, and ι carries the eigenvalues of $R(\mathrm{Frob}_w^{-1})$ to $\psi_w(\varpi_w)\mathrm{N}w^{3/2}\alpha$; so $R \leftrightarrow_\iota \Pi$. Conjugate self-duality. (Od1) for $x \circ D^{\mathrm{u}} = \det \circ \tilde{r}$ says that $r(cg^{-1}c)$ and $r(g)$ have the same characteristic polynomial for $g \in G_L$; the eigenvalues of ${}^t r(cgc)^{-1}$ are those of $r(cg^{-1}c)$, so the absolutely irreducible representations r and $(r^c)^\vee$ have equal determinants (Theorem B.83, (Ch1)) and are isomorphic ((Ch3)): $r^c \cong r^\vee$. With (Tw1) for $m = -3$, $R^c = r^c\kappa^c \cong r^\vee\kappa^{-1} \cdot \kappa\kappa^c = R^\vee\chi_{\mathrm{cyc}}^{-3}$. Now Π^c (with $\Pi_w^c \cong \Pi_{cw}$) has R^c attached, and $\Pi^\vee$ (Satake parameters α^{-1}) has $R^\vee\chi_{\mathrm{cyc}}^{-3}$ attached: So the cuspidal representations Π^c and $\Pi^\vee$ have isomorphic unramified components at almost all places (the characteristic polynomial determines the Satake parameters), and $\Pi^c \cong \Pi^\vee$ by Theorem B.60, (Js1). So Π is regular algebraic, conjugate self-dual and cuspidal with $r \otimes \kappa \leftrightarrow_\iota \Pi$: (Cy2). With Steps 2–4, x is a classical ordinary point for ι and κ. □

Lemma 53.19. *In the setting of Lemma 53.7, let $n \geq 0$ be an integer, $y\colon A \to \overline{\mathbb{Q}}_\ell$ an $\mathcal{O}'$-algebra homomorphism and $x = y \circ \phi_n\colon B_4^{\mathrm{po}} \to \overline{\mathbb{Q}}_\ell$. Then:*

(In1) *x is continuous for the $\mathfrak{m}$-adic topology of B_4^{po}, $\mathfrak{m}$ its maximal ideal, and the ℓ-adic topology of $\overline{\mathbb{Q}}_\ell$;*

(In2) *if moreover $g \mapsto y(r_P(g))$ is continuous on G_L, there are a finite extension E of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ containing $\mathcal{O}'$, with ring of integers $\mathcal{O}_E \supseteq x(B_4^{\mathrm{po}})$, a matrix $u \in \mathrm{GL}_4(E)$ and a continuous homomorphism $\rho\colon \Gamma^0_{\mathscr{H}} \to \mathrm{GL}_4(\mathcal{O}_E)$ such that $\rho(\pi_{\mathscr{H}}(g)) = u\,\lambda_n(g)\,y(r_P(g))\,u^{-1}$ for all $g \in G_L$ and $x \circ D^{\mathrm{u}} = \det \circ \rho$.*

Proof. Symbols introduced here are local to this proof. Write $B^{\mathrm{po}} = B_4^{\mathrm{po}}$, $\pi = \pi_{\mathscr{H}}$, $\Gamma^0 = \Gamma^0_{\mathscr{H}}$.

Step 1: integrality and continuity. For $b \in \mathfrak{m}$ there is a homomorphism $\mathcal{O}'[[T]] \to B^{\mathrm{po}}$, $T \mapsto b$ (completeness); let $z = x(b)$, algebraic over $\mathbb{Q}_\ell$, and $f \in \mathcal{O}'[T]$ primitive with $f(z) = 0$. If $|z| \geq 1$, write $f = Pf_1$ in $\mathcal{O}'[T]$ with P monic having as roots the roots of f of absolute value < 1 (its coefficients are integral and lie in the fraction field of $\mathcal{O}'$, the set of these roots being stable under its Galois group) and f_1 primitive with all roots of absolute value ≥ 1, so that its constant term has the largest absolute value among its coefficients and is a unit; then $f_1(z) = 0$, while f_1 is a unit of $\mathcal{O}'[[T]]$, so its image $f_1(b)$ is a unit of B^{po} and $f_1(z) = x(f_1(b)) \neq 0$: contradiction. So $|x(b)| < 1$ for $b \in \mathfrak{m}$, and $|x(b)| \leq 1$ for all b ($b \in \mathcal{O}' + \mathfrak{m}$). Let $b_1, \dots, b_q$ generate $\mathfrak{m}$, E_0 the finite extension of $\mathbb{Q}_\ell$ generated by the fraction field of $\mathcal{O}'$ and the $x(b_i)$, and $\delta < 1$ the absolute value of its uniformisers. Every element of $\mathfrak{m}^k$ is a combination of monomials of degree k in the b_i with coefficients in B^{po}, so its image has absolute value $\leq \delta^k$: this is (In1); it uses no hypothesis on y. As $\mathcal{O}'[b_1, \dots, b_q]$ is dense in B^{po} (every element is a convergent power series in

the b_i, by Cohen's structure theorem), $x(B^{\mathrm{po}})$ lies in the closure of $\mathcal{O}'[x(b_1),\dots,x(b_q)]$, hence in the ring of integers of E_0, and in that of every finite extension of E_0.

Step 2: a lattice. From now on assume that $g \mapsto y(r_P(g))$ is continuous on G_L (the hypothesis of (In2)). Put $\rho_L(g) = \lambda_n(g)\, y(r_P(g))$ for $g \in G_L$, a continuous homomorphism $G_L \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ (λ_n is continuous). G_L is the countable union of the closed subgroups $\rho_L^{-1}(\mathrm{GL}_4(E'))$, E' over the finite extensions of E_0 in $\overline{\mathbb{Q}}_\ell$ ($\mathrm{GL}_4(E')$ is closed, E' being complete); by Baire's theorem one of them has non-empty interior, hence is open of finite index, and adjoining the entries of ρ_L at finitely many coset representatives gives a finite extension $E \supseteq E_0$ with $\rho_L(G_L) \subseteq \mathrm{GL}_4(E)$. The compact subgroup $\rho_L(G_L)$ stabilises an $\mathcal{O}_E$-lattice of E^4, so there is $u \in \mathrm{GL}_4(E)$ with $u\rho_L(G_L)u^{-1} \subseteq \mathrm{GL}_4(\mathcal{O}_E)$. By Step 1, $E \supseteq \mathcal{O}'$ and $x(B^{\mathrm{po}}) \subseteq \mathcal{O}_E$.

Step 3: factorisation through Γ^0. Let $s \in I_{\bar w} \cap G_L$, $\bar w$ a place of $\overline{\mathbb{Q}}$ not above ℓ, with $\bar\chi_a(s) = 1$ for all a. One of the $\chi_a^{\mathscr{T}}$ is trivial, so (Lk6) and $\kappa(s) = 1$ (Tw2) give $\bar\nu(s) = 1$; then (Lk6) gives $\bar\chi_a^{\mathscr{T}}(s) = \bar\chi_a(s) = 1$ for all a, hence $\bar r_0(s) = 1$, and $r_P(s) = 1$ by (Mq1). $\nu(s)^m = 1$ with $\ell \nmid m$ (Lk3) and $\bar\nu(s) = 1$ give $\nu(s) = 1$ (a root of unity of order prime to ℓ that is $\equiv 1$ modulo the maximal ideal is 1); $t(s) = 0$ and $\ker t \subseteq \ker\gamma$ (Lk2) give $\gamma(s) = 1$. So $\lambda_n(s) = 1$ and $\rho_L(s) = 1$. The conjugates of such s by G_L are of the same kind (inertia at the conjugate place, same values of the $\bar\chi_a$), and so are those by c: $csc \in I_{c\bar w} \cap G_L$ and $\bar\chi_a(csc) = \bar\chi_{5-a}(s)^{-1} = 1$ (Rd5). The subgroup they generate is normal in $G_{L^+} = G_L \cup cG_L$, so its closure is $N_{\mathscr{H}}$; as $\ker\rho_L$ is closed, $N_{\mathscr{H}} \subseteq \ker\rho_L$. Since $\pi|_{G_L}\colon G_L \to \Gamma^0 = G_L/N_{\mathscr{H}}$ is a quotient map, $u\rho_L u^{-1} = \rho \circ \pi|_{G_L}$ for a continuous homomorphism $\rho\colon \Gamma^0 \to \mathrm{GL}_4(\mathcal{O}_E)$.

Step 4: the determinant. For $g \in G_L$, $\mathrm{cp}(g) \in B[X]$ maps to the characteristic polynomial of $r_P(g)$ (Zq2), so by (Cm1)

$$x\big(\chi_{D^{\mathrm{u}}}(\pi(g))\big)(X) = \lambda_n(g)^4\, y(\mathrm{cp}(g))(\lambda_n(g)^{-1}X) = \det\big(X - \lambda_n(g)y(r_P(g))\big) = \det\big(X - \rho(\pi(g))\big).$$

As $\pi(G_L) = \Gamma^0$, the $\overline{\mathbb{Q}}_\ell$-valued determinants $x \circ D^{\mathrm{u}}$ and $\det\circ\rho$ of dimension 4 on Γ^0 have the same characteristic polynomials at all elements of Γ^0, so they are equal by Theorem B.83, (Ch1). This is (In2). □

Lemma 53.20. *Let $\mathscr{H}$ be a rank-4 Hecke datum over a coefficient ring $\mathcal{O}'$, with CM field L and complex conjugation c. Let $\kappa\colon G_L \to \mathcal{O}'^{\times}$ be a continuous character with $\kappa(s)\kappa(csc) = \chi_{\mathrm{cyc}}(s)^{-3}$ for all $s \in G_L$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $(B, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank 4 over $\mathscr{H}$, and $J \subseteq B$ an ideal with (Hf2) and $J \subseteq \mathcal{J}_\kappa^{\mathrm{cl}}$. Let $x\colon B \to \overline{\mathbb{Q}}_\ell$ be a classical ordinary point for ι and κ, with $\tilde r$ as in its definition, such that $\tilde r$ restricted to $\Gamma^0_{\mathscr{H}} \cap \ker\bar\omega$ is absolutely irreducible and there are integers $m_{v,a}$ with $m_{v,a} \geq m_{v,b} + 2$ for $a < b$ and $k \geq 1$ with $\big(x(\psi^{\mathrm{d}}_{v,a}(g))\chi_{\mathrm{cyc}}(g)^{-m_{v,a}}\big)^k = 1$ for all $v \in S_\ell$, a and $g \in I_v$. Then every prime ideal $\mathfrak{q} \subseteq \ker x$ of B contains J.*

Proof. Symbols introduced here are local to this proof. Write $\pi = \pi_{\mathscr{H}}$, $\Gamma^0 = \Gamma^0_{\mathscr{H}}$, $r = \tilde r \circ \pi|_{G_L}$, $d = [L^+ : \mathbb{Q}] = \sum_{v\in S_\ell} d_v$, $\Lambda = \Lambda_{4d}$, and $\mathfrak{p} = \ker x$, a prime of B with $J \subseteq \mathcal{J}_\kappa^{\mathrm{cl}} \subseteq \mathfrak{p}$.

Step 1: irreducibility over $L(\mu_{\ell^\infty})$. The kernel of $\bar\omega$ on G_L is $G' = G_{L(\mu_\ell)}$ (χ_{cyc} has values in $\mathbb{Z}_\ell^\times$ and $\mathbb{Z}_\ell \cap \mathfrak{m}_{\mathcal{O}'} = \ell\mathbb{Z}_\ell$), and $\pi(G') = \Gamma^0 \cap \ker\bar\omega$, so $r|_{G'}$ is absolutely irreducible on $V = \overline{\mathbb{Q}}_\ell^4$. Let $H = G_{L(\mu_{\ell^\infty})}$, closed and normal in G'; $G'/H \cong \mathrm{Gal}(L(\mu_{\ell^\infty})/L(\mu_\ell))$ is isomorphic through χ_{cyc} to a closed subgroup of $1 + \ell\mathbb{Z}_\ell \cong \mathbb{Z}_\ell$ (ℓ odd), so it is pro-ℓ and procyclic, with a topological generator σ. The sum of the H-irreducible subspaces of V is non-zero and G'-stable (H is normal), hence equal to V: $V|_H$ is semisimple. G' permutes its isotypic components $U_1,\dots,U_t$ transitively (the sum over an orbit is G'-stable), all of the same dimension, so $t \mid 4$; the stabiliser of U_1 is closed (r is continuous), of index t and contains H, so t is the index of an open subgroup of the pro-ℓ group G'/H, a power of the odd prime ℓ. Hence $t = 1$: $V \cong W \otimes \overline{\mathbb{Q}}_\ell^m$ as H-modules,

W irreducible, H acting on the first factor. The span of $r(H)$ is $\mathcal{A} = \mathrm{End}(W) \otimes 1$ (Burnside), with commutant $\mathcal{C} = 1 \otimes \mathrm{M}_m(\overline{\mathbb{Q}}_\ell)$, whose commutant is $\mathcal{A}$. Conjugation by $r(g)$, $g \in G'$, preserves $r(H)$, hence $\mathcal{A}$ and $\mathcal{C}$, and on $\mathcal{C}$ it is inner (Skolem–Noether): $r(g)(1 \otimes X)r(g)^{-1} = 1 \otimes m_g X m_g^{-1}$ with $m_g \in \mathrm{GL}_m(\overline{\mathbb{Q}}_\ell)$ unique up to a scalar; then $r(g)(1 \otimes m_g)^{-1}$ commutes with $\mathcal{C}$, so $r(g) = w_g \otimes m_g$ with $w_g \in \mathrm{GL}(W)$. $g \mapsto \mathrm{Ad}(m_g)$ is a homomorphism, trivial on H. Let $\lambda \subseteq \overline{\mathbb{Q}}_\ell^m$ be an eigenline of m_{g_σ} for a lift g_σ of σ; then $m_g\lambda = \lambda$ for every g in the subgroup generated by g_σ and H, which is dense in G'. The set of $g \in G'$ with $r(g)(W \otimes \lambda) = W \otimes \lambda$ is closed and contains this subgroup, so $W \otimes \lambda$ is G'-stable; by irreducibility $m = 1$. So $r|_H$ is irreducible.

Step 2: the tangent space at x has dimension at most $4d$. Let $R_0 = B/\mathfrak{p} \cong x(B)$; by Lemma 53.21 (last assertion) it is a domain, finite over $\mathcal{O}'$, so $K_0 = R_0[1/\ell]$ is its fraction field, a finite (separable, characteristic 0) extension of $K' = \mathcal{O}'[1/\ell]$, and K_0 is the residue field $\kappa(\mathfrak{p})$ of $B_\mathfrak{p}$, embedded in $\overline{\mathbb{Q}}_\ell$ by x. Let $T = \mathrm{Hom}_B(\mathfrak{p}/\mathfrak{p}^2, \overline{\mathbb{Q}}_\ell) = \mathrm{Hom}_{K_0}(\mathfrak{p}B_\mathfrak{p}/\mathfrak{p}^2B_\mathfrak{p}, \overline{\mathbb{Q}}_\ell)$, so that $\dim_{\overline{\mathbb{Q}}_\ell} T = \dim_{K_0} \mathfrak{p}B_\mathfrak{p}/\mathfrak{p}^2B_\mathfrak{p}$. (a) Every $\phi \in T$ is the restriction to $\mathfrak{p}$ of a unique $\mathcal{O}'$-linear $\delta\colon B \to \overline{\mathbb{Q}}_\ell$ with $x + \epsilon\delta$ a ring homomorphism. Existence: let $C = B/\mathfrak{p}^2$ and let C_ϕ be the push-out of the square-zero extension $0 \to \mathfrak{p}/\mathfrak{p}^2 \to C \to R_0 \to 0$ along ϕ, a square-zero extension of R_0 by the R_0-module $\overline{\mathbb{Q}}_\ell$. After inverting ℓ it is a square-zero extension of K_0, finite separable over K', hence formally étale, so it has a K'-algebra section s; then $e \mapsto (\bar{e}, e - s(\bar{e}))$ identifies $C_\phi[1/\ell]$ with $K_0 \oplus \overline{\mathbb{Q}}_\ell\epsilon$, and composing $B \to C \to C_\phi \to C_\phi[1/\ell]$ with x on K_0 gives $x' = x + \epsilon\delta\colon B \to \overline{\mathbb{Q}}_\ell[\epsilon]$ with $\delta|_\mathfrak{p} = \phi$. Uniqueness: the difference of two such δ vanishes on $\mathfrak{p}$, so it is an $\mathcal{O}'$-derivation $R_0 \to \overline{\mathbb{Q}}_\ell$; it extends to a K'-derivation of K_0, which is 0 as K_0/K' is finite separable. (b) For $\phi \in T$ with $x' = x + \epsilon\delta$ as in (a), $v \in S_\ell$ and $1 \le a \le 4$, $x'(\psi^{\mathrm{d}}_{v,a}(g)) = x(\psi^{\mathrm{d}}_{v,a}(g))(1 + \epsilon h_{v,a}(g))$ defines a homomorphism $h_{v,a}\colon D_v \to \overline{\mathbb{Q}}_\ell$, continuous by Lemma 53.21 (last assertion) and continuity of $\psi^{\mathrm{d}}_{v,a}$. It factors through D_v^{ab}, in which the image of I_v is the image of $\mathcal{O}^\times_{L_v^+}$ under the local Artin map; continuous homomorphisms $\mathcal{O}^\times_{L_v^+} \to \overline{\mathbb{Q}}_\ell$ form a space of dimension d_v ($\mathcal{O}^\times_{L_v^+}$ is a finite group times $\mathbb{Z}_\ell^{d_v}$). So $\phi \mapsto (h_{v,a}|_{I_v})_{v,a}$ is a linear map Φ from T to a space of dimension $\sum_v 4d_v = 4d$. (c) Φ is injective: if $\Phi(\phi) = 0$ then $x'(\psi^{\mathrm{d}}_{v,a}(g)) = x(\psi^{\mathrm{d}}_{v,a}(g))$ for $g \in I_v$, and Lemma 53.21 applies with $n = 4$ ($\kappa(s)\kappa(csc) = \chi_{\mathrm{cyc}}(s)^{1-4}$; $r|_{G_{L(\mu_{\ell^\infty})}}$ irreducible by Step 1; the integers $m_{v,a}$ and k of the hypothesis): $x' = x$, so $\delta = 0$ and $\phi = 0$. Hence $\dim_{K_0} \mathfrak{p}B_\mathfrak{p}/\mathfrak{p}^2B_\mathfrak{p} \le 4d$, and $\dim B_\mathfrak{p} \le 4d$ (the dimension of a Noetherian local ring is at most its embedding dimension).

Step 3: $\dim(B/J)_\mathfrak{p} \ge 4d$. Let $A = B/J \ne 0$ (as $J \subseteq \mathfrak{p}$) and $\lambda\colon \Lambda \to A$ as in (Hf2): A is a finite torsion-free Λ-module, so λ is injective. Let $\mathfrak{p}_A = \mathfrak{p}/J$, $\mathfrak{p}_0 = \lambda^{-1}(\mathfrak{p}_A)$, and $\mathfrak{q}_A \subseteq \mathfrak{p}_A$ a minimal prime of A. Its elements are zero-divisors of A (A Noetherian), so $\lambda^{-1}(\mathfrak{q}_A) = 0$ by torsion-freeness, and $\Lambda \subseteq A/\mathfrak{q}_A$ is a finite extension of domains with Λ regular, hence normal. By going-down ([166, Theorem 9.4(ii)]) every chain of primes descending from $\mathfrak{p}_0$ lifts to one descending from $\mathfrak{p}_A/\mathfrak{q}_A$, so $\dim A_{\mathfrak{p}_A} \ge \mathrm{ht}\,\mathfrak{p}_0$. $\Lambda/\mathfrak{p}_0$ embeds in $A/\mathfrak{p}_A \cong R_0$, finite over $\mathcal{O}'$ and containing $\mathcal{O}'$, so $\dim \Lambda/\mathfrak{p}_0 = 1$; Λ is a regular local ring of dimension $4d + 1$, hence catenary with $\mathrm{ht}\,\mathfrak{p}_0 + \dim \Lambda/\mathfrak{p}_0 = 4d + 1$ ([166, Theorem 17.4]). So $\dim(B/J)_\mathfrak{p} = \dim A_{\mathfrak{p}_A} \ge 4d$.

Step 4: conclusion. $(B/J)_\mathfrak{p} = B_\mathfrak{p}/JB_\mathfrak{p}$, so $4d \le \dim B_\mathfrak{p}/JB_\mathfrak{p} \le \dim B_\mathfrak{p} \le 4d$ by Steps 2 and 3. Thus $B_\mathfrak{p}$ has dimension equal to its embedding dimension: it is regular, hence a domain ([166, Theorem 14.3]). If $JB_\mathfrak{p} \ne 0$, it contains a non-zero element f, a non-zero-divisor, and $\dim B_\mathfrak{p}/JB_\mathfrak{p} \le \dim B_\mathfrak{p}/fB_\mathfrak{p} = \dim B_\mathfrak{p} - 1$ ([166, Theorem 13.6]), a contradiction. So $JB_\mathfrak{p} = 0$: for $j \in J$ there is $s \in B \smallsetminus \mathfrak{p}$ with $sj = 0$. If $\mathfrak{q} \subseteq \mathfrak{p}$ is prime, then $s \notin \mathfrak{q}$ and $sj = 0 \in \mathfrak{q}$, so $j \in \mathfrak{q}$. Hence $J \subseteq \mathfrak{q}$. □

Lemma 53.21. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}'$, with CM field L and complex conjugation c, let $(B, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ be a universal polarised ordinary determinant of*

rank n over $\mathscr{H}$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and $\kappa\colon G_L \to \overline{\mathbb{Q}}_\ell^\times$ a continuous character with $\kappa(s)\kappa(csc) = \chi_{\mathrm{cyc}}(s)^{1-n}$ for all $s \in G_L$. Let $x\colon B \to \overline{\mathbb{Q}}_\ell$ be a classical ordinary point for ι and κ, with $\tilde{r}$ as in its definition, and put $r = \tilde{r} \circ \pi_{\mathscr{H}}|_{G_L}$. Assume that the restriction of r to $G_{L(\mu_{\ell^\infty})}$ is irreducible on $\overline{\mathbb{Q}}_\ell^n$, and that there are integers $m_{v,a}$ ($v \in S_\ell$, $1 \le a \le n$) with $m_{v,a} \ge m_{v,b}+2$ for $a < b$ and $k \ge 1$ with $\left(x(\psi^{\mathrm{d}}_{v,a}(g))\chi_{\mathrm{cyc}}(g)^{-m_{v,a}}\right)^k = 1$ for all $v \in S_\ell$, $1 \le a \le n$ and $g \in I_v$. Then every homomorphism of $\mathcal{O}'$-algebras $x'\colon B \to \overline{\mathbb{Q}}_\ell[\epsilon]$ with $x'(b) \equiv x(b)$ modulo ϵ for all $b \in B$ and

$$x'(\psi^{\mathrm{d}}_{v,a}(g)) = x(\psi^{\mathrm{d}}_{v,a}(g)) \qquad (v \in S_\ell,\ 1 \le a \le n,\ g \in I_v)$$

is equal to x. Moreover (without the assumptions on r, $m_{v,a}$ and k) $B/\ker x$ is a finite $\mathcal{O}'$-module, and every homomorphism of $\mathcal{O}'$-algebras $x'\colon B \to \overline{\mathbb{Q}}_\ell[\epsilon]$ with $x' \equiv x$ modulo ϵ is continuous for the $\mathfrak{m}_B$-adic topology of B and the ℓ-adic topology of $\overline{\mathbb{Q}}_\ell[\epsilon] \cong \overline{\mathbb{Q}}_\ell^2$.

Proof. Symbols introduced here are local to this proof. Write $\pi = \pi_{\mathscr{H}}$, $\Gamma^0 = \Gamma^0_{\mathscr{H}}$, $A_\epsilon = \overline{\mathbb{Q}}_\ell[\epsilon]$, $\psi_{v,a} = x \circ \psi^{\mathrm{d}}_{v,a}$, $\psi'_{v,a} = x' \circ \psi^{\mathrm{d}}_{v,a}$, $D' = x' \circ D^{\mathrm{u}}$, and $x' = x + \epsilon\delta$, so that $\delta\colon B \to \overline{\mathbb{Q}}_\ell$ is $\mathcal{O}'$-linear with $\delta(bb') = x(b)\delta(b') + x(b')\delta(b)$.

Step 0: an element with distinct weights. For $v \in S_\ell$ write $\psi_{v,a}|_{I_v} = \chi_{\mathrm{cyc}}^{m_{v,a}}\eta_{v,a}$ with $\eta_{v,a}^k = 1$. The subgroup $I'_v = \bigcap_a \ker \eta_{v,a}$ is open in I_v (each $\eta_{v,a}$ is continuous of finite order), and $\chi_{\mathrm{cyc}}(I_v)$ is open in $\mathbb{Z}_\ell^\times$, so $\chi_{\mathrm{cyc}}(I'_v)$ is an open subgroup of $\mathbb{Z}_\ell^\times$, which is infinite while $\mathbb{Z}_\ell^\times$ has only finitely many roots of unity; choose $g_v \in I'_v$ with $\chi_{\mathrm{cyc}}(g_v)$ not a root of unity. Then $c_{v,a} = \psi_{v,a}(g_v) = \chi_{\mathrm{cyc}}(g_v)^{m_{v,a}}$ are pairwise distinct, and for $a < b$ the character $\psi_{v,a}\psi_{v,b}^{-1}\chi_{\mathrm{cyc}}^{-1}$ takes at g_v the value $\chi_{\mathrm{cyc}}(g_v)^{m_{v,a}-m_{v,b}-1} \ne 1$ (as $m_{v,a} - m_{v,b} - 1 \ge 1$); so $\psi_{v,a} \ne \psi_{v,b}\chi_{\mathrm{cyc}}$ on D_v.

Step 1: x' is continuous. Let $\mathfrak{p} = \ker x$. The domain $B/\mathfrak{p} \cong x(B) \subseteq \overline{\mathbb{Q}}_\ell$ is a complete Noetherian local $\mathcal{O}'$-algebra; by Cohen's structure theorem it is finite over a subring $\mathcal{O}'[[Y_1, \dots, Y_e]]$, and $e = 0$, since otherwise $x(Y_1)$, algebraic over the fraction field of $\mathcal{O}'$, would satisfy $Q(Y_1) = 0$ in $B/\mathfrak{p}$ for a non-zero $Q \in \mathcal{O}'[Y]$, impossible in $\mathcal{O}'[[Y_1, \dots, Y_e]]$. So $B/\mathfrak{p}$ is a finite $\mathcal{O}'$-module, hence so is $C = B/\mathfrak{p}^2$ ($\mathfrak{p}/\mathfrak{p}^2$ is finitely generated over $B/\mathfrak{p}$). As x' vanishes on $\mathfrak{p}^2$ it factors through C; $C/\ell C$ is a finite local ring, so $\mathfrak{m}_B^N$ maps into ℓC for some N, and an $\mathcal{O}'$-linear map from the finite $\mathcal{O}'$-module C to $A_\epsilon \cong \overline{\mathbb{Q}}_\ell^2$ is ℓ-adically continuous. So x' is continuous for the $\mathfrak{m}_B$-adic topology. This step uses only that x' lifts x, and proves the last assertion of the statement.

Step 2: a representation lifting $\tilde{r}$. D' is an A_ϵ-valued determinant of dimension n on Γ^0 (base change of D^{u}), and its reduction modulo ϵ is $x \circ D^{\mathrm{u}} = \det \circ \tilde{r}$ (Cy1), with $\tilde{r}$ absolutely irreducible, so $\tilde{r}\colon \overline{\mathbb{Q}}_\ell[\Gamma^0] \to \mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ is surjective (Burnside). Let $R = A_\epsilon[\Gamma^0]/\ker(D')$ with the determinant $\tilde{D}$ induced by D' (Theorem B.88, (Ck1)). The characteristic polynomials of $\tilde{D}$ are those of D', so $\mathrm{CH}(\tilde{D})$ is the image of $\mathrm{CH}(D') \subseteq \ker(D')$ ((Ck2)), and $\tilde{D}$ is Cayley–Hamilton. A_ϵ is Artinian local, hence henselian, with residue field $\overline{\mathbb{Q}}_\ell$, and the residual determinant is that of the surjection $R \otimes \overline{\mathbb{Q}}_\ell \to \mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ induced by $\tilde{r}$ (its kernel contains the image of $\ker(D')$, by (Ck1) applied to $\det \circ \tilde{r}$). By Theorem B.89 there is an isomorphism of A_ϵ-algebras $\rho\colon R \to \mathrm{M}_n(A_\epsilon)$ with $\tilde{D} = \det \circ \rho$. Let $r''\colon \Gamma^0 \to \mathrm{GL}_n(A_\epsilon)$ be $\gamma \mapsto \rho(\gamma)$. Its reduction modulo ϵ is a surjective, hence irreducible, representation with determinant $\det \circ \tilde{r}$, so it is conjugate to $\tilde{r}$ (Theorem B.83, (Ch3)); conjugating ρ by a lift of the conjugating matrix we may assume $r'' \equiv \tilde{r}$ modulo ϵ. Continuity: choose $\gamma_1, \dots, \gamma_{n^2} \in \Gamma^0$ with $\tilde{r}(\gamma_i)$ a basis of $\mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$; by Nakayama the $r''(\gamma_i)$ form an A_ϵ-basis of $\mathrm{M}_n(A_\epsilon)$, and the Gram matrix $G = (\mathrm{tr}(r''(\gamma_i)r''(\gamma_j)))_{i,j}$ is invertible (its reduction is the Gram matrix of the non-degenerate trace form). Writing $r''(\gamma) = \sum_j a_j(\gamma) r''(\gamma_j)$, the vector $(a_j(\gamma))_j$ is G^{-1} applied to $(\mathrm{tr}(r''(\gamma\gamma_i)))_i = (x'(\mathrm{tr}_{D^{\mathrm{u}}}(\gamma\gamma_i)))_i$, which is continuous in γ by continuity of D^{u} and Step 1. So r'' is continuous.

Step 3: the tangent cocycle. Write $r''(\pi(g)) = (1 + \epsilon z(g))r(g)$ for $g \in G_L$. From $r''(\pi(gh)) = r''(\pi(g))r''(\pi(h))$, $z(gh) = z(g) + r(g)z(h)r(g)^{-1}$: $z\colon G_L \to \mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ is a continuous 1-cocycle for $\mathrm{Ad}\, r$, which is also $\mathrm{Ad}\, r'$ for $r' = r \otimes \kappa$.

Claim (traces determine the class). Let $\sigma\colon G \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be an absolutely irreducible representation of a group G and $u\colon G \to \mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ a 1-cocycle for $\mathrm{Ad}\,\sigma$ with $\mathrm{tr}(u(g)\sigma(g)) = 0$ for all g. Then there is $Y \in \mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ with $u(g) = Y - \sigma(g)Y\sigma(g)^{-1}$. Indeed $w(g) = u(g)\sigma(g)$ satisfies $w(gh) = w(g)\sigma(h) + \sigma(g)w(h)$; extend w and σ linearly to $\overline{\mathbb{Q}}_\ell[G]$, so that $w(ab) = w(a)\sigma(b) + \sigma(a)w(b)$ and $\mathrm{tr}\, w = 0$. If $\sigma(a) = 0$ then $0 = \mathrm{tr}\, w(ab) = \mathrm{tr}(w(a)\sigma(b))$ for all b, and $\sigma(\overline{\mathbb{Q}}_\ell[G]) = \mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ (Burnside) with the trace form non-degenerate gives $w(a) = 0$. So $w = \Delta\circ\sigma$ for a linear $\Delta\colon \mathrm{M}_n \to \mathrm{M}_n$ with $\Delta(XX') = \Delta(X)X' + X\Delta(X')$; every derivation of $\mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ is inner, $\Delta(X) = YX - XY$, and $u(g) = w(g)\sigma(g)^{-1} = Y - \sigma(g)Y\sigma(g)^{-1}$.

Step 4: the polarisation of the lift. By (Od1) for D' (the base change of an identity of characteristic polynomials), the continuous representations $\rho_1(g) = r''(\pi(cgc))$ and $\rho_2(g) = {}^t r''(\pi(g))^{-1}$ of G_L have equal characteristic polynomials: that of $\rho_1(g)$ is $\chi_{D'}(\pi(c\,g\,c)) = \chi_{D'}(\pi(g^{-1}))$, that of $\rho_2(g)$. Modulo ϵ both are absolutely irreducible with the same determinant (Theorem B.83, (Ch1)), hence conjugate ((Ch3)): there is $Q \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ with $r(cgc) = Q\,{}^t r(g)^{-1}Q^{-1}$. Write $\rho_1 = (1+\epsilon u_1)\bar\rho_1$ and $Q\rho_2 Q^{-1} = (1+\epsilon u_2)\bar\rho_1$ with $\bar\rho_1(g) = r(cgc)$; equality of traces gives $\mathrm{tr}((u_1-u_2)(g)\bar\rho_1(g)) = 0$, so by the Claim $u_1 - u_2 = Y - \bar\rho_1 Y \bar\rho_1^{-1}$, and with $Q'' = (1+\epsilon Y)Q$,

$$r''(\pi(cgc)) = Q''\,{}^t r''(\pi(g))^{-1}\,Q''^{-1} \qquad (g \in G_L).$$

Put $r''' = (r''\circ\pi)\otimes\kappa$, a lift of r'. As $\kappa(cgc)\kappa(g) = \chi_{\mathrm{cyc}}(g)^{1-n}$, we have

$$r'''(cgc) = \chi_{\mathrm{cyc}}(g)^{1-n} Q''\,{}^t r'''(g)^{-1} Q''^{-1},$$

and modulo ϵ, $r'(cgc) = \chi_{\mathrm{cyc}}(g)^{1-n} Q\,{}^t r'(g)^{-1} Q^{-1}$. Let V be $\mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ with the action of $G_{L^+} = G_L \sqcup cG_L$ of Theorem B.72 for r' and $\mathbf{P} = Q$: $g\cdot X = r'(g)Xr'(g)^{-1}$ ($g \in G_L$), $c\cdot X = -Q\,{}^t X Q^{-1}$ (an action, as shown there; $Q\,{}^t Q^{-1} = \lambda$ is a scalar with $\lambda^2 = 1$). Comparing ϵ-parts of the relation for r''', with $r''' = (1+\epsilon z)r'$:

$$z(cgc) = Y - r'(cgc)Yr'(cgc)^{-1} - Q\,{}^t z(g) Q^{-1},$$

hence, with $W = c\cdot Y$,

$$c\cdot z(cgc) = z(g) + W - g\cdot W \qquad (g \in G_L).$$

So the class of z in $H^1(G_L, V)$ is fixed by c. As V is a $\mathbb{Q}$-vector space and $[G_{L^+} : G_L] = 2$, restriction $H^1(G_{L^+}, V) \to H^1(G_L, V)^c$ is an isomorphism (inflation–restriction; the cohomology of a group of order 2 with coefficients in a $\mathbb{Q}$-vector space vanishes in positive degree). Hence there is a continuous cocycle $Z\colon G_{L^+} \to V$ whose restriction to G_L is cohomologous to z; subtracting a coboundary we take $Z|_{G_L} = z$.

Step 5: local conditions away from ℓ. Let $\bar w$ be a place of $\overline{\mathbb{Q}}$ above a finite place $w \nmid \ell$ of L^+, $I = I_{\bar w} \cap G_{L^+}$ and $I_0 = \{s \in I \cap G_L : \bar\chi_a(s) = 1,\ 1 \le a \le n\}$, a normal subgroup of finite index of I (the $\bar\chi_a$ have finite image). $I_0 \subseteq N_{\mathscr{H}}$, so $r''\circ\pi$ and r are trivial on I_0, $z|_{I_0} = 0$, and I_0 acts trivially on V (the action of $g \in G_L$ is $\mathrm{Ad}\, r(g)$); so $Z|_I$ is inflated from a cocycle of the finite group I/I_0 with values in a $\overline{\mathbb{Q}}_\ell$-vector space, hence a coboundary (average over I/I_0). So $Z|_D$ lies in H^1_f for every decomposition group D at w. At archimedean places $H^1_f = 0$ and H^1 of a group of order at most 2 with coefficients in V vanishes.

Step 6: local conditions at ℓ. Let $v \in S_\ell$; $D_v = D_{\bar v} \cap G_{L^+} \subseteq G_L$ (Rd2) is a decomposition group at v, identified with $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/L_v^+)$, with inertia I_v. By (Od2) for D' and Step 2, $\mathrm{tr}\big(\rho(\gamma)\prod_{a=1}^n (r''(g_a) - \psi'_{v,a}(g_a))\big) = 0$ for all $\gamma \in \Gamma^0$, $g_a \in D_v$; the $\rho(\gamma)$ span $\mathrm{M}_n(A_\epsilon)$ and the trace form is non-degenerate, so $\prod_{a=1}^n (r''(g_a) - \psi'_{v,a}(g_a)) = 0$ (product with $a = 1$ on the left; $r''(g)$ stands for $r''(\pi(g))$). With all $g_a = g_v$ of Step 0 and $\psi'_{v,a}(g_v) = \psi_{v,a}(g_v) = c_{v,a}$ (hypothesis on

x'), $\prod_a (r''(g_v) - c_{v,a}) = 0$ with distinct $c_{v,a} \in \overline{\mathbb{Q}}_\ell$. Hence $E_a = \prod_{b \neq a} (r''(g_v) - c_{v,b})/(c_{v,a} - c_{v,b})$ are orthogonal idempotents with $\sum_a E_a = 1$ and $r''(g_v)E_a = c_{v,a}E_a$ (Lagrange interpolation). Each $E_a A_\epsilon^n$ is a direct summand, hence free, of rank 1: modulo ϵ it is the $c_{v,a}$-eigenline of $r(g_v)$, whose characteristic polynomial is $\prod_a (X - c_{v,a})$ by (Cy3). Choose generators e_a and let $M \in \mathrm{GL}_n(A_\epsilon)$ have columns $e_1, \dots, e_n$. For $1 \leq j \leq i \leq n$ and $g \in D_v$, apply the identity with $g_a = g_v$ for $a \neq i$ and $g_i = g$ to e_j: the factors with $a > i$ multiply e_j by the unit $\prod_{a>i}(c_{v,j} - c_{v,a})$, and the factors with $a < i$ act invertibly on $\bigoplus_{b \geq i} E_b A_\epsilon^n$ and kill $E_b A_\epsilon^n$ for $b < i$. So the components of $(r''(g) - \psi'_{v,i}(g))e_j$ in $E_b A_\epsilon^n$, $b \geq i$, vanish. For $j < i$ this says that $r''(g)e_j \in \bigoplus_{b \leq j} E_b A_\epsilon^n$; for $j = i$ that the e_i-coefficient of $r''(g)e_i$ is $\psi'_{v,i}(g)$. Thus $T''(g) = M^{-1} r''(g) M$ is upper triangular with diagonal $(\psi'_{v,1}(g), \dots, \psi'_{v,n}(g))$ for $g \in D_v$. Write $M = M_0(1 + \epsilon M_1)$ and $T'' = T + \epsilon T_1$; then $T(g) = M_0^{-1} r(g) M_0$ is upper triangular with diagonal $(\psi_{v,a}(g))_a$ and

$$z_T(g) := T_1(g)T(g)^{-1} = M_0^{-1} z(g) M_0 + \big(T(g) M_1 T(g)^{-1} - M_1\big) \qquad (g \in D_v),$$

a cocycle of $\mathrm{Ad}\, T$ cohomologous to $M_0^{-1} z M_0$, with upper triangular values; its diagonal entries $T_1(g)_{aa}/\psi_{v,a}(g)$ are the ϵ-parts of $\psi'_{v,a}(g)/\psi_{v,a}(g)$, which vanish on I_v by the hypothesis on x'. By Lemma 42.8, items 1 and 3, $T|_{D_v}$ and z_T take values in $\mathrm{M}_n(E)$ for a finite extension E of $\mathbb{Q}_\ell$. The hypotheses of Lemma 44.14 hold for $M = L_v^+$, $\rho = T|_{D_v}$, the integers $m_{v,1} > \cdots > m_{v,n}$ and $u = z_T$ (Step 0 and the hypothesis on $\psi_{v,a}\chi_{\mathrm{cyc}}^{-m_{v,a}}$). So the class of z_T, and hence, transporting by $X \mapsto M_0 X M_0^{-1}$, that of $z|_{D_v} = Z|_{D_v}$, lies in H^1_f; the twist by κ does not change Ad. Every other decomposition group at v is hD_vh^{-1} with $h \in G_{L^+}$, and $Z(hyh^{-1}) = h \cdot Z(y) + (1 - hyh^{-1}) \cdot Z(h)$ for $y \in D_v$, so its class is the transport of that at D_v by the isomorphism $h\cdot$, and lies in H^1_f. By Steps 5 and 6, Z is a Bloch–Kato cocycle for V over L^+.

Step 7: vanishing. By Lemma 42.8, item 1, choose a finite extension E of $\mathbb{Q}_\ell$ containing the entries of $r'(G_L)$ and of Q. The restriction of $r' = r \otimes \kappa$ to $G_{L(\mu_{\ell^\infty})}$ is irreducible (twisting by a character keeps invariant subspaces), $r' \leftrightarrow_\iota \Pi$ for the regular algebraic conjugate self-dual cuspidal Π of (Cy2), $n \geq 2$, and Q satisfies the relation of Theorem B.72 for $F = L$, $F^+ = L^+$; $V = \mathrm{ad}_E\, r' \otimes_E \overline{\mathbb{Q}}_\ell$. By Theorem B.72, $H^1_f(L^+, V) = 0$, so $Z(g) = g \cdot X - X$ for some $X \in V$, and for $g \in G_L$, $z(g) = r(g) X r(g)^{-1} - X$. Then $r''(\pi(g)) = (1 + \epsilon z(g)) r(g) = (1 - \epsilon X) r(g) (1 + \epsilon X)$ for $g \in G_L$, and as $\pi(G_L) = \Gamma^0$, $r'' = (1 - \epsilon X)\tilde{r}(1 + \epsilon X)$ on Γ^0.

Step 8: conclusion. First, $D' = \det \circ r'' = \det \circ \tilde{r} = x \circ D^{\mathrm{u}}$ (equal characteristic polynomials, (Ch1)). Second, for $v \in S_\ell$ the idempotents E_a of Step 6 are polynomials in $r''(g_v)$, so $E_a = (1 - \epsilon X) E_a^0 (1 + \epsilon X)$ with E_a^0 the same polynomials in $r(g_v)$, and by Step 6 (the case $j = i$, which also holds for x in place of x') $E_a r''(g) E_a = \psi'_{v,a}(g) E_a$ and $E_a^0 r(g) E_a^0 = \psi_{v,a}(g) E_a^0$ for $g \in D_v$; conjugating the second by $1 - \epsilon X$ gives $E_a r''(g) E_a = \psi_{v,a}(g) E_a$, and $E_a \neq 0$, so $\psi'_{v,a} = \psi_{v,a}$ on D_v. Let $A \subseteq A_\epsilon$ be the $\mathcal{O}'$-subalgebra generated by $x(B) \cup x'(B)$. It is a finite $\mathcal{O}'$-module (Step 1), with no idempotents other than $0, 1$ (as A_ϵ is local), hence a complete local Noetherian $\mathcal{O}'$-algebra; its residue field is generated by the images of $x(B)$ and $x'(B)$, both $\mathbb{F}_{\mathcal{O}'}$ (an $\mathcal{O}'$-algebra map $B \to$ a field has kernel the maximal ideal $\mathfrak{m}_B$), so $A \in \mathrm{CNL}_{\mathcal{O}'}$, and $x, x' \colon B \to A$ are morphisms of $\mathrm{CNL}_{\mathcal{O}'}$. They classify the same polarised ordinary determinant $(x \circ D^{\mathrm{u}}, (\psi_{v,a})) = (D', (\psi'_{v,a}))$ (continuous by Step 1), so $x = x'$ by the universal property. □

Lemma 53.22. *In the setting of Lemma 53.7, assume moreover that $\mathscr{T}$ is good and that ν satisfies (Lk5). Let ϕ_0 be the morphism given there for $n = 0$ and $J_0 \subseteq B_4^{\mathrm{po}}$ an ideal with the purity property (Hf4). Then every ideal $\mathfrak{J} \subseteq B$ containing $\phi_0(J_0)$ satisfies (Jq4): for every $\mathcal{O}'$-algebra homomorphism $y \colon A \to \overline{\mathbb{Q}}_\ell$ with $y(\mathfrak{J}) = 0$ such that $y(P)$ has weights $(2e+2, 2e, 2, 0)$, every homomorphism $r \colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ with $r|_{G_F} = y \circ r_P$ is weakly pure for ι.*

Proof. Symbols introduced here are local to this proof. Let y and r be as in the statement, $(m_1, m_2, m_3, m_4) = (e + 1, e, 1, 0)$, and $x = y \circ \phi_0 \colon B_4^{\mathrm{po}} \to \overline{\mathbb{Q}}_\ell$ (with $B \subseteq A$ by (Zq1)); then

$x(J_0) \subseteq y(\mathfrak{J}) = 0$. By (Cm1), (Cm2) and (Zq2), for $g \in G_L$ the characteristic polynomial $\chi_{x\circ D^{\mathrm{u}}}(\pi_{\mathscr{H}}(g))$ is that of $\lambda_0(g)\, y(r_P(g)) = \lambda_0(g) r(g)$, and $x(\psi^{\mathrm{d}}_{v,a}(\sigma)) = \lambda_0(\sigma)\, y(\psi^P_{v,a}(\sigma))$ for $\sigma \in D_v$.

Step 1: x is arithmetic in the sense of (Hf4). Fix $v \in \Sigma$ and identify D_v with G_{F_v} ($F_v = L_v^+$). By the weights, $y(\psi^P_{v,a})^2 = \chi_{\mathrm{cyc}}^{2m_a}$ on $U \cap I_v$ for an open U; so $y(\psi^P_{v,a})\chi_{\mathrm{cyc}}^{-m_a}$ is a continuous character of $U \cap I_v$ with values in $\{\pm 1\}$, trivial on an open subgroup. By (Lk5), $\nu\chi_{\mathrm{cyc}}^{-a_v}$ has finite order on I_v for some integer a_v, hence is trivial on an open subgroup. As κ corresponds to an algebraic Hecke character, there are integers $n_{v,\tau}$ with $\kappa(\mathrm{Art}_{F_v}(u)) = \prod_\tau \tau(u)^{n_{v,\tau}}$ for u in an open subgroup of $\mathcal{O}_{F_v}^\times$, τ running over the field embeddings $F_v \to \overline{\mathbb{Q}}_\ell$. Finally $\chi_{\mathrm{cyc}}(\mathrm{Art}_{F_v}(u)) = \mathrm{N}_{F_v/\mathbb{Q}_\ell}(u)^{-1} = \prod_\tau \tau(u)^{-1}$ for $u \in \mathcal{O}_{F_v}^\times$ (Lubin–Tate theory, for the arithmetic normalisation). Hence on an open subgroup of I_v, with σ the image of $\mathrm{Art}_{F_v}(u_\sigma)$,

$$x(\psi^{\mathrm{d}}_{v,a}(\sigma)) = \nu(\sigma)\kappa(\sigma)^{-1}\chi_{\mathrm{cyc}}(\sigma)^{m_a} = \prod_\tau \tau(u_\sigma)^{h_{v,a,\tau}}, \qquad h_{v,a,\tau} = -a_v - n_{v,\tau} - m_a,$$

and $h_{v,a,\tau}$ is strictly increasing in a because $m_1 > m_2 > m_3 > m_4$.

Step 2: the decomposition of (Hf4) is trivial. By (Hf4) for J_0 there are a continuous semisimple $\rho\colon G_L \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ with $\det\circ\rho = x \circ D^{\mathrm{u}} \circ \pi_{\mathscr{H}}|_{G_L}$ and ρ'_i, d_i, m'_i ($1 \le i \le k$) with $\chi_{\rho(g)} = \prod_i \prod_{j<m'_i} \chi_{\chi_{\mathrm{cyc}}(g)^{-j}\rho'_i(g)}$. Suppose $m'_i \ge 2$ for some i. All these representations are continuous on the compact group G_L, so their characteristic polynomials have integral coefficients and can be reduced; let σ_{ij} be the semisimplified reduction of $\chi_{\mathrm{cyc}}^{-j}\rho'_i$ over $\overline{\mathbb{F}}_\ell$. The reduction of $\chi_{\rho(g)} = \chi_{x\circ D^{\mathrm{u}}}(\pi_{\mathscr{H}}(g))$ is $\prod_a (X - \bar\chi_a(g))$ ($x\circ D^{\mathrm{u}}$ lifts $\bar D_4$), so by the Brauer–Nesbitt theorem $\bigoplus_{i,j}\sigma_{ij} \cong \bigoplus_a \bar\chi_a$. Choose a with $\bar\chi_a \subseteq \sigma_{i0}$; then $\bar\omega^{-1}\bar\chi_a \subseteq \sigma_{i1} = \bar\omega^{-1}\sigma_{i0}$ ($\bar\omega$ the reduction of χ_{cyc}), so $\bar\omega^{-1}\bar\chi_a = \bar\chi_b$ for some b on G_L. By (Lk6), $\bar\chi_a/\bar\chi_b = \bar\chi_a^{\mathscr{T}}/\bar\chi_b^{\mathscr{T}}$, which is $\overline{\mathrm{rt}_w}$ for some $w \in \mathbb{Z}^2$ (for $a = b$, $w = 0$; otherwise, as $\chi_a^{\mathscr{T}}\chi_{5-a}^{\mathscr{T}} = \theta$ for both labellings, the ratios are $\chi_1^{\mathscr{T}}/\chi_2^{\mathscr{T}} = \mathrm{rt}_{w_\alpha} = \chi_3^{\mathscr{T}}/\chi_4^{\mathscr{T}}$, $\chi_1^{\mathscr{T}}/\chi_3^{\mathscr{T}} = \mathrm{rt}_{w_\Psi} = \chi_2^{\mathscr{T}}/\chi_4^{\mathscr{T}}$, $\chi_2^{\mathscr{T}}/\chi_3^{\mathscr{T}} = \mathrm{rt}_{w_\beta}$, $\chi_1^{\mathscr{T}}/\chi_4^{\mathscr{T}} = \mathrm{rt}_{w_\theta}$ and their inverses); so $\overline{\mathrm{rt}_w} = \bar\omega$ on G_L, contradicting (Oq3). Hence all $m'_i = 1$, and by (Hf4) there is a finite set Z of finite places of L such that for $w \notin Z$, ρ is unramified at w and any two eigenvalues α, α' of $\rho(\mathrm{Frob}_w)$ satisfy $|\iota(\alpha)| < \mathrm{N}w\,|\iota(\alpha')|$.

Step 3: weak purity. Take $H = G_L$, open in $G_{\mathbb{Q}}$, and M the product of ℓ and the primes below the places in Z. Let $p \nmid M$, $\vec{p}'$ a place above p, $f \ge 1$, and $\sigma \in G_L$ an arithmetic Frobenius element for p^f at $\vec{p}'$; let w be the place of L below $\vec{p}'$, with residue degree f_w over $\mathbb{F}_p$. The image of $D_{\vec{p}'} \cap G_L$ in the Galois group of the residue field extension is the closure of the group generated by $x \mapsto x^{\mathrm{N}w}$, and contains $x \mapsto x^{p^f}$; so $f_w \mid f$, $p^f = \mathrm{N}w^k$ with $k = f/f_w \ge 1$, and $\sigma \in \mathrm{Frob}_w^k(I_{\vec{p}'} \cap G_L)$. As ρ is unramified at w, the eigenvalues of $\rho(\sigma)$ are the α^k, α an eigenvalue of $\rho(\mathrm{Frob}_w)$ (with multiplicity), and those of $r(\sigma)$ are $\lambda_0(\sigma)^{-1}\alpha^k$, since $\rho(\sigma)$ and $\lambda_0(\sigma)r(\sigma)$ have the same characteristic polynomial. For two of them, $z = \lambda_0(\sigma)^{-1}\alpha^k$ and $z' = \lambda_0(\sigma)^{-1}\alpha'^k$, Step 2 gives $|\iota(z)|/|\iota(z')| = (|\iota(\alpha)|/|\iota(\alpha')|)^k < \mathrm{N}w^k = p^f$, which is (45.1). □

Lemma 53.23. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum, $\mathscr{T}$ a fit and deep auxiliary tower for it, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $\mathcal{O}_1, \nu, \gamma, t$ as given by Lemma 53.2, $\mathcal{O}' \supseteq \mathcal{O}_1$ a coefficient ring, $\kappa\colon G_L \to \mathcal{O}'^\times$ a character with (Tw1) for $m = -3$, (Tw2), (Tw4) that is a twist for the datum $\mathscr{H}$ given there, $\mathscr{X} = (B, A, P, \dots)$ a space for $(\mathscr{T}, \mathcal{O}')$, a universal polarised ordinary determinant of rank 4 over $\mathscr{H}$ with ring B_4^{po}, the morphism $\phi_0\colon B_4^{\mathrm{po}} \to B$ of Lemma 53.7, and an ideal $J_0 \subseteq B_4^{\mathrm{po}}$ with the property (Hf3) (for ι and κ). Then every ideal $\mathfrak{J} \subseteq B$ containing $\phi_0(J_0)$ satisfies (Jq5) for ι.*

Proof. Symbols introduced here are local to this proof. Let $\mathfrak{J} \supseteq \phi_0(J_0)$, and let $y\colon A \to \overline{\mathbb{Q}}_\ell$ be an $\mathcal{O}'$-algebra homomorphism with $y(\mathfrak{J}) = 0$ such that $y(P)$ has weights $(2m_1, \dots, 2m_4) =$

$(2e+2, 2e, 2, 0)$, and $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ continuous with $r|_{G_F} = y \circ r_P$, $r(g)$ of similitude $\theta(g)$ for all g, and $r|_{G_L}$ irreducible. Put $r' = \nu\, r|_{G_L}$, continuous and irreducible.

Step 1: automorphy. $y(\phi_0(J_0)) \subseteq y(\mathfrak{J}) = 0$. The tower is fit, hence good, ν satisfies (Lk5), κ is a twist for $\mathscr{H}$, and J_0 has (Hf3); so Lemma 53.24 applies: r' is automorphic of conjugate self-dual type for ι.

Step 2: (Qw1) for r'. Let $w \nmid \ell$ be a finite place of L, $\bar{w} \mid w$, $D = D_{\bar{w}} \cap G_L \supseteq I = I_{\bar{w}} \cap G_L$, and $\bar{\chi} = (\bar{\chi}_1, \dots, \bar{\chi}_4)\colon G_L \to (\mathbb{F}^\times_{\mathcal{O}'})^4$. If $s \in I$ and $\bar{\chi}(s) = 1$: one of $\chi_1^{\mathscr{T}}, \dots, \chi_4^{\mathscr{T}}$ is trivial (the character $\mathbf{1}$ of the datum), so (Lk6) gives $\bar{\nu}\bar{\kappa}^{-1}(s) = 1$ and then $\bar{\chi}_a^{\mathscr{T}}(s) = 1$ for all a; so $\bar{r}_0(s) = 1$, $s \in N$, $r_P(s) = 1$ (Mq1) and $r(s) = 1$. Hence $r|_I = \varphi \circ \bar{\chi}|_I$ for a homomorphism φ from the finite abelian group $\bar{\chi}(I)$ to $\mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$, which is diagonalisable: $r|_I$ is conjugate to $\mathrm{diag}(\varphi_i \circ \bar{\chi})$ for characters φ_i of $\bar{\chi}(I)$. Extend each φ_i to a character φ_i' of the finite abelian group $\bar{\chi}(D) \supseteq \bar{\chi}(I)$. By (Lk3), ν^m is trivial on I for some $m \geq 1$, so $\nu(I)$ is finite. Thus $r'(I)$ is finite and $r'|_I$ is conjugate to $s \mapsto \mathrm{diag}(\Psi_i(s))$ with the characters $\Psi_i = (\varphi_i' \circ \bar{\chi})\, \nu$ of D.

Step 3: (Qw2) for r'. Let $\bar{w}$ be a place of $\overline{\mathbb{Q}}$ above ℓ. It is $g\bar{v}$ for some $v \in \Sigma$ and $g \in G_F$, and $D_v \subseteq G_L$ (v splits in L), so $D_{\bar{w}} \cap G_L = g D_v g^{-1}$ and $I_{\bar{w}} \cap G_L = g I_v g^{-1}$ (G_L is normal in G_F). By (Mq3), (Mq4), $y(n_v)^{-1} r(s) y(n_v)$ is upper triangular with diagonal $(y(\psi^P_{v,a}(s)))_a$ for $s \in D_v$. For $t = gsg^{-1}$, $r'(t) = \nu(t)\, r(g) r(s) r(g)^{-1}$; so r' is triangular on gD_vg^{-1}, conjugated by $r(g)y(n_v)$, with diagonal $(\kappa_a)_a$, $\kappa_a(t) = \nu(t)\, y(\psi^P_{v,a}(g^{-1}tg))$. By the weights there is an open subgroup U of G_F with $y(\psi^P_{v,a}(s))^2 = \chi_{\mathrm{cyc}}(s)^{2m_a}$ for $s \in U \cap I_v$; hence $\epsilon_{ab} = y(\psi^P_{v,a}) y(\psi^P_{v,b})^{-1} \chi_{\mathrm{cyc}}^{-(m_a - m_b)}$ is a continuous character of $U \cap I_v$ with values in $\{\pm 1\}$, so trivial on an open subgroup U_{ab}. As $\chi_{\mathrm{cyc}}(g^{-1}tg) = \chi_{\mathrm{cyc}}(t)$, the character $\kappa_a \kappa_b^{-1} \chi_{\mathrm{cyc}}^{-(m_a - m_b)}$ is $t \mapsto \epsilon_{ab}(g^{-1}tg)$, trivial on the open subgroup $gU_{ab}g^{-1}$ of $I_{\bar{w}} \cap G_L$. This is (Qw2) with $h_{ab} = m_a - m_b$.

Step 4: witness conditions and descent. L is a CM field, r' is irreducible, continuous, automorphic of conjugate self-dual type (Step 1) and satisfies (Qw1), (Qw2) (Steps 2, 3); by Lemma 53.25, $\nu\, r|_{G_L}$ satisfies the witness conditions. By Lemma 53.26 (its hypotheses are those on y and r above), r satisfies (Sy2) for ι. So $\mathfrak{J}$ satisfies (Jq5). □

Lemma 53.24. *In the setting of Lemma 53.7, assume moreover that $\mathscr{T}$ is good, that ν satisfies (Lk5) and that κ is a twist for $\mathscr{H}$; let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism and $J_0 \subseteq B_4^{\mathrm{po}}$ an ideal with the property (Hf3) for ι and κ. Let $y\colon A \to \overline{\mathbb{Q}}_\ell$ be an $\mathcal{O}'$-algebra homomorphism with $y(\phi_0(J_0)) = 0$ such that $y(P)$ has weights $(2e+2, 2e, 2, 0)$, and $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ a continuous homomorphism with $r|_{G_F} = y \circ r_P$ and $r|_{G_L}$ irreducible. Then $r|_{G_L} \otimes \nu$ is automorphic of conjugate self-dual type for ι.*

Proof. Symbols introduced here are local to this proof. Put $x = y \circ \phi_0\colon B_4^{\mathrm{po}} \to \overline{\mathbb{Q}}_\ell$ and $(m_1, \dots, m_4) = (e+1, e, 1, 0)$; $x(J_0) = 0$.

Steps 1–2: integrality and the representation. The map $g \mapsto y(r_P(g)) = r(g)$ is continuous on G_L. By Lemma 53.19 with $n = 0$ there are a finite extension E of $\mathbb{Q}_\ell$ containing $\mathcal{O}'$, whose ring of integers A contains $x(B_4^{\mathrm{po}})$, $u \in \mathrm{GL}_4(E)$ and a continuous homomorphism $\rho\colon \Gamma^0_{\mathscr{H}} \to \mathrm{GL}_4(A)$ with $\rho(\pi_{\mathscr{H}}(g)) = u\, \lambda_0(g)\, r(g)\, u^{-1}$ for $g \in G_L$ and $x \circ D^{\mathrm{u}} = \det \circ \rho$. A is an integral test ring over $\mathcal{O}'$, and x is a homomorphism $B_4^{\mathrm{po}} \to A$. Let $\rho_L = \rho \circ \pi_{\mathscr{H}}|_{G_L}$, continuous and isomorphic to $\lambda_0\, r|_{G_L}$, hence irreducible over $\overline{\mathbb{Q}}_\ell$.

Step 3: the inertial weight. Put $\psi^\circ_{v,a} = x \circ \psi^{\mathrm{d}}_{v,a}|_{I_v} = \lambda_0\, y(\psi^P_{v,a})|_{I_v}$ (Cm2), with values in $A^\times$. Its extension $\phi_{v,a} = \lambda_0\, y(\psi^P_{v,a})$ to D_v is a character (the diagonal of the triangular $n_v^{-1} r(s) n_v$, (Mq4)), congruent to $\bar{\chi}_a$ by (Lk6). On an open subgroup of I_v: $y(\psi^P_{v,a}) = \chi_{\mathrm{cyc}}^{m_a}$ (the weights give $y(\psi^P_{v,a})^2 = \chi_{\mathrm{cyc}}^{2m_a}$, and a continuous character with values in $\{\pm 1\}$ has open kernel), $\nu = \chi_{\mathrm{cyc}}^{a_v}$ for an integer a_v (Lk5), $\kappa(\mathrm{Art}(w)) = \prod_\tau \tau(w)^{n_{v,\tau}}$ for integers $n_{v,\tau}$ (κ is algebraic), and $\chi_{\mathrm{cyc}}(\mathrm{Art}(w)) = \prod_\tau \tau(w)^{-1}$ (Lubin–Tate, arithmetic normalisation), w in the

units of L_v^+ and τ over the field embeddings $L_v^+ \to \overline{\mathbb{Q}}_\ell$; so $\phi_{v,a}(\mathrm{Art}(w)) = \prod_\tau \tau(w)^{h_{v,a,\tau}}$ with $h_{v,a,\tau} = -a_v - n_{v,\tau} - m_a$; so $\phi_{v,a}$ is locally algebraic, hence de Rham, with τ-Hodge–Tate weight $-h_{v,a,\tau} = a_v + n_{v,\tau} + m_a$ (the character $u \mapsto \tau(u)^{-1}$ on inertia being that of weight 1), strictly decreasing in a. Thus $(\psi_{v,a}^\circ)$ is an inertial weight for $\mathscr{H}$ (Rd8), and $\psi_{v,a}^\circ(\psi_{v,b}^\circ)^{-1}\chi_{\mathrm{cyc}}^{-(m_a-m_b)}$ is trivial on an open subgroup of I_v.

Step 4: conclusion. All hypotheses of (Hf3) hold for J_0, $(\psi_{v,a}^\circ)$, A, ρ and $x\colon B_4^{\mathrm{po}} \to A$ ($x(J_0) = 0$, $x(\psi_{v,a}^{\mathrm{d}}(g)) = \psi_{v,a}^\circ(g)$). Hence $\rho_L \otimes \kappa$ is automorphic of conjugate self-dual type for ι, and so is the isomorphic representation $\lambda_0 \kappa\, r|_{G_L} = \nu\, r|_{G_L}$. □

Lemma 53.25. *Let F be a CM field, $n \geq 2$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and $r\colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ an irreducible continuous representation which is automorphic of conjugate self-dual type for ι. Assume:*

(Qw1) *for every finite place $v \nmid \ell$ of F and every place $\bar v \mid v$, $r(I_{\bar v})$ is finite and $r|_{I_{\bar v}}$ is conjugate to $s \mapsto \mathrm{diag}(\Psi_1(s), \dots, \Psi_n(s))$ for some characters $\Psi_1, \dots, \Psi_n$ of $D_{\bar v}$;*

(Qw2) *for every place $v \mid \ell$ of F and every place $\bar v \mid v$, r is triangular on $D_{\bar v}$ with some diagonal $(\kappa_1, \dots, \kappa_n)$ such that for all a, b there is an integer h_{ab} with $\kappa_a \kappa_b^{-1} \chi_{\mathrm{cyc}}^{-h_{ab}}$ trivial on an open subgroup of $I_{\bar v}$.*

Then r satisfies the witness conditions, with $\tau_\pi = 0$ in (Wc2).

Proof. Step 0: the automorphic representation. Let Π be regular algebraic conjugate self-dual cuspidal with $r \leftrightarrow_\iota \Pi$. By Theorem B.62 ($\Pi$ is polarizable with the trivial character), $r_{\ell,\iota}(\Pi)$ is semisimple and, by (Bg1), has the Frobenius characteristic polynomials of r at almost all places; as r is irreducible, $r \cong r_{\ell,\iota}(\Pi)$ (Chebotarev, Brauer–Nesbitt). The absolute value of the central character ω_Π is a continuous homomorphism $\mathbb{A}_F^\times / F^\times \to \mathbb{R}_{>0}$, trivial on the compact group of idele classes of norm one, hence $|\omega_\Pi| = |\cdot|_{\mathbb{A}}^s$ for a real s; from $\Pi^\vee \cong \Pi^c$, $\omega_\Pi^{-1} = \omega_\Pi \circ c$, and as the idelic norm is c-invariant, $s = -s = 0$. So $|\omega_{\Pi_v}(\varpi_v)| = 1$ for every finite place v.

Step 1: a criterion for the window. Let v be a finite place and $\bar v \mid v$, $(V, \rho, N) = \mathrm{WD}(r|_{D_{\bar v}})$ (for $v \mid \ell$, $r|_{D_{\bar v}}$ is de Rham by (Bg2), hence potentially semistable). Suppose $N = 0$ and that $\rho|_{I_{\bar v}}$ has a semisimplification $\bigoplus_i \Psi_i'|_{I_{\bar v}}$ with characters Ψ_i' of $W_{\bar v}$ of finite order on $I_{\bar v}$. Then Π_v has a cuspidal support of characters $\xi_1, \dots, \xi_n$ of $F_v^\times$ with $|\xi_i(\varpi_v)| = 1$ for all i. Indeed, the $\Psi_i' \circ \mathrm{Art}_{F_v}$ are trivial on an open subgroup U of $\mathcal{O}_{F_v}^\times$, so (Bg3) gives characters ξ_i, trivial on U, with Π_v a subquotient of the normalised induction of $\xi_1 \otimes \cdots \otimes \xi_n$. By Theorem B.73 ($v \mid \ell$) or Theorem B.90 ($v \nmid \ell$), $(V, \rho, 0)$ is pure; by (Pu2), resp. (Pv2), the zero map is an isomorphism $\mathrm{gr}_{k+i} \to \mathrm{gr}_{k-i}$ for $i \geq 1$, so these spaces vanish, $V = \mathrm{gr}_k$, and every eigenvalue α of $\rho(\Phi)$, $n(\Phi) = -1$, has $|\iota(\alpha)| = q_v^{k/2}$. By (Bg5) the semisimplification of ρ is $\bigoplus_i \lambda_i$ with $\iota\lambda_i(\mathrm{Art}_{F_v}(\varpi_v)^{-1}) = \xi_i(\varpi_v) q_v^{(n-1)/2}$, and $\lambda_i(\mathrm{Art}_{F_v}(\varpi_v)^{-1})$ is an eigenvalue of $\rho(\Phi)$ for a geometric Frobenius Φ (characters factor through $W_{\bar v}^{\mathrm{ab}}$). So $|\xi_i(\varpi_v)| = q_v^{(k-n+1)/2}$ for all i. The centre of $\mathrm{GL}_n(F_v)$ acts on the normalised induction, hence on Π_v, through $\prod_i \xi_i$, so $\prod_i |\xi_i(\varpi_v)| = |\omega_{\Pi_v}(\varpi_v)| = 1$ (Step 0), and $|\xi_i(\varpi_v)| = 1$.

Step 2: places $v \nmid \ell$. By (Qw1), r is trivial on an open subgroup J of $I_{\bar v}$; in (Wd1), $\exp(t_\ell(\sigma)N) = 1$ with $t_\ell(\sigma) \neq 0$ for some $\sigma \in J$, so $N = 0$, and $\rho|_{I_{\bar v}} = r|_{I_{\bar v}}$ is the sum of the characters $\Psi_i|_{I_{\bar v}}$ of finite order. Step 1 applies.

Step 3: places $v \mid \ell$, the monodromy. Let $v \mid \ell$, $\bar v \mid v$, and let (κ_a) be a diagonal of r on $D = D_{\bar v}$ with $\kappa_a \neq \kappa_b \chi_{\mathrm{cyc}}|_D$ for all a, b (i.e. r generic at v with this diagonal at $\bar v$). The κ_a are the Jordan–Hölder constituents of $r|_D$, so (Qw2), which concerns the multiset of diagonal characters, holds for them. $r|_D$ is potentially semistable ((Bg2)), so (Tj1) of Lemma 48.13 (for F, v, $\bar v$ and this diagonal) gives $N = 0$. By Step 0, $r|_D$ is conjugate to $r_{\ell,\iota}(\Pi)|_D$, so by Theorem B.73

$(V,\rho,0)$ is pure, with the conditions (Pu1), (Pu2), which are those of (Tj2) of Lemma 48.13; hence every eigenvalue of $\rho(\Phi)$ has absolute value $q_v^{k/2}$ under ι, and $\kappa_a^m \neq (\kappa_b\chi_{\mathrm{cyc}})^m$ on D for all a,b and $m \geq 1$.

Step 4: (Wc1). With v, $\bar{v}$, (κ_a) as in Step 3, Step 3 gives $\kappa_a^m \neq (\kappa_b\chi_{\mathrm{cyc}})^m$ on D for all a,b and $m \geq 1$. As $\bar{v} \mid v$ was arbitrary, r is potentially generic at every place at which it is generic: (Wc1).

Step 5: places $v \mid \ell$ where r is potentially generic. Then r is generic at v (take $m=1$), so $N=0$ by Step 3; ρ is trivial on an open subgroup of $I_{\bar{v}}$, so $\rho|_{I_{\bar{v}}}$ factors through a finite group, is semisimple, and its constituents are the restrictions to $I_{\bar{v}}$ of the characters $\mathrm{WD}(\kappa_a)$ of $W_{\bar{v}}$ (WD is exact on potentially semistable representations, [95], applied along the triangular flag), characters of $W_{\bar{v}}$ of finite order on inertia. Step 1 applies.

Step 6: (Wc2). Let π be regular algebraic cuspidal with $r \leftrightarrow_\iota \pi$. At almost all places both π_w and Π_w are unramified with Satake parameters determined by the Frobenius characteristic polynomial of r, so $\pi_w \cong \Pi_w$, and $\pi \cong \Pi$ by (Js1). By Steps 1, 2 and 5, at every finite $v \nmid \ell$, and every $v \mid \ell$ at which r is potentially generic, π_v has a cuspidal support of characters, and every character ξ in it satisfies $|\xi(\varpi_v)| = 1 \in (q_v^{-1/2}, q_v^{1/2})$; cuspidal supports are unique up to order, so this holds for every cuspidal support. This is (Wc2) with $\tau_\pi = 0$. □

Lemma 53.26. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with characters θ and integer e, $\mathscr{T}$ a fit and deep auxiliary tower for it with fields $F \subseteq L$, $\mathcal{O}_1$ and ν as given by Lemma 53.2, $\mathcal{O}' \supseteq \mathcal{O}_1$ a coefficient ring, $\mathscr{X} = (B, A, P, \dots)$ a space for $(\mathscr{T}, \mathcal{O}')$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and $y\colon A \to \overline{\mathbb{Q}}_\ell$ an $\mathcal{O}'$-algebra homomorphism such that $y(P)$ has weights $(2e+2, 2e, 2, 0)$. Let $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ be a continuous homomorphism with $r|_{G_F} = y \circ r_P$, $r(g)$ of similitude $\theta(g)$ for all $g \in G_{\mathbb{Q}}$ and $r|_{G_L}$ irreducible, such that $\nu\, r|_{G_L}$ is automorphic of conjugate self-dual type for ι and satisfies the witness conditions. Then r satisfies (Sy2) for ι.*

Proof. Symbols introduced here are local to this proof. For a CM field $M \subset \overline{\mathbb{Q}}$ the complex conjugation c lies in G_{M^+} (it fixes every totally real field) and not in G_M; for a character υ of G_M put $\upsilon^c(s) = \upsilon(csc)$.

Step 0: characters. By (Sg1), $\xi(c) = -1$ and $\Psi(c) = 1$; $\chi_{\mathrm{cyc}}(c) = -1$; and $\omega_\ell(c) = -1$, since $\omega_\ell(c)^2 = 1$ and $\bar{\omega}_\ell(c) = \bar{\chi}_{\mathrm{cyc}}(c) = -1 \neq 1$ (ℓ odd). Hence $\alpha(c) = -1$ and

$$\theta(c) = \alpha(c)\Psi(c) = -1.$$

Further $\theta = \chi_{\mathrm{cyc}}^{e+1}\theta_{\mathrm{f}}$ with $\theta_{\mathrm{f}} = \omega_\ell^{-1}\xi\chi$ of finite order. Put $\Phi = \theta^{-1}\chi_{\mathrm{cyc}}^{-3} = \chi_{\mathrm{cyc}}^{-(e+4)}\theta_{\mathrm{f}}^{-1}$. For every imaginary quadratic field K fix, by Lemma 53.30 (with $k = -(e+4)$ and the character θ_{f}^{-1} of finite order), a continuous character $\lambda_K\colon G_K \to \overline{\mathbb{Q}}_\ell^\times$, algebraic for ι, with $\lambda_K\lambda_K^c = \Phi$ on G_K and $\lambda_K(I_{\bar{v}} \cap G_K)$ finite for every place $\bar{v}$ above a prime $p \neq \ell$. We use two facts on Hecke characters. (H1) If υ is a character of G_M algebraic for ι, there is an algebraic Hecke character Υ of M with Υ_w unramified and $\Upsilon_w(\varpi_w) = \iota\upsilon(\mathrm{Frob}_w^{-1})$ for all but finitely many finite places w of M: the Hecke character to which υ corresponds, or its inverse, according to the normalisation of the correspondence (inverses of algebraic Hecke characters are algebraic). (H2) A Hecke character Ξ of a number field M that is unramified with $\Xi_w(\varpi_w) = 1$ at every finite place w outside a finite set T is trivial: it is trivial on $M_w^\times$ for these w; given $x \in \mathbb{A}_M^\times$, weak approximation gives $\delta \in M^\times$ close to x at the places in T and the infinite places, and then x is close to $\delta\, x_1$, where x_1 has the components of $\delta^{-1}x$ outside these places and 1 at them; so $\Xi(x) = 1$ by continuity.

Step 1: polarisation, and the condition (Vj3). Let M be a CM field, $K \subseteq M$ an imaginary quadratic field, $\rho = r|_{G_M} \otimes \lambda_K|_{G_M}$, $\mu = \chi_{\mathrm{cyc}}^{-3}|_{G_{M^+}}$, and $\langle x, x' \rangle = {}^t x\, J_4\, r(c)\, x'$ on $\overline{\mathbb{Q}}_\ell^4$, a non-degenerate bilinear form. (a) (ρ,μ) satisfies (Vj1) with c and $\langle\ ,\ \rangle$. For $\sigma \in G_M$, $r(c)r(c\sigma c) =$

$r(\sigma)r(c)$ as $c^2=1$, and $r(\sigma)$ has similitude $\theta(\sigma)$, so

$$\langle\rho(\sigma)x,\rho(c\sigma c)x'\rangle=\lambda_K(\sigma)\lambda_K(c\sigma c)\,{}^t x\,{}^t r(\sigma)J_4 r(\sigma)\,r(c)\,x'=\Phi(\sigma)\theta(\sigma)\langle x,x'\rangle=\mu(\sigma)\langle x,x'\rangle.$$

From ${}^t r(c)J_4r(c)=\theta(c)J_4$ and $r(c)^{-1}=r(c)$ we get ${}^t r(c)J_4=\theta(c)J_4r(c)$, and with ${}^tJ_4=-J_4$, $\langle x',x\rangle={}^t x\,{}^t r(c)\,{}^t J_4\,x'=-\theta(c)\langle x,x'\rangle=\langle x,x'\rangle$ by Step 0. As $-\mu(c)=-\chi_{\mathrm{cyc}}(c)^{-3}=1$, this is $\langle x,x'\rangle=-\mu(c)\langle x',x\rangle$. So the sign that (Vj1) demands for the multiplier χ_{cyc}^{-3} is the sign $-\theta(c)$ of the form given by the similitude; this is where $\theta(c)=-1$ enters. (b) (Vj3) holds for (ρ,μ) over M if and only if ρ is automorphic of conjugate self-dual type for ι. In (Vj3), $\mu\chi_{\mathrm{cyc}}^3=\mathbf{1}$, so the character χ_1 of $\mathbb{A}_{M^+}^\times/(M^+)^\times$ there satisfies $\mathbf{1}\leftrightarrow_\iota\chi_1$: for almost all finite places w of M^+, $\chi_{1,w}$ is unramified with $\chi_{1,w}(\varpi_w)=1$, so $\chi_1=\mathbf{1}$ by (H2). Conversely $\chi_1=\mathbf{1}$ has $\chi_{1,v}(-1)=1=(-1)^4$ at every real place v and $\mathbf{1}\leftrightarrow_\iota\mathbf{1}$, and with it the condition on π in (Vj3) reads $\pi^c\cong\pi^\vee$. So (Vj3) says exactly that $\rho\leftrightarrow_\iota\pi$ for a regular algebraic conjugate self-dual cuspidal automorphic representation π of $\mathrm{GL}_4(\mathbb{A}_M)$. (The twist by λ_K makes the sign condition of (Vj3) hold for every e. With $\rho=r|_{G_M}$ and $\mu=\theta|_{G_{M^+}}$, (Vj1) holds with the same form, but the character of (Vj3) is then forced to be the algebraic Hecke character χ_1 with $\theta\chi_{\mathrm{cyc}}^3\leftrightarrow_\iota\chi_1$, whose ℓ-adic character has Hodge–Tate weight $-(e+4)$ and the value $\theta(c)\chi_{\mathrm{cyc}}(c)^3=1$ on complex conjugations; by the relation between these and $\chi_{1,v}(-1)$ at a real place v ([17], Appendix A.2), $\chi_{1,v}(-1)=(-1)^e$, and the condition $\chi_{1,v}(-1)=(-1)^4$ fails for odd e.)

Step 2: twists with trivial conjugate product. Let M be a CM field, ρ_1 a continuous representation of G_M automorphic of conjugate self-dual type for ι, and υ a character of G_M algebraic for ι with $\upsilon\upsilon^c=\mathbf{1}$. Then $\rho_1\otimes\upsilon$ is automorphic of conjugate self-dual type for ι. Let $\rho_1\leftrightarrow_\iota\Pi$ with Π regular algebraic, conjugate self-dual and cuspidal, and Υ as in (H1). At almost all finite places w the Satake parameters of $(\Pi\otimes(\Upsilon\circ\det))_w$ are the $\zeta\,\Upsilon_w(\varpi_w)$, ζ running over those of Π_w, and ι carries the eigenvalues of $(\rho_1\otimes\upsilon)(\mathrm{Frob}_w^{-1})$ to the $\mathrm{N}w^{3/2}\zeta\cdot\Upsilon_w(\varpi_w)$; so $\rho_1\otimes\upsilon\leftrightarrow_\iota\Pi\otimes(\Upsilon\circ\det)$, which is cuspidal and regular algebraic (twisting by an algebraic Hecke character shifts the infinitesimal character at each complex place by an integral scalar). Let $\Upsilon^c=\Upsilon\circ c$. For almost all w, $c(\varpi_w)$ is a uniformiser at cw and $c\,\mathrm{Frob}_w\,c$ is an arithmetic Frobenius at cw, so $(\Upsilon\Upsilon^c)_w(\varpi_w)=\iota(\upsilon\upsilon^c)(\mathrm{Frob}_w^{-1})=1$; by (H2), $\Upsilon^c=\Upsilon^{-1}$. Hence $(\Pi\otimes(\Upsilon\circ\det))^c\cong\Pi^\vee\otimes(\Upsilon^{-1}\circ\det)=(\Pi\otimes(\Upsilon\circ\det))^\vee$.

Step 3: from L to every imaginary quadratic field. For a CM field M and an imaginary quadratic field $K\subseteq M$ let $\mathrm{A}(M,K)$ be the assertion that $r|_{G_M}\otimes\lambda_K|_{G_M}$ is automorphic of conjugate self-dual type for ι. (R1) If $K_1,K_2\subseteq M$, then $\mathrm{A}(M,K_1)\Leftrightarrow\mathrm{A}(M,K_2)$: $\upsilon=\lambda_{K_2}\lambda_{K_1}^{-1}$ on G_M is algebraic for ι with $\upsilon\upsilon^c=\Phi\Phi^{-1}=\mathbf{1}$; apply Step 2 to υ and to υ^{-1}. (R2) If $K\subseteq F'\subseteq M$ with F', M CM fields, M/F' soluble Galois and $r|_{G_M}$ irreducible, then $\mathrm{A}(F',K)\Leftrightarrow\mathrm{A}(M,K)$: apply Theorem B.78 to F' (not totally real, as it contains K), M, $n=4$, $\rho=r|_{G_{F'}}\otimes\lambda_K|_{G_{F'}}$ and $\mu=\chi_{\mathrm{cyc}}^{-3}|_{G_{F'+}}$. (Vj1) holds by Step 1(a); (Vj2) holds since a subspace stable under $\rho(G_M)$ is stable under $r(G_M)$; (Vj3) is $\mathrm{A}(F',K)$ by Step 1(b) for F'; and (Vj4), which is (Vj3) for $(r|_{G_M}\otimes\lambda_K|_{G_M},\chi_{\mathrm{cyc}}^{-3}|_{G_{M^+}})$ over M, is $\mathrm{A}(M,K)$ by Step 1(b) for M. The start. By Lemma 53.2, L is a CM field with $L^+=F$; $E\subseteq L$, and L is abelian over $\mathbb{Q}$ as F is, so L/E is abelian. The character ν is algebraic for ι: at every place $\bar w$ above ℓ, by (Lk5), $\nu\chi_{\mathrm{cyc}}^{-a}$ (with the integer a given there) has finite order on $I_{\bar w}\cap G_L$, so it is potentially unramified, hence de Rham, and ν is de Rham at $\bar w$; a continuous character of G_L that is de Rham at all places above ℓ is algebraic for ι ([220, Ch. III]; [69, Lemma 4.1.3]). So $\upsilon_0=\lambda_E|_{G_L}\nu^{-1}$ is algebraic for ι, and $\upsilon_0\upsilon_0^c=\Phi\,(\nu\nu^c)^{-1}=\theta^{-1}\chi_{\mathrm{cyc}}^{-3}\cdot\theta\chi_{\mathrm{cyc}}^3=\mathbf{1}$ on G_L by (Lk1). By hypothesis $\nu\,r|_{G_L}$ is automorphic of conjugate self-dual type, so by Step 2 so is $(\nu\,r|_{G_L})\otimes\upsilon_0=r|_{G_L}\otimes\lambda_E|_{G_L}$: $\mathrm{A}(L,E)$ holds. By (R2) for $E\subseteq L$ ($r|_{G_L}$ irreducible), $\mathrm{A}(E,E)$ holds. Now let K' be an imaginary quadratic field with $r|_{G_{K'}}$ irreducible. If $K'=E$, $\mathrm{A}(K',K')$ holds. Otherwise $r|_{G_E}$ is irreducible (as $r|_{G_L}$ is), and Lemma 53.29 (for E and K') gives an imaginary quadratic field $K''\notin\{E,K'\}$ with $r|_{G_{EK''}}$

and $r|_{G_{K'K''}}$ irreducible. For distinct imaginary quadratic fields K_1, K_2 the compositum K_1K_2 is a CM field (a totally imaginary quadratic extension of its real quadratic subfield), Galois of degree 2 over K_1 and over K_2. Hence, by (R2), (R1), (R2), (R2), (R1), (R2) in turn,

$$\mathrm{A}(E,E) \Rightarrow \mathrm{A}(EK'',E) \Rightarrow \mathrm{A}(EK'',K'') \Rightarrow \mathrm{A}(K'',K'') \\ \Rightarrow \mathrm{A}(K''K',K'') \Rightarrow \mathrm{A}(K''K',K') \Rightarrow \mathrm{A}(K',K').$$

In both cases $\rho' = r|_{G_{K'}} \otimes \lambda_{K'}$ is automorphic of conjugate self-dual type: $\rho' \leftrightarrow_\iota \Pi'$ with Π' regular algebraic, conjugate self-dual and cuspidal.

Step 4: automorphy over K'. Let Υ be as in (H1) for $\upsilon = \lambda_{K'}$. At almost all finite places w of K', $r(\mathrm{Frob}_w^{-1}) = \rho'(\mathrm{Frob}_w^{-1})\lambda_{K'}(\mathrm{Frob}_w^{-1})^{-1}$, so ι carries its eigenvalues to the $\mathrm{N}w^{3/2}\zeta\,\Upsilon_w(\varpi_w)^{-1}$, ζ over the Satake parameters of Π'_w; these $\zeta\,\Upsilon_w(\varpi_w)^{-1}$ are the Satake parameters of $(\Pi' \otimes (\Upsilon^{-1} \circ \det))_w$. Hence $r|_{G_{K'}} \leftrightarrow_\iota \Pi' \otimes (\Upsilon^{-1} \circ \det)$, a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_{K'})$. This is the first part of (Sy2).

Step 5: the witness conditions over K'. We apply Lemma 53.25 to the CM field K', $n = 4$ and ρ', which is continuous, irreducible (as $r|_{G_{K'}}$ is) and automorphic of conjugate self-dual type (Step 3). (Qw1): let $v \nmid \ell$ be a finite place of K' above the prime p and $\bar{v} \mid v$. By Lemma 53.27, r maps the inertia group of $\bar{v}$ in $G_{\mathbb{Q}}$ to a finite group and, on it, is conjugate to $s \mapsto \mathrm{diag}(\psi_1(s), \dots, \psi_4(s))$ with characters ψ_i of the decomposition group of $\bar{v}$ in $G_{\mathbb{Q}}$; by Step 0, $\lambda_{K'}$ takes finitely many values on $I_{\bar{v}} = I_{\bar{v}}^{\mathbb{Q}} \cap G_{K'}$. So $\rho'(I_{\bar{v}})$ is finite and $\rho'|_{I_{\bar{v}}}$ is conjugate to $s \mapsto \mathrm{diag}\big(\psi_i(s)\lambda_{K'}(s)\big)_i$, with characters $\psi_i\lambda_{K'}$ of $D_{\bar{v}}$. (Qw2): let $v \mid \ell$ and $\bar{v} \mid v$. By Lemma 53.28, r is triangular on the decomposition group of $\bar{v}$ in $G_{\mathbb{Q}}$ with a diagonal $(\kappa_1, \dots, \kappa_4)$ such that $\kappa_i\chi_{\mathrm{cyc}}^{-m_i}$, $(m_1, \dots, m_4) = (e+1, e, 1, 0)$, is trivial on an open subgroup V_i of its inertia group. With the same conjugating matrix, ρ' is triangular on $D_{\bar{v}}$ with diagonal $(\kappa_i\lambda_{K'})_i$, and $(\kappa_i\lambda_{K'})(\kappa_j\lambda_{K'})^{-1}\chi_{\mathrm{cyc}}^{-(m_i-m_j)} = (\kappa_i\chi_{\mathrm{cyc}}^{-m_i})(\kappa_j\chi_{\mathrm{cyc}}^{-m_j})^{-1}$ is trivial on the open subgroup $V_i \cap V_j \cap G_{K'}$ of $I_{\bar{v}}$. By Lemma 53.25, ρ' satisfies (Wc1) and (Wc2), with $\tau = 0$. Untwisting. For a place $v \mid \ell$ of K' and $\bar{v} \mid v$, the triangular forms of $r|_{G_{K'}}$ and of $\rho' = r|_{G_{K'}} \otimes \lambda_{K'}$ on $D_{\bar{v}}$ correspond (same conjugating matrix), the diagonals differing by the factor $\lambda_{K'}$; the conditions $\kappa_i \neq \kappa_j\chi_{\mathrm{cyc}}$ and $\kappa_i^h \neq (\kappa_j\chi_{\mathrm{cyc}})^h$ involve only the ratios κ_i/κ_j. So $r|_{G_{K'}}$ is generic, resp. potentially generic, at v if and only if ρ' is, and (Wc1) for ρ' gives (Wc1) for $r|_{G_{K'}}$. For (Wc2) we first show $|\Upsilon| = \|\cdot\|^t$ for a real t, $\|\cdot\|$ the idelic norm of K': $|\Upsilon|$ is a continuous homomorphism $\mathbb{A}_{K'}^\times / K'^\times \to \mathbb{R}_{>0}$, trivial on the compact group $\prod_w \mathcal{O}_{K'_w}^\times$; on $K'^\times_\infty = \mathbb{C}^\times$, which is connected, the algebraic character Υ is $z \mapsto z^{n_1}\bar{z}^{n_2}$ with integers n_1, n_2, so $|\Upsilon(z)| = |z|^{n_1+n_2} = \|z\|^{(n_1+n_2)/2}$; with $t = (n_1+n_2)/2$, $|\Upsilon|\,\|\cdot\|^{-t}$ is trivial on $K'^\times$, $\mathbb{C}^\times$ and $\prod_w \mathcal{O}_{K'_w}^\times$, so it factors through the finite class group and takes values in the torsion of $\mathbb{R}_{>0}$, which is trivial. Hence $|\Upsilon_v(\varpi_v)| = q_v^{-t}$ for every finite place v. Now let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_{K'})$ with $r|_{G_{K'}} \leftrightarrow_\iota \pi$ ($r|_{G_{K'}}$ is semisimple, being irreducible). As in Step 2, $\rho' \leftrightarrow_\iota \pi \otimes (\Upsilon \circ \det)$, regular algebraic and cuspidal. Let v be a finite place with $v \nmid \ell$, or $v \mid \ell$ and $r|_{G_{K'}}$ potentially generic at v (equivalently, ρ' potentially generic at v). If π_v is a subquotient of the normalised induction of $\xi_1 \otimes \cdots \otimes \xi_4$, then $\pi_v \otimes (\Upsilon_v \circ \det)$ is a subquotient of that of $\xi_1\Upsilon_v \otimes \cdots \otimes \xi_4\Upsilon_v$, and conversely; so the cuspidal supports of π_v are those of $(\pi \otimes (\Upsilon \circ \det))_v$ multiplied by Υ_v^{-1}. By (Wc2) for ρ', π_v has a cuspidal support consisting of characters, and every character ξ in a cuspidal support of π_v has $q_v^{-1/2} < |\xi(\varpi_v)\Upsilon_v(\varpi_v)| < q_v^{1/2}$, that is $q_v^{t-1/2} < |\xi(\varpi_v)| < q_v^{t+1/2}$. This is (Wc2) for $r|_{G_{K'}}$ with $\tau_\pi = t$. As K' was an arbitrary imaginary quadratic field with $r|_{G_{K'}}$ irreducible, Steps 4 and 5 give (Sy2) for r and ι. □

Lemma 53.27. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with prime ℓ, $\mathscr{T}$ an auxiliary tower for $\mathfrak{D}$ with field F, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring, $\mathscr{X} = (B, A, P, \dots)$ a space for $(\mathscr{T}, \mathcal{O}')$, $y\colon A \to \overline{\mathbb{Q}}_\ell$ an $\mathcal{O}'$-algebra homomorphism and $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ a homomorphism with $r|_{G_F} = y \circ r_P$. Then*

for every prime $p \neq \ell$ and every place $\bar{p} \mid p$ of $\overline{\mathbb{Q}}$: $r(I_{\bar{p}})$ is a finite abelian group commuting with $r(D_{\bar{p}})$, and there are continuous characters $\Psi_1, \dots, \Psi_4 \colon D_{\bar{p}} \to \overline{\mathbb{Q}}_\ell^\times$ such that $r|_{I_{\bar{p}}}$ is conjugate to $s \mapsto \operatorname{diag}(\Psi_1(s), \dots, \Psi_4(s))$.

Proof. Symbols introduced here are local to this proof. Let $H = G_F \cap \ker \bar{r}_0$. It is an open normal subgroup of $G_{\mathbb{Q}}$ (G_F is open and normal as $F/\mathbb{Q}$ is Galois, and $\bar{r}_0$ is a locally constant homomorphism on $G_{\mathbb{Q}}$), and $G_{\mathbb{Q}}/H$ embeds into $\operatorname{Gal}(F/\mathbb{Q}) \times \bar{r}_0(G_{\mathbb{Q}})$, which is abelian ($F/\mathbb{Q}$ abelian, $\bar{r}_0$ diagonal); so the closure of the commutator subgroup of $G_{\mathbb{Q}}$ lies in H.

Let $p \neq \ell$, $\bar{p} \mid p$, $D = D_{\bar{p}} \supseteq I = I_{\bar{p}}$. If $s \in I \cap H$, then $s \in I_{\bar{p}} \cap G_F$ with $\bar{r}_0(s) = 1$ and $\bar{p} \nmid \ell$, so $s \in N$, $r_P(s) = 1$ by (Mq1), and $r(s) = y(r_P(s)) = 1$. Hence $r|_I$ factors through $I/(I \cap H)$, which embeds into the finite abelian group $G_{\mathbb{Q}}/H$: $r(I)$ is finite abelian. For $d \in D$ and $s \in I$, $dsd^{-1}s^{-1}$ lies in I (I is normal in D) and in the commutator subgroup, hence in $I \cap H$, so $r(d)r(s)r(d)^{-1} = r(s)$: $r(I)$ commutes with $r(D)$.

As $r(I)$ is a finite abelian group of matrices over the algebraically closed field $\overline{\mathbb{Q}}_\ell$ of characteristic 0, $V = \overline{\mathbb{Q}}_\ell^4$ is the direct sum of the joint eigenspaces V_χ, χ running over the characters of I (through r) that occur, and $r|_I$ acts on V_χ by χ. Each χ is trivial on the open subgroup $I \cap H$, hence continuous, and satisfies $\chi(dsd^{-1}) = \chi(s)$ for $d \in D$ (as $r(dsd^{-1}) = r(s)$). The quotient D/I is isomorphic to $\widehat{\mathbb{Z}}$, a free profinite group, so the projection $D \to D/I$ has a continuous section on a topological generator: there is $\phi \in D$ such that $t \mapsto \phi^t$, $t \in \widehat{\mathbb{Z}}$, is a continuous homomorphism whose composite with $D \to D/I$ is an isomorphism; then $(s, t) \mapsto s\phi^t$ is a homeomorphism $I \times \widehat{\mathbb{Z}} \to D$. Put $\Psi_\chi(s\phi^t) = \chi(s)$. It is a homomorphism, since $s\phi^t s'\phi^{t'} = s(\phi^t s' \phi^{-t})\phi^{t+t'}$ and $\chi(\phi^t s' \phi^{-t}) = \chi(s')$, and continuous, as χ is. Listing each χ with multiplicity $\dim V_\chi$ gives characters $\Psi_1, \dots, \Psi_4$ of D, and in a basis adapted to $V = \bigoplus_\chi V_\chi$, $r(s) = \operatorname{diag}(\Psi_1(s), \dots, \Psi_4(s))$ for $s \in I$. □

Lemma 53.28. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with prime ℓ and integer e, $\mathscr{T}$ an auxiliary tower for $\mathfrak{D}$ with field F, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring, $\mathscr{X} = (B, A, P, \dots)$ a space for $(\mathscr{T}, \mathcal{O}')$, and $y \colon A \to \overline{\mathbb{Q}}_\ell$ an $\mathcal{O}'$-algebra homomorphism such that $y(P)$ has weights $(2e + 2, 2e, 2, 0)$. Let $r \colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ be a continuous homomorphism with $r|_{G_F} = y \circ r_P$, and put $(m_1, m_2, m_3, m_4) = (e + 1, e, 1, 0)$. Then for every place $\bar{u}$ of $\overline{\mathbb{Q}}$ above ℓ the representation r is triangular on $D_{\bar{u}}$ with a diagonal $(\kappa_1, \dots, \kappa_4)$ of continuous characters of $D_{\bar{u}}$ such that, for $a = 1, \dots, 4$, the character $\kappa_a \chi_{\mathrm{cyc}}^{-m_a}$ is trivial on an open subgroup of $I_{\bar{u}}$.*

Proof. Symbols introduced here are local to this proof.

Step 1: reduction to the marked places. By the definition of an auxiliary tower, $\bar{u} = g\bar{v}$ for some $v \in \Sigma$ and $g \in G_F$; then $D_{\bar{u}} = gD_{\bar{v}}g^{-1}$ and $I_{\bar{u}} = gI_{\bar{v}}g^{-1}$. Suppose $b \in \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$ is such that $b\,r(s)\,b^{-1}$ is upper triangular with diagonal entries $\kappa_a(s)$ for $s \in D_{\bar{v}}$, with continuous characters κ_a of $D_{\bar{v}}$ and open subgroups $V_a \subseteq I_{\bar{v}}$ on which $\kappa_a \chi_{\mathrm{cyc}}^{-m_a}$ is trivial. For $s \in D_{\bar{u}}$, $(b\,r(g)^{-1})\,r(s)\,(b\,r(g)^{-1})^{-1} = b\,r(g^{-1}sg)\,b^{-1}$ is upper triangular with diagonal entries $\kappa_a(g^{-1}sg)$; the characters $s \mapsto \kappa_a(g^{-1}sg)$ of $D_{\bar{u}}$ are continuous, and since $\chi_{\mathrm{cyc}}(g^{-1}sg) = \chi_{\mathrm{cyc}}(s)$, the character $s \mapsto \kappa_a(g^{-1}sg)\chi_{\mathrm{cyc}}(s)^{-m_a}$ is trivial on the open subgroup gV_ag^{-1} of $I_{\bar{u}}$. So it suffices to treat $\bar{u} = \bar{v}$ with $v \in \Sigma$. Write $D = D_{\bar{v}} \supseteq I = I_{\bar{v}}$ (in $G_{\mathbb{Q}}$), so that $D_v = D \cap G_F$ and $I_v = I \cap G_F$.

Step 2: the flag on D_v. Let $P = (r_P, (n_w)_{w \in \Sigma})$. By (Mq3), n_v is lower triangular with diagonal entries 1, so $y(n_v)$ is invertible; put $b = y(n_v)^{-1}$. For $s \in D_v$, $X(s) = b\,r(s)\,b^{-1} = y(n_v^{-1} r_P(s) n_v)$ is upper triangular by (Mq4), with diagonal entries $\psi_a(s) = \psi_{v,a}^{y(P)}(s)$. As $y(P)$ has weights $(2m_1, \dots, 2m_4)$, there is an open subgroup $U \subseteq G_F$ with $\psi_a(s)^2 = \chi_{\mathrm{cyc}}(s)^{2m_a}$ for all a and all $s \in U \cap I_v$.

Step 3: a test element of infinite cyclotomic order. U is open in $G_{\mathbb{Q}}$ (as G_F is), hence of finite index, so $N = \bigcap_{g \in G_{\mathbb{Q}}} gUg^{-1}$ is a finite intersection (the conjugate gUg^{-1} depends only on the coset gU) and is an open normal subgroup of $G_{\mathbb{Q}}$ contained in U. The cyclotomic character maps I onto $\mathbb{Z}_\ell^\times$ ($\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell$ is totally ramified with group $\mathbb{Z}_\ell^\times$), and $N \cap I$ has finite index in I, so $\chi_{\mathrm{cyc}}(N \cap I)$ is a closed subgroup of finite index of $\mathbb{Z}_\ell^\times$, hence open, hence contains $1 + \ell^j \mathbb{Z}_\ell$ for some $j \geq 1$, whose non-trivial elements have infinite order ($1 + \ell\mathbb{Z}_\ell$ is torsion-free, ℓ being odd). Choose $s_0 \in N \cap I$ with $z = \chi_{\mathrm{cyc}}(s_0)$ of infinite order.

Step 4: the flag is stable under D. Let $g \in D$ and $s_1 = g s_0 g^{-1}$. As N is normal in $G_{\mathbb{Q}}$ and I is normal in D, $s_1 \in N \cap I \subseteq U \cap I_v \subseteq D_v$, and $\chi_{\mathrm{cyc}}(s_1) = z$. For $t = 0, 1$ the matrix $M_t = X(s_t)^2$ is upper triangular with diagonal entries $\psi_a(s_t)^2 = d_a$, where $d_a = z^{2m_a}$; the d_a are pairwise distinct, since $m_1 > m_2 > m_3 > m_4$ (as $e \geq 2$) and z has infinite order. Put $B = b\, r(g)\, b^{-1}$; from $g\, s_0^2 = s_1^2\, g$ we get $BM_0 = M_1 B$. We show $B_{ij} = 0$ for $i > j$, by induction on $j - i \in \{-3, -2, -1\}$. Let $i > j$ and assume $B_{i'j'} = 0$ whenever $i' > j'$ and $j' - i' < j - i$. In $(BM_0)_{ij} = \sum_{k \leq j} B_{ik} (M_0)_{kj}$ the terms with $k < j$ vanish ($i > k$ and $k - i < j - i$), so $(BM_0)_{ij} = B_{ij} d_j$; in $(M_1 B)_{ij} = \sum_{k \geq i} (M_1)_{ik} B_{kj}$ the terms with $k > i$ vanish ($k > j$ and $j - k < j - i$), so $(M_1 B)_{ij} = d_i B_{ij}$. Hence $B_{ij}(d_j - d_i) = 0$ and $B_{ij} = 0$. So $b\, r(g)\, b^{-1}$ is upper triangular for every $g \in D$. The diagonal entries $\kappa_a(g) = (b\, r(g)\, b^{-1})_{aa}$ are multiplicative in g (the diagonal of a product of upper triangular matrices is the product of the diagonals) and continuous, as r is: r is triangular on D with diagonal $(\kappa_1, \ldots, \kappa_4)$.

Step 5: the weights. κ_a restricts to ψ_a on D_v. The group $V = U \cap I_v = U \cap I$ is an open subgroup of I, and on V, $(\kappa_a \chi_{\mathrm{cyc}}^{-m_a})^2 = \psi_a^2 \chi_{\mathrm{cyc}}^{-2m_a} = 1$ by Step 2. So $\kappa_a \chi_{\mathrm{cyc}}^{-m_a}$ maps V to $\{\pm 1\}$; its kernel in V is closed of index at most 2, hence open in V and in I, and $\kappa_a \chi_{\mathrm{cyc}}^{-m_a}$ is trivial on it. With Step 1 this proves the lemma. □

Lemma 53.29. *Let ℓ be a prime, $n \geq 1$, $r\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ a homomorphism, and $K, K' \subset \overline{\mathbb{Q}}$ imaginary quadratic fields such that $r|_{G_K}$ and $r|_{G_{K'}}$ are irreducible. Then there is an imaginary quadratic field $K'' \subset \overline{\mathbb{Q}}$ with $K'' \neq K$, $K'' \neq K'$, such that $r|_{G_{KK''}}$ and $r|_{G_{K'K''}}$ are irreducible.*

Proof. Symbols introduced here are local to this proof.

Step 1: restriction to a subgroup of index 2. Let G be a group, $\rho\colon G \to \mathrm{GL}(V)$ an irreducible representation on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space, $H \subseteq G$ a subgroup of index 2 and $\eta\colon G \to \{\pm 1\}$ the character with kernel H. We show: if $\rho|_H$ is reducible, then $\rho \otimes \eta \cong \rho$. Let $W \subseteq V$ be a non-zero proper H-invariant subspace and $g \in G \setminus H$. As H is normal, $\rho(g)W$ is H-invariant, and $\rho(g)\rho(g)W = W$ since $g^2 \in H$; as $G = H \cup gH$, the subspaces $W + \rho(g)W$ and $W \cap \rho(g)W$ are G-invariant. By irreducibility, $W + \rho(g)W = V$, and $W \cap \rho(g)W \neq V$ (it lies in $W \neq V$), so $W \cap \rho(g)W = 0$ and $V = W \oplus \rho(g)W$. Let $A \in \mathrm{GL}(V)$ be 1 on W and -1 on $\rho(g)W$. Every $s \in H$ preserves W, and $\rho(s)\rho(g)W = \rho(g)\rho(g^{-1}sg)W = \rho(g)W$ as $g^{-1}sg \in H$; so $A\rho(s) = \rho(s)A$. Every $s \in G \setminus H$ satisfies $s \in gH$ and $sg \in H$, so $\rho(s)W = \rho(g)W$ and $\rho(s)\rho(g)W = \rho(sg)W = W$; so $A\rho(s) = -\rho(s)A$. Hence $A\rho(s)A^{-1} = \eta(s)\rho(s)$ for all $s \in G$, i.e. A is an isomorphism $\rho \to \rho \otimes \eta$.

Step 2: few characters fix ρ. With G, ρ, V as in Step 1 and $\dim V = n$, the set $X(\rho)$ of homomorphisms $\chi\colon G \to \overline{\mathbb{Q}}_\ell^\times$ with $\rho \otimes \chi \cong \rho$ has at most n^2 elements. Indeed, let G act on $\mathrm{End}(V)$ by $s \cdot B = \rho(s) B \rho(s)^{-1}$. For $\chi \in X(\rho)$ an isomorphism $A_\chi\colon \rho \to \rho \otimes \chi$ is a non-zero $A_\chi \in \mathrm{End}(V)$ with $A_\chi \rho(s) = \chi(s)\rho(s)A_\chi$, i.e. $s \cdot A_\chi = \chi(s)^{-1} A_\chi$. Common eigenvectors of G for pairwise distinct characters are linearly independent, so $|X(\rho)| \leq \dim \mathrm{End}(V) = n^2$.

Step 3: distinct fields give distinct characters. For an imaginary quadratic field $F \neq K$, KF is biquadratic over $\mathbb{Q}$, $G_{KF} = G_K \cap G_F$ has index 2 in G_K; let $\eta_F\colon G_K \to \{\pm 1\}$ be the character with kernel G_{KF}. If $F_1 \neq F_2$ are imaginary quadratic fields different from K with $\eta_{F_1} = \eta_{F_2}$, then $G_{KF_1} = G_{KF_2}$, so $KF_1 = KF_2 =: M$ by Galois theory. Writing $K = \mathbb{Q}(\sqrt{-a})$,

$F_1 = \mathbb{Q}(\sqrt{-b})$ with $a, b > 0$, the three quadratic subfields of M are K, F_1 and the real field $\mathbb{Q}(\sqrt{ab})$; so the only imaginary quadratic subfield of M other than K is F_1, and $F_2 = F_1$, a contradiction. Hence $F \mapsto \eta_F$ is injective on imaginary quadratic fields $F \neq K$, and likewise $F \mapsto \eta'_F$ (the character of $G_{K'}$ with kernel $G_{K'F}$) on those $F \neq K'$.

Step 4: choice of K''. Let B be the set of imaginary quadratic fields $F \notin \{K, K'\}$ with $\eta_F \in X(r|_{G_K})$ or $\eta'_F \in X(r|_{G_{K'}})$. By Steps 2 and 3, $|B| \leq 2n^2$. There are infinitely many imaginary quadratic fields (for instance $\mathbb{Q}(\sqrt{-p})$, p prime), so there is an imaginary quadratic $K'' \notin B \cup \{K, K'\}$. As $\eta_{K''} \notin X(r|_{G_K})$, Step 1 (for $G = G_K$, $H = G_{KK''}$, $\rho = r|_{G_K}$, irreducible by hypothesis) shows that $r|_{G_{KK''}}$ is irreducible; in the same way $r|_{G_{K'K''}}$ is irreducible. □

Lemma 53.30. *Let ℓ be an odd prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $K \subset \overline{\mathbb{Q}}$ an imaginary quadratic field, $c \in G_\mathbb{Q}$ a complex conjugation, k an integer and $\varphi\colon G_\mathbb{Q} \to \overline{\mathbb{Q}}_\ell^\times$ a continuous character of finite order. Then there is a continuous character $\psi\colon G_K \to \overline{\mathbb{Q}}_\ell^\times$, algebraic for ι, such that $\psi(s)\psi(csc) = \chi_{\mathrm{cyc}}(s)^k\varphi(s)$ for all $s \in G_K$, and such that $\psi(I_{\bar{v}} \cap G_K)$ is finite for every place $\bar{v}$ of $\overline{\mathbb{Q}}$ above a prime $p \neq \ell$.*

Proof. Symbols introduced here are local to this proof. Let $\eta_K\colon G_\mathbb{Q} \to \{\pm 1\}$ be the character with kernel G_K; as K is imaginary, $c \notin G_K$ and $\eta_K(c) = -1$.

Step 1: the kernel of the transfer. Let $\mathcal{V}\colon G_\mathbb{Q}^{ab} \to G_K^{ab}$ be the transfer. With the coset representatives $1, c$ of G_K in $G_\mathbb{Q}$ one has $\mathcal{V}(s) = s \cdot csc$ modulo commutators for $s \in G_K$, and $\mathcal{V}(c) = c^2 = 1$. We use from global class field theory ([11], [190]): $\mathrm{Art}_\mathbb{Q}$ and Art_K are surjective, their kernels are the connected components of $C_\mathbb{Q}$ and C_K, and the connected component of C_K is the closure of the image of $K_\infty^\times = \mathbb{C}^\times$; Art_K composed with the map $C_\mathbb{Q} \to C_K$ induced by $\mathbb{A}_\mathbb{Q}^\times \subseteq \mathbb{A}_K^\times$ is $\mathcal{V} \circ \mathrm{Art}_\mathbb{Q}$; and on the image of $\mathbb{R}^\times$ in $C_\mathbb{Q}$, $\mathrm{Art}_\mathbb{Q}$ is the local Artin map $\mathbb{R}^\times \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, which sends -1 to complex conjugation, followed by the inclusion of a decomposition group at infinity.

We claim that $\ker \mathcal{V} = \{1, \bar{c}\}$, where $\bar{c}$ is the image of c (all complex conjugations are conjugate in $G_\mathbb{Q}$, so they have the same image $\bar{c}$ in $G_\mathbb{Q}^{ab}$). We saw $\bar{c} \in \ker \mathcal{V}$. Let $g \in \ker \mathcal{V}$ and write $g = \mathrm{Art}_\mathbb{Q}(a)$ with $a \in \mathbb{A}_\mathbb{Q}^\times$. Then $\mathrm{Art}_K(a) = 1$, so there are $\alpha_i \in K^\times$ and $z_i \in \mathbb{C}^\times$ with $\alpha_i z_i \to a$ in $\mathbb{A}_K^\times$; on finite components, $\alpha_i \to a_f$ in the finite ideles $\mathbb{A}_{K,f}^\times$. Let $x \mapsto \bar{x}$ be the automorphism of $\mathbb{A}_K^\times$ induced by the non-trivial automorphism of K; it fixes $\mathbb{A}_\mathbb{Q}^\times$ and preserves $K^\times$. Then $\beta_i = \bar{\alpha}_i/\alpha_i \in K^\times$ converges to $\bar{a}_f/a_f = 1$ in $\mathbb{A}_{K,f}^\times$. The unit ideles form an open subgroup of $\mathbb{A}_{K,f}^\times$, so for large i, β_i is a unit at every finite place, i.e. $\beta_i \in \mathcal{O}_K^\times$, a finite group as K is imaginary quadratic; a sequence in a finite subset of a Hausdorff group converging to 1 is eventually 1, so $\alpha_i \in \mathbb{Q}^\times$ for large i. Now $\mathbb{Q}^\times$ is discrete in $\mathbb{A}_{\mathbb{Q},f}^\times$ ($\mathbb{Q}^\times \cap \widehat{\mathbb{Z}}^\times = \{\pm 1\}$ and $\widehat{\mathbb{Z}}^\times$ is open), and $\mathbb{A}_{\mathbb{Q},f}^\times$ is a closed subgroup of $\mathbb{A}_{K,f}^\times$ with the induced topology; so the convergent sequence α_i is eventually constant, equal to some $q \in \mathbb{Q}^\times$, and $a_f = q$. Then $b = q^{-1}a \in \mathbb{A}_\mathbb{Q}^\times$ has trivial finite components, so $b \in \mathbb{R}^\times$, and $g = \mathrm{Art}_\mathbb{Q}(b) \in \{1, \bar{c}\}$.

Step 2: the part of finite order. Since $\varphi(c)^2 = \varphi(c^2) = 1$, put $\varphi_1 = \varphi$ if $\varphi(c) = 1$ and $\varphi_1 = \varphi\eta_K$ if $\varphi(c) = -1$; then $\varphi_1(c) = 1$ and $\varphi_1 = \varphi$ on G_K. As a character of $G_\mathbb{Q}^{ab}$, φ_1 is trivial on $\ker \mathcal{V}$ by Step 1. The image $H = \mathcal{V}(G_\mathbb{Q}^{ab})$ is a closed subgroup of G_K^{ab}, and $\mathcal{V}$ induces a continuous bijection $G_\mathbb{Q}^{ab}/\ker \mathcal{V} \to H$ of compact Hausdorff groups, hence a homeomorphism; so $\lambda_0(\mathcal{V}(g)) = \varphi_1(g)$ defines a continuous character λ_0 of H of finite order. Its kernel is open in H, so it contains $H \cap N$ for an open subgroup N of the profinite group G_K^{ab}; λ_0 factors through the subgroup $H/(H \cap N)$ of the finite abelian group G_K^{ab}/N and extends to a character of G_K^{ab}/N with values in the divisible group of roots of unity of $\overline{\mathbb{Q}}_\ell^\times$. This gives a continuous character

$\lambda\colon G_K \to \overline{\mathbb{Q}}_\ell^\times$ of finite order with $\lambda \circ \mathcal{V} = \varphi_1$, so that for $s \in G_K$

$$\lambda(s)\lambda(csc) = \lambda(\mathcal{V}(s)) = \varphi_1(s) = \varphi(s).$$

Step 3: the character. By Lemma 49.11 (ℓ odd) there is a continuous character $\kappa_K\colon G_K \to \overline{\mathbb{Q}}_\ell^\times$, algebraic for ι, with (Ik1)–(Ik3). Put $\psi = \kappa_K^{-k}\lambda$. For $s \in G_K$, by (Ik1) and Step 2,

$$\psi(s)\psi(csc) = \big(\kappa_K(s)\kappa_K(csc)\big)^{-k}\lambda(s)\lambda(csc) = \chi_{\mathrm{cyc}}(s)^k\varphi(s).$$

λ has finite order, so $\iota \circ \lambda \circ \mathrm{Art}_K$ is a Hecke character of finite order, which is algebraic; products and inverses of characters algebraic for ι are algebraic for ι (the corresponding algebraic Hecke characters multiply); so ψ is algebraic for ι. Finally let $\bar{v}$ lie above a prime $p \neq \ell$: κ_K is trivial on $I_{\bar{v}} \cap G_K$ if $p \nmid d_K$ by (Ik2), and of finite order there if $p \mid d_K$ by (Ik3); λ has finite order; so $\psi(I_{\bar{v}} \cap G_K)$ is finite. □

53.3 Propagation in rank four: local charts

This subsection proves the propagation theorem in rank four, Theorem 53.31, as Subsections 49.2–49.4 prove it in ranks two and three, and with two of their lemmas, on chains of components and on one-dimensional primes (Lemma 49.28, Lemma 49.35). Its main input is Proposition 53.32: the completed local rings of a space at residual pairs good in case (III) are flat completions with a presentation that obeys the Euler-characteristic count (Proposition 53.34, Lemma 53.35, Lemma 53.36), and their reducible and coincidence loci have small dimension (Proposition 53.39, Lemma 53.33). The bound for the reducible loci is the core. It goes through reducible tangent spaces graded by a symplectic cocharacter (Definition 53.38, Definition 53.40): they do not grow under degeneration along the grading (Lemma 53.43), and at the diagonal limit pairs at most one odd root contributes to the degree (Lemma 53.44), which gives the tangent bound Lemma 53.42 and with it Lemma 53.41. Off the reducibility ideals a lift of a residual pair is irreducible (Lemma 53.45), and the fibre dimension over the Hecke side is controlled at irreducible primes (Lemma 53.47).

Theorem 53.31. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ a fit and deep auxiliary tower for $\mathfrak{D}$ with $d = [F : \mathbb{Q}]$, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring, $\mathscr{X} = (B, A, P, \dots)$ a space for $(\mathscr{T}, \mathcal{O}')$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $\mathfrak{J} \subseteq B$ an ideal modular for ι, and $C \geq 0$ a Selmer bound for $(\mathscr{T}, \mathcal{O} \to \mathbb{F}_{\mathcal{O}'})$ in degree d with $30(C+1)^2 \leq d$. Let $x_0, x_1\colon A \to \mathbb{F}_{\mathcal{O}'}$ be $\mathcal{O}'$-algebra homomorphisms such that $x_0(P)$ is a residual pair and there is a chain in case (III) from $x_0(P)$ to $x_1(P)$. If $\mathfrak{J}$ is somewhere at $\ker x_1$, then $\mathfrak{J}$ is everywhere at $\ker x_0$.*

Proof. The argument is commutative algebra on the rings of the space and its charts, in the pattern of the unitary Theorem 49.12; it uses no automorphy of the points of $\mathfrak{D}$, nor over L; the only Galois-theoretic and modularity input is Proposition 53.32. Write $\mathbb{F} = \mathbb{F}_{\mathcal{O}'}$, ϖ' for a uniformiser of $\mathcal{O}'$, $\sigma = \#\Sigma$ and $N = 2d + 1 - 2\sigma$. As C is a Selmer bound, $1 \leq \sigma \leq C$, and $d \geq 30(C+1)^2$ gives

$$d + 20C + 1 < N, \qquad 2d + 1 - d/\sigma + 2 + 4\sigma < N. \tag{53.1}$$

Part A: one residual point. Let $x\colon A \to \mathbb{F}$ be an $\mathcal{O}'$-algebra homomorphism such that $x(P)$ is a residual pair good in case (III), and suppose $\mathfrak{J}$ is somewhere at $\ker x$. We show that $\mathfrak{J}$ is everywhere at $\ker x$. Let R, f be as in Proposition 53.32 and $\phi = f|_B$.

(A1) *Minimal primes.* By (Vx1) (f is flat with $f^{-1}(\mathfrak{m}_R) = \ker x$) and going down, (a) for a minimal prime $\mathcal{P}$ of R, $f^{-1}(\mathcal{P})$ is a minimal prime of A contained in $f^{-1}(\mathfrak{m}_R) = \ker x$; (b) every

minimal prime $\eta \subseteq \ker x$ of A is $f^{-1}(\mathcal{P})$ for a minimal prime $\mathcal{P}$ of R (going down from $\mathfrak{m}_R$ gives $\mathcal{P}'$ with $f^{-1}(\mathcal{P}') = \eta$; a minimal $\mathcal{P} \subseteq \mathcal{P}'$ has $f^{-1}(\mathcal{P}) \subseteq \eta$ prime, hence equal). For a prime $\mathcal{P}$ of R and $\eta = f^{-1}(\mathcal{P})$: $\mathfrak{J}A \subseteq \eta$ if and only if $\phi(\mathfrak{J}) \subseteq \mathcal{P}$. So it suffices to show: if $\phi(\mathfrak{J}) \subseteq \mathcal{P}_0$ for one minimal prime $\mathcal{P}_0$ of R, then for all.

(A2) *Chains of minimal primes.* By (Vx2), $R \cong S/I$ with $S = \mathcal{O}'[[X_1, \dots, X_a]]$, a complete Noetherian local domain of dimension $a+1$, and I generated by b elements, which lie in $\mathfrak{m}_S$ (the preimage of $\mathfrak{m}_R$). Since $b + 2 + 2d \le a + 2\sigma$, $\dim S \ge (N+1) + 1 + b$, and Lemma 49.28 with $D = N+1$, $h = b$ applies; through the correspondence between primes of R and primes of S containing I, any two minimal primes $\mathcal{P}, \mathcal{Q}$ of R are joined by minimal primes $\mathcal{P} = \mathcal{P}_0, \dots, \mathcal{P}_k = \mathcal{Q}$ with $\dim R/(\mathcal{P}_{t-1} + \mathcal{P}_t) \ge N+1$. It therefore suffices to show, for minimal primes $\mathcal{P}, \mathcal{Q}$ of R with $\dim R/(\mathcal{P} + \mathcal{Q}) \ge N+1$: if $\phi(\mathfrak{J}) \subseteq \mathcal{P}$, then $\phi(\mathfrak{J}) \subseteq \mathcal{Q}$.

(A3) *A nice prime.* Let $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ be the ideals of (Vx3). For $v \in \Sigma$ and $i \ne j$ let $\mathfrak{c}_{v,i,j}$ be the ideal generated by the $f(\psi^P_{v,i}(s)) - f(\psi^P_{v,j}(s))$, $s \in U_v$, and $\mathfrak{b}_{v,i,j} = \phi(\mathfrak{J})R + \varpi' R + \mathfrak{c}_{v,i,j}$. Let $\mathcal{L}$ be the finite set of minimal primes $\mathfrak{q}$ of the ideals $\mathfrak{b}_{v,i,j}$ (R is Noetherian) that contain no $\mathfrak{a}_k$. For $\mathfrak{q} \in \mathcal{L}$: $\phi^{-1}(\mathfrak{q})$ contains $\mathfrak{J}$, $\mathfrak{m}_{\mathcal{O}'}B$ and the $\psi^B_{v,i}(s) - \psi^B_{v,j}(s)$, $s \in U_v$ (by (Zq2) $\phi(\psi^B_{v,i}(s)) = f(\psi^P_{v,i}(s))$), so (Vx0) gives $\dim B/\phi^{-1}(\mathfrak{q}) \le 2d + 1 - d/\sigma$, and (Vx3)(a) gives $\dim R/\mathfrak{q} \le 2d + 1 - d/\sigma + 2 + 4\sigma < N$ by (53.1). Also $\dim R/\mathfrak{a}_k \le d + 20C + 1 < N$. Now Lemma 49.35, applied to ϖ', N, the ideals $\mathfrak{a}_k$ and the members of $\mathcal{L}$, and $I = \mathcal{P} + \mathcal{Q}$, gives a prime $\mathfrak{p} \supseteq \mathcal{P} + \mathcal{Q}$ with $\dim R/\mathfrak{p} = 1$, $\varpi' \in \mathfrak{p}$, containing no $\mathfrak{a}_k$ and no member of $\mathcal{L}$. Then $\mathfrak{p}$ contains no $\mathfrak{b}_{v,i,j}$: otherwise it would contain a minimal prime $\mathfrak{q}$ of $\mathfrak{b}_{v,i,j}$, which contains some $\mathfrak{a}_k$ (then so does $\mathfrak{p}$) or lies in $\mathcal{L}$. As $\mathfrak{p} \supseteq \mathcal{P} \supseteq \phi(\mathfrak{J})$ and $\varpi' \in \mathfrak{p}$, it follows that $\mathfrak{c}_{v,i,j} \not\subseteq \mathfrak{p}$ for all v, $i \ne j$; by (Vx3)(b) the matrices $f(r_P(g))$ modulo $\mathfrak{p}$ span M_4 over the fraction field of $R/\mathfrak{p}$.

(A4) *The curve point.* The ring $R/\mathfrak{p}$ is a complete Noetherian local domain of dimension 1 with residue field $\mathbb{F}$ and of characteristic ℓ ($\ell \in \varpi'\mathcal{O}'$). Its normalisation is finite over it, hence a complete local normal domain of dimension 1, i.e. a complete discrete valuation ring of characteristic ℓ with finite residue field $k \supseteq \mathbb{F}$, so isomorphic to $k[[t]]$; this gives an injective local homomorphism $j\colon R/\mathfrak{p} \to k[[t]]$, and k is an $\mathcal{O}'$-algebra through $\mathcal{O}' \to \mathbb{F} \subseteq k$. Let $y\colon A \to k[[t]]$ be f followed by $R \to R/\mathfrak{p} \xrightarrow{j} k[[t]]$, an $\mathcal{O}'$-algebra homomorphism. Then: $y(\mathfrak{m}_B) \subseteq tk[[t]]$ (x kills $\mathfrak{m}_B$, as $x|_B$ is an $\mathcal{O}'$-algebra map from the local ring B to $\mathbb{F}$, so $f(\mathfrak{m}_B) \subseteq \mathfrak{m}_R$ by (Vx1), and j is local); $y(\mathfrak{J}) = 0$ ($\phi(\mathfrak{J}) \subseteq \mathfrak{p}$); for all v and $i \ne j$ some $s \in U_v$ has $y(\psi^P_{v,i}(s)) \ne y(\psi^P_{v,j}(s))$ ($\mathfrak{c}_{v,i,j} \not\subseteq \mathfrak{p}$ and j is injective); and $y \circ r_P$ is irreducible over the fraction field of $k[[t]]$, which contains that of $R/\mathfrak{p}$ (A3). By (Jq2), $\mathfrak{J}$ is local at y. The prime $\phi^{-1}(\mathcal{Q})$ of B satisfies $y(\phi^{-1}(\mathcal{Q})) \subseteq j(\mathcal{Q}/\mathfrak{p}) = 0$ since $\mathcal{Q} \subseteq \mathfrak{p}$; hence $\mathfrak{J} \subseteq \phi^{-1}(\mathcal{Q})$, i.e. $\phi(\mathfrak{J}) \subseteq \mathcal{Q}$. With (A1), (A2) this proves Part A.

Part B: along the chain. Let $c_0 = x_0(P), c_1, \dots, c_m = x_1(P)$ be a chain in case (III), with c_{t-1}, c_t linked by a residual pair L_t with values in $\mathbb{F}[X]$ ($1 \le t \le m$). By (Zq5) there are $\mathcal{O}'$-algebra homomorphisms $f_t\colon A \to \mathbb{F}[X]$ with $f_t(P) = L_t$ and $f_t(\mathfrak{m}_B) = 0$. Two $\mathcal{O}'$-algebra homomorphisms $A \to \mathbb{F}$ that kill $\mathfrak{m}_B$ and agree on the entries of P are equal: they agree on B (both are the reduction $B \to B/\mathfrak{m}_B = \mathbb{F}$) and the entries of P generate A over B (Zq1). Every $\mathcal{O}'$-algebra homomorphism $A \to \mathbb{F}$ kills $\mathfrak{m}_B$ (as in (A4)). Put $x^{(t)} = \mathrm{ev}_0 \circ f_{t+1}$ for $t < m$ and $x^{(m)} = x_1$; then $x^{(t)}(P) = c_t$, $x^{(0)} = x_0$ and $\mathrm{ev}_1 \circ f_t = x^{(t)}$ for $1 \le t \le m$. Each c_t is a residual pair good in case (III). We show by descending induction on t that $\mathfrak{J}$ is everywhere at $\ker x^{(t)}$. For $t = m$: by hypothesis $\mathfrak{J}$ is somewhere at $\ker x_1$, and Part A applies. For $t < m$, the ideal $\ker f_{t+1}$ is a prime of A ($\mathbb{F}[X]$ is a domain) contained in $\ker x^{(t)}$ and in $\ker x^{(t+1)}$; a minimal prime $\eta \subseteq \ker f_{t+1}$ of A exists and lies in $\ker x^{(t+1)}$, so it contains $\mathfrak{J}A$ by the induction hypothesis; as $\eta \subseteq \ker x^{(t)}$, $\mathfrak{J}$ is somewhere at $\ker x^{(t)}$, and Part A gives everywhere. For $t = 0$ this is the assertion. □

Proposition 53.32. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ a fit*

and deep auxiliary tower for $\mathfrak{D}$ with field F, index set Σ and $d = [F : \mathbb{Q}]$, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring, $\mathscr{X} = (B, A, P, \mathrm{cp}, (\psi^B_{v,i}))$ a space for $(\mathscr{T}, \mathcal{O}')$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $\mathfrak{J} \subseteq B$ an ideal modular for ι, and $C \geq 0$ a Selmer bound for $(\mathscr{T}, \mathcal{O} \to \mathbb{F}_{\mathcal{O}'})$ in degree d with $30(C+1)^2 \leq d$. Then:

(Vx0) *$d/\#\Sigma$ is an integer, and for every $v \in \Sigma$, all $i \neq j$ and every prime $\mathfrak{q}$ of B containing $\mathfrak{J}$, $\mathfrak{m}_{\mathcal{O}'}B$ and $\psi^B_{v,i}(s) - \psi^B_{v,j}(s)$ for all $s \in U_v$,*

$$\dim B/\mathfrak{q} + d/\#\Sigma \leq 2d + 1.$$

Moreover, for every $\mathcal{O}'$-algebra homomorphism $x\colon A \to \mathbb{F}_{\mathcal{O}'}$ such that $x(P)$ is a residual pair good in case (III), there are $R \in \mathrm{CNL}_{\mathcal{O}'}$ and an $\mathcal{O}'$-algebra homomorphism $f\colon A \to R$ such that:

(Vx1) *R is a flat A-module through f, and $f^{-1}(\mathfrak{m}_R) = \ker x$;*

(Vx2) *there are integers $a, b \geq 0$ with $b + 2 + 2d \leq a + 2\#\Sigma$ and a surjective $\mathcal{O}'$-algebra homomorphism $\mathcal{O}'[[X_1, \dots, X_a]] \to R$ whose kernel is generated by b elements;*

(Vx3) *there are finitely many ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ of R with $\dim R/\mathfrak{a}_k \leq d + 20C + 1$ for all k such that every prime $\mathfrak{p}$ of R containing none of the $\mathfrak{a}_k$ satisfies:*

- *(a) if $\mathfrak{p}$ contains $f(\mathfrak{J})R$, $\mathfrak{m}_{\mathcal{O}'}R$ and, for some $v \in \Sigma$ and $i \neq j$, all $f(\psi^P_{v,i}(s)) - f(\psi^P_{v,j}(s))$ with $s \in U_v$, then $\dim R/\mathfrak{p} \leq \dim B/\phi^{-1}(\mathfrak{p}) + 2 + 4\#\Sigma$, where $\phi\colon B \to A \xrightarrow{f} R$;*
- *(b) either the matrices $f(r_P(g))$ modulo $\mathfrak{p}$, $g \in G_F$, span M_4 over the fraction field of $R/\mathfrak{p}$, or there are $v \in \Sigma$ and $i \neq j$ with $f(\psi^P_{v,i}(s)) - f(\psi^P_{v,j}(s)) \in \mathfrak{p}$ for all $s \in U_v$.*

Proof. (Vx0). A fit tower is good, and an ideal modular for ι satisfies (Jq1): $\mathfrak{m}_B^n \subseteq \mathfrak{J} + \mathfrak{m}_{\mathcal{O}'}B + (\psi^B_{v,i}(s) - 1 : \ v \in \Sigma,\ 1 \leq i \leq 4,\ s \in I_v)$ for some $n \geq 1$. These are the hypotheses of Lemma 53.33, which gives that $d/\#\Sigma$ is an integer and the inequality of (Vx0).

(Vx1)–(Vx3). Under the hypotheses of the statement, Proposition 53.34 gives, for every x as in the statement, R and f with (Yw1)–(Yw3), which are (Vx1)–(Vx3). □

Lemma 53.33. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ a good auxiliary tower for $\mathfrak{D}$ with field F, index set Σ and $d = [F : \mathbb{Q}]$, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring, $\mathscr{X} = (B, A, P, \mathrm{cp}, (\psi^B_{v,i}))$ a space for $(\mathscr{T}, \mathcal{O}')$, and $\mathfrak{J} \subseteq B$ an ideal such that $\mathfrak{m}_B^n \subseteq \mathfrak{J} + \mathfrak{m}_{\mathcal{O}'}B + (\psi^B_{v,i}(s) - 1 : \ v \in \Sigma,\ 1 \leq i \leq 4,\ s \in I_v)$ for some $n \geq 1$. Then $d/\#\Sigma$ is an integer, and for every $v \in \Sigma$, all $i \neq j$ and every prime $\mathfrak{q}$ of B containing $\mathfrak{J}$, $\mathfrak{m}_{\mathcal{O}'}B$ and $\psi^B_{v,i}(s) - \psi^B_{v,j}(s)$ for all $s \in U_v$,*

$$\dim B/\mathfrak{q} + d/\#\Sigma \leq 2d + 1.$$

Proof. Step 0: the local degree. The places of F above ℓ correspond bijectively to the G_F-orbits of places of $\overline{\mathbb{Q}}$ above ℓ, hence to Σ. As $F/\mathbb{Q}$ is Galois, all of them have the same local degree $\delta = [F_u : \mathbb{Q}_\ell]$, and $\#\Sigma \cdot \delta = d$.

Step 1: the local characters. For $u \in \Sigma$ and $1 \leq m \leq 4$, the map $s \mapsto \psi^B_{u,m}(s)$ on D_u is a continuous homomorphism to $B^\times$ for the $\mathfrak{m}_B$-adic topology, trivial on μ_u, with $\psi^B_{u,m}(t) - 1 \in \mathfrak{m}_B$ for $t \in I_u$, and $\psi^B_{u,m}(s)\psi^B_{u,5-m}(s) = \theta(s)$ (image in B). Indeed $B \to A$ is injective (Zq1) and maps $\psi^B_{u,m}(s)$ to $\psi^P_{u,m}(s)$ (Zq2), the diagonal entries of the upper triangular $n_u^{-1}r_P(s)n_u$ (Mq4), which is a homomorphism in s, has similitude $\theta(s)$ (as n_u has similitude 1), and for an upper

triangular Y with ${}^tYJ_4Y = \mu J_4$ the entries $(1,4)$ and $(2,3)$ give $Y_{11}Y_{44} = Y_{22}Y_{33} = \mu$; triviality on μ_u is (Mq4); continuity and the congruence modulo $\mathfrak{m}_B$ to $\bar\chi_m^{\mathscr{T}}(t)$ are (Zq2), and $\bar\chi_m^{\mathscr{T}}(t) = 1$ for $t \in I_u$ by (Oq6).

Step 2: pinning elements. Call R *admissible* if it is a complete Noetherian local ring with finite residue field of characteristic ℓ; then $1+\mathfrak{m}_R$ is a pro-ℓ group. (P1) For every $u \in \Sigma$ there are $s_{u,1}, \dots, s_{u,\delta} \in I_u$ such that every continuous homomorphism $\kappa\colon D_u \to R^\times$, R admissible, with $\kappa(I_u) \subseteq 1+\mathfrak{m}_R$, $\kappa(\mu_u) = 1$ and $\kappa(s_{u,k}) = 1$ for all k, is trivial on I_u. Indeed, κ factors through D_u^{ab}, and by local class field theory the image of I_u in D_u^{ab} is topologically isomorphic to $\mathcal{O}_{F_u}^\times = \mu_{q-1} \times (1+\mathfrak{m}_{F_u})$, with $1+\mathfrak{m}_{F_u} \cong \Theta \times \mathbb{Z}_\ell^\delta$, Θ a finite ℓ-group (q the residue cardinality; structure of the units of a local field). κ kills μ_{q-1} (order prime to ℓ, target pro-ℓ); every $t \in I_u$ whose image lies in Θ has $t^{\ell^a} \in \overline{[D_u, D_u]}$ for some a, so $t \in \mu_u$ and $\kappa(t) = 1$; choose $s_{u,k} \in I_u$ whose images form a $\mathbb{Z}_\ell$-basis of the factor $\mathbb{Z}_\ell^\delta$; by continuity κ kills this factor. (P2) For $v \in \Sigma$ there is $s_0 \in I_v$ such that any two continuous homomorphisms $\kappa, \kappa'\colon D_v \to R^\times$, R admissible, with values in $1+\mathfrak{m}_R$ on I_v, equal on U_v and at s_0, are equal on I_v. Indeed, $\bar r_0(I_v) = 1$ by (Oq6), so $U_v = I_v \cap \ker\chi_{\mathrm{cyc}}$, and $\lambda = \kappa/\kappa'$ factors through $I_v/U_v \cong \chi_{\mathrm{cyc}}(I_v)$, a closed subgroup of $\mathbb{Z}_\ell^\times = \mu_{\ell-1} \times (1+\ell\mathbb{Z}_\ell)$, hence of the form $H_1 \times H_2$ with $H_1 \subseteq \mu_{\ell-1}$ and H_2 closed in $1+\ell\mathbb{Z}_\ell \cong \mathbb{Z}_\ell$, so procyclic; λ kills H_1; take s_0 with $\chi_{\mathrm{cyc}}(s_0) = (1, h_2)$, h_2 a topological generator of H_2; then $\lambda(s_0) = 1$ forces $\lambda = 1$.

Step 3: parameters. Fix v, $i \neq j$, $\mathfrak{q}$ as in the statement. Choose an index a: $a = j$ if $j \neq 5-i$, and otherwise any $a \notin \{i, 5-i\}$; in both cases $\{a, 5-a, i, 5-i\} = \{1,2,3,4\}$. Consider the family x of elements of B:

$$\psi^B_{u,a}(s_{u,k}) - 1 \ (u \in \Sigma,\ k \le \delta), \qquad \psi^B_{u,i}(s_{u,k}) - 1 \ (u \in \Sigma \setminus \{v\},\ k \le \delta), \qquad \psi^B_{v,i}(s_0) - \psi^B_{v,j}(s_0),$$

with $\#\Sigma\delta + (\#\Sigma - 1)\delta + 1 = 2d - \delta + 1$ members, all in $\mathfrak{m}_B$ (Step 1). *Claim: the only prime $\mathfrak{p}$ of B containing $\mathfrak{q}$ and all members of x is $\mathfrak{m}_B$.* Let $\mathfrak{p}$ be such, $R = B/\mathfrak{p}$ (admissible: $B \in \mathrm{CNL}_{\mathcal{O}'}$), and $\kappa_{u,m}$ the composite of $\psi^B_{u,m}$ with $B \to R$: continuous, values in $1+\mathfrak{m}_R$ on I_u, trivial on μ_u, with $\kappa_{u,m}\kappa_{u,5-m} = 1$ on I_u (the image of $\theta(t)$ in R is its residue, as $\mathfrak{m}_{\mathcal{O}'}B \subseteq \mathfrak{p}$, and $\bar\theta(t) = 1$ by (Oq6)). By (P1), $\kappa_{u,a} = 1$ on I_u for all u, and $\kappa_{u,i} = 1$ on I_u for $u \neq v$. By (P2), $\kappa_{v,i} = \kappa_{v,j}$ on I_v ($\mathfrak{p} \supseteq \mathfrak{q}$ contains $\psi^B_{v,i} - \psi^B_{v,j}$ on U_v, and the last member of x). If $a = j$, then $\kappa_{v,i} = \kappa_{v,a} = 1$ on I_v; otherwise $j = 5-i$ and $\kappa_{v,i}^2 = \kappa_{v,i}\kappa_{v,5-i} = 1$ on I_v, and an element $y \in 1+\mathfrak{m}_R$ with $y^2 = 1$ is 1 ($y+1 \equiv 2$ is a unit, ℓ odd). So $\kappa_{u,i} = 1$ and $\kappa_{u,a} = 1$ on I_u for all u, hence also $\kappa_{u,5-i} = \kappa_{u,5-a} = 1$, i.e. all $\psi^B_{u,m}(t) - 1$ ($t \in I_u$) lie in $\mathfrak{p}$. With $\mathfrak{J} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{m}_{\mathcal{O}'}B \subseteq \mathfrak{p}$, the hypothesis gives $\mathfrak{m}_B^n \subseteq \mathfrak{p}$, so $\mathfrak{p} = \mathfrak{m}_B$.

Step 4: the bound. By the claim, the images of the members of x generate an ideal of the Noetherian local ring $B/\mathfrak{q}$ whose radical is the maximal ideal. By Krull's dimension theorem (the dimension of a Noetherian local ring is at most the number of generators of any ideal primary to its maximal ideal), $\dim B/\mathfrak{q} \le 2d - \delta + 1$, i.e. $\dim B/\mathfrak{q} + d/\#\Sigma \le 2d + 1$. □

Proposition 53.34. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ a fit and deep auxiliary tower for $\mathfrak{D}$ with field F, index set Σ and $d = [F:\mathbb{Q}]$, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring, $\mathscr{X} = (B, A, P, \mathrm{cp}, (\psi^B_{v,i}))$ a space for $(\mathscr{T}, \mathcal{O}')$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $\mathfrak{J} \subseteq B$ an ideal modular for ι, and $C \ge 0$ a Selmer bound for $(\mathscr{T}, \mathcal{O} \to \mathbb{F}_{\mathcal{O}'})$ in degree d with $30(C+1)^2 \le d$. Then for every $\mathcal{O}'$-algebra homomorphism $x\colon A \to \mathbb{F}_{\mathcal{O}'}$ such that $x(P)$ is a residual pair good in case (III), there are $R \in \mathrm{CNL}_{\mathcal{O}'}$ and an $\mathcal{O}'$-algebra homomorphism $f\colon A \to R$ such that:*

(Yw1) *R is a flat A-module through f, and $f^{-1}(\mathfrak{m}_R) = \ker x$;*

(Yw2) *there are integers $a, b \ge 0$ with $b + 2 + 2d \le a + 2\#\Sigma$ and a surjective $\mathcal{O}'$-algebra homomorphism $\mathcal{O}'[[X_1, \dots, X_a]] \to R$ whose kernel is generated by b elements;*

(Yw3) *there are finitely many ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ of R with $\dim R/\mathfrak{a}_k \le d + 20C + 1$ for all k such that every prime $\mathfrak{p}$ of R containing none of the $\mathfrak{a}_k$ satisfies:*

- *(a) if $\mathfrak{p}$ contains $f(\mathfrak{J})R$, $\mathfrak{m}_{\mathcal{O}'}R$ and, for some $v \in \Sigma$ and $i \ne j$, all $f(\psi^P_{v,i}(s)) - f(\psi^P_{v,j}(s))$ with $s \in U_v$, then $\dim R/\mathfrak{p} \le \dim B/\phi^{-1}(\mathfrak{p}) + 2 + 4\#\Sigma$, where $\phi\colon B \to A \xrightarrow{f} R$;*
- *(b) either the matrices $f(r_P(g))$ modulo $\mathfrak{p}$, $g \in G_F$, span M_4 over the fraction field of $R/\mathfrak{p}$, or there are $v \in \Sigma$ and $i \ne j$ with $f(\psi^P_{v,i}(s)) - f(\psi^P_{v,j}(s)) \in \mathfrak{p}$ for all $s \in U_v$.*

Proof. Fix x as in the statement and put $\mathfrak{m} = \ker x$. We take for R the completion $\widehat{A} = \varprojlim_n A/\mathfrak{m}^n$ of A at $\mathfrak{m}$ and for $f\colon A \to R$ the canonical homomorphism; (Yw2) and (Yw3) for this pair are Lemma 53.35 and Proposition 53.39, whose hypotheses are those of the statement (Lemma 53.35 without ι and $\mathfrak{J}$). It remains to show that $R \in \mathrm{CNL}_{\mathcal{O}'}$ and (Yw1).

Step 1: A is Noetherian and $\mathfrak{m}$ is maximal with residue field $\mathbb{F}_{\mathcal{O}'}$. By (Zq1), $B \in \mathrm{CNL}_{\mathcal{O}'}$ is Noetherian and A is a B-algebra of finite type, hence a quotient of a polynomial ring in finitely many variables over B; by Hilbert's basis theorem A is Noetherian. The homomorphism x is $\mathcal{O}'$-linear, so its image contains the image of $\mathcal{O}' \to \mathbb{F}_{\mathcal{O}'}$, which is all of $\mathbb{F}_{\mathcal{O}'}$; thus x is surjective, $\mathfrak{m}$ is a maximal ideal and x induces an isomorphism $A/\mathfrak{m} \cong \mathbb{F}_{\mathcal{O}'}$ of $\mathcal{O}'$-algebras. Moreover $x(\mathfrak{m}_{\mathcal{O}'}A) = 0$ since $\mathfrak{m}_{\mathcal{O}'}$ maps to 0 in $\mathbb{F}_{\mathcal{O}'}$; so $\mathfrak{m}_{\mathcal{O}'}A \subseteq \mathfrak{m}$.

Step 2: $R \in \mathrm{CNL}_{\mathcal{O}'}$. Since $\mathfrak{m}$ is maximal, $A/\mathfrak{m}^n = A_{\mathfrak{m}}/\mathfrak{m}^n A_{\mathfrak{m}}$ for every n (every element of $A \setminus \mathfrak{m}$ is a unit modulo $\mathfrak{m}^n$), so R is also the $\mathfrak{m}A_{\mathfrak{m}}$-adic completion of the Noetherian local ring $A_{\mathfrak{m}}$. The completion of a Noetherian local ring at its maximal ideal is a complete Noetherian local ring with maximal ideal $\mathfrak{m}_R = \mathfrak{m}R$ and the same residue field (Matsumura, Commutative Ring Theory, Theorems 8.7, 8.8 and 8.11); thus $R/\mathfrak{m}_R = A/\mathfrak{m} \cong \mathbb{F}_{\mathcal{O}'}$, and the structure map $\mathcal{O}' \to R$ induces this isomorphism on residue fields. It is local, because $\mathfrak{m}_{\mathcal{O}'}$ maps into $\mathfrak{m}R = \mathfrak{m}_R$ by Step 1. Hence $R \in \mathrm{CNL}_{\mathcal{O}'}$.

Step 3: (Yw1). The homomorphism f is the composite $A \to A_{\mathfrak{m}} \to R$. Localisation $A \to A_{\mathfrak{m}}$ is flat, and the completion of the Noetherian ring $A_{\mathfrak{m}}$ at an ideal is flat over it (Matsumura, Theorem 8.8); a composite of flat homomorphisms is flat, so R is a flat A-module through f. Finally, the composite $A \xrightarrow{f} R \to R/\mathfrak{m}_R$ is the projection $A \to A/\mathfrak{m}$ followed by the isomorphism $A/\mathfrak{m} \cong R/\mathfrak{m}_R$ of Step 2, whose kernel is $\mathfrak{m}$; hence $f^{-1}(\mathfrak{m}_R) = \mathfrak{m} = \ker x$.

Steps 2 and 3 with Lemma 53.35 and Proposition 53.39 give R and f with (Yw1)–(Yw3). □

Lemma 53.35. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ a fit and deep auxiliary tower for $\mathfrak{D}$ with field F, index set Σ and $d = [F : \mathbb{Q}]$, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring, $\mathscr{X} = (B, A, P, \mathrm{cp}, (\psi^B_{v,i}))$ a space for $(\mathscr{T}, \mathcal{O}')$, and $C \ge 0$ a Selmer bound for $(\mathscr{T}, \mathcal{O} \to \mathbb{F}_{\mathcal{O}'})$ in degree d with $30(C+1)^2 \le d$. Let $x\colon A \to \mathbb{F}_{\mathcal{O}'}$ be an $\mathcal{O}'$-algebra homomorphism such that $x(P)$ is a residual pair good in case (III), and let R be the completion of A at the maximal ideal $\ker x$. Then there are integers $a, b \ge 0$ with $b + 2 + 2d \le a + 2\#\Sigma$ and a surjective $\mathcal{O}'$-algebra homomorphism $\mathcal{O}'[[X_1, \dots, X_a]] \to R$ whose kernel is generated by b elements.*

Proof. Let $\mathbb{F}'$ be the residue field of $\mathcal{O}'$ and ϖ a uniformiser. A residual pair good in case (III) is in particular a residual pair, and the tower is fit and deep; so Lemma 53.36 applies to x and gives $h^i = h^i(\Gamma; W, b)$ and an integer $0 \le m \le 2\#\Sigma$ with (Yd1)–(Yd3). (The Selmer bound C and goodness of $x(P)$ beyond being a residual pair are not used.)

Step 1: a presentation by a basis of the cotangent space. Let $a = \dim_{\mathbb{F}'} t_R^*$ (finite, R being Noetherian), $y_1, \dots, y_a \in \mathfrak{m}_R$ with classes forming a basis of t_R^*, $S = \mathcal{O}'[[X_1, \dots, X_a]]$ with maximal ideal $\mathfrak{n}$, and $\Pi\colon S \to R$ the continuous $\mathcal{O}'$-algebra homomorphism with $\Pi(X_i) = y_i$. Then $\mathfrak{m}_R = \varpi R + \sum_i y_i R + \mathfrak{m}_R^2$, so $\mathfrak{m}_R = \varpi R + \sum_i y_i R = \mathfrak{n}R$ by Nakayama's lemma; R is

$\mathfrak{n}$-adically separated, S is $\mathfrak{n}$-adically complete and $R/\mathfrak{n}R = \mathbb{F}'$ is generated by 1, so $R = \Pi(S)$ by the complete Nakayama lemma [166, Thm. 8.4]. The kernel is minimal in the sense of Lemma 49.21: if $f = c_0 + \sum_i c_i X_i + q \in \ker\Pi$ with $q \in (X_1, \dots, X_a)^2$, then $c_0 \in \mathfrak{m}_{\mathcal{O}'}$ (reduce modulo $\mathfrak{m}_R$), hence $\sum_i c_i y_i \in \varpi R + \mathfrak{m}_R^2$, and as the classes of the y_i are independent in t_R^*, all $c_i \in \mathfrak{m}_{\mathcal{O}'}$.

Step 2: the relations. Apply Lemma 49.21 to Π with the trivial group $\mathsf{A} = \{0\}$ (every ring is graded with $C = C_0$, every ideal homogeneous, every homomorphism graded) and $d(0) = h^2 + 2\#\Sigma - m \geq 0$. Condition (Ov) then asks: for every surjective $\mathcal{O}'$-algebra homomorphism $p\colon C \to \bar{C}$ of rings of finite length in $\mathrm{CNL}_{\mathcal{O}'}$ with kernel I, $\mathfrak{m}_C I = 0$, $\dim_{\mathbb{F}'} I > d(0)$, and every surjective $\bar{\xi}\colon R \to \bar{C}$, an ideal $I' \subseteq I$ of codimension 1 and a lift $\xi\colon R \to C/I'$ of $\bar{\xi}$; this is (Yd2). So $\ker\Pi$ is generated by $b \leq h^2 + 2\#\Sigma - m$ elements.

Step 3: the count. By (Yd1), $a \geq h^1 - h^0 + 2 - m$, and by (Yd3), $h^2 + 2d \leq h^1 - h^0$. Hence

$$b + 2 + 2d \leq h^2 + 2d + 2\#\Sigma - m + 2 \leq h^1 - h^0 + 2 - m + 2\#\Sigma \leq a + 2\#\Sigma.$$

So Π is a surjection $\mathcal{O}'[[X_1, \dots, X_a]] \to R$ whose kernel is generated by b elements, with $b + 2 + 2d \leq a + 2\#\Sigma$. □

Lemma 53.36. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ a fit and deep auxiliary tower for $\mathfrak{D}$ with field F, index set Σ and $d = [F : \mathbb{Q}]$, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring with residue field $\mathbb{F}'$, $\mathscr{X} = (B, A, P, \mathrm{cp}, (\psi_{v,i}^B))$ a space for $(\mathscr{T}, \mathcal{O}')$, and $x\colon A \to \mathbb{F}'$ an $\mathcal{O}'$-algebra homomorphism such that $c = x(P) = (r_c, (n_{c,v})_{v\in\Sigma})$ is a residual pair. Let R be the completion of A at the maximal ideal $\ker x$ and $t_R^* = \mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_{\mathcal{O}'}R)$. Let $\widetilde{N}$ be the closed normal subgroup of G_F generated by N, $\Gamma = G_F/\widetilde{N}$ with projection π, $W = \mathfrak{sp}_4(\mathbb{F}')$ with G_F acting by $g \cdot X = r_c(g) X r_c(g)^{-1}$, $\mathfrak{b} \subseteq W$ the subspace of upper triangular matrices, and $b_v = n_{c,v}\mathfrak{b}\, n_{c,v}^{-1}$ for $v \in \Sigma$. Then the action of G_F on W factors through a continuous $\mathbb{F}'$-linear action of Γ with open kernel and each b_v is stable under D_v, so that the groups $H^i(\Gamma; W, b)$ for the family $(D_v)_{v\in\Sigma}$ are defined; they are $\mathbb{F}'$-vector spaces, and h^i denotes their dimension over $\mathbb{F}'$. Moreover there is an integer m with $0 \leq m \leq 2\#\Sigma$ such that:*

(Yd1) $\dim_{\mathbb{F}'} t_R^* \geq h^1 - h^0 + 2 - m$;

(Yd2) *for every surjective $\mathcal{O}'$-algebra homomorphism $p\colon C \to \bar{C}$ between rings $C, \bar{C} \in \mathrm{CNL}_{\mathcal{O}'}$ of finite length whose kernel I satisfies $\mathfrak{m}_C I = 0$ and $\dim_{\mathbb{F}'} I > h^2 + 2\#\Sigma - m$, and every $\mathcal{O}'$-algebra homomorphism $\bar{\xi}\colon R \to \bar{C}$, there are an ideal $I' \subseteq I$ of C with $\dim_{\mathbb{F}'} I/I' = 1$ and an $\mathcal{O}'$-algebra homomorphism $\xi\colon R \to C/I'$ whose composite with the projection $C/I' \to \bar{C}$ is $\bar{\xi}$;*

(Yd3) *h^2 is finite and $h^1 - h^0 - h^2 \geq 2d$.*

Proof. Write $\mathbb{F} = \mathbb{F}_{\mathcal{O}}$, $J = J_4$, and a for the structure map $\mathcal{O} \to C$ of an $\mathcal{O}'$-algebra C. For $C \in \mathrm{CNL}_{\mathcal{O}'}$ and an ideal I with $\mathfrak{m}_C I = 0$, I is an $\mathbb{F}'$-vector space, $\mathrm{M}_4(I) = \mathrm{M}_4(\mathbb{F}') \otimes_{\mathbb{F}'} I$, and for $Y \in \mathrm{M}_4(I)$ and invertible $s \in \mathrm{M}_4(C)$ with residue s_0 one has $s(1+Y)s^{-1} = 1 + s_0 Y s_0^{-1}$ and $(1+Y)(1+Y') = 1 + Y + Y'$. Rings of finite length in $\mathrm{CNL}_{\mathcal{O}'}$ are finite (the residue field $\mathbb{F}'$ is finite), and a continuous map to them from a profinite group is locally constant. Let $\mathfrak{t}$, $\mathfrak{n}^-$ be the subspaces of diagonal and of strictly lower triangular matrices in W, and for a root w let W_w be the subspace of W of matrices supported on the entries (i,j) with $\mathrm{wt}_4(i,j) = w$.

Step 0: matrix facts. (a) If $X \in \mathrm{M}_4(C)$ has similitude $\nu \in C^\times$, then $\det(X)^2 = \nu^4$, so X is invertible; if moreover X is upper triangular (or diagonal), then $X_{11}X_{44} = X_{22}X_{33} = \nu$ (the entries $(1,4)$ and $(2,3)$ of ${}^tXJX$). (b) The equation ${}^tXJ + JX = 0$ links the entry (i,j)

only to the entry $(5-j, 5-i)$, which lies in the same one of the three regions (diagonal, strictly upper, strictly lower) and has the same weight ($\lambda_{5-k} = 1-\lambda_k$, $\mu_{5-k} = 1-\mu_k$). Hence $W = \mathfrak{t} \oplus \bigoplus_w W_w$ (w over the roots), $W = \mathfrak{b} \oplus \mathfrak{n}^-$, and $\mathfrak{t} = \{\mathrm{diag}(y_1, y_2, -y_2, -y_1)\}$ has dimension 2. As $J^2 = -1$, $X \in W$ implies ${}^tX \in W$, so transposition exchanges the strictly upper and strictly lower parts of W, and $\dim W = 10$ gives $\dim \mathfrak{b} = 6$, $\dim \mathfrak{n}^- = 4$. Conjugation by a matrix of similitude $\nu \in \mathbb{F}'^\times$ preserves W. (c) *Cayley transform.* Let C be a ring in which 2 is a unit. For strictly lower (resp. upper) triangular $Y \in \mathfrak{sp}_4(C)$, $X = (1+Y)(1-Y)^{-1}$ is lower (resp. upper) unitriangular of similitude 1, and every such X arises from exactly one Y, namely $Y = (X-1)(X+1)^{-1}$. Hence, for a surjection $C \to \bar{C}$ of rings in which 2 is a unit: every lower unitriangular matrix of similitude 1 over $\bar{C}$ lifts to one over C (lift Y entrywise); and every upper triangular (resp. diagonal) matrix $\bar{X}$ of similitude $\bar{\nu}$ over $\bar{C}$ lifts, for any lift $\nu \in C^\times$ of $\bar{\nu}$, to an upper triangular (resp. diagonal) matrix of similitude ν: write $\bar{X} = \bar{D}\bar{U}$ with $\bar{D}$ its diagonal (similitude $\bar{\nu}$ by (a)) and $\bar{U}$ upper unitriangular of similitude 1, lift $\bar{D} = \mathrm{diag}(\bar{d}_1, \bar{d}_2, \bar{\nu}/\bar{d}_2, \bar{\nu}/\bar{d}_1)$ by $\mathrm{diag}(d_1, d_2, \nu/d_2, \nu/d_1)$ and $\bar{U}$ by the Cayley transform. (d) Let $C \to \bar{C}$ be surjective in $\mathrm{CNL}_{\mathcal{O}'}$ with kernel I, $\mathfrak{m}_C I = 0$, $\nu \in C^\times$ with residue $\nu_0 \in \mathbb{F}'$. Every $\bar{X} \in \mathrm{M}_4(\bar{C})$ of similitude $\bar{\nu}$ lifts to some X of similitude ν: take any lift X_0; $E = {}^tX_0 J X_0 - \nu J \in \mathrm{M}_4(I)$ is antisymmetric; choose $Z \in \mathrm{M}_4(I)$ with $JZ = -\frac{1}{2}\nu_0^{-1}E$; then ${}^tZJ + JZ = JZ - {}^t(JZ) = -\nu_0^{-1}E$ and $X = X_0(1+Z)$ has ${}^tXJX = \nu J + E + \nu_0({}^tZJ + JZ) = \nu J$.

Step 1: the action, θ and the μ-conditions. (i) N is stable under conjugation in G_F ($\bar{r}_0$ has abelian image) and $r_c(N) = 1$ by (Mq1); as r_c is a locally constant homomorphism into $\mathrm{GL}_4(\mathbb{F}')$, its kernel is an open normal subgroup containing N, hence $\widetilde{N}$. So the action on W (which preserves W by Step 0 (b), $r_c(g)$ having similitude $\theta(g)$) factors through Γ, is $\mathbb{F}'$-linear and has open kernel. The same argument shows: for every marked ordinary pair (or pair satisfying (Mq1)) $(r, (n_v))$ with values in a finite ring, locally constant, $r(\widetilde{N}) = 1$. (ii) $\theta(\widetilde{N}) = 1$ in $\mathcal{O}$, and $\bar{\omega}_\ell(\widetilde{N}) = 1$. For $g \in N$ (in $I_{\bar{w}}$, $\bar{w} \nmid \ell$, with $\bar{r}_0(g) = 1$): $\chi_{\mathrm{cyc}}(g) = \omega_\ell(g) = 1$ (unramified away from ℓ); $\chi(g) = 1$ by (Oq5) (a fit tower is good); $\bar{\theta}(g) = 1$ gives $\bar{\xi}(g) = 1$, and as ξ has order prime to ℓ, $\xi(g) = 1$. So $\theta(g) = \chi_{\mathrm{cyc}}(g)^{e+1}\omega_\ell(g)^{-1}\xi(g)\chi(g) = 1$; θ being a character, $\theta(\widetilde{N}) = 1$. (iii) Each b_v is D_v-stable: for $s \in D_v$, $\bar{u}(s) = n_{c,v}^{-1} r_c(s) n_{c,v}$ is invertible and upper triangular (Mq4) with a similitude, so $s \cdot b_v = n_{c,v}\bar{u}(s)\mathfrak{b}\,\bar{u}(s)^{-1}n_{c,v}^{-1} = b_v$. All cochain spaces defining $H^i(\Gamma; W, b)$ are $\mathbb{F}'$-spaces, hence so are the H^i. (iv) Fix $v \in \Sigma$. By local class field theory $D_v^{\mathrm{ab}} = D_v/\overline{[D_v, D_v]}$ is the profinite completion of $F_v^\times$, whose torsion subgroup is the finite cyclic group of roots of unity in F_v; so the elements of ℓ-power order in D_v^{ab} form a finite cyclic group, generated by the image of some $s_v \in D_v$, and μ_v is its preimage. For $s \in \mu_v$ and a continuous character γ of D_v, $\gamma(s)^{\ell^k} = 1$ for some k. Applied to χ_{cyc}: $\chi_{\mathrm{cyc}}(s)$ is a torsion element of $\mathbb{Z}_\ell^\times$ of ℓ-power order, hence 1 (ℓ odd); to ω_ℓ, ξ and χ, whose orders on D_v are prime to ℓ (for χ because the tower is deep): they are 1. Hence $\theta(s) = 1$ for $s \in \mu_v$. (v) Let C be a finite $\mathcal{O}'$-algebra and $Q = (r, (n_v))$ a locally constant tuple satisfying (Mq1)–(Mq3) and the upper triangularity in (Mq4). Then each $\psi_{v,i}^Q$ is a locally constant homomorphism $D_v \to C^\times$ (diagonal of an upper triangular representation), hence trivial on $\overline{[D_v, D_v]}$, and $\psi_{v,1}^Q\psi_{v,4}^Q = \psi_{v,2}^Q\psi_{v,3}^Q = a \circ \theta$ on D_v by Step 0 (a) (n_v has similitude 1). So, by (iv): *Q satisfies (Mq4) if and only if $\psi_{v,1}^Q(s_v) = \psi_{v,2}^Q(s_v) = 1$ for all $v \in \Sigma$* (then $\psi_{v,3}^Q(s_v) = \psi_{v,4}^Q(s_v) = \theta(s_v) = 1$, and every $s \in \mu_v$ has image a power of that of s_v in D_v^{ab}).

Step 2: two module facts. (i) $\mathrm{Hom}_\Gamma(W, \mathbb{F}(\bar{\omega}_\ell)) = 0$ ($\mathbb{F}$-linear maps). By (Wq1), (Wq2), $r_c(g) = \Delta(g)(1+U(g))$ with $\Delta(g) = \mathrm{diag}(\bar{\chi}_i^{\mathscr{J}}(g))$ and $U(g)$ supported in $\Sigma_0 \in \{\Sigma_P, \Sigma_M\}$. Let $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ be $f(w) = 2w_1 + 3w_2$ if $\Sigma_0 = \Sigma_M$ and $f(w) = 2w_1 + w_2$ if $\Sigma_0 = \Sigma_P$; then $f < 0$ on Σ_0 For $m \in \mathbb{Z}$ let $M_{<m}$ be the span of the matrix units E_{ij} with $f(\mathrm{wt}_4(i,j)) < m$ ($\mathrm{wt}_4(i,i) = 0$), and $W_{<m} = W \cap M_{<m}$, $W_{\le m} = W_{<m+1}$. Since $\mathrm{wt}_4(i,k) = \mathrm{wt}_4(i,j) + \mathrm{wt}_4(j,k)$, $U(g)$ is nilpotent and $(1+U(g))X(1+U(g))^{-1} \in X + M_{<f(w)}$ for X supported in weight w; and $\Delta(g)X\Delta(g)^{-1} = \overline{\mathrm{rt}_w}(g)X$, because $\chi_1^{\mathscr{J}}\chi_4^{\mathscr{J}} = \chi_2^{\mathscr{J}}\chi_3^{\mathscr{J}}$ ($\theta = \alpha\Psi$) gives $\chi_i^{\mathscr{J}}/\chi_j^{\mathscr{J}} = \mathrm{rt}_{\mathrm{wt}_4(i,j)}$. So

the $W_{\le m}$ are G_F-stable and g acts on $W_{\le m}/W_{<m}$ by $\overline{\mathrm{rt}_w}(g)$ on the image of W_w ($f(w)=m$) and trivially on that of $\mathfrak{t}$ ($m=0$). If $\phi\neq 0$ were a Γ-map $W\to\mathbb{F}(\bar\omega_\ell)$, take m minimal with $\phi(W_{\le m})\neq 0$ and $X\in W_w$ ($f(w)=m$, or $w=0$ and $X\in\mathfrak{t}$) with $\phi(X)\neq 0$: then $\bar\omega_\ell(g)\phi(X)=\phi(g\cdot X)=\overline{\mathrm{rt}_w}(g)\phi(X)$ for all $g\in G_F$, contradicting the first clause of (Oq3). (ii) For a real place u of F and a complex conjugation $c_u\in G_F$ at u: c_u is conjugate in $G_{\mathbb{Q}}$ to c, so $\theta(c_u)=\theta(c)=\chi_{\mathrm{cyc}}(c)\omega_\ell(c)^{-1}\xi(c)\Psi(c)=(-1)(-1)(-1)(1)=-1$ by (Sg1). Thus $\gamma=r_c(c_u)$ has $\gamma^2=1$ and similitude -1, and Lemma 53.37 (over $\mathbb{F}'$, of characteristic $\ell\neq 2$) gives $\dim_{\mathbb{F}'}W^{c_u=1}=4$.

Step 3: (Yd3). Apply Proposition 48.9 with the totally real field F and $K'=G_F\cap\ker\bar r_0$, open normal of index dividing $\#(\mathbb{F}^\times)^4$, prime to ℓ. The closed normal subgroup generated by the $I_{\bar u}\cap K'$, $\bar u\nmid\ell$, is $\widetilde{N}$, since their union is N. The places of F above ℓ correspond to Σ ($v\mapsto$ the place below $\bar v$; every place of $\overline{\mathbb{Q}}$ above ℓ is G_F-conjugate to exactly one $\bar v$), with $D_v=D_{\bar v}\cap G_F$. Take W as an $\mathbb{F}$-space with the action of Step 1 (i) and the D_v-stable b_v (Step 1 (iii)); the hypothesis on Hom is Step 2 (i). So $H^2(\Gamma;W,b)$ is finite-dimensional and, all spaces being $\mathbb{F}'$-spaces whose $\mathbb{F}$-dimensions are $[\mathbb{F}':\mathbb{F}]$ times their $\mathbb{F}'$-dimensions, dividing by $[\mathbb{F}':\mathbb{F}]$ and using $\dim b_v=6$ (Step 0 (b)), $\sum_v[F_v:\mathbb{Q}_\ell]=d$, the d real places of F and Step 2 (ii):

$$h^2+h^0+6d\le h^1+4d.$$

This is (Yd3) (finiteness of h^1 is shown in Step 5).

Step 4: Artinian points of R. Let $\mathfrak{m}=\ker x$, $f_R\colon A\to R$ the canonical map and $P_R=f_R(P)$. Since $A/\mathfrak{m}^k=R/\mathfrak{m}^kR$, $f_R^{-1}(\mathfrak{m}_R)=\mathfrak{m}$ and $R=f_R(A)+\mathfrak{m}_R^k$ for all k. For $C\in\mathrm{CNL}_{\mathcal{O}'}$ of finite length let $\mathcal{D}(C)$ be the set of marked ordinary pairs over $(\mathscr{T},\emptyset)$ with values in C, continuous for $\mathfrak{m}_C$, with reduction c, and $\mathcal{D}'(C)\supseteq\mathcal{D}(C)$ the same with only the upper triangularity in (Mq4). *Claim: $\xi\mapsto\xi(P_R)$ is a bijection from the $\mathcal{O}'$-algebra homomorphisms $R\to C$ onto $\mathcal{D}(C)$.* Every $\mathcal{O}'$-algebra homomorphism from R or B to C or to $\mathbb{F}'$ is local (its composite to $\mathbb{F}'$ is surjective, so its kernel is the maximal ideal). Given ξ, $\xi(P_R)$ is a marked ordinary pair (all conditions are identities between entries, preserved by ring homomorphisms), its reduction is the image of P under $A\to R\to R/\mathfrak{m}_R=A/\mathfrak{m}$, i.e. $x(P)=c$, and it is continuous because $\xi f_R(\mathfrak{m}_BA)\subseteq\mathfrak{m}_C$ and P is continuous for $\mathfrak{m}_BA$ (Zq1). Injectivity: if $\xi_1(P_R)=\xi_2(P_R)$, the homomorphisms $y_j=\xi_jf_R$ agree on the entries of all $r_P(g)$ and n_v, hence on the images in A of the coefficients of $\mathrm{cp}(g)$ and of the $\psi^B_{v,i}(s)$ (Zq2); by (Zq3) $y_1|_B=y_2|_B$, and by the generation in (Zq1) $y_1=y_2$; as $\mathfrak{m}_C^k=0$ for some k and $R=f_R(A)+\mathfrak{m}_R^k$, $\xi_1=\xi_2$. Surjectivity: for $Q\in\mathcal{D}(C)$, (Zq4) gives $y\colon A\to C$ with $y(P)=Q$; the composite $A\to C\to\mathbb{F}'$ sends P to $c=x(P)$, so equals x by the injectivity argument (with $C=\mathbb{F}'$); hence $y(\mathfrak{m})\subseteq\mathfrak{m}_C$, $y(\mathfrak{m}^k)=0$, and y factors through $A/\mathfrak{m}^k=R/\mathfrak{m}^kR$, giving ξ with $\xi(P_R)=Q$. The bijection is compatible with homomorphisms $C\to C'$.

Step 5: tangent spaces, (Yd1). Let $\mathbb{F}'[\varepsilon]$ be the dual numbers. The $\mathcal{O}'$-algebra homomorphisms $R\to\mathbb{F}'[\varepsilon]$ form an $\mathbb{F}'$-vector space T dual to t_R^* (classical; $\xi\mapsto$ the ε-coefficient of ξ on $\mathfrak{m}_R$), finite-dimensional as R is Noetherian. An element of $\mathcal{D}'(\mathbb{F}'[\varepsilon])$ is uniquely $r(g)=(1+\varepsilon z(g))r_c(g)$, $n_v=n_{c,v}(1+\varepsilon y_v)$ with $z(g),y_v\in\mathrm{M}_4(\mathbb{F}')$, and, writing $Y_v=n_{c,v}y_vn_{c,v}^{-1}$ and $\bar u(s)=n_{c,v}^{-1}r_c(s)n_{c,v}$: similitude $a(\theta(g))=\bar\theta(g)$ holds iff $z(g)\in W$; r is a homomorphism iff z is a 1-cocycle; $r(N)=1$ iff $z(\widetilde{N})=0$ (Step 1 (i)), i.e. iff z factors through Γ; continuity iff z is locally constant; (Mq2) iff $z(g_{\mathscr{T}})\in\mathfrak{t}$; (Mq3) iff $y_v\in\mathfrak{n}^-$; and

$$n_v^{-1}r(s)n_v=(1+\varepsilon\,n_{c,v}^{-1}(z(s)+(s-1)Y_v)n_{c,v})\bar u(s),$$

upper triangular iff $z(s)+(s-1)Y_v\in b_v$ ($s\in D_v$). So $\mathcal{D}'(\mathbb{F}'[\varepsilon])$ is in $\mathbb{F}'$-linear bijection with

$$T'=\{(z,(y_v)):\ z\in Z^1(\Gamma,W)\text{ continuous},\ z(\pi(g_{\mathscr{T}}))\in\mathfrak{t},\ y_v\in\mathfrak{n}^-,\\ z(\pi(s))+(s-1)n_{c,v}y_vn_{c,v}^{-1}\in b_v\ \forall v,\ s\in D_v\},$$

and T with the subspace corresponding to $\mathcal{D}(\mathbb{F}'[\varepsilon])$ (Step 4; linearity: the entries of $\xi(P_R)$ are ξ applied to entries of P_R). *The μ-map.* For $t=(z,(y_v))\in T'$ let $\delta_{v,i}(t)$ be the (i,i) entry of $n_{c,v}^{-1}(z(\pi(s_v))+(s_v-1)Y_v)n_{c,v}$. The corresponding pair has $\psi_{v,i}(s_v)=(1+\varepsilon\delta_{v,i}(t))\psi^c_{v,i}(s_v)=1+\varepsilon\delta_{v,i}(t)$ (diagonal of a product of upper triangular matrices; $\psi^c_{v,i}(s_v)=1$ by (Mq4) for c). Let $\mu_T\colon T'\to\mathbb{F}'^{\Sigma\times\{1,2\}}$, $t\mapsto(\delta_{v,i}(t))_{v,i\le 2}$, and $m=\operatorname{rank}\mu_T\le 2\#\Sigma$. By Step 1 (v), $T\cong\ker\mu_T$. Hence $\dim T'=\dim t_R^*+m<\infty$. *Counting T'.* Let $T_0'\supseteq T'$ be defined without the condition at $g_{\mathscr{T}}$, $\mathcal{Z}$ the space of pairs $(z,(X_v))$ with z a continuous cocycle and $z(\pi(s))-(s-1)X_v\in b_v$, and $\mathcal{B}\subseteq\mathcal{Z}$ the pairs $((g-1)X,(X_v))$ with $X_v-X\in b_v$, so $H^1(\Gamma;W,b)=\mathcal{Z}/\mathcal{B}$. The map $T_0'\times\prod_v b_v\to\mathcal{Z}$, $((z,(y_v)),(\beta_v))\mapsto(z,(\beta_v-n_{c,v}y_vn_{c,v}^{-1}))$, is well defined ($b_v$ is D_v-stable) and bijective, because $W=b_v\oplus n_{c,v}\mathfrak{n}^-n_{c,v}^{-1}$ (Step 0 (b), conjugated by $n_{c,v}$). The map $W\times\prod_v b_v\to\mathcal{B}$, $(X,(\beta_v))\mapsto((g-1)X,(X+\beta_v))$, is surjective with kernel $\{(X,(-X)):X\in W^\Gamma\cap\bigcap_v b_v\}\cong H^0(\Gamma;W,b)$. As T_0' is finite-dimensional (T' has codimension at most $\dim W/\mathfrak{t}=8$ in it), so are $\mathcal{Z}$ and H^1, and

$$\dim T_0'=\dim\mathcal{Z}-6\#\Sigma=h^1+(10+6\#\Sigma-h^0)-6\#\Sigma=h^1-h^0+10,\qquad \dim T'\ge h^1-h^0+2.$$

So $\dim_{\mathbb{F}'}t_R^*=\dim T'-m\ge h^1-h^0+2-m$: (Yd1).

Step 6: obstructions, (Yd2). Let $p\colon C\to\bar C$, I, $\bar\xi$ be as in (Yd2), and $\bar Q=\bar\xi(P_R)=(\bar r',(\bar n_v'))\in\mathcal{D}(\bar C)$ (Step 4); its reduction is c. For an ideal $I_0\subseteq I$, $C/I_0\to\bar C$ has kernel I/I_0, killed by $\mathfrak{m}_C$. (i) *Global cochain.* $\bar r'$ is locally constant with $\bar r'(\widetilde{N})=1$ (Step 1 (i)) and $a\circ\theta\colon G_F\to C^\times$ is locally constant with $\theta(\widetilde{N})=1$ (Step 1 (ii)); both factor through Γ. Choose an open normal $U\subseteq\Gamma$ on which both are trivial, representatives of Γ/U (with 1 for U), lifts $\sigma(\gamma)\in\mathrm{M}_4(C)$ of $\bar r'(\gamma)$ of similitude $a(\theta(\gamma))$ (Step 0 (d); $\sigma(1)=1$), and put $\sigma(\gamma u)=\sigma(\gamma)$ for $u\in U$. Define $k(g,g')\in\mathrm{M}_4(I)$ by $\sigma(g)\sigma(g')=(1+k(g,g'))\sigma(gg')$ (both sides lift $\bar r'(gg')$; $\sigma(gg')$ is invertible by Step 0 (a)). As $1+k$ has similitude 1, $k(g,g')\in W\otimes I$; k is locally constant, and expanding $\sigma(g)\sigma(g')\sigma(g'')$ in two ways gives $k(g,g')+k(gg',g'')=g\cdot k(g',g'')+k(g,g'g'')$: a continuous 2-cocycle. (ii) *Local cochains.* Fix v. By Step 0 (c) let $n_v\in\mathrm{M}_4(C)$ be a lower unitriangular lift of $\bar n_v'$ of similitude 1; its residue is $n_{c,v}$. The map $\bar u'(s)=\bar n_v'^{-1}\bar r'(\pi(s))\bar n_v'$ is upper triangular of similitude $a(\theta(s))$; as in (i), using Step 0 (c), choose a locally constant upper triangular lift $u\colon D_v\to\mathrm{M}_4(C)$ of similitude $a\circ\theta$ with $u(1)=1$. Define $k_v(s)\in W\otimes I$ by $\sigma(\pi(s))=(1+k_v(s))n_vu(s)n_v^{-1}$ and the upper triangular $\beta_v(s,s')\in\mathfrak{b}\otimes I$ by $u(s)u(s')=(1+\beta_v(s,s'))u(ss')$. Then

$$\sigma(\pi(s))\sigma(\pi(s'))=\big(1+k_v(s)+s\cdot k_v(s')+n_{c,v}\beta_v(s,s')n_{c,v}^{-1}-k_v(ss')\big)\sigma(\pi(ss')),$$

so $k(\pi(s),\pi(s'))-\big(s\cdot k_v(s')-k_v(ss')+k_v(s)\big)\in b_v\otimes I$. (iii) *The class.* For $\lambda\in I^\vee=\mathrm{Hom}_{\mathbb{F}'}(I,\mathbb{F}')$, $(\lambda\circ k,(\lambda\circ k_v)_v)$ is a pair as in the definition of $H^2(\Gamma;W,b)$; let $o(\lambda)$ be its class, $\mathbb{F}'$-linear in λ. Then $V=\ker o$ has $\dim V\ge\dim I-h^2$. Let $I_1=\bigcap_{\lambda\in V}\ker\lambda$, an ideal with $\dim I/I_1=\dim V$; choose a basis $\lambda_1,\dots,\lambda_q$ of V and $i_1,\dots,i_q\in I$ with $\lambda_a(i_b)=\delta_{ab}$, so that $w\equiv\sum_b(\mathrm{id}\otimes\lambda_b)(w)\otimes i_b$ modulo $W\otimes I_1$ for $w\in W\otimes I$. As $o(\lambda_b)=0$ there are continuous $\beta_b\colon\Gamma\to W$ and $Y_{b,v}\in W$ with $\lambda_b\circ k=d\beta_b$, $(d\beta)(g,g')=g\cdot\beta(g')-\beta(gg')+\beta(g)$, and $\lambda_b(k_v(s))-\beta_b(\pi(s))+(s-1)Y_{b,v}\in b_v$. Put $\beta=\sum_b\beta_b\otimes i_b$, $Y_v=\sum_b Y_{b,v}\otimes i_b$. Then, modulo I_1: $k=d\beta$ and $b_s:=k_v(s)-\beta(\pi(s))+(s-1)Y_v\in b_v\otimes(I/I_1)$. (iv) *A lift without the μ-condition over $C_1=C/I_1$.* Over C_1 put $\sigma'(g)=(1-\beta(g))\sigma(g)$. Then $\sigma'(g)\sigma'(g')=(1-\beta(g)-g\cdot\beta(g')+k(g,g'))\sigma(gg')=\sigma'(gg')$, so $r'(g)=\sigma'(\pi(g))$ is a locally constant homomorphism of similitude $a\circ\theta$ with $r'(N)=1$, lifting $\bar r'$. For $s\in D_v$, $\sigma'(\pi(s))(1+Y_v)n_v=(1-\beta(\pi(s))+k_v(s)+s\cdot Y_v)n_vu(s)=(1+b_s)(1+Y_v)n_vu(s)$, so with $\tilde n_v=(1+Y_v)n_v$, $\tilde n_v^{-1}r'(s)\tilde n_v=(1+n_{c,v}^{-1}b_sn_{c,v})u(s)$ is upper triangular. *Normalising the flag:* write $n_{c,v}^{-1}Y_vn_{c,v}=Y^{\mathfrak{b}}+Y^-$ in $(\mathfrak{b}\oplus\mathfrak{n}^-)\otimes(I/I_1)$ and $n_v'=\tilde n_v(1-Y^{\mathfrak{b}})=n_v(1+Y^-)$: lower unitriangular of similitude 1, and $n_v'^{-1}r'(s)n_v'=(1+Y^{\mathfrak{b}})\tilde n_v^{-1}r'(s)\tilde n_v(1-Y^{\mathfrak{b}})$ is upper triangular. *The element $g_{\mathscr{T}}$:* $\bar r'(g_{\mathscr{T}})$ is diagonal; let D_1 be a diagonal lift over C_1 of similitude $a(\theta(g_{\mathscr{T}}))$ (Step 0 (c)), so $r'(g_{\mathscr{T}})=(1+\kappa)D_1$ with $\kappa=\kappa_{\mathfrak{t}}+\kappa_{\mathrm{off}}\in(\mathfrak{t}\oplus\bigoplus_w W_w)\otimes(I/I_1)$. The residue

$r_c(g_{\mathscr{T}}) = \mathrm{diag}(\bar{\chi}_i^{\mathscr{T}}(g_{\mathscr{T}}))$ ((Mq2), (Wq1)) acts on W_w, $w = \mathrm{wt}_4(i,j)$, by $\bar{\chi}_i^{\mathscr{T}}(g_{\mathscr{T}})/\bar{\chi}_j^{\mathscr{T}}(g_{\mathscr{T}}) \neq 1$ (Oq7); so there is $Y_{\mathscr{T}} \in \bigoplus_w W_w \otimes (I/I_1)$ with $Y_{\mathscr{T}} - g_{\mathscr{T}} \cdot Y_{\mathscr{T}} = -\kappa_{\mathrm{off}}$, and $r'' = (1+Y_{\mathscr{T}})r'(1-Y_{\mathscr{T}})$ has $r''(g_{\mathscr{T}}) = (1+\kappa_{\mathfrak{t}})D_1$ diagonal, similitude $a \circ \theta$, $r''(N) = 1$. Normalise $(1+Y_{\mathscr{T}})n'_v$ as above to a lower unitriangular n''_v of similitude 1 with $n''^{-1}_v r''(s) n''_v$ upper triangular. Thus $Q_1 = (r'', (n''_v)) \in \mathcal{D}'(C_1)$ lifts $\bar{Q}$. (v) *The μ-conditions.* For $t = (z, (y_v)) \in T'$ and $j \in I/I_1$ put $Q_1 \cdot (t,j) = \big((1+z(\pi(g))j)r''(g), (n''_v(1+y_v j))\big)$. The computations of Step 5, with ε replaced by j (using $\mathfrak{m}_{C_1}(I/I_1) = 0$), show $Q_1 \cdot (t,j) \in \mathcal{D}'(C_1)$, lifting $\bar{Q}$, with $\psi_{v,i}(s_v) = \psi_{v,i}^{Q_1}(s_v) + \delta_{v,i}(t)j$ (as $\psi_{v,i}^{Q_1}(s_v) \equiv 1$ modulo $\mathfrak{m}_{C_1}$). Let $e_{v,i} = \psi_{v,i}^{Q_1}(s_v) - 1$ $(i = 1,2)$, which lies in I/I_1 because $\bar{Q}$ satisfies (Mq4). The $\mathbb{F}'$-linear map $(I/I_1)^\vee \to \mathbb{F}'^{\Sigma\times\{1,2\}}/\mu_T(T')$, $\lambda \mapsto$ the class of $(\lambda(e_{v,i}))$, has source of dimension $\dim V \geq \dim I - h^2 > 2\#\Sigma - m$, the dimension of the target; take $\lambda \neq 0$ in its kernel, $t \in T'$ with $\mu_T(t) = (\lambda(e_{v,i}))$, and $j_0 \in I/I_1$ with $\lambda(j_0) = 1$. Then $Q_2 = Q_1 \cdot (t, -j_0)$ has $\psi_{v,i}^{Q_2}(s_v) - 1 = e_{v,i} - \delta_{v,i}(t)j_0 \in \ker\lambda$ for $i = 1,2$. Let $I' \subseteq I$ be the preimage of $\ker\lambda$: an ideal with $\dim I/I' = 1$. The image Q_3 of Q_2 over C/I' has $\psi_{v,1}(s_v) = \psi_{v,2}(s_v) = 1$ for all v, so $Q_3 \in \mathcal{D}(C/I')$ by Step 1 (v), lifting $\bar{Q}$. (vi) By Step 4 there is $\xi\colon R \to C/I'$ with $\xi(P_R) = Q_3$; its composite with $C/I' \to \bar{C}$ sends P_R to $\bar{Q} = \bar{\xi}(P_R)$, hence is $\bar{\xi}$ by the injectivity in Step 4. This proves (Yd2). □

Lemma 53.37. *Let k be a field of characteristic different from 2, $\mathfrak{sp}_4(k) = \{X \in \mathrm{M}_4(k) : {}^tXJ_4 + J_4X = 0\}$, and $\gamma \in \mathrm{GL}_4(k)$ with $\gamma^2 = 1$ and similitude -1, i.e. ${}^t\gamma J_4\gamma = -J_4$. Then $X \mapsto \gamma X\gamma^{-1}$ preserves $\mathfrak{sp}_4(k)$, and its eigenspaces for the eigenvalues -1 and $+1$ have dimensions 6 and 4. For $\gamma = \mathrm{diag}(-1,1,-1,1)$ the (-1)-eigenspace is the sum of the root spaces $\mathfrak{sp}_4(k) \cap \big(\bigoplus_{\mathrm{wt}_4(i,j)=w} kE_{ij}\big)$ of the six odd roots w, and the $(+1)$-eigenspace is the sum of the diagonal part and the root spaces of $\pm w_\Psi$.*

Proof. Preservation. From ${}^t\gamma J_4\gamma = -J_4$: ${}^t\gamma J_4 = -J_4\gamma^{-1}$ and $J_4\gamma = -{}^t\gamma^{-1}J_4$. For $X \in \mathfrak{sp}_4(k)$ and $Y = \gamma X\gamma^{-1}$,

$$ {}^tY J_4 + J_4 Y = {}^t\gamma^{-1}\,{}^tX\,({}^t\gamma J_4) + (J_4\gamma)X\gamma^{-1} = -{}^t\gamma^{-1}\big({}^tXJ_4 + J_4X\big)\gamma^{-1} = 0. $$

The eigenspaces of γ on V are Lagrangian. As $\gamma^2 = 1$ and $2 \in k^\times$, $V = V_+ \oplus V_-$ with $V_\pm = \ker(\gamma \mp 1)$ (projectors $(1 \pm \gamma)/2$). For $v, w \in V_\varepsilon$, $\langle \gamma v, \gamma w\rangle = {}^tv\,{}^t\gamma J_4\gamma\, w = -\langle v, w\rangle$ and $\langle \gamma v, \gamma w\rangle = \varepsilon^2\langle v, w\rangle = \langle v, w\rangle$; so $2\langle v, w\rangle = 0$ and V_ε is isotropic. Isotropic subspaces of a non-degenerate alternating space of dimension 4 have dimension at most 2, and $\dim V_+ + \dim V_- = 4$; so $\dim V_\pm = 2$.

Dimensions. Since ${}^tJ_4 = -J_4$, $X \in \mathfrak{sp}_4(k)$ if and only if J_4X is symmetric (${}^t(J_4X) = -{}^tXJ_4$), so $X \mapsto J_4X$ is an isomorphism of $\mathfrak{sp}_4(k)$ onto the space Sym of symmetric 4×4 matrices, i.e. of symmetric bilinear forms $b_X(v,w) = {}^tvJ_4Xw$ on V. Using $\gamma^{-1} = \gamma$ and $J_4\gamma = -{}^t\gamma J_4$ (the second identity above with ${}^t\gamma^{-1} = {}^t\gamma$),

$$ J_4(\gamma X\gamma^{-1}) = -{}^t\gamma\,(J_4X)\,\gamma, \qquad \text{i.e.} \qquad b_{\gamma X\gamma^{-1}}(v,w) = -b_X(\gamma v, \gamma w). $$

A symmetric bilinear form b on $V_+ \oplus V_-$ is the sum of its restrictions b_{++} to $V_+ \times V_+$, b_{--} to $V_- \times V_-$ and its mixed part b_{+-} (determined by $V_+ \times V_-$), and these three spaces have dimensions 3, 3 and 4. The map $b \mapsto b(\gamma\cdot, \gamma\cdot)$ is $+1$ on b_{++} and b_{--} and -1 on b_{+-}. With the sign, conjugation by γ is -1 on a space of dimension 6 and $+1$ on a space of dimension 4, which together span $\mathfrak{sp}_4(k)$.

The diagonal case. For $\gamma = \mathrm{diag}(\gamma_1,\dots,\gamma_4) = \mathrm{diag}(-1,1,-1,1)$ (similitude -1: ${}^t\gamma J_4\gamma$ multiplies $(J_4)_{ij}$, $i+j=5$, by $\gamma_i\gamma_j = -1$), conjugation multiplies E_{ij} by $\gamma_i\gamma_j$. Write $\gamma_i = (-1)^{\lambda_i}$ with $\lambda = (1,0,1,0)$, the exponents of t_1 in $\mathrm{T}_4(t_1,t_2) = \mathrm{diag}(t_1t_2, t_2, t_1, 1)$; then $\gamma_i\gamma_j = (-1)^{\lambda_i - \lambda_j} = (-1)^{w_1}$ for $w = \mathrm{wt}_4(i,j)$. $\mathfrak{sp}_4(k)$ is the sum of its diagonal part (eigenvalue $+1$)

and of the root spaces, each of which is spanned by matrix units of one weight and hence is an eigenspace for $(-1)^{w_1}$. The roots with w_1 odd are $\pm w_\alpha = \pm(1,0)$, $\pm w_\beta = \pm(-1,1)$, $\pm w_\theta = \pm(1,1)$; those with w_1 even are $\pm w_\Psi = \pm(0,1)$. This is the last assertion (and the dimensions 6, 4 agree with the general count).

Computer check. Computer check C6, listed in Appendix B, checks the diagonal case exactly over $\mathbb{Z}[1/2]$ (the integral root-vector basis and the eigenvalue of each basis vector) and, by an independent null-space computation, the dimensions 6 and 4 for 200 random involutions of similitude -1 over $\mathbb{Q}$ and modulo $3, 5, 7$; it is a check of the proof above, not a part of it. □

Definition 53.38. The **reducibility types** are the nine sets of roots

$$\Sigma_1^{\mathrm{red}} = \{w_\beta, -w_\beta, -w_\alpha, -w_\theta, -w_\Psi\}, \quad \Sigma_2^{\mathrm{red}} = \{w_\beta, -w_\beta, w_\alpha, w_\theta, w_\Psi\},$$
$$\Sigma_3^{\mathrm{red}} = \{w_\theta, -w_\theta, w_\Psi, w_\beta, -w_\alpha\}, \quad \Sigma_4^{\mathrm{red}} = \{w_\theta, -w_\theta, -w_\Psi, -w_\beta, w_\alpha\},$$
$$\Sigma_5^{\mathrm{red}} = \{w_\alpha, -w_\alpha, -w_\theta, -w_\Psi, -w_\beta\}, \quad \Sigma_6^{\mathrm{red}} = \{w_\alpha, -w_\alpha, w_\theta, w_\Psi, w_\beta\},$$
$$\Sigma_7^{\mathrm{red}} = \{w_\Psi, -w_\Psi, w_\beta, -w_\alpha, -w_\theta\}, \quad \Sigma_8^{\mathrm{red}} = \{w_\Psi, -w_\Psi, -w_\beta, w_\alpha, w_\theta\},$$
$$\Sigma_9^{\mathrm{red}} = \Sigma_Y = \{w_\beta, -w_\beta, w_\theta, -w_\theta\}.$$

For a commutative ring R, a set Γ, a map $r\colon \Gamma \to \mathrm{M}_4(R)$ and a set of roots Σ_0, the **reducibility ideal** $\mathrm{Red}_{\Sigma_0}(r) \subseteq R$ is the ideal generated by the entries $r(g)_{ij}$ with $g \in \Gamma$, $i \neq j$ and $\mathrm{wt}_4(i,j) \notin \Sigma_0$. For a ring homomorphism $\varphi\colon R \to K$ to a field, $\varphi \circ r$ is supported in Σ_0 if and only if $\ker\varphi \supseteq \mathrm{Red}_{\Sigma_0}(r)$.

Proposition 53.39. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ a fit and deep auxiliary tower for $\mathfrak{D}$ with field F, index set Σ and $d = [F:\mathbb{Q}]$, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring, $\mathscr{X} = (B, A, P, \mathrm{cp}, (\psi^B_{v,i}))$ a space for $(\mathscr{T}, \mathcal{O}')$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $\mathfrak{J} \subseteq B$ an ideal modular for ι, and $C \geq 0$ a Selmer bound for $(\mathscr{T}, \mathcal{O} \to \mathbb{F}_{\mathcal{O}'})$ in degree d with $30(C+1)^2 \leq d$. Let $x\colon A \to \mathbb{F}_{\mathcal{O}'}$ be an $\mathcal{O}'$-algebra homomorphism such that $x(P)$ is a residual pair good in case (III), R the completion of A at the maximal ideal $\ker x$, $f\colon A \to R$ the canonical homomorphism and $\phi\colon B \to A \xrightarrow{f} R$. Then there are finitely many ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ of R with $\dim R/\mathfrak{a}_k \leq d + 20C + 1$ for all k such that every prime $\mathfrak{p}$ of R containing none of the $\mathfrak{a}_k$ satisfies:*

(a) if $\mathfrak{p}$ contains $f(\mathfrak{J})R$, $\mathfrak{m}_{\mathcal{O}'}R$ and, for some $v \in \Sigma$ and $i \neq j$, all $f(\psi^P_{v,i}(s)) - f(\psi^P_{v,j}(s))$ with $s \in U_v$, then $\dim R/\mathfrak{p} \leq \dim B/\phi^{-1}(\mathfrak{p}) + 2 + 4\#\Sigma$;

(b) either the matrices $f(r_P(g))$ modulo $\mathfrak{p}$, $g \in G_F$, span M_4 over the fraction field of $R/\mathfrak{p}$, or there are $v \in \Sigma$ and $i \neq j$ with $f(\psi^P_{v,i}(s)) - f(\psi^P_{v,j}(s)) \in \mathfrak{p}$ for all $s \in U_v$.

Proof. Put $\mathfrak{m} = \ker x$, $c = x(P)$ and $Q = f(P) = (f \circ r_P, (f(n_v)))$, where $P = (r_P, (n_v))$.

Step 0: the chart. By (Zq1), A is a finitely generated algebra over the Noetherian ring B, hence Noetherian; x is $\mathcal{O}'$-linear, so it is onto $\mathbb{F}_{\mathcal{O}'}$ and $\mathfrak{m}$ is a maximal ideal with $A/\mathfrak{m} = \mathbb{F}_{\mathcal{O}'}$. The completion R of the Noetherian ring A at the maximal ideal $\mathfrak{m}$ is a complete Noetherian local ring with maximal ideal $\mathfrak{m}_R = \mathfrak{m}R$ and residue field $A/\mathfrak{m}$, and $A \xrightarrow{f} R \to R/\mathfrak{m}_R$ is x (Matsumura, Commutative Ring Theory, Theorems 8.7, 8.8 and 8.11); since $\mathfrak{m}_{\mathcal{O}'}A \subseteq \mathfrak{m}$, $R \in \mathrm{CNL}_{\mathcal{O}'}$. A ring homomorphism carries marked ordinary pairs to marked ordinary pairs (the conditions (Mq1)–(Mq4) are equations among the entries, and f is an $\mathcal{O}'$-algebra map), so Q is a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in R for $\mathcal{O} \subseteq \mathcal{O}' \to R$, with $\psi^Q_{v,i} = f \circ \psi^P_{v,i}$. Since $x(\mathfrak{m}_B) = 0$ ($x|_B$ is an $\mathcal{O}'$-algebra map from the local ring B to $\mathbb{F}_{\mathcal{O}'}$), $f(\mathfrak{m}_B A) \subseteq \mathfrak{m}_R$; as P is continuous for $\mathfrak{m}_B A$, every $g \mapsto f(r_P(g)_{ij}) \bmod \mathfrak{m}_R^k$ is locally constant, so Q is continuous for $\mathfrak{m}_R$, and its reduction

is $x(P) = c$. Finally $\Sigma \neq \emptyset$: places of $\overline{\mathbb{Q}}$ above ℓ exist, and each is G_F-conjugate to $\bar{v}$ for some $v \in \Sigma$.

Step 1: the ideals. As c is good in case (III), either (G) c is good, or (A) c is not good, lies in A, and has an entry of weight $-w_\Psi$ or entries of weights $-w_\beta$ and $-w_\theta$. Put $\mathcal{K} = \{1, \dots, 9\}$ in case (G) and $\mathcal{K} = \{1, \dots, 8\}$ in case (A), and let the ideals of the statement be $\mathfrak{a}_k = \mathrm{Red}_{\Sigma_k^{\mathrm{red}}}(f \circ r_P)$, $k \in \mathcal{K}$. By Step 0, Lemma 53.41 applies to c, R and Q ((Rx1) in case (G), (Rx2) in case (A)) and gives $\dim R/\mathfrak{a}_k \leq d + 20C + 1$ for $k \in \mathcal{K}$.

Now let $\mathfrak{p}$ be a prime of R containing no $\mathfrak{a}_k$, $k \in \mathcal{K}$, let K be the fraction field of $R/\mathfrak{p}$, $\pi\colon R \to R/\mathfrak{p} \subseteq K$, and $\bar{r} = \pi \circ f \circ r_P$; $\ker \pi = \mathfrak{p}$.

Step 2: the case $\mathfrak{p} \not\supseteq \mathrm{Red}_{\Sigma_Y}(f \circ r_P)$. We show that the K-span of $\bar{r}(G_F)$ is $\mathrm{M}_4(K)$. In this case $\mathfrak{p}$ contains none of the nine ideals $\mathrm{Red}_{\Sigma_k^{\mathrm{red}}}(f \circ r_P)$ (in case (G) by the choice of $\mathfrak{p}$; in case (A) the ninth by assumption, the others by the choice of $\mathfrak{p}$). A fit tower is good. Lemma 53.45, applied to c, R, Q (Step 0) and $\varphi = \pi$, gives the claim. In case (G) this applies to every $\mathfrak{p}$ as above, because $9 \in \mathcal{K}$.

Step 3: the case $\mathfrak{p} \supseteq \mathrm{Red}_{\Sigma_Y}(f \circ r_P)$. Then we are in case (A). We show that $\bar{r}$ is supported in Σ_Y, has entries of weights $w_\beta, -w_\beta, w_\theta, -w_\theta$, and that for some $v \in \Sigma$ and $(i,j) \in \{(1,2),(2,4)\}$, $f(\psi^P_{v,i}(s)) - f(\psi^P_{v,j}(s)) \in \mathfrak{p}$ for all $s \in D_v$. The support statement is the last sentence of Definition 53.38. For the entries, the four pairs $(k,w) = (1,w_\theta), (2,-w_\theta), (3,-w_\beta), (4,w_\beta)$ satisfy $\Sigma_Y \setminus \Sigma_k^{\mathrm{red}} = \{w\}$.

Fix such (k,w); $k \leq 4$ lies in $\mathcal{K}$, so $\mathfrak{a}_k \not\subseteq \mathfrak{p}$, and some generator of $\mathfrak{a}_k$ is not in $\mathfrak{p}$: there are $g \in G_F$ and $i \neq j$ with $\mathrm{wt}_4(i,j) \notin \Sigma_k^{\mathrm{red}}$ and $\bar{r}(g)_{ij} \neq 0$. As $\bar{r}$ is supported in Σ_Y, $\mathrm{wt}_4(i,j) \in \Sigma_Y \setminus \Sigma_k^{\mathrm{red}} = \{w\}$; so $\bar{r}$ has an entry of weight w. For the coincidence, pick $v \in \Sigma$ (Step 0). Since c lies in A, the image of $f((n_v)_{21})$ in $R/\mathfrak{m}_R$ is $x((n_v)_{21})$, the $(2,1)$ entry of the marking of c at v, which is non-zero, so $f((n_v)_{21})$ is a unit of the local ring R and $\pi(f((n_v)_{21})) \neq 0$. The pair $\pi(Q)$ is a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in K (Step 0, applied to π), its representation $\bar{r}$ is supported in Σ_Y and its marking at v has non-zero $(2,1)$ entry. By Lemma 53.46, $\pi(f(\psi^P_{v,i}(s))) = \pi(f(\psi^P_{v,j}(s)))$ for all $s \in D_v$, for $(i,j) = (1,2)$ or $(i,j) = (2,4)$, i.e. the differences lie in $\ker \pi = \mathfrak{p}$.

Step 4: (b). If $\mathfrak{p} \not\supseteq \mathrm{Red}_{\Sigma_Y}(f \circ r_P)$, Step 2 gives the first alternative of (b) (the matrices $f(r_P(g))$ modulo $\mathfrak{p}$ span M_4 over K). Otherwise Step 3 gives $v \in \Sigma$ and $i \neq j$ with $f(\psi^P_{v,i}(s)) - f(\psi^P_{v,j}(s)) \in \mathfrak{p}$ for all $s \in D_v$, in particular for all $s \in U_v \subseteq D_v$: the second alternative.

Step 5: (a). We show $\dim R/\mathfrak{p} \leq \dim B/\phi^{-1}(\mathfrak{p}) + 2 + 4\#\Sigma$ for every $\mathfrak{p}$ as above, which contains (a). Lemma 53.47 applies to the fit tower $\mathscr{T}$, the space $\mathscr{X}$, c, R, f (Step 0) and $\mathfrak{q} = \mathfrak{p}$, with $\phi = f|_B$ as in the statement: its first alternative holds by Step 2 if $\mathfrak{p} \not\supseteq \mathrm{Red}_{\Sigma_Y}(f \circ r_P)$, its second by Step 3 otherwise.

Steps 1, 4 and 5 give the ideals $\mathfrak{a}_k$, $k \in \mathcal{K}$, with the required properties. □

Definition 53.40. Let $\mathscr{T}$ be an auxiliary tower for a symplectic Eisenstein datum and k a field. Indices i, j run over $\{1,2,3,4\}$, and E_{ij} denotes the matrix unit. A **marked pair** with values in k is $c = (r, (n_v)_{v \in \Sigma})$ with $r\colon G_F \to \mathrm{GL}_4(k)$ a group homomorphism and $n_v \in \mathrm{M}_4(k)$ of similitude 1.

Reducible tangent spaces. For a marked pair $c = (r, (n_v))$ and a set S of roots, the **reducible tangent space** $\mathrm{T}_S(c)$ is the set of pairs $\xi = (x, (y_v)_{v \in \Sigma})$ of a map $x\colon G_F \to \mathfrak{sp}_4(k)$ and elements $y_v \in \mathfrak{sp}_4(k)$ such that:

(Ut1) x is locally constant, $x(gh) = x(g) + r(g)\,x(h)\,r(g)^{-1}$ for all $g, h \in G_F$, $x(g) = 0$ for $g \in N$, and $x(g_{\mathscr{F}})$ is diagonal;

(Ut2) $x(g)\,r(g)$ is supported in S for every $g \in G_F$;

(Ut3) for every $v \in \Sigma$, the entries (i,j) with $i \le j$ of $n_v^{-1} y_v n_v$ vanish;

(Ut4) for every $v \in \Sigma$ and $s \in D_v$, the matrix $n_v^{-1}(x(s) + r(s)\, y_v\, r(s)^{-1} - y_v)n_v$ is upper triangular.

All conditions are k-linear in ξ, so $\mathrm{T}_S(c)$ is a k-vector space under pointwise operations. It is the space of first-order deformations of c that keep the similitude, the conditions at N and $g_{\mathscr{T}}$, the support in S and the ordinary flags at the places of Σ.

Gradings. A **symplectic grading** is $m = (m_1, m_2, m_3, m_4) \in \mathbb{Z}^4$ with $m_1 + m_4 = m_2 + m_3$. A matrix $Y \in \mathrm{M}_4(k)$ is **m-filtered** if $Y_{ij} = 0$ whenever $m_i < m_j$, and its **m-leading part** $\mathrm{gr}_m Y$ is the matrix with entries Y_{ij} where $m_i = m_j$ and 0 elsewhere.

Lemma 53.41. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ a fit and deep auxiliary tower for $\mathfrak{D}$ with field F, $d = [F:\mathbb{Q}]$, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring, $C \ge 0$ a Selmer bound for $(\mathscr{T}, \mathcal{O} \to \mathbb{F}_{\mathcal{O}'})$ in degree d, $\mathscr{X} = (B, A, P, \mathrm{cp}, (\psi^B_{v,i}))$ a space for $(\mathscr{T}, \mathcal{O}')$, $x\colon A \to \mathbb{F}_{\mathcal{O}'}$ an $\mathcal{O}'$-algebra homomorphism such that $c = x(P)$ is a residual pair, R the completion of A at the maximal ideal $\ker x$, $f\colon A \to R$ the canonical homomorphism and $Q = (r_Q, (n_v)) = f(P)$.*

(Rx1) *If c is good, then $\dim R/\mathrm{Red}^{\mathrm{red}}_{\Sigma_k}(r_Q) \le d + 20C + 1$ for $1 \le k \le 9$.*

(Rx2) *If c lies in A and has an entry of weight $-w_\Psi$ or entries of weights $-w_\beta$ and $-w_\theta$, then $\dim R/\mathrm{Red}^{\mathrm{red}}_{\Sigma_k}(r_Q) \le d + 20C + 1$ for $1 \le k \le 8$.*

Proof. Put $k = \mathbb{F}_{\mathcal{O}'}$, $\mathfrak{m} = \ker x$, $c = x(P) = (r_c, (n_v^c))$, $P = (r_P, (n_v))$, $Q = (r_Q, (n_v^Q)) = f(P)$, and let ϖ be a uniformiser of the discrete valuation ring $\mathcal{O}'$. For $u \in R$ write $\bar u$ for its residue in k. Fix $S = \Sigma_t^{\mathrm{red}}$ ($1 \le t \le 9$ for (Rx1), $1 \le t \le 8$ for (Rx2)), $I = \mathrm{Red}_S(r_Q)$, $\bar R = R/I$ and $\pi\colon R \to \bar R$.

Step 0: the completion. By (Zq1), A is of finite type over the Noetherian ring B, hence Noetherian; x is $\mathcal{O}'$-linear, so onto k, and $\mathfrak{m}$ is maximal. Hence R is a complete Noetherian local ring with maximal ideal $\mathfrak{m}_R = \mathfrak{m}R$, residue field k, $u \mapsto \bar u$ restricted along f is x, and f induces $A/\mathfrak{m}^2 \cong R/\mathfrak{m}_R^2$ (see [166, Thms. 8.7 and 8.11]). Q is a marked ordinary pair with values in R (the conditions (Mq1)–(Mq4) are equations in the entries), and its reduction is c. As $x(\mathfrak{m}_B) = 0$ ($x|_B$ is an $\mathcal{O}'$-algebra map from the local ring B to k), $f(\mathfrak{m}_B A) \subseteq \mathfrak{m}_R$; since P is continuous for $\mathfrak{m}_B A$, every $g \mapsto r_Q(g)_{ij} \bmod \mathfrak{m}_R^2$ is locally constant.

Step 1: the unit case. If r_c is not supported in S, some $r_c(g)_{ij} \ne 0$ with $i \ne j$, $\mathrm{wt}_4(i,j) \notin S$; then $r_Q(g)_{ij} \in I$ is a unit of R, $\bar R = 0$ and $\dim \bar R = -\infty$. Assume from now on that r_c is supported in S. Then every generator of I has residue 0, so $I \subseteq \mathfrak{m}_R$ and $\bar R$ is a Noetherian local ring with maximal ideal $\bar{\mathfrak{m}} = \mathfrak{m}_R/I$ and residue field k. Put $t^* = \bar{\mathfrak{m}}/(\bar{\mathfrak{m}}^2 + \varpi \bar R)$, a finite-dimensional k-space.

Step 2: tangent vectors. For $\lambda \in \mathrm{Hom}_k(t^*, k)$ define $D_\lambda\colon \bar R \to k$ by $D_\lambda(u) = \lambda(u - \tilde u \bmod \bar{\mathfrak{m}}^2 + \varpi \bar R)$, where $\tilde u$ is the image in $\bar R$ of any $o \in \mathcal{O}'$ with the same residue as u; two such o differ by an element of $\varpi\mathcal{O}'$, so D_λ is well defined. It is k-linear in λ, vanishes on the image of $\mathcal{O}'$, and $D_\lambda(uu') = \bar u\, D_\lambda(u') + \bar u'\, D_\lambda(u)$.

Hence $\phi_\lambda\colon R \to k[\epsilon]$ ($\epsilon^2 = 0$), $u \mapsto \bar u + \epsilon D_\lambda(\pi(u))$, is an $\mathcal{O}'$-algebra homomorphism with $\phi_\lambda(I) = 0$ and $\phi_\lambda(\mathfrak{m}_R) \subseteq \epsilon k$, so $\phi_\lambda(\mathfrak{m}_R^2) = 0$. Write $D = D_\lambda \circ \pi$, applied to matrices entrywise, and put

$$x_\lambda(g) = D(r_Q(g))\, r_c(g)^{-1}, \qquad y_{\lambda,v} = D(n_v^Q)\, (n_v^c)^{-1},$$

so that $\phi_\lambda(r_Q(g)) = (1 + \epsilon x_\lambda(g)) r_c(g)$ and $\phi_\lambda(n_v^Q) = (1 + \epsilon y_{\lambda,v}) n_v^c$. The map $\Lambda\colon \lambda \mapsto \xi_\lambda = (x_\lambda, (y_{\lambda,v}))$ is k-linear.

Step 3: $\xi_\lambda \in \mathrm{T}_S(c)$. Write $x = x_\lambda$, $y_v = y_{\lambda,v}$, $r = r_c$, $n = n_v^c$. $\mathfrak{sp}_4$: by (Mq1) $r_Q(g)$ has similitude $\theta(g)$, and $\phi_\lambda(\theta(g)) = a(\theta(g)) \in k$; so ${}^t r(1+\epsilon\,{}^t x)J_4(1+\epsilon x)r = {}^t r J_4 r$, i.e. ${}^t r({}^t x J_4 + J_4 x)r = 0$, and $x(g) \in \mathfrak{sp}_4(k)$; likewise $y_v \in \mathfrak{sp}_4(k)$ from (Mq3). (Ut1): x is locally constant by Step 0 (as ϕ_λ kills $\mathfrak{m}_R^2$); applying ϕ_λ to $r_Q(gh) = r_Q(g)r_Q(h)$ gives $(1+\epsilon x(gh))r(gh) = (1+\epsilon(x(g)+r(g)x(h)r(g)^{-1}))r(gh)$; $r_Q(g) = 1$ on N gives $x(N) = 0$; $r_Q(g_{\mathscr{T}})$ and $r(g_{\mathscr{T}})$ are diagonal (Mq2), so $x(g_{\mathscr{T}})$ is diagonal. (Ut2): $(x(g)r(g))_{ij} = D(r_Q(g)_{ij})$, and for $i \neq j$, $\mathrm{wt}_4(i,j) \notin S$, $r_Q(g)_{ij} \in I = \ker\pi$. (Ut3): n_v^Q, n are lower unitriangular (Mq3), so $D(n_v^Q)$ is strictly lower triangular, and so is $n^{-1}y_v n = n^{-1}D(n_v^Q)$. (Ut4): for $s \in D_v$, ϕ_λ applied to (Mq4) says that

$$\big((1+\epsilon y_v)n\big)^{-1}(1+\epsilon x(s))r(s)(1+\epsilon y_v)n = n^{-1}r(s)n + \epsilon\, n^{-1}\big(x(s)+r(s)y_v r(s)^{-1} - y_v\big)n \cdot n^{-1}r(s)n$$

is upper triangular; $n^{-1}r(s)n$ is upper triangular and invertible (Mq4 for c), so $n^{-1}(x(s) + r(s)y_v r(s)^{-1} - y_v)n$ is upper triangular.

Step 4: Λ is injective. Let $\xi_\lambda = 0$. Then $\psi_\lambda = \phi_\lambda \circ f$ and $\psi_0 = \phi_0 \circ f$ are $\mathcal{O}'$-algebra homomorphisms $A \to k[\epsilon]$ that agree on all entries of all $r_P(g)$ and n_v. Their restrictions to B agree on the coefficients of every $\mathrm{cp}(g)$ (these map to the coefficients of the characteristic polynomial of $r_P(g)$, (Zq2)) and on every $\psi_{v,i}^B(s)$ (these map to $(n_v^{-1}r_P(s)n_v)_{ii}$, and $n_v^{-1} = J_4^{-1}\,{}^t n_v J_4$ as n_v has similitude 1); as $k[\epsilon] \in \mathrm{CNL}_{\mathcal{O}'}$, (Zq3) gives $\psi_\lambda|_B = \psi_0|_B$. By (Zq1) the entries generate A over B, so $\psi_\lambda = \psi_0$. Both ϕ_λ, ϕ_0 kill $\mathfrak{m}_R^2$ and $R = f(A) + \mathfrak{m}_R^2$ (Step 0), so $\phi_\lambda = \phi_0$, i.e. $D_\lambda \circ \pi = 0$; as $\pi(\mathfrak{m}_R) = \bar{\mathfrak{m}}$ and $D_\lambda(\mu) = \lambda(\mu \bmod \bar{\mathfrak{m}}^2 + \varpi\bar{R})$ for $\mu \in \bar{\mathfrak{m}}$, $\lambda = 0$. Hence $\dim_k t^* \le \dim_k \mathrm{T}_S(c)$.

Step 5: dimension. The surjection $\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2 \to t^*$ has kernel $(\varpi\bar{R} + \bar{\mathfrak{m}}^2)/\bar{\mathfrak{m}}^2$, the image of the $\bar{R}$-module $\varpi\bar{R}$ generated by $\varpi \in \bar{\mathfrak{m}}$; as $\bar{R}$ acts on $\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2$ through k, this kernel is at most one-dimensional. By Theorem B.157 applied to $\bar{R}$,

$$\dim \bar{R} \le \dim_k \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2 \le \dim_k t^* + 1 \le \dim_k \mathrm{T}_S(c) + 1.$$

Step 6: the bound. Apply Lemma 53.42 with $k = \mathbb{F}_{\mathcal{O}'}$, $a\colon \mathcal{O} \subseteq \mathcal{O}' \to k$ (so $a(\mathfrak{m}_{\mathcal{O}}) = 0$), the Selmer bound C for $(\mathscr{T}, a)$ in degree d, the residual pair c over $(\mathscr{T}, \emptyset)$ and the type S, in which r_c is supported (Step 1). For (Rx1), c is good and (Nb1) gives $\dim \mathrm{T}_S(c) \le d + 20C$. For (Rx2), c lies in A, has an entry of weight $-w_\Psi$ or entries of weights $-w_\beta, -w_\theta$, and $S \neq \Sigma_Y = \Sigma_9^{\mathrm{red}}$ as $t \le 8$; (Nb2) gives the same bound. With Step 5, $\dim R/\mathrm{Red}_S(r_Q) \le d + 20C + 1$. □

Lemma 53.42. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ an auxiliary tower for $\mathfrak{D}$ with field F, $d = [F:\mathbb{Q}]$, k a field, $a\colon \mathcal{O} \to k$ a ring homomorphism with $a(\mathfrak{m}_{\mathcal{O}}) = 0$, $C \ge 0$ a Selmer bound for $(\mathscr{T}, a)$ in degree d, $c = (r, (n_v))$ a residual pair over $(\mathscr{T}, \emptyset)$ with values in k, and S a reducibility type such that r is supported in S.*

(Nb1) *If c is good, then $\dim_k \mathrm{T}_S(c) \le d + 20C$.*

(Nb2) *If c lies in A, has an entry of weight $-w_\Psi$ or entries of weights $-w_\beta$ and $-w_\theta$, and $S \neq \Sigma_Y$, then $\dim_k \mathrm{T}_S(c) \le d + 20C$.*

Proof. Write $\chi_i = \chi_i^{\mathscr{T}}$ and $r_0(g) = \mathrm{diag}(a(\chi_1(g)), \dots, a(\chi_4(g)))$. By (Mq1) $r(g)$ has similitude $a(\theta(g)) \in k^\times$, so $r\colon G_F \to \mathrm{GL}_4(k)$; by (Mq3) each n_v has similitude 1 and is lower unitriangular. So c is a marked pair. By (Wq1), the diagonal of $r(g)$ is that of $r_0(g)$, and $U_v(s) = n_v^{-1}r(s)n_v$ is upper triangular (Mq4) with $U_v(s)_{11} = a(\chi_1(s))$ for $s \in D_v$. As r is supported in S, every root w such that c has an entry of weight w lies in S. Below we exhibit in each case a symplectic grading m and a marked pair $c_0 = (r_0, (n_v^0))$ such that $r(g)$, n_v are m-filtered, $\mathrm{gr}_m r(g) =$

$r_0(g)$, $\mathrm{gr}_m n_v = n_v^0$; then Lemma 53.43 gives $\dim \mathrm{T}_S(c) \le \dim \mathrm{T}_S(c_0)$, and Lemma 53.44 gives $\dim \mathrm{T}_S(c_0) \le d + 19C \le d + 20C$. In all cases an off-diagonal entry (i,j) of $r(g)$ or n_v that may be non-zero has $m_i > m_j$, except where said otherwise; then the matrix is m-filtered and its m-leading part is its diagonal part plus the entries with $m_i = m_j$.

Case 1: c has entries of weights $-w_\Psi$ and one of $w_\beta, -w_\alpha$, or of weights $-w_\alpha$ and $-w_\beta$. Then S satisfies the hypothesis of (Nl1) of Lemma 53.44. Take $n_v^0 = 1$. By (Wq2), either (P) r is supported in $\Sigma_P = \{w_\beta, -w_\alpha, -w_\theta, -w_\Psi\}$ and every n_v in $\Sigma_Q = \{-w_\alpha, -w_\Psi, -w_\theta\}$: take $m = (0,2,1,3)$; or (M) r is supported in $\Sigma_M = \{-w_\alpha, -w_\beta, -w_\Psi, -w_\theta\}$: take $m = (0,1,2,3)$.

So $\mathrm{gr}_m r(g) = r_0(g)$ and $\mathrm{gr}_m n_v = 1$, and (Nl1) applies.

Case 2: c lies in V and has an entry of weight $-w_\alpha$, or entries of weights w_β and $-w_\theta$. Then S satisfies the first hypothesis of (Nl2). Take $m = (0,1,0,1)$ ($m_1 + m_4 = m_2 + m_3 = 1$), $b_v = (n_v)_{31} \ne 0$ and $n_v^0 = 1 + b_v(E_{31} + E_{42})$. r is supported in $\Sigma_V = \{w_\beta, -w_\alpha, -w_\theta\}$, at positions $(2,3),(2,1),(4,3),(4,1)$, all with $m_i - m_j = 1$; the off-diagonal positions with $m_i = m_j$ are those of $\pm w_\Psi$, not in Σ_V, so $\mathrm{gr}_m r(g) = r_0(g)$. n_v is lower unitriangular and supported in Σ_Q, so $(n_v)_{32} = 0$ and its entries at $(2,1),(4,3),(4,1)$ have $m_i - m_j = 1$, those at $(3,1),(4,2)$ have $m_i = m_j$; and $(n_v)_{42} = (n_v)_{31} - (n_v)_{32}(n_v)_{21} = (n_v)_{31}$ by Lemma 52.7, n_v being lower unitriangular of similitude 1. Hence $\mathrm{gr}_m n_v = n_v^0$. For the character hypothesis of (Nl2), let $s \in D_v$ and compare the $(3,1)$ entries of $r(s)n_v = n_v U_v(s)$: row 3 of $r(s)$ has only the diagonal entry (the weights $-w_\Psi, -w_\beta, w_\alpha$ at $(3,1),(3,2),(3,4)$ are not in Σ_V), so the left side is $a(\chi_3(s))(n_v)_{31}$; as $U_v(s)$ is upper triangular and n_v lower unitriangular, the right side is $(n_v)_{31}U_v(s)_{11} = (n_v)_{31}a(\chi_1(s))$. As $(n_v)_{31} \ne 0$, $a(\chi_1(s)) = a(\chi_3(s))$. (Nl2) applies.

(Nb1). A good residual pair is in Case 1 (first two clauses of goodness) or Case 2 (last two clauses; the condition on rt_{w_α} in the fourth clause is not used).

(Nb2). Here r is supported in $\Sigma_A = \{-w_\beta, -w_\Psi, -w_\theta\}$ and $(n_v)_{21} \ne 0$; $S \ne \Sigma_Y$ contains $-w_\Psi$ or both $-w_\beta, -w_\theta$, the hypothesis of (Nl3). Take $m = (0,0,1,1)$ ($m_1 + m_4 = m_2 + m_3 = 1$), $b_v = (n_v)_{21}$ and $n_v^0 = 1 + b_v(E_{21} - E_{43})$. The positions of Σ_A are $(3,2),(3,1),(4,2),(4,1)$, all with $m_i - m_j = 1$; the off-diagonal positions with $m_i = m_j$ carry $\pm w_\alpha \notin \Sigma_A$; so $\mathrm{gr}_m r(g) = r_0(g)$. For the lower unitriangular n_v, the entries at $(2,1),(4,3)$ have $m_i = m_j$, the others below the diagonal $m_i - m_j = 1$, and $(n_v)_{43} = -(n_v)_{21}$ (Lemma 52.7); so $\mathrm{gr}_m n_v = n_v^0$. For the character hypothesis, compare the $(2,1)$ entries of $r(s)n_v = n_v U_v(s)$, $s \in D_v$: row 2 of $r(s)$ has only the diagonal entry (weights $-w_\alpha, w_\beta, w_\Psi$ at $(2,1),(2,3),(2,4)$ are not in Σ_A), so the left side is $a(\chi_2(s))(n_v)_{21}$, and the right side is $(n_v)_{21}U_v(s)_{11} = (n_v)_{21}a(\chi_1(s))$. As $(n_v)_{21} \ne 0$, $a(\chi_1(s)) = a(\chi_2(s))$, and (Nl3) applies. □

Lemma 53.43. *Let $\mathscr{T}$ be an auxiliary tower for a symplectic Eisenstein datum, with index set Σ, k a field, S a set of roots and m a symplectic grading. Let $c = (r, (n_v))$ and $c_0 = (r_0, (n_v^0))$ be marked pairs with values in k such that, for all $g \in G_F$ and $v \in \Sigma$, the matrices $r(g)$ and n_v are m-filtered, $r_0(g) = \mathrm{gr}_m r(g)$ and $n_v^0 = \mathrm{gr}_m n_v$. Then $\dim_k \mathrm{T}_S(c) \le \dim_k \mathrm{T}_S(c_0)$.*

Proof. If $\mathrm{T}_S(c_0)$ is infinite-dimensional there is nothing to prove; assume it is finite-dimensional. Let $s_0 = m_1 + m_4 = m_2 + m_3$, so $m_i + m_{5-i} = s_0$ for all i.

Step 1: degrees of matrices. For $p \in \mathbb{Z}$ call $Y \in \mathrm{M}_4(k)$ *of degree at least p* if $Y_{ij} = 0$ whenever $m_i - m_j < p$, and let $Y^{[p]}$ be the matrix with entries Y_{ij} where $m_i - m_j = p$ and 0 elsewhere; so m-filtered means of degree at least 0, and $\mathrm{gr}_m Y = Y^{[0]}$. Three facts:

(a) If A has degree at least p and B degree at least q, then AB has degree at least $p+q$ and $(AB)^{[p+q]} = A^{[p]}B^{[q]}$. Indeed $(AB)_{ij} = \sum_l A_{il}B_{lj}$, a term can be non-zero only if $m_i - m_l \ge p$ and $m_l - m_j \ge q$, which forces $m_i - m_j \ge p+q$; if $m_i - m_j = p+q$, the non-zero terms have $m_i - m_l = p$ and $m_l - m_j = q$, which is the (i,j) entry of $A^{[p]}B^{[q]}$.

(b) Put $Y^\dagger = J_4^{-1}\,{}^tY\,J_4$. Its (i,j) entry is $\pm Y_{5-j,5-i}$, and $m_{5-j} - m_{5-i} = (s_0 - m_j) - (s_0 - m_i) = m_i - m_j$. Hence $Y \mapsto Y^\dagger$ preserves "degree at least p" and $(Y^\dagger)^{[p]} = (Y^{[p]})^\dagger$. A matrix X lies in $\mathfrak{sp}_4(k)$ if and only if $X^\dagger = -X$, and n has similitude 1 if and only if $n^{-1} = n^\dagger$. Consequently $X \in \mathfrak{sp}_4(k)$ implies $X^{[p]} \in \mathfrak{sp}_4(k)$; and $n_v^{-1} = n_v^\dagger$ is m-filtered with $(n_v^{-1})^{[0]} = (n_v^0)^\dagger = (n_v^0)^{-1}$, as n_v^0 has similitude 1.

(c) The entries of $Y^{[p]}$ are entries of Y or 0; so if a given entry of Y vanishes, the same entry of $Y^{[p]}$ vanishes, and if Y is diagonal (upper triangular) so is $Y^{[p]}$.

Also $r(g)^{-1} = r(g^{-1})$ is m-filtered with $\mathrm{gr}_m\, r(g^{-1}) = r_0(g^{-1}) = r_0(g)^{-1}$.

Step 2: the filtration. Let $P = \{m_i - m_j\}$, a finite set of integers with smallest element $p_- \le 0$ and largest $p_+ \ge 0$. For $p \in \mathbb{Z}$ let $F_p \subseteq \mathrm{T}_S(c)$ be the subspace of $\xi = (x, (y_v))$ such that every $x(g)$ and every y_v has degree at least p. Then $F_{p_-} = \mathrm{T}_S(c)$, $F_{p_+ + 1} = 0$ and $F_{p+1} \subseteq F_p$. For $\xi \in F_p$ put $\lambda_p(\xi) = (g \mapsto x(g)^{[p]}, (y_v^{[p]})_v)$; λ_p is k-linear with kernel F_{p+1}.

Step 3: $\lambda_p(F_p) \subseteq \mathrm{T}_S(c_0)$. Let $\xi = (x, (y_v)) \in F_p$, and write $x^{[p]}(g) = x(g)^{[p]}$. By Step 1(b), $x^{[p]}(g), y_v^{[p]} \in \mathfrak{sp}_4(k)$. (Ut1): $x^{[p]}$ is locally constant (its entries are entries of x or 0); by Step 1(a) applied to $r(g)\,x(h)\,r(g^{-1})$ (degrees $0, p, 0$), the degree-p part of $x(gh) = x(g) + r(g)x(h)r(g)^{-1}$ is $x^{[p]}(gh) = x^{[p]}(g) + r_0(g)\,x^{[p]}(h)\,r_0(g)^{-1}$; $x^{[p]}(g) = 0$ for $g \in N$; and $x^{[p]}(g_{\mathscr{T}})$ is diagonal by Step 1(c). (Ut2): $x(g)r(g)$ has degree at least p with $(x(g)r(g))^{[p]} = x^{[p]}(g)\,r_0(g)$ by Step 1(a); its entries (i,j) with $i \ne j$ and $\mathrm{wt}_4(i,j) \notin S$ vanish, hence those of $x^{[p]}(g)r_0(g)$ vanish (Step 1(c)). (Ut3): by Step 1(a),(b), $n_v^{-1} y_v n_v$ has degree at least p with degree-p part $(n_v^0)^{-1} y_v^{[p]} n_v^0$, whose entries (i,j), $i \le j$, vanish by Step 1(c). (Ut4): for $s \in D_v$, $Z = x(s) + r(s) y_v r(s)^{-1} - y_v$ has degree at least p with $Z^{[p]} = x^{[p]}(s) + r_0(s)\,y_v^{[p]}\,r_0(s)^{-1} - y_v^{[p]}$; so $n_v^{-1} Z n_v$ has degree at least p with degree-p part $(n_v^0)^{-1} Z^{[p]} n_v^0$, which is upper triangular by Step 1(c). Hence $\lambda_p(\xi) \in \mathrm{T}_S(c_0)$.

Step 4: counting. Let $H_p \subseteq \mathrm{T}_S(c_0)$ be the subspace of elements $(x', (y'_v))$ with $x'(g)^{[p]} = x'(g)$ and $y_v'^{[p]} = y'_v$ for all g, v (homogeneous of degree p). By Step 3, $\lambda_p(F_p) \subseteq H_p$. The sum $\sum_{p \in P} H_p$ is direct, since an element of H_p has non-zero entries only at positions (i,j) with $m_i - m_j = p$. Therefore

$$\dim \mathrm{T}_S(c) = \sum_{p=p_-}^{p_+} \dim F_p/F_{p+1} = \sum_p \dim \lambda_p(F_p) \le \sum_{p \in P} \dim H_p \le \dim \mathrm{T}_S(c_0),$$

where $\dim F_p/F_{p+1} = \dim \lambda_p(F_p)$ because $\ker \lambda_p = F_{p+1}$, and $\lambda_p(F_p)$ is zero for $p \notin P$ (then every $Y^{[p]}$ is 0, so $F_p = F_{p+1}$). □

Lemma 53.44. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ an auxiliary tower for $\mathfrak{D}$ with field F and index set Σ, $d = [F : \mathbb{Q}]$, k a field, $a\colon \mathcal{O} \to k$ a ring homomorphism with $a(\mathfrak{m}_\mathcal{O}) = 0$, and $C \ge 0$ a Selmer bound for $(\mathscr{T}, a)$ in degree d. Put $r_0(g) = \mathrm{diag}(a(\chi_1^{\mathscr{T}}(g)), \dots, a(\chi_4^{\mathscr{T}}(g)))$ for $g \in G_F$, let $(b_v)_{v \in \Sigma}$ be non-zero elements of k, and let S be a reducibility type.*

(Nl1) *If S contains $-w_\Psi$ and one of $w_\beta, -w_\alpha$, or contains $-w_\alpha$ and $-w_\beta$, then*

$$\dim_k \mathrm{T}_S(r_0, (1)_v) \le d + 19C.$$

(Nl2) *If S contains $-w_\alpha$, or contains w_β and $-w_\theta$, and $a(\chi_1^{\mathscr{T}}(s)) = a(\chi_3^{\mathscr{T}}(s))$ for all $v \in \Sigma$ and $s \in D_v$, then*

$$\dim_k \mathrm{T}_S(r_0, (1 + b_v(E_{31} + E_{42}))_v) \le d + 19C.$$

(Nl3) *If $S \neq \Sigma_Y$, S contains $-w_\Psi$ or contains $-w_\beta$ and $-w_\theta$, and $a(\chi_1^{\mathscr{T}}(s)) = a(\chi_2^{\mathscr{T}}(s))$ for all $v \in \Sigma$ and $s \in D_v$, then*

$$\dim_k \mathrm{T}_S(r_0, (1 + b_v(E_{21} - E_{43}))_v) \leq d + 19C.$$

In each item the second entry is a family of matrices of similitude 1*, so the pair is a marked pair.*

Proof. Write $\chi_i = \chi_i^{\mathscr{T}}$ and $\rho_w = a \circ \mathrm{rt}_w$. As $a(\ell) = 0$ and ℓ is odd, 2 is invertible in k. Let $c = (r_0, (n_v))$ be the pair of one of the items, with $n_v = 1$, $1 + b_v(E_{31} + E_{42})$ or $1 + b_v(E_{21} - E_{43})$. Let $P = \{(i,j) : i + j \leq 5\}$ (ten positions). The weights at P are: 0 at $(1,1), (2,2)$; $w_\Psi, -w_\Psi$ at $(1,3), (3,1)$; and the six odd roots $w_\alpha, w_\theta, w_\beta, -w_\alpha, -w_\beta, -w_\theta$ at $(1,2), (1,4), (2,3), (2,1), (3,2), (4,1)$, the *odd positions*.

Step 0: the markings and the entries. n_v has similitude 1 by Lemma 52.7: it is lower unitriangular, and $n_{43} = -n_{21}$, $n_{42} = n_{31} - n_{21}n_{32}$ hold for both markings ($1 + b(E_{31} + E_{42})$: $n_{21} = n_{43} = n_{32} = 0$, $n_{42} = n_{31} = b$; $1 + b(E_{21} - E_{43})$: $n_{43} = -n_{21} = -b$, $n_{42} = n_{31} = n_{32} = 0$). The characters satisfy $\chi_i\chi_{5-i} = \theta|_{G_F}$ (the labelling permutes $(\theta, \Psi, \alpha, \mathbf{1})$, whose products $\kappa_1\kappa_4 = \kappa_2\kappa_3 = \theta$ are the similitude of r_0), hence $\chi_i/\chi_j = \mathrm{rt}_{\mathrm{wt}_4(i,j)}$:

Let $\xi = (x, (y_v)) \in \mathrm{T}_S(c)$ and write $x_{ij}(g) = x(g)_{ij}$. As $(r_0(g)Yr_0(g)^{-1})_{ij} = \rho_{\mathrm{wt}_4(i,j)}(g)Y_{ij}$, (Ut1) gives: x_{ij} is locally constant, $x_{ij}(gh) = x_{ij}(g) + \rho_{\mathrm{wt}_4(i,j)}(g)x_{ij}(h)$, $x_{ij}(N) = 0$, and $x_{ij}(g_{\mathscr{T}}) = 0$ for $i \neq j$ (note $\mathrm{wt}_4(i,j) = 0$ only for $i = j$). So $x_{ij} \in Z(\mathrm{wt}_4(i,j))$. If $i \neq j$ and $\mathrm{wt}_4(i,j) \notin S$, then $x_{ij} = 0$: by (Ut2) $0 = (x(g)r_0(g))_{ij} = x_{ij}(g)\, a(\chi_j(g))$ and $a(\chi_j(g)) \in k^\times$.

Step 1: local coboundaries. For $v \in \Sigma$ and a character $\rho\colon D_v \to k^\times$ let $B_v(\rho)$ be the k-space of functions $s \mapsto (1 - \rho(s))t$ on D_v, $t \in k$. Call an odd-position entry x_{ij} *locally trivial* if $x_{ij}|_{D_v} \in B_v(\rho_{\mathrm{wt}_4(i,j)}|_{D_v})$ for every v; then $x_{ij} \in Z^{\mathrm{str}}(\mathrm{wt}_4(i,j))$ by Step 0. For $s \in D_v$ put $Z^v(s) = x(s) + r_0(s)y_v r_0(s)^{-1} - y_v$; then

$$Z^v(s)_{ij} = x_{ij}(s) - \big(1 - \rho_{\mathrm{wt}_4(i,j)}(s)\big)(y_v)_{ij}, \qquad \text{so } Z^v_{ij} \equiv x_{ij}|_{D_v} \bmod B_v(\rho_{\mathrm{wt}_4(i,j)}|_{D_v}),$$

$Z^v(s) \in \mathfrak{sp}_4(k)$, and $n_v^{-1}Z^v(s)n_v$ is upper triangular by (Ut4).

Step 2: the count under a one-odd hypothesis. Suppose there is an odd position (i_0, j_0) such that *every $\xi \in \mathrm{T}_S(c)$ with $x_{i_0j_0} = x_{11} = x_{22} = 0$ has its six odd-position entries locally trivial.* Let $\Phi_1\colon \mathrm{T}_S(c) \to Z(\mathrm{wt}_4(i_0,j_0)) \times Z(0) \times Z(0)$, $\xi \mapsto (x_{i_0j_0}, x_{11}, x_{22})$ (Step 0); as $\mathrm{wt}_4(i_0,j_0)$ is an odd root, its image has dimension at most $(d + C) + C + C$. On $K = \ker \Phi_1$ consider

$$\Phi_2\colon K \to \prod_{(i,j)\in P} M_{ij} \times \prod_{v\in\Sigma} k^{\{(i,j):\, i>j\}}, \qquad \xi \mapsto \Big((x_{ij})_{(i,j)\in P},\ \big((n_v^{-1}y_v n_v)_{ij}\big)_{v,\, i>j}\Big),$$

with $M_{ij} = Z^{\mathrm{str}}(\mathrm{wt}_4(i,j))$ at odd positions and $M_{ij} = Z(\mathrm{wt}_4(i,j))$ at the four positions of weight $0, \pm w_\Psi$; it is well defined by Step 0 and the hypothesis. The target has dimension at most $10C + 6\,\#\Sigma \leq 16C$ (Selmer bound: $\dim Z^{\mathrm{str}}(w) \leq C$ for roots, $\dim Z(w) \leq C$ for $w \in \{0, \pm w_\Psi\}$, $\#\Sigma \leq C$). Φ_2 is injective: an element X of $\mathfrak{sp}_4(k)$ is determined by its entries at P,

so $\Phi_2(\xi) = 0$ forces $x = 0$; and $n_v^{-1}y_v n_v$ has vanishing entries on and above the diagonal by (Ut3), and below it by $\Phi_2(\xi) = 0$, so $y_v = 0$. Hence $\dim \mathrm{T}_S(c) \leq (d + 3C) + 16C = d + 19C$. It remains to verify the one-odd hypothesis in each item; the table of the nine types is in Definition 53.38.

Step 3: (Nl1). Here $n_v = 1$, so $Z^v(s)$ is upper triangular and $Z^v_{ij} = 0$ for $i > j$: the entries at the lower odd positions $(2,1), (3,2), (4,1)$ are locally trivial for every ξ (Step 1). The types

satisfying the hypothesis of (Nl1) are $\Sigma_1^{\mathrm{red}}, \Sigma_5^{\mathrm{red}}, \Sigma_7^{\mathrm{red}}$, and each contains exactly one of the roots $w_\alpha, w_\theta, w_\beta$ of the upper odd positions:

Take (i_0, j_0) the position of that root; the other two upper odd entries vanish by Step 0. So all six odd entries of every ξ with $x_{i_0 j_0} = 0$ are locally trivial.

Step 4: (Nl2). Here $n = n_v = 1 + bX$ with $X = E_{31} + E_{42}$, $b = b_v \neq 0$, $X^2 = 0$, $n^{-1} = 1 - bX$. Let $Z \in \mathfrak{sp}_4(k)$ with $U = n^{-1} Z n$ upper triangular ($U \in \mathfrak{sp}_4(k)$ as n has similitude 1). Then

$$Z_{21} = -b\, Z_{23}, \qquad Z_{41} = -b^2 Z_{23}, \qquad Z_{32} = 2b\, Z_{12} + b^2 Z_{14}.$$

On D_v, $a(\chi_1) = a(\chi_3)$ gives $\rho_{-w_\alpha} = \rho_{w_\beta} = \rho_{-w_\theta} =: \rho^-$ and $\rho_{w_\alpha} = \rho_{w_\theta} = \rho_{-w_\beta} =: \rho^+$

so by Step 1, modulo $B_v(\rho^-)$: $x_{21} \equiv -b\, x_{23}$, $x_{41} \equiv -b^2 x_{23}$; modulo $B_v(\rho^+)$: $x_{32} \equiv 2b\, x_{12} + b^2 x_{14}$ (restrictions to D_v). As $b, 2 \in k^\times$: if one of x_{21}, x_{23}, x_{41} is locally trivial, all three are; if two of x_{12}, x_{14}, x_{32} are, all three are. The types satisfying the hypothesis of (Nl2) are $\Sigma_1^{\mathrm{red}}, \Sigma_3^{\mathrm{red}}, \Sigma_5^{\mathrm{red}}, \Sigma_6^{\mathrm{red}}, \Sigma_7^{\mathrm{red}}, \Sigma_Y$, and the choice

$$(i_0, j_0) = (2,3), (2,3), (1,2), (1,2), (2,3), (1,4)$$

respectively makes one entry of the first triple and two of the second equal to $x_{i_0 j_0}$ or of weight outside S (hence 0 by Step 0):

So every ξ with $x_{i_0 j_0} = 0$ has its six odd entries locally trivial.

Step 5: (Nl3). Here $n = 1 + bX$ with $X = E_{21} - E_{43}$, $b = b_v \neq 0$, $X^2 = 0$. For $Z \in \mathfrak{sp}_4(k)$ with $n^{-1} Z n$ upper triangular,

$$Z_{21} = b^2 Z_{12} + b\,(Z_{11} - Z_{22}), \qquad Z_{32} = 0, \qquad Z_{41} = 0.$$

On D_v, $a(\chi_1) = a(\chi_2)$ gives $\rho_{\pm w_\alpha} = 1$, and $\rho_0 = 1$; as $B_v(1) = 0$, Step 1 gives $Z_{ij}^v = x_{ij}|_{D_v}$ for $(i,j) \in \{(1,1), (2,2), (1,2), (2,1)\}$. Hence for ξ with $x_{11} = x_{22} = 0$: $x_{21}(s) = b^2 x_{12}(s)$ for $s \in D_v$; and for every ξ the entries x_{32}, x_{41} are locally trivial ($Z_{32}^v = Z_{41}^v = 0$). The types satisfying the hypothesis of (Nl3) are $\Sigma_1^{\mathrm{red}}, \Sigma_4^{\mathrm{red}}, \Sigma_5^{\mathrm{red}}, \Sigma_7^{\mathrm{red}}, \Sigma_8^{\mathrm{red}}$; take

$$(i_0, j_0) = (2,3), (1,4), (1,2), (2,3), (1,4)$$

respectively.

Then for ξ with $x_{i_0 j_0} = x_{11} = x_{22} = 0$: $x_{14} = x_{23} = 0$ (one is $x_{i_0 j_0}$, the other has weight outside S); one of x_{12}, x_{21} is 0 (it is $x_{i_0 j_0}$ or has weight outside S), so both vanish on every D_v ($b \neq 0$) and are locally trivial; and x_{32}, x_{41} are locally trivial. This is the one-odd hypothesis, and Step 2 concludes. □

Lemma 53.45. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ a good auxiliary tower for $\mathfrak{D}$ with field F, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring, c a residual pair over $(\mathscr{T}, \emptyset)$ with values in $\mathbb{F}_{\mathcal{O}'}$, $R \in \mathrm{CNL}_{\mathcal{O}'}$, and $Q = (r_Q, (n_v))$ a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in R (for $\mathcal{O} \subseteq \mathcal{O}' \to R$), continuous for $\mathfrak{m}_R$, with reduction c. Let $\varphi\colon R \to K$ be a ring homomorphism to a field such that $\ker\varphi$ contains none of the ideals $\mathrm{Red}_{\Sigma_k^{\mathrm{red}}}(r_Q)$, $1 \leq k \leq 9$. Then the K-span of the matrices $\varphi(r_Q(g))$, $g \in G_F$, is $\mathrm{M}_4(K)$.*

Proof. Write $r = \varphi \circ r_Q\colon G_F \to \mathrm{M}_4(K)$ and let $\mathcal{A}$ be the K-span of $r(G_F)$. We show: if $\mathcal{A} \neq \mathrm{M}_4(K)$, then r is supported in some Σ_k^{red}, i.e. $\ker\varphi \supseteq \mathrm{Red}_{\Sigma_k^{\mathrm{red}}}(r_Q)$ (last sentence of Definition 53.38); this is the contrapositive of the statement.

Step 1: $\mathcal{A}$ is a subalgebra containing the diagonal matrices. r_Q is a homomorphism of monoids, so $r(1) = 1$ and $r(g)r(h) = r(gh)$; hence $\mathcal{A}$ is a K-subalgebra of $\mathrm{M}_4(K)$. By (Mq2),

$X = r(g_{\mathscr{T}})$ is diagonal, with entries $x_i = \varphi(r_Q(g_{\mathscr{T}})_{ii})$. As Q has reduction c and c satisfies (Wq1), $r_Q(g_{\mathscr{T}})_{ii} \equiv \chi_i^{\mathscr{T}}(g_{\mathscr{T}})$ modulo $\mathfrak{m}_R$; by (Oq7) the residues $\bar{\kappa}_i^{\epsilon}(g_{\mathscr{T}})$ are pairwise distinct, so for $i \neq j$ the difference $r_Q(g_{\mathscr{T}})_{ii} - r_Q(g_{\mathscr{T}})_{jj}$ is a unit of the local ring R and $x_i \neq x_j$. Then $E_{ii} = \prod_{j \neq i}(X - x_j)/(x_i - x_j) \in \mathcal{A}$ for every i.

Step 2: the pattern. Let P be the set of pairs (i,j) with $E_{ij} \in \mathcal{A}$. For $Y \in \mathcal{A}$ we have $E_{ii} Y E_{jj} = Y_{ij} E_{ij} \in \mathcal{A}$ and $Y = \sum_{i,j} Y_{ij} E_{ij}$. Hence: $(i,j) \notin P$ if and only if $r(g)_{ij} = 0$ for all $g \in G_F$; $\mathcal{A}$ is spanned by the E_{ij}, $(i,j) \in P$; $(i,i) \in P$; and P is transitive, as $E_{ij}E_{jk} = E_{ik}$. Moreover, with $1' = 4$, $2' = 3$, $3' = 2$, $4' = 1$: if $(i,j) \in P$ then $(j', i') \in P$. Indeed, for $g \in G_F$ the matrix $Y = r(g)$ has similitude $\nu = \varphi(\theta(g))$, a unit, by (Mq1), so ${}^t Y J_4 Y = \nu J_4$ gives $J_4^{-1}\,{}^t Y J_4 = \nu Y^{-1} = \nu\, r(g^{-1}) \in \mathcal{A}$ (as $r(g)r(g^{-1}) = 1$); by linearity $J_4^{-1}\,{}^t Y J_4 \in \mathcal{A}$ for all $Y \in \mathcal{A}$, and $J_4^{-1}\,{}^t E_{ij} J_4 = \pm E_{j'i'}$.

The map $(i,j) \mapsto (j', i')$ preserves weights and exchanges the two entries of each weight $\pm w_\alpha$, $\pm w_\Psi$ and fixes the entries of weight $\pm w_\beta, \pm w_\theta$.

So $(i,j) \in P$ depends only on $\mathrm{wt}_4(i,j)$ for $i \neq j$.

Step 3: a proper pattern lies in a type. Suppose $\mathcal{A} \neq \mathrm{M}_4(K)$; by Step 2 some $(i_0, j_0) \notin P$. Let $U = \{k : (k, j_0) \in P\}$; then $j_0 \in U$ and $i_0 \notin U$. If $a \notin U$ and $b \in U$, then $(a,b) \notin P$: otherwise $(a,b), (b,j_0) \in P$ give $(a,j_0) \in P$ by transitivity. Let W_U be the set of roots w such that every pair (a,b), $a \neq b$, with $\mathrm{wt}_4(a,b) = w$ has $a \in U$ or $b \notin U$. If $i \neq j$ and $\mathrm{wt}_4(i,j) \notin W_U$, some pair (a,b) of the same weight has $a \notin U$, $b \in U$, so $(a,b) \notin P$, hence $(i,j) \notin P$ by the last sentence of Step 2, and $r(g)_{ij} = 0$ for all g. Thus r is supported in W_U. Finally, U is a non-empty proper subset of $\{1,2,3,4\}$, and for each of the fourteen such sets W_U is one of the reducibility types.

Hence r is supported in some Σ_k^{red}, which proves the contrapositive. □

Lemma 53.46. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ an auxiliary tower for $\mathfrak{D}$ with index set Σ, K a field, $a\colon \mathcal{O} \to K$ a ring homomorphism and $P = (r, (n_v))$ a marked ordinary pair over $(\mathscr{T}, \emptyset)$ with values in K (for a) such that r is supported in Σ_Y. If $v \in \Sigma$ and $(n_v)_{21} \neq 0$, then $\psi_{v,1}^P(s) = \psi_{v,2}^P(s)$ for all $s \in D_v$, or $\psi_{v,2}^P(s) = \psi_{v,4}^P(s)$ for all $s \in D_v$.*

Proof. Write $n = n_v$, $V = K^4$ with standard basis $e_1, \dots, e_4$, and $\omega(x,y) = {}^t x J_4 y$; so $\omega(e_i, e_j) = (J_4)_{ij}$ is non-zero only for $\{i,j\} = \{1,4\}$ or $\{2,3\}$. For $s \in D_v$ put $T(s) = n^{-1} r(s) n$, upper triangular with diagonal $\psi_i(s) = \psi_{v,i}^P(s)$ by (Mq4).

Step 1: the flag. Put $F_k = n\langle e_1, \dots, e_k\rangle$. Since $r(s)ne_k = nT(s)e_k = \sum_{j \leq k} T(s)_{jk}\, ne_j$, each F_k is $r(s)$-stable and $r(s)$ acts on F_k/F_{k-1} by $\psi_k(s)$. By (Mq3), n has similitude 1, so $\omega(nx, ny) = \omega(x,y)$; as $\langle e_1\rangle^\perp = \langle e_1, e_2, e_3\rangle$, we get $F_3 = F_1^\perp$.

Step 2: $\psi_1\psi_4 = \psi_2\psi_3$. By (Mq1), $r(s)$ has similitude $\nu = a(\theta(s))$, hence so has $T(s)$. For an upper triangular T with ${}^t T J_4 T = \nu J_4$, the $(1,4)$ entry gives $T_{11}T_{44} = \nu$ and the $(2,3)$ entry gives $T_{22}T_{33} = \nu$.

So $\psi_1(s)\psi_4(s) = \psi_2(s)\psi_3(s)$, and all $\psi_i(s)$ are non-zero ($r(s)r(s^{-1}) = 1$).

Step 3: an eigenplane. The vector $u = ne_1 = (1, x_2, x_3, x_4)$ (first column of n, unit diagonal by (Mq3)) has $x_2 = (n_v)_{21} \neq 0$, and $r(s)u = \psi_1(s)u$ (Step 1, $k = 1$). The off-diagonal entries of weight in $\Sigma_Y = \{\pm w_\beta, \pm w_\theta\}$ are $(2,3), (3,2)$ (weights $\pm w_\beta$) and $(1,4), (4,1)$ (weights $\pm w_\theta$), so every $r(s)$ preserves $V_{14} = \langle e_1, e_4\rangle$ and $V_{23} = \langle e_2, e_3\rangle$. Writing $u = u' + u''$ with $u' = (1,0,0,x_4) \in V_{14}$ and $u'' = (0, x_2, x_3, 0) \in V_{23}$, both non-zero, $r(s)u' = \psi_1(s)u'$ and $r(s)u'' = \psi_1(s)u''$. So $W = \langle u', u''\rangle$ is a plane on which every $r(s)$, $s \in D_v$, acts by the scalar $\psi_1(s)$; W is isotropic for ω ($V_{14} \perp V_{23}$, and each of u', u'' is isotropic as ω is alternating), and $u \in W$, so $W \subseteq u^\perp = F_3$.

Step 4: conclusion. If $W \subseteq F_2$, take $w \in W \setminus F_1$ (possible, $\dim W = 2$): its class in F_2/F_1 is non-zero and $r(s)w = \psi_1(s)w$, so $\psi_2(s) = \psi_1(s)$ for all $s \in D_v$. Otherwise take $w \in W \setminus F_2 \subseteq F_3$: in the same way $\psi_3(s) = \psi_1(s)$ for all $s \in D_v$, and Step 2 gives $\psi_4(s) = \psi_2(s)\psi_3(s)/\psi_1(s) = \psi_2(s)$. The alternative does not depend on s, since W and the F_k do not. □

Lemma 53.47. *Let $\mathfrak{D}$ be a symplectic Eisenstein datum with coefficient ring $\mathcal{O}$, $\mathscr{T}$ a fit auxiliary tower for $\mathfrak{D}$ with field F and index set Σ, $\mathcal{O}' \supseteq \mathcal{O}$ a coefficient ring, $\mathscr{X} = (B, A, P, \mathrm{cp}, (\psi^B_{v,i}))$ a space for $(\mathscr{T}, \mathcal{O}')$, $x\colon A \to \mathbb{F}_{\mathcal{O}'}$ an $\mathcal{O}'$-algebra homomorphism such that $c = x(P)$ is a residual pair over $(\mathscr{T}, \emptyset)$, R the completion of A at the maximal ideal $\ker x$ and $f\colon A \to R$ the canonical homomorphism; put $\phi = f|_B\colon B \to R$. Let $\mathfrak{q}$ be a prime of R, K the fraction field of $R/\mathfrak{q}$ and $\bar{r}\colon G_F \to \mathrm{M}_4(K)$ the composite of $f \circ r_P$ with $R \to R/\mathfrak{q} \subseteq K$. If the K-span of $\bar{r}(G_F)$ is $\mathrm{M}_4(K)$, or $\bar{r}$ is supported in Σ_Y and has entries of weights w_β, $-w_\beta$, w_θ and $-w_\theta$, then*

$$\dim R/\mathfrak{q} \leq \dim B/\phi^{-1}(\mathfrak{q}) + 2 + 4\#\Sigma.$$

Proof. Symbols introduced here are local to this proof. Let $\mathfrak{m} = \ker x$, $\mathfrak{p} = f^{-1}(\mathfrak{q})$ (a prime of A contained in $\mathfrak{m}$) and $\mathfrak{p}_B = \mathfrak{p} \cap B = \phi^{-1}(\mathfrak{q})$ ($B \subseteq A$ by (Zq1)). As $x|_B$ is a surjective homomorphism of the local ring B onto a field, $\mathfrak{m} \cap B = \mathfrak{m}_B$. A is a finitely generated algebra over the Noetherian ring B (Zq1), hence Noetherian.

Step 1: reduction to A. R is the $\mathfrak{m}$-adic completion of the Noetherian local ring $A_{\mathfrak{m}}$, and $R/\mathfrak{p}R$ is the completion of $(A/\mathfrak{p})_{\mathfrak{m}}$; a Noetherian local ring and its completion have the same dimension. Since $\mathfrak{q} \supseteq \mathfrak{p}R$, $\dim R/\mathfrak{q} \leq \dim R/\mathfrak{p}R = \dim(A/\mathfrak{p})_{\mathfrak{m}} = \mathrm{ht}(\mathfrak{m}/\mathfrak{p})$.

Step 2: the dimension inequality. $A/\mathfrak{p}$ is a domain, finitely generated over its subring $B/\mathfrak{p}_B$, a Noetherian domain; $\mathfrak{m}/\mathfrak{p}$ lies over $\mathfrak{m}_B/\mathfrak{p}_B$ and both residue fields are $\mathbb{F}_{\mathcal{O}'}$. By the dimension inequality ([166, Thm. 15.5]), $\mathrm{ht}(\mathfrak{m}/\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{m}_B/\mathfrak{p}_B) + \mathrm{trdeg}_{K_B} K_A$, where $K_B \subseteq K_A$ are the fraction fields of $B/\mathfrak{p}_B \subseteq A/\mathfrak{p}$; and $\mathrm{ht}(\mathfrak{m}_B/\mathfrak{p}_B) = \dim B/\mathfrak{p}_B$ as B is local. So it suffices to show $\mathrm{trdeg}_{K_B} K_A \leq 2 + 4\#\Sigma$.

Step 3: generators. f induces an injection $A/\mathfrak{p} \to R/\mathfrak{q} \subseteq K$; we regard $K_B \subseteq K_A \subseteq K$. The matrices $\bar{r}(g)$ have similitude $\nu(g)$, the image of $\theta(g) \in \mathcal{O}$ (Mq1), so $\nu(g) \in K_B$; $\bar{r}(g_{\mathscr{T}})$ is diagonal (Mq2); its diagonal entries are pairwise distinct, because their reductions modulo $\mathfrak{m}_R \supseteq \mathfrak{q}$ are the residues $\bar{\kappa}^c_i(g_{\mathscr{T}})$ of the residual pair c (Wq1), pairwise distinct by (Oq7). By the hypothesis and Lemma 53.48 there are $s, t \in K$ such that every subfield of K containing s, t, the $\nu(g)$, the traces and the diagonal of $\bar{r}(g_{\mathscr{T}})$ contains all entries of all $\bar{r}(g)$. For $v \in \Sigma$ the matrix n_v is lower triangular with diagonal entries 1 and similitude 1 (Mq3); writing its entries below the diagonal as $a = (n_v)_{21}$, $b = (n_v)_{31}$, $c' = (n_v)_{41}$, $d = (n_v)_{32}$, $e = (n_v)_{42}$, $h = (n_v)_{43}$, the entries $(2,1)$ and $(3,1)$ of ${}^t n_v J_4 n_v = J_4$ give $b - ad - e = 0$ and $-a - h = 0$, so $e = b - ad$ and $h = -a$. Let $T \subseteq K$ consist of s, t and the images of a, b, c', d for all $v \in \Sigma$, so $\#T \leq 2 + 4\#\Sigma$, and let $E \subseteq K$ be the algebraic closure in K of $K_B(T)$, a subfield.

Step 4: $K_A \subseteq E$. The traces of $\bar{r}(g)$ are images of coefficients of $\mathrm{cp}(g) \in B[X]$ (Zq2), so lie in K_B; the diagonal entries of $\bar{r}(g_{\mathscr{T}})$ are roots of the image of the monic $\mathrm{cp}(g_{\mathscr{T}})$, hence algebraic over K_B and in E. So E satisfies the hypotheses of Lemma 53.48, and all entries of all $\bar{r}(g)$ lie in E; so do the images of all entries of all n_v by Step 3. These entries generate A as a B-algebra (Zq1), so $A/\mathfrak{p} \subseteq E$ and $K_A \subseteq E$. Hence $\mathrm{trdeg}_{K_B} K_A \leq \mathrm{trdeg}_{K_B} E = \mathrm{trdeg}_{K_B} K_B(T) \leq \#T \leq 2 + 4\#\Sigma$, and with Steps 1 and 2, $\dim R/\mathfrak{q} \leq \dim B/\phi^{-1}(\mathfrak{q}) + 2 + 4\#\Sigma$. □

Lemma 53.48. *Let K be a field, Γ a group, $r\colon \Gamma \to \mathrm{M}_4(K)$ a homomorphism of monoids and $\nu\colon \Gamma \to K$ a map such that $r(g)$ has similitude $\nu(g)$ for every g, and $g_0 \in \Gamma$ such that $r(g_0)$ is diagonal with pairwise distinct diagonal entries. Assume that the K-span of $r(\Gamma)$ is $\mathrm{M}_4(K)$,*

or that r is supported in Σ_Y and has entries of weights w_β, $-w_\beta$, w_θ and $-w_\theta$. Then there are $s,t \in K$ such that every subfield $E \subseteq K$ containing s, t, all $\nu(g)$, all traces $\operatorname{tr} r(g)$ and the diagonal entries of $r(g_0)$ contains all entries of all $r(g)$, $g \in \Gamma$.

Proof. Symbols introduced here are local to this proof. Write E_{ij} for the matrix units, $D = r(g_0) = \operatorname{diag}(d_1, \dots, d_4)$, and for a subfield E as in the statement let $\mathcal{A}_E$ be the E-linear span of $r(\Gamma)$; as r is a homomorphism of monoids, $\mathcal{A}_E$ is an E-algebra containing $r(1) = 1$ and D, and $\operatorname{tr}(\mathcal{A}_E) \subseteq E$. Since $r(g)r(g^{-1}) = 1$, every $r(g)$ is invertible.

Step 1 (any such E). $E_{ii} = \prod_{j \neq i}(D - d_j)/(d_i - d_j) \in \mathcal{A}_E$, so $r(g)_{ij}E_{ij} = E_{ii}r(g)E_{jj} \in \mathcal{A}_E$ for all g, i, j. Taking traces: $r(g)_{ii} \in E$; $r(g)_{ij}r(h)_{ji} \in E$; and $r(g)_{ij}r(h)_{jk}r(h')_{ki} \in E$ for distinct i, j, k (trace of $r(g)_{ij}E_{ij} \cdot r(h)_{jk}E_{jk} \cdot r(h')_{ki}E_{ki}$). Consequently: (a) if $x \in E$ is a non-zero (i,j)-entry of some $r(g)$, then every (j,i)-entry $r(h)_{ji}$ lies in E; (b) if $x \in E$, $x' \in E$ are non-zero (i,j)- and (j,k)-entries of some $r(g)$, $r(g')$ (i, j, k distinct), then every (k,i)-entry lies in E.

Step 2: first alternative. The K-span of $r(\Gamma)$ is $\mathrm{M}_4(K)$, so $E_{ii}\mathrm{M}_4(K)E_{jj} \neq 0$ shows that for all $i \neq j$ some $r(g)_{ij} \neq 0$. Choose g_{12}, g_{13} with $s = r(g_{12})_{12} \neq 0$, $t = r(g_{13})_{13} \neq 0$. Let J_4 have $(J_4)_{i,\sigma(i)} = \varepsilon_i$ with $\sigma = (1\,4)(2\,3)$, $(\varepsilon_1, \dots, \varepsilon_4) = (1, 1, -1, -1)$, so $J_4^{-1} = -J_4$; from ${}^t r(g) J_4 r(g) = \nu(g) J_4$, $r(g)^{-1} = -\nu(g)^{-1} J_4 \, {}^t r(g) J_4$, i.e. $r(g^{-1})_{ij} = -\nu(g)^{-1}\varepsilon_i\varepsilon_{\sigma(j)} r(g)_{\sigma(j)\sigma(i)}$. Hence $r(g_{13}^{-1})_{24} = -\nu(g_{13})^{-1}t$ and $r(g_{12}^{-1})_{34} = \nu(g_{12})^{-1}s$, non-zero. Now let E be as in the statement. Then $s, t, -\nu(g_{13})^{-1}t, \nu(g_{12})^{-1}s \in E$ are non-zero entries at $(1,2), (1,3), (2,4), (3,4)$, so by (a) all entries at $(2,1), (3,1), (4,2), (4,3)$ lie in E; choosing non-zero such entries (they exist), (a) gives all entries at $(1,2), (1,3), (2,4), (3,4)$. By (b) with $(1,2), (2,4)$: all entries at $(4,1)$; then (a): all entries at $(1,4)$. By (b) with $(3,1), (1,2)$: all entries at $(2,3)$; with $(2,1), (1,3)$: all entries at $(3,2)$. With the diagonal (Step 1), all entries of all $r(g)$ lie in E.

Step 3: second alternative. The weights are $\mathrm{wt}_4(1,4) = w_\theta$, $\mathrm{wt}_4(2,3) = w_\beta$ and their negatives at $(4,1)$, $(3,2)$ (with $\boldsymbol{\lambda} = (1,0,1,0)$, $\boldsymbol{\mu} = (1,1,0,0)$), so r is supported on the diagonal and the positions $(1,4), (4,1), (2,3), (3,2)$, and each of these carries a non-zero entry of some $r(g)$. Choose $s \neq 0$ a $(1,4)$-entry and $t \neq 0$ a $(2,3)$-entry. For E as in the statement, (a) gives all $(4,1)$- and $(3,2)$-entries in E; choosing non-zero ones, (a) gives all $(1,4)$- and $(2,3)$-entries; with the diagonal and the vanishing of the other entries, all entries lie in E. □

A The construction beneath (TCS)

This appendix explains the construction that is expected to produce a theta chain system (TCS), Hypothesis 21.1. It is not part of the proof, and the paragraphs headed "Sketch." are heuristic arguments, not proofs. A theta chain system is stated in Section 21 through its records, the base translate packets of Definition 19.1; this appendix describes the chains of theta lifts that are expected to produce these records. In Alpöge's account the chain is a sequence of iterated theta lifts $\mathrm{U}(X_0) \to \mathrm{U}(X_1) \to \cdots \to \mathrm{U}(X_m)$ between unitary groups, which wins multiplicity m [4], and the see-saw, the Bessel periods and the work of Rallis belong to it. The construction follows the formal development [3], which we have not verified.

We recall what (TCS) asserts; Section 21 reads its statement clause by clause. At every large scale x of a set of scales containing λ-windows, and over prescribed good-host data at that scale (Definition 11.4), (TCS) asks for a base translate packet: one loss datum and one slope record for every translate $u \in A_x$, with constants common to all scales. Clause (Rr1) asks the loss data to satisfy (Lm2) and (Lm5)–(Lm7) of Definition 5.5. Clause (Rr2) makes the numerical data explicit: rank sizes, integer weights in prescribed boxes, the selected good-orbit choices, and orbit corrections with common constants. Clause (Rr3) holds under the eventual bound for Tate–Shafarevich groups. It asks for lower completions, kernel hypotheses and a word bound; for covolume, chain, deep-cover, component and model data; for explicit arrays of divided weights; for unitary Shimura data that satisfy (Aj1) and (Aj3) of Theorem 29.14; and for reference comparison data that meet every hypothesis of Theorem 20.2.

The clauses are consumed by one result, Proposition 21.2. It uses all of (Rr1) and (Rr3), and all of (Rr2) except the exact formula for L_c, of which it needs only an upper bound. It obtains the bounds (Ms1) and (Ms3) for model slopes from Theorem 29.14, and the realisation estimate (Ri1) from Theorem 20.2. Both theorems are proved in the body of the paper. Beneath (TCS) they appear only through their hypotheses, which the construction must meet. The same holds in two further places. The covolume inequality is Theorem 19.26, and the construction supplies the data to which it applies. The local losses are bounded by the Ribet budget Theorem 22.1, and the construction must meet its hypotheses at the members it builds.

The sections follow the construction. Section A.1 shows how three statements combine into (TCS): a class of families with a selection, a table of heights and depths at every scale, and the reading of the packets from the selected members. Section A.2 chooses the tables, so that the orbit shares in (Rr1) are small. Sections A.3 and A.4 build the arithmetic and automorphic material of the chain: theta kernels on a CM elliptic curve, integral at the primes inert in the imaginary quadratic field, and theta lifts along the rows of a grid of hermitian spaces, with separation of Bessel functionals, Rallis towers, the see-saw and an identity for multiplicities. Section A.5 builds the members of the packets from this material in two passes. Sections A.6 and A.7 prove the approximate orthogonality of Poincaré series at the members, which gives the slope records. Section A.8 supplies the covolume data, and Section A.9 attaches the Galois-theoretic records to the members. Where no statement printed here covers a step of the construction, a short sketch says what the step must achieve and why it is plausible.

A.1 What a theta chain system asks for, and how the construction supplies it

The proof of (TCS) from the construction is a composition of three statements, and this section states two of them. Proposition A.1 fixes one class $\mathfrak{A}$ of families together with a selection σ, before λ, the host requirement and ε, and returns, for every small ε, a family of the class over the prescribed host requirement. Proposition A.19, stated in Section A.2, chooses one pair of integer

tables of heights and depths at every scale, before any context or member, and shows that the members built on these tables have the numerical content of (Rr1) and (Rr2). Proposition A.2 reads the packets and their completions from the members that the selection picks: its first item gives (Rr1) and (Rr2), its second the lower completions and kernel data, its third the covolume, chain, deep-cover and model data with (Aj1) and (Aj3), and its fourth the reference data for Theorem 20.2. The choices are made in a fixed order: the class and the selection; then ε; then thresholds; then the profile and the exponents; then the tables; then contexts and members; and last the constants and the tails. The threshold on ε that (TCS) allows may depend on λ and on the host requirement, and the bound $\varepsilon_0(\lambda, G)$ of Proposition A.1 depends on exactly these. The second subsection collects the bookkeeping that lets finitely many separately proved properties hold on the same selected objects.

A.1.1 The three inputs and their composition

The class of Proposition A.1 is defined by a witnessed equivalence, so a selection over it exists automatically. The substance of the statement is that actual design families belong to the class. One row parameter must serve the whole family, and at every real size there must be bundles, linked for that parameter, at the members of the family. The statement also fixes the number of words of every context, $|\mathcal{W}_{x,u}| = \lceil 3\Gamma_{x,u}^2 x^{C_H} \rceil$, which (TCS) uses in its word bound. Proposition A.2 then reads the packets of (TCS) from the selected members. Its second item rests on the packet-level row records of Theorem A.71, and its local defect data on the Galois-theoretic records of Theorem A.105. Its third item asks for the covolume, deep-cover and model data that Section A.8 constructs at realised points, and its fourth for reference data that meet every hypothesis of Theorem 20.2. The sketch after Proposition A.2 explains how these attachments are expected to go.

Proposition A.1. *Let $p = p_A$ be the final parameter of the row-parameter construction. There are one class $\mathfrak{A}$ of parametrised families carrying member objects, a selection σ over it and an E_{all}-selection structure, all defined before λ, the host requirements and ε. Membership in $\mathfrak{A}$ is defined by an equivalence, described in the sketch below, with one linked row parameter per family and built E_{all}-bundles of every real size at the member objects of the family. Here $R_{\mathrm{all}}, E_{\mathrm{all}}$ are a joint slot and bundle predicate for the twenty-six joint row conclusions, and the norm predicate is the predicate $\mathcal{H}_{\mathrm{norm}}$ of the row-parameter construction. The class has the seven duties and the conclusions below.*

Fix $\lambda > 1$ and a predicate $G(s, \phi, \mathfrak{h}, T, t)$ on a scale $s = (x, \varepsilon, J)$, a frame ϕ, a host $\mathfrak{h}$ on that frame, its top set T and its rank t. Assume the following host-existence condition: for every $0 < \varepsilon < 1/52$ there are constants C_{T_2}, $C_8 \geq 1$, $C_5, c_6, C_b > 0$, a common J, a set W containing λ-windows and a real X_H such that every $x \in W$ with $x \geq X_H$ admits a complete choice of the data of the concrete good-host requirement of Definition 11.4 with these constants and with $G((x, \varepsilon, J), \phi_x, \mathfrak{h}_x, T_x, t_x)$. The predicate G depends only on these host data; it prescribes no later constant profile, exponent pair, table, context or member.

There is $\varepsilon_0 = \varepsilon_0(\lambda, G) > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$ with $\varepsilon < 1/52$ and every structural threshold triple $\mathfrak{K}$ there is a family $\mathcal{F} \in \mathfrak{A}$ of parameter ε which is large for $\mathfrak{K}$ at its own design parameter

$$C_H(\mathcal{F}) = C_F(\mathcal{F}) \geq 1.$$

Its scales contain λ-windows, its twists are good, and every retained member satisfies G at its own scale, frame, host, tops and rank. The good-host data with their common constants, the admissible profile, the constants c_D, c_h in their nested eventual ranges and the canonical tables of heights and depths are chosen in the construction, subject to the host requirement G; they

are not prescribed in advance. The profile, the exponent pair and the tables are common to all translates and are fixed before the contexts and members of the family. For the chosen collection there is a cutoff $X \geq X_H$ with

$$I(x) = A_x \quad (x \in W,\ x > X), \qquad I(x) = \varnothing \quad \text{otherwise.}$$

Every retained member uses one context word scale $\Gamma_{x,u}$ in all its word tests, and

$$|\mathcal{W}_{x,u}| = \left\lceil 3\Gamma_{x,u}^{2} x^{C_H(\mathcal{F})} \right\rceil .$$

In particular the word-count bounds (Rk13) of the row linkage hold with coefficient 3.

For this family and the selection fixed above, the following hold.

1. *For every retained scale x and every $u \in A_x$ there are a context $K_{x,u}$, a base point $\beta_{x,u}$ and a bundle $b_{x,u}$ linked to it for p. Their frames, hosts, tops, spacing sets, ranks and auxiliary primes are those chosen in the good-host collection of the family, with translate u. The profile and the top tables are the common ones chosen for this family.*

2. *The base point satisfies $\mathrm{Good}_{\mathrm{E}}(\mathfrak{S}(p)) \wedge R_{\mathrm{all}}$. Its layered datum is admissible, and its size datum and canonical instance form a part whose member has all conclusions (a)–(e) of Proposition A.18. This part is the one used by its linked bundle.*

3. *The family belongs to $\mathfrak{A}$ and $\sigma(\mathcal{F}, x, u) = b_{x,u}$. It carries the seven duties and the profile and member ties of (Rk6) and (Rk7) of the row linkage. The twenty-six joint row conclusions hold for this selection and this class, with their closed-family and eventual quantifiers. In particular the class norm condition is on its standard restriction.*

The class and the selection are chosen before the later local indices and before the conditional Sha assertions are tested. The lower realisation data needed by a row conclusion are those of its selected bundle.

The joint slot and bundle predicate are chosen, before the class and the selection are fixed, so that they also carry the row assertion of the supply of Galois and depth-zero conditions. Consequently every member of the returned family satisfies the Galois preconditions (Gv1)–(Gv19) and the depth-zero tie (Ze1)–(Ze3), including every subclause of (Ze2), on its pair

$$(D_{x,i}, \mathrm{nch}(O_{x,i})), \qquad \mathfrak{o}_{x,i} = (O_{x,i}, \ell_{x,i}).$$

The member data, objects, coefficient rings, local indices and comparison maps are those of this construction; no family, object, part or selection is replaced to obtain these conditions.

The loss datum $\mathfrak{M}_{x,i}$ of each returned member is the canonical numerical reading of its datum $D_{x,i}$, its size datum $\mathcal{S}_{x,i}$ and its supplied canonical instance $I_{x,i}$, with the common orbit-constant tuple $\mathfrak{k}_{\mathrm{orb}}$ and the constant C_e chosen before the tables. Explicitly:

- *its group is the frame group A, its arithmetic datum is that of the host, its tops are the tops T, and its ranks satisfy $(k, m, t_R, e_2, o) = (k, m, t, e, o)$;*
- *its top heights and depths are the zero extensions of the top tables $a^{\mathrm{top}}, b^{\mathrm{top}}$; its weights are*

$$w_j(z) = \mathrm{fill}(I_{x,i}, b^{\mathrm{top}}, j, z) = \tilde{b}_{\tau_z, j}, \qquad \varsigma_j = \frac{k + (2j-1)(t+1)}{2}$$

for the shifts in the integer-weight description;

- *its orbit origins, representative map and full good sets are those of the ordered good-host selection, fixed before the tables; fix one common* $C_N > 0$ *with* $N_{\mathrm{orb}} \le C_N \log x$ *on the common tail, and write*

$$\rho_B = \frac{8nx^{-\varepsilon/8}x^{\varepsilon}L_c}{k} + \frac{C_{\mathrm{band}}C_N(\log x)L_c}{k}.$$

  Its candidate predicate uses this wider band ρ_B*, the threshold* $p_0 = k^2x^{26}$ *and the original order cutoffs. Its orbit constants are the prescribed capped expression in the low regime and, in the high regime, the canonical infimum, with the same endpoint arrays, middle floor* $\lfloor 4S_c \rfloor$*, integer depth ceiling, degree and separation;*

- *its size entries are* S_c, η_S*,* $N = k + m(t+1)$ *and*

$$L_a(\mathfrak{M}_{x,i}) = L_c = C_\Lambda N_{\mathrm{orb}}^2 \log a, \qquad C_{\mathrm{La}} = C_\Lambda.$$

  The no-gain normalization is $C_r = 1$*. The common constants, the wider-band prescription and the full good sets are fixed before the tables, contexts and parts;*

- *its place field is the original large coefficient field* L*, its support is* $\mathrm{supp}(\mathcal{S}_{x,i})$*, and its place weight is* $[L_w : \mathbb{Q}_p]\log p/[L : \mathbb{Q}]$*. Its gain is the member-truncated gain* $\mathrm{gain}_j^{\flat}(w)$*, which vanishes at the dropped primes of the member; its correction and companion terms, served sets and tops, and partial weights are those of the same canonical instance and part. Its correction exponents are* $\epsilon_p = \max(0, \lceil C_e f k^2 \log_p a \rceil)$*, with all eight orbit constants read at their selected prescribed values.*

These assignments define the loss record before the canonical-table conclusions are applied. They do not identify an independently given compatible loss record with this one.

Proof. Sketch. The proof separates a formal part from a constructive one. The class $\mathfrak{A}$ is defined by an equivalence: a family belongs to it exactly when there is one row parameter at which the family has its seven duties and, at every retained member (x, i) and every real size c, a bundle built over the member's own datum and member object $(D_{x,i}, \mathfrak{o}_{x,i})$ that satisfies the joint row predicate fixed beforehand. The selection σ chooses, for each family of the class, one such parameter and one such bundle at the family's size; class and selection are thus defined before λ, the host requirement and ε, and the joint row conclusions apply to them because the duties of a selection are exactly what membership provides. The selection is therefore automatic, and the substance of the statement is that actual families belong to the class.

For this, λ and G are fixed, and $\varepsilon_0(\lambda, G)$ is the minimum of the finitely many positive bounds of the inputs used; by Lemma A.4 their structural restrictions combine into one, by pointwise maxima of thresholds (Definition A.3). In this order one then chooses a good-host collection satisfying G, a profile large for the combined restriction, $c_D \ge 415$, $c_h \ge 402$, and the canonical tables of Proposition A.19, all before any context or member; the members are those of Theorem A.52 for the concrete system of Theorem A.51, and one set of words with an injective code is fixed at each scale before the translates, which gives $|\mathcal{W}_{x,u}| = \lceil 3\Gamma_{x,u}^2 x^{C_H} \rceil$. The numerical fields are elementary: every size parameter of a member is at most x^H with H depending only on c_D, c_h, while the budgets involve $b = e^{x^{\varepsilon}}$, so each inequality reduces to $Cx^A e^{-x^a} \to 0$ or $(\log x)^A/x^a \to 0$ on one common tail; and $N_{\mathrm{orb}} = O_\varepsilon(\log x)$ gives $L_c \le x^{\varepsilon+\varepsilon/100}$. Among the seven duties, the wall duty is the small-share inequality of the chosen tables (Lemma A.17) read at the actual members, the twisting duty holds after the auxiliary prime used for twisting is replaced by the least prime $q_* \equiv 1 \pmod{\ell_0 N_{\mathfrak{h}}}$, and the others are readings retained by

the member construction. Finally, a supply of linked bundles at every real size over the same member datum and object, at one family-wide row parameter, gives membership through the reverse implication of the defining equivalence, and items 1–3 are read at the selected bundles, with Proposition A.18 giving item 2. The argument is bookkeeping; its depth lies entirely in the inputs it cites, namely the member construction, the canonical tables and the supply of linked bundles at every real size.

The good-host collection over which the family is built is chosen in the course of the construction, subject only to the host requirement G. Whether the construction can instead be carried out over a collection of good-host data fixed in advance, as (TCS) prescribes, is open. □

Proposition A.2. *Choose the structural threshold triple used in Proposition A.1 so that it dominates the threshold $\mathfrak{K}_{\mathrm{dg}}$ of the diagonal expansions at the selected packets, before ε, the good-host data, the class and the family are chosen; then the returned design family is large for $\mathfrak{K}_{\mathrm{dg}}$. Fix one complete collection of good-host data. Take the class and the selection of Proposition A.1, the design family over this prescribed collection returned by Proposition A.1 with its member objects, and its tables and constants of Proposition A.19. The construction of Proposition A.1 chooses its own collection of good-host data; whether its class, selection and member objects can be attached to a prescribed complete collection, as is done here, is an open question, and the proposition concerns these data whenever such an attachment exists. Then, after enlarging a common cutoff, the following hold.*

1. *At each retained scale, the loss and slope records of its members, indexed by $u \in A_x$, form a base translate packet P_x with the selected host, frame, tops, spacing set, rank, auxiliary primes and final top profile. These packets satisfy (Rr1) and (Rr2) of Hypothesis 21.1.*

2. *Under the eventual Sha assumption of (Rr3) of that proposition, the same packets admit the twenty-four common constants and compatible lower completion data (Pl1), with all numbered conditions and exclusions stated in (Rr3). Their kernel data satisfy (Kc1)–(Kc4), finite total measure and both scalar inequalities specified there. Their word count has the bound stated there.*

3. *These completions admit the base covolume, chain, deep-cover, component and model-slope realisations required in (Rr3), including its field and metric identifications, common constants, precision data and joint inequalities, and the unitary Shimura data with every hypothesis of (Aj1) and (Aj3) as quantified there.*

4. *The reference data, denominators and local arrays satisfy all hypotheses of Theorem 20.2 on one common tail, with the same records and constants as in (Rr3), including its support, signed-index, Gram, denominator, weighted-defect, frame and main-term identities. The member also satisfies (Ri2), (Ri3) and (Ri4) as demanded there. No field, lattice, metric, map, local value, orbit mass or final selected profile is replaced by a substitute.*

The constants do not depend on the scale or on any local index. Every finite-Sha, word-size and nonzero-map antecedent, and the order of every inner integer choice, are as in (Pl2) and (Rr3).

Proof. Sketch. The statement attaches (TCS), Hypothesis 21.1, to the construction: its packets should be the loss and slope records of the members selected by Proposition A.1, built on the tables of Proposition A.19. Item 1 is read from these two statements: the selected members carry the prescribed frame, host, tops, spacing set, rank and auxiliary primes, and items 1–4 of Proposition A.19 give (Rr1) and (Rr2). For item 2 the structural threshold is first enlarged so

that the returned family is also large for the threshold of the diagonal expansion; at the family's own design exponent $c = C_H = C_F$, Theorem A.71 then gives the row records and the kernel hypotheses, and the word count of Proposition A.1, with coefficient 3 and the context's own word scale, gives the word bound. The member inputs of that theorem are the ties of each selected member with its context (Proposition A.114), its Poincaré frame, its counting operator and its Gauss ranks; and since every twist of a selected base point is good, specialising the twist condition to the exponent 1 gives the good twist data of Definition A.104.

It remains to show four attachments. First, the family must be built over the collection of good-host data prescribed in (TCS), whereas Proposition A.1 chooses its collection in the course of the construction, subject only to the host requirement. Whether that construction can be carried out over a prescribed collection is open. It is expected to be possible because the construction uses the good-host data only as data fixed at the start, and its bound $\varepsilon_0(\lambda, G)$ depends only on λ and the requirement, as the threshold of (TCS) may. Second, every selected member must satisfy the Galois preconditions (Galois representations compatible with the base changes of its automorphic representations, residual irreducibility, depth zero at the auxiliary primes g_1, g_2) and the depth-zero tie; once they hold, Theorem A.105 gives the local defect data and the hypotheses of Theorem 22.1. Heuristically, the members are cohomological automorphic representations of unitary groups over a CM field, for which base change and Galois representations with local–global compatibility are available [153, 231, 67, 56], and the depth-zero types at g_1, g_2 are chosen so that their inertia polynomials pass a test designed to exclude small invariant subspaces of the residual representation. Third, the unitary Shimura data of signature $(k + j - 1, m - j)$, the component sets and the arrays required by (Aj1) and (Aj3) of Theorem 29.14 must be constructed over the realised points of the covolume construction (Proposition A.92), with which (TCS) asks them to be compatible; that theorem only consumes these data. Fourth, the reference lattices, pairings, signed indices and denominators must be constructed so that every hypothesis of Theorem 20.2 holds on the same objects; the period service supplies part of these data, and the rest is a finite list of support, frame and scalar identities that the construction is expected to satisfy by design. Once the four attachments are supplied, the maximum of finitely many cutoffs gives the lower completion of (TCS) on a common tail. □

A.1.2 Restrictions, thresholds and linked rows

Many readers impose lower thresholds on the same family, and each of them is proved separately. Two simple facts let all of them hold at once. A threshold is a restriction of the class of families, and finitely many restrictions intersect in a restriction whose bounds are the pointwise maxima of the given ones (Lemma A.4); so one structural profile meets every closed law. One slot and one bundle predicate are fixed before any class, and a finite conjunction of properties that hold for linked rows on one selection holds on the same selection (Lemma A.5); so the separately proved row services hold on the same selected bundles.

Definition A.3. A **structural profile** carried by a parametrised host family $\mathcal{F}$ consists of a real number $C_H(\mathcal{F})$, its **chosen design parameter**, and four ordered lists of real numbers: $\mathbf{c}_{\mathrm{pre}}(\mathcal{F})$ of length 1, $\mathbf{c}_{\mathrm{out}}(\mathcal{F})$ of length 16, a list of length 9, and $\mathbf{c}_{\mathrm{last}}(\mathcal{F})$ of length 32. For each real u, write $\mathbf{c}_{\mathrm{mid}}(\mathcal{F}; u)$ for the length-10 list obtained by putting u before that length-9 list. In particular $\mathbf{c}_{\mathrm{mid}}(\mathcal{F}; C_H(\mathcal{F}))$ uses the chosen design parameter. The profile and chosen design parameter are part of the data of the family. For a family linked to a row parameter κ in the sense of (Rk6) of the row linkage, its length-9 list is

$$(C_\Gamma, C_{\mathrm{sz}}, C_{\mathrm{sz}}, C_{\mathrm{sz}}, C_{\mathrm{sz}}, C_{\mathrm{sz}}, C_{\mathrm{err}}, C_\Lambda, C_F),$$

namely entries 2 to 10 of $\mathbf{C}(\kappa)^{\mathrm{mid}}(u)$ for every real u. The generic notion of structural profile does not require a linked row parameter.

A **structural threshold** is a triple

$$\mathfrak{K}_{\mathrm{str}} = (k_{\mathrm{pre}}, k_{\mathrm{mid}}, k_{\mathrm{last}}) \in \mathbb{R} \times (\mathbb{R}^{<\infty} \to \mathbb{R}) \times (\mathbb{R}^{<\infty} \to \mathbb{R}).$$

A family with a structural profile is **large for** $\mathfrak{K}_{\mathrm{str}}$ if

$$\begin{aligned} k_{\mathrm{pre}} &\le c \quad \text{for every entry } c \text{ of } \mathbf{c}_{\mathrm{pre}}(\mathcal{F}),\\ k_{\mathrm{mid}}\big(\mathbf{c}_{\mathrm{out}}(\mathcal{F})\big) &\le c \quad \text{for every entry } c \text{ of } \mathbf{c}_{\mathrm{mid}}(\mathcal{F}; C_H(\mathcal{F})),\\ k_{\mathrm{last}}\big(\mathbf{c}_{\mathrm{out}}(\mathcal{F}) \parallel \mathbf{c}_{\mathrm{mid}}(\mathcal{F}; C_H(\mathcal{F}))\big) &\le c \quad \text{for every entry } c \text{ of } \mathbf{c}_{\mathrm{last}}(\mathcal{F}). \end{aligned}$$

A **restriction** $\mathfrak{r}_{\mathrm{str}}$ consists of a real number $\varepsilon_{\mathfrak{r}} > 0$ and a function

$$\mathfrak{K}_{\mathfrak{r}} \colon (0, 1/52) \longrightarrow \mathbb{R} \times (\mathbb{R}^{<\infty} \to \mathbb{R})^2.$$

Here the subscript $\mathfrak{r}$ denotes this fixed restriction. For a class $\mathcal{B}$ of parametrised host families with slope records and structural profiles, its **restriction by** $\mathfrak{r}_{\mathrm{str}}$ is the class $\mathcal{B}[\mathfrak{r}]$ of the same enriched families $\mathcal{F} \in \mathcal{B}$ whose parameter and profile satisfy

$$0 < \varepsilon < \min(\varepsilon_{\mathfrak{r}}, 1/52), \qquad \mathcal{F} \text{ is large for } \mathfrak{K}_{\mathfrak{r}}(\varepsilon).$$

In particular the profile and $C_H(\mathcal{F})$ in this test are the data already carried by $\mathcal{F}$.

The **intersection restriction** of $\mathfrak{r}_{\mathrm{str}}$ and $\mathfrak{r}'_{\mathrm{str}}$ is denoted $\mathfrak{r}_{\mathrm{str}} \vee \mathfrak{r}'_{\mathrm{str}}$. Its upper parameter bound is $\min(\varepsilon_{\mathfrak{r}}, \varepsilon_{\mathfrak{r}'})$. For each $\varepsilon \in (0, 1/52)$, write

$$\mathfrak{K}_{\mathfrak{r}}(\varepsilon) = (k_{\mathrm{pre}}, k_{\mathrm{mid}}, k_{\mathrm{last}}), \qquad \mathfrak{K}_{\mathfrak{r}'}(\varepsilon) = (k'_{\mathrm{pre}}, k'_{\mathrm{mid}}, k'_{\mathrm{last}}).$$

The threshold triple of the intersection restriction has scalar entry $\max(k_{\mathrm{pre}}, k'_{\mathrm{pre}})$ and, for every finite real list c, function values

$$\max\big(k_{\mathrm{mid}}(c), k'_{\mathrm{mid}}(c)\big), \qquad \max\big(k_{\mathrm{last}}(c), k'_{\mathrm{last}}(c)\big).$$

Lemma A.4. *For every class $\mathcal{B}$ of parametrised host families with slope records and structural profiles, and every two restrictions,*

$$\mathcal{B}[\mathfrak{r}_{\mathrm{str}} \vee \mathfrak{r}'_{\mathrm{str}}] = \mathcal{B}[\mathfrak{r}] \cap \mathcal{B}[\mathfrak{r}'].$$

Consequently every nonempty finite intersection of restrictions of $\mathcal{B}$ is a single restriction of $\mathcal{B}$. Its upper parameter bound is the minimum of the given upper bounds, and each of its three threshold entries is the pointwise maximum of the corresponding entries. The same enriched family, including its structural profile and chosen design parameter, occurs in each class in this identity.

Proof. Fix an enriched family $\mathcal{F}$ with parameter ε. The bound $0 < \varepsilon < \min(\varepsilon_{\mathfrak{r}}, \varepsilon_{\mathfrak{r}'}, 1/52)$ is equivalent to the parameter bounds required by both restrictions. For any two real lower bounds a_1, a_2 and any real c, the inequality $\max(a_1, a_2) \le c$ is equivalent to $a_1 \le c$ and $a_2 \le c$.

Apply this equivalence first to every entry of $\mathbf{c}_{\mathrm{pre}}(\mathcal{F})$. Apply it next to every entry of $\mathbf{c}_{\mathrm{mid}}(\mathcal{F}; C_H(\mathcal{F}))$, evaluating both middle threshold functions at the same list $\mathbf{c}_{\mathrm{out}}(\mathcal{F})$. Finally

apply it to every entry of $\mathbf{c}_{\mathrm{last}}(\mathcal{F})$, evaluating both last threshold functions at the same concatenated list

$$\mathbf{c}_{\mathrm{out}}(\mathcal{F}) \parallel \mathbf{c}_{\mathrm{mid}}(\mathcal{F}; C_H(\mathcal{F})).$$

These three equivalences say exactly that $\mathcal{F}$ is large for the intersection threshold if and only if it is large for both original thresholds. Together with membership in $\mathcal{B}$ and the parameter bounds, they prove the displayed class identity.

For a nonempty finite collection, successively apply the two-restriction identity. A finite minimum of positive upper bounds is positive; the minimum and each pointwise maximum are associative. This gives the stated single restriction and the formula for all its entries. □

Lemma A.5. *Fix a parameter p, a finite set I_{row}, and for every $s \in I_{\mathrm{row}}$ a property $H_s(\mathfrak{A}, \sigma)$ of classes and selections. Suppose that for every s the property $\mathrm{Row}(\cdot, \cdot, H_s)$ holds linked at p.*

Put $G = \mathrm{Good}_{\mathrm{E}}(\mathfrak{S}(p))$. There are a large-base slot R over $\mathrm{Built}_{\mathrm{E}}(\mathfrak{S}(p))$ and a bundle predicate E such that

$$\mathrm{RowE}(G \wedge R, \mathrm{Lk}_p, E)$$

and, for every class $\mathfrak{A}$ and every E-selection σ for the slot $G \wedge R$,

$$\bigwedge_{s \in I_{\mathrm{row}}} H_s(\mathfrak{A}, \sigma).$$

More precisely, for any slots R_s and bundle predicates E_s for which the individual linked row properties hold, these conclusions hold with the pointwise conjunctions

$$R = \bigwedge_{s \in I_{\mathrm{row}}} R_s, \qquad E(\kappa, b) = \bigwedge_{s \in I_{\mathrm{row}}} E_s(\kappa, b).$$

The empty conjunctions are the predicates that are always true.

Proof. Outline. The eventual quantifiers and the largeness of triples are closed under finite conjunction (take the minimum of the upper bounds on ε and successive maxima of the thresholds, and unions of the demands), so the pointwise conjunctions R and E give a large-base slot and the row assertion for $G \wedge R$; and an E-selection for $G \wedge R$ is, with the same row parameter and bundles, an E_s-selection for $G \wedge R_s$ for every s, so each H_s holds on the same selection. □

Sketch: the row services at one parameter. The selected bundles must carry, at one row parameter, a finite list of row services: arithmetic services at the auxiliary primes, a period and valuation service at the selected points, stage-zero and counting services, a definite-side service, and the services of the chain of lifts described at the end of Section A.4. Each service is a property of linked bundles that holds eventually in the scale and persists when the parameter is enlarged. Finitely many such properties can therefore be met at one parameter, by enlarging the parameter to the largest of their thresholds (Lemma A.4) and conjoining them on one selection (Lemma A.5). What each service needs is its own supply on the actual bundles, and all of these supplies are read from the same CM data: the abelian variety and the CM type of the host, the theta functions on the CM elliptic curve, and the auxiliary primes g_1, g_2. Heuristically this is why one parameter can serve all of them: the services ask for finitely many explicit facts about one CM configuration, and each such fact holds once the scale is large.

A.2 Small orbit shares: canonical tables of heights and depths

This section chooses, at every scale and before any member is built, a pair of integer tables of heights and depths, and shows that members built on these tables satisfy the numerical clauses of (Rr1) and (Rr2) (Proposition A.19). The difficult clause is (Lm5) of Definition 5.5: it asks that the canonical orbit constants have a small *mean* over a box of heights and depths. A pointwise bound would fail, but a mean only needs a majorant with small mean. Theorem A.9 gives one: a universal majorant whose mean over the box is at most CNN_1/Y (clause (Om1)), and which has the separated rank property (clause (Om2)). The majorant is universal over unramified quadratic settings, and the separated rank property does not depend on the setting (Lemma A.10). This is what allows the tables to be chosen before the members: one pair of tables serves every translate. A finite averaging step then selects the tables (Lemma A.16, Lemma A.17). The numerical clauses of a member over a context are Proposition A.18, and from these numerical data alone the body of the paper derives the linking property (Theorem 18.4, with Lemma 18.5).

A.2.1 A universal majorant for the orbit constants

An unramified quadratic setting carries weight arrays on an orbit and the integral orbit lattices of polynomial representations (Definition A.6). The canonical orbit constant of a weight array, and the separated rank property, are defined in Definition A.7, and the explicit flat depth and level of the majorant in Definition A.8. Theorem A.9 is the main result of the section. Its proof passes from a majorant in one fixed setting to all unramified quadratic settings at once, using that the separated rank property does not depend on the setting (Lemma A.10).

Definition A.6. Let p be a prime and $n \geq 2$, $k \geq 1$ integers. Positions are the residues $s \in \{0, \dots, n-1\}$; a position is **even** or **odd** according to s.

Settings. An **unramified quadratic setting at** (p, n) is a tuple $\mathfrak{t} = (\mathcal{W}, \mathcal{O}, F_{\mathfrak{t}}, K_{\mathfrak{t}}, j, \iota, \sigma, \varpi)$ of discrete valuation rings $\mathcal{W}, \mathcal{O}$ with fraction fields $F_{\mathfrak{t}}, K_{\mathfrak{t}}$, a ring homomorphism $j\colon \mathcal{W} \to \mathcal{O}$, a field homomorphism $\iota\colon F_{\mathfrak{t}} \to K_{\mathfrak{t}}$, a field automorphism σ of $F_{\mathfrak{t}}$ and an element $\varpi \in \mathcal{O}$, such that:

(Qs1) $p \geq 5$ and n is even;

(Qs2) p is a uniformiser of $\mathcal{W}$, $\mathcal{W}$ is complete for the p-adic topology and $|\mathcal{W}/p\mathcal{W}| = p^n$;

(Qs3) $\sigma^n = \mathrm{id}$, and for every $x \in \mathcal{W}$ there is $y \in \mathcal{W}$ with $\sigma(x) = x^p + py$;

(Qs4) $\iota(x) = j(x)$ for $x \in \mathcal{W}$, $\mathcal{O}$ is a finitely generated $\mathcal{W}$-module through j, and $\mathcal{O}$ is complete;

(Qs5) ϖ is a uniformiser of $\mathcal{O}$, $\varpi^2 = p$ and $|\mathcal{O}/\varpi\mathcal{O}| = p^{2n}$;

(Qs6) every integer is a square in $\mathcal{O}$.

Weight arrays. A **weight array** of rank k is a family $\mu = (\mu_s)_{0 \leq s < n}$ of vectors $\mu_s = (\mu_{s,1}, \dots, \mu_{s,k}) \in \mathbb{Z}^k$; it is **dominant** if every μ_s is nonincreasing. Its **twisted entries** are $\mu^\natural_s = \mu_s$ for even s and $\mu^\natural_{s,a} = -\mu_{s,k+1-a}$ $(1 \leq a \leq k)$ for odd s; put $\nu_s(\mu) = (\mu^\natural_{s,a} - \mu^\natural_{s,k})_{1 \leq a \leq k}$ and

$$\mathrm{tp}(\mu) = \sum_{s=0}^{n-1} \mu^\natural_{s,1}.$$

If μ is dominant, every $\mu^\natural_s$ is nonincreasing and every $\nu_s(\mu)$ is a nonincreasing vector of nonnegative integers.

Lattices of one induced module. Let K be the fraction field of a discrete valuation ring $\mathcal{O}$, and $X = (X_{ab})_{1\le a,b\le k}$ a matrix of indeterminates. An $\mathcal{O}$-lattice in K^k is the $\mathcal{O}$-span of the columns of some $g \in \mathrm{GL}_k(K)$, a **basis matrix** of it. For such a lattice Λ and a nonincreasing $\eta \in \mathbb{Z}^k$ put

$$H^0_\Lambda(\eta) = \bigcap_g \det(g)^{-\eta_k} \cdot \{P \in K[X] :\ P(gX) \in \nabla_{\mathcal{O}}(\eta - \eta_k)\},$$

the intersection over all basis matrices g of Λ, where $\nabla_{\mathcal{O}}(\eta-\eta_k) \subseteq \mathcal{O}[X] \subseteq K[X]$ and $P(gX)$ is P evaluated at the matrix product gX.

Orbit spaces and orbit lattices. Fix a setting $\mathfrak{t}$ at (p,n) and a dominant weight array μ of rank k; write $K = K_{\mathfrak{t}}$. Take matrices of indeterminates $X^{(s)}$, $0 \le s < n$, each $k \times k$. For $\mathcal{O}$-submodules $M_s \subseteq K[X]$ let $\bigotimes_s M_s$ be the $\mathcal{O}$-span of the products $\prod_s P_s(X^{(s)})$ with $P_s \in M_s$. The **orbit space** $V_{\mathfrak{t}}(\mu)$ is the K-span of the products $\prod_s P_s(X^{(s)})$ with $P_s \in \nabla_K(\nu_s(\mu))$, and $V_{\mathfrak{t}}(\mu)^* = \mathrm{Hom}_K(V_{\mathfrak{t}}(\mu), K)$. For an $\mathcal{O}$-submodule $M \subseteq V_{\mathfrak{t}}(\mu)$ put $M^\vee = \{\phi \in V_{\mathfrak{t}}(\mu)^* : \phi(M) \subseteq \mathcal{O}\}$.

A **$\mathcal{W}$-lattice** is the $\mathcal{W}$-span $L \subseteq F_{\mathfrak{t}}^k$ of the columns of a matrix in $\mathrm{GL}_k(F_{\mathfrak{t}})$. Its **twist** at the position s is the $\mathcal{O}$-lattice $L^{(s)} \subseteq K^k$ spanned by the vectors $\iota(\sigma^s(v))$, $v \in L$ ($\iota \circ \sigma^s$ applied to each coordinate). The **orbit lattice** of L is

$$H_{\mathfrak{t}}(\mu, L) = V_{\mathfrak{t}}(\mu) \cap \bigotimes_s H^0_{L^{(s)}}(\mu^\natural_s).$$

Edges. An **edge** (L', L) is a pair of $\mathcal{W}$-lattices with $pL \subseteq L' \subseteq L$ and $|L/L'| = p^n$; L' is its source and L its target. Its **midpoint** at the position s is the $\mathcal{O}$-lattice $L'^{(s)} + \varpi L^{(s)}$, and its **edge lattices** are

$$R_{\mathfrak{t}}(\mu; L', L) = \varpi^{-\mathrm{tp}(\mu)}\Big(V_{\mathfrak{t}}(\mu) \cap \bigotimes_s H^0_{L'^{(s)}+\varpi L^{(s)}}(\mu^\natural_s)\Big),$$
$$R^\vee_{\mathfrak{t}}(\mu; L', L) = \varpi^{-\mathrm{tp}(\mu)}\Big(V_{\mathfrak{t}}(\mu) \cap \bigotimes_s H^0_{L'^{(s)}+\varpi L^{(s)}}(\mu^\natural_s)\Big)^\vee.$$

Source and target lattices. For an integer $t_{\mathrm{d}} \ge 0$ (the depth) and a $\mathcal{W}$-lattice L put

$$N^{\mathrm{src}}_{\mathfrak{t}}(\mu, t_{\mathrm{d}}, L) = \varpi^{-t_{\mathrm{d}}} H_{\mathfrak{t}}(\mu, L) \ \cap \bigcap_{(L,L_1)\ \text{edge}} R_{\mathfrak{t}}(\mu; L, L_1),$$
$$N^{\mathrm{tgt}}_{\mathfrak{t}}(\mu, t_{\mathrm{d}}, L) = \varpi^{-t_{\mathrm{d}}} H_{\mathfrak{t}}(\mu, L)^\vee \ \cap \bigcap_{(L_1,L)\ \text{edge}} R^\vee_{\mathfrak{t}}(\mu; L_1, L),$$

the intersections over all edges with source L, respectively with target L. The **edge pairing** is

$$\langle F, G\rangle_\mu = G(p^{\mathrm{tp}(\mu)} F) \qquad (F \in V_{\mathfrak{t}}(\mu),\ G \in V_{\mathfrak{t}}(\mu)^*).$$

Definition A.7. Let $\mathfrak{t}$ be an unramified quadratic setting at (p,n), μ a dominant weight array of rank k, $t_{\mathrm{d}}, D, c_{\mathrm{sep}} \ge 0$ integers and κ a real number.

Separated rank. The array μ has the **separated rank property for** $(t_{\mathrm{d}}, D, \kappa, c_{\mathrm{sep}})$ **in** $\mathfrak{t}$ if $4(D+1) \le c_{\mathrm{sep}}$ and for every $\mathcal{W}$-lattice L there are an integer $M \ge 0$, edges (L'_i, L) with target L and vectors $F_i \in V_{\mathfrak{t}}(\mu)$ $(1 \le i \le M)$ such that:

(Sv1) $F_i \in N^{\mathrm{src}}_{\mathfrak{t}}(\mu, t_{\mathrm{d}}, L'_i)$ for every i;

(Sv2) $F_i \in \varpi^{c_{\mathrm{sep}}} R_{\mathfrak{t}}(\mu; L'_i, L_1)$ for every i and every edge (L'_i, L_1) with source L'_i and $L_1 \ne L$;

(Sv3) there are an integer $n_{\mathrm{rk}} \geq (1-\kappa)\dim_{K_{\mathrm{t}}} V_{\mathrm{t}}(\mu)$, an $\mathcal{O}$-linear map $u\colon \mathcal{O}^{n_{\mathrm{rk}}} \to N_{\mathrm{t}}^{\mathrm{tgt}}(\mu, t_{\mathrm{d}}, L)$ and a matrix $B \in \mathrm{M}_{n_{\mathrm{rk}}\times M}(\mathcal{O})$ with

$$\sum_{i=1}^{M} B_{ji}\,\langle F_i, u(y)\rangle_{\mu} = p^{D} y_j \qquad (y \in \mathcal{O}^{n_{\mathrm{rk}}},\ 1 \leq j \leq n_{\mathrm{rk}}).$$

(In (Sv2) the edges (L_i', L_1) are those with the same source as the i-th edge and different from it.)

Base weight arrays. Let $T \subseteq \{0, \dots, n-1\}$, $\alpha, \beta\colon T \to \mathbb{Z}$ and $c_{\mathrm{md}} \geq 0$ an integer. A weight array μ of rank $k \geq 2$ is a **base array for** $(T, \alpha, \beta, c_{\mathrm{md}})$ if

(Bw1) for $s \in T$: $\mu_{s,1} = \alpha(s)$, $\mu_{s,k} = -\beta(s)$ and $|\mu_{s,a}| \leq c_{\mathrm{md}}$ for $1 < a < k$;

(Bw2) for $s \notin T$: $|\mu_{s,a}| \leq c_{\mathrm{md}}$ for $1 \leq a \leq k$.

The elements of T are the **tops**, $\alpha(s)$ the **heights** and $\beta(s)$ the **depths**.

Canonical orbit constant. For $p, n, k \geq 2$, T, α, β, c_{md} as above and integers $t_{\mathrm{d}}, D, c_{\mathrm{sep}} \geq 0$, let $\mathcal{K}_{p,n,k}(T, \alpha, \beta; c_{\mathrm{md}}, t_{\mathrm{d}}, D, c_{\mathrm{sep}})$ be the set of real $\kappa \geq 0$ such that every dominant base array μ of rank k for $(T, \alpha, \beta, c_{\mathrm{md}})$ has the separated rank property for $(t_{\mathrm{d}}, D, \kappa, c_{\mathrm{sep}})$ in every unramified quadratic setting at (p, n). When this set is nonempty, the **canonical orbit constant** is

$$\kappa^{\mathrm{can}}_{p,n,k}(T, \alpha, \beta; c_{\mathrm{md}}, t_{\mathrm{d}}, D, c_{\mathrm{sep}}) = \inf \mathcal{K}_{p,n,k}(T, \alpha, \beta; c_{\mathrm{md}}, t_{\mathrm{d}}, D, c_{\mathrm{sep}}) \ \geq 0.$$

The parameters $c_{\mathrm{md}}, t_{\mathrm{d}}, D, c_{\mathrm{sep}}$ may themselves be functions of (T, α, β); the constant is then formed with their values at (T, α, β).

Definition A.8. Let p be a prime, $n \geq 2$ and $k \geq 2$ integers, $T \subseteq \{0, \dots, n-1\}$ with $N = |T|$, $G_{\mathrm{gap}}, N_1, c_{\mathrm{md}}, c_{\mathrm{sep}} \geq 0$ integers, Y, a, b' real numbers with $a \geq 1$, and $C_0, C_A, C_t > 0$ real constants. Put

$$a_0 = \lceil a \rceil, \quad \Lambda = C_0 N_1^2 \log a_0, \quad Q = p^{G_{\mathrm{gap}}}, \quad c = c_{\mathrm{sep}} + 2Nc_{\mathrm{md}} + 2nc_{\mathrm{md}}, \quad \ell = \log_p(8a_0 + k - 1) + 1,$$

and let N_{e} be the number of even elements of T. If $N_{\mathrm{e}} \geq 1$, for an odd $y \in T$ let $d^-(y)$ be the least integer $d \geq 1$ such that $(y-d) \bmod n$ is an even element of T, and $y^- = (y - d^-(y)) \bmod n$.

Explicit flat depth. For $A, B\colon T \to \mathbb{Z}_{\geq 0}$ let

$$x(A,B) = \sum_{\substack{y \in T \\ y \text{ odd}}} \min\Big(B(y), \frac{A(y^-)}{p^{d^-(y)}}\Big) + C_t N\Big(\frac{a_0}{p^2} + \frac{b'}{p} + \frac{\Lambda a_0}{pk} + \frac{\Lambda b'}{k} + kQ + Qc\Big) + \frac{N\Lambda b'}{k},$$

the first sum being 0 when $N_{\mathrm{e}} = 0$, and

$$t^{\exp}(A,B) = \max\big(\lceil x(A,B) \rceil, 0\big) \in \mathbb{Z}_{\geq 0}.$$

Explicit level.

$$D_1^{\exp} = N_{\mathrm{e}}\big(C_A Y^2 N_1^2 k^2 \ell + 4Nk \log_p a_0\big) + 8Nk \log_p a_0 + 4N\lceil (k+1)\ell \rceil.$$

When the constants must be shown we write $t^{\exp}_{C_0,C_t}$ and $D^{\exp}_{1,C_A}$.

Theorem A.9. *There is a real $T_0 > 0$ such that for every real $C_0 \geq T_0$ there are reals $T_2 > 0$ and $C > 0$ with the following property for every real $C_H \geq T_2$.*

*Let p be a prime, $n, k \geq 1$ integers, $T \subseteq \{0, \dots, n-1\}$, $G_{\mathrm{gap}}, N_1, c_{\mathrm{md}}, D, c_{\mathrm{sep}} \geq 0$ integers and $Y, Y_{\mathrm{b}}, a, b', b_{\mathrm{lo}}, b_{\mathrm{up}}$ real numbers. Put $N = |T|$, the **orbit scale** $L_c = C_0 N_1^2 \log a$, and*

$$\delta_{\mathrm{w}} = \frac{C_H N_1 L_c}{k} + \frac{2(b_{\mathrm{up}} - b_{\mathrm{lo}})}{b'}.$$

Assume:

1. *$1 \leq Y$, $k^2 Y^2 \leq p$, $1 \leq N_1$ and $2N_1 + 3 \leq k$;*
2. *T has at least two even elements, and for every even $s \in T$, with s' the next even element of T after s in the cyclic order of $\mathbb{Z}/n$ and $g = (s' - s) \bmod n \in \{1, \dots, n-1\}$, one has $g \leq G_{\mathrm{gap}}$, and at most N_1 odd $y \in T$ satisfy $0 < (y - s) \bmod n < g$;*
3. *$8 \log(NY) \leq L_c$;*
4. *$0 < b'$, $b_{\mathrm{lo}} \leq b' \leq b_{\mathrm{up}}$ and $Y_{\mathrm{b}} N L_c b'/k \leq b_{\mathrm{up}} - b_{\mathrm{lo}} \leq b'/2$;*
5. *$C_H \leq Y_{\mathrm{b}}$, $2Y_{\mathrm{b}} N L_c < k$, $kY^2 Y_{\mathrm{b}} N L_c \log a \leq a/b'$ and $C_H N L_c^2 \leq k$;*
6. *$\delta_{\mathrm{w}} < 1/6$, $p < a/b'$, and no odd integer $d \geq 1$ satisfies $(a/b')(1 - 2\delta_{\mathrm{w}}) \leq p^d \leq (a/b')(1 + 2\delta_{\mathrm{w}})$;*
7. *$c_{\mathrm{md}} \leq L_c b'/k$, $C_H N Y^2 (N_1^5 + k^3) \log_p a \leq D$, $8(D+1) \leq c_{\mathrm{sep}}$ and $C_H k^3 N p^{4G_{\mathrm{gap}}} (c_{\mathrm{sep}} + N c_{\mathrm{md}} + n c_{\mathrm{md}}) Y \leq b'$.*

Let $I_a = \mathbb{Z} \cap [\lceil a \rceil, \lfloor a + a/(kY) \rfloor]$ and $I_b = \mathbb{Z} \cap [\lceil b_{\mathrm{lo}} \rceil, \lfloor b_{\mathrm{up}} \rfloor]$. For an odd $y \in T$ let $d^-(y)$ be the least integer $d \geq 1$ such that $(y - d) \bmod n$ is an even element of T (it exists by item 2), and $y^- = (y - d^-(y)) \bmod n$. For $\alpha, \beta \colon T \to \mathbb{Z}$ put $c' = c_{\mathrm{sep}} + CNk \log_p a + 4N c_{\mathrm{md}} + 2n c_{\mathrm{md}}$ and

$$\begin{aligned} t_I(\alpha, \beta) = {} & \sum_{\substack{y \in T \\ y \text{ odd}}} \min\Big(\beta(y), \frac{\alpha(y^-)}{p^{d^-(y)}}\Big) + CNN_1\Big(\frac{a}{p^2} + \frac{b'}{p} + \frac{L_c a}{pk} + \frac{L_c b'}{k}\Big) \\ & + CN\big(p^{G_{\mathrm{gap}}} c' + k p^{G_{\mathrm{gap}}} + k \log_p a + c_{\mathrm{md}}\big) + 2n c_{\mathrm{md}}. \end{aligned}$$

Then there is a function $\kappa \colon \mathbb{Z}^T \times \mathbb{Z}^T \to [0, \infty)$ such that:

(Om1) Mean. *If I_a and I_b are nonempty,*

$$\frac{1}{|I_a|^N |I_b|^N} \sum_{\alpha \in I_a^T} \sum_{\beta \in I_b^T} \kappa(\alpha, \beta) \leq \frac{CNN_1}{Y}.$$

(Om2) Separated rank. *For every unramified quadratic setting $\mathfrak{t}$ at (p, n), all $\alpha \in I_a^T$, $\beta \in I_b^T$, every dominant base array μ of rank k for $(T, \alpha, \beta, c_{\mathrm{md}})$ and every integer $t_{\mathrm{d}} \geq t_I(\alpha, \beta)$, the array μ has the separated rank property for $(t_{\mathrm{d}}, D, \kappa(\alpha, \beta), c_{\mathrm{sep}})$ in $\mathfrak{t}$.*

Proof. Let T_0 be the constant of the orbit majorant in one fixed unramified quadratic setting, whose constants do not depend on the setting, and for $C_0 \geq T_0$ let T_2, C be the constants it provides; we show that the same T_0, T_2, C have the asserted property. Fix $C_H \geq T_2$ and data satisfying items 1–7, and put $N = |T|$.

No setting at (p,n). If there is no unramified quadratic setting at (p,n), let $\kappa = 0$. Then (Om2) holds vacuously, and (Om1) holds because the mean of κ is 0 while $CNN_1/Y \geq 0$ ($C > 0$, $N_1 \geq 1$ and $Y \geq 1$ by item 1).

Some setting $\mathfrak{t}_0$ at (p,n). Otherwise fix one, and let $\kappa = \kappa_{\mathfrak{t}_0} \colon \mathbb{Z}^T \times \mathbb{Z}^T \to [0,\infty)$ be the function given by that majorant for these data and $\mathfrak{t}_0$; its hypotheses are exactly items 1–7 for (C_0, C_H), and its function t_I is the one of this statement (same data and same C). Its item 1 is (Om1). For (Om2), let $\mathfrak{t}$ be any unramified quadratic setting at (p,n), $\alpha \in I_a^T$, $\beta \in I_b^T$, μ a dominant base array of rank k for $(T, \alpha, \beta, c_{\mathrm{md}})$ and $t_{\mathrm{d}} \geq t_I(\alpha,\beta)$ an integer. By item 2 of that majorant, μ has the separated rank property for $(t_{\mathrm{d}}, D, \kappa(\alpha,\beta), c_{\mathrm{sep}})$ in $\mathfrak{t}_0$; by Lemma A.10, applied to the settings $\mathfrak{t}_0$ and $\mathfrak{t}$ at the same (p,n), it has it in $\mathfrak{t}$. □

Lemma A.10. *Let $\mathfrak{t}_1, \mathfrak{t}_2$ be unramified quadratic settings at the same (p,n), μ a dominant weight array of rank k, $t_{\mathrm{d}}, D, c_{\mathrm{sep}} \geq 0$ integers and κ real. If μ has the separated rank property for $(t_{\mathrm{d}}, D, \kappa, c_{\mathrm{sep}})$ in $\mathfrak{t}_1$, then it has it in $\mathfrak{t}_2$.*

Proof. Outline. Two unramified quadratic settings at the same (p,n) are isomorphic, compatibly with j, ι, σ and ϖ: the rings $\mathcal{W}$ are identified by the uniqueness of complete discrete valuation rings with uniformiser p and a given perfect residue field [221], the automorphisms σ by their common reduction, and the fields $K_{\mathfrak{t}}$ as the extension by ϖ, with $\varpi^2 = p$, of the unramified quadratic extension of $F_{\mathfrak{t}}$; this isomorphism transports lattices, edges, orbit and edge lattices, source and target lattices and the edge pairing, and with them the data of (Sv1)–(Sv3). □

A.2.2 Gram windows and the raw step

The majorant in one fixed setting is a local computation, and its local input is one correction step at the identity vertex: a symmetric family of relaxed blocks must be realised, up to a controlled error, by a polynomial with bounded denominator that becomes integral after every substitution off a fixed plane. The keys of this step and a residue-line detector are given in Lemma A.13, and the step itself is Proposition A.14, for which we give a sketch. Every other step of the orbit mean has a complete proof. Two further results enter the tables: the Gram-window theorem for one segment, which is derived from the majorant and bounds the mean of the segment deficiency over a window of heights (Theorem A.11), and symbol spaces of almost full rank (Proposition A.12).

Theorem A.11. *There is a real $T_0 > 0$ such that for every real $C_0 \geq T_0$ there is a real $T_1 \geq 0$ such that for all reals $C_E, C_m \geq T_1$ there are reals $T_2, C_A > 0$ with the following property for every real $C_H \geq T_2$. Let $p, n, k, T, G_{\mathrm{gap}}, N_1, c_{\mathrm{md}}, D, c_{\mathrm{sep}}, Y, Y_{\mathrm{b}}, a, b', b_{\mathrm{lo}}, b_{\mathrm{up}}$ satisfy items 1–7 of Theorem A.9 for (C_0, C_H), $N = |T|$, let a_0, Λ, Q, c be as in Definition A.8, $J = \lfloor \Lambda b'/k \rfloor$, $W_a^+ = \mathbb{Z} \cap [a_0, \lfloor a + a/(kY) \rfloor + J + c_{\mathrm{md}}]$, $W_b^+ = \mathbb{Z} \cap [\lceil b_{\mathrm{lo}} \rceil, \lfloor b_{\mathrm{up}} \rfloor + J + c_{\mathrm{md}}]$,*

$$E = C_E\Big(\frac{a_0}{p^2} + \frac{b'}{p} + \frac{\Lambda N_1 a_0}{pk} + \frac{\Lambda b'}{k} + k^2 N_1 Q^2 c\Big), \qquad m = C_m k N_1 Q c + 12c,$$

$$\ell = \log_p(8a_0 + k - 1) + 1, \qquad D_{\mathrm{seg}} = C_A Y^2 N_1^2 k^2 \ell + 4Nk \log_p a_0.$$

Let $\mathfrak{t}$ be an unramified quadratic setting at (p,n) and $s \in T$ even; let s^+ be the next even element of T after s in the cyclic order of $\mathbb{Z}/n$, and $\sigma(s)$ the set consisting of s^+ and the odd $y \in T$ with $0 < (y-s) \bmod n < (s^+ - s) \bmod n$. For $A \in (W_a^+)^T$, $B \in (W_b^+)^T$ the number $\kappa_s^{\mathrm{seg}}(A, B; D_{\mathrm{seg}})$, formed in $\mathfrak{t}$ with the parameters (a_0, b', Λ, E, m), depends on A only through $A|_{\sigma(s)}$. Then for

every $B \in (W_b^+)^T$

$$\frac{1}{|W_a^+|^{|\sigma(s)|}} \sum_{A|_{\sigma(s)} \in (W_a^+)^{\sigma(s)}} \kappa_s^{\mathrm{seg}}(A, B; D_{\mathrm{seg}}) \le \frac{C_A \, |\sigma(s)| \, N_1}{Y}.$$

Proof. Outline. For fixed heights the segment Fock lattice is the direct sum of the block lattices, so the segment deficiency is at most the sum of the block badnesses divided by the rank; the block badness is then averaged over the heights one digit class at a time, using averaging bounds for the levels of a block and a comparison of masses when one far height is raised, and since the rank is comparable to the mass this gives the mean bound $C_A|\sigma(s)|N_1/Y$. □

Proposition A.12. *There is a real $C_0 > 0$ such that for every real $C_{\mathrm{La}} \ge C_0$ there are reals $C_{\mathrm{rk}}, C_{\mathrm{deg}}, C_{\mathrm{g}} > 0$ with the following property for every real $C \ge C_{\mathrm{g}}$. Let $(\mathfrak{t}, k, r, T, L_1, L_2, c_{\mathrm{md}}, c_0, D, c_{\mathrm{sep}}, \delta, h_{\mathrm{o}}, x, a, Y, \alpha, \beta, \mu)$ be a Type II configuration for C, with $\mathfrak{t}$ at (p, n), let (J, f) be a frame system of rank r for T with parameters $(L_2, N_{\mathrm{tl}}(r, \delta, h_{\mathrm{o}}, x), G_{\mathrm{tl}}(r, \delta, h_{\mathrm{o}}, x))$, and take its symbol data for C_{La}. Let $E' \in \mathbb{Z}^n_{\ge 0}$ with $E'_s \le \min(E_0, d_s(\mu))$ for $s \in T$, and let $N \subseteq \mathcal{O}[x]$ be the $\mathcal{O}$-span of $\mathrm{Gen}(X^{\mathrm{sym}}, d_{E'})$. Then:*

1. *there is an $\mathcal{O}$-submodule $S \subseteq \mathcal{O}[x]$ which is a direct summand of the $\mathcal{O}$-module $\mathcal{O}[x]$ and free of finite rank with an $\mathcal{O}$-basis consisting of elements of $\mathrm{Gen}(X^{\mathrm{sym}}, d_{E'})$, such that*

$$\mathrm{rank}_{\mathcal{O}} S \;\ge\; p^{-nrK}\Big(1 - \frac{C_{\mathrm{rk}}|T|r^2k}{Y}\Big) \prod_{s \in T} \binom{d_s(\mu) + k - 1}{k - 1};$$

2. *every monomial x^e occurring in an element of N has*

$$\mathrm{nd}(e) \;\le\; C_{\mathrm{deg}}|T|\Big(\frac{\ell\, a}{pk} + a\, p^{-L_1} + kQ\Big).$$

Proof. Outline. One selects inside N a direct summand S with a basis of generators whose rank is the dimension of the reduction of N modulo ϖ; the discriminant of the frame system is nonzero modulo ϖ, and multiplication by it maps the reduction of the span of the coarse generators into the reduction of N, so the rank bound follows from the count of coarse generators, while the bound on normal degrees follows from the degree caps of the symbol spaces and the label condition. □

Lemma A.13. *Use an unramified quadratic setting and the bihomogeneous substitution $\mathrm{sub}_g^{\mathrm{bh},(1)}$ ($g \in \mathrm{GL}_k(\mathcal{W})$), with $k \ge 2$, $n \ge 2$. For each position u write $V_{u,a} = x_{u,a}$ and $C_{u,a} = y_{u,a}$ if u is even, and $V_{u,a} = y_{u,a}$ and $C_{u,a} = x_{u,a}$ if u is odd. Let s be even, y odd, $1 \le d < n$ and $y \equiv s + d \pmod n$, and put $q = p^d$. Define*

$$K_{s,y} = \sum_{a=1}^{k} V_{s,a}^q C_{y,a}, \qquad W_{s,y,c} = V_{y,c}V_{s,1}^q - V_{y,1}V_{s,c}^q \quad (2 \le c \le k).$$

Then $\mathrm{sub}_g^{\mathrm{bh},(1)}(K_{s,y})$ and $\mathrm{sub}_g^{\mathrm{bh},(1)}(W_{s,y,c})$ are ϖ-integral for every $g \in \mathrm{GL}_k(\mathcal{W})$. At $g = 1$ they equal, respectively,

$$\varpi\left(V_{s,1}^q C_{y,1} + \varpi^{q-1}\sum_{a=2}^{k} V_{s,a}^q C_{y,a}\right), \qquad \varpi\left(V_{y,c}V_{s,1}^q - \varpi^{q-1}V_{y,1}V_{s,c}^q\right).$$

Let $T \subseteq \{0,\dots,n-1\}$ be the top set, and let S be its set of even elements, with $|S| \geq 2$, let s^+ be the cyclic successor in S, and let $G_s = (s^+ - s) \bmod n \in \{1,\dots,n-1\}$. For $2 \leq c \leq k$ put

$$L_c = \prod_{s\in S} V_{s,1}^{p^{G_s}-1} - \prod_{s\in S} V_{s,c}^{p^{G_s}-1}, \qquad D_{s_0} = V_{s_0,1} \prod_{c=2}^{k} L_c \quad (s_0 \in S).$$

If $\Pi_1(g) \neq \Pi_1(1)$, then $\mathrm{sub}_g^{\mathrm{bh},(1)}(D_{s_0})$ is ϖ-integral. At the identity,

$$\mathrm{sub}_1^{\mathrm{bh},(1)}(L_c) = \prod_{s\in S} V_{s,1}^{p^{G_s}-1} - \varpi^{\sum_{s\in S}(p^{G_s}-1)} \prod_{s\in S} V_{s,c}^{p^{G_s}-1}.$$

All polynomials are formed over the same coefficient ring and in the same variables.

Proof. Outline. Modulo ϖ the substitution sends $V_{u,b}$ to $V_{u,1}$ times an entry of the first row of the reduced inverse matrix at u, and since σ reduces to the p-power Frobenius these entries at y are the q-th powers of those at s, which makes the reduced keys vanish; for the detector, the same Frobenius relations along the even tops show that the substituted detector vanishes modulo ϖ when $\Pi_1(g) \neq \Pi_1(1)$, and the formulas at the identity come from direct substitution. □

Proposition A.14. *There is a real $T_0 > 0$ such that for every real $C_0 \geq T_0$ there is a real $T_1 \geq 0$ such that for every real $C_E \geq T_1$ there are reals $T_2, C_t^* > 0$ with the following property for all reals $C_t \geq C_t^*$ and $C_H \geq T_2$. Let the data $p, n, k, T, \dots, b_{\mathrm{up}}$ satisfy items 1–7 of Theorem A.9 for (C_0, C_H); let $N, a_0, \Lambda, Q, c, J_0, W_a^+, W_b^+, E$ be the quantities formed from these data and C_E in the realisation of symmetric block families at the identity vertex, form $x(A,B)$ with (C_0, C_t), and put*

$$\delta = \frac{C_H N_1 \Lambda}{k} + \frac{2(b_{\mathrm{up}} - b_{\mathrm{lo}})}{b'}, \qquad \kappa_1 = 13 k N_1 Q,$$
$$\mu_j = 12c + j\kappa_1, \quad E_j = (c-j)\kappa_1 \quad (0 \leq j \leq c).$$

Let $\mathfrak{t}$ be an unramified quadratic setting at (p,n), $A \in (W_a^+)^T$, $B \in (W_b^+)^T$, $t_{\mathrm{bh}} = \lfloor x(A,B) - NJ_0 \rfloor$, and form kept blocks, relaxed blocks, clean polynomials and the modules $\mathcal{M}$ with $(a_0, b', \Lambda, E, \delta, c)$ and A, B. Then (the clause is named (Rs2) for reference):

(Rs2) *for every integer $1 \leq j \leq c$, every relaxed tuple β at slack E_j and every family $v = (v_s)_s$ with $v_s \in \Sigma_{\beta_s}^{(\mu_j)}$ there are a polynomial R clean at order $\mathrm{ord}(\beta)$ and elements*

$$\tau_\epsilon \in \mathcal{M}(\mu_{j-\epsilon}, E_{j-\epsilon}, \mathrm{ord}(\beta) + \epsilon) \qquad (1 \leq \epsilon < j)$$

such that

$$\mathrm{sub}_1^{\mathrm{bh},(1)}(R) - \mathrm{sym}_\beta(v) - \sum_{\epsilon=1}^{j-1} \varpi^\epsilon \tau_\epsilon \quad \text{is } \varpi^j\text{-integral.}$$

Proof. Sketch. Clause (Rs2) is one step of a ϖ-adic approximation. The aim is to realise a symmetric family $\mathrm{sym}_\beta(v)$ of integral segment vectors, at the identity vertex, as $\mathrm{sub}_1^{\mathrm{bh},(1)}(F)$ of a bihomogeneous polynomial F that has denominator at most $\varpi^{\mathrm{ord}(\beta)}$ and becomes ϖ^c-integral after every substitution off the plane $\Pi_1(1)$; this is the local input of Theorem A.11 and of the orbit mean (Om1) of Theorem A.9. Given (Rs2), a strong induction on j produces such an F modulo ϖ^c, since each correction τ_ϵ is treated by the step at level $j - \epsilon$; this induction is elementary. The inputs of the step are proved in Lemma A.13: the keys $K_{s,y}$ and $W_{s,y,c}$

are divisible by ϖ after every substitution, with leading terms $\varpi V_{s,1}^{q} C_{y,1}$ and $\varpi V_{y,c} V_{s,1}^{q}$ at the identity up to terms carrying a further factor ϖ^{q-1}, and the detector D_{s_0} is divisible by ϖ off the plane while its value at the identity has a monomial leading term.

The argument should run as follows. (i) Choose an integral basis of standard columns for the polynomial space cut out by the capacity conditions of every prefix of letters, the same basis for all prefixes, whose admissible columns span the integral intersection. (ii) Realise each column generator as a product of boundedly many keys, each divided by ϖ, so that the denominator grows by one per key, in agreement with the order $\mathrm{ord}(\beta)$, and the initial form at the identity is the prescribed column; then multiply by a fixed power of the detector times a fixed standard monomial, which makes the product clean off the plane. (iii) Expand the product: the leading term is $\mathrm{sym}_{\beta}(v)$, and a term in which ϵ factors are replaced by their corrections is ϖ^{ϵ} times a polynomial that must be shown to lie in $\mathcal{M}(\mu_{j-\epsilon}, E_{j-\epsilon}, \mathrm{ord}(\beta)+\epsilon)$. Heuristically, (iii) is what the parameters are calibrated for: passing from level j to $j-\epsilon$ lowers the margin $\mu_j = 12c + j\kappa_1$ and raises the slack $E_j = (c-j)\kappa_1$ by $\epsilon\kappa_1$, with $\kappa_1 = 13kN_1Q$, which is room for the bounded change in degrees and capacities caused by perturbing ϵ factors of bounded degree, so the perturbed terms should again be relaxed tuples in the larger module. Step (i) is an integral form of a standard-monomial basis for spaces of polynomials in vectors and covectors cut out by stability under the operators $\mathsf{E}_{cd}^{(J)}$; over the unramified ring $\mathcal{O}$ such bases are expected to exist, the delicate point being that one basis must serve all prefixes at once. □

A.2.3 One table at every scale

The mean of the canonical orbit constant over the full box follows from the majorant (Corollary A.15). Since the requirement is a mean over a finite box, a finite averaging argument selects one profile, jointly for all the candidate means (Lemma A.16), and hence one canonical profile with uniformly small orbit shares (Lemma A.17). Proposition A.18 records the numerical clauses that a member built over a context satisfies, and Proposition A.19 assembles them into the table at every scale and its identities at the constructed members.

Corollary A.15. *Let T_0 be as in Theorem A.9, $C_0 \geq T_0$, let T_2, C be as there for C_0, and $C_H \geq T_2$. Let $p, n, k, T, G_{\mathrm{gap}}, N_1, c_{\mathrm{md}}, D, c_{\mathrm{sep}}, Y, Y_{\mathrm{b}}, a, b', b_{\mathrm{lo}}, b_{\mathrm{up}}$ satisfy items 1–7 of Theorem A.9, with $N = |T|$, and let $t_{\mathrm{d}} \colon \mathbb{Z}^T \times \mathbb{Z}^T \to \mathbb{Z}_{\geq 0}$ satisfy $t_{\mathrm{d}}(\alpha,\beta) \geq t_I(\alpha,\beta)$ for all $\alpha \in I_a^T$, $\beta \in I_b^T$. Then for all $\alpha \in I_a^T$, $\beta \in I_b^T$ the set $\mathcal{K}_{p,n,k}(T, \alpha, \beta; c_{\mathrm{md}}, t_{\mathrm{d}}(\alpha,\beta), D, c_{\mathrm{sep}})$ is nonempty, so the canonical orbit constant*

$$\kappa^{\mathrm{can}}(\alpha,\beta) = \kappa^{\mathrm{can}}_{p,n,k}(T, \alpha, \beta; c_{\mathrm{md}}, t_{\mathrm{d}}(\alpha,\beta), D, c_{\mathrm{sep}}) \geq 0$$

is defined, and, if I_a and I_b are nonempty,

$$\frac{1}{|I_a|^N |I_b|^N} \sum_{\alpha \in I_a^T} \sum_{\beta \in I_b^T} \kappa^{\mathrm{can}}(\alpha,\beta) \leq \frac{CNN_1}{Y}.$$

Proof. Let $\kappa \colon \mathbb{Z}^T \times \mathbb{Z}^T \to [0,\infty)$ be the function given by Theorem A.9 for these data and constants.

The majorant lies in the canonical set. Fix $\alpha \in I_a^T$ and $\beta \in I_b^T$. Then $\kappa(\alpha,\beta) \geq 0$. Let $\mathfrak{t}$ be an unramified quadratic setting at (p,n) and μ a dominant base array of rank k for $(T, \alpha, \beta, c_{\mathrm{md}})$. The integer $t_{\mathrm{d}}(\alpha,\beta)$ is at least $t_I(\alpha,\beta)$ by hypothesis, so (Om2) of Theorem A.9, applied with this $\mathfrak{t}$, α, β, μ and depth $t_{\mathrm{d}}(\alpha,\beta)$, shows that μ has the separated rank property for $(t_{\mathrm{d}}(\alpha,\beta), D, \kappa(\alpha,\beta), c_{\mathrm{sep}})$ in $\mathfrak{t}$. As $\mathfrak{t}$ and μ were arbitrary,

$$\kappa(\alpha,\beta) \in \mathcal{K}_{p,n,k}(T, \alpha, \beta; c_{\mathrm{md}}, t_{\mathrm{d}}(\alpha,\beta), D, c_{\mathrm{sep}}).$$

This set is therefore nonempty; it consists of nonnegative reals, so its infimum $\kappa^{\mathrm{can}}(\alpha,\beta)$ exists and

$$0 \le \kappa^{\mathrm{can}}(\alpha,\beta) \le \kappa(\alpha,\beta).$$

Averaging. Suppose I_a, I_b nonempty; they are finite, so $I_a^T \times I_b^T$ is a nonempty finite set and $|I_a|^N |I_b|^N > 0$. Summing the pointwise inequality over $I_a^T \times I_b^T$ and dividing by $|I_a|^N |I_b|^N$ gives that the mean of κ^{can} is at most the mean of κ, which is at most CNN_1/Y by (Om1) of Theorem A.9. □

Lemma A.16. *Fix $0 < \epsilon < 1/52$, put*

$$\theta = \epsilon/100, \qquad \gamma = \frac{1}{2} - \frac{11399}{1200}\epsilon,$$

and let $x \ge \max\{1, 48^{1/\gamma}\}$. Put

$$\delta = x^{-\epsilon/8}, \qquad b = \exp(x^{\epsilon}), \qquad a = xb.$$

Let b' be any real number. Let T be a finite set, let $n, k, L_a > 0$ be real numbers and let $m \ge 1$ be an integer, with

$$|T| \le 2n\delta, \qquad n \le x^{2+8\epsilon+\theta}, \qquad m \le x^{\theta}\sqrt{x\delta}.$$

Define the finite sets

$$\mathcal{A} = \left\{ \alpha \in \mathbb{Z}^T : \lceil a \rceil \le \alpha_t \le \left\lfloor a + \frac{a}{kx^{13}} \right\rfloor \text{ for every } t \in T \right\},$$
$$\mathcal{B} = \left\{ \beta \in \mathbb{Z}^T : \lceil b' \rceil \le \beta_t \le \left\lfloor b' + \frac{2n\delta x^{\epsilon} L_a b'}{k} \right\rfloor \text{ for every } t \in T \right\}.$$

Assume $\mathcal{A}, \mathcal{B}$ are nonempty. Let $\mathcal{C}$ be a finite set of primes $p < \exp(x^{\epsilon/3})$. For every $p \in \mathcal{C}$, let $s_p : \mathcal{A} \times \mathcal{B} \to \mathbb{R}_{\ge 0}$ satisfy

$$\frac{1}{|\mathcal{A}||\mathcal{B}|} \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{B}} s_p(\alpha,\beta) \le m|T|x^{-12}.$$

Then there exists one pair $(\alpha^*, \beta^*) \in \mathcal{A} \times \mathcal{B}$ such that

$$\sum_{p \in \mathcal{C}} \frac{\log p}{p} \left(21_{s_p(\alpha^*,\beta^*) > x^{-\epsilon}} + 4 s_p(\alpha^*, \beta^*) \right) \le x^{-9}.$$

In particular, if the same fixed arithmetic orbit data give, for every pair $(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}$, orbit masses $\Sigma_p(\alpha,\beta)$ as in Definition 5.2, with candidate set exactly $\mathcal{C}$, and

$$s_p(\alpha,\beta) = \Sigma_p(\alpha,\beta) \quad (p \in \mathcal{C}),$$

then the selected profile satisfies the small-share inequality (Lm5) of Definition 5.5, with its depths in the displayed short window.

Proof. First, $\gamma > 0$, since $11399/(1200 \cdot 52) < 1/2$. We derive a harmonic bound using only the candidate cutoff. Put $t = x^{\epsilon/3}$ and $M = \lceil \exp(t) \rceil - 1$. Here $t \ge 1$, so $M \ge 1$. Every $p \in \mathcal{C}$ is an integer at most M, and $M < \exp(t)$. Thus, using the elementary integral estimate for the harmonic sum,

$$\sum_{p \in \mathcal{C}} \frac{\log p}{p} \le t \sum_{q=1}^{M} \frac{1}{q} \le t(1 + \log M) \le t(1+t) \le t(2+t).$$

The harmonic estimate holds also for $M = 1$; for $M \geq 2$, it follows from $\sum_{q=2}^{M} q^{-1} \leq \int_1^M u^{-1}\,du$.

Write $Q = |\mathcal{A}||\mathcal{B}| > 0$, and define on the finite product

$$Z(\alpha,\beta) = \sum_{p\in\mathcal{C}} \frac{\log p}{p}\left(\mathbf{1}_{s_p(\alpha,\beta)>x^{-\epsilon}} + s_p(\alpha,\beta)\right).$$

For any $s \geq 0$, one has $\mathbf{1}_{s>x^{-\epsilon}} \leq x^{\epsilon}s$. Exchanging finite sums and using the assumed mean bound at each candidate therefore gives

$$\frac{1}{Q}\sum_{\alpha\in\mathcal{A}}\sum_{\beta\in\mathcal{B}} Z(\alpha,\beta) \leq (x^{\epsilon}+1)m|T|x^{-12}\sum_{p\in\mathcal{C}}\frac{\log p}{p}.$$

The size hypotheses imply

$$m|T| \leq 2x^{5/2+(125/16)\epsilon+2\theta}.$$

Indeed, the two factors of δ, with powers 1 and $1/2$, subtract $3\epsilon/16$ from the 8ϵ contribution in the upper bound for n. For $x \geq 1$, we also have

$$x^{\epsilon}+1 \leq 2x^{\epsilon}, \qquad x^{\epsilon/3}(2+x^{\epsilon/3}) \leq 3x^{2\epsilon/3}.$$

Consequently

$$\begin{aligned}\frac{1}{Q}\sum_{\alpha,\beta} Z(\alpha,\beta) &\leq 12x^{-19/2+(125/16+1/50+5/3)\epsilon}\\ &= 12x^{-19/2+(11399/1200)\epsilon} = 12x^{-9-\gamma} \leq x^{-9}/4.\end{aligned}$$

The last step uses $x^{\gamma} \geq 48$. A nonempty finite set contains an element with value at most its average, so there is one pair (α^*,β^*) with $Z(\alpha^*,\beta^*) \leq x^{-9}/4$. Since $2I+4s \leq 4(I+s)$ for $I, s \geq 0$, its asserted weighted sum is at most $4Z(\alpha^*,\beta^*) \leq x^{-9}$.

This argument also covers $T = \varnothing$: the two product boxes are singletons, and the nonnegative means are then zero. Empty candidate sets cause no difficulty. For the conditional application, substitute the stated equality with the orbit masses into the same finite sum. Box membership gives the short window for the selected integer depths. No arithmetic datum, orbit representative, or profile value is changed after selection. □

Lemma A.17. *Let $T_0 > 0$ be as in Theorem A.9. Fix*

$$0 < \varepsilon < 1/52, \qquad C_0 \geq T_0,$$

and let $T_2, C_M > 0$ be the constants of that theorem for C_0. Fix constants

$$C_H \geq T_2, \quad C_D \geq C_H, \quad C_{\mathrm{band}} \geq 2C_H, \quad C_{\mathrm{t1}} \geq 2C_M, \quad C_{\mathrm{t2}} \geq C_M, \quad C_{\mathrm{orb}} \geq 0.$$

There is $X > 1$, depending only on these fixed constants and ε, with the following property.

Let $x \geq X$, let an arithmetic datum on H, its tops $T \subseteq H$, ordering and orbit representatives be fixed, and put

$$n = |H|, \qquad \theta = \varepsilon/100, \qquad \delta = x^{-\varepsilon/8}.$$

Let $k, m \geq 1$ be integers such that

$$n \leq x^{2+8\varepsilon+\theta}, \quad |T| \leq 2n\delta, \quad x^{9/2+10\varepsilon-\theta} \leq k \leq x^{9/2+10\varepsilon+\theta}, \quad m \leq x^{\theta}\sqrt{x\delta}.$$

Put

$$r = \lceil (32+64\varepsilon)/\varepsilon \rceil + 1, \quad G = G_{\mathrm{orb}}, \quad N_1 = N_{\mathrm{orb}}, \quad N_* = 2n\delta,$$

$$a = x\exp(x^{\varepsilon}), \quad X' = \exp(x^{\varepsilon/3}), \quad b' = a/X', \quad L_c = C_0 N_1^2 \log a.$$

Let $\Delta > 0$ *satisfy* $\log \Delta \le x^{\varepsilon/4}/2$, *put* $c_G = 16k\Delta$, *and suppose*

$$4N_* x^{\varepsilon} L_c/k + C_{\mathrm{band}} N_1 L_c/k \le \rho \le x^{-2}.$$

Form the candidate set $\mathcal{C}$ *with* $p_0 = k^2 x^{26}$. *For each inert prime fix* $G_p \subseteq R_p$, *before any height or depth table. For every high candidate* $p \ge p_0$ *and* $v \in G_p$, *require that* $P_p(v) \subseteq \{0, \dots, n_p - 1\}$, $n_p = \mathrm{ord}(h_p)$, *satisfies (Go2),(Go3) for these same parameters.*

Define the nonempty integer intervals and product boxes

$$I_a = \mathbb{Z} \cap [\lceil a \rceil, \lfloor a + a/(kx^{13}) \rfloor], \quad I_b = \mathbb{Z} \cap [\lceil b' \rceil, \lfloor b' + N_* x^{\varepsilon} L_c b'/k \rfloor], \qquad \mathcal{A} = I_a^T, \quad \mathcal{B} = I_b^T.$$

For every candidate p *and* $v \in G_p$, *put* $n_p = \mathrm{ord}(h_p)$, $P = P_p(v)$, *and* $N_p(v) = |P|$. *For a high candidate define*

$$D_p = \left\lceil C_D (40 n_p \delta) x^{26} (N_1^5 + k^3) \log_p a \right\rceil, \quad c_{\mathrm{sep},p} = 8(D_p + 1),$$

$$c_{\mathrm{cor},p} = c_{\mathrm{sep},p} + C_{\mathrm{t2}}(40 n_p \delta) k \log_p a + 4(40 n_p \delta) c_G + 2 n_p c_G.$$

For $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$, *define*

$$\alpha_{p,v}(i) = \alpha(v + i h_p), \qquad \beta_{p,v}(i) = \beta(v + i h_p) \qquad (i \in P).$$

For odd $i \in P$, *let* $d^-_{p,v}(i)$ *be the least* $d \in \{1, \dots, n_p\}$ *for which* $(i - d) \bmod n_p$ *is an even element of* P, *and put* $d^-_{p,v}(i) = 0$ *if there is none. Set*

$$S_{p,v}(\alpha, \beta) = \sum_{\substack{i \in P,\ i \text{ odd} \\ d^-_{p,v}(i) > 0}} \min\left(\beta_{p,v}(i), \frac{\alpha_{p,v}((i - d^-_{p,v}(i)) \bmod n_p)}{p^{d^-_{p,v}(i)}} \right),$$

$$\begin{aligned} t_{p,v}(\alpha, \beta) = {} & S_{p,v}(\alpha, \beta) + C_{\mathrm{t1}} N_p(v) N_1 \left(\frac{a}{p^2} + \frac{L_c a}{pk} \right) \\ & + C_{\mathrm{t2}} N_p(v) \left(p^G c_{\mathrm{cor},p} + k p^G + k \log_p a + c_G \right) + 2 n_p c_G. \end{aligned}$$

Define

$$\kappa_p(v; \alpha, \beta) = \begin{cases} \min\{1, C_{\mathrm{orb}} N_p(v) r^2 k / x^{26}\}, & p \in \mathcal{C},\ p < p_0, \\ \kappa^{\mathrm{can}}_{p, n_p, k}(P_p(v), \alpha_{p,v}, \beta_{p,v}; \lfloor c_G \rfloor, \lceil t_{p,v}(\alpha, \beta) \rceil, D_p, c_{\mathrm{sep},p}), & p \in \mathcal{C},\ p \ge p_0, \\ 0, & p \notin \mathcal{C}. \end{cases}$$

All these constants are defined and lie in $[0, 1]$. *There is one pair* $(\alpha^*, \beta^*) \in \mathcal{A} \times \mathcal{B}$ *for which*

$$\Sigma_p(\alpha, \beta) = m \sum_{v \in G_p} \kappa_p(v; \alpha, \beta)$$

satisfies

$$\sum_{p \in \mathcal{C}} \frac{\log p}{p} \left(21_{\Sigma_p(\alpha^*, \beta^*) > x^{-\varepsilon}} + 4\Sigma_p(\alpha^*, \beta^*) \right) \le x^{-9}.$$

Thus the selected orbit constants have the small-share inequality (Lm5).

At this same pair, for every high candidate, every $v \in G_p$, *every unramified quadratic setting at* (p, n_p), *and every dominant base array with exactly these local endpoints and middle bound* $\lfloor c_G \rfloor$, *the separated rank property holds with depth* $\lceil t_{p,v}(\alpha^*, \beta^*) \rceil$, *degree* D_p, *constant* $\kappa_p(v; \alpha^*, \beta^*)$ *and separation* $c_{\mathrm{sep},p}$. *All representatives, local origins and numerical choices in this conclusion are the ones fixed above.*

Proof. Outline. On a common tail the height and depth boxes are nonempty and, at every high candidate and good representative, the data satisfy items 1–7 of Theorem A.9 with a depth function at least t_I, so Corollary A.15 bounds the local mean of the canonical constant; this local mean equals the mean over the global tables, the low candidates are bounded pointwise, and Lemma A.16 then selects one pair (α^*, β^*) with the small-share inequality, at which the separated rank property holds as at every pair. □

Proposition A.18. *Let $0 < \varepsilon < \frac{1}{52}$, $\mathbf{C}$ an admissible constant family, $c_D \geq 0$, $c_h \geq 402$, and put $\theta = \varepsilon/100$. There is x_0 such that for every context K with these parameters, scale (x, ε, J) and $x \geq x_0$, with $\varepsilon_z = \varepsilon/J$:*

(a) $0 \leq a(t) \leq H_x(1 + \frac{1}{kx^{13}})$ for every top t, and $\tilde{b}_{\tau,j} \geq W_x$ for every $\tau \in \Phi$ and $1 \leq j \leq m$;

(b) $x^{2+8\varepsilon-\theta} \leq |A| \leq x^{2+8\varepsilon+\theta}$, $m \leq x^{\theta}(x\delta_1)^{1/2}$, $x^{9/2+10\varepsilon-\theta} \leq k \leq x^{9/2+10\varepsilon+\theta}$, $t \leq x^{401+\theta}$, $o \leq x^{\theta}$, and $2^e \leq |A| \leq o\,2^e$;

(c) $\log \Delta_D \leq x^{\varepsilon/4}/2$, $mh_0 > 0$, $\log(mh_0) \leq x^{\varepsilon/4}$, and $\log d \geq x^{\varepsilon_z/5-\varepsilon_z/100}$;

(d) if $X'(\mathfrak{s}) \leq T(\omega)$ for every word ω, then for every assembled datum D over K and every layered datum on D admissible at K, the size datum L attached to it has $S_c = 4k\Delta_D$, $\eta_S = mh_0$, $C_t = 401$, and its support contains every place w with $p(w) \leq X''(\mathfrak{s}, |A|) = x|A|X'(\mathfrak{s})$;

(e) the budget $L_a = \log(a(\mathfrak{s})\,\mathrm{rd}_E)$ of the host field E satisfies $x^{\varepsilon} \leq L_a \leq x^{\varepsilon+\theta}$.

Proof. Take x_0 at least the threshold of the construction of the size datum L, $x_0 \geq 2^{1/\theta}$, $x_0 \geq C_8^{1/\theta}$, $x_0 \geq x_1$ with $x_1 = x_1(\varepsilon)$ the threshold beyond which the sizes of a scale bound the rank entry t, and so large that for $x \geq x_0$: $\log(2x^{c_D}) \leq x^{\varepsilon/4}/2$; $\log(8x^{c_h+1}) \leq x^{\varepsilon/4}$; and $(\log x)^2 \geq 2C_8^2$.

(a) By (Cx3e), $0 < H_x \leq a(t) \leq H_x + H_x/(kx^{13})$. By (Cx3f), $\tilde{b}_{\tau,j} = b^1(\tau) + (j-1)h_0 + \frac{h_0}{2} \geq b^1(\tau)$. At the top place of t, $b^1(\tau) = b(t) + \lfloor B_x/x \rfloor \geq b(t) \geq B_x \geq W_x$ ((Cx3e), $x|A| \geq 1$). At thick places $b^1(\tau) \geq \lceil e^L \rceil \geq e^L \geq B_x \geq W_x$, since $B_x = e^L\, x\, e^{-x^{\varepsilon/3}} \leq e^L$ for $x \geq x_0$ (enlarging x_0 so that $x^{\varepsilon/3} \geq \log x$). At thin places $b^1(\tau) = \lceil W_x \rceil \geq W_x$.

(b) By (Cx3a): $x^{2+8\varepsilon} \leq |A| \leq 2x^{2+8\varepsilon} \leq x^{2+8\varepsilon+\theta}$, $m \leq 2(x\delta_1)^{1/2} \leq x^{\theta}(x\delta_1)^{1/2}$ and $x^{9/2+10\varepsilon} \leq k \leq 2x^{9/2+10\varepsilon} \leq x^{9/2+10\varepsilon+\theta}$, as $x^{\theta} \geq 2$; $t \leq x^{401}$ by the bound of the rank entry in terms of the sizes of a scale, since the rank datum of K has the sizes of (x, ε, J) by (Cx3a) and $x \geq x_1$; $o \leq C_8 \leq x^{\theta}$ by (Cx3b). For the frame (e, A, h, a_0) (Definition 3.11), $h\colon A \to H_e = (\mathbb{Z}/2^{e+2}\mathbb{Z})^{\times}/\{\pm 1\}$ is surjective and $|H_e| = 2^e$, so $2^e \leq |A|$; h is injective on $A_0 = \langle a_0 \rangle$, so $|A_0| \leq 2^e$ and $|A| = o|A_0| \leq o2^e$.

(c) $\Delta_D = \lceil x^{c_D} \rceil \leq 2x^{c_D}$ and $0 < mh_0 \leq 2\sqrt{x} \cdot 2(x^{c_h} + 1) \leq 8x^{c_h+1}$; use the choice of x_0. By (Cx3b) there is Y with $x^{1/J} \leq Y \leq x$ satisfying (Hg6), so $\log d > \frac{1}{2}Y^{\varepsilon/5}(\log Y)^2 \geq \frac{1}{2}x^{\varepsilon_z/5}(\log x)^2 J^{-2}$. Since $J \leq C_8$ (Cx3b) and $(\log x)^2 \geq 2C_8^2$, this is $\geq x^{\varepsilon_z/5} \geq x^{\varepsilon_z/5-\varepsilon_z/100}$.

(d) S_c, η_S, C_t are validity values of L, which keeps those of the size datum over the context, and the support clause is the condition (Rf1) of the refilling of a size datum by a canonical instance, which L satisfies. Finally $X''(\mathfrak{s}, |A|) = a(\mathfrak{s})/w''(\mathfrak{s}, |A|) = x|A|X'(\mathfrak{s})$ by the definition of the derived quantities of a scale.

(e) By Lemma 19.16, $1 \leq \mathrm{rd}_E \leq N_{\mathfrak{h}}$, and by (Hg6) (through (Cx3b)) $\log N_{\mathfrak{h}} \leq 2Y^{\varepsilon/5}(\log Y)^2 \leq 2x^{\varepsilon/5}(\log x)^2$. As $\log a(\mathfrak{s}) = \log x + x^{\varepsilon}$,

$$x^{\varepsilon} \leq L_a \leq x^{\varepsilon} + \log x + 2x^{\varepsilon/5}(\log x)^2 \leq x^{\varepsilon+\theta},$$

the last inequality for $x \ge x_0$ after enlarging x_0 (the left side is $x^{\varepsilon}(1+o(1))$ and $x^{\theta} \to \infty$). $\square$

Proposition A.19. ***Canonical projection convention.*** *The conditional phrase* constructed using these exact tables and their standard canonical instances *below has the following precise meaning. Fix the ordered host group, its arithmetic datum, frame, tops and full good-representative sets before the tables. Use one common tuple* $\mathfrak{k}_{\mathrm{orb}}$*, one* $C_e \ge 0$*,* $C_r = 1$*, and the common short-depth constant* $C_\Lambda = C_{\mathrm{La}}$*. Choose* $C_N > 0$ *with* $N_{\mathrm{orb}} \le C_N \log x$ *on the common tail, and prescribe*

$$\delta = x^{-\varepsilon/8}, \quad N_* = 2n\delta, \quad L_c = C_{\mathrm{La}} N_{\mathrm{orb}}^2 \log a, \qquad \rho_B = \frac{4N_* x^{\varepsilon} L_c}{k} + \frac{C_{\mathrm{band}} C_N (\log x) L_c}{k}.$$

The same wider band ρ_B *is fixed before the tables, contexts and canonical refill; it is used in all candidate and no-gain readings.*

At a returned member, its loss datum means the following explicit projection, not an arbitrary independently chosen record. Its group, arithmetic datum, frame, tops and (k, m, t_R, e_2, o) *are the retained host data* $(A, \mathfrak{A}, T; k, m, t, e, o)$*. Its heights and depths are the selected tables, zero extended off* T *where an expression requires this; its weights are the actual fillings* $\mathrm{fill}_j(z) = \tilde{b}_{\tau(z),j}$ *and its shifts are* $[k + (2j-1)(t+1)]/2$*. Its representative map, orbit origins and full good sets are those of this same ordered host. Its candidates, low capped constants, high canonical infima, piecewise correction degrees and real thresholds are the computed prescriptions of Definition 5.2 and Definition 18.3 and of the refilling of a size datum by a canonical instance, on this instance and tuple, with* $p_0 = k^2 x^{26}$*,* $\rho = \rho_B$*,* $c_G = 4S_c$*, and integer depth* $\lceil t_p(v) \rceil$ *in the high canonical infimum. Its size entries are*

$$N = k + m(t+1), \quad S_c = 4k\Delta_D, \quad \eta = mh_0, \quad L_a = L_c.$$

Its place field is the original large coefficient field L*, its place set is the support of the returned size datum, and*

$$\ell(w) = \frac{[L_w : \mathbb{Q}_{p(w)}]}{[L : \mathbb{Q}]} \log p(w).$$

Its place gain is the truncated member gain; its corrections, companion terms, served sets, served tops and partial weights are read from the same canonical instance and supplied part. Its error exponent is the canonical $\max(0, \lceil C_e f k^2 \log_p a \rceil)$*. These are field assignments and computed-reading conventions, made in the construction before any properties of this proposition are invoked; none assumes the small-share inequality or (Lm2), (Lm5), (Lm6), (Lm7). No field of an arbitrary loss record or of the stored canonical instance is replaced by this convention. For every fixed collection of complete good-host data there are a common tail, one tuple* $\mathfrak{k}_{\mathrm{orb}}$ *of eight nonnegative real constants, one* $C_e \ge 0$*, common* $\varepsilon_z > 0, C_t$*, and at each retained scale one pair of integer height and depth tables, chosen before its contexts or members are constructed. The tables satisfy the prescribed windows and the averaged canonical small-share choice. Whenever the common-profile class and members of Proposition A.1 are constructed using these exact tables and their standard canonical instances, the following properties hold at every retained member.*

1. *Its loss datum has the selected rank datum and rank sizes, with* t_R *the fixed selected rank. Its heights, depths, weights and shifts satisfy the integer weights and boxes for the same* $c_D > 397, c_h \ge 402$ *used in constructing the member. The field in these integer-weight conditions is its host field, and its thick subset is the selected one.*

2. *Its top set, Frobenius translations, orbit representatives and good representatives are the chosen finite good-orbit selection. Its base representative function is the standard orbit representative function of the canonical instance used in its size datum.*

3. *Its correction degrees, orbit thresholds and correction errors satisfy (Nc2)–(Nc4) for the same* $(\mathfrak{k}_{\mathrm{orb}}, C_e)$*. In particular*

$$\epsilon_p = \left\lceil C_e f k^2 \log_p a \right\rceil^+, \qquad L_c = C_{\mathrm{La}} N_{\mathrm{orb}}^2 \log a, \qquad C_{\mathrm{La}} \geq 1.$$

All remaining components of the tuple are used at their actual prescribed values, including the two prime regimes.

4. *Its loss datum has (Lm2), (Lm5), (Lm6) and (Lm7) for* (ε_z, C_t)*; its frame exponent* e_2 *is the selected frame exponent. Its residual density is at most* x^{-2}*. Its served sets, gain, corrections, companion terms, served tops and partial weights are the ones read from the same canonical instance and part.*

The constants and the enlarged cutoff do not depend on the scale or the member. No final top profile, depth, orbit choice, local value or part is altered to obtain these identities.

Proof. Sketch. The loss condition (Lm5) of Definition 5.5 asks for a small *mean* of canonical orbit constants over a height–depth box, not for a pointwise bound. The universal majorant of Theorem A.9 has mean at most CNN_1/Y over the box in every unramified quadratic setting, so a finite averaging argument (Lemma A.16, Lemma A.17) selects at each scale one pair of integer height and depth tables, before any context or member, for which the weighted candidate sum is at most x^{-9}; since the majorant does not depend on the setting, one pair serves every translate. The proof then establishes the numerical part. From Definition 18.3, $N_{\mathrm{orb}} = O_\varepsilon(1+\log x)$, hence $x^\varepsilon \leq L_c \leq x^{\varepsilon+\varepsilon/100}$ and the band width ρ_B is at most x^{-2}; the middle bound is $c_G = 16k\Delta_S$ on both sides; the inert primes of the loss datum are exactly the signal and inert classes of the host; and with the normalisation $C_r = 1$ the no-gain predicate is the disjunction (Dr1)–(Dr3) of Definition 5.2. The member operation changes only divisor lattices, word heights and word lattices, so it keeps the tables, the canonical instance and all orbit readings; under the numerical reading of the member, Proposition A.18 gives (Lm2) and (Lm7) with $C_t = 401$ and $\varepsilon_z = \varepsilon/C_8$, and the identity $\sum_{w|p}[L_w : \mathbb{Q}_p] = [L : \mathbb{Q}]$ gives the place weights of (Lm6). Finally, if the member's good sets, candidates and arrays are those of the selector, the canonical constants agree, the actual masses equal the selector's, and (Lm5) holds for every context built on the one selected pair.

It remains to show that the actual construction realises exactly these readings: its finite group and ordering give the least orbit representatives, its top arrays are the selected tables, its good sets and candidate predicate are those of the selector, and its loss projection (the shifts $\frac{1}{2}(k + (2j-1)(t+1))$, the band, the truncated gain and the place weights) agrees with the loss data of the class, with $C_r = 1$ built into the common profile; and that the dominant arrays, local settings and coefficient rings of the canonical instance are those of the high-prime conclusion of Lemma A.17. Heuristically these are identities between two descriptions of the same finite data: the selector's prescription is modelled on the canonical instance, both are computed from the same host, scale and top numbers, and the member construction receives the tables as input and never reselects them. The one point needing care is that the selector allows any fixed good subset satisfying its finite conditions, so the member's good sets must be shown to be that subset itself rather than another subset with the same properties. The mean bound of Theorem A.9 on which the selection rests depends in turn on the raw step Proposition A.14. □

A.3 Theta kernels on the CM base: integrality at inert primes

In Alpöge's account the vanishing of Stepanov's auxiliary function at the rational points becomes divisibility at inert primes: on a CM curve, for p inert in K, a derivative of order a of the theta

function divided by Ω^a is divisible by p to the power $a/p+a/p^3+\cdots-O(\log a)$ [4]. This section proves the integrality statement that underlies such divisibility. Let $K=\mathbb{Q}(\sqrt{-d})$, and let $\ell\geq 5$ be a prime inert in K. All integrality at ℓ reduces to one statement about one CM elliptic curve: the formal exponential of a good Weierstrass model is integral in the logarithmic coordinates of Lubin and Tate (Proposition A.25). Theta tensors, their torsion translates and the kernel families of a CM base inherit this property, so the theta kernels of a CM base are integral at the primes inert in K (Proposition A.28). That this integrality is the arithmetic source of the gains at inert primes, which offset the local losses in (Rr1), is our reading. The gains of Definition 5.3 are defined abstractly, and the construction carries the reading through the served kernel entries and their column valuations. In this section "inert" means inert in K; it is not the classes of inert primes of the loss data.

A.3.1 Formal groups of CM elliptic curves at inert primes

The CM base is built over the CM elliptic lattice $\Lambda=\delta\,\sigma_K(\mathfrak{c})$; CM bases carry kernel families $e_{B,\chi}$ indexed by characters of the ideals of the CM field E (Definition A.20), and Definition A.21 says what it means for a theta series to be integral in formal-group coordinates at an inert prime. At a prime $\ell\geq 5$ inert in K the curve has supersingular reduction, so its formal group has height two over $\mathbb{Z}_\ell$ and, with its action of $\mathcal{O}_K$, height one as a formal $\mathcal{O}_{K_\ell}$-module. Two classical facts then apply: height-one formal modules over the completed maximal unramified extension are of Lubin–Tate type (Theorem A.22), and commutative formal group laws over a $\mathbb{Q}$-algebra have logarithms (Theorem A.23). With good Weierstrass models over that ring (Proposition A.24) they give Proposition A.25.

Definition A.20. Let $\mathfrak{c}$ be a non-zero ideal of $\mathcal{O}_K$ of norm prime to $2d$ and $\delta\in\mathbb{C}$ with $\delta^{12}=g_2(\sigma_K\mathfrak{c})^3-27g_3(\sigma_K\mathfrak{c})^2$, and put $\Lambda=\delta\,\sigma_K(\mathfrak{c})$; the notation of Definition A.21 is used for this Λ. Put $\alpha(x)=(-1)^{\mathrm{round}(\frac{\sqrt{d}}{2}H(x,x))}$ and, for subgroups $\Theta_1,\Theta_2\subset V_{N,N'}$ with $\Theta_1\cap\Theta_2$ of finite index in Θ_2 and G entire on $V_{N,N'}$,

$$I_{\Theta_1,\Theta_2}G=[\Theta_2:\Theta_1\cap\Theta_2]^{-1/2}\sum_q\alpha(q)\,U_qG,$$

q running over representatives of $\Theta_2/(\Theta_1\cap\Theta_2)$.

(Kb1) CM bases. A **CM base** B over Λ consists of the following data and of an entire function θ_0 on $\mathbb{C}^\Phi$, its **standard theta function**.

- A point $u\in\Lambda$ with $\omega_K u/2\notin\Lambda$ and $(1-\omega_K)u/2\notin\Lambda$. Put $r=[E:K]=|\Phi|$, $A=A(\Lambda)$, and on $\mathbb{C}^\Phi$

$$H_0(z,z')=A^{-1}\sum_{\tau\in\Phi}z_\tau\overline{z'_\tau},\qquad \alpha_0(x)=e^{\pi i(\sqrt{d}/2)H_0(x,x)},$$

$$M_0=\text{the subgroup generated by }\{(w\,\tau(a))_{\tau\in\Phi}:\ w\in\Lambda,\ a\in\mathfrak{a}_0\}.$$

With the one-variable operator $(U_t^{(1)}G)(z)=G(z+t)\exp(-\pi z\bar{t}/A-\pi|t|^2/(2A))$, put

$$F_B(z)=\frac{(U_{u/2}^{(1)}\sigma^*)(z)}{(U_{u/2}^{(1)}\sigma^*)(0)}\qquad(z\in\mathbb{C}).$$

- A K-linear isomorphism $\Gamma_0\colon E^4 \to K^{4r}$, the coordinates of K^{4r} indexed by $\{1,\dots,4\}\times\Phi$, that is an isometry from $\sum_{\nu=1}^{4}\operatorname{Tr}_{E/K}(x_\nu\bar{y}_\nu)$ to the standard form $\sum_j x_j\bar{y}_j$, and the $\mathbb{C}$-linear automorphism g_C of $\mathbb{C}^{\{1,\dots,4\}\times\Phi}$ with

$$g_C\big((\tau(x_\nu))_{\nu,\tau}\big) = \big(\sigma_K((\Gamma_0 x)_j)\big)_j \qquad (x\in E^4).$$

- For every rational prime q that splits in K: a K-linear isometry $\Gamma_q\colon E^4\to K^{4r}$ for the same two forms with $\mathbb{Z}_{(q)}\Gamma_q(\mathfrak{a}_0^4) = \mathbb{Z}_{(q)}\mathcal{O}_K^{4r}$; a prime $\mathfrak{p}_q$ of $\mathcal{O}_K$ above q; $\pi_q\in\mathcal{O}_K$ with $\mathfrak{p}_q^h = \pi_q\mathcal{O}_K$, h the class number of K; and $\rho_q\in\mathbb{C}$ with $\rho_q^{2h} = \sigma_K(\pi_q)/\sigma_K(\bar{\pi}_q)$. Put $y_q = v_{\mathfrak{p}_q}\big(\det_K(\Gamma_q\Gamma_0^{-1})\big)$, $v_{\mathfrak{p}_q}$ the normalised valuation; it is required that $y_q\neq 0$ for only finitely many q, and

$$\kappa = \prod_{q \text{ split},\ y_q\neq 0} \rho_q^{-y_q}.$$

- On $\mathbb{C}^{4r} = \mathbb{C}^{\{1,\dots,4\}\times\Phi}$ put $H_{\mathrm{st}}(z,w) = A^{-1}\sum_j z_j\bar{w}_j$, $\alpha_{\mathrm{st}}(x) = e^{\pi i(\sqrt{d}/2)H_{\mathrm{st}}(x,x)}$, U_t^{st} the translate of Definition A.21 for H_{st}, $F_B^{\otimes 4r}(z) = \prod_j F_B(z_j)$ and $M' = g_C(M_0^4)$, and for G entire on $\mathbb{C}^{4r}$

$$\mathsf{I}_{\Lambda^{4r},M'}G = [M' : \Lambda^{4r}\cap M']^{-1/2}\sum_m \alpha_{\mathrm{st}}(m)\,U_m^{\mathrm{st}}G,$$

  m running over representatives in M' of $M'/(\Lambda^{4r}\cap M')$.

- θ_0 satisfies, for all $x\in M_0$ and $z\in\mathbb{C}^\Phi$,

$$\alpha_0(x)\,\theta_0(z+x)\,e^{-\pi H_0(z,x)-\pi H_0(x,x)/2} = \theta_0(z),$$

  and for all $z_1,\dots,z_4\in\mathbb{C}^\Phi$

$$\prod_{\nu=1}^{4}\theta_0(z_\nu) = \kappa\,\big(\mathsf{I}_{\Lambda^{4r},M'}F_B^{\otimes 4r}\big)\big(g_C(z_1,\dots,z_4)\big).$$

(Kb2) Characters of ideals. A **character of the ideals of** E is a complex function χ on the $\mathcal{O}_E$-submodules of E with $\chi(\mathfrak{a}_1\mathfrak{a}_2) = \chi(\mathfrak{a}_1)\chi(\mathfrak{a}_2)$ whenever $\mathfrak{a}_1\bar{\mathfrak{a}}_1 = \mathfrak{a}_2\bar{\mathfrak{a}}_2 = \mathcal{O}_E$, and $\chi(z\mathcal{O}_E) = \prod_{\tau\in\Phi}\tau(z)^{-1}$ for $z\in E$ with $z\bar{z}=1$.

(Kb3) The kernel family. For a CM base B over Λ with standard theta function θ_0 and a character χ of the ideals of E, put $\theta_0^{\otimes}(z) = \prod_{a,b}\theta_0\big((z_{\tau ab})_{\tau\in\Phi}\big)$ on $V_{N,N'}$; for $L\subset E^N$ let $\det L$ be the $\mathcal{O}_E$-span of the $\det A$ with $A\mathcal{O}_E^N\subseteq L$. The **kernel family** of (B,χ) assigns to every admitted pair (L,L') the function

$$e_{B,\chi}(L,L') = \chi(\det L)^{N'/2}\,\chi(\det L')^{-N/2} \cdot I_{\Theta(L,\mathcal{O}_E^{N'}),\,\Theta(L,L')}\,I_{\Theta(\mathcal{O}_E^N,\mathcal{O}_E^{N'}),\,\Theta(L,\mathcal{O}_E^{N'})}\,\theta_0^{\otimes}$$

on $V_{N,N'}$; when B is fixed it is written e_χ.

Definition A.21. All other notation is local. **Host and CM lattice.** Let $\mathfrak{h} = (q,d,Q_S,\alpha)$ be a host for a frame with $\alpha(-1) = ((-1,-1),0)$, let $E = E_{\mathfrak{h}}$ be its field, read ϵ_2 on $\mathrm{Gal}(E/\mathbb{Q})$ through the isomorphism induced by α, write $x\mapsto\bar{x}$ for the automorphism of E corresponding to $((-1,-1),0)$ (complex conjugation) and E^+ for its fixed field. By (Hr1), d is a prime with

$d \equiv 7 \pmod{16}$. Assume that E is unramified over E^+ at every finite place: for every maximal ideal P of $\mathcal{O}_E$ some $\theta \in \mathcal{O}_E$ has $\theta - \bar{\theta} \notin P$. Fix $s_d \in E$ and a field embedding $\varphi_0 \colon E \hookrightarrow \mathbb{C}$ with

$$s_d^2 = -d, \qquad \sigma(s_d) = \epsilon_2(\sigma)\, s_d \quad (\sigma \in \mathrm{Gal}(E/\mathbb{Q})), \qquad \varphi_0(s_d) = i\sqrt{d};$$

write $\sqrt{-d} = s_d$, $K = \mathbb{Q}(\sqrt{-d}) \subseteq E$ and $\sigma_K = \varphi_0|_K$, through which K is viewed inside $\mathbb{C}$ (so $\sigma_K(\sqrt{-d}) = i\sqrt{d}$), and put $\omega_K = (1+\sqrt{-d})/2$. Let $\mathfrak{b}$ be an integral ideal of $\mathcal{O}_E$ with

$$\mathcal{D}_{E/K} = \mathfrak{b}\,\bar{\mathfrak{b}}, \qquad \mathfrak{b} + \bar{\mathfrak{b}} = \mathcal{O}_E, \qquad \sigma(\mathfrak{b}) = \mathfrak{b} \ \text{ for every } \sigma \in \mathrm{Gal}(E/\mathbb{Q}) \text{ with } \epsilon_2(\sigma) = 1,$$

where $\mathcal{D}_{E/K}$ is the relative different, and put

$$\Phi = \{\tau \colon E \hookrightarrow \mathbb{C} : \ \tau|_K = \sigma_K\}, \qquad \mathfrak{a}_0 = \mathfrak{b}^{-1}.$$

Thus Φ is the full CM type of E over σ_K, and $\mathfrak{a}_0$ is a fractional ideal of E, in general not contained in $\mathcal{O}_E$. Let $\Lambda \subset \mathbb{C}$ be a lattice with $\omega_K\Lambda \subseteq \Lambda$ and $g_2(\Lambda)^3 - 27g_3(\Lambda)^2 = 1$; $A(\Lambda) > 0$ is its covolume.

Lattices and theta lattices. For $N \geq 1$ put $h_N(x,y) = \sum_i \bar{x}_i y_i$ on E^N. A pair (L, L') of $\mathcal{O}_E$-submodules $L \subset E^N$, $L' \subset E^{N'}$ is **admitted** if N, N' are even and each is **self-dual**: finitely generated, spanning, and equal to $\{y : \ h(x,y) \in \mathcal{O}_E \text{ for all } x \text{ in it}\}$. Put

$$Y(L,L') = \mathcal{O}_E\text{-span of } \{aA : \ a \in \mathfrak{a}_0, \ A \in \mathrm{M}_{N'\times N}(E), \ AL \subseteq L'\},$$
$$h_Y(x,y) = \mathrm{Tr}_{E/K} \textstyle\sum_{a,b} x_{ab}\overline{y_{ab}} \in K.$$

Let $V = V_{N,N'} = \mathbb{C}^{\Phi\times N'\times N}$ with coordinates $z_{\tau ab}$, $\iota(A) = (\tau(A_{ab}))_{\tau ab}$, and let $\Theta(L,L') \subset V$ be the subgroup generated by $w\,\iota(A)$, $w \in \Lambda$, $A \in Y(L,L')$ (the **theta lattice**). Put $H(z,w) = A(\Lambda)^{-1}\sum_{\tau ab} z_{\tau ab}\overline{w_{\tau ab}}$ and, for $t \in V$ and $G \colon V \to \mathbb{C}$, $(U_tG)(z) = G(z+t)\exp\bigl(-\pi H(z,t) - \frac{\pi}{2}H(t,t)\bigr)$.

Prime-to-p notions. Let $p \geq 5$ be a prime with $\left(\frac{-d}{p}\right) = -1$. Two $\mathcal{O}_E$-modules M, M_1 (in the same E^N) are **p-locally equal** if $sM \subseteq M_1$ and $sM_1 \subseteq M$ for some integer s prime to p. A point $t \in V$ is **prime-to-p torsion** for $\Theta(L,L')$ if $nt \in \Theta(L,L')$ for some integer $n \geq 1$ prime to p. A family $b_1, \ldots, b_r \in \mathrm{M}_{N'\times N}(E)$ is an **orthonormal p-basis** of $Y = Y(L,L')$ if it is K-linearly independent, $sb_i \in Y$ for some s prime to p, every $y \in Y$ has $sy \in \sum_i \mathcal{O}_K b_i$ for some s prime to p, and $h_Y(b_i, b_j) = \delta_{ij}$.

p-adic series. Let $\lambda \colon \overline{\mathbb{Q}} \to \mathbb{C}_p$ be a field embedding, extended to $\mathbb{C}$ by $\lambda(x) = 0$ for $x \notin \overline{\mathbb{Q}}$. For G entire on V let $\widehat{G}_\lambda \in \mathbb{C}_p[[z_{\tau ab}]]$ be the series whose coefficients are the images under λ of the Taylor coefficients of G at 0. A **good model** is a change of variables C over $\mathbb{C}_p$, with scaling $u \in \mathbb{C}_p^\times$, carrying $y^2 = x^3 - \frac{1}{4}\lambda(g_2(\Lambda))x - \frac{1}{4}\lambda(g_3(\Lambda))$ to a general Weierstrass equation W with coefficients of absolute value ≤ 1 and discriminant 1; $\log_W \in \mathbb{C}_p[[X]]$ is the formal logarithm of W in the parameter $X = -x/y$, with $\log_W(X) = X + O(X^2)$. Let σ^* be the unique odd entire function with $(\sigma^*)'(0) = 1$ and $\sigma^*(z+\omega)\exp\bigl(-\pi z\bar{\omega}/A(\Lambda) - \pi|\omega|^2/(2A(\Lambda))\bigr) = \pm\sigma^*(z)$ for $\omega \in \Lambda$, sign $+$ exactly when $\omega/2 \in \Lambda$; $\widehat{\sigma^*}_\lambda$ its series.

Integrality in formal-group coordinates. A family e assigning to every admitted (L,L') an entire function $e(L,L')$ on $V_{N,N'}$ is **integral in formal-group coordinates at** (p,λ) if for every admitted (L,L'), every series

$$F = \sum_{k=1}^{m} r_k \left(U_{t_k}\, e(L_k, L_k')\right)^\wedge_\lambda$$

with $|r_k| \le 1$, t_k prime-to-p torsion for $\Theta(L, L')$, and (L_k, L_k') admitted and p-locally equal to (L, L') componentwise, every good model C (scaling u, $\varepsilon = u^{-1}$) and every orthonormal p-basis $b_1, \dots, b_r$ of $Y(L, L')$, all coefficients of the series

$$F\Big(\Big(\sum_{i=1}^{r} \varepsilon\, \lambda\big(\tau(b_{i,ab})\big) \log_W(X_i)\Big)_{\tau ab}\Big) \cdot \prod_{i=1}^{r} \frac{\varepsilon X_i}{\widehat{\sigma^*}_\lambda\big(\varepsilon \log_W(X_i)\big)}$$

in $\mathbb{C}_p[[X_1, \dots, X_r]]$ have absolute value at most 1. The family is **integral at inert primes** if this holds for every prime $p \ge 5$ with $\left(\frac{-d}{p}\right) = -1$ and every λ.

Theorem A.22. *All notation is local. Let K_0 be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}$, uniformiser π and residue field of cardinality q. Let $\breve{\mathcal{O}}$ be the valuation ring of the completion of the maximal unramified extension of K_0 in $\mathbb{C}_p$. A* formal $\mathcal{O}$-module *over a commutative $\mathcal{O}$-algebra R is a one-dimensional commutative formal group law $G(X, Y) \in R[[X, Y]]$ with a ring homomorphism $a \mapsto [a]_G$ from $\mathcal{O}$ to the endomorphisms of G over R such that $[a]_G(X) \equiv aX \pmod{X^2}$; over $R = \breve{\mathcal{O}}$ it has* height one *if the coefficient of X^q in $[\pi]_G(X)$ is a unit of $\breve{\mathcal{O}}$.*

(a) Put $\log_\pi(X) = \sum_{k \ge 0} \pi^{-k} X^{q^k} \in K_0[[X]]$. The power series

$$F_\pi(X, Y) = \log_\pi^{-1}\big(\log_\pi(X) + \log_\pi(Y)\big), \qquad [a]_\pi(X) = \log_\pi^{-1}\big(a \log_\pi(X)\big) \quad (a \in \mathcal{O})$$

have coefficients in $\mathcal{O}$, and they form a formal $\mathcal{O}$-module over $\mathcal{O}$ of height one, with $[\pi]_\pi(X) \equiv X^q \pmod{\pi}$.

(b) Every formal $\mathcal{O}$-module G of height one over $\breve{\mathcal{O}}$ is isomorphic over $\breve{\mathcal{O}}$ to F_π: there is $\theta(X) \in \breve{\mathcal{O}}[[X]]$ with $\theta(X) \equiv \varepsilon X \pmod{X^2}$, $\varepsilon \in \breve{\mathcal{O}}^\times$, such that $\theta(F_\pi(X, Y)) = G(\theta(X), \theta(Y))$ and $\theta \circ [a]_\pi = [a]_G \circ \theta$ for all $a \in \mathcal{O}$.

Proof. Outline. Part (a) follows from Hazewinkel's functional-equation lemma applied to $\log_\pi$ [124], and part (b) from the Lubin–Tate uniqueness of the formal module attached to a Lubin–Tate series, together with the classification of formal $\mathcal{O}$-modules by height [124]. □

Theorem A.23. *Let R be a commutative $\mathbb{Q}$-algebra and let $F(X, Y)$ be a commutative formal group law of dimension $d \ge 0$ over R. There is a unique vector of formal power series $\ell_F(X) \in R[[X_1, \dots, X_d]]^d$ with zero constant term and linear term X such that*

$$\ell_F(F(X, Y)) = \ell_F(X) + \ell_F(Y).$$

The formal map ℓ_F has a compositional inverse. If G is a commutative formal group law of dimension e over R and $u : F \to G$ is a homomorphism of formal group laws, then

$$\ell_G(u(X)) = du(0)\, \ell_F(X),$$

where $du(0) \in M_{e,d}(R)$ is the linear term of u.

Proof. Outline. The logarithm is the characteristic-zero logarithm of a commutative formal group law [124], compositionally invertible because its linear term is the identity, and the homomorphism identity and uniqueness follow because an additive formal map over a rational algebra equals its linear term, by Euler's identity applied to its homogeneous components. □

Proposition A.24. *All notation is local. Let $d \geq 1$ be a squarefree integer with $d \equiv 3 \pmod 4$, put $K = \mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$ and $\omega_K = (1 + i\sqrt{d})/2$, so that $\mathcal{O}_K = \mathbb{Z}[\omega_K]$. Let $L \subset \mathbb{C}$ be a lattice with $\omega_K L \subseteq L$, with Weierstrass invariants $g_2(L)$, $g_3(L)$. Let $\ell \geq 5$ be a prime with $\ell \nmid d$ and let $\iota\colon \mathbb{C} \to \mathbb{C}_\ell$ be an isomorphism of fields. Let $\breve{\mathbb{Q}}_\ell \subseteq \mathbb{C}_\ell$ be the closure of the subfield generated by the roots of unity of order prime to ℓ, and $\breve{\mathbb{Z}}_\ell$ its valuation ring. If moreover d is a prime with $d > 3$, then there is $\lambda \in \mathbb{C}_\ell^\times$ such that*

$$a = \lambda^4\, \iota(g_2(L)) \in \breve{\mathbb{Z}}_\ell, \qquad b = \lambda^6\, \iota(g_3(L)) \in \breve{\mathbb{Z}}_\ell, \qquad a^3 - 27b^2 \in \breve{\mathbb{Z}}_\ell^\times.$$

Proof. Outline. An elliptic curve with complex multiplication by $\mathcal{O}_K$ and good reduction away from d exists over $\mathbb{Q}(j(\mathcal{O}_K))$; Galois conjugation over a subfield of the Hilbert class field, which is unramified above ℓ, moves its invariant to $j(L)$, and a short Weierstrass model with unit discriminant at the place of ι is then carried to the invariants of L by a scaling (λ^4, λ^6). □

Proposition A.25. *All notation is local; d, K, ω_K, L, ℓ, ι, $\breve{\mathbb{Q}}_\ell$ and $\breve{\mathbb{Z}}_\ell$ are as in Proposition A.24. For a, b in a field of characteristic zero put $c_2 = a/20$, $c_3 = b/28$ and, for $k \geq 4$,*

$$c_k = \frac{3}{(2k+1)(k-3)} \sum_{m=2}^{k-2} c_m\, c_{k-m},$$

(the coefficients of $\wp(z) = z^{-2} + \sum_{k\geq 2} c_k z^{2k-2}$ for $y^2 = 4x^3 - ax - b$), and

$$u_{a,b}(Z) = Z\left(1 + \sum_{k\geq 2} c_k Z^{2k}\right)\left(1 - \sum_{k\geq 2} (k-1) c_k Z^{2k}\right)^{-1},$$

the formal expansion of $-2\wp(z)/\wp'(z)$. Put $\log_{\mathrm{LT}}(X) = \sum_{j\geq 0} \ell^{-j} X^{\ell^{2j}}$. Assume that $-d$ is not a square modulo ℓ. Let $\lambda \in \mathbb{C}_\ell^\times$ and

$$a = \lambda^4 \iota(g_2(L)), \qquad b = \lambda^6 \iota(g_3(L)), \qquad a, b \in \breve{\mathbb{Z}}_\ell, \qquad a^3 - 27b^2 \in \breve{\mathbb{Z}}_\ell^\times.$$

Then there is $\varepsilon \in \breve{\mathbb{Z}}_\ell^\times$ such that every coefficient of the power series $u_{a,b}(\varepsilon \log_{\mathrm{LT}}(X)) \in \mathbb{C}_\ell[[X]]$ lies in $\breve{\mathbb{Z}}_\ell$.

Proof. Write $F = \breve{\mathbb{Q}}_\ell$, $R = \breve{\mathbb{Z}}_\ell$, $q = \ell^2$, and let E be the curve $y^2 = 4x^3 - ax - b$ over F; the substitution $y = 2Y$ gives $Y^2 = x^3 - \frac{a}{4}x - \frac{b}{4}$, a Weierstrass equation with coefficients in R (2 is a unit, as $\ell \geq 5$) and discriminant $-16(4(-\frac{a}{4})^3 + 27(\frac{b}{4})^2) = a^3 - 27b^2 \in R^\times$, so E has good reduction and this equation is its Néron model over R ([233], Chapter IV, §6: a Weierstrass model that is smooth over R is a Néron model).

The formal group and its exponential. Let $G \in R[[t_1, t_2]]$ be the formal group law of this equation in the parameter $t = -x/Y = -2x/y$ ([234], Chapter IV, §1), with invariant differential dx/y and formal logarithm $\log_G \in F[[t]]$ normalised by $\log_G(t) \equiv t \pmod{t^2}$ ([234], Chapter IV, §4, §5). Over the field F of characteristic zero, the formal Laurent series $x(z) = z^{-2} + \sum_{k\geq 2} c_k z^{2k-2}$ (with the c_k of the statement) and $y(z) = x'(z)$ satisfy $y^2 = 4x^3 - ax - b$ (the recursion of the c_k is the coefficient comparison in $\wp'' = 6\wp^2 - \frac{a}{2}$), and $dx/y = dz$. Hence $t(z) = -2x(z)/y(z) = u_{a,b}(z)$ is a formal parametrisation with $\log_G(u_{a,b}(z)) = z$; that is, $u_{a,b}$ is the compositional inverse of $\log_G$.

The formal module. Over $\mathbb{C}_\ell$, E is isomorphic (by $(x, y) \mapsto (\lambda^{-2}x, \lambda^{-3}y)$) to the image under ι of the curve $y^2 = 4x^3 - g_2(L)x - g_3(L)$, whose endomorphism ring is $\mathcal{O}_K$ acting through

$z \mapsto \alpha z$ on $\mathbb{C}/L$. So $\mathcal{O}_K$ acts on $E_{\mathbb{C}_\ell}$, and $[\alpha]^*(dx/y) = \varphi(\alpha)\,dx/y$ for the embedding $\varphi = \iota|_K \colon K \to \mathbb{C}_\ell$. Since ℓ is unramified in K, $\varphi(K) \subseteq F$. Every endomorphism is defined over a finite Galois extension F' of F; for $\tau \in \mathrm{Gal}(F'/F)$, $\tau([\alpha])$ multiplies dx/y by $\tau(\varphi(\alpha)) = \varphi(\alpha)$, and an endomorphism is determined by its action on the invariant differential in characteristic zero; so $\tau([\alpha]) = [\alpha]$ and $[\alpha]$ is defined over F. By the Néron property it extends to the Néron model, hence gives $[\alpha]_G \in \mathrm{End}_R(G)$ with $[\alpha]_G(t) \equiv \varphi(\alpha)t \pmod{t^2}$. As $\mathrm{End}_R(G)$ contains $\mathbb{Z}_\ell$ (by continuity) and $\mathcal{O} := \mathcal{O}_K \otimes \mathbb{Z}_\ell = \mathbb{Z}_\ell[\omega_K]$ is the unramified quadratic extension $\mathbb{Z}_{\ell^2}$ ($-d$ is not a square modulo ℓ), G is a formal $\mathcal{O}$-module over R (with R an $\mathcal{O}$-algebra through φ), uniformiser $\pi = \ell$, residue cardinality $q = \ell^2$.

Height one. The reduction $\tilde{E}$ over $\overline{\mathbb{F}}_\ell$ is supersingular: its invariant is the reduction of $\iota(j(L))$, and by Deuring's criterion (S. Lang, Elliptic functions, 2nd ed., Chapter 13, §4, Theorem 12) the reduction of an elliptic curve with complex multiplication by $\mathcal{O}_K$ at a prime above ℓ is supersingular when ℓ is inert in K. Hence the formal group of $\tilde{E}$ has height 2, i.e. $[\ell]_G(t) \bmod \ell$ is a series in t^{ℓ^2} whose coefficient of t^{ℓ^2} is nonzero ([234], Chapter IV, §7, and Chapter V, Theorem 3.1); so the coefficient of t^q in $[\pi]_G$ is a unit of R, and G has height one as a formal $\mathcal{O}$-module.

Conclusion. By Theorem A.22(b) there is an isomorphism $\theta \colon F_\pi \to G$ of formal $\mathcal{O}$-modules over R with $\theta(X) \in R[[X]]$ and $\theta(X) \equiv \varepsilon X \pmod{X^2}$, $\varepsilon \in R^\times$, where F_π has logarithm $\log_\pi(X) = \sum_{j\ge 0} \ell^{-j} X^{\ell^{2j}} = \log_{\mathrm{LT}}(X)$ (Theorem A.22(a)). By Theorem A.23 over F, $\log_G(\theta(X)) = \varepsilon \log_{\mathrm{LT}}(X)$; applying the compositional inverse $u_{a,b}$ of $\log_G$ gives $\theta(X) = u_{a,b}(\varepsilon \log_{\mathrm{LT}}(X))$. So every coefficient of $u_{a,b}(\varepsilon \log_{\mathrm{LT}}(X))$ lies in $R = \breve{\mathbb{Z}}_\ell$. □

A.3.2 Theta kernels and the Hecke reading

The one-dimensional statement passes to the standard theta tensors (Lemma A.26). The theta kernels of a CM base are covariant under the unitary group (Proposition A.27), which transports integrality between admitted pairs, and together these give Proposition A.28. For the kernel family of a tied entry the integrality takes the form of a host condition at the inert-class and signal primes ≥ 5 (Lemma A.29); this condition is stronger than integrality of the coefficients. Finally, integrality survives the normalised Hecke reading: the Hecke reading preserves the integral model ball (Lemma A.30), on the integral models of unitary Shimura data of Section A.8 (Proposition A.103).

Lemma A.26. *Let $p \ge 5$ be inert in K and let $\lambda : \overline{\mathbb{Q}} \to \mathbb{C}_p$ be a field embedding. For even nonnegative integers N, N', put $m = N'N$ and, in the ordered matrix coordinates,*

$$Y_{\mathrm{st}} = M_{N'\times N}(A_0), \quad h_Y(x,y) = \mathrm{Tr}_{E/K} \sum_{a,b} x_{ab}\overline{y_{ab}},$$

$$T_{N,N'}(z) = \prod_{a=1}^{N'} \prod_{b=1}^{N} T_B((z_{\tau ab})_\tau), \qquad \Theta_{\mathrm{st}} = M_0^m, \qquad H_{\mathrm{st}} = \bigoplus_{a,b} H_0.$$

Let s satisfy $ns \in \Theta_{\mathrm{st}}$ for some positive integer n prime to p. Let $b_1, \dots, b_d$ be an orthonormal p-basis of Y_{st}; it is a K-basis of $M_{N'\times N}(E)$, spans SY_{st} over $S = O_K \otimes \mathbb{Z}_{(p)}$, and satisfies $h_Y(b_i, b_j) = \delta_{ij}$. Thus $d = rm$.

For every good model W, with scaling u, $\varepsilon_W = u^{-1}$ and normalized logarithm $\log_W$, all coefficients of

$$(U_s^{H_{\mathrm{st}}} T_{N,N'})^\wedge_\lambda \left(\left(\sum_{i=1}^{d} \varepsilon_W \lambda(\tau((b_i)_{ab})) \log_W(X_i) \right)_{\tau ab} \right) \prod_{i=1}^{d} \frac{\varepsilon_W X_i}{\widehat{S_{L\lambda}}(\varepsilon_W \log_W(X_i))}$$

belong to $\mathcal{O}_{\mathbb{C}_p}$. The quotients are read by cancelling their common simple zero. For $m = 0$ the expression is the constant 1*. All theta functions, torsion points, coefficient directions and model corrections are those of the data fixed above.*

Proof. Outline. One extends λ to a field isomorphism $\mathbb{C} \to \mathbb{C}_p$, fixes a good elliptic scaling and a Lubin–Tate unit by Proposition A.24 and Proposition A.25, proves integrality of one copy of T_B in Lubin–Tate logarithmic coordinates of a local frame from its finite-intertwiner identity, takes the product over the $m = N'N$ matrix positions, passes to the given orthonormal p-basis through a quadratic-unitary transition matrix and the Lubin–Tate substitution of Theorem A.22(a), and finally compares with an arbitrary good model using the algebraicity of the Taylor coefficients. □

Proposition A.27. *In the notation of Proposition A.28, let (M, M') be an admitted pair, $M \subset E^N$, $M' \subset E^{N'}$, and let $\gamma \in \mathrm{GL}_N(E)$, $\gamma' \in \mathrm{GL}_{N'}(E)$ satisfy $\bar{\gamma}^{\mathsf{T}}\gamma = 1$, $\bar{\gamma}'^{\mathsf{T}}\gamma' = 1$. Then for every $z \in V_{N,N'}$, writing $z_\tau \in \mathrm{M}_{N'\times N}(\mathbb{C})$ for its τ-component and $\tau(\gamma)$ for the entrywise image,*

$$e_\chi(\gamma M, \gamma' M')(z) = e_\chi(M, M')\Big(\big(\overline{\tau(\gamma')}\big)^{\mathsf{T}}\, z_\tau\, \tau(\gamma)\big)_{\tau\in\Phi}\Big).$$

Proof. Outline. One first proves the composition law $I_{23}I_{12} = I_{13}$ for the finite intertwiners on the one-dimensional theta lines of the period lattices, the phase being 1 by positivity of a local quadratic sum split by an isotropic projection (from the unramified Cartan decomposition); then computes the scalar for changing the standard pair by one unitary factor through the Cayley theta identity, whose sign is fixed by an element of norm -1 modulo a suitable integer; and finally transports an arbitrary pair and cancels the determinant factors by the principal-ideal rule of (Kb2) in Definition A.20. □

Proposition A.28. *Let $\mathfrak{c}$ be a non-zero ideal of $\mathcal{O}_K$ of norm prime to $2d$, $\delta \in \mathbb{C}$ with $\delta^{12} = g_2(\sigma_K\mathfrak{c})^3 - 27g_3(\sigma_K\mathfrak{c})^2$ and $\Lambda = \delta\,\sigma_K(\mathfrak{c})$; let B be a CM base over Λ and χ a character of the ideals of E. Then the kernel family $e_\chi = e_{B,\chi}$ is integral in formal-group coordinates at inert primes in the sense of Definition A.21.*

Proof. Fix a prime $p \geq 5$ with $(\frac{-d}{p}) = -1$ and λ. For an admitted pair (L, L'), a good model (scaling u, $\varepsilon = u^{-1}$) and an orthonormal p-basis $b = (b_1, \dots, b_r)$ of $Y(L, L')$, and $F \in \mathbb{C}_p[[z_{\tau ab}]]$, put

$$T_b(F) = F(Z_b)\cdot P_r, \qquad Z_b = \Big(\sum_i \varepsilon\,\lambda\big(\tau(b_{i,ab})\big)\log_W(X_i)\Big)_{\tau ab},$$
$$P_r = \prod_{i=1}^{r} \frac{\varepsilon X_i}{\widehat{\sigma^*_\lambda}(\varepsilon\log_W(X_i))}.$$

The substituted series have no constant term, so $F \mapsto F(Z_b)$ is a well-defined $\mathbb{C}_p$-linear map (each coefficient is a finite sum). Call F *good for* (L, L') if $T_b(F)$ has all coefficients of absolute value ≤ 1 for every good model and every orthonormal p-basis b of $Y(L, L')$. We must show: every F as in Definition A.21 for the family $e = e_\chi$ is good for (L, L').

Step 1 (integral combinations). By linearity $T_b(\sum_k r_k F_k) = \sum_k r_k T_b(F_k)$; by the ultrametric inequality, if each F_k is good for (L, L') and $|r_k| \leq 1$, then $\sum_k r_k F_k$ is good for (L, L').

Step 2 (p-local equality). Let $L, L_1 \subset E^N$ and $L', L'_1 \subset E^{N'}$ be p-locally equal via integers s, s' prime to p. If $AL \subseteq L'$ then $(ss'A)L_1 \subseteq s'A(L) \subseteq s'L' \subseteq L'_1$; with the symmetric statement,

$ss'\,Y(L,L') \subseteq Y(L_1,L_1')$ and $ss'\,Y(L_1,L_1') \subseteq Y(L,L')$, hence also $ss'\Theta(L,L') \subseteq \Theta(L_1,L_1')$ and conversely. Consequently the two modules Y have the same orthonormal p-bases (linear independence and orthonormality are intrinsic; the two span conditions transfer after multiplying by ss', prime to p), so a series is good for (L,L') if and only if it is good for (L_1,L_1'); and a point is prime-to-p torsion for $\Theta(L,L')$ if and only if it is for $\Theta(L_1,L_1')$.

Step 3 (unitary pull-back). Let $\gamma \in \mathrm{GL}_N(E)$ and $\gamma' \in \mathrm{GL}_{N'}(E)$ satisfy $\bar\gamma^{\mathsf T}\gamma = 1$ and $\bar\gamma'^{\mathsf T}\gamma' = 1$, and put $\pi(z)_\tau = \tau(\bar\gamma'^{\mathsf T})\, z_\tau\, \tau(\gamma)$ on V. Since E is a CM field, $\tau(\bar x) = \overline{\tau(x)}$, so $\tau(\gamma),\tau(\gamma')$ are unitary complex matrices and $H(\pi z,\pi w) = H(z,w)$ ($\sum_{ab} z_{ab}\overline{w_{ab}} = \mathrm{tr}(z\bar w^{\mathsf T})$). (i) $A\gamma M \subseteq \gamma' M'$ if and only if $\bar\gamma'^{\mathsf T} A\gamma M \subseteq M'$, so $x \mapsto \bar\gamma'^{\mathsf T} x\gamma$ maps $Y(\gamma M,\gamma' M')$ onto $Y(M,M')$ ($\mathfrak a_0$-multiples correspond) and π maps $\Theta(\gamma M,\gamma' M')$ onto $\Theta(M,M')$; in particular πt is prime-to-p torsion for $\Theta(M,M')$ when t is for $\Theta(\gamma M,\gamma' M')$. The same map preserves h_Y (by $\gamma\bar\gamma^{\mathsf T} = 1$, $\gamma'\bar\gamma'^{\mathsf T} = 1$ and cyclicity of the trace), K-linear independence and $\mathcal O_K$-spans, so $b_i' = \bar\gamma'^{\mathsf T} b_i \gamma$ is an orthonormal p-basis of $Y(M,M')$ whenever b is one of $Y(\gamma M,\gamma' M')$. (ii) For G entire, $U_t(G\circ\pi) = (U_{\pi t}G)\circ\pi$, by H-invariance. (iii) If G has algebraic Taylor coefficients, those of $G\circ\pi$ are integral polynomials in them and in the algebraic entries of $\tau(\gamma),\tau(\gamma')$, and since λ is a ring homomorphism on $\overline{\mathbb Q}$, $\widehat{(G\circ\pi)}_\lambda(Z_b) = \widehat G_\lambda(Z_{b'})$. Hence: *if $\widehat G_\lambda$ is good for (M,M') and G has algebraic Taylor coefficients, then $\widehat{(G\circ\pi)}_\lambda$ is good for $(\gamma M,\gamma' M')$.*

Step 4 (pairs p-locally equal to the standard pair). Let (M,M') be admitted and p-locally equal to $(\mathcal O_E^N,\mathcal O_E^{N'})$, and t' prime-to-p torsion for $\Theta(M,M')$. A translated kernel at a p-locally standard pair is a p-integral combination of standard translates: $U_{t'}e_\chi(M,M') = \sum_k c_k U_{s_k}\theta_0^{\otimes}$ with c_k algebraic, $|\lambda(c_k)| \le 1$, s_k prime-to-p torsion for the standard theta lattice. Each $U_{s_k}\theta_0^{\otimes}$ has algebraic Taylor coefficients, as every torsion translate of $\theta_0^{\otimes}$ does, so $\big(U_{t'}e_\chi(M,M')\big)^\wedge_\lambda = \sum_k \lambda(c_k)(U_{s_k}\theta_0^{\otimes})^\wedge_\lambda$. Since N,N' are even, the integrality of torsion translates of $\theta_0^{\otimes}$ in formal-group coordinates makes each summand good for the standard pair, Step 2 good for (M,M'), and Step 1 the sum good for (M,M').

Step 5 (assembly). Let (L,L') be admitted and $F = \sum_k r_k (U_{t_k}e_\chi(L_k,L_k'))^\wedge_\lambda$ as in Definition A.21. By Step 1 it suffices that each summand is good for (L,L'); by Step 2 it suffices that it is good for (L_k,L_k'), and t_k is prime-to-p torsion for $\Theta(L_k,L_k')$. So let (L,L') be admitted and t prime-to-p torsion for $\Theta(L,L')$. Every admitted pair is a unitary translate of a p-locally standard one at an inert prime: $L = \gamma M$, $L' = \gamma' M'$ with γ,γ' as in Step 3 and (M,M') admitted and p-locally standard. By Proposition A.27, $e_\chi(L,L') = e_\chi(M,M')\circ\pi$, so by Step 3(ii) $U_t e_\chi(L,L') = \big(U_{\pi t}e_\chi(M,M')\big)\circ\pi$, where πt is prime-to-p torsion for $\Theta(M,M')$ (Step 3(i)). The function $G = U_{\pi t}e_\chi(M,M')$ has algebraic Taylor coefficients (as do all torsion translates of the kernels) and $\widehat G_\lambda$ is good for (M,M') (Step 4); by Step 3(iii), $\big(U_t e_\chi(L,L')\big)^\wedge_\lambda$ is good for (L,L'). □

Lemma A.29. *Let t be a tied kernel entry of an object of an assembled datum D, $p \ge 5$ a prime of class zero or of signal class, and $\lambda\colon \overline{\mathbb Q} \to \mathbb C_p$ a field embedding. Then e_t satisfies the stable host condition at (p,λ).*

Proof. Outline. The class of p forces $\left(\frac{-d}{p}\right) = -1$, so Proposition A.28 makes e_t integral in formal-group coordinates at (p,λ); a good model with its formal logarithm, a Lubin–Tate law and a Frobenius lift give unitarily stable curve data at p; and applying the integrality to every integral combination of e_t and an orthonormal p-basis, then translating it into host-log integrality, gives the stable host condition. □

Lemma A.30. *Let F be the coefficient number field of a unitary moduli system in the setting of the integral-frame lemma for Hecke neighbours, with its fixed complex embedding. Let λ be a finite place of F, and let $\mathcal S_F$ be a space of algebraic sections with natural framed evaluations.*

Its theta-generator degree is the same even integer $n \geq 2$ as in the Hecke reading dictionary. A model frame at a point x consists of a semiabelian $\mathcal{O}_E$-model over a complete characteristic-zero discrete valuation ring O with finite residue field, coefficient embedding $F \hookrightarrow \mathrm{Frac}(O)$ inducing λ, a primitive theta generator nonzero at a special-fibre point, and separate integral bases of the generic eigenspace intersections in its Lie module. Evaluation in such a frame is in $\mathrm{Frac}(O)$.

Write $\mathcal{B}_\lambda \subseteq \mathcal{S}_F$ for the sections whose evaluation belongs to O in every such model frame, allowing every finite extension of the complete coefficient field. Fix a split prime $\mathfrak{P} \neq \bar{\mathfrak{P}}$ over an odd rational prime p different from the residue characteristic of λ, put $Q = N\mathfrak{P}$, and fix a type k and the prescribed Hecke representatives.

Assume the moduli datum and good level are those of the PEL Hecke quotient construction, and $p \nmid N$. The prescribed finite kernels at every model frame satisfy the explicit ideal bounds, isotropy and cardinality conditions of that construction. Take their quotient points, maps, primitive cubical comparisons and transported half levels from that construction, retaining the prescribed transversals. Their marked frames and complex readings satisfy the Hecke reading dictionary. Use on the quotient the canonical forward-trace theta generator constructed from the source model generator. Its quotient tangent frames are the prescribed $p^{-1}df$ of the source frames. An F-linear geometric operator $H : \mathcal{S}_F \to \mathcal{S}_F$ is defined by the finite trace in those frames:

$$\mathrm{ev}_x(Hs) = Q^{-kn} \sum_C \mathrm{ev}_{z_C}(s).$$

Then $H\mathcal{B}_\lambda \subseteq \mathcal{B}_\lambda$. On the same complex row readings its operator is

$$\Phi_{Hs}(h) = \mu_0(\mathfrak{P})^{-k} Q^{-kn/2} \sum_{a \in R_c} \Phi_s(ha).$$

In particular this normalised row operator preserves the image of $\mathcal{B}_\lambda$ under the prescribed reading map.

Proof. Outline. For $s \in \mathcal{B}_\lambda$ and a model frame, the PEL Hecke quotient construction and the forward theta trace identify each quotient frame in the trace with a model frame of the kind defining $\mathcal{B}_\lambda$ over a finite extension, so every $\mathrm{ev}_{z_C}(s)$ is integral there; since Q^{-kn} is a p-power unit, $\mathrm{ev}_x(Hs)$ is integral and lies in $\mathrm{Frac}(O)$, hence in O, and the Hecke reading dictionary with geometric scalar $Q^{-kn/2}$ gives the stated complex operator. □

Sketch: from integral kernels to served entries and gains. The loss data of (Rr1) need, at each inert prime, a lower bound for the valuation of certain kernel columns, the gain of Definition 5.3. The construction is expected to obtain it as follows. The kernel entries of a selected bundle are written in standard normalising series, after a hyperbolic substitution, and the host condition of Lemma A.29 turns integrality in Lubin–Tate coordinates into integral expansions along the Frobenius orbits of the inert primes. The minors of these expansions then have valuations bounded below by orbit sums, and these lower bounds are the gains that the linking inequality uses. The entries with these properties, together with structural, archimedean and jet conditions, are called served kernel entries, and a served selection is a selection whose kernel entries are served. Heuristically the mechanism is Alpöge's divisibility: the gain $\gamma_p = p/(p^2-1) = \sum_{e \text{ odd}} p^{-e}$ underlying Definition 5.3 has the shape of his exponent $a/p + a/p^3 + \cdots$. The delicate step is the comparison of the expansion of the host logarithm with the orbit expansion through one common normalising series: it must show that normalising the series costs no valuation at the inert primes.

A.4 The chain of theta lifts: Bessel periods, Rallis towers, the see-saw and multiplicity

This section constructs the chain of lifts. Along a row of a grid of hermitian spaces over a CM extension E/F, definite and indefinite unitary groups are linked by theta lifts: consecutive discrete series models along the row stand in the theta-pair relation (Definition A.31, Proposition A.32), and the row theta lifts of multiplicity maps converge and factor through a multiplicity map at the end of the row (Proposition A.33). Three properties of the chain are proved, and Alpöge names their ingredients: the see-saw, Bessel periods and Rallis [4]. First, the theta–Bessel functionals of a row separate the trivial eigenspace (Proposition A.38). Second, a linear relation among top tails gives the same relation among bottom periods, by see-saw transport (Proposition A.42). Third, rational period pairings equal spectral inner products up to unit phases (Proposition A.47). In all three, nonvanishing is propagated rather than computed: separation reduces to the nonvanishing of one archimedean Bessel form (Proposition A.35), the ascent carries a nonzero restriction period up to the top (Proposition A.37), and the see-saw carries relations down. The phases of the multiplicity identity have valuation 0 at every finite place, so the comparison loses nothing at any prime. All surjectivity of theta maps passes through one theorem, Theorem A.40, whose inductive step is Rallis' step, under an assumption of the shape of the Rallis inner product formula (Proposition A.41).

A.4.1 Theta pairs along a row

Definition A.31 fixes theta pairs of signed parameters and the theta-pair relation, in both directions and on both branches. Along a row of a grid, regular signed parameters give discrete series models whose neighbours satisfy this relation, with the attached characters (Proposition A.32). The row theta lift composes the lifts along the row, and Proposition A.33 shows that it converges and lands in a multiplicity map at the end of the row. This row of lifts is the chain of Alpöge's account, in our reading; its length is the number of lifts.

Definition A.31. (Xp1) Admissible quadruples. Let $t \geq 1$ be an integer and $b_t = (\frac{t-1}{2} - i)_{0 \leq i < t}$ (the centred block of length t). A quadruple $(\alpha, \beta, \gamma, \delta)$ of finite lists of rational numbers with $L = |\alpha| + |\beta| + |\gamma| + |\delta| \geq 1$ is **admissible for** t if, with $N = L + t$, $p = |\alpha| + |\beta|$ and $q = |\gamma| + |\delta|$:

(a) each of $\alpha, \beta, \gamma, \delta$ is strictly decreasing, and the L entries of $\alpha\beta\gamma\delta$ are pairwise distinct;

(b) the entries of α and γ are positive and those of β and δ are negative;

(c) every entry e of $\alpha\beta\gamma\delta$ lies in $\frac{t+1}{2} + \mathbb{Z}$ and satisfies $|e| \geq \frac{t+1}{2}$;

(d) $|x - y| > N^3$ and $|x - y| \geq 120\, L \min(p, q)^{3/2}$ for every entry x of $\alpha\beta$ and every entry y of $\gamma\delta$.

An admissible quadruple is **of type** $+$ if moreover $|x - y| > N^3$ for every entry x of $\alpha b_t \delta$ and every entry y of $\gamma\beta$; it is **of type** $-$ if moreover $|x - y| > N^3$ for every entry x of $\alpha\delta$ and every entry y of $\gamma b_t \beta$.

(Xp2) Negated signed sets; sign flip. For finite lists ℓ, ℓ' of rational numbers and $c \in \mathbb{Q}$ put

$$[\ell \,|\, \ell']_c = \{(-x - c, +) : x \text{ an entry of } \ell\} \cup \{(-x - c, -) : x \text{ an entry of } \ell'\}.$$

The **sign flip** of a signed parameter S is $S^{\mathrm{fl}} = \{(x, -\epsilon) : (x, \epsilon) \in S\}$.

(Xp3) Theta pairs. Let $t \geq 1$ be an integer, S, S' signed parameters and $c, c' \in \mathbb{Q}$. The pair (S, S') is a **theta pair of step t with exponents** (c, c') if there is a quadruple $(\alpha, \beta, \gamma, \delta)$ (a **witness**) with

$$S = [\alpha\beta \,|\, \gamma\delta]_c$$

and one of:

(+) $(\alpha, \beta, \gamma, \delta)$ is admissible for t of type $+$, and $S' = [\alpha b_t \delta \,|\, \gamma\beta]_{c'}$;

(−) $(\alpha, \beta, \gamma, \delta)$ is admissible for t of type $-$, and $S' = [\alpha\delta \,|\, \gamma b_t \beta]_{c'}$.

(Xp4) The theta-pair relation. Let S, S' be signed parameters and $a_S, a_T \in \mathbb{Q}$. The pair (S, S') satisfies the **theta-pair relation with exponents** (a_S, a_T) if there is an integer $t \geq 1$ such that one of:

(f) (forward) (S, S') is a theta pair of step t with exponents (a_T, a_S), and $|x + a_T| \geq \frac{t+3}{2}$ for every $(x, \epsilon) \in S$;

(b) (backward) (S', S) is a theta pair of step t with exponents (a_S, a_T).

Proposition A.32. *For every real ε and every $C_F \geq 6510$ there is x_0 such that the following holds. Let D be an assembled datum over a context with scale $\mathfrak{s} = (x, \varepsilon, J)$, $x \geq x_0$, with a size datum at $\mathfrak{s}$ satisfying the size-row conditions; let R be real data of D with the level choice (Wr6) at the base tuple, $\mathcal{J}$ a grid for R with grid constants $(1, C_F, b(\mathfrak{s}))$, and $\mathcal{W}$ a Weil datum of $\mathcal{J}$ lying over R, with family θ. Put $M_0 = M_{Y_0}$. Then there are discrete series models M_i of $V_{\mathrm{grid}(i,0)}$ for θ, $1 \leq i \leq m$, such that*

(a) $\sigma_{M_i}(\tau) = \sigma_{\mathrm{grid}(i,0)}(\tau)$ for $0 \leq i \leq m$ and $\tau \in \Phi$;

(b) for $0 \leq i < m$ and $\tau \in \Phi$ there are $a_S, a_T \in \mathbb{Q}$, infinity exponents at τ of the attached characters $\chi_S(p)$ and $\chi_T(p)$ of the pair $p = \mathrm{chi}(i, 0)$, such that the pair $(\Pi_\tau(M_{i+1})^{\mathrm{fl}}, \Pi_\tau(M_i))$ satisfies the theta-pair relation with exponents (a_S, a_T).

Proof. All auxiliary parameters below are local. Put $s = r + 6$, $n_i = r + is$, $q = m/2$ and $N_* = N_{\mathrm{gr}}$. The rank conditions give r odd, s odd, $r \geq 2m + 2$ and $q \leq n_i/4$. From (Jk3) and (R3), $k \geq f \geq m^4$ and $m \geq 44$, so in particular $r > 1800$. Only these numerical consequences and (Jk1) will be needed; the argument is uniform in the scale, so one may take $x_0 = 1$.

Fix τ and abbreviate the gap $g(D, 1)$ by g, the centres $\mathrm{cen}_D(1, \tau, j)$ by c_j, and the entries $v_{\tau,a}, b_{\tau,l}$ by v_a, b_l. Set $\epsilon_l = +$ for odd l and $\epsilon_l = -$ for even l. Define the following local signed sets:

$$P_i = \{(v_a, +) : 1 \leq a \leq k - 1\} \cup \{(-b_l, \epsilon_l) : 1 \leq l \leq m\} \cup \bigcup_{j=m-i+1}^{m} \{(z, +) : z \in \mathrm{bl}(c_j, s)\},$$

where the last union is empty for $i = 0$, let

$$d_i = \begin{cases} -\frac{1}{2} & i \text{ odd}, \\ 0 & i \text{ even}, \end{cases}$$

and put

$$S_i = \{(-z - d_i, \epsilon) : (z, \epsilon) \in P_i\}. \tag{1}$$

The first two parts of P_0 are exactly $\mathrm{SP}_0(\tau)$; thus $S_0 = \Pi_\tau(M_0)$ by (Xl2b)(c).

We check both regularity and the inequalities used below. By (Pa1) all cloud entries v_a are distinct and lie strictly above every $-b_l$. By (Jk1),

$$b_{l+1} - b_l \geq t + 2N_*^3 + 1 \geq s + 2N_*^3 > 2N_*^3, \tag{2}$$

because $t \geq k+m+4 = r+5 = s-1$. The centres are integers and $c_j - c_{j+1} = g$. Their defining floor gives $c_m \geq v_1 + \frac{3}{2}g - \frac{1}{2}$. Each block has length s and radius $(s-1)/2$, and

$$g \geq 2N_*^3 + 2s + 3. \tag{3}$$

Consequently the blocks are disjoint from each other and lie strictly above the cloud. All entries of P_i are therefore distinct integers. Its cardinality is n_i, and its negative signs are precisely those of the $m/2$ even-indexed lower entries.

Entries of opposite sign in P_i differ by more than $2N_*^3$. For two lower entries this is (2). A negative lower entry is $-b_l$ with even $l \geq 2$, whereas every cloud entry is greater than $-b_1$, giving a difference greater than $b_l - b_1 \geq b_2 - b_1$. The positive added blocks lie still higher. These cases exhaust the opposite-sign pairs.

Since s and r are odd, n_i is odd for even i and even for odd i. Thus (1) lies in the Harish-Chandra lattice $(n_i - 1)/2 + \mathbb{Z}$. Enumerate its entries in strictly decreasing order and give each entry its indicated sign. The resulting parameter is regular, has q negative signs and $n_i - q$ positive signs, exactly the signature of $V_{\mathrm{grid}(i,0)}$ in (Jg5),(Jg6). These two signature numbers are positive. By Harish-Chandra's existence theorem for the discrete series of $\mathrm{U}(p,q)$ with a given parameter, it has a discrete series representation.

For each $i \geq 1$ and each τ, use the frame $B_{\mathrm{grid}(i,0)}(\tau)$ from (Jw2)(a). Its sign multiplicities are the signature just computed. Reorder its columns so that their signs match the signed decreasing enumeration of S_i; call the resulting frame B_i'. Then

$$(B_i')^* J_\tau(V_{\mathrm{grid}(i,0)}) B_i' = J_{\epsilon_i}, \qquad \sigma_{\mathrm{grid}(i,0)} B_i' = B_i' J_{\epsilon_i}.$$

Transport the discrete series just obtained by $P_i^{\mathrm{mod}} = (B_i')^{-1}$ to the original hermitian space. Its model involution is

$$(P_i^{\mathrm{mod}})^{-1} J_{\epsilon_i} P_i^{\mathrm{mod}} = \sigma_{\mathrm{grid}(i,0)}.$$

Make these finitely many choices at all archimedean places and use the given family θ to form the models M_i. The equality in (a) at $i = 0$ is (Jw5), and at $i \geq 1$ is the displayed identity.

We now construct the theta-pair witness for every adjacent pair. Fix $i < m$, put $c = c_{m-i}$, $L = n_i$ and $N = L + s$. Then $N \leq N_*$. Every entry z of P_i satisfies

$$c - z \geq g - \tfrac{s-1}{2}. \tag{4}$$

For $i = 0$ this follows from $z \leq v_1$ and the bound on c_m; for $i > 0$, the largest possible entry is $c_{m-i+1} + (s-1)/2$ and the next centre is g higher. Let β be the decreasing list of $z - c$ for positive-sign entries of P_i, and δ the decreasing list of $z - c$ for negative-sign entries; put $\alpha = \gamma = \varnothing$. All these entries are negative integers. Equations (3),(4) give $|z - c| \geq (s+1)/2$, so they lie in $(s+1)/2 + \mathbb{Z}$ and meet the sign and absolute-value conditions of (Xp1).

Opposite-sign differences are greater than $2N_*^3 > N^3$. They also satisfy the second inequality of (Xp1)(d): since $q \leq L/4$ and $N > 1800$,

$$120L \min(L-q, q)^{3/2} = 120Lq^{3/2} \leq 15L^{5/2} \leq 15N^{5/2} < 2N^3 \leq 2N_*^3. \tag{5}$$

For the type $-$ condition, the only additional comparisons are between δ and the centred block b_s. By (4) their distances are at least $g - (s-1) > N_*^3 \geq N^3$. Hence $(\varnothing, \beta, \varnothing, \delta)$ is admissible for s of type $-$.

The attached-character exponents have the required exact values. By (Wr6)(a), $\eta = \eta^C$. The law (Fe4), continuity and density of the diagonal $E^\times$ in the archimedean units give infinity exponent $1/2$ for η at every τ, with the prescribed family θ. By (Jg4), χ_{m-i} has exponent c. The dimensions n_{i+1} and n_i have opposite parity, so the attached characters of $p = \mathrm{chi}(i,0)$ have exponents

$$a_S = c + d_i, \qquad a_T = c + d_{i+1}. \tag{6}$$

Indeed the odd-dimensional space contributes exactly the factor η^{-1}. These are half-integers, as required for infinity exponents.

Finally (1) and the definitions of the two lists give

$$S_i = [\,\beta \,|\, \delta\,]_{a_S}, \qquad S_{i+1}^{\mathrm{fl}} = [\,\delta \,|\, b_s\beta\,]_{a_T}. \tag{7}$$

For the old entries this follows by cancelling the centre c in (6); all old signs are reversed in the second expression. For the newly added block it follows from $\mathrm{bl}(c,s) = c + b_s$ and the invariance of b_s under negation. Thus (7) is branch $-$ of the theta-pair definition for $(S_i, S_{i+1}^{\mathrm{fl}})$ with exponents (a_S, a_T). The backward alternative of (Xp4) says exactly that $(S_{i+1}^{\mathrm{fl}}, S_i)$ satisfies the theta-pair relation with these exponents. This proves (b) for every i, τ, using the same models that establish (a). □

Proposition A.33. *Let D be an assembled master datum, R real data of D, $\mathcal{J}$ a grid for R, $\mathcal{W}$ a Weil datum of $\mathcal{J}$ with family θ, and $\mathcal{M}$ a row-normalised volume datum of $\mathcal{J}$. Let $M_0, \dots, M_m$ be discrete series models of $V_{\mathrm{grid}(0,0)}, \dots, V_{\mathrm{grid}(m,0)}$ for θ such that $\sigma_{M_i}(\tau) = \sigma_{\mathrm{grid}(i,0)}(\tau)$ for $0 \le i \le m$ and $\tau \in \Phi$, and such that for $0 \le i < m$ and $\tau \in \Phi$ there are infinity exponents a_S, a_T at τ of $\chi_S(\mathrm{chi}(i,0))$ and $\chi_T(\mathrm{chi}(i,0))$ for which $\big(\Pi_\tau(M_{i+1})^{\mathrm{fl}}, \Pi_\tau(M_i)\big)$ satisfies the theta-pair relation with exponents (a_S, a_T).*

Then there are, for every $0 \le n \le m$ and every $w \in M_0^\infty$, maps $\rho \mapsto \mathbf{v}_w^n(\rho)$ from the real vectors of $\mathcal{J}$ to M_n^∞ such that for every $T \in \mathrm{Mult}(M_0, \varnothing, 1)$, every coset datum Φ and every $0 \le n \le m$ there is $F_n \in \mathrm{Mult}(M_n, \varnothing, 1)$ with the following property: for every real vector ρ and every $w \in M_0^\infty$ the iterated lift defining the row part $\Theta_{\rho,\Phi}^{(n)}(T(w))$ converges and

$$\Theta_{\rho,\Phi}^{(n)}\big(T(w)\big) = F_n\big(\mathbf{v}_w^n(\rho)\big).$$

In particular ($n = m$) the row theta lift $\Theta_{\rho,\Phi}(T(w)) = \Theta_{\rho,\Phi}^{(m)}(T(w))$ converges and equals $F_m(\mathbf{v}_w^m(\rho))$.

Proof. Outline. The proof is an induction on n, starting from $\mathbf{v}_w^0(\rho) = w$ and $F_0 = T$: at the step from n to $n+1$ the pair $\mathrm{chi}(n,0)$, the models M_n and M_{n+1} (whose signed parameters satisfy the theta-pair relation by hypothesis), the frames of the Weil datum and its coset vectors, which are vacua, satisfy the hypotheses of the one-step lifting proposition for theta pairs, and that proposition supplies $\mathbf{v}_w^{n+1}(\rho)$ and F_{n+1} together with the integrability that gives the convergence and the value of the lift along $\mathrm{right}(n,0)$. □

A.4.2 Bessel periods and separation

Definition A.34 defines theta lifts along paths of the grid, the top Bessel pairing and the theta–Bessel functionals. Proposition A.35 factors these functionals through one archimedean Bessel form, which reduces separation to the nonvanishing of that form, and allows the quantifiers to be exchanged. The nonvanishing is supplied from below: restriction periods of row forms against K-finite core forms do not all vanish (Proposition A.36), and a nonzero restriction

period gives a nonzero top Bessel pairing (Proposition A.37). Together these give separation (Proposition A.38).

Definition A.34. Let $\mathcal{J}$ be a grid for R, $\mathcal{W}$ a Weil datum of $\mathcal{J}$ and $\mathcal{M}$ a volume datum of $\mathcal{J}$. Functions on G_{V_X} are called functions at the node X.

(Jf1) Theta lift along an arrow. Let $a\colon X \to Y$ be an arrow with pair p, f a Schwartz function of a and F a function at X. The **theta lift** $\mathcal{L}_a(f)F$ is the function at Y given by the downward and upward theta lifts (Zw13) of a see-saw tower for the Weil action ω_p on (S_p, T_p), formed with the measure ν_X and the set $\mathcal{F}_X$:

$$\mathcal{L}_a(f)F = \begin{cases} \Lambda^{\downarrow}_{\omega_p}(F; f) & a = \mathrm{right}(i,j) \text{ or } \mathrm{down}(t,i,j) \ (X \text{ the node of } T_p), \\ \Lambda^{\uparrow}_{\omega_p}(F; f) & a = \mathrm{coreDown}(i) \text{ or } \mathrm{col}(t,j) \ (X \text{ the node of } S_p), \\ \overline{\Lambda^{\downarrow}_{\omega_p}(\overline{F}; f)} & a = \mathrm{coreUp}(i), \end{cases}$$

and the lift **converges** at (f, F) if the lift of (Zw13) on the right converges (at $(\overline{F}, f)$ for $\mathrm{coreUp}(i)$).

(Jf2) Paths and iterated lifts. A **path** from X to Y is a finite sequence of arrows $a_1\colon X = X_0 \to X_1, \dots, a_n\colon X_{n-1} \to X_n = Y$; a **pure datum** on it is a family $(f_1, \dots, f_n)$ of Schwartz functions of $a_1, \dots, a_n$. The **iterated lift** of F along a path with pure datum is $\mathcal{L}_{a_n}(f_n) \cdots \mathcal{L}_{a_1}(f_1)F$; it **converges** if each lift $\mathcal{L}_{a_l}(f_l)$ converges at its argument $\mathcal{L}_{a_{l-1}}(f_{l-1}) \cdots \mathcal{L}_{a_1}(f_1)F$. A **total datum** is a finitely supported complex function c on the pure data of the path; its lift is $\sum_d c(d) \cdot$(iterated lift with d) and converges if each lift with d in the support converges. The **row path** is $\mathrm{right}(0,0), \dots, \mathrm{right}(m-1,0)$ from $\mathrm{grid}(0,0)$ to $\mathrm{grid}(m,0)$; the **down path** at (t,i) of length n is $\mathrm{down}(t,i,n-1), \dots, \mathrm{down}(t,i,0)$ from $\mathrm{grid}(i,n)$ to $\mathrm{grid}(i,0)$.

(Jf3) Row theta lift. For a real vector ρ, a coset datum Φ and a function u at $\mathrm{grid}(0,0)$ (that is, on $\mathrm{U}(Y_0)(\mathbb{A}_F)$), the **row theta lift** $\Theta_{\rho,\Phi}(u)$ is the iterated lift of u along the row path with the pure datum $(d_0(\rho,\Phi), \dots, d_{m-1}(\rho,\Phi))$, a function at $\mathrm{grid}(m,0)$; the row theta lift **converges** if this iterated lift converges.

(Jf4) Top Bessel pairing. The **top embedding** $s\colon G_{V_+} \to G_{V_{\mathrm{grid}(m,0)}}$ sends h to the element acting as $\iota_{\mathrm{top}} h \iota_{\mathrm{top}}^{-1}$ on $\iota_{\mathrm{top}}(V_+) \otimes \mathbb{A}$ and as the identity on its orthogonal complement (the element related to h, (Bd4)). For a function φ at $\mathrm{top}(0)$ and a function F at $\mathrm{grid}(m,0)$ the **top Bessel pairing** is the Bessel period

$$\mathcal{B}(\varphi, F) = \mathcal{P}_{d_{\mathrm{top}}}(\varphi, F) = \int_{\mathcal{F}_{\mathrm{top}(0)}} \int_{\mathcal{F}_N} \varphi(h)\, F(n\, s(h))\, \overline{\psi_{d_{\mathrm{top}}}(n)}\, d\nu_N(n)\, d\nu_{\mathrm{top}(0)}(h)$$

of (Bd3) with $(\mu, \mathcal{F}) = (\nu_{\mathrm{top}(0)}, \mathcal{F}_{\mathrm{top}(0)})$, the pair $(\nu_N, \mathcal{F}_N)$, the map s and the character ψ of $\mathcal{W}$; it **converges** at (φ, F) if $n \mapsto \varphi(h) F(n\, s(h)) \overline{\psi_{d_{\mathrm{top}}}(n)}$ is ν_N-integrable on $\mathcal{F}_N$ for every h, and the resulting inner integral is $\nu_{\mathrm{top}(0)}$-integrable on $\mathcal{F}_{\mathrm{top}(0)}$. (This is weaker than the convergence of (Bd3).)

(Jf5) Theta-Bessel functionals. For a real vector ρ, $\xi \in W_+^*$, a column form Ψ, a coset datum Φ and a function u at $\mathrm{grid}(0,0)$, the **theta-Bessel functional** is

$$\ell_{\rho,\xi,\Psi,\Phi}(u) = \mathcal{B}(\varphi_{\xi,\Psi}, \Theta_{\rho,\Phi}(u)),$$

and it **converges** at u if the row theta lift $\Theta_{\rho,\Phi}(u)$ converges and $\mathcal{B}$ converges at $(\varphi_{\xi,\Psi}, \Theta_{\rho,\Phi}(u))$.

(Jf6) Row forms and separating vectors. Let $\mathcal{H}^\infty = \bigotimes_{\tau\in\Phi} \mathcal{H}_\tau^\infty$ (algebraic tensor product) for the model M_{Y_0}, and $E_{\mathrm{triv}} = \mathrm{M}_R^{\mathrm{good}}(P_1)$, the space of realised forms of R at the trivial level. For $w \in \mathcal{H}^\infty$ and a multiplicity map T (multilinear on $\prod_\tau \mathcal{H}_\tau^\infty$) the **row form** $T(w)$ is the function at $\mathrm{grid}(0,0)$ given by the linear extension of T to $\mathcal{H}^\infty$. A pair (ρ,ξ) of a real vector and $\xi \in W_+^*$ **separates** w (for $\mathcal{J},\mathcal{W},\mathcal{M}$) if for every $T \in E_{\mathrm{triv}}$ with $T \neq 0$ there are a column form Ψ and a coset datum Φ such that $\ell_{\rho,\xi,\Psi,\Phi}$ converges at $T(w)$ and $\ell_{\rho,\xi,\Psi,\Phi}(T(w)) \neq 0$. It **separates** w **in span** if for every such T there are a column form Ψ and a finitely supported function c on the coset data such that $\ell_{\rho,\xi,\Psi,\Phi}$ converges at $T(w)$ for every Φ in the support of c and $\sum_\Phi c(\Phi)\,\ell_{\rho,\xi,\Psi,\Phi}(T(w)) \neq 0$.

Proposition A.35. *Let $\mathcal{J}$ be a grid for real data R, $\mathcal{W}$ a Weil datum of $\mathcal{J}$ with family θ, $\mathcal{M}$ a row-normalised volume datum of $\mathcal{J}$, and $w \in \mathcal{H}^\infty$. Assume:*

(Sw1) there are a discrete series model M_B of $V_{\mathrm{grid}(m,0)}$ for θ, with smooth vectors $\mathcal{H}_\tau^\infty(M_B)$, and a map $\rho \mapsto \mathbf{v}(\rho)$ from the real vectors of $\mathcal{J}$ to $\bigotimes_\tau \mathcal{H}_\tau^\infty(M_B)$ such that for every $T \in E_{\mathrm{triv}}$ and every coset datum Φ there is $F' \in \mathrm{Mult}(M_B,\varnothing,1)$ with $\Theta_{\rho,\Phi}(T(w)) = F'(\mathbf{v}(\rho))$ for every real vector ρ (F' extended linearly to the tensor product);

(Sw2) $\ell_{\rho,\xi,\Psi,\Phi}$ converges at $T(w)$ for all $T \in E_{\mathrm{triv}}$, real vectors ρ, $\xi \in W_+^*$, column forms Ψ and coset data Φ;

(Sw3) for every $T \in E_{\mathrm{triv}}$ with $T \neq 0$ there are ρ,ξ,Ψ,Φ with $\ell_{\rho,\xi,\Psi,\Phi}(T(w)) \neq 0$.

Then there are a real vector ρ_0 and $\xi_0 \in W_+^*$ such that (ρ_0,ξ_0) separates w for $(\mathcal{J},\mathcal{W},\mathcal{M})$. More precisely, under (Sw1) alone there is a function a of pairs (ρ,ξ) such that for all $T \in E_{\mathrm{triv}}$, column forms Ψ and coset data Φ there is $l \in \mathbb{C}$ with

$$\ell_{\rho,\xi,\Psi,\Phi}(T(w)) = a(\rho,\xi)\,l \qquad \text{for all real vectors } \rho \text{ and all } \xi \in W_+^*.$$

Proof. Sketch. The proposition is an exchange of quantifiers. Condition (Sw3) gives, for each nonzero $T \in E_{\mathrm{triv}}$, some pair (ρ,ξ) at which a theta–Bessel functional of $T(w)$ is nonzero; the conclusion asks for one pair (ρ_0,ξ_0) that works for every T at once. This follows from the factorisation $\ell_{\rho,\xi,\Psi,\Phi}(T(w)) = a(\rho,\xi)\,l$, in which a depends only on (ρ,ξ) and l only on (T,Ψ,Φ): one nonzero value for a fixed T_0 gives $a(\rho_0,\xi_0) \neq 0$, every nonzero T has $l \neq 0$ for suitable (Ψ,Φ), and the product is then nonzero.

The factorisation is the core of the proof. By (Sw1) the theta lift of $T(w)$ is a multiplicity map F' evaluated at a vector $\mathbf{v}(\rho)$ of one discrete series model, so the functional is the top Bessel pairing $\beta(\xi,v)$ of the scalar form $\varphi_{\xi,\Psi}$ with $F'(v)$. As a function of its components (ξ_τ,v_τ) at one real place τ, the others fixed, β is continuous, invariant under the definite unitary group $\mathrm{U}(V_+)(F_\tau)$ embedded as the stabiliser H_z, and transforms by the nondegenerate character ψ_d under the unipotent group N_d. These properties come from unimodularity, the compactness of $\Gamma_{V_+}\backslash G_{V_+}$ (V_+ being totally definite) and of $(N_d \cap \Gamma_B)\backslash N_d$, and the Banach–Steinhaus theorem; so β is a local archimedean Bessel form for an irreducible finite-dimensional representation of H_z and an irreducible unitary discrete series of $\mathrm{U}(B)(F_\tau)$ (Definition A.81). By the uniqueness of archimedean Bessel models [139] such forms span a space of dimension at most one at each place, so $\beta = l\,\Lambda$, where Λ is the bilinear form equal to $\prod_\tau b_\tau(\xi_\tau,v_\tau)$ on pure tensors for fixed local forms b_τ, and $a(\rho,\xi) = \Lambda(\xi,\mathbf{v}(\rho))$. Thus the dependence on the pair is purely archimedean, and separation reduces to the nonvanishing of one archimedean form; this is how Proposition A.38 uses the proposition. □

Proposition A.36. *For every real ε and every $C_F \geq 6510$ there is x_0 such that the following holds. Let D be an assembled master datum with size data of scale $x \geq x_0$ and parameter ε satisfying the size-row conditions; let R be real data of D with a level choice at the base tuple; let $\mathcal{J}$ be a grid for R with grid constants $(1, C_F, b)$, b the constant of the size data; let $\mathcal{W}$ be a Weil datum of $\mathcal{J}$ with infinite-type family θ and $\mathcal{M}$ a row-normalised volume datum of $\mathcal{J}$. Let M_0 be a discrete series model on Y_0 with the parameters of M_{Y_0}, and M_P a discrete series model on $V_{\mathrm{core}(0)}$ with the discrete-series parameter of $\mathrm{core}(0)$. Then for every nonzero row form T of level $g_1 g_2$ for (θ, M_0) and every nonzero $w \in \bigotimes_\tau \mathcal{H}_\tau^\infty(M_0)$ there are a core form $y \in \mathrm{CF}_\theta(\mathrm{core}(0), M_P)$ and $v \in \mathrm{KF}(M_P)$ with*

$$\mathrm{per}_0\big(y(v), T(w)\big) \neq 0.$$

Proof. Outline. A pack of controlled loss for the extension of Y_0 into $V_{\mathrm{core}(0)}$ and the depth estimate of the grid give a scale s; for a suitable vector w_0 the function $u = T(w_0)$ is bounded, nonzero and Lipschitz (by the loss clauses of the pack), so a short-vector estimate for cusp forms shows that u is large at an s-long point x, the identity term then dominates the pairing of u with a Poincaré series at x, this Poincaré series is the value of a core form, and a transfer lemma for K-finite vectors passes from w_0 to every nonzero w. □

Proposition A.37. *Let D be an assembled master datum whose type primes $p_{\mathrm{de}}, p'_{\mathrm{de}}$ are both prime to $Q_S q$ (the integers Q_S and q of the host of D), let R be real data of D with the level choice (Wr6) at the base tuple, $\mathcal{J}$ a grid for R (any grid constants), $\mathcal{W}$ a Weil datum of $\mathcal{J}$ and $\mathcal{M}$ a row-normalised volume datum of $\mathcal{J}$. Let v be a core volume, $\Psi = (\Psi_i)_{0 \leq i \leq m}$ a column chain for v, and u a row-rapid function at $\mathrm{grid}(0,0)$ with*

$$\mathrm{per}_0(\Psi_0, u) \neq 0.$$

Then there are a real vector ρ and a coset datum Φ of $\mathcal{J}$ with

$$\mathcal{B}\big(\Psi_m, \Theta_{\rho,\Phi}(u)\big) \neq 0.$$

Proof. By the existence theorem for good rung decomposition systems fitting the ends, whose hypotheses are those of the statement, there are a rung decomposition system $\mathfrak{r}$ of $\mathcal{J}$ that fits the ends for $(v, \mathcal{W}, \mathcal{M})$ and line lattices Γ^L such that $\mathfrak{r}$ is good for $(v, \mathcal{W}, \Gamma^L)$. Form the conditions NZB_k and $\mathrm{NZJ}_k(\Gamma^L)$ of (Ay7) for $(\mathcal{W}, \mathcal{M}, v, \mathfrak{r}, \Psi, u)$; the hypotheses of the rung proposition for good rung decomposition systems (with its three parts (a), (b), (c) used below) hold for these data.

NZB_k **holds for** $0 \leq k \leq r-1$**.** We argue by induction on k. For $k = 0$: since $\mathrm{per}_0(\Psi_0, u) \neq 0$, part (a) of the rung proposition gives $\mathrm{NZJ}_0(\Gamma^L)$, and part (b) with $k = 0$ (allowed, as $r \geq 1$) gives NZB_0. Let $0 \leq k \leq r-2$ and assume NZB_k. Part (c) gives $\mathrm{NZJ}_{k+1}(\Gamma^L)$, and part (b) with $k+1 \leq r-1$ gives NZB_{k+1}.

The top rung gives the top Bessel pairing. Take $k = r-1$, so $2k+2 = m$. By NZB_{r-1} there are a real vector ρ and a coset datum Φ with

$$\mathcal{P}^B_{r-1}\big(\overline{\Psi_m}, \overline{\Theta^{(m)}_{\rho,\Phi}(u)}\big) \neq 0.$$

The function Ψ_m at $\mathrm{core}(m) = \mathrm{top}(0)$ has rapid decay because Ψ is a column chain (Jc4), and $\Theta^{(m)}_{\rho,\Phi}(u)$ at $\mathrm{grid}(m,0)$ has rapid decay because u is row-rapid (Jc5). Since $\mathfrak{r}$ fits the ends, the

second clause of (Ay6) applied to $\varphi = \Psi_m$ and $F = \Theta^{(m)}_{\rho,\Phi}(u)$ gives $\mathcal{B}(\Psi_m, \Theta^{(m)}_{\rho,\Phi}(u)) \neq 0$. Finally the row part $\Theta^{(m)}_{\rho,\Phi}(u)$ is the iterated lift of u along $\mathrm{right}(0,0), \dots, \mathrm{right}(m-1,0)$ with the data $d_0(\rho,\Phi), \dots, d_{m-1}(\rho,\Phi)$ (Jc5), that is, along the row path with the row datum, which is the row theta lift $\Theta_{\rho,\Phi}(u)$ of (Jf3). Hence $\mathcal{B}(\Psi_m, \Theta_{\rho,\Phi}(u)) \neq 0$. □

Proposition A.38. *For every real ε and every $C_F \geq 6510$ there is x_0 such that the following holds. Let D be an assembled datum over a context with scale $\mathfrak{s} = (x, \varepsilon, J)$ with $x \geq x_0$, and suppose that D has a size datum at $\mathfrak{s}$ satisfying the size-row conditions. Let R be real data of D with the level choice (Wr6) at the base tuple, $\mathcal{J}$ a grid for R with grid constants $(1, C_F, b(\mathfrak{s}))$, $\mathcal{W}$ a Weil datum of $\mathcal{J}$ lying over R, and $\mathcal{M}$ a row-normalised volume datum of $\mathcal{J}$. Then for every nonzero $w \in \mathcal{H}^\infty$ there are a real vector ρ of $\mathcal{J}$ and $\xi \in W_+^*$ (W_+ for the family of $\mathcal{W}$) such that (ρ, ξ) separates w for $(\mathcal{J}, \mathcal{W}, \mathcal{M})$.*

Proof. **Choice of x_0.** Given ε and $C_F \geq 6510$, let x_1, x_2, x_3 be the thresholds for (ε, C_F) of Proposition A.36, of the row-service lemma for the row chain (whose clauses (Sv1)–(Sv5) are used below) and of the lemma that writes K-finite core forms as combinations of reachable functions, and $x_0 = \max(x_1, x_2, x_3)$. Fix data $D, R, \mathcal{J}, \mathcal{W}, \mathcal{M}$ as in the statement with $x \geq x_0$, let $\theta = \vartheta_\infty$ be the family of $\mathcal{W}$ ((Jw5)), and fix $w \in \mathcal{H}^\infty$, $w \neq 0$.

Step 1: the type primes are prime to $Q_S q$. By (Cx1), $p_\delta, p_{\delta'} \equiv 1 \pmod{N_\mathfrak{h}}$ with $N_\mathfrak{h} = Q_S q d$. A prime ℓ dividing both p_δ and $Q_S q$ would be p_δ itself and divide $N_\mathfrak{h}$, contradicting $p_\delta \equiv 1 \pmod{N_\mathfrak{h}}$. Likewise for $p_{\delta'}$. So the hypothesis on the type primes in Proposition A.37 holds.

Step 2: weak separation at w. Let $T \in E_{\mathrm{triv}}$, $T \neq 0$, and put $u = T(w)$. By (Sv2) of the row-service lemma, $w' \mapsto T(w')$ is a nonzero row form of level $g_1 g_2$ for (θ, M_{Y_0}). By (Sv1) there is a discrete series model M_P of $V_{\mathrm{core}(0)}$ with the core-zero parameter $\Pi_D^{\mathrm{core}}(1,\tau)$ at every τ and with the involutions and frames of $\mathcal{W}$ at core(0). Applying Proposition A.36 with $M_0 = M_{Y_0}$ and this M_P (its discrete-series parameter of core(0) is $\Pi_D^{\mathrm{core}}(1,\tau)$, the grid constants being $(1, C_F, b(\mathfrak{s}))$) gives $y \in \mathrm{CF}_\theta(\mathrm{core}(0), M_P)$ and $u' \in \mathrm{KF}(M_P)$ with

$$\mathrm{per}_0(y(u'), u) \neq 0.$$

A core volume v exists: every $\mathrm{U}(V_{\mathrm{core}(i)})(\mathbb{A}_F)$ is a second countable locally compact group in which the rational points form a discrete subgroup, so it carries a Haar measure and a measurable fundamental domain. By the lemma on K-finite core forms, $y(u') = \sum_{j=1}^{J} c_j F_j$ with $c_j \in \mathbb{C}$ and F_j reachable for v. Each F_j is the first member of a column chain, so it has rapid decay and is bounded (ht ≥ 1); by (Sv3), u is row-rapid, so u, its row part of length 0, has rapid decay and is bounded; and $\nu_{\mathrm{grid}(0,0)}(\mathcal{F}_{\mathrm{grid}(0,0)}) < \infty$, the Haar pair of the row-normalised datum being one of the unitary group $\mathrm{U}(Y_0)$, whose quotient by the rational points has finite volume (its centre is anisotropic). Hence every $h \mapsto F_j(s_0(h))u(h)$ is integrable on $\mathcal{F}_{\mathrm{grid}(0,0)}$, $\mathrm{per}_0(\cdot, u)$ is linear on these functions, and

$$0 \neq \mathrm{per}_0(y(u'), u) = \sum_j c_j \, \mathrm{per}_0(F_j, u),$$

so $\mathrm{per}_0(F_j, u) \neq 0$ for some j. By (Jc4) there are a column chain $\Psi = (\Psi_i)_{0 \leq i \leq m}$ for v with $\Psi_0 = F_j$, a column form Ψ^{c} and $\xi \in W_+^*$ with $\Psi_m = \varphi_{\xi,\Psi^{\mathrm{c}}}$. By Step 1 and Proposition A.37 (any grid constants), applied to v, Ψ and the row-rapid u, there are a real vector ρ and a coset datum Φ with

$$\ell_{\rho,\xi,\Psi^{\mathrm{c}},\Phi}(T(w)) = \mathcal{B}(\varphi_{\xi,\Psi^{\mathrm{c}}}, \Theta_{\rho,\Phi}(u)) \neq 0.$$

This is (Sw3) of Proposition A.35 for w.

Step 3: the remaining hypotheses of the exchange. (Sw2): every $T \in E_{\mathrm{triv}}$ gives a row form of level $g_1 g_2$ (Sv2), so (Sv4) gives the convergence of all $\ell_{\rho,\xi,\Psi,\Phi}$ at $T(w)$. (Sw1): a row form of level $g_1 g_2$ comes from a multiplicity map of M_{Y_0} of level $K_{\mathrm{grid}(0,0)}(g_1 g_2 \mathcal{O}_E)$ with trivial character, which satisfies the conditions of (Hh4) for the empty level set; so every $T \in E_{\mathrm{triv}}$ lies in $\mathrm{Mult}(M_{Y_0}, \varnothing, 1)$ through $T(w')$, and (Sv5) with $\mathbf{v} = \mathbf{v}_w$ gives (Sw1).

Step 4: conclusion. By Proposition A.35 (with $\mathcal{M}$ row-normalised) there is (ρ_0, ξ_0) separating w for $(\mathcal{J}, \mathcal{W}, \mathcal{M})$. □

A.4.3 Rallis towers and the see-saw

Surjectivity of the downward theta maps is organised in Rallis towers (Definition A.39): Hecke parameters at the stages, downward theta maps between them, Petersson structures with eigen-elements, an archimedean step relation, and a condition on the L-values along the chain. Every composite of downward theta maps of a well-formed tower with these two conditions is surjective (Theorem A.40). The inductive step is Rallis' step (Proposition A.41): the adjoint of a downward theta map sends an eigen-element with nonvanishing L-value to an eigen-element with the stepped parameter. The step assumes an identity $\mathrm{RIP}(D)$ of the shape of the Rallis inner product formula [203, 152], and this assumption is carried by Theorem A.40. In the other direction, see-saw transport carries linear relations from the top of the chain to its bottom (Proposition A.42).

Definition A.39. Let D be a master datum with CM extension E/F and CM type Φ.

(Xo1) Hecke parameters. Let S be a finite set of finite places of F and S_E the set of places of E above S. A **parameter family outside** S is a map B assigning to every finite place $w \notin S_E$ of E a finite multiset $B(w)$ of nonzero complex numbers; B is **bounded by** (n, κ) if $|B(w)| \le n$ and $|b| \le \mathrm{N}(w)^{\kappa}$ for every w and every $b \in B(w)$. Let V be a hermitian space with a self-dual lattice Λ and, for a finite place v of F, $K_v \subseteq \mathrm{U}(V)(F_v)$ the stabiliser of $\Lambda \otimes \mathcal{O}_{F_v}$. A function $\phi\colon G_V \to \mathbb{C}$ has **Hecke parameter** B **outside** S (for Λ) if for every finite place $v \notin S$ the function $\overline{\phi}$ is right K_v-invariant and is an eigenfunction of the spherical Hecke algebra $\mathcal{H}(\mathrm{U}(V)(F_v), K_v)$ acting by right convolution, whose eigencharacter has, at every place w of E above v, the Satake parameter $B(w)$ (normalised induction, read after base change to $\mathrm{GL}_{\dim V}(E_w)$). For a Hecke character χ of E unramified at w, $\chi(w)$ is its value at a uniformiser of E_w. For Hecke characters χ_S, χ_T unramified outside S_E and an integer $t \ge 1$ the **stepped parameter** is the parameter family

$$\mathrm{st}_{\chi_S,\chi_T,t}(B)(w) = \{\chi_S(w)\chi_T(w)^{-1} b :\ b \in B(w)\} + \{\chi_S(w)\,\mathrm{N}(w)^{\frac{t-1}{2}-i} :\ 0 \le i < t\} \quad \text{(A.1)}$$

(sum of multisets), and the **partial** L**-function** is $L^S(z, B \times \chi) = \prod_{w \notin S_E} \prod_{b \in B(w)} (1 - b\,\chi(w)\,\mathrm{N}(w)^{-z})^{-1}$ wherever the product converges absolutely.

(Xo2) Rallis tower data. A **Rallis tower datum T** over D consists of integers $n \ge 1$ (the **base rank**), $t \ge 1$ (the **step**), $m \ge 0$ (the **length**), a finite set S of finite places of F, an archimedean family θ, an additive character ψ of $\mathbb{A}_F$, Hecke characters $\chi_S(k), \chi_T(k)$ of E for $0 \le k < m$ (the **step characters**), and:

- for $0 \le j \le m$ a **stage** $s_j = (V_j, M_j, \Lambda_j, J_j, \kappa_j, \mu_j, \mathcal{F}_j)$: a hermitian space V_j of dimension $n + jt$, a discrete series model M_j of V_j, a self-dual lattice Λ_j of V_j, a subgroup $J_j \subseteq G_{V_j,f}$ (the **level**) with a map $\kappa_j\colon J_j \to \mathbb{C}^{\times}$ (the **level character**), and a Haar pair $(\mu_j, \mathcal{F}_j)$ of V_j; its **space of forms** is $\mathcal{C}_j = \mathrm{Mult}(M_j, J_j, \kappa_j)$;

- for $0 \le k < m$ a **step** $p_k = (\omega_k, f_k, \mathrm{It}_k, \mathrm{io}_k)$: a Weil action ω_k on (V_k, V_{k+1}), a function $f_k\colon \mathbb{W}^\circ(V_k, V_{k+1}) \to \mathbb{C}$, a linear map $\mathrm{It}_k\colon \mathcal{P}(V_k, V_{k+1}) \to \mathrm{Hom}_{\mathbb{C}}(M_{k+1}^\infty, M_k^\infty)$ (the **lifting operator**) and a linear map $\mathrm{io}_k\colon \mathcal{C}_{k+1} \to \mathcal{C}_k$ (the **downward theta map**).

For $P \in \mathcal{P}(V_k, V_{k+1})$ write $F_{k,P} = F_P(f_k)$ for the Fock vector of (Wo7) formed at every τ with the pair setup of the models: $B_S = f_{k,\tau}^{-1}$, $\sigma_S = f_{k,\tau}^{-1} J_{\varepsilon_{k,\tau}} f_{k,\tau}$ from M_k and B_T, σ_T likewise from M_{k+1}. The **composites** are $\Theta_0 = \mathrm{id}_{\mathcal{C}_0}$ and $\Theta_{k+1} = \Theta_k \circ \mathrm{io}_k\colon \mathcal{C}_{k+1} \to \mathcal{C}_0$.

(Xo3) Well-formed Rallis tower data. **T** is **well formed** if:

(a) ψ is standard of sign -1; $\chi_S(k)$ and $\chi_T(k)$ are unramified at every finite place of E outside S_E;

(b) (stages) for every j: the Witt index of V_j is less than t; there is an open subgroup $U \subseteq G_{V_j}$ with $U \cap G_{V_j,f} \subseteq J_j$ and $\kappa_j = 1$ on $U \cap G_{V_j,f}$; every element of J_j stabilises $\Lambda_j \otimes \mathcal{O}_{F_v}$ at every finite place v; every $g \in G_{V_j,f}$ with g_v in the stabiliser of $\Lambda_j \otimes \mathcal{O}_{F_v}$ for $v \notin S$ and $g_v = 1$ for $v \in S$ lies in J_j and has $\kappa_j(g) = 1$; κ_j is multiplicative and $|\kappa_j| = 1$;

(c) (Weil data) for every k, ω_k is an oriented Weil datum for $(\psi, V_k, V_{k+1}, \chi_S(k), \chi_T(k))$;

(d) (vacua) f_k is a nonzero oriented theta function with $a_{\tau,z} f_k = 0$ for every τ and z (annihilation operators formed with σ_S, σ_T as above), and f_k is a lattice vacuum for $\Lambda^\circ_{\{v\}}(\Lambda_k, \Lambda_{k+1})$ at every finite place $v \notin S$;

(e) (level law) $\omega_k(g,h) F_{k,P} = \kappa_k(g)\,\kappa_{k+1}(h)^{-1} F_{k,P}$ for all P, $g \in J_k$, $h \in J_{k+1}$;

(f) (coefficient support) for every covolume-one pair (ν, X) of $\mathbb{W}^\circ(V_k, V_{k+1})$, every P and every $g \in G_{V_k}$ such that g_v stabilises $\Lambda_k \otimes \mathcal{O}_{F_v}$ for all finite $v \notin S$ and $g \notin G_{V_k,\infty} J_k$, the **Weil coefficient** $\int_X (\omega_k(g,1)F_{k,P})(x)\,\overline{F_{k,P}(x)}\,d\nu(x)$ vanishes;

(g) (lifting operator) $\mathrm{It}_k \neq 0$, and for every τ: if g lies in the τ-component of $G_{V_k,\infty}$ and $F_{k,P'} = \omega_k(g,1)F_{k,P}$, then $\mathrm{It}_k(P') = \sigma_{k,\tau}(g) \circ \mathrm{It}_k(P)$ ($\sigma_{k,\tau}(g)$ acting on the τ-factor of M_k^∞ through θ_τ and $f_{k,\tau}$); if h lies in the τ-component of $G_{V_{k+1},\infty}$ and $F_{k,P'} = \omega_k(1,h)F_{k,P}$, then $\mathrm{It}_k(P') \circ \sigma_{k+1,\tau}(h) = \mathrm{It}_k(P)$; the same two rules hold infinitesimally (for X in the Lie algebras, with $F_{k,P'}$ the derivative of $r \mapsto \omega_k(\exp rX, 1)F_{k,P}$ at 0 and $\mathrm{It}_k(P') = X \circ \mathrm{It}_k(P)$, respectively of $r \mapsto \omega_k(1, \exp rX)F_{k,P}$ and $\mathrm{It}_k(P') + \mathrm{It}_k(P) \circ X = 0$); and for every P and every vector w of the algebraic tensor product of the $H_{k,\tau}$ there is $c \ge 0$ with $|\langle \mathrm{It}_k(P)(\otimes v), w\rangle| \le c \prod_\tau \|v_\tau\|$ for every vector v of M_{k+1};

(h) (downward theta map) for all $G \in \mathcal{C}_{k+1}$, P, $v \in M_{k+1}^\infty$ and $g \in G_{V_k}$, the function $h \mapsto (\omega_k(g,h)F_{k,P})(0)\,G(v)(h)$ is μ_{k+1}-integrable on $\mathcal{F}_{k+1}$, and

$$\Lambda^{\downarrow}_{\omega_k}\big(G(v); F_{k,P}\big)(g) = \mathrm{io}_k(G)\big(\mathrm{It}_k(P)v\big)(g) \tag{A.2}$$

with the Haar pair $(\mu_{k+1}, \mathcal{F}_{k+1})$ in (Zw13).

(Xo4) Petersson structures, eigen-elements, chain parameters. The **tensor inner product** on M_j^∞ is the hermitian inner product, linear in the first variable, with $\langle \otimes v, \otimes v'\rangle = \prod_\tau \langle v'_\tau, v_\tau\rangle_{H_{j,\tau}}$, the inner products of the $H_{j,\tau}$ being conjugate-linear in the first argument as in (Hh4). A **Petersson structure** on $\mathcal{C}_j$ is a positive definite hermitian form $\langle\cdot,\cdot\rangle_j$ on $\mathcal{C}_j$, linear in the first variable, such that for all $x, y \in \mathcal{C}_j$ and $v, v' \in M_j^\infty$ the function $x(v)\overline{y(v')}$ is μ_j-integrable on $\mathcal{F}_j$ and

$$\int_{\mathcal{F}_j} x(v)\,\overline{y(v')}\,d\mu_j = \langle x, y\rangle_j\,\langle v, v'\rangle. \tag{A.3}$$

An **eigen-element with parameter** B is a nonzero $u \in \mathcal{C}_j$ such that every nonzero value $u(v)$ ($v \in M_j^\infty$) has Hecke parameter B outside S for Λ_j. The **chain parameters** of a parameter family B outside S are $B_0 = B$ and $B_{k+1} = \mathrm{st}_{\chi_S(k),\chi_T(k),t}(B_k)$, the stepped parameter of (Xo1).

(Xo5) The archimedean step relation. The **signed parameter** of a discrete series model M at τ is $\Pi_\tau(M) = \{(\lambda_{\tau,i}, \varepsilon_{\tau,i})\}_i \subseteq \mathbb{R} \times \{+,-\}$ ($+$ for the entries 1 of J_ε). The step k of $\mathbf{T}$ satisfies the **archimedean step relation** if for every $\tau \in \Phi$, with a_S, a_T the infinity exponents of $\chi_S(k), \chi_T(k)$ at τ:

(a) the pair $(\Pi_\tau(M_k), \Pi_\tau(M_{k+1}))$ is a theta pair of step t with exponents (a_T, a_S) that has a witness satisfying branch $(+)$ of (Xp3) of Definition A.31; explicitly, there is a quadruple $(\alpha, \beta, \gamma, \delta)$ admissible for t of type $+$ ((Xp1) there) with

$$\Pi_\tau(M_k) = \{(-x - a_T, +) : x \in \alpha\beta\} \cup \{(-x - a_T, -) : x \in \gamma\delta\},$$
$$\Pi_\tau(M_{k+1}) = \{(-x - a_S, +) : x \in \alpha b_t \delta\} \cup \{(-x - a_S, -) : x \in \gamma\beta\},$$

$b_t = \left(\frac{t-1}{2} - i\right)_{0 \le i < t}$;

(b) (far from the exponent) $|x + a_T| \ge \frac{t+3}{2}$ for every $(x, \epsilon) \in \Pi_\tau(M_k)$, that is, $|e| \ge \frac{t+3}{2}$ for every entry e of $\alpha\beta\gamma\delta$.

$\mathbf{T}$ satisfies the archimedean step relation if every step does.

(Xo6) Right domains, continued values; the chain L-value condition. A **right domain** is an open set $U \subseteq \mathbb{C}$ that contains a half-plane $\{\mathrm{Re}\, z > \sigma\}$ and satisfies $z + r \in U$ for all $z \in U$ and all real $r \ge 0$. Let h be a complex function defined on a half-plane $\{\mathrm{Re}\, z > \sigma_0\}$. A complex number l is a **continued value** of h at $z_* \in \mathbb{C}$ if there are a right domain $U \ni z_*$ and a function $\tilde{h}$ meromorphic on U and holomorphic at z_*, with $\tilde{h}(z_*) = l$ and $\tilde{h} = h$ on some half-plane $\{\mathrm{Re}\, z > \sigma_1\}$. $\mathbf{T}$ satisfies the **chain L-value condition** if for every parameter family B outside S bounded by $(n, \frac{n-1}{2})$ and every $k < m$, the function $z \mapsto L^S(z + \frac{1}{2}, B_k \times \chi_T(k)^{-1})$ (chain parameters of B; partial L-function of (Xo1) for the character $\chi_T(k)^{-1}$, on a right half-plane where its product converges absolutely) has a nonzero continued value at $z = \frac{t}{2}$.

Theorem A.40. *Let D be an assembled datum with* RIP(D) *and let* $\mathbf{T}$ *be a well-formed Rallis tower datum over D, of length m, that satisfies the archimedean step relation and the chain L-value condition. Then for every $0 \le k \le m$ the composite $\Theta_k \colon \mathcal{C}_k \to \mathcal{C}_0$ is surjective.*

Proof. Outline. After Petersson structures are fixed on the finite-dimensional spaces $\mathcal{C}_j$, an induction on k using Rallis' step Proposition A.41 and the chain L-value condition attaches to every eigen-element $u \in \mathcal{C}_0$ with parameter B an eigen-element $x \in \mathcal{C}_k$ with parameter B_k such that $\langle y, x\rangle_k = \langle \Theta_k(y), u\rangle_0$ for all $y \in \mathcal{C}_k$; an element of $\mathcal{C}_0$ orthogonal to $\Theta_k(\mathcal{C}_k)$ is then split into eigen-elements with distinct parameters, these parameters stay distinct along the chain, and the linear independence of eigen-elements with distinct parameters forces the element to vanish. □

Proposition A.41. *Let D be an assembled datum with* RIP(D), $\mathbf{T}$ *a well-formed Rallis tower datum over D with step t and finite set S, and k smaller than its length, such that the step k satisfies the archimedean step relation. Let $\langle\cdot,\cdot\rangle_k$ and $\langle\cdot,\cdot\rangle_{k+1}$ be Petersson structures on $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$, and let $u \in \mathcal{C}_k$ be an eigen-element with parameter B such that $z \mapsto L^S(z + \frac{1}{2}, B \times \chi_T(k)^{-1})$*

has a nonzero continued value at $z = \frac{t}{2}$ (right domains and continued values as in (Xo6)). Then there is an eigen-element $x \in \mathcal{C}_{k+1}$ with parameter $\mathrm{st}_{\chi_S(k),\chi_T(k),t}(B)$ such that

$$\langle y, x\rangle_{k+1} = \langle \mathrm{io}_k(y), u\rangle_k \qquad \text{for every } y \in \mathcal{C}_{k+1}.$$

Proof. Sketch. This is Rallis' step: an eigen-element u at stage k whose partial L-function $L^S(z+\frac{1}{2}, B \times \chi_T(k)^{-1})$ has a nonzero continued value at $z = \frac{t}{2}$ is seen by the downward theta map, through an eigen-element x at stage $k+1$ that represents the functional $y \mapsto \langle \mathrm{io}_k(y), u\rangle_k$ and has the stepped parameter $\mathrm{st}_{\chi_S(k),\chi_T(k),t}(B)$. The space $\mathcal{C}_{k+1}$ is finite-dimensional (automorphic forms of fixed archimedean type and level), so this functional has a Riesz representative x; it remains to show that x is nonzero and an eigen-element with the stepped parameter. The argument should run as in Rallis' method [204]. (i) Form the upward theta lift $\theta(u)$, the integral of u against the theta kernel of a Fock vector $F_{k,P}$. (ii) By the Rallis inner product property $\mathrm{RIP}(D)$, an assumption of the statement, applied to the complex conjugate of u, whose parameter is $\overline{B}$, the Petersson norm of $\theta(u)$ is the continued L-value of the statement times a semilocal doubling zeta integral [199] over the places of S and the archimedean places; the identity assumed as $\mathrm{RIP}(D)$ has the shape of the Rallis inner product formula [152, 132, 101]. (iii) Show that $\theta(u)$ lies in the span of the values of $\mathcal{C}_{k+1}$ at K-finite vectors and is adjoint to io_k through the lifting operator, so that x is a nonzero multiple of $\theta(u)$ once $\theta(u) \neq 0$. (iv) Show by the unramified theta correspondence that $\theta(u)$ has Hecke parameter $\mathrm{st}_{\chi_S(k),\chi_T(k),t}(B)$ outside S.

Heuristically, the L-value in (ii) is nonzero by assumption, and the semilocal integral should be nonzero: at the places of S the level law and the coefficient support (Xo3)(e),(f) of Definition A.39 reduce it to a volume, and at a real place the archimedean step relation (Xo5) places the two discrete series in the archimedean theta correspondence [162], where the doubling integral of the Fock vector against a highest-weight matrix coefficient is expected to be a positive real. Step (iii) needs only that the lift has the discrete series type and the level prescribed by the tower, and step (iv) is the explicit shape of the unramified correspondence: the parameter B twisted by $\chi_S(k)\chi_T(k)^{-1}$, together with the segment $\chi_S(k)(w)\mathrm{N}(w)^{(t-1)/2-i}$, $0 \le i < t$, contributed by the t extra dimensions. Through this step the downward theta maps of a Rallis tower are onto (Theorem A.40). □

Proposition A.42. *Let c be a see-saw tower over D, $m = 2n+2$, and $(t_s)_{s\in\mathcal{T}}$ a family of tuples of c satisfying the see-saw laws; data of t_s carry the superscript s. Let I be a finite set, $s\colon I \to \mathcal{T}$, $\alpha\colon I \to \mathbb{C}$, and for $i \in I$ let $\lambda_i \in \mathrm{Ram}_0^{s(i)}$ and $F_i \in \mathcal{C}_T^{s(i)}$. If*

$$\sum_{i\in I} \alpha_i \, c_{n+1}^{\mathrm{top},s(i)} \ \mathrm{tl}^{s(i)}\big(\mathrm{up}_{n+1}^{s(i)}\lambda_i,\ F_i\big) = 0,$$

then for all $y \in \mathrm{Fr}_0$ and $x \in M_0^\infty$

$$\sum_{i\in I} \alpha_i \, c^{\mathrm{bot},s(i)} \ \mathrm{per}_0^{s(i)}\big(\lambda_i,\ \mathrm{dn}_{m-1}^{s(i)}(\mathrm{io}_T^{s(i)}F_i),\ y,\ x\big) = 0.$$

Proof. **Climbing stays ramified.** For every tuple t of the family, $\lambda \in \mathrm{Ram}_0$ and $k \le n+1$: $\mathrm{up}_k\lambda \in \mathrm{Ram}_{2k}$. Induction on k: if $\mathrm{up}_k\lambda \in \mathrm{Ram}_{2k}$ with $2k+2 \le m$, then S_{2k} preserves Ram_{2k} ($2k \le m$), K_{2k} maps it into Ram_{2k+1} and K_{2k+1} maps that into Ram_{2k+2} ($2k+1 < m$), by (Zw28).

Descent along the double steps. We prove by induction on $k \in \{0, \dots, n\}$ the claim (D_k): for every finite set I', $s' \colon I' \to \mathcal{T}$, $\beta \colon I' \to \mathbb{C}$, $\lambda'_i \in \mathrm{Ram}_0^{s'(i)}$ and $G_i \in \mathcal{C}_{2k}^{s'(i)}$, if

$$\sum_i \beta_i W_{i,k} \, \mathrm{per}_{2k}^{s'(i)}(\mathrm{up}_k \lambda'_i, G_i, y, x) = 0$$

for all $y \in \mathrm{Fr}_{2k}$, $x \in M_{2k}^{\infty}$, then

$$\sum_i \beta_i \Big(\prod_{j<2k} c_{i,j}^I \Big) \mathrm{per}_0^{s'(i)}(\lambda'_i, \mathrm{dn}_{2k} G_i, y, x) = 0$$

for all $y \in \mathrm{Fr}_0$, $x \in M_0^{\infty}$ (constants, W and maps of $t_{s'(i)}$). (D_0) is the hypothesis itself ($W_{i,0} = 1$, $\mathrm{up}_0 = \mathrm{dn}_0 = \mathrm{id}$). Let $k+1 \le n$ and assume (D_k); take data as in (D_{k+1}). Since $W_{i,k+1} = W_{i,k} c_{i,2k}^V c_{i,2k}^K c_{i,2k+1}^K$ and $\mathrm{up}_{k+1} \lambda'_i = K_{2k+1} K_{2k} S_{2k} (\mathrm{up}_k \lambda'_i)$, the hypothesis of ($\mathrm{D}_{k+1}$) is the hypothesis of the double-step lemma (linear relations among periods pass down a double step of the see-saw tower) at $j = 2k$ (a link step, as $2k \le m-4$) for the coefficients $\beta_i W_{i,k}$, the ramified columns $\mathrm{up}_k \lambda'_i \in \mathrm{Ram}_{2k}$ and the $G_i \in \mathcal{C}_{2k+2}$. That lemma gives, for all $y \in \mathrm{Fr}_{2k}$ and x,

$$\sum_i \beta_i W_{i,k} \, c_{i,2k}^I c_{i,2k+1}^I \, \mathrm{per}_{2k}\big(\mathrm{up}_k \lambda'_i, \mathrm{io}_{2k}\mathrm{io}_{2k+1} G_i, y, x\big) = 0,$$

which is the hypothesis of (D_k) for the coefficients $\beta_i c_{i,2k}^I c_{i,2k+1}^I$ and $\mathrm{io}_{2k}\mathrm{io}_{2k+1} G_i \in \mathcal{C}_{2k}$. Its conclusion is that of (D_{k+1}), since $\big(\prod_{j<2k} c_{i,j}^I\big) c_{i,2k}^I c_{i,2k+1}^I = \prod_{j<2k+2} c_{i,j}^I$ and $\mathrm{dn}_{2k}\mathrm{io}_{2k}\mathrm{io}_{2k+1} = \mathrm{dn}_{2k+2}$.

The last two steps. Put $\mu_i = \mathrm{up}_n \lambda_i \in \mathrm{Ram}_{2n} = \mathrm{Ram}_{m-2}$. As $c_{n+1}^{\mathrm{top}} = W_{i,n} c_{i,2n}^V c_{i,2n}^K c_{i,2n+1}^K c^P$ and $\mathrm{up}_{n+1} \lambda_i = K_{2n+1} K_{2n} S_{2n} \mu_i$, the hypothesis is that of the last-steps lemma (linear relations among tails pass down the last two steps of the see-saw tower) for the coefficients $\alpha_i W_{i,n}$, the μ_i and the F_i. It gives, for all $y \in \mathrm{Fr}_{2n}$ and x,

$$\sum_i \alpha_i W_{i,n} \, c_{i,2n}^I c_{i,2n+1}^I \, \mathrm{per}_{2n}\big(\mathrm{up}_n \lambda_i, \mathrm{io}_{2n}\mathrm{io}_T F_i, y, x\big) = 0.$$

(D_n) for the coefficients $\alpha_i c_{i,2n}^I c_{i,2n+1}^I$ and $\mathrm{io}_{2n}\mathrm{io}_T F_i \in \mathcal{C}_{2n}$ gives

$$\sum_i \alpha_i \Big(\prod_{j<2n+2} c_{i,j}^I \Big) \mathrm{per}_0(\lambda_i, \mathrm{dn}_{2n}\mathrm{io}_{2n}\mathrm{io}_T F_i, y, x) = 0.$$

Since $2n+2 = m$, the product is c^{bot} and $\mathrm{dn}_{2n}\mathrm{io}_{2n} = \mathrm{dn}_{2n+1} = \mathrm{dn}_{m-1}$; this is the claim. □

A.4.4 The multiplicity identity

Definition A.43 defines multiplicity data and the unit-phase multiplicity identity, which compares rational period pairings with spectral inner products. Surjectivity for the tower of a true column chain comes from the doubling method (Proposition A.45). The multiplicity core (Proposition A.46) combines the bottom and level hypotheses, the family conditions and nonvanishing member projections into the identity, through the identity at a realisation (Proposition A.44). Eventually every linked bundle then has the identity (Proposition A.47). Because the phases are units at every finite place, the identity transfers period lattices to spectral lattices with no loss at any prime.

Definition A.43. Let L be a number field with a fixed embedding $\sigma \colon L \to \mathbb{C}$; for $\varphi \in L^n$ write $\sigma(\varphi) \in \mathbb{C}^n$ for the vector of the $\sigma(\varphi_i)$. For a finite place w of L let $v_w \colon L^{\times} \to \mathbb{Z}$ be its normalised valuation. Inner products of complex inner product spaces are conjugate-linear in the first argument.

(Mu1) Members and tuples. A **multiplicity datum** $\mathfrak{m}$ over (L,σ) consists of: a finite set $\Omega_{\mathfrak{m}}$ of **members** and a finite set $\mathcal{T}_{\mathfrak{m}}$ of **tuples** with a **base tuple** $t_0 \in \mathcal{T}_{\mathfrak{m}}$; a map $w \mapsto t_{\mathfrak{m}}(w)$ from the finite places of L to $\mathcal{T}_{\mathfrak{m}}$ (the **tuple map**);

(Mu2) Columns. for every member ω an integer $a(\omega) \geq 0$ and a function $n_\omega \colon \mathbb{C}^{a(\omega)} \to \mathbb{R}_{\geq 0}$ (the **column norm** at σ);

(Mu3) Periods. for every tuple t an integer $d(t) \geq 0$, and for every member ω and tuple t a map $\mathrm{Pt}_{\omega,t} \colon L^{a(\omega)} \times L^{d(t)} \to L$ (the **period pairing**);

(Mu4) Row space and row theta maps. a finite-dimensional complex inner product space $\mathcal{E}_{\mathfrak{m}}$ with a subspace $\mathcal{E}^{\circ}_{\mathfrak{m}}$, and for every tuple t a σ-semilinear map $\vartheta_t \colon L^{d(t)} \to \mathcal{E}_{\mathfrak{m}}$ (the **row theta map**: theta lift of the rational row forms read through their rational structure);

(Mu5) Spectral side. a finite-dimensional complex inner product space $H_{\mathfrak{m}}$, a conjugate-linear map $s_{\mathfrak{m}} \colon \mathcal{E}_{\mathfrak{m}} \to H_{\mathfrak{m}}$ and, for every member ω, a $\mathbb{C}$-linear map $P_\omega \colon \mathbb{C}^{a(\omega)} \to H_{\mathfrak{m}}$ (the **member projection**).

A **multiplicity system** for $\mathfrak{m}$ is a tuple (V, a, U, ζ) of a $\mathbb{C}$-linear map $V \colon \mathcal{E}_{\mathfrak{m}} \to \mathcal{E}_{\mathfrak{m}}$, a function $a \colon \Omega_{\mathfrak{m}} \to \mathbb{C}$, matrices $U_\omega \in M_{a(\omega)}(\mathbb{C})$ $(\omega \in \Omega_{\mathfrak{m}})$ and a function $\zeta \colon \Omega_{\mathfrak{m}} \times \mathcal{T}_{\mathfrak{m}} \to \mathbb{C}$ such that:

(Mv1) Isometry on the distinguished subspace. $V(e) \in \mathcal{E}^{\circ}_{\mathfrak{m}}$ and $\|V(e)\| = \|e\|$ for every $e \in \mathcal{E}^{\circ}_{\mathfrak{m}}$;

(Mv2) Column unitarity. $n_\omega(U_\omega x) = n_\omega(x)$ for every ω and $x \in \mathbb{C}^{a(\omega)}$;

(Mv3) Nonzero constants. $a(\omega) \neq 0$ for every ω;

(Mv4) The identity. for every ω, t, $\varphi \in L^{a(\omega)}$ and $F \in L^{d(t)}$:

$$\zeta(\omega,t)\,\sigma\big(\mathrm{Pt}_{\omega,t}(\varphi,F)\big) = a(\omega)\,\big\langle s_{\mathfrak{m}}\big(V(\vartheta_t(F))\big),\, P_\omega\big(U_\omega\,\sigma(\varphi)\big)\big\rangle_{H_{\mathfrak{m}}}. \tag{A.4}$$

It is **unit-phase** if moreover

(Mv5) Unit phases. $\zeta(\omega,t) \neq 0$ for all ω, t; $\zeta(\omega,t_0) = 1$ for all ω; and for every member ω, every finite place w of L and every $k \in L$ with $\sigma(k) = \zeta(\omega, t_{\mathfrak{m}}(w))$, $v_w(k) = 0$;

and **root-of-unity** if instead every $\zeta(\omega,t)$ is a root of unity and $\zeta(\omega,t_0) = 1$ for all ω. The datum $\mathfrak{m}$ satisfies the **unit-phase multiplicity identity** if it admits a unit-phase multiplicity system, and the **root-of-unity multiplicity identity** if it admits a root-of-unity multiplicity system.

Proposition A.44. *Let D be an assembled datum and W a real package over a member object of D such that* $\mathrm{Bot}(W)$ *holds, $\vartheta_t(F) \in \mathcal{E}^{\circ}_{\mathfrak{m}(W)}$ for every tuple t and $F \in L^{d(t)}$,* $\mathrm{Joint}(W)$ *holds and* $\mathrm{PuNZ}(D)$ *holds. Then $\mathfrak{m}(W)$ satisfies the unit-phase multiplicity identity.*

Proof. Sketch. The identity (Mv4) of Definition A.43 compares two pairings attached to the same real package W: the rational period pairing $\mathrm{Pt}_{\omega,t}(\varphi,F)$ of a column vector with a row form, and the spectral inner product of the theta lift $\vartheta_t(F)$, moved by an isometry V of the sharp subspace, with the member projection of the column vector. The argument should run as follows. Choose a central element from the joint condition on W, the corresponding common shift V_z, a conjugate model vector and coordinate isomorphisms on the columns; the package conditions express the column periods through the bottom periods, with coefficients that are

units at every finite place. It remains to show a global period comparison: on every row e of the sharp-level subspace, the bottom-period functional must be a fixed nonzero multiple of $e \mapsto \langle s_W(V_z e), P_\omega(A_\omega \varphi)\rangle$, uniformly in the tuple and the row coefficient, where A_ω is the column transport used for the bottom relations; the transport must carry the frame norm to the member-space norm up to one positive factor, V_z must preserve the sharp subspace and its norm, and the nonvanishing of the member projections, PuNZ(D), must supply an anchor at which the functional is nonzero. Given these, the rest is elementary: if the comparison reads $c_\omega(t)\,\sigma(\mathrm{Pt})\,b_\omega = p_\omega \langle s_W(V_z \vartheta_t(F)), P_\omega(U_\omega \sigma(\varphi))\rangle$ with $p_\omega b_\omega c_\omega(t_0) \neq 0$, then $\zeta(\omega, t) = c_\omega(t)/c_\omega(t_0)$ and $a(\omega) = p_\omega/(b_\omega c_\omega(t_0))$ satisfy (Mv4), $\zeta(\omega, t_0) = 1$, and the valuation clause of (Mv5) is the unit-coefficient property of the bottom relations. Heuristically the comparison should hold because both sides are computed from one theta integral along the chain of lifts: unfolding the spectral inner product against the member projection should produce the bottom period, with a constant made of local factors that does not depend on the tuple, and the realisation is built so that its columns are read through the same rational structure as the periods. The identity is the last step of the multiplicity core Proposition A.46, which gives the multiplicity row Proposition A.47. □

Proposition A.45. *For every column datum w with laws and every later state Z of w,* Onto($D(w, Z)$) *holds.*

Proof. Let $D = D(w, Z)$. By the Rallis inner product property of every assembled datum, obtained by the doubling method, RIP(D) holds ((Mq5) of the package conditions). Since the Rallis inner product property gives onto theta maps, Onto(D) holds. (The argument uses no property of D beyond being an assembled datum: Onto holds for every assembled datum.) □

Proposition A.46. *Let D be an assembled datum and W a real package over a member object of D such that* Bot(W) *and* Lev(W) *hold, the indefinite realisation of W admits a realisation recursion,* FamAll(W) *holds and* PuNZ(D) *holds. Then $\mathfrak{m}(W)$ satisfies the unit-phase multiplicity identity.*

Proof. Step 1: W is served. Every real package is served, so Served(W) holds; this uses only the character frame (Mq9) of W, not the bottom hypotheses nor the realisation recursion.

Step 2: the joint condition. From FamAll(W) and Step 1, the proposition that the family-all and served conditions give the joint condition yields Joint(W) ((Mq10) of the package conditions): a central element z of the finite adelic part of GA(Y_0) and, for every word u, bottom data pinned at z, a map of the member space isometric up to a positive scale and unit coefficients satisfying the bottom relation.

Step 3: the row theta maps land in the sharp level subspace. From Bot(W) and Lev(W), the level proposition for row theta maps gives $\vartheta_t(F) \in \mathcal{E}^\circ_{\mathfrak{m}(W)}$ for every tuple t and every $F \in L^{d(t)}$.

Step 4: the identity. The hypotheses of Proposition A.44 are Bot(W) (assumed), the conclusion of Step 3, Joint(W) (Step 2) and PuNZ(D) (assumed). Hence $\mathfrak{m}(W)$ satisfies the unit-phase multiplicity identity.

Remark. The hypothesis that the indefinite realisation of W admits a realisation recursion is not used. □

Proposition A.47. *Let p be a parameter of a norm-choice extension. Then eventually in the row parameters κ and the context K the following holds: for every base point β at K that is good for p, every $c \in \mathbb{R}$ and every bundle b over $(D(\beta), \mathfrak{o}(\beta), c)$ that is linked to β for p, the multiplicity datum $\mathfrak{m}(b)$ satisfies the unit-phase multiplicity identity. In the terms of (Rk8): the*

predicate "$\mathfrak{m}(b)$ satisfies the unit-phase multiplicity identity" holds eventually at linked bundles for p.

Proof. Step 1: the row filter is closed under finite conjunction. Let $P_1(\kappa, K)$ and $P_2(\kappa, K)$ hold eventually in the row parameters and the context, with data $\varepsilon_0^{(i)}$, threshold families $\mathbf{K}_0^{(i)}$, constants $c_1^{(i)}$, $c_2^{(i)}$ and, for positive κ, $x_0^{(i)}$ $(i = 1, 2)$, as in (Rk1) of the row carrier. Put $\varepsilon_0 = \min(\varepsilon_0^{(1)}, \varepsilon_0^{(2)})$. For $0 < \varepsilon \le \varepsilon_0$ with $\varepsilon < \frac{1}{52}$ and reals $C_8 \ge 1$, C_5, c_6, C_b, let $\mathbf{K}_0$ be the threshold family whose preliminary entry and whose two functions are the maxima of those of $\mathbf{K}_0^{(1)}$ and $\mathbf{K}_0^{(2)}$; largeness ((Lc4) of the definition of contexts) consists of lower bounds on entries, so a constant family large with respect to $\mathbf{K}_0$ is large with respect to both. For such $\mathbf{C}$ take $c_1 = \max(c_1^{(1)}, c_1^{(2)})$, for $c_D \ge c_1$ take $c_2 = \max(c_2^{(1)}, c_2^{(2)})$, and for $c_h \ge c_2$ and positive κ take $x_0 = \max(x_0^{(1)}, x_0^{(2)})$. Then every context $K \models \kappa$ of scale $x \ge x_0$ has $P_1(\kappa, K)$ and $P_2(\kappa, K)$.

Step 2: the eventual properties. By the true-package proposition, eventually at linked bundles for p every linked bundle satisfies the point conditions and has a true real package. In particular the hypothesis of the nonvanishing theorem for member projections, which rests on the convergent Siegel–Weil formula, holds, so eventually at linked bundles for p the datum of the base point satisfies PuNZ. By Step 1, eventually in κ and K both hold at once: for every base point β at K that is good for p, every c and every bundle b over $(D(\beta), \mathfrak{o}(\beta), c)$ linked to β for p, (i) b satisfies the point conditions, (ii) $W = W(b)$ is true and (iii) PuNZ$(D(\beta))$ holds. We show that for every such b the datum $\mathfrak{m}(b)$ satisfies the unit-phase multiplicity identity; this proves the statement in the same filter.

Step 3: one linked bundle. Fix κ, K, β, c, b as in Step 2 and write $D = D(\beta)$, $W = W(b)$. By (ii) there are a column datum w with laws and a later state Z of w with $D = D(w, Z)$, and AllFam(W) holds. By Proposition A.45, Onto(D) holds. By (P10) of (i), $r_S \ge 1$. Hence the proposition on the family conditions of a true real package applies to (w, Z, W) and gives FamAll(W). The family conditions AllFam(W) contain the package hypotheses, among them Bot(W) and Lev(W), and a realisation recursion of the indefinite realisation of W ((Mq3) of the package conditions). Together with PuNZ(D) from (iii), all hypotheses of Proposition A.46 hold, and so $\mathfrak{m}(W)$ satisfies the unit-phase multiplicity identity. Finally $\mathfrak{m}(W(b)) = \mathfrak{m}(b)$, since both are read off the same indefinite realisation Q_b ((Mq1) of the package conditions, (Rk9) of the row carrier). □

Sketch: what the chain supplies to the rows. The three properties of the chain enter the construction as row services at one parameter, as described in the sketch at the end of Section A.1. Separation gives the functional row and the multiplicity identity gives the multiplicity row; the rows of the Hecke kernel datum κ record an eigenvalue law on the same lifts, and a comparison tie fixes the real normalisations of the selected bundle. What must be added to the three properties is their attachment to the selected bundles. The analytic grid on which separation is proved must be chosen above one threshold for all members at once, the constant-function fit must hold on the same grid, and the realisation at which the multiplicity identity is proved must be the one carried by the member. Heuristically each of these is a matter of choosing the auxiliary data of the chain (grids, test functions, base tuples) once the scale is large, since each property holds for every admissible choice and the choices form a nonempty set above a threshold.

A.5 The members: Hida settings and the two passes

A packet has one member for every translate $u \in A_x$, and the loss datum and slope record of the member are read from automorphic objects built at its context. This section builds them. The members come from an abstract producer, a two-pass system with seven named hypotheses (H0)–(H6) (Definition A.48, Theorem A.49). A first pass climbs the levels of a state to a core with the slot Built; a second pass enlarges the field of coefficients while keeping every reference datum. For the concrete system (Definition A.50) five of the hypotheses are proved, namely (H0), (H1), (H2), (H5) and (H6) (Theorem A.51); the export property (H3) and the second pass (H4) are the hypotheses of the member step Theorem A.52, and they are sketched in the last subsection. The automorphic data at a member are called a Hida setting: unitary automorphic data with an archimedean test function and transports between levels, not a p-adic Hida family. The spectral input is classical. Discrete series of $\mathrm{U}(p,q)$ above an explicit gap have integrable lowest-K-type coefficients, Harish-Chandra's formal degree and Schur orthogonality hold, and together they make evaluation at the lowest K-type an exact bijection onto conjugate rows. The count identity comes from the Tamagawa number 2 of unitary groups: volumes are masses (Proposition A.59).

A.5.1 Two-pass systems and the member step

A two-pass system (Definition A.48) consists of contexts, pre-bases, frames and bases with enlargements, stage tuples, cores with slots, assembled data, layered data and demands. Theorem A.49 is pure assembly: from a baseline (H0), largeness of forced data (H1), a first pass (H2), export (H3), a second pass (H4), properties of cores (H5) and layers from cores (H6) it produces, eventually in the context, an assembled datum with layered data and a core with the final slot. The concrete system is Definition A.50, and Theorem A.51 proves for it all hypotheses except (H3) and (H4). The member step Theorem A.52 then gives, over every complete good-host witness and *given* (H3) and (H4), a good context and a member that satisfies the numerical clauses of Proposition A.18 and the member clauses of Proposition A.53.

Definition A.48. An **avoidance map** is a map A from the set of rational primes to the set of finite subsets of $\mathbb{Z}_{\geq 1}$. For avoidance maps A, A' write $A' \leq A$ if $A'(\ell) \subseteq A(\ell)$ for every prime ℓ, and $A \cup A'$ for the map $\ell \mapsto A(\ell) \cup A'(\ell)$; the **empty** avoidance map has all values $\varnothing$.

A **two-pass system** consists of the following data.

(Tp1) Contexts. A set $\mathcal{K}$ of **contexts** and a filter $\mathfrak{E}$ on $\mathcal{K}$. A property of contexts holds **eventually** if the set of contexts having it belongs to $\mathfrak{E}$.

(Tp2) Pre-bases, frames, bases. For every $K \in \mathcal{K}$ a set $\mathcal{P}(K)$ of **pre-bases over** K. For every pre-base P a set $\mathcal{F}(P)$ of **frames**, a set $\mathfrak{X}(P)$, and a set $\mathcal{B}(P)$ of **bases over** P. Every base B carries three subfields $F_{\mathrm{ex}}(B)$, $F_{\mathrm{fix}}(B)$, $F_{\mathrm{L}}(B)$ of $\mathbb{C}$ (the exact, fixed and large coefficient fields). A base B **contains** a pair (S_1, S_2) of finite subsets of $\mathbb{C}$ if $S_1 \subseteq F_{\mathrm{fix}}(B)$ and $S_2 \subseteq F_{\mathrm{L}}(B)$. For $B, B' \in \mathcal{B}(P)$, B' is an **enlargement** of B if $F_{\mathrm{ex}}(B') = F_{\mathrm{ex}}(B)$, $F_{\mathrm{fix}}(B') = F_{\mathrm{fix}}(B)$ and $F_{\mathrm{L}}(B) \subseteq F_{\mathrm{L}}(B')$.

(Tp3) Twisting data. For every base $B \in \mathcal{B}(P)$ a set $\mathcal{T}(B)$ of **twisting data**. Every $T \in \mathcal{T}(B)$ has a finite set $\mathrm{pl}(T)$ of primes, for every $f \in \mathcal{F}(P)$ a **character set** $\Xi(f,T) \subseteq \mathfrak{X}(P)$, and a relation "$T$ **avoids** A" with avoidance maps A, such that T avoids every $A' \leq A$ whenever it avoids A. For every finite set S of primes there is an **untwisted datum** $\upsilon(B,S) \in \mathcal{T}(B)$ with $\mathrm{pl}(\upsilon(B,S)) = S$, avoiding every avoidance map, and with

$\Xi(f, \upsilon(B,S)) = \Xi_0(f)$ for every $f \in \mathcal{F}(P)$, where $\Xi_0(f) \subseteq \mathfrak{X}(P)$ is a set attached to f alone. A twisting datum T is **untwisted** if $T = \upsilon(B, \mathrm{pl}(T))$.

(Tp4) Forced twisting data. For every $K \in \mathcal{K}$, a property "T is **forced at** K" of pairs (B,T) with $B \in \mathcal{B}(P)$, $P \in \mathcal{P}(K)$, $T \in \mathcal{T}(B)$.

(Tp5) Stages, cores, assembled data. A set $\mathcal{D}$ of **assembled data**, and for each $D \in \mathcal{D}$ a set $\mathcal{O}(D)$ of **objects of** D. For every base B a set $\mathcal{X}(B)$ of **stage tuples** with a map $X \mapsto D(X) \in \mathcal{D}$. For $K \in \mathcal{K}$, $B \in \mathcal{B}(P)$ with $P \in \mathcal{P}(K)$, $X \in \mathcal{X}(B)$, $f \in \mathcal{F}(P)$ and $T \in \mathcal{T}(B)$, a set $\mathcal{C}(K,X,f,T)$ of **cores**, each core c with an object $O(c) \in \mathcal{O}(D(X))$. A **slot** is a property of tuples (K,X,f,T,c) with $c \in \mathcal{C}(K,X,f,T)$.

(Tp6) Layered data. For each $D \in \mathcal{D}$ a set $\mathcal{L}(D)$ of **layered data on** D, each $Y \in \mathcal{L}(D)$ with an object $O(Y) \in \mathcal{O}(D)$.

Demands and large data. Let $K \in \mathcal{K}$. A **demand at** K is a quadruple $\rho = (S_1, P_1, A, S_2)$ of maps assigning: to every $P \in \mathcal{P}(K)$, $f \in \mathcal{F}(P)$ and subfield F of $\mathbb{C}$, a finite set $S_1(P,f,F)$ of algebraic numbers in $\mathbb{C}$; to every such (P,f,F) and subfield F_0 of $\mathbb{C}$, a finite set $P_1(P,f,F,F_0)$ of primes and an avoidance map $A(P,f,F,F_0)$; and to every such (P,f,F,F_0) and subset $\Xi \subseteq \mathfrak{X}(P)$, a finite set $S_2(P,f,F,F_0,\Xi)$ of algebraic numbers in $\mathbb{C}$. A triple (B,f,T) with $B \in \mathcal{B}(P)$, $P \in \mathcal{P}(K)$, $f \in \mathcal{F}(P)$, $T \in \mathcal{T}(B)$ **meets** ρ if, with $F = F_{\mathrm{ex}}(B)$ and $F_0 = F_{\mathrm{fix}}(B)$,

$$B \text{ contains } \big(S_1(P,f,F),\, S_2(P,f,F,F_0,\Xi(f,T))\big),$$
$$P_1(P,f,F,F_0) \subseteq \mathrm{pl}(T), \qquad T \text{ avoids } A(P,f,F,F_0).$$

A property of such triples holds **for large data at** K if there is a demand at K such that every triple meeting it has the property.

Theorem A.49. *Fix a two-pass system. Let* BP *be a property of pairs* (B,f) *with* $B \in \mathcal{B}(P)$, $f \in \mathcal{F}(P)$*; let* Q, Q' *be slots; let* Ex *and* τ *be properties of pairs* (D,O) *with* $D \in \mathcal{D}$, $O \in \mathcal{O}(D)$*; let* V *and* M *be properties of assembled data; and let* Π *be a property of triples* (K,D,Y) *with* $K \in \mathcal{K}$, $D \in \mathcal{D}$, $Y \in \mathcal{L}(D)$*. Assume:*

(H0) Baseline. *Eventually in* K*, there are* $P \in \mathcal{P}(K)$ *and* $f_0 \in \mathcal{F}(P)$ *such that for every map* S_1 *from subfields of* $\mathbb{C}$ *to finite sets of algebraic numbers and every map* S_2 *from pairs of subfields of* $\mathbb{C}$ *to finite sets of algebraic numbers there is* $B \in \mathcal{B}(P)$ *with* $\mathrm{BP}(B,f_0)$ *which contains* $\big(S_1(F_{\mathrm{ex}}(B)),\, S_2(F_{\mathrm{ex}}(B), F_{\mathrm{fix}}(B))\big)$*.*

(H1) Forced data are large. *For every* $K \in \mathcal{K}$*, "*T *is forced at* K*" holds for large data* (B,f,T) *at* K*.*

(H2) First pass. *Eventually in* K*, the following holds for large data* (B,f,T) *at* K*: if* $\mathrm{BP}(B,f)$*,* T *is forced at* K *and* T *is untwisted, then there are* $X \in \mathcal{X}(B)$ *and* $c \in \mathcal{C}(K,X,f,T)$ *with* $Q(K,X,f,T,c)$*.*

(H3) Export. *Eventually in* K*, the following holds for large data* (B,f,T) *at* K*: if* $\mathrm{BP}(B,f)$ *and* T *is untwisted, then* $\mathrm{Ex}(D(X),O(c))$ *for every* $X \in \mathcal{X}(B)$ *and every* $c \in \mathcal{C}(K,X,f,T)$ *with* $Q(K,X,f,T,c)$*.*

(H4) Second pass. *Eventually in* K*, the following holds for large data* (B,f,T) *at* K*: if* $\mathrm{BP}(B,f)$ *and* T *is untwisted, then for every* $X \in \mathcal{X}(B)$ *and* $c \in \mathcal{C}(K,X,f,T)$ *with* $Q(K,X,f,T,c)$ *and* $\mathrm{Ex}(D(X),O(c))$*, every avoidance map* A *and every map* S_2 *from*

subsets of $\mathfrak{X}(P)$ to finite sets of algebraic numbers, there are an enlargement $B' \in \mathcal{B}(P)$ of B and $T' \in \mathcal{T}(B')$ with

$$\mathrm{pl}(T') = \mathrm{pl}(T), \qquad T' \text{ avoids } A, \qquad S_2(\Xi(f, T')) \subseteq F_{\mathrm{L}}(B'),$$

and $X' \in \mathcal{X}(B')$, $c' \in \mathcal{C}(K, X', f, T')$ with $Q(K, X', f, T', c')$, $\tau(D(X'), O(c'))$ and $Q'(K, X', f, T', c')$.

(H5) Cores. *For every core $c \in \mathcal{C}(K, X, f, T)$ the assembled datum $D(X)$ has V and M.*

(H6) Layers from cores. *Eventually in K: for every $B \in \mathcal{B}(P)$ with $P \in \mathcal{P}(K)$, $X \in \mathcal{X}(B)$, $f \in \mathcal{F}(P)$, $T \in \mathcal{T}(B)$ and $c \in \mathcal{C}(K, X, f, T)$ such that $D(X)$ has V, there is $Y \in \mathcal{L}(D(X))$ with $O(Y) = O(c)$ and $\Pi(K, D(X), Y)$.*

Then eventually in K there are $D \in \mathcal{D}$ and $Y \in \mathcal{L}(D)$ such that $\Pi(K, D, Y)$, $\tau(D, O(Y))$ and $M(D)$ hold, and there are $P \in \mathcal{P}(K)$, $B \in \mathcal{B}(P)$, $X \in \mathcal{X}(B)$, $f \in \mathcal{F}(P)$, $T \in \mathcal{T}(B)$ and a core $c \in \mathcal{C}(K, X, f, T)$ with $Q'(K, X, f, T, c)$, $D = D(X)$ and $O(Y) = O(c)$.

Proof. Step 1: large data form a filter. Let $K \in \mathcal{K}$ and let $p_1, \dots, p_r$ be properties of triples that hold for large data at K, with demands $\rho_i = (S_1^i, P_1^i, A^i, S_2^i)$. Put $\rho = (S_1, P_1, A, S_2)$ with

$$S_1 = \bigcup_i S_1^i, \quad P_1 = \bigcup_i P_1^i, \quad A = \bigcup_i A^i, \quad S_2 = \bigcup_i S_2^i,$$

the unions taken argument by argument. Finite unions of finite sets of algebraic numbers, of finite sets of primes and of avoidance maps are again of the same kind, so ρ is a demand at K. If (B, f, T) meets ρ, it meets every ρ_i: containment of a pair passes to pairs of subsets; $P_1^i \subseteq P_1 \subseteq \mathrm{pl}(T)$; and $A^i \leq A$, so T avoids A^i by (Tp3). Hence the conjunction $p_1 \wedge \cdots \wedge p_r$ holds for large data at K.

Step 2: one context. Let $\mathcal{K}_0, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_6 \in \mathfrak{E}$ be the sets of contexts at which the statements in (H0), (H2), (H3), (H4), (H6) hold. Their intersection $\mathcal{K}^*$ belongs to $\mathfrak{E}$. Fix $K \in \mathcal{K}^*$. By (H1), (H2)–(H4) at K and Step 1 there is one demand $\rho = (S_1, P_1, A, S_2)$ at K such that every triple (B, f, T) meeting ρ is forced at K and has the three properties stated in (H2), (H3), (H4).

Step 3: a base and the untwisted datum. Let $P \in \mathcal{P}(K)$ and $f_0 \in \mathcal{F}(P)$ be given by (H0) at K. Apply (H0) to the maps

$$S_1'(F) = S_1(P, f_0, F), \qquad S_2'(F, F_0) = S_2(P, f_0, F, F_0, \Xi_0(f_0)),$$

whose values are finite sets of algebraic numbers. This gives $B \in \mathcal{B}(P)$ with $\mathrm{BP}(B, f_0)$ containing $(S_1(P, f_0, F), S_2(P, f_0, F, F_0, \Xi_0(f_0)))$, where $F = F_{\mathrm{ex}}(B)$, $F_0 = F_{\mathrm{fix}}(B)$. Put $T = \upsilon(B, P_1(P, f_0, F, F_0))$. By (Tp3), $\Xi(f_0, T) = \Xi_0(f_0)$, $\mathrm{pl}(T) = P_1(P, f_0, F, F_0)$ and T avoids $A(P, f_0, F, F_0)$. So (B, f_0, T) meets ρ. Moreover $T = \upsilon(B, \mathrm{pl}(T))$ is untwisted, and T is forced at K by the choice of ρ.

Step 4: the two passes. By (H2) applied to (B, f_0, T) there are $X \in \mathcal{X}(B)$ and $c \in \mathcal{C}(K, X, f_0, T)$ with $Q(K, X, f_0, T, c)$. By (H3), $\mathrm{Ex}(D(X), O(c))$. Apply (H4) with the empty avoidance map and the map $S_2 \equiv \varnothing$. It gives an enlargement $B' \in \mathcal{B}(P)$ of B, $T' \in \mathcal{T}(B')$, $X' \in \mathcal{X}(B')$ and $c' \in \mathcal{C}(K, X', f_0, T')$ with $\tau(D(X'), O(c'))$ and $Q'(K, X', f_0, T', c')$.

Step 5: the layered datum. By (H5), $D(X')$ has V and M. Since $K \in \mathcal{K}_6$, (H6) applied to (B', X', f_0, T', c') gives $Y \in \mathcal{L}(D(X'))$ with $O(Y) = O(c')$ and $\Pi(K, D(X'), Y)$. Put $D = D(X')$. Then $\Pi(K, D, Y)$ and $M(D)$ hold, and $\tau(D, O(Y))$ holds because $O(Y) = O(c')$. The core c' over (K, X', f_0, T'), with P, B', X', f_0, T', satisfies Q', $D = D(X')$ and $O(Y) = O(c')$.

So the conclusion holds at every context of $\mathcal{K}^* \in \mathfrak{E}$. Since a filter is closed under supersets, it holds eventually. $\square$

Definition A.50. The **concrete two-pass system** has:

(Tp1) $\mathcal{K}$ the set of contexts and $\mathfrak{E}$ the context filter;

(Tp2) $\mathcal{P}(K)$ the pre-bases over K; $\mathcal{F}(P)$ the frames over P; $\mathfrak{X}(P)$ the Hecke characters of the host field; $\mathcal{B}(P)$ the bases over P, with $F_{\mathrm{ex}}(B), F_{\mathrm{fix}}(B), F_{\mathrm{L}}(B)$ the exact, fixed and large coefficient fields;

(Tp3) $\mathcal{T}(B)$ the twisting data over B, with $\mathrm{pl}(T)$, $\Xi(f,T)$, avoidance, and $\upsilon(B,S)$ the untwisted datum with prime set S; $\Xi_0(f) = \{1, \delta_f, \delta'_f\}$;

(Tp4) "forced at K" as defined for the concrete twisting data;

(Tp5) $\mathcal{D}$ the assembled master data, $\mathcal{O}(D)$ the objects of D ((Xh1)); $\mathcal{X}(B)$ the stage tuples over B with $X \mapsto D(X)$; $\mathcal{C}(K,X,f,T)$ the cores over (K,X,f,T) ((Xh2)) with their objects $O(c)$;

(Tp6) $\mathcal{L}(D)$ the layered data on D with $Y \mapsto O(Y)$ ((Ly) of the definition of layered data).

That these data satisfy the axioms of a two-pass system is Theorem A.51.

Theorem A.51. *1. The concrete two-pass system is a two-pass system.*

2. *Fix $c_0 \in \mathbb{R}$ and put $Q = \mathrm{Built}_{\mathrm{E}}(\mathfrak{S}(\mathcal{N}_{\mathrm{or}}(c_0)))$ (a core c over (K,X,f,T) has Q if $X = (B; C(w), Z)$ for a column datum w over (B,f,T) and a state Z of w at which c is Built with canonical entries for $\mathfrak{S}(\mathcal{N}_{\mathrm{or}}(c_0))$), let V be the volume–mass identity and Π admissibility. Then (H0), (H1), (H2) and (H6) hold for the concrete two-pass system with* BP, Q, V, Π.

3. *Consequently, for every slot Q' and all properties Ex, τ of pairs (D,O) and M of assembled data such that (H3), (H4) and (H5) hold with $\mathrm{BP}, Q, Q', \mathrm{Ex}, \tau, V, M$, the conclusion of the two-pass producer theorem (which gives the conclusion of Theorem A.49 at all contexts with sufficiently large parameters) holds for the concrete two-pass system with $\mathrm{BP}, Q, Q', \mathrm{Ex}, \tau, V, M, \Pi$.*

4. *(H5) holds with V and with M the stability of the spaces $\mathcal{M}_t$ of (Xh2) in the definition of cores; so for this M the conclusion of item 3 holds for every Q', Ex, τ satisfying (H3) and (H4).*

5. *(Adjoined base property.) Let A be a property of (K,B,f,T) that holds eventually at all contexts for large data, and let $Q_A = \mathrm{Built}_{\mathrm{E}}(\mathfrak{S}(p_A))$ with $p_A = (\mathcal{N}_{\mathrm{or}}(c_0)_A, \varnothing, \Gamma_\top)$ the parameter with A adjoined, as in the lemma on adjoining an eventual base condition ($\mathfrak{S}(\mathcal{N}_{\mathrm{or}}(c_0))$ being the slot system of $(\mathcal{N}_{\mathrm{or}}(c_0), \varnothing, \Gamma_\top)$). Then (H0), (H1), (H2), (H5), (H6) hold with $\mathrm{BP}, Q_A, V, M, \Pi$, and for every Q', Ex, τ such that (H3) and (H4) hold with $\mathrm{BP}, Q_A, Q', \mathrm{Ex}, \tau$, the conclusion of the two-pass producer theorem holds with $\mathrm{BP}, Q_A, Q', \mathrm{Ex}, \tau, V, M, \Pi$.*

Proof. 1. (Tp1): $\mathfrak{E}$ is a filter on the set of contexts, by the filter lemma for contexts. (Tp2), (Tp4)–(Tp6) only name sets and maps, which Definition A.50 supplies. (Tp3): if T avoids A and $A' \le A$, then $N(v) \notin A(\ell_w) \supseteq A'(\ell_w)$ for all w and $v \in Q_w$, so T avoids A'. For a finite set S of primes, $\upsilon(B,S)$ ($\mathrm{pl} = S$, $Q_w = \varnothing$, $k_w = 0$, $\psi_w = 1$) is a twisting datum: (Tx1) holds for the trivial character with $k_w = 0$ (order $1 = \ell_w^0$, unramified everywhere, so ramified exactly at $Q_w = \varnothing$, trivial on $\mathbb{A}_{E^+}^\times$ and on every $E_v^\times$), (Tx2) is vacuous, and (Tx3) holds. It avoids every

avoidance map because all Q_w are empty. Its character set is $\{1, \delta_f, \delta'_f\} \cup \{\delta_f, \delta'_f\} = \Xi_0(f)$, since $\xi_{f,v}(w) \in \{\delta_f, \delta'_f\}$. The notion "untwisted" is the one of (Tp3) by definition.

2. (H0) is the baseline proposition for good contexts. (H1) is the lemma that forced twisting data are large. (H2): by Proposition A.54, eventually in K, for large data (B, f, T) at K with $\mathrm{BP}(B, f)$, T forced and untwisted, there are a column datum w over (B, f, T), a state Z of w and a core c over $(K, (B; C(w), Z), f, T)$ Built with canonical entries for $\mathfrak{S}(\mathcal{N}_{\mathrm{or}}(c_0))$ at (w, Z); with $X = (B; C(w), Z) \in \mathcal{X}(B)$ this is $c \in \mathcal{C}(K, X, f, T)$ with $Q(K, X, f, T, c)$. (H6) is the proposition on layered data from cores (it holds at every context, hence eventually).

3. By 1 and 2, all of (H0)–(H6) hold for the concrete two-pass system, whose contexts and filter are the contexts and the context filter, as the two-pass producer theorem requires; that theorem applies.

4. (H5) with these V, M is the core lemma for the concrete two-pass system; then item 3 applies.

5. (H0), (H1), (H5) and (H6) do not involve the slot Q, so they hold as in items 2 and 4. (H2) for Q (item 2) and the eventual largeness of A give (H2) for Q_A by the lemma on adjoining an eventual base condition (first-pass transfer: enlarge the demand by that of A and insert the proof of A into the Built record). With (H3), (H4) for Q_A as assumed, all of (H0)–(H6) hold for the concrete two-pass system with Q_A, and the two-pass producer theorem applies as in item 3. □

Theorem A.52. *Fix $c_0 \in \mathbb{R}$, and let Q be either $\mathrm{Built}_{\mathrm{E}}(\mathfrak{S}(\mathcal{N}_{\mathrm{or}}(c_0)))$ or Q_A for a property A of (K, B, f, T) that holds eventually at all contexts for large data. Let Q', Ex, τ be properties such that (H3) and (H4) hold for the concrete two-pass system with $\mathrm{BP}, Q, Q', \mathrm{Ex}, \tau, V, M$. There is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ with $\varepsilon < \frac{1}{52}$ and all $C_8 \ge 1$, $C_5, c_6, C_b > 0$ there is a threshold family $\mathbf{K}_0$ such that for every admissible $\mathbf{C}$ with this C_8, large with respect to $\mathbf{K}_0$, there is $c_1 \ge 0$ such that for every $c_D \ge c_1$ there is $c_2 \ge 402$ such that for every $c_h \ge c_2$ there is x_0 with the following property.*

Let $x \ge x_0$, $J \ge 1$, and let a complete witness of $\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}, T, t)$ with $\kappa = (C_{T_2}, C_8, C_5, c_6, C_b)$ (C_{T_2} that of $\mathbf{C}$) be given, with its element $u_0 \in A$ (no clause of Definition 11.4 restricts u_0 beyond $u_0 \in A$, so u_0 may be any element of A). Then for all top tables $a, b\colon A \to \mathbb{Z}$ satisfying the windows (Cx3e), every finite set Ω of words with $|\Omega| = \lceil 3\Gamma^2 x^{C_F} \rceil$, $\Gamma = \exp(C_\Gamma m^5 f L)$, every base word $\omega_0 \in \Omega$ and every injective word code $c\colon \Omega \to \{0,1\}^{k-2}$ there are Harish-Chandra parameters and a level choice Lv such that, with the pre-character datum $\mathfrak{P}$ formed from the witness and these Ω, ω_0 by the existence theorem for good pre-character data,

$$K = (x, \varepsilon, J, \mathbf{C}, c_D, c_h, \mathfrak{P}, q_0, a, b, c, \mathrm{Lv})$$

is a context good for (C_5, c_6, C_b) with $X'(\mathfrak{s}) \le T(\omega)$ for every word ω; there are an assembled datum D and layered data $Y \in \mathcal{L}(D)$ with $\Pi(K, D, Y)$, $\tau(D, O(Y))$ and $M(D)$, together with $P, B, X, f, \mathsf{T}, c'$ with $Q'(K, X, f, \mathsf{T}, c')$, $D = D(X)$ and $O(Y) = O(c')$; and there is a record part $(L, I, (N^{\mathrm{gl}}_\omega)_\omega)$ of Y (a size datum L of D, a canonical instance I over L, and $\mathcal{O}_L$-submodules N^{gl}_ω of the divisor spaces of the words ω), whose member satisfies (a)–(e) of Proposition A.18 and (Mc1)–(Mc5) of Proposition A.53; in particular the member is formed from the witness, u_0 and these a, b, c, it satisfies (La4) and M, and, if τ is "all twists good" ((Xh5) in the definition of cores), it satisfies τ with the object $O(Y)$. The member carries the same supplied canonical gate instance I, lattice instance $\mathrm{Gy}(O(Y))$, gathering ties and instance constants $5, 4, 12, 3$, with the parameter-fill, lattice-tie, gathering-filling and forced local-value equations of the record part. Its underlying object remains $O(Y) = O(c')$.

Proof. By item 4 (for $Q = \mathrm{Built}_{\mathrm{E}}(\mathfrak{S}(\mathcal{N}_{\mathrm{or}}(c_0)))$) or item 5 (for $Q = Q_A$) of Theorem A.51, the

conclusion of the two-pass producer theorem holds for the concrete two-pass system with $\mathrm{BP}, Q, Q', \mathrm{Ex}, \tau, V, M, \Pi$. Let ε_0 be its constant, decreased if necessary below $\frac{1}{52}$. Given $\varepsilon, C_8, C_5, c_6, C_b$ take its threshold family $\mathbf{K}_0$; given $\mathbf{C}$ take its c_1 and replace it by $\max(c_1, 0)$; given c_D take its c_2 and replace it by $\max(c_2, 402)$; given c_h let x_0 be the maximum of: its x_0; the thresholds of the existence theorem for good pre-character data (for $\varepsilon, \mathbf{C}, c_D, c_h, C_5, c_6, C_b$), of the existence theorem for level choices (for ε), of the record-part construction and of Proposition A.18 (for $\varepsilon, \mathbf{C}, c_D, c_h$), and of Proposition A.53 (for $\varepsilon, \mathbf{C}, c_D, c_h, C_5, c_6, C_b$). All of these are chosen before x, the witness and the member.

Let $x \geq x_0$, $J \geq 1$, a witness and $a, b, \Omega, \omega_0, c$ as in the statement be given. By the existence theorem for good pre-character data, which holds for all such $a, b, \Omega, \omega_0, c$, there are Harish-Chandra parameters such that $(x, \varepsilon, J, \mathbf{C}, c_D, c_h, \mathfrak{P}, q_0, a, b, c)$ satisfies (Cx1)–(Cx3) with $\Gamma = \exp(C_\Gamma m^5 f L)$ and the goodness clauses for (C_5, c_6, C_b). By the conclusion of the two-pass producer theorem, $d_\omega(\mathfrak{P})$ is a positive integer and $mL \leq \log \theta_\omega$ for every word, with $d^* = d(\mathfrak{P})$, $D^* = D(\mathfrak{P})$. So the existence theorem for level choices gives a level choice $\mathrm{Lv} = (T, \pi_T, d(\mathfrak{P}), D(\mathfrak{P}), \exp(C_\Gamma m^5 f L))$ for $\mathfrak{P}$. Again by the conclusion of the two-pass producer theorem, K is a context, hence a context good for (C_5, c_6, C_b) (the goodness clauses hold), and there are $D, Y, P, B, X, f, \mathsf{T}, c'$ with the listed properties. For every word ω, $m \geq 1$ and (Lv3) give $\log X'(\mathfrak{s}) = x^{\varepsilon/3} \leq x^\varepsilon = L \leq mL \leq \log \theta_\omega \leq \log T(\omega)$, so $X'(\mathfrak{s}) \leq T(\omega)$. Since $\Pi(K, D, Y)$, $x \geq x_0$ and $X'(\mathfrak{s}) \leq T(\omega)$ for all ω, the record-part construction gives a record part $(L, I, (N^{\mathrm{gl}}_\omega)_\omega)$ of Y, and Proposition A.18 gives (a)–(e) for it (the size datum of its member is L with its lattices and lengths set; (a)–(e) concern only K and the scale, top numbers, validity values and support of L, which the member keeps). Finally Proposition A.53 applies to the witness, $a, b, \Omega, \omega_0, c$, the parameters, Lv, K, D, Y and the record part, and gives (Mc1)–(Mc5); with $\Pi(K, D, Y)$ (which contains (La4)), $M(D)$ (M is the stability of the spaces $\mathcal{M}_t$, item 4 of Theorem A.51) and, when τ is "all twists good", $\tau(D, O(Y))$, (Mc5) gives the last assertions.

The conclusion of the record-part construction on the supplied instance is obtained before the member is formed. Use its I, $\mathrm{Gy}(O(Y))$, gathering ties and constants as the readings of the supplied part. The member operation retains these components, the scale, top table, lift weights, fillings and local F-values. Its four displayed equations therefore remain valid, unchanged, on the member. The underlying object and its theta and kernel data are unchanged, so the already established Q', τ and object identities are unaffected. □

Proposition A.53. *Let $0 < \varepsilon < \frac{1}{52}$, $\mathbf{C}$ an admissible constant family, $c_D \geq 0$, $c_h \geq 402$ and $C_5, c_6, C_b > 0$. There is x_0 such that the following holds for every $x \geq x_0$ and $J \geq 1$. Let a complete witness of $\mathfrak{G}(x, \varepsilon, J, \kappa, \mathfrak{h}, T, t)$, $\kappa = (C_{T_2}, C_8, C_5, c_6, C_b)$, be given, with rank datum $\mathfrak{R}$, host data, primes g_1, g_2, p_1, p_2, q_0, spacing set S and $u_0 \in A$; let $a, b, \Omega, \omega_0, c$ and Harish-Chandra parameters be as in the existence theorem for good pre-character data, with pre-character datum $\mathfrak{P}$; let* Lv *be a level choice for $\mathfrak{P}$ such that $K = (x, \varepsilon, J, \mathbf{C}, c_D, c_h, \mathfrak{P}, q_0, a, b, c, \mathrm{Lv})$ is a context and $X'(\mathfrak{s}) \leq T(\omega)$ for every word ω. Let D be an assembled datum over K, Y a layered datum on D with $\Pi(K, D, Y)$, $(L, X, (N^{\mathrm{gl}}_\omega)_\omega)$ a record part of Y as given by the record-part construction, and $(D^\natural, L^\natural, X, Y)$ its member. Then:*

(Mc1) Provenance. *$D^\natural$ is built over $\mathfrak{P}$ (so with the rank datum $\mathfrak{R}$, the host data, the auxiliary primes g_1, g_2, the distinguished primes p_1, p_2 and the words Ω of the witness and of that existence theorem, the word ω carrying the code $c(\omega)$) with the level primes of* Lv*, which contain $\{g_1, g_2, p_1, p_2, q_0\}$; K satisfies (Cx3) with the tops T, the spacing set S and the element u_0 of the witness; and the top numbers of $L^\natural$ at a top $t' \in T$ are $a(t'), b(t')$.*

(Mc2) Host arithmetic. *For $i = 1, 2$ there is $\pi_i \in \mathcal{O}_E \cap K$ with $\pi_i \bar{\pi}_i = g_i$, and $g_i \equiv 1 \pmod{N_\mathfrak{h}}$; p_1, p_2 split completely in E; every level prime of $D^\natural$ (that is, of* Lv*) splits in*

K.

(Mc3) Tame and Minkowski conditions. *(La2)(e) and the Minkowski choice of D_Γ in (La2)(f) hold for $D^\natural$.*

(Mc4) The auxiliary level prime. *q_0 is a level prime of $D^\natural$, $q_0 \equiv 1 \pmod{\ell_0 N_\mathfrak{h}}$ with ℓ_0 as in (Hg5), and*

$$\log q_0 \le L(\mathfrak{s}), \qquad q_0 < X'(\mathfrak{s}),$$
$$q_0 \le W_{\mathrm{tg}}(L^\natural) := \frac{m\, b(\mathfrak{s})\, x^{\varepsilon}}{k\,(m+t)\,\log\bigl(3(b(\mathfrak{s}) + S_c + \eta_S + t + 1)\bigr)},$$

with S_c, η_S those of $L^\natural$ and k, m, t those of $\mathfrak{R}$; moreover $q_0 \notin \{p_1, p_2, g_1, g_2\}$ and $q_0 \neq T(\omega)$ for every word ω.

(Mc5) Tuples, stability, twists. *(La4) holds for $D^\natural$; if the spaces $\mathcal{M}_t$ of D are stable, so are those of $D^\natural$; and if D with the twist packs and the theta record of $O(Y)$ has all twists good, so has $D^\natural$ with the same object.*

Proof. Outline. The member changes only the divisor lattices and word heights of D and the lengths and lattices N_ω of L, so (Mc1), (Mc3) and (Mc5) follow because the conditions involved read only retained data; (Mc2) follows from (Hg5) and the norm equation of the Gauss primes, which makes them split in K; and (Mc4) follows from the size bounds of (Hg5), (Hg6) and the bounds of Proposition A.18 for k, t, S_c, η_S, combined with elementary growth inequalities in x. □

A.5.2 The first pass

The first pass, (H2) for the concrete system, is Proposition A.54: over every forced untwisted datum, the stage steps lead to a core with the slot Built. Its proof is a short assembly of the stage steps, which the following sketch describes.

Sketch: the stage steps of the first pass. A state records, level by level, the data chosen so far: a row of theta series at the first levels, then a saturated state with its coefficient order and normalised theta generators, then a measured Hida setting with transports at the third level, and the fourth level on top of it. At the fourth level a core is assembled and receives the slot Built; theta data with record functions, whose multiplicities are Petersson-isometric, and ambient and divisor data complete it. Each step asks for finitely many new objects (levels, characters, discrete series models, theta lifts at a given level, integral structures) chosen compatibly with the earlier ones. Heuristically each step should hold because the objects it asks for exist individually by classical results: discrete series models of given level and signature, theta lifts of them at controlled level, and integral structures on finite-dimensional spaces of automorphic forms. The work is to make the choices compatible and uniform in the context, and the compatibility conditions are finitely many closed conditions, each met above a threshold. The climb from a saturated state at the third level to a record of level three, and the assembly of the Built core at level four, carry most of this bookkeeping.

Proposition A.54. *Let $c \in \mathbb{R}$ and $\mathfrak{S} = \mathfrak{S}(\mathcal{N}_{\mathrm{or}}(c))$. Eventually in K, for large data (B, f, T) at K: if $\mathrm{BP}(B, f)$, T is forced at K and T is untwisted, then there are a column datum w over (B, f, T), a state Z of w and a core over $(K, (B; C(w), Z), f, T)$ that is Built with canonical entries for $\mathfrak{S}$ at (w, Z).*

Proof. Step 1: stage steps compose. Let p, q, r be properties of tuples (K, B, f, T, w). (a) If p implies q for every tuple, then $p \Rightarrow q$ is a stage step (it holds at every context, with every demand). (b) If $p \Rightarrow q$ and $p \Rightarrow q'$ are stage steps, so is $p \Rightarrow q \wedge q'$; (c) if $p \Rightarrow q$ and $q \Rightarrow r$ are stage steps, so is $p \Rightarrow r$. For (b) and (c): the sets of contexts at which the two statements hold belong to the filter of "eventually", hence so does their intersection; at such a context K the two large-data statements hold for all triples meeting demands $\rho_1 = (S_1^1, P_1^1, A^1, S_2^1)$ and $\rho_2 = (S_1^2, P_1^2, A^2, S_2^2)$. The argumentwise union $\rho = (S_1^1 \cup S_1^2, P_1^1 \cup P_1^2, A^1 \cup A^2, S_2^1 \cup S_2^2)$ is a demand (finite unions of finite sets of algebraic numbers, of finite sets of primes and of avoidance maps), and a triple meeting ρ meets ρ_1 and ρ_2: containment passes to subsets, $P_1^i \subseteq P_1^1 \cup P_1^2 \subseteq \mathrm{pl}(T)$, and $A^i \leq A^1 \cup A^2$, so T avoids A^i by (Tp3). For such a triple with $\mathrm{BP}(B, f)$ and T forced, a column datum with p has q and q' (case (b)), resp. has q and therefore r (case (c)).

Step 2: the chain of stage steps. The start property implies "pinned" (by definition), so by (a) start $\Rightarrow$ pinned is a stage step. By the stage proposition from the start to level three, start $\Rightarrow \mathrm{Has}_3(\mathfrak{S})$; with (b), start $\Rightarrow$ pinned $\wedge\, \mathrm{Has}_3(\mathfrak{S})$. By the same proposition, $\mathfrak{S}$ satisfies the level-three move condition, so the stage proposition from level three to level four (through the archimedean spectral setting) gives pinned $\wedge\, \mathrm{Has}_3(\mathfrak{S}) \Rightarrow \mathrm{Has}_4(\mathfrak{S})$; by (c) and (b), start $\Rightarrow$ pinned $\wedge\, \mathrm{Has}_4(\mathfrak{S})$. By the divisor, ambient and start proposition and the existence of theta data over every level-four record, $\mathrm{AmbEx}(\mathfrak{S})$, $\mathrm{DivEx}(\mathfrak{S})$ and $\mathrm{ThEx}(\mathfrak{S})$ hold, so the proposition constructing the core from level four gives pinned $\wedge\, \mathrm{Has}_4(\mathfrak{S}) \Rightarrow \mathrm{Has}_C(\text{Built with canonical entries for } \mathfrak{S})$. By (c),

$$\text{start} \implies \mathrm{Has}_C(\text{Built with canonical entries for } \mathfrak{S})$$

is a stage step.

Step 3: a column datum with the start property. The statement of Step 2 and the existence of column data with the start property (from the divisor, ambient and start proposition) both hold eventually and for large data; as in Step 1, both hold at every context of one set of the filter, for every triple meeting one demand. For such a triple (B, f, T) with $\mathrm{BP}(B, f)$, T forced and untwisted, the divisor, ambient and start proposition gives a column datum w over (B, f, T) with the start property, and Step 2 gives $\mathrm{Has}_C(\dots)$ at w: a state Z of w and a core over $(K, (B; C(w), Z), f, T)$ that is Built with canonical entries for $\mathfrak{S}$ at (w, Z). □

A.5.3 Discrete series, formal degrees and masses

The spectral input at the members is classical, and the results here are proved. Under the gap hypothesis every Hida setting has the decay property and an integrable archimedean test function (Proposition A.55). Harish-Chandra's existence theorem and formal degree formula for discrete series [119], in the normalisation of the compact dual, are Theorem A.58, and Schur orthogonality holds for multiplicity maps of realised data (Proposition A.57). Together they make evaluation at the vector matching the lowest K-type an exact map of row multiplicity maps onto the conjugate rows (Proposition A.56). Finally, for every based hermitian space of constant signature, volumes are masses (Proposition A.59); this gives (H5) and the count identity.

Proposition A.55. *Let D be an assembled master datum with the gap hypothesis and X a Hida setting for D. Then X has the decay property and its archimedean test function ϕ_X is integrable for μ_∞.*

Proof. Outline. The decay property follows from the decay proposition for Hida settings, and integrability of ϕ_X from the integrability of discrete-series test functions, whose hypotheses

(Harish-Chandra parameters, lowest K-types and the opposite-sign gap of (Sp8)) are checked at every archimedean place. □

Proposition A.56. *Let D be an assembled master datum with the gap hypothesis, $(R, \Phi_2, e_{\mathcal{E}})$ a row package of D whose level characters are those of the tuple μ_0 and such that R has models, X a Hida setting for D whose instance is the standard instance of D and which pins E, and $t \in \mathbf{G}(\mathbb{A})$ a finite point at which R is visible in X. Then there is a unit vector $v_0 \in \prod_\tau \mathcal{H}_\tau^\infty$ such that*

1. *for every $T \in \mathrm{M}_R^{\mathrm{row}}$ and every $f \in L_X^2$ with $f(y) = T(v_0)(\iota_X^{-1}(y)\, t)$ for μ_X-almost every $y \in \mathcal{F}_X$, one has $f \in \overline{\mathrm{H}}_X^{\mathrm{row}}$;*
2. *for every $f \in \overline{\mathrm{H}}_X^{\mathrm{row}}$ there is $T \in \mathrm{M}_R^{\mathrm{row}}$ with $f(y) = T(v_0)(\iota_X^{-1}(y)\, t)$ for μ_X-almost every $y \in \mathcal{F}_X$.*

Proof. Outline. Harish-Chandra's conjugate-parameter and uniqueness theorems give antiunitary intertwiners that define v_0; Schur orthogonality makes the operator of ϕ_X times the level idempotent a projector with the matrix-coefficient identity, which attaches to each row vector an intertwiner from the discrete series, cuspidal because the constant terms of discrete-series matrix coefficients vanish; assertion 2 then follows from the bijection between such intertwiners and multiplicity maps, and assertion 1 from the Schur orthogonality of Proposition A.57 and the same projector. □

Proposition A.57. *Let D be an assembled master datum and $(R, \Phi_2, e_{\mathcal{E}})$ a row package of D such that R has models. Then for every $T \in \mathrm{M}_R^{\mathrm{row}}$:*

1. *for all $v, v' \in \prod_\tau \mathcal{H}_\tau^\infty$ the function $g \mapsto \overline{T(v)(g)}\, T(v')(g)$ is ν_R-integrable on $\mathcal{F}_R$;*
2. *there is $a \in \mathbb{C}$ such that $\int_{\mathcal{F}_R} \overline{T(v)}\, T(v')\, d\nu_R = a \prod_\tau \langle v_\tau, v'_\tau \rangle$ for all $v, v' \in \prod_\tau \mathcal{H}_\tau^\infty$.*

Proof. Outline. One realises the discrete series of the row package as a discrete series model of Y_0, checks condition by condition that every $T \in \mathrm{M}_R^{\mathrm{row}}$ is a multiplicity map of this model with the row level character, and applies Schur orthogonality for multiplicity maps to the Haar pair $(\nu_R, \mathcal{F}_R)$. □

Theorem A.58. *Let $N \geq 1$, $\varepsilon \in \{\pm\}^N$ and $\lambda \in \mathbb{R}^N$ strictly decreasing with $\lambda_a - \frac{N-1}{2} \in \mathbb{Z}$ for all a.*

1. *There is a discrete series of G_ε with parameter λ.*
2. *Let* cd *be a compact-dual Haar family and* sc *a scale for* cd*. Then for every discrete series σ of G_ε with parameter λ, $\mathrm{d}_{\mathrm{HC}}(\lambda)$ is a formal degree of σ for $\mathrm{sc}(N, \varepsilon)\, \mu_\varepsilon$.*

Proof. Outline. Both parts are Harish-Chandra's existence theorem and formal degree formula for discrete series in the compact-dual normalisation, once the scaled measure $\mathrm{sc}(N, \varepsilon)\, \mu_\varepsilon$ is identified with the compact-dual Haar measure by transporting the invariant holomorphic top form along conjugation by the permutation matrix that orders the signs. □

Proposition A.59. *Let E/F be a CM extension of number fields unramified at every finite place,* cd *a compact-dual Haar family and* vol *a function on based hermitian spaces over E such that* $(\mathrm{cd}, \mathrm{vol})$ *has the volume tie. Then $\mathrm{vol}(W) = \mathrm{mass}_E(n)$ for every based hermitian space W over E of rank n with $\mathrm{sig}_\tau(W) = (P, Q)$ for every τ (the same pair (P, Q) for all τ).*

Proof. Let W be a based hermitian space of rank n with $\mathrm{sig}_\tau(W) = (P, Q)$ for every $\tau \in \Phi$, and $W_0 = E^n$ with the standard basis, Gram matrix 1_n and base lattice $\mathcal{O}_E^n$, a definite based hermitian space of rank n. We show $\mathrm{vol}(W) = \mathrm{vol}(W_0)$; since $\mathrm{cd}_{N,0}$ is a probability measure for every N (Sp1), the definite case of the volume–mass identity gives $\mathrm{vol}(W_0) = \mathrm{mass}_E(n)$. Fix Ω_0 as in the local volume formulas for the Gram-twisted top form and let $d = |\Phi| = [F : \mathbb{Q}]$.

The archimedean constants. Let $\tau \in \Phi$. All invariant complex top forms on $\mathfrak{gl}_n(\mathbb{C})$, the common complexified Lie algebra of $\mathrm{U}(P', Q')$ ($P' + Q' = n$), are complex multiples of $\Omega^\tau_{\mathbb{C}}$, and multiplying the form by $z \in \mathbb{C}^\times$ divides all densities against it by $|z|$. Hence (Sp1) says: for every $P' + Q' = n$ there is $c \in (0, \infty)$ with $\mathrm{cd}_{P',Q'} = c\,\nu^\tau_{P',Q'}$ and $\mathrm{cd}_{n,0} = c\,\nu^\tau_{n,0}$. The second equation determines $c = c_\tau$ independently of (P', Q'):

$$\mathrm{cd}_{P',Q'} = c_\tau\,\nu^\tau_{P',Q'} \qquad (P' + Q' = n,\ \tau \in \Phi). \tag{A.5}$$

The Tamagawa measure is a fixed multiple of the volume measure. Let $W' \in \{W, W_0\}$, of signature (P', Q') at every τ, with Gram matrix Γ' and $\delta = \det \Gamma' \in F^\times$. By the volume tie (Vm2) there are a volume measure $\mu_{W'}$ for W' relative to cd and a measurable fundamental domain $\mathcal{F}_{W'}$ of $\mathrm{U}(W')(F)$, and $\mathrm{vol}(W') = \mu_{W'}(\mathcal{F}_{W'})$. Choose A_τ with $A_\tau^* J_{P',Q'} A_\tau = \tau(\Gamma')$ (Sylvester), Borel sets $B_\tau \subseteq \mathrm{U}(P', Q')$, and put $B_\infty = \{h_\infty :\ A_\tau \tau(h_\tau) A_\tau^{-1} \in B_\tau\ (\tau \in \Phi)\}$. The stabiliser $K_{\mathrm{f}}(W') = \prod_{v<\infty} K_v(W')$ of $\widehat{L_0(W')}$ is open in $\mathrm{U}(W')(\mathbb{A}_{F,\mathrm{f}})$. By the construction of the Tamagawa measure (absolute convergence of its local factors and the product formula defining it) with $\omega = \omega_{W'}$, then the local volume formulas for the Gram-twisted top form and (A.5),

$$\begin{aligned}
\tau_{\mathrm{U}(W')}\big(K_{\mathrm{f}}(W') \times B_\infty\big) &= |d_F|^{-n^2/2} L(1,\chi)^{-1} \prod_{v<\infty} \lambda_v |\omega_{W'}|_v(K_v(W')) \\
&\quad \cdot \prod_{\tau\in\Phi} |\omega_{W'}|_\tau\big(\{h_\tau : A_\tau \tau(h_\tau) A_\tau^{-1} \in B_\tau\}\big) \\
&= |d_F|^{-n^2/2} L(1,\chi)^{-1} \prod_{v<\infty} \lambda_v |\omega_{W_0}|_v(K_v(W_0)) \\
&\quad \cdot \prod_{v<\infty} |\delta|_v^n \prod_{\tau\in\Phi} |\tau(\delta)|^n \cdot \prod_{\tau\in\Phi} c_\tau^{-1} \mathrm{cd}_{P',Q'}(B_\tau).
\end{aligned}$$

The splitting of the infinite product is legitimate because $|\delta|_v = 1$ for almost all v, and the product formula for $\delta \in F^\times$ gives $\prod_{v<\infty} |\delta|_v \prod_\tau |\tau(\delta)| = 1$ (the real places of F are the restrictions of $\tau \in \Phi$, each once). With

$$\Lambda = |d_F|^{-n^2/2} L(1,\chi)^{-1} \prod_{v<\infty} \lambda_v |\omega_{W_0}|_v(K_v(W_0)) \prod_{\tau\in\Phi} c_\tau^{-1} \in (0, \infty)$$

(positive by the absolute convergence of the local factors), which depends on E/F, n, Ω_0 and cd but not on W', and by (Vm1),

$$\tau_{\mathrm{U}(W')}\big(K_{\mathrm{f}}(W') \times B_\infty\big) = \Lambda \prod_\tau \mathrm{cd}_{P',Q'}(B_\tau) = \Lambda\,\mu_{W'}\big(K_{\mathrm{f}}(W') \times B_\infty\big).$$

Taking for each B_τ a compact neighbourhood of 1, the set $K_{\mathrm{f}}(W') \times B_\infty$ has positive finite Haar measure, so by uniqueness of Haar measure $\tau_{\mathrm{U}(W')} = \Lambda\,\mu_{W'}$.

Conclusion. The Tamagawa number of a unitary group is 2 [269], [195], [150]: $\tau_{\mathrm{U}(W')}(\mathcal{F}_{W'}) = 2$. Therefore $\mathrm{vol}(W') = \mu_{W'}(\mathcal{F}_{W'}) = 2/\Lambda$ for both $W' = W$ and $W' = W_0$. Hence $\mathrm{vol}(W) = \mathrm{vol}(W_0) = \mathrm{mass}_E(n)$. □

A.5.4 Export and the second pass

The export property (H3) asks for one universal twist function N_0 with the conditions (Ex1) and (Ex2) attached to the actual theta datum (Proposition A.60). The second pass (H4) asks for a simultaneous twist and point lift, a reference-preserving extension of the realised data, and one new core that meets uniform demands (Proposition A.61). Its main input is the reference-preserving coefficient enlargement Proposition A.62. The reason the reference estimates survive the enlargement is that the second pass keeps the geometric core, the reference models, the vacuum, the theta maps and the small rational coordinate space literally, so that the signed indices scale exactly, $I'(v') = e(v'/v)\, I(v)$ (item 5 of Proposition A.62). All three statements are given with sketches.

Proposition A.60. *For an assembled datum D and an object O, let* $\mathrm{Ex}(D, O)$ *mean the following two assertions, for the integral Hecke-form datum E_O of the theta datum* $\mathrm{xd}(O)$.

(Ex1) *There is one function N_0 assigning a nonnegative integer to each twist situation i of D, such that at every nonzero index situation i: if its Bloch–Kato set condition holds, then for every $\epsilon' > 0$, every choice of finite sets $\mathcal{S}(j, b)$ indexing bases of the full-rank Hecke-form subspaces of E_O ($1 \le j \le m$, $b \in \{0, 1\}$), and every finite set A of places of the host field, there are a finite set Q of split places, an integer $k' \ge 0$ and a character ψ of the corresponding ray-class quotient, with values in the k'-th coefficient extension at i, such that every power ψ^a with $\gcd(a, \ell_i) = 1$ is a good twist for these same $N_0, \epsilon', \mathcal{S}, A, Q, k'$.*

(Ex2) *There is one family of finite sets $\mathcal{S}(i, j, b)$ such that for every place w with $\mu_{\mathrm{at}}(w) \neq \mu_0$, the Bloch–Kato set condition holds at the situation $i(w)$ read from* $\mathrm{nch}(O)$, *and the twist input $(E_O, N_0, \mathcal{S})$ is admissible at $i(w)$ in the sense of (Xh5): all its prescribed Hecke-form subspaces have full rank and the corresponding sets $\mathcal{S}(i(w), j, b)$ index bases of them.*

Fix a parameter p whose norm choices carry the full extension law. Put

$$Q = \mathrm{Built}_{\mathrm{E}}(\mathfrak{S}(p)).$$

Then (H3) of Theorem A.49 holds for this Q and Ex*: eventually in the context K, for large data (B, f, T) at K, if* $\mathrm{BP}(B, f)$ *and T is untwisted, then* $\mathrm{Ex}(D(X), O(c))$ *for every stage tuple X over B and every core c with $Q(K, X, f, T, c)$. The same N_0 and the same admissible basis sets are used in (Ex1) and (Ex2).*

Proof. Sketch. The proposition is the export condition (H3) of Theorem A.49 for the concrete two-pass system of Theorem A.51: every core produced by the first pass (Proposition A.54) must carry one twist input, built from its own theta datum, that the second pass can use. Clause (Ex2) is a finiteness and admissibility assertion: at each twisted place the Hecke-form subspaces cut out of the integral Hecke-form datum E_O must have full rank, and finite sets indexing bases of them must be fixed once and for all, together with the Bloch–Kato set condition at the situations $i(w)$. Heuristically this should follow from the integrality of the theta datum: the normalised Hecke operators preserve the integral sections (Lemma A.30), so the Hecke-form subspaces should be lattices of full rank in their rational spans, and finitely many basis indices can then be chosen.

Clause (Ex1) is the substantial one. For one threshold function N_0, fixed in advance, and for every finite set A of places, one needs a finite set Q of split places and one ray-class character ψ of ℓ-power order, ramified only at Q, such that the Bloch–Kato Selmer group [34] of each twisted representation $\psi_s^{-1}\chi_s^{-1}\rho_s$ of a good signature s, with unrestricted conditions at Q, has

normalised length at most $\nu_\ell + \varepsilon_T(s)$, where ν_ℓ is the cyclotomic valuation of the host field and $\sum_s \varepsilon_T(s) \le \ell^{-2}$ (Definition A.104). The argument should run as in the classical constructions of auxiliary twists: by the Chebotarev density theorem [65] there are infinitely many split places outside A with prescribed Frobenius behaviour, and a character ramified at a suitable finite set of them should cut the Selmer group of the twist down to the part forced by the ℓ-power roots of unity, which ν_ℓ measures. That a single ψ serves all its powers ψ^a with $\ell \nmid a$ is heuristically automatic once the coefficients of ρ_s, χ_s and of the local conditions lie in a field linearly disjoint from the values of ψ: the automorphism $\zeta \mapsto \zeta^a$ of the coefficient extension then fixes all the data except ψ, sends ψ to ψ^a, and preserves Selmer lengths.

The local Hecke inputs of this construction (the exact complex reading of the geometric Hecke transform, the preservation of integral sections and the bound for the denominators of theta values) are proved, but they do not by themselves give (Ex1) or (Ex2). It remains to construct the twist with one threshold function valid for all twist situations, to attach it to the actual theta datum and numerical choice of each first-pass core, and to verify the remaining clauses of goodness: admissibility of the character, the Hecke-lattice index and inclusion conditions, and the theta-value and tame-valuation conditions. The Selmer bound obtained in this way is the input from which Proposition A.108 produces the Selmer allowances of the members in Theorem A.105. □

Proposition A.61. *Fix a parameter p whose norm choices carry the full extension law. Fix an arbitrary large-base slot R over $\mathrm{Built}_{\mathrm{E}}(\mathfrak{S}(p))$ in the sense of (Rk7) of the definition of parameters and their slot systems; R is fixed before all choices of the second pass. Put*

$$Q = \mathrm{Built}_{\mathrm{E}}(\mathfrak{S}(p)), \qquad Q' = \mathrm{Good}_{\mathrm{E}}(\mathfrak{S}(p)) \wedge R.$$

For a pair (D, O) put $\tau(D, O)$ equal to the assertion that $(D, \mathrm{nch}(O), \mathrm{xd}(O))$ has all twists good, including the conjugation conditions in (Xh5). Then (H4) of Theorem A.49 holds for these Q, Q', Ex, τ.

Explicitly, eventually in the context K, for large data (B, f, T) at K with $\mathrm{BP}(B, f)$ and T untwisted, the following assertion holds. Given a stage tuple X over B and a core c with

$$Q(K, X, f, T, c), \qquad \mathrm{Ex}(D(X), O(c)),$$

for every avoidance map A and every map S_2 from subsets of $\mathfrak{X}(P)$ to finite sets of algebraic numbers there are an enlargement B' of B, a twisting datum T' over B', a stage tuple X' over B' and a core c' such that

$$\mathrm{pl}(T') = \mathrm{pl}(T), \qquad T' \text{ avoids } A, \qquad S_2(\Xi(f, T')) \subseteq F_{\mathrm{L}}(B'),$$

$$Q(K, X', f, T', c'), \qquad Q'(K, X', f, T', c'), \qquad \tau(D(X'), O(c')).$$

The pre-base and frame are those of B; the enlargement retains the exact and fixed embedded fields and enlarges the ambient coefficient field. The construction of c' keeps the reference models, the vacuum and both theta maps of c, through Proposition A.62. All data in the conclusion belong to the one new core c'. No positivity of the dimensions of the definite form spaces is imposed.

Proof. Sketch. The proposition is the second-pass condition (H4) of Theorem A.49. From a first-pass core c that is Built with canonical entries and satisfies the export property of Proposition A.60, one must produce, after an enlargement of the coefficient field, one new core c' that is again Built, is Good, has the prescribed large-base slot R, and has all twists good. The argument should run in four steps.

First, at each twisted place choose, through (Ex1), one character ψ all of whose powers prime to ℓ are good twists, with its auxiliary split places outside the requested avoidance set, and adjoin its values and the finitely many requested algebraic numbers to the ambient coefficient field, keeping the pre-base, the frame and the exact and fixed fields; this gives the enlargement B' and a twisting datum T' with $\mathrm{pl}(T') = \mathrm{pl}(T)$. Second, transport the realised row data to the enlarged field by Proposition A.62. That transport keeps the geometric core, the reference models, the vacuum and both theta maps, and multiplies the signed reference indices by the ramification indices, so the places of nonzero index correspond, also when the definite form spaces are zero. Third, lift the pinned point and its state over the enlarged field and rebuild, in order, the level-three record (by promoting the kept numerical choice with its original pre-state, level and saturation data), the level-four record, the ambient, divisor and theta data, and from these three inputs the core with canonical entries. Fourth, replace the numerical choice inside this one Built record by one with the good-twist property: by the definition of the slots this gives Good while keeping the original canonical kernel entries, and the conjugation conditions with the same admissible twist input give all twists good. Since R is fixed before all choices, its finite field, prime and avoidance demands can be added to the requests by finite unions, and the new core then has R as well.

The reductions in the third and fourth steps and the finite-demand argument are proved, but they rest on Proposition A.62 and on stage constructions of the first pass whose proofs are not complete, and they need the finiteness statement for the kept numerical choice in a form with arbitrary spectral and geometric arguments. It remains to construct the simultaneous twisting datum and the lift of the point and state over the enlarged field, all attached to the one chosen character, and to assemble one context tail and one large-data demand from the finitely many construction demands. Heuristically the construction is a base change: the enlargement changes only the coefficient field and the twist characters, the geometric objects are defined over the exact field, which does not change, and the twists are chosen at split places outside a finite set, of which there are always enough. □

Proposition A.62. *Eventually at all contexts K, for large data (B', f', T') over K, the following holds for every pinned column datum w' over (B', f', T').*

Let (B, f, T) be over the same context, with the same pre-base and frame, and let w be pinned over (B, f, T) with the same pre-extension as w'. Assume

$$F_{\mathrm{ex}}(B') = F_{\mathrm{ex}}(B), \qquad F_{\mathrm{fix}}(B) \subseteq F_{\mathrm{fix}}(B'), \qquad L = F_{\mathrm{L}}(B) \subseteq L' = F_{\mathrm{L}}(B').$$

Let Z be a state of w, with its exact field datum and group compatibilities, and let W_t be a set of places of L. Suppose realised row data $\mathcal{R}$ over $(D(w, Z), I, T, W_t)$ are given with:

1. *their real data R analytically characterised, their geometric half analytically characterised, their geometric core canonical with the prescribed coefficient ring and frame, their standard definite bridges, and positive-dimensional moduli host spaces;*

2. *the standard row-realisation premises: the right and left Hecke lattices are good with rational inert adjoints; the Hecke, group-unitarity and group-equivariance laws; the block law; both lift tests; the conditional lift-onto law; uniformisation; the full member-lattice law; the level law for the W-spaces; the theta theorem law; the parts law; the auxiliary choice laws; and admissibility at 1 with type separation.*

No positivity of the dimensions of the automorphic form spaces is assumed.

For every state Z' of w' with its exact field datum, the group compatibility with f' and I', and the volume tie, there are realised row data $\mathcal{R}'$ over

$$(D(w', Z'), I', T', W'_t), \qquad W'_t = \{v' \in \mathrm{Pl}(L') : \mathrm{reschar}(v') \in \mathrm{pl}(T')\},$$

whose real data and geometric half are analytically characterised, whose real data R' satisfy the group law (chiW) of (Xl2b)(j) in the definition of realised row data, and whose definite bridges are standard, with the following simultaneous compatibilities.

1. *The exact field F_c, moduli host, level objects, coefficient datum, schemes, all uniformisations, host spaces, dictionaries, functions μ, compact levels, component constants, characters χ_W, base points and lattice maps of the geometric core are the same as for $\mathcal{R}$. Only its embedding into L' is extended. The pinned exact coefficient ring is unchanged up to its unique embedded-field identification.*

2. *The additive character, archimedean embedding family, self-dual host lattices, level matrices, auxiliary group places, placewise group embedding, complex coefficient spaces of the V_i and their representations, fixed column vacua, column Fock bilinear maps and column coefficient matrices are the same as for $\mathcal{R}$. For every embedding $\sigma' : L' \hookrightarrow \mathbb{C}$, the discrete-series model, its action, inner product, signs, frame, extremal-vector indexing and host Hilbert module are the old ones at $\sigma'|_L$. The Haar measures and fundamental domains of the W_i-models are retained; the reference definite-side measures are retained as specified next.*

3. *At the reference tuple μ_0 the column Weil representation, its chosen Schwartz vector, the definite Hilbert model, its Haar measure and fundamental domain, and both theta maps are those of $\mathcal{R}$. The complex readings of the definite form maps agree on every pair of base-tuple coefficient functions with identical embedded polynomial values.*

4. *The integers $n_W(i)$ and the geometric weights and degrees $\kappa_i = (a_{P,i}, a_{M,i}, n_i)$ are unchanged. In the coordinate spaces $L^{n_W(i)} \subseteq (L')^{n_W(i)}$, the small F_c-rational coordinate space is*

   $$\mathrm{FW}_i'^{F_c} = \mathrm{FW}_i^{F_c},$$

   as the image of $\mathrm{FW}_i^{F_c}$ itself, not its L'-span. For x in this space,

   $$j'_{i,\sigma'}(x) = j_{i,\sigma'|_L}(x).$$

   The points defining its local norms and those norms are the same, and the coordinate characters and group actions are extended coefficientwise. For every geometric cusp section s,

   $$e'_i(s) = e_i(s) \quad \text{in } (L')^{n_W(i)}.$$

   Consequently its distinguished complex reading and every coefficient coordinate agree with those of $\mathcal{R}$. The geometric extremal vector at σ' is the old one at $\sigma'|_L$.

5. *Let $v' \mid v$ be finite places, with ramification index $e(v'/v)$. At each row step the definite lift lattice, its hermitian form and its hermitian dual, and the reference theta-image and member lattices used in the signed indices are their $\mathcal{O}_{L'}$-coefficient extensions under the same reading maps. Thus each of the two signed relative indices I_i, J_i satisfies*

   $$I'_i(v') = e(v'/v) I_i(v), \qquad J'_i(v') = e(v'/v) J_i(v).$$

In particular the nonzero-index place set at zero error corresponds exactly:

$$v \in W_1(\mathcal{R}; 0) \quad \Longleftrightarrow \quad v' \in W_1(\mathcal{R}'; 0).$$

Here W_1 is the set obtained from the nonzero reference row indices and their conjugates. The same equivalence holds simultaneously at every place above v.

All the data in these conclusions belong to the one realised row $\mathcal{R}'$.

Proof. Sketch. The proposition says that realised row data survive an enlargement $L \subseteq L'$ of the coefficient field with the same exact field, with every reference object kept unchanged. The argument runs by transport along the inclusion $\iota\colon L \to L'$. Every complex embedding σ' of L' restricts to the embedding $\sigma'\iota$ of L, so all analytic objects at σ' (discrete-series models, frames, inner products and theta maps) can be taken to be the old ones at $\sigma'\iota$. The geometric core, with its exact field, moduli host, uniformisations and level structures, is carried over unchanged, only its embedding being extended; this transport, the preservation of the core and tie laws on the enlarged geometric half, and the retention of the level pins are proved.

The lattice part is pure coefficient extension. At a place $v' \mid v$ the lattices are the $\mathcal{O}_{L'}$-extensions of the old ones under the same reading maps, hermitian duality commutes with this flat extension, and for lattices $M \subseteq N$ the length of $\mathcal{O}_{L'_{v'}} \otimes (N/M)$ is $e(v'/v)$ times the length of N/M. This gives $I'_i(v') = e(v'/v)I_i(v)$ and $J'_i(v') = e(v'/v)J_i(v)$, so the places of nonzero index correspond, in rank zero as well (item 5). The integral reference comparison, the phase and short-theta comparisons, the convergence and cuspidal growth of the global theta integrals, their factorisation through finite relations and the smooth extension at the same level are also proved.

It remains to prove existence. The inner and last theta lifts, the projected chain map and the multiplicity-space theta map must be shown to exist for the enlarged data. The conjugate-linear model identification, the polynomial frame identification, the integer determinant shift and the compact splitting characters must be attached to the retained data. The characterised real data and geometric half must then be assembled simultaneously in one realised row, with one context tail for all the large-data demands. Heuristically nothing new should be needed: the characterisations of the real data and of the geometric half are analytic conditions read through complex embeddings, which all factor through L, so the old objects should satisfy them as they stand, and the only fresh choices, the splitting and auxiliary characters, are finite in number and constrained at finitely many places. This is the mechanism by which the reference estimates survive the second pass: the reference data are kept unchanged, and the signed indices scale by the ramification indices. □

A.6 Approximate orthogonality I: Gram blocks of Poincaré series

The slope inequality of the body of the paper needs the mean slope of the target lattice, not its maximal slope, and this comes from near-orthogonal frames (Theorem 14.2, Theorem 14.8). Alpöge describes the input as a bound for the norm of a suitably projected Poincaré series, an "approximate orthogonality of the columns of the integral structure" [4]. This section and the next supply it at the members. Each member carries a Poincaré frame realisation for every word, and the off-diagonal Gram blocks of these frames unfold exactly into term operators over double cosets (Lemma A.64, Lemma A.65). Near-orthogonality thus becomes an exact lattice-point problem with no error term. The row records (Ro1)–(Ro4) of Theorem 15.3, proved in the body, turn bounds for these operators into the row estimates (Rw1), and each kind of row

has its own control. Loop rows are bounded by Siegel products of local densities (Proposition A.66); non-generic rows by a point sum over B-points (Proposition A.67, with the point sum Proposition A.87 of Section A.7); and the main row needs a *lower* bound, obtained from density domination (Proposition A.68), whose archimedean factor is the engine place theorem A.78 and whose finite-place factor is a local theta majorant (Proposition A.69). Theorem A.71 collects the resulting records and kernel hypotheses at the packets, which is what the second item of Proposition A.2 reads.

A.6.1 Unfolding the Gram blocks

Definition A.63 attaches to a member Hida and Poincaré setting its Poincaré frame realisations. The inner product of two Poincaré series unfolds over double cosets (Lemma A.64), and in orthonormal coefficient bases the Gram blocks of a frame realisation are exactly the term operators of the double cosets (Lemma A.65). Both proofs are standard unfolding, and we give their ideas.

Definition A.63. Let $\mathfrak{X}$ be a member Hida and Poincaré setting and let frame data $(\mathcal{H}, D^*, (D_\omega, \kappa_\omega, A_\omega)_\omega)$ of a frame realisation be given, indexed by the same words ω. For a column system $(h_c)_{c\in R_\omega}$ of ω and $T \in \mathrm{Cu}_\omega$ let $T_c^\sharp \in V_\omega$ $(c \in R_\omega)$ be the vector with $\langle T_c^\sharp, v\rangle = T(v)(1, h_c)$ for all $v \in V_\omega$; the **coefficient product** of this column system is

$$\langle T, T'\rangle_{\mathrm{Cu}_\omega} = \sum_{c\in R_\omega} \langle T_c'^\sharp, T_c^\sharp\rangle_{V_\omega} \qquad (T, T' \in \mathrm{Cu}_\omega).$$

The frame data form a **Poincaré frame realisation** of $\mathfrak{X}$ if:

(Fj1) Trivial rational stabilisers. For every word ω and every $h \in H_\omega(\mathbb{A}_{F,f})$,

$$H_\omega(F) \cap hK_\omega h^{-1} = \{1\}.$$

(Fj2) Poincaré frames. For every word ω one has $\kappa_\omega = c_\omega^{-1}$, and there are a column system of ω and an orthonormal basis $(T_{\omega,i})_{1\le i\le D_\omega}$ of Cu_ω for its coefficient product such that

$$\langle A_\omega e_i, A_{\omega'} e_j\rangle_{\mathcal{H}} = \langle P_\omega T_{\omega,i}, P_{\omega'} T_{\omega',j}\rangle_{\mathcal{L}} \qquad (1 \le i \le D_\omega,\ 1 \le j \le D_{\omega'})$$

for all words ω, ω'.

Lemma A.64. *Let G be a locally compact second countable group with a left Haar measure μ, let $\Gamma \le G$ be a countable subgroup and $S, S' \le \Gamma$ subgroups. Let $R \subseteq \Gamma$ contain exactly one element of every double coset $S\gamma S'$ $(\gamma \in \Gamma)$, and for $r \in R$ put $S_r = S \cap rS'r^{-1}$. Let $D, D' \subseteq \Gamma$ satisfy $\Gamma = \bigsqcup_{\delta\in D} S\delta = \bigsqcup_{\delta\in D'} S'\delta$. Let $F_\Gamma, F_S \subseteq G$ and $F_r \subseteq G$ $(r \in R)$ be Borel sets such that every orbit Γx, Sx, $S_r x$ $(x \in G)$ meets F_Γ, F_S, F_r respectively in exactly one point. Let $f, g\colon G \to \mathbb{C}$ be Borel functions with $f(sx) = f(x)$ for $s \in S$ and $g(s'x) = g(x)$ for $s' \in S'$ $(x \in G)$, such that*

$$\sum_{r\in R} \int_{F_r} |f(x)|\,|g(r^{-1}x)|\,d\mu(x) < \infty \tag{A.6}$$

and $\sum_{\delta\in D'} |g(\delta x)| < \infty$ for every $x \in G$. Put $P_g(x) = \sum_{\delta\in D'} g(\delta x)$.

(i) P_g is Borel, does not depend on the choice of D', and $P_g(\gamma x) = P_g(x)$ for $\gamma \in \Gamma$, $x \in G$.

(ii) For every $r \in R$ the function $x \mapsto \overline{f(x)}\, g(r^{-1}x)$ is integrable on F_r, the function $\overline{f}P_g$ is integrable on F_S, and

$$\int_{F_S} \overline{f(x)}\, P_g(x)\, d\mu(x) = \sum_{r\in R} \int_{F_r} \overline{f(x)}\, g(r^{-1}x)\, d\mu(x),$$

the series converging absolutely.

(iii) If moreover $\sum_{\delta\in D} |f(\delta x)| < \infty$ for every $x \in G$ and $P_f(x) = \sum_{\delta\in D} f(\delta x)$, then $\overline{P_f}P_g$ is integrable on F_Γ and $\int_{F_\Gamma} \overline{P_f(x)}\, P_g(x)\, d\mu(x) = \int_{F_S} \overline{f(x)}\, P_g(x)\, d\mu(x)$.

(iv) Let $f_1, \dots, f_m$ and $g_1, \dots, g_n$ be Borel functions such that every pair $(f, g) = (f_i, g_j)$ satisfies all hypotheses above, including those of (iii). Then the matrix $M = \big(\int_{F_\Gamma} \overline{P_{f_i}}\, P_{g_j}\, d\mu\big)_{i,j}$ and the matrices $Y_r = \big(\int_{F_r} \overline{f_i(x)}\, g_j(r^{-1}x)\, d\mu(x)\big)_{i,j}$ $(r \in R)$ satisfy

$$\sum_{r\in R} \|Y_r\|_{\mathrm{HS}} < \infty \qquad \text{and} \qquad M = \sum_{r\in R} Y_r.$$

Proof. Sketch. This is the classical unfolding of an inner product of two Poincaré series, written with fundamental sets in place of quotient measures, so that no regularity of the quotients is needed. The core of the proof is an enumeration of $S'\backslash\Gamma$ through double cosets: if $\Sigma_r \subseteq S$ contains one element of each coset σS_r, then $(r, \sigma) \mapsto S' r^{-1}\sigma^{-1}$ is a bijection from the pairs with $r \in R$, $\sigma \in \Sigma_r$ onto $S'\backslash\Gamma$, so that

$$P_g(x) = \sum_{r\in R} \sum_{\sigma\in\Sigma_r} g(r^{-1}\sigma^{-1}x).$$

Integrating against $\overline{f}$ over F_S and substituting $x = \sigma y$, using the S-invariance of f and the left invariance of μ, turns the inner sum into an integral over $\bigsqcup_{\sigma\in\Sigma_r} \sigma^{-1}F_S$, which is again a fundamental set for S_r. A second elementary fact replaces this set by F_r: the integral of an invariant function does not depend on the fundamental set, because left multiplication is free and two fundamental sets are countable disjoint unions of translates of pieces of one another. For nonnegative functions the computation is justified by Tonelli's theorem; the summability assumption on $\sum_r \int_{F_r} |f(x)|\, |g(r^{-1}x)|\, d\mu(x)$ then gives absolute convergence and, by dominated convergence, the same computation for $\overline{f}$ and g. Part (iii) folds P_f in the same way, and (iv) is the entrywise version with $\sum_r \|Y_r\|_{\mathrm{HS}} < \infty$. The result is an exact identity with no error term: the matrix of inner products of Poincaré series is the absolutely convergent sum, over the double cosets $S\gamma S'$, of the terms Y_r. This is how near-orthogonality becomes a lattice-point problem in the row estimates of Theorem 15.3. □

Lemma A.65. *Let $\mathfrak{X}$ be a member Hida and Poincaré setting and let frame data of a frame realisation form a Poincaré frame realisation of $\mathfrak{X}$. Let ω, ω' be words, fix column systems $(h_c)_{c\in R_\omega}$ of ω and $(h_{c'})_{c'\in R_{\omega'}}$ of ω' (the same one if $\omega = \omega'$), put $J_\omega T = (T_c^\sharp)_{c\in R_\omega} \in \bigoplus_{c\in R_\omega} V_\omega$ for $T \in \mathrm{Cu}_\omega$, likewise $J_{\omega'}$, and $\mathcal{H}_1 = \mathrm{Hom}(\bigoplus_{c'\in R_{\omega'}} V_{\omega'}, \bigoplus_{c\in R_\omega} V_\omega)$.*

(i) J_ω is a conjugate-linear bijection of Cu_ω onto $\bigoplus_{c\in R_\omega} V_\omega$ with $\langle J_\omega T, J_\omega T'\rangle = \langle T', T\rangle_{\mathrm{Cu}_\omega}$; the coefficient product of ω does not depend on the column system; and $D_\omega = \dim_{\mathbb{C}} \mathrm{Cu}_\omega = |R_\omega| \dim_{\mathbb{C}} V_\omega$.

(ii) Define $\Theta\colon \mathcal{H}_1 \to \mathrm{Hom}(\mathbb{C}^{D_{\omega'}}, \mathbb{C}^{D_\omega})$ by $\langle e_i, \Theta(Z) e_j\rangle = \langle Z\, J_{\omega'} T_{\omega',j},\, J_\omega T_{\omega,i}\rangle$. Then Θ is a conjugate-linear bijection with $\|\Theta(Z)\| = \|Z\|$ for all Z, and $\Theta(1) = 1$ if $\omega = \omega'$. If $Y \in \mathcal{H}_1$ satisfies $\mathcal{K}_{\omega,\omega'}(T, T') = \langle Y J_{\omega'} T', J_\omega T\rangle$ for all $T \in \mathrm{Cu}_\omega$, $T' \in \mathrm{Cu}_{\omega'}$, then

$$\mathrm{Gr}_{\omega\omega'} = \Theta(Y).$$

(iii) For every countable family (y_t) in $\mathcal{H}_1$ with $\sum_t \|y_t\| < \infty$ one has $\sum_t \|\Theta(y_t)\| = \sum_t \|y_t\|$, $\Theta(\sum_t y_t) = \sum_t \Theta(y_t)$ and $\|\sum_t \Theta(y_t)\| = \|\sum_t y_t\|$. If a vector $K \in \mathcal{H}_1$ has a pair expansion at scale ϑ with price C_{pr}, main constant C_{mn}, loop share β_{lp} and non-generic share β_{ng}, with vectors (y_t, z_t), then $\Theta(K)$ has a pair expansion with the same scale, price, constants, shares, terms, rows, labels, row normalisation, label matrix, corank and scalar pairs, with vectors $(\Theta(y_t), z_t)$.

Proof. Sketch. The lemma is a dictionary between two descriptions of the same finite-dimensional data, and once one formula is in place its proof is linear algebra. By the column decomposition (Yt1), every finite adele can be written as $f = a h_c k$ with $a \in H_\omega(F)$, h_c a column representative and $k \in K_\omega$, and the writing is unique because $H_\omega(F) \cap h_c K_\omega h_c^{-1}$ is trivial by (Fj1). The transformation laws of a cusp form $T \in \mathrm{Cu}_\omega$ then give

$$T(v)(s, a h_c k) = \chi_\omega(k)\, \langle T_c^\sharp, \pi_\omega(a_\infty^{-1} s) v \rangle.$$

This formula is the core of the proof. It shows that T is determined by its column vectors $T_c^\sharp$, and conversely, by the uniqueness of the decomposition, that any family $(q_c)_c$ defines a cusp form through it; so J_ω is a conjugate-linear bijection carrying the coefficient product to the inner product of $\bigoplus_c V_\omega$, and the independence of the column system follows from the unitarity of π_ω and $|\chi_\omega| = 1$. Writing an operator Y in the orthonormal bases $J_\omega T_{\omega,i}$ and $J_{\omega'} T_{\omega',j}$ gives Θ, which is a Hilbert–Schmidt isometry by Parseval, and the definition of the Gram matrices through the kernel $\mathcal{K}_{\omega,\omega'}$ gives $\mathrm{Gr}_{\omega\omega'} = \Theta(Y)$. Being additive, real-linear and isometric, Θ carries absolutely summable series term by term, which is (iii): a pair expansion of the kernel operator is a pair expansion of the Gram matrix with the same scale, constants and shares. In this way the expansions obtained from the unfolding Lemma A.64 become statements about the Gram matrices of the Poincaré frame realisations (Definition A.63), to which the row estimates of Theorem 15.3 apply without loss. □

A.6.2 Loop rows and non-generic rows

The terms of the loop rows are bounded by a product of local densities of an integral genus, and Proposition A.66 gives an explicit bound for this Siegel product when the discriminant module is $\mathcal{O}_E/(T)$. The non-generic rows are controlled through family systems, a generic mass bound and a pinned point sum (Proposition A.67); the statement is proved relative to the conditions (Na1)–(Na4), which the members must satisfy.

Proposition A.66. *All symbols introduced here are local to this statement and to the results that cite it. Let E be a CM field with (Um), F its maximal totally real subfield, $f = [F : \mathbb{Q}]$, d_F the discriminant of F, (V, h) a definite hermitian space over E of dimension k, $T \geq 1$ an integer, L an integral lattice in V with $L^\vee / L \simeq \mathcal{O}_E / T\mathcal{O}_E$ as $\mathcal{O}_E$-modules ($T = 1$ is the case $L^\vee = L$), and $a \geq 1$ an integer with $k \geq a^2 + 2$. For a nondegenerate hermitian $a \times a$ matrix Z over E let*

$$\mathrm{dens}(Z) = g_\infty(\Phi_\infty, Z) \prod_{w < \infty} g_w(\mathbf{1}_{(L \otimes \mathcal{O}_{F_w})^a}, Z),$$

with Φ_∞ smooth, compactly supported and equal to 1 on a neighbourhood of $\{X : \mathrm{gram}(X) = Z\}$. For a finite place w of F let q_w be the order of its residue field and v_w its valuation. Then $\mathrm{dens}(Z) = 0$ unless Z is positive definite at every real place τ of F, and in that case

$$\mathrm{dens}(Z) \leq \rho_T\, m_0(Z),$$

where

$$m_0(Z) := \bigl(1+3\cdot 2^{a^2-k}\bigr)^f\,|d_F|^{-(ka-a^2/2)}\prod_{\tau}\frac{(2\pi)^{ka-a(a-1)/2}}{\prod_{j=0}^{a-1}\Gamma(k-j)}\,\tau(\det Z)^{k-a},$$

$$\rho_T := \prod_{w\mid T} q_w^{-a\,v_w(T)}\,\frac{1-q_w^{a^2-k}}{1-q_w^{a^2+1-k}} \;\le\; T^{-fa}\prod_{w\mid T}\bigl(1-q_w^{-1}\bigr)^{-1} \;\le\; 1$$

(the products over the finite places w of F dividing T; $\rho_1=1$). In particular $\mathrm{dens}(Z)\le m_0(Z)$*, and clause (Ok3) of the one-loop amplitude bound for the loop rows holds with $m=m_0$ (and $m_0(Z):=0$ for Z not totally positive definite) whenever its lattice L is integral with $L^\vee/L\simeq \mathcal{O}_E/T\mathcal{O}_E$ for some integer $T\ge 1$ and $k\ge a^2+2$; the majorant m_0 does not depend on T.*

Proof. Sketch. By Siegel's formula [232], in the unitary form of Theorem B.158, the product $\mathrm{dens}(Z)$ is, for $2a<k$, the mass-weighted average, over the genus of L, of the number of representations of Z; it does not depend on the additive character, so one may use the standard one. The bound is obtained by estimating the product place by place. At the real places the density is computed exactly: the Gram map is proper on a definite space, and the archimedean density is $(2\pi)^{ka-a(a-1)/2}\prod_{j<a}\Gamma(k-j)^{-1}\tau(\det Z)^{k-a}$ when Z_τ is positive definite and 0 otherwise. At a finite place w the localised lattice is free over a principal ideal domain, its local dual is the completion of $L^\vee$, and $L_w^\vee/L_w\simeq \mathcal{O}_{E_w}/T\mathcal{O}_{E_w}$. A uniform local estimate, valid for $k\ge a^2+1$ (and $k\ge a^2+2$ at the places dividing T), bounds the density by $q_w^{-v_w(\delta_w)(ka-a^2/2)}(1-q_w^{a^2-k})^{-1}$, with an extra factor $q_w^{-av_w(T)}$ and a slightly weaker Euler factor at $w\mid T$. The core of the proof is this local estimate together with the fact that the product of the local bounds is explicit: the local differents contribute $|d_F|^{-(ka-a^2/2)}$, and the Euler factors give $\zeta_F(k-a^2)\le\zeta(k-a^2)^f\le(1+3\cdot 2^{a^2-k})^f$. The places dividing T are collected in $\rho_T\le 1$, so the majorant m_0 does not depend on T. This is the bound that controls the loop rows: it is the majorant of the one-loop amplitudes from which the loop records of the row estimates Theorem 15.3 are obtained. □

Proposition A.67. *All symbols introduced here are local to this statement and to the results that cite it. Let $r\ge 0$, $k\ge 4r+7$, $A\ge 0$ and $e_N\ge 0$ be integers with*

$$40(r+1)^2+2(2r+2)+5+e_N+6\le A(r+1)^2,$$

put $m=2r+2$, let C_h be real and C_B, $C_{g,0}$ the numbers of Theorem A.86 for $(A_w,C_h)=(A,C_h)$, and let $K,C_j\ge 0$ be real. Let a pair expansion at scale $\vartheta\ge 0$ be given, E a CM field of degree $2f$ with CM type Φ, $T,T',g\ge 1$ integers, P the set of points of length $r+1$ over E, and let sets Cl*, N and maps* $\mathrm{cl}\colon\mathcal{M}_{\mathrm{n}}\to\mathrm{Cl}$, $\mathrm{pt}\colon\mathcal{M}_{\mathrm{n}}\to P$, $\mathrm{nm}\colon\mathcal{M}_{\mathrm{n}}\to N$ *be such that every occupied point is a B-point for (T,T') and (Ng1), (Ng2), (Ng3) hold for*

$$w=\mathrm{wp},\qquad P_{\mathrm{n}}(\gamma)=(TT')^{6f}\max\Bigl(1,\prod_{v\in\Phi}\bigl|\iota_v G_{T,T'}(\gamma)\bigr|\Bigr)^6,\qquad S_w=64^{f(r+1)}.$$

Suppose:

(Na1) Family systems *for every $\gamma\in P_{\mathrm{oc}}$ a family system of type* ngen *at γ for $(\mathcal{M}_{\mathrm{n}},z,\mathrm{cl},\mathrm{pt},\mathrm{nm})$ with parameters r,k is given;*

(Na2) Generic mass *for every $\gamma\in P_{\mathrm{oc}}$, with the places, heights, volume and generic local masses of the system of (Na1), $\prod_{p\in\Pi}H_p\le H(\gamma)$ and $V\prod_{p\in\Pi}n_p^\circ\le K\,H(\gamma)^{e_N}\,n(\gamma)$, $H(\gamma)$ formed with T,T',g;*

(Na3) Point sums *there are $C_g \ge C_{g,0}$, primes g_1, g_2, a real $b \ge 2$ and point-sum data at the places $v \in \Phi$ such that C_g, $(r, k, E, \Phi, g_1, g_2, 2\pi/(\mathrm{rd}_E g_1 g_2), T, T', b)$ and these data satisfy all hypotheses of Theorem A.86, and for every $\gamma \in P_{\mathrm{oc}}$: $J_\gamma \le C_j \prod_{v\in\Phi} \mathrm{os}_v(\tilde\gamma_v)$ $(0\cdot\infty = 0)$, J_γ that of the system of (Na1);*

(Na4) Collapse *with*

$$T_{\mathrm{pt}} = K\,(TT')^{6f}\,C_j\,\exp(C_B m^5 f \log b)\prod_{v\in\Phi}\mathrm{bm}_v,$$

numbers $d, d', D^* > 0$ with $dd'/D^* = \vartheta$ and all further numbers of the collapse lemma for the non-generic row normalisation (by the main-row mass algebra) are given that satisfy the hypotheses of its item (ii) for c_{rw} of the pair expansion, these r, A, f, C_B, S_w, T_{pt} and b, and for constants $C_{\mathrm{St}}, C_T, C'_g \ge 0$ in place of its $C_{\mathrm{St}}, C_T, C_g$.

Then there are functions $J, n_{\mathrm{g}}\colon P \to [0,\infty)$ and $a_2\colon \mathrm{Cl}\times P\times N \to [0,\infty)$ such that $(\mathrm{Cl}, P, N, \mathrm{cl}, \mathrm{pt}, \mathrm{nm}, w, P_{\mathrm{n}}, J, n_{\mathrm{g}}, a_2, S_w, T_{\mathrm{pt}})$ are non-generic row data with parameters r, k having (Ng1)–(Ng4) and (Ng6),

$$c_{\mathrm{rw}}\,S_w\,T_{\mathrm{pt}} \le \exp(C_{\mathrm{ng}}\,m^5 f\log b)\,\vartheta, \qquad C_{\mathrm{ng}} = C_B + C_{\mathrm{St}} + 12C_T + (A/4+1)C'_g + 5,$$

and, for every $\gamma \in P_{\mathrm{oc}}$, data as in Lemma 16.4 with these r, k and $C_0 = 1$ satisfying its items 1 and 2 and (Nv1), (Nv2).

Proof. Outline. The functions J, n_{g} and a_2 are read off the family systems of (Na1) at the occupied points and set to 0 elsewhere; the properties of family systems then give (Ng4) and the data of Lemma 16.4, (Ng6) follows from Proposition A.87 once its hypotheses are checked from (Na2), (Na3) and a weight comparison at B-points, and the bound on $c_{\mathrm{rw}}S_wT_{\mathrm{pt}}$ is item (ii) of the collapse lemma under (Na4). □

A.6.3 The main row and the packet records

The main row needs a lower bound. At the selected members it comes from density domination (Proposition A.68), which combines an archimedean factor, the engine place of Section A.7, with a finite-place factor. The finite-place factor rests on local block kernels: they are positive semidefinite and vanish between points of different modulus (Lemma A.70), and they admit local theta majorants of every block size (Proposition A.69). Theorem A.71 then records the row estimates and the kernel hypotheses at the selected packets; its kernel data use Corollary 14.18 of the body.

Proposition A.68. *In the setting of Theorem A.71, let X be the threshold of the selected family in Proposition A.114, let $x \in W$ with $x > X$, $u \in A_x$, and let $\mathfrak{X}$ be the member Hida and Poincaré setting of u given by Proposition A.114. Fix the arithmetic main-row normalisation on the original completions of the member. Let ω, ω' be words, $(h_c)_c$ a column system of ω, ρ a row labelling of the main-row data of $(\mathfrak{X}, \omega, \omega')$, and let n_{c} be the constant of Siegel's formula for the row representatives of these data ($\mathcal{R} = Q_\omega$); form* al, *σ, ϕ, a^{mem}_ϱ, $\tilde a_\varrho$ and $k^{\mathrm{top}}_{\mathrm{mem}}$ with this n_{c}. Then:*

1. *Γ_w is finite for every top point w, the product defining* al *converges absolutely to a positive number, every eligible term at a top point has a normal representative, and every term t that has a normal representative satisfies $d(t) \le e_{\mathrm{dn}}\,\mathrm{al}$, where $e_{\mathrm{dn}} = \exp\bigl(9f\,2^{2r+2-k}\bigr)$ and $f = [F:\mathbb{Q}]$; call a real $s \ge 0$* admissible *if $d(t) \le s\,\mathrm{al}$ for every eligible term t that has a normal representative (so $s = e_{\mathrm{dn}}$ is admissible);*

2. *for every admissible s, every finite set P of top points and every finite set R of values of ρ, the matrix*
$$\Big[s\,\frac{\mathrm{K}^{\mathrm{top}}(y,y')}{\phi(y)\,\phi(y')} - \sum_{\varrho\in R} a_{\varrho}^{\mathrm{mem}}(y)\,\overline{a_{\varrho}^{\mathrm{mem}}(y')} \Big]_{y,y'\in P}$$
is positive semidefinite;

3. *for every admissible s, every finite set P of points over E of level $r+1$ and every finite set R of values of ρ, the matrix $\big[s\,k_{\mathrm{mem}}^{\mathrm{top}}(v,v') - \sum_{\varrho\in R}\tilde{a}_{\varrho}(v)\overline{\tilde{a}_{\varrho}(v')}\big]_{v,v'\in P}$ is positive semidefinite, and $\tilde{a}_{\varrho}(v)\neq 0$ only if v has level $r+1$.*

Proof. Outline. After the arithmetic facts of the member are collected (the extension E/F is unramified at the finite places, the signature, an element of trace one, the flag and its row representatives, the main-row normalisation, normal representatives of eligible terms and the finiteness of the sets Γ_w), item 1 follows from a local computation of the level factors, whose product converges to al and bounds the densities, item 2 is a positivity statement for main rows whose hypotheses (a partner identity, a fibre identity from Siegel's formula for the row representatives, and equal norms on the supports) are verified for the normalised density $d(t)/\mathrm{al}$, and item 3 is item 2 transported by the sign twist σ. □

Proposition A.69. *Fix the local setting and θ, and put $R = R_\theta$.*

(i) For all integers $s\geq 1$, $m\geq 0$ and every finite set Q of pairs (h,t), h a point of size m and $t\in E^\times$, the kernel $((h,t),(h',t'))\mapsto \mathrm{N}_s(h,h';t,t')$ is positive semidefinite on Q.

(ii) If $s\geq 1$, h,h' are points of one size, $t,t'\in E^\times$ and $\mathrm{N}_s(h,h';t,t')\neq 0$, then $t^{-1}-t'^{-1}\in R$.

(iii) For every integer $s\geq 1$ and every finite set Q of points of one size there is a local theta majorant of block size s on Q.

(iv) For every integer $m\geq 0$ there is a family $(\Theta(x))_{x\in E}$ of complex functions on pairs of points of size m such that $\Theta(x+y)=\Theta(x)$ for $x\in E$, $y\in R$, $\Theta(t)(h,h')=\mathrm{N}_1(h,h';t,t)$ for every $t\in E^\times$ and all points h,h' of size m, and the restriction of Θ to $Q\times Q$ is a local theta majorant of block size 1 on Q for every finite set Q of points of size m.

Proof. Outline. Parts (i) and (ii) are properties of the block kernels (Lemma A.70 and a support lemma for the kernels); for block size 1 the kernel N_1 evaluated at a unit section of E/R is a local theta majorant, which gives (iv) and (iii) for $s=1$, and larger block sizes follow by induction on s, through majorants over the compact cells Ω_j of units and a limit along a subsequence (Bolzano–Weierstrass), the defining conditions being closed and involving finitely many entries. □

Lemma A.70. *Fix the local setting and θ.*

(a) For all integers $s\geq 1$, $m\geq 0$ and every finite set Q of pairs (h,t), h a point of size m and $t\in E^\times$, the kernel $((h,t),(h',t'))\mapsto \mathrm{N}_s(h,h';t,t')$ is positive semidefinite on Q; and N_0 is positive semidefinite on every finite set of points of one size.

(b) If $s\geq 0$, h,h' are points of one size, $t,t'\in E^\times$ and $\mathrm{N}_s(h,h';t,t')\neq 0$ (for $s=0$: $\mathrm{N}_0(h,h')\neq 0$), then $|\det\bar{h}|\,|t|^s = |\det\bar{h}'|\,|t'|^s$. In particular $\mathrm{N}_s(h,h';t,t)\neq 0$ implies $|\det\bar{h}| = |\det\bar{h}'|$.

Proof. Points of size 0 are only $\varnothing$, with $\mathrm{N}_0(\varnothing,\varnothing)=1$; so let the points have size $n\geq 1$.

The local kernel is a Gram kernel. Let P be a finite set of points of size n. The orbit integrals of a point vanish outside finitely many marked classes, so the set C_P of classes $c\in \mathrm{Cl}_{n-1}$ with $\mathrm{B}(w,c)\neq 0$ for some $w\in P$ is finite, and for $w,w'\in P$ the series (Fk4) reduces to

$$\mathrm{K}(w,w')=\sum_{c\in C_P}\frac{\mathrm{B}(w,c)\,\overline{\mathrm{B}(w',c)}}{n_{\mathrm{Aut}}(c)},$$

with $n_{\mathrm{Aut}}(c)$ a positive integer. Hence K is hermitian on P and $\sum_{w,w'\in P}v_w\overline{v_{w'}}\,\mathrm{K}(w,w')=\sum_{c\in C_P}|\sum_w v_w\mathrm{B}(w,c)|^2/n_{\mathrm{Aut}}(c)\geq 0$ for every $v\in\mathbb{C}^P$. This is the second assertion of (a).

(a). The map $(h,t)\mapsto h\cdot(s,t)$ is injective (h is the point with the last block removed and t the entry of that block). On Q, $\mathrm{N}_s(h,h';t,t')=d(h,t)\,\mathrm{K}(h\cdot(s,t),h'\cdot(s,t'))\,d(h',t')$ with $d(h,t)=|t|^{s(s-1)/2}>0$, so the matrix of N_s on Q is $D\,\mathcal{K}\,D$ with D positive diagonal and $\mathcal{K}$ the matrix of K on the image of Q, positive semidefinite by the previous paragraph.

(b). Put $w=h\cdot(s,t)$, $w'=h'\cdot(s,t')$ ($w=h$, $w'=h'$ if $s=0$). If $\mathrm{K}(w,w')\neq 0$, some c has $\mathrm{B}(w,c)\neq 0\neq\mathrm{B}(w',c)$, so by the same finiteness lemma for orbit integrals $|N_{E/F}(\det\bar w)|_F=|N_{E/F}(\det Y_c)|_F=|N_{E/F}(\det\bar w')|_F$. Since $\bar w$ is block diagonal with blocks $\bar h$ and tJ_s, $\det\bar w=\pm\det\bar h\,t^s$, and $|\cdot|$ is multiplicative. □

Theorem A.71. *Fix one complete collection of good-host witnesses, take the selected class, selection, member objects and tables named above, and let P_x ($x\in W$, $x\geq X$) be the packets they form. The construction of Proposition A.1 chooses its own collection of good-host witnesses; whether its selected class, selection and member objects can be attached to a prescribed complete collection, as is done here, is an open question, and the theorem concerns these data whenever such an attachment exists. Assume the eventual Sha hypothesis of (Rr3) of Hypothesis 21.1. Equip every packet member $u\in A_x$ with the lower completion data in which the frame realisation, word scale, Gram blocks $\mathrm{Gr}_{\omega\omega'}$, frame data D_ω, D^* and the row parameters are those determined by the member's selected tied local data: k_{d} the dimension of the definite part of its hermitian space, r_{f} the length of the isotropic flag of its Bessel data, f_∞ the number of archimedean places of its totally real field, $g_1^{\mathrm{aux}},g_2^{\mathrm{aux}}$ its two selected auxiliary primes, and T_{amp} its amplitude threshold at the constant C_{A} below. Let $\mathcal{F}$ be the family these packets form (item 3 of Proposition A.1), taken at its design choice ((Rk13) of the row carrier). The word-size constant of these completions is $C_0=3$, and the word scale Γ_{w} of a member is the word scale of its context.*

Then there are real constants

$$C_{\mathrm{A}}\geq 1,\quad 0<c_{\mathrm{lp}}\leq\tfrac{1}{3},\quad C_{\mathrm{ng}}\geq 0,\quad C_{\mathrm{lb}}\geq 1,\quad A_{\mathrm{lb}}\geq 1,$$

and an integer $C_0^{\mathrm{ng}}\geq 0$, none of which depends on the scale, the member, the words or the local indices, such that for every real $c>0$ there is a real $Y_{\mathrm{X}}(c)$ with the following in the pointwise interpretation (Pl2) at every $x\in W$ with $x\geq Y_{\mathrm{X}}(c)$ and every member $u\in A_x$.

(X1) If the member has word sizes for C_0, then for all words $\omega\neq\omega'$ there are data with (Ro1)–(Ro4) of Theorem 15.3 for the constants $C_0,C_{\mathrm{A}},c_{\mathrm{lp}},C_{\mathrm{ng}},C_{\mathrm{lb}},A_{\mathrm{lb}},C_0^{\mathrm{ng}}$, with $K_0=32C_{\mathrm{lb}}^{32}$ and the member's row parameters.

(X2) The member satisfies the inequalities of (Rw2) with exponent c for $(C_{\mathrm{A}},c_{\mathrm{lp}},C_{\mathrm{ng}})$.

(X3) If the member has word sizes for C_0, then it satisfies the kernel hypotheses of Corollary 14.18 for C_0 and the exponent c: a measure space (X,μ) with $\mu(X)<\infty$, measurable $\mathrm{ls},\mathrm{kd},\mathrm{na},\mathrm{fa}$, a real $\tau_*>0$ and $c_{\mathrm{id}},\epsilon_{\mathrm{N}},\epsilon_{\mathrm{F}},\kappa\geq 0$ with (Kc1)–(Kc4), $\kappa\,\mu(X)<1$ and $1+\epsilon_{\mathrm{N}}+\epsilon_{\mathrm{F}}\leq(1-\kappa\,\mu(X))(1+x^{-c}/12)$.

In particular, for the common exponent $c > 0$ chosen in (Pl3), whatever its size (at the design choice, $c = C_H(\mathcal{F})$ by (Rk13) of the row carrier), the packets satisfy on the tail $x \geq Y_{\mathrm{X}}(c)$ (Ro1)–(Ro4), (Rw2) and the kernel hypotheses of (Pc5) demanded in (Rr3), with one common C_0, one common c and one common $(C_{\mathrm{A}}, c_{\mathrm{lp}}, C_{\mathrm{ng}})$.

Proof. Outline. The constants are fixed first, independently of the exponent, from propositions on the row data, the loop records and the row sizes of the members, and for each $c > 0$ the threshold $Y_{\mathrm{X}}(c)$ is the largest of the thresholds of these propositions and of those on the amplitude and the kernel data; then (X1) combines the row data giving (Ro1), (Ro3), (Ro4) of Theorem 15.3 with a loop-row bound giving (Ro2), (X2) collects the size and amplitude inequalities of (Rw2), and (X3) follows from the kernel data through a comparison corollary for the kernel hypotheses of Corollary 14.18. □

A.7 Approximate orthogonality II: the engine place and Bessel amplitudes

This section supplies the archimedean inputs of Section A.6. The engine place gives the local kernel clauses behind the lower bound for the main row (Ro4); the bounds for Bessel amplitudes feed the loop rows (Ro2) and the point sums of the non-generic rows (Ro3). The place theorem A.78 is an assembly of four steps: positivity on a cut (Proposition A.73), interpolation on a long cut (Lemma A.74), decay of matrix coefficients at a large gap (Theorem A.75), and an energy identity (Proposition A.76, Proposition A.77). It needs only $h > \max(261n^4,\, n^5 + 5(C_b + 3)^2 n^3)$, and $h \geq \frac{1}{2}n^{19}$ suffices when $C_b \leq 11$. At a member the cut is non-empty because the word code of the context is injective, so that distinct words differ by Δ_D in one coordinate (Proposition A.79). The engine place should not be confused with the pointwise Bessel bound at large gap, Theorem A.84, which drives the amplitude bounds of the second subsection.

A.7.1 The engine place

Definition A.72 sets up the engine energies at a place: normalised kernels, jet energies, the Bessel energy, extensions of words and pointwise bounds. The J-pairing inequality gives positivity on the cut (Proposition A.73). Grids with small Lagrange totals interpolate on a long cut (Lemma A.74), and smooth matrix coefficients of discrete series decay polynomially at a large gap (Theorem A.75). Engine places have an energy model (Proposition A.76), from which the limit clause follows (Proposition A.77). The place theorem A.78 assembles these four steps, and Proposition A.79 checks its hypotheses at the engine place of a member at a complex place, for the actual words of one context.

Definition A.72. Let a local kernel datum of rank r be given, with $T = T(z)$, $T' = T(z')$, and a **type** $0 \leq l \leq k-1$ with sign $s = (-1)^{k-l-1}$ in (Pk6). Only (Pk5), (Pk7) and s enter below; the data $F, P, \chi, \mathsf{g}, \mathsf{d}$ of (Pk6) do not. Cut words are those for the cut I_l. Put $\sqrt{x} = 0$ for $x < 0$.

Weight of a local point. For a local point $h = ((s_1, t_1), \dots, (s_a, t_a))$ with block starts $\beta_j = 1 + s_1 + \cdots + s_{j-1}$ put $\mathrm{n_e}(h) = \prod_j \phi_j$ with $\phi_j = (|t_j|^2 - TT')^{k-1}$ if $\beta_j + s_j - 1 = r + 1$ and $s_j = 1$, and $\phi_j = |t_j|^{2s_j(\mathrm{dm}_{\beta_j + s_j - 1} + s_j - 3)}$ otherwise.

Normalised kernels and jet energies. For $\mathrm{nn}\colon \mathbb{R} \to (0, \infty)$ and a word w put $\mathrm{nc}(w) = \prod_{(\pm, y) \in w} \mathrm{nn}(y)$ (over the positions of w; the point of a letter $(-, y+1)$ is $y+1$). For a coefficient Φ of rank r, $0 \leq i \leq r+1$, a word w and local points a, b:

$$\mathrm{NP}_{\mathrm{nn}}(\Phi)_{i,w}(a,b) = \frac{\sqrt{\mathrm{n_e}(a)}\sqrt{\mathrm{n_e}(b)}}{\mathrm{nc}(w)}\, K_{i,w}(\Phi)(a,b), \qquad \mathrm{Ew}_{\mathrm{nn}}(\Phi)_{i,w}(h) = \mathrm{Re}\,\mathrm{NP}_{\mathrm{nn}}(\Phi)_{i,w}(h,h)$$

(the **jet energy** of Φ at h; $K_{i,w}(\Phi)(a,b)=0$ unless a,b have level i). We write $\mathrm{Ew}(\Phi)_{i,w}$ for $\mathrm{nn}\equiv 1$.

Bessel energy. For $c\in\mathbb{C}^{r+1}$ let $\mathrm{bigPt}(c)=((1,c_0),\dots,(1,c_r))$ (a local point of level $r+1$ when all $c_j\neq 0$), and

$$\mathrm{jN2}(\Phi)=\int_{\mathbb{C}^{r+1}}\max\bigl(\mathrm{Ew}(\Phi)_{r+1,\varnothing}(\mathrm{bigPt}(c)),0\bigr)\,dc\in[0,\infty],$$

the lower integral with respect to Lebesgue measure.

Extensions. For a finite $Y\subset\mathbb{R}$ and integers $N,m'\geq 0$, a word w' is a (Y,N,m')**-extension** of a word w if w' is a cut word, $|w'|\leq N$, and $w'=L\,w$ (concatenation) for a word L with $|L|\leq m'$ whose letters are $(+,y)$ or $(-,y+1)$ with $y\in Y$.

Pointwise bound at the place. For $\kappa\geq 0$, Φ satisfies the **pointwise bound with constant** κ if for every local point γ of level $r+1$ with $G(\gamma)=|\det M_{r+1}(\gamma)|^2-TT'|\det M_r(\gamma)|^2>0$: $\max(\mathrm{Ew}(\Phi)_{r+1,\varnothing}(\gamma),0)\leq\kappa\,\mathrm{jN2}(\Phi)$ in $[0,\infty]$.

Block growth bound. For $\omega_1,\omega_2\in\mathbb{C}$ with $\mathrm{Im}(\bar\omega_1\omega_2)\neq 0$ put $L=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ and $L^\vee=\{\sigma\in\mathbb{C}:\exp(2i\,\mathrm{Im}(\sigma\omega))=1\text{ for all }\omega\in L\}$. Let $\mathrm{nn}\colon\mathbb{R}\to(0,\infty)$, $Y\subset\mathbb{R}$ finite and $K,N\geq 0$ integers. A coefficient Φ of rank r satisfies the **block growth bound for** (nn,Y,K,N) if, with $\mathrm{NP}=\mathrm{NP}_{\mathrm{nn}}(\Phi)$ and $\mathrm{Ew}=\mathrm{Ew}_{\mathrm{nn}}(\Phi)$: for all integers $b\geq 1$, $s\geq 2$ with $b+s-1\leq r+1$, every local point h of level $b-1$ and every Y-admissible word w with $|w|+K\leq N$ such that for every (Y,N,K)-extension w' of w the function $t\mapsto\mathrm{Ew}_{b+s-2,w'}(h\cdot(s-1,t))$ is bounded above on $\mathbb{C}^\times$, and for all $\omega_1,\omega_2,z_0\in\mathbb{C}$ with $\mathrm{Im}(\bar\omega_1\omega_2)\neq 0$, there is a function $\kappa\colon\mathbb{C}\to\mathbb{C}$ with

$$\frac{|\,\mathrm{Im}(\bar\omega_1\omega_2)|}{\pi}\sum_{\omega\in L}\max\bigl(\mathrm{Ew}_{b+s-1,w}(h\cdot(s,z_0+\omega)),0\bigr)\leq\sum_{\sigma\in L^\vee}|\kappa(\sigma)|,$$

$$|\kappa(\sigma)|\leq\limsup_{t\to 0,\ t\neq 0}\bigl|\mathrm{NP}_{b+s-2,w}(h\cdot(s-1,t),h\cdot(s-1,(t^{-1}+\sigma)^{-1}))\bigr|\quad(\sigma\in L^\vee)$$

in $[0,\infty]$ ($h\cdot(m,c)$ as in Definition 15.7; as $h\cdot(s,0)$ is not a local point, $\mathrm{Ew}_{b+s-1,w}(h\cdot(s,0))$ is read as 0).

Compact coefficients. A coefficient Φ of rank r is **compact** if it is smooth, there is R with $\|g\|\leq R$ whenever $\Phi(g)\neq 0$ ($\|g\|$ the Frobenius norm of the matrix of g), and there is $\varepsilon>0$ with $G_f(g)\geq\varepsilon$ whenever $\Phi(g)\neq 0$ (G_f of Definition A.81).

Engine places. Let $k\geq 2$ and let a Bessel datum of rank r (Definition A.81) be given with π,π' irreducible of Harish-Chandra parameters $u=(u_0,\dots,u_{k-1})$, $u'=(u'_0,\dots,u'_{k-1})$ (strictly decreasing); let $0\leq l\leq k-1$ and $C_b\geq 0$. Write $\mathfrak{g}_r=\{X:X^*H_r+H_rX=0\}$, $(R_XH)(g)=\frac{d}{d\tau}|_{\tau=0}H(g\exp\tau X)$, $(L_XH)(g)=\frac{d}{d\tau}|_{\tau=0}H(\exp(-\tau X)g)$, and $R_{\mathbf{Z}}=R_{Z_1}\cdots R_{Z_q}$ for a finite sequence $\mathbf{Z}=(Z_1,\dots,Z_q)$ in $\mathfrak{g}_r$. An **engine place of type** l **with constant** C_b on the given datum is a local kernel datum such that:

(EP0) (normalised measures) for every $0\leq\rho\leq r$ the Haar measure dn on N_ρ of its Bessel datum of rank ρ is the **normalised** one: for $\rho=0$ the unit point mass on the trivial group N_0, and for $\rho<r$ the measure on $\mathrm{N}_{\rho+1}$ is the image of the product of Lebesgue measure on $\mathbb{C}^{n_\rho}\times\mathbb{R}$ ($\mathbb{C}=\mathbb{R}^2$) and the normalised measure on N_ρ under $(X,\lambda,\nu)\mapsto\mathrm{Rn}(X,(),\lambda)\,\mathrm{diag}(1,\nu,1)$, where $\mathrm{Rn}(X,u,\lambda)$ is the slice matrix of the local kernels with $j=1$ (blocks of sizes $(1,n_\rho,1)$, u empty);

(EP1) its Bessel datum of rank r has the same z,z',π,π',μ as the given one (its Haar measures may differ, subject to (EP0)), and its sign is $s=(-1)^{k-l-1}$;

(EP2) its top datum is frame-compatible for the parameters $(u_0,\dots,u_{k-1})$, $(u'_0,\dots,u'_{k-1})$, read as $(u_1,\dots,u_k)$, $(u'_1,\dots,u'_k)$ there, with $l_0=l+1$; so its cut is the cut I_l;

(EP3) (the top lift is a differential operator of controlled order and growth) for every cut word w and every $e \in \{0,1\}^w$ there are integers $o, \kappa \geq 0$ with $o \leq C_b|w|n$ and $\kappa \leq C_b(r+1)o$; finitely many smooth maps $E_1, \dots, E_{n_f}$ from $\{g \in \mathrm{G}_r : G_f(g) > 0\}$ to the linear endomorphisms of $\mathrm{Hom}(\mathcal{E}', \mathcal{E})$ such that for every $\varepsilon > 0$ and integer $i \geq 0$ there is C with $\|(L_{X_1} \cdots L_{X_a} R_{X_{a+1}} \cdots R_{X_i} E_j)(g)\| \leq C\|g\|^{3(r+1)i}$ for all j, all $0 \leq a \leq i$, all $X_1, \dots, X_i \in \mathfrak{g}_r$ with entries of modulus at most 1 and all g with $G_f(g) \geq \varepsilon$; and finitely many terms t, each given by a finite sequence $\mathbf{X}_t$ in $\mathfrak{g}_r$, a polynomial P_t of total degree at most κ in the matrix entries of g and their conjugates, integers $a_t, b_t, c_t \geq 0$ and finitely many pairs $(j_{t,1}, \mathbf{Z}_{t,1}), \dots, (j_{t,q}, \mathbf{Z}_{t,q})$ ($1 \leq j_{t,i} \leq n_f$, $\mathbf{Z}_{t,i}$ finite sequences in $\mathfrak{g}_r$) with $|\mathbf{X}_t| + \sum_i |\mathbf{Z}_{t,i}| \leq o$, such that for every smooth coefficient H of rank r and every g with $G_f(g) > 0$

$$(\Theta_{w,e}H)(g) = \sum_t P_t(g)\, D_{r+1}(g)^{-a_t}\, \overline{D_{r+1}(g)}^{\,-b_t}\, G_f(g)^{-c_t} \cdot \big((R_{\mathbf{Z}_{t,1}} E_{j_{t,1}})(g) \circ \cdots \circ (R_{\mathbf{Z}_{t,q}} E_{j_{t,q}})(g)\big)\big((R_{\mathbf{X}_t} H)(g)\big);$$

and, on $\{G_f > 0\}$, $\Theta_{w,e}$ commutes with the translations $H \mapsto \pi(h)\, H(h^{-1}\nu^{-1} \cdot \nu' h')\, \pi'(h')^{-1}$ ($h \in \mathrm{H}_z$, $h' \in \mathrm{H}_{z'}$, $\nu, \nu' \in \mathrm{N}_r$) of smooth coefficients.

The engine energies of the place are those above, formed with its kernels $K_{i,w}$ and its type l.

Proposition A.73. *Let an engine place of type l (with any constant) on a Bessel datum of rank r be given ($k \geq 2$). Let $0 \leq j \leq r$, w a cut word and $ys = (y_1, \dots, y_m) \in \mathbb{R}^m$ with entries pairwise differing by more than 2 such that w^+_{ys} and w^-_{ys} are cut words. Then $q_w(y_a) > 0$ for $1 \leq a \leq m$, and for all families x, x' of nice coefficients of rank $r - j$ indexed by $\{0,1\}^{w^+_{ys}}$ and $\{0,1\}^{w^-_{ys}}$*

$$\big|\mathrm{J}_{j,w}(ys; x, x')\big|^2 \prod_{a=1}^{m} q_w(y_a) \leq \Lambda_{j,w^+_{ys}}(x,x)\, \Lambda_{j,w^-_{ys}}(x',x').$$

Proof. Sketch. The inequality is the Cauchy–Schwarz inequality for a positive semidefinite form, and the work lies in exhibiting that form. The first step is positivity on the cut: because w^+_{ys} is a cut word for the type data of the engine place, the stage function satisfies $q_w(y_a) > 0$ for every a, so $\kappa_L = \prod_a q_w(y_a) > 0$. Next, the three pairings are moved to a radial model. With $v = w^+_{ys}$, $v' = w^-_{ys}$ and R the radial map, the level forms become $\Lambda_{j,v}(x,x) = C_j\, \mathrm{fm}_0(\mathrm{RR}^{(0)}_v \mathrm{R}x, \mathrm{R}x)$ and likewise for v', with one constant $C_j > 0$, and the J-pairing becomes $C_j\, \mathrm{fm}_0(J_L \mathrm{RR}^{(0)}_w \mathrm{R}x, \mathrm{R}x')$. A twisted radial descent supplies a class of radial functions, containing the images of nice coefficients, on which $B(Z, Z') = \mathrm{fm}_0(\mathrm{RR}^{(0)}_v Z, Z')$ is sesquilinear and positive semidefinite. The core of the proof is a pair of identities in the algebra of operators acting on radial functions,

$$\mathrm{Ps}^\dagger_L\, \mathrm{RR}_v = J_L\, \mathrm{RR}_w, \qquad \mathrm{RR}_{v'} = \kappa_L\, \mathrm{Ps}^\dagger_L\, \mathrm{RR}_v\, \mathrm{Ps}_L,$$

obtained by fusing the stages of the word one letter at a time; the separation of the letters by more than 2 ensures that no stage function vanishes at the points where a fusion is performed. With these identities the J-pairing is $C_j B(X, \mathrm{Ps}_L X')$ and the level form of v' is $C_j \kappa_L B(\mathrm{Ps}_L X', \mathrm{Ps}_L X')$, where $X = \mathrm{R}x$ and $X' = \mathrm{R}x'$, so the Cauchy–Schwarz inequality for B gives the claim. In the engine place theorem A.78 this is the step of positivity on the cut. □

Lemma A.74. *Let $y_- < y_+$ be reals with $y_+ - y_- \geq 1$, let $D \geq 0$ and $N \geq K \geq 1$ be integers, $\delta \geq 3$ real, let P be a nonzero real polynomial of degree at most D and $q\colon \mathbb{R} \to \mathbb{R}$ a*

function that is continuous and positive on $[y_-, y_+ - 1]$. *Take the cut* $I = [y_-, y_+]$, *the finite set* $\mathcal{L} = \{y_- + t\delta : \ t \in \mathbb{Z},\ 0 \le t \le (y_+ - 1 - y_-)/\delta\}$, *and let* $\mathcal{W}$ *be the set of cut words* w *with* $|w| \le N$, $\mathrm{wt}^{\mathrm{loc}}(w) > 0$ *and every letter of the form* $(+, y)$ *or* $(-, y+1)$ *with* $y \in \mathcal{L}$; $\mathcal{W}$ *is finite and contains* $\varnothing$. *Grids are formed with* $\mathcal{W}$ *as the set of admitted words. For a word* w *put* $q_w(y) = q(y) \prod_{(+,y')\in w} \frac{y-y'+1}{y-y'} \prod_{(-,y')\in w} \frac{y'-y}{y'-y-1}$ *(products over the positions of* w*).*

(i) Let $d \ge 0$ *be an integer and* $w \in \mathcal{W}$ *with* $|w| + K \le N$ *and*

$$y_+ - y_- \ge 1 + \delta\big(K(d+1) + 3|w| + 2D\big).$$

Then w *has a grid of degree* d, *with all nodes in* $\mathcal{L} \subseteq [y_-, y_+ - 1]$ *and Lagrange total* $\mathrm{Lt} \le \big((2/\delta)^d/d!\big)^K$.

(ii) Put $\beta^2 = 2^N \max_{y_- \le y \le y_+ - 1} |P(y)P(y+1)|/q(y)$. *For every word* w, *every grid of any degree for* w *and every node* y *of it,* $q_w(y) > 0$ *and* $|P(y)P(y+1)| \le \beta^2 q_w(y)$.

In particular, if $y_+ - y_- \ge 1 + \delta(K(2r+1) + 3N + 2D)$, *the datum with these* I, $\mathcal{W}$, q_w *satisfies (Kx1) with* $G_{\mathrm{W}}(j) = ((2/\delta)^{2j}/(2j)!)^K$, $G_{\mathrm{L}}(b,s) = ((2/\delta)^{2(b-1)+s}/(2(b-1)+s)!)^K$, $G_{\mathrm{T}}(s) = ((2/\delta)^{r+1-s}/(r+1-s)!)^K$, *and (Kx6) with* $\beta_j = \beta'_{b,s} = \beta$.

Proof. Outline. For (i), the nodes are chosen among the points of $\mathcal{L}$ that lie at distance more than 3 from every letter value of w and avoid the zeros of $P(y)P(y+1)$, of which there are at least $K(d+1)$, and the Lagrange total factorises over the K blocks and is bounded by the spacing δ of the nodes; for (ii), every factor of $q_w(y)$ coming from a letter of w exceeds $\frac{1}{2}$ because the node is more than 2 away from each letter value, so $q_w(y) \ge 2^{-N} q(y)$; the particular case follows by applying (i) to the degrees occurring in (Kx1). □

Theorem A.75. *Let* $k \ge 1$, $r \ge 0$ *be integers,* $n = 2r + k + 1$, *let* H *be the matrix of signature* $(k+r, r+1)$ *of (Dp4) in the definition of square-integrable representations with Harish-Chandra parameter relative to a sign vector, with* $\mathrm{U}(H)$ *and its Lie algebra* $\mathfrak{u}(H)$ *as there, and* $\|g\| = (\sum_{i,j} |g_{ij}|^2)^{1/2}$. *Let* $\varepsilon \in \{+,-\}^n$ *be a sign vector with* p *entries* $+$ *and* q *entries* $-$, $q_0 = \min(p,q)$, *and*

$$\theta(\varepsilon) = \max\big(npq + 1,\ 16\, n\, q_0 + 8\sqrt{2q_0}\big).$$

Let $\lambda \in \mathbb{R}^n$ *be strictly decreasing with* $\lambda_a - \frac{n-1}{2} \in \mathbb{Z}$, *let* $h > 0$ *with* $|\lambda_a - \lambda_b| \ge h$ *whenever* $\varepsilon_a \ne \varepsilon_b$, *and let* σ *be a unitary representation of* $\mathrm{U}(H)$ *on a Hilbert space* $\mathcal{H}$ *which, after conjugation by a unitary Cayley transform carrying* H *to* $\mathrm{diag}(\varepsilon)$, *is square-integrable with Harish-Chandra parameter* λ *relative to* ε *((Dp5); in particular* σ *is irreducible and square-integrable). Words* w *in* $\mathfrak{u}(H)$ *and* $d\sigma(w)u$ *for smooth* u *are as in (Dp6). Then for every integer* $m \ge 0$ *with* $\theta(\varepsilon) + m < h$ *there are* $C \ge 0$ *and a finite set* $\mathcal{W}$ *of words in* $\mathfrak{u}(H)$ *such that for all smooth vectors* $u, v \in \mathcal{H}$ *and all* $g \in \mathrm{U}(H)$

$$\|g\|^m\, |\langle v, \sigma(g)u\rangle| \le C\Big(\sum_{w\in\mathcal{W}} \|d\sigma(w)u\|\Big)\Big(\sum_{w\in\mathcal{W}} \|d\sigma(w)v\|\Big).$$

Proof. Outline. After a unitary Cayley transform, the proof combines a one-sided decay bound $|\langle w_0, \sigma(y)u\rangle| \le C_1 \|y\|^{-A} S(u)$ for a fixed nonzero K-finite vector w_0 with the Schur orthogonality relations, which express $\langle v, \sigma(g)u\rangle$ as an integral over the group of a product of two such coefficients, and then bounds $\|g\|^m$ times this integral by the Cauchy–Schwarz inequality and the integrability of a negative power of $\|y\|$ over the group. □

Proposition A.76. *Let $C_b \geq 0$, let $r \geq 0$, $k \geq 2$, $N_{\mathrm{tot}} \geq 1$ be integers, $n = 2r + k + 1$, and let $A_{\mathrm{E}}, m_{\mathrm{E}}, A_{\mathrm{T}}, m_{\mathrm{T}}$ be the numbers given for $(C_b, r, k, N_{\mathrm{tot}})$ by the adjointness and transfer propositions for the engine energy system. Let an engine place of type l with constant C_b on a Bessel datum of rank r be given, with admitted words $\mathcal{W}$, an integer N_{top} and the data F, P, χ of (Pk6), and let $[\alpha, \beta] = I_l$. Suppose:*

(Em1) *$u_i - u_0 \in \mathbb{Z}$ and $u_i' - u_0 \in \mathbb{Z}$ for $0 \leq i \leq k-1$;*

(Em2) *χ takes values in $[0,1]$, $\chi = 0$ on $(-\infty, \frac{1}{2}]$ and $\chi = 1$ on $[1, \infty)$;*

(Em3) *there are reals $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ with $\lambda_l < \min(u_l, u_l') + 2$ and, if $l \geq 1$, $\lambda_{l-1} > \max(u_l, u_l') - 2$, such that $P(Y) = \prod_{t=0}^{n-1}(Y + \lambda_t)$ and $\mathrm{qd}(Y + \frac{n-1}{2})F = P(Y)F$ for every $Y \in [\alpha, \beta]$;*

(Em4) *there is $A_0 > 2(k+r)(r+1)$ such that F lies in $\mathcal{D}_r(\max(A_0, A_{\mathrm{E}}, A_{\mathrm{T}}), \max(4, m_{\mathrm{E}}, m_{\mathrm{T}}))$ and, for every $\varepsilon > 0$, $\chi(G_f/\varepsilon)F$ lies in $\mathcal{D}_r(A_0, nN_{\mathrm{top}} + 5)$ (each with some constant);*

(Em5) *every $w \in \mathcal{W}$ with $|w| \leq N_{\mathrm{top}}$ is a cut word;*

(Em6) *$N_{\mathrm{top}} \leq N_{\mathrm{tot}}$, and there is a real $L \geq 16\, e\, N_{\mathrm{tot}}(r+1)^2(1 + \sqrt{C_7})$, $C_7 = e^{28}(1 + C_{\mathrm{res}})^2$, with $6N_{\mathrm{tot}}L \leq \beta - \alpha$ and $nL \leq \beta - \alpha$.*

Then the energy system of the engine place for $(F, (\lambda_0, \ldots, \lambda_{n-1}), N_{\mathrm{tot}}, L)$, formed with this χ, together with $C_{\mathrm{m}} = \Gamma_0^{-1}$ and $\sigma_{\mathrm{m}} = 1$, is an energy model of the level-zero kernels of the datum. Consequently the datum satisfies (Pk14) with

$$J = \Gamma_0^{-1} \int_\Omega \|(\mathrm{R}F)(c,g)\|_{\mathrm{HS}}^2 \, d\mu(c,g) < \infty.$$

Proof. Outline. The proof checks that the energy system of the engine place satisfies (Ey1)–(Ey5) and the hypotheses of the energy limit theorem for energy systems: (Em3) makes $s\,P$ positive on the cut, the box words are cut words whose weights are positive definite, the energy identities are proved for nice approximations of F and passed to the limit, and the level-zero kernels are real by a self-adjointness argument for the Capelli operators; the resulting identity $g^{\mathrm{loc}}_{\varepsilon,0,w}(\varnothing,\varnothing) = \Gamma_0^{-1} E(\varepsilon, w)$ gives an energy model, and (Pk14) follows from Proposition A.77. □

Proposition A.77. *Let a local kernel datum with admitted words $\mathcal{W}$ and integer N_{top} be given. An **energy model** of its level-zero kernels consists of an energy system $\mathcal{S}$ satisfying (Ey1)–(Ey5) and the remaining hypotheses of the energy limit theorem for energy systems, with $N_{\mathrm{tot}} \geq N_{\mathrm{top}}$, and reals $C_{\mathrm{m}}, \sigma_{\mathrm{m}} > 0$, such that every $w \in \mathcal{W}$ with $|w| \leq N_{\mathrm{top}}$, read as a signed word (the letter $(\pm, y)$ as the pair $(\pm, y)$), is admitted for $\mathcal{S}$, has $\mathrm{size}(w) = \mathrm{wt}^{\mathrm{loc}}(w)$, and satisfies*

$$g^{\mathrm{loc}}_{\varepsilon,0,w}(\varnothing,\varnothing) = C_{\mathrm{m}}\, E(\sigma_{\mathrm{m}}\varepsilon, w) \qquad (\varepsilon > 0).$$

If the datum has an energy model, it satisfies (Pk14) with $J = C_{\mathrm{m}}\,\mathrm{jn}$.

Proof. By the energy limit theorem for energy systems, $\mathrm{jn} < \infty$; it is ≥ 0 as the integral of the nonnegative function $\mathrm{dens}(\varnothing)$, so $J = C_{\mathrm{m}}\mathrm{jn}$ is a real ≥ 0. Let $w \in \mathcal{W}$ with $|w| \leq N_{\mathrm{top}} \leq N_{\mathrm{tot}}$. By the same theorem, $E(\varepsilon', w) \to \mathrm{size}(w)\,\mathrm{jn}$ as $\varepsilon' \to 0^+$. As $\varepsilon \to 0^+$, $\varepsilon' = \sigma_{\mathrm{m}}\varepsilon \to 0^+$, so $g^{\mathrm{loc}}_{\varepsilon,0,w}(\varnothing,\varnothing) = C_{\mathrm{m}} E(\sigma_{\mathrm{m}}\varepsilon, w) \to C_{\mathrm{m}}\,\mathrm{size}(w)\,\mathrm{jn} = \mathrm{wt}^{\mathrm{loc}}(w)\,J$. □

Theorem A.78. *Let $r \geq 1$, $k \geq 2$ be integers, $n = 2r + k + 1$, $N_{\mathrm{top}} = 20(r+1)$, $n_{\mathrm{d}} = 5$, $C_b \geq 0$, and let an engine place of type l with constant C_b on a Bessel datum of rank r be given, with the data F, P, χ of (Pk6) and the Harish-Chandra parameters $u = (u_0, \dots, u_{k-1})$, $u' = (u'_0, \dots, u'_{k-1})$ of π, π'; write $I_l = [y_-, y_+]$. Suppose:*

(Pl1) Integrality. *$u_i - u_0 \in \mathbb{Z}$ and $u'_i - u_0 \in \mathbb{Z}$ for $0 \leq i \leq k-1$.*

(Pl2) Cut-off. *χ takes values in $[0,1]$, $\chi = 0$ on $(-\infty, \frac{1}{2}]$ and $\chi = 1$ on $[1, \infty)$.*

(Pl3) Test coefficient and gap. *$F(g) = \sum_{i=1}^{N} P_i\, \sigma(g)\, Q_i$, where σ is an irreducible square-integrable unitary representation of G_r on a Hilbert space $\mathcal{H}$ with Harish-Chandra parameter λ relative to a sign vector $\mathbf{j} \in \{+,-\}^n$ with $k+r$ entries $+$ and $r+1$ entries $-$, and $Q_i \colon \mathcal{E}' \to \mathcal{H}$, $P_i \colon \mathcal{H} \to \mathcal{E}$ are bounded linear maps with $Q_i(\mathcal{E}')$ and $P_i^*(\mathcal{E})$ consisting of smooth vectors; and there is a real h with $|\lambda_a - \lambda_b| \geq h$ whenever $\mathbf{j}_a \neq \mathbf{j}_b$ and*

$$h > \max\bigl(261\, n^4,\ n^5 + 5(C_b+3)^2 n^3\bigr).$$

(Pl4) Casimir polynomial. *There are reals $\mu_0 \geq \mu_1 \geq \cdots \geq \mu_{n-1}$ with $\mu_l < \min(u_l, u'_l) + 2$ and, if $l \geq 1$, $\mu_{l-1} > \max(u_l, u'_l) - 2$, such that $P(Y) = \prod_{t=0}^{n-1}(Y + \mu_t)$ for every $Y \in \mathbb{R}$, and $\mathrm{qd}(Y + \frac{n-1}{2})F = P(Y)F$ for every $Y \in [y_-, y_+]$.*

(Pl5) Long cut. *There is a real $L \geq 16\, e\, N_{\mathrm{top}}(r+1)^2(1+\sqrt{C_7})$ with $6N_{\mathrm{top}}L \leq y_+ - y_-$ and $nL \leq y_+ - y_-$.*

Take as admitted words the finite set $\mathcal{W}$ of Lemma A.74 for $(y_-, y_+, D, N, K, \delta) = (y_-, y_+, n, N_{\mathrm{top}}, 5, 3)$ and the polynomial P, as cut I_l, as functions q_w those of the wave and lift grids on the cut, the constants $G_{\mathrm{W}}, G_{\mathrm{L}}, G_{\mathrm{T}}$ and $\beta_j = \beta'_{b,s} = \beta$ of Lemma A.74 (for this q) and the constant C_{T} of the top-grid weight bound for $N = N_{\mathrm{top}}$, $K = 5$. Then the datum, with $N_{\mathrm{top}} \geq n_{\mathrm{d}}$, has the lower block clauses and satisfies (Pk14); its coefficients Φ_m of (Kx10) are, for each $\varepsilon > 0$, $\Phi_m(g) = s((4 - \|g\|_{\mathrm{F}}^2/(m+1)^2)/3)\chi(G_f(g)/\varepsilon)F(g)$ for any fixed smooth $s \colon \mathbb{R} \to [0,1]$ with $s = 0$ on $(-\infty, 0]$ and $s = 1$ on $[1, \infty)$, and it has an energy model of its level-zero kernels. In particular, if $C_b \leq 11$, the gap hypothesis of (Pl3) holds whenever $h \geq \frac{1}{2}n^{19}$.

Proof. We verify the hypotheses of part (ii) of the archimedean product theorem for local kernel data: (Kx1)–(Kx6), (Kx9), (Kx13); for every $\varepsilon > 0$ nice coefficients Φ_m with (Tn1), (Tn3), (Tn5) (with (Tn6)) and with (Kx11), (Kx12); an energy model of the level-zero kernels. Throughout, $n \geq 5$ (as $r \geq 1$, $k \geq 2$).

The cut and the cut words. By (EP2) the top datum is frame-compatible for (u, u'), read as $(u_1, \dots, u_k)$, $(u'_1, \dots, u'_k)$, with $l_0 = l + 1$; so the cut I_{l+1} of (Ft0), used below for the slice, descent and top clauses, is I_l, and the cut words there are those of the wave and lift grids on the cut. With $m_* = \min(u_l, u'_l)$, $M_* = \max(u_l, u'_l)$ we have $I_l = [2 - M_*, -m_* - 2]$, so $y_+ - y_- = M_* - m_* - 4$.

Sizes of the cut. By (Pl5), $y_+ - y_- \geq nL \geq 16e \cdot 20(r+1)^3 n > 800\, n(r+1)^3$. Hence (S1) $y_+ - y_- \geq 1 + 3(5(2r+1) + 3N_{\mathrm{top}} + 2n) = 196 + 210r + 6n$; (S2) $y_+ - y_- \geq 9(2r+1) + 6N_{\mathrm{top}} + 1 = 138r + 130$; (S3) $M_* - m_* = y_+ - y_- + 4 \geq 5$; each because $800n(r+1)^3 \geq 800n(r+1) > 196 + 210r + 6n$ and $> 138r + 130$.

Positivity of q. Let $y \in [y_-, y_+ - 1]$. The one-letter words $((+, y))$ and $((-, y+1))$ are cut words (one letter, value in I_l), so Proposition A.73 applies with $j = 0$, $w = \varnothing$ (a cut word), $ys = (y)$ (its entries pairwise differ by more than 2 vacuously; $w^+_{ys} = ((+, y))$, $w^-_{ys} = ((-, y+1))$ are cut words) and gives $q(y) = q_{\varnothing}(y) > 0$. Also q is a polynomial, hence continuous.

(Kx1) and (Kx6). By (Pl4), P is a nonzero real polynomial of degree n. Apply Lemma A.74 with $y_- < y_+$ ($y_+ - y_- \geq 1$ by (S3)), $D = n$, $N = N_{\mathrm{top}} \geq K = 5$, $\delta = 3$, this P and this q (continuous and positive on $[y_-, y_+ - 1]$ by the previous paragraph). Its admitted words are $\mathcal{W}$ and its q_w are those of the wave and lift grids on the cut. By (S1) the hypothesis of its final sentence holds, so the datum satisfies (Kx1) with $G_{\mathrm{W}}, G_{\mathrm{L}}, G_{\mathrm{T}}$ and (Kx6) with $\beta_j = \beta'_{b,s} = \beta$. In particular every admitted word is a cut word of length at most N_{top} with $\mathrm{wt}^{\mathrm{loc}} > 0$, $\varnothing \in \mathcal{W}$, and $\mathcal{W}$ is finite.

(Kx2), (Kx3), (Kx13). By the lemma on the slice, wave-isometry and descent clauses of engine places, whose hypotheses are an engine place of type l with constant C_b on a Bessel datum of rank r, $k \geq 2$, and the cut I_{l+1} of (Ft0), which is I_l.

(Kx4). By the Fourier identity for engine places (engine place, $k \geq 2$, cut I_l): (Kx4) holds for every nice Φ, every $0 \leq j \leq r$, every cut word w with $\mathrm{wt}^{\mathrm{loc}}(w) > 0$ and all local points of level j; for $\mathrm{wt}^{\mathrm{loc}}(w) = 0$ (Kx4) asserts nothing.

(Kx5). Let Φ, $0 \leq j \leq r$, a cut word w, $ys \in \mathbb{R}^m$ with $w^{\pm}_{ys}$ cut words and families x, x' of nice coefficients be given. The entries of ys are letter values of distinct positions of the cut word w^+_{ys}, so they pairwise differ by more than 2. Hence Proposition A.73 (engine place of type l, $k \geq 2$, the q_w of the wave and lift grids, which are the q_w of the datum) gives $q_w(y) > 0$ for y in ys and the inequality of (Kx5).

(Kx9). (a) (Kx15) is part (c) of the top-clause lemma for engine places, and (Kx16) is the completeness of the type projections for frame-compatible top data (the top datum is frame-compatible); (b) is part (b) of the top-clause lemma; (d) is its part (a), for every cut word w, with $q_t \leq C_b|w|n$ and $\kappa_e = C_b(C_b+3)(r+1)|w|n$. The hypotheses of the top-clause lemma are an engine place of type l with constant C_b on a Bessel datum of rank $r \geq 1$, $k \geq 2$, and the cut I_{l+1} of (Ft0). (c) is the top-grid weight bound for frame-compatible top data: the top datum is frame-compatible for (u, u') with $l_0 = l+1$, so its u_{l_0}, u'_{l_0} are u_l, u'_l and $M - m = M_* - m_* \geq 5$ by (S3); P is continuous; $N = N_{\mathrm{top}} \geq K = 5$; and every admitted word has length at most N_{top}. So (Kx9)(c) holds with C_{T}.

Decay of the test coefficient. The decay lemma for cut-off discrete-series coefficients applies to the Bessel datum of rank r ($k \geq 1$), σ, λ, $\mathbf{j}$, the gap $h > 0$, the maps P_i, Q_i of (Pl3), and any smooth χ_1 with χ_1 and all its derivatives bounded: then $\chi_1(G_f/\varepsilon)F \in \mathcal{D}_r(A, M)$ whenever $\theta(\mathbf{j}) + \max(\lceil A \rceil, 0) + 2(r+1)M < h$. Both χ (smooth by (Pk6), with values in $[0,1]$ and constant outside $[\frac{1}{2}, 1]$ by (Pl2), so every derivative is continuous with support in $[\frac{1}{2}, 1]$) and $\chi_1 \equiv 1$ qualify. We also use that $\mathcal{D}_r(A, M) \subseteq \mathcal{D}_r(A, M')$ for integers $0 \leq M' \leq M$ (fewer derivatives are bounded), with the same constant.

The truncations. Fix $\varepsilon > 0$ and s as in the statement, and let $\Phi_m = s((4 - \|g\|^2_{\mathrm{F}}/(m+1)^2)/3)F_\varepsilon$; these are the coefficients of the lemma on cut-off truncations and of the lemma on truncations at the top level, and they are nice by the former, whose standing hypothesis holds as χ vanishes on $(-\infty, \frac{1}{2})$ by (Pl2). Every word of $\mathcal{W}^+$ is a cut word of length at most $N_{\mathrm{top}}+3$: the words of $\mathcal{W}$, and the words $w^{\pm}_{\mathbf{n}}$ of decay grids of $w \in \mathcal{W}$, which are cut words of length $|w|+3$ by definition. For a cut word w' the operator $\mathrm{RR}_{w'}$ is defined, since the factors $\mathrm{R}_a(x)$, $\widetilde{\mathrm{R}}_a(x)$ are evaluated at differences of letter values of distinct positions, which exceed 2 in absolute value, so $x \neq 1$.

(Tn1). By the existence of decay grids on a long cut: the cut I_l contains $[y_-, y_+]$, every admitted word is a cut word of length at most N_{top}, and $y_+ - y_- \geq 9(2r+1) + 6N_{\mathrm{top}} + 1$ by (S2).

(Tn5) with (Tn6). Let $w \in \mathcal{W}$, $0 \leq j \leq r$, h_0 a local point of level j and $ys \in \mathbb{R}^{m'}$, $m' \in \{0, 3\}$, with $w' = w^+_{ys}$ a cut word; $|w'| \leq N_{\mathrm{top}} + 3$. Apply the lemma on cut-off truncations to w' and j with $A_{\mathrm{B}} = 2(k+r)(r+1)+1$. Its hypothesis (1) holds since $A_{\mathrm{B}} > 2(k+\rho)(\rho+1)$

for $\rho = r - j \le r$. For (2), its numbers A_1, m_1, A_2, A_0, M_0 are those of part (b) of the gap budget of the truncation clauses with $x = |w'|$ (same A_{B}, same n_ρ), and by that part (b) (with $r \ge 1$, $k \ge 2$ and the sign vector $\mathbf{j}$ of (Pl3)) $\theta(\mathbf{j}) + \lceil A_0 + 1 \rceil + 2(r+1)M_0 < 261n^4 \le h_* < h$; so $F_\varepsilon \in \mathcal{D}_r(A_0+1, M_0)$ by the decay paragraph. Conclusion (a) of the lemma on cut-off truncations is (Tn5) with (Tn6) for w', j, h_0.

(Tn3) at levels $i \le r$. For $w' \in \mathcal{W}^+$ and $j = i$, the same check (with $x = |w'| \le N_{\mathrm{top}}+3$) gives the hypotheses of the lemma on cut-off truncations, whose conclusion (b) is $K_{i,w'}(\Phi_m)(p,q) \to K_{i,w'}(F_\varepsilon)(p,q)$ for local points of level i; for other points both sides vanish.

(Tn3) at the level $r+1$. For $w \in \mathcal{W}^+$ hypothesis (i) of the lemma on truncations at the top level is (Kx9)(d), proved above for every cut word, with $o_w = \max q_t \le \lfloor C_b|w|n \rfloor$ (o_w is an integer) and $\kappa_w = C_b(C_b+3)(r+1)|w|n$. By part (a) of the gap budget of the truncation clauses, with $x = |w| \le N_{\mathrm{top}} + 3$, $\theta(\mathbf{j}) + \lceil A_{\mathrm{orb}} + \kappa_w \rceil + 2(r+1)\lfloor C_b|w|n \rfloor < n^5 + 5(C_b+3)^2 n^3 \le h_* < h$, so $F_\varepsilon \in \mathcal{D}_r(A_{\mathrm{orb}} + \kappa_w, \lfloor C_b|w|n \rfloor) \subseteq \mathcal{D}_r(A_{\mathrm{orb}} + \kappa_w, o_w)$: hypothesis (ii). By the final sentence of the lemma on truncations at the top level, (Tn3) holds at the level $r+1$ for ε and (Φ_m), with the same Φ_m.

(Kx11) and (Kx12) for these Φ_m. The Φ_m are nice and the admitted words are cut words of positive weight, so parts (b) and (c) of the continuity and germ-rate lemma for engine places (engine place of type l with constant C_b, $k \ge 2$) give (Kx11) and (Kx12).

Energy model. Apply Proposition A.76 with $N_{\mathrm{tot}} = N_{\mathrm{top}} = 20(r+1)$ and the numbers $A_{\mathrm{E}}, m_{\mathrm{E}}, A_{\mathrm{T}}, m_{\mathrm{T}}$ of the adjointness and transfer propositions for the engine energy system, for $(C_b, r, k, N_{\mathrm{tot}})$ (they do not depend on C_b): (Em1) is (Pl1); (Em2) is (Pl2); (Em3) is (Pl4) with $\lambda_t = \mu_t$ and $[\alpha, \beta] = [y_-, y_+]$; (Em5) holds as every admitted word is a cut word; (Em6) is (Pl5) with $N_{\mathrm{top}} \le N_{\mathrm{tot}}$. For (Em4) take $A_0 = 2(k+r)(r+1)+1 > 2(k+r)(r+1)$. By the decay budget of the energy model (for $r \ge 0$, $k \ge 2$, $N_{\mathrm{tot}} = 20(r+1)$, this A_0 and the sign vector $\mathbf{j}$ with $k+r$ entries $+$ and $r+1$ entries $-$), $\theta(\mathbf{j}) + \lceil \max(A_0, A_{\mathrm{E}}, A_{\mathrm{T}}) \rceil + 2(r+1)\max(4, m_{\mathrm{E}}, m_{\mathrm{T}}) < 261n^4 < h$, so the decay paragraph with $\chi_1 \equiv 1$ gives $F \in \mathcal{D}_r(\max(A_0, A_{\mathrm{E}}, A_{\mathrm{T}}), \max(4, m_{\mathrm{E}}, m_{\mathrm{T}}))$; and $\theta(\mathbf{j}) + A_0 + 2(r+1)(nN_{\mathrm{top}} + 5) < 261n^4 < h$ by the same lemma, so the decay paragraph with χ gives $F_\varepsilon \in \mathcal{D}_r(A_0, nN_{\mathrm{top}} + 5)$ for every $\varepsilon > 0$. Hence the place has an energy model of its level-zero kernels.

Conclusion. All hypotheses of part (ii) of the archimedean product theorem hold for the datum with admitted words $\mathcal{W}$ and $N_{\mathrm{top}} \ge n_{\mathrm{d}}$, the cut I_l, the q_w and the constants named in the statement; so it has the lower block clauses and satisfies (Pk14), the coefficients of (Kx10) being the Φ_m. If $C_b \le 11$, then $h_* < \frac{1}{2}n^{19}$ by part (c) of the gap budget of the truncation clauses, so $h \ge \frac{1}{2}n^{19}$ gives $h > h_*$. □

Proposition A.79. *In the setting of Theorem A.71 there is a real Y_{ins} with the following property. Let $x \in W$ with $x \ge Y_{\mathrm{ins}}$, let $a \in A_x$ be a member with word sizes for the fixed word-size constant C_0, and let $\mathfrak{X}$ be its member Hida and Poincaré setting determined by the selected tied local data, with assembled datum D. Fix a local choice (φ, ν, m) of $\mathfrak{X}$, distinct member words $\omega \ne \omega'$, a place $\tau \in \Phi$, and a Bessel place formed by the selected tied local data at this choice for (ω, ω') at τ at scale 1, with its given section μ. Put $u = u_{\omega,\tau}$, $u' = u_{\omega',\tau}$, $r = r_S = m_D/2 - 1$ and $n = k + m_D - 1 = 2r + k + 1$, where m_D denotes the rank-datum integer of D.*

Let $a_\tau, b_\tau \in \mathcal{H}^\infty_{X,\tau}$, $\xi_\tau \in \mathcal{E}$, $\xi'_\tau \in \mathcal{E}'$ and $c^{\mathrm{coef}}_\tau \in \mathbb{C}$ be given. Define the prescribed coefficient by

$$F_{\mathrm{given}}(g)\eta = c^{\mathrm{coef}}_\tau \, \langle a_\tau, \sigma^\varphi_\tau(g) b_\tau \rangle \, \xi_\tau \langle \xi'_\tau, \eta \rangle \qquad (\eta \in \mathcal{E}').$$

Then there are an integer $1 \le l \le k-2$ with $c(\omega)_l \ne c(\omega')_l$ (and hence $u_l \ne u'_l$, using zero-based coordinates for u, u'), and an engine place of type l with constant $C_b = 11$ on the Bessel datum of the given place, whose data have $F = F_{\mathrm{given}}$ and whose P, χ satisfy (Pl1)–(Pl5) with the same

representation $\sigma = \sigma^{\varphi}_{\tau}$, $\lambda = \lambda_{\tau}$, $\mathbf{j} = \varepsilon_S$ and $h = n^{19}/2$. In particular $F = \sum_i P_i \sigma Q_i$ with $Q_i(\mathcal{E}')$ and $P_i^*(\mathcal{E})$ consisting of smooth vectors; χ is a cut-off of (Pl2); $P(Y) = \prod_{s=0}^{n-1}(Y + \lambda_{\tau,s})$ for all real Y and $\mathrm{qd}(Y + (n-1)/2)F = P(Y)F$ on the cut; and I_l is long as in (Pl5).

Proof. Outline. By the context tie Proposition A.114 the word code of the context is injective, so two distinct words differ in some bit l with $1 \le l \le k-2$, whence $|u_l - u'_l| = \Delta_D$ and the cut I_l has length $\Delta_D - 4$, which gives (Pl5) on a tail in x; the engine place of type l with $C_b = 11$ is obtained from the existence result for engine places, (Pl1) comes from the half-integrality of u, u', (Pl3) from the square-integrability of σ^{φ}_{τ} with parameter λ_{τ} and the gap of the member parameter, and the order conditions of (Pl4) from its interlacing, the prescribed coefficient is written as one term $P_1 \sigma Q$, the eigen-equation of (Pl4) follows from the centrality of the coefficients of the quantum determinant, and an explicit smooth cut-off gives (Pl2). □

A.7.2 Bessel amplitudes and point sums

At a complex place the Bessel places, Bessel sections, amplitudes and Bessel masses are defined in Definition A.80, and the hermitian space, Bessel subgroups, Bessel transform, big-cell pairing and orbital integrals in Definition A.81. The minor-weighted local amplitude bound of a term is Proposition A.82. Its bound (Aq1) does not depend on the following lemma; its clause for degenerate pairs rests in addition on an absolute bound with size decay, Lemma A.83, which we sketch. The main analytic input is the pointwise Bessel bound at large gap (Theorem A.84), and the constants are uniform on a parameter ball, independently of the place (Proposition A.85). A second bound of the same kind controls the weighted point sum of orbital squares over B-points (Theorem A.86), and this gives the point sum (Ng6) of the non-generic row (Proposition A.87).

Definition A.80. Fix integers $r \ge 0$, $k \ge 1$ and put $n = 2r + k + 1$.

Bessel places. A **Bessel place** of rank r consists of a Bessel datum of rank r (vectors $z, z' \in W$ with $T(z), T(z') > 0$, representations π of H_z on $\mathcal{E}$ and π' of $\mathrm{H}_{z'}$ on $\mathcal{E}'$, the section μ of (Pk4)), a real number $c_\psi > 0$ (the **character constant**) with the characters

$$\psi_z(g) = \exp\bigl(i\,\mathrm{Re}(c_\psi\,\ell_z(g))\bigr), \qquad \psi_{z'}(g) = \exp\bigl(i\,\mathrm{Re}(c_\psi\,\ell_{z'}(g))\bigr)$$

of N_r, a Haar measure dg on G_r, the Haar probability measures dh, dh' on H_z, $\mathrm{H}_{z'}$, a Haar measure dn on N_r, a Haar measure $dg_{\deg}$ on G_r (for formal degrees), and measures ν_S, ν'_S on G_r with

$$\int \phi\, dg = \iiint \phi(hnx)\, dh\, dn\, d\nu_S(x) = \iiint \phi(h'nx)\, dh'\, dn\, d\nu'_S(x) \qquad (\phi \ge 0 \text{ measurable}).$$

Bessel sections. Let a Bessel place be given and $\lambda \in \mathbb{R}^n$ a Harish-Chandra parameter. A **Bessel section at** λ is a tuple $A = (\mathcal{H}, \sigma, B, B', v_0, c_N, \deg)$ where

(Bs1) σ is an irreducible square-integrable unitary representation of G_r on a Hilbert space $\mathcal{H}$ with Harish-Chandra parameter λ relative to ε (after conjugation by a Cayley transform from the standard form of the unitary group to G_r); $\mathcal{H}^{\infty}$ is its space of smooth vectors;

(Bs2) $B\colon \mathcal{H}^{\infty} \to \mathcal{E}$ and $B'\colon \mathcal{H}^{\infty} \to \mathcal{E}'$ are **Bessel maps**: linear, with

$$B(\sigma(hn)v) = \psi_z(n)\,\pi(h)B(v)$$

for $h \in \mathrm{H}_z$, $n \in \mathrm{N}_r$, $v \in \mathcal{H}^{\infty}$ (and likewise for B' with z', π'), and continuous in the sense that $\|B(v)\| \le C \sum_w \|d\sigma(w)v\|$ for some C and finitely many words w in the Lie algebra of G_r;

(Bs3) $v_0 \in \mathcal{H}^\infty$ is nonzero and lies in the lowest K-type of σ;

(Bs4) $c_N > 0$ and, with $\Phi_A(g) = B(\sigma(g)v_0)$ and $\Phi'_A(g) = B'(\sigma(g)v_0)$,

$$\int \|\Phi_A\|^2 \, d\nu_S = c_N \dim \mathcal{E}, \qquad \int \|\Phi'_A\|^2 \, d\nu'_S = c_N \dim \mathcal{E}';$$

(Bs5) $\deg > 0$ is the formal degree of σ with respect to $dg_{\deg}$: $\int \langle w, \sigma(g)v\rangle \overline{\langle w', \sigma(g)v'\rangle} \, dg_{\deg} = \langle v', v\rangle\langle w, w'\rangle / \deg$ for all $v, w, v', w' \in \mathcal{H}$.

Amplitudes and Bessel masses. For a Bessel section A, a measure ν on G_r and $\gamma \in \mathrm{G}_r$ the **amplitude** is

$$J_A(\nu, \gamma) = c_N^{-1} \int \Phi_A(\gamma x) \otimes \Phi'_A(x)^* \, d\nu(x) \in \mathrm{Hom}(\mathcal{E}', \mathcal{E}),$$

its **Bessel mass** is

$$M(A) = \int_{(\mathbb{C}^\times)^r \times \Omega} w(c, \zeta) \, \|J_A(dg, g_{c,\zeta})\|_{\mathrm{HS}}^2 \, dc \, d\zeta \in [0, \infty],$$

and its **unit amplitude** is $j_A(\nu, \gamma) = M(A)^{-1/2} J_A(\nu, \gamma)$ if $0 < M(A) < \infty$, and $j_A(\nu, \gamma) = 0$ otherwise.

Relative points and root measures. Let p be a point over $\mathbb{C}$ of length $r+1$, and let $S_r = \mathrm{diag}((-1)^{r-b})_{0 \le b \le r}$ be the **sign matrix** of the flag. An element $\gamma \in \mathrm{G}_r$ is a **relative point** of p if $G_f(\gamma) \neq 0$ and $\widehat{Z}(\gamma)\, S_r = L \, \mathrm{pm}(p) \, U$ with L lower and U upper unitriangular. A measure μ_R on G_r is a **root measure** of p at γ if $\gamma = n_1 g_0 n_2$ with $n_1, n_2 \in \mathrm{N}_r$, $\widehat{Z}(g_0)\, S_r = \mathrm{pm}(p)$, and μ_R is the image under $x \mapsto n_2^{-1} x n_2$ of a measure μ_0 concentrated on the group $R'_{g_0} = \{s \in \mathrm{N}_r : \ g_0 s g_0^{-1} \in \mathrm{N}_r\}$ whose image under the **right root coordinates** $s \mapsto$ (the e_i-coordinate of $sf'_j)_{(i,j)}$, (i,j) running over the pairs $i < j$ of indices in one block of p, is $(c_\psi/2\pi)^{\#}$ times Lebesgue measure ($\#$ the number of such pairs). Then μ_R is concentrated on $\{s \in \mathrm{N}_r : \ \gamma s \gamma^{-1} \in \mathrm{N}_r\} = \gamma^{-1} R_\gamma \gamma$. For $\theta > 0$, a measure ν on G_r is a **root quotient of scale** θ at γ if for some point p and some root measure μ_R of p at γ

$$\int \phi \, dg = \theta \iint \phi(sx) \, d\mu_R(s) \, d\nu(x) \qquad (\phi \ge 0 \text{ measurable}).$$

Definition A.81. The space and the group. Let $k \ge 1$ and $\rho \ge 0$ be integers and $n = 2\rho + k + 1$. On $V_\rho = \mathbb{C}^n$, with standard basis $e_0, \dots, e_{n-1}$, let H_ρ be the matrix with $(H_\rho)_{a,n-1-a} = (H_\rho)_{n-1-a,a} = 1$ for $a < \rho$, $(H_\rho)_{aa} = 1$ for $\rho \le a < \rho + k$, $(H_\rho)_{\rho+k,\rho+k} = -1$ and all other entries 0, and put $h(v, w) = w^* H_\rho v$ (linear in v). Write $e_a^* = e_{n-1-a}$ $(a < \rho)$, $X_a = \mathrm{span}(e_0, \dots, e_{a-1})$ and $W = \mathrm{span}(e_\rho, \dots, e_{\rho+k})$, of signature $(k, 1)$; so $h(e_a, e_b^*) = \delta_{ab}$, the e_a are isotropic and V_ρ has signature $(k + \rho, \rho + 1)$. Let $\mathrm{G}_\rho = \{g \in \mathrm{GL}_n(\mathbb{C}) : g^* H_\rho g = H_\rho\}$, $\mathrm{N}_\rho = \{g \in \mathrm{G}_\rho : ge_a \in e_a + X_a \ (a < \rho), \ gw \in w + X_\rho \ (w \in W)\}$ (the unipotent radical of the parabolic subgroup stabilising the flag $X_1 \subset \cdots \subset X_\rho$), and for $z \in W$ put $T(z) = -h(z, z)$,

$$\ell_z(g) = \sum_{a=0}^{\rho-2} g_{a,a+1} + (gz)_{\rho-1}, \qquad \psi_z(g) = \exp(\ell_z(g) - \overline{\ell_z(g)}) \qquad (\ell_z = 0 \text{ if } \rho = 0),$$

and $\mathrm{H}_z = \{g \in \mathrm{G}_\rho : ge_a = e_a, \ ge_a^* = e_a^* \ (a < \rho), \ gz = z\}$. The space $V_{\rho'}$ for $\rho' \le \rho$ is identified with $\mathrm{span}(e_{\rho-\rho'}, \dots, e_{n-1-\rho+\rho'}) \subseteq V_\rho$ and $\mathrm{G}_{\rho'}$ with the subgroup of G_ρ fixing e_a, e_a^* for $a < \rho - \rho'$; W is the same space at every ρ'.

Bessel data. A **Bessel datum** of rank ρ consists of

(Pk1) vectors $z, z' \in W$ with $T(z) > 0$ and $T(z') > 0$ (then $z^{\perp} \cap W$ is positive definite and $\mathrm{H}_z \cong \mathrm{U}(k)$ is compact);

(Pk2) finite-dimensional Hilbert spaces $\mathcal{E}, \mathcal{E}'$ and continuous unitary representations π of H_z on $\mathcal{E}$ and π' of $\mathrm{H}_{z'}$ on $\mathcal{E}'$;

(Pk3) Haar measures dg on G_ρ and dn on N_ρ and the Haar probability measures dh, dh' on H_z, $\mathrm{H}_{z'}$;

(Pk4) a smooth map μ from $\Omega = \{\zeta \in \mathbb{C} : |\zeta|^2 > T(z)T(z')\}$ to the unitary group $\mathrm{U}(W, h)$ with $h(\mu(\zeta)z', z) = \zeta$.

A **coefficient** is a function $F \colon \mathrm{G}_\rho \to \mathrm{Hom}(\mathcal{E}', \mathcal{E})$.

Bessel transform. For a coefficient F and $g \in \mathrm{G}_\rho$,

$$(\mathcal{J}F)(g) = \int_{\mathrm{H}_z} \int_{\mathrm{N}_\rho} \int_{\mathrm{H}_{z'}} \int_{\mathrm{N}_\rho} \psi_z(n)\, \overline{\psi_{z'}(n')}\, \pi(h)\, F(h^{-1}n^{-1}gn'h')\, \pi'(h')^{-1}\, dn'\, dh'\, dn\, dh. \tag{A.7}$$

The big cell and the pairing. For $c \in (\mathbb{C}^\times)^\rho$ and $\zeta \in \Omega$ let $g_{c,\zeta} \in \mathrm{G}_\rho$ be given by $g_{c,\zeta} e_a = c_a e_a^*$, $g_{c,\zeta} e_a^* = \bar{c}_a^{-1} e_a$ $(a < \rho)$ and $g_{c,\zeta}|_W = \mu(\zeta)$, and put $w(c,\zeta) = \prod_{a<\rho} |c_a|^{2(n-2a-2)}\, (|\zeta|^2 - T(z)T(z'))^{k-1}$. The **big-cell pairing** of coefficients F, F' is

$$\beta(F, F') = \int_{(\mathbb{C}^\times)^\rho \times \Omega} w(c,\zeta)\, \big\langle (\mathcal{J}F')(g_{c,\zeta}), (\mathcal{J}F)(g_{c,\zeta}) \big\rangle_{\mathrm{HS}}\, dc\, d\zeta \tag{A.8}$$

(Lebesgue measure on $\mathbb{C}^{\rho+1} = \mathbb{R}^{2\rho+2}$); it is linear in F and conjugate-linear in F'.

Invariants of a double coset. Put $f_a = f'_a = e_a$ $(a < \rho)$, $f_\rho = z$, $f'_\rho = z'$, and for $g \in \mathrm{G}_\rho$ let $\widehat{Z}(g)$ be the $(\rho+1) \times (\rho+1)$ matrix $\widehat{Z}(g)_{ab} = h(g f'_b, f_a)$, $D_l(g)$ $(0 \le l \le \rho+1)$ its leading principal $l \times l$ minor $(D_0 = 1)$ and

$$G_f(g) = |D_{\rho+1}(g)|^2 - T(z)T(z')\, |D_\rho(g)|^2. \tag{A.9}$$

Orbital integrals. Let $\gamma \in \mathrm{G}_\rho$, $R_\gamma = \{m \in \mathrm{N}_\rho : \gamma^{-1} m \gamma \in \mathrm{N}_\rho\}$ and dm a Haar measure on R_γ. A **cut** at γ is a measurable $\vartheta \ge 0$ on $\mathrm{H}_z \times \mathrm{N}_\rho \times \mathrm{H}_{z'} \times \mathrm{N}_\rho$ with $\int_{R_\gamma} \vartheta(h, mn, h', \gamma^{-1} m \gamma n')\, dm = 1$ for all (h, n, h', n'). The **orbital integral** of F at (γ, dm, ϑ) is

$$\mathcal{O}_\gamma(F) = \int \vartheta(h,n,h',n')\, \psi_z(n)\, \overline{\psi_{z'}(n')}\, \pi(h)\, F(h^{-1}n^{-1}\gamma n' h')\, \pi'(h')^{-1}\, dn'\, dh'\, dn\, dh. \tag{A.10}$$

Base frames and representatives. For $v \in V_\rho$ let v_W be its component in W and $\mathrm{pr}_z(v) = v_W - \frac{h(v_W, z)}{h(z,z)} z$. A **base frame** is a pair (ν', m_0) with $\nu' \in W$, $h(\nu', z') = 0$, $h(\nu', \nu') = 1$, and $m_0 \in \mathrm{G}_\rho$ fixing all e_a, e_a^* $(a < \rho)$ with $m_0 z' = \sqrt{T(z')/T(z)}\, z$. For γ with $\det \widehat{Z}(\gamma) \neq 0$ put $\mathrm{cof}_R(\gamma) = \sum_b \mathrm{adj}(\widehat{Z}(\gamma))_{b\rho}\, \gamma f'_b$ and $\mathrm{cof}_L(\gamma) = \sum_a \overline{\mathrm{adj}(\widehat{Z}(\gamma))_{\rho a}}\, f_a$. An element $\gamma \in \mathrm{G}_\rho$ is **framed** by (ν', m_0) if $\mathrm{pr}_z(\mathrm{cof}_R(\gamma)) = a \det \widehat{Z}(\gamma)\, m_0 \nu'$ and $\mathrm{pr}_{z'}(\gamma^{-1}\mathrm{cof}_L(\gamma)) = a' \overline{\det \widehat{Z}(\gamma)}\, \nu'$ for some reals $a, a' > 0$, and $\mathrm{pr}_z(u) = m_0\, \mathrm{pr}_{z'}(\gamma^{-1} u)$ for every u with $h(u, f_a) = h(u, \gamma f'_a) = 0$ for all $a \le \rho$.

Proposition A.82. *Let $C_h, C_n, C_1 \ge 0$. Let C_a, C_b, C_k and the lower bound $\max(C_{g,0}, C_{g,1})$ for C_g be as in the assembly of the clauses (Am2), (Am3) on the parameter ball for (C_h, C_n, C_1) and $C_M = 0$, and let C_g be a real at least this bound and at least the constant $C_{g,0}$ of part (b) of the construction of the Bessel place of a member for $(C_h, C_u) = (C_h, C_n)$. Let $C_K \ge 0$. Let C_{gf} be the constant of the change of scale of a point-sum datum by a torus translate. Then the following*

holds for every assembled master datum D with rank datum $(\dots;k,m,t)$, $n=k+m-1$, $r=r_S$, with $C_n\le m^2$, $C_g\le m^7$ and $8C_{gf}\le n^{23/2}$, and every $\tau\in\Phi$ at which D satisfies (Zn1)–(Zn4).

Let u,u' be Harish-Chandra parameters of D at τ satisfying (P2) for C_g, let $b\ge\max(2,u_0,u_0',b_{\tau,m})$, let $T,T',c_\psi,v,\theta>0$ satisfy (Mp1) with $C_T=C_c=C_h$, and let local data of D at τ for (u,u',T,T',v,θ) be given, with character constant c_ψ and representation π on $\mathcal{E}$. Let reals g,L,C_l,C_g',C_S be such that $(v,g,b,L,C_l,C_g',C_S,C_K,\theta)$ satisfy the conditions of the constant clause (Am4) for Bessel amplitude data at large gap, and $TT'\theta^2\le 2^{r+1}b^{C_Sm^2}$. Put $R=C_nkm^3$, let $\mathcal{P}\subseteq\lambda_\tau+\mathbb{Z}^n$ with $\lambda_\tau\in\mathcal{P}$ and, for $\lambda\in\mathcal{P}$, Bessel sections A_λ at λ; put $A_0=A_{\lambda_\tau}$. Let p be a point over $\mathbb{C}$ of length $r+1$, γ a relative point of p, $G=G_f(\gamma)$ (then $G>0$), and ν a root quotient at γ of a scale θ' with $b^{-C_1m^2}\le\theta'\le b^{C_1m^2}$. Assume the decomposition condition (As3) of that assembly for these data, $\mu_0=\lambda_\tau$ and R; the conditions (As1), (As2) and (As4) are not assumed. (As3) need not be assumed if the conditions (Dz1) and (Dz2) for decomposition data on a parameter ball hold for these data, $\mu_0=\lambda_\tau$ and R. Put

$$X=\max(G,G^{-1})\prod_{l\in\mathrm{be}(p)}\max\bigl(|D_l(p)|,|D_l(p)|^{-1}\bigr)^{2(m+4)},$$

$$\Lambda=b^{(C_b+11C_a)m^4+C_Km^2}\max(1,G^{-1})^{10m^2}\,X,$$

$B=b^{C_km^2}(e\,n^3)^{(r+1)/2}$ and $t=t(T',\mathrm{vU}(n)\,\alpha\,\deg(A_0),u')$. Then $|D_l(\gamma)|=|D_l(p)|$ for $0\le l\le r+1$, and for every integer $0\le l\le r+1$ and every integer $q\ge0$ with $q\,l\le C_nkm^3$,

$$n_0(T)\,G^k\,\|J_{A_0}(\nu,\gamma)\|_{\mathrm{HS}}^2\,|D_l(\gamma)|^{2q}\le\Lambda\,t^{-k}\,\dim\mathcal{E}\;B^{2q}.$$

Moreover, if $n\le b^c$ for a real $c\ge0$, then $B\le b^{(C_k+1+2c)m^2}$; and if the place satisfies the compact-dual normalisation (Cd) of the formal degree, then $t=t(T',\deg(A_0),u')$. In particular, at a complex place τ of archimedean amplitude records with $n_\tau=n_0(T)G^k$, $h_\tau=\|J_{A_0}(\nu,\gamma)\|_{\mathrm{HS}}^2$, $d_\tau=\dim\mathcal{E}$, $\lambda_\tau=\Lambda$, $t_\tau=t$, $D_{\tau,l}=|D_l(\gamma)|$, $Q\le C_nkm^3$ and $b^{(C_k+1+2c)m^2}\le H$ (in place of the constant B of the records), the inequality of (Aq1) holds at τ.

Degenerate-pair clause. *Let $A>0$. There is a real $C_d\ge1$ such that the following holds for the same data whenever, in addition,*

(Dg1) *$T'/T\le A$;*

(Dg2) *$k^{3r}g^m\le T'\,t$, with t as above;*

(Dg3) *$g\ge1$, $v\ge1$, $k\ge C_d$ and $n\le b^c$.*

Put $\kappa=1$ if $D_r(\gamma)=0$, and otherwise $c_\gamma=|D_{r+1}(\gamma)|/(|D_r(\gamma)|\sqrt{TT'})$ (then $c_\gamma>1$, as $G>0$) and $\kappa=(1+c_\gamma)/\sqrt{c_\gamma^2-1}$. With $\mathrm{be}(p)$ and $D_l(p)$ as above put

$$\Lambda_d=b^{C_dm^4}\max(G,G^{-1})^{C_dm^2}\prod_{l\in\mathrm{be}(p)}\max\bigl(|D_l(p)|,|D_l(p)|^{-1}\bigr)^{C_dm}$$

$$\cdot\Bigl(b^{C_dm}\max_{l\in\mathrm{be}(p)}\max\bigl(1,|D_l(p)|^{-1}\bigr)\Bigr)^{2C_dr^3},$$

$$\mathrm{sz}_d=\prod_{l=1}^{r+1}\max\bigl(1,|D_l(\gamma)|/b^{C_dm^2}\bigr).$$

Then for every integer q with $0\le q\le C_nkm^2$,

$$n_0(T)\,G^k\,\|J_{A_0}(\nu,\gamma)\|_{\mathrm{HS}}^2\,\kappa^{2k}\,T^{-k}\le\Lambda_d\Bigl(\frac{C_d}{k}\Bigr)^k\dim\mathcal{E}\;\mathrm{sz}_d^{-2q/(r+1)}.$$

In particular, at a complex place τ of degenerate-pair records with the identifications above, $T_1 = T$, $\kappa_\tau = \kappa$, $\mu_\tau = \Lambda_d$, $Q_d \le C_n k m^2$ and $H \ge b^{C_d m^2}$, the inequality of (Aq5) holds at τ for this C_d. All constants above depend only on (C_h, C_n, C_1, C_K) and c (and C_d also on A), not on D, τ, the local data or the sections.

Proof. Outline. Leading principal minors are unchanged by unitriangular factors, so $|D_l(\gamma)| = |D_l(p)|$ and $G > 0$; the bound (Aq1) then follows from a minor-weighted one-level amplitude bound, whose clauses are supplied by the Bessel place of a member (mass comparability and the constant clause), a weight bound at the point p and the assembly of (Am2), (Am3) on the parameter ball, while the degenerate-pair clause combines this bound, with size decay, with the absolute bound Lemma A.83 through a numerical lemma for degenerate pairs. □

Lemma A.83. *Let $C_h, C_n, C_1, C_K \ge 0$, $c \ge 0$ and $A > 0$. There are reals $C_{ab} \ge 2$ and $C_{hb} \ge 0$, depending only on (C_h, C_n, C_1, C_K), c and A, such that the following holds. Let D, τ, k, m, n, $r = r_S$, $u, u', b, T, T', c_\psi, v, \theta$, the local data with representations π on $\mathcal{E}$ and π' on $\mathcal{E}'$, g, L, C_l, C'_g, C_S, $\mathcal{P}$, the sections A_λ, $A_0 = A_{\lambda_\tau}$, the point p, the relative point γ of p and the root quotient ν satisfy all hypotheses of Proposition A.82 for (C_h, C_n, C_1, C_K) (with (As3), or (Dz1) and (Dz2)), together with $T'/T \le A$, $v \ge 1$ and $n \le b^c$. If $D_r(\gamma) \ne 0$, then for every integer q with $0 \le q \le C_n k m^2$,*

$$\big\| J_{A_0}(\nu, \gamma) \big\|_{\mathrm{op}} \le P_{ab} \Big(\frac{C_{ab}}{k\, v} \Big)^{r(k-2-r)} |D_r(\gamma)|^{-(k-2-r)}\, \mathrm{sz}_{ab}^{-q/(r+1)},$$

where $\| \cdot \|_{\mathrm{op}}$ is the operator norm for the Hilbert norms of $\mathcal{E}'$ and $\mathcal{E}$,

$$P_{ab} = \Big(b^{C_{ab} m} \max_{l \in \mathrm{be}(p)} \max(1, |D_l(p)|^{-1}) \Big)^{C_{ab} r^3}, \qquad \mathrm{sz}_{ab} = \prod_{l=1}^{r+1} \max(1, |D_l(\gamma)| / b^{C_{hb} m^2}).$$

Proof. Sketch. The lemma is used only in the degenerate-pair clause of Proposition A.82; the main bound (Aq1) of that proposition does not depend on it. Its role is to control the amplitude when the pair is nearly degenerate, that is when $G = G_f(\gamma)$ is small and the factor $\max(G, G^{-1})$ makes the non-degenerate bound useless. The bound sought involves only the minor $D_r(\gamma)$ and not G; combined with $G = TT' |D_r(\gamma)|^2 (c_\gamma^2 - 1)$ and with the non-degenerate bound with size decay, it gives the degenerate-pair inequality, and this combination is carried out in the proof of Proposition A.82.

The argument should run in two steps. First, an absolute bound at every parameter λ of the ball $\mathcal{P}$: since $J_A(\nu, \gamma) = c_N^{-1} \int \Phi_A(\gamma x) \otimes \Phi'_A(x)^*\, d\nu(x)$ (Definition A.80), its operator norm is at most $c_N^{-1} \int \|\Phi_A(\gamma x)\|\, \|\Phi'_A(x)\|\, d\nu(x)$, and it remains to bound this integral by moments of majorants of the two Bessel functions, using the leading minors $D_l(p)$, $l \in \mathrm{be}(p)$, of the point and the normal form of the root measure. Second, size decay: a bound that is uniform over the parameters of the ball should convert into decay in the minors $D_l(\gamma)$, since a unit shift of the parameter changes the amplitude by an operator given by a Pieri rule whose size is governed by these minors; a mechanism of this kind already produces the factor $|D_l(\gamma)|^{2q}$ in the first part of Proposition A.82. Heuristically, the shape of the bound is that of lowest-K-type coefficients of discrete series whose parameter has size about k: the dependence on γ through the power $|D_r(\gamma)|^{-(k-2-r)}$ reflects their homogeneity in the leading minor, the factor $(C_{ab}/(kv))^{r(k-2-r)}$ is of the factorial size produced by Gamma-function normalisations such as the formal degree and the constant c_N, and the dependence on b and on the minors of p is polynomial, as in P_{ab}. It remains to prove the moment bounds and the passage to size decay with constants depending only on (C_h, C_n, C_1, C_K), c and A. □

Theorem A.84. *For a point p over $\mathbb{C}$ of length $r+1$ put $G_{T,T'}(p) = |D_{r+1}(p)|^2 - TT'|D_r(p)|^2 \in \mathbb{R}$. For $\mu, \mu' \in \mathbb{R}^n$ put $\|\mu - \mu'\|_1 = \sum_s |\mu_s - \mu'_s|$, and say that μ has gaps $\geq g$ if $|\mu_s - \mu_{s'}| \geq g$ for all $s \neq s'$.*

For every $C_h \geq 0$ there are constants $C_b \geq 0$ and $C_{g,0}$ such that for all $C_g \geq C_{g,0}$ the following holds. Let $r \geq 0$, $k \geq 2$ be integers, $b \geq 2$, and let a point-sum datum of rank r with scale $\kappa = 1$ be given whose representation σ has Harish-Chandra parameter λ and whose coefficient is F. Assume that there is $\lambda_0 \in \mathbb{R}^n$ such that

(Bp1) *λ_0 is strictly decreasing with $\lambda_{0,s} - \frac{n-1}{2} \in \mathbb{Z}$ for all s, and the standard conditions (Xb1)–(Xb5) for (C_h, C_g, b) hold with λ_0 in place of λ;*

(Bp2) *λ_0 has gaps $\geq 200\,\|\lambda - \lambda_0\|_1$.*

Then for every point p over $\mathbb{C}$ of length $r+1$ with $G_{T,T'}(p) \neq 0$,

$$\max\bigl(\mathrm{pw}_{T,T'}(p), 0\bigr)\,\mathrm{os}(p) \leq b^{C_b m^4}\,\max\bigl(1, G_{T,T'}(p)^{-1}\bigr)^{10m^2}\,\mathrm{bm}$$

in $[0,\infty]$ (with $0 \cdot \infty = 0$; for $G_{T,T'}(p) < 0$ the maximum is 1). In particular, if $C_u \geq 0$, $C'_g \geq 200\,C_u$, λ_0 has gaps $\geq C'_g\,k\,m^3$ and $\|\lambda - \lambda_0\|_1 \leq C_u\,k\,m^3$, then (Bp2) holds, and the bound holds with $\max(1, G_{T,T'}(p)^{-1})^{Cm^2}$ for every $C \geq 10$.

Proof. Outline. The bound is deduced from a general eigen-decay bound for point sums, whose hypotheses are checked from the standard conditions (Xb1)–(Xb5): they make b large compared with n and the gaps of λ_0 large compared with m^3, the coefficient has eigen-decay because the parameter λ has a gap far above the decay threshold $\theta(\varepsilon)$ of discrete-series coefficients, and it satisfies the big-cell regularity bound by the regularity of discrete-series coefficients at a large gap; for $G_{T,T'}(p) < 0$ the orbital square vanishes. □

Proposition A.85. *Let $C_h, C_a \geq 0$, let C_b, $C_{g,0}$ be the constants of Theorem A.84 for C_h, let $C_g \geq C_{g,0}$, $B \geq 0$, $R \geq 0$. Let $r \geq 0$, $k \geq 2$ be integers, $n = 2r + k + 1$, $m = 2r + 2$, $b \geq 2$, and let a Bessel place of rank r be given, with $T = T(z)$, $T' = T(z')$. Let $\mu_0 \in \mathbb{R}^n$, let p be a point over $\mathbb{C}$ of length $r+1$ with $\mathrm{pw}_{T,T'}(p) > 0$, γ a relative point of p and ν a root quotient at γ. Let $\mathcal{P} \subseteq \mu_0 + \mathbb{Z}^n$ be a set with $\mu_0 \in \mathcal{P}$ and, for every $\lambda \in \mathcal{P}$, let A_λ be a Bessel section at λ. Put $\delta(\lambda) = \|\lambda - \mu_0\|_1$ and $j(\lambda) = j_{A_\lambda}(\nu, \gamma)$, an element of the space $\mathrm{Hom}(\mathcal{E}', \mathcal{E})$ with the Hilbert–Schmidt norm, and for $\lambda \in \mathcal{P}$ and $0 \leq l \leq r+1$ let $\mathcal{G}_l(\lambda)$ be the set of l-element subsets $I \subseteq \{0, \ldots, n-1\}$ with $\lambda + \varepsilon_I \in \mathcal{P}$. Assume:*

(F1) *for every $\lambda \in \mathcal{P}$ with $\delta(\lambda) \leq R$, the section A_λ satisfies (Ab1) and (Ab2) for (C_h, C_g, C_a, b), p, γ and ν;*

(F2) *for every integer $0 \leq l \leq r+1$ and every $\lambda \in \mathcal{P}$ with $\delta(\lambda) + l \leq R$, the sections $A_0 = A_\lambda$ and $A_I = A_{\lambda + \varepsilon_I}$ ($I \in \mathcal{G}_l(\lambda)$) satisfy (Me1)–(Me3) for this l, $\mathcal{G} = \mathcal{G}_l(\lambda)$ and B.*

Then the data $(\mathcal{P}, \delta, \mu_0, j)$ satisfy (Am2) with

$$w = \mathrm{pw}_{T,T'}(p), \qquad \beta_1 = b^{(C_b + 11C_a)m^4}\,\max\bigl(1, G_f(\gamma)^{-1}\bigr)^{10m^2},$$

and (Am3) with $D_l = D_l(p)$ ($0 \leq l \leq r+1$), the radius R and the constant B. The numbers β_1 (for fixed $G_f(\gamma)$) and B do not depend on λ, on the place, or on the sections.

Proof. **(Am2).** Let $\lambda \in \mathcal{P}$ with $\delta(\lambda) \le R$. By (F1), the pointwise bound for unit amplitudes at a Bessel place applies to the Bessel section A_λ at λ, the point p, the relative point γ and the root quotient ν, and gives $\mathrm{pw}_{T,T'}(p)^+ \|j_{A_\lambda}(\nu,\gamma)\|_{\mathrm{HS}}^2 \le \beta_1$. Since $\mathrm{pw}_{T,T'}(p) > 0$, $\mathrm{pw}_{T,T'}(p)^+ = w$, so $w\|j(\lambda)\|^2 \le \beta_1$. This is (Am2); $w > 0$ as required there.

(Am3). Let $0 \le l \le r+1$ and $\lambda \in \mathcal{P}$ with $\delta(\lambda) + l \le R$. By (F2), the minor expansion of unit amplitudes at a Bessel place applies with $A_0 = A_\lambda$, $\mathcal{G} = \mathcal{G}_l(\lambda)$, $A_I = A_{\lambda+\varepsilon_I}$, and gives numbers a_I ($I \in \mathcal{G}_l(\lambda)$) and the sign $\epsilon_{r,l} \in \{1,-1\}$ with $\sum_I |\epsilon_{r,l} a_I| \le B$ and

$$D_l(p)\, j(\lambda) = D_l(p)\, j_{A_\lambda}(\nu,\gamma) = \sum_{I \in \mathcal{G}_l(\lambda)} \epsilon_{r,l} a_I\, j_{A_{\lambda+\varepsilon_I}}(\nu,\gamma) = \sum_{I \in \mathcal{G}_l(\lambda)} \epsilon_{r,l} a_I\, j(\lambda+\varepsilon_I).$$

The coefficients of (Am3) are the numbers $\epsilon_{r,l} a_I$. The family $(\lambda+\varepsilon_I)_{I \in \mathcal{G}_l(\lambda)}$ is finite (at most $\binom{n}{l}$ members), lies in $\mathcal{P}$ by the definition of $\mathcal{G}_l(\lambda)$, and $\delta(\lambda+\varepsilon_I) \le \delta(\lambda) + \|\varepsilon_I\|_1 = \delta(\lambda) + l$ by the triangle inequality, as ε_I has exactly l entries equal to 1 and the others 0. (If $\mathcal{G}_l(\lambda)$ is empty, the identity reads $D_l(p) j(\lambda) = 0$, an expansion with no terms.) This is (Am3).

Uniformity. β_1 is built from C_b, C_a, m, b and $G_f(\gamma)$ only, and C_b depends only on C_h by Theorem A.84; B is the given constant. Neither depends on λ, on the place or on the sections. □

Theorem A.86. *All symbols introduced here are local to this statement and to the results that cite it. For all real $A_w \ge 0$ and C_h there are real numbers C_B and $C_{g,0}$ with the following property. Let $C_g \ge C_{g,0}$, let $r \ge 0$ and $k \ge 2r+5$ be integers with k even, $m = 2r+2$, $n = 2r+k+1$, let E be a CM field of degree $2f$ with CM type Φ, let $g_1 \ne g_2$ be primes and $\kappa = 2\pi/(\mathrm{rd}_E\, g_1 g_2)$, and let $T, T' \ge 1$ be integers and $b \ge 2$ real. Suppose that at every $v \in \Phi$ a point-sum datum of rank r with scale κ is given, with $T(z_v) = T$, $T(z_v') = T'$, satisfying the standard conditions (Xb1)–(Xb5) for (C_h, C_g, b), whose Harish-Chandra parameter has the form $\lambda^{(v)} = (v_0, \dots, v_{k-2}, -b_0, \dots, -b_{m-1})$ with integers v_i, b_j, and with $|\lambda_s^{(v)} - \lambda_{s'}^{(v)}| \ge C_g m^4$ for $s \ne s'$. Then*

$$\sum_\gamma \mathrm{wp}(\gamma)\, |\mathrm{pw}_{T,T'}(\gamma)|\, \mathrm{nw}_{T,T'}(\gamma)^{A_w(r+1)^2} \prod_{v \in \Phi} \mathrm{os}_v(\tilde\gamma_v) \;\le\; \exp(C_B\, m^5 f \log b) \prod_{v \in \Phi} \mathrm{bm}_v,$$

the sum over all B-points γ for (T,T') (a sum of terms in $[0,\infty]$, with $0 \cdot \infty = 0$).

Proof. Outline. For each B-point the leading minors of its turned local images are those of the point rescaled by powers of rd_E, so that at each place the weights are bounded by a power of one leading minor; a one-place Bessel bound then estimates each local factor by bm_v up to a negative power of $\iota_v G_{T,T'}(\gamma)$, these powers are absorbed by $|N_{E/\mathbb{Q}}(G_{T,T'}(\gamma))| \ge 1$, and the sum over the B-points is controlled by a weighted count of B-points. □

Proposition A.87. *All symbols introduced here are local to this statement and to the results that cite it. Let $A \ge 0$ be an integer and C_h real, and let C_B, $C_{g,0}$ be the numbers of Theorem A.86 for $(A_w, C_h) = (A, C_h)$. Let non-generic row data be given for a pair expansion or, more generally, data as in Definition 15.2 with a countable set S of terms in place of $\mathcal{M}_{\mathrm{n}}$ (as in (i) of Theorem 15.5), and let $C_g \ge C_{g,0}$, $(r, k, E, \Phi, g_1, g_2, \kappa, T, T', b)$ and point-sum data at the places $v \in \Phi$ satisfy all hypotheses of Theorem A.86, with r and k those of these data. Let $C_w, C_j \ge 0$ and suppose:*

(Xb6) Point tie *there is an injective map $\gamma \mapsto \hat\gamma$ from P_{oc} to the set of B-points for (T,T');*

(Xb7) Weight tie $w(\gamma) P_{\mathrm{n}}(\gamma) n_{\mathrm{g}}(\gamma) \le C_w\, \mathrm{wp}(\hat\gamma)\, |\mathrm{pw}_{T,T'}(\hat\gamma)|\, \mathrm{nw}_{T,T'}(\hat\gamma)^{A(r+1)^2}$ *for $\gamma \in P_{\mathrm{oc}}$;*

(Xb8) Jacobian tie $J(\gamma) \le C_j \prod_{v\in\Phi} \mathrm{os}_v(\tilde{\hat{\gamma}}_v)$ *for* $\gamma \in P_{\mathrm{oc}}$ *(with* $0\cdot\infty = 0$*);*

(Xb9) Normalisation $C_w C_j \exp(C_B m^5 f \log b) \prod_{v\in\Phi} \mathrm{bm}_v \le T_{\mathrm{pt}}$.

Then these data have the point sum (Ng6).

Proof. Let $F \subseteq P_{\mathrm{oc}}$ be finite. For $\gamma \in F$ the four factors $w(\gamma)$, $P_{\mathrm{n}}(\gamma)$, $n_{\mathrm{g}}(\gamma)$, $J(\gamma)$ are nonnegative, so by (Xb7) and (Xb8)

$$w(\gamma)P_{\mathrm{n}}(\gamma)n_{\mathrm{g}}(\gamma)J(\gamma) \le C_w C_j\, a(\hat{\gamma}),$$
$$a(\beta) = \mathrm{wp}(\beta)\,|\mathrm{pw}_{T,T'}(\beta)|\,\mathrm{nw}_{T,T'}(\beta)^{A(r+1)^2} \prod_{v\in\Phi} \mathrm{os}_v(\tilde{\beta}_v) \in [0,\infty]$$

(if $C_j = 0$ then $J(\gamma) = 0$ by (Xb8), and the bound holds with the convention $0\cdot\infty = 0$). By (Xb6) the B-points $\hat{\gamma}$, $\gamma \in F$, are distinct, and the terms $a(\beta)$ are nonnegative, so $\sum_{\gamma\in F} a(\hat{\gamma})$ is at most the sum of $a(\beta)$ over all B-points β for (T,T'). By Theorem A.86 (with $A_w = A$) that sum is at most $\exp(C_B m^5 f \log b)\prod_v \mathrm{bm}_v$. With (Xb9),

$$\sum_{\gamma\in F} w(\gamma)P_{\mathrm{n}}(\gamma)n_{\mathrm{g}}(\gamma)J(\gamma) \le C_w C_j \exp(C_B m^5 f \log b) \prod_{v\in\Phi} \mathrm{bm}_v \le T_{\mathrm{pt}}.$$

(In particular the left side is finite.) As F was arbitrary, this is (Ng6). □

A.8 Covolumes: deep covers, short generators and heights

Alpöge lists the computation of covolumes among the inputs of the proof [4]; in our reading this item covers the covolume bound of the body, Theorem 19.26, with the model slopes of Theorem 29.14 that feed it. This section proves no new covolume inequality. The construction supplies the data on which the inequality of the body applies: chain data, deep-cover data (Definition 19.8), component and height data (Definition 19.10) and model-slope data, at realised points of the constructed families (Proposition A.92). Proposition A.90 applies the covolume bound to the constructed families, and Lemma A.91 turns it into the covolume row. The data rest on two mechanisms. Deep lattices are eigenlattices of small groups of level automorphisms, and therefore have short generators. Heights of components are bounded below by the logarithm of the size of short generators and above by Bergman volumes, and an effective arithmetic Hilbert–Samuel inequality joins the two bounds.

A.8.1 The covolume row

Definition A.88 defines the covolume class of constructed families, and Definition A.89 the Petersson lattices of a uniformised moduli datum with their component invariants: rank, degree, covering degree and height. Proposition A.90 applies Theorem 19.26, together with Theorem 19.24, Theorem 19.20 and Theorem 19.18 of the body, to these families; the data it needs are Proposition A.92, and Lemma A.91 links the resulting bound to a row parameter.

Definition A.88. (Yv1) Largeness at a design parameter. Let $\mathcal{F}$ be a parametrised host family with a structural profile, $\mathfrak{K} = (k_{\mathrm{pre}}, k_{\mathrm{mid}}, k_{\mathrm{last}})$ a structural threshold and u a real number. Then $\mathcal{F}$ is **large for** $(\mathfrak{K}, u)$ if

$$\begin{aligned} k_{\mathrm{pre}} &\le c \quad \text{for every entry } c \text{ of } \mathbf{c}_{\mathrm{pre}}(\mathcal{F}),\\ k_{\mathrm{mid}}\big(\mathbf{c}_{\mathrm{out}}(\mathcal{F})\big) &\le c \quad \text{for every entry } c \text{ of } \mathbf{c}_{\mathrm{mid}}(\mathcal{F};u),\\ k_{\mathrm{last}}\big(\mathbf{c}_{\mathrm{out}}(\mathcal{F}) \parallel \mathbf{c}_{\mathrm{mid}}(\mathcal{F};u)\big) &\le c \quad \text{for every entry } c \text{ of } \mathbf{c}_{\mathrm{last}}(\mathcal{F}). \end{aligned}$$

Thus $\mathcal{F}$ is large for $\mathfrak{K}$ in the sense of Definition A.3 exactly when it is large for $(\mathfrak{K}, C_H(\mathcal{F}))$. The real u is called a **design parameter**; it need not be the chosen one.

(Yv2) The covolume clause. Let b be a bundle. The base level of $\mathrm{Cx}_3(b)$ carries a positive integer $D(b)$, the dimension of the realisation, and for every lift j a frame $X_j(b)$ (in the convention of Definition 19.5) and a lattice $M_j(b)$ of the size required by that frame. The indefinite scale of the face $\mathrm{fc}(z_s)$ carries real numbers $\ell^{\mathrm{m}}_j(b)$ for every lift j, the **covolume main terms**, and a real number $\ell^{\mathrm{u}}(b)$, the **covolume unit**; for reals C, C_H the numerical tuple $\mathrm{num}(b; C, C_H)$ has the **covolume entries** $\ell_{3,j} = \ell^{\mathrm{m}}_j(b) + C\,\ell^{\mathrm{u}}(b)$, which do not involve C_H (the design parameter enters $\mathrm{num}(b; C, C_H)$ only through its height entry). The **clause L_3 of $\mathrm{Cx}_3(b)$ at base level** holds at $\mathrm{num}(b; C, C_H)$ if

$$\frac{\mathrm{lcv}\big(M_j(b), X_j(b)\big)}{D(b)} \le \ell^{\mathrm{m}}_j(b) + C\,\ell^{\mathrm{u}}(b) \qquad \text{for every lift } j.$$

In particular whether it holds does not depend on C_H.

(Yv3) The covolume class. $\mathcal{B}_{\mathrm{cv}}$ is the class of the parametrised host families $\mathcal{F}$ of the construction of the lattice side, with their member data $D_{x,i}$, member objects $\mathfrak{o}_{x,i}$ and size constant $C_{\mathrm{sz}}(\mathcal{F})$, with slope records, structural profiles and covolume records, that satisfy the seven duties and the conditions of the standard restriction, in which, at every member (x, i):

(i) $\mathrm{Pt}_{x,i}$ is the set of points of the construction at $(D_{x,i}, \mathfrak{o}_{x,i}, C_{\mathrm{sz}}(\mathcal{F}))$ that satisfy its **point conditions** and its **arithmetic ties** (the size condition on the coefficient field, and the Green series tie of the member datum $D_{x,i}$); for every bundle b over $(D_{x,i}, \mathfrak{o}_{x,i}, C_{\mathrm{sz}}(\mathcal{F}))$ and every $t \in \mathrm{Cmp}_2(b)$ the point package $\mathrm{pt}(b, t)$ is a point of the construction at this member;

For a point $\mathfrak{z} \in \mathrm{Pt}_{x,i}$, write $R = R_{\mathfrak{z}}$ for its carried real data, $T = T_{\mathfrak{z}}$ for its geometric core and $\mathcal{H} = \mathcal{H}_{\mathfrak{z}}$ for its geometric half, in the sense of the conditions (Xl2a)–(Xl2c) on the levels of a state. The following are clauses of these point conditions on the same carried data.

The **size condition on the coefficient field** in the arithmetic ties above means the following. Put $D = D_{x,i}$ and let $F_c \subseteq \mathbb{C}$ be the coefficient field of the carried geometric core T. Write $\mathcal{O}_D$ for the coefficient ring of the member datum and $\varphi_D : \mathcal{O}_D \hookrightarrow \mathbb{C}$ for its fixed complex embedding. The carried coefficient-ring identification is an isomorphism

$$e_D : \mathcal{O}_{F_c} \xrightarrow{\sim} \mathcal{O}_D, \qquad \varphi_D(e_D(u)) = u \quad (u \in \mathcal{O}_{F_c}),$$

with u read through the given inclusion $F_c \subseteq \mathbb{C}$. Consequently F_c is the rationality field:

$$F_c = \mathbb{Q}(\varphi_D(\mathcal{O}_D)).$$

With f, r and $\ell = x^{\varepsilon}$ the ranks and logarithmic scale of the member size datum, put

$$P_0 = 19 f r^2 \ell + 198 f^2 \ell, \qquad d_c = [F_c : \mathbb{Q}].$$

The two field-size conditions are

$$\log d_c \le P_0, \qquad \frac{\log |\operatorname{disc}(F_c)|}{d_c} \le P_0.$$

They concern the field F_c itself, not the larger realisation coefficient field L. These conditions are imposed on the points and assert no existence of points or fields.

The real data R have the level choice of (Wr6) of the Weil laws of the real data, with the lattices, level elements, frame primes and characters already carried by the point. The real data R satisfy their characterisation laws: the base-tuple level and character laws, group reductions, record, family, model, chain and theta-lift laws specified in the hypotheses of the analytic characterisation of geometric halves; the core characters χ_W are those of R, and the core's two imaginary-quadratic primes are those of the character frame of R. This clause does not impose the separate rank inequality in that proposition. The primitive rows satisfy (U4) and the full uniformity law (U10) of stage-3 states, for their carried coordinate maps, action and character. In particular, for every $a < m$ and $\sigma : L \to \mathbb{C}$, $j_{a,\sigma}$ sends finite L-linearly independent families to complex-linearly independent families, its image spans $H_{W_a,\sigma}$, and its intertwining with the level action is, through the carried Hecke model,

$$j_{a,\sigma}(\rho_{\mathrm{FW}}(g)x) = \sigma(\chi_a^{\mathrm{FW}}(g))\, g \cdot j_{a,\sigma}(x).$$

The geometric half $\mathcal{H}$ also satisfies the condition (GeoAct) of (Xl2e) on the levels of a state, with its F_c-linear action on the cusp space, its top character and its notion of integral forms through integral framed points. These are conditions on the points, not conclusions about projectors or slopes.

The primitive-row clauses of these point conditions also require, for every lift $1 \le j \le m$, $d_{\mathfrak{z},j} = D_{\mathfrak{z}} > 0$ and that the coordinate pullback $M_{\mathfrak{z},j}$ of (Rk11) of the row carrier be a finitely generated $\mathcal{O}_L$-module with L-span $L^{d_{\mathfrak{z},j}}$. These clauses are part of the point conditions defining $\mathrm{Pt}_{x,i}$; they assert neither the existence of such a point nor that every point of the construction satisfies them. Finally $\mathcal{H}$ satisfies the condition (JEvBar) of (Xl2e) on the levels of a state: put $\sigma_c = \sigma|_{F_c}$ and let $\bar\sigma_c$ be its complex conjugate. For every $a < m$ and σ with σ_c not good, $\bar\sigma_c$ is good and $V_{a+1,\bar\sigma_c} = W_a$; there are a vector family $v' = (v'_\rho)_\rho$ of $M_{W_a,\sigma}$ with $\|v'_\rho\| = 1$ at every archimedean place ρ of the maximal real host field, and a σ_c-semilinear map $J : \mathrm{Cusp}_{a+1}(\kappa_a) \to H_{W_a,\sigma}$, such that

$$\mathrm{ev}_{a,\sigma,v'}(J(f)) = \overline{\phi_{a+1,\bar\sigma_c,\kappa_a}(f)} \qquad (f \in \mathrm{Cusp}_{a+1}(\kappa_a)).$$

Here the maps, models, weights and evaluations are those of R, T and $\mathcal{H}$. These clauses assert no existence of a point and do not replace its real data or its geometric half.

(ii) the covolume record of a point $\mathfrak{z}$ consists of its realisation lattices $M_{\mathfrak{z},j}$ at base level for all lifts $1 \le j \le m$ with their archimedean frames, the common dimension $D_{\mathfrak{z}}$ of the realisation (not the rank of $M_{\mathfrak{z},j}$), and its realisation places and weights; and if $\mathfrak{z} = \mathrm{pt}(b,t)$, then for all reals C, C_H the clause L_3 of $\mathrm{Cx}_3(b)$ at base level holds at $\mathrm{num}(b; C, C_H)$ if and only if $\mathrm{lcv}(M_{\mathfrak{z},j}, X_{\mathfrak{z},j})/D_{\mathfrak{z}} \le \mathrm{mt}_j(\mathfrak{z}) + C\,\mathrm{ue}(x,i)$ for every $j \in \{1,\dots,m\}$.

Definition A.89. A **uniformised moduli datum** $\mathfrak{m} = (\mathfrak{u}, (U_\sigma, V_\sigma, \mathrm{dict}_\sigma)_\sigma)$ consists of a unitary Shimura datum $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$, $r = p+q$, and, for every embedding $\sigma\colon F \to \mathbb{C}$, a uniformisation U_σ of $\mathrm{Sh} \otimes_\sigma \mathbb{C}$ whose components k carry frame data (Xl2d), a hermitian space V_σ over E of rank r, and a **dictionary** $\mathrm{dict}_\sigma(g) = (k(g), \mathrm{fr}(g))$ assigning to $g \in \mathrm{U}(V_\sigma)(\mathbb{A})$ a component of U_σ and a frame $\mathrm{fr}(g) \in \mathcal{D}_{k(g)}$. The geometric core at a stage j ($\mathcal{X}_j$ over F_c with $U_{j,\sigma}$, $V_{j,\sigma}$, $\mathrm{dict}_{j,\sigma}$ of (Xl2a)) and a deep cover $\mathcal{Y} = (F', \delta, (U_\sigma, \mathrm{dict}_\sigma)_\sigma, K_f)$ at V (Sh_2 over F',

$V_\sigma = V$; written $\mathfrak{m}_{\mathcal{Y}}$) are uniformised moduli data. For $\sigma\colon F \to \mathbb{C}$ put $\sigma^{\mathrm{g}} = \sigma$ if σ is good and $\sigma^{\mathrm{g}} = \bar\sigma$ otherwise. Fix $\mathfrak{m}$ and a real number $\xi > 0$, the **Lie parameter**.

(Vy1) Cusp forms of integral weight. An **integral weight** is a pair $G = (a, n)$ of a function $a\colon \mathrm{Ix} \to \mathbb{Z}$ and an even integer n; put $|a| = \sum_{(\tau,i)\in \mathrm{Ix}} a(\tau, i)$. Algebraic forms of weight G, the F-space $\mathrm{Alg}_F(G)$, valuations and integral forms are defined by (Zj2)–(Zj4) read with these integer exponents. For a component k of U_σ the **frame weight** $\lambda_k(G)$ is given by $\lambda_k(G)(\tau, i) = a(\tau, i - p_0) + n/2$ for $i > p_0$ and $\lambda_k(G)(\tau, i) = -a(\bar\tau, p_0 + 1 - i) - n/2$ for $i \le p_0$ (embeddings read through σ, as for $\lambda_k(a, n)$ in (Xl2d)). The **cusp space** $\mathrm{Cusp}_{\mathfrak{m}}(G) \subseteq \mathrm{Alg}_F(G)$ is the F-subspace of the f whose readings $(f_\sigma(k, \cdot))_k$ satisfy the cusp condition (Hh7) for the frame weights $(\lambda_k(G))_k$ at every good σ; for the geometric core at stage j and a weight (a, n) of (Zj2) it is $\mathrm{Cusp}_j(a, n)$ of (Xl2a). Its integral forms are $\Lambda_{\mathfrak{m}}(G) = \mathrm{Cusp}_{\mathfrak{m}}(G)^{\mathrm{int}}$.

(Vy2) Petersson norms. For $\sigma\colon F \to \mathbb{C}$ let $(\nu_\sigma, \Omega_\sigma)$ be a Haar pair of $\mathrm{U}(V_{\sigma^{\mathrm{g}}})$ with $0 < \nu_\sigma(\Omega_\sigma) < \infty$, and let $\nu^1_\sigma = \nu_\sigma(\Omega_\sigma)^{-1}\nu_\sigma|_{\Omega_\sigma}$. The **Petersson norm** at σ of $f \in \mathrm{Alg}_F(G)$ is

$$\|f\|_{G,\sigma} = (2\pi\xi)^{-|a|/2}\Big(\int_{\Omega_\sigma} \big\|f_{\sigma^{\mathrm{g}}}(k(g), \cdot)\big\|_{\lambda_{k(g)}(G)}\big(\mathrm{fr}(g)\big)^2\, d\nu^1_\sigma(g)\Big)^{1/2},$$

with the dictionary $\mathrm{dict}_{\sigma^{\mathrm{g}}}(g) = (k(g), \mathrm{fr}(g))$, and $\|f\|_{G,\sigma} = 0$ if the integral is infinite.

(Vy3) Petersson lattices. For an F-subspace $C \subseteq \mathrm{Cusp}_{\mathfrak{m}}(G)$ and a family $N = (N_\sigma)_\sigma$ of functions $C \to [0, \infty)$, the pair (C^{int}, N), $C^{\mathrm{int}} = C \cap \Lambda_{\mathfrak{m}}(G)$, is **good** if C is finite-dimensional, C with the norms N_v (one for the fixed embedding v of each infinite place of F) is a hermitian space over F with $N_{\bar\sigma}(f) = N_\sigma(f)$, and C^{int} is a finitely generated $\mathcal{O}_F$-module spanning C; it is then a hermitian lattice. The **Petersson lattice** of weight G is $(\Lambda_{\mathfrak{m}}(G), (\|\cdot\|_{G,\sigma})_\sigma)$; $\mathfrak{m}$ is **good at** G if it is good; $\mathrm{rk}_{\mathfrak{m}}(G) = \mathrm{rk}\,\Lambda_{\mathfrak{m}}(G)$, and $\widehat{\mu}_{\mathfrak{m}}(G) = \widehat{\mu}(\Lambda_{\mathfrak{m}}(G))$ when $\mathfrak{m}$ is good at G and $\Lambda_{\mathfrak{m}}(G) \neq 0$.

(Vy4) Components; rank and degree on a component. $\mathrm{Comp}(\mathfrak{m})$ is the set of irreducible components of $\mathrm{Sh} \otimes_{K_{\mathsf{L}}} F$. A framed S-point (x, u, s) **lies on** $Z \in \mathrm{Comp}(\mathfrak{m})$ if every point of $\mathrm{Spec}\, S$ is mapped into Z by x, and **off** Z if none is. A form f is **supported on** Z if $f(y) = 0$ for every framed point y off Z; $\mathrm{Cusp}_{\mathfrak{m},Z}(G)$ is the space of cusp forms supported on Z and $\Lambda_{\mathfrak{m},Z}(G)$ its set of integral forms. A frame (k, x) lies on Z if $y_{k,x}$ does. The **Petersson norm on** Z, $\|f\|_{G,\sigma,Z}$, is defined as in (Vy2) with ν^1_σ replaced by $\nu_\sigma(\Omega_{\sigma,Z})^{-1}\nu_\sigma|_{\Omega_{\sigma,Z}}$, $\Omega_{\sigma,Z} = \{g \in \Omega_\sigma : \mathrm{dict}_{\sigma^{\mathrm{g}}}(g) \text{ lies on } Z\}$ (and $\|f\|_{G,\sigma,Z} = 0$ if $\nu_\sigma(\Omega_{\sigma,Z}) = 0$). $\mathfrak{m}$ is **good at** G **on** Z if $(\Lambda_{\mathfrak{m},Z}(G), (\|\cdot\|_{G,\sigma,Z})_\sigma)$ is good (Vy3). The **rank** and the **degree** on Z are $\mathrm{rc}_Z(G) = \mathrm{rk}\,\Lambda_{\mathfrak{m},Z}(G)$ and $\mathrm{gc}_Z(G) = \mathrm{rk}\,\Lambda_{\mathfrak{m},Z}(G) \cdot \widehat{\mu}(\Lambda_{\mathfrak{m},Z}(G))$ ($\mathrm{gc}_Z(G) = 0$ if the rank is 0 or $\mathfrak{m}$ is not good at G on Z).

(Vy5) Covering degree and height of a component. The **flag dimension** is $D_{\mathrm{fl}} = |\Phi_{\mathsf{H}}|\, r(r-1)/2$. For $Z \in \mathrm{Comp}(\mathfrak{m})$ let $\kappa(Z)$ be its function field and Ω_Z the field of fractions of $\kappa(Z)[X_{\tau,i,j} : \tau\colon E \to \mathbb{C},\ 1 \le i < j \le m_\tau]$. A **generic framed point** of Z is a framed Ω_Z-point $y_Z = (x_Z, u, s)$ with x_Z the generic point of Z for which there is a framed $\kappa(Z)$-point (x^0, u^0, s^0) with x^0 the generic point of Z and $u_{\tau,i} = u^0_{\tau,i} + \sum_{j>i} X_{\tau,i,j}u^0_{\tau,j}$ for all $(\tau, i) \in \mathrm{Ix}$; one generic framed point y_Z is fixed for every Z. For a basis $b = (b_1, \dots, b_N)$ of $C = \mathrm{Cusp}_{\mathfrak{m},Z}(G)$ the **image ideal** $I_Z(G; b) \subseteq F[x_1, \dots, x_N]$ is the ideal generated by the homogeneous P with $P(b_1(y_Z), \dots, b_N(y_Z)) = 0$. The **covering degree** $\mathrm{dg}_Z(G)$ is the integer δ such that $I_Z(G; b)$ has a Chow form over F in dimension D_{fl} and degree δ for some basis b, and 0 if there is none. For a polynomial f in blocks $u_i = (u_{i,1}, \dots, u_{i,N})$ and an $N \times n'$ matrix M let $f \circ M$ be obtained by substituting $\sum_l M_{jl}u'_{i,l}$ for $u_{i,j}$; $|g|_v$ is the maximum

of the normalised absolute values at the finite place v of the coefficients of g; and for g in $D_{\mathrm{fl}}+1$ blocks of N complex variables $m(g)=\int\log|g(z_0,\dots,z_{D_{\mathrm{fl}}})|\,d\mu(z_0)\cdots d\mu(z_{D_{\mathrm{fl}}})$, μ the invariant probability measure on the unit sphere of $\mathbb{C}^N$. The **height** $\mathrm{ht}_Z(G)$ of Z is a real number h (one fixed choice if several exist, and 0 if none exists) for which there are: a basis b of C; an $N\times n'$ matrix w over F whose columns span F^N and for which the elements $\sum_j w_{jl}b_j$ $(1\le l\le n')$ generate $\Lambda_{\mathfrak{m},Z}(G)$ over $\mathcal{O}_F$; matrices $A_\sigma\in\mathrm{GL}_N(\mathbb{C})$ with $|A_\sigma^{-1}(\sigma(c_j))_j|=\|\sum_j c_jb_j\|_{G,\sigma,Z}$ for all $c\in F^N$ and $|A_{\bar\sigma}^{-1}\bar z|=|A_\sigma^{-1}z|$ for $z\in\mathbb{C}^N$; and a Chow form f over F of $I_Z(G;b)$ in dimension $D=D_{\mathrm{fl}}$ and degree δ, such that

$$(D+1)\,\delta\,h=\frac{1}{[F:\mathbb{Q}]}\Big(\sum_{v\text{ finite}}\log|f\circ w|_v+\sum_{\sigma\colon F\to\mathbb{C}}\Big(m(f^\sigma\circ A_\sigma)+\frac{(D+1)\,\delta}{2}\sum_{l=1}^{N-1}\frac{1}{l}\Big)\Big),$$

f^σ obtained by applying σ to the coefficients.

(Vy6) Lattice of a deep cover over core forms. Let $\mathcal{Y}$ be a deep cover at V of the unitary Shimura datum of a moduli scheme $\mathcal{X}$ over $F_c\subseteq F'$. A form $g\in\mathrm{Alg}_{F'}(G)$ on Sh_2 is a **pull-back** of a form f on $\mathcal{X}$ if $g(y_2)=f(y)$ for every algebraically closed field $k\supseteq F'$ and all framed k-points y_2 of Sh_2 and y of $\mathcal{X}$ such that y_2 lies over y. For a set B of forms on $\mathcal{X}$, the **lattice of $\mathcal{Y}$ over** B at G is

$$\Lambda_{\mathcal{Y}}(G;B)=\Lambda_{\mathfrak{m}_{\mathcal{Y}}}(G)\cap\mathrm{span}_{F'}\{g:\ g\text{ is a pull-back of some }f\in B\},$$

with the Petersson norms of $\mathfrak{m}_{\mathcal{Y}}$.

Proposition A.90. *There is $\varepsilon_1>0$ and, for every $\varepsilon\in(0,\varepsilon_1)$, a structural threshold $\mathfrak{K}(\varepsilon)$, such that for every family $\mathcal{F}\in\mathcal{B}_{\mathrm{cv}}$ with parameter ε and every real number C_H for which $\mathcal{F}$ is large for $(\mathfrak{K}(\varepsilon),C_H)$ there are real numbers $C_0\ge0$ and x_0 such that for every real $x\ge x_0$ with $x>x_1$, every $i\in I(x)$, every $\mathfrak{z}\in\mathrm{Pt}_{x,i}$, every real $C\ge C_0$ and every $j\in\{1,\dots,m\}$*

$$\frac{\mathrm{lcv}(M_{\mathfrak{z},j},X_{\mathfrak{z},j})}{D_{\mathfrak{z}}}\le\mathrm{mt}_j(\mathfrak{z})+C\,\mathrm{ue}(x,i).$$

In particular (take $C_H=C_H(\mathcal{F})$), for a restriction $\mathfrak{r}_{\mathrm{str}}$ with $\varepsilon_{\mathfrak{r}}=\varepsilon_1$ and $\mathfrak{K}_{\mathfrak{r}}(\varepsilon)=\mathfrak{K}(\varepsilon)$ for $\varepsilon<\min(\varepsilon_1,1/52)$, every family in $\mathcal{B}_{\mathrm{cv}}[\mathfrak{r}]$ has the covolume bound.

Proof. Let ε_1 and $\mathfrak{K}(\varepsilon)$, $0<\varepsilon<\varepsilon_1$, be those of Proposition A.92. Fix $\varepsilon\in(0,\varepsilon_1)$, a family $\mathcal{F}\in\mathcal{B}_{\mathrm{cv}}$ of parameter ε and a real C_H for which $\mathcal{F}$ is large for $(\mathfrak{K}(\varepsilon),C_H)$. By Proposition A.92 (with $u=C_H$), $\mathcal{F}$ carries chain data, deep-cover data, component data and model-slope data with (Su1)–(Su5); let $C_6,C_{\mathrm{top}},C_{\mathrm{F}},C',C_4\ge0$ be the constants named there. The design parameter C_H is used only in this step.

Step 1: (Ms1)–(Ms4). $\mathcal{F}$ carries chain, deep-cover, component and model-slope data, and by (Su5) the data (Aj1)–(Aj3) are given and hold at every real $x>x_1$, every $i\in I(x)$, every $\mathfrak{z}\in\mathrm{Pt}_{x,i}$ and every lift $j<m$; these are the hypotheses of Theorem 29.14. That theorem gives a re-choice of the model slopes $\mathrm{ns}(Z)$, all other entries of the model-slope data kept, after which $\mathcal{F}$ has (Ms1), and (Ms3) for its constant C_{ms}, at all large points with $j<m$. The re-choice changes neither the chain, deep-cover and component data nor the entries $E,M,B,L,\mathrm{wn},\mathrm{wsp}$; (Ms4) reads only T^{d}, B, rd_E, L, wn, wsp and $\mathrm{ue}(x,i)$, so by (Su4) it still holds with C_4. (Ms2) holds at every point with $j<m$: by (Aj1) the datum level is $N=M$ and at most $4|\Phi|n^2$ primes divide $2\,\mathrm{disc}(E)\,M$, where $\Phi=\Phi_{\mathsf{H}}$ and $n=p+q$; by the definition of the threshold in Definition 21.3 and $g>0$, $4|\Phi|n^2\le4|\Phi|n^2(1+\|u\|_\infty/g)=b_0(u,g)$, and $b_0(u,g)<\lfloor B\rfloor+1$ by (Aj2); as the

number of these primes and $\lfloor B \rfloor$ are integers, the number is at most $\lfloor B \rfloor$. Neither E, M, B nor (Aj1), (Aj2) is changed by the re-choice of the model slopes. Put $C_{\mathrm{M}} = \max(C_{\mathrm{ms}}, C_4) \geq 0$. Both (Ms3) and (Ms4) remain true when their constant is increased to C_{M}:

Taking as threshold the maximum of e and of the thresholds of (Ms1), (Ms3) and of (Su4), $\mathcal{F}$ has (Ms1)–(Ms4) for the constant C_{M}.

Step 2: (Ht1)–(Ht5). At every point with $j < m$ put $e_* = C_{\mathrm{M}}(\mathrm{lz} \cdot \mathrm{wn} + \mathrm{wsp}) + 2\, c^{\mathrm{HT}}_{E,M}(B)\, \mathrm{wn}$. Then $e_* \geq 0$: $C_{\mathrm{M}} \geq 0$, $\mathrm{lz} \cdot \mathrm{wn} + \mathrm{wsp} \geq 0$ (Step 1), $\mathrm{wn} \geq 0$, and $c^{\mathrm{HT}}_{E,M}(B) \geq 0$ as a finite sum of terms $\big(\frac{1}{p-1} + \frac{40\, v_p(\mathrm{disc}(E))}{[E:\mathbb{Q}]} + \delta_{p,2}\big) \log p \geq 0$. So the component data may be taken with $e^{\mathrm{ms}} = e_*$; the entry e^{ms} occurs in none of (Lc1)–(Lc6), (Dp0)–(Dp5), (Ht1), (Ht2), (Ht4), (Ht5), (Ms1)–(Ms4), which therefore still hold. By Theorem 19.18, applied with (Ms1)–(Ms4) for C_{M} and with (Ht1), (Ht2), (Ht4), (Ht5) for C' from (Su3), $\mathcal{F}$ has (Ht1)–(Ht5) for the constant $C_{\mathrm{ht}} = \max\big(C', C_{\mathrm{M}}(C_{\mathrm{M}} + 120)\big) \geq 0$.

Step 3: (Dp6). $\mathcal{F}$ carries chain, deep-cover and component data with (Ht1)–(Ht5) for $C_{\mathrm{ht}} \geq 0$, so by Theorem 19.20 it has (Dp6) with the constant $C_{\mathrm{U}} = 8C_{\mathrm{ht}} + 4$.

Step 4: (Lc7). By (Su2) and Step 3, $\mathcal{F}$ carries chain and deep-cover data with (Dp0)–(Dp6), (Dp0) for C_{top}, (Dp5) for C_{F} and (Dp6) for C_{U}. By Theorem 19.24 it has (Lc7) with the constant $C_7 = \max\big(C_{\mathrm{top}}, 2(C_{\mathrm{F}} + C_{\mathrm{U}}) + 9\big)$.

Step 5: the bound. By (Su1) and Step 4, $\mathcal{F}$ carries chain data with (Lc1)–(Lc7), (Lc6) for C_6 and (Lc7) for C_7. By Theorem 19.26 it has the covolume bound with $C_0 = C_6 + C_7 + 1 \geq 0$: there is x_0 such that the displayed inequality holds for every real $x \geq x_0$ with $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$, every $C \geq C_0$ and every lift j. This is the assertion for $\mathcal{F}$ and C_H.

The restricted class. Let $\mathfrak{r}_{\mathrm{str}}$ be as stated and $\mathcal{F} \in \mathcal{B}_{\mathrm{cv}}[\mathfrak{r}]$. By Definition A.3, its parameter satisfies $0 < \varepsilon < \min(\varepsilon_1, 1/52)$ and $\mathcal{F}$ is large for $\mathfrak{K}_{\mathfrak{r}}(\varepsilon) = \mathfrak{K}(\varepsilon)$, that is, by (Yv1) of Definition A.88, large for $(\mathfrak{K}(\varepsilon), C_H(\mathcal{F}))$. The assertion with $C_H = C_H(\mathcal{F})$ gives $C_0 \geq 0$ and x_0 as in the definition of the covolume bound in Definition 19.5. □

Lemma A.91. *Let $p = (\mathcal{N}, S, \Gamma)$ be a parameter such that every norm choice of $\mathcal{N}$ extends with its law. Assume the ray-class-field statement and the four published real-group inputs (discrete-series cover, Harish–Chandra modules, rapid decay of cusp forms, Casselman–Wallach globalisation), and that the services (S1) h_{PA} (primitivity), (S3) h_{DivG} (divisor rows), (S8)* good *(goodness) and (S9)* geo *(geometric weak ties) hold linked at p. Then the covolume row* $\mathrm{EvWC}(\top, L_3)$ *holds linked at p. Explicitly: there are a large-base slot R over* $\mathrm{Built}_{\mathrm{E}}(\mathfrak{S}(p))$ *and a bundle predicate E with* $\mathrm{RowE}(\mathrm{Good}_{\mathrm{E}}(\mathfrak{S}(p)) \wedge R, \mathrm{Lk}_p, E)$ *such that for every class $\mathfrak{A}$ and every E-selection σ over $\mathfrak{A}$ there is $\varepsilon_1 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_1$ there is a threshold triple $\mathfrak{K}$ with the following property: for every $\mathcal{F} \in \mathfrak{A}^{\mathrm{st}}$ of parameter ε and every real C_H for which $\mathcal{F}$ is large for $(\mathfrak{K}, C_H)$, there is $C_0 \geq 0$ such that for every $C \geq C_0$ there is x_0 such that the clause L_3 of* $\mathrm{Cx}_3(\sigma(\mathcal{F}, x, i))$ *at base level holds at* $\mathrm{num}(\sigma(\mathcal{F}, x, i); C, C_H)$ *for all members (x, i) of $\mathcal{F}$ with $x \geq x_0$.*

Proof. Outline. The point properties of the selection, together with the inclusion of the families in $\mathcal{B}_{\mathrm{cv}}$, show that the selected points are realised points at all large members; the covolume bound Proposition A.90, applied with a common refinement of the two structural thresholds involved, then gives the covolume inequality at these points, which by (ii) of (Yv3) of Definition A.88 is the clause L_3 at base level. □

Proposition A.92. *There is $\varepsilon_1 > 0$ and, for every $\varepsilon \in (0, \varepsilon_1)$, a structural threshold $\mathfrak{K}(\varepsilon)$, such that every family $\mathcal{F} \in \mathcal{B}_{\mathrm{cv}}$ of parameter $\varepsilon \in (0, \varepsilon_1)$ that is large for $(\mathfrak{K}(\varepsilon), u)$ for some real u carries chain data, deep-cover data, component data and model-slope data with the following properties.*

(Su1) *(Lc1)–(Lc6) hold, (Lc6) for some constant* $C_6 \geq 0$.

(Su2) *(Dp0)–(Dp5) hold, (Dp0) for some constant* $C_{\mathrm{top}} \geq 0$ *and (Dp5) for some constant* $C_{\mathrm{F}} \geq 0$.

(Su3) *There is a real* $C' \geq 0$ *such that the inequalities (Ht1), (Ht2), (Ht4), (Ht5) hold with the constant* C' *for all large points with* $j < m$.

(Su4) *There is a real* $C_4 \geq 0$ *such that (Ms4) holds with the constant* C_4 *for all large points with* $j < m$.

(Su5) *For every real* $x > x_1$, *every* $i \in I(x)$, *every* $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ *and every lift* $j < m$, *the data (Aj1), (Aj2), (Aj3) are given and hold for these component data and model-slope data.*

Proof. Outline. One takes the intersection of the structural thresholds of four supply results for the covolume data (for the main terms, the bad parts, the lattice slopes and the deep covers, the last being Proposition A.93); the realised covolume objects of a point then give chain data, which satisfy (Lc1)–(Lc6) at large points, and the deep-cover, component and model-slope data supplied by Proposition A.93 give (Su2)–(Su5). □

A.8.2 Deep covers and short generators

The main terms, the top lift and the deep-cover, component and model-slope data over the realised core lattices are Proposition A.93. On a covering cover, the deep lattice is the eigenlattice of a character of a small group of level automorphisms that keep integral forms and Petersson norms (Proposition A.94). A section whose translates have a non-vanishing group determinant gives descent maps into the eigenlattice (Lemma A.95), and with these the deep lattice is spanned by vectors of norm at most $\exp(P\varkappa)$ (Proposition A.96).

Proposition A.93. *There is* $\varepsilon_1 > 0$ *and, for every* $\varepsilon \in (0, \varepsilon_1)$, *a structural threshold* $\mathfrak{K}(\varepsilon)$, *such that every family* $\mathcal{F} \in \mathcal{B}_{\mathrm{cv}}$ *of parameter* ε *that is large for* $(\mathfrak{K}(\varepsilon), u)$ *for some real* u *has the following properties.*

(Hq1) Size of the main terms *There is a real* A *such that for all large points* $|\mathrm{mt}_j(\mathfrak{z})| \leq x^A\, b(x)$.

(Hq2) Top lift *There is a real* $C_{\mathrm{top}} \geq 0$ *such that for all large points at which* $L^{\mathrm{c}}_{\mathfrak{z},m}$ *is a non-zero hermitian lattice,* $2\,\widehat{\mu}(L^{\mathrm{c}}_{\mathfrak{z},m}) \geq -\mathrm{mt}_m(\mathfrak{z}) - C_{\mathrm{top}}\,\mathrm{ue}(x,i)$.

(Hq3) Deep covers *For every real* $x > x_1$, *every* $i \in I(x)$, *every* $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ *and every lift* $j < m$ *the data fields of deep-cover data are given, such that (Dp1), (Dp2), (Dp3) hold for all large points with* $j < m$, *(Dp5) holds for all large points with* $j < m$ *with some constant* $C_{\mathrm{F}} \geq 0$, *and* $2\,\widehat{\mu}(B_{\mathfrak{z},j}) - 2\,b(x) \leq 2\,\widehat{\mu}(L^{\mathrm{c}}_{\mathfrak{z},j})$ *with* $B_{\mathfrak{z},j} \neq 0$ *for all large points with* $j < m$ *at which* $L^{\mathrm{c}}_{\mathfrak{z},j}$ *is a non-zero hermitian lattice.*

(Hq4) Components and model slopes *With these deep-cover data the family carries component data and model-slope data such that: there is a real* $C' \geq 0$ *for which (Ht1), (Ht2), (Ht4), (Ht5) hold with the constant* C' *for all large points with* $j < m$; *there is a real* $C_4 \geq 0$ *for which (Ms4) holds with the constant* C_4 *for all large points with* $j < m$; *and for every real* $x > x_1$, *every* $i \in I(x)$, *every* $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ *and every lift* $j < m$ *the data (Aj1), (Aj2), (Aj3) are given and hold for these component data and model-slope data.*

Proof. Outline. One intersects the structural thresholds of five supply results for the data of a point (sizes and divided weights, covering covers, rank comparisons, heights, which is Proposition A.102, and an arithmetic Hilbert–Samuel bound); at a large pair the deep-cover data are taken from the cusp spaces and Petersson lattices of a covering cover $\mathcal{Y}$ at the shifted weights, with deep lattice $B_{\mathcal{Y}}$ and the descent maps of the cover, the component data from the irreducible components at the twisted weight, and (Hq1)–(Hq4) are then read off from these five results. □

Proposition A.94. *There is $\varepsilon_1 > 0$ and, for every $\varepsilon \in (0, \varepsilon_1)$, a structural threshold $\mathfrak{K}(\varepsilon)$, such that for every family $\mathcal{F} \in \mathcal{B}_{\mathrm{cv}}$ of parameter ε that is large for $(\mathfrak{K}(\varepsilon), u)$ for some real u there are real numbers $C \geq 0$ and x_0 such that for every real $x \geq x_0$ with $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$ with rank datum (k, m, t) and derived ranks f, r, every lift $j < m$ and every covering cover $\mathcal{Y}$ of $\mathfrak{z}$ at j, with $\mathfrak{m} = \mathfrak{m}_{\mathcal{Y}}$ with field F', $z = f\, r\, x^{\varepsilon}$, the core weight κ and the weight constant $c_0 = c_0(j)$ of $\mathfrak{z}$ at j, $W = w(c_0 h, 0)$ and $B_{\mathcal{Y}} = \Lambda_{\mathcal{Y}}(\kappa; \mathrm{Typ}^{\mathrm{c}}_{\mathfrak{z},j})$, there are a finite group H, for every integral weight G a map ρ_G from H to the F'-linear endomorphisms of $\mathrm{Alg}_{F'}(G)$, and a homomorphism $\chi\colon H \to F'^{\times}$ such that:*

1. *for all integral weights G, G', all $\gamma, \gamma' \in H$, $y \in \mathrm{Alg}_{F'}(G)$ and $y' \in \mathrm{Alg}_{F'}(G')$: $\rho_G(\gamma\gamma') = \rho_G(\gamma') \circ \rho_G(\gamma)$, $\rho_G(1)$ is the identity, and $\rho_{G+G'}(\gamma)(yy') = \rho_G(\gamma)(y)\, \rho_{G'}(\gamma)(y')$, yy' being the product of algebraic forms;*
2. *for every integral weight G and every $\gamma \in H$: $\rho_G(\gamma)$ maps $\mathrm{Cusp}_{\mathfrak{m}}(G)$ into itself and $\Lambda_{\mathfrak{m}}(G)$ into itself, and $\|\rho_G(\gamma)y\|_{G,\sigma} = \|y\|_{G,\sigma}$ for every $y \in \mathrm{Cusp}_{\mathfrak{m}}(G)$ and every embedding $\sigma\colon F' \to \mathbb{C}$;*
3. *every value of χ is a root of unity, and $\log|H| \leq 45\, z^2$;*
4. *$B_{\mathcal{Y}} = \{y \in \Lambda_{\mathfrak{m}}(\kappa) :\ \rho_\kappa(\gamma)y = \chi(\gamma)y$ for every $\gamma \in H\}$;*
5. *there is $\vartheta \in \Lambda_{\mathfrak{m}}(W)$ such that the group determinant $D_\vartheta = \det\big(\rho_W(\gamma_1\gamma_2)\vartheta\big)_{\gamma_1,\gamma_2 \in H}$, an algebraic form of weight $|H|\, W$, satisfies $D_\vartheta\, y \neq 0$ for every non-zero finite sum y of algebraic forms of integral weights, and such that $\|y{\cdot}\rho_W(\gamma)\vartheta\|_{G+W,\sigma} \leq \exp(C\, z^9)\, \|y\|_{G,\sigma}$ for every $\gamma \in H$, every integral weight G, every $y \in \mathrm{Cusp}_{\mathfrak{m}}(G)$ with $y{\cdot}\rho_W(\gamma)\vartheta \in \mathrm{Cusp}_{\mathfrak{m}}(G{+}W)$ and every embedding $\sigma\colon F' \to \mathbb{C}$.*

Proof. Sketch. The covering cover $\mathcal{Y}$ is a moduli scheme of deeper level: its level is $g_1 g_2$ times a cofactor prime to $g_1 g_2$, and a type frame identifies the level structure modulo the type primes with $(\mathbb{Z}/g_v)^r$. The argument should take for H the finite group of level automorphisms that reduce to the unipotent level group modulo $g_1 g_2$, acting on algebraic forms by pull-back along the change of level structure. Pull-back of forms is multiplicative, which gives (1); it preserves the cusp condition and integrality, and it preserves Petersson norms because these are defined by invariant Haar measures, which gives (2). For χ one takes the generic character of H built from non-trivial additive characters modulo g_1 and g_2, so that its values are roots of unity; the order of H is at most $(g_1 g_2)^{O(r^2)}$, and the size conditions on the auxiliary primes should give $\log|H| \leq 45 z^2$, which is (3). Item (4) is Galois descent along the level cover: a form on the cover descends to a form of type χ on the core exactly when H acts on it through χ, so the integral pull-backs of the core type forms are the integral χ-eigenforms. Item (5) asks for one integral form ϑ of weight W whose translates have a non-degenerate group determinant. Heuristically such forms exist because H acts faithfully on the function field of the cover: by the normal basis theorem some form has translates whose group matrix is invertible (the group determinant factors over the irreducible representations of H, after Dedekind and Frobenius), so $D_\vartheta \neq 0$ is

the avoidance of a single hypersurface of degree $|H|$ in the space of forms of weight W, and a Siegel-lemma argument [37] in the lattice of integral cusp forms, whose covolume is controlled by the flat degree of the cover, should produce such a ϑ of controlled size. The bound $\exp(Cz^9)$ for multiplication by its translates should then follow from a comparison between the supremum of ϑ and its Petersson norm. Items (1)–(4) are what Proposition A.96 uses, and (1), (2) and (5) are the conditions under which Lemma A.95 constructs descent maps into the eigenlattice of (4). □

Lemma A.95. *Let $\mathfrak{m}$ be a uniformised moduli datum with field F' and a Lie parameter, let G_- and W be integral weights, $G = G_- + W$ and $G_+ = G + W$, and suppose that $\mathfrak{m}$ is good at G and at G_+. Let H be a finite group, for every integral weight G' let $\rho_{G'}$ be a map from H to the F'-linear endomorphisms of $\mathrm{Alg}_{F'}(G')$, and let $\chi\colon H \to F'^{\times}$ be a homomorphism whose values are roots of unity, such that:*

1. *$\rho_{G'}(1)$ is the identity, $\rho_{G'}(\gamma\gamma') = \rho_{G'}(\gamma')\circ\rho_{G'}(\gamma)$ and $\rho_{G'+G''}(\gamma)(yy') = \rho_{G'}(\gamma)(y)\,\rho_{G''}(\gamma)(y')$ for all integral weights G', G'', all $\gamma, \gamma' \in H$, $y \in \mathrm{Alg}_{F'}(G')$, $y' \in \mathrm{Alg}_{F'}(G'')$;*
2. *for $G' \in \{W, G\}$ and $\gamma \in H$, $\rho_{G'}(\gamma)$ maps $\mathrm{Cusp}_{\mathfrak{m}}(G')$ into itself and $\Lambda_{\mathfrak{m}}(G')$ into itself, and $\|\rho_G(\gamma)y\|_{G,\sigma} = \|y\|_{G,\sigma}$ for $y \in \mathrm{Cusp}_{\mathfrak{m}}(G)$ and every embedding $\sigma\colon F' \to \mathbb{C}$.*

Let $\vartheta \in \Lambda_{\mathfrak{m}}(W)$ be such that $D_\vartheta = \det(\rho_W(\gamma_1\gamma_2)\vartheta)_{\gamma_1,\gamma_2\in H}$ satisfies $D_\vartheta\, y \neq 0$ for every non-zero finite sum y of algebraic forms of integral weights; let $B \geq 0$ be such that $\|y \cdot \rho_W(\gamma)\vartheta\|_{G,\sigma} \leq B\,\|y\|_{G_-,\sigma}$ for every $\gamma \in H$, every $y \in \mathrm{Cusp}_{\mathfrak{m}}(G_-)$ with $y\cdot\rho_W(\gamma)\vartheta \in \mathrm{Cusp}_{\mathfrak{m}}(G)$ and every σ; and let b be a real number with $|H|^3B^2 \leq e^{2b}$. Put $\Lambda^\chi = \{y \in \Lambda_{\mathfrak{m}}(G) :\ \rho_G(\gamma)y = \chi(\gamma)y \text{ for every } \gamma \in H\}$. Then, with $n = |H|$, there are F'-linear maps $\phi_1, \dots, \phi_n\colon \mathrm{Cusp}_{\mathfrak{m}}(G_-) \to \mathrm{Cusp}_{\mathfrak{m}}(G)$ such that $\phi_k(\Lambda_{\mathfrak{m}}(G_-)) \subseteq \Lambda^\chi$ for every k; $y = 0$ whenever $\phi_k(y) = 0$ for all k; $\sum_{k=1}^n \|\phi_k(y)\|^2_{G,\sigma} \leq e^{2b}\|y\|^2_{G_-,\sigma}$ for every $y \in \mathrm{Cusp}_{\mathfrak{m}}(G_-)$ and every σ; and $n \operatorname{rk}\Lambda^\chi \leq \mathrm{rk}_{\mathfrak{m}}(G_+)$.

Proof. **Products and the graded algebra.** We use the commutative algebra of finite sums of algebraic forms, with the weight grading and the pointwise product. The given maps on its homogeneous pieces extend linearly to an anti-action on this algebra by algebra automorphisms: the inverse of $\rho(\gamma)$ is $\rho(\gamma^{-1})$, and the product law is hypothesis (1). Products of two cusp forms are cusp forms at the sum weight. To read this also for forms not supported on a single component, decompose each form by the finitely many open and closed irreducible components of the smooth moduli scheme. Restriction followed by zero extension is a natural algebraic form and preserves the cusp condition on each uniformisation component. Products on different components vanish, and the products on each single component are cusp forms, since products of cusp forms on a component are cusp forms. Summing gives the assertion. Products of integral forms are integral: their valuations at every integral framed point add. These facts apply at weights $G_- + W = G$ and $G + W = G_+$.

Write $\vartheta_h = \rho_W(h)\vartheta$. Hypothesis (2) gives $\vartheta_h \in \Lambda_{\mathfrak{m}}(W)$ for every h. For $y \in \mathrm{Cusp}_{\mathfrak{m}}(G_-)$ define

$$\phi_h(y) = \sum_{\gamma\in H} \chi(\gamma)^{-1}\rho_G(\gamma)(y\vartheta_h).$$

Products and cusp stability show that these are F'-linear maps to $\mathrm{Cusp}_{\mathfrak{m}}(G)$. Enumerate H once to obtain the requested $\phi_1, \dots, \phi_n$.

The character and integral lattice. For $\delta \in H$, the anti-action law and the substitution

$\eta = \gamma\delta$ give

$$\begin{aligned}\rho_G(\delta)\phi_h(y) &= \sum_{\gamma\in H} \chi(\gamma)^{-1}\rho_G(\gamma\delta)(y\vartheta_h)\\ &= \chi(\delta)\sum_{\eta\in H}\chi(\eta)^{-1}\rho_G(\eta)(y\vartheta_h) = \chi(\delta)\phi_h(y).\end{aligned}$$

A root of unity and its inverse are algebraic integers, so every $\chi(\gamma)^{-1}$ belongs to $\mathcal{O}_{F'}$. If y is integral, each product $y\vartheta_h$ is integral, and its image under $\rho_G(\gamma)$ is integral by hypothesis (2). The sum is consequently integral and has the displayed character. This proves $\phi_h(\Lambda_{\mathfrak{m}}(G_-)) \subseteq \Lambda^{\chi}$.

Joint injectivity by an adjugate. The product and anti-action laws give, as an identity in the algebra of forms,

$$\phi_h(y) = \sum_{\gamma\in H} (\rho_W(h\gamma)\vartheta)\chi(\gamma)^{-1}\rho_{G_-}(\gamma)y.$$

Let M be the matrix with rows h, columns γ, and entries $M_{h,\gamma} = \rho_W(h\gamma)\vartheta$. Thus the vector of the $\phi_h(y)$ is Mv, where $v_\gamma = \chi(\gamma)^{-1}\rho_{G_-}(\gamma)y$. The determinant of this matrix is exactly D_ϑ, using the same enumeration for rows and columns. If every $\phi_h(y) = 0$, multiplication by the adjugate yields $D_\vartheta v_\gamma = 0$ for every γ. Each v_γ is an algebraic form, so the stated non-zero-divisor hypothesis gives $v_\gamma = 0$. At $\gamma = 1$ this is $y = 0$. No cusp stability at weight G_- for the action is needed in this step.

The simultaneous norm bound. Fix y and an embedding σ. Every $y\vartheta_h$ is a cusp form of weight G, so the conditional multiplication bound in the hypothesis applies to it. Since G is good, its Petersson norm is a Hermitian norm and satisfies the triangle inequality. The isometry in hypothesis (2) and $|\sigma(\chi(\gamma))| = 1$ therefore give

$$\|\phi_h(y)\|_{G,\sigma} \le \sum_{\gamma\in H}\|\rho_G(\gamma)(y\vartheta_h)\|_{G,\sigma} = n\|y\vartheta_h\|_{G,\sigma} \le nB\|y\|_{G_-,\sigma}.$$

All quantities are nonnegative. Squaring and summing over the n choices of h proves

$$\sum_h \|\phi_h(y)\|^2_{G,\sigma} \le n^3B^2\|y\|^2_{G_-,\sigma} \le e^{2b}\|y\|^2_{G_-,\sigma}.$$

This uses only the given values of the source norm and its multiplication bound, not goodness at G_-.

The rank count. Define an $\mathcal{O}_{F'}$-linear map

$$\Psi : (\Lambda^{\chi})^H \longrightarrow \Lambda_{\mathfrak{m}}(G_+), \qquad \Psi((f_h)_h) = \sum_{h\in H} f_h\vartheta_h.$$

Products of cusp forms and integral forms justify this codomain. Suppose its value is zero. Apply the algebraic action $\rho_{G_+}(\gamma)$ to the equality. The eigenvector equations for the f_h, the product law, and the anti-action law give

$$0 = \chi(\gamma)\sum_{h\in H} f_h\,\rho_W(h\gamma)\vartheta \qquad (\gamma\in H).$$

The scalar $\chi(\gamma)$ is invertible. Hence $M^{\mathsf{T}}(f_h)_h = 0$ for the same matrix M as above. The adjugate again yields $D_\vartheta f_h = 0$, and the non-zero-divisor hypothesis forces every $f_h = 0$. Thus Ψ is injective.

Goodness at G makes $\Lambda_{\mathfrak{m}}(G)$ a finitely generated torsion-free $\mathcal{O}_{F'}$-module; its submodule Λ^{χ} is finitely generated and torsion-free as well, since $\mathcal{O}_{F'}$ is Noetherian. Goodness at G_+ gives the same property for the target. Tensor the injection Ψ with the fraction field F'; localisation is exact. Taking dimensions gives

$$n \operatorname{rk} \Lambda^{\chi} \le \operatorname{rk} \Lambda_{\mathfrak{m}}(G_+) = \operatorname{rk}_{\mathfrak{m}}(G_+).$$

This proves every assertion, including zero source or eigenlattice ranks. □

Proposition A.96. *There is $\varepsilon_1 > 0$ and, for every $\varepsilon \in (0, \varepsilon_1)$, a structural threshold $\mathfrak{K}(\varepsilon)$, such that for every family $\mathcal{F} \in \mathcal{B}_{\mathrm{cv}}$ of parameter ε that is large for $(\mathfrak{K}(\varepsilon), u)$ for some real u there is a real number x_0 such that for every real $x \ge x_0$ with $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$, every lift $j < m$ and every covering cover $\mathcal{Y}$ of $\mathfrak{z}$ at j, with $\mathfrak{m} = \mathfrak{m}_{\mathcal{Y}}$ with field F', the weight data of $\mathfrak{z}$ at j and $B_{\mathcal{Y}} = \Lambda_{\mathcal{Y}}(\kappa; \mathrm{Typ}^{\mathrm{c}}_{\mathfrak{z},j})$: there is a set $s \subseteq B_{\mathcal{Y}}$ with $\|y\|_{\kappa,\sigma} \le e^{P_{\mathfrak{z}}\varkappa}$ for every $y \in s$ and every embedding $\sigma\colon F' \to \mathbb{C}$, and $B_{\mathcal{Y}} \subseteq \operatorname{span}_{F'} s$.*

Proof. Outline. By Proposition A.94, $B_{\mathcal{Y}}$ is the χ-eigenlattice of a finite group H acting isometrically on the integral cusp forms of weight κ, with $\log|H| \le 45z^2$; a bound for minimal generators gives integral generators of the cusp space of norm at most $\exp(Cz^2\varkappa)$, an eigenlattice lemma turns them into generators of $B_{\mathcal{Y}}$ of norm at most $|H|\exp(Cz^2\varkappa)$, and the size condition $z^{12} \le P_{\mathfrak{z}}$ bounds the exponent by $P_{\mathfrak{z}}\varkappa$. □

A.8.3 Heights and Bergman volumes

A short spanning family of a lattice over a number field gives short integral generators (Lemma A.98), and the height of a component is bounded below by minus the logarithm of a bound for its short generators (Lemma A.97). From above, the Bergman volume of a component is bounded by its covering degree times its height plus a shift (Proposition A.99); Hirzebruch–Mumford proportionality makes the unit volumes of the components equal (Proposition A.100); and an effective arithmetic Hilbert–Samuel inequality on each component, with explicit parameters, joins the two sides (Lemma A.101). The heights of the components then follow from their volume slopes, the tower datum and the budget (Proposition A.102). The last statement provides integral models of unitary Shimura data with level, arc, transition and Siegel conditions, and a dominating model over the ring of integers (Proposition A.103); these are the models on which the Hecke reading of Section A.3 acts.

Lemma A.97. *Let $\mathfrak{m}$ be a uniformised moduli datum with coefficient field F and Lie parameter $\xi > 0$, $Z \in \mathrm{Comp}(\mathfrak{m})$ and G an integral weight such that $\mathfrak{m}$ is good at G on Z and $\mathrm{dg}_Z(G) > 0$. Let $\Lambda \ge 1$ and let $w_1, \dots, w_n \in \mathrm{Cusp}_{\mathfrak{m},Z}(G)$ satisfy $\Lambda_{\mathfrak{m},Z}(G) = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_n$ and $\|w_l\|_{G,\sigma,Z} \le \Lambda$ for $1 \le l \le n$ and every embedding $\sigma\colon F \to \mathbb{C}$. Then*

$$\mathrm{ht}_Z(G) + \log \Lambda \ge 0.$$

Proof. If no height exists, the convention gives $\mathrm{ht}_Z(G) = 0$, and the assertion follows from $\Lambda \ge 1$. Otherwise fix data b, w', A_σ, f as in (Vy5) for which $h = \mathrm{ht}_Z(G)$ satisfies the defining identity there. Write $N = \dim_F \mathrm{Cusp}_{\mathfrak{m},Z}(G)$, $D = D_{\mathrm{fl}}$, $\delta = \mathrm{dg}_Z(G) > 0$, and $B = D + 1$. The nonzero Chow form f has degree δ in each of its B blocks. In particular $N \ge 1$.

The given short generators may be used in the finite part. Let $W \in \mathrm{M}_{N,n}(F)$ have as its lth column the coordinates of w_l in b. Since the w_l generate the integral lattice over $\mathbb{Z}$,

they also generate it over $\mathcal{O}_F$: their $\mathcal{O}_F$-span lies in that lattice and contains its $\mathbb{Z}$-span. The columns of both W and w' therefore generate the same $\mathcal{O}_F$-module. Consequently $W = w'U$ and $w' = WV$ for rectangular matrices U, V over $\mathcal{O}_F$. Substitution of an integral linear combination of variables cannot increase a finite Gauss norm: expand each monomial, noting that every resulting coefficient is an integral linear combination of the original coefficients. Applying this to both identities gives

$$|f \circ W|_v = |f \circ w'|_v \qquad (v \text{ finite}).$$

Moreover the columns of W span F^N, since the lattice spans the cusp space by goodness. The norm identity for A_σ gives

$$|A_\sigma^{-1}\sigma(W_l)| = \|w_l\|_{G,\sigma,Z} \le \Lambda. \tag{1}$$

Here W_l denotes column l. Thus it suffices to prove the following polynomial estimate, with the same matrices A_σ and W:

$$\frac{1}{[F:\mathbb{Q}]}\left(\sum_{v \text{ finite}} \log|f \circ W|_v + \sum_\sigma \left(m(f^\sigma \circ A_\sigma) + \frac{B\delta}{2}H_{N-1}\right)\right) \ge -B\delta \log\Lambda, \quad H_{N-1} = \sum_{j=1}^{N-1}\frac{1}{j}. \tag{2}$$

A directional leading coefficient gives the sharp sphere-mean inequality. We give the analytic calculation used in (2). Let p be a nonzero complex polynomial, homogeneous separately in B blocks of N variables, and choose a nonzero vector $v \in \mathbb{C}^N$ and a block a. Expand

$$p(z_1, \dots, z_a + tv, \dots, z_B) = \sum_{j=0}^{l} t^j q_j(z), \qquad q_l \ne 0.$$

Put $q = q_l$ and $r = |v| > 0$. The polynomial q is invariant under translation of block a by a multiple of v: replacing z_a by $z_a + sv$ replaces t by $t + s$ and leaves the leading coefficient unchanged. It is multihomogeneous, and its total degree is that of p minus l.

Put $u = v/r$. For a sphere point z_a, rotate its component along u by $e^{i\theta}$ and leave its orthogonal component fixed. This unitary map preserves the invariant sphere probability measure. For fixed z the resulting value of p is a polynomial in $e^{i\theta}$ whose leading coefficient is $q(z)(\langle u, z_a\rangle/r)^l$, except on a set of sphere measure zero. Jensen's one-variable formula, in its elementary form saying that the circle mean of the logarithm of the absolute value of a polynomial is at least the logarithm of the absolute value of its leading coefficient, therefore gives, after averaging over all blocks,

$$m(p) \ge m(q) - l\log r - \frac{l}{2}H_{N-1}. \tag{3}$$

For completeness, the last constant is exact. The mean of $\log|\langle u, z_a\rangle|$ on the unit sphere of $\mathbb{C}^N$ is $-H_{N-1}/2$. For $N = 1$ the absolute value is 1. For $N > 1$ its squared absolute value has density $(N-1)(1-t)^{N-2}\,dt$ on $(0,1)$; integration of $\frac{1}{2}\log t$ against this density gives the stated value. The logarithms of nonzero polynomials are integrable on these products of spheres, so these averages and the interchange with the circle average are legitimate. One can see the integrability by passing, using homogeneity, to projective coordinate charts: there the logarithm differs from that of a nonzero holomorphic polynomial by a smooth function. Local integrability of the latter logarithm and a finite chart cover give the assertion. Thus no value assigned on a zero locus affects (3).

Induction combines the sphere estimate with the finite Gauss norms. For a nonzero polynomial p over F, multihomogeneous of degrees $a_1, \dots, a_B$, put $e = a_1 + \cdots + a_B$ and

$$J(p) = \frac{1}{[F:\mathbb{Q}]}\left(\sum_{v \text{ finite}} \log|p \circ W|_v + \sum_\sigma m(p^\sigma \circ A_\sigma)\right).$$

The full rank of W makes the linear substitution injective on polynomial rings, so $p \circ W \neq 0$. We prove by induction on e that

$$J(p) \geq -e\left(\log \Lambda + \frac{1}{2} H_{N-1}\right). \tag{4}$$

If $e = 0$, p is a nonzero constant; the product formula gives $J(p) = 0$.

Suppose $e > 0$. Some block and some column W_l give a nonconstant polynomial in t after the translation of that block by tW_l. Indeed, if every such translation left p unchanged, the fact that the columns span F^N would make p independent of that block; applying this to every block would make it constant. Choose such a block and column, and let q be the leading coefficient in t, of degree $l_0 \geq 1$. Then $q \neq 0$ is multihomogeneous of total degree $e - l_0$.

At a finite place,

$$|q \circ W|_v \leq |p \circ W|_v. \tag{5}$$

In fact, after substituting W in every block, the translation by tW_l is the replacement of the lth variable of the chosen block by that variable plus t. This substitution has integer coefficients, and extraction of a coefficient cannot increase its Gauss norm. This proves (5), including the possible binomial coefficients.

At each embedding apply (3) to $p^\sigma \circ A_\sigma$ and the vector $v_\sigma = A_\sigma^{-1}\sigma(W_l)$. This vector is nonzero because $W_l \neq 0$ and A_σ is invertible; by (1) its norm is at most Λ. The leading coefficient is precisely $q^\sigma \circ A_\sigma$. Therefore

$$m(p^\sigma \circ A_\sigma) \geq m(q^\sigma \circ A_\sigma) - l_0\left(\log \Lambda + \frac{1}{2} H_{N-1}\right). \tag{6}$$

Sum (5) and (6), divide by $[F : \mathbb{Q}]$, and use that the number of embeddings of F is $[F : \mathbb{Q}]$. The inductive hypothesis for q yields

$$J(p) \geq J(q) - l_0\left(\log \Lambda + \frac{1}{2} H_{N-1}\right) \geq -e\left(\log \Lambda + \frac{1}{2} H_{N-1}\right),$$

which proves (4).

The harmonic correction in the height cancels exactly. Apply (4) to the Chow form f, whose total degree is $e = B\delta$. Adding $B\delta H_{N-1}/2$ gives (2). By the first step the finite terms are the terms defining the height h, with the same Petersson norm matrices A_σ. Hence

$$B\delta h \geq -B\delta \log \Lambda.$$

Since $B = D + 1 > 0$ and $\delta > 0$, division proves $h + \log \Lambda \geq 0$. The calculation uses the Hilbert norms only through (1); it introduces neither a pointwise bound for cusp forms nor a factor depending on the number of generators. □

Lemma A.98. *Let K be a number field of degree $d = [K : \mathbb{Q}]$ and root discriminant $\mathrm{rd}_K = |d_K|^{1/d}$, let V be a finite-dimensional K-vector space and $L \subseteq V$ a finitely generated $\mathcal{O}_K$-submodule, and put $m = \dim_K(KL)$. For every embedding $\sigma\colon K \to \mathbb{C}$ let $N_\sigma\colon V \to \mathbb{R}$ satisfy, for all $a \in K$ and $x, y \in V$,*

$$N_\sigma(x) \geq 0, \qquad N_\sigma(ax) = |\sigma(a)|\, N_\sigma(x), \qquad N_\sigma(x + y) \leq N_\sigma(x) + N_\sigma(y).$$

Let $A \in \mathbb{R}$ and $s \subseteq L$ with $\mathrm{span}_K\, s = V$ and $N_\sigma(y) \leq A$ for every $y \in s$ and every σ. Then there are an integer $0 \leq n \leq 2dm$, elements $w_1, \ldots, w_n \in V$ and a real number Λ with

$$L = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_n, \qquad N_\sigma(w_l) \leq \Lambda\ (1 \leq l \leq n,\ \text{all } \sigma), \qquad 1 \leq \Lambda \leq \max(1,\ d^4\, m\, \mathrm{rd}_K\, A).$$

Proof. The inclusion $s \subseteq L$ and the spanning hypothesis imply $KL = V$, so $\dim_K V = m$. Homogeneity with the scalar 0 gives $N_\sigma(0) = 0$. If $m = 0$, then $V = L = 0$; take $n = 0$ and $\Lambda = 1$. This proves the assertion for every real A in that case. Henceforth $m \geq 1$. Then s is nonempty, and its bound together with nonnegativity gives $A \geq 0$.

Choose $y_1, \dots, y_m \in s$ forming a K-basis of V. Since a number field has a $\mathbb{Q}$-basis of algebraic integers whose conjugates are bounded in terms of the root discriminant (not necessarily an integral basis of $\mathcal{O}_K$), there are d elements $a_1, \dots, a_d \in \mathcal{O}_K$, linearly independent over $\mathbb{Q}$, with

$$|\sigma(a_i)| \leq \sqrt{d}\,\mathrm{rd}_K \qquad (1 \leq i \leq d, \text{ every } \sigma).$$

They form a $\mathbb{Q}$-basis of K. Consequently the $h = dm$ vectors $a_i y_j$ form a $\mathbb{Q}$-basis of V. Enumerate them as $e_1, \dots, e_h$. They belong to L, and homogeneity gives the simultaneous bounds

$$N_\sigma(e_l) \leq B, \qquad B = \sqrt{d}\,\mathrm{rd}_K A \geq 0.$$

We construct an integral basis of L with controlled coordinates in this particular rational basis. Since $\mathcal{O}_K$ is finitely generated as an abelian group and L is finitely generated over $\mathcal{O}_K$, L is finitely generated as an abelian group. Every element of a fixed finite generating family has rational coordinates in (e_l). Clearing their denominators gives an integer $H \geq 1$ such that

$$\sum_{l=1}^{h} \mathbb{Z} e_l \;\subseteq L\; \subseteq \sum_{l=1}^{h} H^{-1}\mathbb{Z} e_l.$$

For $0 \leq i \leq h$, put

$$V_i = \mathrm{span}_{\mathbb{Q}}(e_1, \dots, e_i), \qquad L_i = L \cap V_i.$$

Thus $L_0 = 0$ and $L_h = L$. For $i \geq 1$, let $\pi_i : V_i \to \mathbb{Q}$ be the coefficient of e_i. The group $\pi_i(L_i)$ is a subgroup of $H^{-1}\mathbb{Z}$ containing $\mathbb{Z}$, because $e_i \in L_i$. It therefore equals $q_i\mathbb{Z}$ for a positive rational number $q_i \leq 1$. Choose a lift $v_i \in L_i$ with $\pi_i(v_i) = q_i$. Subtract integer multiples of $e_1, \dots, e_{i-1}$ to obtain a vector $w_i \in L_i$ of the form

$$w_i = q_i e_i + \sum_{j<i} c_{ij} e_j, \qquad |c_{ij}| \leq \tfrac{1}{2}.$$

Indeed each rational coordinate can be reduced to $[-1/2, 1/2]$ by subtraction of an integer; these subtractions preserve membership in L_i and its i-th coordinate.

For any $x \in L_i$, its i-th coordinate is an integer multiple kq_i. Hence $x - kw_i \in L_{i-1}$. Conversely L_{i-1} and w_i are contained in L_i, and no nonzero integer multiple of w_i belongs to V_{i-1} because $q_i > 0$. Thus

$$L_i = L_{i-1} \oplus \mathbb{Z} w_i.$$

Induction shows that $w_1, \dots, w_h$ are a $\mathbb{Z}$-basis of L.

For a rational scalar c, every field embedding has $|\sigma(c)| = |c|$. Applying the triangle inequality and homogeneity to the displayed expression for w_i therefore gives, for every σ,

$$N_\sigma(w_i) \leq q_i N_\sigma(e_i) + \sum_{j<i} |c_{ij}| N_\sigma(e_j) \leq \left(1 + \frac{i-1}{2}\right) B \leq hB = d^{3/2} m\,\mathrm{rd}_K A.$$

Since $d \geq 1$ and $A \geq 0$, this is at most $d^4 m\,\mathrm{rd}_K A$. Take

$$n = h = dm, \qquad \Lambda = \max(1, d^4 m\,\mathrm{rd}_K A).$$

Then $0 \leq n \leq 2dm$, the vectors generate L over $\mathbb{Z}$, and every required seminorm is at most Λ. The argument also applies when $A = 0$: all the displayed seminorm bounds are then zero, while $\Lambda = 1$. This completes the proof. □

Proposition A.99. *There is $\varepsilon_1 > 0$ and, for every $\varepsilon \in (0, \varepsilon_1)$, a structural threshold $\mathfrak{K}(\varepsilon)$, such that for every family $\mathcal{F} \in \mathcal{B}_{\mathrm{cv}}$ of parameter ε that is large for $(\mathfrak{K}(\varepsilon), u)$ for some real u there is a real number x_0 such that for every real $x \geq x_0$ with $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$, every lift $j < m$ and every covering cover $\mathcal{Y}$ of $\mathfrak{z}$ at j, with coefficient field F' and $\mathfrak{m} = \mathfrak{m}_{\mathcal{Y}}$ taken with the Lie parameter ξ_{h}, the following holds. Let $Z \in \mathrm{Comp}(\mathfrak{m})$, let λ be a weight array of the unitary Shimura datum of $\mathfrak{m}$, $G = w(\lambda, 0)$, let $w_1, \dots, w_n \in \Lambda_{\mathfrak{m},Z}(G)$ and let Λ_2, t be real numbers. Suppose that $\mathfrak{m}$ is good at G on Z, is birational on Z at G and has no base locus at G on Z; that $\mathrm{dg}_Z(G) > 0$; that $\Lambda_{\mathfrak{m},Z}(G)$ is the set of $\mathbb{Z}$-linear combinations of $w_1, \dots, w_n$; that $\|w_l\|_{G,\sigma,Z} \leq \Lambda_2$ for $1 \leq l \leq n$ and every embedding $\sigma\colon F' \to \mathbb{C}$; that $\Lambda_2 \geq 1$; and that $\log \Lambda_2 \leq t$. Then*

$$\mathrm{bv}_Z(G; t) \leq \max\bigl(0,\ \mathrm{dg}_Z(G)\,(\mathrm{ht}_Z(G) + t)\bigr).$$

Proof. Sketch. The inequality bounds an arithmetic volume by an arithmetic self-intersection, and the argument should run as follows. The generators $w_1, \dots, w_n$ define a map from the flag space over Z to projective space; since $\mathfrak{m}$ is birational on Z at G and has no base locus there, this map is defined everywhere and is birational onto its image X_Z, a variety of dimension $D = D_{\mathrm{fl}}$ and degree $\mathrm{dg}_Z(G) > 0$. In particular the image ideal has a Chow form, and its Mahler measure for the Petersson metrics gives a height in the sense of (Vy5) of Definition A.89, so that $\mathrm{ht}_Z(G)$ is a genuine height and not the conventional value 0; this has to be verified as part of the proof. Equip $\mathcal{O}(1)$ on the closure of X_Z over $\mathcal{O}_{F'}$ with the metric induced by the Bergman kernel of the Petersson norm and shift it by t. By the reproducing property $|w_l(y)|^2 \leq \|w_l\|^2_{G,\sigma,Z}\,\mathrm{Berg}(y)$, every generator has pointwise norm at most $\Lambda_2 \leq e^t$, so after the shift the bundle is generated by sections of norm at most 1 at every place and is therefore nef. Its arithmetic self-intersection is computed from the Chow form by the comparison of Bost, Gillet and Soulé [48], and with the normalisation of (Vy5) it is at most $[F' : \mathbb{Q}](D+1)\,\mathrm{dg}_Z(G)\,(\mathrm{ht}_Z(G) + t)$. The small lattice forms of weight $8kG$, being integral and bounded by e^{8kt} times the $8k$-th power of the Bergman kernel, inject into the small sections of the $8k$-th power of the shifted bundle on a normal model dominating X_Z. For a nef hermitian line bundle the arithmetic volume equals the self-intersection (arithmetic Hilbert–Samuel [111], [1], [276], and its extension to nef bundles [182], [272]), and dividing by the normalisation of bv_Z gives the bound. Under the same conditions Lemma A.97 gives $\mathrm{ht}_Z(G) + t \geq \mathrm{ht}_Z(G) + \log \Lambda_2 \geq 0$, so under these conditions the maximum is attained by its second term. □

Proposition A.100. *There is $\varepsilon_1 > 0$ and, for every $\varepsilon \in (0, \varepsilon_1)$, a structural threshold $\mathfrak{K}(\varepsilon)$, such that for every family $\mathcal{F} \in \mathcal{B}_{\mathrm{cv}}$ of parameter ε that is large for $(\mathfrak{K}(\varepsilon), u)$ for some real u there is a real number x_0 such that for every real $x \geq x_0$ with $x > x_1$, every $i \in I(x)$, every $\mathfrak{z} \in \mathrm{Pt}_{x,i}$, every lift $j < m$ and every covering cover $\mathcal{Y}$ of $\mathfrak{z}$ at j, with $\mathfrak{m} = \mathfrak{m}_{\mathcal{Y}}$ taken with the Lie parameter ξ_{h} and unitary Shimura datum with CM type Φ:*

1. *for every $Z \in \mathrm{Comp}(\mathfrak{m})$ and every weight array λ such that $i \mapsto \lambda(\tau, i)$ is strictly decreasing for every $\tau \in \Phi$, $\mathrm{vol}_Z(w(\lambda, 0)) = \mathrm{td}_Z(\lambda)$;*
2. *$v_0(Z) > 0$ for every $Z \in \mathrm{Comp}(\mathfrak{m})$, and $v_0(Z) = v_0(Z')$ for all $Z, Z' \in \mathrm{Comp}(\mathfrak{m})$.*

Proof. Sketch. This is Hirzebruch–Mumford proportionality [188], read on the flag space over a component. The flag space over Z has dimension $D_{\mathrm{fl}} = |\Phi|\,r(r-1)/2$, which is the dimension of the full flag variety of each factor of the compact dual, and a weight $w(\lambda, 0)$ defines an automorphic line bundle on it whose direct image to Z is the automorphic vector bundle of

highest weight λ. On a smooth projective toroidal compactification [7], [87], [154], cusp forms of weight $k\,w(\lambda,0)$ supported on Z are the sections of the k-th power of the canonical extension twisted down by the divisor at infinity, and this twist changes their number only by $O(k^{D_{\mathfrak{fl}}-1})$. For strictly decreasing λ the canonical extension should be nef and big, as it is on the compact dual where the line bundle is ample, so that by asymptotic Riemann–Roch $\mathrm{vol}_Z(w(\lambda,0))$ is its top self-intersection. Mumford's theorem expresses this Chern number as the volume of Z times the corresponding Chern number on the compact dual, which is the degree $\mathrm{vdm}(\lambda)$ of the line bundle on the flag variety; the factor is independent of λ, and evaluating at the standard array u_{std} identifies it with $v_0(Z)$, which gives (1). The factor is positive because the volume of an arithmetic quotient is positive, which is the first half of (2). The equality of the unit volumes of different components should follow because the components are permuted by Galois conjugation and by Hecke translations, both of which preserve top Chern numbers of automorphic bundles. □

Lemma A.101. *There is $\varepsilon_1>0$ and, for every $\varepsilon\in(0,\varepsilon_1)$, a structural threshold $\mathfrak{K}(\varepsilon)$, such that for every family $\mathcal{F}\in\mathcal{B}_{\mathrm{cv}}$ of parameter ε that is large for $(\mathfrak{K}(\varepsilon),u)$ for some real u there is a real number x_0 such that for every real $x\ge x_0$ with $x>x_1$, every $i\in I(x)$, every $\mathfrak{z}\in\mathrm{Pt}_{x,i}$ with rank datum of derived rank r, every lift $j<m$ and every covering cover $\mathcal{Y}$ of $\mathfrak{z}$ at j, with $\mathfrak{m}=\mathfrak{m}_{\mathcal{Y}}$ taken with the Lie parameter ξ_{h} and the weight data of $\mathfrak{z}$ at j (Φ the CM type of the core datum; write $D=D_{\mathfrak{fl}}$, $P=P_{\mathfrak{z}}$): for every divided weight u_1 there is a function $w\colon\Phi\to\mathbb{Z}$ with $|w(\tau)|\le 3r$ for all τ such that, with the weight array $\lambda_T(\tau,i)=u_M(\tau,i)-w(\tau)$:*

1. *$\mathfrak{m}$ is good at $w(\lambda_T,0)$ on every $Z\in\mathrm{Comp}(\mathfrak{m})$;*
2. *for every $Z\in\mathrm{Comp}(\mathfrak{m})$ with $\delta=\mathrm{dg}_Z(w(u_1,0))\ge 1$ and $d=\mathrm{rc}_Z(w(\lambda_T,0))>0$: Z has a height at $w(u_1,0)$ (so that $h=\mathrm{ht}_Z(w(u_1,0))$ is one), $\delta\le e^P$, the number $z_H=1-N^{\mathrm{b}}\delta/d$ satisfies $0\le z_H\le 1$, and there are real numbers $n,\Lambda,\beta\ge 1$ and J with $\log n\le P$, $\log\beta\le P$, $\log\Lambda\le P(\varkappa/T+1)$, $J\le 4\big(DP(\varkappa/T+1)+TP\big)+2P(\varkappa/T)D$ and, with $g=\mathrm{gc}_Z(w(\lambda_T,0))$,*

$$(1-z_H)\Big(Th-3(D+1)|h|-J-26D^2\log(\varrho T)-(T+D+1)\log\beta-(T+D^2)(G+2)\Big)\\ -z_H\big(T\log(2\varrho\sqrt{\beta})+2P\varkappa\big)\le\frac{g}{d},$$

where $\varrho=\Lambda\big(nD\delta(2+|h|+\log\Lambda)\big)^{26}$ and $G=\frac{1}{2}(D+1)(\log n+1)+1$.

Proof. Sketch. The statement is an effective arithmetic Hilbert–Samuel inequality on each component: up to explicit errors, the normalised arithmetic degree g/d of the Petersson lattice at the high weight $w(\lambda_T,0)$ is at least T times the height h of Z at the divided weight u_1. The weight λ_T is chosen so that, up to a parallel shift w with $|w|\le 3r$, it is the weight $\kappa_{\mathfrak{fl}}+u_2+(D+T)u_1$ of a fixed form of weight $\kappa_{\mathfrak{fl}}+u_2+Du_1$ multiplied by monomials of degree T in forms of weight u_1; this weight identity at the point $\mathfrak{z}$ is where the shift w comes from. The argument should run as follows. On a smooth projective toroidal compactification of the flag space over Z on which the relevant Hodge bundle is generated [7], [154], the forms of weight u_1 (and u_2) are birational, so the image ideal has a Chow form of dimension D and degree δ, which gives the existence of the height h. The products of degree T should span a sublattice of rank $N^{\mathrm{b}}\delta$, the proportion $1-z_H$ of the rank d, and an effective arithmetic Hilbert–Samuel theorem bounds its slope below by Th minus the displayed error terms; the inputs are Kawamata–Viehweg vanishing [143], [263], L^2 extension [194], [79], GAGA [219], and the theory of Chow forms and heights [48], [197]. The complementary part of the lattice, of proportion z_H, is controlled only through short generators:

the lattices at the intermediate weights are spanned by vectors of Petersson norm at most $e^{P\varkappa}$, the frame norms are bounded by Λ and the Bergman kernel by β, which accounts for the term $z_H\big(T\log(2\varrho\sqrt{\beta})+2P\varkappa\big)$. Since arithmetic degrees are additive in an exact sequence of hermitian lattices, g/d is the average of the two slopes with weights $1-z_H$ and z_H, and combining the two bounds gives (2), once the image degree is identified with dg_Z, the Chow height with ht_Z, and $\log \mathrm{dg}_Z(w(u_1,0))\le P$ is shown. Item (1) is the regularity of the Petersson lattice at $w(\lambda_T,0)$ on each component, the same goodness that the covering domain provides at the other weights of the construction. The inequality is used component by component in the effective Hilbert–Samuel bound for the covering cover, which compares the heights of the components with the arithmetic degrees of the Petersson lattices. □

Proposition A.102. *There is $\varepsilon_1>0$ and, for every $\varepsilon\in(0,\varepsilon_1)$, a structural threshold $\mathfrak{K}(\varepsilon)$, such that for every family $\mathcal{F}\in\mathcal{B}_{\mathrm{cv}}$ of parameter ε that is large for $(\mathfrak{K}(\varepsilon),u)$ for some real u there are real numbers $C\ge 0$ and x_0 such that for every real $x>x_1$, every $i\in I(x)$, every $\mathfrak{z}\in\mathrm{Pt}_{x,i}$ and every lift $j<m$, with the weight data of $\mathfrak{z}$ at j:*

1. *if $x\ge x_0$: for every covering cover $\mathcal{Y}$ of $\mathfrak{z}$ at j, with $\mathfrak{m}_{\mathcal{Y}}$ taken with the Lie parameter ξ_{h} of the core data, and every divided weight u_1 there are model-slope entries E, $M\ge 1$, $B\ge 128$, L, $\mathrm{wn}\ge 0$, $\mathrm{wsp}\ge 0$ and real numbers $\mathrm{vs}(Z)$, $Z\in\mathrm{Comp}(\mathfrak{m}_{\mathcal{Y}})$, such that (Aj1)–(Aj3) hold at $(\mathfrak{z},j)$ for $\mathcal{Z}=\mathrm{Comp}(\mathfrak{m}_{\mathcal{Y}})$, $D^{\mathrm{fl}}=D_{\mathrm{fl}}$, these volume slopes and these entries, and, with $h_Z=\mathrm{ht}_Z(w(u_1,0))$ and $\rho=\rho(u_1)$ there are reals $c^{\mathrm{s}},c^{\mathrm{c}}\ge 0$ such that for every $Z\in\mathrm{Comp}(\mathfrak{m}_{\mathcal{Y}})$*

$$h_Z+c^{\mathrm{s}}\ge 0,\qquad (1-\rho)\max\big(0,\mathrm{vs}(Z)+c^{\mathrm{s}}\big)\le h_Z+c^{\mathrm{s}}+c^{\mathrm{c}},$$

and $T\,c^{\mathrm{c}}\le C\,\mathrm{ue}(x,i)$, $c^{\mathrm{s}}\big(6D_{\mathrm{fl}}+10+2\rho T\big)\le C\,\mathrm{ue}(x,i)$ and $T\big(\mathrm{lz}\cdot\mathrm{wn}+\mathrm{wsp}\big)\le C\,\mathrm{ue}(x,i)$;

2. *(at every point) there are a finite non-empty set $\mathcal{Z}$, an integer $D^{\mathrm{fl}}\ge 0$, real numbers $\mathrm{vs}(Z)$ ($Z\in\mathcal{Z}$) and model-slope entries E, $M\ge 1$, $B\ge 128$, L, $\mathrm{wn}\ge 0$, $\mathrm{wsp}\ge 0$ for which (Aj1)–(Aj3) hold at $(\mathfrak{z},j)$.*

Proof. Outline. By the proportionality Proposition A.100 and a comparison of image degrees, the covering degree at the divided weight is positive and at most $\mathrm{td}_Z(u_1')/(1-\rho)$; short generators of the Petersson lattices give c^{s} with $h_Z+c^{\mathrm{s}}\ge 0$ by Lemma A.97, and a multiplicity constant together with a height cost gives c^{c}; the height inequality then follows by bounding the counting volume by a Bergman volume and this by Proposition A.99, the volume slopes being the suprema of the slope sets so obtained. □

Proposition A.103. *Let $\mathfrak{u}=(E,\mathsf{H},\mathsf{U},p,q,\mathsf{L},\mathrm{Sh},F)$ be a unitary Shimura datum with $p+q\ge 1$, level $\mathsf{L}=(N,\mathfrak{n},A)$ good at every prime and $A=1$, and let a system of moduli schemes $(\mathrm{Sh}_{\mathsf{L}'})$ extending $\mathfrak{u}$ be given. Then there are:*

1. *for every prime ℓ and every ℓ-adic place v, a family $(\mathcal{Y}^{v}_{\mathsf{L}',F'})$, indexed by the levels L' for H good at ℓ and the coefficient fields F' for L', of integral models of $\mathrm{Sh}_{\mathsf{L}'}$ over $\mathcal{O}_{F',v}$, every member of which satisfies (Gl1) and (Gl4), and which satisfies (Gl6) and (Gl7);*

2. *an integral model $\mathcal{Y}$ of Sh over $\mathcal{O}_F$ satisfying (Gl8), and satisfying (Gl9) with respect to the families of part 1.*

Proof. Sketch. The statement asks for integral models of the unitary Shimura varieties at every prime, including the primes of bad reduction, in the semiabelian form of Definition 31.2; it is the arithmetic setting in which the normalised Hecke reading Lemma A.30 takes place. At a prime ℓ and an ℓ-adic place v, the argument should run as follows. Choose an auxiliary odd prime $n \neq \ell$ and a power $m = n^{j'} \geq 3$, and use the moduli of principally polarised abelian schemes with symplectic level-m structure, which is smooth over $\mathbb{Z}_{(\ell)}[e^{2\pi i/m}]$, as a Siegel family. For each level good at ℓ, map the moduli scheme with auxiliary level m to it, take the normalisation of the closure of the image in a toroidal compactification of the Siegel family, and descend along the auxiliary level; this gives normal flat models whose transition maps between levels are finite, which is (Gl6), and whose comparison with the Siegel family is built in, which is (Gl7). The level-structure conditions (Gl1) are functoriality of this construction together with the extension of prime-to-ℓ torsion over the abelian locus. The arcs (Gl4) come from Mumford's construction of degenerating abelian varieties from degeneration data [187], [87], and their existence and uniqueness over complete discrete valuation rings rest on Grothendieck's semistable reduction theorem [227]. At primes of good reduction these are the arithmetic toroidal compactifications of [154]; at the remaining finitely many primes the normalisation construction is the one of [156] (compare [155]). Finally the model over $\mathcal{O}_F$ is obtained by gluing a smooth model over $\mathcal{O}_F[1/N]$ with normalisations at the finitely many remaining primes, chosen to dominate the local models; then its generic fibre descends, which is (Gl8), and the domination is (Gl9). □

A.9 Attachment at the members: Galois representations and the Ribet budget

The local losses of the lattices are bounded in the body by the Ribet budget for one coefficient extension, Theorem 22.1, together with its whole local theory. Alpöge's "congruences between lifts and forms" [4] are, in our reading, this budget. Beneath (TCS) no new local theory is needed: what is needed is to attach the hypotheses of the budget to the actual members. Theorem A.105 does this relative to four inputs, good cuts (Proposition A.106), the member Ribet supply (Proposition A.107), the Selmer budget of the chosen twists (Proposition A.108) and bounds at the generator primes (Proposition A.109), and it requires good twist data (Definition A.104). The budget is asked of one global coefficient extension, so the members never need roots of unity in their coefficients. The p-adic lifting theory needed at the members is proved (Theorem A.111). What the attachment needs beyond it is arithmetic of the actual members: Galois representations at the points of the members (Proposition A.110), position data at large primes, the corner design of the hub, and the Selmer clause of twist goodness. The last subsection treats the ties of a selected member with its context.

A.9.1 The Ribet budget at the members

Definition A.104 defines good twist data of a parametrised host family. Theorem A.105 shows that the actual members carry the records, the budget data and the Galois Ribet items of Theorem 22.1, given its four inputs. The four inputs, and the existence of hub data for member data with the Galois preconditions (Proposition A.110), are stated with sketches. The Galois representations attached to cohomological automorphic representations of unitary groups [231, 67] are the expected source of the hub data.

Definition A.104. A parametrised host family $\mathcal{F}$ has **good twist data** if there is a real x_0 such that for every $x \geq x_0$ with $x > x_1(\mathcal{F})$ and every member (x, i), with $D = D_{x,i}$, $\nu = \mathrm{nch}(O_{x,i})$ and $\vartheta = \mathrm{xd}(O_{x,i})$:

1. ν satisfies the conjugation conditions: for every place w with $\mu_{\mathrm{at}}(w) \neq \mu_0$ the normalised twist packs of ν at $\bar{w}$ and w coincide, and $\epsilon(\bar{w}) \neq \epsilon(w)$ when $\bar{w} \neq w$;

2. there is a twist input $\mathcal{I}$ for D whose integral Hecke-form datum is that of ϑ such that for every place w with $\mu_{\mathrm{at}}(w) \neq \mu_0$, $\mathcal{I}$ is admissible at $i(w)$ and the twist $(\epsilon(w), Q(w), k'(w), \psi'_w)$ is good for $\mathcal{I}$ at $i(w)$.

Theorem A.105. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with threshold x_1 and good twist data, such that at every member (x,i), $x > x_1$, the member datum $D_{x,i}$ with its numerical choice has the Galois preconditions and the depth-zero tie. Then $\mathcal{F}$ has local defect data formed from its actual members as in Proposition A.106, with (Ld3) and (Ld4), together with local bound records, Ribet records, budget data, inert budget data, Galois Ribet items at every point (j,w,r,π) of every member (x,i), $x > x_1$, real numbers $c_{\mathrm{sp}}, c_{\mathrm{in}}, c_{\mathrm{sm}}, C_{\mathrm{bud}} \geq 1$ and a real x_0, such that for every $x \geq x_0$ with $x > x_1$ and every member (x,i) with finite Sha lengths there are an integer $N_{x,i} \geq 1$ with $4 \mid N_{x,i}$ and integers $S^{\mathrm{rm}}_{x,i}, \eta^{\mathrm{rm}}_{x,i} \geq 0$ for which, with $\mathcal{F}'$ the coefficient extension of $\mathcal{F}$ along $\mathcal{O}_r \subseteq \mathcal{O}_r(N_{x,i})$ at the local components of these members (and along $\mathcal{O}_r \subseteq \mathcal{O}_r$ elsewhere), the Galois Ribet items of $\mathcal{F}'$ satisfy (Ub1), (Ub2), (Ub3°), (Ub4)–(Ub10) at every point of (x,i), the Ribet records have (Rf3), and the local defect data and local bound records have (Lq2), (Lq4), (Lq5), (Lq6) and (Lq8).*

Proof. Local defect data. By Proposition A.106, $\mathcal{F}$ has local defect data formed from its actual members with (Ld3) and (Ld4), for any choice of the error allowances $\mathrm{ex}_{x,i}$; these are fixed in the last step and (Ld3), (Ld4) do not involve them. Every twisted place $w \in W_{x,i}$ is the place $w(a)$ of a twisted local index, so $\mu_{\mathrm{at}}(w) \neq \mu_0$.

Items, records and budget data. By Proposition A.107, applied to these local defect data (its hypotheses are those of the theorem), there are local bound records with $\mathrm{sl}(r,\pi) \leq \lambda^{\mathrm{BK}}_{x,i}(w)$ for $r \in R_{x,i}(j,w)$ and $\pi \in Z_r$, Ribet records with (Rf3), budget data, inert budget data, Galois Ribet items and constants $c_{\mathrm{sp}}, c_{\mathrm{in}}, c_{\mathrm{sm}}, C_{\mathrm{bud}} \geq 1$, x_0 and integers $N_{x,i}$, $S^{\mathrm{rm}}_{x,i}$, $\eta^{\mathrm{rm}}_{x,i}$ such that the Galois Ribet items of the coefficient extension $\mathcal{F}'$ along $\mathcal{O}_r \subseteq \mathcal{O}_r(N_{x,i})$ satisfy (Ub1), (Ub2), (Ub3°), (Ub4)–(Ub10). The local bound records of $\mathcal{F}'$ are those of $\mathcal{F}$ (Definition 17.4), and of their entries these conditions read only sl, lt and $\mathrm{la}_{x,i}$; the Selmer allowances $\mathrm{sa}_{x,i}$ and the auxiliary allowances $\mathrm{ga}_{x,i}$ may therefore be replaced below without affecting them or (Rf3).

Selmer allowances: (Lq2) and (Lq6). By Proposition A.108 (good twist data) there are $\nu_{x,i}(\ell) \geq 0$ and C_1 with items 1 and 2 there for all large members. Put $\mathrm{sa}_{x,i}(\ell) = \nu_{x,i}(\ell)$. For a large member with finite Sha lengths, $w \in W_{x,i}$ with $\ell = \ell_{x,i}(w) \notin G_{x,i}$, $r \in R_{x,i}(j,w)$ and $\pi \in Z_r$: since $\mu_{\mathrm{at}}(w) \neq \mu_0$, item 1 applies at w and gives

$$\mathrm{sl}(r,\pi) \leq \lambda^{\mathrm{BK}}_{x,i}(w) \leq \nu_{x,i}(\ell) + \ell^{-2} = \mathrm{sa}_{x,i}(\ell) + \ell^{-2},$$

which is (Lq2). As $\nu_{x,i} \geq 0$, item 2 gives $\sum_{\ell \in P_{x,i} \setminus G_{x,i}} \log \ell \cdot \mathrm{sa}_{x,i}(\ell) \leq \sum_{\ell} \log \ell \cdot \nu_{x,i}(\ell) \leq C_1\, f\, m\, b\, x^{2\varepsilon}$, which is (Lq6).

Auxiliary allowances: (Lq4) and (Lq8). By Proposition A.109, applied to the same local defect data, there are $\mathrm{ga}_{x,i}$ with (Lq4) and (Lq8); take these as the auxiliary allowances.

Error allowances: (Lq5). Define $\mathrm{ex}_{x,i}$ by (13.7) from $\mathrm{sa}_{x,i}$, $\mathrm{la}_{x,i}$ and $\mathrm{ga}_{x,i}$ (and $\mathrm{ex}_{x,i}(\ell) = 0$ for $\ell \notin P_{x,i}$); it takes values in $[0,\infty)$ since $\mathrm{sa}, \mathrm{ga} \geq 0$ and $\mathrm{la}_{x,i} = a_{x,i} \geq 0$ on $P_{x,i} \setminus G_{x,i}$ by (Ub9). Then (Lq5) holds by definition. Finally take for x_0 the largest of the thresholds above. □

Proposition A.106. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with threshold x_1, such that at every member (x,i), $x > x_1$, the member datum $D_{x,i}$ with its numerical choice has the Galois preconditions and the depth-zero tie. Then $\mathcal{F}$ has local defect*

data, with arbitrary error allowances, formed from its actual members: at (x,i) the twisted places are the places $w(a)$ of the twisted local indices a of $D_{x,i}$ (those with $\mu(a) \neq \mu_0$: the local type differs from the base type), with residue characteristic ℓ_a and weight the log weight times the ramification index; the local components at (j,w) are the Hecke lattices $(\mathcal{O}_a, \mathcal{Z}_a, \mathbb{T}_a, M_a)$ of the twisted indices a with $(j(a), w(a)) = (j,w)$; the good set of a is Z_a^{gd}; the kept set $\widehat{Z}_a$ contains Z_a^{gd} and no element of $Z_a \setminus Z_a^{\mathrm{gd}}$; $N_{\mathrm{lev}}(x,i)$ is the order $n_{x,i}^{\mathrm{rw}}$ of the finite group of rows of $D_{x,i}$; and these local defect data have (Ld3) and (Ld4). Moreover, at every member (x,i), $x > x_1$, kept sets $\widehat{Z}_a \subseteq \mathcal{Z}_a$ are given at every local index a of $D_{x,i}$ with $Z_a^{\mathrm{gd}} \neq \emptyset$ (for the twisted ones those above; for the others sets with the same two properties: containing Z_a^{gd} and no element of $Z_a \setminus Z_a^{\mathrm{gd}}$), such that

(Kn) ***Kept points are tame, monodromy-free and squarefree at the auxiliary primes*** *for every such a, every automorphic representation Π at a with $\mathrm{pt}_a(\Pi) \in \widehat{Z}_a$, every $k \in \{1,2\}$ and every place $w \mid g_k$ of the host field E, with n the dimension of the local parameter of Π_w and q_w the cardinality of the residue field of w: (i) for every $\sigma \in I_w$ in the wild set at w, the inertia polynomial of Π_w at σ is $(X-1)^n$; (ii) Π_w has zero monodromy; (iii) for every $\gamma \in I_w$ such that γ and the wild set at w topologically generate I_w, every root z of the inertia polynomial of Π_w at γ of multiplicity at least $\mathrm{thr}(w,n)$ satisfies $z^{q_w-1} = 1$; here $\mathrm{thr}(w,n) = 2$ if w lies above a prime p that splits completely in E and $n = 1$ or $16n^2 < p$, and $\mathrm{thr}(w,n) = 1$ otherwise.*

Proof. Sketch. The local defect data are read off the member: the twisted places, the Hecke lattices of the twisted local indices, the good sets Z_a^{gd} and the order $n_{x,i}^{\mathrm{rw}} = \prod_{k=1}^{2} |\mathrm{GL}_r(\mathbb{F}_{g_k})|^f$ of the finite group G_0 of rows, so that only the kept sets and the properties (Ld3), (Ld4) and (Kn) need an argument. The cut should come from the good idempotent e_0 of the depth-zero tie, a central idempotent of $\mathbb{C}[G_0]$ formed from the series elements of the good tuples of labels. For a finite group of order n and a central idempotent e of its complex group algebra, ne has coefficients in $\mathbb{Z}[\zeta_n]$ and therefore lifts to a central element of $\mathcal{O}[G_0]$ as soon as $\mathcal{O}$ contains a primitive n-th root of unity, without inverting n; the depth-zero tie supplies such a root in every $\mathcal{O}_a$, and a central action on M_a commuting with $\mathbb{T}_a$. By the series localisation (Ze3), e_0 acts on the type spaces of each eigensystem by 1 or by 0, so the image α of $n_{x,i}^{\mathrm{rw}} e_0$ acts by $n_{x,i}^{\mathrm{rw}}$ on the eigenvectors of the good eigensystems and by 0 off the kept set, the set of eigensystems on which e_0 does not vanish; this is (Ld4) with $N_{\mathrm{lev}}(x,i) = n_{x,i}^{\mathrm{rw}}$. The condition (Kn) is the depth-zero tie read at the kept points: a kept point has a good label, which is squarefree and avoids the paired character, and for such labels the tie gives zero monodromy, trivial action of wild inertia, and an inertia polynomial at the auxiliary places whose repeated roots come only from the paired character and so lie in the multiplicative group of the residue field, which is (iii) when the threshold is 2. For (Ld3), localisation uniqueness leaves at most one twisted index at each pair (j,w), and theta injectivity in the Galois preconditions makes $\sum_{x \in Z_a^{\mathrm{gd}}} m_a(x)$ the rank of the good part, which is at most the dimension of the image of the theta map of the twisted part; by the rank condition this equals the corresponding dimension for the base part. It remains to show that this dimension is the zero-space rank $\mathrm{rk}_j(x,i)$ recorded in the numerical data of the family; heuristically this holds because the numerical data of a member are computed from the same base part, but the identification is not among the conditions of the statement. (Ld3) and (Ld4) are among the properties read by Lemma A.113 and Theorem A.105. □

Proposition A.107. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with threshold x_1 and good twist data, such that every member datum $D_{x,i}$, $x > x_1$, with its numerical choice has the Galois preconditions and the depth-zero tie, and let local defect data*

of $\mathcal{F}$ be formed from its actual members as in Proposition A.106. Then there are local bound records with $\mathrm{sl}(r,\pi) \le \lambda^{\mathrm{BK}}_{x,i}(w)$ for $r \in R_{x,i}(j,w)$ and $\pi \in Z_r$, Ribet records, budget data with $\{2,3\} \cap (P_{x,i} \setminus G_{x,i}) \subseteq \mathcal{P}^{\mathrm{sm}}_{x,i}$, inert budget data, Galois Ribet items at every point (j,w,r,π) of every member (x,i), $x > x_1$, real numbers $c_{\mathrm{sp}}, c_{\mathrm{in}}, c_{\mathrm{sm}}, C_{\mathrm{bud}} \ge 1$ and a real x_0 such that the hypotheses of Theorem 22.1 hold: for every $x \ge x_0$ with $x > x_1$ and every member (x,i) with finite Sha lengths there are an integer $N_{x,i} \ge 1$ with $4 \mid N_{x,i}$ and integers $S^{\mathrm{rm}}_{x,i}, \eta^{\mathrm{rm}}_{x,i} \ge 0$ for which, with $\mathcal{F}'$ the coefficient extension of $\mathcal{F}$ along $\mathcal{O}_r \subseteq \mathcal{O}_r(N_{x,i})$ at the local components of these members (and along $\mathcal{O}_r \subseteq \mathcal{O}_r$ elsewhere), the Galois Ribet items of $\mathcal{F}'$ satisfy (Ub1), (Ub2), (Ub3°), (Ub4)–(Ub10) at every point of (x,i); and the Ribet records have (Rf3).

Proof. Sketch. The statement asks that the conditions (Ub1)–(Ub10) of the Ribet budget theorem 22.1 hold at the actual members, after one coefficient extension $\mathcal{O}_r \subseteq \mathcal{O}_r(N_{x,i})$ per member. The argument starts from the hub data of the members (whose existence is Proposition A.110): a supply statement for the records attached to hub data, itself resting on further member inputs (the corner design, a Selmer comparison and quantitative inputs), gives local bound records, Ribet records with (Rf3), budget data, inert budget data and Galois Ribet items with (Ub1), (Ub3°), (Ub8) and (Ub9). Given these, Lemma 22.17 shows that the same conditions, together with the local conditions (Ll1) and (Ll2), survive the coefficient extension; this step is complete. It remains to show (Ub2), (Ub4)–(Ub7′) and (Ub10) for the items of the extended family. Except for (Ub6), these are local conditions on the Galois representations at the member points: their Hodge jumps at the places above ℓ lie in windows of length at most the size H of the budget data, the representations are crystalline or de Rham with prescribed jumps, and at primes of split and of small type there are column data with the column properties. Heuristically these should hold because the jumps are read from the regular infinitesimal characters and the weights and character exponents fixed in the construction, which are polynomially bounded in x, and because local–global compatibility at $\ell = p$ [18], [58] turns the local components of the members at places above ℓ into the corresponding p-adic Hodge-theoretic conditions. Condition (Ub10), at the kept points that are not good, should follow from the non-vanishing criterion Proposition 22.7, whose first two conditions hold at such points, once the cut is formed with the kept sets of Proposition A.106. Condition (Ub6) concerns the large primes and has a different character: it needs, at each member, congruence and position data for the large primes (low jumps, base sizes, and avoidance of an exceptional set of primes), from which the vanishing of the local losses at large primes Proposition 28.1 applies; no construction of these data at the members is known yet, and this is the main missing input. The primes 2 and 3 need no separate treatment, since they are placed among the small primes $\mathcal{P}^{\mathrm{sm}}_{x,i}$ and are covered by (Ub7). □

Proposition A.108. *For a member (x,i) and a place w of the coefficient field with $\mu_{\mathrm{at}}(w) \ne \mu_0$, put $\psi_w = \mathrm{side}_{\epsilon(w)}(\psi'_w) \circ \mathrm{Art}_{i(w),Q(w)}$ and $\psi_{w,s} = \mathrm{side}_{\mathrm{side}(s)}(\psi_w)$, with the side convention of (Xh5) in the definition of cores over a stage tuple, and let $\lambda^{\mathrm{BK}}_{x,i}(w) \in [0,\infty]$ be the largest, over the signature triples s at $i(w)$ that are good for the Bloch–Kato reduction, of the normalised length of the Bloch–Kato Selmer group with the coefficient module, local conditions and action $\psi^{-1}_{w,s}\chi_s^{-1}\rho_s$ of that reduction (and $\lambda^{\mathrm{BK}}_{x,i}(w) = 0$ if there is no good signature triple at $i(w)$); and let $\ell(w)$ be the residue characteristic of w. Let $\mathcal{F}$ be a parametrised host family with slope records and defect data and with good twist data. Then there are reals $\nu_{x,i}(\ell) \ge 0$, for the members (x,i) and primes ℓ, and a real C such that for all large members:*

1. *$\lambda^{\mathrm{BK}}_{x,i}(w) \le \nu_{x,i}(\ell(w)) + \ell(w)^{-2}$ for every w with $\mu_{\mathrm{at}}(w) \ne \mu_0$;*
2. *$\sum_\ell \log \ell \cdot \nu_{x,i}(\ell) \le C\, f\, m\, b\, x^{2\varepsilon}$.*

Proof. Sketch. The argument should take for $\nu_{x,i}(\ell)$ the cyclotomic valuation $\mathrm{cyv}_\ell(E_{x,i}; tr)$ of the host field of the member for the exponent $M = tr$, where t is odd and $r = k + m - 1$ is odd: the largest k with $\chi_\ell(\sigma)^M \equiv 1 \pmod{\ell^k}$ for all $\sigma \in G_{E_{x,i}}$, so that $\ell^{\nu_{x,i}(\ell)}$ is the order of the invariants of $G_{E_{x,i}}$ on $\mathbb{Q}_\ell/\mathbb{Z}_\ell(M)$. This is the natural size of the part of a Bloch–Kato Selmer group of a twist that the choice of twist cannot remove. Item 2 is then elementary: for a host satisfying the normalisation (Hn3) of Definition 7.5 with $4M + 1 < \min(q, d)$, the cyclotomic valuation vanishes at every prime not dividing Q_S and is at most $v_\ell(Q_S) + v_\ell(M)$ otherwise, so $\sum_\ell \log \ell \cdot \nu_{x,i}(\ell) \le \log Q_S + \log M \le 2 \log n + \log t + \log r$ by the bound on Q_S in Definition 7.2, and the size condition (Lm7) of Definition 5.5 bounds this by $C \log x$, which is at most $C\, f\, m\, b\, x^{2\varepsilon}$. What this needs, and what remains to be shown, is that the host of each member datum is a host of the kind just described, with (Hn3), (Fr4) and $4tr + 1 < \min(q, d)$ as for the host of a context, and with ranks equal to those of the loss data of the member; heuristically this holds because the members are produced over contexts whose hosts have these properties (Theorem A.52). Item 1 is the Selmer clause of the goodness of the chosen twist: for each signature triple s good for the Bloch–Kato reduction there is $\varepsilon_T(s) \ge 0$, with $\sum_s \varepsilon_T(s) \le \ell^{-2}$, such that the normalised Selmer length of the twist of the representation of s by $\psi_w^{\pm 1}$ is at most $\nu_{x,i}(\ell) + \varepsilon_T(s)$. Taking the largest over the good s gives item 1, since each $\varepsilon_T(s)$ is at most ℓ^{-2}. The twist is the one already chosen with the good twist data (Definition A.104), specialised to the exponent 1, so nothing new is chosen; what remains is to verify that the goodness of a twist, unfolded, contains exactly this clause with ν_ℓ the cyclotomic valuation above. □

Proposition A.109. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with threshold x_1, such that every member datum $D_{x,i}$, $x > x_1$, with its numerical choice has the Galois preconditions and the depth-zero tie, and let local defect data of $\mathcal{F}$ be formed from its actual members as in Proposition A.106. Then there are functions $\mathrm{ga}_{x,i}$ from the set of primes to $[0, \infty)$ such that (Lq4) and (Lq8) hold.*

Proof. Sketch. The statement bounds the cut defects at the two auxiliary primes $\ell \in G_{x,i} = \{g_1, g_2\}$, where the depth-zero structure of the members is concentrated. The argument should run as follows, with absolute constants $C_T = 40$ and $C = 8C_T$. At every twisted index a over $\ell \in G_{x,i}$ one proves a paired-character bound: the local quantity $\eta_{\mathrm{loc}}(a)$ is at most $\sum_\pi m(\pi)$, over the good eigensystems π, times a sum of a polynomial in n_V, η, t, f, r, the Selmer length s_π, the lengths $\mathrm{len}_\ell(E, e)$ for $0 < e < t$, $v_\ell(h_E)$, and valuations of the discriminant and of roots of unity at the tame places. Its ingredients are: lifting exponents at the places above ℓ from the tame potentially crystalline lifting theorem A.111, in its form for Ribet fibre products at places of degree one (at the places above ℓ, $E_w = \mathbb{Q}_\ell$), applied to the twisted good points; local losses from Theorem 25.3, with the local Bloch–Kato count Theorem B.43 and local Tate duality Theorem B.29; Galois Ribet data at the index with (Gd1)–(Gd5), to be obtained from a good Galois family by a choice of idempotent and Burnside's theorem; the Ribet-type bound Theorem 22.11 with the class-number bound; and the central cut operator, which contributes $v_\ell(n^{\mathrm{rw}}_{x,i})$. That contribution is at most $2fr^2$: $v_g(|\mathrm{GL}_r(\mathbb{F}_g)|) = r(r-1)/2$, and for the other auxiliary prime g', $v_g(g'^i - 1) \le i \log g' / \log g \le 2i$ because $\log g' \le 2 \log g$, so that $v_g(n^{\mathrm{rw}}_{x,i}) \le f\big(r(r-1)/2 + r(r+1)\big) \le 2fr^2$. Three inputs remain to be established at the members: the index lemma by which the cut operator contributes exactly the term $v_\ell(n^{\mathrm{rw}}_{x,i})$ (the arithmetic of the bound $2fr^2$ is elementary, as just shown), the Galois Ribet data at the index, and, above all, the local input of the lifting step, namely that the twisted good points become crystalline over one tame Galois extension, with inertia acting through at most $n_V + 1$ characters of finite order prime to ℓ and with Hodge jumps in a prescribed window. Heuristically the last is what depth zero means: the depth-zero tie gives zero monodromy and tame inertia at the places above g_k,

and by local–global compatibility at $\ell = p$ [18], [58] the corresponding Galois representations are potentially crystalline with tame inertial type. Specialised at the hub data of the member, once its parameters are shown to be polynomially bounded in x and the multiplicity bound (Ld3) is available, the paired-character bound gives (Lq4) with $\mathrm{ga}_{x,i}(\ell) = C x^{C} \eta_{\mathrm{sz}}(x,i)$, and (Lq8) is the sum of $\log \ell \cdot \mathrm{ga}_{x,i}(\ell)$ over the two auxiliary primes, which is small against $f\, m\, b\, x^{2\varepsilon}$ because $b = e^{x^{\varepsilon}}$ grows faster than any polynomial in x. □

Proposition A.110. *Let D be a member datum and ν a numerical choice of D such that (D,ν) has the Galois preconditions. Then D has hub data.*

Proof. Sketch. Hub data consist of integral Galois representations at all eigensystems of every local index, and of their complements at the theta eigensystems, and their construction should follow the standard route from automorphic to Galois representations. For every local index a and every $x \in \mathcal{Z}_a$ choose an automorphic representation Π with $\mathrm{pt}_a(\Pi) = x$. Its base change to GL_{n_V} over E is an isobaric sum of conjugate self-dual cuspidal representations with regular infinitesimal character, so it has an attached Galois representation ρ_Π [231], [67], compatible with Π at every finite place; this is the compatibility that the admissibility of the member datum asks for. By local–global compatibility and the convention that all statements at a use one coefficient ring $\mathcal{O}_a$ with fixed embeddings, the characteristic polynomial of ρ_Π at a Frobenius element at every place split out of a is the Hecke polynomial $H_{a,x,w}$. By compactness of G_E there is a G_E-stable $\mathcal{O}_a$-lattice, which gives $\mathbf{r}_{a,x}\colon G_E \to \mathrm{GL}_{n_V}(\mathcal{O}_a)$; after semisimplification over K_a the Frobenius polynomials are unchanged, by the Brauer–Nesbitt theorem and the Chebotarev density theorem B.18, and the large-coefficient condition ensures that the irreducible constituents are defined over K_a. This is (Zh1). At a theta eigensystem $\pi \in Z_a$ the base-change list contains the entry $(1, \mu_a \otimes \eta^{-1}, t)$, a character repeated with the shifts $|\cdot|^{(t-1)/2-b}$; on the Galois side these give the characters $\chi_{a,c} = \chi_{a,1}\varepsilon_a^{1-c}$, and the provisions of the Galois preconditions assert accordingly that $\mathbf{r}_{a,\pi} \otimes K_a$ splits as a representation of dimension r plus $\chi_{a,1}, \dots, \chi_{a,t}$, and a G_E-stable lattice in the first summand gives the complement $\boldsymbol{\rho}_{a,\pi}$, which is (Zh2). What remains is to make precise, at each eigensystem of the member, which Galois representation is attached to it, compatibly with the chosen coefficient embeddings; once this is fixed, each step above is standard. □

A.9.2 Lifting and cut congruences

The quantitative tame potentially crystalline lifting theorem (Theorem A.111) is the p-adic input of the bounds at the generator primes. The cut congruence modules at the actual members are bounded for every Hecke lattice (Proposition A.112); the commuting-endomorphism hypotheses in the statement are not used in the proof. The local defect data formed from the actual members then have the properties (Ld1)–(Ld9) that the budget needs (Lemma A.113).

Theorem A.111. *Let $C \geq 40$ be an integer, p a prime, $L/\mathbb{Q}_p$ a finite extension with ring of integers O, and $K'/\mathbb{Q}_p$ a finite Galois extension with maximal unramified subfield K'_0. Put $I' = G_{K'_0}$ and choose $\mathrm{Fr} \in G_{\mathbb{Q}_p}$ inducing arithmetic Frobenius on K'_0. Let Σ be a finite set of continuous characters $I' \to L^\times$, trivial on $G_{K'}$, of finite order prime to p, containing 1, and stable under conjugation by Fr^{-1}. Define*

$$r = \max_{\chi,\chi' \in \Sigma} \min\{ s \geq 1 : (\chi/\chi')(\mathrm{Fr}^{-s} g \mathrm{Fr}^{s}) = (\chi/\chi')(g) \text{ for every } g \in I' \}.$$

Let $f : T_1 \to T_2$ be a surjective continuous O-linear $G_{\mathbb{Q}_p}$-equivariant map of finite free O-lattices.

Write $V_i = T_i[1/p]$. Suppose both V_i are crystalline over the same K', and

$$D_{\mathrm{cris},K'}(V_i) = \sum_{\chi\in\Sigma} D_{\mathrm{cris},K'}(V_i)^{I'=\chi} \qquad (i=1,2).$$

Let $N,\eta,H\in\mathbb{N}$ satisfy $N\geq 2$, $1\leq\eta<H$, and $|\Sigma|\leq N$. Suppose a finite set $U\subset\mathbb{Z}$ contains 0, satisfies

$$U\subset[-\eta,H], \qquad |U|\leq N+1,$$

and contains every labelled filtration jump of both representations. Here a filtration jump is an integer j with nonzero gr^j of the corresponding de Rham filtration, with no Hodge–Tate index shift. If

$$C(HN^2+N^3)<p,$$

then

$$p^{C[Nr+\eta(N+r)]}H^1_f(\mathbb{Q}_p,T_2)\subseteq \mathrm{im}\big(H^1_f(\mathbb{Q}_p,T_1)\to H^1_f(\mathbb{Q}_p,T_2)\big).$$

For clarity, $H^1_f(\mathbb{Q}_p,T)$ denotes the subgroup of continuous integral cohomology whose image in $H^1(G_{\mathbb{Q}_p},B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}T[1/p])$ is zero. Thus coboundaries in the integral cohomology are those of vectors in T, not arbitrary vectors in $T[1/p]$.

Proof. Outline. After a finite coefficient extension, the two representations and the kernel of f are crystalline over a common tamely ramified extension $K=F(\pi_0)$ with $\pi_0^e=-p$, $e=p^{(p-1)!}-1$, with distinguished inertia supported on at most N characters; a class of $H^1_f(\mathbb{Q}_p,T_2)$ is realised in the crystalline realisation of the exact sequence $0\to\ker f\to T_1\to T_2\to 0$, lifted to T_1 by an obstruction-lifting lemma at an explicit power of p, and turned back into an integral cocycle, which gives the inclusion with an exponent at most $7[Nr+\eta(N+r)]\leq C[Nr+\eta(N+r)]$; a descent of coefficients returns to O. □

Proposition A.112. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with threshold x_1, such that at every member (x,i), $x>x_1$, the member datum $D_{x,i}$ with its numerical choice has the Galois preconditions and the depth-zero tie. Then local defect data of $\mathcal{F}$ formed from its actual members as in Proposition A.106 have the property (Ld6).*

Proof. Let local defect data of $\mathcal{F}$ be formed from its actual members as in Proposition A.106. Let (x,i) be a member with $x>x_1$, $1\leq j\leq m$, $w\in W_{x,i}$ with $\ell=\ell_{x,i}(w)\notin G_{x,i}$, and $r\in R_{x,i}(j,w)$. By the definition of local defect data, $(\mathcal{O}_r,\mathcal{Z}_r,\mathbb{T}_r,M_r)$ is a Hecke lattice over ℓ and $Z_r\subseteq\widehat{Z}_r\subseteq\mathcal{Z}_r$. Since cut congruence modules are bounded by the congruence ideals,

$$\mathrm{df}^{\mathrm{cut}}(r)=\mathrm{df}^{\widehat{Z}_r}_{Z_r}(M_r)\leq\sum_{\pi\in Z_r}m_r(\pi)\,\mathrm{cg}^{\widehat{Z}_r}_{Z_r}(\pi)=\sum_{\pi\in Z_r}m_r(\pi)\,\mathrm{cg}(r,\pi).$$

Since $N_{\mathrm{lev}}(x,i)\geq 1$ is an integer, $v_\ell(N_{\mathrm{lev}}(x,i))\geq 0$, so each summand is at most $m_r(\pi)(\mathrm{cg}(r,\pi)+v_\ell(N_{\mathrm{lev}}(x,i)))$ (in $[0,\infty]$, with the usual conventions for these defects). This is the inequality of (Ld6) at (x,i,j,w,r); as it holds at every member with $x>x_1$, (Ld6) holds with $x_0=x_1+1$. □

Lemma A.113. *Let $\mathcal{F}$ be a parametrised host family with slope records and defect data, with threshold x_1 and good twist data, such that at every member (x,i), $x>x_1$, the member datum $D_{x,i}$ with its numerical choice has the Galois preconditions and the depth-zero tie. Then $\mathcal{F}$ has local defect data formed from its actual members as in Proposition A.106 with the properties (Ld3), (Ld4), (Ld7), (Ld8) and (Ld9).*

Proof. **Step 1: the records.** By Theorem A.105, applied to $\mathcal{F}$ (its hypotheses are exactly those of the lemma), $\mathcal{F}$ has local defect data formed from its actual members with (Ld3) and (Ld4), together with local bound records, Ribet records, budget data, inert budget data, Galois Ribet items at every point (j, w, r, π) of every member (x, i) with $x > x_1$, real numbers $c_{\mathrm{sp}}, c_{\mathrm{in}}, c_{\mathrm{sm}}, C_{\mathrm{bud}} \geq 1$ and a real x_0, such that for every $x \geq x_0$ with $x > x_1$ and every member with finite Sha lengths there are integers $N_{x,i}$, $S^{\mathrm{rm}}_{x,i}$, $\eta^{\mathrm{rm}}_{x,i}$ for which the Galois Ribet items of the coefficient extension along $\mathcal{O}_r \subseteq \mathcal{O}_r(N_{x,i})$ satisfy (Ub1), (Ub2), (Ub3°), (Ub4)–(Ub10), the Ribet records have (Rf3), and the local defect data and local bound records have (Lq2), (Lq4), (Lq5), (Lq6) and (Lq8). We fix these objects for the rest of the proof.

Step 2: the Ribet bound by prime type. The objects of Step 1 are the data required by Theorem 22.1: a family with local defect data, local bound records, Ribet records, budget data and inert budget data; Galois Ribet items at every point of every member beyond x_1; constants $c_{\mathrm{sp}}, c_{\mathrm{in}}, c_{\mathrm{sm}}, C_{\mathrm{bud}} \geq 1$; and a real x_0 beyond which, at every member with finite Sha lengths, the items of the coefficient extension along $\mathcal{O}_r \subseteq \mathcal{O}_r(N_{x,i})$ satisfy (Ub1), (Ub2), (Ub3°), (Ub4)–(Ub10). Its item 2 gives that the Ribet records have (Rf1), (Rf2), (Rf4) and (Rf5); its item 3 gives (Lq3) and (Lq7) (the latter with the constant $19C_3^3$ named there).

Step 3: lattice index bounds. With (Rf3) from Step 1, the Ribet records have (Rf1)–(Rf5), so Theorem 13.13 gives (Lq1).

Step 4: congruence ideals and auxiliary primes. The local bound records now have (Lq1) (Step 3), (Lq2) (Step 1), (Lq3) (Step 2), (Lq4) and (Lq5) (Step 1); so Theorem 13.16 gives (Ld7) and (Ld8).

Step 5: the error budget. The local bound records have (Lq5), (Lq6), (Lq8) (Step 1) and (Lq7) (Step 2); so Theorem 13.17 gives (Ld9).

The local defect data of Step 1 are formed from the actual members and have (Ld3), (Ld4) (Step 1), (Ld7), (Ld8) (Step 4) and (Ld9) (Step 5). □

A.9.3 The ties of a selected member

A selected member must be tied to the context from which it was built: its concrete context, Hida setting, words, Harish-Chandra parameters and thresholds must be the original ones (Proposition A.114). This tie is used at several places above, for instance in the engine place of a member (Proposition A.79).

Proposition A.114. *Fix any such selected family with its prescribed complete good-host witness collection, its threshold X, one admissible common constant profile $\mathbf{C}$, and one choice of*

$$0 < \varepsilon < 1/52, \qquad c_D > 397, \qquad c_h \geq \max\{402, c_2\}$$

and prescribed J, all fixed before the scale, member and word indices. Here c_2 is one threshold furnished, for $(\varepsilon, \mathbf{C}, c_D)$, by the size conditions for the threshold configurations of contexts, chosen once for these common data before c_h and before (x, u). These are the common profile and exponent choices of the selected family, not choices depending on (x, u).

For every $x \in W$ with $x > X$ and every $u \in A_x$ there are a context $K_{x,u}$, a concrete base $B_{x,u}$ whose pre-base is over $K_{x,u}$, and a stage tuple $\mathcal{X}_{x,u}$ over $B_{x,u}$, with the following identifications of objects.

1. *The context has scale (x, ε, J) and exactly $\mathbf{C}, c_D, c_h$. Its host, frame, tops, spacing set, rank and auxiliary primes are the prescribed ones at x with the member's translate u; the prescribed top tables and common profile are retained. Put $D_{x,u} = D(\mathcal{X}_{x,u})$ and let $\mathbf{F}_{x,u}$ be*

*its stage-*0 *column family. The setting determined by the member's selected tied local data is a member Hida and Poincaré setting for* $(D_{x,u}, \mathbf{F}_{x,u})$*, with its data and normalizations.*

2. *The member words, word code, Gauss primes and Harish–Chandra arrays, and those of* $D_{x,u}$*, are those of the pre-character datum of this* $K_{x,u}$*, with (Cx2) of the definition of contexts. In particular the word tuple used in (Hj5) is this tuple, not the separate row tuple. The shared row entries* v_τ, b_τ *are the context's entries.*

3. *Under the identification of the member's archimedean place set with* Φ*,*

$$k_{\mathrm{d}} = k, \qquad r_{\mathrm{f}} = m/2 - 1, \qquad f_\infty = |\Phi|, \qquad g_1^{\mathrm{aux}} = g_1, \quad g_2^{\mathrm{aux}} = g_2.$$

For every ordered pair of distinct member words, the amplitude threshold configuration of the member is the threshold configuration of this context at $(\omega, \omega'; l(\omega, \omega'))$*. Thus*

$$\Delta = \lceil x^{c_D} \rceil, \quad \varrho = \tfrac{1}{2}|d_E|^{1/[E:\mathbb{Q}]}, \quad \kappa = \pi/\varrho, \quad g = g_1 g_2, \quad \nu_\tau^{\mathrm{last}} = 0,$$

the first weights are $u_{\omega,\tau,1}$*, the nodes are* $b_{\tau,j}$*, the top flag and top weight are the top data of the context, the word vectors are* $(c(\omega)_1, \dots, c(\omega)_{k-2}, 0)$ *and likewise for* ω'*, and the norms are the same* $T(\omega), T(\omega')$*. The cut index satisfies* $1 \le l \le k-2$ *and* $c(\omega)_l \ne c(\omega')_l$*.*

These identifications hold without a condition on word sizes; the constant $C_0 = 3$ *and the word-size conditions enter only the results that use them. The hyperbolic flag rank* r_{f} *is distinct from the total row rank* $k + m - 1$*. No decay, term estimate, amplitude bound or other class duty is part of this conclusion.*

Proof. Sketch. The statement identifies objects rather than estimating them: the member of the selected family at (x, u) is to be the member produced over one concrete context, with every datum of that context carried over unchanged. For a member produced by Theorem A.52 (under the export and second-pass properties that theorem assumes) the identifications should hold by construction. That theorem forms the context $K = (x, \varepsilon, J, \mathbf{C}, c_D, c_h, \mathfrak{P}, q_0, a, b, c, \mathrm{Lv})$ from the prescribed good-host witness at x, with the translate u as its base element, the top tables a, b, the word set, the base word and an injective word code c, and it chooses the Harish-Chandra parameters and the level; the assembled datum, its stage tuple and the member are then built over this K. So the host, frame, tops, spacing set, rank and auxiliary primes of item 1 are those of the witness, and the words, code, Gauss primes and Harish-Chandra arrays of item 2 are those of the pre-character datum of K. Item 3 is the definition of the threshold configuration of a context at a pair of words, evaluated at K: the values $\Delta = \lceil x^{c_D} \rceil$, $\varrho = \frac{1}{2}|d_E|^{1/[E:\mathbb{Q}]}$, $\kappa = \pi/\varrho$, $g = g_1 g_2$ and the rank $(k, \frac{m}{2} - 1)$ over Φ are those of that definition, and because the word code is injective two distinct words differ at some coordinate $l \le k - 2$, which gives the cut index. The order of choices is compatible: the threshold c_2 attached to threshold configurations depends only on $(\varepsilon, \mathbf{C}, c_D)$, and Theorem A.52 likewise fixes its own lower bound for c_h after c_D, so that c_h can be taken above both before the scale and the member are chosen. The selection retains these objects, since it follows from the construction of Proposition A.1 that the selected members are the produced members themselves, with the setting determined by their selected tied local data; there is no analytic content here, every datum named being an input or an output of the producer. The good-host collection over which the family is built is, however, chosen in the course of that construction. Whether the identifications can instead be obtained over a collection of good-host data fixed in advance, as (TCS) prescribes, is open. The identification is what Proposition A.79 uses to make the cut of the engine place non-empty, and it is also read by the loop-row and amplitude bounds at the members. □

Sketch: the remaining ties of a selected member. Besides its context, a selected member must be tied to its Poincaré frame, to its counting operator and to its Gauss ranks, and the members of a parametrised host family must carry the member fields that the family's definition names. Each tie asserts that an object constructed in one part of the construction is literally the object used in another: the member built by the two passes of Section A.5 is the one whose frame enters the Gram blocks of Section A.6, and whose counting operator and level ranks enter the main row. Heuristically these ties hold by construction, because the selection of Section A.1 retains the member data, objects and comparison maps of the construction and replaces none of them (Proposition A.1); what remains is to read each field of the later statements off the retained data, with the constants of the context fixed before the thresholds that depend on them.

B Quoted results

This appendix states the results that the paper quotes from the literature, 174 in all, grouped by field in Subsections B.1–B.10. In place of a proof, each statement is followed by its sources and, for most statements, the translation from the notation of the sources to ours. A quoted result is used as it is stated in its source; where the statement printed here is a special case of the result in the source or a standard reformulation of it, the note after the statement says so. The opening of each subsection says what is quoted there and in which Parts and sections it is used. Subsection B.11 lists the computations cited in proofs.

B.1 Analytic number theory

The first five statements are the classical facts on Dirichlet L-functions: the analytic continuation of $L(s,\chi)$ with the Laurent expansion of ζ at 1 (Theorem B.1), the non-vanishing of $L(1,\chi)$ for non-principal χ (Theorem B.2), Chebyshev's bound $\prod_{p\le n} p < 4^n$ (Lemma B.3), and the functional equations of ζ and of $L(s,\chi)$ for primitive χ (Theorem B.4, Theorem B.12). All five are used in Part I, three of them again in Part II, and Chebyshev's bound also in Part III. Part VII uses the functional equations, the non-vanishing and Chebyshev's bound, together with the values of Dirichlet L-functions at non-positive integers as generalised Bernoulli numbers (Theorem B.11), for the minus part (Section 37). The other six statements concern primes in arithmetic progressions and a possible exceptional real zero: the Landau–Page theorem, an effective bound for a real zero, Montgomery's zero-density estimate, Linnik's theorem on the least prime, and the theorems of Thorner and Zaman that carry the exceptional zero as a secondary main term, in the prime number theorem for progressions and in the Chebotarev density theorem. They are used in Part II, for almost all quadratic twists (Section 10) and for good hosts (Section 11).

Theorem B.1. *Let $N \ge 1$ and let χ be a Dirichlet character modulo N. The function defined for $\operatorname{Re} s > 1$ by $\sum_{n\ge1} \chi(n) n^{-s}$ extends to a meromorphic function $L(s,\chi)$ on $\mathbb{C}$. If $\chi \ne \chi_{0,N}$, this function is entire. If $\chi = \chi_{0,N}$, its only pole is a simple pole at $s = 1$, with residue $\varphi(N)/N$. In particular $\zeta(s)$ is holomorphic on $\mathbb{C}\setminus\{1\}$, and $\zeta(s) - \frac{1}{s-1}$ extends to an entire function whose value at $s = 1$ is γ:*

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|) \qquad (s \to 1).$$

Proof. [180] (the analytic continuation of $\zeta(s)$ with its Laurent expansion at $s = 1$, and of $L(s,\chi)$), [75], [260]. Notation as in the references: $L(s,\chi)$ for a character χ modulo N, with

Euler factors 1 at the primes dividing N; for $\chi = \chi_{0,N}$ one has $L(s,\chi_{0,N}) = \zeta(s)\prod_{p\mid N}(1-p^{-s})$, whence the residue $\prod_{p\mid N}(1-p^{-1}) = \varphi(N)/N$. $\square$

Theorem B.2. *Let $N \geq 1$ and let χ be a Dirichlet character modulo N with $\chi \neq \chi_{0,N}$. Then $L(1,\chi) \neq 0$.*

Proof. Dirichlet's theorem, [180] (the non-vanishing of $L(1,\chi)$ in the proof of Dirichlet's theorem on primes in arithmetic progressions), [75]. Notation as in the references; $L(1,\chi)$ is the value at $s = 1$ of the entire function of Theorem B.1, equal to the conditionally convergent series $\sum_{n\geq 1}\chi(n)/n$. $\square$

Lemma B.3. *For every integer $n \geq 1$, $\prod_{p\leq n} p < 4^n$, the product running over the primes $p \leq n$. Consequently, for every real $x \geq 0$,*

$$\sum_{p\leq x} \log p \leq x \log 4.$$

Proof. See [118, Chapter XXII] (the bound $\vartheta(n) < 2n\log 2$, by the argument of Erdős with the middle binomial coefficient); see also [180] (Chebyshev estimates, Chapter 2). The second form follows from the first: for $x \geq 1$ the sum equals $\log\prod_{p\leq\lfloor x\rfloor} p < \lfloor x\rfloor \log 4 \leq x\log 4$, and for $0 \leq x < 2$ it is 0. $\square$

Theorem B.4. *The function $\xi(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$ is holomorphic on $\mathbb{C}\setminus\{0,1\}$ and satisfies*

$$\xi(1-s) = \xi(s) \qquad (s \in \mathbb{C}\setminus\{0,1\}).$$

Moreover $\zeta(0) = -\tfrac{1}{2}$.

Proof. Riemann's functional equation, [180] (Chapter 10), [75] (Chapter 8), [260] (Chapter II). Notation as in the references; ξ here is the completed zeta function without the factor $\tfrac{1}{2}s(s-1)$ of Riemann's normalisation, so that it has simple poles at $s = 0$ and $s = 1$. The value $\zeta(0) = -\tfrac{1}{2}$ follows from the functional equation and the residue 1 of ζ at $s = 1$. $\square$

Theorem B.5. *There is an absolute constant $c_{\mathrm{P}} > 0$ such that for every $q \geq 3$, every real primitive Dirichlet character χ modulo q and every real β with $0 < \beta < 1$ and $L(\beta,\chi) = 0$,*

$$1 - \beta \geq \frac{c_{\mathrm{P}}}{q^{1/2}\log^2 q}.$$

Proof. [75] (Chapter 14: the bound $\beta < 1 - c\,q^{-1/2}\log^{-2} q$, from the lower bound $L(1,\chi) \gg q^{-1/2}$ given by the class number formula and the bound $L'(\sigma,\chi) \ll \log^2 q$ for $1 - 1/\log q \leq \sigma \leq 1$); an explicit form with $c_{\mathrm{P}} = 40$ for $q \geq 3$ is in M. A. Bennett, G. Martin, K. O'Bryant and A. Rechnitzer, *Explicit bounds for primes in arithmetic progressions*, Illinois J. Math. 62 (2018), Proposition 1.11. The notation is that of the references. $\square$

Theorem B.6. *There is an absolute constant $c_{\mathrm{LP}} > 0$ such that for every real $Z \geq 3$ there is at most one real primitive Dirichlet character $\chi \neq \mathbf{1}$ of modulus at most Z for which $L(s,\chi)$ has a real zero β with*

$$1 - \frac{c_{\mathrm{LP}}}{\log Z} < \beta < 1.$$

Proof. This is the theorem of Landau and Page; see [75] (Chapter 14), [180] (Chapter 11), [134] (Chapter 5), and E. Landau (1918) and A. Page, *On the number of primes in an arithmetic progression*, Proc. London Math. Soc. (2) 39 (1935). It follows from Landau's inequality: for distinct real primitive characters $\chi_1 \neq \chi_2$ (both $\neq \mathbf{1}$) of moduli q_1, q_2 with real zeros β_1, β_2, one has $\min(\beta_1, \beta_2) < 1 - c/\log(q_1 q_2)$; take $c_{\mathrm{LP}} = c/2$. The notation is that of the references, and the constant is effective. $\square$

Theorem B.7. *For an integer $M \geq 2$, an integer b and real X let $\vartheta(X; M, b) = \sum \log p$, the sum over the primes $p \leq X$ with $p \equiv b \pmod M$.*

1. *For every integer $M \geq 2$ the product $\prod_{\chi \bmod M} L(s, \chi)$ has no zero in the region $\operatorname{Re} s \geq 1 - 1/(13 \log(M(|\operatorname{Im} s| + 3)))$, except possibly one real zero $\beta_1 < 1$. If it exists, it is called the* exceptional zero modulo M*; there is then exactly one real non-principal character χ_1 modulo M with $L(\beta_1, \chi_1) = 0$.*

2. *There are absolute, effectively computable constants $c_4 > 0$ and A_4 such that for every integer $M \geq 2$, every integer b with $\gcd(b, M) = 1$ and every real $X \geq M^{12}$,*

$$\Big|\vartheta(X; M, b) - \frac{X - \chi_1(b) X^{\beta_1}/\beta_1}{\varphi(M)}\Big| \leq A_4\Big(e^{-c_4 \log X/\log M} + e^{-c_4 (\log X)^{3/5} (\log\log X)^{-1/5}}\Big) \frac{X - \chi_1(b) X^{\beta_1}/\beta_1}{\varphi(M)},$$

where the term $\chi_1(b) X^{\beta_1}/\beta_1$ is omitted (in both places) when there is no exceptional zero modulo M.

Proof. [259]: item 1 is the statement on the exceptional zero in Section 1.1 (from McCurley's zero-free region for $\prod_{\chi \bmod q} L(s, \chi)$), item 2 is Corollary 1.4 with Remark 1.5 (numbering of arXiv:2108.10878v2). Notation: $q \mapsto M$, $a \mapsto b$, $x \mapsto X$; the left side of Corollary 1.4 is $\vartheta(X; M, b)$; its main term $\lambda x/\varphi(q)$ with $\lambda = \lambda(x, q, a, x) = 1 - \chi_1(a) x^{\beta_1 - 1}/\beta_1$ (Remark 1.5; $\lambda = 1$ if β_1 does not exist) is $(X - \chi_1(b) X^{\beta_1}/\beta_1)/\varphi(M)$; the implied constant of its O-term is A_4. That $\beta_1 < 1$ is the classical non-vanishing of $L(s, \chi)$ on $\operatorname{Re} s \geq 1$. $\square$

Theorem B.8. *For a Dirichlet character χ and reals α, T let $N(\alpha, T, \chi)$ be the number of zeros ρ of $L(s, \chi)$, counted with multiplicity, with $\alpha \leq \operatorname{Re} \rho \leq 1$ and $|\operatorname{Im} \rho| \leq T$. There is an absolute constant C_{M} such that for all $\frac{1}{2} \leq \alpha < 1$, $Q \geq 1$ and $T \geq 2$,*

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{\chi \bmod q} N(\alpha, T, \chi) \leq C_{\mathrm{M}} \, (Q^2 T)^{3(1-\alpha)/(2-\alpha)} \, (\log(QT))^9,$$

where $\sum^*$ runs over the primitive characters modulo q.

Proof. [179, Theorem 12.2]; the large sieve density estimates go back to [35]. The notation is that of the reference: $\sum^*$ runs over primitive characters, and $N(\alpha, T, \chi)$ is the number of zeros in the rectangle $\alpha \leq \sigma \leq 1$, $|t| \leq T$. $\square$

Theorem B.9. *There is an absolute real number $A \geq 1$ such that, for every integer $M > 2$ and every integer a with $\gcd(a, M) = 1$, there is a prime p satisfying*

$$p \equiv a \pmod M, \qquad p \leq M^A.$$

Proof. This is Linnik's theorem on the least prime in an arithmetic progression [163]. Here M denotes the modulus, a a reduced residue class, and A an absolute exponent. $\square$

Theorem B.10. *Let L/F be a Galois extension of number fields with group G, let $C \subseteq G$ be a conjugacy class, and for $x \geq 2$ let $\pi_C(x) = \pi_C(x, L/F)$ be the number of prime ideals $\mathfrak{p}$ of F with $\mathrm{N}_{F/\mathbb{Q}}\mathfrak{p} \leq x$ that are unramified in L and whose Artin symbol is C. Let $H \subseteq G$ be an abelian subgroup with $C \cap H \neq \varnothing$, choose $g_C \in C \cap H$, let $K = L^H$, and identify each character χ of $H = \mathrm{Gal}(L/K)$ with a primitive Hecke character of K of conductor $\mathfrak{f}_\chi$ (class field theory). Put*

$$\mathcal{Q} = \max_{\chi \in \widehat{H}} \mathrm{N}_{K/\mathbb{Q}}\mathfrak{f}_\chi, \qquad Q = D_K\, \mathcal{Q}\, n_K^{\,n_K}, \qquad \mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

1. *At most one real character $\chi_1 \in \widehat{H}$ has a Hecke L-function $L(s, \chi_1, L/K)$ with a real zero β_1 such that $1 - \frac{1}{8 \log Q} < \beta_1 < 1$; if it exists, β_1 is called the Landau–Siegel zero. Put $\theta_1 = \chi_1(g_C)$ if β_1 exists and $\theta_1 = 0$ otherwise.*

2. *There are absolute, effectively computable constants $c_2, c_3, A_5 > 0$ such that, if $x \geq Q^{c_2}$,*

$$\pi_C(x) = \frac{|C|}{|G|}\big(\mathrm{Li}(x) - \theta_1 \,\mathrm{Li}(x^{\beta_1})\big)\big(1 + E\big),$$

$$|E| \leq A_5\Big(\exp\Big[-\frac{c_3 \log x}{\log Q}\Big] + \exp\Big[-\frac{(c_3 \log x)^{1/2}}{n_K^{1/2}}\Big]\Big),$$

   the term $\theta_1 \,\mathrm{Li}(x^{\beta_1})$ being omitted if β_1 does not exist.

3. *(Stark's bound.) There is an absolute constant $c_S > 0$ such that, if $\chi \in \widehat{H}$ is real and $L(\beta, \chi, L/K) = 0$ for a real $\beta = 1 - \lambda/\log Q$ with $\lambda > 0$, then $\lambda \geq c_S Q^{-2}$.*

Proof. [258]: item 1 is the statement attributed to Stark in Section 1.2 (before Theorem 1.4), together with Theorem 3.1 (the exceptional zero is real, simple and attached to a real Hecke character); item 2 is Theorem 1.4 (with Theorem 1.1 for the constants c_2, c_3; A_5 is the implied constant of its O-term); item 3 is Theorem 3.3 (Stark's bound, from Stark, Invent. Math. 23 (1974), Theorem 1'). The numbering is that of arXiv:1803.02823v3. The notation is that of the source, with the implied constants named. $\square$

Theorem B.11. *Let $f \geq 1$ and let χ be a primitive Dirichlet character modulo f (for $f = 1$, $\chi = \chi_{0,1}$ and $L(s, \chi) = \zeta(s)$). For every integer $k \geq 1$,*

$$L(1 - k, \chi) = -\frac{B_{k,\chi}}{k}, \qquad B_{k,\chi} = f^{\,k-1} \sum_{a=1}^{f} \chi(a)\, B_k\Big(\frac{a}{f}\Big),$$

where $L(1-k, \chi)$ is the value of the meromorphic continuation, which is holomorphic at $1 - k$.

Proof. [266], Chapter 4, Theorem 4.2, with Proposition 4.1 for the expression of $B_{k,\chi}$ through Bernoulli polynomials. Washington's n is our k; for $f = 1$ his $B_{k,1}$ is $B_k(1)$, so that $\zeta(0) = -B_1(1) = -\frac{1}{2}$. $\square$

Theorem B.12. *Let $q \geq 3$ and let χ be a primitive Dirichlet character modulo q. Let $a \in \{0, 1\}$ with $\chi(-1) = (-1)^a$, let $\tau(\chi) = \sum_{m=1}^{q} \chi(m) e^{2\pi i m/q}$ be its Gauss sum, and put*

$$\xi(s, \chi) = \Big(\frac{q}{\pi}\Big)^{(s+a)/2} \Gamma\Big(\frac{s + a}{2}\Big) L(s, \chi).$$

Then $\xi(s,\chi)$ is entire, $|\tau(\chi)| = q^{1/2}$, and with $W(\chi) = \tau(\chi)/(i^a q^{1/2})$, so that $|W(\chi)| = 1$,

$$\xi(s,\chi) = W(\chi)\,\xi(1-s,\overline{\chi}) \qquad (s \in \mathbb{C}).$$

Proof. [75] (Chapter 9, the functional equation in the form $\xi(1-s,\overline{\chi}) = \frac{i^a q^{1/2}}{\tau(\chi)}\xi(s,\chi)$, and Chapter 2 for $|\tau(\chi)| = q^{1/2}$), [180] (Chapter 10). Notation as in the references, except that the root number is written $W(\chi) = \tau(\chi)/(i^a q^{1/2})$, the inverse of Davenport's factor $i^a q^{1/2}/\tau(\chi)$ (the two have modulus 1). □

B.2 Algebraic number theory and class field theory

From class field theory and the arithmetic of number fields the paper quotes ray class fields (Theorem B.13), used for good hosts in Part II; the unit theorem and the conductor–discriminant formula (Theorem B.14, Theorem B.15), used for the joint packet supply in Part IV (Section 19); decomposition in cyclotomic fields and global reciprocity (Theorem B.16, Theorem B.17), used in the Ribet budget theorem (Section 22); and the Chebotarev density theorem (Theorem B.18), used in Parts V and VII and in Appendix E. The arithmetic of cyclotomic fields, namely the theorem of von Staudt and Clausen, Gauss sums with Stickelberger's factorisation, the Davenport–Hasse relation, Stickelberger's theorem and reciprocity for the Brauer group of $\mathbb{Q}$, enters the minus part of the Sha bound (Section 37); the theorem of von Staudt and Clausen is used again in Section 51 and Stickelberger's theorem in Section 44. Steinitz's classification of finitely generated modules over a Dedekind domain and Borel's ranks of the K-groups of a ring of integers (Theorem B.24, Theorem B.25) are used for the second cohomology of Tate twists (Section 40). The characters of CM fields of Barnet-Lamb, Gee, Geraghty and Taylor (Lemma B.26) are used in Parts VIII and IX, and the Hasse norm theorem and Landherr's classification of hermitian forms (Theorem B.27, Theorem B.28) in Appendix D (Section D.4).

Theorem B.13. *Let K be a number field and $\mathfrak{m}$ a non-zero ideal of $\mathcal{O}_K$. Let $I_K(\mathfrak{m})$ be the group of fractional ideals of K coprime to $\mathfrak{m}$, and $P_K(\mathfrak{m})$ its subgroup of principal ideals (α) with $\alpha \equiv 1 \pmod{}^{\times}\mathfrak{m}$. Put $\mathrm{Cl}_K(\mathfrak{m}) = I_K(\mathfrak{m})/P_K(\mathfrak{m})$, let h_K be the class number of K and $\mathcal{O}_{K,\mathfrak{m}}^{\times}$ the group of units $u \in \mathcal{O}_K^{\times}$ with $u \equiv 1 \pmod{\mathfrak{m}}$.*

1. *$\mathrm{Cl}_K(\mathfrak{m})$ is finite, of order*
$$|\mathrm{Cl}_K(\mathfrak{m})| = \frac{h_K\,|(\mathcal{O}_K/\mathfrak{m})^{\times}|}{[\mathcal{O}_K^{\times} : \mathcal{O}_{K,\mathfrak{m}}^{\times}]}.$$
2. *There is a finite abelian extension L/K, the ray class field of K modulo $\mathfrak{m}$, unramified at every prime ideal of K not dividing $\mathfrak{m}$, such that the Artin map $\mathfrak{p} \mapsto \left(\frac{L/K}{\mathfrak{p}}\right)$ on the prime ideals $\mathfrak{p} \nmid \mathfrak{m}$, extended multiplicatively to $I_K(\mathfrak{m})$, is surjective onto $\mathrm{Gal}(L/K)$ with kernel $P_K(\mathfrak{m})$. In particular a prime ideal $\mathfrak{p} \nmid \mathfrak{m}$ has trivial Artin symbol in L/K if and only if $\mathfrak{p} \in P_K(\mathfrak{m})$.*
3. *Through this isomorphism every character of $\mathrm{Gal}(L/K)$ is a character of $\mathrm{Cl}_K(\mathfrak{m})$, and the primitive Hecke character inducing it has conductor dividing $\mathfrak{m}$; in particular its norm is at most $\mathrm{N}\mathfrak{m}$.*

Proof. This is classical class field theory (Takagi's existence theorem and Artin's reciprocity law in ideal-theoretic form); see [190], Ch. VI. Item 1 is the exact sequence for the ray class group

modulo $\mathfrak{m}$ (with a modulus without infinite places); item 2 is the existence of the ray class field and the reciprocity isomorphism $\mathrm{Cl}_K(\mathfrak{m}) \cong \mathrm{Gal}(L/K)$ given by the Artin symbol; item 3 is the definition of the conductor of a character of $\mathrm{Cl}_K(\mathfrak{m})$, the smallest modulus through which it factors, which divides $\mathfrak{m}$. Neukirch writes $J^{\mathfrak{m}}$, $P^{\mathfrak{m}}$ and $\mathrm{Cl}_K^{\mathfrak{m}}$ for $I_K(\mathfrak{m})$, $P_K(\mathfrak{m})$ and $\mathrm{Cl}_K(\mathfrak{m})$. $\square$

Theorem B.14. *Let K be a number field with r_1 real embeddings and r_2 conjugate pairs of complex embeddings. Its roots of unity form a finite cyclic group $\mu(K)$, and*

$$\mathcal{O}_K^{\times} \simeq \mu(K) \times \mathbb{Z}^{r_1+r_2-1}$$

as abelian groups.

Proof. This is Dirichlet's unit theorem, including the torsion subgroup identification; see [190, Chapter I]. $\square$

Theorem B.15. *Let L/K be a finite abelian extension of number fields with Galois group G. For each $\chi \in \widehat{G} = \mathrm{Hom}(G, \mathbb{C}^{\times})$, let $\mathfrak{f}_0(\chi)$ be the finite part of the conductor of the finite-order Hecke character obtained by composing χ with the Artin map for L/K. Then the relative discriminant ideal is*

$$\mathfrak{d}_{L/K} = \prod_{\chi \in \widehat{G}} \mathfrak{f}_0(\chi).$$

For $K = \mathbb{Q}$, this says that the absolute discriminant of L is the product of the positive integer conductors of the corresponding primitive Dirichlet characters.

Proof. This is the conductor–discriminant formula, [173, Chapter V, Theorem 3.27]. The ideal denoted $\mathrm{disc}(L/K)$ there is $\mathfrak{d}_{L/K}$ here, and its factor $f_0(\chi \circ \phi_{L/K})$ is the finite conductor ideal $\mathfrak{f}_0(\chi)$. Over $\mathbb{Q}$, the norm of that ideal is the conductor of the corresponding primitive Dirichlet character, so taking ideal norms gives the last assertion. $\square$

Theorem B.16. *Let $n \geq 1$, $L = \mathbb{Q}(\zeta_n)$, and identify $\mathrm{Gal}(L/\mathbb{Q})$ with $(\mathbb{Z}/n\mathbb{Z})^{\times}$ by $\sigma_u(\zeta_n) = \zeta_n^u$. For a rational prime p, write $n = p^a n'$, $p \nmid n'$. At any prime of L above p, the inertia and decomposition groups are respectively*

$$I_p = \{u : u \equiv 1 \pmod{n'}\}, \qquad D_p = \{u : u \equiv p^b \pmod{n'} \text{ for some } b \geq 0\}.$$

In particular the ramification index is $\varphi(p^a)$; $L/\mathbb{Q}$ is unramified outside n.

For every intermediate field $\mathbb{Q} \subseteq E \subseteq L$, restriction sends these groups onto the inertia and decomposition groups at the restricted prime of E. Thus those groups are the images of I_p, D_p in $\mathrm{Gal}(E/\mathbb{Q})$. All notation is local.

Proof. These are the classical cyclotomic decomposition formulas. The p-power cyclotomic factor is totally ramified at p; the prime-to-p factor is unramified, with arithmetic Frobenius acting by $\zeta_{n'} \mapsto \zeta_{n'}^p$. The Chinese remainder identification gives the displayed groups. Restriction of decomposition and inertia groups in a finite Galois tower gives the final assertion. See [175, §6, Proposition 6.2 and Theorem 6.4]. $\square$

Theorem B.17. *Let L/F be a finite abelian extension of number fields. There is a continuous surjective homomorphism*

$$\mathrm{rec}_{F,L} : C_F \longrightarrow \mathrm{Gal}(L/F), \qquad \ker \mathrm{rec}_{F,L} = N_{L/F} C_L.$$

Its pullback to $\mathbb{A}_F^\times$ kills $F^\times$. For every place v of F and $w \mid v$, its restriction to $F_v^\times$ is the local reciprocity map into $D_w = \mathrm{Gal}(L_w/F_v)$, followed by inclusion in $\mathrm{Gal}(L/F)$; its local kernel is $N_{L_w/F_v} L_w^\times$. At finite v, local units map onto inertia. At an unramified finite place, units map to 1 and a uniformiser maps to arithmetic Frobenius. At a real place becoming complex, positive elements map to 1 and negative elements to complex conjugation; at a split real or complex place the local map is trivial.

Proof. This is global Artin reciprocity and its local compatibility: [173, Chapter V, Proposition 5.2 and Theorem 5.3; Chapter I, Theorem 1.1 and 1.6]. The real normalization follows from $N_{\mathbb{C}/\mathbb{R}}\mathbb{C}^\times = \mathbb{R}_{>0}$. □

Theorem B.18. *Let L/F be a finite Galois extension of number fields and $C \subseteq \mathrm{Gal}(L/F)$ a subset stable under conjugation. Then the set of finite places v of F unramified in L whose Frobenius class in $\mathrm{Gal}(L/F)$ is contained in C has Dirichlet density $\#C/[L:F]$.*

Proof. [65]; in the form stated (Dirichlet density): [190], Chapter VII, Theorem 13.4. For a conjugation-stable C apply the theorem to each conjugacy class contained in C and add the densities. □

Theorem B.19. *Let the Bernoulli numbers $B_j \in \mathbb{Q}$ be defined by $t/(e^t-1) = \sum_{j\ge 0} B_j t^j/j!$. Then $B_0 = 1$, $B_1 = -\tfrac{1}{2}$, $B_j = 0$ for every odd $j \ge 3$, and for every even $j \ge 2$*

$$B_j + \sum_{\substack{p \text{ prime} \\ (p-1)\mid j}} \frac{1}{p} \in \mathbb{Z}.$$

Proof. [266], Chapter 5, Theorem 5.10 (von Staudt–Clausen); the values B_0, B_1 and the vanishing for odd $j \ge 3$ follow from the generating function, since $t/(e^t-1) + t/2$ is even ([266], Chapter 4). Notation as in the reference. □

Theorem B.20. *Let $m \ge 2$ be an integer, $K = \mathbb{Q}(\mu_m) \subset \overline{\mathbb{Q}}$, and for $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ let $\sigma_t \in \mathrm{Gal}(K/\mathbb{Q})$ be the automorphism with $\sigma_t(\zeta) = \zeta^{\tilde t}$ for $\zeta \in \mu_m$, where $\tilde t \in \{1,\dots,m-1\}$ is the representative of t. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ not dividing m, p its residue characteristic, $k = \mathcal{O}_K/\mathfrak{p}$ and $q = |k|$, so that $m \mid q-1$. Let $\chi_\mathfrak{p} \colon k^\times \to \mu_m$ be the m-th power residue character: $\chi_\mathfrak{p}(x \bmod \mathfrak{p})$ is the element of μ_m congruent to $x^{(q-1)/m}$ modulo $\mathfrak{p}$, for $x \in \mathcal{O}_K \setminus \mathfrak{p}$. Fix a primitive p-th root of unity $\zeta_p \in \overline{\mathbb{Q}}$ and put, for $j \in \mathbb{Z}$,*

$$\mathrm{g}_\mathfrak{p}(j) = \sum_{c \in k^\times} \chi_\mathfrak{p}(c)^{-j}\, \zeta_p^{\mathrm{Tr}_{k/\mathbb{F}_p}(c)} \in \mathbb{Q}(\mu_{mp}).$$

Then:

1. *if $m \nmid j$, then $\mathrm{g}_\mathfrak{p}(j)\,\mathrm{g}_\mathfrak{p}(-j) = \chi_\mathfrak{p}(-1)^j\, q$;*
2. *$\mathrm{g}_\mathfrak{p}(1)^m \in K$, and as ideals of $\mathcal{O}_K$*

$$\mathrm{g}_\mathfrak{p}(1)^m\, \mathcal{O}_K = \prod_{t \in (\mathbb{Z}/m\mathbb{Z})^\times} \sigma_t^{-1}(\mathfrak{p})^{\tilde t}.$$

Proof. [266], Chapter 6 (Gauss sums and Stickelberger's theorem; item 1 is the classical evaluation $g(\alpha)g(\alpha^{-1}) = \alpha(-1)q$ for a nontrivial multiplicative character α of a finite field of q elements, here $\alpha = \chi_{\mathfrak{p}}^{-j}$; item 2 is Stickelberger's factorisation $(g^m) = \mathfrak{p}^{m\theta}$ with $\theta = \sum_t \{\tilde{t}/m\}\sigma_t^{-1}$, so that $m\theta = \sum_t \tilde{t}\,\sigma_t^{-1}$). Notation: the Gauss sum attached to $\mathfrak{p}$ in the references is $\mathrm{g}_{\mathfrak{p}}(1)$ here, the sum over $k^\times$ being the sum over k with the convention that the character vanishes at 0; texts differ in naming the power residue character or its inverse, and the normalisation stated here is the one in which the character $\chi_{\mathfrak{p}}^{-1}$ produces the ideal $\mathfrak{p}^{\theta}$. $\square$

Theorem B.21. *Let k be a finite field of characteristic p, k'/k a finite extension of degree s, $\zeta_p \in \overline{\mathbb{Q}}$ a primitive p-th root of unity, and for a finite field $\kappa \supseteq \mathbb{F}_p$ and a character $\alpha\colon \kappa^\times \to \overline{\mathbb{Q}}^\times$ put $\mathrm{g}_\kappa(\alpha) = \sum_{c\in\kappa^\times} \alpha(c)\,\zeta_p^{\mathrm{Tr}_{\kappa/\mathbb{F}_p}(c)}$. If α is a nontrivial character of $k^\times$, then*

$$-\mathrm{g}_{k'}(\alpha\circ \mathrm{N}_{k'/k}) = \bigl(-\mathrm{g}_k(\alpha)\bigr)^s.$$

Proof. The Davenport–Hasse relation (H. Davenport and H. Hasse, *Die Nullstellen der Kongruenzzetafunktionen in gewissen zyklischen Fällen*, J. Reine Angew. Math. 172 (1935)); see also [266], Chapter 6, and the chapter on Gauss and Jacobi sums of Ireland and Rosen, *A Classical Introduction to Modern Number Theory*. Notation as in the references, with the additive character $c \mapsto \zeta_p^{\mathrm{Tr}_{\kappa/\mathbb{F}_p}(c)}$ of κ; the trace of k' is the trace of k composed with $\mathrm{Tr}_{k'/k}$. $\square$

Theorem B.22. *Let ℓ be an odd prime and $R \geq 1$. For a place v of $\mathbb{Q}$ write $\mathrm{res}_v\colon H^2(G_{\mathbb{Q}}, \mu_{\ell^R}) \to H^2(G_{\mathbb{Q}_v}, \mu_{\ell^R})$ for the restriction to a decomposition group at v. Then:*

1. *$H^2(G_{\mathbb{R}}, \mu_{\ell^R}) = 0$, and for every prime p the map $\mathrm{inv}_p\colon H^2(G_{\mathbb{Q}_p}, \mu_{\ell^R}) \to \mathbb{Q}/\mathbb{Z}$, the composite of the map to the Brauer group $\mathrm{Br}(\mathbb{Q}_p)$ induced by $\mu_{\ell^R} \subset \overline{\mathbb{Q}}_p^\times$ with the Hasse invariant, is injective;*
2. *for every $\xi \in H^2(G_{\mathbb{Q}}, \mu_{\ell^R})$, $\mathrm{res}_p(\xi) = 0$ for all but finitely many primes p, and*

$$\sum_p \mathrm{inv}_p(\mathrm{res}_p\,\xi) = 0.$$

In particular, if $\mathrm{res}_v(\xi) = 0$ for every place $v \neq v_0$, then $\mathrm{res}_{v_0}(\xi) = 0$.

Proof. Kummer theory and Hilbert's Theorem 90 identify $H^2(G_K, \mu_{\ell^R})$ with $\mathrm{Br}(K)[\ell^R]$ for $K = \mathbb{Q}, \mathbb{Q}_p, \mathbb{R}$, compatibly with restriction [222]; the Hasse invariant $\mathrm{Br}(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z}$ is an isomorphism, $\mathrm{Br}(\mathbb{R})$ has order 2, and the sequence $0 \to \mathrm{Br}(\mathbb{Q}) \to \bigoplus_v \mathrm{Br}(\mathbb{Q}_v) \to \mathbb{Q}/\mathbb{Z} \to 0$ is exact (Albert, Brauer, Hasse, Noether) [190], [172]. Item 1: $H^2(G_{\mathbb{R}}, \mu_{\ell^R})$ is killed by 2 and by ℓ^R. Item 2 is the exactness in the middle restricted to ℓ^R-torsion. The last sentence follows from item 2 and the injectivity in item 1. Notation as in the references. $\square$

Theorem B.23. *Let $Q \geq 3$ be an integer with $Q \not\equiv 2 \pmod 4$, $K = \mathbb{Q}(\zeta_Q)$, $G = \mathrm{Gal}(K/\mathbb{Q})$, and for b prime to Q let $\sigma_b \in G$ be the automorphism with $\sigma_b(\zeta_Q) = \zeta_Q^b$. Put*

$$\theta_Q = \sum_{\substack{1\leq b\leq Q\\ \gcd(b,Q)=1}} \Bigl\{\frac{b}{Q}\Bigr\}\sigma_b^{-1} \in \mathbb{Q}[G],$$

where $\{y\}$ is the fractional part of y. Then for every integer a prime to Q the element $(a-\sigma_a)\theta_Q$ lies in $\mathbb{Z}[G]$ and annihilates the ideal class group of K.

Proof. [266], Chapter 6, Lemma 6.9 (integrality of $(a-\sigma_a)\theta_Q$) and Theorem 6.10 (Stickelberger's theorem: if $\beta \in \mathbb{Z}[G]$ and $\beta\theta_Q \in \mathbb{Z}[G]$, then $\beta\theta_Q$ annihilates the ideal class group of $\mathbb{Q}(\zeta_Q)$), applied with $\beta = a - \sigma_a$. Notation as in the reference (m there is Q, c there is a). □

Theorem B.24. *Let $\mathcal{O}$ be a Dedekind domain, $n \geq 1$, and $\mathfrak{a}_1, \dots, \mathfrak{a}_n$, $\mathfrak{b}_1, \dots, \mathfrak{b}_n$ nonzero fractional ideals of $\mathcal{O}$. Then $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n \cong \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_n$ as $\mathcal{O}$-modules if and only if $\mathfrak{a}_1 \cdots \mathfrak{a}_n$ and $\mathfrak{b}_1 \cdots \mathfrak{b}_n$ have the same class in the ideal class group of $\mathcal{O}$.*

Proof. [245] and [246]. Standard modern accounts: [176, §1] ($\mathfrak{a} \oplus \mathfrak{b} \cong \mathcal{O} \oplus \mathfrak{a}\mathfrak{b}$, and the class of the product is an invariant through the top exterior power); [71, §4]. Notation as in these sources. □

Theorem B.25. *Let L be a number field and $m \geq 2$. The group $K_m(\mathcal{O}_L)$ is finitely generated, and $\dim_{\mathbb{Q}} K_m(\mathcal{O}_L) \otimes \mathbb{Q}$ is 0 if m is even, $r_1(L) + r_2(L)$ if $m \equiv 1 \pmod 4$, and $r_2(L)$ if $m \equiv 3 \pmod 4$.*

Proof. [39] (the stable real cohomology of the arithmetic groups $\mathrm{SL}_n(\mathcal{O}_L)$), translated into ranks of K-groups through the rational Hurewicz isomorphism onto the primitives of $H_\bullet(\mathrm{GL}(\mathcal{O}_L);\mathbb{Q})$; finite generation: D. Quillen, *Finite generation of the groups K_i of rings of algebraic integers*, Lecture Notes in Math. 341 (1973), 179–198. See also C. Weibel, *The K-book*, Ch. VI (Borel's theorem on the ranks). Borel's r_1, r_2 agree with ours. □

Lemma B.26. *Let ℓ be a prime, $F \subset \overline{\mathbb{Q}}$ an imaginary CM field with maximal totally real subfield F^+, $c \in G_{F^+}$ a complex conjugation, and S a finite set of finite places of F containing every place above ℓ, with $S^c = S$ and every $v \in S$ unramified over F^+. For $v \in S$ fix a place $\bar{v}$ of $\overline{\mathbb{Q}}$ above v, choosing $\overline{v^c} = c\bar{v}$ when $v^c \neq v$, and put $G_{F_v} = D_{\bar{v}} \subseteq G_F$, $I_{F_v} = I_{\bar{v}}$. For characters ψ_v of G_{F_v} ($v \in S$) define the character $\psi_{v^c}^c$ of G_{F_v} by $\psi_{v^c}^c(g) = \psi_{v^c}(cgc)$ if $v^c \neq v$, and by $\psi_{v^c}^c(g) = \psi_v(sgs^{-1})$ if $v^c = v$, where $s \in G_{F^+} \setminus G_F$ is any element with $s\bar{v} = \bar{v}$. Let $\chi\colon G_{F^+} \to \overline{\mathbb{Q}}_\ell^\times$ be a continuous character taking the same value on all complex conjugations in G_{F^+}, and for $v \in S$ let $\psi_v\colon G_{F_v} \to \overline{\mathbb{Q}}_\ell^\times$ be a continuous character, de Rham if $v \mid \ell$, such that*

$$(\psi_v\, \psi_{v^c}^c)|_{I_{F_v}} = \chi|_{I_{F_v}} \qquad (v \in S).$$

Then there is a continuous character $\theta\colon G_F \to \overline{\mathbb{Q}}_\ell^\times$ with $\theta(g)\,\theta(cgc) = \chi(g)$ for all $g \in G_F$, and $\theta|_{I_{F_v}} = \psi_v|_{I_{F_v}}$ for all $v \in S$.

Proof. [17], Lemma A.2.5, part (1) (Appendix A.2, "Building fields and characters"). *Notation.* Their $\theta\theta^c = \chi|_{G_F}$ is $\theta(g)\theta(cgc) = \chi(g)$; their condition "$\chi(c_v)$ is independent of $v \mid \infty$" is the condition on complex conjugations. Their ψ_{vc}^c, a character of G_{F_v}, is ψ_{vc} transported by c; with $G_{F_{v^c}} = D_{c\bar{v}} = cD_{\bar{v}}c$ this is $g \mapsto \psi_{v^c}(cgc)$, and when $v^c = v$ the group $D_{c\bar{v}}$ is $hD_{\bar{v}}h^{-1}$ for $h \in G_F$ with $h\bar{v} = c\bar{v}$, so the transport is $g \mapsto \psi_v(h^{-1}cgch)$, which is the formula of the statement with $s = h^{-1}c$ (fixing $\bar{v}$, not in G_F); another choice of s differs by an element of $D_{\bar{v}} \cap G_F$, whose conjugation does not change a character of the abelianisation. The statement is part (1) as printed, nothing added. □

Theorem B.27. *Let F be a number field and $\delta, x \in F^\times$. Suppose that for every real embedding $\sigma\colon F \to \mathbb{R}$ there are $a, b \in \mathbb{R}$ with $\sigma(x) = a^2 - \sigma(\delta)b^2$, and that for every finite place v of F there are $a, b \in F_v$ with $x = a^2 - \delta b^2$ in F_v. Then there are $a, b \in F$ with $x = a^2 - \delta b^2$.*

Proof. This is Hasse's norm theorem [123] for the extension $F(\sqrt{\delta})/F$, which is cyclic of degree at most 2; see [190], Chapter VI, Section 4. *Notation.* If δ is not a square in F, then $a^2 - \delta b^2$ is

the norm of $a+b\sqrt{\delta}$ from $E = F(\sqrt{\delta})$, and the local hypothesis at a place v says that x is a norm from $E \otimes_F F_v$; at a complex place this holds for every x, so complex places are omitted. If $\delta = d^2$ is a square in F, the conclusion holds without hypotheses: $x = a^2 - \delta b^2$ with $a = (x+1)/2$, $b = (x-1)/(2d)$. □

Theorem B.28. *Let E be a CM field and $G, G' \in \mathrm{M}_r(E)$ nondegenerate hermitian matrices ($\bar{G}^\top = G$, $\bar{G}'^\top = G'$). Suppose that for every embedding $\tau\colon E \to \mathbb{C}$ the hermitian matrices $\tau(G)$ and $\tau(G')$ have the same signature, and that $\det G' = z\bar{z} \det G$ for some $z \in E$. Then there is $\gamma \in \mathrm{GL}_r(E)$ with*

$$\bar{x}^\top \bar{\gamma}^\top G \gamma\, y = \bar{x}^\top G'\, y \qquad \text{for all } x, y \in E^r,$$

that is, the hermitian forms of G and G' are isometric.

Proof. Landherr's theorem [158] (Abh. Math. Sem. Univ. Hamburg 11 (1936), 245–248); see [214], Chapter 10: for a quadratic extension E/F of number fields, two nondegenerate hermitian forms over E are isometric if and only if they have the same dimension, the same discriminant in $F^\times/\mathrm{N}_{E/F}(E^\times)$, and the same signature at every real place of F that does not split in E. Only the sufficiency is quoted. *Notation.* Here F is the maximal totally real subfield of E, so every real place of F is complex in E and every embedding τ of E restricts to a real place of F; the signature at that place is the signature of $\tau(G)$, which does not depend on the choice of τ above it, since the other choice gives the complex conjugate of $\tau(G)$, whose signature is the same. The determinant of a hermitian matrix lies in $F^\times$, and the condition $\det G' = z\bar{z} \det G$ says that the discriminants agree modulo $\mathrm{N}_{E/F}(E^\times)$ (with $z \neq 0$, as both determinants are nonzero). The isometry is the displayed identity, that is $\bar{\gamma}^\top G \gamma = G'$. □

B.3 Galois cohomology and Iwasawa theory

This subsection quotes the duality theorems and Euler characteristic formulas of Tate and Poitou, finiteness theorems for continuous cohomology, and three further finiteness theorems. Local Tate duality with the local Euler characteristic formula (Theorem B.29) is used in Parts V, VII, VIII and IX and in Appendix E; Tate's global Euler characteristic formula with the end of the Poitou–Tate sequence (Theorem B.37) is used in Parts VIII and IX and in Appendix E; and local duality and the Euler characteristic for ℓ-adic representations (Theorem B.36) are used in Part VIII and in Section 47. Tate's theorems on continuous cohomology of ℓ-adic lattices, its finite generation and its description as an inverse limit (Theorem B.30, Theorem B.31, Lemma B.33), are used in Parts V and VII, and Poitou–Tate duality for finite modules (Theorem B.34) in Part VII and Appendix E. Part VII also uses Kato's finiteness of Selmer groups of modular forms away from the centre (Theorem B.32, Section 38) and the vanishing of the Iwasawa μ-invariant of abelian fields by Ferrero and Washington (Theorem B.35, Section 39). The finite presentation theorem of Shafarevich and Koch and the nine-term Poitou–Tate sequence (Theorem B.38, Theorem B.39) are used for the Selmer bounds at Taylor–Wiles levels (Section E.4).

Theorem B.29. *Let F be a number field, v a finite place of F, ℓ a prime, $r \geq 1$, and M a finite $\mathbb{Z}/\ell^r\mathbb{Z}$-module with a continuous action of G_{F_v}. Then:*

1. *$H^i(F_v, M)$ is finite for every $i \geq 0$, and $H^i(F_v, M) = 0$ for $i \geq 3$;*
2. *$H^2(F_v, \mu_{\ell^r}) \cong \mathbb{Z}/\ell^r\mathbb{Z}$, and for $i = 0, 1, 2$ the cup product*

$$H^i(F_v, M) \times H^{2-i}(F_v, M^\vee) \longrightarrow H^2(F_v, \mu_{\ell^r})$$

is a perfect pairing of finite groups;

3. *if $v \nmid \ell$ and I_v acts trivially on M, then $H^1_{\mathrm{ur}}(F_v, M)$ and $H^1_{\mathrm{ur}}(F_v, M^\vee)$ are the exact annihilators of each other under the pairing in item 2 with $i = 1$;*

4. *(local Euler characteristic formula) $\#H^1(F_v, M) = \#H^0(F_v, M)\,\#H^2(F_v, M)\,(\#M)^{[F_v:\mathbb{Q}_\ell]}$ if $v \mid \ell$, and $\#H^1(F_v, M) = \#H^0(F_v, M)\,\#H^2(F_v, M)$ if $v \nmid \ell$.*

Proof. This is Tate's local duality theorem and Tate's local Euler characteristic formula, [249]; in the form stated: [172], Chapter I, Corollary 2.3 (item 1 and the perfect pairing of item 2, with $H^2(F_v, \mu_{\ell^r}) = \mathrm{Br}(F_v)[\ell^r] \cong \mathbb{Z}/\ell^r\mathbb{Z}$ by the invariant map), Theorem 2.6 (item 3) and Theorem 2.8 (item 4); see also [222], Chapter II, Section 5. Notation: Milne's K is F_v, his M^D is $M^\vee$, and his H^1_{nr} is H^1_{ur}. □

Theorem B.30. *Let G be a profinite group and ℓ a prime such that $H^i(G, M)$ is finite for every $i \geq 0$ and every finite discrete G-module M of ℓ-power order. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$ and T a finitely generated $\mathcal{O}$-module with a continuous $\mathcal{O}$-linear action of G (ℓ-adic topology on T). Then for every $i \geq 0$ the continuous cochain cohomology group $H^i(G, T)$ is a finitely generated $\mathcal{O}$-module.*

Proof. J. Tate, *Relations between K_2 and Galois cohomology*, Invent. Math. 36 (1976), 257–274, Section 2: for a finitely generated $\mathbb{Z}_\ell$-module T with continuous action and finite $H^i(G, T/\ell T)$, the groups $H^i(G, T)$ are finitely generated $\mathbb{Z}_\ell$-modules; see also [191], Chapter II, Section 7. This gives the statement. Indeed, T is finitely generated over $\mathbb{Z}_\ell$ because $\mathcal{O}$ is; a finitely generated $\mathbb{Z}_\ell$-module on which $\mathcal{O}$ acts is a finitely generated $\mathcal{O}$-module; and Tate's hypothesis is the case $M = T/\ell T$ of ours. For $G = G_{F_v}$ with F_v a finite extension of $\mathbb{Q}_p$ (any p) the hypothesis holds by Theorem B.29(1). □

Theorem B.31. *Let G be a profinite group, ℓ a prime, T a finitely generated $\mathbb{Z}_\ell$-module with a continuous $\mathbb{Z}_\ell$-linear action of G (ℓ-adic topology on T), and $T_n = T/\ell^n T$ for $n \geq 1$, a finite discrete G-module. Let $i \geq 1$ and suppose that $H^{i-1}(G, T_n)$ is finite for every $n \geq 1$. Then the map*

$$H^i(G, T) \longrightarrow \varprojlim_n H^i(G, T_n)$$

induced by the projections $T \to T_n$, from continuous cochain cohomology to the inverse limit along the maps induced by $T_{n+1} \to T_n$, is an isomorphism.

Proof. J. Tate, *Relations between K_2 and Galois cohomology*, Invent. Math. 36 (1976), 257–274, Section 2 (Proposition 2.2 and the exact sequence $0 \to \varprojlim_n{}^1 H^{i-1}(G, T_n) \to H^i(G, T) \to \varprojlim_n H^i(G, T_n) \to 0$ preceding it); the $\varprojlim^1$ of an inverse system of finite groups vanishes. See also [137], Section 2, and [191], Chapter II, Section 7. Notation: Tate's M and $M/\ell^n M$ are T and T_n. □

Theorem B.32. *Let $f = \sum_{n \geq 1} a_n q^n$ be a normalised newform of weight $k \geq 2$, level N and Nebentypus ε_f ($f|_k\gamma = \varepsilon_f(d) f$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, so that the Euler factor of $L(f, s)$ at $\ell \nmid N$ is $(1 - a_\ell \ell^{-s} + \varepsilon_f(\ell)\ell^{k-1-2s})^{-1}$), let $F_f = \mathbb{Q}(a_n : n \geq 1) \subset \mathbb{C}$, let λ be a finite place of F_f above the prime p, with completion F_λ and ring of integers $\mathcal{O}_\lambda$, and let $V_{F_\lambda}(f)$ be the two-dimensional F_λ-linear representation of $G_\mathbb{Q}$ attached to f in [141], 6.3. For $r \in \mathbb{Z}$ write $V_{F_\lambda}(f)(r) = V_{F_\lambda}(f) \otimes \chi_{\mathrm{cyc}}^r$.*

1. *For every prime $\ell \nmid Np$, $V_{F_\lambda}(f)$ is unramified at ℓ and $\det(1 - \mathrm{Frob}_\ell^{-1} u \mid V_{F_\lambda}(f)) = 1 - a_\ell u + \varepsilon_f(\ell)\ell^{k-1}u^2$, where Frob_ℓ is an arithmetic Frobenius at ℓ.*

2. *Let $K \subset \overline{\mathbb{Q}}$ be a finite abelian extension of $\mathbb{Q}$ and r an integer with $1 \le r \le k-1$ and $r \ne k/2$. Then $\mathrm{Sel}(K,T)$ is finite for every $G_{\mathbb{Q}}$-stable $\mathcal{O}_\lambda$-lattice T of $V_{F_\lambda}(f)(r)$.*

Proof. Part 2 is [141], Theorem 14.2(1), for any prime p and any finite place λ; Kato's Selmer group $\mathrm{Sel}(K,T) \subseteq H^1(K, T \otimes \mathbb{Q}/\mathbb{Z})$ ([141], 14.1) imposes at each place one local condition of the form of Definition 22.2, so the group $\mathrm{Sel}(K,T)$ of Definition 22.2 (conditions at every decomposition group) is a subgroup of Kato's and is finite. Part 1 is [141], (14.10.3), recalled there as known (Carayol, Saito): $\det(1 - \mathrm{Fr}_\ell^{-1} u \,;\, H^0(\mathbb{Q}_\ell^{\mathrm{ur}}, V_{F_\lambda}(f))) = 1 - a_\ell u + \varepsilon(\ell)\ell^{k-1}u^2$ for ℓ not below λ, with Fr_ℓ the arithmetic Frobenius; for $\ell \nmid N$ the right side has degree 2, so the inertia invariants are all of $V_{F_\lambda}(f)$. Notation: Kato's κ is χ_{cyc}, his ε is ε_f, his p is the residue characteristic of λ. □

Lemma B.33. *Let G be a profinite group and $(A_n)_{n\ge 1}$ an inverse system of finite discrete G-modules, with continuous actions and surjective G-equivariant transition maps $A_{n+1} \to A_n$; let $A = \varprojlim_n A_n$ with the inverse-limit topology. For every $i \ge 1$ there is an exact sequence*

$$0 \to \varprojlim_n{}^1 H^{i-1}(G, A_n) \to H^i(G, A) \to \varprojlim_n H^i(G, A_n) \to 0,$$

in which $H^i(G,A)$ is continuous cochain cohomology and the second map is induced by the projections $A \to A_n$. In particular, if $H^{i-1}(G, A_n)$ is finite for every n, the natural map $H^i(G,A) \to \varprojlim_n H^i(G, A_n)$ is an isomorphism.

Proof. [191], Chapter II, Section 7, Theorem 2.7.5 and Corollary 2.7.6; the same statement for continuous cochains is in [252, Section 2]. The last sentence holds because an inverse system of finite groups satisfies the Mittag-Leffler condition, so its $\varprojlim^1$ vanishes. Notation as in the references; $H^i(G,A)$ is $H^i_{\mathrm{cts}}(G,A)$ there. □

Theorem B.34. *Let F be a number field with no real place, ℓ a prime, S a finite set of places of F containing the places above ℓ and the archimedean places, $r \ge 1$, and M a finite $\mathbb{Z}/\ell^r\mathbb{Z}$-module with a continuous action of $G_{F,S}$; put $M^\vee = \mathrm{Hom}(M, \mu_{\ell^r})$. Then:*

1. *$H^i(G_{F,S}, M)$ is finite for every $i \ge 0$;*
2. *$\mathrm{Sha}^1_S(F,M)$ and $\mathrm{Sha}^2_S(F, M^\vee)$ are finite and there is a perfect pairing*

$$\mathrm{Sha}^1_S(F,M) \times \mathrm{Sha}^2_S(F, M^\vee) \to \mathbb{Q}/\mathbb{Z};$$

   in particular $\#\mathrm{Sha}^1_S(F,M) = \#\mathrm{Sha}^2_S(F,M^\vee)$;
3. *$H^i(G_{F,S}, M) = 0$ for every $i \ge 3$.*

Proof. [172], Ch. I, Theorem 4.10: parts (a), (b) and (c) there; in (c) the groups $H^i(G_{F,S}, M)$ for $i \ge 3$ are identified with the sum of the $H^i(F_v, M)$ over the real places v, which is empty here. See also [191], Chapter VIII, Section 6 (Poitou–Tate duality) and Section 3 (cohomological dimension of $G_{F,S}$). Milne's K, M^D and $\mathrm{Sha}^i(G_S, M)$ are our F, $M^\vee$ and $\mathrm{Sha}^i_S(F,M)$; the hypothesis that $\#M$ is a unit outside S holds because S contains the places above ℓ. The statement is specialised to fields without real places, which is the case needed here; the prime $\ell = 2$ is then covered without modified cohomology at the real places. □

Theorem B.35. *Let p be a prime, K a finite abelian extension of $\mathbb{Q}$, K_∞/K the cyclotomic $\mathbb{Z}_p$-extension, K_n its subfield with $[K_n : K] = p^n$, and A_n the p-Sylow subgroup of the ideal class group of K_n. Let $\mu, \lambda \geq 0$ and ν be the Iwasawa invariants of K_∞/K, so that $\#A_n = p^{\mu p^n + \lambda n + \nu}$ for all sufficiently large n. Then $\mu = 0$.*

Proof. [90], for every prime p; another proof is due to Sinnott [235]; see also [266], Chapter 7. The invariants are those of Iwasawa's theorem ([266], Chapter 13). The prime p is that of the cited works. □

Theorem B.36. *Let ℓ be a prime, M a finite extension of $\mathbb{Q}_\ell$ with absolute Galois group G_M, and U a finite-dimensional $\mathbb{Q}_\ell$-vector space with a continuous linear action of G_M; write $h^i(U) = \dim_{\mathbb{Q}_\ell} H^i(G_M, U)$ (continuous cochains) and $U^*(1) = \mathrm{Hom}_{\mathbb{Q}_\ell}(U, \mathbb{Q}_\ell) \otimes \chi_{\mathrm{cyc}}$. Then the groups $H^i(G_M, U)$ are finite-dimensional, they vanish for $i \geq 3$, and*

$$h^0(U) - h^1(U) + h^2(U) = -[M : \mathbb{Q}_\ell] \dim_{\mathbb{Q}_\ell} U, \qquad h^2(U) = h^0\big(U^*(1)\big).$$

Proof. J. Tate, *Duality theorems in Galois cohomology over number fields* [249] (local Euler characteristic formula and local duality for finite modules), passed to ℓ-adic coefficients through a G_M-stable $\mathbb{Z}_\ell$-lattice $T \subset U$ and the finiteness of $H^i(G_M, T/\ell^n T)$ (continuous cohomology of T is the inverse limit, and $H^i(G_M, U) = H^i(G_M, T) \otimes \mathbb{Q}$). This is the form used by Bloch and Kato [34], p. 356 (proof of (3.8)), who cite J.-P. Serre, *Cohomologie galoisienne*, II.5.7, for the Euler characteristic, and local duality for the second formula. □

Theorem B.37. *Let F be a number field and ℓ a prime; assume that ℓ is odd or that F has no real place. Let S be a finite set of places of F containing the places above ℓ and the archimedean places, $r \geq 1$, and M a finite $\mathbb{Z}/\ell^r\mathbb{Z}$-module with a continuous action of $G_{F,S}$. For $v \in S$, $H^i(F_v, M)$ is formed through a decomposition group at v and its map to $G_{F,S}$. Then:*

1. *$H^i(G_{F,S}, M)$ is finite for every $i \geq 0$;*
2. *(Tate's global Euler characteristic formula)*

$$\frac{\#H^0(G_{F,S}, M) \cdot \#H^2(G_{F,S}, M)}{\#H^1(G_{F,S}, M)} = \prod_{v|\infty} \frac{\#H^0(F_v, M)}{|\#M|_v};$$

3. *the sequence*

$$H^2(G_{F,S}, M) \longrightarrow \bigoplus_{v \in S,\ v \nmid \infty} H^2(F_v, M) \longrightarrow H^0(G_{F,S}, M^\vee)^* \longrightarrow 0,$$

whose first map is the sum of the restrictions and * the Pontryagin dual, is exact.

Proof. [172], Ch. I, Theorem 4.10 (finiteness, and the Poitou–Tate nine-term exact sequence, whose last three terms are those of item 3) and Theorem 5.1 (the Euler characteristic); see also [191, Ch. VIII]. Notation: Milne's K, G_S, M^D are F, $G_{F,S}$, $M^\vee$; his hypothesis that $\#M$ is a unit outside S holds since S contains the places above ℓ. In the nine-term sequence the archimedean places contribute $\hat{H}^2(F_v, M)$, which vanishes under the hypothesis: for ℓ odd because $\mathrm{Gal}(\mathbb{C}/F_v)$ has order at most 2 and $\#M$ is odd, and when F has no real place because that group is trivial at every archimedean place; this is why they are omitted in item 3. Items 1

and 2 hold in Milne's theorems for every prime ℓ; when F has no real place every archimedean factor of item 2 is $\#M/|\#M|_v = (\#M)^{-1}$, also for $\ell = 2$ (compare the example in the footnote to Theorem 5.1 of loc. cit.). Theorem 5.1 of [172], Ch. I, assumes only that S is a finite nonempty set of places containing the archimedean ones and that the order of M is invertible in the ring of S-integers; it carries no hypothesis on ℓ, its archimedean factors use the ordinary invariants $H^0(F_v, M)$, and its footnote treats $M = \mathbb{Z}/2\mathbb{Z}$, where a complex place contributes $2/4$. The Poitou–Tate sequence behind item 3 is stated for every prime, with Tate cohomology at the archimedean places, in [174]. □

Theorem B.38. *Let K be a number field, ℓ an odd prime and S a finite set of places of K containing all places above ℓ. Let $K_S(\ell)$ be the maximal pro-ℓ extension of K (in a fixed algebraic closure) unramified at all finite places outside S, and $G_S(\ell) = \mathrm{Gal}(K_S(\ell)/K)$. Then $H^1(G_S(\ell), \mathbb{Z}/\ell)$ and $H^2(G_S(\ell), \mathbb{Z}/\ell)$ are finite; that is, the pro-ℓ group $G_S(\ell)$ is finitely generated and has finitely many defining relations.*

Proof. See [147] (Shafarevich–Koch theorem on the generator and relation ranks of $G_S(p)$); [191, Chapter X, §7]. For odd ℓ the archimedean places do not ramify in pro-ℓ extensions, so including them in S or not gives the same group. □

Theorem B.39. *Let F be a number field and ℓ a prime; assume that ℓ is odd or that F has no real place. Let S be a finite set of places of F containing the places above ℓ and the archimedean places, $S_f \subseteq S$ its finite places, $r \geq 1$, and M a finite $\mathbb{Z}/\ell^r\mathbb{Z}$-module with a continuous action of $G_{F,S}$. For $v \in S_f$, $H^i(F_v, M)$ is formed through a decomposition group at v and its map to $G_{F,S}$. Write $H^i(M) = H^i(G_{F,S}, M)$, $P^i(M) = \prod_{v \in S_f} H^i(F_v, M)$, * for the Pontryagin dual, and $\lambda^i_M \colon H^i(M) \to P^i(M)$ for the product of the restriction maps. Then:*

1. *$H^i(M)$ and $H^i(F_v, M)$ ($v \in S_f$) are finite for every $i \geq 0$;*
2. *$\mathrm{Sha}^1_S(F, M)$ and $\mathrm{Sha}^2_S(F, M^\vee)$ are finite, and the cup product followed by the sum of the local invariants induces a perfect pairing $\mathrm{Sha}^1_S(F, M) \times \mathrm{Sha}^2_S(F, M^\vee) \to \mathbb{Q}/\mathbb{Z}$;*
3. *the sequence*

$$\begin{aligned} 0 \to H^0(M) \xrightarrow{\lambda^0_M} P^0(M) \xrightarrow{\beta^0} H^2(M^\vee)^* \\ \to H^1(M) \xrightarrow{\lambda^1_M} P^1(M) \xrightarrow{\beta^1} H^1(M^\vee)^* \\ \to H^2(M) \xrightarrow{\lambda^2_M} P^2(M) \xrightarrow{\beta^2} H^0(M^\vee)^* \to 0 \end{aligned}$$

is exact, where $\beta^i((x_v)_v)(y) = \sum_{v \in S_f} \mathrm{inv}_v(x_v \cup \mathrm{res}_v(y))$ for $y \in H^{2-i}(M^\vee)$, the cup product being taken with values in $H^2(F_v, \mu_{\ell^r})$.

Proof. This is the Poitou–Tate theorem, [172], Chapter I, Theorem 4.10: (a) gives item 1 (with Corollary 2.3 for the local groups), (b) item 2, and (c) the nine-term exact sequence of item 3; see also [191], Chapter VIII, Theorems 8.6.7 and 8.6.10. Milne's K, G_S, M^D are F, $G_{F,S}$, $M^\vee$ here. His hypothesis that $\#M$ is a unit outside S holds since S contains the places above ℓ. In Milne's sequence the archimedean places contribute the Tate cohomology groups $\hat{H}^i(F_v, M)$, $i = 0, 1, 2$, and the Tate–Shafarevich groups are formed over all of S. Under the hypothesis these archimedean groups vanish (for ℓ odd, $\mathrm{Gal}(\mathbb{C}/F_v)$ has order at most 2 and $\#M$ is odd, so the Tate cohomology is killed by 2 and by ℓ^r; when F has no real place the group is trivial at every archimedean place), which is why the products run over S_f only. The maps β^i are those of Milne's sequence: the duals of $\lambda^{2-i}_{M^\vee}$ composed with local Tate duality. □

B.4 p-adic Hodge theory

Part V is the main user of p-adic Hodge theory. Its Wach modules, relay classes and losses at the places above ℓ (Sections 23–25) rest on Fontaine's ring A_{inf} with the map θ and B_{dR}^{+} (Theorem B.40), on the theorem of Bhatt and Scholze that crystalline lattices come from prismatic F-crystals (Theorem B.41), and on the invariants of $\mathbb{C}_p$ over the cyclotomic tower of Tate and Ax–Sen–Tate (Theorem B.42); all three are used in Appendix C as well. The finite parts of Bloch and Kato, with their dimensions, their orthogonality under local duality and the unramified case (Theorem B.43, Theorem B.44, Theorem B.46, Theorem B.49, Theorem B.50, Theorem B.56), and Fontaine's stability of crystalline representations (Theorem B.45) are used in Part V and in Parts VIII and IX. Fontaine's (φ,Γ)-modules (Theorem B.51, Theorem B.52) are used for the lossless local conditions at large primes (Section 28), and the formula of Cherbonnier and Colmez for the dual exponential and Tate's computation of the first cohomology of the twists of $\mathbb{C}_p$ (Theorem B.48, Theorem B.47) in Section 25. Outside Parts V, VIII and IX, the bounds of Fargues for the Hodge–Tate map (Theorem B.53, Theorem B.54) are used for linked stages in Part VI (Section 35), Fontaine's weak admissibility (Theorem B.55) in Section 38, and Kisin's crystalline lifting rings (Theorem B.57) in Section 47.

Theorem B.40. *Let p be a prime, $[\cdot]\colon \mathcal{O}_{\mathbb{C}_p^{\flat}} \to A_{\mathrm{inf}}$ the Teichmüller map, $x \mapsto x^{\sharp}$ the map $\mathcal{O}_{\mathbb{C}_p^{\flat}} \to \mathcal{O}_{\mathbb{C}_p}$, $\varepsilon \in \mathcal{O}_{\mathbb{C}_p^{\flat}}$ the element defined by ζ (so $(\varepsilon^{1/p^k})^{\sharp} = \zeta_{p^k}$), $p^{\flat} \in \mathcal{O}_{\mathbb{C}_p^{\flat}}$ an element with $(p^{\flat})^{\sharp} = p$, $q = [\varepsilon^{1/p}]$ and $\xi_0 = \sum_{k=0}^{p-1} q^k$. The group $G_{\mathbb{Q}_p}$ acts on $\mathcal{O}_{\mathbb{C}_p^{\flat}}$ and, by functoriality of Witt vectors, on A_{inf}, with $\sigma[x] = [\sigma x]$.*

(Ai1) *A_{inf} is an integral domain; θ is a surjective $G_{\mathbb{Q}_p}$-equivariant ring homomorphism with $\theta([x]) = x^{\sharp}$; $\ker\theta = \xi_0 A_{\mathrm{inf}}$; and $(p,\xi_0) = (p,[p^{\flat}])$ as ideals of A_{inf}.*

(Ai2) *A_{inf} is separated and complete for the topology defined by the ideals $(p,[p^{\flat}])^n$, $n \geq 1$; the Teichmüller map is continuous for this topology and the valuation topology of $\mathcal{O}_{\mathbb{C}_p^{\flat}}$; and the action of $G_{\mathbb{Q}_p}$ on A_{inf} is continuous for it.*

(Ai3) *For every $b \geq 1$ the natural map $A_{\mathrm{inf}}[1/p]/\xi_0^b A_{\mathrm{inf}}[1/p] \to B_{\mathrm{dR}}^{+}/\xi_0^b B_{\mathrm{dR}}^{+}$ is an isomorphism; B_{dR}^{+} is a complete discrete valuation ring, and the kernel of the extension $\theta\colon B_{\mathrm{dR}}^{+} \to \mathbb{C}_p$ is $\xi_0 B_{\mathrm{dR}}^{+} = t B_{\mathrm{dR}}^{+}$; $\overline{\mathbb{Q}}_p \subset B_{\mathrm{dR}}^{+}$ compatibly with θ, and $\sigma(t) = \chi_{\mathrm{cyc}}(\sigma)t$.*

(Ai4) *(de Rham comparison) If F is a finite extension of $\mathbb{Q}_p$ and V a de Rham representation of G_F, $D = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_F}$ with $\mathrm{Fil}^j D = D \cap (t^j B_{\mathrm{dR}}^{+} \otimes_{\mathbb{Q}_p} V)$, then the natural map $B_{\mathrm{dR}} \otimes_F D \to B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ is a G_F-equivariant isomorphism (G_F acting trivially on D) which maps $\sum_{j\in\mathbb{Z}} t^{k-j} B_{\mathrm{dR}}^{+} \otimes_F \mathrm{Fil}^j D$ onto $t^k B_{\mathrm{dR}}^{+} \otimes_{\mathbb{Q}_p} V$ for every $k \in \mathbb{Z}$.*

Proof. Fontaine [94], Sections 1–3 (the ring $R = \mathcal{O}_{\mathbb{C}_p^{\flat}}$, $W(R)$, θ and its kernel generated by an element of the form ξ_0, the weak topology, B_{dR}^{+} and t) and [96] (admissibility, the filtered comparison for de Rham representations); see also [52] (the chapters on A_{inf}, B_{dR} and de Rham representations). Fontaine's R and ε are ours; the ideal equality $(p,\xi_0) = (p,[p^{\flat}])$ holds because modulo p, $\xi_0 \equiv [(\varepsilon-1)/(\varepsilon^{1/p}-1)]$ and this element of R has the valuation of $p^{\flat}$. □

Theorem B.41. *Let p be a prime and K a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$. An* object over $\mathcal{O}_K$ *is a bounded prism (A,I) in the sense of Bhatt–Scholze together with a ring homomorphism $\mathcal{O}_K \to A/I$; a morphism $f\colon (A,I) \to (B,J)$ is a homomorphism of δ-rings with $f(I) \subseteq J$ inducing an $\mathcal{O}_K$-algebra map $A/I \to B/J$. Let T be a finite free $\mathbb{Z}_p$-module of rank d with a continuous $\mathbb{Z}_p$-linear action of G_K such that $T[1/p]$ is crystalline. Then there is a datum $\mathcal{E}$ consisting of:*

(Bs1) *for every object (A, I) over $\mathcal{O}_K$, a finite projective A-module $\mathcal{E}(A)$ of rank d and an A-linear isomorphism $\varphi_{\mathcal{E},A}\colon (A \otimes_{\varphi_A,A} \mathcal{E}(A))[1/I] \to \mathcal{E}(A)[1/I]$;*

(Bs2) *for every morphism $f\colon (A, I) \to (B, J)$ over $\mathcal{O}_K$, an f-semilinear map $\mathcal{E}(f)\colon \mathcal{E}(A) \to \mathcal{E}(B)$ whose B-linear extension $B \otimes_{f,A} \mathcal{E}(A) \to \mathcal{E}(B)$ is an isomorphism, with $\mathcal{E}(\mathrm{id}) = \mathrm{id}$, $\mathcal{E}(g \circ f) = \mathcal{E}(g) \circ \mathcal{E}(f)$, and compatible with the isomorphisms $\varphi_{\mathcal{E},A}$, $\varphi_{\mathcal{E},B}$;*

such that the following holds for the evaluation at Fontaine's ring. Put $\mu_\varepsilon = [\varepsilon] - 1$, $\tilde{\xi} = \sum_{k=0}^{p-1}[\varepsilon]^k = \varphi(\mu_\varepsilon)/\mu_\varepsilon$ and $\tilde{\theta} = \theta \circ \varphi^{-1}\colon A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_p}$. Then $(A_{\mathrm{inf}}, \tilde{\xi}A_{\mathrm{inf}})$ with $\mathcal{O}_K \subseteq \mathcal{O}_{\mathbb{C}_p} \cong A_{\mathrm{inf}}/\tilde{\xi}A_{\mathrm{inf}}$ (via $\tilde{\theta}$) is an object over $\mathcal{O}_K$, every $\sigma \in G_K$ acts on it by a morphism over $\mathcal{O}_K$, and with $M = \mathcal{E}(A_{\mathrm{inf}})$, $\varphi_M = \varphi_{\mathcal{E},A_{\mathrm{inf}}}$ and the semilinear action $\sigma \mapsto \mathcal{E}(\sigma)$ of G_K on M:

(Bs3) *there is an isomorphism of $W(\mathbb{C}_p^\flat)$-modules $M \otimes_{A_{\mathrm{inf}}} W(\mathbb{C}_p^\flat) \cong T \otimes_{\mathbb{Z}_p} W(\mathbb{C}_p^\flat)$, equivariant for G_K (acting diagonally on the right) and carrying $\varphi_M \otimes \varphi$ to $1 \otimes \varphi$, which restricts to an isomorphism $M[1/\mu_\varepsilon] \cong T \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[1/\mu_\varepsilon]$.*

(Bs4) *Let $B'^{+} = \varprojlim_b A_{\mathrm{inf}}[1/p]/\tilde{\xi}^b A_{\mathrm{inf}}[1/p]$ be the completion of $A_{\mathrm{inf}}[1/p]$ at $\ker\tilde{\theta}$ and $B'_{\mathrm{dR}} = B'^{+}[1/\tilde{\xi}]$, with G_K acting through its action on A_{inf}. The image of μ_ε in B'^{+} is a unit, so (Bs3) induces an isomorphism $M \otimes_{A_{\mathrm{inf}}} B'_{\mathrm{dR}} \cong T \otimes_{\mathbb{Z}_p} B'_{\mathrm{dR}}$ carrying $M \otimes_{A_{\mathrm{inf}}} B'^{+}$ onto $T \otimes_{\mathbb{Z}_p} B'^{+}$. Under it, the B'^{+}-submodule generated by the images of the elements $\varphi_M(1 \otimes m) \in M[1/\tilde{\xi}]$, $m \in M$, equals the B'^{+}-submodule of $T \otimes_{\mathbb{Z}_p} B'_{\mathrm{dR}}$ generated by $(T \otimes_{\mathbb{Z}_p} B'_{\mathrm{dR}})^{G_K}$.*

(Bs5) *The datum is functorial in T: if T' is another such module with datum $\mathcal{E}'$, every G_K-equivariant $\mathbb{Z}_p$-linear map $f\colon T \to T'$ induces A-linear maps $\mathcal{E}(f)_A\colon \mathcal{E}(A) \to \mathcal{E}'(A)$, for all objects (A, I) over $\mathcal{O}_K$, such that $\mathcal{E}'(g) \circ \mathcal{E}(f)_A = \mathcal{E}(f)_B \circ \mathcal{E}(g)$ for every morphism $g\colon (A, I) \to (B, J)$ over $\mathcal{O}_K$, $\varphi_{\mathcal{E}',A} \circ (1 \otimes \mathcal{E}(f)_A) = \mathcal{E}(f)_A \circ \varphi_{\mathcal{E},A}$ after inverting I, $\mathcal{E}(\mathrm{id}_T)_A = \mathrm{id}$, $\mathcal{E}(f' \circ f)_A = \mathcal{E}(f')_A \circ \mathcal{E}(f)_A$, and the isomorphisms (Bs3) for T and T' carry $\mathcal{E}(f)_{A_{\mathrm{inf}}} \otimes 1$ to $f \otimes 1$.*

(Bs6) *Full faithfulness: the maps of (Bs5) exist for arbitrary data. Let T_0, T_1 be finite free $\mathbb{Z}_p$-modules with continuous G_K-actions, crystalline after inverting p, and let $\mathcal{E}_0$, $\mathcal{E}_1$ be data as in (Bs1), (Bs2), each with an isomorphism $c_\nu\colon \mathcal{E}_\nu(A_{\mathrm{inf}}) \otimes_{A_{\mathrm{inf}}} W(\mathbb{C}_p^\flat) \cong T_\nu \otimes_{\mathbb{Z}_p} W(\mathbb{C}_p^\flat)$ with the properties of (Bs3). Then every G_K-equivariant $\mathbb{Z}_p$-linear $f\colon T_0 \to T_1$ induces A-linear maps $\mathcal{E}(f)_A\colon \mathcal{E}_0(A) \to \mathcal{E}_1(A)$, for all objects (A, I) over $\mathcal{O}_K$, compatible with the maps $\mathcal{E}_\nu(g)$ of (Bs2) and with the Frobenius isomorphisms, such that $c_1 \circ (\mathcal{E}(f)_{A_{\mathrm{inf}}} \otimes 1) = (f \otimes 1) \circ c_0$.*

Proof. Bhatt and Scholze [31]: the datum $\mathcal{E}$ is a prismatic F-crystal in vector bundles on the absolute prismatic site of $\mathrm{Spf}(\mathcal{O}_K)$ (their Definition 1.1 and 4.1: (Bs1) and (Bs2) are the definition of a vector bundle with Frobenius on the ringed site); its existence with étale realisation T is the essential surjectivity in their Theorem 5.6 (Theorem 1.2 of the introduction), stated for T finite free over $\mathbb{Z}_p$ with $T[1/p]$ crystalline. The prism $(A_{\mathrm{inf}}, \ker\tilde{\theta})$ with $\ker\tilde{\theta} = ([p]_q)$, $q = [\varepsilon]$, $\mu = q - 1$, is the perfect prism of $\mathcal{O}_{\mathbb{C}_p}$ (their §5, standing notation before Theorem 5.2), and M is the Breuil–Kisin–Fargues module of the restriction of $\mathcal{E}$ to $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p})$, with its G_K-equivariant structure from the descent along $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p}) \to \mathrm{Spf}(\mathcal{O}_K)$. (Bs3) is the construction following their Theorem 5.2: $T = (M \otimes W(\mathbb{C}_p^\flat))^{\varphi=1}$, Artin–Schreier theory gives $M \otimes W(\mathbb{C}_p^\flat) \cong T \otimes W(\mathbb{C}_p^\flat)$ extending the identity of T on φ-invariants, and it restricts to $M[1/\mu] \cong T \otimes A_{\mathrm{inf}}[1/\mu]$ by [30], Lemma 4.26; the G_K-equivariance is used in the proof of their Proposition 5.3. (Bs4) is the commutativity of the diagram of their Construction 5.5, which they deduce from Theorem 5.2 and their Remark 5.4: for an F-crystal on $\mathrm{Spf}(\mathcal{O}_K)$ with crystalline étale realisation T, the lattice

$F = \varphi^*M \otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{dR}}$ attached to its restriction to $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p})$ in the construction following Theorem 5.2 (the image of φ^*M under the Frobenius, inside $M \otimes B_{\mathrm{dR}} \cong T \otimes B_{\mathrm{dR}}$, the isomorphism coming from $M[1/\mu] \cong T \otimes A_{\mathrm{inf}}[1/\mu]$ because μ is invertible at the Hodge–Tate point) is $D_{\mathrm{dR}}(V) \otimes_K B^+_{\mathrm{dR}}$, the image of $B^+_{\mathrm{dR}} \otimes_K (V \otimes B_{\mathrm{dR}})^{G_K}$ under the natural map; their B^+_{dR} (Notation 5.1: the completion of $A_{\mathrm{inf}}[1/p]$ at $\tilde{\theta}$, uniformiser $[p]_q$) is our B'^+. (Bs5) is the full faithfulness in their Theorem 5.6 (Theorem 1.2: the étale realisation is an equivalence of categories between prismatic F-crystals in vector bundles on $\mathcal{O}_K$ and lattices in crystalline representations of G_K): a map of realisations is the realisation of a unique map of F-crystals, i.e. a compatible family of A-linear maps commuting with Frobenius and base change, and the identification (Bs3) is natural in the crystal. (Bs6) is the same full faithfulness applied to $\mathcal{E}_0$, $\mathcal{E}_1$: since c_ν has the properties of (Bs3), it identifies the étale realisation $(\mathcal{E}_\nu(A_{\mathrm{inf}}) \otimes W(\mathbb{C}_p^\flat))^{\varphi=1}$ of $\mathcal{E}_\nu$ G_K-equivariantly with T_ν; so f is a map of étale realisations, hence the realisation of a unique map of F-crystals $\mathcal{E}_0 \to \mathcal{E}_1$, whose components are the maps $\mathcal{E}(f)_A$, and $c_1 \circ (\mathcal{E}(f)_{A_{\mathrm{inf}}} \otimes 1) = (f \otimes 1) \circ c_0$ holds on φ-invariants, hence everywhere by $W(\mathbb{C}_p^\flat)$-linearity. *Notation.* Their $q, \mu, \tilde{\xi}, \tilde{\theta}$ are our $[\varepsilon]$, μ_ε, $\tilde{\xi}$, $\tilde{\theta}$, with ε the element of Theorem B.40, $\varepsilon^\sharp = 1$. □

Theorem B.42. *Let p be a prime, $F \subset \overline{\mathbb{Q}}_p$ a finite extension of $\mathbb{Q}_p$, $F_\infty = F(\mu_{p^\infty})$, $\widehat{F}_\infty$ the closure of F_∞ in $\mathbb{C}_p$, $\mathfrak{m}_{F_\infty}$ the maximal ideal of the valuation ring of F_∞, and $H_F = \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\infty) \subseteq G_F$, acting on $\mathbb{C}_p$ by continuity.*

1. *(Ax–Sen–Tate) For every closed subgroup H of $G_{\mathbb{Q}_p}$, $\{c \in \mathbb{C}_p : hc = c \text{ for all } h \in H\}$ is the closure in $\mathbb{C}_p$ of the fixed field of H in $\overline{\mathbb{Q}}_p$. In particular the H_F-invariants of $\mathbb{C}_p$ form $\widehat{F}_\infty$, and the G_F-invariants form F.*
2. *(Tate) If $j \neq 0$ is an integer and $c \in \mathbb{C}_p$ satisfies $\chi_{\mathrm{cyc}}(g)^j\, gc = c$ for all $g \in G_F$, then $c = 0$.*
3. *(Tate) For every finite extension $L \subset \overline{\mathbb{Q}}_p$ of F_∞ with valuation ring $\mathcal{O}_L$, $\mathrm{Tr}_{L/F_\infty}(\mathcal{O}_L) \supseteq \mathfrak{m}_{F_\infty}$.*

Proof. Tate [251], Section 3: item 3 is the statement that the cyclotomic tower is deeply ramified (the different of a finite extension of the layers tends to the unit ideal), proved for a totally ramified $\mathbb{Z}_p$-extension; F_∞ is such an extension of $F(\mu_{p^n})$ for n large, and a finite extension of F_∞ is a finite extension of that $\mathbb{Z}_p$-extension. Item 2 is $H^0(G_F, \mathbb{C}_p(j)) = 0$ for $j \neq 0$, in the notation $\mathbb{C}_p(j) = \mathbb{C}_p \otimes \mathbb{Z}_p(j)$, where an invariant $c \otimes \epsilon^{\otimes j}$ is exactly a solution of $\chi_{\mathrm{cyc}}(g)^j gc = c$. Item 1 for closed subgroups of $G_{\mathbb{Q}_p}$ in general is due to Ax [15], Sen and Tate; see also [52], Section 2 and the chapter on Sen theory, for all three items. □

Theorem B.43. *Let p be a prime, K a finite extension of $\mathbb{Q}_p$ and V a finite-dimensional continuous $\mathbb{Q}_p$-linear representation of G_K which is de Rham. Put $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ and $\mathrm{Fil}^0 D_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V) \cap (\mathrm{Fil}^0 B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$. Then*

$$\dim_{\mathbb{Q}_p} H^1_f(G_K, V) = \dim_{\mathbb{Q}_p} H^0(G_K, V) + \dim_{\mathbb{Q}_p} (D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)).$$

Proof. [34], Corollary 3.8.4. *Notation.* Their K, V, $H^1_f(K, V)$ (the kernel of $H^1(K, V) \to H^1(K, B_{\mathrm{crys}} \otimes V)$) are ours; their $D_{\mathrm{dR}}(V)/D^0_{\mathrm{dR}}(V)$ is $D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)$. □

Theorem B.44. *Let ℓ be a prime and F_v a finite extension of $\mathbb{Q}_\ell$, with absolute Galois group G_{F_v} and inertia subgroup I_{F_v}. Then $H^1_f(G_{F_v}, \mathbb{Q}_\ell) \subseteq H^1(G_{F_v}, \mathbb{Q}_\ell) = \mathrm{Hom}_{\mathrm{cont}}(G_{F_v}, \mathbb{Q}_\ell)$ consists exactly of the homomorphisms that vanish on I_{F_v}.*

Proof. [34], Example 3.9 (case $r = 0$: "$H^1_f(K, \mathbb{Q}_p) \subset \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \mathbb{Q}_p)$ are the unramified homomorphisms"). *Notation.* Their K, p are F_v, ℓ; their H^1_f is the kernel of $H^1(K, V) \to H^1(K, B_{\mathrm{crys}} \otimes V)$, which is the local condition of Definition 22.2 at a place above ℓ. □

Theorem B.45. *Let p be a prime and K a finite extension of $\mathbb{Q}_p$. If V and V' are crystalline finite-dimensional continuous $\mathbb{Q}_p$-linear representations of G_K, then every G_K-stable subspace of V, every quotient of V by a G_K-stable subspace, $V \oplus V'$, $V \otimes_{\mathbb{Q}_p} V'$ and the dual V^* are crystalline.*

Proof. [94]; [52], Theorem 5.2.1 (the category of B-admissible representations of an (F, G)-regular ring B is stable under subobjects, quotients, direct sums, tensor products and duals) with $B = B_{\mathrm{cris}}$, which is $(\mathbb{Q}_p, G_K)$-regular ([52], §9.1). *Notation.* Here B_{cris}-admissible means crystalline in the sense fixed in Section 13. □

Theorem B.46. *Let p be a prime and F a finite extension of $\mathbb{Q}_p$ with maximal unramified subfield F_0.*

1. *For every integer $i \geq 1$, $H^0(G_F, t^i B^+_{\mathrm{dR}}) = H^1(G_F, t^i B^+_{\mathrm{dR}}) = 0$.*
2. *Let V be a de Rham representation of G_F on a finite-dimensional $\mathbb{Q}_p$-vector space, $V^*(1) = \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p(1))$, $H^1_f(F, V) = \ker\big(H^1(G_F, V) \to H^1(G_F, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)\big)$ and $H^1_g(F, V) = \ker\big(H^1(G_F, V) \to H^1(G_F, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)\big)$. Then $H^1_f(F, V) \subseteq H^1_g(F, V)$ and*

$$\dim_{\mathbb{Q}_p} H^1_g(F, V) = \dim_{\mathbb{Q}_p} H^1_f(F, V) + \dim_{\mathbb{Q}_p} D_{\mathrm{cris}}\big(V^*(1)\big)^{\varphi=1}.$$

Proof. Item 2: Bloch and Kato [34], Corollary 3.8.4 (with their definitions of H^1_f and H^1_g in Section 3; the inclusion $H^1_f \subseteq H^1_g$ comes from $B_{\mathrm{cris}} \subset B_{\mathrm{dR}}$). Item 1 is the consequence of Tate's theorem $H^0(G_F, \mathbb{C}_p(j)) = H^1(G_F, \mathbb{C}_p(j)) = 0$ for $j \neq 0$ [251], applied to the graded pieces $t^j B^+_{\mathrm{dR}}/t^{j+1} B^+_{\mathrm{dR}} \cong \mathbb{C}_p(j)$, $j \geq i$, and passage to the limit; it is used in this form in [34], Section 3, and stated in [52]. *Notation.* Their K is our F. □

Theorem B.47. *Let p be a prime, F a finite extension of $\mathbb{Q}_p$, $j \in \mathbb{Z}$, and $c\colon G_F \to \mathbb{C}_p$ a continuous map with $c(\sigma\tau) = c(\sigma) + \chi_{\mathrm{cyc}}(\sigma)^j \sigma\big(c(\tau)\big)$ for $\sigma, \tau \in G_F$.*

1. *If $j \neq 0$, there is $y \in \mathbb{C}_p$ with $c(\sigma) = \chi_{\mathrm{cyc}}(\sigma)^j \sigma(y) - y$ for all $\sigma \in G_F$.*
2. *If $j = 0$, there are $\alpha \in F$ and $y \in \mathbb{C}_p$ with $c(\sigma) = \alpha \log \chi_{\mathrm{cyc}}(\sigma) + \sigma(y) - y$ for all $\sigma \in G_F$.*

Proof. Tate, *p-divisible groups*, Driebergen 1966 (published 1967) [251], Section 3.3, Theorem 2: $H^1_{\mathrm{cont}}(G_F, \mathbb{C}_p(j)) = 0$ for $j \neq 0$, and $H^1_{\mathrm{cont}}(G_F, \mathbb{C}_p)$ is a one-dimensional F-vector space spanned by the class of the homomorphism $\log \chi_{\mathrm{cyc}}$. Notation: $\mathbb{C}_p(j)$ is $\mathbb{C}_p$ with the action $\sigma \mapsto \chi_{\mathrm{cyc}}(\sigma)^j \sigma$; a continuous 1-cocycle is a map c as in the statement, and it is zero in H^1 exactly when it has the form $\sigma \mapsto \chi_{\mathrm{cyc}}(\sigma)^j \sigma(y) - y$. □

Theorem B.48. *Let p be a prime, K_0 a finite unramified extension of $\mathbb{Q}_p$, $e \geq 0$ an integer, $T = \mathbb{Z}_p(-e)$ with basis e_{-e}, on which G_{K_0} acts by χ^{-e}_{cyc}, and $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$; V is crystalline. Let $f \in \mathcal{O}_{K_0}[[X]]$ with $\psi(f(\pi)) = f(\pi)$, so that $y = f(\pi) \otimes e_{-e} \in \mathbf{D}(T)^{\psi=1}$, and let $n \geq 1$. Then:*

1. *(Fontaine; Cherbonnier–Colmez II.1; [25], Theorem II.8.) The classes $h^1_{K_m,V}(y)$, $m \geq 1$, form an element of $H^1_{\mathrm{Iw}}(K_0, T) = \varprojlim_{\mathrm{cor}} H^1(K_m, T)$; in particular $h^1_{K_n,V}(y)$ is the image of a class in $H^1(K_n, T)$.*

2. *(Cherbonnier–Colmez IV.2.1; [25], Theorem II.6.)*
$$\exp^*_{K_n,V^*(1)}\big(h^1_{K_n,V}(y)\big) = p^{-n}\,\partial_V\big(\varphi^{-n}(y)\big).$$

3. *(Kato's formula; [25], Proposition II.5, applied over K_n as in the proof of Theorem II.6.) The map $D_{\mathrm{dR}}(V|_{G_{K_n}}) \to H^1(K_n, B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} V)$, $x \mapsto [\sigma \mapsto \log\chi_{\mathrm{cyc}}(\sigma)\,x]$, is an isomorphism, and $\exp^*_{K_n,V^*(1)}$ is the composite of $H^1(K_n,V) \to H^1(K_n, B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} V)$ with its inverse.*

Proof. Quoted from L. Berger, *Bloch and Kato's exponential map: three explicit formulas*, Doc. Math. Extra Vol. Kato (2003), 99–129 [25]: item 1 is Theorem II.8 (due to Fontaine, proved by Cherbonnier and Colmez, J. Amer. Math. Soc. 12 (1999), II.1) together with the sentence preceding it (for $y \in \mathbf{D}(T)^{\psi=1}$ the $h^1_{K_m,V}(y)$ are compatible under corestriction and define an element of $H^1_{\mathrm{Iw}}(K,T)$); item 2 is Theorem II.6, which the text attributes to Cherbonnier and Colmez, loc. cit., IV.2.1, for $y \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1} \supset \mathbf{D}(V)^{\psi=1}$ (inclusion as in Proposition I.5 there), every crystalline V of G_F, $F = W(k)[1/p]$, and every $n \geq 1$; item 3 is Proposition II.5, which the proof of Theorem II.6 applies over F_n. The map h^1 is that of Proposition I.5 (with the normalisation $\log^0_p(\chi(\gamma))$; for Γ_n with torsion, i.e. $p = 2$ and $n = 1$, by corestriction, §I.3 there). Notation: Berger's F, F_n, $\varepsilon^{(n)}$, χ are K_0, K_n, ζ_{p^n}, χ_{cyc}; the statement is the case $V = \mathbb{Q}_p(-e)$, $y \in \mathbf{D}(T)^{\psi=1}$ of the printed results. The criterion for $\psi(f(\pi)) = f(\pi)$ is the one printed in the proof of Lemma II.1 there. □

Theorem B.49. *Let ℓ be a prime, M a finite extension of $\mathbb{Q}_\ell$ with absolute Galois group G_M, and V a de Rham representation of G_M on a finite-dimensional $\mathbb{Q}_\ell$-vector space; put $V^*(1) = \mathrm{Hom}_{\mathbb{Q}_\ell}(V,\mathbb{Q}_\ell)\otimes\chi_{\mathrm{cyc}}$. Let*
$$\langle\ ,\ \rangle\colon H^1(G_M,V)\times H^1\big(G_M,V^*(1)\big) \longrightarrow H^2\big(G_M,\mathbb{Q}_\ell(1)\big)$$
be the cup product followed by the evaluation map $V\otimes_{\mathbb{Q}_\ell} V^*(1) \to \mathbb{Q}_\ell(1)$. Then $H^1_f(G_M,V)$ and $H^1_f(G_M,V^*(1))$ are exact annihilators of each other: a class $c \in H^1(G_M,V)$ lies in $H^1_f(G_M,V)$ if and only if $\langle c,y\rangle = 0$ for every $y \in H^1_f(G_M,V^*(1))$, and symmetrically.

Proof. [34], Proposition 3.8 (case $l = p$, under the hypothesis made there that V is de Rham). *Notation.* Their K, p, V, V^* are M, ℓ, V, $\mathrm{Hom}_{\mathbb{Q}_\ell}(V,\mathbb{Q}_\ell)$; their $H^1_f(K,V)$ is the kernel of $H^1(K,V) \to H^1(K, B_{\mathrm{crys}}\otimes V)$, ours; the pairing is the cup product into $H^2(K,\mathbb{Q}_p(1)) \cong \mathbb{Q}_p$. □

Theorem B.50. *Let p be a prime, F a finite extension of $\mathbb{Q}_p$, L a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$, and $\mathcal{D}_L = L/\mathcal{O}_L$. For an $\mathcal{O}_L$-module X write $X(1) = X\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(1)$. Let T be a free $\mathcal{O}_L$-module of finite rank with a continuous $\mathcal{O}_L$-linear action of G_F, $V = T\otimes_{\mathcal{O}_L} L$, and*
$$T^* = \mathrm{Hom}_{\mathcal{O}_L}(T,\mathcal{O}_L(1)), \qquad V^* = \mathrm{Hom}_L(V,L(1)) = T^*\otimes_{\mathcal{O}_L} L,$$
$$W^* = V^*/T^* = \mathrm{Hom}_{\mathcal{O}_L}(T,\mathcal{D}_L(1)),$$
with G_F acting by $(gf)(x) = g\,f(g^{-1}x)$, W^* discrete. Let $H^1_f(F,T)$ be the inverse image of $H^1_f(F,V)$ under the natural map $H^1(F,T) \to H^1(F,V)$, and let $H^1_f(F,W^*)$ be the image of $H^1_f(F,V^*)$ under the natural map $H^1(F,V^*) \to H^1(F,W^*)$. Then:

1. *the local invariant gives isomorphisms*
$$H^2(F,\mathcal{O}_L(1)) \cong \mathcal{O}_L, \qquad H^2(F,L(1)) \cong L, \qquad H^2(F,\mathcal{D}_L(1)) \cong \mathcal{D}_L,$$
compatible with the maps induced by $\mathcal{O}_L \subset L \to \mathcal{D}_L$; and the cup products composed with the evaluation maps give $\mathcal{O}_L$-bilinear perfect pairings
$$H^1(F,V)\times H^1(F,V^*) \to L, \qquad H^1(F,T)\times H^1(F,W^*) \to \mathcal{D}_L;$$

2. *if V is de Rham, $H^1_f(F,T)$ and $H^1_f(F,W^*)$ are exact orthogonal complements of each other under the second pairing.*

Proof. Item 1 is local Tate duality with $\mathcal{O}_L$-coefficients, [209], Chapter I, Theorem 4.1 (with $\Phi = L$, $\mathcal{O} = \mathcal{O}_L$, $\mathbf{D} = \mathcal{D}_L$, $i = 1$), which refers to [172], Corollary I.2.3, and [222], II.5.2. Item 2 is [209], Chapter I, Proposition 4.3(i), case (c) (residue characteristic p, with the chosen subspaces $H^1_f(F,V)$, $H^1_f(F,V^*)$, which are orthogonal complements of each other for V de Rham by [34], Proposition 3.8, as recorded with coefficients in L in [209], Chapter I, Remark 7.1; the $\mathbb{Q}_p$-form is Theorem B.49), with $H^1_f(F,T)$ and $H^1_f(F,W^*)$ defined as inverse image and image as in Definition I.3.4 there. *Notation.* Rubin's K, T, V, T^*, V^*, W^*, $\mathbf{D}$, $\langle\ ,\ \rangle_K$ are our F, T, V, T^*, V^*, W^*, $\mathcal{D}_L$ and the pairings of item 1; "orthogonal complements" there means exact annihilators of each other. □

Theorem B.51. *Let p be an odd prime, K a finite unramified extension of $\mathbb{Q}_p$ with residue field k_K, $\mathcal{O}_K = W(k_K)$, and $w\colon \mathcal{O}_K \to W(\mathbb{C}_p^\flat)$ the map of Witt vectors induced by the embedding of k_K into $\mathcal{O}_{\mathbb{C}_p^\flat}$ lifting $k_K \subset \mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}$; $W(\mathbb{C}_p^\flat)$ carries its Witt vector Frobenius φ and the action of G_K. Put $\pi = [\varepsilon] - 1$. Then:*

(Fi1) The rings *Let $w(\mathcal{O}_K)[[\pi]] \subseteq A_{\mathrm{inf}}$ be the ring of the series $\sum_{m\geq 0} w(a_m)\pi^m$, $a_m \in \mathcal{O}_K$ (convergent for the topology of (Ai2), as $\pi \in (p,[p^\flat])$). The p-adic closure $\mathcal{O}_{\mathcal{E}_K}$ in $W(\mathbb{C}_p^\flat)$ of $w(\mathcal{O}_K)[[\pi]][1/\pi]$ is a complete discrete valuation ring with uniformiser p and residue field $\mathbf{E}_K$ (reduction modulo p being the reduction $W(\mathbb{C}_p^\flat) \to \mathbb{C}_p^\flat$, under which $\pi \mapsto \bar{\pi}$); it is stable under φ and G_K, and H_K acts trivially on it. The p-adic closure $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$ in $W(\mathbb{C}_p^\flat)$ of the maximal unramified extension of $\mathcal{O}_{\mathcal{E}_K}$ in $W(\mathbb{C}_p^\flat)$ is stable under φ and G_K, $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}/p\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} = \mathbf{E}^{\mathrm{sep}} \subset \mathbb{C}_p^\flat$, and $(\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}})^{H_K} = \mathcal{O}_{\mathcal{E}_K}$.*

(Fi2) Representations *For a $\mathbb{Z}_p$-module T of finite type with a continuous $\mathbb{Z}_p$-linear action of G_K put $\mathbf{D}(T) = (\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} T)^{H_K}$, with $\varphi\otimes 1$ and the induced action of Γ_K. The functor $T \mapsto \mathbf{D}(T)$ is exact; if T is free of rank d, then $\mathbf{D}(T)$ is a free $\mathcal{O}_{\mathcal{E}_K}$-module of rank d and the natural map $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_K}} \mathbf{D}(T) \to \mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} T$ is a G_K-equivariant isomorphism commuting with φ; if $pT = 0$, $\mathbf{D}(T) = (\mathbf{E}^{\mathrm{sep}} \otimes_{\mathbb{F}_p} T)^{H_K}$ is the module of Theorem B.52.*

(Fi3) Étale φ-modules *If M is a free $\mathcal{O}_{\mathcal{E}_K}$-module of finite rank d with a φ-semilinear map $\varphi_M\colon M \to M$ whose linearisation $\mathcal{O}_{\mathcal{E}_K} \otimes_{\varphi,\mathcal{O}_{\mathcal{E}_K}} M \to M$ is an isomorphism, then $T(M) = (\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_K}} M)^{\varphi=1}$ is a free $\mathbb{Z}_p$-module of rank d and the natural map $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} T(M) \to \mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}_K}} M$ is an isomorphism commuting with φ.*

Proof. This is Fontaine's theory of (φ,Γ)-modules [93], Part A: (Fi3) is the equivalence between $\mathbb{Z}_p$-representations of the absolute Galois group of a field of characteristic p and étale φ-modules over a Cohen ring of it (§A.1), applied to $\mathbf{E}_K$, whose Cohen ring is $\mathcal{O}_{\mathcal{E}_K}$ with the unramified closure $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$ (§A.1, A.3.1); (Fi2) is the equivalence between $\mathbb{Z}_p$-representations of finite type of G_K and étale (φ,Γ_K)-modules over $\mathcal{O}_{\mathcal{E}_K}$, an exact functor (§A.3.4), the case $pT = 0$ being the reduction $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}/p = \mathbf{E}^{\mathrm{sep}}$; (Fi1) is the explicit description of these rings inside $W(\mathrm{Fr}\,R) = W(\mathbb{C}_p^\flat)$ for K unramified, where $\mathcal{O}_{\mathcal{E}_K}$ is the p-adic completion of $\mathcal{O}_K[[\pi]][1/\pi]$ (§A.3.1; also [26], §I). *Notation.* Fontaine's R, $\mathcal{O}_{\mathcal{E}}$, $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$, $D(V)$, $\pi = [\varepsilon] - 1$ are $\mathcal{O}_{\mathbb{C}_p^\flat}$, $\mathcal{O}_{\mathcal{E}_K}$, $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$, $\mathbf{D}(T)$, π. □

Theorem B.52. *Let p be an odd prime, K a finite extension of $\mathbb{Q}_p$, $K_\infty = K(\mu_{p^\infty})$, $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. Let $\mathbf{E}_K$ be the field of norms of K_∞/K, a complete discretely valued field of characteristic p with the Frobenius $\varphi(x) = x^p$ and a continuous action*

of Γ_K commuting with φ, and $\mathbf{E}^{\mathrm{sep}}$ a separable closure of $\mathbf{E}_K$, with its continuous action of G_K commuting with φ, under which $H_K \cong \mathrm{Gal}(\mathbf{E}^{\mathrm{sep}}/\mathbf{E}_K)$ and $(\mathbf{E}^{\mathrm{sep}})^{H_K} = \mathbf{E}_K$. If K is unramified over $\mathbb{Q}_p$ with residue field k_K, then $\mathbf{E}_K = k_K((\bar\pi))$ with $\varphi(\bar\pi) = \bar\pi^p$ and $g(\bar\pi) = (1+\bar\pi)^{\chi_{\mathrm{cyc}}(g)} - 1$ for $g \in \Gamma_K$, where $\bar\pi = \varepsilon - 1$ for a compatible system $\varepsilon = (\zeta_{p^m})_m$ of primitive p^m-th roots of unity.

*An **étale (φ,Γ_K)-module** over $\mathbf{E}_K$ is a finite-dimensional $\mathbf{E}_K$-vector space D with a φ-semilinear map $\varphi_D\colon D \to D$ whose image spans D over $\mathbf{E}_K$, and a continuous semilinear action of Γ_K commuting with φ_D; morphisms are the $\mathbf{E}_K$-linear maps commuting with φ_D and with Γ_K. For a finite-dimensional $\mathbb{F}_p$-vector space V with a continuous linear action of G_K (discrete topology) put*

$$\mathbf{D}(V) = \left(\mathbf{E}^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V\right)^{H_K},$$

with $\varphi_{\mathbf{D}(V)} = \varphi \otimes 1$ and the action of $\Gamma_K = G_K/H_K$ induced by the diagonal action of G_K. Then:

(Fe1) *$\mathbf{D}(V)$ is an étale (φ,Γ_K)-module with $\dim_{\mathbf{E}_K} \mathbf{D}(V) = \dim_{\mathbb{F}_p} V$, and the natural map $\mathbf{E}^{\mathrm{sep}} \otimes_{\mathbf{E}_K} \mathbf{D}(V) \to \mathbf{E}^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V$ is a G_K-equivariant isomorphism commuting with φ;*

(Fe2) *$V \mapsto \mathbf{D}(V)$ is an exact equivalence of categories from the category of such representations (morphisms: G_K-equivariant $\mathbb{F}_p$-linear maps) to the category of étale (φ,Γ_K)-modules over $\mathbf{E}_K$, with quasi-inverse $D \mapsto \left(\mathbf{E}^{\mathrm{sep}} \otimes_{\mathbf{E}_K} D\right)^{\varphi=1}$.*

Proof. This is Fontaine's theorem [93], Part A, §A.3.4 (Théorème A.3.4.3), in the case of representations killed by p (Fontaine's construction in characteristic 0 through Γ_K is made for $p \neq 2$): Fontaine proves the equivalence between $\mathbb{Z}_p$-representations of finite type of G_K and étale (φ,Γ_K)-modules over $\mathcal{O}_{\mathcal{E}_K}$, and $\mathcal{O}_{\mathcal{E}_K}/p = \mathbf{E}_K$, $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}/p = \mathbf{E}^{\mathrm{sep}}$; the case killed by p is the underlying statement for étale φ-modules over $\mathbf{E}_K$ (Fontaine, §A.1), to which the action of Γ_K is added by functoriality. The field of norms is that of Fontaine and Wintenberger [97]; its description for K unramified is Fontaine, §A.3.1. *Notation.* Fontaine's E_K, E^{sep}, $D(V)$, $\Phi\Gamma^{\mathrm{et}}(E_K)$ are $\mathbf{E}_K$, $\mathbf{E}^{\mathrm{sep}}$, $\mathbf{D}(V)$ and the étale (φ,Γ_K)-modules above. □

Theorem B.53. *Let p be a prime and K an extension of $\mathbb{Q}_p$, complete for a valuation $v\colon K \to \mathbb{R}\cup\{+\infty\}$ with $v(p) = 1$ (no hypothesis on the residue field; v need not be discrete); let $\overline{K}$ be an algebraic closure of K, v the extended valuation and $\mathcal{O}_{\overline{K}}$ its valuation ring. Let Γ be a commutative finite flat group scheme over $\mathcal{O}_K$ of order a power of p, with Cartier dual Γ^D and ω_{Γ^D} the $\mathcal{O}_K$-module of invariant differentials of Γ^D. For $x \in \Gamma(\mathcal{O}_{\overline{K}}) = \mathrm{Hom}(\Gamma^D_{\mathcal{O}_{\overline{K}}}, \mathbb{G}_m)$ let $\alpha_\Gamma(x) = x^*\frac{dT}{T} \in \omega_{\Gamma^D} \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}$ (the Hodge–Tate map). Then the cokernel of*

$$\alpha_\Gamma \otimes 1\colon\ \Gamma(\mathcal{O}_{\overline{K}}) \otimes_{\mathbb{Z}} \mathcal{O}_{\overline{K}} \longrightarrow \omega_{\Gamma^D} \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}$$

is killed by every element of $\mathcal{O}_{\overline{K}}$ of valuation $\geq \frac{1}{p-1}$: for every $a \in \mathcal{O}_{\overline{K}}$ with $v(a) \geq \frac{1}{p-1}$, $a\,(\omega_{\Gamma^D} \otimes \mathcal{O}_{\overline{K}})$ is contained in the $\mathcal{O}_{\overline{K}}$-span of $\alpha_\Gamma(\Gamma(\mathcal{O}_{\overline{K}}))$.

Proof. Fargues, La filtration canonique des points de torsion des groupes p-divisibles [89], Section 5.4, Théorème 3 (p. 935), deduced there from Théorème 2 and attributed to Théorème 6 of Fargues, La filtration de Harder–Narasimhan des schémas en groupes finis et plats, J. reine angew. Math. 645 (2010); standing notation of the paper (p. 911 and p. 911–912): K complete for a real valuation with $v(p) = 1$, no hypothesis on the residue field, finite flat group schemes commutative of order a power of p, $\alpha_G\colon G(\mathcal{O}_{\overline{K}}) \to \omega_{G^D} \otimes \mathcal{O}_{\overline{K}}$. Notation: Fargues writes G for Γ. □

Theorem B.54. *Let p be a prime and K an extension of $\mathbb{Q}_p$, complete for a valuation $v\colon K \to \mathbb{R}\cup\{+\infty\}$ with $v(p)=1$ (no hypothesis on the residue field); let C be the completion of an algebraic closure of K, $\mathcal{O}_C$ its ring of integers, and v the extended valuation. Let H be a p-divisible group over $\mathcal{O}_K$ with Cartier dual H^D, Tate module $T_p(H)$, and Hodge–Tate map $\alpha_H\colon T_p(H)\to \omega_{H^D}\otimes\mathcal{O}_C$, $x\mapsto (x^D)^*\frac{dT}{T}$. Then the cohomology of the complex of $\mathcal{O}_C$-modules*

$$0 \longrightarrow \mathrm{Lie}(H)\otimes\mathcal{O}_C(1) \xrightarrow{{}^t(\alpha_{H^D}\otimes 1)(1)} T_p(H)\otimes_{\mathbb{Z}_p}\mathcal{O}_C \xrightarrow{\alpha_H\otimes 1} \omega_{H^D}\otimes\mathcal{O}_C \longrightarrow 0$$

is killed by every element of $\mathcal{O}_C$ of valuation $\geq \frac{1}{p-1}$. In particular the left map is injective, the composite is zero, and every $\kappa\in\mathcal{O}_C$ with $v(\kappa)\geq\frac{1}{p-1}$ satisfies $\kappa\,(\omega_{H^D}\otimes\mathcal{O}_C)\subseteq \mathrm{im}(\alpha_H\otimes 1)$.

Proof. Fargues, La filtration canonique des points de torsion des groupes p-divisibles [89], Théorème 2 (Section 5.3.2), which is Théorème II.1.1 of [88]; the finite flat version (cokernel of $\alpha_G\otimes 1$ for a finite flat group scheme G over $\mathcal{O}_K$ killed by elements of valuation $\geq\frac{1}{p-1}$) is Théorème 3 of [89]. Notation: Fargues writes $\omega_H^\vee$ for $\mathrm{Lie}(H)$, $\mathcal{O}_{\widehat{\overline{K}}}$ for $\mathcal{O}_C$. □

Theorem B.55. *Let L be a finite unramified extension of $\mathbb{Q}_p$ with Frobenius automorphism σ, and V a crystalline representation of G_L on a finite-dimensional $\mathbb{Q}_p$-vector space. Then $D=D_{\mathrm{cris}}(V)=(B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}V)^{G_L}$ is an L-vector space of dimension $\dim_{\mathbb{Q}_p}V$ with a bijective σ-semilinear endomorphism φ (from $\varphi\otimes 1$), the inclusion $B_{\mathrm{cris}}\subset B_{\mathrm{dR}}$ induces an isomorphism $D\cong D_{\mathrm{dR}}(V)=(B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}V)^{G_L}$, through which D carries the exhaustive separated decreasing filtration $\mathrm{Fil}^iD=(\mathrm{Fil}^iB_{\mathrm{dR}}\otimes V)^{G_L}$, and D is **weakly admissible**: for every φ-stable L-subspace $D'\subseteq D$, with the induced filtration $\mathrm{Fil}^iD'=D'\cap\mathrm{Fil}^iD$,*

$$t_H(D')\leq t_N(D'),\qquad \text{with equality for } D'=D,$$

where $t_H(D')=\sum_{i\in\mathbb{Z}} i\dim_L(\mathrm{Fil}^iD'/\mathrm{Fil}^{i+1}D')$ and $t_N(D')=v_p(\det A)$ for the matrix A of $\varphi|_{D'}$ in any L-basis of D' ($v_p(p)=1$; this does not depend on the basis).

Proof. [96], Proposition 5.4.2 (admissible filtered φ-modules are weakly admissible; here with $N=0$), with the definition of t_H, t_N in Section 4 there and the equivalence of crystalline representations with admissible filtered φ-modules (Section 5); see also [70] (the converse, not used) and [52], Theorem 9.3.4 and Section 8.2. For L unramified, $L_0=L$ and $D_{\mathrm{cris}}(V)\otimes_{L_0}L=D_{\mathrm{cris}}(V)$. Independence of the basis: a change of basis by P replaces A by $P^{-1}A\sigma(P)$, and $v_p(\det\sigma(P))=v_p(\det P)$. □

Theorem B.56. *Let ℓ be a prime, M a finite extension of $\mathbb{Q}_\ell$ with absolute Galois group G_M, and V a de Rham representation of G_M on a finite-dimensional $\mathbb{Q}_\ell$-vector space. Let $D_{\mathrm{dR}}(V)=(B_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell}V)^{G_M}$ with the filtration $\mathrm{Fil}^iD_{\mathrm{dR}}(V)=(\mathrm{Fil}^iB_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell}V)^{G_M}$, and let $H^1_f(G_M,V)$ be the kernel of $H^1(G_M,V)\to H^1(G_M,B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V)$. Then*

$$\dim_{\mathbb{Q}_\ell}H^1_f(G_M,V)=\dim_{\mathbb{Q}_\ell}(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0D_{\mathrm{dR}}(V))+\dim_{\mathbb{Q}_\ell}H^0(G_M,V).$$

Proof. [34], Corollary 3.8.4 ("In particular,

$$\dim_{\mathbf{Q}_p}H^1_f(K,V)=\dim_{\mathbf{Q}_p}(DR(V)/DR(V)^0)+\dim_{\mathbf{Q}_p}H^0(K,V)",$$

for V a de Rham representation of $\mathrm{Gal}(\overline{K}/K)$, K finite over $\mathbf{Q}_p$). *Notation.* Their K, p, $DR(V)$, $DR(V)^0$ are M, ℓ, $D_{\mathrm{dR}}(V)$, $\mathrm{Fil}^0D_{\mathrm{dR}}(V)$; their $H^1_f(K,V)$ is the kernel of $H^1(K,V)\to H^1(K,B_{\mathrm{crys}}\otimes V)$, which is the local condition of Definition 22.2 at a place above ℓ (a class lies in the kernel exactly when a representing cocycle c satisfies $c(g)\otimes 1=gy-y$ for some $y\in V\otimes B_{\mathrm{cris}}$). □

Theorem B.57. *Let ℓ be a prime, M a finite extension of $\mathbb{Q}_\ell$ with absolute Galois group G_M, and $\mathcal{O}' \subset \overline{\mathbb{Q}}_\ell$ a coefficient ring with residue field $\mathbb{F}'$ whose fraction field contains the image of every field embedding $M \to \overline{\mathbb{Q}}_\ell$. Let $n \geq 1$, $\bar\rho\colon G_M \to \mathrm{GL}_n(\mathbb{F}')$ continuous, and $(R^\square, \rho^\square)$ the universal lifting ring of $\bar\rho$: $R^\square$ is a complete local Noetherian $\mathcal{O}'$-algebra with residue field $\mathbb{F}'$, $\rho^\square\colon G_M \to \mathrm{GL}_n(R^\square)$ is a continuous lift of $\bar\rho$, and for every complete local Noetherian $\mathcal{O}'$-algebra B with residue field $\mathbb{F}'$ every continuous lift of $\bar\rho$ to $\mathrm{GL}_n(B)$ is $f \circ \rho^\square$ for exactly one local $\mathcal{O}'$-algebra homomorphism $f\colon R^\square \to B$. For every field embedding $\tau\colon M \to \overline{\mathbb{Q}}_\ell$ let H_τ be a multiset of n integers. Then there is an ideal $\mathfrak{a} \subseteq R^\square$ such that $R^\square/\mathfrak{a}$ is reduced and has no ℓ-torsion, and such that for every valuation ring A of a finite extension of the fraction field of $\mathcal{O}'$ inside $\overline{\mathbb{Q}}_\ell$ and every $\mathcal{O}'$-algebra homomorphism $x\colon R^\square \to A$ with $x(\mathfrak{m}_{R^\square}) \subseteq \mathfrak{m}_A$: $x(\mathfrak{a}) = 0$ if and only if the representation $x \circ \rho^\square$ of G_M is crystalline and has τ-Hodge–Tate weights H_τ for every τ.*

Proof. See [17], Section 1.4 (the rings $R^\square_{\mathcal{O},\bar\rho,\{H_\tau\},\mathrm{cris}}$), whose existence is deduced there from [144], Corollary 2.6.2 and Theorem 2.5.5(2).

The published statement. With $\mathcal{O}$ the ring of integers of a finite extension L of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ containing the image of each embedding $K \to \overline{\mathbb{Q}}_\ell$, $K/\mathbb{Q}_\ell$ finite, $\bar\rho\colon G_K \to \mathrm{GL}_n(\mathbb{F})$ and n-element multisets $\{H_\tau\}$ of integers: the universal lifting ring $R^\square_{\mathcal{O},\bar\rho}$ has a quotient $R^\square_{\mathcal{O},\bar\rho,\{H_\tau\},\mathrm{cris}}$, uniquely characterised by being reduced without ℓ-torsion and by the property that a $\overline{\mathbb{Q}}_\ell$-point of $R^\square_{\mathcal{O},\bar\rho}$ factors through it if and only if it corresponds to a representation $G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which is de Rham with $\mathrm{HT}_\tau = H_\tau$ for all τ and crystalline.

Comparison of notation. Their K, L, $\mathcal{O}$, $\mathbb{F}$ are our M, the fraction field of $\mathcal{O}'$, $\mathcal{O}'$, $\mathbb{F}'$; $\mathfrak{a}$ is the kernel of $R^\square \to R^\square_{\mathcal{O}',\bar\rho,\{H'_\tau\},\mathrm{cris}}$. In [17] the cyclotomic character has Hodge–Tate weight -1, so their multisets are $H'_\tau = \{-h : h \in H_\tau\}$. A homomorphism x as in the statement is a $\overline{\mathbb{Q}}_\ell$-point (it is local and lands in the valuation ring of a finite extension), and crystalline representations are de Rham. □

B.5 Automorphic forms and Galois representations

The automorphic input is used in Parts VII–IX and Appendix E. Its core is the Galois representations attached to automorphic representations, with local–global compatibility: Deligne's representations of newforms (Theorem B.59), the conjugate self-dual and polarizable cases of Shin, Chenevier–Harris, Barnet-Lamb–Gee–Geraghty–Taylor and Caraiani, with purity (Theorem B.58, Theorem B.62, Theorem B.73, Theorem B.90), Thorne's ordinary criterion (Theorem B.94) and the corresponding statements for GL_2 over $\mathbb{Q}$ and over totally real fields, together with the strong multiplicity one of Jacquet and Shalika (Theorem B.60); they are used in the counts of the plus part (Section 38), which also quotes the classical theory of newforms, Clozel's theorems on rationality and cohomological representations, Borel's injectivity of cuspidal cohomology and the valence formula, and in most sections of Part IX and in Appendix E. Part VIII uses the vanishing of adjoint Bloch–Kato Selmer groups of Newton and Thorne and of Thorne (Theorem B.68, Theorem B.69, Theorem B.70, Theorem B.72) in the one-character bounds (Sections 42 and 44), and Mazur's representability of framed deformations (Theorem B.71) for its deformation rings, which Part IX and Appendix E use as well. Part IX moves automorphy between fields by base change, descent and automorphic induction (Arthur and Clozel, Langlands, Barnet-Lamb–Gee–Geraghty–Taylor, Gelbart and Jacquet), and it quotes Katz's modular forms over rings and Hida's nearly ordinary families of Hilbert modular forms. Chenevier's theory of determinants (Theorem B.83 and the related statements of Chenevier and of Bellaïche–Chenevier) is used in Sections 50 and 53 and in Appendix E, which quotes in addition Geraghty's control theorems for ordinary forms on definite unitary groups (Theorem B.91,

Theorem B.98), the local Langlands correspondence for GL_n, Labesse's base change for unitary groups and the description of the discrete spectrum of GL_n by Moeglin and Waldspurger.

Theorem B.58. *Let E be a CM field, $n \geq 1$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and π a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ that is regular algebraic and conjugate self-dual. Then there is a continuous semisimple representation $r_{\ell,\iota}(\pi)\colon G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that for every finite place $v \nmid \ell$ of E at which π_v is unramified, every place $\bar{v}$ of $\overline{\mathbb{Q}}$ above v and every arithmetic Frobenius σ at $\bar{v}$: $r_{\ell,\iota}(\pi)$ is trivial on the inertia subgroup $I_{\bar{v}}$, and*

$$\iota\Big(\det\big(X - r_{\ell,\iota}(\pi)(\sigma^{-1})\big)\Big) = \prod_{\alpha}(X - q_v^{(n-1)/2}\alpha),$$

α running over the Satake parameters of π_v with multiplicity. In particular $r_{\ell,\iota}(\pi) \leftrightarrow_\iota \pi$.

Proof. [67], Theorem 3.2.3 (all n), and [231], Theorem 1.2 (n odd, or a mild condition at infinity for n even), building on [121] and [69]. For cuspidal π, regular algebraic in the sense of [68] is the same as cohomological, the hypothesis of these theorems ([68], Lemma 3.14). *Notation.* The theorems give, at every finite place $v \nmid \ell$, the compatibility $\iota\,\mathrm{WD}(r_{\ell,\iota}(\pi)|_{G_{E_v}})^{\text{F-ss}} \prec \mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2})$ (equality of semisimplifications, the monodromy of the left side dominated by that of the right), with rec normalised as in [121] (uniformisers to geometric Frobenius elements). For unramified π_v, contained in the normalised induction of the unramified character with values $\alpha_1, \ldots, \alpha_n$ at ϖ_v, the right side is unramified with zero monodromy and geometric Frobenius acting with eigenvalues $q_v^{(n-1)/2}\alpha_i$. So the Weil–Deligne representation of $r_{\ell,\iota}(\pi)$ at v has zero monodromy and an inertia action, through a finite quotient, whose semisimplification is trivial; it is therefore trivial, $r_{\ell,\iota}(\pi)$ is trivial on $I_{\bar{v}}$ for every $\bar{v} \mid v$, and σ^{-1} (a geometric Frobenius) has the stated characteristic polynomial. □

Theorem B.59. *Let $g = \sum_{n\geq 1} a_n q^n$ be a normalised newform of weight $k \geq 2$, level L and character ε.*

(Dg1) *For every prime ℓ and every isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ there is a continuous representation $\rho_{g,\iota}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that for every prime $p \nmid \ell L$ and every place $\bar{p}$ of $\overline{\mathbb{Q}}$ above p, $\rho_{g,\iota}$ is trivial on $I_{\bar{p}}$ and $\iota(\det(X - \rho_{g,\iota}(\sigma^{-1}))) = X^2 - a_pX + \varepsilon(p)p^{k-1}$ for every arithmetic Frobenius σ at $\bar{p}$.*

(Dg2) *For every prime $p \nmid L$, both roots of $X^2 - a_pX + \varepsilon(p)p^{k-1}$ have absolute value $p^{(k-1)/2}$.*

Proof. (Dg1): [76], Théorème 6.1 (for $k = 2$ the representation on the Tate module of the Jacobian, Eichler–Shimura and Igusa). Deligne's representation is cut out of étale cohomology, where the trace of geometric Frobenius is a_p; in the other common normalisation (arithmetic Frobenius with characteristic polynomial $X^2 - a_pX + \varepsilon(p)p^{k-1}$) the contragredient has the property stated, since the eigenvalues of $\rho^\vee(\sigma^{-1})$ are those of $\rho(\sigma)^{-1}$ inverted, i.e. those of $\rho(\sigma)$. (Dg2): [77], Théorème 8.2, with the reduction in [76] of the Ramanujan–Petersson conjecture to the Weil conjectures. □

Theorem B.60. *Let F be a number field and $n \geq 1$.*

(Js1) *If π, π' are cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ with $\pi_v \cong \pi'_v$ for all but finitely many places v of F, then $\pi \cong \pi'$.*

(Js2) *If π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ whose central character is unitary, and v is a finite place at which π_v is unramified, then every Satake parameter α of π_v satisfies $q_v^{-1/2} < |\alpha| < q_v^{1/2}$.*

Proof. [136]: (Js1) is the strong multiplicity one theorem (part II, Theorem 4.4; see also [198]); (Js2) is part I, Corollary 2.5, for the unitary cuspidal representation π (a cuspidal automorphic representation with unitary central character is unitary). *Notation.* Their Hecke matrix of π_v has as eigenvalues the Satake parameters, the values at ϖ_v of the unramified characters whose normalised induction contains π_v. □

Theorem B.61. *Let $k \geq 2$ and $L \geq 1$ be integers, and let $S_k(\Gamma_1(L))$ be the space of cusp forms of weight k for $\Gamma_1(L)$.*

(Nw1) *Distinct normalised newforms of weight k whose levels divide L (of any characters) are linearly independent elements of $S_k(\Gamma_1(L))$.*

(Nw2) *The Fourier coefficients a_n of a normalised newform are algebraic integers.*

(Nw3) *If $g = \sum_{n\geq 1} a_n q^n$ is a normalised newform of weight k, level L' and character ε, and $\sigma \in \mathrm{Aut}(\mathbb{C})$, then $g^\sigma = \sum_{n\geq 1} \sigma(a_n) q^n$ is a normalised newform of weight k, level L' and character $\sigma \circ \varepsilon$.*

Proof. (Nw1): the newform theory of [13] and, for $\Gamma_0(L)$ with character (hence for $\Gamma_1(L)$, the direct sum of these spaces over the characters modulo L), [161]: $S_k(\Gamma_1(L))$ is the direct sum, over the normalised newforms g of level $L' \mid L$, of the spans of the forms $g(dz)$, $d \mid L/L'$; in particular the newforms themselves are linearly independent. (Nw2) and (Nw3): [229], Theorem 3.52 and the discussion following it ($S_k(\Gamma_1(L))$ has a basis of forms with rational integral Fourier coefficients, stable under the Hecke operators; $\mathrm{Aut}(\mathbb{C})$ acts on q-expansions compatibly with the Hecke operators and the diamond operators, preserving the space of newforms of each level). □

Theorem B.62. *Let F be a CM field with maximal totally real subfield F^+ and complex conjugation c, $n \geq 1$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and π a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which is **polarizable**: there is a continuous character $\chi\colon \mathbb{A}_{F^+}^\times/(F^+)^\times \to \mathbb{C}^\times$, with $\chi_v(-1)$ independent of the infinite place v of F^+, such that $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathrm{N}_{F/F^+} \circ \det)$. Then there is a continuous semisimple representation $r_{\ell,\iota}(\pi)\colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that, for every finite place v of F and every place $\bar{v}$ of $\overline{\mathbb{Q}}$ above v, writing $r_{\bar{v}}$ for the restriction of $r_{\ell,\iota}(\pi)$ to $D_{\bar{v}}$ (identified with G_{F_v}), $W_{\bar{v}} \subseteq D_{\bar{v}}$ for its Weil group and $\mathcal{O}_v = \mathcal{O}_{F_v}$:*

(Bg1) *if $v \nmid \ell$ and π_v is unramified, then $r_{\ell,\iota}(\pi)$ is trivial on $I_{\bar{v}}$ and $\iota(\det(X - r_{\ell,\iota}(\pi)(\sigma^{-1}))) = \prod_\alpha (X - q_v^{(n-1)/2}\alpha)$ for every arithmetic Frobenius σ at $\bar{v}$, α running over the Satake parameters of π_v with multiplicity; in particular $r_{\ell,\iota}(\pi) \leftrightarrow_\iota \pi$;*

(Bg2) *if $v \mid \ell$, then $r_{\bar{v}}$ is de Rham, and for every field embedding $\tau\colon F_v \to \overline{\mathbb{Q}}_\ell$ over $\mathbb{Q}_\ell$ the τ-Hodge–Tate weights of $r_{\bar{v}}$ are the n numbers $z - \frac{n-1}{2}$, z running over the infinitesimal character of π at the complex embedding $\iota \circ \tau|_F$ of F;*

(Bg3) *if $U \subseteq \mathcal{O}_v^\times$ is a subgroup and there are characters $\Psi_1, \ldots, \Psi_n$ of $W_{\bar{v}}$, each trivial on the image of $\mathrm{Art}_{F_v}(U)$ in $W_{\bar{v}}^{\mathrm{ab}}$, such that the semisimplification of the restriction to $I_{\bar{v}}$ of the representation ρ of $\mathrm{WD}(r_{\bar{v}})$ is isomorphic to $\bigoplus_i \Psi_i|_{I_{\bar{v}}}$ (for $v \mid \ell$, $\mathrm{WD}(r_{\bar{v}})$ is defined by*

(Bg2), since de Rham representations are potentially semistable), then π_v is isomorphic to an irreducible subquotient of the normalised parabolic induction from the upper triangular Borel subgroup of $\mathrm{GL}_n(F_v)$ of $\xi_1 \otimes \cdots \otimes \xi_n$, for some characters $\xi_1, \dots, \xi_n$ of $F_v^\times$ trivial on U;

(Bg4) *if $v \mid \ell$ and π_v is unramified, then $r_{\bar v}$ is crystalline, $\mathrm{WD}(r_{\bar v}) = (V, \rho, N)$ has $N = 0$ and ρ trivial on $I_{\bar v}$, and $\iota(\det(X - \rho(\Phi))) = \prod_\alpha (X - q_v^{(n-1)/2}\alpha)$ for every $\Phi \in W_{\bar v}$ with $n(\Phi) = -1$ (a geometric Frobenius), α as in (Bg1);*

(Bg5) *if π_v is isomorphic to an irreducible subquotient of the normalised parabolic induction from the upper triangular Borel subgroup of $\mathrm{GL}_n(F_v)$ of $\xi_1 \otimes \cdots \otimes \xi_n$, for characters $\xi_1, \dots, \xi_n$ of $F_v^\times$, then the semisimplification of the representation ρ of $\mathrm{WD}(r_{\bar v})$ is isomorphic to $\lambda_1 \oplus \cdots \oplus \lambda_n$, where λ_i is the character of $W_{\bar v}$ with $\iota(\lambda_i(\mathrm{Art}_{F_v}(x)^{-1})) = \xi_i(x)\,|x|_v^{(1-n)/2}$ for $x \in F_v^\times$ (characters of $W_{\bar v}$ factor through $W_{\bar v}^{\mathrm{ab}}$, identified with $F_v^\times$ by Art_{F_v});*

(Bg7) *if $v \nmid \ell$, then the Frobenius-semisimplification of $\mathrm{WD}(r_{\bar v})$, transported to $\mathbb{C}$ by ι, is isomorphic to $\mathrm{rec}_{F_v}(\pi_v \otimes |\det|_v^{(1-n)/2})$, where rec_{F_v} is the local Langlands correspondence of Harris–Taylor (uniformisers corresponding to geometric Frobenius elements); in particular the monodromy operators of $\mathrm{WD}(r_{\bar v})$ and of $\mathrm{rec}_{F_v}(\pi_v)$ are conjugate.*

Proof. [17], Theorem 2.1.1 (in the published version, which incorporates [56] and [58]), together with [18], Theorem 1.1, and [58], Theorem 1.1, for places above ℓ at which π_v has no Iwahori fixed vector.

The published statements. For a regular algebraic cuspidal polarized automorphic representation (π, χ) of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight a (that is, π_∞ has the infinitesimal character of the dual of the algebraic representation Ξ_a with highest weights $a_{\tau'}$, $\tau' \colon F \to \mathbb{C}$), Theorem 2.1.1 of [17] gives a continuous semisimple $r_{\ell,\iota}(\pi)$ with: (2) for $v \nmid \ell$, $\iota\mathrm{WD}(r_{\ell,\iota}(\pi)|_{G_{F_v}})^{F\text{-}ss} \cong \mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2})$; (3) $r_{\ell,\iota}(\pi)$ is de Rham and $\mathrm{HT}_\tau = \{a_{\iota\tau,1} + n - 1, a_{\iota\tau,2} + n - 2, \dots, a_{\iota\tau,n}\}$, in the convention in which the cyclotomic character has Hodge–Tate weight -1; (4) for $v \mid \ell$ with π_v having an Iwahori fixed vector, the isomorphism of (2) holds, $r_{\ell,\iota}(\pi)$ is semistable at v and crystalline if π_v is unramified. Here rec is the local Langlands correspondence of [121], normalised through the Artin map sending uniformisers to geometric Frobenius elements. For an arbitrary $v \mid \ell$, [18] gives $\mathrm{WD}(r_{\ell,\iota}(\pi)|_{G_{F_v}})^{\mathrm{ss}} \cong \iota^{-1}\mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2})^{\mathrm{ss}}$ (its Theorem 1.1, for (π, χ) regular algebraic, essentially conjugate self-dual and cuspidal over an imaginary CM field, which is our polarizability; there the Frobenius semisimplification of the left side is $\prec$ the right side, which contains the isomorphism of semisimplifications), and the isomorphism up to Frobenius semisimplification when π has Shin-regular weight (automatic for n odd); [58], Theorem 1.1, gives the latter for all π with $\pi^\vee \cong \pi \circ c$. Only the semisimplified form is used in (Bg1)–(Bg5); (Bg7) is (2) itself, for $v \nmid \ell$ (with $\mathrm{WD}(\cdot)^{F\text{-}ss}$ the Frobenius-semisimplification, which keeps the monodromy operator; twisting by the unramified character $|\det|_v^{(1-n)/2}$ does not change it, by (Ln2) of Theorem B.92). In [17], a polarized automorphic representation is a pair (π, χ) as in our hypothesis; the further normalisation $\chi_v(-1) = (-1)^n$ for F imaginary is achieved by replacing χ by $\chi\delta_{F/F^+}$, which does not change $\chi \circ \mathrm{N}_{F/F^+}$. Regular algebraic means there that π_∞ has the infinitesimal character of an irreducible algebraic representation, which is the notion fixed in Section 13.

Notation. (Bg2): the infinitesimal character of $\Xi_a^\vee$ at $\iota\tau$ is $\{-(a_{\iota\tau,i} + \frac{n+1}{2} - i)\}_i = \{\frac{n-1}{2} - h : h \in \mathrm{HT}_\tau\}$ (Definition 38.10: V_μ has infinitesimal character $\{\mu_i + \frac{n+1}{2} - i\}$, its dual the negatives); changing the sign convention of Hodge–Tate weights gives $\{z - \frac{n-1}{2}\}$. (Bg1): if

π_v is unramified and contained in the normalised induction of unramified characters μ_i with $\mu_i(\varpi_v) = \alpha_i$, then $\mathrm{rec}(\pi_v \otimes |\det|^{(1-n)/2}) = \bigoplus_i \mu_i |\cdot|^{(1-n)/2} \circ \mathrm{Art}^{-1}$ is unramified with $N = 0$ and a geometric Frobenius acting with eigenvalues $q_v^{(n-1)/2}\alpha_i$; by (2), $\mathrm{WD}(r)$ has $N = 0$ and trivial inertia action, so r is trivial on $I_{\bar v}$ by (Wd1), and σ^{-1} is a geometric Frobenius. (Bg4): the same computation with (4). (Bg3): the correspondence rec maps characters ξ of $F_v^\times$ to $\xi\circ\mathrm{Art}^{-1}$, supercuspidal representations of $\mathrm{GL}_m(F_v)$ to irreducible m-dimensional representations of W_{F_v}, and is compatible with supercuspidal supports: if π_v is an irreducible subquotient of the normalised induction of supercuspidals $\sigma_1 \otimes \cdots \otimes \sigma_k$, then $\mathrm{rec}(\pi_v)^{\mathrm{ss}} \cong \bigoplus_j \mathrm{rec}(\sigma_j)$ ([121], through the classification of [275]; [125]). Under the hypothesis of (Bg3), every irreducible constituent $\rho_j = \mathrm{rec}(\sigma_j)$ of the semisimplification (twisted by the unramified $|\cdot|^{(1-n)/2}$) has a restriction to $I_{\bar v}$ containing some $\Psi_i|_{I_{\bar v}}$; then $\mathrm{Hom}_{I_{\bar v}}(\Psi_i, \rho_j) \neq 0$ is stable under $W_{\bar v}$, so $\rho_j \otimes \Psi_i^{-1}$ has nonzero inertia invariants stable under $W_{\bar v}$, is unramified by irreducibility, hence one-dimensional (an irreducible representation of $W_{\bar v}/I_{\bar v} \cong \mathbb{Z}$). So each σ_j is a character ξ_j of $F_v^\times$, π_v is a subquotient of the normalised induction of $\xi_1 \otimes \cdots \otimes \xi_n$, and $\{\xi_j \circ \mathrm{Art}^{-1}|_I\} = \{\Psi_i|_I\}$ as multisets; as Art maps $\mathcal{O}_v^\times$ onto the image of inertia in $W_{\bar v}^{\mathrm{ab}}$ (in either normalisation, the two differing by $x \mapsto x^{-1}$), each ξ_j is trivial on U. (Bg5): by (2) for $v \nmid \ell$, and by [18], Theorem 1.1, for $v \mid \ell$, $\mathrm{WD}(r_{\bar v})^{\mathrm{ss}} \cong \iota^{-1}\mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2})^{\mathrm{ss}}$ (a Weil–Deligne representation and its Frobenius semisimplification have the same semisimplification). The supercuspidal support of $\pi_v \otimes |\det|_v^{(1-n)/2}$ is $(\xi_i|\cdot|_v^{(1-n)/2})_i$, so by the compatibility of rec with supercuspidal supports the right side is $\bigoplus_i \iota^{-1}\mathrm{rec}(\xi_i|\cdot|_v^{(1-n)/2})$, and $\mathrm{rec}(\xi)$ is the character λ with $\lambda(\mathrm{Art}_{F_v}(x)^{-1}) = \xi(x)$: the normalisation of [121] sends uniformisers to geometric Frobenius elements, that of Art_{F_v} to arithmetic ones. □

Theorem B.63. *Let K be an imaginary quadratic field, $n \geq 1$, π a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then there is a regular algebraic cuspidal automorphic representation π' of $\mathrm{GL}_n(\mathbb{A}_K)$ with $\pi'_f \cong {}^\sigma\pi_f = \pi_f \otimes_{\mathbb{C},\sigma} \mathbb{C}$, whose infinitesimal character at $\sigma \circ \phi$ equals that of π at ϕ for both embeddings $\phi\colon K \to \mathbb{C}$.*

Proof. [68], Théorème 3.13: for a regular algebraic cuspidal π of GL_n over a number field, cohomological for the coefficient system V (Lemme 3.14), and $\sigma \in \mathrm{Aut}(\mathbb{C})$, there is a cuspidal representation π', cohomological for ${}^\sigma V$, with $\pi'_f \cong {}^\sigma\pi_f$ (and π_f is defined over a number field). *Notation.* Here ${}^\sigma V$ is the algebraic representation whose factor at the embedding $\sigma \circ \phi$ is the factor of V at ϕ; by the cohomological criterion the infinitesimal character of π' at $\sigma \circ \phi$ is that of the dual of this factor, i.e. that of π at ϕ, and π' is regular algebraic. □

Theorem B.64. *Let K be an imaginary quadratic field, $n \geq 1$, $\mu, \mu' \in \mathbb{Z}^n$ dominant and $U \subseteq \mathrm{GL}_n(\mathbb{A}_{K,f})$ an open compact subgroup. If S is a finite set of pairwise non-isomorphic cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_K)$, each cohomological for $V_{\mu,\mu'}$ and with $\pi_f^U \neq 0$, then $|S| \leq h^*(U, V_{\mu,\mu'})$.*

Proof. [40], Corollary 5.5: for an arithmetic group the map from cuspidal cohomology to the full cohomology is injective. *Notation.* The locally symmetric space $\mathrm{GL}_n(K)\backslash(G_\infty/K_\infty \times \mathrm{GL}_n(\mathbb{A}_{K,f})/U)$ is the disjoint union of the $\Gamma_j(U)\backslash G_\infty/K_\infty$, whose cohomology with coefficients in the local system of $V_{\mu,\mu'}$ is $H^*(\Gamma_j(U), V_{\mu,\mu'})$ (complex coefficients, G_∞/K_∞ contractible, stabilisers finite); its cuspidal cohomology is $\bigoplus_\pi m(\pi)\, H^*(\mathfrak{g}, K_\infty; \pi_\infty \otimes V_{\mu,\mu'}) \otimes \pi_f^U$ over the isomorphism classes of cuspidal π ([68], §3; $m(\pi) = 1$ for GL_n, [228]), and each $\pi \in S$ contributes at least 1 to its dimension. □

Theorem B.65. *Let K be an imaginary quadratic field with embeddings $\tau, \bar\tau$, $n \geq 1$, and $\mu, \mu' \in \mathbb{Z}^n$ dominant. Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$ whose infinitesimal*

character at τ is that of the dual of V_μ and at $\bar\tau$ that of the dual of $V_{\mu'}$. Then π is cohomological for $V_{\mu,\mu'}$.

Proof. [68], Lemme 3.14 (with §3.5): a cuspidal automorphic representation of GL_n over a number field whose infinitesimal character is that of the dual of an irreducible algebraic representation V has archimedean component the essentially tempered generic representation with that infinitesimal character (it is generic by Shalika's theorem and its parameter is bounded by [136]), and that representation has nonzero $(\mathfrak{g}, K_\infty)$-cohomology with coefficients in V. *Notation.* Here $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ through $(\tau, \bar\tau)$, whose irreducible algebraic representations are the $V_{\mu,\mu'}$; the matching of the central characters on $\mathbb{R}_{>0} \subset K_\infty$ follows from the equality of infinitesimal characters, $\mathbb{R}_{>0}$ being connected. □

Theorem B.66. *Let $g = \sum_{n\geq 1} a_n q^n$ be a normalised newform of weight $k \geq 2$, level L and character ε, let ℓ be a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ a continuous semisimple representation such that, for all but finitely many primes p and every place $\bar p \mid p$ of $\overline{\mathbb{Q}}$, ρ is trivial on $I_{\bar p}$ and $\iota(\det(X - \rho(\sigma^{-1}))) = X^2 - a_p X + \varepsilon(p)p^{k-1}$ for every arithmetic Frobenius σ at $\bar p$. For a prime v write ρ_v for the restriction of ρ to the decomposition group of a place of $\overline{\mathbb{Q}}$ above v, identified with $G_{\mathbb{Q}_v}$.*

(Lc1) *For every prime $v \neq \ell$, $\mathrm{a}(\rho_v) = v_v(L)$.*

(Lc2) *ρ_ℓ is potentially semistable and $\mathrm{a}(\rho_\ell) = v_\ell(L)$.*

Proof. Proof of (Lc1). By [59], Théorème (A), for the base field $\mathbb{Q}$: for $v \neq \ell$ the Frobenius-semisimplified Weil–Deligne representation of the ℓ-adic representation attached to g at v corresponds, under the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_v)$, to the local component π_v of the automorphic representation of g; the local correspondence preserves conductors, and the conductor exponent of π_v is $v_v(L)$ by Casselman's theory of the newvector ([61] treats the unramified case; the newform level is treated in Casselman, *On some results of Atkin and Lehner*, Math. Ann. 201 (1973)).

Proof of (Lc2). By the main theorem of [213], the restriction to $G_{\mathbb{Q}_\ell}$ of the ℓ-adic representation attached to g is potentially semistable (Tsuji) and the Weil–Deligne representation attached to it by Fontaine [95] corresponds to π_ℓ under the local Langlands correspondence; the conductors are compared as for (Lc1). Frobenius semisimplification does not change $\mathrm{a}(\cdot)$, which depends only on the restriction to inertia and on N.

The representation ρ. The representation attached to g in these papers is Deligne's ([76], in the normalisation of (Dg1) of Theorem B.59); it is irreducible (Ribet, *Galois representations attached to eigenforms with Nebentypus*, 1977), hence semisimple, and by the Chebotarev density theorem and the Brauer–Nesbitt theorem every continuous semisimple ρ with the stated Frobenius characteristic polynomials is isomorphic to it; isomorphic representations have isomorphic restrictions and Weil–Deligne representations. □

Theorem B.67. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index m containing -1 and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, let $k \geq 0$ be an even integer and $F = \sum_{n\geq 0} b_n q^n$ $(q = e^{2\pi i z})$ a nonzero modular form of weight k for Γ. Then $b_n \neq 0$ for some integer n with $0 \leq n \leq km/12$.*

Proof. By the valence formula ([229], Proposition 2.16), for nonzero F the sum over the points P of the compactified curve $\Gamma\backslash\mathfrak{H}^*$ of the orders $\nu_P(F)$, measured in local parameters and divided by the order of the stabiliser at elliptic points, equals $km/12$, where $m = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = [\mathrm{PSL}_2(\mathbb{Z}) :$

$\Gamma/\{\pm1\}]$ as $-1 \in \Gamma$. All terms are nonnegative for holomorphic F; at the cusp ∞, whose width is 1 because the stabiliser of ∞ in $\Gamma/\{\pm1\}$ contains, hence is generated by, the image of $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, the local parameter is q and $\nu_\infty(F) = \min\{n : b_n \neq 0\}$. Hence $\min\{n : b_n \neq 0\} \leq km/12$. □

Theorem B.68. *Let $f = \sum_{n\geq1} a_n q^n$ be a normalised newform of weight $k \geq 2$ and character ε, and let η' be a quadratic Dirichlet character such that $\eta'(p)a_p = a_p$ for all but finitely many primes p; let $K' \subset \overline{\mathbb{Q}}$ be the quadratic field with $G_{K'} = \ker\eta'$. Let ℓ be a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $E \subset \overline{\mathbb{Q}}_\ell$ a finite extension of $\mathbb{Q}_\ell$ and $V\colon G_\mathbb{Q} \to \mathrm{GL}_2(E)$ a continuous representation, semisimple over $\overline{\mathbb{Q}}_\ell$, such that for all but finitely many primes p, V is unramified at p and $\iota(\det(X - V(\sigma^{-1}))) = X^2 - a_pX + \varepsilon(p)p^{k-1}$ for every arithmetic Frobenius σ at p. If $K' \not\subseteq \mathbb{Q}(\mu_{\ell^\infty})$, then $H^1_f(\mathbb{Q}, \mathrm{ad}_E V) = 0$, where $\mathrm{ad}_E V$ is $\mathrm{M}_2(E)$ with g acting by $X \mapsto V(g)XV(g)^{-1}$.*

Proof. This is [192], Theorem 5.4, case (2) ("π has CM by an extension K/F, and $K \not\subset F(\zeta_{p^\infty})$"), for the totally real field $F = \mathbb{Q}$, any prime ℓ and any ι.

Their π is the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by f, twisted by $|\det|^{(2-k)/2}$ so that it is regular algebraic with $r_{\pi,\iota}$ having the stated characteristic polynomials of geometric Frobenius elements; "π has CM by $K'/\mathbb{Q}$" means $\pi \cong \pi \otimes \eta'$, which holds exactly when $\eta'(p)a_p = a_p$ for almost all p (strong multiplicity one for GL_2, [136]); their K is K'. As in Theorem B.69, $V \otimes_E \overline{\mathbb{Q}}_\ell \cong r_{\pi,\iota}$ by Chebotarev and Brauer–Nesbitt ([220], I.2.2), and $H^1_f(\mathbb{Q}, \mathrm{ad}_E V)$ injects into $H^1_f(\mathbb{Q}, \mathrm{ad}\, V \otimes_E \overline{\mathbb{Q}}_\ell)$ (an E-linear projection $\overline{\mathbb{Q}}_\ell \to E$ retracts the inclusion equivariantly). The local conditions of [192] are those of Definition 22.2. □

Theorem B.69. *Let $f = \sum_{n\geq1} a_n q^n$ be a normalised newform of weight $k \geq 2$ and character ε without complex multiplication, that is, there is no quadratic Dirichlet character η' with $\eta'(p)a_p = a_p$ for all but finitely many primes p. Let ℓ be a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $E \subset \overline{\mathbb{Q}}_\ell$ a finite extension of $\mathbb{Q}_\ell$ and $V\colon G_\mathbb{Q} \to \mathrm{GL}_2(E)$ a continuous representation, semisimple over $\overline{\mathbb{Q}}_\ell$, such that for all but finitely many primes p, V is unramified at p and $\iota(\det(X - V(\sigma^{-1}))) = X^2 - a_pX + \varepsilon(p)p^{k-1}$ for every arithmetic Frobenius σ at p. Then $H^1_f(\mathbb{Q}, \mathrm{ad}_E V) = 0$, where $\mathrm{ad}_E V$ is $\mathrm{M}_2(E)$ with g acting by $X \mapsto V(g)XV(g)^{-1}$.*

Proof. This is [192], Theorem B(1), for the totally real field $F = \mathbb{Q}$, any prime ℓ and any ι ("no additional conditions are required").

Their π is the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by f, twisted by $|\det|^{(2-k)/2}$ so that it is regular algebraic with $r_{\pi,\iota}$ having the stated characteristic polynomials of geometric Frobenius elements; π is non-CM in their sense ($\pi \not\cong \pi \otimes \eta'$ for every quadratic character η') exactly when f is without complex multiplication as defined here, by strong multiplicity one for GL_2 ([136]); $V \otimes_E \overline{\mathbb{Q}}_\ell \cong r_{\pi,\iota}$ by Chebotarev and Brauer–Nesbitt ([220], I.2.2), both being semisimple with the same characteristic polynomials of Frobenius elements at almost all primes; and $H^1_f(\mathbb{Q}, \mathrm{ad}_E V)$ injects into $H^1_f(\mathbb{Q}, \mathrm{ad}\, V \otimes_E \overline{\mathbb{Q}}_\ell)$ (an E-linear projection $\overline{\mathbb{Q}}_\ell \to E$ retracts the inclusion equivariantly). □

Theorem B.70. *Let $F \subset \overline{\mathbb{Q}}$ be a CM field with maximal totally real subfield F^+, $c \in G_{F^+}$ a complex conjugation, $n \geq 1$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and π a regular algebraic conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let $E \subset \overline{\mathbb{Q}}_\ell$ be a finite extension of $\mathbb{Q}_\ell$ and $r\colon G_F \to \mathrm{GL}_n(E)$ a continuous representation, absolutely irreducible, with $r \leftrightarrow_\iota \pi$, and let $\mathbf{P} \in \mathrm{GL}_n(E)$ satisfy*

$$r(cgc) = \chi_{\mathrm{cyc}}(g)^{1-n}\, \mathbf{P}\, {}^t r(g)^{-1}\, \mathbf{P}^{-1} \qquad (g \in G_F).$$

Then the formulas $g\cdot X = r(g)Xr(g)^{-1}$ ($g\in G_F$) and $c\cdot X = -\mathbf{P}\,{}^tX\,\mathbf{P}^{-1}$ define a continuous E-linear action of G_{F^+} on $\mathrm{M}_n(E)$; write $\mathrm{ad}_E\, r$ for this representation. Assume that every element of $r(G_F)$ has all its eigenvalues in E, and let $\mathbf{Z}\subseteq \mathrm{GL}_{n,E}$ be the Zariski closure of $r(G_{F(\mu_{\ell^\infty})})$, with identity component $\mathbf{Z}^\circ$. If $\mathbf{Z}^\circ$ acts absolutely irreducibly on E^n and $\mathbf{Z}^\circ(\overline{\mathbb{Q}}_\ell)$ contains an element with n distinct eigenvalues, then

$$H^1_f(F^+, \mathrm{ad}_E\, r) = 0.$$

For an imaginary quadratic field $F = K$ one has $F^+ = \mathbb{Q}$, and the conclusion reads $H^1_f(\mathbb{Q}, \mathrm{ad}_E\, r) = 0$.

Proof. This is [192], Theorem A (for the CM field F; any prime ℓ; no hypothesis on the residual representation), combined with Lemma 2.28(2) of the same paper (if the identity component of the Zariski closure of H contains regular semisimple elements and acts absolutely irreducibly on E^n, then H is enormous, under the standing assumption of that lemma that the characteristic polynomials of the elements of H split over E), applied to $H = r(G_{F(\mu_{\ell^\infty})})$ after conjugating r into GL_n of the ring of integers of E (the image is compact).

Notation. "Of unitary type" in [192] means $\pi^c\cong\pi^\vee$; their $r_{\pi,\iota}$ satisfies $r^c_{\pi,\iota}\cong r^\vee_{\pi,\iota}\otimes\chi_{\mathrm{cyc}}^{1-n}$ on G_F (multiplier $\epsilon^{1-n}\delta^n_{F/F^+}$), and G_{F^+} acts on $\mathrm{ad}\, r_{\pi,\iota} = \mathrm{End}(V)$ through conjugation on G_F and by $X\mapsto -X^*$ for complex conjugation, X^* the adjoint for the pairing $\langle x,y\rangle = {}^t x\,\mathbf{P}^{-1}y$, so that $X^* = \mathbf{P}\,{}^tX\,\mathbf{P}^{-1}$.

The action. Applying the displayed relation twice shows that $\mathbf{P}\,{}^t\mathbf{P}^{-1}$ commutes with $r(G_F)$, hence is a scalar λ by absolute irreducibility, with $\lambda^2=1$; so $c\cdot(c\cdot X)=X$, and $c\cdot(g\cdot(c\cdot X)) = r(cgc)Xr(cgc)^{-1}$; as $G_{F^+} = G_F\sqcup cG_F$ this defines an action.

Comparison with $r_{\pi,\iota}$. An absolutely irreducible r with $r\leftrightarrow_\iota\pi$ is isomorphic over $\overline{\mathbb{Q}}_\ell$ to $r_{\pi,\iota}$ (Chebotarev and Brauer–Nesbitt, [220], I.2.2), and two E-forms of an absolutely irreducible representation that are isomorphic over $\overline{\mathbb{Q}}_\ell$ are isomorphic over E (the space of intertwiners is defined over E); the matrix $\mathbf{P}$ is unique up to a scalar (Schur), so $\mathrm{ad}_E\, r$ depends only on r up to isomorphism, and its Bloch–Kato Selmer group, whether formed with coefficients E or $\overline{\mathbb{Q}}_\ell$, injects into the one formed with $\overline{\mathbb{Q}}_\ell$ (an E-linear projection $\overline{\mathbb{Q}}_\ell\to E$ retracts the inclusion equivariantly). The condition on $\mathbf{Z}^\circ(\overline{\mathbb{Q}}_\ell)$ is the condition "contains regular semisimple elements" of Lemma 2.28: the regular semisimple locus is open, and $\mathbf{Z}^\circ(E)$ is Zariski dense in the connected group $\mathbf{Z}^\circ$ over the field E of characteristic 0. The group H^1_f of [192] is the Bloch–Kato group (unramified classes at $v\nmid\ell$, kernel of $H^1\to H^1(\cdot\otimes B_{\mathrm{cris}})$ at $v\mid\ell$), as in Definition 22.2. □

Theorem B.71. *Let ℓ be a prime and $\mathcal{O}$ a coefficient ring with residue field of characteristic ℓ. A profinite group Γ satisfies* Mazur's condition Φ_ℓ *if $\mathrm{Hom}_{\mathrm{cont}}(\Gamma',\mathbb{F}_\ell)$ is finite for every open subgroup $\Gamma'\subseteq\Gamma$.*

1. *If Γ satisfies Φ_ℓ and $\bar\rho\colon\Gamma\to\mathrm{GL}_n(\mathbb{F}_\mathcal{O})$ is continuous, the functor on $\mathrm{CNL}_\mathcal{O}$ sending A to the set of continuous homomorphisms $\rho\colon\Gamma\to\mathrm{GL}_n(A)$ reducing to $\bar\rho$ is represented by an object $R^\square$ of $\mathrm{CNL}_\mathcal{O}$ and a universal lift $\rho^\square\colon\Gamma\to\mathrm{GL}_n(R^\square)$.*
2. *For a number field F and a finite set S of places of F, $G_{F,S}$ satisfies Φ_ℓ.*

Proof. [167, §1.1–1.2]: item 2 is Mazur's observation that $\mathrm{Hom}(\Gamma',\mathbb{F}_\ell)$ for an open $\Gamma'\subseteq G_{F,S}$ classifies abelian extensions of exponent ℓ of a number field unramified outside S, finite in number by class field theory (Hermite–Minkowski); item 1 is the framed form of his existence theorem (Schlessinger's criteria hold for the functor of lifts without any irreducibility hypothesis, and Φ_ℓ makes its tangent space $Z^1(\Gamma,\mathrm{ad}\,\bar\rho)$ finite-dimensional, so the pro-representing ring is

Noetherian). Notation: $R^{\square}$, $\rho^{\square}$ as there; $\mathrm{CNL}_{\mathcal{O}}$ is Mazur's category of complete Noetherian local $\mathcal{O}$-algebras with residue field $\mathbb{F}_{\mathcal{O}}$. □

Theorem B.72. *Let $F \subset \overline{\mathbb{Q}}$ be a CM field with maximal totally real subfield F^+, $c \in G_{F^+}$ a complex conjugation, $n \geq 2$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and π a regular algebraic conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let $E \subset \overline{\mathbb{Q}}_\ell$ be a finite extension of $\mathbb{Q}_\ell$ and $r\colon G_F \to \mathrm{GL}_n(E)$ a continuous representation with $r \leftrightarrow_\iota \pi$, and let $\mathbf{P} \in \mathrm{GL}_n(E)$ satisfy*

$$r(cgc) = \chi_{\mathrm{cyc}}(g)^{1-n}\,\mathbf{P}\,{}^t r(g)^{-1}\,\mathbf{P}^{-1} \qquad (g \in G_F).$$

Assume that the restriction of r to $G_{F(\mu_{\ell^\infty})}$ is irreducible on $\overline{\mathbb{Q}}_\ell^n$. Then the formulas $g \cdot X = r(g)Xr(g)^{-1}$ $(g \in G_F)$ and $c \cdot X = -\mathbf{P}\,{}^t X\,\mathbf{P}^{-1}$ define a continuous E-linear action of G_{F^+} on $\mathrm{M}_n(E)$, and, writing $\mathrm{ad}_E\, r$ for this representation,

$$H^1_f(F^+, \mathrm{ad}_E\, r) = 0 \quad \textit{and} \quad H^1_f(F^+, \mathrm{ad}_E\, r \otimes_E \overline{\mathbb{Q}}_\ell) = 0.$$

For an imaginary quadratic field $F = K$ one has $F^+ = \mathbb{Q}$, and the conclusion reads $H^1_f(\mathbb{Q}, \mathrm{ad}_E\, r) = 0 = H^1_f(\mathbb{Q}, \mathrm{ad}_E\, r \otimes_E \overline{\mathbb{Q}}_\ell)$.

Proof. This is [257], Theorem 6.1 (stated as Theorem A in the introduction), with $p = \ell$: if F is a CM number field, $n \geq 2$, (π, χ) is a regular algebraic, cuspidal, polarized automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$ is an isomorphism and $r_{\pi,\iota}|_{G_{F(\zeta_{p^\infty})}}$ is irreducible, then $H^1_f(F^+, \mathrm{ad}\, r_{\pi,\iota}) = 0$; no hypothesis on the residual representation, on the prime, or on F/F^+ (the reductions to F/F^+ everywhere unramified and $[F^+ : \mathbb{Q}]$ even are made inside its proof).

Notation. A regular algebraic conjugate self-dual cuspidal π ($\pi^c \cong \pi^\vee$) gives the polarized pair $(\pi, \delta^n_{F/F^+})$ in the sense of [17], Section 2.1, to which [257] refers (the sign condition at the real places holds as $\delta_{F/F^+,v}(-1) = -1$ for every real place v of F^+, F being totally imaginary); then $r^c_{\pi,\iota} \cong r^\vee_{\pi,\iota} \otimes \chi_{\mathrm{cyc}}^{1-n}$ on G_F, and G_{F^+} acts on $\mathrm{ad}\, r_{\pi,\iota} = \mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ by conjugation through $r_{\pi,\iota}$ on G_F and by $X \mapsto -\mathbf{P}\,{}^t X\,\mathbf{P}^{-1}$ for c, the same extension as in [192] (which [257] takes over unchanged, Section 6.3). The group H^1_f there is the Bloch–Kato group (unramified classes at $v \nmid \ell$, the kernel of $H^1 \to H^1(\cdot \otimes B_{\mathrm{cris}})$ at $v \mid \ell$), as in Definition 22.2.

Comparison of r with $r_{\pi,\iota}$. The restriction of r to $G_{F(\mu_{\ell^\infty})}$ is irreducible over $\overline{\mathbb{Q}}_\ell$, so r is absolutely irreducible. An absolutely irreducible r with $r \leftrightarrow_\iota \pi$ is isomorphic over $\overline{\mathbb{Q}}_\ell$ to $r_{\pi,\iota}$ (equal characteristic polynomials of Frobenius elements at almost all places, Chebotarev and Brauer–Nesbitt, [220], I.2.2); so $r_{\pi,\iota}|_{G_{F(\zeta_{\ell^\infty})}}$ is irreducible and the theorem applies to π. We check that the formulas define an action. Applying the displayed relation twice shows that $\mathbf{P}\,{}^t\mathbf{P}^{-1}$ commutes with $r(G_F)$, hence is a scalar λ with $\lambda^2 = 1$ (Schur), so $c \cdot (c \cdot X) = X$; and $c \cdot (g \cdot (c \cdot X)) = r(cgc)Xr(cgc)^{-1}$; as $c \notin G_F$ (F is totally imaginary) and $[G_{F^+} : G_F] = 2$, $G_{F^+} = G_F \sqcup cG_F$ and this defines an action, continuous because r is. If $u \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ conjugates r to $r_{\pi,\iota}$, then $u\mathbf{P}\,{}^t u$ satisfies the displayed relation for $r_{\pi,\iota}$, and by Schur it is a scalar multiple of the matrix used for $r_{\pi,\iota}$; the action $c \cdot X = -\mathbf{P}\,{}^t X\mathbf{P}^{-1}$ does not change when $\mathbf{P}$ is scaled, so $X \mapsto uXu^{-1}$ is a G_{F^+}-isomorphism $\mathrm{ad}_E\, r \otimes_E \overline{\mathbb{Q}}_\ell \cong \mathrm{ad}\, r_{\pi,\iota}$, and $H^1_f(F^+, \mathrm{ad}_E\, r \otimes_E \overline{\mathbb{Q}}_\ell) = 0$. Finally an E-linear projection $\overline{\mathbb{Q}}_\ell \to E$ retracting the inclusion is G_{F^+}-equivariant on M_n (the action is E-linear), and the Bloch–Kato conditions are functorial, so $H^1_f(F^+, \mathrm{ad}_E\, r)$ injects into $H^1_f(F^+, \mathrm{ad}_E\, r \otimes_E \overline{\mathbb{Q}}_\ell) = 0$. □

Theorem B.73. *Let F be a CM field, $n \geq 2$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and π a regular algebraic conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let $v \mid \ell$ be a place of F, $\bar{v} \mid v$ a place of $\overline{\mathbb{Q}}$, and $(V, \rho, N) = \mathrm{WD}(r_{\ell,\iota}(\pi)|_{D_{\bar{v}}})$ (defined, since the restriction is de Rham by (Bg2), hence potentially semistable). Let $\Phi \in W_{\bar{v}}$ with $n(\Phi) = -1$. Then (V, ρ, N)*

is pure*: there are a real number k and subspaces $\mathrm{Fil}_j V$ ($j \in k + \mathbb{Z}$) stable under $\rho(W_{\bar v})$, with $\mathrm{Fil}_{j-1}V \subseteq \mathrm{Fil}_j V$, $\mathrm{Fil}_j V = 0$ for j small, $\mathrm{Fil}_j V = V$ for j large and $N(\mathrm{Fil}_j V) \subseteq \mathrm{Fil}_{j-2}V$, such that, writing $\mathrm{gr}_j = \mathrm{Fil}_j V/\mathrm{Fil}_{j-1}V$:*

(Pu1) *for every j, every eigenvalue α of the endomorphism of gr_j induced by $\rho(\Phi)$ is algebraic over $\mathbb{Q}$ and satisfies $|\sigma(\alpha)| = q_v^{j/2}$ for every field embedding $\sigma\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$;*

(Pu2) *for every integer $i \geq 1$, N^i induces an isomorphism $\mathrm{gr}_{k+i} \to \mathrm{gr}_{k-i}$.*

Proof. This follows from [58], Theorem 1.1 and the paragraph following it, with [56], Theorem 1.2, and [255], Section 1 and Lemma 1.4.

The published statements. Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, $n \geq 2$, with $\Pi^\vee \cong \Pi \circ c$ and Π_∞ cohomological for an irreducible algebraic representation, and $R_\ell(\Pi)$ its Galois representation (Shin, Chenevier–Harris). [58], Theorem 1.1: for every place $y \mid \ell$ of F, $\mathrm{WD}(R_\ell(\Pi)|_{G_{F_y}})^{F\text{-ss}} \cong \iota^{-1}\mathcal{L}_{n,F_y}(\Pi_y)$, the local Langlands correspondence in the geometric normalisation (for n odd this is due to [18], as Π then has Shin-regular weight). [56], Theorem 1.2: Π is tempered at every finite place. As stated in [58], p. 1, temperedness of Π_y makes $\iota^{-1}\mathcal{L}_{n,F_y}(\Pi_y)$ pure of some weight in the sense of [255] (Lemma 1.4(3) there), so the Frobenius semisimplification of $\mathrm{WD}(R_\ell(\Pi)|_{G_{F_y}})$ is pure, and so is $\mathrm{WD}(R_\ell(\Pi)|_{G_{F_y}})$ itself ([255], Lemma 1.4(1)).

Definitions of [255], Section 1. A Weil–Deligne representation (V, r, N) of W_{F_v} satisfies $r(\sigma)Nr(\sigma)^{-1} = |\mathrm{Art}^{-1}(\sigma)|N$ with the Artin map sending uniformisers to geometric Frobenius elements; for a geometric Frobenius Φ this is $\rho(\Phi)N\rho(\Phi)^{-1} = q_v^{-1}N$, which is (38.5) with $n(\Phi) = -1$. A Weil q-number of weight j is an $\alpha \in \overline{\mathbb{Q}}_\ell$ with $\sigma(\alpha)\overline{\sigma(\alpha)} = q^j$ for every embedding σ (so α is algebraic); (V, r, N) is strictly pure of weight j if every eigenvalue of $r(\Phi)$ is a Weil q_v-number of weight j (for some, equivalently every, lift Φ of Frobenius), mixed if it has an increasing filtration by Weil–Deligne subrepresentations with strictly pure graded pieces, the j-th of weight j (then $N(\mathrm{Fil}_j) \subseteq \mathrm{Fil}_{j-2}$), and pure of weight k if it is mixed with all weights in $k + \mathbb{Z}$ and $N^i\colon \mathrm{gr}_{k+i} \to \mathrm{gr}_{k-i}$ is an isomorphism for every integer $i \geq 1$. This is (Pu1)–(Pu2). The functor WD of [255] at $\ell = p$ is Fontaine's, as in (Wd2).

Comparison of notation. Regular algebraic cuspidal representations are those with cohomological archimedean component ([68], Section 3), and conjugate self-dual means $\pi^\vee \cong \pi^c$, so π satisfies the hypotheses on Π. $R_\ell(\Pi)$ and $r_{\ell,\iota}(\pi)$ are semisimple and attached to π at almost all places ((Bg1) of Theorem B.62), hence isomorphic by the Chebotarev density theorem and the Brauer–Nesbitt theorem; isomorphic representations have isomorphic Weil–Deligne representations, and $D_{\bar v}$ is identified with G_{F_v}. □

Theorem B.74. *Let E/F be a cyclic extension of number fields of prime degree, σ a generator of $\mathrm{Gal}(E/F)$, $n \geq 1$, and Π a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\Pi^\sigma \cong \Pi$, where Π^σ is the representation $g \mapsto \Pi(\sigma(g))$. Then there is a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that for every place v of F and every place w of E above v:*

(Bc1) *if v splits completely in E, then $\Pi_w \cong \pi_v$, where E_w and F_v are identified;*

(Bc2) *if v is archimedean, the Langlands parameter of Π_w is the restriction to the Weil group of E_w of the Langlands parameter of π_v.*

Proof. See [9], Chapter 3, Theorems 4.2 and 5.1, and for $n = 2$ [159]. The statements there are as follows. For E/F cyclic of prime degree, every cuspidal π of $\mathrm{GL}_n(\mathbb{A}_F)$ has a base change

$\Pi = \pi_E$, an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ whose local component at every place $w \mid v$ is the local base change of π_v; a cuspidal Π with $\Pi^\sigma \cong \Pi$ is the base change of exactly $[E:F]$ representations π, all cuspidal and twists of one another by the characters of $\mathbb{A}_F^\times / F^\times \mathrm{N}_{E/F}\mathbb{A}_E^\times$. In the notation used here: at a place v split in E, $E \otimes_F F_v = \prod_{w\mid v} E_w$ with $E_w = F_v$, and the local base change of π_v is π_v at each w, which is (Bc1); at an archimedean place the local base change is defined by restriction of Langlands parameters to the Weil group of E_w ([9], Chapter 1), which is (Bc2). Only the existence of one cuspidal π with (Bc1), (Bc2) is stated; the twist ambiguity is not used. □

Theorem B.75. *Let F be a totally real field, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and π a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Then there is a continuous representation $r\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ with $r \leftrightarrow_\iota \pi$.*

Proof. See [59], [253], [33] (Galois representations attached to Hilbert modular eigenforms of weights $k_v \geq 2$ of equal parity: Carayol when $[F:\mathbb{Q}]$ is odd or π is discrete series at some finite place, Taylor and Blasius–Rogawski in the remaining cases), and in the present normalisation [17], Theorem 2.1.1 (2), which allows F totally real: for a regular algebraic cuspidal polarized (π,χ), $\iota\mathrm{WD}(r_{l,\iota}(\pi)|_{G_{F_v}})^{F\text{-ss}} \cong \mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2})$ for $v \nmid \ell$. For GL_2, $\pi^\vee \cong \pi \otimes (\omega_\pi^{-1} \circ \det)$ with ω_π the central character, so (π,ω_π) is polarized in the sense of op. cit., Section 2.1, once $\omega_{\pi,v}(-1)$ is independent of the real place v; this holds because π_v is a twist $D_{k_v} \otimes |\det|^{t_v}$ of a discrete series, $\omega_{\pi,v}(-1) = (-1)^{k_v}$, regular algebraicity gives $t_v - k_v/2 \in \mathbb{Z}$, and $2t_v$ is independent of v because ω_π is trivial on a subgroup of finite index of the totally positive units (Dirichlet's unit theorem). At an unramified $v \nmid \ell$, π_v is the irreducible unramified subquotient of the normalised induction of unramified μ_1, μ_2, $\mathrm{rec}(\pi_v \otimes |\det|^{-1/2}) = \mu_1|\cdot|^{-1/2} \oplus \mu_2|\cdot|^{-1/2}$ (composed with the inverse Artin map, uniformisers going to geometric Frobenius elements in the normalisation of op. cit.) with $N = 0$, so r is unramified at v and a geometric Frobenius σ^{-1} has the eigenvalues $\iota^{-1}(q_v^{1/2}\mu_i(\varpi_v))$; this is $r \leftrightarrow_\iota \pi$. □

Theorem B.76. *Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Then there are a normalised newform $f = \sum_{n\geq 1} a_n q^n$ of some weight $k \geq 2$, level N and character ε, and an integer j, such that for every prime $p \nmid N$ the component π_p is unramified and*

$$\prod_{\alpha} (X - p^{1/2}\alpha) = X^2 - p^j a_p X + \varepsilon(p)\, p^{2j+k-1},$$

α running over the Satake parameters of π_p with multiplicity.

Proof. By [103], Section 5, together with [135] and Casselman's theory of the newvector (W. Casselman, *On some results of Atkin and Lehner*, Math. Ann. 201 (1973)), a normalised newform f of weight $k \geq 2$, level N and character ε generates a cuspidal automorphic representation $\pi(f)$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with $\pi(f)_\infty \cong D_k$, the discrete series of weight k with central character trivial on $\mathbb{R}_{>0}$, unramified at every $p \nmid N$ and with $L(s,\pi(f)) = L(s + \frac{k-1}{2}, f)$; conversely every cuspidal automorphic representation π_0 with $\pi_{0,\infty} \cong D_k$ is $\pi(f)$ for exactly one normalised newform f of weight k.

Notation. (i) Write $\pi = \pi_0 \otimes |\det|^t$ with $t \in \mathbb{R}$ such that the central character of π_0 is unitary. The representation π_0 is cuspidal and unitary, hence $\pi_{0,\infty}$ is unitary and generic. Regular algebraicity in the sense of [68] says that the Langlands parameter of $\pi_\infty \otimes |\det|^{-1/2}$ restricted to $\mathbb{C}^\times$ is $z^{p_1}\bar z^{q_1} \oplus z^{p_2}\bar z^{q_2}$ with integers p_i, q_i and $p_1 \neq p_2$. An irreducible principal series of $\mathrm{GL}_2(\mathbb{R})$ has $p_i = q_i$, so regularity forces $p_1 - p_2$ to be a nonzero integer, i.e. the ratio of the two inducing characters is $|\cdot|^s$ times a sign with $\mathrm{Re}\, s$ a nonzero integer, which is not unitary for an irreducible principal series (the complementary series has $|\mathrm{Re}\, s| < 1$). Hence

$\pi_{0,\infty} \cong D_k$ for some $k \geq 2$, and then the exponents are $p = k/2 + t - 1$, $q = t - k/2$ (and their exchange), so $t - k/2 \in \mathbb{Z}$. (ii) For $p \nmid N$, $L_p(s, \pi(f)) = (1 - a_p p^{-(k-1)/2} p^{-s} + \varepsilon(p) p^{-2s})^{-1}$, so the Satake parameters α_0, β_0 of $\pi(f)_p$ satisfy $\alpha_0 + \beta_0 = a_p p^{-(k-1)/2}$ and $\alpha_0 \beta_0 = \varepsilon(p)$; those of $\pi_p = \pi(f)_p \otimes |\det|^t$ are $p^{-t}\alpha_0, p^{-t}\beta_0$. With $j = 1 - t - k/2 \in \mathbb{Z}$, the numbers $p^{1/2}\alpha$ have sum $a_p p^{1-t-k/2} = p^j a_p$ and product $\varepsilon(p) p^{1-2t} = \varepsilon(p) p^{2j+k-1}$. (If a reference lifts the complex conjugate form instead, the statement holds with that newform.) □

Theorem B.77. *Let $f = \sum_{n \geq 1} a_n q^n$ be a normalised newform of weight $k \geq 2$, level N and character ε, ℓ a prime, $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and $\rho \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ a continuous semisimple representation such that for all but finitely many primes p and every place $\bar{p} \mid p$, ρ is trivial on $I_{\bar{p}}$ and $\iota(\det(X - \rho(\sigma^{-1}))) = X^2 - a_p X + \varepsilon(p) p^{k-1}$ for every arithmetic Frobenius σ at $\bar{p}$. Then $\rho|_{D_\ell}$ is Hodge–Tate with Hodge–Tate weights 0 and $1 - k$.*

Proof. By [86], the ℓ-adic representation attached to an eigenform of weight $k \geq 2$ (realised in the étale cohomology of a modular curve with coefficients in the $(k-2)$-nd symmetric power of the first relative cohomology of the universal elliptic curve, where geometric Frobenius elements have trace a_p) carries a Hodge–Tate structure of weights 0 and $k-1$ up to the sign convention (in the normalisation of [86]: Hodge–Tate weights 0 and $k+1$ for an eigenform of weight $k+2 \geq 2$). *Notation.* This representation is Deligne's $\rho_{f,\iota}$ of (Dg1) in Theorem B.59. By the hypothesis on ρ and (Dg1) of Theorem B.59, ρ and $\rho_{f,\iota}$ have the same characteristic polynomial at every arithmetic Frobenius element at all but finitely many primes; these elements are dense in $G_{\mathbb{Q}}$ (Chebotarev density theorem), so by continuity the characteristic polynomials agree on all of $G_{\mathbb{Q}}$, and by the Brauer–Nesbitt theorem ρ, being semisimple, is isomorphic to the semisimplification of $\rho_{f,\iota}$, which is Hodge–Tate with the same weights as $\rho_{f,\iota}$ (on Hodge–Tate representations $\mathbb{C}_\ell \otimes (\cdot)$ is exact and the weights of a representation are those of its composition factors). The sign is fixed by the determinant: $\iota \det \rho(\sigma^{-1}) = \varepsilon(p) p^{k-1} = \varepsilon(p) \chi_{\mathrm{cyc}}(\sigma^{-1})^{1-k}$, so $\det \rho = \varepsilon' \chi_{\mathrm{cyc}}^{1-k}$ with ε' of finite order (Chebotarev), of Hodge–Tate weight $1 - k$ with χ_{cyc} of weight 1. The two weights 0 and $\pm(k-1)$ must add up to it. □

Theorem B.78. *Let $F \subset \overline{\mathbb{Q}}$ be a CM field that is not totally real, with maximal totally real subfield F^+, let $\delta_F \colon G_{F^+} \to \{\pm 1\}$ be the character with kernel G_F, and let $M \supseteq F$ be a CM field such that M/F is a soluble Galois extension; M^+ is the maximal totally real subfield of M. Let $n \geq 1$, ℓ a prime, $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and $r \colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, $\mu \colon G_{F^+} \to \overline{\mathbb{Q}}_\ell^\times$ continuous homomorphisms such that*

(Vj1) *there are a complex conjugation $c \in G_{F^+}$ and a non-degenerate bilinear form $\langle\ ,\ \rangle$ on $\overline{\mathbb{Q}}_\ell^n$ with $\langle x, y\rangle = -\mu(c)\langle y, x\rangle$ and $\langle r(\sigma)x, r(c\sigma c)y\rangle = \mu(\sigma)\langle x, y\rangle$ for all $\sigma \in G_F$ and $x, y \in \overline{\mathbb{Q}}_\ell^n$;*

(Vj2) *$r|_{G_M}$ is irreducible.*

Then the following are equivalent:

(Vj3) *there are a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ and a continuous character $\chi \colon \mathbb{A}_{F^+}^\times / (F^+)^\times \to \mathbb{C}^\times$ with $\chi_v(-1) = (-1)^n$ at every real place v of F^+ and $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathrm{N}_{F/F^+} \circ \det)$, such that $r \leftrightarrow_\iota \pi$ and $\mu \chi_{\mathrm{cyc}}^{n-1} \leftrightarrow_\iota \chi$;*

(Vj4) *the same holds for M, M^+, $r|_{G_M}$ and $\mu|_{G_{M^+}}$ in place of F, F^+, r and μ.*

In particular, if n is odd and

1. *there are a complex conjugation $c \in G_{F^+}$ and a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ on $\overline{\mathbb{Q}}_\ell^n$ with $\langle r(\sigma)x, r(c\sigma c)y\rangle = \chi_{\mathrm{cyc}}(\sigma)^{1-n}\langle x, y\rangle$ for all $\sigma \in G_F$ and $x, y \in \overline{\mathbb{Q}}_\ell^n$;*

2. $r|_{G_M}$ *is irreducible;*

3. $r|_{G_M} \cong r_{\ell,\iota}(\Pi)$ *for a regular algebraic conjugate self-dual cuspidal automorphic representation* Π *of* $\mathrm{GL}_n(\mathbb{A}_M)$*,*

then $r \cong r_{\ell,\iota}(\pi)$ *for a regular algebraic conjugate self-dual cuspidal automorphic representation* π *of* $\mathrm{GL}_n(\mathbb{A}_F)$*.*

Proof. This is [17], Lemma 2.2.2 (proved there in the same way as Lemma 4.2.2 of [69]): for M/F soluble Galois with M CM and (r,μ) a polarised ℓ-adic representation of G_F with $r|_{G_M}$ irreducible, (r,μ) is automorphic if and only if $(r|_{G_M}, \mu|_{G_{M^+}})$ is.

The notation of op. cit. (Section 2.1 there). A polarised ℓ-adic representation of G_F is a pair (r,μ) of continuous homomorphisms with a non-degenerate pairing such that $\langle x,y\rangle = \varepsilon\langle y,x\rangle$ and $\langle r(\sigma)x, r(c\sigma c)y\rangle = \mu(\sigma)\langle x,y\rangle$, and, F being imaginary, $\varepsilon = -\mu(c)$: this is (Vj1). A polarised automorphic representation is a pair (π,χ) with χ a continuous character of $\mathbb{A}_{F^+}^\times/(F^+)^\times$, $\chi_v(-1)$ independent of $v \mid \infty$, $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathrm{N}_{F/F^+} \circ \det)$ and, F being imaginary, $\chi_v(-1) = (-1)^n$ for all $v \mid \infty$. (r,μ) is automorphic if $(r,\mu) \cong (r_{\ell,\iota}(\pi), r_{\ell,\iota}(\chi)\chi_{\mathrm{cyc}}^{1-n})$ for a regular algebraic cuspidal polarised (π,χ), with $r_{\ell,\iota}(\pi)$ as in op. cit., Theorem 2.1.1 (the representation of Theorem B.62), and $r_{\ell,\iota}(\chi)$ the ℓ-adic character of the algebraic character χ.

From $\cong$ *to* $\leftrightarrow_\iota$. If $r \cong r_{\ell,\iota}(\pi)$, then $r \leftrightarrow_\iota \pi$ by local-global compatibility at the unramified places (Theorem B.62). Conversely, if $r \leftrightarrow_\iota \pi$ with r irreducible (as $r|_{G_M}$ is, by (Vj2)), then r and $r_{\ell,\iota}(\pi)$ have the same characteristic polynomials on a dense set of Frobenius elements (Chebotarev), hence on all of G_F; so $r_{\ell,\iota}(\pi)^{\mathrm{ss}} \cong r$ (Brauer–Nesbitt), and a representation whose semisimplification is irreducible is irreducible, so $r_{\ell,\iota}(\pi) \cong r$. For characters, $\leftrightarrow_\iota$ and equality agree by the Chebotarev density theorem and continuity. So (Vj3) and (Vj4) are the automorphy of (r,μ) and of $(r|_{G_M}, \mu|_{G_{M^+}})$.

The special case. Take $\mu = \chi_{\mathrm{cyc}}^{1-n}\delta_F$. Then $\mu(c) = (-1)^{1-n}\cdot(-1) = -1$ for n odd, so the form of (Vj1) is symmetric: hypothesis 1 is (Vj1), and hypothesis 2 is (Vj2). In (Vj3) (and (Vj4)) $\mu\chi_{\mathrm{cyc}}^{n-1} = \delta_F$ forces χ to be the quadratic character of F/F^+, which satisfies $\chi_v(-1) = -1 = (-1)^n$ and $\chi \circ \mathrm{N}_{F/F^+} = 1$; so (Vj3) says that $r \leftrightarrow_\iota \pi$ with π regular algebraic, cuspidal and conjugate self-dual, and likewise over M, as $\mu|_{G_{M^+}} = \chi_{\mathrm{cyc}}^{1-n}\delta_M$. Hypothesis 3 gives (Vj4) (the representation $r_{\ell,\iota}(\Pi)$ of Theorem B.58 is attached to Π at almost all places), and (Vj3) gives $r \cong r_{\ell,\iota}(\pi)$ with $r_{\ell,\iota}(\pi)$ as in Theorem B.58, by the argument of the previous paragraph. □

Theorem B.79. *Let* $N \geq 3$*,* $\zeta_N = e^{2\pi i/N} \in \mathbb{C}$*,* $R_N = \mathbb{Z}[1/N, \zeta_N]$*, let* X_N *be the compactified moduli scheme over* R_N *of elliptic curves with a full level* N *structure of Weil-pairing determinant* ζ_N*,* $C \subset X_N$ *its divisor of cusps and* ω *the Hodge bundle. For an* R_N*-algebra* B *and* $k \in \mathbb{Z}$ *put* $M_k(N;B) = H^0(X_N \otimes_{R_N} B, \omega^{\otimes k})$*. Then:*

(Kt1) X_N *is a smooth proper curve over* R_N *with geometrically connected fibres,* C *is finite étale over* R_N*, and* $\omega^{\otimes 2} \cong \Omega^1_{X_N/R_N}(C)$ *(Kodaira–Spencer). Evaluation on the Tate curve with its canonical differential and its level* N *structures gives, at each cusp, an injective* q*-expansion map* $M_k(N;B) \to B[[q^{1/N}]]$*; the* ***value*** *of* f *at the cusp is the constant coefficient. The group* $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ *acts on* X_N *over* R_N *through the level structures, compatibly with* ω*; this action permutes the cusps and acts on each* $M_k(N;B)$*.*

(Kt2) *Through the complex uniformisation,* $M_k(N;\mathbb{C})$ *is the space of holomorphic modular forms of weight* k *for* $\Gamma(N)$*; the element* $\gamma \bmod N$ *of* $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$*,* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$*, acts by* $f \mapsto f|_k\gamma$*; the* q*-expansions of* f *at the cusps are the Fourier expansions in* $e^{2\pi i\tau/N}$ *of the functions* $f|_k\gamma$*,* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$*, the expansion at the standard cusp* ∞ *being that of* f *itself. If*

$B \subset \mathbb{C}$ is a subring containing R_N and $f \in M_k(N;\mathbb{C})$ has its expansion at ∞ in $B[[q^{1/N}]]$, then f lies in $M_k(N;B)$, and all q-expansions of f lie in $B[[q^{1/N}]]$ (q-expansion principle).

(Kt3) *If $k \geq 2$, then for every homomorphism of R_N-algebras $B \to B'$ the natural map $M_k(N;B) \otimes_B B' \to M_k(N;B')$ is an isomorphism.*

(Kt4) *Let F be an algebraically closed field of characteristic $p \nmid N$ with an R_N-algebra structure, and let $\mathcal{R} = \bigoplus_{k\in\mathbb{Z}} H^0(Y, \omega^{\otimes k})$, $Y = (X_N \setminus C) \otimes F$. The Hasse invariant $A \in M_{p-1}(N;F)$ is fixed by $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and all its q-expansions equal 1; a homogeneous element of $\mathcal{R}$ is determined by its weight and any one of its q-expansions; and the kernel of the q-expansion map of $\mathcal{R}$ at any cusp is the ideal $(A-1)\mathcal{R}$. Consequently, if $f \in M_k(N;F)$ and $g \in M_{k'}(N;F)$, $k \geq k'$, have the same nonzero q-expansion at one cusp, then $k \equiv k' \pmod{p-1}$ and $f = A^{(k-k')/(p-1)}g$.*

Proof. For (Kt1)–(Kt3), see N. Katz, *p-adic properties of modular schemes and modular forms*, Modular functions of one variable III, Lecture Notes in Math. 350 (1973), Chapter 1 (the moduli problem of level $N \geq 3$ and its compactification, §1.2; q-expansions and the q-expansion principle, §1.6; Theorem 1.7.1: for $n \geq 3$ and $k \geq 2$, $H^0(X,\omega^k) \otimes K \to H^0(X, \omega^k \otimes K)$ is an isomorphism for every $\mathbb{Z}[1/n]$-module K, which gives (Kt3) after base change to R_N), and Appendix A1 (Kodaira–Spencer); the complex uniformisation and the identification of q-expansions with Fourier expansions of $f|_k\gamma$ are in loc. cit., A1.2 and §1.6, and [76]. For (Kt4), see N. Katz, *A result on modular forms in characteristic p*, Modular functions of one variable V, Lecture Notes in Math. 601 (1977), 53–61, §I (for any p and any $N \geq 3$ prime to p, over an algebraically closed field: the definition of A through $\eta^p = A(E,\omega,\alpha)\eta$ in $H^1(E,\mathcal{O}_E)$, which does not involve the level structure, whence the invariance; all q-expansions of A are 1; a homogeneous element is determined by its weight and any one q-expansion; the kernels of the q-expansion maps are $(A-1)\mathcal{R}$). *The consequence in (Kt4).* If $k \not\equiv k' \pmod{p-1}$, then $f - g$ lies in the kernel $(A-1)\mathcal{R}$; since $A - 1 \equiv 0$ in $\mathcal{R}/(A-1)\mathcal{R}$, this quotient is graded by $\mathbb{Z}/(p-1)\mathbb{Z}$ and the q-expansion map factors injectively through it. The forms f and g have nonzero images (their q-expansion is nonzero) in different graded pieces, while $f - g$ has image 0, a contradiction. So $k \equiv k'$, and f and $A^{(k-k')/(p-1)}g$ are homogeneous of weight k with the same q-expansion at that cusp, hence equal. □

Theorem B.80. *Let $g = \sum_{n\geq 1} a_n q^n$ be a normalised newform of weight $k \geq 2$, level L_g and character ε, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, and $\rho^{\mathrm{ar}} = \rho_{g,\iota}^\vee$, so that $\iota \det(X - \rho^{\mathrm{ar}}(\sigma)) = X^2 - a_p X + \varepsilon(p)p^{k-1}$ for every prime $p \nmid \ell L_g$ and arithmetic Frobenius σ at a place above p, and $\det \rho^{\mathrm{ar}} = \varepsilon\chi_{\mathrm{cyc}}^{k-1}$. Then:*

(Gn1) *The representation ρ^{ar} is irreducible.*

(Gn2) *If $\ell \nmid L_g$ and $\iota^{-1}(a_\ell)$ is an ℓ-adic unit, let α be the root of $X^2 - \iota^{-1}(a_\ell)X + \iota^{-1}(\varepsilon(\ell))\ell^{k-1}$ that is an ℓ-adic unit. Then for every place $\bar{\ell}$ of $\overline{\mathbb{Q}}$ above ℓ, ρ^{ar} is triangular on $D_{\bar{\ell}}$ with a diagonal (δ_1,δ_2) such that δ_2 is trivial on $I_{\bar{\ell}}$ and $\delta_2(\sigma) = \alpha$ for every arithmetic Frobenius σ at $\bar{\ell}$.*

(Gn3) *If $p \neq \ell$ is a prime with $p \mid L_g$ and $v_p(L_g) = v_p(\operatorname{cond}\varepsilon)$, then for every place $\bar{p}$ above p the restriction of ρ^{ar} to $D_{\bar{p}}$ is conjugate to $s \mapsto \operatorname{diag}(\nu_1(s),\nu_2(s))$ for characters ν_1,ν_2 of $D_{\bar{p}}$ with ν_1 trivial on $I_{\bar{p}}$, and $|\iota\nu_1(\sigma)| = |\iota\nu_2(\sigma)| = p^{(k-1)/2}$ for every $\sigma \in D_{\bar{p}}$ acting on the residue field as $x \mapsto x^p$.*

Proof. For (Gn1), see K. Ribet, *Galois representations attached to eigenforms with Nebentypus*, Modular functions of one variable V, Lecture Notes in Math. 601 (1977), Theorem 2.3 (irreducibility of $\rho_{g,\iota}$, hence of its contragredient). For (Gn2), see [270], Theorem 2.1.4 (the case $F = \mathbb{Q}$, with ρ^{ar} the representation of op. cit., whose arithmetic Frobenius traces are the a_p). For g ordinary at the place of ι, the restriction to a decomposition group at ℓ is an extension of an unramified character, sending an arithmetic Frobenius to the unit root α, by a character. This is triangular on $D_{\bar{\ell}}$ in the sense of Definition 38.4 with δ_2 the unramified quotient. For another place $\gamma\bar{\ell}$, conjugate by $\rho^{\mathrm{ar}}(\gamma)$. For (Gn3), see H. Carayol, [59], Théorème (A): for $p \neq \ell$ the Frobenius-semisimplified Weil–Deligne representation of $\rho^{\mathrm{ar}}|_{D_{\bar{p}}}$ corresponds under the local Langlands correspondence (suitably normalised) to the local component π_p of the automorphic representation of g. By Casselman's theory of the conductor (*On some results of Atkin and Lehner*, Math. Ann. 201 (1973)), π_p has conductor exponent $v_p(L_g)$ and central character of conductor exponent $v_p(\operatorname{cond}\varepsilon)$; when these are equal and positive, π_p is a principal series $\pi(\mu_1, \mu_2)$ with μ_1 unramified and μ_2 ramified, so the Weil–Deligne representation is a sum of two characters with trivial monodromy, one of them unramified, and $\rho^{\mathrm{ar}}|_{D_{\bar{p}}}$, which then has finite image on inertia, is conjugate to the corresponding diagonal representation. In the unitary normalisation π_p is unitary. An irreducible unitary principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ with $\mu_1\mu_2^{-1}$ ramified is not a complementary series (those have $\mu_1\mu_2^{-1} = |\cdot|^{\pm 2s}$, [135]), so $|\mu_1(p)| = |\mu_2(p)|$, whence $|\iota\nu_1(\sigma)| = |\iota\nu_2(\sigma)|$; as $|\iota(\nu_1\nu_2)(\sigma)| = |\iota(\varepsilon\chi_{\mathrm{cyc}}^{k-1})(\sigma)| = p^{k-1}$, both equal $p^{(k-1)/2}$. □

Theorem B.81. *Let E/F be a cyclic extension of number fields of prime degree, $n \geq 1$, and Π a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. There is an isobaric automorphic representation $\mathrm{BC}_{E/F}(\Pi)$ of $\mathrm{GL}_n(\mathbb{A}_E)$ such that:*

(Ac1) *for every finite place v of F unramified in E at which Π_v is unramified with Satake parameters $\{\alpha_1, \dots, \alpha_n\}$, and every place $w \mid v$ of E with residue degree f, $\mathrm{BC}_{E/F}(\Pi)_w$ is unramified with Satake parameters $\{\alpha_1^f, \dots, \alpha_n^f\}$;*

(Ac2) *for every archimedean place $w \mid v$, the Langlands parameter of $\mathrm{BC}_{E/F}(\Pi)_w$ is the restriction of that of Π_v to the Weil group of E_w;*

(Ac3) *$\mathrm{BC}_{E/F}(\Pi)$ is cuspidal if and only if $\Pi \not\cong \Pi \otimes (\omega \circ \det)$ for every Hecke character $\omega \neq 1$ of F trivial on $\mathrm{N}_{E/F}(\mathbb{A}_E^\times)$.*

Proof. See [9, Chapter 3, Theorems 4.2 and 5.1] (global base change for E/F cyclic of prime degree, for representations induced from cuspidal ones, with local components the local base changes at all places; the base change of a cuspidal Π is cuspidal unless $\Pi \cong \Pi \otimes \omega$ for a nontrivial character ω of $F^\times \mathrm{N}_{E/F}(\mathbb{A}_E^\times)\backslash\mathbb{A}_F^\times$, in which case it is induced from cuspidal representations of smaller groups). Local base change of unramified representations [9, Chapter 1, §4–§6] and at archimedean places (restriction of Langlands parameters) gives (Ac1) and (Ac2). □

Theorem B.82. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. There is an isobaric automorphic representation $\mathrm{Ad}(\pi)$ of $\mathrm{GL}_3(\mathbb{A}_\mathbb{Q})$ such that:*

(Gj1) *for every prime p at which π_p is unramified with Satake parameters $\{\alpha, \beta\}$, $\mathrm{Ad}(\pi)_p$ is unramified with Satake parameters $\{\alpha/\beta, 1, \beta/\alpha\}$;*

(Gj2) *the Langlands parameter of $\mathrm{Ad}(\pi)_\infty$ is the composite of that of π_∞ with the adjoint representation $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}(\mathfrak{sl}_2(\mathbb{C})) \cong \mathrm{GL}_3(\mathbb{C})$;*

(Gj3) *$\mathrm{Ad}(\pi)$ is cuspidal if and only if there is no Hecke character $\omega \neq 1$ of $\mathbb{Q}$ with $\omega^2 = 1$ and $\pi \otimes (\omega \circ \det) \cong \pi$.*

Proof. See [104, Theorem 9.3] (existence of the lift with local components the local lifts at every place, and the cuspidality criterion: $\mathrm{Ad}(\pi)$ is cuspidal if and only if π is not monomial, i.e. $\pi \ncong \pi \otimes \omega$ for every quadratic character $\omega \neq 1$), with the local lifts of [104, §3] (for an unramified principal series the induced representation from $(\mu_1\mu_2^{-1}, 1, \mu_2\mu_1^{-1})$, which gives (Gj1); at the archimedean place the lift is defined through Langlands' parametrisation, which gives (Gj2)). □

Theorem B.83. *Let $n \geq 1$. For a commutative ring A, a group G, an A-valued determinant D of dimension n on G and $x \in A[G]$ write $\chi_D(x)(X) = D(X - x) = \sum_{i=0}^{n}(-1)^i\Lambda_i(x)X^{n-i}$.*

(Ch1) *(Amitsur's formula.) For $m \geq 1$, $1 \leq i \leq n$ and $g_1, \ldots, g_m \in G$, $\Lambda_i(t_1g_1 + \cdots + t_mg_m) \in A[t_1, \ldots, t_m]$ is a polynomial with integer coefficients, depending only on n, m and i, in the elements $\Lambda_j(w)$ ($1 \leq j \leq n$), w running over the finite products of the g_k. Consequently D is the base change of a determinant with values in the subring of A generated by the $\Lambda_i(g)$, $g \in G$; and two A-valued determinants of dimension n on G with $\chi_D(g) = \chi_{D'}(g)$ for all $g \in G$ are equal.*

(Ch2) *(The universal continuous determinant.) Let k be a finite field of characteristic ℓ, $W(k)$ its ring of Witt vectors, G a profinite group and $\bar{D}$ a continuous k-valued determinant of dimension n on G. There are a profinite local $W(k)$-algebra $A(\bar{D})$ with residue field k and a continuous $A(\bar{D})$-valued determinant $D(\bar{D})$ of dimension n on G lifting $\bar{D}$, such that for every profinite local $W(k)$-algebra A with residue field k and every continuous A-valued determinant D of dimension n on G lifting $\bar{D}$ there is exactly one continuous $W(k)$-algebra homomorphism $f\colon A(\bar{D}) \to A$ with $f \circ D(\bar{D}) = D$. If G satisfies Φ_ℓ, then $A(\bar{D})$ is Noetherian.*

(Ch3) *(Determinants over algebraically closed fields.) If k is an algebraically closed field and D a k-valued determinant of dimension n on G, there is a semisimple representation $\rho\colon G \to \mathrm{GL}_n(k)$ with $D = \det \circ \rho$, unique up to isomorphism.*

(Ch4) *(Gluing.) Let G be a compact topological group, A a compact topological ring, $(A_i)_{i\in I}$ a family of topological rings, $\iota\colon A \to \prod_{i\in I} A_i$ a continuous injective ring homomorphism, and D_i a continuous A_i-valued determinant of dimension n on G for each i. If for every g in a dense subset $X \subseteq G$ the family $(\chi_{D_i}(g))_i$ of characteristic polynomials lies in $\iota(A)[t]$, then there is a continuous A-valued determinant D of dimension n on G such that D_i is the base change of D along the i-th component of ι for every i.*

Proof. All references are to [66]. (Ch1) is Amitsur's formula (1.4), (1.5) of §1.10 (Lemma 1.12 (ii)) with Corollary 1.14 and its proof; that a homogeneous polynomial law is determined by its values $D(t_1g_1 + \cdots + t_mg_m)$ on the generic elements is recalled in §1.1 and §1.5 (the functions $D[\alpha]$), and $D(t_1g_1 + \cdots + t_mg_m) = \Lambda_n(t_1g_1 + \cdots + t_mg_m)$. (Ch2) is Proposition 3.3 with Lemma 3.2 (the functor on profinite local $W(k)$-algebras with residue field k) and Proposition 3.7 (Noetherian under Mazur's condition, called (F) there); continuity of an A-valued determinant for profinite A means continuity of the maps $\Lambda_i\colon G \to A$ (§2.30). (Ch3) is Theorem 2.12 (with $R = k[G]$). (Ch4) is Example 2.32 (glueing determinants), whose proof uses Corollary 1.14 and the closedness of $\iota(A)$ (A compact); continuity of D there means continuity of the Λ_i (§2.30). In Chenevier's notation, d is n, p is ℓ and σ_i is Λ_i. □

Theorem B.84. *Let F be a totally real field, p an odd prime, $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$ an isomorphism, I the set of embeddings $\sigma\colon F \to \overline{\mathbb{Q}}_p$ (identified through ι with the real places), $t = \sum_{\sigma\in I}\sigma \in \mathbb{Z}[I]$, $\mathfrak{r}_p = \mathcal{O}_F \otimes \mathbb{Z}_p$, N an integral ideal prime to p, and $\mathcal{O} \subset \overline{\mathbb{Q}}_p$ a coefficient ring containing $\sigma(\mathcal{O}_F)$*

for all σ. Put $Z(N) = \varprojlim_\alpha \mathrm{Cl}_F(Np^\alpha)$ (ray class groups), $G = Z(N) \times \mathfrak{r}_p^\times$, $G = W \times G_{\mathrm{tor}}$ a decomposition with W free of finite rank over $\mathbb{Z}_p$ and G_{tor} finite, $A = \mathcal{O}[[W]]$, and $\mathcal{N}\colon Z(N) \to \mathbb{Z}_p^\times$ the p-adic cyclotomic character (via class field theory). An arithmetic weight *is $(n, v) \in \mathbb{Z}[I]^2$ with $n \geq 0$, $v \geq 0$ and $n + 2v = [n + 2v]\,t$, $[n + 2v] \in \mathbb{Z}$; it has $k = n + 2t$. For such (n, v), $\alpha \geq 1$ and finite-order characters ψ of $\mathrm{Cl}_F(Np^\alpha)$ and ψ' of $(\mathcal{O}_F/p^\alpha)^\times$, let $P_{n,v,\psi,\psi'}\colon \mathcal{O}[[G]] \to \overline{\mathbb{Q}}_p$ be the $\mathcal{O}$-algebra homomorphism induced by $(z, a) \mapsto \psi(z)\psi'(a)\mathcal{N}(z)^{[n+2v]}a^v$, $a^v = \prod_\sigma \sigma(a)^{v_\sigma}$, and let $S_{k,w}(Np^\alpha, \psi', \psi; \mathbb{C})$ be Hida's space of holomorphic cusp forms on $\mathrm{GL}_2(\mathbb{A}_F)$ of weight (k, w), level $V_1(N) \cap U(p^\alpha)$ and characters (ψ, ψ'), with the Hecke operators $T(\mathfrak{q})$, $\mathfrak{q} \nmid Np$, $T(x)$ ($0 \neq x \in \mathfrak{r}_\mathfrak{p}$, $\mathfrak{p} \mid p$; double coset of $\mathrm{diag}(1, x)$), and $\langle\mathfrak{q}\rangle$, and its nearly ordinary part (where $x^{-v}T(x)$ acts invertibly for x a uniformiser). Then there is a compact $\mathcal{O}[[G]]$-algebra $\mathbf{h} = \mathbf{h}^{\mathrm{n.ord}}(N; \mathcal{O})$, topologically generated over $\mathcal{O}[[G]]$ by elements $T(\mathfrak{q})$, $T(x)$, with:*

(Hf1) $\mathbf{h}$ *is a finitely generated torsion-free A-module; every irreducible component of* $\operatorname{Spec}\mathbf{h}$ *is* $\operatorname{Spec}\mathbb{I}$ *for an integral domain $\mathbb{I}$ finite and torsion-free over A.*

(Hf2) *If $\lambda\colon \mathbf{h} \to \overline{\mathbb{Q}}_p$ is an $\mathcal{O}$-algebra homomorphism whose restriction to $\mathcal{O}[[G]]$ is $P_{n,v,\psi,\psi'}$ for an arithmetic weight (n, v), then there is a non-zero common eigenform $f \in S_{k,w}(Np^\alpha, \psi', \psi; \mathbb{C})$ in the nearly ordinary part on which every $T(\mathfrak{q})$, $T(x)$, $\langle\mathfrak{q}\rangle$ acts by $\iota\lambda(\cdot)$, the operators $T(x)$ normalised by x^{-v}. Conversely, every nearly ordinary common eigenform in some $S_{k,w}(Np^\alpha, \psi', \psi; \mathbb{C})$ defines such a λ.*

(Hf3) *For λ as in (Hf2) there is a unique semisimple continuous $\sigma_\lambda\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, unramified outside Np, with $\det(1 - \sigma_\lambda(\mathrm{Frob}_\mathfrak{q})X) = 1 - \lambda(T(\mathfrak{q}))X + \lambda(\langle\mathfrak{q}\rangle)\mathrm{N}\mathfrak{q}X^2$ for $\mathfrak{q} \nmid Np$. For every $\mathfrak{p} \mid p$ and every decomposition group $D_\mathfrak{p}$ there are characters ϵ, δ of $D_\mathfrak{p}$ with $\sigma_\lambda|_{D_\mathfrak{p}} \cong \left(\begin{smallmatrix}\epsilon & * \\ 0 & \delta\end{smallmatrix}\right)$ and $\delta(\mathrm{rec}_\mathfrak{p}(x)) = \lambda_f(x^{-v}T(x))$ for $0 \neq x \in \mathfrak{r}_\mathfrak{p}$, where $\lambda_f(h)$ is the eigenvalue of the classical operator h on f and $\mathrm{rec}_\mathfrak{p}$ is local reciprocity normalised by $\mathcal{N}(\mathrm{rec}_\mathfrak{p}(u)) = \mathrm{N}_{F_\mathfrak{p}/\mathbb{Q}_p}(u)^{-1}$ for $u \in \mathfrak{r}_\mathfrak{p}^\times$.*

Proof. (Hf1): [127], Theorem 2.4, as restated in [128], p. 118 ("$\mathbf{h}^{\mathrm{n.ord}}(S; \mathcal{O})$ is of finite type and torsion free as A-module") and Remark (i) on p. 119 (irreducible components $\mathrm{Spec}(\mathbb{I})$, $\mathbb{I}$ finite and torsion-free over A), and in [129], p. 336. (Hf2): [129], Corollary 3.3 ($\bar{S}^{\mathrm{n.ord}}(N)[P_{n,v,\psi,\psi'}] = S_{k,w}^{\mathrm{n.ord}}(Np^\alpha, \psi', \psi; \mathcal{O})$, the character $G \ni (z, a) \mapsto \psi(z)\psi'(a)\mathcal{N}(z)^{[n+2v]}a^v$), its proof on p. 336 (control isomorphism $\mathbf{h}_P^{\mathrm{n.ord}}/P\mathbf{h}_P^{\mathrm{n.ord}} \cong \mathbf{h}_{k,w}^{\mathrm{n.ord}}(Np^\alpha, \psi', \psi; K)$ from [127], Theorem 2.4; Hida notes that the argument there, written for ψ' trivial, applies to general characters), the duality Theorem 3.1, and p. 337 ("each specialization $\mathbf{f}_P$ is a common eigenform and is classical"); the independence of the weight (3.2) = [127], Theorem 2.3, for all (n, v) with $n \geq 0$, $v \geq 0$, $n + 2v \in \mathbb{Z}t$; weights $k = n + 2t$, $w = t - v$ in the normalisation of [129] ($w = v + k - t$ with the automorphy factor $\det^{-w}(cz + d)^k$ in [128], p. 117). (Hf3): [128], Theorem I (i)–(iv) (for any continuous λ into an integral domain in $\mathcal{C}$ of characteristic $\neq 2$, in particular a coefficient ring) and Proposition 2.3 (for λ on $\mathbf{h}_{k,w}^{\mathrm{n.ord}}(S(p^\alpha); \mathcal{O})$, $n \geq 0$, $v \geq 0$, $n + 2v \in \mathbb{Z}t$: δ corresponds to $\xi_\mathfrak{p}$ with $\lambda(x^{-v}T(x)) = \xi_\mathfrak{p}(x)$); the normalisation of local reciprocity is that of [130], Theorem 3.1(4) ($\mathcal{N}([u, F_\mathfrak{p}]) = N_{F_\mathfrak{p}/\mathbb{Q}_p}(u)^{-1}$, $\mathcal{N}(\mathrm{Frob}_l) = l$). For parallel weight the local shape is Wiles' theorem as stated in [236], (3.2), whose $T(\mathfrak{q})$, $S(\mathfrak{q})$ are Shimura's operators for $\mathfrak{q} \nmid \mathfrak{n}$ (op. cit., p. 30) and whose $T_0(\mathfrak{p}_i) = (\lambda^{(\mathfrak{p}_i)})^{-v}T(\mathfrak{p}_i)$ is the normalised operator above. As for notation, Hida's $\mathfrak{r}$, $\mathfrak{r}_p$, $\mathbf{G}$, $\mathbf{W}$, $\mathbf{A}$, $\mathcal{N}$ are $\mathcal{O}_F$, $\mathfrak{r}_p$, G, W, A, $\mathcal{N}$; his π, $\phi_\mathfrak{q}$ are σ_λ, $\mathrm{Frob}_\mathfrak{q}$. □

Theorem B.85. *Let F be a totally real field, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and π a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Then there is a continuous semisimple $r_{\ell,\iota}(\pi)\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ with $r_{\ell,\iota}(\pi) \leftrightarrow_\iota \pi$ such that for every finite place $v \nmid \ell$: if π_v is isomorphic to the irreducible normalised induction of characters μ_1, μ_2 of $F_v^\times$, then*

$r_{\ell,\iota}(\pi)|_{I_v} \cong \nu_1 \oplus \nu_2$ *for characters* ν_i *of* I_v *of finite order with* $\iota \circ \nu_i \circ \mathrm{Art}_{F_v} = \mu_i^{-1}$ *on* $\mathcal{O}_{F_v}^\times$*, where* Art_{F_v} *is the Artin map normalised to send uniformisers to geometric Frobenius elements.*

Proof. By [17], Theorem 2.1.1 (2), which allows F totally real, for a regular algebraic cuspidal polarized (π, χ) there is a continuous semisimple $r_{\ell,\iota}(\pi)$ with $\iota\mathrm{WD}(r_{\ell,\iota}(\pi)|_{G_{F_v}})^{F\text{-ss}} \cong \mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2})$ for $v \nmid \ell$ (Carayol, Taylor, Blasius–Rogawski for $n = 2$: [59], [253]). For GL_2 every regular algebraic cuspidal π is polarized by $\chi = \omega_\pi$, the central character: $\pi^\vee \cong \pi \otimes (\omega_\pi^{-1} \circ \det)$, and $\omega_{\pi,v}(-1)$ is independent of the real place v. Indeed $\pi_v \cong D_{k_v} \otimes |\det|^{t_v}$ is a twist of a discrete series, $\omega_{\pi,v}(-1) = (-1)^{k_v}$, regular algebraicity gives $t_v - k_v/2 \in \mathbb{Z}$, and $2t_v$ is independent of v because ω_π is trivial on a subgroup of finite index of the totally positive units (Dirichlet's unit theorem); so the parity of k_v is independent of v. If π_v is the irreducible normalised induction of μ_1, μ_2, then $\mathrm{rec}(\pi_v \otimes |\det|^{-1/2}) = (\mu_1|\cdot|^{-1/2} \oplus \mu_2|\cdot|^{-1/2}) \circ \mathrm{Art}_{F_v}^{-1}$ with monodromy $N = 0$; the Weil–Deligne representation of a representation with $N = 0$ has the same restriction to inertia as the representation (Grothendieck's monodromy theorem), Frobenius-semisimplification does not change the restriction to inertia, and $|\cdot|$ is trivial on units. In the normalisation of op. cit. (uniformisers to geometric Frobenius) this gives the stated ν_i. The relation $r_{\ell,\iota}(\pi) \leftrightarrow_\iota \pi$ is the unramified case. □

Theorem B.86. *Let* F *be a totally real field of degree* n *with narrow class group* $\mathrm{Cl}^+(F)$*, let* $\mathfrak{n}$ *be a non-zero integral ideal,* $\kappa \geq 1$ *an integer and* ψ *a character of finite order of the narrow ray class group modulo* $\mathfrak{n}$ *(extended by* $\psi(\mathfrak{a}) = 0$ *for* $\mathfrak{a}$ *not prime to* $\mathfrak{n}$*). Let* $M_\kappa(\mathfrak{n}, \psi) \supseteq S_\kappa(\mathfrak{n}, \psi)$ *be Shimura's spaces of Hilbert modular forms and cusp forms of parallel weight* κ*, level* $\mathfrak{n}$ *and character* ψ*: tuples* $f = (f_\lambda)_{\lambda \in \mathrm{Cl}^+(F)}$ *of holomorphic functions on* $\mathfrak{H}^n$*, attached to fixed representatives* $\mathfrak{t}_\lambda$ *prime to a given finite set of primes, with Fourier expansions* $f_\lambda(z) = a_\lambda(0) + \sum_{b \in \mathfrak{t}_\lambda,\, b \gg 0} a_\lambda(b) e_F(bz)$*,* normalised coefficients $c(\mathfrak{m}, f) = a_\lambda(b)\mathrm{N}(\mathfrak{t}_\lambda)^{-\kappa/2}$ *for* $\mathfrak{m} = (b)\mathfrak{t}_\lambda^{-1}$ *(*$\mathfrak{m}$ *a non-zero integral ideal) and* constant terms $c_\lambda(0, f) = a_\lambda(0)\mathrm{N}(\mathfrak{t}_\lambda)^{-\kappa/2}$*. Then:*

(Hb1) $M_\kappa(\mathfrak{n}, \psi)$ *is finite-dimensional over* $\mathbb{C}$*, and an* $f \in M_\kappa(\mathfrak{n}, \psi)$ *with* $c(\mathfrak{m}, f) = 0$ *for all non-zero integral* $\mathfrak{m}$ *is* 0*.*

(Hb2) *For every prime* $\mathfrak{q}$ *there is an endomorphism* $T(\mathfrak{q})$ *of* $M_\kappa(\mathfrak{n}, \psi)$ *preserving* $S_\kappa(\mathfrak{n}, \psi)$*; these endomorphisms commute, and for all non-zero integral* $\mathfrak{m}$

$$c(\mathfrak{m}, f|T(\mathfrak{q})) = c(\mathfrak{m}\mathfrak{q}, f) + \psi(\mathfrak{q})\mathrm{N}\mathfrak{q}^{\kappa-1} c(\mathfrak{m}\mathfrak{q}^{-1}, f),$$

with $c(\mathfrak{m}\mathfrak{q}^{-1}, f) = 0$ *if* $\mathfrak{q} \nmid \mathfrak{m}$*. For* $\mathfrak{q} \mid \mathfrak{n}$ *one writes* $U(\mathfrak{q}) = T(\mathfrak{q})$*.*

(Hb3) *If* $f \in M_\kappa(\mathfrak{n}, \psi)$ *and* $g \in M_{\kappa'}(\mathfrak{n}', \psi')$ *(same representatives* $\mathfrak{t}_\lambda$*), then* $fg = (f_\lambda g_\lambda)_\lambda \in M_{\kappa+\kappa'}(\mathfrak{n} \cap \mathfrak{n}', \psi\psi')$*.*

(Hb4) *Let* η, ψ_1 *be (possibly imprimitive) finite-order characters of the narrow ray class groups modulo integral ideals* $\mathfrak{a}, \mathfrak{b}$*, with signs* $q, r \in (\mathbb{Z}/2\mathbb{Z})^n$*,* $q + r \equiv (\kappa, \ldots, \kappa)$*. There is* $E_\kappa(\eta, \psi_1) \in M_\kappa(\mathfrak{a}\mathfrak{b}, \eta\psi_1)$ *with*

$$c(\mathfrak{m}, E_\kappa(\eta, \psi_1)) = \sum_{\mathfrak{r} \mid \mathfrak{m}} \eta(\mathfrak{m}\mathfrak{r}^{-1})\psi_1(\mathfrak{r})\mathrm{N}\mathfrak{r}^{\kappa-1}$$

for all non-zero integral $\mathfrak{m}$*; if* $\kappa > 1$*, then* $c_\lambda(0, E_\kappa(\eta, \psi_1)) = 2^{-n}\eta^{-1}(\mathfrak{t}_\lambda)L(\psi_1\eta^{-1}, 1-\kappa)$ *if* $\mathfrak{a} = 1$ *and* 0 *otherwise; if* $\kappa = 1$ *and* $\mathfrak{a} \neq 1 \neq \mathfrak{b}$*, then* $c_\lambda(0, E_1(\eta, \psi_1)) = 0$ *for all* λ*. Here* $L(\chi, s)$ *is the L-function of* χ *with the Euler factors at the primes dividing the modulus removed.*

(Hb5) *Let $\kappa \geq 2$ and let $f \in M_\kappa(\mathfrak{n}, \psi)$, $f \notin S_\kappa(\mathfrak{n}, \psi)$, be an eigenvector of every $T(\mathfrak{q})$ with eigenvalue $\lambda_f(\mathfrak{q})$. Then there are finite-order characters η_1, η_2 of narrow ray class groups, with $\eta_1\eta_2(\mathfrak{q}) = \psi(\mathfrak{q})$ for $\mathfrak{q} \nmid \mathfrak{n}$, such that $\lambda_f(\mathfrak{q}) = \eta_1(\mathfrak{q}) + \eta_2(\mathfrak{q})\mathrm{N}\mathfrak{q}^{\kappa-1}$ for $\mathfrak{q} \nmid \mathfrak{n}$, and $\lambda_f(\mathfrak{q}) \in \{0,\ \eta_1(\mathfrak{q}),\ \eta_2(\mathfrak{q})\mathrm{N}\mathfrak{q}^{\kappa-1}\}$ for $\mathfrak{q} \mid \mathfrak{n}$, where $\eta_i(\mathfrak{q}) = 0$ if η_i is ramified at $\mathfrak{q}$.*

Proof. See [230], Section 2 (the spaces $M_\kappa(\mathfrak{n}, \psi)$, $S_\kappa(\mathfrak{n}, \psi)$, the normalised coefficients $C(\mathfrak{m}, f)$, the Hecke operators and their action on $C(\mathfrak{m}, f)$, formula (2.23)) and Proposition 3.4 (existence of the Eisenstein series); [73], Section 2.1 (definitions, (47)–(49): normalised coefficients $c(\mathfrak{m}, f) = C(\mathfrak{m}, f)$ and constant terms $c_\lambda(0, f)$) and Proposition 2.1 (formulas (50)–(52): Fourier coefficients and constant terms of $E_k(\eta, \psi)$, imprimitive characters allowed, constant term 0 in weight one when both moduli are $\neq 1$). For (Hb1), finite dimensionality and the q-expansion principle are in [230], Section 2; for parallel weight $\kappa \geq 1$ a form all of whose non-constant coefficients vanish has constant components f_λ, and a constant satisfying (47) of [73] with a factor $(cz + d)^{-\kappa}$, $c \neq 0$, vanishes. For (Hb3), the automorphy factor $\det(\gamma)^{\kappa/2}(cz + d)^{-\kappa}$ is multiplicative in κ, and the character condition on Γ_λ and the condition $S(\mathfrak{m})f = \psi(\mathfrak{m})f$ are multiplicative (adelically: products of functions on $\mathrm{GL}_2(\mathbb{A}_F)$). For (Hb5), the space $M_\kappa(\mathfrak{n}, \psi)$ is the direct sum of $S_\kappa(\mathfrak{n}, \psi)$ and the Hecke-stable space spanned by the translates $E_\kappa(\eta_1, \eta_2)|B(\mathfrak{c})$ with $\eta_1\eta_2 = \psi$, $\mathrm{cond}(\eta_1)\mathrm{cond}(\eta_2)\mathfrak{c} \mid \mathfrak{n}$, on whose summand for (η_1, η_2) the operators $T(\mathfrak{q})$, $\mathfrak{q} \nmid \mathfrak{n}$, act by $\eta_1(\mathfrak{q}) + \eta_2(\mathfrak{q})\mathrm{N}\mathfrak{q}^{\kappa-1}$ and $U(\mathfrak{q})$, $\mathfrak{q} \mid \mathfrak{n}$, has eigenvalues among 0, $\eta_1(\mathfrak{q})$, $\eta_2(\mathfrak{q})\mathrm{N}\mathfrak{q}^{\kappa-1}$ ([230], Sections 2–3; the same classification is the input of [236], Section 3.4). *Notation.* Shimura's $C(\mathfrak{m}, f)$ is $c(\mathfrak{m}, f)$; Dasgupta–Darmon–Pollack's $E_k(\eta, \psi)$ is $E_\kappa(\eta, \psi_1)$ here. □

Theorem B.87. *Let M/L be a cyclic extension of number fields of prime degree p, σ a generator of $\mathrm{Gal}(M/L)$, $n \geq 1$, and π a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_M)$; write π^σ for $g \mapsto \pi(\sigma(g))$. There is an isobaric automorphic representation $\mathrm{AI}_{M/L}(\pi)$ of $\mathrm{GL}_{np}(\mathbb{A}_L)$ such that:*

(Ai1) *for every finite place v of L unramified in M such that π_w is unramified for every place $w \mid v$ of M, $\mathrm{AI}_{M/L}(\pi)_v$ is unramified, and its multiset of Satake parameters is the union of those of the π_w, $w \mid v$, if v splits completely in M, and is $\{\gamma :\ \gamma^p = \alpha\}$, α running over the Satake parameters of π_w (each γ once for each α), if v is inert in M with $w \mid v$;*

(Ai2) *for every complex place v of L, the Langlands parameter of $\mathrm{AI}_{M/L}(\pi)_v$ is the direct sum of the Langlands parameters of the π_w, $w \mid v$;*

(Ai3) *$\mathrm{AI}_{M/L}(\pi)$ is cuspidal if and only if $\pi \not\cong \pi^\sigma$.*

Proof. See [9, Chapter 3, §6] (automorphic induction from $\mathrm{GL}_n(\mathbb{A}_M)$ to $\mathrm{GL}_{np}(\mathbb{A}_L)$ for M/L cyclic of prime degree, derived there from base change; secondary sources number the theorem 6.2). The statement there is as follows. There is an automorphic representation $\Pi = I_M^L(\pi)$, induced from cuspidal, whose local component at every place v is the local automorphic induction of $(\pi_w)_{w|v}$, i.e. whose local Langlands parameter is the induction of the parameter of π_w from the Weil group of M_w to that of L_v when v does not split, and the sum over $w \mid v$ when v splits; and Π is cuspidal exactly when π is not isomorphic to its Galois conjugates (for prime p, $\pi \cong \pi^{\sigma^i}$ for some $0 < i < p$ if and only if $\pi \cong \pi^\sigma$). In the notation used here: at an unramified inert v the induction of the unramified character of W_{M_w} with $\mathrm{Frob}_w \mapsto \alpha$ (and $\mathrm{Frob}_w = \mathrm{Frob}_v^p$) has Frob_v-eigenvalues the p-th roots of α, which is (Ai1) for inert v; complex places of L split completely in M, which is (Ai2). □

Theorem B.88. *Let A be a commutative ring, R an A-algebra and $D\colon R \to A$ a determinant of dimension d. Let*

$$\ker(D) = \{r \in R :\ D_T(1+rr') = 1 \text{ for every commutative } A\text{-algebra } T \text{ and every } r' \in R \otimes_A T\},$$

and let $\mathrm{CH}(D)$ be the two-sided ideal of R generated by the coefficients of $\chi_{D'}(t_1 r_1 + \cdots + t_m r_m)(t_1 r_1 + \cdots + t_m r_m) \in R[t_1, \ldots, t_m]$, for $m \geq 1$, $r_1, \ldots, r_m \in R$ and D' the base change of D to $A[t_1, \ldots, t_m]$.

(Ck1) *$\ker(D)$ is also the set of $r \in R$ with $D_T(1 + r'r) = 1$ for all T and $r' \in R \otimes_A T$; it is a two-sided ideal of R, and it is the largest two-sided ideal K of R such that $D = \tilde{D} \circ \pi$ for the projection $\pi\colon R \to R/K$ and a determinant $\tilde{D}$ of dimension d on the A-algebra R/K.*

(Ck2) *$\mathrm{CH}(D) \subseteq \ker(D)$.*

Proof. This is [66, Lemma 1.19 (i), (ii) and Lemma 1.21], stated there for polynomial laws $P \in \mathcal{M}_A^d(R, S)$ that are multiplicative and homogeneous of degree d; (Ck1) is the case $S = A$, and the factorisation $\tilde{D}$ is again homogeneous of degree d and multiplicative (loc. cit., remark after Lemma 1.18). □

Theorem B.89. *Let A be a henselian local commutative ring with residue field k, R an A-algebra and $D\colon R \to A$ a determinant of dimension d which is Cayley–Hamilton, that is $\mathrm{CH}(D) = 0$. Assume that the residual determinant $D \otimes_A k\colon R \otimes_A k \to k$ is the determinant of a surjective homomorphism of k-algebras $R \otimes_A k \to \mathrm{M}_d(k)$. Then there is an isomorphism of A-algebras $\rho\colon R \to \mathrm{M}_d(A)$ with $D = \det \circ \rho$.*

Proof. By [66], Theorem 2.22(i), if D is Cayley–Hamilton and A is henselian, and the residual determinant is split and absolutely irreducible, then there is an A-algebra isomorphism $\rho\colon R \to \mathrm{M}_d(A)$ with $D = \det \circ \rho$. By the remark after Definition 2.19 there, a determinant $R \otimes_A k \to k$ is split and absolutely irreducible if and only if it is the determinant of a surjective k-representation $R \otimes_A k \to \mathrm{M}_d(k)$, which is the hypothesis stated here. □

Theorem B.90. *Let F be a CM field, $n \geq 2$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and π a regular algebraic conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let $v \nmid \ell$ be a finite place of F, $\bar{v} \mid v$ a place of $\overline{\mathbb{Q}}$, $(V, \rho, N) = \mathrm{WD}(r_{\ell,\iota}(\pi)|_{D_{\bar{v}}})$ and $\Phi \in W_{\bar{v}}$ with $n(\Phi) = -1$. Then (V, ρ, N) is pure: there are a real number k and $\rho(W_{\bar{v}})$-stable subspaces $\mathrm{Fil}_j V$ ($j \in k + \mathbb{Z}$), increasing in j, zero for j small and equal to V for j large, with $N(\mathrm{Fil}_j V) \subseteq \mathrm{Fil}_{j-2} V$, such that, with $\mathrm{gr}_j = \mathrm{Fil}_j V / \mathrm{Fil}_{j-1} V$:*

(Pv1) *every eigenvalue α of the endomorphism of gr_j induced by $\rho(\Phi)$ is algebraic over $\mathbb{Q}$ with $|\sigma(\alpha)| = q_v^{j/2}$ for every field embedding $\sigma\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$;*

(Pv2) *for every integer $i \geq 1$, N^i induces an isomorphism $\mathrm{gr}_{k+i} \to \mathrm{gr}_{k-i}$.*

Proof. This follows from [56], Theorem 1.1 and Theorem 1.2, with [255], Section 1 and Lemma 1.4.

The published statements. [56], Theorem 1.1: let $n \geq 2$, L a CM field, Π cuspidal automorphic of $\mathrm{GL}_n(\mathbb{A}_L)$ with $\Pi^\vee \cong \Pi \circ c$ and Π_∞ cohomological for an irreducible algebraic representation, $R_\ell(\Pi)$ its Galois representation (Shin, Chenevier–Harris); for $p \neq \ell$ and a place $y \mid p$ of L, $\mathrm{WD}(R_\ell(\Pi)|_{G_{L_y}})^{F\text{-ss}} \cong \iota^{-1}\mathcal{L}_{n,L_y}(\Pi_y)$, the local Langlands correspondence in the geometric normalisation. Theorem 1.2 (same hypotheses): Π is tempered at every finite place.

As stated on p. 2 of loc. cit., temperedness makes $\iota^{-1}\mathcal{L}_{n,L_y}(\Pi_y)$ pure ([255], Lemma 1.4(3)), so $\mathrm{WD}(R_\ell(\Pi)|_{G_{L_y}})^{F\text{-ss}}$ is pure, and then so is $\mathrm{WD}(R_\ell(\Pi)|_{G_{L_y}})$ ([255], Lemma 1.4(1)). The definitions of pure, mixed and strictly pure Weil–Deligne representations of [255], Section 1, give (Pv1), (Pv2), as in the published form recorded in Theorem B.73 for the places above ℓ; the functor WD for $p \neq \ell$ there is Grothendieck's, which is (Wd1).

Comparison of notation. Regular algebraic cuspidal means cohomological at infinity ([68], Section 3) and conjugate self-dual means $\pi^\vee \cong \pi^c$, so π satisfies the hypotheses on Π. $R_\ell(\Pi)$ and $r_{\ell,\iota}(\pi)$ are semisimple with the same Frobenius characteristic polynomials at almost all places ((Bg1) of Theorem B.62), hence isomorphic (Chebotarev, Brauer–Nesbitt), and $D_{\bar v}$ is identified with G_{F_v}. □

Theorem B.91. *Let ℓ be a prime, $n \geq 2$, F an imaginary CM field with maximal totally real subfield F^+ such that every place of F^+ above ℓ splits in F, and $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ containing the image of every embedding $F \to \overline{\mathbb{Q}}_\ell$. Let $\mathbb{G}$ be a unitary group over $\mathcal{O}_{F^+}$ attached to an involution of the second kind on $\mathrm{M}_n(F)$ and an order of $\mathrm{M}_n(F)$ stable under it and maximal at the places split over F^+, such that $\mathbb{G}(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact and $\mathbb{G}$ is quasi-split at every finite place of F^+. Let T be a finite set of finite places of F^+, all split in F, containing the set S_ℓ of places above ℓ, and $U = \prod_v U_v \subseteq \mathbb{G}(\mathbb{A}_{F^+}^\infty)$ a compact open subgroup with $U_v = \mathbb{G}(\mathcal{O}_{F_v^+})$ for $v \in S_\ell$ and for every $v \notin T$ split in F. For $1 \leq b \leq c$ let $U(\ell^{b,c}) \subseteq U$ be the subgroup whose components at $v \in S_\ell$ are the groups $\mathrm{Iw}(\tilde v^{b,c})$; let $S^{\mathrm{ord}}(U(\ell^\infty), \mathcal{K}/\mathcal{O}) = \varinjlim_c S^{\mathrm{ord}}(U(\ell^{c,c}), \mathcal{K}/\mathcal{O})$ be the space of ordinary forms of trivial weight with coefficients in $\mathcal{K}/\mathcal{O}$, $\mathcal{K} = \mathcal{O}[1/\ell]$, and $\mathcal{S}$ its Pontryagin dual; let $\mathfrak{l} = \prod_{v \in S_\ell} v$ (an ideal of $\mathcal{O}_{F^+}$) and, for $b \geq 1$, let $T_n(\mathfrak{l}^b)$ and $T_n(\ell^b)$ be the kernels of the reductions $T_n(\mathcal{O}_{F^+} \otimes \mathbb{Z}_\ell) \to T_n(\mathcal{O}_{F^+}/\mathfrak{l}^b)$ and $T_n(\mathcal{O}_{F^+} \otimes \mathbb{Z}_\ell) \to T_n(\mathcal{O}_{F^+}/\ell^b)$ (T_n the diagonal torus; $T_n(\ell^b) \subseteq T_n(\mathfrak{l}^b)$, with equality if ℓ is unramified in F^+); let $\Lambda^+ = \mathcal{O}[[T_n(\mathcal{O}_{F^+} \otimes \mathbb{Z}_\ell)]]$, $\Lambda_b^{\mathfrak{l}} = \mathcal{O}[[T_n(\mathfrak{l}^b)]]$ and $\Lambda_b = \mathcal{O}[[T_n(\ell^b)]]$, so that $\Lambda_b \subseteq \Lambda_b^{\mathfrak{l}} \subseteq \Lambda^+$, acting on $\mathcal{S}$ through the diamond operators $\langle u \rangle$; and let $\mathbb{T}^{\mathrm{ord}}(U)$ be the Λ^+-subalgebra of $\mathrm{End}_{\mathcal{O}}(\mathcal{S})$ generated by the Hecke operators $T_w^{(j)}$, $(T_w^{(n)})^{-1}$ at the places w of F above places $v \notin T$ split in F, and the operators $U_{\tilde v}^{(j)}$, $v \in S_\ell$, $1 \leq j \leq n$. If $b_0 \geq 1$ and $U(\ell^{b_0,b_0})$ is sufficiently small, then:*

(Gy1) *$\mathcal{S}$ is a free $\Lambda_{b_0}^{\mathfrak{l}}$-module of finite rank;*

(Gy2) *$\mathbb{T}^{\mathrm{ord}}(U)$ is commutative and finite as a $\Lambda_{b_0}^{\mathfrak{l}}$-module, and, if $\mathcal{S} \neq 0$, $\Lambda_{b_0}^{\mathfrak{l}} \to \mathbb{T}^{\mathrm{ord}}(U)$ is injective;*

(Gy3) *(classicality; objects of §2.2 and §2.6 of [105]) let λ be a dominant weight for $\mathbb{G}$, w_0 the longest element of the Weyl group, ν the weight with components $(n-1, n-2, \dots, 0)$ at every embedding, give $\mathbb{T}^{\mathrm{ord}}(U)$ the $\Lambda_1^{\mathfrak{l}}$-structure $u \mapsto (w_0\nu)^{-1}(u)\langle u \rangle$ ($u \in T_n(\mathfrak{l})$), and its restriction to Λ_1. For a character $\alpha\colon T_n(\mathfrak{l}) \to \overline{\mathbb{Q}}_\ell^\times$ of finite order let $\mathfrak{P}_{\lambda,\alpha}^{\mathfrak{l}}$ be the kernel of the homomorphism $\Lambda_1^{\mathfrak{l}} \to \overline{\mathbb{Q}}_\ell$ induced by the character $\alpha\,(w_0\nu)^{-1}(w_0\lambda)^{-1}$; for a character $\alpha\colon T_n(\ell^1) \to \overline{\mathbb{Q}}_\ell^\times$ of finite order let $\mathfrak{P}_{\lambda,\alpha}$ be the kernel of the homomorphism $\Lambda_1 \to \overline{\mathbb{Q}}_\ell$ induced by the restriction of $\alpha\,(w_0\nu)^{-1}(w_0\lambda)^{-1}$ to $T_n(\ell^1)$. For $r \geq 1$, α a character of finite order of $T_n(\mathfrak{l})$ or of $T_n(\ell^1)$, and a field $A \supseteq \mathcal{O}$ containing the values of α, let $S_\lambda^{\mathrm{ord}}(U(\ell^{r,r}), \alpha, A)$ be the subspace of the ordinary forms of weight λ and level $U(\ell^{r,r})$ with coefficients in A on which the operator $\langle u \rangle$ of weight λ acts by $\alpha(u)$ for every u in the domain of α, and $\mathbb{T}_\lambda^{\mathrm{ord}}(U(\ell^{r,r}), \alpha, A)$ the A-algebra of endomorphisms of it generated by the operators of weight λ. Now let $\alpha\colon T_n(\mathfrak{l}) \to \overline{\mathbb{Q}}_\ell^\times$ be a character of finite order, $r \geq 1$ with $T_n(\mathfrak{l}^r) \subseteq \ker\alpha$, $\mathfrak{P} = \mathfrak{P}_{\lambda,\alpha}^{\mathfrak{l}}$ and $\mathcal{K}'$ the fraction field of $\Lambda_1^{\mathfrak{l}}/\mathfrak{P}$. Then the action on $S_\lambda^{\mathrm{ord}}(U(\ell^{r,r}), \alpha, \mathcal{K}')$ induces a surjection of finite $\mathcal{K}'$-algebras $\mathbb{T}^{\mathrm{ord}}(U) \otimes_{\Lambda_1^{\mathfrak{l}}} (\Lambda_1^{\mathfrak{l}})_{\mathfrak{P}}/\mathfrak{P} \to$*

$\mathbb{T}^{\mathrm{ord}}_\lambda(U(\ell^{r,r}),\alpha,\mathcal{K}')$, *and its kernel is nilpotent. If* α *is trivial and* $\gamma\colon T_n(\mathcal{O}_{F^+}/\mathfrak{l})\to\mathcal{O}^\times$ *is a character, write* $\mathbb{T}^{\mathrm{ord}}(U)_\gamma$ *for the maximal quotient of* $\mathbb{T}^{\mathrm{ord}}(U)$ *on which* $\langle u\rangle=\gamma(u)$ *for* $u\in T_n(\mathcal{O}_{F^+}/\mathfrak{l})$ *(Teichmüller representatives,* $\Lambda^+=\Lambda_1^{\mathfrak{l}}[T_n(\mathcal{O}_{F^+}/\mathfrak{l})]$*); then the action on the ordinary forms of weight* λ *and Iwahori level* $U(\ell^{0,1})$ *on which* $T_n(\mathcal{O}_{F^+}/\mathfrak{l})$ *acts through* $\gamma\,\omega^{w_0\lambda}$ *(*$\omega^{w_0\lambda}$ *the Teichmüller lift of the reduction of* $w_0\lambda$*) induces a surjection* $\mathbb{T}^{\mathrm{ord}}(U)_\gamma\otimes_{\Lambda_1^{\mathfrak{l}}}(\Lambda_1^{\mathfrak{l}})_{\mathfrak{P}}/\mathfrak{P}\to\mathbb{T}^{\mathrm{ord}}_\lambda(U(\ell^{0,1}),\gamma\omega^{w_0\lambda},\mathcal{K}')$ *with nilpotent kernel (*$\mathfrak{P}=\mathfrak{P}^{\mathfrak{l}}_{\lambda,1}$*).*

For (Gy4)–(Gy6), Art *is normalised to send uniformisers to geometric Frobenius elements,* ε *is the* ℓ*-adic cyclotomic character, Hodge–Tate weights are normalised so that* ε *has weight* -1*,* rec *is the local Langlands correspondence of Harris–Taylor, and for an irreducible admissible representation* σ *of* $\mathrm{GL}_n(F_w)$*,* $r_\ell(\sigma)$ *is the* ℓ*-adic representation of* G_{F_w} *attached to* $\mathrm{rec}(\sigma^\vee\otimes|\det|^{(1-n)/2})$*;* $\iota_w\colon\mathbb{G}(F_v^+)\to\mathrm{GL}_n(F_w)$ *is the isomorphism at a place* w *of* F *above a split* v*, and* $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$ *an isomorphism.*

(Gy4) *(descent) Suppose that* F/F^+ *is unramified at all finite places and* $[F^+:\mathbb{Q}]\geq 2$*. For every regular algebraic conjugate self-dual cuspidal automorphic representation* Π *of* $\mathrm{GL}_n(\mathbb{A}_F)$ *there are a dominant weight* λ *for* $\mathbb{G}$ *and an irreducible constituent* π *of the* $\mathbb{G}(\mathbb{A}_{F^+}^\infty)$*-representation* $S_\lambda(\overline{\mathbb{Q}}_\ell)$ *such that* $\pi_v\cong\iota^{-1}\Pi_w\circ\iota_w$ *for every finite place* v *of* F^+ *which splits as* ww^c *in* F*, and* π_v *has a non-zero vector fixed by a hyperspecial maximal compact subgroup of* $\mathbb{G}(F_v^+)$ *for every place* v *inert in* F *at which* Π *is unramified.*

(Gy5) *(Galois representations) For every dominant weight* λ *and every irreducible constituent* π *of* $S_\lambda(\overline{\mathbb{Q}}_\ell)$ *there is a continuous semisimple representation* $r_\pi\colon G_F\to\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ *such that: (i) for every finite place* $v\notin S_\ell$ *of* F^+ *which splits as* ww^c *in* F*,* $r_\pi|^{\mathrm{ss}}_{G_{F_w}}\cong\big(r_\ell(\pi_v\circ\iota_w^{-1})^\vee\otimes\varepsilon^{1-n}\big)^{\mathrm{ss}}$*; in particular, if* π_v *is unramified, then* $r_\pi|^{\mathrm{ss}}_{G_{F_w}}$ *is unramified and* $r_\pi(\mathrm{Frob}_w)$ *has characteristic polynomial* $\sum_{j=0}^n(-1)^j(\mathrm{N}w)^{j(j-1)/2}t_w^{(j)}X^{n-j}$*, where* $t_w^{(0)}=1$ *and* $t_w^{(j)}$ *is the eigenvalue of* $T_w^{(j)}$ *on* $\pi_v^{\mathbb{G}(\mathcal{O}_{F_v^+})}$*; (ii)* $r_\pi^c\cong r_\pi^\vee\varepsilon^{1-n}$*; (iii)* r_π *is unramified above every place* v *inert in* F *at which* π_v *has a vector fixed by a hyperspecial maximal compact subgroup; (iv) for every place* $w\mid\ell$ *of* F *and embedding* $\tau\colon F\to\overline{\mathbb{Q}}_\ell$ *giving rise to* w*,* $r_\pi|_{G_{F_w}}$ *is potentially semistable with* τ*-Hodge–Tate weights* $\lambda_{\tau,j}+n-j$*,* $1\leq j\leq n$*.*

(Gy6) *(ordinary forms of regular weight) Let* λ *be a dominant weight for* $\mathbb{G}$ *which is regular (for every* $v\in S_\ell$ *and* $1\leq j<n$ *some* τ *giving rise to* $\tilde{v}$ *has* $\lambda_{\tau,j}>\lambda_{\tau,j+1}$*),* π *an irreducible constituent of* $S_\lambda(\overline{\mathbb{Q}}_\ell)$ *with* $\pi^{U(\ell^{0,1})}\cap S^{\mathrm{ord}}_\lambda(U(\ell^{0,1}),\mathcal{O})\neq 0$*, and* $v\in S_\ell$*. Then* $r_\pi|_{G_{F_{\tilde{v}}}}$ *is crystalline and conjugate to an upper triangular representation with diagonal* $(\psi_{\tilde{v},1},\varepsilon^{-1}\psi_{\tilde{v},2},\dots,\varepsilon^{-(n-1)}\psi_{\tilde{v},n})$*, where* $\psi_{\tilde{v},j}(\mathrm{Art}_{F_{\tilde{v}}}(x))=\prod_\tau\tau(x)^{-\lambda_{\tau,n-j+1}}$ *for* $x\in\mathcal{O}_{F_{\tilde{v}}}^\times$ *(*τ *running over the embeddings* $F_{\tilde{v}}\to\overline{\mathbb{Q}}_\ell$*) and* $\psi_{\tilde{v},j}(\mathrm{Art}_{F_{\tilde{v}}}(\varpi_{\tilde{v}}))=u^{(j)}/u^{(j-1)}$*, with* $u^{(0)}=1$ *and* $u^{(j)}$ *the eigenvalue of* $U^{(j)}_{\lambda,\varpi_{\tilde{v}}}$ *on the line* $\pi^{U(\ell^{0,1})}\cap S^{\mathrm{ord}}_\lambda(U(\ell^{0,1}),\mathcal{O})$*.*

(Gy7) *(local ordinary theory and the operators at weight* λ*) (a) Let* E *be a finite extension of* $\mathbb{Q}_p$ *(any prime* p*) with uniformiser* ϖ*,* $B_n=T_nN_n$ *the upper triangular Borel subgroup,* $\delta_{B_n}(t)=|\det(\mathrm{ad}\,t\,|\,\mathrm{Lie}\,N_n(E))|$*,* $\mathrm{Iw}(\varpi^{b,c})\subseteq\mathrm{GL}_n(\mathcal{O}_E)$ *(*$0\leq b\leq c$*) the matrices upper triangular modulo* ϖ^c *and unipotent modulo* ϖ^b*,* $T_n(\varpi^b)$ *the diagonal matrices* $\equiv 1\bmod\varpi^b$*,* $\alpha_\varpi^{(j)}=\mathrm{diag}(\varpi 1_j,1_{n-j})$ *and* $U_\varpi^{(j)}=[N_n(\mathcal{O}_E)\alpha_\varpi^{(j)}N_n(\mathcal{O}_E)]$*. For a smooth admissible representation* σ *of* $\mathrm{GL}_n(E)$ *over an algebraically closed field of characteristic* 0*, with* $N_n(E)$*-coinvariants* σ_{N_n}*, and* $0\leq b\leq c$*,* $c>0$*: the natural map* $\sigma^{\mathrm{Iw}(\varpi^{b,c})}\to(\sigma_{N_n})^{T_n(\varpi^b)}$ *is surjective, its kernel is the subspace on which some* $U_\varpi^{(j)}$ *acts non-invertibly, it is* $T_n(\mathcal{O}_E)$*-equivariant, and* $U_\varpi^{(j)}$ *corresponds to* $\delta_{B_n}(\alpha_\varpi^{(j)})^{-1}\sigma_{N_n}(\alpha_\varpi^{(j)})$*. (b) For* Π *regular algebraic*

cuspidal on $\mathrm{GL}_n(\mathbb{A}_F)$ *of weight* $\iota\lambda$ *(*λ_τ *indexed by the embeddings* $\tau\colon F \to \overline{\mathbb{Q}}_\ell$*), a place* $w \mid \ell$ *and* $b > 0$*, the operators* $U^{(j)}_{\lambda,\varpi_w} = \prod_\tau \prod_{i=1}^{j} \tau(\varpi_w)^{-\lambda_{\tau,n-i+1}} U^{(j)}_{\varpi_w}$ *(*τ *running over the embeddings inducing* w*) act on* $(\iota^{-1}\Pi_w)^{\mathrm{Iw}(\varpi_w^{b,b})}$*; its* ordinary part *is the largest subspace stable under them on which all their eigenvalues are* ℓ*-adic units,* $(\iota^{-1}\Pi_w)^{\mathrm{ord}}$ *is the direct limit over* b*,* Π *is* ι-ordinary at w *if it is non-zero, and then it has dimension* 1*. (c) The surjections of (Gy3) are induced by the isomorphism of Hecke algebras which sends* $T^{(j)}_w$ *to* $T^{(j)}_w$*,* $U^{(j)}_{\tilde{v}}$ *to* $U^{(j)}_{\lambda,\varpi_{\tilde{v}}} = (w_0\lambda)(\alpha^{(j)}_{\varpi_{\tilde{v}}})^{-1} U^{(j)}_{\varpi_{\tilde{v}}}$ *and* $\langle u\rangle$ *to* $(w_0\lambda)(u)^{-1}\langle u\rangle$*, the operators on the right acting on forms of weight* λ*.*

Proof. This is [105], Proposition 2.5.3 (freeness of the dual of the space of ordinary forms of level $U(\ell^\infty)$ over Λ_{b_0}) and Corollary 2.5.4 (the ordinary Hecke algebras are finite faithful Λ_{b_0}-algebras, equal to the Λ^+-subalgebras of $\mathrm{End}(S^{\mathrm{ord}}(U(\ell^\infty), \mathcal{K}/\mathcal{O}))$ generated by the operators listed), in the numbering of the 2010 version of that paper; commutativity is Lemma 2.3.3 there. (Gy3) is Lemma 2.6.4 there (first surjection; the renormalised Λ-structure is Definition 2.6.2, $\wp_{\lambda,\alpha}$ is Definition 2.6.3(2), and the Hecke algebra $\widetilde{\mathbb{T}}^{T,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\ell^{r,r}), \alpha, K')$ there, with $R = \emptyset$, is our $\mathbb{T}^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \alpha, \mathcal{K}')$); the passage between $\widetilde{\mathbb{T}}$ and $\mathbb{T}$ (the latter generated by fewer operators) is immediate there (*the analogous results for* $\mathbb{T}$ *follow immediately*). The action on the dual $\mathcal{S}$ is through the duality of Remark 2.4.6 there, under which $\mathrm{End}_\mathcal{O}$ of a module and of its dual are identified. As for notation, Geraghty's K, $\mathcal{O}$, F, F^+, G, l, $\lambda = 0$, $R = \emptyset$ are our $\mathcal{K}$, $\mathcal{O}$, F, F^+, $\mathbb{G}$, ℓ; his ideal $\mathfrak{l}$ (the product of the places in S_l, §2.3), $T_n(\mathfrak{l}^b)$, $\Lambda = \mathcal{O}[[T_n(\mathfrak{l})]]$ and $\Lambda_b = \mathcal{O}[[T_n(\mathfrak{l}^b)]]$ (Definition 2.5.1) are our $\mathfrak{l}$, $T_n(\mathfrak{l}^b)$, $\Lambda^{\mathfrak{l}}_1$, $\Lambda^{\mathfrak{l}}_b$, and his $\wp_{\lambda,\alpha}$ (Definition 2.6.3(2), α a character of finite order of $T_n(\mathfrak{l})$, r with $T_n(\mathfrak{l}^r) \subseteq \ker\alpha$) is our $\mathfrak{P}^{\mathfrak{l}}_{\lambda,\alpha}$; his $S^{\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(l^{r,r}), \alpha, A)$ is the maximal subspace on which $\langle u\rangle = \alpha(u)$ for all $u \in T_n(\mathfrak{l})$ (paragraph before Definition 2.6.3); the subgroups $\mathrm{Iw}(\tilde{v}^{b,c})$ are unipotent modulo $\varpi^b_{\tilde{v}}$ (§2.1), so our $U(\ell^{b,c})$ is his $U(l^{b,c})$. The groups $T_n(\ell^b)$, the algebras Λ_b and the ideals $\mathfrak{P}_{\lambda,\alpha} \subseteq \Lambda_1$ (reduction modulo the rational ℓ^b) are notation of ours, not Geraghty's; they enter no claim of this quotation; his $\mathbb{T}^{T,\mathrm{ord}}_0(U(l^\infty), \mathcal{O})$ is our $\mathbb{T}^{\mathrm{ord}}(U)$. The last sentence of (Gy3) is the remark following the proof of Lemma 2.6.4 there ("Taking α to be the trivial character in the lemma ..."), for the algebras with fixed character γ of Definition 2.6.3 and the paragraph before it. (Gy4) is Proposition 2.2.7 there (a consequence of Corollaire 5.3 and Théorème 5.4 of J.-P. Labesse, *Changement de base CM et séries discrètes*; Théorème 5.4 there is stated for E/F unramified at all finite places and U quasi-split at the finite inert places, with Labesse's standing hypothesis $[F : \mathbb{Q}] \geq 2$, which we add), combined with Lemma 2.2.5 there (the isomorphism of $S_\lambda(\overline{\mathbb{Q}}_\ell)$ with $\mathrm{Hom}_{\mathbb{G}(F^+_\infty)}(\xi^\vee_{\lambda,\iota}, \mathcal{A})$, $\mathcal{A}$ semisimple) to turn the automorphic π with $\pi_\infty \cong \xi^\vee_{\lambda,\iota}$ into a constituent of $S_\lambda(\overline{\mathbb{Q}}_\ell)$. (Gy5) is Proposition 2.7.2 there, parts 1, 2, 3 and 5 with $R = \emptyset$ (proof: Labesse, Corollaire 5.3, and Chenevier–Harris, Theorem 3.2.5); the characteristic polynomial in (i) is the unramified case of part 1 written in Hecke eigenvalues, as in property 2 of Proposition 2.7.3 there (whose proof derives it from Proposition 2.7.2 by the argument of Proposition 3.4.2 of Clozel–Harris–Taylor); the potential semistability in (iv) is part 4. (Gy6) is Corollary 2.7.8 part 1 there, with Definition 2.7.1(a) for regularity and Lemma 2.7.5 for the line. Normalisations: Section 2.7 there (local Langlands as in §3.1 of Clozel–Harris–Taylor, geometric Frobenius, ε of Hodge–Tate weight -1, as the proof of Corollary 2.7.8 shows: $\chi_{\tilde{v},j} = \varepsilon^{-(j-1)}\psi_{\tilde{v},j}$ has Hodge–Tate number $\lambda_{\tau,n-j+1} + (j-1)$). (Gy7)(a) is Lemma 5.1.1 there (with the notation of the start of §5.1: δ_{B_n}, $\mathrm{Iw}(v^{b,c})$, $T_n(v^b)$, $\alpha^{(j)}_{\varpi_v}$; the map is the composite of the inclusion into $\sigma^{N_n(\mathcal{O})}$ and the projection, hence $T_n(\mathcal{O})$-equivariant); (Gy7)(b) is the definition before Definition 5.1.2, Definition 5.1.2 and Lemma 5.1.3(2) there; (Gy7)(c) is the description of ϕ_λ before Definition 2.6.2 there ($\phi_\lambda(T_w) = T_w$, $\phi_\lambda(U^{(j)}_{0,\varpi}) = U^{(j)}_{\lambda,\varpi}$, $\phi_\lambda(\langle u\rangle) = (w_0\lambda)(u^{-1})\langle u\rangle$), through which Lemma 2.6.4 is stated, with $U^{(j)}_{\lambda,\varpi} = (w_0\lambda)(\alpha^{(j)}_\varpi)^{-1} U^{(j)}_\varpi$

as in the proof of Lemma 2.7.5 there. The places $v \in T \smallsetminus S_\ell$ carry an arbitrary level U_v (loc. cit. §2.3: "we do not specify U_v for $v \in T - (S_l \cup R)$"). □

Theorem B.92. *Let K be a finite extension of $\mathbb{Q}_p$, W_K its Weil group, $|\cdot|$ the normalised absolute value of K, and $\mathrm{Art}_K\colon K^\times \to W_K^{\mathrm{ab}}$ the local Artin map, normalised so that uniformisers go to geometric Frobenius elements. There is a family of bijections rec_K, one for each $n \geq 1$, from the set of isomorphism classes of irreducible smooth representations of $\mathrm{GL}_n(K)$ over $\mathbb{C}$ to the set of isomorphism classes of n-dimensional Frobenius-semisimple Weil–Deligne representations of W_K over $\mathbb{C}$, such that:*

(Ln1) $\mathrm{rec}_K(\chi) = \chi \circ \mathrm{Art}_K^{-1}$ *for every smooth character χ of $K^\times = \mathrm{GL}_1(K)$;*

(Ln2) $\mathrm{rec}_K(\pi \otimes (\chi \circ \det)) = \mathrm{rec}_K(\pi) \otimes \mathrm{rec}_K(\chi)$ *for every π and every smooth character χ of $K^\times$;*

(Ln3) $\det \mathrm{rec}_K(\pi) = \mathrm{rec}_K(\omega_\pi)$, *where ω_π is the central character of π, and* $\mathrm{rec}_K(\pi^\vee) = \mathrm{rec}_K(\pi)^\vee$;

(Ln4) $L(s, \pi_1 \times \pi_2) = L(s, \mathrm{rec}_K(\pi_1) \otimes \mathrm{rec}_K(\pi_2))$ *and* $\varepsilon(s, \pi_1 \times \pi_2, \psi) = \varepsilon(s, \mathrm{rec}_K(\pi_1) \otimes \mathrm{rec}_K(\pi_2), \psi)$ *for all π_1, π_2 and every non-trivial additive character ψ of K;*

(Ln5) *π is supercuspidal if and only if* $\mathrm{rec}_K(\pi)$ *is irreducible (in particular has monodromy 0);*

(Ln6) *(compatibility with the Langlands classification) if π is the Langlands quotient of a multisegment $(\Delta_1, \ldots, \Delta_r)$, i.e. the unique irreducible quotient of the normalised parabolic induction of $Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_r)$ with the segments ordered so that Δ_i does not precede Δ_j for $i < j$, where $Q(\Delta)$ is the essentially square-integrable representation attached to the segment Δ, then* $\mathrm{rec}_K(\pi) \cong \bigoplus_{i=1}^r \mathrm{rec}_K(Q(\Delta_i))$.

The family is uniquely determined by (Ln1), (Ln2), (Ln3), (Ln4) on supercuspidal representations together with (Ln6).

Proof. Existence: [121], Theorem A (properties (Ln1)–(Ln5) for supercuspidal representations, and the extension to all irreducible smooth representations through the Bernstein–Zelevinsky and Zelevinsky classification, Introduction and Section VII); independently [125]; uniqueness by Henniart's characterisation through L- and ε-factors of pairs. The extension to all representations by (Ln6) and the resulting properties (Ln1)–(Ln4) for all π are explained in Harris–Taylor, loc. cit., and in T. Wedhorn, *The local Langlands correspondence for* $\mathrm{GL}(n)$ *over p-adic fields*, ICTP Lecture Notes 2008, Sections 3 and 4. Normalisation: Harris and Taylor normalise Art_K so that uniformisers correspond to geometric Frobenius elements, as here; their rec is the map of this statement. □

Theorem B.93. *Let F be a finite extension of $\mathbb{Q}_p$, $n \geq 1$, and* rec *the local Langlands correspondence of Harris–Taylor for $\mathrm{GL}_n(F)$ over $\mathbb{C}$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and a representation π of $\mathrm{GL}_n(F)$ on a complex vector space X, let π^σ be the representation on $X \otimes_{\mathbb{C},\sigma} \mathbb{C}$, and for a Weil–Deligne representation (r, N) on Y let $\sigma(r, N)$ be $(r \otimes 1, N \otimes 1)$ on $Y \otimes_{\mathbb{C},\sigma} \mathbb{C}$. Then for every irreducible smooth representation π of $\mathrm{GL}_n(F)$ over $\mathbb{C}$ and every $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\mathrm{rec}\big(\pi^\sigma \otimes |\det|^{(1-n)/2}\big) \cong \sigma\Big(\mathrm{rec}\big(\pi \otimes |\det|^{(1-n)/2}\big)\Big).$$

Proof. See Caraiani, Emerton, Gee, Geraghty, Paškūnas and Shin, *Patching and the p-adic local Langlands correspondence*, Cambridge J. Math. 4 (2016), §1.5 (the map $r_p(\pi) = \mathrm{rec}_p(\pi \otimes |\det|^{(1-n)/2})$ is independent of the choice of the isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$ and commutes with automorphisms of $\mathbb{C}$), who cite C. Breuil and P. Schneider, *First steps towards p-adic Langlands functoriality*, J. Reine Angew. Math. 610 (2007), §4. The property goes back to Henniart's characterisation of rec. □

Theorem B.94. *Let F be an imaginary CM field, $n \geq 1$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $\lambda = (\lambda_\tau)_\tau$ with $\lambda_\tau = (\lambda_{\tau,1} \geq \cdots \geq \lambda_{\tau,n}) \in \mathbb{Z}^n$ for every embedding $\tau\colon F \to \overline{\mathbb{Q}}_\ell$, and Π a regular algebraic conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\iota\lambda$ (Π_∞ has the infinitesimal character of the dual of the irreducible algebraic representation of $\prod_\tau \mathrm{GL}_n$ of highest weight $(\lambda_{\iota^{-1}\sigma})_\sigma$).*

(To1) *There is a continuous semisimple $r_\iota(\Pi)\colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, unique up to isomorphism, with $r_\iota(\Pi)^c \cong r_\iota(\Pi)^\vee \varepsilon^{1-n}$, de Rham at the places above ℓ with $\mathrm{HT}_\tau(r_\iota(\Pi)) = \{\lambda_{\tau,n}, \lambda_{\tau,n-1}+1, \ldots, \lambda_{\tau,1}+n-1\}$, and such that for* every *finite place w of F*

$$\mathrm{WD}\big(r_\iota(\Pi)|_{G_{F_w}}\big)^{\text{F-ss}} \cong \mathrm{rec}^T(\iota^{-1}\Pi_w).$$

(To2) *Let $w \mid \ell$, ϖ_w a uniformiser of F_w, e_w the absolute ramification index of F_w, and v_ℓ the valuation with $v_\ell(\ell) = 1$. Then Π is ι-ordinary at w (in the sense of [105], Definition 5.1.2, recalled in (Gy7) of Theorem B.91) if and only if there are smooth characters $\chi_{w,1}, \ldots, \chi_{w,n}\colon F_w^\times \to \overline{\mathbb{Q}}_\ell^\times$ such that $\iota^{-1}\Pi_w$ is a subquotient of the normalised induction $\text{n-Ind}_{B_n}^{\mathrm{GL}_n(F_w)}(\chi_{w,1} \otimes \cdots \otimes \chi_{w,n})$ and, for $1 \leq i \leq n$,*

$$v_\ell(\chi_{w,i}(\varpi_w)) = \frac{1}{e_w} \sum_{\tau\colon F_w \to \overline{\mathbb{Q}}_\ell} \Big(\lambda_{\tau,n+1-i} - \frac{n-1}{2} + i - 1\Big);$$

the tuple $(\chi_{w,1}, \ldots, \chi_{w,n})$ is then uniquely determined by $\iota^{-1}\Pi_w$.

(To3) *If Π is ι-ordinary at $w \mid \ell$, with $(\chi_{w,i})$ as in (To2), then $r_\iota(\Pi)|_{G_{F_w}}$ is conjugate to an upper triangular representation with diagonal characters $\psi_1, \ldots, \psi_n$ (in this order) such that $\psi_i(\mathrm{Art}_{F_w}(z)) = \chi_{w,i}(z) \prod_\tau \tau(z)^{-(\lambda_{\tau,n-i+1}+i-1)}$ for all $z \in \mathcal{O}_{F_w}^\times$.*

Proof. See [256], Theorem 2.2 (for (To1), with $\chi = \delta_{F/F^+}^n$, i.e. RACSDC; existence by [67], Theorem 3.2.3, local-global compatibility with monodromy by [56] for $w \nmid \ell$ and [58] for $w \mid \ell$; uniqueness by Chebotarev), Lemma 2.3 (for (To2), with the remark following it on the uniqueness of the tuple; Thorne proves it from Lemma 5.1.1 of [105]) and Theorem 2.4 (for (To3), formula (2.1) there). The normalisations are those of §1 of [256]: local Artin maps send uniformisers to geometric Frobenius elements, $\mathrm{HT}_\tau(\varepsilon) = \{-1\}$, $\mathrm{rec}_{F_w}^T(\pi) = \mathrm{rec}_{F_w}(\pi \otimes |\cdot|^{(1-n)/2})$. *Notation.* Thorne's v, π, $r_\iota(\pi)$ are our w, Π, $r_\iota(\Pi)$; his weight $\iota\lambda$ with $(\iota\lambda)_\tau = \lambda_{\iota^{-1}\tau}$ is ours. □

Theorem B.95. *Let F be a number field and $\mathbb{A}_F$ its adeles.*

(Mw1) *Let $m, k \geq 1$ and let τ be a unitary cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$. The normalised parabolic induction to $\mathrm{GL}_{km}(\mathbb{A}_F)$, from the standard parabolic subgroup of type $(m, \ldots, m)$, of $\tau|\det|^{(k-1)/2} \otimes \tau|\det|^{(k-3)/2} \otimes \cdots \otimes \tau|\det|^{(1-k)/2}$ has a unique irreducible quotient $\mathrm{Sp}(\tau, k)$, and $\mathrm{Sp}(\tau, k)$ occurs in the discrete spectrum of $\mathrm{GL}_{km}(\mathbb{A}_F)$. Conversely, every irreducible automorphic representation of $\mathrm{GL}_N(\mathbb{A}_F)$ in the discrete spectrum is isomorphic to $\mathrm{Sp}(\tau, k)$ for a unique pair (τ, k) with $km = N$.*

(Mw2) *Let $r \geq 1$, and for $1 \leq i \leq r$ let τ_i be a unitary cuspidal automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$ and $k_i \geq 1$; let $\Pi = \mathrm{Sp}(\tau_1, k_1) \boxplus \cdots \boxplus \mathrm{Sp}(\tau_r, k_r)$ be the isobaric sum, an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, $n = \sum_i k_i m_i$. Then for every finite place w of F, with $K = F_w$ and $|\cdot|$ the normalised absolute value of K,*

$$\mathrm{rec}_K(\Pi_w) \cong \bigoplus_{i=1}^{r} \bigoplus_{a=0}^{k_i-1} \mathrm{rec}_K(\tau_{i,w}) \otimes \mathrm{rec}_K\big(|\cdot|^{(k_i-1)/2-a}\big) \tag{B.1}$$

as Weil–Deligne representations (monodromy operators included).

Proof. (Mw1) is the main theorem of [178] (the residual spectrum of GL_N, with the cuspidal spectrum the case $k = 1$; uniqueness of (τ, k) from the cuspidal support, Jacquet–Shalika). For (Mw2), by [178] the representation $\mathrm{Sp}(\tau, k)$ is the restricted tensor product of its local components, and at a finite place w the component $\mathrm{Sp}(\tau, k)_w$ is the unique irreducible quotient of the corresponding local induced representation of $\tau_w|\det|^{(k-1)/2} \otimes \cdots \otimes \tau_w|\det|^{(1-k)/2}$, with τ_w irreducible, generic (Shalika) and unitary. By the Bernstein–Zelevinsky classification and the Jacquet–Shalika bound, write τ_w as the irreducible induced representation of essentially square-integrable $Q(\Delta_j)|\det|^{e_j}$ with $|e_j| < \frac{1}{2}$. Then this quotient is the Langlands quotient of the multisegment formed by the shifted segments $\Delta_j|\cdot|^{e_j+(k-1)/2-a}$, so (Ln6) of Theorem B.92 and (Ln2) give $\mathrm{rec}_K(\mathrm{Sp}(\tau,k)_w) \cong \bigoplus_a \mathrm{rec}_K(\tau_w) \otimes \mathrm{rec}_K(|\cdot|^{(k-1)/2-a})$ (the L-parameter ϕ_ψ attached to the Arthur parameter $\tau \boxtimes \mathrm{Sym}^{k-1}$). The local component of an isobaric sum is the local isobaric sum (Langlands; Jacquet–Shalika, *On Euler products and the classification of automorphic forms II*, 1981), the irreducible subquotient of the induced representation of the components whose Langlands parameter is the direct sum of theirs (Langlands classification and its compatibility with rec_K, [121]). *Notation.* Our $\mathrm{Sp}(\tau, k)$ is the residual representation attached to (τ, k) in [178]; our rec_K is that of Theorem B.92. □

Theorem B.96. *Let F be a totally real field with $[F : \mathbb{Q}] \geq 2$, let E/F be a totally imaginary quadratic extension, and let U be an inner form of the quasi-split unitary group in $n \geq 1$ variables for E/F. Let σ occur in the discrete automorphic spectrum of U, with σ_∞ discrete series. Assume property $(*)$: for a pseudo-coefficient f_∞ of σ_∞, the integer traces $\mathrm{tr}\,\sigma'_\infty(f_\infty)$, for discrete automorphic σ' with $\sigma'_f \cong \sigma_f$, have the same sign as $\mathrm{tr}\,\sigma_\infty(f_\infty)$.*

Then there are positive integers $n_1, \ldots, n_s$ with $\sum_i n_i = n$ and discrete automorphic representations π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ with $\pi_i^c \cong \pi_i^\vee$ such that $\pi = \boxplus_i \pi_i$ is a weak base change of σ. Here $\pi_v = \bigotimes_{w|v} \pi_w$ is viewed on $(\mathrm{Res}_{E/F}\mathrm{GL}_n)(F_v)$. At every split finite place v the correspondence $\sigma_v \mapsto \pi_v$ is local base change. At every finite v where E/F and U are unramified and σ_v is spherical, π_v is its unramified local base change. At infinity π_∞ is cohomological, with the coefficient system of the archimedean base-change construction, and its infinitesimal character is the base change of that of σ_∞. If, in addition, every π_i is cuspidal, then π_∞ itself is the archimedean base change of σ_∞.

Property $(*)$ is automatic when $U(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact.

Proof. This is [153], Corollaire 5.3, with the standing hypotheses at the beginning of Section 5 and the compactness remark immediately preceding the corollary. The notation θ_{n_i}-invariant there means $\pi_i^c \cong \pi_i^\vee$. Weak base change is defined in Section 4.10; the cohomological coefficient system is the one in Lemma 4.4 of Section 4.6. As in loc. cit., the last archimedean conclusion is stated only under the additional hypothesis that every discrete summand is cuspidal. The statement makes no splitting assumption at a chosen coefficient prime, no quasi-splitness hypothesis at the finite places and no hypothesis that E/F is unramified at the finite places. □

Theorem B.97. *Let F be a number field. Here a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ is an irreducible representation of the form $\tau \otimes |\det|^t$ with $t \in \mathbb{R}$ and τ an irreducible subrepresentation of the space of cusp forms on $\mathrm{GL}_m(F)\backslash\mathrm{GL}_m(\mathbb{A}_F)$ with unitary central character. Let $(\sigma_i)_{i\in I}$ and $(\sigma'_j)_{j\in J}$ be finite families of cuspidal automorphic representations of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$ and $\mathrm{GL}_{m'_j}(\mathbb{A}_F)$ respectively ($m_i, m'_j \geq 1$). Suppose that for all but finitely many finite places w of F all $\sigma_{i,w}$ and $\sigma'_{j,w}$ are unramified and*

$$\bigsqcup_{i\in I} \alpha_w(\sigma_i) = \bigsqcup_{j\in J} \alpha_w(\sigma'_j)$$

as multisets. Then there is a bijection $s\colon I \to J$ with $m'_{s(i)} = m_i$ and $\sigma'_{s(i)} \cong \sigma_i$ for every $i \in I$.

Proof. This is the classification of isobaric automorphic representations of Jacquet and Shalika, [136], part II, Theorem 4.4: two isobaric sums $\boxplus_i \sigma_i$ and $\boxplus_j \sigma'_j$ of cuspidal automorphic representations (real twists $|\det|^t$ allowed) whose local components are isomorphic at almost all places have the same constituents up to order. *Notation.* At a finite place w where all constituents are unramified, the local component of $\boxplus_i \sigma_i$ is the unramified representation of $\mathrm{GL}_{\sum m_i}(F_w)$ whose Satake parameters are $\bigsqcup_i \alpha_w(\sigma_i)$, and an unramified representation is determined by its Satake parameters; so the hypothesis says that the two isobaric sums have isomorphic local components at almost all places. For a single cuspidal representation on each side this is strong multiplicity one, (Js1) of Theorem B.60. □

Theorem B.98. *Let ℓ, n, F, F^+, $\mathcal{O}$ (with uniformiser ϖ), $\mathbb{G}$ and T be as in Theorem B.91, and $\mathcal{K} = \mathcal{O}[1/\ell]$. For a compact open subgroup $U' \subseteq \mathbb{G}(\mathbb{A}_{F^+}^\infty)$ and an $\mathcal{O}$-module A let $S(U', A)$ be the $\mathcal{O}$-module of functions $\mathbb{G}(F^+)\backslash\mathbb{G}(\mathbb{A}_{F^+}^\infty)/U' \to A$ (automorphic forms of trivial weight). U' is* sufficiently small *if for some place v of F^+ its projection to $\mathbb{G}(F_v^+)$ contains no element of finite order other than* 1.

(Gk1) *Let A be an $\mathcal{O}$-algebra, $U'' \subseteq \mathbb{G}(\mathbb{A}_{F^+}^\infty)$ a sufficiently small compact open subgroup whose components at the places $v \in S_\ell$ lie in $\mathbb{G}(\mathcal{O}_{F_v^+})$, and $U' \subseteq U''$ an open normal subgroup. Then $S(U', A)$, with the finite group U''/U' acting by right translation, $(gf)(x) = f(xg)$, is a finite free $A[U''/U']$-module.*

(Gk2) *Let U be as in Theorem B.91 and A one of $\mathcal{O}$, $\mathcal{K}/\mathcal{O}$, $\mathcal{O}/\varpi^\gamma$, $\varpi^{-\gamma}\mathcal{O}/\mathcal{O}$ ($\gamma \geq 1$). For all $1 \leq b \leq c$ the inclusion $S^{\mathrm{ord}}(U(\ell^{b,b}), A) \to S^{\mathrm{ord}}(U(\ell^{b,c}), A)$ is an isomorphism.*

(Gk3) *(module-level classicality) Let U, b_0, $\mathcal{S}$, $\mathfrak{l}$, $T_n(\mathfrak{l}^b)$, $\Lambda_1^{\mathfrak{l}}$ and the $\Lambda_1^{\mathfrak{l}}$-structure given by $u \mapsto (w_0\nu)^{-1}(u)\langle u\rangle$ ($u \in T_n(\mathfrak{l})$) be as in Theorem B.91, λ a dominant weight, $\alpha\colon T_n(\mathfrak{l}) \to \overline{\mathbb{Q}}_\ell^\times$ a character of finite order, $r \geq 1$ with $T_n(\mathfrak{l}^r) \subseteq \ker\alpha$, and $\mathfrak{P}_{\lambda,\alpha}^{\mathfrak{l}} \subseteq \Lambda_1^{\mathfrak{l}}$ as in (Gy3); suppose that $\mathcal{O}$ contains the values of α, so that $\Lambda_1^{\mathfrak{l}}/\mathfrak{P}_{\lambda,\alpha}^{\mathfrak{l}} = \mathcal{O}$ and $\mathcal{K}' = \mathcal{K}$. Let $S_\lambda^{\mathrm{ord}}(U(\ell^{r,r}), \alpha, \mathcal{K})$ be the space of ordinary forms of weight λ and level $U(\ell^{r,r})$ on which $\langle u\rangle$ acts by $\alpha(u)$ for all $u \in T_n(\mathfrak{l})$, as in (Gy3). Then there is an isomorphism of $\mathcal{K}$-vector spaces*

$$\mathcal{S}_{\mathfrak{P}}/\mathfrak{P}\mathcal{S}_{\mathfrak{P}} \cong \mathrm{Hom}_{\mathcal{K}}(S_\lambda^{\mathrm{ord}}(U(\ell^{r,r}), \alpha, \mathcal{K}), \mathcal{K}) \qquad (\mathfrak{P} = \mathfrak{P}_{\lambda,\alpha}^{\mathfrak{l}},\ \mathcal{S}_{\mathfrak{P}} = \mathcal{S} \otimes_{\Lambda_1^{\mathfrak{l}}} (\Lambda_1^{\mathfrak{l}})_{\mathfrak{P}})$$

which carries the action of $T_w^{(j)}$, $U_{\tilde{v}}^{(j)}$, $\langle u\rangle$ on the left to the transposes of the operators assigned to them by (Gy7)(c) on the right, and which is equivariant for right translation by every $g \in \mathbb{G}(\mathbb{A}_{F^+}^\infty)$ with $g_v = 1$ for $v \in S_\ell$ and $gU(\ell^{r,r})g^{-1} = U(\ell^{r,r})$ (acting on both sides, the left side dually to right translation of ordinary forms of level $U(\ell^\infty)$ when g normalises all $U(\ell^{c,c})$).

(Gk4) *(forms of weight λ) Let U be as in Theorem B.91 and λ a dominant weight for $\mathbb{G}$. Let M_λ be the finite free $\mathcal{O}$-module of [105], Definition 2.2.3, with its action ξ_λ of $\prod_{v\in S_\ell}\mathbb{G}(\mathcal{O}_{F_v^+})$, and $\epsilon_\lambda\colon M_\lambda\to\mathcal{O}$ the natural $\mathcal{O}$-linear projection onto the lowest weight space, equivariant for $\prod_{v\in S_\ell}\mathrm{Iw}(\tilde{v}^{0,1})$ acting on $\mathcal{O}$ through the character $w_0\lambda$. For an $\mathcal{O}$-module A and a compact open subgroup $U'\subseteq\mathbb{G}(\mathbb{A}_{F^+}^\infty)$ whose components at the places of S_ℓ lie in $\mathbb{G}(\mathcal{O}_{F_v^+})$, let $S_\lambda(U',A)$ be the $\mathcal{O}$-module of functions $f\colon\mathbb{G}(F^+)\backslash\mathbb{G}(\mathbb{A}_{F^+}^\infty)\to M_\lambda\otimes_{\mathcal{O}}A$ with $\xi_\lambda(u_{S_\ell})f(xu)=f(x)$ for all $u\in U'$ and x (u_{S_ℓ} the components at S_ℓ; for $\lambda=0$ this is $S(U',A)$), with the operators $T_w^{(j)}$, $(T_w^{(n)})^{-1}$, $U_{\lambda,\varpi_{\tilde{v}}}^{(j)}$ and $\langle u\rangle$ ($u\in T_n(\mathcal{O}_{F^+}\otimes\mathbb{Z}_\ell)$) of weight λ of (Gy7)(c) on the levels $U(\ell^{b,c})$; for A one of $\mathcal{O}$, $\mathcal{K}/\mathcal{O}$, $\mathcal{O}/\varpi^\gamma$, $\varpi^{-\gamma}\mathcal{O}/\mathcal{O}$ ($\gamma\geq 1$) the* ordinary part $S_\lambda^{\mathrm{ord}}(U(\ell^{b,c}),A)=e\,S_\lambda(U(\ell^{b,c}),A)$ *is the image of the ordinary idempotent $e=\lim_r U(\ell)^{r!}$, $U(\ell)=\prod_{v\in S_\ell}\prod_{j=1}^n U_{\lambda,\varpi_{\tilde{v}}}^{(j)}$, of the (finite) Hecke algebra of that level, and is the largest $\mathcal{O}$-submodule on which every $U_{\lambda,\varpi_{\tilde{v}}}^{(j)}$ acts invertibly; $S_\lambda^{\mathrm{ord}}(U(\ell^{b,c}),\mathcal{K})=S_\lambda^{\mathrm{ord}}(U(\ell^{b,c}),\mathcal{O})\otimes_{\mathcal{O}}\mathcal{K}$; and $S_\lambda^{\mathrm{ord}}(U(\ell^\infty),\mathcal{K}/\mathcal{O})=\varinjlim_c S_\lambda^{\mathrm{ord}}(U(\ell^{c,c}),\mathcal{K}/\mathcal{O})$. Then:*

- *(a) if U' is sufficiently small, then $S_\lambda(U',A)=S_\lambda(U',\mathcal{O})\otimes_{\mathcal{O}}A$;*
- *(b) for $b\geq 1$ the inclusions identify $S_\lambda^{\mathrm{ord}}(U(\ell^{b,b}),\mathcal{K}/\mathcal{O})$ with the submodule of $S_\lambda^{\mathrm{ord}}(U(\ell^\infty),\mathcal{K}/\mathcal{O})$ fixed by the operators $\langle u\rangle$, $u\in T_n(\mathfrak{l}^b)$, compatibly with the operators, where $\mathfrak{l}=\prod_{v\in S_\ell}v$ (an ideal of $\mathcal{O}_{F^+}$) and $T_n(\mathfrak{l}^b)$ is the kernel of reduction $T_n(\mathcal{O}_{F^+}\otimes\mathbb{Z}_\ell)\to T_n(\mathcal{O}_{F^+}/\mathfrak{l}^b)$;*
- *(c) the maps $f\mapsto\epsilon_\lambda\circ f$ induce an isomorphism $\lambda_*\colon S_\lambda^{\mathrm{ord}}(U(\ell^\infty),\mathcal{K}/\mathcal{O})\to S^{\mathrm{ord}}(U(\ell^\infty),\mathcal{K}/\mathcal{O})$ which commutes with the $T_w^{(j)}$, carries $U_{\lambda,\varpi_{\tilde{v}}}^{(j)}$ to the operator $U_{\tilde{v}}^{(j)}$ of trivial weight, and satisfies $\lambda_*(\langle u\rangle f)=(w_0\lambda)(u)\,\langle u\rangle\lambda_*(f)$ for $u\in T_n(\mathcal{O}_{F^+}\otimes\mathbb{Z}_\ell)$;*
- *(d) the space $S_\lambda(\overline{\mathbb{Q}}_\ell)=\bigcup_{U'}S_\lambda(U',\overline{\mathbb{Q}}_\ell)$ (U' over the compact open subgroups as above), with $\mathbb{G}(\mathbb{A}_{F^+}^\infty)$ acting by $(g\cdot f)(x)=\xi_\lambda(g_{S_\ell})f(xg)$ (ξ_λ extended to $\prod_{v\in S_\ell}\mathbb{G}(F_v^+)$ on $M_\lambda\otimes\overline{\mathbb{Q}}_\ell$), is a semisimple admissible representation, and $S_\lambda(U',\overline{\mathbb{Q}}_\ell)$ is its subspace of U'-invariants.*

Proof. All references are to [105]. (Gk1) is Lemma 2.2.6 there (proved as Lemma 3.3.1 of [69]), for the trivial weight $\lambda=0$ and no places carrying a character ($R=\varnothing$), where $S_0(U',A)$ is the space of functions of the statement and the hypothesis "$u_\ell\in\mathbb{G}(\mathcal{O}_{F^+,\ell})$ for every $u\in U''$" is our condition on the components at S_ℓ; (Gk2) is Lemma 2.5.2 there, for the same data. (Gk3) is the first display of the proof of Lemma 2.6.4 there (with γ trivial and $\mathcal{K}'=\mathcal{K}$, as arranged there by enlarging $\mathcal{K}$), combined with the isomorphism $S_\lambda^{\mathrm{ord}}(U(\ell^{r,r}),\alpha,\mathcal{K}/\mathcal{O})^\vee\otimes_{\mathcal{O}}\mathcal{K}\cong\mathrm{Hom}_{\mathcal{K}}(S_\lambda^{\mathrm{ord}}(U(\ell^{r,r}),\alpha,\mathcal{K}),\mathcal{K})$ displayed after it; its compatibility with the Hecke operators is the one used there to obtain the surjection of (Gy3). As for the equivariance, every map in that chain (restriction to the forms of level $U(\ell^{r,r})$ on which $T_n(\mathfrak{l})$ acts through α, the twist ϕ_λ of weight λ at the places of S_ℓ before Definition 2.6.2 there, Pontryagin duality, and $-\otimes\mathcal{K}$) is natural and changes only the components at S_ℓ or the coefficients, so it commutes with right translation by an element g as stated, which acts on forms of every level $U(\ell^{b,c})$ it normalises by $f\mapsto f(\cdot\,g)$. (Gk4): the objects are those of Definition 2.2.4 there ($R=\varnothing$), §2.4 (for the four coefficient modules listed: the idempotent $e=\lim U(\ell)^{r!}$ before Definition 2.4.2, Definition 2.4.2 and the sentence after it, "the largest $\mathcal{O}$-submodule on which each of the operators $U_{\lambda,\varpi_{\tilde{v}}}^{(j)}$ acts invertibly"; the space with coefficients in $\mathcal{K}$ is used in the proof of Lemma 2.6.4 there as $S^{\mathrm{ord}}(\mathcal{O})\otimes\mathcal{K}$, which we take as its definition) and §2.6 (the morphism $\lambda\colon M_\lambda\to\mathcal{O}(w_0\lambda)$, "evaluation at the identity or projection to the lowest weight space"); (a) is the paragraph after Lemma 2.2.5 there ("if U is sufficiently small then $S_{\lambda,\{\chi_v\}}(U,A)=S_{\lambda,\{\chi_v\}}(U,\mathcal{O})\otimes A$"); (b) is the first display of the proof of Proposition 2.5.3 there, which rests on Lemma 2.5.2 there for

weight λ (the operators on the limit being those of the finite levels, the double cosets having common representatives in $\mathbb{G}(F_\ell^+)$ for all levels, proof of Lemma 2.5.2); (c) is Proposition 2.6.1 there (the isomorphism λ_*, defined in its proof by $f \mapsto \lambda \circ f$ on $S_\lambda(U(\ell^{c,c}), \varpi^{-r}\mathcal{O}/\mathcal{O})$, $c \geq r$, and the direct limit over c); (d) is Definition 2.2.4 there (the action $(g \cdot f)(h) = g_{S_l} f(hg)$ on $S_\lambda(A)$ for a $\mathcal{K}$-module A, and $S_\lambda(U, A)$ its U-invariants) with Lemma 2.2.5 there and the sentence after it (the space of automorphic forms is semisimple admissible, so $S_\lambda(\overline{\mathbb{Q}}_\ell)$ is). Here U, $U(\ell^{b,c})$, S^{ord} are those of Theorem B.91. □

Theorem B.99. *Let k be a finite field of characteristic ℓ, $W(k)$ its ring of Witt vectors, G a profinite group, $\bar{k}$ an algebraic closure of k, and $\bar{\rho}\colon G \to \mathrm{GL}_d(\bar{k})$ a continuous semisimple representation such that $\det(1 + t\bar{\rho}(g)) \in k[t]$ for all $g \in G$. Let $H_0 \subseteq \ker\bar{\rho}$ be the smallest closed normal subgroup of G contained in $\ker\bar{\rho}$ such that $\ker\bar{\rho}/H_0$ is a pro-ℓ group. Let A be a commutative profinite local $W(k)$-algebra with residue field k and D a continuous A-valued determinant of dimension d on G whose reduction modulo $\mathfrak{m}_A$ is $\det\circ\bar{\rho}$. Then D factors through $A[G] \to A[G/H_0]$: there is a determinant D' on G/H_0 with D equal to D' composed with this map.*

Proof. See [66, §3.1 and Lemma 3.8] (there p is the characteristic of k). The proof there reduces to finite A and a finite quotient of G; the radical of the finite ring $A[G]/\ker D$ is the kernel of the map to $\bar{\rho}(k[G])$, so $\ker\bar{\rho}$ maps into the ℓ-group $1 + \mathrm{Rad}$. □

Theorem B.100. *Let A be a henselian local ring in which 2 is invertible, S an A-algebra which is integral over A, τ an A-linear map $S \to S$ with $\tau^2 = \mathrm{id}_S$ which is either a ring homomorphism or a ring anti-homomorphism, and $I \subseteq \mathrm{rad}(S)$ a two-sided ideal with $\tau(I) = I$. Let $\epsilon_1, \ldots, \epsilon_k$ be orthogonal idempotents of S/I such that the set $\{\epsilon_1, \ldots, \epsilon_k\}$ is stable under the map induced by τ on S/I. Then there are orthogonal idempotents $e_1, \ldots, e_k$ of S with e_i lifting ϵ_i for each i and such that the set $\{e_1, \ldots, e_k\}$ is stable under τ.*

Proof. This is [21], Lemma 1.8.2. (In the statement of loc. cit. the stable set is written as a subset of R; it is a subset of S/I, on which τ acts since $\tau(I) = I$.) □

Theorem B.101. *Let A be a commutative ring, R an A-algebra, S a commutative A-algebra and $D\colon R \to S$ an S-valued determinant of dimension d (a multiplicative A-polynomial law $R \to S$, homogeneous of degree d). For a commutative A-algebra C and $r \in R \otimes_A C$ define $\Lambda_i(r) \in S \otimes_A C$ by $D(t - r) = \sum_{i=0}^{d}(-1)^i\Lambda_i(r)t^{d-i}$ in $(S \otimes_A C)[t]$, and $\Lambda_i = 0$ for $i > d$.*

(Am1) *$D(1 + rr') = D(1 + r'r)$ for all $r, r' \in R$ (and likewise after any base change $A \to C$).*

(Am2) *(Amitsur's formula.) For $m \geq 1$, $i \geq 0$ and $r_1, \ldots, r_m \in R$, $\Lambda_i(t_1r_1 + \cdots + t_mr_m) \in S[t_1, \ldots, t_m]$ equals $P_{m,i}((\Lambda_j(w))_{j,w})$, where w runs over the finite products of the r_k and $P_{m,i}$ is a polynomial with integer coefficients in the variables (t_k) and $(\Lambda_j(w))$ which depends only on m and i (not on A, R, S, D or d).*

Proof. This is [66], Lemma 1.12 (i) and (ii), with Amitsur's formula in the explicit form (1.5) there: $\Lambda_i(t_1r_1 + \cdots + t_mr_m) = \sum_{\ell(w)=i}\epsilon(w)\Lambda(w)$, the sum over the words w of length i in the letters t_kr_k, with $\epsilon(w) \in \{\pm 1\}$ and $\Lambda(w) = \prod_k \Lambda_{l_k}(w_k)$ for the Lyndon factorisation $w = w_1^{l_1} \cdots w_q^{l_q}$, which depend only on the word. □

Theorem B.102. *Let A be a commutative ring, R an A-algebra and $D\colon R \to A$ a determinant of dimension d.*

(Cd1) *Assume that* $\operatorname{Spec}(A)$ *is connected and let* $e \in R$ *be an idempotent. The polynomial map* $D_e\colon eRe \to A$, $x \mapsto D(x+1-e)$, *is a determinant on the* A*-algebra* eRe *(with unit* e*) of some dimension* $r(e) \le d$*; one has* $r(e) + r(1-e) = d$*, and the restriction of* D *to the* A*-subalgebra* $eRe \oplus (1-e)R(1-e)$ *is the product determinant* $x \mapsto D_e(ex)\, D_{1-e}((1-e)x)$*. If* D *is Cayley–Hamilton, so is* D_e*.*

(Cd2) *If* D *is Cayley–Hamilton of dimension* 1*, then the structure map* $A \to R$ *is an isomorphism and* D *is the identity.*

Proof. (Cd1) is [66], Lemma 2.4 (1)–(3); (Cd2) is [66], Lemma 2.6. □

B.6 Shimura varieties, abelian varieties and moduli

All these results are used in Part VI and Appendix D.Lan's fine PEL moduli spaces with liftable full level, their Kodaira–Spencer isomorphism, their comparison with the complex uniformisation and their projective toroidal compactifications (Theorem B.105, Theorem B.106, Theorem B.114, Theorem B.108), and the degeneration data of Faltings–Chai and Lan with Mumford's construction (Theorem B.109, Theorem B.111, Theorem B.112), are used for the integral models with uniformised degenerations (Section 32) and for good origins (Section 33). The extension of semiabelian homomorphisms, the theorem of the cube, Moret-Bailly's cubical structures and his theorem on the group $K(L)$, Grothendieck's theorems on semistable reduction, Grothendieck–Messing theory, Serre's tensor construction, Zarhin's trick, quotients by finite subgroup schemes and the potentially good reduction of elliptic curves with complex multiplication are used in Sections 31–34 and 35. Appendix D uses, besides several of these, Lefschetz's very ampleness theorem, Rapoport's freeness theorem, the Hodge and conjugate filtrations of de Rham cohomology in characteristic p, the universal vector extension of Mazur and Messing, and the openness and closedness of the polarisation locus (Sections D.2 and D.5).

Theorem B.103. *Let* $A \to S$ *be an abelian scheme over a* $\mathbb{Q}$*-scheme. There is a normalized Poincaré line bundle* $\mathcal{P}_A$ *on* $A \times_S A^\vee$*, rigidified on both zero sections. For a homomorphism* $Q : A \to A^\vee$*, put*

$$\mathcal{P}_Q = (\mathrm{id}_A \times Q)^* \mathcal{P}_A.$$

Normalized biextensions on $A \times_S A$ *are classified by these homomorphisms; equality of the homomorphisms gives a unique normalized biextension isomorphism. In particular, for a rigidified line bundle* $\mathcal{M}$*,*

$$\Lambda(\mathcal{M}) = m^*\mathcal{M} \otimes \mathrm{pr}_1^*\,\mathcal{M}^{-1} \otimes \mathrm{pr}_2^*\,\mathcal{M}^{-1} \simeq \mathcal{P}_{\phi_{\mathcal{M}}}.$$

The line bundle $\mathcal{M}$ *has a unique compatible cubical structure. It is relatively ample if and only if* $\phi_{\mathcal{M}}$ *is a polarisation. These statements and normalized identifications commute with arbitrary base change. The bundle* $\mathcal{P}_Q$ *is bilinear in its two arguments, and interchanging the arguments replaces* Q *by its dual* $Q^\vee$*, using* $A = A^{\vee\vee}$*.*

In addition, the following integral clause holds. Let R *be a complete discrete valuation ring and* A/R *an abelian scheme with identity* e*. Every line bundle* $\mathcal{L}$ *rigidified by* $e^*\mathcal{L} \simeq R$ *has a unique compatible normalized cubical structure: a trivialization*

$$\bigotimes_{I \subseteq \{1,2,3\}} m_I^*\mathcal{L}^{(-1)^{3-|I|}} \simeq \mathcal{O}_{A^3}, \qquad m_\varnothing = e,$$

with the usual symmetry and cocycle identities, equal to the identity on a coordinate-zero face under the rigidifications. Formation of this structure commutes with base change, in particular to the fraction field and valued extensions.

Proof. These are the normalized Poincaré universal property, the theorem of the cube for abelian schemes, and the polarisation criterion for a line bundle. They are quoted at their standard relative statements: [186], the theorem of the cube and duality theory; [181], Chapter I, 2.6; and [154], Construction 1.3.2.7, Lemma 1.3.2.10, Proposition 1.3.2.15 and Proposition 3.2.3.1.

For the integral clause apply the theorem of the cube in its rigidified cubical form over an abelian scheme over a complete discrete valuation ring, [181], Chapter I, §2. The rigidification fixes the coordinate-zero normalization; the symmetry and cocycle conditions characterize its unique compatible cubical trivialization. This uniqueness also identifies every base change with the corresponding normalized cubical structure. No Poincaré classification over a non-$\mathbb{Q}$ base beyond this cubical clause is asserted here. □

Theorem B.104. *Let S be a noetherian normal scheme, let $U \subseteq S$ be a dense open subscheme, and let G, H be semiabelian schemes over S. Every homomorphism of U-group schemes*

$$f_U : H_U \longrightarrow G_U$$

extends uniquely to a homomorphism of S-group schemes $f : H \to G$.

Proof. This is the semiabelian homomorphism extension theorem, [87, Chapter I, Proposition 2.7]; see also [154, Proposition 3.3.1.5]. The base is noetherian and normal, the subscheme is open and dense, and both group schemes are semiabelian, exactly as required by that theorem. □

Theorem B.105. *Let E be a CM field and $M \neq 0$ a finite projective $\mathcal{O}_E$-module with a perfect alternating $\mathbb{Z}$-pairing ψ. Suppose*

$$\psi(ax, y) = \psi(x, \bar{a}y), \quad J \in \mathrm{End}_{E\otimes\mathbb{R}}(M_{\mathbb{R}}), \quad J^2 = -1,$$
$$\psi(Jx, Jy) = \psi(x, y), \quad \psi(Jx, x) > 0 \ (x \neq 0).$$

Set

$$g = \tfrac{1}{2}\operatorname{rank}_{\mathbb{Z}} M, \quad P_a(T) = \det_{\mathbb{C}}(T - a \mid (M_{\mathbb{R}}, J)), \quad F_{\mathrm{ref}} = \mathbb{Q}(\textit{coefficients of } P_a : \ a \in \mathcal{O}_E).$$

Fix $N \geq 3$. The following functor on F_{ref}-schemes is represented by a smooth separated quasi-projective scheme with its universal tuple.

Over S, its objects are tuples $(A, \phi_{\mathrm{pol}}, \iota, \alpha_N, \nu_N)$, where A/S is an abelian scheme of dimension g, ϕ_{pol} is a principal polarisation, ι is a unital $\mathcal{O}_E$-action, and

$$\phi_{\mathrm{pol}}\iota(a) = \iota(\bar{a})^{\vee}\phi_{\mathrm{pol}}, \qquad \det(T - \iota(a) \mid \mathrm{Lie}(A/S)) = P_a(T) \quad (a \in \mathcal{O}_E).$$

The level consists of $\mathcal{O}_E$-linear $\alpha_N : (M/NM)_S \xrightarrow{\sim} A[N]$ and $\nu_N : (\mathbb{Z}/N)_S \xrightarrow{\sim} \mu_{N,S}$ satisfying

$$e_N^{\phi_{\mathrm{pol}}}(\alpha_N(x), \alpha_N(y)) = \nu_N(\psi(x, y) \bmod N).$$

It is required to lift to compatible analogous pairs (α_m, ν_m) over a tower of finite étale surjections $S_m \to S$, indexed by $N \mid m$, with $S_N = S$. For $N \mid l \mid m$ the transition $S_m \to S_l$ is finite étale surjective and the pair at m reduces to the pair at l: multiplication by m/l on $A[m]$, reduction on M/mM, and the (m/l)th power on μ_m. Isomorphisms preserve every component of the tuple; the functor assigns isomorphism classes.

Proof. This is the full-level PEL representability theorem of [154], Definition 1.4.1.2, Theorem 1.4.1.11, Corollary 1.4.1.12 and Corollary 7.2.3.10, with the liftability condition of Definition

1.3.6.2 and with no allowed positive residue characteristics. The lattice is self-dual, so the polarisation is principal.

Here is the sign convention in this formulation. In the Tate-twisted convention use the pairing

$$\langle x, y\rangle_{\mathrm{tw}} = -2\pi i\,\psi(x, y), \qquad h(a + bi) = a + bJ.$$

Then the real form associated to $\langle x, h(i)y\rangle_{\mathrm{tw}}$ is $-\psi(x, Jy)$, whose value at (x, x) is $\psi(Jx, x) > 0$. The $+i$ subspace of $h(i)$ has determinant $P_a(T)$ as stated. Inverting every cyclotomic trivialisation converts the twisted pairing convention into the displayed pairing condition and preserves all tower transition equations. Thus the determinant and positivity conventions agree. The level is taken over the entire reflex field; no single cyclotomic component is selected. □

Theorem B.106. *Let $E, M, \psi, J, N, F_{\mathrm{ref}}$ be as in Theorem B.105, and let X/F_{ref} be its fine full-level moduli scheme, with universal tuple $(A, \phi_{\mathrm{pol}}, \iota, \alpha_N, \nu_N)$. Put*

$$\omega_A = \mathrm{Lie}(A/X)^\vee, \qquad \omega_{A^\vee} = \mathrm{Lie}(A^\vee/X)^\vee.$$

Let $\mathcal{R}$ be the coherent subsheaf of $\omega_A \otimes_{\mathcal{O}_X} \omega_{A^\vee}$ generated locally by

$$\phi_{\mathrm{pol}}^*(y) \otimes z - \phi_{\mathrm{pol}}^*(z) \otimes y \quad (y, z \in \omega_{A^\vee})$$

and

$$\iota(a)^*(x) \otimes y - x \otimes (\iota(a)^\vee)^*(y) \quad (a \in \mathcal{O}_E,\ x \in \omega_A,\ y \in \omega_{A^\vee}).$$

The Kodaira–Spencer morphism induces an isomorphism

$$(\omega_A \otimes_{\mathcal{O}_X} \omega_{A^\vee})/\mathcal{R} \xrightarrow{\sim} \Omega^1_{X/F_{\mathrm{ref}}}.$$

Proof. This is the universal PEL Kodaira–Spencer theorem, [154], Definition 2.3.5.1 and Proposition 2.3.5.2, applied to the identity morphism of the smooth moduli scheme. The identity is flat and étale, so the indicated morphism is an isomorphism. An integral basis of $\mathcal{O}_E$ suffices for the endomorphism relations, giving the stated coherent subsheaf.

The negative Tate-twisted pairing and inversion of the cyclotomic trivialisations in Theorem B.105 identify its functor with this full-level moduli problem, leaving the universal abelian scheme, polarisation and action unchanged. Thus the coherent-sheaf relations agree exactly with those displayed here. □

Theorem B.107. *Let S be a normal locally noetherian scheme, $G \to S$ a semiabelian scheme, and let U be either an open subscheme of S that is schematically dense in S or, when S is irreducible, its generic point. Write $G_U = G \times_S U$.*

1. *Every rigidified line bundle on G carries exactly one cubical structure compatible with its rigidification, and every isomorphism of rigidified line bundles on G respects these cubical structures (the forgetful functor from cubical line bundles on G to rigidified line bundles on G is an equivalence of categories).*

2. *Restriction from cubical line bundles on G to cubical line bundles on G_U is fully faithful: every isomorphism of cubical line bundles on G_U between the restrictions of two cubical line bundles on G is the restriction of exactly one isomorphism of cubical line bundles on G.*

3. *If $\mathcal{L}_U$ is a cubical line bundle on G_U whose underlying line bundle is symmetric, $\mathcal{L}_U \cong [-1]^*\mathcal{L}_U$, then $\mathcal{L}_U$ is isomorphic, as a cubical line bundle, to the restriction of a cubical line bundle on G.*

Proof. Moret-Bailly, *Pinceaux de variétés abéliennes* [181]: (1) is Ch. I, 2.6; (2) is Ch. II, 3.2.1–3.2.3; (3) is Ch. II, 3.3. We read them in the form of [154]: (1) is Proposition 3.2.3.1 under its condition 2 (S normal; the fibres of G over the maximal points of S are extensions of abelian varieties by tori), (2) is Proposition 3.3.2.2(1), (3) is Theorem 3.3.2.3 under its condition 2. Notation: Lan's cubical $\mathbb{G}_m$-torsors are the cubical line bundles of Section 29.1; the trivialisation of $\Theta(\mathcal{L})$ at the origin of G^3 is a rigidification of $\mathcal{L}$, which is the rigidification meant in (1). Lan states these results for a normal base scheme; we use them only for locally noetherian ones. □

Theorem B.108. *Let E, (M, ψ, J) and $N \geq 3$ be as in Theorem B.105, with reflex field $F_{\mathrm{ref}} \subset \mathbb{C}$, and let $\mathcal{M}_N$ be the full-level scheme of Theorem B.105 over F_{ref}, with universal tuple $(A, \phi_{\mathrm{pol}}, \iota, \alpha_N, \nu_N)$. Let Σ be a compatible collection of admissible rational polyhedral cone decomposition data for $\mathcal{M}_N$ which is projective and smooth in the sense of [156] (Def. 2.1, 2.2, 2.7: each cone decomposition satisfies Cond. 6.2.5.25 of Lan's book [154], and Σ satisfies Cond. 6.3.3.2 there), induced by a compatible collection* pol *of polarisation functions, and let $d \geq 1$ be divisible by the integers d_0 of [156] (Prop. 4.19, 5.1). Let p be a prime with $N/p^{v_p(N)} \geq 3$. Then there are a normal scheme $\mathcal{X}_p$, projective and flat over $\mathcal{O}_{F_{\mathrm{ref}},(p)}$, an open subscheme $\mathcal{X}_p^\circ \subseteq \mathcal{X}_p$ dense in every fibre over $\mathcal{O}_{F_{\mathrm{ref}},(p)}$ with $\mathcal{X}_p^\circ \otimes F_{\mathrm{ref}} = \mathcal{M}_N$, a proper surjection $\mathcal{X}_p \to \mathcal{X}_p^{\min}$ onto a normal scheme projective and flat over $\mathcal{O}_{F_{\mathrm{ref}},(p)}$, and a semiabelian scheme $G_p \to \mathcal{X}_p$ with a polarisation of its restriction to $\mathcal{M}_N$ and a ring homomorphism $\iota_p \colon \mathcal{O}_E \to \mathrm{End}(G_p)$, such that:*

1. *over $\mathcal{M}_N$, (G_p, ι_p) with the polarisation is $(A, \iota, \phi_{\mathrm{pol}})$;*

2. *$\mathcal{X}_p$ is the disjoint union of locally closed strata, each flat over $\mathcal{O}_{F_{\mathrm{ref}},(p)}$, normal, and with dense characteristic-0 fibre; $\mathcal{X}_p^\circ$ is one stratum; the formal completion of $\mathcal{X}_p$ along a stratum is canonically isomorphic to Lan's formal scheme $\mathfrak{X}_{\Phi,\delta,\sigma}$ of a representative (Φ, δ, σ) of the stratum, and the pull-back of G_p to it is the Mumford family over $\mathfrak{X}_{\Phi,\delta,\sigma}$ (Lan's book, Def. 6.2.5.28);*

3. *the Hodge bundle $\omega_p = \det e^*\Omega^1_{G_p/\mathcal{X}_p}$ is the pull-back of an ample invertible sheaf on $\mathcal{X}_p^{\min}$;*

4. *there is an effective Cartier divisor D_p on $\mathcal{X}_p$ with $(D_p)_{\mathrm{red}} = \mathcal{X}_p \setminus \mathcal{X}_p^\circ$ such that $\mathcal{O}(-D_p)$ is relatively ample over $\mathcal{X}_p^{\min}$; its ideal is the pull-back of the ideal of the schematic closure in $\mathcal{X}_p^{\min}$ of the closed subscheme of $\mathcal{X}_p^{\min} \otimes F_{\mathrm{ref}}$ defined by Lan's characteristic-0 ideal $\mathcal{J}_{d\,\mathrm{pol}}$, which is defined from Σ,* pol *and d on the characteristic-0 minimal compactification;*

5. *(extension property) let S be an irreducible noetherian normal $\mathcal{O}_{F_{\mathrm{ref}},(p)}$-scheme with generic point η, $x \colon \eta \to \mathcal{M}_N$ a morphism, and suppose that there is a degenerating family of type $\mathcal{M}_N$ over S in the sense of Lan's book, Def. 5.3.2.1 (a semiabelian scheme with $\mathcal{O}_E$-action, a homomorphism to its dual and a level structure over η) whose restriction to η defines the morphism x. Then x extends (necessarily uniquely) to a morphism $S \to \mathcal{X}_p$ over $\mathcal{O}_{F_{\mathrm{ref}},(p)}$ if Lan's condition holds at every geometric point $\bar{s}$ of S: for every complete discrete valuation ring V with algebraically closed residue field and every dominant morphism $\mathrm{Spec}\, V \to S$ centred at $\bar{s}$, after a finite extension of $\mathrm{Frac}\, V$ the valuation of the degeneration datum of the pulled-back family lies in one cone of Σ;*

6. *(characteristic 0) $\mathcal{X}_p \otimes F_{\mathrm{ref}} \to \mathcal{X}_p^{\min} \otimes F_{\mathrm{ref}}$ is Lan's characteristic-0 toroidal compactification $\mathcal{M}_\Sigma^{\mathrm{tor}}$ of $\mathcal{M}_N$ for Σ with its map to the minimal compactification (Lan's book, Thm. 6.4.1.1*

and 7.2.4.1, with $\square = \emptyset$), the identification restricting to the identity of $\mathcal{M}_N$; in particular $D_p \otimes F_{\mathrm{ref}}$ is the divisor on $\mathcal{M}_\Sigma^{\mathrm{tor}}$ of the pull-back of $\mathcal{J}_{d\,\mathrm{pol}}$, the same for every p.

Proof. Quoted from K.-W. Lan, Integral models of toroidal compactifications with projective cone decompositions [156], Theorem 6.1 and its proof, Construction 3.12, Corollaries 4.23, 6.6, 6.7 and Proposition 5.15; the setting is that of Lan, Compactifications of PEL-type Shimura varieties in ramified characteristics [155], Sect. 2, with a single lattice ($J = \{j_0\}$, $(g_{j_0}, L_{j_0}) = (1, M)$) and $\mathcal{O} = \mathcal{O}_E$.

Location of each clause. Main clause: Theorem 6.1 (normal scheme projective and flat over $\vec{\mathsf{S}}_0 = \operatorname{Spec} \mathcal{O}_{F_0,(p)}$, F_0 the reflex field, containing $\vec{\mathsf{M}}_{\mathcal{H}}$ as an open fibrewise dense subscheme, with a tautological degenerating family). Item 1: 6.1(1). Item 2: 6.1(3) (strata flat over $\vec{\mathsf{S}}_0$ and normal; $\vec{\mathsf{M}}_{\mathcal{H}}$ the open stratum), the proof of Proposition 5.15 (the characteristic-0 fibre of each stratum is dense in it), 6.1(4) (completions and Mumford families). Item 3: 6.1(2): $\omega_J^{\mathrm{tor}} \cong \omega_{j_0}^{\otimes a_{j_0}}$ with $\omega_{j_0} = \det \mathrm{Lie}_{G_{j_0}}^\vee$ is the pull-back of the ample $\omega_J^{\min}$, and ω_{j_0} descends to an invertible sheaf $\omega_{j_0}^{\min}$ with $(\omega_{j_0}^{\min})^{\otimes a_{j_0}} \cong \omega_J^{\min}$; a positive power of $\omega_{j_0}^{\min}$ being ample, $\omega_{j_0}^{\min}$ is ample (EGA II 4.5.6), and $\omega_p = \omega_{j_0}$ is its pull-back. Item 4: Corollary 6.7 and its proof (the pull-back of $\vec{\mathcal{J}}_{\mathcal{H},d\,\mathrm{pol}}$, defined in Construction 3.12 as the ideal of the schematic closure of the subscheme defined by $\mathcal{J}_{\mathcal{H},d\,\mathrm{pol}}$, is $\mathcal{O}(-\mathsf{D})$). Item 5: 6.1(6), which asserts the extension under the condition ("if"; no converse is stated there). Item 6: Construction 3.12 (by Theorem 3.10, i.e. Lan's book Thm. 7.3.3.4, for Σ projective and smooth the characteristic-0 toroidal compactification is the normalised blow-up $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},d_0\mathrm{pol}}$, the generic fibre of the p-integral one), Corollary 4.23 with Corollary 6.6 (Σ smooth satisfying Cond. 6.3.3.2, $\Sigma' = \Sigma$: the canonical morphism (4.24) from Lan's characteristic-0 $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},d\,\mathrm{pol}}$ is an isomorphism); normalised blow-ups commute with the flat base change $\mathcal{O}_{F_0,(p)} \to F_0$, and the restriction of $\vec{\mathcal{J}}$ to the generic fibre is $\mathcal{J}$, so the generic fibre of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\,\mathrm{pol}}$ is $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},d\,\mathrm{pol}}$; the characteristic-0 objects do not involve p.

Notation. Lan's PEL datum $(\mathcal{O}, \star, L, \langle\,,\rangle, h_0)$ is $(\mathcal{O}_E,\ a \mapsto \bar{a},\ M,\ 2\pi i\,\psi,\ J)$; the level $\mathcal{H}$ is the principal congruence subgroup $\mathcal{U}(N) \subseteq G(\widehat{\mathbb{Z}})$; Lan's reflex field F_0 is F_{ref}; Lan's $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}}$, $\vec{\mathsf{M}}_{\mathcal{H}}$, $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$, D, (G, λ, i) are $\mathcal{X}_p$, $\mathcal{X}_p^\circ$, $\mathcal{X}_p^{\min}$, D_p, $(G_p, \text{polarisation}, \iota_p)$. Lan's standing hypothesis (Theorem 6.1, Construction 3.12) is that the image $\mathcal{H}^p$ of $\mathcal{H}$ in $G(\widehat{\mathbb{Z}}^p)$ is neat; here $\mathcal{H}^p$ is the principal congruence subgroup of level $N/p^{v_p(N)} \geq 3$, which is neat (Lan's book, Def. 1.4.1.8, Serre's lemma). Over F_{ref}, Lan's $\mathsf{M}_{\mathcal{H}}$ is the moduli problem of Lan's book, Def. 1.4.1.4 (with $\square = \emptyset$), whose level-$\mathcal{U}(N)$ structures are the symplectic-liftable principal level-N structures (α_N, ν_N); this is the functor of Theorem B.105, so $\mathsf{M}_{\mathcal{H}} = \mathcal{M}_N$. Lan's Cond. 1.4.3.10 on L is assumed in his Sect. 2; with $\mathcal{O} = \mathcal{O}_E$ maximal it holds. □

Theorem B.109. *Let R be a noetherian normal domain, complete with respect to an ideal I with $\operatorname{rad}(I) = I$; put $S = \operatorname{Spec} R$, $S_0 = \operatorname{Spec} R/I$, η the generic point, and write ^{for} for formal completion along S_0.*

1. *(Raynaud extension.) Let G be a semiabelian scheme over S such that $G \times_S S_0$ is an extension of an abelian scheme A_0 by an isotrivial torus T_0. Then there are, uniquely up to unique isomorphism, an abelian scheme A and a torus T over S and an extension $0 \to T \to \widetilde{G} \to A \to 0$ whose formal completion is the extension $0 \to T^{\mathrm{for}} \to G^{\mathrm{for}} \to A^{\mathrm{for}} \to 0$ of G^{for} (the Raynaud extension of G). The natural functor*

$$\mathrm{CUB}_S(\widetilde{G}, \mathbb{G}_{m,S}) \longrightarrow \mathrm{CUB}_{S^{\mathrm{for}}}(G^{\mathrm{for}}, \mathbb{G}_{m,S^{\mathrm{for}}}), \qquad \mathcal{N} \mapsto \mathcal{N}^{\mathrm{for}},$$

from cubical $\mathbb{G}_m$-torsors on the scheme $\widetilde{G}$ over S to cubical $\mathbb{G}_m$-torsors on the formal scheme G^{for}, induced by the isomorphism $\widetilde{G}^{\mathrm{for}} \cong G^{\mathrm{for}}$, is fully faithful, and its essential

image contains all cubical $\mathbb{G}_m$-torsors on G^{for} coming from a cubical $\mathbb{G}_m$-torsor on A^{for}. In particular, if all tori over S_0 are isotrivial, this functor is an equivalence of categories.

2. *(Association of degeneration data and Mumford's construction.) Let $\mathrm{DEG}_{\mathrm{ample}}$ be the category of pairs $(G, \mathcal{L})$ of a semiabelian scheme G over S as in (1) with abelian generic fibre G_η and an ample cubical invertible sheaf $\mathcal{L}$ on G, rigidified along the unit section, such that $\mathcal{L}^{\mathrm{for}}$ is in the essential image of the functor of (1) (a rigidified invertible sheaf on G has a unique cubical structure). Let $\mathrm{DD}_{\mathrm{ample}}$ be the category of ample degeneration data $(A, X, Y, \phi, c, c^\vee, \widetilde{\mathcal{L}}, \tau, \psi)$: an extension $\widetilde{G}$ of an abelian scheme A by a torus with character group X, given by $c\colon X \to A^\vee$; an étale sheaf Y of free abelian groups of the same rank with $c^\vee\colon Y \to A$; an injective $\phi\colon Y \to X$ with finite cokernel; a cubical ample invertible sheaf $\widetilde{\mathcal{L}}$ on $\widetilde{G}$ inducing a polarisation λ_A of A with $\lambda_A c^\vee = c\phi$; a trivialisation τ of the biextension $(c^\vee \times c)^*\mathcal{P}_A^{-1}$ over η, symmetric with respect to ϕ in the sense that $(\mathrm{id}_Y \times \phi)^*\tau$ is symmetric (equivalently a period homomorphism $\iota\colon Y_\eta \to \widetilde{G}_\eta$ over $c^\vee$); and a cubical trivialisation ψ of $\iota^*\widetilde{\mathcal{L}}_\eta^{-1}$ compatible with $\tau \circ (\mathrm{id}_Y \times \phi)$ (this compatibility forces the symmetry of τ); with the positivity conditions, read after a finite étale surjective base change over which X and Y are constant (none if they are constant over S), through the cubical isomorphism $\widetilde{\mathcal{L}} \cong \pi^*\mathcal{M}$, with a cubical invertible sheaf $\mathcal{M}$ on A, determined by a cubical trivialisation s of $i^*\widetilde{\mathcal{L}}$ (where $i\colon T \to \widetilde{G}$ is the torus and $\pi\colon \widetilde{G} \to A$ the projection), so that $\tau(y, \phi(y))$ is a trivialisation of $(c^\vee(y) \times c\phi(y))^*\mathcal{P}_A^{-1}$ and $\psi(y)$ one of $\mathcal{M}(c^\vee(y))^{-1}$ over η: the* condition on τ, *that $\tau(y, \phi(y))$ extends over S for every y and is 0 modulo I for $y \neq 0$; and the* condition on ψ, *that $\psi(y)$ extends over S for every y and that, for every $n \geq 1$, $\psi(y)$ is 0 modulo I^n for all but finitely many y. The condition on τ does not involve s; the condition on ψ does, and an object of $\mathrm{DD}_{\mathrm{ample}}$ is required to satisfy it for one choice of s. For a given s, the condition on ψ implies the condition on τ; if moreover $\mathcal{M}$ is symmetric and, for every y, $\psi(-y)$ corresponds to $\psi(y)$ under an isomorphism $[-1]^*\mathcal{M} \cong \mathcal{M}$ (we say that ψ is* even *through s), then the condition on τ implies the condition on ψ for this s. Then the Fourier expansions of the sections of the $\mathcal{L}^{\otimes k}$ attach to every $(G, \mathcal{L}) \in \mathrm{DEG}_{\mathrm{ample}}$, functorially in isomorphisms, a tuple $F_{\mathrm{ample}}(G, \mathcal{L}) = (A, X, Y, \phi, c, c^\vee, \widetilde{\mathcal{L}}, \tau, \psi)$ as above, with $\widetilde{G}$ the Raynaud extension of G and $\widetilde{\mathcal{L}}^{\mathrm{for}} \cong \mathcal{L}^{\mathrm{for}}$, whose τ satisfies the condition on τ; if ψ is even through some s, then $F_{\mathrm{ample}}(G, \mathcal{L})$ is an object of $\mathrm{DD}_{\mathrm{ample}}$. Mumford's construction is a functor $M_{\mathrm{ample}}\colon \mathrm{DD}_{\mathrm{ample}} \to \mathrm{DEG}_{\mathrm{ample}}$ with $F_{\mathrm{ample}}(M_{\mathrm{ample}}(\mathfrak{d})) \cong \mathfrak{d}$ for every object $\mathfrak{d}$ of $\mathrm{DD}_{\mathrm{ample}}$, and $M_{\mathrm{ample}}(F_{\mathrm{ample}}(G, \mathcal{L})) \cong (G, \mathcal{L})$ whenever $F_{\mathrm{ample}}(G, \mathcal{L})$ is an object of $\mathrm{DD}_{\mathrm{ample}}$; thus F_{ample} and M_{ample} are quasi-inverse equivalences between $\mathrm{DD}_{\mathrm{ample}}$ and the full subcategory of $\mathrm{DEG}_{\mathrm{ample}}$ of the $(G, \mathcal{L})$ with $F_{\mathrm{ample}}(G, \mathcal{L})$ in $\mathrm{DD}_{\mathrm{ample}}$, which contains every $(G, \mathcal{L})$ whose ψ is even through some s. (Ampleness for symmetric sheaves.) Let G be a semiabelian scheme over S with G_η an abelian variety, and $\mathcal{L}$ a cubical invertible sheaf on G with $\mathcal{L}_\eta$ nondegenerate (its homomorphism $\lambda_{\mathcal{L}_\eta}\colon G_\eta \to G_\eta^\vee$ is an isogeny). If $\mathcal{L}_\eta$ is symmetric, then $\mathcal{L}$ is ample if and only if $\mathcal{L}_\eta$ is ample, and if and only if $\lambda_{\mathcal{L}_\eta}$ is a polarisation.*

3. *($\mathcal{O}$-structures.) Let $\mathcal{O}$ be an order in a finite-dimensional semisimple $\mathbb{Q}$-algebra with positive involution $\star$. The equivalence of (2), in its polarised form $\mathrm{DEG}_{\mathrm{pol}} \simeq \mathrm{DD}_{\mathrm{pol}}$, induces an equivalence $\mathrm{DD}_{\mathrm{PE},\mathcal{O}} \simeq \mathrm{DEG}_{\mathrm{PE},\mathcal{O}}$ between triples (G, λ, i) with (G, λ) a polarised degeneration and $i\colon \mathcal{O} \to \mathrm{End}_S(G)$ restricting to an $\mathcal{O}$-structure of (G_η, λ_η), and polarised degeneration data with an $\mathcal{O}$-action i_A on A and $\mathcal{O}$-module structures on X and Y compatible with c, $c^\vee$, ϕ, λ_A and τ.*

4. *(Fourier expansions.) Let $(G, \mathcal{L}) \in \mathrm{DEG}_{\mathrm{ample}}$ with $F_{\mathrm{ample}}(G, \mathcal{L}) = (A, X, Y, \phi, c, c^\vee, \widetilde{\mathcal{L}}, \tau, \psi)$, $\pi\colon \widetilde{G} \to A$, and suppose X, Y constant and $\widetilde{\mathcal{L}} \cong \pi^*\mathcal{M}$ as cubical sheaves for a cubical*

invertible sheaf $\mathcal{M}$ on A. For $\chi \in X$ put $\mathcal{O}_\chi = (\mathrm{Id}, c(\chi))^*\mathcal{P}_A$ and $\mathcal{M}_\chi = \mathcal{M} \otimes \mathcal{O}_\chi$. Then $\Gamma(G^{\mathrm{for}}, \mathcal{L}^{\mathrm{for}})$ is the I-adic completion of $\bigoplus_{\chi \in X} \Gamma(A, \mathcal{M}_\chi)$ (the decomposition into weights of the torus), which defines K-linear maps $\sigma_\chi \colon \Gamma(G_\eta, \mathcal{L}_\eta) \to \Gamma(A_\eta, \mathcal{M}_{\chi,\eta})$, and: (a) $\sigma_\chi \neq 0$ for every $\chi \in X$; (b) for all $y \in Y$, $\chi \in X$, $\psi(y)\tau(y,\chi) \circ T^*_{c^\vee(y)} \circ \sigma_\chi = \sigma_{\chi+\phi(y)}$, where $\psi(y)\tau(y,\chi)$ is read, by $\mathcal{M}_\chi(c^\vee(y))^{-1} \cong \mathcal{M}(c^\vee(y))^{-1} \otimes \mathcal{O}_\chi(c^\vee(y))^{-1}$, as an isomorphism $T^*_{c^\vee(y)}\mathcal{M}_{\chi,\eta} \to \mathcal{M}_{\chi+\phi(y),\eta}$; the sections $\psi(y)$, $\tau(y,\chi)$ are the unique ones satisfying (b) and the normalisation $\tau(y,0) = 1$ (the inverse of $\tau(y,0)$ is the isomorphism $c^\vee(y)^*\mathcal{O}_A \cong \mathcal{O}_S$), which implies $\psi(0) = 1$ (the rigidification) and $\tau(0,\chi) = 1$; replacing $\mathcal{M}$ by $\mathcal{M} \otimes \mathcal{O}_{\chi_0}$ shifts the indices of the σ_χ by χ_0 and changes neither τ nor ψ as a trivialisation of $\iota^*\widetilde{\mathcal{L}}_\eta^{-1}$; (c) for $n \geq 1$, $F_{\mathrm{ample}}(G, \mathcal{L}^{\otimes n}) = (A, X, Y, n\phi, c, c^\vee, \widetilde{\mathcal{L}}^{\otimes n}, \tau, \psi^n)$; (d) $\deg \lambda_\eta = [X : \phi(Y)]^2 \deg \lambda_A$ for the polarisations λ_η of G_η and λ_A of A induced by $\mathcal{L}$, $\widetilde{\mathcal{L}}$.

5. *(Torsion of Mumford quotients at points.) Let $(A, X, Y, \phi, c, c^\vee, \widetilde{\mathcal{L}}, \tau, \psi) \in \mathrm{DD}_{\mathrm{ample}}$ with X, Y constant and $\widetilde{\mathcal{L}} = \pi^*\mathcal{M}$ for an ample invertible sheaf $\mathcal{M}$ on A, where $\pi \colon \widetilde{G} \to A$, such that ψ, read through $\widetilde{\mathcal{L}} = \pi^*\mathcal{M}$, satisfies the condition on ψ of (2) (the tuple $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ is then an object of $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$). Let $(\widetilde{P}, \widetilde{\mathcal{L}})$ be a relatively complete model of it (in the sense of [154], Def. 4.5.1.2), and G the Mumford quotient of $\widetilde{G}$ by Y attached to it ([154], Constr. 4.5.2.17, Def. 4.5.2.18). For $y \in Y$, $\chi \in X$ let $I_{y,\chi} \subseteq K$ be the invertible R-submodule defined by $\tau(y,\chi)$. For $s \in S$ put $Y_s = \{z \in Y : I_{z,\phi(z)} \cdot \mathcal{O}_{S,s} = \mathcal{O}_{S,s}\}$. Then $Y_s = \{z \in Y : \iota(z) \in \widetilde{G}(\mathrm{Spec}\, \mathcal{O}_{S,s})\}$, and for every integer $n \geq 1$ there is a natural exact sequence of group schemes over $\kappa(s)$*

$$0 \to \widetilde{G}[n]_s \to G[n]_s \to \tfrac{1}{n}Y_s/Y_s \to 0.$$

Proof. [87], Ch. II, Thm. 6.2, and Ch. III; [154], Ch. 3–5. The statement is the text printed in Lan's book (numbering and pages of the book): (1) is the setting of Sect. 3.3.3 (p. 165; there R is only a noetherian integral domain), Prop. 3.3.3.6 and Def. 3.3.3.8 (p. 166; the ample cubical sheaf on G assumed in Prop. 3.3.3.6 exists because S is normal, Remark 3.3.3.9) and Prop. 3.3.3.11 (p. 167); (2) is the setting of Sect. 4.1 (p. 175), Def. 4.2.1.1 (p. 176), Lemma 4.2.1.7 and Defs. 4.2.1.10–4.2.1.13 (pp. 177–180), Thm. 4.2.1.14 (p. 180), Thm. 4.4.16 (p. 215) and Cor. 4.5.4.9 (p. 246), and its last two sentences are Lemma 4.2.1.6 (p. 177) under the assumptions of Lemma 4.2.1.4 (p. 176); (3) is Defs. 5.1.1.2–5.1.1.3 and Thm. 5.1.1.4 (pp. 286–287); (4) is Sect. 4.3.1 ((4.3.1.2)–(4.3.1.7), Remark 4.3.1.1, Lemma 4.3.1.8, pp. 191–193), Cor. 4.3.4.2 (p. 209) and Lemma 3.4.4.3 (p. 174); (5) is Def. 4.5.1.1 (p. 216), Def. 4.5.1.2 (pp. 216–217), Constr. 4.5.2.17 and Def. 4.5.2.18 (pp. 232–233), Prop. 4.5.3.11 and Cor. 4.5.3.12 (pp. 241–242; cf. Faltings–Chai III, Prop. 5.10, Cor. 5.11, and Mumford 1972, Cor. 4.11), with $I_{y,\chi}$ as in Def. 4.2.4.6 (pp. 189–190); in (2), the symmetry of τ is Def. 4.2.1.13, item 6, and the remark that the compatibility of ψ forces it is on pp. 178–179, where the positivity conditions are stated after the base change named (Defs. 4.2.1.10–4.2.1.11, Lemma 4.2.1.12); the isomorphism $\widetilde{\mathcal{L}} \cong \pi^*\mathcal{M}$ through which ψ is read is the one determined by a cubical trivialisation s (p. 178 and Sect. 4.2.3, pp. 184–185, where the cubical trivialisations form a torsor under X and two choices of $\mathcal{M}$ differ by some $\mathcal{O}_\chi$, Lemma 4.2.3.1), and Def. 4.2.1.13 (p. 180) requires both positivity conditions. From Lemma 4.2.1.12 (p. 179; proof in Sect. 4.2.4, pp. 189–190) (2) uses two implications: from the condition on ψ to the condition on τ, for every s; and from the condition on τ to the condition on ψ when ψ is even through s. In the even case $\psi(y)\psi(-y) = \tau(y, \phi(y))$ (by (4.2.3.4), p. 188, at $(y, -y)$, with $\psi(0) = 1$) and $I_y = I_{-y}$ (Defs. 4.2.4.5–4.2.4.6, pp. 189–190) give $I_y^2 = I_{y,\phi(y)}$, so $v(\psi(y)) = \frac{1}{2}v(\tau(y, \phi(y)))$ for every valuation v of K, a quadratic form in y, which is the form in which the printed proof argues; in particular $I_y \subseteq R$ by Lemma 4.2.4.2

(p. 189). That $F_{\mathrm{ample}}(G,\mathcal{L})$ satisfies the condition on τ is Prop. 4.3.1.9, item 5 (p. 194), proved on pp. 195–196 from the Fourier expansion and (4.2.4.7) for any choice of $\mathcal{M}$; with the even case this gives the condition on ψ, hence membership in $\mathrm{DD}_{\mathrm{ample}}$, when ψ is even through some s. That the condition on ψ involves s: replacing s by its product with a character $\chi_0 \in X$ shifts the weights by χ_0 (p. 185; Remark 4.3.1.1, p. 192) and multiplies $\psi(y)$, read through the new $\mathcal{M}$, by $\tau(y,\chi_0)^{\pm 1}$, while τ does not change (Lemma 4.3.1.14, p. 196); on the Tate curve over $k[[t]]$ with $q=t^2$, $\psi(y)=t^{y^2}$ becomes t^{y^2-2yj}, not integral at $y=j\neq 0$. In (5), $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ is Def. 4.5.1.1 (p. 216), whose objects carry the positivity condition on ψ read through the given $\mathcal{M}$. Notation map: Lan's $G^\natural$ is $\widetilde{G}$, his $\mathcal{L}^\natural$ is $\widetilde{\mathcal{L}}$, his CUB the categories of cubical torsors; ι (period map), τ, ψ, X, Y, ϕ as above; in the principally polarised case ϕ is an isomorphism and Y is identified with X, $c^\vee$ with c via λ_A. □

Theorem B.110. *Let S be a locally noetherian normal scheme and $0\to\mathbb{T}\xrightarrow{i}\widetilde{G}\xrightarrow{\pi}A\to 0$ an exact sequence of smooth commutative group schemes with connected fibres over S, with $\mathbb{T}$ a split torus and A an abelian scheme.*

1. *Every cubical $\mathbb{G}_m$-torsor on $\widetilde{G}$ is isomorphic, as a cubical torsor, to $\pi^*\mathcal{M}$ for a cubical $\mathbb{G}_m$-torsor $\mathcal{M}$ on A.*
2. *The category $\mathrm{CUB}_S(A,\mathbb{G}_{m,S})$ is equivalent to the category of pairs $(\mathcal{L},s)$, where $\mathcal{L}$ is an object of $\mathrm{CUB}_S(\widetilde{G},\mathbb{G}_{m,S})$ and s is a trivialisation of $i^*\mathcal{L}$ in $\mathrm{CUB}_S(\mathbb{T},\mathbb{G}_{m,S})$ (equivalently, in the category $\mathrm{EXT}_S(\mathbb{T},\mathbb{G}_{m,S})$ of commutative group extensions of $\mathbb{T}$ by $\mathbb{G}_{m,S}$).*

Proof. K.-W. Lan, Arithmetic compactifications of PEL-type Shimura varieties, LMS Monographs 36 (2013) [154], Sect. 3.2.5 (pp. 160–161), after Moret-Bailly, Pinceaux de variétés abéliennes [181], Ch. I, 7.2.1–7.2.4. Item 1 is Cor. 3.2.5.7 (p. 161: for S locally noetherian and normal and a finite étale surjection $S'\to S$ splitting the torus, $\mathrm{EXT}_{S'}(\mathbb{G}_{m,S'},\mathbb{G}_{m,S'})=0$, and hence every cubical $\mathbb{G}_m$-torsor over $\widetilde{G}_{S'}$ comes from $A_{S'}$), applied with the identity of S as the splitting cover. Item 2 is Prop. 3.2.5.4 (p. 160), stated by Lan for any torus over any base scheme and an exact sequence of smooth commutative group schemes with connected geometric fibres, here specialised; the parenthesis is Prop. 3.2.5.3 (p. 160: $\mathrm{CUB}_S(\mathbb{T},\mathbb{G}_{m,S})$ is equivalent to $\mathrm{EXT}_S(\mathbb{T},\mathbb{G}_{m,S})$). The same numbers hold in Lan's Harvard thesis (2008). Notation map: Lan's T, G, H are $\mathbb{T}$, $\widetilde{G}$, A. □

Theorem B.111. *Let R, I, S, η be as in Theorem B.109, and let $(G,\mathcal{L}_1)$, $(G,\mathcal{L}_2)$ be objects of $\mathrm{DEG}_{\mathrm{ample}}$ with the same semiabelian scheme G, with $F_{\mathrm{ample}}(G,\mathcal{L}_k)=(A,X,Y,\phi_k,c,c^\vee,\widetilde{\mathcal{L}}_k,\tau_k,\psi_k)$ ($k=1,2$; the entries $A,X,Y,c,c^\vee$ depend only on G), and let $\lambda_k\colon G_\eta\to G_\eta^\vee$ be the polarisation induced by $\mathcal{L}_{k,\eta}$.*

1. *If $N_1\lambda_1=N_2\lambda_2$ for some positive integers N_1,N_2, then $\tau_1=\tau_2$.*
2. *Let $(G,\mathcal{L})$ be a third object of $\mathrm{DEG}_{\mathrm{ample}}$ with $\mathcal{L}\cong\mathcal{L}_1\otimes\mathcal{L}_2$ and*

$$F_{\mathrm{ample}}(G,\mathcal{L})=(A,X,Y,\phi,c,c^\vee,\widetilde{\mathcal{L}},\tau,\psi).$$

Then $\widetilde{\mathcal{L}}\cong\widetilde{\mathcal{L}}_1\otimes\widetilde{\mathcal{L}}_2$. If $\tau_1=\tau_2$, then $\phi=\phi_1+\phi_2$, $\tau=\tau_1=\tau_2$ and $\psi=\psi_1\psi_2$ (read through this isomorphism).

Proof. See [154] (the printed book, London Math. Soc. Monographs 36, Princeton 2013), Sect. 4.3.4: item (1) is Proposition 4.3.4.5 (p. 211), item (2) is Lemma 4.3.4.1 (p. 209; after Faltings–Chai, Degeneration of abelian varieties, Ch. II, Lemma 6.1); the isomorphism $\widetilde{\mathcal{L}} \cong \widetilde{\mathcal{L}}_1 \otimes \widetilde{\mathcal{L}}_2$ is the first paragraph of the printed proof of Lemma 4.3.4.1 ($\widetilde{\mathcal{L}}_k \cong \pi^*\mathcal{M}_k$ gives $\widetilde{\mathcal{L}} \cong \pi^*(\mathcal{M}_1 \otimes \mathcal{M}_2)$), and it is the identification through which the printed equality $\psi = \psi_1\psi_2$ is read. (In Lan's Harvard thesis, 2008, the same results are Lemma 4.3.4.1 and Proposition 4.3.4.6.) Notation: Lan's objects of $\mathrm{DEG}_{\mathrm{ample}}(R, I)$ (book, Definition 4.2.1.1) with their data as in his Corollary 4.3.1.16 and Theorem 4.2.1.14 are the objects of $\mathrm{DEG}_{\mathrm{ample}}$ of Theorem B.109(2) with F_{ample}; Lan's $\mathcal{L}^\natural$ is our $\widetilde{\mathcal{L}}$; his $\lambda_k \colon G \to G^\vee$ is the unique extension of the polarisation of G_η (his Convention 4.2.1.5 and Proposition 3.3.1.5), so $N_1\lambda_1 = N_2\lambda_2$ on G is equivalent to the same equality on G_η. □

Theorem B.112. *Let R, I, S, η, K be as in Theorem B.109. For an object $\mathfrak{d} = (A, X, Y, \phi, c, c^\vee, \widetilde{\mathcal{L}}, \tau, \psi)$ of $\mathrm{DD}_{\mathrm{ample}}$ let $i \colon T \to \widetilde{G}$ and $\pi \colon \widetilde{G} \to A$ be the torus and the projection of the extension defined by c, and for an integer $n \geq 1$ put*

$$\mathfrak{d}^{(n)} = (A,\ X,\ Y,\ (2n+1)\phi,\ c,\ c^\vee,\ \widetilde{\mathcal{L}}^{\otimes n+1} \otimes [-1]^*\widetilde{\mathcal{L}}^{\otimes n},\ \tau,\ \psi^{n+1}[-1]^*\psi^n).$$

A split object *is a tuple $\mathfrak{e} = (A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ with X and Y constant, $\mathcal{M}$ an ample invertible sheaf on A, and $\bar{\mathfrak{e}} = (A, X, Y, \phi, c, c^\vee, \pi^*\mathcal{M}, \tau, \psi)$ an object of $\mathrm{DD}_{\mathrm{ample}}$ (the* image *of $\mathfrak{e}$), such that ψ, read through the identity of $\pi^*\mathcal{M}$, satisfies the condition on ψ of Theorem B.109(2); for $y \in Y$ let $I_y \subseteq K$ be the invertible R-submodule defined by $\psi(y)$, a trivialisation over η of $\mathcal{M}(c^\vee(y))^{-1}$, and for $n \geq 1$ put*

$$\mathfrak{e}^{(n)} = (A,\ \mathcal{M}^{\otimes n+1} \otimes [-1]^*\mathcal{M}^{\otimes n},\ X,\ Y,\ (2n+1)\phi,\ c,\ c^\vee,\ \tau,\ \psi^{n+1}[-1]^*\psi^n).$$

A star *is a finite subset $\mathrm{St} \subseteq X$ generating X with $0 \in \mathrm{St}$ and $-\mathrm{St} = \mathrm{St}$; a split object satisfies* Condition 4.5.1.6 for St *if $I_y \cdot I_{y,\alpha} \subseteq R$ for all $y \in Y$ and all $\alpha \in \mathrm{St}$.*

1. *(Descent of $\widetilde{\mathcal{L}}$ to A.) Let $\mathfrak{d} \in \mathrm{DD}_{\mathrm{ample}}$ with X constant. Every cubical trivialisation s of $i^*\widetilde{\mathcal{L}}$ determines a cubical invertible sheaf $\mathcal{M}$ on A and a cubical isomorphism $\widetilde{\mathcal{L}} \cong \pi^*\mathcal{M}$, both depending uniquely on the choice of s, and $\mathcal{M}$ is ample because $\widetilde{\mathcal{L}}$ is.*

2. *(Twisting.) (a) For every $\mathfrak{d} \in \mathrm{DD}_{\mathrm{ample}}$ there is an integer $n_0 > 0$ such that $\mathfrak{d}^{(n)}$ is an object of $\mathrm{DD}^*_{\mathrm{ample}}$ (item (3)), in particular of $\mathrm{DD}_{\mathrm{ample}}$, for every $n \geq n_0$. (b) For every split object $\mathfrak{e}$ and every star St there is an integer $n_1 > 0$ such that $\mathfrak{e}^{(n)}$ satisfies Condition 4.5.1.6 for St for every $n \geq n_1$. (c) If a split object satisfies Condition 4.5.1.6 for a star St, then the pair $(\widetilde{P}_{(\phi,\psi),\mathrm{St}}, \widetilde{\mathcal{L}})$ of [154], Constr. 4.5.1.5, is a relatively complete model of it.*

3. *(Mumford's construction on split objects.) Fix a star St_0, and let $\mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}}$ be the category of split objects satisfying Condition 4.5.1.6 for St_0. The Mumford quotient attached to the relatively complete model of (2)(c) defines a functor $M^{\mathrm{split},*}_{\mathrm{ample}} \colon \mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}} \to \mathrm{DEG}_{\mathrm{ample}}$, $\mathfrak{e} \mapsto (G, \mathcal{L})$, with G that Mumford quotient. Let $\mathrm{DD}^*_{\mathrm{ample}}$ be the full subcategory of $\mathrm{DD}_{\mathrm{ample}}$ of the objects $\mathfrak{d}$ for which there are a finite étale surjection $S' = \operatorname{Spec} R' \to S$ (with $I' = \mathrm{rad}(I \cdot R')$) over which X and Y are constant and an ample invertible sheaf $\mathcal{M}'$ on $A \otimes_R R'$ with $\widetilde{\mathcal{L}} \otimes_R R' \cong \pi^*\mathcal{M}'$, such that $(A, \mathcal{M}', X, Y, \phi, c, c^\vee, \tau, \psi) \otimes_R R'$ is an object of $\mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}}(R', I')$. Then $M^{\mathrm{split},*}_{\mathrm{ample}}$ extends to a functor $M^*_{\mathrm{ample}} \colon \mathrm{DD}^*_{\mathrm{ample}} \to \mathrm{DEG}_{\mathrm{ample}}$ compatible with finite étale surjective base changes: for every $\mathfrak{e}$ in $\mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}}$ the image $\bar{\mathfrak{e}}$ lies in $\mathrm{DD}^*_{\mathrm{ample}}$ and $M^*_{\mathrm{ample}}(\bar{\mathfrak{e}}) \cong M^{\mathrm{split},*}_{\mathrm{ample}}(\mathfrak{e})$. The functor M_{ample} extends M^*_{ample}: $M_{\mathrm{ample}}(\mathfrak{d}) \cong M^*_{\mathrm{ample}}(\mathfrak{d})$ for every $\mathfrak{d}$ in $\mathrm{DD}^*_{\mathrm{ample}}$.*

4. *(Compatibility with the forgetful functors.) There is a functor M from the category of the tuples $(A, X, Y, c, c^\vee, \tau)$ underlying objects of $\mathrm{DD}_{\mathrm{ample}}$, with their isomorphisms, to the category of semiabelian schemes over S, with isomorphisms of group schemes, such that M_{ample} and M are compatible under the forgetful functors $(A, X, Y, \phi, c, c^\vee, \widetilde{\mathcal{L}}, \tau, \psi) \mapsto (A, X, Y, c, c^\vee, \tau)$ and $(G, \mathcal{L}) \mapsto G$. In particular, if two objects of $\mathrm{DD}_{\mathrm{ample}}$ have the same $(A, X, Y, c, c^\vee, \tau)$, the semiabelian schemes of their images under M_{ample} are isomorphic as group schemes over S.*

5. *(Independence of the Mumford quotient.) For $j = 1, 2$ let $\mathfrak{e}_j = (A, \mathcal{M}_j, X, Y, \phi_j, c, c^\vee, \tau, \psi_j)$ be split objects with the same $(A, X, Y, c, c^\vee, \tau)$, each with a relatively complete model, and G_j the attached Mumford quotients, with the identifications $G_j^{\mathrm{for}} \cong \widetilde{G}^{\mathrm{for}}$ of [154], Constr. 4.5.2.17. Then G_1 and G_2 are group schemes over S, and there is a unique homomorphism $G_1 \to G_2$ of group schemes over S whose formal completion is the identity of $\widetilde{G}^{\mathrm{for}}$; it is an isomorphism.*

Proof. [154] (numbering and pages of the book), following Mumford 1972 and [87], Ch. III. Split objects are the objects of $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R, I)$ of Def. 4.5.1.1 (p. 216: tuples as in Def. 4.2.1.13 with the positivity conditions for ψ and τ, ψ being viewed as a trivialisation of $(c^\vee)^*\mathcal{M}^{-1}$), with the natural functor to $\mathrm{DD}_{\mathrm{ample}}(R, I)$ given there; I_y and $I_{y,\chi}$ are as in Defs. 4.2.4.5–4.2.4.6; stars and Condition 4.5.1.6 as in Constr. 4.5.1.5 and Cond. 4.5.1.6 (pp. 217–219). (1) is the paragraph of p. 178 preceding the proof of Lemma 4.2.1.8 ("By [112, XI, 1.11], $\mathcal{M}$ is ample because $\mathcal{L}^\natural$ is", Raynaud, Faisceaux amples sur les schémas en groupes, LNM 119), with Prop. 3.2.5.4 (p. 160) and Cor. 3.2.5.7 (p. 161): over the normal noetherian S with X constant, $i^*\widetilde{\mathcal{L}}$ is trivial and its cubical trivialisations correspond to the pairs $(\mathcal{M}, \widetilde{\mathcal{L}} \cong \pi^*\mathcal{M})$. (2)(a) is Lemma 4.5.4.5 (p. 244) with the second tuple taken trivial, which that lemma allows ("an object in $\mathrm{DD}_{\mathrm{IS}}(R, I)$ such that all of ϕ_2, $\mathcal{L}_2^\natural$, and ψ_2 are trivial"); (2)(b) is Cor. 4.5.1.8 (pp. 221–222), first tuple, with $\mathcal{M}'$ trivial; (2)(c) is Prop. 4.5.1.15 (p. 224). (3) is Def. 4.5.4.1 (p. 242), (4.5.4.2), Def. 4.5.4.3 and Prop. 4.5.4.4 (p. 243), and Cor. 4.5.4.9 (p. 246: M_{IS} extends M^*_{ample} and restricts to M_{ample}); that the image of an object of $\mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}}$ lies in $\mathrm{DD}^*_{\mathrm{ample}}$ is Def. 4.5.4.3 with $S' = S$ (the identity is a finite étale surjection and $\mathrm{rad}(I) = I$), which is what "extends" in Prop. 4.5.4.4 presupposes. (4) is Thm. 4.4.16 (p. 215: M_{ample}, M_{pol}, M_{IS}, M "are compatible with each other under the natural forgetful functors"; $M\colon \mathrm{DD}(R, I) \to \mathrm{DEG}(R, I)$, $(A, X, Y, c, c^\vee, \tau) \mapsto G$, morphisms of both categories being isomorphisms over S respecting all structures), with Cor. 4.5.4.9; the last sentence applies M to the identity of the common tuple. (5) is Thm. 4.5.3.6 (p. 237: a unique homomorphism with given formal completion) applied to $\mathrm{Id}_{\widetilde{G}}$ and Id_Y, Cor. 4.5.3.7 (p. 238: G depends up to isomorphism only on $(A, X, Y, c, c^\vee, \tau)$, independently of ϕ, ψ, $\mathcal{M}$ and the relatively complete model, by a canonical isomorphism inducing the identity of the formal completions) and Cor. 4.5.3.8 (p. 239: G is a group scheme); cf. Mumford 1972, Thm. 4.6, Cors. 4.7–4.8, and Faltings–Chai III, Thm. 5.5, Cors. 5.6–5.7. Notation map: Lan's $G^\natural$, $\mathcal{L}^\natural$, $\bigstar$ are $\widetilde{G}$, $\widetilde{\mathcal{L}}$, St; his $P^\natural_{(\phi,\psi),\bigstar}$ is $\widetilde{P}_{(\phi,\psi),\mathrm{St}}$. □

Theorem B.113. *Let R be a strictly henselian discrete valuation ring with fraction field K and residue field k, K^s a separable closure of K, $I = \mathrm{Gal}(K^s/K)$, ℓ a prime invertible in k, A an abelian variety over K, and $\mathcal{G}$ a semiabelian scheme over R with an isomorphism $\mathcal{G}_K \cong A$, whose special fibre $\mathcal{G}_k$ is an extension of an abelian variety of dimension a by a torus of dimension t. Then $(\sigma - 1)^2 = 0$ on the Tate module $T_\ell(A)$ for every $\sigma \in I$, and the module of invariants $T_\ell(A)^I$ is a free $\mathbb{Z}_\ell$-module of rank $2a + t$.*

Proof. A. Grothendieck, *Modèles de Néron et monodromie*, Exposé IX of [227]: the identity component of the Néron model of A is $\mathcal{G}$ (a semiabelian model is the identity component of the

Néron model; see also [45], Section 7.4), $T_\ell(A)^I$ is identified with T_ℓ of the identity component of the special fibre of the Néron model, of rank $2a+t$ (Exposé IX, Section 2), and semiabelian reduction is equivalent to the action of I on $T_\ell(A)$ being unipotent of level at most two, $(\sigma-1)^2=0$ (Exposé IX, Proposition 3.5). □

Theorem B.114. *Let (E,M,ψ,J) and $N\geq 3$ be as in Theorem B.105, let $\mathcal{M}_{N,\mathbb{C}}=\mathcal{M}_N\otimes_{F_{\mathrm{ref}}}\mathbb{C}$, formed with the inclusion $F_{\mathrm{ref}}\subseteq\mathbb{C}$, and $\mathcal{M}_{N,\mathbb{C}}^{\mathrm{an}}$ its complex analytic space. Let G be the group scheme over $\mathbb{Z}$ with*

$$G(R)=\{(\gamma,c)\in \mathrm{GL}_{\mathcal{O}_E\otimes R}(M\otimes R)\times R^\times:\ \psi(\gamma x,\gamma y)=c\,\psi(x,y)\},\qquad \nu(\gamma,c)=c,$$

written γ, and $\mathcal{H}_N=\ker\big(G(\widehat{\mathbb{Z}})\to G(\mathbb{Z}/N\mathbb{Z})\big)$. Let $X=\{\gamma J\gamma^{-1}:\gamma\in G(\mathbb{R})\}$, identified with $G(\mathbb{R})/\mathrm{Stab}_{G(\mathbb{R})}(J)$ and given the complex structure for which $J'\mapsto\ker(J'+i)\subseteq M_{\mathbb{C}}$ maps each connected component of X biholomorphically onto an open subset of $G(\mathbb{C})/P_0(\mathbb{C})$, P_0 the stabiliser of $\ker(J+i)$; let X_0 be the connected component containing J, $G(\mathbb{Q})_0$ its stabiliser in $G(\mathbb{Q})$, and $\Gamma^{(1)}=\mathcal{H}_N\cap G(\mathbb{Q})_0$, acting on X_0 by conjugation. For $J'\in X_0$ let $A_{J'}=(M_{\mathbb{R}},J')/M$ with the $\mathcal{O}_E$-action ι induced by that of M, let $\phi_{J'}$ be the polarisation with Riemann form ψ on $H_1(A_{J'},\mathbb{Z})=M$, and

$$x(J')=(A_{J'},\phi_{J'},\iota,\alpha_N,\nu_N),\qquad \alpha_N(x)=N^{-1}x \bmod M,\quad \nu_N(c)=\exp(-2\pi i\,c/N).$$

Then:

(Lan1) *For $J'\in X_0$, $\phi_{J'}$ is a principal polarisation, $e_n^{\phi_{J'}}(n^{-1}x,n^{-1}y)=\exp(-2\pi i\,\psi(x,y)/n)$ for all $n\geq 1$ and $x,y\in M$, and $x(J')$ is an object of $\mathcal{M}_N(\mathbb{C})$.*

(Lan2) *$\Gamma^{(1)}$ acts on X_0 without fixed points; $(x(J'))_{J'\in X_0}$ is a holomorphic family; and $J'\mapsto x(J')$ induces an isomorphism of complex analytic spaces from $\Gamma^{(1)}\backslash X_0$ onto an open and closed complex analytic subspace of $\mathcal{M}_{N,\mathbb{C}}^{\mathrm{an}}$, along which the universal tuple pulls back to the family $(x(J'))_{J'\in X_0}$.*

Proof. K.-W. Lan, *Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties*, J. reine angew. Math. 664 (2012), 163–228, Sections 2.2–2.5, cited from the author's compilation of the paper (errata corrected; page numbers are those of the compilation). (Lan1): Section 2.2, p. 10 (the Weil pairing e^{λ_h} on $G_h[n]=\frac{1}{n}L/L$ is realised as $(x,y)\mapsto\mathbf{e}(n\langle x,y\rangle)$; Remark 2.2.1: the sign is a convention) and Section 2.3, p. 11 (the canonical $\hat{\alpha}_h\colon L\otimes\widehat{\mathbb{Z}}\to \mathrm{T}\,G_h$ matches $\langle\,,\rangle$ with e^{λ_h}, its $\mathcal{H}$-orbit is an integral level structure, and $(G_h,\lambda_h,i_h,\alpha_{h,\mathcal{H}})$ is an object of $\mathsf{M}_{\mathcal{H}}(\mathbb{C})$); λ_h is induced by $\mathrm{sgn}(h)$ times the identity (Section 2.2, p. 9; p. 14). (Lan2): end of Section 2.4, p. 14 (the tuples form a holomorphic family over $X\times G(\mathbb{A}^\infty)/\mathcal{H}$); (2.5.2), Assumption 2.5.4 and the text after it, p. 15 ($\mathrm{Sh}_{\mathcal{H}}=G(\mathbb{Q})\backslash X\times G(\mathbb{A}^\infty)/\mathcal{H}=\coprod_{i\in I}\Gamma^{(g_i)}\backslash X_0$, I finite, $\Gamma^{(g_i)}=g_i\mathcal{H}g_i^{-1}\cap G(\mathbb{Q})_0$ acting without fixed points, for any g_i with $G(\mathbb{A}^\infty)=\coprod_i G(\mathbb{Q})_0 g_i\mathcal{H}$; the family descends to $\mathrm{Sh}_{\mathcal{H}}$); Lemma 2.5.6, the paragraph after it and Definition 2.5.7, p. 16 ($\mathrm{Sh}_{\mathcal{H}}$ is isomorphic, as a complex analytic space and compatibly with the families, to $\mathsf{M}_{\mathcal{H},\mathrm{an},L\otimes\mathbb{Q}}$, the analytification of the open and closed subscheme of $\mathsf{M}_{\mathcal{H},\mathbb{C}}$ of the components over which $(H_1(G_s,\mathbb{Q}),\langle\,,\rangle_{\lambda_s},h_0)\cong(L\otimes\mathbb{Q},\langle\,,\rangle,h_0)$). (Lan2) is this isomorphism on the piece of (2.5.2) for the representative $g_1=1$; the pieces are open and closed in $\mathrm{Sh}_{\mathcal{H}}$ since $G(\mathbb{A}^\infty)/\mathcal{H}$ is discrete, and on it the family is $(G_h,\lambda_h,i_h,\alpha_{h,\mathcal{H}})$, $h\in X_0$. Complex structure of X: Section 2.3, pp. 10–11 ($X=G(\mathbb{R})/\mathcal{U}_\infty$; its connected components embed as open complex submanifolds of $G(\mathbb{C})/P_{h_0}(\mathbb{C})$ through $h\mapsto\mathsf{P}_h=\{\sqrt{-1}x-h(\sqrt{-1})x\}$, Lemma 2.1.2), and $\mathsf{P}_h=\ker(h(\sqrt{-1})+\sqrt{-1})$. *Notation.* Lan's PEL-type $\mathcal{O}$-lattice $(L,\langle\,,\rangle,h_0)$ (Section 1.1, p. 4) is $(M,-2\pi\sqrt{-1}\,\psi,J)$ with $\mathcal{O}=\mathcal{O}_E$ and $h_0(\sqrt{-1})=J$: his condition (2)(a) is $\psi(Jx,Jy)=\psi(x,y)$

with $\psi(ax,y)=\psi(x,\bar a y)$, and (2)(b), $-\sqrt{-1}\langle x,h_0(\sqrt{-1})x\rangle = 2\pi\,\psi(Jx,x)>0$, is the positivity of Theorem B.105; then $\langle\,,\rangle$ is $\mathbb{Z}(1)$-valued and $\mathbf{e}(n\langle n^{-1}x,n^{-1}y\rangle)=\exp(-2\pi\sqrt{-1}\,\psi(x,y)/n)$. The set $\square$ of primes is empty (Condition 1.2.5 and the sentence after it, p. 6), $F_0=F_{\mathrm{ref}}$ with its inclusion into $\mathbb{C}$, $\mathcal{H}=\mathcal{H}_N$, neat as $N\geq 3$ (Remark 1.1.4). Lan's moduli problem $\mathsf{M}_{\mathcal{H}}$ (Definition 1.2.2, Theorem 1.2.4) is $\mathcal{M}_N$ of Theorem B.105: the integral level-$\mathcal{H}_N$ structures of type $(L\otimes\widehat{\mathbb{Z}},\langle\,,\rangle)$ are the liftable pairs (α_N,ν_N) ([154], Definition 1.3.6.2 and Lemma 1.3.6.6), they force the polarisation to be principal since $\langle\,,\rangle$ is perfect on L, and in characteristic 0 the determinantal condition given by $(L\otimes\mathbb{R},\langle\,,\rangle,h_0)$ is $\det(T-\iota(a)\mid\mathrm{Lie})=P_a(T)$ (p. 10: $\mathrm{Lie}\,G_h\cong V_0$). On X_0 one has $\mathrm{sgn}(h)=1$: the sign of the definite form $-\sqrt{-1}\langle x,h(\sqrt{-1})x\rangle$ is locally constant on X (definiteness is an open condition, and $X\to\mathrm{End}(L\otimes\mathbb{R})$ is continuous) and is $+1$ at h_0; so λ_h has Riemann form ψ, and $(G_h,\lambda_h,i_h,\alpha_{h,\mathcal{H}})$ is $x(J')$ for $J'=h(\sqrt{-1})$, the canonical $\hat\alpha_h$ reducing to α_N and the pairing to ν_N. Not used here: the members of the family at $g\neq 1$ (modification by $\mathbb{Q}^\times$-isogenies, p. 11) and the components of $\mathsf{M}_{\mathcal{H},\mathbb{C}}$ outside the rational class of the datum, which pp. 1–2 of the paper attribute to the failure of the Hasse principle. $\square$

Theorem B.115. *Let R be a strictly henselian discrete valuation ring with fraction field K, residue field k and $I=\mathrm{Gal}(K^s/K)$, ℓ a prime invertible in k, A an abelian variety over K with a principal polarisation λ, and $\mathcal{G}$ a semiabelian scheme over R with $\mathcal{G}_K\cong A$; let $\mathcal{T}\subseteq\mathcal{G}_k$ be the maximal subtorus of the special fibre. The specialisation maps identify $T_\ell(\mathcal{G}_k)$ with $T_\ell(A)^I$ (via $\mathcal{G}[\ell^n](R)\cong\mathcal{G}_k[\ell^n](k)$ and $\mathcal{G}[\ell^n](R)\subseteq A[\ell^n](K^s)$), hence $T_\ell(\mathcal{T})$ with a submodule $T_\ell(A)_{\mathrm{t}}$ of $T_\ell(A)^I$. Then, for the Weil pairing $e^\lambda\colon T_\ell(A)\times T_\ell(A)\to\mathbb{Z}_\ell(1)$ of λ,*

$$T_\ell(A)_{\mathrm{t}}=\{u\in T_\ell(A):\ e^\lambda(u,u')=1 \text{ for all } u'\in T_\ell(A)^I\}.$$

Proof. A. Grothendieck, *Modèles de Néron et monodromie*, Exposé IX of [227], the orthogonality theorem ("théorème d'orthogonalité"): for semistable reduction, the toric part of $T_\ell(A)$ and the invariants $T_\ell(A')^I$ of the dual are exact orthogonals of each other under the Weil pairing $T_\ell(A)\times T_\ell(A')\to\mathbb{Z}_\ell(1)$; the identification of $T_\ell(A)^I$ with the Tate module of the special fibre of the Néron model, whose identity component is $\mathcal{G}$ ([45], 7.4), is in the same exposé, Section 2. With $A'=A$ via the principal polarisation λ one obtains the statement. $\square$

Theorem B.116. *Let $S\hookrightarrow\tilde S$ be a closed immersion of affine schemes defined by an ideal $\mathcal{I}$ with $\mathcal{I}^2=0$.*

1. *Let A be an abelian scheme over S and $\tilde A$ an abelian scheme over $\tilde S$ with $\tilde A\times_{\tilde S}S=A$. For every abelian scheme $\tilde A'$ over $\tilde S$ with $\tilde A'\times_{\tilde S}S=A$ there is a canonical isomorphism $c_{\tilde A'}\colon H^1_{\mathrm{dR}}(\tilde A'/\tilde S)\to H^1_{\mathrm{dR}}(\tilde A/\tilde S)$ reducing to the identity of $H^1_{\mathrm{dR}}(A/S)$, and $\tilde A'\mapsto c_{\tilde A'}(\omega_{\tilde A'})$ is a bijection from the set of isomorphism classes of such liftings $\tilde A'$ of A (isomorphisms reducing to the identity of A) onto the set of $\mathcal{O}_{\tilde S}$-submodules $M\subseteq H^1_{\mathrm{dR}}(\tilde A/\tilde S)$ with $H^1_{\mathrm{dR}}(\tilde A/\tilde S)/M$ projective and $M\otimes_{\mathcal{O}_{\tilde S}}\mathcal{O}_S=\omega_A$ in $H^1_{\mathrm{dR}}(A/S)$.*
2. *Let $\tilde A,\tilde B$ be abelian schemes over $\tilde S$, $A=\tilde A\times_{\tilde S}S$, $B=\tilde B\times_{\tilde S}S$, and $f\colon A\to B$ a homomorphism. There is a canonical morphism $f^*\colon H^1_{\mathrm{dR}}(\tilde B/\tilde S)\to H^1_{\mathrm{dR}}(\tilde A/\tilde S)$ reducing to $H^1_{\mathrm{dR}}(f)$, and f lifts to a homomorphism $\tilde A\to\tilde B$ if and only if $f^*(\omega_{\tilde B})\subseteq\omega_{\tilde A}$. At most one homomorphism $\tilde A\to\tilde B$ lifts f.*

Proof. K.-W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, London Math. Soc. Monographs 36, Princeton 2013, Section 2.1.6: the canonical morphisms are Proposition 2.1.6.4 (with Proposition 2.1.6.2), the Hodge sequence is (2.1.6.6), and parts 1 and 2 (the

existence criterion) are Propositions 2.1.6.8 and 2.1.6.9, p. 114 ("an analogue of a weaker form of the main theorem in Grothendieck–Messing theory"; cf. [171], [168]). *Notation.* Lan states part 1 in the first de Rham homology $H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$, the dual of $H^1_{\mathrm{dR}}(\tilde{A}/\tilde{S})$, with the submodule $\mathrm{Lie}^{\vee}_{\tilde{A}'^{\vee}/\tilde{S}}$ (the kernel of $H_1^{\mathrm{dR}} \to \mathrm{Lie}_{\tilde{A}'/\tilde{S}}$, (2.1.6.7)); passing to annihilators under the perfect pairing $H_1^{\mathrm{dR}} \times H^1_{\mathrm{dR}} \to \mathcal{O}_{\tilde{S}}$ turns it into $\omega_{\tilde{A}'}$ and his condition on M into the one above (an annihilator of a submodule with projective quotient has projective quotient, and annihilators commute with $\otimes\mathcal{O}_S$ for such submodules). Lan's set $\mathrm{Lift}(A; S \hookrightarrow \tilde{S})$ (Definition 2.1.2.1) consists of the isomorphism classes of smooth liftings with an identification of the reduction; a smooth lifting of A has a section lifting the zero section ($\tilde{S}$ affine, smoothness), and with it is an abelian scheme, and two abelian liftings that are isomorphic as liftings are isomorphic as abelian schemes after composing with a translation by a section reducing to zero (rigidity; Mumford–Fogarty–Kirwan, loc. cit. below, Ch. 6, §1); so the bijection is one of liftings that are abelian schemes, as stated. Lan's Proposition 2.1.6.9 lifts f to a morphism of schemes $\tilde{f}$; then $\tilde{f} - \tilde{f}(0)$ (subtraction of the section $\tilde{f}(0)$, which reduces to the zero section) is a morphism of abelian schemes preserving the zero sections, hence a homomorphism, and it still lifts f; conversely a homomorphism lifting f is a morphism lifting f. Uniqueness: if two homomorphisms $\tilde{f}_1, \tilde{f}_2$ lift f, the homomorphism $\tilde{f}_1 - \tilde{f}_2$ is zero on A, which has the same underlying space as $\tilde{A}$, so by the rigidity lemma (D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory*, Prop. 6.1 and Cor. 6.4) it factors through the zero section on every connected component of $\tilde{S}$, i.e. $\tilde{f}_1 = \tilde{f}_2$. □

Theorem B.117. *Let $k \subset \mathbb{C}$ be an imaginary quadratic field with ring of integers $\mathcal{O}_k$.*

1. *(Algebraicity.) If Z is an elliptic curve over $\mathbb{C}$ with $\mathrm{End}(Z) \cong \mathcal{O}_k$, then $j(Z)$ is an algebraic number; hence there are a number field $M \subset \mathbb{C}$ and an elliptic curve Z_M over M with $Z_M \otimes_M \mathbb{C} \cong Z$, and M may be chosen so that every endomorphism of $Z_M \otimes_M \mathbb{C}$ is defined over M.*

2. *(Potentially good reduction.) If M is a number field and Z_M an elliptic curve over M whose endomorphism ring over $\overline{M}$ is an order in an imaginary quadratic field, then Z_M has potentially good reduction at every finite place of M: there is a finite extension M'/M such that $Z_M \otimes_M M'$ has good reduction at every finite place of M', that is, extends to an abelian scheme over $\mathcal{O}_{M'}$.*

Proof. Item 1: the algebraicity of $j(Z)$ (indeed $k(j(Z))$ is the Hilbert class field of k) is the first main theorem of complex multiplication for elliptic curves (Deuring; Shimura–Taniyama), [233], Chapter II, §4, Theorem II.4.3; an elliptic curve over $\mathbb{C}$ with algebraic j-invariant is defined over $\mathbb{Q}(j)$, and its endomorphisms, finitely many generators of a $\mathbb{Z}$-module of rank 2, are defined over a finite extension (each is defined over $\overline{\mathbb{Q}}$ because homomorphisms between abelian varieties over an algebraically closed field do not change under extension to a larger algebraically closed field). Item 2: J.-P. Serre and J. Tate [224], §4, Theorem 6 (an abelian variety with complex multiplication, over a field with a discrete valuation with finite residue field, has potentially good reduction at every place), and [233], Chapter II, §6, for elliptic curves; the global extension M' is obtained since Z_M has good reduction outside finitely many places and good reduction persists under finite extensions; an elliptic curve over M' with good reduction at every finite place is the generic fibre of its Néron model, an abelian scheme over $\mathcal{O}_{M'}$ ([45], §1.4). The notation agrees with that of the sources. □

Theorem B.118. *Let R be a commutative ring which is free of finite rank as a $\mathbb{Z}$-module, S a scheme, A an abelian scheme over S with a ring homomorphism $\iota\colon R \to \mathrm{End}_S(A)$, and M a finitely generated projective R-module.*

(i) The functor $T \mapsto M \otimes_R A(T)$ on S-schemes is represented by an abelian scheme $M \otimes_R A$ over S. For R-linear maps $f\colon M \to N$ of finitely generated projective R-modules and R-linear homomorphisms $\phi\colon A \to B$ of abelian schemes with R-action there is a homomorphism $f \otimes \phi\colon M \otimes_R A \to N \otimes_R B$, given on T-points by $m \otimes a \mapsto f(m) \otimes \phi(a)$; in particular a commutative ring P acting R-linearly on M acts on $M \otimes_R A$ through M.

(ii) There is a canonical isomorphism of $\mathcal{O}_S$-modules $\mathrm{Lie}_S(M \otimes_R A) \cong M \otimes_R \mathrm{Lie}_S(A)$, functorial in M and A.

Since $M \otimes_R A$ is defined by the functor of (i), its base change along any morphism $S' \to S$ is $M \otimes_R A_{S'}$, compatibly with (i) and (ii).

Proof. B. Conrad, Gross–Zagier revisited, in: Heegner points and Rankin L-series, MSRI Publ. 49, Cambridge University Press (2004), 67–163, §7 (representability of $T \mapsto M \otimes_R A(T)$ by an abelian scheme, after Serre); Z. Amir-Khosravi, [5], §1, the definition of the Serre tensor construction (with the action of a commuting ring P through M and the maps $f \otimes \phi$) and the lemma of the subsection on the Lie algebra and the Tate module (the canonical isomorphism $\mathrm{Lie}_S(M \otimes_R A) \cong M \otimes_R \mathrm{Lie}_S(A)$, proved from $\mathrm{Lie}_S(A)(U) = \ker(A(U[\epsilon]) \to A(U))$ and the flatness of M). Amir-Khosravi allows non-commutative R and right modules; here R is commutative. Notation: identical. □

Theorem B.119. *Let R be a complete noetherian normal local domain, $S = \operatorname{Spec} R$ with generic point η and closed fibre S_0, and G a semiabelian scheme over S with abelian generic fibre, with Raynaud extension $0 \to \mathbb{T} \to \widetilde{G} \xrightarrow{\pi} A \to 0$. For a closed subgroup scheme $H \subseteq G$, flat and quasi-finite over S, let H^f be its largest subscheme finite over S (an open and closed subgroup), $H^\mu \subseteq H^f$ the unique finite flat subgroup of multiplicative type extending $H \cap \mathbb{T}_0$ ($\mathbb{T}_0$ the maximal torus of the closed fibre G_0), and $H^{ab} = H^f/H^\mu$.*

1. *For every $m \geq 1$ there are canonical isomorphisms of group schemes over S*

$$G[m]^\mu \cong \mathbb{T}[m], \qquad G[m]^f \cong \widetilde{G}[m], \qquad G[m]^{ab} \cong A[m],$$

   characterised by inducing on formal completions the isomorphisms given by $G^{\mathrm{for}} \cong \widetilde{G}^{\mathrm{for}}$.

2. *Let $\mathcal{L}$ be a cubical $\mathbb{G}_m$-torsor on G_η with $K(\mathcal{L})$ finite which admits a cubical extension over G (this holds if S is regular). Then $K(\mathcal{L})$ extends to a unique closed subgroup K_S of G flat and quasi-finite over S (its schematic closure); there is a unique alternating pairing $e_S^{\mathcal{L}}\colon K_S \times_S K_S \to \mathbb{G}_{m,S}$ extending $e^{\mathcal{L}}$; and (orthogonality) K_S^f is the annihilator of K_S^μ under $e_S^{\mathcal{L}}$, and the induced pairing on K_S^{ab} is a perfect duality.*

3. *If moreover the cubical torsor on $\widetilde{G}$ whose formal completion is that of the cubical extension of $\mathcal{L}$ is $\pi^*\mathcal{M}$ for a cubical $\mathbb{G}_m$-torsor $\mathcal{M}$ on A, then K_S^{ab} corresponds under the isomorphisms of (1) to $K(\mathcal{M}) \subseteq A$, and the pairing induced on it by $e_S^{\mathcal{L}}$ is $e^{\mathcal{M}}$.*

Proof. Moret-Bailly, Pinceaux de variétés abéliennes [181], Ch. IV, Théorème 2.4, in the form of K.-W. Lan, Arithmetic compactifications of PEL-type Shimura varieties (Harvard thesis, 2008), Theorem 3.4.2.6 (items 1, 2, 4, 5) with the definitions of Section 3.4.1 ((3.4.1.1)–(3.4.1.5)) and the isomorphisms (3.4.2.1)–(3.4.2.3) (after Grothendieck, SGA 7 I, Exp. IX, 7.3); the orthogonality is Grothendieck's theorem of orthogonality (SGA 7 I, Exp. IX, 2.4) in Moret-Bailly's form, which has no hypothesis on the residue characteristic. Notation: Lan writes $K_S^\natural$ for the abelian part transported to A, T for $\mathbb{T}$. □

Theorem B.120. *Let S be a scheme and A an abelian scheme over S whose dual abelian scheme $A^\vee$ exists, and let $\lambda\colon A \to A^\vee$ be a polarisation, that is, a homomorphism whose base change to every geometric point $\bar{s}$ of S is the homomorphism $\phi_{\mathcal{L}}$ of an ample line bundle $\mathcal{L}$ on $A_{\bar{s}}$. Then the abelian scheme $(A \times_S A^\vee)^4$ admits a principal polarisation, that is, a polarisation which is an isomorphism onto its dual.*

Proof. This is Zarhin's trick: [273] for abelian varieties over a field, and [181] for abelian schemes over a base, where the same construction (a principal polarisation of $(A \times_S A^\vee)^4$ built from λ and the endomorphism of A^4 given by a quaternion matrix with integer entries a, b, c, d, $a^2 + b^2 + c^2 + d^2 \equiv -1$ modulo an integer killing $\ker\lambda$) works over S. Notation: $A^\vee$, $\phi_{\mathcal{L}}$ as in Section 29.1. □

Theorem B.121. *Let k be an algebraically closed field, X an abelian variety over k and $K \subseteq X$ a finite subgroup scheme. Then there are an abelian variety Y over k and a homomorphism $p\colon X \to Y$ which is finite, flat and surjective, such that*

$$K \times_k X \longrightarrow X \times_Y X, \qquad (h, x) \longmapsto (x + h, x),$$

is an isomorphism.

Proof. [186], §12, Theorem 1 (quotient of a variety by a free action of a finite group scheme) and its application to a finite subgroup scheme of an abelian variety in the same section (X/K is an abelian variety); for reduced K, §7, Theorem 4. □

Theorem B.122. *Let k be a field, X an abelian variety over k and $\mathcal{M}$ an ample line bundle on X. Then for every integer $n \geq 3$ the line bundle $\mathcal{M}^{\otimes n}$ is very ample over k: it is generated by $H^0(X, \mathcal{M}^{\otimes n})$, and for a k-basis $s_0, \ldots, s_N$ of $H^0(X, \mathcal{M}^{\otimes n})$ the morphism $X \to \mathbb{P}^N_k$ defined by $s_0, \ldots, s_N$ is a closed immersion. In particular this holds for a complex abelian variety.*

Proof. **Algebraically closed k.** This is Lefschetz's theorem, [186], Section 17 (Theorem of Lefschetz), where the base field is an arbitrary algebraically closed field; for $k = \mathbb{C}$ see also [32], Theorem 4.5.1. Notation: X, $\mathcal{M}$ are X, L there.

Arbitrary k. Let K be an algebraic closure of k, and write X_K, $\mathcal{M}_K$ for the base changes. Then X_K is an abelian variety over K. The line bundle $\mathcal{M}_K$ is ample: some power $\mathcal{M}^{\otimes m}$ has sections defining a closed immersion $X \to \mathbb{P}^M_k$, and its base change $X_K \to \mathbb{P}^M_K$ is a closed immersion defined by the pulled-back sections of $\mathcal{M}_K^{\otimes m}$. By the algebraically closed case, $\mathcal{M}_K^{\otimes n}$ is very ample over K.

Put $V = H^0(X, \mathcal{M}^{\otimes n})$. The morphism $\operatorname{Spec} K \to \operatorname{Spec} k$ is flat and X is quasi-compact and separated, so flat base change [239] gives $V \otimes_k K \cong H^0(X_K, \mathcal{M}_K^{\otimes n})$; hence a k-basis $s_0, \ldots, s_N$ of V is a K-basis of $H^0(X_K, \mathcal{M}_K^{\otimes n})$.

Global generation. Let $\mathcal{C}$ be the cokernel of the evaluation map $V \otimes_k \mathcal{O}_X \to \mathcal{M}^{\otimes n}$, a coherent sheaf. Pull-back is right exact, so the pull-back of $\mathcal{C}$ to X_K is the cokernel of the evaluation map over K, which is zero because $\mathcal{M}_K^{\otimes n}$ is generated by its global sections. The projection $X_K \to X$ is flat and surjective, so each local ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X_K,x'}$ (x' over x) is faithfully flat, and $\mathcal{C}_x \otimes \mathcal{O}_{X_K,x'} = 0$ forces $\mathcal{C}_x = 0$. Hence $\mathcal{C} = 0$.

Closed immersion. Let $\varphi\colon X \to \mathbb{P}^N_k$ be the morphism defined by $s_0, \ldots, s_N$. Its base change along $\mathbb{P}^N_K \to \mathbb{P}^N_k$ is the morphism $X_K \to \mathbb{P}^N_K$ defined by the same sections, which is a closed immersion because they form a basis of the complete linear system of the very ample $\mathcal{M}_K^{\otimes n}$. The morphism $\mathbb{P}^N_K \to \mathbb{P}^N_k$ is the base change of $\operatorname{Spec} K \to \operatorname{Spec} k$, which is flat, surjective and quasi-compact; so $\{\mathbb{P}^N_K \to \mathbb{P}^N_k\}$ is an fpqc covering. Being a closed immersion is fpqc local on the base [240], so φ is a closed immersion. □

Theorem B.123. *Let K be a number field with ring of integers $\mathcal{O}_K$, let k be a perfect field of characteristic $p > 0$ with ring of Witt vectors $W(k)$, and let X be an abelian variety of dimension g over k with a ring homomorphism $m\colon \mathcal{O}_K \to \mathrm{End}_k(X)$, $m(1) = \mathrm{id}$. Then $H^1_{\mathrm{cris}}(X/W(k))$ is a free $\mathcal{O}_K \otimes_{\mathbb{Z}} W(k)$-module, and $H^1_{\mathrm{dR}}(X/k) = H^1_{\mathrm{cris}}(X/W(k))/p$ is a free $\mathcal{O}_K \otimes_{\mathbb{Z}} k$-module; both have rank $2g/[K:\mathbb{Q}]$. No hypothesis is made on the ramification of p in K.*

Proof. [205, Lemme 1.3] (stated there for an arbitrary number field F with ring of integers $\mathfrak{o}$, an abelian variety over a perfect field of characteristic $p > 0$ with an action of $\mathfrak{o}$: $H^1_{\mathrm{cris}}(X)$ is a free $\mathfrak{o} \otimes W(k)$-module), and the paragraph following its proof (the reduction modulo p of $H^1_{\mathrm{cris}}(X)$ is H^1_{DR}, hence a free $\mathfrak{o} \otimes k$-module). *Notation.* Rapoport's F and $\mathfrak{o}$ are our K and $\mathcal{O}_K$. The rank is forced by $\mathrm{rk}_{W(k)} H^1_{\mathrm{cris}} = 2g$ and $\mathrm{rk}_{\mathbb{Z}} \mathcal{O}_K = [K:\mathbb{Q}]$. Rapoport's proof uses only that $\mathcal{O}_K \otimes W(k)$ is a Dedekind ring (a finite product of discrete valuation rings, ramified or not) over which H^1_{cris} is torsion free, and that the trace of $m(a)$ is rational; it makes no use of an unramifiedness hypothesis. (The condition that fails at ramified p is Rapoport's condition (*) of Définition 1.1 on $\mathrm{Lie}(X)$, which is not part of this statement.) □

Theorem B.124. *Let p be a prime, S a scheme with $p\mathcal{O}_S = 0$, $F_S\colon S \to S$ its absolute Frobenius, and $f\colon A \to S$ an abelian scheme of relative dimension g. Put $\omega_{A/S} = f_*\Omega^1_{A/S}$.*

(i) The $\mathcal{O}_S$-modules $\omega_{A/S}$, $R^1 f_* \mathcal{O}_A$ and $H^1_{\mathrm{dR}}(A/S) = R^1 f_* \Omega^\bullet_{A/S}$ are locally free of ranks g, g, $2g$, and their formation commutes with every base change $S' \to S$.

(ii) (Hodge filtration.) There is an exact sequence $0 \to \omega_{A/S} \to H^1_{\mathrm{dR}}(A/S) \to R^1 f_* \mathcal{O}_A \to 0$.

(iii) (Conjugate filtration.) The conjugate spectral sequence $E_2^{a,b} = F_S^* R^a f_* \Omega^b_{A/S} \Rightarrow H^{a+b}_{\mathrm{dR}}(A/S)$ gives an exact sequence

$$0 \to F_S^* R^1 f_* \mathcal{O}_A \to H^1_{\mathrm{dR}}(A/S) \to F_S^* \omega_{A/S} \to 0.$$

(iv) Both sequences are functorial: a homomorphism $u\colon A \to B$ of abelian schemes over S induces, by pull-back, a morphism from the sequence (ii) (resp. (iii)) of B to that of A, acting on H^1_{dR} by u^* and on the outer terms of (iii) by F_S^* of the maps u^* on $R^1 f_* \mathcal{O}$ and on ω.

(v) (Duality.) Let $A^\vee$ be the dual abelian scheme and $\mathrm{Lie}(A^\vee/S)$ its Lie algebra along the zero section. There is a canonical isomorphism $\lambda_A\colon \mathrm{Lie}(A^\vee/S) \to R^1 f_* \mathcal{O}_A$, compatible with base change, and natural: for a homomorphism $u\colon A \to B$ of abelian schemes over S with transpose $u^\vee\colon B^\vee \to A^\vee$, $\lambda_A \circ \mathrm{Lie}(u^\vee) = u^* \circ \lambda_B$.

Proof. (i) and (ii): [28, Proposition 2.5.2 (i), (iii)] (local freeness and base change of $R^q f_* \Omega^p$ and of H^i_{dR}, degeneration of the Hodge–de Rham spectral sequence at E_1) and Lemme 2.5.3 (the exact sequence (2.5.3.1) $0 \to \omega_A \to H^1_{\mathrm{DR}}(A/S) \to R^1 f_* \mathcal{O}_A \to 0$ of locally free modules commuting with base change; there $\omega_A = e^* \Omega^1_{A/S}$, which equals $f_* \Omega^1_{A/S}$ as $\Omega^1_{A/S} \cong f^* \omega_A$, p. 100); over a perfect field: [193, Corollary 5.11]. (v): [28, 5.1.1] (the isomorphism λ_A from the tangent sheaf of $A^\vee$ along the zero section to $R^1 f_* \mathcal{O}_A$, defined through line bundles on $A[\varepsilon]$; see also p. 100), Lemme 5.1.4 and Théorème 5.1.6 (in whose proof $\Psi^0_A = \lambda_A$); naturality is the functoriality (5.1.3.3) of the Poincaré bundle, $(\mathrm{Id}_{B^\vee} \times u)^* \mathcal{L}_B \cong (u^\vee \times \mathrm{Id}_A)^* \mathcal{L}_A$. (iii): the conjugate spectral sequence and the identification of its E_2 terms $F^*_{\mathrm{abs}}(R^a \pi_* \Omega^b_{X/S})$, under the hypothesis that the $R^a \pi_* \Omega^b$ commute with base change, are [142, Theorem 7.4] (via Cartier's isomorphism, Theorem 7.2); the hypothesis (7.4.0) is (i). In degree one the sequence of low

degree terms is $0 \to E_2^{1,0} \to H^1_{\mathrm{dR}} \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0}$, and $H^1_{\mathrm{dR}} \to E_2^{0,1} = F_S^*\omega_{A/S}$ is surjective because on the fibre at every point s (all terms commute with base change by (i)) the ranks are $g + g = 2g$. (iv): functoriality of the de Rham complex, of the Hodge filtration and of the Cartier isomorphism. *Notation.* Katz's π, X, F_{abs} are f, A, F_S. □

Theorem B.125. *Let S be a scheme and $f\colon A \to S$ an abelian scheme with unit section e and dual abelian scheme $A^\vee$. Put $\omega_A = e^*\Omega^1_{A/S}$, let $W(M)$ denote the vector group $T \mapsto \Gamma(T, M \otimes_{\mathcal{O}_S} \mathcal{O}_T)$ of a quasi-coherent $\mathcal{O}_S$-module M, let $A_1 \subset A$ be the first infinitesimal neighbourhood of e, and $\Delta^1(A) \subset A \times_S A$ that of the diagonal, with projections p_1, p_2, and $\tau\colon A_1 \to \Delta^1(A)$ the morphism with $p_1\tau = e \circ \pi$, $p_2\tau =$ the inclusion ($\pi\colon A_1 \to S$ the structure map). For an S-scheme T, a* rigidified extension *of A_T by $\mathbb{G}_m$ is an extension $0 \to \mathbb{G}_{m,T} \to E \to A_T \to 0$ of fppf sheaves of commutative groups together with a morphism of T-pointed T-schemes $r\colon (A_T)_1 \to E$ lifting the inclusion $(A_T)_1 \subset A_T$; their isomorphism classes form a group under Baer sum, and* $\mathrm{Extrig}(A, \mathbb{G}_m)$ *is the Zariski sheaf associated with $T \mapsto$ this group. Then:*

(a) For every quasi-coherent $\mathcal{O}_S$-module M, every homomorphism of sheaves of groups $A \to W(M)$ is zero; and $f_*\mathcal{O}_A = \mathcal{O}_S$ universally.

(b) For an extension $0 \to \mathbb{G}_m \to E \to A \to 0$, E is representable and $E \to A$ is smooth. $\mathrm{Extrig}(A, \mathbb{G}_m)$ *is represented by a smooth commutative S-group scheme, and there is an exact sequence*

$$0 \to W(\omega_A) \to \mathrm{Extrig}(A, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m) \to 0,$$

whose second map forgets the rigidification and whose first map sends η to the trivial extension $\mathbb{G}_m \times A$ with the rigidification given by the pointed morphism $A_1 \to \mathbb{G}_m$ corresponding to η under $\mathrm{Hom}_{S\text{-pointed}}(A_1, \mathbb{G}_m) \cong \omega_A$. Moreover $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m) \cong A^\vee = \underline{\mathrm{Pic}}^0(A)$, an extension corresponding to the class of the line bundle of its underlying $\mathbb{G}_m$-torsor.

(c) Let $0 \to \omega_A \to E(A^\vee) \to A^\vee \to 0$ be the universal extension of $A^\vee$ by a vector group. The canonical morphism $E(A^\vee) \to \mathrm{Extrig}(A, \mathbb{G}_m)$ is an isomorphism, functorial in A; both $E(A^\vee)$ and $\mathrm{Extrig}(A, \mathbb{G}_m)$ are compatible with arbitrary base change.

(d) The $\mathcal{O}_S$-modules $H^1_{\mathrm{dR}}(A/S) = R^1 f_*\Omega^\bullet_{A/S}$ and $R^q f_*\Omega^p_{A/S}$ are locally free, their formation commutes with base change, and the Hodge–de Rham spectral sequence degenerates at E_1.

(e) Let S be affine. Every rigidified extension (E, r) of A by $\mathbb{G}_m$ carries a unique integrable connection ∇ on the $\mathbb{G}_m$-torsor E over A, compatible with the group structure, whose pull-back $\tau^*(\nabla)$ carries the unit section of π^*e^*E to r; this defines an isomorphism from the group $\mathrm{Ext}^\natural(A, \mathbb{G}_m)$ of isomorphism classes of extensions with such connections onto $\mathrm{Extrig}(A, \mathbb{G}_m)(S)$. On the trivial extension, a connection given by a unit $1 + w$ of $\mathcal{O}_{\Delta^1(A)}$, $w \in \Gamma(A, \Omega^1_{A/S})$, corresponds to the rigidification given by the unit $1 + \tau^*w$, and $\tau^*\colon \Gamma(A, \Omega^1_{A/S}) \to \Gamma(S, \omega_A)$ is an isomorphism.

(f) Let $P^\natural$ be the Zariski sheaf associated with $S' \mapsto \{$isomorphism classes of line bundles with integrable connection on $A_{S'}\}$, and $E^\natural = P^\natural \times_{\underline{\mathrm{Pic}}(A)} A^\vee$. Then $P^\natural(S) \cong \mathbb{H}^1(A, \Omega^{\bullet\times}_{A/S})$, $\Omega^{\bullet\times}_{A/S} = [\mathcal{O}_A^\times \xrightarrow{d\log} \Omega^1_{A/S} \xrightarrow{d} \Omega^2_{A/S} \to \cdots]$, by sending $(\mathcal{L}, \nabla)$ to the Čech class of its transition functions and connection forms; the split exact sequence of complexes $0 \to \Omega^\bullet_{A/S} \to \Omega^{\bullet\times}_{A_{S[\varepsilon]}/S[\varepsilon]} \to \Omega^{\bullet\times}_{A/S} \to 0$ induces a canonical isomorphism $H^1_{\mathrm{dR}}(A/S) \cong \mathrm{Lie}(P^\natural)$, and hence $H^1_{\mathrm{dR}}(A/S) \cong \mathrm{Lie}(E^\natural)$, since $\mathrm{Lie}(A^\vee) \to \mathrm{Lie}(\underline{\mathrm{Pic}}(A))$ is an isomorphism; $E^\natural$ is a smooth group scheme. Forgetting the connection gives an exact sequence $0 \to \omega_A \to$

$P^\natural \to \underline{\mathrm{Pic}}(A) \to \underline{\mathbb{H}}^2(\Omega^{\geq 1}_{A/S})$ *compatible with* $0 \to \mathbb{H}^1(\Omega^{\geq 1}_{A/S}) \to \mathbb{H}^1(\Omega^{\bullet\times}_{A/S}) \to H^1(A, \mathcal{O}_A^\times)$*, the kernel consisting of the trivial bundle with the connections given by* $\Gamma(A, \Omega^1_{A/S}) = \Gamma(S, \omega_A)$*.*

(g) Forgetting the horizontal isomorphism $s^*\mathcal{L} \cong p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}$ *is an isomorphism* $\mathrm{Ext}^\natural(A, \mathbb{G}_m) \to E^\natural$*. Together with (e) and (f) this gives an isomorphism between* $H^1_{\mathrm{dR}}(A/S)$ *and the Lie algebra of* $\mathrm{Extrig}(A, \mathbb{G}_m)$*.*

Proof. B. Mazur, W. Messing, *Universal extensions and one dimensional crystalline cohomology*, Lecture Notes in Math. 370, Springer 1974 [168], Chapter I: (a) Corollary (1.5), p. 7, and (1.9)(b), p. 9; (b) (2.6) and (2.6.1), p. 21 (the dual abelian scheme A^* exists and is isomorphic to $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$, citing Oort and Serre there), the first map as in the proof of Proposition (2.2.1), pp. 15–16, and (1.2), p. 3; the description of $\underline{\mathrm{Ext}}^1$ by torsors and line bundles, (4.1.6), p. 48, and p. 51; (c) Proposition (2.6.7), p. 24, with the base-change remark in its proof; (d) (4.1.1), p. 44; (e) Section 3, standing hypothesis "over an affine base S", p. 26; construction (3.2.1), pp. 34–36; Proposition (3.2.3)(a), p. 36, proof pp. 36–42, the trivial extension in item 3), p. 40; (f) (4.1.2), (4.1.3), p. 46, Proposition (4.1.4), pp. 46–47, Lemma (4.1.5), p. 47, Definition (4.1.6) and Proposition (4.1.7), p. 48; (g) (4.2) and Proposition (4.2.1), pp. 48–49; the conclusion is announced at the head of Section 4, p. 44 ("We shall establish an isomorphism between $H^1_{\mathrm{DR}}(A)$ and the lie algebra of $\mathrm{Extrig}(A, \mathbb{G}_m)$"). *Notation.* Mazur–Messing write A^* for $A^\vee$, $\underline{\omega}_A$ for ω_A, $\mathrm{Inf}^1(A)$ for A_1, $\tau_1(\Omega^\cdot_A)$ for $\Omega^{\geq 1}_{A/S}$, Ω^*_A for $\Omega^{\bullet\times}_{A/S}$, $\mathrm{Ext}^\natural$ (with a natural sign) for $\mathrm{Ext}^\natural$, $\underline{P}^\natural$ for $P^\natural$. □

Theorem B.126. *Let S be a locally noetherian scheme, A an abelian scheme over S and $\lambda\colon A \to A^\vee$ a homomorphism of S-group schemes which is symmetric ($\lambda^\vee = \lambda$ under the canonical identification $A \cong A^{\vee\vee}$). Then the set of points $s \in S$ such that, for a geometric point $\bar{s}$ over s, $\lambda_{\bar{s}} = \phi_{\mathcal{M}}$ for an ample line bundle $\mathcal{M}$ on $A_{\bar{s}}$, is open and closed in S. In particular, if S is connected and this holds at one point, then λ is a polarisation of A over S (it holds at every point).*

Proof. See D. Mumford, J. Fogarty, F. Kirwan, Geometric Invariant Theory, 3rd ed., Springer 1994, Ch. 6, §2, Proposition 6.10 (this statement has not been checked against the text of loc. cit.). Notation: $A^\vee$ is Mumford's $\hat{A}$, $\phi_{\mathcal{M}}$ is Mumford's $\phi_{\mathcal{M}}$ ($= \Lambda(\mathcal{M})$ read as a homomorphism to the dual). The last sentence is the case of a connected base: a non-empty open and closed subset of a connected space is the whole space; "polarisation over S" is the fibrewise condition (as in [154], printed book, Definition 1.3.2.16 with Proposition 1.3.2.15(1), p. 65; in Lan's 2008 thesis these are Definitions 1.3.2.19–1.3.2.20 with Proposition 1.3.2.18(1)). □

B.7 Arakelov geometry and heights

Bost's slope inequality for hermitian lattices (Lemma B.127) is used for the slope inequality of Part III (Section 14), and the comparison of Arakelov degree with Minkowski covolume (Theorem B.128) for the reference budget of Part IV (Section 20). The model slopes and good root towers of Part VI rest on the heights of projective schemes of Burgos Gil, Philippon and Sombra (Proposition B.129, Proposition B.132) and on Zhang's arithmetic Hilbert–Samuel theorem and semipositivity (Theorem B.130, Theorem B.131), in Sections 29–31; the facts of good towers (Section 34) use Moret-Bailly models with torsion sections and the Faltings height of elliptic curves with complex multiplication (Theorem B.133, Theorem B.134). The remaining statements are used in Appendix D: Banaszczyk's transference theorem for base settings (Section D.1); Zhang's absolute Siegel lemma, Moret-Bailly's key formula and Autissier's matrix lemma for the cusp mean (Section D.2); and Mourougane's arithmetic Segre classes and the arithmetic Chow rings of Burgos Gil, Kramer and Kühn for flat determinants (Section D.4).

Lemma B.127. *Let K be a number field, let $\overline{E}$ and $\overline{F}$ be hermitian lattices over $\mathcal{O}_K$ of positive rank, and let $\varphi\colon E_K \to F_K$ be a non-zero K-linear map with $\varphi(E) \subseteq F$. Put*

$$h(\varphi) = \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma\colon K \to \mathbb{C}} \log \|\varphi_\sigma\|,$$

where $\|\varphi_\sigma\|$ is the operator norm of $\varphi_\sigma\colon E \otimes_\sigma \mathbb{C} \to F \otimes_\sigma \mathbb{C}$ for the hermitian norms of $\overline{E}$ and $\overline{F}$ at σ. Then

$$\widehat{\mu}_{\min}(\overline{E}) \leq \widehat{\mu}_{\max}(\overline{F}) + h(\varphi),$$

and if φ is injective, $\widehat{\mu}_{\max}(\overline{E}) \leq \widehat{\mu}_{\max}(\overline{F}) + h(\varphi)$.

Proof. Bost's slope inequality [46], in the form of [47], §4.1.4, Proposition 4.5, (4.11): for hermitian vector bundles $\overline{E'}$, $\overline{F}$ over $\operatorname{Spec}\mathcal{O}_K$ with $\operatorname{rk} E' > 0$ and an injective K-linear map $\psi\colon E'_K \to F_K$,

$$\widehat{\mu}(\overline{E'}) \leq \widehat{\mu}_{\max}(\overline{F}) + h(\overline{E'}, \overline{F}, \psi),$$

where $h(\overline{E'}, \overline{F}, \psi) = [K:\mathbb{Q}]^{-1}\bigl(\sum_{\mathfrak{p}} \log \|\psi\|_{\mathfrak{p}} + \sum_\sigma \log \|\psi_\sigma\|\bigr)$ (Bost's (4.10)); Proposition 4.6 there is the version for filtered bundles. Notation: $\widehat{\mu}$, $\widehat{\mu}_{\max}$, $\widehat{\mu}_{\min}$ are those of Section 14 (Bost's normalised slopes, §4.1.1). The two forms stated follow at once. If $\psi(E') \subseteq F$, then $\|\psi\|_{\mathfrak{p}} \leq 1$ for every $\mathfrak{p}$, so $h(\overline{E'}, \overline{F}, \psi) \leq h(\psi)$, the archimedean height of the statement.

- Injective φ: let $E' \subseteq E$ be a non-zero submodule with the induced norms and $\widehat{\mu}(\overline{E'}) = \widehat{\mu}_{\max}(\overline{E})$. The restriction $\psi = \varphi|_{E'_K}$ is injective, $\psi(E') \subseteq F$, and $\|\psi_\sigma\| \leq \|\varphi_\sigma\|$; so $\widehat{\mu}_{\max}(\overline{E}) \leq \widehat{\mu}_{\max}(\overline{F}) + h(\varphi)$.
- Non-zero φ: let $E' = E/(E \cap \ker\varphi)$, a torsion-free quotient of positive rank with the quotient norms, and $\psi\colon E'_K \to F_K$ the injective map induced by φ. Then $\psi(E') = \varphi(E) \subseteq F$ and $\|\psi_\sigma\| = \|\varphi_\sigma\|$ (the quotient norm by the kernel preserves operator norms), so $\widehat{\mu}_{\min}(\overline{E}) \leq \widehat{\mu}(\overline{E'}) \leq \widehat{\mu}_{\max}(\overline{F}) + h(\varphi)$.

See also H. Chen, *Explicit uniform estimation of rational points I. Estimation of heights*, Ann. Inst. Fourier 59 (2009), Section 2. □

Theorem B.128. *Let K be a totally complex number field, let $N = [K:\mathbb{Q}] = 2s$, and choose one embedding $v\colon K \hookrightarrow \mathbb{C}$ from each complex-conjugate pair. Let $\overline{M}$ be a hermitian $\mathcal{O}_K$-lattice of rank $r \geq 1$, with norms at the conjugate embeddings obtained by complex conjugation. In*

$$\bigoplus_{v|\infty} M \otimes_{\mathcal{O}_K, v} \mathbb{C}$$

use the real Euclidean inner product $\operatorname{Re}\sum_{v|\infty}\langle\,\cdot,\cdot\,\rangle_v$. Let $\operatorname{covol}(\overline{M})$ be the covolume of the diagonal image of M for this metric. Then

$$\operatorname{covol}(\overline{M}) = 2^{-rs}\,|\operatorname{disc}(K)|^{r/2} \exp\bigl(-\widehat{\deg}(\overline{M})\bigr).$$

Proof. This is the classical degree–Minkowski-covolume formula for hermitian lattices; see [183]. Here the Arakelov degree is unnormalised, each complex pair contributes one complex summand to the Euclidean space, and that summand has its unweighted real Euclidean metric. □

Proposition B.129. *Let $\mathcal{S}$ be a projective scheme of finite type over* $\operatorname{Spec}\mathbb{Z}$. *A hermitian line bundle on $\mathcal{S}$ is* ***semipositive*** *if its metric is the uniform limit of a sequence of smooth semipositive metrics, and* ***DSP*** *if it is isometric to $\overline{\mathcal{A}} \otimes \overline{\mathcal{A}}'^{\otimes -1}$ for semipositive hermitian line bundles $\overline{\mathcal{A}}$, $\overline{\mathcal{A}}'$ on $\mathcal{S}$. If $\mathcal{S}$ has relative dimension $m-1$ over* $\operatorname{Spec}\mathbb{Z}$ *and $\overline{\mathcal{M}}_1, \dots, \overline{\mathcal{M}}_m$ are DSP hermitian line bundles on $\mathcal{S}$, then $h_{\overline{\mathcal{M}}_1,\dots,\overline{\mathcal{M}}_m}(\mathcal{S}) \in \mathbb{R}$ denotes the height of $\mathcal{S}$ with respect to them in the sense of Burgos Gil, Philippon and Sombra.*

Let $\pi\colon \mathcal{W} \to \mathcal{V}$ be a morphism between two projective schemes of finite type over $\operatorname{Spec}\mathbb{Z}$ *of relative dimensions $d+b-1$ and $b-1$, respectively, with $b, d \geq 0$. Let $\overline{\mathcal{L}}_i$, $i = 1, \dots, d$, and $\overline{\mathcal{H}}_j$, $j = 1, \dots, b$, be DSP hermitian line bundles on $\mathcal{W}$ and $\mathcal{V}$, respectively. Then*

$$h_{\pi^*\overline{\mathcal{H}}_1,\dots,\pi^*\overline{\mathcal{H}}_b,\overline{\mathcal{L}}_1,\dots,\overline{\mathcal{L}}_d}(\mathcal{W}) = \deg_{\mathcal{L}_1,\dots,\mathcal{L}_d}(\mathcal{W}_\eta)\, h_{\overline{\mathcal{H}}_1,\dots,\overline{\mathcal{H}}_b}(\mathcal{V}),$$

where $\mathcal{W}_\eta$ denotes the fibre of $\mathcal{W}$ over the generic point η of $\mathcal{V}$ and $\deg_{\mathcal{L}_1,\dots,\mathcal{L}_d}(\mathcal{W}_\eta)$ its degree with respect to the restrictions of $\mathcal{L}_1, \dots, \mathcal{L}_d$. In particular, if π is not dominant, then $h_{\pi^*\overline{\mathcal{H}}_1,\dots,\pi^*\overline{\mathcal{H}}_b,\overline{\mathcal{L}}_1,\dots,\overline{\mathcal{L}}_d}(\mathcal{W}) = 0$.

Proof. [109, Proposition 2.3] (numbered as in arXiv:1408.3222). *Notation.* Here π, $\mathcal{W}$, $\mathcal{V}$, b, d, $\overline{\mathcal{L}}_i$, $\overline{\mathcal{H}}_j$, $\mathcal{W}_\eta$, deg and h are those of loc. cit.; its "finitely generated projective schemes over $\operatorname{Spec}(\mathbb{Z})$" are rendered as projective schemes of finite type over $\operatorname{Spec}\mathbb{Z}$. Hermitian line bundles and semipositive metrics are as in §1 of loc. cit. (with reference to Definition 1.4.1 of Burgos Gil, Philippon and Sombra, Arithmetic geometry of toric varieties. Metrics, measures and heights, Astérisque 360 (2014), and to Maillot's Definition 4.5.5); DSP (difference of semipositive) is as in §2 of loc. cit., with reference to the same Definition 1.4.1; the heights are the global heights of that Astérisque volume, which for semipositive metrics on arithmetic varieties are the arithmetic intersection numbers in the sense of Gillet and Soulé (Introduction of loc. cit.). The proof in loc. cit. reduces by linearity and continuity to ample line bundles with smooth semipositive metrics and proceeds by induction on d through the arithmetic Bézout formula; the case $d = 0$ is Moriwaki, Arithmetic height functions over finitely generated fields, Invent. Math. 140 (2000), Proposition 1.3(2), for dominant π, and Theorem 1.5.11(2) of the Astérisque volume in general. □

Theorem B.130. *Let $\mathcal{X}$ be an integral scheme of dimension $D+1$, projective and flat over* $\operatorname{Spec}\mathbb{Z}$ *(its generic fibre may be singular). A Hermitian line bundle $\overline{\mathcal{M}}$ on $\mathcal{X}$ has a* smooth *metric if its pull-back along every holomorphic map from a polydisc to $\mathcal{X}(\mathbb{C})$ is smooth (for instance the restriction of a smooth metric on an ambient complex manifold), and is* relatively semipositive *if (i) $\deg(\mathcal{M}|_C) \geq 0$ for every closed integral curve C contained in a fibre of $\mathcal{X} \to \operatorname{Spec}\mathbb{Z}$, and (ii) for every holomorphic map f from a polydisc to $\mathcal{X}(\mathbb{C})$ the curvature form $\frac{\partial\bar\partial}{\pi i}\log\|l\|$ of $f^*\overline{\mathcal{M}}$ (l a local invertible section) is semipositive. For a lattice Γ in a finite-dimensional real normed space V put $\chi(\Gamma) = -\log \operatorname{vol}(V/\Gamma)$, for the Haar measure giving the unit ball volume* 1.

Let $\overline{\mathcal{L}}$, $\overline{\mathcal{N}}$ be Hermitian line bundles on $\mathcal{X}$ with smooth metrics such that $\mathcal{L}_\mathbb{Q}$ is ample and $\overline{\mathcal{L}}$ is relatively semipositive. Then, as $N \to \infty$,

$$\chi_{\sup}(\Gamma(\mathcal{X}, \mathcal{L}^{\otimes N} \otimes \mathcal{N})) = \frac{N^{D+1}}{(D+1)!}\,\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{D+1}) + o(N^{D+1}),$$

where $\chi_{\sup}$ is χ of the lattice $\Gamma(\mathcal{X}, \mathcal{L}^{\otimes N} \otimes \mathcal{N})$ in $\Gamma(\mathcal{X}_\mathbb{R}, \mathcal{L}^{\otimes N} \otimes \mathcal{N})$ with the sup norm over $\mathcal{X}(\mathbb{C})$, and $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{D+1})$ is computed on a generic resolution of singularities of $\mathcal{X}$ (it does not depend on the resolution).

Proof. [276], Theorem (1.4), with the definitions (1.1)–(1.3) there. Notation: Zhang's arithmetic variety of dimension d is our $\mathcal{X}$ with $d = D+1$; his $c_1(L)^d$ is $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{D+1})$; his $\chi_{\sup}$ is χ for

the sup norm as above. Zhang states that the case of a regular generic fibre and a relatively ample $\overline{\mathcal{L}}$ with positive curvature is due to Gillet and Soulé (his Lemma (1.5)) and reduces the general case to it by generic resolution and Minkowski's theorem. □

Theorem B.131. *(A) Let X be a projective variety over $\mathbb{C}$ and L an ample line bundle on X with a continuous metric $\|\cdot\|$. The metric is called **semiample** if for every $\varepsilon > 0$ there is $n \geq 1$ such that for every $x \in X(\mathbb{C})$ there is a non-zero section l of $L^{\otimes n}$ with $\|l\|_{\sup} \leq e^{n\varepsilon}\|l\|(x)$. If there are an embedding i of X into a compact complex manifold Y and a Hermitian line bundle $\overline{M}$ on Y with M ample and semipositive curvature form such that $i^*\overline{M}$ is isometric to $(L, \|\cdot\|)$, then the metric is semiample.*

*(B) Let $\mathcal{X}$ be an arithmetic variety of absolute dimension $D+1$ and $\overline{\mathcal{L}}$ a Hermitian line bundle on $\mathcal{X}$ with smooth metric such that $\mathcal{L}_{\mathbb{Q}}$ is ample, $\overline{\mathcal{L}}$ is relatively semipositive, and the metric of $\overline{\mathcal{L}}$ on $\mathcal{L}_{\mathbb{C}}$ is semiample (i.e. $\overline{\mathcal{L}}$ is **relatively semiample**). If $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}) \cdot [D_x]) \geq 0$ for the Zariski closure D_x of every $x \in \mathcal{X}(\overline{\mathbb{Q}})$, then $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{D+1}) \geq 0$.*

Proof. [276]: (A) is the archimedean case of Theorem (3.5), with the definition (3.1) of a semiample metrized line bundle (the equivalent form with sections is the one stated in (3.1)); (B) is Lemma (5.4), under the assumptions of Theorem (5.2) ($L_{\mathbb{Q}}$ ample, $\overline{L}$ relatively semiample with smooth metric), with the definition of relatively semiample in (5.1). Notation: Zhang's arithmetic variety of dimension d is $\mathcal{X}$ with $d = D+1$; his height $h_L(x) = \deg(\overline{L}|_{D_x})/\deg(D_x)$; his $c_1(\overline{L})^d$ is $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}})^{D+1})$. For an arithmetic variety over $\mathbb{Z}$ the archimedean metric is a conjugation-invariant metric over $\mathbb{C}$ (Zhang's $K_0 = \mathbb{R}$, $K = \mathbb{C}$). □

Proposition B.132. *Let K be $\mathbb{R}$, $\mathbb{C}$ or a field complete with respect to the absolute value of a non-trivial discrete valuation, with valuation ring K°, uniformiser ϖ and residue field k; let X be a projective variety over K and X^{an} its analytification (for $K = \mathbb{R}$, the complex space $X^{\mathrm{an}}_{\mathbb{C}}$).*

1. ***(Local heights.)** To each d-dimensional cycle Y of X and each family $(\overline{L}_i, s_i)$, $i = 0, \ldots, d$, of DSP metrized line bundles with rational sections meeting Y properly, the local height $h_{\overline{L}_0,\ldots,\overline{L}_d}(Y; s_0, \ldots, s_d) \in \mathbb{R}$ is determined inductively by $h(\varnothing) = 0$ and*

$$h_{\overline{L}_0,\ldots,\overline{L}_d}(Y; s_0, \ldots, s_d) = h_{\overline{L}_0,\ldots,\overline{L}_{d-1}}(Y \cdot \operatorname{div} s_d; s_0, \ldots, s_{d-1}) - \int_{X^{\mathrm{an}}} \log \|s_d\|_d \, c_1(\overline{L}_0) \wedge \cdots \wedge c_1(\overline{L}_{d-1}) \wedge \delta_Y.$$

   For smooth metrics in the Archimedean case, $c_1(\overline{L}) = \partial\bar{\partial} \log \|s\|^2$ for a non-vanishing local section s, and $\delta_Y(\omega) = (2\pi i)^{-d} \int_{Y^{\mathrm{an}}} \omega$.

2. ***(Algebraic metrics.)** If K is non-Archimedean, Y is prime, and the metrics of the $\overline{L}_i$ are algebraic, induced by proper models $(\mathcal{X}, \mathcal{L}_i, e_i)$ over K° with a common $\mathcal{X}$, then the recursion in (1) reads*

$$h_{\overline{L}_0,\ldots,\overline{L}_d}(Y; s_0, \ldots, s_d) = h_{\overline{L}_0,\ldots,\overline{L}_{d-1}}(Y \cdot \operatorname{div} s_d; s_0, \ldots, s_{d-1}) - \frac{\log|\varpi|}{e_0 \cdots e_d} \sum_{V \in \widetilde{\mathcal{Y}}_o^{(0)}} \operatorname{ord}_V(s_d^{\otimes e_d}) \deg_{\mathcal{L}_0,\ldots,\mathcal{L}_{d-1}}(V),$$

   where $\widetilde{\mathcal{Y}}$ is the normalisation of the closure $\mathcal{Y}$ of Y in $\mathcal{X}$ and $\widetilde{\mathcal{Y}}_o^{(0)}$ the set of irreducible components of its special fibre.

3. ***(Global heights.)*** *Let X be a projective variety over $\mathbb{Q}$, with the places v of $\mathbb{Q}$ and the absolute values $|\cdot|_\infty$ and $|\cdot|_p$, $|p|_p = p^{-1}$, all with weight 1. Let $\overline{L}_0, \dots, \overline{L}_d$ be line bundles on X with DSP metrics at every place whose metrics at all but finitely many primes are induced by one proper model over the ring of S-integers. For a d-dimensional cycle Y of X and rational sections $s_0, \dots, s_d$ meeting Y properly, $\sum_v h_{\overline{L}_{0,v},\dots,\overline{L}_{d,v}}(Y; s_0, \dots, s_d)$ is a real number that does not depend on the sections; it is the global height $h_{\overline{L}_0,\dots,\overline{L}_d}(Y)$.*

Proof. [108] (Burgos Gil, Philippon, Sombra, Arithmetic geometry of toric varieties. Metrics, measures and heights, Astérisque 360 (2014)), Chapter 1. Item (1) is the definition of local heights, Definition 1.4.11 (number as cited in [109], §2), with the normalisations of §1.1 (definition of the first Chern form and of the signed measure $c_1(\overline{L}_0) \wedge \cdots \wedge c_1(\overline{L}_{d-1}) \wedge \delta_Y$ for smooth metrics; the remark reducing varieties over $\mathbb{R}$ to $X_\mathbb{C}$). Item (2) is the Remark following Definition 1.4.11, obtained there from the description of algebraic metrics and of the measure $c_1(\overline{L}_0)\wedge\cdots\wedge c_1(\overline{L}_{d-1})\wedge\delta_Y$ attached to algebraic metrics (§1.3, Definition of that measure through the normalisation $\widetilde{\mathcal{Y}}$). Item (3) is the definition of the global height over an adelic field (§1.5), with $\mathbb{Q}$ as an adelic field with weights 1; the independence of the sections holds by the product formula and the theorem on the properties of local heights, item (3) (change of the section by a rational function changes the local height by $\log|f(Z)|_v$), and every cycle is integrable for such quasi-algebraic metrics (§1.5, proposition on quasi-algebraic metrics). *Notation.* Here ϖ, $\widetilde{\mathcal{Y}}_o^{(0)}$, $\deg_{\mathcal{L}_0,\dots,\mathcal{L}_{d-1}}$ and ord_V are those of loc. cit. □

Theorem B.133. *Let K_0 be a number field, A an abelian variety of dimension $g \geq 1$ over K_0, L_A a symmetric ample line bundle on A defining a principal polarisation, $r_0 \in L_A(0)$ non-zero, and $r \geq 2$ an **even** integer.*

1. *(Existence.) There are a finite extension K/K_0 with $A[r^2] \subseteq A(K)$ and a Moret-Bailly model of $(A_K, L_A \otimes K, r_0 \otimes 1, A[r^2](K))$ over $\mathcal{O}_K$.*

2. *(Base change.) If $(\mathcal{G}, i, \mathcal{M}, (\|\cdot\|_\sigma), \varphi, u, (\varepsilon_P)_{P\in F})$ is a Moret-Bailly model of $(A_K, L_A \otimes K, r_0 \otimes 1, F)$ over $\mathcal{O}_K$ and K'/K is finite, the base changes of all the data form a Moret-Bailly model of $(A_{K'}, L_A\otimes K', r_0\otimes 1, F)$ over $\mathcal{O}_{K'}$, whose Moret-Bailly lattice is $\Lambda\otimes_{\mathcal{O}_K}\mathcal{O}_{K'}$, Λ the Moret-Bailly lattice of the given model.*

3. *(Independence.) Two Moret-Bailly models of $(A_K, L_A \otimes K, r_0 \otimes 1, F)$ over $\mathcal{O}_K$ have the same Moret-Bailly lattice in $H^0(A_K, L_A \otimes K)$.*

4. *(Translates.) Let $(\mathcal{G}, i, \mathcal{M}, (\|\cdot\|_\sigma), \varphi, u, (\varepsilon_x)_{x\in A[r^2](K)})$ be a Moret-Bailly model of $(A_K, L_A\otimes K, r_0 \otimes 1, A[r^2](K))$ over $\mathcal{O}_K$. There are isomorphisms of line bundles on $\mathcal{G}$*

$$j\colon [r]^*\mathcal{M} \to \mathcal{M}^{\otimes r^2}, \qquad i_x\colon t_{\varepsilon_x}^*\mathcal{M}^{\otimes r^2} \to \mathcal{M}^{\otimes r^2} \quad (x \in A[r^2](K)),$$

t_{ε_x} the translation by the section ε_x, which are isometric for the cubist metrics (and their tensor powers) at every $\sigma\colon K \to \mathbb{C}$, and such that, with $\psi_x(s) = i_x\big(t_{\varepsilon_x}^* j([r]^* s)\big)$, for every set $\Sigma \subseteq A[r^2](K)$ of representatives of $A[r^2]/A[r]$ the map

$$H^0(\mathcal{G}_K, \mathcal{M}_K)^\Sigma \to H^0(\mathcal{G}_K, \mathcal{M}_K^{\otimes r^2}), \qquad (s_x)_{x\in\Sigma} \mapsto \sum_{x\in\Sigma} \psi_x(s_x),$$

is an isomorphism of K-vector spaces.

Proof. F. Pazuki, Theta height and Faltings height, Bull. Soc. Math. France 140 (2012), 19–49 (arXiv:0907.1458): item 1 is Theorem 3.4 (i) for $F = A_{r^2}(\overline{\mathbb{Q}})$ together with Definition 3.1 (the sections ε_P, $P \in F$, force $F \subseteq A(K)$) and Remark 3.3; item 2 is Theorem 3.4 (iv), the lattice statement being the isometric isomorphism $j^*\pi_*\overline{\mathcal{L}} \to \tilde{\pi}_*\tilde{\overline{\mathcal{L}}}$ there; item 3 is Theorem 3.4 (iii) (the canonical isomorphisms $(\pi_*\mathcal{L})_{\overline{\mathbb{Q}}} \simeq H^0(A,L) \simeq (\pi'_*\mathcal{L}')_{\overline{\mathbb{Q}}}$ extend to an isometric isomorphism $\pi_*\overline{\mathcal{L}} \simeq \pi'_*\overline{\mathcal{L}'}$); item 4 is Proposition 5.1 (i) (the isomorphism $j\colon [r]^*\mathcal{L} \simeq \mathcal{L}^{\otimes r^2}$ compatible with the rigidification, §2.3.1, and a good choice of the $i_x\colon t_x^*\mathcal{L}^{\otimes r^2} \simeq \mathcal{L}^{\otimes r^2}$ extend to isometric isomorphisms of Hermitian line bundles over $\mathcal{G}$) together with Proposition 2.5 (the map $\Phi = \sum_{x\in\Sigma}\psi_x$ is an isomorphism; proof by the irreducibility of the representation of the theta group of $L^{\otimes r^2}$, Mumford, On the equations defining abelian varieties I, Invent. Math. 1 (1966)). Pazuki attributes Theorem 3.4 to J.-B. Bost, Intrinsic heights of stable varieties and abelian varieties, Duke Math. J. 82 (1996), §4.3.2, and L. Moret-Bailly [181]. Notation: Pazuki's MB-model $(\pi\colon \mathcal{A} \to \operatorname{Spec}\mathcal{O}_K, i, \overline{\mathcal{L}}, \varphi, (\varepsilon_P))$ is our $(\mathcal{G}, i, \mathcal{M}$ with the cubist metrics$, \varphi, (\varepsilon_P))$; his φ is compatible with the rigidification ("since $\mathcal{L}_K$ is cubist, it is automatically rigidified", §5.1), which is our generator u of (Mb3); his $\pi_*\mathcal{L}$ with the identification of its generic fibre with $H^0(A,L)$ is our Moret-Bailly lattice; his level r is even throughout (§2.3.1, Definition 3.2, §5.1). □

Theorem B.134. *Let K be an imaginary quadratic field of discriminant D_K, χ_K the quadratic Dirichlet character of K (the Kronecker symbol $(D_K/\cdot)$, primitive modulo $|D_K|$) and $L(s,\chi_K)$ its Dirichlet L-function. Let $k \subseteq \overline{\mathbb{Q}}$ be a number field and A an elliptic curve over k whose ring of $\overline{\mathbb{Q}}$-endomorphisms is isomorphic to $\mathcal{O}_K$ and whose Néron model $\mathcal{A} \to \operatorname{Spec}\mathcal{O}_k$ is an abelian scheme. For a non-zero rational section s of $\pi_*\Omega^1_{\mathcal{A}/\mathcal{O}_k}$ put*

$$h^{\mathrm{Falt}}(A) = \frac{1}{[k:\mathbb{Q}]}\sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(s)\log \mathrm{N}(\mathfrak{p}) - \frac{1}{2[k:\mathbb{Q}]}\sum_{\sigma\colon k\to\mathbb{C}} \log\Big|\int_{A^\sigma(\mathbb{C})} s^\sigma \wedge \overline{s^\sigma}\Big|$$

(independent of s). Then

$$h^{\mathrm{Falt}}(A) = -\frac{1}{2}\,\frac{L'(0,\chi_K)}{L(0,\chi_K)} - \frac{1}{4}\log|D_K| - \frac{1}{2}\log(2\pi).$$

Proof. This is [8], Theorem A (p. 2; proved as Theorem 9.5.5), in the case $d = 1$: $E = K$, $F = \mathbb{Q}$, $D_F = 1$, χ the quadratic character of $K/\mathbb{Q}$ with its L-function without archimedean factors, and the left side the average of $h^{\mathrm{Falt}}_{(K,\Phi)}$ over the two CM types of K. The normalisation of the Faltings height is their Definition 9.1.4 (with h^{Falt}_f and h^{Falt}_∞ as displayed above), and by their Theorem 9.1.5 (Colmez, Théorème 0.3 of *Périodes des variétés abéliennes à multiplication complexe*, Ann. of Math. 1993) $h^{\mathrm{Falt}}_{(K,\Phi)} = h^{\mathrm{Falt}}(A)$ for every A with complex multiplication by $\mathcal{O}_K$ of type Φ. *Notation.* Here A with its $\mathcal{O}_K$-action has one of the two CM types Φ, and A with the action precomposed with complex conjugation of K has the other, so both values $h^{\mathrm{Falt}}_{(K,\Phi)}$ equal $h^{\mathrm{Falt}}(A)$ and so does their average. For $d = 1$ the statement is the Chowla–Selberg formula in Colmez's form (their §1.1). □

Theorem B.135. *Let $n \geq 1$, let $\mathbb{R}^n$ carry its Euclidean inner product $u\cdot v$ and norm $\|\cdot\|$, let $\Gamma \subset \mathbb{R}^n$ be a lattice (the subgroup generated by n linearly independent vectors) and $\Gamma^* = \{u \in \mathbb{R}^n : u\cdot v \in \mathbb{Z}$ for all $v \in \Gamma\}$ its dual lattice. For $1 \leq i \leq n$ let $\lambda_i(\Gamma) = \min\{s > 0 : \dim \operatorname{span}_{\mathbb{R}}(\Gamma \cap \{\|v\| \leq s\}) \geq i\}$ be the i-th successive minimum. Then*

$$\lambda_i(\Gamma)\,\lambda_{n-i+1}(\Gamma^*) \leq n \qquad (1 \leq i \leq n).$$

Proof. [16]: the quantity $\xi_n = \sup_\Gamma \max_{1\le i\le n} \lambda_i(\Gamma)\lambda_{n-i+1}(\Gamma^*)$ satisfies $\xi_n \le n$ for every n. Notation: the paper writes L, L^* for Γ, Γ^*; the definitions of the dual lattice and of the successive minima are those of the paper. □

Theorem B.136. *Let K be a number field and $\overline{M} = (M, (\|\cdot\|_\sigma)_\sigma)$ a Hermitian lattice over $\mathcal{O}_K$ of rank $n \ge 1$. For a finite extension K'/K let $\overline{M}_{K'}$ be the Hermitian lattice $M \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ over $\mathcal{O}_{K'}$ with the norms $\|\cdot\|_{\sigma'} = \|\cdot\|_{\sigma'|_K}$ (extended to $M \otimes_{\mathcal{O}_K,\sigma'|_K} \mathbb{C}$), and for $0 \ne v \in M \otimes_{\mathcal{O}_K} K'$ let $\overline{M}_{K'}(v) = \big(K'v \cap (M \otimes_{\mathcal{O}_K} \mathcal{O}_{K'})\big)$ with the induced norms (a Hermitian lattice of rank one). Put $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Then for every $\varepsilon > 0$ there are a finite extension K'/K and $0 \ne v \in M \otimes_{\mathcal{O}_K} K'$ with*

$$\widehat{\mu}\big(\overline{M}_{K'}(v)\big) \;\ge\; \widehat{\mu}(\overline{M}) - \tfrac{1}{2}\,(H_n - 1) - \varepsilon.$$

Proof. This is the first-minimum case of Zhang's theorem on the successive minima of heights over $\overline{\mathbb{Q}}$ [276] (see also Roy–Thunder, An absolute Siegel's lemma, J. reine angew. Math. 476 (1996)), with the exact constant of Gaudron and Rémond (É. Gaudron, G. Rémond, Corps de Siegel, J. reine angew. Math. 726 (2017), 187–247, §5.2), in the form stated in É. Gaudron, Minima and slopes of rigid adelic spaces, Chapter II of Arakelov Geometry and Diophantine Applications, Lecture Notes in Math. 2276 (2021), §4.2: for a rigid adelic space E over an algebraic extension K of $\mathbb{Q}$, $c_I(n,K) = \sup_{\dim E=n} \lambda_1(E) H(E)^{-1/n} = \sup_{\dim E=n} \lambda_1(E) \exp \mu(E)$, where $\lambda_1(E) = \inf\{H_E(x) : x \in E \smallsetminus \{0\}\}$, $H_E(x) = \exp \int_{V(K)} \log \|x\|_{E,v}\, d\sigma(v)$ and $\mu(E) = -\frac{1}{n} \log H(E)$; and $c_I(n,\overline{\mathbb{Q}}) = \exp(\frac{H_n-1}{2})$ (op. cit. p. 61, case (2), attributed to [Gaudron–Rémond, §5.2]). Hence $\lambda_1(E) \le \exp(\frac{H_n-1}{2} - \mu(E))$ for every E of dimension n over $\overline{\mathbb{Q}}$. *Notation.* The Hermitian lattice $\overline{M}$, extended to $\overline{\mathbb{Q}}$, is the rigid adelic space E with the lattice norms at the finite places and $\|\cdot\|_\sigma$ at the infinite ones (Gaudron's measure σ gives each embedding weight $1/[K' : \mathbb{Q}]$, slopes and heights being invariant under scalar extension, op. cit. p. 43); $\mu(E) = \widehat{\mu}(\overline{M})$; for v as in the statement, $\log H_E(v) = -\widehat{\mu}(\overline{M}_{K'}(v))$ (product formula applied to a generator of the rank-one lattice, normalised by $[K' : \mathbb{Q}]$); $\lambda_1(E)$ is the infimum of $H_E(v)$ over all $v \ne 0$, each defined over a finite extension, whence the ε. □

Theorem B.137. *Let K_0 be a number field, A an abelian variety of dimension $g \ge 1$ over K_0 and L a symmetric ample line bundle on A defining a principal polarisation. Then there are a finite extension K/K_0 and the following data:*

- *a smooth separated commutative group scheme $\pi\colon \mathcal{G} \to \operatorname{Spec} \mathcal{O}_K$ of finite type, every connected component of every fibre of which is an extension of an abelian variety by a torus, with an isomorphism $\mathcal{G}_K \cong A_K$;*
- *a line bundle $\mathcal{M}$ on $\mathcal{G}$ with a cubical structure, and an isomorphism $\varphi\colon \mathcal{M}|_{\mathcal{G}_K} \cong L_K$;*
- *for every $\sigma\colon K \to \mathbb{C}$ the **cubist metric** $\|\cdot\|_\sigma$ on $\mathcal{M}_\sigma$: the Hermitian metric for which the trivialisation of $\Theta(\mathcal{M})_\sigma$ given by the cubical structure has norm 1; for it $\varepsilon^*\mathcal{M}$ (ε the unit section) with the induced metrics is isometric to $\mathcal{O}_K$ with the trivial metrics, and the curvature form of $\|\cdot\|_\sigma$ is invariant under translations of $A_\sigma(\mathbb{C})$;*
- *a closed subscheme of $\mathcal{G}$, finite and flat over $\mathcal{O}_K$, whose generic fibre corresponds to $A_K[2]$ under $\mathcal{G}_K \cong A_K$;*

such that for every finite extension K'/K, with $\mathcal{G}' = \mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K'}$ and $\mathcal{M}'$ the pull-back of $\mathcal{M}$: the $\mathcal{O}_{K'}$-module $E' = H^0(\mathcal{G}', \mathcal{M}')$ is projective of rank one, $E' \otimes_{\mathcal{O}_{K'}} K' = H^0(A_{K'}, \mathcal{M}|_{A_{K'}})$, and

the Hermitian lattice $\overline{E'}$ *with the norms*

$$\|s\|^2_{L^2,\sigma'} = \int_{A_{\sigma'}(\mathbb{C})} \|s(x)\|^2_{\sigma'}\, d\mu_{\sigma'}(x) \qquad (\sigma'\colon K' \to \mathbb{C},\ \mu_{\sigma'} \text{ the Haar probability measure})$$

satisfies

$$\frac{\widehat{\deg}\,\overline{E'}}{[K':\mathbb{Q}]} = -\frac{1}{2}\,h_F(A) - \frac{g}{4}\log(2\pi).$$

Here $h_F(A)$ *is the stable Faltings height in Deligne's normalisation: for a finite extension* K''/K_0 *and a semiabelian scheme* $\mathcal{S}$ *over* $\mathcal{O}_{K''}$ *with generic fibre* $A_{K''}$, $h_F(A) = \widehat{\deg}\,\overline{\omega}_{\mathcal{S}}/[K'':\mathbb{Q}]$, *where* $\omega_{\mathcal{S}} = \varepsilon^*\Omega^g_{\mathcal{S}/\mathcal{O}_{K''}}$ *carries the norms* $\|\alpha\|^2_{\sigma''} = \frac{i^{g^2}}{(2\pi)^g}\int_{A_{\sigma''}(\mathbb{C})} \alpha\wedge\overline{\alpha}$; *it does not depend on* K'' *and* $\mathcal{S}$.

Proof. F. Pazuki, Theta height and Faltings height, Bull. Soc. Math. France 140 (2012), 19–49 (arXiv:0907.1458), Definition 3.1 (an MB-model of $(A, L, \varnothing)$: in particular a subscheme of $\mathcal{A}$, finite and flat over $\mathcal{O}_K$, whose generic fibre is the Mumford group $K(L^2)$, which is $A[2]$ for a principal polarisation), Theorem 3.4 (i) existence of an MB-model $(\pi\colon \mathcal{A} \to \operatorname{Spec}\mathcal{O}_K, i, \overline{\mathcal{L}}, \varphi)$, (iv) its base change to $\mathcal{O}_{K'}$ is again an MB-model, (v) Arakelov slope: $\pi_*\overline{\mathcal{L}}$ is semistable and $\widehat{\deg}\,\pi_*\overline{\mathcal{L}}/([K:\mathbb{Q}]\rho(L)) = -\frac{1}{2}h_F(A) + \frac{1}{4}\log(\rho(L)/(2\pi)^g)$, here with $\rho(L) = c_1(L)^g/g! = 1$ (principal polarisation); §3.1 for semistable group schemes, cubist Hermitian line bundles (the section of $\mathcal{D}_3(\overline{\mathcal{L}})$ of norm one; $\varepsilon^*\overline{\mathcal{L}} \simeq \overline{\mathcal{O}}$; translation invariant c_1), local freeness of $\pi_*\mathcal{L}$ (of rank $\rho(L)$) and the L^2 metric for the normalised Haar measure; Definition 2.1 and §2.2 for h_F in Deligne's normalisation. Pazuki attributes (i)–(v) to J.-B. Bost, Intrinsic heights of stable varieties and abelian varieties, Duke Math. J. 82 (1996), 21–70, §4.3.2, resting on L. Moret-Bailly, Pinceaux de variétés abéliennes [181] and Sur l'équation fonctionnelle de la fonction thêta de Riemann, Compositio Math. 75 (1990). In the case of good reduction it is [46, Théorème 4.2 ii)]. Notation: Pazuki's $\pi_*\mathcal{L}$, $\overline{\mathcal{L}}$, i are E' (over K'), $(\mathcal{M}, \|\cdot\|_\sigma)$ and the isomorphism $\mathcal{G}_K \cong A_K$; the identification of the generic fibre of $\pi_*\mathcal{L}$ with $H^0(A, L)$ is his Theorem 3.4 (iii) together with flat base change. □

Theorem B.138. *Let* K *be a number field and* A *an abelian variety of dimension* $g \geq 1$ *over* K *with a principal polarisation. For* $\sigma\colon K \to \mathbb{C}$ *write* $A_\sigma(\mathbb{C}) = \operatorname{Lie}(A_\sigma)/\Pi_\sigma$, *let* H_σ *be the positive definite Hermitian form on* $\operatorname{Lie}(A_\sigma)$ *whose imaginary part on* Π_σ *is the Riemann form of the polarisation, and* $\rho_\sigma = \min_{0\neq l\in\Pi_\sigma} H_\sigma(l,l)^{1/2}$. *Then for every* $\varepsilon \in]0,1[$

$$h_F(A) - \frac{g}{2}\log\pi + \frac{g}{2}\log\Big(\frac{2\pi^2}{\varepsilon}\Big) \;\geq\; \frac{(1-\varepsilon)\pi}{6[K:\mathbb{Q}]}\sum_{\sigma\colon K\to\mathbb{C}} \frac{1}{\rho_\sigma^2}.$$

Proof. P. Autissier [14, Corollaire 1.4] (a consequence of his Théorème 1.1, the lower bound $h_{\mathrm{Fa}}(A) \geq \frac{1}{[K:\mathbb{Q}]}\sum_\sigma\big(\frac{\pi}{6\rho_\sigma^2} + g\ln(\kappa\rho_\sigma\sqrt{g})\big)$ with ρ_σ truncated at $\sqrt{\pi/(3g)}$, which rests on Bost's lower bound for h_{Fa}, proved in the appendix of [102]). Notation: Autissier's diamètre d'injectivité $\rho(A_\sigma; L_\sigma) = \min_{\gamma\neq 0}\sqrt{H_L(\gamma;\gamma)}$ with H_L the Riemann form on T_{A_σ} is our ρ_σ; his height h_{Fa} is the stable Faltings height with the norm $\|\varphi\|^2 = \frac{i^{g^2}}{2^g}\int\varphi\wedge\overline{\varphi}$ on ω; Deligne's normalisation uses $\frac{i^{g^2}}{(2\pi)^g}$, so $\|\cdot\|_{\mathrm{Del}} = \pi^{-g/2}\|\cdot\|_{\mathrm{Aut}}$ and $h_{\mathrm{Fa}} = h_F - \frac{g}{2}\log\pi$, which gives the displayed form. □

Theorem B.139. *Let* $\mathcal{Y}$ *be a regular arithmetic variety over* $\mathbb{Z}$, $\overline{W} = (W, h)$ *a Hermitian vector bundle of rank* 2 *on* $\mathcal{Y}$ *(smooth metric invariant under complex conjugation),* $\pi\colon \mathbb{P}(W) \to \mathcal{Y}$ *the bundle of lines in* W, $\mathcal{S} \subseteq \pi^*W$ *the tautological line with the metric induced by* h, *and*

$\hat\zeta = \widehat{c}_1(\overline{\mathcal{S}}^\vee) \in \widehat{\mathrm{CH}}^1(\mathbb{P}(W))_\mathbb{Q}$. *Let* $c_1 = c_1(W,h)$, $c_2 = c_2(W,h)$ *be the Chern forms on* $\mathcal{Y}(\mathbb{C})$ *and define forms* s'_m *of type* (m,m) *and* R_m *of type* $(m-1,m-1)$ *by*

$$\sum_{m\ge 0} s'_m\, t^m = \left(1 + c_1 t + c_2 t^2\right)^{-1}, \qquad R_0 = 0, \qquad R_m = -\sum_{p+q=m-1} \frac{s'_p\, s'_q}{q+1} \quad (m \ge 1), \tag{B.2}$$

and classes $\hat s_m \in \widehat{\mathrm{CH}}^m(\mathcal{Y})_\mathbb{Q}$ *by* $\sum_m \hat s_m t^m = \left(1 + \widehat{c}_1(\overline{W})t + \widehat{c}_2(\overline{W})t^2\right)^{-1}$. *Then for every* $m \ge 0$

$$\pi_*(\hat\zeta^{\,1+m}) = \hat s_m - a(R_m) \qquad \text{in } \widehat{\mathrm{CH}}^m(\mathcal{Y})_\mathbb{Q}. \tag{B.3}$$

Proof. This is [184], Theorem 4 (§8.1),

$$\widehat{c}_t(\overline{E}^\star)\cdot \widehat{s}_t(\overline{E}) = 1 \quad \text{with} \quad \widehat{s}_m = \widehat{s}'_m + a(R_m)$$
$$\text{and} \quad \widehat{s}'_m = \widehat{\pi}_*(\widehat{c}_1(\mathcal{O}_{\overline{E}}(1))^{r-1+m}),$$

in rank $r = 2$, together with Theorem 3 and Proposition 6 there, for the base $\operatorname{Spec}\mathbb{Z}$ ($K = \mathbb{Q}$). *Notation.* Mourougane's $\mathbb{P}(E)$ is the variety of hyperplanes of E; we take $E = W^\vee$ with the dual metric, so that $\mathbb{P}(E)$ is the bundle of lines in W, $\mathcal{O}_{\overline{E}}(1)$ (quotient of π^*E with the quotient metric) is $\overline{\mathcal{S}}^\vee$, and $\overline{E}^\star = \overline{W}$. His total Segre form satisfies $\left(\sum t^m c_m(E^\star,h)\right)\left(\sum t^m s'_m(E,h)\right) = 1$ pointwise (Proposition 6), which is the first identity of (B.2). His $R_t = \sum_{m\ge0} t^m R_{m+1} = s'_t S_t$ with $S_{m+1} = -\sum_{a+b+c=m} \mathcal{H}^{r-1+c}_{r-1-a-b} c_a(E^\star,h) s^b_c(E,h)$ (Theorem 3); for $r = 2$ only $a = b = 0$ contributes ($\mathcal{H}^{\cdot}_0 = 0$ and $\mathcal{H}^{\cdot}_a$ is an empty sum for $a < 0$), $\mathcal{H}^{1+m}_1 = \mathcal{H}_1 \binom{1}{1}\binom{1+m}{1}^{-1} = \frac{1}{m+1}$ and $s^0_m = s'_m$, so $S_{m+1} = -\frac{1}{m+1} s'_m$ and $R_{m+1} = -\sum_{p+q=m} \frac{s'_p s'_q}{q+1}$, which is (B.2) and the second display of the rank-2 example on p. 21 of the paper. Degree 0: $\widehat{s}'_0 = \pi_*(\hat\zeta) = 1$.

Framework. The push-forward π_* of the statement and the rules $a(\eta)\cdot x = a(\eta \wedge \omega(x))$ and $\widehat{\deg}\, a(\eta) = \frac12 \int \eta$ used with it are (GA2), (GA3) and (GA5) of Lemma D.89, which applies to π ($\mathcal{Y}$ and $\mathbb{P}(W)$ are regular, flat and projective over $\mathbb{Z}$, π is smooth of relative dimension 1) and to the structure morphisms, and which is proved from [106]; its push-forward is the push-forward of Gillet and Soulé ([110]) that [184], §8, takes as known. The arithmetic Chern classes $\widehat{c}_1(\overline{W})$, $\widehat{c}_2(\overline{W})$ of the statement, and $\hat\zeta = \widehat{c}_1(\overline{\mathcal{S}}^\vee)$ (rank 1), are the arithmetic Chern classes of Gillet and Soulé, which [184], §8, takes as known; they are the classes of Lemma D.90, obtained from [107]. □

Proposition B.140. *Let* $\mathbb{Z}$ *carry its embedding into* $\mathbb{C}$ *and complex conjugation. Call a regular scheme* X*, flat and quasi-projective over* $\operatorname{Spec}\mathbb{Z}$*, of relative dimension* d*, an arithmetic variety over* $\mathbb{Z}$*; put* $X_\infty = X(\mathbb{C})$ *and let* F_∞ *be complex conjugation on* X_∞*. For such* X *there are groups* $\widehat{\mathrm{CH}}^p(X, \mathcal{D}_{\log})$ *of classes* $[y, \mathfrak{g}_y]$ *of pairs of a* p*-codimensional cycle* y *and a Green form* $\mathfrak{g}_y = (\omega_y, \widetilde{g}_y)$ *for the class of* y*, with maps*

$$\zeta[y,\mathfrak{g}_y] = [y] \in \mathrm{CH}^p(X), \qquad \omega[y,\mathfrak{g}_y] = \omega_y, \qquad \mathrm{a}(\widetilde{x}) = [0, \mathrm{a}(\widetilde{x})] \quad (\widetilde{x} \in \widetilde{\mathcal{D}}^{2p-1}_{\log}(X,p)),$$

where $\widetilde{\mathcal{D}}^{2p-1}_{\log}(X,p) = \{x \in E^{p-1,p-1}_{\log,\mathbb{R}}(X_\infty)(p-1) : F^*_\infty x = \bar x\}/(\operatorname{Im}\partial + \operatorname{Im}\bar\partial)$*,* $E_{\log}$ *denoting forms with logarithmic singularities at infinity and* $(p-1)$ *the twist by* $(2\pi i)^{p-1}$*, and where on Green objects* $\mathrm{a}(\widetilde{x}) = (-\mathrm{d}_\mathcal{D}x, -\widetilde{x})$ *with* $\mathrm{d}_\mathcal{D}x = -2\partial\bar\partial x$ *in this degree. Then:*

(BK1) $\widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\log})_\mathbb{Q}$ *is a commutative and associative* $\mathbb{Q}$*-algebra with unit, compatible with* ω *and* ζ*. If* y *and* z *are cycles whose generic fibres intersect properly,* $[y,\mathfrak{g}_y]\cdot[z,\mathfrak{g}_z]$ *is the class of* $((y_\mathbb{Q}\cdot z_\mathbb{Q}, \mathfrak{g}_y * \mathfrak{g}_z), [y\cdot z]_{\mathrm{fin}})$*; and if* $\mathfrak{g}_y = \mathrm{a}(\widetilde{x})$*, then* $\mathfrak{g}_y * \mathfrak{g}_z = \mathrm{a}(\widetilde{x \wedge \omega_z})$.

(BK2) *For a morphism $f\colon X \to Y$ of arithmetic varieties over $\mathbb{Z}$ there is a pull-back*

$$f^*\colon \widehat{\mathrm{CH}}^p(Y, \mathcal{D}_{\log}) \to \widehat{\mathrm{CH}}^p(X, \mathcal{D}_{\log}),$$

compatible with the pull-back of forms via ω and of cycles via ζ, with $f^* \circ \mathrm{a} = \mathrm{a} \circ f^*$ (on the right, the pull-back of forms), with $(g \circ f)^* = f^* \circ g^*$, and inducing an algebra morphism on $\widehat{\mathrm{CH}}^*(\cdot, \mathcal{D}_{\log})_{\mathbb{Q}}$.

(BK3) *Let $f\colon X \to Y$ be a proper morphism of arithmetic varieties over $\mathbb{Z}$ of relative dimension e such that $f_{\mathbb{R}}$ is a smooth proper morphism of proper varieties. Then there is a push-forward $f_*\colon \widehat{\mathrm{CH}}^p(X, \mathcal{D}_{\log}) \to \widehat{\mathrm{CH}}^{p-e}(Y, \mathcal{D}_{\log})$, compatible with the push-forward of forms via ω and of cycles via ζ, with $(g \circ f)_* = g_* \circ f_*$ for another such g, and*

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta) \in \widehat{\mathrm{CH}}^{p+q-e}(Y, \mathcal{D}_{\log})_{\mathbb{Q}} \qquad (\alpha \in \widehat{\mathrm{CH}}^p(Y, \mathcal{D}_{\log}),\ \beta \in \widehat{\mathrm{CH}}^q(X, \mathcal{D}_{\log})).$$

This f_* is the push-forward of the abstract theory for the covariant f-pseudo-morphism $f_*\mathcal{D}_{\log,X} \xleftarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{D}_{\log,Y}(-e)[-2e]$, where $\mathcal{F}(U,p) = \varinjlim_{\overline{Y},D} \mathcal{D}(E_{X\times_Y \overline{Y}}(\log X \times_Y D), p)$ (limit over the compactifications $\overline{Y}$ of U dominating Y with normal crossing boundary $D = \overline{Y} \setminus U$), u is the natural morphism, an isomorphism on global sections, and $v(\omega) = f_!\omega = (2\pi i)^{-e} \int_f \omega$; and $f_* \circ \mathrm{a} = \mathrm{a} \circ f_{\#}$, $\omega \circ f_* = f_{\#} \circ \omega$, with $f_{\#} = v \circ u^{-1}$ on global sections.

(BK4) *The assignment $[y, (\omega_y, \widetilde{g}_y)] \mapsto [y, 2(2\pi i)^{d-p+1}[g_y]_X]$, where $[g]_X(\eta) = (2\pi i)^{-d} \int_{X_\infty} \eta \wedge g$, induces a well-defined map $\Psi\colon \widehat{\mathrm{CH}}^p(X, \mathcal{D}_{\log}) \to \widehat{\mathrm{CH}}^p(X)$ to the arithmetic Chow group of Gillet and Soulé, compatible with products and pull-backs; if $X_{\mathbb{R}}$ is projective, Ψ is an isomorphism, compatible with push-forwards. The current $g = 2(2\pi i)^{d-p+1}[g_y]_X$ is a Green current for y in the sense of Gillet and Soulé: $F_\infty^* g = (-1)^{p-1} g$ and $g(dd^c\eta) = \int_{X_\infty} \eta \wedge \omega - \int_{\widetilde{Y}} \imath^*\eta$ for all test forms η, with $\omega = (2\pi i)^{-p}\omega_y$ ($\imath\colon \widetilde{Y} \to X_\infty$ a resolution of the support of y, extended linearly).*

(BK5) *For $S = \operatorname{Spec}\mathbb{Z}$, the arithmetic degree $\widehat{\deg}\colon \widehat{\mathrm{CH}}^1(S, \mathcal{D}_{\log}) \to \mathbb{R}$, induced by*

$$\Big(\sum_j n_j \mathfrak{p}_j, (0, \widetilde{g})\Big) \mapsto \sum_j n_j \log|\mathbb{Z}/\mathfrak{p}_j| + g,$$

is an isomorphism, compatible under Ψ with the arithmetic degree of Gillet and Soulé.

Proof. This is [106] (J. I. Burgos Gil, J. Kramer, U. Kühn, Cohomological arithmetic Chow rings), specialised to the arithmetic ring $\mathbb{Z}$ (an arithmetic ring there: subrings of a number field with that field as fraction field), with the Gillet complex $\mathcal{D}_{\log}$ (contravariant $\mathcal{D}_{\log}$-arithmetic Chow groups). Locations, by the labels of the printed text: the definitions of arithmetic ring, arithmetic variety, $\widehat{\mathrm{CH}}^p$ (def:gacg) and of ζ, a, ω in the chapter on abstract arithmetic Chow groups; a on truncated relative cohomology in the section on truncated relative cohomology groups; $\mathrm{d}_{\mathcal{D}}$ and the product of the Deligne algebra in degrees $2p$, $2p-1$ in the review of Deligne algebras; $\widetilde{\mathcal{D}}^{2p-1}_{\log}(X,p)$ at the beginning of the section on contravariant $\mathcal{D}_{\log}$-arithmetic Chow rings. (BK1): the theorem on the multiplicative structure of $\widehat{\mathrm{CH}}^*(X, \mathcal{D}_{\log})_{\mathbb{Q}}$ there, formula (eq:6) with Theorem thm:2, and Proposition prop:explicit (i). (BK2): the theorem on inverse images in that section, and Theorem thm:6 (iii) of the abstract theory ($\omega \circ f^* = f^{\#} \circ \omega$, $\zeta \circ f^* = f^* \circ \zeta$, $f^* \circ \mathrm{a} = \mathrm{a} \circ f^{\#}$), with $f^{\#}$ the pull-back of forms for the complexes $\mathcal{D}_{\log}$. (BK3): the theorem on direct images of contravariant arithmetic Chow rings (stated there as

a consequence of Theorem thm:properpushlog and Theorem thm:18), the construction of $\mathcal{F}$, u, v preceding Theorem thm:properpushlog, Definitions def:21 and def:22 (pseudo-morphisms and the induced direct image), and Theorem thm:18 (iii) of the abstract theory. (BK4): the Lemma and its proof preceding Theorem thm:24, Theorem thm:24, and equation eq:26 for $[\cdot]_X$. (BK5): the paragraph on the arithmetic degree map (eq:ar-deg), with the remark that the map a of loc. cit. is minus that of Gillet and Soulé and that the factor 2 is explained by Theorem thm:24. *Notation.* Their arithmetic Chow groups are written $\widehat{\mathrm{CH}}$ here, their map from forms to arithmetic Chow groups is written a here, their X_F for $F = \mathbb{Q}$ is $X_{\mathbb{Q}}$; their general arithmetic ring A is specialised to $\mathbb{Z}$. □

Proposition B.141. *Let X be a regular scheme, flat and projective over* $\operatorname{Spec}\mathbb{Z}$ *(an arithmetic variety over $\mathbb{Z}$ in the sense of Proposition B.140), of relative dimension d, and take the empty normal crossing divisor on $X_{\mathbb{R}}$. A hermitian vector bundle $\overline{E} = (E, h)$ on X is a vector bundle E with a smooth hermitian metric h on E_∞ over $X_\infty = X(\mathbb{C})$, invariant under complex conjugation. For a local frame ξ of E_∞ let $K(\xi) = \bar\partial(\partial h(\xi)\cdot h(\xi)^{-1})$ be the curvature matrix and, for a symmetric polynomial ϕ, let $\phi(E,h) = \phi(-K(\xi))$ be the Chern form (independent of ξ, closed, with component of degree k in $E^{k,k}\cap E^{2k}_{\mathbb{R}}(k)$, where (k) is the twist by $(2\pi i)^k$). Then:*

(BC1) *The natural inclusion $\mathcal{D}_{\log}\subseteq\mathcal{D}_{\mathrm{l,ll}}$ induces an isomorphism*

$$\iota\colon \widehat{\mathrm{CH}}^*(X,\mathcal{D}_{\log})\to\widehat{\mathrm{CH}}^*(X,\mathcal{D}_{\mathrm{l,ll}}),$$

compatible with inverse images, intersection products and arithmetic degrees; and

$$\widehat{\mathrm{CH}}^*(X,\mathcal{D}_{\mathrm{l,ll}}) = \widehat{\mathrm{CH}}^*(X,\mathcal{D}_{\mathrm{ls}}).$$

(BC2) *The assignment $[y,(\omega_y,\widetilde{g}_y)]\mapsto[y,2(2\pi i)^{d-p+1}[g_y]_X]$ induces an isomorphism*

$$\Psi\colon \widehat{\mathrm{CH}}^p(X,\mathcal{D}_{\mathrm{ls}})\to\widehat{\mathrm{CH}}^p(X)$$

onto the arithmetic Chow group of Gillet and Soulé, compatible with products and pull-backs.

(BC3) *For every $k\geq 0$ and every hermitian vector bundle $\overline{E}$ of rank n on X there is a class $\widehat{c}_k(\overline{E})\in\widehat{\mathrm{CH}}^k(X,\mathcal{D}_{\mathrm{ls}})$ (the class $\widehat{\phi}(\overline{E})$ for ϕ the k-th elementary symmetric polynomial e_k in n variables) such that:*

1. *$f^*\widehat{c}_k(\overline{E}) = \widehat{c}_k(f^*\overline{E})$ for every morphism $f\colon X'\to X$ of regular schemes flat and projective over $\mathbb{Z}$;*
2. *if $\overline{E} = \overline{L}_1\oplus\cdots\oplus\overline{L}_n$ is an orthogonal direct sum of hermitian line bundles, then $\widehat{c}_k(\overline{E}) = e_k(\widehat{c}_1(\overline{L}_1),\dots,\widehat{c}_1(\overline{L}_n))$;*
3. *for hermitian line bundles $\overline{L}$, $\overline{M}$: $\widehat{c}_1(\overline{L}\otimes\overline{M}) = \widehat{c}_1(\overline{L})+\widehat{c}_1(\overline{M})$;*
4. *$\omega(\widehat{c}_k(\overline{E})) = e_k(-K(\xi))$, the Chern form of degree k;*
5. *$\Psi(\widehat{c}_k(\overline{E}))$ is the k-th arithmetic Chern class of $\overline{E}$ defined by Gillet and Soulé.*

(BC4) *The forgetful map $\zeta\colon\widehat{\mathrm{CH}}^*(X,\mathcal{D}_{\mathrm{ls}})\to\mathrm{CH}^*(X)$ is compatible with inverse images and with Chern classes: $\zeta(\widehat{c}_k(\overline{E})) = c_k(E)$.*

(BC5) *The sequence $H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(p))\xrightarrow{\mathrm{a}}\widehat{\mathrm{CH}}^p(X,\mathcal{D}_{\mathrm{l,ll}})\xrightarrow{(\zeta,-\omega)}\mathrm{CH}^p(X)\oplus\mathrm{Z}\mathcal{D}^{2p}_{\mathrm{l,ll}}(X,p)$ is exact, where $H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(p)) = H^{2p-1}(\mathcal{D}_{\mathrm{l,ll}}(X,p))$ is the subgroup of $\widetilde{\mathcal{D}}^{2p-1}_{\mathrm{l,ll}}(X,p)$ of classes of $\mathrm{d}_{\mathcal{D}}$-closed elements and* a *is its restriction.*

Proof. This is [107] (J. I. Burgos Gil, J. Kramer, U. Kühn, Arithmetic characteristic classes of automorphic vector bundles), specialised to the arithmetic ring $\mathbb{Z}$, to the empty normal crossing divisor and to smooth metrics; the paper's arithmetic varieties are regular (its section on non-regular varieties says so by contrast). Locations, by the labels of the printed text. (BC1): the paragraph defining the log-log arithmetic Chow groups (the natural inclusion $\mathcal{D}_{\log} \to \mathcal{D}_{\mathrm{l,ll}}$), Theorem thm:7 (the induced morphism, compatible with inverse images, intersection products and arithmetic degrees; an isomorphism when the divisor is empty and X is projective), Remark rem:3 and the Remark after Theorem thm:11 (X projective: $\mathcal{D}_{\mathrm{ls}} = \mathcal{D}_{\mathrm{l,ll}}$ and the groups agree). (BC2): Theorem thm:8. (BC3): Theorem thm:11 with $\phi = e_k$, $B = \mathbb{Z}$ and empty divisor (a log-singular metric along the empty divisor is a smooth metric): items Functoriality, Normalization, Twist by a line bundle (with $n = 1$, $\phi = T$: $\phi(T_1+T) = T_1+T$), Compatibility with characteristic forms, Compatibility with the definition of Gillet and Soulé; the Chern forms and their basic properties in the subsection on Chern forms for hermitian metrics (Theorem thm:20). (BC5): Theorem thm:16 (second exact sequence), with the identification of H^{2p-1} of the complex $\mathcal{D}_{\mathrm{l,ll}}$ with Deligne–Beilinson cohomology as in the paragraph on Green objects with values in $\mathcal{D}_{\mathrm{l,ll}}$; the same exact sequence in the normalisation of Gillet and Soulé is [110], Theorem 3.3.5 (ii), p. 128, with $H^{p-1,p-1}(X_{\mathbb{R}})$ identified on p. 130 with $\mathrm{Ker}(dd^c)$ on $\widetilde{A}^{p-1,p-1}(X_{\mathbb{R}})$ (read in print). (BC4): the list in the section "Some remarks on the properties" (items on the inverse image of $\widehat{c}_r$ and on the forgetful morphism ζ, "compatible with inverse images and Chern classes"). Remark on the grounding: for the empty divisor the classes are defined in the printed proof as Ψ^{-1} of the classes of Gillet and Soulé (H. Gillet, C. Soulé, Characteristic classes for algebraic vector bundles with hermitian metric I, II, Ann. of Math. 131 (1990)), and "all the properties stated in the theorem can be checked as in" that paper; (BC4) is stated in the summary list without a separate proof. *Notation.* Their arithmetic Chow groups are written $\widehat{\mathrm{CH}}$ here, their arithmetic Chern classes $\widehat{c}_r$, and their log-log and log-singular complexes $\mathcal{D}_{\mathrm{l,ll}}$ and $\mathcal{D}_{\mathrm{ls}}$. □

B.8 Algebraic and complex geometry

Part VI and Appendix D use resolution of singularities, in Kollár's form of Hironaka's theorem (Theorem B.142), and the asymptotic Hodge theory of a polarised variation of Hodge structure along a normal crossings divisor: Schmid's nilpotent orbit theorem, the Hodge norm estimates of Kashiwara and of Cattani, Kaplan and Schmid, and Griffiths's seminegativity of the last Hodge quotient (Theorem B.145, Theorem B.146, Theorem B.147, Theorem B.144). These are used for good origins, the facts of good towers and base settings (Sections 33, 34 and D.1). The other statements of algebraic geometry (finite free quotients, nef and big line bundles, GAGA, localized Chern characters, flattening by blow-up, quotients by finite locally free group schemes and norms of ample line bundles) are used in Part VI, four of them for linked stages (Section 35). Part IX quotes four facts of commutative algebra on Noetherian local rings: Chevalley's lemma, the dimension formula, Grothendieck's connectedness theorem and the bound of the Krull dimension by the embedding dimension (Theorem B.154, Theorem B.155, Theorem B.156, Theorem B.157), used in Sections 49, 50 and 53.

Theorem B.142. *Let k be a field of characteristic zero. A divisor $E = \sum_i E^i$ on a smooth variety X over k is a* ***simple normal crossing divisor*** *if every E^i is smooth and for every point $x \in X$ there are local coordinates $z_1, \dots, z_n$ in the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{x,X}$ such that for every i either $x \notin E^i$, or $E^i = (z_{c(i)} = 0)$ in a neighbourhood of x, with $c(i) \neq c(i')$ for $i \neq i'$.*

(A) (Strong resolution.) Let X be a quasi-projective variety over k. There is a birational and projective morphism $\Pi\colon R(X) \to X$ such that $R(X)$ is smooth, Π is an isomorphism over the smooth locus X^{ns}, and $\Pi^{-1}(\mathrm{Sing}\, X)$ is a divisor with simple normal crossings.

(B) (Principalisation.) Let X be a smooth scheme of finite type over k and $I \subseteq \mathcal{O}_X$ an ideal sheaf that is not zero on any irreducible component of X. There is a finite sequence of blow-ups $\Pi\colon X_r \to X_{r-1} \to \cdots \to X_0 = X$ with smooth centres (so that X_r is smooth and Π is projective) such that $\Pi^* I \subseteq \mathcal{O}_{X_r}$ is the ideal sheaf of a simple normal crossing divisor and Π is an isomorphism over $X \setminus \operatorname{cosupp} I$.

Proof. [131] (Main Theorems I and II, the original theorem); in the strong form stated here [148], Chapter 3: (A) is Theorem 3.27 (Resolution of singularities, II), (B) is Theorem 3.35 (Principalization, IV), items (1)–(3) with $E = \varnothing$, and the definition of simple normal crossing divisors is Definition 3.24; the remark that a composite of blow-ups with smooth centres has smooth source and is birational and projective follows Theorem 3.26 there. Notation as in loc. cit.; $\operatorname{cosupp} I$ is the closed subset where $I \neq \mathcal{O}_X$. □

Theorem B.143. *Let a finite group Γ act freely on a quasi-projective scheme X of finite type over a field K. Then the fppf quotient $Y = X/\Gamma$ is a quasi-projective K-scheme, and $X \to Y$ is a finite étale surjective Γ-torsor.*

Proof. The scheme-quotient and torsor assertions are the finite constant-group case of [241], Lemma 66.14.2. The constant group scheme is finite étale, so its torsor is finite étale. Write $\pi : X \to Y$ for this torsor. Finite type over K descends along this finite étale surjective cover, so Y is of finite type over K.

For completeness, quasi-projectivity follows by the ample-norm argument. Choose a very ample line bundle $\mathcal{B}$ on X and form

$$\mathcal{N} = \bigotimes_{\gamma \in \Gamma} \gamma^* \mathcal{B}.$$

Permuting the factors gives a canonical linearisation. Descent for invertible sheaves along the torsor gives a line bundle $\mathcal{C}$ on Y whose pullback is $\mathcal{N}$.

We verify the affine-section criterion for its ampleness. For any closed point $y \in Y$ and open neighbourhood U of y, there are an integer $n > 0$ and a section s of $\mathcal{B}^{\otimes n}$ nonzero on the finite fibre over y, vanishing on $X \setminus \pi^{-1}(U)$, and with affine nonvanishing locus. Indeed embed X in projective space by $\mathcal{B}$, take its projective closure, and choose a homogeneous polynomial vanishing on the boundary and on the closure of $X \setminus \pi^{-1}(U)$, but nonzero on that finite fibre. Such a polynomial of sufficiently large degree exists by interpolation on a finite closed subscheme disjoint from that closed set. Its projective nonvanishing locus intersected with the closure is affine and lies inside X.

The norm $\bigotimes_\gamma \gamma^* s$ is invariant, so descends to a section of $\mathcal{C}^{\otimes n}$. Its nonvanishing preimage is the intersection of the translates of the affine nonvanishing locus of s. This is affine because X is separated; it is invariant and lies in $\pi^{-1}(U)$. Its quotient is affine, with coordinate ring the invariants, by the affine quotient construction. Thus sections of positive powers of $\mathcal{C}$ give affine neighbourhoods of all closed points, refining every open neighbourhood. Since Y is of finite type over a field, closed points detect nonempty closed subsets, and this proves the affine-section criterion for ampleness. Hence Y has an ample line bundle and is quasi-projective over K. □

Theorem B.144. *Let S be a complex manifold and $(\mathcal{H}_\mathbb{Z}, \mathcal{F}^\bullet, Q)$ a polarised variation of Hodge structure of weight w on S, $\mathcal{H} = \mathcal{H}_\mathbb{Z} \otimes \mathcal{O}_S$. Let a be the largest integer with $\mathcal{F}^a = \mathcal{H}$ and give the holomorphic vector bundle $\mathcal{E} = \mathcal{H}/\mathcal{F}^{a+1}$ the Hodge metric, i.e. the metric of the Hodge form $Q(C\cdot, \bar{\cdot})$ transported by the isomorphism $\mathcal{H}^{a,w-a} \to \mathcal{E}$ from the Hodge summand $\mathcal{H}^{a,w-a}$ (the orthogonal complement of $\mathcal{F}^{a+1}$). Then $\mathcal{E}$ is Griffiths seminegative: for every holomorphic section s of $\mathcal{E}$ on an open subset of S, the function $\log \|s\|^2$ is plurisubharmonic where $s \neq 0$.*

Proof. [114, §7], curvature of the Hodge bundles; in print: [216, Lemma (7.18), p. 271] (curvature of F^p/F^{p+1} with the metric $i^{2p-w}Q(\cdot,\bar{\cdot})$ transported from $\mathcal{H}^{p,w-p}$), and Lemma (7.19), p. 272; see also [196]. The curvature of the Hodge bundle $\mathcal{H}^{p,w-p}$ is $-\sigma_p^*\wedge\sigma_p-\sigma_{p+1}\wedge\sigma_{p+1}^*$ up to the sign conventions, where σ_p is the second fundamental form (the Higgs field) $\mathcal{H}^{p,w-p}\to\mathcal{H}^{p-1,w-p+1}\otimes\Omega^1$; for the last quotient ($p=a$: $\mathcal{H}^{a-1,w-a+1}=0$, so $\sigma_a=0$) only the term $-\sigma_{a+1}\wedge\sigma_{a+1}^*$ survives, so $(\Theta(X,\bar X)v,v)=-\|\sigma_{a+1}^*(\bar X)v\|^2\le 0$ for every $(1,0)$-vector X and $v\in\mathcal{E}$ (Schmid, loc. cit., the last display of the proof of (7.18)). For a holomorphic section $s\neq 0$ with Chern connection D, Schmid's identity (7.13), $\partial\bar\partial\|s\|^2=(Ds,Ds)-(\Theta s,s)$, gives

$$\partial\bar\partial\log\|s\|^2(X,\bar X)=\frac{\|s\|^2\|D_Xs\|^2-|(D_Xs,s)|^2}{\|s\|^4}-\frac{(\Theta(X,\bar X)s,s)}{\|s\|^2}\ge 0$$

by the Cauchy–Schwarz inequality, so $\log\|s\|^2$ is plurisubharmonic. Notation: $\mathcal{F}^\bullet$ is the Hodge filtration, C the Weil operator. □

Theorem B.145. *Let $1\le a\le d$, $\Delta=\{z\in\mathbb{C}:|z|<1\}$, $\Delta_\rho=\{|z|<\rho\}$, $V=\Delta^d$, $V^*=\{t\in V:t_1\cdots t_a\neq 0\}$ and $D=V\smallsetminus V^*$. Let $(\mathbb{H},\mathcal{F}^\bullet,Q)$ be a polarised variation of Hodge structure of weight w on V^*: $\mathbb{H}$ a local system of free $\mathbb{Z}$-modules of finite rank, Q a flat non-degenerate $(-1)^w$-symmetric bilinear form on $\mathbb{H}$, and $\mathcal{F}^\bullet$ a decreasing filtration of $\mathcal{H}=\mathbb{H}\otimes_{\mathbb{Z}}\mathcal{O}_{V^*}$ by holomorphic subbundles satisfying Griffiths transversality and inducing at every point a Hodge structure of weight w polarised by Q. Let H be the space of flat sections of $\mathbb{H}\otimes\mathbb{C}$ over the universal cover of V^*, T_j ($1\le j\le a$) the monodromy of the loop $\theta\mapsto(\frac12,\dots,\frac12e^{i\theta},\dots,\frac12,0,\dots,0)$ ($e^{i\theta}$ in the j-th place, $\frac12$ in the other first a places) acting on H, and assume every T_j unipotent; the T_j commute, and we put $N_j=\log T_j$. For $s\in H$ let $\tilde s=\exp(-\sum_{j=1}^a\frac{\log t_j}{2\pi i}N_j)s$, a single-valued holomorphic section of $\mathcal{H}\otimes\mathbb{C}$ over V^*; Deligne's canonical extension $\widehat{\mathcal{H}}$ of $\mathcal{H}\otimes\mathbb{C}$ is the holomorphic vector bundle on V with frame $\tilde s_1,\dots,\tilde s_r$ for a basis $s_1,\dots,s_r$ of H, and its fibre at every $x\in D$ is identified with H by $\tilde s(x)\mapsto s$. Then there is $\rho\in(0,1]$ such that:*

1. *each $\mathcal{F}^p|_{V^*\cap\Delta_\rho^d}$ extends to a holomorphic subbundle $\widehat{\mathcal{F}}^p$ of $\widehat{\mathcal{H}}|_{\Delta_\rho^d}$;*
2. *with $F_0^p\subseteq H$ the fibre of $\widehat{\mathcal{F}}^p$ at 0, $N=N_1+\cdots+N_a$ and W the monodromy weight filtration of N on H centred at w, the pair $(W,F_0^\bullet)$ is a mixed Hodge structure on H (rational structure $H_{\mathbb{Q}}$ the flat sections of $\mathbb{H}\otimes\mathbb{Q}$), with $NF_0^p\subseteq F_0^{p-1}$ and $NW_k\subseteq W_{k-2}$, polarised by N: for every $k\ge 0$ the pure Hodge structure of weight $w+k$ induced on the primitive part $P_{w+k}=\ker(N^{k+1}\colon\mathrm{Gr}^W_{w+k}\to\mathrm{Gr}^W_{w-k-2})$ is polarised by the form $Q(\cdot,N^k\cdot)$.*

For $a=1$ this is the case of one punctured coordinate with parameters $t_2,\dots,t_d$.

Proof. Clause 1 is the nilpotent orbit theorem of W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. 22 (1973), 211–319, Theorem 4.12 (several variables; unpunctured parameters as in E. Cattani, A. Kaplan and W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. (2) 123 (1986), 457–535, Section 1): the untwisted period map $\exp(-\sum_j\frac{\log t_j}{2\pi i}N_j)\mathcal{F}^\bullet$ extends holomorphically over V, which is the statement that the $\mathcal{F}^p$ extend to subbundles of $\widehat{\mathcal{H}}$. In this form it is used, for $V=\Delta^l$ and the divisor $\{z_1\cdots z_l=0\}$ with unipotent monodromies, in M. Kashiwara, *The asymptotic behavior of a variation of polarized Hodge structure*, Publ. RIMS 21 (1985), 853–875, (4.1.5)–(4.1.6). For clause 2, restrict to the disc $\phi(s)=(s,\dots,s,0,\dots,0)$ (s in the first a places): the pulled-back variation on Δ^* has monodromy $T_1\cdots T_a=\exp N$, and $\phi^*\tilde s=\exp(-\frac{\log s}{2\pi i}N)s$ is its canonical-extension frame, so its limit Hodge filtration is $F_0^\bullet$; clause 2 is then Schmid's theorem that the limit mixed Hodge structure in one variable is polarised by N (loc. cit., Theorem 6.16 with Lemma 6.4, printed there

for the cohomology of a family of polarised projective manifolds over Δ^*; for an abstract polarised variation on Δ^* the same statement is printed in [212], "Overview of the results", subsection "The limiting mixed Hodge structure", theorems "Limiting $\mathfrak{sl}_2$-Hodge structure" (items 1–3: each primitive subspace $\ker N^{k+1}$ of the k-th weight space is polarised by the hermitian pairing $Q(\cdot, N^k\cdot)$) and "Limiting mixed Hodge structure", for polarised complex variations with the hermitian form attached to Q; see also Cattani–Kaplan–Schmid, loc. cit., Section 2). Notation: Q is Schmid's polarisation S, W his $W(N)$ shifted to be centred at w, F_0 his limit filtration F_∞. □

Theorem B.146. *Let $1 \le a \le d$, $V = \Delta^d$, $V^* = \{t \in V : t_1 \cdots t_a \neq 0\}$, and let $(\mathbb{H}, \mathcal{F}^\bullet, Q)$ be a polarised variation of Hodge structure on V^* whose monodromies $T_1, \dots, T_a$ around the divisors $\{t_j = 0\}$ are unipotent; $N_j = \log T_j$. For $t \in V^*$ let $|u|_t^2 = Q(C_t u, \bar u)$ be the Hodge norm on $\mathbb{H}_t \otimes \mathbb{C}$ (C_t the Weil operator) and $y_j(t) = -\log|t_j|$. For a numbering $(j_1, \dots, j_a)$ of $\{1, \dots, a\}$ and $1 \le \nu \le a$ let $W(\nu)$ be the monodromy weight filtration (centred at 0: $NW_k \subseteq W_{k-2}$ and $N^k\colon \mathrm{Gr}_k \cong \mathrm{Gr}_{-k}$) of $N = \sum_{\mu \ge \nu} N_{j_\mu}$. Then there is $M > 0$ such that for every subanalytic set $U \subseteq V^*$ relatively compact in V, every $\varepsilon > 0$, every flat section u of $\mathbb{H} \otimes \mathbb{C}$ over U and all integers $p_1, \dots, p_a$ with $u \in \bigcap_\nu W_{p_\nu}(\nu)$ there is $C > 0$ with*

$$|u|_t^2 \;\le\; C\, y_{j_1}^{p_1} \Big(\frac{y_{j_2}}{y_{j_1}}\Big)^{p_2} \cdots \Big(\frac{y_{j_a}}{y_{j_{a-1}}}\Big)^{p_a}$$

for all $t \in U$ with $y_{j_\nu}(t) > M$ ($1 \le \nu \le a$) and $y_{j_\nu}(t)/y_{j_{\nu-1}}(t) > \varepsilon$ ($2 \le \nu \le a$).

Proof. M. Kashiwara [140, Theorem 3.4.1 (p. 870)], applied with $X = V$, $Y = \{t_1 \cdots t_a = 0\}$ (a normal crossing hypersurface), $x_0 = 0$, $\alpha = \{1, \dots, a\}$, $f_j = t_j$, and his $\alpha_\nu = \{j_\mu : \mu \ge \nu\}$; there the estimate is stated with $\sim$ (bounded above and below by positive multiples), of which only the upper bound is stated here. The same estimates are due independently to E. Cattani, A. Kaplan and W. Schmid [64, Section 5] (see [140, p. 855]). Notation: Kashiwara's $|u|_x$, $y_\nu = -\log|f_{j_\nu}|$, $W_p(\alpha_\nu)$ are $|u|_t$, y_{j_ν}, $W_p(\nu)$. □

Theorem B.147. *In the setting of Theorem B.146, fix a numbering $(j_1, \dots, j_a)$ of $\{1, \dots, a\}$. The filtrations $\widehat{W}(\nu) = W(\nu) \otimes \mathcal{O}$ of $\widehat{\mathcal{H}}$ ($1 \le \nu \le a$; flat filtrations, sections of $\widehat{\mathcal{H}}$ being combinations of the frame $\tilde s$) form a distributive family. Let $\widehat{\mathcal{H}} = \bigoplus_{p \in \mathbb{Z}^a} U_p$ be a splitting into C^∞ subbundles near 0 with $\widehat{W}_k(\nu) = \bigoplus_{p:\, p_\nu \le k} U_p$ for all ν, k, and $\|\cdot\|$ a C^∞ hermitian norm on $\widehat{\mathcal{H}}$ near 0. Then for every $\varepsilon > 0$ there are $C \ge 1$ and a neighbourhood V' of 0 such that for every $t \in V' \cap V^*$ with $y_{j_\nu}(t)/y_{j_{\nu-1}}(t) > \varepsilon$ ($2 \le \nu \le a$) and every $u = \sum_p u_p \in \widehat{\mathcal{H}}_t$, $u_p \in U_{p,t}$,*

$$C^{-1} \sum_p m_p(t) \|u_p\|^2 \;\le\; |u|_t^2 \;\le\; C \sum_p m_p(t) \|u_p\|^2, \qquad m_p = y_{j_1}^{p_1} \Big(\frac{y_{j_2}}{y_{j_1}}\Big)^{p_2} \cdots \Big(\frac{y_{j_a}}{y_{j_{a-1}}}\Big)^{p_a}.$$

Proof. M. Kashiwara [140, Theorem 3.4.2 (p. 870)], with $X = V$, $Y = \{t_1 \cdots t_a = 0\}$, $x_0 = 0$, $\alpha = \{1, \dots, a\}$, $\alpha_\nu = \{j_\mu : \mu \ge \nu\}$; for unipotent monodromy his $E^+(\mathcal{O} \otimes H_\mathbb{C})$ (characterised in 3.3.1 by residues with eigenvalues λ, $0 \le \operatorname{Re}\lambda < 1$) is Deligne's canonical extension $\widehat{\mathcal{H}}$, and $E^+(\mathcal{O} \otimes W(\alpha_\nu))$ is $\widehat{W}(\nu)$. Kashiwara's $\sim$ is, by his Definition 1.9.3 with (1.9.5), the two-sided comparison displayed, with one constant for all u and all points. Same estimates: [64, Section 5]. □

Theorem B.148. *Let X be an irreducible complex projective variety of dimension n and L a nef line bundle on X. Then:*

1. $h^0(X, L^{\otimes m}) = \dfrac{(L^n)}{n!}\, m^n + O(m^{n-1})$ *as* $m \to \infty$;

2. L *is big if and only if* $(L^n) > 0$;

3. *if* L *is big and* P *is an effective Cartier divisor on* X*, then* $H^0(X, L^{\otimes m} \otimes \mathcal{O}_X(-P)) \neq 0$ *for all sufficiently large* m.

Proof. (1) [160, I, Corollary 1.4.41]; (2) [160, I, Theorem 2.2.16]; (3) [160, I, Proposition 2.2.6] (Kodaira's lemma). Notation: $L^{\otimes m} = \mathcal{O}_X(mD)$ for a Cartier divisor D of L; $(L^n) = (D^n)$. □

Theorem B.149. *Let* X *be a reduced projective scheme over* $\mathbb{C}$ *(a closed reduced subscheme of some* $\mathbb{P}^N_{\mathbb{C}}$*, not necessarily irreducible),* X^{an} *the associated complex analytic space, and* $\mathcal{F}$ *a coherent* $\mathcal{O}_X$*-module with analytification* $\mathcal{F}^{\mathrm{an}} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}$*. Then for every* $i \geq 0$ *the natural map*

$$H^i(X, \mathcal{F}) \longrightarrow H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$$

is bijective. In particular, for an invertible sheaf $\mathcal{M}$ *on* X*, every holomorphic section of* $\mathcal{M}^{\mathrm{an}}$ *over* X^{an} *is the analytification of exactly one section of* $\mathcal{M}$ *over* X.

Proof. This is Serre's comparison theorem [219], Théorème 1 (Section 12), stated there for coherent algebraic sheaves on projective algebraic varieties over $\mathbb{C}$ in the sense of Serre's *Faisceaux algébriques cohérents* (closed subvarieties of a projective space, with the Zariski topology); a reduced projective scheme over $\mathbb{C}$ with a coherent sheaf is such a variety with such a sheaf, and the map of the statement is Serre's map ε. The last sentence is the case $i = 0$, $\mathcal{F} = \mathcal{M}$. □

Theorem B.150. *Let* S *be a noetherian universally catenary scheme with a dimension function* δ*, let* Y *be a quasi-compact scheme locally of finite type over* S*,* $i\colon Z \to Y$ *a closed immersion, and* E *an object of* $D(\mathcal{O}_Y)$ *represented by a bounded complex of finite locally free* $\mathcal{O}_Y$*-modules with* $E|_{Y \smallsetminus Z} \cong 0$*. Bivariant classes* $A^p(Z \to Y)$ *are as in the Stacks Project, Definition 42.33.1: for every* $Y' \to Y$ *locally of finite type, maps* $c \cap -\colon \mathrm{CH}_k(Y') \to \mathrm{CH}_{k-p}(Y' \times_Y Z)$ *compatible with proper push-forward, flat pull-back and the Gysin maps of zero schemes of sections of invertible modules. Then there are classes* $P_p(Z \to Y, E) \in A^p(Z \to Y)$*,* $p \geq 0$*, depending only on the isomorphism class of* E*; put* $\mathrm{ch}(Z \to Y, E) = \sum_{p \geq 0} P_p(Z \to Y, E)/p!$ *and* $\mathrm{ch}_p = P_p/p!$*. They satisfy:*

(L1) Push-forward $i_* \circ \mathrm{ch}_p(Z \to Y, E) = \mathrm{ch}_p(E)$*, the ordinary Chern character of* E*; in particular, for* $Z = Y$ *the localized Chern character is the ordinary one.*

(L2) Restriction *for* $f\colon Y' \to Y$ *locally of finite type,* $\mathrm{ch}_p(f^{-1}Z \to Y', Lf^*E)$ *is the restriction of* $\mathrm{ch}_p(Z \to Y, E)$ *to* Y'*, i.e.* $\mathrm{ch}_p(f^{-1}Z \to Y', Lf^*E) \cap \alpha = \mathrm{ch}_p(Z \to Y, E) \cap \alpha$ *for* $\alpha \in \mathrm{CH}_*(Y')$.

(L3) Change of support *if* $Z \subseteq Z' \subseteq Y$ *are closed subschemes, then* $\mathrm{ch}_p(Z' \to Y, E) = (Z \to Z')_* \circ \mathrm{ch}_p(Z \to Y, E)$.

(L4) Additivity *for a distinguished triangle* $E_1 \to E_2 \to E_3 \to E_1[1]$ *of such objects with* $E_1|_{Y \smallsetminus Z} \cong 0 \cong E_3|_{Y \smallsetminus Z}$*:* $\mathrm{ch}(Z \to Y, E_2) = \mathrm{ch}(Z \to Y, E_1) + \mathrm{ch}(Z \to Y, E_3)$.

(L5) Multiplicativity *for closed subschemes* Z_1, Z_2 *and such objects* F_1, F_2 *with* $F_j|_{Y \smallsetminus Z_j} \cong 0$*:* $\mathrm{ch}(Z_1 \cap Z_2 \to Y, F_1 \otimes^{\mathbf{L}} F_2) = \mathrm{ch}(Z_1 \to Y, F_1)\, \mathrm{ch}(Z_2 \to Y, F_2)$ *(product of bivariant classes).*

Proof. [244, Chapter 42 (Chow Homology)]: Definition 42.33.1 (bivariant classes), Lemma 42.50.2 with equation (42.50.2.1) (construction; (L1)), Definition 42.50.3 (the localized Chern character), Lemma 42.50.4 ((L2)), Lemma 42.50.8 ((L3)), Lemma 42.52.5 ((L4)), Lemma 42.52.6 ((L5)). Hypothesis (3) of Situation 42.50.1 holds here because E is a bounded complex of finite locally free modules, and Y is quasi-compact. □

Theorem B.151. *Let S be a quasi-compact and quasi-separated scheme, $U \subseteq S$ a quasi-compact open subscheme, $f\colon X \to S$ a morphism of finite presentation and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module of finite type such that $\mathcal{F}|_{f^{-1}(U)}$ is of finite presentation and flat over U. Then there is a U-admissible blow-up $b\colon S' \to S$ (the blow-up of S in a closed subscheme of finite presentation disjoint from U) such that the strict transform of $\mathcal{F}$, i.e. the quotient of the pull-back of $\mathcal{F}$ to $X \times_S S'$ by its subsheaf of sections supported in the preimage of $S \setminus U$, is of finite presentation and flat over S'.*

Proof. [243, Theorem 38.30.7, Tag 0815], a result of Raynaud and Gruson [206, première partie, Théorème 5.2.2]. Notation: identical; the case used later is $X = S$, where flat and of finite presentation means locally free of finite rank. □

Theorem B.152. *Let S be a scheme, G a finite locally free S-group scheme and X an S-scheme with an action $\alpha\colon G \times_S X \to X$ of G which is free, i.e. the morphism $G \times_S X \to X \times_S X$, $(g, x) \mapsto (\alpha(g, x), x)$, is a monomorphism, and such that for every point x of X the image of $G \times_S \operatorname{Spec} \kappa(x) \to X$, $g \mapsto \alpha(g, x)$, is contained in an affine open subscheme of X. Then there is a morphism of S-schemes $p\colon X \to Y$ with $p \circ \alpha = p \circ \mathrm{pr}_2$ which is finite, locally free and surjective, and such that*

$$G \times_S X \longrightarrow X \times_Y X, \qquad (g, x) \longmapsto (\alpha(g, x), x),$$

is an isomorphism.

Proof. [226], Exp. V, Th. 4.1 (existence of the quotient of a finite locally free groupoid whose orbits lie in affine open subschemes; for a free action, part (iv): p finite locally free and $G \times_S X \cong X \times_Y X$). The same statement is Theorem (4.16)(i)–(ii) of Ch. IV of the book draft *Abelian varieties* of B. Edixhoven, G. van der Geer and B. Moonen. □

Theorem B.153. *Let Y be a scheme, $f\colon X \to Y$ a quasi-compact morphism and $g\colon X' \to X$ a finite surjective morphism such that $g_*\mathcal{O}_{X'}$ is a locally free $\mathcal{O}_X$-module. Then for every invertible $\mathcal{O}_{X'}$-module $\mathcal{L}'$ ample relative to $f \circ g$, its norm $\mathrm{N}_{X'/X}(\mathcal{L}')$ ([81], §6.5) is ample relative to f.*

Proof. [81], Proposition (6.6.1) (case $g_*\mathcal{O}_{X'}$ locally free). □

Theorem B.154. *Let $(A, \mathfrak{m})$ be a complete Noetherian local ring, M a finitely generated A-module and $M_1 \supseteq M_2 \supseteq \cdots$ a decreasing sequence of A-submodules of M with $\bigcap_{t \geq 1} M_t = 0$. Then for every $k \geq 1$ there is $t \geq 1$ with $M_t \subseteq \mathfrak{m}^k M$.*

Proof. This is Chevalley's lemma; see [274, Ch. VIII, §5, Theorem 13] (for ideals of a complete semi-local ring), and, in the stated form for finitely generated modules over a complete local ring, e.g. [170, Lemma 9.7]. The notation agrees with that of these references. □

Theorem B.155. *Let A be a Noetherian domain, $B \supseteq A$ a domain that is finitely generated as an A-algebra, P a prime of B and $\mathfrak{p} = P \cap A$.*

1. $\operatorname{ht} P + \operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(P) \leq \operatorname{ht} \mathfrak{p} + \operatorname{trdeg}_A B$.

2. *If A is universally catenary, equality holds in 1.*

3. *Every complete Noetherian local ring, and every quotient of one, is universally catenary.*

Proof. Item 1 is the dimension inequality and item 2 the dimension formula of Ratliff, [166, Theorems 15.5 and 15.6]. Item 3: a complete Noetherian local ring is a quotient of a regular local ring by Cohen's structure theorem [166, Theorem 29.4], a regular local ring is Cohen–Macaulay, a Cohen–Macaulay ring is universally catenary [166, Theorem 17.9], and a quotient of a universally catenary ring is universally catenary (finitely generated algebras over the quotient are finitely generated over the ring). Notation is that of the reference. □

Theorem B.156. *For a Noetherian topological space T of finite dimension, the* connectedness dimension *is*

$$c(T) = \min\{\dim Z : \ Z \subseteq T \text{ closed}, \ T \setminus Z \text{ disconnected}\},$$

where the empty space counts as disconnected and has dimension -1; for $T \neq \emptyset$ the subdimension $\operatorname{sdim} T$ *is the minimum of the dimensions of the irreducible components of T. For a ring C write $c(C) = c(\operatorname{Spec} C)$ and $\operatorname{sdim} C = \operatorname{sdim} \operatorname{Spec} C$. Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and let $a_1, \dots, a_h \in \mathfrak{m}$. Then*

$$c(R/(a_1, \dots, a_h)) \ \geq \ \min\{c(R), \ \operatorname{sdim} R - 1\} - h.$$

Proof. See Grothendieck [225], Exposé XIII, Théorème 2.1; Brodmann–Sharp [53], Theorem 19.2.9, where the right side is $\min\{c(R), \operatorname{sdim} R - 1\} - \operatorname{ara}(\mathfrak{a})$ with $\operatorname{ara}(\mathfrak{a})$ the arithmetic rank of $\mathfrak{a} = (a_1, \dots, a_h)$ (the least number of elements generating an ideal with the radical of $\mathfrak{a}$); here $\operatorname{ara}(\mathfrak{a}) \leq h$. The notation is that of these references, with the convention of Brodmann–Sharp, Chapter 19, that the empty space is disconnected of dimension -1. □

Theorem B.157. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field $\kappa = A/\mathfrak{m}$. Then $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ is finite and the Krull dimension of A satisfies $\dim A \leq \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.*

Proof. This is the dimension theorem for Noetherian local rings: $\dim A$ equals the least number of generators of an $\mathfrak{m}$-primary ideal, and $\mathfrak{m}$ is generated by $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ elements by Nakayama's lemma. See M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra (1969), Corollary 11.15, and [166], Theorem 13.4 and the remarks following it. The notation there is the same. □

B.9 Representation theory and groups

Siegel's formula for unitary groups in Weil's convergent range (Theorem B.158) is used for the row estimates of Part III (Section 15). Strong approximation, Minkowski's lemma on torsion-free congruence subgroups and the reduction theory of Borel and Harish-Chandra (Theorem B.159, Theorem B.160, Theorem B.165) are used for good root towers in Part VI and Appendix D. Part VII uses the principal series of GL_n over a p-adic field and the theorem of Borel and Serre for the plus part (Section 38), and van der Kallen's homological stability and the theorem of Quillen and Milnor–Moore for the second cohomology of Tate twists (Section 40). Appendix E uses the principal series again, the zero-monodromy types of Schneider and Zink, the Bernstein presentation of the Iwahori–Hecke algebra and the braid relations in the pro-p Iwahori Hecke ring (Sections E.1 and E.5).

Theorem B.158. *Let E/F be a CM extension, let V be a nondegenerate hermitian space over E of dimension n with form $\langle\,,\rangle$, let $G=\mathrm{U}(V)$, and let $a\geq 1$ be an integer with $2a<n$. For an F-algebra R write $V_R=V\otimes_F R$ and let* $\mathrm{gram}\colon V_R^a\to \mathrm{Herm}_a(R)$, $\mathrm{gram}(x)=(\langle x_i,x_j\rangle)_{i,j}$, *be the Gram map,* Herm_a *denoting hermitian $a\times a$ matrices over $E\otimes_F R$. For a place w of F let dx_w be the Haar measure on $V_{F_w}^a$ that is self-dual for the pairing $(x,y)\mapsto \psi_w(\mathrm{tr}_{E_w/F_w}\sum_i\langle x_i,y_i\rangle)$, and db_w the Haar measure on* $\mathrm{Herm}_a(F_w)$ *that is self-dual for $(b,b')\mapsto\psi_w(\mathrm{tr}\,bb')$. Let $\Phi=\Phi_\infty\otimes\bigotimes_{w<\infty}\Phi_w$, where Φ_∞ is a smooth compactly supported function on $V_{F_\infty}^a$, $F_\infty=F\otimes_{\mathbb{Q}}\mathbb{R}$, each Φ_w is a Schwartz–Bruhat function on $V_{F_w}^a$, and Φ_w is the characteristic function of $(L\otimes\mathcal{O}_{F_w})^a$ for all but finitely many w, for an $\mathcal{O}_E$-lattice $L\subset V$ (an* admissible test function*). Let $T\in\mathrm{Herm}_a(F)$ be nondegenerate. Then:*

1. *for every Haar measure μ on $G(\mathbb{A}_F)$, the quotient $[G]$ has finite volume $\mu([G])$;*
2. *for every place w (with ∞ standing for the product of the archimedean places), the image under* gram *of the measure $\Phi_w\,dx_w$, restricted to the open set of x with* $\det\mathrm{gram}(x)$ *invertible, has a continuous density $b\mapsto g_w(\Phi_w,b)$ with respect to db_w on the open set of nondegenerate matrices, and the product $\prod_{w<\infty}g_w(\Phi_w,T)$ converges;*
3. *the function $\vartheta_T(g)=\sum_{X\in V^a,\ \mathrm{gram}(X)=T}\Phi(g^{-1}X)$ on $[G]$ is integrable, and*

$$\frac{1}{\mu([G])}\int_{[G]}\vartheta_T(g)\,d\mu(g)=g_\infty(\Phi_\infty,T)\prod_{w<\infty}g_w(\Phi_w,T).$$

Proof. [268], Chapter VI, section 52, Theorem 5 and formula (41), together with Chapter III, section 37, Proposition 6 and formula (29), and Chapter IV, section 42. In Weil's notation take $k=F$, the algebra with involution $\mathrm{End}_E(V)$ with its adjoint involution, and $X=V^a$. This is type (I_2) with $\epsilon=1/2$, so his condition (B) is $n>2a$. His invariant map is $x\mapsto x^*hx$, namely the displayed Gram map. The auxiliary local hypothesis of Theorem 5 holds at a finite place: condition (B) gives a nonempty full-rank zero-Gram fibre there by section 38, and Witt extension makes the full local unitary group transitive on each full-rank Gram fibre. If r is the Witt index of V, then $r\leq n/2$ and hence $n-r>a$; Proposition 8 in section 51 gives absolute convergence. Formula (41), with its normalized Haar specialization, gives the asserted mean as the product of the local relative-gauge integrals. Proposition 6 identifies these with continuous pushforward densities; section 42 gives absolute convergence of the product. Nondegenerate T forces every tuple in its fibre to have full rank, so its full-rank sum is ϑ_T.

The following calculation identifies the measures, with no unspecified local scalar. Choose rational F-bases on $X=V^a$ and $Y=\mathrm{Herm}_a$. Let dx_w^0,db_w^0 be the coordinate measures obtained from the ψ_w-self-dual measure on F_w. For the two nondegenerate F-bilinear pairings displayed in the statement let B_X,B_Y be their matrices in these bases. Fourier inversion and linear change of variables give

$$dx_w=|\det B_X|_w^{1/2}dx_w^0,\qquad db_w=|\det B_Y|_w^{1/2}db_w^0.$$

Let g_w^0 be the pushforward density for the same Gram map with the coordinate measures dx_w^0,db_w^0. Consequently the relative density for the measures in the statement is

$$g_w(\Phi_w,T)=\left|\frac{\det B_X}{\det B_Y}\right|_w^{1/2}g_w^0(\Phi_w,T).$$

Both determinants lie in $F^\times$. The product formula makes the product of the displayed positive factors equal to one, including the archimedean factors; only finitely many finite factors

differ from one. Thus the product of Weil's relative-gauge densities equals the product in the statement, also at dyadic and split places. This is a measure comparison, with no Weil-index factor.

Weil uses the Haar measure for which the right quotient has volume one. Inversion transfers it to the left quotient because $\mathrm{U}(V)$ is unimodular, replacing the action uX by $g^{-1}X$ and preserving total volume. For any Haar μ, its normalization is $\mu/\mu([G])$, so the mean is unchanged. Quotient finiteness (item 1) is the classical finite-volume theorem for unitary groups (Borel–Harish-Chandra). Smooth compactly supported archimedean test functions belong to the Schwartz space used there, as do the given finite factors. For hermitian b, b', $\mathrm{tr}(bb') \in F_w$, so the matrix pairing also equals $\frac{1}{2}\operatorname{Tr}_{E_w/F_w}\mathrm{tr}(bb')$, if written in that alternative convention. □

Theorem B.159. *Let K be a number field, $\mathbb{A}_K$ its ring of adeles, and H a connected, simply connected, absolutely almost simple linear algebraic group over K. Let S be a finite set of places of K, and let $\mathbb{A}_K^S$ be the ring of adeles of K with the components at the places of S omitted, so that $H(K)$ embeds diagonally in $H(\mathbb{A}_K^S)$. If the group $H_S = \prod_{v\in S} H(K_v)$ is not compact, then $H(K)$ is dense in $H(\mathbb{A}_K^S)$.*

Proof. M. Kneser, Strong approximation, in: Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math. 9 (1966) [146]; V. P. Platonov, Izv. Akad. Nauk SSSR 33 (1969); stated in this form (for global fields, with the converse) as Theorem 2.3 of A. S. Rapinchuk, Strong approximation for algebraic groups, in: Arithmetic, geometry, cryptography and coding theory 2009, Contemp. Math. 521 (2010), 15–32; see also [200], §7.4. Notation: Rapinchuk's G, $\mathbb{A}_S$, G_S are our H, $\mathbb{A}_K^S$, H_S; only the sufficiency half is quoted. □

Theorem B.160. *Let $n \geq 1$ and $m \geq 3$ be integers. Then the kernel of the reduction map $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$ is torsion-free. Equivalently: if $A \in \mathrm{GL}_n(\mathbb{Z})$ has finite order and $A - 1_n \in m\,\mathrm{M}_n(\mathbb{Z})$, then $A = 1_n$.*

Proof. This is Minkowski's lemma (H. Minkowski, Zur Theorie der positiven quadratischen Formen, J. Reine Angew. Math. 101 (1887)), in the form printed in [223], Lecture I, Lemma 1: "If $m \geq 3$, the kernel of $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$ is torsion free." The second sentence restates it: an element of finite order of the kernel is trivial. The hypothesis $m \geq 3$ cannot be dropped: -1_n has order 2 and lies in the kernel for $m = 2$. □

Theorem B.161. *Let F_v be a non-archimedean local field with ring of integers $\mathcal{O}_v$, $n \geq 1$, $G = \mathrm{GL}_n(F_v)$, $B = TN$ the upper triangular Borel subgroup with diagonal torus T and unipotent radical N, $\bar{N}$ the lower triangular unipotent subgroup and δ_B the modulus character of B. For characters $\xi_1, \dots, \xi_n$ of $F_v^\times$ let $I(\xi_1, \dots, \xi_n)$ be the normalised induction: the smooth functions $f\colon G \to \mathbb{C}$ with $f(tug) = \delta_B^{1/2}(t)\prod_i \xi_i(t_i) f(g)$ for $t = \mathrm{diag}(t_1, \dots, t_n) \in T$, $u \in N$, G acting by right translation. For a smooth representation V of G, V_N is its Jacquet module (the coinvariants of N), a smooth representation of T.*

(Ps1) *Every irreducible subquotient of $I(\xi_1, \dots, \xi_n)$ is isomorphic to a subrepresentation of $I(\xi_{s(1)}, \dots, \xi_{s(n)})$ for some permutation s of $\{1, \dots, n\}$.*

(Ps2) *If $\xi_i\xi_j^{-1} \neq |\cdot|_v$ for all $i \neq j$, then $I(\xi_1, \dots, \xi_n)$ is irreducible.*

(Ps3) *(Jacquet's lemma) If $K \subseteq G$ is a compact open subgroup with $K = (K \cap \bar{N})(K \cap T)(K \cap N)$, then for every admissible representation V of G the projection $V \to V_N$ maps V^K onto $(V_N)^{K\cap T}$.*

Proof. (Ps1): [27], Theorem 2.9 (an irreducible subquotient of a representation induced from a cuspidal representation of a Levi subgroup embeds into the induction of a Weyl conjugate; here the Levi subgroup is T, whose characters are cuspidal, and the Weyl group S_n acts by permuting the ξ_i). (Ps2): [275], Theorem 4.2 (a product of segments that are pairwise not linked is irreducible; the one-element segments $\{\xi_i\}$, $\{\xi_j\}$ are linked exactly when $\xi_j = \xi_i|\cdot|_v^{\pm 1}$), equivalently [27], Theorem 4.2 for GL_n. (Ps3): [62], Theorem 3.3.3. *Notation.* The normalised induction of [27] is $I(\xi_1,\dots,\xi_n)$ (functions transforming on the left, $\delta_B^{1/2}$ included). □

Theorem B.162. *Let F be a number field with ring of integers $\mathcal{O}_F$, $n \geq 1$, and Γ a torsion-free subgroup of $\mathrm{GL}_n(F)$ commensurable with $\mathrm{GL}_n(\mathcal{O}_F)$. Then there are integers $d, r_0, \dots, r_d \geq 0$ and an exact sequence of $\mathbb{C}[\Gamma]$-modules*

$$0 \to \mathbb{C}[\Gamma]^{r_d} \to \cdots \to \mathbb{C}[\Gamma]^{r_1} \to \mathbb{C}[\Gamma]^{r_0} \to \mathbb{C} \to 0,$$

where $\mathbb{C}$ carries the trivial action.

Proof. [43], §11: Γ is an arithmetic subgroup of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$; it acts freely and properly on the contractible Borel–Serre bordification $\overline{X}$ of its symmetric space with compact quotient, a compact manifold with corners, so $\Gamma\backslash\overline{X}$ is a finite $K(\Gamma,1)$ complex and Γ is of type (FL) (loc. cit., 11.4). The cellular chain complex of the universal cover of a finite $K(\Gamma,1)$ CW complex, tensored with $\mathbb{C}$, is the stated resolution (r_q the number of q-cells). □

Theorem B.163. *Let R be a commutative ring of Bass stable rank at most $s \geq 1$. For all integers $i \geq 0$ and $n \geq 2i + s + 2$, the map $H_i(\mathrm{GL}_n(R),\mathbb{Z}) \to H_i(\mathrm{GL}_{n+1}(R),\mathbb{Z})$ induced by $\mathrm{stab}_{n,n+1}$ is an isomorphism.*

Proof. [262, Theorem 4.11]: surjectivity and injectivity of $H_i(\mathrm{GL}_n(R)) \to H_i(\mathrm{GL}_{n+1}(R))$ for n beyond $\max(2i, i+s-1)$, respectively $\max(2i+1, i+s)$ (restated by M. Schlichting [215] as bijectivity of $H_i(\mathrm{GL}_{n-1}) \to H_i(\mathrm{GL}_n)$ for $n-1 \geq \max(2i, s+i)$). The range $n \geq 2i+s+2$ stated here lies inside each of these forms, whether the published inequalities are strict or not. Notation: van der Kallen's $\mathrm{sr}(R)$ is the stable rank; for a commutative noetherian ring of Krull dimension d, $\mathrm{sr}(R) \leq d+1$ (Bass's stable range theorem [19]), so a Dedekind domain has stable rank at most 2. □

Theorem B.164. *Let R be a commutative ring, $\mathrm{GL}(R) = \varinjlim_n \mathrm{GL}_n(R)$ along the stabilisations, $\mathcal{H} = H_\bullet(\mathrm{GL}(R);\mathbb{Q})$ and $\iota_n \colon H_\bullet(\mathrm{GL}_n(R);\mathbb{Q}) \to \mathcal{H}$ the natural maps. Then:*

1. *there is a unique product μ on $\mathcal{H}$ with $\mu(\iota_n x \otimes \iota_m y) = \iota_{n+m}(\oplus_*(x \times y))$, where $\oplus \colon \mathrm{GL}_n(R) \times \mathrm{GL}_m(R) \to \mathrm{GL}_{n+m}(R)$ is the block sum and $\times$ the homology cross product; with it and the coproduct Δ induced by the diagonal maps, $\mathcal{H}$ is a connected graded commutative and cocommutative Hopf algebra over $\mathbb{Q}$;*
2. *$\mathcal{H}$ is the free graded-commutative algebra on its graded subspace $\mathcal{P}$ of primitive elements, and $\mathcal{P}_m \cong K_m(R) \otimes \mathbb{Q}$ for $m \geq 1$ (through the rational Hurewicz map of $B\mathrm{GL}(R)^+$).*

Proof. D. Quillen's plus construction [202], which gives

$$H_\bullet(\mathrm{GL}(R)) = H_\bullet(B\mathrm{GL}(R)^+) \quad \text{and} \quad K_m(R) = \pi_m(B\mathrm{GL}(R)^+),$$

with the H-space structure of $B\mathrm{GL}(R)^+$ induced by block sum ([164]); [177] (a connected commutative cocommutative Hopf algebra over $\mathbb{Q}$ is the free graded-commutative algebra on

its primitives, and for a connected H-space the rational Hurewicz map is an isomorphism onto the primitives). See also [267, Ch. IV]. Well-definedness of μ on the colimit: the two ways of stabilising a block sum differ by conjugation by a permutation matrix, and inner automorphisms act trivially on group homology. □

Theorem B.165. *The notation introduced here is local to this result and to the results that apply it. Let H be a linear algebraic group over $\mathbb{Q}$. A subgroup $\Gamma \subseteq H(\mathbb{Q})$ is* arithmetic *if for some faithful $\mathbb{Q}$-representation $H \hookrightarrow \mathrm{GL}(V_{\mathbb{Q}})$ and some lattice $L \subset V_{\mathbb{Q}}$ (a finitely generated subgroup spanning $V_{\mathbb{Q}}$ over $\mathbb{Q}$) it is commensurable with $\{\gamma \in H(\mathbb{Q}) : \gamma L = L\}$.*

1. *For every faithful $\mathbb{Q}$-representation $H \hookrightarrow \mathrm{GL}(V_{\mathbb{Q}})$ and every lattice $L \subset V_{\mathbb{Q}}$, the stabiliser $\{\gamma \in H(\mathbb{Q}) : \gamma L = L\}$ is arithmetic; every arithmetic subgroup is a discrete subgroup of $H(\mathbb{R})$.*
2. *If H is connected and has no nontrivial character defined over $\mathbb{Q}$, then for every arithmetic subgroup Γ the group $H(\mathbb{R})$ is unimodular and every Borel fundamental domain of Γ in $H(\mathbb{R})$ (for the action by left multiplication) has finite Haar measure.*
3. *If H is connected and reductive and $P \subseteq H$ is a parabolic subgroup defined over $\mathbb{Q}$, then for every arithmetic subgroup Γ the double coset space $\Gamma\backslash H(\mathbb{Q})/P(\mathbb{Q})$ is finite.*

Proof. **Quotation.** Parts (1) and (2): [42] (independence of arithmeticity of the representation and the lattice; discreteness; the finite-volume criterion: $\Gamma\backslash H(\mathbb{R})$ has finite invariant volume if and only if the identity component of H has no nontrivial $\mathbb{Q}$-character). Part (3), the finiteness of the number of cusps: [38]; also A. Borel, Introduction aux groupes arithmétiques (Hermann, 1969), and V. Platonov and A. Rapinchuk, Algebraic groups and number theory (Academic Press, 1994), Chapter 4. Unimodularity in (2): a locally compact group containing a lattice is unimodular.

Notation map. We use it for $H = \mathrm{Res}_{K/\mathbb{Q}} H'$, H' a linear algebraic group over a number field K: then $H(\mathbb{Q}) = H'(K)$, $H(\mathbb{R}) = H'(K \otimes_{\mathbb{Q}} \mathbb{R})$, a parabolic K-subgroup P' of H' gives the parabolic $\mathbb{Q}$-subgroup $\mathrm{Res}_{K/\mathbb{Q}} P'$, and the $\mathbb{Q}$-characters of H correspond to the K-characters of H'; a K-vector space V with an $\mathcal{O}_K$-lattice L is a $\mathbb{Q}$-vector space with the lattice L. □

Theorem B.166. *Let F be a finite extension of $\mathbb{Q}_p$, $n \geq 1$, $G = \mathrm{GL}_n(F)$, $K = \mathrm{GL}_n(\mathcal{O}_F)$, $I_F \subseteq W_F$ the inertia and Weil groups, and* rec *the local Langlands correspondence of Harris–Taylor between irreducible smooth representations of G over $\mathbb{C}$ and n-dimensional Frobenius-semisimple Weil–Deligne representations (r, N) of W_F over $\mathbb{C}$. Let $\tau \colon I_F \to \mathrm{GL}_n(\mathbb{C})$ be an* ***inertial type****: a representation with open kernel which extends to W_F. Then there is a finite-dimensional smooth irreducible representation $\sigma(\tau)$ of K over $\mathbb{C}$ such that for every irreducible smooth representation π of G over $\mathbb{C}$: the restriction $\pi|_K$ contains a copy of $\sigma(\tau)$ if and only if $\mathrm{rec}(\pi)|_{I_F} \cong \tau$ and $N = 0$ on $\mathrm{rec}(\pi)$; in this case $\pi|_K$ contains exactly one copy of $\sigma(\tau)$.*

Proof. See [57, Theorem 3.7] (stated there over $\overline{\mathbb{Q}}_p$ through a fixed isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$; the underlying results are over $\mathbb{C}$). It summarises [217, Proposition 6.2 and §6] ($\sigma(\tau) = \sigma_{\mathcal{P}_{\max}}(\lambda)$ for the maximal partition-valued function), and [74, Theorem 4.1]. The two conditions on $\mathrm{rec}(\pi)$ are unchanged when rec is replaced by $\pi \mapsto \mathrm{rec}(\pi \otimes |\det|^s)$ for any s, since this twists $\mathrm{rec}(\pi)$ by an unramified character, which is trivial on I_F and does not change N. So the statement does not depend on the normalisation of rec. □

Theorem B.167. *Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, uniformiser ϖ and residue field of order q; let $G = \mathrm{GL}_n(K)$, $\mathrm{Iw} \subseteq \mathrm{GL}_n(\mathcal{O}_K)$ the subgroup of matrices that are*

*upper triangular modulo ϖ, and $\mathcal{H} = C_c(\mathrm{Iw}\backslash G/\mathrm{Iw})$ the algebra of compactly supported bi-*Iw*-invariant functions $G \to \mathbb{C}$ under convolution for the Haar measure giving* Iw *measure* 1. *For $g \in G$ put $T_g = 1_{\mathrm{Iw}g\mathrm{Iw}}$; for $w \in S_n$ let $T_w = T_g$ with g the permutation matrix of w; for $\lambda \in \mathbb{Z}^n$ put $\varpi^\lambda = \mathrm{diag}(\varpi^{\lambda_1}, \dots, \varpi^{\lambda_n})$, call λ dominant if $\lambda_1 \geq \cdots \geq \lambda_n$, and put $\langle \rho, \lambda \rangle = \sum_{i=1}^n \frac{n+1-2i}{2}\lambda_i$. For $1 \leq a < n$ let $s_a = (a\ a{+}1) \in S_n$, and let $\ell(w)$ be the number of inversions of w. Then:*

(IH1) *(Iwahori–Matsumoto) $T_{s_a}^2 = (q-1)T_{s_a} + q \cdot 1$, and $T_{s_a w} = T_{s_a}T_w$ whenever $\ell(s_a w) = \ell(w) + 1$; the elements T_g, g running over representatives of* Iw$\backslash G/$Iw*, form a basis of $\mathcal{H}$.*

(IH2) *(Bernstein) There is a unique homomorphism of groups $\lambda \mapsto \Theta_\lambda$ from $\mathbb{Z}^n$ to $\mathcal{H}^\times$ with $\Theta_\lambda = q^{-\langle \rho, \lambda \rangle} T_{\varpi^\lambda}$ for dominant λ; its linear extension $\Theta\colon \mathbb{C}[\mathbb{Z}^n] \to \mathcal{H}$ is an injective homomorphism of algebras, and $(\lambda, w) \mapsto \Theta_\lambda T_w$ is a bijection from $\mathbb{Z}^n \times S_n$ onto a basis of $\mathcal{H}$.*

(IH3) *(Bernstein relation) For $1 \leq a < n$, $\alpha^\vee = e_a - e_{a+1}$ and every $\mu \in \mathbb{Z}^n$,*

$$T_{s_a}\Theta_\mu = \Theta_{s_a(\mu)}T_{s_a} + (1-q)\,\frac{\Theta_{s_a(\mu)} - \Theta_\mu}{1 - \Theta_{-\alpha^\vee}},$$

the fraction being the element $\Theta(z)$ for the Laurent polynomial $z = (x^{s_a(\mu)} - x^\mu)/(1 - x^{-\alpha^\vee})$.

(IH4) *(Bernstein's centre) $\Theta(\mathbb{C}[\mathbb{Z}^n]^{S_n})$ is the centre of $\mathcal{H}$.*

Proof. See [116], Sections 1 and 2, for a split group G with split maximal torus A and Borel subgroup $B = AN$, the Iwahori subgroup I being the inverse image of $B(k)$ in $G(\mathcal{O})$; the results are originally due to Iwahori–Matsumoto (Publ. Math. IHÉS 25, 1965) for (IH1) and to Bernstein, as presented in G. Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. 2 (1989), Section 3, for (IH2)–(IH4). *Notation.* Here $G = \mathrm{GL}_n$, A the diagonal torus, N the upper triangular unipotent matrices, so that $I = \mathrm{Iw}$, $X_*(A) = \mathbb{Z}^n$ with $\pi^\mu = \varpi^\mu$; a coweight μ is dominant for B exactly when $\mu_1 \geq \cdots \geq \mu_n$ (it is the condition $\pi^\mu(I \cap N)\pi^{-\mu} \subseteq I \cap N$ of the cited text, Section 1); ρ is half the sum of the roots $e_i - e_j$, $i < j$, so $\langle \rho, \mu \rangle$ is as stated; the simple coroots are $e_a - e_{a+1}$ with simple reflections s_a. In the cited text Θ_λ is defined through the universal unramified principal series and shown (Remark after Lemma *bases*) to equal $q^{\langle \rho, -\lambda_1 + \lambda_2 \rangle} T_{\pi^{\lambda_1}} T_{\pi^{\lambda_2}}^{-1}$ for $\lambda = \lambda_1 - \lambda_2$ with λ_1, λ_2 dominant, which is the homomorphism of (IH2). Lemma *bases* gives $\mathbb{C}[X_*] \otimes \mathcal{H}_0 \cong \mathcal{H}$, $\pi^\mu \otimes h \mapsto \Theta_\mu h$, with $\mathcal{H}_0$ spanned by the T_w, $w \in S_n$, which is the basis statement of (IH2). Finally, (IH3) is the displayed Bernstein relation there, and (IH4) is the theorem of Section 2 that R^W is the centre of $\mathcal{H}$. $\square$

Theorem B.168. *Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, uniformiser ϖ and residue field of order q. Let $G = \mathrm{GL}_n(K)$,* Iw $\subseteq \mathrm{GL}_n(\mathcal{O}_K)$ *the matrices upper triangular modulo ϖ, and* $\mathrm{Iw}_p \subseteq$ Iw *those that are moreover unipotent modulo ϖ. Let $\mathcal{N} \subseteq G$ be the group of monomial matrices and, for $g \in \mathcal{N}$, $\ell(g)$ the integer with $|\mathrm{Iw}g\mathrm{Iw}/\mathrm{Iw}| = q^{\ell(g)}$. Let $\mathcal{H}^{(1)}$ be the ring of compactly supported bi-Iw_p-invariant functions $G \to \mathbb{Z}$ under convolution for the Haar measure giving* Iw_p *measure* 1, *and put $T_g = 1_{\mathrm{Iw}_p g \mathrm{Iw}_p}$. Then $G = \bigsqcup_g \mathrm{Iw}_p g \mathrm{Iw}_p$ with g running over $\mathcal{N}$ modulo the diagonal matrices $\equiv 1$ modulo ϖ, $|\mathrm{Iw}_p g \mathrm{Iw}_p/\mathrm{Iw}_p| = q^{\ell(g)}$, and for $g, g' \in \mathcal{N}$ with $\ell(gg') = \ell(g) + \ell(g')$*

$$T_g T_{g'} = T_{gg'}.$$

Proof. See [264], Theorem 1, the Iwahori–Matsumoto presentation of the pro-p Iwahori Hecke ring of a split reductive group (braid relations), specialised to GL_n; for the Iwahori subgroup itself these are the relations of N. Iwahori and H. Matsumoto (Publ. Math. IHÉS 25, 1965). *Notation.* The pro-p Iwahori subgroup is Iw_p, the group $W(1)$ of loc. cit. is $\mathcal{N}$ modulo the diagonal matrices $\equiv 1 \bmod \varpi$, and its length is the length of the image in the extended affine Weyl group, which equals $\log_q |\mathrm{Iw} g \mathrm{Iw}/\mathrm{Iw}|$. $\square$

B.10 Combinatorics and other

Amitsur's formula for the determinant of a sum of matrices (Theorem B.169) is used in the Ribet budget theorem (Section 22). Artin's basis of the polynomial ring over the symmetric polynomials, the hook-content formula and the classical expansions of Schur polynomials (Theorem B.170, Theorem B.171, Theorem B.172) are used for the model slopes of Part VI (Section 29), and the degree of the complete flag variety (Theorem B.173) for the properties of good root towers (Section 30). Borsuk's nerve theorem (Theorem B.174) is used for the second cohomology of Tate twists in Part VII (Section 40).

Theorem B.169. *Let C be a commutative ring, $N, k \geq 1$ integers, $A_1, \dots, A_k \in \mathrm{M}_N(C)$, and $t_1, \dots, t_k$ commuting indeterminates. For a word $w = i_1 i_2 \cdots i_p$ in the alphabet $\{1, \dots, k\}$ put $A_w = A_{i_1} A_{i_2} \cdots A_{i_p}$ and $t_w = t_{i_1} t_{i_2} \cdots t_{i_p}$. Then, in the ring of formal power series $C[[t_1, \dots, t_k]]$,*

$$\det\Big(1 - \sum_{i=1}^{k} t_i A_i\Big) = \prod_{w} \det(1 - t_w A_w),$$

the product running over all Lyndon words w in the alphabet $\{1, \dots, k\}$ (the non-empty words that are strictly smaller, for the lexicographic order, than each of their proper suffixes); it converges in the $(t_1, \dots, t_k)$-adic topology, since t_w has degree equal to the length of w and there are finitely many words of each length.

Proof. [6]; short proof from Lyndon's factorisation of the free monoid by C. Reutenauer and M.-P. Schützenberger [207]; the form used here is [92, (1.12)], applied to the matrices $t_i A_i$ with entries in $C[[t_1, \dots, t_k]]$. The identity is one of formal power series with coefficients that are integral polynomials in the entries of the A_i, so it holds over every commutative ring. $\square$

Theorem B.170. *Let $n \geq 1$, let $\Lambda_n \subseteq \mathbb{R}[y_1, \dots, y_n]$ be the subring of symmetric polynomials and $P_k = \sum_{i=1}^{n} y_i^k$.*

(a) $\mathbb{R}[y_1, \dots, y_n]$ is a free Λ_n-module with basis the monomials $y_1^{e_1} \cdots y_n^{e_n}$ with $0 \leq e_i \leq i - 1$ for every i.

(b) $\Lambda_n = \mathbb{R}[P_1, \dots, P_n]$.

Proof. (a) is E. Artin's basis theorem ([10]; it holds over any commutative ring of coefficients). (b) is the fundamental theorem on symmetric polynomials ($\Lambda_n = \mathbb{R}[e_1, \dots, e_n]$ for the elementary symmetric polynomials) combined with Newton's identities $k e_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} P_i$, which express $e_1, \dots, e_n$ as polynomials in $P_1, \dots, P_n$ over $\mathbb{Q}$. Notation: y_i are the variables of one block of the Chern-root ring in the application, P_k its power sums. $\square$

Theorem B.171. *Let $n \geq 1$ and $w = (w_1, \dots, w_n)$ indeterminates. For $e = (e_1, \dots, e_n) \in \mathbb{N}^n$ let $a_e(w) = \det(w_i^{e_j})_{1 \leq i,j \leq n}$, and let $\delta_n = (n-1, n-2, \dots, 1, 0)$, so that $a_{\delta_n}(w) = \prod_{1 \leq i < i' \leq n} (w_i - w_{i'})$. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ be a partition with at most n parts, $|\lambda| = \sum_i \lambda_i$.*

(a) $a_{\lambda+\delta_n}$ *is divisible by* a_{δ_n} *in* $\mathbb{Z}[w]$*, and the quotient, the* Schur polynomial

$$s_\lambda(w) = a_{\lambda+\delta_n}(w)/a_{\delta_n}(w),$$

equals $\sum_T w^T$*, the sum over the semistandard Young tableaux* T *of shape* λ *with entries in* $\{1,\dots,n\}$*, where* $w^T = \prod_i w_i^{\#\{\text{entries of } T \text{ equal to } i\}}$*. In particular* s_λ *is homogeneous of degree* $|\lambda|$ *and all its coefficients are non-negative integers.*

(b) (Hook-content formula.) $s_\lambda(1,\dots,1) = \prod_{x\in\lambda} \dfrac{n+c(x)}{h(x)}$*, where for the cell* $x=(i,j)$ *of the diagram of* λ *(row* i*, column* j*) the content is* $c(x) = j-i$ *and the hook length is* $h(x) = \lambda_i - j + \lambda'_j - i + 1$*,* λ' *the conjugate partition.*

Proof. Classical; see [165], Chapter I: (a) is the definition of s_λ as a quotient of alternants, Ch. I, (3.1), together with the combinatorial description, Ch. I, (5.12) (Jacobi's bialternant formula and Littlewood's tableau formula); (b) is Ch. I, §3, Example 4, at $q=1$ (Stanley's hook-content formula). Notation: w plays the role of the negatives of the entries of one block of a weight in the application; s_λ is used only inside the calculus of the Chern-root ring. □

Theorem B.172. *Let* $m, M \geq 1$*, and let* $u=(u_1,\dots,u_m)$*,* $w=(w_1,\dots,w_M)$ *be indeterminates. For a partition* λ *with more than* m *parts put* $s_\lambda(u)=0$*;* λ' *is the conjugate partition. For partitions* $\mu\subseteq\lambda$ *let* $s_{\lambda/\mu}(w) = \sum_T w^T$*, the sum over the semistandard tableaux* T *of skew shape* λ/μ *with entries in* $\{1,\dots,M\}$*; in particular all coefficients of* $s_{\lambda/\mu}$ *are non-negative integers, and* $s_{\lambda/\lambda}=1$*.*

(a) (Basis.) For every $r\geq 0$ *the* $s_\lambda(u)$ *with* $|\lambda|=r$ *and at most* m *parts form a basis of the real vector space of symmetric polynomials in* u*, homogeneous of degree* r*.*

(b) (Power sums.) For $r\geq 1$*,* $\sum_{i=1}^m u_i^r = \sum_{j=0}^{r-1}(-1)^j\, s_{(r-j,1^j)}(u)$*, where* $(r-j,1^j)$ *is the hook with first row* $r-j$ *and* j *further rows of length one.*

(c) (Dual Cauchy identity.) $\prod_{i=1}^m\prod_{j=1}^M(1+u_iw_j) = \sum_\lambda s_\lambda(u)\, s_{\lambda'}(w)$*, the sum over all partitions* λ *(only finitely many terms are non-zero).*

(d) (Branching.) For every partition λ*,* $s_\lambda(u_1,\dots,u_m,w_1,\dots,w_M) = \sum_{\mu\subseteq\lambda} s_\mu(u)\, s_{\lambda/\mu}(w)$*.*

(e) (Littlewood–Richardson positivity.) For partitions λ,μ*,* $s_\lambda(u)s_\mu(u) = \sum_\nu c^\nu_{\lambda\mu}\, s_\nu(u)$ *with non-negative integers* $c^\nu_{\lambda\mu}$ *(the sum over* ν *with* $|\nu| = |\lambda|+|\mu|$*).*

Proof. Classical; see [165], Chapter I: (a) is (3.2)–(3.3) (the Schur polynomials in m variables with at most m parts form a basis of the symmetric polynomials); (b) is §3, Example 11 (the expansion of p_r in hook Schur functions, a case of the Murnaghan–Nakayama rule), specialised to m variables, where the s_λ with more than m parts vanish; (c) is (4.3'); (d) is (5.9) with the tableau formula (5.12) for skew Schur functions; (e) is the Littlewood–Richardson rule (9.2), or (5.3) with the positivity of skew Schur functions. Notation: u will be the shifted entries of one side of one block of a weight, w a vector of ones. □

Theorem B.173. *Let* $n\geq 2$*,* $G=\mathrm{SL}_n(\mathbb{C})$*,* $B\subseteq G$ *the Borel subgroup of upper triangular matrices and* $N = n(n-1)/2 = \dim G/B$*. Let* $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n$ *with* $\lambda_1>\lambda_2>\cdots>\lambda_n$ *(a regular dominant weight, taken modulo* $\mathbb{Z}(1,\dots,1)$*), let* V_λ *be the irreducible representation*

of G with highest weight λ, $v_\lambda \in V_\lambda$ a highest weight vector, $e_\lambda \colon G/B \to \mathbb{P}(V_\lambda)$, $gB \mapsto [g\,v_\lambda]$, and $\mathcal{L}_\lambda = e_\lambda^* \mathcal{O}(1)$. Then e_λ is a closed embedding and

$$\int_{G/B} c_1(\mathcal{L}_\lambda)^N = N! \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j}{j - i}.$$

Proof. [201]: the Proposition of the section on degrees of Schubert varieties ($\deg_\lambda(X_w) = \ell(w)!\,\mathfrak{D}_w(\lambda)$ for dominant λ, with $\deg_\lambda(X_w) = \langle [X_w] \cdot c_1(\mathcal{L}_\lambda)^{\ell(w)} \rangle$ and e_λ an embedding for regular λ), applied to the longest element $w_\circ$ ($X_{w_\circ} = G/B$, $\ell(w_\circ) = N$), and the Corollary $\mathfrak{D}_{w_\circ}(y) = \prod_{\alpha>0}(y, \alpha^\vee)/(\rho, \alpha^\vee)$ of the section on the asymptotic expression for degrees (there attributed to Stembridge; it is Weyl's dimension formula plus the asymptotics of $\dim V_{k\lambda}$; the formula goes back to Borel and Hirzebruch). Notation: for SL_n the positive roots are $\alpha = \epsilon_i - \epsilon_j$ $(i < j)$, $(\lambda, \alpha^\vee) = \lambda_i - \lambda_j$ and $(\rho, \alpha^\vee) = j - i$. □

Theorem B.174. *Let $C_1, \dots, C_q$ ($q \ge 1$) be nonempty compact convex subsets of a finite-dimensional real vector space. Let $\mathcal{K}$ be the nerve of the family: the abstract simplicial complex with vertex set $\{1, \dots, q\}$ whose simplices are the nonempty subsets $J \subseteq \{1, \dots, q\}$ with $\bigcap_{j \in J} C_j \neq \emptyset$. Then the geometric realisation of $\mathcal{K}$ is homotopy equivalent to $C_1 \cup \cdots \cup C_q$.*

Proof. [44] (the nerve theorem: a compact metric space that is the union of finitely many closed sets all of whose nonempty intersections are absolute retracts has the homotopy type of the nerve). The convex case stated here is the special case in which every nonempty intersection is a nonempty compact convex set of a Euclidean space, hence an absolute retract. The C_j are the closed sets of the cover, and $\mathcal{K}$ is Borsuk's nerve. □

B.11 Computer checks cited in proofs

The proofs of nine results cite nine computer checks, C1–C9, listed below in the order in which they are first cited, each with what it computes and the results whose proofs cite it. In the proofs of Lemma 51.12, Lemma 51.15 and Lemma 52.5 the computation is a step of the proof: the first two establish identities between explicit matrices with polynomial entries, and the third is a finite case check, whose reduction to finitely many cases is proved in the text. The other six computations accompany steps that are proved in the text and check them by a second route; for Theorem D.87, the closed form of the sphere moments is Lemma D.94. The code and its documentation are available from the authors.

- C1, cited in Theorem 34.20: the inequality $8\pi e^\gamma < 49$ in exact rational arithmetic from $\gamma < 0.5773$ and $\pi < 3.1416$; and, for $E = \mathbb{Q}(\sqrt{-d})$ with $d \in \{7, 23, 71, 103, 199\}$ and $p, q \le 12$, a numerical comparison of the flat height with $\frac{p-q}{2}$ times the Chowla–Selberg average of Lemma 34.21, and of the bound (Fd3).
- C2, cited in Lemma 34.21 and Theorem 34.20: the normalisations in the period of an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-d})$, numerically for seven discriminants, against the eta-quotient form of the Chowla–Selberg formula and against a published Faltings height.
- C3, cited in Lemma 51.12: clauses (Ie3)–(Ie5), that is, the explicit form of the induced embedding on pairs of lower triangular matrices, the similitude of the flag matrices and the triangular shape after conjugation by them, as identities of polynomials with integer coefficients after clearing denominators; with a second implementation in exact rational arithmetic at random rational points.

- C4, cited in Lemma 51.15: clauses (Te3)–(Te5), together with the Kronecker-product form of the tensor embedding and its similitude formula, as polynomial identities, by two independent implementations.
- C5, cited in Lemma 52.5: under the sign hypotheses (h1)–(h8), (m1)–(m4) on two integers A, B, one of the six conditions that become (Zc1)–(Zc6) holds; exhaustively over $(A, B) \in [-2, 2]^2$ and all 2^8 assignments, with a second method by exact Fourier–Motzkin elimination.
- C6, cited in Lemma 53.37: for an involution of similitude -1 acting on $\mathfrak{sp}_4$, the diagonal case exactly over $\mathbb{Z}[1/2]$, and the dimensions 6 and 4 of the eigenspaces for 200 random involutions over $\mathbb{Q}$ and modulo 3, 5 and 7.
- C7, cited in Theorem D.87 and Lemma D.94: the sphere-moment integrals of the fibre square in closed form, by generating functions.
- C8, cited in Lemma D.88: the symbolic steps of the push-forward of the powers of the relative tangent class along a $\mathbb{P}^1$-bundle, and its coefficients for $N \leq 60$.
- C9, cited in Lemma D.94: the moments S_k for $1 \leq k \leq 12$ in exact rational arithmetic, by two methods that share no step (the double integral against Q_k, and the Funk–Hecke expansion in Legendre polynomials on $S^2 \times S^2$), with a direct quadrature for $k \leq 3$.

C Weight words along Frobenius orbits

This appendix proves the three colength bounds on which the row bound Proposition 26.2 of Section 26 rests, one for each configuration of weight word treated at the primes of inert type: Proposition C.1 for alternating words with a thin site (Section C.1), Proposition C.52 for alternating words that are thick at every site (Section C.2), and Proposition C.77 for long Frobenius orbits whose thick sites form isolated clusters (Section C.3). No other result of the paper uses the appendix. Its proofs are long and their interest is combinatorial: they decide, site by site along the Frobenius orbit, at what cost a crystalline class can be lifted along a surjection of lattices.

The pattern is that of the lifting exponent of Section 24, carried out along longer arcs of sites. The sites are divided by *cuts*, a class is split into relay classes at the cuts (Proposition 24.4), and each relay class is lifted along the arc that ends at its cut, with the level data of the arc recording the weight word (Subsection 23.1). For alternating words with a thin site this gives first a killing exponent with few exceptional classes: $\ell^e H^1_f(D, T_2)$ lies in $\Phi_* H^1_f(D, T_1)$ plus the span of at most $4nm$ classes, with e of size h/ℓ^2 plus lower-order terms (Proposition C.2). On long orbits with isolated thick clusters it gives a lifting exponent of size $h/(\ell-1)^{j+1} + (G+1)(j+1) + \log_\ell h$, with no exceptional classes (Proposition C.79). In both cases the colength follows by the generator count Lemma 27.3 and Lemma 27.14. For words that are thick everywhere the lifting is done at the down sites, with an exponent of order H'/ℓ^2, on a submodule reached in at most six steps of bounded generation and three steps of bounded length (Proposition C.57). The appendix uses the Wach modules, levels, relay classes and lifting lemmas of Sections 23 and 24, the two counting lemmas of Section 27, and the Γ-weight separation lift of Section 28.

C.1 Alternating weight words: splitting along arcs

This section proves Proposition C.1, the colength bound for an alternating weight word with a thin site, which is item 1 of the row bound Proposition 26.2 in Section 26. The colength follows

from a killing exponent with at most $4nm$ exceptional classes (Proposition C.2) by the counting lemmas 27.3 and Lemma 27.14. The cuts are the sites c for which $c+1$ is thin; a crystalline class is split into relay classes at the cuts, and the relay class at each cut is lifted along the arc that ends there (Proposition C.4). The lifting of one graded section is proved separately for arcs of length one, two and three (Proposition C.25). Subsection C.1.1 sets up the arcs, Subsection C.1.2 treats one cut, and Subsections C.1.3 and C.1.4 lift graded sections along arcs of each length.

C.1.1 Colength bounds and the arc setup

The subsection states Proposition C.1 and Proposition C.2 and reduces the exponent to the arc part Proposition C.4; when b is below $160\,\mathrm{Sg}$, the first-order lifting exponent is at most $160\,\mathrm{Sg}$ (Lemma C.3). The arc setup Proposition C.5 writes every crystalline class as the class of an extension by the trivial lattice whose unit lifts to the Wach module at every cut up to ℓ^{r_0}, with $r_0 \le 3\,\mathrm{Sg}$. It rests on Wach lifting along onto maps of lattices (Proposition C.6): a site vector lifts up to an explicit power of ℓ, computed from the killing indices of the cokernel along the Frobenius cycle (Lemma C.7). Proposition C.6 is used again in Sections C.2 and C.3.

Proposition C.1. *Let $\ell \ge 11$ be a prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of degree n, L as in Definition 23.1, $(\tau_a)_{a\in\mathbb{Z}/n}$ a Frobenius numbering, $\Phi\colon V_1 \to V_2$ an onto map of crystalline lattices $T_1 \to T_2$ over F with coefficients L, and $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ a weight word for Φ. Let $M_a, h_a \ge 0$ be integers such that every Hodge jump of V_1 at τ_a lies in $[-M_a, h_a]$, and $e = e(M_\bullet, h_\bullet)$. Suppose that some site is thin, that $\epsilon_{a+1} \ne \epsilon_a$ for every a, that no three consecutive sites $a, a+1, a+2$ are all thick, and that $20w \le b$, $20\eta \le b$, $20S \le b$. Then the colength of Φ is at most*

$$68\,c_{\mathrm{ur}}\Big((n+1)\,d_2\Big(\frac{h}{\ell^2} + \frac{w+\eta}{\ell} + S + 1 + \log_\ell(2h) + \eta\,\beta_{320}\Big) + n\,m\,e\Big).$$

Proof. Put $D = G_F$, $H = H^1_f(D, T_2)$ and $H' = \Phi_* H^1_f(D, T_1) \subseteq H$; the colength of Φ is $\mathrm{nl}(H/H')$.

The three inputs. (i) V_2 is crystalline, hence de Rham, so by Lemma 27.3 H is generated by at most $g = (n+1)d_2$ elements. (ii) The hypotheses of the statement are those of Proposition C.2, which gives an integer e_0 with $0 \le e_0 \le 55\,c_{\mathrm{ur}}\,\mathrm{Sg} + 13S + \eta\beta_{320}$, $\mathrm{Sg} = h/\ell^2 + (w+S+\eta)/\ell + \log_\ell(2h) + 1$, and a finite $Z \subseteq H$ with $|Z| \le 4nm$ and $\ell^{e_0} H \subseteq H' + \sum_{z\in Z} \mathcal{O}_L z$. (iii) ℓ is odd, F is unramified, L contains $\sigma(F(\mu_\ell))$ for every embedding σ, V_1, V_2 are crystalline, $\Phi(T_1) = T_2$, and every Hodge jump of V_1 at τ_a lies in $[-M_a, h_a]$; so Proposition 24.1, applied with $a^-_{\tau_a} = M_a$ and $a^+_{\tau_a} = h_a$, gives $\ell^e H \subseteq H'$ with $e = e(M_\bullet, h_\bullet)$; in particular $\ell^e z \in H'$ for every $z \in Z$.

The colength. By Lemma 27.14 with these g, e_0, Z and $e' = e$,

$$\mathrm{nl}(H/H') \le (n+1)\,d_2\,e_0 + 4nm\,e.$$

Put $X = h/\ell^2 + (w+\eta)/\ell + S + 1 + \log_\ell(2h) + \eta\beta_{320}$. Every term of X is non-negative ($h \ge 1$, so $\log_\ell(2h) > 0$), and $(w+S+\eta)/\ell \le (w+\eta)/\ell + S$, so $\mathrm{Sg} \le X - \eta\beta_{320}$, hence, as $c_{\mathrm{ur}} \ge 1$, $e_0 \le 55c_{\mathrm{ur}}(X - \eta\beta_{320}) + \eta\beta_{320} + 13S \le 55c_{\mathrm{ur}}X + 13X \le 68\,c_{\mathrm{ur}}X$. Hence

$$\mathrm{nl}(H/H') \le 68\,c_{\mathrm{ur}}\,(n+1)\,d_2\,X + 4\,n\,m\,e \le 68\,c_{\mathrm{ur}}\big((n+1)\,d_2\,X + n\,m\,e\big),$$

which is the assertion. □

Proposition C.2. *Let $\ell \geq 11$ be a prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of degree n, $D = G_F$, L with ring of integers $\mathcal{O}_L$ as in Definition 23.1, $(\tau_a)_{a\in\mathbb{Z}/n}$ a Frobenius numbering, $\Phi\colon V_1 \to V_2$ an onto map of crystalline lattices $T_1 \to T_2$ over F with coefficients L, and $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ a weight word for Φ. Suppose that some site is thin, that $\epsilon_{a+1} \neq \epsilon_a$ for every a, that no three consecutive sites $a, a+1, a+2$ are all thick, and that $20w \leq b$, $20\eta \leq b$, $20S \leq b$. Put*

$$\mathrm{Sg} = \frac{h}{\ell^2} + \frac{w + S + \eta}{\ell} + \log_\ell(2h) + 1.$$

Then there are an integer e with $0 \leq e \leq 55\, c_{\mathrm{ur}}\,\mathrm{Sg} + 13S + \eta\,\beta_{320}$ and a finite subset $Z \subseteq H^1_f(D, T_2)$ with $|Z| \leq 4nm$ such that

$$\ell^e\, H^1_f(D, T_2) \subseteq \Phi_* H^1_f(D, T_1) + \sum_{z\in Z} \mathcal{O}_L z.$$

Proof. Put $H = H^1_f(D, T_2)$ and $H_1 = \Phi_* H^1_f(D, T_1)$. Since all terms of Sg are non-negative ($h \geq 1$), $\mathrm{Sg} \geq 1 + \log_\ell(2h) \geq 1$.

Case 1: some site is thick and $b < 160\,\mathrm{Sg}$. By Lemma C.3 there is an integer e' with $0 \leq e' \leq 19\, c_{\mathrm{ur}}\,\mathrm{Sg} \leq 55\, c_{\mathrm{ur}}\,\mathrm{Sg}$ and $\ell^{e'} H \subseteq H_1$; take $e = e'$ and $Z = \emptyset$.

Case 2: no site is thick, or $160\,\mathrm{Sg} \leq b$. The hypotheses of Proposition C.4 hold; let C, len_c, J_c, κ_c be as there, and r_0, σ, E, X_c, Z_c the data it provides. We first record three facts about the cuts.

(a) *The arcs partition the orbit:* $\sum_{c\in C} \mathrm{len}_c = n$. Some site a is thin, so $a - 1 \in C \neq \emptyset$; for $c \in C$ the integer $d = n$ satisfies $c - d = c \in C$, so $1 \leq \mathrm{len}_c \leq n$. Put $A_c = \{c - t :\ 0 \leq t < \mathrm{len}_c\}$, a set of len_c residues. Every $a \in \mathbb{Z}/n$ lies in some A_c: let $t \geq 0$ be least with $a + t \in C$ ($t \leq n - 1$); then $c = a + t \in C$, and for $1 \leq d \leq t$ the residue $c - d = a + (t - d)$ is not in C by minimality, so $t < \mathrm{len}_c$ and $a = c - t \in A_c$. The sets are disjoint: if $c - t = c' - t'$ with $t < \mathrm{len}_c$, $t' < \mathrm{len}_{c'}$ and $t < t'$, then $c = c' - (t' - t)$ with $1 \leq t' - t < \mathrm{len}_{c'}$, so $c \notin C$, a contradiction; hence $t = t'$ and $c = c'$.

(b) $\mathrm{len}_c \leq 3$. For $1 \leq d < \mathrm{len}_c$, $c - d \notin C$, so the site $c - d + 1$ is not thin, hence thick by (Ow3). Thus the $\mathrm{len}_c - 1$ consecutive sites $c - \mathrm{len}_c + 2, \ldots, c$ are thick; as no three consecutive sites are thick, $\mathrm{len}_c \leq 3$.

(c) $\kappa_c \leq 13S + 3\,\mathrm{Sg}$. Each W_a consists of S consecutive integers i with $1 \leq |i| \leq s_a + S \leq h$ (because $s_a > \eta \geq 0$; and $s_a + S \leq b + \eta + S$ at a thick site, $s_a + S \leq w + S$ at a thin one, both $\leq h$). For $k \geq 1$ at most $S/\ell^k + 1$ of them are divisible by ℓ^k, and none if $\ell^k > h$; so $\sum_{i\in W_a} v_\ell(i) = \sum_{k\geq 1} \#\{i \in W_a :\ \ell^k \mid i\} \leq S/(\ell - 1) + \log_\ell h$. The set J_c is a union of $\mathrm{len}_c \leq 3$ such windows, so $|J_c| \leq \mathrm{len}_c S$, $0 \notin J_c$, and

$$\kappa_c \leq \mathrm{len}_c + \mathrm{len}_c(\mathrm{len}_c + 1)S + \frac{3S}{\ell - 1} + 3\log_\ell h \leq 3 + 12S + \frac{3S}{10} + 3\log_\ell(2h) \leq 13S + 3\,\mathrm{Sg}.$$

The exponent and the exceptional classes. Let $\kappa = \max_{c\in C} \kappa_c$, $e = r_0 + 3\sigma + \kappa + E$ and $Z = \bigcup_{c\in C} Z_c \subseteq H$. For $x \in H$ write $\ell^{r_0+2\sigma} x = \sum_c x_c$ with $x_c \in X_c$; then $\ell^{\sigma+\kappa+E} x_c = \ell^{\kappa-\kappa_c}\, \ell^{\sigma+\kappa_c+E} x_c \in H_1 + \sum_{z\in Z_c} \mathcal{O}_L z$, hence $\ell^e x \in H_1 + \sum_{z\in Z} \mathcal{O}_L z$. By (b) and (a), $|Z| \leq \sum_c m\,\mathrm{len}_c(\mathrm{len}_c + 1) \leq 4m \sum_c \mathrm{len}_c = 4nm$. By the bounds of Proposition C.4 and (c),

$$0 \leq e \leq 3\,\mathrm{Sg} + 9\,\mathrm{Sg} + 13S + 3\,\mathrm{Sg} + 40\,\mathrm{Sg} + \eta\beta_{320} = 55\,\mathrm{Sg} + 13S + \eta\beta_{320}.$$

As $c_{\mathrm{ur}} \geq 1$, in both cases $0 \leq e \leq 55\, c_{\mathrm{ur}}\,\mathrm{Sg} + 13S + \eta\beta_{320}$, $|Z| \leq 4nm$, $Z \subseteq H$, and $\ell^e H \subseteq H_1 + \sum_{z\in Z} \mathcal{O}_L z$. □

Lemma C.3. *Let $\ell \geq 11$ be a prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of degree n, $D = G_F$, L as in Definition 23.1, $(\tau_a)_{a\in\mathbb{Z}/n}$ a Frobenius numbering, $\Phi\colon V_1 \to V_2$ an onto map of crystalline lattices $T_1 \to T_2$ over F with coefficients L, and $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ a weight word for Φ with $20w \leq b$, $20\eta \leq b$, $20S \leq b$, and some site thick. Put*

$$\mathrm{Sg} = \frac{h}{\ell^2} + \frac{w + S + \eta}{\ell} + \log_\ell(2h) + 1,$$

and suppose $b < 160\,\mathrm{Sg}$. Let $(M'_a, h'_a) = (\eta, s_a + S)$ if $\epsilon_a = 1$ and $(M'_a, h'_a) = (s_a + S, \eta)$ if $\epsilon_a = -1$. Then $e' = e(M'_\bullet, h'_\bullet)$ is defined, $0 \leq e' \leq 19\, c_{\mathrm{ur}}\,\mathrm{Sg}$, and

$$\ell^{e'} H^1_f(D, T_2) \subseteq \Phi_* H^1_f(D, T_1).$$

Proof. The windows and the lifting. Since some site is thick, $h = b + \eta + S$. Every size satisfies $s_a \leq b+\eta$: at a thick site by (Ow3), at a thin site because $s_a \leq w \leq b/20$. By (Ow1) every Hodge jump of V_1 at τ_a lies in $[s_a, s_a+S]\cup[-\eta, \eta] \subseteq [-M'_a, h'_a]$ if $\epsilon_a = 1$, and in $[-s_a-S, -s_a]\cup[-\eta, \eta] \subseteq [-M'_a, h'_a]$ if $\epsilon_a = -1$. So e' is defined, and Proposition 24.1 (with $p = \ell$ odd, $K = F$ unramified, L containing $\sigma(F(\mu_\ell))$ for every σ, V_1, V_2 crystalline, $\Phi(T_1) = T_2$, $a^-_{\tau_a} = M'_a$, $a^+_{\tau_a} = h'_a$) gives $\ell^{e'} H^1_f(D, T_2) \subseteq \Phi_* H^1_f(D, T_1)$, with $e' = e_{\mathrm{A}} = \lfloor c_{\mathrm{ur}}(2 + \mathrm{wt}/\ell + \log_\ell(\mathrm{wt} + 1))\rfloor \geq 0$.

The size of e'. With $a^-_{\tau_a} = M'_a$, $a^+_{\tau_a} = h'_a$ we have $\mathrm{wt} = \max_a(s_a + S + \eta) \leq h + \eta$, as $s_a + S \leq b + \eta + S = h$. Since $\eta \leq b/20 \leq h/20$, $\mathrm{wt} \leq \frac{21}{20}h$; and $\mathrm{wt} + 1 \leq h + \eta + 1 \leq 2h$ because $h \geq b + \eta \geq 1 + \eta$. Since all terms of Sg are non-negative ($h \geq 1$), $\mathrm{Sg} \geq 1 + \log_\ell(2h) \geq 1$. Now $h = b + \eta + S \leq \frac{11}{10}b < \frac{11}{10} \cdot 160\,\mathrm{Sg} = 176\,\mathrm{Sg}$, and $\ell \geq 11$, so

$$\frac{\mathrm{wt}}{\ell} \leq \frac{21}{20} \cdot \frac{176}{11}\,\mathrm{Sg} = \frac{84}{5}\,\mathrm{Sg}, \qquad 2 + \log_\ell(\mathrm{wt} + 1) \leq 1 + \big(1 + \log_\ell(2h)\big) \leq 2\,\mathrm{Sg}.$$

Hence $e' \leq c_{\mathrm{ur}}\big(2 + \mathrm{wt}/\ell + \log_\ell(\mathrm{wt} + 1)\big) \leq c_{\mathrm{ur}}\big(\frac{84}{5} + 2\big)\mathrm{Sg} < 19\, c_{\mathrm{ur}}\,\mathrm{Sg}$. □

Proposition C.4. *Let $\ell \geq 11$ be a prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of degree n, $D = G_F$, L with ring of integers $\mathcal{O}_L$ as in Definition 23.1, $(\tau_a)_{a\in\mathbb{Z}/n}$ a Frobenius numbering, $\Phi\colon V_1 \to V_2$ an onto map of crystalline lattices $T_1 \to T_2$ over F with coefficients L, and $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ a weight word for Φ. Suppose that some site is thin, that $\epsilon_{a+1} \neq \epsilon_a$ for every a, that no three consecutive sites are all thick, that $20w \leq b$, $20\eta \leq b$, $20S \leq b$, and that either no site is thick or $160\,\mathrm{Sg} \leq b$, where $\mathrm{Sg} = h/\ell^2 + (w + S + \eta)/\ell + \log_\ell(2h) + 1$. Let $C = \{c \in \mathbb{Z}/n :\ c + 1$ is thin$\}$ (the* cuts*); for $c \in C$ let len_c be the least integer $d \geq 1$ with $c - d \in C$, let*

$$J_c = \bigcup_{0\leq t<\mathrm{len}_c} W_{c-t}, \qquad W_a = \begin{cases} \{i \in \mathbb{Z} :\ -s_a - S \leq i < -s_a\} & \text{if } \epsilon_a = 1, \\ \{i \in \mathbb{Z} :\ s_a \leq i < s_a + S\} & \text{if } \epsilon_a = -1, \end{cases}$$

and $\kappa_c = \mathrm{len}_c + \sum_{i\in J_c}\big(\mathrm{len}_c + 1 + v_\ell(i)\big)$. Then there are integers $r_0, \sigma, E \geq 0$ with

$$r_0 \leq 3\,\mathrm{Sg}, \qquad \sigma \leq 3\,\mathrm{Sg}, \qquad E \leq 40\,\mathrm{Sg} + \eta\,\beta_{320},$$

and, for every $c \in C$, an $\mathcal{O}_L$-submodule $X_c \subseteq H^1_f(D, T_2)$ and a finite subset $Z_c \subseteq H^1_f(D, T_2)$ with $|Z_c| \leq m\,\mathrm{len}_c(\mathrm{len}_c + 1)$, such that

$$\ell^{r_0+2\sigma} H^1_f(D, T_2) \subseteq \sum_{c\in C} X_c \qquad \text{and} \qquad \ell^{\sigma+\kappa_c+E} X_c \subseteq \Phi_* H^1_f(D, T_1) + \sum_{z\in Z_c} \mathcal{O}_L z \quad (c \in C).$$

Proof. **Parameters.** Let $(M_a, h_a)_a$ be the window of Proposition C.5: $(\eta, s_a + S)$ if $\epsilon_a = 1$, $(s_a + S, \eta)$ if $\epsilon_a = -1$; it contains the Hodge jumps of V_1, V_2 and of every extension V_Y produced there. Let h_B, M, m_X, c', H_c, Π_c be as in Proposition C.25 and σ_B as in Proposition C.10. Some site is thin and its size exceeds $\eta \geq 0$, so $w \geq 1$ and $h_B \geq 1$. Put $G = g = w + S$ and $m' = R = m_X$, and $j = 1$. For $c \in C$ the site $c+1$ is thin, so $M_{c+1} \leq \max(\eta, s_{c+1} + S) \leq w + S = G$ (sizes exceed η); every $M_a \leq h_B$ (at a thick site $s_a + S \leq b + \eta + S$), and $m_X = (\ell - 1)M > 2h_B \geq \max(g, 1 + \max_a M_a)$. So G, g, m', R satisfy the hypotheses of Proposition 24.4 (for T_2 and any Y of Proposition C.5) and of Proposition C.10; I_c and $\rho_{c,1}$ are formed for $\mathbf{N}_2 = \mathbf{N}(T_2)$ with these parameters, and $\sigma' = \sigma_\ell(w + S, m_X, 1) = \lfloor (w + S + M)\ell/(\ell - 1)^2 \rfloor + 1 \leq \sigma_B$ (Lemma 24.6).

$\sigma_B \leq 3\,\mathrm{Sg}$**.** With $q = \ell/(\ell - 1)^2$ ($\ell \geq 11$): $q \leq \min(0.11, 1.21/\ell)$ and $q/(\ell - 1) \leq 1.331/\ell^2$ (equalities at $\ell = 11$, ratios decreasing). As $M \leq 2h_B/(\ell - 1) + 1$,

$$\sigma_B \leq (2(w + S) + M)q + 2 \leq \frac{2.42\,(w + S)}{\ell} + \frac{2.662\,h_B}{\ell^2} + 2.11 \leq 3\,\mathrm{Sg},$$

since $h_B \leq h$ ($h_B = h$ if some site is thick, $h_B = w + S < h$ otherwise) and $\log_\ell(2h) \geq 0$. Also $\mathrm{Sg} \geq 1$.

Step 1 (relay classes are Γ-invariant). Let $w_0 \in I_c$ and $u \in \mathbb{Z}_\ell^\times$. Then $\gamma_u w_0 \in I_c$ (Lemma C.16 (d)) and $\rho_{c,1}(\gamma_u w_0) = \rho_{c,1}(w_0)$. Indeed, $\gamma_u W_{w_0} \in X\mathbf{N}_E$ and $(\gamma_v - 1)\gamma_u W_{w_0} = \gamma_u(\gamma_v - 1)w_0 = (\gamma_v - 1)\gamma_u w_0$ for all v, so $W_{\gamma_u w_0} = \gamma_u W_{w_0}$ by the uniqueness in (Hf1). If τ is a relay correction of w_0 with V, then $\gamma_u V - W_{\gamma_u w_0} = \gamma_u(V - W_{w_0}) \in X^R\mathbf{N}_E$ and $\gamma_u\tau - \ell^{\sigma'}\varphi(\gamma_u V) = \gamma_u(\tau - \ell^{\sigma'}\varphi(V)) \in X^{m'}\lambda_1^g P_1(\mathbf{N}_{2,c})[1/\ell]$, because γ_u commutes with φ, preserves $P_1(\mathbf{N}_{2,c})$ and multiplies X and $\lambda_1 = Q_0$ by units (Lemma C.16 (b)); so $\gamma_u\tau$ is a relay correction of $\gamma_u w_0$, and $\rho_{c,1}(\gamma_u w_0) = \mathrm{cl}(\gamma_u y)$ with $y = \ell^{\sigma'}\varphi(w_0) - \tau$, a Herr vector of precision m' (Ry2). Now $\gamma_u y$ is a Herr vector of precision m', $(\gamma_u - 1)y = (\varphi - 1)(z_u(y))$ with $z_u(y) \in X^{m'}\mathbf{N}_2$ and $(\gamma_v - 1)z_u(y) \in X^{m'}\mathbf{N}_2$ for all v; by (Hc4) and (Hc3) of Lemma 23.28, $\mathrm{cl}(\gamma_u y) = \mathrm{cl}(y) + \mathrm{cl}((\varphi - 1)z_u(y)) = \mathrm{cl}(y) = \rho_{c,1}(w_0)$.

Step 2 (splitting). Let $x \in H^1_f(D, T_2)$, and let $r_0 \leq 3\,\mathrm{Sg}$, $0 \to T_2 \xrightarrow{\iota'} Y \xrightarrow{\pi} \mathcal{O}_L \to 0$ with $[Y] = x$ and $x_c \in \mathbf{N}(Y)_c$ ($c \in C$) be as in Proposition C.5; r_0 does not depend on x. The cuts satisfy the hypotheses of Proposition 24.4: $C \neq \emptyset$, and len_c is the least $d \geq 1$ with $c - d \in C$; also $j = 1 \leq \mathrm{len}_c$. So there are $\tau'_c \in P_1(\mathbf{N}_{2,c})$ and $w_c \in I_c$ with

$$\mathbf{N}(\iota')(w_c) = \varphi^{e-1}(X_{c'}) - \ell^{\sigma'}x_c, \qquad X_{c'} = \ell^{\sigma'}\varphi(x_{c'}) - \mathbf{N}(\iota')(\tau'_{c'}), \qquad \ell^{r_0 + 2\sigma'}x = \sum_{c \in C} \rho_{c,1}(w_c),$$

where $e = \mathrm{len}_c$. Put $\bar{w}_c = (\ell - 1)^{-1}\sum_{u \in \mu_{\ell-1}} \gamma_u w_c \in I_c$; it is Δ-invariant (that is, fixed by γ_u for $u \in \mu_{\ell-1}$), and $\rho_{c,1}(\bar{w}_c) = \rho_{c,1}(w_c)$ by Step 1 and (Ry4). Hence $\ell^{r_0 + 2\sigma'}x = \sum_c \rho_{c,1}(\bar{w}_c)$.

Step 3 (the vectors $\bar{w}_c$ satisfy the hypotheses of Proposition C.10). Fix $c \in C$, $e = \mathrm{len}_c$. Since $\mathbf{N}(\iota')$ is $\mathbf{A}_{F,L}$-linear and commutes with φ, $X_{c'} \in P_1(\mathbf{N}(Y)_{c'})$, so $\varphi^{e-1}(X_{c'}) \in P_e(\mathbf{N}(Y)_{c'})$ (as $\varphi^{e-1}(a\varphi(y)) = \varphi^{e-1}(a)\varphi^e(y)$), and $\ell^{\sigma'}x_c \in \mathbf{N}(Y)_c$. Both modules are γ_u-stable and $\mathbf{N}(\iota')$ commutes with γ_u, so $\mathbf{N}(\iota')(\bar{w}_c) = p - y$ with $p \in P_e(\mathbf{N}(Y)_{c'})$, $y \in \mathbf{N}(Y)_c$. (i) *Arc membership.* Every Hodge jump of V_Y at τ_a lies in $[-M_a, h_a] \subseteq [-(s_a + S), s_a + S] \subseteq [-H_c(a), H_c(a)]$ ($s_a + S \leq w + S$ at thin sites, $\leq h_B$ everywhere, as in Lemma C.16 (1)); so $(H_c(a), H_c(a))_a$ is a window of V_Y, and Lemma C.50 (Ae2) (with c', e, $H = H_c$) gives $\Pi_c\bar{w}_c \in P_e(\mathbf{N}_{2,c'})$. (ii) *Pole bounds.* Let $\mathbf{n}$ be a frame at c' of $\mathbf{N}_2$ fixed by the γ_u, $u \in \mu_{\ell-1}$, and let $\mathbf{n}_Y$ be any frame at c' of $\mathbf{N}(Y)$ (it is free by (Wm1)). Lemma C.51 (with c', e, $a = c$, $H = H_c$, $m = m_X \geq 2H_c(c - s)$) gives $t \geq 0$ and an arc coordinate vector of $\ell^t\bar{w}_c$ for $\mathbf{n}$, e, H_c, m_X that is pole-bounded by B^c, provided that $\varphi^e(\varepsilon_{c'+3}D_{\mathrm{cris}}(V_2)) \cap \varphi^{s+4}(\mathrm{Fil}^{s_{c-s}+S+1}D_{\mathrm{cris}}(V_2)) = 0$ for every $s < e$ with $\epsilon_{c-s} = 1$. This holds: by the remark in Definition 23.3, $\varphi^{-(s+4)}$ maps $\varphi^e(\varepsilon_{c'+3}D_{\mathrm{cris}}(V_2))$ into the τ_{c-s}-component $\varepsilon_{c-s-1}D_{\mathrm{cris}}(V_2)$, whose filtration is that of $D_{\mathrm{dR},\tau_{c-s}}(V_2)$ (F unramified, V_2

crystalline); as $\epsilon_{c-s}=1$, every jump there is $\le h_{c-s}=s_{c-s}+S$, so $\varepsilon_{c-s-1}\mathrm{Fil}^{s_{c-s}+S+1}D_{\mathrm{cris}}(V_2)=0$, and the intersection is 0 because φ is bijective on $D_{\mathrm{cris}}(V_2)$ and Fil^u is an $F\otimes L$-submodule. With $\bar w_c\in I_c$ and Δ-invariant, all hypotheses of Proposition C.10 on $w=\bar w_c$ hold.

Step 4 (conclusion). Let $E\ge 0$ and Z_c ($c\in C$) be as in Proposition C.10, and put $E^+=\max(E,\sigma_B)$; then $E^+\le\max(40\,\mathrm{Sg}+\eta\beta_{320},3\,\mathrm{Sg})=40\,\mathrm{Sg}+\eta\beta_{320}$ and $|Z_c|\le m\,\mathrm{len}_c(\mathrm{len}_c+1)$. Let X_c be the $\mathcal{O}_L$-span of the classes $\rho_{c,1}(w_0)$, w_0 running over the vectors satisfying the hypotheses of Proposition C.10 at c. For such w_0, $\Pi_c w_0\in P_e(\mathbf{N}_{2,c'})$ and $\Pi_c P_e(\mathbf{N}_{2,c'})\subseteq\mathbf{N}_{2,c}$ (Lemma 23.15 (2), window (H_c,H_c)), so $\Pi_c^2 w_0\in\mathbf{N}_2$ and $\rho_{c,1}(w_0)\in H^1_f(D,T_2)$ by (Ry7); thus $X_c\subseteq H^1_f(D,T_2)$, and Proposition C.10 gives $\ell^{\sigma_B+\kappa_c+E^+}X_c\subseteq\Phi_*H^1_f(D,T_1)+\sum_{z\in Z_c}\mathcal{O}_L z$. By Steps 2 and 3, $\ell^{r_0+2\sigma'}x\in\sum_c X_c$ for every $x\in H^1_f(D,T_2)$, hence $\ell^{r_0+2\sigma_B}H^1_f(D,T_2)\subseteq\sum_c X_c$ as $\sigma'\le\sigma_B$. The assertion follows with r_0, the exponent $\sigma_B\le 3\,\mathrm{Sg}$ in place of σ, and E^+ in place of E. □

Proposition C.5. *Let ℓ, F, n, $D=G_F$, L, (τ_a), $\Phi\colon V_1\to V_2$ ($T_1\to T_2$) and the weight word $(b,w,S,\eta,m,(s_a),(\epsilon_a))$ satisfy the hypotheses of Proposition C.25. For $a\in\mathbb{Z}/n$ put $(M_a,h_a)=(\eta,\ s_a+S)$ if $\epsilon_a=1$ and $(M_a,h_a)=(s_a+S,\ \eta)$ if $\epsilon_a=-1$, and $\varepsilon_a=(\iota\otimes\mathrm{id})(e_a)\in A_{\mathrm{inf}}\otimes_{\mathbb{Z}_\ell}\mathcal{O}_L$. Then every Hodge jump of V_1 and of V_2 at τ_a lies in $[-M_a,h_a]$, and there is an integer r_0 with $0\le r_0\le 3\,\mathrm{Sg}$ such that for every $x\in H^1_f(D,T_2)$ there are a pair (V_Y,Y) and an exact sequence $0\to T_2\xrightarrow{\iota'}Y\xrightarrow{\pi}\mathcal{O}_L\to 0$ of G_F-equivariant $\mathcal{O}_L$-linear maps with $[Y]=x$, such that every Hodge jump of V_Y at τ_a lies in $[-M_a,h_a]$ ($a\in\mathbb{Z}/n$), and for every $c\in C$ there is $x_c\in\mathbf{N}(Y)_c$ with*

$$(1\otimes\pi)\big(\iota_Y(x_c)\big)=\ell^{r_0}\varepsilon_c.$$

Proof. **Windows.** Every Hodge jump of V_1 and of V_2 at τ_a lies in $[-M_a,h_a]$ by Lemma C.9 (1).

The extension. Let $x\in H^1_f(D,T_2)$ ($=H^1_f(F,T_2)$ of Proposition 23.27: both are the inverse image of $H^1_f(D,V_2)$). Lemma 23.26 gives a pair (V_Y,Y) and $0\to T_2\to Y\to\mathcal{O}_L\to 0$ with $[Y]=x$ whose jumps at τ_a lie in $[-M_a,h_a]$.

Lifting the unit. Let r_c ($c\in\mathbb{Z}/n$) be the integers of Proposition C.6 formed with the window $(M_a,h_a)_a$; they depend only on ℓ and the weight word. Let $\mathbf{1}\in\mathbf{N}(\mathcal{O}_L)$ be as in Proposition C.8, and $c\in C$. Apply Proposition C.6 ($\ell\ge 5$) to the surjection $\pi\colon Y\to\mathcal{O}_L$, the window (M_a,h_a) of V_Y and $y=e_c\mathbf{1}\in\mathbf{N}(\mathcal{O}_L)_c$: there is $x'_c\in\mathbf{N}(Y)_c$ with $\mathbf{N}(\pi)(x'_c)=\ell^{r_c}e_c\mathbf{1}$. By (Ex3) and (Ex2) (writing $e_c=\sum_i f_i\otimes\lambda_i$ with $f_i\in\mathcal{O}_F$, $\lambda_i\in\mathcal{O}_L$),

$$(1\otimes\pi)\big(\iota_Y(x'_c)\big)=\iota_{\mathcal{O}_L}\big(\mathbf{N}(\pi)(x'_c)\big)=\ell^{r_c}\,(\iota\otimes\mathrm{id})(e_c)\,\iota_{\mathcal{O}_L}(\mathbf{1})=\ell^{r_c}\varepsilon_c.$$

Let $r_0=\max_{c\in C}r_c$ and $x_c=\ell^{r_0-r_c}x'_c$. By Lemma C.9 (2), $r_c\le 3\,\mathrm{Sg}$ for every $c\in C$, so $0\le r_0\le 3\,\mathrm{Sg}$. □

Proposition C.6. *Let $\ell\ge 5$ be a prime, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a field K with $F\subseteq K\subseteq F(\mu_\ell)$ as in Proposition 23.7, $d=|\Delta_K|$, and $f\colon T\to T''$ a G_K-equivariant $\mathcal{O}_L$-linear surjection of the lattices of pairs (V,T), (V'',T'') over K. Let $M_a,h_a\ge 0$ be integers such that every Hodge jump of V at every embedding of K over τ_a lies in $[-M_a,h_a]$ ($a\in\mathbb{Z}/n$), and put $H(a)=M_a+h_a$, $H_{\max}=\max_a H(a)$, $c_H=\lceil 1+\log_\ell(H_{\max}+1)\rceil$ and, for $c\in\mathbb{Z}/n$,*

$$s_c=\sum_{a\ge 0}\frac{H(c+a+1)+c_H+3}{d^a}\in(0,\infty],\qquad r_c=\Big\lfloor\min(\lfloor s_c\rfloor,H_{\max})\frac{\ell}{d(\ell-1)}\Big\rfloor+c_H$$

($s_c=\infty$ when $d=1$, and then the minimum is $H_{\max}$). Then for every $c\in\mathbb{Z}/n$ and every $y\in\mathbf{N}(T'')_c$ there is $x\in\mathbf{N}(T)_c$ with $\mathbf{N}(f)(x)=\ell^{r_c}y$. For $K=F$ (pairs, G_F, the embedding τ_a itself, $d=\ell-1$) the constants are $s_c=\sum_{a\ge0}(H(c+a+1)+c_H+3)/(\ell-1)^a$ and $r_c=\lfloor\min(\lfloor s_c\rfloor,H_{\max})\ell/(\ell-1)^2\rfloor+c_H$.

Proof. The data ℓ, F, L, (τ_a), K, f, $(M_a, h_a)_a$ satisfy the hypotheses of Lemma C.7, and d, $H(a)$, $H_{\max}$, c_H, s_c, r_c are defined there by the same formulas as here. Let $c \in \mathbb{Z}/n$. By (Ki2) of Lemma C.7, ℓ^{r_c} is killing at c, that is, $\ell^{r_c}\mathbf{N}(T'')_c \subseteq \mathbf{N}(f)(\mathbf{N}(T)_c)$. Hence for $y \in \mathbf{N}(T'')_c$ there is $x \in \mathbf{N}(T)_c$ with $\mathbf{N}(f)(x) = \ell^{r_c}y$. □

Lemma C.7. *Let $\ell \geq 5$ be a prime, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a field K with $F \subseteq K \subseteq F(\mu_\ell)$ as in Proposition 23.7, $d = |\Delta_K|$ and $e = (\ell-1)/d$, and $f\colon T \to T''$ a G_K-equivariant $\mathcal{O}_L$-linear surjection of the lattices of pairs (V,T), (V'',T'') over K. Let $M_a, h_a \geq 0$ be integers such that every Hodge jump of V at every embedding of K over τ_a lies in $[-M_a, h_a]$ ($a \in \mathbb{Z}/n$), and put $H(a) = M_a + h_a$, $H_{\max} = \max_a H(a)$, $c_H = \lceil 1 + \log_\ell(H_{\max}+1)\rceil$ and, for $c \in \mathbb{Z}/n$,*

$$s_c = \sum_{a\geq 0} \frac{H(c+a+1)+c_H+3}{d^a} \in (0,\infty], \qquad r_c = \Big\lfloor \min(\lfloor s_c\rfloor, H_{\max})\frac{\ell}{d(\ell-1)}\Big\rfloor + c_H$$

($s_c = \infty$ when $d = 1$, and then the minimum is $H_{\max}$). Write $\mathbf{N}''_c = \mathbf{N}(T'')_c$ and $\mathbf{N}'_c = \mathbf{N}(f)(\mathbf{N}(T)_c)$, and call $t \in e_c\mathbf{A}_{F,L}$ killing at c *if $t\,\mathbf{N}''_c \subseteq \mathbf{N}'_c$. The **killing index** at c is*

$$\sigma_c = \min\{s \geq 0 :\ X^s \text{ is killing at } c\}.$$

Then σ_c is well defined, and for every $c \in \mathbb{Z}/n$:

(Ki1) *$\sigma_c \leq \min(\lfloor s_c\rfloor, H_{\max})$, and for every integer $m \geq 1$*

$$\sigma_c \leq \sum_{a=0}^{m-1} \frac{H(c+a+1)+c_H+3}{d^a} + \frac{\sigma_{c+m}}{d^m} \leq \sum_{a=0}^{m-1} \frac{H(c+a+1)+c_H+3}{d^a} + \frac{H_{\max}}{d^m}.$$

(Ki2) *For all integers $0 \leq j \leq \sigma_c$ and $D \geq \mathrm{dsc}_{\ell,d}(j,\sigma_c)$, $\ell^D X^j$ is killing at c. In particular ℓ^{r_c} is killing at c.*

(Ki3) *Let $\pi \in e_c\mathbf{A}_{F,L}$ with $\pi = X^d v$ for a unit v of $e_c\mathbf{A}_{F,L}$, $E \in \mathbb{Z}_\ell[Y]$ monic of degree e with all other coefficients in $\ell\mathbb{Z}_\ell$ (for instance E_Δ of Definition 23.23 with $\Delta = \Delta_K$, by Lemma 23.24(Rr3)), and $A_c = \lceil \sigma_c/d\rceil$. Then $A_c \leq H_{\max}/d+1$, and π^{A_c} and $E(\pi)^{\lceil A_c/e\rceil + r_c}$ are killing at c.*

(Ki4) *Let $t \in e_c\mathbf{A}_{F,L}$ be killing at c and let $y \in \mathbf{N}''_c$, and suppose $\gamma_u(t) = t$ and $\gamma_u(y) = y$ for all $u \in \Delta_K$. Then there is $x \in \mathbf{N}(T)_c$ with $\gamma_u(x) = x$ for all $u \in \Delta_K$ and $\mathbf{N}(f)(x) = ty$.*

For $K = F$ (G_F, pairs, the embedding τ_a itself, $d = \ell-1$, $e = 1$, $\Delta_F = \mu_{\ell-1}$, $\mathrm{dsc}_{\ell,\ell-1} = \mathrm{dsc}_\ell$) and $E(Y) = Y + \ell$ this is the statement with $s_c = \sum_a (H(c+a+1)+c_H+3)/(\ell-1)^a$, $r_c = \lfloor \min(\lfloor s_c\rfloor, H_{\max})\ell/(\ell-1)^2\rfloor + c_H$, $\pi = X^{\ell-1}v$, $A_c = \lceil \sigma_c/(\ell-1)\rceil$, $A_c \leq H_{\max}/(\ell-1)+1$ and $(\pi+\ell)^{A_c+r_c}$.

Proof. **Setting.** Put $\mathbf{N} = \mathbf{N}(T'')$ and $\mathbf{N}' = \mathbf{N}(f)(\mathbf{N}(T))$, so $\mathbf{N}'_a = e_a\mathbf{N}' = \mathbf{N}(f)(\mathbf{N}(T)_a)$ and $\mathbf{N}_a = \mathbf{N}''_a$. By (Ex1), $\mathbf{N}(T)$ is a Wach module over K with window $(M_a,h_a)_a$; by Lemma 23.25 (V is crystalline, hence de Rham) every Hodge jump of V'' at an embedding of K over τ_a is one of V at that embedding, so by (Ex1) $\mathbf{N}$ is a Wach module over K with the same window. By (Ex3), $\mathbf{N}(f)$ is $\mathbf{A}_{F,L}$-linear and commutes with φ (extended to $\mathbf{N}(T)[1/Q]$) and with every γ_u, $u \in \Gamma_K$. Hence $(\gamma_u - 1)\mathbf{N}' = \mathbf{N}(f)((\gamma_u-1)\mathbf{N}(T)) \subseteq X\mathbf{N}'$ for all $u \in \Gamma_K$ by (Wm3) for $\mathbf{N}(T)$, and applying $\mathbf{N}(f)$ to (Wm2) for $\mathbf{N}(T)$ gives $Q^{h_a}\mathbf{N}'_a \subseteq \mathbf{A}_{F,L}\varphi(\mathbf{N}'_{a-1})$ for all a; so Lemma 23.20 (over K) applies to $\mathbf{N}$, $\mathbf{N}'$. By Proposition 23.18 (over K), $X^{H_{\max}}$ is killing at every a, so σ_a is well defined and $\sigma_a \leq H_{\max}$. Put $k_H = c_H + 3$.

Step 1: $\sigma_{a-1} \le H(a) + k_H + \sigma_a/d$ for every a. Let $s = \max(\sigma_a, 1)$ and $b = c_H + 1$. Then $X^s \mathbf{N}_a \subseteq \mathbf{N}'_a$, and $s \le H_{\max} + 1$ gives $2 + \log_\ell s \le 2 + \log_\ell(H_{\max} + 1) \le 1 + c_H = b$. Item 4 of Lemma 23.20 gives $\sigma_{a-1} \le M_a + h_a + \lceil s/d \rceil + c_H + 1$. Finally $\lceil s/d \rceil \le \sigma_a/d + 1$: for $\sigma_a = 0$ both sides are 1, and for $\sigma_a \ge 1$, $\lceil \sigma_a/d \rceil \le \sigma_a/d + (d-1)/d$. Hence $\sigma_{a-1} \le H(a) + c_H + 2 + \sigma_a/d$.

Step 2: (Ki1). By induction on $m \ge 1$, Step 1 at the indices $c+1, \dots, c+m$ gives the first inequality of (Ki1): for $m = 1$ it is Step 1 at $a = c+1$, and the step from m to $m+1$ inserts $\sigma_{c+m} \le H(c+m+1) + k_H + \sigma_{c+m+1}/d$. The second inequality follows from $\sigma_{c+m} \le H_{\max}$. If $d \ge 2$, letting $m \to \infty$ gives $\sigma_c \le s_c$ (the series converges, its terms being at most $(H_{\max} + k_H) d^{-a}$), and since σ_c is an integer, $\sigma_c \le \lfloor s_c \rfloor$; if $d = 1$, $s_c = \infty$. In both cases $\sigma_c \le H_{\max}$.

Step 3: (Ki2), descent to powers of ℓ. Apply Lemma 23.21 with $\Gamma = \Gamma_K$, a closed subgroup of $\mathbb{Z}_\ell^\times$ containing $1 + \ell\mathbb{Z}_\ell$ of index $|\Delta_K| = d$ (Lemma 23.24(Rr7)), $D = N = \mathbf{N}_c$ (free by (Wm1), so X is injective) and $N' = \mathbf{N}'_c$: by (Wm3) $(\gamma_u - 1)\mathbf{N}_c \subseteq X\mathbf{N}_c$, and $(\gamma_u - 1)\mathbf{N}'_c = e_c(\gamma_u - 1)\mathbf{N}' \subseteq X\mathbf{N}'_c$ ($\gamma_u(e_c) = e_c$), for $u \in \Gamma_K$; X^{σ_c} kills $\mathbf{N}_c/\mathbf{N}'_c$. Item 1 gives the first claim of (Ki2). For $j = 0$, item 2 gives $\mathrm{dsc}_{\ell,d}(0, \sigma_c) \le \lfloor \sigma_c \ell/(d(\ell-1)) \rfloor$, and by (Ki1)

$$\mathrm{dsc}_{\ell,d}(0, \sigma_c) \le \Big\lfloor \min(\lfloor s_c \rfloor, H_{\max}) \frac{\ell}{d(\ell-1)} \Big\rfloor \le r_c,$$

so ℓ^{r_c} is killing at c.

Step 4: (Ki3). First, $A_c \le \sigma_c/d + 1 \le H_{\max}/d + 1$ by (Ki1). Since $\mathbf{N}'_c$ and $\mathbf{N}_c$ are $e_c\mathbf{A}_{F,L}$-modules, every multiple wt ($w \in e_c\mathbf{A}_{F,L}$) of a killing element t is killing, and so is every X^k with $k \ge \sigma_c$. Now $\pi^{A_c} = v^{A_c} X^{dA_c}$ with $dA_c \ge \sigma_c$, so π^{A_c} is killing. Write $E(Y) = Y^e + \ell g(Y)$ with $g \in \mathbb{Z}_\ell[Y]$, put $B = \lceil A_c/e \rceil$ and $N = B + r_c$, and expand

$$E(\pi)^N = \sum_{i=0}^{N} \binom{N}{i} \pi^{ei}\, \ell^{N-i} g(\pi)^{N-i}.$$

For $i \ge B$, $ei \ge A_c$ and the term is a multiple of π^{A_c}; for $i < B$, $N - i > r_c$ and the term is a multiple of ℓ^{r_c}, killing by (Ki2). The killing elements form an additive group, so $E(\pi)^N$ is killing. (For $e = 1$, $E(Y) = Y + \ell$: $B = A_c$ and this is $(\pi + \ell)^{A_c + r_c}$.)

Step 5: (Ki4), Δ_K-averaging. The group Δ_K is finite, of order d dividing $\ell - 1$, so $d \in \mathcal{O}_L^\times$. Choose $x_0 \in \mathbf{N}(T)_c$ with $\mathbf{N}(f)(x_0) = ty$ and put

$$x = \frac{1}{d} \sum_{u \in \Delta_K} \gamma_u(x_0).$$

Each γ_u ($u \in \Gamma_K \supseteq \Delta_K$) is semilinear and preserves $\mathbf{N}(T)$ (definition of a Wach module over K), and $\gamma_u(e_c) = e_c$, so $x \in \mathbf{N}(T)_c$; as Δ_K is a group and $\gamma_u \gamma_{u'} = \gamma_{uu'}$, $\gamma_{u'}(x) = x$ for $u' \in \Delta_K$. Since $\mathbf{N}(f)$ commutes with γ_u (Ex3) and $\gamma_u(ty) = \gamma_u(t)\gamma_u(y) = ty$,

$$\mathbf{N}(f)(x) = \frac{1}{d} \sum_{u \in \Delta_K} \gamma_u(ty) = ty.$$

□

Proposition C.8. *Let ℓ, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ be as in Proposition 23.7, and let $(L, \mathcal{O}_L)$ be the trivial pair (G_F acting trivially). Then there is $\mathbf{1} \in \mathbf{N}(\mathcal{O}_L)$ with $\iota_{\mathcal{O}_L}(\mathbf{1}) = 1 \otimes 1$ in $A_{\mathrm{inf}}[1/\mu_\varepsilon] \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$.*

Proof. By (Ex5) of Proposition 23.7 (with $K = F$, $s = 1$, ℓ odd) there are a datum $\mathcal{E}_1$ given by Theorem B.41 (for F and $p = \ell$) for the trivial lattice $T_1 = \mathcal{O}_L$ (a free $\mathbb{Z}_\ell$-module of rank

$[L:\mathbb{Q}_\ell]$ with trivial action) and an isomorphism c_1 with the properties of (Bs3) for it such that $\mathbf{N}(\mathcal{O}_L)=\mathcal{E}_1(\mathbf{A}_F)$ and $\iota_{\mathcal{O}_L}=c_1\circ\mathcal{E}_1(\iota)$, where $\iota\colon \mathbf{A}_F\to A_{\mathrm{inf}}$ is the morphism over $\mathcal{O}_F$ of Lemma 23.8.

Let $\mathcal{E}_0(A)=A\otimes_{\mathbb{Z}_\ell}\mathcal{O}_L$ for every object (A,I) over $\mathcal{O}_F$, with $\varphi_{\mathcal{E}_0,A}(a\otimes(b\otimes\lambda))=a\varphi_A(b)\otimes\lambda$ and $\mathcal{E}_0(g)=g\otimes 1$. This is a datum as in (Bs1), (Bs2): $\mathcal{E}_0(A)$ is free of rank $[L:\mathbb{Q}_\ell]$, $\varphi_{\mathcal{E}_0,A}$ is an isomorphism already before inverting I, the linearisation of $g\otimes 1$ is the isomorphism $B\otimes_{g,A}(A\otimes\mathcal{O}_L)\cong B\otimes\mathcal{O}_L$, and all compatibilities are clear. The identity c_0 of $W(\mathbb{C}_\ell^\flat)\otimes_{\mathbb{Z}_\ell}\mathcal{O}_L$ has the properties of (Bs3) for $T_0=\mathcal{O}_L$: it is G_F-equivariant (G_F acts trivially on $\mathcal{O}_L$), carries $\varphi_{\mathcal{E}_0,A_{\mathrm{inf}}}\otimes\varphi=\varphi\otimes 1$ to $1\otimes\varphi$ on $\mathcal{O}_L\otimes W(\mathbb{C}_\ell^\flat)$ (the same map), and restricts to the identity of $A_{\mathrm{inf}}[1/\mu_\varepsilon]\otimes\mathcal{O}_L$.

By (Bs6) of Theorem B.41, applied to $\mathrm{id}\colon T_0=\mathcal{O}_L\to T_1=\mathcal{O}_L$, there are maps $\mathcal{E}(\mathrm{id})_A\colon \mathcal{E}_0(A)\to\mathcal{E}_1(A)$ with $\mathcal{E}_1(g)\circ\mathcal{E}(\mathrm{id})_A=\mathcal{E}(\mathrm{id})_B\circ\mathcal{E}_0(g)$ for morphisms $g\colon A\to B$ over $\mathcal{O}_F$, and $c_1\circ(\mathcal{E}(\mathrm{id})_{A_{\mathrm{inf}}}\otimes 1)=c_0$. Put $\mathbf{1}=\mathcal{E}(\mathrm{id})_{\mathbf{A}_F}(1\otimes 1)\in\mathcal{E}_1(\mathbf{A}_F)=\mathbf{N}(\mathcal{O}_L)$. Then, with $g=\iota$,

$$\iota_{\mathcal{O}_L}(\mathbf{1})=c_1\big(\mathcal{E}_1(\iota)\,\mathcal{E}(\mathrm{id})_{\mathbf{A}_F}(1\otimes 1)\big)=c_1\big(\mathcal{E}(\mathrm{id})_{A_{\mathrm{inf}}}(\iota(1)\otimes 1)\big)=c_0(1\otimes 1)=1\otimes 1.$$

□

Lemma C.9. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1\to V_2$ ($T_1\to T_2$) and the weight word $(b,w,S,\eta,m,(s_a),(\epsilon_a))$ satisfy the hypotheses of Proposition C.25. For $a\in\mathbb{Z}/n$ put $(M_a,h_a)=(\eta,\ s_a+S)$ if $\epsilon_a=1$ and $(M_a,h_a)=(s_a+S,\ \eta)$ if $\epsilon_a=-1$. Then:*

1. *every Hodge jump of V_1 and of V_2 at τ_a lies in $[-M_a,h_a]$ ($a\in\mathbb{Z}/n$);*
2. *the integers r_c of Proposition C.6 formed with the window $(M_a,h_a)_a$ satisfy $r_c\le 3\,\mathrm{Sg}$ for every $c\in C$.*

Proof. **1.** By (Ow1), a jump of V_1 at τ_a lies in $[s_a,s_a+S]\cup[-\eta,\eta]\subseteq[-\eta,s_a+S]$ if $\epsilon_a=1$ and in $[-s_a-S,-s_a]\cup[-\eta,\eta]\subseteq[-s_a-S,\eta]$ if $\epsilon_a=-1$, i.e. in $[-M_a,h_a]$; the jumps of V_2 are among those of V_1 (Lemma 24.2 (1)).

2. Here $H(a)=M_a+h_a=s_a+S+\eta$, $H_{\max}=\max_a H(a)$, $c_H=\lceil 1+\log_\ell(H_{\max}+1)\rceil$, $s_c=\sum_{a\ge 0}(H(c+a+1)+c_H+3)/(\ell-1)^a$ and $r_c=\lfloor\min(\lfloor s_c\rfloor,H_{\max})\,\ell/(\ell-1)^2\rfloor+c_H$. Let $c\in C$; then $c+1$ is thin, so $H(c+1)\le w+S+\eta$ by (Ow3). Every $H(a)\le h+\eta$: at a thick site $s_a\le b+\eta$ and $h=b+\eta+S$; at a thin site $s_a\le w$, and $w+S\le h$ (if some site is thick, $w\le b$ because $20w\le b$; otherwise $h=w+S+\eta+1$). Also $h\ge\eta+1$ ($b\ge 1$), so $H_{\max}+1\le h+\eta+1\le 2h$ and $c_H\le 2+\log_\ell(2h)$. Hence, with the geometric series ($\ell\ge 11$),

$$s_c\le(w+S+\eta)+(c_H+3)+\frac{h+\eta+c_H+3}{\ell-2}.$$

Put $q=\ell/(\ell-1)^2$; for $\ell\ge 11$, $q\le\min(0.11,\ 1.21/\ell)$ and $q/(\ell-2)\le 1.5/\ell^2$ (both hold at $\ell=11$ and the left sides times ℓ, resp. ℓ^2, decrease in ℓ). Therefore

$$r_c\le q\,s_c+c_H\le\frac{1.21(w+S+\eta)}{\ell}+\frac{1.5(h+\eta)}{\ell^2}+(c_H+3)\Big(0.11+\frac{1.5}{121}\Big)+c_H.$$

Here $1.5(h+\eta)/\ell^2\le 1.5h/\ell^2+0.14\,\eta/\ell$, and $(c_H+3)\cdot 0.1224+c_H\le 1.1224\,c_H+0.368\le 2.62+1.1224\log_\ell(2h)$. Comparing term by term with $3\,\mathrm{Sg}=3h/\ell^2+3(w+S+\eta)/\ell+3\log_\ell(2h)+3$ (and $\log_\ell(2h)\ge 0$) gives $r_c\le 3\,\mathrm{Sg}$. □

C.1.2 Relay classes at a cut and the universal part

The result of this subsection is Proposition C.10: the relay classes at one cut lift along the onto map up to the killer valuation, modulo few exceptional classes; it is used in Proposition C.4. Its main input is the universal part at a cut (Proposition C.11), assembled from four packages: a Δ-invariant frame at the start of the arc adapted to the high and low jumps (Proposition C.12); killer-polynomial denominators for the coordinates of arc relay vectors off the low coordinates (Proposition C.13); an exceptional family of at most $m\,\mathrm{len}(\mathrm{len}+1)$ vectors on the low coordinates (Proposition C.21); and the realisation of coordinate vectors that are Hodge-integral modulo γ by arc relay vectors (Proposition C.22). The remaining lemmas are a calculus of arc coordinates and levels, and a membership criterion along an arc (Proposition C.24).

Proposition C.10. *Let ℓ, F, n, $D = G_F$, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25, and put $\mathbf{N}_\nu = \mathbf{N}(T_\nu)$. Let $M_a, h_a \geq 0$ ($a \in \mathbb{Z}/n$) be integers such that every Hodge jump of V_1 at τ_a lies in $[-M_a, h_a]$ (by Lemma 24.2 (1) and (Ex1), $(M_a, h_a)_a$ is then a window of both $\mathbf{N}_1$ and $\mathbf{N}_2$), and let G, g, m', R be integers with $G \geq M_{c+1}$ for every $c \in C$, $g \geq G$, $m' \geq 1 + \max_a M_a$ and $R \geq \max(m', g)$; put $\sigma_B = \lceil (2(w+S) + m_X/(\ell-1))\,\ell/(\ell-1)^2 \rceil + 1$, and form I_c, $\rho_{c,1}$ for $\mathbf{N}_2$ (and I_c^1, $\rho_{c,1}^1$ for $\mathbf{N}_1$) with $j = 1$ and these G, g, m', R. For $c \in C$ let $B^c = (B_s^c)_{s<\mathrm{len}_c}$ be $B_s^c = s_{c-s} + S$ if $\epsilon_{c-s} = 1$ and $B_s^c = \infty$ if $\epsilon_{c-s} = -1$.*

Then there are an integer E with $0 \leq E \leq 40\,\mathrm{Sg} + \eta\,\beta_{320}$ and, for every $c \in C$, a finite subset $Z_c \subseteq H^1_f(D, T_2)$ with $|Z_c| \leq m\,\mathrm{len}_c(\mathrm{len}_c+1)$, such that the following holds. Let $c \in C$ and $w \in I_c$ with $\gamma_u(w) = w$ for $u \in \mu_{\ell-1}$, such that $\Pi_c\, w \in P_{\mathrm{len}_c}(\mathbf{N}_{2,c'})$ and, for every frame $\mathbf{n}$ at c' of $\mathbf{N}_2$ consisting of vectors fixed by γ_u ($u \in \mu_{\ell-1}$), some $\ell^t w$ ($t \geq 0$) has an arc coordinate vector for $\mathbf{n}$, len_c, H_c and m_X that is pole-bounded by B^c. Then

$$\ell^{\,\sigma_B + \kappa_c + \max(E, \sigma_B)} \rho_{c,1}(w) \in \Phi_* H^1_f(D, T_1) + \sum_{z \in Z_c} \mathcal{O}_L z.$$

Proof. **Conventions.** Whenever a cut $c \in C$ occurs, $e = \mathrm{len}_c$ and $c' = c - e$; so $c' + e = c$, and $c' \in C$, i.e. the site $c'+1$ is thin. Put $R_c = e_c \mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ (a domain), $\Pi = \Pi_c = \prod_{s<e} Q_s^{H_c(c-s)}$ and $U = P_e(\mathbf{N}_{2,c'})$. Unless said otherwise, a *coordinate vector* is an arc coordinate vector for $\mathbf{N}_2$, a frame at c' named in the text, e, H_c and m_X. We write $\rho = \rho_{c,1}$ and $\rho^1 = \rho^1_{c,1}$. Here $H^1(F,T) = H^1(D,T)$ for $D = G_F$, and $H^1_f(F,T)$ of Proposition 23.27 and $H^1_f(D,T)$ are both the inverse image of $H^1_f(D,V)$.

Steps 1 and 2 (window and calculus of coordinate vectors). By Lemma C.16 (1), every Hodge jump of V_2 at τ_a lies in $[-H_c(a), H_c(a)]$ and

$$\Pi\, U \subseteq \mathbf{N}_{2,c}, \qquad \Pi\, \mathbf{N}_{2,c} \subseteq U. \tag{1}$$

Below, (a) linearity, (b) Γ-equivariance, (c) vanishing and (d) the relay-class facts refer to the parts of Lemma C.16, for the frame named in the text.

Step 3 (the data at the cut; E and Z_c). Let $E = E_{\mathrm{arc}}$ of Proposition C.25 (it does not depend on c), so $0 \leq E \leq 40\,\mathrm{Sg} + \eta\,\beta_{320}$, and $E^+ = \max(E, \sigma_B)$. Let $c \in C$. Proposition C.11 gives a Δ-invariant frame $\mathbf{n}$ at c' with crystalline frame (d_k), a set $\mathcal{R}$ with $|\mathcal{R}| \leq me$, and $J_1, \ldots, J_{g_0}$, $g_0 \leq me(e+1)$, as in (U1)–(U5). By (U5), applied to J_i, there is $y_i \in I_c$ with $\Pi y_i \in U$ such that $\ell^{\sigma_B} J_i$ is a coordinate vector of y_i. Put $Z_c = \{\rho(y_1), \ldots, \rho(y_{g_0})\}$, so $|Z_c| \leq m\,\mathrm{len}_c(\mathrm{len}_c + 1)$. By (1), $\Pi^2 y_i \in \mathbf{N}_2$, and Π^2 is a finite product of the Q_s, so $\rho(y_i) \in H^1_f$ by (Ry7). Z_c depends only on c.

Step 4 (the killer polynomial). Let $w_0 \in I_c$ be the vector w of the statement and $x = \rho(w_0)$. Its hypotheses are those of (U3) for the Δ-invariant frame $\mathbf{n}$, which gives a Δ-invariant coordinate vector f of $\ell^{\sigma_B} w_0$ with the properties listed there. Let $P = P_{J_c,e}$ (formed with u_0), $v = P(\gamma_{u_0})(\ell^{\sigma_B} w_0) \in I_c$ and $g_P = P(\gamma_{u_0})f$; by (b), g_P is a coordinate vector of v and $\Pi v \in U$; g_P is Δ-invariant because the γ_u commute. The sets W_a consist of integers of absolute value at least $s_a > \eta \geq 0$, so $0 \notin J_c$, and Lemma 23.22 (2) (with $p = \ell$ odd, $m = v_\ell(u_0 - 1) = 1$) gives $v_\ell(P(1)) = e + \sum_{i \in J_c}(e + 1 + v_\ell(i)) = \kappa_c$; write $P(1) = \vartheta \ell^{\kappa_c}$ with $\vartheta \in \mathbb{Z}_\ell^\times$. By (d),

$$\rho(v) = P(1)\, \ell^{\sigma_B} x = \vartheta\, \ell^{\sigma_B + \kappa_c} x. \qquad (2)$$

Step 5 (splitting off the low coordinates). Let g_P'' have entries $g_{P,k}$ for $k \notin \mathcal{R}$ and 0 for $k \in \mathcal{R}$, and $g_P^{\mathcal{R}} = g_P - g_P''$. Both are Δ-invariant, have $\Pi(g_P'')_k, \Pi(g_P^{\mathcal{R}})_k \in R_c$ (as $\Pi g_{P,k} \in R_c$ by (b)), and are Hodge-integral modulo γ by (U3). By (U5) there are $u'', u^{\mathcal{R}} \in I_c$ with $\Pi u'', \Pi u^{\mathcal{R}} \in U$ such that $\ell^{\sigma_B} g_P''$, $\ell^{\sigma_B} g_P^{\mathcal{R}}$ are coordinate vectors of u'', $u^{\mathcal{R}}$. By (a), 0 is a coordinate vector of $\ell^{\sigma_B} v - u'' - u^{\mathcal{R}} \in I_c$, whose product with Π lies in U; by (d) and (2),

$$\rho(u'') + \rho(u^{\mathcal{R}}) = \ell^{\sigma_B} \rho(v) = \vartheta\, \ell^{2\sigma_B + \kappa_c} x. \qquad (3)$$

Step 6 (the low coordinates give Z_c). $g_P^{\mathcal{R}}$ satisfies the hypotheses of (U4), which gives $o \in \mathcal{O}_L^{g_0}$ and $h \in \mathcal{H}_{\mathbf{n}}$ with $g_P^{\mathcal{R}} - \sum_i o_i J_i = X^{\ell-1} h$. As h is Hodge-integral, Proposition C.23 gives $x_h \in \mathbf{N}_{2,c}$ such that $\ell^{\sigma_B} h$ is a coordinate vector of x_h; $\Pi x_h \in U$ by (1). By (a), 0 is a coordinate vector of $u^{\mathcal{R}} - \sum_i o_i y_i - X^{\ell-1} x_h$, which lies in I_c (as $X^{\ell-1} x_h \in X\mathbf{N}_{2,c}$) and whose product with Π lies in U. By (d) and (Ry5),

$$\rho(u^{\mathcal{R}}) = \sum_i o_i\, \rho(y_i) \in \sum_{z \in Z_c} \mathcal{O}_L z. \qquad (4)$$

Step 7 (lifting the high coordinates). Let $k \notin \mathcal{R}$. By (U3), $g_{P,k}$ is admissible at c in the sense of Proposition C.25 (Δ-invariant, $\nu_c g_{P,k} \in R_c$, and the congruence conditions at the $s < e$ with $\epsilon_{c-s} = -1$), and by (U1) the index k satisfies the filtration condition there. So Proposition C.25 gives $w_1^{(k)} \in I_c^1$, a finite product λ_k' of the Q_s and $z_k \in X\mathbf{N}_{2,c}$ with $\lambda_k' w_1^{(k)} \in \mathbf{N}_1$, $\Pi\, \mathbf{N}(\Phi)(w_1^{(k)}) \in U$, and $\ell^E g_{P,k} \mathbf{e}_k$ a coordinate vector of $\mathbf{N}(\Phi)(w_1^{(k)}) - z_k$. Put $u_1 = \ell^{E^+ - E} \sum_{k \notin \mathcal{R}} w_1^{(k)} \in I_c^1$, $\lambda' = \prod_{k \notin \mathcal{R}} \lambda_k'$ and $z = \ell^{E^+ - E} \sum_{k \notin \mathcal{R}} z_k \in X\mathbf{N}_{2,c}$. Then $\lambda' u_1 \in \mathbf{N}_1$, and by (a) and the linearity of $\mathbf{N}(\Phi)$, $\ell^{E^+} g_P''$ is a coordinate vector of $\mathbf{N}(\Phi)(u_1) - z$; it is also one of $\ell^{E^+ - \sigma_B} u''$. Hence 0 is a coordinate vector of $D_0 = \ell^{E^+ - \sigma_B} u'' - \mathbf{N}(\Phi)(u_1) + z$, and $\Pi D_0 \in U$ (for z by (1)). By (c), $D_0 \in X\mathbf{N}_{2,c}$, so $\ell^{E^+ - \sigma_B} u'' - \mathbf{N}(\Phi)(u_1) \in X\mathbf{N}_{2,c}$. Now Lemma 24.9 (window $(M_a, h_a)_a$ for both lattices; $j = 1$ and G, g, m', R as in the statement) gives

$$\ell^{E^+ - \sigma_B} \rho(u'') = \Phi_* \rho^1(u_1) \in \Phi_* H_f^1(D, T_1). \qquad (5)$$

Step 8. Multiplying (3) by $\ell^{E^+ - \sigma_B}$ and using (4), (5):

$$\vartheta\, \ell^{\sigma_B + \kappa_c + E^+} x = \ell^{E^+ - \sigma_B} \rho(u'') + \ell^{E^+ - \sigma_B} \rho(u^{\mathcal{R}}) \in \Phi_* H_f^1(D, T_1) + \sum_{z \in Z_c} \mathcal{O}_L z,$$

and dividing by the unit ϑ gives the assertion, with $E^+ = \max(E, \sigma_B)$. □

Proposition C.11. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25; put $\mathbf{N} = \mathbf{N}(T_2)$, $r = \dim_L V_2$, and fix*

$u_0 \in \mathbb{Z}_\ell^\times$ *with* $v_\ell(u_0-1)=1$. *Let* $c \in C$, $e = \mathrm{len}_c$, $P_c = P_{J_c,e}$ *(formed with* $u = u_0$*), and in* $e_c\mathbf{A}_{F,L}$

$$\mu_c = \prod_{\substack{s<e\\ \epsilon_{c-s}=1}} Q_s^{s_{c-s}+S}, \quad \delta_c = \prod_{\substack{s<e\\ \epsilon_{c-s}=-1}} Q_s^{s_{c-s}}, \quad \nu_c^+ = \prod_{\substack{s<e\\ \epsilon_{c-s}=-1}} Q_s^{s_{c-s}+S}, \quad \nu_c = \prod_{\substack{s<e\\ \epsilon_{c-s}=1}} Q_s^{s_{c-s}}.$$

For a frame $\mathbf{n}$ *at* c' *of* $\mathbf{N}$ *call* $f \in \mathcal{K}_c^r$ Hodge-integral *if* $\Pi_c f_k \in e_c\mathbf{A}_{F,L}$ *for all* k *and* $\ell^\kappa f$ *is Hodge-graded (for* $\mathbf{n}$ *and* e*) for some* $\kappa \geq 0$*; these vectors form an* $e_c\mathbf{A}_{F,L}$*-module* $\mathcal{H}_\mathbf{n}$. *Call* f Hodge-integral modulo γ *if* $(\gamma_{u_0}-1)f \in X^{\ell-1}\mathcal{H}_\mathbf{n}$. *Then there are a* Δ*-invariant frame* $\mathbf{n}$ *at* c' *of* $\mathbf{N}$*, with crystalline frame* (d_k)*, and a set* $R \subseteq \{1,\dots,r\}$ *with* $|R| \leq m\,e$*, such that:*

(U1) High coordinates *for* $k \notin R$ *and* $s < e$ *with* $\epsilon_{c-s} = 1$*:* $\varphi^e(d_k) \in \varphi^{s+4}\big(\mathrm{Fil}^{\,s_{c-s}} D_{\mathrm{cris}}(V_2)\big)$*;*

(U2) Low coordinates *for* $s < e$ *with* $\epsilon_{c-s} = -1$*:*

$$\varphi^{s+4}\big(\varepsilon_{c-s-1}\mathrm{Fil}^{-\eta} D_{\mathrm{cris}}(V_2)\big) \subseteq \sum_{k\in R} L\,\varphi^e(d_k);$$

(U3) Coordinates of arc relay vectors *for every* Δ*-invariant* $w \in e_c\mathbf{N}[1/\lambda_\infty]$ *with* $(\gamma_u - 1)(w) \in X\mathbf{N}$ *for all* $u \in \mathbb{Z}_\ell^\times$, $\Pi_c w \in P_e(\mathbf{N}_{c'})$, *and some* $\ell^t w$ *having an arc coordinate vector for* $\mathbf{n}$, e, H_c, m_X *that is pole-bounded by* B^c*, there is a* Δ*-invariant arc coordinate vector* f *of* $\ell^{\sigma_B} w$ *for* $\mathbf{n}$, e, H_c, m_X *such that for every* $k \notin R$*:* $\mu_c f_k \in e_c\mathbf{A}_{F,L}$*;* $\mu_c(\gamma_{u_0}-1)f_k \in X^{\ell-1}\delta_c\, e_c\mathbf{A}_{F,L}$*;* $\nu_c(\gamma_{u_0}-1)P_c(\gamma_{u_0})f_k \in X^{\ell-1}\nu_c^+\, e_c\mathbf{A}_{F,L}$*;* $\nu_c P_c(\gamma_{u_0})f_k \in e_c\mathbf{A}_{F,L}$*; and for every* $s < e$ *with* $\epsilon_{c-s} = -1$ *there are* $\kappa \geq 0$ *and* $x \in \mathcal{O}_L$ *with* $\nu_c(\ell^\kappa P_c(\gamma_{u_0})f_k - e_c(1\otimes x)) \in Q_s^{s_{c-s}+S} e_c\mathbf{A}_{F,L}$*; and the vector whose* k*-th entry is* $P_c(\gamma_{u_0})f_k$ *for* $k \notin R$ *and* 0 *for* $k \in R$*, and the vector whose* k*-th entry is* 0 *for* $k \notin R$ *and* $P_c(\gamma_{u_0})f_k$ *for* $k \in R$*, are both Hodge-integral modulo* γ*;*

(U4) Exceptional lattice *there are* $g_0 \leq m\,e\,(e+1)$ Δ*-invariant vectors* $J_1,\dots,J_{g_0} \in \mathcal{K}_c^r$*, each with* $\Pi_c J_{i,k} \in e_c\mathbf{A}_{F,L}$*,* $J_{i,k} = 0$ *for* $k \notin R$*, and Hodge-integral modulo* γ*, such that for every* Δ*-invariant* $f \in \mathcal{K}_c^r$ *with* $\Pi_c f_k \in e_c\mathbf{A}_{F,L}$ *for all* k*,* $f_k = 0$ *for* $k \notin R$*, and* f *Hodge-integral modulo* γ*, there is* $o \in \mathcal{O}_L^{g_0}$ *with* $f - \sum_i o_i J_i \in X^{\ell-1}\mathcal{H}_\mathbf{n}$*;*

(U5) Realisation *for every* Δ*-invariant* $f \in \mathcal{K}_c^r$ *with* $\Pi_c f_k \in e_c\mathbf{A}_{F,L}$ *for all* k *that is Hodge-integral modulo* γ*, there is a* Δ*-invariant* $y \in e_c\mathbf{N}[1/\lambda_\infty]$ *with* $(\gamma_u - 1)(y) \in X\mathbf{N}$ *for all* $u \in \mathbb{Z}_\ell^\times$ *and* $\Pi_c y \in P_e(\mathbf{N}_{c'})$ *such that* $\ell^{\sigma_B} f$ *is an arc coordinate vector of* y *for* $\mathbf{n}$, e, H_c, m_X.

Proof. Let $c \in C$ and $e = \mathrm{len}_c$. By Proposition C.12 there is a low frame $(\mathbf{n}, R)$ at c: a Δ-invariant frame $\mathbf{n}$ at c' of $\mathbf{N}$, with crystalline frame (d_k), and $R \subseteq \{1,\dots,r\}$ with $|R| \leq m\,e$, satisfying (U1) and (U2). For this low frame, Proposition C.13 is (U3) and Proposition C.21 is (U4); and Proposition C.22, applied to the Δ-invariant frame $\mathbf{n}$, is (U5). □

Proposition C.12. *Let* ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ *(*$T_1 \to T_2$*) and the weight word* $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ *satisfy the hypotheses of Proposition C.25; put* $\mathbf{N} = \mathbf{N}(T_2)$ *and* $r = \dim_L V_2$. *Let* $c \in C$ *and* $e = \mathrm{len}_c$. *A* low frame at c *is a pair* $(\mathbf{n}, R)$ *of a* Δ*-invariant frame* $\mathbf{n}$ *at* c' *of* $\mathbf{N}$*, with crystalline frame* (d_k)*, and a set* $R \subseteq \{1,\dots,r\}$ *with* $|R| \leq m\,e$*, such that*

(U1) *for* $k \notin R$ *and* $s < e$ *with* $\epsilon_{c-s} = 1$*:* $\varphi^e(d_k) \in \varphi^{s+4}\big(\mathrm{Fil}^{\,s_{c-s}} D_{\mathrm{cris}}(V_2)\big)$*;*

(U2) *for* $s < e$ *with* $\epsilon_{c-s} = -1$*:* $\varphi^{s+4}\big(\varepsilon_{c-s-1}\mathrm{Fil}^{-\eta} D_{\mathrm{cris}}(V_2)\big) \subseteq \sum_{k\in R} L\,\varphi^e(d_k)$.

Then a low frame at c exists.

Proof. Write $D = D_{\mathrm{cris}}(V_2)$, $R_0 = e_{c'}\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$, $\Delta = \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$, and ε_b for the idempotents of D (Lemma 23.10).

Step 1: a Δ-invariant frame exists. $\mathbf{N}$ is a Wach module ((Ex1) of Proposition 23.7), so by Lemma 24.15 $\mathbf{N}_{c'}$ has a Δ-invariant frame $\mathbf{n}^0$; let (d_k^0) be its crystalline frame.

Step 2: changing the frame by a constant matrix. Let $g = (g_{kj}) \in \mathrm{GL}_r(\mathcal{O}_L)$ and $n_k = \sum_j g_{kj} n_j^0$ (with g_{kj} acting as $e_{c'}(1 \otimes g_{kj})$). Then $\mathbf{n}$ is a frame at c', Δ-invariant because γ_δ fixes the coefficients. The horizontal section through n_k is $\sum_j g_{kj} H_j^0$ (it lies in $n_k + X\mathbf{N}_E$ and is γ_u-fixed; uniqueness in (Hf2) of Lemma 23.9), and Ev is linear over the constants $\mathcal{O}_L$ ((Ev2), (Ev3) of Lemma 23.11: $\mathrm{Ev}(fW) = e(f)\mathrm{Ev}(W)$ and $e(e_{c'}(1 \otimes \lambda))$ acts on $d_j^0 \in \varepsilon_{c'+3}D$ as λ). Hence $d_k = \sum_j g_{kj} d_j^0$ and $\varphi^e(d_k) = \sum_j g_{kj}\varphi^e(d_j^0)$, as φ is L-linear.

Step 3: the subspaces. By Lemma 23.10 (2), (d_k^0) is an L-basis of $\varepsilon_{c'+3}D$. The Frobenius of D is bijective, L-linear, and maps $\varepsilon_b D$ onto $\varepsilon_{b+1}D$; so $\beta_k^0 = \varphi^e(d_k^0)$ is an L-basis of $D' = \varepsilon_{c+3}D$, and $D_0 = \bigoplus_k \mathcal{O}_L\beta_k^0$ is a lattice in D'. For $s < e$ put $a = c - s$ and

$$W_s = \varphi^{s+4}\big(\varepsilon_{a-1}\mathrm{Fil}^{\,s_a}D\big)\ (\epsilon_a = 1), \qquad G_s = \varphi^{s+4}\big(\varepsilon_{a-1}\mathrm{Fil}^{-\eta}D\big)\ (\epsilon_a = -1),$$

L-subspaces of D' ($\mathrm{Fil}^u D$ is an $F \otimes L$-submodule, so $\varepsilon_{a-1}\mathrm{Fil}^u D \subseteq \mathrm{Fil}^u D \cap \varepsilon_{a-1}D$). By the remark in Definition 23.3, $\varepsilon_{a-1}D$ is the τ_a-component of $D = D_{\mathrm{dR}}(V_2)$, so $\dim_L \varepsilon_{a-1}\mathrm{Fil}^u D = \dim_L \mathrm{Fil}^u D_{\mathrm{dR},\tau_a}(V_2)$. Every Hodge jump of V_2 at τ_a is one of V_1 (Lemma 24.2 (1)). If $\epsilon_a = 1$ these lie in $[s_a, s_a+S] \cup [-\eta, \eta]$ by (Ow1) of Definition 23.1, and $s_a > \eta$; so the jumps below s_a are the low ones, of total multiplicity at most m by (Ow2), and $\mathrm{codim}_{D'} W_s \le m$. If $\epsilon_a = -1$ they lie in $[-s_a - S, -s_a] \cup [-\eta, \eta]$ with $-s_a < -\eta$; so the jumps $\ge -\eta$ are the low ones and $\dim G_s \le m$. Put $W = \bigcap_{\epsilon_{c-s}=1} W_s$ and $G = \sum_{\epsilon_{c-s}=-1} G_s$ (over $s < e$); then $\mathrm{codim}\, W + \dim G \le m\,e$.

Step 4: an adapted lattice basis. $W_0 = W \cap D_0$ is saturated in D_0, so D_0/W_0 is a finitely generated torsion-free, hence free, $\mathcal{O}_L$-module and $D_0 = W_0 \oplus C_0$ with $\mathrm{rank}\, C_0 = \mathrm{codim}\, W$. Put $Z = G + LC_0$. Then $Z \cap D_0 = C_0 \oplus (Z \cap W_0)$ (if $z = w + c$ with $c \in C_0 \subseteq Z$, then $w \in Z$), and $Z \cap W_0$ is saturated in W_0, so $W_0 = (Z \cap W_0) \oplus W_1$. Hence $D_0 = (Z \cap D_0) \oplus W_1$, and $\mathrm{rank}(Z \cap D_0) = \dim Z \le \dim G + \mathrm{codim}\, W \le m\,e$. Choose an $\mathcal{O}_L$-basis (β_k) of D_0 whose first $|R|$ members, indexed by a set R, form a basis of $Z \cap D_0$ and whose others form a basis of W_1; write $\beta_k = \sum_j g_{kj}\beta_j^0$ with $g \in \mathrm{GL}_r(\mathcal{O}_L)$.

Step 5: the low frame. Let $\mathbf{n}$ be the frame of Step 2 for this g. It is a Δ-invariant frame at c' with $\varphi^e(d_k) = \beta_k$, and $|R| \le m\,e$. (U1): for $k \notin R$, $\beta_k \in W_1 \subseteq W \subseteq W_s \subseteq \varphi^{s+4}(\mathrm{Fil}^{s_{c-s}}D)$ for every $s < e$ with $\epsilon_{c-s} = 1$. (U2): for $s < e$ with $\epsilon_{c-s} = -1$, $G_s \subseteq G \subseteq Z = L(Z \cap D_0) = \sum_{k \in R} L\beta_k$, since $Z \cap D_0$ is a lattice in Z. So $(\mathbf{n}, R)$ is a low frame at c. □

Proposition C.13. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word satisfy the hypotheses of Proposition C.25; put $\mathbf{N} = \mathbf{N}(T_2)$, $r = \dim_L V_2$. Let $c \in C$, $e = \mathrm{len}_c$, and $(\mathbf{n}, R)$ a low frame at c. Then for every Δ-invariant $w \in e_c\mathbf{N}[1/\lambda_\infty]$ with $(\gamma_u - 1)(w) \in X\mathbf{N}$ for all $u \in \mathbb{Z}_\ell^\times$, $\Pi_c w \in P_e(\mathbf{N}_{c'})$, and some $\ell^t w$ having an arc coordinate vector for $\mathbf{n}$, e, H_c, m_X that is pole-bounded by B^c, there is a Δ-invariant arc coordinate vector f of $\ell^{\sigma_B} w$ for $\mathbf{n}$, e, H_c, m_X such that for every $k \notin R$: $\mu_c f_k \in e_c\mathbf{A}_{F,L}$; $\mu_c(\gamma_{u_0} - 1)f_k \in X^{\ell-1}\delta_c\, e_c\mathbf{A}_{F,L}$; $\nu_c(\gamma_{u_0} - 1)P_c(\gamma_{u_0})f_k \in X^{\ell-1}\nu_c^+\, e_c\mathbf{A}_{F,L}$; $\nu_c P_c(\gamma_{u_0})f_k \in e_c\mathbf{A}_{F,L}$; and for every $s < e$ with $\epsilon_{c-s} = -1$ there are $\kappa \ge 0$ and $x \in \mathcal{O}_L$ with $\nu_c(\ell^\kappa P_c(\gamma_{u_0})f_k - e_c(1 \otimes x)) \in Q_s^{s_{c-s}+S} e_c\mathbf{A}_{F,L}$; and the vector whose k-th entry is $P_c(\gamma_{u_0})f_k$ for $k \notin R$ and 0 for $k \in R$, and the vector whose k-th entry is 0 for $k \notin R$ and $P_c(\gamma_{u_0})f_k$ for $k \in R$, are both Hodge-integral modulo γ.*

Proof. Throughout, w_0 denotes the vector w of the statement; for $s < e$ write $a_s = c - s$, and define $\alpha, \alpha' \in \mathbb{Z}^e$ by $\alpha_s = -(s_{a_s} + S)$, $\alpha'_s = -(s_{a_s} + S)$ if $\epsilon_{a_s} = 1$, and $\alpha_s = 0$, $\alpha'_s = s_{a_s}$ if

$\epsilon_{a_s} = -1$; thus $I_\alpha = \mu_c^{-1}\mathcal{R}$ and $I_{\alpha'} = \mu_c^{-1}\delta_c\mathcal{R}$. Since Π divides a power of λ_e, every $h \in \mathcal{K}_c$ with $\Pi h \in \mathcal{R}$ lies in $\mathcal{R}[1/\lambda_e]$. We use the coordinate calculus of Lemma C.16 ((1) window, (a) linearity, (b) equivariance) and Lemma C.17; its hypothesis $m_X \geq 2H_c(x)$ holds because $m_X = (\ell-1)(\lfloor 2h_B/(\ell-1)\rfloor+1) > 2h_B$ and $h_B \geq H_c(x)$ for all x ($w+S \leq b+\eta+S$ as $20w \leq b$).

Step 0: windows and low coordinates. Fix $s < e$ and $a = a_s$. By (Ow1) and Lemma 24.2 (1), every Hodge jump of V_2 at τ_a lies in $[s_a, s_a+S]\cup[-\eta,\eta]$ if $\epsilon_a = 1$ and in $[-s_a-S,-s_a]\cup[-\eta,\eta]$ if $\epsilon_a = -1$; recall $s_a > \eta$. By the remark in Definition 23.3, for $\mathbf{v} \in \mathcal{O}_L^r$ and $u \in \mathbb{Z}$ the condition $\delta_e(\mathbf{v}) \in \varphi^{s+4}(\mathrm{Fil}^u D_2)$ says that $\varphi^{-(s+4)}\delta_e(\mathbf{v})$, an element of $\varepsilon_{a-1}D_2$, lies in $\varepsilon_{a-1}\mathrm{Fil}^u D_2$, whose jumps are the Hodge jumps of V_2 at τ_a. Hence:

(W1) $\varepsilon_{a-1}\mathrm{Fil}^u D_2 = 0$ for $u > s_a + S$;

(W2) $\varepsilon_{a-1}\mathrm{Fil}^u D_2 = \varepsilon_{a-1}D_2$ for $u \leq -(s_a+S)$; so $\delta_e(\mathbf{v}) \in \varphi^{s+4}(\mathrm{Fil}^u D_2)$ for all $\mathbf{v}$ and all $u \leq -(s_a+S)$, in particular for $u = -H_c(a)$ (Lemma C.16 (1));

(W3) if $\epsilon_a = -1$ and $u > -s_a$, then $\varepsilon_{a-1}\mathrm{Fil}^u D_2 \subseteq \varepsilon_{a-1}\mathrm{Fil}^{-\eta} D_2$ (there is no jump in $[-s_a+1, -\eta-1]$).

By Lemma 23.10 (2) the d_k form an L-basis of $\varepsilon_{c'+3}D_2$, so the $\varphi^e(d_k)$ are L-linearly independent. Let $(g_i, \mathbf{v}_i, \alpha_i)$ be a section decomposition, $k \notin R$, and i with $v_{i,k} \neq 0$. We claim:

(V) if $\delta_e(\mathbf{v}_i) \in \varphi^{s+4}(\mathrm{Fil}^{-\alpha_i(s)} D_2)$, then $\alpha_i(s) \geq -(s_a+S)$ when $\epsilon_a = 1$, and $\alpha_i(s) \geq s_a$ when $\epsilon_a = -1$.

If $\epsilon_a = 1$ and $\alpha_i(s) < -(s_a+S)$, (W1) gives $\delta_e(\mathbf{v}_i) = 0$, so $\mathbf{v}_i = 0$ by independence, a contradiction. If $\epsilon_a = -1$ and $\alpha_i(s) < s_a$, then $-\alpha_i(s) > -s_a$ and (W3) and (U2) give $\delta_e(\mathbf{v}_i) \in \varphi^{s+4}(\varepsilon_{a-1}\mathrm{Fil}^{-\eta} D_2) \subseteq \sum_{k' \in R} L\varphi^e(d_{k'})$, so $v_{i,k} = 0$ by independence, a contradiction.

We also use: (Δ-*invariant multiples of* X) if $h \in X\mathcal{R}$ is Δ-invariant then $h \in \pi_0\mathcal{R}$ and h/π_0 is Δ-invariant. Indeed, if $h \neq 0$ has lowest term $a_j X^j$ ($j \geq 1$), then $\gamma_\omega(h)$ has lowest term $a_j\omega^j X^j$ ($\gamma_\omega(X) = \omega X + O(X^2)$), so $\omega^j = 1$ for all $\omega \in \mu_{\ell-1}$, $(\ell-1) \mid j$ and $h \in X^{\ell-1}\mathcal{R} = \pi_0\mathcal{R}$; as π_0 is Δ-invariant and $\mathcal{R}$ a domain, h/π_0 is Δ-invariant.

Step 1: a Δ-invariant coordinate vector. By Lemma C.14 (frame $\mathbf{n}$ at c', e, H_c, m_X; $\sigma_B \geq (2H_c(c'+1)+m_X/(\ell-1))\ell/(\ell-1)^2$ as $H_c(c'+1) = w+S$; $\Pi w_0 \in U$) there is a coordinate vector f^1 of $\ell^{\sigma_B}w_0$ with $\Pi f_k^1 \in \mathcal{R}$. Put $f = (\ell-1)^{-1}\sum_{\omega\in\mu_{\ell-1}}\gamma_\omega(f^1)$. By (b) each $\gamma_\omega(f^1)$ is a coordinate vector of $\gamma_\omega(\ell^{\sigma_B}w_0) = \ell^{\sigma_B}w_0$, so by (a) f is a coordinate vector of $\ell^{\sigma_B}w_0$; f is Δ-invariant, and $\Pi f_k \in \mathcal{R}$ since $\gamma_\omega(\Pi) \in \Pi\mathcal{R}^\times$.

Step 2: restriction of poles, $\mu_c f_k \in \mathcal{R}$ for $k \notin R$. Let f^0 be a coordinate vector of $\ell^t w_0$ that is pole-bounded by B^c, with decomposition $(g_i, \mathbf{v}_i, \alpha_i)$ as in (Gs2). By Lemma C.17 (1), $\ell^t f$ and $\ell^{\sigma_B}f^0$ are coordinate vectors of $\ell^{\sigma_B+t}w_0$, so by Lemma C.17 (2) $\ell^t f_k - \ell^{\sigma_B}f_k^0 \in X\Pi\mathcal{R} \subseteq \mathcal{R}$. Fix $k \notin R$ and i with $v_{i,k} \neq 0$, and $s < e$. If $\epsilon_{a_s} = 1$: either $\alpha_i(s) \geq 0$, or the Fil-condition of (Gs2) holds and (V) gives $\alpha_i(s) \geq -(s_{a_s}+S)$. If $\epsilon_{a_s} = -1$: if $\alpha_i(s) < 0$, the Fil-condition holds and (V) gives $\alpha_i(s) \geq s_{a_s} > 0$, impossible; so $\alpha_i(s) \geq 0$. Hence $\alpha_i \geq \alpha$, $g_i \in I_\alpha$ and $f_k^0 = \sum_i g_i v_{i,k} \in I_\alpha$. As $\alpha \leq 0$, $\mathcal{R} \subseteq I_\alpha$, so $\ell^t f_k \in I_\alpha$; since $f_k \in \mathcal{R}[1/\lambda_e]$, item 2 of Lemma C.18 (with $c = \ell^t$) gives $f_k \in I_\alpha$, i.e. $\mu_c f_k \in \mathcal{R}$.

Step 3: $(\gamma-1)f = \pi_0 G$ with $G \in \mathcal{H}_{\mathbf{n}}$. Since γ fixes e_c, $z = (\gamma-1)w_0 \in X\mathbf{N}\cap e_c\mathbf{N} = X\mathbf{N}_c$, and z is Δ-invariant ($\gamma_\omega\gamma = \gamma\gamma_\omega$). In a Δ-invariant $\mathcal{R}$-basis of $\mathbf{N}_c$ (Lemma 24.15) the coordinates of z are Δ-invariant elements of $X\mathcal{R}$, hence in $\pi_0\mathcal{R}$ with Δ-invariant quotients (Step 0); so $z = \pi_0 y_0$ with $y_0 \in \mathbf{N}_c$ Δ-invariant. By Lemma C.16 (1), $\Pi y_0 \in U$, and Lemma 23.13 (2) (frame $\mathbf{n}$ at c', e, H_c, m_X) gives $t_1, \kappa_1 \geq 0$ and a coordinate vector g^1 of $\ell^{t_1}y_0$ with $\ell^{\kappa_1}g^1$ Hodge-graded. Put $g = (\ell-1)^{-1}\sum_\omega \gamma_\omega(g^1)$: as in Step 1 it is a Δ-invariant coordinate vector of $\ell^{t_1}y_0$, and $\ell^{\kappa_1}g$

is Hodge-graded, because $(\gamma_\omega(g_i), \mathbf{v}_i, \alpha_i)$ is a section decomposition of $\gamma_\omega(\ell^{\kappa_1} g^1)$ with the same $\mathbf{v}_i$ ($\gamma_\omega(I_\eta) = I_\eta$ by Lemma 23.8 (2)), and sums and $\mathbb{Z}_\ell$-multiples of Hodge-graded vectors are Hodge-graded (concatenate decompositions).

By (a), (b), $(\gamma-1)f$ is a coordinate vector of $\ell^{\sigma_B}(\gamma-1)w_0 = \ell^{\sigma_B}\pi_0 y_0$; by Lemma C.17 (1), $\pi_0 g$ is a coordinate vector of $\ell^{t_1}\pi_0 y_0$. So $\ell^{t_1}(\gamma-1)f$ and $\ell^{\sigma_B}\pi_0 g$ are coordinate vectors of $\ell^{\sigma_B+t_1}\pi_0 y_0$, and $d = \ell^{t_1}(\gamma-1)f - \ell^{\sigma_B}\pi_0 g$ has entries $d_k \in X\Pi\mathcal{R}$ (Lemma C.17 (2)). The d_k are Δ-invariant, so $d_k = \pi_0 d_k'$ with $d_k' \in \mathcal{R}$ (Step 0); then $X^{\ell-1}(v d_k') = d_k \in \Pi\mathcal{R} = I_{(H_c(c-s))_s}$, and item 2 of Lemma C.18 (with $c = X^{\ell-1}$) gives $v d_k' \in \Pi\mathcal{R}$, so $d_k' \in \Pi\mathcal{R}$. Put

$$G = \ell^{-t_1}\big(\ell^{\sigma_B} g + d'\big) \in \mathcal{K}_c^r, \qquad \text{so that} \qquad (\gamma-1)f = \pi_0 G.$$

We have $G \in \mathcal{H}_{\mathbf{n}}$: first, $y = \Pi G_k$ satisfies $\ell^{t_1} y = \ell^{\sigma_B}\Pi g_k + \Pi d_k' \in \mathcal{R}$ and $\pi_0 y = \Pi(\gamma-1)f_k = \gamma(\Pi f_k)\cdot\Pi/\gamma(\Pi) - \Pi f_k \in \mathcal{R}$; writing $r_1 = \ell^{t_1} y \in \mathcal{R}$, the series $X^{\ell-1} v\, r_1 = \ell^{t_1}\pi_0 y$ has all coefficients in $\ell^{t_1}\mathcal{O}_L$, hence so has r_1 (v a unit, $X^{\ell-1}$ shifts coefficients), and $y \in \mathcal{R}$. Second, $\ell^{\kappa_1+t_1} G = \ell^{\sigma_B}(\ell^{\kappa_1} g) + \ell^{\kappa_1} d'$ is Hodge-graded: $\ell^{\kappa_1} g$ is, and d' has the section decomposition $(d_k', \mathbf{e}_k, (H_c(a_s))_s)_k$ ($d_k' \in \Pi\mathcal{R} = Q^{(H_c(a_s))_s}\mathcal{R}$), which satisfies (Gs1) by (W2).

Step 4: restriction of the γ-difference, $\mu_c(\gamma-1)f_k \in X^{\ell-1}\delta_c\mathcal{R}$ for $k \notin R$. Let $(g_i, \mathbf{v}_i, \alpha_i)$ be a decomposition as in (Gs1) of $\ell^\kappa G$, $\kappa = \kappa_1 + t_1$, and $k \notin R$. For every i with $v_{i,k} \neq 0$ and every $s < e$, (V) gives $\alpha_i(s) \geq \alpha_s'$; so $\ell^\kappa G_k = \sum_i g_i v_{i,k} \in I_{\alpha'}$, and, as $G_k \in \mathcal{R}[1/\lambda_e]$ ($\Pi G_k \in \mathcal{R}$), item 2 of Lemma C.18 gives $G_k \in I_{\alpha'}$. Hence $\mu_c(\gamma-1)f_k = \pi_0\mu_c G_k \in \pi_0\delta_c\mathcal{R} = X^{\ell-1}\delta_c\mathcal{R}$.

Step 5: the operator $P_c(\gamma)$. Let $k \notin R$. Apply Lemma C.20 to f_k, with $e = \mathrm{len}_c$, S, $\epsilon_s = \epsilon_{a_s}$, $b_s = s_{a_s} \geq 1$ (sizes exceed $\eta \geq 0$), $J = J_c$ (the union of the windows W_{a_s}, as required there), u_0, and $P = P_{J_c,e} = P_c$; then its μ, δ, ν^+, ν are $\mu_c, \delta_c, \nu_c^+, \nu_c$. Its hypotheses are Steps 2 and 4, and f_k is Δ-invariant (Step 1). Its items 1, 2, 3 give $\nu_c P_c(\gamma) f_k \in \mathcal{R}$, $\nu_c(\gamma-1)P_c(\gamma)f_k \in X^{\ell-1}\nu_c^+\mathcal{R}$, and, for $s < e$ with $\epsilon_{a_s} = -1$, $\kappa \geq 0$ and $x \in \mathcal{O}_L$ with $\nu_c(\ell^\kappa P_c(\gamma)f_k - e_c(1\otimes x)) \in Q_s^{s_{a_s}+S}\mathcal{R}$.

Step 6: the two split vectors. Let f^{out} have entries $P_c(\gamma)f_k$ for $k \notin R$ and 0 for $k \in R$, and $f^{\mathrm{in}} = P_c(\gamma)f - f^{\mathrm{out}}$. By Step 5, $(\gamma-1)f^{\mathrm{out}} = X^{\ell-1}h$ with $h_k \in \nu_c^{-1}\nu_c^+\mathcal{R}$ for $k \notin R$ and $h_k = 0$ for $k \in R$. Then $h \in \mathcal{H}_{\mathbf{n}}$: $\Pi h_k \in \mathcal{R}$ because ν_c has exponent $s_{a_s} \leq H_c(a_s)$ at Q_s; and $(h_k, \mathbf{e}_k, \beta')_{k\notin R}$, with $\beta_s' = -s_{a_s}$ if $\epsilon_{a_s} = 1$ and $\beta_s' = s_{a_s}+S$ if $\epsilon_{a_s} = -1$, is a section decomposition satisfying (Gs1): at $\epsilon_{a_s} = 1$, $\delta_e(\mathbf{e}_k) = \varphi^e(d_k) \in \varphi^{s+4}(\mathrm{Fil}^{s_{a_s}} D_2)$ by (U1) ($k \notin R$); at $\epsilon_{a_s} = -1$, by (W2). So f^{out} is Hodge-integral modulo γ. Next, $\mathcal{H}_{\mathbf{n}}$ is stable under every γ_u (if $\Pi h_k \in \mathcal{R}$ then $\Pi\gamma_u(h_k) = \gamma_u(\Pi h_k)\,\Pi/\gamma_u(\Pi) \in \mathcal{R}$; section decompositions are transported as in Step 3), and $\gamma_u(\pi_0) \in \pi_0\mathcal{R}^\times$ ($\gamma_u(X) \in X\mathcal{R}^\times$, Lemma 23.8 (2)); so $\pi_0\mathcal{H}_{\mathbf{n}}$ is stable under every polynomial in γ, and

$$(\gamma-1)P_c(\gamma)f = P_c(\gamma)((\gamma-1)f) = P_c(\gamma)(\pi_0 G) \in \pi_0\mathcal{H}_{\mathbf{n}} = X^{\ell-1}\mathcal{H}_{\mathbf{n}}$$

by Step 3. Hence $(\gamma-1)f^{\mathrm{in}} = (\gamma-1)P_c(\gamma)f - (\gamma-1)f^{\mathrm{out}} \in X^{\ell-1}\mathcal{H}_{\mathbf{n}}$, and f^{in} is Hodge-integral modulo γ.

Steps 1–6 give a Δ-invariant coordinate vector f of $\ell^{\sigma_B} w_0$ with all asserted properties. □

Lemma C.14. *Let ℓ be an odd prime, F, L, (τ_a) and a pair (V,T) as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$ of rank r. Let $c' \in \mathbb{Z}/n$, $e \geq 1$, $c = c'+e$, $H\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$, $\Pi = \Lambda_{c,e}(H)$, let $m \geq 1$ with $m \geq 2H(c-s)$ for $0 \leq s < e$, and let σ be an integer with $\sigma \geq \big(2H(c'+1)+m/(\ell-1)\big)\ell/(\ell-1)^2$. Let $\mathbf{n}$ be a frame at c' of $\mathbf{N}$. Then for every $y \in e_c\mathbf{N}[1/\lambda_\infty]$ with $\Pi y \in P_e(\mathbf{N}_{c'})$ there is an arc coordinate vector f of $\ell^\sigma y$ for $\mathbf{n}$, e, H, m with $\Pi f_k \in e_c\mathbf{A}_{F,L}$ for all k; one can take the admissible exponent $\kappa = s_\ell(2m-2)$ and $\kappa' = 0$.*

Proof. Write $R_c = e_c\mathbf{A}_{F,L}$, $U = P_e(\mathbf{N}_{c'})$, $s = s_\ell$, $E = 2H(c'+1)$, and $\Lambda' = e_{c'}(\mathcal{O}_F \otimes \mathcal{O}_L)$.

Step 1: superadditivity. $s(a)+s(b)\le s(a+b)$ for $a,b\ge 0$: with $J(k)=\lfloor k/(\ell-1)\rfloor$ we have $J(a)+J(b)\le J(a+b)$, and $v_\ell(J_1!)+v_\ell(J_2!)\le v_\ell((J_1+J_2)!)$ (the binomial coefficient is an integer), while $J\mapsto J+v_\ell(J!)$ is nondecreasing.

Step 2: the inverse of the horizontal matrix. Let $h=(h_{kj})$ be the matrix with $H_k=\sum_j h_{kj}n_j$ (horizontal sections through the frame), $h=\sum_{i\ge 0}h_iX^i$ with $h_i\in M_r(e_{c'}E)$, $h_0=I$ ((Hf2) of Lemma 23.9) and $\ell^{s(i)}h_i\in M_r(\Lambda')$ ((Hf3), applied to an $\mathbf{A}_{F,L}$-basis of $\mathbf{N}$ extending $\mathbf{n}$ by frames at the other sites; $H_k\in e_{c'}\mathbf{N}_E$ by the uniqueness in (Hf2), as $e_{c'}$ commutes with every γ_u). Put $g=h^{-1}=\sum_{k\ge 0}(I-h)^k$ (convergent, $I-h\in XM_r(e_{c'}E[[X]])$), $g=\sum_i g_iX^i$. Each g_i is a sum of products $\pm h_{i_1}\cdots h_{i_k}$ with $i_1+\cdots+i_k=i$, $i_j\ge 1$, so $\ell^{s(i)}g_i\in M_r(\Lambda')$ by Step 1. Put $U_m=\sum_{i<m}h_iX^i$, $G_m=\sum_{i<m}g_iX^i$; then $G_mU_m=I+\sum_{m\le i\le 2m-2}w_iX^i$ with $w_i=\sum_{i_1+i_2=i,\ i_1,i_2<m}g_{i_1}h_{i_2}$, so $\ell^{s(i)}w_i\in M_r(\Lambda')$. Let $\kappa=s(2m-2)$; it is admissible and $\kappa\ge s(i)$ for $i\le 2m-2$; and $T_\kappa(H_k)=\sum_j\ell^\kappa(U_m)_{kj}n_j$.

Step 3: scalars. By Lemma C.15, for $0\le i<m$ there are $\alpha_i,\beta_i\in\mathbb{Z}_\ell[[X]]$ with $\ell^\sigma\varphi^e(X)^i=\ell^{s(i)}\alpha_i+D\beta_i$, $D=(X\lambda_{e-1})^mQ_{e-1}^E$, and $D\in X\Pi^2\mathbb{Z}_\ell[[X]]$ (for $s=e-1$, $c-s=c'+1$ and the exponent of Q_{e-1} in Π^2 is E; for $s<e-1$, Q_s^m divides λ_{e-1}^m and $m\ge 2H(c-s)$). Also $\varphi^e(X)^i=(X\lambda_e)^i\in X\Pi^2\mathbb{Z}_\ell[[X]]$ for $i\ge m$.

Step 4: the coordinate vector. Write $\Pi y=\sum_j b_j\varphi^e(n_j)$ with $b_j\in R_c$ (U is generated by the $\varphi^e(n_j)$), and $a_j=b_j/\Pi\in\mathcal{K}_c$. Define

$$f_k=\sum_j a_j\sum_{i<m}\varphi^e\big(\ell^{s(i)}(g_i)_{jk}\big)\alpha_i,$$

so $\Pi f_k=\sum_j b_j(\cdots)\in R_c$. By Step 3, $f=a(\ell^\sigma\varphi^e(G_m)-D\,B)$ with $B_{jk}=\sum_{i<m}\varphi^e((g_i)_{jk})\beta_i$ (row vector $a=(a_j)$, φ^e applied entrywise). Hence

$$f\cdot\ell^\kappa\varphi^e(U_m)=\ell^{\sigma+\kappa}a+a\,\mathcal{E},\qquad \mathcal{E}=\ell^{\sigma+\kappa}\sum_{m\le i\le 2m-2}\varphi^e(w_i)\varphi^e(X)^i-D\,\ell^\kappa B\,\varphi^e(U_m).$$

Every entry of $\mathcal{E}$ lies in $X\Pi^2R_c$: in the first sum $\ell^\kappa\varphi^e(w_i)$ is integral ($\kappa\ge s(i)$) and $\varphi^e(X)^i\in X\Pi^2$ ($i\ge m$); in the second, $\ell^\kappa B\varphi^e(U_m)$ is a sum of terms $\ell^\kappa\varphi^e(g_{i_1}h_{i_2})\beta_{i_1}\varphi^e(X)^{i_2}$ with $i_1,i_2<m$, integral because $s(i_1)+s(i_2)\le s(2m-2)=\kappa$ (Step 1), and $D\in X\Pi^2$. As $\varphi_{\mathbf{N}}$ is φ-semilinear, $\varphi^e(T_\kappa(H_k))=\sum_{j'}\ell^\kappa\varphi^e(U_m)_{kj'}\varphi^e(n_{j'})$, so

$$\sum_k\Pi f_k\,\varphi^e(T_\kappa(H_k))=\sum_{j'}\Big(\ell^{\sigma+\kappa}b_{j'}+\sum_j b_j\mathcal{E}_{jj'}\Big)\varphi^e(n_{j'})=\ell^\kappa\Pi(\ell^\sigma y)+\sum_{j,j'}b_j\mathcal{E}_{jj'}\varphi^e(n_{j'}),$$

and the last sum lies in $X\Pi^2U$. So $\ell^\kappa\Pi(\ell^\sigma y)-\sum_k\Pi f_k\varphi^e(T_\kappa(H_k))\in X\Pi^2U$: f is an arc coordinate vector of $\ell^\sigma y$ with $(\kappa,0)$, and $\Pi f_k\in R_c$. □

Lemma C.15. *Let ℓ be an odd prime, $m\ge 1$ and $E\ge 0$ integers, and σ an integer with*

$$\sigma\ \ge\ \Big(E+\frac{m}{\ell-1}\Big)\frac{\ell}{(\ell-1)^2}.$$

Then for every integer i with $0\le i<m$,

$$\ell^\sigma(XQ)^i\ \in\ \ell^{s_\ell(i)}\,\mathbb{Z}_\ell[[X]]+X^mQ^E\,\mathbb{Z}_\ell[[X]],$$

and consequently, for every integer $e\ge 1$,

$$\ell^\sigma\varphi^e(X)^i\ \in\ \ell^{s_\ell(i)}\,\mathbb{Z}_\ell[[X]]+(X\lambda_{e-1})^mQ_{e-1}^E\,\mathbb{Z}_\ell[[X]].$$

Proof. Write $R = \mathbb{Z}_\ell[[X]]$, $s = s_\ell(i)$ and $K = E + m/(\ell-1)$, so that $\sigma \geq K\ell/(\ell-1)^2$. Since $\varphi(X) = (1+X)^\ell - 1$,

$$Q = \sum_{k=1}^{\ell} \binom{\ell}{k} X^{k-1} = X^{\ell-1} + \ell\alpha, \qquad \alpha \in \mathbb{Z}_\ell[X],$$

because ℓ divides $\binom{\ell}{k}$ for $1 \leq k \leq \ell-1$.

Case $s \leq \sigma$. Then $\ell^\sigma (XQ)^i = \ell^s \cdot \ell^{\sigma-s}(XQ)^i \in \ell^s R$.

Case $s > \sigma$. Put $d = s - \sigma \geq 1$. From $K\ell/(\ell-1)^2 \leq \sigma < s \leq i\ell/(\ell-1)^2$ we get $i > K \geq E$, so $z = i - E \geq 1$ and $(XQ)^i = Q^E X^i Q^z$. By the binomial theorem

$$X^i Q^z = \sum_{t=0}^{z} \binom{z}{t} \ell^t \alpha^t X^{\,i+(\ell-1)(z-t)}.$$

A term with $t \geq d$ lies in $\ell^d R$, so ℓ^σ times it lies in $\ell^s R$. A term with $t \leq d-1$ has X-adic order at least $i+(\ell-1)(z-d+1)$, and this is at least m: indeed, with $s \leq i\ell/(\ell-1)^2$, $\sigma \geq K\ell/(\ell-1)^2$ and $m = (\ell-1)(K-E)$,

$$\begin{aligned}(i + (\ell-1)(z-d+1)) - m \;&\geq\; i + (\ell-1)(i-E+1) - \frac{(i-K)\ell}{\ell-1} - (\ell-1)(K-E) \\ &= (i-K)\frac{\ell(\ell-2)}{\ell-1} + K + \ell - 1 > 0,\end{aligned}$$

as $i > K$ and $\ell \geq 3$. So such a term lies in $X^m R$. Hence $\ell^\sigma X^i Q^z \in \ell^s R + X^m R$, and multiplying by Q^E gives $\ell^\sigma (XQ)^i \in \ell^s R + X^m Q^E R$.

Consequence. φ is a ring endomorphism of R fixing ℓ, with $\varphi^{e-1}(XQ) = \varphi^e(X)$, $\varphi^{e-1}(Q) = Q_{e-1}$ and $\varphi^{e-1}(X) = X\lambda_{e-1}$ (induction: $\varphi^{k+1}(X) = \varphi^k(XQ) = \varphi^k(X)Q_k$). Applying φ^{e-1} to the first assertion gives the second. □

Lemma C.16. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25; put $\mathbf{N}_2 = \mathbf{N}(T_2)$. Let $c \in C$, $e = \mathrm{len}_c$, $R_c = e_c \mathbf{A}_{F,L}$, $\Pi = \Pi_c$, $U = P_e(\mathbf{N}_{2,c'})$, and let $\mathbf{n}$ be a frame at c' of $\mathbf{N}_2$; in (a)–(d) a* coordinate vector *is an arc coordinate vector for $\mathbf{N}_2$, $\mathbf{n}$, e, H_c and m_X.*

(1) Window *For every a, every Hodge jump of V_1 and of V_2 at τ_a has absolute value at most $s_a + S \leq H_c(a)$. Hence $(H_c(a), H_c(a))_a$ is a window of $\mathbf{N}_2$ as in (Ex1), and $\Pi U \subseteq \mathbf{N}_{2,c}$, $\Pi \mathbf{N}_{2,c} \subseteq U$.*

(a) Linearity *If f, f' are coordinate vectors of $y, y' \in e_c\mathbf{N}_2[1/\lambda_\infty]$ and $o, o' \in R_c$, then $of + o'f'$ is a coordinate vector of $oy + o'y'$.*

(b) Equivariance *Let $u \in \mathbb{Z}_\ell^\times$. Then $\gamma_u(Q_s) \in Q_s\,\mathbb{Z}_\ell[[X]]^\times$ ($s \geq 0$), $\gamma_u(X) \in X\,\mathbb{Z}_\ell[[X]]^\times$ and $\gamma_u(U) = U$; if f is a coordinate vector of y, then $\gamma_u(f) = (\gamma_u f_k)_k$ is a coordinate vector of $\gamma_u(y)$. Hence for $P \in \mathbb{Z}_\ell[Y]$, $P(\gamma_u)f$ is a coordinate vector of $P(\gamma_u)y$, and $\Pi P(\gamma_u)y \in U$ if $\Pi y \in U$.*

(c) Vanishing *If $y \in e_c\mathbf{N}_2[1/\lambda_\infty]$, $\Pi y \in U$ and 0 is a coordinate vector of y, then $y \in X\mathbf{N}_{2,c}$.*

(d) Relay classes *Let I_c and $\rho = \rho_{c,1}$ be formed for $\mathbf{N}_2$ with a window and parameters G, g, m', R as in Lemma 24.5 ($j = 1$), and $u_0 \in \mathbb{Z}_\ell^\times$. Then I_c is a γ_u-stable $\mathcal{O}_L$-module containing $X\mathbf{N}_{2,c}$; if $y \in I_c$, $\Pi y \in U$ and 0 is a coordinate vector of y, then $\rho(y) = 0$; and $\rho(P(\gamma_{u_0})y) = P(1)\rho(y)$ for $y \in I_c$ and $P \in \mathbb{Z}_\ell[Y]$.*

(e) The source lattice *The last two assertions of (1) and items (a), (b), (c) hold with* $\mathbf{N}_1 = \mathbf{N}(T_1)$ *in place of* $\mathbf{N}_2$ *(frames of* $\mathbf{N}_1$*,* $U = P_e(\mathbf{N}_{1,c'})$*, coordinate vectors for* $\mathbf{N}_1$*).*

Proof. **(1).** A jump of V_2 at τ_a is a jump of V_1 (Lemma 24.2 (1)); by (Ow1) its absolute value is at most $\max(s_a+S,\eta) = s_a+S$ (sizes exceed η). By (Ow3), $s_a \le w$ at a thin site and $s_a \le b+\eta$ at a thick one; so $s_a+S \le w+S = H_c(c'+1)$ at the thin site $c'+1$, and $s_a+S \le h_B$ everywhere ($h_B = b+\eta+S$ if some site is thick; otherwise all sites are thin and $h_B = w+S$). By (Ex1) of Proposition 23.7, $\mathbf{N}_2$ is a Wach module with window $(H_c(a), H_c(a))_a$, and Lemma 23.15 (2) gives the two inclusions.

(a). If κ is admissible, so is every $\kappa_1 \ge \kappa$, and $T_{\kappa_1}(H_k) = \ell^{\kappa_1-\kappa}T_\kappa(H_k)$; multiplying the defining relation by $\ell^{\kappa_1-\kappa}\ell^{\kappa_1'-\kappa'}$ shows that it holds for all $\kappa_1 \ge \kappa$, $\kappa_1' \ge \kappa'$. So f, f' satisfy it with a common pair, and the relation for $of + o'f'$ follows as $X\Pi^2 U$ is an R_c-module; also $\Pi(of_k + o'f_k') \in R_c$.

(b). $\gamma_u(X) = X\varepsilon_u$ with $\varepsilon_u = ((1+X)^u - 1)/X$ of constant term u, and $\varphi^{s+1}(X) = X\lambda_{s+1}$; as γ_u commutes with φ, $\gamma_u(\lambda_{s+1}) = \lambda_{s+1}\varphi^{s+1}(\varepsilon_u)/\varepsilon_u$, so $\gamma_u(Q_s) = Q_s\varphi^{s+1}(\varepsilon_u)/\varphi^s(\varepsilon_u)$, a unit multiple ($\varphi$ preserves constant terms). Hence $\gamma_u(\Pi) = \Pi\upsilon$ with υ a unit, and $\gamma_u(U) = U$ (γ_u commutes with φ, preserves $\mathbf{N}_{2,c'}$ and R_c). Let f be a coordinate vector of y with (κ,κ'). Applying γ_u to the defining relation and dividing by υ,

$$\ell^{\kappa'}\Big(\ell^{\kappa}\Pi\gamma_u(y) - \sum_k \Pi\gamma_u(f_k)\varphi^e\big(\gamma_u T_\kappa(H_k)\big)\Big) \in X\Pi^2 U.$$

By definition

$$T_\kappa(H_k) - \ell^\kappa H_k = -\sum_j \sum_{i\ge m_X} \ell^\kappa h_{kj,i} X^i n_j \in X^{m_X}\mathbf{N}_E,$$

and $\gamma_u(H_k) = H_k$ (Lemma 23.9 (Hf2)); so $\gamma_u T_\kappa(H_k) - T_\kappa(H_k) = (\gamma_u - 1)(T_\kappa(H_k) - \ell^\kappa H_k) \in X^{m_X}\mathbf{N}_E \cap \mathbf{N}_{2,c'} = X^{m_X}\mathbf{N}_{2,c'}$ (coefficients in the basis $\mathbf{n}$, which is also a basis of $e_{c'}\mathbf{N}_E$ over $e_{c'}E[[X]]$ by (Hf2), as $n_k \in H_k + X\mathbf{N}_E$). Further $\varphi^e(X^{m_X}\mathbf{N}_{2,c'}) \subseteq (X\lambda_e)^{m_X}U \subseteq X\Pi^2 U$, because $m_X = (\ell-1)M > 2h_B \ge 2H_c(c-s)$ for $s < e$ and $m_X \ge 1$, and $\Pi\gamma_u(f_k) = \upsilon^{-1}\gamma_u(\Pi f_k) \in R_c$. So $\gamma_u(f)$ satisfies the defining relation for $\gamma_u(y)$ with the same (κ,κ'). The rest follows with (a).

(c). The defining relation reads $\ell^{\kappa+\kappa'}\Pi y \in X\Pi^2 U$; by saturation (Lemma 23.15 (3), $f = X\Pi^2$, $f' = 1$), $\Pi y = X\Pi^2 p$ with $p \in U$; as $e_c\mathbf{N}_2[1/\lambda_\infty]$ is torsion free over the domain R_c, $y = X\Pi p \in X\,\Pi U \subseteq X\mathbf{N}_{2,c}$ by (1).

(d). I_c is an $\mathcal{O}_L$-module (Ry4) and contains $X\mathbf{N}_{2,c}$, on which ρ vanishes (Ry5); it is γ_u-stable, as $(\gamma_{u'} - 1)\gamma_u y = \gamma_u(\gamma_{u'} - 1)y \in X\mathbf{N}_2$. The second assertion follows from (c). For the third, Lemma 23.22 (3) (module $e_c\mathbf{N}_2[1/\lambda_\infty]$, endomorphism γ_{u_0}, stable submodule $X\mathbf{N}_{2,c}$, $f = 1$; $(\gamma_{u_0} - 1)y \in X\mathbf{N}_2 \cap e_c\mathbf{N}_2[1/\lambda_\infty] = X\mathbf{N}_{2,c}$) gives $P(\gamma_{u_0})y - P(1)y \in X\mathbf{N}_{2,c}$, and (Ry4), (Ry5) give $\rho(P(\gamma_{u_0})y) = P(1)\rho(y)$.

(e). By (1), every Hodge jump of V_1 at τ_a has absolute value at most $H_c(a)$, so (Ex1) and Lemma 23.15 (2) give the two inclusions for $\mathbf{N}_1$; the proofs of (a), (b), (c) use of $\mathbf{N}_2$ only these inclusions, (Hf2) of Lemma 23.9, Lemma 23.15 (3) and $m_X > 2h_B$, all of which hold for $\mathbf{N}_1$, and go through unchanged. □

Lemma C.17. *Let* ℓ*,* F*,* L*,* (τ_a) *and a pair* (V,T) *be as in Proposition 23.7,* $\mathbf{N} = \mathbf{N}(T)$ *of rank* r*,* $c \in \mathbb{Z}/n$*,* $e \ge 0$*,* $a = c+e$*,* $H\colon \mathbb{Z}/n \to \mathbb{Z}_{\ge 0}$*,* $\Pi = \Lambda_{a,e}(H)$*,* $m \ge 1$ *with* $m \ge 2H(a-s)$ *for* $s < e$*, and* $\mathbf{n}$ *a frame of* $\mathbf{N}$ *at* c*;* coordinate vector *means arc coordinate vector for* $\mathbf{n}$*,* e*,* H*,* m*.*

1. *If* f *is a coordinate vector of* $y \in \mathbf{N}[1/\lambda_\infty]$ *and* $o \in e_a\mathbf{A}_{F,L}$*, then* of *is a coordinate vector of* oy*; if* f' *is a coordinate vector of* y'*, then* $f+f'$ *is a coordinate vector of* $y+y'$*.*

2. *If f and f' are coordinate vectors of the same element, then $f_k - f'_k \in X\Pi\, e_a\mathbf{A}_{F,L}$ for every $k \le r$.*

Proof. Write $R = e_a\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ and $U = P_e(\mathbf{N}_c)$, an $\mathbf{A}_{F,L}$-module.

Item 1. Multiplying (23.1) for f by o gives (23.1) for of and oy with the same κ, κ', and $\Pi o f_k \in R$. For the sum, let (κ_1, κ'_1), (κ_2, κ'_2) be the exponents of f, f', and $\kappa = \max(\kappa_1, \kappa_2)$, $\kappa' = \max(\kappa'_1, \kappa'_2)$. An integer $\ge$ an admissible one is admissible, and $T_\kappa(H_k) = \ell^{\kappa-\kappa_i} T_{\kappa_i}(H_k)$ by the definition of the truncation; multiplying the relation for f by $\ell^{\kappa'-\kappa'_1}\ell^{\kappa-\kappa_1}$ (an element of $\mathcal{O}_L$) gives (23.1) for f with (κ, κ'), likewise for f', and adding gives it for $f + f'$ and $y + y'$.

Item 2. By item 1 (with $o = -1$), $f - f'$ is a coordinate vector of 0. By (Ac4) of Lemma 23.14 there are $t \ge 0$ and $g_k \in R$ with $\ell^t(f_k - f'_k) = X\Pi g_k$. As $\Pi f_k, \Pi f'_k \in R$, $X\Pi^2 g_k = \ell^t \Pi(f_k - f'_k) \in \ell^t R$. Now $G = X\Pi^2$ is a monic polynomial in X, so its reduction modulo a uniformiser ϖ of $\mathcal{O}_L$ is a nonzero element of the domain $(\mathcal{O}_L/\varpi)[[X]]$; hence $Gh \in \varpi R$ implies $h \in \varpi R$, and by induction $Gh \in \varpi^j R$ implies $h \in \varpi^j R$ (R is a domain). So $g_k \in \ell^t R$ and $f_k - f'_k = X\Pi(g_k/\ell^t) \in X\Pi R$. □

Lemma C.18. *Let ℓ be an odd prime, L a finite extension of $\mathbb{Q}_\ell$, $R = \mathcal{O}_L[[X]]$ with γ_u ($u \in \mathbb{Z}_\ell^\times$) acting $\mathcal{O}_L$-linearly by $X \mapsto (1+X)^u - 1$, $\mathcal{K}$ the fraction field of R, and $e \ge 1$. For $\alpha \in \mathbb{Z}^e$ put $I_\alpha = \prod_{s<e} Q_s^{\alpha_s} R \subseteq \mathcal{K}$, and for $t < e$ put $R^{(t)} = R[1/\prod_{s<e,\, s\ne t} Q_s] \subseteq \mathcal{K}$.*

1. (Membership level by level.) *For $h \in \mathcal{K}$ and $\alpha \in \mathbb{Z}^e$: $h \in I_\alpha$ if and only if $h \in Q_t^{\alpha_t} R^{(t)}$ for every $t < e$. In particular $I_\alpha \cap I_\beta = I_{\max(\alpha,\beta)}$ (componentwise maximum) for $\alpha, \beta \in \mathbb{Z}^e$.*

2. (Saturation.) *Let $c \in R$ be such that $c(x_0) \ne 0$ for every root x_0 of every Q_s, $s < e$ (for instance $c \in \mathcal{O}_L \setminus \{0\}$, or $c = X^j$ with $j \ge 0$). If $h \in R[1/\lambda_e]$ and $ch \in I_\alpha$, then $h \in I_\alpha$.*

3. (Invariant residues are constants.) *Let $t < e$, $u_0 \in \mathbb{Z}_\ell^\times$ with $v_\ell(u_0 - 1) = 1$, and $h \in R^{(t)}$ with $\gamma_\delta(h) = h$ for every $\delta \in \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$ and $(\gamma_{u_0} - 1)(h) \in Q_t^N R^{(t)}$ for an integer $N \ge 1$. Then there are an integer $\kappa \ge 0$ and $x \in \mathcal{O}_L$ with $\ell^\kappa h - x \in Q_t^N R^{(t)}$.*

Proof. Evaluation. For a root $x_0 = \zeta - 1$ of Q_t we have $|x_0| < 1$, so $g \mapsto g(x_0)$ is a ring homomorphism $\mathrm{ev}_{x_0} \colon R \to \mathcal{O}_L[\zeta] \subset \overline{\mathbb{Q}}_\ell$. For $s \ne t$ the roots of Q_s have a different order, so $Q_s(x_0) \ne 0$; hence ev_{x_0} extends to a ring homomorphism $R^{(t)} \to \overline{\mathbb{Q}}_\ell$. Moreover $\mathrm{ev}_{x_0}(c) = c \ne 0$ for $c \in \mathcal{O}_L \setminus \{0\}$, and $\mathrm{ev}_{x_0}(X^j) = x_0^j \ne 0$ since $\zeta \ne 1$.

Step 1: a divisibility fact. Let $t < e$, $a \ge 0$, $g \in R$, and let $z \in R$ with $z(x_0) \ne 0$ for every root x_0 of Q_t (for instance a product of an element c as in item 2 and of powers of the Q_s, $s \ne t$, $s < e$). If $Q_t^a \mid zg$ in R, then $Q_t^a \mid g$. Indeed, for $a \ge 1$ write $zg = Q_t^a w$; evaluating at a root x_0 of Q_t gives $z(x_0)g(x_0) = 0$ with $z(x_0) \ne 0$, so g vanishes at all roots of Q_t and $g = Q_t g'$ with $g' \in R$; as R is a domain, $zg' = Q_t^{a-1} w$, and induction on a gives $Q_t^{a-1} \mid g'$. Consequently, if $a_t \ge 0$ and $Q_t^{a_t} \mid g$ for every $t < e$, then $\prod_{t<e} Q_t^{a_t} \mid g$: by induction on the number of levels already collected, if $g = \prod_{t\in T} Q_t^{a_t} y$ with $y \in R$ and $t' \notin T$, then $Q_{t'}^{a_{t'}} \mid (\prod_{t\in T} Q_t^{a_t}) y$, so $Q_{t'}^{a_{t'}} \mid y$ by the first part.

Item 1. If $h = \prod_s Q_s^{\alpha_s} y$ with $y \in R$, then for each t the factor $\prod_{s\ne t} Q_s^{\alpha_s} y$ lies in $R^{(t)}$, so $h \in Q_t^{\alpha_t} R^{(t)}$. Conversely, suppose $h = Q_t^{\alpha_t} y_t$ with $y_t \in R^{(t)}$ for every $t < e$. Choose $N \ge 0$ with $N \ge -\alpha_t$ and $(\lambda^{(t)})^N y_t \in R$ for every t, and put $g = \lambda_e^N h$. Since $\lambda_e = Q_t \lambda^{(t)}$,

$$g = Q_t^{N+\alpha_t} (\lambda^{(t)})^N y_t \in Q_t^{N+\alpha_t} R \qquad (t < e),$$

with $N+\alpha_t \geq 0$. In particular $g \in R$, and by Step 1, $g \in \prod_t Q_t^{N+\alpha_t} R$; hence $h = \lambda_e^{-N} g \in I_\alpha$. For the last claim: $I_{\max(\alpha,\beta)} \subseteq I_\alpha \cap I_\beta$ because $Q_s^{\max(\alpha_s,\beta_s)-\alpha_s} \in R$; and if $h \in I_\alpha \cap I_\beta$, then for each t, $h \in Q_t^{\alpha_t} R^{(t)}$ and $h \in Q_t^{\beta_t} R^{(t)}$, so $h \in Q_t^{\max(\alpha_t,\beta_t)} R^{(t)}$, and $h \in I_{\max(\alpha,\beta)}$ by the first part.

Item 2. Write $h = \lambda_e^{-N} g$ with $g \in R$ and $N \geq 0$ so large that $N + \alpha_t \geq 0$ for all t. Then $cg = \lambda_e^N ch \in I_{\alpha+(N,\dots,N)}$, so $Q_t^{N+\alpha_t} \mid cg$ in R for each t (the other factors $Q_s^{N+\alpha_s}$ lie in R). By Step 1 (with $z = c$, which does not vanish at the roots of Q_t by hypothesis), $Q_t^{N+\alpha_t} \mid g$ for every t, hence $g \in \prod_t Q_t^{N+\alpha_t} R$ and $h \in I_\alpha$.

Item 3. *Transport of γ_u.* For $g \in R$, $\gamma_u(g) = g\big((1+X)^u - 1\big)$, and evaluating the series $(1+X)^u = \sum_k \binom{u}{k} X^k$ at $x_0 = \zeta - 1$ gives ζ^u (for $u \in \mathbb{Z}_{\geq 0}$ this is the binomial theorem; both sides are continuous in $u \in \mathbb{Z}_\ell$, the right side because ζ^u depends only on $u \bmod \ell^{t+1}$). Hence $\mathrm{ev}_{\zeta-1} \circ \gamma_u = \mathrm{ev}_{\zeta^u - 1}$ on R, and, both sides being ring homomorphisms and $\gamma_u(Q_s) \in Q_s R^\times$ (Lemma 23.8(2)), also on $R^{(t)}$; here ζ^u is again of exact order ℓ^{t+1}.

One value at all roots. By hypothesis $(\gamma_{u_0} - 1)(h) \in Q_t^N R^{(t)} \subseteq Q_t R^{(t)}$ and $(\gamma_\delta - 1)(h) = 0$ lie in $Q_t R^{(t)}$, which every ev_{x_0} kills. So $h(\zeta^u - 1) = h(\zeta - 1)$ for every ζ of exact order ℓ^{t+1} and every u in the subgroup $U \subseteq \mathbb{Z}_\ell^\times$ generated by u_0 and $\mu_{\ell-1}$. The image of U in $(\mathbb{Z}/\ell^{t+1})^\times$ is everything: reduction modulo ℓ maps $\mu_{\ell-1}$ bijectively onto $\mathbb{F}_\ell^\times$, so its image has order $\ell - 1$; by Lemma 23.22(1) (with $m = 1$), $v_\ell(u_0^k - 1) = 1 + v_\ell(k)$ for $k \neq 0$, so u_0 has order ℓ^t modulo ℓ^{t+1}; the two orders are coprime, so the image of U has order $(\ell-1)\ell^t = |(\mathbb{Z}/\ell^{t+1})^\times|$. Since every ζ' of exact order ℓ^{t+1} is ζ^u for some u prime to ℓ, h takes one value y_0 at all roots of Q_t.

The value lies in L. Write $h = g/(\lambda^{(t)})^N$ with $g \in R$. Let $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell / L)$. It is continuous on the complete field $L(\zeta)$ and fixes the coefficients of g (in $\mathcal{O}_L$) and of $\lambda^{(t)}$ (in $\mathbb{Z}_\ell$), so $\tau(h(\zeta - 1)) = h(\tau(\zeta) - 1) = y_0$, as $\tau(\zeta)$ has exact order ℓ^{t+1}. Hence $y_0 \in L$, and there is $\kappa \geq 0$ with $x = \ell^\kappa y_0 \in \mathcal{O}_L$.

Conclusion. $h' = \ell^\kappa h - x \in R^{(t)}$ vanishes at every root of Q_t. Choose M with $g' = (\lambda^{(t)})^M h' \in R$; g' vanishes at every root of Q_t, so $g' \in Q_t R$ by Lemma 23.8(3), and $h' = (\lambda^{(t)})^{-M} g' \in Q_t R^{(t)}$.

Higher order. Put $u' = u_0^{\ell^e}$; by Lemma 23.22(1), $v_\ell(u' - 1) = 1 + e$, so $u' \equiv 1 \bmod \ell^e$, and $\gamma_{u_0}^{\ell^e} = \gamma_{u'}$. Each γ_u maps $Q_t^a R^{(t)}$ onto itself ($\gamma_u(Q_s) \in Q_s R^\times$, Lemma 23.8(2)). We show $h' \in Q_t^k R^{(t)}$ for $k = 1, \dots, N$ by induction; let $1 \leq k < N$ and $h' \in Q_t^k R^{(t)}$. Then $(\gamma_{u_0} - 1)(h') = \ell^\kappa (\gamma_{u_0} - 1)(h) \in Q_t^{k+1} R^{(t)}$ (x is fixed by γ_{u_0}), so $(\gamma_{u'} - 1)(h') = \big(\sum_{j<\ell^e} \gamma_{u_0}^j\big)(\gamma_{u_0} - 1)(h') \in Q_t^{k+1} R^{(t)}$. Choose $M' \geq 0$ with $h' \in I_\eta$, $\eta_t = k$, $\eta_s = -M'$ ($s \neq t$) (item 1); by item 2 of Lemma C.19, $(\gamma_{u'} - u'^k)(h') \in I_{\eta + \mathbf{1}_t} \subseteq Q_t^{k+1} R^{(t)}$, where $\mathbf{1}_t$ is the t-th unit vector. Subtracting, $c\,h' \in Q_t^{k+1} R^{(t)}$ with $c = u'^k - 1 \in \mathbb{Z}_\ell$, $c \neq 0$ ($v_\ell(c) = 1 + e + v_\ell(k)$ by Lemma 23.22(1)). By item 1, $ch' \in I_{\eta'}$ with $\eta'_t = k+1$ and $\eta'_s = -M''$ for $s \neq t$, M'' large; by item 2, $h' \in I_{\eta'} \subseteq Q_t^{k+1} R^{(t)}$. $\square$

Lemma C.19. *Let ℓ be an odd prime, F and L as in Definition 23.2, $R = \mathcal{O}_L[[X]]$ with γ_u acting $\mathcal{O}_L$-linearly by $X \mapsto (1+X)^u - 1$, $e \geq 1$, and for $\alpha = (\alpha_0, \dots, \alpha_{e-1}) \in \mathbb{Z}^e$ let $I_\alpha = \prod_{s<e} Q_s^{\alpha_s} R \subseteq R[1/\lambda_e]$; $\delta_t \in \mathbb{Z}^e$ is the t-th unit vector.*

1. *If $\mathbf{N}$ is a Wach module over F with coefficients L, $u \in \mathbb{Z}_\ell^\times$, $P \in \mathbb{Z}_\ell[Y]$ and $w \in \mathbf{N}$ with $(\gamma_{u,\mathbf{N}} - 1)w \in X\mathbf{N}$, then $P(\gamma_{u,\mathbf{N}})w - P(1)w \in X\mathbf{N}$.*

2. *Let $u' \in \mathbb{Z}_\ell^\times$ with $u' \equiv 1 \bmod \ell^e$. Then $\gamma_{u'}(I_\alpha) = I_\alpha$ and $(\gamma_{u'} - u'^{\alpha_t})(I_\alpha) \subseteq I_{\alpha+\delta_t}$ for every $t < e$ and every $\alpha \in \mathbb{Z}^e$.*

3. *Let $u \in 1 + \ell\mathbb{Z}_\ell$, $J \subset \mathbb{Z}$ finite, $t < e$, $\alpha \in \mathbb{Z}^e$ and $f \in I_\alpha$. If $\alpha_t \in J \setminus \{0\}$ then $P_{J,e}(\gamma_u) f \in I_{\alpha+\delta_t}$; if $\alpha_t = 0$ then $P_{J,e}(\gamma_u)(\gamma_u f - f) \in I_{\alpha+\delta_t}$; and for every $N \geq 0$, $\prod_{i=\alpha_t}^{\alpha_t+N-1} (\gamma_u^{\ell^e} - u^{i\ell^e}) f \in I_{\alpha+N\delta_t}$.*

Proof. **Item 1.** $\gamma_{u,\mathbf{N}}$ is a $\mathbb{Z}_\ell$-linear endomorphism of $\mathbf{N}$, and $X\mathbf{N}$ is stable under it, since $\gamma_{u,\mathbf{N}}(Xy) = \gamma_u(X)\gamma_{u,\mathbf{N}}(y)$ and $\gamma_u(X) \in X\mathbf{A}_F$ (Lemma 23.8(2)). Apply Lemma 23.22(3) with $M = \mathbf{N}$, $A = \gamma_{u,\mathbf{N}}$, $M' = X\mathbf{N}$, $f = 1$, $Q = P$.

Item 2. By Lemma 23.8(2), $\gamma_{u'}(Q_s) = Q_s v_s$ with $v_s \in \mathbb{Z}_\ell[[X]]^\times$. Let $g = \prod_{s<e} Q_s^{\alpha_s}$ and $y \in R$. Then $\gamma_{u'}(gy) = g\, v\, \gamma_{u'}(y)$ with $v = \prod_s v_s^{\alpha_s} \in R^\times$; as $\gamma_{u'}$ is bijective on R, $\gamma_{u'}(I_\alpha) = I_\alpha$, and $(\gamma_{u'} - u'^{\alpha_t})(gy) = g\,\Psi$ with $\Psi = v\gamma_{u'}(y) - u'^{\alpha_t} y \in R$. Let $x_0 = \zeta - 1$ with ζ a primitive ℓ^{t+1}-th root of unity, a root of Q_t; evaluation at x_0 is a ring homomorphism $R \to \mathcal{O}_L[\zeta]$. Since $t+1 \le e$, $u' \equiv 1 \bmod \ell^{t+1}$ and $(1+x_0)^{u'} - 1 = \zeta^{u'} - 1 = x_0$; hence $\gamma_{u'}(y)(x_0) = y(x_0)$. For $s \neq t$, $Q_s(x_0) \neq 0$ (the roots of the Q_s are the $\zeta' - 1$ with ζ' of exact order ℓ^{s+1}), so $v_s(x_0) = Q_s(x_0)/Q_s(x_0) = 1$. For $s = t$, differentiate the identity $Q_t((1+X)^{u'} - 1) = Q_t(X)v_t(X)$ and evaluate at x_0: $Q_t'(x_0)\, u'(1+x_0)^{u'-1} = Q_t'(x_0)v_t(x_0)$, and $Q_t'(x_0) \neq 0$ (simple root), $(1+x_0)^{u'-1} = \zeta^{u'-1} = 1$; so $v_t(x_0) = u'$. Therefore $\Psi(x_0) = u'^{\alpha_t} y(x_0) - u'^{\alpha_t} y(x_0) = 0$ for every root x_0 of Q_t, and by Lemma 23.8(3), $\Psi \in Q_t R$. Hence $(\gamma_{u'} - u'^{\alpha_t})(gy) \in gQ_tR = I_{\alpha+\delta_t}$.

Item 3. Put $u' = u^{\ell^e}$; since $u \equiv 1 \bmod \ell$, $u' \equiv 1 \bmod \ell^{e+1}$, and $\gamma_u^{\ell^e} = \gamma_{u'}$, $u^{i\ell^e} = u'^{\,i}$. All operators below are polynomials in γ_u, hence commute, and each preserves every I_β (item 2 for γ_u requires only $\gamma_u(Q_s) \in Q_s R^\times$, which is Lemma 23.8(2)). If $\alpha_t = i_0 \in J \setminus \{0\}$, then

$$P_{J,e}(\gamma_u) f = \Big(\prod_{i \in J \setminus \{0, i_0\}} (\gamma_{u'} - u'^{\,i}) \Big) \Big(\sum_{k < \ell^e} \gamma_u^k \Big) (\gamma_{u'} - u'^{\,i_0}) f,$$

and $(\gamma_{u'} - u'^{\,i_0}) f \in I_{\alpha+\delta_t}$ by item 2. If $\alpha_t = 0$, then $(\sum_{k<\ell^e} \gamma_u^k)(\gamma_u - 1) = \gamma_{u'} - 1 = \gamma_{u'} - u'^{\,0}$, so $P_{J,e}(\gamma_u)(\gamma_u - 1) f = \prod_{i \in J \setminus \{0\}} (\gamma_{u'} - u'^{\,i})\,(\gamma_{u'} - 1) f \in I_{\alpha+\delta_t}$ by item 2. For the last claim apply item 2 successively with $\alpha_t, \alpha_t + 1, \ldots, \alpha_t + N - 1$, the factors being applied in increasing order of i. □

Lemma C.20. *Let ℓ be an odd prime, L a finite extension of $\mathbb{Q}_\ell$, $R = \mathcal{O}_L[[X]]$ with γ_u ($u \in \mathbb{Z}_\ell^\times$) acting $\mathcal{O}_L$-linearly by $X \mapsto (1+X)^u - 1$ and extended to the fraction field $\mathcal{K}$ of R, and fix $u_0 \in \mathbb{Z}_\ell^\times$ with $v_\ell(u_0 - 1) = 1$; write $\gamma = \gamma_{u_0}$. Let $e \ge 1$ and $S \ge 0$ be integers, and for $s < e$ let $\epsilon_s \in \{1, -1\}$ and $b_s \ge 1$ be integers. Put*

$$\mu = \prod_{\substack{s<e \\ \epsilon_s = 1}} Q_s^{b_s+S}, \quad \delta = \prod_{\substack{s<e \\ \epsilon_s = -1}} Q_s^{b_s}, \quad \nu^+ = \prod_{\substack{s<e \\ \epsilon_s = -1}} Q_s^{b_s+S}, \quad \nu = \prod_{\substack{s<e \\ \epsilon_s = 1}} Q_s^{b_s},$$

let $J \subset \mathbb{Z}$ be a finite set containing $\{i : -b_s - S \le i < -b_s\}$ for every $s < e$ with $\epsilon_s = 1$ and $\{i : b_s \le i < b_s + S\}$ for every $s < e$ with $\epsilon_s = -1$, and let $P = P_{J,e}$ (formed with $p = \ell$, $u = u_0$). Let $f \in \mathcal{K}$ with

$$\mu f \in R \qquad \text{and} \qquad \mu\,(\gamma - 1)(f) \in X^{\ell-1}\delta R.$$

Then:

1. *$\nu\, P(\gamma)(f) \in R$;*
2. *$\nu\,(\gamma - 1)P(\gamma)(f) \in X^{\ell-1}\nu^+ R$;*
3. *if moreover $\gamma_\omega(f) = f$ for every $\omega \in \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$, then for every $s < e$ with $\epsilon_s = -1$ there are an integer $\kappa \ge 0$ and $x \in \mathcal{O}_L$ with*

$$\nu\,(\ell^\kappa P(\gamma)(f) - x) \in Q_s^{b_s+S} R.$$

Proof. Define $\alpha, \beta, \alpha', \beta' \in \mathbb{Z}^e$ by

$$(\alpha_s, \beta_s, \alpha'_s, \beta'_s) = \begin{cases} (-b_s - S,\ -b_s,\ -b_s - S,\ -b_s) & \text{if } \epsilon_s = 1, \\ (0,\ 0,\ b_s,\ b_s + S) & \text{if } \epsilon_s = -1. \end{cases}$$

Then $I_\alpha = \mu^{-1}R$, $I_{\alpha'} = \mu^{-1}\delta R$, $I_\beta = \nu^{-1}R$, $I_{\beta'} = \nu^{-1}\nu^+R$. The hypotheses say $f \in I_\alpha$ and $(\gamma-1)(f) = X^{\ell-1}g$ with $g \in I_{\alpha'}$; items 1 and 2 say $P(\gamma)(f) \in I_\beta$ and $(\gamma-1)P(\gamma)(f) \in X^{\ell-1}I_{\beta'}$.

Step 0: stability. For $u \in \mathbb{Z}_\ell^\times$, $\gamma_u(Q_s) \in Q_s R^\times$ and $\gamma_u(X) \in XR^\times$ (Lemma 23.8(2)), and γ_u is bijective on R; hence $\gamma_u(I_\eta) = I_\eta$ and $\gamma_u(X^{\ell-1}I_\eta) = X^{\ell-1}I_\eta$ for every η. So every $Z(\gamma)$, $Z \in \mathbb{Z}_\ell[Y]$, maps I_η and $X^{\ell-1}I_\eta$ into themselves.

Step 1: one factor raises one level. Put $u' = u_0^{\ell^e}$. By Lemma 23.22(1), $v_\ell(u'-1) = 1+e$, so $u' \equiv 1 \bmod \ell^{e+1}$; and $\gamma^{\ell^e} = \gamma_{u'}$, $u_0^{i\ell^e} = u'^{\,i}$. Let $\eta \in \mathbb{Z}^e$ and $t < e$. Then

$$(\gamma_{u'} - u'^{\,\eta_t})(I_\eta) \subseteq I_{\eta+\mathbf{1}_t} \qquad \text{and} \qquad (\gamma_{u'} - u'^{\,\eta_t})(X^{\ell-1}I_\eta) \subseteq X^{\ell-1}I_{\eta+\mathbf{1}_t}.$$

The first inclusion is item 2 of Lemma C.19. For the second, write $\gamma_{u'}(X^{\ell-1}) = X^{\ell-1}z$ with $z \in R^\times$ (Step 0). At a root $x_0 = \zeta - 1$ of Q_t (ζ of exact order ℓ^{t+1}, $t+1 \le e$) we have $\gamma_{u'}(X)(x_0) = \zeta^{u'} - 1 = x_0 \neq 0$, so $z(x_0) = 1$; hence $z - 1 \in Q_t R$ by Lemma 23.8(3). For $h \in I_\eta$,

$$(\gamma_{u'} - u'^{\,\eta_t})(X^{\ell-1}h) = X^{\ell-1}\Big((\gamma_{u'} - u'^{\,\eta_t})(h) + (z-1)\,\gamma_{u'}(h)\Big),$$

where the first summand lies in $I_{\eta+\mathbf{1}_t}$ by the first inclusion and the second in $Q_t I_\eta = I_{\eta+\mathbf{1}_t}$ by Step 0.

Step 2: a window raises one level by its length. Let M_η stand for I_η or for $X^{\ell-1}I_\eta$ throughout. If $t < e$, $N \ge 0$ and $\{\eta_t, \eta_t+1, \dots, \eta_t+N-1\} \subseteq J \setminus \{0\}$, then $P(\gamma)(M_\eta) \subseteq M_{\eta+N\mathbf{1}_t}$. Indeed, by the definition of $P_{J,e}$,

$$P(Y) = \prod_{i=\eta_t}^{\eta_t+N-1} (Y^{\ell^e} - u_0^{i\ell^e}) \cdot P'(Y)$$

with $P' \in \mathbb{Z}_\ell[Y]$ the product of the remaining factors; $P'(\gamma)$ maps M_η into itself (Step 0), and the factors $\gamma_{u'} - u'^{\,i}$, applied in increasing order of i, map $M_{\eta+(i-\eta_t)\mathbf{1}_t}$ into $M_{\eta+(i-\eta_t+1)\mathbf{1}_t}$ by Step 1 (the t-th coordinate of $\eta + (i-\eta_t)\mathbf{1}_t$ is i). All these operators are polynomials in γ and commute.

Item 1. Let $t < e$. If $\epsilon_t = 1$, then $\{\alpha_t, \dots, \alpha_t + S - 1\} = \{-b_t - S, \dots, -b_t - 1\} \subseteq J \setminus \{0\}$, so by Step 2 (with $M = I$, $N = S$) $P(\gamma)(f) \in I_{\alpha+S\mathbf{1}_t}$, whose t-th coordinate is β_t; hence $P(\gamma)(f) \in Q_t^{\beta_t} R^{(t)}$ (item 1 of Lemma C.18, easy direction). If $\epsilon_t = -1$, then $\beta_t = \alpha_t = 0$ and $P(\gamma)(f) \in I_\alpha \subseteq Q_t^0 R^{(t)}$ by Step 0. By item 1 of Lemma C.18, $P(\gamma)(f) \in I_\beta$, that is $\nu P(\gamma)(f) \in R$.

Item 2. $(\gamma-1)P(\gamma)(f) = P(\gamma)((\gamma-1)(f)) = P(\gamma)(X^{\ell-1}g)$ with $g \in I_{\alpha'}$. Let $t < e$. The window $\{\alpha'_t, \dots, \alpha'_t + S - 1\}$ is $\{-b_t - S, \dots, -b_t - 1\}$ if $\epsilon_t = 1$ and $\{b_t, \dots, b_t + S - 1\}$ if $\epsilon_t = -1$; it lies in $J \setminus \{0\}$ (in the second case because $b_t \ge 1$). By Step 2 (with $M = X^{\ell-1}I$, $N = S$), $(\gamma-1)P(\gamma)(f) = X^{\ell-1}h$ with $h \in I_{\alpha'+S\mathbf{1}_t}$, where $h = X^{1-\ell}(\gamma-1)P(\gamma)(f) \in \mathcal{K}$ does not depend on t. The t-th coordinate of $\alpha' + S\mathbf{1}_t$ is β'_t, so $h \in Q_t^{\beta'_t} R^{(t)}$ for every t, and $h \in I_{\beta'}$ by item 1 of Lemma C.18. Thus $\nu(\gamma-1)P(\gamma)(f) = X^{\ell-1}\nu h \in X^{\ell-1}\nu^+ R$.

Item 3. Fix $s < e$ with $\epsilon_s = -1$, put $b = b_s$ and $F = P(\gamma)(f)$. For $\omega \in \mu_{\ell-1}$, $\gamma_\omega \gamma = \gamma_{\omega u_0} = \gamma\gamma_\omega$, so $\gamma_\omega(F) = P(\gamma)(\gamma_\omega f) = F$. By item 1, $F \in I_\beta \subseteq Q_s^0 R^{(s)} = R^{(s)}$ ($\beta_s = 0$). By item 2, $(\gamma-1)(F) \in X^{\ell-1}I_{\beta'} \subseteq I_{\beta'} \subseteq Q_s^{b+S} R^{(s)}$. Item 3 of Lemma C.18 (with $t = s$, $h = F$, $N = b + S$) gives $\kappa \ge 0$ and $x \in \mathcal{O}_L$ with $H = \ell^\kappa F - x \in Q_s^{b+S} R^{(s)}$. Since $\beta \le 0$, $x \in R \subseteq I_\beta$, so $H \in I_\beta$, hence $H \in Q_t^{\beta_t} R^{(t)}$ for $t \neq s$; with $H \in Q_s^{b+S} R^{(s)}$, item 1 of Lemma C.18 gives $H \in I_{\beta+(b+S)\mathbf{1}_s} = \nu^{-1} Q_s^{b+S} R$, that is $\nu(\ell^\kappa P(\gamma)(f) - x) \in Q_s^{b+S} R$. □

Proposition C.21. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25; put $\mathbf{N} = \mathbf{N}(T_2)$, $r = \dim_L V_2$. Let $c \in C$, $e = \mathrm{len}_c$, and $(\mathbf{n}, R)$ a low frame at c. Then there are $g_0 \le m\,e\,(e+1)$ Δ-invariant vectors $J_1, \dots, J_{g_0} \in \mathcal{K}_c^r$, each with $\Pi_c J_{i,k} \in e_c \mathbf{A}_{F,L}$ for all k, $J_{i,k} = 0$ for $k \notin R$, and Hodge-integral modulo γ, such that for every Δ-invariant $f \in \mathcal{K}_c^r$ with $\Pi_c f_k \in e_c \mathbf{A}_{F,L}$ for all k, $f_k = 0$ for $k \notin R$, and f Hodge-integral modulo γ, there is $o \in \mathcal{O}_L^{g_0}$ with $f - \sum_i o_i J_i \in X^{\ell-1}\mathcal{H}_{\mathbf{n}}$.*

Proof. Put $Z = X^{\ell-1}\mathcal{H}_{\mathbf{n}}$, let $\mathcal{M}$ be the set of Δ-invariant $f \in \mathcal{K}_c^r$ with $\Pi f_k \in \mathcal{R}$ for all k and $f_k = 0$ for $k \notin R$, and $\mathcal{M}_\gamma \subseteq \mathcal{M}$ the subset of those f that are Hodge-integral modulo γ. Both are $\mathcal{O}_L$-modules ($\mathcal{H}_{\mathbf{n}}$ is an $\mathcal{R}$-module and γ is $\mathcal{O}_L$-linear). We must find $g_0 \le m\,e\,(e+1)$ elements $J_1, \dots, J_{g_0}$ of $\mathcal{M}_\gamma$ such that every $f \in \mathcal{M}_\gamma$ lies in $\sum_i \mathcal{O}_L J_i + Z$.

Step 1: three properties of Z. (i) $X^{\ell-1}\Pi\mathcal{R}^r \subseteq Z$: for $d \in \Pi\mathcal{R}^r$, $\Pi d_k \in \mathcal{R}$, and $(d_k, \mathbf{e}_k, (H_s)_s)_k$ is a section decomposition with $\delta_e(\mathbf{e}_k) \in \varphi^{s+4}(\mathrm{Fil}^{-H_s} D_{\mathrm{cris}}(V_2))$, because every Hodge jump of V_2 at τ_{c-s} has absolute value at most H_s (Lemma C.16 (1)), so that by the remark in Definition 23.3 the filtration step $-H_s$ of the relevant component is the whole component. (ii) $\gamma_u(Z) = Z$ for $u \in \mathbb{Z}_\ell^\times$: if $\Pi h_k \in \mathcal{R}$ then $\Pi\gamma_u(h_k) = \gamma_u(\Pi h_k)\,\Pi/\gamma_u(\Pi) \in \mathcal{R}$, a section decomposition $(g_i, \mathbf{v}_i, \alpha_i)$ of h gives the decomposition $(\gamma_u(g_i), \mathbf{v}_i, \alpha_i)$ of $\gamma_u(h)$ ($\gamma_u(I_\eta) = I_\eta$), and $\gamma_u(X^{\ell-1}) \in X^{\ell-1}\mathcal{R}^\times$ (Lemma 23.8 (2)); the same holds for $\gamma_u^{-1} = \gamma_{u^{-1}}$. (iii) If $f \in \mathcal{M}$, $c \in \mathcal{O}_L \setminus \{0\}$ and $cf \in Z$, then $f \in Z$: write $cf = X^{\ell-1}h$ with $h \in \mathcal{H}_{\mathbf{n}}$ and put $h' = h/c$, so $f = X^{\ell-1}h'$. For each k, $y = \Pi h'_k$ satisfies $X^{\ell-1}y = \Pi f_k \in \mathcal{R}$ and $r_1 = cy = \Pi h_k \in \mathcal{R}$; then $X^{\ell-1}r_1 \in c\mathcal{R}$, so every coefficient of r_1 lies in $c\mathcal{O}_L$ and $y \in \mathcal{R}$. If $\ell^\kappa h$ is Hodge-graded and $\ell^j/c \in \mathcal{O}_L$, then $\ell^{\kappa+j}h' = (\ell^j/c)\ell^\kappa h$ is Hodge-graded (multiply the $\mathbf{v}_i$ by ℓ^j/c). So $h' \in \mathcal{H}_{\mathbf{n}}$ and $f \in Z$.

Step 2: $A = \mathcal{M}/(\mathcal{M} \cap Z)$ is a finitely generated torsion-free $\mathcal{O}_L$-module. Let $P_0 = X^{\ell-1}\Pi^2$, a monic polynomial congruent to $X^{\deg P_0}$ modulo ℓ (Lemma 23.8 (3)); by the Weierstrass division theorem $\mathcal{R}/P_0\mathcal{R} \cong \mathcal{O}_L[X]/(P_0)$ is free of rank $\deg P_0$ over $\mathcal{O}_L$. The map $\mathcal{M} \to \bigoplus_{k\in R} \mathcal{R}/P_0\mathcal{R}$, $f \mapsto (\Pi f_k \bmod P_0)_k$, is $\mathcal{O}_L$-linear, and its kernel consists of f with $f_k \in X^{\ell-1}\Pi\mathcal{R}$ for all k, hence lies in Z by (i). So A is a quotient of a submodule of a finitely generated module over the noetherian ring $\mathcal{O}_L$, hence finitely generated; it is torsion-free by (iii).

Step 3: the Chinese remainder vectors. For $s < e$ the polynomials $Q_s^{H_s}$ and $\Pi'_s = \prod_{t<e,\, t\neq s} Q_t^{H_t}$ have no common root (Lemma 23.8 (3)), so $A_s Q_s^{H_s} + B_s\Pi'_s = 1$ with $A_s, B_s \in L[X]$; fix $\kappa_c \ge 0$ with $\ell^{\kappa_c}A_s, \ell^{\kappa_c}B_s \in \mathcal{O}_L[X]$ for all s. Then $u_s^0 = \ell^{\kappa_c}B_s\Pi'_s \in \mathcal{R}$ satisfies $u_s^0 - \ell^{\kappa_c} \in Q_s^{H_s}\mathcal{R}$ and $u_s^0 \in Q_t^{H_t}\mathcal{R}$ for $t \neq s$. Put $u_s = (\ell-1)^{-1}\sum_{\omega\in\mu_{\ell-1}} \gamma_\omega(u_s^0)$; it is Δ-invariant and has the same two properties, since each γ_ω maps $Q_t^a\mathcal{R}$ onto itself and fixes ℓ^{κ_c}. Put $\tilde\Pi = (\ell-1)^{-1}\sum_\omega \gamma_\omega(\Pi)$. Each $\gamma_\omega(\Pi) = \Pi v_\omega$ with $v_\omega \in \mathcal{R}^\times$, and $v_\omega(0) = 1$ because γ_ω preserves constant terms and $\Pi(0) = \ell^{\sum_s H_s} \neq 0$; so $\tilde\Pi = \Pi\tilde w$ with $\tilde w(0) = 1$, $\tilde w \in \mathcal{R}^\times$, and $\tilde\Pi$ is Δ-invariant. The $N_0 = |R|(e+1) \le m\,e\,(e+1)$ vectors

$$C_{k,s} = u_s\,\mathbf{e}_k \ (k \in R,\ s < e), \qquad C_{k,e} = \tilde\Pi\,\mathbf{e}_k \ (k \in R)$$

lie in $\mathcal{M}$.

Step 4: a multiple of each $f \in \mathcal{M}_\gamma$. Let $M_0 = \max_s H_s$, $J = \{i \in \mathbb{Z} :\ -M_0 \le i < M_0\}$ and $P = P_{J,e}$ (formed with $p = \ell$, $u = u_0$); $P(1) \neq 0$ by Lemma 23.22 (2). We claim: for every $f \in \mathcal{M}_\gamma$ there is $K \ge 0$ with $\ell^K P(1) f \in \sum_{k,j} \mathcal{O}_L C_{k,j} + Z$.

(a) By Lemma 23.22 (3) there is $Q_1 \in \mathbb{Z}_\ell[Y]$ with $P(\gamma) - P(1) = Q_1(\gamma)(\gamma-1)$; as $(\gamma-1)f \in Z$ and Z is γ-stable (ii), $P(\gamma)f - P(1)f \in Z$.

(b) Fix $k \in R$ and $h = f_k$; then $h \in I_{-H}$ ($H = (H_s)_s$) and h is Δ-invariant. Put $u' = u_0^{\ell^e}$, so $\gamma^{\ell^e} = \gamma_{u'}$ and $P(\gamma) = \prod_{i\in J\setminus\{0\}}(\gamma_{u'} - u'^i)\cdot\sum_{j<\ell^e}\gamma^j$. All polynomials in γ commute and map each I_η into itself (Lemma 23.8 (2)). Let $t < e$. Since $\{-H_t, \dots, -1\} \subseteq J \setminus \{0\}$, the last clause

of item 3 of Lemma C.19 (with $\alpha = -H$, $N = H_t$), applied after the remaining factors of P, gives $F = P(\gamma)h \in I_{-H+H_t\mathbf{1}_t} \subseteq Q_t^0 R^{(t)}$ ($\mathbf{1}_t$ the t-th unit vector); by item 1 of Lemma C.18, $F \in I_0 = \mathcal{R}$. Next, $(\sum_{j<\ell^e} \gamma^j)(\gamma - 1) = \gamma_{u'} - u'^0$, so

$$(\gamma - 1)F = \prod_{i \in J} (\gamma_{u'} - u'^i)\, h,$$

and as $\{-H_t, \dots, H_t - 1\} \subseteq J$ the same clause (with $N = 2H_t$) gives $(\gamma - 1)F \in I_{-H+2H_t\mathbf{1}_t} \subseteq Q_t^{H_t} R^{(t)}$. F is Δ-invariant (γ_ω commutes with γ). If $H_t \geq 1$, item 3 of Lemma C.18 (with $N = H_t$) gives $\kappa_t \geq 0$ and $x_t \in \mathcal{O}_L$ with $\ell^{\kappa_t} F - x_t \in Q_t^{H_t} R^{(t)}$; if $H_t = 0$ put $\kappa_t = 0$, $x_t = 0$. Let $\kappa = \max_t \kappa_t$ and

$$y = \ell^{\kappa_c + \kappa} F - \sum_{t<e} \ell^{\kappa - \kappa_t} x_t\, u_t \in \mathcal{R}.$$

For each t, by the properties of the $u_{t'}$, $y \equiv \ell^{\kappa_c + \kappa - \kappa_t}(\ell^{\kappa_t} F - x_t) \equiv 0$ modulo $Q_t^{H_t} R^{(t)}$; by item 1 of Lemma C.18, $y \in I_H = \Pi\mathcal{R}$. So $y = \tilde{\Pi}\rho$ with $\rho \in \mathcal{R}$, and ρ is Δ-invariant (y, $\tilde{\Pi}$ are, $\mathcal{R}$ is a domain). The element $\rho - \rho(0) \in X\mathcal{R}$ is Δ-invariant; if non-zero with lowest term $a_j X^j$, then γ_ω multiplies that term by ω^j, so $(\ell - 1) \mid j$; hence $\rho - \rho(0) \in X^{\ell-1}\mathcal{R}$, and $y - \rho(0)\tilde{\Pi} \in X^{\ell-1}\Pi\mathcal{R}$. Thus $\ell^{K_k} P(\gamma) f_k \in \sum_{s<e} \mathcal{O}_L u_s + \mathcal{O}_L \tilde{\Pi} + X^{\ell-1}\Pi\mathcal{R}$ with $K_k = \kappa_c + \kappa$.

(c) For $k \notin R$, $f_k = 0$ and $P(\gamma) f_k = 0$. With $K = \max_{k \in R} K_k$, (b) and (i) give $\ell^K P(\gamma) f \in \sum_{k,j} \mathcal{O}_L C_{k,j} + Z$, and (a) gives the claim.

Step 5: exact generation over the principal ideal domain $\mathcal{O}_L$. Let $B \subseteq A$ be the image of $\mathcal{M}_\gamma$ and $T \subseteq A$ the image of $\sum_{k,j} \mathcal{O}_L C_{k,j}$; T is generated by N_0 elements. By Step 2, B is finitely generated and torsion-free, hence free of finite rank g_0. By Step 4, for every $b \in B$ there is $c_b = \ell^K P(1) \neq 0$ with $c_b b \in T$ (the difference of $\ell^K P(1) f$ and an element of $\sum \mathcal{O}_L C_{k,j}$ lies in $\mathcal{M} \cap Z$). Since A is torsion-free, $A \to A \otimes_{\mathcal{O}_L} L$ is injective, and $B \otimes L$, $T \otimes L$ are subspaces of $A \otimes L$ (L is flat) with $B \otimes L \subseteq T \otimes L$, as $b \otimes 1 = c_b^{-1}(c_b b \otimes 1)$. Hence $g_0 = \dim_L(B \otimes L) \leq \dim_L(T \otimes L) \leq N_0 \leq m\, e\, (e + 1)$, using $|R| \leq m\, e$. Let $b_1, \dots, b_{g_0}$ be an $\mathcal{O}_L$-basis of B and $J_i \in \mathcal{M}_\gamma$ lifts. Each J_i is Δ-invariant, has $\Pi J_{i,k} \in \mathcal{R}$, $J_{i,k} = 0$ for $k \notin R$, and is Hodge-integral modulo γ. For f as in the statement, $f \in \mathcal{M}_\gamma$, its image is $\sum_i o_i b_i$ with $o \in \mathcal{O}_L^{g_0}$, and $f - \sum_i o_i J_i \in \mathcal{M} \cap Z \subseteq X^{\ell-1}\mathcal{H}_{\mathbf{n}}$. □

Proposition C.22. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word satisfy the hypotheses of Proposition C.25; let $\nu \in \{1, 2\}$ and put $\mathbf{N} = \mathbf{N}(T_\nu)$, $r = \dim_L V_\nu$ (Hodge-integral vectors, as defined in Proposition C.11, formed for frames of $\mathbf{N}$ with $D_{\mathrm{cris}}(V_\nu)$). Let $c \in C$, $e = \mathrm{len}_c$, and $\mathbf{n}$ a Δ-invariant frame at c' of $\mathbf{N}$. Then for every Δ-invariant $f \in \mathcal{K}_c^r$ with $\Pi_c f_k \in e_c \mathbf{A}_{F,L}$ for all k that is Hodge-integral modulo γ (for $\mathbf{n}$), there is a Δ-invariant $y \in e_c \mathbf{N}[1/\lambda_\infty]$ with $(\gamma_u - 1)(y) \in X\mathbf{N}$ for all $u \in \mathbb{Z}_\ell^\times$ and $\Pi_c y \in P_e(\mathbf{N}_{c'})$ such that $\ell^{\sigma_B} f$ is an arc coordinate vector of y for $\mathbf{n}$, e, H_c, m_X.*

Proof. Write $U = P_e(\mathbf{N}_{c'})$, $\Pi = \Pi_c$, and *coordinate vector* for arc coordinate vector for $\mathbf{n}$, e, H_c, m_X. We use the calculus of Lemma C.16 for the frame $\mathbf{n}$ (for $\nu = 1$ by its item (e)), and Proposition C.23 for the same ν: (a) coordinate vectors are $e_c \mathbf{A}_{F,L}$-linear in the vector; (b) if f is a coordinate vector of y then $\gamma_u(f)$ is one of $\gamma_u(y)$, and $\gamma_u(U) = U$, $\gamma_u(\Pi) \in \Pi\, \mathbb{Z}_\ell[[X]]^\times$; (c) if 0 is a coordinate vector of y and $\Pi y \in U$, then $y \in X\mathbf{N}_c$.

Let f be as in the statement. By (R2) of Proposition C.23 there is $y_0 \in e_c \mathbf{N}[1/\lambda_\infty]$ with $\Pi y_0 \in U$ such that $\ell^{\sigma_B} f$ is a coordinate vector of y_0. Put $y = (\ell - 1)^{-1} \sum_{\delta \in \mu_{\ell-1}} \gamma_\delta(y_0)$. Then y is Δ-invariant, $\Pi y \in U$ by (b), and, as f is Δ-invariant, (a) and (b) show that $\ell^{\sigma_B} f = (\ell - 1)^{-1} \sum_\delta \gamma_\delta(\ell^{\sigma_B} f)$ is a coordinate vector of y.

The γ-condition for u_0. As f is Hodge-integral modulo γ, $(\gamma_{u_0} - 1) f = X^{\ell-1} h$ with $h \in \mathcal{H}_{\mathbf{n}}$, i.e. $\Pi h_k \in e_c \mathbf{A}_{F,L}$ for all k and $\ell^\kappa h$ is Hodge-graded for some κ. By (R1) of Proposition C.23

there is $x_h \in \mathbf{N}_c$ such that $\ell^{\sigma_B} h$ is a coordinate vector of x_h. By (a) and (b), $(\gamma_{u_0}-1)(\ell^{\sigma_B} f) = \ell^{\sigma_B} X^{\ell-1} h$ is a coordinate vector of $(\gamma_{u_0}-1)y$ and of $X^{\ell-1}x_h$, so 0 is a coordinate vector of $D = (\gamma_{u_0}-1)y - X^{\ell-1}x_h$. Moreover $\Pi D \in U$: $\Pi(\gamma_{u_0}-1)y \in U$ by (b), and $\Pi x_h \in U$ because $\Pi \mathbf{N}_c \subseteq U$ (Lemma C.16 (1)). By (c), $D \in X\mathbf{N}_c$, hence $(\gamma_{u_0}-1)y \in X\mathbf{N}$. Also $(\gamma_\delta - 1)y = 0$ for $\delta \in \mu_{\ell-1}$.

All u. Let S be the set of $u \in \mathbb{Z}_\ell^\times$ with $(\gamma_u - 1)y \in X\mathbf{N}$. Then S is a closed subgroup of $\mathbb{Z}_\ell^\times$: it is a subgroup because γ_u preserves $X\mathbf{N}$; it is closed because, writing $y = x/f'$ with $x \in \mathbf{N}$ and f' a product of the Q_s, $f'(\gamma_w - 1)y \to 0$ in $\mathbf{N}$ as $w \to 1$ (Lemma 23.29 (Gc3)) and $f'X\mathbf{N}$ is closed in $\mathbf{N}$. The set S contains $\mu_{\ell-1}$ and u_0, and u_0 generates a dense subgroup of $1 + \ell\mathbb{Z}_\ell$ ($v_\ell(u_0 - 1) = 1$, ℓ odd); so $S = \mathbb{Z}_\ell^\times = \mu_{\ell-1} \times (1 + \ell\mathbb{Z}_\ell)$. Thus y has all asserted properties. □

Proposition C.23. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25; let $\nu \in \{1, 2\}$ and put $\mathbf{N} = \mathbf{N}(T_\nu)$ and $r = \dim_L V_\nu$ (Hodge-graded vectors formed with $D_{\mathrm{cris}}(V_\nu)$). Let $c \in C$, $e = \mathrm{len}_c$, and let $\mathbf{n}$ be a frame at c' of $\mathbf{N}$ with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$. Let $f \in \mathcal{K}_c^r$ with $\Pi_c f_k \in e_c \mathbf{A}_{F,L}$ for every k.*

(R1) Lattice vectors *If $\ell^\kappa f$ is Hodge-graded (for $\mathbf{n}$ and e) for some $\kappa \geq 0$, there is $x \in \mathbf{N}_c$ such that $\ell^{\sigma_B} f$ is an arc coordinate vector of x for $\mathbf{n}$, e, H_c and m_X.*

(R2) Pull-back vectors *There is $y \in e_c\mathbf{N}[1/\lambda_\infty]$ with $\Pi_c y \in P_e(\mathbf{N}_{c'})$ such that $\ell^{\sigma_B} f$ is an arc coordinate vector of y for $\mathbf{n}$, e, H_c and m_X.*

(R3) Sub-arcs *Let $1 \leq e_1 \leq e$, $a = c' + e_1$ and $\Pi' = \Lambda_{a,e_1}(H_c)$. Then (R1) and (R2) hold with (c, e, Π_c) replaced by (a, e_1, Π'): for $f \in \mathcal{K}_a^r$ with $\Pi' f_k \in e_a \mathbf{A}_{F,L}$ for every k there is $y \in e_a\mathbf{N}[1/\lambda_\infty]$ with $\Pi' y \in P_{e_1}(\mathbf{N}_{c'})$ such that $\ell^{\sigma_B} f$ is an arc coordinate vector of y for $\mathbf{n}$, e_1, H_c and m_X; and if $\ell^\kappa f$ is Hodge-graded (for $\mathbf{n}$ and e_1) for some $\kappa \geq 0$, then $y \in \mathbf{N}_a$.*

Proof. Write $\Pi = \Pi_c$, $R_c = e_c\mathbf{A}_{F,L}$, $U = P_e(\mathbf{N}_{c'})$, $m = m_X$, $E = 2(w + S)$ and $s(i) = s_\ell(i)$ as in Lemma 23.9; *coordinate vector* means arc coordinate vector for $\mathbf{n}$, e, H_c, m_X. We have $e \geq 1$ (Proposition C.4). Let $H_k = \sum_j h_{kj} n_j$, $h_{kj} = \sum_{i \geq 0} h_{kj,i} X^i \in e_{c'} E[[X]]$, be the horizontal sections through the frame.

Step 1: an admissible exponent. Extend $\mathbf{n}$ by frames at the other sites to an $\mathbf{A}_{F,L}$-basis of $\mathbf{N}$; the horizontal section through n_k lies in $e_{c'}\mathbf{N}_E$ (uniqueness in (Hf2) of Lemma 23.9, as $e_{c'}$ commutes with every γ_u). By (Hf3) of Lemma 23.9, $\ell^{s(i)} h_{kj,i} \in e_{c'}(\mathcal{O}_F \otimes \mathcal{O}_L)$ for all k, j, i. As s_ℓ is nondecreasing, $\kappa := s(m-1)$ is admissible and $\kappa \geq s(i)$ for $i < m$.

Step 2: the scalar divisibility. By definition $\sigma_B \geq (E + m/(\ell-1))\ell/(\ell-1)^2$, so Lemma C.15 (with $\sigma = \sigma_B$) gives, for $0 \leq i < m$, elements $a_i, b_i \in \mathbb{Z}_\ell[[X]]$ with

$$\ell^{\sigma_B} \varphi^e(X)^i = \ell^{s(i)} a_i + D\, b_i, \qquad D = (X\lambda_{e-1})^m Q_{e-1}^E.$$

Moreover $D \in X\Pi^2\, \mathbb{Z}_\ell[[X]]$: $\Pi^2 = \prod_{s<e} Q_s^{2H_c(c-s)}$; for $s = e-1$ we have $c - s = c' + 1$ and $2H_c(c'+1) = E$; for $s < e-1$ the factor Q_s^m of λ_{e-1}^m suffices, since $m = (\ell-1)(\lfloor 2h_B/(\ell-1)\rfloor + 1) > 2h_B \geq 2H_c(c-s)$; and X divides X^m.

Step 3: (R2). Let f be as in the statement. Put

$$z_k = \sum_j \sum_{i<m} \varphi^e\big(\ell^{s(i)} h_{kj,i}\big)\, a_i\, \varphi^e(n_j) \in U, \qquad u_k = \sum_j \sum_{i<m} \ell^{\kappa - s(i)} \varphi^e\big(\ell^{s(i)} h_{kj,i}\big)\, b_i\, \varphi^e(n_j) \in U,$$

using $\varphi^e(e_{c'}(\mathcal{O}_F\otimes\mathcal{O}_L))\subseteq \mathbf{A}_{F,L}$. Since $\Pi f_k\in R_c$ and Π is a product of the Q_s, $f_k\in R_c[1/Q_s : s\ge 0]$, so $y:=\sum_k f_k z_k\in e_c\mathbf{N}[1/\lambda_\infty]$, and $\Pi y=\sum_k(\Pi f_k)z_k\in U$. As $\varphi_{\mathbf{N}}$ is φ-semilinear,

$$\ell^{\sigma_B}\varphi^e\big(T_\kappa(H_k)\big)=\sum_j\sum_{i<m}\ell^{\kappa-s(i)}\varphi^e\big(\ell^{s(i)}h_{kj,i}\big)\,\ell^{\sigma_B}\varphi^e(X)^i\,\varphi^e(n_j)=\ell^\kappa z_k+D\,u_k.$$

Multiplying by Πf_k and summing over k,

$$\ell^\kappa\Pi y-\sum_k\Pi(\ell^{\sigma_B}f_k)\varphi^e(T_\kappa(H_k))=-D\sum_k(\Pi f_k)u_k\in X\Pi^2U,$$

by Step 2. Together with $\Pi(\ell^{\sigma_B}f_k)\in R_c$, this says that $\ell^{\sigma_B}f$ is a coordinate vector of y (with κ and $\kappa'=0$).

Step 4: (R1). Let $\ell^{\kappa_0}f$ be Hodge-graded and let y be as in Step 3. Multiplying the functions g_i of a section decomposition of $\ell^{\kappa_0}f$ by ℓ^{σ_B} gives a section decomposition of $\ell^{\kappa_0}(\ell^{\sigma_B}f)$ with the same $\mathbf{v}_i$ and α_i, so $\ell^{\kappa_0}(\ell^{\sigma_B}f)$ is Hodge-graded. By the membership criterion Proposition C.24 (with $t=0$ and the coordinate vector $\ell^{\sigma_B}f$ of y), $y\in\mathbf{N}_c$. So $x=y$ has the asserted property.

Step 5: (R3). Let e_1, a, Π' and f be as in (R3). Steps 1–4 go through with (c,e,Π) replaced by (a,e_1,Π'). Step 1 does not involve e. In Step 2 take $D=(X\lambda_{e_1-1})^mQ^E_{e_1-1}$ (Lemma C.15 holds for every $e_1\ge 1$); then $D\in X\Pi'^2\mathbb{Z}_\ell[[X]]$, since $\Pi'^2=\prod_{s<e_1}Q_s^{2H_c(a-s)}$, for $s=e_1-1$ the site $a-s$ is $c'+1$ with $2H_c(c'+1)=E$, and for $s<e_1-1$ the factor Q_s^m suffices as $m>2h_B\ge 2H_c(a-s)$. Step 3 is unchanged with φ^{e_1} and $U'=P_{e_1}(\mathbf{N}_{c'})$ (it gives $y=\sum_k f_kz_k\in e_a\mathbf{N}[1/\lambda_\infty]$ with $\Pi'y\in U'$), and Step 4 uses the sub-arc clause of Proposition C.24, which gives $y\in\mathbf{N}_a$. □

Proposition C.24. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1\to V_2$ ($T_1\to T_2$) and the weight word $(b,w,S,\eta,m,(s_a),(\epsilon_a))$ satisfy the hypotheses of Proposition C.25; let $\nu\in\{1,2\}$ and put $\mathbf{N}=\mathbf{N}(T_\nu)$. Let $c\in C$, $e=\mathrm{len}_c$, and let $\mathbf{n}$ be a frame at c' of $\mathbf{N}$, with crystalline frames formed in $D_{\mathrm{cris}}(V_\nu)$, with $\gamma_u(n_k)=n_k$ for $u\in\mu_{\ell-1}\subset\mathbb{Z}_\ell^\times$. Let $y\in e_c\mathbf{N}[1/\lambda_\infty]$ with $\Pi_c y\in P_e(\mathbf{N}_{c'})$, let $t\ge 0$, and let f be an arc coordinate vector of $\ell^t y$ for $\mathbf{n}$, e, H_c and m_X. If $\ell^\kappa f$ is Hodge-graded (for $\mathbf{n}$ and e) for some $\kappa\ge 0$, then $y\in\mathbf{N}_c$.*

Sub-arcs. *The same holds for the initial sub-arcs of the arc: let $1\le e_1\le e$, $a=c'+e_1$ and $\Pi'=\Lambda_{a,e_1}(H_c)$. If $y\in e_a\mathbf{N}[1/\lambda_\infty]$ with $\Pi' y\in P_{e_1}(\mathbf{N}_{c'})$, $t\ge 0$, f is an arc coordinate vector of $\ell^t y$ for $\mathbf{n}$, e_1, H_c and m_X, and $\ell^\kappa f$ is Hodge-graded (for $\mathbf{n}$ and e_1) for some $\kappa\ge 0$, then $y\in\mathbf{N}_a$.*

Proof. Write $c'=c-e$ (the site of the frame), $\Pi=\Pi_c=\Lambda_{c,e}(H_c)$, $m=m_X$, $D=D_{\mathrm{cris}}(V_\nu)$, $B=B^+_{\mathrm{dR}}$, and $h_s=H_c(c-s)$ for $s<e$; *coordinate vector* means arc coordinate vector for $\mathbf{n}$, e, H_c, m. We have $e\ge 1$ (Proposition C.4).

Step 1: $y\in\mathbf{N}_c[1/\lambda_e]$. Since $\varphi_{\mathbf{N}}$ maps $\mathbf{N}$ into $\mathbf{N}[1/Q]$ and $\varphi(Q_{s'})=Q_{s'+1}$, $\varphi^e_{\mathbf{N}}(\mathbf{N}_{c'})\subseteq\mathbf{N}[1/\lambda_e]$, so $P_e(\mathbf{N}_{c'})\subseteq\mathbf{N}[1/\lambda_e]$. As Π is a product of the Q_s with $s<e$, it is a unit of $\mathbf{A}_{F,L}[1/\lambda_e]$, and $y=\Pi^{-1}(\Pi y)\in\mathbf{N}[1/\lambda_e]$ (inside $\mathbf{N}[1/\lambda_\infty]$; $\mathbf{N}$ is free by (Wm1)); with $y=e_cy$, $y\in\mathbf{N}_c[1/\lambda_e]$.

Step 2: two inequalities. (a) $m\ge 2h_s$ for $s<e$: $m=(\ell-1)M>2h_B$, and $H_c\le h_B$ because $w+S\le h_B$ (if some site is thick, $h_B=b+\eta+S$ and $w\le b/20$). (b) Every Hodge jump u of V_ν at τ_x, $x=c-s$, $s<e$, satisfies $u\ge -H_c(x)$. By Lemma 24.2 (1) it suffices to take $\nu=1$. By (Ow1), $u\ge -s_x-S$ or $|u|\le\eta$. Recall $\eta<s_a$ for all a and that some site is thin, so $\eta<w\le h_B$. If $x=c'+1$, then x is thin because $c'\in C$, so $s_x\le w$ and $u\ge -(w+S)=-H_c(x)$. If $x\ne c'+1$: a thin x has $s_x+S\le w+S\le h_B$, and a thick x has $s_x+S\le b+\eta+S=h_B$ (thick sites exist then); so $u\ge -h_B=-H_c(x)$. Consequently, with $\epsilon=\epsilon_{c-s-1}$: by Definition 23.3 the filtration $\epsilon\,\mathrm{Fil}^\bullet D$ of ϵD is the Hodge filtration of V_ν at τ_{c-s}; it is decreasing and exhaustive and has no jump below $-h_s$, so $\epsilon\,\mathrm{Fil}^{-h_s}D=\epsilon D$.

Step 3: the level images of y. Fix $s < e$ and apply Lemma 23.6 with c', e, $a = c$, s, the frame $\mathbf{n}$ at c' and $(V,T) = (V_\nu, T_\nu)$; let λ_s, Λ_s, $R^+ = \epsilon(B \otimes_{\mathbb{Q}_\ell} L)$, $D_k = \varphi^{e-s-4}(d_k)$ and $\Lambda_D = \sum_k R^+ D_k$ be as there. We show $\Lambda_s(y) \in \epsilon(B \otimes_{\mathbb{Q}_\ell} V_\nu)$. *(i)* $\xi_0^{h_s}\Lambda_D \subseteq \epsilon(B \otimes V_\nu)$. By (Lm5) and Step 2, $D_k \in \epsilon D = \epsilon\,\mathrm{Fil}^{-h_s} D \subseteq t^{-h_s} B \otimes V_\nu$ (definition of Fil in Definition 23.3); as $t \in \xi_0 B^\times$, $\xi_0^{h_s} D_k \in \epsilon(B \otimes V_\nu)$, a module over R^+. *(ii) The coordinate relation.* By hypothesis there are an admissible κ, an integer $\kappa' \geq 0$ and $p \in P_e(\mathbf{N}_{c'})$ with

$$\ell^{\kappa'}\Big(\ell^{\kappa+t}\Pi y - \sum_k \Pi f_k\, \varphi^e(T_\kappa(H_k))\Big) = X\Pi^2 p.$$

By (Lm1), $\lambda_s(\Pi) = \prod_{s'<e} \lambda_s(Q_{s'})^{h_{s'}} = \xi_0^{h_s} w$ with $w \in B^\times$, and $\lambda_s(X) \in B^\times$; as $\Pi f_k \in e_c \mathbf{A}_{F,L}$, $\lambda_s(\Pi f_k) \in R^+$, so $\lambda_s(f_k) \in \xi_0^{-h_s} R^+$. By (Lm5), $\Lambda_s(\varphi^e(T_\kappa H_k)) = \ell^\kappa D_k + \xi_0^m \delta_k$ with $\delta_k \in \Lambda_D$, and $\Lambda_s(p) \in \Lambda_D$. Applying the λ_s-semilinear map Λ_s and dividing by $\ell^{\kappa+\kappa'+t}\lambda_s(\Pi)$ (invertible in $B_{\mathrm{dR}} \otimes L$),

$$\Lambda_s(y) = \ell^{-t} \sum_k \lambda_s(f_k) D_k + \ell^{-\kappa-t} \sum_k \lambda_s(f_k)\, \xi_0^m \delta_k + \ell^{-\kappa-\kappa'-t} \lambda_s(X)\lambda_s(\Pi)\Lambda_s(p).$$

The second term lies in $\xi_0^{m-h_s}\Lambda_D \subseteq \xi_0^{h_s}\Lambda_D$ (Step 2 (a)), the third in $\xi_0^{h_s}\Lambda_D$; both lie in $\epsilon(B \otimes V_\nu)$ by (i) (ℓ is a unit of B). *(iii) The Hodge-graded term.* Let $(g_i, \mathbf{v}_i, \alpha_i)_{i\in I}$ be a section decomposition of $\ell^{\kappa_0} f$ with $\delta_e(\mathbf{v}_i) \in \varphi^{s+4}(\mathrm{Fil}^{-\alpha_i(s)} D)$ for all i (the given $\kappa \geq 0$ with $\ell^\kappa f$ Hodge-graded is called κ_0 here). Then $\ell^{\kappa_0}\sum_k \lambda_s(f_k) D_k = \sum_i \lambda_s(g_i) \sum_k v_{i,k} D_k$. The Frobenius of D is $\varphi \otimes 1$ on $B_{\mathrm{cris}} \otimes V_\nu$, hence L-linear and bijective on D, so $\sum_k v_{i,k} D_k = \varphi^{-(s+4)}(\sum_k v_{i,k}\varphi^e(d_k)) = \varphi^{-(s+4)}(\delta_e(\mathbf{v}_i)) \in \mathrm{Fil}^{-\alpha_i(s)} D \subseteq t^{-\alpha_i(s)} B \otimes V_\nu$. Write $g_i = Q^{\alpha_i} g_i'$ with $g_i' \in e_c \mathbf{A}_{F,L}$; by (Lm1), $\lambda_s(Q^{\alpha_i}) = \xi_0^{\alpha_i(s)} w_i$ with $w_i \in B^\times$ and $\lambda_s(g_i') \in R^+$. Hence $\lambda_s(g_i)\sum_k v_{i,k} D_k \in \xi_0^{\alpha_i(s)} t^{-\alpha_i(s)} R^+(B \otimes V_\nu) = \epsilon(B \otimes V_\nu)$, and the first term lies in $\epsilon(B \otimes V_\nu)$. So $\Lambda_s(y) \in \epsilon_{c-s-1}(B \otimes_{\mathbb{Q}_\ell} V_\nu)$ for every $s < e$.

Step 4: conclusion. By Step 1, $y \in \mathbf{N}_c[1/\lambda_e]$, and by Step 3 its images under all Λ_s, $s < e$, lie in $\epsilon_{c-s-1}(B \otimes V_\nu)$. The lattice criterion (Lm4) of Lemma 23.6 (with $a = c$) gives $y \in \mathbf{N}_c$.

Sub-arcs. Let e_1, a, Π', y, t, f, κ be as in the last assertion. Steps 1–4 go through with (c, e, Π) replaced by (a, e_1, Π'), for these reasons. Step 1 uses only that Π' is a product of the Q_s with $s < e_1$ and $P_{e_1}(\mathbf{N}_{c'}) \subseteq \mathbf{N}[1/\lambda_{e_1}]$. In Step 2, $h_s = H_c(a-s)$ for $s < e_1$, the sites $a - s$ ($s < e_1$) are $c'+1, \ldots, a$, and both (a) and (b) were proved for every site x with $H_c(x)$ in place of h_s (the site $c'+1$ being thin, the others bounded by h_B). Step 3 applies Lemma 23.6 with c', e_1, a and $s < e_1$, and uses $\lambda_s(\Pi') = \xi_0^{h_s} w$ with $w \in B^\times$ by (Lm1), the coordinate relation for e_1, and the section decomposition of $\ell^\kappa f$ for e_1. Step 4 applies (Lm4) with a and e_1. Hence $y \in \mathbf{N}_a$. □

C.1.3 Lifting graded sections along arcs of length one and two

Proposition C.25 lifts one graded section along an arc, according to the shape of the arc; it is used in Proposition C.10. This subsection states it and proves the cases of arcs of length one (Proposition C.26) and two (Proposition C.32). The shared tools are a lift of sections that are constant to high order at every site of the arc (Proposition C.27), a lift of sections with poles of the Hodge size at the up sites and zeros at the down sites (Lemma C.30), and lifts of filtered constants by window separation or by a Hodge gap (Lemma C.33, Lemma C.35). The lemma on the Δ-invariant variable (Lemma C.34) is used again in Section C.2.

Proposition C.25. *Let $\ell \geq 11$ be a prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of degree n, L as in Definition 23.1, $(\tau_a)_{a\in\mathbb{Z}/n}$ a Frobenius numbering, $\Phi\colon V_1 \to V_2$ an onto map of crystalline*

lattices $T_1 \to T_2$ over F with coefficients L, and $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ a weight word for Φ. Suppose that some site is thin, that $\epsilon_{a+1} \neq \epsilon_a$ for every a, that no three consecutive sites are all thick, that $20w \le b$, $20\eta \le b$, $20S \le b$, and that either no site is thick or $160\,\mathrm{Sg} \le b$. Put $\mathbf{N}_\nu = \mathbf{N}(T_\nu)$, let $h_B = b + \eta + S$ if some site is thick and $h_B = w + S$ otherwise, $M = \lfloor 2h_B/(\ell-1) \rfloor + 1$ and $m_X = (\ell-1)M$. For $c \in C$ write $e = \mathrm{len}_c$, $c' = c - e$, let $H_c \colon \mathbb{Z}/n \to \mathbb{Z}_{\ge 0}$ be $H_c(c'+1) = w + S$ and $H_c(x) = h_B$ for $x \neq c'+1$, put $\Pi_c = \Lambda_{c,e}(H_c)$, and

$$\nu_c = \prod_{s<e,\ \epsilon_{c-s}=1} Q_s^{s_{c-s}} \ \in e_c \mathbf{A}_{F,L}.$$

Call $g \in \mathcal{K}_c$ admissible at c *if $\gamma_u(g) = g$ for $u \in \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$, $\nu_c g \in e_c \mathbf{A}_{F,L}$, and for every $s < e$ with $\epsilon_{c-s} = -1$ there are $\kappa \ge 0$ and $x \in \mathcal{O}_L$ with $\nu_c(\ell^\kappa g - e_c(1 \otimes x)) \in Q_s^{s_{c-s}+S} e_c \mathbf{A}_{F,L}$.*

Then there is an integer E_{arc} with $0 \le E_{\mathrm{arc}} \le 40\,\mathrm{Sg} + \eta\,\beta_{320}$ such that the following holds for every $c \in C$ (with e, c' as above). Let $\mathbf{n} = (n_k)_{k \le r}$ be a frame at c' of $\mathbf{N}_2$ with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1}$, with crystalline frame (d_k), let $k_0 \le r$ be an index with

$$\varphi^e(d_{k_0}) \in \varphi^{s+4}\big(\mathrm{Fil}^{\,s_{c-s}} D_{\mathrm{cris}}(V_2)\big) \qquad \text{for every } s < e \text{ with } \epsilon_{c-s} = 1,$$

and let g be admissible at c. Then there are $w_1 \in I_c$ (formed for $\mathbf{N}_1$), a finite product λ' of the Q_s ($s \ge 0$) and $z \in X\mathbf{N}_{2,c}$ such that $\lambda' w_1 \in \mathbf{N}_1$, $\Pi_c\,\mathbf{N}(\Phi)(w_1) \in P_e(\mathbf{N}_{2,c'})$, and the vector $\ell^{E_{\mathrm{arc}}} g\,\mathbf{e}_{k_0}$ ($\mathbf{e}_{k_0}$ the k_0-th unit vector) is an arc coordinate vector of $\mathbf{N}(\Phi)(w_1) - z$ for $\mathbf{n}$, e, H_c and m_X.

Proof. **Arc lengths.** Let $c \in C$; for $d \ge 1$, $c - d \in C$ means that the site $c - d + 1$ is thin, so $c - d \notin C$ means that it is thick (Ow3). If $\mathrm{len}_c \ge 4$, then $c-1, c-2, c-3 \notin C$ and the three consecutive sites $c, c-1, c-2$ are thick, which is excluded. Hence $\mathrm{len}_c \in \{1, 2, 3\}$. If $\mathrm{len}_c = 3$, the sites c and $c-1$ are thick, so some site is thick and $160\,\mathrm{Sg} \le b$ by hypothesis.

A common exponent. Let E_1, E_2 be the exponents of Proposition C.26 and Proposition C.32; if some cut has length 3, let E_3, E_4 be those of Proposition C.37 and Proposition C.39 (their extra hypothesis $160\,\mathrm{Sg} \le b$ holds by the previous paragraph), and otherwise $E_3 = E_4 = 0$. Put $E_{\mathrm{arc}} = \max(E_1, E_2, E_3, E_4)$; then $0 \le E_{\mathrm{arc}} \le 40\,\mathrm{Sg} + \eta\,\beta_{320}$.

The lifting. Let $c \in C$, $e = \mathrm{len}_c$, and $\mathbf{n}$, k_0, g as in the statement. According as $e = 1$, $e = 2$, $e = 3$ with $\epsilon_{c-2} = 1$, or $e = 3$ with $\epsilon_{c-2} = -1$, the result Proposition C.26, Proposition C.32, Proposition C.37 or Proposition C.39, with exponent E_i, gives $w_1 \in I_c$ (for $\mathbf{N}_1$), a finite product λ' of the Q_s and $z \in X\mathbf{N}_{2,c}$ with $\lambda' w_1 \in \mathbf{N}_1$, $\Pi_c \mathbf{N}(\Phi)(w_1) \in P_e(\mathbf{N}_{2,c'})$, and $\ell^{E_i} g\,\mathbf{e}_{k_0}$ an arc coordinate vector of $\mathbf{N}(\Phi)(w_1) - z$. Let $t = E_{\mathrm{arc}} - E_i \ge 0$. Multiplying the defining relation of arc coordinate vectors (Definition 23.3) by ℓ^t shows that $\ell^{E_{\mathrm{arc}}} g\,\mathbf{e}_{k_0}$ is an arc coordinate vector of $\ell^t(\mathbf{N}(\Phi)(w_1) - z) = \mathbf{N}(\Phi)(\ell^t w_1) - \ell^t z$ (and $\Pi_c \ell^{E_{\mathrm{arc}}} g \in e_c \mathbf{A}_{F,L}$ as $\Pi_c \ell^{E_i} g$ is). Since I_c is an $\mathcal{O}_L$-module (Lemma 24.5 (Ry4)), $\ell^t w_1 \in I_c$; also $\lambda' \ell^t w_1 \in \mathbf{N}_1$, $\Pi_c \mathbf{N}(\Phi)(\ell^t w_1) \in P_e(\mathbf{N}_{2,c'})$ and $\ell^t z \in X\mathbf{N}_{2,c}$. So $\ell^t w_1$, λ', $\ell^t z$ have the asserted properties. □

Proposition C.26. *Let ℓ, F, n, L, (τ_a), $\Phi \colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25. Then there is an integer E with $0 \le E \le 40\,\mathrm{Sg} + \eta\,\beta_{320}$ such that the following holds for every $c \in C$ with $\mathrm{len}_c = 1$ (write $e = 1$). Let $\mathbf{n} = (n_k)_{k \le r}$ be a frame at c' of $\mathbf{N}_2$ with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1}$, with crystalline frame (d_k), let $k_0 \le r$ be an index with $\varphi^e(d_{k_0}) \in \varphi^{s+4}(\mathrm{Fil}^{\,s_{c-s}} D_{\mathrm{cris}}(V_2))$ for every $s < e$ with $\epsilon_{c-s} = 1$, and let g be admissible at c. Then there are $w_1 \in I_c$ (formed for $\mathbf{N}_1$), a finite product λ' of the Q_s ($s \ge 0$) and $z \in X\mathbf{N}_{2,c}$ such that $\lambda' w_1 \in \mathbf{N}_1$, $\Pi_c\,\mathbf{N}(\Phi)(w_1) \in P_e(\mathbf{N}_{2,c'})$, and $\ell^E g\,\mathbf{e}_{k_0}$ is an arc coordinate vector of $\mathbf{N}(\Phi)(w_1) - z$ for $\mathbf{n}$, e, H_c and m_X.*

Proof. Write $R_c = e_c\mathbf{A}_{F,L}$ and *coordinate vector* for arc coordinate vector for $\mathbf{N}_2$, $\mathbf{n}$, $e = 1$, H_c, m_X.

Exponent. Let $E_C \le 6\,\mathrm{Sg}$ be the integer of Proposition C.27 and $E_N \le 6\,\mathrm{Sg}$ the integer of Lemma C.30, and put $E = \max(E_C, E_N)$; then $0 \le E \le 6\,\mathrm{Sg} \le 40\,\mathrm{Sg} + \eta\,\beta_{320}$.

Raising the exponent. If the conclusion holds for c with some $E' \le E$ in place of E (data w_1, λ', z), it holds with E: replace w_1, z by $\ell^{E-E'}w_1, \ell^{E-E'}z$; I_c is an $\mathcal{O}_L$-module ((Ry4) of Lemma 24.5), and $\ell^{E-E'}$ times a coordinate vector of y is one of $\ell^{E-E'}y$ (Lemma C.16 (a)).

Let $c \in C$ with $\mathrm{len}_c = 1$, so $c' = c - 1 \in C$ and the site $c = c' + 1$ is thin; let $\mathbf{n}$, k_0, g be as in the statement. As $e = 1$, the only $s < e$ is $s = 0$, with the site c.

Case $\epsilon_c = -1$. Then $\nu_c = 1$, so admissibility says: $\gamma_u(g) = g$ for $u \in \mu_{\ell-1}$, $g \in R_c$, and $\ell^\kappa g - e_c(1\otimes x) \in Q_0^{s_c+S}R_c$ for some $\kappa \ge 0$, $x \in \mathcal{O}_L$. These are the hypotheses of Proposition C.27 for c (with $t_0 = s_c + S$), which gives the conclusion with E_C.

Case $\epsilon_c = 1$. Then $\nu_c = Q_0^{s_c}$ and $\nu_c^+ = 1$ (no site of the arc has sign -1), so admissibility gives $\nu_c g \in \nu_c^+ R_c$; and the hypothesis on k_0 is the one of Lemma C.30. That lemma gives $w_1 \in I_c$ with $\Pi_c\,\mathbf{N}(\Phi)(w_1) \in P_1(\mathbf{N}_{2,c'})$ such that $\ell^{E_N} g\,\mathbf{e}_{k_0}$ is a coordinate vector of $\mathbf{N}(\Phi)(w_1)$. With $\lambda' = 1$ (so $\lambda' w_1 = w_1 \in \mathbf{N}_1$) and $z = 0$ this is the conclusion with E_N, and raising the exponent gives it with E. □

Proposition C.27. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25. Then there is an integer E_C with $0 \le E_C \le 6\,\mathrm{Sg}$ such that the following holds for every $c \in C$ (with $e = \mathrm{len}_c$, $c' = c - e$). Let $\mathbf{n} = (n_k)_{k\le r}$ be a frame at c' of $\mathbf{N}_2$ with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1}$, let $k_0 \le r$, and let $g \in e_c\mathbf{A}_{F,L}$ with $\gamma_u(g) = g$ for $u \in \mu_{\ell-1}$ such that for every $s < e$ there are $\kappa \ge 0$ and $x \in \mathcal{O}_L$ with*

$$\ell^\kappa g - e_c(1\otimes x) \in Q_s^{t_s}\,e_c\mathbf{A}_{F,L}, \qquad t_s = s_{c-s} + S \text{ if } \epsilon_{c-s} = -1, \quad t_s = \eta \text{ if } \epsilon_{c-s} = 1.$$

Then there are $w_1 \in I_c$ (formed for $\mathbf{N}_1$), a finite product λ' of the Q_s ($s \ge 0$) and $z \in X\mathbf{N}_{2,c}$ such that $\lambda' w_1 \in \mathbf{N}_1$, $\Pi_c\,\mathbf{N}(\Phi)(w_1) \in P_e(\mathbf{N}_{2,c'})$, and $\ell^{E_C} g\,\mathbf{e}_{k_0}$ is an arc coordinate vector of $\mathbf{N}(\Phi)(w_1) - z$ for $\mathbf{n}$, e, H_c and m_X.

Filtered constants. *Let $c \in C$ (with $e = \mathrm{len}_c$, $c' = c - e$), let $\mathbf{n} = (n_k)_{k\le r}$ be a frame at c' of $\mathbf{N}_2$ with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1}$, let $k_0 \le r$, let $\mathbf{n}^{(1)} = (n_m^{(1)})_{m\le r_1}$ ($r_1 = \dim_L V_1$) be a frame at c' of $\mathbf{N}_1$ with $\gamma_u(n_m^{(1)}) = n_m^{(1)}$ for $u \in \mu_{\ell-1}$, with crystalline frame $(d_m^{(1)})$, and let $\mu \in M_{r_1\times r}(\mathcal{O}_L)$ be the matrix of $\mathbf{N}(\Phi)$ for $\mathbf{n}^{(1)}$ and $\mathbf{n}$. Let U be a set of integers $s < e$ with $\epsilon_{c-s} = 1$, put $\nu_U = \prod_{s\in U} Q_s^{s_{c-s}}$, and for $s < e$ put $t_s = s_{c-s} + S$ if $\epsilon_{c-s} = -1$ and $t_s = \eta$ if $\epsilon_{c-s} = 1$. Let $E' \ge 0$ be an integer and $\mathbf{c} = (c_m)_{m\le r_1} \in \mathcal{O}_L^{r_1}$ with*

$$\mathbf{c}\,\mu = \ell^{E'}\mathbf{e}_{k_0}, \qquad \sum_m c_m\,\varphi^e(d_m^{(1)}) \in \varphi^{s+4}\big(\mathrm{Fil}^{\,s_{c-s}} D_{\mathrm{cris}}(V_1)\big) \quad \text{for every } s \in U,$$

and let $g \in \mathcal{K}_c$ with $\gamma_u(g) = g$ for $u \in \mu_{\ell-1}$ and $\nu_U g \in e_c\mathbf{A}_{F,L}$, such that for every $s < e$ with $s \notin U$ there are $\kappa \ge 0$ and $x \in \mathcal{O}_L$ with $\nu_U(\ell^\kappa g - e_c(1\otimes x)) \in Q_s^{t_s}e_c\mathbf{A}_{F,L}$. (For U the set of all $s < e$ with $\epsilon_{c-s} = 1$ these conditions on g say that g is admissible at c.) Then there are $w_1 \in I_c$ (formed for $\mathbf{N}_1$) and a finite product λ' of the Q_s ($s \ge 0$) such that $\lambda' w_1 \in \mathbf{N}_1$, $\Pi_c\,\mathbf{N}(\Phi)(w_1) \in P_e(\mathbf{N}_{2,c'})$, and $\ell^{\sigma_B+E'} g\,\mathbf{e}_{k_0}$ is an arc coordinate vector of $\mathbf{N}(\Phi)(w_1)$ for $\mathbf{n}$, e, H_c and m_X.

Proof. Write $R_c = e_c\mathbf{A}_{F,L}$, $\Pi = \Pi_c$, $\psi = \mathbf{N}(\Phi)\colon \mathbf{N}_1 \to \mathbf{N}_2$, $r_1 = \dim_L V_1$, and fix u_0 as in Proposition C.11.

Exponents. Let $r_{c''}$ ($c'' \in \mathbb{Z}/n$) be the exponents of Proposition C.6 for the surjection $\Phi\colon T_1 \to T_2$ and the window (M_a, h_a) of Lemma C.9, which contains the Hodge jumps of V_1 by

Lemma C.9 (1); by Lemma C.9 (2), $r_{c''} \le 3\,\mathrm{Sg}$ for $c'' \in C$. Put $E_C = \sigma_B + \max_{c''\in C} r_{c''}$; then $0 \le E_C \le 6\,\mathrm{Sg}$, since $\sigma_B \le 3\,\mathrm{Sg}$.

It suffices to prove the conclusion at c with some $E' \le E_C$ in place of E_C: if $\ell^{E'} g\,\mathbf{e}_{k_0}$ is an arc coordinate vector of $\psi(w_1) - z$, then $\ell^{E_C} g\,\mathbf{e}_{k_0}$ is one of $\psi(\ell^{E_C-E'} w_1) - \ell^{E_C-E'} z$ (Lemma C.16 (a)), and $\ell^{E_C-E'} w_1 \in I_c$ by (Ry4) of Lemma 24.5. We take $E' = \sigma_B + r_{c'}$.

Fix $c \in C$, $e = \mathrm{len}_c$, $c' = c - e \in C$, and $\mathbf{n}$, k_0, g as in the statement.

Step 1: a frame of $\mathbf{N}_1$ and lifted constants. $\mathbf{N}_1$ is a Wach module ((Ex1) of Proposition 23.7), so by Lemma 24.15 $\mathbf{N}_{1,c'}$ has a Δ-invariant frame $\mathbf{n}^1 = (n^1_m)_{m\le r_1}$; let (d^1_m) be its crystalline frame, in $D_{\mathrm{cris}}(V_1)$. Let $\mu \in M_{r_1\times r}(\mathcal{O}_L)$ be the matrix of Lemma C.29 (1) for $f = \Phi$ and the frames $\mathbf{n}^1$, $\mathbf{n}$ at c': $\mu_{mk} = M_{mk}(0)$, where $\psi(n^1_m) = \sum_k M_{mk} n_k$. By Proposition C.6 there is $x_1 \in \mathbf{N}_{1,c'}$ with $\psi(x_1) = \ell^{r_{c'}} n_{k_0}$. Write $x_1 = \sum_m a_m n^1_m$ and $\mathbf{c} = (a_m(0))_m \in \mathcal{O}_L^{r_1}$. Comparing coordinates, $\sum_m a_m M_{mk} = \ell^{r_{c'}}\delta_{kk_0}$, and taking constant terms (a ring homomorphism $e_{c'}\mathbf{A}_{F,L} \to \mathcal{O}_L$),

$$\textstyle\sum_m \mathbf{c}_m \mu_{mk} = \ell^{r_{c'}}\delta_{kk_0} \qquad (k \le r).$$

Step 2: the coordinate vector in $\mathbf{N}_1$ is Hodge-integral modulo γ. Put $f^1 = (g\,\mathbf{c}_m)_{m\le r_1} \in \mathcal{K}_c^{r_1}$. It is Δ-invariant, and $\Pi f^1_m \in R_c$ as $g \in R_c$. By Lemma C.28 (with $K_s = t_s$), $(\gamma_{u_0} - 1)g = X^{\ell-1}h_0$ with $h_0 \in R_c$ and $\ell^\kappa h_0 \in \prod_{s<e} Q_s^{t_s} R_c$. So $(\gamma_{u_0} - 1)f^1 = X^{\ell-1}h$ with $h = (h_0\mathbf{c}_m)_m$, $\Pi h_m \in R_c$, and $\ell^\kappa h$ has the one-term section decomposition $(\ell^\kappa h_0, \mathbf{c}, \alpha)$ with $\alpha(s) = t_s \ge 0$. It is Hodge-graded: $\delta_e(\mathbf{c})$ lies in $\varepsilon_{c+3}D_{\mathrm{cris}}(V_1) = \varphi^{s+4}(\varepsilon_{c-s-1}D_{\mathrm{cris}}(V_1))$, and $\varepsilon_{c-s-1}\mathrm{Fil}^{-t_s}D_{\mathrm{cris}}(V_1) = \varepsilon_{c-s-1}D_{\mathrm{cris}}(V_1)$, because every Hodge jump of V_1 at τ_{c-s} is $\ge -t_s$ by (Ow1) of Definition 23.1 (in $[-s_{c-s} - S, -s_{c-s}] \cup [-\eta, \eta]$ if $\epsilon_{c-s} = -1$, in $[s_{c-s}, s_{c-s} + S] \cup [-\eta, \eta]$ if $\epsilon_{c-s} = 1$; $\varepsilon_{c-s-1}D_{\mathrm{cris}}$ is the τ_{c-s}-component, remark in Definition 23.3). Hence $h \in \mathcal{H}_{\mathbf{n}^1}$ and f^1 is Hodge-integral modulo γ (for $\mathbf{n}^1$).

Step 3: realisation in $\mathbf{N}_1$. By Proposition C.22 with $\nu = 1$ there is a Δ-invariant $y \in e_c\mathbf{N}_1[1/\lambda_\infty]$ with $(\gamma_u - 1)(y) \in X\mathbf{N}_1$ for all u (so $y \in I_c$, formed for $\mathbf{N}_1$) and $\Pi y \in P_e(\mathbf{N}_{1,c'})$, such that $\ell^{\sigma_B} f^1$ is an arc coordinate vector of y for $\mathbf{n}^1$, e, H_c, m_X. By definition of $\mathbf{N}_1[1/\lambda_\infty]$ there is a finite product λ' of the Q_s with $\lambda' y \in \mathbf{N}_1$.

Step 4: transport to $\mathbf{N}_2$. By Lemma C.29 (2) (with $a = c$, $H = H_c$, $m = m_X \ge 1$, and $m_X > 2h_B \ge 2H_c(c - s)$), $\Pi\psi(y) = \psi(\Pi y) \in P_e(\mathbf{N}_{2,c'})$, and $\ell^{\sigma_B} f^1\mu$ is an arc coordinate vector of $\psi(y)$ for $\mathbf{n}$, e, H_c, m_X. By Step 1, $(\ell^{\sigma_B} f^1\mu)_k = \ell^{\sigma_B} g \sum_m \mathbf{c}_m\mu_{mk} = \ell^{\sigma_B + r_{c'}} g\,\delta_{kk_0}$, i.e. $\ell^{\sigma_B} f^1\mu = \ell^{\sigma_B + r_{c'}} g\,\mathbf{e}_{k_0}$. So $w_1 = y$, λ' and $z = 0$ have the asserted properties with the exponent $\sigma_B + r_{c'}$.

Filtered constants. Let c, $\mathbf{n}$, k_0, $\mathbf{n}^{(1)}$, μ, E', $\mathbf{c}$, g be as in that paragraph of the statement; write $\nu = \nu_U$, t_s as in the statement, and $T = \prod_{s<e,\ s\notin U} Q_s^{t_s}$. By hypothesis, $\gamma_\delta(g) = g$ for $\delta \in \mu_{\ell-1}$, $\nu g \in R_c$, and for each $s < e$ with $s \notin U$ there are $\kappa_s \ge 0$, $x_s \in \mathcal{O}_L$ and $p_s \in R_c$ with $\nu(\ell^{\kappa_s} g - x_s) = Q_s^{t_s} p_s$; let $\kappa = \max_s \kappa_s$ ($\kappa = 0$ if there is no such s).

Step F1: the γ-difference of g. Put $G = (\gamma_{u_0} - 1)(g) \in \mathcal{K}_c$. As $\mathbb{Z}_\ell^\times$ is abelian, $\gamma_\delta(G) = G$ for $\delta \in \mu_{\ell-1}$. For $u \in \mathbb{Z}_\ell^\times$ and $s \ge 0$, $\gamma_u(Q_s) = Q_s \upsilon_{u,s}$ with $\upsilon_{u,s} \in \mathbb{Z}_\ell[[X]]^\times$ (Lemma C.16 (b)); hence $\gamma_u(\nu) = \nu\upsilon_u$ with $\upsilon_u \in R_c^\times$, and $\nu G = \upsilon_{u_0}^{-1}\gamma_{u_0}(\nu g) - \nu g \in R_c$. Since $Q_s(0) = \ell \ne 0$, ν is a unit of $L[[X]]$, so $g = (\nu g)\nu^{-1}$ and G lie in $L[[X]]$, on which γ_u acts by the same substitution $X \mapsto (1+X)^u - 1$. The constant term of G is $g(0) - g(0) = 0$. If $G \ne 0$, let $a_i X^i$ ($a_i \ne 0$, $i \ge 1$) be its term of least degree and δ a generator of $\mu_{\ell-1}$; the coefficient of X^i in $\gamma_\delta(G)$ is $\delta^i a_i$, so $\delta^i = 1$ and $(\ell - 1) \mid i$. Hence $\nu G \in R_c \cap X^{\ell-1}L[[X]] = X^{\ell-1}R_c$; write $\nu G = X^{\ell-1}k$ with $k \in R_c$. For $s < e$ with $s \notin U$,

$$\ell^{\kappa_s}\nu G = \nu(\gamma_{u_0} - 1)(\nu^{-1}Q_s^{t_s}p_s) = Q_s^{t_s}(\upsilon_{u_0}^{-1}\upsilon_{u_0,s}^{t_s}\gamma_{u_0}(p_s) - p_s) \in Q_s^{t_s}R_c.$$

$R_c \cong \mathcal{O}_L[[X]]$ is a unique factorisation domain, and X and the Q_s ($s \ge 0$) are pairwise coprime in it: $Q_s = \Phi_{\ell^{s+1}}(1 + X)$ is a distinguished polynomial whose roots $\zeta - 1$ (ζ of exact order ℓ^{s+1})

are simple, so by Weierstrass preparation its prime factors in R_c are those of Q_s in $\mathcal{O}_L[X]$, and different s (and X) have no common root. Hence $\ell^\kappa \nu G \in X^{\ell-1} T R_c$ and, cancelling $X^{\ell-1}$ in the domain R_c, $\ell^\kappa k \in T R_c$. Put $h_0 = k/\nu$, so that $(\gamma_{u_0} - 1)(g) = X^{\ell-1} h_0$.

Step F2: Hodge-integral modulo γ. Put $f^1 = (g\, c_m)_{m \le r_1} \in \mathcal{K}_c^{r_1}$; it is Δ-invariant. We have $\Pi/\nu \in R_c$, i.e. $H_c(c-s) \ge s_{c-s}$ for $s \in U$: for $s = e-1$ the site $c'+1$ is thin ($c' \in C$), so $s_{c'+1} \le w \le H_c(c'+1)$; otherwise $H_c(c-s) = h_B$, and $s_{c-s} \le w \le h_B$ (thin site) or $s_{c-s} \le b + \eta \le h_B$ (thick site; then $h_B = b + \eta + S$). Hence $\Pi f^1_m = (\Pi/\nu)(\nu g) c_m \in R_c$. Next, $(\gamma_{u_0} - 1) f^1 = X^{\ell-1} h$ with $h = (h_0 c_m)_m$, $\Pi h_m = (\Pi/\nu) k\, c_m \in R_c$, and $\ell^\kappa h$ has the one-term section decomposition $(\ell^\kappa h_0, \mathbf{c}, \alpha)$ with $\alpha(s) = -s_{c-s}$ if $s \in U$ and $\alpha(s) = t_s$ if $s \notin U$, since $\ell^\kappa h_0 = \ell^\kappa k/\nu \in (T/\nu) R_c = Q^\alpha R_c$. It is Hodge-graded for $\mathbf{n}^{(1)}$: at the levels $s \in U$ this is the hypothesis on $\mathbf{c}$; at a level $s \notin U$, $\varphi^{-(s+4)}(\sum_m c_m \varphi^e(d_m^{(1)}))$ lies in the τ_{c-s}-component $\varepsilon_{c-s-1} D_{\mathrm{cris}}(V_1)$ (remark in Definition 23.3), which equals $\varepsilon_{c-s-1} \mathrm{Fil}^{-t_s} D_{\mathrm{cris}}(V_1)$ because every Hodge jump of V_1 at τ_{c-s} is $\ge -t_s$: by (Ow1) it lies in $[s_{c-s}, s_{c-s}+S] \cup [-\eta, \eta]$ if $\epsilon_{c-s} = 1$ and in $[-s_{c-s} - S, -s_{c-s}] \cup [-\eta, \eta]$ if $\epsilon_{c-s} = -1$, and $\eta < s_{c-s}$. So $h \in \mathcal{H}_{\mathbf{n}^{(1)}}$, and f^1 is Hodge-integral modulo γ (for $\mathbf{n}^{(1)}$).

Step F3: realisation and transport. By Proposition C.22 with $\nu = 1$ and the Δ-invariant frame $\mathbf{n}^{(1)}$ there is $y \in e_c \mathbf{N}_1[1/\lambda_\infty]$ with $(\gamma_u - 1)(y) \in X \mathbf{N}_1$ for all $u \in \mathbb{Z}_\ell^\times$ (so $y \in I_c$, formed for $\mathbf{N}_1$) and $\Pi y \in P_e(\mathbf{N}_{1,c'})$, such that $\ell^{\sigma_B} f^1$ is an arc coordinate vector of y for $\mathbf{n}^{(1)}$, e, H_c, m_X; let λ' be a finite product of the Q_s with $\lambda' y \in \mathbf{N}_1$. By Lemma C.29 (2) (with $m_X > 2h_B \ge 2H_c(c-s)$), $\Pi\psi(y) = \psi(\Pi y) \in P_e(\mathbf{N}_{2,c'})$ and $\ell^{\sigma_B} f^1 \mu$ is an arc coordinate vector of $\psi(y)$ for $\mathbf{n}$, e, H_c, m_X; its k-th entry is $\ell^{\sigma_B} g\, (\mathbf{c}\mu)_k = \ell^{\sigma_B + E'} g\, \delta_{kk_0}$. So $w_1 = y$ and λ' have the asserted properties. □

Lemma C.28. *Let ℓ be an odd prime, F, L, (τ_a) as in Definition 23.2, $c \in \mathbb{Z}/n$, $R = e_c \mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$, $u_0 \in \mathbb{Z}_\ell^\times$, $e \ge 0$ and integers $K_0, \dots, K_{e-1} \ge 0$. Let $g \in R$ with $\gamma_\delta(g) = g$ for every $\delta \in \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$, such that for every $s < e$ there are $\kappa_s \ge 0$ and $x_s \in \mathcal{O}_L$ with $\ell^{\kappa_s} g - e_c(1 \otimes x_s) \in Q_s^{K_s} R$. Then $(\gamma_{u_0} - 1)(g) = X^{\ell-1} h_0$ with $h_0 \in R$, and $\ell^\kappa h_0 \in \prod_{s<e} Q_s^{K_s} R$ for $\kappa = \max_s \kappa_s$ ($\kappa = 0$ if $e = 0$).*

Proof. Write $G = (\gamma_{u_0} - 1)(g)$. The operators γ_u are $\mathcal{O}_L$-linear ring automorphisms of R with $\gamma_u(X) = h_u = (1+X)^u - 1 = uX + O(X^2)$, and $\gamma_{uv} = \gamma_u \gamma_v$; as $\mathbb{Z}_\ell^\times$ is abelian, $\gamma_\delta(G) = (\gamma_{u_0} - 1)(\gamma_\delta g) = G$ for $\delta \in \mu_{\ell-1}$.

Divisibility by $X^{\ell-1}$. The constant term of G is $g(0) - g(0) = 0$, as $h_{u_0}(0) = 0$. If $G \ne 0$, let $a_i X^i$ ($a_i \ne 0$, $i \ge 1$) be its term of least degree, and δ a generator of the cyclic group $\mu_{\ell-1}$. The coefficient of X^i in $\gamma_\delta(G) = G(h_\delta)$ is $\delta^i a_i$, so $\delta^i = 1$ ($\mathcal{O}_L$ is a domain), $(\ell - 1) \mid i$ and $i \ge \ell - 1$. Hence $G = X^{\ell-1} h_0$ with $h_0 \in R$.

Divisibility by $Q_s^{K_s}$. For $u \in \mathbb{Z}_\ell^\times$, $\gamma_u(Q_s) = Q_s \upsilon_{u,s}$ with a unit $\upsilon_{u,s}$ of $\mathbb{Z}_\ell[[X]]$ (Lemma C.16 (b), first assertion, a statement about $\mathbb{Z}_\ell[[X]]$ only). Write $\ell^{\kappa_s} g - x_s = Q_s^{K_s} q$ with $q \in R$ (x_s standing for $e_c(1 \otimes x_s)$, fixed by γ_{u_0}). Then

$$\ell^{\kappa_s} G = (\gamma_{u_0} - 1)(Q_s^{K_s} q) = Q_s^{K_s} (\upsilon_{u_0,s}^{K_s} \gamma_{u_0}(q) - q) \in Q_s^{K_s} R,$$

so $\ell^\kappa G \in Q_s^{K_s} R$ for every $s < e$.

Combination. $R \cong \mathcal{O}_L[[X]]$ is a regular local ring, hence a unique factorisation domain. $Q_s = \Phi_{\ell^{s+1}}(1+X)$ has the distinct simple roots $\zeta - 1$, ζ of exact order ℓ^{s+1}, so by Weierstrass preparation its prime factors in R are those of the distinguished polynomial Q_s, and they are prime to X and to the prime factors of $Q_{s'}$ for $s' \ne s$ (different roots, all in the maximal ideal of $\mathcal{O}_{\overline{L}}$). Hence for every prime element π of R, π divides at most one of $X^{\ell-1}, Q_0^{K_0}, \dots, Q_{e-1}^{K_{e-1}}$, and $v_\pi(\ell^\kappa G) \ge v_\pi(X^{\ell-1} \prod_s Q_s^{K_s})$. So $\ell^\kappa G \in X^{\ell-1} \prod_{s<e} Q_s^{K_s} R$, and cancelling $X^{\ell-1}$ in the domain R gives $\ell^\kappa h_0 \in \prod_{s<e} Q_s^{K_s} R$. □

Lemma C.29. *Let ℓ, F, L, (τ_a) be as in Proposition 23.7, let (V,T) and (V',T') be two pairs as there, of ranks $r=\dim_L V$ and $r'=\dim_L V'$, let $f\colon T\to T'$ be G_F-equivariant and $\mathcal{O}_L$-linear, and put $\mathbf{N}=\mathbf{N}(T)$, $\mathbf{N}'=\mathbf{N}(T')$, $\psi=\mathbf{N}(f)$. Fix $c\in\mathbb{Z}/n$, frames $\mathbf{n}=(n_k)_{k\le r}$ of $\mathbf{N}$ and $\mathbf{n}'=(n'_j)_{j\le r'}$ of $\mathbf{N}'$ at c, with horizontal sections H_k, H'_j and crystalline frames d_k, d'_j.*

1. *There is a unique matrix $\mu=(\mu_{kj})\in M_{r\times r'}(\mathcal{O}_L)$ with $\psi(H_k)=\sum_j \mu_{kj}H'_j$ for all k (ψ extended $E[[X]]$-linearly to $\mathbf{N}_E\to\mathbf{N}'_E$, and $\mathcal{O}_L$ acting through $e_c(1\otimes\mathcal{O}_L)$); μ_{kj} is the constant term of the n'_j-coordinate of $\psi(n_k)$, and $(1\otimes f)(d_k)=\sum_j\mu_{kj}d'_j$, where $1\otimes f\colon B_{\mathrm{cris}}\otimes_{\mathbb{Z}_\ell}T\to B_{\mathrm{cris}}\otimes_{\mathbb{Z}_\ell}T'$.*
2. *Let $e\ge 0$, $a=c+e$, $H\colon\mathbb{Z}/n\to\mathbb{Z}_{\ge 0}$, $\Pi=\Lambda_{a,e}(H)$ and $m\ge 1$ with $m\ge 2H(a-s)$ for $s<e$. Then $\psi(P_e(\mathbf{N}_c))\subseteq P_e(\mathbf{N}'_c)$, and if $g\in\mathcal{K}_a^r$ is an arc coordinate vector of $y\in\mathbf{N}[1/\lambda_\infty]$ for $\mathbf{n},e,H,m$, then $g\mu=\big(\sum_k g_k\mu_{kj}\big)_{j\le r'}$ is an arc coordinate vector of $\psi(y)$ for $\mathbf{n}',e,H,m$.*

Proof. By (Ex3) of Proposition 23.7, ψ is $\mathbf{A}_{F,L}$-linear and commutes with φ and every γ_u; so $\psi(\mathbf{N}_c)\subseteq\mathbf{N}'_c$, and its $E[[X]]$-linear extension to $\mathbf{N}_E$ (and its extension to $\mathbf{N}[1/\lambda_\infty]$) commutes with the operators $\gamma_u(h\otimes x)=(h\circ h_u)\otimes\gamma_u(x)$ (and with φ).

1. Write $\psi(n_k)=\sum_j M_{kj}n'_j$ with $M_{kj}\in e_c\mathbf{A}_{F,L}$. Extending $\mathbf{n}$, $\mathbf{n}'$ by frames at the other sites, (Hf2) of Lemma 23.9 shows that (H'_j) is an $e_cE[[X]]$-basis of $e_c\mathbf{N}'_E$, so $\psi(H_k)=\sum_j c_{kj}H'_j$ with unique $c_{kj}\in e_cE[[X]]$. As $\psi(H_k)$ and the H'_j are fixed by every γ_u, $c_{kj}\circ h_u=c_{kj}$ for every $u\in\mathbb{Z}_\ell^\times$. Take $u=1+\ell$, not a root of unity: if $c_{kj}=\sum_i a_iX^i$ had a nonzero coefficient a_i with $i\ge 1$, the least such i would give $a_iu^i=a_i$ as the coefficient of X^i (since $h_u=uX+O(X^2)$), which is impossible. So c_{kj} is a constant. Reducing modulo $X\mathbf{N}'_E$, where $H_k\equiv n_k$ and $H'_j\equiv n'_j$, gives $c_{kj}=M_{kj}(0)\in e_c(\mathcal{O}_F\otimes\mathcal{O}_L)\cong\mathcal{O}_L$; put $\mu_{kj}=c_{kj}$. For the crystalline frames: ψ maps $\mathbf{N}_R$ into $\mathbf{N}'_R$, and $\mathrm{Ev}'(\psi(W))=(1\otimes f)(\mathrm{Ev}(W))$ for $W\in\mathbf{N}_R$, because for $W=h\otimes x$ both sides equal $e(h)\cdot\varphi^3\big((\rho\otimes 1)((1\otimes f)\iota_T(x))\big)$ by (Ev3) of Lemma 23.11 and $\iota_{T'}\circ\psi=(1\otimes f)\circ\iota_T$ ((Ex3)), $1\otimes f$ being $B_{\mathrm{cris}}\otimes\mathcal{O}_L$-linear. As Ev' is linear over the constants (for $\lambda\in\mathcal{O}_L$, $e(e_c(1\otimes\lambda))$ acts as λ on $\mathrm{Ev}'(e_c\mathbf{N}'_R)$, by (Ev2)), $(1\otimes f)(d_k)=\mathrm{Ev}'(\sum_j\mu_{kj}H'_j)=\sum_j\mu_{kj}d'_j$.

2. $\psi(\varphi^e(x))=\varphi^e(\psi(x))$ and $\mathbf{A}_{F,L}$-linearity give $\psi(P_e(\mathbf{N}_c))\subseteq P_e(\mathbf{N}'_c)$. Let g be an arc coordinate vector of y with exponents (κ,κ'). Admissible exponents form an upward closed set, $T_{\kappa_1}=\ell^{\kappa_1-\kappa}T_\kappa$ for $\kappa_1\ge\kappa$, and the defining relation persists after multiplication by $\ell^{\kappa_1-\kappa}$; so we may assume κ admissible for both $\mathbf{n}$ and $\mathbf{n}'$. Applying ψ to the defining relation,

$$\ell^{\kappa'}\Big(\ell^{\kappa}\Pi\,\psi(y)-\sum_k \Pi g_k\,\varphi^e\big(\psi(T_\kappa(H_k))\big)\Big)\in X\Pi^2P_e(\mathbf{N}'_c).$$

Now $T_\kappa(H_k)-\ell^\kappa H_k\in X^m\mathbf{N}_E$ and $T_\kappa(H'_j)-\ell^\kappa H'_j\in X^m\mathbf{N}'_E$ by definition of the truncation, so by part 1

$$\delta_k:=\psi(T_\kappa(H_k))-\sum_j\mu_{kj}T_\kappa(H'_j)=\psi\big(T_\kappa(H_k)-\ell^\kappa H_k\big)-\sum_j\mu_{kj}\big(T_\kappa(H'_j)-\ell^\kappa H'_j\big)\in X^m\mathbf{N}'_E.$$

Also $\delta_k\in\mathbf{N}'_c$, since $T_\kappa(H_k)\in\mathbf{N}_c$, $T_\kappa(H'_j)\in\mathbf{N}'_c$ and μ_{kj} is integral; comparing coordinates in the basis $\mathbf{n}'$ (also an $e_cE[[X]]$-basis of $e_c\mathbf{N}'_E$), $\delta_k\in X^m\mathbf{N}'_c$. Hence $\varphi^e(\delta_k)\in\varphi^e(X)^mP_e(\mathbf{N}'_c)=(X\lambda_e)^mP_e(\mathbf{N}'_c)\subseteq X\Pi^2P_e(\mathbf{N}'_c)$, as $m\ge 1$ and $\lambda_e^m=\prod_{s<e}Q_s^m$ with $m\ge 2H(a-s)$. Since $\Pi g_k\in e_a\mathbf{A}_{F,L}$, substituting gives

$$\ell^{\kappa'}\Big(\ell^{\kappa}\Pi\,\psi(y)-\sum_j\Pi(g\mu)_j\,\varphi^e\big(T_\kappa(H'_j)\big)\Big)\in X\Pi^2P_e(\mathbf{N}'_c),$$

and $\Pi(g\mu)_j=\sum_k(\Pi g_k)\mu_{kj}\in e_a\mathbf{A}_{F,L}$. So $g\mu$ is an arc coordinate vector of $\psi(y)$. □

Lemma C.30. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25. Then there is an integer E_N with $0 \le E_N \le 6\,\mathrm{Sg}$ such that the following holds for every $c \in C$ (with $e = \mathrm{len}_c$, $c' = c - e$). Let $\mathbf{n} = (n_k)_{k \le r}$ be a frame at c' of $\mathbf{N}_2$ with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1}$, with crystalline frame (d_k), let $k_0 \le r$ be an index with $\varphi^e(d_{k_0}) \in \varphi^{s+4}(\mathrm{Fil}^{s_{c-s}} D_{\mathrm{cris}}(V_2))$ for every $s < e$ with $\epsilon_{c-s} = 1$, and let $g \in \mathcal{K}_c$ with*

$$\nu_c\, g \in \nu_c^+\, e_c \mathbf{A}_{F,L}.$$

Then there is $w_1 \in \mathbf{N}_{1,c}$ with $w_1 \in I_c$ (formed for $\mathbf{N}_1$) and $\Pi_c\, \mathbf{N}(\Phi)(w_1) \in P_e(\mathbf{N}_{2,c'})$ such that $\ell^{E_N} g\, \mathbf{e}_{k_0}$ is an arc coordinate vector of $\mathbf{N}(\Phi)(w_1)$ for $\mathbf{n}$, e, H_c and m_X.

Proof. Write $R_c = e_c \mathbf{A}_{F,L}$ and *coordinate vector* for arc coordinate vector for $\mathbf{N}_2$, $\mathbf{n}$, e, H_c, m_X.

The exponent. Let (M_a, h_a) be the window of Lemma C.9; it contains every Hodge jump of V_1 at τ_a (Lemma C.9 (1)). Let r_c ($c \in \mathbb{Z}/n$) be the exponents of Proposition C.6 for the surjection $\Phi\colon T_1 \to T_2$ and this window ($\ell \ge 11$); they depend only on the window and c, and $r_c \le 3\,\mathrm{Sg}$ for $c \in C$ by Lemma C.9 (2). Put $r^* = \max_{c \in C} r_c$. Also $\sigma_B \le 3\,\mathrm{Sg}$:

Put $E_N = r^* + \sigma_B$; then $0 \le E_N \le 6\,\mathrm{Sg}$.

The lift. Let c, $\mathbf{n}$, k_0, g be as in the statement. By definition of δ_e, $\delta_e(\mathbf{e}_{k_0}) = \varphi^e(d_{k_0}) \in \varphi^{s+4}(\mathrm{Fil}^{s_{c-s}} D_{\mathrm{cris}}(V_2))$ for every $s < e$ with $\epsilon_{c-s} = 1$, and $\nu_c g \in \nu_c^+ R_c$. By Lemma C.31 (with $\mathbf{c} = \mathbf{e}_{k_0}$), $f = g\, \mathbf{e}_{k_0}$ satisfies $\Pi_c f_k \in R_c$ for all k and is Hodge-graded. By (R1) of Proposition C.23 there is $x_2 \in \mathbf{N}_{2,c}$ such that $\ell^{\sigma_B} f$ is a coordinate vector of x_2. By Proposition C.6 there is $x_1 \in \mathbf{N}_{1,c}$ with $\mathbf{N}(\Phi)(x_1) = \ell^{r_c} x_2$. Put $w_1 = \ell^{r^* - r_c} x_1 \in \mathbf{N}_{1,c}$, so $\mathbf{N}(\Phi)(w_1) = \ell^{r^*} x_2$. Then:

- $w_1 \in I_c$, since $(\gamma_u - 1)\mathbf{N}_1 \subseteq X\mathbf{N}_1$ by (Wm3) ($\mathbf{N}_1$ is a Wach module by (Ex1) of Proposition 23.7);
- $\Pi_c\, \mathbf{N}(\Phi)(w_1) = \ell^{r^*} \Pi_c x_2 \in \Pi_c \mathbf{N}_{2,c} \subseteq P_e(\mathbf{N}_{2,c'})$ by Lemma C.16 (1), $P_e(\mathbf{N}_{2,c'})$ being an $\mathcal{O}_L$-module;
- $\ell^{r^*} \cdot \ell^{\sigma_B} f = \ell^{E_N} g\, \mathbf{e}_{k_0}$ is a coordinate vector of $\ell^{r^*} x_2 = \mathbf{N}(\Phi)(w_1)$ by Lemma C.16 (a).

□

Lemma C.31. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25; put $\mathbf{N} = \mathbf{N}(T_2)$ and $r = \dim_L V_2$. Let $c \in C$, $e = \mathrm{len}_c$, let $\mathbf{n}$ be a frame at c' of $\mathbf{N}$ with crystalline frame (d_k), let $\mathbf{c} \in \mathcal{O}_L^r$ with*

$$\delta_e(\mathbf{c}) \in \varphi^{s+4}(\mathrm{Fil}^{s_{c-s}} D_{\mathrm{cris}}(V_2)) \qquad \text{for every } s < e \text{ with } \epsilon_{c-s} = 1,$$

and let $g \in \mathcal{K}_c$ with $\nu_c g \in \nu_c^+\, e_c \mathbf{A}_{F,L}$. Then the vector $f = (g\, c_k)_{k \le r}$ (c_k acting as $e_c(1 \otimes c_k)$) satisfies $\Pi_c f_k \in e_c \mathbf{A}_{F,L}$ for every k and is Hodge-graded (for $\mathbf{n}$ and e).

Proof. Define $\alpha\colon \{0, \dots, e-1\} \to \mathbb{Z}$ by $\alpha(s) = s_{c-s} + S$ if $\epsilon_{c-s} = -1$ and $\alpha(s) = -s_{c-s}$ if $\epsilon_{c-s} = 1$. Then $Q^\alpha = \prod_{s<e} Q_s^{\alpha(s)} = \nu_c^+ / \nu_c$, so $g = Q^\alpha a$ with $a = \nu_c g / \nu_c^+ \in e_c \mathbf{A}_{F,L}$, and $(g, \mathbf{c}, \alpha)$ is a one-term section decomposition of f.

Hodge-graded. We check $\delta_e(\mathbf{c}) \in \varphi^{s+4}(\mathrm{Fil}^{-\alpha(s)} D_{\mathrm{cris}}(V_2))$ for every $s < e$. If $\epsilon_{c-s} = 1$, then $-\alpha(s) = s_{c-s}$ and this is the hypothesis. Let $\epsilon_{c-s} = -1$, so $-\alpha(s) = -s_{c-s} - S$. By the remark in Definition 23.3, $\varphi^{-(s+4)}(\delta_e(\mathbf{c}))$ lies in $\varepsilon_{c-s-1} D_{\mathrm{cris}}(V_2)$, the component of the embedding τ_{c-s}, which is $D_{\mathrm{dR},\tau_{c-s}}(V_2)$ with the filtration $\varepsilon_{c-s-1}\mathrm{Fil}^u D_{\mathrm{cris}}(V_2) = \mathrm{Fil}^u D_{\mathrm{cris}}(V_2) \cap \varepsilon_{c-s-1} D_{\mathrm{cris}}(V_2)$ (F is unramified and V_2 crystalline, so $D_{\mathrm{dR}}(V_2) = D_{\mathrm{cris}}(V_2)$ as filtered $F \otimes_{\mathbb{Q}_\ell} L$-modules, and Fil^u is an $F \otimes L$-submodule). Every Hodge jump of V_2 at τ_{c-s} is one of V_1 (Lemma 24.2 (1)),

hence lies in $[-s_{c-s}-S,-s_{c-s}]\cup[-\eta,\eta]$ by (Ow1), hence is $\geq -s_{c-s}-S$ because $\eta < s_{c-s}$ (sizes exceed η in a weight word). The filtration is decreasing and exhaustive, so if $\mathrm{Fil}^{-s_{c-s}-S}$ were not the whole component there would be a jump $a < -s_{c-s}-S$; hence $\varepsilon_{c-s-1}\mathrm{Fil}^{-s_{c-s}-S}D_{\mathrm{cris}}(V_2) = \varepsilon_{c-s-1}D_{\mathrm{cris}}(V_2)$ and $\delta_e(\mathbf{c}) \in \varphi^{s+4}(\mathrm{Fil}^{-\alpha(s)}D_{\mathrm{cris}}(V_2))$. So f is Hodge-graded.

Integrality. $\Pi_c = \prod_{s<e} Q_s^{H_c(c-s)}$, so $\Pi_c f_k = \prod_{s<e} Q_s^{H_c(c-s)+\alpha(s)}\, a\, e_c(1\otimes c_k)$, and it suffices that $H_c(c-s)+\alpha(s) \geq 0$ for every $s < e$. This is clear if $\epsilon_{c-s} = -1$. Let $\epsilon_{c-s} = 1$. If $s = e-1$, the site $c-s = c'+1$ is thin ($c' \in C$), so $s_{c-s} \leq w \leq w+S = H_c(c'+1)$. Otherwise $H_c(c-s) = h_B$, and $s_{c-s} \leq h_B$: a thin site has $s_{c-s} \leq w$, a thick one $s_{c-s} \leq b+\eta$ (Ow3), and $h_B = b+\eta+S$ if some site is thick, $h_B = w+S$ otherwise. Hence $\Pi_c f_k \in e_c\mathbf{A}_{F,L}$. □

Proposition C.32. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b,w,S,\eta,m,(s_a),(\epsilon_a))$ satisfy the hypotheses of Proposition C.25. Then there is an integer E with $0 \leq E \leq 40\,\mathrm{Sg} + \eta\,\beta_{320}$ such that the following holds for every $c \in C$ with $\mathrm{len}_c = 2$ (write $e = 2$). Let $\mathbf{n} = (n_k)_{k\leq r}$ be a frame at c' of $\mathbf{N}_2$ with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1}$, with crystalline frame (d_k), let $k_0 \leq r$ be an index with $\varphi^e(d_{k_0}) \in \varphi^{s+4}(\mathrm{Fil}^{\,s_{c-s}}D_{\mathrm{cris}}(V_2))$ for every $s < e$ with $\epsilon_{c-s} = 1$, and let g be admissible at c. Then there are $w_1 \in I_c$ (formed for $\mathbf{N}_1$), a finite product λ' of the Q_s ($s \geq 0$) and $z \in X\mathbf{N}_{2,c}$ such that $\lambda' w_1 \in \mathbf{N}_1$, $\Pi_c\,\mathbf{N}(\Phi)(w_1) \in P_e(\mathbf{N}_{2,c'})$, and $\ell^E g\,\mathbf{e}_{k_0}$ is an arc coordinate vector of $\mathbf{N}(\Phi)(w_1) - z$ for $\mathbf{n}$, e, H_c and m_X.*

Proof. Write $R_c = e_c\mathbf{A}_{F,L}$, $\psi = \mathbf{N}(\Phi)$, $\beta = \beta_{320}$, and *coordinate vector* for arc coordinate vector for $\mathbf{N}_2$, $\mathbf{n}$, $e = 2$, H_c, m_X. Since $m_X = (\ell-1)M > 2h_B \geq 2H_c(x)$ for every site x, Lemma C.17 applies to these coordinate vectors.

Exponents. Let $E_C \leq 6\,\mathrm{Sg}$ and $E_N \leq 6\,\mathrm{Sg}$ be the integers of Proposition C.27 and Lemma C.30, and $E_0 = \max(E_C, E_N)$. Let $(M_a, h_a)_a$ be the window of Lemma C.9, so $H(a) = M_a + h_a = s_a + S + \eta$, and let $H_{\max}$, c_H and r_x ($x \in \mathbb{Z}/n$) be formed with it as in Proposition C.6 for the surjection $\Phi\colon T_1 \to T_2$; by Lemma C.9, every Hodge jump of V_1 at τ_a lies in $[-M_a, h_a]$ and $r_x \leq 3\,\mathrm{Sg}$ for $x \in C$. Also $\sigma_B \leq 3\,\mathrm{Sg}$.

Put $E = \lfloor 12\,\mathrm{Sg}\rfloor + \eta\beta$; then $0 \leq E \leq 40\,\mathrm{Sg} + \eta\beta_{320}$. Below, each case gives the conclusion with an integer exponent $E'' \leq 12\,\mathrm{Sg} + \eta\beta$, hence $E'' \leq E$.

Three ways of combining.

(a) Raising. If the conclusion holds for g with an exponent $E'' \leq E$ (data w_1, λ', z), it holds with E (data $\ell^{E-E''}w_1, \lambda', \ell^{E-E''}z$): I_c is an $\mathcal{O}_L$-module (Lemma 24.5 (Ry4)), and ℓ^t times a coordinate vector of an element is one of ℓ^t times that element (Lemma C.17 item 1).

(b) Sums. If (w_j, λ'_j, z_j) give the conclusion for g_j ($j = 1, 2$) with the same exponent E'', then $(w_1+w_2, \lambda'_1\lambda'_2, z_1+z_2)$ gives it for g_1+g_2: I_c is an $\mathcal{O}_L$-module, $\lambda'_1\lambda'_2(w_1+w_2) \in \mathbf{N}_1$ as $\mathbf{N}_1$ is an $\mathbf{A}_{F,L}$-module, $P_2(\mathbf{N}_{2,c'})$ and $X\mathbf{N}_{2,c}$ are groups, and coordinate vectors add (Lemma C.17 item 1).

(c) Descent. Suppose $\ell^K g = x_2 + f_2$ with $K \geq 0$, $x_2 \in \mathcal{O}_L$ (acting as $e_c(1\otimes x_2)$) and $f_2 \in \mathcal{K}_c$ with $\nu_c f_2 \in \nu_c^+ R_c$ (ν_c^+ as in Proposition C.11). The constant $x_2 \in R_c$ is fixed by every γ_u and satisfies the hypotheses of the first part of Proposition C.27 (with $\kappa = 0$ and $x = x_2$ at every level), which gives the conclusion for x_2 with E_C. By Lemma C.30 (its hypothesis on k_0 is the one of the statement) there is $w_N \in I_c \cap \mathbf{N}_{1,c}$ with $\Pi_c\psi(w_N) \in P_2(\mathbf{N}_{2,c'})$ such that $\ell^{E_N} f_2 \mathbf{e}_{k_0}$ is a coordinate vector of $\psi(w_N)$, i.e. the conclusion for f_2 with E_N ($\lambda' = 1$, $z = 0$). By (a) (with E_0 in place of E) and (b), the conclusion holds for $\ell^K g$ with E_0, i.e. for g with $E_0 + K$.

(d) Filtered constants. By (Ex1) of Proposition 23.7, $\mathbf{N}_1$ is a Wach module, so by Lemma 24.15 $\mathbf{N}_{1,c'}$ has a frame $\mathbf{n}^{(1)} = (n_m^{(1)})_{m\leq r_1}$ fixed by γ_u, $u \in \mu_{\ell-1}$; let $(d_m^{(1)})$

be its crystalline frame and μ the matrix of ψ for $\mathbf{n}^{(1)}$ and $\mathbf{n}$ (Lemma C.29 (1)). If $\mathbf{c} \in \mathcal{O}_L^{r_1}$ and $E' \geq 0$ satisfy the hypotheses of the clause "Filtered constants" of Proposition C.27 for c, $\mathbf{n}$, k_0, $\mathbf{n}^{(1)}$ and U the set of all $s < 2$ with $\epsilon_{c-s} = 1$ (for this U the conditions on g there are admissibility), then that clause gives the conclusion for the admissible g with the exponent $\sigma_B + E'$ ($z = 0$).

The arc. Let $c \in C$ with $\mathrm{len}_c = 2$, and $\mathbf{n}$, k_0, g as in the statement. As $c - 1 \notin C$, the site c is thick; as $c' = c - 2 \in C$, the site $a := c - 1 = c' + 1$ is thin (Ow3). So some site is thick, $h = h_B = b + \eta + S$, and $s_c \leq b + \eta \leq h$, $s_a \leq w \leq h$; also $H_{\max} \leq b + 2\eta + S = h + \eta \leq 1.05\, h$ (as $20\eta \leq b$) and $c_H \leq 2 + \log_\ell(H_{\max} + 1) \leq 2 + \log_\ell(2h)$. Signs alternate, so $\epsilon_a = -\epsilon_c$. The site a is thin with the thick neighbour c, so $\beta = 1$ as soon as $|\ell s_a - b| \leq 320\, Z$, $Z = \ell\eta + \ell S + b/\ell + s_a + \ell \log_\ell(2h)$. We use $\log_\ell(2h) \leq \mathrm{Sg} - 1$ and $\mathrm{Sg} \geq 1$.

Case 1: $\epsilon_c = 1$ and $\ell(s_a + S + \eta) + \eta < s_c$. By Proposition C.6, ℓ^{r_c} is killing at c for ψ (in the sense of Lemma C.7). The hypothesis on k_0 at $s = 0$ reads $\varphi^2(d_{k_0}) \in \varphi^4(\mathrm{Fil}^{s_c} D_{\mathrm{cris}}(V_2))$. So Lemma C.33 (with $r = r_c$ and the frames $\mathbf{n}$, $\mathbf{n}^{(1)}$) gives $\mathbf{c} \in \mathcal{O}_L^{r_1}$ with $\mathbf{c}\mu = \ell^{2\sigma_B + r_c}\mathbf{e}_{k_0}$ and $\sum_m c_m \varphi^2(d_m^{(1)}) \in \varphi^4(\mathrm{Fil}^{s_c} D_{\mathrm{cris}}(V_1))$. As $s = 0$ is the only level with $\epsilon_{c-s} = 1$, these are the hypotheses of (d) with $E' = 2\sigma_B + r_c$; the exponent is $3\sigma_B + r_c \leq 12\,\mathrm{Sg}$.

Case 2: $\epsilon_c = 1$ and $s_c \leq \ell(s_a + S + \eta) + \eta$. Here $\nu_c = Q_0^{s_c}$ and $\nu_c^+ = Q_1^y$ with $y = s_a + S \geq 1$. By admissibility $\nu_c g \in R_c$ and $\nu_c(\ell^\kappa g - x) \in Q_1^y R_c$ for some $\kappa \geq 0$, $x \in \mathcal{O}_L$. Let $K = K_\uparrow(s_c, y)$ of Lemma C.36 (Cd2) (with $s = 0$); then $Q_0^{s_c} \notin \varpi\ell^K R_c + Q_1^y R_c$, and $Q_1^y \notin \varpi R_c$ (Q_1 is a monic polynomial). So (Cd1) (with $A = \nu_c$, $N = Q_1^y$, $f = g$) gives $\ell^K g = x_2 + f_2$ with $x_2 \in \mathcal{O}_L$ and $\nu_c f_2 \in \nu_c^+ R_c$, and (c) gives the conclusion with $E_0 + K$. If $s_c < \ell y$, then $K = 0$. Otherwise $\ell y \leq s_c \leq \ell(y + \eta) + \eta < \ell(y + \eta + 1)$, so the bound of (Cd2) with $\eta + 1$ in place of η gives $K \leq \eta + 2 + \log_\ell s_c \leq \eta + 1 + \mathrm{Sg}$; and $\beta = 1$, since $\ell s_a \leq s_c - \ell S \leq b + \eta$ and $\ell s_a \geq s_c - \ell S - \ell\eta - \eta \geq b - \ell S - \ell\eta - \eta$ give $|\ell s_a - b| \leq \ell S + \ell\eta + \eta \leq 2Z$. In both cases $K \leq \eta\beta + 2\,\mathrm{Sg}$, and the exponent is at most $8\,\mathrm{Sg} + \eta\beta$.

Case 3: $\epsilon_c = -1$ and $A_a + \eta + 1 \leq s_a$, where $A_a = \lceil \sigma_a/(\ell - 1) \rceil$ and σ_a is the killing index of ψ at a (Lemma C.7). Then $\epsilon_a = 1$, and the hypothesis on k_0 at $s = 1$, $\varphi^2(d_{k_0}) \in \varphi^5(\mathrm{Fil}^{s_a} D_{\mathrm{cris}}(V_2))$, gives $\varphi(d_{k_0}) \in \varphi^4(\mathrm{Fil}^{s_a} D_{\mathrm{cris}}(V_2))$, as φ is bijective on $D_{\mathrm{cris}}(V_2)$. So Lemma C.35 (for c, $a = c' + 1$, the frames $\mathbf{n}$, $\mathbf{n}^{(1)}$) gives $\mathbf{c}$ with $\mathbf{c}\mu = \ell^{2\sigma_B}\mathbf{e}_{k_0}$ and $\sum_m c_m \varphi(d_m^{(1)}) \in \varphi^4(\mathrm{Fil}^{s_a} D_{\mathrm{cris}}(V_1))$; applying the L-linear map φ, $\sum_m c_m \varphi^2(d_m^{(1)}) \in \varphi^5(\mathrm{Fil}^{s_a} D_{\mathrm{cris}}(V_1))$. As $s = 1$ is the only level with $\epsilon_{c-s} = 1$, (d) applies with $E' = 2\sigma_B$; the exponent is $3\sigma_B \leq 9\,\mathrm{Sg}$.

Case 4: $\epsilon_c = -1$ and $s_a \leq A_a + \eta$. Here $\nu_c = Q_1^y$ with $y = s_a \geq 1$ and $\nu_c^+ = Q_0^B$ with $B = s_c + S \geq 1$; by admissibility $\nu_c g \in R_c$ and $\nu_c(\ell^\kappa g - x) \in Q_0^B R_c$ for some κ, x. Let $K = K_\downarrow(y, B)$ of Lemma C.36 (Cd3) (with $s = 0$); then $Q_1^y \notin \varpi\ell^K R_c + Q_0^B R_c$, and (Cd1) (with $A = \nu_c$, $N = Q_0^B$) gives $\ell^K g = x_2 + f_2$ with $\nu_c f_2 \in \nu_c^+ R_c$; (c) gives the conclusion with $E_0 + K$. If $\ell y < B$, then $K = 0$. Otherwise $K \leq y - B/\ell + 1 + \log_\ell y$ by (Cd3). By (Ki1) of Lemma C.7 with $m = 1$ (for $\Phi\colon T_1 \to T_2$ and the window above; $a + 1 = c$), $\sigma_a \leq H(c) + c_H + 3 + H_{\max}/(\ell - 1)$, with $H(c) = s_c + S + \eta = B + \eta$; and $y \leq A_a + \eta \leq \sigma_a/(\ell - 1) + 1 + \eta$. Hence $K \leq \eta + 2.5\,\mathrm{Sg}$:

If $\eta \geq 1$ and $K > 0$, then $\beta = 1$: indeed $\ell s_a = \ell y \geq B \geq b$, and $\ell s_a \leq \sigma_a + \sigma_a/(\ell - 1) + \ell + \ell\eta$, which gives $0 \leq \ell s_a - b \leq 3Z$:

So $K \leq \eta\beta + 2.5\,\mathrm{Sg}$ in all cases, and the exponent is at most $8.5\,\mathrm{Sg} + \eta\beta$.

Conclusion. The four cases cover all $c \in C$ with $\mathrm{len}_c = 2$; each gives the conclusion with an integer exponent at most $12\,\mathrm{Sg} + \eta\beta$, hence at most E, and (a) raises it to E. □

Lemma C.33. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25; put $\mathbf{N}_\nu = \mathbf{N}(T_\nu)$ and $r_\nu = \dim_L V_\nu$. Let*

$c \in C$ with $\mathrm{len}_c = 2$ and $\epsilon_c = 1$, put $c' = c-2$, and suppose

$$\ell\,(s_{c-1} + S + \eta) + \eta < s_c.$$

Let $r \geq 0$ be an integer such that ℓ^r is killing at c for $\mathbf{N}(\Phi)$. Let $\mathbf{n} = (n_k)_{k \leq r_2}$ and $\mathbf{n}^{(1)} = (n^{(1)}_m)_{m \leq r_1}$ be frames at c' of $\mathbf{N}_2$ and $\mathbf{N}_1$, with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$ and all k, with crystalline frames (d_k) and $(d^{(1)}_m)$, let $\mu \in M_{r_1 \times r_2}(\mathcal{O}_L)$ be the matrix of $\mathbf{N}(\Phi)$ for $\mathbf{n}^{(1)}$ and $\mathbf{n}$, and let $k_0 \leq r_2$ with

$$\varphi^2(d_{k_0}) \in \varphi^4\big(\mathrm{Fil}^{\,s_c} D_{\mathrm{cris}}(V_2)\big).$$

Then there is $\mathbf{c} = (c_m)_{m \leq r_1} \in \mathcal{O}_L^{r_1}$ with

$$\mathbf{c}\,\mu = \ell^{2\sigma_B + r}\mathbf{e}_{k_0} \qquad \text{and} \qquad \sum_m c_m\,\varphi^2\big(d^{(1)}_m\big) \in \varphi^4\big(\mathrm{Fil}^{\,s_c} D_{\mathrm{cris}}(V_1)\big);$$

in particular $(1 \otimes \Phi)\big(\sum_m c_m d^{(1)}_m\big) = \ell^{2\sigma_B + r} d_{k_0}$.

Proof. Write $R = e_c \mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$, $Q = Q_0$, $b_0 = s_c$, $w' = s_{c-1}$, $y = w' + S + \eta$, and let π_0, q, ω, ω_1, R^Δ, $\mathcal{K}^\Delta$ and the average $f \mapsto \bar f$ be as in Lemma C.34. A *coordinate vector* (for $\mathbf{N}_\nu$) is an arc coordinate vector for $\mathbf{n}$ (resp. $\mathbf{n}^{(1)}$), $e = 2$, H_c and m_X. The site $c - 1 = c' + 1$ is thin ($c' \in C$) and $\epsilon_{c-1} = -1$ (signs alternate), so $w' \leq w$; $H_c(c) = h_B$, $H_c(c-1) = w + S$ and $\Pi = \Pi_c = Q^{h_B} Q_1^{w+S}$. Moreover $b_0 + S \leq h_B$ (if c is thick, $b_0 \leq b + \eta$ and $h_B = b + \eta + S$; if c is thin, $b_0 \leq w$ and $h_B \geq w + S$) and $\eta < w' \leq w$ (as $\eta < s_a$ for all a). By Lemma C.16 (1), the Hodge jumps of V_1 and V_2 at τ_x have absolute value at most $s_x + S$; by (Ow1), those of V_1 at τ_c lie in $[b_0, b_0 + S] \cup [-\eta, \eta]$ and those at τ_{c-1} in $[-w' - S, -w'] \cup [-\eta, \eta]$. Recall (Definition 23.3) that for $\mathbf{v} \in L^{r_\nu}$ the vector $\varphi^{-(s+4)}(\delta_2(\mathbf{v}))$ lies in the component $\varepsilon D_{\mathrm{cris}}(V_\nu)$ of τ_{c-s}, on which $\varepsilon\,\mathrm{Fil}^\bullet$ is the Hodge filtration at τ_{c-s}, and $\mathrm{Fil}^u D_{\mathrm{cris}}$ is stable under $F \otimes L$.

Step 1: realising the section. Put $g_2 = q^{-b_0}\varphi(q)^{w'+S} \in \mathcal{K}^\Delta$. By (Dv2), (Dv2') of Lemma C.34, $g_2 \in Q^{-b_0} Q_1^{w'+S} R$ and $\Pi g_2 \in Q^{h_B - b_0} Q_1^{w+S+w'+S} R \subseteq R$. The section decomposition $(g_2, \mathbf{e}_{k_0}, \alpha)$, $\alpha(0) = -b_0$, $\alpha(1) = w' + S$, shows that $g_2 \mathbf{e}_{k_0}$ is Hodge-graded: $\delta_2(\mathbf{e}_{k_0}) = \varphi^2(d_{k_0}) \in \varphi^4(\mathrm{Fil}^{b_0})$ by hypothesis, and $\varphi^{-5}(\varphi^2(d_{k_0}))$ lies in the component of τ_{c-1}, where V_2 has no Hodge jump below $-(w' + S)$, so it lies in $\mathrm{Fil}^{-(w'+S)}$. By Proposition C.23 (R1) ($\nu = 2$, the $\mu_{\ell-1}$-fixed frame $\mathbf{n}$) there is $y_2 \in \mathbf{N}_{2,c}$ such that $\ell^{\sigma_B} g_2 \mathbf{e}_{k_0}$ is a coordinate vector of y_2.

Step 2: lifting and coordinates. As ℓ^r is killing at c, $\ell^r y_2 = \mathbf{N}(\Phi)(x_0)$ with $x_0 \in \mathbf{N}_{1,c}$. By Lemma C.16 (e), $\Pi x_0 \in P_2(\mathbf{N}_{1,c'})$. By Lemma C.14 (with c', $e = 2$, H_c, $m_X \geq 2H_c(x)$ and $\sigma = \sigma_B \geq (2H_c(c'+1) + m_X/(\ell-1))\ell/(\ell-1)^2)$ there is a coordinate vector f of $\ell^{\sigma_B} x_0$ with $\Pi f_m \in R$, and by Lemma 23.13 (2) there are $t, \kappa \geq 0$ and a coordinate vector f' of $\ell^t x_0$ with $\ell^\kappa f'$ Hodge-graded.

Step 3: the two congruences. By Lemma C.29 (2), $f\mu$ is a coordinate vector of $\ell^{\sigma_B + r} y_2$, as is $\ell^{2\sigma_B + r} g_2 \mathbf{e}_{k_0}$ (Lemma C.17 item 1); and $\ell^t f$, $\ell^{\sigma_B} f'$ are coordinate vectors of $\ell^{t+\sigma_B} x_0$. By Lemma C.17 item 2,

$$f\mu - \ell^{2\sigma_B + r} g_2 \mathbf{e}_{k_0} \in (X\Pi R)^{r_2}, \qquad \ell^{t+\kappa} f = \sum_i g_i \mathbf{v}_i + X\Pi \mathbf{h},$$

where $\mathbf{h} \in R^{r_1}$, $g_i \in Q^{\alpha_i(0)} Q_1^{\alpha_i(1)} R$, $\mathbf{v}_i \in \mathcal{O}_L^{r_1}$,

$$\delta_2(\mathbf{v}_i) \in \varphi^4\big(\mathrm{Fil}^{-\alpha_i(0)} D_{\mathrm{cris}}(V_1)\big) \cap \varphi^5\big(\mathrm{Fil}^{-\alpha_i(1)} D_{\mathrm{cris}}(V_1)\big)$$

($\ell^{-\sigma_B} g_i$ forming a section decomposition of $\ell^\kappa f'$).

Step 4: bounded poles. If $\mathbf{v}_i \neq 0$ then $-\alpha_i(0) \leq b_0+S$ and $-\alpha_i(1) \leq \eta$. Indeed $\varphi^{-4}\delta_2(\mathbf{v}_i)$ lies in the component of τ_c and in $\mathrm{Fil}^{-\alpha_i(0)}$, and that component of Fil^u is 0 for $u > b_0+S$ (no jump above b_0+S); likewise $\varphi^{-5}\delta_2(\mathbf{v}_i)$ lies in the component of τ_{c-1}, whose Fil^u is 0 for $u > \eta$. If either inequality failed, $\delta_2(\mathbf{v}_i) = \varphi^2(\sum_m v_{i,m} d_m^{(1)}) = 0$; as φ is bijective on $D_{\mathrm{cris}}(V_1)$ and the $d_m^{(1)}$ are linearly independent over L (Lemma 23.10 item 1), $\mathbf{v}_i = 0$. We drop the terms with $\mathbf{v}_i = 0$.

Step 5: clearing the poles. Put $\Psi = q^{b_0+S}\varphi(q)^\eta \in R^\Delta$, so $\Psi \in Q^{b_0+S}Q_1^\eta R^\times$. By Step 4, $\Psi g_i \in Q^{b_0+S+\alpha_i(0)}Q_1^{\eta+\alpha_i(1)}R \subseteq R$, and $\Psi X\Pi\mathbf{h} \in R^{r_1}$; hence $\ell^{t+\kappa}\Psi f_m \in R$. Also $\Psi f_m = G^{-1}A_m$ with $G = Q^{h_B-b_0-S}Q_1^{w+S-\eta}$ (exponents ≥ 0) and $A_m = (\Psi\Pi^{-1}G)\Pi f_m \in R$. So $\ell^{t+\kappa}A_m \in GR$; as G is a monic polynomial, R/GR is ℓ-torsion free (reduce modulo a uniformiser in the domain $(\mathcal{O}_L/\varpi)[[X]]$ and induct), so $A_m \in GR$ and $\Psi f_m \in R$.

Step 6: averaging. Put $\Phi_m = \overline{\Psi f_m} = \Psi \bar{f}_m \in R^\Delta$. Averaging the relations of Step 3 multiplied by Ψ, by (Dv3) and (Dv2') (using $\Psi g_2 = q^S\varphi(q)^y$ and $\Psi X\Pi R = XQ^{h_B+b_0+S}Q_1^{w+S+\eta}R$):

$$\Phi\mu - \ell^{2\sigma_B+r}q^S\varphi(q)^y\mathbf{e}_{k_0} \in (\mathcal{E})^{r_2}, \qquad \ell^{t+\kappa}\Phi - \sum_i \Psi\bar{g}_i\,\mathbf{v}_i \in (\mathcal{E})^{r_1},$$
$$\mathcal{E} = \pi_0 q^{h_B+b_0+S}\varphi(q)^{w+S+\eta}R^\Delta,$$

with $\Psi\bar{g}_i \in q^{b_0+S+\alpha_i(0)}\varphi(q)^{\eta+\alpha_i(1)}R^\Delta$ (both exponents ≥ 0).

Step 7: reading the coefficient of q^J, $J = S+\ell y$. In $R^\Delta = \mathcal{O}_L[[q]]$ (Lemma C.34 (Dv1)) let λ_J be the coefficient of q^J. Every element of $q^N R^\Delta$ has $\lambda_J = 0$ if $N > J$. Since $J = S+\ell y < S+b_0 \leq h_B+b_0+S$ and $\mathcal{E} \subseteq q^{h_B+b_0+S}R^\Delta$, $\lambda_J(\mathcal{E}) = 0$. By (Dv2'), $\varphi(q)^y \equiv (\sum_{j\geq\ell} a_j q^j)^y$ modulo ϖ, so $\theta := \lambda_J(q^S\varphi(q)^y)$, the coefficient of $q^{\ell y}$ in $\varphi(q)^y$, satisfies $\theta \equiv a_\ell^y$ and is a unit of $\mathcal{O}_L$. Put $\mathbf{c} = \theta^{-1}(\lambda_J(\Phi_m))_m \in \mathcal{O}_L^{r_1}$. The first relation gives $\mathbf{c}\mu = \ell^{2\sigma_B+r}\mathbf{e}_{k_0}$. The second gives $\ell^{t+\kappa}\theta\,\mathbf{c} = \sum_i \lambda_J(\Psi\bar{g}_i)\mathbf{v}_i$, where $\lambda_J(\Psi\bar{g}_i) = 0$ unless $b_0+S+\alpha_i(0) \leq J$, i.e. $-\alpha_i(0) \geq b_0-\ell y \geq \eta+1$ (hypothesis). For the remaining i, $\varphi^{-4}\delta_2(\mathbf{v}_i) \in \mathrm{Fil}^{\eta+1}D_{\mathrm{cris}}(V_1)$, hence $z := \varphi^{-4}(\sum_m c_m\varphi^2(d_m^{(1)})) \in \mathrm{Fil}^{\eta+1}D_{\mathrm{cris}}(V_1)$, and z lies in the component of τ_c.

Step 8: the Hodge gap. On the component of τ_c the Hodge filtration of V_1 has no jump in $[\eta+1, b_0-1]$ (by (Ow1), as recalled at the start), so its $\mathrm{Fil}^{\eta+1}$ and Fil^{b_0} agree and $z \in \mathrm{Fil}^{b_0}D_{\mathrm{cris}}(V_1)$. Finally $(1\otimes\Phi)(d_m^{(1)}) = \sum_k \mu_{mk}d_k$ (Lemma C.29 (1)) gives $(1\otimes\Phi)(\sum_m c_m d_m^{(1)}) = \ell^{2\sigma_B+r}d_{k_0}$. □

Lemma C.34. *Let ℓ be an odd prime, L a finite extension of $\mathbb{Q}_\ell$, $R = \mathcal{O}_L[[X]]$ with field of fractions $\mathcal{K}$, and for $u \in \mathbb{Z}_\ell^\times$ let γ_u be the $\mathcal{O}_L$-algebra automorphism of R (and of $\mathcal{K}$) with $\gamma_u(X) = (1+X)^u - 1$. Call $f \in \mathcal{K}$ Δ-*invariant *if $\gamma_\delta(f) = f$ for every $\delta \in \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$, write R^Δ, $\mathcal{K}^\Delta$ for the Δ-invariant elements, and put $q = \pi_0 + \ell$.*

(Dv1) Variable *The $\mathcal{O}_L$-algebra homomorphism $\mathcal{O}_L[[T]] \to R$, $T \mapsto q$, is injective with image R^Δ; so is $T \mapsto \pi_0$. Every $f \in R^\Delta$ with $f(0) = 0$ lies in $\pi_0 R^\Delta$.*

(Dv2) The ideal of Q *$q = Q\omega$ with $\omega \in R^\times$. Consequently $Q^\alpha R \cap \mathcal{K}^\Delta = q^\alpha R^\Delta$ for every $\alpha \in \mathbb{Z}$, and $XQ^N R \cap R^\Delta = \pi_0 q^N R^\Delta$ for every integer $N \geq 0$.*

(Dv2') The ideal of Q_1 *$\varphi(q)$ is Δ-invariant and $\varphi(q) = Q_1\omega_1$ with $\omega_1 = \varphi(\omega) \in R^\times$ ($Q_1 = \varphi(Q)$, φ as in Definition 23.2). For $\alpha, \beta \in \mathbb{Z}$ and integers $N, M \geq 0$: $Q^\alpha Q_1^\beta R \cap \mathcal{K}^\Delta = q^\alpha\varphi(q)^\beta R^\Delta$ and $XQ^N Q_1^M R \cap R^\Delta = \pi_0 q^N \varphi(q)^M R^\Delta$. Writing $\varphi(q) = \sum_j a_j q^j$ by (Dv1), $a_j \in \varpi\mathcal{O}_L$ for $j < \ell$ and $a_\ell \in \mathcal{O}_L^\times$ (ϖ a uniformiser of $\mathcal{O}_L$).*

(Dv3) Averaging *For $\delta \in \mu_{\ell-1}$, integers $i \geq 0$ and $\alpha, \beta \in \mathbb{Z}$, $\gamma_\delta(X^i Q^\alpha Q_1^\beta R) = X^i Q^\alpha Q_1^\beta R$. Hence the average $\bar{f} = (\ell-1)^{-1} \sum_{\delta \in \mu_{\ell-1}} \gamma_\delta(f)$ of an element $f \in X^i Q^\alpha Q_1^\beta R$ lies in $X^i Q^\alpha Q_1^\beta R \cap \mathcal{K}^\Delta$, and $\bar{f} = f$ if f is Δ-invariant.*

Proof. Write $\mathfrak{m}$ for the maximal ideal of R, ϖ for a uniformiser of $\mathcal{O}_L$ and $\bar{h}$ for the reduction of $h \in R$ modulo ϖ. We apply Lemma 23.24 with $\Delta = \mu_{\ell-1}$, so $e = 1$, $\pi_\Delta = \pi_0$ and, by (Rr3), $q_\Delta = \pi_0 + \ell = q$. By (Rr1), π_0 is Δ-invariant and $\pi_0 = X^{\ell-1} v$ with $v \in \mathbb{Z}_\ell[[X]]^\times$; hence q is Δ-invariant and $\bar{q} = X^{\ell-1}\bar{v}$ has X-adic order exactly $\ell - 1$.

A divisibility criterion. *If $h \in R$ has $\bar{h}$ of X-adic order exactly $\ell-1$ and $h(\zeta - 1) = 0$ for every primitive ℓ-th root of unity ζ, then $h \in QR^\times$.* By Weierstrass preparation $h = Pu$ with $u \in R^\times$ and $P \in \mathcal{O}_L[X]$ distinguished of degree $\ell - 1$. For each of the $\ell - 1$ elements $x = \zeta - 1$ (of positive valuation in $\mathcal{O}_{L(\zeta)}$) the value $u(x)$ is a unit, so $P(x) = 0$. Thus the monic polynomial $Q = \Phi_\ell(1 + X)$, whose roots are these $\ell - 1$ distinct elements, divides P in $L(\zeta)[X]$, hence in $\mathcal{O}_L[X]$ (division by a monic polynomial); both are monic of degree $\ell - 1$, so $P = Q$.

(Dv2). By (Rr4) of Lemma 23.24, $q = q_\Delta = Q\omega$ with $\omega = w_\Delta \in \mathbb{Z}_\ell[[X]]^\times \subseteq R^\times$. If $f \in Q^\alpha R$ is Δ-invariant, then $q^{-\alpha} f = \omega^{-\alpha} Q^{-\alpha} f \in R$ and is Δ-invariant, so $f \in q^\alpha R^\Delta$; the converse inclusion is clear. If $f \in XQ^N R \cap R^\Delta$, then $g = q^{-N} f \in XR$ is Δ-invariant with $g(0) = 0$, so $g \in \pi_0 R^\Delta$ by (Dv1); conversely $\pi_0 q^N R^\Delta \subseteq XQ^N R$ as $\pi_0 \in XR$.

(Dv2'). The ring endomorphism φ of R ($X \mapsto (1+X)^\ell - 1$, identity on $\mathcal{O}_L$) commutes with every γ_u, since both sides send $1 + X$ to $(1+X)^{\ell u}$; so $\varphi(q)$ is Δ-invariant, and $\varphi(q) = \varphi(Q)\varphi(\omega) = Q_1\omega_1$ with $\omega_1(0) = \omega(0)$ a unit. The two equalities of ideals follow as in (Dv2), dividing by the Δ-invariant element $q^\alpha \varphi(q)^\beta \in Q^\alpha Q_1^\beta R^\times$ (resp. $q^N \varphi(q)^M$) and using (Dv1). For the coefficients: reduction modulo ϖ maps $R^\Delta = \mathcal{O}_L[[q]]$ to $(\mathcal{O}_L/\varpi)[[X]]$ by $\sum_j b_j q^j \mapsto \sum_j \bar{b}_j \bar{q}^j$, and as $\bar{q} = X^{\ell-1}\bar{v}$, the X-adic order of the image is $(\ell-1)\min\{j : \bar{b}_j \neq 0\}$. For $\varphi(q) = Q_1\omega_1$ the image is $\bar{Q}_1\bar{\omega}_1$, of X-adic order $(\ell-1)\ell$: indeed $Q_1 = \sum_{i<\ell}(1+X)^{i\ell}$ and $(1+X)^\ell \equiv 1 + X^\ell$ modulo ℓ, so $\bar{Q}_1 = ((1+X^\ell)^\ell - 1)/X^\ell = X^{(\ell-1)\ell}$. Hence $\bar{a}_j = 0$ for $j < \ell$ and $\bar{a}_\ell \neq 0$.

(Dv1). By (Rr2) of Lemma 23.24, $T \mapsto \pi_0$ is injective with image R^Δ, and $R^\Delta \cap XR = \pi_0 R^\Delta$; an $f \in R^\Delta$ with $f(0) = 0$ lies in XR, hence in $\pi_0 R^\Delta$. *The variable q.* Since $\ell \in \varpi\mathcal{O}_L$, the substitutions $T \mapsto T + \ell$ and $T \mapsto T - \ell$ are mutually inverse continuous automorphisms of $\mathcal{O}_L[[T]]$; composing with $T \mapsto \pi_0$ gives the assertion for $q = \pi_0 + \ell$.

(Dv3). $\gamma_\delta(X) = (1+X)^\delta - 1 = X(\delta + X(\dots)) \in XR^\times$, as $\delta \in \mathbb{Z}_\ell^\times$. Next, $\overline{\gamma_\delta(Q)} = \gamma_\delta(\bar{Q})$ with $\bar{Q} = X^{\ell-1}$, so $\overline{\gamma_\delta(Q)}$ has X-adic order $\ell - 1$; and for $x = \zeta - 1$ as above, $\gamma_\delta(Q)(x) = Q((1+x)^\delta - 1) = \Phi_\ell(\zeta^n) = 0$ with $n \equiv \delta$, $\ell \nmid n$. By the criterion $\gamma_\delta(Q) \in QR^\times$, and applying φ (which commutes with γ_δ and maps $R^\times$ to $R^\times$), $\gamma_\delta(Q_1) \in Q_1 R^\times$. Hence $\gamma_\delta(X^i Q^\alpha Q_1^\beta) \in X^i Q^\alpha Q_1^\beta R^\times$, and $\gamma_\delta(X^i Q^\alpha Q_1^\beta R) = X^i Q^\alpha Q_1^\beta R$ (the inclusion for γ_δ and $\gamma_{\delta^{-1}}$). As $\ell - 1 \in \mathcal{O}_L^\times$, the average of $f \in X^i Q^\alpha Q_1^\beta R$ lies in $X^i Q^\alpha Q_1^\beta R$, and it is Δ-invariant because $\delta \mapsto \gamma_\delta$ is a homomorphism on the group $\mu_{\ell-1}$. □

Lemma C.35. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25; put $\mathbf{N}_\nu = \mathbf{N}(T_\nu)$ and $r_\nu = \dim_L V_\nu$. Let $c \in C$, $c' = c - \mathrm{len}_c$ and $a = c' + 1$, and suppose that $\epsilon_a = 1$ and*

$$A_a + \eta + 1 \leq s_a, \qquad A_a = \lceil \sigma_a/(\ell-1) \rceil,$$

where σ_a is the killing index of $\mathbf{N}(\Phi)$ at a (formed for $\Phi|_{T_1} : T_1 \to T_2$). Let $\mathbf{n} = (n_k)_{k \leq r_2}$ and $\mathbf{n}^{(1)} = (n_m^{(1)})_{m \leq r_1}$ be frames at c' of $\mathbf{N}_2$ and $\mathbf{N}_1$, with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$ and all k, with crystalline frames (d_k) and $(d_m^{(1)})$, let $\mu \in M_{r_1 \times r_2}(\mathcal{O}_L)$ be the matrix of $\mathbf{N}(\Phi)$ for $\mathbf{n}^{(1)}$ and

$\mathbf{n}$, *and let* $k_0 \le r_2$ *with*

$$\varphi(d_{k_0}) \in \varphi^4\big(\mathrm{Fil}^{s_a} D_{\mathrm{cris}}(V_2)\big).$$

Then there is $\mathbf{c} = (c_m)_{m \le r_1} \in \mathcal{O}_L^{r_1}$ *with*

$$\mathbf{c}\,\mu = \ell^{2\sigma_B}\mathbf{e}_{k_0} \qquad \text{and} \qquad \sum_m c_m\,\varphi\big(d_m^{(1)}\big) \in \varphi^4\big(\mathrm{Fil}^{s_a} D_{\mathrm{cris}}(V_1)\big),$$

where $\mathbf{e}_{k_0}$ *is the* k_0*-th unit vector; in particular* $(1\otimes\Phi)\big(\sum_m c_m d_m^{(1)}\big) = \ell^{2\sigma_B} d_{k_0}$.

Proof. Write $R = e_a\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$, $\mathcal{K} = \mathcal{K}_a$, $H = H_c$, $\Pi' = \Lambda_{a,1}(H) = Q^{w+S}$ (as $H(c'+1) = w+S$), $A = A_a$, $s = s_a$, and let π_0, $q = \pi_0 + \ell$, R^Δ, $\mathcal{K}^\Delta$ and the average $f \mapsto \bar f$ be as in Lemma C.34. A *coordinate vector* (for $\mathbf{N}_\nu$) is an arc coordinate vector for the frame $\mathbf{n}$ (resp. $\mathbf{n}^{(1)}$), $e = 1$, H and m_X; this is the length-one initial sub-arc $c' \to a$ of the arc of c. We have $s \le w$ because $c'+1$ is thin ($c' \in C$), $m_X = (\ell-1)M > 2h_B \ge 2H(x)$ for every site x, and $\sigma_B \ge \big(2H(c'+1) + m_X/(\ell-1)\big)\ell/(\ell-1)^2$.

Step 1: coordinates. By Lemma C.17 (with c', $e=1$, H, m_X): coordinate vectors of one element agree modulo $X\Pi' R$ entrywise, and of is a coordinate vector of oz when f is one of z and $o \in R$.

Step 2: realising the pole. Put $g_2 = q^{-s} \in \mathcal{K}^\Delta$. By Lemma C.34 (Dv2), $q = Q\omega$ with $\omega \in R^\times$, so $\Pi' g_2 = Q^{w+S-s}\omega^{-s} \in R$, and $(g_2, \mathbf{e}_{k_0}, \alpha)$ with $\alpha(0) = -s$ is a section decomposition of $g_2\mathbf{e}_{k_0}$ with $\delta_1(\mathbf{e}_{k_0}) = \varphi(d_{k_0}) \in \varphi^4(\mathrm{Fil}^s D_{\mathrm{cris}}(V_2))$; so $g_2\mathbf{e}_{k_0}$ is Hodge-graded for $\mathbf{n}$ and $e=1$. By Proposition C.23 (R3) with $e_1 = 1$ (for $\nu = 2$ and the $\mu_{\ell-1}$-fixed frame $\mathbf{n}$) there is $y \in \mathbf{N}_{2,a}$ such that $\ell^{\sigma_B} g_2\mathbf{e}_{k_0}$ is a coordinate vector of y.

Step 3: killing the cokernel. By Lemma C.34 and Lemma 28.15 item 1, $\pi_0 = X^{\ell-1}v$ with v a unit, so by (Ki3) of Lemma C.7 π_0^A is killing at a: there is $x_0 \in \mathbf{N}_{1,a}$ with $\mathbf{N}(\Phi)(x_0) = \pi_0^A y$. By Lemma C.16 (1) every Hodge jump of V_1 at τ_a has absolute value at most $H(a)$, so by (Ex1) of Proposition 23.7 and (Wm2), $\Pi'\mathbf{N}_{1,a} = Q^{H(a)}\mathbf{N}_{1,a} \subseteq \mathbf{A}_{F,L}\varphi(\mathbf{N}_{1,c'}) = P_1(\mathbf{N}_{1,c'})$. Hence Lemma C.14 (with $e=1$, H, m_X, $\sigma = \sigma_B$) gives a coordinate vector f of $\ell^{\sigma_B}x_0$ (for $\mathbf{N}_1$) with $\Pi' f_m \in R$, and Lemma 23.13 (2) gives $t, \kappa \ge 0$ and a coordinate vector f' of $\ell^t x_0$ with $\ell^\kappa f'$ Hodge-graded.

Step 4: the two congruences. By Lemma C.29 (2), $f\mu$ is a coordinate vector (for $\mathbf{N}_2$) of $\mathbf{N}(\Phi)(\ell^{\sigma_B}x_0) = \ell^{\sigma_B}\pi_0^A y$, and so is $\ell^{2\sigma_B}\pi_0^A g_2\mathbf{e}_{k_0}$ by Steps 1 and 2. Also $\ell^t f$ and $\ell^{\sigma_B} f'$ are coordinate vectors of $\ell^{t+\sigma_B}x_0$. By Step 1,

$$f\mu - \ell^{2\sigma_B}\pi_0^A q^{-s}\mathbf{e}_{k_0} \in (X\Pi' R)^{r_2}, \qquad \ell^{t+\kappa} f = \sum_i g_i\mathbf{v}_i + X\Pi'\mathbf{h}, \tag{C.1}$$

where $(\ell^{-\sigma_B}g_i, \mathbf{v}_i, \alpha_i)_i$ is a section decomposition of $\ell^\kappa f'$ with $\delta_1(\mathbf{v}_i) \in \varphi^4(\mathrm{Fil}^{-\alpha_i(0)} D_{\mathrm{cris}}(V_1))$, so $g_i \in Q^{\alpha_i(0)}R$, and $\mathbf{h} \in R^{r_1}$.

Step 5: averaging over Δ. Apply the average entrywise. Since π_0, q are Δ-invariant and μ, $\mathbf{v}_i$ are constant, Lemma C.34 (Dv3) turns (C.1) into the same relations for $\bar f$, $\bar g_i \in Q^{\alpha_i(0)}R \cap \mathcal{K}^\Delta$ and $\overline{X\Pi'\mathbf{h}} \in (XQ^{w+S}R \cap \mathcal{K}^\Delta)^{r_1}$, and $\bar f_m \in Q^{-(w+S)}R \cap \mathcal{K}^\Delta$. Multiply by q^{w+S} and put $\Phi_m = q^{w+S}\bar f_m$. By (Dv2), $\Phi_m \in R^\Delta$, $q^{w+S}\bar g_i \in q^{w+S+\alpha_i(0)}R^\Delta$, and the two error terms lie in $XQ^{2(w+S)}R \cap R^\Delta = \pi_0 q^{2(w+S)}R^\Delta$:

$$\Phi\mu - \ell^{2\sigma_B}(q-\ell)^A q^{w+S-s}\mathbf{e}_{k_0} \in \big(q^{2(w+S)}R^\Delta\big)^{r_2}, \qquad \ell^{t+\kappa}\Phi - \sum_i q^{w+S}\bar g_i\mathbf{v}_i \in \big(q^{2(w+S)}R^\Delta\big)^{r_1}. \tag{C.2}$$

Step 6: reading one coefficient. By (Dv1), $R^\Delta = \mathcal{O}_L[[q]]$, a subring of the field $L((q))$ of Laurent series, which also contains $q^j R^\Delta$ for $j < 0$; let λ_J be the coefficient of q^J, an L-linear

map $L((q)) \to L$ with $\lambda_J(\mathcal{O}_L[[q]]) \subseteq \mathcal{O}_L$, for

$$J = A + w + S - s.$$

Since $s \le w$ and $A < s$, $0 \le J < w + S \le 2(w + S)$, so λ_J kills $q^{2(w+S)}R^\Delta$. Put $c_m = \lambda_J(\Phi_m) \in \mathcal{O}_L$. The first relation of (C.2) gives $\mathbf{c}\mu = \ell^{2\sigma_B}\lambda_J((q-\ell)^A q^{w+S-s})\mathbf{e}_{k_0} = \ell^{2\sigma_B}\mathbf{e}_{k_0}$, as the coefficient of q^A in $(q-\ell)^A$ is 1. The second gives

$$\ell^{t+\kappa}\mathbf{c} = \sum_i \lambda_J(q^{w+S}\bar{g}_i)\mathbf{v}_i,$$

and $\lambda_J(q^{w+S}\bar{g}_i) = 0$ unless $w + S + \alpha_i(0) \le J$, i.e. $-\alpha_i(0) \ge s - A \ge \eta + 1$. For the remaining i, $\delta_1(\mathbf{v}_i) \in \varphi^4(\mathrm{Fil}^{-\alpha_i(0)}D_{\mathrm{cris}}(V_1)) \subseteq \varphi^4(\mathrm{Fil}^{\eta+1}D_{\mathrm{cris}}(V_1))$. As δ_1 is L-linear and $\mathrm{Fil}^{\eta+1}$ is an L-subspace, $z := \varphi^{-4}(\sum_m c_m\varphi(d_m^{(1)})) \in \mathrm{Fil}^{\eta+1}D_{\mathrm{cris}}(V_1)$.

Step 7: the Hodge gap. By Definition 23.3, z lies in the component $\varepsilon_{a-1}D_{\mathrm{cris}}(V_1)$ of the embedding τ_a, on which $\varepsilon_{a-1}\mathrm{Fil}^\bullet D_{\mathrm{cris}}(V_1)$ is the Hodge filtration of V_1 at τ_a ($\mathrm{Fil}^u D_{\mathrm{cris}}$ is stable under $F \otimes L$, so $z = \varepsilon_{a-1}z \in \varepsilon_{a-1}\mathrm{Fil}^{\eta+1}$). By (Ow1) with $\epsilon_a = 1$, the Hodge jumps of V_1 at τ_a lie in $[s, s+S] \cup [-\eta, \eta]$, so this filtration has no jump in $[\eta+1, s-1]$ and $\varepsilon_{a-1}\mathrm{Fil}^{\eta+1} = \varepsilon_{a-1}\mathrm{Fil}^s$. Hence $z \in \mathrm{Fil}^s D_{\mathrm{cris}}(V_1)$, i.e. $\sum_m c_m\varphi(d_m^{(1)}) \in \varphi^4(\mathrm{Fil}^s D_{\mathrm{cris}}(V_1))$. Finally, by Lemma C.29 (1), $(1 \otimes \Phi)(d_m^{(1)}) = \sum_k \mu_{mk}d_k$, so $(1 \otimes \Phi)(\sum_m c_m d_m^{(1)}) = \sum_k(\mathbf{c}\mu)_k d_k = \ell^{2\sigma_B}d_{k_0}$. □

Lemma C.36. *Let ℓ be a prime, L a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_L$ and uniformiser ϖ, $R = \mathcal{O}_L[[X]]$ and $\mathcal{K}$ its field of fractions.*

(Cd1) Descent *Let $A, N \in R$ with $N \notin \varpi R$, and let $K \ge 0$ be an integer with*

$$A \notin \varpi\ell^K R + NR.$$

Let $f \in \mathcal{K}$ with $Af \in R$, and suppose there are an integer $\kappa \ge 0$ and $x \in \mathcal{O}_L$ with $A(\ell^\kappa f - x) \in NR$. Then $\ell^K x \in \ell^\kappa\mathcal{O}_L$; with $x_2 \in \mathcal{O}_L$ defined by $\ell^\kappa x_2 = \ell^K x$ and $f_2 = \ell^K f - x_2$,

$$\ell^K f = x_2 + f_2, \qquad Af_2 \in NR.$$

Moreover, if $N' \in R$, $\kappa' \ge 0$ and $x' \in \mathcal{O}_L$ satisfy $A(\ell^{\kappa'}f - x') \in N'R$, then $A(\ell^{\kappa'}f_2 - (\ell^K x' - \ell^{\kappa'}x_2)) \in N'R$; and if γ is an $\mathcal{O}_L$-algebra automorphism of R (extended to $\mathcal{K}$) with $\gamma(f) = f$, then $\gamma(f_2) = f_2$.

(Cd2) Pole at level s, congruence at level $s+1$ *Let $s \ge 0$, $y \ge 1$ and $b \ge 0$ be integers, and put $K_\uparrow(b, y) = 0$ if $b < \ell y$, and*

$$K_\uparrow(b, y) = a - y + 1 + v_\ell\binom{a}{y-1}, \qquad a = \lceil b/\ell \rceil,$$

if $b \ge \ell y$. Then $Q_s^b \notin \varpi\ell^{K_\uparrow(b,y)}R + Q_{s+1}^y R$. If moreover $b \ge 1$ and $b < \ell(y + \eta)$ for an integer $\eta \ge 0$, then $K_\uparrow(b, y) \le \eta + 1 + \log_\ell b$.

(Cd3) Pole at level $s+1$, congruence at level s *Let $s \ge 0$ and $y, B \ge 1$ be integers, and put $K_\downarrow(y, B) = 0$ if $\ell y < B$, and*

$$K_\downarrow(y, B) = y - i_1 + v_\ell\binom{y}{i_1}, \qquad i_1 = \lceil B/\ell \rceil - 1,$$

if $B \le \ell y$. Then $Q_{s+1}^y \notin \varpi\ell^{K_\downarrow(y,B)}R + Q_s^B R$, and $K_\downarrow(y, B) \le \max(0,\, y - B/\ell + 1 + \log_\ell y)$.

Proof. Write v_ϖ for the normalised valuation of L, $e_L = v_\varpi(\ell)$, $k = \mathcal{O}_L/\varpi$, and $\bar{h} \in k[[X]]$ for the reduction of $h \in R$. We have $\varpi\ell^K R = \varpi^{e_L K+1} R$.

Three facts on R.

(i) *If $G \in R \setminus \varpi R$, $j \geq 0$ and $h \in R$ with $Gh \in \varpi^j R$, then $h \in \varpi^j R$; hence $\varpi^j h \in GR$ implies $h \in GR$.* Induction on j, the case $j = 0$ being trivial. If $Gh \in \varpi^j R$ with $j \geq 1$, then $\bar{G}\bar{h} = 0$ in the domain $k[[X]]$ with $\bar{G} \neq 0$, so $h = \varpi h'$; as R is a domain, $Gh' \in \varpi^{j-1}R$ and $h' \in \varpi^{j-1}R$ by induction. For the second assertion write $\varpi^j h = G\beta$; then $\beta = \varpi^j\beta'$ and $h = G\beta'$.

(ii) *Let $G \in \mathcal{O}_L[X]$ be a distinguished polynomial (monic, the other coefficients in $\varpi\mathcal{O}_L$) of degree $d \geq 1$. If $c \in \mathcal{O}_L$ and $c \in \varpi^j R + GR$, then $v_\varpi(c) \geq j$.* Write $c = \varpi^j\alpha + G\beta$ and, by Weierstrass division, $\alpha = Gq + r$ with $r \in \mathcal{O}_L[X]$ of degree $< d$. Then $c - \varpi^j r = G(\beta + \varpi^j q)$ is a polynomial of degree $< d$; by the uniqueness in Weierstrass division it is its own remainder, so $\beta + \varpi^j q = 0$ and $c = \varpi^j r$, whence $c = \varpi^j r(0)$ with $r(0) \in \mathcal{O}_L$.

(iii) *Q_s is a distinguished polynomial of degree $(\ell-1)\ell^s$ with $Q_s(0) = \ell$, and $Q_s^\ell = Q_{s+1} + \ell W_s$ with $W_s \in R^\times$.* Indeed $Q_s = \sum_{i=0}^{\ell-1}(1+X)^{i\ell^s} \in \mathbb{Z}[X]$ is monic of degree $(\ell-1)\ell^s$ with $Q_s(0) = \ell$; in $\mathbb{F}_\ell[X]$, $(1+X)^{\ell^s} = 1 + X^{\ell^s}$, so $Q_s \equiv \big((1+X^{\ell^s})^\ell - 1\big)/X^{\ell^s} \equiv X^{(\ell-1)\ell^s}$. Hence $Q_s^\ell \equiv X^{(\ell-1)\ell^{s+1}} \equiv Q_{s+1}$ modulo ℓ, i.e. $Q_s^\ell - Q_{s+1} = \ell W_s$ with $W_s \in \mathbb{Z}[X]$, and evaluating at 0 gives $W_s(0) = \ell^{\ell-1} - 1 \in \mathbb{Z}_\ell^\times$, so W_s is a unit of R.

An expansion bound. *Let G be as in (ii), $B \geq 1$, $0 \leq m_0 < m_1 < \cdots < m_n$ integers with $m_0 < B$, $c_i \in \mathcal{O}_L \setminus \{0\}$ and $u_i \in R^\times$, and $H = \sum_{i=0}^n c_i u_i G^{m_i}$. Let $\mu = \min\{v_\varpi(c_i) : m_i < B\}$. Then $H \notin \varpi^{\mu+1}R + G^B R$.* Suppose the contrary. Let i_0 be the least index with $m_{i_0} < B$ and $v_\varpi(c_{i_0}) = \mu$. For $i < i_0$ we have $m_i < B$ and $v_\varpi(c_i) > \mu$, so these terms lie in $\varpi^{\mu+1}R$; the terms with $m_i \geq B$ lie in $G^B R$. Hence $G^{m_{i_0}} Z \in \varpi^{\mu+1}R + G^B R$, where

$$Z = c_{i_0}u_{i_0} + \sum_{i>i_0,\ m_i<B} c_i u_i G^{m_i - m_{i_0}} \ \in \ c_{i_0}u_{i_0} + GR.$$

Write $G^{m_{i_0}} Z = \varpi^{\mu+1}\alpha + G^B\beta$. Then $G^{m_{i_0}}(Z - G^{B-m_{i_0}}\beta) \in \varpi^{\mu+1}R$, and by (i) applied m_{i_0} times ($G \notin \varpi R$ as G is monic) $Z - G^{B-m_{i_0}}\beta \in \varpi^{\mu+1}R$. As $B - m_{i_0} \geq 1$, $Z \in \varpi^{\mu+1}R + GR$, so $c_{i_0}u_{i_0} \in \varpi^{\mu+1}R + GR$ and, multiplying by $u_{i_0}^{-1}$, $c_{i_0} \in \varpi^{\mu+1}R + GR$. By (ii), $v_\varpi(c_{i_0}) \geq \mu + 1$, a contradiction.

(Cd1). Write $A(\ell^\kappa f - x) = N\beta$ with $\beta \in R$. First, $\ell^K x \in \ell^\kappa \mathcal{O}_L$: this is clear if $x = 0$. Otherwise let $a = v_\varpi(x)$, $x = \varpi^a u$ with $u \in \mathcal{O}_L^\times$, and suppose $a + e_L K < e_L\kappa$. From $\varpi^a uA = \ell^\kappa(Af) - N\beta$ and $Af \in R$ we get $\varpi^a uA = \varpi^{e_L\kappa}\alpha - N\beta$ for some $\alpha \in R$; so $N\beta \in \varpi^a R$, hence $\beta = \varpi^a\beta'$ by (i), and $uA = \varpi^{e_L\kappa - a}\alpha - N\beta'$. Thus $A \in \varpi^{e_L\kappa-a}R + NR \subseteq \varpi^{e_L K+1}R + NR = \varpi\ell^K R + NR$, as $e_L\kappa - a \geq e_L K + 1$; this contradicts the hypothesis. So $x_2 \in \mathcal{O}_L$ is defined, and $\ell^K f = x_2 + f_2$ by definition. Next,

$$\ell^\kappa(Af_2) = \ell^K\,\ell^\kappa Af - \ell^\kappa x_2 A = \ell^K A(\ell^\kappa f - x) = \ell^K N\beta \in NR,$$

and $Af_2 = \ell^K Af - x_2 A \in R$; by (i) (with $\varpi^{e_L\kappa}$ in place of ℓ^κ, which differs by a unit) $Af_2 \in NR$. For the second congruence,

$$A\big(\ell^{\kappa'} f_2 - \ell^K x' + \ell^{\kappa'} x_2\big) = A\big(\ell^{K+\kappa'} f - \ell^K x'\big) = \ell^K A(\ell^{\kappa'} f - x') \in N'R.$$

Finally $\gamma(f_2) = \ell^K\gamma(f) - \gamma(x_2) = \ell^K f - x_2 = f_2$, since γ fixes $\mathcal{O}_L$.

(Cd2). If $b < \ell y$, suppose $Q_s^b \in \varpi R + Q_{s+1}^y R$; reducing modulo ϖ and using (iii), $X^{(\ell-1)\ell^s b} \in X^{(\ell-1)\ell^{s+1}y}k[[X]]$, impossible as $(\ell-1)\ell^s b < (\ell-1)\ell^{s+1}y$. Let $b \geq \ell y$ and $a = \lceil b/\ell \rceil$, so $\ell a \geq b$

and $a \geq y$. Since $Q_s^{\ell a} = Q_s^{\ell a - b} Q_s^b$, it suffices to show $Q_s^{\ell a} \notin \varpi \ell^K R + Q_{s+1}^y R$ with $K = K_\uparrow(b, y)$. By (iii) and the binomial theorem,

$$Q_s^{\ell a} = (Q_{s+1} + \ell W_s)^a = \sum_{i=0}^{a} \binom{a}{i} \ell^{a-i} W_s^{a-i} Q_{s+1}^i.$$

Apply the expansion bound with $G = Q_{s+1}$ (distinguished by (iii)), $m_i = i$, $c_i = \binom{a}{i}\ell^{a-i} \neq 0$, $u_i = W_s^{a-i}$ and $B = y$: since $y - 1 \leq a$, $\mu \leq v_\varpi(c_{y-1}) = e_L(a - y + 1 + v_\ell\binom{a}{y-1}) = e_L K$, so $Q_s^{\ell a} \notin \varpi^{\mu+1} R + Q_{s+1}^y R \supseteq \varpi^{e_L K + 1} R + Q_{s+1}^y R$. For the bound, let $1 \leq b < \ell(y + \eta)$. If $b < \ell y$ then $K_\uparrow = 0 \leq \eta + 1 + \log_\ell b$. Otherwise $a = \lceil b/\ell \rceil \leq y + \eta$ (as $b/\ell < y + \eta$), so $a - y + 1 \leq \eta + 1$, and $1 \leq a \leq b$. By Legendre's formula $v_\ell\binom{a}{j} = \sum_{t \geq 1} (\lfloor a/\ell^t \rfloor - \lfloor j/\ell^t \rfloor - \lfloor (a-j)/\ell^t \rfloor)$ for $0 \leq j \leq a$; every summand lies in $\{0, 1\}$ and vanishes when $\ell^t > a$, so

$$v_\ell \binom{a}{j} \leq \lfloor \log_\ell a \rfloor \qquad (a \geq 1,\ 0 \leq j \leq a). \tag{C.3}$$

Hence $K_\uparrow(b, y) \leq \eta + 1 + \log_\ell a \leq \eta + 1 + \log_\ell b$.

(Cd3). If $\ell y < B$, suppose $Q_{s+1}^y \in \varpi R + Q_s^B R$; modulo ϖ this gives $X^{(\ell-1)\ell^{s+1} y} \in X^{(\ell-1)\ell^s B} k[[X]]$, impossible as $\ell y < B$. Let $B \leq \ell y$; then $i_1 = \lceil B/\ell \rceil - 1$ satisfies $0 \leq i_1 \leq y - 1$ and $\ell i_1 < B$. By (iii),

$$Q_{s+1}^y = (Q_s^\ell - \ell W_s)^y = \sum_{i=0}^{y} \binom{y}{i} (-\ell)^{y-i} W_s^{y-i} Q_s^{\ell i}.$$

Apply the expansion bound with $G = Q_s$, $m_i = \ell i$, $c_i = \binom{y}{i}(-\ell)^{y-i} \neq 0$, $u_i = W_s^{y-i}$ and B ($m_0 = 0 < B$): $\mu \leq v_\varpi(c_{i_1}) = e_L(y - i_1 + v_\ell\binom{y}{i_1}) = e_L K_\downarrow(y, B)$, which gives the assertion as in (Cd2). Since $i_1 \geq B/\ell - 1$ and, by (C.3), $v_\ell\binom{y}{i_1} \leq \log_\ell y$, we get $K_\downarrow(y, B) \leq y - B/\ell + 1 + \log_\ell y$. □

C.1.4 Lifting graded sections along arcs of length three

This subsection completes Proposition C.25 with the arcs of length three. An arc whose first site is up is treated by Proposition C.37. An arc whose first site has sign -1 (Proposition C.39) takes most of the subsection: it rests on filtered constants by a content bound (Lemma C.40) and on a scalar-multiple lift in the band case (Proposition C.41), whose proof uses a near-horizontal pull-back vector at the up site and its first-level lift (Proposition C.43, Proposition C.44), a Frobenius first lift pushed to the last site (Proposition C.48), and the resulting section vector (Proposition C.49). The lift of a Frobenius-square vector by separation of Γ-weights (Lemma C.45) and the letter condition (Lemma C.46) proved here are used again in Section C.2.

Proposition C.37. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25, and suppose $160\,\mathrm{Sg} \leq b$. Then there is an integer E with $0 \leq E \leq 40\,\mathrm{Sg} + \eta\,\beta_{320}$ such that the following holds for every $c \in C$ with $\mathrm{len}_c = 3$ and $\epsilon_{c-2} = 1$ (write $e = 3$). Let $\mathbf{n} = (n_k)_{k \leq r}$ be a frame at c' of $\mathbf{N}_2$ with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1}$, with crystalline frame (d_k), let $k_0 \leq r$ be an index with $\varphi^e(d_{k_0}) \in \varphi^{s+4}(\mathrm{Fil}^{s_{c-s}} D_{\mathrm{cris}}(V_2))$ for every $s < e$ with $\epsilon_{c-s} = 1$, and let g be admissible at c. Then there are $w_1 \in I_c$ (formed for $\mathbf{N}_1$), a finite product λ' of the Q_s ($s \geq 0$) and $z \in X\mathbf{N}_{2,c}$ such that $\lambda' w_1 \in \mathbf{N}_1$, $\Pi_c\,\mathbf{N}(\Phi)(w_1) \in P_e(\mathbf{N}_{2,c'})$, and $\ell^E g\,\mathbf{e}_{k_0}$ is an arc coordinate vector of $\mathbf{N}(\Phi)(w_1) - z$ for $\mathbf{n}$, e, H_c and m_X.*

Proof. Write $R_c = e_c\mathbf{A}_{F,L}$, $\psi = \mathbf{N}(\Phi)$, $\beta = \beta_{320}$, $e = 3$, and *coordinate vector* for arc coordinate vector for $\mathbf{N}_2$, $\mathbf{n}$, e, H_c, m_X ($m_X > 2h_B \geq 2H_c(x)$, so Lemma C.17 applies).

Exponents and combining. Let $E_C, E_N \leq 6\,\mathrm{Sg}$ be the integers of Proposition C.27 and Lemma C.30, $E_0 = \max(E_C, E_N)$; $\sigma_B \leq 3\,\mathrm{Sg}$.

Let (M_a, h_a) be the window of Lemma C.9, $H(x) = s_x + S + \eta$, and $H_{\max}$, c_H formed with it as in Proposition C.6. Put $E = \lfloor 13\,\mathrm{Sg} \rfloor + \eta\beta \leq 40\,\mathrm{Sg} + \eta\beta_{320}$. We combine data by three rules, each a direct check: (a) if the conclusion holds for g with an exponent $E'' \leq E$ (data w_1, λ', z), it holds with E (data $\ell^{E-E''}w_1, \lambda', \ell^{E-E''}z$), because I_c is an $\mathcal{O}_L$-module (Lemma 24.5 (Ry4)) and coordinate vectors are R_c-linear (Lemma C.17 item 1); (b) data for g_1 and g_2 with the same exponent add to data for $g_1 + g_2$ (sum of the w_j and of the z_j, product of the λ'_j; Lemma C.17 item 1); (c) if $\ell^K g = x_2 + f_2$ with $x_2 \in \mathcal{O}_L$ and $\nu_c f_2 \in \nu_c^+ R_c$, the first part of Proposition C.27 for the constant x_2 (with $\kappa = 0$, $x = x_2$ at every level) and Lemma C.30 for f_2 (its hypothesis on k_0 is the one of the statement) give, by (a) and (b), the conclusion for g with the exponent $E_0 + K$.

The arc. Let $c \in C$ with $\mathrm{len}_c = 3$ and $\epsilon_{c-2} = 1$, and $\mathbf{n}$, k_0, g as in the statement. As $c-1, c-2 \notin C$, the sites c and $c-1$ are thick; as $c' = c-3 \in C$, the site $a := c-2 = c'+1$ is thin. Signs alternate, so $\epsilon_a = 1$, $\epsilon_{c-1} = -1$, $\epsilon_c = 1$. Hence $h = h_B = b + \eta + S$, and with $b_0 = s_c \leq b + \eta$, $t = s_a$ ($\eta < t \leq w$) and $B = s_{c-1} + S$ ($b \leq B \leq h$):

$$\nu_c = Q_0^{b_0} Q_2^t, \qquad \nu_c^+ = Q_1^B.$$

By admissibility, $\gamma_\delta(g) = g$ for $\delta \in \mu_{\ell-1}$, $\nu_c g \in R_c$, and $\nu_c(\ell^\kappa g - x) \in Q_1^B R_c$ for some $\kappa \geq 0$, $x \in \mathcal{O}_L$. Also $H_{\max} \leq h + \eta \leq 1.05h$ and $c_H \leq 2 + \log_\ell(2h)$; we use $\log_\ell(2h) \leq \mathrm{Sg} - 1$, $\mathrm{Sg} \geq 1$ and $160\,\mathrm{Sg} \leq b$.

The mixed descent. Let $a_0 = \lceil b_0/\ell \rceil$, $X_0 = t - (B - a_0)/\ell + 1 + \log_\ell t$ and $K = \lceil \frac{\ell}{\ell-1} \max(0, X_0) \rceil$. Then $a_0 + K < B$:

By Lemma C.38 item 1, $\nu_c \notin \varpi\ell^K R_c + Q_1^B R_c$, and $Q_1^B \notin \varpi R_c$ (Q_1 is monic). So (Cd1) of Lemma C.36 (with $A = \nu_c$, $N = Q_1^B$, $f = g$) gives $x_2 \in \mathcal{O}_L$ and $f_2 = \ell^K g - x_2$ with $\nu_c f_2 \in \nu_c^+ R_c$.

Case 1: $A_a + \eta + 1 \leq s_a$, $A_a = \lceil \sigma_a/(\ell-1) \rceil$, σ_a the killing index of ψ at a. Let s_{fr} and h_1 be as in Lemma C.38 item 2 for (t, B, η), put $s_i = \min(K, s_{\mathrm{fr}})$ and $h_\star = \ell^{s_{\mathrm{fr}} - s_i} h_1$; so $h_\star$ is Δ-invariant, $Q_2^t h_\star \in R_c$ and $Q_2^t(\ell^{s_i} h_\star - 1) \in Q_1^B Q_0^\eta R_c$. Put $e_4 = K - s_i$, $f_a = x_2 h_\star$ and $f_3 = \ell^{e_4} g - f_a$. *The part f_3.* We have $\nu_c f_3 \in R_c$ and $\ell^{s_i}\nu_c f_3 = \nu_c(\ell^K g - x_2) - x_2 Q_0^{b_0} Q_2^t(\ell^{s_i} h_\star - 1) = \nu_c f_2 - x_2 Q_0^{b_0} Q_2^t(\ell^{s_i} h_\star - 1) \in Q_1^B R_c$. Writing $\ell^{s_i}\nu_c f_3 = Q_1^B \beta'$, the element $Q_1^B \beta'$ lies in $\ell^{s_i} R_c$; reducing modulo a uniformiser in the domain $(\mathcal{O}_L/\varpi)[[X]]$ repeatedly ($Q_1^B \notin \varpi R_c$) gives $\beta' \in \ell^{s_i} R_c$, so $\nu_c f_3 \in \nu_c^+ R_c$. By Lemma C.30 the conclusion holds for f_3 with E_N. *The part f_a (filtered constants with $U = \{2\}$).* Let $\mathbf{n}^{(1)}$ be a frame at c' of $\mathbf{N}_1$ fixed by γ_u, $u \in \mu_{\ell-1}$ (Lemma 24.15; $\mathbf{N}_1$ is a Wach module by (Ex1) of Proposition 23.7), with crystalline frame $(d_m^{(1)})$, and μ the matrix of ψ for $\mathbf{n}^{(1)}$, $\mathbf{n}$. The hypothesis on k_0 at $s = 2$ gives $\varphi(d_{k_0}) \in \varphi^4(\mathrm{Fil}^{s_a} D_{\mathrm{cris}}(V_2))$ (φ bijective), so Lemma C.35 (for c, $a = c'+1$) gives $\mathbf{c}$ with $\mathbf{c}\mu = \ell^{2\sigma_B}\mathbf{e}_{k_0}$ and $\sum_m c_m \varphi(d_m^{(1)}) \in \varphi^4(\mathrm{Fil}^{s_a} D_{\mathrm{cris}}(V_1))$; applying the L-linear φ^2, $\sum_m c_m \varphi^3(d_m^{(1)}) \in \varphi^{2+4}(\mathrm{Fil}^{s_{c-2}} D_{\mathrm{cris}}(V_1))$. The function f_a is Δ-invariant, $\nu_{\{2\}} f_a = x_2 Q_2^t h_\star \in R_c$, and $\nu_{\{2\}}(\ell^{s_i} f_a - x_2) = x_2 Q_2^t(\ell^{s_i} h_\star - 1)$ lies in $Q_1^B R_c$ ($s = 1$, $t_1 = B$) and in $Q_0^\eta R_c$ ($s = 0$, $t_0 = \eta$). So the clause "Filtered constants" of Proposition C.27 with $U = \{2\}$, $E' = 2\sigma_B$ gives the conclusion for f_a with $3\sigma_B$. By (a), (b) with $E_1 = \max(E_N, 3\sigma_B) \leq 9\,\mathrm{Sg}$, the conclusion holds for $f_3 + f_a = \ell^{e_4} g$ with E_1, i.e. for g with $E_1 + e_4$. If $s_i = K$, $e_4 = 0$. Otherwise $s_i = s_{\mathrm{fr}} \geq t - B/\ell - \eta/\ell - 1$, and if $X_0 \leq 0$ then $K = 0$; so $e_4 \leq 3.3\,\mathrm{Sg}$:

The exponent is at most $12.3\,\mathrm{Sg}$.

Case 2: $s_a \leq A_a + \eta$**.** By (c), the conclusion holds with $E_0 + K$. By (Ki1) of Lemma C.7 with $m = 1$ (for $\Phi\colon T_1 \to T_2$ and the window above; $a+1 = c-1$), $\sigma_a \leq H(c-1) + c_H + 3 + H_{\max}/(\ell-1)$

with $H(c-1) = B + \eta$, and $t \le \sigma_a/(\ell-1) + 1 + \eta$. Hence $K \le \eta + 3.9\,\mathrm{Sg}$:

If $\eta \ge 1$ and $K > 0$, then $\beta = 1$: $X_0 > 0$ gives $\ell s_a > B - a_0 - \ell(1 + \log_\ell t)$, so $b - \ell s_a < h/\ell + 1 + \ell + \ell \log_\ell(2h) \le 2Z$, and $\ell s_a \le \sigma_a + \sigma_a/(\ell-1) + \ell + \ell\eta$ gives $\ell s_a - b \le 3Z$, where $Z = \ell\eta + \ell S + b/\ell + s_a + \ell \log_\ell(2h)$; and the thin site a has the thick neighbour $c-1$.

So $K \le \eta\beta + 3.9\,\mathrm{Sg}$, and the exponent is at most $9.9\,\mathrm{Sg} + \eta\beta$.

Conclusion. The two cases cover all $c \in C$ with $\mathrm{len}_c = 3$ and $\epsilon_{c-2} = 1$; each gives an integer exponent at most $13\,\mathrm{Sg} + \eta\beta$, hence at most E, and (a) raises it to E. □

Lemma C.38. *Let ℓ be an odd prime, L a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_L$ and uniformiser ϖ, $R = \mathcal{O}_L[[X]]$ and $\mathcal{K}$ its field of fractions.*

1. Mixed pole. *Let $b_0 \ge 0$, $t \ge 1$, $B \ge 1$ and $K \ge 0$ be integers, $a = \lceil b_0/\ell \rceil$, with $a + K < B$ and $K \ge K_\downarrow(t, B - a - K)$. Then*

$$Q_0^{b_0} Q_2^t \notin \varpi\ell^K R + Q_1^B R.$$

   The hypothesis $K \ge K_\downarrow(t, B-a-K)$ holds for every integer $K \ge \frac{\ell}{\ell-1}\max(0,\ t - (B - a)/\ell + 1 + \log_\ell t)$ with $a + K < B$.

2. Fractions. *Let $t, B, \eta \ge 0$ be integers and $s_{\mathrm{fr}} = \max(0,\ \lceil t - B/\ell \rceil - \lceil \eta/\ell \rceil)$. Then $Q_2^t \in \ell^{s_{\mathrm{fr}}} R + Q_1^B Q_0^\eta R$, and there is a Δ-invariant $h \in \mathcal{K}$ with*

$$Q_2^t h \in R \qquad \text{and} \qquad Q_2^t(\ell^{s_{\mathrm{fr}}} h - 1) \in Q_1^B Q_0^\eta R.$$

Proof. Write $e_L = v_\varpi(\ell)$, so $\varpi\ell^K R = \varpi^{e_L K + 1} R$.

Two facts. (A) *For $s \ge 0$, $Q_s^\ell = Q_{s+1} + \ell W_s$ with $W_s \in R^\times$. $Q_s = \sum_{i<\ell}(1+X)^{i\ell^s} \in \mathbb{Z}[X]$* is monic of degree $(\ell-1)\ell^s$ with $Q_s(0) = \ell$; in $\mathbb{F}_\ell[X]$, $(1+X)^{\ell^s} = 1 + X^{\ell^s}$, so $Q_s \equiv X^{(\ell-1)\ell^s}$ and $Q_s^\ell \equiv X^{(\ell-1)\ell^{s+1}} \equiv Q_{s+1}$ modulo ℓ. Hence $W_s = (Q_s^\ell - Q_{s+1})/\ell \in \mathbb{Z}[X]$, and $W_s(0) = \ell^{\ell-1} - 1 \in \mathbb{Z}_\ell^\times$, so $W_s \in R^\times$. (B) *If $G \in R \setminus \varpi R$, $j \ge 0$, $y \in R$ and $Gy \in \varpi^j R$, then $y \in \varpi^j R$:* for $j \ge 1$, the reductions satisfy $\bar{G}\bar{y} = 0$ in the domain $(\mathcal{O}_L/\varpi)[[X]]$ with $\bar{G} \ne 0$, so $y = \varpi y'$ with $Gy' \in \varpi^{j-1}R$ (R is a domain), and induction on j applies. The monic polynomials Q_s^N lie in $R \setminus \varpi R$.

Item 1. Suppose $Q_0^{b_0} Q_2^t \in \varpi\ell^K R + Q_1^B R$, and put $N = a + K < B$. By (A) and the binomial theorem,

$$Q_1^N = (Q_0^\ell - \ell W_0)^N = \sum_{j=0}^{N} \binom{N}{j} Q_0^{\ell j} (-\ell W_0)^{N-j}.$$

For $j \ge a$, $Q_0^{\ell j} \in Q_0^{b_0} R$ as $\ell j \ge \ell a \ge b_0$; for $j < a$, $N - j \ge K + 1$. So $Q_1^N = Q_0^{b_0} g_1 + \ell^{K+1} g_2$ with $g_1, g_2 \in R$, and

$$Q_1^N Q_2^t = g_1\, Q_0^{b_0} Q_2^t + \ell^{K+1} g_2 Q_2^t \in \varpi\ell^K R + Q_1^B R,$$

since $\ell^{K+1} \in \varpi\ell^K R$. Write $Q_1^N Q_2^t = \varpi\ell^K \alpha + Q_1^B \beta$. Then $Q_1^N (Q_2^t - Q_1^{B-N}\beta) \in \varpi^{e_L K + 1} R$, and by (B) $Q_2^t \in \varpi\ell^K R + Q_1^{B-N} R$. Since $K \ge K_\downarrow(t, B-N)$, this contradicts (Cd3) of Lemma C.36 (with $s = 1$, $y = t$ and $B - N \ge 1$ in place of B), which gives $Q_2^t \notin \varpi\ell^{K_\downarrow(t,B-N)} R + Q_1^{B-N} R \supseteq \varpi\ell^K R + Q_1^{B-N} R$.

For the last assertion, let $x_0 = t - (B-a)/\ell + 1 + \log_\ell t$ and $K \ge \frac{\ell}{\ell-1}\max(0, x_0)$ with $a + K < B$. Then $K \ge 0$ and $K - K/\ell \ge x_0$, i.e. $K \ge t - (B - a - K)/\ell + 1 + \log_\ell t$; by (Cd3), $K_\downarrow(t, B-a-K) \le \max(0,\ t - (B-a-K)/\ell + 1 + \log_\ell t) \le K$.

Item 2. Put $\lambda = \lceil \eta/\ell \rceil$. For $j \geq 0$, (A) gives $Q_1^j = \sum_{k \leq j} \binom{j}{k} Q_0^{\ell k} (-\ell W_0)^{j-k}$; the terms with $k \geq \lambda$ lie in $Q_0^\eta R$ ($\ell k \geq \eta$), those with $k < \lambda$ in $\ell^{j-\lambda+1} R$, so

$$Q_1^j \in Q_0^\eta R + \ell^{\max(0,\, j-\lambda+1)} R.$$

By (A), $Q_2^t = \sum_{i \leq t} \binom{t}{i} (-\ell W_1)^{t-i} Q_1^{\ell i}$. A term with $\ell i < B$ lies in $\ell^{t-i} R$, and the integer $t - i$ exceeds $t - B/\ell$, so $t - i \geq \lceil t - B/\ell \rceil \geq s_{\mathrm{fr}}$ (and $t - i \geq 0$). A term with $\ell i \geq B$ lies in $\ell^{t-i} Q_1^B Q_1^{\ell i - B} R \subseteq Q_1^B Q_0^\eta R + \ell^{\, t-i+\max(0, \ell i - B - \lambda + 1)} R$, and

$$t - i + (\ell i - B - \lambda + 1) = t - B + (\ell - 1) i - \lambda + 1 \geq t - B/\ell - \lambda + 1,$$

using $i \geq B/\ell$; as the left side is an integer, it is $\geq \lceil t - B/\ell \rceil - \lambda + 1 > s_{\mathrm{fr}}$ unless $s_{\mathrm{fr}} = 0$, and $t - i \geq 0$. Hence $Q_2^t \in \ell^{s_{\mathrm{fr}}} R + Q_1^B Q_0^\eta R$. Write $Q_2^t = \ell^{s_{\mathrm{fr}}} F + Q_1^B Q_0^\eta M$ with $F, M \in R$, and put $h_0 = F/Q_2^t \in \mathcal{K}$; then $Q_2^t h_0 = F \in R$ and $Q_2^t(\ell^{s_{\mathrm{fr}}} h_0 - 1) = -Q_1^B Q_0^\eta M$. For $u \in \mathbb{Z}_\ell^\times$ and $s \geq 0$, $\gamma_u(Q_s) = Q_s \upsilon_{u,s}$ with $\upsilon_{u,s} \in \mathbb{Z}_\ell[[X]]^\times$ (Lemma C.16 (b), first assertion, a statement about $\mathbb{Z}_\ell[[X]]$). Hence for $\delta \in \mu_{\ell-1}$, $Q_2^t \gamma_\delta(h_0) = \upsilon_{\delta,2}^{-t} \gamma_\delta(F) \in R$ and $Q_2^t(\ell^{s_{\mathrm{fr}}} \gamma_\delta(h_0) - 1) = \upsilon_{\delta,2}^{-t} \gamma_\delta(Q_2^t(\ell^{s_{\mathrm{fr}}} h_0 - 1)) \in Q_1^B Q_0^\eta R$. Put $h = (\ell - 1)^{-1} \sum_{\delta \in \mu_{\ell-1}} \gamma_\delta(h_0)$ ($\ell - 1 \in \mathcal{O}_L^\times$). It is Δ-invariant, as $\mu_{\ell-1}$ is a group and $\gamma_{\delta\delta'} = \gamma_\delta \gamma_{\delta'}$, and averaging the two memberships (since $\ell^{s_{\mathrm{fr}}} h - 1 = (\ell - 1)^{-1} \sum_\delta (\ell^{s_{\mathrm{fr}}} \gamma_\delta(h_0) - 1)$) gives $Q_2^t h \in R$ and $Q_2^t(\ell^{s_{\mathrm{fr}}} h - 1) \in Q_1^B Q_0^\eta R$. □

Proposition C.39. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25, and suppose $160\,\mathrm{Sg} \leq b$. Then there is an integer E with $0 \leq E \leq 40\,\mathrm{Sg} + \eta\,\beta_{320}$ such that the following holds for every $c \in C$ with $\mathrm{len}_c = 3$ and $\epsilon_{c-2} = -1$ (write $e = 3$). Let $\mathbf{n} = (n_k)_{k \leq r}$ be a frame at c' of $\mathbf{N}_2$ with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1}$, with crystalline frame (d_k), let $k_0 \leq r$ be an index with $\varphi^e(d_{k_0}) \in \varphi^{s+4}(\mathrm{Fil}^{\,s_{c-s}} D_{\mathrm{cris}}(V_2))$ for every $s < e$ with $\epsilon_{c-s} = 1$, and let g be admissible at c. Then there are $w_1 \in I_c$ (formed for $\mathbf{N}_1$), a finite product λ' of the Q_s ($s \geq 0$) and $z \in X\mathbf{N}_{2,c}$ such that $\lambda' w_1 \in \mathbf{N}_1$, $\Pi_c\, \mathbf{N}(\Phi)(w_1) \in P_e(\mathbf{N}_{2,c'})$, and $\ell^E g\, \mathbf{e}_{k_0}$ is an arc coordinate vector of $\mathbf{N}(\Phi)(w_1) - z$ for $\mathbf{n}$, e, H_c and m_X.*

Proof. Write $R_c = e_c \mathbf{A}_{F,L}$, $\psi = \mathbf{N}(\Phi)$, $\beta = \beta_{320}$, $e = 3$, and *coordinate vector* for arc coordinate vector for $\mathbf{N}_2$, $\mathbf{n}$, e, H_c, m_X ($m_X > 2h_B \geq 2H_c(x)$, so Lemma C.17 applies).

Exponents and combining. Let $E_C \leq 6\,\mathrm{Sg}$ be the integer of Proposition C.27; $\sigma_B \leq 3\,\mathrm{Sg}$.

Let (M_a, h_a) be the window of Lemma C.9, $H(x) = s_x + S + \eta$, and $H_{\max}$, c_H formed with it as in Proposition C.6. Put $E = \lfloor 22\,\mathrm{Sg} \rfloor + \eta\beta \leq 40\,\mathrm{Sg} + \eta\beta_{320}$. We combine data by three rules, each a direct check: (a) if the conclusion holds for g with an exponent $E'' \leq E$ (data w_1, λ', z), it holds with E (data $\ell^{E-E''} w_1, \lambda', \ell^{E-E''} z$), because I_c is an $\mathcal{O}_L$-module (Lemma 24.5 (Ry4)) and coordinate vectors are R_c-linear (Lemma C.17 item 1); (b) data for g_1 and g_2 with the same exponent add to data for $g_1 + g_2$ (sum of the w_j and of the z_j, product of the λ'_j; Lemma C.17 item 1); (c) data for g with an exponent E'', multiplied by $u \in \mathcal{O}_L$ (the w_1 and z multiplied by u), are data for ug with the exponent E'', by the same two facts.

The arc. Let $c \in C$ with $\mathrm{len}_c = 3$ and $\epsilon_{c-2} = -1$, and $\mathbf{n}$, k_0, g as in the statement. As $c - 1, c - 2 \notin C$, the sites c and $c - 1$ are thick; as $c' = c - 3 \in C$, the site $c - 2 = c' + 1$ is thin. Signs alternate, so $\epsilon_{c-1} = 1$, $\epsilon_c = -1$. Hence $h = h_B = b + \eta + S$, and with $b_1 = s_{c-1}$ ($b \leq b_1 \leq b + \eta$), $s_2 = s_{c-2}$ ($\eta < s_2 \leq w$) and $y = s_2 + S + \eta$, the only up level is $s = 1$ and $\nu_c = Q_1^{b_1}$. By admissibility, $\gamma_\delta(g) = g$ for $\delta \in \mu_{\ell-1}$, $Q_1^{b_1} g \in R_c$, and there are $\kappa_0, \kappa_2 \geq 0$, $x', x'' \in \mathcal{O}_L$ with

$$Q_1^{b_1}(\ell^{\kappa_0} g - x') \in Q_0^{s_c + S} R_c, \qquad Q_1^{b_1}(\ell^{\kappa_2} g - x'') \in Q_2^{s_2 + S} R_c.$$

Also $H_{\max} \le h+\eta \le 1.05h$ and $c_H \le 2+\log_\ell(2h)$; we use $h \le 1.1b$, $\log_\ell(2h) \le \mathrm{Sg}-1$, $\mathrm{Sg} \ge 1$ and $160\,\mathrm{Sg} \le b$. The hypothesis on k_0 at $s=1$ reads $\varphi^3(d_{k_0}) \in \varphi^5(\mathrm{Fil}^{b_1} D_{\mathrm{cris}}(V_2))$, i.e. $\varphi^2(d_{k_0}) \in \varphi^4(\mathrm{Fil}^{b_1} D_{\mathrm{cris}}(V_2))$ (φ is bijective).

Case C: $\ell y \le b_1$. Let $a=c-1$, σ_a the killing index of ψ at a, $A=\lceil \sigma_a/(\ell-1)\rceil$, and

$$t = \Big\lfloor \frac{A+\eta-1+\ell(1+\log_\ell y)}{\ell-1} \Big\rfloor + 1.$$

Then $t \ge 1$ and $(\ell-1)t > A+\eta-1+\ell(1+\log_\ell y)$. By (Ki1) of Lemma C.7 with $m=1$ (for $\Phi\colon T_1 \to T_2$ and the window above; $a+1=c$), $\sigma_a \le H(c)+c_H+3+H_{\max}/(\ell-1)$ with $H(c)=s_c+S+\eta \le h+\eta$; this gives $A+\eta+t \le b_1$ and $t \le 2.3\,\mathrm{Sg}$:

Let $\mathbf{n}^{(1)}$ be a frame at c' of $\mathbf{N}_1$ fixed by γ_u, $u \in \mu_{\ell-1}$ (Lemma 24.15; $\mathbf{N}_1$ is a Wach module by (Ex1) of Proposition 23.7), with crystalline frame $(d_m^{(1)})$, and μ the matrix of ψ for $\mathbf{n}^{(1)}$, $\mathbf{n}$. By Lemma C.40 there is $\mathbf{c} \in \mathcal{O}_L^{r_1}$ with $\mathbf{c}\mu = \ell^{2\sigma_B+t}\mathbf{e}_{k_0}$ and $\sum_m c_m \varphi^2(d_m^{(1)}) \in \varphi^4(\mathrm{Fil}^{b_1} D_{\mathrm{cris}}(V_1))$; applying the L-linear φ, $\sum_m c_m \varphi^3(d_m^{(1)}) \in \varphi^{1+4}(\mathrm{Fil}^{s_{c-1}} D_{\mathrm{cris}}(V_1))$. With $U=\{1\}$ (the set of all up levels), $\nu_U = \nu_c$ and the conditions on g in the clause "Filtered constants" of Proposition C.27 are the admissibility of g. That clause with $E' = 2\sigma_B + t$ gives the conclusion for g with the exponent $3\sigma_B + t \le 11.3\,\mathrm{Sg}$ (and $z=0$), and (a) raises it to E.

Case D: $b_1 < \ell y$. *Descent at the level of the first site.* Let $K = K_\uparrow(b_1, s_2+S)$ as in (Cd2) of Lemma C.36 (with $s=1$); then $Q_1^{b_1} \notin \varpi\ell^K R_c + Q_2^{s_2+S} R_c$, and $Q_2^{s_2+S} \notin \varpi R_c$ (Q_2 is monic). So (Cd1) of Lemma C.36 (with $A = Q_1^{b_1}$, $N = Q_2^{s_2+S}$, $f=g$, $\kappa=\kappa_2$, $x=x''$) gives $x_2 \in \mathcal{O}_L$ and $f_2 = \ell^K g - x_2$ with $Q_1^{b_1} f_2 \in Q_2^{s_2+S} R_c$, $\gamma_\delta(f_2) = f_2$ for $\delta \in \mu_{\ell-1}$, and, from the congruence at level 0, $Q_1^{b_1}(\ell^{\kappa_0} f_2 - (\ell^K x' - \ell^{\kappa_0} x_2)) \in Q_0^{s_c+S} R_c$. *Bound for K.* If $b_1 < \ell(s_2+S)$ then $K=0$. Otherwise $b_1 \ge 1$ and $b_1 < \ell(s_2+S+\eta)$, so by (Cd2) $K \le \eta+1+\log_\ell b_1 \le \eta+\mathrm{Sg}$; and then $\beta=1$: the thin site $c-2$ has the thick neighbour $c-1$, and $\ell s_2 \le b_1 - \ell S \le b+\eta$, $\ell s_2 > b_1 - \ell(S+\eta) \ge b - \ell(S+\eta)$, so $|\ell s_2 - b| \le \ell\eta + \ell S \le 320(\ell\eta + \ell S + b/\ell + s_2 + \ell\log_\ell(2h))$. In both cases $K \le \eta\beta + \mathrm{Sg}$. *The two lifts.* By Proposition C.41 (its hypotheses on c, $\mathbf{n}$, k_0 are those above, and on f_2 were just checked, with $\kappa=\kappa_0$, $x_0 = \ell^K x' - \ell^{\kappa_0} x_2$) there are $e_D \le 6\,\mathrm{Sg}$, $t_3 \in \mathcal{O}_L \setminus \{0\}$ with $v_\ell(t_3) \le 14\,\mathrm{Sg}$, and data for $t_3 f_2$ with the exponent e_D. The constant $t_3 x_2 \in \mathcal{O}_L$ satisfies the hypothesis of the first part of Proposition C.27 (with $\kappa=0$, $x = t_3 x_2$ at every level), which gives data for it with the exponent E_C. By (a) and (b), with $e_0 = \max(e_D, E_C) \le 6\,\mathrm{Sg}$, there are data for $t_3 x_2 + t_3 f_2 = t_3 \ell^K g$ with the exponent e_0. *Rescaling.* Let $N = \lceil v_\ell(t_3)\rceil + K$ and $u = \ell^N/(t_3\ell^K)$; $v_\ell(u) \ge 0$, so $u \in \mathcal{O}_L$. By (c) there are data for $u\,t_3\ell^K g = \ell^N g$ with the exponent e_0, i.e. for g with the exponent $e_0 + N \le 6\,\mathrm{Sg} + 14\,\mathrm{Sg} + 1 + \eta\beta + \mathrm{Sg} \le 22\,\mathrm{Sg} + \eta\beta$, an integer; so it is at most E, and (a) raises it to E.

Conclusion. Cases C and D cover all $c \in C$ with $\mathrm{len}_c = 3$ and $\epsilon_{c-2} = -1$, each with the exponent E. □

Lemma C.40. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25; put $\mathbf{N}_\nu = \mathbf{N}(T_\nu)$ and $r_\nu = \dim_L V_\nu$. Let $c \in C$ with $\mathrm{len}_c = 3$ and $\epsilon_{c-2} = -1$, put $c' = c-3$, $a = c-1$, $b_1 = s_{c-1}$, $y = s_{c-2} + S + \eta$ and $A = A_a = \lceil \sigma_a/(\ell-1)\rceil$, where σ_a is the killing index of $\mathbf{N}(\Phi)$ at a, and suppose*

$$\ell\, y \le b_1.$$

Let $t \ge 1$ be an integer with

$$A+\eta+t \le b_1 \qquad \text{and} \qquad (\ell-1)\,t > A+\eta-1+\ell(1+\log_\ell y).$$

Let $\mathbf{n} = (n_k)_{k \le r_2}$ and $\mathbf{n}^{(1)} = (n_m^{(1)})_{m \le r_1}$ be frames at c' of $\mathbf{N}_2$ and $\mathbf{N}_1$, with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$ and all k, with crystalline frames (d_k) and $(d_m^{(1)})$, let $\mu \in M_{r_1 \times r_2}(\mathcal{O}_L)$ be the

matrix of $\mathbf{N}(\Phi)$ *for* $\mathbf{n}^{(1)}$ *and* $\mathbf{n}$*, and let* $k_0 \le r_2$ *with*

$$\varphi^2(d_{k_0}) \in \varphi^4\big(\mathrm{Fil}^{\,b_1} D_{\mathrm{cris}}(V_2)\big).$$

Then there is $\mathbf{c} = (c_m)_{m \le r_1} \in \mathcal{O}_L^{r_1}$ *with*

$$\mathbf{c}\,\mu = \ell^{2\sigma_B + t}\mathbf{e}_{k_0} \qquad \text{and} \qquad \sum_m c_m\,\varphi^2(d_m^{(1)}) \in \varphi^4\big(\mathrm{Fil}^{\,b_1} D_{\mathrm{cris}}(V_1)\big);$$

in particular $(1 \otimes \Phi)\big(\sum_m c_m d_m^{(1)}\big) = \ell^{2\sigma_B + t} d_{k_0}$.

Proof. Write $R = e_a \mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$, $Q = Q_0$, $s_2 = s_{c-2}$, $H = H_c$, and let π_0, $q = \pi_0 + \ell$, ω, ω_1, R^Δ, $\mathcal{K}^\Delta$ and the average $f \mapsto \bar{f}$ be as in Lemma C.34. A *coordinate vector* (for $\mathbf{N}_\nu$) is an arc coordinate vector for $\mathbf{n}$ (resp. $\mathbf{n}^{(1)}$), $e = 2$, H and m_X: this is the initial sub-arc $c' \to c' + 2 = a$ of the arc of c, with level 0 at the site $a = c - 1$ and level 1 at the site $c - 2$; for $\mathbf{v} \in L^{r_\nu}$ the vector $\varphi^{-(s+4)}(\delta_2(\mathbf{v}))$ lies in the component $\varepsilon D_{\mathrm{cris}}(V_\nu)$ of τ_{a-s}, on which $\varepsilon\,\mathrm{Fil}^\bullet$ is the Hodge filtration at τ_{a-s} (Definition 23.3), and $\mathrm{Fil}^u D_{\mathrm{cris}}$ is stable under $F \otimes L$.

Step 0: the arc. As $c-1, c-2 \notin C$ and $c' \in C$, the sites c, $c-1$ are thick and $c - 2 = c' + 1$ is thin; signs alternate, so $\epsilon_{c-1} = 1$, $\epsilon_c = -1$. Hence $b \le b_1 \le b + \eta$, $\eta < s_2 \le w$, $h_B = b + \eta + S \ge b_1 + S$, $H(c-1) = h_B$, $H(c-2) = w + S$ and $\Pi' := \Lambda_{a,2}(H) = Q^{h_B} Q_1^{w+S}$. By (Ow1), the Hodge jumps of V_1 at τ_{c-1} lie in $[b_1, b_1 + S] \cup [-\eta, \eta]$ and those at τ_{c-2} in $[-s_2 - S, -s_2] \cup [-\eta, \eta]$; by Lemma C.16 (1) every Hodge jump of V_1, V_2 at τ_x has absolute value at most $s_x + S \le H(x)$. Also $y \ge s_2 \ge 1$, and $m_X > 2h_B \ge 2H(x)$, so Lemma C.17 applies.

Step 1: realising the section. Put $g_2 = q^{-b_1}\varphi(q)^{s_2+S} \in \mathcal{K}^\Delta$. By (Dv2), (Dv2') of Lemma C.34, $g_2 \in Q^{-b_1} Q_1^{s_2+S} R$ and $\Pi' g_2 \in Q^{h_B - b_1} Q_1^{w+S+s_2+S} R \subseteq R$. The section decomposition $(g_2, \mathbf{e}_{k_0}, \alpha)$ with $\alpha(0) = -b_1$, $\alpha(1) = s_2 + S$ shows that $g_2\mathbf{e}_{k_0}$ is Hodge-graded: $\delta_2(\mathbf{e}_{k_0}) = \varphi^2(d_{k_0}) \in \varphi^4(\mathrm{Fil}^{b_1})$ by hypothesis, and $\varphi^{-5}(\varphi^2(d_{k_0}))$ lies in the component of τ_{c-2}, where V_2 has no Hodge jump below $-(s_2 + S)$, hence in $\mathrm{Fil}^{-(s_2+S)}$. By Proposition C.23 (R3) with $e_1 = 2$ ($\nu = 2$, the $\mu_{\ell-1}$-fixed frame $\mathbf{n}$) there is $y_2 \in \mathbf{N}_{2,a}$ such that $\ell^{\sigma_B} g_2 \mathbf{e}_{k_0}$ is a coordinate vector of y_2.

Step 2: killing the cokernel; coordinates. By Lemma 28.15 item 1, $\pi_0 = X^{\ell-1} v$ with v a unit, so by (Ki3) of Lemma C.7 π_0^A is killing at a: there is $x_0 \in \mathbf{N}_{1,a}$ with $\mathbf{N}(\Phi)(x_0) = \pi_0^A y_2$. By Step 0, $(H(x))_x$ is a window for $\mathbf{N}_1$ as in (Ex1) of Proposition 23.7, so (Wm2) gives $Q^{H(x)}\mathbf{N}_{1,x} \subseteq \mathbf{A}_{F,L}\varphi(\mathbf{N}_{1,x-1})$ for every x; applying this at $x = a$ and, after φ (with $\varphi(Q) = Q_1$), at $x = c - 2$,

$$\Pi'\mathbf{N}_{1,a} = Q_1^{H(c-2)} Q^{H(c-1)} \mathbf{N}_{1,a} \subseteq \mathbf{A}_{F,L}\varphi\big(Q^{H(c-2)}\mathbf{N}_{1,c-2}\big) \subseteq \mathbf{A}_{F,L}\varphi^2(\mathbf{N}_{1,c'}) = P_2(\mathbf{N}_{1,c'}).$$

Hence Lemma C.14 (with c', $e = 2$, H, m_X and $\sigma = \sigma_B \ge (2H(c'+1) + m_X/(\ell-1))\ell/(\ell-1)^2$) gives a coordinate vector f of $\ell^{\sigma_B} x_0$ with $\Pi' f_m \in R$, and Lemma 23.13 (2) gives $t', \kappa \ge 0$ and a coordinate vector f' of $\ell^{t'} x_0$ with $\ell^\kappa f'$ Hodge-graded.

Step 3: the two congruences. By Lemma C.29 (2), $f\mu$ is a coordinate vector of

$$\mathbf{N}(\Phi)(\ell^{\sigma_B} x_0) = \ell^{\sigma_B}\pi_0^A y_2,$$

and so is $\ell^{2\sigma_B}\pi_0^A g_2 \mathbf{e}_{k_0}$ (Lemma C.17 item 1); $\ell^{t'} f$ and $\ell^{\sigma_B} f'$ are coordinate vectors of $\ell^{t' + \sigma_B} x_0$. By Lemma C.17 item 2,

$$f\mu - \ell^{2\sigma_B}\pi_0^A g_2 \mathbf{e}_{k_0} \in (X\Pi' R)^{r_2}, \qquad \ell^{t'+\kappa} f = \sum_i g_i \mathbf{v}_i + X\Pi'\mathbf{h},$$

where $\mathbf{h} \in R^{r_1}$, $g_i \in Q^{\alpha_i(0)}Q_1^{\alpha_i(1)}R$, $\mathbf{v}_i \in \mathcal{O}_L^{r_1}$ and

$$\delta_2(\mathbf{v}_i) \in \varphi^4(\mathrm{Fil}^{-\alpha_i(0)}D_{\mathrm{cris}}(V_1)) \cap \varphi^5(\mathrm{Fil}^{-\alpha_i(1)}D_{\mathrm{cris}}(V_1))$$

(the $\ell^{-\sigma_B}g_i$ forming a section decomposition of $\ell^\kappa f'$).

Step 4: bounded poles. If $\mathbf{v}_i \neq 0$ then $-\alpha_i(0) \leq b_1 + S$ and $-\alpha_i(1) \leq \eta$: $\varphi^{-4}\delta_2(\mathbf{v}_i)$ lies in the component of τ_{c-1}, whose Fil^u is 0 for $u > b_1 + S$, and $\varphi^{-5}\delta_2(\mathbf{v}_i)$ in the component of τ_{c-2}, whose Fil^u is 0 for $u > \eta$ (Step 0); if either inequality failed, $\varphi^2(\sum_m v_{i,m}d_m^{(1)}) = \delta_2(\mathbf{v}_i) = 0$, so $\mathbf{v}_i = 0$, as φ is bijective and the $d_m^{(1)}$ are linearly independent (Lemma 23.10 item 1). We drop the terms with $\mathbf{v}_i = 0$.

Step 5: clearing the poles. Put $\Psi = q^{b_1+S}\varphi(q)^\eta \in R^\Delta$, so $\Psi \in Q^{b_1+S}Q_1^\eta R^\times$. By Step 4, $\Psi g_i \in R$, and $\Psi X\Pi'\mathbf{h} \in R^{r_1}$; so $\ell^{t'+\kappa}\Psi f_m \in R$. Also $\Psi f_m = G^{-1}A_m$ with $G = Q^{h_B-b_1-S}Q_1^{w+S-\eta}$ (exponents ≥ 0 by Step 0) and $A_m = (\Psi\Pi'^{-1}G)\,\Pi' f_m \in R$. So $\ell^{t'+\kappa}A_m \in GR$; as G is a monic polynomial, R/GR is ℓ-torsion free (reduce modulo a uniformiser in the domain $(\mathcal{O}_L/\varpi)[[X]]$ and induct), so $A_m \in GR$ and $\Psi f_m \in R$.

Step 6: averaging. Put $\Phi_m = \overline{\Psi f_m} = \Psi\bar{f}_m \in R^\Delta$. Multiply the relations of Step 3 by Ψ and average. Since $\Psi\pi_0^A g_2 = (q-\ell)^A q^S\varphi(q)^y$, $\Psi X\Pi' R = XQ^{h_B+b_1+S}Q_1^{w+S+\eta}R$, and μ, $\mathbf{v}_i$ are constant, (Dv3) and (Dv2') give

$$\Phi\mu - \mathcal{F}\,\mathbf{e}_{k_0} \in \mathcal{E}^{r_2}, \qquad \ell^{t'+\kappa}\Phi - \sum_i \Psi\bar{g}_i\,\mathbf{v}_i \in \mathcal{E}^{r_1},$$
$$\mathcal{F} = \ell^{2\sigma_B}(q-\ell)^A q^S\varphi(q)^y,$$

with $\mathcal{E} = \pi_0 q^{h_B+b_1+S}\varphi(q)^{w+S+\eta}R^\Delta$ and $\Psi\bar{g}_i \in q^{b_1+S+\alpha_i(0)}\varphi(q)^{\eta+\alpha_i(1)}R^\Delta$ (both exponents ≥ 0).

Step 7: a coefficient of small valuation. In $R^\Delta = \mathcal{O}_L[[q]]$ (Lemma C.34 (Dv1)) let λ_j be the coefficient of q^j, and put $B_0 = b_1 + S - \eta$. We claim that $\ell^{2\sigma_B+t} \nmid \lambda_j(\mathcal{F})$ for some $j < B_0$. Suppose not. Since $\ell^{2\sigma_B}$ is a non-zero element of the domain $\mathcal{O}_L$, every coefficient λ_j, $j < B_0$, of $\mathcal{G} = (q-\ell)^A q^S\varphi(q)^y$ is divisible by ℓ^t, i.e. $\mathcal{G} \in \ell^t R^\Delta + q^{B_0}R^\Delta$. Expanding $q^{A+t-1} = ((q-\ell)+\ell)^{A+t-1}$ by the binomial theorem, the terms with ℓ^i, $i \geq t$, lie in $\ell^t\mathcal{O}_L[q]$ and the others contain $(q-\ell)^{A+t-1-i}$ with $A+t-1-i \geq A$; so $q^{A+t-1} \in (q-\ell)^A\mathcal{O}_L[q] + \ell^t\mathcal{O}_L[q]$, and hence $q^{A+t-1+S}\varphi(q)^y \in \ell^t R^\Delta + q^{B_0}R^\Delta$. Its coefficient of $q^{j+A+t-1+S}$ is $\lambda_j(\varphi(q)^y)$, so with $B = B_0 - (A+t-1+S) = b_1 - \eta - A - t + 1 \geq 1$ (hypothesis),

$$\varphi(q)^y \in \ell^t R^\Delta + q^B R^\Delta.$$

By (Dv2), (Dv2'), $\varphi(q)^y = Q_1^y\omega_1^y$ and $q^B = Q^B\omega^B$ with units ω, ω_1; multiplying by ω_1^{-y}, $Q_1^y \in \ell^t R + Q^B R \subseteq \varpi\ell^{t-1}R + Q^B R$. By (Cd3) of Lemma C.36 (with $s = 0$, $y \geq 1$, $B \geq 1$), $Q_1^y \notin \varpi\ell^K R + Q^B R$ for $K = K_\downarrow(y, B) \leq \max(0,\, y - B/\ell + 1 + \log_\ell y)$; so $t - 1 < K$, i.e. $t \leq K$. But $\ell y \leq b_1$ gives $y - B/\ell \leq (\eta + A + t - 1)/\ell$, so

$$K \leq \max\Big(0,\ \frac{\eta + A + t - 1}{\ell} + 1 + \log_\ell y\Big) < t,$$

the last inequality being $t \geq 1$ and the hypothesis $(\ell-1)t > A + \eta - 1 + \ell(1 + \log_\ell y)$. This contradiction proves the claim.

Step 8: reading the coefficient. Fix $j < B_0$ as in Step 7. As $\mathcal{O}_L$ is a discrete valuation ring and $\ell^{2\sigma_B+t} \nmid \lambda_j(\mathcal{F})$, there is $u \in \mathcal{O}_L$ with $u\,\lambda_j(\mathcal{F}) = \ell^{2\sigma_B+t}$. Every element of $q^N R^\Delta$ has $\lambda_j = 0$ if $N > j$; since $j < B_0 \leq h_B + b_1 + S$, $\lambda_j(\mathcal{E}) = 0$. Put $\mathbf{c} = u\,(\lambda_j(\Phi_m))_m \in \mathcal{O}_L^{r_1}$. The first relation of Step 6 gives $\mathbf{c}\mu = u\lambda_j(\mathcal{F})\mathbf{e}_{k_0} = \ell^{2\sigma_B+t}\mathbf{e}_{k_0}$. The second gives $\ell^{t'+\kappa}\mathbf{c} = u\sum_i \lambda_j(\Psi\bar{g}_i)\mathbf{v}_i$, and $\lambda_j(\Psi\bar{g}_i) = 0$ unless $b_1 + S + \alpha_i(0) \leq j < b_1 + S - \eta$, i.e. $-\alpha_i(0) \geq \eta + 1$.

For the remaining i, $\varphi^{-4}\delta_2(\mathbf{v}_i) \in \mathrm{Fil}^{\eta+1}D_{\mathrm{cris}}(V_1)$; as δ_2 is L-linear and $\mathrm{Fil}^{\eta+1}$ is an L-subspace, $z := \varphi^{-4}(\sum_m c_m\varphi^2(d_m^{(1)})) \in \mathrm{Fil}^{\eta+1}D_{\mathrm{cris}}(V_1)$, and z lies in the component of τ_{c-1}.

Step 9: the Hodge gap. On the component of τ_{c-1} the Hodge filtration of V_1 has no jump in $[\eta+1, b_1-1]$ (Step 0), so there $\mathrm{Fil}^{\eta+1} = \mathrm{Fil}^{b_1}$ and $z \in \mathrm{Fil}^{b_1}D_{\mathrm{cris}}(V_1)$. Finally $(1\otimes\Phi)(d_m^{(1)}) = \sum_k \mu_{mk}d_k$ (Lemma C.29 (1)) gives $(1\otimes\Phi)(\sum_m c_m d_m^{(1)}) = \ell^{2\sigma_B+t}d_{k_0}$. □

Proposition C.41. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.25, and suppose $160\,\mathrm{Sg} \le b$. Let $c \in C$ with $\mathrm{len}_c = 3$, $\epsilon_{c-2} = -1$ and*

$$s_{c-1} < \ell\,(s_{c-2} + S + \eta)$$

(write $e = 3$, $R_c = e_c\mathbf{A}_{F,L}$). Let $\mathbf{n} = (n_k)_{k\le r}$ be a frame at c' of $\mathbf{N}_2$ with $\gamma_u(n_k) = n_k$ for $u \in \mu_{\ell-1}$, with crystalline frame (d_k), and $k_0 \le r$ with $\varphi^3(d_{k_0}) \in \varphi^5(\mathrm{Fil}^{\,s_{c-1}}D_{\mathrm{cris}}(V_2))$. Let $f_2 \in \mathcal{K}_c$ with $\gamma_u(f_2) = f_2$ for $u \in \mu_{\ell-1}$,

$$Q_1^{s_{c-1}}f_2 \in Q_2^{s_{c-2}+S}R_c,$$

and such that $Q_1^{s_{c-1}}(\ell^\kappa f_2 - e_c(1\otimes x_0)) \in Q_0^{s_c+S}R_c$ for some integer $\kappa \ge 0$ and some $x_0 \in \mathcal{O}_L$. Then there are an integer e_D with $0 \le e_D \le 6\,\mathrm{Sg}$ and $t_3 \in \mathcal{O}_L \setminus \{0\}$ with $v_\ell(t_3) \le 14\,\mathrm{Sg}$, and there are $w_1 \in I_c$ (formed for $\mathbf{N}_1$), a finite product λ' of the Q_s ($s \ge 0$) and $z \in X\mathbf{N}_{2,c}$ such that $\lambda' w_1 \in \mathbf{N}_1$, $\Pi_c\,\mathbf{N}(\Phi)(w_1) \in P_e(\mathbf{N}_{2,c'})$, and $\ell^{e_D}t_3 f_2\,\mathbf{e}_{k_0}$ is an arc coordinate vector of $\mathbf{N}(\Phi)(w_1) - z$ for $\mathbf{n}$, e, H_c and m_X.

Proof. Write $R_c = e_c\mathbf{A}_{F,L}$, $\psi = \mathbf{N}(\Phi)$, $b_1 = s_{c-1}$, $t = s_{c-2}+S$, $B = s_c + S$, and let $\kappa_0 \ge 0$, $x_0 \in \mathcal{O}_L$ be the integer and the constant of the hypothesis, so $Q_1^{b_1}f_2 \in Q_2^t R_c$ and $Q_1^{b_1}(\ell^{\kappa_0}f_2 - e_c(1\otimes x_0)) \in Q_0^B R_c$. Fix $u_0 = 1+\ell$.

The arc. Since $\mathrm{len}_c = 3$, $c-1, c-2 \notin C$ and $c' = c-3 \in C$ (Proposition C.4), so the sites c, $c-1$ are thick and $c-2 = c'+1$ is thin; hence $b \le b_1 \le b+\eta$, $b \le s_c \le b+\eta$, $h = b+\eta+S$, and $B \le h$, $b_1 \le 2h$, $t \ge 1$. Recall $\mathrm{Sg} = h/\ell^2 + (w+S+\eta)/\ell + \log_\ell(2h) + 1 \ge 1$, $\ell \ge 11$, and $\sigma_B \le 3\,\mathrm{Sg}$:

The four pieces. By Proposition C.43 there are an admissible κ and λ with its conclusion. By Proposition C.44, $s_0\lambda \in \mathbf{N}_{2,c-1}$ and there are $E \le 8\,\mathrm{Sg}$ and n_1 with $\psi(n_1) = \ell^E s_0\lambda$. By Proposition C.48 there are $\sigma_1 \le B\ell/(\ell-1)^2$, $y_1 \in P_1(\mathbf{N}_{1,c-1}) \cap I_c$ with $Q_0^B y_1 \in \mathbf{N}_{1,c}$, g_1 and κ_1 with

$$A_1 := \ell^{\kappa_1}g_1(\ell^{b_1+\kappa}\psi(y_1) - \ell^{\sigma_1+E+\sigma_B+t}\Theta) \in Q_0^B P_1, \qquad \Theta = \varphi^3(T_\kappa(H_{k_0})),\ P_1 = P_1(\mathbf{N}_{2,c-1}).$$

By Proposition C.49 (its hypothesis $s_0\lambda \in \mathbf{N}_{2,c-1}$ was just obtained), $V = f_2\varphi(\lambda) \in P_1$, $\Pi_c f_2 \in R_c$, $\varphi(\lambda) \in P_3(\mathbf{N}_{2,c'})$, $\ell^{\sigma_B}f_2\mathbf{e}_{k_0}$ is an arc coordinate vector of V, and there are g_2, κ_2 with $A_2 := \ell^{\kappa_2}g_2(\ell^{\kappa+\kappa_0}V - \ell^{\sigma_B}x_0\Theta) \in Q_0^B P_1$.

The scalar. Put $D_1 = \kappa_0 + \sigma_1 + E + t$. If $x_0 = 0$ or $v_\ell(x_0) \ge D_1 - b_1$, let $t_3 = 1$ and $B' = x_0\ell^{b_1-D_1}$ (in $\mathcal{O}_L$: if $x_0 \ne 0$, $v_\ell(B') = v_\ell(x_0) + b_1 - D_1 \ge 0$). Otherwise $x_0 \ne 0$ and $D_1 - b_1 > v_\ell(x_0) \ge 0$; let $t_3 = \ell^{D_1-b_1}/x_0$ and $B' = 1$. In both cases $t_3 \in \mathcal{O}_L \setminus \{0\}$, $B' \in \mathcal{O}_L$ and

$$t_3 x_0 \ell^{b_1} = B'\ell^{D_1}.$$

Valuation of t_3. In the first case $v_\ell(t_3) = 0$. In the second, $v_\ell(t_3) = D_1 - b_1 - v_\ell(x_0)$, and Lemma C.42 (with $f = f_2$, $\kappa = \kappa_0$; $b_1, t, B \ge 1$) gives $v_\ell(x_0) \ge \kappa_0 + t - B/(\ell(\ell-1)) - K$ with $K = K_\downarrow(b_1, B) \le \max(0,\, b_1 - B/\ell + 1 + \log_\ell b_1)$. Hence

$$v_\ell(t_3) \le \sigma_1 + E + \frac{B}{\ell(\ell-1)} + \max\Big(-b_1,\ -\frac{B}{\ell} + 1 + \log_\ell b_1\Big).$$

So $v_\ell(t_3) \le 14\,\mathrm{Sg}$.

The difference lies in the lattice at c. Let $D = t_3 V - B'\psi(y_1) \in P_1$ (ψ commutes with φ, so $\psi(P_1(\mathbf{N}_{1,c-1})) \subseteq P_1$). Expanding, and using $t_3 x_0 \ell^{b_1} = B'\ell^{D_1}$ to cancel the two multiples of $\ell^{\sigma_B}\Theta$,

$$t_3 \ell^{b_1+\kappa_1} g_1 A_2 - B'\ell^{\kappa_0+\kappa_2} g_2 A_1 = \ell^{\kappa+\kappa_0+b_1+\kappa_1+\kappa_2} g_1 g_2\, D,$$

so $\ell^k g_1 g_2 D \in Q_0^B P_1$ with $k = \kappa + \kappa_0 + b_1 + \kappa_1 + \kappa_2$. The module P_1 is free over R_c on $\varphi(n_1'), \dots, \varphi(n_r')$ for a basis (n_j') of $\mathbf{N}_{2,c-1}$: these generate it, and P_1 has rank r because it contains $Q^{h_c}\mathbf{N}_{2,c}$ by (Wm2) (window of Lemma C.9 (1), a window of $\mathbf{N}_2$ by (Ex1) of Proposition 23.7); a surjection $R_c^r \to P_1$ onto a torsion-free module of rank r is injective. Write $D = \sum_j a_j \varphi(n_j')$; then $\ell^k g_1 g_2 a_j \in Q_0^B R_c$, and $\ell^k g_1 g_2$ does not vanish at the roots of Q_0 (those of Q_s, $s \ge 1$, are other roots of unity minus 1), so $a_j \in Q_0^B R_c$ by Lemma C.18 item 2 ($e = 1$). Thus $D \in Q_0^B P_1$. At the down site c the window has $M_c = s_c + S = B$, so (Wm2) gives $Q_0^B \varphi(\mathbf{N}_{2,c-1}) \subseteq \mathbf{N}_{2,c}$ and $D \in \mathbf{N}_{2,c}$.

The lift. By Proposition C.6 (for the surjection Φ and the window of Lemma C.9) there is $x_1 \in \mathbf{N}_{1,c}$ with $\psi(x_1) = \ell^{r_c} D$, and $r_c \le 3\,\mathrm{Sg}$ by Lemma C.9 (2). Put

$$w_1 = x_1 + \ell^{r_c} B' y_1, \qquad \lambda' = Q_0^B, \qquad z = 0, \qquad e_D = r_c + \sigma_B.$$

Then $\psi(w_1) = \ell^{r_c}(D + B'\psi(y_1)) = \ell^{r_c} t_3 V$. Further: $w_1 \in I_c$, as $x_1 \in \mathbf{N}_{1,c} \subseteq I_c$ by (Wm3), $y_1 \in I_c$ and I_c is an $\mathcal{O}_L$-module (Lemma 24.5 (Ry4)); $\lambda' w_1 = Q_0^B x_1 + \ell^{r_c} B' Q_0^B y_1 \in \mathbf{N}_1$; $\Pi_c \psi(w_1) = \ell^{r_c} t_3 (\Pi_c f_2) \varphi(\lambda) \in P_3(\mathbf{N}_{2,c'})$, since $\Pi_c f_2 \in R_c$ and $\varphi(\lambda) \in P_3(\mathbf{N}_{2,c'})$; and by Lemma C.16 (a) with $o = \ell^{r_c} t_3 \in R_c$, the vector $\ell^{r_c} t_3 \cdot \ell^{\sigma_B} f_2 \mathbf{e}_{k_0} = \ell^{e_D} t_3 f_2 \mathbf{e}_{k_0}$ is an arc coordinate vector of $\psi(w_1) = \psi(w_1) - z$. Finally $0 \le e_D \le 3\,\mathrm{Sg} + 3\,\mathrm{Sg} = 6\,\mathrm{Sg}$. □

Lemma C.42. *Let ℓ be an odd prime, L a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_L$, v_ℓ the valuation of L with $v_\ell(\ell) = 1$, $R = \mathcal{O}_L[[X]]$ and $\mathcal{K}$ its field of fractions. Let $b_1, t, B \ge 1$ and $\kappa \ge 0$ be integers, $f \in \mathcal{K}$ and $x_0 \in \mathcal{O}_L \setminus \{0\}$ with*

$$Q_1^{b_1} f \in Q_2^t R \qquad \text{and} \qquad Q_1^{b_1}(\ell^\kappa f - x_0) \in Q_0^B R.$$

Then

$$v_\ell(x_0) \;\ge\; \kappa + t - \frac{B}{\ell(\ell-1)} - K_\downarrow(b_1, B) \;\ge\; \kappa + t - \frac{B}{\ell(\ell-1)} - \max\Big(0,\; b_1 - \frac{B}{\ell} + 1 + \log_\ell b_1\Big).$$

Proof. Let ϖ be a uniformiser of $\mathcal{O}_L$ and $e = v_\varpi(\ell)$, so that $\ell\mathcal{O}_L = \varpi^e \mathcal{O}_L$ and $v_\ell = v_\varpi / e$ on $\mathcal{O}_L$. Two facts about R: $R/\varpi R = k_L[[X]]$ is a domain, so ϖ is a prime element of R; and $Q_0 = \Phi_\ell(1+X) \equiv X^{\ell-1} \pmod{\ell}$, so $Q_0 \notin \varpi R$. Hence if ϖ^j divides $Q_0^B h$ in R then ϖ^j divides h (induction on j).

Step 1: Q_2 modulo ℓ. Put $N = \ell(\ell-1)$. As $Q_0 = ((1+X)^\ell - 1)/X$, we have $(1+X)^\ell = 1 + XQ_0$. Let $Z = (1+X)^{\ell^2} - 1 = (1+XQ_0)^\ell - 1$; since ℓ divides $\binom{\ell}{j}$ for $0 < j < \ell$, $Z \equiv (XQ_0)^\ell \pmod{\ell R}$. Further

$$Q_2 = \Phi_\ell(1+Z) = \frac{(1+Z)^\ell - 1}{Z} = \sum_{j=1}^{\ell} \binom{\ell}{j} Z^{j-1} \equiv Z^{\ell-1} \pmod{\ell R},$$

so $Q_2 \equiv (XQ_0)^N \pmod{\ell R}$, i.e. $Q_2 = \ell g + Q_0^N h$ with $g, h \in R$.

Step 2: Q_2^t modulo Q_0^B. Put $A = \max(0, t - \lceil B/N \rceil + 1)$. By the binomial theorem $Q_2^t = \sum_{i=0}^t \binom{t}{i} (Q_0^N h)^i (\ell g)^{t-i}$; a term with $Ni \ge B$ lies in $Q_0^B R$, and a term with $Ni < B$ has

$i \le \lceil B/N \rceil - 1$, so it lies in $\ell^{t-i}R \subseteq \ell^A R$. Hence $Q_2^t \in \ell^A R + Q_0^B R$. Since $\lceil B/N \rceil < B/N + 1$, in both cases of the maximum $A \ge t - B/N$.

Step 3. Write $Q_1^{b_1} f = Q_2^t U_1$ and $Q_1^{b_1}(\ell^\kappa f - x_0) = Q_0^B U_2$ with $U_1, U_2 \in R$. Then $x_0 Q_1^{b_1} = \ell^\kappa Q_2^t U_1 - Q_0^B U_2 \in \ell^{\kappa+A} R + Q_0^B R$ by Step 2. Write $x_0 = u\varpi^n$ with $u \in \mathcal{O}_L^\times$, $n \ge 0$, and put $K = K_\downarrow(b_1, B) \ge 0$. We claim $n \ge e(\kappa + A - K)$. If $n \ge e(\kappa + A)$ this is clear. Otherwise $m = e(\kappa + A) - n \ge 1$ and $u\varpi^n Q_1^{b_1} = \varpi^{n+m} g' + Q_0^B h'$ with $g', h' \in R$. Then ϖ^n divides $Q_0^B h'$, hence $h' = \varpi^n h''$ by the two facts above, and cancelling ϖ^n in the domain R gives $uQ_1^{b_1} = \varpi^m g' + Q_0^B h''$, so $Q_1^{b_1} \in \varpi^m R + Q_0^B R$. By (Cd3) of Lemma C.36 (with $s = 0$, $y = b_1$), $Q_1^{b_1} \notin \varpi \ell^K R + Q_0^B R = \varpi^{1+eK} R + Q_0^B R$. Hence $m < 1 + eK$, i.e. $m \le eK$, which is the claim.

Conclusion. $v_\ell(x_0) = n/e \ge \kappa + A - K \ge \kappa + t - B/(\ell(\ell-1)) - K$, and $K \le \max(0,\, b_1 - B/\ell + 1 + \log_\ell b_1)$ by (Cd3) of Lemma C.36. □

Proposition C.43. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$), the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$, c, the frame $\mathbf{n} = (n_k)_{k \le r}$ at $c' = c - 3$ of $\mathbf{N}_2$ and k_0 satisfy the hypotheses of Proposition C.41; let H_{k_0} be the horizontal section through n_{k_0}, $G_0 = w + S$, $u_0 \in \mathbb{Z}_\ell^\times$ with $v_\ell(u_0 - 1) = 1$, and*

$$J = \varphi(\pi_0^M(\pi_0 + \ell)^{2G_0}) \in \mathbb{Z}_\ell[[X]],$$

so that J is Δ-invariant and $J = X^{m_X} Q_0^{m_X} Q_1^{2G_0} \varepsilon$ with a unit ε of $\mathbb{Z}_\ell[[X]]$. Then there are an integer $\kappa \ge 0$ admissible for $\mathbf{n}$ and m_X, and a Δ-invariant $\lambda \in P_2(\mathbf{N}_{2,c'})$, such that $(\gamma_{u_0} - 1)\lambda = Jy$ for a Δ-invariant $y \in P_2(\mathbf{N}_{2,c'})$, and

$$\ell^{\kappa'}\big(\ell^\kappa \lambda - \ell^{\sigma_B} \varphi^2(T_\kappa(H_{k_0}))\big) \in J\, P_2(\mathbf{N}_{2,c'}) \qquad \text{for some integer } \kappa' \ge 0.$$

Proof. Write $\mathbf{N} = \mathbf{N}_2$, $R = \mathcal{O}_L[[X]]$ for a site ring, $P_e = P_e(\mathbf{N}_{c'})$, $\gamma = \gamma_{u_0}$, $H = H_{k_0}$, $n = n_{k_0}$, $m = m_X$ and $J_1 = \pi_0^M(\pi_0 + \ell)^{2G_0}$, so $J = \varphi(J_1)$.

Elementary facts. (i) By Lemma 28.15 item 1, $\pi_0 = X^{\ell-1} v$ and $\pi_0 + \ell = Qw$ with units v, w of $\mathbb{Z}_\ell[[X]]$, and π_0 is Δ-invariant, hence so is $\pi_0 + \ell$; so $J_1 = X^m Q_0^{2G_0} \varepsilon_1$ ($m = (\ell-1)M$) with the unit $\varepsilon_1 = v^M w^{2G_0}$, $J_1 = \pi_0^M(\pi_0 + \ell)^{2G_0}$ is Δ-invariant as a product of Δ-invariant elements, and $J = \varphi(J_1) = X^m Q_0^m Q_1^{2G_0} \varphi(\varepsilon_1)$ since $\varphi(X) = XQ_0$, $\varphi(Q_0) = Q_1$. By Lemma C.16 (b), $\gamma_u(J_1) \in J_1 \mathbb{Z}_\ell[[X]]^\times$ for every u. (ii) P_e is stable under every γ_u (γ_u commutes with φ by (Wm3) and preserves $\mathbf{N}$ and $\mathbf{A}_{F,L}$), and $\varphi(P_1) \subseteq P_2$ since $\varphi(a\varphi(x)) = \varphi(a)\varphi^2(x)$. (iii) P_1 is free over $e_{c'+1}\mathbf{A}_{F,L}$ on $\varphi(n_1), \dots, \varphi(n_r)$: these generate it, and P_1 contains $Q^{h_{c'+1}} \mathbf{N}_{c'+1}$ by (Wm2) (the window of Lemma C.9 (1) is a window of $\mathbf{N}$ by (Ex1) of Proposition 23.7), so it has rank r, and a surjection from a free module of rank r onto a torsion-free module of rank r is injective. (iv) Saturation: if $x \in P_1$ and $\ell^k x \in J_1 P_1$, then $x \in J_1 P_1$. Indeed, coordinatewise in the basis of (iii), $\ell^k a \in X^m Q_0^{2G_0} R$ with $a \in R$ gives $a = X^m a'$ with $a' \in R$ (compare coefficients) and $\ell^k a' \in Q_0^{2G_0} R$, hence $a' \in Q_0^{2G_0} R$ by Lemma C.18 item 2 ($e = 1$, $c = \ell^k$). (v) For $k \ge 0$, $X^k \mathbf{N}_E \cap \mathbf{N}_{c'}[1/\ell] = X^k \mathbf{N}_{c'}[1/\ell]$ and $X^k \mathbf{N}_E \cap \mathbf{N}_{c'} = X^k \mathbf{N}_{c'}$ (compare coordinates in the basis (n_j)); and $\varphi(X^m \mathbf{N}_{c'}[1/\ell]) = X^m Q_0^m \varphi(\mathbf{N}_{c'})[1/\ell] \subseteq J_1 P_1[1/\ell]$ because $m \ge 2G_0$ ($m > 2h_B \ge 2G_0$).

A relay correction. The window has $M_a \le s_a + S \le h_B$ for all a and $M_{c'+1} = s_{c-2} + S \le G_0$ (the site $c - 2 = c' + 1$ is thin with sign -1). Apply Lemma 24.5 to $\mathbf{N}$ at the site c' with $j = 1$, $G = G_0$, $g = 2G_0$, $m = m_X \ge 1 + \max_a M_a$ and $R' = m_X$; let $\sigma = \sigma_\ell(2G_0, m_X, 1)$, so $\sigma \le \sigma_B$ (compare the definitions: $\sigma_B = \lceil (2G_0 + m_X/(\ell-1))\ell/(\ell-1)^2 \rceil + 1$). By (Wm3), $n \in I_{c'}$. By (Hf2) of Lemma 23.9, $H \in n + X\mathbf{N}_E$ and $\gamma_u(H) = H$ for all u; so $n - H \in X\mathbf{N}_E$ has $(\gamma_u - 1)(n - H) = (\gamma_u - 1)(n)$, and $W_n = n - H$ by the uniqueness in (Hf1). By (Ry1) there are $\tau \in P_1$ and $V \in \mathbf{N}_{c'}[1/\ell]$ with $V - (n - H) \in X^m \mathbf{N}_E$ and $\tau - \ell^\sigma \varphi(V) \in J_1 P_1[1/\ell]$ (here

$X^m\lambda_1^{2G_0}P_1 = J_1P_1$ by (i)). Fix an integer κ admissible for $\mathbf{n}$ and m (it exists: finitely many coefficients $h_{kj,i}$, $i<m$, each in E). Put

$$\lambda_0 = \ell^{\sigma_B}\varphi(n) - \ell^{\sigma_B-\sigma}\tau \ \in P_1.$$

Approximation. First, $\lambda_0 - \ell^{\sigma_B}\varphi(n-V) = \ell^{\sigma_B-\sigma}\big(\ell^{\sigma}\varphi(V)-\tau\big) \in J_1P_1[1/\ell]$. Second, $\ell^{\kappa}(n-V) - T_\kappa(H) = -\ell^{\kappa}(V-(n-H)) + \big(\ell^{\kappa}H - T_\kappa(H)\big)$ lies in $X^m\mathbf{N}_E$ (the second term by the definition of T_κ) and in $\mathbf{N}_{c'}[1/\ell]$, hence in $X^m\mathbf{N}_{c'}[1/\ell]$ by (v), and its image under φ lies in $J_1P_1[1/\ell]$. Together, $\ell^{\kappa}\lambda_0 - \ell^{\sigma_B}\varphi(T_\kappa(H)) \in J_1P_1[1/\ell]$; as it lies in P_1, there is $\kappa'\geq 0$ with

$$A := \ell^{\kappa'}\big(\ell^{\kappa}\lambda_0 - \ell^{\sigma_B}\varphi(T_\kappa(H))\big) \in J_1P_1.$$

Δ-*average.* Put $\bar\lambda_0 = (\ell-1)^{-1}\sum_{\delta\in\mu_{\ell-1}}\gamma_\delta(\lambda_0) \in P_1$ ($\ell-1$ is a unit), which is Δ-invariant. For each δ, $\gamma_\delta(T_\kappa(H)) - T_\kappa(H) = \gamma_\delta\big(T_\kappa(H)-\ell^{\kappa}H\big) - \big(T_\kappa(H)-\ell^{\kappa}H\big) \in X^m\mathbf{N}_E \cap \mathbf{N}_{c'} = X^m\mathbf{N}_{c'}$ ($\gamma_\delta H = H$, and γ_δ preserves $X^m\mathbf{N}_E$ as $\gamma_\delta(X)\in X\mathbb{Z}_\ell[[X]]^\times$). Averaging $\gamma_\delta(A)\in J_1P_1$ (by (i), (ii)) and using (v) gives $\ell^{\kappa'}\big(\ell^{\kappa}\bar\lambda_0 - \ell^{\sigma_B}\varphi(T_\kappa(H))\big) \in J_1P_1$.

The γ-clause. Apply $\gamma-1$ to the last membership: $(\gamma-1)(J_1P_1)\subseteq J_1P_1$ by (i), (ii), and $(\gamma-1)T_\kappa(H) = (\gamma-1)\big(T_\kappa(H)-\ell^{\kappa}H\big) \in X^m\mathbf{N}_E\cap\mathbf{N}_{c'} = X^m\mathbf{N}_{c'}$, whose image under φ lies in J_1P_1 by (v); γ commutes with φ. Hence $\ell^{\kappa'+\kappa}(\gamma-1)\bar\lambda_0 \in J_1P_1$, and by (iv) $(\gamma-1)\bar\lambda_0 = J_1y_0$ with $y_0\in P_1$, unique as P_1 is torsion-free. Since γ commutes with every γ_δ, $(\gamma-1)\bar\lambda_0$ is Δ-invariant, and so is J_1; uniqueness gives $\gamma_\delta(y_0)=y_0$.

Conclusion. Put $\lambda=\varphi(\bar\lambda_0)$ and $y=\varphi(y_0)$; both lie in P_2 by (ii) and are Δ-invariant (φ commutes with γ_δ). Then $(\gamma-1)\lambda = \varphi(J_1y_0) = Jy$, and applying φ to the averaged approximation gives $\ell^{\kappa'}\big(\ell^{\kappa}\lambda - \ell^{\sigma_B}\varphi^2(T_\kappa(H))\big) \in \varphi(J_1)\varphi(P_1) \subseteq J\,P_2$. □

Proposition C.44. *Let ℓ, F, n, L, (τ_a), Φ, the weight word, c, $\mathbf{n}$, k_0, u_0 be as in Proposition C.43, and let κ and λ satisfy the conclusion of Proposition C.43. Put $b_1 = s_{c-1}$, $t = s_{c-2}+S$ and*

$$s_0 = Q_1^{t}\,Q_0^{-b_1} \in \mathcal{K}_{c-1}.$$

Then $s_0\lambda \in \mathbf{N}_{2,c-1}$, and there are an integer E with $0\leq E\leq 8\,\mathrm{Sg}$ and $n_1\in\mathbf{N}_{1,c-1}$ with

$$\mathbf{N}(\Phi)(n_1) = \ell^{E}s_0\lambda.$$

Proof. Write $\mathbf{N}=\mathbf{N}_2$, $R = e_{c-1}\mathbf{A}_{F,L}$, $b_1=s_{c-1}$, $t=s_{c-2}+S$, $G_0 = w+S$, $\varsigma=\pi_0+\ell$, $\gamma=\gamma_{u_0}$, and $T = T_\kappa(H_{k_0})$. By the conclusion of Proposition C.43 there are $\kappa'\geq 0$, $y''\in P_2(\mathbf{N}_{c'})$ and a Δ-invariant $y\in P_2(\mathbf{N}_{c'})$ with $\ell^{\kappa'}(\ell^{\kappa}\lambda - \ell^{\sigma_B}\varphi^2(T)) = Jy''$ and $(\gamma-1)\lambda = Jy$, where $J = \varphi(\pi_0)^M\varphi(\varsigma)^{2G_0} = X^{m_X}Q_0^{m_X}Q_1^{2G_0}\cdot(\text{unit})$.

The arc and the numbers. As $\mathrm{len}_c = 3$, the sites c, $c-1$ are thick and $c-2 = c'+1$ is thin, with $\epsilon_{c-1}=1$; so $b\leq b_1\leq b+\eta$, $\eta < s_{c-2}\leq w$, $h = h_B = b+\eta+S \leq 1.1b$ (as $20\eta, 20S\leq b$), $\max_a s_a \leq b+\eta$, hence $H'\leq h+\eta\leq 1.05h$, and $\mathrm{Sg} = h/\ell^2 + (w+S+\eta)/\ell + \log_\ell(2h)+1$ with $160\,\mathrm{Sg}\leq b$, $\ell\geq 11$. Put $a=\lceil a_{\mathrm{R}}\rceil$. Then:

(N1) $b_1 \geq 2a_{\mathrm{R}}+\eta+1$;

(N2) $\frac{3a_{\mathrm{R}}+S+2\eta}{\ell-1} + \log_\ell(2H') + 1 \leq 8\,\mathrm{Sg}$;

(N3) $m_X \geq a + b_1 + G_0$.

Units. By Lemma 28.15 item 1, $\pi_0 = X^{\ell-1}v$ and $\varsigma = Q_0 w$ with units v, w of $\mathbb{Z}_\ell[[X]]$, and π_0, ς are Δ-invariant; so are $\varphi(\pi_0)$, $\varphi(\varsigma)$ (φ commutes with every γ_δ). Put $u = \varphi(w)^t w^{-b_1}$, a unit, and $s = \varphi(\varsigma)^t \varsigma^{-b_1} = s_0 u$, which is Δ-invariant. Further $\varphi(\pi_0) = (XQ_0)^{\ell-1}\varphi(v) = \pi_0 \varsigma^{\ell-1}\varepsilon$ with the unit $\varepsilon = v^{-1}w^{1-\ell}\varphi(v)$, which is Δ-invariant as a quotient of Δ-invariant elements of the domain $\mathbb{Z}_\ell[[X]]$.

(a) Membership. Consider the initial sub-arc $e_1 = 2$ of the arc of c: $c' + 2 = c - 1$ and $\Pi' = \Lambda_{c-1,2}(H_c) = Q_0^{h_B} Q_1^{G_0}$ ($H_c(c-1) = h_B$, $H_c(c'+1) = w + S$). The element $s_0\lambda$ lies in $e_{c-1}\mathbf{N}[1/\lambda_\infty]$, and $\Pi' s_0\lambda = Q_0^{h_B - b_1} Q_1^{G_0+t}\lambda \in P_2(\mathbf{N}_{c'})$ since $b_1 \le h_B$. Let $f = \ell^{\sigma_B} s_0 \mathbf{e}_{k_0}$; then $\Pi' f_{k_0} = \ell^{\sigma_B} Q_0^{h_B - b_1} Q_1^{G_0+t} \in R$, and

$$\ell^{\kappa'}(\ell^\kappa \Pi' s_0\lambda - \Pi' f_{k_0}\varphi^2(T)) = \Pi' s_0\, Jy'' \in X\Pi'^2 P_2(\mathbf{N}_{c'}),$$

because $\Pi' s_0 J = X^{m_X} Q_0^{m_X + h_B - b_1} Q_1^{3G_0+t} \cdot (\text{unit})$ and $X\Pi'^2 = XQ_0^{2h_B}Q_1^{2G_0}$, with $m_X \ge 1$ and $m_X \ge h_B + b_1$ ($m_X > 2h_B$). As κ is admissible for $\mathbf{n}$ and m_X, f is an arc coordinate vector of $s_0\lambda$ for $\mathbf{n}$, 2, H_c, m_X. It is Hodge-graded, with the one-term section decomposition $(g, \mathbf{v}, \alpha) = (\ell^{\sigma_B} s_0, \mathbf{e}_{k_0}, (-b_1, t))$, $g \in Q_0^{-b_1} Q_1^t R$: here $\delta_2(\mathbf{e}_{k_0}) = \varphi^2(d_{k_0})$, and (level 0) $\varphi^2(d_{k_0}) \in \varphi^4(\mathrm{Fil}^{b_1} D_{\mathrm{cris}}(V_2))$ by the hypothesis $\varphi^3(d_{k_0}) \in \varphi^5(\mathrm{Fil}^{s_{c-1}} D_{\mathrm{cris}}(V_2))$ of Proposition C.41 (φ is bijective on $D_{\mathrm{cris}}(V_2)$); (level 1) $\varphi^{-5}(\varphi^2(d_{k_0}))$ lies in the component $\varepsilon_{c-3} D_{\mathrm{cris}}(V_2)$ of the embedding τ_{c-2} (Lemma 23.10 and the remark in Definition 23.3), on which Fil^{-t} is everything, as every Hodge jump of V_2 at τ_{c-2} is $\ge -(s_{c-2} + S) = -t$ (Lemma C.16 (1)). By the Sub-arcs paragraph of Proposition C.24 ($\nu = 2$, $e_1 = 2$, $t = 0$, $\kappa = 0$; $\mathbf{n}$ is fixed by γ_u, $u \in \mu_{\ell-1}$), $s_0\lambda \in \mathbf{N}_{c-1}$. Hence $x := s\lambda = u\, s_0\lambda \in \mathbf{N}_{c-1}$, and x is Δ-invariant (s and λ are).

(b) The γ-clause. Put $x' = \lambda \in e_{c-1}\mathbf{N}[1/\lambda_\infty]$; then $\varsigma^{b_1} x = \varphi(\varsigma)^t x'$. Since $\varphi(\pi_0)^M = \pi_0^M \varsigma^{m_X}\varepsilon^M$ ($m_X = (\ell-1)M$),

$$\varphi(\varsigma)^t(\gamma - 1)(x') = \varphi(\varsigma)^t Jy = \pi_0 \varsigma^{a+b_1} z, \qquad z = \pi_0^{M-1}\varsigma^{m_X - a - b_1}\varepsilon^M \varphi(\varsigma)^{t+2G_0} y,$$

where $M \ge 1$ and $m_X - a - b_1 \ge G_0$ by (N3). By (Ry0) of Lemma 24.5 (for $\mathbf{N}$ with the window of Lemma C.9 (1), a window by (Ex1) of Proposition 23.7; site c', $j = 2$, $G = G_0 \ge M_{c-2} = t$, $M_{c-1} = \eta$; $g = 2G_0$, $m = R = m_X$), $(Q_0Q_1)^{G_0} P_2(\mathbf{N}_{c'}) \subseteq \mathbf{N}_{c-1}$; as $\varsigma^{G_0}\varphi(\varsigma)^{G_0} = (Q_0Q_1)^{G_0} \cdot (\text{unit})$ and the remaining factors of z lie in R, $z \in \mathbf{N}_{c-1}$; and z is Δ-invariant, all its factors being so.

(c) The lift. Apply Lemma C.45 with $i = c - 1$: $\ell \ge 11$ is prime, F is unramified, (τ_a) is a Frobenius numbering, Φ is an onto map of crystalline lattices with the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ (hypotheses of Proposition C.25), $\epsilon_{c-1} = 1$ and $s_{c-1} = b_1 \ge 2a_{\mathrm{R}} + \eta + 1$ by (N1); u_0 is the element of Proposition C.43, as allowed by Lemma 28.15. With x, x' and z from (a), (b) ($s_{i-1} + S = t$, $s_i = b_1$) it gives an integer E with $0 \le E \le \frac{3a_{\mathrm{R}} + S + 2\eta}{\ell - 1} + \log_\ell(2H') + 1 \le 8\,\mathrm{Sg}$ (N2) and $y_1 \in \mathbf{N}_{1,c-1}$ with $\mathbf{N}(\Phi)(y_1) = \ell^E x$. Put $n_1 = u^{-1}y_1 \in \mathbf{N}_{1,c-1}$; by R-linearity, $\mathbf{N}(\Phi)(n_1) = \ell^E u^{-1} s\lambda = \ell^E s_0\lambda$. □

Lemma C.45. *Let $\ell \ge 11$ be a prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of degree n, L as in Definition 23.1, $(\tau_a)_{a\in\mathbb{Z}/n}$ a Frobenius numbering, $\Phi\colon V_1 \to V_2$ an onto map of crystalline lattices $T_1 \to T_2$ over F with coefficients L, $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ a weight word for Φ, and*

$$a_{\mathrm{R}} = \frac{H'(2\ell - 1)}{(\ell - 1)^2} + 4 + \log_\ell(2H').$$

Let $i \in \mathbb{Z}/n$ with $\epsilon_i = 1$ and $s_i \ge 2a_{\mathrm{R}} + \eta + 1$, form π_0 and u_0 as in Lemma 28.15, and put $a = \lceil a_{\mathrm{R}} \rceil$. There is an integer E with

$$0 \le E \le \frac{3a_{\mathrm{R}} + S + 2\eta}{\ell - 1} + \log_\ell(2H') + 1$$

such that the following holds. Let $x \in \mathbf{N}(T_2)_i$ be Δ-invariant, let $x' \in e_i\mathbf{N}(T_2)[1/\lambda_\infty]$ with $(\pi_0+\ell)^{s_i}x = \varphi(\pi_0+\ell)^{\,s_{i-1}+S}x'$, and suppose that there is a Δ-invariant $z \in \mathbf{N}(T_2)_i$ with $\varphi(\pi_0+\ell)^{\,s_{i-1}+S}(\gamma_{u_0}-1)(x') = \pi_0(\pi_0+\ell)^{a+s_i}z$. Then there is a Δ-invariant $y \in \mathbf{N}(T_1)_i$ with $\mathbf{N}(\Phi)(y) = \ell^E x$.

Proof. Write $\varsigma = \pi_0 + \ell$ (local to this proof). By Lemma 28.15 (1), π_0 is Δ-invariant, $e_i\pi_0 = X^{\ell-1}v$ and $\varsigma = Q\kappa$ with units v, κ of $\mathbb{Z}_\ell[[X]]$; so ς is Δ-invariant and invertible in $\mathbf{A}_{F,L}[1/Q]$.

Step 0: numbers. Since $a_{\mathrm{R}} \geq 4$, $a \geq 1$; and $a < a_{\mathrm{R}} + 1$, so $2a + \eta < 2a_{\mathrm{R}} + \eta + 2 \leq s_i + 1$ by the hypothesis $s_i \geq 2a_{\mathrm{R}} + \eta + 1$; as all are integers, $2a + \eta \leq s_i$. Put $J_1 = \{s_i - a + 1, \dots, s_i\}$, $J_2 = \{s_i + 1, \dots, s_i + S + a\} \cup \{1 - \eta, \dots, \eta + a\}$ and

$$E = \Big\lfloor \frac{3a + S + 2\eta - 1}{\ell - 1} + \log_\ell(s_i + \eta) \Big\rfloor.$$

Then $E \geq 0$ (as $a \geq 1$ and $s_i + \eta \geq 1$), and $E \leq \frac{3a_{\mathrm{R}}+3+S+2\eta-1}{\ell-1} + \log_\ell(2H') \leq \frac{3a_{\mathrm{R}}+S+2\eta}{\ell-1} + \log_\ell(2H') + 1$, using $s_i + \eta \leq H' \leq 2H'$ and $\ell \geq 11$. By Lemma 28.15 (4) (with a, $b = s_i$, S, η; $2a + \eta \leq s_i$), $J_1 \cap J_2 = \emptyset$ and for every $j_0 \in J_1$ the integer $\sum_{j \in J_1 \setminus \{j_0\}} v_\ell(j_0 - j) + \sum_{j \in J_2} v_\ell(j_0 - j)$ is at most the real number in the floor, hence at most E.

Step 1: ς^a is killing at i for Φ. Every Hodge jump of V_1 at τ_c lies in $[-M_c, h_c]$ by (Ow1) (for $\epsilon_c = 1$ the jumps lie in $[s_c, s_c + S] \cup [-\eta, \eta] \subseteq [-\eta, s_c + S]$, for $\epsilon_c = -1$ in $[-s_c - S, -s_c] \cup [-\eta, \eta] \subseteq [-(s_c + S), \eta]$). Apply Lemma C.7 to the surjection $\Phi\colon T_1 \to T_2$ ($\ell \geq 5$) with this window: $H(c) = s_c + S + \eta \leq H'$, so $H_{\max} \leq H'$ and $c_H \leq 2 + \log_\ell(H' + 1) \leq 2 + \log_\ell(2H')$ ($H' \geq 1$). By (Ki3) with $\pi = e_i\pi_0$, $(e_i\pi_0 + \ell)^{A_i + r_i}$ is killing at i, where

$$A_i + r_i \leq \frac{H'}{\ell - 1} + 1 + \frac{H'\ell}{(\ell-1)^2} + 2 + \log_\ell(2H') = \frac{H'(2\ell - 1)}{(\ell-1)^2} + 3 + \log_\ell(2H') < a_{\mathrm{R}} \leq a.$$

As $\mathbf{N}(\Phi)(\mathbf{N}(T_1)_i)$ is an $e_i\mathbf{A}_{F,L}$-module, $e_i\varsigma^a = \varsigma^{\,a - A_i - r_i}(e_i\pi_0 + \ell)^{A_i + r_i}$ is killing at i; it is Δ-invariant.

Step 2: the letters of T_1. By Step 1 and (Ki4) (applied to $t = e_i\varsigma^a$ and x), there is $x_1 \in \mathbf{N}_1^\Delta$ with $\mathbf{N}(\Phi)(x_1) = \varsigma^a x$. Put $z_1 = \varsigma^{-a}x_1 \in e_i\mathbf{N}(T_1)[1/Q]$; it is Δ-invariant, $Q^a z_1 = \kappa^{-a}x_1 \in \mathbf{N}(T_1)$, and $\mathbf{N}(\Phi)(z_1) = x$ ($\mathbf{N}(\Phi)$ is $\mathbf{A}_{F,L}[1/Q]$-linear). The Hodge jumps j of V_1 at τ_i lie in $[s_i, s_i + S] \cup [-\eta, \eta]$ (Ow1, $\epsilon_i = 1$), so $j + k \in J_2$ for $1 \leq k \leq a$. By Lemma C.46 (pair (V_1, T_1), site i, $Y = a \geq 1$, $J = J_2$), $P_{J_2}(\nabla)z_1 \in \mathbf{N}(T_1)_i$; it is Δ-invariant, being a $\mathbb{Z}_\ell$-linear combination of the $\nabla^k z_1 \in \mathbf{N}_{1,\infty}^\Delta$ (Lemma 28.15 (1)). Since $\mathbf{N}(\Phi) \circ \nabla = \nabla \circ \mathbf{N}(\Phi)$ on $\mathbf{N}_{1,\infty}^\Delta$,

$$P_{J_2}(\nabla)\,x = \mathbf{N}(\Phi)\big(P_{J_2}(\nabla)z_1\big) \in \mathbf{N}(\Phi)(\mathbf{N}_1^\Delta).$$

Step 3: the top layers. By Lemma C.47 (for the pair (V_2, T_2) at the site i, with its integers a, b, h taken as a, s_i, $s_{i-1} + S$) there is a Δ-invariant $y' \in \mathbf{N}(T_2)_i$ with $P_{J_1}(\nabla)x = \varsigma^a y'$. By Step 1 and (Ki4), $\varsigma^a y' = \mathbf{N}(\Phi)(y_1)$ with $y_1 \in \mathbf{N}_1^\Delta$. So $P_{J_1}(\nabla)x \in \mathbf{N}(\Phi)(\mathbf{N}_1^\Delta)$.

Step 4: separation. We have $x \in \mathbf{N}_2^\Delta$, the sets J_1, J_2 are finite and disjoint, E satisfies the valuation condition of Lemma 28.15 (3) (Step 0), and $P_{J_1}(\nabla)x$, $P_{J_2}(\nabla)x$ lie in $\mathbf{N}(\Phi)(\mathbf{N}_1^\Delta)$ (Steps 2, 3). By Lemma 28.15 (3), $\ell^E x = \mathbf{N}(\Phi)(y)$ for some $y \in \mathbf{N}_1^\Delta$, a Δ-invariant element of $\mathbf{N}(T_1)_i$. □

Lemma C.46. *Let ℓ, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ be as in Proposition 23.7, (V, T) a pair, $c \in \mathbb{Z}/n$, and form u_0, $c(j)$, P_J and ∇ as in Lemma 28.15 (for $T_1 = T_2 = T$). Let $Y \geq 1$ be an integer and $J \subset \mathbb{Z}$ a finite set containing $a + k$ for every Hodge jump a of V at τ_c and every integer $1 \leq k \leq Y$. If $z \in e_c\mathbf{N}(T)[1/Q]$ is Δ-invariant and $Q^Y z \in \mathbf{N}(T)$, then $P_J(\nabla)z \in \mathbf{N}(T)_c$.*

Proof. Let J_c be the set of Hodge jumps of V at τ_c (finite) and $Z = \{a+k : a \in J_c,\ 1 \le k \le Y\}$; by hypothesis $Z \subseteq J$. Let $x_1, \dots, x_m$ be an enumeration of Z without repetition, and put

$$y_0 = z, \qquad y_i = (\nabla - c(x_i))(y_{i-1}) \quad (1 \le i \le m).$$

By item 1 of Lemma 28.15, ∇ is an $\mathcal{O}_L$-linear map of $\mathbf{N}_\infty^\Delta$ into itself and $c(x_i) \in \mathbb{Z}_\ell$, so by induction every y_i lies in $\mathbf{N}_\infty^\Delta \subseteq e_c\mathbf{N}[1/Q] \subseteq e_c\mathbf{N}[1/\lambda_\infty]$ (the second inclusion because $Q = Q_0$ is among the inverted elements and $\mathbf{N}_c$ is free over the domain $e_c\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ by (Wm1), so the localisation maps are injective). By the defining property of ∇,

$$\pi_0(y_i + c(x_i)\, y_{i-1}) = \pi_0\, \nabla y_{i-1} = (\gamma_{u_0} - 1)(y_{i-1}) \qquad (1 \le i \le m),$$

and $Q^Y y_0 = Q^Y z \in \mathbf{N}$. Apply Lemma 24.8 to the pair (V, T), the site c, the finite set J_c (it contains every Hodge jump of V at τ_c), $u = u_0$ (which lies in $1 + \ell\mathbb{Z}_\ell$ since $v_\ell(u_0 - 1) = 1$; the numbers $c(x)$ of that lemma for $u = u_0$ are ours), the integer Y and the enumeration $x_1, \dots, x_m$ of $\{j + k : j \in J_c,\ 1 \le k \le Y\} = Z$: it gives $y_m \in \mathbf{N}$. As $y_m = e_c y_m$, $y_m \in e_c\mathbf{N} = \mathbf{N}_c$, and y_m is Δ-invariant, so $y_m \in \mathbf{N}^\Delta$.

The operators $\nabla - c(x)$ are polynomials in the one operator ∇ and commute, so $y_m = P_Z(\nabla)z$; since $Z \subseteq J$, $P_J = P_{J\setminus Z}\, P_Z$ and

$$P_J(\nabla)z = P_{J\setminus Z}(\nabla)(y_m).$$

By item 1 of Lemma 28.15, ∇ maps $\mathbf{N}^\Delta$ into itself; hence so does the polynomial $P_{J\setminus Z}(\nabla)$ with coefficients in $\mathbb{Z}_\ell$, and $P_J(\nabla)z \in \mathbf{N}^\Delta \subseteq \mathbf{N}(T)_c$.

Remark. No comparison of $\mathbf{N}(T)$ with Berger's Wach module is needed: Lemma 24.8 computes the action of γ_{u_0} on the pole layers directly through the de Rham expansion at the site c. Rationally, the statement for $Y = 1$ is consistent with the description of the action of Γ_1 on N_V/q in [24] (eigenvalues given by the Hodge–Tate weights), twisted by the first pole layer. □

Lemma C.47. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ be as in Proposition 23.7, (V, T) a pair, $i \in \mathbb{Z}/n$ and $a, b, h \ge 0$ integers. Form π_0, u_0, $c(j)$, P_J and ∇ as in Lemma 28.15 for $\mathbf{N}(T)$ at the site i (with $T_1 = T_2 = T$), and put $J_1 = \{b - a + 1, \dots, b\}$. Let $x \in \mathbf{N}(T)_i$ be Δ-invariant, let $x' \in e_i\mathbf{N}(T)[1/\lambda_\infty]$ with $(\pi_0 + \ell)^b x = \varphi(\pi_0 + \ell)^h x'$, and suppose that there is a Δ-invariant $z \in \mathbf{N}(T)_i$ with*

$$\varphi(\pi_0 + \ell)^h\, (\gamma_{u_0} - 1)(x') = \pi_0\, (\pi_0 + \ell)^{a+b}\, z.$$

Then there is a Δ-invariant $y \in \mathbf{N}(T)_i$ with $P_{J_1}(\nabla)\, x = (\pi_0 + \ell)^a y$.

Proof. Write $\gamma = \gamma_{u_0}$ and $\varsigma = \pi_0 + \ell$. All elements of $\mathbb{Z}_\ell[[X]]$ act on the site i through e_i. By Lemma 28.15 (1), $\pi_0 = X^{\ell-1}v$ and $\varsigma = Qw$ with units v, w of $\mathbb{Z}_\ell[[X]]$, and π_0, ς are Δ-invariant. Since $v_\ell(u_0 - 1) = 1$, $u_0 \equiv 1 \bmod \ell$. The module $\mathbf{N}(T)_i$ is free over the domain R ((Wm1)), so $e_i\mathbf{N}(T)[1/\lambda_\infty]$ is free over the domain $R[1/Q_s : s \ge 0]$ and every non-zero element of R is a non-zero-divisor on it.

Step 1: the residue map. Let ζ be a primitive ℓ-th root of unity and $\mathcal{O}' = \mathcal{O}_L \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[\zeta]$, a free $\mathcal{O}_L$-module with basis $(\zeta - 1)^k$, $k < \ell - 1$; it is ℓ-torsion-free, and $\zeta - 1$ is a non-zero-divisor on it ($\mathcal{O}'$ is flat over the domain $\mathbb{Z}_\ell[\zeta]$). As $(\zeta - 1)^{\ell-1} \in \ell\mathcal{O}'$, there is a unique continuous $\mathcal{O}_L$-algebra homomorphism $\varrho\colon R \to \mathcal{O}'$ with $\varrho(X) = \zeta - 1$. Then:

1. $\ker\varrho = QR = \varsigma R$, and $\ker\varrho \cap R^\Delta = \varsigma R^\Delta$. Indeed $\varrho(Q) = \Phi_\ell(\zeta) = 0$; conversely, if $\varrho(f) = 0$, Weierstrass division by the distinguished polynomial Q of degree $\ell - 1$ gives $f = Qg + r$ with $r \in \mathcal{O}_L[X]$ of degree $< \ell - 1$, and $r(\zeta - 1) = 0$ forces $r = 0$ by the choice of basis. If moreover $f = \varsigma g'$ is Δ-invariant, then $\varsigma\gamma_\delta(g') = \gamma_\delta(f) = f = \varsigma g'$ and $g' \in R^\Delta$, R being a domain.

2. $\varrho\circ\gamma = \varrho$: both are continuous $\mathcal{O}_L$-algebra maps, and $\varrho(\gamma X) = \zeta^{u_0} - 1 = \zeta - 1$ (approximate u_0 by integers $\equiv 1 \bmod \ell$).

3. $\varrho(\varphi(f)) = f(0)$ for $f \in \mathbb{Z}_\ell[[X]]$: $\varrho \circ \varphi$ is a continuous ring map sending X to $\zeta^\ell - 1 = 0$.

4. $\varrho(\pi_0) = -\ell$, since $\varrho(\varsigma) = \varrho(Qw) = 0$.

Δ-invariant multiples of X. If $f \in XR$ is Δ-invariant and non-zero with lowest term $c_k X^k$ ($k \geq 1$), the lowest term of $\gamma_\delta(f)$ is $c_k \delta^k X^k$, so $\delta^k = 1$ for all $\delta \in \mu_{\ell-1}$, $(\ell-1) \mid k$, and $f = \pi_0 g$ with $g \in R$, Δ-invariant as above. Hence

$$(XR)^\Delta = \pi_0 R^\Delta. \qquad (1)$$

As $\gamma(X) \in XR$, γ preserves constant terms; it commutes with every γ_δ ((Wm3) on R: $\gamma_u\gamma_v = \gamma_{uv}$).

Step 2: three units. (a) For $g \in R^\Delta$, $\gamma(g) - g$ is Δ-invariant, has constant term 0 and $\varrho(\gamma g - g) = 0$ (item 2). By (1), $\gamma(g) - g = \pi_0 g_1$ with $g_1 \in R^\Delta$; then $-\ell\varrho(g_1) = 0$, so $\varrho(g_1) = 0$ and $g_1 \in \varsigma R^\Delta$ (item 1): $\gamma(g) - g = \pi_0\varsigma\tilde{g}$ with $\tilde{g} \in R^\Delta$.

(b) Let $\eta_0 = \gamma(X)/X = ((1+X)^{u_0} - 1)/X \in \mathbb{Z}_\ell[[X]]$, a unit with $\eta_0(0) = u_0$. From $Q = \varphi(X)/X$ and $\gamma\varphi = \varphi\gamma$ on $\mathbb{Z}_\ell[[X]]$ (both ring maps, equal on X): $X\eta_0\gamma(Q) = \gamma(\varphi X) = \varphi(X\eta_0) = XQ\varphi(\eta_0)$, so $\gamma(Q) = Q\varphi(\eta_0)\eta_0^{-1}$ and

$$\gamma(\varsigma) = t\,\varsigma, \qquad t = \varphi(\eta_0)\,\eta_0^{-1}\,\gamma(w)\,w^{-1} \in \mathbb{Z}_\ell[[X]]^\times.$$

The unit t is Δ-invariant ($\varsigma t = \gamma(\varsigma)$ is, and R is a domain). By item 3, $\varrho(\varphi(\eta_0)) = \eta_0(0) = u_0$; by item 2, $(\zeta - 1)\varrho(\eta_0) = \varrho(\gamma X) = \zeta - 1$, so $\varrho(\eta_0) = 1$; and $\varrho(\gamma w) = \varrho(w)$, a unit. Hence $\varrho(t) = u_0$, and $t(0) = 1$ (compare constant terms in $\gamma(\varsigma) = t\varsigma$, $\varsigma(0) = \ell$).

(c) For $k \geq 0$, $t^k - 1$ is Δ-invariant with constant term 0, so by (1) $t^k = 1 + \pi_0 T_k$ with $T_k \in R^\Delta$, and $-\ell\varrho(T_k) = u_0^k - 1$. The element $\varphi(t) \in \mathbb{Z}_\ell[[X]]$ is Δ-invariant (φ commutes with the γ_δ) and $\gamma(\varphi\varsigma) = \varphi(\gamma\varsigma) = \varphi(t)\varphi(\varsigma)$; $\varphi(t)^h - 1$ has constant term $t(0)^h - 1 = 0$, so $\varphi(t)^h = 1 + \pi_0 A$ with $A \in R^\Delta$, and $-\ell\varrho(A) = \varrho(\varphi(t))^h - 1 = t(0)^h - 1 = 0$, so $\varrho(A) = 0$.

Step 3: ∇x. Apply the semilinear γ to $\varsigma^b x = \varphi(\varsigma)^h x'$ and use the hypothesis on $(\gamma - 1)(x')$:

$$t^b\varsigma^b\gamma(x) = \varphi(t)^h\varphi(\varsigma)^h(x' + (\gamma-1)x') = \varphi(t)^h(\varsigma^b x + \pi_0\varsigma^{a+b}z).$$

Cancelling ς^b gives $t^b\gamma(x) = \varphi(t)^h(x + \pi_0\varsigma^a z)$. Insert $\gamma(x) = x + \pi_0\nabla x$, $t^b = 1 + \pi_0 T_b$, $\varphi(t)^h = 1 + \pi_0 A$, and cancel π_0:

$$\nabla x = e_1 x + \varsigma^a z_1, \qquad e_1 = t^{-b}(A - T_b) \in R^\Delta, \quad z_1 = t^{-b}\varphi(t)^h z \in \mathbf{N}^\Delta.$$

By Step 2, $\ell\varrho(e_1) = u_0^{-b}(0 + (u_0^b - 1)) = 1 - u_0^{-b} = \ell c(b)$, so $\varrho(e_1) = c(b)$ ($\mathcal{O}'$ is ℓ-torsion-free).

Step 4: two product rules. For $n \in \mathbf{N}^\Delta$ and $k \geq 0$, $\gamma(\varsigma^k n) - \varsigma^k n = \varsigma^k((t^k - 1)\gamma(n) + \pi_0\nabla n)$, so

$$\nabla(\varsigma^k n) = \varsigma^k(T_k\gamma(n) + \nabla n), \qquad T_k\gamma(n) + \nabla n \in \mathbf{N}^\Delta,$$

as γ preserves $\mathbf{N}(T)_i$ and commutes with the γ_δ, and ∇ preserves $\mathbf{N}^\Delta$ (Lemma 28.15 (1)). For $g \in R^\Delta$ and $f = \varsigma^k g$: $\gamma(f) = t^k\varsigma^k\gamma(g)$ and, by Step 2 (a), (c), $\gamma(f) - f = \varsigma^k((t^k - 1)\gamma(g) + \gamma(g) - g) = \pi_0\varsigma^k(T_k\gamma(g) + \varsigma\tilde{g})$; hence $\pi_0\nabla(fx) = (\gamma f - f)x + \pi_0\gamma(f)\nabla x$ and, by Step 3,

$$\nabla(\varsigma^k g x) = \varsigma^k(T_k\gamma(g) + \varsigma\tilde{g} + t^k\gamma(g)e_1)x + \varsigma^a \cdot \varsigma^k t^k\gamma(g)z_1.$$

Step 5: induction on the layers. For $k \geq 0$ let $\mathcal{V}_k$ be the set of $y = \varsigma^k g x + \varsigma^a n$ with $g \in R^\Delta$, $n \in \mathbf{N}^\Delta$. We claim $(\nabla - c(b-k))(\mathcal{V}_k) \subseteq \mathcal{V}_{k+1}$. For such y, Step 4 gives

$$(\nabla - c(b-k))y = \varsigma^k Y x + \varsigma^a n', \qquad Y = T_k\gamma(g) + \varsigma\tilde{g} + t^k\gamma(g)e_1 - c(b-k)g \in R^\Delta,$$

with $n' = \varsigma^k t^k \gamma(g) z_1 + T_a\gamma(n) + \nabla n - c(b-k)n \in \mathbf{N}^\Delta$. By Steps 1–3, $\varrho(\gamma g) = \varrho(g)$ and

$$\ell\varrho(Y) = \varrho(g)\big((1-u_0^k) + u_0^k(1-u_0^{-b}) - (1-u_0^{k-b})\big) = 0,$$

so $\varrho(Y) = 0$ and $Y = \varsigma Y'$ with $Y' \in R^\Delta$ (item 1). Hence $(\nabla - c(b-k))y = \varsigma^{k+1}Y'x + \varsigma^a n' \in \mathcal{V}_{k+1}$.

Conclusion. We have $x \in \mathcal{V}_0$ ($g = 1$, $n = 0$). Put $y_0 = x$ and $y_{k+1} = (\nabla - c(b-k))y_k$ for $0 \leq k < a$; by the claim $y_k \in \mathcal{V}_k$, so $y_a = \varsigma^a(gx + n)$ with $g \in R^\Delta$, $n \in \mathbf{N}^\Delta$. As $\{b - k : 0 \leq k < a\} = J_1$ and the operators $\nabla - c(j)$ are polynomials in the one $\mathcal{O}_L$-linear operator ∇ on $\mathbf{N}^\Delta$, $y_a = P_{J_1}(\nabla)x$. So $y = gx + n$ is a Δ-invariant element of $\mathbf{N}(T)_i$ with $P_{J_1}(\nabla)x = (\pi_0 + \ell)^a y$. □

Proposition C.48. *Let ℓ, F, n, L, (τ_a), Φ, the weight word, c, $\mathbf{n}$, k_0, u_0 be as in Proposition C.43, let κ and λ satisfy the conclusion of Proposition C.43, and let E and n_1 satisfy the conclusion of Proposition C.44 for these κ, λ. Put $b_1 = s_{c-1}$, $t = s_{c-2} + S$, $B = s_c + S$. Then there are an integer $\sigma_1 \geq 0$ with*

$$\sigma_1 \leq \frac{B\,\ell}{(\ell-1)^2}$$

and $y_1 \in P_1(\mathbf{N}_{1,c-1})$ with $y_1 \in I_c$ and $Q_0^B y_1 \in \mathbf{N}_{1,c}$, and there are a finite product g_1 of the Q_s with $s \geq 1$ and an integer $\kappa_1 \geq 0$ with

$$\ell^{\kappa_1} g_1\Big(\ell^{\,b_1+\kappa}\,\mathbf{N}(\Phi)(y_1) - \ell^{\,\sigma_1+E+\sigma_B+t}\,\varphi^3\big(T_\kappa(H_{k_0})\big)\Big) \in Q_0^B\,P_1(\mathbf{N}_{2,c-1}).$$

Proof. Write $\psi = \mathbf{N}(\Phi)$, $G_0 = w + S$, $T = T_\kappa(H_{k_0})$, $H = H_{k_0}$, and $\mathbf{N}_{\nu,E}$ for the module $\mathbf{N}_E$ of $\mathbf{N}_\nu$. As in Proposition C.49, $c-1$ is an up site and $c-2$, c are down sites of the arc, so the window has $M_c = s_c + S = B$, $M_{c-1} = \eta \leq G_0$, $M_{c-2} = t \leq G_0$, and $B \leq h_B < m_X$.

Conventions on $\mathbf{N}_E$. Every Q_s is a unit of $E[[X]]$ ($Q_s(0) = \ell$), so $\mathbf{N}[1/\lambda_\infty] \subseteq \mathbf{N}_E$, φ extends to a φ-semilinear map of $\mathbf{N}_E$ commuting with every γ_u and mapping $X^k\mathbf{N}_E$ into $X^k\mathbf{N}_E$ ($\varphi(X) = XQ_0$), and ψ extends $E[[X]]$-linearly, commuting with φ and the γ_u (Proposition 23.7). For a site component and $k \geq 0$, $X^k\mathbf{N}_E \cap \mathbf{N}_a = X^k\mathbf{N}_a$ (compare coordinates in a basis). Multiplication by a product of the Q_s preserves $X^k\mathbf{N}_E$.

Step 1: the horizontal section through $s_0\lambda$. By the conclusion of Proposition C.43, $\ell^\kappa\lambda - \ell^{\sigma_B}\varphi^2(T) \in \ell^{-\kappa'}JP_2(\mathbf{N}_{2,c'}) \subseteq X\mathbf{N}_{2,E}$ ($J \in X\mathbb{Z}_\ell[[X]]$), and $\varphi^2(T) - \ell^\kappa\varphi^2(H) = \varphi^2(T - \ell^\kappa H) \in \varphi^2(X^{m_X}\mathbf{N}_{2,E}) \subseteq X^{m_X}\mathbf{N}_{2,E}$. Hence $\lambda - \ell^{\sigma_B}\varphi^2(H) \in X\mathbf{N}_{2,E}$. With $s_0 = Q_1^t Q_0^{-b_1}$ (a unit of $E[[X]]$ with constant term ℓ^{t-b_1}), $s_0\lambda - \ell^{t-b_1}\lambda = (s_0 - \ell^{t-b_1})\lambda \in X\mathbf{N}_{2,E}$. So the element $\ell^{t-b_1+\sigma_B}\varphi^2(H)$, which is invariant under every γ_u (H is, and γ_u commutes with φ), lies in $s_0\lambda + X\mathbf{N}_{2,E}$; as $s_0\lambda \in \mathbf{N}_{2,c-1}$ (Proposition C.44), it is the horizontal section through $s_0\lambda$ by the uniqueness in (Hf2) of Lemma 23.9.

Step 2: the horizontal section through n_1. Let $H^{(1)}$ be the horizontal section of $\mathbf{N}_1$ through n_1 ((Hf2) for the Wach module $\mathbf{N}_1$). Then $\psi(H^{(1)})$ is γ_u-invariant and lies in $\psi(n_1) + X\mathbf{N}_{2,E} = \ell^E s_0\lambda + X\mathbf{N}_{2,E}$, so by uniqueness and Step 1

$$\ell^{b_1}\psi(H^{(1)}) = \ell^{E+t+\sigma_B}\varphi^2(H).$$

Step 3: the lift. Put $\sigma_1 = s_\ell(B-1)$; then $\sigma_1 \le (B-1)\ell/(\ell-1)^2 \le B\ell/(\ell-1)^2$ (Lemma 23.9). By (Hf3) there is $x_B \in \mathbf{N}_{1,c-1}$ with $\ell^{\sigma_1}H^{(1)} - x_B \in X^B\mathbf{N}_{1,E}$. Put $y_1 = \varphi(x_B) \in P_1(\mathbf{N}_{1,c-1})$. By (Wm2) for $\mathbf{N}_1$ with the window of Lemma C.9 ((Ex1) of Proposition 23.7), $Q^{M_c}\varphi(\mathbf{N}_{1,c-1}) \subseteq \mathbf{N}_{1,c}$, so $Q_0^B y_1 \in \mathbf{N}_{1,c}$. For every u, $(\gamma_u - 1)x_B = (\gamma_u - 1)(x_B - \ell^{\sigma_1}H^{(1)}) \in X^B\mathbf{N}_{1,E} \cap \mathbf{N}_{1,c-1} = X^B\mathbf{N}_{1,c-1}$, hence $(\gamma_u - 1)y_1 = \varphi((\gamma_u-1)x_B) \in X^B Q_0^B\varphi(\mathbf{N}_{1,c-1}) \subseteq X^B\mathbf{N}_{1,c} \subseteq X\mathbf{N}_1$ ($B \ge 1$); so $y_1 \in I_c$.

Step 4: the congruence. Let

$$D' = \ell^{b_1+\kappa}\psi(x_B) - \ell^{\sigma_1+E+t+\sigma_B}\varphi^2(T).$$

By Step 2, $\ell^{b_1+\kappa}\psi(x_B) - \ell^{\kappa+\sigma_1+E+t+\sigma_B}\varphi^2(H) = \ell^{b_1+\kappa}\psi(x_B - \ell^{\sigma_1}H^{(1)}) \in X^B\mathbf{N}_{2,E}$, and $\ell^\kappa\varphi^2(H) - \varphi^2(T) \in X^{m_X}\mathbf{N}_{2,E} \subseteq X^B\mathbf{N}_{2,E}$; so $D' \in X^B\mathbf{N}_{2,E}$. By (Ry0) of Lemma 24.5 (site c', $j=2$, $G = G_0$, $g = 2G_0$, $m = R = m_X$), $(Q_0Q_1)^{G_0}P_2(\mathbf{N}_{2,c'}) \subseteq \mathbf{N}_{2,c-1}$; as $\varphi^2(T) \in P_2(\mathbf{N}_{2,c'})$ and $\psi(x_B) \in \mathbf{N}_{2,c-1}$, we get $(Q_0Q_1)^{G_0}D' \in \mathbf{N}_{2,c-1} \cap X^B\mathbf{N}_{2,E} = X^B\mathbf{N}_{2,c-1}$. Apply φ and put $g_1 = (Q_1Q_2)^{G_0}$, $\kappa_1 = 0$:

$$g_1\Big(\ell^{b_1+\kappa}\psi(y_1) - \ell^{\sigma_1+E+\sigma_B+t}\varphi^3(T)\Big) = \varphi\big((Q_0Q_1)^{G_0}D'\big) \in X^B Q_0^B\varphi(\mathbf{N}_{2,c-1}) \subseteq Q_0^B P_1(\mathbf{N}_{2,c-1}),$$

using $\psi\circ\varphi = \varphi\circ\psi$. □

Proposition C.49. *Let ℓ, F, n, L, (τ_a), Φ, the weight word, c, $\mathbf{n}$, k_0 and f_2 satisfy the hypotheses of Proposition C.41, with $\kappa_0 \ge 0$ and $x_0 \in \mathcal{O}_L$ such that $Q_1^{s_{c-1}}(\ell^{\kappa_0}f_2 - e_c(1\otimes x_0)) \in Q_0^{s_c+S}R_c$. Let u_0 be as in Proposition C.43, let κ and λ satisfy the conclusion of Proposition C.43, and suppose $s_0\lambda \in \mathbf{N}_{2,c-1}$. Put $B = s_c + S$ and $V = f_2\,\varphi(\lambda)$. Then $\Pi_c f_2 \in R_c$, $\varphi(\lambda) \in P_3(\mathbf{N}_{2,c'})$, $V \in P_1(\mathbf{N}_{2,c-1})$, the vector $\ell^{\sigma_B}f_2\,\mathbf{e}_{k_0}$ is an arc coordinate vector of V for $\mathbf{n}$, 3, H_c and m_X, and there are a finite product g_2 of the Q_s with $s \ge 1$ and an integer $\kappa_2 \ge 0$ with*

$$\ell^{\kappa_2}g_2\Big(\ell^{\,\kappa+\kappa_0}\,V - \ell^{\sigma_B}x_0\,\varphi^3(T_\kappa(H_{k_0}))\Big) \in Q_0^B\,P_1(\mathbf{N}_{2,c-1}).$$

Proof. Write $\mathbf{N} = \mathbf{N}_2$, $R_a = e_a\mathbf{A}_{F,L}$, $b_1 = s_{c-1}$, $t = s_{c-2} + S$, $G_0 = w + S$, $T = T_\kappa(H_{k_0})$, $P_1 = P_1(\mathbf{N}_{c-1})$. The sites $c-1$ and c are thick and $c-2$ is thin (as $\mathrm{len}_c = 3$), with signs $+1$, -1, -1 at $c-1$, c, $c-2$; so $b_1 \le b + \eta \le h_B$, $B \le h_B$, $\eta < s_{c-2} \le w$, and the window has $M_{c-1} = \eta \le G_0$, $M_{c-2} = t \le G_0$.

Three facts. (i) By Lemma 28.15 item 1 and $\varphi(X) = XQ_0$, $\varphi^s(Q_0) = Q_s$: $J = X^{m_X}Q_0^{m_X}Q_1^{2G_0}\varepsilon$ and $\varphi(J) = X^{m_X}Q_0^{m_X}Q_1^{m_X}Q_2^{2G_0}\varphi(\varepsilon)$ with units $\varepsilon, \varphi(\varepsilon)$. Since $m_X > 2h_B$, $B \le h_B$ and $\Pi_c = Q_0^{h_B}Q_1^{h_B}Q_2^{G_0}$, this gives $\varphi(J) \in Q_0^B R_c$ and $\varphi(J) \in X\Pi_c^2 R_c$. (ii) By (Ry0) of Lemma 24.5 (for $\mathbf{N}$ with the window of Lemma C.9 (1), a window by (Ex1) of Proposition 23.7; site c', $j=2$, $G = G_0$, which bounds M_{c-2} and M_{c-1}, $g = 2G_0$, $m = R = m_X \ge 1 + \max_a M_a$), $(Q_0Q_1)^{G_0}P_2(\mathbf{N}_{c'}) \subseteq \mathbf{N}_{c-1}$; applying φ, $(Q_1Q_2)^{G_0}\varphi(x) \in \varphi(\mathbf{N}_{c-1}) \subseteq P_1$ for $x \in P_2(\mathbf{N}_{c'})$. (iii) $\varphi(P_e(\mathbf{N}_{c'})) \subseteq P_{e+1}(\mathbf{N}_{c'})$, since $\varphi(a\varphi^e(x)) = \varphi(a)\varphi^{e+1}(x)$.

The data. By hypothesis there are $u, u' \in R_c$ with $Q_1^{b_1}f_2 = Q_2^t u$ and $Q_1^{b_1}(\ell^{\kappa_0}f_2 - x_0) = Q_0^B u'$ (here x_0 stands for $e_c(1\otimes x_0)$). By the conclusion of Proposition C.43 there are $\kappa' \ge 0$ and $y' \in P_2(\mathbf{N}_{c'})$ with $\ell^{\kappa'}(\ell^\kappa\lambda - \ell^{\sigma_B}\varphi^2(T)) = Jy'$; applying φ,

$$\ell^{\kappa'}(\ell^\kappa\varphi(\lambda) - \ell^{\sigma_B}\varphi^3(T)) = \varphi(J)\varphi(y'), \qquad \varphi(y') \in P_3(\mathbf{N}_{c'}). \qquad (*)$$

Integrality. $\Pi_c f_2 = Q_0^{h_B}Q_1^{h_B-b_1}Q_2^{G_0+t}u \in R_c$ as $b_1 \le h_B$. $\varphi(\lambda) \in P_3(\mathbf{N}_{c'})$ by (iii). With $s_0 = Q_1^tQ_0^{-b_1}$ at the site $c-1$ we have $\varphi(s_0) = Q_2^tQ_1^{-b_1}$, so $f_2 = \varphi(s_0)u$ and $V = u\,\varphi(s_0\lambda) \in R_c\,\varphi(\mathbf{N}_{c-1}) \subseteq P_1$, as $s_0\lambda \in \mathbf{N}_{c-1}$.

Coordinates. By $(*)$, $\ell^{\kappa'}\big(\ell^{\kappa}\Pi_c V - \Pi_c\,\ell^{\sigma_B} f_2\,\varphi^3(T)\big) = (\Pi_c f_2)\,\varphi(J)\,\varphi(y') \in X\Pi_c^2 P_3(\mathbf{N}_{c'})$ by (i). As $\Pi_c \ell^{\sigma_B} f_2 \in R_c$ and κ is admissible for $\mathbf{n}$ and m_X, this is (23.1) of Definition 23.3 for $y = V$, $e = 3$, $H = H_c$, $m = m_X$ and the vector $\ell^{\sigma_B} f_2 \mathbf{e}_{k_0}$ (the sum over k has the single term $k = k_0$).

Congruence. Put $g_2 = Q_1^{b_1}(Q_1Q_2)^{G_0}$ and $\kappa_2 = \kappa'$. Since $\ell^{\kappa_0} f_2 = x_0 + Q_1^{-b_1} Q_0^B u'$, $(*)$ gives

$$\ell^{\kappa'}\big(\ell^{\kappa+\kappa_0} V - \ell^{\sigma_B} x_0 \varphi^3(T)\big) = x_0\,\varphi(J)\varphi(y') + \ell^{\kappa'+\kappa} Q_1^{-b_1} Q_0^B u'\,\varphi(\lambda).$$

Multiplying by g_2: the first term becomes $x_0 Q_1^{b_1}\varphi(J)\cdot(Q_1Q_2)^{G_0}\varphi(y') \in Q_0^B P_1$ by (i), (ii); the second becomes $\ell^{\kappa'+\kappa} Q_0^B u' \cdot (Q_1Q_2)^{G_0}\varphi(\lambda) \in Q_0^B P_1$ by (ii). Hence $\ell^{\kappa_2} g_2\big(\ell^{\kappa+\kappa_0} V - \ell^{\sigma_B} x_0 \varphi^3(T)\big) \in Q_0^B P_1$. □

Lemma C.50. *Let ℓ, F, L, (τ_a) and $0 \to T' \xrightarrow{\iota'} T \xrightarrow{\pi} T'' \to 0$ be as in Lemma 23.33, with $\mathbf{N}(\iota')$, $\mathbf{N}(\pi)$ extended linearly to $\mathbf{N}(\,\cdot\,)[1/\lambda_\infty]$. Let $c \in \mathbb{Z}/n$, $e \geq 0$, $a = c + e$.*

(Ae1) Left exactness $\mathbf{N}(\iota')\big(P_e(\mathbf{N}(T')_c)\big) = P_e(\mathbf{N}(T)_c) \cap \ker \mathbf{N}(\pi)$. *In particular, if $x \in \mathbf{N}(T')[1/\lambda_\infty]$ and $\mathbf{N}(\iota')(x) \in P_e(\mathbf{N}(T)_c)$, then $x \in P_e(\mathbf{N}(T')_c)$.*

(Ae2) Arc membership *Let $(M_a, h_a)_a$ be a window of V as in (Ex1) and $H\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$ with $H \geq h$ pointwise. If $x \in \mathbf{N}(T')[1/\lambda_\infty]$ and $\mathbf{N}(\iota')(x) \in P_e(\mathbf{N}(T)_c) + \mathbf{N}(T)_a$, then $\Lambda_{a,e}(H)\,x \in P_e(\mathbf{N}(T')_c)$.*

Proof. Write $N' = \mathbf{N}(T')$, $N = \mathbf{N}(T)$, $N'' = \mathbf{N}(T'')$, $R_b = e_b \mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$.

(Ae1). By Lemma 23.33, the sequence of R_c-modules $0 \to N'_c \to N_c \to N''_c$ (maps $\mathbf{N}(\iota')$, $\mathbf{N}(\pi)$) is exact. The ring map $\varphi^e\colon R_c \to R_a$ is, under the identifications with $\mathcal{O}_L[[X]]$, the substitution $X \mapsto (1+X)^{\ell^e} - 1$; as $(1+X)^{\ell^e} - 1$ is a distinguished polynomial of degree ℓ^e, Weierstrass division shows that R_a is a free R_c-module with basis $1, X, \dots, X^{\ell^e - 1}$. So $R_a \otimes_{\varphi^e, R_c} (-)$ is exact, and $0 \to R_a \otimes N'_c \to R_a \otimes N_c \to R_a \otimes N''_c$ is exact. By Lemma 23.15(1), $g \otimes x \mapsto g\varphi^e(x)$ identifies these three modules with $P_e(N'_c)$, $P_e(N_c)$, $P_e(N''_c)$; since $\mathbf{N}(\iota')$ and $\mathbf{N}(\pi)$ are $\mathbf{A}_{F,L}$-linear and commute with φ (Ex3), the identifications carry $1 \otimes \mathbf{N}(\iota')$ and $1 \otimes \mathbf{N}(\pi)$ to the restrictions of $\mathbf{N}(\iota')$ and $\mathbf{N}(\pi)$. Hence $0 \to P_e(N'_c) \to P_e(N_c) \to P_e(N''_c)$ is exact, which is the first assertion. For the second: $\mathbf{N}(\pi)(\mathbf{N}(\iota')(x)) = 0$ (Lemma 23.33, extended to the localisation), so $\mathbf{N}(\iota')(x) = \mathbf{N}(\iota')(x')$ with $x' \in P_e(N'_c)$, and $x = x'$ because $\mathbf{N}(\iota')$ remains injective on $N'[1/\lambda_\infty]$ (localisation is exact).

(Ae2). Put $\Lambda = \Lambda_{a,e}(H) \in \mathbb{Z}_\ell[[X]]$. Then $\Lambda P_e(N_c) \subseteq P_e(N_c)$ (it is an $\mathbf{A}_{F,L}$-module) and $\Lambda N_a \subseteq P_e(N_c)$ by Lemma 23.15(2). Hence $\mathbf{N}(\iota')(\Lambda x) = \Lambda\,\mathbf{N}(\iota')(x) \in P_e(N_c)$, and (Ae1) gives $\Lambda x \in P_e(N'_c)$. □

Lemma C.51. *Let ℓ, F, L, (τ_a) be as in Proposition 23.7; let (V', T'), (V, T) be pairs and $0 \to T' \xrightarrow{\iota'} T \xrightarrow{\pi} \mathcal{O}_L \to 0$ an exact sequence of G_F-equivariant $\mathcal{O}_L$-linear maps ($\mathcal{O}_L$ the trivial representation); put $\mathbf{N}' = \mathbf{N}(T')$, $\mathbf{N} = \mathbf{N}(T)$, and extend $\mathbf{N}(\iota')$ linearly to $\mathbf{N}'[1/\lambda_\infty]$. Let $c \in \mathbb{Z}/n$, $e \geq 0$, $a = c + e$, $H\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$, $\Pi = \Lambda_{a,e}(H)$, and $m \geq 1$ an integer with $m \geq 2H(a - s)$ for $0 \leq s < e$. Let $\mathbf{n}'$, $\mathbf{n}$ be frames at c of $\mathbf{N}'$, $\mathbf{N}$, with crystalline frames (d'_k), (d_k). Let $w \in \mathbf{N}'[1/\lambda_\infty]$ with*

$$\Pi\, w \in P_e(\mathbf{N}'_c) \qquad \text{and} \qquad \mathbf{N}(\iota')(w) = p - x, \quad p \in P_e(\mathbf{N}_c),\ x \in \mathbf{N}_a.$$

Then there are $t \geq 0$ and an arc coordinate vector f of $\ell^t w$ (for $\mathbf{N}'$, $\mathbf{n}'$) that is polar and is pole-bounded by every $B = (B_s)_{s<e}$, $B_s \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, such that

$$\varphi^e\big(\varepsilon_{c+3} D_{\mathrm{cris}}(V')\big) \cap \varphi^{s+4}\big(\mathrm{Fil}^{B_s+1} D_{\mathrm{cris}}(V')\big) = 0 \qquad \text{for every } s < e \text{ with } B_s < \infty.$$

Proof. By Lemma 23.14, $\mathbf{N}'$ has (Ac1) for $\mathbf{n}'$ and $\mathbf{N}$ has (Ac2) and (Ac4) for $\mathbf{n}$, at (c,e,H,m); by Lemma 23.13 (2), $\mathbf{N}$ has (Ac3) for $\mathbf{n}$ at (c,e,H,m). Write r', r for the ranks, $D'=D_{\mathrm{cris}}(V')$, $D=D_{\mathrm{cris}}(V)$, δ'_e, δ_e for the maps of Definition 23.3 built from (d'_k), (d_k), and T'_κ, T_κ for the truncations of the horizontal sections H'_k, H_k through n'_k, n_k. For a row vector $\mathbf{v}\in\mathcal{K}_a^{r'}$ and an $r'\times r$ matrix o write $\mathbf{v}o\in\mathcal{K}_a^{r}$.

Step 0: arc coordinate vectors are linear in y. If κ is admissible, so is every $\kappa_1\geq\kappa$, and $T_{\kappa_1}(H_k)=\ell^{\kappa_1-\kappa}T_\kappa(H_k)$; multiplying the defining relation of arc coordinate vectors by $\ell^{\kappa_1-\kappa}$ shows that an arc coordinate vector for (κ,κ') is one for every $(\kappa_1,\kappa'_1)\geq(\kappa,\kappa')$. Hence (a) if f is an arc coordinate vector of y, then $\ell^u f$ is one of $\ell^u y$ ($u\geq 0$; multiply the defining relation of arc coordinate vectors by ℓ^u), and (b) if f_1, f_2 are arc coordinate vectors of y_1, y_2, then, taking common (κ,κ') and subtracting, f_1-f_2 is one of y_1-y_2. The same holds for $\mathbf{N}'$.

Step 1: the frames are compared by a constant matrix, and arc coordinate vectors are transported. $\mathbf{N}(\iota')$ is $\mathbf{A}_{F,L}$-linear, commutes with φ and the γ_u (Proposition 23.7 (Ex3)), and extends to $\mathbf{N}'_E\to\mathbf{N}_E$ commuting with the γ_u of Lemma 23.9. Write $\mathbf{N}(\iota')(n'_k)=\sum_j\mu_{kj}n_j$ with $\mu_{kj}\in e_c\mathbf{A}_{F,L}$, and let $o_{kj}\in\mathcal{O}_L$ be given by $\mu_{kj}\equiv e_c(1\otimes o_{kj})\bmod X$. The element $\mathbf{N}(\iota')(H'_k)$ is fixed by every γ_u and lies in $\mathbf{N}(\iota')(n'_k)+X\mathbf{N}_E=\sum_j o_{kj}n_j+X\mathbf{N}_E$; so does $\sum_j o_{kj}H_j$, since γ_u fixes the constants $e_c(1\otimes o_{kj})$. By the uniqueness in Lemma 23.9 (Hf2),

$$\mathbf{N}(\iota')(H'_k)=\sum_j o_{kj}H_j\qquad(k\leq r'). \tag{C.4}$$

Let κ be admissible for both frames and $\Delta_k=\mathbf{N}(\iota')(T'_\kappa(H'_k))-\sum_j o_{kj}T_\kappa(H_j)\in\mathbf{N}_c$. By the definition of truncations and (C.4), $\Delta_k=\mathbf{N}(\iota')\big(T'_\kappa(H'_k)-\ell^\kappa H'_k\big)+\sum_j o_{kj}\big(\ell^\kappa H_j-T_\kappa(H_j)\big)\in X^m\mathbf{N}_E$. In the basis $\mathbf{n}$, $\mathbf{N}_c\cap X^m\mathbf{N}_E$ consists of the vectors with coordinates in $e_c\mathbf{A}_{F,L}\cap X^m e_cE[[X]]=X^m e_c\mathbf{A}_{F,L}$, so $\Delta_k\in X^m\mathbf{N}_c$, and since $\varphi^e(X)=X\lambda_e$,

$$\varphi^e(\Delta_k)\in X^m\lambda_e^m P_e(\mathbf{N}_c)\subseteq X\Pi^2\,P_e(\mathbf{N}_c),$$

because $m\geq 1$ and $\lambda_e^m=\prod_{s<e}Q_s^m$ is divisible by $\Pi^2=\prod_{s<e}Q_s^{2H(a-s)}$ in $\mathbb{Z}_\ell[[X]]$ ($m\geq 2H(a-s)$). Now let f be an arc coordinate vector of $y'\in\mathbf{N}'[1/\lambda_\infty]$ for (κ,κ'), κ admissible for both frames (Step 0). Applying $\mathbf{N}(\iota')$ to the defining relation of arc coordinate vectors, which maps $X\Pi^2P_e(\mathbf{N}'_c)$ into $X\Pi^2P_e(\mathbf{N}_c)$, and using $\varphi^e(e_c(1\otimes o)z)=e_a(1\otimes o)\varphi^e(z)$,

$$\ell^{\kappa'}\Big(\ell^\kappa\Pi\,\mathbf{N}(\iota')(y')-\sum_j\Pi(fo)_j\,\varphi^e(T_\kappa(H_j))\Big)=\ell^{\kappa'}\sum_k\Pi f_k\,\varphi^e(\Delta_k)+\big(\text{element of }X\Pi^2P_e(\mathbf{N}_c)\big),$$

and $\Pi f_k\in e_a\mathbf{A}_{F,L}$ gives $\Pi f_k\varphi^e(\Delta_k)\in X\Pi^2P_e(\mathbf{N}_c)$. As $\Pi(fo)_j\in e_a\mathbf{A}_{F,L}$, *$fo$ is an arc coordinate vector of $\mathbf{N}(\iota')(y')$.*

Step 2: the crystalline frames are compared by the same matrix; consequences. Let $D(\iota')$ be the restriction of $1\otimes\iota'\colon B_{\mathrm{cris}}\otimes_{\mathbb{Z}_\ell}T'\to B_{\mathrm{cris}}\otimes_{\mathbb{Z}_\ell}T$ to $D'\to D$, and $D(\pi)\colon D\to D_{\mathrm{cris}}(L)$ likewise. Since $0\to T'\to T\to\mathcal{O}_L\to 0$ is split as a sequence of $\mathbb{Z}_\ell$-modules, $0\to B\otimes T'\to B\otimes T\to B\otimes\mathcal{O}_L\to 0$ is exact for every $\mathbb{Z}_\ell$-module B; the maps $1\otimes\iota'$, $1\otimes\pi$ are G_F-equivariant, $B_{\mathrm{cris}}\otimes\mathcal{O}_L$-linear and commute with $\varphi=\varphi\otimes 1$. By Lemma 23.11 (Ev3) and Proposition 23.7 (Ex3), on pure tensors $\mathrm{Ev}(g\otimes\mathbf{N}(\iota')z)=e(g)\varphi^3\big((\rho\otimes 1)((1\otimes\iota')\iota_{T'}(z))\big)=(1\otimes\iota')\mathrm{Ev}'(g\otimes z)$, so $\mathrm{Ev}\circ\mathbf{N}(\iota')=(1\otimes\iota')\circ\mathrm{Ev}'$. Also $\mathrm{Ev}(gW)=e(g)\mathrm{Ev}(W)$ for $g\in\mathbf{A}_{F,L}$, as e is a ring homomorphism; with $H_j=e_cH_j$ this gives $\mathrm{Ev}(e_c(1\otimes o)H_j)=(1\otimes o)e(e_c)\mathrm{Ev}(H_j)=o\,d_j$. Applying Ev to (C.4) ($H'_k\in\mathbf{N}'_R$ by Lemma 23.10):

$$D(\iota')(d'_k)=\sum_j o_{kj}d_j,\qquad\text{hence}\qquad D(\iota')(\delta'_e(\mathbf{v}))=\delta_e(\mathbf{v}o)\quad(\mathbf{v}\in L^{r'}). \tag{C.5}$$

(i) δ_e is injective on L^r: the d_j are L-linearly independent (Lemma 23.10 (2)) and φ^e is L-linear and injective on $B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V$ (φ is injective on B_{cris}). The same holds for δ'_e. (ii) $\mathbf{v} \mapsto \mathbf{v}o$ is injective on $L^{r'}$: if $\mathbf{v}o = 0$ then $D(\iota')(\sum_k v_k d'_k) = 0$ by (C.5), so $\sum_k v_k d'_k = 0$ and $\mathbf{v} = 0$. Hence o has rank r' over L, and there are $q \geq 0$ and an $r \times r'$ matrix P over $\mathcal{O}_L$ with $oP = \ell^q I_{r'}$.

Step 3: polar vectors of T come from T', with the filtration. Let $\mathbf{v} \in \mathcal{O}_L^r$, and suppose $\delta_e(\mathbf{v}) \in \varphi^{s_0+4}(\mathrm{Fil}^{u_0} D)$ for some $s_0 < e$ and $u_0 \geq 1$. We show: there is $\tilde{\mathbf{v}} \in L^{r'}$ with $\tilde{\mathbf{v}}o = \mathbf{v}$, and for all $s < e$, $u \in \mathbb{Z}$, if $\delta_e(\mathbf{v}) \in \varphi^{s+4}(\mathrm{Fil}^u D)$ then $\delta'_e(\tilde{\mathbf{v}}) \in \varphi^{s+4}(\mathrm{Fil}^u D')$.

First, $(t^u B_{\mathrm{dR}}^+)^{G_F} = 0$ for $u \geq 1$: if $z = t^u b$ is G_F-invariant, then $\chi_{\mathrm{cyc}}(g)^u g\theta(b) = \theta(b)$ for all g, so $\theta(b) = 0$ by Theorem B.42 (2), i.e. $z \in t^{u+1} B_{\mathrm{dR}}^+$; by induction $z \in \bigcap_k t^k B_{\mathrm{dR}}^+ = 0$ (Theorem B.40). As G_F acts trivially on L, $\mathrm{Fil}^u D_{\mathrm{cris}}(L) \subseteq (t^u B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_\ell} L)^{G_F} = (t^u B_{\mathrm{dR}}^+)^{G_F} \otimes_{\mathbb{Q}_\ell} L = 0$ for $u \geq 1$.

Write $\delta_e(\mathbf{v}) = \varphi^{s_0+4}(z_0)$, $z_0 \in \mathrm{Fil}^{u_0} D$. Then $D(\pi)(z_0) \in \mathrm{Fil}^{u_0} D_{\mathrm{cris}}(L) = 0$, so $D(\pi)(\delta_e(\mathbf{v})) = \varphi^{s_0+4}(D(\pi)z_0) = 0$, and by exactness and G_F-equivariance $\delta_e(\mathbf{v}) = D(\iota')(y)$ with $y \in D'$. On D', a finite-dimensional $\mathbb{Q}_\ell$-space, φ is injective, hence bijective; let $y_0 = \varphi^{-e}(y)$. Then $\varphi^e(D(\iota')y_0) = \delta_e(\mathbf{v}) = \varphi^e(\sum_k v_k d_k)$, so $D(\iota')(y_0) = \sum_k v_k d_k \in \varepsilon_{c+3} D$. Write $y_0 = \sum_b \varepsilon_b y_0$ (ε_b the orthogonal idempotents of Lemma 23.10, which commute with $D(\iota')$); for $b \neq c+3$, $D(\iota')(\varepsilon_b y_0) = \varepsilon_b \sum_k v_k d_k = 0$, so $\varepsilon_b y_0 = 0$. Thus $y_0 \in \varepsilon_{c+3} D'$, which has L-basis (d'_k) by Lemma 23.10 (2): $y_0 = \sum_k \tilde{v}_k d'_k$, $y = \delta'_e(\tilde{\mathbf{v}})$, and $\delta_e(\tilde{\mathbf{v}}o) = D(\iota')(y) = \delta_e(\mathbf{v})$ by (C.5), so $\tilde{\mathbf{v}}o = \mathbf{v}$ by Step 2 (i). Finally let $\delta_e(\mathbf{v}) = \varphi^{s+4}(z_1)$ with $z_1 \in \mathrm{Fil}^u D$, and $z'_1 = \varphi^{-(s+4)}(y) \in D'$; then $\varphi^{s+4}(D(\iota')z'_1) = \varphi^{s+4}(z_1)$, so $D(\iota')(z'_1) = z_1$. Choose a $\mathbb{Q}_\ell$-basis of V extending one of $\iota'(V')$: the coordinates of $z_1 = (1 \otimes \iota')(z'_1)$ in B_{dR} lie in $\mathrm{Fil}^u B_{\mathrm{dR}}$ and are those of z'_1 on the basis of V'; so $z'_1 \in \mathrm{Fil}^u B_{\mathrm{dR}} \otimes V'$, i.e. $z'_1 \in \mathrm{Fil}^u D'$ and $\delta'_e(\tilde{\mathbf{v}}) = \varphi^{s+4}(z'_1) \in \varphi^{s+4}(\mathrm{Fil}^u D')$.

Step 4: one linear relation among the coordinates. By (Ac1) for $\mathbf{N}'$ there are t_1 and an arc coordinate vector F of $\ell^{t_1} w$; by Step 1, Fo is one of $\ell^{t_1}(p - x)$. By (Ac2) there are t_2 and an arc coordinate vector G of $\ell^{t_2} p$ with $G_j \in e_a \mathbf{A}_{F,L}$. Since $\mathbf{N}(\iota')$ is $\mathbf{A}_{F,L}$-linear and commutes with φ, it maps $P_e(\mathbf{N}'_c)$ into $P_e(\mathbf{N}_c)$; so $\Pi x = \Pi p - \mathbf{N}(\iota')(\Pi w) \in P_e(\mathbf{N}_c)$, and by (Ac3) there are t_3, κ_3 and an arc coordinate vector f^x of $\ell^{t_3} x$ with a section decomposition

$$\ell^{\kappa_3} f_j^x = \sum_{i \in I} g_i v_{i,j}, \qquad g_i \in Q^{\alpha_i} e_a \mathbf{A}_{F,L}, \quad \delta_e(\mathbf{v}_i) \in \varphi^{s+4}\big(\mathrm{Fil}^{-\alpha_i(s)} D\big) \ (s < e).$$

By Step 0, $\ell^{t_2+t_3} Fo - \ell^{t_1+t_3} G + \ell^{t_1+t_2} f^x$ is an arc coordinate vector of $\ell^{t_1+t_2+t_3}(\mathbf{N}(\iota')(w) - p + x) = 0$, so by (Ac4) there are $t_4 \geq 0$ and $\epsilon \in (X \Pi e_a \mathbf{A}_{F,L})^r$ with

$$\ell^{t_4+t_2+t_3} Fo = \epsilon + \ell^{t_4+t_1+t_3} G - \ell^{t_4+t_1+t_2} f^x. \tag{C.6}$$

Step 5: separating the polar terms. Let $I_{\mathrm{p}} = \{i \in I : \ \alpha_i(s) < 0 \text{ for some } s < e\}$. For $i \notin I_{\mathrm{p}}$, $\alpha_i \geq 0$ and $g_i \in e_a \mathbf{A}_{F,L}$. For $i \in I_{\mathrm{p}}$, Step 3 applies to $\mathbf{v}_i$ (with $u_0 = -\alpha_i(s_0) \geq 1$) and gives $\tilde{\mathbf{v}}_i \in L^{r'}$ with $\tilde{\mathbf{v}}_i o = \mathbf{v}_i$ and $\delta'_e(\tilde{\mathbf{v}}_i) \in \varphi^{s+4}(\mathrm{Fil}^{-\alpha_i(s)} D')$ for all $s < e$. Choose $b \geq 0$ with $\mathbf{r}_i = \ell^b \tilde{\mathbf{v}}_i \in \mathcal{O}_L^{r'}$ for all $i \in I_{\mathrm{p}}$, and put

$$S_j^{\mathrm{np}} = \sum_{i \notin I_{\mathrm{p}}} g_i v_{i,j} \in e_a \mathbf{A}_{F,L}, \qquad \mathcal{P}_k = \sum_{i \in I_{\mathrm{p}}} g_i r_{i,k}, \qquad \text{so} \quad (\mathcal{P}o)_j = \ell^b\big(\ell^{\kappa_3} f_j^x - S_j^{\mathrm{np}}\big).$$

Multiplying (C.6) by ℓ^{κ_3+b} and inserting this,

$$\big(\ell^{\kappa_3+b+t_4+t_2+t_3} F + \ell^{t_4+t_1+t_2} \mathcal{P}\big) o = \mathcal{R},$$
$$\mathcal{R} = \ell^{\kappa_3+b} \epsilon + \ell^{\kappa_3+b+t_4+t_1+t_3} G - \ell^{b+t_4+t_1+t_2} S^{\mathrm{np}} \in (e_a \mathbf{A}_{F,L})^r.$$

Multiplying on the right by the matrix P of Step 2 ($oP = \ell^q I$) and putting $N_0 = q + \kappa_3 + b + t_4 + t_2 + t_3$:

$$\ell^{N_0} F = \mathcal{R}P - \ell^{q+t_4+t_1+t_2} \mathcal{P}, \qquad \mathcal{R}P \in (e_a \mathbf{A}_{F,L})^{r'}. \tag{C.7}$$

Step 6: the conclusion. Let $f = \ell^{N_0} F$; by Step 0 (a) it is an arc coordinate vector of $\ell^t w$, $t = N_0 + t_1$. By (C.7), f has the section decomposition consisting of the terms $\big((\mathcal{R}P)_k, \mathbf{e}_k, 0\big)$, $k \le r'$ ($\mathbf{e}_k$ the unit vectors), and $\big(g_i, -\ell^{q+t_4+t_1+t_2}\mathbf{r}_i, \alpha_i\big)$, $i \in I_{\mathrm{p}}$; discard the terms whose vector is 0. For the first kind $\alpha = 0$. For a term of the second kind with vector $\mathbf{v}' \neq 0$ and $s < e$ with $\alpha_i(s) < 0$: $\delta'_e(\mathbf{v}')$ is a non-zero (Step 2 (i)) L-multiple of $\delta'_e(\tilde{\mathbf{v}}_i)$, so $\delta'_e(\mathbf{v}') \in \varphi^{s+4}(\mathrm{Fil}^{-\alpha_i(s)} D')$. Hence f is polar. Let B be as in the statement and $B_s < \infty$. Since $\delta'_e(\mathbf{v}') = \varphi^e(\sum_k v'_k d'_k) \in \varphi^e(\varepsilon_{c+3} D')$ is non-zero, the hypothesis on B excludes $\delta'_e(\mathbf{v}') \in \varphi^{s+4}(\mathrm{Fil}^{B_s+1} D')$; as the filtration is decreasing, $-\alpha_i(s) \le B_s$. So the same decomposition shows that f is pole-bounded by B. $\square$

C.2 Thick alternating words

This section proves Proposition C.52, the colength bound for words of even length that are thick at every site, with signs alternating by parity; it is item 2 of the row bound Proposition 26.2. Since no site is thin, the cuts of Section C.1 are not available, and the down sites take their place. After a bounded power of ℓ and n generator steps of a free cover, every crystalline class is a sum of relay classes of Wach vectors at the down sites (Proposition C.53, Subsection C.2.1); these vectors lift along the onto map modulo X up to ℓ^e with e of order H'/ℓ^2, on a submodule reached in at most six generator steps and three length steps (Proposition C.57, Subsection C.2.2). The colength then follows from Lemma C.76.

C.2.1 Splitting of crystalline classes for thick alternating words

The subsection states Proposition C.52 and proves the splitting Proposition C.53, from unit lifts at the down sites (Proposition C.54) and a second lift with the pull-back condition (Proposition C.56). It also states the lifting result Proposition C.57, which is proved in Subsection C.2.2.

Proposition C.52. *There is an absolute constant $c_{\mathrm{C}} \ge 1$ with the following property. Let $\ell \ge 11$ be a prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of even degree n, L with ring of integers $\mathcal{O}_L$ as in Definition 23.1, $(\tau_a)_{a\in\mathbb{Z}/n}$ a Frobenius numbering, $\Phi\colon V_1 \to V_2$ an onto map of crystalline lattices $T_1 \to T_2$ over F with coefficients L, and $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ a weight word for Φ. Suppose that there is $d \in \{0,1\}$ such that, for the representatives $0 \le a < n$, $\epsilon_a = -1$ if and only if $a + d$ is even, and that, with*

$$a_{\mathrm{R}} = \frac{H'(2\ell-1)}{(\ell-1)^2} + 4 + \log_\ell(2H'),$$

every $s_a \ge 2a_{\mathrm{R}} + \eta + 1$ and every $s_a \ge (H' - \eta)/2$. Then the colength of Φ is at most

$$c_{\mathrm{C}}\Big((n+1)\, d_2\Big(\frac{H'}{\ell^2} + S + \frac{\eta}{\ell} + 1 + \log_\ell(2H')\Big) + n\, m\Big(1 + \frac{H'}{\ell} + \log_\ell(2H')\Big)\Big).$$

Proof. Put

$$B = \frac{H'}{\ell^2} + S + \frac{\eta}{\ell} + 1 + \log_\ell(2H'), \qquad \lambda = d_2 S + 2m\Big(\frac{H'+1}{\ell-1} + 1\Big),$$

let c_1 and c_4 be the absolute constants of Proposition C.53 and Proposition C.57, c_{ur} that of Proposition 24.1, and $c_{\mathrm{C}} = \max(c_1 + c_4 + 2,\ 8c_{\mathrm{ur}} + 4)$. Here $H' = \max_a s_a + S + \eta \ge 1$, since $s_a > \eta \ge 0$. Let $\mathcal{H} = H^1_f(D, T_2)$ and $\mathcal{Y} = \Phi_* H^1_f(D, T_1) \subseteq \mathcal{H}$; the colength of Φ is $\mathrm{nl}(\mathcal{H}/\mathcal{Y})$.

Step 0: down sites and windows. Let $C = \{c \in \mathbb{Z}/n :\ \epsilon_c = -1\}$. Since n is even, the parity of $a+d$ is well defined for $a \in \mathbb{Z}/n$ and changes from a to $a+1$; so by hypothesis $|C| = n/2$,

and $\epsilon_{c+1} = 1$ for every $c \in C$. Put $(M_a, h_a) = (s_a + S, \eta)$ if $\epsilon_a = -1$ and $(M_a, h_a) = (\eta, s_a + S)$ if $\epsilon_a = 1$. By (Ow1), every Hodge jump of V_1 at τ_a lies in $[-M_a, h_a]$. The same holds for V_2: $\Phi(V_1)$ contains $\Phi(T_1) = T_2$, which spans V_2, so Φ is onto and $V_2 \cong V_1/\ker\Phi$; for de Rham representations the functors $V \mapsto \mathrm{gr}^i D_{\mathrm{dR}}(V)$ are exact (each is left exact, and the sum of their dimensions is $\dim V$ on de Rham representations), and they respect the decomposition along the embeddings τ_a, so the multiplicity of each Hodge jump of V_2 at τ_a is at most its multiplicity for V_1. Thus $(M_a, h_a)_a$ is a window of both V_1 and V_2 as in (Ex1), it is the window used in Proposition C.53 and Proposition C.57, and $\max_a M_a = \max_a s_a + S$ (as $s_a > \eta$), $M_{c+1} = \eta$ for $c \in C$, $\max_a(M_a + h_a) = H'$.

Step 1: a free cover. The representation V_2 is crystalline, hence de Rham, of dimension d_2 over L, and F has degree n over $\mathbb{Q}_\ell$; by Lemma 27.3, $\mathcal{H}$ is generated by $r \le (n+1)d_2$ elements. Let $P = \mathcal{O}_L^r$ and $\psi\colon P \to \mathcal{H}$ the surjection sending the standard basis to these generators.

Step 2: splitting at the down sites. By Proposition C.53 (its hypotheses are those of the present statement) there are integers $0 \le s \le c_1 B$, $g \ge \eta$, $m_{\mathrm{R}} \ge 1 + \max_a(s_a + S)$, $R \ge \max(m_{\mathrm{R}}, g)$, submodules $P = P_0 \supseteq \cdots \supseteq P_{k_0}$ with $k_0 \le n$ and each P_i/P_{i+1} generated by at most m elements, and $\mathcal{O}_L$-linear maps $w_c\colon P_{k_0} \to I_c^2$ ($c \in C$; I_c^ν and $\rho_{c,1}^\nu$ formed for $\mathbf{N}(T_\nu)$ with c, $j = 1$, $G = \eta$, g, m_{R}, R) satisfying (Vw1) and (Vw2).

Step 3: lifting at the down sites, one after the other. Enumerate $C = \{c_1, \dots, c_{n/2}\}$ and put $P^{(0)} = P_{k_0}$. For $i = 1, \dots, n/2$ apply Proposition C.57 at the down site c_i to P, the submodule $P^{(i-1)} \subseteq P_{k_0}$ and the restriction of w_{c_i} to $P^{(i-1)}$; its hypotheses hold by (Vw1). It gives an integer $0 \le e_{c_i} \le c_4 B$ and a chain from $P^{(i-1)}$ down to a submodule $P^{(i)}$ with at most 6 steps whose quotients are generated by at most m elements and at most 3 steps whose quotients have normalised length at most λ, such that for every $f \in P^{(i)}$ there are $u \in I_{c_i}^1$ and a finite product λ' of the Q_s with $\lambda' u \in \mathbf{N}(T_1)$ and $\ell^{e_{c_i}} w_{c_i}(f) - \mathbf{N}(\Phi)(u) \in X\mathbf{N}(T_2)_{c_i}$. Concatenating the chain of Step 2 with these $n/2$ chains gives submodules $P = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_k = P^{(n/2)}$ and a set I of indices (the steps of Step 2 and the generator steps of Step 3) with

$$|I| \le n + 6 \cdot \tfrac{n}{2} = 4n, \qquad k - |I| \le 3 \cdot \tfrac{n}{2},$$

P_i/P_{i+1} generated by at most m elements for $i \in I$ and $\mathrm{nl}(P_i/P_{i+1}) \le \lambda$ for $i \notin I$.

Step 4: the killing exponent. Let $e^* = \max_{c \in C} e_c \le c_4 B$ and $f \in P_k$. For $c \in C$, $f \in P^{(i)}$ for the index i with $c = c_i$, so there are u_c, λ_c' as in Step 3. Apply Lemma 24.9 with $w = w_c(f) \in I_c^2$, $u = u_c \in I_c^1$, $k = e_c$ and the parameters c, $j = 1$, $G = \eta$, g, m_{R}, R: its hypotheses hold by Step 0 ($g \ge G$; $m_{\mathrm{R}} \ge 1 + \max_a M_a$ for both windows; $R \ge \max(m_{\mathrm{R}}, g)$; $M_{c+1} = \eta \le G$), and $\lambda_c' u_c \in \mathbf{N}(T_1)$. Hence $\ell^{e_c}\rho_{c,1}^2(w_c(f)) \in \mathcal{Y}$, and so $\ell^{e^*}\rho_{c,1}^2(w_c(f)) \in \mathcal{Y}$. By (Vw2), $\ell^{s+e^*}\psi(f) = \pm\sum_{c \in C} \ell^{e^*}\rho_{c,1}^2(w_c(f)) \in \mathcal{Y}$. Thus $\ell^e\psi(P_k) \subseteq \mathcal{Y}$ with $e = s + e^* \le (c_1 + c_4)B$.

Step 5: the uniform exponent. Apply Proposition 24.1 with $p = \ell$ (odd), $K = F$ (unramified), the field L (it contains $\sigma(F(\mu_\ell))$ for every embedding σ, by Definition 23.1), V_1, V_2 crystalline, the map Φ with $\Phi(T_1) = T_2$, and $a_{\tau_a}^- = M_a$, $a_{\tau_a}^+ = h_a$ (the τ_a are all the embeddings of F into L; Step 0). Then $\mathrm{wt} = H'$ and $\ell^{e'}\mathcal{H} \subseteq \mathcal{Y}$ with the integer $e' = e_{\mathrm{A}} \le c_{\mathrm{ur}}\big(2 + H'/\ell + \log_\ell(H' + 1)\big) \le 2c_{\mathrm{ur}}\big(1 + H'/\ell + \log_\ell(2H')\big)$, using $H' + 1 \le 2H'$.

Step 6: counting. Lemma C.76, applied to $\mathcal{H} \supseteq \mathcal{Y}$, the surjection ψ from the free module P of rank r, the chain and the set I of Step 3, the integers e, e' and the real λ ((Fc1), (Fc2) by Step 3, (Fc3) by Step 4, (Fc4) by Step 5), gives

$$\mathrm{nl}(\mathcal{H}/\mathcal{Y}) \le r\,e + |I|\,m\,e' + (k - |I|)\,\lambda \le (n+1)d_2(c_1 + c_4)B + 8c_{\mathrm{ur}}\,n\,m\Big(1 + \frac{H'}{\ell} + \log_\ell(2H')\Big) + \frac{3n}{2}\lambda.$$

With the side calculation, the colength is at most

$$\Big(c_1+c_4+\tfrac{3}{2}\Big)(n+1)\,d_2\,B+(8c_{\mathrm{ur}}+4)\,n\,m\Big(1+\frac{H'}{\ell}+\log_\ell(2H')\Big)$$
$$\le c_{\mathrm{C}}\Big((n+1)\,d_2\,B+n\,m\Big(1+\frac{H'}{\ell}+\log_\ell(2H')\Big)\Big),$$

which is the assertion. □

Proposition C.53. *There is an absolute constant $c_1\ge 1$ with the following property. Let ℓ, F, n, L, $(\tau_a)_{a\in\mathbb{Z}/n}$, $\Phi\colon V_1\to V_2$ (with lattices T_1,T_2) and the weight word $(b,w,S,\eta,m,(s_a),(\epsilon_a))$ satisfy the hypotheses of Proposition C.52, let $C=\{c\in\mathbb{Z}/n:\ \epsilon_c=-1\}$ be the set of* down sites*, and put*

$$B=\frac{H'}{\ell^2}+S+\frac{\eta}{\ell}+1+\log_\ell(2H').$$

Then there are integers $0\le s\le c_1B$, $g\ge\eta$, $m_{\mathrm{R}}\ge 1+\max_a(s_a+S)$ and $R\ge\max(m_{\mathrm{R}},g)$ such that, with I_c and $\rho_{c,1}\colon I_c\to H^1(F,T_2)$ formed for $\mathbf{N}(T_2)$ with c, $j=1$, $G=\eta$, g, m_{R} (in the role of m) and R, the following holds. For every free $\mathcal{O}_L$-module P of finite rank and every $\mathcal{O}_L$-linear map $\psi\colon P\to H^1_f(F,T_2)$ there are submodules $P=P_0\supseteq P_1\supseteq\cdots\supseteq P_k$ with $k\le n$ and every P_i/P_{i+1} generated by at most m elements, and $\mathcal{O}_L$-linear maps $w_c\colon P_k\to I_c$ $(c\in C)$, such that for every $f\in P_k$:

(Vw1) *for every $c\in C$: $\gamma_\delta(w_c(f))=w_c(f)$ for all $\delta\in\mu_{\ell-1}\subseteq\mathbb{Z}_\ell^\times$, and $Q^\eta w_c(f)\in P_1(\mathbf{N}(T_2)_{c-1})$;*

(Vw2) *$\ell^s\psi(f)=\pm\sum_{c\in C}\rho_{c,1}\big(w_c(f)\big)$ in $H^1(F,T_2)$, for one of the two signs (which may depend on f).*

Proof. Let c_{U} and c_{S} be the absolute constants of Proposition C.54 and Proposition C.56, and put $c_1=c_{\mathrm{U}}+c_{\mathrm{S}}$. Let s_1 be the integer of Proposition C.54, and s_2, g, m_{R}, R the integers of Proposition C.56; put $s=s_1+s_2$. Then $0\le s\le c_{\mathrm{U}}B+c_{\mathrm{S}}B=c_1B$, and $g\ge\eta$, $m_{\mathrm{R}}\ge 1+\max_a(s_a+S)$, $R\ge\max(m_{\mathrm{R}},g)$, so I_c and $\rho_{c,1}$ are the objects of the statement. Let P and ψ be given.

Step 1: the first lift. By Proposition C.54 there are $P=P_0\supseteq\cdots\supseteq P_{k_1}$ with $k_1\le n/2$, every P_i/P_{i+1} generated by at most m elements, such that every $f\in P_{k_1}$ has property (Ul) with the exponent s_1.

Step 2: the second lift. Apply Proposition C.56 with this s_1, P, ψ and $P'=P_{k_1}$: there are $P_{k_1}=P'_0\supseteq\cdots\supseteq P'_{k_2}$ with $k_2\le n/2$, every quotient generated by at most m elements, such that every $f\in P'_{k_2}$ admits $w_c\in I_c$ $(c\in C)$ with (Sl1) and (Sl2). Put $k=k_1+k_2\le n$ and $P_{k_1+i}=P'_i$ for $0\le i\le k_2$; this is a chain $P=P_0\supseteq\cdots\supseteq P_k$ as in the statement.

Step 3: linear choice. As a submodule of the free module P of finite rank over the discrete valuation ring $\mathcal{O}_L$, P_k is free of finite rank; let $f_1,\dots,f_t$ be a basis. For each i choose, by Step 2, elements $w_c^{(i)}\in I_c$ $(c\in C)$ with (Sl1) and a sign $\sigma_i\in\{1,-1\}$ with $\ell^s\psi(f_i)=\sigma_i\sum_{c\in C}\rho_{c,1}(w_c^{(i)})$. Define $w_c\colon P_k\to I_c$ by

$$w_c\Big(\sum_i a_i f_i\Big)=\sum_i a_i\sigma_i\,w_c^{(i)}\qquad(a_i\in\mathcal{O}_L);$$

it is well defined and $\mathcal{O}_L$-linear, and takes values in I_c because I_c is an $\mathcal{O}_L$-submodule of $e_c\mathbf{N}(T_2)[1/\lambda_\infty]$ ((Ry4) of Lemma 24.5).

(Vw1). Let $f = \sum_i a_i f_i \in P_k$ and $c \in C$. The map γ_δ is semilinear over the automorphism γ_δ of $\mathbf{A}_{F,L}$, which is the identity on the coefficients $\mathcal{O}_F \otimes \mathcal{O}_L$; so γ_δ is $\mathcal{O}_L$-linear and $\gamma_\delta(w_c(f)) = \sum_i a_i \sigma_i \gamma_\delta(w_c^{(i)}) = w_c(f)$ by (Sl1). Since $P_1(\mathbf{N}(T_2)_{c-1})$ is an $\mathbf{A}_{F,L}$-submodule, $Q^\eta w_c(f) = \sum_i a_i \sigma_i Q^\eta w_c^{(i)} \in P_1(\mathbf{N}(T_2)_{c-1})$ by (Sl1).

(Vw2). By the $\mathcal{O}_L$-linearity of $\rho_{c,1}$ ((Ry4)) and of ψ, and $\sigma_i^2 = 1$,

$$\sum_{c\in C} \rho_{c,1}(w_c(f)) = \sum_i a_i \sigma_i \sum_{c\in C} \rho_{c,1}(w_c^{(i)}) = \sum_i a_i \sigma_i^2\, \ell^s \psi(f_i) = \ell^s \psi(f),$$

which is (Vw2) with the sign + for every f. □

Proposition C.54. *There is an absolute constant $c_{\mathrm{U}} \geq 1$ with the following property. Let ℓ, F, n, L, $(\tau_a)_{a\in\mathbb{Z}/n}$, $\Phi\colon V_1 \to V_2$ (with lattices T_1, T_2) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.52; let C, $(M_a, h_a)_a$ and B be as in Proposition C.53, and $\varepsilon_a = (\iota \otimes \mathrm{id})(e_a) \in A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$. Then there is an integer s_1 with $0 \leq s_1 \leq c_{\mathrm{U}} B$ such that for every free $\mathcal{O}_L$-module P of finite rank and every $\mathcal{O}_L$-linear map $\psi\colon P \to H^1_f(F, T_2)$ there are submodules $P = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_k$ with $k \leq n/2$ and every P_i/P_{i+1} generated by at most m elements, such that every $f \in P_k$ has the following property **(Ul)** with the exponent s_1:*

(Ul) *there are a pair (V_Y, Y) and an exact sequence $0 \to T_2 \xrightarrow{\iota'} Y \xrightarrow{\pi} \mathcal{O}_L \to 0$ of G_F-equivariant $\mathcal{O}_L$-linear maps ($\mathcal{O}_L$ the trivial representation) with $[Y] = \psi(f)$, such that every Hodge jump of V_Y at τ_a lies in $[-M_a, h_a]$ ($a \in \mathbb{Z}/n$), and for every $c \in C$ an element $x_c \in \mathbf{N}(Y)_c$ with $\gamma_\delta(x_c) = x_c$ for all $\delta \in \mu_{\ell-1} \subseteq \mathbb{Z}_\ell^\times$ and*

$$(1 \otimes \pi)\big(\iota_Y(x_c)\big) = \ell^{\,s_1} \varepsilon_c.$$

Proof. **Step 0: the killer sets and s_1.** Put $a = \lceil a_{\mathrm{R}} \rceil \geq 1$. For $b \in \mathbb{Z}/n$ let $\mathcal{X}_b$ be the set of Hodge jumps of V_2 at τ_b together with 0; for $c \in C$ let $J_c = \{x + k : \ x \in \mathcal{X}_c,\ 1 \leq k \leq a\}$ and $J_c' = J_c \setminus \{0\}$. Since $c(0) = 0$, Lemma 28.15 (2) gives $v_\ell(c(j)) = v_\ell(j)$ for $j \neq 0$, so $P_{J_c'}(0) = \prod_{j\in J_c'}(-c(j)) = \ell^{t_c} \upsilon_c$ with $\upsilon_c \in \mathbb{Z}_\ell^\times$ and $t_c = \sum_{j\in J_c'} v_\ell(j)$. Put $s_1 = \max_{c\in C} t_c$ and $c_{\mathrm{U}} = 6$. By (Ow1) and Lemma 24.2 (1), $\mathcal{X}_c \subseteq [-s_c - S, -s_c] \cup [-\eta, \eta]$, and $s_c \geq 2a_{\mathrm{R}} + \eta + 1 > a$; so $J_c \subseteq [-s_c - S + 1, -s_c + a] \cup [-\eta + 1, \eta + a]$, the first interval consisting of negative integers. Hence $0 \leq s_1 \leq c_{\mathrm{U}} B$. The Hodge jumps of V_2 at τ_c in $[-a, -1]$ lie in $[-\eta, -1]$ (as $s_c > a$), so by (Ow2)

$$\mu_c := \sum_{j=-a}^{-1} \dim_L \mathrm{gr}^j D_{\mathrm{dR},\tau_c}(V_2) \leq m. \qquad (1)$$

Step 1: one extension for all of P. Let P, ψ be given, $p_1, \ldots, p_t$ a basis of P. By Lemma 23.26, for each i there is an extension $0 \to T_2 \xrightarrow{\iota_i'} Y_i \xrightarrow{\pi_i} \mathcal{O}_L \to 0$ of pairs with $[Y_i] = \psi(p_i)$; fix $\tilde{e}_i \in Y_i$ with $\pi_i(\tilde{e}_i) = 1$ and let $\kappa_i(\sigma) = \iota_i'^{-1}(\sigma\tilde{e}_i - \tilde{e}_i)$. Let $\mathcal{Y} = (\bigoplus_i Y_i)/K$ with $K = \{(\iota_i'(t_i))_i : \sum_i t_i = 0\}$, with $\iota'\colon T_2 \to \mathcal{Y}$ induced by any ι_i' and $\pi\colon \mathcal{Y} \to P$, $(y_i) \mapsto \sum_i \pi_i(y_i) p_i$. Since K is saturated in $\bigoplus_i T_2$, which is saturated in $\bigoplus_i Y_i$, $\mathcal{Y}$ is a G_F-stable lattice in $V_{\mathcal{Y}} = \mathcal{Y} \otimes \mathbb{Q}$, a quotient of $\bigoplus_i V_{Y_i}$, hence crystalline (Theorem B.45); and $0 \to T_2 \to \mathcal{Y} \to P \to 0$ is exact, P carrying the trivial action. By exactness and strict compatibility with the filtrations of D_{dR} on de Rham representations (Fontaine), every Hodge jump of $V_{\mathcal{Y}}$, and of any extension of L by V_2, at τ_a lies in $\mathcal{X}_a$ (jumps of V_2, and 0), hence in $[-M_a, h_a]$ (Step 0 and Lemma 24.2 (1)). For $p \in P$ let $j_p\colon \mathcal{O}_L \to P$, $1 \mapsto p$, and $e_c^\circ \otimes p = \mathbf{N}(j_p)(e_c^\circ)$; by (Ex3) and Lemma 24.3, $\iota_P(e_c^\circ \otimes p) = \varepsilon_c \otimes p$, so $p \mapsto e_c^\circ \otimes p$ is $\mathcal{O}_L$-linear (ι_P is injective), and $e_c^\circ \otimes p$ is fixed by all γ_u (by (Ex2), as G_F fixes $\varepsilon_c \otimes p$). In the same way every identity between images of elements under $\mathbf{N}$ of $\mathcal{O}_L$-linear

combinations of maps is checked after ι, where $\mathbf{N}(g)$ becomes $1 \otimes g$; we use this without further mention (it gives $\mathbf{N}(g+g') = \mathbf{N}(g) + \mathbf{N}(g')$ and $\mathbf{N}(0) = 0$).

Step 2: lifts with a Q^a-pole. Let $\mathbf{N}' = \mathbf{N}(\pi)(\mathbf{N}(\mathcal{Y})) \subseteq \mathbf{N}(P)$ and $H = \max_a(M_a + h_a) = H'$. By Proposition 23.18, X^H kills $\mathbf{N}(P)/\mathbf{N}'$; $(\gamma_u - 1)\mathbf{N}(P) \subseteq X\mathbf{N}(P)$ and $(\gamma_u - 1)\mathbf{N}' = \mathbf{N}(\pi)((\gamma_u - 1)\mathbf{N}(\mathcal{Y})) \subseteq X\mathbf{N}'$ by (Wm3), and X is injective on $\mathbf{N}(P)$ (Wm1). By Lemma 23.21 (1), (2), ℓ^d kills $\mathbf{N}(P)/\mathbf{N}'$ with $d = \lfloor H\ell/(\ell-1)^2 \rfloor$; and $\pi_0^A = X^{(\ell-1)A} v^A$ kills it for $A = \lceil H/(\ell-1) \rceil$. In $q^a = \sum_{i=0}^{a} \binom{a}{i} \pi_0^i \ell^{a-i}$ every term with $i \geq A$ kills, and every term with $i < A$ has $a - i \geq a - A + 1 \geq d$, because $d + A - 1 \leq H(\frac{\ell}{(\ell-1)^2} + \frac{1}{\ell-1}) = \frac{H'(2\ell-1)}{(\ell-1)^2} \leq a_{\mathrm{R}} \leq a$. So q^a kills $\mathbf{N}(P)/\mathbf{N}'$. For $c \in C$ and each i choose $x'_{c,i} \in \mathbf{N}(\mathcal{Y})_c$ with $\mathbf{N}(\pi)(x'_{c,i}) = q^a(e_c^\circ \otimes p_i)$ and put $\tilde{x}_{c,i} = q^{-a}\mathrm{av}(x'_{c,i}) \in e_c\mathbf{N}(\mathcal{Y})[1/Q]$. Since q and $e_c^\circ \otimes p_i$ are Δ-invariant and $\mathbf{N}(\pi)$ commutes with γ_δ: $\tilde{x}_{c,i}$ is Δ-invariant, $Q^a\tilde{x}_{c,i} = w_q^{-a}\mathrm{av}(x'_{c,i}) \in \mathbf{N}(\mathcal{Y})$, and $\mathbf{N}(\pi)(\tilde{x}_{c,i}) = e_c^\circ \otimes p_i$.

Step 3: the killer. Let $y_{c,i} = P_{J'_c}(\nabla)\tilde{x}_{c,i}$, with ∇ of Lemma 28.15 (1) for $\pi\colon \mathcal{Y} \to P$ at the site c. By Lemma 28.15 (1), $y_{c,i}$ is Δ-invariant and $Q^a y_{c,i} \in \mathbf{N}(\mathcal{Y})$. Since $\mathcal{X}_c$ contains every Hodge jump of $V_{\mathcal{Y}}$ at τ_c, Lemma C.46 (with its parameter a) gives $P_{J_c}(\nabla)\tilde{x}_{c,i} \in \mathbf{N}(\mathcal{Y})_c$. If $0 \notin J_c$, then $y_{c,i} = P_{J_c}(\nabla)\tilde{x}_{c,i} \in \mathbf{N}(\mathcal{Y})$; if $0 \in J_c$, then $P_{J_c}(Z) = Z\,P_{J'_c}(Z)$ and $(\gamma_{u_0} - 1)y_{c,i} = \pi_0\nabla y_{c,i} = \pi_0 P_{J_c}(\nabla)\tilde{x}_{c,i} \in \mathbf{N}(\mathcal{Y})$. In both cases $(\gamma_{u_0} - 1)y_{c,i} \in \mathbf{N}(\mathcal{Y})$. As ∇ commutes with $\mathbf{N}(\pi)$ and $\nabla(e_c^\circ \otimes p_i) = 0$ (it solves $\pi_0\nabla x = (\gamma_{u_0} - 1)x = 0$; uniqueness),

$$\mathbf{N}(\pi)(y_{c,i}) = P_{J'_c}(0)\, e_c^\circ \otimes p_i = \ell^{t_c} v_c\, e_c^\circ \otimes p_i. \qquad (2)$$

Step 4: reference lifts and the passage. Let $N = \max_a r_a$, r_a the integers of Proposition C.6 for $\pi\colon \mathcal{Y} \to P$ and the window $(M_a, h_a)_a$ ($\ell \geq 11$). Choose $\epsilon'_{c,i} \in \mathbf{N}(\mathcal{Y})_c$ with $\mathbf{N}(\pi)(\epsilon'_{c,i}) = \ell^N e_c^\circ \otimes p_i$ and put $\epsilon_{c,i} = \mathrm{av}(\epsilon'_{c,i})$, Δ-fixed with the same image. The element $\ell^N y_{c,i} - \ell^{t_c} v_c \epsilon_{c,i} \in e_c\mathbf{N}(\mathcal{Y})[1/Q]$ has $\mathbf{N}(\pi)$-image 0 by (2), so it equals $\mathbf{N}(\iota')(u_{c,i})$ for a unique $u_{c,i} \in e_c\mathbf{N}_2[1/Q]$ (Lemma 23.33, after clearing the power of Q: all modules are free site by site). Moreover $\mathbf{N}(\iota')$ is injective on $\mathbf{N}_2[1/Q]$ and $v \in \mathbf{N}_2[1/Q]$ with $\mathbf{N}(\iota')(v) \in \mathbf{N}(\mathcal{Y})$ lies in $\mathbf{N}_2$ (if $\mathbf{N}(\iota')(v) = y \in \mathbf{N}(\mathcal{Y})$ then $\mathbf{N}(\pi)(y) = 0$, $y = \mathbf{N}(\iota')(v')$ with $v' \in \mathbf{N}_2$ by Lemma 23.33, and $v = v'$). Hence: $Q^a u_{c,i} \in \mathbf{N}_2$; $u_{c,i}$ is Δ-fixed; and $(\gamma_{u_0} - 1)u_{c,i} \in \mathbf{N}_2$, because $(\gamma_{u_0} - 1)\epsilon_{c,i} \in \mathbf{N}(\mathcal{Y})$ and $(\gamma_{u_0} - 1)y_{c,i} \in \mathbf{N}(\mathcal{Y})$ (Step 3). Define $\omega_c\colon P \to e_c\mathbf{N}_2[1/Q]$ by $\omega_c(\sum_i a_i p_i) = \sum_i a_i u_{c,i}$; it is $\mathcal{O}_L$-linear and every value satisfies (Ps1) with its parameter equal to a, (Ps2), (Ps3) of Lemma C.55 at the site c. Enumerate $C = \{c_1, \ldots, c_k\}$, $k = n/2$, put $P_0 = P$ and $P_j = \{f \in P_{j-1} : \ \omega_{c_j}(f) \in \mathbf{N}_2\}$. By Lemma C.55 for (V_2, T_2), the site c_j, its parameter equal to a and the restriction of ω_{c_j} to the finitely generated module P_{j-1}, each P_j is a submodule and P_{j-1}/P_j is generated by at most $\mu_{c_j} \leq m$ elements by (1). So $P = P_0 \supseteq \cdots \supseteq P_k$ with $k \leq n/2$, and $\omega_c(f) \in \mathbf{N}_2$ for $f \in P_k$, $c \in C$.

Step 5: integral lifts in $\mathcal{Y}$. Let $f = \sum_i a_i p_i \in P_k$ and $c \in C$, and put $y_c(f) = \sum_i a_i y_{c,i}$, $\epsilon_c(f) = \sum_i a_i \epsilon_{c,i}$. Then $\ell^N y_c(f) = \ell^{t_c} v_c \epsilon_c(f) + \mathbf{N}(\iota')(\omega_c(f)) \in \mathbf{N}(\mathcal{Y})_c$. Since $Q^a y_c(f) \in \mathbf{N}(\mathcal{Y})_c$, the saturation (3) of Lemma 23.15 (with $e = 0$, $U = \mathbf{N}(\mathcal{Y})_c$, and its f, f' equal to $1, Q^a$) gives $y_c(f) \in \mathbf{N}(\mathcal{Y})_c$. Put $X_c(f) = \ell^{s_1 - t_c} v_c^{-1} y_c(f) \in \mathbf{N}(\mathcal{Y})_c$: it is Δ-fixed and, by (2) and linearity, $\mathbf{N}(\pi)(X_c(f)) = \ell^{s_1} e_c^\circ \otimes f$.

Step 6: the extension of f and (Ul). Let $Y = \{(y, \lambda) \in \mathcal{Y} \oplus \mathcal{O}_L : \ \pi(y) = \lambda f\}$, with $\iota'_Y(t) = (\iota'(t), 0)$ and $\pi_Y(y, \lambda) = \lambda$. It is a G_F-stable lattice in a subrepresentation of the crystalline $V_{\mathcal{Y}} \oplus L$, hence a pair (Theorem B.45), and $0 \to T_2 \to Y \to \mathcal{O}_L \to 0$ is exact (π is onto). With $\tilde{e} = (\bar{y}_0, 1)$, $\bar{y}_0$ the image of $(a_i\tilde{e}_i)_i$ (so $\pi(\bar{y}_0) = f$), $\sigma\tilde{e} - \tilde{e} = (\iota'(\sum_i a_i\kappa_i(\sigma)), 0)$, so $[Y] = \sum_i a_i[Y_i] = \psi(f)$ (Lemma 23.32). The representation V_Y is an extension of L by V_2, so its Hodge jumps at τ_a lie in $[-M_a, h_a]$ (Step 1). The sequence $0 \to Y \xrightarrow{\mathrm{inc}} \mathcal{Y} \oplus \mathcal{O}_L \xrightarrow{h} P \to 0$, $h(y, \lambda) = \pi(y) - \lambda f$, is an exact sequence of lattices of pairs; by Lemma 23.33, $\ker \mathbf{N}(h) = \mathbf{N}(\mathrm{inc})(\mathbf{N}(Y))$. With ι_1, ι_2 the inclusions of the summands, the element $Z = \mathbf{N}(\iota_1)(X_c(f)) + \mathbf{N}(\iota_2)(\ell^{s_1} e_c^\circ)$ satisfies

$\mathbf{N}(h)(Z) = \mathbf{N}(\pi)(X_c(f)) - \ell^{s_1} e_c^\circ \otimes f = 0$, so $Z = \mathbf{N}(\mathrm{inc})(x_c)$ with $x_c \in \mathbf{N}(Y)$, and $x_c = e_c x_c \in \mathbf{N}(Y)_c$ ($Z = e_c Z$, $\mathbf{N}(\mathrm{inc})$ injective and $\mathbf{A}_{F,L}$-linear). It is Δ-fixed, as Z is and $\mathbf{N}(\mathrm{inc})$ is injective and commutes with γ_δ. Finally $\pi_Y = \mathrm{pr}_2 \circ \mathrm{inc}$, so $\mathbf{N}(\pi_Y)(x_c) = \mathbf{N}(\mathrm{pr}_2)(Z) = \ell^{s_1} e_c^\circ$, and by (Ex3) and Lemma 24.3, $(1 \otimes \pi_Y)(\iota_Y(x_c)) = \iota_{\mathcal{O}_L}(\ell^{s_1} e_c^\circ) = \ell^{s_1} \varepsilon_c$. This is (Ul) for f with the exponent s_1. □

Lemma C.55. *Let ℓ, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ be as in Proposition 23.7, (V, T) a pair and $\mathbf{N} = \mathbf{N}(T)$ with a window $(M_a, h_a)_a$ as in (Ex1). Let $c \in \mathbb{Z}/n$, let $k \geq 0$ be an integer, put*

$$\mu = \sum_{j=-k}^{-1} \dim_L \left(\mathrm{Fil}^j D_{\mathrm{dR},\tau_c}(V) / \mathrm{Fil}^{j+1} D_{\mathrm{dR},\tau_c}(V) \right)$$

(the number of Hodge jumps of V at τ_c in $[-k, -1]$, counted with multiplicity), and fix $u_0 \in \mathbb{Z}_\ell^\times$ with $v_\ell(u_0 - 1) = 1$. Let P be a finitely generated $\mathcal{O}_L$-module and $\omega \colon P \to e_c \mathbf{N}[1/Q]$ an $\mathcal{O}_L$-linear map such that for every $f \in P$:

(Ps1) Pole *$Q^k \omega(f) \in \mathbf{N}$;*

(Ps2) Δ-fixed *$\gamma_\delta(\omega(f)) = \omega(f)$ for every $\delta \in \mu_{\ell-1} \subseteq \mathbb{Z}_\ell^\times$;*

(Ps3) Integral defect *$(\gamma_{u_0} - 1)(\omega(f)) \in \mathbf{N}$.*

Then $P' = \{ f \in P : \ \omega(f) \in \mathbf{N} \}$ is an $\mathcal{O}_L$-submodule of P and P/P' is generated by at most μ elements.

Proof. **Step 0: level data of length one.** Let $r = \dim_L V$. Choose a frame $(n_k)_{k \leq r}$ at $c - 1$ (an $e_{c-1}\mathbf{A}_{F,L}$-basis of $\mathbf{N}_{c-1}$, which exists by (Wm1)) and arc level data of the identity map $T \to T$ with cut c, length $\mathrm{len} = 1$, $H(c) = k + h_c$ and arbitrary values of the remaining parameters (they do not enter (Lp1)–(Lp3), (Lp8)). Then $\Pi = Q_0^{H(c)} = Q^{k+h_c}$ and $W = \{ w \in \mathbf{N}[1/\lambda_\infty] : \ Q^{k+h_c} w \in P_1(\mathbf{N}_{c-1}) \}$. By Lemma 23.17, (Lp1), (Lp2) and (Lp8) hold at $s = 0$, and by Lemma 23.5 (2) with $\mathrm{len} = 1$: for $w \in W$, $w \in \mathbf{N}_c$ if and only if $\mathrm{lv}(w)$ is de Rham integral at 0. By the definition of the levels, ϵ_0 is the idempotent of the embedding τ_c, the D_k form an L-basis of $\epsilon_0 D_{\mathrm{cris}}(V)$, and $v \mapsto \sum_k v_k D_k$ maps $\mathrm{Fil}(u)$ isomorphically onto $\epsilon_0 \mathrm{Fil}^u D_{\mathrm{cris}}(V)$. Since F is unramified and V crystalline, $D_{\mathrm{cris}}(V) = D_{\mathrm{dR}}(V)$, and $\epsilon_0 D_{\mathrm{dR}}(V)$ with the induced filtration is $D_{\mathrm{dR},\tau_c}(V)$: the map $\tau_c \otimes \mathrm{id} \colon F \otimes_{\mathbb{Q}_\ell} L \to L$ is the projection onto the factor cut out by ϵ_0, so $L \otimes_{F \otimes L, \tau_c \otimes \mathrm{id}} D_{\mathrm{dR}}(V) = \epsilon_0 D_{\mathrm{dR}}(V)$ (Definition 13.7), and the filtrations correspond because each $\mathrm{Fil}^j D_{\mathrm{dR}}(V)$ is an $F \otimes L$-submodule, whose image is $\epsilon_0 \mathrm{Fil}^j D_{\mathrm{dR}}(V)$. Hence

$$\dim_L \mathrm{Fil}(-k)/\mathrm{Fil}(0) = \sum_{j=-k}^{-1} \dim_L \mathrm{gr}^j D_{\mathrm{dR},\tau_c}(V) = \mu. \tag{1}$$

Step 1: the vectors lie in the arc domain. Let $f \in P$ and $y = \omega(f)$. Since e_c commutes with Q and with γ_{u_0} ($\gamma_u(e_a) = e_a$), the elements $Q^k y$ and $z = (\gamma_{u_0} - 1)(y)$ lie in $e_c \mathbf{N}[1/Q] \cap \mathbf{N} = \mathbf{N}_c$ by (Ps1), (Ps3). By (Wm2), $Q^{h_c} \mathbf{N}_c \subseteq \mathbf{A}_{F,L} \varphi(\mathbf{N}_{c-1}) = P_1(\mathbf{N}_{c-1})$. Hence $Q^{k+h_c} y = Q^{h_c}(Q^k y) \in P_1(\mathbf{N}_{c-1})$, so $y \in W$; and every element x of $\mathbf{N}_c$ lies in W, since $Q^{k+h_c} x \in Q^k P_1(\mathbf{N}_{c-1}) \subseteq P_1(\mathbf{N}_{c-1})$. So y, $Q^k y$, $z \in W$, and the last two lie in $\mathbf{N}_c$. Moreover $y \in \mathbf{N}$ if and only if $y \in \mathbf{N}_c$, as $y = e_c y$.

Step 2: the expansion. By (Lp1), y is algebraic at level 0; write $\mathrm{lv}(y) = \sum_i a_i(f) t^i$ with $a_i(f) \in K^r$, $a_i(f) = 0$ for $i < -(k + h_c)$ (Lemma 23.17). The level map $\Lambda_0 = \varphi^{-1} \circ \iota_T$ is additive and $\Lambda_0(\lambda x) = \epsilon_0 (1 \otimes \lambda) \Lambda_0(x)$ for $\lambda \in \mathcal{O}_L$ ((Ex2)), and the D_k form a basis over $R_0 \supseteq K((t))$; so each $f \mapsto a_i(f)$ is $\mathcal{O}_L$-linear.

Step 3: the constant coefficient is in L^r. Let $\delta \in \mu_{\ell-1}$. By (Lp2) and (Ps2), $\sum_i a_i t^i = \mathrm{lv}(\gamma_\delta y) = \sum_i \delta^i \sigma_\delta(a_i) t^i$, so $\sigma_\delta(a_0) = a_0$. For $u' \in 1+\ell\mathbb{Z}_\ell$, $\sigma_{u'} = \mathrm{id}$ on K by (Lp8) at $s=0$. Every $u \in \mathbb{Z}_\ell^\times$ is $\delta u'$ with $\delta \in \mu_{\ell-1}$, $u' \in 1+\ell\mathbb{Z}_\ell$, and $\sigma_u = \sigma_\delta \circ \sigma_{u'}$ (both are induced by $g_\delta g_{u'}$, where $\chi(g_\delta)=\delta$, $\chi(g_{u'})=u'$). So a_0 is fixed by every σ_u, and $a_0 \in L^r$ by (Lp8).

Step 4: the non-constant coefficients. As $u_0 \in 1+\ell\mathbb{Z}_\ell$, $\sigma_{u_0} = \mathrm{id}$, and by (Lp2) $\mathrm{lv}(z) = \sum_i (u_0^i - 1) a_i t^i$. Since $z \in W \cap \mathbf{N}_c$, $\mathrm{lv}(z)$ is de Rham integral (Step 0): $(u_0^i-1)a_i \in K \otimes_L \mathrm{Fil}(-i)$ for every i. For $i \neq 0$, $u_0^i \neq 1$ ($1+\ell\mathbb{Z}_\ell$ is torsion free, ℓ being odd, and $u_0 \neq 1$), and $K \otimes_L \mathrm{Fil}(-i)$ is a K-submodule of K^r; so

$$a_i \in K \otimes_L \mathrm{Fil}(-i) \qquad (i \neq 0). \tag{2}$$

Step 5: the pole bounds the constant coefficient. The de Rham integral vectors at level 0 form a $K[[t]]$-submodule of $K((t))^r$: if $g = \sum_i g_i t^i$ is de Rham integral and $b = \sum_{p \geq 0} b_p t^p \in K[[t]]$, the t^i-coefficient of bg is $\sum_{p \geq 0} b_p g_{i-p}$, and $g_{i-p} \in K \otimes_L \mathrm{Fil}(p-i) \subseteq K \otimes_L \mathrm{Fil}(-i)$ since Fil is decreasing. By (Lp1), $\lambda_0(Q) = \lambda_0(Q_0) = t\,e$ with $e \in K[[t]]^\times$. As $Q^k y \in W \cap \mathbf{N}_c$, $\mathrm{lv}(Q^k y) = t^k e^k \mathrm{lv}(y)$ is de Rham integral, hence so is $t^k \mathrm{lv}(y) = e^{-k}\mathrm{lv}(Q^k y)$, i.e. $a_i \in K \otimes_L \mathrm{Fil}(-i-k)$ for all i. For an L-subspace $\mathcal{X} \subseteq L^r$ one has $(K \otimes_L \mathcal{X}) \cap L^r = \mathcal{X}$ inside $K^r = K \otimes_L L^r$ (choose an L-basis of the finite-dimensional L-algebra K containing 1 and compare the coefficients of 1). With Step 3,

$$a_0(f) \in \mathrm{Fil}(-k). \tag{3}$$

Step 6: the membership test. By Step 1 and Step 0, $\omega(f) \in \mathbf{N}$ iff $\mathrm{lv}(y)$ is de Rham integral, i.e. iff $a_i \in K \otimes_L \mathrm{Fil}(-i)$ for all i. By (2) this holds iff $a_0 \in K \otimes_L \mathrm{Fil}(0)$, and by Step 3 and Step 5 iff $a_0(f) \in \mathrm{Fil}(0)$. By (3) and Step 2,

$$\beta \colon P \to \mathrm{Fil}(-k)/\mathrm{Fil}(0), \qquad f \mapsto a_0(f) \bmod \mathrm{Fil}(0),$$

is a well-defined $\mathcal{O}_L$-linear map, and $P' = \ker \beta$. So P' is a submodule and $P/P' \cong \beta(P)$.

Step 7: the count. $\beta(P)$ is a finitely generated $\mathcal{O}_L$-submodule of the L-vector space $\mathrm{Fil}(-k)/\mathrm{Fil}(0)$, of dimension μ by (1). It is torsion free, hence free over the discrete valuation ring $\mathcal{O}_L$, and an $\mathcal{O}_L$-basis of it is L-linearly independent (clear denominators), so its rank is at most μ. Hence P/P' is generated by at most μ elements. □

Proposition C.56. *There is an absolute constant $c_{\mathrm{S}} \geq 1$ with the following property. Let ℓ, F, n, L, $(\tau_a)_{a \in \mathbb{Z}/n}$, $\Phi \colon V_1 \to V_2$ (with lattices T_1, T_2) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.52, and let C, $(M_a, h_a)_a$ and B be as in Proposition C.53. Then there are integers $0 \leq s_2 \leq c_{\mathrm{S}} B$, $g \geq \eta$, $m_{\mathrm{R}} \geq 1 + \max_a(s_a + S)$ and $R \geq \max(m_{\mathrm{R}}, g)$ such that, with I_c and $\rho_{c,1} \colon I_c \to H^1(F, T_2)$ formed for $\mathbf{N}(T_2)$ with c, $j=1$, $G=\eta$, g, m_{R} (in the role of m) and R, the following holds. Let $s_1 \geq 0$ be an integer, P a free $\mathcal{O}_L$-module of finite rank, $\psi \colon P \to H^1_f(F, T_2)$ an $\mathcal{O}_L$-linear map and $P' \subseteq P$ a submodule every element of which has property (Ul) of Proposition C.54 with the exponent s_1. Then there are submodules $P' = P'_0 \supseteq P'_1 \supseteq \cdots \supseteq P'_k$ with $k \leq n/2$ and every P'_i/P'_{i+1} generated by at most m elements, such that for every $f \in P'_k$ there are $w_c \in I_c$ ($c \in C$) with*

(Sl1) *for every $c \in C$: $\gamma_\delta(w_c) = w_c$ for all $\delta \in \mu_{\ell-1} \subseteq \mathbb{Z}_\ell^\times$, and $Q^\eta w_c \in P_1\big(\mathbf{N}(T_2)_{c-1}\big)$;*

(Sl2) *$\ell^{s_1+s_2}\psi(f) = \pm \sum_{c \in C} \rho_{c,1}(w_c)$ in $H^1(F, T_2)$, for one of the two signs.*

Proof. **Step 0: the constants.** Put $c_{\mathrm{S}} = 8$, $g = \eta$, $m_{\mathrm{R}} = 1 + \max_a(s_a + S)$, $R = m_{\mathrm{R}}$, $\sigma = \sigma_\ell(g, m_{\mathrm{R}}, 1)$ and $s_2 = 2\sigma$. Since every $s_a > \eta$, $m_{\mathrm{R}} \geq g$, so $R \geq \max(m_{\mathrm{R}}, g)$, and $g \geq \eta$. Hence $0 \leq s_2 \leq 8B = c_{\mathrm{S}} B$. Let I_c, $\rho_{c,1}$ be formed for $\mathbf{N}_2$ with c, $j=1$, $G=\eta$, g, m_{R}, R.

Step 1: the configuration. By the parity hypothesis of Proposition C.52 (n even, $\epsilon_a = -1$ iff $a + d$ is even), for $c \in C$ we have $c-1, c+1 \notin C$ and $c-2 \in C$; so $\mathrm{len}_c = 2$ satisfies the condition of Proposition 24.4, and $j = 1 \le \mathrm{len}_c$. For $c \in C$, $c+1$ is an up site, so $M_{c+1} = \eta = G$, and $h_c = \eta$. By (Ow1), every Hodge jump of V_1 at τ_a lies in $[-(s_a + S), \eta]$ if $\epsilon_a = -1$ and in $[-\eta, s_a + S]$ if $\epsilon_a = 1$ (as $s_a > \eta$), hence in $[-M_a, h_a]$; by Lemma 24.2 (1) the same holds for V_2. Also $m_{\mathrm{R}} \ge \max(g, 1 + \max_a M_a)$ and $R \ge m_{\mathrm{R}}$. So for every extension Y as in (Ul), $(M_a, h_a)_a$ is a window of V_2 and of V_Y, and Proposition 24.4 applies to $0 \to T_2 \to Y \to \mathcal{O}_L \to 0$ with C, $\mathrm{len} \equiv 2$, $j = 1$, $G = \eta$, g, $m = m_{\mathrm{R}}$, R and $r = s_1$.

Step 2: units and reference lifts. Let Y, ι', π, $(x_c)_{c\in C}$ be as in (Ul). (a) By Lemma 24.3, (Ex3) and the injectivity of $\iota_{\mathcal{O}_L}$ (Proposition 23.7), $\iota_{\mathcal{O}_L}(\mathbf{N}(\pi)(x_c)) = (1 \otimes \pi)(\iota_Y(x_c)) = \ell^{s_1}\varepsilon_c = \iota_{\mathcal{O}_L}(\ell^{s_1} e_c^\circ)$, so $\mathbf{N}(\pi)(x_c) = \ell^{s_1} e_c^\circ$. Similarly $\varphi(e_a^\circ) = e_{a+1}^\circ$ (as $\iota_{\mathcal{O}_L}(\varphi e_a^\circ) = (\varphi \otimes 1)(\varepsilon_a) = (\iota \otimes \mathrm{id})(\varphi(e_a)) = \varepsilon_{a+1}$, ι commuting with φ by Lemma 23.8) and $\gamma_u(e_a^\circ) = e_a^\circ$ (by (Ex2), $\iota_{\mathcal{O}_L}(\gamma_{\chi(\sigma)} e_a^\circ) = \sigma(\varepsilon_a) = \varepsilon_a$, since G_F fixes $\iota(\mathcal{O}_F) \subseteq W(k)$ and acts trivially on $\mathcal{O}_L$). (b) Let $N = \max_a r_a$ with r_a the integers of Proposition C.6 for the window $(M_a, h_a)_a$; they depend only on ℓ and the window. By Proposition C.6 ($\ell \ge 11$), applied to π and $\ell^{N-r_a} e_a^\circ$, there is $\varepsilon'_a \in \mathbf{N}(Y)_a$ with $\mathbf{N}(\pi)(\varepsilon'_a) = \ell^N e_a^\circ$; then $\epsilon_a^Y := \mathrm{av}(\varepsilon'_a) \in \mathbf{N}(Y)_a$ is Δ-fixed and $\mathbf{N}(\pi)(\epsilon_a^Y) = \ell^N e_a^\circ$, since $\mathbf{N}(\pi)$ commutes with γ_δ and $\gamma_\delta(e_a^\circ) = e_a^\circ$.

Step 3: exactness facts. Let Y be as in (Ul). (E1) Let $\mathcal{S}$ be a multiplicative set of finite products of the Q_s. Then $\mathbf{N}(\iota')$ is injective on $\mathbf{N}_2[1/\mathcal{S}]$, its image is the kernel of $\mathbf{N}(\pi)$ on $\mathbf{N}(Y)[1/\mathcal{S}]$, and if $v \in \mathbf{N}_2[1/\mathcal{S}]$ with $\mathbf{N}(\iota')(v) \in \mathbf{N}(Y)$ then $v \in \mathbf{N}_2$. Indeed, all three modules are free over $\mathbf{A}_{F,L}$ site by site ((Wm1)), hence $\mathcal{S}$-torsion free; if $\mathbf{N}(\pi)(y/f) = 0$ then $\mathbf{N}(\pi)(y) = 0$, so $y \in \mathbf{N}(\iota')(\mathbf{N}_2)$ by Lemma 23.33; injectivity is that of Lemma 23.33 after clearing f; and if $\mathbf{N}(\iota')(v) = y \in \mathbf{N}(Y)$, then $\mathbf{N}(\pi)(y) = 0$, $y = \mathbf{N}(\iota')(v')$ with $v' \in \mathbf{N}_2$, and $v = v'$. (E2) For $c \in \mathbb{Z}/n$: if $y \in P_1(\mathbf{N}(Y)_{c-1})$ and $\mathbf{N}(\pi)(y) = 0$, then $y = \mathbf{N}(\iota')(v)$ with $v \in P_1(\mathbf{N}_{2,c-1})$. Indeed, $e_c\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ is finite free (of rank ℓ, with basis $(1+X)^i$, $0 \le i < \ell$) over its subring $\varphi(e_{c-1}\mathbf{A}_{F,L}) = \mathcal{O}_L[[(1+X)^\ell - 1]]$ by Weierstrass division, so $R \otimes_\varphi -$ ($R = e_c\mathbf{A}_{F,L}$) is exact; applied to the exact sequence $0 \to \mathbf{N}_{2,c-1} \to \mathbf{N}(Y)_{c-1} \to \mathbf{N}(\mathcal{O}_L)_{c-1}$ (Lemma 23.33) and transported by the isomorphisms $R \otimes_\varphi \mathbf{N}_{c-1} \cong P_1(\mathbf{N}_{c-1})$ of Lemma 23.15 (1), which commute with $\mathbf{N}(\iota')$ and $\mathbf{N}(\pi)$ (these commute with φ), it gives exactness of $0 \to P_1(\mathbf{N}_{2,c-1}) \to P_1(\mathbf{N}(Y)_{c-1}) \to P_1(\mathbf{N}(\mathcal{O}_L)_{c-1})$, and the last module lies in $\mathbf{N}(\mathcal{O}_L)[1/\lambda_\infty]$.

Step 4: Δ-averaging of the class splitting. (a) *Herr classes are Δ-invariant.* Let y be a Herr vector of precision m_{R} for $\mathbf{N}_2$, $\delta \in \Delta$ and $g \in G_F$ with $\chi_{\mathrm{cyc}}(g) = \delta$. As γ_δ commutes with φ and all γ_u ((Wm3), (Gc0)) and $\gamma_\delta(X^{m_{\mathrm{R}}}\mathbf{N}_2) = X^{m_{\mathrm{R}}}\mathbf{N}_2$ ($\gamma_\delta(X) \in X\,\mathbb{Z}_\ell[[X]]^\times$), $\gamma_\delta y$ is a Herr vector and $z_u(\gamma_\delta y) = \gamma_\delta z_u(y)$ by uniqueness in Lemma 23.28. If $(\varphi \otimes 1 - 1)(b) = \iota_{T_2}(y)$, then $(\varphi \otimes 1 - 1)(gb) = g\,\iota_{T_2}(y) = \iota_{T_2}(\gamma_\delta y)$ by (Gc0). For $\sigma' \in G_F$, using $\chi_{\mathrm{cyc}}(g^{-1}\sigma' g) = \chi_{\mathrm{cyc}}(\sigma')$ and (Gc0), with $\kappa = c_{y,b}$,

$$c_{\gamma_\delta y, gb}(\sigma') = g\,\iota_{T_2}(z_{\chi(\sigma')}(y)) - (\sigma' - 1)gb = g(\kappa(g^{-1}\sigma' g) + (g^{-1}\sigma' g - 1)b) - \sigma' gb + gb = g\,\kappa(g^{-1}\sigma' g).$$

The cocycle relation gives $\kappa(\sigma' g) = \kappa(g) + g\,\kappa(g^{-1}\sigma' g)$ and $\kappa(\sigma' g) = \kappa(\sigma') + \sigma'\kappa(g)$, so $c_{\gamma_\delta y, gb}(\sigma') = \kappa(\sigma') + (\sigma' - 1)\kappa(g)$ with $\kappa(g) \in T_2$ ((Hc1)): $\mathrm{cl}(\gamma_\delta y) = \mathrm{cl}(y)$. (b) *Relay classes are Δ-invariant.* Let $w \in I_c$, let τ be a relay correction of w ((Ry1)), and let V be as in the definition of a relay correction. Since $\gamma_\delta(e_c) = e_c$ and $\gamma_\delta(Q_s) = \varphi^s(\gamma_\delta Q) \in Q_s\mathbf{A}_F^\times$ (Lemma 23.8 (2)), γ_δ preserves $e_c\mathbf{N}_2[1/\lambda_\infty]$, $\mathbf{N}_{2,c}[1/\ell]$, $P_1(\mathbf{N}_{2,c})$ and $X^{m_{\mathrm{R}}}\lambda_1^g P_1(\mathbf{N}_{2,c})[1/\ell]$; on $\mathbf{N}_{2,E}$ it commutes with every γ_u and preserves $X^k\mathbf{N}_{2,E}$ (Lemma 23.9). Hence $(\gamma_u - 1)\gamma_\delta w = \gamma_\delta(\gamma_u - 1)w \in X\mathbf{N}_2$, so $\gamma_\delta w \in I_c$; $W_{\gamma_\delta w} = \gamma_\delta W_w$ by the uniqueness in (Hf1); and $\gamma_\delta V$, $\gamma_\delta\tau$ satisfy the two conditions of a relay correction of $\gamma_\delta w$ (apply γ_δ to those for w, V, τ). By (Ry2), (Ry3) and (a), $\rho_{c,1}(\gamma_\delta w) = \mathrm{cl}(\gamma_\delta(\ell^\sigma\varphi(w) - \tau)) = \mathrm{cl}(\ell^\sigma\varphi(w) - \tau) = \rho_{c,1}(w)$. By (Ry4), $\mathrm{av}(w) \in I_c$, $\mathrm{av}(w)$ is Δ-fixed, and $\rho_{c,1}(\mathrm{av}(w)) = \rho_{c,1}(w)$. (c) *Averaged splitting.* Let Y, (x_c) be as in

(Ul). Proposition 24.4, applied with $j=1$, gives (in its statement) $\tau'_c \in P_1(\mathbf{N}_{2,c})$ and $w'_c \in I_c$ with its two identities and its class identity. Put $\tau_c = \mathrm{av}(\tau'_c) \in P_1(\mathbf{N}_{2,c})$, $w_c = \mathrm{av}(w'_c)$ and $X_c = \ell^\sigma\varphi(x_c) - \mathbf{N}(\iota')(\tau_c) = \mathrm{av}(\ell^\sigma\varphi(x_c) - \mathbf{N}(\iota')(\tau'_c))$ (x_c is Δ-fixed; $\mathbf{N}(\iota')$ and φ commute with γ_δ). Applying av to the identities of Proposition 24.4 and using (b):

$$\begin{gathered}\mathbf{N}(\iota')(w_c) = \varphi(X_{c-2}) - \ell^\sigma x_c, \qquad (\gamma_u - 1)(X_c) \in \mathbf{N}(\iota')(X^{m_{\mathrm{R}}}\mathbf{N}_2),\\ \ell^{\,s_1+2\sigma}[Y] = \sum_{c\in C}\rho_{c,1}(w_c),\end{gathered} \tag{1}$$

and w_c, τ_c, X_c are Δ-fixed.

Step 5: the vectors that must become integral. With Y, x_c, ϵ^Y_a, w_c, X_c as in Steps 2 and 4, for $c \in C$: (i) There is a unique $u_c \in e_{c+1}\mathbf{N}_2[1/Q]$ with $\mathbf{N}(\iota')(u_c) = \ell^N X_c - \ell^{s_1+\sigma}\epsilon^Y_{c+1}$, and $Q^\eta u_c \in \mathbf{N}_2$, u_c is Δ-fixed, $(\gamma_{u_0}-1)(u_c) \in \mathbf{N}_2$. Indeed $\varphi(x_c) \in e_{c+1}\mathbf{N}(Y)[1/Q]$ and $\tau_c \in Q^{-\eta}\mathbf{N}_{2,c+1}$ by (Ry0) ($G=\eta$, $\lambda_1 = Q$), so the right side lies in $e_{c+1}\mathbf{N}(Y)[1/Q]$, and by Step 2 its image under $\mathbf{N}(\pi)$ is $\ell^{N+\sigma}\varphi(\ell^{s_1}e^\circ_c) - \ell^{s_1+\sigma}\ell^N e^\circ_{c+1} = 0$; (E1) gives u_c. By (Wm2) at the up site $c+1$, $Q^\eta\varphi(x_c) \in \mathbf{N}(Y)_{c+1}$, so $Q^\eta X_c \in \mathbf{N}(Y)$ and $\mathbf{N}(\iota')(Q^\eta u_c) \in \mathbf{N}(Y)$, whence $Q^\eta u_c \in \mathbf{N}_2$ by (E1). The Δ-fixedness of u_c follows from that of X_c and ϵ^Y_{c+1} and from injectivity. Finally $(\gamma_{u_0}-1)\epsilon^Y_{c+1} \in \mathbf{N}(Y)$ ((Wm3)) has $\mathbf{N}(\pi)$-image $(\gamma_{u_0}-1)(\ell^N e^\circ_{c+1}) = 0$, so it lies in $\mathbf{N}(\iota')(\mathbf{N}_2)$ (Lemma 23.33); with (1), $\mathbf{N}(\iota')((\gamma_{u_0}-1)u_c) \in \mathbf{N}(\iota')(\mathbf{N}_2)$, and $(\gamma_{u_0}-1)u_c \in \mathbf{N}_2$ by (E1). (ii) There is $v_c \in P_1(\mathbf{N}_{2,c-1})$ with $\ell^N Q^\eta w_c = Q^\eta\varphi(u_{c-2}) + v_c$. Indeed, by (1) and (i) for $c-2 \in C$, $\mathbf{N}(\iota')(\ell^N Q^\eta w_c) = Q^\eta\varphi(\mathbf{N}(\iota')u_{c-2}) + y_c$ with $y_c = Q^\eta(\ell^{s_1+\sigma}\varphi(\epsilon^Y_{c-1}) - \ell^{N+\sigma}x_c)$. Here $\varphi(\epsilon^Y_{c-1}) \in P_1(\mathbf{N}(Y)_{c-1})$ and $Q^\eta x_c \in Q^{h_c}\mathbf{N}(Y)_c \subseteq P_1(\mathbf{N}(Y)_{c-1})$ ((Wm2), $h_c = \eta$), so $y_c \in P_1(\mathbf{N}(Y)_{c-1})$, and $\mathbf{N}(\pi)(y_c) = Q^\eta(\ell^{s_1+\sigma+N}e^\circ_c - \ell^{N+\sigma+s_1}e^\circ_c) = 0$. By (E2), $y_c = \mathbf{N}(\iota')(v_c)$ with $v_c \in P_1(\mathbf{N}_{2,c-1})$, and the identity follows from the injectivity in (E1).

Step 6: data on a basis. Let s_1, P, ψ, P' be given. As a submodule of a free module of finite rank over the discrete valuation ring $\mathcal{O}_L$, P' is free; let $f_1,\dots,f_t$ be a basis. For each i choose, by (Ul) for f_i, an extension Y_i with $[Y_i] = \psi(f_i)$ and $x^{(i)}_c$, and then by Steps 2, 4, 5 the elements $\epsilon^{(i)}_a$, $w^{(i)}_c$, $X^{(i)}_c$, $u^{(i)}_c$, $v^{(i)}_c$. For $c \in C$ define the $\mathcal{O}_L$-linear maps

$$\omega_c\colon P' \to e_{c+1}\mathbf{N}_2[1/Q], \quad \omega_c\Big(\sum_i a_i f_i\Big) = \sum_i a_i u^{(i)}_c, \qquad w_c\colon P' \to I_c, \quad w_c\Big(\sum_i a_i f_i\Big) = \sum_i a_i w^{(i)}_c$$

(I_c is an $\mathcal{O}_L$-module by (Ry4)). By Step 5 (i), and since $\mathbf{N}_2$ is an $\mathcal{O}_L$-module and γ_δ, γ_{u_0} are $\mathcal{O}_L$-linear, every $\omega_c(f)$ satisfies (Ps1) with its parameter equal to η, (Ps2) and (Ps3) of Lemma C.55 at the site $c+1$.

Step 7: the passage chain. Enumerate $C = \{c_1,\dots,c_k\}$, $k = |C| = n/2$. Put $P'_0 = P'$ and $P'_j = \{f \in P'_{j-1} :\ \omega_{c_j}(f) \in \mathbf{N}_2\}$. By Lemma C.55, applied to (V_2, T_2) with its window, the site c_j+1, its parameter equal to η, the finitely generated module P'_{j-1} and the restriction of ω_{c_j}, P'_j is a submodule and P'_{j-1}/P'_j is generated by at most $\mu_j = \sum_{i=-\eta}^{-1}\dim_L \mathrm{gr}^i D_{\mathrm{dR},\tau_{c_j+1}}(V_2) \le m$ elements, the inequality by (Ow2). So $P' = P'_0 \supseteq \cdots \supseteq P'_k$ with $k \le n/2$, and $\omega_c(f) \in \mathbf{N}_2$ for every $f \in P'_k$ and $c \in C$.

Step 8: (Sl1) and (Sl2). Let $f = \sum_i a_i f_i \in P'_k$ and put $w_c = w_c(f)$. Each $w^{(i)}_c$ is Δ-fixed, so w_c is. By Step 5 (ii), $\ell^N Q^\eta w_c = Q^\eta\varphi(\omega_{c-2}(f)) + \sum_i a_i v^{(i)}_c$. As $c-2 \in C$, $\omega_{c-2}(f) \in \mathbf{N}_2 \cap e_{c-1}\mathbf{N}_2[1/Q] = \mathbf{N}_{2,c-1}$, so $\varphi(\omega_{c-2}(f)) \in P_1(\mathbf{N}_{2,c-1})$; hence $\ell^N Q^\eta w_c \in P_1(\mathbf{N}_{2,c-1})$. Since $w_c \in e_c\mathbf{N}_2[1/\lambda_\infty]$, there is a finite product f' of the Q_s with $f'Q^\eta w_c \in \mathbf{N}_{2,c}$, so $Q^{h_c}f'Q^\eta w_c \in P_1(\mathbf{N}_{2,c-1})$ ((Wm2)), and $Q^\eta w_c$ lies in $P_1(\mathbf{N}_{2,c-1})\otimes_R \mathrm{Frac}(R)$; the saturation (3) of Lemma 23.15 (with $U = P_1(\mathbf{N}_{2,c-1})$ and its f equal to 1) gives $Q^\eta w_c \in P_1(\mathbf{N}_{2,c-1})$. This is (Sl1). By (Ry4), the linearity of ψ and (1) for each Y_i,

$$\sum_{c\in C}\rho_{c,1}(w_c) = \sum_i a_i \sum_{c\in C}\rho_{c,1}(w^{(i)}_c) = \sum_i a_i\,\ell^{\,s_1+2\sigma}\psi(f_i) = \ell^{\,s_1+s_2}\psi(f),$$

which is (Sl2) with the sign +. □

Proposition C.57. *There is an absolute constant $c_4 \geq 1$ with the following property. Let ℓ, F, n, L, $(\tau_a)_{a\in\mathbb{Z}/n}$, $\Phi\colon V_1 \to V_2$ (with lattices T_1, T_2) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.52, put*

$$B = \frac{H'}{\ell^2} + S + \frac{\eta}{\ell} + 1 + \log_\ell(2H'), \qquad \lambda = d_2 S + 2m\Big(\frac{H'+1}{\ell-1} + 1\Big),$$

and let $c \in \mathbb{Z}/n$ with $\epsilon_c = -1$. Write I_c^ν for the set I_c formed for $\mathbf{N}(T_\nu)$. Then there is an integer $0 \leq e \leq c_4 B$ with the following property. Let P be a free $\mathcal{O}_L$-module of finite rank, $P' \subseteq P$ a submodule and $w\colon P' \to I_c^2$ an $\mathcal{O}_L$-linear map such that for every $f \in P'$

$$\gamma_\delta(w(f)) = w(f) \ \ (\delta \in \mu_{\ell-1} \subseteq \mathbb{Z}_\ell^\times) \qquad \text{and} \qquad Q^\eta w(f) \in P_1(\mathbf{N}(T_2)_{c-1}).$$

Then there are submodules $P' = P'_0 \supseteq P'_1 \supseteq \cdots \supseteq P'_k$ and a set $J \subseteq \{0, \dots, k-1\}$ with $|J| \leq 6$ and $k - |J| \leq 3$, such that P'_i/P'_{i+1} is generated by at most m elements for $i \in J$ and $\mathrm{nl}(P'_i/P'_{i+1}) \leq \lambda$ for $i \notin J$, and such that for every $f \in P'_k$ there are $u \in I_c^1$ and a finite product λ' of the Q_s ($s \geq 0$) with $\lambda' u \in \mathbf{N}(T_1)$ and

$$\ell^e\, w(f) - \mathbf{N}(\Phi)(u) \in X\, \mathbf{N}(T_2)_c.$$

Proof. Let $c_{4,1}$ and $c_{4,2}$ be the absolute constants of Proposition C.59 and Proposition C.75, and put $c_4 = c_{4,1} + c_{4,2}$. Given the data and the down site c, let $e_2 \leq c_{4,1}B$ and $e_6 \leq c_{4,2}B$ be the integers these two statements provide, and $e = e_2 + e_6 \leq c_4 B$. Throughout, $\mathbf{N}(\Phi)$ is extended $\mathbf{A}_{F,L}$-linearly to $\mathbf{N}(T_1)[1/\lambda_\infty]$; it is additive and $\mathcal{O}_L$-linear.

First stage. Let P, P' and w be as in the statement. They satisfy the hypotheses of Proposition C.59, which gives a chain $P' = P'_0 \supseteq \cdots \supseteq P'_{k_1}$ with at most 3 steps whose quotients are generated by at most m elements and at most 1 step whose quotient has normalised length at most λ, and an $\mathcal{O}_L$-linear map $y\colon P'_{k_1} \to I_c^1$ such that for $f \in P'_{k_1}$, $\lambda'_f y(f) \in \mathbf{N}(T_1)$ for a finite product λ'_f of the Q_s, and

$$w'(f) := \ell^{e_2} w(f) - \mathbf{N}(\Phi)(y(f)) \in \mathbf{N}(T_2)_c$$

is fixed by every γ_δ, $\delta \in \mu_{\ell-1}$. The map $w'\colon P'_{k_1} \to \mathbf{N}(T_2)_c$ is $\mathcal{O}_L$-linear, as w, y and $\mathbf{N}(\Phi)$ are.

Second stage. The module P, its submodule P'_{k_1} and the map w' satisfy the hypotheses of Proposition C.75, which gives a chain $P'_{k_1} \supseteq \cdots \supseteq P'_k$ with at most 3 steps of the first kind and at most 2 of the second, such that for every $f \in P'_k$ there are $x_f \in I_c^1$ and a finite product λ''_f of the Q_s with $\lambda''_f x_f \in \mathbf{N}(T_1)$ and $\ell^{e_6} w'(f) - \mathbf{N}(\Phi)(x_f) \in X\mathbf{N}(T_2)_c$.

Assembly. The concatenated chain $P' = P'_0 \supseteq \cdots \supseteq P'_k$ has at most $3 + 3 = 6$ steps of the first kind and at most $1 + 2 = 3$ of the second. Let $f \in P'_k$ and $u = \ell^{e_6} y(f) + x_f$. By (Ry4) of Lemma 24.5, I_c^1 is an $\mathcal{O}_L$-submodule, so $u \in I_c^1$; the product $\lambda' = \lambda'_f \lambda''_f$ of the Q_s satisfies $\lambda' u = \ell^{e_6} \lambda''_f(\lambda'_f y(f)) + \lambda'_f(\lambda''_f x_f) \in \mathbf{N}(T_1)$, since $\mathbf{N}(T_1)$ is an $\mathbf{A}_{F,L}$-module. Finally

$$\ell^e w(f) - \mathbf{N}(\Phi)(u) = \ell^{e_6}\big(\ell^{e_2} w(f) - \mathbf{N}(\Phi)(y(f))\big) - \mathbf{N}(\Phi)(x_f) = \ell^{e_6} w'(f) - \mathbf{N}(\Phi)(x_f) \in X\mathbf{N}(T_2)_c.$$

□

C.2.2 Lifting at a down site

This subsection proves Proposition C.57 in two stages, after three generator steps and one length step (Proposition C.59) and after three generator steps and two length steps (Proposition C.75). The tools are the crystalline value of a Wach vector (Definition C.58) and the Frobenius-square congruence $\ell^{\sigma}\varphi^2(\pi_0)^y \in \ell^{\lceil y\ell/(\ell-1)\rceil} + \pi_0(\pi_0+\ell)^h$, with σ about $h/(\ell-1)^2$ (Lemma C.63), which enters the Frobenius-square truncations; with them Fil-filtered values from two sites back lift at an up site (Proposition C.72). The length steps are bounded by a dual length lemma at a down site (Proposition C.65), and Lemma C.76 turns generator steps, length steps and a killing exponent into a colength. The lemmas C.68 and Lemma C.69 are used again in Section C.3.

Definition C.58. Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a pair (V,T) be as in the existence of Wach modules over unramified fields, and $\mathbf{N} = \mathbf{N}(T)$. For $x \in \mathbf{N}$ let $H_x \in x + X\mathbf{N}_E$ be the horizontal section through x, i.e. the unique element of $x + X\mathbf{N}_E$ fixed by every γ_u, $u \in \mathbb{Z}_\ell^\times$. The **crystalline value** of x is

$$\mathrm{Cv}_T(x) = \mathrm{Ev}(H_x) \ \in \ B_{\mathrm{cris}} \otimes_{\mathbb{Z}_\ell} T.$$

We write Cv for Cv_T when T is clear.

Proposition C.59. *There is an absolute constant $c_{4,1} \geq 1$ with the following property. Let ℓ, F, n, L, $(\tau_a)_{a\in\mathbb{Z}/n}$, $\Phi\colon V_1 \to V_2$ (with lattices T_1, T_2) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.52, form B and λ as in Proposition C.57, and let $c \in \mathbb{Z}/n$ with $\epsilon_c = -1$. Then there is an integer $0 \leq e_2 \leq c_{4,1}B$ with the following property. Let P be a free $\mathcal{O}_L$-module of finite rank, $P' \subseteq P$ a submodule and $w\colon P' \to I_c^2$ an $\mathcal{O}_L$-linear map such that for every $f \in P'$, $\gamma_\delta(w(f)) = w(f)$ for all $\delta \in \mu_{\ell-1} \subseteq \mathbb{Z}_\ell^\times$ and $Q^\eta w(f) \in P_1(\mathbf{N}(T_2)_{c-1})$. Then there are submodules $P' = P'_0 \supseteq P'_1 \supseteq \cdots \supseteq P'_k$, a set $J \subseteq \{0,\dots,k-1\}$ with $|J| \leq 3$ and $k - |J| \leq 1$, such that P'_i/P'_{i+1} is generated by at most m elements for $i \in J$ and $\mathrm{nl}(P'_i/P'_{i+1}) \leq \lambda$ for $i \notin J$, and an $\mathcal{O}_L$-linear map $y\colon P'_k \to I_c^1$ such that for every $f \in P'_k$: $\gamma_\delta(y(f)) = y(f)$ for $\delta \in \mu_{\ell-1}$, $\lambda' y(f) \in \mathbf{N}(T_1)$ for some finite product λ' of the Q_s ($s \geq 0$), and*

$$\ell^{e_2} w(f) - \mathbf{N}(\Phi)(y(f)) \in \mathbf{N}(T_2)_c \quad \text{and is fixed by every } \gamma_\delta,\ \delta \in \mu_{\ell-1}.$$

Proof. Put $M = s_c + S$, $h'' = s_{c-2} + S$, $q = \pi_0 + \ell$ ($q(0) = \ell$), $D = D_{\mathrm{cris}}(V_2)$, $\mathbf{N} = \mathbf{N}(T_2)$, let E be the integer of Proposition C.72 for the up site $c-1$, κ that of Proposition C.71 for c, and

$$e_2 = e_{\mathrm{H}} + \kappa - h'' = e_{\mathrm{H}} + s_\ell(k-1) + E + \sigma_{\mathrm{G}} - s_{c-1}.$$

Step 0: signs, windows, level data. As n is even and the signs alternate, $\epsilon_{c-1} = 1$, $\epsilon_{c-2} = -1$. By Lemma 24.2 (1) and (Ow1), the Hodge jumps of V_2 at τ_c lie in $[-M, -s_c] \cup [-\eta, \eta]$ and those at τ_{c-1} in $[s_{c-1}, s_{c-1}+S] \cup [-\eta, \eta]$. Fix arc level data of (V_2, T_2) with cut c, length 1, $H(c) = \eta$ and a frame at $c-1$. The level-0 expansion of an element of $e_c\mathbf{N}[1/\lambda_1]$ algebraic at level 0 is the coordinate vector of its image under $\Lambda_0 = \varphi^{-1} \circ \iota_{T_2}$ in the level frame, so it depends on the cut, the length and the frame but not on H; for such an element v we write $\Lambda_0(v) = \sum_i t^i \delta_i(v)$, $\delta_i(v) \in K_{(0)} \otimes_L \epsilon_{c-1}D$, as in Proposition C.60. For $f \in P'$, $w(f) \in W$ and, by Proposition C.60 (Ct1), $\delta_i(w(f)) = 0$ for $i < -\eta$, $\delta_i(w(f)) \in K_{(0)} \otimes_L \mathrm{Fil}^{-i}D$ for $i \neq 0$, $\delta_0(w(f)) \in \epsilon_{c-1}D$; these vectors are $\mathcal{O}_L$-linear in f.

Step 1: the constant term, linearly. The module P' is a submodule of a free module of finite rank over the principal ideal domain $\mathcal{O}_L$, hence free with a finite basis (f_j). For each j, Proposition C.60 (Ct2) gives a Δ-invariant $x_j \in \mathbf{N}_{c-2}$ with $\ell^{e_{\mathrm{H}}}\delta_0(w(f_j)) - \varphi^{-2}(\mathrm{Cv}(x_j)) \in$

$\mathrm{Fil}^{-\eta}D$. Let $\xi\colon P' \to \mathbf{N}_{c-2}^{\Delta}$ be the $\mathcal{O}_L$-linear map with $\xi(f_j) = x_j$. As Cv is $\mathcal{O}_L$-linear ((Cv1), (Cv2)) and φ^{-2} is L-linear on D, for every $f \in P'$

$$\ell^{e_{\mathrm{H}}}\delta_0(w(f)) - \varphi^{-2}\big(\mathrm{Cv}(\xi(f))\big) \in \mathrm{Fil}^{-\eta}D. \qquad (1)$$

Step 2: the dual length lemma at $c-2$. The site $c-2$ is down. With Σ_{c-2} from Proposition C.65 at $c-2$, let $P_1' = \xi^{-1}(\pi_0\mathbf{N}_{c-2}^{\Delta} + Y_{c-2} + \sum_{\Sigma_{c-2}} \mathcal{O}_L y)$ and $P_2' = \xi^{-1}(\pi_0\mathbf{N}_{c-2}^{\Delta} + Y_{c-2})$. The quotient P'/P_1' embeds in $\mathbf{N}_{c-2}^{\Delta}/(\pi_0\mathbf{N}_{c-2}^{\Delta} + Y_{c-2} + \sum_{\Sigma_{c-2}} \mathcal{O}_L y)$, so $\mathrm{nl}(P'/P_1') \le \lambda$; P_1'/P_2' embeds in a module generated by the images of the at most m elements of Σ_{c-2}, so it is generated by at most m elements.

Step 3: one Frobenius step back. Let $f \in P_2'$. Then $\xi(f) = \pi_0 n + \sum_j o_j q^{h''} g_j \varphi(x_j'')$ with $n \in \mathbf{N}_{c-2}^{\Delta}$, $o_j \in \mathcal{O}_L$, $g_j \in \mathbf{A}_{F,L}$ and $x_j'' \in \mathbf{N}_{c-3}$ Δ-invariant, each summand lying in $\mathbf{N}_{c-2}$ (Proposition C.65). By (Cv2), $\mathrm{Cv}(\pi_0 n) = 0$ ($\pi_0(0) = 0$); by (Cv4) and (Cv1), $\mathrm{Cv}(q^{h''} g_j \varphi(x_j'')) = \ell^{h''}\varphi^3(\iota_\Lambda(g_j(0)))\varphi(\mathrm{Cv}(x_j'')) = \ell^{h''}\varphi(\mathrm{Cv}(\varphi^{-1}(g_j(0))x_j''))$ (φ is bijective on Λ and ι is φ-equivariant). Hence $\mathrm{Cv}(\xi(f)) \in \ell^{h''}\varphi(\mathrm{Cv}(\mathbf{N}_{c-3}^{\Delta}))$, and $\alpha(f) = \ell^{-h''}\varphi^{-1}(\mathrm{Cv}(\xi(f)))$ defines an $\mathcal{O}_L$-linear map $\alpha\colon P_2' \to \mathrm{Cv}(\mathbf{N}_{c-3}^{\Delta}) \subseteq \epsilon_c D$ ((Cv1)), with

$$\varphi^{-2}(\mathrm{Cv}(\xi(f))) = \ell^{h''}\varphi^{-1}(\alpha(f)). \qquad (2)$$

Step 4: the Hodge condition at τ_{c-1}. We have $\varphi^{-2}(\alpha(f)) \in \epsilon_{c-2}D = D_{\mathrm{dR},\tau_{c-1}}(V_2)$. Let $\beta(f)$ be its class modulo $\epsilon_{c-2}\mathrm{Fil}^{s_{c-1}}D$ and $P_3' = P_2' \cap \ker\beta$. The target of β has L-dimension the number, with multiplicity, of Hodge jumps of V_2 at τ_{c-1} below s_{c-1}; these lie in $[-\eta,\eta]$ (Step 0, $s_{c-1} > \eta$), so by (Ow2) the dimension is at most m. The quotient P_2'/P_3' embeds in it, so it is torsion free of rank at most m, hence free and generated by at most m elements.

Step 5: realisation along Φ, linearly. The module P_3' is free; let (f_j') be a basis. For each j, $\alpha(f_j') = \mathrm{Cv}(x_j')$ for a Δ-invariant $x_j' \in \mathbf{N}_{c-3}$ (an $\mathcal{O}_L$-combination of Δ-invariant vectors), with $\varphi^{-2}(\mathrm{Cv}(x_j')) \in \mathrm{Fil}^{s_{c-1}}D$ (Step 4). By Proposition C.71 at c there is a Δ-invariant $y_j = \varphi(\tilde{y}_j)$, $\tilde{y}_j \in \mathbf{N}(T_1)_{c-1}$, with $Q^M y_j \in \mathbf{N}(T_1)_c$, $(\gamma_u - 1)y_j \in X\mathbf{N}(T_1)$ for all u, $\mathbf{N}(\Phi)(y_j)$ algebraic at level 0, and, in vector form (the level frame being an L-basis of $\epsilon_{c-1}D$), $\delta_i(\mathbf{N}(\Phi)y_j) = 0$ for $i < 0$ and for $0 < i < M$, and $\delta_0(\mathbf{N}(\Phi)y_j) = \ell^{\kappa}\varphi^{-1}(\mathrm{Cv}(x_j')) = \ell^{\kappa}\varphi^{-1}(\alpha(f_j'))$. Let $y\colon P_3' \to e_c\mathbf{N}(T_1)[1/\lambda_\infty]$ be the $\mathcal{O}_L$-linear map with $y(f_j') = y_j$. All these properties are $\mathcal{O}_L$-linear, so for every $f \in P_3'$: $y(f)$ is Δ-invariant, $Q^M y(f) \in \mathbf{N}(T_1)$ (so $\lambda' = Q_0^M$ works), $(\gamma_u - 1)y(f) \in X\mathbf{N}(T_1)$, so $y(f) \in I_c^1$; $\mathbf{N}(\Phi)(y(f))$ lies in the arc domain with $H = 0$, hence in W ($Q^\eta P_1 \subseteq P_1$); and, writing $\delta_i'(f) = \delta_i(\mathbf{N}(\Phi)(y(f)))$,

$$\delta_i'(f) = 0 \ \ (i < 0 \text{ or } 0 < i < M), \qquad \delta_0'(f) = \ell^{\kappa}\varphi^{-1}(\alpha(f)). \qquad (3)$$

Step 6: the constant term. For $f \in P_3'$, multiply (1) by ℓ^{κ} and use (2), (3): $\ell^{\kappa+e_{\mathrm{H}}}\delta_0(w(f)) - \ell^{h''}\delta_0'(f) \in \mathrm{Fil}^{-\eta}D$; dividing by $\ell^{h''}$ ($\mathrm{Fil}^{-\eta}D$ is an L-subspace),

$$\ell^{e_2}\delta_0(w(f)) - \delta_0'(f) \in \epsilon_{c-1}\mathrm{Fil}^{-\eta}D.$$

Let $\beta'(f)$ be its class in $\epsilon_{c-1}\mathrm{Fil}^{-\eta}D/\epsilon_{c-1}\mathrm{Fil}^{0}D$, and $P_4' = P_3' \cap \ker\beta'$. The target has dimension the number of Hodge jumps of V_2 at τ_c in $[-\eta,-1]$, at most m by (Ow2); as in Step 4, P_3'/P_4' is generated by at most m elements.

Step 7: membership. Let $f \in P_4'$ and $w'' = \ell^{e_2}w(f) - \mathbf{N}(\Phi)(y(f)) \in W$. Then $\Lambda_0(w'') = \sum_i t^i\big(\ell^{e_2}\delta_i(w(f)) - \delta_i'(f)\big)$, and its t^i-coefficient lies in $K_{(0)} \otimes_L \mathrm{Fil}^{-i}D$ for every i: for $i < 0$ and $0 < i < M$ by Step 0 and (3); for $i = 0$ because $f \in \ker\beta'$; for $i \ge M$ because every Hodge jump of V_2 at τ_c is $\ge -M \ge -i$, so the τ_c-component of $\mathrm{Fil}^{-i}D$ is all of $\epsilon_{c-1}D$. So $\mathrm{lv}_0(w'')$ is de Rham

integral, and by Lemma 23.5 (2) (length 1), $w'' \in \mathbf{N}(T_2)_c$. It is fixed by every γ_δ, $\delta \in \mu_{\ell-1}$: $w(f)$ and $y(f)$ are, and $\mathbf{N}(\Phi)$ commutes with the γ_u (Ex3).

Step 8: the chain and the exponent. The chain $P' = P'_0 \supseteq P'_1 \supseteq P'_2 \supseteq P'_3 \supseteq P'_4$ with $k = 4$ and $J = \{1, 2, 3\}$ ($|J| = 3$, $k - |J| = 1$) has the required properties, with y restricted to P'_4. For the exponent: by Proposition C.60, $e_{\mathrm{H}} \le (s_c+\eta)/(\ell-1)^2 + s_{c-1} + S - s_c/\ell + 3 + \log_\ell(s_{c-1}+S)$ and $e_{\mathrm{H}} \ge \sigma_{\mathrm{H}} + s_{c-1} + S + 1 - \lceil s_c/\ell \rceil$; $k = (\ell-1)\lceil M/(\ell-1)\rceil \le M + \ell - 2$, so $\lceil s_c/\ell\rceil - 1 \le \lceil M/(\ell-1)\rceil - 1 = \lfloor (k-1)/(\ell-1)\rfloor \le s_\ell(k-1) \le (k-1)\ell/(\ell-1)^2 \le M\ell/(\ell-1)^2 + 1$. Hence $e_2 \ge \sigma_{\mathrm{H}} + S + E + \sigma_{\mathrm{G}} \ge 0$, and, with $E \le (3a_{\mathrm{R}} + S + 2\eta)/(\ell-1) + \log_\ell(2H') + 1$, $\sigma_{\mathrm{G}} \le 2H'/(\ell-1)^2 + 2$, $s_c \le H'$, $s_{c-1} + S \le 2H'$, $\ell \ge 11$:

$$e_2 \le \frac{13.8\, H'}{\ell^2} + 1.21\, S + \frac{2.31\, \eta}{\ell} + 2.3 \log_\ell(2H') + 8.2 \le 14B.$$

So the statement holds with $c_{4,1} = 14$. □

Proposition C.60. *Let ℓ, F, n, L, $(\tau_a)_{a\in\mathbb{Z}/n}$, $\Phi\colon V_1 \to V_2$ (with lattices T_1, T_2) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.52, let $c \in \mathbb{Z}/n$ with $\epsilon_c = -1$, $D = D_{\mathrm{cris}}(V_2)$, and put*

$$y = s_{c-1} + S, \qquad \sigma_{\mathrm{H}} = \Big\lceil \frac{s_c + \eta}{(\ell-1)^2} \Big\rceil + 1, \qquad e_{\mathrm{H}} = \sigma_{\mathrm{H}} + y + 1 - \lceil s_c/\ell \rceil + \lfloor \log_\ell y \rfloor.$$

Let $w \in I_c^2$ be Δ-invariant with $Q^\eta w \in P_1(\mathbf{N}(T_2)_{c-1})$. Fix arc level data of (V_2, T_2) with cut c, length 1, $H(c) = \eta$ and a frame at $c-1$; then $w \in W$, and writing $\mathrm{lv}_0(w) = \sum_i t^i v_i$ ($v_i \in K_{(0)}^r$) put $\delta_i = \sum_k v_{i,k} D_k^{(0)} \in K_{(0)} \otimes_L \epsilon_{c-1} D$. Then:

(Ct1) *$\delta_i = 0$ for $i < -\eta$; $\delta_i \in K_{(0)} \otimes_L \mathrm{Fil}^{-i} D$ for $i \ne 0$; and $\delta_0 \in \epsilon_{c-1} D$;*

(Ct2) *there is a Δ-invariant $x \in \mathbf{N}(T_2)_{c-2}$ with*

$$\ell^{e_{\mathrm{H}}} \delta_0 - \varphi^{-2}(\mathrm{Cv}_{T_2}(x)) \in \mathrm{Fil}^{-\eta} D.$$

Moreover $0 \le e_{\mathrm{H}} \le \dfrac{s_c + \eta}{(\ell-1)^2} + s_{c-1} + S - \dfrac{s_c}{\ell} + 3 + \log_\ell(s_{c-1} + S)$.

Proof. Put $M = s_c + S$, $q = \pi_0 + \ell$, $\mathbf{N} = \mathbf{N}(T_2)$.

Step 0: signs, windows, sizes. As n is even and the signs alternate by parity, $\epsilon_{c-1} = 1$. By Lemma 24.2 (1), (Ow1) and (Ex1), the Hodge jumps of V_2 at τ_c lie in $[-M, -s_c] \cup [-\eta, \eta]$, and $\mathbf{N}$ has window (η, y) at the up site $c-1$, so (Wm2) gives $Q^y \mathbf{N}_{c-1} \subseteq \mathbf{A}_{F,L}\varphi(\mathbf{N}_{c-2})$. By the hypotheses $s_{c-1} \ge (H' - \eta)/2$ and $s_c \le H' - S - \eta$, so $s_c \le 2s_{c-1} \le \ell y$; also $y \ge s_{c-1} \ge 1$ and $s_c \ge 1$.

Step 1: the expansion, (Ct1). Here $\Pi = Q^\eta$ and $Q^\eta w \in P_1(\mathbf{N}_{c-1})$, so $w \in W$; by Lemma 23.17, w is algebraic at level 0 and $\mathrm{lv}_0(w) \in t^{-\eta} K_{(0)}[[t]]^r$, whence $\delta_i = 0$ for $i < -\eta$. For $u \in \mathbb{Z}_\ell^\times$, $(\gamma_u - 1)w \in X\mathbf{N} \cap e_c\mathbf{N}[1/\lambda_\infty] \subseteq \mathbf{N}_c$ (definition of I_c) and $(\gamma_u - 1)w \in W$ ((Lp2)); by Lemma 23.5 (2), $\mathrm{lv}_0((\gamma_u - 1)w)$ is de Rham integral, and by (Lp2) its t^i-coefficient is $u^i \sigma_u(v_i) - v_i$. Take $u = u_0$ with $v_\ell(u_0 - 1) = 1$: $\sigma_{u_0} = \mathrm{id}$ on $K_{(0)}$ by (Lp8), so $(u_0^i - 1)v_i \in K_{(0)} \otimes_L \mathrm{Fil}_0(-i)$, and $u_0^i \ne 1$ for $i \ne 0$ ($1 + \ell\mathbb{Z}_\ell$ is torsion free); hence $\delta_i \in K_{(0)} \otimes_L \mathrm{Fil}^{-i} D$ for $i \ne 0$. For $\delta \in \mu_{\ell-1}$, $\gamma_\delta w = w$ gives $\sigma_\delta(v_0) = v_0$ by (Lp2); $\sigma_u = \mathrm{id}$ for $u \in 1 + \ell\mathbb{Z}_\ell$ (Lp8), and $\sigma_{\delta u} = \sigma_\delta \sigma_u$ (both are induced by products of elements of G_F with these characters); so v_0 is fixed by every σ_u, $v_0 \in L^r$ by (Lp8), and $\delta_0 \in \epsilon_{c-1} D$.

Step 2: a Frobenius square. Put $P_c = \varphi(q)^y q^\eta$, Δ-invariant ((Dv2')), and $P = P_c w$. By (Dv2), $q^\eta w = \omega^\eta Q^\eta w \in P_1(\mathbf{N}_{c-1})$, the $\mathbf{A}_{F,L}$-span of $\varphi(\mathbf{N}_{c-1})$; for $n' \in \mathbf{N}_{c-1}$, $\varphi(q)^y \varphi(n') =$

$\varphi(q^y n')$ with $q^y n' \in Q^y \mathbf{N}_{c-1} \subseteq \mathbf{A}_{F,L}\varphi(\mathbf{N}_{c-2})$ (Step 0), so $\varphi(q^y n')$ lies in the $\mathbf{A}_{F,L}$-span of $\varphi^2(\mathbf{N}_{c-2})$. Let $(e'_m)_{m\le r}$ be a Δ-invariant basis of $\mathbf{N}_{c-2}$ (Lemma 24.15). As φ is semilinear, $P = \sum_m a_m \varphi^2(e'_m)$ with $a_m \in e_c\mathbf{A}_{F,L}$; since P and the $\varphi^2(e'_m)$ are Δ-invariant and $\gamma_\delta(a\varphi^2(e'_m)) = \gamma_\delta(a)\varphi^2(e'_m)$, we may replace a_m by $(\ell-1)^{-1}\sum_{\delta\in\mu_{\ell-1}}\gamma_\delta(a_m)$; so the a_m are Δ-invariant.

Step 3: dual truncation. Apply Lemma C.62 with $i = c$, the basis (e'_m), $h = s_c + \eta$ (so $\sigma = \sigma_{\mathrm{H}}$): Δ-invariant $Z_{mk} \in e_c\mathbf{A}_{F,L}$ with $\ell^{\sigma_{\mathrm{H}}}\Lambda_0(\varphi^2(e'_m)) - \sum_k \lambda_0(Z_{mk})D_k \in t^h\Lambda_D$, $D_k = \varphi^{-2}(\mathrm{Cv}(e'_k))$. The D_k are an L-basis of $\epsilon_{c-1}D$ ((Cv3): the $\mathrm{Cv}(e'_k)$ are a basis of $\epsilon_{c+1}D$, and φ^{-2} is bijective between components). Since $\Lambda_0(aW') = \lambda_0(a)\Lambda_0(W')$ and $\lambda_0(e_c\mathbf{A}_{F,L}) \subseteq K_{(0)}[[t]] \subseteq B^+_{\mathrm{dR}} \otimes L$ ((Lp1)),

$$\ell^{\sigma_{\mathrm{H}}}\Lambda_0(P) - \sum_k \lambda_0(b_k)D_k \in t^h\Lambda_D, \qquad b_k = \sum_m a_m Z_{mk} \ \ (\Delta\text{-invariant}).$$

Step 4: coefficients modulo $\mathrm{Fil}^{-\eta}$. By (Lp1) and (Dv2), $\lambda_0(q) = tu$ with $u \in K_{(0)}[[t]]^\times$, and by (Dv2') $\lambda_0(\varphi(q)) = \lambda_0(Q_1)\lambda_0(\omega_1)$ is a unit of $K_{(0)}[[t]]$; so $\lambda_0(P_c) = \sum_j p_j t^j$ with $p_j = 0$ for $j < \eta$ and $p_\eta = \lambda_0(\varphi(q))(0)^y u(0)^\eta$ a unit of $K_{(0)}$. Since $\Lambda_0(w) = \sum_i t^i\delta_i$ and $\Lambda_0(P) = \lambda_0(P_c)\Lambda_0(w)$, the vector of Step 3 equals $\sum_k g_k D_k$ with $g_k \in K_{(0)}((t))$, and it lies in $t^h\Lambda_D = \bigoplus_k t^h R^+ D_k$ ($R^+ = \epsilon_{c-1}(B^+_{\mathrm{dR}} \otimes L)$). Hence all t^R-coefficients with $R < h = s_c + \eta$ vanish:

$$\ell^{\sigma_{\mathrm{H}}} \sum_{-\eta\le i\le R-\eta} p_{R-i}\delta_i = \sum_k \beta_{k,R}D_k, \qquad \beta_{k,R} = [t^R]\lambda_0(b_k) \qquad (R < s_c + \eta).$$

Every $i \ne 0$ occurring has $-\eta \le i < s_c$. If $i \le \eta$, $\mathrm{Fil}^{-i}D \subseteq \mathrm{Fil}^{-\eta}D$. If $\eta < i < s_c$, there are no Hodge jumps of V_2 at τ_c in $[-i, -\eta-1]$ (Step 0), so the τ_c-components (ϵ_{c-1}-components) of $\mathrm{Fil}^{-i}D$ and $\mathrm{Fil}^{-\eta}D$ agree. By (Ct1), modulo $K_{(0)} \otimes_L \epsilon_{c-1}\mathrm{Fil}^{-\eta}D$:

$$\ell^{\sigma_{\mathrm{H}}} p_R\, \delta_0 \equiv \sum_k \beta_{k,R}D_k \qquad (R < s_c + \eta). \qquad (*)$$

Step 5: coordinates. Let $\bar{D} = \epsilon_{c-1}D/\epsilon_{c-1}\mathrm{Fil}^{-\eta}D$, $\bar{D}_k$, $\bar{\delta}_0$ the images, and M' the $\mathcal{O}_L$-span of the $\bar{D}_k$; it is finitely generated and torsion free over the principal ideal domain $\mathcal{O}_L$, so free with a basis $g_1, \dots, g_p$, which is L-linearly independent; write $\bar{D}_k = \sum_j m_{kj}g_j$, $m_{kj} \in \mathcal{O}_L$. Extend (g_j) to an L-basis of $\bar{D}$; its image is a $K_{(0)}$-basis of $K_{(0)} \otimes_L \bar{D}$. By $(*)$ at $R = \eta$, $\bar{\delta}_0 = \ell^{-\sigma_{\mathrm{H}}}p_\eta^{-1}\sum_k \beta_{k,\eta}\bar{D}_k$ lies in the $K_{(0)}$-span of the g_j; as $\bar{\delta}_0 \in \bar{D}$ has coordinates in L, $\bar{\delta}_0 = \sum_j v_j g_j$ with $v_j \in L$. Inserting this in $(*)$ and comparing g_j-coordinates,

$$\ell^{\sigma_{\mathrm{H}}} v_j\, p_R = \sum_k m_{kj}\beta_{k,R} = [t^R]\lambda_0(\alpha_j), \qquad \alpha_j = \sum_k m_{kj}b_k \qquad (R < s_c + \eta),$$

that is, $\lambda_0(\alpha_j) - \theta_j\lambda_0(\varphi(q)^y q^\eta) \in t^{s_c+\eta}K_{(0)}[[t]]$ with $\theta_j = \ell^{\sigma_{\mathrm{H}}}v_j \in L$ and $\alpha_j \in e_c\mathbf{A}_{F,L}$ Δ-invariant.

Step 6: integrality, (Ct2). By Lemma C.61 with y, $b = s_c \le \ell y$ (Step 0) and η: $\ell^{h_{\mathrm{s}}}\theta_j \in \mathcal{O}_L$, $h_{\mathrm{s}} = y + 1 - \lceil s_c/\ell\rceil + \lfloor \log_\ell y\rfloor$. So $o_j = \ell^{e_{\mathrm{H}}}v_j = \ell^{h_{\mathrm{s}}}\theta_j \in \mathcal{O}_L$ and $\ell^{e_{\mathrm{H}}}\bar{\delta}_0 = \sum_j o_j g_j \in M'$, say $= \sum_k o'_k\bar{D}_k$ with $o'_k \in \mathcal{O}_L$. Put $x = \sum_k o'_k e'_k$, a Δ-invariant element of $\mathbf{N}_{c-2}$. By (Cv2) for constants and the L-linearity of φ^{-2} on D, $\varphi^{-2}(\mathrm{Cv}(x)) = \sum_k o'_k D_k$, so $\ell^{e_{\mathrm{H}}}\delta_0 - \varphi^{-2}(\mathrm{Cv}(x)) \in \epsilon_{c-1}\mathrm{Fil}^{-\eta}D \subseteq \mathrm{Fil}^{-\eta}D$.

Step 7: the exponent. We have $\sigma_{\mathrm{H}} \le (s_c+\eta)/(\ell-1)^2 + 2$ and $-\lceil s_c/\ell\rceil \le -s_c/\ell$, so $e_{\mathrm{H}} \le (s_c+\eta)/(\ell-1)^2 + s_{c-1} + S - s_c/\ell + 3 + \log_\ell(s_{c-1}+S)$; and $e_{\mathrm{H}} \ge \sigma_{\mathrm{H}} + y + 1 - \lceil s_c/\ell\rceil \ge 0$ since $s_c/\ell \le y$ (Step 0) and y is an integer, so $\lceil s_c/\ell\rceil \le y$. □

Lemma C.61. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ be as in Proposition 23.7, let $c \in \mathbb{Z}/n$, and let λ_0 and $K_{(0)}$ be the level-0 map and the level coefficient ring of arc level data with cut c (they depend*

only on c); L is a subring of $K_{(0)}$ through ϵ_0. Put $q = \pi_0 + \ell$. Let $y, b \geq 1$ and $\eta \geq 0$ be integers, and

$$K = \min(y, \lceil b/\ell \rceil - 1), \qquad h = y - K + \lfloor \log_\ell y \rfloor.$$

If $\alpha \in e_c \mathbf{A}_{F,L}$ is Δ-invariant, $\theta \in L$, and

$$\lambda_0(\alpha) - \theta\, \lambda_0(\varphi(q)^y q^\eta) \in t^{\,b+\eta} K_{(0)}[[t]],$$

then $\ell^h \theta \in \mathcal{O}_L$. If moreover $b \leq \ell y$, then $h = y + 1 - \lceil b/\ell \rceil + \lfloor \log_\ell y \rfloor \leq y - b/\ell + 1 + \log_\ell y$.

Proof. Let $R = e_c \mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ and R^Δ its Δ-invariant elements.

Step 1: expansions in q. By (Dv1), $\alpha = A(q)$ for a unique $A = \sum_k a_k T^k \in \mathcal{O}_L[[T]]$; for every $N \geq 0$, writing $A = \sum_{k<N} a_k T^k + T^N A_N$ with $A_N \in \mathcal{O}_L[[T]]$ and applying the homomorphism $T \mapsto q$, we get $\alpha = \sum_{k<N} a_k q^k + q^N \alpha_N$ with $\alpha_N = A_N(q) \in R^\Delta$. The element $\varphi(q)$ is Δ-invariant (as φ commutes with the γ_δ) and lies in $\mathbb{Z}_\ell[[X]]$, so by (Dv1) for $\mathbb{Z}_\ell[[X]]$ there is $g = \sum_j g_j T^j \in \mathbb{Z}_\ell[[T]]$ with $\varphi(q) = g(q)$.

Step 2: the shape of g. (a) $g_0 = \ell$. Fix a primitive ℓ-th root of unity ζ in an algebraic closure of $\mathbb{Q}_\ell$ and let $\mathrm{ev}\colon \mathbb{Z}_\ell[[X]] \to \mathbb{Z}_\ell[\zeta]$, $f \mapsto f(\zeta - 1)$; it is a ring homomorphism, continuous for the (ℓ, X)-adic topology since $\zeta - 1$ has positive valuation. By (Dv2), $q = Q\omega$, and $\mathrm{ev}(Q) = \Phi_\ell(\zeta) = 0$ ($Q = \Phi_\ell(1 + X)$), so $\mathrm{ev}(q) = 0$ and $\mathrm{ev}(\varphi(q)) = \mathrm{ev}(g(q)) = g_0$ (the series $\sum_j g_j q^j$ converges (ℓ, X)-adically, $q(0) = \ell$). On the other hand $\varphi(q)(X) = q((1+X)^\ell - 1)$, so $\mathrm{ev}(\varphi(q)) = q(\zeta^\ell - 1) = q(0) = \pi_0(0) + \ell = \ell$, because $\pi_0(0) = \sum_{\delta \in \mu_{\ell-1}} 1 - (\ell - 1) = 0$. (b) $g \equiv T^\ell \pmod{\ell \mathbb{Z}_\ell[[T]]}$. For $f \in \mathbb{Z}_\ell[[X]]$, $\varphi(f) = f((1 + X)^\ell - 1) \equiv f(X^\ell) \equiv f^\ell$ modulo $\ell \mathbb{Z}_\ell[[X]]$ (the coefficients of f modulo ℓ lie in $\mathbb{F}_\ell$, fixed by the ℓ-th power map). Hence $(g - T^\ell)(q) = \varphi(q) - q^\ell \in \ell \mathbb{Z}_\ell[[X]]$; it is Δ-invariant, so $\ell^{-1}(g - T^\ell)(q)$ is a Δ-invariant element of $\mathbb{Z}_\ell[[X]]$, equal to $g'(q)$ with $g' \in \mathbb{Z}_\ell[[T]]$ by (Dv1); injectivity of $T \mapsto q$ gives $g - T^\ell = \ell g'$. By (a) and (b), $g = \ell G + T^\ell$ with $G \in \mathbb{Z}_\ell[[T]]$ and $G(0) = g_0/\ell = 1$. Therefore

$$g^y = \sum_{k=0}^{y} \binom{y}{k} \ell^{\,y-k} G^{\,y-k}\, T^{\ell k}. \qquad (1)$$

Step 3: comparison of coefficients. By (Lp1), $\lambda_0(Q) \in t\, K_{(0)}[[t]]^\times$ and $\lambda_0(R) \subseteq K_{(0)}[[t]]$; ω is a unit of R, so $\lambda_0(\omega)$ is a unit and $\lambda_0(q) = tu$ with $u \in K_{(0)}[[t]]^\times$. Put $N = b + \eta$ and write $g^y T^\eta = \sum_{k<N} c_k T^k + T^N r$ with $c_k \in \mathbb{Z}_\ell$, $r \in \mathbb{Z}_\ell[[T]]$, so that $\varphi(q)^y q^\eta = \sum_{k<N} c_k q^k + q^N \rho$ with $\rho = r(q) \in R^\Delta$. By Step 1,

$$\lambda_0(\alpha) - \theta \lambda_0(\varphi(q)^y q^\eta) = \sum_{k<N} (a_k - \theta c_k) u^k t^k + t^N u^N (\lambda_0(\alpha_N) - \theta \lambda_0(\rho)),$$

and the last term lies in $t^N K_{(0)}[[t]]$. So $\sum_{k<N} (a_k - \theta c_k) u^k t^k \in t^N K_{(0)}[[t]]$. By induction on $k < N$: the coefficient of t^k is $(a_k - \theta c_k) u(0)^k$ plus terms from indices $k' < k$, which vanish; $u(0)$ is a unit of $K_{(0)}$ and $K_{(0)}$ is an L-vector space, so $a_k = \theta c_k$. In particular, for $0 \leq j < b$,

$$\theta \cdot [T^j]\, g^y = \theta\, c_{\eta+j} = a_{\eta+j} \in \mathcal{O}_L. \qquad (2)$$

Step 4: a coefficient of small valuation. We have $K \geq 0$ and $\ell K < b$ (if $K = \lceil b/\ell \rceil - 1$ this is $\ell(\lceil b/\ell \rceil - 1) < b$; if $K = y < \lceil b/\ell \rceil - 1$ then $\ell y < b$). Let $\mu = \min_{0 \leq k \leq K} v_\ell\big(\binom{y}{k} \ell^{y-k}\big)$ and k_1 the least k attaining it. In (1) the terms with $k > K$ are 0 (if $K = y$) or lie in $T^{\ell \lceil b/\ell \rceil} \mathbb{Z}_\ell[[T]] \subseteq T^b \mathbb{Z}_\ell[[T]]$; so modulo T^b, $\ell^{-\mu} g^y \equiv \sum_{k \leq K} \ell^{-\mu} \binom{y}{k} \ell^{y-k} G^{y-k} T^{\ell k}$, a series with coefficients in $\mathbb{Z}_\ell$. Reduce it modulo ℓ: the terms $k < k_1$ vanish (their scalar has valuation $> \mu$),

the terms $k > k_1$ only involve T^j with $j \geq \ell k > \ell k_1$, and the term k_1 contributes to $T^{\ell k_1}$ the residue of $\ell^{-\mu}\binom{y}{k_1}\ell^{y-k_1}G(0)^{y-k_1}$, a unit ($G(0)=1$). Since $\ell k_1 \leq \ell K < b$, the coefficient $[T^{\ell k_1}]g^y$ has valuation exactly μ. By (2) with $j = \ell k_1$, $v_\ell(\theta) \geq -\mu$, so $\ell^\mu\theta \in \mathcal{O}_L$.

Step 5: the exponent. We have $\mu \leq v_\ell\big(\binom{y}{K}\big) + y - K$, and by Kummer's theorem $v_\ell\big(\binom{y}{K}\big)$ is the number of carries when K and $y-K$ are added in base ℓ; a carry out of the leading digit of y is impossible, so there are at most $\lfloor \log_\ell y \rfloor$ carries, that is, the number of base-ℓ digits of y minus 1. Hence $\mu \leq h$ and $\ell^h\theta = \ell^{h-\mu}\ell^\mu\theta \in \mathcal{O}_L$. If $b \leq \ell y$, then $\lceil b/\ell \rceil - 1 < y$, so $K = \lceil b/\ell \rceil - 1$ and $h = y + 1 - \lceil b/\ell \rceil + \lfloor \log_\ell y \rfloor \leq y + 1 - b/\ell + \log_\ell y$. □

Lemma C.62. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a pair (V,T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$, $D = D_{\mathrm{cris}}(V)$, $i \in \mathbb{Z}/n$, and $(e'_m)_{m\leq r}$ a basis of $\mathbf{N}_{i-2}$ over $e_{i-2}\mathbf{A}_{F,L}$ consisting of Δ-invariant vectors. Let $h \geq 1$ and $\sigma = \lceil h/(\ell-1)^2 \rceil + 1$. Let $\Lambda_0 = \varphi^{-1}\circ\iota_T$ on $\mathbf{N}[1/\lambda_2]$, $\lambda_0 = \varphi^{-1}\circ(\iota\otimes\mathrm{id})$ on $\mathbf{A}_{F,L}[1/\lambda_2]$, and $\Lambda_D = \epsilon_{i-1}(B^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_\ell} L)\cdot\epsilon_{i-1}D$. Then there are Δ-invariant $Z_{mk} \in e_i\mathbf{A}_{F,L}$ $(m,k\leq r)$ such that for every m*

$$\ell^\sigma\,\Lambda_0(\varphi^2(e'_m)) - \sum_{k\leq r}\lambda_0(Z_{mk})\,\varphi^{-2}(\mathrm{Cv}(e'_k)) \in t^h\Lambda_D.$$

Proof. Put $\varsigma = \pi_0 + \ell$, $D_k = \varphi^{-2}(\mathrm{Cv}(e'_k))$, and identify $e_{i-2}E[[X]] \cong L[[X]]$, $e_i\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$.

Step 1: the basis in the horizontal frame. Let H'_k be the horizontal section through e'_k ((Hf2)); the H'_k form an $E[[X]]$-basis of $e_{i-2}\mathbf{N}_E$. Write $H'_k = \sum_m h_{km}e'_m$ with $h_{km} \in L[[X]]$; then $h \equiv I$ modulo X ($H'_k \in e'_k + X\mathbf{N}_E$), and $\ell^{s_\ell(j)}[X^j]h_{km} \in \mathcal{O}_L$ for all j by (Hf3). The h_{km} are Δ-invariant: $\gamma_\delta(H'_k)$ is fixed by every γ_u (the γ_u commute) and lies in $\gamma_\delta(e'_k)+X\mathbf{N}_E = e'_k+X\mathbf{N}_E$, so $\gamma_\delta(H'_k) = H'_k$ by the uniqueness in (Hf2); compare coordinates in the Δ-invariant basis (e'_m). The function s_ℓ is superadditive: with $J_a = \lfloor a/(\ell-1)\rfloor$, $J_a + J_b \leq J_{a+b}$, $v_\ell(J_a!) + v_\ell(J_b!) \leq v_\ell((J_a+J_b)!)$ (a binomial coefficient is an integer), and $J \mapsto J + v_\ell(J!)$ is non-decreasing; so $s_\ell(a) + s_\ell(b) \leq s_\ell(a+b)$. Hence $\mathcal{S} = \{f \in L[[X]] :\ \ell^{s_\ell(j)}[X^j]f \in \mathcal{O}_L \text{ for all } j\}$ is a subring of $L[[X]]$ (for $[X^j](fg) = \sum_{a+b=j} f_a g_b$, $\ell^{s_\ell(j)}f_a g_b = \ell^{s_\ell(j)-s_\ell(a)-s_\ell(b)}(\ell^{s_\ell(a)}f_a)(\ell^{s_\ell(b)}g_b)$), closed under X-adically convergent sums. As $I - h \in X\,M_r(\mathcal{S})$, the inverse $m = h^{-1} = \sum_{p\geq 0}(I-h)^p$ converges X-adically and lies in $M_r(\mathcal{S})$; its entries are Δ-invariant (those of h are, and γ_δ is a ring automorphism). Thus $e'_m = \sum_k m_{mk}H'_k$.

Step 2: π_0-expansions. Let $f \in \mathcal{S}$ be Δ-invariant and $K \geq 1$. If $f' \in XL[[X]]$ is Δ-invariant and $f' \neq 0$, its lowest term aX^j becomes $a\delta^j X^j$ under γ_δ ($\gamma_\delta(X) = \delta X + O(X^2)$), so $(\ell-1) \mid j$ and $j \geq \ell - 1$; as $\pi_0 = X^{\ell-1}v$ with v a unit of $\mathbb{Z}_\ell[[X]]$ and $v(0) = 1/(\ell-2)!$ ((Rr1)), $f'/\pi_0 \in L[[X]]$, and it is Δ-invariant (π_0 is). Recursively ($f_0 = f$, $b_j = f_j(0)$, $f_{j+1} = (f_j - b_j)/\pi_0$),

$$f = \sum_{j<K} b_j\pi_0^j + \pi_0^K f_K, \qquad b_j \in L,\ f_K \in L[[X]]\ \Delta\text{-invariant}.$$

Comparing coefficients of $X^{(\ell-1)j}$ in $f = \sum_{j'<j} b_{j'}\pi_0^{j'} + \pi_0^j f_j$ gives

$$b_j = v(0)^{-j}\Big([X^{(\ell-1)j}]f - \sum_{j'<j} b_{j'}[X^{(\ell-1)j}]\pi_0^{j'}\Big),$$

with $[X^{(\ell-1)j}]\pi_0^{j'} \in \mathbb{Z}_\ell$ and $v(0) \in \mathbb{Z}_\ell^\times$. By induction on j and monotonicity of s_ℓ, $\ell^{s_\ell((\ell-1)j)}b_{j'} \in \mathcal{O}_L$ for $j' \leq j$. Finally $s_\ell((\ell-1)j) = j + v_\ell(j!) \leq j + (j-1)/(\ell-1) \leq e(j)$ for $j \geq 1$ (Legendre), and both are 0 for $j = 0$; so $\ell^{e(j)}b_j \in \mathcal{O}_L$.

Step 3: the matrix Z. By Lemma C.63 choose $\alpha_j, \beta_j \in \mathcal{R}$ with $\ell^\sigma\varphi^2(\pi_0)^j = \ell^{e(j)}\alpha_j + \pi_0\varsigma^h\beta_j$ ($j \geq 0$). Let $K_0 = \lceil h/(\ell-1)\rceil \geq 1$, expand each m_{mk} by Step 2 with $K = K_0$: $m_{mk} =$

$\sum_{j<K_0} b_{mk,j}\pi_0^j + \pi_0^{K_0} f_{mk}$, and put

$$Z_{mk} = \sum_{j<K_0} (\ell^{e(j)} b_{mk,j})\alpha_j \in \mathcal{O}_L[[X]] \cong e_i \mathbf{A}_{F,L},$$

Δ-invariant since the α_j are.

Step 4: level 0. The map $\mathrm{ev}^{(1)}$ is a ring homomorphism $E[[X]] \to B^+_{\mathrm{dR}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$: it is $\varphi \circ \iota_\Lambda$ on coefficients followed by the substitution $X \mapsto \varphi(\mu)$, and $\theta(\varphi(\mu)) = 0$, so the substitution is continuous from the X-adic to the t-adic topology (ℓ is invertible in B^+_{dR}). On $\mathbf{A}_{F,L}$ it equals $\varphi \circ (\iota \otimes \mathrm{id}) = \lambda_0 \circ \varphi^2$ (continuous ring maps agreeing on Λ and on X, ι being φ-equivariant), and for $n \in \mathbf{N}_{i-2}$, (Ex2) gives $\Lambda_0(\varphi^2(n)) = \varphi(\iota_T(n)) = G^{(1)}(n)$ (expand n in a frame). By the definition of $G^{(1)}$ through coordinates, $G^{(1)}(fW) = \mathrm{ev}^{(1)}(f)G^{(1)}(W)$ for $f \in E[[X]]$, $W \in e_{i-2}\mathbf{N}_E$, and by Lemma 23.13 (1) with $q = 1$, $G^{(1)}(H'_k) = \varphi^{-2}(\mathrm{Ev}(H'_k)) = D_k$ ($\mathrm{Cv}(e'_k) = \mathrm{Ev}(H'_k)$ by Definition C.58). Hence, by Step 1,

$$\Lambda_0(\varphi^2(e'_m)) = G^{(1)}(e'_m) = \sum_k \mathrm{ev}^{(1)}(m_{mk})\, D_k.$$

By Steps 2 and 3, with $\mathrm{ev}^{(1)}(\pi_0^j) = \lambda_0(\varphi^2(\pi_0)^j)$,

$$\ell^\sigma \mathrm{ev}^{(1)}(m_{mk}) - \lambda_0(Z_{mk}) = \lambda_0(\varsigma)^h\, \lambda_0(\pi_0) \sum_{j<K_0} b_{mk,j}\lambda_0(\beta_j) + \ell^\sigma\, \mathrm{ev}^{(1)}(\pi_0)^{K_0}\, \mathrm{ev}^{(1)}(f_{mk}).$$

By (Lp1) for arc level data with cut i, $\lambda_0(Q) \in tK_{(0)}[[t]]$ and $\lambda_0(e_i\mathbf{A}_{F,L}) \subseteq K_{(0)}[[t]] \subseteq B^+_{\mathrm{dR}} \otimes L$; with $\varsigma = Qw$, w a unit (Lemma 28.15 (1)), the first term lies in $t^h(B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} L)$ ($b_{mk,j} \in L$). As $\pi_0 = X^{\ell-1}v$ and $\mathrm{ev}^{(1)}(X) = \varphi(\mu) \in \ker\theta = tB^+_{\mathrm{dR}}$, $\mathrm{ev}^{(1)}(\pi_0) \in t^{\ell-1}(B^+_{\mathrm{dR}} \otimes \mathcal{O}_L)$, so the second term lies in $t^{(\ell-1)K_0}(B^+_{\mathrm{dR}} \otimes \mathcal{O}_L) \subseteq t^h(B^+_{\mathrm{dR}} \otimes L)$. Since $\mathrm{Cv}(e'_k) \in \epsilon_{i+1}D$ (Lemma C.64 (Cv1)), $D_k \in \epsilon_{i-1}D$, and multiplying by D_k,

$$\ell^\sigma \Lambda_0(\varphi^2(e'_m)) - \sum_k \lambda_0(Z_{mk})D_k = \sum_k (\ell^\sigma \mathrm{ev}^{(1)}(m_{mk}) - \lambda_0(Z_{mk}))D_k \in t^h\Lambda_D.$$

□

Lemma C.63. *Let ℓ be an odd prime, $\mathcal{R} = \mathbb{Z}_\ell[[X]]^\Delta$ the ring of Δ-invariant elements of $\mathbb{Z}_\ell[[X]]$, $\varsigma = \pi_0 + \ell$, let $h \geq 1$ be an integer and*

$$\sigma = \Big\lceil \frac{h}{(\ell-1)^2} \Big\rceil + 1, \qquad e(y) = \Big\lceil \frac{y\,\ell}{\ell-1} \Big\rceil \quad (y \geq 0).$$

Then $\varphi^2(\pi_0) \in \mathcal{R}$ and, for every integer $y \geq 0$,

$$\ell^\sigma \varphi^2(\pi_0)^y \in \ell^{\,e(y)}\mathcal{R} + \pi_0\varsigma^h\mathcal{R}.$$

Proof. **Step 1: the ring.** The map φ commutes with every γ_δ on $\mathbb{Z}_\ell[[X]]$ (both are ring endomorphisms, and $\varphi\gamma_\delta(X) = (1+X)^{\ell\delta} - 1 = \gamma_\delta\varphi(X)$), so $\varphi(\mathcal{R}) \subseteq \mathcal{R}$; $\pi_0, \varsigma \in \mathcal{R}$ (Lemma 28.15 (1)). If $f \in \mathbb{Z}_\ell[[X]]$ and $gf \in \mathcal{R}$ for some non-zero $g \in \mathcal{R}$, then $f \in \mathcal{R}$ ($\mathbb{Z}_\ell[[X]]$ is a domain); in particular units of $\mathbb{Z}_\ell[[X]]$ lying in $\mathcal{R}$ have inverses in $\mathcal{R}$.

Step 2: $\varphi(\pi_0) = U\pi_0\varsigma^{\ell-1}$ and $\varphi(\varsigma) = \varsigma^\ell + \ell a_1$ with $U \in \mathcal{R}^\times$, $a_1 \in \mathcal{R}$. By Lemma 28.15 (1), $\pi_0 = X^{\ell-1}v$ and $\varsigma = Qw_0$ with units v, w_0; as $\varphi(X) = XQ$, $\varphi(\pi_0) = X^{\ell-1}Q^{\ell-1}\varphi(v) = \pi_0\varsigma^{\ell-1}U$ with $U = v^{-1}w_0^{1-\ell}\varphi(v)$ a unit, which lies in $\mathcal{R}$ by Step 1. Modulo ℓ, $\varphi(f)(X) =$

$f((1+X)^\ell - 1) \equiv f(X^\ell) \equiv f(X)^\ell$ for $f \in \mathbb{Z}_\ell[[X]]$ (coefficients in $\mathbb{F}_\ell$), so $\varphi(\pi_0) = \pi_0^\ell + \ell a_0$ with $a_0 \in \mathbb{Z}_\ell[[X]]$, and $a_0 \in \mathcal{R}$ by Step 1 (as $\ell a_0 \in \mathcal{R}$). From $\pi_0 = \varsigma - \ell$, $\pi_0^\ell = \varsigma^\ell - \ell d_0$ with $d_0 = -\sum_{k\ge 1} \binom{\ell}{k} \ell^{k-1}(-1)^k \varsigma^{\ell-k} \in \mathcal{R}$. Hence $\varphi(\varsigma) = \varphi(\pi_0) + \ell = \varsigma^\ell + \ell a_1$ with $a_1 = a_0 - d_0 + 1 \in \mathcal{R}$. Consequently

$$\varphi^2(\pi_0) = \varphi(U)\varphi(\pi_0)\varphi(\varsigma)^{\ell-1} = U'\pi_0 \varsigma^{\ell-1}\varphi(\varsigma)^{\ell-1}, \qquad U' = \varphi(U)U \in \mathcal{R}^\times,$$

and $\varphi^2(\pi_0) \in \mathcal{R}$.

Step 3: an inequality. Let $y \ge 1$ with $e(y) > \sigma$. Then

$$e(y) - \sigma + \Big\lfloor \frac{h-1}{\ell} \Big\rfloor \le (\ell - 1)y. \qquad (*)$$

Indeed $e(y) < y\ell/(\ell-1) + 1$ and $\sigma \ge h/(\ell-1)^2 + 1$; so $e(y) > \sigma$ gives $y\ell/(\ell-1) > h/(\ell-1)^2$, i.e. $y > h/(\ell(\ell-1))$, and the left side of $(*)$ is $< y\ell/(\ell-1) - h/(\ell-1)^2 + h/\ell$. Now

$$(\ell-1)y - \frac{y\ell}{\ell-1} = y\,\frac{\ell^2 - 3\ell + 1}{\ell - 1} > \frac{h(\ell^2 - 3\ell + 1)}{\ell(\ell-1)^2} = \frac{h}{\ell} - \frac{h}{(\ell-1)^2},$$

using $\ell^2 - 3\ell + 1 > 0$ and $(\ell-1)^2 - \ell = \ell^2 - 3\ell + 1$. So the left side of $(*)$, an integer, is $< (\ell-1)y$.

Step 4: the congruence. If $e(y) \le \sigma$ (this includes $y = 0$), then

$$\ell^\sigma \varphi^2(\pi_0)^y = \ell^{e(y)} \cdot \ell^{\sigma - e(y)} \varphi^2(\pi_0)^y \in \ell^{e(y)}\mathcal{R}.$$

Let $e(y) > \sigma$, so $y \ge 1$, and put $N = (\ell-1)y$. By Step 2,

$$\varphi^2(\pi_0)^y = U'^y \pi_0^y \varsigma^N \sum_{k=0}^{N} \binom{N}{k} \varsigma^{\ell k} \ell^{N-k} a_1^{N-k}.$$

A term with $N + \ell k \ge h$ lies in $\pi_0 \varsigma^h \mathcal{R}$ ($y \ge 1$). A term with $N + \ell k \le h - 1$ has $k \le (h-1)/\ell$, so $k \le \lfloor (h-1)/\ell \rfloor$ and, by $(*)$, $N - k \ge e(y) - \sigma$: after multiplication by ℓ^σ it lies in $\ell^{\sigma + N - k}\mathcal{R} \subseteq \ell^{e(y)}\mathcal{R}$. Summing, $\ell^\sigma \varphi^2(\pi_0)^y \in \ell^{e(y)}\mathcal{R} + \pi_0 \varsigma^h \mathcal{R}$. □

Lemma C.64. *Let ℓ, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ and a pair (V, T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$, $d = \dim_L V$, $D = D_{\mathrm{cris}}(V)$, and let $\varphi = \varphi \otimes 1$ on $B_{\mathrm{cris}} \otimes_{\mathbb{Z}_\ell} T$; an element $\lambda \in \mathcal{O}_L$ acts on $B_{\mathrm{cris}} \otimes_{\mathbb{Z}_\ell} T$ through T. Then:*

(Cv1) $\mathrm{Cv}(\mathbf{N}) \subseteq D$, Cv *is additive, and* $\mathrm{Cv}(cx) = \varphi^3(\iota_\Lambda(c))\,\mathrm{Cv}(x)$ *for $c \in \Lambda$, $x \in \mathbf{N}$; in particular* Cv *is $\mathcal{O}_L$-linear and* $\mathrm{Cv}(\mathbf{N}_a) \subseteq \epsilon_{a+3} D$.

(Cv2) $\mathrm{Cv}(fx) = \varphi^3(\iota_\Lambda(f(0)))\,\mathrm{Cv}(x)$ *for $f \in \mathbf{A}_{F,L}$ with constant term $f(0) \in \Lambda$ and $x \in \mathbf{N}$; in particular* $\mathrm{Cv}(X\mathbf{N}) = 0$.

(Cv3) *For $a \in \mathbb{Z}/n$ and an $e_a\mathbf{A}_{F,L}$-basis $(n_{a,i})_{i \le d}$ of $\mathbf{N}_a$, the vectors $\mathrm{Cv}(n_{a,i})$ form an L-basis of $\epsilon_{a+3}D$, and $\mathrm{Cv}(\mathbf{N}_a) = \bigoplus_{i \le d} \mathcal{O}_L\,\mathrm{Cv}(n_{a,i})$; so $\mathrm{Cv}(\mathbf{N}_a)$ is an $\mathcal{O}_L$-lattice in $\epsilon_{a+3}D$.*

(Cv4) *If $x \in \mathbf{N}$, $\kappa \in \mathbf{A}_{F,L}$ and $\kappa\,\varphi_{\mathbf{N}}(x) \in \mathbf{N}$, then $\mathrm{Cv}(\kappa\,\varphi_{\mathbf{N}}(x)) = \varphi^3(\iota_\Lambda(\kappa(0)))\,\varphi(\mathrm{Cv}(x))$; in particular $\mathrm{Cv}(\varphi_{\mathbf{N}}(x)) = \varphi(\mathrm{Cv}(x))$ if $\varphi_{\mathbf{N}}(x) \in \mathbf{N}$. More generally, if $x_1, \dots, x_q \in \mathbf{N}$ and $\kappa_1, \dots, \kappa_q \in \mathbf{A}_{F,L}$ with $\sum_j \kappa_j \varphi_{\mathbf{N}}(x_j) \in \mathbf{N}$, then*

$$\mathrm{Cv}\Big(\sum_j \kappa_j \varphi_{\mathbf{N}}(x_j)\Big) = \sum_j \varphi^3(\iota_\Lambda(\kappa_j(0)))\,\varphi(\mathrm{Cv}(x_j)).$$

(Cv5) *For a pair (V',T') and a G_F-equivariant $\mathcal{O}_L$-linear map $f\colon T\to T'$, $\mathrm{Cv}_{T'}(\mathbf{N}(f)(x)) = (1\otimes f)(\mathrm{Cv}_T(x))$ for $x\in\mathbf{N}$.*

Proof. For $x\in\mathbf{N}$ write H_x for the horizontal section through x ((Hf2)).

Preliminaries on H_x. (i) $H_{x+y}=H_x+H_y$ and $H_{cx}=cH_x$ for $c\in\Lambda$: the right-hand sides are fixed by every γ_u (which fixes Λ) and lie in $x+y+X\mathbf{N}_E$, resp. $cx+X\mathbf{N}_E$; apply the uniqueness in (Hf2). (ii) $H_{Xy}=0$ for $y\in\mathbf{N}$, since 0 is γ_u-invariant and lies in $Xy+X\mathbf{N}_E$. (iii) $H_x\in\mathbf{N}_R$: by (Hf3) its coordinates in a basis of $\mathbf{N}$ satisfy $\ell^{s_\ell(k)}a_k\in\Lambda$, and $s_\ell(k)\le k\ell/(\ell-1)^2\le 2k$. (iv) From the defining formula (Ev3) and the multiplicativity of e (Ev2), $\mathrm{Ev}(gW)=e(g)\mathrm{Ev}(W)$ for $g\in R$, $W\in\mathbf{N}_R$, and $e(c)=\varphi^3(\iota_\Lambda(c))$ for $c\in\Lambda\subseteq\mathbf{A}_{F,L}$ (Ev2). (v) $\varphi^3(\iota_\Lambda(e_a))=\iota_\Lambda(\varphi^3(e_a))=\epsilon_{a+3}$, as ι is φ-equivariant (Lemma 23.8 (4)) and $\varphi(e_b)=e_{b+1}$.

(Cv1), (Cv2). Additivity and the formula for $c\in\Lambda$ follow from (i) and (iv); for $\lambda\in\mathcal{O}_L$, $\varphi^3(\iota_\Lambda(1\otimes\lambda))=1\otimes\lambda$, since $\iota_\Lambda=\iota|_{\mathcal{O}_F}\otimes\mathrm{id}$ and φ acts on the first factor. For $\sigma\in G_F$, (Ev3) and the invariance of H_x give $\sigma(\mathrm{Cv}(x))=\mathrm{Ev}(\gamma_{\chi_{\mathrm{cyc}}(\sigma)}H_x)=\mathrm{Cv}(x)$, so $\mathrm{Cv}(x)\in(B_{\mathrm{cris}}\otimes_{\mathbb{Q}_\ell}V)^{G_F}=D$. For $x\in\mathbf{N}_a$, $x=e_ax$ and (v) give $\mathrm{Cv}(x)=\epsilon_{a+3}\mathrm{Cv}(x)$. For $f\in\mathbf{A}_{F,L}$ write $f=f(0)+Xg$ with $g\in\mathbf{A}_{F,L}$; by (i), (ii), $H_{fx}=f(0)H_x+H_{X(gx)}=f(0)H_x$, and (iv) gives (Cv2).

(Cv3). Let $x=\sum_i f_in_{a,i}$ with $f_i\in e_a\mathbf{A}_{F,L}$. By (Cv1) and (Cv2),

$$\mathrm{Cv}(x)=\sum_i\varphi^3(\iota_\Lambda(f_i(0)))\,\mathrm{Cv}(n_{a,i}).$$

Here $f_i(0)\in e_a\Lambda$, and $e_a\Lambda=e_a(1\otimes\mathcal{O}_L)$ (under $\Lambda\cong\prod_b\mathcal{O}_L$ of Definition 23.2, $e_a(y\otimes1)=e_a(1\otimes\tau_a(y))$); so $\varphi^3(\iota_\Lambda(f_i(0)))=\epsilon_{a+3}(1\otimes\lambda_i)$ with $\lambda_i\in\mathcal{O}_L$ by (v), every $\lambda_i\in\mathcal{O}_L$ occurring (take $f_i=e_a(1\otimes\lambda_i)$). Hence $\mathrm{Cv}(\mathbf{N}_a)=\sum_i\mathcal{O}_L\,\mathrm{Cv}(n_{a,i})$. By definition $\mathrm{Cv}(n_{a,i})=\mathrm{Ev}(H_{a,i})$ with $H_{a,i}$ the horizontal section through $n_{a,i}$, and by Lemma 23.10 (2) these form a basis of $\epsilon_{a+3}D$ over $\epsilon_{a+3}(F\otimes_{\mathbb{Q}_\ell}L)\cong L$, the isomorphism being $\epsilon_{a+3}(1\otimes\lambda)\leftrightarrow\lambda$. So the sum is direct and spans $\epsilon_{a+3}D$ over L.

(Cv4). Since $\kappa\varphi_{\mathbf{N}}(e_cx)=e_{c+1}\kappa\varphi_{\mathbf{N}}(x)\in\mathbf{N}$ and both sides are additive in x, we may assume $x\in\mathbf{N}_c$. Fix bases (n_k) of $\mathbf{N}_c$ and (n'_l) of $\mathbf{N}_{c+1}$, with horizontal sections H_k, H'_l and $d_k=\mathrm{Ev}(H_k)$, $d'_l=\mathrm{Ev}(H'_l)$; write $x=\sum_k f_kn_k$ and $\kappa\varphi_{\mathbf{N}}(x)=\sum_l g_ln'_l$ with $f_k\in e_c\mathbf{A}_{F,L}$, $g_l\in e_{c+1}\mathbf{A}_{F,L}$. By (i), (ii), $H_x=\sum_k f_k(0)H_k$ and $H_{\kappa\varphi x}=\sum_l g_l(0)H'_l$.

$H_{\kappa\varphi x}=\kappa(0)\varphi(H_x)$. Extend φ semilinearly to $\mathbf{N}_E$ (Q is a unit of $E[[X]]$, as $Q(0)=\ell$). The γ_u commute with φ on $\mathbf{N}_E$ ((Wm3), and $\varphi(g)\circ h_u=\varphi(g\circ h_u)$ for $g\in E[[X]]$, both being ring maps that agree on E and on X), so $\varphi(H_x)$ is γ_u-invariant; and $H_x-x\in X\mathbf{N}_E$ gives $\varphi(H_x)-\varphi_{\mathbf{N}}(x)\in\varphi(X)\mathbf{N}_E\subseteq X\mathbf{N}_E$. As $\kappa-\kappa(0)\in X\mathbf{A}_{F,L}$, also $\kappa(0)\varphi(H_x)-\kappa\varphi_{\mathbf{N}}(x)\in X\mathbf{N}_E$, and $\kappa(0)\varphi(H_x)$ is γ_u-invariant (γ_u fixes Λ). By uniqueness in (Hf2), $\kappa(0)\varphi(H_x)=H_{\kappa\varphi x}$.

$G^{(3)}(\varphi W)=G^{(4)}(W)$ *for* $W\in e_c\mathbf{N}_E$. The map $\mathrm{ev}^{(q)}$ is the continuous ring homomorphism $E[[X]]\to B^+_{\mathrm{dR}}\otimes\mathcal{O}_L$ that is $\varphi^q\circ\iota_\Lambda$ on E and sends X to $\varphi^q(\mu)$; on $\mathbf{A}_{F,L}$ it equals $\varphi^q\circ\iota_L$ (the tails of both series beyond degree b lie in $\varphi^q(\mu)^b(A_{\mathrm{inf}}\otimes\mathcal{O}_L)$, and $\varphi^q(\mu)\in\ker\theta$ because $\theta(\varphi^q(\mu))=\theta([\varepsilon^{\ell^q}])-1=0$), hence also on $\mathbf{A}_{F,L}[1/Q]$, Q being a unit of $E[[X]]$. As $\iota_L(\varphi X)=\varphi(\mu)$ and $\iota_\Lambda\varphi=\varphi\iota_\Lambda$, the homomorphisms $\mathrm{ev}^{(3)}\circ\varphi$ and $\mathrm{ev}^{(4)}$ agree on E and on X, so $\mathrm{ev}^{(3)}\circ\varphi=\mathrm{ev}^{(4)}$. By (Wm2), $\varphi_{\mathbf{N}}(n_k)=\sum_l A_{kl}n'_l$ with $A_{kl}\in e_{c+1}\mathbf{A}_{F,L}[1/Q]$, and by (Ex2) (extended to $\mathbf{N}[1/Q]$) $\varphi(\iota_T(n_k))=\iota_T(\varphi_{\mathbf{N}}n_k)=\sum_l\iota_L(A_{kl})\iota_T(n'_l)$. For $W=\sum_k c_kn_k$,

$$\begin{aligned}G^{(3)}(\varphi W)&=\sum_l\mathrm{ev}^{(3)}\Big(\sum_k\varphi(c_k)A_{kl}\Big)\varphi^3(\iota_Tn'_l)=\sum_k\mathrm{ev}^{(4)}(c_k)\,\varphi^3\Big(\sum_l\iota_L(A_{kl})\iota_T(n'_l)\Big)\\&=\sum_k\mathrm{ev}^{(4)}(c_k)\varphi^4(\iota_Tn_k)=G^{(4)}(W).\end{aligned}$$

Conclusion. $G^{(q)}(gW) = \mathrm{ev}^{(q)}(g)G^{(q)}(W)$ for $g \in E[[X]]$ (from the definition, $\mathrm{ev}^{(q)}$ being multiplicative), and $\mathrm{ev}^{(q)}(c) = \varphi^q(\iota_\Lambda(c))$ for $c \in \Lambda$. By Lemma 23.13 (1), $G^{(3)}(H'_l) = d'_l$ and $G^{(4)}(H_k) = \varphi(d_k)$. Hence, in $B_{\mathrm{dR}} \otimes_{\mathbb{Z}_\ell} T$, using (Cv3) at $c+1$ and at c and the φ-semilinearity of φ on $B_{\mathrm{cris}} \otimes T$,

$$\begin{aligned}\mathrm{Cv}(\kappa\varphi_{\mathbf{N}}x) &= \sum_l \varphi^3(\iota_\Lambda(g_l(0)))d'_l = G^{(3)}(H_{\kappa\varphi x}) = \varphi^3(\iota_\Lambda(\kappa(0)))\, G^{(3)}(\varphi H_x)\\ &= \varphi^3(\iota_\Lambda(\kappa(0)))\, G^{(4)}(H_x) = \varphi^3(\iota_\Lambda(\kappa(0)))\sum_k \varphi^4(\iota_\Lambda(f_k(0)))\varphi(d_k)\\ &= \varphi^3(\iota_\Lambda(\kappa(0)))\,\varphi(\mathrm{Cv}(x)).\end{aligned}$$

Both ends lie in $D \subseteq B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V$, and $B_{\mathrm{cris}} \to B_{\mathrm{dR}}$ is injective; so (Cv4) holds for one term. *Sums.* Let $s = \sum_j \kappa_j\varphi_{\mathbf{N}}(x_j) \in \mathbf{N}$. Since $e_{c+1}s = \sum_j \kappa_j\varphi_{\mathbf{N}}(e_c x_j) \in \mathbf{N}$ and both sides are additive over the sites, we may assume all $x_j \in \mathbf{N}_c$. As above, $\sum_j \kappa_j(0)\varphi(H_{x_j})$ is fixed by every γ_u and congruent to s modulo $X\mathbf{N}_E$, so it equals H_s; writing $s = \sum_l g_l n'_l$, $\mathrm{Cv}(s) = \sum_l \varphi^3(\iota_\Lambda(g_l(0)))d'_l = G^{(3)}(H_s) = \sum_j \varphi^3(\iota_\Lambda(\kappa_j(0)))G^{(4)}(H_{x_j}) = \sum_j \varphi^3(\iota_\Lambda(\kappa_j(0)))\varphi(\mathrm{Cv}(x_j))$, by the identities established above.

(Cv5). $\mathbf{N}(f)$ is $\mathbf{A}_{F,L}$-linear and commutes with the γ_u (Ex3); its extension $\mathbf{N}(f)\otimes\mathrm{id}\colon \mathbf{N}_E \to \mathbf{N}(T')_E$ commutes with the γ_u of Lemma 23.9 and maps $X\mathbf{N}_E$ into $X\mathbf{N}(T')_E$ and $\mathbf{N}_R$ into $\mathbf{N}(T')_R$. So $\mathbf{N}(f)(H_x)$ is γ_u-invariant in $\mathbf{N}(f)(x) + X\mathbf{N}(T')_E$, i.e. equals $H_{\mathbf{N}(f)x}$. For $g \in R$, $y \in \mathbf{N}$: $\mathrm{Ev}(g \otimes \mathbf{N}(f)y) = e(g)\varphi^3\rho(\iota_{T'}\mathbf{N}(f)y) = e(g)\varphi^3\rho((1\otimes f)\iota_T y) = (1\otimes f)\big(e(g)\varphi^3\rho(\iota_T y)\big)$, since $1\otimes f$ is $B_{\mathrm{cris}}\otimes\mathcal{O}_L$-linear and commutes with $\varphi^3\otimes 1$ and $\rho\otimes 1$. By additivity, $\mathrm{Cv}_{T'}(\mathbf{N}(f)x) = \mathrm{Ev}(\mathbf{N}(f)H_x) = (1\otimes f)\mathrm{Ev}(H_x)$. □

Proposition C.65. *Let ℓ, F, n, L, $(\tau_a)_{a\in\mathbb{Z}/n}$, $\Phi\colon V_1 \to V_2$ (with lattices T_1, T_2) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.52, and let $c \in \mathbb{Z}/n$ with $\epsilon_c = -1$. Let $\mathbf{N}_c^\Delta$ be the set of Δ-invariant elements of $\mathbf{N}(T_2)_c$, and Y_c the $\mathcal{O}_L$-submodule of $e_c\mathbf{N}(T_2)[1/Q]$ generated by the elements $(\pi_0+\ell)^{s_c+S}g\,\varphi(x)$ with $g \in \mathbf{A}_{F,L}$ and $x \in \mathbf{N}(T_2)_{c-1}$ both Δ-invariant. Then $Y_c \subseteq \mathbf{N}_c^\Delta$, and there is a set $\Sigma \subseteq \mathbf{N}_c^\Delta$ with at most m elements such that*

$$\mathrm{nl}\Big(\mathbf{N}_c^\Delta \Big/ \big(\pi_0\mathbf{N}_c^\Delta + Y_c + \textstyle\sum_{y\in\Sigma}\mathcal{O}_L y\big)\Big) \le d_2 S + 2m\Big(\frac{H'+1}{\ell-1}+1\Big) = \lambda.$$

Proof. Write $M = s_c + S$, $r = d_2$, $\varsigma = \pi_0 + \ell = Qw_0$ (w_0 a unit of $\mathbb{Z}_\ell[[X]]$, Lemma 28.15 (1); ς is Δ-invariant and $w_0(0) = 1$ since $\varsigma(0) = \ell = Q(0)$), and $R = e_c\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ with its Δ-invariant elements R^Δ.

Step 0: jumps and window. By Lemma 24.2 (1) and (Ow1) ($\epsilon_c = -1$), every Hodge jump of V_2 at τ_c lies in $[-M, -s_c] \cup [-\eta, \eta]$, two disjoint intervals as $s_c > \eta$; by (Ow2) the jumps in $[-\eta, \eta]$, counted with multiplicity, number $m' \le m$. By (Ex1) with a window having $M_c = M$, (Wm2) gives $Q^M\varphi(\mathbf{N}(T_2)_{c-1}) \subseteq \mathbf{N}(T_2)_c$, hence $\varsigma^M g\varphi(x) \in \mathbf{N}(T_2)_c$ for $g \in \mathbf{A}_{F,L}$, $x \in \mathbf{N}(T_2)_{c-1}$; it is Δ-invariant when g and x are, because φ commutes with the γ_δ (Wm3). So $Y_c \subseteq \mathbf{N}_c^\Delta$.

Step 1: reduction modulo π_0. By Lemma 24.15, $\mathbf{N}(T_2)_c$ has a Δ-invariant R-basis $(e_l)_{l\le r}$; by uniqueness of coordinates, $\mathbf{N}_c^\Delta = \bigoplus_l R^\Delta e_l$. Let $\mathrm{red}\colon \mathbf{N}_c^\Delta \to \mathcal{O}_L^r$, $\sum_l f_l e_l \mapsto (f_l(0))_l$ (via $R/XR \cong \mathcal{O}_L$). It is surjective (the constants $e_c(1\otimes\lambda)$ are Δ-invariant), and its kernel is $\bigoplus_l (XR)^\Delta e_l = \pi_0\mathbf{N}_c^\Delta$, because $(XR)^\Delta = \pi_0 R^\Delta$: if $f \in XR$ is Δ-invariant with lowest term $a_i X^i$, the lowest term of $\gamma_\delta(f)$ is $a_i\delta^i X^i$, so $(\ell-1) \mid i$ and $f = \pi_0 f_1$ (π_0 is $X^{\ell-1}$ times a unit), f_1 Δ-invariant as π_0 is and R is a domain. Hence, for any Σ, the quotient in the statement is isomorphic to $\mathcal{O}_L^r/\mathrm{red}(Y_c + \sum_{y\in\Sigma}\mathcal{O}_L y)$, and if $C \in M_r(\mathcal{O}_L)$ has its columns in $\mathrm{red}(Y_c + \sum_\Sigma \mathcal{O}_L y)$

and $\det C \neq 0$, then the normalised length is at most $\mathrm{nl}(\mathcal{O}_L^r/C\mathcal{O}_L^r) = v_\ell(\det C)$ (elementary divisors over the discrete valuation ring $\mathcal{O}_L$).

Step 2: the columns. By Lemma C.66 for (V_2, T_2) at c there are a Δ-invariant basis $(e'_k)_{k\le r}$ of $\mathbf{N}(T_2)_{c-1}$ and integers j_k with $\varphi^{-3}(\mathrm{Cv}(e'_k)) \in \mathrm{Fil}^{j_k} D_{\mathrm{cris}}(V_2)$, each Hodge jump u at τ_c occurring as j_k exactly with its multiplicity. Call k *small* if $|j_k| \le \eta$ (there are m' of them) and *big* otherwise ($j_k \in [-M, -s_c]$). For big k put $\mathrm{col}_k = \varsigma^M \varphi(e'_k) \in Y_c$. For small k apply Lemma C.67 to (V_2, T_2), c, M (every jump at τ_c is $\ge -M$), $u = j_k$ and $k_0 = (\ell-1)\lceil (M+\eta)/(\ell-1)\rceil \ge M + \eta \ge M + j_k$, $k_0 \ge 1$: with $s = s_\ell(k_0 - 1)$ there is a Δ-invariant $\tilde{x}_k \in \mathbf{N}(T_2)_{c-1}$ with $\tilde{x}_k - \ell^s e'_k \in X\mathbf{N}(T_2)_{c-1}$ and $\mathrm{col}_k = \varsigma^{-j_k}\varphi(\tilde{x}_k) \in \mathbf{N}_c^\Delta$. Let $\Sigma = \{\mathrm{col}_k : \ k \text{ small}\}$, of at most $m' \le m$ elements.

Step 3: the determinant. Let $P \in M_r(R)$ with $\varsigma^M\varphi(e'_k) = \sum_l P_{lk} e_l$. By Lemma C.70 (applied with (e_l), (e'_k) and M), the matrix $P^Q = w_0^{-M} P$ of the $Q^M\varphi(e'_k)$ has $\det P^Q = vQ^w$ with $v \in R^\times$ and $w = \sum_k (M + j_k)$ (the j_k run over the jumps with multiplicity). At $X = 0$: $P(0) = P^Q(0)$ and $\det P(0) = v(0)\ell^w$ with $v(0) \in \mathcal{O}_L^\times$. For small k write $\tilde{x}_k = \sum_m z_{mk} e'_m$, $z_{mk} \in e_{c-1}\mathbf{A}_{F,L}$ with $z_{mk}(0) = \ell^s \delta_{mk} e_{c-1}$. Then $\varsigma^M\varphi(\tilde{x}_k) = \sum_l (\sum_m P_{lm}\varphi(z_{mk}))e_l$, and the coordinates $c_{lk} \in R$ of col_k satisfy $\varsigma^{M+j_k} c_{lk} = \sum_m P_{lm}\varphi(z_{mk})$ ($M + j_k \ge M - \eta > 0$). At $X = 0$, using $\varphi(z_{mk})(0) = \varphi(z_{mk}(0)) = \ell^s \delta_{mk} e_c$: $\ell^{M+j_k} c_{lk}(0) = \ell^s P_{lk}(0)$. Hence the matrix C with columns $\mathrm{red}(\mathrm{col}_k)$ is $C = P(0)\,\mathrm{diag}(d_k)$ with $d_k = 1$ for big k and $d_k = \ell^{\,s-M-j_k}$ for small k, and

$$v_\ell(\det C) = w + \sum_{k \text{ small}} (s - M - j_k) = \sum_{k \text{ big}} (M + j_k) + m's \le (r - m')S + m's,$$

since $0 \le M + j_k \le S$ for big k. In particular $\det C \neq 0$.

Step 4: the bound. Put $q = \lceil (M+\eta)/(\ell-1)\rceil$, so $k_0 - 1 = (\ell-1)q - 1$ and $J = \lfloor (k_0 - 1)/(\ell-1)\rfloor = q - 1$; by Legendre's formula $s = J + v_\ell(J!) \le J\ell/(\ell-1)$, and $J < (M+\eta)/(\ell-1) \le H'/(\ell-1)$ ($M + \eta = s_c + S + \eta \le H'$). So $s \le H'\ell/(\ell-1)^2 \le 2H'/(\ell-1) \le 2\big((H'+1)/(\ell-1) + 1\big)$. With Steps 1 and 3 and $m' \le m$, $r = d_2$, the normalised length is at most $d_2 S + 2m\big((H'+1)/(\ell-1) + 1\big) = \lambda$. □

Lemma C.66. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a pair (V, T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$ of rank r, $D = D_{\mathrm{cris}}(V)$ and $c \in \mathbb{Z}/n$. Then there are a Δ-invariant $e_{c-1}\mathbf{A}_{F,L}$-basis $(e'_k)_{k\le r}$ of $\mathbf{N}_{c-1}$ and integers $j_1, \dots, j_r$ such that*

$$\varphi^{-3}(\mathrm{Cv}(e'_k)) \in \mathrm{Fil}^{j_k} D \qquad (k \le r),$$

and for every integer u the number of k with $j_k = u$ is the multiplicity

$$\dim_L(\mathrm{Fil}^u D_{\mathrm{dR},\tau_c}(V)/\mathrm{Fil}^{u+1} D_{\mathrm{dR},\tau_c}(V))$$

of u as a Hodge jump of V at τ_c.

Proof. **The lattice.** Put $\epsilon = \epsilon_{c-1}$. By (Cv1) and (Cv3), $\mathrm{Cv}(\mathbf{N}_{c-1})$ is an $\mathcal{O}_L$-lattice in $\epsilon_{c+2}D$ (an $\mathcal{O}_L$-submodule, free with an L-basis of $\epsilon_{c+2}D$ as basis), and $\mathrm{Cv}(x) = 0$ for $x \in \mathbf{N}_{c-1}$ only if $x \in X\mathbf{N}_{c-1}$: in a basis (n_k), $\mathrm{Cv}(\sum_k f_k n_k) = \sum_k \epsilon_{c+2}(1 \otimes \lambda_k)\mathrm{Cv}(n_k)$ with $f_k(0) = e_{c-1}(1 \otimes \lambda_k)$, and the $\mathrm{Cv}(n_k)$ are L-independent. The Frobenius φ of D commutes with the action of $\mathcal{O}_L$ (which is through T) and maps $\epsilon_b D$ bijectively onto $\epsilon_{b+1}D$; so $\Lambda = \varphi^{-3}(\mathrm{Cv}(\mathbf{N}_{c-1}))$ is an $\mathcal{O}_L$-lattice in ϵD, and $\varphi^{-3} \circ \mathrm{Cv}$ induces an isomorphism $\mathbf{N}_{c-1}/X\mathbf{N}_{c-1} \cong \Lambda$. As in Definition 23.3, $\epsilon = \epsilon_{c-1}$ is the idempotent of the embedding τ_c: for $y \in \mathcal{O}_F$, $\iota(y) = \varphi(y)$ and $e_{c-1}(y \otimes 1) = e_{c-1}(1 \otimes \tau_{c-1}(y))$, so F acts on ϵD through $\tau_{c-1} \circ \varphi^{-1} = \tau_c$; hence $\epsilon D = D_{\mathrm{dR},\tau_c}(V)$ (V is crystalline, $D_{\mathrm{cris}} = D_{\mathrm{dR}}$) with the filtration $\epsilon\mathrm{Fil}^u D$.

An adapted basis of Λ. For $u \in \mathbb{Z}$ let $\Lambda_u = \Lambda \cap \epsilon \mathrm{Fil}^u D$. Each Λ_u is saturated in Λ (Λ/Λ_u embeds in $\epsilon D/\epsilon\mathrm{Fil}^u D$, hence is torsion free), so over the discrete valuation ring $\mathcal{O}_L$ it is a direct summand of Λ and of every $\Lambda_{u'}$, $u' \le u$, containing it; its rank is $\dim_L \epsilon\mathrm{Fil}^u D$ (every $v \in \epsilon\mathrm{Fil}^u D$ has $\ell^N v \in \Lambda$). The filtration is finite: $\Lambda_u = \Lambda$ for $u \ll 0$ and $\Lambda_u = 0$ for $u \gg 0$. Choosing, for u from large to small, a complement of Λ_{u+1} in Λ_u and a basis of it, we get a basis $(f_k)_{k \le r}$ of Λ and integers j_k (f_k chosen at step $u = j_k$) with $f_k \in \Lambda_{j_k} \subseteq \epsilon\mathrm{Fil}^{j_k} D$, and the number of k with $j_k = u$ is $\operatorname{rank}\Lambda_u - \operatorname{rank}\Lambda_{u+1} = \dim_L(\epsilon\mathrm{Fil}^u D/\epsilon\mathrm{Fil}^{u+1}D)$, the multiplicity of u as a Hodge jump at τ_c.

Lifting. Choose $e''_k \in \mathbf{N}_{c-1}$ with $\varphi^{-3}(\mathrm{Cv}(e''_k)) = f_k$. Their images in $\mathbf{N}_{c-1}/X\mathbf{N}_{c-1} \cong \Lambda \cong \mathcal{O}_L^r$ form a basis; as $\mathbf{N}_{c-1}$ is free of rank r over the local ring $e_{c-1}\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$ ((Wm1)), Nakayama's lemma shows that (e''_k) generates $\mathbf{N}_{c-1}$, hence is a basis (a surjective endomorphism of a finitely generated module is bijective). Put $e'_k = (\ell-1)^{-1}\sum_{\delta\in\mu_{\ell-1}} \gamma_\delta(e''_k) \in \mathbf{N}_{c-1}$ ($\ell - 1$ is a unit of $\mathbb{Z}_\ell$); it is Δ-invariant since $\gamma_{\delta'}\gamma_\delta = \gamma_{\delta'\delta}$. For $x \in \mathbf{N}$ and $\delta \in \mu_{\ell-1}$, $\gamma_\delta(H_x)$ is fixed by every γ_u (they commute) and lies in $\gamma_\delta(x) + X\mathbf{N}_E$, so $H_{\gamma_\delta x} = \gamma_\delta(H_x)$ and, by (Ev3), $\mathrm{Cv}(\gamma_\delta x) = \sigma(\mathrm{Cv}(x)) = \mathrm{Cv}(x)$ for $\sigma \in G_F$ with $\chi_{\mathrm{cyc}}(\sigma) = \delta$, as $\mathrm{Cv}(x) \in D$ is G_F-invariant. Hence $\varphi^{-3}(\mathrm{Cv}(e'_k)) = f_k \in \mathrm{Fil}^{j_k} D$, and (e'_k) is a basis of $\mathbf{N}_{c-1}$ by the same Nakayama argument. □

Lemma C.67. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a pair (V,T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$, $D = D_{\mathrm{cris}}(V)$, $c \in \mathbb{Z}/n$, and let $M \ge 0$, u and $k \ge 1$ be integers such that every Hodge jump of V at τ_c is at least $-M$, $(\ell-1) \mid k$ and $k \ge M + u$. Let $x \in \mathbf{N}_{c-1}$ with $\varphi^{-3}(\mathrm{Cv}(x)) \in \mathrm{Fil}^u D$. Then there is $\tilde{x} \in \mathbf{N}_{c-1}$ with $\tilde{x} - \ell^{s_\ell(k-1)}x \in X\mathbf{N}_{c-1}$ and*

$$(\pi_0 + \ell)^{-u}\,\varphi(\tilde{x}) \in \mathbf{N}_c;$$

if x is Δ-invariant, $\tilde{x}$ can be chosen Δ-invariant, and then $(\pi_0+\ell)^{-u}\varphi(\tilde{x})$ is Δ-invariant.

Proof. Write $\varsigma = \pi_0 + \ell$ and $s = s_\ell(k-1)$. By Lemma 28.15 (1), $\varsigma = Qw_0$ with w_0 a unit of $\mathbb{Z}_\ell[[X]]$, so ς is invertible in $\mathbf{A}_{F,L}[1/Q]$, and ς is Δ-invariant.

Step 1: the truncation. By (Hf3) (applied to H_x, which lies in $e_{c-1}\mathbf{N}_E$ by uniqueness in (Hf2)) there is $x_k \in \mathbf{N}$ with $\ell^s H_x - x_k \in X^k\mathbf{N}_E$; replacing x_k by $e_{c-1}x_k$ we may take $x_k \in \mathbf{N}_{c-1}$. If x is Δ-invariant, put $\tilde{x} = (\ell-1)^{-1}\sum_{\delta\in\mu_{\ell-1}}\gamma_\delta(x_k)$; otherwise $\tilde{x} = x_k$. Since every γ_δ fixes H_x and preserves $X^k\mathbf{N}_E$, in both cases $\ell^s H_x - \tilde{x} \in X^k\mathbf{N}_E$, so by (Hf3) $(\gamma_{u_0}-1)(\tilde{x}) \in X^k\mathbf{N}$ (for the u_0 fixed in Definition 23.4), and $\tilde{x} - \ell^s x \in \mathbf{N} \cap X\mathbf{N}_E = X\mathbf{N}$ (as $H_x - x \in X\mathbf{N}_E$). In the Δ-invariant case $\tilde{x}$ is Δ-invariant ($\gamma_{\delta'}\gamma_\delta = \gamma_{\delta'\delta}$), hence so is $\varsigma^{-u}\varphi(\tilde{x})$ (φ commutes with the γ_δ).

Step 2: level data. Fix arc level data for (V,T) with cut c, length 1 (so $c_0 = c-1$), $H(c) = \max(u,0)$ (and $H = 0$ elsewhere), $\Pi = Q^{H(c)}$, a frame $(n_j)_{j\le r}$ of $\mathbf{N}_{c-1}$, parameters $j = 1$, $M_{\mathrm{t}} = k/(\ell-1)$ in the role of M, $g = 0$, and arbitrary $G, S, \eta, \kappa_0, b_0$. Write $\tilde{x} - \sum_j a_j n_j \in X\mathbf{N}_{c-1}$ with $a \in \mathcal{O}_L^r$ (via $e_{c-1}\mathbf{A}_{F,L}/X \cong \mathcal{O}_L$). The element $w = \Pi^{-1}\varphi(\tilde{x})$ satisfies $\Pi w = \varphi(\tilde{x}) \in P_1(\mathbf{N}_{c-1})$, so $w \in W$, and $(\gamma_{u_0}-1)\tilde{x} \in X^{(\ell-1)M_{\mathrm{t}}}\mathbf{N}_{c-1}$ with $(\ell-1)M_{\mathrm{t}} = k \ge 0 = 3g$. By Lemma C.68 ((Lp6)), $\lambda_0(\Pi)\mathrm{lv}_0(w) - a \in t^k K_{(0)}[[t]]^r$.

Step 3: the level-0 expansion of $g = \varsigma^{-u}\varphi(\tilde{x})$. g lies in W: $\Pi g = Q^{H(c)}\varsigma^{-u}\varphi(\tilde{x})$ equals $w_0^{-u}\varphi(\tilde{x})$ if $u \ge 0$ and $\varsigma^{-u}\varphi(\tilde{x})$ if $u < 0$, an element of $\mathbf{A}_{F,L}\varphi(\mathbf{N}_{c-1}) = P_1(\mathbf{N}_{c-1})$. As $\Lambda_0(fy) = \lambda_0(f)\Lambda_0(y)$, $\mathrm{lv}_0(g) = \lambda_0(\varsigma)^{-u}\lambda_0(\Pi)\mathrm{lv}_0(w)$. By Lemma 23.17 ((Lp1)), $\lambda_0(Q) \in tK_{(0)}[[t]]^\times$ and $\lambda_0(w_0) \in K_{(0)}[[t]]$ with $\lambda_0(w_0)\lambda_0(w_0^{-1}) = 1$, so $\lambda_0(\varsigma) = t\vartheta$ with $\vartheta \in K_{(0)}[[t]]^\times$. Hence

$$\mathrm{lv}_0(g) = t^{-u}\vartheta^{-u}(a+\rho), \qquad \rho \in t^k K_{(0)}[[t]]^r.$$

Step 4: de Rham integrality. The level frame is $D_j^{(0)} = \varphi^{-3}(\mathrm{Ev}(H_{n_j})) = \varphi^{-3}(\mathrm{Cv}(n_j))$. By (Cv1), (Cv2), $\mathrm{Cv}(\tilde{x}) = \sum_j a_j \mathrm{Cv}(n_j)$ (the a_j acting through T, which commutes with φ),

and $\mathrm{Cv}(\tilde{x}) = \ell^s \mathrm{Cv}(x)$; so $\sum_j a_j D_j^{(0)} = \ell^s \varphi^{-3}(\mathrm{Cv}(x)) \in \mathrm{Fil}^u D$, i.e. $a \in \mathrm{Fil}_0(u)$. The coefficient of t^i in $t^{-u}\vartheta^{-u}a$ is a $K_{(0)}$-multiple of a and vanishes for $i < -u$; for $i \geq -u$, $-i \leq u$ and $\mathrm{Fil}_0(u) \subseteq \mathrm{Fil}_0(-i)$ (Fil is decreasing). The coefficient of t^i in $t^{-u}\vartheta^{-u}\rho$ vanishes for $i < k-u$, and for $i \geq k-u \geq M$ we have $\mathrm{Fil}_0(-i) = L^r$: the $D_j^{(0)}$ lie in $\epsilon_0 D = \epsilon_{c-1} D = D_{\mathrm{dR},\tau_c}(V)$ (Definition 23.4: ϵ_0 is the idempotent of τ_c), whose filtration satisfies $\mathrm{Fil}^v = \epsilon_{c-1} D$ for $v \leq -M$, every Hodge jump at τ_c being $\geq -M$. So $\mathrm{lv}_0(g)$ is de Rham integral at 0, and by Lemma 23.5 (2) ((Lp3) with length 1), $g \in \mathbf{N}_c$. □

Lemma C.68. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ be as in Proposition 23.7, (V,T) a pair, $\mathbf{N} = \mathbf{N}(T)$, and fix arc level data for (V,T) (cut c, length* len*, H, frame $\mathbf{n}$ at $c_0 = c-$*len*, parameters j, M, g) with $(\ell-1)M \geq 3g$. Then (Lp6) holds: if $x \in \mathbf{N}_{c_0}$, $(\gamma_{u_0}-1)x \in X^{(\ell-1)M}\mathbf{N}_{c_0}$, $x - \sum_k a_k n_k \in X\mathbf{N}_{c_0}$ with $a \in \mathcal{O}_L^r$, and $w \in W$ with $\Pi w = \varphi^{\mathrm{len}}(x)$, then for every $s <$* len

$$\lambda_s(\Pi)\,\mathrm{lv}_s(w) - a \in t^{(\ell-1)M} K_{(s)}[[t]]^r \subseteq \lambda_s(f')\,K_{(s)}[[t]]^r, \qquad \text{so } \mathrm{lv}_s(w) - \lambda_s(\Pi)^{-1}a \in \mathcal{T}_s.$$

Proof. Put $m = (\ell-1)M$, $K = K_{(s)}$, $q =$ len $- s - 1$, and let H_k be the horizontal sections through the n_k ($E[[X]]$-basis of $e_{c_0}\mathbf{N}_E$, Lemma 23.9 (Hf2)).

Step 1: $Y = x - \sum_k a_k H_k \in X^m\mathbf{N}_E$. $Y \in X\mathbf{N}_E$, since $x - \sum_k a_k n_k \in X\mathbf{N}$ and $H_k - n_k \in X\mathbf{N}_E$; and $(\gamma_{u_0}-1)Y = (\gamma_{u_0}-1)x \in X^m\mathbf{N}_E$ (a_k is constant and H_k is γ-fixed). If $Y = X^i \sum_k y_k H_k$ with $y_k \in E[[X]]$ and $1 \leq i < m$, then, as $\gamma_{u_0}(f) = f \circ h_{u_0}$ on $E[[X]]$ with $h_{u_0}(X) \in u_0 X + X^2\mathbb{Z}_\ell[[X]]$,

$$(\gamma_{u_0}-1)Y \equiv (u_0^i - 1)X^i \sum_k y_k(0) H_k \bmod X^{i+1}\mathbf{N}_E;$$

since $u_0 \in 1 + \ell\mathbb{Z}_\ell$ is not a root of unity (ℓ odd), $u_0^i - 1 \neq 0$, so all $y_k(0) = 0$ and $Y \in X^{i+1}\mathbf{N}_E$. By induction, $Y \in X^m\mathbf{N}_E$: $Y = \sum_k y_k H_k$ with $y_k \in X^m e_{c_0} E[[X]]$.

Step 2: the level. By (Eq3) of Lemma C.69, $\lambda_s(\Pi)\Lambda_s(w) = \Lambda_s(\varphi^{\mathrm{len}}(x)) = G^{(q)}(x)$, and by Lemma 23.13 (1) and the semilinearity of $G^{(q)}$ ($G^{(q)}(gY) = \mathrm{ev}^{(q)}(g)G^{(q)}(Y)$ by its definition, $\mathrm{ev}^{(q)}$ being a ring homomorphism by (Eq1) there),

$$G^{(q)}(x) = \sum_k \mathrm{ev}^{(q)}(a_k) G^{(q)}(H_k) + \sum_k \mathrm{ev}^{(q)}(y_k) G^{(q)}(H_k) = \sum_k (a_k + \mathrm{ev}^{(q)}(y_k)) D_k^{(s)},$$

where, by (Eq3) of Lemma C.69, $\mathrm{ev}^{(q)}(a_k) = \epsilon_s a_k$ (the constant $a_k \in e_{c_0}(1 \otimes \mathcal{O}_L)$) and $\mathrm{ev}^{(q)}(y_k) \in t^m K[[t]]$ ($y_k \in X^m e_{c_0} E[[X]]$). Hence $\lambda_s(\Pi)\mathrm{lv}_s(w) - a \in t^m K[[t]]^r$.

Step 3: the truncation ideal. Since $\varphi^k(X) = X\lambda_k$, $f' = \varphi^{\mathrm{len}-j}(X^m \prod_{s'<j} Q_{s'}^{3g}) = X^m \prod_{s'<\mathrm{len}-j} Q_{s'}^m \prod_{\mathrm{len}-j\leq s'<\mathrm{len}} Q_{s'}^{3g}$. By Lemma 23.17 ((Lp1): $\lambda_s(X)$, $\lambda_s(Q_{s'})$ units for $s' \neq s$, $\lambda_s(Q_s) \in tK[[t]]^\times$), $\lambda_s(f') \in t^{e_s}K[[t]]^\times$ with $e_s = m$ if $s <$ len $- j$ and $e_s = 3g$ otherwise; in both cases $e_s \leq m$. So $t^m K[[t]]^r \subseteq \lambda_s(f')K[[t]]^r$, which is the claim. □

Lemma C.69. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a pair (V,T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$, $B = B_{\mathrm{dR}}^+$, $\iota_L = \iota \otimes \mathrm{id}\colon \mathbf{A}_{F,L} \to A_{\mathrm{inf}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$, and $q \geq 0$ an integer.*

(Eq1) *$\mathrm{ev}^{(q)}\colon E[[X]] \to B \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$ is a ring homomorphism, $\mathrm{ev}^{(q)}(X^m f) = \varphi^q(\mu)^m \mathrm{ev}^{(q)}(f)$ for $m \geq 0$, and $\mathrm{ev}^{(q)}(g) = (\varphi^q \otimes 1)(\iota_L(g))$ for $g \in \mathbf{A}_{F,L}[1/\ell]$.*

(Eq2) *For $c \in \mathbb{Z}/n$ and $x \in \mathbf{N}_c$, $G^{(q)}(x) = (\varphi^q \otimes 1)(\iota_T(x))$, $G^{(q)}$ formed with a frame at c.*

(Eq3) *Fix arc level data for (V,T) (cut c, length len, $c_0 = c-\mathrm{len}$), let $s < \mathrm{len}$ and $q = \mathrm{len}-s-1$. Then $\Lambda_s(\varphi^{\mathrm{len}}(x)) = G^{(q)}(x)$ for $x \in \mathbf{N}_{c_0}$, and $\lambda_s(\varphi^{\mathrm{len}}(a)) = \mathrm{ev}^{(q)}(a)$ for $a \in e_{c_0}\mathbf{A}_{F,L}[1/\ell]$. For $f = \sum_i b_i X^i \in e_{c_0}E[[X]]$ the elements $\varphi^q(\iota_\Lambda(b_i))$ lie in $\epsilon_s L \subseteq K_{(s)}$ and*

$$\mathrm{ev}^{(q)}(f) = \sum_{i\geq 0} \varphi^q(\iota_\Lambda(b_i))(\exp(\ell^q t)-1)^i \ \in K_{(s)}[[t]];$$

if moreover $f \in X^m E[[X]]$, then $\mathrm{ev}^{(q)}(f) \in t^m K_{(s)}[[t]]$.

Proof. **(Eq1).** The map $a \mapsto \varphi^q(\iota_\Lambda(a))$ is a ring homomorphism $E \to (A_{\mathrm{inf}}\otimes\mathcal{O}_L)[1/\ell] \subseteq B\otimes\mathcal{O}_L$ (ι_Λ is the restriction of $\iota\otimes\mathrm{id}$ with ℓ inverted). By Theorem B.40 (Ai1), $\theta(\varphi^q(\mu)) = \theta([\varepsilon^{\ell^q}])-1 = (\varepsilon^{\ell^q})^\sharp - 1 = 0$, so $\varphi^q(\mu)$ lies in the maximal ideal $\ker\theta = tB$ of the complete discrete valuation ring B ((Ai3)); and $B\otimes\mathcal{O}_L$ is free of finite rank over B, hence t-adically complete and separated. So the series defining $\mathrm{ev}^{(q)}$ converges, and $\mathrm{ev}^{(q)}$ is the substitution homomorphism $X \mapsto \varphi^q(\mu)$ over $a \mapsto \varphi^q(\iota_\Lambda(a))$: modulo t^b it agrees, on series and on their products, with the ring homomorphism given by the truncation at degree b, for every b. The identity for $X^m f$ is read off the series. Let $g \in \mathbf{A}_{F,L} = \Lambda[[X]]$ and $b \geq 1$, and write $g = \sum_{i<b} a_i X^i + X^b g_b$ with $a_i \in \Lambda$, $g_b \in \mathbf{A}_{F,L}$. As ι_L is a ring homomorphism equal to ι_Λ on Λ with $\iota_L(X) = \iota(X) = \mu$ (Lemma 23.8),

$$(\varphi^q\otimes 1)(\iota_L(g)) = \sum_{i<b} \varphi^q(\iota_\Lambda(a_i))\varphi^q(\mu)^i + \varphi^q(\mu)^b(\varphi^q\otimes 1)(\iota_L(g_b)) \equiv \mathrm{ev}^{(q)}(g) \bmod t^b(B\otimes\mathcal{O}_L).$$

As b is arbitrary, $\mathrm{ev}^{(q)}(g) = (\varphi^q\otimes 1)(\iota_L(g))$; for $g \in \mathbf{A}_{F,L}[1/\ell]$ multiply by a power of ℓ.

(Eq2). Let $\mathbf{n} = (n_j)$ be a frame at c, an $e_c\mathbf{A}_{F,L}$-basis of $\mathbf{N}_c$, and $x = \sum_j g_j n_j$ with $g_j \in e_c\mathbf{A}_{F,L}$. By (Ex2), $\iota_T(gy) = \iota_L(g)\iota_T(y)$ for $g \in \mathbf{A}_{F,L}$, $y \in \mathbf{N}$ ($\mathbf{A}_{F,L} = \mathbf{A}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$). Hence, by the definition of $G^{(q)}$ and (Eq1),

$$G^{(q)}(x) = \sum_j \mathrm{ev}^{(q)}(g_j)\,(\varphi^q\otimes 1)(\iota_T(n_j)) = (\varphi^q\otimes 1)\Big(\sum_j \iota_L(g_j)\iota_T(n_j)\Big) = (\varphi^q\otimes 1)(\iota_T(x)).$$

(Eq3). By (Lm2) of Lemma 23.6, applied with the site c_0 in place of its c, $e = \mathrm{len}$ and $a = c$ (its Λ_s is ours), $\Lambda_s(\varphi^{\mathrm{len}}(x)) = (\varphi^q\otimes 1)(\iota_T(x))$ for $x \in \mathbf{N}$, which is $G^{(q)}(x)$ for $x \in \mathbf{N}_{c_0}$ by (Eq2). By (Lm1) there, $\lambda_s(\varphi^{\mathrm{len}}(a)) = (\varphi^q\otimes 1)(\iota_L(a))$ for $a \in \mathbf{A}_{F,L}$, hence for $a \in \mathbf{A}_{F,L}[1/\ell]$, and this is $\mathrm{ev}^{(q)}(a)$ by (Eq1). Now let $f = \sum_i b_i X^i \in e_{c_0}E[[X]]$. Since $e_{c_0}E = e_{c_0}(1\otimes L)$ and $\iota_\Lambda(e_{c_0}) = \epsilon_{c_0}$ (Lemma 23.10), $\iota_\Lambda(b_i) \in \epsilon_{c_0}L$; as ι commutes with φ (Lemma 23.8) and $(\varphi\otimes\mathrm{id})(e_b) = e_{b+1}$ ($\tau_{b+1}\circ\varphi = \tau_b$ for the Frobenius numbering), $\varphi^q(\epsilon_{c_0}) = \epsilon_{c_0+q} = \epsilon_{c-s-1} = \epsilon_s$, so $c_i := \varphi^q(\iota_\Lambda(b_i)) \in \epsilon_s L \subseteq K_{(s)}$. In B, $[\varepsilon] = \exp(t)$ ($t = \log[\varepsilon]$), so $\varphi^q(\mu) = [\varepsilon]^{\ell^q} - 1 = \exp(\ell^q t) - 1 \in t\,\mathbb{Q}_\ell[[t]]$. Hence $(\exp(\ell^q t)-1)^i \in t^i\mathbb{Q}_\ell[[t]]$, the series $\sum_i c_i(\exp(\ell^q t)-1)^i$ defines an element of $K_{(s)}[[t]]$ (each coefficient is a finite sum), and it agrees with $\mathrm{ev}^{(q)}(f) \in R_s^+$ modulo $t^b R_s^+$ for every b, both being congruent to $\sum_{i<b} c_i(\exp(\ell^q t)-1)^i$ (the identification of Definition 23.4 maps $t^b K_{(s)}[[t]]$ into $t^b R_s^+$). As R_s^+ is t-adically separated, the two are equal. Finally, for $f = X^m f_1$, $\mathrm{ev}^{(q)}(f) = (\exp(\ell^q t)-1)^m \mathrm{ev}^{(q)}(f_1) \in t^m K_{(s)}[[t]]$ by (Eq1). □

Lemma C.70. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a pair (V,T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$ of rank $r = \dim_L V$, $a \in \mathbb{Z}/n$, and $M \geq 0$ an integer such that every Hodge jump of V at τ_a is at least $-M$. For $j \in \mathbb{Z}$ let $m_j = \dim_L(\mathrm{Fil}^j D_{\mathrm{dR},\tau_a}(V)/\mathrm{Fil}^{j+1}D_{\mathrm{dR},\tau_a}(V))$ be the multiplicity of j as a Hodge jump at τ_a. Let $(e_k)_{k\leq r}$ be an $e_a\mathbf{A}_{F,L}$-basis of $\mathbf{N}_a$ and $(e'_k)_{k\leq r}$ an $e_{a-1}\mathbf{A}_{F,L}$-basis of $\mathbf{N}_{a-1}$. Then $Q^M\varphi(e'_k) \in \mathbf{N}_a$ for every k, and the matrix $P \in M_r(e_a\mathbf{A}_{F,L})$ with $Q^M\varphi(e'_k) = \sum_m P_{mk}e_m$ satisfies*

$$\det P = v\,Q^w, \qquad v \in (e_a\mathbf{A}_{F,L})^\times, \qquad w = \sum_{j\in\mathbb{Z}}(M+j)\,m_j = rM + \sum_j j\,m_j.$$

Proof. Put $R = e_a\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$, a unique factorisation domain (a regular local ring).

Step 1: $\det P$ divides a power of Q. Apply (Ex1) with the window $M_a = M$ and M_b, h_b ($b \in \mathbb{Z}/n$, h_a included) larger than all Hodge jumps. By (Wm2), $Q^M\varphi(\mathbf{N}_{a-1}) \subseteq \mathbf{N}_a$, so $P \in M_r(R)$; and $Q^{h_a}\mathbf{N}_a \subseteq \mathbf{A}_{F,L}\varphi(\mathbf{N}_{a-1}) = \sum_k R\,\varphi(e'_k)$ (φ is semilinear), so there is $P' \in M_r(R)$ with $Q^{h_a+M}e_m = \sum_k P'_{km}Q^M\varphi(e'_k)$, i.e. $PP' = Q^{h_a+M}I$, and $\det P \cdot \det P' = Q^{r(h_a+M)}$.

Step 2: the prime factors of Q. L contains a primitive ℓ-th root of unity ζ (it contains $\sigma(F(\mu_\ell))$), and $Q = \Phi_\ell(1+X) = \prod_{k=1}^{\ell-1}\pi_k$ with $\pi_k = X - (\zeta^k - 1)$. Each π_k is prime in R ($R/\pi_k R \cong \mathcal{O}_L$), and π_k, $\pi_{k'}$ are not associate for $k \neq k'$ (their zeros $\zeta^k - 1 \neq \zeta^{k'} - 1$ in the maximal ideal of $\mathcal{O}_L$ differ). By Step 1 and unique factorisation, $\det P = v\prod_k \pi_k^{w_k}$ with $v \in R^\times$ and integers $w_k \geq 0$.

Step 3: the exponents w_k are equal. Let $u \in \mathbb{Z}_\ell^\times$ and $\gamma = \gamma_u$. With $\eta_u = \gamma(X)/X = ((1+X)^u - 1)/X$, a unit of $\mathbb{Z}_\ell[[X]]$, and $\gamma\varphi = \varphi\gamma$ on $\mathbb{Z}_\ell[[X]]$: $X\eta_u\gamma(Q) = \gamma(\varphi X) = \varphi(X\eta_u) = XQ\varphi(\eta_u)$, so $\gamma(Q) = Q\varphi(\eta_u)\eta_u^{-1}$ and $\gamma(Q)/Q$ is a unit. Let $G \in \mathrm{GL}_r(R)$, $G' \in \mathrm{GL}_r(e_{a-1}\mathbf{A}_{F,L})$ be the matrices of γ in the bases (e_m), (e'_k) (γ is bijective, with inverse $\gamma_{u^{-1}}$, by (Wm3)). Computing $\gamma(Q^M\varphi(e'_k))$ once as $\gamma(Q)^M\varphi(\gamma e'_k)$ ((Wm3)) and once from $Q^M\varphi(e'_k) = \sum_m P_{mk}e_m$ gives $G\,\gamma(P) = (\gamma(Q)/Q)^M P\,\varphi(G')$, hence

$$\det G \cdot \gamma(\det P) = (\gamma(Q)/Q)^{Mr}\ \det P \cdot \varphi(\det G'),$$

and $\gamma(\det P)$ and $\det P$ generate the same ideal of R. Now $\gamma(\pi_k) = (1+X)^u - \zeta^k$; let $k' \in \{1,\dots,\ell-1\}$ with $k'u \equiv k \bmod \ell$. Then $\gamma(\pi_k)$ vanishes at $X = \zeta^{k'} - 1$ (as $\zeta^{k'u} = \zeta^k$, u being an ℓ-adic limit of integers $\equiv u \bmod \ell$), so $\gamma(\pi_k) = \pi_{k'}g$ with $g \in R$; comparing constant terms, $-(\zeta^{k'}-1)g(0) = 1 - \zeta^k$, and both $\zeta^{k'} - 1$ and $1 - \zeta^k$ have valuation $1/(\ell-1)$, so $g(0)$ and g are units. Hence $\gamma(\det P) = \gamma(v)\prod_k \pi_{k'(k)}^{w_k} \cdot (\text{unit})$, and unique factorisation gives $w_{k'(k)} = w_k$, i.e. $w_k = w_{ku}$ (indices mod ℓ) for every u. As u runs through $\mathbb{Z}_\ell^\times$, u mod ℓ runs through $\mathbb{F}_\ell^\times$, so all w_k equal one integer w, and $\det P = vQ^w$.

Step 4: the level map at τ_a. Let $\lambda = \varphi^{-1}\circ\iota_L$ on $\mathbf{A}_{F,L}$ and $\Lambda = \varphi^{-1}\circ\iota_T$ on $\mathbf{N}$ (φ is bijective on A_{inf}). Since $\theta(\varphi^{-1}(\mu)) = \zeta_\ell - 1 \neq 0$ for the primitive root $\zeta_\ell = (\varepsilon^{1/\ell})^\sharp$, $\varphi^{-1}(\mu)$ is a unit of B^+_{dR} (Ai3), so λ takes values in $B^+_{\mathrm{dR}} \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$ and Λ in $B^+_{\mathrm{dR}} \otimes_{\mathbb{Z}_\ell} T = B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V$; by (Ex2), $\Lambda(fx) = \lambda(f)\Lambda(x)$ and $\Lambda(\varphi x) = \iota_T(x)$ for $f \in \mathbf{A}_{F,L}$, $x \in \mathbf{N}$. Further $\lambda(Q) = \varphi^{-1}(\sum_{k<\ell}[\varepsilon]^k) = \sum_{k<\ell}[\varepsilon^{1/\ell}]^k = \xi_0$, a uniformiser of B^+_{dR} with $\xi_0 B^+_{\mathrm{dR}} = tB^+_{\mathrm{dR}}$ (Ai1, Ai3). Let $\epsilon = \lambda(e_a) = \iota_\Lambda(\varphi^{-1}(e_a)) = \epsilon_{a-1}$. For $y \in \mathcal{O}_F$, $\iota(y) = \varphi(y)$ in $W(k) \subset A_{\mathrm{inf}}$ and $e_{a-1}(y\otimes 1) = e_{a-1}(1\otimes\tau_{a-1}(y))$; so on the ϵ-component $\varphi(y)\otimes 1$ acts as $1\otimes\tau_{a-1}(y)$, i.e. $F \subset B_{\mathrm{dR}}$ acts through $\tau_{a-1}\circ\varphi^{-1} = \tau_a$. Hence, for $D = D_{\mathrm{cris}}(V) = D_{\mathrm{dR}}(V)$ (V crystalline), $\epsilon D = D_{\mathrm{dR},\tau_a}(V)$ with its filtration $\epsilon\mathrm{Fil}^j D$. Put $R^+ = \epsilon(B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} L)$, on which ξ_0 is a non-zero-divisor and not a unit.

(i) $(\Lambda(e_m))_m$ is an R^+-basis of $\epsilon(B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V)$. Apply φ^{-1} to $A_{\mathrm{inf}}[1/\mu]\cdot\iota_T(\mathbf{N}) = A_{\mathrm{inf}}[1/\mu]\otimes T$ (Ex2) and extend scalars to B^+_{dR}: $B^+_{\mathrm{dR}} \otimes T$ is spanned over $B^+_{\mathrm{dR}} \otimes \mathcal{O}_L$ by Λ of an $\mathbf{A}_{F,L}$-basis of $\mathbf{N} = \bigoplus_b \mathbf{N}_b$; multiplying by $\epsilon = \lambda(e_a)$ kills $\Lambda(\mathbf{N}_b)$ for $b \neq a$ and fixes $\Lambda(e_m)$. So the r vectors $\Lambda(e_m)$ span $\epsilon(B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V) = R^+ \otimes_L V$, free of rank r over R^+; a surjective endomorphism of a finitely generated module is bijective, so they form a basis.

(ii) $(\Lambda(\varphi e'_k))_k$ is an R^+-basis of $\Lambda_D = \bigoplus_i R^+ D_i^{\mathrm{ad}}$, where (D_i^{ad}) is an L-basis of ϵD adapted to the filtration: $D_i^{\mathrm{ad}} \in \epsilon\mathrm{Fil}^{j_i}D$ and $\epsilon\mathrm{Fil}^j D$ is spanned by the D_i^{ad} with $j_i \geq j$. Let H'_k be the horizontal section through e'_k and $D_k = G^{(0)}(H'_k)$, which equals $\varphi^{-3}(\mathrm{Ev}(H'_k))$ by Lemma 23.13 (1); by Lemma 23.10 (2) the $\mathrm{Ev}(H'_k)$ form an L-basis of $\epsilon_{a+2}D$, so the D_k form an L-basis of ϵD (φ maps $\epsilon_b D$ bijectively onto $\epsilon_{b+1}D$), and $\sum_k R^+ D_k = \Lambda_D$. Writing $H'_k = \sum_j h_{kj}e'_j$ with $h \equiv I \bmod X$ ((Hf2)), $D_k = \sum_j \mathrm{ev}^{(0)}(h_{kj})\,\iota_T(e'_j)$, and $\mathrm{ev}^{(0)}(h) \equiv I$ modulo ξ_0 because $\mathrm{ev}^{(0)}(X) = \mu$ and $\theta(\mu) = 0$; so $\mathrm{ev}^{(0)}(h)$ is invertible over $B^+_{\mathrm{dR}} \otimes \mathcal{O}_L$ and the $\iota_T(e'_j) = \Lambda(\varphi e'_j)$ are an R^+-basis of Λ_D (the D_k are one, being L-independent in $R^+ \otimes_L \epsilon D$).

(iii) $\epsilon(B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V) = \bigoplus_i t^{-j_i} R^+ D_i^{\mathrm{ad}}$. By (Ai4) with $k=0$ and the ϵ-component ($\epsilon(B^+_{\mathrm{dR}} \otimes_F \mathrm{Fil}^j D) = R^+ \otimes_L \epsilon\mathrm{Fil}^j D$): $\epsilon(B^+_{\mathrm{dR}} \otimes V) = \sum_j t^{-j} R^+ \otimes_L \epsilon\mathrm{Fil}^j D = \bigoplus_i (\sum_{j \le j_i} t^{-j} R^+) D_i^{\mathrm{ad}} = \bigoplus_i t^{-j_i} R^+ D_i^{\mathrm{ad}}$, the sum being direct since the D_i^{ad} are independent over $R^+[1/t]$ ((Ai4)).

Step 5: $w = rM + \sum_j j\, m_j$**.** Apply Λ to $Q^M \varphi(e'_k) = \sum_m P_{mk} e_m$:

$$\xi_0^M \Lambda(\varphi e'_k) = \sum_m \lambda(P_{mk}) \Lambda(e_m).$$

By (ii) and (iii), the basis $(\Lambda(\varphi e'_k))$ is $C \cdot (D_i^{\mathrm{ad}})$ with $C \in \mathrm{GL}_r(R^+)$, $(D_i^{\mathrm{ad}}) = \mathrm{diag}(t^{j_i}) \cdot (t^{-j_i} D_i^{\mathrm{ad}})$, and $(t^{-j_i} D_i^{\mathrm{ad}}) = C' \cdot (\Lambda(e_m))$ with $C' \in \mathrm{GL}_r(R^+)$ by (i), (iii). Comparing coefficients in the basis $(\Lambda(e_m))$ and taking determinants,

$$\lambda(\det P) = \det \lambda(P) = \xi_0^{rM}\, t^{\sum_i j_i} \cdot (\text{unit of } R^+) = \xi_0^{\,rM + \sum_i j_i} \cdot (\text{unit of } R^+).$$

On the other hand $\lambda(\det P) = \lambda(v) \xi_0^w$ with $\lambda(v)$ a unit. As ξ_0 is a non-zero-divisor and not a unit of R^+, $w = rM + \sum_i j_i$. Finally $\sum_i j_i = \sum_j j \, \dim_L(\epsilon\mathrm{Fil}^j D / \epsilon\mathrm{Fil}^{j+1} D) = \sum_j j\, m_j$ (the number of i with $j_i = j$ is that dimension), and $\sum_j m_j = r$. □

Proposition C.71. *Let ℓ, F, n, L, $(\tau_a)_{a \in \mathbb{Z}/n}$, $\Phi\colon V_1 \to V_2$ (with lattices T_1, T_2) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.52, and let $c \in \mathbb{Z}/n$ with $\epsilon_c = -1$. Put $M = s_c + S$, $k = (\ell-1)\lceil M/(\ell-1)\rceil$, let E be the integer of Proposition C.72 for the up site $c-1$, and*

$$\kappa = s_\ell(k-1) + E + \sigma_{\mathrm{G}} + s_{c-2} + S - s_{c-1}.$$

Let $x \in \mathbf{N}(T_2)_{c-3}$ be Δ-invariant with $\varphi^{-2}(\mathrm{Cv}_{T_2}(x)) \in \mathrm{Fil}^{s_{c-1}} D_{\mathrm{cris}}(V_2)$. Then there is a Δ-invariant $\tilde{y} \in \mathbf{N}(T_1)_{c-1}$ such that $y = \varphi(\tilde{y})$ satisfies $Q^M y \in \mathbf{N}(T_1)_c$ and $(\gamma_u - 1)(y) \in X\mathbf{N}(T_1)$ for every $u \in \mathbb{Z}_\ell^\times$, and such that, for arc level data of (V_2, T_2) with cut c, length 1, $H = 0$ and any frame $(n_j)_{j \le r}$ at $c-1$, $\mathbf{N}(\Phi)(y) \in W$ and

$$\mathrm{lv}_0(\mathbf{N}(\Phi)(y)) - \ell^\kappa a \in t^M K_{(0)}[[t]]^r,$$

where $a \in L^r$ is given by $\sum_j a_j\, \varphi^{-3}(\mathrm{Cv}_{T_2}(n_j)) = \varphi^{-1}(\mathrm{Cv}_{T_2}(x))$.

Proof. **Step 0.** As n is even and the signs alternate by parity, $\epsilon_{c-1} = 1$. By Lemma 24.2 (1) and (Ow1), the Hodge jumps of V_1 and V_2 at τ_c lie in $[-M, \eta]$; by (Ex1) and (Wm2), $Q^M \varphi(\mathbf{N}(T_\nu)_{c-1}) \subseteq \mathbf{N}(T_\nu)_c$ ($\nu = 1, 2$).

Step 1: lifting at the up site $c-1$**.** By Proposition C.72 at $i = c-1$ (so $i - 2 = c - 3$), there is a Δ-invariant $y_1 \in \mathbf{N}(T_1)_{c-1}$ with $(1 \otimes \Phi)(\mathrm{Cv}_{T_1}(y_1)) = \ell^{E + \sigma_{\mathrm{G}} + s_{c-2} + S - s_{c-1}} \varphi^2(\mathrm{Cv}_{T_2}(x))$.

Step 2: truncation. Let $s = s_\ell(k-1)$. By (Hf3) there is $y_k \in \mathbf{N}(T_1)_{c-1}$ with $\ell^s H_{y_1} - y_k \in X^k \mathbf{N}(T_1)_E$ ($H_{y_1} \in e_{c-1}\mathbf{N}(T_1)_E$, replace y_k by $e_{c-1} y_k$). Put $\tilde{y} = (\ell-1)^{-1} \sum_{\delta \in \mu_{\ell-1}} \gamma_\delta(y_k)$; as the γ_δ fix H_{y_1} and preserve $X^k \mathbf{N}(T_1)_E$, also $\ell^s H_{y_1} - \tilde{y} \in X^k \mathbf{N}(T_1)_E$, so (Hf3) gives $(\gamma_u - 1)\tilde{y} \in X^k \mathbf{N}(T_1)$ for every u, and $\tilde{y} - \ell^s y_1 \in X\mathbf{N}(T_1)$. The element $\tilde{y}$ is Δ-invariant.

Step 3: $y = \varphi(\tilde{y})$**.** By Step 0, $Q^M y \in \mathbf{N}(T_1)_c$. For $u \in \mathbb{Z}_\ell^\times$, $(\gamma_u - 1)y = \varphi((\gamma_u - 1)\tilde{y}) \in \varphi(X)^k \varphi(\mathbf{N}(T_1)_{c-1}) = X^k Q^{k-M} \cdot Q^M \varphi(\mathbf{N}(T_1)_{c-1}) \subseteq X\mathbf{N}(T_1)$ ((Wm3); $k \ge M \ge 1$). The element y is Δ-invariant.

Step 4: the level-0 **expansion.** The map $\mathbf{N}(\Phi)$ is $\mathbf{A}_{F,L}$-linear and commutes with φ and the γ_u (Ex3); so $z := \mathbf{N}(\Phi)(\tilde{y}) \in \mathbf{N}(T_2)_{c-1}$ has $(\gamma_{u_0} - 1)z \in X^k \mathbf{N}(T_2)_{c-1}$, $z - \ell^s \mathbf{N}(\Phi)(y_1) \in X\mathbf{N}(T_2)$, and $\mathbf{N}(\Phi)(y) = \varphi(z)$. With $\Pi = 1$, $\varphi(z) \in P_1(\mathbf{N}(T_2)_{c-1})$, so $\mathbf{N}(\Phi)(y) \in W$. Take the parameters $j = 1$, $k/(\ell-1)$ in the role of M, $g = 0$ (so $(\ell-1) \cdot k/(\ell-1) = k \ge 3g$). Write

$z-\sum_j a'_j n_j \in X\mathbf{N}(T_2)_{c-1}$ with $a' \in \mathcal{O}_L^r$. By Lemma C.68 ((Lp6)), $\mathrm{lv}_0(\varphi(z))-a' \in t^k K_{(0)}[[t]]^r \subseteq t^M K_{(0)}[[t]]^r$. The level frame is $\varphi^{-3}(\mathrm{Cv}_{T_2}(n_j))$ (length 1), and by (Cv1), (Cv2), (Cv5),

$$\sum_j a'_j \varphi^{-3}(\mathrm{Cv}(n_j)) = \varphi^{-3}(\mathrm{Cv}(z)) = \ell^s \varphi^{-3}((1\otimes\Phi)\mathrm{Cv}_{T_1}(y_1)) = \ell^\kappa \varphi^{-1}(\mathrm{Cv}_{T_2}(x)),$$

so $a' = \ell^\kappa a$, the level frame being an L-basis of $\epsilon_{c-1}D_{\mathrm{cris}}(V_2)$, which contains $\varphi^{-1}(\mathrm{Cv}(x))$ ($\mathrm{Cv}(x) \in \epsilon_c D$, Lemma C.64 (Cv1)). □

Proposition C.72. *Let ℓ, F, n, L, $(\tau_a)_{a\in\mathbb{Z}/n}$, $\Phi\colon V_1 \to V_2$ (with lattices T_1, T_2) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.52, let $i \in \mathbb{Z}/n$ with $\epsilon_i = 1$, and put $\sigma_{\mathrm{G}} = \lceil 2H'/(\ell-1)^2 \rceil + 1$. There is an integer E with*

$$0 \le E \le \frac{3a_{\mathrm{R}} + S + 2\eta}{\ell - 1} + \log_\ell(2H') + 1$$

such that for every Δ-invariant $y \in \mathbf{N}(T_2)_{i-2}$ with $\varphi^{-2}\big(\mathrm{Cv}_{T_2}(y)\big) \in \mathrm{Fil}^{s_i} D_{\mathrm{cris}}(V_2)$ there is a Δ-invariant $y_1 \in \mathbf{N}(T_1)_i$ with

$$(1\otimes\Phi)\big(\mathrm{Cv}_{T_1}(y_1)\big) = \ell^{\,E+\sigma_{\mathrm{G}}+s_{i-1}+S-s_i}\,\varphi^2\big(\mathrm{Cv}_{T_2}(y)\big).$$

Proof. Let E be the integer given by Lemma C.45 for the site i; it satisfies the stated bound. Let y be as in the statement. By Lemma C.73 there are Δ-invariant $x, z \in \mathbf{N}(T_2)_i$ and $x' \in e_i\mathbf{N}(T_2)[1/\lambda_\infty]$ with $(\pi_0+\ell)^{s_i}x = \varphi(\pi_0+\ell)^{s_{i-1}+S}x'$, $\varphi(\pi_0+\ell)^{s_{i-1}+S}(\gamma_{u_0}-1)(x') = \pi_0(\pi_0+\ell)^{a+s_i}z$ and $\mathrm{Cv}_{T_2}(x) = \ell^{\sigma_{\mathrm{G}}+s_{i-1}+S-s_i}\varphi^2(\mathrm{Cv}_{T_2}(y))$. These are the hypotheses of Lemma C.45 on (x, x', z), which therefore gives a Δ-invariant $y_1 \in \mathbf{N}(T_1)_i$ with $\mathbf{N}(\Phi)(y_1) = \ell^E x$. By Lemma C.64 (Cv5) and (Cv1) ($\mathcal{O}_L$-linearity),

$$(1\otimes\Phi)\big(\mathrm{Cv}_{T_1}(y_1)\big) = \mathrm{Cv}_{T_2}\big(\mathbf{N}(\Phi)(y_1)\big) = \ell^E\,\mathrm{Cv}_{T_2}(x) = \ell^{\,E+\sigma_{\mathrm{G}}+s_{i-1}+S-s_i}\,\varphi^2\big(\mathrm{Cv}_{T_2}(y)\big).$$

□

Lemma C.73. *Let ℓ, F, n, L, $(\tau_a)_{a\in\mathbb{Z}/n}$, $\Phi\colon V_1 \to V_2$ (with lattices T_1, T_2) and the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ satisfy the hypotheses of Proposition C.52, let $i \in \mathbb{Z}/n$ with $\epsilon_i = 1$, form π_0 and u_0 as in Lemma 28.15, and put*

$$a = \lceil a_{\mathrm{R}} \rceil, \qquad \sigma_{\mathrm{G}} = \Big\lceil \frac{2H'}{(\ell-1)^2} \Big\rceil + 1.$$

Let $y \in \mathbf{N}(T_2)_{i-2}$ be Δ-invariant with $\varphi^{-2}\big(\mathrm{Cv}_{T_2}(y)\big) \in \mathrm{Fil}^{s_i} D_{\mathrm{cris}}(V_2)$. Then there are Δ-invariant $x, z \in \mathbf{N}(T_2)_i$ and $x' \in e_i\mathbf{N}(T_2)[1/\lambda_\infty]$ with

$$(\pi_0+\ell)^{s_i}x = \varphi(\pi_0+\ell)^{\,s_{i-1}+S}x', \qquad \varphi(\pi_0+\ell)^{\,s_{i-1}+S}(\gamma_{u_0}-1)(x') = \pi_0(\pi_0+\ell)^{a+s_i}z,$$

$$\mathrm{Cv}_{T_2}(x) = \ell^{\,\sigma_{\mathrm{G}}+s_{i-1}+S-s_i}\,\varphi^2\big(\mathrm{Cv}_{T_2}(y)\big).$$

Here the exponent may be negative; $D_{\mathrm{cris}}(V_2)$ is an L-vector space.

Proof. Write $\varsigma = \pi_0 + \ell = Qw_0$ (w_0 a unit of $\mathbb{Z}_\ell[[X]]$, ς Δ-invariant, Lemma 28.15 (1)), $h = 2H'$, $h' = s_{i-1} + S$, $\mathbf{N} = \mathbf{N}(T_2)$, $D = D_{\mathrm{cris}}(V_2)$.

Step 0: signs and windows. As n is even and the signs alternate by parity, $\epsilon_{i-1} = -1$ and $\epsilon_{i-2} = 1$. By Lemma 24.2 (1) and (Ow1), every Hodge jump of V_2 at τ_a lies in $[-(s_a+S), \eta]$

if $\epsilon_a = -1$ and in $[-\eta, s_a + S]$ if $\epsilon_a = 1$; so by (Ex1) $\mathbf{N}$ is a Wach module with $M_{i-1} = h'$, $M_i = \eta$, and (Wm2) gives $Q^{h'}\varphi(\mathbf{N}_{i-2}) \subseteq \mathbf{N}_{i-1}$ and $Q^{\eta}\varphi(\mathbf{N}_{i-1}) \subseteq \mathbf{N}_i$. Hence $\varphi(\varsigma)^{h'}P_2(\mathbf{N}_{i-2}) \subseteq \mathbf{A}_{F,L}\varphi(\mathbf{N}_{i-1})$ and

$$\varsigma^{\eta}\varphi(\varsigma)^{h'}P_2(\mathbf{N}_{i-2}) \subseteq \mathbf{N}_i. \qquad \text{(W)}$$

Numbers: $s_i + \eta \le H'$; $a = \lceil a_{\mathrm{R}} \rceil < a_{\mathrm{R}} + 1 \le (s_i - \eta + 1)/2 \le H'$ (as $s_i \ge 2a_{\mathrm{R}} + \eta + 1$); so $h - a - s_i \ge \eta$ and $h - s_i \ge \eta$. Also $\sigma_{\mathrm{G}} = \lceil h/(\ell-1)^2 \rceil + 1$.

Step 1: the truncation. By Lemma C.74 (pair (V_2, T_2), site i, this h, $\sigma = \sigma_{\mathrm{G}}$) there is a Δ-invariant $z_0 \in P_2(\mathbf{N}_{i-2})$ with (Ft1)–(Ft3). Put $x' = z_0 \in e_i\mathbf{N}[1/\lambda_2] \subseteq e_i\mathbf{N}[1/\lambda_\infty]$ and $x = \varsigma^{-s_i}\varphi(\varsigma)^{h'}z_0 \in e_i\mathbf{N}[1/\lambda_2]$ (ς is invertible in $\mathbf{A}_{F,L}[1/Q]$). Then $\varsigma^{s_i}x = \varphi(\varsigma)^{h'}x'$, and x is Δ-invariant.

Step 2: $x \in \mathbf{N}_i$. Fix arc level data for (V_2, T_2) with cut i, length 2 ($c_0 = i-2$), $H(i) = s_i$, $H(i-1) = 0$, so $\Pi = Q^{s_i}$, a frame (e'_m) at $i-2$, and any parameters. Then $\Pi x = w_0^{-s_i}\varphi(\varsigma)^{h'}z_0 \in P_2(\mathbf{N}_{i-2})$ and $\Pi z_0 \in P_2(\mathbf{N}_{i-2})$, so $x, z_0 \in W$. The level frame is $D_m^{(s)} = \varphi^{-2-s}(\mathrm{Cv}(e'_m))$; $\epsilon_0 = \epsilon_{i-1}$ is the idempotent of τ_i and ϵ_1 that of τ_{i-1} (Definition 23.4), so Fil_0, Fil_1 read the Hodge filtrations of V_2 at τ_i, τ_{i-1}. By Lemma 23.17 ((Lp1)): $\lambda_0(Q) \in tK_{(0)}[[t]]^\times$, $\lambda_0(Q_1) \in K_{(0)}[[t]]^\times$, $\lambda_1(Q) \in K_{(1)}[[t]]^\times$, $\lambda_1(Q_1) \in tK_{(1)}[[t]]^\times$, and λ_s maps units of $e_i\mathbf{A}_{F,L}$ to units of $K_{(s)}[[t]]$; as $\varphi(\varsigma) = Q_1\varphi(w_0)$, $\lambda_0(\varsigma) = t\vartheta_0$, $\lambda_0(\varphi\varsigma) = \vartheta'_0$, $\lambda_1(\varsigma) = \vartheta_1$, $\lambda_1(\varphi\varsigma) = t\vartheta'_1$ with units ϑ.

Level 0. Let $a_y \in L^r$ with $\sum_m a_{y,m}D_m^{(0)} = \varphi^{-2}(\mathrm{Cv}(y))$. By (Ft3) (with $\Lambda_D = \bigoplus_m R_0^+ D_m^{(0)}$) and uniqueness of coordinates in the R_0-basis $(D_m^{(0)})$, $\mathrm{lv}_0(z_0) - \ell^{\sigma_{\mathrm{G}}}a_y \in K_{(0)}((t))^r \cap t^h(R_0^+)^r = t^hK_{(0)}[[t]]^r$ (if $f = \sum_{j \ge j_0} f_j t^j$ with $f_{j_0} \ne 0$, $j_0 < h$, lay in $t^hR_0^+$, then $\theta(t^{-j_0}f) = \theta(f_{j_0}) = 0$, contradicting injectivity of θ on $K_{(0)}$). Hence

$$\mathrm{lv}_0(x) = t^{-s_i}\vartheta_0^{-s_i}\vartheta_0'^{h'}\big(\ell^{\sigma_{\mathrm{G}}}a_y + t^h g\big), \qquad g \in K_{(0)}[[t]]^r.$$

The hypothesis $\varphi^{-2}(\mathrm{Cv}(y)) \in \mathrm{Fil}^{s_i}D$ says $a_y \in \mathrm{Fil}_0(s_i)$. The coefficient of t^j coming from a_y vanishes for $j < -s_i$ and is a $K_{(0)}$-multiple of $a_y \in \mathrm{Fil}_0(s_i) \subseteq \mathrm{Fil}_0(-j)$ for $j \ge -s_i$. The coefficient coming from $t^h g$ vanishes for $j < h - s_i$, and for $j \ge h - s_i \ge \eta$, $\mathrm{Fil}_0(-j) = L^r$, every jump at τ_i being $\ge -\eta$. So $\mathrm{lv}_0(x)$ is de Rham integral at 0.

Level 1. We have $\mathrm{lv}_1(x) = \vartheta_1^{-s_i}t^{h'}\vartheta_1'^{h'}\mathrm{lv}_1(z_0)$, and $\mathrm{lv}_1(z_0) \in K_{(1)}[[t]]^r$ by (Lp4) ($\lambda_1(\Pi)$ is a unit). Its coefficients of t^j vanish for $j < h'$, and for $j \ge h' = s_{i-1} + S$, $\mathrm{Fil}_1(-j) = L^r$, every jump at τ_{i-1} being $\ge -(s_{i-1} + S)$. So $\mathrm{lv}_1(x)$ is de Rham integral at 1, and by Lemma 23.5 (2) ((Lp3)), $x \in \mathbf{N}_i$.

Step 3: the Γ-defect. By (Ft1), $(\gamma_{u_0} - 1)(x') = \pi_0\varsigma^h q$ with $q \in P_2(\mathbf{N}_{i-2})$, so $\varphi(\varsigma)^{h'}(\gamma_{u_0} - 1)(x') = \pi_0\varsigma^{a+s_i}z$ with $z = \varsigma^{h-a-s_i-\eta} \cdot \varsigma^{\eta}\varphi(\varsigma)^{h'}q \in \mathbf{N}_i$ by (W) and $h - a - s_i \ge \eta$. The element z is Δ-invariant: $(\gamma_{u_0} - 1)x'$ is (γ_{u_0} commutes with the γ_δ), and z is its product with $\varphi(\varsigma)^{h'}(\pi_0\varsigma^{a+s_i})^{-1}$, a quotient of Δ-invariant elements, in the module $e_i\mathbf{N}[1/\lambda_2]$, free over a domain.

Step 4: the crystalline value. By (Ft2), $z_0 = \ell^{\sigma_{\mathrm{G}}}\varphi^2(y) + Xp$ with $p \in P_2(\mathbf{N}_{i-2})$. Put $u_1 = \varsigma^{h'}\varphi(y) \in \mathbf{N}_{i-1}$ (Step 0). Then

$$\varsigma^{s_i+\eta}x = \varsigma^{\eta}\varphi(\varsigma)^{h'}z_0 = \ell^{\sigma_{\mathrm{G}}}\varsigma^{\eta}\varphi(u_1) + X \cdot \varsigma^{\eta}\varphi(\varsigma)^{h'}p,$$

with $\varsigma^{\eta}\varphi(u_1) \in \mathbf{N}_i$ (Step 0) and $\varsigma^{\eta}\varphi(\varsigma)^{h'}p \in \mathbf{N}_i$ by (W). Apply Cv, using Lemma C.64 (Cv1), (Cv2) ($\mathrm{Cv}(X\mathbf{N}) = 0$, $\varsigma(0) = \ell$) and (Cv4) twice (with $\kappa = \varsigma^{\eta}$ at i and $\kappa = \varsigma^{h'}$ at $i-1$; $\varphi^3 \circ \iota_\Lambda$ fixes ℓ): $\ell^{s_i+\eta}\mathrm{Cv}(x) = \ell^{\sigma_{\mathrm{G}}}\ell^{\eta}\varphi(\mathrm{Cv}(u_1)) = \ell^{\sigma_{\mathrm{G}}+\eta+h'}\varphi^2(\mathrm{Cv}(y))$. Hence $\mathrm{Cv}(x) = \ell^{\sigma_{\mathrm{G}}+s_{i-1}+S-s_i}\varphi^2(\mathrm{Cv}(y))$ in the L-vector space D. □

Lemma C.74. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ and a pair (V,T) be as in Proposition 23.7, $\mathbf{N} = \mathbf{N}(T)$, $D = D_{\mathrm{cris}}(V)$, $i \in \mathbb{Z}/n$, $P_2 = P_2(\mathbf{N}_{i-2})$, $\varsigma = \pi_0 + \ell$, and let $h \ge 1$ and $\sigma = \lceil h/(\ell-1)^2 \rceil + 1$. Let*

$\Lambda_0 = \varphi^{-1} \circ \iota_T$ *on* $\mathbf{N}[1/\lambda_2]$ *(values in* $B_{\mathrm{dR}} \otimes_{\mathbb{Z}_\ell} T$*) and* $\Lambda_D = \epsilon_{i-1}(B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} L) \cdot \epsilon_{i-1} D$*. For every* Δ*-invariant* $y \in \mathbf{N}_{i-2}$ *there is a* Δ*-invariant* $z \in P_2$ *with*

(Ft1) $(\gamma_u - 1)(z) \in \pi_0 \varsigma^h P_2$ *for every* $u \in \mathbb{Z}_\ell^\times$*;*

(Ft2) $z - \ell^\sigma \varphi^2(y) \in X P_2$*;*

(Ft3) $\Lambda_0(z) - \ell^\sigma \varphi^{-2}\big(\mathrm{Cv}(y)\big) \in t^h \Lambda_D$*.*

Proof. Throughout $\gamma = \gamma_u$ for a fixed $u \in \mathbb{Z}_\ell^\times$; $\gamma(X) = X\eta_u$ with η_u a unit of $\mathbb{Z}_\ell[[X]]$, so $\gamma(\pi_0)$ is π_0 times a unit ($\pi_0 = X^{\ell-1}v$, Lemma 28.15 (1)); $\varsigma = Qw_0$ with w_0 a unit, and $\gamma(Q) = Q\varphi(\eta_u)\eta_u^{-1}$ (from $Q = \varphi(X)/X$), so $\gamma(\pi_0\varsigma^h) = \pi_0\varsigma^h u_\gamma$ with u_γ a unit of $\mathbb{Z}_\ell[[X]]$.

Step 1: P_2 is free on φ^2 of a basis. By Lemma 24.15, $\mathbf{N}_{i-2}$ has a Δ-invariant basis $(e'_m)_{m \le r}$ over $e_{i-2}\mathbf{A}_{F,L}$. Since φ is semilinear, $P_2 = \sum_m R\,\varphi^2(e'_m)$ with $R = e_i\mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$. By (Wm2) at $i-1$ and i, the $R[1/\lambda_2]$-span of the $\varphi^2(e'_m)$ is $e_i\mathbf{N}[1/\lambda_2]$, free of rank r over the domain $R[1/\lambda_2]$; r generators of a free module of rank r form a basis, so the $\varphi^2(e'_m)$ are R-independent and $P_2 = \bigoplus_m R\varphi^2(e'_m)$. The map γ preserves P_2 ($\gamma(g\varphi^2(n)) = \gamma(g)\varphi^2(\gamma n)$, (Wm3)).

Step 2: the π_0-expansion of H_y. The section $H_y \in e_{i-2}\mathbf{N}_E$ is Δ-invariant ($\gamma_\delta H_y$ is horizontal through $\gamma_\delta y = y$). Write $H_y = \sum_m f_m e'_m$; the $f_m \in e_{i-2}E[[X]] \cong L[[X]]$ are Δ-invariant, and by (Hf3) the coefficient $a_{m,j}$ of X^j satisfies $\ell^{s_\ell(j)} a_{m,j} \in \mathcal{O}_L$. A Δ-invariant $f \in XL[[X]]$ is π_0 times a Δ-invariant series (lowest-degree argument: the lowest term aX^j of f becomes $a\delta^j X^j$ under γ_δ, so $(\ell-1) \mid j$). Hence, recursively, $f_m = \sum_{k<K} b_{mk}\pi_0^k + \pi_0^K f_{m,K}$ for every K, with constants $b_{mk} \in L$ and Δ-invariant $f_{m,K} \in L[[X]]$, and $b_{mk} = v(0)^{-k}\big(a_{m,(\ell-1)k} - \sum_{k'<k} b_{mk'}\,[X^{(\ell-1)k}]\pi_0^{k'}\big)$, where $[X^j]$ denotes the coefficient and the $[X^j]\pi_0^{k'}$ lie in $\mathbb{Z}_\ell$. By induction on k and monotonicity of s_ℓ, $\ell^{s_\ell((\ell-1)k)} b_{mk'} \in \mathcal{O}_L$ for $k' \le k$; and $s_\ell((\ell-1)k) = k + v_\ell(k!) \le k + (k-1)/(\ell-1) \le e(k)$ ($k \ge 1$; both are 0 for $k = 0$). Put $\hat{c}_k = \sum_m b_{mk} e'_m$, $c_k = \ell^{e(k)}\hat{c}_k \in \mathbf{N}_{i-2}$ (Δ-invariant) and $R_K = \sum_m f_{m,K} e'_m \in e_{i-2}\mathbf{N}_E$, so that

$$H_y = \sum_{k<K} \pi_0^k \hat{c}_k + \pi_0^K R_K \qquad (K \ge 0), \qquad \hat{c}_0 - y \in \mathbf{N} \cap X\mathbf{N}_E = X\mathbf{N}.$$

Step 3: the vector z. By Lemma C.63 choose, for each $k \ge 0$, $\alpha_k, \beta_k \in \mathcal{R}$ with $\ell^\sigma \varphi^2(\pi_0)^k = \ell^{e(k)}\alpha_k + \pi_0\varsigma^h\beta_k$, taking $\alpha_0 = \ell^\sigma$, $\beta_0 = 0$, and, for $(\ell-1)k \ge h$, $\alpha_k = 0$: this is possible because $\varphi(\pi_0) = \varphi(X)^{\ell-1}\varphi(v) = \pi_0\varsigma^{\ell-1}U$ with $U = v^{-1}w_0^{1-\ell}\varphi(v)$, so $\varphi^2(\pi_0) \in \pi_0\varsigma^{\ell-1}\mathcal{R}$ and $\varphi^2(\pi_0)^k \in \pi_0\varsigma^{(\ell-1)k}\mathcal{R}$ for $k \ge 1$ (the quotient lies in $\mathcal{R}$: φ commutes with every γ_δ on $\mathbb{Z}_\ell[[X]]$, as $\varphi\gamma_\delta(X) = (1+X)^{\ell\delta} - 1 = \gamma_\delta\varphi(X)$, so $\varphi(\mathcal{R}) \subseteq \mathcal{R}$ and $\varphi^2(\pi_0)^k \in \mathcal{R}$; and if $f \in \mathbb{Z}_\ell[[X]]$ and $gf \in \mathcal{R}$ for a non-zero $g \in \mathcal{R}$, here $g = \pi_0\varsigma^{(\ell-1)k}$, then $g(\gamma_\delta(f) - f) = \gamma_\delta(gf) - gf = 0$, so $\gamma_\delta(f) = f$ for every $\delta \in \Delta$, $\mathbb{Z}_\ell[[X]]$ being a domain). For $k \ge 1$, evaluating at $X = 0$ ($\pi_0(0) = 0$) gives $\ell^{e(k)}\alpha_k(0) = 0$, so $\alpha_k(0) = 0$. Let $K_0 = \lceil h/(\ell-1) \rceil$ and

$$z = \sum_{k<K_0} \alpha_k\,\varphi^2(c_k) \ \in P_2,$$

Δ-invariant ($\alpha_k \in \mathcal{R}$, φ commutes with the γ_δ). *(Ft2):* $z - \alpha_0\varphi^2(c_0) \in XP_2$ as $\alpha_k \in X\mathbb{Z}_\ell[[X]]$ for $k \ge 1$, and $\alpha_0\varphi^2(c_0) - \ell^\sigma\varphi^2(y) = \ell^\sigma\varphi^2(\hat{c}_0 - y) \in \varphi^2(X)P_2 \subseteq XP_2$.

Step 4: an integral identity. Fix $K \ge K_0$ and put $y_K = \sum_{k<K} \ell^{e(K)-e(k)}\pi_0^k c_k = \ell^{e(K)}\sum_{k<K}\pi_0^k\hat{c}_k \in \mathbf{N}_{i-2}$ (e is non-decreasing). Then

$$\begin{aligned}\ell^\sigma\varphi^2(y_K) &= \sum_{k<K} \ell^{e(K)-e(k)}\big(\ell^{e(k)}\alpha_k + \pi_0\varsigma^h\beta_k\big)\varphi^2(c_k) \\ &= \ell^{e(K)}z + \pi_0\varsigma^h p_K, \qquad p_K = \sum_{k<K} \ell^{e(K)-e(k)}\beta_k\varphi^2(c_k) \in P_2. \qquad \text{(I)}\end{aligned}$$

Step 5: (Ft1). As $\gamma H_y = H_y$, Step 2 gives $(\gamma-1)y_K = -\ell^{e(K)}(\gamma-1)(\pi_0^K R_K) = -\ell^{e(K)}\pi_0^K\big((\gamma\pi_0/\pi_0)^K\gamma(R_K)-R_K\big)$, an element of $\mathbf{N}\cap\pi_0^K\mathbf{N}_E = \mathbf{N}\cap X^{(\ell-1)K}\mathbf{N}_E = X^{(\ell-1)K}\mathbf{N} = \pi_0^K\mathbf{N}$; write $(\gamma-1)y_K = \pi_0^K m_K$, $m_K\in\mathbf{N}_{i-2}$. Apply $\gamma-1$ to (I) (γ commutes with φ):

$$\ell^{e(K)}(\gamma-1)z = \ell^\sigma\varphi^2(\pi_0)^K\varphi^2(m_K) - \pi_0\varsigma^h\big(u_\gamma\gamma(p_K)-p_K\big) = \pi_0\varsigma^h q_K,\qquad q_K\in P_2,$$

since $\varphi^2(\pi_0)^K\in\pi_0\varsigma^{(\ell-1)K}\mathcal{R}\subseteq\pi_0\varsigma^h\mathcal{R}$ ($(\ell-1)K\geq h$). In the basis of Step 1 write $(\gamma-1)z=\sum_m a_m\varphi^2(e'_m)$ and $q_K=\sum_m b_m\varphi^2(e'_m)$ ($a_m,b_m\in R$): $\ell^{e(K)}a_m=\pi_0\varsigma^h b_m$. In the unique factorisation domain $R\cong\mathcal{O}_L[[X]]$ a uniformiser ϖ of $\mathcal{O}_L$ is prime and does not divide $\pi_0\varsigma^h$ (modulo ϖ it is $X^{(\ell-1)(h+1)}$ times a unit, as $\varsigma\equiv\pi_0$ modulo ℓ); so $\ell^{e(K)}\mid b_m$, $a_m\in\pi_0\varsigma^h R$, and $(\gamma-1)z\in\pi_0\varsigma^h P_2$.

Step 6: (Ft3). Put $\epsilon=\epsilon_{i-1}$, $R^+=\epsilon(B^+_{\mathrm{dR}}\otimes L)$. For $n\in\mathbf{N}_{i-2}$, (Ex2) gives $\Lambda_0(\varphi^2 n)=\varphi(\iota_T(n))=G^{(1)}(n)$ (with $\mathrm{ev}^{(1)}=\varphi\circ\iota_L$ on $\mathbf{A}_{F,L}$, both being continuous ring maps equal on Λ and on X), and $\Lambda_0(gw)=\lambda_0(g)\Lambda_0(w)$ with $\lambda_0=\varphi^{-1}\circ\iota_L$. Let (H'_m) be the horizontal sections through (e'_m); by Lemma 23.13 (1), $G^{(1)}(H'_m)=\varphi^{-2}(\mathrm{Cv}(e'_m))=:D_m$, an L-basis of ϵD (Lemma C.64 (Cv3), φ bijective between components), so $\Lambda_D=\bigoplus_m R^+D_m$. As $e'_m=\sum_k(h^{-1})_{mk}H'_k$ with $h^{-1}\in\mathrm{GL}_r(E[[X]])$ and $\mathrm{ev}^{(1)}(E[[X]])\subseteq B^+_{\mathrm{dR}}\otimes\mathcal{O}_L$, $G^{(1)}(W)\in\Lambda_D$ for every $W\in e_{i-2}\mathbf{N}_E$, and $\Lambda_0(P_2)\subseteq\Lambda_D$. Moreover, $\theta(\mathrm{ev}^{(1)}(X))=\theta(\varphi(\mu))=0$, so $\mathrm{ev}^{(1)}(\pi_0)\in t^{\ell-1}B^+_{\mathrm{dR}}$; $\lambda_0(\varsigma)=\lambda_0(Q)\lambda_0(w_0)$ with $\lambda_0(Q)=\varphi^{-1}(\sum_{k<\ell}[\varepsilon]^k)=\xi_0\in tB^+_{\mathrm{dR}}$ (Theorem B.40 (Ai3)); and $H_y=\sum_m y_m(0)H'_m$ for $y=\sum_m y_m e'_m$ (both sides horizontal through elements congruent modulo X), so $G^{(1)}(H_y)=\sum_m\varphi(\iota_\Lambda(y_m(0)))D_m=\varphi^{-2}(\mathrm{Cv}(y))$ (Lemma C.64 (Cv2)). Apply Λ_0 to (I):

$$\ell^{e(K)}\Lambda_0(z)=\ell^\sigma G^{(1)}(y_K)-\lambda_0(\pi_0)\lambda_0(\varsigma)^h\Lambda_0(p_K),$$
$$G^{(1)}(y_K)=\ell^{e(K)}\big(G^{(1)}(H_y)-\mathrm{ev}^{(1)}(\pi_0)^K G^{(1)}(R_K)\big).$$

The last term lies in $t^{(\ell-1)K}\Lambda_D\subseteq t^h\Lambda_D$ and $\lambda_0(\pi_0)\lambda_0(\varsigma)^h\Lambda_0(p_K)\in t^h\Lambda_D$. Dividing by $\ell^{e(K)}$ (a unit of B^+_{dR}): $\Lambda_0(z)-\ell^\sigma\varphi^{-2}(\mathrm{Cv}(y))\in t^h\Lambda_D$. □

Proposition C.75. *There is an absolute constant $c_{4,2}\geq 1$ with the following property. Let ℓ, F, n, L, $(\tau_a)_{a\in\mathbb{Z}/n}$, $\Phi\colon V_1\to V_2$ (with lattices T_1,T_2) and the weight word $(b,w,S,\eta,m,(s_a),(\epsilon_a))$ satisfy the hypotheses of Proposition C.52, form B and λ as in Proposition C.57, and let $c\in\mathbb{Z}/n$ with $\epsilon_c=-1$. Then there is an integer $0\leq e_6\leq c_{4,2}B$ with the following property. Let P be a free $\mathcal{O}_L$-module of finite rank, $P'\subseteq P$ a submodule and $w\colon P'\to\mathbf{N}(T_2)_c$ an $\mathcal{O}_L$-linear map with $\gamma_\delta(w(f))=w(f)$ for all $f\in P'$ and $\delta\in\mu_{\ell-1}\subseteq\mathbb{Z}_\ell^\times$. Then there are submodules $P'=P'_0\supseteq P'_1\supseteq\cdots\supseteq P'_k$ and a set $J\subseteq\{0,\dots,k-1\}$ with $|J|\leq 3$ and $k-|J|\leq 2$, such that P'_i/P'_{i+1} is generated by at most m elements for $i\in J$ and $\mathrm{nl}(P'_i/P'_{i+1})\leq\lambda$ for $i\notin J$, and such that for every $f\in P'_k$ there are $x\in I_c^1$ and a finite product λ' of the Q_s ($s\geq 0$) with $\lambda' x\in\mathbf{N}(T_1)$ and*

$$\ell^{e_6}w(f)-\mathbf{N}(\Phi)(x)\in X\,\mathbf{N}(T_2)_c.$$

Proof. Put $M=s_c+S$, $b'=s_{c-1}+S$, $h''=s_{c-2}+S$, $\varsigma=\pi_0+\ell=Qw_0$ (w_0 a unit, $\varsigma(0)=\ell$), $D=D_{\mathrm{cris}}(V_2)$, $\mathbf{N}=\mathbf{N}(T_2)$, and $e_6=E+\sigma_{\mathrm{G}}+S$ with E the integer of Proposition C.72 for the up site $c-1$.

Step 0: signs, windows, three facts. As n is even and the signs alternate, $\epsilon_{c-1}=1$, $\epsilon_{c-2}=-1$. By Lemma 24.2 (1), (Ow1) and (Ex1), $\mathbf{N}(T_\nu)$ ($\nu=1,2$) is a Wach module with window (s_a+S,η) at down sites and (η,s_a+S) at up sites; so (Wm2) gives $Q^M\varphi(\mathbf{N}(T_\nu)_{c-1})\subseteq\mathbf{N}(T_\nu)_c$, $Q^{b'}\mathbf{N}_{c-1}\subseteq\mathbf{A}_{F,L}\varphi(\mathbf{N}_{c-2})$, $Q^{h''}\varphi(\mathbf{N}_{c-3})\subseteq\mathbf{N}_{c-2}$. (F1) For a site a, $\{x\in\mathbf{N}_a^\Delta:\mathrm{Cv}(x)=0\}=\pi_0\mathbf{N}_a^\Delta$: by (Cv3), $\mathrm{Cv}(x)=0$ iff $x\in X\mathbf{N}_a$, and in a Δ-invariant basis (Lemma 24.15)

the Δ-invariant elements of $X\mathbf{N}_a$ are those with coordinates in $(XR)^\Delta = \pi_0 R^\Delta$ (lowest-degree argument, $R = e_a\mathbf{A}_{F,L}$). (F2) If a is a down site and $y \in Y_a$, then $\mathrm{Cv}(y) = \ell^{s_a+S}\varphi(\mathrm{Cv}(y'))$ for some $y' \in \mathbf{N}^\Delta_{a-1}$: write $y = \sum_j o_j \varsigma^{s_a+S} g_j \varphi(x_j)$ ($o_j \in \mathcal{O}_L$, g_j, x_j Δ-invariant); each term lies in $\mathbf{N}_a$, and by (Cv4) and (Cv1),

$$\mathrm{Cv}(\varsigma^{s_a+S} g_j \varphi(x_j)) = \ell^{s_a+S}\varphi^3(\iota_\Lambda(g_j(0)))\varphi(\mathrm{Cv}(x_j)) = \ell^{s_a+S}\varphi(\mathrm{Cv}(\varphi^{-1}(g_j(0))x_j))$$

(φ is bijective on Λ, and ι is φ-equivariant); take $y' = \sum_j o_j\varphi^{-1}(g_j(0))x_j$, Δ-invariant as the constants are. (F3) For $y_1 \in \mathbf{N}^\Delta_{c-1}$ there is $y_2 \in \mathbf{N}^\Delta_{c-2}$ with $\ell^{b'}\mathrm{Cv}(y_1) = \varphi(\mathrm{Cv}(y_2))$: with a Δ-invariant basis (e''_k) of $\mathbf{N}_{c-2}$, $\varsigma^{b'} y_1 = \sum_k g_k\varphi(e''_k)$ with $g_k \in \mathbf{A}_{F,L}$; by (Cv2) and the sum form of (Cv4), $\ell^{b'}\mathrm{Cv}(y_1) = \sum_k \varphi^3(\iota_\Lambda(g_k(0)))\varphi(\mathrm{Cv}(e''_k)) = \varphi(\mathrm{Cv}(y_2))$ with $y_2 = \sum_k \varphi^{-1}(g_k(0))e''_k$.

Step 1: the dual length lemma at c. Let Σ_c be as in Proposition C.65 at c, $P'_1 = w^{-1}(\pi_0\mathbf{N}^\Delta_c + Y_c + \sum_{\Sigma_c}\mathcal{O}_L y)$ and $P'_2 = w^{-1}(\pi_0\mathbf{N}^\Delta_c + Y_c)$ (all submodules of P are finitely generated, $\mathcal{O}_L$ being principal). The quotient P'/P'_1 embeds in $\mathbf{N}^\Delta_c/(\pi_0\mathbf{N}^\Delta_c + Y_c + \sum_{\Sigma_c}\mathcal{O}_L y)$, so $\mathrm{nl}(P'/P'_1) \le \lambda$; P'_1/P'_2 embeds in a module generated by the images of Σ_c, so it is generated by at most m elements.

Step 2: linear crystalline data. For $f \in P'_2$, $\mathrm{Cv}(w(f)) \in \ell^M\varphi(\mathrm{Cv}(\mathbf{N}^\Delta_{c-1}))$ by (F1), (F2). As φ is injective and D is torsion free, there are unique $\alpha_1(f) \in \mathrm{Cv}(\mathbf{N}^\Delta_{c-1})$ with $\ell^M\varphi(\alpha_1(f)) = \mathrm{Cv}(w(f))$ and, by (F3), $\alpha_2(f) \in \mathrm{Cv}(\mathbf{N}^\Delta_{c-2})$ with $\varphi(\alpha_2(f)) = \ell^{b'}\alpha_1(f)$; by uniqueness α_1, α_2 are $\mathcal{O}_L$-linear. By (F1), setting $\psi(f)$ equal to the class of any $y_2 \in \mathbf{N}^\Delta_{c-2}$ with $\mathrm{Cv}(y_2) = \alpha_2(f)$ gives a well-defined $\mathcal{O}_L$-linear map $\psi\colon P'_2 \to \mathbf{N}^\Delta_{c-2}/\pi_0\mathbf{N}^\Delta_{c-2}$.

Step 3: the dual length lemma at $c-2$. With Σ_{c-2} from Proposition C.65 at $c-2$, let $P'_3 = \psi^{-1}\big((\pi_0\mathbf{N}^\Delta_{c-2} + Y_{c-2} + \sum_{\Sigma_{c-2}}\mathcal{O}_L y)/\pi_0\mathbf{N}^\Delta_{c-2}\big)$ and $P'_4 = \psi^{-1}\big((\pi_0\mathbf{N}^\Delta_{c-2} + Y_{c-2})/\pi_0\mathbf{N}^\Delta_{c-2}\big)$. As in Step 1, $\mathrm{nl}(P'_2/P'_3) \le \lambda$ and P'_3/P'_4 is generated by at most m elements. For $f \in P'_4$ choose $y_2 \in Y_{c-2}$ with $\mathrm{Cv}(y_2) = \alpha_2(f)$ ((F1)); by (F2), $\alpha_2(f) = \ell^{h''}\varphi(\mathrm{Cv}(y_3))$ with $y_3 \in \mathbf{N}^\Delta_{c-3}$. Put $\alpha_3(f) = \ell^{-h''}\varphi^{-1}(\alpha_2(f))$, $\mathcal{O}_L$-linear in f, so $\alpha_3(f) = \mathrm{Cv}(y_3)$ for some Δ-invariant $y_3 = y_3(f) \in \mathbf{N}_{c-3}$.

Step 4: the Hodge condition at τ_{c-1}. We have $\varphi^{-2}(\alpha_3(f)) \in \epsilon_{c-2}D = D_{\mathrm{dR},\tau_{c-1}}(V_2)$ ($\mathrm{Cv}(\mathbf{N}_{c-3}) \subseteq \epsilon_c D$). Let $\beta(f)$ be its class modulo $\epsilon_{c-2}\mathrm{Fil}^{s_{c-1}}D$ and $P'_5 = \ker\beta \cap P'_4$. The target has L-dimension the number of Hodge jumps of V_2 at τ_{c-1} below s_{c-1}, with multiplicity; these lie in $[-\eta,\eta]$ ($\epsilon_{c-1} = 1$, (Ow1) via Lemma 24.2 (1), $s_{c-1} > \eta$), so by (Ow2) the dimension is $\le m$. The quotient P'_4/P'_5 embeds in it, so it is torsion free of rank $\le m$, hence free and generated by at most m elements.

Step 5: lifting. Let $f \in P'_5$ and $y_3 = y_3(f)$. Then $\varphi^{-2}(\mathrm{Cv}(y_3)) \in \mathrm{Fil}^{s_{c-1}}D$. By Proposition C.72 at the up site $c-1$ there is a Δ-invariant $z \in \mathbf{N}(T_1)_{c-1}$ with $(1\otimes\Phi)\mathrm{Cv}(z) = \ell^{k'}\varphi^2(\mathrm{Cv}(y_3))$, $k' = E + \sigma_{\mathrm{G}} + h'' - s_{c-1}$. Put $x = \varsigma^M\varphi(z) \in \mathbf{N}(T_1)_c$ (Step 0): $(\gamma_u - 1)x \in X\mathbf{N}(T_1)$ by (Wm3), so $x \in I^1_c$, with $\lambda' = 1$. By (Cv4), (Cv5) ($\mathbf{N}(\Phi)$ commutes with φ),

$$\mathrm{Cv}(\mathbf{N}(\Phi)x) = \ell^M\varphi((1\otimes\Phi)\mathrm{Cv}(z)) = \ell^{M+k'}\varphi^3(\mathrm{Cv}(y_3)),$$
$$\ell^{b'}\mathrm{Cv}(w(f)) = \ell^M\varphi^2(\alpha_2(f)) = \ell^{M+h''}\varphi^3(\mathrm{Cv}(y_3)).$$

Since $b' + k' = e_6 + h''$, $\ell^{h''}\big(\ell^{e_6}\mathrm{Cv}(w(f)) - \mathrm{Cv}(\mathbf{N}(\Phi)x)\big) = 0$, so $\mathrm{Cv}(\ell^{e_6}w(f) - \mathbf{N}(\Phi)x) = 0$ and, by (Cv3), $\ell^{e_6}w(f) - \mathbf{N}(\Phi)(x) \in X\mathbf{N}(T_2)_c$.

Step 6: the chain and the exponent. The chain $P' = P'_0 \supseteq \cdots \supseteq P'_5$ with $k = 5$, $J = \{1,3,4\}$ ($|J| = 3$, $k - |J| = 2$) has the required properties. Finally, with $\ell \ge 11$, $a_{\mathrm{R}} = H'(2\ell-1)/(\ell-1)^2 + 4 + \log_\ell(2H')$, $(2\ell-1)/(\ell-1)^3 \le 2/(\ell-1)^2 \le 2.42/\ell^2$ and $\ell/(\ell-1) \le 1.1$:

$$e_6 \le \frac{3a_{\mathrm{R}} + S + 2\eta}{\ell-1} + \log_\ell(2H') + 1 + \frac{2H'}{(\ell-1)^2} + 2 + S \le \frac{10H'}{\ell^2} + 2S + \frac{2.2\eta}{\ell} + 2\log_\ell(2H') + 4.2 \le 10B,$$

so the statement holds with $c_{4,2} = 10$ and $e_6 \ge 0$. □

Lemma C.76. *Let ℓ be a prime, $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_\ell$, H an $\mathcal{O}$-module, $H' \subseteq H$ a submodule, P a free $\mathcal{O}$-module of finite rank r and $\psi\colon P \to H$ a surjective $\mathcal{O}$-linear map. Let $P = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_k$ be submodules, $I \subseteq \{0, \dots, k-1\}$, $m \geq 0$ and $e, e' \geq 0$ integers and $\lambda \geq 0$ a real number such that*

(Fc1) *for $i \in I$ the module P_i/P_{i+1} is generated by at most m elements;*

(Fc2) *for $i \notin I$, $\mathrm{nl}(P_i/P_{i+1}) \leq \lambda$;*

(Fc3) *$\ell^e \psi(P_k) \subseteq H'$;*

(Fc4) *$\ell^{e'} H \subseteq H'$.*

Then H/H' has finite length and

$$\mathrm{nl}(H/H') \leq r\,e + |I|\,m\,e' + (k - |I|)\,\lambda.$$

Proof. Let $Q = H/H'$, $\rho\colon P \to Q$ the composite of ψ with the quotient map (surjective), and $Q_i = \rho(P_i)$, so that $Q = Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_k$. For $i < k$, ρ induces a surjection $P_i/P_{i+1} \to Q_i/Q_{i+1}$, because $\rho(P_{i+1}) = Q_{i+1}$.

For $i \in I$, Q_i/Q_{i+1} is generated by at most m elements by (Fc1) and is killed by $\ell^{e'}$, since $\ell^{e'} Q = 0$ by (Fc4); so it is a quotient of $(\mathcal{O}/\ell^{e'})^m$ and has length at most $m\,e'\,e_{\mathcal{O}}$. For $i \notin I$, Q_i/Q_{i+1} is a quotient of P_i/P_{i+1}, so its length is at most $\lambda\,e_{\mathcal{O}}$ by (Fc2). Finally P_k is a submodule of the free module P of rank r over the principal ideal domain $\mathcal{O}$, hence free of rank at most r; so $Q_k = \rho(P_k)$ is generated by at most r elements, and it is killed by ℓ^e by (Fc3); it has length at most $r\,e\,e_{\mathcal{O}}$.

Lengths are additive along the finite filtration $Q = Q_0 \supseteq \cdots \supseteq Q_k \supseteq 0$, so Q has finite length at most $\big(r\,e + |I|\,m\,e' + (k - |I|)\lambda\big)e_{\mathcal{O}}$; divide by $e_{\mathcal{O}}$. □

C.3 Long Frobenius orbits with isolated thick clusters

This section proves Proposition C.77, item 3 of the row bound Proposition 26.2: on a long Frobenius orbit whose thick sites form isolated clusters, the colength carries $h/(\ell-1)^{j+1}$ in place of h/ℓ. It follows from the lifting exponent Proposition C.79, with no exceptional classes. The thick sites admit a cut configuration of order j, in which the j sites after each cut are thin and each arc contains at most two thick sites (Definition C.78, Lemma C.80), and the crystalline classes lift up to the arc exponent of the configuration (Proposition C.82), which is at most $80\big(h/(\ell-1)^{j+1} + g\,\mathrm{len}_{\max} + \log_\ell h\big)$ (Lemma C.81). The arc theorem is proved from arc level data (Proposition C.84, Proposition C.106, Proposition C.107): relay vectors are killed by polynomials in γ at the target and at the source of the arc, and the constants that remain at the down levels are lifted (Proposition C.85, Proposition C.94).

Proposition C.77. *There is an absolute constant $c_{\mathrm{A}} \geq 1$ with the following property. Let $\ell \geq 11$ be a prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of degree n, L as in Definition 23.1, $(\tau_a)_{a \in \mathbb{Z}/n}$ a Frobenius numbering, $\Phi\colon V_1 \to V_2$ an onto map of crystalline lattices $T_1 \to T_2$ over F with coefficients L, and $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ a weight word for Φ. Let $j \geq 1$ be an integer with $j + 1 \leq n$ and $c_{\mathrm{A}}(j+1)(G+1) \leq b$, and suppose that every thick site a satisfies one of:*

- *none of $a-1, \dots, a-j$ is thick;*

- *there is $1 \le d \le j$ such that $a-d$ is thick and none of the sites $a-k$, $1 \le k \le d+j$, $k \ne d$, is thick.*

Then the colength of Φ is at most

$$c_{\mathrm{A}}\,(n+1)\,d_2\Big(\frac{h}{(\ell-1)^{j+1}} + (G+1)(j+1) + \log_\ell h\Big).$$

Proof. Let $c_{\mathrm{E}} \ge 1$ be the absolute constant of Proposition C.79; we show that $c_{\mathrm{A}} = c_{\mathrm{E}}$ has the property. Let the data be as in the statement, with $c_{\mathrm{A}}(j+1)(G+1) \le b$. Put $D = G_F$, $H = H^1_f(D, T_2)$ and $H_1 = \Phi_* H^1_f(D, T_1)$; the colength of Φ is $\mathrm{nl}(H/H_1)$.

(i) V_2 is crystalline, hence de Rham, so by Lemma 27.3 H is generated by at most $(n+1)d_2$ elements. (ii) The hypotheses are those of Proposition C.79, which gives an integer e with $0 \le e \le c_{\mathrm{E}}(h/(\ell-1)^{j+1} + (G+1)(j+1) + \log_\ell h)$ and $\ell^e H \subseteq H_1$. (iii) By Lemma 27.14 with $g = (n+1)d_2$, this e, $Z = \emptyset$ and $e' = 0$, H/H_1 has finite length and

$$\mathrm{nl}(H/H_1) \le (n+1)\,d_2\,e \le c_{\mathrm{A}}\,(n+1)\,d_2\Big(\frac{h}{(\ell-1)^{j+1}} + (G+1)(j+1) + \log_\ell h\Big).$$

□

Definition C.78. Let $n \ge 1$ and $j \ge 1$ be integers and $\Theta \subseteq \mathbb{Z}/n$ a set (the *thick* sites; the others are *thin*).

Cut configurations. A **cut configuration of order** j for Θ is a non-empty set $\mathcal{C} \subseteq \mathbb{Z}/n$ (the *cuts*) with integers len_c ($c \in \mathcal{C}$) such that, for every $c \in \mathcal{C}$:

(Cc1) the sites $c+1, \dots, c+j$ are thin;

(Cc2) $j \le \mathrm{len}_c \le n$, $c - \mathrm{len}_c \in \mathcal{C}$, and $c - k \notin \mathcal{C}$ for $1 \le k < \mathrm{len}_c$;

(Cc3) at most two of the sites $c, c-1, \dots, c-\mathrm{len}_c+1$ (the **arc** of c) are thick.

Its **maximal length** is $\mathrm{len}_{\max} = \max_{c\in\mathcal{C}} \mathrm{len}_c$.

The arc exponent. Let moreover $\ell \ge 3$ be a prime, $g, h \ge 1$ integers, $H\colon \mathbb{Z}/n \to \mathbb{Z}_{\ge 0}$ a function with maximum $H_{\max}$, and $(\mathcal{C}, (\mathrm{len}_c))$ a cut configuration of order j for some Θ. Put

$$\sigma_{\mathrm{E}} = \Big\lceil \Big(3g + \frac{\lfloor (h+2g)/(\ell-1)\rfloor + 1}{\ell^{j-1}}\Big)\frac{\ell}{(\ell-1)^2} + 1\Big\rceil, \qquad c_H = \lceil 1 + \log_\ell(H_{\max}+1)\rceil, \tag{C.8}$$

$$\rho_{\mathrm{E}} = \max_{c\in\mathcal{C}}\Big(\Big\lfloor \min(\lfloor s_c\rfloor, H_{\max})\frac{\ell}{(\ell-1)^2}\Big\rfloor + c_H\Big), \qquad s_c = \sum_{a\ge 0}\frac{H(c+a+1)+c_H+3}{(\ell-1)^a}, \tag{C.9}$$

(the series converges, H being bounded). Let $\mathcal{I}(g,h)$ be the set of integer intervals $\{i \in \mathbb{Z} :\ a \le i < a+k\}$ with $a \in \mathbb{Z}$, $-2h \le a$, $0 \le k \le 3g$ and $a + k \le 2h+1$, and put

$$\kappa_{\mathrm{E}} = \max_{I_1,I_2,I_3\in\mathcal{I}(g,h)}\Big(\mathrm{len}_{\max} + \sum_{i\in(I_1\cup I_2\cup I_3)\setminus\{0\}}\big(\mathrm{len}_{\max} + 1 + v_\ell(i)\big)\Big), \tag{C.10}$$

with v_ℓ the ℓ-adic valuation. The **initial exponent** and the **arc exponent** are

$$a_{\mathrm{E}} = 2\sigma_{\mathrm{E}} + \rho_{\mathrm{E}} + 2g\,\mathrm{len}_{\max}, \qquad e_{\mathrm{E}} = 2\rho_{\mathrm{E}} + 4\sigma_{\mathrm{E}} + 2a_{\mathrm{E}} + 2\kappa_{\mathrm{E}}; \tag{C.11}$$

they depend on $(\ell, j, g, h, H, \mathcal{C}, (\mathrm{len}_c))$.

Proposition C.79. *There is an absolute constant $c_{\mathrm{E}} \geq 1$ with the following property. Let $\ell \geq 11$ be a prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of degree n, $D = G_F$, L as in Definition 23.1, $(\tau_a)_{a\in\mathbb{Z}/n}$ a Frobenius numbering, $\Phi\colon V_1 \to V_2$ an onto map of crystalline lattices $T_1 \to T_2$ over F with coefficients L, and $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ a weight word for Φ. Let $j \geq 1$ be an integer with $j+1 \leq n$ and $c_{\mathrm{E}}(j+1)(G+1) \leq b$, and suppose that every thick site a satisfies one of:*

- *none of $a-1, \dots, a-j$ is thick;*
- *there is $1 \leq d \leq j$ such that $a-d$ is thick and none of the sites $a-k$, $1 \leq k \leq d+j$, $k \neq d$, is thick.*

Then there is an integer e with

$$0 \leq e \leq c_{\mathrm{E}}\Big(\frac{h}{(\ell-1)^{j+1}} + (G+1)(j+1) + \log_\ell h\Big) \qquad \text{and} \qquad \ell^e\, H^1_f(D, T_2) \subseteq \Phi_* H^1_f(D, T_1).$$

Proof. We show that $c_{\mathrm{E}} = 240$ has the property. Let the data be as in the statement with $240(j+1)(G+1) \leq b$; so $b \geq 480(G+1)$ and $w, S, \eta \leq G \leq b/480$. Let Θ be the set of thick sites, and g, h_*, H as in Proposition C.82.

A cut configuration with short arcs. The hypothesis on the thick sites is (Cl1)/(Cl2) of Lemma C.80, and $j+1 \leq n$; that lemma gives a cut configuration $(\mathcal{C}, (\mathrm{len}_c))$ of order j for Θ with $\mathrm{len}_{\max} \leq 3j+1$.

The numerical input. We have $1 \leq g \leq G+1$, and $1 \leq h_* \leq h$: if $\Theta \neq \emptyset$ then $h_* = b+\eta+S = h$, and otherwise $h_* = \max(G, 1) \leq G+1 = h$. Moreover $H \leq 2h_*$ everywhere ($G \leq h_*$: if $\Theta \neq \emptyset$ because $w \leq b$, otherwise because $G \leq g$), and for $c \in \mathcal{C}$, $1 \leq k \leq j$, the site $c+k$ is thin by (Cc1), so $H(c+k) = 2G \leq 2g$. Hence Lemma C.81 applies to $(\ell, n, j, g, h_*, H, \mathcal{C}, (\mathrm{len}_c))$: with $X_* = h_*/(\ell-1)^{j+1}$ and $\Lambda_* = \log_\ell h_*$,

$$a_{\mathrm{E}} \leq 5X_* + 2g\,\mathrm{len}_{\max} + g + 2\Lambda_* + 8, \qquad e_{\mathrm{E}} \leq 80(X_* + g\,\mathrm{len}_{\max} + \Lambda_*).$$

The hypotheses (Ae1), (Ae2) of Proposition C.82. (Ae1): a thick site has $s_a \geq b > G+1 \geq g$. (Ae2) concerns two thick sites, so we may assume $\Theta \neq \emptyset$; then $h_* = b+\eta+S \leq \frac{481}{480}b \leq 2b$, and all thick sizes lie in $[b, b+\eta] \subseteq [b, 2b]$. Hence for thick $c-s$, $c-t$: $s_{c-s} \leq 2b < 11b \leq \ell s_{c-t}$ and $2s_{c-s} \leq 4b \leq \ell s_{c-t}$. It remains to show $2a_{\mathrm{E}} < s_{c-t}$, for which $a_{\mathrm{E}} < b/2$ suffices. Now $5X_* \leq 5h_*/100 \leq \frac{5\cdot 481}{48000}b < 0.051\,b$; $2g\,\mathrm{len}_{\max} + g + 8 \leq 2(G+1)(3j+1) + 9(G+1) \leq (6+\frac{9}{2})(G+1)(j+1) \leq \frac{10.5}{240}b < 0.044\,b$ (using $j+1 \geq 2$); and $2\Lambda_* \leq 2\log_{11}(2b) \leq 0.1\,b$, because $b \geq 480$, the inequality holds at $b = 480$ and $0.1\,b - 2\log_{11}(2b)$ increases for $b \geq 480$. So $a_{\mathrm{E}} < 0.2\,b < b/2$.

Conclusion. By Proposition C.82, $\ell^e H^1_f(D, T_2) \subseteq \Phi_* H^1_f(D, T_1)$ with $e = e_{\mathrm{E}}$, an integer ≥ 0 (all terms of (C.11) are non-negative). Since $X_* \leq h/(\ell-1)^{j+1}$, $\Lambda_* \leq \log_\ell h$ and $g\,\mathrm{len}_{\max} \leq (G+1)(3j+1) \leq 3(G+1)(j+1)$,

$$e \leq 80\Big(\frac{h}{(\ell-1)^{j+1}} + 3(G+1)(j+1) + \log_\ell h\Big) \leq 240\Big(\frac{h}{(\ell-1)^{j+1}} + (G+1)(j+1) + \log_\ell h\Big).$$

□

Lemma C.80. *Let $n, j \geq 1$ be integers with $j+1 \leq n$, and $\Theta \subseteq \mathbb{Z}/n$ a set such that every $t \in \Theta$ satisfies one of:*

(Cl1) *none of $t-1, \dots, t-j$ lies in Θ;*

(Cl2) *there is* $1 \le d \le j$ *with* $t - d \in \Theta$ *such that* $t - k \notin \Theta$ *for every* $1 \le k \le d + j$ *with* $k \ne d$.

Then Θ *has a cut configuration* $(\mathcal{C}, (\mathrm{len}_c))$ *of order* j *with* $\mathrm{len}_{\max} \le 3j + 1$.

Proof. Call $e \in \Theta$ an *end* if none of $e + 1, \dots, e + j$ lies in Θ. We use two elementary facts.

Splitting a length. Every integer $N \ge j$ is a sum of integers in $[j, 2j - 1]$: write $N = qj + r$ with $q \ge 1$, $0 \le r < j$, and take $q - 1$ parts j and one part $j + r$.

Every $t \in \Theta$ *has an end among* $t, t+1, \dots, t+j$. If none of $t+1, \dots, t+j$ is in Θ, t is an end. Otherwise pick $1 \le k \le j$ with $t + k \in \Theta$; we claim $t + k$ is an end. If not, $a = t + k + k' \in \Theta$ for some $1 \le k' \le j$. As $a - k' \in \Theta$ with $k' \le j$, (Cl1) fails for a, so (Cl2) holds for a with some d; if $d \ne k'$, then $1 \le k' \le d + j$ and $a - k' \in \Theta$ contradict (Cl2), so $d = k'$. But then $a - (k + k') = t \in \Theta$ with $1 \le k + k' \le d + j$ and $k + k' \ne d$, contradicting (Cl2).

Case 1: $\Theta = \emptyset$. (Cc1) and (Cc3) hold for every choice of cuts. If $n \le 3j + 1$, take $\mathcal{C} = \{0\}$, $\mathrm{len}_0 = n$: (Cc2) holds since $j < n$, $0 - n = 0$, and $-k \ne 0$ in $\mathbb{Z}/n$ for $1 \le k < n$. If $n > 3j + 1$, split $n = d_1 + \cdots + d_r$ with parts $d_i \in [j, 2j - 1]$ ($r \ge 2$), put $p_0 = 0$, $p_i = d_1 + \cdots + d_i$, $\mathcal{C} = \{p_i \bmod n : \ 0 \le i < r\}$ and $\mathrm{len}_{p_i} = d_i$ for $1 \le i \le r$ (where $p_r \equiv p_0$). The p_i, $0 \le i < r$, are distinct modulo n since $0 \le p_i < n$; between p_{i-1} and p_i there is no other cut, so (Cc2) holds, and $\mathrm{len}_{\max} \le 2j - 1$.

Case 2: $\Theta \ne \emptyset$. By the fact above there is an end. List the ends as integers $e_1 < e_2 < \cdots < e_q$ with $e_q - e_1 < n$ (representatives of their classes along one turn of the cycle) and put $e_{q+1} = e_1 + n$; the gaps $\gamma_i = e_{i+1} - e_i$ satisfy $\gamma_i \ge 1$ and $\sum_i \gamma_i = n$. For each i consider the integer interval $(e_i, e_{i+1}]$, whose last point is an end and which contains no other end.

- *Thick sites of* $(e_i, e_{i+1}]$ *lie in* $[e_{i+1} - j, e_{i+1}]$*, and there are at most two.* Let $t \in (e_i, e_{i+1}]$ be (the class of) a thick site. By the fact above some $t + k$, $0 \le k \le j$, is an end; as $t + k > e_i$ and e_{i+1} is the first end after e_i, $t + k \ge e_{i+1}$, so $t \ge e_{i+1} - j$. If $t_1 < t_2 < t_3$ were thick sites in $[e_{i+1} - j, e_{i+1}]$, then for $a = t_3$ the site $a - (t_3 - t_2)$ is thick with $t_3 - t_2 \le j$, so (Cl2) holds with $d = t_3 - t_2$ (as in the proof of the fact above), and $a - (t_3 - t_1) = t_1$ is thick with $t_3 - t_1 \le j \le d + j$, $t_3 - t_1 \ne d$: a contradiction.

- $\gamma_i \ge j + 1$. The site e_{i+1} is thick, and $e_i + 1, \dots, e_i + j$ are thin since e_i is an end; so $e_{i+1} \notin \{e_i + 1, \dots, e_i + j\}$ (as classes), and $e_{i+1} > e_i$ gives $e_{i+1} \ge e_i + j + 1$.

Now place cuts in each interval $(e_i, e_{i+1}]$. If $\gamma_i \le 3j+1$, the only cut in it is e_{i+1}, with $\mathrm{len}_{e_{i+1}} = \gamma_i$. If $\gamma_i > 3j + 1$, then $\gamma_i - 2j - 1 \ge j$; split $\gamma_i - 2j - 1 = d_1 + \cdots + d_r$ with parts in $[j, 2j - 1]$ and take the cuts $p_k = e_i + d_1 + \cdots + d_k$ ($1 \le k \le r$, so $p_r = e_{i+1} - 2j - 1$) and e_{i+1}, with $\mathrm{len}_{p_k} = d_k$ and $\mathrm{len}_{e_{i+1}} = 2j + 1$. Let $\mathcal{C}$ be the set of classes of all cuts so placed; it contains the ends, so it is non-empty. The cuts, taken over one turn $e_1 < \cdots \le e_{q+1}$, are strictly increasing integers in $(e_1, e_1 + n]$, hence distinct modulo n, and len_c is the distance from c back to the preceding cut. So:

- (Cc2): $\mathrm{len}_c \ge j$ (parts are $\ge j$, $\gamma_i \ge j+1$, $2j+1 \ge j$), $\mathrm{len}_c \le \gamma_i \le n$, $c - \mathrm{len}_c$ is the preceding cut, and no cut lies strictly between; moreover $\mathrm{len}_c \le \max(3j + 1, 2j - 1, 2j + 1) = 3j + 1$.

- (Cc1): for an end e_{i+1} by definition of an end. For an inner cut $p_k \le e_{i+1} - 2j - 1$ of $(e_i, e_{i+1}]$, the sites $p_k + 1, \dots, p_k + j$ lie in $(e_i, e_{i+1} - j - 1]$, which contains no thick site by the first item.

- (Cc3): the arc of a cut c is the interval $(c - \mathrm{len}_c, c]$, contained in some $(e_i, e_{i+1}]$; it contains at most the (at most two) thick sites of that interval.

□

Lemma C.81. *Let $\ell \geq 11$ be a prime, $n, j, g, h \geq 1$ integers, $H\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$ with $H(a) \leq 2h$ for all a, and $(\mathcal{C}, (\mathrm{len}_c))$ a cut configuration of order j for some $\Theta \subseteq \mathbb{Z}/n$ such that $H(c+k) \leq 2g$ for all $c \in \mathcal{C}$ and $1 \leq k \leq j$. Put $X = h/(\ell-1)^{j+1}$ and $\Lambda = \log_\ell h$. Then*

$$a_{\mathrm{E}} \leq 5X + 2g\,\mathrm{len}_{\max} + g + 2\Lambda + 8 \qquad \text{and} \qquad e_{\mathrm{E}} \leq 80(X + g\,\mathrm{len}_{\max} + \Lambda).$$

Proof. We use repeatedly that $x \mapsto x/(x-1)^2$, $x \mapsto x/((x-1)(x-2))$ and $x \mapsto x/(x-2)$ decrease on $(2,\infty)$, so for $\ell \geq 11$: $\ell/(\ell-1)^2 \leq \frac{11}{100}$, $\ell/((\ell-1)(\ell-2)) \leq \frac{11}{90}$, $\ell/(\ell-2) \leq \frac{11}{9}$, $(\ell-1)^{j+1} \geq 100$; and $\log_\ell 2 \leq \log_{11} 2 < 0.29$, $\log_\ell 3 < 0.46$. Also $\Lambda \geq 0$ as $h \geq 1$, and $\mathrm{len}_{\max} \geq j \geq 1$ by (Cc2).

The relay exponent: $\sigma_{\mathrm{E}} \leq \frac{11}{10}X + \frac{36}{100}g + \frac{211}{100}$. Let $M = \lfloor (h+2g)/(\ell-1) \rfloor + 1 \leq (h+2g)/(\ell-1) + 1$. Then $3g\,\ell/(\ell-1)^2 \leq \frac{33}{100}g$, and

$$\frac{M\,\ell}{\ell^{j-1}(\ell-1)^2} \leq \frac{(h+2g)\,\ell^{2-j}}{(\ell-1)^3} + \frac{\ell^{2-j}}{(\ell-1)^2} = \frac{h+2g}{(\ell-1)^{j+1}}\Big(\frac{\ell}{\ell-1}\Big)^{2-j} + \frac{\ell^{2-j}}{(\ell-1)^2} \leq \frac{11}{10}\cdot\frac{h+2g}{(\ell-1)^{j+1}} + \frac{11}{100},$$

since $(\ell/(\ell-1))^{2-j} \leq \ell/(\ell-1) \leq \frac{11}{10}$ for $j \geq 1$ and $\ell^{2-j} \leq \ell$. With $2g/(\ell-1)^{j+1} \leq \frac{2}{100}g$ and $\lceil y \rceil < y+1$, (C.8) gives $\sigma_{\mathrm{E}} < \frac{33}{100}g + \frac{11}{10}X + \frac{22}{1000}g + \frac{11}{100} + 2$, which implies the claim.

The remainder exponent: $\rho_{\mathrm{E}} \leq \frac{245}{100}X + \frac{245}{1000}g + \frac{113}{100}\Lambda + \frac{313}{100}$. Since $H_{\max} \leq 2h$ and $2h+1 \leq 3h$, $c_H \leq 2 + \log_\ell(3h) \leq \frac{246}{100} + \Lambda$. Fix $c \in \mathcal{C}$. For $0 \leq a \leq j-1$ the site $c+a+1 = c+k$ has $1 \leq k \leq j$, so $H(c+a+1) \leq 2g$; for $a \geq j$, $H(c+a+1) \leq 2h$. Hence

$$s_c \leq (2g + c_H + 3)\sum_{a \geq 0}(\ell-1)^{-a} + 2h\sum_{a \geq j}(\ell-1)^{-a} = (2g + c_H + 3)\frac{\ell-1}{\ell-2} + \frac{2h}{(\ell-1)^{j-1}(\ell-2)}.$$

As $0 \leq \min(\lfloor s_c \rfloor, H_{\max}) \leq s_c$, the term of (C.9) at c is at most

$$s_c\frac{\ell}{(\ell-1)^2} + c_H \leq (2g + c_H + 3)\frac{\ell}{(\ell-1)(\ell-2)} + 2X\frac{\ell}{\ell-2} + c_H \leq \frac{22}{90}g + \frac{101}{90}c_H + \frac{33}{90} + \frac{22}{9}X.$$

Inserting the bound for c_H gives the claim.

The killer exponent: $\kappa_{\mathrm{E}} \leq \mathrm{len}_{\max} + 9g\,\mathrm{len}_{\max} + \frac{99}{10}g + 3\Lambda + \frac{87}{100}$. Let $I_1, I_2, I_3 \in \mathcal{I}(g,h)$ and $J^* = (I_1 \cup I_2 \cup I_3) \setminus \{0\}$. Each I_t has at most $3g$ elements, all in $[-2h, 2h]$ (as $a \geq -2h$ and $a + k \leq 2h+1$), so $|J^*| \leq 9g$. For one $I = I_t$ with $k \leq 3g$ elements: a non-zero i with $\ell^u \mid i$ has $\ell^u \leq |i| \leq 2h$, so only $1 \leq u \leq U = \lfloor \log_\ell(2h) \rfloor$ contribute to

$$\sum_{i \in I \setminus 0} v_\ell(i) = \sum_{u \geq 1} \#\{i \in I \setminus 0 :\ \ell^u \mid i\} \leq \sum_{u=1}^{U}\Big(\frac{k}{\ell^u} + 1\Big) \leq \frac{k}{\ell-1} + \log_\ell(2h) \leq \frac{3g}{10} + \frac{29}{100} + \Lambda,$$

a set of k consecutive integers containing at most $k/\ell^u + 1$ multiples of ℓ^u. As $v_\ell \geq 0$, $\sum_{i \in J^*} v_\ell(i) \leq \frac{9}{10}g + \frac{87}{100} + 3\Lambda$, and the bracket in (C.10) is at most $\mathrm{len}_{\max} + 9g(\mathrm{len}_{\max}+1) + \frac{9}{10}g + \frac{87}{100} + 3\Lambda$.

Assembly. By (C.11),

$$a_{\mathrm{E}} = 2\sigma_{\mathrm{E}} + \rho_{\mathrm{E}} + 2g\,\mathrm{len}_{\max} \leq \frac{465}{100}X + \frac{965}{1000}g + \frac{113}{100}\Lambda + \frac{735}{100} + 2g\,\mathrm{len}_{\max},$$

which gives the first bound. Then $e_{\mathrm{E}} = 2\rho_{\mathrm{E}} + 4\sigma_{\mathrm{E}} + 2a_{\mathrm{E}} + 2\kappa_{\mathrm{E}}$ is at most

$$\frac{186}{10}X + 22\,g\,\mathrm{len}_{\max} + 2\,\mathrm{len}_{\max} + \frac{2366}{100}g + \frac{1052}{100}\Lambda + \frac{3114}{100}.$$

Since $g, \mathrm{len}_{\max} \geq 1$, the terms $2\,\mathrm{len}_{\max}$, $23.66\,g$ and 31.14 are at most $(2 + 23.66 + 31.14)\,g\,\mathrm{len}_{\max}$, so $e_{\mathrm{E}} \leq 18.6X + 78.8\,g\,\mathrm{len}_{\max} + 10.52\Lambda \leq 80(X + g\,\mathrm{len}_{\max} + \Lambda)$. □

Proposition C.82. *Let $\ell \geq 11$ be a prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of degree n, $D = G_F$, L with ring of integers $\mathcal{O}_L$ as in Definition 23.1, $(\tau_a)_{a\in\mathbb{Z}/n}$ a Frobenius numbering, $\Phi\colon V_1 \to V_2$ an onto map of crystalline lattices $T_1 \to T_2$ over F with coefficients L, and $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ a weight word for Φ. Let $\Theta \subseteq \mathbb{Z}/n$ be the set of thick sites, $G = w+S+\eta$, $g = \max(G, 1)$, $h_* = b+\eta+S$ if $\Theta \neq \emptyset$ and $h_* = g$ otherwise, and $H\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$, $H(a) = 2h_*$ for $a \in \Theta$ and $H(a) = 2G$ for $a \notin \Theta$. Let $j \geq 1$ and let $(\mathcal{C}, (\mathrm{len}_c))$ be a cut configuration of order j for Θ; form a_{E} and e_{E} with $(\ell, j, g, h_*, H, \mathcal{C}, (\mathrm{len}_c))$. Suppose:*

(Ae1) *$s_a > g$ for every $a \in \Theta$;*

(Ae2) *for every $c \in \mathcal{C}$ and all integers $0 \leq s < t < \mathrm{len}_c$ with $c-s, c-t \in \Theta$: if $\epsilon_{c-t} = -1$ and $\epsilon_{c-s} = 1$, then $s_{c-s} < \ell\, s_{c-t}$; if $\epsilon_{c-t} = 1$ and $\epsilon_{c-s} = -1$, then $2s_{c-s} \leq \ell\, s_{c-t}$ and $2a_{\mathrm{E}} < s_{c-t}$.*

Then

$$\ell^{\,e_{\mathrm{E}}} H^1_f(D, T_2) \subseteq \Phi_* H^1_f(D, T_1).$$

Proof. We enlarge the coefficients and descend. Fix an algebraic closure of L, a primitive ℓ^n-th root of unity ζ in it, and put $L' = L(\zeta)$, with ring of integers $\mathcal{O}_{L'}$. As a finitely generated torsion-free module over the discrete valuation ring $\mathcal{O}_L$, $\mathcal{O}_{L'}$ is free; fix a basis $\beta_1, \dots, \beta_d$, $d = [L' : L]$; it is also an L-basis of L'.

Step 1: the enlarged data. $L' \supseteq L$ contains $\sigma(F(\mu_\ell))$ for every embedding σ, so L' is admissible in Definition 23.1; every $\mathbb{Q}_\ell$-embedding of F into L' lands in L, so $\tau'_a = \iota \circ \tau_a$ ($\iota\colon L \subseteq L'$) is a Frobenius numbering for L'. Put $V'_\nu = V_\nu \otimes_L L'$, $T'_\nu = T_\nu \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$ and $\Phi' = \Phi \otimes \mathrm{id}$, with D acting on the first factor. The maps

$$\theta_V\colon V_\nu^{\oplus d} \to V'_\nu, \quad (x_i)_i \mapsto \sum_i x_i \otimes \beta_i, \qquad \theta_T\colon T_\nu^{\oplus d} \to T'_\nu,$$

are D-equivariant isomorphisms of L-spaces, respectively $\mathcal{O}_L$-modules, compatible with $T_\nu \subseteq V_\nu$ and with $\Phi^{\oplus d}$ and Φ'. Hence V'_ν is crystalline as a $\mathbb{Q}_\ell$-representation (a direct sum of crystalline ones), T'_ν is a D-stable $\mathcal{O}_{L'}$-lattice in V'_ν, and $\Phi'(T'_1) = \Phi(T_1) \otimes \mathcal{O}_{L'} = T'_2$: Φ' is an onto map of crystalline lattices over F with coefficients L'.

Step 2: the weight word is unchanged. Since L' is a free L-module of finite rank with trivial D-action, taking D-invariants commutes with $- \otimes_L L'$, so $D_{\mathrm{dR}}(V'_\nu) = D_{\mathrm{dR}}(V_\nu) \otimes_L L'$ with $\mathrm{Fil}^i D_{\mathrm{dR}}(V'_\nu) = \mathrm{Fil}^i D_{\mathrm{dR}}(V_\nu) \otimes_L L'$; the idempotent of $F \otimes_{\mathbb{Q}_\ell} L'$ belonging to τ'_a is the image of the idempotent of $F \otimes_{\mathbb{Q}_\ell} L$ belonging to τ_a. Therefore $D_{\mathrm{dR},\tau'_a}(V'_\nu) = D_{\mathrm{dR},\tau_a}(V_\nu) \otimes_L L'$ as filtered spaces, and

$$\dim_{L'}(\mathrm{Fil}^i/\mathrm{Fil}^{i+1}) D_{\mathrm{dR},\tau'_a}(V'_\nu) = \dim_L(\mathrm{Fil}^i/\mathrm{Fil}^{i+1}) D_{\mathrm{dR},\tau_a}(V_\nu) \qquad (i \in \mathbb{Z}).$$

So V'_ν has at τ'_a exactly the Hodge jumps of V_ν at τ_a, with the same graded dimensions, and $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ is a weight word for Φ' with respect to (τ'_a): (Ow1) and (Ow2) transfer by the display, (Ow3) involves only the sizes. The thick sites, G, g, h_*, H, the cut configuration, a_{E}, e_{E}, (Ae1) and (Ae2) depend only on ℓ, n, the weight word, j and $(\mathcal{C}, (\mathrm{len}_c))$, and are the same for Φ'. Since $\zeta \in L'$, Proposition C.83 applies to Φ':

$$\ell^{\,e_{\mathrm{E}}} H^1_f(D, T'_2) \subseteq \Phi'_* H^1_f(D, T'_1).$$

Step 3: descent. Continuous cohomology commutes with finite direct sums, and $H^1_f(D, -)$ of Definition 22.2 is additive in the representation (it is the kernel of the map to $H^1(D, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} -)$,

both functors being additive). Hence θ_V and θ_T induce isomorphisms $H^1(D,T_\nu)^{\oplus d}\cong H^1(D,T'_\nu)$ and $H^1(D,V_\nu)^{\oplus d}\cong H^1(D,V'_\nu)$ compatible with the maps from lattice to space, which carry $H^1_f(D,V_\nu)^{\oplus d}$ onto $H^1_f(D,V'_\nu)$; taking inverse images, $H^1_f(D,T_\nu)^{\oplus d}\cong H^1_f(D,T'_\nu)$, and under these isomorphisms Φ'_* is $(\Phi_*)^{\oplus d}$. Let $x\in H^1_f(D,T_2)$ and let $x'\in H^1_f(D,T'_2)$ correspond to $(x,0,\dots,0)$. By Step 2 there is $y'\in H^1_f(D,T'_1)$ with $\ell^{e_{\mathrm{E}}}x'=\Phi'_*(y')$; writing y' as $(y_1,\dots,y_d)$ with $y_i\in H^1_f(D,T_1)$ and comparing first components gives $\ell^{e_{\mathrm{E}}}x=\Phi_*(y_1)\in\Phi_*H^1_f(D,T_1)$. □

Proposition C.83. *Let $\ell\geq 11$ be a prime, F a finite unramified extension of $\mathbb{Q}_\ell$ of degree n, $D=G_F$, L with ring of integers $\mathcal{O}_L$ as in Definition 23.1, and suppose moreover that L contains a primitive ℓ^n-th root of unity. Let $(\tau_a)_{a\in\mathbb{Z}/n}$ be a Frobenius numbering, $\Phi\colon V_1\to V_2$ an onto map of crystalline lattices $T_1\to T_2$ over F with coefficients L, and $(b,w,S,\eta,m,(s_a),(\epsilon_a))$ a weight word for Φ. Let $\Theta\subseteq\mathbb{Z}/n$ be the set of thick sites, $G=w+S+\eta$, $g=\max(G,1)$, $h_*=b+\eta+S$ if $\Theta\neq\emptyset$ and $h_*=g$ otherwise, and $H\colon\mathbb{Z}/n\to\mathbb{Z}_{\geq 0}$, $H(a)=2h_*$ for $a\in\Theta$ and $H(a)=2G$ for $a\notin\Theta$. Let $j\geq 1$ and let $(\mathcal{C},(\mathrm{len}_c))$ be a cut configuration of order j for Θ; form a_{E} and e_{E} with $(\ell,j,g,h_*,H,\mathcal{C},(\mathrm{len}_c))$. Suppose:*

(Ae1) *$s_a>g$ for every $a\in\Theta$;*

(Ae2) *for every $c\in\mathcal{C}$ and all integers $0\leq s<t<\mathrm{len}_c$ with $c-s,c-t\in\Theta$: if $\epsilon_{c-t}=-1$ and $\epsilon_{c-s}=1$, then $s_{c-s}<\ell\, s_{c-t}$; if $\epsilon_{c-t}=1$ and $\epsilon_{c-s}=-1$, then $2s_{c-s}\leq\ell\, s_{c-t}$ and $2a_{\mathrm{E}}<s_{c-t}$.*

Then

$$\ell^{\,e_{\mathrm{E}}}H^1_f(D,T_2)\subseteq\Phi_*H^1_f(D,T_1).$$

Proof. We verify the hypothesis (Ae3) of Proposition C.84; its other hypotheses are those of the statement, and its conclusion is the claim.

Let $c\in\mathcal{C}$ and $\nu\in\{1,2\}$. The representation V_ν has finitely many Hodge jumps at each τ_a, so there are integers $M_a,h_a\geq 0$ with all of them in $[-M_a,h_a]$; by (Ex1), $\mathbf{N}(T_\nu)$ is a Wach module with window $(M_a,h_a)_a$, and by (Wm1) its site component at $c-\mathrm{len}_c$ is free over $e_{c-\mathrm{len}_c}\mathbf{A}_{F,L}$; a basis is a frame $\mathbf{n}^\nu$ at $c-\mathrm{len}_c$. By (Cc2), $1\leq j\leq\mathrm{len}_c\leq n$. Moreover $h_*\geq g$: if no site is thick, $h_*=g$; otherwise, for a thick site a, (Ow3) and (Ae1) give $h_*=b+\eta+S\geq s_a>g$. Hence Proposition C.107 (for the odd prime ℓ, the weight word, the cut c, the length len_c and j) applies to the frames $\mathbf{n}^1$, $\mathbf{n}^2$: the arc level data of Φ along the word at c have the arc level properties (Lp1)–(Lp8), (Lh1)–(Lh2). This is (Ae3), and Proposition C.84 gives $\ell^{e_{\mathrm{E}}}H^1_f(D,T_2)\subseteq\Phi_*H^1_f(D,T_1)$. □

Proposition C.84. *Let $\ell\geq 11$, F, n, $D=G_F$, L, $(\tau_a)_{a\in\mathbb{Z}/n}$, $\Phi\colon V_1\to V_2$ ($T_1\to T_2$), the weight word $(b,w,S,\eta,m,(s_a),(\epsilon_a))$, Θ, G, g, h_*, H, j, the cut configuration $(\mathcal{C},(\mathrm{len}_c))$, a_{E}, e_{E} and the hypotheses (Ae1), (Ae2) be as in Proposition C.83 (in particular L contains a primitive ℓ^n-th root of unity). Suppose moreover:*

(Ae3) *for every $c\in\mathcal{C}$ there are frames at $c-\mathrm{len}_c$ of $\mathbf{N}(T_1)$ and $\mathbf{N}(T_2)$ for which the arc level data of Φ along the word at the cut c of length len_c (with this j) have the arc level properties (Lp1)–(Lp8), (Lh1)–(Lh2).*

Then

$$\ell^{\,e_{\mathrm{E}}}H^1_f(D,T_2)\subseteq\Phi_*H^1_f(D,T_1).$$

Proof. We verify the hypothesis (Ae4) of Proposition C.106, whose other hypotheses are those of the statement and whose conclusion is the claim. Let $c \in \mathcal{C}$. By (Ae3) there are frames at $c - \mathrm{len}_c$ for which the arc level data of Φ along the word at the cut c of length len_c have the arc level properties. Their arc domains are the sets W_ν^c of Proposition C.106: both are defined by $\Lambda_{c,\mathrm{len}_c}(H_c)\, w \in P_{\mathrm{len}_c}(\mathbf{N}(T_\nu)_{c-\mathrm{len}_c})$ with the same function H_c of the word, and do not depend on the frames. Hence Proposition C.85 at c gives the statement of (Ae4) at c. As c was arbitrary, (Ae4) holds and Proposition C.106 gives $\ell^{e_{\mathrm{E}}} H_f^1(D, T_2) \subseteq \Phi_* H_f^1(D, T_1)$. □

Proposition C.85. *Let $\ell \geq 11$, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$), the weight word, Θ, G, g, h_*, H, j, $(\mathcal{C}, (\mathrm{len}_c))$, a_{E} and the hypotheses (Ae1), (Ae2) be as in Proposition C.83, and form $\sigma_{\mathrm{E}}, \rho_{\mathrm{E}}, \kappa_{\mathrm{E}}$ with the same data. Let $c \in \mathcal{C}$ and suppose that for some frames at $c - \mathrm{len}_c$ the arc level data of Φ along the word at the cut c of length len_c have the arc level properties. Let W_ν ($\nu = 1, 2$) be their arc domains and*

$$I_c(T_\nu) = \{w \in e_c \mathbf{N}(T_\nu)[1/\lambda_\infty] :\ (\gamma_u - 1)(w) \in X\mathbf{N}(T_\nu) \text{ for all } u \in \mathbb{Z}_\ell^\times\}.$$

Then for every $w \in W_2 \cap I_c(T_2)$ there are $w' \in W_1 \cap I_c(T_1)$, $\epsilon \in \mathcal{O}_L^\times$ and an integer $0 \leq k \leq 2\kappa_{\mathrm{E}}$ with

$$\mathbf{N}(\Phi)(w') - \epsilon\, \ell^{\,2\sigma_{\mathrm{E}} + 2a_{\mathrm{E}} + \rho_{\mathrm{E}} + k}\, w \in X\, \mathbf{N}(T_2)_c.$$

Proof. Write $\mathrm{len} = \mathrm{len}_c$, $\gamma = \gamma_{u_0}$, $\mathbf{N}_\nu = \mathbf{N}(T_\nu)$, $e_1 = 2\sigma_{\mathrm{E}} + 2a_{\mathrm{E}} + \rho_{\mathrm{E}}$, and let κ_s, $b_s = s_{c-s}$, $H(c-s)$ be the kinds, sizes and heights of the arc level data along the word.

Step 0: windows and parameters. Let $(M_a, h_a) = (\eta, \max(s_a + S, \eta))$ if $\epsilon_a = 1$ and $(\max(s_a + S, \eta), \eta)$ if $\epsilon_a = -1$. By (Ae1), Lemma C.102 applies: by its item 1 this is a window of V_1 and of V_2; by its items 2 and 3, $h_* \geq g$, $G, S, \eta \leq g$ ($g = \max(G, 1)$, $G = w + S + \eta$), and $b_s = s_{c-s} > g$ for every thick $c - s$. Moreover $h_{c-s} \leq H(c-s)$ for $s < \mathrm{len}$: the height $H(c-s)$ is $g + G$, $g + s_{c-s} + S$ or $g + \eta$ at a thin site, a thick site with $\epsilon_{c-s} = 1$ or a thick site with $\epsilon_{c-s} = -1$, and there h_{c-s} is at most G, equal to $s_{c-s} + S$, or equal to η (item 2). Put $E = \mathrm{len}_{\max} - 1 \geq \mathrm{len} - 1$.

Step 1: the killing sets and their valuations. Let J_2 be the set of i with $\mathrm{Nb}_s(i + g) \neq \mathrm{Nb}_s(i)$ for some $s < \mathrm{len}$, and J_1 the set of i with $\mathrm{Fil}_s^1(-i - g) \neq \mathrm{Fil}_s^1(-i)$ for some $s < \mathrm{len}$. From the kind families: a thin s contributes $[-g, -1]$, an up s contributes $[-b_s - g, -b_s - 1] \cup [-g, -1]$, a down s contributes $[-g, -1] \cup [b_s - g, b_s - 1]$. From (Lp7) for T_1 (Fil_s^1 decreasing): $\mathrm{Fil}_s^1(-i-g) \neq \mathrm{Fil}_s^1(-i)$ if and only if some jump u satisfies $-i - g \leq u \leq -i - 1$, i.e. $i \in [-u - g, -u - 1]$; jumps with $|u| \leq G$ (thin sites, and the parts $[-\eta, \eta]$, $\eta \leq G$) give subsets of $[-G - g, G - 1]$, an up site gives $[-b_s - S - g, -b_s - 1]$, a down site $[b_s - g, b_s + S - 1]$. By (Cc3) at most two sites of the arc are thick. Hence each of J_1, J_2 lies in a union $I_1 \cup I_2 \cup I_3$ of three intervals of $\mathcal{I}(g, h_*)$: the intervals above have at most $3g$ elements ($S \leq g$, $G \leq g$), start at $\geq -2h_*$ and end at $\leq 2h_*$ (for $[-G - g, G - 1]$ use $G \leq g \leq h_*$), because $b_s + S + g \leq h_* + g \leq 2h_*$ for thick s ($b_s \leq b + \eta$, $h_* = b + \eta + S$ when $\Theta \neq \emptyset$). Put $P = P_{J_2,E}$, $P' = P_{J_1,E}$. Since $u_0 \in 1 + \ell\mathbb{Z}_\ell$ with $v_\ell(u_0 - 1) = 1$ and ℓ is odd, $v_\ell(1 - u_0^m) = 1 + v_\ell(m)$ for $m \neq 0$; with $P_{J,E}(1) = \ell^E \prod_{i \in J \setminus 0}(1 - u_0^{i\ell^E})$,

$$v_\ell(P_{J,E}(1)) = E + \sum_{i \in J \setminus \{0\}} \big(1 + E + v_\ell(i)\big) \leq \mathrm{len}_{\max} + \sum_{i \in (I_1 \cup I_2 \cup I_3) \setminus \{0\}} \big(\mathrm{len}_{\max} + 1 + v_\ell(i)\big) \leq \kappa_{\mathrm{E}}.$$

Hence $P(1)P'(1) = \epsilon \ell^k$ with $\epsilon \in \mathbb{Z}_\ell^\times$ and $0 \leq k \leq 2\kappa_{\mathrm{E}}$.

Step 2: Δ-averaging. Let $w \in W_2 \cap I_c(T_2)$ and $\bar{w} = (\ell - 1)^{-1} \sum_{\delta \in \mu_{\ell-1}} \gamma_\delta(w)$. By (Lp2), $\bar{w} \in W_2$; $\gamma_\delta \bar{w} = \bar{w}$; $(\gamma_u - 1)\bar{w} = (\ell - 1)^{-1} \sum_\delta \gamma_\delta(\gamma_u - 1)w \in X\mathbf{N}_2$ (as $\gamma_\delta(X\mathbf{N}_2) \subseteq X\mathbf{N}_2$); and $\bar{w} - w = (\ell - 1)^{-1} \sum_\delta (\gamma_\delta - 1)w \in e_c X\mathbf{N}_2 = X\mathbf{N}_{2,c}$.

Step 3: killing at the target. Lemma C.86 for T_2 (window of Step 0, g, E, J_2) applied to $\bar{w}$ gives $x' = P(\gamma)\bar{w} \in W_2$, Δ-invariant, with $(\gamma_u - 1)x' \in X\mathbf{N}_2$ for all u, $x' - P(1)\bar{w} \in X\mathbf{N}_{2,c}$, and $\mathrm{lv}_s^{(2)}(x') \in L^{r_2} + \{\text{kind integral}\}$ for all $s < \mathrm{len}$.

Step 4: lifting to the source. Proposition C.92 gives $z \in W_1$ with $\mathbf{N}(\Phi)z = \ell^{e_1}x'$ and $v_s \in L^{r_1}$ with $\lambda_s(Q_s)^g(\mathrm{lv}_s^{(1)}(z) - v_s)$ de Rham integral. Replace z by $\bar{z} = (\ell-1)^{-1}\sum_\delta \gamma_\delta(z) \in W_1$: since $\mathbf{N}(\Phi)$ commutes with the γ_δ and x' is Δ-invariant, $\mathbf{N}(\Phi)\bar{z} = \ell^{e_1}x'$; by (Lp2), $\mathrm{lv}_s^{(1)}(\bar{z}) - v_s = (\ell-1)^{-1}\sum_\delta \gamma_\delta^{(s)}(\mathrm{lv}_s^{(1)}(z) - v_s)$, and $\gamma_\delta^{(s)}(\lambda_s(Q_s)^g f) = \lambda_s(\gamma_\delta Q_s)^g \gamma_\delta^{(s)}(f)$ with $\gamma_\delta(Q_s) = Q_s \cdot (\text{unit of } \mathbb{Z}_\ell[[X]])$ (Lemma 23.8), and $\gamma_\delta^{(s)}$ and multiplication by units of $K_{(s)}[[t]]$ preserve de Rham integrality (the de Rham family is increasing). Hence $\lambda_s(Q_s)^g\gamma_\delta^{(s)}(f) = \lambda_s(\text{unit})^{-g}\gamma_\delta^{(s)}(\lambda_s(Q_s)^g f)$ is de Rham integral for $f = \mathrm{lv}_s^{(1)}(z) - v_s$, using (Lp1) ($\lambda_s$ of a unit of $e_c\mathbf{A}_{F,L}$ is a unit of $K_{(s)}[[t]]$). So $\bar{z}$ has the same properties and is Δ-invariant.

Step 5: killing at the source. Item 2 of Lemma C.91 for T_1 (g, E, J_1) gives $w' = P'(\gamma)\bar{z} \in W_1$ with $(\gamma_u - 1)w' \in X\mathbf{N}_{1,c}$ for all u; as $w' \in W_1 \subseteq e_c\mathbf{N}_1[1/\lambda_\infty]$, $w' \in W_1 \cap I_c(T_1)$.

Step 6: the congruence. $\mathbf{N}(\Phi)$ is $\mathbf{A}_{F,L}$-linear and commutes with γ, so $\mathbf{N}(\Phi)w' = \ell^{e_1}P'(\gamma)x'$. Since $(\gamma-1)x' \in X\mathbf{N}_2$, $P'(\gamma)x' - P'(1)x' \in X\mathbf{N}_{2,c}$: write $P' = \sum_k p'_k Y^k$ with $p'_k \in \mathbb{Z}_\ell$ ($P' = P_{J_1,E}$ and $u_0 \in \mathbb{Z}_\ell$); since $\gamma(X) \in X\mathbf{A}_{F,L}$, $\gamma(X\mathbf{N}_2) \subseteq X\mathbf{N}_2$, so $(\gamma^k - 1)x' = \sum_{m<k}\gamma^m(\gamma - 1)x' \in X\mathbf{N}_2$ and $P'(\gamma)x' - P'(1)x' = \sum_k p'_k(\gamma^k - 1)x' \in X\mathbf{N}_2$; as $x' \in W_2 \subseteq e_c\mathbf{N}_2[1/\lambda_\infty]$, this difference lies in $e_cX\mathbf{N}_2 = X\mathbf{N}_{2,c}$. With Steps 2 and 3,

$$\begin{aligned}\mathbf{N}(\Phi)w' - \ell^{e_1}P'(1)P(1)\,w \\ = \ell^{e_1}\left[(P'(\gamma)x' - P'(1)x') + P'(1)(x' - P(1)\bar{w}) + P'(1)P(1)(\bar{w} - w)\right] \in X\mathbf{N}_{2,c},\end{aligned}$$

and $P'(1)P(1) = \epsilon\ell^k$ with $\epsilon \in \mathbb{Z}_\ell^\times \subseteq \mathcal{O}_L^\times$, $0 \le k \le 2\kappa_{\mathrm{E}}$ (Step 1). This is the assertion. □

Lemma C.86. *Let ℓ, F, L, (τ_a) be as in Proposition 23.7, (V,T) a pair, $\mathbf{N} = \mathbf{N}(T)$ with a window $(M_a, h_a)_a$ as in (Ex1), and fix arc level data for (V,T) (cut c, length* len*, H with $H(c-s) \ge h_{c-s}$ for $s <$* len*, frame, parameters G, S, η, kinds κ_s, sizes b_s) satisfying (Lp1), (Lp2), (Lp7), (Lp8) for every $s <$* len*. Let $g \ge 0$ with $G, S, \eta \le g$ and $g < b_s$ whenever κ_s is not thin, $E \ge$* len $- 1$*, and $J \subset \mathbb{Z}$ finite containing $\{i :\ \mathrm{Nb}_s(i+g) \ne \mathrm{Nb}_s(i)\}$ for every $s <$* len*; put $P = P_{J,E}$. Let $w \in W$ with $(\gamma_u - 1)(w) \in X\mathbf{N}$ for all $u \in \mathbb{Z}_\ell^\times$ and $\gamma_\delta(w) = w$ for all $\delta \in \mu_{\ell-1}$, and $x' = P(\gamma_{u_0})(w)$. Then $x' \in W$, $(\gamma_u - 1)(x') \in X\mathbf{N}$ for all u, $\gamma_\delta(x') = x'$ for all $\delta \in \mu_{\ell-1}$,*

$$x' - P(1)\,w \in X\mathbf{N}_c, \qquad \textit{and} \qquad \mathrm{lv}_s(x') \in L^r + \{\textit{kind integral vectors at } s\} \quad (s < \mathrm{len}).$$

Proof. Write $\gamma = \gamma_{u_0}$. The coefficients of P lie in $\mathbb{Z}_\ell$ ($u_0 \in \mathbb{Z}_\ell$), so x' is a $\mathbb{Z}_\ell$-linear combination of the $\gamma^k w$.

The vector x'. By (Lp2) W is γ_u-stable, and it is an $\mathbf{A}_{F,L}$-module, so $x' \in W$. Since $\gamma_u(X) = X(u + X\cdots)$, $\gamma_u(X\mathbf{N}) \subseteq X\mathbf{N}$; hence $(\gamma^k - 1)w = \sum_{m<k}\gamma^m(\gamma - 1)w \in X\mathbf{N}$, and $x' - P(1)w = \sum_k p_k(\gamma^k - 1)w \in X\mathbf{N}$ for $P = \sum_k p_k Y^k$; as $w \in W \subseteq e_c\mathbf{N}[1/\lambda_\infty]$, this difference lies in $e_cX\mathbf{N} = X\mathbf{N}_c$. The γ_u commute, so $(\gamma_u - 1)x' = P(\gamma)(\gamma_u - 1)w \in X\mathbf{N}$ and $\gamma_\delta x' = x'$.

De Rham integrality up to a constant. Fix $s <$ len. For $u \in \mathbb{Z}_\ell^\times$, $(\gamma_u - 1)w \in W \cap X\mathbf{N}_c$ (as above), so by item 3 of Lemma C.89 (with $g' = 0$, $k = 1$; its hypothesis is $H(c-s) \ge h_{c-s}$) $\mathrm{lv}_s((\gamma_u - 1)w)$ is de Rham integral, and by (Lp2) it equals $(\gamma_u^{(s)} - 1)\mathrm{lv}_s(w)$. Item 3 of Lemma C.88, with $\mathcal{X}(i) = L^r$ for all i and $\mathcal{X}'(i) = \mathrm{Fil}_s(-i)$, gives $v \in L^r$ such that $e' := \mathrm{lv}_s(w) - v$ is de Rham integral at s.

Rounding and killing. By Lemma C.90 ((Lp7) holds), $t^g e'$ is kind integral. By (Lp1), $\lambda_s(Q_s) = t\eta$ with $\eta \in K_{(s)}[[t]]^\times$, and the kind family is increasing, so $\lambda_s(Q_s)^g e' = \eta^g t^g e'$ is

kind integral (multiplication by $a = \sum_{k\geq 0} a_k t^k \in K_{(s)}[[t]]$ preserves Ξ_s of an increasing family $\mathcal{X}$: for $f = \sum_i f_i t^i \in \Xi_s(\mathcal{X})$ the t^i-coefficient of af is the finite sum $\sum_{k\geq 0} a_k f_{i-k}$, and $f_{i-k} \in K_{(s)} \otimes_L \mathcal{X}(i-k) \subseteq K_{(s)} \otimes_L \mathcal{X}(i)$, a $K_{(s)}$-subspace). By (Lp2), $\mathrm{lv}_s(w) = \mathrm{lv}_s(\gamma_\delta w) = \gamma_\delta^{(s)} \mathrm{lv}_s(w)$, so $e := \mathrm{lv}_s(w) = v + e'$ is Δ-invariant. Item 2 of Lemma C.87 (family Nb_s, $E \geq s$) gives $P(\gamma_{u_0}^{(s)})e \in L^r + \Xi_s(\mathrm{Nb}_s)$. Finally, lv_s is $\mathbb{Z}_\ell$-linear and $\mathrm{lv}_s(\gamma^k w) = (\gamma_{u_0}^{(s)})^k \mathrm{lv}_s(w)$ by (Lp2), so $\mathrm{lv}_s(x') = P(\gamma_{u_0}^{(s)})e$. □

Lemma C.87. *Fix arc level data and $s <$ len; write $K = K_{(s)}$, $\gamma_u = \gamma_u^{(s)}$ on $K((t))^r$ and $\gamma = \gamma_{u_0}$. Call $f \in K((t))^r$ Δ-invariant if $\gamma_\delta f = f$ for every $\delta \in \mu_{\ell-1} \subset \mathbb{Z}_\ell^\times$. Let $E \geq s$.*

1. *Let $\mathcal{X}' \subseteq \mathcal{X}$ be families of L-subspaces of L^r and $J \subset \mathbb{Z}$ finite with $\mathcal{X}(i) = \mathcal{X}'(i)$ for $i \notin J$. If $f \in \Xi_s(\mathcal{X})$ is Δ-invariant, there is $d \in \mathcal{X}(0)$ with $\sum_{k<\ell^E} \sigma_{u_0}^k(f^{(0)}) = \ell^E d$ and*

$$P_{J,E}(\gamma) f - P_{J,E}(1)\, d \in \Xi_s(\mathcal{X}').$$

2. *Let $\mathcal{X}$ be an increasing family ($\mathcal{X}(i) \subseteq \mathcal{X}(i+1)$), $g \geq 0$, and $J \supseteq \{i :\ \mathcal{X}(i+g) \neq \mathcal{X}(i)\}$ finite. If $e \in K((t))^r$ is Δ-invariant and $e = v + e'$ with $v \in L^r$ and $\lambda_s(Q_s)^g e' \in \Xi_s(\mathcal{X})$, then $P_{J,E}(\gamma)e \in L^r + \Xi_s(\mathcal{X})$.*

Proof. **Item 1.** By item 2 of Lemma C.88, for every i,

$$(P_{J,E}(\gamma)f)^{(i)} = \prod_{i' \in J\setminus\{0\}} (u_0^{i\ell^E} - u_0^{i'\ell^E}) \sum_{k<\ell^E} u_0^{ik} \sigma_{u_0}^k(f^{(i)}).$$

Each $K \otimes_L \mathcal{X}(i)$ is stable under the σ_u (which act on the factor K). For $i \in J\setminus\{0\}$ the coefficient is 0. For $i \notin J$, $i \neq 0$, it lies in $K \otimes \mathcal{X}(i) = K \otimes \mathcal{X}'(i)$. For $i = 0$ it is $\prod_{i'}(1 - u_0^{i'\ell^E})\, S$ with $S = \sum_{k<\ell^E} \sigma_{u_0}^k(f^{(0)}) \in K \otimes \mathcal{X}(0)$. By (Lp8), $\sigma_{u_0}^{\ell^s} = \mathrm{id}$, hence $\sigma_{u_0}^{\ell^E} = \mathrm{id}$ and $\sigma_{u_0}(S) = S$. Δ-invariance of f at t^0 reads $\sigma_\delta(f^{(0)}) = f^{(0)}$, and σ_δ commutes with σ_{u_0}, so $\sigma_\delta(S) = S$. Now σ_u depends only on u modulo ℓ^{s+1} (it acts on $\zeta_{\ell^{s+1}}$ by $\zeta \mapsto \zeta^u$ and trivially on F and L), and $\mu_{\ell-1}$ together with u_0 generates $(\mathbb{Z}/\ell^{s+1})^\times$ ($v_\ell(u_0 - 1) = 1$, so u_0 generates $(1+\ell\mathbb{Z})/(1+\ell^{s+1}\mathbb{Z})$); hence S is fixed by every σ_u. Writing $S = \sum_m \kappa_m \otimes x_m$ for an L-basis (x_m) of $\mathcal{X}(0)$, each $\kappa_m \in K$ is fixed by every σ_u, hence lies in L by (Lp8). So $S \in \mathcal{X}(0)$, and $d = \ell^{-E} S \in \mathcal{X}(0)$. Finally $P_{J,E}(1) = \ell^E \prod_{i'}(1 - u_0^{i'\ell^E})$ (item 2 of Lemma C.88), so the t^0-coefficient of $P_{J,E}(\gamma)f - P_{J,E}(1)d$ is 0, while its other coefficients are those of $P_{J,E}(\gamma)f$. Hence it lies in $\Xi_s(\mathcal{X}')$.

Item 2. By (Lp1), $\lambda_s(Q_s) = t\eta$ with $\eta \in K[[t]]^\times$. For an increasing family $\mathcal{Y}$ and $c = \sum_{k\geq 0} c_k t^k \in K[[t]]$, $f \in \Xi_s(\mathcal{Y})$ implies $cf \in \Xi_s(\mathcal{Y})$, since $(cf)^{(i)} = \sum_k c_k f^{(i-k)}$ and $K \otimes \mathcal{Y}(i-k) \subseteq K \otimes \mathcal{Y}(i)$. Hence $t^g e' = \eta^{-g} \lambda_s(Q_s)^g e' \in \Xi_s(\mathcal{X})$, i.e. $e' \in \Xi_s(\mathcal{Y})$ with $\mathcal{Y}(i) = \mathcal{X}(i+g)$. The family $\mathcal{Y}$ contains $\mathcal{X}$ ($\mathcal{X}$ increasing) and differs from it only on J. Constants are fixed by every γ_u (σ_u fixes L and $u^0 = 1$), so $e' = e - v$ is Δ-invariant. Item 1 for $(\mathcal{Y}, \mathcal{X}, J)$ gives $d \in \mathcal{Y}(0) \subseteq L^r$ with $P_{J,E}(\gamma)e' - P_{J,E}(1)d \in \Xi_s(\mathcal{X})$, and $P_{J,E}(\gamma)v = P_{J,E}(1)v$ by item 2 of Lemma C.88. Therefore $P_{J,E}(\gamma)e = P_{J,E}(1)(v+d) + \big(P_{J,E}(\gamma)e' - P_{J,E}(1)d\big) \in L^r + \Xi_s(\mathcal{X})$. □

Lemma C.88. *Fix arc level data (any pair, cut, length, frame, parameters) and $s <$ len; write $K = K_{(s)}$, $\gamma_u = \gamma_u^{(s)}$ on $K((t))^r$, $\gamma = \gamma_{u_0}$, and for $f \in K((t))^r$ let $f^{(i)} \in K^r$ be its t^i-coefficient.*

1. *For an integer $k \geq 0$: $(\gamma^k f)^{(i)} = u_0^{ik} \sigma_{u_0}^k(f^{(i)})$. If $E \geq s$, then $(\gamma^{\ell^E} f)^{(i)} = u_0^{i\ell^E} f^{(i)}$, and for a polynomial $P \in L[Y]$, $(P(\gamma^{\ell^E})f)^{(i)} = P(u_0^{i\ell^E}) f^{(i)}$.*

2. *(Killing polynomials) Let $E \geq s$, $J \subset \mathbb{Z}$ finite, and $P_{J,E}(Y) = \prod_{i' \in J \setminus \{0\}} (Y^{\ell^E} - u_0^{i'\ell^E}) \cdot \sum_{k<\ell^E} Y^k \in L[Y]$. Then*

$$(P_{J,E}(\gamma)f)^{(i)} = \prod_{i' \in J\setminus\{0\}} (u_0^{i\ell^E} - u_0^{i'\ell^E}) \sum_{k<\ell^E} u_0^{ik} \sigma_{u_0}^k(f^{(i)});$$

in particular $(P_{J,E}(\gamma)f)^{(i)} = 0$ for $i \in J \setminus \{0\}$, and $P_{J,E}(\gamma)v = P_{J,E}(1)\, v$ for a constant $v \in L^r$, with $P_{J,E}(1) = \ell^E \prod_{i' \in J\setminus\{0\}} (1 - u_0^{i'\ell^E})$.

3. *(Constants) Let $\mathcal{X}' \subseteq \mathcal{X}$ be families of L-subspaces of L^r (i.e. $\mathcal{X}'(i) \subseteq \mathcal{X}(i)$ for all i). If $f \in \Xi_s(\mathcal{X})$ and $(\gamma_u - 1)f \in \Xi_s(\mathcal{X}')$ for every $u \in \mathbb{Z}_\ell^\times$, then there is $d \in \mathcal{X}(0)$ with $f - d \in \Xi_s(\mathcal{X}')$.*

Proof. By definition $\gamma_u(\sum_i a_i t^i) = \sum_i u^i \sigma_u(a_i) t^i$ coordinatewise, and σ_u is a ring automorphism of K fixing $L = \epsilon_s L$ (by Definition 23.4 it is induced by any $g \in G_F$ with $\chi(g) = u$, which fixes F, ϵ_s and L and acts on $K = \epsilon_s(F(\zeta_{\ell^{s+1}}) \otimes_{\mathbb{Q}_\ell} L)$ through its action on $F(\zeta_{\ell^{s+1}})$; taking gh with $\chi(h) = v$ gives $\sigma_u\sigma_v = \sigma_{uv}$), so γ_u is L-linear and $\gamma_u\gamma_v = \gamma_{uv}$.

1. Induction on k gives $(\gamma^k f)^{(i)} = u_0^{ik}\sigma_{u_0}^k(f^{(i)})$. For $E \geq s$, $\sigma_{u_0}^{\ell^E} = (\sigma_{u_0}^{\ell^s})^{\ell^{E-s}} = \mathrm{id}$ by (Lp8) (Lemma 23.17), so γ^{ℓ^E} multiplies the t^i-coefficient by the scalar $u_0^{i\ell^E} \in \mathbb{Z}_\ell$; a polynomial in γ^{ℓ^E} with coefficients in L then acts on it by the value of the polynomial at that scalar.

2. $P_{J,E}(\gamma) = Q(\gamma^{\ell^E}) \circ S(\gamma)$ with $Q(Z) = \prod_{i'}(Z - u_0^{i'\ell^E})$ and $S(Y) = \sum_{k<\ell^E} Y^k$; both are L-linear and act coefficientwise by 1, which gives the formula. For $i \in J \setminus \{0\}$ the factor $i' = i$ vanishes. For a constant $v \in L^r$ only $i = 0$ occurs, σ_{u_0} fixes v, and the coefficient is $\prod_{i'}(1 - u_0^{i'\ell^E}) \cdot \ell^E v = P_{J,E}(1)v$.

3. Let $i \neq 0$. Put $u = 1 + \ell^{s+1}$; then $\sigma_u = \mathrm{id}$ (σ_u depends only on $u \bmod \ell^{s+1}$), and $u^i \neq 1$ since $u \in 1 + \ell\mathbb{Z}_\ell$ is not a root of unity (ℓ odd). So $((\gamma_u - 1)f)^{(i)} = (u^i - 1)f^{(i)} \in K \otimes_L \mathcal{X}'(i)$ gives $f^{(i)} \in K \otimes_L \mathcal{X}'(i)$. Let $i = 0$ and let $\mathcal{G}$ be the finite group of the σ_u ($u \in \mathbb{Z}_\ell^\times$), of order $N = \ell^s(\ell - 1)$, invertible in L. The hypothesis gives $\sigma(f^{(0)}) - f^{(0)} \in K \otimes_L \mathcal{X}'(0)$ for $\sigma \in \mathcal{G}$, with $\mathcal{G}$ acting on $K \otimes_L \mathcal{X}(0)$ through the first factor. Put $d = N^{-1} \sum_{\sigma\in\mathcal{G}} \sigma(f^{(0)})$. Then $d \in (K \otimes_L \mathcal{X}(0))^{\mathcal{G}} = K^{\mathcal{G}} \otimes_L \mathcal{X}(0) = \mathcal{X}(0)$ (choose an L-basis of $\mathcal{X}(0)$; $K^{\mathcal{G}} = L$ by (Lp8)), and $f^{(0)} - d = N^{-1} \sum_\sigma (f^{(0)} - \sigma(f^{(0)})) \in K \otimes_L \mathcal{X}'(0)$. Hence $f - d \in \Xi_s(\mathcal{X}')$. □

Lemma C.89. *Let ℓ, F, L, (τ_a) be as in Proposition 23.7, (V,T) a pair, $\mathbf{N} = \mathbf{N}(T)$ a Wach module with window $(M_a, h_a)_a$ by (Ex1), and fix arc level data for (V,T) (cut c, length $1 \leq \mathrm{len} \leq n$, H, frame). Let $g' \geq 0$ be an integer with $H(c-s) \geq g' + h_{c-s}$ for all $s < \mathrm{len}$. Then:*

1. *$\lambda_{\mathrm{len}}^{-g'} \mathbf{N}_c \subseteq W$; in particular $\mathbf{N}_c \subseteq W$;*
2. *for $x \in W$: $\lambda_{\mathrm{len}}^{g'} x \in \mathbf{N}_c$ if and only if $t^{g'} \mathrm{lv}_s(x)$ is de Rham integral at s for every $s < \mathrm{len}$;*
3. *for $x \in W$ and $k \geq 0$: $x \in X^k \mathbf{N}_c$ if and only if $X^{-k}x \in W$ and $\mathrm{lv}_s(x)$ is de Rham integral at s for every $s < \mathrm{len}$.*

Proof. Write $K = K_{(s)}$. *De Rham integrality is stable under $K[[t]]$*: if f is de Rham integral at s and $c = \sum_{j\geq 0} c_j t^j \in K[[t]]$, the t^i-coefficient of cf is $\sum_{j\geq 0} c_j f^{(i-j)} \in \sum_j K \otimes_L \mathrm{Fil}_s(-i+j) \subseteq K \otimes_L \mathrm{Fil}_s(-i)$ (Fil_s is decreasing); so multiplication by a unit of $K[[t]]$ preserves de Rham integrality in both directions. Recall from Lemma 23.17 ((Lp1)) that $\lambda_s(X) \in K[[t]]^\times$, $\lambda_s(Q_{s'}) \in K[[t]]^\times$ for $s' \neq s$ and $\lambda_s(Q_s) \in tK[[t]]^\times$, so $\lambda_s(\lambda_{\mathrm{len}}) \in tK[[t]]^\times$; and $\mathrm{lv}_s(ax) = \lambda_s(a)\mathrm{lv}_s(x)$ for $a \in e_c \mathbf{A}_{F,L}[1/\lambda_{\mathrm{len}}]$ and x algebraic at level s ($\Lambda_s(ax) = \lambda_s(a)\Lambda_s(x)$).

1. Let $y \in \mathbf{N}_c$ and $x = \lambda_{\mathrm{len}}^{-g'} y$. Define $H'' \colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$ by $H''(c-s) = H(c-s) - g'$ for $s < \mathrm{len}$ (these sites are distinct, $\mathrm{len} \leq n$) and $H''(a) = h_a$ at the other sites; then $H'' \geq h$ pointwise and, as $\lambda_{\mathrm{len}} = \prod_{s<\mathrm{len}} Q_s$, $\Pi x = \Lambda_{c,\mathrm{len}}(H'')y \in P_{\mathrm{len}}(\mathbf{N}_{c-\mathrm{len}})$ by Lemma 23.15 (2). So $x \in W$. For $g' = 0$ this is $\mathbf{N}_c \subseteq W$.

2. Let $x \in W$. Then $\lambda_{\mathrm{len}}^{g'} x \in W$ ($P_{\mathrm{len}}(\mathbf{N}_{c-\mathrm{len}})$ is an $\mathbf{A}_{F,L}$-module), and $\mathrm{lv}_s(\lambda_{\mathrm{len}}^{g'} x) = \lambda_s(\lambda_{\mathrm{len}})^{g'} \mathrm{lv}_s(x) = v\, t^{g'} \mathrm{lv}_s(x)$ with $v \in K[[t]]^\times$. By Lemma 23.5 (2), $\lambda_{\mathrm{len}}^{g'} x \in \mathbf{N}_c$ iff all $\mathrm{lv}_s(\lambda_{\mathrm{len}}^{g'} x)$ are de Rham integral, iff all $t^{g'} \mathrm{lv}_s(x)$ are.

3. If $x \in X^k \mathbf{N}_c$, then $X^{-k} x \in \mathbf{N}_c \subseteq W$ by 1, and $\mathrm{lv}_s(x) = \lambda_s(X)^k \mathrm{lv}_s(X^{-k} x)$ is de Rham integral by Lemma 23.5 (2). Conversely, if $X^{-k} x \in W$ and all $\mathrm{lv}_s(x)$ are de Rham integral, then so are the $\mathrm{lv}_s(X^{-k} x) = \lambda_s(X)^{-k} \mathrm{lv}_s(x)$, and Lemma 23.5 (2) gives $X^{-k} x \in \mathbf{N}_c$. □

Lemma C.90. *Fix arc level data for a pair (any cut, length, H, frame) with parameters G, S, η, kinds κ_s and sizes b_s, and let $g \geq 0$ be an integer with $G, S, \eta \leq g$ and $g < b_s$ whenever κ_s is not thin. Let $s < \mathrm{len}$ and suppose Fil_s satisfies (Lp7). Then for every $i \in \mathbb{Z}$*

$$\mathrm{Nb}_s(i-g) \subseteq \mathrm{Fil}_s(-i) \subseteq \mathrm{Nb}_s(i+g),$$

and consequently, for $f \in K_{(s)}((t))^r$: if f is kind integral at s, then $t^g f$ is de Rham integral at s; if f is de Rham integral at s, then $t^g f$ is kind integral at s.

Proof. Write $\mathrm{Fil} = \mathrm{Fil}_s$, $b = b_s$, and let J be its set of jumps. Since Fil is decreasing, equal to L^r for $u \ll 0$ and to 0 for $u \gg 0$, and constant between consecutive jumps: $\mathrm{Fil}(u) = L^r$ for $u \leq \min J$, $\mathrm{Fil}(u) = 0$ for $u > \max J$, and $\mathrm{Fil}(u) = \mathrm{Fil}(u')$ if no jump lies in $[u, u')$ (if $J = \emptyset$, Fil is constant, equal to L^r and to 0, so $r = 0$ and every claim is trivial). We use $u \leq u' \Rightarrow \mathrm{Fil}(u) \supseteq \mathrm{Fil}(u')$ throughout.

Thin ($J \subseteq [-G, G]$, $\mathrm{Nb}(k) = 0$ for $k < 0$, L^r for $k \geq 0$). Left: if $i - g \geq 0$ then $-i \leq -g \leq -G \leq \min J$, so $\mathrm{Fil}(-i) = L^r$. Right: if $i + g < 0$ then $-i > g \geq \max J$, so $\mathrm{Fil}(-i) = 0$.

Up ($J \subseteq [b, b+S] \cup [-\eta, \eta]$, $U = \mathrm{Fil}(b)$; $\mathrm{Nb}(k) = 0$ for $k < -b$, U for $-b \leq k < 0$, L^r for $k \geq 0$). Left: if $i - g \geq 0$, $-i \leq -g \leq -\eta \leq \min J$ and $\mathrm{Fil}(-i) = L^r$; if $-b \leq i - g < 0$, then $-i \leq b - g \leq b$ and $\mathrm{Fil}(b) \subseteq \mathrm{Fil}(-i)$. Right: if $-b \leq i + g < 0$, then $-i > g \geq \eta$; if $-i \geq b$, $\mathrm{Fil}(-i) \subseteq \mathrm{Fil}(b)$, and if $-i < b$ no jump lies in $[-i, b)$ (it would exceed η and be below b), so $\mathrm{Fil}(-i) = \mathrm{Fil}(b)$; if $i + g < -b$, then $-i > b + g \geq b + S \geq \max J$ and $\mathrm{Fil}(-i) = 0$.

Down ($J \subseteq [-b-S, -b] \cup [-\eta, \eta]$, $U = \mathrm{Fil}(-\eta)$; $\mathrm{Nb}(k) = 0$ for $k < 0$, U for $0 \leq k < b$, L^r for $k \geq b$). Left: if $i - g \geq b$, $-i \leq -b - g \leq -b - S \leq \min J$ and $\mathrm{Fil}(-i) = L^r$; if $0 \leq i - g < b$, $-i \leq -g \leq -\eta$ and $\mathrm{Fil}(-\eta) \subseteq \mathrm{Fil}(-i)$. Right: if $0 \leq i + g < b$, then $g - b < -i \leq g$; if $-i \geq -\eta$, $\mathrm{Fil}(-i) \subseteq \mathrm{Fil}(-\eta)$, and if $-i < -\eta$ no jump lies in $[-i, -\eta)$ (it would be $> g - b \geq -b$, hence not in $[-b-S, -b]$, and $< -\eta$), so $\mathrm{Fil}(-i) = \mathrm{Fil}(-\eta)$; if $i + g < 0$, then $-i > g \geq \eta \geq \max J$ ($b > 0$, so $[-b-S, -b]$ lies below 0) and $\mathrm{Fil}(-i) = 0$.

In each case the remaining instances are trivial ($\mathrm{Nb} = 0$ on the left, $\mathrm{Nb} = L^r$ on the right). Finally, the t^i-coefficient of $t^g f$ is the t^{i-g}-coefficient $f^{(i-g)}$ of f. If f is kind integral, $f^{(i-g)} \in K \otimes_L \mathrm{Nb}(i-g) \subseteq K \otimes_L \mathrm{Fil}(-i)$; if f is de Rham integral, $f^{(i-g)} \in K \otimes_L \mathrm{Fil}(-(i-g)) \subseteq K \otimes_L \mathrm{Nb}(i)$ (the right inclusion at $i - g$). □

Lemma C.91. *Let ℓ, F, L, (τ_a), (V, T), $\mathbf{N} = \mathbf{N}(T)$ with a window (M_a, h_a) and arc level data (cut c, length* len*, H with $H(c-s) \geq h_{c-s}$ for $s <$* len*) satisfying (Lp1), (Lp2), (Lp7), (Lp8) be as in Lemma C.86. Then:*

1. *$(\gamma_u - 1)(W) \subseteq XW$ for every $u \in \mathbb{Z}_\ell^\times$.*

2. *Let $g \geq 0$, $E \geq \mathrm{len} - 1$, $J \subset \mathbb{Z}$ finite containing $\{i : \mathrm{Fil}_s(-i-g) \neq \mathrm{Fil}_s(-i)\}$ for every $s < \mathrm{len}$, and $P = P_{J,E}$. Let $z \in W$ with $\gamma_\delta(z) = z$ for $\delta \in \mu_{\ell-1}$ such that for every $s < \mathrm{len}$ there is $v_s \in L^r$ with $\lambda_s(Q_s)^g(\mathrm{lv}_s(z) - v_s)$ de Rham integral at s. Then $w' = P(\gamma_{u_0})(z)$ lies in W and $(\gamma_u - 1)(w') \in X\mathbf{N}_c$ for every $u \in \mathbb{Z}_\ell^\times$.*

Proof. **Item 1.** Write $P_W = P_{\mathrm{len}}(\mathbf{N}_{c_0})$, $c_0 = c - \mathrm{len}$, and $\gamma = \gamma_u$. For $n \in \mathbf{N}_{c_0}$, $\gamma\varphi^{\mathrm{len}}(n) - \varphi^{\mathrm{len}}(n) = \varphi^{\mathrm{len}}((\gamma-1)n) \in \varphi^{\mathrm{len}}(X\mathbf{N}_{c_0})$ ((Wm3)), and $\varphi^{\mathrm{len}}(X) = X\lambda_{\mathrm{len}} \in X\mathbf{A}_{F,L}$; for $f \in \mathbf{A}_{F,L}$, $\gamma(f) - f \in X\mathbf{A}_{F,L}$. Hence $(\gamma-1)P_W \subseteq XP_W$ (using $\gamma(fp) - fp = \gamma(f)(\gamma p - p) + (\gamma f - f)p$). For each s, $\gamma(Q_s)$ vanishes at the roots of Q_s (Lemma 23.8), so $\gamma(Q_s) = Q_s v_s$ with $v_s \in \mathbb{Z}_\ell[[X]]$; at $X = 0$, $\gamma(Q_s)(0) = Q_s(0) \neq 0$, so $v_s(0) = 1$ and v_s is a unit with $v_s \equiv 1 \bmod X$. Hence $\gamma(\Pi) = \Pi v$ with $v \equiv 1 \bmod X$ a unit. For $w \in W$ put $p = \Pi w \in P_W$; then

$$\Pi(\gamma w - w) = v^{-1}\gamma(p) - p = v^{-1}(\gamma p - p) + (v^{-1} - 1)p \in XP_W,$$

so $X^{-1}(\gamma w - w) \in W$.

Item 2. By (Lp2) and item 1, $w' \in W$ and $\gamma_\delta w' = w'$. Fix $s < \mathrm{len}$ and put $e = \mathrm{lv}_s(z)$, $e' = e - v_s$; e is Δ-invariant by (Lp2). The family $\mathcal{X}(i) = \mathrm{Fil}_s(-i)$ is increasing (Fil_s is decreasing, (Lp7)), and $\lambda_s(Q_s)^g e' \in \Xi_s(\mathcal{X})$; item 2 of Lemma C.87 gives $\mathrm{lv}_s(w') = P(\gamma_{u_0}^{(s)})e \in L^r + \Xi_s(\mathcal{X})$ (using (Lp2) for $\mathrm{lv}_s(\gamma^k z)$). For $u \in \mathbb{Z}_\ell^\times$, $\gamma_u^{(s)}$ fixes L^r and maps $\Xi_s(\mathcal{X})$ into itself (it multiplies the t^i-coefficient by u^i and applies σ_u, which preserves $K_{(s)} \otimes \mathcal{X}(i)$). Hence $\mathrm{lv}_s((\gamma_u - 1)w') = (\gamma_u^{(s)} - 1)\mathrm{lv}_s(w')$ is de Rham integral at s, for every s. By item 1, $X^{-1}(\gamma_u - 1)w' \in W$, and item 3 of Lemma C.89 (with $g' = 0$, $k = 1$) gives $(\gamma_u - 1)w' \in X\mathbf{N}_c$. □

Proposition C.92. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$), the weight word, Θ, G, g, h_*, H, j, $(\mathcal{C}, (\mathrm{len}_c))$, $\sigma_{\mathrm{E}}, \rho_{\mathrm{E}}, a_{\mathrm{E}}$, (Ae1), (Ae2), $c \in \mathcal{C}$ and the arc level data of Φ along the word at the cut c of length len_c, with the arc level properties, be as in Proposition C.85; $r_\nu = \dim_L V_\nu$. Let $x' \in W_2$ with $\gamma_\delta(x') = x'$ for $\delta \in \mu_{\ell-1}$ such that for every $s < \mathrm{len}_c$, $\mathrm{lv}_s^{(2)}(x') \in L^{r_2} + \{$kind integral vectors at $s\}$. Then there is $z \in W_1$ with*

$$\mathbf{N}(\Phi)(z) = \ell^{\,2\sigma_{\mathrm{E}} + 2a_{\mathrm{E}} + \rho_{\mathrm{E}}}\, x'$$

such that for every $s < \mathrm{len}_c$ there is $v_s \in L^{r_1}$ with $\lambda_s(Q_s)^g(\mathrm{lv}_s^{(1)}(z) - v_s)$ de Rham integral at s.

Proof. Write $\mathrm{len} = \mathrm{len}_c$ and $e_1 = 2\sigma_{\mathrm{E}} + 2a_{\mathrm{E}}$, and let κ_s be the kinds of the arc level data along the word.

Step 1: kind integral vectors and constants. For each $s < \mathrm{len}$ the kind integral vectors at s form an additive group, stable under multiplication by elements of $\mathbb{Z}_\ell$: the condition is that the t^i-coefficient lies in the L-subspace $K_{(s)} \otimes_L \mathrm{Nb}_s(i)$, for every i. The same holds for de Rham integral vectors, and $\mathrm{lv}_s^{(\nu)}$ is additive and $\mathbb{Z}_\ell$-linear (Λ_s is, and the expansion in the R_s-basis $(D_k^{(s)})_k$ is unique). If κ_s is thin or up, then $\mathrm{Nb}_s(i) = L^{r_2}$ for $i \geq 0$, so every constant $v \in L^{r_2}$ (a single t^0-coefficient in $K_{(s)} \otimes_L L^{r_2}$) is kind integral at s.

Step 2: no down level. Suppose κ_s is thin or up for every $s < \mathrm{len}$. By the hypothesis and Step 1, $\mathrm{lv}_s^{(2)}(x')$ is kind integral at every $s < \mathrm{len}$. Lemma C.93 gives $z' \in W_1$ with $\mathbf{N}(\Phi)(z') = \ell^{\rho_{\mathrm{E}}} x'$ and $\lambda_s(Q_s)^g \mathrm{lv}_s^{(1)}(z')$ de Rham integral for all s. Then $z = \ell^{e_1} z' \in W_1$ satisfies $\mathbf{N}(\Phi)(z) = \ell^{e_1 + \rho_{\mathrm{E}}} x'$, and the assertion holds with $v_s = 0$.

Step 3: a down level. Suppose κ_s is down for some $s < \mathrm{len}$. Proposition C.94 gives $z_0 \in W_1$ and $v_s \in L^{r_1}$ with $\lambda_s(Q_s)^g(\mathrm{lv}_s^{(1)}(z_0) - v_s)$ de Rham integral and $\mathrm{lv}_s^{(2)}(x'')$ kind integral for every $s < \mathrm{len}$, where $x'' = \ell^{e_1} x' - \mathbf{N}(\Phi)(z_0)$. Here $x'' \in W_2$, as W_2 is an $\mathbf{A}_{F,L}$-module and

$\mathbf{N}(\Phi)(W_1) \subseteq W_2$ (Definition 23.4). Lemma C.93 applied to x'' gives $z'' \in W_1$ with $\mathbf{N}(\Phi)(z'') = \ell^{\rho_{\mathrm{E}}} x''$ and $\lambda_s(Q_s)^g \mathrm{lv}_s^{(1)}(z'')$ de Rham integral for all s. Put $z = \ell^{\rho_{\mathrm{E}}} z_0 + z'' \in W_1$. Then

$$\mathbf{N}(\Phi)(z) = \ell^{\rho_{\mathrm{E}}} \mathbf{N}(\Phi)(z_0) + \ell^{\rho_{\mathrm{E}}} \big(\ell^{e_1} x' - \mathbf{N}(\Phi)(z_0)\big) = \ell^{\,2\sigma_{\mathrm{E}} + 2a_{\mathrm{E}} + \rho_{\mathrm{E}}} x',$$

and, with $v_s' = \ell^{\rho_{\mathrm{E}}} v_s \in L^{r_1}$,

$$\lambda_s(Q_s)^g \big(\mathrm{lv}_s^{(1)}(z) - v_s'\big) = \ell^{\rho_{\mathrm{E}}}\, \lambda_s(Q_s)^g \big(\mathrm{lv}_s^{(1)}(z_0) - v_s\big) + \lambda_s(Q_s)^g\, \mathrm{lv}_s^{(1)}(z'')$$

is a sum of two de Rham integral vectors at s, hence de Rham integral (Step 1). This is the assertion. □

Lemma C.93. *Let $\ell \geq 11$, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$), the weight word $(b, w, S, \eta, m, (s_a), (\epsilon_a))$, Θ, G, g, h_*, H, j, $(\mathcal{C}, (\mathrm{len}_c))$ and (Ae1) be as in Proposition C.83, and form ρ_{E} with the same data. Let $c \in \mathcal{C}$, $\mathrm{len} = \mathrm{len}_c$, and suppose that for some frames at $c - \mathrm{len}$ the arc level data of Φ along the word at the cut c of length len have the arc level properties; let W_1, W_2 be their arc domains. If $x \in W_2$ and $\mathrm{lv}_s^{(2)}(x)$ is kind integral at s for every $s < \mathrm{len}$, then there is $z \in W_1$ with*

$$\mathbf{N}(\Phi)(z) = \ell^{\,\rho_{\mathrm{E}}}\, x$$

such that $\lambda_s(Q_s)^g\, \mathrm{lv}_s^{(1)}(z)$ is de Rham integral at s for every $s < \mathrm{len}$.

Proof. Write $\mathbf{N}_\nu = \mathbf{N}(T_\nu)$, $\lambda = \lambda_{\mathrm{len}} = \prod_{s<\mathrm{len}} Q_s$, and let κ_s, $b_s = s_{c-s}$ and $H_c(c-s)$ $(= g + G$, $g + s_{c-s} + S$, $g + \eta$ for κ_s thin, up, down) be the kinds, sizes and heights of the arc level data along the word.

Step 0: a window and the parameters. Put $(M_a, h_a) = (\eta, \max(s_a + S, \eta))$ if $\epsilon_a = 1$ and $(M_a, h_a) = (\max(s_a + S, \eta), \eta)$ if $\epsilon_a = -1$. By (Ow1) every Hodge jump of V_1 at τ_a lies in $[-M_a, h_a]$, and by item 1 of Lemma 24.2 so does every Hodge jump of V_2. At a thin site $s_a \leq w$ (Definition 23.1), so $M_a, h_a \leq \max(w + S, \eta) \leq G$; at a thick site $s_a > g \geq G \geq \eta$ by (Ae1), and $s_a \leq b + \eta$ by (Ow3), so $\max(s_a + S, \eta) = s_a + S \leq h_*$. Consequently:

1. $H_c(c-s) \geq g + h_{c-s}$ for $s < \mathrm{len}$: at a thin site $h \leq G$ and $H_c = g + G$; at a thick site with $\epsilon = 1$, $h = s_{c-s} + S$ and $H_c = g + s_{c-s} + S$; with $\epsilon = -1$, $h = \eta$ and $H_c = g + \eta$;
2. $M_a + h_a \leq H(a)$ for every a: $M_a + h_a \leq 2G = H(a)$ at a thin site, and $M_a + h_a \leq h_* + \eta \leq 2h_* = H(a)$ at a thick site ($\eta \leq g < s_a \leq h_*$);
3. $G, S, \eta \leq g$ (as $g = \max(G, 1)$, $G = w + S + \eta$), and $b_s = s_{c-s} > g$ whenever κ_s is not thin (then $c - s$ is thick; (Ae1)).

Step 1: from kind integral levels to the lattice. Let $s < \mathrm{len}$. By (Lp7) for T_2 and item 3 of Step 0, Lemma C.90 applies at s: since $\mathrm{lv}_s^{(2)}(x)$ is kind integral, $t^g \mathrm{lv}_s^{(2)}(x)$ is de Rham integral at s. By item 1 of Step 0 and item 2 of Lemma C.89 for T_2 with $g' = g$ (and the window of Step 0), $y := \lambda^g x \in \mathbf{N}_{2,c}$.

Step 2: lifting. $\Phi\colon T_1 \to T_2$ is a G_F-equivariant $\mathcal{O}_L$-linear surjection and $\ell \geq 5$; by Step 0 every Hodge jump of V_1 at τ_a lies in $[-M_a, h_a]$. Proposition C.6 (with $f = \Phi$, the window (M_a, h_a), the function $a \mapsto M_a + h_a$) gives $m \in \mathbf{N}_{1,c}$ with $\mathbf{N}(\Phi)(m) = \ell^{r_c} y$, where r_c is formed with $a \mapsto M_a + h_a$. By item 2 of Step 0 this function is bounded by H pointwise; the sum s_c, the maximum and c_H are monotone in the function, so $r_c \leq \rho_{\mathrm{E}}$ (definition of ρ_{E} in Definition C.78). Put $m' = \ell^{\rho_{\mathrm{E}} - r_c} m \in \mathbf{N}_{1,c}$.

Step 3: the lift z. Let $z = \lambda^{-g} m' \in e_c \mathbf{N}_1[1/\lambda_{\mathrm{len}}]$. By item 1 of Step 0 and item 1 of Lemma C.89 for T_1 with $g' = g$, $z \in W_1$. The map $\mathbf{N}(\Phi)$ is $\mathbf{A}_{F,L}$-linear, hence commutes with multiplication by $\lambda^{\pm g}$ on the localisation, and

$$\mathbf{N}(\Phi)(z) = \lambda^{-g} \ell^{\rho_{\mathrm{E}} - r_c} \mathbf{N}(\Phi)(m) = \lambda^{-g} \ell^{\rho_{\mathrm{E}}} \lambda^g x = \ell^{\rho_{\mathrm{E}}} x.$$

Step 4: the levels of z. Let $s < \mathrm{len}$. Since $m' \in \mathbf{N}_{1,c} \subseteq W_1$ (item 1 of Lemma C.89 with $g' = 0$), (Lp3) for T_1 shows that $\mathrm{lv}_s^{(1)}(m')$ is de Rham integral at s. The level map satisfies $\Lambda_s(aw) = \lambda_s(a)\Lambda_s(w)$, and the expansion in the R_s-basis $(D_k^{(s)})_k$ is unique, so $\mathrm{lv}_s^{(1)}(z) = \lambda_s(\lambda)^{-g}\, \mathrm{lv}_s^{(1)}(m')$ and

$$\lambda_s(Q_s)^g\, \mathrm{lv}_s^{(1)}(z) = u^{-g}\, \mathrm{lv}_s^{(1)}(m'), \qquad u = \lambda_s\Big(\prod_{s' < \mathrm{len},\, s' \neq s} Q_{s'} \Big) \in K_{(s)}[[t]]^\times$$

by (Lp1). Multiplication by an element $v = \sum_{i \geq 0} v_i t^i$ of $K_{(s)}[[t]]$ preserves de Rham integrality: Fil_s^1 is decreasing ((Lp7)), so if the t^j-coefficient f_j of f lies in $K_{(s)} \otimes_L \mathrm{Fil}_s^1(-j)$ for all j, then the t^i-coefficient $\sum_{j \leq i} v_{i-j} f_j$ of vf (a finite sum, f being a Laurent series) lies in $K_{(s)} \otimes_L \mathrm{Fil}_s^1(-i)$. Applied with $v = u^{-g}$, $\lambda_s(Q_s)^g \mathrm{lv}_s^{(1)}(z)$ is de Rham integral at s. □

Proposition C.94. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$), the weight word, Θ, G, g, h_*, H, j, $(\mathcal{C}, (\mathrm{len}_c))$, $\sigma_{\mathrm{E}}, a_{\mathrm{E}}$, (Ae1), (Ae2), $c \in \mathcal{C}$ and the arc level data of Φ along the word at the cut c of length len_c, with the arc level properties, be as in Proposition C.92; $r_\nu = \dim_L V_\nu$. Suppose that κ_s is down for some $s < \mathrm{len}_c$. Let $x' \in W_2$ with $\gamma_\delta(x') = x'$ for $\delta \in \mu_{\ell-1}$ such that for every $s < \mathrm{len}_c$, $\mathrm{lv}_s^{(2)}(x') \in L^{r_2} + \{$kind integral vectors at $s\}$. Then there is $z_0 \in W_1$ such that for every $s < \mathrm{len}_c$:*

1. *there is $v_s \in L^{r_1}$ with $\lambda_s(Q_s)^g\big(\mathrm{lv}_s^{(1)}(z_0) - v_s\big)$ de Rham integral at s;*
2. *$\mathrm{lv}_s^{(2)}\big(\ell^{2\sigma_{\mathrm{E}} + 2a_{\mathrm{E}}} x' - \mathbf{N}(\Phi)(z_0)\big)$ is kind integral at s.*

Proof. Write $\mathrm{len} = \mathrm{len}_c$, $\Pi = \prod_{s < \mathrm{len}} Q_s^{H(c-s)}$ (the same for $\nu = 1, 2$), $A = e_c \mathbf{A}_{F,L}$, $a = a_{\mathrm{E}}$, and let κ_s, $b_s = s_{c-s}$, $H(c-s)$, $M = \lfloor (h_* + 2g)/(\ell - 1) \rfloor + 1$ and j be the data of the arc level data along the word.

Step 0: parameters. $\sigma := \sigma_\ell(3g, (\ell-1)M, j) = \lfloor (3g + M/\ell^{j-1})\ell/(\ell-1)^2 \rfloor + 1 \leq \sigma_{\mathrm{E}}$ (Definition C.78, with $h = h_*$). By (Ae1), items 2 and 3 of Lemma C.102 apply: $h_* \geq g$; $G, S, \eta \leq g$; $b_s = s_{c-s} > g \geq 1$ for thick $c - s$; and $H(c-s) = g + b_s + S \geq b_s$ for an up level (Definition 23.4). By (Cc2) $c_0 = c - \mathrm{len} \in \mathcal{C}$ and $j \leq \mathrm{len}$, and by (Cc1) for c_0 the sites $c_0 + 1, \ldots, c_0 + j$ are thin; so Lemma C.105 applies for $\nu = 1, 2$: every truncation small vector is de Rham integral and kind integral. $\mathcal{T}_s^\nu$ is a $K_{(s)}[[t]]$-submodule (its definition is a $K_{(s)}[[t]]$-linear condition), so $o\,\mathcal{T}_s^1 \subseteq \mathcal{T}_s^2$, where o is the matrix of the level map (Lh1) of Definition 23.4. The kind and the de Rham integral vectors form additive groups stable under $\mathbb{Z}_\ell$ and, the families being increasing in i, under multiplication by $K_{(s)}[[t]]$. Finally $\rho_{\mathrm{E}} \leq a$ ($a = 2\sigma_{\mathrm{E}} + \rho_{\mathrm{E}} + 2g\,\mathrm{len}_{\max}$).

Step 1: the word. By (Cc3) at most two sites of the arc are thick, and κ_s is up or down exactly when $c - s$ is thick; by hypothesis some κ_s is down. So exactly one of the following holds: (D) one level s_1 is down, all others thin; (DD) two levels are down, all others thin; (DU) a down level s_1 and an up level $s_0 < s_1$, all others thin; (UD) a down level s_0 and an up level $s_1 > s_0$, all others thin. In case (DU), clause 1 of (Ae2) (with $s = s_0$, $t = s_1$) gives $b_{s_0} < \ell\, b_{s_1}$.

Step 2: partner of x'. By item 1 of Lemma C.95 for T_2 there is $y \in A^{r_2}$ with $\hat{y}_s - \ell^\sigma \mathrm{lv}_s^{(2)}(x') \in \mathcal{T}_s^2$ for all $s < \mathrm{len}$. By hypothesis and Step 0, $\hat{y}_s \in L^{r_2} + \{$kind integral at $s\}$ for all s.

Step 3: constants of y and their lifts. We find $\lambda, \lambda' \in \mathcal{O}_L^{r_2}$, a scalar $\theta \in A[1/\lambda_{\mathrm{len}}]$ with $\Pi\theta \in A$, and $v, v' \in \mathcal{O}_L^{r_1}$ such that, for every $s < \mathrm{len}$,

(a) $\hat{y}_s - \lambda - \lambda_s(\theta)\lambda'$ is kind integral;

(b) $ov = \ell^{2a}\lambda$, $ov' = \ell^{2a}\lambda'$;

(c) $\lambda_s(\theta)v' \in L^{r_1} + \lambda_s(Q_s)^{-g}\{\text{de Rham integral at } s\}$.

In cases (D), (DU): Lemma C.97 gives λ; put $\theta = 0$, $\lambda' = v' = 0$. In case (DD): Lemma C.98 gives $f \in Q_{s_1}^{b_{s_1}} A$ with $f - \varpi^\rho \in Q_{s_0}^{b_{s_0}} A$ and λ, λ'; put $\theta = f$. In case (UD): Lemma C.99 gives $\mathfrak{c}, \alpha, \beta$ with $P_1 = Q_{s_1}^{b_{s_1}} = \varpi^{\mathfrak{c}}\alpha + Q_{s_0}^{b_{s_0}}\beta$, λ, and $\lambda' \in U_{s_1}^{2,L} \cap \mathcal{O}_L^{r_2}$; put $\theta = \alpha/P_1$ ($\Pi\theta = \alpha\,\Pi/P_1 \in A$ as $b_{s_1} \le H(c - s_1)$). For (b): v is $\ell^{2a-\rho_{\mathrm{E}}}$ times the vector of Lemma C.101 for λ; in case (DD) v' likewise for λ'; in case (UD) $v' \in U_{s_1}^{1,L} \cap \mathcal{O}_L^{r_1}$ is given by Proposition C.103. For (c), with $\lambda_s(Q_s)^g = t^g \cdot \mathrm{unit}$ ((Lp1)) and Step 0: at a thin level, $\lambda_s(\theta) \in K[[t]]$ (for (UD) $\lambda_s(P_1)$ is a unit there), and $t^g K[[t]]v'$ is de Rham integral since $\mathrm{Fil}_s^1(u) = L^{r_1}$ for $u \le -G$ ((Lp7)) and $g \ge G$. In case (DD) at s_1: $\lambda_{s_1}(f) \in t^{b_{s_1}}K[[t]]$, and $t^{g+b_{s_1}}K[[t]]v'$ is de Rham integral ($\mathrm{Fil}_{s_1}^1(u) = L^{r_1}$ for $u \le -b_{s_1} - S$, $S \le g$); at s_0: $\lambda_{s_0}(f) \in \varpi^\rho + t^{b_{s_0}}K[[t]]$, so take the constant $\varpi^\rho v'$. In case (UD) at s_1: $\lambda_{s_1}(\theta) \in t^{-b_{s_1}}K[[t]]$ and $v' \in U_{s_1}^{1,L} = \mathrm{Fil}_{s_1}^1(b_{s_1}) \subseteq \mathrm{Fil}_{s_1}^1(u)$ for $u \le b_{s_1}$, so $t^g\lambda_{s_1}(\theta)v'$ is de Rham integral; at s_0: $\alpha/P_1 = \varpi^{-\mathfrak{c}}(1 - Q_{s_0}^{b_{s_0}}\beta/P_1)$ and $\lambda_{s_0}(P_1)$ is a unit, so $\lambda_{s_0}(\theta) \in \varpi^{-\mathfrak{c}} + t^{b_{s_0}}K[[t]]$; take the constant $\varpi^{-\mathfrak{c}}v'$ and use the down-level argument of case (DD).

Step 4: the vector z_0. Let $y_1 = \Pi(v + \theta v') \in A^{r_1}$, so $\hat{y}_{1,s} = v + \lambda_s(\theta)v'$. By item 2 of Lemma C.95 for T_1 there is $z \in W_1$ with $\mathrm{lv}_s^{(1)}(z) - \ell^\sigma \hat{y}_{1,s} \in \mathcal{T}_s^1$. Put $z_0 = \ell^{2\sigma_{\mathrm{E}} - 2\sigma} z \in W_1$.

Step 5: item 1. $\mathrm{lv}_s^{(1)}(z_0) = \ell^{2\sigma_{\mathrm{E}}-\sigma}(v + \lambda_s(\theta)v') + \tau$ with $\tau \in \mathcal{T}_s^1$; by (c) and Step 0 (truncation small vectors are de Rham integral; multiplication by $\lambda_s(Q_s)^g \in K[[t]]$ preserves this), there is $v_s \in L^{r_1}$ with $\lambda_s(Q_s)^g(\mathrm{lv}_s^{(1)}(z_0) - v_s)$ de Rham integral.

Step 6: item 2. By (Lh1), (b) and Step 0, $\mathrm{lv}_s^{(2)}(\mathbf{N}(\Phi)(z)) = o\,\mathrm{lv}_s^{(1)}(z) \in \ell^{\sigma+2a}(\lambda + \lambda_s(\theta)\lambda') + \mathcal{T}_s^2$. By Step 2, $\ell^{\sigma+2a}\ell^\sigma \mathrm{lv}_s^{(2)}(x') \in \ell^{\sigma+2a}\hat{y}_s + \mathcal{T}_s^2$. Hence

$$\mathrm{lv}_s^{(2)}\big(\ell^{2\sigma+2a}x' - \mathbf{N}(\Phi)(z)\big) \in \ell^{\sigma+2a}\big(\hat{y}_s - \lambda - \lambda_s(\theta)\lambda'\big) + \mathcal{T}_s^2,$$

which is kind integral by (a) and Step 0. Multiplying by $\ell^{2\sigma_{\mathrm{E}}-2\sigma}$: $\mathrm{lv}_s^{(2)}(\ell^{2\sigma_{\mathrm{E}}+2a}x' - \mathbf{N}(\Phi)(z_0))$ is kind integral for every $s < \mathrm{len}$. The Δ-invariance of x' is not used. □

Lemma C.95. *Let ℓ, F, L, (τ_a) be as in Proposition 23.7, (V,T) a pair of rank r, $\mathbf{N} = \mathbf{N}(T)$, and fix arc level data for (V,T) (cut c, length* len*, H, Π, frame $\mathbf{n}$ at $c_0 = c - \mathrm{len}$, parameters g, M, j with $1 \le j \le \mathrm{len}$). Put $\sigma = \sigma_\ell(3g, (\ell-1)M, j)$. Then:*

1. *for every $w \in W$ there is $y \in (e_c\mathbf{A}_{F,L})^r$ with $\hat{y}_s - \ell^\sigma \mathrm{lv}_s(w) \in \mathcal{T}_s$ for every $s < \mathrm{len}$;*
2. *for every $y \in (e_c\mathbf{A}_{F,L})^r$ there is $w \in W$ with $\mathrm{lv}_s(w) - \ell^\sigma\hat{y}_s \in \mathcal{T}_s$ for every $s < \mathrm{len}$.*

Proof. Put $m = (\ell-1)M$, $K = K_{(s)}$, $q_s = \mathrm{len} - s - 1 \ge 0$ for $s < \mathrm{len}$, and let $H_k = \sum_l h_{kl}n_l$ be the horizontal sections through the frame ($h_{kl} \in e_{c_0}E[[X]]$, Lemma 23.9 (Hf2)).

Step 0: denominators. Let $\mathcal{R}$ be the set of $\sum_i a_i X^i \in E[[X]]$ with $\ell^{s_\ell(i)}a_i \in \Lambda$ for all i. The function $s_\ell(i) = J + v_\ell(J!)$, $J = \lfloor i/(\ell-1)\rfloor$, is superadditive ($\lfloor a/(\ell-1)\rfloor + \lfloor b/(\ell-1)\rfloor \le \lfloor (a+b)/(\ell-1)\rfloor$, $v_\ell(J_1!) + v_\ell(J_2!) \le v_\ell((J_1+J_2)!)$, and $J \mapsto J + v_\ell(J!)$ is increasing) with $s_\ell(0) = 0$, so $\mathcal{R}$ is a subring of $E[[X]]$, closed under X-adic limits. By (Hf3) $h = (h_{kl}) \in M_r(\mathcal{R})$,

and $h \equiv 1 \bmod X$ ($H_k \in n_k + X\mathbf{N}_E$); hence $h = 1 - Z$ with $Z \in X\,M_r(\mathcal{R})$ and $h' = h^{-1} = \sum_{i\geq 0} Z^i \in M_r(\mathcal{R})$, i.e. $n_l = \sum_k h'_{lk} H_k$.

Step 1: truncated relay of one coordinate. Let $R \geq m$ and $u = \sum_i a_i X^i \in e_{c_0}\mathcal{R}$. Then

$$\ell^\sigma \varphi^j(u) = a(u) + X^m \lambda_j^{3g}\, b(u) + \ell^\sigma \varphi^j\big(X^R u_R\big), \qquad a(u) \in \mathbf{A}_{F,L},\ b(u) \in \mathbf{A}_{F,L}[1/\ell],$$

where $u_R = \sum_{i\geq R} a_i X^{i-R} \in E[[X]]$. Indeed $\varphi^j(X) = X\lambda_j$ and φ^j acts on coefficients by an automorphism of Λ, so $\ell^\sigma\varphi^j(\sum_{i<R} a_i X^i) = \sum_{i<R} \ell^\sigma\varphi^j(a_i)(X\lambda_j)^i$, and (Rt1) of Lemma 24.6 (with $3g$ in place of g; $\ell^{s_\ell(i)}\varphi^j(a_i) \in \Lambda$) writes each of these finitely many terms as an element of $\Lambda[[X]]$ plus an element of $X^m\lambda_j^{3g}\Lambda[[X]][1/\ell]$.

Step 2: levels of Frobenius pull-backs. For $Y = \sum_l c_l n_l \in e_{c_0}\mathbf{N}_E$ and $s < \mathrm{len}$, $G^{(q_s)}(Y) = \sum_l \mathrm{ev}^{(q_s)}(c_l)\varphi^{q_s}(\iota_T(n_l))$ (Lemma 23.13); for $Y \in \mathbf{N}_{c_0}$, $\Lambda_s(\varphi^{\mathrm{len}}(Y)) = G^{(q_s)}(Y)$, and $\lambda_s(\varphi^{\mathrm{len}}(a)) = \mathrm{ev}^{(q_s)}(a)$ for $a \in e_{c_0}\mathbf{A}_{F,L}[1/\ell]$ ((Eq3) of Lemma C.69, with $q = q_s$). By Lemma 23.13 (1), $G^{(q_s)}(H_k) = \varphi^{q_s-3}(d_k) = D_k^{(s)}$. Hence

$$\Lambda_s(\varphi^{\mathrm{len}}(n_l)) = \sum_k \mathrm{ev}^{(q_s)}(h'_{lk})\, D_k^{(s)},$$

so $\mathrm{lv}_s(\varphi^{\mathrm{len}}(n_l)) = (\mathrm{ev}^{(q_s)}(h'_{lk}))_k$; as $\varphi^{\mathrm{len}}(n_l) \in P_{\mathrm{len}}(\mathbf{N}_{c_0}) \subseteq W$ (Lemma C.96 item 1), this lies in $K((t))^r$ by (Lp1), and also in $(R_s^+)^r$; since $K((t)) \cap t^i R_s^+ = t^i K[[t]]$ (θ is injective on K), these coordinates lie in $K[[t]]$. Likewise, for $u \in e_{c_0}\mathcal{R}$, $\mathrm{ev}^{(q_s)}(X^R u_R) \in t^R R_s^+$ ($\mathrm{ev}^{(q)}(X) = \varphi^q(\mu) \in tB_{\mathrm{dR}}^+$), and whenever it lies in $K((t))$ it lies in $t^R K[[t]]$.

Step 3: the truncation ideal. As $\varphi^k(X) = (1+X)^{\ell^k} - 1 = X\prod_{s'<k} Q_{s'}$ and $\varphi^k(Q_{s'}) = Q_{s'+k}$,

$$f' = \varphi^{\mathrm{len}-j}(X^m\lambda_j^{3g}) = X^m \prod_{s'<\mathrm{len}-j} Q_{s'}^m \prod_{\mathrm{len}-j\leq s'<\mathrm{len}} Q_{s'}^{3g}.$$

By (Lp1) (Lemma 23.17) $\lambda_s(X)$ and $\lambda_s(Q_{s'})$ for $s' \neq s$ are units of $K[[t]]$ and $\lambda_s(Q_s) \in tK[[t]]^\times$, so $\lambda_s(f') \in t^{e_s}K[[t]]^\times$ with $e_s = m$ or $e_s = 3g$, and $e_s \leq m \leq R$ ($3g \leq m$); so $t^R K[[t]] \subseteq \lambda_s(f')K[[t]]$, and $\lambda_s(\varphi^{\mathrm{len}-j}(X^m\lambda_j^{3g}b)) = \lambda_s(f')\lambda_s(\varphi^{\mathrm{len}-j}(b)) \in \lambda_s(f')K[[t]]$ for $b \in \mathbf{A}_{F,L}[1/\ell]$ ((Lp1)). By definition, $f \in \mathcal{T}_s$ if and only if $\lambda_s(\Pi)f \in \lambda_s(f')K[[t]]^r$.

Item 1. Let $w \in W$, so $\Pi w = \sum_l y_l^0 \varphi^{\mathrm{len}}(n_l)$ with $y_l^0 \in e_c\mathbf{A}_{F,L}$ ($\mathbf{n}$ generates $\mathbf{N}_{c_0}$). By Step 2, $\lambda_s(\Pi)\mathrm{lv}_s(w)_k = \sum_l \lambda_s(y_l^0)\,\mathrm{ev}^{(q_s)}(h'_{lk})$. Apply Step 1 to $u = h'_{lk}$ and put $y_k = \sum_l \varphi^{\mathrm{len}-j}(a(h'_{lk}))\, y_l^0 \in e_c\mathbf{A}_{F,L}$. Applying $\varphi^{\mathrm{len}-j}$ to the identity of Step 1 and then λ_s (via $\mathrm{ev}^{(q_s)}$ for the last term),

$$\ell^\sigma \mathrm{ev}^{(q_s)}(h'_{lk}) = \lambda_s(\varphi^{\mathrm{len}-j}(a(h'_{lk}))) + \lambda_s(f')\lambda_s(\varphi^{\mathrm{len}-j}(b(h'_{lk}))) + \ell^\sigma \mathrm{ev}^{(q_s)}\big(X^R(h'_{lk})_R\big);$$

the first two terms lie in $K[[t]]$, hence so does the third, which therefore lies in $t^R K[[t]]$ (Step 2). Multiplying by $\lambda_s(y_l^0) \in K[[t]]$ and summing over l, $\ell^\sigma\lambda_s(\Pi)\mathrm{lv}_s(w) - \lambda_s(y) \in \lambda_s(f')K[[t]]^r$ (Step 3), i.e. $\hat{y}_s - \ell^\sigma \mathrm{lv}_s(w) \in \mathcal{T}_s$.

Item 2. Let $y \in (e_c\mathbf{A}_{F,L})^r$. Apply Step 1 to $u = h_{kl}$, put $I_k = \sum_l a(h_{kl})\varphi^j(n_l) \in P_j(\mathbf{N}_{c_0})$, and let $w = \Pi^{-1}\sum_k y_k \varphi^{\mathrm{len}-j}(I_k)$. Since $\varphi^{\mathrm{len}-j}$ maps $P_j(\mathbf{N}_{c_0})$ into $P_{\mathrm{len}}(\mathbf{N}_{c_0})$, $\Pi w \in P_{\mathrm{len}}(\mathbf{N}_{c_0})$ and $w \in W$. Since $\varphi^{\mathrm{len}-j}(I_k) = \sum_l \varphi^{\mathrm{len}-j}(a(h_{kl}))\varphi^{\mathrm{len}}(n_l)$, Step 2 gives $\mathrm{lv}_s(\varphi^{\mathrm{len}-j}(I_k)) = \sum_l \lambda_s(\varphi^{\mathrm{len}-j}(a(h_{kl})))\,\mathrm{lv}_s(\varphi^{\mathrm{len}}(n_l))$. Exactly as in item 1 (Step 1 for $u = h_{kl}$, $\varphi^{\mathrm{len}-j}$, then λ_s and $\mathrm{ev}^{(q_s)}$, which agree on φ^{len} of elements of $e_{c_0}\mathbf{A}_{F,L}[1/\ell]$), $\lambda_s(\varphi^{\mathrm{len}-j}(a(h_{kl}))) \in \ell^\sigma \mathrm{ev}^{(q_s)}(h_{kl}) + \lambda_s(f')K[[t]] + t^R K[[t]]$, where $\mathrm{ev}^{(q_s)}(h) = \mathrm{ev}^{(q_s)}(h')^{-1}$ has entries in $K[[t]]$ because $\mathrm{ev}^{(q_s)}(h') \in M_r(K[[t]])$ (Step 2) and $\mathrm{ev}^{(q_s)}(h') - 1 \in M_r(K[[t]] \cap tR_s^+) = M_r(tK[[t]])$ ($h' \equiv 1 \bmod X$); the term $\ell^\sigma\mathrm{ev}^{(q_s)}(X^R(h_{kl})_R)$ is then in $K[[t]] \cap t^R R_s^+ = t^R K[[t]]$. The coordinates $\mathrm{lv}_s(\varphi^{\mathrm{len}}(n_l)) = (\mathrm{ev}^{(q_s)}(h'_{lk'}))_{k'}$ lie in $K[[t]]$ (Step 2), and $\mathrm{ev}^{(q_s)}$ is a ring homomorphism ((Eq1) of Lemma C.69)

with $hh' = 1$, so $\sum_l \mathrm{ev}^{(q_s)}(h_{kl})\mathrm{ev}^{(q_s)}(h'_{lk'}) = \delta_{kk'}$. Hence $\mathrm{lv}_s(\varphi^{\mathrm{len}-j}(I_k)) \in \ell^\sigma e_k + \lambda_s(f')K[[t]]^r + t^R K[[t]]^r = \ell^\sigma e_k + \lambda_s(f')K[[t]]^r$ (Step 3), e_k the k-th unit vector. Therefore $\lambda_s(\Pi)\mathrm{lv}_s(w) = \sum_k \lambda_s(y_k)\,\mathrm{lv}_s(\varphi^{\mathrm{len}-j}(I_k)) \in \ell^\sigma\lambda_s(y) + \lambda_s(f')K[[t]]^r$, i.e. $\mathrm{lv}_s(w) - \ell^\sigma\hat{y}_s \in \mathcal{T}_s$. □

Lemma C.96. *Let ℓ, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ be as in Proposition 23.7, let (V',T'), (V,T), (V'',T'') be pairs with an exact sequence $0 \to T' \xrightarrow{\iota'} T \xrightarrow{\pi} T'' \to 0$ of G_F-equivariant $\mathcal{O}_L$-linear maps, and let $(M_a,h_a)_a$ be a window of V as in (Ex1). Let $c \in \mathbb{Z}/n$, $1 \le e \le n$, and $H\colon \mathbb{Z}/n \to \mathbb{Z}_{\ge 0}$ with $H(c-s) \ge h_{c-s}$ for $0 \le s < e$. For a pair (V°,T°) put*

$$W(T^\circ) = \{w \in \mathbf{N}(T^\circ)[1/\lambda_\infty] :\ \Lambda_{c,e}(H)\,w \in P_e(\mathbf{N}(T^\circ)_{c-e})\}.$$

Then:

1. *$W(T^\circ)$ is an $\mathbf{A}_{F,L}$-submodule of $\mathbf{N}(T^\circ)[1/\lambda_\infty]$ containing $P_e(\mathbf{N}(T^\circ)_{c-e})$;*
2. *$\mathbf{N}(T)_c \subseteq W(T)$;*
3. *if $u \in \mathbf{N}(T')[1/\lambda_\infty]$ and $\mathbf{N}(\iota')(u) \in W(T)$, then $u \in W(T')$.*

Proof. Write

$$\Lambda = \Lambda_{c,e}(H) \in \mathbb{Z}_\ell[[X]] \quad \text{and} \quad P(T^\circ) = P_e(\mathbf{N}(T^\circ)_{c-e}),$$

an $\mathbf{A}_{F,L}$-submodule of $\mathbf{N}(T^\circ)[1/\lambda_\infty]$ by definition.

Item 1. For $f \in \mathbf{A}_{F,L}$ and $w \in W(T^\circ)$, $\Lambda f w = f\Lambda w \in P(T^\circ)$; and $\Lambda P(T^\circ) \subseteq P(T^\circ)$.

Item 2. Let $H'\colon \mathbb{Z}/n \to \mathbb{Z}_{\ge 0}$ equal H at the e distinct sites $c-s$ ($0 \le s < e$; distinct as $e \le n$) and h elsewhere. Then $H' \ge h$ pointwise and $\Lambda_{c,e}(H') = \prod_{s<e} Q_s^{H(c-s)} = \Lambda$. By (Ex1), $\mathbf{N}(T)$ is a Wach module with window (M_a,h_a), and item 2 of Lemma 23.15 (with its c,e,a equal to $c-e,e,c$) gives $\Lambda\,\mathbf{N}(T)_c \subseteq P(T)$.

Item 3. Put $R_b = e_b\mathbf{A}_{F,L}$. By item 1 of Lemma 23.20 (a statement about the rings only), R_b is a free module over the subring $\varphi(R_{b-1})$ (basis $1,X,\dots,X^{\ell-1}$), and φ is injective; by transitivity R_c is free over $\varphi^e(R_{c-e})$, so R_c is flat as an R_{c-e}-module via φ^e. By Lemma 23.33 the sequence

$$0 \to \mathbf{N}(T')_{c-e} \xrightarrow{\mathbf{N}(\iota')} \mathbf{N}(T)_{c-e} \xrightarrow{\mathbf{N}(\pi)} \mathbf{N}(T'')_{c-e}$$

is exact (multiply the exact sequence of the lemma by e_{c-e}), hence so is its base change along $\varphi^e\colon R_{c-e} \to R_c$. Each $\mathbf{N}(T^\circ)$ is a Wach module for some window ((Ex1); the Hodge jumps of V° form a finite set), so by item 1 of Lemma 23.15 the base change of $\mathbf{N}(T^\circ)_{c-e}$ is identified with $P(T^\circ)$ by $g\otimes x \mapsto g\varphi^e(x)$, compatibly with $\mathbf{N}(\iota')$ and $\mathbf{N}(\pi)$, which commute with φ (Ex3). Hence

$$0 \to P(T') \xrightarrow{\mathbf{N}(\iota')} P(T) \xrightarrow{\mathbf{N}(\pi)} P(T'')$$

is exact. Now let u be as in item 3 and $v = \Lambda\mathbf{N}(\iota')(u) = \mathbf{N}(\iota')(\Lambda u) \in P(T)$. Since $\mathbf{N}(\pi)\circ\mathbf{N}(\iota') = 0$ (Lemma 23.33, extended to the localisations), $\mathbf{N}(\pi)(v) = 0$, so $v = \mathbf{N}(\iota')(p)$ with $p \in P(T')$. Finally $\mathbf{N}(\iota')$ is injective on $\mathbf{N}(T')[1/\lambda_\infty]$ (Lemma 23.33 and exactness of localisation), so $\Lambda u = p \in P(T')$, i.e. $u \in W(T')$. □

Lemma C.97. *Let ℓ, F, L, (τ_a) be as in Proposition 23.7, (V,T) a pair of rank r, and fix arc level data for (V,T) (cut c, length* len*, H, $\Pi = \prod_{s<\mathrm{len}} Q_s^{H(c-s)}$, frame, parameters, kinds κ_s and sizes b_s). For $y \in (e_c\mathbf{A}_{F,L})^r$ and $s <$ len put $\hat{y}_s = \lambda_s(\Pi)^{-1}\lambda_s(y) \in K_{(s)}((t))^r$. Suppose that there is $s_1 <$ len with κ_{s_1} down and $b_{s_1} \ge 1$, and that one of the following holds:*

(D) *κ_s is thin for every $s < \mathrm{len}$, $s \neq s_1$;*

(DU) *there is $s_0 < s_1$ with κ_{s_0} up, $b_{s_0} \leq H(c - s_0)$ and $b_{s_0} < \ell\, b_{s_1}$, and κ_s is thin for every $s < \mathrm{len}$, $s \notin \{s_0, s_1\}$.*

If $y \in (e_c\mathbf{A}_{F,L})^r$ and $\hat{y}_s \in L^r + \{$kind integral vectors at $s\}$ for every $s < \mathrm{len}$, then there is $\lambda \in \mathcal{O}_L^r$ such that $\hat{y}_s - \lambda$ is kind integral at s for every $s < \mathrm{len}$.

Proof. Identify $e_c\mathbf{A}_{F,L} = e_c(\mathcal{O}_F \otimes \mathcal{O}_L)[[X]]$ with $\mathcal{O}_L[[X]]$ ($e_c(\mathcal{O}_F \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L) \cong \mathcal{O}_L$), and let ϖ be a uniformiser of $\mathcal{O}_L$ and k_L its residue field. By Lemma 23.17, (Lp1) and (Lp5) hold and $\lambda_s(\Pi) \in t^{H(c-s)}K_{(s)}[[t]]^\times$; in particular $\lambda_s(\Pi)$ is invertible in $K_{(s)}((t))$ and may be cancelled. Each λ_s is a ring homomorphism sending a constant $a \in \mathcal{O}_L$ to $a \in L \subseteq K_{(s)}$; hence for an $\mathcal{O}_L$-linear form $\phi\colon \mathcal{O}_L^r \to \mathcal{O}_L$, applied coordinatewise, $\lambda_s(\phi(y)) = \phi(\lambda_s(y)) = \lambda_s(\Pi)\,\phi(\hat{y}_s)$, with ϕ extended $K_{(s)}((t))$-linearly. Put $b_1 = b_{s_1}$, $U = U_{s_1}^L$, $U_\mathcal{O} = U \cap \mathcal{O}_L^r$; in case (D) put $b_0 = 0$ and let s_0 be any index (the factor $Q_{s_0}^{b_0} = 1$ below), in case (DU) $b_0 = b_{s_0}$.

Facts on $\mathcal{O}_L[[X]]$. By Lemma 23.8, $Q_s = \Phi_{\ell^{s+1}}(1 + X)$ is a monic polynomial of degree $\ell^s(\ell - 1)$ with $Q_s \equiv X^{\ell^s(\ell-1)} \bmod \ell$, hence modulo ϖ: a distinguished polynomial, and so are its powers. (F1) For a distinguished polynomial P, $\mathcal{O}_L[[X]]/(P)$ is a free $\mathcal{O}_L$-module (Weierstrass division), so $\ell^N f \in P\,\mathcal{O}_L[[X]]$ or $\varpi^N f \in P\,\mathcal{O}_L[[X]]$ implies $f \in P\,\mathcal{O}_L[[X]]$. (F2) For $s \neq s'$ and $h, h' \geq 0$, Q_s^h and $Q_{s'}^{h'}$ have no common prime factor in the unique factorisation domain $\mathcal{O}_L[[X]]$: a common prime factor is not associated to ϖ (Q_s is monic), so by Weierstrass preparation it is associated to a distinguished polynomial of positive degree whose roots in the open unit disc are common roots of $\Phi_{\ell^{s+1}}(1 + X)$ and $\Phi_{\ell^{s'+1}}(1 + X)$; there are none. Hence if f is divisible by $Q_s^{h_s}$ for each s in a finite set, it is divisible by their product.

Step 1: the constant at s_1. By hypothesis $\hat{y}_{s_1} = d + k$ with $d \in L^r$ and k kind integral at s_1: as κ_{s_1} is down, the t^i-coefficient of k is 0 for $i < 0$ and lies in $K_{(s_1)} \otimes_L U$ for $0 \leq i < b_1$. It suffices to find $\lambda \in \mathcal{O}_L^r$ with $d - \lambda \in U$. Indeed, then $\hat{y}_{s_1} - \lambda = (d - \lambda) + k$ has t^0-coefficient in $K_{(s_1)} \otimes_L U$ ($b_1 \geq 1$) and the other coefficients of k, so it is kind integral at s_1. For $s \neq s_1$ the kind is thin or up and $\mathrm{Nb}_s(i) = L^r$ for $i \geq 0$, so constants are kind integral at s; kind integral vectors form an additive group (a coefficientwise condition by subspaces), so $\hat{y}_s - \lambda$ is kind integral at s by the hypothesis.

Step 2: reduction to linear forms. The submodule $U_\mathcal{O}$ is saturated in $\mathcal{O}_L^r$, so $\mathcal{O}_L^r/U_\mathcal{O}$ is torsion free, hence free, and there is a basis $e_1, \dots, e_r$ of $\mathcal{O}_L^r$ with $e_1, \dots, e_u$ a basis of $U_\mathcal{O}$, hence an L-basis of U. Let $\phi_1, \dots, \phi_r$ be the dual coordinate forms. If $\phi_i(d) \in \mathcal{O}_L$ for $u < i \leq r$, then $\lambda = \sum_{i>u} \phi_i(d)e_i \in \mathcal{O}_L^r$ and $d - \lambda \in U$. So it suffices to show: *for every $\mathcal{O}_L$-linear form ϕ on $\mathcal{O}_L^r$ vanishing on $U_\mathcal{O}$ (hence, extended, on U), $\phi(d) \in \mathcal{O}_L$.*

Step 3: divisibility. Fix such ϕ and put $f = \phi(y) \in \mathcal{O}_L[[X]]$. For a thin s: $\hat{y}_s \in K_{(s)}[[t]]^r$ (constants and kind integral vectors have no negative powers of t), so $\lambda_s(f) \in t^{H(c-s)}K_{(s)}[[t]]$. In case (DU), at s_0: the kind family of an up level vanishes below $-b_0$, so $\hat{y}_{s_0} \in t^{-b_0}K_{(s_0)}[[t]]^r$ and $\lambda_{s_0}(f) \in t^{H(c-s_0)-b_0}K_{(s_0)}[[t]]$. At s_1: $\phi(k)$ has vanishing t^i-coefficients for $i < b_1$ (ϕ kills $K_{(s_1)} \otimes_L U$), so

$$\phi(\hat{y}_{s_1}) = \phi(d) + t^{b_1}h, \qquad h \in K_{(s_1)}[[t]],$$

and $\lambda_{s_1}(f) \in t^{H(c-s_1)}K_{(s_1)}[[t]]$. By (Lp5) and (F1), f is divisible by $Q_s^{H(c-s)}$ for $s \neq s_0$ and by $Q_{s_0}^{H(c-s_0)-b_0}$; by (F2), $f = \Pi' w$ with $w \in \mathcal{O}_L[[X]]$ and $\Pi' = \Pi\, Q_{s_0}^{-b_0} \in \mathcal{O}_L[[X]]$.

Step 4: the constant is integral. From $\lambda_{s_1}(\Pi')\lambda_{s_1}(w) = \lambda_{s_1}(f) = \lambda_{s_1}(\Pi')\lambda_{s_1}(Q_{s_0})^{b_0}(\phi(d) + t^{b_1}h)$ and the cancellation of $\lambda_{s_1}(\Pi')$ (a divisor of $\lambda_{s_1}(\Pi)$, and $\lambda_{s_1}(Q_{s_0}) \in K_{(s_1)}[[t]]$ by (Lp1)),

$$\lambda_{s_1}(w - Q_{s_0}^{b_0}\phi(d)) \in t^{b_1}K_{(s_1)}[[t]].$$

Choose $k \geq 0$ with $c_0 := \varpi^k \phi(d) \in \mathcal{O}_L$. Then $\lambda_{s_1}(\varpi^k w - Q_{s_0}^{b_0} c_0) \in t^{b_1} K_{(s_1)}[[t]]$, and (Lp5) with (F1) give $m \in \mathcal{O}_L[[X]]$ with

$$\varpi^k w - Q_{s_0}^{b_0} c_0 = Q_{s_1}^{b_1} m.$$

Suppose $\phi(d) \notin \mathcal{O}_L$, i.e. $c_0 = \varpi^e u$ with $u \in \mathcal{O}_L^\times$ and $0 \leq e < k$. Then $\varpi^e(\varpi^{k-e} w - u Q_{s_0}^{b_0}) \in Q_{s_1}^{b_1} \mathcal{O}_L[[X]]$, so by (F1) $\varpi^{k-e} w - u Q_{s_0}^{b_0} = Q_{s_1}^{b_1} m'$ with $m' \in \mathcal{O}_L[[X]]$. Reduce modulo ϖ in $k_L[[X]]$: since $k - e \geq 1$,

$$-\bar{u}\, X^{\ell^{s_0}(\ell-1)b_0} = X^{\ell^{s_1}(\ell-1)b_1}\, \bar{m}', \qquad \bar{u} \neq 0.$$

The left side has X-adic order exactly $\ell^{s_0}(\ell-1)b_0$, the right side order at least $\ell^{s_1}(\ell-1)b_1$; hence $\ell^{s_0} b_0 \geq \ell^{s_1} b_1$. In case (D), $b_0 = 0 < \ell^{s_1} b_1$; in case (DU), $s_1 \geq s_0 + 1$ gives $\ell^{s_1} b_1 \geq \ell^{s_0} \cdot \ell b_1 > \ell^{s_0} b_0$. Either way a contradiction, so $\phi(d) \in \mathcal{O}_L$. By Step 2 this gives λ, and by Step 1 the assertion. □

Lemma C.98. *Let ℓ, F, L, (τ_a) be as in Proposition 23.7, (V, T) a pair of rank r, and fix arc level data for (V, T) (cut c, length* len*, H, Π, frame, parameters, kinds and sizes). Identify $e_c \mathbf{A}_{F,L}$ with $\mathcal{O}_L[[X]]$, let ϖ be a uniformiser of $\mathcal{O}_L$, and suppose that there are $s_0 \neq s_1$ below* len *with $\kappa_{s_0}, \kappa_{s_1}$ down, $b_0 = b_{s_0} \geq 1$, $b_1 = b_{s_1} \geq 1$, and κ_s thin for every other $s <$* len*. Then:*

1. *the ideal $(Q_{s_0}^{b_0}, Q_{s_1}^{b_1})$ of $\mathcal{O}_L[[X]]$ meets $\mathcal{O}_L$ in $\varpi^\rho \mathcal{O}_L$ for an integer $\rho \geq 0$, and there is $f \in Q_{s_1}^{b_1} \mathcal{O}_L[[X]]$ with $f - \varpi^\rho \in Q_{s_0}^{b_0} \mathcal{O}_L[[X]]$;*
2. *for every such f and every $y \in (e_c \mathbf{A}_{F,L})^r$ with $\hat{y}_s \in L^r + \{$kind integral vectors at $s\}$ for every $s <$* len*, there are $\lambda, \lambda' \in \mathcal{O}_L^r$ such that $\hat{y}_s - \lambda - \lambda_s(f)\lambda'$ is kind integral at s for every $s <$* len*.*

Proof. Write $A = \mathcal{O}_L[[X]]$, $P_i = Q_{s_i}^{b_i}$, $U_i = U_{s_i}^L$ and $U_{i,\mathcal{O}} = U_i \cap \mathcal{O}_L^r$ $(i = 0, 1)$. We use the following facts: the Q_s are distinguished polynomials (Lemma 23.8); (F1) $A/(P)$ is $\mathcal{O}_L$-free for distinguished P (Weierstrass division), so $\varpi^N f \in PA$ implies $f \in PA$; (F2) powers of distinct Q_s are coprime in the unique factorisation domain A (a common prime factor is not associated to ϖ, as Q_s is monic, so by Weierstrass preparation it is associated to a distinguished polynomial of positive degree whose roots are common roots of $\Phi_{\ell^{s+1}}(1 + X)$ and $\Phi_{\ell^{s'+1}}(1 + X)$, and there are none); $\lambda_s(\Pi) \in t^{H(c-s)} K_{(s)}[[t]]^\times$ and (Lp1), (Lp5) hold (Lemma 23.17). For a saturated submodule $U_{\mathcal{O}} \subseteq \mathcal{O}_L^r$ fix a basis $e_1, \ldots, e_r$ of $\mathcal{O}_L^r$ whose first u members are a basis of $U_{\mathcal{O}}$, with dual forms $\phi_1, \ldots, \phi_r$; then $z \in A^r$ lies in $U_{\mathcal{O}} \otimes A + P\, A^r$ if and only if $\phi_i(z) \in PA$ for $i > u$ (write $z = \sum_i \phi_i(z) e_i$). A constant vector of $U_{\mathcal{O}} \otimes A + PA^r$ (P distinguished of positive degree) lies in $U_{\mathcal{O}}$, since a constant in PA is 0 (uniqueness of the Weierstrass remainder).

Item 1. The polynomials P_0, P_1 are distinguished without common root, so their resultant is a nonzero element of $\mathcal{O}_L$ lying in $(P_0, P_1)\mathcal{O}_L[X]$. Hence $(P_0, P_1) \cap \mathcal{O}_L$ is a nonzero ideal $\varpi^\rho \mathcal{O}_L$. Writing $\varpi^\rho = aP_0 + bP_1$ $(a, b \in A)$, $f = bP_1$ has the two properties.

Step 1: Π divides y. At a thin or down level, kind integral vectors and constants have no negative powers of t, so $\hat{y}_s \in K_{(s)}[[t]]^r$ and $\lambda_s(y_k) \in t^{H(c-s)} K_{(s)}[[t]]$ for every coordinate and every $s <$ len. By (Lp5), (F1), (F2), $y = \Pi x$ with $x \in A^r$, and $\hat{y}_s = \lambda_s(x)$ (cancellation).

Step 2: integral constants at each down level. Let $i \in \{0, 1\}$, $s = s_i$, and write $\lambda_s(x) = d_i + k_i$ with $d_i \in L^r$ and k_i kind integral: its t^j-coefficients vanish for $j < 0$ and lie in $K_{(s)} \otimes_L U_i$ for $0 \leq j < b_i$. For an $\mathcal{O}_L$-linear form ϕ vanishing on $U_{i,\mathcal{O}}$, $\lambda_s(\phi(x)) - \phi(d_i) \in t^{b_i} K_{(s)}[[t]]$; choosing k with $\varpi^k \phi(d_i) \in \mathcal{O}_L$, (Lp5) and (F1) give $\varpi^k \phi(x) - \varpi^k \phi(d_i) \in Q_s^{b_i} A$, and comparing Weierstrass remainders modulo Q_s (degree ≥ 1) shows that $\phi(d_i)$ is the constant term of the remainder of $\phi(x)$, an element of $\mathcal{O}_L$. Hence there is $\lambda_i \in \mathcal{O}_L^r$ with $d_i - \lambda_i \in U_i$: $U_{i,\mathcal{O}}$ is saturated in $\mathcal{O}_L^r$, so some basis $e_1, \ldots, e_r$ of $\mathcal{O}_L^r$ has $e_1, \ldots, e_u$ a basis of $U_{i,\mathcal{O}}$, hence an L-basis of U_i; the

dual forms ϕ'_j, $j > u$, vanish on $U_{i,\mathcal{O}}$, so $\phi'_j(d_i) \in \mathcal{O}_L$, and $\lambda_i = \sum_{j>u} \phi'_j(d_i) e_j$ works. Then $\lambda_s(x - \lambda_i) = (d_i - \lambda_i) + k_i$, so $\lambda_s(\phi_j(x - \lambda_i)) \in t^{b_i} K_{(s)}[[t]]$ for the forms ϕ_j ($j > u$) of an adapted basis of $U_{i,\mathcal{O}}$, and by (Lp5), (F1) $\phi_j(x - \lambda_i) \in P_i A$. Hence

$$x - \lambda_i \in U_{i,\mathcal{O}} \otimes A + P_i A^r \qquad (i = 0, 1).$$

Step 3: the two constants. Let $N = U_{0,\mathcal{O}} + U_{1,\mathcal{O}} \subseteq \mathcal{O}_L^r$ and $M = \mathcal{O}_L^r / N$, a finitely generated $\mathcal{O}_L$-module, and $\delta = \lambda_0 - \lambda_1$. Subtracting the two relations of Step 2, the image $\bar{\delta}$ of δ in $M \otimes_{\mathcal{O}_L} A = M[[X]]$ lies in $(P_0, P_1) \cdot M[[X]]$. We claim $M \cap (P_0, P_1) M[[X]] = \varpi^\rho M$. Decompose $M \cong \bigoplus_m \mathcal{O}_L / \varpi^{k_m}$ ($k_m \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$); both sides are compatible with the decomposition, so it suffices to treat $M = \mathcal{O}_L / \varpi^k$, i.e. to show: if $c \in \mathcal{O}_L$ and $c \in I + \varpi^k A$ with $I = (P_0, P_1)$, then $c \in \varpi^{\min(\rho,k)} \mathcal{O}_L$. Suppose $c = \varpi^e u$ with u a unit and $e < \min(\rho, k)$. In $R = A/I$ we get $\varpi^e \in \varpi^k R \subseteq \varpi^{e+1} R$, so $\varpi^e(1 - \varpi r) = 0$ in R for some r; R is a quotient of the local ring A with ϖ in its maximal ideal, so $1 - \varpi r$ is a unit and $\varpi^e = 0$ in R, i.e. $\varpi^e \in I \cap \mathcal{O}_L = \varpi^\rho \mathcal{O}_L$, contradicting $e < \rho$. The inclusion $\varpi^\rho M \subseteq (P_0, P_1) M[[X]]$ is clear. Hence $\bar{\delta} \in \varpi^\rho M$: there are $\lambda' \in \mathcal{O}_L^r$, $u_0 \in U_{0,\mathcal{O}}$, $u_1 \in U_{1,\mathcal{O}}$ with $\lambda_0 - \lambda_1 = \varpi^\rho \lambda' + u_0 + u_1$. Put $\lambda = \lambda_1 + u_1 \in \mathcal{O}_L^r$.

Step 4: the levels. Let $\mu = x - \lambda - f\lambda' \in A^r$; then $\hat{y}_s - \lambda - \lambda_s(f)\lambda' = \lambda_s(\mu)$. At s_1: $\mu = (x - \lambda_1) - u_1 - f\lambda' \in U_{1,\mathcal{O}} \otimes A + P_1 A^r$, as $f \in P_1 A$. At s_0: $\mu = (x - \lambda_0) + u_0 - (f - \varpi^\rho)\lambda' \in U_{0,\mathcal{O}} \otimes A + P_0 A^r$. For $i = 0, 1$, λ_{s_i} maps $U_{i,\mathcal{O}} \otimes A$ into $K_{(s_i)}[[t]] \otimes_L U_i$ and $P_i A^r$ into $t^{b_i} K_{(s_i)}[[t]]^r$ ((Lp1): $\lambda_{s_i}(Q_{s_i}) \in t K_{(s_i)}[[t]]$), so $\lambda_{s_i}(\mu)$ has no negative powers of t and has t^j-coefficients in $K_{(s_i)} \otimes_L U_i$ for $j < b_i$: it is kind integral at s_i (down kind). At a thin level, $\lambda_s(\mu) \in K_{(s)}[[t]]^r$ is kind integral. This proves item 2. □

Lemma C.99. *Let ℓ, F, L, (τ_a) be as in Proposition 23.7, (V, T) a pair of rank r, and fix arc level data for (V, T) (cut c, length* len*, H, Π, frame, parameters, kinds and sizes). Identify $e_c \mathbf{A}_{F,L}$ with $A = \mathcal{O}_L[[X]]$ and let ϖ be a uniformiser of $\mathcal{O}_L$. Suppose that there are $s_0 \neq s_1$ below* len *with κ_{s_1} up, $1 \leq b_1 = b_{s_1} \leq H(c - s_1)$, κ_{s_0} down, $b_0 = b_{s_0} \geq 1$, and κ_s thin for every other $s <$* len*; put $P_i = Q_{s_i}^{b_i}$ and $U_{1,\mathcal{O}} = U_{s_1}^L \cap \mathcal{O}_L^r$. Then:*

1. *there are an integer $\mathfrak{c} \geq 0$ and $\alpha, \beta \in A$ with $P_1 = \varpi^{\mathfrak{c}} \alpha + P_0 \beta$ and $P_1 \notin (\varpi^{\mathfrak{c}+1}, P_0)$;*
2. *for all such $\mathfrak{c}, \alpha, \beta$ and every $y \in (e_c \mathbf{A}_{F,L})^r$ with $\hat{y}_s \in L^r +$ {kind integral vectors at s} for every $s <$* len*, there are $\lambda \in \mathcal{O}_L^r$ and $\lambda' \in U_{1,\mathcal{O}}$ such that*

$$\hat{y}_s - \lambda - \lambda_s(\alpha) \lambda_s(P_1)^{-1} \lambda'$$

is kind integral at s for every $s <$ len*.*

Proof. We use: (F1) $A/(P)$ is $\mathcal{O}_L$-free for a distinguished polynomial P (Weierstrass division; the Q_s and their powers are distinguished, Lemma 23.8); (F2) powers of distinct Q_s are coprime in the unique factorisation domain A (by Weierstrass preparation a common prime factor would give a common root of $\Phi_{\ell^{s+1}}(1+X)$ and $\Phi_{\ell^{s'+1}}(1+X)$, $s \neq s'$, in the open unit disc); (Lp1), (Lp5) and $\lambda_s(\Pi) \in t^{H(c-s)} K_{(s)}[[t]]^\times$ (Lemma 23.17); for a saturated $U_{\mathcal{O}} \subseteq \mathcal{O}_L^r$ with adapted basis and dual forms ϕ_j, $z \in U_{\mathcal{O}} \otimes A + PA^r$ if and only if $\phi_j(z) \in PA$ for the forms ϕ_j vanishing on $U_{\mathcal{O}}$. Multiplication by a unit of $K_{(s)}[[t]]$ preserves kind integrality, the kind families being increasing in i (as for de Rham integrality in Lemma C.93). Put $U_0 = U_{s_0}^L$, $U_1 = U_{s_1}^L$, $F_{\mathcal{O}} = U_0 \cap \mathcal{O}_L^r$, $C = U_{1,\mathcal{O}}$.

Item 1. We have $P_1 \in (\varpi^0, P_0) = A$. If $P_1 \in (\varpi^N, P_0)$ for all N, then the image of P_1 in $A/(P_0)$, a finitely generated free $\mathcal{O}_L$-module, lies in $\bigcap_N \varpi^N (A/P_0) = 0$, so $P_0 \mid P_1$, contradicting (F2) ($\deg P_0 \geq 1$). Take $\mathfrak{c}$ maximal with $P_1 \in (\varpi^{\mathfrak{c}}, P_0)$ and write $P_1 = \varpi^{\mathfrak{c}} \alpha + P_0 \beta$.

Steps 1 and 2: division and the up level. The hypotheses of Lemma C.100 hold: κ_{s_1} is up with $1 \le b_1 \le H(c-s_1)$, every other level is thin or down, and $\hat{y}_s \in L^r +$ {kind integral vectors at s} for every s. So, with $\Pi' = \Pi/P_1 \in A$, $y = \Pi' x$ with $x \in A^r$, $\hat{y}_s = \lambda_s(P_1)^{-1}\lambda_s(x)$ for every s, and $x = u + P_1 z$ with $u \in C \otimes A$, $z \in A^r$.

Step 3: the down level. Write $\hat{y}_{s_0} = d_0 + k_0$ with $d_0 \in L^r$ and k_0 kind integral (down), and choose $k \ge 0$ with $d' = \varpi^k d_0 \in \mathcal{O}_L^r$. For a form ϕ vanishing on $F_{\mathcal{O}}$: $\lambda_{s_0}(\phi(x)) = \lambda_{s_0}(P_1)(\phi(d_0) + \phi(k_0))$ and $\phi(k_0) \in t^{b_0} K_{(s_0)}[[t]]$, so by (Lp5), (F1)

$$\varpi^k \phi(x) - P_1\, \phi(d') \in P_0 A. \tag{$*$}$$

Step 4: the key divisibility. Let ϕ vanish on $F_{\mathcal{O}}$ with $\phi(C) \subseteq \varpi^n \mathcal{O}_L$, and $m = \min(n, \mathfrak{c})$. Then $\varpi^{k+m} \mid \varpi^{\mathfrak{c}}\phi(d')$. Indeed, by Step 2, $\phi(u) = \varpi^n g$ with $g \in A$, and $(*)$ reads $\varpi^{k+n} g + P_1\varpi^k\phi(z) - P_1\phi(d') \in P_0 A$. Modulo the ideal $J = (\varpi^{k+m}, P_0)$: $\varpi^{k+n} g \in J$, and $P_1\varpi^k\phi(z) = \varpi^{\mathfrak{c}+k}\alpha\phi(z) + P_0\beta\varpi^k\phi(z) \in J$; hence $P_1\phi(d') \in J$. If $\phi(d') = \varpi^e v$ with v a unit and $e + \mathfrak{c} < k + m$, then in $R = A/(P_0)$, which is ϖ-torsion free (F1), $\bar{P}_1\varpi^e \in \varpi^{k+m}R \subseteq \varpi^{e+\mathfrak{c}+1}R$, so $\bar{P}_1 \in \varpi^{\mathfrak{c}+1}R$, i.e. $P_1 \in (\varpi^{\mathfrak{c}+1}, P_0)$, a contradiction. So $e + \mathfrak{c} \ge k + m$ (or $\phi(d') = 0$).

Step 5: the constants. The submodule $F_{\mathcal{O}}$ is saturated, so $M = \mathcal{O}_L^r/F_{\mathcal{O}}$ is free; let $\bar{C}$ be the image of C. By the stacked basis theorem there are a basis $f_1, \dots, f_{m'}$ of M, an integer q and $a_1, \dots, a_q \ge 0$ with $\varpi^{a_1} f_1, \dots, \varpi^{a_q} f_q$ a basis of $\bar{C}$; let ψ_j be the dual coordinates, viewed as forms on $\mathcal{O}_L^r$ vanishing on $F_{\mathcal{O}}$. Then $\psi_j(C) = \varpi^{a_j}\mathcal{O}_L$ for $j \le q$ and $\psi_j(C) = 0$ for $j > q$. Step 4 with $n = a_j$ ($j \le q$), resp. n arbitrary ($j > q$, so $m = \mathfrak{c}$), gives: if $j > q$ or $a_j \ge \mathfrak{c}$, then $\varpi^k \mid \psi_j(d')$, and we put $\psi_j(d') = \varpi^k l_j$, $\mu_j = 0$; if $j \le q$ and $a_j < \mathfrak{c}$, then $\varpi^{\mathfrak{c}}\psi_j(d') = \varpi^{k+a_j}\mu_j$ with $\mu_j \in \mathcal{O}_L$, and we put $l_j = 0$. Let $\lambda \in \mathcal{O}_L^r$ lift $\sum_j l_j f_j$ and $\lambda' \in C$ lift $\sum_{j \le q} \mu_j \varpi^{a_j} f_j \in \bar{C}$. Comparing coordinates,

$$\varpi^{\mathfrak{c}} d' - \varpi^{k+\mathfrak{c}}\lambda - \varpi^k \lambda' \in F_{\mathcal{O}}, \qquad \text{so} \qquad \psi(\varpi^{\mathfrak{c}} d') = \varpi^{k+\mathfrak{c}}\psi(\lambda) + \varpi^k \psi(\lambda') \tag{$**$}$$

for every form ψ vanishing on $F_{\mathcal{O}}$.

Step 6: the levels. Put $\nu = x - P_1\lambda - \alpha\lambda' \in A^r$; then $\hat{y}_s - \lambda - \lambda_s(\alpha)\lambda_s(P_1)^{-1}\lambda' = \lambda_s(P_1)^{-1}\lambda_s(\nu)$. At a thin level s, $\lambda_s(P_1)$ is a unit ($s \ne s_1$, (Lp1)) and $\lambda_s(\nu) \in K_{(s)}[[t]]^r$: kind integral. At s_1: $\nu = (u - \alpha\lambda') + P_1(z - \lambda)$ with $u - \alpha\lambda' \in C \otimes A$; so $\lambda_{s_1}(P_1)^{-1}\lambda_{s_1}(\nu) \in t^{-b_1} K_{(s_1)}[[t]] \otimes_L U_1 + K_{(s_1)}[[t]]^r$, which is kind integral for the up kind (coefficients in U_1 from t^{-b_1} to t^{-1}, nothing below). At s_0: for a form ϕ vanishing on $F_{\mathcal{O}}$, by $(*)$, $(**)$ and $P_1 - \varpi^{\mathfrak{c}}\alpha = P_0\beta$,

$$\varpi^{k+\mathfrak{c}}\phi(\nu) \equiv P_1\big(\varpi^{\mathfrak{c}}\phi(d') - \varpi^{k+\mathfrak{c}}\phi(\lambda)\big) - \varpi^{k+\mathfrak{c}}\alpha\phi(\lambda') = \varpi^k\phi(\lambda')\,(P_1 - \varpi^{\mathfrak{c}}\alpha) \equiv 0 \pmod{P_0 A},$$

so $\phi(\nu) \in P_0 A$ by (F1), and $\nu \in F_{\mathcal{O}} \otimes A + P_0 A^r$. Hence $\lambda_{s_0}(\nu)$ has no negative powers and t^j-coefficients in $K_{(s_0)} \otimes_L U_0$ for $j < b_0$ ((Lp1): $\lambda_{s_0}(P_0) \in t^{b_0} K_{(s_0)}[[t]]$): it is kind integral (down), and so is its product with the unit $\lambda_{s_0}(P_1)^{-1}$. This proves item 2. □

Lemma C.100. *Let ℓ, F, L, (τ_a) be as in Proposition 23.7, (V,T) a pair of rank r, and fix arc level data for (V,T) (cut c, length* len*, H, Π, frame, parameters, kinds and sizes). Identify $e_c\mathbf{A}_{F,L}$ with $A = \mathcal{O}_L[[X]]$. Suppose that $s_1 <$* len *has κ_{s_1} up and $1 \le b_1 = b_{s_1} \le H(c - s_1)$, and that κ_s is thin or down for every $s <$* len*, $s \ne s_1$. Put $P_1 = Q_{s_1}^{b_1}$, $\Pi' = \Pi/P_1 \in A$ and $C = U_{s_1}^L \cap \mathcal{O}_L^r$. Let $y \in A^r$ with $\hat{y}_s \in L^r + \{$kind integral vectors at $s\}$ for every $s <$* len*. Then $y = \Pi' x$ with $x \in A^r$, $\hat{y}_s = \lambda_s(P_1)^{-1}\lambda_s(x)$ for every $s <$* len*, and*

$$x \in C \otimes_{\mathcal{O}_L} A + P_1 A^r.$$

Proof. We use two classical facts on A. (F1) For a distinguished polynomial $P \in A$, such as a power of some Q_s (Lemma 23.8), A/PA is a free $\mathcal{O}_L$-module (Weierstrass division); hence $\varpi^N f \in PA$ implies $f \in PA$, ϖ a uniformiser of $\mathcal{O}_L$. (F2) Powers of distinct Q_s have no common prime factor in the unique factorisation domain A: by Weierstrass preparation such a factor would give a common root of $\Phi_{\ell^{s+1}}(1+X)$ and $\Phi_{\ell^{s'+1}}(1+X)$, $s \neq s'$, in the open unit disc. By Lemma 23.17 (applied to $\Phi = \mathrm{id}_T$) the arc level data have (Lp1) and (Lp5), and $\lambda_s(\Pi) \in t^{H(c-s)} K_{(s)}[[t]]^\times$.

Division. The kind family at a thin or down level vanishes in negative degrees, and at the up level s_1 below $-b_1$; constants have no negative powers of t. So $\hat{y}_s \in K_{(s)}[[t]]^r$ for $s \neq s_1$ and $\hat{y}_{s_1} \in t^{-b_1} K_{(s_1)}[[t]]^r$. Hence, for every coordinate y_k, $\lambda_s(y_k) = \lambda_s(\Pi)\hat{y}_{s,k}$ lies in $t^{H(c-s)}K_{(s)}[[t]]$ for $s \neq s_1$ and in $t^{H(c-s_1)-b_1}K_{(s_1)}[[t]]$ for $s = s_1$. By (Lp5) and (F1), y_k is divisible by $Q_s^{H(c-s)}$ ($s \neq s_1$) and by $Q_{s_1}^{H(c-s_1)-b_1}$, and by (F2) by their product Π'. So $y = \Pi' x$ with $x \in A^r$, and $\hat{y}_s = \lambda_s(\Pi)^{-1}\lambda_s(\Pi')\lambda_s(x) = \lambda_s(P_1)^{-1}\lambda_s(x)$.

The up level. C is saturated in $\mathcal{O}_L^r$, hence a direct summand: choose a basis of $\mathcal{O}_L^r$ extending a basis of C, and let ϕ_j be the dual coordinate forms vanishing on C. A vector $z \in A^r$ lies in $C \otimes A + P_1 A^r$ if and only if $\phi_j(z) \in P_1 A$ for all these j. Now $\lambda_{s_1}(x) = \lambda_{s_1}(P_1)\hat{y}_{s_1}$ with $\lambda_{s_1}(P_1) \in t^{b_1}K_{(s_1)}[[t]]^\times$ ((Lp1)), and the t^i-coefficients of $\hat{y}_{s_1}$ with $i < 0$ lie in $K_{(s_1)} \otimes_L U_{s_1}^L$ (the constant contributes only in degree 0, and the up kind family is $U_{s_1}^L$ in degrees $-b_1 \leq i < 0$). So $\lambda_{s_1}(\phi_j(x)) \in t^{b_1}K_{(s_1)}[[t]]$, whence $\phi_j(x) \in P_1 A$ by (Lp5) and (F1). Thus $x \in C \otimes A + P_1 A^r$. □

Lemma C.101. *Let $\ell \geq 11$, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$), the weight word, Θ, G, g, h_*, H, j, $(\mathcal{C}, (\mathrm{len}_c))$, (Ae1), ρ_{E}, $c \in \mathcal{C}$, $\mathrm{len} = \mathrm{len}_c$ and frames $\mathbf{n}^1, \mathbf{n}^2$ at $c_0 = c - \mathrm{len}$ of $\mathbf{N}(T_1)$, $\mathbf{N}(T_2)$ be as in Lemma C.93, and let $o \in M_{r_2 \times r_1}(\mathcal{O}_L)$ be the frame matrix of Φ for these frames. Then for every $v \in \mathcal{O}_L^{r_2}$ there is $v_1 \in \mathcal{O}_L^{r_1}$ with $o\, v_1 = \ell^{\rho_{\mathrm{E}}} v$.*

Proof. Write $\mathbf{N}_\nu = \mathbf{N}(T_\nu)$. By (Cc2), $c_0 \in \mathcal{C}$. By (Ae1) and items 1 and 3 of Lemma C.102, the window (M_a, h_a), which is $(\eta, \max(s_a + S, \eta))$ for $\epsilon_a = 1$ and $(\max(s_a + S, \eta), \eta)$ for $\epsilon_a = -1$, bounds the Hodge jumps of V_1, $M_a + h_a \leq H(a)$ for all a, and therefore the exponent r_{c_0} of Proposition C.6 for Φ, this window and the site c_0 satisfies $r_{c_0} \leq \rho_{\mathrm{E}}$. Indeed, r_{c_0} is formed with the function $a \mapsto M_a + h_a$, which is bounded by H pointwise; the sum s_{c_0}, the maximum and c_H of Proposition C.6 are monotone in the function; and ρ_{E} is the maximum over $\mathcal{C} \ni c_0$ of the exponent formed with H (Definition C.78). Let $v \in \mathcal{O}_L^{r_2}$ and $y = \sum_l v_l n_l^2 \in \mathbf{N}_{2,c_0}$. By Proposition C.6 there is $x \in \mathbf{N}_{1,c_0}$ with $\mathbf{N}(\Phi)(x) = \ell^{r_{c_0}} y$. As $\mathbf{n}^1$ is a basis and $e_{c_0}\mathbf{A}_{F,L}/X \cong \mathcal{O}_L$, there is $a \in \mathcal{O}_L^{r_1}$ with $x - \sum_k a_k n_k^1 \in X\mathbf{N}_{1,c_0}$. Since $\mathbf{N}(\Phi)$ is $\mathbf{A}_{F,L}$-linear, it maps $X\mathbf{N}_{1,c_0}$ into $X\mathbf{N}_{2,c_0}$, and by the definition of o,

$$\ell^{r_{c_0}} \sum_l v_l n_l^2 = \mathbf{N}(\Phi)(x) \equiv \sum_k a_k \sum_l o_{lk} n_l^2 = \sum_l (oa)_l\, n_l^2 \pmod{X\mathbf{N}_{2,c_0}}.$$

The images of the n_l^2 form an $\mathcal{O}_L$-basis of $\mathbf{N}_{2,c_0}/X\mathbf{N}_{2,c_0}$, so $oa = \ell^{r_{c_0}} v$, and $v_1 = \ell^{\rho_{\mathrm{E}} - r_{c_0}} a$ satisfies $o v_1 = \ell^{\rho_{\mathrm{E}}} v$. □

Lemma C.102. *Let ℓ, F, L, $(\tau_a)_{a \in \mathbb{Z}/n}$ be as in Proposition 23.7, $\Phi\colon V_1 \to V_2$ an onto map of crystalline lattices $T_1 \to T_2$ over F with coefficients L, and $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ a weight word for Φ. Let $\Theta \subseteq \mathbb{Z}/n$ be the set of thick sites, $G = w + S + \eta$, $g = \max(G, 1)$, $h_* = b + \eta + S$ if $\Theta \neq \emptyset$ and $h_* = g$ otherwise, and $H(a) = 2h_*$ for $a \in \Theta$, $H(a) = 2G$ for $a \notin \Theta$. Suppose that $s_a > g$ for every $a \in \Theta$. Put*

$$(M_a, h_a) = \begin{cases} (\eta, \max(s_a + S, \eta)) & \text{if } \epsilon_a = 1, \\ (\max(s_a + S, \eta), \eta) & \text{if } \epsilon_a = -1. \end{cases}$$

Then:

1. *for $\nu = 1, 2$ and every a, every Hodge jump of V_ν at τ_a lies in $[-M_a, h_a]$;*
2. *$G, S, \eta \le g$; at a thin site a, $M_a, h_a \le G$; at a thick site a, $\eta \le g < s_a$, $\max(s_a + S, \eta) = s_a + S \le h_*$, and $\{M_a, h_a\} = \{\eta, s_a + S\}$;*
3. *$h_* \ge g$, and $M_a \le h_*$, $h_a \le h_*$ and $M_a + h_a \le H(a)$ for every a.*

Proof. **Item 1.** Let $\epsilon_a = 1$. By (Ow1) every Hodge jump of V_1 at τ_a lies in $[s_a, s_a + S] \cup [-\eta, \eta]$; since $s_a > \eta \ge 0$, this set is contained in $[-\eta, \max(s_a + S, \eta)] = [-M_a, h_a]$. The case $\epsilon_a = -1$ is symmetric: $[-s_a - S, -s_a] \cup [-\eta, \eta] \subseteq [-\max(s_a + S, \eta), \eta]$. By item 1 of Lemma 24.2 every Hodge jump of V_2 at τ_a is one of V_1, so it lies in the same interval.

Item 2. $G = w + S + \eta$ with $w, S, \eta \ge 0$, so $S, \eta \le G \le g$. At a thin site $s_a \le w$ by (Ow3), so $\max(s_a + S, \eta) \le \max(w + S, \eta) \le G$, and the other entry of (M_a, h_a) is $\eta \le G$. At a thick site $s_a > g$ by hypothesis, so $\eta \le g < s_a \le s_a + S$ and $\max(s_a + S, \eta) = s_a + S$; by (Ow3) $s_a \le b + \eta$, so $s_a + S \le b + \eta + S = h_*$ (here $\Theta \ne \emptyset$). Then $\{M_a, h_a\} = \{\eta, s_a + S\}$ by the definition.

Item 3. If $\Theta \ne \emptyset$, a thick site a has $g < s_a \le h_*$ by item 2; otherwise $h_* = g$. At a thin site $M_a, h_a \le G \le g \le h_*$ and $M_a + h_a \le 2G = H(a)$. At a thick site $M_a, h_a \in \{\eta, s_a + S\}$ with $\eta \le g \le h_*$ and $s_a + S \le h_*$, so $M_a, h_a \le h_*$ and $M_a + h_a = \eta + s_a + S \le 2h_* = H(a)$. □

Proposition C.103. *Let ℓ, F, n, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$), the weight word, Θ, G, g, h_*, H, j, $(\mathcal{C}, (\mathrm{len}_c))$, a_{E}, (Ae1), (Ae2), $c \in \mathcal{C}$ and the arc level data of Φ along the word at the cut c of length len_c, with the arc level properties, be as in Proposition C.94. Suppose that there are $s_0 < s_1 < \mathrm{len}_c$ with κ_{s_0} down, κ_{s_1} up, and κ_s thin for every other $s < \mathrm{len}_c$. Then for every $\lambda' \in U^{2,L}_{s_1} \cap \mathcal{O}_L^{r_2}$ there is $v' \in U^{1,L}_{s_1} \cap \mathcal{O}_L^{r_1}$ with $o\, v' = \ell^{2a_{\mathrm{E}}} \lambda'$.*

Proof. Write $\mathrm{len} = \mathrm{len}_c$, $A = e_c \mathbf{A}_{F,L} \cong \mathcal{O}_L[[X]]$, $\Pi = \prod_{s < \mathrm{len}} Q_s^{H(c-s)}$, $b_i = b_{s_i}$, $P_i = Q_{s_i}^{b_i}$, $\Pi' = \Pi / P_1 \in A$ ($b_1 \le H(c - s_1) = g + b_1 + S$), $C_\nu = U^{\nu,L}_{s_1} \cap \mathcal{O}_L^{r_\nu}$, $\sigma = \sigma_\ell(3g, (\ell - 1)M, j) \le \sigma_{\mathrm{E}}$ and $\alpha = 2\sigma + \rho_{\mathrm{E}}$. By (Ae1) and items 2 and 3 of Lemma C.102, $h_* \ge g$, $G, S, \eta \le g$ and $b_s = s_{c-s} > g$ for thick $c - s$. By (Cc2), $c_0 = c - \mathrm{len} \in \mathcal{C}$ and $j \le \mathrm{len}$, and by (Cc1) for c_0 the sites $c_0 + 1, \dots, c_0 + j$ are thin; so Lemma C.105 applies for $\nu = 1, 2$: every truncation small vector is de Rham integral and kind integral. $\mathcal{T}_s^\nu$ is a $K_{(s)}[[t]]$-submodule (its definition is a $K_{(s)}[[t]]$-linear condition), so $o\,\mathcal{T}_s^1 \subseteq \mathcal{T}_s^2$. The kind and the de Rham integral vectors form additive groups stable under $\mathbb{Z}_\ell$ and, the families being increasing in i, under multiplication by $K_{(s)}[[t]]$ (the t^i-coefficient of a product is a finite sum of coefficients of lower index).

Step 1: the core relation. Let $e \in C_2$, $\mu\colon \mathcal{O}_L^{r_2} \to \mathcal{O}_L$ an $\mathcal{O}_L$-linear form and $a \in \mathcal{O}_L$ with $\mu(o\, v) \in a\mathcal{O}_L$ for all $v \in C_1$. We show

$$\mu(e)\, \ell^\alpha \lambda^{2g} P_0 \in aA + P_1 A. \qquad (*)$$

(i) Let $y_2 = \Pi' P_0 e \in A^{r_2}$; $\hat{y}_{2,s} = \lambda_s(P_0)\lambda_s(P_1)^{-1} e$. By (Lp1) this lies in $K[[t]]e$ at a thin level, in $t^{-b_1} K[[t]]e$ at s_1 (with $e \in U^{2,L}_{s_1}$), in $t^{b_0} K[[t]]e$ at s_0: it is kind integral at every level. (ii) Item 2 of Lemma C.95 for T_2 gives $w \in W_2$ with $\mathrm{lv}_s^{(2)}(w) - \ell^\sigma \hat{y}_{2,s} \in \mathcal{T}_s^2$; so $\mathrm{lv}_s^{(2)}(w)$ is kind integral. (iii) Lemma C.93 gives $z \in W_1$ with $\mathbf{N}(\Phi)(z) = \ell^{\rho_{\mathrm{E}}} w$ and $\lambda_s(Q_s)^g \mathrm{lv}_s^{(1)}(z)$ de Rham integral. (iv) Item 1 of Lemma C.95 for T_1 gives $y_1 \in A^{r_1}$ with $\hat{y}_{1,s} - \ell^\sigma \mathrm{lv}_s^{(1)}(z) \in \mathcal{T}_s^1$. (v) Let $\delta = o\, y_1 - \ell^\alpha y_2$. By (Lh1), $\hat{\delta}_s = o\, \hat{y}_{1,s} - \ell^\alpha \hat{y}_{2,s} \in \ell^\sigma \mathrm{lv}_s^{(2)}(\mathbf{N}(\Phi) z) - \ell^{\sigma + \rho_{\mathrm{E}}} \mathrm{lv}_s^{(2)}(w) + \mathcal{T}_s^2 = \mathcal{T}_s^2$, so $\lambda_s(\delta) \in \lambda_s(f') K[[t]]^{r_2}$, and $\lambda_s(f') \in t^{e_s} K[[t]]^\times$ with $e_s = (\ell - 1)M$ for $s < \mathrm{len} - j$ and $e_s = 3g$ otherwise: as $\varphi^k(X) = (1 + X)^{\ell^k} - 1 = X \prod_{s' < k} Q_{s'}$ and $\varphi^k(Q_{s'}) = Q_{s' + k}$, $f' = X^{(\ell-1)M} \prod_{s' < \mathrm{len} - j} Q_{s'}^{(\ell-1)M} \prod_{\mathrm{len} - j \le s' < \mathrm{len}} Q_{s'}^{3g}$, and by (Lp1) (Lemma 23.17) $\lambda_s(X)$ and $\lambda_s(Q_{s'})$

for $s' \neq s$ are units of $K[[t]]$ while $\lambda_s(Q_s) \in tK[[t]]^\times$. In both cases $e_s \geq H(c-s)$: $(\ell-1)M > h_* + 2g > g + h_* \geq H(c-s)$ (as $b_s + S \leq h_*$ by (Ow3) and $G \leq g \leq h_*$), and for $s \geq \mathrm{len} - j$ the site $c-s$ is thin ((Cc1) for c_0), $H(c-s) = g + G \leq 3g$. By (Lp5) and two classical facts on A (Weierstrass division: A/PA is $\mathcal{O}_L$-free for a distinguished polynomial P, such as a power of Q_s (Lemma 23.8), so $\varpi^N f \in PA$ implies $f \in PA$; and powers of distinct Q_s have no common prime factor in the unique factorisation domain A, since by Weierstrass preparation such a factor would give a common root of $\Phi_{\ell^{s+1}}(1+X)$ and $\Phi_{\ell^{s'+1}}(1+X)$ in the open unit disc), $\delta = \Pi d$ with $d \in A^{r_2}$. (vi) Let $Y = \lambda^{2g} y_1$. Then $\hat{Y}_s = \lambda_s(\lambda)^{2g}\hat{y}_{1,s}$ with $\lambda_s(\lambda)^{2g} = t^{2g}\cdot$unit ((Lp1)); $t^{2g}\mathrm{lv}_s^{(1)}(z) = t^g \cdot (t^g \mathrm{lv}_s^{(1)}(z))$ is kind integral by Lemma C.90 (de Rham integral times t^g), and so is t^{2g} times a truncation small vector; hence $\hat{Y}_s$ is kind integral at every level. By Lemma C.100 for T_1 (κ_{s_1} up with $1 \leq b_1 \leq H(c-s_1)$, every other level thin or down, $Y \in A^{r_1}$ and $\hat{Y}_s$ kind integral, with constants 0), $Y = \Pi' x$ with $x \in C_1 \otimes A + P_1 A^{r_1}$. Since $\mu \circ o$ maps C_1 into $a\mathcal{O}_L$, $\mu(o\,Y) = \Pi'\mu(o\,x) \in \Pi'(aA + P_1 A)$. (vii) Now $\lambda^{2g}\mu(o\,y_1) = \mu(o\,Y)$ and $\mu(o\,y_1) = \ell^\alpha \mu(y_2) + \mu(\delta) = \Pi'(\ell^\alpha P_0 \mu(e) + P_1\mu(d))$. Hence $\Pi'\lambda^{2g}\ell^\alpha P_0\mu(e) \in \Pi'(aA + P_1 A)$, and cancelling $\Pi' \neq 0$ in the domain A gives $(*)$.

Step 2: the valuation bound. If $\mu(e) = 1$ in Step 1, then $(*)$ and Proposition C.104 (with $s_0 < s_1$, $b_0 \geq 1$; $2(\alpha + 2g\,\mathrm{len}) \leq 2a_\mathrm{E} < b_1$ and $2b_0 \leq \ell b_1$ by (Ae2), clause 2, since $c - s_1$ is thick with $\epsilon = 1$ and $c - s_0$ thick with $\epsilon = -1$; $\alpha + 2g\,\mathrm{len} \leq 2\sigma_\mathrm{E} + \rho_\mathrm{E} + 2g\,\mathrm{len}_{\max} = a_\mathrm{E}$) give $a \mid \ell^{2a_\mathrm{E}}$.

Step 3: the lift. C_2 is saturated in $\mathcal{O}_L^{r_2}$, hence a direct summand; let $\pi\colon \mathcal{O}_L^{r_2} \to C_2$ be a projection. By (Lh2) (Lemma 23.17), $o(C_1) \subseteq C_2$. By the stacked basis theorem there are a basis $f_1, \ldots, f_m$ of C_2, an integer $q \leq m$ and $d_1, \ldots, d_q \geq 0$ such that $\varpi^{d_1} f_1, \ldots, \varpi^{d_q} f_q$ is a basis of $o(C_1)$. Let μ_i be the i-th coordinate of π in this basis, so $\mu_i(f_i) = 1$, $\mu_i(o(C_1)) = \varpi^{d_i}\mathcal{O}_L$ for $i \leq q$ and $= 0$ for $i > q$. For $i > q$, Step 2 with $e = f_i$, $\mu = \mu_i$ and $a = \varpi^N$ gives $\varpi^N \mid \ell^{2a_\mathrm{E}}$ for every N, which is impossible; so $q = m$. For $i \leq m$, Step 2 with $a = \varpi^{d_i}$ gives $\varpi^{d_i} \mid \ell^{2a_\mathrm{E}}$, hence $\ell^{2a_\mathrm{E}} f_i \in o(C_1)$. Therefore $\ell^{2a_\mathrm{E}} C_2 \subseteq o(C_1)$: for every $\lambda' \in C_2$ there is $v' \in C_1$ with $o\,v' = \ell^{2a_\mathrm{E}}\lambda'$. □

Proposition C.104. *Let ℓ be an odd prime, L a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_L$, $A = \mathcal{O}_L[[X]]$, $Q_s = \Phi_{\ell^{s+1}}(1+X)$ and $\lambda_e = \prod_{s<e} Q_s$. Let $\mathrm{len} \geq 1$, $g, \alpha \geq 0$, $s_0 < s_1 < \mathrm{len}$ and $b_0 \geq 1$, $b_1 \geq 1$ be integers with*

$$2(\alpha + 2g\,\mathrm{len}) < b_1, \qquad 2b_0 \leq \ell\, b_1.$$

If $a \in \mathcal{O}_L$ and $\ell^\alpha \lambda_{\mathrm{len}}^{2g} Q_{s_0}^{b_0} \in aA + Q_{s_1}^{b_1} A$, then a divides $\ell^{2(\alpha+2g\,\mathrm{len})}$ in $\mathcal{O}_L$.

Proof. Let v be the valuation of $\mathbb{C}_\ell$ with $v(\ell) = 1$, fix $\mathcal{O}_L \subset \mathbb{C}_\ell$, put $\beta = \alpha + 2g\,\mathrm{len}$, and write $F = \ell^\alpha \lambda_{\mathrm{len}}^{2g} Q_{s_0}^{b_0}$, a polynomial. Suppose $F = aG + Q_{s_1}^{b_1} B$ with $G, B \in A$ and, for a contradiction, $v(a) > 2\beta$ (this includes $a = 0$, $v(a) = \infty$).

Roots. $Q_s = \Phi_{\ell^{s+1}}(1+X) = \prod_\zeta (X - (\zeta - 1))$ over the primitive ℓ^{s+1}-th roots of unity ζ, of degree $\ell^s(\ell-1)$ (Lemma 23.8). Fix a root $\rho = \zeta_1 - 1$ of Q_{s_1} and put $r = v(\rho) = 1/(\ell^{s_1}(\ell-1))$. For a root $\tau = \zeta' - 1$ of Q_s, $\rho - \tau = \zeta_1 - \zeta' = \zeta'(\zeta_1\zeta'^{-1} - 1)$: if $s < s_1$, $\zeta_1\zeta'^{-1}$ has order ℓ^{s_1+1} and $v(\rho - \tau) = r$; if $s > s_1$, $v(\rho - \tau) = 1/(\ell^s(\ell-1))$; if $s = s_1$, $v(\rho - \tau) \geq \min(v(\rho), v(\tau)) = r$.

Discs. For $0 < R \leq r$ let $D_R = \{x \in \mathbb{C}_\ell : \ v(x - \rho) \geq R\}$; then $v(x) \geq R > 0$ on D_R, so every $H \in \mathcal{O}_L[[X]]$ converges on D_R with $v(H(x)) \geq 0$. For a monic $Q = \prod_\tau (X - \tau)$ and $x \in D_R$, $v(Q(x)) \geq \phi_Q(R) := \sum_\tau \min(R, v(\rho - \tau))$, since $v(x - \tau) \geq \min(v(x-\rho), v(\rho - \tau))$; and if $x = \rho + y$ with $v(y) = R$ and the reduction of y/π_R (π_R a fixed element of valuation R) avoids the finitely many residues of $-(\rho - \tau)/\pi_R$ with $v(\rho - \tau) = R$, then $v(x - \tau) = \min(R, v(\rho - \tau))$ for every τ. As the residue field of $\mathbb{C}_\ell$ is infinite, one $x_R \in D_R$ achieves equality simultaneously for all roots of all the Q_s, $s < \mathrm{len}$.

The estimate. Let $R = \theta r$ with $0 < \theta \le 1$ rational and write $\phi_s = \phi_{Q_s}(R)$. By the root distances: $\phi_{s_1} = \ell^{s_1}(\ell-1)\theta r = \theta$; for $s < s_1$, $\phi_s = \ell^s(\ell-1)\theta r = \theta\ell^{s-s_1} \le 1$, in particular $\phi_{s_0} \le \theta/\ell$; for $s > s_1$, $\phi_s \le \ell^s(\ell-1)/(\ell^s(\ell-1)) = 1$. Hence at $x = x_R$,

$$v(F(x)) = \alpha + 2g\sum_{s<\mathrm{len}} \phi_s + b_0\phi_{s_0} \le \beta + \theta b_0/\ell, \qquad v(Q_{s_1}(x)^{b_1}B(x)) \ge b_1\theta, \qquad v(aG(x)) \ge v(a).$$

From $F(x) = aG(x) + Q_{s_1}(x)^{b_1}B(x)$, $v(F(x)) \ge \min(v(a), b_1\theta)$. Now $b_1 - b_0/\ell \ge b_1/2 > \beta$ (as $2b_0 \le \ell b_1$ and $2\beta < b_1$), so $\theta_0 = \beta/(b_1 - b_0/\ell) < 1$, and $\beta + \theta_0 b_0/\ell = \beta b_1/(b_1 - b_0/\ell) \le 2\beta < v(a)$. Choose a rational $\theta \in (\theta_0, 1]$ so close to θ_0 that $\beta + \theta b_0/\ell < v(a)$. Then $b_1\theta > \beta + \theta b_0/\ell$ (equivalent to $\theta(b_1 - b_0/\ell) > \beta$), so $v(F(x)) \le \beta + \theta b_0/\ell < \min(v(a), b_1\theta) \le v(F(x))$, a contradiction. Hence $v(a) \le 2\beta$, i.e. a divides $\ell^{2\beta}$ in the discrete valuation ring $\mathcal{O}_L$. □

Lemma C.105. *Let Φ, the weight word, $G = w + S + \eta$, $g = \max(G, 1)$ and h_* be as in Proposition C.107, with $h_* \ge g$; let $c \in \mathbb{Z}/n$, $1 \le j \le \mathrm{len} \le n$ with the sites $c - \mathrm{len} + 1, \dots, c - \mathrm{len} + j$ thin, and fix the arc level data of Φ along the word at the cut c of length* len *(any frames), for $\nu = 1$ or 2, with (Lp7). Then for every $s <$* len*:*

1. *every $f \in \mathcal{T}_s$ is de Rham integral and kind integral at s;*
2. *constants $v \in L^r$ satisfy $\gamma_u^{(s)}v = v$, and $\gamma_u^{(s)}$ ($u \in \mathbb{Z}_\ell^\times$) maps de Rham integral vectors to de Rham integral vectors and kind integral vectors to kind integral vectors.*

Proof. Write $K = K_{(s)}$, $H = H(c-s)$, $b_s = s_{c-s}$, $M = \lfloor (h_* + 2g)/(\ell-1) \rfloor + 1$, so $(\ell-1)M > h_* + 2g$.

1. As $\varphi^k(X) = (1+X)^{\ell^k} - 1 = X\prod_{s'<k} Q_{s'}$ and $\varphi^k(Q_{s'}) = Q_{s'+k}$,

$$f' = \varphi^{\mathrm{len}-j}(X^{(\ell-1)M}\prod_{s'<j} Q_{s'}^{3g}) = X^{(\ell-1)M}\prod_{s'<\mathrm{len}-j} Q_{s'}^{(\ell-1)M}\prod_{\mathrm{len}-j\le s'<\mathrm{len}} Q_{s'}^{3g};$$

by (Lp1) (Lemma 23.17) $\lambda_s(X)$ and $\lambda_s(Q_{s'})$ for $s' \ne s$ are units of $K[[t]]$ and $\lambda_s(Q_s) \in tK[[t]]^\times$. Hence $\lambda_s(f') \in t^eK[[t]]^\times$ with $e = (\ell-1)M$ if $s < \mathrm{len} - j$ and $e = 3g$ if $s \ge \mathrm{len} - j$; and $\lambda_s(\Pi) \in t^HK[[t]]^\times$ (Lemma 23.17). So $f \in \mathcal{T}_s$ means $f \in t^mK[[t]]^r$ with $m = e - H$, i.e. $f^{(i)} = 0$ for $i < m$. It suffices that $\mathrm{Fil}_s(-i) = L^r$ for $i \ge m$, i.e. $-m \le \min J$ (J the jumps; $\mathrm{Fil}_s(u) = L^r$ for $u \le \min J$), and $\mathrm{Nb}_s(i) = L^r$ for $i \ge m$, i.e. $m \ge 0$ (thin, up) or $m \ge b_s$ (down). By (Lp7) and (Ow3) ($b_s \le b + \eta$ at thick sites, so $b_s + S \le h_*$):

- $s \ge \mathrm{len} - j$: the site $c - s$ is thin, $H = g + G$, $e = 3g$, $m = 2g - G \ge g \ge G \ge -\min J$, and $m \ge 0$.
- $s < \mathrm{len} - j$, thin: $H = g + G$, $m > h_* + g - G \ge g \ge G \ge -\min J$.
- up: $H = g + b_s + S$, $m > h_* + g - b_s - S \ge g \ge \eta \ge -\min J$ (jumps in $[b_s, b_s + S] \cup [-\eta, \eta]$), and $m \ge 0$.
- down: $H = g + \eta$, $m > h_* + g - \eta = b + S + g \ge b_s + S + g - \eta \ge b_s + S \ge -\min J$ (jumps in $[-b_s - S, -b_s] \cup [-\eta, \eta]$; $g \ge \eta$), and $m \ge b_s$.

2. $\gamma_u^{(s)}$ multiplies the t^i-coefficient by u^i and applies σ_u coordinatewise; σ_u fixes L, so constants are fixed, and it preserves $K \otimes_L \mathcal{X}$ for every L-subspace $\mathcal{X} \subseteq L^r$ (acting on the first factor), so it preserves $\Xi_s(\mathcal{X})$ for every family $\mathcal{X}$, in particular the de Rham and kind families. □

Proposition C.106. *Let $\ell \geq 11$, F, n, $D = G_F$, L, (τ_a), $\Phi\colon V_1 \to V_2$ ($T_1 \to T_2$), the weight word, Θ, G, g, h_*, H, j, $(\mathcal{C}, (\mathrm{len}_c))$, a_{E}, e_{E} and the hypotheses (Ae1), (Ae2) be as in Proposition C.83, and form $\sigma_{\mathrm{E}}, \rho_{\mathrm{E}}, \kappa_{\mathrm{E}}$ with the same data. For $c \in \mathcal{C}$ let $H_c\colon \mathbb{Z}/n \to \mathbb{Z}_{\geq 0}$ be the function of the arc level data along the word at c ($H_c(c-s) = g + G$, $g + s_{c-s} + S$, $g + \eta$ for $c - s$ thin, thick with $\epsilon_{c-s} = 1$, thick with $\epsilon_{c-s} = -1$), and for $\nu = 1, 2$*

$$W_\nu^c = \{w \in \mathbf{N}(T_\nu)[1/\lambda_\infty] :\ \Lambda_{c,\mathrm{len}_c}(H_c)\, w \in P_{\mathrm{len}_c}(\mathbf{N}(T_\nu)_{c-\mathrm{len}_c})\},$$
$$I_c(T_\nu) = \{w \in e_c\mathbf{N}(T_\nu)[1/\lambda_\infty] :\ (\gamma_u - 1)w \in X\mathbf{N}(T_\nu)\ \forall u\}.$$

Suppose:

(Ae4) *for every $c \in \mathcal{C}$ and every $w \in W_2^c \cap I_c(T_2)$ there are $w' \in W_1^c \cap I_c(T_1)$, $\epsilon \in \mathcal{O}_L^\times$ and an integer $0 \leq k \leq 2\kappa_{\mathrm{E}}$ with $\mathbf{N}(\Phi)(w') - \epsilon\, \ell^{\,2\sigma_{\mathrm{E}} + 2a_{\mathrm{E}} + \rho_{\mathrm{E}} + k}\, w \in X\mathbf{N}(T_2)_c$.*

Then

$$\ell^{\,e_{\mathrm{E}}} H_f^1(D, T_2) \subseteq \Phi_* H_f^1(D, T_1).$$

Proof. Throughout, ℓ, F, L, (τ_a) satisfy the hypotheses of Proposition 23.7, and H_f^1 of a lattice is the inverse image of H_f^1 of its representation.

Step 0: a common window. Define (M_a, h_a) by $(M_a, h_a) = (\eta, \max(s_a + S, \eta))$ if $\epsilon_a = 1$ and $(M_a, h_a) = (\max(s_a + S, \eta), \eta)$ if $\epsilon_a = -1$. By (Ow1) every Hodge jump of V_1 at τ_a lies in $[-M_a, h_a]$; by item 1 of Lemma 24.2 so does every Hodge jump of V_2; and $0 \in [-M_a, h_a]$. We record: (i) $h_{c-s} \leq H_c(c-s)$ for all $c \in \mathcal{C}$, $0 \leq s < \mathrm{len}_c$: at a thin site $s_a \leq w$, so $M_a, h_a \leq \max(w + S, \eta) \leq G \leq g + G$; at a thick site with $\epsilon = 1$, $s_a > g \geq G \geq \eta$ by (Ae1), so $h_a = s_a + S \leq g + s_a + S$; with $\epsilon = -1$, $h_a = \eta \leq g + \eta$. (ii) $M_{c+s} \leq G$ for $c \in \mathcal{C}$, $1 \leq s \leq j$, as these sites are thin (Cc1). (iii) $h_* \geq g$: if $\Theta \neq \emptyset$, a thick a has $g < s_a \leq b + \eta \leq h_*$ by (Ae1), (Ow3); otherwise $h_* = g$. Hence $M_a \leq \max(G, h_*) \leq h_*$ at every site (thick: $s_a + S \leq b + \eta + S = h_*$). (iv) $M_a + h_a \leq H(a)$: at a thick site $M_a + h_a \leq h_* + \eta \leq 2h_*$; at a thin site $M_a + h_a \leq 2G$.

Step 1: the parameters. Put $M' = \lfloor (h_* + 2g)/(\ell - 1) \rfloor + 1$, $m = (\ell - 1)M'$, $R = m$ and $g' = 3g$. Then $m > h_* + 2g$, so $m \geq h_* + 2g + 1 \geq \max(3g, 1 + \max_a M_a)$ by (iii), $g' \geq G$, and $\sigma := \sigma_\ell(g', m, j) = \lfloor (3g + M'/\ell^{j-1})\ell/(\ell - 1)^2 \rfloor + 1 \leq \sigma_{\mathrm{E}}$ since $m/N_j = M'/\ell^{j-1}$ and $\lfloor y \rfloor + 1 \leq \lceil y + 1 \rceil$. With (ii) and $j \leq \mathrm{len}_c$ (Cc2), the hypotheses of Lemma 24.5 hold for $\mathbf{N}(T_1)$ and $\mathbf{N}(T_2)$ with the window (M_a, h_a) and the parameters $c \in \mathcal{C}$, j, G, g', m, R; let $\rho_{c,j}^{(\nu)}$ be the relay classes of $\mathbf{N}(T_\nu)$. Their domain is $I_c(T_\nu)$.

Step 2: the extension and its vectors x_c. Let $x \in H_f^1(D, T_2)$. By Lemma 23.26 there is an exact sequence $0 \to T_2 \xrightarrow{\iota'} T \xrightarrow{\pi} \mathcal{O}_L \to 0$ of lattices of pairs with $[T] = x$, whose Hodge jumps are 0 or jumps of V_2; so (M_a, h_a) is a window of V and of V_2. Let $c \in \mathcal{C}$. By Lemma 24.3 there is $e_c^\circ \in \mathbf{N}(\mathcal{O}_L)_c$ with $\iota_{\mathcal{O}_L}(e_c^\circ) = \varepsilon_c$. By Proposition C.6 for the surjection π, with the window (M_a, h_a) of V, there is $x_c' \in \mathbf{N}(T)_c$ with $\mathbf{N}(\pi)(x_c') = \ell^{r_c} e_c^\circ$, where r_c is formed with $H^V(a) = M_a + h_a$. As $H^V \leq H$ pointwise by (iv), and s_c, the maximum and c_H are monotone in the function, $r_c \leq \rho_{\mathrm{E}}$. Put $x_c = \ell^{\rho_{\mathrm{E}} - r_c} x_c'$; by (Ex3), $(1 \otimes \pi)(\iota_T(x_c)) = \iota_{\mathcal{O}_L}(\mathbf{N}(\pi)x_c) = \ell^{\rho_{\mathrm{E}}}\varepsilon_c$.

Step 3: splitting. Proposition 24.4 (with $T' = T_2$, $C = \mathcal{C}$, the lengths len_c, which satisfy its hypotheses by (Cc2), the window (M_a, h_a) of V_2 and V, the parameters of Step 1 and $r = \rho_{\mathrm{E}}$) gives $\tau_c' \in P_j(\mathbf{N}(T_2)_c)$ and $w_c \in I_c(T_2)$ with

$$\mathbf{N}(\iota')(w_c) = \varphi^{\mathrm{len}_c - j}(X_{c-\mathrm{len}_c}) - \ell^\sigma x_c, \qquad X_{c'} = \ell^\sigma \varphi^j(x_{c'}) - \mathbf{N}(\iota')(\tau_{c'}'),$$

and $\ell^{\rho_{\mathrm{E}} + 2\sigma} x = \sum_{c \in \mathcal{C}} \rho_{c,j}^{(2)}(w_c)$.

Step 4: $w_c \in W_2^c$**.** Apply Lemma C.96 to $0 \to T_2 \to T \to \mathcal{O}_L \to 0$ with the window (M_a, h_a) of V, $e = \mathrm{len}_c \le n$ and H_c (allowed by (i)); its $W(T_\nu)$, $W(T)$ are W_ν^c and the analogous set for T. Write $c' = c - \mathrm{len}_c$. Since $\varphi^{\mathrm{len}_c - j}\varphi^j(x_{c'}) = \varphi^{\mathrm{len}_c}(x_{c'})$ and, for $\tau'_{c'} = \sum_i f_i \varphi^j(n_i)$ ($n_i \in \mathbf{N}(T_2)_{c'}$), $\varphi^{\mathrm{len}_c - j}(\tau'_{c'}) = \sum_i \varphi^{\mathrm{len}_c - j}(f_i)\varphi^{\mathrm{len}_c}(n_i)$, while $\mathbf{N}(\iota')$ commutes with φ, we get $\varphi^{\mathrm{len}_c - j}(X_{c'}) \in P_{\mathrm{len}_c}(\mathbf{N}(T)_{c'})$. By items 1 and 2 of Lemma C.96, $\mathbf{N}(\iota')(w_c) \in W(T)$, and by item 3, $w_c \in W_2^c$. So $w_c \in W_2^c \cap I_c(T_2)$.

Step 5: lifting each relay class. By (Ae4) there are $w'_c \in W_1^c \cap I_c(T_1)$, $\epsilon_c \in \mathcal{O}_L^\times$ and $0 \le k_c \le 2\kappa_{\mathrm{E}}$ with $\mathbf{N}(\Phi)(w'_c) - \epsilon_c \ell^{e'+k_c} w_c \in X\mathbf{N}(T_2)_c$, $e' = 2\sigma_{\mathrm{E}} + 2a_{\mathrm{E}} + \rho_{\mathrm{E}}$. By (Ry6) for Φ, $\mathbf{N}(\Phi)(w'_c) \in I_c(T_2)$ and $\rho^{(2)}_{c,j}(\mathbf{N}(\Phi)w'_c) = \Phi_* \rho^{(1)}_{c,j}(w'_c)$; by (Ry5) and (Ry4), $\rho^{(2)}_{c,j}(\mathbf{N}(\Phi)w'_c) = \epsilon_c \ell^{e'+k_c}\rho^{(2)}_{c,j}(w_c)$. Moreover $\rho^{(1)}_{c,j}(w'_c) \in H^1_f(F, T_1)$ by (Ry7): $\Lambda_{c,\mathrm{len}_c}(H_c)w'_c \in P_{\mathrm{len}_c}(\mathbf{N}(T_1)_{c'})$, and by item 2 of Lemma 23.15 (window (M_a, h_a) of V_1) $\Lambda_{c,\mathrm{len}_c}(M)P_{\mathrm{len}_c}(\mathbf{N}(T_1)_{c'}) \subseteq \mathbf{N}(T_1)_c$, so $f w'_c \in \mathbf{N}(T_1)$ for the product $f = \Lambda_{c,\mathrm{len}_c}(M)\Lambda_{c,\mathrm{len}_c}(H_c)$ of the Q_s. Multiplying by $\epsilon_c^{-1}\ell^{2\kappa_{\mathrm{E}}-k_c}$,

$$\ell^{\,e'+2\kappa_{\mathrm{E}}}\rho^{(2)}_{c,j}(w_c) \in \Phi_* H^1_f(D, T_1).$$

Step 6: the exponent. Summing over c with Step 3, $\ell^{\rho_{\mathrm{E}}+2\sigma+e'+2\kappa_{\mathrm{E}}}x \in \Phi_* H^1_f(D, T_1)$, an $\mathcal{O}_L$-submodule. Since $\sigma \le \sigma_{\mathrm{E}}$,

$$\rho_{\mathrm{E}} + 2\sigma + 2\sigma_{\mathrm{E}} + 2a_{\mathrm{E}} + \rho_{\mathrm{E}} + 2\kappa_{\mathrm{E}} \le 2\rho_{\mathrm{E}} + 4\sigma_{\mathrm{E}} + 2a_{\mathrm{E}} + 2\kappa_{\mathrm{E}} = e_{\mathrm{E}},$$

so $\ell^{e_{\mathrm{E}}}x \in \Phi_* H^1_f(D, T_1)$. □

Proposition C.107. *Let ℓ be an odd prime, F, L, $(\tau_a)_{a\in\mathbb{Z}/n}$ as in Proposition 23.7, $\Phi\colon V_1 \to V_2$ an onto map of crystalline lattices $T_1 \to T_2$ over F with coefficients L, and $(b, w, S, \eta, m, (s_a), (\epsilon_a))$ a weight word for Φ, with $G = w + S + \eta$, $g = \max(G, 1)$, and $h_* = b + \eta + S$ if some site is thick, $h_* = g$ otherwise. Suppose $h_* \ge g$ (this holds if no site is thick, or if $s_a > g$ for every thick site a). Let $c \in \mathbb{Z}/n$, $1 \le \mathrm{len} \le n$ and $1 \le j \le \mathrm{len}$. Then for all frames $\mathbf{n}^1$, $\mathbf{n}^2$ at $c - \mathrm{len}$ of $\mathbf{N}(T_1)$, $\mathbf{N}(T_2)$, the arc level data of Φ along the word at the cut c of length* len *(with this j) have the arc level properties (Lp1)–(Lp8) and (Lh1)–(Lh2).*

Proof. (Lp1), (Lp2), (Lp4), (Lp5), (Lp8), (Lh1) and (Lh2) hold for any arc level data of Φ by Lemma 23.17 (Φ is G_F-equivariant and $\mathcal{O}_L$-linear). (Lp3) holds for $\nu = 1, 2$ by Lemma 23.5 (2), applied to the pairs (V_ν, T_ν), which are crystalline. (Lp7) is Lemma 24.2 (2). For (Lp6): along the word $M = \lfloor (h_* + 2g)/(\ell - 1) \rfloor + 1$, so $(\ell - 1)M > h_* + 2g \ge 3g$; Lemma C.68 applies to (V_ν, T_ν) and gives (Lp6). (If $s_a > g$ for every thick a and some site is thick, then for a thick a, $h_* = b + \eta + S \ge s_a > g$ by (Ow3).) □

D Technical facts on good root towers

This appendix proves the facts about good root towers that Part VI states and uses but whose proofs are long and technical. Every result it supplies is used in Part VI, at one of four places. The facts of good towers Proposition 34.2 take from it the flat determinant heights (Proposition D.65), the proportional analytic square systems (Proposition D.73) and geometric proportionality with covolumes (Proposition D.79), the identity that writes the geometric intersection numbers of the root models as covolumes of principal congruence subgroups times flag degrees (Section D.4). The Hodge-type base settings Proposition 34.35 take their existence, the base slope identity, the cusp mean bound with the corrected point heights, and positivity (Proposition D.1, Proposition D.19, Proposition D.21, Proposition D.56; Sections D.1–D.3). The existence theorem Proposition 31.14 takes nefness at primes (Gi4) (Theorem D.57), the fibre-square

identities (Fq1) (Theorem D.85) and the reference comparisons (Rh1), (Rh2), (Vl) with the wall inputs (Wd), (Wp) derived from them (Section D.6). And the torsion systems of Section 35 take the test-point facts (Proposition D.97, Section D.5).

The methods are those of Part VI: arithmetic intersection theory with log-log singular metrics and their regularisations, Hodge theory near the boundary of toroidal compactifications, integral p-adic Hodge theory of abelian schemes, and heights of points. A reader who accepts the statements just listed can read Part VI without this appendix. Apart from results of Part VI, the appendix uses one lemma of Part I (Lemma 3.5) and one of Part III (Lemma 16.27).

D.1 Base settings: existence and the slope identity

This section proves two of the four parts of the Hodge-type base settings Proposition 34.35 of Section 34: at every admissible stage and component a good tower has a base setting with regularisation limits (Proposition D.1), and the base slope identity holds (Proposition D.19). Through Proposition 34.35 and Proposition 34.2 they supply the base slope settings from which the slope at the test class (Rt6) is proved (Theorem 31.45). A base setting consists of a base variety with a Hodge-type line, log-log regularisations and cusp data (Definition 31.10, Definition 31.12, Definition 31.11); Subsection D.1.1 constructs it with its cusp weight, Subsection D.1.2 corrects and regularises the weight, and Subsection D.1.3 proves the slope identity.

D.1.1 Existence of base settings and the cusp weight

The subsection states Proposition D.1 and proves the bound for the cusp weight on which the cusp mean rests: the cusp weight of a base setting has mean at most the cusp mean constant against the Hodge measure (Lemma D.2), a lemma used again in Section D.2. The other lemmas are estimates for polarised lattices with CM action over a fundamental domain of the archimedean unitary group: the covolume of an isotropic sublattice, a scaling law for isotropic vectors, counts of short vectors, and the mean of the logarithm of the first minimum.

Proposition D.1. *Let $\mathfrak{u}$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. Then for every admissible stage B and every $Z \in \mathrm{Cp}$ there is a base setting of $\mathcal{T}$ at (B, Z) (in particular its cover has the coordinate charts of (Gb1), and its Hodge-type line is integrally $\bigotimes_\tau (\det \mathcal{V}_{(\tau,-)})^\vee$ with the dual Hodge metric, (Gb2), (Gb3)) such that:*

- *(a) the limit self-intersection $\mathrm{ar}^{\mathrm{b}}(B)$ exists and the setting has regularisation limits (Mz1)–(Mz4) at B;*
- *(b) for all integers $1 \le i_1 \le i_2$ there is $t_1 > 0$ such that the first Chern form $c_1(\overline{\mathcal{M}}_{B,i_1,i_2,t})$ is semipositive on X^{b} for $0 < t \le t_1$;*
- *(c) there is $t_0 > 0$ such that $\psi_{t'} \ge \log h^{\mathrm{H}} + \frac{1}{2}U - \log h^{\mathrm{ref}'}$ on $X^{\mathrm{b},\circ}$ for $0 < t' \le t_0$;*
- *(d) the setting is joined to the stage root model $\mathcal{X}_{B,Z}$;*
- *(e) the setting satisfies the hypotheses (i) and (ii) of Lemma 34.36.*

Proof. Let B be an admissible stage and $Z \in \mathrm{Cp}$; let $\mathcal{Y}_Z$ be the component of the fixed data at Z (it meets $j(\mathrm{Sh}_F)$), $Z_c = j^{-1}(\mathcal{Y}_Z)$, $S'_B = \{\ell \le B \text{ prime} : \ell \notin \mathrm{Bd}\}$ and $S = \mathrm{Bd} \cup S'_B$. By (Go5), Bd is the set of primes dividing $2\,\mathrm{disc}(E)N'$, so S contains every prime dividing $\mathrm{disc}(E)$, and $\mathrm{Bd} \cap S'_B = \varnothing$. Write c for complex conjugation on complex points of schemes over $\mathbb{Z}$.

Step 1: the cover. Proposition 33.9 for $\mathcal{Y}_c = \mathcal{Y}_Z$ and S gives a cover $f\colon \mathcal{X} \to \mathcal{Y}$ with function field Ω, projective over $\operatorname{Spec}\mathbb{Z}$, such that: the generic point of $\mathcal{X}$ maps to that of $\mathcal{Y}_Z$ and Ω is finite over $\mathbf{k}(Z_c)$; $\mathcal{X}_{\mathbb{Q}}$ is smooth and the complement Δ of the abelian locus in $\mathcal{X}_{\mathbb{Q}}$ is the support of a simple normal crossing divisor, which we take reduced; for $\ell \in S$ the ℓ^4-torsion of $G_{\mathcal{X}}$ is rational over Ω, the Hodge lattices at (S,R), $R \equiv 4$, are Hodge bundles $\mathcal{V}_s^S$, and Hodge–Tate freeness holds at every point over ℓ. The generic point of $\mathcal{X}$ lies over $\mathbb{Q}$, so $\mathcal{O}_{\mathcal{X}}$ is torsion free and $\mathcal{X}$ is flat over $\mathbb{Z}$; being proper with dense image, $\mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ is surjective. $\dim \mathcal{X}_{\mathbb{Q}} = \operatorname{trdeg}_{\mathbb{Q}}\Omega = \operatorname{trdeg}_F \mathbf{k}(Z_c) = \dim \operatorname{Sh}_F = |\Phi|pq = e$ (by (Fh2) the flag bundle has dimension $|\Phi|pq$ plus the fibre dimension $|\Phi|(p(p-1)+q(q-1))/2$), so $\mathcal{X}$ has absolute dimension $e+1$. As $\mathcal{X}$ is proper over $\mathbb{Z}$ and $\operatorname{Spec}\mathcal{O}_F$, $\mathcal{Y}$ are separated over $\mathbb{Z}$, $\mathcal{X}$ is proper over $\mathcal{O}_F$ and over $\mathcal{Y}$.

Step 2: the Hodge lattices at $(\mathrm{Bd}, 4)$ *are Hodge bundles.* Put $d = \operatorname{disc}(E)$ and $A = \mathcal{O}_F[1/d]$. Let $e_s \in E \otimes_{\mathbb{Q}} F \cong \prod_{\sigma\in\operatorname{Hom}(E,F)} F$ (the isomorphism $a\otimes b \mapsto (\sigma(a)b)_\sigma$; $\sigma(E) \subseteq F$ for all σ, (Fh1)) be the idempotent of the factor σ_s ($s \mapsto \sigma_s$ is bijective onto $\operatorname{Hom}(E,F)$). The e_s lie in $\mathcal{O}_E \otimes_{\mathbb{Z}} A$: let $\omega_1,\dots,\omega_m$ be a $\mathbb{Z}$-basis of $\mathcal{O}_E$ ($m = [E:\mathbb{Q}] = |\operatorname{Hom}(E,F)|$) and $\omega_1^\vee,\dots,\omega_m^\vee$ the dual basis for the trace form, $\operatorname{Tr}_{E/\mathbb{Q}}(\omega_i^\vee\omega_j) = \delta_{ij}$, a $\mathbb{Z}$-basis of the inverse different $\mathfrak{D}_E^{-1}$. For the square matrices $M = (\sigma(\omega_j))_{\sigma,j}$ and $N = (\sigma(\omega_i^\vee))_{\sigma,i}$ the trace relations say $N^{\mathsf{T}}M = 1$, hence $MN^{\mathsf{T}} = 1$, i.e. $\sum_i \sigma'(\omega_i^\vee)\sigma(\omega_i) = \delta_{\sigma\sigma'}$; so $\sum_i \omega_i^\vee \otimes \sigma_s(\omega_i)$ maps to the unit vector at σ_s and equals e_s. The integral ideal $\mathfrak{D}_E$ has norm $|d|$ ([190, III (2.9)]) and contains its norm (which annihilates $\mathcal{O}_E/\mathfrak{D}_E$), so $d \in \mathfrak{D}_E$, $\mathfrak{D}_E^{-1} \subseteq d^{-1}\mathcal{O}_E$, and $e_s \in d^{-1}\mathcal{O}_E \otimes \mathcal{O}_F \subseteq \mathcal{O}_E \otimes A$. Let $U_2 \subseteq \mathcal{X}$ be the complement of the fibres over the primes of Bd; on U_2, d is invertible, so $\operatorname{Lie}(G_{\mathcal{X}})|_{U_2} = \bigoplus_s e_s\operatorname{Lie}(G_{\mathcal{X}})|_{U_2}$ with locally free summands (direct summands of the locally free $\operatorname{Lie}(G_{\mathcal{X}})|_{U_2}$; finitely generated projective modules over local rings are free), and $(e_s\operatorname{Lie}(G_{\mathcal{X}}))_x = e_s\operatorname{Lie}_x = \operatorname{Lie}_x \cap \operatorname{Li}_s = \mathrm{dR}_{s,x}$ (if $v \in \operatorname{Lie}_x \cap \operatorname{Li}_s$ then $v = e_s v$). Let U_1 be the complement of the fibres over the primes of S'_B. The Hodge lattice at $(\mathrm{Bd},4)$ has stalk $\mathrm{HT}_{s,x}(\ell^4) = V_{s,x}^S$ at x over $\ell \in \mathrm{Bd}$ (these x lie in U_1) and $\mathrm{dR}_{s,x}$ elsewhere; on U_1 it is given by $\mathcal{V}_s^S|_{U_1}$, on U_2 by $e_s\operatorname{Lie}(G_{\mathcal{X}})|_{U_2}$, and on $U_1\cap U_2$ both have the stalks $\mathrm{dR}_{s,x}$. Subsheaves of the constant sheaf Li_s on the integral scheme $\mathcal{X}$ with the same stalks coincide, and $U_1 \cup U_2 = \mathcal{X}$; so they glue to a locally free subsheaf $\mathcal{V}_s$ of rank $\operatorname{rk}(s)$ with the stalks of the Hodge lattice at $(\mathrm{Bd},4)$: these are Hodge bundles. On $\mathcal{X}_{\mathbb{Q}}$, $\mathcal{V}_s = \mathcal{V}_s^S = \operatorname{Lie}_s(G_{\mathcal{X}_{\mathbb{Q}}})$.

Step 3: the embedding. Let $\mathcal{L} = \bigotimes_\tau (\det \mathcal{V}_{(\tau,-)})^\vee$ and $\mathcal{L}^S = \bigotimes_\tau (\det \mathcal{V}^S_{(\tau,-)})^\vee$, invertible subsheaves of the constant sheaf on the one-dimensional Ω-space $\bigotimes_\tau (\det \operatorname{Li}_{(\tau,-)})^\vee$, equal outside the fibres over S'_B. Locally $\mathcal{L} = \mathcal{O}\lambda$, $\mathcal{L}^S = \mathcal{O}g\lambda$ with $g \in \Omega^\times$; the sheaf $\mathcal{O}g \subseteq \Omega$ does not depend on the generators, so there is an invertible subsheaf $\mathcal{O}(D) \subseteq \Omega$ with $\mathcal{O}(D)\mathcal{L} = \mathcal{L}^S$, equal to $\mathcal{O}_{\mathcal{X}}$ outside the fibres over S'_B. These fibres are pairwise disjoint and closed, so $\mathcal{O}(D) = \prod_{\ell\in S'_B} \mathcal{O}(D_\ell)$, where $\mathcal{O}(D_\ell) \subseteq \Omega$ equals $\mathcal{O}(D)$ off the fibres over $S'_B \smallsetminus \{\ell\}$ and $\mathcal{O}_{\mathcal{X}}$ off the fibre over ℓ. At x over $\ell \in S'_B$, if b_τ is a basis of $\mathcal{V}_{(\tau,-),x}$ and $b_\tau T_\tau$ one of $\mathcal{V}^S_{(\tau,-),x} = \mathrm{HT}_{(\tau,-),x}(\ell^4)$, then $(\det(b_\tau T_\tau))^\vee = (\det T_\tau)^{-1}(\det b_\tau)^\vee$, so $\mathcal{O}(D_\ell)_x = (\prod_\tau \det T_\tau)^{-1}\mathcal{O}_x$.

Fix a closed immersion $i_0\colon \mathcal{X} \to \mathbb{P}(\mathbb{Z}^{I_0})$ and $\mathcal{A} = i_0^*\mathcal{O}(1)$. By Serre's theorem ($\mathcal{X}$ projective over the noetherian ring $\mathbb{Z}$) there is $m \geq 1$ such that $\mathcal{N}_1 = \mathcal{L}\otimes\mathcal{A}^m$ and $\mathcal{N}_\ell = \mathcal{O}(D_\ell)\otimes\mathcal{A}^m$ ($\ell \in S'_B$) are generated by global sections, and finitely many suffice (the global sections form finitely generated $\mathbb{Z}$-modules). Let $K = \{0,1\} \sqcup S'_B$, $\mathcal{N}_0 = \mathcal{A}$, and for $k \in K$ let $(\sigma_{k,i})_{i\in I_k}$ be generating global sections of $\mathcal{N}_k$ (for $k=0$ the pulled-back coordinates). They define $i_k\colon \mathcal{X} \to \mathbb{P}(\mathbb{Z}^{I_k})$ with $i_k^*\mathcal{O}(1) \cong \mathcal{N}_k$, the coordinate $Y_{k,i}$ going to $\sigma_{k,i}$, and $\iota = (i_k)_k\colon \mathcal{X} \to \mathbb{P} = \prod_k \mathbb{P}(\mathbb{Z}^{I_k})$. Since $\mathrm{pr}_0 \circ \iota = i_0$ is a closed immersion and pr_0 is separated, ι is a closed immersion; put $\mathcal{X}^{\mathrm{b}} = \iota(\mathcal{X})$, which we identify with $\mathcal{X}$. With the unit vectors e_k, $a^{\mathrm{H}} = e_1 - me_0$ and $v_\ell = e_\ell - me_0$ we get isomorphisms $\alpha\colon \mathcal{O}(a^{\mathrm{H}})|_{\mathcal{X}} \to \mathcal{L}$ and $\beta_\ell\colon \mathcal{O}(v_\ell)|_{\mathcal{X}} \to \mathcal{O}(D_\ell)$. Fixing a non-zero rational section ρ_k of each $\mathcal{N}_k$, $y_{k,i} = \sigma_{k,i}/\rho_k \in \Omega$ are representing coordinates.

(Hy1) and (Gb1): $\mathcal{X}^{\mathrm{b}}$ is a closed integral subscheme of $\mathbb{P}$, flat over $\mathbb{Z}$, of absolute dimension $e+1$, with smooth generic fibre (Step 1). The map $y_{\mathbb{C}} = \iota_{\mathbb{C}}$ is a bijection from the complex points of $\mathcal{X}$ (over all embeddings of F) onto X^{b}, and a biholomorphism; put $\partial X^{\mathrm{b}} = y_{\mathbb{C}}(\Delta(\mathbb{C}))$, the image of the complex points outside the abelian locus: a closed subset, stable under c (Δ is defined over $\mathbb{Q}$), and a divisor with simple normal crossings (the base change to $\mathbb{C}$ of a simple normal crossing divisor of the smooth $\mathbb{Q}$-variety $\mathcal{X}_{\mathbb{Q}}$). The first clauses of (Gb1) are Step 1 and Step 2 ($R(\ell) = 4$ for $\ell \in S$). The rational map given by y is ι: birational onto $\mathcal{X}^{\mathrm{b}}$ and defined everywhere. Charts: for c' with $y_{c'} \neq 0$, $\mathcal{X}'_{c'} = \iota^{-1}(\prod_k D_+(Y_{k,c'_k}))$ is affine (closed in an affine scheme) with ring the image of $\mathbb{Z}[Y_{k,i}/Y_{k,c'_k}]$, which is $\mathbb{Z}[y]_{c'} \subseteq \Omega$; every point lies in some $\mathcal{X}'_{c'}$ (a coordinate of each factor does not vanish there). A point x' of $\mathcal{X}'_{c'}$ has $\mathbb{Z}[y]_{c'} \subseteq \mathcal{O}_{x'}$. Conversely, if $\mathbb{Z}[y]_{c'} \subseteq \mathcal{O}_{x'}$, the induced morphism $\operatorname{Spec}\mathcal{O}_{x'} \to \mathcal{X}'_{c'} \subseteq \mathcal{X}$ and the canonical one agree at the generic point; their equaliser is a closed subscheme of the integral scheme $\operatorname{Spec}\mathcal{O}_{x'}$ ($\mathcal{X}$ is separated) through which $\operatorname{Spec}\Omega$ factors, so its ideal vanishes in $\Omega \supseteq \mathcal{O}_{x'}$ and the two morphisms are equal: $x' \in \mathcal{X}'_{c'}$. Hence the points lying in the chart c' form the affine open $\mathcal{X}'_{c'} = \mathcal{X}_{c'}$ with ring $\mathbb{Z}[y]_{c'}$.

Step 4: Hodge data, (Hy2), (Hy3), (Gb2)–(Gb4). For $\tau \in \Phi$ let $\mathcal{E}_\tau = (\mathrm{Lie}_{(\tau,+)} \oplus \mathrm{Lie}_{(\tau,-)})(G_{\mathcal{X}_{\mathbb{Q}}})^\vee$, of rank $p+q=n$ (the pieces are direct summands of the locally free $\mathrm{Lie}(G_{\mathcal{X}_{\mathbb{Q}}})$, of the generic ranks), and $\mathcal{F}_\tau$ the annihilator of $\mathrm{Lie}_{(\tau,-)}$, of rank $p=j$; their generic fibres are those of (Gb2), and $\mathcal{E}_\tau/\mathcal{F}_\tau = \mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X}_{\mathbb{Q}}})^\vee = \mathcal{V}^\vee_{(\tau,-)}|_{\mathcal{X}_{\mathbb{Q}}}$ (Step 2). At $w = y_{\mathbb{C}}(z) \in X^{\mathrm{b},\circ}$ let $h_\tau(w)$ be the metric dual to the restriction of h_z to $\mathrm{Lie}_{(\tau,+)}(G_{\mathcal{X},z}) \oplus \mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X},z})$, smooth in w by (Fh4); its quotient metric on $\mathcal{E}_\tau/\mathcal{F}_\tau$ is dual to $h_z|_{\mathrm{Lie}_{(\tau,-)}}$. The isomorphism of (Hy2) is $\alpha_{\mathbb{Q}}$ followed by $\mathcal{L}_{\mathbb{Q}} = \bigotimes_\tau \det(\mathcal{E}_\tau/\mathcal{F}_\tau)$. (Gb3): on the affine chart $\mathcal{X}_{c'}$ the locally free $\mathcal{V}_{(\tau,-)}$ are free on the members of a finite cover by principal opens $D(f)$, $f \in \mathbb{Z}[y]_{c'}$, with frames b_τ; there $\bigotimes_\tau(b^\vee_{\tau,1} \wedge \cdots \wedge b^\vee_{\tau,q})$ generates $\mathcal{L}$, and $\prod_k y_{k,c'_k}^{a^{\mathrm{H}}_k}$ generates $\mathcal{O}(a^{\mathrm{H}})$ on $\mathcal{X}_{c'}$, so α maps it to $u\bigotimes_\tau(b^\vee_{\tau,1} \wedge \cdots \wedge b^\vee_{\tau,q})$ with u a unit on $D(f)$. (Hy3), (Gb4): S'_B is the set S of primes of the Hodge–Tate twist (Hy3), and $g_\ell = \beta_\ell^{-1}(1)$, the rational section of $\mathcal{O}(v_\ell)|_{\mathcal{X}}$ corresponding to the rational section 1 of $\mathcal{O}(D_\ell) \subseteq \Omega$. For any chart c', $g_\ell = y_{c'}^{v_\ell} \cdot \beta_\ell(y_{c'}^{v_\ell})^{-1}$ with $\beta_\ell(y_{c'}^{v_\ell})^{-1} \in \Omega^\times = \operatorname{Frac}\mathbb{Z}[y]_{c'}$ a quotient of two multihomogeneous polynomials of equal multidegree, so g_ℓ is a quotient of multihomogeneous polynomials whose degrees differ by v_ℓ. At x over ℓ in the chart c', $\beta_\ell(y_{c'}^{v_\ell})$ generates $\mathcal{O}(D_\ell)_x = (\prod_\tau \det T_\tau)^{-1}\mathcal{O}_x$ (Step 3), i.e. equals $u(\prod_\tau \det T_\tau)^{-1}$ with $u \in \mathcal{O}_x^\times$, so $g_\ell/y_{c'}^{v_\ell} = u^{-1}\prod_\tau \det T_\tau$: this is (Gb4). At x not over ℓ, $\mathcal{O}(D_\ell)_x = \mathcal{O}_x$ and $g_\ell/y_{c'}^{v_\ell}$ is a unit, so the divisor of g_ℓ is supported in the fibre over ℓ. For this B, $S_B = S'_B$.

Step 5: auxiliary line, (Hy5), (Gb6). Let $a' = (1,\dots,1)$ and $h'(\hat{w}) = \prod_k \|\hat{w}_k\|_2$ (the ℓ^2 norm of the coordinates of the factor k), a smooth cone function of multidegree a' invariant under c. Its first Chern form $\Theta = dd^c \log h'$ on X^{b} is the sum of the pull-backs of the Fubini–Study forms of the factors, each ≥ 0, and the one of the factor 0 is positive as i_0 is an immersion: Θ is a Kähler form. Put $k^{\mathrm{A}} = 0$: for $x \in \mathcal{X}^{\mathrm{b}}(\overline{\mathbb{Q}})$, $h_{\overline{\mathcal{A}}'}(D_x) = \sum_k h_{(\mathcal{O}(e_k),\|\cdot\|_2)}(D_x)$, and each term is ≥ 0. Indeed, let $K = \mathbf{k}(D_x)$ and $\nu\colon \operatorname{Spec}\mathcal{O}_K \to D_x$ the normalisation, finite and birational, hence surjective (its image is closed and dense). Apply Lemma 30.13 with $b=1$, $d=0$, the morphism ν and $\overline{\mathcal{H}}_1 = (\mathcal{O}(e_k), \|\cdot\|_2)|_{D_x}$: $\operatorname{Spec}\mathcal{O}_K$ (finite over $\mathbb{Z}$) and D_x (closed in $\mathbb{P}$) are integral, flat and projective over $\mathbb{Z}$ of absolute dimension 1; their generic fibres are spectra of the number field K, hence smooth over $\mathbb{Q}$; the metric is smooth (the restriction of the cone metric $\|\hat{w}_k\|_2$); and $c = [K : \mathbf{k}(D_x)] = 1$. It gives $h_{(\mathcal{O}(e_k),\|\cdot\|_2)}(D_x) = \widehat{\deg}\,\nu^*(\mathcal{O}(e_k), \|\cdot\|_2)$, where $\nu^*\mathcal{O}(e_k)$ is the pull-back of $\mathcal{O}(1)$ with its ℓ^2 metric along the extension $\operatorname{Spec}\mathcal{O}_K \to \mathbb{P}(\mathbb{Z}^{I_k})$ of $\mathrm{pr}_k(x)$. Take homogeneous coordinates $(x_i)_{i\in I_k}$ of $\mathrm{pr}_k(x)$ in K with $x_{i_0} = 1$ for some i_0. At a maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$ put $\mu_{\mathfrak{p}} = \min_i \mathrm{ord}_{\mathfrak{p}}(x_i) \leq \mathrm{ord}_{\mathfrak{p}}(x_{i_0}) = 0$; the line is generated there by

Y_j for any j with $\mathrm{ord}_{\mathfrak{p}}(x_j) = \mu_{\mathfrak{p}}$, so the rational section $Y_{i_0} = (x_{i_0}/x_j)Y_j$ has order $-\mu_{\mathfrak{p}} \geq 0$ at $\mathfrak{p}$; at an embedding $\sigma\colon K \to \mathbb{C}$, $\|Y_{i_0}\|_\sigma = |\sigma(x_{i_0})|/\|\sigma(x)\|_2 = \|\sigma(x)\|_2^{-1}$. Computing the arithmetic degree of a hermitian line bundle on $\operatorname{Spec}\mathcal{O}_K$ with the non-zero rational section Y_{i_0} gives $\widehat{\deg} = \sum_{\mathfrak{p}}(-\mu_{\mathfrak{p}})\log N\mathfrak{p} + \sum_\sigma \log\|\sigma(x)\|_2 \geq 0$, as $\|\sigma(x)\|_2 \geq |\sigma(x_{i_0})| = 1$.

Step 6: the boundary section and the cusp data. $\mathcal{O}(\Delta)$ and its canonical section s_Δ are defined over $\mathbb{Q}$; complex conjugation lifts to an antiholomorphic involution $\tilde{c}$ of the total space of $\mathcal{O}(\Delta)_{\mathbb{C}}$ over c with $\tilde{c}\circ s_\Delta = s_\Delta \circ c$. For a smooth hermitian metric $\|\cdot\|_0$, $(\frac{1}{2}(\|\cdot\|_0^2 + \|\tilde{c}(\cdot)\|_0^2))^{1/2}$ is a smooth metric invariant under $\tilde{c}$; scaling it (compactness) we get $\varsigma = s_\Delta$ with $\|\varsigma\| \leq 1$, $\|\varsigma\| \circ c = \|\varsigma\|$, zero divisor $\Delta_{\mathbb{C}}$ (reduced, simple normal crossings, support ∂X^{b}). This is hypothesis (i) of Lemma D.17; $\mathrm{rW} = e - 2\log\|\varsigma\|$ is invariant under c.

Cusp data: $O = \mathcal{O}_E$; for $w \in X^{\mathrm{b},\circ}$ let $z = y_{\mathbb{C}}^{-1}(w)$, in the abelian locus, over $\kappa\colon F \to \mathbb{C}$, and $G_z = G_{\mathcal{X},z}$. Put $\varepsilon_{w,\tau} = \kappa\circ\bar{\tau}$ (the character of $\mathcal{O}_E$ on $\mathrm{Lie}_{(\tau,-)}(G_z)$, (Fh1)), $\Lambda_w = H_1(G_z(\mathbb{C}),\mathbb{Z}) \subset \mathrm{Lie}(G_z)$ with the action ι, $\langle v, v'\rangle_w = \mathrm{Im}\, H_z(v,v')$ (alternating, integral, unimodular as the polarisation is principal, Definition 29.5), $\|v\|_w^2 = H_z(v,v)$ (positive for $v \neq 0$; $\langle v,v'\rangle_w^2 \leq |H_z(v,v')|^2 \leq \|v\|_w^2\|v'\|_w^2$), and $\lambda_{w,\tau}(v) \in \mathbb{C}^q$ the coordinates, in a fixed H_z-orthonormal basis of $\mathrm{Lie}_{(\tau,-)}(G_z)$, of the component $v_{(\tau,-)}$ of v in the decomposition (Fh1). Then $\lambda_{w,\tau}(av) = \varepsilon_{w,\tau}(a)\lambda_{w,\tau}(v)$. The pieces $\mathrm{Lie}_s(G_z)$ are pairwise H_z-orthogonal: for $u \in \mathrm{Lie}_s$, $u' \in \mathrm{Lie}_{s'}$ and $a \in \mathcal{O}_E$, the Rosati condition $H_z(\iota(a)u,u') = H_z(u,\iota(\bar{a})u')$ gives $\kappa\sigma_s(a)H_z(u,u') = \overline{\kappa\sigma_{s'}(\bar{a})}H_z(u,u') = \kappa\sigma_{s'}(a)H_z(u,u')$ (complex conjugation commutes with every embedding of the CM field E), and $\kappa\sigma_s \neq \kappa\sigma_{s'}$ for $s \neq s'$; hence $\sum_\tau|\lambda_{w,\tau}(v)|^2 = \sum_\tau H_z(v_{(\tau,-)},v_{(\tau,-)}) \leq H_z(v,v)$. Independence: let $I \subseteq \Lambda_w$ be an isotropic $\mathcal{O}_E$-submodule, $v_1,\dots,v_\rho \in I$ $\mathcal{O}_E$-linearly independent, and $I_{\mathbb{R}} \subseteq \mathrm{Lie}(G_z)$ the real span of I. If $u = iu'$ with $u,u' \in I_{\mathbb{R}}$, then $H_z(u',u') = \mathrm{Im}\, H_z(iu',u') = \mathrm{Im}\, H_z(u,u') = 0$, so $u' = 0$: $I_{\mathbb{R}} \cap iI_{\mathbb{R}} = 0$, and the $\mathcal{O}_E$-equivariant $\mathbb{C}$-linear map $\mu\colon I\otimes_{\mathbb{Z}}\mathbb{C} \to \mathrm{Lie}(G_z)$, $v\otimes a \mapsto av$, is injective ($I\otimes\mathbb{R} \cong I_{\mathbb{R}}$ as $\Lambda_w\otimes\mathbb{R} \cong \mathrm{Lie}(G_z)$). For $\sigma = \kappa\circ\bar{\tau}$, μ maps the σ-eigenspace $(I\otimes\mathbb{C})_\sigma = (I\otimes\mathbb{Q})\otimes_{E,\sigma}\mathbb{C}$ into $\mathrm{Lie}_{(\tau,-)}(G_z)$ and the σ-component of $v\otimes 1$ to $v_{(\tau,-)}$. The v_a are E-linearly independent in $I\otimes\mathbb{Q}$, so their σ-components are $\mathbb{C}$-linearly independent, hence so are the $(v_a)_{(\tau,-)}$ and the $\lambda_{w,\tau}(v_a)$. Finally fix, for every non-zero saturated isotropic $\mathcal{O}_E$-submodule I for which one exists, an $\mathcal{O}_E$-linearly independent family v^I spanning a subgroup of finite index. This gives (Hu1), the cusp weight U of (Hu2), and (Gb5) with the identity isomorphism.

Step 7: the log-log regularisations, (Hy4) and hypothesis (ii). Let $h^{\mathrm{ref}}(\hat{w}) = \|\hat{w}_1\|_2\|\hat{w}_0\|_2^{-m}$, a smooth cone function of multidegree a^{H} invariant under c; $\theta = dd^c\log h^{\mathrm{ref}}$, $\varphi = \log(h^{\mathrm{H}}/h^{\mathrm{ref}})$ on $X^{\mathrm{b},\circ}$, invariant under c by Lemma D.9, applied to the data of Steps 3–6 completed by h^{ref}, $h^{\mathrm{ref}'} = h^{\mathrm{ref}}$ and the provisional regularisations $\varphi_t = 0$, $\psi_{t'} = 0$: these data satisfy (Hy1)–(Hy5), (Hu1)–(Hu3) and (Gb1)–(Gb6) (Steps 3–6), so they form a base setting of $\mathcal{T}$ at (B,Z), and h^{H}, U do not depend on the regularisations. Put $\mathcal{M}_\tau = (\det\mathcal{V}_{(\tau,-)})^\vee$, so $\mathcal{L} = \bigotimes_\tau \mathcal{M}_\tau$. By (Gb3) and (Gb2) the metric $|\cdot|/h^{\mathrm{H}}$ corresponds under α to $\bigotimes_\tau\|\cdot\|_{\mathrm{Hdg},\tau}$ with $\|(b_{\tau,1}\wedge\cdots\wedge b_{\tau,q})^\vee\|_{\mathrm{Hdg},\tau}^2 = \det\Gamma(b_\tau)^{-1}$. Choose smooth hermitian metrics $\|\cdot\|_{\mathrm{ref},\tau}$ on $\mathcal{M}_{\tau,\mathbb{C}}$ whose tensor product corresponds to $|\cdot|/h^{\mathrm{ref}}$ (arbitrary for all τ but one, which is then determined); then $\varphi = \sum_\tau\varphi_\tau$ with $\varphi_\tau = \log\|\cdot\|_{\mathrm{ref},\tau} - \log\|\cdot\|_{\mathrm{Hdg},\tau}$. Apply Lemma 33.15, case (i), to the cover $\mathcal{X}$ of Proposition 33.9 for $\mathcal{Y}_Z$ and S, the label $s = (\tau,-)$, $j = \mathrm{rk}(s) = q$ (so $\gamma = \mathrm{id}$ and $P = X^{\mathrm{b}}$), and ς: as $\mathcal{V}^S_{(\tau,-)}$ and $\mathcal{V}_{(\tau,-)}$ agree on $\mathcal{X}_{\mathbb{Q}}$ (Step 2), its line bundle is $\mathcal{M}_{\tau,\mathbb{C}}$ with the Hodge metric $\|\cdot\|_{\mathrm{Hdg},\tau}$, and it gives that φ_τ satisfies hypotheses 1 and 2 of Theorem 33.22 on X^{b} with a constant $C_{5,\tau}$. Then φ satisfies hypothesis 1 with $C_5 = \sum_\tau C_{5,\tau}$ (φ is smooth, $h^{\mathrm{ref}}e^{\varphi}$ corresponds to the tensor product of the psh metrics $\|\cdot\|_{\mathrm{ref},\tau}e^{-\varphi_\tau}$, so its local potentials are sums of psh functions, and the bounds add). For hypothesis 2, for a point x take the charts z^τ on U^τ given for the φ_τ, a relatively compact neighbourhood U_x of x in $\bigcap_\tau U^\tau$ and $z = z^{\tau_0}|_{U_x}$; the Jacobian matrices of the changes of charts are bounded on U_x by some $c_x \geq 1$, so $\|d\varphi\|_z \leq c_x C_5 \mathrm{rW}^{C_5}\exp(\mathrm{rW}/2)$ on $U_x \cap X^{\mathrm{b},\circ}$ (as $\mathrm{rW} \geq e$). Finitely many $U_{x_1},\dots,U_{x_r}$ cover X^{b}, and hypothesis 2 holds with

$C_5' = C_5 \max_i c_{x_i}$ and the chart of a U_{x_i} containing the point. Now Theorem 33.22 on X^{b} with Θ, ς, $(\mathcal{O}(a^{\mathrm{H}})|_{X^{\mathrm{b}}}, h^{\mathrm{ref}})$, φ and C_5' gives $C \geq 1$ (enlarging C), $0 < t_0 < 1$ and $\varphi^T(t)$.

The functions $\varphi^T(t) \circ c$ have the same properties: the pointwise ones because φ and rW are invariant under c; the C^2-convergence because c preserves the compact subsets of $X^{\mathrm{b},\circ}$; and $\theta + dd^c(\varphi^T(t) \circ c) + \varepsilon(t)\Theta = -c^*(\theta + dd^c\varphi^T(t) + \varepsilon(t)\Theta) \geq 0$, since $dd^c(g \circ c) = -c^* dd^c g$ for smooth g, $\log h^{\mathrm{ref}}$ and $\log h'$ are invariant (so $c^*\theta = -\theta$, $c^*\Theta = -\Theta$), and the pull-back by the antiholomorphic c of a semipositive $(1,1)$-form is seminegative. Put $\varphi_t = \frac{1}{2}(\varphi^T(t) + \varphi^T(t) \circ c)$ for $0 < t \leq t_0$ and $\varphi_t = \varphi_{t_0}$ for $t > t_0$: smooth, invariant under c, and with all the properties (they are preserved by averaging: inequalities, the positivity of $\theta + dd^c(\cdot) + \varepsilon(t)\Theta$, limits). This is (Hy4). Shrink $t_0 \leq e^{-e}$ and put $\varepsilon(t) = C \log\log(1/t)/\log(1/t) \geq 0$, $R(t) = \log(1/t) \geq e$, $\delta(t) = t^{1/3} + C\varepsilon(t)$, $C' = \max(C, C_5')$: (R2) is $-C_5' \log \mathrm{rW} \leq \varphi \leq \varphi_t \leq C$; (R3) is $\varphi_t \leq \varphi + t^{1/3} + \varepsilon(t)$ on $\{\mathrm{rW} \leq R(t)\}$; (R4) is $\varphi_t \leq \varphi_{t'} + C\varepsilon(t') \leq \varphi_{t'} + \delta(t')$ for $t \leq t' \leq t_0$; (R5) is the C^2-convergence; and $\theta + dd^c\varphi_t + \varepsilon(t)\Theta \geq 0$. So hypothesis (ii) of Lemma D.17 holds.

Step 8: the bounded regularisations, (Hu3), clauses (b), (c). Let $h^{\mathrm{ref}'} = h^{\mathrm{ref}}$, take the φ_t of Step 7, and provisionally $\psi_{t'} = 0$. Steps 3–7 give (Hy1)–(Hy5), (Hu1)–(Hu3) and (Gb1)–(Gb6): this is a base setting of $\mathcal{T}$ at (B, Z), whose cover is proper over $\mathcal{O}_F$ (Step 1). By Lemma D.9, $\varphi' = \log h^{\mathrm{H}} + \frac{1}{2}U - \log h^{\mathrm{ref}'}$ is invariant under c, and Lemma D.13 (whose hypotheses and conclusions do not involve the regularisations of the setting) gives $C_R \geq 0$, $0 < t_0' < 1$ and smooth functions $\psi_{t'}$ invariant under c with (Rw1)–(Rw3); we take them (and $\psi_{t'} = \psi_{t_0'}$ for $t' > t_0'$) as the bounded regularisations. Definition 31.11 does not restrict them, so this is again a base setting of $\mathcal{T}$ at (B, Z). (c) is the first part of (Rw2). (b): on the cone over X^{b} each g_ℓ is holomorphic without zeros (Step 4), so $\log|g_\ell|$ is pluriharmonic and $c_1(\overline{\mathcal{M}}_{B,i_1,i_2,t}) = i_2(\theta + dd^c\psi_t) + i_1\Theta \geq (i_1 - i_2 C_R/\log(1/t))\Theta \geq 0$ for $0 < t \leq t_1 = \min(t_0', \exp(-i_2 C_R/i_1))$, by (Rw1).

Step 9: clauses (a) and (e). We check the hypotheses of Lemma D.17. (i) is Step 6 and (ii) is Step 7. (iii): by Lemma D.10, near every point of the abelian locus over some κ, with holomorphic frames b_τ, $\frac{1}{2}\sum_\tau \log\det\Gamma(b_\tau) + \frac{1}{2}U \circ y_{\mathbb{C}}$ is continuous; the first sum is smooth, and $y_{\mathbb{C}}$ is a biholomorphism, so U is continuous on $X^{\mathrm{b},\circ}$. For the bound, let $x \in \partial X^{\mathrm{b}}$, a complex point of $\mathcal{X}$ over some κ, of the complex fibre $X_{\mathbb{C}} = \mathcal{X} \otimes_\kappa \mathbb{C}$ (projective, with $G' = G_{\mathcal{X}}$ and the abelian locus, case (B) of Lemma 33.16). As $\Delta_{\mathbb{C}}$ is a reduced simple normal crossing divisor, there are a neighbourhood V of x and a biholomorphic $t\colon V \to \Delta^d$ with $t(x) = 0$, $t(V \cap X^{\mathrm{b},\circ}) = \{t_1 \cdots t_a \neq 0\}$, $a \geq 1$, and (shrinking V) $\|\varsigma\| = |t_1 \cdots t_a| e^{-\chi}$ with χ smooth and bounded on V. With $L = 1 + \sum_{j \leq a} |\log|t_j||$, on $V \cap X^{\mathrm{b},\circ}$ we get $\mathrm{rW} = e + 2\sum_{j \leq a} |\log|t_j|| + 2\chi \geq 2(L-1) - 2\sup_V|\chi|$, so $L \leq c_V\,\mathrm{rW}$ for a constant $c_V \geq 1$, and $1 + \log L \leq (2 + \log c_V)\log\mathrm{rW}$ as $\mathrm{rW} \geq e$. Lemma D.16 gives C_x and a neighbourhood V_x' of x with $U \leq C_x(1 + \log L) \leq C_x(2 + \log c_V)\log\mathrm{rW}$ on $V_x' \cap X^{\mathrm{b},\circ}$. Finitely many V_x' cover the compact ∂X^{b}; on the compact complement $K' \subseteq X^{\mathrm{b},\circ}$ of their union, $U \leq \max_{K'} U \leq (\max_{K'} U)\log\mathrm{rW}$. So $U \leq C_U \log\mathrm{rW}$ on $X^{\mathrm{b},\circ}$ for some $C_U \geq 1$; enlarging C in (ii) to $\max(C', C_U)$ keeps (ii). (iv): by (Rw2), $|\psi_{t'}| \leq C_R$, and by (Rw3), $\psi_{t'} \to \varphi'$ pointwise on $X^{\mathrm{b},\circ}$.

Lemma D.17 gives that $\mathrm{ar}^{\mathrm{b}}(B)$ exists, (Mz1)–(Mz3) at B, and $\int_{X^{\mathrm{b},\circ}}(\omega^{\mathrm{H}})^e = \deg^{\mathrm{H}}$. By Lemma D.2 (an admissible B is ≥ 2), $\int_{X^{\mathrm{b},\circ}}(\omega^{\mathrm{H}})^e > 0$; so $\deg^{\mathrm{H}} \neq 0$, and Lemma D.17 gives (Mz1)–(Mz4) at B: clause (a). Clause (e) is Steps 6 and 7.

Step 10: clause (d). $\mathcal{X}$ is proper over $\operatorname{Spec}\mathbb{Z}$, $R(\ell) = 4$ for $\ell \in \mathrm{Bd} \cup S_B'$, and the Hodge lattices at $(\mathrm{Bd} \cup S_B', 4)$ are Hodge bundles (Step 1). By (Go3)(1), $\mathcal{X}_{B,Z}$ is a normal cover $f_{B,Z}$ of $\mathcal{Y}$ whose generic point maps to that of $\mathcal{Y}_Z$; it is a root model, a closed subscheme of a multiprojective space, hence proper over $\mathbb{Z}$ and over $\mathcal{Y}$. Let Ω_1 be its function field and $K_0 = \mathbf{k}(\mathcal{Y}_Z)$. As Ω is finite over K_0, $\Omega_1 \otimes_{K_0} \Omega$ is a non-zero finite Ω_1-algebra; its quotient by a maximal ideal is a field Ω'', finite over Ω_1, containing Ω_1 and Ω compatibly over K_0. Let $\mathcal{X}_0''$ be the normalisation

of $\mathcal{X}_{B,Z}$ in Ω'', finite over $\mathcal{X}_{B,Z}$ ($\mathcal{X}_{B,Z}$ is of finite type over $\mathbb{Z}$, hence excellent). The inclusion $\Omega \subseteq \Omega''$ over K_0 defines a rational map $\mathcal{X}_0'' \dashrightarrow \mathcal{X}$ over $\mathcal{Y}$; let $\Gamma \subseteq \mathcal{X}_0'' \times_{\mathcal{Y}} \mathcal{X}$ be the closure of its graph, an integral scheme with function field Ω'', and $\mathcal{X}''$ its normalisation (finite over Γ). The projection $\Gamma \to \mathcal{X}_0''$ is proper (base change of the proper $\mathcal{X} \to \mathcal{Y}$, restricted to a closed subscheme) and birational, and $\Gamma \to \mathcal{X}$ is proper (base change of the proper $\mathcal{X}_0'' \to \mathcal{Y}$). So $\phi\colon \mathcal{X}'' \to \mathcal{X}_0'' \to \mathcal{X}_{B,Z}$ and $\phi'\colon \mathcal{X}'' \to \mathcal{X}$ are proper and dominant (both map the generic point to the generic point), $\mathcal{X}''$ with $f'' = f_{B,Z} \circ \phi = f \circ \phi'$ (the two agree as Γ lies over $\mathcal{Y}$) is a normal integral scheme whose generic point maps to that of $\mathcal{Y}_Z$, with Ω'' finitely generated over K_0: a cover of $\mathcal{Y}$. So the setting is joined to the stage root model $\mathcal{X}_{B,Z}$: clause (d).

All clauses hold for the base setting constructed, which proves the proposition. □

Lemma D.2. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2, and put $\mathrm{rd}_E = |\mathrm{disc}(E)|^{1/[E:\mathbb{Q}]}$. Let $B \geq 2$ be an integer, $Z \in \mathrm{Cp}$, and let a base setting of $\mathcal{T}$ at (B, Z) be given. Then $(\omega^{\mathrm{H}})^e$ is a positive measure on $X^{\mathrm{b},\circ}$ with $0 < \int_{X^{\mathrm{b},\circ}} (\omega^{\mathrm{H}})^e < \infty$, U is integrable for it, and*

$$\int_{X^{\mathrm{b},\circ}} U\,(\omega^{\mathrm{H}})^e \;\leq\; |\Phi|\,q\,(6 + \log(2|\Phi|\,\mathrm{rd}_E) + \log(2|\Phi|\,q\,\mathrm{rd}_E)) \int_{X^{\mathrm{b},\circ}} (\omega^{\mathrm{H}})^e.$$

Proof. Write $f_0 = |\Phi|$, so $[E:\mathbb{Q}] = 2f_0$, and $C_0 = f_0 q(6 + \log(2f_0 \mathrm{rd}_E) + \log(2f_0 q\,\mathrm{rd}_E))$. The base setting has type (Φ, n, p), so $e = |\Phi|pq$ and $j' = q$.

Step 1 (the intrinsic cusp weight). For a complex point z' of Sh_F (over any embedding of F) let $A' = G^{\mathrm{un}}_{z'}$, $\Lambda' = H_1(A'(\mathbb{C}), \mathbb{Z}) \subset \mathrm{Lie}(A')$ with its $\mathcal{O}_E$-action and $H' = H_{z'}$ (Definition 29.6, (Fh4)). For a non-zero $\mathcal{O}_E$-submodule $I \subseteq \Lambda'$ on which $\mathrm{Im}\,H'$ vanishes put $\mathrm{N}'(I) = \det\big(H'(b_i, b_k)\big)_{i,k}^{1/2}\,\mathrm{rd}_E^{-f_0\rho}$, with (b_i) a $\mathbb{Z}$-basis of I and $2f_0\rho$ the $\mathbb{Z}$-rank of I; it does not depend on the basis. Put $U'(z') = \max\big(0, \sup_I - \log \mathrm{N}'(I)\big)$, over the non-zero saturated such I. Let $\Psi_{z'}$ be the set of the embeddings by which $\mathcal{O}_E$ acts on the pieces $\mathrm{Lie}_{(\tau,-)}(A')$, $\tau \in \Phi$; it is a CM type (for z' over the embedding κ of F its members are the $\kappa \circ \sigma_{(\tau,-)} = \kappa \circ \bar\tau = \overline{\kappa \circ \tau}$, and $\{\kappa \circ \tau : \tau \in \Phi\}$ is a CM type as Φ is one and κ is injective), and each such piece has dimension q (Kottwitz condition). By Lemma D.3 applied to $(\mathrm{Lie}(A'), H', \Lambda')$ with $\Psi = \Psi_{z'}$ and $m = q$ (the hypotheses: H' is positive definite and hermitian, $\mathcal{O}_E$ preserves Λ' and $H'(ax, y) = H'(x, \bar a y)$, as $\mathrm{Im}\,H'$ is the Riemann form of a polarisation whose Rosati involution is complex conjugation, Definition 29.5 (Ud4)): for every I as above and every $\mathcal{O}_E$-independent family $v_1, \dots, v_\rho \in I$ spanning a subgroup of finite index k_v, $\mathrm{N}'(I) = k_v^{-1} \prod_\tau \det\big(H'((v_a)_{(\tau,-)}, (v_b)_{(\tau,-)})\big)_{a,b}$, the components taken in the pieces $(\tau, -)$, and

$$-\log \mathrm{N}'(I) \leq f_0 q\Big(2\max\big(0, \log \tfrac{1}{\lambda_1'}\big) + \log(2f_0 q\,\mathrm{rd}_E)\Big), \qquad \lambda_1' = \min_{0 \neq x \in \Lambda'} H'(x,x)^{1/2}. \tag{1}$$

Now let z be a complex point of the abelian locus of $\mathcal{X}$ with $w = y_{\mathbb{C}}(z) \in X^{\mathrm{b},\circ}$ and $z' = f(z)$; then $G_{\mathcal{X},z} = G^{\mathrm{un}}_{z'}$ with $H_z = H_{z'}$ (Definition 29.6, (Fh6)). Under the isomorphism $\Lambda_w \cong \Lambda'$ of (Gb5), the saturated isotropic $\mathcal{O}_E$-submodules of Λ_w correspond to those of Λ'; every non-zero one has an $\mathcal{O}_E$-independent family of finite index (a maximal $\mathcal{O}_E$-independent subset of a torsion-free finitely generated module), and $\overline{\lambda_{w,\tau}(v_a)}^{\mathsf{T}} \lambda_{w,\tau}(v_b) = H'((v_b)_{(\tau,-)}, (v_a)_{(\tau,-)})$, whose determinant is that of the complex conjugate of the positive definite matrix above. Hence $\mathrm{N}_w(I) = \mathrm{N}'(I')$ for corresponding I, I', and

$$U(w) = U'(f(z)). \tag{2}$$

Step 2 (the Hodge form). Let ω' be the first Chern form of the holomorphic line bundle $\bigotimes_{\tau\in\Phi}(\det \mathrm{Lie}_{(\tau,-)}(G^{\mathrm{un}}))^\vee$ on the complex points of Sh_F, with the metric dual to the determinant of the restriction of $h=\pi H$ (normalised as ω^{H}). Let z be a smooth complex point of $\mathcal{X}$ in the abelian locus with $y_{\mathbb{C}}(z)\in X^{\mathrm{b},\circ}$, and $D(f')$ a principal open set of a chart as in (Gb3) containing z. At complex points the Hodge lattice is the de Rham lattice, so a frame $(b_{\tau,i})$ of $\mathcal{V}_{(\tau,-)}$ on $D(f')$ is a holomorphic frame of $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X}})=f^*\mathrm{Lie}_{(\tau,-)}(G^{\mathrm{un}})$ near z, and by (Gb3) the section $s=\prod_k y_{k,c'_k}^{a_k^{\mathrm{H}}}$ has $\|s\|=|u|\prod_\tau\|b_{\tau,1}\wedge\cdots\wedge b_{\tau,q}\|_h^{-1}$ along $y_{\mathbb{C}}$, with u a holomorphic unit. As $\log|u|^2$ is pluriharmonic,

$$y_{\mathbb{C}}^*\omega^{\mathrm{H}}=-\tfrac{1}{2\pi}i\partial\bar\partial\log(y_{\mathbb{C}}^*\|s\|^2)=f^*\omega'\quad\text{near } z. \tag{3}$$

Step 3 (from $X^{\mathrm{b},\circ}$ to Z). The rational map y is birational from $\mathcal{X}$ onto $\mathcal{X}^{\mathrm{b}}$ (Gb1), and both have dimension $e+1$, so the function field Ω of $\mathcal{X}$ has transcendence degree $e=\dim Z$ over $\mathbb{Q}$; as the generic point of $\mathcal{X}$ maps to that of $\mathcal{Y}_Z$ and Ω is finitely generated over its residue field (a cover, Definition 29.6 (Fh5)), Ω is a finite extension, of some degree $d\geq 1$, of the function field of Z. In characteristic 0 there are dense Zariski open subsets $Z^\circ\subseteq Z$ over which f is finite étale of degree d, and $\mathcal{X}^\circ\subseteq f^{-1}(Z^\circ)$ in the generic fibre of $\mathcal{X}$, smooth, mapped by y isomorphically onto an open subset of $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$. Let X, X°, $Z(\mathbb{C})$, $Z^\circ(\mathbb{C})$ be the sets of complex points (over every embedding of F). The complements of X° in X, of $y_{\mathbb{C}}(X^\circ)$ in X^{b}, of $Z^\circ(\mathbb{C})$ in $Z(\mathbb{C})$ and of $X^\circ\cap y_{\mathbb{C}}^{-1}(X^{\mathrm{b},\circ})$ in X° (the trace of the divisor ∂X^{b}) are contained in proper analytic subsets, hence are null for every smooth top form. By (Gb1), $\partial X^{\mathrm{b}}=y_{\mathbb{C}}(X\smallsetminus\text{abelian locus})$, so $y_{\mathbb{C}}^{-1}(X^{\mathrm{b},\circ})$ lies in the abelian locus, and $X^\circ\subseteq f^{-1}(Z^\circ)$ lies in it as well. On X°, f is a d-sheeted covering of $Z^\circ(\mathbb{C})$ (open in $f^{-1}(Z^\circ)(\mathbb{C})$ with null complement) and $y_{\mathbb{C}}$ a biholomorphism onto its image. So, for every Borel $\varphi\colon X^{\mathrm{b},\circ}\to[0,\infty]$ with $\varphi\circ y_{\mathbb{C}}=\varphi'\circ f$ on $X^\circ\cap y_{\mathbb{C}}^{-1}(X^{\mathrm{b},\circ})$, by change of variables and (3),

$$\int_{X^{\mathrm{b},\circ}}\varphi\,(\omega^{\mathrm{H}})^e=\int_{X^\circ\cap y_{\mathbb{C}}^{-1}(X^{\mathrm{b},\circ})}(\varphi'\circ f)\,f^*\omega'^{\,e}=d\int_{Z(\mathbb{C})}\varphi'\,\omega'^{\,e}. \tag{4}$$

By (2), (4) applies to $\varphi=U$, $\varphi'=U'$, and to $\varphi=1$, $\varphi'=1$ (once $\omega'^{\,e}\geq 0$ is shown in Step 4; then $(\omega^{\mathrm{H}})^e=y_{\mathbb{C}*}f^*\omega'^{\,e}\geq 0$ on the dense open $y_{\mathbb{C}}(X^\circ)\cap X^{\mathrm{b},\circ}$, hence everywhere on $X^{\mathrm{b},\circ}$ by continuity).

Step 4 (one piece). We have $Z(\mathbb{C})=\bigsqcup_\kappa Z_\kappa(\mathbb{C})$ over the embeddings $\kappa\colon F\to\mathbb{C}$, and by (SU6) and (SU2) of Proposition 33.35 (whose hypotheses are among those of Proposition 34.2; $p+q\geq 1$) $Z_\kappa(\mathbb{C})=u_k(\mathrm{Per}_\sigma(L_k))$ for one $k\in I_\kappa$, with u_k inducing a biholomorphism from $\Gamma_k\backslash\mathrm{Per}_\sigma(L_k)$, $\Gamma_k=\Gamma(L_k,\mathfrak{n})$, acting freely and properly discontinuously. Fix κ, k, put $L=L_k$, $\psi=\psi_k$, $\Phi'=\{\rho\colon E\to\mathbb{C}:\rho(r)=\kappa(r_{\mathbb{C}})\}=\{\kappa\circ\tau:\tau\in\Phi\}$, and let Per, G, G^1, $\omega^{\mathrm{H}}_{\mathrm{Per}}$ be the objects of Lemma 33.38 for $(L_{\mathbb{R}},\psi,\Phi')$. ($L_{\mathbb{R}}$ is free of rank n over $E\otimes\mathbb{R}$ since $L\otimes\widehat{\mathbb{Z}}\cong M\otimes\widehat{\mathbb{Z}}$, (SU2).) (a) We have $\mathrm{Per}_\sigma(L)=\mathrm{Per}$ as complex manifolds. The characteristic polynomial of $a\in\mathcal{O}_E$ on $(L_{\mathbb{R}},J)$ is $\prod_\rho(T-\rho(a))^{\delta_\rho(J)}$, $\delta_\rho(J)$ the dimension of the subspace on which a acts by $\rho(a)$; by Lemma 32.3 and the computation of $\sigma(P_a)$ in (SU4), $\sigma(P_a)=\prod_{\rho\in\Phi'}(T-\rho(a))^p(T-\bar\rho(a))^q$; for a a primitive element the $\rho(a)$ are distinct, so the determinant condition of (PU2) is exactly the dimension condition of (Pd1), and the other conditions coincide. For the complex structure: $K(J)=\ker(J+i)$, and $J\mapsto K(J)$ is holomorphic by (PU2); so $J\mapsto(K(J)\cap W_\rho)_{\rho\in\Phi'}$ is holomorphic (intersection with fixed subspaces in constant dimension q), and it is the open embedding of (Pd1); a holomorphic bijection between manifolds of the same dimension is biholomorphic. (b) We have $u_k^*\omega'=\omega^{\mathrm{H}}_{\mathrm{Per}}$. By (SU3), $u_k^*G^{\mathrm{un}}$ is $A_J=(L_{\mathbb{R}},J)/L$ with $H_1=L$ and Riemann form ψ, so the first Chern class of the polarisation is $H_J(x,y)=\psi(Jx,y)+i\psi(x,y)$, and $h=\pi H_J$; at a point over κ the piece $\mathrm{Lie}_{(\tau,-)}$ is the subspace where a acts by $\kappa(\bar\tau(a))=\overline{(\kappa\circ\tau)(a)}$, i.e. $\mathrm{Lie}_{(\kappa\circ\tau,-)}(J)$ of (Pd2), and $\tau\mapsto\kappa\circ\tau$ is a bijection $\Phi\to\Phi'$.

So the metrised line bundle defining ω' pulls back to that of Lemma 33.38 (4). By (4) of that lemma, $\omega^{\mathrm{H}}_{\mathrm{Per}}$ is G-invariant and positive definite, so $\omega'^{\,e} > 0$ on $Z_\kappa(\mathbb{C})$, and G^1 acts transitively on Per with compact stabilisers. (c) The measure. Fix $J_0 \in \mathrm{Per}$ with stabiliser K_0 in G (G preserves ψ and commutes with E, so acts on Per; K_0 preserves the definite spaces of J_0 and is compact). The push-forward of a Haar measure μ_G of G ($\cong \prod_{\rho\in\Phi'} \mathrm{U}(W_\rho)$, unimodular) along $g \mapsto gJ_0$ is a G^1-invariant Radon measure on $\mathrm{Per} = G^1/(G^1 \cap K_0)$, hence (uniqueness of invariant measures on a quotient by a compact subgroup) equals $c_2^{-1}(\omega^{\mathrm{H}}_{\mathrm{Per}})^e$ for some $c_2 > 0$. Let $F_0 \colon \mathrm{Per} \to [0,\infty]$ be Borel and invariant under the group Γ^{f} of $\mathcal{O}_E$-linear automorphisms of L preserving ψ, and $\tilde{F}_0(g) = F_0(gJ_0)$. Let $\Omega \subseteq \mathrm{Per}$ be a Borel fundamental domain of Γ_k (free and properly discontinuous action of a countable group); then $D' = \{g \in G : gJ_0 \in \Omega\}$ is a Borel fundamental domain of Γ_k acting on G by left multiplication (freeness on Per), and

$$\int_{\Gamma_k\backslash\mathrm{Per}} F_0\,(\omega^{\mathrm{H}}_{\mathrm{Per}})^e = \int_\Omega F_0\,(\omega^{\mathrm{H}}_{\mathrm{Per}})^e = c_2 \int_G \mathbf{1}_\Omega(gJ_0) F_0(gJ_0)\,d\mu_G(g) = c_2 \int_{D'} \tilde{F}_0\,d\mu_G.$$

By (SU1), Γ^{f} is the stabiliser of L in $\mathrm{U}(V_L,h_L)(E^+)$, and Γ_k is the kernel of its action on $L/\mathfrak{n}L$, of finite index i_k. With $\Gamma^{\mathrm{f}} = \bigsqcup_{l\le i_k} \Gamma_k\gamma_l$ and D a Borel fundamental domain of Γ^{f} in G (Lemma D.4), $\bigsqcup_l \gamma_l D$ is one of Γ_k (Γ^{f} acts freely on G); as the integral of a left-Γ_k-invariant function over a fundamental domain does not depend on the domain, and $\tilde{F}_0$ is left Γ^{f}-invariant and μ_G left invariant,

$$\int_{\Gamma_k\backslash\mathrm{Per}} F_0\,(\omega^{\mathrm{H}}_{\mathrm{Per}})^e = c_2\, i_k \int_D \tilde{F}_0\,d\mu_G. \tag{5}$$

(d) The bound. Here G is the group $G_\infty = \prod_v \mathrm{U}(V_v,h_v)$ of Lemma D.4 for (V_L,h_L,L) (an E-linear map preserves ψ if and only if it preserves h_L, the trace form being non-degenerate, (SU1)). The real decomposition $L_\mathbb{R} = \bigoplus_v V_v$ is ψ-orthogonal and J_0-stable (J_0 is E-linear and the idempotents of $E^+ \otimes \mathbb{R}$ are self-adjoint for ψ), and on V_v the positive quadratic form $y \mapsto \psi(J_0y,y)$ is invariant under the unit complex scalars $c \in E \otimes_{E^+,v} \mathbb{R} = \mathbb{C}$ ($\psi(J_0cy,cy) = \psi(J_0y,\bar{c}c\,y)$), so it is $h_v^+(y,y)$ for a positive definite hermitian form h_v^+ on V_v. For $g \in G$ put $J_g = gJ_0g^{-1}$ and $H_g = H_{J_g}$; then $H_g(x,x) = \psi(J_0g^{-1}x, g^{-1}x) = \|x\|_g^2$. We apply Lemma D.5 to $(\Lambda,W,h) = (L,V_L,h_L)$, $N = n$, $\Psi = \{\bar{\rho} : \rho \in \Phi'\}$, $m = q$ ($3q+1 \le n$), $\Gamma = \Gamma^{\mathrm{f}}$, $\mu = \mu_G$: (Fm1) holds as $\mathcal{O}_E$ commutes with J_g, $\mathrm{Im}\,H_g = \psi$ is integral on L, $H_g(ax,y) = H_g(x,\bar{a}y)$, and for $\rho \in \Phi'$ the subspace of $(L_\mathbb{R},J_g)$ on which a acts by $\bar{\rho}(a)$ is $\mathrm{Lie}_{(\rho,-)}(J_g)$, of dimension q ($J_g \in \mathrm{Per}$); (Fm2) holds with $\kappa = 1$; (Fm3): if $\psi(ax,y) = \mathrm{Tr}_{E/\mathbb{Q}}(r\,a\,h_L(x,y)) = 0$ for all $a \in \mathcal{O}_E$ then $h_L(x,y) = 0$. By Lemma D.8 for E there are $\mathbb{Q}$-independent $a_1,\dots,a_{2f_0} \in \mathcal{O}_E$ with $|\sigma(a_i)| \le \mathrm{vr} := \sqrt{2f_0}\,\mathrm{rd}_E$; $\mathrm{vr} \ge 1$ as $\mathrm{rd}_E \ge 1$. With $r' = \mathrm{vr}^{-1/2}$ and $\lambda_1(g) = \min_{0\ne x\in L} H_g(x,x)^{1/2}$, Lemma D.5 gives $0 < \mu_G(D) < \infty$, λ_1 Borel and positive, and $\int_D \log^+(r'/\lambda_1)\,d\mu_G \le 3\mu_G(D)$. Since $\log^+(1/\lambda_1) \le \log^+(r'/\lambda_1) + \frac{1}{2}\log\mathrm{vr}$ and $\log\mathrm{vr} \le \log(2f_0\mathrm{rd}_E)$, (1) at $u_k(J_g)$ (where $\Lambda' = L$, $H' = H_g$, $\lambda_1' = \lambda_1(g)$) gives

$$U'(u_k(J_g)) \le f_0q(2\log^+(r'/\lambda_1(g)) + \log(2f_0\mathrm{rd}_E) + \log(2f_0q\,\mathrm{rd}_E)),$$

using that the right side of (1) is ≥ 0 ($2f_0q\,\mathrm{rd}_E \ge 1$). The function $F_0 = U' \circ u_k$ is Borel on Per (a supremum over the countably many submodules $I \subseteq L$ of the continuous functions $J \mapsto -\log \mathrm{N}'(I)$, then a maximum with 0), and Γ^{f}-invariant ($\gamma \in \Gamma^{\mathrm{f}}$ is an $\mathcal{O}_E$-isometry $(L,H_J) \to (L,H_{\gamma J\gamma^{-1}})$); likewise $F_0 = 1$. Integrating over D and using (5) for both,

$$\int_{\Gamma_k\backslash\mathrm{Per}} U' \circ u_k\,(\omega^{\mathrm{H}}_{\mathrm{Per}})^e \le c_2 i_k\, C_0\,\mu_G(D) = C_0 \int_{\Gamma_k\backslash\mathrm{Per}} (\omega^{\mathrm{H}}_{\mathrm{Per}})^e,$$

and the right side is finite and positive.

Step 5 (conclusion). By (b), u_k carries $(\omega^{\mathrm{H}}_{\mathrm{Per}})^e$ on $\Gamma_k\backslash\mathrm{Per}$ to $\omega'^{\,e}$ on $Z_\kappa(\mathbb{C})$, so Step 4 gives $\int_{Z_\kappa(\mathbb{C})} U'\omega'^{\,e} \le C_0 \int_{Z_\kappa(\mathbb{C})} \omega'^{\,e}$, with $0 < \int_{Z_\kappa(\mathbb{C})} \omega'^{\,e} < \infty$, for every κ. Summing over κ and applying (4) with $\varphi = U$ and $\varphi = 1$: $\int_{X^{\mathrm{b},\circ}} (\omega^{\mathrm{H}})^e = d \int_{Z(\mathbb{C})} \omega'^{\,e} \in (0,\infty)$, U is integrable, and $\int_{X^{\mathrm{b},\circ}} U(\omega^{\mathrm{H}})^e = d \int_{Z(\mathbb{C})} U'\omega'^{\,e} \le C_0 \int_{X^{\mathrm{b},\circ}} (\omega^{\mathrm{H}})^e$. □

Lemma D.3. *Let E be a CM field of degree $2f$ with complex conjugation c, $\mathrm{rd}_E = |\mathrm{disc}(E)|^{1/(2f)}$, and Ψ a CM type of E (a set of f field embeddings $E \to \mathbb{C}$ containing exactly one of $\sigma, \bar\sigma = \overline{\sigma}$ for every embedding σ). Let V be a finite-dimensional complex vector space with a positive definite Hermitian form H (linear in the first variable), $\Lambda \subset V$ a lattice (a discrete subgroup spanning V over $\mathbb{R}$), and $\iota\colon \mathcal{O}_E \to \mathrm{End}_{\mathbb{C}}(V)$ a ring homomorphism with $\iota(a)\Lambda \subseteq \Lambda$ and $H(\iota(a)x, y) = H(x, \iota(c(a))y)$ for $a \in \mathcal{O}_E$, $x, y \in V$. For an embedding $\sigma\colon E \to \mathbb{C}$ let $V_\sigma = \{x : \iota(a)x = \sigma(a)x\ \forall a\}$ and $x \mapsto x_\sigma$ the projection of $V = \bigoplus_\sigma V_\sigma$ onto V_σ; suppose $\dim V_\tau = m \ge 1$ for every $\tau \in \Psi$. Put $\lambda_1 = \min\{H(\lambda,\lambda)^{1/2} : 0 \ne \lambda \in \Lambda\}$. Let $I \subseteq \Lambda$ be a non-zero $\mathcal{O}_E$-submodule with $\mathrm{Im}\, H(v,w) = 0$ for all $v, w \in I$, and let $v_1, \dots, v_\rho \in I$ be $\mathcal{O}_E$-linearly independent and span an $\mathcal{O}_E$-submodule of finite index k_v in I. Put $G_\tau = \big(H((v_a)_\tau, (v_b)_\tau)\big)_{a,b\le\rho}$ and*

$$\mathrm{N}(I) = \frac{1}{k_v} \prod_{\tau\in\Psi} \det G_\tau.$$

Then $\rho \le m$, each G_τ is positive definite, the $\mathbb{Z}$-rank of I is $2f\rho$, and for every $\mathbb{Z}$-basis $b_1, \dots, b_{2f\rho}$ of I

$$\det\big(H(b_i, b_k)\big)_{i,k} = \big(\mathrm{N}(I)\,\mathrm{rd}_E^{f\rho}\big)^2;$$

moreover

$$-\log \mathrm{N}(I) \le f\,m\,\Big(2\max\big(0, \log\tfrac{1}{\lambda_1}\big) + \log(2fm\,\mathrm{rd}_E)\Big).$$

Proof. Eigen-decomposition. Since $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{C} \cong \prod_{\sigma\colon E\to\mathbb{C}} \mathbb{C}$, $V = \bigoplus_\sigma V_\sigma$. If $x \in V_\sigma$, $y \in V_{\sigma'}$, then $\sigma(a)H(x,y) = H(\iota(a)x, y) = H(x, \iota(c(a))y) = \overline{\sigma'(c(a))}H(x,y) = \sigma'(a)H(x,y)$, since $\overline{\sigma'(c(a))} = \sigma'(a)$ for a CM field; so the V_σ are pairwise orthogonal and $H(x,y) = \sum_\sigma H(x_\sigma, y_\sigma)$.

Conjugate components of isotropic vectors. For $v, w \in I$ put $h_\sigma = H(v_\sigma, w_\sigma)$. For $a \in \mathcal{O}_E$, $\iota(a)v \in I$, so $\sum_\sigma \sigma(a) h_\sigma = H(\iota(a)v, w)$ is real; hence $\sum_\sigma \sigma(a)h_\sigma = \sum_\sigma \overline{\sigma(a)}\,\overline{h_\sigma} = \sum_\sigma \sigma(a)\overline{h_{\bar\sigma}}$ for all $a \in \mathcal{O}_E$. By Dedekind's independence of the distinct characters $\sigma|_{\mathcal{O}_E}$, $h_{\bar\sigma} = \overline{h_\sigma}$. In particular $G_{\bar\tau} = \overline{G_\tau}$ for $\tau \in \Psi$, with G_σ defined as G_τ for every σ.

The quadratic form on the span. The ring $E_{\mathbb{R}} = E \otimes_{\mathbb{Q}} \mathbb{R}$ acts on $V = \Lambda \otimes \mathbb{R}$, every σ extends to $E_{\mathbb{R}} \to \mathbb{C}$, and $E_{\mathbb{R}} \to \mathbb{C}^\Psi$, $e \mapsto (\tau(e))_\tau$, is an $\mathbb{R}$-isomorphism. Let $\Phi\colon E_{\mathbb{R}}^\rho \to V$, $(e_a) \mapsto \sum_a e_a v_a$. It is injective: $E^\rho \to \Lambda \otimes \mathbb{Q}$ is injective because the v_a are $\mathcal{O}_E$-independent, and tensoring with $\mathbb{R}$ over $\mathbb{Q}$ preserves injectivity. Its image is $I_{\mathbb{R}}$ (the $\mathcal{O}_E$-span of the v_a has finite index in I), on which H is real because $\mathrm{Im}\, H$ vanishes on $I \times I$ and is $\mathbb{R}$-bilinear. Since $(\Phi e)_\sigma = \sum_a \sigma(e_a)(v_a)_\sigma$, writing $u_\tau = (\tau(e_a))_a \in \mathbb{C}^\rho$ and $Q_\tau(u) = \sum_{a,b} u_a \overline{u_b}\,(G_\tau)_{ab}$ (real, as G_τ is Hermitian), the previous step gives

$$q(e) = H(\Phi e, \Phi e) = \sum_{\tau\in\Psi} \big(Q_\tau(u_\tau) + \overline{Q_\tau(u_\tau)}\big) = 2\sum_{\tau\in\Psi} Q_\tau(u_\tau).$$

As Φ is injective and H positive definite, q is positive definite; the u_τ are independent coordinates, so every Q_τ, i.e. every G_τ, is positive definite. Hence $\det G_\tau > 0$, the vectors $(v_1)_\tau, \dots, (v_\rho)_\tau$ are linearly independent in V_τ, and $\rho \le m$; also $\mathrm{N}(I) > 0$.

The Gram determinant. Use coordinates $(\mathrm{Re}\, u_\tau, \mathrm{Im}\, u_\tau)_\tau$ on $E_{\mathbb{R}}^\rho \cong \mathbb{R}^{2f\rho}$. The real symmetric matrix of the quadratic form $2Q_\tau$ on $\mathbb{C}^\rho = \mathbb{R}^{2\rho}$ has determinant $2^{2\rho}\det(G_\tau)^2$ (the realification

of a Hermitian matrix G has determinant $|\det G|^2$). The image of $\mathcal{O}_E$ in $\mathbb{C}^\Psi = \mathbb{R}^{2f}$ is a lattice of covolume $2^{-f}|\mathrm{disc}(E)|^{1/2}$ for Lebesgue measure (the standard computation of the covolume of $\mathcal{O}_E$ in its Minkowski space, E being totally imaginary), so $\mathcal{O}_E^\rho$ has covolume $2^{-f\rho}|\mathrm{disc}(E)|^{\rho/2}$. For a lattice M with $\mathbb{Z}$-basis matrix B and a form with matrix A, the Gram determinant is $\det(B^{\mathsf{T}}AB) = \mathrm{covol}(M)^2 \det A$; hence the Gram determinant of a $\mathbb{Z}$-basis of $\Phi(\mathcal{O}_E^\rho) = \sum_a \mathcal{O}_E v_a$ for H is

$$2^{-2f\rho}|\mathrm{disc}(E)|^\rho \prod_{\tau\in\Psi} 2^{2\rho}\det(G_\tau)^2 = \Big(\mathrm{rd}_E^{f\rho}\prod_{\tau\in\Psi}\det G_\tau\Big)^2,$$

using $|\mathrm{disc}(E)| = \mathrm{rd}_E^{2f}$. Passing to the overlattice I of index k_v divides the Gram determinant by k_v^2, which gives the identity; the $\mathbb{Z}$-rank of I is that of $\mathcal{O}_E^\rho$, namely $2f\rho$.

The lower bound. Put $n = 2f\rho$. The covolume of I in the Euclidean space $(I_\mathbb{R}, H)$ is $\mathrm{N}(I)\,\mathrm{rd}_E^{f\rho}$, and every non-zero vector of $I \subseteq \Lambda$ has length at least $\lambda_1 > 0$ (Λ is discrete). A ball of radius s contains a cube of side $2s/\sqrt{n}$; by Minkowski's convex body theorem, if $(2s/\sqrt{n})^n > 2^n\,\mathrm{N}(I)\,\mathrm{rd}_E^{f\rho}$ the ball contains a non-zero point of I, so $\lambda_1 \le s$. Hence $\lambda_1 \le \sqrt{n}\,(\mathrm{N}(I)\mathrm{rd}_E^{f\rho})^{1/n}$, i.e.

$$-\log \mathrm{N}(I) \le n\log\frac{\sqrt{n}}{\lambda_1} + f\rho\log \mathrm{rd}_E = f\rho\Big(2\log\frac{1}{\lambda_1} + \log(2f\rho) + \log \mathrm{rd}_E\Big).$$

Finally $\rho \le m$, $2\log\frac{1}{\lambda_1} \le 2\max(0, \log\frac{1}{\lambda_1})$ and $\log(2f\rho\,\mathrm{rd}_E) \le \log(2fm\,\mathrm{rd}_E)$ with $\log(2f\rho\,\mathrm{rd}_E) \ge 0$ ($\mathrm{rd}_E \ge 1$), which gives the bound. □

Lemma D.4. *The notation introduced here is local to this result and to the results that apply it. Let E be a CM field with complex conjugation $a \mapsto \bar a$ and maximal totally real subfield E^+, and $f = [E^+ : \mathbb{Q}]$. Let V be an E-vector space of dimension $n \ge 1$ with a nondegenerate hermitian form h (E-linear in the first variable), $\Lambda \subset V$ an $\mathcal{O}_E$-lattice (a finitely generated $\mathcal{O}_E$-submodule spanning V), and $\Gamma = \{\gamma \in \mathrm{U}(V,h)(E^+) : \gamma\Lambda = \Lambda\}$. For every real place v of E^+ fix a complex place w of E above v, let $V_v = V \otimes_{E,w} \mathbb{C}$ carry the hermitian form h_v induced by h, and fix a positive definite hermitian form h_v^+ on V_v. Put $G_\infty = \prod_v \mathrm{U}(V_v, h_v)$, in which Γ is embedded diagonally, let μ be a Haar measure on G_∞, and for $g = (g_v)_v \in G_\infty$ and $y \in V$ put*

$$\|y\|_g = \Big(\sum_v h_v^+(g_v^{-1}y,\, g_v^{-1}y)\Big)^{1/2}.$$

Let $S \subseteq \Lambda$ be a set of nonzero vectors with $h(y,y) = 0$ for all $y \in S$, such that $\gamma S = S$ for all $\gamma \in \Gamma$ and the index $[\,Ey \cap \Lambda : \mathcal{O}_E y\,]$ is bounded on S. For $t > 0$ and $g \in G_\infty$ put $c_S(t,g) = \#\{y \in S :\ \|y\|_g < t\} \in [0,\infty]$.

Then Γ is a discrete subgroup of G_∞ and has a Borel fundamental domain D (a Borel set whose translates γD, $\gamma \in \Gamma$, are pairwise disjoint and cover G_∞). For every Borel fundamental domain D of Γ the mass

$$m(t) = \int_D c_S(t,g)\,d\mu(g)$$

does not depend on D, is finite for every $t > 0$, and satisfies $m(\theta t) = \theta^{\,2f(n-1)}\,m(t)$ for all $\theta, t > 0$.

Proof. **Step 0: the group.** Let $H = \mathrm{Res}_{E^+/\mathbb{Q}}\mathrm{U}(V,h)$, a linear algebraic group over $\mathbb{Q}$ acting faithfully on the $\mathbb{Q}$-vector space V; $H(\mathbb{Q}) = \mathrm{U}(V,h)(E^+)$ and $H(\mathbb{R}) = G_\infty$, because $E \otimes_{E^+,v} \mathbb{R} \cong \mathbb{C}$ through w. The group Γ is the stabiliser in $H(\mathbb{Q})$ of the lattice Λ of V, so it is arithmetic and discrete in G_∞ by Theorem B.165(1); it is countable and acts freely on G_∞ by left multiplication. Cover G_∞ by countably many open sets $U_1, U_2, \ldots$ with $U_i \cap \gamma U_i = \emptyset$ for $\gamma \ne 1$ (possible by discreteness); then $D_0 = \bigcup_i (U_i \setminus \bigcup_{j<i} \Gamma U_j)$ is a Borel fundamental domain.

For a Borel function $\varphi \geq 0$ with $\varphi(\gamma g) = \varphi(g)$ ($\gamma \in \Gamma$) and two Borel fundamental domains D, D', left invariance of μ gives $\int_D \varphi = \sum_\gamma \int_{D\cap\gamma D'} \varphi = \sum_\gamma \int_{\gamma^{-1}D\cap D'} \varphi = \int_{D'} \varphi$. The function $c_S(t,\cdot) = \sum_{y\in S} \mathbf{1}\{g : \|y\|_g < t\}$ is a countable sum of indicators of open sets, hence Borel, and it is left Γ-invariant, since $\|y\|_{\gamma g} = \|\gamma^{-1}y\|_g$ and γ^{-1} permutes S. So $m(t)$ does not depend on D.

If $S = \emptyset$ then $m = 0$. Otherwise h is isotropic, so $n \geq 2$; let C bound the index on S.

Step 1: S is a finite union of Γ-orbits. (a) $\mathrm{U}(V,h)(E^+)$ acts transitively on the isotropic lines of V (Witt's theorem for hermitian forms), so the set of isotropic lines is $H(\mathbb{Q})/P(\mathbb{Q})$ with $P = \mathrm{Res}_{E^+/\mathbb{Q}}P'$, P' the stabiliser of one isotropic line, a parabolic E^+-subgroup of the connected reductive group $\mathrm{U}(V,h)$. By Theorem B.165(3) there are isotropic lines $\ell_1,\dots,\ell_k$ such that every isotropic line is $\gamma\ell_i$ for some $\gamma \in \Gamma$ and some i. (b) Fix $\ell = \ell_i$ and put $M = \ell \cap \Lambda$, a finitely generated torsion-free $\mathcal{O}_E$-module spanning ℓ. Choose an isotropic line ℓ' with $h(\ell,\ell') \neq 0$ (it exists as h is nondegenerate) and put $W_0 = (\ell+\ell')^\perp$, so $V = \ell \oplus \ell' \oplus W_0$. For $c \in E^\times$ let t_c act by c on ℓ, by $\bar{c}^{-1}$ on ℓ' and trivially on W_0; then $h(t_c y, t_c y') = c\,\overline{\bar{c}^{-1}}\,h(y,y') = h(y,y')$ for $y \in \ell$, $y' \in \ell'$, and the other pairings are unchanged, so $t_c \in \mathrm{U}(V,h)(E^+)$. The lattice $\Lambda' = (\ell\cap\Lambda)\oplus(\ell'\cap\Lambda)\oplus(W_0\cap\Lambda)$ has finite index in Λ; choose an integer $N \geq 1$ with $N\Lambda \subseteq \Lambda'$, and let $U' = \{c \in \mathcal{O}_E^\times : c \equiv 1 \bmod N^2\mathcal{O}_E\}$, a subgroup of finite index of $\mathcal{O}_E^\times$. For $c \in U'$ also $\bar{c}^{-1} \equiv 1 \bmod N^2\mathcal{O}_E$, so $(t_c - 1)\Lambda' \subseteq N^2\Lambda'$ and, for $\lambda \in \Lambda$, $(t_c-1)\lambda = N^{-1}(t_c-1)(N\lambda) \in N\Lambda' \subseteq \Lambda$; thus $t_c\Lambda \subseteq \Lambda$, likewise $t_{c^{-1}}\Lambda \subseteq \Lambda$, and $t_c \in \Gamma$. For $y \in S \cap \ell$ the submodule $\mathcal{O}_E y \subseteq M$ has index $d \leq C$, so it contains $dM \supseteq C!\,M$; as $M/C!\,M$ is finite, only finitely many submodules occur. If $\mathcal{O}_E y = \mathcal{O}_E y'$, then $y' = uy$ with $u \in \mathcal{O}_E^\times$ ($y' = cy$, $y = c'y'$ with $c, c' \in \mathcal{O}_E$, so $cc' = 1$). Hence $S \cap \ell$ lies in finitely many sets $\mathcal{O}_E^\times y_s$, each a finite union of sets $U'uy_s = \{t_c(uy_s) : c \in U'\}$, each contained in one Γ-orbit. (c) Every $y \in S$ spans an isotropic line, so $\gamma y \in \ell_i$ for some $\gamma \in \Gamma$ and i, and $\gamma y \in S$. By (b), $S = \Gamma y_1 \sqcup \cdots \sqcup \Gamma y_r$ for finitely many $y_j \in S$.

Step 2: unfolding. Fix $y = y_j$, let Γ_y and Q_y be the stabilisers of y in Γ and in G_∞, and for $z = (z_v) \in \prod_v V_v$ put $\|z\|_+ = (\sum_v h_v^+(z_v,z_v))^{1/2}$, so $\|y\|_g = \|g^{-1}y\|_+$. The function $\varphi(g) = \mathbf{1}\{\|g^{-1}y\|_+ < t\}$ is left Q_y-invariant. Let R be a set of representatives of $\Gamma_y\backslash\Gamma$; then $D_y = \bigcup_{\rho\in R}\rho D$ is a Borel fundamental domain of Γ_y, and

$$\int_D \sum_{y'\in\Gamma y} \mathbf{1}\{\|y'\|_g < t\}\,d\mu(g) = \sum_{\rho\in R}\int_D \varphi(\rho g)\,d\mu(g) = \sum_{\rho\in R}\int_{\rho D}\varphi\,d\mu = \int_{D_y}\varphi\,d\mu,$$

using $\|\rho^{-1}y\|_g = \|y\|_{\rho g}$, that $\rho \mapsto \rho^{-1}y$ is a bijection from R onto Γy, and left invariance of μ.

Step 3: Γ_y has finite covolume in Q_y. Let $Q_y' \subseteq \mathrm{U}(V,h)$ be the stabiliser of y, an algebraic group over E^+; Q_y is the group of real points of $\mathrm{Res}_{E^+/\mathbb{Q}}Q_y'$, and Γ_y is the stabiliser of Λ in its rational points, hence arithmetic (Theorem B.165(1)). Take $\ell = Ey$ and ℓ', W_0 as in Step 1(b). An element of Q_y' preserves $y^\perp = \ell \oplus W_0$, fixes y, acts trivially on $V/y^\perp$ (since $h(gz,y) = h(gz,gy) = h(z,y)$), and induces a unitary automorphism of $y^\perp/\ell \cong W_0$; this gives a surjection $Q_y' \to \mathrm{U}(W_0)$, split by letting $\mathrm{U}(W_0)$ act trivially on $\ell\oplus\ell'$, whose kernel acts trivially on the graded pieces of $\ell \subset y^\perp \subset V$ and is therefore unipotent, hence connected. So Q_y' is connected. A character of Q_y' defined over E^+ is trivial on the unipotent kernel, and on $\mathrm{U}(W_0)$ it factors through $\det\colon \mathrm{U}(W_0) \to \mathrm{U}(1)_{E/E^+}$ (the derived group is semisimple); the torus $\mathrm{U}(1)_{E/E^+}$ has no nontrivial E^+-character, since complex conjugation acts on its character group $\mathbb{Z}$ by -1. (If $W_0 = 0$ the quotient is trivial.) By Theorem B.165(2), applied to the restriction of scalars $\mathrm{Res}_{E^+/\mathbb{Q}}Q_y'$ as in its proof, Q_y is unimodular and every Borel fundamental domain of Γ_y in Q_y has finite measure vol_y for a fixed Haar measure μ_y of Q_y. Likewise G_∞ is unimodular ($\mathrm{U}(V,h)$ is connected and its characters factor through det in the same way).

Step 4: the orbit and its invariant measure. For each v let $O_v = \{z \in V_v : z \neq 0,\ h_v(z,z) = 0\}$ and $O = \prod_v O_v$. By Witt's theorem $\mathrm{U}(V_v,h_v)$ acts transitively on O_v, so $g \mapsto g^{-1}y$ induces a continuous bijection $Q_y\backslash G_\infty \to O$, which is a homeomorphism (the orbit

map of a transitive continuous action of a second countable locally compact group is open), and right translation by k corresponds to $z \mapsto k^{-1}z$. As G_∞ and Q_y are unimodular, $Q_y\backslash G_\infty$ carries a G_∞-invariant Radon measure $\bar\mu_y$, unique up to a positive factor, normalised so that $\int_{G_\infty} F\,d\mu = \int_{Q_y\backslash G_\infty}\int_{Q_y} F(qg)\,d\mu_y(q)\,d\bar\mu_y(g)$ for Borel $F \ge 0$; for left Q_y-invariant Borel $\varphi \ge 0$, taking $F = \mathbf{1}_{D_y}\varphi$ gives $\int_{D_y}\varphi\,d\mu = \int_{Q_y\backslash G_\infty}\varphi(g)\,\mu_y(\{q \in Q_y : qg \in D_y\})\,d\bar\mu_y(g)$, and for each g the set $\{q \in Q_y : qg \in D_y\} = Q_y \cap D_y g^{-1}$ is a Borel fundamental domain of $\Gamma_y \subseteq Q_y$ in Q_y (as D_y is one in G_∞), of measure vol_y by Step 0 applied in Q_y; hence $\int_{D_y}\varphi\,d\mu = \mathrm{vol}_y \int_{Q_y\backslash G_\infty}\varphi\,d\bar\mu_y$.

Now we build an invariant measure on O. Let $q_v(z) = h_v(z,z)$, a real quadratic form on $V_v \cong \mathbb{R}^{2n}$, and λ_v the Lebesgue measure of the euclidean structure $\mathrm{Re}\,h_v^+$. For $z \ne 0$, $dq_v(z)(u) = 2\,\mathrm{Re}\,h_v(z,u) \ne 0$ for some u, as h_v is nondegenerate; so O_v is a closed smooth hypersurface of $V_v \setminus \{0\}$. Let ν_v be the measure on O_v of the Leray form: the restriction to O_v of any $(2n-1)$-form ω, defined near a point of O_v, with $dq_v \wedge \omega = d\lambda_v$; the restriction does not depend on the choice, since two choices differ by $dq_v \wedge \beta$. Then:

- $g \in \mathrm{U}(V_v, h_v)$ preserves q_v and λ_v (g is $\mathbb{C}$-linear with $|\det g| = 1$, from $g^* h_v g = h_v$), so $g^*\omega$ is again such a form and g preserves ν_v;
- the dilation $\delta_\theta(z) = \theta z$ ($\theta > 0$) maps O_v onto itself with $\delta_\theta^* d\lambda_v = \theta^{2n} d\lambda_v$ and $\delta_\theta^* q_v = \theta^2 q_v$, so $\theta^{-2n+2}\delta_\theta^*\omega$ is such a form and $\nu_v(\delta_\theta A) = \theta^{2n-2}\nu_v(A)$;
- with $|z|_v = h_v^+(z,z)^{1/2}$, the shell $\{z \in O_v : t/2 \le |z|_v < t\}$ has compact closure in O_v, hence finite measure $c_v(t)$, and the ball $\{z \in O_v : |z|_v < t\}$ is the disjoint union of its images under $\delta_{2^{-k}}$, $k \ge 0$, so its measure is $\sum_{k\ge 0} 2^{-k(2n-2)} c_v(t) < \infty$, as $n \ge 2$.

The product $\nu = \bigotimes_v \nu_v$ is a G_∞-invariant Radon measure on O; transported to $Q_y\backslash G_\infty$ it is G_∞-invariant, so there is $c_y > 0$ with $\int_{Q_y\backslash G_\infty}\Phi(g^{-1}y)\,d\bar\mu_y(g) = c_y\int_O \Phi\,d\nu$ for Borel $\Phi \ge 0$ on O.

Step 5: conclusion. Let $B_t = \{z \in O : \|z\|_+ < t\}$. By Steps 1–4,

$$m(t) = \sum_{j=1}^r \int_{D_{y_j}} \mathbf{1}\{\|g^{-1}y_j\|_+ < t\}\,d\mu(g) = \Big(\sum_{j=1}^r \mathrm{vol}_{y_j} c_{y_j}\Big)\nu(B_t).$$

Here $\nu(B_t) \le \prod_v \nu_v\{z \in O_v : |z|_v < t\} < \infty$, and $B_{\theta t} = \prod_v \delta_\theta$ applied to B_t, so $\nu(B_{\theta t}) = \theta^{f(2n-2)}\nu(B_t)$, there being f real places. Hence $m(t) < \infty$ and $m(\theta t) = \theta^{2f(n-1)} m(t)$. □

Lemma D.5. *The notation introduced here is local to this result and to the results that apply it. Let E be a CM field of degree $2f$ with complex conjugation $a \mapsto \bar a$ and maximal totally real subfield E^+, and Ψ a CM type of E. Let Λ be a finitely generated torsion-free $\mathcal{O}_E$-module, $W = \Lambda \otimes_{\mathcal{O}_E} E \supset \Lambda$, $N = \dim_E W$, and h a nondegenerate hermitian form on W (E-linear in the first variable). Let Γ, the places v of E^+ with $W_v = W \otimes_{E,w} \mathbb{C}$, positive definite hermitian forms h_v^+, the group G_∞, a Haar measure μ and the norms $\|\cdot\|_g$ ($g \in G_\infty$) be as in Lemma D.4 for $(V,h,\Lambda) = (W,h,\Lambda)$. Let $m \ge 1$ be an integer with $3m+1 \le N$, and suppose that for every $g \in G_\infty$ a complex structure J_g and a positive definite Hermitian form H_g on $\Lambda_\mathbb{R} = \Lambda \otimes_\mathbb{Z} \mathbb{R}$ are given such that:*

(Fm1) *$(\Lambda_\mathbb{R}, J_g)$ with $H = H_g$, the lattice Λ and the action ι of $\mathcal{O}_E$ on $\Lambda_\mathbb{R}$ by the module structure satisfies the hypotheses of Lemma D.6 with this m: each $\iota(a)$ is J_g-linear, $\mathrm{Im}\,H_g(\Lambda,\Lambda) \subseteq \mathbb{Z}$, $H_g(\iota(a)x,y) = H_g(x,\iota(\bar a)y)$, and $\dim V_\tau = m$ for $\tau \in \Psi$;*

(Fm2) *there is $\kappa > 0$, independent of g, with $H_g(x,x) = \kappa\,\|x\|_g^2$ for all $g \in G_\infty$ and $x \in \Lambda$;*

(Fm3) *for $g \in G_\infty$ and $x, y \in \Lambda$: if $\mathrm{Im}\,H_g(\iota(a)x,y) = 0$ for all $a \in \mathcal{O}_E$, then $h(x,y) = 0$.*

Let $\mathrm{vr} > 0$ *be such that there are* $a_1, \dots, a_{2f} \in \mathcal{O}_E$*, linearly independent over* $\mathbb{Q}$*, with* $|\sigma(a_i)| \le \mathrm{vr}$ *for all* i *and all field embeddings* $\sigma\colon E \to \mathbb{C}$*; put* $r = \mathrm{vr}^{-1/2}$ *and* $\lambda_1(g) = \min\{H_g(x,x)^{1/2} : 0 \ne x \in \Lambda\}$*. Then every Borel fundamental domain* D *of* Γ *in* G_∞ *satisfies* $0 < \mu(D) < \infty$*, the function* λ_1 *is Borel and positive, and*

$$\int_D \log^+(r/\lambda_1(g))\, d\mu(g) \;\le\; 3\,\mu(D).$$

Proof. **Step 0: the domain.** Let $H = \mathrm{Res}_{E^+/\mathbb{Q}}\mathrm{U}(W,h)$, so $H(\mathbb{R}) = G_\infty$ and Γ is the stabiliser in $H(\mathbb{Q})$ of the lattice Λ of the $\mathbb{Q}$-vector space W, an arithmetic subgroup (Theorem B.165(1)). The group $\mathrm{U}(W,h)$ is connected, and an E^+-character of it is trivial on the semisimple derived group and so factors through $\det\colon \mathrm{U}(W,h) \to \mathrm{U}(1)_{E/E^+}$, a torus without nontrivial E^+-characters (complex conjugation acts on its character group $\mathbb{Z}$ by -1). By Theorem B.165(2) and its notation map, $\mu(D) < \infty$. Since $G_\infty = \bigcup_{\gamma\in\Gamma} \gamma D$ is a countable union and $\mu(\gamma D) = \mu(D)$, $\mu(D) > 0$.

Step 1: the counting function. Put $C = \lfloor (\mathrm{vr}\sqrt{2f})^{2f} \rfloor$ and

$$S = \{y \in \Lambda :\; y \ne 0,\; h(y,y) = 0,\; [\,Ey \cap \Lambda : \mathcal{O}_E y\,] \le C\}.$$

An element $\gamma \in \Gamma$ preserves Λ and h and maps $Ey \cap \Lambda$ onto $E\gamma y \cap \Lambda$ and $\mathcal{O}_E y$ onto $\mathcal{O}_E \gamma y$, so $\gamma S = S$. For $t > 0$ and $g \in G_\infty$ put

$$n(t,g) = \#\{y \in S :\; H_g(y,y)^{1/2} < t\} = \#\{y \in S :\; \|y\|_g < t/\sqrt{\kappa}\}$$

by (Fm2). By Lemma D.4 (with $n = N$ and the set S), $n(t,\cdot)$ is Borel and integrable on D, and $M(t) = \int_D n(t,g)\, d\mu(g)$ satisfies $M(\theta t) = \theta^{2f(N-1)} M(t)$ for $\theta, t > 0$. By (Fm2), $\lambda_1(g) = \sqrt{\kappa}\, \inf_{0\ne x\in\Lambda} \|x\|_g$ is an infimum of countably many continuous functions of g, hence upper semicontinuous and Borel; it is positive and attained by Lemma D.6, which applies by (Fm1).

Step 2: the hypotheses of Lemma D.7. We apply it to $X = D$ with the probability measure $\mu|_D/\mu(D)$, $\alpha = \lambda_1$, $a = 2fm \ge 1$, the exponent $2f(N-1)$ in place of its κ (and $2f(N-1) \ge 2f \cdot 3m = 3a$ as $N - 1 \ge 3m$), r, and n; integrability and the scaling (3) are Step 1. Fix g and write $|x| = H_g(x,x)^{1/2}$.

(1) If $t \le \lambda_1(g)$, every $y \in S$ is a nonzero vector of Λ, so $|y| \ge \lambda_1(g) \ge t$ and $n(t,g) = 0$.

(2), upper bound. If $\lambda_1(g) < t \le r$, then $n(t,g) \le \#\{x \in \Lambda : x \ne 0,\; |x| < t\} \le (3t/\lambda_1(g))^{2fm}$ by Lemma D.6(2).

(2), lower bound. Let $\lambda_1(g) < t \le r$ and $x \in \Lambda$ with $|x| = \lambda_1(g)$. As $|x| < r$, Lemma D.6(1) with $v = w = x$ and $b = 1$ gives $\mathrm{Im}\, H_g(\iota(a)x, x) = 0$ for all a, so $h(x,x) = 0$ by (Fm3). Let $M = Ex \cap \Lambda$, a free $\mathbb{Z}$-module of rank $2f$ spanning Ex; it is saturated in Λ, so $M \otimes \mathbb{R} \subseteq \Lambda_\mathbb{R}$ is a euclidean space for $|\cdot|$, in which every nonzero element of M has length at least $|x|$. By Minkowski's first theorem, $|x| \le \sqrt{2f}\,\mathrm{covol}(M)^{1/(2f)}$. The vectors $a_i x = \iota(a_i)x$ are linearly independent over $\mathbb{Z}$ ($y \mapsto yx$ is injective on E) and lie in $\mathcal{O}_E x$, and, as a acts on W_v by the scalar $w(a)$ and g_v is $\mathbb{C}$-linear,

$$|a_i x|^2 = \kappa \sum_v |w(a_i)|^2\, h_v^+(g_v^{-1}x, g_v^{-1}x) \le \mathrm{vr}^2\, \kappa\, \|x\|_g^2 = \mathrm{vr}^2 |x|^2.$$

By Hadamard's inequality, $\mathrm{covol}(\mathcal{O}_E x) \le \mathrm{covol}(\bigoplus_i \mathbb{Z} a_i x) \le \prod_i |a_i x| \le (\mathrm{vr}|x|)^{2f}$. Hence

$$[M : \mathcal{O}_E x] = \frac{\mathrm{covol}(\mathcal{O}_E x)}{\mathrm{covol}(M)} \le \frac{(\mathrm{vr}|x|)^{2f}}{(|x|/\sqrt{2f})^{2f}} = (\mathrm{vr}\sqrt{2f})^{2f},$$

and, the index being an integer, it is at most C. So $x \in S$ with $|x| < t$, and $n(t,g) \ge 1$.

Step 3: conclusion. By Lemma D.7, $\log^+(r/\lambda_1)$ is integrable on D and

$$\mu(D)^{-1}\int_D \log^+(r/\lambda_1)\,d\mu \le 3.$$

□

Lemma D.6. *Let E be a CM field of degree $2f$ with complex conjugation $a \mapsto \bar{a}$, and Ψ a CM type of E. Let V be a finite-dimensional complex vector space with a positive definite Hermitian form H (linear in the first variable), $\Lambda \subset V$ a lattice with $\operatorname{Im} H(\Lambda,\Lambda) \subseteq \mathbb{Z}$, and $\iota\colon \mathcal{O}_E \to \operatorname{End}_{\mathbb{C}}(V)$ a ring homomorphism with $\iota(a)\Lambda \subseteq \Lambda$ and $H(\iota(a)x,y) = H(x,\iota(\bar{a})y)$ for $a \in \mathcal{O}_E$, $x,y \in V$. Suppose $\dim V_\tau = m \ge 1$ for every $\tau \in \Psi$ (notation of Lemma D.3). Write $\|v\| = H(v,v)^{1/2}$ and $\lambda_1 = \min\{\|v\| : 0 \ne v \in \Lambda\}$. Let $\mathrm{vr} > 0$ be such that there are $a_1,\dots,a_{2f} \in \mathcal{O}_E$, linearly independent over $\mathbb{Q}$, with $|\sigma(a_i)| \le \mathrm{vr}$ for all i and all field embeddings $\sigma\colon E \to \mathbb{C}$, and put $r = \mathrm{vr}^{-1/2}$. Then $\lambda_1 > 0$ is attained, and:*

1. *for $v,w \in \Lambda$ with $\|v\| < r$, $\|w\| < r$ and all $a,b \in \mathcal{O}_E$, $\operatorname{Im} H(\iota(a)v,\iota(b)w) = 0$;*
2. *for $\lambda_1 < t \le r$ the set $\{v \in \Lambda : v \ne 0,\ \|v\| < t\}$ is finite and non-empty, and*

$$\#\{v \in \Lambda :\ v \ne 0,\ \|v\| < t\} \ \le\ \Big(\frac{3t}{\lambda_1}\Big)^{2fm}.$$

Proof. **Step 0: λ_1.** Λ is discrete, so the ball $\{\|v\| \le R\}$ contains finitely many points of Λ for every R; hence the minimum over the non-zero points is attained and positive.

Step 1: the action on norms. For embeddings $\sigma \ne \sigma'$ and $x \in V_\sigma$, $y \in V_{\sigma'}$, $a \in \mathcal{O}_E$: $\sigma(a)H(x,y) = H(\iota(a)x,y) = H(x,\iota(\bar{a})y) = \overline{\sigma'(\bar{a})}H(x,y) = \sigma'(a)H(x,y)$, since $\sigma'(\bar{a}) = \overline{\sigma'(a)}$ for a CM field. Choosing a with $\sigma(a) \ne \sigma'(a)$ gives $H(x,y) = 0$: the decomposition $V = \bigoplus_\sigma V_\sigma$ is H-orthogonal. Hence for $a \in \mathcal{O}_E$ and $v \in V$, $\|\iota(a)v\|^2 = \sum_\sigma |\sigma(a)|^2\|v_\sigma\|^2 \le \max_\sigma |\sigma(a)|^2\,\|v\|^2$.

Step 2: (1). Let $\|v\|,\|w\| < r$. By Cauchy–Schwarz and Step 1, $|\operatorname{Im} H(\iota(a_i)v,w)| \le \|\iota(a_i)v\|\,\|w\| \le \mathrm{vr}\,\|v\|\,\|w\| < \mathrm{vr}\,r^2 = 1$, and the value is an integer, so it is 0. Let $c \in \mathcal{O}_E$. Since the a_i form a $\mathbb{Q}$-basis of E, there are an integer $N \ge 1$ and integers n_i with $Nc = \sum_i n_i a_i$; as ι is additive and $\operatorname{Im} H$ is biadditive, $N\operatorname{Im} H(\iota(c)v,w) = \sum_i n_i \operatorname{Im} H(\iota(a_i)v,w) = 0$. For $a,b \in \mathcal{O}_E$, $\operatorname{Im} H(\iota(a)v,\iota(b)w) = \operatorname{Im} H(\iota(\bar{b}a)v,w) = 0$ by the case $c = \bar{b}a$.

Step 3: the span of the short vectors. Let $\lambda_1 < t \le r$ and $S_t = \{v \in \Lambda : v \ne 0,\ \|v\| < t\}$; it is finite by Step 0 and non-empty since it contains a vector of norm λ_1. Let I be the $\mathcal{O}_E$-submodule of Λ generated by S_t: a non-zero finitely generated torsion-free $\mathcal{O}_E$-module. By (1) and biadditivity, $\operatorname{Im} H$ vanishes on $I \times I$. A maximal $\mathcal{O}_E$-linearly independent family $v_1,\dots,v_\rho$ in I spans a submodule of finite index (for every $x \in I$ some non-zero multiple cx, $c \in \mathcal{O}_E$, lies in it, and I is finitely generated, so a non-zero integer M with MI inside it exists, and I/MI is finite). By Lemma D.3, $\rho \le m$ and I is a free $\mathbb{Z}$-module of rank $k = 2f\rho \le 2fm$.

Step 4: packing. Let W be the real span of I, of real dimension k (I is discrete in V, so a $\mathbb{Z}$-basis of I is $\mathbb{R}$-linearly independent), with the Euclidean norm $\|\cdot\|$ and its Lebesgue measure. Distinct points $u \ne u'$ of I satisfy $\|u-u'\| \ge \lambda_1$, so the open balls of radius $\lambda_1/2$ in W around the points of $S_t \cup \{0\}$ are disjoint, and they lie in the ball of radius $t + \lambda_1/2$ around 0. Comparing volumes,

$$\#S_t + 1 \le \Big(\frac{t+\lambda_1/2}{\lambda_1/2}\Big)^k = \Big(1+\frac{2t}{\lambda_1}\Big)^k \le \Big(\frac{3t}{\lambda_1}\Big)^k \le \Big(\frac{3t}{\lambda_1}\Big)^{2fm},$$

using $1 < t/\lambda_1$ for the middle inequality and $3t/\lambda_1 > 1$, $k \le 2fm$ for the last. This proves (2). □

Lemma D.7. *Let (X,μ) be a probability space, $\alpha\colon X\to(0,\infty)$ measurable, $a\ge 1$ an integer, $\kappa\ge 3a$ and $r>0$ real numbers, and let $n\colon(0,\infty)\times X\to[0,\infty)$ be a function such that $n(t,\cdot)$ is measurable and integrable for every $t>0$, and such that for all $t>0$ and $z\in X$:*

1. *$n(t,z)=0$ if $t\le\alpha(z)$;*
2. *$1\le n(t,z)\le\bigl(3t/\alpha(z)\bigr)^a$ if $\alpha(z)<t\le r$;*
3. *$M(t)=\int_X n(t,z)\,d\mu(z)$ satisfies $M(\theta t)=\theta^\kappa M(t)$ for all $\theta>0$.*

Then $M(r)\le 2\cdot 36^a$; for every $x\ge 0$

$$\mu\{z:\ \alpha(z)<e^{-x}r\}\le\min\bigl(1,\ 2\cdot 36^a e^{-\kappa x}\bigr);$$

and $\log^+(r/\alpha)$ is integrable with

$$\int_X\log^+\bigl(r/\alpha(z)\bigr)\,d\mu(z)\le 3.$$

Proof. For $t>0$ put $A(t)=\{z\in X:\ \alpha(z)<t\}$, a measurable set.

Step 1: mass of $A(t)$. Let $0<t\le r$. By (2), $n(t,z)\ge 1$ for $z\in A(t)$, and $n\ge 0$, so $\mu(A(t))\le\int_X n(t,z)\,d\mu(z)=M(t)$. By (3) with $\theta=t/r$, $M(t)=(t/r)^\kappa M(r)$. Hence

$$\mu(A(t))\le\min\bigl(1,(t/r)^\kappa M(r)\bigr)\qquad(0<t\le r).$$

Step 2: the mean at r. By (1), $n(r,z)=0$ for $z\notin A(r)$, and by (2), $n(r,z)\le(3r/\alpha(z))^a$ for $z\in A(r)$. Since $\alpha>0$, $A(r)$ is the disjoint union of the measurable shells $S_k=\{z:\ 2^{-k-1}r\le\alpha(z)<2^{-k}r\}$, $k\ge 0$. On S_k we have $(3r/\alpha)^a\le(3\cdot 2^{k+1})^a=6^a2^{ka}$, and $S_k\subseteq A(2^{-k}r)$, so by Step 1 $\mu(S_k)\le\min(1,2^{-\kappa k}M(r))$. By monotone convergence,

$$M(r)\le\sum_{k\ge 0}6^a2^{ka}\mu(S_k)\le 6^a(1+2^a)+6^aM(r)\sum_{k\ge 2}2^{(a-\kappa)k}.$$

So $M(r)\le 2\cdot 12^a+\frac12 M(r)$. As $M(r)<\infty$ ($n(r,\cdot)$ is integrable), $M(r)\le 4\cdot 12^a\le 2\cdot 36^a$, the last inequality because $2\cdot 36^a/(4\cdot 12^a)=3^a/2\ge\frac32$.

Step 3: the tail. For $x\ge 0$ put $t=e^{-x}r\le r$; Step 1 and Step 2 give $\mu(A(e^{-x}r))\le\min(1,e^{-\kappa x}M(r))\le\min(1,2\cdot 36^a e^{-\kappa x})$.

Step 4: the mean of the logarithm. The function $f=\log^+(r/\alpha)$ is measurable and non-negative, and for $x\ge 0$ we have $\{f>x\}=\{\alpha<e^{-x}r\}=A(e^{-x}r)$. By the layer-cake formula and Step 3 with $C=4\cdot 12^a\ge 1$ (Step 2),

$$\int_X f\,d\mu=\int_0^\infty\mu(A(e^{-x}r))\,dx\le\int_0^\infty\min(1,Ce^{-\kappa x})\,dx=\frac{\log C}{\kappa}+\frac{1}{\kappa}.$$

Hence f is integrable and $\int_X f\,d\mu<1.63\le 3$. □

Lemma D.8. *Let K be a number field of degree n and $\mathrm{rd}_K=|\mathrm{disc}(K)|^{1/n}$. There are $a_1,\dots,a_n\in\mathcal{O}_K$, linearly independent over $\mathbb{Q}$, such that for every i*

$$\sum_{\sigma\colon K\to\mathbb{C}}|\sigma(a_i)|^2\le n\,\mathrm{rd}_K^2;$$

in particular $\quad|\sigma(a_i)|\le\sqrt{n}\,\mathrm{rd}_K\le n\,\mathrm{rd}_K$ *for every field embedding $\sigma\colon K\to\mathbb{C}$.*

Proof. The Euclidean structure. Let $K_{\mathbb{R}} = K \otimes_{\mathbb{Q}} \mathbb{R}$; every field embedding $\sigma\colon K \to \mathbb{C}$ extends to an $\mathbb{R}$-algebra homomorphism $\sigma\colon K_{\mathbb{R}} \to \mathbb{C}$, and the map $K_{\mathbb{R}} \to \prod_\sigma \mathbb{C}$ is injective. Put

$$\langle x, y\rangle = \sum_\sigma \sigma(x)\overline{\sigma(y)} \qquad (x, y \in K_{\mathbb{R}}).$$

The terms of a real σ are real and the terms of σ and $\bar\sigma$ are complex conjugate, so $\langle\,,\rangle$ is real valued; it is $\mathbb{R}$-bilinear, symmetric and positive definite, so $(K_{\mathbb{R}}, \langle\,,\rangle)$ is a Euclidean space of dimension n, isometric to $\mathbb{R}^n$, with $\|x\|^2 = \sum_\sigma |\sigma(x)|^2$. Let c be the $\mathbb{R}$-algebra automorphism of $K_{\mathbb{R}} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ that conjugates the complex factors; then $\sigma(c(y)) = \overline{\sigma(y)}$ for every σ, so $\langle x, y\rangle = \mathrm{Tr}_{K_{\mathbb{R}}/\mathbb{R}}(x\,c(y))$ and $\|c(y)\| = \|y\|$. The image Γ of $\mathcal{O}_K$ is a lattice in $K_{\mathbb{R}}$.

The dual lattice has minimum at least $\sqrt{n}/\mathrm{rd}_K$. Let $0 \neq y \in \Gamma^*$ and $t = c(y)$. Then $\mathrm{Tr}(xt) \in \mathbb{Z}$ for all $x \in \mathcal{O}_K$. If $\omega_1, \dots, \omega_n$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$ and $\omega_1^\vee, \dots, \omega_n^\vee \in K$ the dual basis for the (non-degenerate) trace form, then $t = \sum_j \mathrm{Tr}(\omega_j t)\,\omega_j^\vee$ by non-degeneracy of the trace form of $K_{\mathbb{R}}$; hence $t \in K$, and t lies in the inverse different $\mathfrak{d}_K^{-1} = \{u \in K : \mathrm{Tr}(u\mathcal{O}_K) \subseteq \mathbb{Z}\}$, $t \neq 0$. So $t\mathcal{O}_K = \mathfrak{d}_K^{-1}\mathfrak{a}$ with $\mathfrak{a}$ an integral ideal, and

$$\prod_\sigma |\sigma(t)| = |N_{K/\mathbb{Q}}(t)| = N(\mathfrak{d}_K)^{-1}N(\mathfrak{a}) \geq N(\mathfrak{d}_K)^{-1} = |\mathrm{disc}(K)|^{-1}.$$

By the inequality of arithmetic and geometric means,

$$\|y\|^2 = \|t\|^2 = \sum_\sigma |\sigma(t)|^2 \geq n\Big(\prod_\sigma |\sigma(t)|^2\Big)^{1/n} \geq n\,|\mathrm{disc}(K)|^{-2/n} = \frac{n}{\mathrm{rd}_K^2}.$$

Thus $\lambda_1(\Gamma^*) \geq \sqrt{n}/\mathrm{rd}_K$.

Transference. By Banaszczyk's transference theorem (W. Banaszczyk, Math. Ann. 296 (1993), 625–635) with $i = n$ (applied after an isometry $K_{\mathbb{R}} \cong \mathbb{R}^n$, which maps Γ^* to the dual of the image of Γ), $\lambda_n(\Gamma) \leq n/\lambda_1(\Gamma^*) \leq \sqrt{n}\,\mathrm{rd}_K$. As Γ is discrete, the set $\Gamma \cap \{\|v\| \leq \lambda_n(\Gamma)\}$ is finite and spans $K_{\mathbb{R}}$, so it contains n elements $a_1, \dots, a_n$ that are linearly independent over $\mathbb{R}$, hence over $\mathbb{Q}$, with $\sum_\sigma |\sigma(a_i)|^2 = \|a_i\|^2 \leq n\,\mathrm{rd}_K^2$. Each term of the sum is at most the sum, which gives $|\sigma(a_i)| \leq \sqrt{n}\,\mathrm{rd}_K$, and $\sqrt{n} \leq n$. □

D.1.2 The corrected weight: symmetry, plurisubharmonicity and regularisation

The Hodge-type metric and the cusp weight of a base setting are invariant under complex conjugation (Lemma D.9); the corrected weight is plurisubharmonic and continuous on the open part, locally bounded up to the boundary, and has bounded regularisations (Lemma D.10, Lemma D.15, Lemma D.13); and the cusp weight grows at most like $\log\log$ near the boundary (Lemma D.16). With log-log regularisation of the Hodge weight, the arithmetic intersection numbers of the regularised metrics then have limits (Lemma D.18, Lemma D.17), by the convergence of mixed Monge–Ampère integrals of Section 31. These are the regularisation limits asked for in Proposition D.1.

Lemma D.9. *Let a base setting of $\mathcal{T}$ at (B, Z) be given. Then $X^{\mathrm{b},\circ}$ is stable under complex conjugation, and for every point $\hat w$ of the affine cone over $X^{\mathrm{b},\circ}$, with image $w \in X^{\mathrm{b},\circ}$,*

$$h^{\mathrm{H}}(\bar{\hat w}) = h^{\mathrm{H}}(\hat w), \qquad U(\bar w) = U(w).$$

In particular the corrected weight $\log h^{\mathrm{H}} + \frac{1}{2}U$ is invariant under complex conjugation, and so is $\log h^{\mathrm{H}} + \frac{1}{2}U - \log h^{\mathrm{ref}'}$ on $X^{\mathrm{b},\circ}$ for every cone metric $h^{\mathrm{ref}'}$ of multidegree a^{H}.

Proof. Step 1: conjugate points of the cover. The complex points of the $\mathbb{Z}$-schemes $\mathcal{X}$ and $\mathcal{X}^{\mathrm{b}}$ carry complex conjugation $z \mapsto \bar{z} = z \circ \mathrm{Spec}(c)$, and $y_{\mathbb{C}}(\bar{z}) = \overline{y_{\mathbb{C}}(z)}$ because y is given by elements of the function field and the charts of (Gb1) are defined over $\mathbb{Z}$. If z lies over κ, then $\bar{z}$ lies over $c \circ \kappa = \bar{\kappa}$. The abelian locus $f^{-1}(j(\mathrm{Sh}_F))$ is a subscheme, so it is stable under conjugation; hence so is its complement, and by (Gb1) $\partial X^{\mathrm{b}} = y_{\mathbb{C}}(\text{complement})$ and $X^{\mathrm{b},\circ}$ are stable. Let z be in the abelian locus, $A = G_{\mathcal{X},z}$; then $G_{\mathcal{X},\bar{z}} = A^c$, and the canonical bijection $\gamma\colon A(\mathbb{C}) \to A^c(\mathbb{C})$ is an antiholomorphic isomorphism of real Lie groups. Its differential $d\gamma\colon \mathrm{Lie}(A) \to \mathrm{Lie}(A^c)$ is a $\mathbb{C}$-antilinear isomorphism with $\gamma \circ \exp = \exp \circ d\gamma$, so it maps $H_1(A(\mathbb{C}),\mathbb{Z}) = \ker \exp$ isomorphically onto $H_1(A^c(\mathbb{C}),\mathbb{Z})$. It commutes with the action of $\mathcal{O}_E$ (the endomorphisms of A^c are the base changes of those of A). If $a \in \mathcal{O}_E$ acts on $x \in \mathrm{Lie}_s(A)$ by $\kappa(\sigma_s(a))$, it acts on $d\gamma(x)$ by $\overline{\kappa(\sigma_s(a))} = \bar{\kappa}(\sigma_s(a))$, the character of $\mathcal{O}_E$ on $\mathrm{Lie}_s(A^c)$ ($\bar{z}$ lies over $\bar{\kappa}$); as both decompositions are direct sums over the labels, $d\gamma(\mathrm{Lie}_s(A)) = \mathrm{Lie}_s(A^c)$ for every label s. The polarisation of A^c is the base change of that of A; γ^* carries its line bundle to the complex conjugate of $\mathcal{L}_z$, so $E_{\bar{z}}(d\gamma x, d\gamma x') = -E_z(x,x')$ on H_1, hence on $\mathrm{Lie}(A)$ by $\mathbb{R}$-linearity, and with $i\,d\gamma(x) = d\gamma(-ix)$:

$$\begin{aligned} H_{\bar{z}}(d\gamma x, d\gamma x') &= E_{\bar{z}}(d\gamma(-ix), d\gamma x') + iE_{\bar{z}}(d\gamma x, d\gamma x') \\ &= E_z(ix, x') - iE_z(x,x') = \overline{H_z(x,x')}. \end{aligned} \tag{1}$$

Finally, for a section b of $\mathrm{Lie}(G_{\mathcal{X}})$ over an open subset of $\mathcal{X}$ containing z (hence $\bar{z}$), $b(\bar{z}) = d\gamma(b(z))$, and for a regular function u there, $u(\bar{z}) = \overline{u(z)}$.

Step 2: the Hodge-type metric. Let $w \in X^{\mathrm{b},\circ}$, z with $y_{\mathbb{C}}(z) = w$ (by (Gb1) it lies in the abelian locus), a chart c' in which z lies, and $D(f)$, the unit u and the frames $b_{\tau,i}$ of $\mathcal{V}_{(\tau,-)}$ of (Gb3) with $z \in D(f)$; at complex points of the abelian locus these frames are frames of $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X}})$. As $D(f)$ is an open subscheme, $\bar{z} \in D(f)$, and $\bar{w} = y_{\mathbb{C}}(\bar{z})$. By Step 1 and (1), the Gram matrix $\big(h_{\bar{z}}(b_{\tau,i}(\bar{z}), b_{\tau,k}(\bar{z}))\big)$ is the complex conjugate of $\big(h_z(b_{\tau,i}(z), b_{\tau,k}(z))\big)$; both are positive definite hermitian, so their determinants are equal positive reals, and $\|b_{\tau,1} \wedge \cdots \wedge b_{\tau,q}\|_{h_{\bar{z}}}(\bar{z}) = \|b_{\tau,1} \wedge \cdots \wedge b_{\tau,q}\|_{h_z}(z)$. With $|u(\bar{z})| = |u(z)|$, (Gb3) gives $\|\sigma_{c'}\|(\bar{w}) = \|\sigma_{c'}\|(w)$ for $\sigma_{c'} = \prod_k y_{k,c'_k}^{a_k^{\mathrm{H}}}$ in the metric of the Hodge-type line. The polynomial $\sigma_{c'}$ has integer coefficients, so $|\sigma_{c'}(\bar{\hat{w}})| = |\sigma_{c'}(\hat{w})| \neq 0$ for a cone point $\hat{w}$ over w, and $h^{\mathrm{H}}(\bar{\hat{w}}) = |\sigma_{c'}(\bar{\hat{w}})|/\|\sigma_{c'}\|(\bar{w}) = h^{\mathrm{H}}(\hat{w})$.

Step 3: the cusp weight. Keep w, z over κ, A, γ. Put $\Psi = \{\kappa \circ \bar{\tau} : \tau \in \Phi\}$; it is a CM type of E, since $\kappa \circ \bar{\tau} = \overline{\kappa \circ \tau}$ (κ commutes with complex conjugation on the CM field $\tau(E)$) and $\{\kappa \circ \tau : \tau \in \Phi\}$ is a CM type (Φ is one and κ is injective), and $\mathrm{Lie}(A)_{\kappa\circ\bar{\tau}} = \mathrm{Lie}_{(\tau,-)}(A)$ has dimension $q \geq 1$. So Lemma D.3 applies to $(V,H,\Lambda,\iota) = (\mathrm{Lie}(A), H_z, H_1(A(\mathbb{C}),\mathbb{Z}), \mathcal{O}_E)$ and Ψ. Let $I \subseteq \Lambda_w$ be a non-zero saturated isotropic $\mathcal{O}_E$-submodule; it has an $\mathcal{O}_E$-linearly independent family of finite index (a maximal $\mathcal{O}_E$-linearly independent subset of the torsion-free finitely generated module I), so the fixed family v^I of (Hu1) exists. Let $I_z \subseteq H_1(A(\mathbb{C}),\mathbb{Z})$ be its image under the isomorphism of (Gb5) at w (through z); $\mathrm{Im}\, H_z$ vanishes on it. As $\lambda_{w,\tau}$ is carried to orthonormal coordinates on $\mathrm{Lie}_{(\tau,-)}(A)$, $\overline{\lambda_{w,\tau}(v_a^I)}^{\mathsf{T}} \lambda_{w,\tau}(v_b^I) = H_z\big((v_b)_{(\tau,-)}, (v_a)_{(\tau,-)}\big)$ for the images v_a of v_a^I, a hermitian matrix with the determinant of its transpose; so $\mathrm{N}_w(I)$ is the quantity $\mathrm{N}(I_z)$ of Lemma D.3, and that lemma gives, for any $\mathbb{Z}$-basis $b_1,\dots,b_{2f\rho}$ of I_z ($2f = [E:\mathbb{Q}]$, $2f\rho$ the $\mathbb{Z}$-rank of I),

$$\begin{aligned} \mathrm{N}_w(I) &= \det\big(H_z(b_i,b_k)\big)_{i,k}^{1/2}\, \mathrm{rd}_E^{-f\rho} \\ &= \det\big(\mathrm{Re}\, H_z(b_i,b_k)\big)_{i,k}^{1/2}\, \mathrm{rd}_E^{-f\rho}, \end{aligned} \tag{2}$$

the second equality because $\mathrm{Im}\, H_z = 0$ on I_z. By Step 1, $d\gamma$ is an isomorphism of $\mathcal{O}_E$-modules $H_1(A(\mathbb{C}),\mathbb{Z}) \to H_1(A^c(\mathbb{C}),\mathbb{Z})$ with $E_{\bar{z}} \circ (d\gamma \times d\gamma) = -E_z$; so it maps the non-zero saturated E_z-isotropic $\mathcal{O}_E$-submodules bijectively onto the non-zero saturated $E_{\bar{z}}$-isotropic ones, preserving

$\mathbb{Z}$-ranks, and by (1) $\operatorname{Re} H_{\bar z}(d\gamma b_i, d\gamma b_k) = \operatorname{Re} H_z(b_i, b_k)$. Composing with the isomorphisms of (Gb5) at w (through z) and at $\bar w$ (through $\bar z$) gives a bijection $I \mapsto I'$ between the non-zero saturated isotropic $\mathcal{O}_E$-submodules of Λ_w and of $\Lambda_{\bar w}$, and (2) at $\bar w$ (with $\bar z$ over $\bar\kappa$ and the CM type $\{\bar\kappa \circ \bar\tau\}$) gives $\mathrm{N}_{\bar w}(I') = \mathrm{N}_w(I)$. Hence the sets $\{-\log \mathrm{N}_w(I)\}$ and $\{-\log \mathrm{N}_{\bar w}(I')\}$ coincide, $s(\bar w) = s(w)$ and $U(\bar w) = U(w)$ by (Hu2).

Step 4. By Steps 2 and 3, $\log h^{\mathrm{H}} + \frac{1}{2}U$ is invariant under conjugation on the cone over $X^{\mathrm{b},\circ}$, and a cone metric satisfies $h^{\mathrm{ref}'}(\bar{\hat w}) = h^{\mathrm{ref}'}(\hat w)$ by definition; the difference, a function on $X^{\mathrm{b},\circ}$ (equal multidegrees), is invariant. □

Lemma D.10. *Let a base setting with cusp data be given, X the set of complex points of its cover $\mathcal{X}$ over one embedding $\kappa\colon F \to \mathbb{C}$ (as a complex manifold near the points considered), $W \subseteq X$ open, and for $\tau \in \Phi$ let b_τ be a holomorphic frame of $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X}})^{\mathrm{an}}$ on W. Then the function*

$$u(z) = \frac{1}{2}\sum_{\tau\in\Phi} \log\det \Gamma(b_\tau(z)) + \frac{1}{2}\,U(y_{\mathbb{C}}(z))$$

is plurisubharmonic and continuous on $W \cap$ (abelian locus). (By (Gb3), u is the corrected weight $\log h^{\mathrm{H}} + \frac{1}{2}U$ read in the frame b, up to the logarithm of the modulus of a holomorphic unit.)

Proof. Both properties are local. Fix $z_0 \in W$ in the abelian locus (an open set) and a connected, simply connected open neighbourhood $W' \subseteq W$ of z_0 inside the abelian locus. Put $w_0 = y_{\mathbb{C}}(z_0)$ and $\Lambda = \Lambda_{w_0}$ with $\langle\cdot,\cdot\rangle_{w_0}$ and the families of (Hu1) at w_0, as in Lemma D.11. For $z \in W'$ let $\psi_z\colon \Lambda_w \to H_1(G_{\mathcal{X},z}(\mathbb{C}),\mathbb{Z})$ be the isomorphism of (Gb5) at $w = y_{\mathbb{C}}(z)$, and let $\phi_z\colon \Lambda \to H_1(G_{\mathcal{X},z}(\mathbb{C}),\mathbb{Z})$ ($z \in W'$) be obtained from ψ_{z_0} by parallel transport. As the $\mathcal{O}_E$-action and the Riemann form $\operatorname{Im} H_z$ of the polarisation are flat on the local system of first homology, the ϕ_z are as in Lemma D.11. Write $L(z) = \sum_{\tau\in\Phi} \log\det\Gamma(b_\tau(z))$, a continuous function ($\Gamma(b_\tau(z))$ is positive definite and smooth in z).

Step 1: the cusp weight through the flat identification. Let $z \in W'$ and $g_z = \phi_z^{-1} \circ \psi_z\colon \Lambda_w \to \Lambda$, an isomorphism of $\mathcal{O}_E$-modules carrying $\langle\cdot,\cdot\rangle_w$ to $\langle\cdot,\cdot\rangle_{w_0}$; it maps the non-zero saturated isotropic $\mathcal{O}_E$-submodules of Λ_w bijectively onto those of Λ, and the family of (Hu1) at w for I to an $\mathcal{O}_E$-linearly independent family in $g_z I$ spanning a submodule of the same index. By (Gb5), $\lambda_{w,\tau}(v)$ is the coordinate vector of $\psi_z(v)_{(\tau,-)}$ in an H_z-orthonormal basis, so $\overline{\lambda_{w,\tau}(v_a)}^{\mathsf{T}}\lambda_{w,\tau}(v_b)$ is the H_z-Gram entry of $\phi_z(g_z v_a)_{(\tau,-)}, \phi_z(g_z v_b)_{(\tau,-)}$, and $\mathrm{N}_w(I)$ is the expression defining N_z^ϕ evaluated on the family $(g_z v_a^I)_a$ of $g_z I$, which spans an $\mathcal{O}_E$-submodule of index $\iota(I)$ in $g_z I$. This expression does not depend on the family: let $v = (v_a)$, $v' = (v'_a)$ be two $\mathcal{O}_E$-linearly independent families in a saturated isotropic I' spanning $\mathcal{O}_E$-submodules M, M' of finite indices ι, ι'. Then $v' = Av$ for some $A \in \mathrm{GL}_\rho(E)$, and since E acts on $\mathrm{Lie}_{(\tau,-)}$ through $\kappa\circ\bar\tau$, the Gram matrix of the $(\tau,-)$-components of $\phi_z(v')$ is $\kappa\bar\tau(A)\,\Gamma\,\kappa\bar\tau(A)^*$ with Γ that of $\phi_z(v)$. As $\kappa\circ\bar\tau = \overline{\kappa\circ\tau}$ (κ commutes with complex conjugation on the CM field $\tau(E)$) and $\{\kappa\circ\tau : \tau\in\Phi\}$ is a CM type (Φ is one, κ is injective), $\prod_{\tau\in\Phi}|\kappa\bar\tau(\det A)|^2 = |\mathrm{N}_{E/\mathbb{Q}}(\det A)|$; and $\iota'/\iota = [M : M'] = |\mathrm{N}_{E/\mathbb{Q}}(\det A)|$ (generalised index of the free $\mathcal{O}_E$-lattices $M = \mathcal{O}_E^\rho$ and $M' = AM$; clear denominators by an integer d, using $[\mathcal{O}_E^\rho : B\mathcal{O}_E^\rho] = |\mathrm{N}_{E/\mathbb{Q}}(\det B)|$ for $B \in M_\rho(\mathcal{O}_E)$). The two factors cancel. Hence $\mathrm{N}_w(I) = \mathrm{N}_z^\phi(g_z I)$, and

$$U(w) = \max\Big(0,\ \sup_I\big(-\log \mathrm{N}_z^\phi(I)\big)\Big),$$

the supremum over the non-zero saturated isotropic I admitting a family (the others have $\mathrm{N} = 1$ and contribute 0).

Step 2: a finite maximum. By Lemma D.12 there are an open $V \ni z_0$, $V \subseteq W'$, and a finite set F with $\mathrm{N}_z^\phi(I) > 1$, i.e. $-\log \mathrm{N}_z^\phi(I) < 0$, for $z \in V$ and $I \notin F$. So on V the supremum in

Step 1 exceeds 0 only through $I \in F$, and

$$2u(z) = L(z) + U(y_{\mathbb{C}}(z)) = \max\Big(L(z),\ \max_{I\in F}\big(L(z) - \log \mathrm{N}_z^{\phi}(I)\big)\Big) \qquad (z \in V).$$

Step 3: conclusion. By Lemma D.11 (for $I = 0$ and for $I \in F$) each of the finitely many functions in this maximum is plurisubharmonic and continuous on W'. A maximum of finitely many continuous plurisubharmonic functions is continuous and plurisubharmonic. So u is plurisubharmonic and continuous on V; as z_0 was arbitrary, on $W \cap$ (abelian locus). □

Lemma D.11. *Let a base setting with cusp data be given, X the set of complex points of its cover $\mathcal{X}$ over one embedding $\kappa\colon F \to \mathbb{C}$, $W' \subseteq X$ a connected, simply connected open subset of the abelian locus, and for $\tau \in \Phi$ let b_τ be a holomorphic frame of $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X}})^{\mathrm{an}}$ on W'. Let $w_0 \in X^{\mathrm{b},\circ}$ and write $\Lambda = \Lambda_{w_0}$, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{w_0}$, with the families v_a^I and the indices $\iota(I)$ of (Hu1) at w_0. Let $\phi_z\colon \Lambda \to H_1(G_{\mathcal{X},z}(\mathbb{C}),\mathbb{Z})$ $(z \in W')$ be isomorphisms of $\mathcal{O}_E$-modules carrying $\langle\cdot,\cdot\rangle$ to $\mathrm{Im}\, H_z$ and obtained from one of them by parallel transport in the local system of first homology. For a saturated isotropic $\mathcal{O}_E$-submodule $I \neq 0$ of Λ admitting the family $(v_a^I)_{a\le\rho}$ of (Hu1) put*

$$\mathrm{N}_z^{\phi}(I) = \frac{1}{\iota(I)} \prod_{\tau\in\Phi} \det\Big(H_z\big(\phi_z(v_b^I)_{(\tau,-)}, \phi_z(v_a^I)_{(\tau,-)}\big)\Big)_{a,b\le\rho},$$

where $v_{(\tau,-)}$ is the component of $v \in \mathrm{Lie}(G_{\mathcal{X},z})$ in $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X},z})$, and put $\mathrm{N}_z^{\phi}(0) = 1$. Then for every such I (and for $I = 0$) the function $\mathrm{N}^{\phi}(I)$ is continuous and positive on W', and

$$z \mapsto \sum_{\tau\in\Phi} \log\det\Gamma(b_\tau(z)) - \log \mathrm{N}_z^{\phi}(I)$$

is plurisubharmonic on W'.

Proof. Write $\Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, Q for the $\mathbb{C}$-bilinear extension of $\langle\cdot,\cdot\rangle$, $m = \dim \mathrm{Lie}(G_{\mathcal{X},z})$, and for a subgroup $I \subseteq \Lambda$ put $I_{\mathbb{C}} = I \otimes \mathbb{C}$ and $I^{\perp} = \{v \in \Lambda_{\mathbb{C}} : Q(v, I_{\mathbb{C}}) = 0\}$. As ϕ_z is $\mathcal{O}_E$-linear and carries $\langle\cdot,\cdot\rangle$ to $\mathrm{Im}\, H_z$, (Ud4) gives $\langle av, w\rangle = \langle v, \bar{a}w\rangle$ on Λ. Let I be 0 or a non-zero saturated isotropic $\mathcal{O}_E$-submodule with its family $v_a = v_a^I$ $(a \le \rho;\ \rho = 0$ for $I = 0)$; as $\mathcal{O}_E$ has rank $2|\Phi|$, $\dim I_{\mathbb{C}} = 2|\Phi|\rho$.

Step 1: the variation of Hodge structure of the family. Let $\beta_z\colon \Lambda_{\mathbb{C}} \to \mathrm{Lie}(G_{\mathcal{X},z})$ be the $\mathbb{C}$-linear extension of ϕ_z, where $H_1(G_{\mathcal{X},z}(\mathbb{C}),\mathbb{Z})$ is the kernel of the exponential map $\mathrm{Lie}(G_{\mathcal{X},z}) \to G_{\mathcal{X},z}(\mathbb{C})$. For $v \in \Lambda$, $z \mapsto \phi_z(v)$ is a holomorphic section of $\mathrm{Lie}(G_{\mathcal{X}})^{\mathrm{an}}$: the exponential map of the family over W' is holomorphic and a local biholomorphism, so near each point its kernel is the image of the zero section under a holomorphic local inverse, and parallel transport in the local system of first homology is the continuation of these local sections. So β is a holomorphic map $\Lambda_{\mathbb{C}} \otimes \mathcal{O}_{W'} \to \mathrm{Lie}(G_{\mathcal{X}})^{\mathrm{an}}$, surjective at every z ($\phi_z(\Lambda)$ spans Lie over $\mathbb{R}$), and $\mathcal{K}_z = \ker\beta_z$ is a holomorphic subbundle of rank m. Fix z, let J be the complex structure of $\mathrm{Lie}(G_{\mathcal{X},z})$ transported to $\Lambda_{\mathbb{R}}$ by ϕ_z and $E_0 = \langle\cdot,\cdot\rangle$ on $\Lambda_{\mathbb{R}}$; as H_z is hermitian with imaginary part E_0, $H_z(\phi_z v, \phi_z w) = E_0(Jv, w) + iE_0(v, w)$, $E_0(Jv, Jw) = E_0(v, w)$ and $E_0(Jv, w)$ is symmetric. Then

$$\mathcal{K}_z = \{v + iJv :\ v \in \Lambda_{\mathbb{R}}\}, \qquad \overline{\mathcal{K}_z} = \{v - iJv :\ v \in \Lambda_{\mathbb{R}}\}, \qquad \beta_z(v - iJv) = 2\phi_z(v),$$

$$Q(\mathcal{K}_z, \mathcal{K}_z) = 0, \qquad iQ(u, \bar{w}) = \tfrac{1}{2}H_z(\beta_z u, \beta_z w) \quad (u, w \in \overline{\mathcal{K}_z}).$$

So $\Lambda_{\mathbb{C}} = \mathcal{K}_z \oplus \overline{\mathcal{K}_z}$ with $\mathcal{K}_z$ isotropic and $iQ(u, \bar{u}) > 0$ for $0 \neq u \in \overline{\mathcal{K}_z}$: with the Weil operator C acting on $H^{a,b}$ by i^{a-b} (the convention of Theorem B.145 and Theorem B.144: the Hodge form

$Q(C\cdot,\bar{\cdot})$ is positive definite), $(\Lambda,\mathcal{K},-Q)$ is a polarised variation of Hodge structure of weight -1 on W' with $H^{-1,0}=\overline{\mathcal{K}}$, $H^{0,-1}=\mathcal{K}=F^0$ (Griffiths transversality is empty for a filtration with two steps) , its last quotient $\Lambda_{\mathbb{C}}\otimes\mathcal{O}/\mathcal{K}$ is $\mathrm{Lie}(G_{\mathcal{X}})^{\mathrm{an}}$ via β, and its Hodge metric is $iQ(u,\bar{u})$ on $\overline{\mathcal{K}}$.

Step 2: the quotient by I. (a) $I_{\mathbb{C}}\cap\overline{\mathcal{K}_z}=0=I_{\mathbb{C}}\cap\mathcal{K}_z$: for $u\in I_{\mathbb{C}}\cap\overline{\mathcal{K}_z}$, $\bar{u}\in I_{\mathbb{C}}$, so $Q(u,\bar{u})=0$ by isotropy and $u=0$ by positivity; conjugate for $\mathcal{K}_z$. (b) $\Lambda'=(I^{\perp}\cap\Lambda)/I$ is a free $\mathbb{Z}$-module (I is saturated), an $\mathcal{O}_E$-module ($I^{\perp}$ is $\mathcal{O}_E$-stable as $\langle av,w\rangle=\langle v,\bar{a}w\rangle$), with $\Lambda'_{\mathbb{C}}=I^{\perp}/I_{\mathbb{C}}$ and the non-degenerate form Q' induced by Q (as $(I^{\perp})^{\perp}=I_{\mathbb{C}}$). Put $\mathcal{K}'_z=((\mathcal{K}_z\cap I^{\perp})+I_{\mathbb{C}})/I_{\mathbb{C}}$. The map $I_{\mathbb{C}}\to\mathcal{K}_z^*$, $y\mapsto Q(y,\cdot)$, is injective (its kernel lies in $\mathcal{K}_z^{\perp}=\mathcal{K}_z$, a Lagrangian subspace, and $I_{\mathbb{C}}\cap\mathcal{K}_z=0$), so $\mathcal{K}_z\cap I^{\perp}$ has dimension $m-2|\Phi|\rho$, is a holomorphic subbundle (kernel of a holomorphic surjection of constant rank), and by (a) maps isomorphically onto $\mathcal{K}'_z$, of half the dimension $2m-4|\Phi|\rho$ of $\Lambda'_{\mathbb{C}}$. The subspace $\mathcal{K}'_z$ is Q'-isotropic. For $u\in\overline{\mathcal{K}_z}\cap I^{\perp}$ and $y\in I_{\mathbb{C}}$, $Q(u+y,\bar{u}+\bar{y})=Q(u,\bar{u})$, so $iQ'(x,\bar{x})>0$ for every $0\neq x\in\overline{\mathcal{K}'_z}$ (and $x=[u]\neq0$ iff $u\neq0$, by (a)); an $x\in\mathcal{K}'_z\cap\overline{\mathcal{K}'_z}$ has $x,\bar{x}\in\mathcal{K}'_z$, so $Q'(x,\bar{x})=0$ and $x=0$. Hence (by the same computation as in Step 1) $(\Lambda',\mathcal{K}',-Q')$ is again a polarised variation of Hodge structure of weight -1 on W', with Hodge metric $iQ'(x,\bar{x})$ on $\overline{\mathcal{K}'}$. For $I=0$ it is the variation of Step 1.

(c) Let $P_z=\beta_z(I_{\mathbb{C}})$, a holomorphic subbundle of $\mathrm{Lie}(G_{\mathcal{X}})^{\mathrm{an}}$ of rank $2|\Phi|\rho$ (β_z is injective on $I_{\mathbb{C}}$ by (a)). The map $I^{\perp}\to\mathrm{Lie}(G_{\mathcal{X},z})/P_z$, $v\mapsto\beta_z v \bmod P_z$, has kernel $I^{\perp}\cap(\mathcal{K}_z+I_{\mathbb{C}})=(\mathcal{K}_z\cap I^{\perp})+I_{\mathbb{C}}$, so it induces a holomorphic isomorphism $\Lambda'_{\mathbb{C}}/\mathcal{K}'\to\mathrm{Lie}(G_{\mathcal{X}})^{\mathrm{an}}/P$ (both of rank $m-2|\Phi|\rho$). It multiplies squared norms by 2π, for the Hodge metric on the left and the quotient metric of $h_z=\pi H_z$ on the right: let $u\in\overline{\mathcal{K}_z}\cap I^{\perp}$ and $y\in I_{\mathbb{C}}$, $y=y_1+y_2$ with $y_1\in\mathcal{K}_z$, $y_2\in\overline{\mathcal{K}_z}$, so $\beta_z y=\beta_z y_2$; then $Q(u,\bar{y}_2)=Q(u,\bar{y})-Q(u,\bar{y}_1)=0$ ($u\in I^{\perp}$, $\bar{y}\in I_{\mathbb{C}}$; $\overline{\mathcal{K}_z}$ isotropic), so $H_z(\beta_z u,\beta_z y)=0$ by Step 1. Thus $\beta_z(\overline{\mathcal{K}_z}\cap I^{\perp})$ is H_z-orthogonal to P_z, hence equal to $P_z^{\perp}$ (dimensions), and the quotient norm of $\beta_z u \bmod P_z$ is $h_z(\beta_z u,\beta_z u)=\pi H_z(\beta_z u,\beta_z u)=2\pi\, iQ'([u],[\bar{u}])$, the class $[u]\in\overline{\mathcal{K}'_z}$ representing $u \bmod \mathcal{K}'_z$.

Step 3: the pieces $(\tau,-)$. The Lie_s are mutually H_z-orthogonal: for $x\in\mathrm{Lie}_s$, $y\in\mathrm{Lie}_{s'}$, $s\neq s'$, choose $a\in\mathcal{O}_E$ with $\sigma_s(a)\neq\sigma_{s'}(a)$; applying (Ud4) to x and to ix gives $H_z(ax,y)=H_z(x,\bar{a}y)$, i.e. $\sigma_s(a)H_z(x,y)=\overline{\sigma_{s'}(\bar{a})}H_z(x,y)=\sigma_{s'}(a)H_z(x,y)$, so $H_z(x,y)=0$. As β_z is $\mathcal{O}_E$-linear, $P_z=\bigoplus_s P_{z,s}$ with $P_{z,s}=P_z\cap\mathrm{Lie}_s=\beta_z\big((I_{\mathbb{C}})_{\sigma_s}\big)$, $(I_{\mathbb{C}})_{\sigma}$ the subspace on which $\mathcal{O}_E$ acts by σ, and $\mathrm{Lie}/P=\bigoplus_s\mathrm{Lie}_s/P_s$ orthogonally, the quotient metric restricting to the quotient metric of $h_z|_{\mathrm{Lie}_s}$ on each summand. For $\tau\in\Phi$ write $V_\tau=\mathrm{Lie}_{(\tau,-)}$, $P_\tau=P_{(\tau,-)}$ and $p_{\tau,a}(z)=\phi_z(v_a)_{(\tau,-)}$, the image under β_z of the $\bar{\tau}$-component of v_a. These are holomorphic in z (components of holomorphic sections along a holomorphic direct sum) and form a basis of $P_{z,\tau}$: the v_a are an E-basis of $I\otimes\mathbb{Q}$, so their $\bar{\tau}$-components are a basis of $(I_{\mathbb{C}})_{\bar{\tau}}=I\otimes_{\mathcal{O}_E,\bar{\tau}}\mathbb{C}$, and β_z is injective on $I_{\mathbb{C}}$. Hence $\rho\le q$, and $\mathrm{N}_z^{\phi}(I)=\iota(I)^{-1}\prod_\tau\det\big(H_z(p_{\tau,b},p_{\tau,a})\big)_{a,b}$ is a product of Gram determinants of linearly independent vectors: positive, and continuous in z.

Step 4: separating the cusp. Fix τ and $k=q-\rho$. The canonical holomorphic isomorphism $\bigwedge^q V_\tau\cong\bigwedge^\rho P_\tau\otimes\bigwedge^k(V_\tau/P_\tau)$, $x_1\wedge\cdots\wedge x_\rho\wedge y_1\wedge\cdots\wedge y_k\mapsto(x_1\wedge\cdots\wedge x_\rho)\otimes(\bar{y}_1\wedge\cdots\wedge\bar{y}_k)$ ($x_i\in P_\tau$, $\bar{y}_j$ the class of y_j), is isometric for the metrics induced by h_z on V_τ and P_τ and the quotient metric on V_τ/P_τ (the y_j may be taken in $P_\tau^{\perp}$, and then the Gram matrix is block diagonal). Define c_τ by $b_{\tau,1}\wedge\cdots\wedge b_{\tau,q}=(p_{\tau,1}\wedge\cdots\wedge p_{\tau,\rho})\otimes c_\tau$: a nowhere vanishing holomorphic section of $\bigwedge^k(V_\tau/P_\tau)$ on W', as $p_{\tau,1}\wedge\cdots\wedge p_{\tau,\rho}$ is holomorphic and nowhere zero. Taking squared norms, $\det\Gamma(b_\tau)=\pi^\rho\det\big(H_z(p_{\tau,b},p_{\tau,a})\big)\,\|c_\tau\|^2$, and

$$\sum_{\tau\in\Phi}\log\det\Gamma(b_\tau(z))-\log\mathrm{N}_z^{\phi}(I)=\log\iota(I)+\rho|\Phi|\log\pi+\sum_{\tau\in\Phi}\log\|c_\tau(z)\|^2$$

(for $I=0$: $\rho=0$, $\iota=1$, $c_\tau=b_{\tau,1}\wedge\cdots\wedge b_{\tau,q}$).

Step 5: Griffiths. It suffices that each $\log\|c_\tau\|^2$ is plurisubharmonic. If $k=0$, c_τ is a nowhere vanishing holomorphic function and $\log|c_\tau|^2$ is pluriharmonic. Let $k\ge1$. By Step

3, $\bigwedge^k(V_\tau/P_\tau)$ is a holomorphic line subbundle of $\bigwedge^k(\mathrm{Lie}/P)$ with the induced metric, and by Step 2(c) $\bigwedge^k(\mathrm{Lie}/P)$ is isomorphic to $\bigwedge^k(\Lambda'_{\mathbb C}/\mathcal K')$ with squared norms multiplied by $(2\pi)^k$, the latter with the metric induced by the Hodge metric. Consider $(\bigwedge^k\Lambda', F^\bullet, (-1)^kQ'_k)$, where $(-1)^kQ'_k(x_1\wedge\cdots\wedge x_k, y_1\wedge\cdots\wedge y_k) = \det\bigl(-Q'(x_i,y_j)\bigr)$ is the form induced by the polarisation $-Q'$ of Step 2; here $F^p = \sum_{p_1+\cdots+p_k=p} F'^{p_1}\wedge\cdots\wedge F'^{p_k}$ ($F'^{-1}=\Lambda'_{\mathbb C}$, $F'^0=\mathcal K'$, $F'^1=0$) and $Q'_k(x_1\wedge\cdots\wedge x_k, y_1\wedge\cdots\wedge y_k) = \det\bigl(Q'(x_i,y_j)\bigr)$. It is a polarised variation of Hodge structure of weight $-k$ on W'.

Since $\dim\overline{\mathcal K'} = m - 2|\Phi|\rho \ge k$ (it contains a copy of V_τ/P_τ), the largest a with $F^a = \bigwedge^k\Lambda'_{\mathbb C}$ is $a=-k$; F^{-k+1} is the kernel of $\bigwedge^k\Lambda'_{\mathbb C}\to\bigwedge^k(\Lambda'_{\mathbb C}/\mathcal K')$, so the last quotient of Theorem B.144 is $\bigwedge^k(\Lambda'_{\mathbb C}/\mathcal K')$ with the metric induced by the Hodge metric of $\Lambda'_{\mathbb C}/\mathcal K'$ (on $H^{-k,0} = \bigwedge^k\overline{\mathcal K'}$ the Hodge form is the Gram determinant of $iQ'(x,\bar y)$). By Theorem B.144 on $S=W'$, the logarithm of the squared Hodge norm of the holomorphic, nowhere vanishing section c_τ is plurisubharmonic, and $\log\|c_\tau\|^2$ differs from it by the constant $k\log(2\pi)$. Summing over $\tau\in\Phi$ proves the claim. $\square$

Lemma D.12. *In the setting of Lemma D.11, every $z_0\in W'$ has an open neighbourhood $V\subseteq W'$ with a finite set F of non-zero saturated isotropic $\mathcal O_E$-submodules of Λ such that $\mathrm N_z^\phi(I)>1$ for every $z\in V$ and every non-zero saturated isotropic $\mathcal O_E$-submodule $I\notin F$ admitting a family as in (Hu1).*

Proof. For $z\in W'$ let $\gamma_z(v,w) = \operatorname{Re}H_z(\phi_z v,\phi_z w)$, a positive definite inner product on $\Lambda_{\mathbb R} = \Lambda\otimes\mathbb R$ (as $\phi_z\otimes\mathbb R$ is an $\mathbb R$-isomorphism onto $\mathrm{Lie}(G_{\mathcal X,z})$). It depends continuously on z: H_z is a smooth metric on $\mathrm{Lie}(G_{\mathcal X})^{\mathrm{an}}$, and for fixed v the vectors $\phi_z(v)$ form a holomorphic section, the exponential map of the family being a local biholomorphism whose kernel is, near each point, the image of the zero section under a holomorphic local inverse, continued by parallel transport. For a subgroup $M\subseteq\Lambda$ of rank $n\ge1$ with $\mathbb Z$-basis $e_1,\dots,e_n$ put $\mathrm{covol}_\gamma(M) = \det\bigl(\gamma(e_i,e_j)\bigr)^{1/2}$, and let D_E be the absolute value of the discriminant of E.

Step 1: the cusp norm is a covolume. We show: for every non-zero saturated isotropic $\mathcal O_E$-submodule I with a family $v_1,\dots,v_\rho$ as in (Hu1) and every $z\in W'$,

$$\mathrm N_z^\phi(I) = D_E^{-\rho/2}\,\mathrm{covol}_{\gamma_z}(I).$$

First, the Lie_s are mutually H_z-orthogonal: for $x\in\mathrm{Lie}_s$, $y\in\mathrm{Lie}_{s'}$, $s\ne s'$, pick $a\in\mathcal O_E$ with $\sigma_s(a)\ne\sigma_{s'}(a)$; (Ud4) applied to x and to ix gives $H_z(ax,y) = H_z(x,\bar a y)$, i.e. $\sigma_s(a)H_z(x,y) = \sigma_{s'}(a)H_z(x,y)$. Let $x,y\in I_{\mathbb R}$ and write x_s for the component of $\phi_z(x)$ in Lie_s. As I is an isotropic $\mathcal O_E$-module and ϕ_z carries $\langle\cdot,\cdot\rangle$ to $\operatorname{Im}H_z$, $\operatorname{Im}H_z(\phi_z(ax),\phi_z(y))=0$ for $a\in\mathcal O_E$, hence (by $\mathbb R$-linearity in a) for all $a\in E\otimes\mathbb R$:

$$\operatorname{Im}\sum_{\tau\in\Phi}\bigl(\tau(a)\,H_z(x_{(\tau,+)},y_{(\tau,+)}) + \overline{\tau(a)}\,H_z(x_{(\tau,-)},y_{(\tau,-)})\bigr) = 0.$$

As $a\mapsto(\tau(a))_{\tau\in\Phi}$ maps $E\otimes\mathbb R$ onto $\mathbb C^\Phi$, we may take $\tau(a)=1$ and then $\tau(a)=i$ at one τ and 0 at the others; this gives $H_z(x_{(\tau,+)},y_{(\tau,+)}) = \overline{H_z(x_{(\tau,-)},y_{(\tau,-)})}$, so

$$\gamma_z(x,y) = 2\sum_{\tau\in\Phi}\operatorname{Re}H_z(x_{(\tau,-)},y_{(\tau,-)}) \qquad (x,y\in I_{\mathbb R}).$$

The v_a are a basis of $I_{\mathbb R}$ over $E\otimes\mathbb R$; write $x=\sum_a x_a v_a$ ($x_a\in E\otimes\mathbb R$) and $w_{\tau,a}=\tau(x_a)$, which identifies $I_{\mathbb R}$ with $(\mathbb C^\rho)^\Phi$. With $p_{\tau,a} = \phi_z(v_a)_{(\tau,-)}$ we have $x_{(\tau,-)} = \sum_a\overline{w_{\tau,a}}\,p_{\tau,a}$, so $\gamma_z(x,x) = 2\sum_\tau\sum_{a,b}\overline{w_{\tau,a}}w_{\tau,b}\,G_\tau[a,b]$ with $G_\tau = (H_z(p_{\tau,a},p_{\tau,b}))_{a,b}$: an orthogonal sum over τ of 2 times a positive definite hermitian form on $\mathbb C^\rho$ with determinant $\det G_\tau$, whose determinant as a real quadratic form on $\mathbb R^{2\rho}$ is $(2^\rho\det G_\tau)^2$. The subgroup $M=\bigoplus_a\mathcal O_E v_a$ corresponds to $\mathcal O_E^\rho$

embedded by $(\tau)_{\tau\in\Phi}$, of covolume $(2^{-|\Phi|}D_E^{1/2})^\rho$ for Lebesgue measure on $(\mathbb{C}^\Phi)^\rho$ (Minkowski). Hence

$$\mathrm{covol}_{\gamma_z}(M) = (2^{-|\Phi|}D_E^{1/2})^\rho \prod_{\tau\in\Phi} 2^\rho \det G_\tau = D_E^{\rho/2} \prod_{\tau\in\Phi} \det G_\tau,$$

and $\mathrm{covol}_{\gamma_z}(I) = \mathrm{covol}_{\gamma_z}(M)/\iota(I)$. Since $\det G_\tau$ equals the determinant in the definition of $\mathrm{N}_z^\phi(I)$ (the transposed Gram matrix), the identity follows.

Step 2: uniform comparison. Let $B \subseteq W'$ be a compact neighbourhood of z_0 and V its interior. By continuity of $z \mapsto \gamma_z$ and compactness of B times the unit sphere of γ_{z_0}, there is $C \geq 1$ with $\gamma_z \geq C^{-1}\gamma_{z_0}$ for $z \in B$. For a subgroup of rank n the Gram determinants then satisfy $\mathrm{covol}_{\gamma_z} \geq C^{-n/2}\mathrm{covol}_{\gamma_{z_0}}$ (for positive definite forms $A \geq B$ on $\mathbb{R}^n$, $\det A \geq \det B$). With $2m' = \mathrm{rk}_{\mathbb{Z}}\Lambda$ we have $n \leq 2m'$ and $\rho \leq m'$ (as $\mathrm{rk}_{\mathbb{Z}} I = 2|\Phi|\rho \leq 2m'$). So if $z \in V$ and $\mathrm{N}_z^\phi(I) \leq 1$, then by Step 1

$$\mathrm{covol}_{\gamma_{z_0}}(I) \leq C^{m'}\,\mathrm{covol}_{\gamma_z}(I) \leq C^{m'} D_E^{\rho/2} \leq T := C^{m'} D_E^{m'/2}.$$

Step 3: finiteness. For a saturated subgroup $I \neq 0$ of rank n with $\mathbb{Z}$-basis $e_1, \ldots, e_n$, the vector $\omega_I = e_1 \wedge \cdots \wedge e_n$ of the lattice $\bigwedge^n \Lambda$ is determined by I up to sign, its norm for the inner product induced by γ_{z_0} on $\bigwedge^n \Lambda_{\mathbb{R}}$ is $\mathrm{covol}_{\gamma_{z_0}}(I)$, and $I = \{v \in \Lambda : v \wedge \omega_I = 0\}$ ($v \wedge \omega_I = 0$ iff v lies in the real span of I, and Λ meets this span in I by saturation). A lattice has only finitely many vectors of norm at most T, and n takes finitely many values. Hence the set F of non-zero saturated isotropic $\mathcal{O}_E$-submodules I admitting a family with $\mathrm{covol}_{\gamma_{z_0}}(I) \leq T$ is finite, and by Step 2, $\mathrm{N}_z^\phi(I) > 1$ for every $z \in V$ and every such $I \notin F$. □

Lemma D.13. *Let a base setting of $\mathcal{T}$ at (B, Z) be given whose cover $\mathcal{X}$ is proper over $\mathcal{O}_F$, and let $h^{\mathrm{ref}'}$ be a smooth cone function of multidegree a^{H}. Put*

$$\varphi' = \log h^{\mathrm{H}} + \tfrac{1}{2}U - \log h^{\mathrm{ref}'} \quad \text{on } X^{\mathrm{b},\circ}, \qquad \theta = dd^c \log h^{\mathrm{ref}'}, \qquad \Theta = dd^c \log h'$$

(the last two the first Chern forms on X^{b}). Then there are $C \geq 0$, $0 < t_0 < 1$ and smooth functions $\psi_{t'}$ on X^{b} $(0 < t' \leq t_0)$ such that

(Rw1) $\theta + dd^c\psi_{t'} + \frac{C}{\log(1/t')}\Theta \geq 0$ *on* X^{b};

(Rw2) $\psi_{t'} \geq \varphi'$ *on* $X^{\mathrm{b},\circ}$ *and* $|\psi_{t'}| \leq C$ *on* X^{b};

(Rw3) $\psi_{t'}(w) \to \varphi'(w)$ *as* $t' \to 0^+$, *for every* $w \in X^{\mathrm{b},\circ}$.

If moreover φ' is invariant under complex conjugation, the $\psi_{t'}$ can be chosen invariant under complex conjugation.

Proof. Let X be the set of all complex points of $\mathcal{X}$ (over every embedding $\kappa\colon F \to \mathbb{C}$, as in (Gb1)) with its analytic structure, the disjoint union over the finitely many κ of the complex points of $\mathcal{X} \otimes_{\mathcal{O}_F,\kappa} \mathbb{C}$; as $\mathcal{X}$ is proper over $\mathcal{O}_F$, each of these is compact, hence so is X, so the continuous surjection $y_{\mathbb{C}}\colon X \to X^{\mathrm{b}}$ of (Gb1) is proper. Put $Y^\circ = y_{\mathbb{C}}^{-1}(X^{\mathrm{b},\circ})$; by (Gb1) it lies in the abelian locus, and $y_{\mathbb{C}}\colon Y^\circ \to X^{\mathrm{b},\circ}$ is proper and surjective, hence a closed map and a quotient map. We check the hypotheses of Lemma D.14 for X^{b}, Θ, θ, $Z = \partial X^{\mathrm{b}}$ and $\varphi = \varphi'$: X^{b} is a compact complex manifold (Hy1); Θ is a Kähler form, being closed, real and positive (Hy5); θ is a smooth closed real $(1,1)$-form ($h^{\mathrm{ref}'}$ is smooth); ∂X^{b} is a divisor with normal crossings, a closed analytic subset with empty interior. It remains to show that φ' is bounded and locally of the form (plurisubharmonic) $-q$ with $dd^c q = \theta$.

Step 1: the corrected weight in a chart. Let $z \in X$. By (Gb1), z lies in a chart c', and by (Gb3) in a principal open subset $D(f)$ of $\mathcal{X}_{c'}$ with a unit u_f and frames b_τ of $\mathcal{V}_{(\tau,-)}$ on $D(f)$; on the abelian locus these are holomorphic frames of $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X}})^{\mathrm{an}}$, and the polynomial $\sigma_{c'} = \prod_k y_{k,c'_k}^{a^{\mathrm{H}}_k}$ of multidegree a^{H} does not vanish at the points of $y_{\mathbb{C}}(D(f)(\mathbb{C}))$ (they lie in the chart). On the open set $\{\sigma_{c'} \neq 0\} \subseteq X^{\mathrm{b}}$ put $q_{c'} = \log h^{\mathrm{ref}'} - \log|\sigma_{c'}| = -\log\|\sigma_{c'}\|_{h^{\mathrm{ref}'}}$, a smooth function (homogeneous of degree 0 on the cone) with $dd^c q_{c'} = \theta$, since $\log|\sigma_{c'}|$ is pluriharmonic where $\sigma_{c'} \neq 0$. For $z \in D(f)(\mathbb{C}) \cap Y^\circ$ and $w = y_{\mathbb{C}}(z)$, (Gb3) gives $\|\sigma_{c'}\|_{h^{\mathrm{H}}}(w) = |u_f(z)| \prod_\tau \det\Gamma(b_\tau(z))^{-1/2}$, i.e. $\log h^{\mathrm{H}} = \log|\sigma_{c'}| - \log|u_f(z)| + \frac{1}{2}\sum_\tau \log\det\Gamma(b_\tau(z))$, and therefore

$$\varphi'(w) + q_{c'}(w) = u(z) - \log|u_f(z)|, \qquad (*)$$

with $u = \frac{1}{2}\sum_\tau \log\det\Gamma(b_\tau) + \frac{1}{2}U \circ y_{\mathbb{C}}$ the function of Lemma D.10.

Step 2: boundedness. For $x \in X$ take $D(f) \ni x$ as in Step 1. By Lemma D.15 there are an open neighbourhood W'_x of x, which we may take relatively compact in $D(f)(\mathbb{C})$ with $\overline{y_{\mathbb{C}}(W'_x)} \subseteq \{\sigma_{c'} \neq 0\}$, and C_x with $|u| \leq C_x$ on W'_x in the abelian locus; as u_f is continuous and nowhere zero on $D(f)(\mathbb{C})$ and $q_{c'}$ is continuous near $\overline{y_{\mathbb{C}}(W'_x)}$, $|\log|u_f|| + |q_{c'} \circ y_{\mathbb{C}}| \leq C'_x$ on W'_x. Finitely many W'_x cover the compact X. Every $w \in X^{\mathrm{b},\circ}$ is $y_{\mathbb{C}}(z)$ for some $z \in Y^\circ$, and z lies in one of these W'_x; by $(*)$, $|\varphi'(w)| \leq A_0 := \max_x(C_x + C'_x)$.

Step 3: continuity. By $(*)$ and the continuity of u on the abelian locus (Lemma D.10), $\varphi' \circ y_{\mathbb{C}} = u - \log|u_f| - q_{c'} \circ y_{\mathbb{C}}$ is continuous on each $D(f)(\mathbb{C}) \cap Y^\circ$, hence on Y°; as $y_{\mathbb{C}} \colon Y^\circ \to X^{\mathrm{b},\circ}$ is a quotient map, φ' is continuous on $X^{\mathrm{b},\circ}$.

Step 4: plurisubharmonicity. The map y restricts to a birational morphism of the generic fibres onto $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ (Gb1), which is an isomorphism over a dense Zariski open subset; let $A \subseteq X^{\mathrm{b}}$ be the complex points of its complement, a nowhere dense closed analytic subset. Let $w_0 \in X^{\mathrm{b},\circ}$, choose $z_0 \in y_{\mathbb{C}}^{-1}(w_0)$ and a chart c' in which z_0 lies, so $\sigma_{c'}(w_0) \neq 0$, and let $N \ni w_0$ be a connected open subset of $X^{\mathrm{b},\circ} \cap \{\sigma_{c'} \neq 0\}$. Put $v = \varphi' + q_{c'}$ on N. On $N \smallsetminus A$, v is plurisubharmonic: let $w \in N \smallsetminus A$, z its unique preimage, and c'', $D(f'')$, $u_{f''}$, frames as in Step 1 for z; then $\sigma_{c''}(w) \neq 0$, and near w, by $(*)$ for c'', $v = (u - \log|u_{f''}|) \circ y_{\mathbb{C}}^{-1} + (q_{c'} - q_{c''})$, where $q_{c'} - q_{c''} = \log|\sigma_{c''}/\sigma_{c'}|$ is pluriharmonic where both polynomials (of the same multidegree) are non-zero; here $y_{\mathbb{C}}^{-1}$ is biholomorphic onto an open subset of the manifold $X \smallsetminus y_{\mathbb{C}}^{-1}(A)$, u is plurisubharmonic there (Lemma D.10) and $\log|u_{f''}|$ is pluriharmonic. Moreover v is continuous on N (Step 3) and bounded (Step 2). By the extension theorem, $v|_{N \smallsetminus A}$ extends to a plurisubharmonic function on N whose value at $a \in A \cap N$ is $\limsup_{x \to a,\, x \notin A} v(x) = v(a)$ (A nowhere dense, v continuous). So $v = \varphi' + q_{c'}$ is plurisubharmonic on N, with $q_{c'}$ smooth and $dd^c q_{c'} = \theta$; this is the local hypothesis of Lemma D.14 with $q = q_{c'}$.

Step 5. Lemma D.14 applies with the bound A_0 of Step 2 and gives C, t_0 and smooth $\psi_{t'}$ with (B1)–(B3), which are (Rw1)–(Rw3) ($\delta(t') = C/\log(1/t')$). Complex conjugation σ of $X^{\mathrm{b}} = \mathcal{X}^{\mathrm{b}}(\mathbb{C})$ is an antiholomorphic involution with $\sigma(\partial X^{\mathrm{b}}) = \partial X^{\mathrm{b}}$ (Hy1); cone functions are invariant under conjugation, so $\log h^{\mathrm{ref}'} \circ \sigma = \log h^{\mathrm{ref}'}$ and $\log h' \circ \sigma = \log h'$, whence $\sigma^*\theta = -\theta$ and $\sigma^*\Theta = -\Theta$ ($\sigma^*\partial\bar\partial = \bar\partial\partial\sigma^*$). If $\varphi' \circ \sigma = \varphi'$, the last assertion of Lemma D.14 gives invariant $\psi_{t'}$. □

Lemma D.14. *Let X be a compact complex manifold with a Kähler form Θ, θ a smooth closed real $(1,1)$-form on X, $Z \subseteq X$ a closed analytic subset with empty interior, $X^\circ = X \smallsetminus Z$, and $\varphi \colon X^\circ \to \mathbb{R}$ a function with $|\varphi| \leq A$ such that every point of X° has an open neighbourhood W and a smooth q on W with $dd^c q = \theta$ and $q + \varphi$ plurisubharmonic on W. Then there are $C \geq 0$, $0 < t_0 < 1$ and smooth functions ψ_t on X $(0 < t \leq t_0)$ such that, with $\delta(t) = C/\log(1/t)$:*

(B1) $\theta + dd^c\psi_t + \delta(t)\Theta \geq 0$ on X;

(B2) $\varphi \le \psi_t$ *on* X° *and* $|\psi_t| \le C$ *on* X*;*

(B3) $\psi_t(x) \to \varphi(x)$ *as* $t \to 0^+$*, for every* $x \in X^\circ$*.*

If moreover $\sigma\colon X \to X$ *is an antiholomorphic involution with* $\sigma(Z) = Z$*,* $\sigma^*\theta = -\theta$*,* $\sigma^*\Theta = -\Theta$ *and* $\varphi \circ \sigma = \varphi$*, the* ψ_t *can be chosen with* $\psi_t \circ \sigma = \psi_t$*.*

Proof. Write $\ell = \log(1/t)$; we take $t \le e^{-e}$, so $\ell \ge e$ and $t \le 1/\ell$ (as $t\ell \le 1/e$).

Step 1: charts. Every $p \in X$ has holomorphic coordinates w centred at p, defined on $\{|w| < 3r_p\}$ for some $r_p > 0$, on which there are smooth q, ψ with $dd^c q = \theta$, $dd^c\psi = \Theta$ (a closed real $(1,1)$-form on a polydisc is dd^c-exact). Put $\kappa_p = w/r_p$. By compactness finitely many of these, $\kappa_j\colon U_j \to \Delta_3$ $(1 \le j \le N)$ with potentials q_j of θ, satisfy $X = \bigcup_j U_j^1$, $U_j^r = \kappa_j^{-1}(\Delta_r)$. Let Q bound $|q_j|$ on $\kappa_j^{-1}(\bar\Delta_2)$ for all j, and put $A' = A + Q$.

Step 2: extension across Z*.* On $U_j \cap X^\circ$, $q_j + \varphi$ is plurisubharmonic: near each point it differs from $q + \varphi$ of the hypothesis by the pluriharmonic function $q_j - q$ (a real function with $dd^c = 0$). The set $Z_j = \kappa_j(Z \cap U_j)$ is a closed analytic subset of Δ_3 with empty interior, hence closed and pluripolar, and $(q_j + \varphi) \circ \kappa_j^{-1}$ is bounded above near it. By the removable singularity theorem of Grauert and Remmert ([145], Ch. 2) it extends to a plurisubharmonic function u_j on Δ_3, given on Z_j by $u_j(z) = \limsup_{y \to z,\, y \notin Z_j}(q_j + \varphi)(\kappa_j^{-1}y)$. Let $\tilde\varphi = \varphi$ on X° and $\tilde\varphi(x) = \limsup_{y \to x,\, y \in X^\circ} \varphi(y)$ on Z; then $|\tilde\varphi| \le A$, $u_j = (q_j + \tilde\varphi) \circ \kappa_j^{-1}$ on Δ_3, and $|u_j| \le A'$ on Δ_2.

Step 3: local regularisation. Let $M_j(z,r) = \max_{\bar\Delta(z,r)} u_j$ for $z \in \Delta_{5/4}$, $0 < r \le 1/2$; then $-A' \le M_j(z,r) \le A'$. Let $\Lambda > 1$ and $\ell \ge 2\log(2\Lambda)$. With $a = \log(t/\Lambda) < b = \log(\Lambda t) < c = \log(1/2)$ we have $b - a = 2\log\Lambda$ and $c - b \ge \ell/2$, and the convexity of $\sigma \mapsto M_j(z, e^\sigma)$ (Lemma 33.24(P1)) gives

$$M_j(z,\Lambda t) - M_j(z, t/\Lambda) \le (b-a)\frac{M_j(z,1/2) - M_j(z,\Lambda t)}{c-b} \le \frac{8A'\log\Lambda}{\ell}. \tag{1}$$

For $t \le 1/8$ put $u_{j,t} = u_j * \rho_t$ on $\Delta_{5/4}$. By Lemma 33.24(P2) it is smooth and plurisubharmonic with $u_j \le u_{j,t} \le M_j(\cdot,t)$; by (P3), (P1) and (1) for $\Lambda = 8$,

$$u_{j,t}(z) \ge M_j(z,2t) - K_\rho(M_j(z,2t) - M_j(z,t/4)) \ge M_j(z,t) - \frac{8K_\rho A'\log 8}{\ell}. \tag{2}$$

Step 4: comparison of charts. Put $f_{t,j} = u_{j,t} \circ \kappa_j - q_j$ on $U_j^{5/4}$: smooth, with $\theta + dd^c f_{t,j} \ge 0$ since $q_j + f_{t,j} = u_{j,t} \circ \kappa_j$ is plurisubharmonic. Let Λ_0, G, t_* be the constants of Lemma 33.26 for θ, the charts κ_j and the q_j; it applies to $\tilde\varphi$, which is bounded above. Let $t_0 \le \min(e^{-e}, 1/8, t_*)$ with $\log(1/t_0) \ge 2\log(16\Lambda_0)$. For $x \in U_j^{5/4} \cap U_k^{5/4}$ and $0 < t \le t_0$, by (2) for j and k, Lemma 33.26 and (1) for $\Lambda = \Lambda_0$ (if $\Lambda_0 = 1$ the oscillation is 0),

$$|f_{t,j}(x) - f_{t,k}(x)| \le \frac{16K_\rho A'\log 8}{\ell} + \frac{8A'\log\Lambda_0}{\ell} + 2G\Lambda_0 t \le \frac{C'}{\ell} =: \delta_0(t),$$

with $C' = 16K_\rho A'\log 8 + 8A'\log\Lambda_0 + 2G\Lambda_0 + 1$, using $t \le 1/\ell$. Moreover $t \le \delta_0(t)$.

Step 5: gluing and bounds. Let K be the constant of Lemma 33.25 for (X, Θ, θ) and the charts κ_j, and let F_t be the function of Lemma 33.25(G1) for $\delta = \delta_0(t)$ and $(f_{t,j})_j$. Then F_t is smooth, $\theta + dd^c F_t + K\delta_0(t)\Theta \ge 0$, and $f_{t,j} \le F_t \le f_{t,j} + 3\delta_0(t)$ on $U_j^{5/4}$. Let $x \in X$, $x \in U_j^1$, $z = \kappa_j x$. Then

$$\tilde\varphi(x) = u_j(z) - q_j(x) \le f_{t,j}(x) \le F_t(x) \le M_j(z,t) - q_j(x) + 3\delta_0(t),$$

so $-A \le F_t \le A' + Q + 3C'$, and $\varphi \le F_t$ on X°. By (P1), $M_j(z,t) \to u_j(z)$ as $t \to 0^+$, so $F_t(x) \to \tilde{\varphi}(x)$ for every $x \in X$, in particular $F_t \to \varphi$ on X°. With $C = \max(KC', A+2Q+3C')$ and $\psi_t = F_t$, (B1)–(B3) hold.

Step 6: the involution. Put $\psi_t = \frac{1}{2}(F_t + F_t \circ \sigma)$, smooth. As σ is antiholomorphic, $\sigma^* \partial = \bar{\partial} \sigma^*$, so $dd^c(F_t \circ \sigma) = -\sigma^* dd^c F_t$ and

$$\theta + dd^c(F_t \circ \sigma) + \delta(t)\Theta = -\sigma^*\big(\theta + dd^c F_t + \delta(t)\Theta\big) \ge 0,$$

because σ^* maps a semipositive real $(1,1)$-form $\sum i\alpha_k \wedge \bar{\alpha}_k$ (α_k of type $(1,0)$) to $\sum i\beta_k \wedge \bar{\beta}_k$ with $\beta_k = \sigma^* \alpha_k$ of type $(0,1)$, which is seminegative. Averaging keeps (B1). Since $\sigma(X^\circ) = X^\circ$ and $\varphi \circ \sigma = \varphi$: $\varphi(x) = \varphi(\sigma x) \le F_t(\sigma x)$, and $F_t(\sigma x) \to \varphi(\sigma x) = \varphi(x)$; so (B2), (B3) hold for ψ_t, and $\psi_t \circ \sigma = \psi_t$. □

Lemma D.15. *Let a base setting with cusp data be given, X a complex fibre of its cover, $x \in X$, W an open neighbourhood of x and b_τ ($\tau \in \Phi$) holomorphic frames of $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X}})^{\mathrm{an}}$ on W, and let u be as in Lemma D.10. Then there are an open neighbourhood $W' \subseteq W$ of x and $C \ge 0$ with $|u| \le C$ on $W' \cap$ (abelian locus). (By (Gb3) this says that the corrected weight minus the logarithm of a smooth reference cone function is bounded near every point of X^{b}.)*

Proof. If x lies in the abelian locus, u is continuous near x by Lemma D.10, hence bounded on a neighbourhood. Let x lie outside the abelian locus. Choose a chart $t\colon V \to \Delta^d$ at x, $V \subseteq W$, as in Lemma 33.16, with $V^* = V \cap$ (abelian locus) $= \{t_1 \cdots t_a \ne 0\}$, $a \ge 1$, and put $L = 1 + \sum_{j \le a} |\log |t_j||$ on V^*.

Upper bound. By Lemma 33.17 (applied to each frame b_τ, of q sections linearly independent at x), $\Gamma(b_\tau(z)) \le C_1 I_q$ near x, so $\frac{1}{2}\sum_\tau \log\det\Gamma(b_\tau(z)) \le \frac{1}{2}|\Phi| q \log C_1$; by Lemma D.16, $U(y_{\mathbb{C}}(z)) \le C_2(1 + \log L(z))$ near x. Hence there are $0 < r < e^{-1}$ and C_4 with $\Delta_r = \{|t_j| < r \ \forall j\} \subseteq V$ and $u \le C_4(1 + \log L)$ on $\Delta_r \cap V^*$. Fix $0 < \rho < r$; the distinguished boundary $T_\rho = \{|t_j| = \rho \ \forall j\}$ is a compact subset of V^*, so $M_\rho = \max_{T_\rho} u$ is finite (u is continuous on V^*). For $\varepsilon > 0$ the function $v_\varepsilon = u + \varepsilon \sum_{j \le a} \log|t_j|$ is plurisubharmonic on $\Delta_r \cap V^*$ (Lemma D.10; $\log|t_j|$ is pluriharmonic there), and $v_\varepsilon \le C_4(1 + \log L) - \varepsilon(L-1)$ is bounded above on $\Delta_r \cap V^*$, since $\log L = o(L)$. By the extension theorem v_ε extends to a plurisubharmonic function on Δ_r, and by the maximum principle on $\bar{\Delta}_\rho$, for $z \in \Delta_\rho \cap V^*$,

$$u(z) + \varepsilon \sum_{j \le a} \log|t_j(z)| \ \le \ \max_{T_\rho} v_\varepsilon = \max_{T_\rho} u + \varepsilon\, a \log\rho \ \le \ M_\rho.$$

Letting $\varepsilon \to 0$ gives $u \le M_\rho$ on $\Delta_\rho \cap V^*$.

Lower bound. As x is not in the abelian locus, $G_{\mathcal{X},x}$ is not abelian, so its torus part has positive dimension and the vanishing cycles I at x (Lemma 33.18) are non-zero. By (Cu), $U(y_{\mathbb{C}}(z)) \ge -\log \mathrm{N}_z(I)$ on V^*, so $u(z) \ge \frac{1}{2}\log\big(\prod_\tau \det\Gamma(b_\tau(z))/\mathrm{N}_z(I)\big)$, which by (Cn) tends to $\frac{1}{2}\log\nu'$ ($\nu' > 0$) as $z \to x$ in V^*; so u is bounded below on $V'' \cap V^*$ for a neighbourhood V'' of x. With $W' = \Delta_\rho \cap V''$ both bounds hold. □

Lemma D.16. *In this setting put $L(z) = 1 + \sum_{j=1}^a |\log|t_j(z)||$ for $z \in V^*$. Then there are $C \ge 0$ and a neighbourhood V' of x such that*

$$U(y_{\mathbb{C}}(z)) \ \le \ C\,(1 + \log L(z)) \qquad (z \in V' \cap V^*).$$

Proof. **Step 1: the cusp weight by the first minimum.** Let $z \in V^*$ and $w = y_{\mathbb{C}}(z)$. By (Gb5) of Definition 31.11 the cusp lattice Λ_w is identified with $\Lambda_z = H_1(G_{\mathcal{X},z}(\mathbb{C}),\mathbb{Z}) \subset \mathrm{Lie}(G_{\mathcal{X},z})$, $\langle\cdot,\cdot\rangle_w$ with $\mathrm{Im}\, H_z$, and the cusp norm $\mathrm{N}_w(J)$ of a non-zero saturated isotropic $\mathcal{O}_E$-submodule J with $\frac{1}{\iota(J)}\prod_\tau \det\big(H_z((v_a^J)_{\sigma_\tau},(v_b^J)_{\sigma_\tau})\big)$ (since $\overline{\lambda_{w,\tau}(v)}^{\mathsf{T}}\lambda_{w,\tau}(v') = H_z(v'_{\sigma_\tau}, v_{\sigma_\tau})$ in orthonormal coordinates, and a hermitian matrix and its transpose have the same determinant). We apply Lemma D.3 with $V = \mathrm{Lie}(G_{\mathcal{X},z})$, $H = H_z$, the lattice Λ_z, the $\mathcal{O}_E$-action (adjoint for H_z to the conjugate action, by the Rosati condition of the polarisation), $\Psi = \{\sigma_\tau : \tau \in \Phi\}$, a CM type (its members are the $\kappa\circ\bar\tau = \overline{\kappa\circ\tau}$, κ the embedding of F over which z lies, and $\{\kappa\circ\tau : \tau\in\Phi\}$ is a CM type as Φ is one and κ is injective), with $\dim V_{\sigma_\tau} = q$, and to the family v^J of index $\iota(J)$ in J. It gives $-\log \mathrm{N}_w(J) \le fq(2\max(0,\log\frac{1}{\lambda_1(z)}) + \log(2fq\,\mathrm{rd}_E))$, $\lambda_1(z)^2 = \min\{H_z(\lambda,\lambda) : 0\neq\lambda\in\Lambda_z\}$, for every such J; hence, by (Hu2),

$$U(w) \le fq\Big(2\max\big(0,\log\tfrac{1}{\lambda_1(z)}\big) + \log(2fq\,\mathrm{rd}_E)\Big).$$

Step 2: Hodge norms of lattice vectors. For $\lambda\in\Lambda_z$ (a real vector of $\Lambda_z\otimes\mathbb{C}$) write $\lambda = f_0 + g_0$ with $f_0\in\mathcal{F}_z^0$ and $g_0\in\overline{\mathcal{F}_z^0}$, the h-orthogonal complement of $\mathcal{F}_z^0$; then $g_0 = \overline{f_0}$, $h(f_0,f_0) = h(g_0,g_0)$, and λ and g_0 have the same image λ in $\mathrm{Lie}(G_{\mathcal{X},z})$, so $|\lambda|_t^2 = h(\lambda,\lambda) = 2h(g_0,g_0) = H_z(\lambda,\lambda)$. So $\lambda_1(z)^2$ is the minimum of the Hodge norms $|\lambda|_t^2$ over the non-zero integral vectors.

Step 3: the lower bound $\lambda_1(z)^2 \ge c\,L(z)^{-4}$. Identify V with Δ^d by t, put $y_j = -\log|t_j|$ and choose a neighbourhood with $|t_j|\le e^{-1}$, so $y_j\ge 1$. For each numbering $(j_1,\dots,j_a)$ the filtrations $W(\nu)$ of Theorem B.147 form a distributive family of filtrations of the constant space $H = H_{\mathbb{Z}}\otimes\mathbb{C}$, so they have a common splitting $H = \bigoplus_p U_p$ (linear algebra of distributive families); it is constant, hence a C^∞ splitting of $\widehat{\mathcal{H}}$ in the canonical frame $\tilde s$; let $\|\cdot\|$ be the hermitian norm making the frame $\tilde s$ orthonormal. As $N'_\nu = N_{j_1}+\cdots+N_{j_\nu}$ has square 0, $W(\nu)$ has at most three non-zero graded pieces, with consecutive indices $c_\nu-1, c_\nu, c_\nu+1$ for its centre $c_\nu\in\{-1,0\}$; so every index p_ν occurring is ≥ -2. Apply Theorem B.147 with $\varepsilon = \frac12$ to each of the finitely many numberings, and take the largest constant C_1 and the intersection V_1 of the neighbourhoods. For $t\in V_1\cap V^*$ choose the numbering with $y_{j_1}\le\cdots\le y_{j_a}$ (so the ratios $y_{j_\nu}/y_{j_{\nu-1}}\ge 1 > \varepsilon$); every factor of $m_p(t)$ has base ≥ 1 and exponent ≥ -2, so $m_p(t)\ge y_{j_1}^{-2}\prod_{\nu\ge2}(y_{j_\nu}/y_{j_{\nu-1}})^{-2} = y_{j_a}^{-2}\ge L(t)^{-2}$, and, with $c_2>0$ such that $\sum_p\|u_p\|^2\ge c_2\|u\|^2$ for the finitely many constant splittings,

$$|u|_t^2 \;\ge\; C_1^{-1}L(t)^{-2}\sum_p\|u_p\|^2 \;\ge\; C_1^{-1}c_2\,L(t)^{-2}\|u\|^2 \qquad (u\in\widehat{\mathcal{H}}_t).$$

Let $0\neq\lambda\in\Lambda_t$, a flat section on a slit neighbourhood; on the branch with $0\le\mathrm{Re}\,z_j<1$, $z_j = \frac{\log t_j}{2\pi i}$, $\lambda = \sum_k c_k s_k$ with $c\in\mathbb{Z}^{2g}\smallsetminus\{0\}$ in a flat $\mathbb{Z}$-basis s of $H_{\mathbb{Z}}$, and $s_k = \exp(\sum_j z_jN_j)\tilde s_k$. Since $N_iN_j = 0$ for all i,j ($N_\kappa^2=0$ for every κ and the N_j commute), $\exp(-\sum_j z_jN_j) = 1-\sum_j z_jN_j$ has operator norm at most $1+C_2\sum_j|z_j|\le C_3L(t)$ in the coordinates of $\tilde s$, as $|z_j|\le 1+y_j/(2\pi)$. Hence $\|\lambda\|\ge|c|/(C_3L(t))\ge 1/(C_3L(t))$, and

$$\lambda_1(t)^2 \;\ge\; C_1^{-1}c_2C_3^{-2}\,L(t)^{-4}.$$

Step 4. By Steps 1 and 3, on $V' = V_1$, $2\max(0,\log\frac{1}{\lambda_1}) \le \max\big(0, 4\log L + \log(C_1C_3^2/c_2)\big)$, and $U(y_{\mathbb{C}}(z))\le C(1+\log L(z))$ with $C = fq\big(4+\log^+(C_1C_3^2/c_2)+\log(2fq\,\mathrm{rd}_E)\big)$. □

Lemma D.17. *Let a Hodge-type base setting with cusp data be given; put $\Theta = dd^c\log h'$ (a Kähler form on X^{b} by (Hy5)), $\theta = dd^c\log h^{\mathrm{ref}}$ and $\varphi = \log(h^{\mathrm{H}}/h^{\mathrm{ref}})$ on $X^{\mathrm{b},\circ}$. Suppose:*

(i) *there is a holomorphic section ς of a smooth hermitian line bundle on X^{b} with $\|\varsigma\| \le 1$ whose zero divisor has simple normal crossings and support ∂X^{b}; put* $\mathrm{rW} = e - 2\log\|\varsigma\|$ *on $X^{\mathrm{b},\circ}$;*

(ii) *φ is smooth on $X^{\mathrm{b},\circ}$, and there are $C \ge 1$, $0 < t_0 < 1$ and functions $\delta, \varepsilon\colon (0,t_0] \to [0,\infty)$, $R\colon (0,t_0] \to [e,\infty)$ with $\delta(t), \varepsilon(t) \to 0$, $R(t) \to \infty$ as $t \to 0^+$, such that $u = \varphi$, $u_t = \varphi_t$ satisfy (R2)–(R5) and $\theta + dd^c\varphi_t + \varepsilon(t)\Theta \ge 0$ on X^{b} for $0 < t \le t_0$;*

(iii) *U is continuous on $X^{\mathrm{b},\circ}$ and $U \le C \log \mathrm{rW}$ there;*

(iv) *$|\psi_{t'}| \le C$ on X^{b} for small $t' > 0$, and $\psi_{t'} \to \varphi' = \log h^{\mathrm{H}} + \frac{1}{2}U - \log h^{\mathrm{ref}'}$ pointwise on $X^{\mathrm{b},\circ}$ as $t' \to 0^+$.*

Then for every integer B: the limit self-intersection $\mathrm{ar}^{\mathrm{b}}(B)$ *exists and the setting has (Mz1), (Mz2), (Mz3) at B. Moreover $(\omega^{\mathrm{H}})^e$ and $U(\omega^{\mathrm{H}})^e$ are integrable on $X^{\mathrm{b},\circ}$, $\int_{X^{\mathrm{b},\circ}}(\omega^{\mathrm{H}})^e = \deg^{\mathrm{H}}$, and if $\deg^{\mathrm{H}} \ne 0$ the setting has regularisation limits (Mz1)–(Mz4) at B.*

Proof. Write $X = X^{\mathrm{b}}$, $X^\circ = X^{\mathrm{b},\circ}$; ∂X is the zero set of ς, a proper analytic subset, hence null. $X = \mathcal{X}^{\mathrm{b}}(\mathbb{C})$ for the arithmetic variety $\mathcal{X}^{\mathrm{b}}$ (closed integral in a multiprojective space, flat over $\mathbb{Z}$, of dimension $e+1$, smooth generic fibre: (Hy1)), and $e \ge 1$.

Step 0: the regularisation data. Let $I = \{\mathrm{H}, \mathrm{A}\}$ with

$$(\theta_{\mathrm{H}}, u_{\mathrm{H}}, u_{\mathrm{H},t}) = (\theta, \varphi, \varphi_t), \qquad (\theta_{\mathrm{A}}, u_{\mathrm{A}}, u_{\mathrm{A},t}) = (\Theta, 0, 0).$$

Shrinking t_0 so that $\varepsilon \le 1 \le C$ on $(0,t_0]$, (R1) holds for H by (ii), and for A as $\Theta \ge 0$; (R2)–(R5) hold by (ii), resp. trivially (as $\mathrm{rW} \ge e$). The φ_t are invariant under complex conjugation (Hy4). On the cone over X, $h^0_{\mathrm{H}} = h^{\mathrm{ref}} \prod_{\ell \in S_B} |g_\ell|$ is a smooth cone function of multidegree a^{H}_B, invariant under conjugation (h^{ref} is a cone metric and the g_ℓ are quotients of polynomials with rational coefficients), with $c_1 = \theta$; and h' is one of multidegree a' with $c_1 = \Theta$. The metric of $\overline{\mathcal{H}}^{\mathrm{b}}_{B,t}$ is $h^0_{\mathrm{H}} e^{\varphi_t}$, that of $\overline{\mathcal{N}}_{B,i_1,i_2,t}$ is $(h^0_{\mathrm{H}})^{i_2} h'^{i_1} e^{i_2\varphi_t}$, with first Chern forms $\theta + dd^c\varphi_t$ and $i_2(\theta + dd^c\varphi_t) + i_1\Theta$. On X°, $\omega^{\mathrm{H}} = dd^c \log h^{\mathrm{H}} = \theta + dd^c\varphi$.

Step 1: mixed limits and $\mathrm{ar}^{\mathrm{b}}(B)$. For $0 \le k \le e+1$ apply Lemma D.18 with $D = e$, the data of Step 0, $\mathcal{M}_1 = \cdots = \mathcal{M}_{e+1-k} = \mathcal{O}(a^{\mathrm{H}}_B)|_{\mathcal{X}^{\mathrm{b}}}$ with h^0_{H} and $n = (1,0)$, and $\mathcal{M}_{e+2-k} = \cdots = \mathcal{M}_{e+1} = \mathcal{O}(a')|_{\mathcal{X}^{\mathrm{b}}}$ with h' and $n = (0,1)$: the limit

$$L_k = \lim_{t \to 0^+} \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{H}}^{\mathrm{b}}_{B,t})^{e+1-k} \widehat{c}_1(\overline{\mathcal{A}}')^k)$$

exists. In particular $\mathrm{ar}^{\mathrm{b}}(B) = L_0$ exists.

Step 2: (Mz1). For $1 \le i_1 \le i_2$, $c_1(\overline{\mathcal{N}}_{B,i_1,i_2,t}) = i_2(\theta + dd^c\varphi_t + \varepsilon(t)\Theta) + (i_1 - i_2\varepsilon(t))\Theta \ge 0$ as soon as $\varepsilon(t) \le i_1/i_2$, which holds for $0 < t \le t_1$ with some $t_1 > 0$ since $\varepsilon(t) \to 0$.

Step 3: (Mz3). Since $\widehat{c}_1(\overline{\mathcal{N}}_{B,i_1,i_2,t}) = i_2\widehat{c}_1(\overline{\mathcal{H}}^{\mathrm{b}}_{B,t}) + i_1\widehat{c}_1(\overline{\mathcal{A}}')$, multilinearity gives

$$\mathrm{tN}(i_1,i_2,t) = \sum_{k=0}^{e+1} \binom{e+1}{k} \Big(\frac{i_1}{i_2}\Big)^k \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{H}}^{\mathrm{b}}_{B,t})^{e+1-k} \widehat{c}_1(\overline{\mathcal{A}}')^k)$$

$$\longrightarrow \mathrm{tN}^\infty(i_1,i_2) = \sum_{k=0}^{e+1} \binom{e+1}{k} \Big(\frac{i_1}{i_2}\Big)^k L_k.$$

With $\omega_t = \theta + dd^c\varphi_t + \frac{i_1}{i_2}\Theta$ and $\omega_\infty = \omega^{\mathrm{H}} + \frac{i_1}{i_2}\Theta$ on X°, $\omega_t^e = \sum_k \binom{e}{k}(\frac{i_1}{i_2})^k (\theta + dd^c\varphi_t)^{e-k} \wedge \Theta^k$, each term a form μ_t of Lemma 31.35 for the data of Step 0 (indices H and A), with limit form

$\mu = (\omega^{\mathrm{H}})^{e-k} \wedge \Theta^k$. By (iii), $g_t = g = \frac{1}{2}U$ is continuous with $|g| \le C \log \mathrm{rW}$ ($U \ge 0$), so Lemma 31.35(2) gives $V^{\mathrm{b}}(t) = \int_{X^\circ} \frac{1}{2}U\,\omega_t^e \to V^{\mathrm{b},\infty}(i_1,i_2) = \int_{X^\circ} \frac{1}{2}U\,\omega_\infty^e$, the latter absolutely convergent by Lemma 31.35(1) ($\frac{1}{2}U \le C \log \mathrm{rW}$). (That $V^{\mathrm{b}}(t)$ exists is shown in Step 4.)

Step 4: (Mz2). Fix $1 \le i_1 \le i_2$ and $0 < t \le t_0$ with $c_1(\overline{\mathcal{N}}_{B,i_1,i_2,t}) \ge 0$, so that ω_t^e is a positive smooth measure on the compact X, of finite mass. $V^{\mathrm{b}}(t)$ exists: $0 \le \frac{1}{2}U \le C \log \mathrm{rW}$, $\omega_t^e \le c\,\Theta^e$ for some c (a smooth top form on a compact manifold), and $(\log \mathrm{rW})\Theta^e$ is integrable by Lemma 31.35(1) (index A only). The integrand of $I(t,t')$ is $F_{t'} = \log(h^{\mathrm{ref}'}/h^{\mathrm{ref}}) + \psi_{t'} - \varphi_t$, a function on X (equal multidegrees), with $|F_{t'}| \le \max_X |\log(h^{\mathrm{ref}'}/h^{\mathrm{ref}})| + C + \max_X |\varphi_t|$ independent of t' (continuous functions on the compact X), and by (iv), on X°, $F_{t'} \to \log(h^{\mathrm{ref}'}/h^{\mathrm{ref}}) + \varphi' - \varphi_t = \log h^{\mathrm{H}} + \frac{1}{2}U - \log h^{\mathrm{ref}} - \varphi_t$, the integrand of $V(t)$, which is therefore bounded by the same constant. As ∂X is null, dominated convergence gives that $V(t)$ exists and $I(t,t') \to V(t)$ as $t' \to 0^+$. Finally the integrand of $V(t)$ minus that of $V^{\mathrm{b}}(t)$ is $\varphi - \varphi_t \le 0$ by (R2), and $\omega_t^e \ge 0$, so $V(t) \le V^{\mathrm{b}}(t)$. This holds with $t_2 = t_0$.

Step 5: the Hodge mass and (Mz4). Lemma 31.35(1) with the index H taken e times gives that $(\omega^{\mathrm{H}})^e$ and $(\log \mathrm{rW})(\omega^{\mathrm{H}})^e$ are absolutely integrable on X°, hence so is $U(\omega^{\mathrm{H}})^e$ by (iii), and $\int_{X^\circ}(\omega^{\mathrm{H}})^e = \int_X \theta^e = \deg(c_1(\mathcal{O}(a^{\mathrm{H}}))^e \cap [\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}]) = \deg^{\mathrm{H}}$ (θ represents the first Chern class of $\mathcal{O}(a^{\mathrm{H}})$ on the complex points of $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$). By Step 3 with $(i_1,i_2) = (1,m)$, $\mathrm{tN}^\infty(1,m) = \sum_k \binom{e+1}{k} m^{-k} L_k \to L_0 = \mathrm{ar}^{\mathrm{b}}(B)$ as $m \to \infty$, and

$$V^{\mathrm{b},\infty}(1,m) = \sum_{k=0}^{e} \binom{e}{k} m^{-k} \int_{X^\circ} \tfrac{1}{2}U\,(\omega^{\mathrm{H}})^{e-k} \wedge \Theta^k \longrightarrow \tfrac{1}{2}\int_{X^\circ} U\,(\omega^{\mathrm{H}})^e,$$

each integral being absolutely convergent by Lemma 31.35(1) and (iii). If $\deg^{\mathrm{H}} \ne 0$, then, U being integrable for $(\omega^{\mathrm{H}})^e$, the correction mean is $\mathrm{m}^{\mathrm{b}} = \frac{1}{\deg^{\mathrm{H}}}\int_{X^\circ} U(\omega^{\mathrm{H}})^e$ (Definition 31.12), so the limit is $\frac{1}{2}\mathrm{m}^{\mathrm{b}} \deg^{\mathrm{H}}$: this is (Mz4). □

Lemma D.18. *Let $\mathcal{X}$ be an arithmetic variety (an integral scheme, projective and flat over $\mathbb{Z}$) of dimension $D+1$, $D \ge 1$, with $\mathcal{X}_{\mathbb{Q}}$ smooth, and let $X = \mathcal{X}(\mathbb{C})$ carry a Kähler form Θ and data ς, ∂X, X°, rW as in Lemma 31.35. Let I be a finite set, $0 < t_0 < 1$, and let θ_i, u_i, $u_{i,t}$ ($i \in I$, $0 < t \le t_0$) satisfy (R1)–(R5) of Lemma 31.35 for some $C \ge 1$ and functions δ, R as there, with every $u_{i,t}$ invariant under complex conjugation. Let $\mathcal{M}_1, \dots, \mathcal{M}_{D+1}$ be line bundles on $\mathcal{X}$ with smooth metrics h_k^0 invariant under complex conjugation, and $n_{k,i} \in \mathbb{Z}$ with $c_1(\mathcal{M}_k, h_k^0) = \alpha_k := \sum_{i\in I} n_{k,i}\theta_i$. Put $w_{k,t} = \sum_i n_{k,i}u_{i,t}$, $w_k = \sum_i n_{k,i}u_i$, $h_k(t) = h_k^0 e^{w_{k,t}}$ and $\alpha_k^\infty = \sum_i n_{k,i}(\theta_i + dd^c u_i)$ on X°. Then for every l the form $w_l \bigwedge_{k<l} \alpha_k^\infty \wedge \bigwedge_{k>l} \alpha_k$ is absolutely integrable on X°, and*

$$\lim_{t\to 0^+} \widehat{\deg}\Big(\prod_{k=1}^{D+1} \widehat{c}_1(\mathcal{M}_k, h_k(t))\Big) = \widehat{\deg}\Big(\prod_{k=1}^{D+1} \widehat{c}_1(\mathcal{M}_k, h_k^0)\Big) + \sum_{l=1}^{D+1} \int_{X^\circ} w_l \bigwedge_{k<l} \alpha_k^\infty \wedge \bigwedge_{k>l} \alpha_k.$$

In particular the limit exists.

Proof. Step 0: fixed forms as regularised data. For every $i \in I$ add an index i^0 with $\theta_{i^0} = \theta_i$ and $u_{i^0} = u_{i^0,t} = 0$, and let $I' = I \sqcup \{i^0 : i \in I\}$. As X is compact and Θ positive definite, there is $C_1 \ge C$ with $\theta_i + C_1\Theta \ge 0$ for all $i \in I$. Then the data indexed by I' satisfy (R1)–(R5) with C_1, δ, R: for $i \in I$, (R1) holds as $C_1\Theta - C\Theta \ge 0$ and (R2) as $C \le C_1$; for i^0, (R1) is the choice of C_1, (R2) reads $-C_1 \log \mathrm{rW} \le 0 \le 0 \le C_1$, true as $\mathrm{rW} \ge e$, and (R3)–(R5) are trivial.

Step 1: telescoping. Fix $0 < t \le t_0$ and put $\alpha_{k,t} = \sum_i n_{k,i}(\theta_i + dd^c u_{i,t}) = c_1(\mathcal{M}_k, h_k(t))$, a smooth form on X. For $0 \le l \le D+1$ let $A_l(t)$ be the arithmetic intersection number of the $\widehat{c}_1(\mathcal{M}_k, h_k(t))$ for $k \le l$ and the $\widehat{c}_1(\mathcal{M}_k, h_k^0)$ for $k > l$ (all metrics smooth and invariant

under conjugation, as the $u_{i,t}$ are); $A_0(t)$ does not depend on t, and $A_{D+1}(t)$ is the number in the statement. $A_l(t)$ and $A_{l-1}(t)$ differ only in the metric of $\mathcal{M}_l$, so by symmetry and the change-of-metric formula

$$A_l(t) - A_{l-1}(t) = \int_X w_{l,t}\ \beta_{l,t}, \qquad \beta_{l,t} = \bigwedge_{k<l} \alpha_{k,t} \wedge \bigwedge_{k>l} \alpha_k.$$

Step 2: expansion. By multilinearity, $w_{l,t}\beta_{l,t}$ is a finite sum, with integer coefficients independent of t (products of the $n_{k,i}$), of terms $u_{j,t}\,\mu_{\underline{i},t}$ with $j \in I$, $\underline{i} = (i_1, \dots, i_D) \in I'^D$ (indices in I for the factors $k < l$, indices i^0 for the factors $k > l$) and $\mu_{\underline{i},t} = \bigwedge_s(\theta_{i_s} + dd^c u_{i_s,t})$. The same expansion with $u_{j,t}, u_{i,t}$ replaced by u_j, u_i writes $w_l \bigwedge_{k<l} \alpha_k^\infty \wedge \bigwedge_{k>l} \alpha_k$ as the same combination of the terms $u_j\,\mu_{\underline{i}}$, $\mu_{\underline{i}} = \bigwedge_s(\theta_{i_s} + dd^c u_{i_s})$ on X°. The boundary ∂X, the zero set of ς, is a proper analytic subset and null, so $\int_X = \int_{X^\circ}$ for the smooth forms of Step 1.

Step 3: the limit. By (R2), $-C_1 \log \mathrm{rW} \le u_j \le u_{j,t} \le C_1 \le C_1 \log \mathrm{rW}$ on X°, so $|u_{j,t}|, |u_j| \le C_1 \log \mathrm{rW}$; by (R5), $u_{j,t} \to u_j$ uniformly on compact subsets of X°, and all are continuous there. By Lemma 31.35(1) for the data indexed by I' (with $D_v = \varnothing$), $(\log \mathrm{rW})\,\mu_{\underline{i}}$ is absolutely integrable on X°, hence so is $u_j\mu_{\underline{i}}$, which proves the integrability claim. By Lemma 31.35(2) with $g_t = u_{j,t}$, $g = u_j$, $C' = C_1$, $\int_{X^\circ} u_{j,t}\,\mu_{\underline{i},t} \to \int_{X^\circ} u_j\,\mu_{\underline{i}}$ as $t \to 0^+$. Summing the finitely many terms, $A_l(t) - A_{l-1}(t) \to \int_{X^\circ} w_l \bigwedge_{k<l} \alpha_k^\infty \wedge \bigwedge_{k>l} \alpha_k$ for every l, and $A_{D+1}(t) = A_0 + \sum_{l=1}^{D+1}(A_l(t) - A_{l-1}(t))$ converges to the stated value. □

D.1.3 The base slope identity

Proposition D.19 identifies the arithmetic degree of the test class on the root model with a slope of the base setting, which is clause (Bs1) of the base slope settings. The proof places the stage root model and a base setting joined to it on a common model (Lemma D.20) and compares the two families of metrics there with the one-model comparison of Section 31 (Lemma 31.34).

Proposition D.19. *Let* $\mathfrak{u}$, $\mathcal{Y}$ *and the root tower* $\mathcal{T}$ *of type* (Φ, n, p) *with its geometric realisation* $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ *and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2, and let* $\mathcal{T}'$ *be a root tower of type* (Φ, n, p) *with the components and the bad primes of* $\mathcal{T}$ *such that* $\widehat{\deg}'_{B,Z}(F) = \mathrm{pp}(\mathcal{X}_{B,Z}; F)$ *for every integer* $B \ge 2$, *every* $Z \in \mathrm{Cp}$ *and every* $F \in \mathcal{R}$, *where* $\widehat{\deg}'$ *are the arithmetic degrees of* $\mathcal{T}'$, *and such that* $\mathcal{T}'$ *with the geometric realisation of* $\mathcal{T}$ *has (Gi1). Let* B *be an admissible stage,* $Z \in \mathrm{Cp}$, *and let a base setting of* $\mathcal{T}$ *at* (B, Z) *be given which is joined to the stage root model, satisfies the hypotheses (i) and (ii) of Lemma 34.36, and whose limit self-intersection* $\mathrm{ar}^{\mathrm{b}}(B)$ *exists. Then* $\deg^{\mathrm{H}} > 0$, *so (Bs0) holds, and the setting satisfies the slope identity (Bs1) for* $\mathcal{T}'$ *at* (B, Z).

Proof. Let B be an admissible stage, $Z \in \mathrm{Cp}$, and fix a base setting as in the statement, with cover $\mathcal{X}$, base variety $\mathcal{X}^{\mathrm{b}}$ and $e = |\Phi|pq$. The type is (Φ, n, p), so the cut is $\kappa = p$ and $D^{\mathrm{gr}} = e$; as $q \le p$ (Definition 31.11), $n - \kappa \le \kappa$ and $\mathrm{pl} = \sum_{\tau\in\Phi} \sum_{i=p+1}^{n} x_{\tau,i}$. Let $r_0 = D - e = |\Phi|(p(p-1) + q(q-1))/2$ and write $\mathrm{fb} = \prod_{j=1}^{r_0} \ell_j$ with $\ell_j = x_{\tau_j,i'_j} - x_{\tau_j,i_j}$ the factors of the fibre class. Write $\mathcal{X}_1 = \mathcal{X}_{B,Z}$, $\delta_1 = \delta_{\mathcal{X}_1}$, $h_r(t)$ for its root metrics. The hypothesis on $\mathcal{T}'$ refers to $\mathrm{pp}(\mathcal{X}_1; \cdot)$, so $\mathcal{X}_1$ has convergent regularisation and $\mathrm{pp}(\mathcal{X}_1; F) = \lim_{t\to 0^+} \mathrm{ai}_t(\mathcal{X}_1; F)$ for every $F \in \mathcal{R}$ (a finite combination of monomials).

Step 1: $\deg^{\mathrm{H}} > 0$. By Lemma D.2, $0 < \int_{X^{\mathrm{b},\circ}} (\omega^{\mathrm{H}})^e < \infty$; by Lemma 34.36 (hypotheses (i), (ii)) this integral equals $\deg^{\mathrm{H}}$. So $\deg^{\mathrm{H}} > 0$, and with the existence of $\mathrm{ar}^{\mathrm{b}}(B)$ this is (Bs0).

Step 2: the common model. Take $\widetilde{\mathcal{X}}$, p_1, e_1, β, W, λ, t_0, $\mathcal{M}_{\mathrm{pl}}$ with $h^{(1)}(t)$, $h^{(2)}(t)$ and the complex data of Lemma D.20. For $1 \le j \le r_0$ let $\mathcal{N}_j = p_1^*(\mathcal{O}(e_{(\tau_j,i'_j)} - e_{(\tau_j,i_j)})|_{\mathcal{X}_1})$ with the

metric $h_{\mathcal{N}_j}(t) = p_1^*\big(h_{(\tau_j,i'_j)}(t)/h_{(\tau_j,i_j)}(t)\big)$, and $\overline{\mathcal{N}}_j(t) = (\mathcal{N}_j, h_{\mathcal{N}_j}(t))$. By part 1 of Lemma D.20 and additivity of first Chern forms, $\mathcal{N}_j$ with $h^{(1)} = h^{(2)} = h_{\mathcal{N}_j}(t)$ satisfies hypothesis 1 of Lemma 31.34 (integer coefficients, independent of t), and hypothesis 2 with $g = 0$.

Step 3: the test class on the common model. Let $0 < t \le t_0$. Expanding $\mathrm{pl}^{e+1}\mathrm{fb}$ (homogeneous of degree $e + 1 + r_0 = D + 1$) into monomials and using multilinearity and additivity of $\widehat{c}_1$,

$$\delta_1\,\mathrm{ai}_t(\mathcal{X}_1;\mathrm{pl}^{e+1}\mathrm{fb}) = \widehat{\deg}\Big(\widehat{c}_1\big(\mathcal{O}(e_{\mathrm{pl}}), h_{\mathrm{pl}}(t)\big)^{e+1}\prod_{j=1}^{r_0}\widehat{c}_1\big(\mathcal{O}(e_{(\tau_j,i'_j)} - e_{(\tau_j,i_j)}), h_{(\tau_j,i'_j)}(t)/h_{(\tau_j,i_j)}(t)\big)\Big)$$

on $\mathcal{X}_1$, with e_{pl}, $h_{\mathrm{pl}}(t)$ as in Lemma D.20. The $D+1$ hermitian line bundles in this product have smooth metrics (cone metrics restricted to $\mathcal{X}_1$), and their pull-backs by p_1 are $(\mathcal{M}_{\mathrm{pl}}, h^{(1)}(t))$ and the $\overline{\mathcal{N}}_j(t)$. Apply Lemma 30.13 with $b = D + 1$, $d = 0$ and the morphism $p_1\colon \widetilde{\mathcal{X}}$ and $\mathcal{X}_1$ are arithmetic varieties of absolute dimension $D + 1$ with smooth generic fibres (Lemma D.20, and Lemma 31.37 for the component Z), and p_1 is surjective and generically finite of degree $e_1 = [\mathbf{k}(\widetilde{\mathcal{X}}) : \mathbf{k}(\mathcal{X}_1)]$ (Lemma D.20). Hence

$$e_1\delta_1\,\mathrm{ai}_t(\mathcal{X}_1;\mathrm{pl}^{e+1}\mathrm{fb}) = \widehat{\deg}\Big(\widehat{c}_1\big(\mathcal{M}_{\mathrm{pl}}, h^{(1)}(t)\big)^{e+1}\prod_j\widehat{c}_1\big(\overline{\mathcal{N}}_j(t)\big)\Big). \tag{A}$$

Step 4: replacement in the limit. Apply Lemma 31.34 on $\widetilde{\mathcal{X}}$ (dimension $D + 1$, smooth generic fibre) with the data of Lemma D.20, $\pi = \mathrm{id}$, $d = 1$, and the $D + 1$ line bundles $\mathcal{M}_1 = \cdots = \mathcal{M}_{e+1} = \mathcal{M}_{\mathrm{pl}}$ with the families $h^{(1)}(t)$, $h^{(2)}(t)$ (part 2 of Lemma D.20) and $\mathcal{M}_{e+1+j} = \mathcal{N}_j$ with both families equal to $h_{\mathcal{N}_j}(t)$ (Step 2). It gives

$$\lim_{t\to 0^+}\Big(\widehat{\deg}\Big(\widehat{c}_1\big(\mathcal{M}_{\mathrm{pl}}, h^{(1)}(t)\big)^{e+1}\prod_j\widehat{c}_1\big(\overline{\mathcal{N}}_j(t)\big)\Big) - \widehat{\deg}\Big(\widehat{c}_1\big(\mathcal{M}_{\mathrm{pl}}, h^{(2)}(t)\big)^{e+1}\prod_j\widehat{c}_1\big(\overline{\mathcal{N}}_j(t)\big)\Big)\Big) = 0.$$

By the definition of $h^{(2)}(t)$, λ is an isometry from $(\mathcal{M}_{\mathrm{pl}}, h^{(2)}(t))$ onto $\beta^*\overline{\mathcal{H}}^{\mathrm{b}}_{B,t}$, so the second term is $\widehat{\deg}(\widehat{c}_1(\beta^*\overline{\mathcal{H}}^{\mathrm{b}}_{B,t})^{e+1}\prod_j\widehat{c}_1(\overline{\mathcal{N}}_j(t)))$.

Step 5: the arithmetic factor on the base. Let $\widetilde{\mathcal{X}}_\eta$ be the fibre of β over the generic point η of $\mathcal{X}^{\mathrm{b}}$, and

$$c = \deg\big(c_1(\mathcal{N}_1|_{\widetilde{\mathcal{X}}_\eta})\cdots c_1(\mathcal{N}_{r_0}|_{\widetilde{\mathcal{X}}_\eta})\cap[\widetilde{\mathcal{X}}_\eta]\big) \in \mathbb{Z},$$

which does not depend on t. Apply Lemma 30.13 with $b = e + 1$, $d = r_0$ and the morphism β: $\widetilde{\mathcal{X}}$ is an arithmetic variety of absolute dimension $D + 1 = b + d$ with smooth generic fibre and β is a morphism to $\mathcal{X}^{\mathrm{b}}$ (Lemma D.20), an arithmetic variety of absolute dimension $e + 1 = b$ with smooth generic fibre ((Hy1)); $\overline{\mathcal{H}}_1 = \cdots = \overline{\mathcal{H}}_{e+1} = \overline{\mathcal{H}}^{\mathrm{b}}_{B,t}$ has a smooth metric ((Hy4)), and $\overline{\mathcal{L}}_j = \overline{\mathcal{N}}_j(t)$ $(1 \le j \le r_0)$ have smooth metrics (pull-backs by p_1 of quotients of cone metrics). It gives

$$\widehat{\deg}\Big(\widehat{c}_1(\beta^*\overline{\mathcal{H}}^{\mathrm{b}}_{B,t})^{e+1}\prod_j\widehat{c}_1\big(\overline{\mathcal{N}}_j(t)\big)\Big) = c\,\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{H}}^{\mathrm{b}}_{B,t})^{e+1}).$$

So the second term of Step 4 tends to $c\,\mathrm{ar}^{\mathrm{b}}(B)$ as $t \to 0^+$ (the limit exists by hypothesis). With (A) and Step 4,

$$e_1\delta_1\,\mathrm{pp}(\mathcal{X}_1;\mathrm{pl}^{e+1}\mathrm{fb}) = c\,\mathrm{ar}^{\mathrm{b}}(B). \tag{Ar}$$

Step 6: the same factor on generic fibres. We first show

$$\beta_{\mathbb{Q}*}\Big(c_1(\mathcal{N}_{1,\mathbb{Q}})\cdots c_1(\mathcal{N}_{r_0,\mathbb{Q}})\cap[\widetilde{\mathcal{X}}_{\mathbb{Q}}]\Big) = c\,[\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}]. \tag{C}$$

This is Lemma 30.25 over $k=\mathbb{Q}$ for $\beta_{\mathbb{Q}}\colon \widetilde{\mathcal{X}}_{\mathbb{Q}}\to\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ and the line bundles $\mathcal{N}_{j,\mathbb{Q}}$: $\widetilde{\mathcal{X}}_{\mathbb{Q}}$ and $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ are integral projective $\mathbb{Q}$-schemes (generic fibres of arithmetic varieties) of dimensions $D=e+r_0$ and e, and the fibre of $\beta_{\mathbb{Q}}$ over the generic point of $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ is $\widetilde{\mathcal{X}}_\eta$ (as $\mathbf{k}(\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}})=\mathbf{k}(\mathcal{X}^{\mathrm{b}})$), so the factor of that lemma is the c of Step 5. Next, by multilinearity, $p_{1,\mathbb{Q}*}[\widetilde{\mathcal{X}}_{\mathbb{Q}}]=e_1[\mathcal{X}_{1,\mathbb{Q}}]$ and the projection formula for $p_{1,\mathbb{Q}}$,

$$e_1\delta_1\,\mathrm{gi}(\mathcal{X}_1;\mathrm{pl}^e\mathrm{fb})=\deg\Big(c_1(\mathcal{M}_{\mathrm{pl},\mathbb{Q}})^e\,c_1(\mathcal{N}_{1,\mathbb{Q}})\cdots c_1(\mathcal{N}_{r_0,\mathbb{Q}})\cap[\widetilde{\mathcal{X}}_{\mathbb{Q}}]\Big).$$

By λ, $\mathcal{M}_{\mathrm{pl},\mathbb{Q}}\cong\beta_{\mathbb{Q}}^*\mathcal{O}(a_B^{\mathrm{H}})$, and on $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ the sections g_ℓ ($\ell\in S'_B$) have no zeros or poles ((Hy3): their divisors lie over ℓ), so $\mathcal{O}(a_B^{\mathrm{H}})|_{\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}}\cong\mathcal{O}(a^{\mathrm{H}})|_{\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}}$. By the projection formula for the proper morphism $\beta_{\mathbb{Q}}$ [98] (Proposition 2.5(c)), the invariance of degrees under proper push-forward and (C),

$$e_1\delta_1\,\mathrm{gi}(\mathcal{X}_1;\mathrm{pl}^e\mathrm{fb})=\deg(c_1(\mathcal{O}(a^{\mathrm{H}}))^e\cap c\,[\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}])=c\,\deg^{\mathrm{H}}.$$

As $\mathrm{pl}^e\mathrm{fb}$ is homogeneous of degree D, (Gi1) for $\mathcal{T}'$ gives $\mathrm{gi}(\mathcal{X}_1;\mathrm{pl}^e\mathrm{fb})=\mathrm{vol}(Z)\deg^{\mathrm{fl}}(\mathrm{pl}^e\mathrm{fb})$, so

$$e_1\delta_1\,\mathrm{vol}(Z)\deg^{\mathrm{fl}}(\mathrm{pl}^e\mathrm{fb})=c\,\deg^{\mathrm{H}}. \tag{G}$$

Step 7: the slope identity. By Step 1, $\deg^{\mathrm{H}}>0$, so (G) gives $c=e_1\delta_1\mathrm{vol}(Z)\deg^{\mathrm{fl}}(\mathrm{pl}^e\mathrm{fb})/\deg^{\mathrm{H}}$, and (Ar), $e_1\delta_1>0$ and the hypothesis $\widehat{\deg}'_{B,Z}=\mathrm{pp}(\mathcal{X}_1;\cdot)$ give

$$\widehat{\deg}'_{B,Z}(\mathrm{pl}^{e+1}\mathrm{fb})=\mathrm{vol}(Z)\deg^{\mathrm{fl}}(\mathrm{pl}^e\mathrm{fb})\,\frac{\mathrm{ar}^{\mathrm{b}}(B)}{\deg^{\mathrm{H}}}=(D^{\mathrm{gr}}+1)\,\mathrm{vol}(Z)\deg^{\mathrm{fl}}\big(\mathrm{pl}^{D^{\mathrm{gr}}}\mathrm{fb}\big)\,\mathrm{sl}^{\mathrm{b}}(B),$$

by (31.7) and $D^{\mathrm{gr}}=e$. This is (Bs1) for $\mathcal{T}'$ at (B,Z). □

Lemma D.20. *Let $\mathfrak{u}=(E,\mathsf{H},\mathsf{U},p,q,\mathsf{L},\mathrm{Sh},F)$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ,n,p) with its geometric realisation $(\nu_{\mathcal{T}},(\mathcal{X}_{B,Z}))$ and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2, and put $D=|\Phi|\,n(n-1)/2$. Let $B\geq 2$ be an integer, $Z\in\mathrm{Cp}$, and let a base setting of $\mathcal{T}$ at (B,Z) be given which is joined to the stage root model $\mathcal{X}_{B,Z}$ and satisfies the hypotheses (i) and (ii) of Lemma 34.36. Write $h_r(t)$ ($r\in\mathrm{Rt}$) for the root metrics of $\mathcal{X}_{B,Z}$, and put*

$$e_{\mathrm{pl}}=\sum_{\tau\in\Phi}\sum_{i=p+1}^{n}e_{(\tau,i)},\qquad h_{\mathrm{pl}}(t)=\prod_{\tau\in\Phi}\prod_{i=p+1}^{n}h_{(\tau,i)}(t),$$

so that $h_{\mathrm{pl}}(t)$ is a metric on $\mathcal{O}(e_{\mathrm{pl}})|_{\mathcal{X}_{B,Z}}$. Then there are:

- *an arithmetic variety $\widetilde{\mathcal{X}}$ of absolute dimension $D+1$ with $\widetilde{\mathcal{X}}_{\mathbb{Q}}$ smooth, a surjective morphism $p_1\colon\widetilde{\mathcal{X}}\to\mathcal{X}_{B,Z}$, generically finite of some degree $e_1\geq 1$, and a surjective morphism $\beta\colon\widetilde{\mathcal{X}}\to\mathcal{X}^{\mathrm{b}}$;*
- *on the compact complex manifold $W=\widetilde{\mathcal{X}}(\mathbb{C})$ a Kähler form Θ and data ς, ∂W, W°, rW as in Lemma 31.35, with $\beta(W^\circ)\subseteq X^{\mathrm{b},\circ}$; a finite set I, a number $0<t_0<1$ and data $\theta_i,u_i,u_{i,t}$ ($i\in I$, $0<t\leq t_0$) satisfying (R1)–(R5) on (W,Θ,ς) for some $C\geq 1$ and functions δ, R;*
- *an isomorphism of line bundles $\lambda\colon p_1^*(\mathcal{O}(e_{\mathrm{pl}})|_{\mathcal{X}_{B,Z}})\to\beta^*(\mathcal{O}(a_B^{\mathrm{H}})|_{\mathcal{X}^{\mathrm{b}}})$ on $\widetilde{\mathcal{X}}$,*

such that, with $\pi=\mathrm{id}_W$, $d=1$ and $N=N'=\varnothing$ in Lemma 31.34:

1. *for every $r\in\mathrm{Rt}$ the line bundle $p_1^*(\mathcal{O}(e_r)|_{\mathcal{X}_{B,Z}})$ with $h^{(1)}(t)=h^{(2)}(t)=p_1^*h_r(t)$ ($0<t\leq t_0$) satisfies hypotheses 1 and 2 of Lemma 31.34;*

2. the line bundle $\mathcal{M}_{\mathrm{pl}} = p_1^*(\mathcal{O}(e_{\mathrm{pl}})|_{\mathcal{X}_{B,Z}})$ *with*

$$h^{(1)}(t) = p_1^* h_{\mathrm{pl}}(t), \qquad h^{(2)}(t) = \lambda^*\beta^*\Big(h^{\mathrm{ref}} e^{\varphi_t} \prod_{\ell \in S'_B} |g_\ell|\Big) \qquad (0 < t \le t_0),$$

the second being the pull-back along β*, transported by* λ*, of the metric of* $\overline{\mathcal{H}}^{\mathrm{b}}_{B,t}$*, satisfies hypotheses 1 and 2 of Lemma 31.34;*

all these metrics being smooth and invariant under complex conjugation.

Proof. Write $\mathcal{X}_1 = \mathcal{X}_{B,Z}$, $f_1\colon \mathcal{X}_1 \to \mathcal{Y}$ for its structure map, Ω_1 for its function field, $\mathcal{A}_1$ for its abelian locus, and $\pi_Z\colon \mathcal{X}_1(\mathbb{C}) \to X_Z$ for the realisation (Go4) of its root origin. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be the cover of the base setting, with function field Ω, abelian locus $\mathcal{A}_X$ and coordinates y; let C_{b}, t_{b}, δ_{b}, ε, R_{b} be the data of hypothesis (ii), ς_{b} the section of hypothesis (i), rW_{b} the corresponding function on $X^{\mathrm{b},\circ}$, $\Theta_{\mathrm{b}} = dd^c \log h'$, $\theta = dd^c \log h^{\mathrm{ref}}$ and $\varphi = \log(h^{\mathrm{H}}/h^{\mathrm{ref}})$. Let $f''\colon \mathcal{X}'' \to \mathcal{Y}$, ϕ, ϕ' be the common cover of the joined clause.

Step 1: the stage root model. By Lemma 31.37, $S_B = \mathrm{Bd} \cup S'_B$ contains every prime dividing $\mathrm{disc}(E)$; $\mathcal{X}_{1,\mathbb{Q}}$ is smooth; $\mathcal{X}_1$ carries the data (Go3)(1)–(4) of a root origin on the fixed data at Z for (S_B, R) with $R \equiv 4$; $\varphi_* = 0 = \varphi_*^T$ for Sys_Z; π_Z satisfies hypotheses 1–3 of Lemma 31.21; a point of $\mathcal{X}_1(\mathbb{C})$ lies in $\mathcal{A}_1$ if and only if its image lies in X_Z°; and on $\pi_Z^{-1}(X_Z^\circ)$

$$h^{\mathrm{ref}}_{(\tau,i)} e^{\varphi_{(\tau,i)} \circ \pi_Z} = \prod_{k \le i} (h^{\mathrm{sing}}_{(\tau,k)})^{-1} \qquad ((\tau,i) \in \mathrm{Rt}). \tag{S}$$

By (Go3)(2), $\mathcal{X}_1$ satisfies (Hp1), (Hp2) of Lemma 31.29 at $(S_B, 4)$, and its ℓ^4-torsion ($\ell \in S_B$) is defined over Ω_1.

Step 2: the base side. By the coordinate-chart clause of (Gb1), every point of $\mathcal{X}$ lies in a chart $\mathcal{X}_{c'} = \mathrm{Spec}\,\mathbb{Z}[y]_{c'}$, and on $\mathcal{X}_{c'}$ the coordinates define a morphism into the standard chart $\{\prod_k Y_{k,c'_k} \ne 0\}$ of $\mathbb{P}$; these glue to a morphism $y\colon \mathcal{X} \to \mathbb{P}$, birational onto $\mathcal{X}^{\mathrm{b}}$ by (Gb1). As $\mathcal{X}$ is proper over $\mathbb{Z}$ (joined clause) and $\mathbb{P}$ is separated, y is proper, so $y(\mathcal{X}) = \mathcal{X}^{\mathrm{b}}$. Let $T = y(\mathcal{X}_{\mathbb{Q}} \smallsetminus \mathcal{A}_X)$, a closed subset of $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ ($y_{\mathbb{Q}}$ is proper). Every complex point of T lifts to a complex point of $\mathcal{X}_{\mathbb{Q}} \smallsetminus \mathcal{A}_X$ (its fibre is a non-empty scheme of finite type over $\mathbb{C}$), so by (Gb1) $T(\mathbb{C}) = \partial X^{\mathrm{b}}$; in particular $T \ne \mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$. The Hodge lattices of $\mathcal{X}$ at $(S_B, 4)$ are Hodge bundles $\mathcal{V}_s^X$ (joined clause: $R \equiv 4$), the ℓ^4-torsion ($\ell \in S_B$) is defined over Ω and the images of the Hodge–Tate spans in the duals of the Li_s are locally free (Gb1); so $\mathcal{X}$ satisfies (Hp1), (Hp2) at $(S_B, 4)$. At a point x over no prime of S'_B the Hodge lattice at $(S_B, 4)$ is that at $(\mathrm{Bd}, 4)$, so $\mathcal{V}^X_{s,x} = \mathcal{V}_{s,x}$, $\mathcal{V}_s$ the Hodge bundle at (Bd, R) of (Gb2)–(Gb4). Let ι be the isomorphism of (Hy2) on generic fibres, which identifies rational sections of $\mathcal{O}(a^{\mathrm{H}})$ with elements of $\bigotimes_\tau \det(\mathrm{Li}^\vee_{(\tau,-)}) = \bigotimes_\tau (\det \mathrm{Li}_{(\tau,-)})^\vee$ (Gb2), and define on rational sections

$$\Lambda_X(P) = \iota\Big(P \Big/ \prod_{\ell \in S'_B} g_\ell\Big).$$

Claim: Λ_X *is an isomorphism of invertible sheaves* $y^*\mathcal{O}(a^{\mathrm{H}}_B) \to \bigotimes_\tau (\det \mathcal{V}^X_{(\tau,-)})^\vee$. Let $x \in \mathcal{X}$, c' a chart containing x, and $D(f_0) \ni x$, u, $(b_{\tau,i})_i$ as in (Gb3). Near x, $y^*\mathcal{O}(a^{\mathrm{H}}_B)$ is generated by $\sigma = \prod_k y_{k,c'_k}^{(a^{\mathrm{H}}_B)_k} = \prod_k y_{k,c'_k}^{a^{\mathrm{H}}_k} \cdot \prod_\ell y_{c'}^{v_\ell}$, and by (Gb3)

$$\Lambda_X(\sigma) = u \cdot \bigotimes_\tau (b_{\tau,1} \wedge \cdots \wedge b_{\tau,q})^\vee \cdot \prod_{\ell \in S'_B} (g_\ell / y_{c'}^{v_\ell})^{-1}.$$

For $\ell \in S'_B$ the Cartier divisor of g_ℓ is supported over ℓ (Hy3), so $g_\ell / y_{c'}^{v_\ell}$ is a unit at every point of $\mathcal{X}_{c'}$ not over ℓ. If x lies over no prime of S'_B, $\Lambda_X(\sigma)$ is a unit times a generator of

$\bigotimes_\tau(\det \mathcal{V}_{(\tau,-),x})^\vee = \bigotimes_\tau(\det \mathcal{V}^X_{(\tau,-),x})^\vee$. If x lies over $\ell_0 \in S'_B$ (then over no other prime), by (Gb4) $g_{\ell_0}/y_{c'}^{v_{\ell_0}}$ is a unit times $\prod_\tau \det T_\tau$, where a basis $(h_{\tau,i})_i$ of $\mathrm{HT}_{(\tau,-),x}(\ell_0^4) = \mathcal{V}^X_{(\tau,-),x}$ is $(b_{\tau,i})_i$ multiplied by T_τ; then $h_{\tau,1}\wedge\cdots\wedge h_{\tau,q} = \det T_\tau \cdot b_{\tau,1}\wedge\cdots\wedge b_{\tau,q}$, so $\Lambda_X(\sigma)$ is a unit times $\bigotimes_\tau(h_{\tau,1}\wedge\cdots\wedge h_{\tau,q})^\vee$, a generator of $\bigotimes_\tau(\det \mathcal{V}^X_{(\tau,-),x})^\vee$. As both sheaves are invertible subsheaves of the same constant sheaf, the claim follows. *Metric.* Let z' be a complex point of $\mathcal{A}_X$ with $w = y_{\mathbb{C}}(z') \in X^{\mathrm{b},\circ}$. The fibre of $\mathcal{V}^X_s$ at z' is $\mathrm{Lie}_s(G_{\mathcal{X},z'})$ (over $\mathbb{Q}$ the Hodge lattices are the de Rham lattices, direct summands of Lie cut out by the idempotents of $\mathcal{O}_E\otimes\mathbb{Q}$). By (Gb2), (Gb3), ι carries the metric of the Hodge-type line at w to the tensor product of the metrics dual to the determinants of $h_{z'}$ on the $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X},z'})$: the generator $\prod_k y_{k,c'_k}^{a^{\mathrm{H}}_k}$ has norm $|u(z')|\prod_\tau \|b_{\tau,1}(z')\wedge\cdots\wedge b_{\tau,q}(z')\|^{-1}_{h_{z'}}$, which is the dual determinant norm of its image. Since $\|P\|_{h^{\mathrm{H}}}\prod|g_\ell| = \|P/\prod_\ell g_\ell\|_{h^{\mathrm{H}}}$, the fibre of Λ_X at z' is an isometry from the metric with cone function $h^{\mathrm{H}}\prod_{\ell\in S'_B}|g_\ell|$ at w to the dual determinant Hodge metric.

Step 3: the common model. The morphism $(\phi,\phi')\colon \mathcal{X}''\to\mathcal{X}_1\times_{\mathbb{Z}}\mathcal{X}$ has proper source, so its scheme-theoretic image $\mathcal{X}_3$ is a closed integral subscheme, proper over $\mathbb{Z}$; its function field Ω_3 is the subfield of $\mathbf{k}(\mathcal{X}'')$ generated by $\phi^*\Omega_1$ and $\phi'^*\Omega$. The two morphisms $f_1\circ\mathrm{pr}_1$, $f\circ\mathrm{pr}_2\colon\mathcal{X}_3\to\mathcal{Y}$ agree after composition with $\mathcal{X}''\to\mathcal{X}_3$ ($f''=f_1\circ\phi=f\circ\phi'$); their equaliser is a closed subscheme ($\mathcal{Y}$ is separated) through which $\mathcal{X}''\to\mathcal{X}_3$ factors, hence all of $\mathcal{X}_3$. The generic points of $\mathcal{X}_1$ and $\mathcal{X}$ map to the generic point of $\mathcal{Y}_Z$; $\mathcal{X}$ has absolute dimension $e+1=|\Phi|pq+1$ ((Hy1), y birational), as has $\mathcal{Y}_Z$ (its generic fibre Z has dimension $|\Phi|pq$), so Ω is finite over $\mathbf{k}(\mathcal{Y}_Z)$, and $\Omega_3=\Omega_1\Omega$ is finite over Ω_1. By Chow's lemma there is a projective birational surjective $\mathcal{X}'_3\to\mathcal{X}_3$ with $\mathcal{X}'_3$ integral and projective over $\mathbb{Z}$; let $\mathcal{X}''_3$ be its normalisation (finite). Write $\beta_3\colon\mathcal{X}''_3\to\mathcal{X}^{\mathrm{b}}$ for the composite with $y\circ\mathrm{pr}_2$ and $\mathcal{A}_3$ for the abelian locus of $\mathcal{X}''_3$ for $f_1\circ\mathrm{pr}_1$. By Theorem B.142(A) there is a projective birational $\mathcal{R}\to\mathcal{X}''_{3,\mathbb{Q}}$ with $\mathcal{R}$ smooth and integral. Let $\mathcal{U}\subseteq\mathcal{R}$ be the preimage of the open set $\mathcal{A}_3\cap\beta_3^{-1}(\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}\smallsetminus T)$; it contains the generic point (which maps to the generic point of $j(Z)$ and of $\mathcal{X}^{\mathrm{b}}$, and $T\neq\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$), so the ideal $I_{\mathcal{R}}$ of the reduced closed set $\mathcal{R}\smallsetminus\mathcal{U}$ is non-zero. By Theorem B.142(B) there is a composite of blow-ups with smooth centres $\Pi\colon\mathcal{R}'\to\mathcal{R}$, an isomorphism over $\mathcal{U}$, with $\mathcal{R}'$ smooth and $\Pi^*I_{\mathcal{R}}$ the ideal of a simple normal crossing divisor B' with support $\mathcal{R}'\smallsetminus\Pi^{-1}(\mathcal{U})$. The composite $\mathcal{R}'\to\mathcal{X}''_{3,\mathbb{Q}}$ is projective and birational, so $\mathcal{R}'$ is a closed integral subscheme of the generic fibre of $\mathbb{P}^N_{\mathbb{Z}}\times_{\mathbb{Z}}\mathcal{X}''_3$ for some N; let $\mathcal{X}_0$ be its closure there and $\widetilde{\mathcal{X}}$ the normalisation of $\mathcal{X}_0$. Then $\widetilde{\mathcal{X}}$ is integral, normal, projective over $\mathbb{Z}$ (Hartshorne II Ex. 4.9) and flat (torsion-free), with $\widetilde{\mathcal{X}}_{\mathbb{Q}}\cong\mathcal{R}'$ ($\mathcal{R}'$ is smooth, hence normal) and function field Ω_3; so it is an arithmetic variety with smooth generic fibre, of absolute dimension $\dim\mathcal{X}_1=D+1$. Let $p_1\colon\widetilde{\mathcal{X}}\to\mathcal{X}_1$ and $p'\colon\widetilde{\mathcal{X}}\to\mathcal{X}$ be the composites with the projections, and $\beta=y\circ p'$. They are dominant and proper, hence surjective; p_1 is generically finite of degree $e_1=[\Omega_3:\Omega_1]\geq 1$. With $\widetilde{f}=f_1\circ p_1=f\circ p'$, $\widetilde{\mathcal{X}}$ is a cover of $\mathcal{Y}$, and its abelian locus $\mathcal{A}$ is $p_1^{-1}(\mathcal{A}_1)=p'^{-1}(\mathcal{A}_X)$. Moreover $\widetilde{\mathcal{X}}_{\mathbb{Q}}\smallsetminus(\mathcal{A}\cap\beta^{-1}(\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}\smallsetminus T))$ is the support of B'.

Step 4: the Hodge bundles and λ. By Steps 1 and 2 and Lemma 31.29(1), applied to p_1 and to p' (both dominant, compatible with $\widetilde{f}$, and S_B contains the primes dividing $\mathrm{disc}(E)$), the Hodge lattices of $\widetilde{\mathcal{X}}$ at $(S_B,4)$ are Hodge bundles $\widetilde{\mathcal{V}}_s=\mathcal{O}_{\widetilde{\mathcal{X}}}\mathcal{V}^1_s=\mathcal{O}_{\widetilde{\mathcal{X}}}\mathcal{V}^X_s\subseteq\widetilde{\mathrm{Li}}_s$, with $p_1^*\mathcal{V}^1_s\cong\widetilde{\mathcal{V}}_s\cong p'^*\mathcal{V}^X_s$, where $\mathcal{V}^1_s$ are those of $\mathcal{X}_1$ (both equal the Hodge lattices of the cover $\widetilde{\mathcal{X}}$). By Lemma 31.29(2) the pull-back of the integral flag of $\mathcal{X}_1$ of (Go3)(3) is an integral flag $(\widetilde{\mathcal{V}}_{s,k})_k$ of $\widetilde{\mathcal{X}}$ whose root lines are $\widetilde{\mathcal{L}}_r\cong p_1^*\mathcal{L}_r$. For a root (τ,i) with $i>p$ the label is $(\tau,-)$, the step $n+1-i$ and the sign -1 (Fh3); so, using the root sections $\varrho_r\colon\mathcal{O}(e_r)|_{\mathcal{X}_1}\to\mathcal{L}_r$ of (Go3)(3) and the canonical isomorphism $\det\widetilde{\mathcal{V}}_{(\tau,-)}\cong\bigotimes_{k=1}^q\widetilde{\mathcal{V}}_{(\tau,-),k}/\widetilde{\mathcal{V}}_{(\tau,-),k-1}$ for a full flag with locally free steps and invertible quotients,

we get an isomorphism

$$\kappa\colon p_1^*(\mathcal{O}(e_{\mathrm{pl}})|_{\mathcal{X}_1}) \xrightarrow{\otimes p_1^*\varrho_{(\tau,i)}} \bigotimes_\tau \bigotimes_{i>p} \widetilde{\mathcal{L}}_{(\tau,i)} = \bigotimes_\tau \bigotimes_{k=1}^{q} \big(\widetilde{\mathcal{V}}_{(\tau,-),k}/\widetilde{\mathcal{V}}_{(\tau,-),k-1}\big)^{-1} \cong \bigotimes_\tau (\det \widetilde{\mathcal{V}}_{(\tau,-)})^\vee.$$

On the other side $p'^*\Lambda_X$ is an isomorphism $\beta^*\mathcal{O}(a_B^{\mathrm{H}}) = p'^*y^*\mathcal{O}(a_B^{\mathrm{H}}) \to p'^*\bigotimes_\tau(\det \mathcal{V}^X_{(\tau,-)})^\vee = \bigotimes_\tau(\det \widetilde{\mathcal{V}}_{(\tau,-)})^\vee$. Put $\lambda = (p'^*\Lambda_X)^{-1}\circ\kappa$, an isomorphism of line bundles on $\widetilde{\mathcal{X}}$.

Step 5: λ is an isometry of the singular Hodge metrics on the good locus. Let $\mathcal{U}' = \mathcal{A}\cap\beta^{-1}(\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}\smallsetminus T)$ and $z\in\mathcal{U}'(\mathbb{C})$; then $p_1(z)\in\mathcal{A}_1(\mathbb{C})$, $z' = p'(z)\in\mathcal{A}_X(\mathbb{C})$, $\beta(z) = y_{\mathbb{C}}(z')\in X^{\mathrm{b},\circ}$ (as $T(\mathbb{C}) = \partial X^{\mathrm{b}}$), and $G_{\mathcal{X}_1,p_1(z)} = G_{\widetilde{\mathcal{X}},z} = G_{\mathcal{X},z'} = G^{\mathrm{un}}_{\widetilde{f}(z)}$. The fibre of $\widetilde{\mathcal{V}}_s$ at z is $\mathrm{Lie}_s(G_{\widetilde{\mathcal{X}},z})$, compatibly with the fibres of $\mathcal{V}^1_s$ at $p_1(z)$ and of $\mathcal{V}^X_s$ at z'. By (Go3)(4) and (Fh6), $p_1^*\varrho_{(\tau,i)}$ carries the metric with cone function $p_1^*h^{\mathrm{sing}}_{(\tau,i)}$ to the Hodge metric of the root line, i.e. the dual of the quotient norm of h_z on $\widetilde{\mathcal{V}}_{(\tau,-),k}/\widetilde{\mathcal{V}}_{(\tau,-),k-1}$ at z ($k = n+1-i$). For a basis $b_1,\dots,b_q$ of $\mathrm{Lie}_{(\tau,-)}(G_z)$ adapted to the flag with Gram matrix Γ, (Fh4) gives $\prod_{k=1}^q\|b_k \bmod V_{k-1}\| = \prod_k(\det\Gamma_{\le k}/\det\Gamma_{\le k-1})^{1/2} = (\det\Gamma)^{1/2} = \|b_1\wedge\cdots\wedge b_q\|$; so the fibre of κ at z carries $p_1^*h^{\mathrm{sing}}_{\mathrm{pl}}$, $h^{\mathrm{sing}}_{\mathrm{pl}} = \prod_\tau\prod_{i>p}h^{\mathrm{sing}}_{(\tau,i)}$, to the dual determinant Hodge metric. By Step 2 the fibre of $p'^*\Lambda_X$ at z carries the metric with cone function $\beta^*(h^{\mathrm{H}}\prod_\ell|g_\ell|)$ to the same metric. Hence, on $\mathcal{U}'(\mathbb{C})$,

$$p_1^*h^{\mathrm{sing}}_{\mathrm{pl}} = \lambda^*\beta^*\Big(h^{\mathrm{H}}\prod_{\ell\in S'_B}|g_\ell|\Big). \tag{1}$$

Step 6: the complex data. Put $W = \widetilde{\mathcal{X}}(\mathbb{C}) = \mathcal{R}'(\mathbb{C})$, a compact complex manifold of dimension D, Θ the restriction of a Fubini–Study form of a projective embedding of $\mathcal{R}'$, $F_1 = \pi_Z\circ p_1\colon W\to X_Z$ and $\beta_{\mathbb{C}}\colon W\to X^{\mathrm{b}}$, both holomorphic. By Step 1, $F_1^{-1}(X_Z^\circ) = \mathcal{A}(\mathbb{C})$. Let $\varsigma = F_1^*\varsigma_Z\otimes\beta^*\varsigma_{\mathrm{b}}$, a holomorphic section of the tensor product of the pulled-back hermitian line bundles, $\|\varsigma\|\le 1$; its zero set is $(W\smallsetminus\mathcal{A}(\mathbb{C}))\cup\beta^{-1}(\partial X^{\mathrm{b}}) = W\smallsetminus\mathcal{U}'(\mathbb{C})$, the support of the simple normal crossing divisor $B'_{\mathbb{C}}$, so its zero divisor ∂W has simple normal crossings. Then $W^\circ = \mathcal{U}'(\mathbb{C})$, $\beta(W^\circ)\subseteq X^{\mathrm{b},\circ}$, $F_1(W^\circ)\subseteq X_Z^\circ$, and

$$\mathrm{rW} = \mathrm{rW}_Z\circ F_1 - 2\log\|\beta^*\varsigma_{\mathrm{b}}\| \ge \mathrm{rW}_Z\circ F_1, \qquad \mathrm{rW}\ge \mathrm{rW}_{\mathrm{b}}\circ\beta \qquad \text{on } W^\circ,$$

as both logarithms are ≤ 0. *Root side.* By Lemma 31.32 applied to Sys_Z and F_1 there is $C_W\ge 1$ such that $F_1^*\theta_a$, $\varphi_a\circ F_1$, $\varphi_a^T(t)\circ F_1$ (a fundamental) satisfy (R1)–(R5) on (W,Θ,ς) with C_W, $\delta_Z(t) = t^{1/3} + (1+C_Z)\varepsilon_Z(t)$, $R_Z(t) = \log(1/t)$, $0<t\le t_1 = \min(t_Z, e^{-e})$, where ε_Z is the function $\varepsilon(t)$ of (Go1) for Sys_Z; and $F_1^*\theta_a + dd^c(\varphi_a^T(t)\circ F_1) = F_1^*\eta_a(t)$. *Base side.* Put $\theta_{\mathrm{b}} = \beta^*\theta$, $u_{\mathrm{b}} = \varphi\circ\beta$ on W° (smooth, as φ is smooth on $X^{\mathrm{b},\circ}$ by (ii)) and $u_{\mathrm{b},t} = \varphi_t\circ\beta$ on W. Choose $0<t_2\le\min(t_1,t_{\mathrm{b}})$ with $\varepsilon(t)\le 1$ and $R_{\mathrm{b}}(t)\ge e$ for $0<t\le t_2$ (possible as $\varepsilon(t)\to 0$, $R_{\mathrm{b}}(t)\to\infty$), and $C_1\ge 1$ with $\beta^*\Theta_{\mathrm{b}}\le C_1\Theta$ on W (compactness; Θ is positive definite). For $0<t\le t'\le t_2$: (R1) $\theta_{\mathrm{b}}+dd^cu_{\mathrm{b},t}+C_1\Theta\ge\beta^*(\theta+dd^c\varphi_t+\varepsilon(t)\Theta_{\mathrm{b}})\ge 0$, as pull-backs of semipositive forms are semipositive; (R2) on W°, $-C_{\mathrm{b}}\log\mathrm{rW}\le -C_{\mathrm{b}}\log(\mathrm{rW}_{\mathrm{b}}\circ\beta)\le u_{\mathrm{b}}\le u_{\mathrm{b},t}\le C_{\mathrm{b}}$; (R3) on $\{\mathrm{rW}\le R(t)\}$ with $R(t)\le R_{\mathrm{b}}(t)$ we have $\mathrm{rW}_{\mathrm{b}}\circ\beta\le R_{\mathrm{b}}(t)$, so $u_{\mathrm{b},t}\le u_{\mathrm{b}}+\delta_{\mathrm{b}}(t)$; (R4) holds pointwise; (R5) for a compact $K\subseteq W^\circ$, $\beta(K)$ is a compact subset of $X^{\mathrm{b},\circ}$, so $\varphi_t\to\varphi$ with first and second derivatives uniformly on $\beta(K)$ and, by the chain rule with the derivatives of β bounded on a compact neighbourhood of K, $u_{\mathrm{b},t}\to u_{\mathrm{b}}$ with first and second derivatives uniformly on K. *The family.* Let I be the set of fundamental indices together with one further index b, carrying $(\theta_{\mathrm{b}},u_{\mathrm{b}},u_{\mathrm{b},t})$; $t_0 = t_2$, $C = \max(C_W,C_{\mathrm{b}},C_1)$, $\delta = \max(\delta_Z,\delta_{\mathrm{b}})$, $R = \min(R_Z,R_{\mathrm{b}})$. Then $\delta(t)\to 0$, $R(t)\to\infty$, $R\ge e$, and (R1)–(R5) hold for every $i\in I$ (enlarging C and δ and shrinking R keeps them).

Step 7: hypothesis 1. Let $r = (\tau, i) \in \mathrm{Rt}$, $a = (\tau, i)$, $a^- = (\tau, i-1)$. The multidegree of r is that of a^- minus that of a, and hypotheses 2 and 3 of Lemma 31.21 for π_Z (Step 1) give $h_r(t) = h^{\mathrm{ref}}_{a^-} e^{\varphi^T_{a^-}(t)\circ\pi_Z} / h^{\mathrm{ref}}_a e^{\varphi^T_a(t)\circ\pi_Z}$ and

$$c_1\big(p_1^*\mathcal{O}(e_r), p_1^* h_r(t)\big) = F_1^*\big(\eta_{a^-}(t) - \eta_a(t)\big),$$

a combination of the forms $\theta_i + dd^c u_{i,t}$ ($i \in I$) with coefficients in $\{0, \pm 1\}$ independent of t. This is hypothesis 1 for part 1. Multiplying over $i = p+1, \dots, n$ (the product telescopes),

$$c_1\big(\mathcal{M}_{\mathrm{pl}}, p_1^* h_{\mathrm{pl}}(t)\big) = \sum_{\tau\in\Phi} F_1^*\big(\eta_{(\tau,p)}(t) - \eta_{(\tau,n)}(t)\big)$$

($p \geq 1$, so (τ, p) is a fundamental index). For $h^{(2)}(t)$: the divisor of g_ℓ lies over ℓ, so g_ℓ is a nowhere vanishing holomorphic section of $\mathcal{O}(v_\ell)$ over X^{b} and $\log|g_\ell|$ is pluriharmonic on the cone; as λ is an isomorphism of line bundles, $c_1(\mathcal{M}_{\mathrm{pl}}, h^{(2)}(t)) = \beta^* c_1(\overline{\mathcal{H}}^{\mathrm{b}}_{B,t}) = \beta^*(\theta + dd^c\varphi_t) = \theta_{\mathrm{b}} + dd^c u_{\mathrm{b},t}$. So hypothesis 1 holds for part 2. All metrics are smooth; those of $\mathcal{X}_1$ are invariant under complex conjugation, the metric of $\overline{\mathcal{H}}^{\mathrm{b}}_{B,t}$ is, being that of a hermitian line bundle (the convention behind $\widehat{\deg}$ is fixed in Section 14), and p_1, β, λ are defined over $\mathbb{Z}$.

Step 8: hypothesis 2. In part 1 the two metrics coincide; take $g_t = 0$. In part 2, $g_t = \log(h^{(1)}(t)/h^{(2)}(t))$ is a smooth function on W. For a fundamental index a put $\Delta_a(t) = \varphi_a^T(t) - \varphi_a$ on X_Z°; by (Go1)(1),(2), $0 \leq \Delta_a(t) \leq C_Z + C_5 \log \mathrm{rW}_Z$, and by (Go1)(3), $\Delta_a(t) \to 0$ uniformly on compact subsets of X_Z°. Dividing the regularised identity (hypothesis 3 of Lemma 31.21) by (S) for (τ, n) and (τ, p) gives, on $\pi_Z^{-1}(X_Z^\circ)$, $\log\big(h_{\mathrm{pl}}(t)/h^{\mathrm{sing}}_{\mathrm{pl}}\big) = \sum_\tau \big(\Delta_{(\tau,p)}(t) - \Delta_{(\tau,n)}(t)\big)\circ\pi_Z$. On $X^{\mathrm{b},\circ}$, $\log\big(h^{\mathrm{ref}} e^{\varphi_t}/h^{\mathrm{H}}\big) = \varphi_t - \varphi$. With (1), on W°

$$g_t = \sum_{\tau\in\Phi} \big(\Delta_{(\tau,p)}(t) - \Delta_{(\tau,n)}(t)\big)\circ F_1 - (\varphi_t - \varphi)\circ\beta.$$

By (R2) for the base, $0 \leq \varphi_t - \varphi \leq C_{\mathrm{b}} + C_{\mathrm{b}} \log \mathrm{rW}_{\mathrm{b}}$; with $\mathrm{rW}_Z \circ F_1 \leq \mathrm{rW}$, $\mathrm{rW}_{\mathrm{b}} \circ \beta \leq \mathrm{rW}$ and $\log \mathrm{rW} \geq 1$ on W°, $|g_t| \leq \big(2|\Phi|(C_Z + C_5) + 2C_{\mathrm{b}}\big) \log \mathrm{rW}$ there. For a compact $K \subseteq W^\circ$, $F_1(K) \subseteq X_Z^\circ$ and $\beta(K) \subseteq X^{\mathrm{b},\circ}$ are compact, so $g_t \to 0$ uniformly on K by (Go1)(3) and (R5) for the base. This is hypothesis 2 for part 2, with $\pi = \mathrm{id}$. □

D.2 The cusp mean and corrected point heights

This section proves Proposition D.21, the third part of Proposition 34.35: the cusp mean bound (Qk1) and the corrected point heights (Qk2) of the base settings of a good tower, with $(r, L) = (\mathrm{rd}_E, L_{\mathsf{H}})$. Through Proposition 34.2 they give the clauses (Bs4) and (Bs5) used in the proof of (Rt6). The core is Lemma D.22, a lower bound for the corrected height of every algebraic point. Such a point carries a principally polarised abelian scheme with $\mathcal{O}_E$-action, and its height is bounded through the maximal slope of the Lie lattice (Subsection D.2.1), the flat saturation of the Lie eigenspaces (Subsection D.2.2), and the polarised de Rham module over the base (Subsection D.2.3). Several of these results are used again in Sections D.3, D.4 and D.6.

D.2.1 Heights of algebraic points and the height dichotomy

After the statements of Proposition D.21 and Lemma D.22, this subsection proves the bound for the maximal slope of the Lie lattice of a principally polarised abelian scheme, logarithmic in the dimension and the height (Proposition D.27); it is used again for the reference lower bound in Section D.6. The input is a lower bound for the Hodge lengths of integral tangent

vectors (Proposition D.29), proved with theta functions: a theta section of small mean-square norm (Proposition D.30), its sup norms on average over the complex places (Proposition D.32), integral spanning sections (Proposition D.33), and the order of vanishing of jets (Lemma D.35). The quoted inputs are Moret-Bailly's key formula and his models with torsion sections [181].

Proposition D.21. *There is a real number $C_{\mathrm{h}} \geq 0$ with the following property. Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2, and put $\mathrm{rd}_E = |\mathrm{disc}(E)|^{1/[E:\mathbb{Q}]}$ and $L_{\mathsf{H}} = \mathrm{Re}\,(L'/L)(1, \chi_d)$, where $\mathbb{Q}(\sqrt{-d}) \subseteq E$ is the imaginary quadratic field of the host. Let B be an admissible stage, $Z \in \mathrm{Cp}$, and let a base setting of $\mathcal{T}$ at (B, Z) be given. If the setting satisfies the hypotheses (i) and (ii) of Lemma 34.36, then its cusp data have the cusp mean bound (Qk1) for $r = \mathrm{rd}_E$. If there is $t_0 > 0$ such that $\psi_{t'} \geq \log h^{\mathrm{H}} + \frac{1}{2}U - \log h^{\mathrm{ref}'}$ on $X^{\mathrm{b},\circ}$ for $0 < t' \leq t_0$, then its cusp data have corrected point heights (Qk2) at B for $(r, L) = (\mathrm{rd}_E, L_{\mathsf{H}})$ and the constant C_{h}.*

Proof. Let $C_{\mathrm{h}} = C_{\mathrm{in}}$ be the constant of Lemma D.22. Let the data be as in the statement.

The root discriminant. By (Ud1), E contains $K = \mathbb{Q}(\sqrt{-d})$ with d a prime, $d \equiv 7 \pmod 8$, so $\mathrm{disc}(K) = -d$ and $d \geq 7$. From $|\mathrm{disc}(E)| = |\mathrm{disc}(K)|^{[E:K]}\,\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{d}_{E/K}) \geq d^{[E:K]}$ we get $\mathrm{rd}_E \geq d^{1/2} \geq \sqrt{7} > 2$, so (Qk1) and (Qk2) are defined for $r = \mathrm{rd}_E$.

(Qk1). Suppose the setting satisfies (i) and (ii) of Lemma 34.36. An admissible B is $\geq 128 \geq 2$. By Lemma D.2, U is integrable for $(\omega^{\mathrm{H}})^e$ on $X^{\mathrm{b},\circ}$ and $\int_{X^{\mathrm{b},\circ}} U(\omega^{\mathrm{H}})^e \leq C_0 \int_{X^{\mathrm{b},\circ}} (\omega^{\mathrm{H}})^e$ with $C_0 = |\Phi|\,q\,(6 + \log(2|\Phi|\mathrm{rd}_E) + \log(2|\Phi|q\,\mathrm{rd}_E)) \geq 0$. By Lemma 34.36, applied to the Hodge-type base setting underlying the given one, $\int_{X^{\mathrm{b},\circ}} (\omega^{\mathrm{H}})^e = \deg^{\mathrm{H}}$. As $j' = q$, this is (Qk1) for $r = \mathrm{rd}_E$.

(Qk2). From $q \geq 1$ and $3q + 1 \leq p + q$ we have $1 \leq q \leq p$; B admissible means $B \geq 128$, $B \geq 4|\Phi|n^2$ and $|\mathrm{Bd}| \leq B$. Let t_0 be as in the statement and $0 < t' \leq t_0$. Then $\psi_{t'} \geq \log h^{\mathrm{H}} + \frac{1}{2}U - \log h^{\mathrm{ref}'}$ on $X^{\mathrm{b},\circ}$, and Lemma D.22 gives, for every $x \in \mathcal{X}^{\mathrm{b}}(\overline{\mathbb{Q}})$ with Zariski closure D_x, $h_{\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'}}(D_x) \geq -C_{\mathrm{h}}\,|\Phi|\,q\,(\log(B\,\mathrm{rd}_E) + |L_{\mathsf{H}}|)\,[\mathbf{k}(D_x) : \mathbb{Q}]$. With $j' = q$ this is (Qk2) at B for $(r, L) = (\mathrm{rd}_E, L_{\mathsf{H}})$, the constant C_{h} and this t_0. □

Lemma D.22. *There is a real number $C_{\mathrm{in}} \geq 0$ with the following property. Let $\mathfrak{u}$ be a unitary Shimura datum with $\Phi = \Phi_{\mathsf{H}}$, $n = p + q$, $1 \leq q \leq p$, put $\mathrm{rd}_E = |\mathrm{disc}(E)|^{1/[E:\mathbb{Q}]}$ and $L_{\mathsf{H}} = \mathrm{Re}\,(L'/L)(1, \chi_d)$, and let a base setting of a root tower $\mathcal{T}$ at (B, Z) with cusp data be given, where $B \geq 128$, $B \geq 4|\Phi|n^2$ and $|\mathrm{Bd}| \leq B$. Let $t' > 0$ be such that $\psi_{t'} \geq \log h^{\mathrm{H}} + \frac{1}{2}U - \log h^{\mathrm{ref}'}$ on $X^{\mathrm{b},\circ}$. Then for every $x \in \mathcal{X}^{\mathrm{b}}(\overline{\mathbb{Q}})$, with Zariski closure D_x,*

$$h_{\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'}}(D_x) \;\geq\; -C_{\mathrm{in}}\,|\Phi|\,q\,\big(\log(B\,\mathrm{rd}_E) + |L_{\mathsf{H}}|\big)\,[\mathbf{k}(D_x) : \mathbb{Q}].$$

Proof. Let C_{ll} be the constant of Proposition D.29 (as in Lemma D.26) and C_{fs} that of Proposition D.36; put $C_1 = (C_{\mathrm{ll}} + \frac{1}{2})(2 + \log(3 + 2C_{\mathrm{fs}}))$ and $C_{\mathrm{in}} = C_1 + 28$, an absolute constant. Put $\Xi = \log B + \log \mathrm{rd}_E + |L_{\mathsf{H}}|$; as $B \geq 128$ and $\mathrm{rd}_E \geq 1$, $\Xi \geq \log B \geq 1$, and $\Xi = \log(B\,\mathrm{rd}_E) + |L_{\mathsf{H}}|$. Let $K = \mathbf{k}(D_x)$ and $\xi = \bar{y}^{-1}(x)$.

Step 1: the point. Let L be a number field containing K and the image of F under the homomorphism $F \to \mathbf{k}(\xi) \to \overline{\mathbb{Q}}$ (so L contains the image of every embedding of E into $\overline{\mathbb{Q}}$, as $F \supseteq \tau(E)$ for all τ). By Lemma D.24 there is $P\colon \mathrm{Spec}\,\mathcal{O}_L \to \mathcal{X}$ as in its item 2, and $\mathcal{X}$ is noetherian. Both properties of P in item 2 persist when P is composed with $\mathrm{Spec}\,\mathcal{O}_{L'} \to \mathrm{Spec}\,\mathcal{O}_L$ for a finite extension L'/L; so L may be enlarged at any point below. Put $\mathcal{A} = P^* G_{\mathcal{X}}$ and σ_τ as in Lemma D.23.

Step 2: from the corrected line to the untwisted line. By Lemma D.25 (its hypotheses: $\mathcal{X}$ locally noetherian, and P with y defined at the image of the generic point with value x and $P(\mathfrak{p})$ in a chart for every $\mathfrak{p}$), its last assertion gives

$$h_{\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'}}(D_x) \geq h_{(\mathcal{O}(a^{\mathrm{H}}),h^\sharp)}(D_x) - |\Phi|\, q\,(\log B + 11)[K:\mathbb{Q}], \qquad h^\sharp = h^{\mathrm{ref}'} e^{\psi_{t'}},$$

and by hypothesis $\log h^\sharp \geq \log h^{\mathrm{H}} + \frac{1}{2}U$ on the cone over $X^{\mathrm{b},\circ}$. By its second assertion, $\sum_{\ell \leq B} c_\ell \leq \log B + 11$; for a prime $\ell > B$, $c_\ell \leq 2\log \ell/\ell < 2\log B/B$, since $\log u/u$ decreases for $u \geq e$; as $|\mathrm{Bd}| \leq B$, $\sum_{\ell \in \mathrm{Bd}} c_\ell \leq 3\log B + 11$. It therefore suffices to find Hermitian metrics ν as in Lemma D.23(i) (for L and P, possibly after enlarging L) with

$$\sum_{\tau \in \Phi} d_{\sigma_\tau}(\nu) \leq (C_1 + 2)\,|\Phi|\, q\, \Xi, \tag{$*$}$$

for then Lemma D.23(i) and the two displays give $h_{\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'}}(D_x) \geq -|\Phi|\, q\,((C_1+2)\Xi + 4\log B + 22)[K:\mathbb{Q}] \geq -(C_1+28)\,|\Phi|\, q\,\Xi\,[K:\mathbb{Q}]$, using $\log B \leq \Xi$ and $1 \leq \Xi$.

Step 3: the interior case. Suppose ξ lies in the abelian locus. As $U \geq 0$, $h^\sharp \geq h^{\mathrm{H}}$ on the cone over $X^{\mathrm{b},\circ}$, so by Lemma D.23(ii) the metrics $\nu_e = h_{z_e}$ satisfy the hypothesis of (i), $\mathcal{A}$ is a principally polarised abelian scheme with $\mathcal{O}_E$-action over $\mathcal{O}_L$ of relative dimension $g = |\Phi| n$ (Kottwitz condition, (Ud4)), and $d_{\sigma_\tau}(\nu) = d_{\sigma_\tau}$. We apply Proposition D.36 to $\mathcal{A}$, for each $\tau \in \Phi$, with $s_1 = \sigma_\tau$ and $s_2 = \sigma_\tau \circ c$ (c the complex conjugation of E). Its hypotheses: the Rosati involution maps $\iota(a)$ to $\iota(\bar{a})$ by the first isomorphism of (Ud4); with ρ the image of $r_{\mathbb{C}} \in K_{\mathsf{L}} \subseteq F$ in L, $\rho^2 = -d$, and for $\gamma\colon L \to k$ algebraically closed the Kottwitz condition of (Ud4) at the geometric point γ gives the eigenspace dimensions $p' = p$, $q' = q$; s_1 is injective; and for every $e\colon L \to \mathbb{C}$, $e \circ s_1$ is (the restriction of) a field embedding ψ of the CM field E, so $e \circ s_2(a) = \psi(\bar{a}) = \overline{\psi(a)}$. As $\mathrm{rk}\,\mathrm{Lie}_{s_1} + \mathrm{rk}\,\mathrm{Lie}_{s_2} = q + p$ and $1 + \log \mathrm{rd}_E + |L_{\mathsf{H}}| \leq 2\Xi$, $\sum_\tau |d_{\sigma_\tau} - d_{\sigma_\tau \circ c}| \leq 2C_{\mathrm{fs}}\, g\, \Xi$. Then Lemma D.26, with $\Psi = \{\sigma_\tau\}$, $C_{\mathrm{f}} = 2C_{\mathrm{fs}}$ (every embedding $E \to L$ lies in exactly one of Ψ, $\bar{\Psi}$; $g \leq 4|\Phi|n^2 \leq B \leq e^\Xi$), gives $\sum_\tau d_{\sigma_\tau} \leq C_1\,|\Phi|\, q\, \Xi$, which is $(*)$.

Step 4: the boundary case; the torus and the abelian part. Suppose ξ does not lie in the abelian locus. By Lemma 34.5, after enlarging L (Step 1) we have the split maximal torus T of $\mathcal{A}_L$ with $\mathcal{O}_E$-module $X = X_*(T)$ and, if $T \neq \mathcal{A}_L$, the scheme $\mathcal{B}$ and $\varphi\colon \mathcal{A} \to \mathcal{B}$, with (Ai1), (Ai2). Every z_e lies over ξ, so outside the abelian locus, and by (Gb1) and Definition 31.10 the complement of the abelian locus is near z_e a divisor with normal crossings; so Lemma 33.16 (case (B), on the complex fibre of $\mathcal{X}$ containing z_e, which is projective as $\bar{y}$ is an isomorphism onto $\mathcal{X}^{\mathrm{b}} \subseteq \mathbb{P}$) applies at z_e with a chart V_e, $V_e^* = V_e \cap$ (abelian locus), and with it Lemma 33.18 and Lemma 34.7. By Lemma 34.6 (1), (3) at the geometric points over ξ, $r_T = \dim_E(X \otimes \mathbb{Q}) \geq 1$, $\dim T = r_T[E:\mathbb{Q}]$, and, if $T \neq \mathcal{A}_L$, $\mathcal{B}_L$ satisfies the Kottwitz condition of signature $(p - r_T, q - r_T)$; in particular $r_T \leq q$, and the σ_τ-eigenspace of $\mathrm{Lie}(T)$ has dimension r_T. Fix $\mathcal{O}_E$-linearly independent $\delta_1, \ldots, \delta_{r_T} \in X$, $k = [X : \sum_a \mathcal{O}_E \delta_a]$.

Step 5: the metrics ν_e. Fix e. By Lemma 34.7 and (Ai2), $\widehat{P}_{z_e}$ maps the vanishing cycles I_e at z_e isomorphically onto $2\pi i\, X_*(T_{z_e}) = 2\pi i\, X$, $\mathcal{O}_E$-equivariantly ($\widehat{P}$ is equivariant), and $M_{z_e} = \mathrm{Lie}(G_{\mathcal{X},z_e})/\mathrm{Lie}(T_{z_e})$. Let $u_a \in I_e$ with $\widehat{P}_{z_e}(u_a) = 2\pi i\, \delta_a$; they are $\mathcal{O}_E$-linearly independent and span a subgroup of index k in I_e. Choose an ι-stable complement C_e of $\mathrm{Lie}(T_{z_e})$ (a complement in each eigenspace), and let ν_e be the Hermitian metric for which $\mathrm{Lie}(T_{z_e}) \perp C_e$, the eigenspaces of $\mathrm{Lie}(T_{z_e})$ are mutually orthogonal, on each eigenspace $\mathrm{Lie}(T_{z_e})_\sigma$ the vectors $(\delta_a)_\sigma$ are orthogonal of squared length $(4\pi)^{-1}k^{1/(r_T|\Phi|)}$, and $C_e \cong M_{z_e}$ carries πH_0. Then

$$\prod_{\tau \in \Phi} \det\big(\nu_e((\delta_a)_{\sigma_\tau}, (\delta_b)_{\sigma_\tau})\big) = k\,(4\pi)^{-r_T|\Phi|}, \qquad \prod_{\tau \in \Phi} \det\big(\nu_e((2\pi i\delta_a)_{\sigma_\tau}, (2\pi i\delta_b)_{\sigma_\tau})\big) = k\,\pi^{r_T|\Phi|},$$

and the quotient metric of ν_e on M_{z_e} (and on each eigenspace of it) is πH_0.

Step 6: the hypothesis of Lemma D.23(i) at e. For $\tau \in \Phi$ choose, by (Ce) of Lemma 33.18, a holomorphic frame

$$b_\tau = (\widehat{P}((u_1)_{\sigma_\tau}), \dots, \widehat{P}((u_{r_T})_{\sigma_\tau}), b_{\tau,r_T+1}, \dots, b_{\tau,q})$$

of $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X}})^{\mathrm{an}} = \mathrm{Lie}_{\sigma_\tau}$ near z_e (the first vectors are linearly independent at z_e; complete them), and let ς be the local generator of $\mathcal{O}(a^{\mathrm{H}})$ near w_e corresponding under θ to the dual of $\bigotimes_\tau \det b_\tau$. For $z \in V_e^*$, by (Gb3) of Definition 31.11 (the norm of the Hodge-type line is dual to the determinant of h_z on the pieces $(\tau,-)$), $\|\varsigma\|_{h^{\mathrm{H}}}(z) = \prod_\tau \det \Gamma(b_\tau(z))^{-1/2}$, Γ the πH_z-Gram matrix. The vanishing cycles I_e are non-zero ($\dim T_{z_e} > 0$), so by (Cu) of Lemma 33.18, $\mathrm{N}_z(I_e)$ may be computed with the family u, $\mathrm{N}_z(I_e) = k^{-1}\prod_\tau \det\big(H_z(\widehat{P}((u_a)_{\sigma_\tau}), \widehat{P}((u_b)_{\sigma_\tau}))\big)$, and $U \geq -\log \mathrm{N}_z(I_e)$ at $y_{\mathbb{C}}(z)$. Hence

$$\|\varsigma\|_{h^\sharp}(z) \leq \|\varsigma\|_{h^{\mathrm{H}}}(z)\, e^{-U(z)/2} \leq \Big(\frac{\prod_\tau \det \Gamma(b_\tau(z))}{\mathrm{N}_z(I_e)}\Big)^{-1/2} = \Big(k\,\pi^{r_T|\Phi|}\prod_\tau \frac{\det \Gamma(b_\tau(z))}{\det \Gamma(\widehat{P}((u)_{\sigma_\tau})(z))}\Big)^{-1/2}.$$

By (Ce) and (Cv) of Lemma 33.18, the last ratio tends, as $z \to z_e$ in V_e^*, to

$$\nu_{\sigma_\tau} = \det\big(\pi H_0(\bar b_{\tau,i}(z_e), \bar b_{\tau,l}(z_e))\big)_{i,l>r_T}.$$

Since $h^\sharp$ is a smooth cone function on all of X^{b} and V_e^* is dense near z_e,

$$\|\varsigma\|_{h^\sharp}(w_e) \leq \big(k\pi^{r_T|\Phi|}\prod_\tau \nu_{\sigma_\tau}\big)^{-1/2}.$$

On the other side, by Gram–Schmidt along $\mathrm{Lie}(T_{z_e}) \perp C_e$ and Step 5, $\det\big(\nu_e(b_{\tau,i}(z_e), b_{\tau,l}(z_e))\big) = \det\big(\nu_e((2\pi i\delta_a)_{\sigma_\tau}, (2\pi i\delta_b)_{\sigma_\tau})\big)\,\nu_{\sigma_\tau}$, using $\widehat{P}((u_a)_\sigma)(z_e) = (2\pi i\delta_a)_\sigma$; so the norm of ς for the dual of the determinant metric of ν_e is $\big(k\pi^{r_T|\Phi|}\prod_\tau \nu_{\sigma_\tau}\big)^{-1/2} \geq \|\varsigma\|_{h^\sharp}(w_e)$. This is the hypothesis of Lemma D.23(i) at e.

Step 7: the degrees. Fix τ, $s = \sigma_\tau$. Let $K_\tau = \mathrm{Lie}_s(\mathcal{A}) \cap \mathrm{Lie}(T)$, saturated in $\mathrm{Lie}_s(\mathcal{A})$ of rank r_T, and Q_τ the quotient, a projective $\mathcal{O}_L$-module of rank $q - r_T$. For the metrics induced by ν on K_τ and the quotient metrics on Q_τ, $\widehat{\deg}\,(\mathrm{Lie}_s(\mathcal{A}), \nu) = \widehat{\deg}\,K_\tau + \widehat{\deg}\,Q_\tau$. By (Ai1), $K_\tau \subseteq (X \otimes_{\mathbb{Z}} \mathcal{O}_L) \cap V_s$ in the notation of Lemma D.55, a lattice of the same rank, so $\widehat{\deg}\,K_\tau$ is at most the degree of the latter, and Lemma D.55(a) with $\Psi = \{\sigma_\tau\}$ (for every e, $\{e \circ \sigma_\tau\}$ is a CM type, as in Step 3) and $c_e = (4\pi)^{-r_T|\Phi|}$ (Step 5) gives $\sum_\tau \widehat{\deg}\,K_\tau \leq \frac{1}{2} r_T|\Phi| \log(4\pi)[L:\mathbb{Q}] \leq 2|\Phi| q\,\Xi\,[L:\mathbb{Q}]$. If $T = \mathcal{A}_L$, $Q_\tau = 0$. Otherwise $d\varphi$ maps Q_τ injectively into $\mathrm{Lie}_s(\mathcal{B})$, of the same rank $q - r_T$, and by (Ai2) and Step 5 the quotient metric is the metric of $\overline{\mathrm{Lie}}(\mathcal{B})$; so $\widehat{\deg}\,Q_\tau \leq \widehat{\deg}\,\overline{\mathrm{Lie}}_s(\mathcal{B})$. Now Step 3 applies without change to $\mathcal{B}$ in place of $\mathcal{A}$, with $(p', q') = (p - r_T, q - r_T)$ (Step 4), $g_{\mathcal{B}} = |\Phi|(n - 2r_T) \leq g$, and the Rosati condition from (Ai2) and (Cv) ($E^{\mathrm{ab}} = \mathrm{Im}\, H_0$ is $\mathcal{O}_E$-adjoint): Proposition D.36 gives $\sum_\tau |d_{\sigma_\tau}(\mathcal{B}) - d_{\sigma_\tau \circ c}(\mathcal{B})| \leq 2C_{\mathrm{fs}} g_{\mathcal{B}} \Xi$, and Lemma D.26 gives $\sum_\tau \widehat{\deg}\,\overline{\mathrm{Lie}}_{\sigma_\tau}(\mathcal{B}) \leq C_1|\Phi|(q - r_T)\,\Xi\,[L:\mathbb{Q}]$. Adding, $\sum_\tau d_{\sigma_\tau}(\nu) \leq (C_1 + 2)|\Phi| q\,\Xi$, which is $(*)$. $\square$

Lemma D.23. *Let a base setting of $\mathcal{T}$ at (B, Z) be given, and put $c_2 = 2\log 2$ and $c_\ell = \log \ell/(\ell - 1)$ for odd primes ℓ. Then the isomorphism of (Hy2) extends uniquely to an isomorphism of invertible sheaves $\theta\colon \bar y^* \mathcal{O}(a^{\mathrm{H}}) \to \bigotimes_{\tau \in \Phi} (\det \mathcal{V}_{(\tau,-)})^\vee$ on $\mathcal{X}$.*

Let $x \in \mathcal{X}^{\mathrm{b}}(\overline{\mathbb{Q}})$, with Zariski closure D_x and residue field $K = \mathbf{k}(D_x)$, let $L \supseteq K$ be a number field and $P\colon \mathrm{Spec}\,\mathcal{O}_L \to \mathcal{X}$ a morphism as in Lemma D.24(2), and put $\mathcal{A} = P^* G_{\mathcal{X}}$, a semiabelian scheme over $\mathcal{O}_L$ with the induced $\mathcal{O}_E$-action ι. For $\tau \in \Phi$ let $\sigma_\tau\colon \mathcal{O}_E \to \mathcal{O}_L$ be $\bar\tau$ followed by the map $\mathcal{O}_F \to \mathcal{O}_L$ induced by P, and $\mathrm{Lie}_{\sigma_\tau}(\mathcal{A}) = \{v \in \mathrm{Lie}(\mathcal{A}) : \iota(a)v = \sigma_\tau(a)v \text{ for all } a \in \mathcal{O}_E\}$. For a field embedding $e\colon L \to \mathbb{C}$ let z_e be the complex point of $\mathcal{X}$ given by e and P, and $w_e = y_{\mathbb{C}}(z_e) \in X^{\mathrm{b}}$; then $\mathrm{Lie}(\mathcal{A}) \otimes_{\mathcal{O}_L, e} \mathbb{C} = \mathrm{Lie}(G_{\mathcal{X}, z_e})$, which contains $\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X},z_e}) = P^*\mathcal{V}_{(\tau,-)} \otimes_{\mathcal{O}_L, e} \mathbb{C}$.

(i) (Every point.) Let $\nu = (\nu_e)_e$ be Hermitian metrics on the spaces $\mathrm{Lie}(G_{\mathcal{X},z_e})$, and $h^\sharp$ a smooth cone function of multidegree a^{H}, such that for every e the norm of $h^\sharp$ on the fibre of $\mathcal{O}(a^{\mathrm{H}})$ at w_e is at most the norm that corresponds under θ to the dual of the determinant metric of ν_e on $\bigotimes_\tau \det \mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X},z_e})$. Put $d_{\sigma_\tau}(\nu) = \widehat{\deg}\,(\mathrm{Lie}_{\sigma_\tau}(\mathcal{A}),\nu)/[L:\mathbb{Q}]$, the Arakelov degree for the metrics induced by ν. Then

$$h_{(\mathcal{O}(a^{\mathrm{H}}),\,h^\sharp)|_{\mathcal{X}^{\mathrm{b}}}}(D_x) \;\geq\; -\Big(\sum_{\tau\in\Phi} d_{\sigma_\tau}(\nu) + |\Phi|\,q\sum_{\ell\in\mathrm{Bd}} c_\ell\Big)[K:\mathbb{Q}].$$

(ii) (Interior points.) Suppose that $\bar y^{-1}(x)$ lies in the abelian locus (equivalently, every complex point of $\mathcal{X}^{\mathrm{b}}$ over x lies in $X^{\mathrm{b},\circ}$). Then $\mathcal{A}$ is a principally polarised abelian scheme with $\mathcal{O}_E$-action over $\mathcal{O}_L$, and for every smooth cone function $h^\sharp$ of multidegree a^{H} with $h^\sharp \geq h^{\mathrm{H}}$ on the cone over $X^{\mathrm{b},\circ}$ the hypothesis of (i) holds with $\nu_e = h_{z_e}$, the metric of $\overline{\mathrm{Lie}}(\mathcal{A})$ at e. Hence, with $d_{\sigma_\tau} = \widehat{\deg}\,\overline{\mathrm{Lie}}_{\sigma_\tau}(\mathcal{A})/[L:\mathbb{Q}]$,

$$h_{(\mathcal{O}(a^{\mathrm{H}}),\,h^\sharp)|_{\mathcal{X}^{\mathrm{b}}}}(D_x) \;\geq\; -\Big(\sum_{\tau\in\Phi} d_{\sigma_\tau} + |\Phi|\,q\sum_{\ell\in\mathrm{Bd}} c_\ell\Big)[K:\mathbb{Q}].$$

Proof. **Step 0: the extension θ.** By (Gb3), on every member $D(f)$ of the finite covers of the charts $\mathcal{X}_{c'}$, which cover $\mathcal{X}$ by (Gb1), the generator $\prod_k y_{k,c'_k}^{a_k^{\mathrm{H}}}$ of $\bar y^*\mathcal{O}(a^{\mathrm{H}})$ corresponds under the isomorphism of (Hy2) to a unit times a generator of $\bigotimes_\tau(\det \mathcal{V}_{(\tau,-)})^\vee$. Hence that isomorphism extends to an isomorphism θ of invertible sheaves on $\mathcal{X}$; it is unique, as $\mathcal{X}$ is integral and the abelian locus is dense.

Step 1: the point and the line at P. Write ξ for the image under P of the generic point of $\operatorname{Spec}\mathcal{O}_L$, so that $\tilde x = \bar y\circ P\colon \operatorname{Spec}\mathcal{O}_L \to \mathcal{X}^{\mathrm{b}}$ extends x. As $\tilde x$ factors through the normalisation of D_x, $[L:K]\,h_{(\mathcal{O}(a^{\mathrm{H}}),h^\sharp)}(D_x) = \widehat{\deg}\,\tilde x^*(\mathcal{O}(a^{\mathrm{H}}),h^\sharp)$. The scheme $\mathcal{A} = P^*G_{\mathcal{X}} = P^*f^*G_{\mathcal{Y}}$ is semiabelian over $\mathcal{O}_L$ with $\mathcal{O}_E$-action and a cubical line bundle (Definition 29.6, (Fh5)), and $\mathrm{Lie}(\mathcal{A}) = P^*\mathrm{Lie}(G_{\mathcal{X}})$, since the Lie algebra commutes with base change; in particular $\mathrm{Lie}(\mathcal{A})\otimes_{\mathcal{O}_L,e}\mathbb{C} = \mathrm{Lie}(G_{\mathcal{X},z_e})$. Pulling back θ gives $\tilde x^*\mathcal{O}(a^{\mathrm{H}}) \cong \bigotimes_\tau(\det M_\tau)^\vee$ with $M_\tau = P^*\mathcal{V}_{(\tau,-)}$, a projective $\mathcal{O}_L$-module of rank q, and $M_\tau\otimes_{\mathcal{O}_L,e}\mathbb{C} = \mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X},z_e})$ by (Gb2). Under the hypothesis of (i), $\|\cdot\|_{h^\sharp}$ is at every e at most the norm of $\bigotimes_\tau(\det\overline{M}_\tau)^\vee$, where $\overline{M}_\tau$ carries the metrics induced by ν; computing both degrees with one rational section,

$$\widehat{\deg}\,\tilde x^*(\mathcal{O}(a^{\mathrm{H}}),h^\sharp) \;\geq\; \sum_{\tau\in\Phi}\widehat{\deg}\,(\det\overline{M}_\tau)^\vee = -\sum_{\tau\in\Phi}\widehat{\deg}\,\overline{M}_\tau.$$

Step 2: M_τ inside the Lie lattice. The stalk $\mathcal{V}_{(\tau,-),\xi}$ is $\mathrm{dR}_{(\tau,-),\xi} = \mathrm{Lie}_\xi \cap \mathrm{Li}_{(\tau,-)}$ (ξ lies over no prime). Since $\mathbb{Q}\subseteq\mathcal{O}_\xi$, $\mathcal{O}_E\otimes\mathcal{O}_\xi$ is a product of copies of $\mathcal{O}_\xi$ indexed by the embeddings, so $\mathrm{Lie}_\xi = \bigoplus_s \mathrm{dR}_{s,\xi}$ and $\mathcal{V}_{(\tau,-),\xi}\otimes\mathbf{k}(\xi)\to\mathrm{Lie}(G_{\mathcal{X},\xi})$ is injective with image the $\bar\tau$-eigenspace. Hence the map $M_\tau\to M_\tau\otimes L = \mathcal{V}_{(\tau,-),\xi}\otimes_{\mathcal{O}_\xi}L\to\mathrm{Lie}_\xi\otimes_{\mathcal{O}_\xi}L = \mathrm{Lie}(\mathcal{A}_L)$ is injective (M_τ is torsion free), and its image (again written M_τ) is a lattice in the σ_τ-eigenspace $\mathrm{Lie}_{\sigma_\tau}(\mathcal{A}_L)$, because $\iota(a)$ acts on $\mathcal{V}_{(\tau,-)}$ by $\bar\tau(a)$ and both have dimension q. For a maximal ideal $\mathfrak{p}$ the localisation $(M_\tau)_\mathfrak{p}$ is the image of $\mathcal{V}_{(\tau,-),P(\mathfrak{p})}\otimes\mathcal{O}_{L,\mathfrak{p}}$ under $P^\sharp$, and $\mathrm{Lie}_{\sigma_\tau}(\mathcal{A})_\mathfrak{p} = \mathrm{Lie}(\mathcal{A})_\mathfrak{p}\cap\mathrm{Lie}_{\sigma_\tau}(\mathcal{A}_L)$. Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}_L$ over the prime p and $x' = P(\mathfrak{p})$, a point over p (in the abelian locus or not).

- If $p\notin\mathrm{Bd}$, then $\mathcal{V}_{(\tau,-),x'} = \mathrm{dR}_{(\tau,-),x'}\subseteq\mathrm{Lie}_{x'}$, so $(M_\tau)_\mathfrak{p}\subseteq\mathrm{Lie}_{\sigma_\tau}(\mathcal{A})_\mathfrak{p}$.

- If $p=\ell\in\mathrm{Bd}$, then $\mathcal{V}_{(\tau,-),x'}=\mathrm{HT}_{(\tau,-),x'}(\ell^{R(\ell)})$. We apply Lemma 35.32 at x' with $p=\ell$, $R=R(\ell)$: $\mathcal{X}$ is noetherian by Lemma D.24(1); the $\ell^{R(\ell)}$-torsion of $G_{\mathcal{X}}$ is rational over Ω and $\mathrm{ht}_{x'}$ is free of rank $\dim \mathrm{Li}$ at every point over ℓ by (Gb1) (Hodge–Tate freeness); that lemma does not require $G_{\mathcal{X}}$ to be abelian at x'. Its part (3) gives $\varpi_\ell\,\mathcal{V}_{(\tau,-),x'}\subseteq \mathrm{dR}_{(\tau,-),x'}$, so $P^\sharp(\varpi_\ell)(M_\tau)_{\mathfrak{p}}\subseteq \mathrm{Lie}_{\sigma_\tau}(\mathcal{A})_{\mathfrak{p}}$. Moreover $P^\sharp(\zeta)$ is a primitive ℓ-th root of unity in L for odd ℓ, so $-\log|P^\sharp(\varpi_\ell)|_{\mathfrak{p}}=[L_{\mathfrak{p}}:\mathbb{Q}_\ell]\,\frac{\log\ell}{\ell-1}$, and $-\log|4|_{\mathfrak{p}}=[L_{\mathfrak{p}}:\mathbb{Q}_2]\,2\log 2$ for $\ell=2$; in both cases this is $[L_{\mathfrak{p}}:\mathbb{Q}_\ell]\,c_\ell$.

Let $N_\tau\subseteq \mathrm{Lie}_{\sigma_\tau}(\mathcal{A}_L)$ be the $\mathcal{O}_L$-lattice equal to $\mathrm{Lie}_{\sigma_\tau}(\mathcal{A})_{\mathfrak{p}}$ at $\mathfrak{p}$ not over Bd and to

$$P^\sharp(\varpi_\ell)^{-1}\mathrm{Lie}_{\sigma_\tau}(\mathcal{A})_{\mathfrak{p}}$$

at $\mathfrak{p}$ over $\ell\in\mathrm{Bd}$, with the metrics induced by ν. Then $M_\tau\subseteq N_\tau$ have the same rank and metrics, so $\widehat{\deg}\,\overline{M}_\tau=\widehat{\deg}\,\overline{N}_\tau-\log\#(N_\tau/M_\tau)\le\widehat{\deg}\,\overline{N}_\tau$, and

$$\widehat{\deg}\,\overline{N}_\tau=\widehat{\deg}\,(\mathrm{Lie}_{\sigma_\tau}(\mathcal{A}),\nu)+q\sum_{\ell\in\mathrm{Bd}}\sum_{\mathfrak{p}\mid\ell}-\log|P^\sharp(\varpi_\ell)|_{\mathfrak{p}}=\widehat{\deg}\,(\mathrm{Lie}_{\sigma_\tau}(\mathcal{A}),\nu)+q\,[L:\mathbb{Q}]\sum_{\ell\in\mathrm{Bd}}c_\ell,$$

using $\sum_{\mathfrak{p}\mid\ell}[L_{\mathfrak{p}}:\mathbb{Q}_\ell]=[L:\mathbb{Q}]$ and that multiplying a rank-q lattice at $\mathfrak{p}$ by π^{-1}, $|\pi|_{\mathfrak{p}}<1$, raises its degree by $-q\log|\pi|_{\mathfrak{p}}$.

Step 3: conclusion of (i). By Steps 1 and 2,

$$[L:K]\,h_{(\mathcal{O}(a^{\mathrm{H}}),h^\sharp)}(D_x)\ge-\sum_{\tau\in\Phi}\widehat{\deg}\,(\mathrm{Lie}_{\sigma_\tau}(\mathcal{A}),\nu)-|\Phi|\,q\,[L:\mathbb{Q}]\sum_{\ell\in\mathrm{Bd}}c_\ell;$$

dividing by $[L:K]$ and using $[L:\mathbb{Q}]/[L:K]=[K:\mathbb{Q}]$ gives (i).

Step 4: (ii). Suppose $\bar y^{-1}(x)$ lies in the abelian locus. Then $\xi=\bar y^{-1}(x)$, and the generic fibre of $\mathcal{A}$ is the base change of the fibre of $G_{\mathcal{X}}$ at ξ, which lies over $j(\mathrm{Sh}_F)$, so it is an abelian variety with ample, principally polarising line bundle (Definition 29.5); so $\mathcal{A}$ is as in Definition 31.6. The equivalence in the hypothesis: by (Gb1) ∂X^{b} is the image under $y_{\mathbb{C}}=\bar y_{\mathbb{C}}$ of the complex points of $\mathcal{X}$ outside the abelian locus, $\bar y$ is an isomorphism, and a complex point over ξ lies in the abelian locus if and only if ξ does. Every z_e lies over ξ, hence in the abelian locus, and $w_e\in X^{\mathrm{b},\circ}$. By (Gb2), (Gb3) and the convention $\|\sigma\|=|\sigma|/h$, the metric given by h^{H} at w_e is, under θ, the dual of the determinant of h_{z_e} on $\bigotimes_\tau\det\mathrm{Lie}_{(\tau,-)}(G_{\mathcal{X},z_e})$. Since $h^\sharp\ge h^{\mathrm{H}}$ there, $\|\cdot\|_{h^\sharp}\le\|\cdot\|_{h^{\mathrm{H}}}$ at w_e, which is the hypothesis of (i) for $\nu_e=h_{z_e}=\pi H_{z_e}$, the metric of $\overline{\mathrm{Lie}}(\mathcal{A})$ at e (Definition 31.6). Then $d_{\sigma_\tau}(\nu)=d_{\sigma_\tau}$, and (i) gives (ii). □

Lemma D.24. *Let a base setting of $\mathcal{T}$ at (B,Z) be given, with cover $\mathcal{X}$, function field Ω and representing coordinates y. For an index c' with $y_{c'}\neq0$ let $\mathbb{Z}[y]_{c'}$, the points lying in the chart c' and the affine open subset $\mathcal{X}_{c'}$ be as in the coordinate-chart clause of (Gb1). Then:*

1. *there is a unique isomorphism of schemes $\bar y\colon\mathcal{X}\to\mathcal{X}^{\mathrm{b}}$ that induces on function fields the identification of $\mathbf{k}(\mathcal{X}^{\mathrm{b}})$ with Ω given by y; it maps $\mathcal{X}_{c'}$ onto $\mathcal{X}^{\mathrm{b}}\cap\{Y_{c'}\neq0\}$, where $Y_{c'}=\prod_k Y_{k,c'_k}$ in the multihomogeneous coordinates Y of $\mathbb{P}$. In particular $\mathcal{X}$ is of finite type over $\mathbb{Z}$, hence noetherian, and the rational map y is defined at every point of $\mathcal{X}$;*
2. *for every $x\in\mathcal{X}^{\mathrm{b}}(\overline{\mathbb{Q}})$, with Zariski closure D_x and residue field $K_x=\mathbf{k}(D_x)$, and every number field $L\supseteq K_x$, there is a morphism $P\colon\operatorname{Spec}\mathcal{O}_L\to\mathcal{X}$ such that y is defined at the image of the generic point and sends it to x, and such that for every maximal ideal $\mathfrak{p}$ of $\mathcal{O}_L$ the image $P(\mathfrak{p})$ lies in some chart.*

So every algebraic point of $\mathcal{X}^{\mathrm{b}}$ satisfies the lifting hypothesis of Lemma D.25, with $L=K_x$.

Proof. **Step 1: the charts of $\mathcal{X}^{\mathrm{b}}$.** For an index c' write $\mathbb{A}_{c'} = \{Y_{c'} \neq 0\} = \operatorname{Spec} \mathbb{Z}[Y_{k,i}/Y_{k,c'_k}]$ for the standard affine chart of $\mathbb{P}$ and $\mathcal{X}^{\mathrm{b}}_{c'} = \mathcal{X}^{\mathrm{b}} \cap \mathbb{A}_{c'}$, an open subset of the integral scheme $\mathcal{X}^{\mathrm{b}}$. If $y_{c'} = 0$ in Ω, the generic point of $\mathcal{X}^{\mathrm{b}}$ is not in $\mathbb{A}_{c'}$, so the open set $\mathcal{X}^{\mathrm{b}}_{c'}$ contains no generic point and is empty. If $y_{c'} \neq 0$, then $\mathcal{X}^{\mathrm{b}}_{c'}$ is a non-empty closed integral subscheme of $\mathbb{A}_{c'}$ whose generic point is the Ω-point $Y_{k,i}/Y_{k,c'_k} \mapsto y_{k,i}/y_{k,c'_k}$; its ideal is therefore the kernel of the evaluation $\mathbb{Z}[Y_{k,i}/Y_{k,c'_k}] \to \Omega$ (a function on an integral affine scheme vanishes if and only if it vanishes at the generic point), and $\mathcal{X}^{\mathrm{b}}_{c'} = \operatorname{Spec} \mathbb{Z}[y]_{c'}$. The sets $\mathbb{A}_{c'}$ cover $\mathbb{P}$, so the $\mathcal{X}^{\mathrm{b}}_{c'}$ with $y_{c'} \neq 0$ cover $\mathcal{X}^{\mathrm{b}}$.

Step 2: the isomorphism. For $y_{c'} \neq 0$ the affine open subscheme $\mathcal{X}_{c'}$ has ring $\mathbb{Z}[y]_{c'}$ by the coordinate-chart clause of (Gb1), so the identity of $\mathbb{Z}[y]_{c'}$ gives an isomorphism $\bar{y}_{c'} \colon \mathcal{X}_{c'} \to \mathcal{X}^{\mathrm{b}}_{c'}$, which induces the identity of Ω on function fields. On $\mathcal{X}_{c'} \cap \mathcal{X}_{c''}$ the two morphisms $\bar{y}_{c'}, \bar{y}_{c''}$ to the separated scheme $\mathcal{X}^{\mathrm{b}}$ agree at the generic point (both send it to the generic point of $\mathcal{X}^{\mathrm{b}}$ by the identity of Ω); since $\mathcal{X}^{\mathrm{b}}$ is of finite type over $\mathbb{Z}$ they agree on a non-empty open subset, and two morphisms from a reduced scheme to a separated scheme that agree on a dense open subset are equal. By the coordinate-chart clause of (Gb1) every point lies in some chart, so the $\mathcal{X}_{c'}$ cover $\mathcal{X}$, so the $\bar{y}_{c'}$ glue to a morphism $\bar{y} \colon \mathcal{X} \to \mathcal{X}^{\mathrm{b}}$. We have $\bar{y}^{-1}(\mathcal{X}^{\mathrm{b}}_{c'}) = \mathcal{X}_{c'}$: the inclusion $\supseteq$ holds by construction, and if $\bar{y}(x') \in \mathcal{X}^{\mathrm{b}}_{c'}$ then $\mathbb{Z}[y]_{c'} \subseteq \mathcal{O}_{\mathcal{X}^{\mathrm{b}},\bar{y}(x')}$, and the local homomorphism $\bar{y}^\sharp$, which is the inclusion of subrings of Ω, gives $\mathbb{Z}[y]_{c'} \subseteq \mathcal{O}_{\mathcal{X},x'}$, i.e. $x' \in \mathcal{X}_{c'}$. So over each member $\mathcal{X}^{\mathrm{b}}_{c'}$ of an open cover of $\mathcal{X}^{\mathrm{b}}$ the morphism $\bar{y}$ restricts to the isomorphism $\bar{y}_{c'}$; hence $\bar{y}$ is an isomorphism. Uniqueness: a morphism inducing the identity of Ω agrees with $\bar{y}$ at the generic point, hence everywhere, by the argument above. Since $\mathcal{X}^{\mathrm{b}}$ is of finite type over $\mathbb{Z}$, so is $\mathcal{X}$; and y is defined at every point, being the composite of $\bar{y}$ with the inclusion $\mathcal{X}^{\mathrm{b}} \subseteq \mathbb{P}$.

Step 3: integral points of $\mathcal{X}^{\mathrm{b}}$. Let x and $L \supseteq K_x$ be given; x defines a morphism $\operatorname{Spec} L \to \mathcal{X}^{\mathrm{b}}$, with multihomogeneous coordinates $w = (w_{k,i})$, $w_{k,i} \in L$, not all zero in each factor k. For c' with $w_{c'} = \prod_k w_{k,c'_k} \neq 0$ put $\mathcal{O}_L[w]_{c'} = \mathcal{O}_L[\, w_{k,i}/w_{k,c'_k} \,] \subseteq L$. Let T be the finite set of maximal ideals $\mathfrak{p}$ with $|w_{k,i}/w_{k,c'_k}|_{\mathfrak{p}} > 1$ for some (k,i). Since the class group of L is finite, there is $s \in \mathcal{O}_L \smallsetminus \{0\}$ whose prime divisors are exactly the members of T (take s generating $(\prod_{\mathfrak{p} \in T} \mathfrak{p})^h$, h the class number). Both $\mathcal{O}_L[w]_{c'}$ and $\mathcal{O}_L[1/s]$ are rings between $\mathcal{O}_L$ and L, hence each is the intersection of the $\mathcal{O}_{L,\mathfrak{p}}$ containing it; a maximal ideal $\mathfrak{p}$ has $\mathcal{O}_{L,\mathfrak{p}} \supseteq \mathcal{O}_L[w]_{c'}$ if and only if $\mathfrak{p} \notin T$, if and only if $\mathcal{O}_{L,\mathfrak{p}} \supseteq \mathcal{O}_L[1/s]$. So $\mathcal{O}_L[w]_{c'} = \mathcal{O}_L[1/s]$, and $V_{c'} = \operatorname{Spec} \mathcal{O}_L[w]_{c'}$ is the open subset $D(s)$ of $\operatorname{Spec} \mathcal{O}_L$: the generic point and the maximal ideals not in T. Since x lies in $\mathcal{X}^{\mathrm{b}}_{c'} = \operatorname{Spec} \mathbb{Z}[y]_{c'}$, the ring homomorphism $\mathbb{Z}[y]_{c'} \to L$ of x sends $y_{k,i}/y_{k,c'_k}$ to $w_{k,i}/w_{k,c'_k}$ and takes values in $\mathcal{O}_L[w]_{c'}$; this gives a morphism $\tilde{x}_{c'} \colon V_{c'} \to \mathcal{X}^{\mathrm{b}}_{c'}$ extending x. Every maximal ideal $\mathfrak{p}$ lies in some $V_{c'}$: choose, for every k, c'_k with $|w_{k,c'_k}|_{\mathfrak{p}} = \max_i |w_{k,i}|_{\mathfrak{p}}$. On overlaps the $\tilde{x}_{c'}$ agree at the generic point, hence agree (as in Step 2: $\operatorname{Spec} \mathcal{O}_L$ is reduced and $\mathcal{X}^{\mathrm{b}}$ separated). They glue to $\tilde{x} \colon \operatorname{Spec} \mathcal{O}_L \to \mathcal{X}^{\mathrm{b}}$ extending x, with $\tilde{x}(\mathfrak{p}) \in \mathcal{X}^{\mathrm{b}}_{c'}$ whenever $\mathfrak{p} \in V_{c'}$.

Step 4: conclusion. Put $P = \bar{y}^{-1} \circ \tilde{x}$. The generic point of $\operatorname{Spec} \mathcal{O}_L$ goes to $\bar{y}^{-1}(x)$, where y is defined (Step 2) with value $\bar{y}(\bar{y}^{-1}(x)) = x$. For a maximal ideal $\mathfrak{p}$ and c' with $\mathfrak{p} \in V_{c'}$, $\tilde{x}(\mathfrak{p}) \in \mathcal{X}^{\mathrm{b}}_{c'}$, so $P(\mathfrak{p}) \in \bar{y}^{-1}(\mathcal{X}^{\mathrm{b}}_{c'}) = \mathcal{X}_{c'}$ by Step 2: $P(\mathfrak{p})$ lies in the chart c'. With $L = K_x$ this is the hypothesis of Lemma D.25, whose remaining hypothesis ($\mathcal{X}$ locally noetherian) holds by item 1 (Step 2). □

Lemma D.25. *Let a base setting of $\mathcal{T}$ at (B, Z) be given, with cover $\mathcal{X}$, and suppose that $\mathcal{X}$ is locally noetherian. A point x' of $\mathcal{X}$ lies in a chart c' as in the coordinate-chart clause of (Gb1), that is, $y_{k,i}/y_{k,c'_k} \in \mathcal{O}_{\mathcal{X},x'}$ for all (k,i). Let $x \in \mathcal{X}^{\mathrm{b}}(\overline{\mathbb{Q}})$ with Zariski closure D_x and residue field $K = \mathbf{k}(D_x)$, and suppose that there are a number field $L \supseteq K$ and a morphism $P \colon \operatorname{Spec} \mathcal{O}_L \to \mathcal{X}$ such that the rational map y is defined at the image of the generic point and sends it to x, and such that for every maximal ideal $\mathfrak{p}$ of $\mathcal{O}_L$ the image $x_{\mathfrak{p}} \in \mathcal{X}$ of $\mathfrak{p}$ lies in some chart. Put $c_2 = 2 \log 2$ and $c_\ell = \log \ell/(\ell - 1)$ for odd primes ℓ. Then for every multidegree a*

and every smooth cone function h of multidegree a,

$$-|\Phi|\,q\Big(\sum_{\ell\in S'_B} c_\ell\Big)[K:\mathbb{Q}] \;\le\; h_{(\mathcal{O}(a+\sum_{\ell\in S'_B} v_\ell),\, h\prod_{\ell\in S'_B}|g_\ell|)}(D_x) - h_{(\mathcal{O}(a),h)}(D_x) \;\le\; 0,$$

the line bundles being restricted to $\mathcal{X}^{\mathrm{b}}$; and

$$\sum_{\ell\in S'_B} c_\ell \;\le\; \sum_{\ell\le B} c_\ell \;\le\; \log B + 11.$$

In particular, for a base setting with cusp data, every $t'>0$ and x as above, $h_{\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'}}(D_x) \ge h_{(\mathcal{O}(a^{\mathrm{H}}),\, h^{\mathrm{ref}'}e^{\psi_{t'}})}(D_x) - |\Phi|\,q\,(\log B+11)\,[K:\mathbb{Q}]$.

Proof. **Step 1: heights at x by local terms.** Let $(\mathcal{O}(b),h_b)$ be a line bundle of $\mathbb{P}$ with a smooth cone function, restricted to $\mathcal{X}^{\mathrm{b}}$. The L-point x extends uniquely to $\tilde{x}\colon \operatorname{Spec}\mathcal{O}_L\to\mathcal{X}^{\mathrm{b}}$, since $\mathbb{P}$ is proper, and $\tilde{x}$ factors through the normalisation of D_x, so $[L:K]\,h_{(\mathcal{O}(b),h_b)}(D_x) = \widehat{\deg}\,\tilde{x}^*(\mathcal{O}(b),h_b)$. Let σ be a quotient of multihomogeneous polynomials of multidegree difference b which is defined and non-zero at x. At a maximal ideal $\mathfrak{p}$, choose c with $|y_{k,c_k}(x)|_{\mathfrak{p}} = \max_i|y_{k,i}(x)|_{\mathfrak{p}}$ for every k; then $\tilde{x}$ maps $\operatorname{Spec}\mathcal{O}_{L,\mathfrak{p}}$ into the standard affine chart $\{y_c\neq 0\}$ of $\mathbb{P}$, on which the monomial $y_c^b=\prod_k y_{k,c_k}^{b_k}$ generates $\mathcal{O}(b)$. Hence

$$[L:K]\,h_{(\mathcal{O}(b),h_b)}(D_x) = \sum_{\mathfrak{p}} -\log\Big|\frac{\sigma}{y_c^b}(x)\Big|_{\mathfrak{p}} + \sum_{e\colon L\to\mathbb{C}} -\log\frac{|\sigma(e(x))|}{h_b(e(x))},$$

and the $\mathfrak{p}$-term does not depend on the choice of a maximising c. **Step 2: the difference.** Apply Step 1 to $(\mathcal{O}(a),h)$ with σ and to the twisted bundle with the section $\sigma\prod_\ell g_\ell$, which is defined and non-zero at x since the divisors of the g_ℓ are vertical. At the archimedean places $|\sigma\prod g_\ell|/(h\prod|g_\ell|)=|\sigma|/h$, so these terms cancel. Since max over monomials of multidegree $a+\sum v_\ell$ is attained at the same c for all multidegrees,

$$[L:K]\,\big(h_{\mathrm{tw}}(D_x)-h(D_x)\big) = \sum_{\ell\in S'_B}\sum_{\mathfrak{p}} -\log|f_{\ell,\mathfrak{p}}(x)|_{\mathfrak{p}},\qquad f_{\ell,\mathfrak{p}} = g_\ell/y_{c(\mathfrak{p})}^{v_\ell},$$

with $c(\mathfrak{p})$ any maximising index at $\mathfrak{p}$. **Step 3: $\mathfrak{p}\nmid\ell$.** On $\mathcal{X}^{\mathrm{b}}\cap\{y_c\neq0\}$ the rational function $f_{\ell,\mathfrak{p}}$ is a local equation of the Cartier divisor $\operatorname{div}(g_\ell)$, which is supported over ℓ; so it is a unit at every point of this chart not over ℓ. The closed point of $\tilde{x}$ at $\mathfrak{p}$ lies over $p\neq\ell$; hence $|f_{\ell,\mathfrak{p}}(x)|_{\mathfrak{p}}=1$. **Step 4: $\mathfrak{p}\mid\ell$, $\ell\in S'_B$.** By hypothesis $x_{\mathfrak{p}}=P(\mathfrak{p})$ lies in a chart c'. The local homomorphism $P^\sharp\colon\mathcal{O}_{\mathcal{X},x_{\mathfrak{p}}}\to\mathcal{O}_{L,\mathfrak{p}}$ sends $y_{k,i}/y_{k,c'_k}$ to its value at x (the map y is defined at the generic point of P with image x), so $|y_{k,i}(x)/y_{k,c'_k}(x)|_{\mathfrak{p}}\le1$ and c' is maximising at $\mathfrak{p}$; take $c(\mathfrak{p})=c'$. The point $x_{\mathfrak{p}}$ lies over ℓ and in the chart c', so by (Gb4) $f_{\ell,\mathfrak{p}}=u\prod_{\tau\in\Phi}\det T_\tau$ with $u\in\mathcal{O}^\times_{\mathcal{X},x_{\mathfrak{p}}}$, where T_τ is the matrix of a basis of $\mathrm{HT}_{(\tau,-),x_{\mathfrak{p}}}(\ell^{R(\ell)})$ in a basis of $\mathcal{V}_{(\tau,-),x_{\mathfrak{p}}}$, a $q\times q$ matrix over Ω. Since $\ell\notin\mathrm{Bd}$, $\mathcal{V}_{(\tau,-),x_{\mathfrak{p}}}=\mathrm{dR}_{(\tau,-),x_{\mathfrak{p}}}$ by (Fh5). We apply Lemma 35.32 with $p=\ell$, $R=R(\ell)$ at $x_{\mathfrak{p}}$: $\mathcal{X}$ is a locally noetherian cover; every $\ell^{R(\ell)}$-torsion point of $G_{\mathcal{X}}$ over an algebraic closure of Ω is defined over Ω by (Gb1); and $\mathrm{ht}_{x_{\mathfrak{p}}}$ is locally free, hence free, of rank $g=\dim\mathrm{Li}$ by the Hodge–Tate freeness of (Gb1). It gives $\varpi_\ell\in\mathcal{O}_{\mathcal{X},x_{\mathfrak{p}}}$, $\mathrm{dR}_{(\tau,-)}\subseteq\mathrm{HT}_{(\tau,-)}(\ell^{R(\ell)})$ and $\varpi_\ell\,\mathrm{HT}_{(\tau,-)}(\ell^{R(\ell)})\subseteq\mathrm{dR}_{(\tau,-)}$ at $x_{\mathfrak{p}}$. The first inclusion says that T_τ^{-1} has entries in $\mathcal{O}_{\mathcal{X},x_{\mathfrak{p}}}$, the second that $\varpi_\ell T_\tau$ has. Hence

$$f_{\ell,\mathfrak{p}}^{-1}\in\mathcal{O}_{\mathcal{X},x_{\mathfrak{p}}},\qquad \varpi_\ell^{q|\Phi|}f_{\ell,\mathfrak{p}}\in\mathcal{O}_{\mathcal{X},x_{\mathfrak{p}}}.$$

Apply $P^\sharp$. The image ζ' of the primitive ℓ-th root of unity ζ of Lemma 35.32 satisfies $\Phi_\ell(\zeta')=0$ for the ℓ-th cyclotomic polynomial, and $\Phi_\ell(1)=\ell\neq0$ in L, so ζ' is a primitive ℓ-th root of

unity and $P^\sharp(\varpi_\ell) \neq 0$, with $-\log|P^\sharp(\varpi_\ell)|_{\mathfrak{p}} = [L_{\mathfrak{p}} : \mathbb{Q}_\ell]\, c_\ell$ (for ℓ odd, $\zeta' - 1$ has ℓ-adic valuation $1/(\ell-1)$; for $\ell = 2$, $\varpi_2 = 4$). Both images are integral and their product is $P^\sharp(\varpi_\ell)^{q|\Phi|} \neq 0$; and $P^\sharp(f_{\ell,\mathfrak{p}}) = f_{\ell,\mathfrak{p}}(x)$. Therefore

$$-q|\Phi|\,[L_{\mathfrak{p}} : \mathbb{Q}_\ell]\, c_\ell \;\leq\; -\log|f_{\ell,\mathfrak{p}}(x)|_{\mathfrak{p}} \;\leq\; 0.$$

Step 5: summation. Summing Step 4 over $\mathfrak{p} \mid \ell$ ($\sum_{\mathfrak{p}|\ell}[L_{\mathfrak{p}} : \mathbb{Q}_\ell] = [L : \mathbb{Q}]$), Step 3 over the other $\mathfrak{p}$, and then over $\ell \in S'_B$, Step 2 gives $-q|\Phi|\,[L : \mathbb{Q}] \sum_{\ell \in S'_B} c_\ell \leq [L : K](h_{\mathrm{tw}}(D_x) - h(D_x)) \leq 0$; divide by $[L : K]$, using $[L : \mathbb{Q}]/[L : K] = [K : \mathbb{Q}]$. **Step 6: the sum over primes.** Write $c_\ell = \sum_{m \geq 1} \log \ell/\ell^m$ for odd ℓ and $c_2 = \log 2 + \sum_{m \geq 1} \log 2/2^m$. Splitting at $\ell^m \leq B$,

$$\sum_{\ell \leq B} c_\ell \leq \log 2 + \sum_{n \leq B} \frac{\Lambda(n)}{n} + \sum_{\ell \leq B} \log \ell \sum_{m:\ \ell^m > B} \ell^{-m}.$$

For $\ell \leq B$ let m_0 be least with $\ell^{m_0} > B$; then $\sum_{m \geq m_0} \ell^{-m} = \ell^{-m_0}\ell/(\ell-1) \leq 2/B$, so the last sum is at most $(2/B)\sum_{n \leq B} \Lambda(n) \leq 6$ by Lemma 3.5(2), and the middle sum is at most $\log B + 3 + \log 2$ by Lemma 3.5(3). Hence $\sum_{\ell \leq B} c_\ell \leq \log B + 9 + 2\log 2 \leq \log B + 11$, and $S'_B \subseteq \{\ell \leq B\}$. **Step 7: the corrected line.** By Definition 31.12, $\overline{\mathcal{H}}^{\mathrm{c}}_{B,t'}$ is the twisted bundle of the lemma for $a = a^{\mathrm{H}}$ and the smooth cone function $h = h^{\mathrm{ref}'} e^{\psi_{t'}}$ (with $S_B = S'_B$), so the last assertion is the lower inequality together with Step 6. □

Lemma D.26. *Let E be a CM field with complex conjugation c, L a number field containing the image of every field embedding $E \to \overline{\mathbb{Q}}$ of E into an algebraic closure of L, and $\mathcal{A}$ a principally polarised abelian scheme with $\mathcal{O}_E$-action ι over $\mathcal{O}_L$ of relative dimension g. For a field embedding $\sigma\colon E \to L$ put*

$$\mathrm{Lie}_\sigma(\mathcal{A}) = \{x \in \mathrm{Lie}(\mathcal{A}) :\ \iota(a)x = \sigma(a)x \text{ for all } a \in \mathcal{O}_E\}, \qquad d_\sigma = \frac{\widehat{\deg}\,\overline{\mathrm{Lie}}_\sigma(\mathcal{A})}{[L : \mathbb{Q}]}, \qquad \bar\sigma = \sigma \circ c.$$

Let Ψ be a set of embeddings $E \to L$ such that every embedding $E \to L$ lies in exactly one of Ψ and $\bar\Psi = \{\bar\sigma : \sigma \in \Psi\}$, and put $r_\Psi = \sum_{\sigma \in \Psi} \mathrm{rk}\,\mathrm{Lie}_\sigma(\mathcal{A})$. Let $C_{\mathrm{f}} \geq 0$ and $\Xi \geq 1$ be real numbers with

$$g \leq e^\Xi \qquad \text{and} \qquad \sum_{\sigma \in \Psi} |d_\sigma - d_{\bar\sigma}| \leq C_{\mathrm{f}}\, g\, \Xi.$$

Then

$$\sum_{\sigma \in \Psi} d_\sigma \;\leq\; (C_{\mathrm{ll}} + \tfrac{1}{2})(2 + \log(3 + C_{\mathrm{f}}))\, r_\Psi\, \Xi.$$

Proof. Write $V = \overline{\mathrm{Lie}}(\mathcal{A})$, $\widehat{\mu}_{\max} = \widehat{\mu}_{\max}(V)$, $h = -\widehat{\deg}(V)/[L : \mathbb{Q}]$, so that $h^+(\mathcal{A}) = \max(h, 0)$, and put $x = \sum_{\sigma \in \Psi} d_\sigma$, $y = \sum_{\sigma \in \Psi} d_{\bar\sigma}$, $C_{\mathrm{s}} = C_{\mathrm{ll}} + \frac{1}{2} \geq 0$.

The eigen-submodules have total degree at most that of V: $x + y \leq -h$. Since L contains all conjugates of E, $\mathcal{O}_E \otimes_{\mathbb{Z}} L \cong \prod_{\sigma\colon E \to L} L$ by $a \otimes b \mapsto (\sigma(a)b)_\sigma$, and the corresponding idempotents decompose $\mathrm{Lie}(\mathcal{A}) \otimes_{\mathcal{O}_L} L = \bigoplus_\sigma W_\sigma$ with W_σ the σ-eigenspace of ι. As $\mathrm{Lie}(\mathcal{A})$ is torsion-free, $\mathrm{Lie}_\sigma(\mathcal{A}) = \mathrm{Lie}(\mathcal{A}) \cap W_\sigma$, a saturated submodule whose L-span is W_σ (every $w \in W_\sigma$ has a non-zero integer multiple in $\mathrm{Lie}(\mathcal{A})$). Hence $M = \sum_\sigma \mathrm{Lie}_\sigma(\mathcal{A})$, the sum over all embeddings $\sigma\colon E \to L$, i.e. over $\Psi \sqcup \bar\Psi$, is a direct sum and a submodule of $\mathrm{Lie}(\mathcal{A})$ of full rank g, so $\mathrm{Lie}(\mathcal{A})/M$ is finite and $\widehat{\deg}\,\overline{M} = \widehat{\deg}\,V - \log\#(\mathrm{Lie}(\mathcal{A})/M) \leq \widehat{\deg}\,V$, $\overline{M}$ with the induced norms. For a direct sum $M = M_1 \oplus M_2$ of projective $\mathcal{O}_L$-modules with norms induced from one Hermitian norm at each complex place, $\det M = \det M_1 \otimes \det M_2$, and at every complex place the norm of $s_1 \wedge s_2$ ($s_i \in \det M_i \otimes \mathbb{C}$) is at most $\|s_1\|\,\|s_2\|$ by Fischer's inequality for the Gram matrix of the

union of bases (the determinant of a positive definite block matrix is at most the product of the determinants of its diagonal blocks); hence $\widehat{\deg}\,\overline{M} \geq \widehat{\deg}\,\overline{M}_1 + \widehat{\deg}\,\overline{M}_2$. By induction over the summands,

$$[L:\mathbb{Q}]\,(x+y) = \sum_{\sigma} \widehat{\deg}\,\overline{\mathrm{Lie}}_\sigma(\mathcal{A}) \leq \widehat{\deg}\,\overline{M} \leq \widehat{\deg}\,V = -[L:\mathbb{Q}]\,h.$$

The conjugate degrees nearly agree: $x - y \leq C_{\mathrm{f}} g \Xi$. Indeed $x - y = \sum_{\sigma\in\Psi}(d_\sigma - d_{\bar\sigma}) \leq \sum_{\sigma\in\Psi} |d_\sigma - d_{\bar\sigma}| \leq C_{\mathrm{f}} g \Xi$ by hypothesis.

Case $h \geq C_{\mathrm{f}} g \Xi$ (large height). Adding the two inequalities, $2x \leq -h + C_{\mathrm{f}} g \Xi \leq 0$, so $x \leq 0$, which is at most the right-hand side of the claim (all its factors are ≥ 0).

Case $h < C_{\mathrm{f}} g \Xi$ (small height). Then $\max(h,0) \leq C_{\mathrm{f}} g \Xi$, and since $g \geq 1$, $\Xi \geq 1$, so $g\Xi \geq 1$,

$$2 + g + h^+(\mathcal{A}) \leq 2g\Xi + g\Xi + C_{\mathrm{f}} g \Xi = (3 + C_{\mathrm{f}})\, g\, \Xi.$$

With $\log g \leq \Xi$ and $\log \Xi \leq \Xi$ this gives $\log(2+g+h^+(\mathcal{A})) \leq \log(3+C_{\mathrm{f}}) + 2\Xi \leq (2+\log(3+C_{\mathrm{f}}))\Xi$. By the logarithmic slope bound Proposition D.27,

$$\widehat{\mu}_{\max} \leq \mu^*, \qquad \mu^* = C_{\mathrm{s}}(2 + \log(3 + C_{\mathrm{f}}))\Xi \ \geq 0.$$

For $\sigma \in \Psi$ with $\mathrm{Lie}_\sigma(\mathcal{A}) \neq 0$, the normalised slope of $\overline{\mathrm{Lie}}_\sigma(\mathcal{A})$ is at most $\widehat{\mu}_{\max}$ by the definition of the maximal slope, so $d_\sigma \leq \mathrm{rk}\,\mathrm{Lie}_\sigma(\mathcal{A})\,\widehat{\mu}_{\max} \leq \mathrm{rk}\,\mathrm{Lie}_\sigma(\mathcal{A})\,\mu^*$ (using $\widehat{\mu}_{\max} \leq \mu^*$ and $\mathrm{rk} \geq 0$); for $\mathrm{Lie}_\sigma(\mathcal{A}) = 0$ both sides vanish. Summing over Ψ gives $x \leq r_\Psi\, \mu^*$, the claim.

Remark on the use of the slope bound. Only the second case uses Proposition D.27, and only there is it essential that the bound is logarithmic in the height: a bound linear in h^+ would give $x \leq C\, r_\Psi\, g\, \Xi$ instead. □

Proposition D.27. *For every CM field E, every number field L and every principally polarised abelian scheme $\mathcal{A}$ with $\mathcal{O}_E$-action over $\mathcal{O}_L$ of relative dimension g,*

$$\widehat{\mu}_{\max}\big(\overline{\mathrm{Lie}}(\mathcal{A})\big) \ \leq \ (C_{\mathrm{ll}} + \tfrac{1}{2}) \log\big(2 + g + h^+(\mathcal{A})\big).$$

Proof. Write $\overline{M} = \overline{\mathrm{Lie}}(\mathcal{A})$ and $Y = \log(2+g+h^+(\mathcal{A}))$. It suffices to show $\widehat{\mu}(\overline{F}) \leq C_{\mathrm{ll}} Y + \frac{1}{2}\log r + \varepsilon$ for every non-zero submodule $F \subseteq M$ of rank r and every $\varepsilon > 0$, since $r \leq g$ gives $\frac{1}{2}\log r \leq \frac{1}{2} Y$.

A large line in F. By Zhang's absolute Siegel lemma [276] (Theorem B.136, applied to the Hermitian lattice $\overline{F}$ of rank $r \geq 1$ over $\mathcal{O}_L$) there are a finite extension K'/L and $0 \neq v \in F \otimes_{\mathcal{O}_L} K'$ such that the rank-one Hermitian lattice $\overline{N} = \overline{F}_{K'}(v) = K'v \cap (F \otimes_{\mathcal{O}_L} \mathcal{O}_{K'})$ satisfies $\widehat{\mu}(\overline{N}) \geq \widehat{\mu}(\overline{F}) - \frac{1}{2}(H_r - 1) - \varepsilon$. Since $H_r - 1 = \sum_{k=2}^{r} \frac{1}{k} \leq \int_1^r \frac{dt}{t} = \log r$,

$$\widehat{\mu}(\overline{N}) \geq \widehat{\mu}(\overline{F}) - \tfrac{1}{2}\log r - \varepsilon. \tag{D.1}$$

Making the line free. Let K'' be the Hilbert class field of K'. By the principal ideal theorem every ideal of $\mathcal{O}_{K'}$ becomes principal in $\mathcal{O}_{K''}$, so the projective rank-one module $N \otimes_{\mathcal{O}_{K'}} \mathcal{O}_{K''}$ (isomorphic to an ideal of $\mathcal{O}_{K'}$ extended to $\mathcal{O}_{K''}$) is free; let y be a generator. With the norms $\|\cdot\|_{\sigma''} = \|\cdot\|_{\sigma''|_{K'}}$, the normalised slope does not change under the extension $\mathcal{O}_{K'} \to \mathcal{O}_{K''}$ (the Arakelov degree is multiplied by $[K'' : K']$, as is $[K' : \mathbb{Q}]$), and for a free Hermitian lattice of rank one $\widehat{\deg} = -\sum_{\sigma''} \log \|y\|_{\sigma''}$. Hence

$$\widehat{\mu}(\overline{N}) = -\frac{1}{[K'':\mathbb{Q}]} \sum_{\sigma'' :\, K'' \to \mathbb{C}} \log \|y\|_{\sigma''}.$$

The length bound after base change. Let $\mathcal{A}'' = \mathcal{A} \times_{\mathcal{O}_L} \mathcal{O}_{K''}$. It is a semiabelian scheme over $\mathcal{O}_{K''}$ with the induced $\mathcal{O}_E$-action and the pulled-back rigidified symmetric cubical line bundle, ample on the generic fibre $A_{K''}$ (an abelian variety of dimension g) with the base-changed principal polarisation; so $\mathcal{A}''$ is a principally polarised abelian scheme with $\mathcal{O}_E$-action over $\mathcal{O}_{K''}$. Since $\mathcal{A}$ is smooth, its Lie algebra commutes with base change, $\mathrm{Lie}(\mathcal{A}'') = \mathrm{Lie}(\mathcal{A}) \otimes_{\mathcal{O}_L} \mathcal{O}_{K''}$, and for $\sigma'' \colon K'' \to \mathbb{C}$ the complex torus, period lattice and Riemann form of $A_{K'',\sigma''}$ are those of $A_{\sigma''|_L}$, so the Hodge norm at σ'' is the Hodge norm at $\sigma''|_L$. Hence $\overline{\mathrm{Lie}}(\mathcal{A}'') = \overline{M} \otimes_{\mathcal{O}_L} \mathcal{O}_{K''}$ as Hermitian lattices, and by the invariance of the normalised degree $h^+(\mathcal{A}'') = h^+(\mathcal{A})$. The vector y lies in $N \otimes \mathcal{O}_{K''} \subseteq F \otimes \mathcal{O}_{K''} \subseteq \mathrm{Lie}(\mathcal{A}'')$ and is non-zero. By Proposition D.29 for $\mathcal{A}''$ and y,

$$\widehat{\mu}(\overline{N}) = -\frac{1}{[K'' : \mathbb{Q}]} \sum_{\sigma''} \log \|y\|_{\sigma''} \le C_{\mathrm{ll}} \log\bigl(2 + g + h^+(\mathcal{A}'')\bigr) = C_{\mathrm{ll}} Y.$$

Conclusion. With (D.1), $\widehat{\mu}(\overline{F}) \le C_{\mathrm{ll}} Y + \frac{1}{2}\log r + \varepsilon$. Letting $\varepsilon \to 0$ and taking the supremum over F gives $\widehat{\mu}_{\max}(\overline{M}) \le C_{\mathrm{ll}} Y + \frac{1}{2}\log g \le (C_{\mathrm{ll}} + \frac{1}{2}) Y$. □

Definition D.28. Let $(\mathcal{A}, \mathcal{L}_{\mathcal{A}}, r)$ be a principally polarised abelian scheme with $\mathcal{O}_E$-action over $\mathcal{O}_L$, of relative dimension g, and $\sigma \colon L \to \mathbb{C}$. Write $\mathcal{L}_\sigma$ for the line bundle on A_σ obtained from $\mathcal{L}_{\mathcal{A}}|_A$ by base change along σ, $r_\sigma \in \mathcal{L}_\sigma(0)$ for the rigidification, $\mathrm{Lie}(A_\sigma) = \mathrm{Lie}(\mathcal{A}) \otimes_{\mathcal{O}_L,\sigma} \mathbb{C}$, x_σ for the image of $x \in \mathrm{Lie}(\mathcal{A})$, and $\pi_\sigma \colon \mathrm{Lie}(A_\sigma) \to A_\sigma(\mathbb{C})$ for the exponential map, with kernel Π_σ.

Theta trivialisation. The **theta trivialisation** $\vartheta_{\mathcal{A},\sigma}$ is the nowhere vanishing holomorphic section of $\pi_\sigma^* \mathcal{L}_\sigma$ on $\mathrm{Lie}(A_\sigma)$ with $\vartheta_{\mathcal{A},\sigma}(0) = r_\sigma$ and

$$\vartheta_{\mathcal{A},\sigma}(w + l) = \Bigl(\alpha_\sigma(l) \exp\bigl(\pi H_\sigma(w,l) + \tfrac{\pi}{2} H_\sigma(l,l)\bigr)\Bigr)^{-1} \vartheta_{\mathcal{A},\sigma}(w) \qquad (w \in \mathrm{Lie}(A_\sigma),\ l \in \Pi_\sigma) \quad \text{(D.2)}$$

in the fibre $\mathcal{L}_\sigma(\pi_\sigma(w))$, where $(H_\sigma, \alpha_\sigma)$ are the Appell–Humbert data of $\mathcal{L}_\sigma$ (the Hermitian form of $\mathcal{L}_\sigma$ is H_σ, since its imaginary part on Π_σ is the Riemann form of the principal polarisation, and $\alpha_\sigma \colon \Pi_\sigma \to \mathbb{C}^\times$ is a semicharacter for it).

Theta functions and canonical norms. For $n \ge 1$ and $s \in H^0(A_\sigma, \mathcal{L}_\sigma^{\otimes n})$ the **theta function** of s is the entire function θ_s on $\mathrm{Lie}(A_\sigma)$ with $\pi_\sigma^* s = \theta_s \, \vartheta_{\mathcal{A},\sigma}^{\otimes n}$, and the **canonical norm** of s at $\pi_\sigma(w)$ is

$$\|s\|(\pi_\sigma(w)) = |\theta_s(w)| \exp\bigl(-\tfrac{\pi n}{2} H_\sigma(w,w)\bigr), \qquad \text{(D.3)}$$

which does not depend on the choice of w by (D.2). Put $\|s\|_{\sup,\sigma} = \sup_{A_\sigma(\mathbb{C})} \|s\|$ and $\|s\|_{L^2,\sigma} = \bigl(\int_{A_\sigma(\mathbb{C})} \|s\|^2 \, d\mu_\sigma\bigr)^{1/2}$, with μ_σ the Haar probability measure. For $s \in H^0(\mathcal{A}, \mathcal{L}_{\mathcal{A}}^{\otimes n})$ or $s \in H^0(A, \mathcal{L}_{\mathcal{A}}|_A^{\otimes n})$ these are taken for its image $s_\sigma \in H^0(A_\sigma, \mathcal{L}_\sigma^{\otimes n})$.

Jets along a tangent vector. For $v \in \mathrm{Lie}(A_\sigma)$ and $k \ge 0$ put $\partial_v^k \theta_s(0) = \frac{d^k}{d\tau^k} \theta_s(\tau v)\big|_{\tau=0}$ ($\tau \in \mathbb{C}$).

Proposition D.29. *There is an absolute constant $C_{\mathrm{ll}} \ge 0$ such that for every CM field E, every number field L, every principally polarised abelian scheme $\mathcal{A}$ with $\mathcal{O}_E$-action over $\mathcal{O}_L$, of relative dimension g, and every $0 \ne x \in \mathrm{Lie}(\mathcal{A})$*

$$-\frac{1}{[L : \mathbb{Q}]} \sum_{\sigma \colon L \to \mathbb{C}} \log \|x\|_\sigma \ \le\ C_{\mathrm{ll}} \log\bigl(2 + g + h^+(\mathcal{A})\bigr).$$

Proof. Write $h = h^+(\mathcal{A})$, $Y = 2 + g + h \ge 3$, and for a number field $K \supseteq L$ and a function f on the embeddings $K \to \mathbb{C}$, $\mathrm{mean}_K f = \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma \colon K \to \mathbb{C}} f(\sigma)$. We show the bound with

$$C_{\mathrm{ll}} = 6 + \tfrac{1}{2}\log 27 + C_{\mathrm{key}} + C_{\sup}\bigl(1 + \log(1 + C_{\mathrm{key}})\bigr).$$

Step 0: base change. For a finite extension K/L the base change $\mathcal{A}_K$ is a principally polarised abelian scheme with $\mathcal{O}_E$-action over $\mathcal{O}_K$; $\mathrm{Lie}(\mathcal{A}_K) = \mathrm{Lie}(\mathcal{A}) \otimes_{\mathcal{O}_L} \mathcal{O}_K$, and at $\sigma\colon K \to \mathbb{C}$ the complex torus, lattice, Riemann form, line bundle and rigidification of $\mathcal{A}_K$ are those of $\mathcal{A}$ at $\sigma|_L$. Hence the Hodge norms, theta functions and canonical norms at σ are those at $\sigma|_L$, $h^+(\mathcal{A}_K) = h$, and, each $\sigma_0\colon L \to \mathbb{C}$ having $[K : L]$ extensions, $\mathrm{mean}_K \log \|x \otimes 1\| = \mathrm{mean}_L \log \|x\|$.

Step 1: the sections. By Proposition D.30 there are L_1/L and $\theta \in H^0(\mathcal{A}_{L_1}, \mathcal{L})$, non-zero on the generic fibre and lying in the Moret-Bailly lattice (Definition 34.8), with $\mathrm{mean}_{L_1} \log \|\theta\|_{L^2} \le \Phi := h/2 + C_{\mathrm{key}} g$ (Step 0 for h). Both Proposition D.32 and Proposition D.33 below are applied to this θ on $\mathcal{A}_{L_1}$, and their hypothesis that θ lies in the Moret-Bailly lattice is this clause. Put $M_\sigma = \|\theta\|_{\sup,\sigma}$, positive since θ_{θ_σ} is a non-zero entire function; by Proposition D.32 ($\Phi \ge 0$),

$$\mathrm{mean}_{L_1} \log M \le \Phi + C_{\sup}\, g \log(2 + g + \Phi). \tag{D.4}$$

Let K_0 be the least even integer with $K_0 \ge 2 + g^2 + h$, $m = 3K_0$ (even, as K_0 is), $n' = 3K_0^2$ and $n = m^2 = 3n'$. By Proposition D.33 (for the even integer $m \ge 2$) there are $L_2 \supseteq L_1$ and $s_1, \dots, s_r \in H^0(\mathcal{A}_{L_2}, \mathcal{L}^{\otimes n})$, $r = m^{2g}$, with $\|s_j\|_{\sup,\sigma} \le M'_\sigma := M_{\sigma|_{L_1}}$ that span $H^0(A_\sigma, \mathcal{L}_\sigma^{\otimes n})$ for every $\sigma\colon L_2 \to \mathbb{C}$. Write $\mathcal{A}_2 = \mathcal{A}_{L_2}$, $x_2 = x \otimes 1 \in \mathrm{Lie}(\mathcal{A}_2)$, $d = [L_2 : \mathbb{Q}]$ and $\ell_\sigma = \|x_2\|_\sigma$; $\ell_\sigma > 0$ because $\mathrm{Lie}(\mathcal{A}_2)$ is projective, so $x_{2,\sigma} \ne 0$, and H_σ is positive definite.

Step 2: the jet filtration. Let j_k be the maps of Lemma D.34 for $\mathcal{A}_2$, x_2 and n. For $c \in L_2^r$ put $s_c = \sum_j c_j s_j$ and $J_k(c) = j_k(s_c)$; let $E_0 = L_2^r$, $E_i = \{c : J_k(c) = 0 \ (k < i)\}$, and $\phi_i = J_i|_{E_i}$, so $E_{i+1} = \ker \phi_i$. For $\sigma\colon L_2 \to \mathbb{C}$ write $E_{i,\sigma} = E_i \otimes_\sigma \mathbb{C} \subseteq \mathbb{C}^r$ and $\phi_{i,\sigma}$; these are cut out by the $J_{k,\sigma}$. By Lemma D.34(2), for $c \in \mathbb{C}^r$: $c \in E_{i,\sigma}$ if and only if $\partial^k \theta_{s_c}(0) = 0$ for $k < i$, and then $\phi_{i,\sigma}(c) = \partial^i \theta_{s_c}(0)$, where ∂ is taken along $x_{2,\sigma}$ and $s_c = \sum_j c_j s_{j,\sigma}$.

Step 3: every step has rank one up to n'. Let $0 \le i \le n'$ and fix σ. By Lemma D.35 ($v = x_{2,\sigma} \ne 0$, $n' \ge 1$) there is $s \in H^0(A_\sigma, \mathcal{L}_\sigma^{\otimes 3n'})$ with $\partial^k \theta_s(0) = 0$ for $k < i$ and $\partial^i \theta_s(0) \ne 0$. As $3n' = n$, the spanning property gives $s = s_c$ for some $c \in \mathbb{C}^r$; by Step 2, $c \in E_{i,\sigma}$ and $\phi_{i,\sigma}(c) \ne 0$. So $\phi_i \ne 0$, and $\dim E_i - \dim E_{i+1} = 1$ for $0 \le i \le n'$.

Step 4: the archimedean bound. Let $c \in E_{i,\sigma}$ and $F(\tau) = \theta_{s_c}(\tau x_{2,\sigma}) = \sum_j c_j \theta_{s_j}(\tau x_{2,\sigma})$. By the definition of the canonical norm (Definition D.28), $|\theta_{s_j}(w)| = \|s_j\|(\pi_\sigma(w)) e^{\pi n H_\sigma(w,w)/2} \le M'_\sigma e^{\pi n H_\sigma(w,w)/2}$, and $\pi H_\sigma(\tau x_2, \tau x_2) = |\tau|^2 \ell_\sigma^2$; by Cauchy–Schwarz, with $|c|^2 = \sum_j |c_j|^2$,

$$|F(\tau)| \le \sqrt{r}\, M'_\sigma\, |c|\, \exp(n\ell_\sigma^2 |\tau|^2/2) \qquad (\tau \in \mathbb{C}).$$

Cauchy's estimate on $|\tau| = R$ gives $|F^{(i)}(0)| \le i!\, R^{-i} \max_{|\tau|=R} |F|$. For $i = 0$ this gives $|F(0)| \le \sqrt{r} M'_\sigma |c|$ ($R \to 0$). For $1 \le i \le n'$ take $R^2 = i/(n\ell_\sigma^2)$ and use $i! \le i^i$, $i \le n'$:

$$|\phi_{i,\sigma}(c)| \le i! \Big(\frac{n\ell_\sigma^2}{i}\Big)^{i/2} e^{i/2} \sqrt{r} M'_\sigma |c| \le A_i(\sigma)\, |c|, \qquad A_i(\sigma) := (enn')^{i/2} \ell_\sigma^i \sqrt{r}\, M'_\sigma \quad (0 \le i \le n'). \tag{D.5}$$

Step 5: the slope inequality along the filtration. Let $\overline{\Lambda} = \mathcal{O}_{L_2}^r$ with the norms $|c|_\sigma = (\sum_j |\sigma(c_j)|^2)^{1/2}$; $\widehat{\deg}\, \overline{\Lambda} = 0$. Put $\Lambda_i = \Lambda \cap E_i$, saturated with $\Lambda_i \otimes L_2 = E_i$, and $Q_i = \Lambda_i/\Lambda_{i+1}$ ($0 \le i \le n'$), projective of rank one by Step 3, with the quotient norms of $\overline{\Lambda}_i$. Additivity of degrees gives

$$0 = \widehat{\deg}\, \overline{\Lambda} = \sum_{i=0}^{n'} \widehat{\deg}\, \overline{Q}_i + \widehat{\deg}\, \overline{\Lambda}_{n'+1} \le \sum_{i=0}^{n'} \widehat{\deg}\, \overline{Q}_i.$$

For $c \in \Lambda_i$ the section s_c lies in $H^0(\mathcal{A}_2, \mathcal{L}^{\otimes n})$ and $J_k(c) = 0$ for $k < i$, so $\phi_i(c) \in \mathcal{O}_{L_2}$ by Lemma D.34(1). Hence ϕ_i induces $\psi_i\colon Q_i \to \mathcal{O}_{L_2}$, non-zero by Step 3 and therefore injective. For

$y \in Q_i \otimes_\sigma \mathbb{C}$ and any lift $c \in E_{i,\sigma}$, $\psi_{i,\sigma}(y) = \phi_{i,\sigma}(c)$ since $\phi_{i,\sigma}$ vanishes on $E_{i+1,\sigma}$; minimising (D.5) over the lifts, $|\psi_{i,\sigma}(y)| \le A_i(\sigma)\|y\|_\sigma$. For $0 \neq y \in Q_i$: $\#(Q_i/\mathcal{O}_{L_2} y) = \#\big(\psi_i(Q_i)/\psi_i(y)\mathcal{O}_{L_2}\big) \le |N_{L_2/\mathbb{Q}}(\psi_i(y))| = \prod_\sigma |\sigma\psi_i(y)|$ and $\log\|y\|_\sigma \ge \log|\sigma\psi_i(y)| - \log A_i(\sigma)$, so $\widehat{\deg}\,\overline{Q}_i \le \sum_\sigma \log A_i(\sigma)$. Therefore

$$0 \le \sum_{i=0}^{n'} \sum_{\sigma\colon L_2 \to \mathbb{C}} \log A_i(\sigma). \tag{D.6}$$

Step 6: the numbers. With $S_\ell = \sum_\sigma \log \ell_\sigma$, $S_M = \sum_\sigma \log M'_\sigma$, $B = (enn')^{1/2}$ and $T = \sqrt{r} = m^g$, the right side of (D.6) is $\frac{n'(n'+1)}{2}(d\log B + S_\ell) + (n'+1)(d\log T + S_M)$. Dividing by $dn'(n'+1)/2$ and using Step 0 ($S_\ell/d = \mathrm{mean}_L \log\|x\|$), $S_M/d = \mathrm{mean}_{L_1} \log M$ and (D.4):

$$-\mathrm{mean}_L \log\|x\| \le \tfrac{1}{2}\log(enn') + \frac{2}{n'}\Big(g\log m + \Phi + C_{\mathrm{sup}}\, g \log(2+g+\Phi)\Big).$$

Now $2+g^2+h \le K_0 \le 4+g^2+h \le Y^2$ (the last since $Y^2 = (2+g+h)^2 \ge 4+g^2+h$), so $K_0 \ge g$ and $K_0 \ge h$. Each term: $\frac{1}{2}\log(enn') = \frac{1}{2}\log(27eK_0^4) \le \frac{1}{2}(1+\log 27) + 4\log Y$; $\frac{2}{n'} g\log m = \frac{2g\log(3K_0)}{3K_0^2} \le \frac{2g}{K_0} \le \frac{2g}{2+g^2} \le 1$; $\frac{2}{n'}\Phi = \frac{h+2C_{\mathrm{key}}g}{3K_0^2} \le \frac{1}{3} + C_{\mathrm{key}}$; and, as $2 \le 2+g+\Phi \le (1+C_{\mathrm{key}})Y$ and $g \le K_0^2$, $\frac{2}{n'} C_{\mathrm{sup}} g\log(2+g+\Phi) \le C_{\mathrm{sup}}\big(\log(1+C_{\mathrm{key}}) + \log Y\big)$. Since $\log Y \ge \log 3 \ge 1$, the sum of these is at most $C_{\mathrm{ll}} \log Y$. □

Proposition D.30. *There is an absolute constant $C_{\mathrm{key}} \ge 0$ such that for every CM field E, every number field L and every principally polarised abelian scheme $(\mathcal{A}, \mathcal{L}_\mathcal{A}, r)$ with $\mathcal{O}_E$-action over $\mathcal{O}_L$ of relative dimension g there are a finite extension L'/L and a section $\theta \in H^0(\mathcal{A}', \mathcal{L}_{\mathcal{A}'})$ of the base change $\mathcal{A}'$ of $\mathcal{A}$ to $\mathcal{O}_{L'}$, non-zero on the generic fibre and lying in the Moret-Bailly lattice, with*

$$\frac{1}{[L':\mathbb{Q}]} \sum_{\sigma'\colon L' \to \mathbb{C}} \log\|\theta\|_{L^2,\sigma'} \;\le\; \frac{h^+(\mathcal{A})}{2} + C_{\mathrm{key}}\, g.$$

Proof. We prove the statement with $C_{\mathrm{key}} = \frac{1}{4}\log(2\pi)$. Let $(\mathcal{A}, \mathcal{L}_\mathcal{A}, r)$ be given over $\mathcal{O}_L$, of relative dimension $g \ge 1$, and put $A = \mathcal{A}_L$, $L_A = \mathcal{L}_\mathcal{A}|_A$: a symmetric ample line bundle defining a principal polarisation. Write $r_0 \in e^*\mathcal{L}_\mathcal{A}$ for the generator with $r(r_0) = 1$; its image in $\mathcal{L}_\sigma(0)$ is the rigidification r_σ.

Step 1: the key formula after a field extension. Apply Theorem B.137 to $K_0 = L$ and (A, L_A): it gives $K \supseteq L$, $\pi\colon \mathcal{G} \to \mathrm{Spec}\,\mathcal{O}_K$, $\mathcal{M}$, $\varphi\colon \mathcal{M}|_{\mathcal{G}_K} \cong L_A \otimes_L K$ and the cubist metrics. The $\mathcal{O}_K$-module $E = H^0(\mathcal{G}, \mathcal{M})$ is projective of rank one (the theorem with $K' = K$), hence isomorphic to a fractional ideal $\mathfrak{a}$ of K. Let L' be the Hilbert class field of K; by the principal ideal theorem (Furtwängler) $\mathfrak{a}\mathcal{O}_{L'}$ is principal. Since $\mathcal{O}_{L'}$ is flat over $\mathcal{O}_K$ and $\mathcal{G}$ is quasi-compact and separated, flat base change gives $E' = H^0(\mathcal{G}', \mathcal{M}') = E \otimes_{\mathcal{O}_K} \mathcal{O}_{L'} \cong \mathfrak{a}\mathcal{O}_{L'}$, a free $\mathcal{O}_{L'}$-module of rank one; let θ_0 be a generator. For a free Hermitian lattice of rank one with generator θ_0 one has $\widehat{\deg}\,\overline{E'} = -\sum_{\sigma'} \log\|\theta_0\|_{L^2,\sigma'}$, so Theorem B.137 (with $K' = L'$) gives

$$\frac{1}{[L':\mathbb{Q}]} \sum_{\sigma'\colon L' \to \mathbb{C}} \log\|\theta_0\|_{L^2,\sigma'} = \frac{1}{2}\, h_F(A) + \frac{g}{4}\log(2\pi), \tag{D.7}$$

where $\|\cdot\|_{L^2,\sigma'}$ is formed with the cubist metric. As $E' \otimes L' = H^0(A_{L'}, \mathcal{M}|_{A_{L'}})$ and $\theta_0 \neq 0$, the image of θ_0 on the generic fibre is non-zero.

Step 2: a Moret-Bailly model over $\mathcal{O}_{L'}$. By Theorem B.137 the pull-back $\varepsilon^*\mathcal{M}$ with the cubist metrics is isometric to $\mathcal{O}_K$ with the trivial metrics; let u_0 be the generator of norm 1 at every place and u its base change, a generator of $\varepsilon^*\mathcal{M}'$ of norm 1 at every $\sigma'\colon L' \to \mathbb{C}$. The

isomorphism $\varphi' = \varphi \otimes L'$ maps u to $a\,(r_0 \otimes 1)$ for some $a \in L'^{\times}$; put $\psi = a^{-1}\varphi'$, so $\psi(u) = r_0 \otimes 1$. Then $(\mathcal{G}', i', \mathcal{M}', (\|\cdot\|_{\sigma'})_{\sigma'}, \psi, u)$, with $i'\colon \mathcal{G}'_{L'} \cong A_{L'}$ the base change of the isomorphism of Theorem B.137, is a Moret-Bailly model of $(A_{L'}, L_A \otimes L', r_0 \otimes 1, \varnothing)$ over $\mathcal{O}_{L'}$: (Mb1) and (Mb2) hold by Theorem B.137 and base change (smoothness, separatedness, finite type, the structure of the fibres, the cubical structure and its trivialisation of norm 1 are preserved); (Mb3) holds by construction; (Mb4) is empty; (Mb5) holds for the base change of the finite flat subscheme of Theorem B.137 with generic fibre $A[2]$. Rescaling φ' by the constant a^{-1} does not change (D.7), which concerns θ_0 and the cubist metric only.

Step 3: the section θ. Let $\mathcal{A}'$ be the base change of $\mathcal{A}$ to $\mathcal{O}_{L'}$ and apply Lemma D.31 (with $K = L'$, $F = \varnothing$, $N = 1$) to the model of Step 2 and $s = \theta_0$: $\theta = \theta_0^{\flat}$ lies in $H^0(\mathcal{A}', \mathcal{L}_{\mathcal{A}'})$, its restriction to $A_{L'}$ is $\psi(\theta_0|_{\mathcal{G}'_{L'}})$ transported by i', which is non-zero by Step 1, and $\|\theta\|_{L^2,\sigma'} = \|\theta_0\|_{L^2,\sigma'}$ for every σ'. By definition of the Moret-Bailly lattice of the model of Step 2, this restriction lies in that lattice, so θ lies in the Moret-Bailly lattice (Definition 34.8 with $K = L'$).

Step 4: the Faltings height and the Lie lattice. Compute $h_F(A)$ with the semiabelian model $\mathcal{S} = \mathcal{A}$ over $\mathcal{O}_L$. Since $e^*\Omega^1_{\mathcal{A}/\mathcal{O}_L}$ is the dual of $\mathrm{Lie}(\mathcal{A}) = e^*T_{\mathcal{A}/\mathcal{O}_L}$, $\omega_{\mathcal{A}} = e^*\Omega^g = \det(\mathrm{Lie}(\mathcal{A}))^{\vee}$. Compare the metrics at $\sigma\colon L \to \mathbb{C}$: let $e_1, \dots, e_g$ be an orthonormal basis of $\mathrm{Lie}(A_\sigma)$ for H_σ, $z_1, \dots, z_g$ the dual coordinates and $\alpha = dz_1 \wedge \cdots \wedge dz_g$. Then $i^{g^2}\alpha \wedge \overline{\alpha} = 2^g\, d\lambda$, with λ the Lebesgue measure of the Euclidean structure $\mathrm{Re}\, H_\sigma$:

The covolume of Π_σ for H_σ equals $\chi(A, L_A) = 1$ ([46, 2.1.2]: the covolume of the period lattice for the Hermitian form of the Riemann form of L is $\chi(A, L)$; here the polarisation is principal). Hence $\|\alpha\|_\sigma^2 = 2^g/(2\pi)^g = \pi^{-g}$ in Deligne's normalisation. For the norm πH_σ on $\mathrm{Lie}(A_\sigma)$ the vectors $e_k/\sqrt{\pi}$ are orthonormal, and the dual norm of α on $\det \mathrm{Lie}(A_\sigma)$ is $|\alpha(e_1/\sqrt{\pi} \wedge \cdots \wedge e_g/\sqrt{\pi})| = \pi^{-g/2}$. So $\overline{\omega}_{\mathcal{A}} \cong \det(\overline{\mathrm{Lie}}(\mathcal{A}))^{\vee}$ isometrically, and

$$h_F(A) = \frac{\widehat{\deg}\,\overline{\omega}_{\mathcal{A}}}{[L:\mathbb{Q}]} = -\frac{\widehat{\deg}\,\overline{\mathrm{Lie}}(\mathcal{A})}{[L:\mathbb{Q}]} \le h^+(\mathcal{A}).$$

Conclusion. The scheme $\mathcal{A}'$ is the base change of $\mathcal{A}$ to $\mathcal{O}_{L'}$, $\theta \in H^0(\mathcal{A}', \mathcal{L}_{\mathcal{A}'})$ is non-zero on the generic fibre and lies in the Moret-Bailly lattice (Step 3), and by Step 3, (D.7) and Step 4,

$$\frac{1}{[L':\mathbb{Q}]} \sum_{\sigma'} \log \|\theta\|_{L^2,\sigma'} = \frac{1}{2}\, h_F(A) + \frac{g}{4}\log(2\pi) \le \frac{h^+(\mathcal{A})}{2} + C_{\mathrm{key}}\, g.$$

□

Lemma D.31. *Let $(\mathcal{A}, \mathcal{L}_{\mathcal{A}}, r)$ be a principally polarised abelian scheme with $\mathcal{O}_E$-action over $\mathcal{O}_L$, $A = \mathcal{A}_L$, and $r_0 \in \mathcal{L}_{\mathcal{A}}(0)$ the generator with $r(r_0) = 1$. Let K/L be a finite extension, $\mathcal{A}_K$ the base change of $\mathcal{A}$ to $\mathcal{O}_K$, $F \subseteq A(K)$ finite, and $(\mathcal{G}, i, \mathcal{M}, (\|\cdot\|_\sigma)_\sigma, \varphi, u, (\varepsilon_P)_{P\in F})$ a Moret-Bailly model of $(A_K, \mathcal{L}_{\mathcal{A}}|_{A_K}, r_0 \otimes 1, F)$ over $\mathcal{O}_K$; let $\mathcal{G}^0 \subseteq \mathcal{G}$ be its identity component. Then:*

1. *there is an isomorphism of group schemes $j\colon \mathcal{A}_K \to \mathcal{G}^0$ with $i \circ j_K = \mathrm{id}_{A_K}$;*
2. *for every integer $N \ge 1$ there is an isomorphism of line bundles $\psi_N\colon j^*(\mathcal{M}^{\otimes N}|_{\mathcal{G}^0}) \to \mathcal{L}_{\mathcal{A}_K}^{\otimes N}$ on $\mathcal{A}_K$ whose restriction to the generic fibre is $\varphi^{\otimes N}$ (transported by $j_K = i^{-1}$), and for every $\sigma\colon K \to \mathbb{C}$ it carries the N-th power of the cubist metric $\|\cdot\|_\sigma$ to the canonical norm on $\mathcal{L}_\sigma^{\otimes N}$;*
3. *consequently, for $s \in H^0(\mathcal{G}, \mathcal{M}^{\otimes N})$ the section $s^{\flat} = \psi_N(j^*(s|_{\mathcal{G}^0}))$ lies in $H^0(\mathcal{A}_K, \mathcal{L}_{\mathcal{A}_K}^{\otimes N})$, its restriction to A_K is $\varphi^{\otimes N}(s|_{\mathcal{G}_K})$ transported by i, and for every σ the canonical norm of $s^{\flat}$ at each point of $A_\sigma(\mathbb{C})$ equals the cubist norm of s at the corresponding point of $\mathcal{G}_\sigma(\mathbb{C})$; in particular $\|s^{\flat}\|_{\sup,\sigma}$ and $\|s^{\flat}\|_{L^2,\sigma}$ are the sup and L^2 norms of s for the cubist metric.*

Proof. (1). The identity component $\mathcal{G}^0$ is the open subgroup scheme whose fibres are the identity components of the fibres of $\mathcal{G}$ (SGA 3, Exp. VI_B, 3.10, for smooth group schemes); its fibres are connected extensions of abelian varieties by tori, so it is a semiabelian scheme over $\mathcal{O}_K$, with generic fibre $\mathcal{G}_K$ (connected, being an abelian variety by i). The base change $\mathcal{A}_K$ is a semiabelian scheme with generic fibre A_K. A semiabelian scheme over a Dedekind scheme whose generic fibre is an abelian variety is the identity component of the Néron model of that abelian variety ([227], Exp. IX; [45], §7.4). Applying this to $\mathcal{A}_K$ and to $\mathcal{G}^0$ (with generic fibre identified with A_K by i), both are identified with the identity component of the Néron model of A_K, compatibly with the generic fibres; this gives j with $i \circ j_K = \mathrm{id}$.

(2), the integral isomorphism. Write $\mathcal{N} = j^*(\mathcal{M}^{\otimes N}|_{\mathcal{G}^0})$. On the generic fibre, $(i^{-1})^*\varphi^{\otimes N}$ is an isomorphism $\mathcal{N}|_{A_K} \to \mathcal{L}_{\mathcal{A}}^{\otimes N}|_{A_K}$; regard it as a rational section ψ of $\mathcal{N}^\vee \otimes \mathcal{L}_{\mathcal{A}_K}^{\otimes N}$ on $\mathcal{A}_K$, regular and nowhere zero on the generic fibre. The scheme $\mathcal{A}_K$ is regular (smooth over the Dedekind ring $\mathcal{O}_K$) and its closed fibres are smooth and connected, hence irreducible; so the divisor of ψ is a combination of closed fibres: $\mathrm{div}(\psi) = \pi^*\Delta$ for a divisor Δ on $\operatorname{Spec}\mathcal{O}_K$, $\pi\colon \mathcal{A}_K \to \operatorname{Spec}\mathcal{O}_K$ the structure map. Pulling back along the unit section e (with $\pi \circ e = \mathrm{id}$), $\mathrm{div}(e^*\psi) = \Delta$. Now $j \circ e = \varepsilon$, so $e^*\mathcal{N} = \varepsilon^*\mathcal{M}^{\otimes N}$ is generated by $u^{\otimes N}$, and $e^*\psi(u^{\otimes N}) = \varphi(u)^{\otimes N} = (r_0 \otimes 1)^{\otimes N}$ by (Mb3), which generates $e^*\mathcal{L}_{\mathcal{A}_K}^{\otimes N}$ because r is an isomorphism $e^*\mathcal{L}_{\mathcal{A}} \cong \mathcal{O}$. Hence $e^*\psi$ maps a generator to a generator, $\Delta = 0$, and ψ is a nowhere vanishing regular section, i.e. an isomorphism $\psi_N\colon \mathcal{N} \to \mathcal{L}_{\mathcal{A}_K}^{\otimes N}$ with $e^*\psi_N(u^{\otimes N}) = (r_0 \otimes 1)^{\otimes N}$.

(2), the metrics. Fix $\sigma\colon K \to \mathbb{C}$ and write $X = A_\sigma(\mathbb{C})$, identified with $\mathcal{G}_\sigma(\mathbb{C})$ by i. Let $\|\cdot\|'$ be the metric on $\mathcal{L}_\sigma^{\otimes N}$ obtained from the N-th power of the cubist metric through ψ_N, and $\|\cdot\|$ the canonical norm. Both curvature forms are translation invariant:

- for $\|\cdot\|'$: let ω be the curvature form of the cubist metric on $\mathcal{M}_\sigma$ (that of $\|\cdot\|'$ is $N\omega$). The cubical trivialisation of $\Theta(\mathcal{M})_\sigma$ is holomorphic, nowhere zero and of norm 1, so the curvature of $\Theta(\mathcal{M})_\sigma$ with the induced metric vanishes: $\sum_{\varnothing \neq I \subseteq \{1,2,3\}} (-1)^{|I|+1} m_I^*\omega = 0$ on X^3. Pull this identity back along $X^2 \to X^3$, $(x,y) \mapsto (x,y,c)$, for fixed $c \in X$; pull-backs along constant maps vanish, so with $D_c = t_c^*\omega - \omega$ one gets $m^*D_c = p_1^*D_c + p_2^*D_c$ on X^2 (m the addition, p_1, p_2 the projections). Restricting to $X \times \{y\}$ gives $t_y^*D_c = D_c$ for every y: D_c is translation invariant. It is also exact, since t_c is homotopic to the identity. A translation-invariant form on a complex torus has constant coefficients, hence is harmonic for a flat metric, and an exact harmonic form is 0; so $t_c^*\omega = \omega$ for every c.
- for $\|\cdot\|$: on $\mathrm{Lie}(A_\sigma)$, for a local holomorphic section s, $-\log\|s\|^2 = -\log|\theta_s|^2 + \pi N H_\sigma(w,w)$ with θ_s holomorphic, so the curvature form is $\frac{i}{2\pi}\partial\bar\partial\big(\pi N H_\sigma(w,w)\big)$, which has constant coefficients.

The function $f = \|\cdot\|'/\|\cdot\|$ is smooth and positive on X, and $dd^c \log f$ is the difference of the two curvature forms. Both represent $c_1(\mathcal{L}_\sigma^{\otimes N})$ and are translation invariant, hence harmonic; cohomologous harmonic forms are equal, so $dd^c \log f = 0$, $\log f$ is pluriharmonic on the compact connected manifold X and therefore constant (maximum principle). At the origin, $\|(r_\sigma)^{\otimes N}\|' = \|u\|_\sigma^N = 1$ by (Mb3), and $\|(r_\sigma)^{\otimes N}\| = |\theta_s(0)| = 1$ for a local section s with $s(0) = r_\sigma^{\otimes N}$, since $\vartheta_{\mathcal{A},\sigma}(0) = r_\sigma$. Hence $f \equiv 1$.

(3). $s|_{\mathcal{G}^0}$ is a section of $\mathcal{M}^{\otimes N}$ over $\mathcal{G}^0$, so $s^\flat$ is a global section of $\mathcal{L}_{\mathcal{A}_K}^{\otimes N}$ over $\mathcal{A}_K$; on the generic fibre $\mathcal{G}_K^0 = \mathcal{G}_K$ and ψ_N is $\varphi^{\otimes N}$, which gives the restriction. The equality of the pointwise norms is (2), and the sup and L^2 norms follow, the Haar probability measures corresponding under the group isomorphism i_σ. □

Proposition D.32. *There is an absolute constant $C_{\sup} \geq 0$ such that for every CM field E, every number field L, every principally polarised abelian scheme $(\mathcal{A}, \mathcal{L}_{\mathcal{A}}, r)$ with $\mathcal{O}_E$-action over*

$\mathcal{O}_L$ of relative dimension g, every $\theta \in H^0(\mathcal{A}, \mathcal{L}_\mathcal{A})$ non-zero on the generic fibre and lying in the Moret-Bailly lattice, and every real $\Phi \geq 0$ with $\frac{1}{[L:\mathbb{Q}]} \sum_\sigma \log \|\theta\|_{L^2,\sigma} \leq \Phi$,

$$\frac{1}{[L:\mathbb{Q}]} \sum_{\sigma\colon L \to \mathbb{C}} \log \|\theta\|_{\sup,\sigma} \;\leq\; \Phi + C_{\sup}\, g \log(2 + g + \Phi).$$

Proof. We prove the statement with $C_{\sup} = 4$. Let θ and Φ be as in the statement, $A = \mathcal{A}_L$, r_0 the generator of $e^*\mathcal{L}_\mathcal{A}$ with $r(r_0) = 1$, and $b_\sigma = \|\theta\|_{L^2,\sigma}$; $b_\sigma > 0$ and $\|\theta\|_{\sup,\sigma} < \infty$ since θ_{θ_σ} is a non-zero entire function and the canonical norm is continuous on the compact $A_\sigma(\mathbb{C})$. For a finite extension K'/L and a function f of the embeddings $L \to \mathbb{C}$, the mean over $\sigma'\colon K' \to \mathbb{C}$ of $f(\sigma'|_L)$ equals the mean over $\sigma\colon L \to \mathbb{C}$ of $f(\sigma)$, each σ having $[K' : L]$ extensions; the canonical norms of θ at σ' are those at $\sigma'|_L$.

Step 1: $h_F(A) \leq 2\Phi - \frac{g}{2}\log(2\pi)$. As θ lies in the Moret-Bailly lattice, there are a finite extension K/L and a Moret-Bailly model $\mathfrak{M}$ of $(A_K, L_A \otimes K, r_0 \otimes 1, \varnothing)$ over $\mathcal{O}_K$ whose lattice Λ contains $\theta|_{A_K}$, where $L_A = \mathcal{L}_\mathcal{A}|_A$. Apply Theorem B.137 to $K_0 = L$ and (A, L_A); it gives a field $K_1 \supseteq L$ and data $(\mathcal{G}_1, \mathcal{M}_1, \varphi_1)$ with cubist metrics and a finite flat subscheme with generic fibre $A[2]$. Let K' be a finite extension of L containing (copies of) K and K_1. By Theorem B.133 (2), the base change $\mathfrak{M}'$ of $\mathfrak{M}$ to $\mathcal{O}_{K'}$ is a Moret-Bailly model with lattice $\Lambda \otimes \mathcal{O}_{K'} \ni \theta|_{A_{K'}}$. On the other side, by Theorem B.137 $\varepsilon^*\mathcal{M}_1$ with the cubist metrics is isometric to $\mathcal{O}_{K_1}$ with the trivial metrics; let u be the base change to $\mathcal{O}_{K'}$ of its generator of norm one, write $\varphi_1 \otimes K'(u) = a\,(r_0 \otimes 1)$ with $a \in K'^\times$, and put $\psi = a^{-1}(\varphi_1 \otimes K')$. The base changes $\mathcal{G}_1'$, $\mathcal{M}_1'$ of $\mathcal{G}_1$, $\mathcal{M}_1$, with the base-changed isomorphism of generic fibres, cubist metrics and finite flat subscheme, together with (ψ, u), satisfy (Mb1), (Mb2), (Mb5) (these properties are preserved by base change) and (Mb3) (by construction), (Mb4) being empty: they form a Moret-Bailly model $\mathfrak{M}_1'$ of $(A_{K'}, L_A \otimes K', r_0 \otimes 1, \varnothing)$ over $\mathcal{O}_{K'}$, whose lattice is $\psi(E')$ with $E' = H^0(\mathcal{G}_1', \mathcal{M}_1')$. By Theorem B.133 (3) the two lattices coincide, so $\theta|_{A_{K'}} = \psi(t|_{\mathcal{G}'_{1,K'}})$ for some $0 \neq t \in E'$. By Lemma D.31 (3) for $\mathfrak{M}_1'$ and $N = 1$, $\|\theta\|_{L^2,\sigma'}$ equals the L^2 norm of t for the cubist metric, for every $\sigma'\colon K' \to \mathbb{C}$. By Theorem B.137 (with its K' our K', rescaling φ_1 not affecting E' and its norms),

$$[K' : \mathbb{Q}]\Big(-\frac{1}{2}h_F(A) - \frac{g}{4}\log(2\pi)\Big) = \widehat{\deg}\,\overline{E'} = \log \#(E'/\mathcal{O}_{K'}t) - \sum_{\sigma'} \log \|\theta\|_{L^2,\sigma'} \geq -[K' : \mathbb{Q}]\,\Phi,$$

using the remark on means and the hypothesis on Φ. This is the claim.

Step 2: the sup norm at one place. Fix $\sigma\colon L \to \mathbb{C}$; write $V = \mathrm{Lie}(A_\sigma)$, $H = H_\sigma$, $\Pi = \Pi_\sigma$, $\rho = \rho_\sigma = \min_{0 \neq l \in \Pi} H(l,l)^{1/2}$, θ_σ for the theta function of θ at σ and $N(w) = |\theta_\sigma(w)| e^{-\frac{\pi}{2}H(w,w)}$, the canonical norm of θ at $\pi_\sigma(w)$. Let λ be the Lebesgue measure of the Euclidean structure $\mathrm{Re}\,H$ on $V \cong \mathbb{R}^{2g}$. The covolume of Π for λ is $\chi(A, L_A) = 1$ ([46, 2.1.2]; the polarisation is principal), so the Haar probability measure of $A_\sigma(\mathbb{C})$ is the image of λ restricted to a fundamental domain, and for every measurable $B \subseteq V$ on which π_σ is injective and every $w_0 \in V$, $\int_B N(w_0 + w)^2\, d\lambda(w) \leq b_\sigma^2$. Fix $w_0 \in V$ and put $f(w) = \theta_\sigma(w_0 + w)\exp\big(-\pi H(w, w_0) - \frac{\pi}{2}H(w_0, w_0)\big)$, an entire function ($H$ is $\mathbb{C}$-linear in its first argument). From $H(w_0 + w, w_0 + w) = H(w,w) + 2\,\mathrm{Re}\,H(w, w_0) + H(w_0, w_0)$,

$$|f(w)| = N(w_0 + w)\, e^{\frac{\pi}{2}H(w,w)}.$$

Let $R = \frac{1}{2}\min(\rho, 1)$ and $B_R = \{w : H(w,w) < R^2\}$. Since $|f|^2$ is plurisubharmonic, hence subharmonic on $\mathbb{R}^{2g}$, it satisfies the sub-mean-value inequality on B_R. If $w, w' \in B_R$ and $w - w' \in \Pi$, then $H(w - w', w - w')^{1/2} < 2R \leq \rho$, so $w = w'$: π_σ is injective on $w_0 + B_R$. With

$\lambda(B_R) = \pi^g R^{2g}/g!$:

$$N(w_0)^2 = |f(0)|^2 \le \frac{1}{\lambda(B_R)} \int_{B_R} N(w_0+w)^2 e^{\pi H(w,w)}\, d\lambda(w) \le \frac{g!\, e^{\pi R^2}}{\pi^g R^{2g}}\, b_\sigma^2.$$

Taking the supremum over w_0 and using $\log g! \le g \log g$, $\pi R^2/2 \le \pi/8 < 1$, $\pi^{-g} \le 1$ and $\log(1/R) = \log 2 + \frac{1}{2}\log\max(1,\rho^{-2}) \le \log 2 + \frac{1}{2}\log(1+\rho^{-2})$:

$$\log \|\theta\|_{\sup,\sigma} \le \log b_\sigma + 1 + \frac{g}{2}\log g + g\log 2 + \frac{g}{2}\log(1+\rho_\sigma^{-2}). \tag{D.8}$$

Step 3: the mean over the places. By Theorem B.138 for A over L with $\varepsilon = \frac{1}{2}$ (the forms H_σ there are ours), $\frac{\pi}{12}\,\mathrm{mean}_\sigma \rho_\sigma^{-2} \le h_F(A) - \frac{g}{2}\log\pi + \frac{g}{2}\log(4\pi^2) = h_F(A) + \frac{g}{2}\log(4\pi)$, and by Step 1

$$\mathrm{mean}_\sigma\, \rho_\sigma^{-2} \le \frac{12}{\pi}\Big(2\Phi + \frac{g}{2}\log 2\Big) \le \frac{24}{\pi}\Phi + \frac{6}{\pi}g.$$

Average (D.8) over σ, use $\mathrm{mean}_\sigma \log b_\sigma \le \Phi$ and Jensen's inequality for the concave log:

$$\mathrm{mean}_\sigma \log(1+\rho_\sigma^{-2}) \le \log(1+\mathrm{mean}_\sigma \rho_\sigma^{-2}) \le \log\big(8(2+g+\Phi)\big) \le 3\log(2+g+\Phi),$$

the last step since $\log 8 \le 2\log 3 \le 2\log(2+g+\Phi)$. With $Y = \log(2+g+\Phi) \ge \log 3 \ge 1$, $\log g \le Y$ and $\log 2 \le Y$:

$$\mathrm{mean}_\sigma \log \|\theta\|_{\sup,\sigma} \le \Phi + gY + \frac{g}{2}Y + gY + \frac{3g}{2}Y = \Phi + 4g\log(2+g+\Phi).$$

□

Proposition D.33. *Let E be a CM field, L a number field, $(\mathcal{A}, \mathcal{L}_\mathcal{A}, r)$ a principally polarised abelian scheme with $\mathcal{O}_E$-action over $\mathcal{O}_L$ of relative dimension g, $\theta \in H^0(\mathcal{A}, \mathcal{L}_\mathcal{A})$ non-zero on the generic fibre and lying in the Moret-Bailly lattice, and $(M_\sigma)_{\sigma\colon L\to\mathbb{C}}$ positive reals with $\|\theta\|_{\sup,\sigma} \le M_\sigma$ for every σ. For every even integer $m \ge 2$ there are a finite extension L'/L and sections $s_1, \dots, s_{m^{2g}} \in H^0(\mathcal{A}', \mathcal{L}_{\mathcal{A}'}^{\otimes m^2})$ of the base change $\mathcal{A}'$ of $\mathcal{A}$ to $\mathcal{O}_{L'}$ such that for every $\sigma'\colon L' \to \mathbb{C}$*

$$\|s_j\|_{\sup,\sigma'} \le M_{\sigma'|_L} \quad (1 \le j \le m^{2g}),$$

and the images of $s_1, \dots, s_{m^{2g}}$ span $H^0(A'_{\sigma'}, \mathcal{L}_{\sigma'}^{\otimes m^2})$ over $\mathbb{C}$.

Proof. Write $A = \mathcal{A}_L$, $L_A = \mathcal{L}_\mathcal{A}|_A$, $r_0 \in \mathcal{L}_\mathcal{A}(0)$ for the generator with $r(r_0) = 1$, and, for a finite extension K/L, θ_K for the restriction of θ to A_K. By Definition 31.6, A is an abelian variety of dimension $g \ge 1$ and L_A is symmetric, ample and defines a principal polarisation. Fix an even integer $m \ge 2$.

Step 1: one model carrying θ and the torsion sections. As θ lies in the Moret-Bailly lattice, there are a finite extension K/L and a Moret-Bailly model $\mathfrak{M}$ of $(A_K, L_A \otimes K, r_0 \otimes 1, \varnothing)$ over $\mathcal{O}_K$ whose Moret-Bailly lattice Λ contains θ_K. By Theorem B.133 (1), applied with L in place of the base field, to A, L_A, $r_0 \ne 0$ and the even integer $r = m \ge 2$ (its hypotheses are those just listed), there are a finite extension K_1/L with $A[m^2] \subseteq A(K_1)$ and a Moret-Bailly model $\mathfrak{M}_1$ of $(A_{K_1}, L_A \otimes K_1, r_0 \otimes 1, A[m^2](K_1))$ over $\mathcal{O}_{K_1}$. Let L' be a finite extension of L containing K and K_1 (a compositum). Then $A[m^2](L') = A[m^2](K_1)$, all m^2-torsion being K_1-rational. By Theorem B.133 (2), the base change $\mathfrak{M}_{L'}$ of $\mathfrak{M}$ is a Moret-Bailly model of $(A_{L'}, L_A \otimes L', r_0 \otimes 1, \varnothing)$ with Moret-Bailly lattice $\Lambda \otimes_{\mathcal{O}_K} \mathcal{O}_{L'}$, which contains $\theta_{L'} = \theta_K \otimes 1$; and the base change

$$\mathfrak{M}' = (\mathcal{G}, i, \mathcal{M}, (\|\cdot\|_{\sigma'})_{\sigma'}, \varphi, u, (\varepsilon_x)_{x \in A[m^2](L')})$$

of $\mathfrak{M}_1$ is a Moret-Bailly model of $(A_{L'}, L_A \otimes L', r_0 \otimes 1, A[m^2](L'))$ over $\mathcal{O}_{L'}$. Forgetting the sections ε_x, the data of $\mathfrak{M}'$ form a Moret-Bailly model of $(A_{L'}, L_A \otimes L', r_0 \otimes 1, \varnothing)$ (condition (Mb4) is empty for $F = \varnothing$, the others are unchanged), with the same Moret-Bailly lattice. By Theorem B.133 (3) this lattice equals that of $\mathfrak{M}_{L'}$, so it contains $\theta_{L'}$: there is $s_0 \in H^0(\mathcal{G}, \mathcal{M})$ with $\varphi(s_0|_{\mathcal{G}_{L'}}) \circ i^{-1} = \theta_{L'}$.

Step 2: the norm of s_0. Let $\mathcal{A}'$ be the base change of $\mathcal{A}$ to $\mathcal{O}_{L'}$. Apply Lemma D.31 to $\mathcal{A}$, the extension L'/L, the finite set $F = A[m^2](L') \subseteq A(L')$ and the model $\mathfrak{M}'$: for every $N \geq 1$ and $s \in H^0(\mathcal{G}, \mathcal{M}^{\otimes N})$ it gives $s^\flat \in H^0(\mathcal{A}', \mathcal{L}_{\mathcal{A}'}^{\otimes N})$ whose restriction to $A_{L'}$ is $\varphi^{\otimes N}(s|_{\mathcal{G}_{L'}})$ transported by i, and whose canonical norm at each point $i(y)$ of $A_{\sigma'}(\mathbb{C})$ equals the cubist norm $\|s\|_{\sigma'}(y)$, for every $\sigma' \colon L' \to \mathbb{C}$ and $y \in \mathcal{G}_{\sigma'}(\mathbb{C})$. For $N = 1$ and $s = s_0$, the sections $s_0^\flat$ and $\theta \otimes 1 \in H^0(\mathcal{A}', \mathcal{L}_{\mathcal{A}'})$ have the same restriction $\theta_{L'}$ to the generic fibre; since $\mathcal{A}'$ is flat over $\mathcal{O}_{L'}$, restriction to the generic fibre is injective on $H^0(\mathcal{A}', \mathcal{L}_{\mathcal{A}'})$, so $s_0^\flat = \theta \otimes 1$. The complex torus, the Hermitian form, the line bundle and the rigidification of $\mathcal{A}'$ at σ' are those of $\mathcal{A}$ at $\sigma'|_L$, hence so are the theta trivialisation and the canonical norms (Definition D.28). Therefore, for every σ' and $y \in \mathcal{G}_{\sigma'}(\mathbb{C})$,

$$\|s_0\|_{\sigma'}(y) = \|\theta \otimes 1\|(i(y)) \leq \|\theta\|_{\sup, \sigma'|_L} \leq M_{\sigma'|_L}. \tag{D.9}$$

Step 3: the translates. Let j, i_x and ψ_x ($x \in A[m^2](L')$) be the isomorphisms and maps of Theorem B.133 (4) for the model $\mathfrak{M}'$ and $r = m$, and let $\Sigma \subseteq A[m^2](L')$ be a set of representatives of $A[m^2]/A[m]$; it has $m^{4g}/m^{2g} = m^{2g}$ elements. Since $[m]$, t_{ε_x}, j and i_x are defined over $\mathcal{G}$, $\psi_x(s_0) = i_x(t_{\varepsilon_x}^* j([m]^* s_0)) \in H^0(\mathcal{G}, \mathcal{M}^{\otimes m^2})$, and its restriction to $\mathcal{G}_{L'}$ is $\psi_x(s_0|_{\mathcal{G}_{L'}})$. Put $s_x = \psi_x(s_0)^\flat \in H^0(\mathcal{A}', \mathcal{L}_{\mathcal{A}'}^{\otimes m^2})$ (Step 2 with $N = m^2$) for $x \in \Sigma$, and number them $s_1, \ldots, s_{m^{2g}}$.

Step 4: the sup norms. Fix σ' and $y \in \mathcal{G}_{\sigma'}(\mathbb{C})$, and write ε_x also for the point $\varepsilon_x(\sigma') \in \mathcal{G}_{\sigma'}(\mathbb{C})$. The metric on $t_{\varepsilon_x}^* \mathcal{M}^{\otimes m^2}$ is the pull-back metric, and that on $[m]^*\mathcal{M}$ the pull-back of the cubist metric; as i_x and j are isometric (Theorem B.133 (4)),

$$\|\psi_x(s_0)\|_{\sigma'}(y) = \big\|j([m]^* s_0)\big\|_{\sigma'}(y + \varepsilon_x) = \|[m]^* s_0\|_{\sigma'}(y + \varepsilon_x) = \|s_0\|_{\sigma'}(my + m\varepsilon_x) \leq M_{\sigma'|_L}$$

by (D.9). By Step 2 the canonical norm of s_x at $i(y)$ is $\|\psi_x(s_0)\|_{\sigma'}(y)$; as y runs over $\mathcal{G}_{\sigma'}(\mathbb{C})$, $i(y)$ runs over $A_{\sigma'}(\mathbb{C})$, so $\|s_x\|_{\sup, \sigma'} \leq M_{\sigma'|_L}$.

Step 5: spanning. The maps φ and i identify $H^0(\mathcal{G}_{L'}, \mathcal{M}_{L'})$ with $H^0(A_{L'}, L_A \otimes L')$, which is one-dimensional and contains $\theta_{L'} \neq 0$ (θ is non-zero on the generic fibre); hence $H^0(\mathcal{G}_{L'}, \mathcal{M}_{L'}) = L'\, s_0|_{\mathcal{G}_{L'}}$. By Theorem B.133 (4), $(c_x s_0|_{\mathcal{G}_{L'}})_{x \in \Sigma} \mapsto \sum_{x \in \Sigma} c_x \psi_x(s_0|_{\mathcal{G}_{L'}})$ ($c_x \in L'$) is an isomorphism onto $H^0(\mathcal{G}_{L'}, \mathcal{M}_{L'}^{\otimes m^2})$, so the $\psi_x(s_0)|_{\mathcal{G}_{L'}}$, $x \in \Sigma$, form an L'-basis of it. Applying the isomorphisms $\varphi^{\otimes m^2}$ and i, the restrictions of the s_x to $A_{L'}$ (Step 2) form an L'-basis of $H^0(A_{L'}, L_A^{\otimes m^2} \otimes L')$. By flat base change along σ', $H^0(A_{\sigma'}, \mathcal{L}_{\sigma'}^{\otimes m^2}) = H^0(A_{L'}, L_A^{\otimes m^2} \otimes L') \otimes_{L', \sigma'} \mathbb{C}$, so the images of $s_1, \ldots, s_{m^{2g}}$ form a $\mathbb{C}$-basis of $H^0(A_{\sigma'}, \mathcal{L}_{\sigma'}^{\otimes m^2})$; in particular they span it. □

Lemma D.34. *Let $(\mathcal{A}, \mathcal{L}_{\mathcal{A}}, r)$ be a principally polarised abelian scheme with $\mathcal{O}_E$-action over $\mathcal{O}_L$, $x \in \mathrm{Lie}(\mathcal{A})$ and $n \geq 1$; put $W_n = H^0(A, \mathcal{L}_{\mathcal{A}}|_A^{\otimes n})$. There are L-linear maps $j_k \colon W_n \to L$ ($k \geq 0$) such that:*

1. *if $s \in H^0(\mathcal{A}, \mathcal{L}_{\mathcal{A}}^{\otimes n})$, $i \geq 0$ and $j_k(s) = 0$ for all $k < i$, then $j_i(s) \in \mathcal{O}_L$;*
2. *for every $\sigma \colon L \to \mathbb{C}$, every $s \in H^0(A_\sigma, \mathcal{L}_\sigma^{\otimes n}) = W_n \otimes_{L, \sigma} \mathbb{C}$ and every $i \geq 0$, with $j_{k,\sigma} = j_k \otimes_\sigma \mathbb{C}$: $j_{k,\sigma}(s) = 0$ for all $k < i$ if and only if $\partial_{x_\sigma}^k \theta_s(0) = 0$ for all $k < i$, and in that case $j_{i,\sigma}(s) = \partial_{x_\sigma}^i \theta_s(0)$.*

Proof. Let X be the invariant vector field on $\mathcal{A}$ with value x along the unit section e; it is a global section of $T_{\mathcal{A}/\mathcal{O}_L}$, i.e. an $\mathcal{O}_L$-derivation of $\mathcal{O}_\mathcal{A}$, and its restriction to A is the invariant vector field of x_L.

Definition of the jets. Since $\mathcal{L}_\mathcal{A}|_A$ is invertible near 0, there is a section ϵ of it on an open neighbourhood of 0 in A generating it there, and after scaling $\epsilon(0) = r$. For $s \in W_n$ the quotient s/ϵ^n is regular at 0; put $j_k(s) = (X^k(s/\epsilon^n))(0) \in L$. These maps are L-linear.

Item 2. Fix σ. The vector field X_σ on $A_\sigma(\mathbb{C})$ is invariant with value x_σ at 0, and its integral curve through 0 is $\tau \mapsto \pi_\sigma(\tau x_\sigma)$; by invariance $\frac{d}{d\tau}\varphi(\pi_\sigma(\tau x_\sigma)) = (X_\sigma\varphi)(\pi_\sigma(\tau x_\sigma))$ for holomorphic φ near 0, hence by induction $(X_\sigma^k\varphi)(0) = \frac{d^k}{d\tau^k}\varphi(\pi_\sigma(\tau x_\sigma))|_{\tau=0}$. So $j_{k,\sigma}(s) = F^{(k)}(0)$ for $F(\tau) = (s/\epsilon^n)(\pi_\sigma(\tau x_\sigma))$. Near 0 write $\pi_\sigma^*\epsilon = \eta\,\vartheta$ with η holomorphic and $\eta(0) = 1$ (both equal r_σ at 0). Then $\pi_\sigma^* s = \theta_s \vartheta^{\otimes n} = ((s/\epsilon^n)\circ\pi_\sigma)\eta^n\vartheta^{\otimes n}$, so $\theta_s(\tau x_\sigma) = F(\tau)U(\tau)$ with $U(\tau) = \eta(\tau x_\sigma)^n$ holomorphic near 0 and $U(0) = 1$. If $F^{(k)}(0) = 0$ for $k < i$, the Leibniz rule gives $(FU)^{(k)}(0) = 0$ for $k < i$ and $(FU)^{(i)}(0) = F^{(i)}(0)U(0) = F^{(i)}(0)$; conversely, if $(FU)^{(k)}(0) = 0$ for $k < i$, the same argument applied to $F = (FU)\cdot U^{-1}$ gives $F^{(k)}(0) = 0$ for $k < i$. This is item 2.

Item 1. Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}_L$, $R = \mathcal{O}_{L,\mathfrak{p}}$, and $P = e(\mathfrak{p}) \in \mathcal{A}$. Choose a generator $\epsilon_\mathfrak{p}$ of $\mathcal{L}_\mathcal{A}$ on an open neighbourhood $N_\mathfrak{p}$ of P. Open sets are stable under generisation, so $N_\mathfrak{p}$ contains the image of $\operatorname{Spec} R$ under e, and $e^*\epsilon_\mathfrak{p}$ generates $e^*\mathcal{L}_\mathcal{A} \otimes R \cong R$; after multiplying by a unit of R, $e^*\epsilon_\mathfrak{p} = r$. For $s \in H^0(\mathcal{A}, \mathcal{L}_\mathcal{A}^{\otimes n})$ we have $s/\epsilon_\mathfrak{p}^n \in \mathcal{O}_{\mathcal{A},P}$, hence $X^a(s/\epsilon_\mathfrak{p}^n) \in \mathcal{O}_{\mathcal{A},P}$, and

$$j_a^\mathfrak{p}(s) := e^*\big(X^a(s/\epsilon_\mathfrak{p}^n)\big) \in R \qquad (a \geq 0);$$

its image in L is $(X^a(s/\epsilon_\mathfrak{p}^n))(0)$ computed on A. Near 0 in A, $\epsilon = \beta\,\epsilon_\mathfrak{p}$ with β regular and invertible at 0, $\beta(0) = 1$. Then $s/\epsilon^n = (s/\epsilon_\mathfrak{p}^n)\,\beta^{-n}$ and the Leibniz rule for the derivation X gives

$$j_k(s) = j_k^\mathfrak{p}(s) + \sum_{a<k} \binom{k}{a} j_a^\mathfrak{p}(s)\,(X^{k-a}\beta^{-n})(0).$$

If $j_k(s) = 0$ for $k < i$, induction on $k < i$ gives $j_k^\mathfrak{p}(s) = 0$ for $k < i$, and then $j_i(s) = j_i^\mathfrak{p}(s) \in R$. As this holds for every $\mathfrak{p}$, $j_i(s) \in \bigcap_\mathfrak{p} \mathcal{O}_{L,\mathfrak{p}} = \mathcal{O}_L$. □

Lemma D.35. *Let $(\mathcal{A}, \mathcal{L}_\mathcal{A}, r)$ be a principally polarised abelian scheme with $\mathcal{O}_E$-action over $\mathcal{O}_L$, $\sigma\colon L \to \mathbb{C}$, $0 \neq v \in \operatorname{Lie}(A_\sigma)$, and let $n' \geq 1$ and $0 \leq i \leq n'$ be integers. Then there is $s \in H^0(A_\sigma, \mathcal{L}_\sigma^{\otimes 3n'})$ with*

$$\partial_v^k\theta_s(0) = 0 \quad (0 \leq k < i), \qquad \partial_v^i\theta_s(0) \neq 0.$$

Proof. The line bundle $\mathcal{L}_\sigma$ is ample, being the base change of the ample $\mathcal{L}_\mathcal{A}|_A$ along the field embedding σ. By Lefschetz's theorem [32] (Theorem B.122), $\mathcal{L}_\sigma^{\otimes 3}$ is very ample, so its complete linear system defines a closed embedding j of A_σ; in particular it is base point free and dj is injective on the tangent space $T_0A_\sigma = \operatorname{Lie}(A_\sigma)$.

Two sections. Choose $s_0 \in H^0(A_\sigma, \mathcal{L}_\sigma^{\otimes 3})$ with $s_0(0) \neq 0$. On the open set $\{s_0 \neq 0\}$ the functions s/s_0 ($s \in H^0(\mathcal{L}_\sigma^{\otimes 3})$) are the affine coordinates of j, so injectivity of dj at 0 gives s with $d(s/s_0)_0(v) \neq 0$; then $s_1 = s - (s/s_0)(0)\,s_0$ satisfies $s_1(0) = 0$ and $d(s_1/s_0)_0(v) = d(s/s_0)_0(v) \neq 0$.

Translation to theta functions. Write $\pi_\sigma^* s_a = \theta_a\vartheta^{\otimes 3}$ ($a = 0, 1$). As ϑ vanishes nowhere, $\theta_0(0) \neq 0$, $\theta_1(0) = 0$, and near 0 we have $(s_1/s_0)\circ\pi_\sigma = \theta_1/\theta_0$. The exponential map π_σ is a local biholomorphism at 0 with differential the identity of $\operatorname{Lie}(A_\sigma)$, so $d(\theta_1/\theta_0)_0(v) = d(s_1/s_0)_0(v) \neq 0$. For $f(\tau) = \theta_1(\tau v)$ and $u(\tau) = \theta_0(\tau v)$ this gives $f(0) = 0$, $u(0) \neq 0$ and, since $f(0) = 0$, $f'(0) = u(0)\,\frac{d}{d\tau}(f/u)(0) = u(0)\,d(\theta_1/\theta_0)_0(v) \neq 0$.

The section. Put $s = s_1^{\otimes i} \otimes s_0^{\otimes(n'-i)} \in H^0(A_\sigma, \mathcal{L}_\sigma^{\otimes 3n'})$. Then $\pi_\sigma^* s = \theta_1^i \theta_0^{n'-i} \vartheta^{\otimes 3n'}$, so $\theta_s = \theta_1^i \theta_0^{n'-i}$ and $\theta_s(\tau v) = f(\tau)^i u(\tau)^{n'-i} = \tau^i q(\tau)$ with q entire and $q(0) = f'(0)^i u(0)^{n'-i} \neq 0$. Hence $\partial_v^k \theta_s(0) = 0$ for $k < i$ and $\partial_v^i \theta_s(0) = i!\, q(0) \neq 0$. □

D.2.2 Flat saturation and flat generators of the determinant line

Flat saturation (Proposition D.36) says that the complex-conjugate Lie eigenspaces of an abelian variety of unitary signature have nearly equal degrees, with an error controlled by rd_E and L_{H}; it is an input of Lemma D.22. The proof realises the abelian variety as a point of a unitary Shimura variety (Proposition D.37) and evaluates a generator of the τ-determinant line there, which is a flat generator with archimedean and ℓ-adic norms (Proposition D.38, Proposition D.39), so that its flat height is given by Theorem 34.20. Proposition D.38 is used again in Section D.4.

Proposition D.36. *There is a real number $C_{\mathrm{fs}} \geq 0$ with the following property. Let E be a CM field with complex conjugation c and a CM host $\mathsf{H} = (d, r, r_{\mathbb{C}})$, put $\mathrm{rd}_E = |\mathrm{disc}(E)|^{1/[E:\mathbb{Q}]}$ and $L_{\mathsf{H}} = \mathrm{Re}\,(L'/L)(1, \chi_d)$, let L be a number field and $\mathcal{A}$ a principally polarised abelian scheme with $\mathcal{O}_E$-action ι over $\mathcal{O}_L$, with generic fibre A, such that the Rosati involution of the principal polarisation of A maps $\iota(a)$ to $\iota(c(a))$ for $a \in \mathcal{O}_E$. Suppose that there are $\rho \in L$ with $\rho^2 = -d$ and integers $p', q' \geq 0$ such that for every algebraically closed field k, every ring homomorphism $\gamma\colon L \to k$ and every ring homomorphism $\sigma\colon \mathcal{O}_E \to k$, the σ-eigenspace $\{x : \iota(a)x = \sigma(a)x \ \forall a \in \mathcal{O}_E\}$ of $\mathrm{Lie}(A) \otimes_{L,\gamma} k$ has dimension p' if $\sigma(r) = \gamma(\rho)$ and dimension q' otherwise. Let $s_1, s_2\colon \mathcal{O}_E \to \mathcal{O}_L$ be ring homomorphisms with s_1 injective and $e \circ s_2 = \overline{e \circ s_1}$ for every field embedding $e\colon L \to \mathbb{C}$. For $i = 1, 2$ put $\mathrm{Lie}_{s_i}(\mathcal{A}) = \{x \in \mathrm{Lie}(\mathcal{A}) : \iota(a)x = s_i(a)x \ \forall a \in \mathcal{O}_E\}$ and $d_{s_i} = \widehat{\deg}\,\overline{\mathrm{Lie}}_{s_i}(\mathcal{A})/[L : \mathbb{Q}]$. Then*

$$|d_{s_1} - d_{s_2}| \ \leq \ C_{\mathrm{fs}}\,(\mathrm{rk}\,\mathrm{Lie}_{s_1}(\mathcal{A}) + \mathrm{rk}\,\mathrm{Lie}_{s_2}(\mathcal{A}))\,(1 + \log \mathrm{rd}_E + |L_{\mathsf{H}}|).$$

Proof. We prove the bound with $C_{\mathrm{fs}} = \frac{3}{2}$. Write $\mathrm{rk}_i = \mathrm{rk}\,\mathrm{Lie}_{s_i}(\mathcal{A})$.

Step 1: the labels. As $r^2 = -d$ in $\mathcal{O}_E$, $s_1(r)^2 = -d = \rho^2$ in the field L, so $s_1(r) = \pm\rho$. If $s_1(r) = -\rho$, then for every $e\colon L \to \mathbb{C}$, $e(s_2(r)) = \overline{e(s_1(r))} = -\overline{e(\rho)} = e(\rho)$, because $e(\rho)^2 = -d$ makes $e(\rho)$ purely imaginary; so $s_2(r) = \rho$, and s_2 is injective (as $e \circ s_2 = \overline{e \circ s_1}$). The hypotheses and the conclusion are symmetric in (s_1, s_2), so we may assume $s_1(r) = \rho$. By the signature hypothesis (for $k = \overline{L}$, and since $\mathrm{Lie}_{s_i}(\mathcal{A}) \otimes_{\mathcal{O}_L} L$ is the s_i-eigenspace of $\mathrm{Lie}(A)$), $\mathrm{rk}_1 = p'$ and $\mathrm{rk}_2 = q'$ ($s_2(r) = -\rho \neq \rho$). If $p' + q' = 0$ both lattices are zero and there is nothing to prove; let $p' + q' \geq 1$.

Step 2: the moduli point. By Proposition D.37 there are a unitary Shimura datum $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p', q', \mathsf{L}, \mathrm{Sh}, F)$ with level good at every prime and split components, an integral model $(\mathcal{Y}, j, G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ over $\mathcal{O}_F$, a finite extension $K \supseteq L$ with $\beta\colon F \to K$, $\beta(r_{\mathbb{C}}) = \rho$, and $y \in \mathrm{Sh}_F(K)$ with an $\mathcal{O}_E$-linear, polarisation-compatible isomorphism $G_y \cong A \otimes_L K$. The maps $\beta \circ \sigma$, $\sigma \in \mathrm{Hom}(E, F)$, are $[E : \mathbb{Q}]$ distinct embeddings $E \to K$ (F contains every conjugate of E), hence all of them; so $s_1 = \beta \circ \tau$ on $\mathcal{O}_E$ for a unique $\tau \in \mathrm{Hom}(E, F)$, and $\beta(\tau(r)) = s_1(r) = \rho = \beta(r_{\mathbb{C}})$ gives $\tau(r) = r_{\mathbb{C}}$, i.e. $\tau \in \Phi_{\mathsf{H}}$ and $\sigma_{(\tau,+)} = \tau$. For $e'\colon K \to \mathbb{C}$ extending $e\colon L \to \mathbb{C}$, $e' \circ \beta$ restricted to the CM field $\tau(E)$ commutes with complex conjugation, so $e'(\beta(\bar\tau(a))) = \overline{e'(\beta(\tau(a)))} = \overline{e(s_1(a))} = e(s_2(a))$; hence $s_2 = \beta \circ \sigma_{(\tau,-)}$.

Step 3: the flat generator. The point $j(y)$ lies in an irreducible component $\mathcal{Y}'$ of $\mathcal{Y}$ meeting $j(\mathrm{Sh}_F)$; put $U = j^{-1}(\mathcal{Y}' \times_{\mathcal{O}_F} F)$, so $y \in U(K)$. By Proposition D.38 (components split, level good at every prime, $n = p' + q' \geq 1$), for a nowhere vanishing section ξ of $\det \mathrm{Lie}_{(\tau,+)} \otimes (\det \mathrm{Lie}_{(\tau,-)})^{-1}$ on $\mathcal{Y}'_F$ the frame function φ_ξ on the connected component U is a flat generator

over a connected component Y of $U(\mathbb{C})$ with some constants (e,c), and has archimedean norms N_s at every $s\colon F\to\mathbb{C}$ and ℓ-adic norms N_κ at every $\kappa\colon F\to\mathbb{C}_\ell$, every ℓ.

Step 4: the product formula. Let $\mathcal{G}=\mathcal{A}\times_{\mathcal{O}_L}\mathcal{O}_K$, a semiabelian scheme over $\mathcal{O}_K$ with $\mathcal{O}_E$-action whose generic fibre is identified with G_y. Its eigen-submodules $\mathcal{L}_\pm$ for $\beta\circ\sigma_{(\tau,\pm)}=s_1,s_2$ are $\mathrm{Lie}_{s_1}(\mathcal{A})\otimes_{\mathcal{O}_L}\mathcal{O}_K$ and $\mathrm{Lie}_{s_2}(\mathcal{A})\otimes_{\mathcal{O}_L}\mathcal{O}_K$ (kernels of finitely many $\mathcal{O}_L$-linear maps commute with the flat base change). As the isomorphism respects the principal polarisations, the Hodge inner product $h_{y^t}=\pi H_{y^t}$ at $t\colon K\to\mathbb{C}$ is the norm $\pi H_{t|_L}$ of the Lie lattice $\overline{\mathrm{Lie}}(\mathcal{A})$ (Definition 31.6). The arithmetic degree of a Hermitian lattice is multiplied by $[K:L]$ under base change to $\mathcal{O}_K$, so $d_{s_i}=\widehat{\deg}\,\overline{\mathcal{L}}_\pm/[K:\mathbb{Q}]$. Lemma D.54, applied to $\mathfrak{u}$, τ, U, φ_ξ, K, β, y and $\mathcal{G}$, gives

$$d_{s_1}-d_{s_2}=-\frac{1}{[F:\mathbb{Q}]}\Big(\sum_s N_s+\sum_\ell\sum_\kappa N_\kappa\Big)=h^{\mathrm{fl}}(\varphi_\xi),$$

the flat height of (Fd1) in Theorem 34.20.

Step 5: the bound. By (Fd3) of Theorem 34.20, applied to φ_ξ (a flat generator over Y with all local norms, U a union of connected components),

$$|d_{s_1}-d_{s_2}|=|h^{\mathrm{fl}}(\varphi_\xi)|\le n\Big(\frac{3}{2}\log\mathrm{rd}_E+\frac{|L_{\mathsf{H}}|}{2}\Big)\le\frac{3}{2}\,(\mathrm{rk}_1+\mathrm{rk}_2)(1+\log\mathrm{rd}_E+|L_{\mathsf{H}}|),$$

since $n=p'+q'=\mathrm{rk}_1+\mathrm{rk}_2$ and $\log\mathrm{rd}_E\ge0$. This is the claim with $C_{\mathrm{fs}}=\frac{3}{2}$. □

Proposition D.37. *Let E be a CM field unramified over its maximal totally real subfield at every finite place, with complex conjugation c and a CM host $\mathsf{H}=(d,r,r_{\mathbb{C}})$; let L be a number field and $\mathcal{A}$ a principally polarised abelian scheme with $\mathcal{O}_E$-action ι over $\mathcal{O}_L$, with generic fibre A, such that the Rosati involution of the principal polarisation of A maps $\iota(a)$ to $\iota(c(a))$. Suppose that there are $\rho\in L$ with $\rho^2=-d$ and integers $p',q'\ge0$ with $p'+q'\ge1$ such that for every algebraically closed field k, every ring homomorphism $\gamma\colon L\to k$ and every ring homomorphism $\sigma\colon\mathcal{O}_E\to k$, the σ-eigenspace of $\mathrm{Lie}(A)\otimes_{L,\gamma}k$ has dimension p' if $\sigma(r)=\gamma(\rho)$ and q' otherwise. Then there are a unitary Shimura datum $\mathfrak{u}=(E,\mathsf{H},\mathsf{U},p',q',\mathsf{L},\mathrm{Sh},F)$ whose level is good at every prime and whose components split, an integral model $(\mathcal{Y},j,G_{\mathcal{Y}},\iota_{\mathcal{Y}},\mathcal{L}_{\mathcal{Y}})$ of Sh over $\mathcal{O}_F$, a finite extension K of L with a field embedding $\beta\colon F\to K$ such that $\beta(r_{\mathbb{C}})=\rho$, and a point $y\in\mathrm{Sh}_F(K)$ (an F-morphism $\mathrm{Spec}\,K\to\mathrm{Sh}_F$ through β), together with an $\mathcal{O}_E$-linear isomorphism of the fibre G_y of the universal abelian scheme with $A\otimes_L K$ that carries the principal polarisation of G_y to that of $A\otimes_L K$.*

Proof. Put $n=p'+q'$ and $g=\dim A$.

Step 1: the unitary datum. The elements $\pm\rho$ are the roots of T^2+d in L, and every embedding of L into $\mathbb{C}$ sends ρ to $\pm r_{\mathbb{C}}$, its complex conjugate to the other sign; so there is $\sigma\colon L\to\mathbb{C}$ with $\sigma(\rho)=r_{\mathbb{C}}$. Write $A_\sigma(\mathbb{C})=\mathrm{Lie}(A_\sigma)/M$, $M=H_1(A_\sigma(\mathbb{C}),\mathbb{Z})$, with the $\mathcal{O}_E$-action ι and the Riemann form $\langle\,,\rangle$ of the principal polarisation: a perfect alternating $\mathbb{Z}$-pairing on M with $\langle Jx,Jy\rangle=\langle x,y\rangle$ and $\langle Jx,x\rangle>0$ for $x\ne0$, where J is multiplication by i on $M_{\mathbb{R}}=\mathrm{Lie}(A_\sigma)$, and $\langle ax,y\rangle=\langle x,\bar a y\rangle$ by the Rosati hypothesis. The lattice M is a finitely generated torsion-free, hence projective, $\mathcal{O}_E$-module. By the signature hypothesis (with $k=\mathbb{C}$, $\gamma=\sigma$), the τ-eigenspace of $\mathrm{Lie}(A_\sigma)$ has dimension p' for $\tau\in\Phi_{\mathsf{H}}$ ($\tau(r)=r_{\mathbb{C}}=\sigma(\rho)$) and q' for $\tau\notin\Phi_{\mathsf{H}}$, and $\mathrm{Lie}(A_\sigma)$ is their sum ($E\otimes\mathbb{C}$ is split); so $g=\frac12[E:\mathbb{Q}]n$, M has $\mathcal{O}_E$-rank n, and for $a\in\mathcal{O}_E$ the $\mathbb{C}$-trace of a on $(M_{\mathbb{R}},J)$ is $t(a)=\sum_{\tau\in\Phi_{\mathsf{H}}}(p'\tau(a)+q'\overline{\tau(a)})$; hence the $\mathbb{R}$-traces of a and of Ja on $M_{\mathbb{R}}$ are $2\,\mathrm{Re}\,t(a)$ and $2\,\mathrm{Re}(i\,t(a))=-2\,\mathrm{Im}\,t(a)$. So $\mathsf{U}=(M,\langle\,,\rangle)$ is a unitary datum of signature (p',q') for H (Ud2), and $\det_{\mathbb{C}}(T-\iota(a)\mid\mathrm{Lie}(A_\sigma))=P_a(T)$, the polynomial of Theorem B.105 for $(E,M,\langle\,,\rangle,J)$.

Step 2: the level. The field $\mathbb{Q}(\sqrt{-d})$ has infinitely many split primes (Dirichlet); choose two distinct split primes $p_1, p_2 \geq 3$ and put $N = p_1 p_2$. As $-d \equiv 1 \pmod 4$, $\mathbb{Z}[w_0]$ is the ring of integers of $\mathbb{Q}(\sqrt{-d}) \subseteq E$ and $p_i = \mathfrak{q}_i \bar{\mathfrak{q}}_i$ with $\mathfrak{q}_i + \bar{\mathfrak{q}}_i = \mathbb{Z}[w_0]$. Put $\mathfrak{n} = \mathfrak{q}_1 \mathfrak{q}_2 \mathcal{O}_E$: it is generated by $\mathfrak{n} \cap \mathbb{Z}[w_0] \supseteq \mathfrak{q}_1 \mathfrak{q}_2$, $\mathfrak{n}\bar{\mathfrak{n}} = N\mathcal{O}_E$ and $\mathfrak{n} + \bar{\mathfrak{n}} = \mathcal{O}_E$. The level $\mathsf{L} = (N, \mathfrak{n}, 1)$ is good at every prime ($N/\ell^{v_\ell(N)} \in \{N, p_1, p_2\}$ is ≥ 3), and $K_{\mathsf{L}} = \mathbb{Q}(r_{\mathbb{C}})$.

Step 3: the moduli scheme and the point. By Proposition 32.10 (E unramified over E^+, level good at every prime, $A = 1$) there is a moduli scheme Sh of type (U, L); by Proposition 32.9 there is a finite étale surjection $\theta \circ \varpi \colon \mathcal{M}_N \to \mathrm{Sh}$ from the full-level scheme of Theorem B.105 (base changed to K_{L}) under which o^{un} pulls back to $(A^{\mathrm{un}}, \iota, \mathcal{L}_{\phi_{\mathrm{pol}}}, \eta, 0)$, whose cubical line bundle has the universal principal polarisation ϕ_{pol}. Let $K_0 \supseteq L$ be a finite extension containing the coordinates of all points of $A[N]$ over an algebraic closure, a primitive N-th root of unity, and regard K_0 as a K_{L}-algebra by $r_{\mathbb{C}} \mapsto \rho$. Fix an embedding $\sigma_0 \colon K_0 \to \mathbb{C}$ extending σ. Through σ_0, $A[m](\overline{K_0}) \cong A_\sigma(\mathbb{C})[m] = \frac{1}{m}M/M \cong M/mM$ for all m, $\mathcal{O}_E$-linearly, and the Weil pairing of the principal polarisation is $\zeta_m^{\langle x, y\rangle}$ for the compatible system $\zeta_m = e^{\epsilon 2\pi i/m}$, with a sign $\epsilon \in \{\pm 1\}$ fixed by the normalisation of e_m. Taking α_N the resulting isomorphism $M/NM \cong A[N]$ over K_0, $\nu_N(1) = \sigma_0^{-1}(\zeta_N)$ and, for $N \mid m$, the analogous (α_m, ν_m) over the finite extension of K_0 generated by the m-torsion and μ_m (finite étale surjective over $\operatorname{Spec} K_0$), the tuple $(A_{K_0}, \phi, \iota, \alpha_N, \nu_N)$ is an object of the functor of Theorem B.105 over K_0: $\phi\iota(a) = \iota(\bar{a})^\vee \phi$ is the Rosati hypothesis, and $\det(T - \iota(a) \mid \mathrm{Lie}(A_{K_0})) = P_a(T)$ because P_a has coefficients in $F_{\mathrm{ref}} \subseteq K_{\mathsf{L}}$ (Lemma 32.3), read in K_0 through $r_{\mathbb{C}} \mapsto \rho$, and the two polynomials agree after the injective σ_0, which sends ρ back to $r_{\mathbb{C}}$ (Step 1). It gives $y' \in \mathcal{M}_N(K_0)$, and $y_0 = \theta \circ \varpi \circ y' \in \mathrm{Sh}(K_0)$ with $G_{y_0} \cong A_{K_0}$, $\mathcal{O}_E$-linearly and compatibly with the principal polarisations.

Step 4: the coefficient field. Let $F_0 \subset \mathbb{C}$ be a coefficient field for L (e.g. the Galois closure of E adjoined $e^{2\pi i/N}$ and $r_{\mathbb{C}}$). The scheme Sh_{F_0} is smooth of finite type; for each of its finitely many irreducible components Z_i let $F_i \subseteq \mathbf{k}(Z_i)$ be the (finite) algebraic closure of F_0, and let $F \subset \mathbb{C}$ be a finite Galois extension of $\mathbb{Q}$ containing F_0 and an image over F_0 of every F_i. Then F is a coefficient field, and the components of Sh_F split: every component of Sh_F is a component of some $Z_i \times_{F_0} F$, whose components correspond to the factors F of $F_i \otimes_{F_0} F$ and have function fields $\mathbf{k}(Z_i) \otimes_{F_i} F$; as $\mathbf{k}(Z_i)$ is separable over F_i (characteristic 0) with F_i algebraically closed in it, $\mathbf{k}(Z_i)$ is regular over F_i, so $\mathbf{k}(Z_i) \otimes_{F_i} F$ is a field in which F is algebraically closed. Choose a finite extension K of K_0 with an embedding $\beta \colon F \to K$ extending the K_{L}-structure ($\beta(r_{\mathbb{C}}) = \rho$): take K generated over K_0 by a root of a primitive element of F over K_{L}. The point y_0 and β give $y \in \mathrm{Sh}_F(K) = (\mathrm{Sh} \times_{K_{\mathsf{L}}} F)(K)$ over β, with $G_y \cong A \otimes_L K$ as in Step 3. The datum $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p', q', \mathsf{L}, \mathrm{Sh}, F)$ is a unitary Shimura datum with level good at every prime and split components.

Step 5: the integral model. By Proposition 32.1 ($n \geq 1$, level good at every prime, $A = 1$) there is an integral model of Sh over $\mathcal{O}_F$ (with uniformised degenerations). □

Proposition D.38. *Let $\mathfrak{u}$ be a unitary Shimura datum with $n = p + q \geq 1$ whose components split, let $(\mathcal{Y}, j, G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ be an integral model of* Sh *over $\mathcal{O}_F$, let $\mathcal{Y}'$ be an irreducible component of $\mathcal{Y}$ meeting $j(\mathrm{Sh}_F)$, $\mathcal{Y}'_F = \mathcal{Y}' \times_{\mathcal{O}_F} F$, $G' = G_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{Y}'_F$ with the induced $\mathcal{O}_E$-action, $U = j^{-1}(\mathcal{Y}'_F) \subseteq \mathrm{Sh}_F$, and $\tau \in \Phi_{\mathsf{H}}$. Put*

$$\mathcal{D} = \det \mathrm{Lie}_{(\tau,+)}(G') \otimes \left(\det \mathrm{Lie}_{(\tau,-)}(G')\right)^{-1}.$$

Then $\mathcal{D}$ is invertible and has a global section on $\mathcal{Y}'_F$ that vanishes at no point of $\mathcal{Y}'_F$. For such a section ξ, a field k with an embedding $F \to k$, $y \in U(k)$ and a frame (v, w) at y, define $\varphi_\xi(y; v, w) \in k^\times$ by

$$\xi(j(y)) = \varphi_\xi(y; v, w)\,(v_1 \wedge \cdots \wedge v_p) \otimes (w_1 \wedge \cdots \wedge w_q)^{-1},$$

the fibre of $\mathcal{D}$ at $j(y)$ being identified with $\det \mathrm{Lie}_{(\tau,+)}(G_y) \otimes (\det \mathrm{Lie}_{(\tau,-)}(G_y))^{-1}$ through the isomorphism of j^*G' with the universal abelian scheme. Then for every such ξ:

1. *U is a connected component of Sh_F, φ_ξ is a frame function on U, and every global section of $\mathcal{D}$ on $\mathcal{Y}'_F$ that vanishes at no point is $x\xi$ for some $x \in F^\times$;*
2. *for every connected component Y of $U(\mathbb{C})$ and every $z_0 \in Y$ there are $\mathcal{O}_E$-linearly independent $e \in \Lambda^n_{z_0}$ and $c \in \mathbb{C}^\times$ such that φ_ξ is a flat generator over Y with constants (e, c);*
3. *φ_ξ has an archimedean norm N_s at every embedding $s\colon F \to \mathbb{C}$;*
4. *for every prime ℓ, φ_ξ has an ℓ-adic norm N_κ at every ring homomorphism $\kappa\colon F \to \mathbb{C}_\ell$.*

Proof. Existence. By Proposition 34.16, $\mathcal{D}$ is invertible and has a global section vanishing at no point. Fix such a ξ.

Clause 1. As $\mathcal{Y}$ is normal, its irreducible components are pairwise disjoint, so $\mathcal{Y}'$ is open and closed in $\mathcal{Y}$, integral, normal and projective over $\mathcal{O}_F$; hence X_F is open and closed in $\mathcal{Y} \times_{\mathcal{O}_F} F$, and it is integral, normal and projective over F. So $U = j^{-1}(X_F)$ is open and closed in Sh_F, and it is irreducible, being identified by j with the dense open subset $X_F \cap j(\mathrm{Sh}_F)$ of X_F; a non-empty irreducible open and closed subset is a connected component. For $y \in U(k)$, $\xi(j(y))$ is a non-zero element of the one-dimensional k-vector space $\det \mathrm{Lie}_{(\tau,+)}(G_y) \otimes (\det \mathrm{Lie}_{(\tau,-)}(G_y))^{-1}$, of which $(v_1 \wedge \cdots \wedge v_p) \otimes (w_1 \wedge \cdots \wedge w_q)^{-1}$ is a basis, so $\varphi_\xi(y; v, w) \in k^\times$ is well defined. Since $(Av)_1 \wedge \cdots \wedge (Av)_p = \det A\,(v_1 \wedge \cdots \wedge v_p)$ and likewise for Bw,

$$\varphi_\xi(y; Av, Bw)\,\frac{\det A}{\det B} = \varphi_\xi(y; v, w),$$

which is the frame law of (Yf2); compatibility with a field homomorphism $f\colon k \to k'$ holds because the fibre of ξ at the induced point is the image of $\xi(j(y))$ under f and the wedge of (f_*v, f_*w) is the image of the wedge of (v, w). So φ_ξ is a frame function on U. If ξ' is another global section of $\mathcal{D}$ vanishing at no point, then $\xi' = h\xi$ with $h \in H^0(X_F, \mathcal{O}_{X_F})^\times$. As X_F is proper and integral over F, $H^0(X_F, \mathcal{O}_{X_F})$ is a field, finite over F and contained in the function field $\mathbf{k}(X_F) = \mathbf{k}(U)$. The components split and U is a component of Sh_F, so F is algebraically closed in $\mathbf{k}(U)$ (Ud6); hence $H^0(X_F, \mathcal{O}_{X_F}) = F$ and $h \in F^\times$.

The comparison at an embedding $s\colon F \to \mathbb{C}$. The scheme U is smooth over F (Ud5) and integral, and F is algebraically closed in $\mathbf{k}(U)$; as F has characteristic zero, $U \otimes_{F,s} \mathbb{C}$ is an irreducible smooth complex variety, so the complex manifold $Y_s = U^s(\mathbb{C})$ of points of U over s is connected. Let Φ be a function as in Lemma 33.21 for (U, s, Y_s), and let X_s be the connected component of $(X_F \otimes_{F,s} \mathbb{C})^{\mathrm{an}}$ containing Y_s. By Lemma 34.17, X_s is a compact connected normal complex space and there is a nowhere vanishing holomorphic section ϑ of $(\mathcal{D} \otimes_{F,s} \mathbb{C})^{\mathrm{an}}$ on X_s with $\vartheta(z) = \Phi(z; v, w)\,(v_1 \wedge \cdots \wedge v_p) \otimes (w_1 \wedge \cdots \wedge w_q)^{-1}$ for $z \in Y_s$. The analytification ξ_s of $\xi \otimes_{F,s} 1$ also vanishes nowhere on X_s, so $\xi_s = \kappa_s \vartheta$ with κ_s a nowhere vanishing holomorphic function on X_s; a holomorphic function on a compact connected complex space is constant (maximum principle), so $\kappa_s \in \mathbb{C}^\times$. Comparing coefficients of $(v_1 \wedge \cdots \wedge v_p) \otimes (w_1 \wedge \cdots \wedge w_q)^{-1}$ at $z \in Y_s$,

$$\varphi_\xi(z; v, w) = \kappa_s\,\Phi(z; v, w) \qquad (z \in Y_s,\ (v, w) \text{ a frame at } z).$$

Clause 3. By clause 3 of Lemma 33.21 there is $c_s > 0$ with $|\Phi(z; v, w)|^2 \det \Gamma(v)(\det \Gamma(w))^{-1} = c_s$ for all $z \in Y_s$ and all frames. Every $y \in U(\mathbb{C})$ over s lies in Y_s, so

$$\log|\varphi_\xi(y; v, w)| + \tfrac{1}{2}\log\det\Gamma(v) - \tfrac{1}{2}\log\det\Gamma(w) = \log|\kappa_s| + \tfrac{1}{2}\log c_s$$

for all such y and all frames: $N_s = \log|\kappa_s| + \frac{1}{2}\log c_s$ is an archimedean norm of φ_ξ at s.

Clause 2. Let s be the inclusion; then $Y_s = U(\mathbb{C})$ is its only connected component Y. Let $z_0 \in Y$. The lattice Λ_{z_0} has $\mathbb{Z}$-rank $2\dim G_{z_0} = [E:\mathbb{Q}]\,n$ (Kottwitz condition of (Ud4)) and is an $\mathcal{O}_E$-module, so $\Lambda_{z_0}\otimes\mathbb{Q}$ is an E-vector space of dimension n and contains n E-linearly independent elements, which may be taken in Λ_{z_0}; let e be such a tuple ($\mathcal{O}_E$-linearly independent). Clause 4 of Lemma 33.21 gives $c \in \mathbb{C}^\times$ with $\Phi(z;v,w)\det\binom{C}{R} = c$ for all $z,\gamma,(v,w),C,B,R$ as in (Yf4); hence $\varphi_\xi(z;v,w)\det\binom{C}{R} = \kappa_s c$, and φ_ξ is a flat generator over Y with constants $(e,\kappa_s c)$.

Clause 4. This is Proposition D.39, whose hypotheses (components split, ξ vanishing nowhere) hold; it does not need the level to be good at any prime. □

Proposition D.39. *Let $\mathfrak{u}$ be a unitary Shimura datum with $n = p+q \geq 1$ whose components split, let $(\mathcal{Y}, j, G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ be an integral model of* Sh *over $\mathcal{O}_F$, $\mathcal{Y}'$ an irreducible component of $\mathcal{Y}$ meeting $j(\mathrm{Sh}_F)$, $G' = G_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{Y}'_F$, $U = j^{-1}(\mathcal{Y}'_F)$, $\tau \in \Phi_{\mathsf{H}}$, and ξ a global section of $\det\mathrm{Lie}_{(\tau,+)}(G') \otimes (\det\mathrm{Lie}_{(\tau,-)}(G'))^{-1}$ on $\mathcal{Y}'_F$ vanishing at no point. Let φ_ξ be the frame function on U with $\xi(j(y)) = \varphi_\xi(y;v,w)\,(v_1\wedge\cdots\wedge v_p)\otimes(w_1\wedge\cdots\wedge w_q)^{-1}$ for every field k with an embedding $F \to k$, $y \in U(k)$ and frame (v,w) at y. Then for every prime ℓ and every ring homomorphism $\kappa\colon F \to \mathbb{C}_\ell$, φ_ξ has an ℓ-adic norm at κ.*

Proof. Fix ℓ and $\kappa\colon F \to \mathbb{C}_\ell$; let $\mathfrak{l} = \{x \in \mathcal{O}_F : |\kappa(x)| < 1\}$, a maximal ideal over ℓ, $\mathcal{O} = \mathcal{O}_{F,\mathfrak{l}}$ the localisation, a discrete valuation ring with $|\kappa| \leq 1$ on $\mathcal{O}$, and $\pi \in \mathcal{O}_F$ a uniformiser of $\mathcal{O}$. To avoid a clash with the open immersion j, the map called j in (Yf3) is written λ below.

Step 1: the model over $\mathcal{O}$. The irreducible components of the normal noetherian scheme $\mathcal{Y}$ are disjoint, so $\mathcal{Y}'$ is open and closed in $\mathcal{Y}$: a normal integral scheme, projective and flat over $\mathcal{O}_F$. Put $X = \mathcal{Y}' \times_{\mathcal{O}_F} \mathcal{O}$, whose points are those of $\mathcal{Y}'$ over $\mathfrak{l}$ or over no prime, with the same local rings: X is normal, integral, proper and flat over $\mathcal{O}$, with generic fibre $X_K = \mathcal{Y}'_F$ and special fibre X_s, non-empty since $X \to \operatorname{Spec}\mathcal{O}$ is proper and dominant. As $\mathcal{Y}'$ meets the dense open set $j(\mathrm{Sh}_F)$ of $\mathcal{Y}\times F$, the generic point of X lies in $j(U)$ and $G_X = G_{\mathcal{Y}}|_X$ has abelian generic fibre. Let v be an ℓ-adic place with $F\cap\mathcal{O}_v = \mathcal{O}$ (extend the valuation of $\mathcal{O}$ to $\mathbb{C}$). Then $\mathcal{O}_{F,v} = \mathcal{O}$, and $\mathcal{Y}\times_{\mathcal{O}_F}\mathcal{O}$ with the restrictions of j, $G_{\mathcal{Y}}$, $\iota_{\mathcal{Y}}$, $\mathcal{L}_{\mathcal{Y}}$ is an integral model of Sh over $\mathcal{O}_{F,v}$: normality, projectivity and flatness are preserved by the localisation, and the generic fibre, j and the abelian locus are unchanged. By Lemma D.53, the semiabelian structure $(G_X,\iota,\mathcal{L})$ satisfies (PF).

Step 2: the special fibre is connected. $A = H^0(X,\mathcal{O}_X)$ is a finite $\mathcal{O}$-algebra and a domain, and by flat base change $A\otimes_{\mathcal{O}} F = H^0(\mathcal{Y}'_F,\mathcal{O})$, a finite F-algebra inside the function field of $\mathcal{Y}'_F$, which is $\mathbf{k}(Z)$ for the component Z of Sh_F with $j(Z)$ dense in $\mathcal{Y}'_F$. Its elements are algebraic over F, and F is algebraically closed in $\mathbf{k}(Z)$ (components split), so $A\otimes F = F$; A, integral over the integrally closed $\mathcal{O}$ and inside F, is $\mathcal{O}$. By Zariski's connectedness theorem [82, 4.3.1] the fibres of $X \to \operatorname{Spec}\mathcal{O}$ are connected; in particular X_s is.

Step 3: the line bundle. By Steps 1 and 2, Corollary D.41 applies to $(G_X,\iota,\mathcal{L})$ over X (a point of residue characteristic ℓ exists), with $\tau\colon \mathcal{O}_E \to \mathcal{O}_F \subseteq \Gamma(X,\mathcal{O}_X)$: $\mathcal{H} = \mathrm{H}^\natural(G_X,\mathcal{L})$ is locally free of rank $n' = 2g/[E:\mathbb{Q}]$ over $\mathcal{O}_E\otimes\mathcal{O}_X$ and $\mathcal{D}_X = \det\mathcal{H}_\tau$ is an invertible sheaf on X; by its (iii) on X_K, $\mathcal{D}_X|_{X_K} \cong \det\mathrm{Lie}_{(\tau,+)}(G') \otimes (\det\mathrm{Lie}_{(\tau,-)}(G'))^{-1}$, by an isomorphism compatible with pull-back to points. Let σ_0 be the section of $\mathcal{D}_X|_{X_K}$ corresponding to ξ, a rational section of $\mathcal{D}_X$ generating it at every point of X_K.

Step 4: degree 0 on the curves of X_s. Let $C \subseteq X_s$ be an integral closed subscheme of dimension one and $\nu\colon \tilde{C} \to C$ its normalisation, an integral normal curve, proper and hence projective over $\kappa(\mathfrak{l})$, of characteristic ℓ; let $c\colon \tilde{C} \to X$ be the composite. By Corollary D.41 (ii), $c^*\mathcal{H}_\tau$ is the quotient H_σ of $H = c^*\mathcal{H}$ for $\sigma = c^*\tau$, and by Corollary D.51 (ii) (its hypotheses

are those of Step 1) it is a vector bundle of degree 0. Hence $\deg \nu^*(\mathcal{D}_X|_C) = \deg c^*\mathcal{D}_X = \deg\det H_\sigma = 0$.

Step 5: constant vertical order. By Steps 1, 2 and 4, Lemma D.50 applies to X, $\mathcal{D}_X$, σ_0 and π: there are $M \geq 1$ and $a \in \mathbb{Z}$ such that $\theta = \sigma_0^M \pi^{-a}$ generates $\mathcal{D}_X^{\otimes M}$ at every point of X.

Step 6: evaluation at an integral frame. Let V, k, λ, β, $y \in U(k)$, $\mathcal{G}$ and the integral frame (v, w) be as in (Yf3), with $\lambda \circ \beta = \kappa$. Then $\beta(\mathcal{O}) \subseteq V$, and $\beta(\pi) \in \mathfrak{m}_V$. By properness of X over $\mathcal{O}$ (valuative criterion), the point $j(y)\colon \operatorname{Spec} k \to X_K$ extends to $z\colon \operatorname{Spec} V \to X$. The semiabelian schemes z^*G_X and $\mathcal{G}$ over V have generic fibres identified $\mathcal{O}_E$-linearly with G_y ((Ud7) and the datum of (Yf3)); by Theorem B.104 (normal base $\operatorname{Spec} V$, dense open $\operatorname{Spec} k$) this identification and its inverse extend to homomorphisms, mutually inverse and $\mathcal{O}_E$-linear by the uniqueness of extensions. So $\operatorname{Lie}(\mathcal{G}) \cong \operatorname{Lie}(z^*G_X) =: L$, and v, w are V-bases of $L[\sigma]$ and $L[\bar\sigma]$, where $\sigma = \beta\circ\tau$, $\bar\sigma = \beta\circ\bar\tau$, in the notation of Lemma D.49. Put $N = z^*\mathcal{H} = \mathrm{H}^\natural(z^*G_X, z^*\mathcal{L})$ (Lemma D.44 (6)): free of rank n' over $\mathcal{O}_E \otimes V$, with the equivariant Hodge sequence $0 \to L^\vee \to N \to L \to 0$, a acting on $L^\vee$ by $\operatorname{Lie}(\iota(\bar a))^\vee$ (Lemma D.43 (2), Lemma D.44 (4)), and $N_\sigma = z^*\mathcal{H}_\tau$, $\det N_\sigma = z^*\mathcal{D}_X$ (Corollary D.41 (ii)). The hypotheses of Lemma D.49 hold: k has characteristic 0, V residue characteristic ℓ, the $[E:\mathbb{Q}]$ embeddings $E \to k$ are the $\beta\circ\tau'$, $\tau' \in \operatorname{Hom}(E,F)$, and they map $\mathcal{O}_E$ into $\beta(\mathcal{O}_F) \subseteq V$; and $r_{\beta\tau'} = p$ if $\tau' \in \Phi_{\mathsf{H}}$, q otherwise (Kottwitz condition at y, as $L \otimes k = \operatorname{Lie}(G_y)$). In particular $n' = p + q = n$.

Let b be a V-basis of N_σ. As $z^*\theta$ generates $(\det N_\sigma)^{\otimes M}$, $z^*\theta = u\,(b_1 \wedge \cdots \wedge b_n)^{\otimes M}$ with $u \in V^\times$. Writing the generic value $z^*\sigma_0 = c\, b_1 \wedge \cdots \wedge b_n$ with $c \in k^\times$, we get $c^M = u\,\beta(\pi)^a$, so $\log|\lambda(c)| = \frac{a}{M}\log|\kappa(\pi)|$ ($|\lambda(u)| = 1$ as $V = \{|\lambda| \leq 1\}$). On the other hand, the identification of Step 3 at $j(y)$ is the one of Corollary D.41 (iii) over k, which is the exact sequence of Lemma D.49; it carries $\xi(j(y)) = \varphi_\xi(y; v, w)\,(v_1 \wedge \cdots \wedge v_p) \otimes (w_1 \wedge \cdots \wedge w_q)^{-1}$ to $\varphi_\xi(y; v, w)\, w_1^* \wedge \cdots \wedge w_q^* \wedge \tilde v_1 \wedge \cdots \wedge \tilde v_p = \varphi_\xi(y; v, w)\,\delta\, b_1 \wedge \cdots \wedge b_n$, with δ as in Lemma D.49. Hence $\varphi_\xi(y; v, w) = c/\delta$.

Step 7: the norm. The absolute value $|\lambda(\cdot)|$ on k has valuation ring V, so $\log|\lambda(x)|$ is a negative multiple of the valuation $v(x)$, and by Lemma D.49

$$\log|\lambda(\delta)| = \sum_{s \in S_\sigma,\ s \neq \sigma} r_s \log|\lambda(s(\varpi) - \sigma(\varpi))| = \sum_{\tau' \in T_\kappa} r_{\tau'} \log|\kappa(\tau'(\varpi) - \tau(\varpi))|,$$

where $\varpi \in \mathcal{O}_E$ is a uniformiser at the prime $\{a \in \mathcal{O}_E :\ |\kappa(\tau(a))| < 1\}$, T_κ is the set of $\tau' \in \operatorname{Hom}(E,F)$, $\tau' \neq \tau$, with $|\kappa(\tau'(a) - \tau(a))| < 1$ for all $a \in \mathcal{O}_E$, and $r_{\tau'} = p$ for $\tau' \in \Phi_{\mathsf{H}}$, q otherwise (for $s = \beta\tau'$, $\lambda(s(x)) = \kappa(\tau'(x))$, and $s(a) - \sigma(a) \in \mathfrak{m}_V$ iff $|\kappa(\tau'(a) - \tau(a))| < 1$). Therefore

$$\log|\lambda(\varphi_\xi(y; v, w))| = \frac{a}{M}\log|\kappa(\pi)| - \sum_{\tau' \in T_\kappa} r_{\tau'} \log|\kappa(\tau'(\varpi) - \tau(\varpi))|,$$

a number that depends only on κ (through $\mathfrak{l}$, X, M, a, π, ϖ), not on $(V, \lambda, \beta, y, \mathcal{G}, v, w)$. It is the ℓ-adic norm N_κ of φ_ξ at κ. The second term vanishes when ℓ is unramified in E.

Remark. No condition on the level is used; the splitting of the components is used in Step 2. □

D.2.3 The polarised de Rham module over the base

The subsection develops the polarised vector extension and the polarised de Rham module of a semiabelian scheme with a rigidified cubical line bundle (Definition D.40). Its main results are that the module is locally free over $\mathcal{O}_E$ on the base (Corollary D.41) and that on every complete curve in a fibre of positive characteristic it has $\mathcal{O}_E$-equivariant Hodge and conjugate sequences and is locally free over $\mathcal{O}_E$ (Corollary D.42). The second is the input of the weighted flag inequality behind nefness at primes in Section D.3; the vertical orders along fibres (Lemma D.50) are used for flat determinants in Section D.4.

Definition D.40. Let S be a scheme, $G \to S$ a semiabelian scheme of relative dimension g with unit section e, and $\mathcal{L}$ a rigidified line bundle on G with a cubical structure. Write $\mathcal{L}_x = x^*\mathcal{L}$ for $x \in G(T)$, T an S-scheme, and $\Lambda_{(x,y)} = (x,y)^*\Lambda(\mathcal{L}) = \mathcal{L}_{x+y} \otimes \mathcal{L}_x^{-1} \otimes \mathcal{L}_y^{-1}$. Let $\omega_G = e^*\Omega^1_{G/S}$, $\mathrm{Lie}(G) = \mathcal{H}om(\omega_G, \mathcal{O}_S)$, and let $G_1 \subset G$ be the first infinitesimal neighbourhood of e (ideal $\mathcal{I}^2$, $\mathcal{I}$ the ideal of e), so that $\mathcal{O}_{G_1} = \mathcal{O}_S \oplus \omega_G$.

Canonical structures of Λ. The rigidification $\mathcal{L}_0 \cong \mathcal{O}$ gives canonical trivialisations of $\Lambda_{(x,0)}$ and $\Lambda_{(0,y)}$; the equality $\Lambda_{(x,y)} = \Lambda_{(y,x)}$ (commutativity of G) is the **symmetry**; and since $\Lambda_{(x+x',y)} \otimes \Lambda_{(x,y)}^{-1} \otimes \Lambda_{(x',y)}^{-1} = \Theta(\mathcal{L})_{(x,x',y)}$, the cubical structure gives the **partial composition** $\Lambda_{(x,y)} \otimes \Lambda_{(x',y)} \cong \Lambda_{(x+x',y)}$.

The polarised vector extension. The polarised vector extension $E^\natural(G,\mathcal{L})$ is the functor on S-schemes T whose T-points are the pairs (x,t) of $x \in G(T)$ and a trivialisation t of the line bundle $(x \circ \mathrm{pr}_1, \mathrm{pr}_2)^*\Lambda(\mathcal{L})$ on $T \times_S G_1$ that equals the canonical trivialisation of $\Lambda_{(x,0)}$ on $T \times_S \{e\}$, with the composition $(x,t)\cdot(x',t') = (x+x', t\cdot t')$, $t\cdot t'$ the image of $t \otimes t'$ under the partial composition. The forgetful map $(x,t) \mapsto x$ is the **projection** $E^\natural(G,\mathcal{L}) \to G$; its kernel consists of the trivialisations $u \cdot t_0$ of $\Lambda_{(0,\cdot)}|_{T\times G_1}$, t_0 the canonical one and $u = 1+\eta$ with $\eta \in \Gamma(T, \omega_G \otimes \mathcal{O}_T)$.

The polarised de Rham module. For an open $U \subseteq S$ let $U[\varepsilon] = U \times \operatorname{Spec}\mathbb{Z}[\varepsilon]/(\varepsilon^2)$. The **polarised de Rham module** $\mathrm{H}^\natural(G,\mathcal{L})$ is the Lie algebra of $E^\natural(G,\mathcal{L})$: the sheaf $U \mapsto \ker(E^\natural(G,\mathcal{L})(U[\varepsilon]) \to E^\natural(G,\mathcal{L})(U))$, an $\mathcal{O}_S$-module with $c \in \mathcal{O}(U)$ acting through $\varepsilon \mapsto c\varepsilon$. Its **Hodge maps** are

$$\omega_G \to \mathrm{H}^\natural(G,\mathcal{L}), \quad \eta \mapsto (0, (1+\varepsilon\eta)t_0), \qquad \mathrm{H}^\natural(G,\mathcal{L}) \to \mathrm{Lie}(G), \quad (v,t) \mapsto v.$$

Pairing. For sections $h = (v,t)$ over $U[\varepsilon]$ and $h' = (v',t')$ over $U[\varepsilon']$ of $\mathrm{H}^\natural(G,\mathcal{L})$, work over $U[\varepsilon,\varepsilon'] = U \times \operatorname{Spec}\mathbb{Z}[\varepsilon,\varepsilon']/(\varepsilon^2, \varepsilon'^2)$: v' is a point of G_1, so $t(v')$ trivialises $\Lambda_{(v,v')}$, and likewise $t'(v)$ trivialises $\Lambda_{(v',v)} = \Lambda_{(v,v')}$. The **pairing** $\langle h, h'\rangle \in \mathcal{O}(U)$ is defined by

$$t'(v) = \bigl(1 + \varepsilon\varepsilon'\langle h, h'\rangle\bigr)\, t(v'). \tag{D.10}$$

Endomorphism actions. Let $\iota\colon \mathcal{O} \to \mathrm{End}_S(G)$ be a ring homomorphism from a ring $\mathcal{O}$ with an involution $a \mapsto \bar a$. An **adjunction system** for $(G,\iota,\mathcal{L})$ is a family $\epsilon = (\epsilon_a)_{a\in\mathcal{O}}$ of isomorphisms of line bundles $\epsilon_a\colon (\iota(a),1)^*\Lambda(\mathcal{L}) \to (1,\iota(\bar a))^*\Lambda(\mathcal{L})$ on $G \times_S G$ that are the identity on $G \times_S \{e\}$ and on $\{e\} \times_S G$ under the canonical trivialisations. Given one, $a \in \mathcal{O}$ acts on $E^\natural(G,\mathcal{L})$ by

$$a\cdot(x,t) = \bigl(\iota(a)x,\ \epsilon_a^{-1}((1\times\iota(\bar a)|_{G_1})^*t)\bigr),$$

where $\iota(\bar a)$ maps G_1 into G_1; $\rho^\natural(a)$ denotes the induced endomorphism of $\mathrm{H}^\natural(G,\mathcal{L})$.

Conjugate map. Suppose $p\mathcal{O}_S = 0$ for a prime p, with absolute Frobenius F_S, and that $E^\natural(G,\mathcal{L})$ is represented by a commutative group scheme flat over S. Its Verschiebung $V\colon E^\natural(G,\mathcal{L})^{(p)} \to E^\natural(G,\mathcal{L})$ induces on Lie algebras the **conjugate map**

$$\tilde\alpha = \mathrm{Lie}(V)\colon F_S^*\mathrm{H}^\natural(G,\mathcal{L}) \to \mathrm{H}^\natural(G,\mathcal{L}).$$

Corollary D.41. *Let E be a CM field with complex conjugation $a \mapsto \bar a$, $\mathcal{X}$ a normal integral noetherian scheme with a point of positive residue characteristic, and $(G,\iota,\mathcal{L})$ a semiabelian structure over $\mathcal{X}$ of relative dimension g, with abelian generic fibre, satisfying (PF). Let $H = \mathrm{H}^\natural(G,\mathcal{L})$, with $a \in \mathcal{O}_E$ acting by $\rho^\natural(a)$. Then $[E:\mathbb{Q}]$ divides $2g$; put $n = 2g/[E:\mathbb{Q}]$.*

(i) H is locally free of rank n as a module over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{X}}$.

(ii) *For every ring homomorphism $\tau\colon \mathcal{O}_E \to \Gamma(\mathcal{X},\mathcal{O}_\mathcal{X})$, the quotient $H_\tau = H/\sum_{a\in\mathcal{O}_E}(\rho^\natural(a) - \tau(a))H$ is locally free of rank n over $\mathcal{O}_\mathcal{X}$, so $\det H_\tau$ is an invertible sheaf; for every morphism $c\colon \mathcal{X}' \to \mathcal{X}$, c^*H_τ is the $c^*\tau$-quotient of $c^*H = \mathrm{H}^\natural(G_{\mathcal{X}'},\mathcal{L}_{\mathcal{X}'})$ with the action $c^*\rho^\natural$.*

(iii) *Let $U \subseteq \mathcal{X}$ be open with $\mathbb{Q} \subseteq \Gamma(U,\mathcal{O}_U)$, τ as in (ii), $\bar\tau(a) = \tau(\bar a)$, and for $s \in \{\tau,\bar\tau\}$ let $\mathrm{Lie}[s] \subseteq \mathrm{Lie}(G)|_U$ be the subsheaf of sections x with $\mathrm{Lie}(\iota(a))x = s(a)x$ for all a. Then $\mathrm{Lie}[\tau]$ and $\mathrm{Lie}[\bar\tau]$ are locally direct summands of $\mathrm{Lie}(G)|_U$, and the Hodge maps induce an exact sequence*

$$0 \to \mathrm{Lie}[\bar\tau]^\vee \to H_\tau|_U \to \mathrm{Lie}[\tau] \to 0,$$

in which the first map sends λ to the class of any $\eta \in \omega_G$ with $\eta|_{\mathrm{Lie}[\bar\tau]} = \lambda$ and the second is induced by the projection $H \to \mathrm{Lie}(G)$. Hence $\det H_\tau|_U \cong \det \mathrm{Lie}[\tau] \otimes (\det \mathrm{Lie}[\bar\tau])^{-1}$, the isomorphism sending $(w_1^* \wedge \cdots \wedge w_q^*) \wedge (\tilde v_1 \wedge \cdots \wedge \tilde v_p)$ to $(v_1 \wedge \cdots \wedge v_p) \otimes (w_1 \wedge \cdots \wedge w_q)^{-1}$ for local bases v of $\mathrm{Lie}[\tau]$ and w of $\mathrm{Lie}[\bar\tau]$, the dual basis w^* and lifts $\tilde v_i \in H_\tau$ of v_i.

Proof. For a point z write $H(z) = H \otimes \kappa(z)$ and $A_z = \mathcal{O}_E \otimes_\mathbb{Z} \kappa(z)$.

Step 1: points of positive characteristic. Let z have residue characteristic $p > 0$ and let k be an algebraic closure of $\kappa(z)$. By Lemma D.44 (6), $H(z) \otimes_{\kappa(z)} k = \mathrm{H}^\natural(G_k,\mathcal{L}_k)$ with the action defined by the pulled-back adjunction system, and by (PF) at z the hypotheses of Lemma D.45 hold for $(G_k,\iota,\mathcal{L}_k)$; by its (1), $H(z)\otimes k$ is free over $A_z \otimes_{\kappa(z)} k$, of some rank r, and $r[E:\mathbb{Q}] = \dim_k H(z)\otimes k = 2g$. So $[E:\mathbb{Q}] \mid 2g$ and $r = n$. Write $A_z = \prod_i A_i$ with A_i local Artinian; $A_z \to A_z \otimes k$ is faithfully flat and $H(z)\otimes_{A_z}(A_z\otimes k) = H(z)\otimes k$ is flat over $A_z\otimes k$, so $H(z)$ is flat, hence projective, over A_z, and each $H(z)_i = H(z)\otimes_{A_z} A_i$ is free over the local ring A_i, of rank n since $H(z)_i \otimes k$ is free of rank n over $A_i \otimes k \neq 0$. Thus $H(z) \cong A_z^n$. Lift a basis to $h_1,\dots,h_n$ in the stalk H_z; the map $(\mathcal{O}_E\otimes\mathcal{O}_{\mathcal{X},z})^n \to H_z$, $(b_m) \mapsto \sum_m \rho^\natural(b_m)h_m$, is surjective by Nakayama's lemma, and both sides are free $\mathcal{O}_{\mathcal{X},z}$-modules of rank $n[E:\mathbb{Q}] = 2g$, so it is an isomorphism; it extends to an isomorphism over an open neighbourhood of z.

Step 2: points of characteristic 0. By hypothesis there is a point of positive characteristic; the generic point η of the integral scheme $\mathcal{X}$ lies in the neighbourhood of Step 1, so $H(\eta) \cong (E\otimes_\mathbb{Q}\kappa(\eta))^n$. Let z have residue characteristic 0, and let $a \in \mathcal{O}_E$ generate E over $\mathbb{Q}$, with minimal polynomial $P \in \mathbb{Q}[T]$ (separable, of degree $[E:\mathbb{Q}]$). The characteristic polynomial $\chi \in \mathcal{O}_{\mathcal{X},z}[T]$ of $\rho^\natural(a)$ on the free $\mathcal{O}_{\mathcal{X},z}$-module H_z equals P^n in $\kappa(\eta)[T]$ (on $E\otimes\kappa(\eta) = \kappa(\eta)[T]/(P)$ the characteristic polynomial of a is P), hence in $\mathcal{O}_{\mathcal{X},z}[T] \subseteq \kappa(\eta)[T]$, and so the characteristic polynomial of a on $H(z)$ is P^n. Write $P = f_1\cdots f_t$ with $f_i \in \kappa(z)[T]$ irreducible, distinct (P separable); then $A_z = \prod_i L_i$, $L_i = \kappa(z)[T]/(f_i)$, $H(z) \cong \prod_i L_i^{m_i}$, and the characteristic polynomial of a on it is $\prod_i f_i^{m_i} = \prod_i f_i^n$; so $m_i = n$ for all i and $H(z) \cong A_z^n$. The Nakayama argument of Step 1 gives freeness of rank n near z. This proves (i).

(ii). Locally $H \cong (\mathcal{O}_E\otimes\mathcal{O}_\mathcal{X})^n$, and $(\mathcal{O}_E\otimes\mathcal{O}_\mathcal{X})/\sum_a(a\otimes 1 - 1\otimes\tau(a)) = \mathcal{O}_\mathcal{X}$; so $H_\tau \cong \mathcal{O}_\mathcal{X}^n$ locally. The quotient is a cokernel, and pull-back is right exact, so c^*H_τ is the $c^*\tau$-quotient of c^*H, which is $\mathrm{H}^\natural(G_{\mathcal{X}'},\mathcal{L}_{\mathcal{X}'})$ with the action $c^*\rho^\natural$ by Lemma D.44 (6).

(iii). Over U, $E\otimes_\mathbb{Q}\Gamma(U,\mathcal{O})$ contains the idempotent e_τ, image under $1\otimes\tau$ of the separability idempotent of $E\otimes_\mathbb{Q}E$; it satisfies $(a\otimes 1)e_\tau = \tau(a)e_\tau$ and $\mu_\tau(e_\tau) = 1$, where $\mu_\tau(a\otimes c) = \tau(a)c$, and $\ker\mu_\tau = (1-e_\tau)$. Therefore, for every $E\otimes\mathcal{O}_U$-module M, we have $M/\sum_a(a-\tau(a))M = M/(1-e_\tau)M \cong e_\tau M = \{m :\ am = \tau(a)m\ \forall a\}$, and $M \mapsto e_\tau M$ is exact. Applied to the equivariant Hodge sequence (Lemma D.44 (4)) this gives the exact sequence $0 \to e_\tau\omega_G \to e_\tau H \cong H_\tau \to e_\tau\mathrm{Lie}(G) \to 0$, with $e_\tau\mathrm{Lie}(G) = \mathrm{Lie}[\tau]$ a direct summand of $\mathrm{Lie}(G)|_U$; likewise $\mathrm{Lie}[\bar\tau] = e_{\bar\tau}\mathrm{Lie}(G)$. On $\omega_G = \mathrm{Lie}(G)^\vee$ the element $\sum_i a_i\otimes c_i$ acts by the transpose of $\sum_i c_i\mathrm{Lie}(\iota(\bar a_i))$, so e_τ acts by the transpose of $\bar e_\tau$, the image of e_τ under $a\otimes c \mapsto \bar a\otimes c$. Hence $e_\tau\omega_G$ is the image of the transpose of the projection $e_{\bar\tau}$, i.e. restriction to $\mathrm{Lie}[\bar\tau]$ identifies $e_\tau\omega_G$ with $\mathrm{Lie}[\bar\tau]^\vee$, and the

class in H_τ of $\eta \in \omega_G$ is that of $e_\tau \eta$, which depends only on $\eta|_{\mathrm{Lie}[\bar\tau]}$. The determinant of a short exact sequence of locally free sheaves gives $\det H_\tau|_U \cong \det \mathrm{Lie}[\bar\tau]^\vee \otimes \det \mathrm{Lie}[\tau]$, with the stated effect on local bases, and $\det(\mathrm{Lie}[\bar\tau]^\vee) = (\det \mathrm{Lie}[\bar\tau])^{-1}$ via $w_1^* \wedge \cdots \wedge w_q^* \mapsto (w_1 \wedge \cdots \wedge w_q)^{-1}$. □

Corollary D.42. *Let E be a CM field with complex conjugation $a \mapsto \bar a$, $\mathcal{X}$ a normal integral noetherian scheme and $(G, \iota, \mathcal{L})$ a semiabelian structure over $\mathcal{X}$ whose fibre at the generic point of $\mathcal{X}$ is an abelian variety, and such that*

(PF) for every point $z \in \mathcal{X}$ of positive residue characteristic and an algebraic closure k of $\kappa(z)$, there are an exact sequence $0 \to T \to G_k \xrightarrow{q} B \to 0$ of a torus T and an abelian variety B over k and a line bundle $\mathcal{M}$ on B defining a principal polarisation, with $\mathcal{L}|_{G_k} \cong q^*\mathcal{M}$.

Then $(G, \iota, \mathcal{L})$ has exactly one adjunction system ϵ, and $a \mapsto \rho^\natural(a)$ is a ring homomorphism $\mathcal{O}_E \to \mathrm{End}(\mathrm{H}^\natural(G, \mathcal{L}))$. Let p be a prime, κ a field of characteristic p, C an integral normal projective curve over κ with absolute Frobenius F, and $c\colon C \to \mathcal{X}$ a morphism. Put $L = c^*\mathrm{Lie}(G)$ with $\mathcal{O}_E$ acting by $\rho_L(a) = c^*\mathrm{Lie}(\iota(a))$, $L^\vee = c^*\omega_G$ with a acting by $\rho_L(\bar a)^\vee$, and $H = c^*\mathrm{H}^\natural(G, \mathcal{L})$ with $\rho_H(a) = c^*\rho^\natural(a)$. Then L, $L^\vee$, H are vector bundles on C of ranks g, g, $2g$, and:

(i) the pulled-back Hodge maps form an $\mathcal{O}_E$-equivariant exact sequence $0 \to L^\vee \to H \to L \to 0$;

(ii) $E^\natural(G_C, \mathcal{L}_C)$ is represented by a smooth commutative group scheme over C, $\tilde\alpha$ vanishes on $F^*L^\vee$, and the map $\alpha\colon F^*L \to H$ it induces fits into an $\mathcal{O}_E$-equivariant exact sequence $0 \to F^*L \xrightarrow{\alpha} H \to F^*(L^\vee) \to 0$, the outer terms carrying $a \mapsto F^*(\rho_L(a))$ and $a \mapsto F^*(\rho_L(\bar a)^\vee)$;

(iii) H is locally free as a module over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{O}_C$.

In particular $(L^\vee, L, H)$ satisfies hypotheses (a) and (b) of Lemma D.52 (with $M_0 = L^\vee$, $M_1 = L$, $K = E$), and (L, H) satisfies the hypotheses of Lemma D.63 and of Lemma D.62.

Proof. **The action and the pull-back.** As $\mathcal{X}$ is normal integral noetherian with abelian generic fibre, $(G, \iota, \mathcal{L})$ has exactly one adjunction system ϵ and $\rho^\natural$ is a ring homomorphism ((1), (2) of Lemma D.44). By (6) of Lemma D.44 and (1), (2) of Lemma D.43, $H = c^*\mathrm{H}^\natural(G, \mathcal{L}) = \mathrm{H}^\natural(G_C, \mathcal{L}_C)$ with the action $c^*\rho^\natural$ defined by the adjunction system $c^*\epsilon$, with the pulled-back Hodge maps and pairing; $E^\natural(G_C, \mathcal{L}_C)$ is a smooth commutative group scheme over C, and L, $L^\vee$, H are vector bundles of ranks g, g, $2g$. Since $p\mathcal{O}_C = 0$, the conjugate map of $(G_C, \mathcal{L}_C)$ is defined, vanishes on $F^*L^\vee$ and induces $\alpha\colon F^*L \to H$ ((4) of Lemma D.43).

(i). The sequence $0 \to L^\vee \to H \to L \to 0$ is exact by (2) of Lemma D.43 over C, and equivariant for the stated actions by (4) of Lemma D.44 over C.

Pointwise input. Let $P \in C$ be a closed point, $z = c(P)$, and k an algebraic closure of $\kappa(P)$; it contains an algebraic closure of $\kappa(z)$, and the data of (PF) at z base change to k (an extension of G_k by a torus with abelian quotient B_k, and $\mathcal{L}_k \cong q^*\mathcal{M}_k$ with $\phi_{\mathcal{M}_k}$ an isomorphism). By (6) of Lemma D.44, the fibre of $(H, c^*\rho^\natural, \langle\,,\rangle, \alpha)$ at P, base changed to k, is $(\mathrm{H}^\natural(G_k, \mathcal{L}_k), \rho^\natural, \langle\,,\rangle, \alpha)$ for the pulled-back adjunction system. By Lemma D.45: (a) $H \otimes k$ is free over $\mathcal{O}_E \otimes k$; (b) $\alpha \otimes k$ is injective; (c) $\langle \alpha u, \alpha u' \rangle = 0$ on $F^*L \otimes k$.

(ii). By (b), $\alpha(P)\colon (F^*L)(P) \to H(P)$ is injective at every closed point P (injectivity descends along the field extension $\kappa(P) \to k$). A map of vector bundles on C that is injective on the fibres at all closed points is injective with locally free cokernel (locally a basis of F^*L maps to part of a basis of H by Nakayama's lemma); so $\alpha(F^*L)$ is a subbundle of rank g. The section $(u, u') \mapsto \langle \alpha u, \alpha u' \rangle$ of $(F^*L)^\vee \otimes (F^*L)^\vee$ vanishes in the fibre at every closed point by (c), hence vanishes (C is reduced). Define $\beta\colon H \to (F^*L)^\vee = F^*(L^\vee)$ by $\beta(h)(u) = \langle \alpha u, h \rangle$. Since

the pairing identifies H with $H^\vee$ ((3) of Lemma D.43) and α is a subbundle, β is surjective and $\ker\beta$ is a subbundle of rank $2g-g=g$. By isotropy $\alpha(F^*L)\subseteq\ker\beta$; a subbundle of H of rank g contained in a subbundle of rank g is locally a direct summand of it of full rank, so $\alpha(F^*L)=\ker\beta$ and $0\to F^*L\xrightarrow{\alpha}H\xrightarrow{\beta}F^*(L^\vee)\to 0$ is exact. Equivariance: $\rho_H(a)\circ\alpha=\alpha\circ F^*\rho_L(a)$ by (5) of Lemma D.44; and by (3) of Lemma D.44 (applied to $\bar a$),

$$\beta(\rho_H(a)h)(u)=\langle\alpha u,\rho_H(a)h\rangle=\langle\rho_H(\bar a)\alpha u,h\rangle=\langle\alpha(F^*\rho_L(\bar a)u),h\rangle=\big(F^*(\rho_L(\bar a)^\vee)\beta(h)\big)(u),$$

so β is equivariant for $a\mapsto F^*(\rho_L(\bar a)^\vee)$ on $F^*(L^\vee)$.

(iii). Let P be a closed point, $A_P=\mathcal{O}_E\otimes_{\mathbb{Z}}\kappa(P)=\prod_i A_i$ with A_i local Artinian. By (a), $H(P)\otimes_{\kappa(P)}k\cong(A_P\otimes k)^r$ for some r. Each factor $H(P)_i$ becomes free of rank r over the faithfully flat extension $A_i\otimes k$ of A_i; so it is flat over the local Artinian ring A_i, hence free, necessarily of rank r. Thus $H(P)$ is free of rank r over A_P, and $r[E:\mathbb{Q}]=\dim_{\kappa(P)}H(P)=2g$. Lift a basis to $v_1,\dots,v_r$ in the stalk H_P; the map $(\mathcal{O}_E\otimes\mathcal{O}_{C,P})^r\to H_P$, $(b_j)\mapsto\sum b_jv_j$, is surjective by Nakayama's lemma and is a surjection of free $\mathcal{O}_{C,P}$-modules of the same rank $2g$, hence an isomorphism; it extends to an isomorphism over an open neighbourhood of P. These neighbourhoods cover C.

Consequences. Items (i) and (ii) are hypothesis (a) of Lemma D.52 with $M_0=L^\vee$, $M_1=L$ (the outer terms of the second sequence carrying F^* of the actions on M_1 and M_0), and (iii) is its hypothesis (b). Items (i) and (ii) are the two equivariant exact sequences of Lemma D.63 ($\mathcal{O}_E$ acting on $L^\vee$ by $a\mapsto\rho_L(\bar a)^\vee$). Item (i) and the injectivity of α give the hypotheses of Lemma D.62. □

Lemma D.43. *Let S be a scheme, $G\to S$ a semiabelian scheme of relative dimension g and $\mathcal{L}$ a rigidified line bundle on G with a cubical structure. Let $W(\omega_G)$ be the vector group $T\mapsto\Gamma(T,\omega_G\otimes_{\mathcal{O}_S}\mathcal{O}_T)$.*

1. *$W(\omega_G)$ acts on $E^\natural(G,\mathcal{L})$ over G by $\eta\cdot(x,t)=(x,(1+\eta)t)$, making the projection $E^\natural(G,\mathcal{L})\to G$ a torsor under $W(\omega_G)\times_S G$, trivial locally on G for the Zariski topology. Hence $E^\natural(G,\mathcal{L})$ is represented by a smooth commutative group scheme over S, and $0\to W(\omega_G)\to E^\natural(G,\mathcal{L})\to G\to 0$ is exact, with $\eta\mapsto(0,(1+\eta)t_0)$ the first map. Its formation commutes with every base change $S'\to S$.*

2. *$\mathrm{H}^\natural(G,\mathcal{L})$ is a locally free $\mathcal{O}_S$-module of rank $2g$, the Hodge maps form an exact sequence $0\to\omega_G\to\mathrm{H}^\natural(G,\mathcal{L})\to\mathrm{Lie}(G)\to 0$, and the formation of $\mathrm{H}^\natural(G,\mathcal{L})$ and of this sequence commutes with base change.*

3. *The pairing $\langle\,,\rangle$ is a well-defined $\mathcal{O}_S$-bilinear pairing on $\mathrm{H}^\natural(G,\mathcal{L})$ inducing an isomorphism $\mathrm{H}^\natural(G,\mathcal{L})\to\mathrm{H}^\natural(G,\mathcal{L})^\vee$; for $\eta\in\omega_G$ and $h=(v,t)$, $\langle h,(0,(1+\varepsilon'\eta)t_0)\rangle=\eta(v)$.*

4. *If $p\mathcal{O}_S=0$ for a prime p, then $\tilde\alpha$ vanishes on $F_S^*\omega_G\subseteq F_S^*\mathrm{H}^\natural(G,\mathcal{L})$, so it induces*

$$\alpha\colon F_S^*\mathrm{Lie}(G)\to\mathrm{H}^\natural(G,\mathcal{L}),$$

and the composite of α with $\mathrm{H}^\natural(G,\mathcal{L})\to\mathrm{Lie}(G)$ is the map

$$\mathrm{Lie}(V_G)\colon F_S^*\mathrm{Lie}(G)=\mathrm{Lie}(G^{(p)})\to\mathrm{Lie}(G).$$

Proof. The cubical structure of the rigidified line bundle $\mathcal{L}$ makes $\Lambda(\mathcal{L})$ a symmetric biextension of $G\times_S G$ by $\mathbb{G}_m$ whose two partial compositions are associative and commutative, compatible with each other, with the symmetry and with the canonical trivialisations along $G\times\{e\}$ and

$\{e\} \times G$ (Moret-Bailly, Pinceaux de variétés abéliennes, Astérisque 129, Ch. I, 2.5; Breen, Fonctions thêta et théorème du cube, LNM 980, §2). For an S-scheme T write $T[\omega] = T \times_S G_1 = \mathrm{Spec}_T(\mathcal{O}_T \oplus \omega_T)$, $\omega_T = \omega_G \otimes \mathcal{O}_T$; it is a square-zero thickening of T.

(1) Torsor and group structure. Let $x \in G(T)$ and $N = (x \circ \mathrm{pr}_1, \mathrm{pr}_2)^*\Lambda(\mathcal{L})$ on $T[\omega]$. Two trivialisations t, t' of N agreeing on $T \times \{e\}$ differ by a unit u of $\mathcal{O}_T \oplus \omega_T$ with image 1 in $\mathcal{O}_T$, that is $u = 1 + \eta$ with $\eta \in \Gamma(T, \omega_T)$; conversely $(1+\eta)t$ is again such a trivialisation. So $W(\omega_G)(T)$ acts simply transitively on the fibre of $E^\natural(G, \mathcal{L})(T) \to G(T)$ over x whenever this fibre is non-empty. It is non-empty locally on T: a point of T has an affine neighbourhood $\mathrm{Spec}\, A$ with A local after localising, $A \oplus \Gamma(\omega_T)$ is then local, so N is trivial near the point; if t trivialises N there, $t|_{T\times\{e\}} = c\, t_{\mathrm{can}}$ with $c \in \mathcal{O}_T^\times$, and $c^{-1}t$ agrees with the canonical trivialisation. Applied to the universal point $x = \mathrm{id}_G$ ($T = G$) this shows that $E^\natural(G, \mathcal{L}) \to G$ is a $W(\omega_G)$-torsor trivial on a Zariski cover of G; hence it is represented by a scheme locally isomorphic over G to $G \times_S W(\omega_G)$, which is smooth over S ($W(\omega_G)$ is locally an affine space $\mathbb{A}^g_S$). The composition of Definition D.40 is defined because the partial composition maps the canonical trivialisations at (x, e) and (x', e) to that at $(x+x', e)$; it is associative and commutative because the first partial composition is, its unit is $(0, t_0)$ (t_0 the canonical trivialisation of $\Lambda_{(0,\cdot)}$, which is neutral for the first partial composition), and (x, t) has the inverse $(-x, t^{-1})$, t^{-1} read through $\Lambda_{(x,y)} \otimes \Lambda_{(-x,y)} \cong \Lambda_{(0,y)} \overset{t_0}{\cong} \mathcal{O}$. On the fibre over 0 the composition is $(0, (1+\eta)t_0) \cdot (0, (1+\eta')t_0) = (0, (1+\eta)(1+\eta')t_0) = (0, (1+\eta+\eta')t_0)$ since $\eta\eta' = 0$ in $\mathcal{O}_T \oplus \omega_T$; so $\eta \mapsto (0, (1+\eta)t_0)$ is an injective homomorphism $W(\omega_G) \to E^\natural$ onto the kernel of the projection, and the projection is surjective as a map of Zariski sheaves by the local existence above. The action of $W(\omega_G)$ is translation by this subgroup, because $(x, t) \cdot (0, (1+\eta)t_0) = (x, (1+\eta)t)$ (t_0 neutral). All constructions are pull-backs of $\Lambda(\mathcal{L})$, G_1 and ω_G, which commute with base change.

(2) The Lie algebra. $E^\natural(G, \mathcal{L})$ is smooth over S, so its Lie algebra is locally free with formation commuting with base change, and the exact sequence of smooth group schemes in (1) gives an exact sequence of Lie algebras $0 \to \mathrm{Lie}\, W(\omega_G) \to \mathrm{H}^\natural \to \mathrm{Lie}(G) \to 0$: left exactness is that of kernels; surjectivity holds because $E^\natural \to G$ has a section over a Zariski neighbourhood of e in G (local triviality of the torsor), whose differential at e splits the sequence. Finally $\mathrm{Lie}\, W(\omega_G) = \omega_G$ via $\eta \mapsto \varepsilon\eta$, which is the first Hodge map $\eta \mapsto (0, (1+\varepsilon\eta)t_0)$; so the rank is $g + g = 2g$.

(3) The pairing. Work locally on S. On $S[\omega] \times_S G_1 = G_1 \times_S G_1 = \mathrm{Spec}(\mathcal{O}_S \oplus \omega_G \oplus \omega_G \oplus \omega_G \otimes \omega_G)$ choose, by the argument of (1), a trivialisation s of $\Lambda(\mathcal{L})$ that agrees with the canonical trivialisations on $G_1 \times \{e\}$ and $\{e\} \times G_1$ (first trivialise, then divide by the unit $c(1 + \eta_1 \otimes 1)(1 + 1 \otimes \eta_2)$ measuring the defect on the two axes, which agree at (e, e)). Pulling back by the symmetry, $s(y, x) = (1+\sigma)s(x, y)$ with $\sigma \in \omega_G \otimes \omega_G$, since the defect is a unit equal to 1 on both axes. Let $h = (v, t) \in \mathrm{H}^\natural(U)$, so $v \in \mathrm{Lie}(G)(U)$ is a point of G_1 over $U[\varepsilon]$ and t trivialises $\Lambda_{(v,\cdot)}$ on $U[\varepsilon][\omega]$ with value t_0 modulo ε. As $s(v, \cdot)$ is another trivialisation with the same properties, (1) gives a unique $w \in \omega_G(U)$ with $t = (1 + \varepsilon w)s(v, \cdot)$; the map $h \mapsto (v, w)$ is an isomorphism $\mathrm{H}^\natural|_U \cong \mathrm{Lie}(G)|_U \oplus \omega_G|_U$ of $\mathcal{O}_U$-modules (both sides have the structure induced by $\varepsilon \mapsto c\varepsilon$). For $h' = (v', t')$, $t' = (1 + \varepsilon' w')s(v', \cdot)$, evaluate at the points of G_1: a function $w \in \omega_G$ takes the value $\varepsilon' w(v')$ at v', and σ takes the value $\varepsilon\varepsilon'\sigma(v, v')$ at (v, v'). Hence

$$t'(v) = (1 + \varepsilon\varepsilon' w'(v))\, s(v', v) = (1 + \varepsilon\varepsilon' w'(v))(1 + \varepsilon\varepsilon'\sigma(v, v'))\, s(v, v'),$$
$$t(v') = (1 + \varepsilon\varepsilon' w(v'))\, s(v, v'),$$

so (D.10) holds with the unique value

$$\langle h, h' \rangle = w'(v) - w(v') + \sigma(v, v').$$

This is $\mathcal{O}_U$-bilinear, and in the decomposition $\mathrm{Lie} \oplus \omega$ it has the block matrix $\begin{pmatrix} \sigma & -1 \\ 1 & 0 \end{pmatrix}$ with respect to the duality of $\mathrm{Lie}(G)$ and ω_G, which is invertible; so $h \mapsto \langle h, \cdot \rangle$ is an isomorphism onto the dual. For $h' = (0, (1 + \varepsilon'\eta)t_0)$ we have $v' = 0$, $w' = \eta$ (as $s(0, \cdot) = t_0$), and $\langle h, h' \rangle = \eta(v)$.

(4) The conjugate map. For a commutative group scheme X flat over S with $p\mathcal{O}_S = 0$, the Verschiebung $V_X \colon X^{(p)} \to X$ is functorial in X and satisfies $V_X \circ F_{X/S} = [p]_X$ (SGA 3, Exp. VII$_\mathrm{A}$, 4.3). For $X = W(\omega_G)$, locally $\mathbb{G}_a^g$, $[p] = 0$ and $F_{X/S}$ is faithfully flat, so $V_{W(\omega_G)} = 0$. Functoriality for the inclusion $i \colon W(\omega_G) \to E^\natural$ gives $V_{E^\natural} \circ i^{(p)} = i \circ V_{W(\omega_G)} = 0$, so $\tilde{\alpha} = \mathrm{Lie}(V_{E^\natural})$ vanishes on $\mathrm{Lie}(W(\omega_G)^{(p)}) = F_S^*\omega_G$ and induces α on $F_S^*\mathrm{H}^\natural / F_S^*\omega_G = F_S^*\mathrm{Lie}(G)$ by (2) (pull-back by F_S is right exact and preserves the locally split sequence). Functoriality for the projection $\pi \colon E^\natural \to G$ gives $\pi \circ V_{E^\natural} = V_G \circ \pi^{(p)}$, whence the last claim on Lie algebras. □

Lemma D.44. *Let E be a CM field with complex conjugation $a \mapsto \bar{a}$, S a normal integral noetherian scheme with generic point η, and $(G, \iota, \mathcal{L})$ a semiabelian structure over S with G_η an abelian variety.*

1. *$(G, \iota, \mathcal{L})$ has exactly one adjunction system ϵ.*
2. *With ϵ, every $a \in \mathcal{O}_E$ acts on $E^\natural(G, \mathcal{L})$ by a group endomorphism, and $a \mapsto \rho^\natural(a)$ is a ring homomorphism $\mathcal{O}_E \to \mathrm{End}_{\mathcal{O}_S}(\mathrm{H}^\natural(G, \mathcal{L}))$.*
3. *$\langle \rho^\natural(a)h, h' \rangle = \langle h, \rho^\natural(\bar{a})h' \rangle$ for all a and all sections h, h'.*
4. *The Hodge maps are $\mathcal{O}_E$-equivariant when a acts on ω_G by $\mathrm{Lie}(\iota(\bar{a}))^\vee$ and on $\mathrm{Lie}(G)$ by $\mathrm{Lie}(\iota(a))$.*
5. *If $p\mathcal{O}_S = 0$ for a prime p, then $\rho^\natural(a) \circ \alpha = \alpha \circ F_S^*\mathrm{Lie}(\iota(a))$ for all a.*
6. *For every morphism $c \colon S' \to S$, $c^*\epsilon$ is an adjunction system for the pull-back $(G_{S'}, \iota_{S'}, \mathcal{L}_{S'})$, and with it $\mathrm{H}^\natural(G_{S'}, \mathcal{L}_{S'}) = c^*\mathrm{H}^\natural(G, \mathcal{L})$ with the action $c^*\rho^\natural$, the pulled-back Hodge maps, pairing and (if $p\mathcal{O}_{S'} = 0$) conjugate map; (3), (4), (5) hold over S'.*

Proof. The rigidified line bundle $\mathcal{L}$ is symmetric with a cubical structure (Definition 31.2), so $\Lambda(\mathcal{L})$ is a symmetric biextension of $G \times_S G$ by $\mathbb{G}_m$ whose two partial compositions are associative and commutative, compatible with each other, with the symmetry and with the canonical trivialisations along $G \times \{e\}$ and $\{e\} \times G$ ([181, Ch. I, 2.5]; [51, §2]). These are the partial compositions, symmetry and canonical trivialisations used below.

Units. Let $n \geq 1$ and $X = G^n = G \times_S \cdots \times_S G$. Then X is smooth over S with geometrically integral fibres (smooth connected group schemes over fields), flat over the integral scheme S, so every irreducible component of X dominates S and meets X_η, which is irreducible; and X is normal. Hence X is integral and $\Gamma(X, \mathcal{O}_X) \subseteq \Gamma(X_\eta, \mathcal{O}) = \kappa(\eta)$, the last equality because $X_\eta = G_\eta^n$ is a proper geometrically integral $\kappa(\eta)$-variety. So every unit u on X is constant along the fibres of $X \to S$: $u = u(e)$, the pull-back of its value along the unit section, a unit of $\mathcal{O}(S)$. Consequently: *two isomorphisms between the same two line bundles on G^n that agree along the unit section are equal.* This is the uniqueness principle used below; in each application both isomorphisms are composites of partial compositions, symmetries and the ϵ_a, all of which are the canonical identifications at the unit section, so they agree there.

(1) By the definition of a semiabelian structure there is an isomorphism $\varphi_a \colon (\iota(a), 1)^*\Lambda(\mathcal{L}) \to (1, \iota(\bar{a}))^*\Lambda(\mathcal{L})$. Its restrictions to $G \times \{e\}$ and $\{e\} \times G$, read through the canonical trivialisations, are units of $\mathcal{O}(G)$, hence equal to the constant $c = \varphi_a(e, e) \in \mathcal{O}(S)^\times$ by the Units step ($n = 1$).

So $\epsilon_a = c^{-1}\varphi_a$ is the identity on both axes. Two such differ by a unit on G^2 equal to 1 at (e,e), which is 1. Hence ϵ exists and is unique.

(2) Compatibilities of ϵ. By the uniqueness principle on G^2 or G^3, for $a, b \in \mathcal{O}_E$ and points x, x', y:

(c1) $\epsilon_1 = \mathrm{id}$ and $\epsilon_{ab} = \epsilon_b(x, \bar{a}y) \circ \epsilon_a(bx, y)$ as maps $\Lambda_{(abx,y)} \to \Lambda_{(x,\overline{ab}y)}$;

(c2) ϵ_a commutes with the first partial composition: $\Lambda_{(ax,y)} \otimes \Lambda_{(ax',y)} \to \Lambda_{(a(x+x'),y)} \xrightarrow{\epsilon_a} \Lambda_{(x+x',\bar{a}y)}$ equals $\epsilon_a \otimes \epsilon_a$ followed by the first partial composition;

(c3) ϵ_{a+b} composed with the first partial composition $\Lambda_{(ax,y)} \otimes \Lambda_{(bx,y)} \to \Lambda_{((a+b)x,y)}$ equals $\epsilon_a \otimes \epsilon_b$ followed by the second partial composition $\Lambda_{(x,\bar{a}y)} \otimes \Lambda_{(x,\bar{b}y)} \to \Lambda_{(x,\overline{(a+b)}y)}$;

(c4) with sym the symmetry, $\epsilon_{\bar{a}} \circ \mathrm{sym} \circ \epsilon_a = \mathrm{sym}$ as maps $\Lambda_{(ax,z)} \to \Lambda_{(z,ax)}$.

By (c2) the action $a \cdot (x,t) = (ax, \epsilon_a^{-1}(t(\bar{a}\,\cdot)))$ is a group endomorphism of $E^\natural$, and by (c1) 1 acts trivially and $(ab)\cdot = a \cdot (b\cdot)$: indeed $a \cdot (b \cdot (x,t)) = (abx, \epsilon_a(bx,\cdot)^{-1}\epsilon_b(x,\bar{a}\,\cdot)^{-1}t(\bar{b}\bar{a}\,\cdot))$. For additivity let $(x,t) \in E^\natural(T)$. Extend t, locally on T, to a trivialisation $\tilde{t}$ of $\Lambda_{(x,\cdot)}$ on an open neighbourhood U of the unit section in G_T, equal to the canonical one at e. The unit $D(y_1,y_2) = \tilde{t}(y_1+y_2)/(\tilde{t}(y_1)\cdot\tilde{t}(y_2))$ (product by the second partial composition) on a neighbourhood of (e,e) in G_T^2 is 1 on both axes, so $D-1$ lies in the product of the ideals of $\{e\}\times G$ and $G \times \{e\}$ (locally G is smooth and the two ideals are generated by complementary parts of a regular sequence). Along $G_1 \to G^2$, $y \mapsto (\bar{a}y, \bar{b}y)$, both ideals pull back into the ideal of e in G_1, whose square is 0; so $t((\bar{a}+\bar{b})y) = t(\bar{a}y)\cdot t(\bar{b}y)$ for $y \in G_1$. With (c3) this gives $(a+b)\cdot(x,t) = a\cdot(x,t)\cdot b\cdot(x,t)$ in the group $E^\natural$. Taking Lie algebras, $\rho^\natural$ is a ring homomorphism.

(3) Let $h = (v,t)$ over $U[\varepsilon]$, $h' = (v',t')$ over $U[\varepsilon']$, and put $A = t(\bar{a}v') \in \Lambda_{(v,\bar{a}v')}$, $B = t'(av) \in \Lambda_{(v',av)}$. By the definitions, $\rho^\natural(a)h = (av, \epsilon_a^{-1}t(\bar{a}\,\cdot))$ and $\rho^\natural(\bar{a})h' = (\bar{a}v', \epsilon_{\bar{a}}^{-1}t'(a\,\cdot))$, so (D.10) reads

$$\mathrm{sym}(B) = (1+\varepsilon\varepsilon' c_1)\,\epsilon_a^{-1}(A), \qquad \epsilon_{\bar{a}}^{-1}(B) = (1+\varepsilon\varepsilon' c_2)\,\mathrm{sym}(A),$$

with $c_1 = \langle \rho^\natural(a)h, h'\rangle$ and $c_2 = \langle h, \rho^\natural(\bar{a})h'\rangle$. By (c4), $\mathrm{sym}\circ\epsilon_a\circ\mathrm{sym} = \epsilon_{\bar{a}}^{-1}$; applying $\mathrm{sym}\circ\epsilon_a$ to the first equation gives $\epsilon_{\bar{a}}^{-1}(B) = (1+\varepsilon\varepsilon' c_1)\mathrm{sym}(A)$, so $c_1 = c_2$.

(4) For $\eta \in \omega_G$, $\rho^\natural(a)(0,(1+\varepsilon\eta)t_0) = (0, \epsilon_a^{-1}((1+\varepsilon\eta(\bar{a}\,\cdot))t_0)) = (0,(1+\varepsilon\,\iota(\bar{a})^*\eta)t_0)$, because ϵ_a is the identity on $\{e\}\times G$ and t_0 is canonical; and $\iota(\bar{a})^*$ on $\omega_G = e^*\Omega^1$ is $\mathrm{Lie}(\iota(\bar{a}))^\vee$. The projection $(v,t) \mapsto v$ carries the action to $v \mapsto \iota(a)v$, i.e. $\mathrm{Lie}(\iota(a))$.

(5) The Verschiebung is functorial for the endomorphism $a\cdot$ of $E^\natural$ ([226, Exp. VII$_\mathrm{A}$, 4.3]), so $\rho^\natural(a)\circ\tilde{\alpha} = \tilde{\alpha}\circ F_S^*\rho^\natural(a)$; restricted to $F_S^*\mathrm{Lie}(G)$ through (4) and (4) of Lemma D.43 this is the claim.

(6) The conditions defining an adjunction system (isomorphism of the pulled-back line bundles, identity on the axes) are preserved by pull-back, and all constructions of Definition D.40 are pull-backs (Lemma D.43, (1), (2)). The proofs of (3)–(5) use only the identities (c1)–(c4), which pull back, and Lemma D.43. □

Lemma D.45. *Let E be a CM field with complex conjugation $a \mapsto \bar{a}$, k an algebraically closed field of characteristic $p > 0$, G a semiabelian variety over k with a ring homomorphism $\iota\colon \mathcal{O}_E \to \mathrm{End}_k(G)$, $\mathcal{L}$ a rigidified line bundle on G with a cubical structure, and ϵ an adjunction system for $(G,\iota,\mathcal{L})$. Suppose there are an exact sequence $0 \to T \to G \xrightarrow{q} B \to 0$ with T a torus and B an abelian variety, and a line bundle $\mathcal{M}$ on B whose polarisation $\phi_\mathcal{M}$ is an isomorphism, with $\mathcal{L} \cong q^*\mathcal{M}$. Then:*

1. $\mathrm{H}^\natural(G,\mathcal{L})$, *with a acting by* $\rho^\natural(a)$, *is a free* $\mathcal{O}_E\otimes_{\mathbb{Z}}k$*-module;*

2. $\alpha\colon F_k^*\mathrm{Lie}(G)\to\mathrm{H}^\natural(G,\mathcal{L})$ *is injective;*

3. $\langle\alpha(u),\alpha(u')\rangle=0$ *for all* $u,u'\in F_k^*\mathrm{Lie}(G)$.

Proof. Write $R=\mathcal{O}_E\otimes_{\mathbb{Z}}k$, a finite k-algebra, hence semilocal. We use Rosenlicht's theorem: on a connected smooth algebraic group X over k every unit is $c\chi$ with $c\in k^\times$ and $\chi\in\mathrm{Hom}(X,\mathbb{G}_m)$. For $X=X_1\times\cdots\times X_n$ ($n\geq 2$) a unit equal to 1 on each locus $\{x_i=e\}$ is therefore 1 (writing it as $c\prod\chi_i(x_i)$ and restricting to these loci).

Step 1: $\Lambda(\mathcal{L})=(q\times q)^*\Lambda(\mathcal{M})$ as biextensions. Give $\mathcal{M}$ a rigidification and its cubical structure, and scale the isomorphism $\mathcal{L}\cong q^*\mathcal{M}$ by a constant so that it respects the rigidifications. The cubical structure of $\mathcal{L}$ and the pull-back of that of $\mathcal{M}$ are two trivialisations of $\Theta(\mathcal{L})$ on G^3 that agree with the canonical trivialisation (given by the rigidification) on each locus $\{x_i=e\}$; their ratio is a unit on G^3 equal to 1 there, hence 1. So the induced isomorphism $\Lambda(\mathcal{L})\cong(q\times q)^*\Lambda(\mathcal{M})$ respects the partial compositions, the symmetry and the canonical trivialisations; we use it as an identification.

Step 2: actions on T and B. For $a\in\mathcal{O}_E$, $q(\iota(a)(T))$ is a torus in the abelian variety B (image of a torus), hence trivial; so $\iota(a)T\subseteq T$, and ι induces ring homomorphisms $\iota_T\colon\mathcal{O}_E\to\mathrm{End}(T)$, $\iota_B\colon\mathcal{O}_E\to\mathrm{End}(B)$ with $q\circ\iota(a)=\iota_B(a)\circ q$.

Step 3: ϵ descends to B. For $a\in\mathcal{O}_E$ the source and target of ϵ_a are the pull-backs along $q\times q$ of $N_1=(\iota_B(a),1)^*\Lambda(\mathcal{M})$ and $N_2=(1,\iota_B(\bar a))^*\Lambda(\mathcal{M})$, and so carry the canonical $T\times T$-linearisations of pull-backs. Comparing ϵ_a at $(x+s,y+s')$ with ϵ_a at (x,y) through these linearisations gives a unit u on $T\times T\times G\times G$, with $u=1$ where $s=s'=e$; and $u(s,e,x,e)=1=u(e,s',e,y)$, because ϵ_a is the identity on $G\times\{e\}$ and on $\{e\}\times G$, where both bundles are canonically trivial compatibly with the linearisations. Writing $u=c\,\chi(s)\chi'(s')\chi_1(x)\chi_2(y)$ these three facts force $u=1$. So ϵ_a is invariant under the $T\times T$-torsor $q\times q\colon G\times G\to B\times B$, and by faithfully flat descent it is the pull-back of an isomorphism $\epsilon_{B,a}\colon N_1\to N_2$, the identity on both axes: $\epsilon_B=(\epsilon_{B,a})_a$ is an adjunction system for $(B,\iota_B,\mathcal{M})$.

Step 4: the subgroup E'. Let $E'=G\times_B E^\natural(B,\mathcal{M})$ and $j\colon E'\to E^\natural(G,\mathcal{L})$, $j(x,(qx,t'))=(x,t'\circ q)$, where $t'\circ q$ is the pull-back of t' along $1\times q|_{G_1}\colon\{x\}\times G_1\to\{qx\}\times B_1$ (Step 1). The map $q|_{G_1}$ corresponds to $\mathcal{O}_{B_1}=k\oplus\omega_B\to k\oplus\omega_G$, injective because $0\to\omega_B\to\omega_G\to\omega_T\to 0$ is exact (dual of $0\to\mathrm{Lie}\,T\to\mathrm{Lie}\,G\to\mathrm{Lie}\,B\to 0$); so pull-back of trivialisations along it is injective and j is injective. It is a homomorphism (Step 1: partial compositions correspond) and $\mathcal{O}_E$-equivariant for the actions defined by ϵ and by ϵ_B on the second factor (ι on the first), by Step 3 and $q\circ\iota(\bar a)=\iota_B(\bar a)\circ q$. Over each x, j is equivariant for $W(\omega_B)\subseteq W(\omega_G)$. Hence on Lie algebras, with $\mathrm{H}^\natural_G=\mathrm{H}^\natural(G,\mathcal{L})$, $\mathrm{H}^\natural_B=\mathrm{H}^\natural(B,\mathcal{M})$, there are $\mathcal{O}_E$-equivariant exact sequences

$$0\to\mathrm{Lie}\,T\to\mathrm{Lie}\,E'\xrightarrow{\pi}\mathrm{H}^\natural_B\to 0,\qquad 0\to\mathrm{Lie}\,E'\xrightarrow{dj}\mathrm{H}^\natural_G\to\omega_T\to 0.$$

The first is exact because $E'\to E^\natural(B,\mathcal{M})$ is smooth surjective with kernel $T\times\{0\}$. The second is exact because dj is injective, $dj(\mathrm{Lie}\,E')\cap\omega_G=\omega_B$ (the fibres over 0), so $\omega_T=\omega_G/\omega_B$ injects into the cokernel, and dimensions agree: $\dim\mathrm{Lie}\,E'=\dim T+2\dim B$ and $\dim\mathrm{H}^\natural_G=2\dim G$ by (2) of Lemma D.43. Here $\mathcal{O}_E$ acts on $\mathrm{Lie}\,T$ by $\mathrm{Lie}\,\iota_T(a)$, on $\mathrm{H}^\natural_B$ by $\rho^\natural_B(a)$ (from ϵ_B) and on ω_T by $\iota_T(\bar a)^*$ (as on ω_G, (4) of Lemma D.44 whose proof of (4) uses only that ϵ_a is the identity on $\{e\}\times G$).

Step 5: freeness (1). T is split (k algebraically closed), and $\mathrm{Lie}\,T=X_*(T)\otimes_{\mathbb{Z}}k$ equivariantly, $X_*(T)$ the cocharacter group with $\mathcal{O}_E$ acting through ι_T. A nonzero $a\in\mathcal{O}_E$ divides the nonzero integer $N_{E/\mathbb{Q}}(a)$ in $\mathcal{O}_E$, which acts injectively on the free $\mathbb{Z}$-module $X_*(T)$; so $X_*(T)$ is a

finitely generated torsion-free, hence projective, $\mathcal{O}_E$-module, isomorphic to $\mathcal{O}_E^{r-1} \oplus \mathfrak{a}$ for an ideal $\mathfrak{a}$ (Steinitz). Then $\mathfrak{a} \otimes_{\mathbb{Z}} k$ is an invertible module over the semilocal ring R, hence free, and $\operatorname{Lie} T$ is free over R. Its k-dual ω_T with the transposed action is free: $\operatorname{Hom}_k(R,k) \cong \operatorname{Hom}_{\mathbb{Z}}(\mathcal{O}_E,\mathbb{Z}) \otimes k$ is invertible (inverse different), hence free of rank one over R; and composing an action with the ring automorphism $a \otimes c \mapsto \bar{a} \otimes c$ of R preserves freeness. By (iii) of Lemma D.46 (for ϵ_B), θ identifies $\mathrm{H}^\natural_B$ with $H^1_{\mathrm{dR}}(B/k)$ on which a acts by $\iota_B(\bar{a})^*$; by Rapoport's lemma (Rapoport, Compositio Math. 36 (1978), Lemme 1.3;) $H^1_{\mathrm{dR}}(B/k)$ is free over R for $a \mapsto \iota_B(a)^*$, hence also for $a \mapsto \iota_B(\bar{a})^*$. Extensions of free (projective) R-modules split, so the two sequences of Step 4 give that $\mathrm{H}^\natural_G$ is free.

Step 6: injectivity (2). By (4) of Lemma D.43, α is induced by $\tilde{\alpha} = \operatorname{Lie}(V_{E^\natural})$. Since $\operatorname{Lie} E' \to \operatorname{Lie} G$ is surjective, $F_k^*\operatorname{Lie} G$ is the image of $F_k^*\operatorname{Lie} E'$, and by functoriality of the Verschiebung for j, $\alpha(F_k^*\operatorname{Lie} G) = dj(\operatorname{Lie}(V_{E'})(F_k^*\operatorname{Lie} E')) \subseteq dj(\operatorname{Lie} E')$. Functoriality for $T \to E'$, $s \mapsto (s,(0,t_0))$, shows that α restricted to $F_k^*\operatorname{Lie} T$ is $\operatorname{Lie}(V_T)$, an isomorphism onto $\operatorname{Lie} T$: for $T \cong \mathbb{G}_m^r$, $F_{T/k}$ is identified with $[p]$ under $T^{(p)} = T$, and $V_T \circ F_{T/k} = [p]$ with $F_{T/k}$ an epimorphism gives $V_T = \mathrm{id}$. Functoriality for $E' \to E^\natural(B,\mathcal{M})$ shows $\pi \circ \alpha = \alpha_B \circ F_k^*\operatorname{Lie}(q)$. Now α_B is injective: by (i), (ii) of Lemma D.46, $\theta \circ \alpha_B$ is $F^*_{B/k}$ restricted to $F_k^* H^1_{\mathrm{dR}}(B/k)/F_k^*\omega_B = F_k^* R^1 f_* \mathcal{O}_B$, which is injective by Lemma D.48. In the diagram with exact rows $0 \to F_k^*\operatorname{Lie} T \to F_k^*\operatorname{Lie} G \to F_k^*\operatorname{Lie} B \to 0$ and $0 \to \operatorname{Lie} T \to \operatorname{Lie} E' \to \mathrm{H}^\natural_B \to 0$ and vertical maps $\operatorname{Lie}(V_T)$ (bijective), α, α_B (injective), α is injective.

Step 7: isotropy (3). For $h, h' \in \operatorname{Lie} E'$, $\langle dj(h), dj(h')\rangle = \langle \pi h, \pi h'\rangle_B$: if $\pi h = (qv,t)$ and $\pi h' = (qv',t')$ then $dj(h) = (v, t\circ q)$, $dj(h') = (v', t' \circ q)$, and $(t' \circ q)(v) = t'(qv)$, $(t \circ q)(v') = t(qv')$ in $\Lambda(\mathcal{L})_{(v,v')} = \Lambda(\mathcal{M})_{(qv,qv')}$ with the same symmetry (Step 1), so (D.10) gives the same scalar. As $\alpha(F_k^*\operatorname{Lie} G) \subseteq dj(\operatorname{Lie} E')$ (Step 6), $\langle \alpha u, \alpha u'\rangle = \langle \alpha_B(qu), \alpha_B(qu')\rangle_B = 0$ by Lemma D.47 (for B and $\mathcal{M}$). □

Lemma D.46. *Let $f\colon A \to S$ be an abelian scheme of relative dimension g and $\mathcal{L}$ a rigidified line bundle on A with a cubical structure whose polarisation $\phi_{\mathcal{L}}\colon A \to A^\vee$ is an isomorphism. Then there is an isomorphism of $\mathcal{O}_S$-modules $\theta\colon \mathrm{H}^\natural(A,\mathcal{L}) \to H^1_{\mathrm{dR}}(A/S)$, compatible with base change in S, such that:*

(i) θ is the identity on $\omega_{A/S}$ (embedded by the Hodge map, resp. by the Hodge sequence), and induces on the quotients the isomorphism $\lambda_A \circ \operatorname{Lie}(\phi_{\mathcal{L}})\colon \operatorname{Lie}(A/S) \to R^1 f_* \mathcal{O}_A$;

(ii) if $p\mathcal{O}_S = 0$ for a prime p, with $F_{A/S}\colon A \to A^{(p)}$ the relative Frobenius and $H^1_{\mathrm{dR}}(A^{(p)}/S) = F_S^* H^1_{\mathrm{dR}}(A/S)$, then $\theta \circ \tilde{\alpha} = F^*_{A/S} \circ F_S^* \theta$;

(iii) if $\iota\colon \mathcal{O} \to \operatorname{End}_S(A)$ is a ring homomorphism from a commutative ring with involution $a \mapsto \bar{a}$, and ϵ is an adjunction system for $(A,\iota,\mathcal{L})$, then $\theta \circ \rho^\natural(a) = \iota(\bar{a})^* \circ \theta$ for every $a \in \mathcal{O}$.

Proof. Write $\operatorname{Extrig}(A) = \operatorname{Extrig}(A, \mathbb{G}_m)$, $W(\omega_A)$ for the vector group of $\omega_A = \omega_{A/S}$, and $E^\natural = E^\natural(A,\mathcal{L})$. For a homomorphism $u\colon A \to B$ of abelian schemes, u maps A_1 into B_1, and $u^\#\colon \operatorname{Extrig}(B) \to \operatorname{Extrig}(A)$ denotes pull-back of rigidified extensions along u; it induces on $\underline{\operatorname{Ext}}^1(-,\mathbb{G}_m) = \underline{\operatorname{Pic}}^0$ the pull-back of line bundles, that is the transpose $u^\vee$ (by (b) of Theorem B.125). We use: (U) a unit on $A_T \times_T A_T$ (T an S-scheme) that equals 1 on $A_T \times \{e\}$ is 1, because $A \times_S A$ is an abelian scheme, so its units over T come from T ((a) of Theorem B.125).

Step 1: the rigidified extension attached to a point of $E^\natural$. By the biextension axioms of $\Lambda(\mathcal{L})$ (Moret-Bailly, Pinceaux de variétés abéliennes, Astérisque 129, Ch. I, 2.5), for $x \in A(T)$ the line bundle $M_x = (x \circ \mathrm{pr}_1, \mathrm{pr}_2)^*\Lambda(\mathcal{L})$ on A_T, $y \mapsto \Lambda_{(x,y)}$, with the partial composition in the

second variable (the partial composition of Definition D.40 transported by the symmetry) and the canonical trivialisation of $\Lambda_{(x,0)}$, makes its $\mathbb{G}_m$-torsor $M_x^\times$ an extension of A_T by $\mathbb{G}_m$; the first partial composition $c\colon M_x \otimes M_{x'} \cong M_{x+x'}$ is compatible with the second, that is, it is an isomorphism of extensions from the contracted product $M_x^\times \wedge M_{x'}^\times$ (the Baer sum) onto $M_{x+x'}^\times$. For $(x,t) \in E^\natural(T)$, t trivialises M_x on $(A_T)_1$ and equals the canonical trivialisation at e: it is a morphism of T-pointed T-schemes $(A_T)_1 \to M_x^\times$ lifting the inclusion, a rigidification. Put

$$\Psi_{\mathcal{L}}(x,t) = \text{the class of } (M_x^\times, t) \in \mathrm{Extrig}(A)(T).$$

Step 2: $\Psi_{\mathcal{L}}$ is an isomorphism of group schemes, compatible with base change. It is a homomorphism: $(x,t)\cdot(x',t') = (x+x', t\cdot t')$ with $t\cdot t' = c(t\otimes t')$, and c is an isomorphism of extensions carrying the rigidification $t\otimes t'$ of the Baer sum to $t\cdot t'$. On the subgroups $W(\omega_A)$: $\Psi_{\mathcal{L}}(0,(1+\eta)t_0)$ is the trivial extension (M_0 trivialised by t_0) with the rigidification $1+\eta$, which is the image of η under the first map of (b) of Theorem B.125; so $\Psi_{\mathcal{L}}$ is the identity of $W(\omega_A)$. On the quotients: $M_x \cong t_x^*\mathcal{L}\otimes\mathcal{L}^{-1}\otimes f_T^*\mathcal{L}_x^{-1}$, whose class in $\underline{\mathrm{Pic}}^0(A)(T) = A^\vee(T)$ is $\phi_{\mathcal{L}}(x)$ by the definition of $\phi_{\mathcal{L}}$; by (b) of Theorem B.125 the projection $\mathrm{Extrig}(A) \to \underline{\mathrm{Ext}}^1(A,\mathbb{G}_m) = A^\vee$ sends $\Psi_{\mathcal{L}}(x,t)$ to the class of M_x, so $\Psi_{\mathcal{L}}$ induces $\phi_{\mathcal{L}}$ on the quotients. The rows $0\to W(\omega_A)\to E^\natural \to A\to 0$ (Lemma D.43 (1)) and $0\to W(\omega_A)\to \mathrm{Extrig}(A)\to A^\vee\to 0$ ((b)) are exact and $\phi_{\mathcal{L}}$ is an isomorphism, so $\Psi_{\mathcal{L}}$ is an isomorphism of sheaves, hence of the representing smooth group schemes. It is formed by pull-backs, so it commutes with base change in S.

Step 3: the de Rham isomorphism μ_A and its naturality. For S affine let

$$\mu_A\colon \mathrm{Lie}\,\mathrm{Extrig}(A) \to H^1_{\mathrm{dR}}(A/S)$$

be the isomorphism of (g) of Theorem B.125: a rigidified extension (E,r) of $A_{S[\varepsilon]}$ reducing to the trivial one receives the unique connection ∇ of (e) (over $S[\varepsilon]$), the class of the line bundle with connection $(\mathcal{L}_E,\nabla)$ in $\mathbb{H}^1(\Omega^{\bullet\times}_{A_{S[\varepsilon]}/S[\varepsilon]})$ lies in the kernel of reduction, and (f) identifies that kernel with $H^1_{\mathrm{dR}}(A/S)$. (N) For a homomorphism $u\colon A\to B$, $\mu_A\circ\mathrm{Lie}(u^\#) = u^*\circ\mu_B$: u maps $A_1\to B_1$ and $\Delta^1(A)\to\Delta^1(B)$ compatibly with the two maps τ, so $u^*\nabla$ is an integrable connection on $u^\#E$, compatible with the group structure, whose pull-back by τ carries the unit section to u^*r; by the uniqueness in (e) it is the connection of $u^\#(E,r)$, and Čech classes and the split sequence of (f) commute with pull-back along u. (BC) For a morphism $S'\to S$ of affine schemes, $\mu_{A_{S'}}$ is the base change of μ_A (same uniqueness argument, with (c) and (d) of Theorem B.125). By (BC) for affine opens, the μ glue to an isomorphism μ_A over any S, satisfying (N) and (BC).

Step 4: θ. Put $\theta = \mu_A\circ\mathrm{Lie}(\Psi_{\mathcal{L}})\colon \mathrm{H}^\natural(A,\mathcal{L}) \to H^1_{\mathrm{dR}}(A/S)$, an isomorphism of $\mathcal{O}_S$-modules by Steps 2 and 3, compatible with base change in S by Step 2 and (BC).

(i). Let η be a section of ω_A over an affine open U. By Step 2, $\mathrm{Lie}(\Psi_{\mathcal{L}})$ maps the Hodge image $(0,(1+\varepsilon\eta)t_0)$ to the trivial extension over $U[\varepsilon]$ with the rigidification $1+\varepsilon\eta$. By (e), the corresponding connection on the trivial torsor is given by the unit $1+w$ with $\tau^*w = \varepsilon\eta$, that is $w = \varepsilon\tilde\eta$ where $\tilde\eta\in\Gamma(A_U,\Omega^1)$ is the form with $\tau^*\tilde\eta = e^*\tilde\eta = \eta$; its connection form is $\varepsilon\tilde\eta$ and its transition functions are 1. Under the split sequence of (f) (in degree 1, $w\mapsto\varepsilon w$) this class is the class of $(0,\tilde\eta)$, the image of $\tilde\eta\in H^0(A,\Omega^1) = \mathbb{H}^1(\Omega^{\geq 1})$ in H^1_{dR}, which is the image of η under the first map of the Hodge sequence ($\omega_{A/S} = f_*\Omega^1_{A/S} \cong e^*\Omega^1_{A/S}$). So θ is the identity on $\omega_{A/S}$. On the quotients, the second Hodge map $(v,t)\mapsto v$ corresponds under $\Psi_{\mathcal{L}}$ to $\mathrm{Lie}\,\mathrm{Extrig}(A)\to\mathrm{Lie}(A^\vee)$, which induces $\mathrm{Lie}(\phi_{\mathcal{L}})$ (Step 2); and by (f), (g) the projection $H^1_{\mathrm{dR}}(A/S)\to R^1f_*\mathcal{O}_A$ (forgetting the connection forms) corresponds to $\mathrm{Lie}(E^\natural)\to\mathrm{Lie}(A^\vee)\cong\mathrm{Lie}(\underline{\mathrm{Pic}}(A))\cong R^1f_*\mathcal{O}_A$, the last map sending a line bundle on $A_{U[\varepsilon]}$ with transition functions $1+\varepsilon g_{ij}$ to the class of (g_{ij}) (degree 0 of the split sequence), which is λ_A. Hence θ induces $\lambda_A\circ\mathrm{Lie}(\phi_{\mathcal{L}})$ on the quotients.

(ii). Let $p\mathcal{O}_S = 0$. By (c), $\mathrm{Extrig}(A)^{(p)} = \mathrm{Extrig}(A^{(p)})$, and $(A^\vee)^{(p)} = (A^{(p)})^\vee$. We show $V_{\mathrm{Extrig}(A)} = F^\#_{A/S}$ as homomorphisms $\mathrm{Extrig}(A^{(p)})\to\mathrm{Extrig}(A)$. Both are compatible with the

projections to $A^\vee$: by functoriality of the Verschiebung (SGA 3, Exp. VII$_\mathrm{A}$, 4.3) the first induces $V_{A^\vee/S}$, the second induces $F^\vee_{A/S}$, and $F^\vee_{A/S} = V_{A^\vee/S}$ (the transpose of the relative Frobenius is the Verschiebung of the dual; Mumford, Abelian varieties, §15). Both vanish on $W(\omega_{A^{(p)}})$: the Verschiebung of the vector group W is 0 (since $V_W \circ F_{W/S} = [p]_W = 0$ and $F_{W/S}$ is faithfully flat, W being locally $\mathbb{G}_a^g$), and $F_{A/S}$ maps A_1 into the unit section of $A^{(p)}$ (the ideal of the unit section of $A^{(p)}$ is generated locally by the $g \otimes 1$ with g in the ideal $\mathcal{I}$ of e, and $F^*_{A/S}(g \otimes 1) = g^p \in \mathcal{I}^2$), so $F^\#_{A/S}$ pulls every rigidification of the trivial extension back to the trivial one. Hence the difference $V_{\mathrm{Extrig}(A)} - F^\#_{A/S}$ is a homomorphism into $W(\omega_A)$ vanishing on $W(\omega_{A^{(p)}})$, so it factors through a homomorphism $(A^{(p)})^\vee \to W(\omega_A)$, which is 0 by (a). Now $\Psi_\mathcal{L}$ is an isomorphism, so $\Psi_\mathcal{L} \circ V_{E^\natural} = V_{\mathrm{Extrig}(A)} \circ \Psi^{(p)}_\mathcal{L}$, and $\Psi^{(p)}_\mathcal{L} = \Psi_{\mathcal{L}^{(p)}}$ under $E^\natural(A,\mathcal{L})^{(p)} = E^\natural(A^{(p)}, \mathcal{L}^{(p)})$ (Step 2, Lemma D.43 (1)). Taking Lie algebras and using (N) for $u = F_{A/S}$ and (BC) for F_S (with $H^1_\mathrm{dR}(A^{(p)}/S) = F_S^* H^1_\mathrm{dR}(A/S)$, Theorem B.124 (i)):

$$\theta \circ \tilde\alpha = \mu_A \circ \mathrm{Lie}(F^\#_{A/S}) \circ \mathrm{Lie}(\Psi_{\mathcal{L}^{(p)}}) = F^*_{A/S} \circ \mu_{A^{(p)}} \circ \mathrm{Lie}(\Psi_{\mathcal{L}^{(p)}}) = F^*_{A/S} \circ F_S^* \theta.$$

(iii). Let $a \in \mathcal{O}$, $b = \iota(\bar a)$, $(x,t) \in E^\natural(T)$. Restricted to $\{x\} \times A_T$, ϵ_a is an isomorphism of line bundles $\epsilon_{a,x} \colon M_{\iota(a)x} \to b^* M_x$ (as $\Lambda_{(x,by)} = (b^* M_x)_y$), equal to the identity at $y = e$ under the canonical trivialisations. It respects the extension structures: the defect between $\epsilon_{a,x}$ composed with the composition of $M_{\iota(a)x}$ and the composition of b^*M_x composed with $\epsilon_{a,x} \otimes \epsilon_{a,x}$ is a unit on $A_T \times_T A_T$ equal to 1 on $A_T \times \{e\}$, hence 1 by (U). Since b is a homomorphism, b^*M_x with the pulled-back composition is the extension $b^\# M_x^\times$, and $(1 \times b|_{A_1})^* t$ is the pulled-back rigidification. So $\epsilon_{a,x}$ is an isomorphism of rigidified extensions from $(M^\times_{\iota(a)x}, \epsilon_a^{-1}((1\times b|_{A_1})^* t))$ to $b^\#(M_x^\times, t)$, that is $\Psi_\mathcal{L}(a \cdot (x,t)) = \iota(\bar a)^\# \Psi_\mathcal{L}(x,t)$. Taking Lie algebras and using (N) for $u = \iota(\bar a)$: $\theta \circ \rho^\natural(a) = \mu_A \circ \mathrm{Lie}(\iota(\bar a)^\#) \circ \mathrm{Lie}(\Psi_\mathcal{L}) = \iota(\bar a)^* \circ \theta$. □

Lemma D.47. *Let p be a prime, S a scheme with $p\mathcal{O}_S = 0$ and absolute Frobenius F_S, $f \colon A \to S$ an abelian scheme, and $\mathcal{L}$ a rigidified line bundle on A with a cubical structure. Then $\langle \alpha(u), \alpha(u') \rangle = 0$ for all sections u, u' of $F_S^* \mathrm{Lie}(A)$.*

Proof. **Notation.** Put $A' = A^{(p)} = A \times_{S,F_S} S$ with the pulled-back rigidified line bundle $\mathcal{L}'$ and cubical structure, $\Lambda = \Lambda(\mathcal{L})$ on $A \times_S A$, $\Lambda' = \Lambda(\mathcal{L}')$ on $A' \times_S A'$, $F = F_{A/S} \colon A \to A'$, $V = V_{A/S} \colon A' \to A$, $E = E^\natural(A,\mathcal{L})$, $E' = E^\natural(A',\mathcal{L}') = E^{(p)}$ (Lemma D.43 (1)), and $\pi \colon E \to A$, $\pi' \colon E' \to A'$ the projections. Let c be the partial composition of Λ (first variable), ς the symmetry, $c^\mathrm{t} = \varsigma \circ c \circ (\varsigma \otimes \varsigma)$ the partial composition in the second variable, can the canonical trivialisations of $\Lambda_{(x,0)}$ and $\Lambda_{(0,y)}$ (they agree on $\Lambda_{(0,0)}$ and ς preserves them), and $c', \varsigma', c'^\mathrm{t}, \mathrm{can}'$ the same for Λ'. We use:

(B) Λ is a symmetric biextension of $A \times_S A$ by $\mathbb{G}_m$ (Moret-Bailly [181], Ch. I, 2.5): c, c^t are associative, commutative and compatible, and can is a unit for both; for $x \in A(T)$, $\Lambda^\times_{(x,\cdot)}$ with c^t is an extension of A_T by $\mathbb{G}_m$ with unit section can; likewise after pull-back along homomorphisms. Iteration gives isomorphisms $c_p \colon \Lambda^{\otimes p}_{(x,y)} \to \Lambda_{(px,y)}$ and $c_p^\mathrm{t} \colon \Lambda^{\otimes p}_{(x,y)} \to \Lambda_{(x,py)}$ mapping $\mathrm{can}^{\otimes p}$ to can.

(R) If $X \to S$ is an abelian scheme (here: a fibre product over S of copies of A and A') and T an S-scheme, a unit on X_T equal to 1 along the unit section is 1, because by (a) it is the pull-back of a unit on T.

(FV) Let X be a commutative group scheme flat over a scheme T with $p\mathcal{O}_T = 0$. Then $V_{X/T} \circ F_{X/T} = [p]_X$, $F_{X/T} \circ V_{X/T} = [p]_{X^{(p)}}$, and V is functorial in X (SGA 3, Exp. VII$_\mathrm{A}$, 4.3).

If $x \in X(R)$, R a T-algebra, reduces to the unit modulo an ideal J with $j^p = 0$ for all $j \in J$, then $F_{X/T}(x)$ is the unit, since $F_{X/T}(x)^*(g \otimes 1) = (x^*g)^p = 0$ for g in the ideal of the unit section and the $g \otimes 1$ generate the ideal of the unit section of $X^{(p)}$; hence $x^p = V_{X/T}(F_{X/T}(x))$ is the unit. The same computation shows that F maps A_1 (where $J = \mathcal{I}/\mathcal{I}^2$) into the unit section of A'.

(Fl) F is an epimorphism in the category of S-schemes, being faithfully flat and quasi-compact (fpqc descent of morphisms): it is finite and surjective (F^* makes $\mathcal{O}_A$ integral over $\mathcal{O}_{A'}$, and F is a universal homeomorphism), and flat by the fibrewise criterion of flatness (EGA IV_3, 11.3.10), since on each fibre it is a finite surjective morphism of regular schemes of the same dimension (EGA IV_2, 6.1.5).

Step 1: $(V \times 1)^*\Lambda \cong (1 \times F)^*\Lambda'$. Let $N_1 = (V \times 1)^*\Lambda$ and $N_2 = (1 \times F)^*\Lambda'$ on $A' \times_S A$. We show there is an isomorphism $\gamma\colon N_1 \to N_2$ of line bundles which maps can to can$'$ along $A' \times \{e\}$ and is compatible with the first partial compositions c (at (Vy_1, Vy_2, z)) and c' (at (y_1, y_2, Fz)). For $y \in A'(T)$, $N_{1,y} = \Lambda_{(Vy,\cdot)}$ with c^{t} and $N_{2,y} = F^*\Lambda'_{(y,\cdot)}$ with the pull-back of c'^{t} are extensions of A_T by $\mathbb{G}_m$ (B); by (b) their classes define morphisms $\xi_1, \xi_2\colon A' \to A^\vee$. For $x \in A(T)$: $N_{1,Fx} = \Lambda_{(px,\cdot)} \cong \Lambda^{\otimes p}_{(x,\cdot)}$ by $V \circ F = [p]_A$ and c_p; and $N_{2,Fx}$ is the restriction of $(F \times F)^*\Lambda' = \Lambda(F^*\mathcal{L}')$ (F is a homomorphism) to $\{x\} \times A_T$, where $F^*\mathcal{L}' \cong \mathcal{L}^{\otimes p}$ (the pull-back of $\mathcal{L}$ along the absolute Frobenius of A, by $\ell \mapsto \ell^{\otimes p}$), so $N_{2,Fx} \cong \Lambda^{\otimes p}_{(x,\cdot)}$ as well. By (b) the class of an extension depends only on its line bundle, so $\xi_1 \circ F = \xi_2 \circ F$, and $\xi_1 = \xi_2$ by (Fl). Hence (for $T = A'$, $y = \mathrm{id}$) the extensions N_1 and N_2 of $A_{A'}$ by $\mathbb{G}_m$ are isomorphic locally on A'. Two isomorphisms of extensions differ by a homomorphism $A_T \to \mathbb{G}_m$, a unit equal to 1 along the unit section, hence 1 by (R); so the local isomorphisms are unique and glue to an isomorphism of extensions $\gamma\colon N_1 \to N_2$. It maps the unit section can to the unit section can$'$ (note $F(e) = e$). The two isomorphisms $\gamma \circ c$ and $c' \circ (\gamma \otimes \gamma)$ from $N_{1,(y_1,z)} \otimes N_{1,(y_2,z)}$ to $N_{2,(y_1+y_2,z)}$ differ by a unit on $A' \times_S A' \times_S A$ that is 1 at the unit section (all maps send can to can$'$ there), hence 1 by (R).

Step 2: a homomorphism $\sigma\colon A' \to E$ lifting V. Let $y \in A'(T)$. By (FV), F maps $(A_T)_1$ into the unit section, so N_2 restricted to $\{y\} \times (A_T)_1$ is the pull-back of $\Lambda'_{(y,e)}$ and is trivialised by can$'$. Put $t_y = \gamma^{-1}(\mathrm{can}')$ there: a trivialisation of $\Lambda_{(Vy,\cdot)}$ on $T \times_S A_1$, equal to can at e (Step 1). So $\sigma(y) = (Vy, t_y) \in E(T)$; this is functorial in T, hence a morphism $\sigma\colon A' \to E$ with $\pi \circ \sigma = V$. It is a homomorphism: $t_{y_1} \cdot t_{y_2} = c(t_{y_1} \otimes t_{y_2}) = \gamma^{-1}(c'(\mathrm{can}' \otimes \mathrm{can}')) = \gamma^{-1}(\mathrm{can}') = t_{y_1+y_2}$ by Step 1 and (B).

Step 3: $V_E = \sigma \circ \pi'$, hence $\alpha = \mathrm{Lie}(\sigma)$. Both are homomorphisms $E' \to E$. By functoriality of V (FV) for π, $\pi \circ V_E = V \circ \pi' = \pi \circ \sigma \circ \pi'$. On $W(\omega_{A'}) = W(\omega_A)^{(p)} \subset E'$: V_E restricts to $i \circ V_{W(\omega_A)} = 0$ (i the inclusion; $V_{W(\omega_A)} \circ F_{W(\omega_A)/S} = [p] = 0$ with $F_{W(\omega_A)/S}$ faithfully flat, $W(\omega_A)$ being locally $\mathbb{G}_a^g$), and $\sigma \circ \pi'$ restricts to 0. So $d = V_E - \sigma \circ \pi'$ takes values in $\ker \pi = W(\omega_A)$ and vanishes on $W(\omega_{A'})$; as $E' \to A'$ is an epimorphism with kernel $W(\omega_{A'})$ (Lemma D.43 (1)), d factors through a homomorphism $A' \to W(\omega_A)$, which is 0 by (a). Thus $V_E = \sigma \circ \pi'$ and $\tilde{\alpha} = \mathrm{Lie}(V_E) = \mathrm{Lie}(\sigma) \circ \mathrm{Lie}(\pi')$. Since $\mathrm{Lie}(\pi')$ is the pull-back by F_S of the second Hodge map of $\mathrm{H}^\natural(A, \mathcal{L})$ (Lemma D.43 (2), base change), the map α of (4) of Lemma D.43 is $\mathrm{Lie}(\sigma)\colon \mathrm{Lie}(A') = F_S^*\mathrm{Lie}(A) \to \mathrm{H}^\natural(A, \mathcal{L})$. Concretely, for an open $U \subseteq S$ and $u \in \mathrm{Lie}(A')(U) = \ker(A'(U[\varepsilon]) \to A'(U))$, $\alpha(u) = \sigma(u) = (Vu, t_u)$.

Step 4: two isomorphisms $(V \times V)^*\Lambda \to \Lambda'^{\otimes p}$ agree. On $A' \times_S A'$ let $Q = (V \times V)^*\Lambda$, so $Q_{(y,y')} = \Lambda_{(Vy,Vy')}$, and note γ at (y, Vy') maps it to $\Lambda'_{(y,FVy')} = \Lambda'_{(y,py')}$ by (FV). Define

$$\Gamma_1 = (c'^{\mathrm{t}}_p)^{-1} \circ \gamma_{(y,Vy')}\colon Q_{(y,y')} \to \Lambda'^{\otimes p}_{(y,y')}, \qquad \Gamma_2 = \varsigma'^{\otimes p} \circ (c'^{\mathrm{t}}_p)^{-1} \circ \gamma_{(y',Vy)} \circ \varsigma\colon Q_{(y,y')} \to \Lambda'^{\otimes p}_{(y,y')},$$

where in Γ_2 the symmetry ς maps $\Lambda_{(Vy,Vy')}$ to $\Lambda_{(Vy',Vy)}$, γ maps this to $\Lambda'_{(y',py)}$, and $(c_p'^{\mathrm{t}})^{-1}$ maps that to $\Lambda'^{\otimes p}_{(y',y)}$. Both are isomorphisms of line bundles on $A' \times_S A'$, and at (e,e) both send can to $\mathrm{can}'^{\otimes p}$ (Step 1, (B), and ς, ς' preserve the canonical trivialisations). So $\Gamma_2 \circ \Gamma_1^{-1}$ is a unit on $A' \times_S A'$ equal to 1 at the unit section, and $\Gamma_1 = \Gamma_2 =: \Gamma$ by (R).

Step 5: p-th powers of first-order trivialisations. Let $U \subseteq S$ be an affine open, $R = \mathcal{O}(U)[\varepsilon,\varepsilon']$ with $\varepsilon^2 = \varepsilon'^2 = 0$, and $u \in A'(U[\varepsilon])$, $u' \in A'(U[\varepsilon'])$ reducing to e. Then $pu' = e$ and $pu = e$ by (FV) ($[p] = V \circ F$ and $F(u') = e$, with $J = (\varepsilon')$). The R-module $\Lambda'_{(u,u')}$ is invertible and reduces modulo the nilpotent ideal $(\varepsilon,\varepsilon')$ to $\Lambda'_{(e,e)} \cong \mathcal{O}(U)$, so it is free, and every lift of a generator modulo (ε') (resp. (ε)) is a generator.

(P1) If w generates $\Lambda'_{(u,u')}$ and $w \equiv \mathrm{can}'_{(u,e)}$ modulo ε', then $c_p'^{\mathrm{t}}(w^{\otimes p}) = \mathrm{can}'_{(u,e)}$. Indeed, let X be the $\mathbb{G}_m$-torsor of $\Lambda'_{(u,\cdot)}$ on $A'_{U[\varepsilon]}$, a commutative group scheme over $U[\varepsilon]$ under c'^{t} with unit $\mathrm{can}'_{(u,e)}$ (B), smooth as a $\mathbb{G}_m$-torsor over an abelian scheme. The point $x = (u', w) \in X(R)$ reduces to the unit modulo $J = (\varepsilon')$, and $j^p = 0$ for $j \in J$; by (FV) x^p is the unit, and $x^p = (pu', c_p'^{\mathrm{t}}(w^{\otimes p}))$.

(P2) Symmetrically (exchange ε and ε', u and u'), if $\tilde{w}$ generates $\Lambda'_{(u',u)}$ and $\tilde{w} \equiv \mathrm{can}'_{(u',e)}$ modulo ε, then $c_p'^{\mathrm{t}}(\tilde{w}^{\otimes p}) = \mathrm{can}'_{(u',e)}$.

(P3) Two generators w_1, w_2 of $\Lambda'_{(u,u')}$ that both reduce to $\mathrm{can}'_{(e,e)}$ modulo $(\varepsilon,\varepsilon')$ have $w_1^{\otimes p} = w_2^{\otimes p}$: $w_2 = \zeta w_1$ with $\zeta = 1 + \varepsilon a + \varepsilon' b + \varepsilon\varepsilon' c$, and $\zeta^p = 1$, since $p\mathcal{O}_U = 0$, $(\varepsilon a + \varepsilon' b + \varepsilon\varepsilon' c)^2 = 2ab\,\varepsilon\varepsilon'$ and the cube is 0 (for $p = 2$, $\zeta^2 = 1 + 2ab\,\varepsilon\varepsilon' = 1$).

The generators w of (P1) and $\varsigma'(\tilde{w})$, $\tilde{w}$ as in (P2), both reduce to $\mathrm{can}'_{(e,e)}$ modulo $(\varepsilon,\varepsilon')$, so by (P3)

$$\text{(P4)} \qquad w^{\otimes p} = \varsigma'(\tilde{w})^{\otimes p} = \varsigma'^{\otimes p}(\tilde{w}^{\otimes p}).$$

Step 6: conclusion. Let u, u' be sections of $F_S^*\mathrm{Lie}(A) = \mathrm{Lie}(A')$ over U, read as in Step 5, and put $h = \alpha(u) = (v, t_u)$, $h' = \alpha(u') = (v', t_{u'})$ with $v = Vu$, $v' = Vu'$ (Step 3); v' is a point of A_1 with $Fv' = FVu' = pu' = e$, and likewise $Fv = e$. By the definition of t_u (Step 2), $\gamma(t_u(v')) = \mathrm{can}'_{(u,e)}$, so by (P1)

$$\Gamma_1\big(t_u(v')\big) = (c_p'^{\mathrm{t}})^{-1}(\mathrm{can}'_{(u,e)}) = w^{\otimes p}.$$

Likewise $\gamma(t_{u'}(v)) = \mathrm{can}'_{(u',e)}$, so by (P2) and (P4)

$$\Gamma_2\big(\varsigma(t_{u'}(v))\big) = \varsigma'^{\otimes p}\big((c_p'^{\mathrm{t}})^{-1}(\mathrm{can}'_{(u',e)})\big) = \varsigma'^{\otimes p}(\tilde{w}^{\otimes p}) = w^{\otimes p}.$$

As $\Gamma_1 = \Gamma_2$ is an isomorphism (Step 4), $\varsigma(t_{u'}(v)) = t_u(v')$ in $\Lambda_{(v,v')}$. This is (D.10) with $\langle h, h'\rangle = 0$. The pairing is $\mathcal{O}_S$-bilinear and local on S, so $\langle \alpha(u), \alpha(u')\rangle = 0$ for all sections u, u'. □

Lemma D.48. *Let p be a prime, S a scheme with $p\mathcal{O}_S = 0$ and absolute Frobenius F_S, $f\colon A \to S$ an abelian scheme, and $F_{A/S}\colon A \to A^{(p)} = A \times_{S,F_S} S$ the relative Frobenius. Identify $H^1_{\mathrm{dR}}(A^{(p)}/S) = F_S^* H^1_{\mathrm{dR}}(A/S)$, $\omega_{A^{(p)}/S} = F_S^*\omega_{A/S}$ and $R^1 f_*^{(p)}\mathcal{O}_{A^{(p)}} = F_S^* R^1 f_*\mathcal{O}_A$ by base change. Then the pull-back*

$$F_{A/S}^*\colon F_S^* H^1_{\mathrm{dR}}(A/S) \to H^1_{\mathrm{dR}}(A/S)$$

is the composite of the Hodge projection $F_S^* H^1_{\mathrm{dR}}(A/S) \to F_S^* R^1 f_*\mathcal{O}_A$ (the second map of (ii) for $A^{(p)}$) and the first map $F_S^* R^1 f_*\mathcal{O}_A \to H^1_{\mathrm{dR}}(A/S)$ of the conjugate sequence (iii). In particular $\ker F_{A/S}^* = F_S^*\omega_{A/S}$, and $F_{A/S}^*$ maps $F_S^* R^1 f_*\mathcal{O}_A$ isomorphically onto the first term of the conjugate sequence.

Proof. Write $\pi\colon A^{(p)} \to S$ and $F = F_{A/S}$, a finite (hence affine) morphism. The complex $F_*\Omega^\bullet_{A/S}$ is a complex of $\mathcal{O}_{A^{(p)}}$-modules with $\mathcal{O}_{A^{(p)}}$-linear differential, and pull-back of forms is a map of complexes $F^*\colon \Omega^\bullet_{A^{(p)}/S} \to F_*\Omega^\bullet_{A/S}$ computing $F^*_{A/S}$ on H^1_{dR} after $R^1\pi_*$ (as F is affine, $R^1\pi_*(F_*\Omega^\bullet_{A/S}) = H^1_{\mathrm{dR}}(A/S)$).

F^* factors through $\mathcal{O}[0]$. Locally $\Omega^1_{A^{(p)}/S}$ is generated by the differentials $d(g\otimes 1)$ of functions g on A, and $F^*(g\otimes 1) = g^p$, so $F^*d(g\otimes 1) = d(g^p) = pg^{p-1}dg = 0$. Hence F^* vanishes on $\Omega^{\geq 1}$ and factors as

$$\Omega^\bullet_{A^{(p)}/S} \longrightarrow \mathcal{O}_{A^{(p)}}[0] \xrightarrow{c} F_*\Omega^\bullet_{A/S}, \qquad c(h) = F^*h,$$

where the first arrow is the quotient by the subcomplex $\Omega^{\geq 1}$ and c is a map of complexes because $d(F^*h) = 0$ (as just shown, F^*h is locally a sum of p-th powers times functions of S).

The two factors. Applying $R^1\pi_*$, the first arrow gives the Hodge projection $H^1_{\mathrm{dR}}(A^{(p)}/S) \to R^1\pi_*\mathcal{O}_{A^{(p)}}$, which by (ii) and (i) for $A^{(p)}$ is surjective with kernel $\omega_{A^{(p)}/S} = F_S^*\omega_{A/S}$. For the second, c identifies $\mathcal{O}_{A^{(p)}}$ with the sheaf $\mathcal{H}^0(F_*\Omega^\bullet_{A/S})$ (the Cartier isomorphism in degree 0, Katz, Nilpotent connections and the monodromy theorem, Theorem 7.2), and the conjugate spectral sequence of Katz, Theorem 7.4, is the spectral sequence of hypercohomology of $F_*\Omega^\bullet_{A/S}$ for its canonical filtration, $E_2^{a,b} = R^a\pi_*\mathcal{H}^b(F_*\Omega^\bullet_{A/S})$; its edge map $E_2^{1,0} \to H^1_{\mathrm{dR}}(A/S)$ is induced by the inclusion $\mathcal{H}^0[0] = \tau_{\leq 0}F_*\Omega^\bullet \to F_*\Omega^\bullet$, that is, by c. By (iii) of Theorem B.124 this edge map $F_S^*R^1f_*\mathcal{O}_A = R^1\pi_*\mathcal{O}_{A^{(p)}} \to H^1_{\mathrm{dR}}(A/S)$ is the injective first map of the conjugate sequence.

Conclusion. $F^*_{A/S}$ is the surjection with kernel $F_S^*\omega_{A/S}$ followed by the injection onto the first term of the conjugate sequence; both claims follow. □

Lemma D.49. *Let E be a number field with an automorphism $a \mapsto \bar a$ of order dividing 2, V a discrete valuation ring with valuation v, maximal ideal $\mathfrak{m}$, fraction field k of characteristic 0 and residue field of characteristic $\ell > 0$, such that there are $[E:\mathbb{Q}]$ field embeddings $E \to k$ and $s(\mathcal{O}_E) \subseteq V$ for each of them. For an embedding $\sigma\colon E \to k$ put $\bar\sigma(a) = \sigma(\bar a)$, let S_σ be the set of embeddings s with $s(a) - \sigma(a) \in \mathfrak{m}$ for all $a \in \mathcal{O}_E$, and let $\varpi_\sigma \in \mathcal{O}_E$ be a uniformiser of $\mathcal{O}_E$ at the prime $\mathcal{O}_E \cap \sigma^{-1}(\mathfrak{m})$. Let L be a free V-module of finite rank with a ring homomorphism $\rho_L\colon \mathcal{O}_E \to \mathrm{End}_V(L)$, $r_s = \dim_k\{x \in L\otimes k :\ \rho_L(a)x = s(a)x\ \forall a\}$ for each embedding s, and $L^\vee = \mathrm{Hom}_V(L,V)$ with a acting by $\rho_L(\bar a)^\vee$. Let N be a module over $\mathcal{O}_E\otimes_{\mathbb{Z}} V$, free of rank n, with an $\mathcal{O}_E$-equivariant exact sequence $0 \to L^\vee \to N \to L \to 0$ of V-modules. For a V-module M with $\mathcal{O}_E$-action write $M_\sigma = M/\sum_a(a - \sigma(a))M$ and $M[\sigma] = \{x \in M :\ ax = \sigma(a)x\ \forall a\}$. Let $v_1,\dots,v_p$ be a V-basis of $L[\sigma]$, $w_1,\dots,w_q$ a V-basis of $L[\bar\sigma]$, $w_1^*,\dots,w_q^*$ the dual basis of $(L[\bar\sigma]\otimes k)^\vee$, and $\tilde v_i \in N_\sigma \otimes k$ lifts of v_i; regard $w_j^* \in N_\sigma\otimes k$ through the exact sequence*

$$0 \to (L[\bar\sigma]\otimes k)^\vee \to N_\sigma \otimes k \to L[\sigma]\otimes k \to 0$$

induced by the given one (the first map: restriction of functionals, inverted). Then N_σ is free of rank $p+q = n$ over V, and for a V-basis b of N_σ the element $\delta \in k^\times$ with $w_1^*\wedge\cdots\wedge w_q^*\wedge\tilde v_1\wedge\cdots\wedge\tilde v_p = \delta\, b_1\wedge\cdots\wedge b_n$ satisfies

$$v(\delta) = \sum_{s\in S_\sigma,\ s\neq\sigma} r_s\, v\big(s(\varpi_\sigma) - \sigma(\varpi_\sigma)\big).$$

In particular $v(\delta) = 0$ if the prime $\mathcal{O}_E\cap\sigma^{-1}(\mathfrak{m})$ is unramified over ℓ.

Proof. **Conventions.** A *lattice* in a finite-dimensional k-space W is a finitely generated V-submodule spanning W; it is free (V is a principal ideal domain). For lattices Λ,Λ' in W let

$[\Lambda' : \Lambda] = v(\det g)$, g the matrix of a basis of Λ in a basis of Λ'; it is independent of the bases, $[\Lambda'' : \Lambda] = [\Lambda'' : \Lambda'] + [\Lambda' : \Lambda]$, $[\Lambda' : \Lambda]$ is the length of Λ'/Λ if $\Lambda \subseteq \Lambda'$, and if $W_1 \subseteq W$ is a subspace with $\Lambda \cap W_1$, $\Lambda' \cap W_1$ lattices in W_1 then $[\Lambda' : \Lambda] = [\Lambda' \cap W_1 : \Lambda \cap W_1] + [\bar{\Lambda}' : \bar{\Lambda}]$, bars denoting images in W/W_1 (block-triangular matrices). For a finitely generated V-module Q let $t(Q)$ be the length of its torsion submodule.

Step 1: reduction to complete V. Let $\hat{V}$ be the completion, $\hat{k}$ its fraction field. $V \to \hat{V}$ is faithfully flat, so base change preserves the freeness of N over $\mathcal{O}_E \otimes \hat{V}$, the exact sequence, the kernels $L[\sigma]$, $L[\bar{\sigma}]$ (kernels commute with flat base change), the quotient N_σ (a cokernel), the dimensions r_s, δ, v, $\mathfrak{m}$-congruences and S_σ. So we may assume V complete.

Step 2: the factor of σ. Let $\mathfrak{p} = \mathcal{O}_E \cap \sigma^{-1}(\mathfrak{m})$, $E_\mathfrak{p}$ the completion, $O_0 \subseteq \mathcal{O}_{E_\mathfrak{p}}$ the ring of integers of its maximal unramified subextension, of degree f over $\mathbb{Z}_\ell$, e the ramification index, and ζ a root of unity of order prime to ℓ with $O_0 = \mathbb{Z}_\ell[\zeta]$. *The embeddings.* An embedding $s\colon E \to k$ maps $\mathcal{O}_E$ into V, so it is continuous for the $\mathfrak{p}_s$-adic topology, $\mathfrak{p}_s = \mathcal{O}_E \cap s^{-1}(\mathfrak{m})$, and extends ($k$ is complete) to a $\mathbb{Q}_\ell$-embedding $E_{\mathfrak{p}_s} \to k$; distinct s give distinct extensions, E being dense in $E_{\mathfrak{p}_s}$. Since $E_{\mathfrak{p}'}$ has at most $[E_{\mathfrak{p}'} : \mathbb{Q}_\ell]$ embeddings into k over $\mathbb{Q}_\ell$ and $\sum_{\mathfrak{p}'|\ell} [E_{\mathfrak{p}'} : \mathbb{Q}_\ell] = [E : \mathbb{Q}]$ is the number of the s, every $\mathbb{Q}_\ell$-embedding $E_\mathfrak{p} \to k$ comes from exactly one s, and there are ef of them. As $\mathbb{Q}_\ell(\zeta)$ has at most f embeddings into k and each has at most $[E_\mathfrak{p} : \mathbb{Q}_\ell(\zeta)] = e$ extensions to $E_\mathfrak{p}$, each of the f embeddings of $\mathbb{Q}_\ell(\zeta)$ occurs, with exactly e extensions; these take e distinct values at ϖ_σ, an embedding of $E_\mathfrak{p} = \mathbb{Q}_\ell(\zeta, \varpi_\sigma)$ being determined by its values at ζ and ϖ_σ. All these values are integral over $\mathbb{Z}_\ell$, hence lie in V. Then $\mathcal{O}_E \otimes_\mathbb{Z} V = \prod_{\mathfrak{p}'|\ell} \mathcal{O}_{E_{\mathfrak{p}'}} \otimes_{\mathbb{Z}_\ell} V$ (V is a complete $\mathbb{Z}_\ell$-algebra); the minimal polynomial g of ζ over $\mathbb{Z}_\ell$ is separable modulo ℓ; its roots in V (images of ζ under the embeddings) are distinct modulo $\mathfrak{m}$, so $O_0 \otimes V = V[x]/(g) \cong V^f$ by the Chinese remainder theorem. As $\mathcal{O}_{E_\mathfrak{p}} = O_0[\varpi_\sigma]$ with ϖ_σ a root of an Eisenstein polynomial over O_0 of degree e, the factor R_σ of $\mathcal{O}_E \otimes V$ on which $\mu_\sigma\colon a \otimes c \mapsto \sigma(a)c$ is non-zero is $R_\sigma = V[t]/(P)$, $t = \varpi_\sigma \otimes 1$, $P = \prod_{s \in S}(t - s(\varpi_\sigma))$, where S is the set of the e embeddings s inducing $\mathfrak{p}$ and the same embedding of O_0 as σ; these are exactly the s with $s(a) \equiv \sigma(a) \bmod \mathfrak{m}$ for all $a \in \mathcal{O}_E$ (same prime and same map on the residue field O_0/ℓ), so $S = S_\sigma$, and the roots $s(\varpi_\sigma)$ are distinct. Let $\varepsilon \in \mathcal{O}_E \otimes V$ be the idempotent of R_σ and $\pi_s = s(\varpi_\sigma)$. Then $\ker \mu_\sigma$ is generated by $1 - \varepsilon$ and $\varepsilon(t - \pi_\sigma)$.

Step 3: a formula for one module. Let M be a free V-module of finite rank with an $\mathcal{O}_E$-action, $d_s = \dim_k(M \otimes k)[s]$, $M' = \varepsilon M$, $T = t - \pi_\sigma$ on M'. By Step 2, $M_\sigma = M'/TM'$. On $M'_k = M' \otimes k$, t acts semisimply with eigenvalues π_s ($s \in S_\sigma$, distinct roots of P) and $M'_k = \bigoplus_{s \in S_\sigma} M_k[s]$; so $M[\sigma] = \ker(T|_{M'})$. Let e_σ be the projection of M'_k onto $M_k[\sigma]$ along $Y = \bigoplus_{s \neq \sigma} M_k[s] = T(M'_k)$, $e_\sigma M = e_\sigma(M')$, $A = M[\sigma]$, $B = M' \cap Y$. We claim

$$[e_\sigma M : A] + t(M_\sigma) = \sum_{s \in S_\sigma,\ s \neq \sigma} d_s\, v(\pi_s - \pi_\sigma). \tag{$*$}$$

The map $M' \to e_\sigma M$, $m \mapsto e_\sigma m$, is onto with kernel B, and m maps into A if and only if $m - e_\sigma m \in M' \cap Y = B$; so $e_\sigma M/A \cong M'/(A \oplus B)$. Next $TM' \subseteq B$, M'/B is torsion-free of rank $d_\sigma = \operatorname{rank} M'/TM'$, so the torsion of $M_\sigma = M'/TM'$ is B/TM'. T maps M' onto TM' with kernel A and is injective on B, so $M'/(A \oplus B) \cong TM'/TB$. Hence the left side of ($*$) is $\operatorname{length}(TM'/TB) + \operatorname{length}(B/TM') = [B : TB] = v(\det(T|_Y)) = \sum_{s \neq \sigma} d_s v(\pi_s - \pi_\sigma)$.

Step 4: N_σ. $N \cong (\mathcal{O}_E \otimes V)^n$ and $(\mathcal{O}_E \otimes V)_\sigma = V$, so $N_\sigma \cong V^n$; it has no torsion, so $N_\sigma \to N_\sigma \otimes k = e_\sigma N_k$ is injective with image $e_\sigma N$. Write $K \subseteq N$ for the image of $L^\vee$, so $K_k \cong L_k^\vee$ and $L = N/K$. The exact sequence of the statement is e_σ applied to $0 \to K_k \to N_k \to L_k \to 0$: $e_\sigma L_k = L_k[\sigma]$, and a acts on $K_k = L_k^\vee$ by transposes of $\rho_L(\bar{a})$, so $e_\sigma K_k$ consists of the functionals supported on $L_k[\bar{\sigma}]$ (on $L_k[s]$, $\rho_L(\bar{a}) = s(\bar{a})$, and $s(\bar{a}) = \sigma(a)$ for all a if and only if $s = \bar{\sigma}$), i.e. $e_\sigma K_k \cong L_k[\bar{\sigma}]^\vee$ by restriction. Rank count: $n = r_\sigma + r_{\bar{\sigma}} = p + q$.

Step 5: the index. Let Λ be the lattice spanned by w^* and $\tilde{v}$ in $e_\sigma N_k$; then $v(\delta) = [e_\sigma N : \Lambda]$, independent of the lifts $\tilde{v}_i$ (another choice changes the basis by a unipotent matrix). Filter by $W_1 = e_\sigma K_k$. Images in $e_\sigma N_k/W_1 = L_k[\sigma]$: that of Λ is $L[\sigma]$, that of $e_\sigma N$ is $e_\sigma L$ (projection commutes with $N \to L$). Intersections with W_1: $\Lambda \cap W_1$ is spanned by w^*, which is $e_\sigma K$: restriction to the direct summand $L[\bar{\sigma}] = L \cap L_k[\bar{\sigma}]$ of L maps $L^\vee$ onto $L[\bar{\sigma}]^\vee$. And $e_\sigma N \cap W_1 = K'$, the kernel of $e_\sigma N \to e_\sigma L$. Hence

$$v(\delta) = [K' : e_\sigma K] + [e_\sigma L : L[\sigma]].$$

Right exactness of $M \mapsto M_\sigma$ gives the exact sequence $K_\sigma \to N_\sigma \to L_\sigma \to 0$; the image of K_σ in $N_\sigma = e_\sigma N$ is $e_\sigma K$, and $e_\sigma N \to e_\sigma L$ is $N_\sigma \to L_\sigma$ followed by the quotient by the torsion of L_σ. So K' is the preimage of the torsion of L_σ, and $K'/e_\sigma K$ is isomorphic to that torsion: $[K' : e_\sigma K] = t(L_\sigma)$. By $(*)$ for $M = L$, $v(\delta) = t(L_\sigma) + [e_\sigma L : L[\sigma]] = \sum_{s \in S_\sigma, s \neq \sigma} r_s \, v(\pi_s - \pi_\sigma)$. If $\mathfrak{p}$ is unramified over ℓ, $e = 1$, $S_\sigma = \{\sigma\}$ and the sum is empty. □

Lemma D.50. *Let $\mathcal{O}$ be a discrete valuation ring with uniformiser ϖ, fraction field K and residue field κ, and let X be a normal integral scheme with a proper morphism $X \to \operatorname{Spec} \mathcal{O}$ whose image contains the generic point; put $X_K = X \times_{\mathcal{O}} K$ and $X_s = X \times_{\mathcal{O}} \kappa$, and suppose that X_s is connected. Let $\mathcal{L}$ be an invertible sheaf on X of degree 0 on every curve of X_s: for every integral closed subscheme $C \subseteq X_s$ of dimension one, with normalisation $\nu\colon \tilde{C} \to C$ (a normal integral curve, proper over κ), $\deg \nu^* \mathcal{L} = 0$; this holds for instance if $\mathcal{L}|_{X_s} \cong \mathcal{O}_{X_s}$. Let σ be a non-zero rational section of $\mathcal{L}$ (an element of the stalk at the generic point) that generates $\mathcal{L}$ at every point of X_K. Then there are integers $M \geq 1$ and a such that $\sigma^M \varpi^{-a}$ generates $\mathcal{L}^{\otimes M}$ at every point of X.*

Proof. **Step 1: the components of the special fibre.** X is of finite type over the noetherian ring $\mathcal{O}$, hence noetherian. As X is integral and dominates $\operatorname{Spec} \mathcal{O}$, ϖ is a non-zero element of the function field, and X_s is the closed subset $V(\varpi)$. If $X_s = \varnothing$, every point lies in X_K and $M = 1$, $a = 0$ will do; assume $X_s \neq \varnothing$. Let $X_1, \dots, X_t$ be the irreducible components of X_s (finitely many), with generic points $\xi_1, \dots, \xi_t$. By Krull's principal ideal theorem each ξ_i has codimension one in X, so $\mathcal{O}_{X,\xi_i}$ is a normal noetherian local domain of dimension one, a discrete valuation ring with valuation v_i. Trivialise $\mathcal{L}$ at ξ_i and let $n_i \in \mathbb{Z}$ be the valuation of σ in this trivialisation (independent of the choice), and $m_i = v_i(\varpi) \geq 1$.

Step 2: the choice of M and a. Choose j with $n_j/m_j = \min_i n_i/m_i$, and put $M = m_1 \cdots m_t$ and $a = n_j \prod_{i \neq j} m_i$, so that $a/M = n_j/m_j$. Let $\theta = \sigma^M \varpi^{-a}$, a rational section of $\mathcal{L}^{\otimes M}$. Its order at ξ_i is $M n_i - a m_i = M m_i (\frac{n_i}{m_i} - \frac{n_j}{m_j}) \geq 0$, with equality for $i = j$. Every point of codimension one of X either lies in X_K, where σ generates and ϖ is a unit, so that θ has order 0 there, or is one of the ξ_i. Hence θ has non-negative order at every point of codimension one. Since X is normal and noetherian, a rational function with non-negative order at every point of codimension one through x lies in $\mathcal{O}_{X,x}$ (the local ring is the intersection of its localisations at height-one primes); applied in local trivialisations, θ is a global section of $\mathcal{L}^{\otimes M}$, and it generates $\mathcal{L}^{\otimes M}$ at x as soon as its order is 0 at every point of codimension one through x.

Step 3: all orders vanish. Let I be the set of indices i such that the order of θ at ξ_i is positive. It therefore suffices to show that I is empty. Put $Z = \bigcup_{i \in I} X_i$. First, θ fails to generate $\mathcal{L}^{\otimes M}$ exactly at the points of Z. Indeed, let $y \in X_i$ with $i \in I$; in a trivialisation of $\mathcal{L}^{\otimes M}$ near y, θ is an element $h \in \mathcal{O}_{X,y}$ with $v_i(h) > 0$; as ξ_i specialises to y, $\mathcal{O}_{X,\xi_i}$ is the localisation of $\mathcal{O}_{X,y}$ at the prime $\mathfrak{p}_i$ of ξ_i, and $v_i(h) > 0$ means $h \in \mathfrak{p}_i \mathcal{O}_{X,\xi_i} \cap \mathcal{O}_{X,y} = \mathfrak{p}_i \subseteq \mathfrak{m}_y$, so θ vanishes at y. If $y \notin Z$, the points of codimension one through y are points of X_K and generic points $\xi_{i'}$ with $i' \notin I$, at which θ has order 0, so θ generates at y by Step 2.

Suppose now $I \neq \varnothing$. Its complement J contains j. The closed sets Z and $Z' = \bigcup_{i' \in J} X_{i'}$ are non-empty and cover the connected space X_s, so $Z \cap Z'$ is a non-empty closed subset of X_s; as X_s is of finite type over κ, $Z \cap Z'$ contains a closed point x of X_s, and $x \in X_{i'}$ for some $i' \in J$.

A curve through x not inside Z. Give $X_{i'}$ its reduced structure, an integral scheme of finite type over κ in which x is a closed point, and let $A = \mathcal{O}_{X_{i'},x}$, a noetherian local domain with maximal ideal $\mathfrak{m}$, and $\mathfrak{a} \subseteq A$ the ideal of the reduced closed subscheme $Z \cap X_{i'}$ at x. The generic point $\xi_{i'}$ is not in Z (the X_i are distinct irreducible components and $i' \notin I$), so $\mathfrak{a} \neq 0$; choose $0 \neq g \in \mathfrak{a}$; then $g \in \mathfrak{m}$ as $x \in Z$, and $d = \dim A \geq 1$ as $x \neq \xi_{i'}$. Since A is a domain and $g \neq 0$, every chain of primes containing g extends by the prime 0, so $\dim A/gA \leq d-1$, and $\dim A/gA \geq d-1$ by Krull's height theorem; hence $\dim A/gA = d-1$. Choose $g_1, \ldots, g_{d-1} \in \mathfrak{m}$ whose images form a system of parameters of A/gA, and put $\mathfrak{b} = (g_1, \ldots, g_{d-1})$; then $\dim A/(\mathfrak{b}+gA) = 0$. By Krull's height theorem $\dim A/\mathfrak{b} \geq d-(d-1) = 1$, and $\dim A/\mathfrak{b} \leq \dim A/(\mathfrak{b}+gA)+1 = 1$ (again by Krull's principal ideal theorem, now in $A/\mathfrak{b}$). Let P be a prime of A minimal over $\mathfrak{b}$ with $\dim A/P = 1$. If $g \in P$, then $P \supseteq \mathfrak{b} + gA$ and $\dim A/P \leq 0$; so $g \notin P$. Let C be the closure in $X_{i'}$ of the point corresponding to P, with its reduced structure: an integral closed subscheme of X_s containing x, with $\dim \mathcal{O}_{C,x} = \dim A/P = 1$, hence of dimension one (C is integral of finite type over κ and x is a closed point). Its generic point η_C is not in Z, because $g \in \mathfrak{a}$ and $g \notin P$.

The degree. Let $\nu\colon \tilde{C} \to C$ be the normalisation, finite and surjective (C is of finite type over a field), and θ_C the pull-back of θ, a global section of $\nu^*\mathcal{L}^{\otimes M}$. At the generic point of $\tilde{C}$, which maps to $\eta_C \notin Z$, θ_C generates; so $\theta_C \neq 0$ and its divisor is effective. Let $\tilde{x} \in \nu^{-1}(x)$, a closed point of $\tilde{C}$; as θ vanishes at $x \in Z$, θ_C vanishes at $\tilde{x}$. Hence $M \deg \nu^*\mathcal{L} = \deg \nu^*\mathcal{L}^{\otimes M} \geq [\kappa(\tilde{x}) : \kappa] \operatorname{ord}_{\tilde{x}}(\theta_C) > 0$, contradicting $\deg \nu^*\mathcal{L} = 0$. (If $\mathcal{L}|_{X_s} \cong \mathcal{O}_{X_s}$, then $\nu^*\mathcal{L} \cong \mathcal{O}_{\tilde{C}}$ has degree 0, which is the instance named in the statement.) Hence $I = \varnothing$, every order of θ at a point of codimension one is 0, and by Step 2 $\theta = \sigma^M \varpi^{-a}$ generates $\mathcal{L}^{\otimes M}$ everywhere. □

Corollary D.51. *Let E be a CM field, $\mathcal{X}$ a normal integral noetherian scheme and $(G, \iota, \mathcal{L})$ a semiabelian structure over $\mathcal{X}$ with abelian generic fibre satisfying (PF). Let p be a prime, κ a field of characteristic p, C an integral normal projective curve over κ, $c\colon C \to \mathcal{X}$ a morphism and $H = c^*\mathrm{H}^\natural(G, \mathcal{L})$ with $\rho_H = c^*\rho^\natural$. Then:*

(i) for every finite field $k \subseteq \Gamma(C, \mathcal{O}_C)$ and every idempotent e of $k \otimes_{\mathbb{Z}} \mathcal{O}_E$ we have $\deg eH = 0$;

(ii) for every ring homomorphism $\sigma\colon \mathcal{O}_E \to \Gamma(C, \mathcal{O}_C)$ the quotient $H_\sigma = H/\sum_a (\rho_H(a) - \sigma(a))H$ is a vector bundle on C with $\deg H_\sigma = 0$.

Proof. By Corollary D.42 applied to c, with $L = c^*\mathrm{Lie}(G)$ and $L^\vee = c^*\omega_G$ (actions as there), (i) and (ii) there give $\mathcal{O}_E$-equivariant exact sequences $0 \to L^\vee \to H \to L \to 0$ and $0 \to F^*L \to H \to F^*(L^\vee) \to 0$, the outer terms of the second carrying F^* of the actions on L and $L^\vee$: this is hypothesis (a) of Lemma D.52 with $K = E$, $M_0 = L^\vee$, $M_1 = L$. Item (iii) there is hypothesis (b). So (i) and (ii) of Lemma D.52 give the two claims. □

Lemma D.52. *Let p be a prime, κ a field of characteristic p, C an integral normal projective curve over κ with absolute Frobenius $F\colon C \to C$, and K a number field with ring of integers $\mathcal{O}_K$. For a vector bundle V on C, an $\mathcal{O}_K$-action on V is a ring homomorphism $\rho_V\colon \mathcal{O}_K \to \mathrm{End}_{\mathcal{O}_C}(V)$; then F^*V carries the $\mathcal{O}_K$-action $a \mapsto F^*(\rho_V(a))$. Let H, M_0, M_1 be vector bundles on C with $\mathcal{O}_K$-actions such that*

(a) there are $\mathcal{O}_K$-equivariant exact sequences of $\mathcal{O}_C$-modules

$$0 \to M_0 \to H \to M_1 \to 0, \qquad 0 \to F^*M_1 \to H \to F^*M_0 \to 0.$$

Then:

(i) for every finite field $k \subseteq \Gamma(C, \mathcal{O}_C)$ and every idempotent e of $k \otimes_{\mathbb{Z}} \mathcal{O}_K$, acting on H by $c \otimes a \mapsto c\,\rho_H(a)$, the image eH is a vector bundle and $\deg eH = 0$;

(ii) if moreover

(b) H is locally free as a module over $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathcal{O}_C$,

then for every ring homomorphism $\sigma\colon \mathcal{O}_K \to \Gamma(C, \mathcal{O}_C)$ the quotient

$$H_\sigma = H \Big/ \sum_{a \in \mathcal{O}_K} (\rho_H(a) - \sigma(a))H$$

is a vector bundle on C and $\deg H_\sigma = 0$.

Proof. Degrees are taken over κ; recall $\deg W = \deg \det W$ and that deg is additive in short exact sequences of vector bundles.

The twisted algebra. Let $k \subseteq \Gamma(C, \mathcal{O}_C)$ be a finite field, $|k| = p^f$, and let $A = k \otimes_{\mathbb{Z}} \mathcal{O}_K$, a finite commutative k-algebra, and $\varphi = \mathrm{Frob}_k \otimes \mathrm{id}$, a ring automorphism of A with $\varphi^f = \mathrm{id}$. For a vector bundle V with $\mathcal{O}_K$-action, A acts on V by $\rho_V(c \otimes a) = c\,\rho_V(a)$ ($c \in k$ acting as a global function). The maps in (a) are $\mathcal{O}_C$-linear and $\mathcal{O}_K$-equivariant, hence A-equivariant.

Frobenius twist of the action. For V with $\mathcal{O}_K$-action and $y \in A$,

$$\rho_{F^*V}(\varphi(y)) = F^*(\rho_V(y)). \tag{D.11}$$

Both sides are additive in y; for $y = c \otimes a$, F^* of multiplication by the function c is multiplication by c^p, so $F^*(c\rho_V(a)) = c^p F^*(\rho_V(a)) = \rho_{F^*V}(c^p \otimes a)$.

Images of idempotents. For an idempotent $e \in A$ put $eV = \rho_V(e)V$. Then $V = eV \oplus (1-e)V$, so eV is a vector bundle; an A-equivariant exact sequence is the direct sum of its e-part and its $(1-e)$-part, so its e-part $0 \to eV' \to eV \to eV'' \to 0$ is exact. Applying F^* to $V = eV \oplus (1-e)V$ and using (D.11), $F^*(eV) = \varphi(e)F^*V$.

Degree zero of the idempotent parts. For every vector bundle W, $\det F^*W = F^* \det W \cong (\det W)^{\otimes p}$, so $\deg F^*W = p \deg W$. Let $e \in A$ be idempotent; so is $\varphi(e)$. The e-part of the first sequence of (a) and the $\varphi(e)$-part of the second give

$$\deg eH = \deg eM_0 + \deg eM_1, \qquad \deg \varphi(e)H = \deg F^*(eM_1) + \deg F^*(eM_0) = p \deg eH.$$

By induction $\deg \varphi^j(e)H = p^j \deg eH$ for all $j \geq 0$; for $j = f$, $\varphi^f(e) = e$ gives $(p^f - 1)\deg eH = 0$, so

$$\deg eH = 0 \quad \text{for every idempotent } e \in A. \tag{D.12}$$

This is (i).

From the idempotent to the σ-quotient: proof of (ii). The ring $\Gamma(C, \mathcal{O}_C)$ is a field (C is integral and proper over κ), so $\sigma(\mathcal{O}_K) \cong \mathcal{O}_K / \ker \sigma$ is a domain; $p \in \ker \sigma$ and $\mathcal{O}_K/p$ is finite, so $k := \sigma(\mathcal{O}_K)$ is a finite field contained in $\Gamma(C, \mathcal{O}_C)$; take A and φ for this k. Let $\sigma_A\colon A \to k$, $c \otimes a \mapsto c\sigma(a)$, a surjective k-algebra homomorphism with kernel $\mathfrak{m}$, a maximal ideal. Modulo the elements $1 \otimes a - \sigma(a) \otimes 1$ every $c \otimes a$ is congruent to $c\sigma(a) \otimes 1$, so these elements generate $\mathfrak{m}$ and $\mathfrak{m}H = \sum_a (\rho_H(a) - \sigma(a))H$, i.e. $H_\sigma = H/\mathfrak{m}H$. The Artinian ring A is the product of its localisations at its maximal ideals ([12, Theorem 8.7]); let e be the idempotent with $eA \cong A_{\mathfrak{m}}$. Then $1 - e \in \mathfrak{m}$, eA is local with maximal ideal $\mathfrak{n} = e\mathfrak{m}$ and residue field k, and $\mathfrak{n}^N = 0$ for

some N. Put $\mathfrak{n}^0 = eA$. As $(1-e)H = (1-e)^2H \subset \mathfrak{m}H$, we have $\mathfrak{m}H = \mathfrak{n}H \oplus (1-e)H$ and $H_\sigma = eH/\mathfrak{n}H$.

Since $k \subset \Gamma(C, \mathcal{O}_C)$, $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathcal{O}_C = A \otimes_k \mathcal{O}_C$, so by (b) every point of C has an open neighbourhood U with $H|_U \cong (A \otimes_k \mathcal{O}_U)^r$ as $A \otimes_k \mathcal{O}_U$-modules ($r$ is constant as C is connected). Then $\mathfrak{n}^i H|_U \cong (\mathfrak{n}^i \otimes_k \mathcal{O}_U)^r$, since $\mathcal{O}_U$ is flat over k. For $0 \le i < N$ multiplication gives a well-defined $\mathcal{O}_C$-linear map

$$\mu_i \colon (\mathfrak{n}^i/\mathfrak{n}^{i+1}) \otimes_k H_\sigma \to \mathfrak{n}^i H/\mathfrak{n}^{i+1}H$$

(it kills $\mathfrak{n}^i \otimes \mathfrak{n}H$, and for $i \ge 1$ also $\mathfrak{n}^i \otimes (1-e)H$, because $\mathfrak{n}^i(1-e) = 0$), and on U it is the isomorphism $(\mathfrak{n}^i/\mathfrak{n}^{i+1}) \otimes_k \mathcal{O}_U^r \to ((\mathfrak{n}^i/\mathfrak{n}^{i+1}) \otimes_k \mathcal{O}_U)^r$. Hence $H_\sigma|_U \cong \mathcal{O}_U^r$, so H_σ is a vector bundle of rank r, and $eH = \mathfrak{n}^0 H \supset \mathfrak{n}^1 H \supset \cdots \supset \mathfrak{n}^N H = 0$ is a filtration by vector bundles with graded pieces $\mathfrak{n}^i H/\mathfrak{n}^{i+1}H \cong H_\sigma^{\oplus d_i}$, $d_i = \dim_k \mathfrak{n}^i/\mathfrak{n}^{i+1}$. By additivity and (D.12),

$$0 = \deg eH = \Big(\sum_{i<N} d_i\Big) \deg H_\sigma = (\dim_k eA) \deg H_\sigma,$$

and $\dim_k eA \ge 1$, so $\deg H_\sigma = 0$. □

Lemma D.53. *Let $(\mathcal{Y}, j, G_\mathcal{Y}, \iota_\mathcal{Y}, \mathcal{L}_\mathcal{Y})$ be either an integral model of $\mathrm{Sh}_{\mathsf{L}'}$ over $\mathcal{O}_{F',v}$ for an ℓ-adic place v, or a global integral model of Sh over $\mathcal{O}_F$. Then for every point $z \in \mathcal{Y}$ of positive residue characteristic and every algebraically closed field k containing $\kappa(z)$ there are an exact sequence $0 \to T \to G_{\mathcal{Y},k} \xrightarrow{q} B \to 0$ of a torus T and an abelian variety B over k and a line bundle $\mathcal{M}$ on B whose polarisation $\phi_\mathcal{M}$ is an isomorphism, with $\mathcal{L}_\mathcal{Y}|_{G_{\mathcal{Y},k}} \cong q^*\mathcal{M}$.*

Proof. **Local models.** Let $\mathcal{Y}$ be an integral model over $O = \mathcal{O}_{F',v}$ and $z \in \mathcal{Y}$. Put $\mathcal{O} = \mathcal{O}_{\mathcal{Y},z}$, a normal local domain ($\mathcal{Y}$ is normal), essentially of finite type over the excellent ring O, hence excellent; its completion $\widehat{\mathcal{O}}$ is therefore normal, hence a domain, and of characteristic 0 since $\mathcal{O} \subseteq \widehat{\mathcal{O}}$ and $\mathcal{Y}$ is flat over O. By (Gl2) (Lemma 32.13) there is a finite separable extension $k' = \kappa(z)[t]/(f)$ over which $G_{\mathcal{Y},z}$ is an extension of an abelian variety by a split torus. Lift f to a monic $\tilde{f} \in \widehat{\mathcal{O}}[t]$ and put $R = \widehat{\mathcal{O}}[t]/(\tilde{f})$: it is finite étale over $\widehat{\mathcal{O}}$ (the discriminant is a unit), local with residue field k' (f irreducible), complete (finite over a complete local ring), normal (étale over a normal ring), hence a complete noetherian normal local domain of characteristic 0. Let $\xi \colon \operatorname{Spec} R \to \mathcal{Y}$ be the composite map; it sends the closed point to z. Its generic point maps to the generic point of the unique irreducible component of $\mathcal{Y}$ through z (R is flat over $\mathcal{O}$), which lies in $\mathcal{Y} \otimes F'$ (flatness over O) and hence in the dense open $j(\mathrm{Sh}_{\mathsf{L}',F'})$, where $G_\mathcal{Y}$ is abelian by (Ud7). The torus part of $\xi^* G_\mathcal{Y}$ over the residue field k' is split. So (Gl3) (Proposition 32.14) applies: $\xi^*(G_\mathcal{Y}, \iota_\mathcal{Y}, \mathcal{L}_\mathcal{Y})$ is uniformised by degeneration data $(B_R, \mathcal{L}_{B_R}, \iota_B, X, c, \psi)$ over $(R, \mathfrak{m}_R)$, where $\mathcal{L}_{B_R}$ is a line bundle on the abelian scheme B_R defining a principal polarisation. Uniformisation for $e = 0$ identifies $G_{\mathcal{Y},z} \otimes_{\kappa(z)} k'$ with the reduction $\widetilde{G}_{k'}$ of the Raynaud extension, an extension $0 \to T_{k'} \to \widetilde{G}_{k'} \xrightarrow{\pi} B_{k'} \to 0$ of $B_{k'} = B_R \otimes k'$ by the split torus with character group X, and $\mathcal{L}_\mathcal{Y} \otimes k'$ with $\pi^*(\mathcal{L}_{B_R} \otimes k')$. For an algebraically closed $k \supseteq \kappa(z)$ choose a $\kappa(z)$-embedding $k' \to k$ ($k'/\kappa(z)$ is finite separable) and base change: $T = T_k$, $B = B_k$, $q = \pi_k$, $\mathcal{M} = \mathcal{L}_{B_R} \otimes k$, whose polarisation is the base change of an isomorphism, hence an isomorphism.

Global models. Let $\mathcal{Y}$ be global over $\mathcal{O}_F$ and $z \in \mathcal{Y}$ of residue characteristic $\ell > 0$, lying over the prime $\mathfrak{p}$ of $\mathcal{O}_F$. Extending the valuation of $\mathcal{O}_{F,\mathfrak{p}}$ to $\mathbb{C}_\ell$ gives an ℓ-adic place v with $\mathcal{O}_{F,v} = \mathcal{O}_{F,\mathfrak{p}}$, and z is a point of $\mathcal{Y} \otimes \mathcal{O}_{F,v}$. By (Gl9) there are $f \colon \mathcal{Y} \otimes \mathcal{O}_{F,v} \to \mathcal{Y}^v_{\mathsf{L},F}$ and a morphism over f of semiabelian structures, i.e. an isomorphism $G_\mathcal{Y} \otimes \mathcal{O}_{F,v} \cong f^* G_{\mathcal{Y}^v}$ carrying $f^*\mathcal{L}_{\mathcal{Y}^v}$ to $\mathcal{L}_\mathcal{Y}$. A field $k \supseteq \kappa(z)$ contains $\kappa(f(z))$, and the claim at z is the claim at $f(z)$ for the local model $\mathcal{Y}^v_{\mathsf{L},F}$ over $\mathcal{O}_{F,v}$, proved above. □

Lemma D.54. *Let $\mathfrak{u}$ be a unitary Shimura datum, $\tau \in \Phi_{\mathsf{H}}$, $U \subseteq \mathrm{Sh}_F$ a union of connected components and φ a frame function on U which has an archimedean norm N_s at every embedding $s\colon F \to \mathbb{C}$ and an ℓ-adic norm N_κ at every ring homomorphism $\kappa\colon F \to \mathbb{C}_\ell$, for every prime ℓ. Let K be a number field with an embedding $\beta\colon F \to K$, $y \in U(K)$, and suppose that there are a semiabelian scheme $\mathcal{G}$ over $\mathcal{O}_K$ (for instance an abelian scheme) with an action $\iota_{\mathcal{G}}\colon \mathcal{O}_E \to \mathrm{End}(\mathcal{G})$ and an $\mathcal{O}_E$-linear isomorphism of its generic fibre with G_y. For $\pm$ let $\mathcal{L}_\pm = \{x \in \mathrm{Lie}(\mathcal{G}) : \iota_{\mathcal{G}}(a)x = \beta(\sigma_{(\tau,\pm)}(a))x \ \forall a \in \mathcal{O}_E\}$, a projective $\mathcal{O}_K$-module whose generic fibre is $\mathrm{Lie}_{(\tau,\pm)}(G_y)$, with the Hermitian norms $\|x\|_t^2 = h_{y^t}(x,x)$ at every embedding $t\colon K \to \mathbb{C}$ (y^t the complex point induced by t), and $d_\pm = \widehat{\deg}\,\overline{\mathcal{L}}_\pm/[K:\mathbb{Q}]$. Then $N_\kappa = 0$ for all κ above all but finitely many primes ℓ, and*

$$d_+ - d_- = -\frac{1}{[F:\mathbb{Q}]}\Big(\sum_{s\colon F\to\mathbb{C}} N_s + \sum_{\ell} \sum_{\kappa\colon F\to\mathbb{C}_\ell} N_\kappa\Big).$$

Proof. Degrees by local norms. For a projective $\mathcal{O}_K$-module M of rank r with Hermitian norms and a non-zero $e \in \det M \otimes K$, define at a ring homomorphism $j\colon K \to \mathbb{C}_\ell$ with valuation ring $V_j = \{a \in K : |j(a)| \le 1\}$ (a discrete valuation ring, the localisation of $\mathcal{O}_K$ at the prime $\mathfrak{p}_j = \{|j(a)| < 1\}$) the number $\log\|e\|_j = \log|j(c)|$, where $e = c\,e_0$ with e_0 a V_j-basis of $\det M \otimes V_j$ ($c \in K^\times$, well defined as basis changes are units of V_j). The $[K_\mathfrak{p} : \mathbb{Q}_\ell]$ homomorphisms j inducing $\mathfrak{p}$ each give $\log|j(c)| = -v_\mathfrak{p}(c)\log N\mathfrak{p}/[K_\mathfrak{p}:\mathbb{Q}_\ell]$, so the usual formula for the arithmetic degree reads

$$\widehat{\deg}\,\overline{M} = -\sum_{t\colon K\to\mathbb{C}} \log\|e\|_t - \sum_{\ell}\sum_{j\colon K\to\mathbb{C}_\ell} \log\|e\|_j,$$

with finitely many non-zero finite terms, independent of e by the product formula.

Local values of φ. Fix a frame (v,w) at y, v a K-basis of $\mathcal{L}_+ \otimes K = \mathrm{Lie}_{(\tau,+)}(G_y)$ and w of $\mathcal{L}_- \otimes K$, and put $e_+ = v_1 \wedge \cdots \wedge v_p$, $e_- = w_1 \wedge \cdots \wedge w_q$, $x = \varphi(y; v, w) \in K^\times$ (Yf2).

- At $j\colon K \to \mathbb{C}_\ell$, put $\kappa = j \circ \beta$. Base change of $\mathcal{G}$ to V_j is a semiabelian scheme over V_j with $\mathcal{O}_E$-action and generic fibre G_y, and its eigen-submodules of the Lie algebra are $\mathcal{L}_\pm \otimes V_j$ (kernels of finitely many $\mathcal{O}_K$-linear maps $\iota_{\mathcal{G}}(a) - \beta(\sigma_{(\tau,\pm)}(a))$, a in a $\mathbb{Z}$-basis of $\mathcal{O}_E$, commute with the flat base change $\mathcal{O}_K \to V_j$); they are free over the discrete valuation ring V_j. Choose V_j-bases $v' = Av$, $w' = Bw$ ($A \in \mathrm{GL}_p(K)$, $B \in \mathrm{GL}_q(K)$); (v', w') is an integral frame (Yf3), so $\log|j(\varphi(y; v', w'))| = N_\kappa$. By (Yf2), $\varphi(y; v', w')\det A = x \det B$, and $e_+ = (\det A)^{-1}(v_1' \wedge \cdots \wedge v_p')$, so $\log\|e_+\|_j = -\log|j(\det A)|$, $\log\|e_-\|_j = -\log|j(\det B)|$. Hence
$$\log|j(x)| = N_{j\circ\beta} + \log\|e_-\|_j - \log\|e_+\|_j.$$
- At $t\colon K \to \mathbb{C}$, put $s = t \circ \beta$. By (Yf2) applied to t, $t(x) = \varphi(y^t; t_*v, t_*w)$, and by (Yf3), with $\det\Gamma(t_*v) = \|e_+\|_t^2$ and $\det\Gamma(t_*w) = \|e_-\|_t^2$ (Gram determinants for h_{y^t}),
$$\log|t(x)| = N_s + \log\|e_-\|_t - \log\|e_+\|_t.$$

Finiteness. For all but finitely many $\mathfrak{p}$, x is a unit at $\mathfrak{p}$ and v, w are V_j-bases ($A = 1$, $B = 1$ possible), so $N_{j\circ\beta} = \log|j(x)| = 0$; every $\kappa\colon F \to \mathbb{C}_\ell$ is of the form $j \circ \beta$ (extend κ to K, $\mathbb{C}_\ell$ being algebraically closed), and N_κ does not depend on the extension. So $N_\kappa = 0$ above all but finitely many ℓ.

Product formula. Summing over all $t\colon K \to \mathbb{C}$ and all $j\colon K \to \mathbb{C}_\ell$, all ℓ, the left-hand sides add to $\sum_t \log|t(x)| + \sum_\ell \sum_j \log|j(x)| = 0$. Every $s\colon F \to \mathbb{C}$ has exactly $[K : \beta F]$ extensions t,

and every $\kappa\colon F \to \mathbb{C}_\ell$ exactly $[K:\beta F]$ extensions j. With the formula for the degree applied to $\overline{\mathcal{L}}_+$ and $\overline{\mathcal{L}}_-$,

$$0 = [K:\beta F]\Big(\sum_s N_s + \sum_\ell \sum_\kappa N_\kappa\Big) + \widehat{\deg}\,\overline{\mathcal{L}}_+ - \widehat{\deg}\,\overline{\mathcal{L}}_-.$$

Dividing by $[K:\mathbb{Q}] = [K:\beta F][F:\mathbb{Q}]$ gives the claim. □

Lemma D.55. *The notation introduced here is local to this result and to the results that apply it. Let E be a CM field of degree $2f$, L a number field containing the image of every field embedding of E into an algebraic closure of L, and Ψ a set of f ring homomorphisms $\mathcal{O}_E \to \mathcal{O}_L$ such that for every field embedding $e\colon L \to \mathbb{C}$ the set $\{e \circ s :\ s \in \Psi\}$ contains exactly one of $\psi, \bar\psi$ for every embedding $\psi\colon E \to \mathbb{C}$. Let X be a finitely generated torsion-free $\mathcal{O}_E$-module with $r = \dim_E(X \otimes \mathbb{Q}) \geq 1$, put $V = X \otimes_{\mathbb{Z}} L$ and $M = X \otimes_{\mathbb{Z}} \mathcal{O}_L \subseteq V$, with $\mathcal{O}_E$ acting through X, and for a ring homomorphism $s\colon \mathcal{O}_E \to \mathcal{O}_L$ let $V_s = \{v \in V :\ av = s(a)v \text{ for all } a \in \mathcal{O}_E\}$ and $v \mapsto v_s$ the projection of $V = \bigoplus_s V_s$ onto V_s.*

(a) Let $\delta_1, \dots, \delta_r \in X$ be $\mathcal{O}_E$-linearly independent, $k = [X : \sum_a \mathcal{O}_E\delta_a]$, and for every $e\colon L \to \mathbb{C}$ let ν_e be a Hermitian metric on $V \otimes_{L,e} \mathbb{C}$ and $c_e > 0$ with

$$\prod_{s\in\Psi} \det\big(\nu_e((\delta_a)_s, (\delta_b)_s)\big)_{a,b\leq r} = k\,c_e.$$

Then the $\mathcal{O}_L$-modules $K_s = M \cap V_s$ ($s \in \Psi$), with the metrics induced by ν, satisfy

$$\sum_{s\in\Psi} \widehat{\deg}\,(K_s, \nu) \ \leq\ -\frac{1}{2} \sum_{e\colon L\to\mathbb{C}} \log c_e.$$

(b) The quantity $k^{-1}\prod_{s\in\Psi}\det\big(\nu_e((\delta_a)_s,(\delta_b)_s)\big)_{a,b}$ does not depend on the choice of the $\mathcal{O}_E$-linearly independent family $\delta_1, \dots, \delta_r$ in X.

Proof. **The eigen-decomposition.** As L contains the image of every embedding of E, $E \otimes_{\mathbb{Q}} L \cong \prod_s L$ over the $2f$ ring homomorphisms $s\colon \mathcal{O}_E \to \mathcal{O}_L$ (equivalently the field embeddings $E \to L$), so $V = (X \otimes \mathbb{Q}) \otimes_E (E \otimes_{\mathbb{Q}} L) = \bigoplus_s V_s$ with $V_s = X \otimes_{\mathcal{O}_E,s} L$ of dimension r, and the projection $v \mapsto v_s$ maps $x \otimes \lambda$ ($x \in X$, $\lambda \in L$) to $x \otimes_s \lambda$. Since X is finitely generated and torsion-free over the Dedekind ring $\mathcal{O}_E$, it is projective, so $M_s = X \otimes_{\mathcal{O}_E,s} \mathcal{O}_L$ is a projective $\mathcal{O}_L$-module of rank r and maps injectively into $X \otimes_{\mathcal{O}_E,s} L = V_s$; we identify it with its image, which is the image of M under $v \mapsto v_s$.

(a). Fix $s \in \Psi$. If $v \in K_s = M \cap V_s$, then $v = v_s \in M_s$; so $K_s \subseteq M_s$. Since M spans V over L and $M \cap V_s \supseteq \varpi M_s$ for a non-zero integer ϖ annihilating the cokernel of $M \to \bigoplus_s M_s$ (both are lattices in V), K_s has rank r, and with the same metrics

$$\widehat{\deg}\,(K_s, \nu) = \widehat{\deg}\,(M_s, \nu) - \log \#(M_s/K_s) \leq \widehat{\deg}\,(M_s, \nu).$$

Put $X_\delta = \sum_a \mathcal{O}_E\delta_a$, free with basis $\delta_1, \dots, \delta_r$, and $D_s = X_\delta \otimes_{\mathcal{O}_E,s} \mathcal{O}_L = \bigoplus_a \mathcal{O}_L(\delta_a)_s \subseteq M_s$. The ring $\mathcal{O}_L$ is torsion-free, hence flat, over $\mathcal{O}_E$ via s, so $M_s/D_s \cong (X/X_\delta) \otimes_{\mathcal{O}_E,s} \mathcal{O}_L$. The finite $\mathcal{O}_E$-module X/X_δ of order k has a composition series with factors $\mathcal{O}_E/\mathfrak{p}$, and $(\mathcal{O}_E/\mathfrak{p}) \otimes_{\mathcal{O}_E,s} \mathcal{O}_L = \mathcal{O}_L/\mathfrak{p}\mathcal{O}_L$ has order $\mathrm{N}\mathfrak{p}^{[L:s(E)]}$; by flatness the orders multiply, so $\#(M_s/D_s) = k^{[L:E]}$. By the formula for $\widehat{\deg}$ with the basis $(\delta_a)_s$,

$$\widehat{\deg}\,(M_s, \nu) = [L:E]\log k - \frac{1}{2}\sum_e \log\det\big(\nu_e((\delta_a)_s,(\delta_b)_s)\big)_{a,b}.$$

Summing over the f elements of Ψ and using the hypothesis,

$$\sum_{s\in\Psi}\widehat{\deg}\,(K_s,\nu)\le f[L:E]\log k-\frac12\sum_e\bigl(\log k+\log c_e\bigr)=-\frac12\sum_e\log c_e,$$

since $\sum_e 1=[L:\mathbb{Q}]=2f[L:E]$.

(b). Let $\delta'_1,\dots,\delta'_r$ be another $\mathcal{O}_E$-linearly independent family in X, with $k'=[X:X_{\delta'}]$. Both families are E-bases of $X\otimes\mathbb{Q}$, so $\delta'_a=\sum_b A_{ab}\delta_b$ with $A\in\mathrm{GL}_r(E)$, and $(\delta'_a)_s=\sum_b s(A_{ab})(\delta_b)_s$. Hence $\det\bigl(\nu_e((\delta'_a)_s,(\delta'_b)_s)\bigr)=|e(s(\det A))|^2\det\bigl(\nu_e((\delta_a)_s,(\delta_b)_s)\bigr)$, and as $\{e\circ s: s\in\Psi\}$ is a CM type, $\prod_{s\in\Psi}|e(s(\det A))|^2=\prod_\psi|\psi(\det A)|=|\mathrm{N}_{E/\mathbb{Q}}(\det A)|$, the product over all embeddings $\psi\colon E\to\mathbb{C}$. On the other side, $X_{\delta'}$ is the image of X_δ under the E-linear map with matrix A in the basis δ, whose determinant as a $\mathbb{Q}$-linear map is $\mathrm{N}_{E/\mathbb{Q}}(\det A)$; comparing covolumes of the lattices X_δ, $X_{\delta'}$ of $X\otimes\mathbb{R}$, $k'=k\,|\mathrm{N}_{E/\mathbb{Q}}(\det A)|$. The two factors cancel. □

D.3 Positivity and nefness at primes

This section proves nefness at primes (Gi4) for good towers (Theorem D.57), used in the proof of Proposition 31.14 for (Gt2), and the ampleness and vertical nefness of the perturbed bundles of the base settings (Proposition D.56), the last part of Proposition 34.35. At the primes of S_B nefness comes from the full flags of a trivial bundle of Section 35 (Proposition 35.66); at the other primes it comes from a Frobenius slope estimate and a weighted flag inequality for the Lie algebra pulled back to a curve in characteristic p (Proposition D.58, Corollary D.61), which rest on the polarised de Rham module of Section D.2.

Proposition D.56. *Let $\mathfrak{u}$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ,n,p) with its geometric realisation and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. Let B be an admissible stage, $Z\in\mathrm{Cp}$, and let a base setting of $\mathcal{T}$ at (B,Z) be given which is joined to the stage root model. Then for all integers $1\le i_1\le i_2$ and every $t>0$ the line bundle $\mathcal{M}_{B,i_1,i_2,t,\mathbb{Q}}$ on $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ is ample, and $\deg(\mathcal{M}_{B,i_1,i_2,t}|_\Gamma)\ge0$ for every closed integral curve $\Gamma\subseteq\mathcal{X}^{\mathrm{b}}$ contained in a fibre of $\mathcal{X}^{\mathrm{b}}\to\operatorname{Spec}\mathbb{Z}$.*

Proof. Write $\mathcal{L}_{\mathcal{X}}=\bigotimes_{\tau\in\Phi}(\det\mathcal{V}^{S_B}_{(\tau,-)})^\vee$ on $\mathcal{X}$ and $\mathcal{L}_{B,Z}=\bigotimes_\tau(\det\mathcal{V}^{B,Z}_{(\tau,-)})^\vee$ on $\mathcal{X}_{B,Z}$, the Hodge bundles at $(S_B,4)$ of the two covers. Since $\mathcal{M}_{B,i_1,i_2,t}=\mathcal{O}(i_2a^{\mathrm{H}}_B+i_1a')|_{\mathcal{X}^{\mathrm{b}}}$ does not depend on t, and $\mathcal{O}(a')$, $a'=(1,\dots,1)$ (Gb6), is very ample on the multiprojective space, hence ample on $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$ and of degree ≥0 on every closed integral curve, it suffices to show: $\deg\mathcal{O}(a^{\mathrm{H}}_B)|_\Gamma\ge0$ for every closed integral curve $\Gamma\subseteq\mathcal{X}^{\mathrm{b}}$ that lies in a fibre over a prime or in $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$. Then $i_2a^{\mathrm{H}}_B$ is nef on the fibres and on $\mathcal{X}^{\mathrm{b}}_{\mathbb{Q}}$, and $\mathcal{M}_{B,i_1,i_2,t,\mathbb{Q}}$, nef plus $i_1\ge1$ times ample, is ample.

Step 1: the cover is the base variety. Every point of $\mathcal{X}$ lies in a chart c', and $\mathcal{X}_{c'}$ is affine with ring $\mathbb{Z}[y]_{c'}$ (Gb1), a quotient of the coordinate ring of the standard affine open $U_{c'}=\{Y_{c'}\ne0\}$ of the multiprojective space; so y is a morphism on $\mathcal{X}$ and $\mathcal{X}_{c'}\to U_{c'}$ is a closed immersion. If $y(x')\in U_{c'}$, the ratios $y_{k,i}/y_{k,c'_k}$ are regular at x', so $x'\in\mathcal{X}_{c'}$; hence $y^{-1}(U_{c'})=\mathcal{X}_{c'}$ and y is an immersion. As $\mathcal{X}$ is proper over $\mathbb{Z}$, it is a closed immersion, and its image is closed, integral and contains the image of the generic point, which is dense in $\mathcal{X}^{\mathrm{b}}$ (y is birational onto $\mathcal{X}^{\mathrm{b}}$); so $y\colon\mathcal{X}\to\mathcal{X}^{\mathrm{b}}$ is an isomorphism, compatible with the fibres over primes and the generic fibres. We identify $\mathcal{X}=\mathcal{X}^{\mathrm{b}}$.

Step 2: the twisted Hodge-type line is $\mathcal{L}_{\mathcal{X}}$. Let x be a point of $\mathcal{X}$, c' a chart and $D(f)\ni x$ as in (Gb3). By (Gb3) the isomorphism of (Hy2) maps the generator $y_{c'}^{a^{\mathrm{H}}}=\prod_k y_{k,c'_k}^{a^{\mathrm{H}}_k}$ of $\mathcal{O}(a^{\mathrm{H}})$ on $D(f)$ to a generator of $\bigotimes_\tau(\det\mathcal{V}_{(\tau,-)})^\vee$ ($\mathcal{V}$ the Hodge bundles at (Bd,R)). Compose with multiplication by $\prod_{\ell\in S'_B}g_\ell^{-1}$, a rational isomorphism $\mathcal{O}(a^{\mathrm{H}}_B)\to\mathcal{O}(a^{\mathrm{H}})$ (S_B of (Hy3) is S'_B here,

(Gb4)). It sends the generator $y_{c'}^{a_B^{\mathrm{H}}} = y_{c'}^{a^{\mathrm{H}}} \prod_\ell y_{c'}^{v_\ell}$ to $y_{c'}^{a^{\mathrm{H}}} \prod_\ell (g_\ell / y_{c'}^{v_\ell})^{-1}$. If x lies over no $\ell \in S'_B$, each $g_\ell / y_{c'}^{v_\ell}$ is a unit at x (its divisor lies over ℓ), and $\mathcal{V}^{S_B}_{(\tau,-),x} = \mathcal{V}_{(\tau,-),x}$; if x lies over $\ell \in S'_B$, then by (Gb4) $g_\ell / y_{c'}^{v_\ell}$ is a unit times $\prod_\tau \det T_\tau$, the others are units, and a generator of $\bigotimes_\tau (\det \mathcal{V}_{(\tau,-),x})^\vee$ times $\prod_\tau (\det T_\tau)^{-1}$ generates $\bigotimes_\tau (\det \mathcal{V}^{S_B}_{(\tau,-),x})^\vee$ ($\mathcal{V}^{S_B}_{(\tau,-),x} = \mathrm{HT}_{(\tau,-),x}(\ell^4)$ has the basis T_τ times a basis of $\mathcal{V}_{(\tau,-),x}$). So the composite rational isomorphism $\mathcal{O}(a_B^{\mathrm{H}})|_{\mathcal{X}} \to \bigotimes_\tau (\det \mathrm{Li}_{(\tau,-)})^\vee$ maps local generators to local generators of $\mathcal{L}_{\mathcal{X}}$ at every point: $\mathcal{O}(a_B^{\mathrm{H}})|_{\mathcal{X}} \cong \mathcal{L}_{\mathcal{X}}$.

Step 3: the root model. On $\mathcal{X}_{B,Z}$ the integral flag of (Go3)(3) gives, for $s = (\tau, -)$, $\det \mathcal{V}_s^{B,Z} \cong \bigotimes_{k=1}^{q} \mathcal{V}_{s,k}^{B,Z} / \mathcal{V}_{s,k-1}^{B,Z}$ (determinants are multiplicative in exact sequences of locally free sheaves), and by (Fh3) the roots (τ, i) with $i > p$ have label $(\tau, -)$, step $n + 1 - i$ and sign -1, so $\bigotimes_{i>p} \mathcal{L}_{(\tau,i)} \cong (\det \mathcal{V}^{B,Z}_{(\tau,-)})^{-1}$. With the root sections, $\mathcal{L}_{B,Z} \cong \mathcal{O}(E_-)|_{\mathcal{X}_{B,Z}}$, $E_- = \sum_\tau \sum_{i>p} e_{(\tau,i)}$. Let μ_- be the weight with $\mu_-(\tau, i) = 0$ for $i \le p$ and -1 for $i > p$: it is dominant, and $\sum_r c(\mu_-, 0)_r e_r = E_-$, $c(\mu_-, 0)_* = 0$.

(a) Vertical curves. Let $W \subseteq \mathcal{X}_{B,Z} \otimes \mathbb{F}_\ell$ be a closed integral curve. If $\ell \le B$ or $\ell \in \mathrm{Bd}$, (Gi4) applies to μ_-; otherwise $\ell > B \ge 4|\Phi| n^2$, so $(\ell - 2)(\mu_-(a, p) - \mu_-(a, p + 1)) = \ell - 2 \ge 4|\Phi| = 4|\Phi| \|\mu_-\|_\infty$ and μ_- is above the nef threshold at ℓ. By Theorem D.57, $c(\mu_-, 0)$ is nef at ℓ: $\deg \mathcal{L}_{B,Z}|_W = \sum_r c(\mu_-, 0)_r \deg \mathcal{O}(e_r)|_W \ge 0$.

(b) Curves of the generic fibre. Let ρ be the weight $\rho(\tau, i) = -i$ and $\nu = \nu_{\mathcal{T}}$. For $0 < \delta \le 1$ and $0 < s \le \delta / (2\|\nu\|_\infty + 1)$ the weight $\lambda = \mu_- + \delta\rho$ has $\|\lambda\|_1 \le A := |\Phi|(q + n^2)$, and $\lambda_s = \lambda - s\nu$ is dominant (consecutive differences $\ge \delta - 2s\|\nu\|_\infty > 0$). By Lemma 31.24 there is $s_0 > 0$, depending on A only, such that $c(\lambda_s, s)$ is semipositive at $t(s)$ on $\mathcal{X}_{B,Z}$ for such $s \le s_0$: the function $\sum_r c(\lambda_s, s)_r \log h_r(t(s))$ is plurisubharmonic along the cone over $\mathcal{X}_{B,Z}(\mathbb{C})$. Let W be a closed integral curve of $\mathcal{X}_{B,Z} \otimes \mathbb{Q}$ and $\widetilde{W}$ the normalisation of $W(\mathbb{C})$; the pull-back to $\widetilde{W}$ of the smooth form $\sum_r c(\lambda_s, s)_r c_1(\mathcal{O}(e_r), h_r(t(s)))$ is semipositive (its local potentials are restrictions of the plurisubharmonic function above along holomorphic maps into the cone with Zariski tangent derivative), so $\sum_r c(\lambda_s, s)_r \deg \mathcal{O}(e_r)|_W \ge 0$. Letting $s \to 0^+$ and then $\delta \to 0^+$, $c(\lambda_s, s) \to c(\mu_-, 0)$ and $\deg \mathcal{L}_{B,Z}|_W \ge 0$.

Step 4: the common cover. The Hodge lattices of $\mathcal{X}$ and of $\mathcal{X}_{B,Z}$ at $(S_B, 4)$ are Hodge bundles (joined; (Go3)(2)), their Hodge–Tate spans have free images in the duals of the Li_s ((Gb1); (Go3)(2)), and $S_B \supseteq \mathrm{Bd}$ contains every prime dividing $\mathrm{disc}(E)$ ((Go5)). By Lemma 31.29(1), applied to ϕ and to ϕ', the Hodge lattices of $\mathcal{X}''$ at $(S_B, 4)$ form Hodge bundles $\mathcal{V}''_s$ with $\phi^* \mathcal{V}_s^{B,Z} \cong \mathcal{V}''_s \cong \phi'^* \mathcal{V}_s^{S_B}$; hence $\phi'^* \mathcal{L}_{\mathcal{X}} \cong \phi^* \mathcal{L}_{B,Z}$.

Step 5: descent. Let $\Gamma \subseteq \mathcal{X}$ be a closed integral curve in the fibre over a prime ℓ or in $\mathcal{X}_{\mathbb{Q}}$. As ϕ' is proper and dominant, it is surjective; a closed point of the generic fibre of $\phi'^{-1}(\Gamma) \to \Gamma$ has a closure $\Gamma'' \subseteq \mathcal{X}''$, a closed integral curve mapping finitely onto Γ, in the same fibre over $\operatorname{Spec} \mathbb{Z}$. By Steps 2 and 4, $[\mathbf{k}(\Gamma'') : \mathbf{k}(\Gamma)] \deg \mathcal{O}(a_B^{\mathrm{H}})|_\Gamma = \deg \phi'^* \mathcal{L}_{\mathcal{X}}|_{\Gamma''} = \deg \phi^* \mathcal{L}_{B,Z}|_{\Gamma''}$. The image $\phi(\Gamma'')$ lies in the corresponding fibre of $\mathcal{X}_{B,Z}$ (ϕ is compatible with the maps to $\mathcal{Y}$) and is a point or a closed integral curve W; in the first case the pull-back of $\mathcal{L}_{B,Z}$ to Γ'' is trivial and the degree is 0, in the second it is $[\mathbf{k}(\Gamma'') : \mathbf{k}(W)] \deg \mathcal{L}_{B,Z}|_W \ge 0$ by Step 3. Hence $\deg \mathcal{O}(a_B^{\mathrm{H}})|_\Gamma \ge 0$, which proves the claim and the proposition. □

Theorem D.57. *Let $\mathfrak{u}$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. Then the realised tower has the nefness at primes (Gi4).*

Proof. Fix $B \ge 2$, $c \in \mathrm{Cp}$, a prime ℓ and a dominant weight μ such that $\ell \le B$, or $\ell \in \mathrm{Bd}$, or μ is above the nef threshold at ℓ; let $\mathcal{X} = \mathcal{X}_{B,c}$. By (Go5) the good origin contains a root origin of $\mathcal{X}$ on the fixed data at c, and $\mathcal{X} \subseteq \mathbb{P}$ is projective over $\mathbb{Z}$. We must show

$\sum_r c(\mu,0)_r \deg(\mathcal{O}(e_r)|_W) \geq 0$ for every integral closed curve $W \subseteq \mathcal{X}_{\mathbb{F}_\ell}$.

Case $\ell \notin S_B$. Then $\ell > B$ and $\ell \notin \mathrm{Bd}$, so by hypothesis μ is above the nef threshold at ℓ, i.e. $4|\Phi|\|\mu\|_\infty \leq (\ell-2)(\mu_{a,p} - \mu_{a,p+1})$ for every a (the cut of a tower of type (Φ,n,p) is $\kappa = p$), and Proposition D.58 gives that $c(\mu,0)$ is nef at ℓ.

Case $\ell \in S_B$. Fix $\tau \in \Phi$ and an integral closed curve $W \subseteq \mathcal{X}_{\mathbb{F}_\ell} = \mathcal{X}_\ell$; W is proper over $\mathbb{F}_\ell$. Write $d_i = \deg(\mathcal{O}(e_{(\tau,i)})|_W)$ and $D_k = d_1 + \cdots + d_k$. Let $(\mathcal{U}_{\tau,k})_k$ be the chain of Proposition 35.66 and $\nu\colon \widetilde{W} \to W \hookrightarrow \mathcal{X}_\ell$ the normalisation of W composed with the inclusion. As the quotients $\mathcal{U}_{\tau,k}/\mathcal{U}_{\tau,k-1}$ are locally free, pulling back preserves the exact sequences, so $\nu^*\mathcal{U}_{\tau,k} \subseteq \mathcal{O}_{\widetilde{W}}^n$ is a locally free submodule of rank k with locally free quotient, and $\det \nu^*\mathcal{U}_{\tau,k} \cong \bigotimes_{i\leq k} \nu^*\mathcal{O}(e_{(\tau,i)})$. Since $\nu^*\mathcal{U}_{\tau,k}$ is locally a direct summand of $\mathcal{O}_{\widetilde{W}}^n$, the induced map $\det \nu^*\mathcal{U}_{\tau,k} \to \bigwedge^k \mathcal{O}_{\widetilde{W}}^n \cong \mathcal{O}_{\widetilde{W}}^{\binom{n}{k}}$ is injective, so one of its $\binom{n}{k}$ coordinates is a non-zero map $\det \nu^*\mathcal{U}_{\tau,k} \to \mathcal{O}_{\widetilde{W}}$, i.e. a non-zero global section of $(\det \nu^*\mathcal{U}_{\tau,k})^{-1}$. Hence $-D_k = \deg_{\widetilde{W}}(\det \nu^*\mathcal{U}_{\tau,k})^{-1} \geq 0$, i.e. $D_k \leq 0$ for $1 \leq k \leq n$; for $k = n$, $\det \nu^*\mathcal{U}_{\tau,n} = \det \mathcal{O}_{\widetilde{W}}^n$ is trivial, so $D_n = 0$. As μ is dominant, $\mu_{\tau,k} - \mu_{\tau,k+1} \geq 0$, and by summation by parts

$$\sum_{i=1}^{n} \mu_{\tau,i}\, d_i = \sum_{k=1}^{n-1} (\mu_{\tau,k} - \mu_{\tau,k+1})\, D_k + \mu_{\tau,n} D_n \leq 0.$$

Summing over τ and using $c(\mu,0)_{(\tau,i)} = -\mu_{\tau,i}$, $c(\mu,0)_* = 0$: $\sum_r c(\mu,0)_r \deg(\mathcal{O}(e_r)|_W) = -\sum_\tau \sum_i \mu_{\tau,i} d_i \geq 0$.

In both cases $c(\mu,0)$ is nef at ℓ on $\mathcal{X}_{B,c}$, which is (Gi4). □

Proposition D.58. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ,n,p) with its geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,c}))$ and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. Let $B \geq 2$, $c \in \mathrm{Cp}$, let ℓ be a prime with $\ell \notin S_B$, and let μ be a dominant weight above the nef threshold at ℓ, i.e. $4|\Phi|\,\|\mu\|_\infty \leq (\ell-2)(\mu_{a,p} - \mu_{a,p+1})$ for every $a \in \Phi$. Then $c(\mu,0)$ is nef at ℓ on $\mathcal{X}_{B,c}$.*

Proof. Write $\mathcal{X} = \mathcal{X}_{B,c}$, $n = p+q$, and let $\ell \notin S_B$ and μ be as in the statement. By definition we must show $\sum_{r \in \mathrm{Rt}^+} c(\mu,0)_r \deg(\mathcal{O}(e_r)|_W) \geq 0$ for every integral closed curve $W \subseteq \mathcal{X}_{\mathbb{F}_\ell}$; as $c(\mu,0)_* = 0$ and $c(\mu,0)_{a,i} = -\mu_{a,i}$, this is

$$\sum_{\tau \in \Phi} \sum_{i=1}^{n} \mu_{\tau,i} \deg(\mathcal{O}(e_{(\tau,i)})|_W) \leq 0. \tag{D.13}$$

The setting at ℓ. The good origin contains a tower origin, so Bd is the set of primes dividing $2\,\mathrm{disc}(E)\,N'$ (Go5) and $\mathrm{Bd} \subseteq S_B$; hence $\ell > 2$, and every prime dividing $\mathrm{disc}(E)$ lies in S_B. The scheme $\mathcal{X} \subseteq \mathbb{P}$ is projective over $\mathbb{Z}$, so W is projective over $\mathbb{F}_\ell$; let $\nu\colon C \to \mathcal{X}$ be its normalisation composed with the inclusion, an integral normal projective curve over a finite field, and recall $\deg(\mathcal{N}|_W) = \deg \nu^*\mathcal{N}$ for a line bundle $\mathcal{N}$. By (Go3)(3) the root line $\mathcal{L}_r$ of the integral flag of the Hodge bundles is isomorphic to $\mathcal{O}(e_r)|_{\mathcal{X}}$, so $\deg(\mathcal{O}(e_r)|_W) = \deg \nu^*\mathcal{L}_r$.

The tie. Lemma D.59 applies ($\mathcal{X}$ carries a root origin, $\ell \notin S_B$, the primes dividing $\mathrm{disc}(E)$ lie in S_B). It gives a finite field $k \subseteq \Gamma(C, \mathcal{O}_C)$ with $k \otimes \mathcal{O}_E \cong k^\Sigma$, the bijection $s \mapsto \tilde{\sigma}_s$ from the labels onto Σ, and with $L = \mathrm{Lie}(G_C) = \nu^*\mathrm{Lie}(G_{\mathcal{X}})$: $\nu^*\mathcal{V}_s = L_{\tilde{\sigma}_s}$ and full flags $W_{\tau,i} = \nu^*\mathcal{V}_{(\tau,+),i}$ $(0 \leq i \leq p)$ of $L_{\tilde{\sigma}_{(\tau,+)}}$ and $V_{\tau,i} = \nu^*\mathcal{V}_{(\tau,-),i}$ $(0 \leq i \leq q)$ of $L_{\tilde{\sigma}_{(\tau,-)}}$, with, by (Fh3),

$$\deg \nu^*\mathcal{L}_{(\tau,i)} = \deg W_{\tau,i} - \deg W_{\tau,i-1}\ (i \leq p), \qquad \deg \nu^*\mathcal{L}_{(\tau,i)} = -\big(\deg V_{\tau,n+1-i} - \deg V_{\tau,n-i}\big)\ (i > p).$$

Put $T = \{\tilde{\sigma}_{(\tau,+)} : \tau \in \Phi\}$; since $\tilde{\sigma}_{(\tau,-)} = \tilde{\sigma}_{(\tau,+)} \circ {}^-$, $\Sigma = T \sqcup \bar{T}$ and $|T| = |\Phi|$.

The weighted flag inequality. The scheme $\mathcal{X}$ is normal, integral and noetherian, and $(G_{\mathcal{X}}, \iota_{\mathcal{X}}, \mathcal{L}_{\mathcal{X}})$, pulled back from $\mathcal{Y}$, is a semiabelian structure whose generic fibre is abelian (the generic point of $\mathcal{X}$ maps into $j(\mathrm{Sh}_F)$, where $G_{\mathcal{Y}}$ is abelian, (Ud7)); it satisfies (PF) by Lemma D.60. So Corollary D.61 applies to $\nu\colon C \to \mathcal{X}$ with the prime $\ell > 2$: the weighted flag inequality (ii) of Lemma D.63 holds for L, with $j = p$, $q = q$, the subsheaves $W_{\tau,i}$, $V_{\tau,i}$ above (indexed by $\tilde\sigma_{(\tau,+)} \in T$; $W_{\tau,0} = V_{\tau,0} = 0$, $W_{\tau,p} = L_{\tilde\sigma_{(\tau,+)}}$, $V_{\tau,q} = L_{\overline{\tilde\sigma_{(\tau,+)}}}$), the weights $\mu_{\tau,1} \geq \cdots \geq \mu_{\tau,n}$ (μ is dominant), $M = \|\mu\|_\infty$ and $g = \min_\tau(\mu_{\tau,p} - \mu_{\tau,p+1})$: the threshold of the lemma, $4|T|M \leq (\ell-2)g$, is the hypothesis $4|\Phi|\,\|\mu\|_\infty \leq (\ell-2)(\mu_{a,p} - \mu_{a,p+1})$ for every a. The inequality reads

$$\sum_{\tau\in\Phi}\Big(\sum_{i=1}^{p}\mu_{\tau,i}(\deg W_{\tau,i} - \deg W_{\tau,i-1}) + \sum_{i'=1}^{q}\mu_{\tau,n+1-i'}\big(\deg V_{\tau,i'-1} - \deg V_{\tau,i'}\big)\Big) \leq 0.$$

With $i = n+1-i'$ the second inner sum is

$$\sum_{i=p+1}^{n}\mu_{\tau,i}(\deg V_{\tau,n-i} - \deg V_{\tau,n+1-i}) = \sum_{i>p}\mu_{\tau,i}\deg\nu^*\mathcal{L}_{(\tau,i)},$$

and the first is $\sum_{i\leq p}\mu_{\tau,i}\deg\nu^*\mathcal{L}_{(\tau,i)}$. So the left side is the left side of (D.13), which is therefore ≤ 0: $c(\mu,0)$ is nef at ℓ on $\mathcal{X}$. □

Lemma D.59. *Let* $\mathfrak{u}$, $\mathcal{Y}$, *a root tower* $\mathcal{T}$ *with geometric realisation,* $B \geq 2$ *and* $c \in \mathrm{Cp}$ *be such that* $\mathcal{X} = \mathcal{X}_{B,c}$ *carries a root origin (Go3), with Hodge bundles* $\mathcal{V}_s$ *at* (S_B, R), *an integral flag* $(\mathcal{V}_{s,j})_j$ *of them and root lines* $\mathcal{L}_r$; *suppose every prime dividing* $\mathrm{disc}(E)$ *lies in* S_B. *Let* $\ell \notin S_B$ *be a prime,* $W \subseteq \mathcal{X} \otimes \mathbb{F}_\ell$ *an integral closed subscheme of dimension one, proper over a field,* $\nu\colon C \to \mathcal{X}$ *the normalisation of* W *composed with the inclusion,* $G_C = \nu^*G_{\mathcal{X}}$ *with its* $\mathcal{O}_E$*-action and* $\mathrm{Lie}(G_C)$. *Then:*

(i) W *lies over one closed point* v *of* $\mathrm{Spec}\,\mathcal{O}_F$, *and* $k = \kappa(v) \subseteq \Gamma(C,\mathcal{O}_C)$; *with* $\Sigma = \mathrm{Hom}(\mathcal{O}_E, k)$, *the map* $k \otimes_{\mathbb{Z}} \mathcal{O}_E \to k^\Sigma$, $\lambda\otimes a \mapsto (\lambda\, t(a))_t$, *is bijective, and* $s \mapsto \tilde\sigma_s = (\sigma_s \bmod v)$ *is a bijection from the labels onto* Σ *with* $\tilde\sigma_{(\tau,-)} = \tilde\sigma_{(\tau,+)} \circ {}^-$;

(ii) for every label s, $\nu^*\mathcal{V}_s = e_{\tilde\sigma_s}\mathrm{Lie}(G_C)$ *as subbundles of* $\mathrm{Lie}(G_C)$, *where* $e_t \in k\otimes\mathcal{O}_E$ *is the idempotent of* $t \in \Sigma$;

(iii) $(\nu^*\mathcal{V}_{s,j})_j$ *is a full flag of* $e_{\tilde\sigma_s}\mathrm{Lie}(G_C)$, *and for every root* r,

$$\nu^*\mathcal{L}_r = \big(\nu^*\mathcal{V}_{s(r),\mathrm{st}(r)}/\nu^*\mathcal{V}_{s(r),\mathrm{st}(r)-1}\big)^{\otimes \mathrm{sg}(r)}.$$

Proof. **(i).** The image of W in $\mathrm{Spec}\,\mathcal{O}_F$ is irreducible and lies in the finite fibre over ℓ, so it is one closed point v; the structure map gives $\kappa(v) \to \Gamma(W,\mathcal{O}_W) \to \Gamma(C,\mathcal{O}_C)$. Let $\omega_1,\dots,\omega_d$ be a $\mathbb{Z}$-basis of $\mathcal{O}_E$, $d = [E:\mathbb{Q}]$. The labels s give the d embeddings $\sigma_s\colon E \to F$ (Fh1), and $\det(\sigma_s(\omega_i))^2 = \mathrm{disc}(E)$, which is prime to ℓ (every prime dividing it is in S_B) and hence a unit in $\mathcal{O}_{F,v}$. Reducing modulo v, the matrix $(\tilde\sigma_s(\omega_i))$ is invertible over k; so the $\tilde\sigma_s$ are pairwise distinct (d distinct elements of Σ, which has at most $\dim_k(k\otimes\mathcal{O}_E) = d$ elements: a bijection), and $k\otimes\mathcal{O}_E \to k^\Sigma$ is a k-linear map between d-dimensional spaces with invertible matrix in the bases $(1\otimes\omega_i)$ and the unit vectors, hence bijective. Finally, $\sigma_{(\tau,-)} = \bar\tau = \tau\circ{}^-$ by (Fh1).

The idempotents at points over ℓ. Let $x \in \mathcal{X}$ lie over ℓ. Its residue field has characteristic ℓ, so $\operatorname{disc}(E)$ is a unit in $\mathcal{O}_x \supseteq \mathcal{O}_F$, and by the same determinant $\mathcal{O}_E \otimes \mathcal{O}_x \to \prod_s \mathcal{O}_x$, $a \otimes b \mapsto (\sigma_s(a)b)_s$, is an isomorphism; let $e_s \in \mathcal{O}_E \otimes \mathcal{O}_x$ be its idempotents. On $\mathrm{Li} = \mathrm{Lie}(G_{\mathcal{X},\Omega})$, e_s is the projection onto Li_s. As $G_{\mathcal{X}}$ is smooth, $\mathrm{Lie}(G_{\mathcal{X}})$ is locally free and $\mathcal{X}$ integral, so $\mathrm{Lie}(G_{\mathcal{X}})_x \to \mathrm{Li}$ is injective with image Lie_x; it is an $\mathcal{O}_E \otimes \mathcal{O}_x$-module, so $\mathrm{Lie}_x = \bigoplus_s e_s\mathrm{Lie}_x$ and $e_s\mathrm{Lie}_x = \mathrm{Lie}_x \cap \mathrm{Li}_s = \mathrm{dR}_{s,x}$ (if $y \in \mathrm{Lie}_x \cap \mathrm{Li}_s$ then $y = e_s y$). As $\ell \notin S_B$, the Hodge lattice at x is $V_{s,x}^{S_B} = \mathrm{dR}_{s,x}$ (Fh5). Hence the stalks at x of the coherent subsheaves $\mathcal{V}_s$ and $e_s\mathrm{Lie}(G_{\mathcal{X}})$ of the constant sheaf Li_s coincide (the latter is defined on the open set where $\operatorname{disc}(E)$ is invertible); finitely many local generators of each lie in the other at x, hence on a neighbourhood of x. So $\mathcal{V}_s = e_s\mathrm{Lie}(G_{\mathcal{X}})$ on an open set $U \supseteq \mathcal{X} \otimes \mathbb{F}_\ell$.

(ii). The morphism ν factors through U. The formation of Lie of a smooth group scheme commutes with base change, so $\nu^*\mathrm{Lie}(G_{\mathcal{X}}) = \mathrm{Lie}(G_C)$ compatibly with $\mathcal{O}_E$, and e_s maps to the idempotent of $k \otimes \mathcal{O}_E$ whose $\tilde{\sigma}_{s'}$-component is the reduction of $\delta_{ss'}$, i.e. to $e_{\tilde{\sigma}_s}$. Pulling back the decomposition $\mathrm{Lie}(G_{\mathcal{X}})|_U = \bigoplus_s e_s\mathrm{Lie}(G_{\mathcal{X}})|_U$ gives $\nu^*\mathcal{V}_s = \nu^*(e_s\mathrm{Lie}(G_{\mathcal{X}})) = e_{\tilde{\sigma}_s}\mathrm{Lie}(G_C)$, a direct summand.

(iii). An integral flag is a chain of locally free submodules with invertible step quotients (Fh2), i.e. of subbundles; pull-back preserves the exact sequences

$$0 \to \mathcal{V}_{s,j-1} \to \mathcal{V}_{s,j} \to \mathcal{V}_{s,j}/\mathcal{V}_{s,j-1} \to 0$$

of vector bundles, so $(\nu^*\mathcal{V}_{s,j})_j$ is a full flag of $\nu^*\mathcal{V}_s$ with step quotients $\nu^*(\mathcal{V}_{s,j}/\mathcal{V}_{s,j-1})$, and ν^* commutes with tensor powers and duals of line bundles; this is the formula for $\nu^*\mathcal{L}_r$ by (Fh3). □

Lemma D.60. *Let $\mathcal{Y}$ be an integral model of* Sh *over $\mathcal{O}_F$ with uniformised degenerations and $f\colon \mathcal{X} \to \mathcal{Y}$ a cover, with the pulled-back semiabelian structure $(G_{\mathcal{X}}, \iota_{\mathcal{X}}, \mathcal{L}_{\mathcal{X}})$. Then for every point $z \in \mathcal{X}$ of positive residue characteristic and every algebraically closed field $k \supseteq \kappa(z)$ there are an exact sequence $0 \to T \to G_{\mathcal{X},k} \xrightarrow{q} B \to 0$ of a torus T and an abelian variety B over k and a line bundle $\mathcal{M}$ on B whose polarisation $\phi_{\mathcal{M}}$ is an isomorphism, with $\mathcal{L}_{\mathcal{X}}|_{G_{\mathcal{X},k}} \cong q^*\mathcal{M}$.*

Proof. Let $z \in \mathcal{X}$ have residue characteristic $\ell > 0$ and let $k \supseteq \kappa(z)$ be algebraically closed. Then $y = f(z) \in \mathcal{Y}$ has residue characteristic ℓ and $\kappa(y) \subseteq \kappa(z) \subseteq k$; let $\mathfrak{p}$ be the prime of $\mathcal{O}_F$ below y. Extending the valuation of $\mathcal{O}_{F,\mathfrak{p}}$ to $\mathbb{C}_\ell$ gives an ℓ-adic place v with $\mathcal{O}_{F,v} = \mathcal{O}_{F,\mathfrak{p}}$, and y is a point of $\mathcal{Y} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v}$. By (Ug1) there is an integral model $\mathcal{Y}^v$ of Sh over $\mathcal{O}_{F,v}$, and by (Ug2) a morphism $f_v\colon \mathcal{Y} \otimes \mathcal{O}_{F,v} \to \mathcal{Y}^v$ with a morphism over f_v of semiabelian structures, i.e. an isomorphism $G_{\mathcal{Y}} \otimes \mathcal{O}_{F,v} \cong f_v^*G_{\mathcal{Y}^v}$ compatible with the $\mathcal{O}_E$-actions and carrying $f_v^*\mathcal{L}_{\mathcal{Y}^v}$ to $\mathcal{L}_{\mathcal{Y}}$. The point $f_v(y)$ has residue characteristic ℓ and $\kappa(f_v(y)) \subseteq \kappa(y) \subseteq k$, so the local case of Lemma D.53 (an integral model over $\mathcal{O}_{F,v}$, with $F' = F$ and $\mathsf{L}' = \mathsf{L}$) gives $0 \to T \to G_{\mathcal{Y}^v,k} \xrightarrow{q} B \to 0$ and $\mathcal{M}$ with $\phi_{\mathcal{M}}$ an isomorphism and $\mathcal{L}_{\mathcal{Y}^v}|_{G_{\mathcal{Y}^v,k}} \cong q^*\mathcal{M}$. The fibres over k satisfy $G_{\mathcal{X},k} = G_{\mathcal{Y},k} \cong G_{\mathcal{Y}^v,k}$ (base change along $\operatorname{Spec} k \to \mathcal{X} \to \mathcal{Y} \to \mathcal{Y}^v$) with $\mathcal{L}_{\mathcal{X}}|_{G_{\mathcal{X},k}} \cong \mathcal{L}_{\mathcal{Y}^v}|_{G_{\mathcal{Y}^v,k}}$; transporting T, q and $\mathcal{M}$ along this isomorphism gives the claim. □

Corollary D.61. *Let E be a CM field, $\mathcal{X}$ a normal integral noetherian scheme and $(G, \iota, \mathcal{L})$ a semiabelian structure over $\mathcal{X}$ with abelian generic fibre satisfying (PF). Let $p > 2$ be a prime, κ a field of characteristic p, C an integral normal projective curve over κ, $c\colon C \to \mathcal{X}$ a morphism, and $L = c^*\mathrm{Lie}(G)$ with $\rho_L(a) = c^*\mathrm{Lie}(\iota(a))$. Then the degrees of the subsheaves of L have a maximum $P \geq 0$ with $(p-2)P \leq -\deg L$, and for every finite field $k \subseteq \Gamma(C, \mathcal{O}_C)$ and every T as in Lemma D.63 (with $K = E$ and complex conjugation as involution), $\deg L_\tau = \deg L_{\bar{\tau}}$ for $\tau \in T$ and the weighted flag inequality (ii) of Lemma D.63 holds for L.*

Proof. By (i) and (ii) of Corollary D.42, applied to c, the bundles L, $L^\vee = c^*\omega_G$ (with a acting by $\rho_L(\bar a)^\vee$) and $H = c^*\mathrm{H}^\natural(G,\mathcal{L})$ (with $c^*\rho^\natural$) form $\mathcal{O}_E$-equivariant exact sequences $0 \to L^\vee \to H \to L \to 0$ and $0 \to F^*L \xrightarrow{\alpha} H \to F^*(L^\vee) \to 0$, the outer terms of the second carrying F^* of the actions on L and $L^\vee$. The first sequence and the injective map α are (a), (b) of Lemma D.62, which gives $P \geq 0$ and $(p-2)P \leq \deg L^\vee = -\deg L$. The two sequences are the hypotheses of Lemma D.63 with $K = E$, which gives $\deg L_\tau = \deg L_{\bar\tau}$ and the weighted flag inequality. □

Lemma D.62. *Let $p > 2$ be a prime, κ a field of characteristic p, C an integral normal projective curve over κ with absolute Frobenius $F\colon C \to C$, and L, H vector bundles on C. Suppose that*

(a) there is an exact sequence of $\mathcal{O}_C$-modules

$$0 \to L^\vee \to H \to L \to 0,$$

where $L^\vee = \mathcal{H}om_{\mathcal{O}_C}(L, \mathcal{O}_C)$;

(b) there is an $\mathcal{O}_C$-linear map $\alpha\colon F^*L \to H$ which is injective at the generic point of C.

Then the degrees of the subsheaves of L have a maximum $P \geq 0$, and

$$(p-2)\,P \leq \deg L^\vee.$$

In particular $\deg L^\vee \geq 0$, and $(p-2)\deg U \leq \deg L^\vee$ for every subsheaf $U \subseteq L$.

Proof. Degrees are over κ. The degrees of the subsheaves of a vector bundle on C are bounded above (existence of the Harder–Narasimhan filtration), and the zero subsheaf has degree 0; let $P \geq 0$ be the maximum and $U_0 \subseteq L$ a subsheaf with $\deg U_0 = P$. Its saturation contains it with torsion quotient, so has degree $\geq P$; hence we may take U_0 saturated, i.e. L/U_0 locally free and U_0 locally a direct summand of L.

The Frobenius image. As U_0 is locally a direct summand, $F^*U_0 \to F^*L$ is injective, and $\deg F^*U_0 = \deg\det F^*U_0 = \deg(\det U_0)^{\otimes p} = pP$. The composite $F^*U_0 \to F^*L \xrightarrow{\alpha} H$ is injective at the generic point by (b), hence injective (its kernel is a torsion subsheaf of the vector bundle F^*U_0). Let $V \subseteq H$ be its image; $V \cong F^*U_0$, so $\deg V = pP$.

Splitting V along (a). Let $V_1 = V \cap L^\vee$ and $V_2 \subseteq L$ the image of V in L; then $0 \to V_1 \to V \to V_2 \to 0$ is exact, so $\deg V = \deg V_1 + \deg V_2$, and $\deg V_2 \leq P$ by the choice of P. Let V_1' be the saturation of V_1 in $L^\vee$ and $Q = L^\vee/V_1'$, a vector bundle. Dualising $L^\vee \to Q \to 0$ gives an injection $Q^\vee \to L^{\vee\vee} = L$, so $-\deg Q = \deg Q^\vee \leq P$, and

$$\deg V_1 \leq \deg V_1' = \deg L^\vee - \deg Q \leq \deg L^\vee + P.$$

Conclusion. $pP = \deg V \leq \deg L^\vee + 2P$, i.e. $(p-2)P \leq \deg L^\vee$. As $p > 2$ and $P \geq 0$, $\deg L^\vee \geq 0$, and for a subsheaf $U \subseteq L$, $(p-2)\deg U \leq (p-2)P \leq \deg L^\vee$. □

Lemma D.63. *Let $p > 2$ be a prime, κ a field of characteristic p, C an integral normal projective curve over κ with absolute Frobenius F, K a number field with ring of integers $\mathcal{O}_K$ and a ring involution $a \mapsto \bar a$ of $\mathcal{O}_K$. Let L and H be vector bundles on C with $\mathcal{O}_K$-actions ρ_L, ρ_H; let $\mathcal{O}_K$ act on $L^\vee$ by $a \mapsto \rho_L(\bar a)^\vee$. Suppose there are $\mathcal{O}_K$-equivariant exact sequences*

$$0 \to L^\vee \to H \to L \to 0, \qquad 0 \to F^*L \to H \to F^*(L^\vee) \to 0.$$

Let $k \subseteq \Gamma(C, \mathcal{O}_C)$ be a finite field such that, with Σ the set of ring homomorphisms $\mathcal{O}_K \to k$, the map $k \otimes_{\mathbb{Z}} \mathcal{O}_K \to k^\Sigma$, $c \otimes a \mapsto (c\,s(a))_s$, is bijective; for $s \in \Sigma$ let $e_s \in k \otimes \mathcal{O}_K$ be the idempotent mapping to the s-th unit vector, $L_s = e_s L$ (acting by $c \otimes a \mapsto c\rho_L(a)$) and $\bar s = s \circ {}^-$. Let $T \subseteq \Sigma$ with $\Sigma = T \sqcup \{\bar\tau : \tau \in T\}$.

(i) $\deg L_\tau = \deg L_{\bar\tau}$ *for every* $\tau \in T$.

(ii) Let $j, q \geq 1$*; for each* $\tau \in T$ *let* $W_{\tau,0}, \dots, W_{\tau,j}$ *and* $V_{\tau,0}, \dots, V_{\tau,q}$ *be subsheaves of* L *with* $W_{\tau,0} = V_{\tau,0} = 0$*,* $W_{\tau,j} = L_\tau$*,* $V_{\tau,q} = L_{\bar\tau}$*, and let* $\mu_{\tau,1} \geq \mu_{\tau,2} \geq \cdots \geq \mu_{\tau,j+q}$ *be real numbers. Let* $M, g \in \mathbb{R}$ *with* $|\mu_{\tau,i}| \leq M$ *and* $g \leq \mu_{\tau,j} - \mu_{\tau,j+1}$ *for all* τ, i*, and*

$$4\,|T|\,M \leq (p-2)\,g.$$

Then

$$\sum_{\tau\in T}\Big(\sum_{i=1}^{j}\mu_{\tau,i}\big(\deg W_{\tau,i} - \deg W_{\tau,i-1}\big) + \sum_{i=1}^{q}\mu_{\tau,j+q+1-i}\big(\deg V_{\tau,i-1} - \deg V_{\tau,i}\big)\Big) \leq 0.$$

Proof. Write $A = k\otimes_{\mathbb{Z}}\mathcal{O}_K \cong k^{\Sigma}$; for a vector bundle V with $\mathcal{O}_K$-action, A acts by $c\otimes a \mapsto c\rho_V(a)$, $V = \bigoplus_s e_s V$, and the e_s-part of an equivariant exact sequence is exact.

The pieces of $L^\vee$. Since $a \mapsto \bar a$ is an involution, $(1\otimes{}^-)(e_s) = e_{\bar s}$: for $s' \in \Sigma$, $s'_A((1\otimes{}^-)e_s) = (s'\circ{}^-)_A(e_s)$ is 1 if $s'\circ{}^- = s$, i.e. $s' = \bar s$, and 0 otherwise. The action of $c\otimes a$ on $L^\vee$ is $c\rho_L(\bar a)^\vee = \rho_L(c\otimes\bar a)^\vee$, so e_s acts on $L^\vee$ as $\rho_L(e_{\bar s})^\vee$, whose image is $(e_{\bar s}L)^\vee = L^\vee_{\bar s}$ (dualise $L = e_{\bar s}L \oplus (1-e_{\bar s})L$).

(i). The sequences of the statement are (a) of Lemma D.52 with $M_0 = L^\vee$, $M_1 = L$, so $\deg e_\tau H = 0$ by (i) there. The e_τ-part of the first sequence is $0 \to L^\vee_{\bar\tau} \to e_\tau H \to L_\tau \to 0$, hence $0 = -\deg L_{\bar\tau} + \deg L_\tau$.

The slope bound. The first sequence is (a) of Lemma D.62, and the first map of the second sequence is an injective $\mathcal{O}_C$-linear map $F^*L \to H$, which is (b) there. So the degrees of subsheaves of L have a maximum $P \geq 0$ and

$$(p-2)P \leq \deg L^\vee = -\deg L. \tag{D.14}$$

By (i), with $D_\tau = \deg L_\tau$, $\deg L = \sum_{s\in\Sigma}\deg L_s = \sum_{\tau\in T}(\deg L_\tau + \deg L_{\bar\tau}) = 2\sum_\tau D_\tau$.

(ii): Abel summation for one τ. If $T = \emptyset$ there is nothing to prove; otherwise $M \geq |\mu_{\tau,1}| \geq 0$, so $(p-2)g \geq 4|T|M \geq 0$ and $g \geq 0$. Fix τ; write $\mu_i = \mu_{\tau,i}$, $w_i = \deg W_{\tau,i}$, $v_i = \deg V_{\tau,i}$, so $w_0 = v_0 = 0$, $w_j = D_\tau$, $v_q = \deg L_{\bar\tau} = D_\tau$ by (i), and $w_i, v_i \leq P$. Summation by parts gives

$$\sum_{i=1}^{j}\mu_i(w_i - w_{i-1}) = \mu_j D_\tau + \sum_{i=1}^{j-1}(\mu_i - \mu_{i+1})w_i \leq \mu_j D_\tau + (\mu_1 - \mu_j)P,$$

and, with $\nu_i = \mu_{j+q+1-i}$ (so $\nu_1 \leq \cdots \leq \nu_q = \mu_{j+1}$),

$$\sum_{i=1}^{q}\nu_i(v_{i-1} - v_i) = -\nu_q D_\tau + \sum_{i=1}^{q-1}(\nu_{i+1} - \nu_i)v_i \leq -\mu_{j+1}D_\tau + (\mu_{j+1} - \mu_{j+q})P,$$

since all the coefficients $\mu_i - \mu_{i+1}$, $\nu_{i+1} - \nu_i$ are ≥ 0. Let $G = \mu_j - \mu_{j+1} \geq g$ and S_τ the sum of the two left sides. Then

$$S_\tau \leq GD_\tau + (\mu_1 - \mu_{j+q} - G)P = gD_\tau + (G-g)(D_\tau - P) + (\mu_1 - \mu_{j+q} - g)P \leq gD_\tau + 2MP,$$

because $(G-g) \geq 0$ and $D_\tau - P \leq 0$ (L_τ is a subsheaf of L), $\mu_1 - \mu_{j+q} \leq 2M$, and $gP \geq 0$.

(ii): summing over τ. By the last two displays and (D.14),

$$\begin{aligned}\sum_{\tau\in T} S_\tau &\leq g\sum_\tau D_\tau + 2|T|MP = -\tfrac{g}{2}\deg L^\vee + \tfrac{1}{2}\,(4|T|M)\,P\\ &\leq -\tfrac{g}{2}\deg L^\vee + \tfrac{1}{2}(p-2)gP \leq -\tfrac{g}{2}\deg L^\vee + \tfrac{g}{2}\deg L^\vee = 0,\end{aligned}$$

using $4|T|M \leq (p-2)g$, $P \geq 0$ and $g \geq 0$. The constant 4 is $2\cdot 2$: one factor 2 from $\deg L = 2\sum_\tau D_\tau$, one from the two boundary terms μ_1 and μ_{j+q} of the Abel summation. □

D.4 Flat determinants, square systems and covolumes

This section proves four statements used in Part VI. Three are facts of good towers in Proposition 34.2: the base determinant heights are flat heights of flat generators up to bad-prime terms (Proposition D.65, Subsection D.4.1), which gives (Db); good towers have proportional analytic square systems (Proposition D.73); and they have geometric proportionality with covolumes (Vp) (Proposition D.79, Subsection D.4.2). The fourth is the fibre-square identities (Fq1) (Theorem D.85, Subsection D.4.3), used in the proof of Proposition 31.14 for (Gt4). (Vp) is the place where covolumes enter the volumes of the towers: the geometric intersection numbers of the root models are sums of covolumes of principal congruence subgroups of lattices in one genus, times flag degrees.

D.4.1 Base determinant heights as flat heights

Proposition D.65 follows from flat determinant lines on the stage-two root models, whose heights split into a flat height and local terms (Proposition D.66). The flat height is computed by Theorem 34.20; the local terms are controlled because the flat generator has constant order along the fibre over every prime of the constant field (Proposition D.67), and its order at a point over a prime is compared with its ℓ-adic norm (Proposition D.69). The comparison uses the Hodge–Tate sandwich of Section 35, and the vertical orders use the full flags of a trivial bundle from the same section.

Definition D.64. Let $\mathcal{X} \subseteq \mathbb{P}$ be a root model of type (Φ, n), and write $\widehat{\mathcal{X}} \subseteq \widehat{\mathbb{C}}$ for the affine cone over $\mathcal{X}(\mathbb{C})$. For a cone metric h of multidegree e, $c_1(h)$ denotes the first Chern form of the Hermitian line $(\mathcal{O}(e), h)|_{\mathcal{X}}$ on $\mathcal{X}(\mathbb{C})$ (for $h = h_r(t)$: $c_1(\overline{\mathcal{L}}_r(t))$).

(Dl1) Flat lines. Let $e_A \in \mathbb{Z}^K$, let h_A be a cone metric of multidegree e_A, and put $\overline{\mathcal{A}} = (\mathcal{O}(e_A), h_A)|_{\mathcal{X}}$. Let F be a number field. The line $\overline{\mathcal{A}}$ is **flat over** F if the structure morphism $\mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ factors through $\operatorname{Spec}\mathcal{O}_F$, and there are an integer $M \geq 1$, multihomogeneous polynomials f, g with integer coefficients, of multidegrees d_f, d_g with $d_f - d_g = Me_A$, neither vanishing identically on $\mathcal{X}$, such that the rational section f/g of $\mathcal{O}(Me_A)|_{\mathcal{X}}$ generates $\mathcal{O}(Me_A)|_{\mathcal{X}}$ at every point of $\mathcal{X}$ (it has neither zeros nor poles), and positive real numbers κ_s $(s\colon F \to \mathbb{C})$ with $\kappa_{\bar{s}} = \kappa_s$ and

$$|f(z)| = \bigl(\kappa_s\, h_A(z)\bigr)^M\, |g(z)| \qquad \text{for every } z \in \widehat{\mathcal{X}} \text{ over a point of } \mathcal{X}(\mathbb{C}) \text{ lying over } s.$$

Equivalently, the section f/g has the constant norm κ_s^M on the part of $\mathcal{X}(\mathbb{C})$ over s. The **height** of $\overline{\mathcal{A}}$ (with these data) is

$$\operatorname{ht}(\overline{\mathcal{A}}) = -\frac{1}{[F:\mathbb{Q}]} \sum_{s\colon F \to \mathbb{C}} \log \kappa_s.$$

(Dl2) Flat determinant lines. Let $a \in \Phi$. A **flat determinant line of $\mathcal{X}$ at** a is a line $\overline{\mathcal{A}} = (\mathcal{O}(e_A), h_A)|_{\mathcal{X}}$ flat over some number field, with data as in (Dl1), such that $e_A = \sum_{i=1}^{n} e_{(a,i)}$, together with functions $v_t\colon \widehat{\mathbb{C}} \to \mathbb{R}$ $(t > 0)$, smooth and invariant under $(\lambda_k) \in (\mathbb{C}^\times)^K$, such that

$$\prod_{i=1}^{n} h_{(a,i)}(t)(z) = h_A(z)\, e^{-v_t(z)} \qquad (t > 0,\ z \in \widehat{\mathcal{X}}),$$

and, for all roots $r_1, \dots, r_D \in \mathrm{Rt}$,

$$\lim_{t \to 0^+} \int_{\mathcal{X}(\mathbb{C})} v_t\, c_1(\overline{\mathcal{L}}_{r_1}(t)) \wedge \cdots \wedge c_1(\overline{\mathcal{L}}_{r_D}(t)) = 0.$$

Its height is the height of $\overline{\mathcal{A}}$.

Proposition D.65. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2 (so every $\mathcal{X}_{B,Z}$ has convergent regularisation, by Theorem 31.20). Let $\mathcal{T}'$ be a root tower of type (Φ, n, p) with the components and the bad primes of $\mathcal{T}$ such that $\widehat{\deg}'_{2,Z}(F) = \mathrm{pp}(\mathcal{X}_{2,Z}; F)$ for every $Z \in \mathrm{Cp}$ and every $F \in \mathcal{R}$, where $\widehat{\deg}'$ are the arithmetic degrees of $\mathcal{T}'$, and such that $\mathcal{T}'$ with the geometric realisation of $\mathcal{T}$ has (Gi1). Then $\mathfrak{u}$ and $\mathcal{T}'$ satisfy the hypotheses 1–3 of Lemma 34.19: for every $Z \in \mathrm{Cp}$ and every $a \in \Phi$ there are a base determinant height $\mathrm{hf}(Z, a)$ with the stage-two degree identity 1, a flat generator with all its norms as in 2, and real numbers d_ℓ, v_ℓ ($\ell \in \mathrm{Bd}$) as in 3.*

Proof. Here $\Phi = \Phi_{\mathsf{H}}$ and $n = p + q$ (Proposition 34.2). Write $\mathrm{vol}'(Z)$ for the volumes of $\mathcal{T}'$. Fix $Z \in \mathrm{Cp}$ and $a \in \Phi$, and put $\mathcal{X} = \mathcal{X}_{2,Z}$. By Proposition D.66 there are U, Y, z_0, a flat generator φ over Y for $\tau = a$ with constants (e, c) and all its norms N_s, N_κ, a flat determinant line $\overline{\mathcal{A}}$ of $\mathcal{X}$ at a, and real numbers d_ℓ ($\ell \in \mathrm{Bd}$) as in its items 1–3. Put $\mathrm{hf}(Z, a) = \mathrm{ht}(\overline{\mathcal{A}})$.

Hypothesis 1. $\mathcal{X}$ has convergent regularisation by Theorem 31.20. Let $Q \in \mathcal{R}$ be homogeneous of degree D. Then $\mathrm{ps}_{a,1} Q \in \mathcal{R}$, and by the hypothesis on $\mathcal{T}'$, by Lemma D.72, and by (Gi1) for $\mathcal{T}'$ with the geometric realisation of $\mathcal{T}$,

$$\widehat{\deg}'_{2,Z}(\mathrm{ps}_{a,1} Q) = \mathrm{pp}(\mathcal{X}; \mathrm{ps}_{a,1} Q) = \mathrm{ht}(\overline{\mathcal{A}})\, \mathrm{gi}(\mathcal{X}; Q) = \mathrm{hf}(Z, a)\, \mathrm{vol}'(Z)\, \deg^{\mathrm{fl}}(Q).$$

Hypothesis 2. $\tau = a \in \Phi_{\mathsf{H}}$, U, Y, z_0 and φ with its norms are as required (item 1 of Proposition D.66).

Hypothesis 3. For $\ell \in \mathrm{Bd}$ put $v_\ell = w_\ell / ([E : \mathbb{Q}] \log \ell)$. The different $\mathfrak{d}_E$ is an integral ideal, and every ring homomorphism $j \colon E \to \mathbb{C}_\ell$ maps $\mathcal{O}_E$ into the closed unit ball, so $|\mathfrak{d}_E|_j \le 1$, $w_\ell \ge 0$ and $v_\ell \ge 0$. Item 3 of Proposition D.66 gives $|d_\ell| \le n\bigl(c_\ell \log \ell + \frac{w_\ell}{[E:\mathbb{Q}]}\bigr) = (n\, c_\ell + n\, v_\ell) \log \ell$ and the decomposition of $\mathrm{hf}(Z, a) = \mathrm{ht}(\overline{\mathcal{A}})$ required in 3. It remains to bound $\sum_{\ell \in \mathrm{Bd}} v_\ell \log \ell = \frac{1}{[E:\mathbb{Q}]} \sum_{\ell \in \mathrm{Bd}} w_\ell$. For a prime $\mathfrak{P}$ of E above ℓ, with ramification index $e_{\mathfrak{P}}$ and residue degree $f_{\mathfrak{P}}$, exactly $e_{\mathfrak{P}} f_{\mathfrak{P}}$ homomorphisms $j \colon E \to \mathbb{C}_\ell$ induce $\mathfrak{P}$, and for each of them $|\mathfrak{d}_E|_j = \ell^{-v_{\mathfrak{P}}(\mathfrak{d}_E)/e_{\mathfrak{P}}}$, since $\mathfrak{d}_E$ is dense in $\mathfrak{d}_E \mathcal{O}_{E_{\mathfrak{P}}} = \mathfrak{P}^{v_{\mathfrak{P}}(\mathfrak{d}_E)} \mathcal{O}_{E_{\mathfrak{P}}}$. Hence w_ℓ is the logarithm of the ℓ-part of the absolute norm $\mathrm{N}(\mathfrak{d}_E)$:

All w_ℓ are ≥ 0 and $\mathrm{N}(\mathfrak{d}_E) = |\mathrm{disc}(E)|$ (Dedekind), so

$$\sum_{\ell \in \mathrm{Bd}} v_\ell \log \ell \le \frac{1}{[E : \mathbb{Q}]} \sum_{\ell} w_\ell = \frac{\log |\mathrm{disc}(E)|}{[E : \mathbb{Q}]} = \log \mathrm{rd}_E.$$

So $\mathrm{hf}(Z, a)$, φ with its norms, and d_ℓ, v_ℓ ($\ell \in \mathrm{Bd}$) satisfy hypotheses 1–3 of Lemma 34.19 at (Z, a). As Z and a were arbitrary, this proves the proposition. □

Proposition D.66. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. For a prime ℓ put $c_2 = 2$, $c_\ell = \frac{1}{\ell - 1}$ if $\ell \ge 3$, and $w_\ell = -\sum_{j \colon E \to \mathbb{C}_\ell} \log |\mathfrak{d}_E|_j$. Then for every $Z \in \mathrm{Cp}$ and every $a \in \Phi$ there are:*

1. *a union $U \subseteq \mathrm{Sh}_F$ of connected components, a connected component Y of $U(\mathbb{C})$, $z_0 \in Y$, and a flat generator φ over Y for $\tau = a$ with constants (e, c) that has an archimedean norm N_s at every embedding $s \colon F \to \mathbb{C}$ and an ℓ-adic norm N_κ at every ring homomorphism $\kappa \colon F \to \mathbb{C}_\ell$, for every prime ℓ;*

2. *a flat determinant line* $\overline{\mathcal{A}}$ *of the root model* $\mathcal{X}_{2,Z}$ *at* a*;*

3. *real numbers* d_ℓ *(*$\ell \in \mathrm{Bd}$*) with* $|d_\ell| \le n\big(c_\ell \log \ell + \frac{w_\ell}{[E:\mathbb{Q}]}\big)$ *and*

$$\mathrm{ht}(\overline{\mathcal{A}}) = -\frac{1}{[F:\mathbb{Q}]}\Big(\sum_{s\colon F\to\mathbb{C}} N_s + \sum_{\ell}\sum_{\kappa\colon F\to\mathbb{C}_\ell} N_\kappa\Big) + \sum_{\ell\in\mathrm{Bd}} d_\ell$$

(the sum over all primes ℓ *has finitely many non-zero terms, by Theorem 34.20).*

Proof. Fix $Z \in \mathrm{Cp}$ (we write $c = Z$) and $a \in \Phi$, and put $\mathcal{X} = \mathcal{X}_{2,c}$ with function field Ω and representing coordinates y, $e_A = \sum_{i=1}^n e_{(a,i)}$, and $\mathcal{D}_{\mathcal{X}} = \det \mathcal{V}_{(a,+)} \otimes (\det \mathcal{V}_{(a,-)})^{-1}$.

Step 1: clause 1. By (Go2), $\mathcal{Y}_c$ is an irreducible component of $\mathcal{Y}$ meeting $j(\mathrm{Sh}_F)$. By Proposition D.38 (components split, level good at every prime, $n \ge 1$) there is a nowhere vanishing section ξ of $\det \mathrm{Lie}_{(a,+)} \otimes (\det \mathrm{Lie}_{(a,-)})^{-1}$ on $\mathcal{Y}_c \times_{\mathcal{O}_F} F$; with $U = Z_c$, a connected component of Sh_F, any connected component Y of $U(\mathbb{C})$ and $z_0 \in Y$, the frame function $\varphi = \varphi_\xi$ is a flat generator over Y for $\tau = a$ with constants (e, c) and has archimedean norms N_s at every $s\colon F \to \mathbb{C}$ and ℓ-adic norms N_κ at every $\kappa\colon F \to \mathbb{C}_\ell$, every ℓ. By (Fd1) of Theorem 34.20, $N_\kappa = 0$ above all but finitely many ℓ. Two facts on these norms. (i) $N_{\bar s} = N_s$: complex conjugation carries $U(\mathbb{C})$ over s to $U(\mathbb{C})$ over $\bar s$, frames to frames, φ to $\overline{\varphi}$ (ξ is defined over F, (Yf2)) and the Hodge forms to their conjugates, so the Gram determinants are unchanged. (ii) N_κ depends only on the prime $\mathfrak{l}$ of $\mathcal{O}_F$ induced by κ, by Lemma D.70, for every $\mathfrak{l}$ at which data of (Yf3) exist; they exist at every $\mathfrak{l}$, at the points z of Step 6 (the base change of $G_{\mathcal{X}}$ along z with bases of its free Lie eigenmodules); write $N_{\mathfrak{l}}$.

Step 2: the rational section. By (Fh3), (Fh6) and the root sections (Go3)(3), $\mathcal{D}_{\mathcal{X}} \cong \bigotimes_i \mathcal{L}_{(a,i)}$ and $P \mapsto \varrho_A P(y)$, $\varrho_A = \prod_i \varrho_{(a,i)}$, is an isomorphism $\mathcal{O}(e_A)|_{\mathcal{X}} \to \mathcal{D}_{\mathcal{X}}$; at every complex point z of the abelian locus and every cone point $\tilde z$ over it, (Go3)(4) and the Gram-determinant formula of (Fh4) give $\|\varrho_A P(y)\|^{\mathrm{Hdg}}(z) = |P(\tilde z)|/\prod_i h^{\mathrm{sing}}_{(a,i)}(\tilde z)$, the Hodge norm of $\mathcal{D}_{\mathcal{X}}$ being the product of the step-quotient norms. Write $u_\xi = \varrho_A \theta$ with $\theta \in \Omega^\times$ and $\theta = F_0(y)/G_0(y)$ for multihomogeneous F_0, G_0 with integer coefficients and $\deg F_0 - \deg G_0 = e_A$ (the elements of Ω are ratios of polynomials in y of equal multidegree; multiply by a monomial of multidegree e_A).

Step 3: the vertical orders. By Proposition D.67 ($\mathrm{Bd} = S_2$ contains every prime dividing $\mathrm{disc}(E)$), $\mathcal{X} \to \mathrm{Spec}\,\mathbb{Z}$ factors through $\mathrm{Spec}\,\mathcal{O}_{F'}$ with connected fibres, u_ξ generates $\mathcal{D}_{\mathcal{X}}$ at every point over no prime, and for each maximal ideal $\mathfrak{p}$ of $\mathcal{O}_{F'}$ with a uniformiser $\pi_\mathfrak{p} \in F'$ there are $M_\mathfrak{p} \ge 1$, $a_\mathfrak{p}$ with $u_\xi^{M_\mathfrak{p}} \pi_\mathfrak{p}^{-a_\mathfrak{p}}$ generating $\mathcal{D}_{\mathcal{X}}^{\otimes M_\mathfrak{p}}$ at every point over $\mathfrak{p}$; $r_\mathfrak{p} = a_\mathfrak{p}/M_\mathfrak{p}$ is the common ratio of the order of u_ξ and of $\pi_\mathfrak{p}$ along the components of the fibre, hence independent of choices. The closed set where u_ξ does not generate $\mathcal{D}_{\mathcal{X}}$ lies over no point of characteristic 0; its image in $\mathrm{Spec}\,\mathcal{O}_{F'}$ is closed ($\mathcal{X}$ is proper) and misses the generic point, hence is a finite set P_0, and $r_\mathfrak{p} = 0$ for $\mathfrak{p} \notin P_0$ (then $M_\mathfrak{p} = 1$, $a_\mathfrak{p} = 0$ will do). Let h be the class number of F', $M' = \prod_{\mathfrak{p}\in P_0} M_\mathfrak{p}$, $M = hM'$; then $Mr_\mathfrak{p} = h\,(M'/M_\mathfrak{p})a_\mathfrak{p} \in h\mathbb{Z}$, so $\prod_{\mathfrak{p}\in P_0} \mathfrak{p}^{Mr_\mathfrak{p}}$ is the h-th power of a fractional ideal, hence principal, $= (\alpha)$ with $\alpha \in F'^\times$. The rational section $u_\xi^M \alpha^{-1}$ of $\mathcal{D}_{\mathcal{X}}^{\otimes M}$ generates at every point: over $\mathfrak{p} \in P_0$ it is the $(M/M_\mathfrak{p})$-th power of $u_\xi^{M_\mathfrak{p}} \pi_\mathfrak{p}^{-a_\mathfrak{p}}$ times $\pi_\mathfrak{p}^{Mr_\mathfrak{p}} \alpha^{-1}$, a unit of $\mathcal{O}_{F',\mathfrak{p}}$; over $\mathfrak{p} \notin P_0$ and over no prime, u_ξ generates and α is a unit. Write $\alpha = A_0(y)/B_0(y)$ with A_0, B_0 multihomogeneous of equal multidegree with integer coefficients, and put $f = F_0^M B_0$, $g = G_0^M A_0$. Then f, g have integer coefficients, $\deg f - \deg g = Me_A$, neither vanishes on $\mathcal{X}$, and the rational section f/g of $\mathcal{O}(Me_A)|_{\mathcal{X}}$, which corresponds to $u_\xi^M \alpha^{-1}$ under the M-th power of the isomorphism of Step 2, generates $\mathcal{O}(Me_A)|_{\mathcal{X}}$ at every point.

Step 4: the metric and (Dl1). For $s\colon F' \to \mathbb{C}$ put $\kappa_s = |s(\alpha)|^{-1/M} e^{N_{s|_F}}$; then $\kappa_{\bar s} = \kappa_s$ by (i). At a cone point $\tilde z$ over a point of $\mathcal{X}(\mathbb{C})$ over s, choose integer multihomogeneous

f', g' with $f'/g' = f/g$ in Ω and $g'(\tilde z) \neq 0$ (possible as f/g generates); then $f'(\tilde z) \neq 0$ and $h_A(\tilde z) = \kappa_s^{-1}(|f'(\tilde z)|/|g'(\tilde z)|)^{1/M}$ is independent of the choice (two choices agree on a dense open set), smooth, positive, homogeneous of multidegree e_A and invariant under conjugation; extend $\log(h_A/h_0)$, for the smooth cone metric $h_0(z) = \prod_k \|z_k\|^{e_{A,k}}$, from the closed submanifold $\mathcal{X}(\mathbb{C})$ ($\mathcal{X} \otimes \mathbb{Q}$ is smooth) to a smooth conjugation-invariant function on $\prod_k \mathbb{P}(\mathbb{C}^{I_k})$, which makes h_A a cone metric of multidegree e_A. By construction $|f(\tilde z)| = (\kappa_s h_A(\tilde z))^M |g(\tilde z)|$ at every cone point over s, so $\overline{\mathcal{A}} = (\mathcal{O}(e_A), h_A)|_{\mathcal{X}}$ is flat over F' (Dl1). At a complex point z of the abelian locus over s, with y^s the induced point of U over $s|_F$: u_ξ at z is $\xi(j(y^s))$, of Hodge norm $|\varphi(y^s; v, w)| \det \Gamma(v)^{1/2} \det \Gamma(w)^{-1/2} = e^{N_{s|_F}}$ (Yf3); by Step 2,

$$\frac{|f(\tilde z)|}{|g(\tilde z)| \prod_i h^{\mathrm{sing}}_{(a,i)}(\tilde z)^M} = \|u_\xi^M \alpha^{-1}\|^{\mathrm{Hdg}}(z) = \frac{e^{M N_{s|_F}}}{|s(\alpha)|},$$

so $h_A = \prod_i h^{\mathrm{sing}}_{(a,i)}$ at all cone points over the abelian locus.

Step 5: (Dl2), clause 2. By Lemma D.71 (root origin of $\mathcal{X}$ realised on $\mathcal{X}(\mathbb{C})$ without resolution, h_A as just shown), $v_t = \log h_A - \sum_i \log h_{(a,i)}(t)$ is smooth and invariant, $\prod_i h_{(a,i)}(t) = h_A e^{-v_t}$, and the limits of (Dl2) vanish. So $\overline{\mathcal{A}}$ with these data is a flat determinant line of $\mathcal{X}_{2,Z}$ at a.

Step 6: the height. By (Dl1) and the product formula $\sum_s \log|s(\alpha)| = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(\alpha) \log N\mathfrak{p} = M \sum_{\mathfrak{p}} r_{\mathfrak{p}} \log N\mathfrak{p}$, and since every $s_0 \colon F \to \mathbb{C}$ has $[F' : F]$ extensions to F',

$$\mathrm{ht}(\overline{\mathcal{A}}) = -\frac{1}{[F' : \mathbb{Q}]} \sum_s \log \kappa_s = -\frac{1}{[F : \mathbb{Q}]} \sum_{s_0} N_{s_0} + \frac{1}{[F' : \mathbb{Q}]} \sum_{\mathfrak{p}} r_{\mathfrak{p}} \log N\mathfrak{p}.$$

Fix a prime ℓ and $\mathfrak{p} \mid \ell$ with ramification index $e_{\mathfrak{p}}$ and residue degree $f_{\mathfrak{p}}$ over ℓ. The fibre of $\mathcal{X}$ over $\mathfrak{p}$ is non-empty ($\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{F'}$ is proper and dominant, hence surjective) and the regular locus contains the generic points of its components (normality), so it has a regular closed point x; by Lemma D.68, applied with Δ the closed set of points mapping outside the open locus of $\mathcal{Y}$ over which $G_{\mathcal{Y}}$ is abelian, there is a point $z \colon \operatorname{Spec} \mathcal{O}_{k,\mathfrak{P}} \to \mathcal{X}$ with centre x whose generic point lies in the abelian locus ((Ud7)), with $\vartheta \colon k \to \mathbb{C}_\ell$; $\vartheta|_{F'}$ induces $\mathfrak{p}$, so $\log|\vartheta(\pi_{\mathfrak{p}})| = -\log \ell / e_{\mathfrak{p}}$, and $\kappa = \vartheta \circ \beta$ induces $\mathfrak{l} = \mathfrak{p} \cap \mathcal{O}_F$. At x, $u_\xi^{M_{\mathfrak{p}}} \pi_{\mathfrak{p}}^{-a_{\mathfrak{p}}}$ is a unit times a generator. Put $\delta_{\mathfrak{p}} = N_{\mathfrak{l}} + r_{\mathfrak{p}} \log \ell / e_{\mathfrak{p}}$. Proposition D.69 gives: $\delta_{\mathfrak{p}} = 0$ if $\ell \notin \mathrm{Bd}$ (then $\ell \notin S_2$ and $\ell \nmid \mathrm{disc}(E)$); and if $\ell \in \mathrm{Bd}$, for every non-zero $d \in \mathfrak{d}_E$, $|\delta_{\mathfrak{p}}| \leq n(c_\ell \log \ell - \log|\kappa(a(d))|)$. Choose d with $|\kappa(a(d))| = |\mathfrak{d}_E|_{\kappa \circ a}$ (the supremum in Definition 33.13 is attained: the values are a discrete set of powers of ℓ bounded by 1). As E is Galois over $\mathbb{Q}$, any two embeddings $j, j' \colon E \to \mathbb{C}_\ell$ satisfy $j' = j \circ \gamma$ with $\gamma \in \mathrm{Gal}(E/\mathbb{Q})$, and $\gamma(\mathfrak{d}_E) = \mathfrak{d}_E$, so $|\mathfrak{d}_E|_j$ does not depend on j and $-\log|\mathfrak{d}_E|_{\kappa \circ a} = w_\ell / [E : \mathbb{Q}]$. Hence $|\delta_{\mathfrak{p}}| \leq n(c_\ell \log \ell + w_\ell / [E : \mathbb{Q}])$ for $\ell \in \mathrm{Bd}$. Now $r_{\mathfrak{p}} \log N\mathfrak{p} = r_{\mathfrak{p}} f_{\mathfrak{p}} \log \ell = e_{\mathfrak{p}} f_{\mathfrak{p}} (\delta_{\mathfrak{p}} - N_{\mathfrak{l}})$. The ring homomorphisms $F \to \mathbb{C}_\ell$ inducing $\mathfrak{l}$ number $[F_{\mathfrak{l}} : \mathbb{Q}_\ell]$, all with the norm $N_{\mathfrak{l}}$ by (ii), and $\sum_{\mathfrak{p} \mid \mathfrak{l}} e_{\mathfrak{p}} f_{\mathfrak{p}} = [F' : F]\,[F_{\mathfrak{l}} : \mathbb{Q}_\ell]$; so

$$\frac{1}{[F' : \mathbb{Q}]} \sum_{\mathfrak{p} \mid \ell} r_{\mathfrak{p}} \log N\mathfrak{p} = -\frac{1}{[F : \mathbb{Q}]} \sum_{\kappa \colon F \to \mathbb{C}_\ell} N_\kappa + d_\ell, \qquad d_\ell = \frac{1}{[F' : \mathbb{Q}]} \sum_{\mathfrak{p} \mid \ell} e_{\mathfrak{p}} f_{\mathfrak{p}}\, \delta_{\mathfrak{p}}.$$

For $\ell \notin \mathrm{Bd}$, $d_\ell = 0$; for $\ell \in \mathrm{Bd}$, as $\sum_{\mathfrak{p} \mid \ell} e_{\mathfrak{p}} f_{\mathfrak{p}} = [F' : \mathbb{Q}]$, $|d_\ell| \leq n(c_\ell \log \ell + w_\ell / [E : \mathbb{Q}])$. Summing over ℓ (finitely many non-zero terms: $r_{\mathfrak{p}} = 0$ off P_0, and $N_\kappa = 0$ above almost all ℓ) gives the formula of clause 3. □

Proposition D.67. *Let $\mathfrak{u}$, $\mathcal{Y}$, $\mathcal{T}$, B, c, $\mathcal{X} = \mathcal{X}_{B,c}$ with its root origin, τ, ξ and u_ξ be as in Proposition D.69, suppose moreover that the level L is good at every prime and that every prime*

dividing $\operatorname{disc}(E)$ *lies in* S_B*, and put* $\mathcal{D}_{\mathcal{X}} = \det \mathcal{V}_{(\tau,+)} \otimes (\det \mathcal{V}_{(\tau,-)})^{-1}$*, an invertible sheaf on* $\mathcal{X}$ *whose stalk at the generic point contains* u_ξ*. Let* F' *be the algebraic closure of* $\mathbb{Q}$ *in* Ω*. Then* F' *is a number field,* $\mathcal{O}_{F'} = H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$*, the fibres of* $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{F'}$ *are connected, and for every maximal ideal* $\mathfrak{p}$ *of* $\mathcal{O}_{F'}$ *and every* $\pi \in F'$ *of* $\mathfrak{p}$*-adic valuation one there are integers* $M \geq 1$ *and* a *such that* $u_\xi^M \pi^{-a}$ *generates* $\mathcal{D}_{\mathcal{X}}^{\otimes M}$ *at every point of* $\mathcal{X}$ *over* $\mathfrak{p}$*, and at every point over no prime.*

Proof. **Step 1: the constant field.** $\mathcal{X}$ is proper over $\mathbb{Z}$, so $A = H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a finite $\mathbb{Z}$-algebra; it is a domain contained in Ω, and integrally closed in its fraction field because $\mathcal{X}$ is normal. Every $\alpha \in F'$ has a non-zero integer multiple $N\alpha$ integral over $\mathbb{Z}$, hence over every local ring of $\mathcal{X}$, hence in every local ring (normality), so $N\alpha \in A$. Thus $\operatorname{Frac}(A) = F'$, F' is finite over $\mathbb{Q}$, and A, integrally closed and finite over $\mathbb{Z}$, is $\mathcal{O}_{F'}$. The morphism $g\colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}_{F'}$ is proper with $g_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\operatorname{Spec} \mathcal{O}_{F'}}$, so its fibres are connected by Zariski's connectedness theorem (EGA III, 4.3.1). Fix $\mathfrak{p}$ over the prime ℓ and π as in the statement; put $\mathcal{O} = \mathcal{O}_{F',\mathfrak{p}}$ and $X = \mathcal{X} \times_{\mathcal{O}_{F'}} \mathcal{O}$, a normal integral scheme, proper over $\mathcal{O}$, dominating $\operatorname{Spec} \mathcal{O}$, with connected special fibre X_s (the fibre of g over $\mathfrak{p}$). Its points are the points of $\mathcal{X}$ over $\mathfrak{p}$ and those over no prime, and π is a uniformiser of $\mathcal{O}$.

Step 2: the generic fibre. Let x be a point of $\mathcal{X}$ over no prime. By (Fh5) the stalk of $\mathcal{V}_s$ at x is $\mathrm{dR}_{s,x} = \mathrm{Lie}_x \cap \mathrm{Li}_s$; since $F \subseteq \mathcal{O}_x$ contains every conjugate of E, the idempotents of $E \otimes_{\mathbb{Q}} F = \prod_{\mathrm{Hom}(E,F)} F$ make it the direct summand $\mathrm{Lie}_s(G_{\mathcal{X}})_x$, and $\mathrm{Lie}_s(G_{\mathcal{X}}) = f^*\mathrm{Lie}_s(G_{\mathcal{Y}})$ near x. The image $f(x)$ lies in $\mathcal{Y}_c \times_{\mathcal{O}_F} F$ (it is a specialisation of the generic point of $\mathcal{Y}_c$ over no prime), so $\mathcal{D}_{\mathcal{X},x} = (f^*\mathcal{D})_x$ with $\mathcal{D}$ as in Proposition D.69, and u_ξ is the germ of $f^*\xi$, which generates there because ξ vanishes nowhere. So u_ξ generates $\mathcal{D}_{\mathcal{X}}$ at every point over no prime; in particular its order at every point of codimension one over no prime is 0, and the last assertion holds for every M and a (π is a unit there).

Step 3: the case $\ell \in S_B$. By (Fh3), (Fh6) and the determinant of a full flag, $\mathcal{D}_{\mathcal{X}} \cong \bigotimes_{i=1}^n \mathcal{L}_{(\tau,i)}$ (the roots (τ,i) with $i \leq p$ give the step quotients of the integral flag of $\mathcal{V}_{(\tau,+)}$, those with $i > p$ the inverses of the step quotients of $\mathcal{V}_{(\tau,-)}$), and by the root sections of (Go3)(3), $\mathcal{L}_{(\tau,i)} \cong \mathcal{O}(e_{(\tau,i)})|_{\mathcal{X}}$. By Proposition 35.66, on $\mathcal{X}_\ell = \mathcal{X} \times \operatorname{Spec} \mathbb{F}_\ell$ the sheaves $\mathcal{O}(e_{(\tau,i)})|_{\mathcal{X}_\ell}$ are the graded pieces of a full flag of $\mathcal{O}_{\mathcal{X}_\ell}^n$, so their tensor product is $\det \mathcal{O}_{\mathcal{X}_\ell}^n \cong \mathcal{O}_{\mathcal{X}_\ell}$. As $\ell \in \mathfrak{p}$, X_s is a closed subscheme of $\mathcal{X}_\ell$, and $\mathcal{D}_{\mathcal{X}}|_{X_s} \cong \mathcal{O}_{X_s}$. By Step 2, u_ξ generates $\mathcal{D}_{\mathcal{X}}$ on the generic fibre of X. Lemma D.50, applied to X, $\mathcal{L} = \mathcal{D}_{\mathcal{X}}|_X$, $\sigma = u_\xi$ and the uniformiser π, gives the integers M, a.

Step 4: the case $\ell \notin S_B$: orders at the components. Then $\ell \nmid \operatorname{disc}(E)$ by hypothesis. $X_s = V(\pi)$; let $X_1, \dots, X_t$ be its irreducible components, with generic points ξ_i of codimension one (Krull), local rings that are discrete valuation rings (normality) with valuations v_i, $m_i = v_i(\pi) \geq 1$, and n_i the order of u_ξ at ξ_i in a trivialisation of $\mathcal{D}_{\mathcal{X}}$. Fix i. The regular locus of $\mathcal{X}$ is open ($\mathcal{X}$ is of finite type over $\mathbb{Z}$, hence excellent) and contains ξ_i (normal schemes are regular in codimension one), so $W_i = X_i \cap \mathrm{Reg}(\mathcal{X}) \smallsetminus \bigcup_{k \neq i} X_k$ is a non-empty open subset of X_i; let $x \in W_i$ be a closed point (it exists, X_i being of finite type over a finite field). Near x the only codimension-one points through x are ξ_i and points over no prime (points over another prime of $\mathcal{O}_{F'}$ do not specialise to x); as $\mathcal{O}_{\mathcal{X},x}$ is regular, hence factorial, $u_\xi^{m_i}\pi^{-n_i}$, whose order is $m_in_i - n_im_i = 0$ at ξ_i and 0 at the other height-one primes of $\mathcal{O}_{\mathcal{X},x}$ (Step 2, and π is a unit over no prime), generates $\mathcal{D}_{\mathcal{X}}^{\otimes m_i}$ at x. Let $\Delta \subseteq \mathcal{X}$ be the closed set of points whose image under f lies outside the open locus of $\mathcal{Y}$ over which $G_{\mathcal{Y}}$ is abelian; it does not contain the generic point of $\mathcal{X}$, whose image lies in $j(\mathrm{Sh}_F)$. By Lemma D.68 there are a number field k, a maximal ideal $\mathfrak{P}$ of $\mathcal{O}_k$ over ℓ, a morphism $z\colon \operatorname{Spec} \mathcal{O}_{k,\mathfrak{P}} \to \mathcal{X}$ with closed point x and generic point $\eta \notin \Delta$ over no prime, and $\vartheta\colon k \to \mathbb{C}_\ell$ with $\mathcal{O}_{k,\mathfrak{P}} = \{|\vartheta| \leq 1\}$. By (Ud7), $f(\eta) \in j(\mathrm{Sh}_F)$: η lies in the abelian locus. Let $\kappa = \vartheta \circ \beta$ as in Proposition D.69; by Proposition D.38(4) (components split, level good at every prime), φ_ξ has an ℓ-adic norm N_κ at κ. Proposition D.69(1), with $V = \mathcal{O}_{k,\mathfrak{P}}$,

$m = m_i$, $\nu = n_i$ and π, gives $\frac{n_i}{m_i}\log|\vartheta(\pi_z)| = N_\kappa$, π_z the image of π in k.

Step 5: independence of i. The composite $\mathcal{O}_{F'} \to \mathcal{O}_{\mathcal{X},x} \to \mathcal{O}_{k,\mathfrak{P}}$ maps $\mathfrak{p}$ into $\mathfrak{P}$, so $\vartheta|_{F'}$ induces $\mathfrak{p}$ and $\log|\vartheta(\pi_z)| = -\frac{\log\ell}{e_\mathfrak{p}}$, $e_\mathfrak{p}$ the ramification index of $\mathfrak{p}$ over ℓ, independent of i. Likewise κ induces on F the prime $\mathfrak{l} = \mathfrak{p} \cap \mathcal{O}_F$ for every i, and N_κ depends only on $\mathfrak{l}$ by Lemma D.70 (all these norms exist by Proposition D.38(4), and data of (Yf3) exist at the point z: the base change of $G_\mathcal{X}$ along z with bases of its free Lie eigenmodules). Hence $r = n_i/m_i = -e_\mathfrak{p}N_\kappa/\log\ell$ is the same rational number for all i.

Step 6: conclusion for $\ell \notin S_B$. Put $M = m_1 \cdots m_t$ and $a = rM \in \mathbb{Z}$ ($rM = n_1\prod_{k\neq 1} m_k$). The rational section $\theta = u_\xi^M \pi^{-a}$ of $\mathcal{D}_\mathcal{X}^{\otimes M}$ has order $Mn_i - am_i = 0$ at every ξ_i and order 0 at every point of codimension one over no prime (Step 2). A point x' of $\mathcal{X}$ over $\mathfrak{p}$ lies in X, and every height-one prime of $\mathcal{O}_{\mathcal{X},x'}$ is one of these; since the local ring of a normal noetherian domain is the intersection of its localisations at height-one primes, θ and θ^{-1} are regular at x' in a trivialisation, i.e. θ generates $\mathcal{D}_\mathcal{X}^{\otimes M}$ at x'. □

Lemma D.68. *Let X be an integral scheme of finite type over $\mathbb{Z}$ whose function field has characteristic 0, let $\Delta \subseteq X$ be a closed subset not containing the generic point of X, let ℓ be a prime and x a closed point of X with $\ell \in \mathfrak{m}_x$ at which X is regular. Then there are a number field k, a maximal ideal $\mathfrak{P}$ of $\mathcal{O}_k$ over ℓ and a morphism $z\colon \operatorname{Spec}\mathcal{O}_{k,\mathfrak{P}} \to X$ that maps the closed point to x and the generic point to a point of $X \smallsetminus \Delta$ lying over no prime. For every field homomorphism $\vartheta\colon k \to \mathbb{C}_\ell$ inducing $\mathfrak{P}$ (such ϑ exist), $\mathcal{O}_{k,\mathfrak{P}} = \{a \in k : |\vartheta(a)| \leq 1\}$.*

Proof. Let $A = \mathcal{O}_{X,x}$, a regular local ring with maximal ideal $\mathfrak{m}$, of dimension d. As the function field has characteristic 0, $\ell \neq 0$ in A; as $\ell \in \mathfrak{m}$, $d \geq 1$. Let $I \subseteq A$ be the ideal of Δ near x (so that a prime $\mathfrak{q}$ of A is a point of Δ iff $I \subseteq \mathfrak{q}$); $I \neq 0$ because Δ does not contain the generic point. Put $J = \ell I \neq 0$; as A is a domain, $\dim A/J \leq d-1$.

Step 1: parameters. We choose $t_1, \ldots, t_{d-1} \in \mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent and with $\dim A/(J + (t_1, \ldots, t_i)) \leq d-1-i$ for $0 \leq i \leq d-1$. Suppose $t_1, \ldots, t_{i-1}$ chosen ($i \leq d-1$) and let $Q_1, \ldots, Q_s$ be the minimal primes of $J + (t_1, \ldots, t_{i-1})$ different from $\mathfrak{m}$. The ideal $\mathfrak{m}^2 + (t_1, \ldots, t_{i-1})$ does not contain $\mathfrak{m}$, since $\dim \mathfrak{m}/\mathfrak{m}^2 = d > i-1$, and no Q_j contains $\mathfrak{m}$. By prime avoidance (all ideals but one prime), there is $t_i \in \mathfrak{m}$ outside $\mathfrak{m}^2 + (t_1, \ldots, t_{i-1})$ and outside every Q_j. Then t_i lies in no minimal prime of $J + (t_1, \ldots, t_{i-1})$ of dimension ≥ 1, so $\dim A/(J + (t_1, \ldots, t_i)) \leq \max(0, \dim A/(J + (t_1, \ldots, t_{i-1})) - 1) \leq d-1-i$. After $d-1$ steps, $J + (t_1, \ldots, t_{d-1})$ is $\mathfrak{m}$-primary.

Step 2: the curve. Since $t_1, \ldots, t_{d-1}$ is part of a regular system of parameters of the regular local ring A, $\mathfrak{q} = (t_1, \ldots, t_{d-1})$ is prime and $V = A/\mathfrak{q}$ is a regular local ring of dimension 1, a discrete valuation ring. If $\ell \in \mathfrak{q}$ or $I \subseteq \mathfrak{q}$, then $J \subseteq \mathfrak{q}$ and $J + (t_1, \ldots, t_{d-1}) \subseteq \mathfrak{q} \neq \mathfrak{m}$ would not be $\mathfrak{m}$-primary; so $\ell \notin \mathfrak{q}$ and $I \not\subseteq \mathfrak{q}$. Let $\eta \in X$ be the point corresponding to $\mathfrak{q}$; then $\eta \notin \Delta$, and η lies over no prime: $\ell \neq 0$ in V, and a prime $p \neq \ell$ is a unit in A (both p and ℓ in $\mathfrak{m}$ would give $1 \in \mathfrak{m}$). Let k be the fraction field of V, the residue field of η, a field of characteristic 0. The closure C of η in X is integral, of finite type over $\mathbb{Z}$, of dimension 1 ($\dim \mathcal{O}_{C,x} = \dim V = 1$ and x is a closed point of X of finite type over $\mathbb{Z}$) and dominates $\operatorname{Spec}\mathbb{Z}$; so its generic fibre over $\mathbb{Q}$ has dimension 0 and k is a finite extension of $\mathbb{Q}$, a number field. The discrete valuation ring $V \subset k$ contains $\mathbb{Z}$, hence its integral closure $\mathcal{O}_k$, and so $V = \mathcal{O}_{k,\mathfrak{P}}$ for $\mathfrak{P} = \mathfrak{m}_V \cap \mathcal{O}_k$, which lies over ℓ as $\ell \in \mathfrak{m}_V$. Let z be the composite $\operatorname{Spec} V \to \operatorname{Spec} A \to X$; it maps the closed point to x and the generic point to η.

Step 3: the embedding. The completion $k_\mathfrak{P}$ is a finite extension of $\mathbb{Q}_\ell$, so it embeds over $\mathbb{Q}_\ell$ into $\mathbb{C}_\ell$; the composite $\vartheta\colon k \to \mathbb{C}_\ell$ induces $\mathfrak{P}$. For any such ϑ, $|\vartheta(\cdot)|$ is an absolute value on k whose valuation ring is the localisation of $\mathcal{O}_k$ at $\mathfrak{P}$, i.e. $\{a \in k : |\vartheta(a)| \leq 1\} = \mathcal{O}_{k,\mathfrak{P}}$. □

Proposition D.69. *Let $\mathfrak{u}$ be a unitary Shimura datum with $n = p + q$, $p, q \geq 1$, whose components split, $\mathcal{Y}$ an integral model of* Sh *over $\mathcal{O}_F$, $\mathcal{T}$ a root tower of type $(\Phi_{\mathsf{H}}, n, p)$ with a geometric realisation, $B \geq 2$, $c \in \mathrm{Cp}$, and let $\mathcal{X} = \mathcal{X}_{B,c}$ carry a root origin on fixed data at c, with cover $f\colon \mathcal{X} \to \mathcal{Y}$, function field Ω and Hodge bundles $\mathcal{V}_s$ at (S_B, R). Let $\tau \in \Phi_{\mathsf{H}}$, let ξ be a global section, vanishing at no point, of $\mathcal{D} = \det \mathrm{Lie}_{(\tau,+)}(G') \otimes (\det \mathrm{Lie}_{(\tau,-)}(G'))^{-1}$ on $\mathcal{Y}_c \times_{\mathcal{O}_F} F$ (G' the restriction of $G_{\mathcal{Y}}$), with frame function φ_ξ on Z_c, and let*

$$u_\xi \in \det \mathrm{Li}_{(\tau,+)} \otimes_\Omega \left(\det \mathrm{Li}_{(\tau,-)}\right)^{-1}$$

be the value of ξ at the generic point of $\mathcal{Y}_c$, transported to Ω along f (the generic point of $\mathcal{X}$ maps to that of $\mathcal{Y}_c$, and $G_{\mathcal{X}} = f^*G_{\mathcal{Y}}$). Let ℓ be a prime, V a discrete valuation ring with fraction field k, $\vartheta\colon k \to \mathbb{C}_\ell$ a field homomorphism with $V = \{a \in k : |\vartheta(a)| \leq 1\}$, and $z\colon \operatorname{Spec} V \to \mathcal{X}$ a morphism whose generic point maps into the abelian locus; let x_0 be the image of the closed point (it lies over ℓ), $\beta\colon F \to k$ the homomorphism induced by $\mathcal{O}_F \to \mathcal{O}_{\mathcal{X},x_0} \to V$, $\kappa = \vartheta \circ \beta$, and suppose that φ_ξ has an ℓ-adic norm N_κ at κ. Let $b^+ = (b_1^+, \dots, b_p^+)$ and $b^- = (b_1^-, \dots, b_q^-)$ be bases of the free $\mathcal{O}_{\mathcal{X},x_0}$-modules $\mathcal{V}_{(\tau,+),x_0}$ and $\mathcal{V}_{(\tau,-),x_0}$, and $a \in \Omega^\times$ with $u_\xi = a\,(b_1^+ \wedge \cdots \wedge b_p^+) \otimes (b_1^- \wedge \cdots \wedge b_q^-)^{-1}$. Suppose that there are integers $m \geq 1$ and ν and an element $\pi \in \Omega^\times$ algebraic over $\mathbb{Q}$ with $a^m \pi^{-\nu} \in \mathcal{O}_{\mathcal{X},x_0}^\times$; π is a unit at the image η of the generic point of Spec V*, and we write $\kappa(\pi)$ for ϑ of its image under $\mathcal{O}_{\mathcal{X},\eta} \to k$ (for $\pi \in F^\times$ this is $\kappa(\pi)$ in the previous sense). Put $c_2 = 2$ and $c_\ell = \frac{1}{\ell - 1}$ for $\ell \geq 3$. Then:*

1. *if $\ell \notin S_B$ and $\ell \nmid \operatorname{disc}(E)$, then $\frac{\nu}{m} \log |\kappa(\pi)| = N_\kappa$;*
2. *if $\ell \in S_B$, then for every non-zero $d \in \mathfrak{d}_E$*

$$\left|\frac{\nu}{m} \log |\kappa(\pi)| - N_\kappa\right| \leq n(c_\ell \log \ell - \log |\kappa(\tau(d))|).$$

Proof. **Step 0: notation.** Write $\mathcal{O} = \mathcal{O}_{\mathcal{X},x_0}$ and $z^\sharp\colon \mathcal{O} \to V$ for the local homomorphism given by z. As $|\vartheta(\ell)| < 1$, ℓ lies in the maximal ideal of V, hence in that of $\mathcal{O}$: x_0 lies over ℓ. Let $\eta \in \mathcal{X}$ be the image of the generic point of $\operatorname{Spec} V$, a generalisation of x_0, so $\mathcal{O} \subseteq \mathcal{O}_\eta \subseteq \Omega$, and let $z_\eta^\sharp\colon \mathcal{O}_\eta \to k$ be the induced homomorphism, which extends $z^\sharp$. The point η lies in the abelian locus $f^{-1}(j(\mathrm{Sh}_F))$, hence in characteristic 0: every prime is a unit in $\mathcal{O}_\eta$, so $F = \mathcal{O}_F \otimes \mathbb{Q} \subseteq \mathcal{O}_\eta$. Since $\operatorname{Spec} k$ is reduced and j is an immersion whose image contains $f(\eta)$, the k-point $f \circ z$ at the generic point is $j \circ y$ for a unique $y \in Z_c(k)$ (F-morphisms through β), and $G_y = y^*G^{\mathrm{un}}$ is identified $\mathcal{O}_E$-linearly with the fibre of $G_{\mathcal{X}} = f^*G_{\mathcal{Y}}$ at η base changed to k. Put $\mathcal{G} = z^*G_{\mathcal{X}}$, a semiabelian scheme over V with $\mathcal{O}_E$-action whose generic fibre is G_y. As $\mathrm{Lie}(G_{\mathcal{X}})$ is locally free and formation of Lie commutes with base change, $\mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}_{x_0} \otimes_{\mathcal{O}} V$ inside $\mathrm{Lie}(G_y) = \mathrm{Lie}_\eta \otimes_{\mathcal{O}_\eta} k$; for $e \in \mathrm{Lie}_\eta$ write z^*e for its image in $\mathrm{Lie}(G_y)$, so that z^* maps Lie_{x_0} into $\mathrm{Lie}(\mathcal{G})$. For a label s put

$$I_s = \mathrm{Lie}(\mathcal{G}) \cap \mathrm{Lie}_s(G_y),$$

the V-module $\mathrm{Lie}_s(\mathcal{G})$ of (Yf3) for $s = (\tau, \pm)$. It is saturated in the free V-module $\mathrm{Lie}(\mathcal{G})$ and spans $\mathrm{Lie}_s(G_y)$, so it is free of rank $\dim_k \mathrm{Lie}_s(G_y) = \mathrm{rk}(s)$ (Kottwitz condition).

Step 1: the pulled-back Hodge lattices. The basis $b^\pm$ of the stalk $\mathcal{V}_{(\tau,\pm),x_0}$ of the locally free sheaf $\mathcal{V}_{(\tau,\pm)}$ is a basis of $\mathcal{V}_{(\tau,\pm),\eta} = \mathcal{V}_{(\tau,\pm),x_0} \otimes_{\mathcal{O}} \mathcal{O}_\eta$. The point η lies over no prime, so $\mathcal{V}_{s,\eta} = \mathrm{dR}_{s,\eta} = \mathrm{Lie}_\eta \cap \mathrm{Li}_s$. As $F \subseteq \mathcal{O}_\eta$ contains every conjugate of E, $E \otimes_{\mathbb{Q}} F = \prod_{\sigma \in \mathrm{Hom}(E,F)} F$ supplies idempotents ϵ_σ acting on Lie_η with $\mathrm{Lie}_\eta \cap \mathrm{Li}_s = \epsilon_{\sigma_s}\mathrm{Lie}_\eta$, a direct summand; its base change to k is $\epsilon_{\sigma_s}\mathrm{Lie}(G_y) = \mathrm{Lie}_s(G_y)$. Hence $z^*b^\pm$ is a k-basis of $\mathrm{Lie}_{(\tau,\pm)}(G_y)$, that is, a frame at y (Yf1). Let Λ_s be the V-span of $z^*\mathcal{V}_{s,x_0}$ in $\mathrm{Lie}_s(G_y)$; for $s = (\tau, \pm)$ it is free with basis $z^*b^\pm$.

Step 2: the value of φ_ξ in the Hodge frame. Put $\theta = a^m\pi^{-\nu} \in \mathcal{O}^\times$. As π and π^{-1} are algebraic over $\mathbb{Q} \subseteq \mathcal{O}_\eta$, they are integral over $\mathcal{O}_\eta$ after multiplication by suitable non-zero integers, which are units of $\mathcal{O}_\eta$; so they lie in the normal domain $\mathcal{O}_\eta$, and $\pi \in \mathcal{O}_\eta^\times$. Hence $a^m \in \mathcal{O}_\eta^\times$; so a and a^{-1} are integral over the normal domain $\mathcal{O}_\eta$ ($\mathcal{X}$ is normal by (Go3)(1)), and $a \in \mathcal{O}_\eta^\times$. Near η (on the part of the abelian locus over no prime), $\mathcal{V}_s = \mathrm{Lie}_s(G_\mathcal{X}) = f^*\mathrm{Lie}_s(G')$ by Step 1 and the base change of the idempotents, so $\det \mathcal{V}_{(\tau,+)} \otimes (\det \mathcal{V}_{(\tau,-)})^{-1}$ and $f^*\mathcal{D}$ have the same stalk at η; the section $f^*\xi$ and u_ξ agree at the generic point, hence u_ξ is the germ of $f^*\xi$ at η, and its image at η base changed to k is $\xi(j(y))$ (the fibre of $\mathcal{D}$ at $j(y)$ being identified with $\det \mathrm{Lie}_{(\tau,+)}(G_y) \otimes (\det \mathrm{Lie}_{(\tau,-)}(G_y))^{-1}$ as in Proposition D.38). Applying $z_\eta^\sharp$ to $u_\xi = a\,(\wedge b^+) \otimes (\wedge b^-)^{-1}$ gives

$$\varphi_\xi(y; z^*b^+, z^*b^-) = a_z, \qquad a_z = z_\eta^\sharp(a), \qquad a_z^m = z^\sharp(\theta)\,\pi_z^\nu, \qquad \pi_z = z_\eta^\sharp(\pi), \quad \vartheta(\pi_z) = \kappa(\pi).$$

As $z^\sharp(\theta) \in V^\times$, $|\vartheta(z^\sharp(\theta))| = 1$, and therefore

$$\log|\vartheta(a_z)| = \frac{\nu}{m}\log|\kappa(\pi)|.$$

Step 3: comparison with an integral frame. Let (v, w) be an integral frame of $\mathcal{G}$: bases of $I_{(\tau,+)}$ and $I_{(\tau,-)}$ (Step 0). Write $v = M_+\, z^*b^+$ and $w = M_-\, z^*b^-$ with $M_+ \in \mathrm{GL}_p(k)$, $M_- \in \mathrm{GL}_q(k)$. By the frame law (Yf2), $\varphi_\xi(y; v, w)\det M_+ = \varphi_\xi(y; z^*b^+, z^*b^-)\det M_-$. The data $(V, k, \vartheta, \beta, y, \mathcal{G}, (v, w))$ satisfy the conditions of (Yf3) ($\vartheta \circ \beta = \kappa$, $\mathcal{G}$ semiabelian over V with an $\mathcal{O}_E$-linear isomorphism of its generic fibre with G_y, and Z_c a connected component of Sh_F on which φ_ξ is a frame function by Proposition D.38(1)), so $\log|\vartheta(\varphi_\xi(y; v, w))| = N_\kappa$. With Step 2,

$$N_\kappa - \frac{\nu}{m}\log|\kappa(\pi)| = \log|\vartheta(\det M_-)| - \log|\vartheta(\det M_+)| \qquad (*).$$

Step 4: clause 1. Let $\ell \notin S_B$ and $\ell \nmid \mathrm{disc}(E)$. Then $\mathcal{V}_{s,x_0} = \mathrm{dR}_{s,x_0} = \mathrm{Lie}_{x_0} \cap \mathrm{Li}_s$. Every prime dividing $\mathrm{disc}(E)$ is different from ℓ, hence a unit in $\mathcal{O}$ (it is coprime to ℓ, which lies in the maximal ideal). The $\mathcal{O}_F$-linear map $\mathcal{O}_E \otimes_\mathbb{Z} \mathcal{O}_F \to \prod_{\sigma\in\mathrm{Hom}(E,F)} \mathcal{O}_F$, $a \otimes b \mapsto (\sigma(a)b)_\sigma$, between free modules of rank $[E : \mathbb{Q}]$, has determinant whose square is $\pm\mathrm{disc}(E)$; so after base change to $\mathcal{O}$ it is an isomorphism, and $\mathcal{O}_E \otimes \mathcal{O}$ contains idempotents ϵ_σ with $(a \otimes 1)\epsilon_\sigma = \sigma(a)\epsilon_\sigma$ and $\sum_\sigma \epsilon_\sigma = 1$. They act on Lie_{x_0}, on $\mathrm{Lie}(\mathcal{G})$ and on $\mathrm{Lie}(G_y)$ compatibly with z^*, and ϵ_{σ_s} is the projection of $\mathrm{Lie}(G_y)$ onto $\mathrm{Lie}_s(G_y)$ along the other eigenspaces. Hence $\mathrm{Lie}_{x_0} \cap \mathrm{Li}_s = \epsilon_{\sigma_s}\mathrm{Lie}_{x_0}$ and $I_s = \epsilon_{\sigma_s}\mathrm{Lie}(\mathcal{G})$, and

$$\Lambda_s = V \cdot z^*(\epsilon_{\sigma_s}\mathrm{Lie}_{x_0}) = \epsilon_{\sigma_s}(\mathrm{Lie}_{x_0} \otimes_\mathcal{O} V) = I_s.$$

So $z^*b^\pm$ and (v, w) are bases of the same V-modules, $M_\pm \in \mathrm{GL}(V)$, $|\vartheta(\det M_\pm)| = 1$, and $(*)$ gives clause 1.

Step 5: clause 2, the lower inclusion. Let $\ell \in S_B$, $R = R(\ell)$. By Lemma 35.32, applied to the locally noetherian cover $\mathcal{X}$ (a closed subscheme of a multiprojective space over $\mathbb{Z}$) at x_0 with $p = \ell$ (its last assertion: $\mathcal{X}$ carries a root origin and $\ell \in S_B$), $\varpi_\ell \in \mathcal{O}$, and with $\mathcal{V}_{s,x_0} = \mathrm{HT}_{s,x_0}(\ell^R)$:

$$\varpi_\ell\,\tau(d)\,\mathcal{V}_{(\tau,+),x_0} \subseteq \mathrm{dR}_{(\tau,+),x_0}, \qquad \varpi_\ell\,\mathcal{V}_{(\tau,-),x_0} \subseteq \mathrm{dR}_{(\tau,-),x_0} \qquad (d \in \mathfrak{d}_E),$$

using $\sigma_{(\tau,+)} = \tau$. As $\mathrm{dR}_{s,x_0} \subseteq \mathrm{Lie}_{x_0}$ and z^* maps Lie_{x_0} into $\mathrm{Lie}(\mathcal{G})$ and $\mathrm{Li}_s \cap \mathrm{Lie}_\eta$ into $\mathrm{Lie}_s(G_y)$, $z^*\mathrm{dR}_{s,x_0} \subseteq I_s$. Put $\varpi = z^\sharp(\varpi_\ell)$; then $\varpi\,\beta(\tau(d))\,\Lambda_{(\tau,+)} \subseteq I_{(\tau,+)}$ and $\varpi\,\Lambda_{(\tau,-)} \subseteq I_{(\tau,-)}$. The value of ϖ: for $\ell = 2$, $\varpi = 4$ and $-\log|\vartheta(\varpi)| = 2\log 2$; for ℓ odd, $\varpi = \zeta' - 1$ with $\zeta' = z_\eta^\sharp(\zeta)$ a root of the ℓ-th cyclotomic polynomial Φ_ℓ in k, and $\zeta' \neq 1$ since $\Phi_\ell(1) = \ell \neq 0$ in k; so $\vartheta(\zeta')$ is a primitive ℓ-th root of unity in $\mathbb{C}_\ell$ and $-\log|\vartheta(\varpi)| = \frac{\log\ell}{\ell-1}$. In both cases $-\log|\vartheta(\varpi)| = c_\ell\log\ell$.

Step 6: clause 2, the upper inclusion $I_s \subseteq \Lambda_s$. Let $s = (\tau, \pm)$ and $r = \mathrm{rk}(s)$. By (Go3)(2) the image h_s of $\mathrm{ht}_{x_0} \subseteq \omega_{x_0}$ under restriction to Li_s is a free $\mathcal{O}$-module of rank $r = \dim_\Omega \mathrm{Li}_s$; choose $w_1, \dots, w_r \in \mathrm{ht}_{x_0}$ whose restrictions $h_1, \dots, h_r$ form a basis of h_s, hence an Ω-basis of the dual of Li_s, and let $e_1, \dots, e_r \in \mathrm{Li}_s$ be the dual basis. For $e \in \mathrm{Li}_s$, $e = \sum_i h_i(e) e_i$, so $\mathrm{HT}'_{s,x_0} = \{e \in \mathrm{Li}_s : h(e) \in \mathcal{O} \text{ for all } h \in h_s\} = \bigoplus_i \mathcal{O} e_i$. Moreover $\mathrm{HT}'_{s,x_0} \subseteq \mathcal{V}_{s,x_0}$: for $s = (\tau, -)$ they are equal, and for $s = (\tau, +)$, if $e \in \mathrm{HT}'_{s,x_0}$ and $d \in \mathfrak{d}_E \subseteq \mathcal{O}_E$, then $\tau(d) \in \mathcal{O}_F$ maps into $\mathcal{O}$ and $\tau(d)e \in \mathrm{HT}'_{s,x_0}$, so $e \in \mathrm{HT}_{s,x_0}(\ell^R)$. In particular $e_i \in \mathcal{V}_{s,x_0} \subseteq \mathrm{Lie}_\eta$ (Step 1). Each $w_i \in \omega_{x_0} = \mathrm{Hom}_{\mathcal{O}}(\mathrm{Lie}_{x_0}, \mathcal{O})$ induces $z^* w_i = w_i \otimes V \in \mathrm{Hom}_V(\mathrm{Lie}(\mathcal{G}), V)$, extended k-linearly to $\mathrm{Lie}(G_y)$; it is also the base change of $w_i \otimes \mathcal{O}_\eta \in \mathrm{Hom}(\mathrm{Lie}_\eta, \mathcal{O}_\eta)$, so $(z^* w_i)(z^* e_j) = z_\eta^\sharp(w_i(e_j)) = z_\eta^\sharp(h_i(e_j)) = \delta_{ij}$. Thus $z^* e_1, \dots, z^* e_r$ is a k-basis of the r-dimensional space $\mathrm{Lie}_s(G_y)$ with dual functionals $z^* w_i$, and every $e \in I_s$ satisfies $e = \sum_i (z^* w_i)(e)\, z^* e_i$ with $(z^* w_i)(e) \in V$, because $e \in \mathrm{Lie}(\mathcal{G})$. Hence $I_s \subseteq V \cdot z^* \mathrm{HT}'_{s,x_0} \subseteq \Lambda_s$.

Step 7: clause 2, the determinants. By Step 6, $v \in I^p_{(\tau,+)} \subseteq \Lambda^p_{(\tau,+)}$, so M_+ has entries in V and $|\vartheta(\det M_+)| \le 1$. Let $d \in \mathfrak{d}_E$ be non-zero and $\delta = \varpi\, \beta(\tau(d))$; $\delta \neq 0$ as $\varpi \neq 0$ (Step 5) and $\tau(d) \neq 0$. By Step 5, $\delta\, z^* b^+ = N_+ v$ with $N_+ \in \mathrm{M}_p(V)$; then $\delta \cdot 1_p = N_+ M_+$, so $\delta^p = \det N_+ \det M_+$ and $|\vartheta(\det M_+)| \ge |\vartheta(\delta)|^p$. In the same way $|\vartheta(\varpi)|^q \le |\vartheta(\det M_-)| \le 1$. Since $\tau(d)$ is integral over $\mathbb{Z}$, $-\log|\kappa(\tau(d))| \ge 0$, and by Step 5 $-\log|\vartheta(\delta)| = c_\ell \log \ell - \log|\kappa(\tau(d))|$. Therefore, by $(*)$,

$$-q\, c_\ell \log \ell \;\le\; N_\kappa - \frac{\nu}{m} \log|\kappa(\pi)| \;\le\; p(c_\ell \log \ell - \log|\kappa(\tau(d))|),$$

and both bounds are at most $n(c_\ell \log \ell - \log|\kappa(\tau(d))|)$ in absolute value. This is clause 2. □

Lemma D.70. *Let $\mathfrak{u}$ be a unitary Shimura datum, $\tau \in \Phi_{\mathsf{H}}$, $U \subseteq \mathrm{Sh}_F$ a union of connected components and φ a frame function on U. Let ℓ be a prime and $\kappa, \kappa' \colon F \to \mathbb{C}_\ell$ ring homomorphisms inducing the same prime $\mathfrak{l}$ of $\mathcal{O}_F$. Suppose that φ has ℓ-adic norms N_κ at κ and $N_{\kappa'}$ at κ', and that there is at least one tuple $(V, k, j, \beta, y, \mathcal{G}, (v, w))$ as in (Yf3) with $j \circ \beta = \kappa$ (a discrete valuation ring with fraction field k, $j \colon k \to \mathbb{C}_\ell$ with valuation ring V, $y \in U(k)$ over β, a semiabelian model $\mathcal{G}$ of G_y over V with $\mathcal{O}_E$-action and an integral frame). Then $N_\kappa = N_{\kappa'}$.*

Proof. The homomorphisms κ, κ' extend by continuity to embeddings over $\mathbb{Q}_\ell$ of the completion $F_{\mathfrak{l}}$, a finite extension of $\mathbb{Q}_\ell$, into $\overline{\mathbb{Q}}_\ell \subset \mathbb{C}_\ell$. Two such embeddings differ by an element of $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell / \mathbb{Q}_\ell)$; it preserves $|\cdot|$ (uniqueness of the extension of the absolute value) and so extends by continuity to an automorphism γ of $\mathbb{C}_\ell$ with $|\gamma(x)| = |x|$ for all x and $\kappa' = \gamma \circ \kappa$. Let $(V, k, j, \beta, y, \mathcal{G}, (v, w))$ be a tuple as in the statement. Then $(V, k, \gamma \circ j, \beta, y, \mathcal{G}, (v, w))$ satisfies the conditions of (Yf3) for κ': $\{a \in k : |\gamma(j(a))| \le 1\} = \{a : |j(a)| \le 1\} = V$ and $\gamma \circ j \circ \beta = \kappa'$. Hence $N_{\kappa'} = \log|\gamma(j(\varphi(y; v, w)))| = \log|j(\varphi(y; v, w))| = N_\kappa$. □

Lemma D.71. *Let $\mathcal{X} = \mathcal{X}_{B,c}$ carry a root origin on fixed data at c, with a realisation $\pi \colon \mathcal{X}(\mathbb{C}) \to X_c^{\mathrm{an}}$ of Sys_c defined on $\mathcal{X}(\mathbb{C})$ itself, $\mathcal{X} \otimes \mathbb{Q}$ being smooth. Let $a \in \Phi$, $e_A = \sum_{i=1}^n e_{(a,i)}$, and let h_A be a cone metric of multidegree e_A with $h_A(\tilde z) = \prod_{i=1}^n h_{(a,i)}^{\mathrm{sing}}(\tilde z)$ at every point $\tilde z$ of the affine cone over a complex point of the abelian locus. For $t > 0$ put $v_t = \log h_A - \sum_{i=1}^n \log h_{(a,i)}(t)$ on the affine cone $\widehat{\mathbb{C}}$. Then v_t is smooth and invariant under $(\mathbb{C}^\times)^K$, $\prod_i h_{(a,i)}(t) = h_A e^{-v_t}$, and for all roots $r_1, \dots, r_D$*

$$\lim_{t \to 0^+} \int_{\mathcal{X}(\mathbb{C})} v_t\, c_1(\overline{\mathcal{L}}_{r_1}(t)) \wedge \cdots \wedge c_1(\overline{\mathcal{L}}_{r_D}(t)) = 0.$$

Proof. **Step 1: v_t.** h_A and $\prod_i h_{(a,i)}(t)$ are smooth, positive and homogeneous of the same multidegree e_A, so their logarithmic difference is smooth and invariant under $(\mathbb{C}^\times)^K$, and $\prod_i h_{(a,i)}(t) = h_A e^{-v_t}$ by definition.

Step 2: v_t through the realisation. For a fundamental index $b = (\sigma, i)$ write $H_{(\sigma,i)}(t) = \prod_{k \le i} h_{(\sigma,k)}(t)$ and $H_{(\sigma,0)} = 1$. By (Go4), for $0 < t \le t_0$, $H_{(\sigma,i)}(t)^{-1} = h^{\mathrm{ref}}_{(\sigma,i)} e^{\varphi^T_{(\sigma,i)}(t) \circ \pi}$ at the cone points of $\mathcal{X}$, and on cone points over $\pi^{-1}(X^\circ)$, which by (Go4) is the set of complex points of the abelian locus, $h^{\mathrm{ref}}_{(\sigma,i)} e^{\varphi_{(\sigma,i)} \circ \pi} = \prod_{k \le i} (h^{\mathrm{sing}}_{(\sigma,k)})^{-1}$. Dividing the two identities for $(\sigma, i) = (a, n)$ and using the hypothesis on h_A,

$$v_t = \log \frac{\prod_i h^{\mathrm{sing}}_{(a,i)}}{\prod_i h_{(a,i)}(t)} = g_t \circ \pi, \qquad g_t = \varphi^T_{(a,n)}(t) - \varphi_{(a,n)} \text{ on } X^\circ \qquad (0 < t \le t_0)$$

at all cone points over $\pi^{-1}(X^\circ)$ (here v_t, being $(\mathbb{C}^\times)^K$-invariant, is read as a function on $\mathcal{X}(\mathbb{C})$).

Step 3: the root forms. By (Go4) the metric of the fundamental index b with cone function $H_b(t)^{-1}$ is the pull-back by π of $(\mathcal{M}_b, h^{\mathrm{ref}}_b e^{\varphi^T_b(t)})$, whose Chern form is $\Omega_b(t) = \theta_b + dd^c \varphi^T_b(t)$, θ_b the Chern form of $(\mathcal{M}_b, h^{\mathrm{ref}}_b)$; put $\Omega_{(\sigma,0)}(t) = 0$. Since $h_{(\sigma,k)}(t) = H_{(\sigma,k)}(t)/H_{(\sigma,k-1)}(t)$, the root line $\overline{\mathcal{L}}_{(\sigma,k)}(t)$ is the fundamental line of $(\sigma, k-1)$ tensored with the inverse of that of (σ, k), and

$$c_1(\overline{\mathcal{L}}_{(\sigma,k)}(t)) = \pi^*\big(\Omega_{(\sigma,k-1)}(t) - \Omega_{(\sigma,k)}(t)\big) \qquad (0 < t \le t_0).$$

Expanding the product by multilinearity, $\bigwedge_j c_1(\overline{\mathcal{L}}_{r_j}(t))$ is a signed sum of at most 2^D forms $\pi^*\mu_t$, $\mu_t = \bigwedge_{j=1}^D \Omega_{b_j}(t)$ for fundamental indices $b_j = (\sigma_j, i_j)$ with $i_j \ge 1$ (terms containing some $\Omega_{(\sigma,0)} = 0$ vanish). It suffices to show $\int_{\mathcal{X}(\mathbb{C})} v_t \, \pi^*\mu_t \to 0$ for each such μ_t.

Step 4: transfer to X. The complement of $\pi^{-1}(X^\circ)$ in $\mathcal{X}(\mathbb{C})$ is the set of complex points outside the abelian locus, a proper closed analytic subset, hence of measure zero; so is the preimage $\pi^{-1}(\Sigma)$ of the exceptional analytic subset $\Sigma \subset X$ of (Go4), and Σ itself. Off these sets π is a local biholomorphism with exactly $\deg_{B,c}$ preimages over every point. By Step 2 and the change of variables for such a covering,

$$\int_{\mathcal{X}(\mathbb{C})} v_t \, \pi^*\mu_t = \deg_{B,c} \int_{X^\circ} g_t \, \mu_t \qquad (0 < t \le t_0),$$

the right side being absolutely convergent because $|g_t| \le C'' \log \mathrm{rW}$ (Step 5) and $\log \mathrm{rW}$ is integrable against the smooth form μ_t (it is a sum of terms $\log(-\log|z_k|^2)$ plus a bounded function in local coordinates adapted to ∂X).

Step 5: the convergence. Apply Lemma 31.35 to X, Θ, ς, with $v = 1$ a section of the trivial bundle ($D_v = \varnothing$, $G = 0$), I the fundamental indices (σ, i), $i \ge 1$, θ_b, $u_b = \varphi_b$, $u_{b,t} = \varphi^T_b(t)$, $t_0' = \min(t_0, e^{-e})$, $\delta(t) = t^{1/3} + (1 + C)\varepsilon(t)$, $R(t) = \log(1/t)$ (so $R(t) \ge e$ and δ, R tend to 0 and ∞), and the constant $C''' = \max(C_5, C, \sup_{0<t\le t_0'} \varepsilon(t), 1)$. Its hypotheses are (Go1): (R1) is the plurisubharmonicity of $h^{\mathrm{ref}}_b e^{\varphi^T_b(t)}$ twisted by $\varepsilon(t)\Theta$, i.e. $\Omega_b(t) + \varepsilon(t)\Theta \ge 0$; (R2) is $-C_5 \log \mathrm{rW} \le \varphi_b \le \varphi^T_b(t) \le C$; (R3) is $\varphi^T_b(t) \le \varphi_b + t^{1/3} + \varepsilon(t)$ where $\mathrm{rW} \le \log(1/t)$; (R4) is $\varphi^T_b(t) \le \varphi^T_b(t') + C\varepsilon(t')$; (R5) is (Go1)(3). The functions g_t are continuous on X° with $0 \le g_t \le C + C_5 \log \mathrm{rW} \le (C + C_5) \log \mathrm{rW}$ (as $\log \mathrm{rW} \ge 1$), and $g_t \to 0$ uniformly on compact subsets of X° by (Go1)(3). By Lemma 31.35(2) with $g = 0$, $\int_{X^\circ} g_t \, \mu_t \to 0$, and by Steps 3 and 4 the limit of the statement is 0. □

Lemma D.72. *Let $\mathcal{X}$ be a root model of type (Φ, n) with convergent regularisation, let $a \in \Phi$, and let $\overline{\mathcal{A}}$ be a flat determinant line of $\mathcal{X}$ at a, of height $\mathrm{ht}(\overline{\mathcal{A}})$. Then*

$$\mathrm{pp}(\mathcal{X}; \mathrm{ps}_{a,1} Q) = \mathrm{ht}(\overline{\mathcal{A}}) \, \mathrm{gi}(\mathcal{X}; Q)$$

for every $Q \in \mathcal{R}$ homogeneous of degree D.

Proof. Both sides are linear in Q, so it suffices to take a monomial $Q = x_{r_1}\cdots x_{r_D}$ with roots $r_1,\dots,r_D \in \mathrm{Rt}$. Take the data of (Dl1) and (Dl2) for $\overline{\mathcal{A}}$: the number field F, M, f, g, (κ_s), h_A and (v_t). Fix $t>0$ and write $c_1^Q(t) = c_1(\overline{\mathcal{L}}_{r_1}(t)) \wedge \cdots \wedge c_1(\overline{\mathcal{L}}_{r_D}(t))$ and $\widehat{c}^{\,Q}(t) = \widehat{c}_1(\overline{\mathcal{L}}_{r_1}(t))\cdots\widehat{c}_1(\overline{\mathcal{L}}_{r_D}(t))$.

Step 1: the product line. No factor of $\mathrm{ps}_{a,1}Q$ is the auxiliary variable, so by (29.2) and the additivity of $\widehat{c}_1$ under tensor products,

$$\mathrm{ai}_t(\mathcal{X};\mathrm{ps}_{a,1}Q) = \frac{1}{\delta_{\mathcal{X}}}\sum_{i=1}^{n}\widehat{\deg}\big(\widehat{c}_1(\overline{\mathcal{L}}_{(a,i)}(t))\,\widehat{c}^{\,Q}(t)\big) = \frac{1}{\delta_{\mathcal{X}}}\widehat{\deg}\big(\widehat{c}_1(\overline{\mathcal{B}}_t)\,\widehat{c}^{\,Q}(t)\big),$$

where $\overline{\mathcal{B}}_t = \bigotimes_i \overline{\mathcal{L}}_{(a,i)}(t) = (\mathcal{O}(e_A), \prod_i h_{(a,i)}(t))|_{\mathcal{X}}$, using $e_A = \sum_i e_{(a,i)}$ (Dl2).

Step 2: the section f/g. Choose any cone metric h_g of multidegree d_g (a product of powers of Fubini–Study cone functions) and put $h_f = (\prod_i h_{(a,i)}(t))^M h_g$, a cone metric of multidegree $Me_A + d_g = d_f$. Then $\overline{\mathcal{B}}_t^{\otimes M} \cong (\mathcal{O}(d_f),h_f)\otimes(\mathcal{O}(d_g),h_g)^{\otimes -1}$ isometrically on $\mathcal{X}$, so by multilinearity $M\,\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{B}}_t)\widehat{c}^{\,Q}(t))$ is the difference of the intersection numbers of $(\mathcal{O}(d_f),h_f)$ and of $(\mathcal{O}(d_g),h_g)$ against $\widehat{c}^{\,Q}(t)$. The global sections f and g do not vanish identically on the integral scheme $\mathcal{X}$, so their divisors meet $\mathcal{X}$ properly, and Lemma 30.17 (on the regular ambient $\mathbb{P}$ of $\mathcal{X}$, with $\mathcal{Z}=\mathcal{X}$; all bundles here are restrictions of cone metrics, i.e. smooth metrics on $\mathbb{P}$) applies to both: the height of $\mathcal{X}$ is the contribution of the Weil divisor of the section (horizontal heights and $\log p$ times vertical degrees) minus $\int_{\mathcal{X}(\mathbb{C})}\log\|s\|\,c_1\wedge\cdots\wedge c_1$. Since f/g generates $\mathcal{O}(Me_A)|_{\mathcal{X}}$ at every point, $\mathrm{div}(f)=\mathrm{div}(g)$ as Cartier divisors on $\mathcal{X}$, so the two Weil divisors coincide and the two divisor contributions cancel. With $\log\|f\|_{h_f}-\log\|g\|_{h_g} = \log|f|-\log|g|-M\sum_i\log h_{(a,i)}(t)$ we get

$$M\,\widehat{\deg}\big(\widehat{c}_1(\overline{\mathcal{B}}_t)\,\widehat{c}^{\,Q}(t)\big) = -\int_{\mathcal{X}(\mathbb{C})}\Big(\log\frac{|f|}{|g|} - M\sum_{i=1}^{n}\log h_{(a,i)}(t)\Big)c_1^Q(t),$$

the integrand being a function on $\mathcal{X}(\mathbb{C})$ (it is invariant under the scaling of the cone, as $d_f-d_g = Me_A$).

Step 3: the integrand. $\mathcal{X}(\mathbb{C}) = \bigsqcup_{s\colon F\to\mathbb{C}}\mathcal{X}_s(\mathbb{C})$ with $\mathcal{X}_s = \mathcal{X}\otimes_{\mathcal{O}_F,s}\mathbb{C}$, as $\mathcal{X}\to\operatorname{Spec}\mathbb{Z}$ factors through $\operatorname{Spec}\mathcal{O}_F$. On $\mathcal{X}_s(\mathbb{C})$, by (Dl1) and (Dl2),

$$\log\frac{|f|}{|g|} - M\sum_i\log h_{(a,i)}(t) = M\log\kappa_s + M\log h_A - M\big(\log h_A - v_t\big) = M\log\kappa_s + Mv_t.$$

Hence

$$\widehat{\deg}\big(\widehat{c}_1(\overline{\mathcal{B}}_t)\,\widehat{c}^{\,Q}(t)\big) = -\sum_s\log\kappa_s\int_{\mathcal{X}_s(\mathbb{C})}c_1^Q(t) - \int_{\mathcal{X}(\mathbb{C})}v_t\,c_1^Q(t).$$

Step 4: the geometric degrees. The integral of the Chern forms of smooth metrics over $\mathcal{X}_s(\mathbb{C})$ is the intersection number $g_s = \deg(c_1(\mathcal{O}(e_{r_1}))\cdots c_1(\mathcal{O}(e_{r_D}))\cap[\mathcal{X}_s])$, independent of t. The schemes $\mathcal{X}_s$ are the base changes of the F-variety $\mathcal{X}_F$ along the embeddings s, which are permuted transitively by the automorphisms of $\mathbb{C}$, and the bundles $\mathcal{O}(e_r)$ are defined over $\mathbb{Z}$; so $g_s = g$ does not depend on s, and $[F:\mathbb{Q}]\,g = \sum_s g_s = \deg(c_1(\mathcal{O}(e_{r_1}))\cdots c_1(\mathcal{O}(e_{r_D}))\cap[\mathcal{X}_{\mathbb{Q}}]) = \delta_{\mathcal{X}}\,\mathrm{gi}(\mathcal{X};Q)$. Therefore, by Steps 1–3 and the definition of $\mathrm{ht}(\overline{\mathcal{A}})$ in (Dl1),

$$\mathrm{ai}_t(\mathcal{X};\mathrm{ps}_{a,1}Q) = \mathrm{ht}(\overline{\mathcal{A}})\,\mathrm{gi}(\mathcal{X};Q) - \frac{1}{\delta_{\mathcal{X}}}\int_{\mathcal{X}(\mathbb{C})}v_t\,c_1^Q(t).$$

Step 5: the limit. As $t\to 0^+$ the last integral tends to 0 by (Dl2), and $\mathrm{ai}_t(\mathcal{X};\mathrm{ps}_{a,1}Q)\to\mathrm{pp}(\mathcal{X};\mathrm{ps}_{a,1}Q)$ because $\mathcal{X}$ has convergent regularisation (Definition 30.1; $\mathrm{ps}_{a,1}Q$ is a sum of monomials of $\mathcal{R}^+$). This gives the identity for monomials, hence for all Q. □

D.4.2 Proportional square systems and covolumes

Flag models carry analytic square systems for every Levi reflection (Proposition D.74), and with the comparison of root origins of Section 31 this gives Proposition D.73. Geometric proportionality with covolumes Proposition D.79 has two halves: the geometric intersection forms of the root models are proportional to the flag degree (Proposition D.80), by a flag-volume identity (Lemma D.81) and the Weyl proportionality of Section 33 (Proposition D.82); and the geometric volumes are sums of covolumes of principal congruence subgroups of lattices in one genus (Proposition D.83), computed through the complex uniformisation of the moduli scheme.

Proposition D.73. *Let $\mathfrak{u}$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. Then the realised tower (whose root models have convergent regularisation by Theorem 31.20) has proportional analytic square systems: for every integer $B \geq 2$, every $Z \in \mathrm{Cp}$ and every Levi reflection s there are a root model $\mathcal{X}'$ of type (Φ, n) with convergent regularisation and a real number $m > 0$ with $\mathrm{pp}(\mathcal{X}';\cdot) = m\,\mathrm{pp}(\mathcal{X}_{B,Z};\cdot)$ and $\mathrm{gi}(\mathcal{X}';\cdot) = m\,\mathrm{gi}(\mathcal{X}_{B,Z};\cdot)$ on $\mathcal{R}$, and a square system for s on $\mathcal{X}'$ with (Fx5), (Fx6), (Fx7) and an analytic square system.*

Proof. Let $B \geq 2$, $Z \in \mathrm{Cp}$ and s a Levi reflection; write $c = Z$, $\mathcal{X} = \mathcal{X}_{B,Z}$, and let $\mathcal{Y}_c$ be the component of $\mathcal{Y}$ of the fixed data at c (Go2), (Go5). The hypotheses of Proposition 34.2 include that $\mathcal{Y}$ has uniformised degenerations and $p, q \geq 1$ ($q \geq 1$ and $p \geq 2q+1$).

Step 1: the flag model. The set S_B contains Bd, the primes dividing $2\,\mathrm{disc}(E)\,N'$ (Go5), hence every prime dividing $\mathrm{disc}(E)$. By Proposition D.74 for the fixed data at c and $S = S_B$ there is a root model $\mathcal{X}'$ of type (Φ, n) with clauses 1–3 there; by clause 3 it carries, for the given s, a square system with (Fx5), (Fx6), (Fx7) and an analytic square system.

Step 2: $\mathcal{X}$ satisfies the hypotheses of Proposition 31.27 for (S_B, R) with $R(\ell) = 4$. This is Lemma 31.37 (with Sys_c, $d = \deg_{B,c}$ and the realisation π of (Go4)).

Step 3: the comparison. By Step 1, clauses 1 and 2 of Proposition D.74, $\mathcal{X}'$ satisfies the same hypotheses for the same fixed data and the same (S_B, R). The set S_B contains every prime dividing $\mathrm{disc}(E)$ (Step 1). By Proposition 31.27 with $S = S_B$, $\mathcal{X}_1 = \mathcal{X}'$, $\mathcal{X}_2 = \mathcal{X}$, $c_1 = c_2 = c$ and the identity twist ($g = \mathrm{id}$, $\mathrm{tw} = \mathrm{id}$, $\psi = \mathrm{id}$, so $\sigma_g = \mathrm{id}$ and tw_F maps Z_c onto itself), both have convergent regularisation and, by its parts 1 and 2, $\mathrm{pp}(\mathcal{X}';F) = \mathrm{pp}(\mathcal{X};F)$, $\mathrm{gi}(\mathcal{X}';F) = \mathrm{gi}(\mathcal{X};F)$ for every $F \in \mathcal{R}$. Hence $\mathcal{X}'$ with the factor $m = 1$ and the square system of Step 1 is a proportional square model of $\mathcal{X}$ for s (Definition 31.9). As B, Z and s were arbitrary, the realised tower has proportional analytic square systems. □

Proposition D.74. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$ be a unitary Shimura datum with $p, q \geq 1$, $\Phi = \Phi_{\mathsf{H}}$, $n = p + q$, let $\mathcal{Y}$ be an integral model of* Sh *over $\mathcal{O}_F$ with uniformised degenerations, let fixed data at an irreducible component $\mathcal{Y}_c$ of $\mathcal{Y}$ be given, and let S be a finite set of primes containing every prime dividing* $\mathrm{disc}(E)$. *Then there is a root model $\mathcal{X}'$ of type (Φ, n) such that:*

1. *$\mathcal{X}'_{\mathbb{Q}}$ is smooth, and $\mathcal{X}'$ carries the data (Go3)(1)–(4) of a root origin on the fixed data at c for S with $R(\ell) = 4$ for every $\ell \in S$;*

2. *there are an analytic system* Sys' *(Go1) with $\varphi_* = 0 = \varphi_*^T$ and a holomorphic map $\pi' \colon \mathcal{X}'(\mathbb{C}) \to X^{\mathrm{an}\prime}$ satisfying hypotheses 1–3 of Lemma 31.21 for some integer $d \geq 1$, such that a point of $\mathcal{X}'(\mathbb{C})$ lies in the abelian locus if and only if its image lies in the open part $X^{\circ\prime}$, and on $\pi'^{-1}(X^{\circ\prime})$ the singular weights are the Hodge metric: $h^{\mathrm{ref}}_{(\tau,i)} e^{\varphi_{(\tau,i)} \circ \pi'} = \prod_{k \leq i} (h^{\mathrm{sing}}_{(\tau,k)})^{-1}$ for $(\tau, i) \in \mathrm{Rt}$;*

3. *for every Levi reflection s there is a square system for s on $\mathcal{X}'$ with open parts and limit metrics (Fx5), fibre charts with (Fx6) and (Fx7), and an analytic square system (Sw1)–(Sw3).*

Proof. Let $\mathcal{X}^{\mathrm{b}}$ be a Hodge base for $\mathcal{Y}_c$ and S (Proposition 33.9; it needs only that S contains the primes dividing $\mathrm{disc}(E)$), and $\mathcal{X}'$ the flag bundle of its Hodge bundles with the multiprojective immersion and the data of (Go3)(1)–(4) of Lemma 33.8, made a root model of type (Φ, n) by Proposition 33.12, item 1.

Clause 1. $\mathcal{X}'_{\mathbb{Q}}$ is smooth by Lemma 33.8, item 1, and Lemma 33.8, item 2, gives the data of (Go3)(1)–(4) on the fixed data at c for S with $R(\ell) = 4$ for every $\ell \in S$ (with $\delta_{\mathcal{X}'} = [F : \mathbb{Q}]\deg$, as fixed in Proposition 33.12).

Clause 2. Proposition 33.12, item 2, gives the analytic system Sys' with $\varphi_* = 0 = \varphi_*^T$ and $\pi' = \mathrm{id}\colon \mathcal{X}'(\mathbb{C}) \to X^{\mathrm{an}\prime} = \mathcal{X}'(\mathbb{C})$ satisfying hypotheses 1–3 of Lemma 31.21 with $d = 1$; its open part $X^{\circ\prime}$ is the abelian locus, so a point lies in the abelian locus if and only if its image lies in $X^{\circ\prime}$; and on $\pi'^{-1}(X^{\circ\prime})$, $h^{\mathrm{ref}}_{(\tau,i)} e^{\varphi_{(\tau,i)} \circ \pi'} = \prod_{k \le i} (h^{\mathrm{sing}}_{(\tau,k)})^{-1}$.

Clause 3. Let $s = s_{\tau,i}$ be a Levi reflection and $(k_0, \mathcal{X}^{(2)}, g, h)$ as in Lemma D.75: it satisfies (Fx2) and (Fx3), and $(e_\Xi)_{k_0} = -2$. The factor k_0 is the Plücker factor $k_{s',j}$ with $s' = (\tau, +)$, $j = i$ if $i < p$ and $s' = (\tau, -)$, $j = n - i$ if $i > p$, and $\{r_1, r_2\} = \{(\tau, i), (\tau, i+1)\}$ is the set of the roots of label s' and steps j, $j+1$ (by (Fh3) of Definition 29.6: for $i < p$ both roots have label $(\tau, +)$ and steps i, $i+1$; for $i > p$ label $(\tau, -)$ and steps $n+1-i$, $n-i$). By Proposition 33.12, item 3, for every $t > 0$ the cone metrics $h_r(t)$, $r \notin \{r_1, r_2\}$, and $h_{r_1}(t) h_{r_2}(t)$ are from the base at k_0. So (Fx1) holds, and $(k_0, \mathcal{X}^{(2)}, g, h)$ is a square system for s on $\mathcal{X}'$. By Proposition D.76 it has open parts and limit metrics (Fx5), fibre charts with (Fx6) and (Fx7), and an analytic square system (Sw1)–(Sw3). One root model $\mathcal{X}'$ serves all Levi reflections. □

Lemma D.75. *In the situation of Lemma 33.8, let $s_{\tau,i}$ be a Levi reflection, and let $k_0 = k_{(\tau,+),i}$ if $i < p$ and $k_0 = k_{(\tau,-),n-i}$ if $i > p$. Then:*

1. *$(e_\Xi)_{k_0} = -2$, $(e_{r_1} + e_{r_2})_{k_0} = 0$, and $(e_r)_{k_0} = 0$ for every root $r \notin \{r_1, r_2\}$;*
2. *there are a closed integral subscheme $\mathcal{X}^{(2)} \subseteq \mathbb{P}^{(2)} = \mathbb{P} \times \mathbb{P}(\mathbb{Z}^{I_{k_0}})$ and multihomogeneous polynomials g, h with integer coefficients in the coordinates of $\mathbb{P}^{(2)}$ satisfying (Fx2) and (Fx3) for $\mathcal{X}'$, $s_{\tau,i}$ and k_0; here $\mathcal{X}^{(2)}$ is the image of $\mathcal{X}' \times_{\beta_{s,j}} \mathcal{X}'$ under the map*

$$(x, x'') \mapsto (x, \text{the } k_0\text{-coordinates of } x''),$$

Δ is the image of the diagonal, and g/h is a 2×2 minor of the k_0-coordinates of the two points divided by a rational section pulled back from $\mathcal{X}'_{s,j}$ (with s, j the label and step of k_0); more precisely, $g = (z_{k_0,a} z'_b - z_{k_0,b} z'_a) H(z)$ and $h = G(z)$ for some $a, b \in I_{k_0}$ and multihomogeneous G, H in the coordinates of $\mathbb{P}$ with $\deg G - \deg H = e_\delta := 2e_{\mathcal{H}} - d_{s,j-1} - d_{s,j+1}$ ($e_{\mathcal{H}}$, $d_{s,k}$ the base and fundamental multidegrees of that Plücker datum, whose fundamental root sections $P \mapsto \varrho_{s,k} P(y) w_{s,k}\colon \mathcal{O}(d_{s,k})|_{\mathcal{X}'} \to \wedge^k \mathcal{V}'_{s,k}$ are isomorphisms by Lemma 33.8(d)), such that under these root sections and $F \mapsto F(y)\colon \mathcal{O}(e_{\mathcal{H}})|_{\mathcal{X}'} \to \beta^* \mathcal{H}^m$ the rational section G/H of $\mathcal{O}(e_\delta)|_{\mathcal{X}'}$ is the section δ^{loc} of $(\wedge^{j-1} \mathcal{V}'_{s,j-1})^{-1} \otimes (\wedge^{j+1} \mathcal{V}'_{s,j+1})^{-1} \otimes \beta^* \mathcal{H}^{2m}$, $\gamma \otimes \gamma' \mapsto \det\big(\sigma_{s,j,c}(\gamma \wedge u_l)\big)_{l=1,2;\, c=a,b}$ for $\gamma' = \gamma \wedge u_1 \wedge u_2$, with $\sigma_{s,j,\cdot}$ the sections of the Plücker datum defining the factor k_0.

Proof. Write (s, j) for the label and step of k_0: $s = (\tau, +)$, $j = i$ if $i < p$; $s = (\tau, -)$, $j = n - i$ if $i > p$. In both cases $1 \le j < \mathrm{rk}(s)$, and $\{r_1, r_2\}$ is the set of the two roots of label s with steps

j and $j+1$: for $i<p$, $\mathrm{st}(r_1)=j$, $\mathrm{st}(r_2)=j+1$, signs $+1$; for $i>p$, $\mathrm{st}(r_1)=n+1-i=j+1$, $\mathrm{st}(r_2)=j$, signs -1.

Item 1. By Lemma 33.8(c), $(e_r)_{k_0}=0$ for $r\notin\{r_1,r_2\}$, and for $i<p$: $(e_{r_1})_{k_0}=-1$, $(e_{r_2})_{k_0}=1$; for $i>p$: $(e_{r_1})_{k_0}=\mathrm{sg}=-1$, $(e_{r_2})_{k_0}=-\mathrm{sg}=1$. In both cases $(e_\Xi)_{k_0}=-2$ and $(e_{r_1}+e_{r_2})_{k_0}=0$. Moreover, with $e^\flat$ denoting e with its k_0-entry replaced by 0, both cases give

$$e_\Xi^\flat = 2e_{\mathcal{H}} - d_{s,j-1} - d_{s,j+1} \tag{XI}$$

.

The fibre square. Let $Q=\mathcal{V}'_{s,j+1}/\mathcal{V}'_{s,j-1}$, pulled back from $\mathcal{X}'_{s,j}$, locally free of rank 2; $\beta_{s,j}$ is the bundle of lines of Q (Lemma 33.8(b)). Let $\mathcal{X}''=\mathcal{X}'\times_{\mathcal{X}'_{s,j}}\mathcal{X}'$ with projections $\mathrm{pr}_0,\mathrm{pr}_1$; via pr_0 it is the bundle of lines of pr_0^*Q over $\mathcal{X}'$, so it is integral, normal, projective over $\mathbb{Z}$, flat over $\mathbb{Z}$, of absolute dimension $D+2$. The composite of $w\wedge\cdot\colon\wedge^{j-1}\mathcal{V}'_{s,j-1}\otimes Q\to\wedge^j\mathcal{V}_s$ (injective with locally free cokernel) and of the map $\wedge^j\mathcal{V}_s\to(\mathcal{H}^m)^{I_{k_0}}$ dual to the surjection given by the $\sigma_{s,j,\cdot}$ is injective with locally free cokernel; so the k_0-component embeds the bundle of lines of Q linearly, and $\mathcal{X}''\to\mathcal{X}'\times\mathbb{P}(\mathbb{Z}^{I_{k_0}})$, $(x,x'')\mapsto(x,\iota_{k_0}(x''))$, is a closed immersion over $\mathcal{X}'$. Composed with the closed immersion of $\mathcal{X}'$ it gives a closed immersion $\mathcal{X}''\to\mathbb{P}^{(2)}$; let $\mathcal{X}^{(2)}$ be its image. By Lemma 33.8(a), all coordinates of x'' outside k_0 equal those of x, so $q_1(\mathcal{X}^{(2)})$ is the image of pr_1, i.e. $\mathcal{X}'$; and $q_0(\mathcal{X}^{(2)})=\mathcal{X}'$. The fibre of q_0 over the generic point of $\mathcal{X}'$ is the image of the fibre of $\beta_{s,j}$ through it, a line by Lemma 33.8(b). This is (Fx2).

The diagonal and the minor. Let $\Delta\subseteq\mathcal{X}^{(2)}$ be the image of the diagonal section of pr_0: closed, integral, of absolute dimension $D+1$, and on it z' and z_{k_0} are the same point of $\mathbb{P}(\mathbb{Z}^{I_{k_0}})$, so the minors $z_{k_0,a}z'_b-z_{k_0,b}z'_a$ vanish. Over a point of $\mathcal{X}'_{s,j}$ choose local bases: by the injectivity above and since the $\sigma_{s,j,\cdot}$ span every fibre of the dual, at the generic point the vectors $(\sigma_{s,j,a}(w_{s,j-1}\wedge v))_{a\in I_{k_0}}$, $v\in\{v_{s,j},v_{s,j+1}\}$, are linearly independent; fix $a,b\in I_{k_0}$ with

$$\delta=\det\begin{pmatrix}\sigma_{s,j,a}(w_{s,j-1}\wedge v_{s,j}) & \sigma_{s,j,b}(w_{s,j-1}\wedge v_{s,j})\\ \sigma_{s,j,a}(w_{s,j-1}\wedge v_{s,j+1}) & \sigma_{s,j,b}(w_{s,j-1}\wedge v_{s,j+1})\end{pmatrix}\neq 0$$

in Ω' (the values lie in $\mathcal{H}^m\otimes\Omega'=\Omega'$). The same determinant, formed with local generators, is the section δ^{loc} of $N=(\wedge^{j-1}\mathcal{V}'_{s,j-1})^{-2}\otimes(\det Q)^{-1}\otimes\mathcal{H}^{2m}=(\wedge^{j-1}\mathcal{V}'_{s,j-1})^{-1}\otimes(\wedge^{j+1}\mathcal{V}'_{s,j+1})^{-1}\otimes\mathcal{H}^{2m}$ given by the determinant of the map $\wedge^{j-1}\mathcal{V}'_{s,j-1}\otimes Q\to(\mathcal{H}^m)^2$ through the coordinates a,b; it is pulled back from $\mathcal{X}'_{s,j}$. Through the fundamental root sections (isomorphisms by Lemma 33.8(d)) and $F\mapsto F(y)$, an isomorphism $\mathcal{O}(e_{\mathcal{H}})|_{\mathcal{X}'}\to\beta^*\mathcal{H}^m$ (the base factors of Definition 33.7), $N\cong\mathcal{O}(e_\delta)|_{\mathcal{X}'}$ with $e_\delta=2e_{\mathcal{H}}-d_{s,j-1}-d_{s,j+1}$, and δ^{loc} corresponds to the rational section $c\,\delta$ of $\mathcal{O}(e_\delta)$ for a constant $c\in\Omega'^{\times}$ (a product of the $\varrho_{s,k}^{\pm1}$). Every element of Ω' is a quotient of multihomogeneous integer polynomials of equal multidegree evaluated at the representing coordinates, so $c\,\delta=G(y)/H(y)$ with G,H multihomogeneous, $\deg G-\deg H=e_\delta$, $G(y),H(y)\neq0$; then the divisor of the rational section G/H of $\mathcal{O}(e_\delta)|_{\mathcal{X}'}$ is $\mathrm{div}(\delta^{\mathrm{loc}})$. Put

$$g=(z_{k_0,a}z'_b-z_{k_0,b}z'_a)\,H(z),\qquad h=G(z),$$

polynomials in the coordinates of $\mathbb{P}^{(2)}$ (G, H in the coordinates of the first factor).

The divisor identity. Let μ be the restriction of the minor to $\mathcal{X}''$, a section of $q_0^*\mathcal{O}(\mathbf{1}_{k_0})\otimes\mathcal{O}(1)_{\mathrm{last}}$. At a point (x,x'') over a point of $\mathcal{X}'_{s,j}$, with x, x'' the lines ℓ, ℓ'' of Q and w a local generator of $\wedge^{j-1}\mathcal{V}'_{s,j-1}$, the k_0-coordinates of x and x'' are the values of the $\sigma_{s,j,\cdot}$ on $w\wedge u$, $w\wedge u''$ for generators u of ℓ, u'' of ℓ''; so μ corresponds to $(w\otimes u)\otimes(w\otimes u'')\mapsto\delta^{\mathrm{loc}}\cdot(u\wedge u'')$, where $u\wedge u''$ is the canonical section of $\ell^{-1}\otimes\ell''^{-1}\otimes\det Q$ (multiplicativity of 2×2 determinants). Locally over $\mathcal{X}'_{s,j}$, with Q free on f_1,f_2 and affine fibre coordinates $\ell=\langle f_1+tf_2\rangle$, $\ell''=\langle f_1+t''f_2\rangle$,

$u \wedge u'' = (t''-t) f_1 \wedge f_2$ (and likewise at $t=\infty$), whose divisor is the diagonal, reduced. Hence, as Cartier (equivalently Weil, $\mathcal{X}''$ being normal) divisors on $\mathcal{X}'' \cong \mathcal{X}^{(2)}$,

$$\begin{aligned}\operatorname{div}(\mu) &= [\Delta] + q_0^*\operatorname{div}(\delta^{\mathrm{loc}}),\\ \operatorname{div}(g|_{\mathcal{X}^{(2)}}) &= [\Delta] + q_0^*\operatorname{div}(\delta^{\mathrm{loc}}) + q_0^*\operatorname{div}(H),\\ \operatorname{div}(h|_{\mathcal{X}^{(2)}}) &= q_0^*\operatorname{div}(G) = q_0^*\operatorname{div}(\delta^{\mathrm{loc}}) + q_0^*\operatorname{div}(H),\end{aligned}$$

so $\operatorname{div}(g|_{\mathcal{X}^{(2)}}) = \operatorname{div}(h|_{\mathcal{X}^{(2)}}) + [\Delta]$; neither g nor h vanishes identically on $\mathcal{X}^{(2)}$. Degrees: $\deg g - \deg h = (\mathbf{1}_{k_0} - e_\delta,\, 1) \in \mathbb{Z}^K \times \mathbb{Z}$, while $q_0^* e_\Xi + q_1^* e_\Xi = (e_\Xi + e_\Xi^\flat, (e_\Xi)_{k_0}) = (2e_\Xi^\flat - 2\cdot\mathbf{1}_{k_0}, -2)$ by item 1; by (XI), $e_\delta = e_\Xi^\flat$, so $2(\deg g - \deg h) = -(q_0^* e_\Xi + q_1^* e_\Xi)$. This is (Fx3). □

Proposition D.76. *In the situation of Proposition 33.12, let s be a Levi reflection and $(k_0, \mathcal{X}^{(2)}, g, h)$ the square system for s of Lemma D.75 on the root model $\mathcal{X}'$ of Proposition 33.12, item 1. Then, with X° the abelian locus of $\mathcal{X}'(\mathbb{C})$, $W^\circ = q_0^{-1}(X^\circ) \cap \mathcal{X}^{(2)}(\mathbb{C})$ and the limit metrics $h_r(0) = h_r^{\mathrm{sing}}$ on the cone over X° ($r \in \mathrm{Rt}$), the square system has open parts and limit metrics (Fx5), fibre charts with (Fx6) and (Fx7), and an analytic square system (Sw1)–(Sw3).*

Proof. Put $X = \mathcal{X}'(\mathbb{C})$ and $W = \mathcal{X}^{(2)}(\mathbb{C})$.

(Fx5)–(Fx7). By Lemma D.77, X°, W° and the limit metrics $h_r(0) = h_r^{\mathrm{sing}}$ satisfy (Fx5), and there are fibre charts satisfying (Fx6) and (Fx7).

Compact models and (Sw2). By Lemma D.78, item 1, $(X, \Theta', \varsigma, \mathrm{id}, 1)$ is a compact model of X° with $X \smallsetminus \partial X = X^\circ$, $(W, \Theta_W, q_0^*\varsigma, \mathrm{id}, 1)$ is a compact model of W°, and $p_j = q_j|_W$ $(j = 0, 1)$ satisfy (Sw2).

(Sw1). Let Λ be the set of fundamental indices of Sys', a finite set, and for $\lambda \in \Lambda$ put $\gamma_\lambda = \theta_\lambda$, $f_\lambda = \varphi_\lambda$ on X° and $f_{\lambda,t} = \varphi_\lambda^T(t)$ on X $(0 < t \le t_1)$, where $t_1 = \min(t_0, e^{-e}) \in (0,1)$. By Lemma D.78, item 2, these families satisfy (R1)–(R5) on (X, Θ', ς) for some constant ≥ 1 and the functions δ, R of that item. For a root $r = (\tau, i)$ and $\lambda \in \Lambda$ put $b_{r,\lambda} = -1$ if $\lambda = (\tau, i)$, $b_{r,\lambda} = 1$ if $i \ge 2$ and $\lambda = (\tau, i-1)$, and $b_{r,\lambda} = 0$ otherwise (both (τ, i) and, for $i \ge 2$, $(\tau, i-1)$ are fundamental indices, as $a(r)$ is one for every root). Then $\sum_\lambda b_{r,\lambda}\gamma_\lambda = \theta_{a(r)^-} - \theta_{a(r)}$, $\sum_\lambda b_{r,\lambda} f_\lambda = \varphi_{a(r)^-} - \varphi_{a(r)}$ and $\sum_\lambda b_{r,\lambda} f_{\lambda,t} = \varphi_{a(r)^-}^T(t) - \varphi_{a(r)}^T(t)$, with the convention $\theta_{(\tau,0)} = 0$, $\varphi_{(\tau,0)} = \varphi_{(\tau,0)}^T = 0$: these are the forms and functions θ_r, u_r, $u_{r,t}$ of Lemma D.78. By its item 2, $\omega_{r,t} = \theta_r + dd^c u_{r,t}$ on X for $0 < t \le t_1$ and $\omega_{r,0} = \theta_r + dd^c u_r$ on X°, which is $\widetilde{X}^\circ \cap \varrho_X^{-1}(X^\circ)$ for $\varrho_X = \mathrm{id}$. This is (Sw1) with t_1 in place of t_0.

(Sw3). Lemma D.78, item 3, gives a holomorphic section v of a smooth hermitian line bundle on W and $\kappa_v \in \mathbb{R}$ with $\|v\| \le 1$, reduced zero divisor $\Delta(\mathbb{C})$ with no component in $\partial W = q_0^{-1}(\partial X)$ and $\Delta(\mathbb{C}) + (\partial W)_{\mathrm{red}}$ of simple normal crossings, and the identities of (Sw3) for the $u_{r,t}$, u_r above, $p_j = q_j$ and $\varrho_W = \mathrm{id}$, for $0 < t \le t_1$ and for $t = 0$.

So the compact models, p_0, p_1, v and κ_v form an analytic square system (Sw1)–(Sw3) for the square system. □

Lemma D.77. *In the situation of Lemma D.75, let $s_{\tau,i}$ be a Levi reflection with square system $(k_0, \mathcal{X}^{(2)}, g, h)$, (s, j) the label and step of k_0, $X = \mathcal{X}'(\mathbb{C})$, $W = \mathcal{X}^{(2)}(\mathbb{C})$, X° the abelian locus of X, $W^\circ = q_0^{-1}(X^\circ) \cap W$, and $h_r(0) = h_r^{\mathrm{sing}}$ on the affine cone over X° ($r \in \mathrm{Rt}$). Then X°, W° and the $h_r(0)$ satisfy (Fx5), and there are fibre charts $(U_n, B_n, G_n, (\alpha_{n,\beta})_\beta, \Phi_n, \Phi_n^{(2)})_{n\ge1}$ satisfying (Fx6) and (Fx7). Here $G_n(y)$ is the Hodge Gram matrix of a holomorphic frame of the rank-two bundle $\mathcal{V}'_{s,j+1}/\mathcal{V}'_{s,j-1}$ with its quotient Hodge metric, and $\Phi_n(z,y)$ is the flag whose step j is the line spanned by the first frame vector plus z times the second.*

Proof. **The Hodge metric of the ξ-line.** In both cases ($i < p$, signs $+1$, $\mathrm{st}(r_1) = j$, $\mathrm{st}(r_2) = j+1$; $i > p$, signs -1, $\mathrm{st}(r_1) = j+1$, $\mathrm{st}(r_2) = j$; by (Fh3) of Definition 29.6, since $r_1 = (\tau, i)$, $r_2 = (\tau, i+1)$ and $i \neq p$) the ξ-line is

$$\Xi = \mathcal{L}_{r_1} \otimes \mathcal{L}_{r_2}^{-1} = (\wedge^j \mathcal{V}'_{s,j})^{\otimes 2} \otimes (\wedge^{j-1} \mathcal{V}'_{s,j-1})^{-1} \otimes (\wedge^{j+1} \mathcal{V}'_{s,j+1})^{-1},$$

isometrically for the Hodge metrics, with $\varrho_\Xi := \varrho_{r_1}/\varrho_{r_2} = \varrho_{s,j}^2/(\varrho_{s,j-1}\varrho_{s,j+1})$ and $\lambda_{r_1}/\lambda_{r_2} = w_{s,j}^2/(w_{s,j-1}w_{s,j+1})$. By (Go3)(4) and multiplicativity, for every rational section $R = A/B$ of $\mathcal{O}(e_\Xi)$ (A, B multihomogeneous) and every cone point $\tilde{x}$ over $x \in X^\circ$ with $A(\tilde{x})B(\tilde{x}) \neq 0$,

$$\log h_\Xi(0)(\tilde{x}) = \log|R(\tilde{x})| - \log\|\varrho_\Xi R(y)\lambda_{r_1}\lambda_{r_2}^{-1}\|^{\mathrm{Hdg}}(x), \tag{HX}$$

and likewise for every root line and for $\mathcal{L}_{r_1} \otimes \mathcal{L}_{r_2}$ with $h_{r_1}(0)h_{r_2}(0)$.

(Fx5). X is a complex manifold ($\mathcal{X}'_{\mathbb{Q}}$ smooth, Lemma 33.8, item 1) and $W \cong \mathcal{X}''(\mathbb{C})$ is a $\mathbb{P}^1$-bundle over X (Lemma D.75), so all points are smooth; $X^\circ = \beta^{-1}$ of the abelian locus of the base is open, hence so is W°. A point of W° is a pair of flags with the same image in $\mathcal{X}'_{s,j}$, hence in $\mathcal{X}^{\mathrm{b}}$; so $q_1(W^\circ) \subseteq X^\circ$, and $q_0(W^\circ) \subseteq X^\circ$ by definition. On the cone over X°, $h_r^{\mathrm{sing}}(\tilde{x}) = |P(\tilde{x})|/\|\varrho_r P(y)\lambda_r\|^{\mathrm{Hdg}}(x)$ for a local section P with $P(\tilde{x}) \neq 0$ (this is (Go3)(4), which holds by Lemma 33.8, item 2, read at a section P with $P(\tilde{x}) \neq 0$, so that $\varrho_r P(y)\lambda_r$ generates $\mathcal{L}_r$ at x by (Go3)(3)): a quotient of a positive smooth function and the norm of a holomorphic nowhere-vanishing section in the Hodge metric, smooth on the abelian locus by (Fh4); so $h_r(0)$ is positive and C^∞, and it has multidegree e_r (Definition 29.7, (Go3)): $h_r(0)((\lambda_k z_k)_k) = \prod_k |\lambda_k|^{e_{r,k}} h_r(0)(z)$.

The charts. $\mathcal{X}'_{s,j}(\mathbb{C})$ is a complex manifold of dimension $D-1$ ($\beta_{s,j}$ is a $\mathbb{P}^1$-bundle, Lemma 33.8(b)), and its abelian part $Y^\circ = \beta_{s,j}(X^\circ)$ is open and second countable. Choose countably many holomorphic charts $\kappa_n \colon U_n \to Y^\circ$ ($U_n \subseteq \mathbb{C}^{D-1}$ open, κ_n biholomorphic onto an open set) covering Y°, so small that on $\kappa_n(U_n)$ the bundle Q has a holomorphic frame $f_{n,1}, f_{n,2}$ and $\wedge^{j-1}\mathcal{V}'_{s,j-1}$ a holomorphic generator γ_n (both bundles are pulled back from $\mathcal{X}'_{s,j}$), and put $B_n = \kappa_n^{-1}(\kappa_n(U_n) \smallsetminus \bigcup_{m<n} \kappa_m(U_m))$, a Borel set. Let $u(z,y) = f_{n,1}(y) + z f_{n,2}(y)$ and let $\Phi_n(z,y)$ be the point of X over $\kappa_n(y)$ whose step j is the preimage in step $j+1$ of the line $\mathbb{C}u(z,y) \subseteq Q_{\kappa_n(y)}$; $\Phi_n^{(2)}(z_0,z_1,y)$ the image in W of $(\Phi_n(z_0,y), \Phi_n(z_1,y)) \in \mathcal{X}''(\mathbb{C})$. In the trivialisation of $\mathbb{P}(Q)$ by the frame these are the standard affine charts $z \mapsto [1:z]$ of $\mathbb{P}^1$, resp. of $\mathbb{P}^1 \times \mathbb{P}^1$, times κ_n: holomorphic, injective, with invertible differential, hence biholomorphic onto open subsets, contained in X°, resp. W°. By construction $q_0\Phi_n^{(2)}(z_0,z_1,y) = \Phi_n(z_0,y)$, and $q_1\Phi_n^{(2)}(z_0,z_1,y) = \Phi_n(z_1,y)$ because all coordinates of $\Phi_n(z_1,y)$ outside k_0 equal those of $\Phi_n(z_0,y)$ (Lemma 33.8(a)). The sets $\kappa_n(B_n)$ partition Y°, so the sets $\Phi_n(\mathbb{C} \times B_n) = \beta_{s,j}^{-1}(\kappa_n(B_n)) \smallsetminus \{\text{step } j = \text{line of } f_{n,2}\}$ are pairwise disjoint and cover X° up to the union of countably many pieces of the closed complex hypersurfaces $\{\text{step } j = \text{line of } f_{n,2}\}$ over $\kappa_n(U_n)$, a null set; likewise for W° with two hypersurfaces per chart. Let $G_n(y)$ be the Hermitian matrix with $u^* G_n(y) u = |u_1 f_{n,1}(y) + u_2 f_{n,2}(y)|_Q^2$ ($u \in \mathbb{C}^2$): positive definite and C^∞ in y (the Hodge metric is smooth on the abelian locus), so that by polarisation $|u_0^* G_n(y) u_1| = |\langle u(z_0,y), u(z_1,y)\rangle_Q|$ for $u_l = (1, z_l)^{\mathsf{T}}$, and $\det G_n(y) = |f_{n,1} \wedge f_{n,2}|_Q^2$.

(Fx6), the ξ-potential. Over $\mathbb{C} \times U_n$ the section $\tau = (\gamma_n \wedge u)^{\otimes 2} \otimes \gamma_n^{-1} \otimes (\gamma_n \wedge f_{n,1} \wedge f_{n,2})^{-1}$ of Ξ (pulled back by Φ_n) is holomorphic and nowhere zero, and by the norm relations for Gram determinants ($\|w \wedge u\| = \|w\|\,|u|_Q$ and $\|w \wedge u \wedge u'\| = \|w\|\,|u \wedge u'|_Q$)

$$\log\|\tau\|^{\mathrm{Hdg}} = 2\log|u|_Q - \log|f_{n,1} \wedge f_{n,2}|_Q = \log\big(u(z)^* G_n(y) u(z)\big) - \tfrac{1}{2}\log\det G_n(y) = -\mathrm{xp}_{G_n(y)}(z),$$

with $u(z) = (1,z)^{\mathsf{T}}$. Let $\tilde{\Phi}$ be a local holomorphic lift of Φ_n to the affine cone near (z^0, y^0), and $R = A/B$ a rational section of $\mathcal{O}(e_\Xi)$ with $A(\tilde{\Phi})B(\tilde{\Phi}) \neq 0$ near this point (a monomial in

coordinates non-zero at the point). Then $\varrho_\Xi R(y)\lambda_{r_1}\lambda_{r_2}^{-1}$ is a holomorphic generator of Ξ near the point ((Go3)(3)), so equals $\phi\tau$ with ϕ holomorphic and nowhere zero, and by (HX)

$$\log h_\Xi(0)\circ\tilde\Phi - \mathrm{xp}_{G_n(y)}(z) = \log|R(\tilde\Phi)| - \log|\phi|,$$

the real part of a logarithm of a nowhere-vanishing holomorphic function: pluriharmonic.

(Fx6), the base lines. For $r\notin\{r_1,r_2\}$ the root line $\mathcal{L}_r$ and its Hodge metric are built from steps other than step j of s, and $\mathcal{L}_{r_1}\otimes\mathcal{L}_{r_2} = (\wedge^{j+1}\mathcal{V}'_{s,j+1}\otimes(\wedge^{j-1}\mathcal{V}'_{s,j-1})^{-1})^{\pm1}$ with the product of the Hodge metrics; so for every $\beta\in\mathrm{Rt}_s^{\mathrm{b}}$ the Hermitian line $\overline{\mathcal{L}}_\beta$ is pulled back by $\beta_{s,j}$ from $\mathcal{X}'_{s,j}$ (over the abelian part). Shrinking the charts we may assume it has a holomorphic generator $\epsilon_{n,\beta}$ on $\kappa_n(U_n)$; put $\alpha_{n,\beta}(y) = -\log\|\epsilon_{n,\beta}(\kappa_n(y))\|^{\mathrm{Hdg}}$, a C^∞ function. As above, the root section of a monomial R is $\phi_\beta\,\epsilon_{n,\beta}$ with ϕ_β holomorphic nowhere zero, and (HX) for β gives $\log h_\beta(0)\circ\tilde\Phi - \alpha_{n,\beta}(y) = \log|R(\tilde\Phi)| - \log|\phi_\beta|$, pluriharmonic. This completes (Fx6).

(Fx7). Let $y\in U_n$, $z_0\neq z_1$ with $gh\neq0$ at $\Phi_n^{(2)}(z_0,z_1,y)$; write $x=\Phi_n(z_0,y)$, $u_l=u(z_l,y)$, $w=\gamma_n(\kappa_n(y))$, $\omega_l = w\wedge u_l\in\wedge^j V_j$ at $\Phi_n(z_l,y)$. Fix a monomial E of multidegree $e_{\mathcal{H}}$ in the base coordinates not vanishing at x and a base cone point $\tilde b$; write $\sigma_{s,j,c}(\omega_l) = s_c^{(l)}E(y)(x)$ ($s_c^{(l)}\in\mathbb{C}$, $c\in I_{k_0}$). As G_0 is invariant under the scalings of the cone, we may evaluate it at the lift $\tilde x$ with base coordinates $\tilde b$ and k_0-coordinates $(s_c^{(0)})_c$, and $\tilde x_1$ the same with $(s_c^{(1)})_c$ (the lift of q_1). Take c with $s_c^{(l)}\neq0$ and the section $P_j = E/Y_{k_0,c}$ of $\mathcal{O}(d_{s,j})$: its fundamental root section is the vector $\gamma\in\wedge^jV_j$ with $\sigma_{s,j,c}(\gamma)=E(y)$, i.e. $\omega_l/s_c^{(l)}$, so $|P_j(\tilde x_l)|/\|\gamma\| = |E(\tilde b)|/\|\omega_l\|$. Fix also sections $P_{j\pm1}$ of $\mathcal{O}(d_{s,j\pm1})$ (free of k_0) not vanishing at x, with fundamental root sections $\gamma_{j\pm1}$. Applying (HX) to $R=P_j^2/(P_{j-1}P_{j+1})$ at $\tilde x_0$ and $\tilde x_1$:

$$\log h_\Xi(0)(\tilde x_l) = 2\log|E(\tilde b)| - 2\log\|\omega_l\| - \log\frac{|P_{j-1}P_{j+1}(\tilde b)|}{\|\gamma_{j-1}\|\,\|\gamma_{j+1}\|}\qquad(l=0,1).$$

By the normalisation of Lemma D.75, $g=\min\cdot H$, $h=G$ with G/H the section δ^{loc} under the root sections; with $\gamma_{j-1}=\mu' w$ and $w\wedge u_0\wedge u_1=\mu\gamma_{j+1}$, bilinearity of the 2×2 determinant gives $\min(\tilde x_0,\tilde x_1) = s_a^{(0)}s_b^{(1)} - s_b^{(0)}s_a^{(1)} = \delta^{\mathrm{loc}}(w\otimes w\wedge u_0\wedge u_1)/E(y)^2$ and $\delta^{\mathrm{loc}}(\gamma_{j-1}\otimes\gamma_{j+1}) = \frac{G(\tilde b)}{H(\tilde b)}\cdot\frac{P_{j-1}P_{j+1}(\tilde b)}{E(\tilde b)^2}E(y)^2$. Hence

$$\log|g/h|(\tilde x_0,\tilde x_1) = \log(\|w\|\,\|w\wedge u_0\wedge u_1\|) + \log\frac{|P_{j-1}P_{j+1}(\tilde b)|}{\|\gamma_{j-1}\|\,\|\gamma_{j+1}\|} - 2\log|E(\tilde b)|$$

. Adding $\frac12\log h_\Xi(0)(\tilde x_0)+\frac12\log h_\Xi(0)(\tilde x_1)$, the terms in E and $P_{j\pm1}$ cancel and

$$\begin{aligned} G_0(\Phi_n^{(2)}(z_0,z_1,y)) &= \log\frac{\|w\|\,\|w\wedge u_0\wedge u_1\|}{\|w\wedge u_0\|\,\|w\wedge u_1\|} = \log\frac{|u_0\wedge u_1|_Q}{|u_0|_Q\,|u_1|_Q}\\ &= \tfrac12\log\Big(1-\frac{|\langle u_0,u_1\rangle_Q|^2}{|u_0|_Q^2|u_1|_Q^2}\Big) = \mathrm{ls}_{G_n(y)}(z_0,z_1)\end{aligned}$$

(Gram determinant of two vectors), using $|u_l|_Q^2 = u_l^*G_n(y)u_l$ and $|\langle u_0,u_1\rangle_Q| = |u_0^*G_n(y)u_1|$. This is (Fx7). □

Lemma D.78. *In the situation of Lemma D.77, on the flag model of Lemma 33.8, in either of its cases (i) and (ii), with the data of Proposition 33.12 in the same case, put $X=\mathcal{X}'(\mathbb{C})$, $W=\mathcal{X}^{(2)}(\mathbb{C})$, and for a root $r=(\tau,i)$ write $a(r)=(\tau,i)$, $a(r)^-=(\tau,i-1)$ (with $\theta_{(\tau,0)}=0$, $\varphi_{(\tau,0)}=\varphi^T_{(\tau,0)}=0$), $\theta_r=\theta_{a(r)^-}-\theta_{a(r)}$, $u_r=\varphi_{a(r)^-}-\varphi_{a(r)}$, $u_{r,t}=\varphi^T_{a(r)^-}(t)-\varphi^T_{a(r)}(t)$. Then:*

1. *$(X, \Theta', \varsigma, \mathrm{id}, 1)$ is a compact model of X°, and $(W, \Theta_W, q_0^*\varsigma, \mathrm{id}, 1)$, for any Kähler form Θ_W on W, is a compact model of W°, with $\mathrm{rW}_W = \mathrm{rW}_X \circ q_0 = \mathrm{rW}_X \circ q_1$ on W°; with $p_j = q_j|_W$ (Sw2) holds;*

2. *with $t_1 = \min(t_0, e^{-e})$, $\delta(t) = t^{1/3} + (1+C)\varepsilon(t)$, $R(t) = \log(1/t)$, the families $(\theta_a, \varphi_a, (\varphi_a^T(t))_{0<t\le t_1})$, a running over the fundamental indices, satisfy (R1)–(R5) on (X, Θ', ς) for some constant; and for every root r, $\omega_{r,t} = \theta_r + dd^c u_{r,t}$ on X $(0 < t \le t_1)$ and $\omega_{r,0} = \theta_r + dd^c u_r$ on X°;*

3. *there are a holomorphic section v of a smooth hermitian line bundle on W and $\kappa_v \in \mathbb{R}$ such that $\|v\| \le 1$, the zero divisor of v is the reduced smooth divisor $\Delta(\mathbb{C})$, no component of it lies in $q_0^{-1}(\partial X)$, $\Delta(\mathbb{C}) + q_0^{-1}(\partial X)_{\mathrm{red}}$ has simple normal crossings, and the identities of (Sw3) hold with these $u_{r,t}$, u_r, $p_j = q_j$ and $\varrho_W = \mathrm{id}$, for $0 < t \le t_1$ and for $t = 0$.*

Proof. **Item 1.** X is a compact complex manifold ($\mathcal{X}'_{\mathbb{Q}}$ smooth and projective) with the Kähler form Θ', and ς is the section of Proposition 33.12 (pulled back from the base, $\|\varsigma\| \le 1$, simple normal crossing zero divisor with support the complement of X°). With $\varrho_X = \mathrm{id}$, $d_X = 1$ and $N = N' = \varnothing$ (no singular points) the covering condition is trivial and $\varrho_X^{-1}(X^\circ) = X \smallsetminus \partial X$. $W \cong \mathcal{X}''(\mathbb{C})$ is a $\mathbb{P}^1$-bundle over X via q_0 (Lemma D.75), a projective manifold, so it carries a Kähler form Θ_W; q_0 is a submersion, so the zero divisor of $q_0^*\varsigma$ (with the pulled-back metric, $\|q_0^*\varsigma\| \le 1$) is the pull-back of a simple normal crossing divisor by a submersion, again one (local coordinates pull back to part of a coordinate system), with complement $q_0^{-1}(X^\circ) = W^\circ$. As ς and its metric are pulled back from the base and q_0, q_1 agree after projection to the base, $\|\varsigma\| \circ q_0 = \|\varsigma\| \circ q_1$, so $\mathrm{rW}_W = \mathrm{rW}_X \circ q_j$ on W° and $q_j(W^\circ) \subseteq X^\circ$ (Lemma D.77, (Fx5)). This gives the compact models and (Sw2) with equality.

Item 2. Lemma 31.32 with $F = \mathrm{id}_X$ (so $\mathrm{rW} \circ F = \mathrm{rW}$) gives (R1)–(R5) for the families of the fundamental indices with δ, R, t_1, and $\theta_a + dd^c\varphi_a^T(t) = \eta_a(t) = c_1(\mathcal{M}_a, h_a^{\mathrm{ref}} e^{\varphi_a^T(t)})$. By hypothesis 3 of Lemma 31.21 in Proposition 33.12, item 2, $\prod_{k\le i} h_{(\tau,k)}(t)^{-1} = h^{\mathrm{ref}}_{(\tau,i)} e^{\varphi^T_{(\tau,i)}(t)}$ for $0 < t \le t_0$, so $h_r(t) = h^{\mathrm{ref}}_{a(r)^-} e^{\varphi^T_{a(r)^-}(t)} / (h^{\mathrm{ref}}_{a(r)} e^{\varphi^T_{a(r)}(t)})$ and $\omega_{r,t} = \eta_{a(r)^-}(t) - \eta_{a(r)}(t) = \theta_r + dd^c u_{r,t}$ (first Chern forms are additive under tensor products). On X°, item 2 of Proposition 33.12 gives $\prod_{k\le i}(h^{\mathrm{sing}}_{(\tau,k)})^{-1} = h^{\mathrm{ref}}_{(\tau,i)} e^{\varphi_{(\tau,i)}}$, so in the same way $\omega_{r,0} = \frac{i}{\pi}\partial\bar\partial \log h_r^{\mathrm{sing}} = \theta_r + dd^c u_r$ (the normalisation of dd^c is the one for which $c_1(\mathcal{M}, he^\varphi) = c_1(\mathcal{M}, h) + dd^c\varphi$, as in Lemma 31.32).

Item 3. Let $h^{\mathrm{ref}}_\Xi = \frac{h^{\mathrm{ref}}_{a(r_1)^-} h^{\mathrm{ref}}_{a(r_2)}}{h^{\mathrm{ref}}_{a(r_1)} h^{\mathrm{ref}}_{a(r_2)^-}}$, a smooth positive cone function of multidegree e_Ξ; by item 2, $\log h_\Xi(t) = \log h^{\mathrm{ref}}_\Xi + u_{r_1,t} - u_{r_2,t}$ for $0 < t \le t_1$ on the cone over X, and the same with $u_{r_1} - u_{r_2}$ for $t = 0$ over X°. Let $\mathcal{M} = \mathcal{O}(\deg g - \deg h)|_W$ with the smooth metric $\|P\|_0 = |P|\,(h^{\mathrm{ref}}_\Xi \circ q_0 \cdot h^{\mathrm{ref}}_\Xi \circ q_1)^{1/2}$ on the cone (the exponent of the multidegree matches: $2(\deg g - \deg h) = -(q_0^* e_\Xi + q_1^* e_\Xi)$ by (Fx3)). Since $\mathrm{div}(g|_W) = \mathrm{div}(h|_W) + [\Delta]$ ((Fx3), on the smooth complex points), g/h is a holomorphic section of $\mathcal{M}$ with zero divisor $\Delta(\mathbb{C})$, reduced. Put $\kappa_v = \log\max(1, \max_W \|g/h\|_0)$ (W compact), $\|\cdot\| = e^{-\kappa_v}\|\cdot\|_0$ and $v = g/h$: then $\|v\| \le 1$ and $\frac12 \log\|v\|^2 + \kappa_v = \log|g| - \log|h| + \frac12 \sum_j \log h^{\mathrm{ref}}_\Xi \circ q_j$, so by (31.4)

$$G_t = \tfrac12 \log\|v\|^2 + \kappa_v + \tfrac12 \sum_{j=0,1} (u_{r_1,t} - u_{r_2,t}) \circ q_j$$

off $\Delta \cup \{gh = 0\}$ for $0 < t \le t_1$, and the same with $u_{r_1} - u_{r_2}$ on W° for $t = 0$. $\Delta(\mathbb{C})$ is the image of the diagonal section of the $\mathbb{P}^1$-bundle $q_0 \colon W \to X$: a smooth hypersurface meeting every fibre transversally in one point; near any of its points, in a trivialisation $W \supseteq U \times \mathbb{P}^1$ with $\Delta = \{w = \phi(x)\}$, the functions $(x, w - \phi(x))$ form coordinates in which $q_0^{-1}(\partial X) = \{x_1 \cdots x_{m'} = 0\}$ and

$\Delta = \{w - \phi(x) = 0\}$: simple normal crossings. Each component of $\Delta(\mathbb{C})$ maps isomorphically onto a component of X by q_0, which is not contained in ∂X, so it is not contained in $q_0^{-1}(\partial X)$. $\square$

Proposition D.79. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. Then the realised tower has geometric proportionality with covolumes (Vp) over E.*

Proof. By Proposition D.83, applied to the same data, there are a hermitian space (V, h) over E of dimension n, indefinite at some real place of E^+, a neat non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_E$, an integer $m \geq 1$, a real number $c_0 > 0$ and, for every $Z \in \mathrm{Cp}$, lattices $\Lambda_{Z,1}, \dots, \Lambda_{Z,m}$ of V, all of them in one genus, such that

$$c_0\,(-1)^D\,\mathrm{gi}(\mathcal{X}_{2,Z}; x^\delta) = \sum_{i=1}^{m} \mu_\infty\big(\Gamma(\Lambda_{Z,i}, \mathfrak{a})\backslash G^1_\infty(V)\big) \qquad (Z \in \mathrm{Cp}). \tag{1}$$

These are data as required in (Vu): one space of dimension n indefinite at some real place, one neat ideal, one m, one c_0, and lattices in one genus. We show that they satisfy the identity of (Vp). Let $B \geq 2$ be an integer, $Z \in \mathrm{Cp}$ and $F \in \mathcal{R}$ homogeneous of degree D. By Proposition D.80, applied to the same data,

$$\mathrm{gi}(\mathcal{X}_{B,Z}; F) = (-1)^D\,\mathrm{gi}(\mathcal{X}_{2,Z}; x^\delta)\,\deg^{\mathrm{fl}}(F).$$

Multiplying by c_0 and inserting (1),

$$c_0\,\mathrm{gi}(\mathcal{X}_{B,Z}; F) = \big(c_0\,(-1)^D\,\mathrm{gi}(\mathcal{X}_{2,Z}; x^\delta)\big)\deg^{\mathrm{fl}}(F) = \Big(\sum_{i=1}^{m} \mu_\infty\big(\Gamma(\Lambda_{Z,i}, \mathfrak{a})\backslash G^1_\infty(V)\big)\Big)\deg^{\mathrm{fl}}(F),$$

which is the identity of (Vp) for (B, Z, F). Since B, Z and F were arbitrary and the data $(V, h), \mathfrak{a}, m, c_0, (\Lambda_{Z,i})$ do not depend on them, the realised tower has geometric proportionality with covolumes (Vp) over E. $\square$

Proposition D.80. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2, and put $x^\delta = \prod_{a\in\Phi}\prod_{i=1}^{n} x_{a,i}^{n-i} \in \mathcal{R}$, a monomial of degree D with $\deg^{\mathrm{fl}}(x^\delta) = (-1)^D$. Then for every integer $B \geq 2$, every $Z \in \mathrm{Cp}$ and every $F \in \mathcal{R}$ homogeneous of degree D,*

$$\mathrm{gi}(\mathcal{X}_{B,Z}; F) = (-1)^D\,\mathrm{gi}(\mathcal{X}_{2,Z}; x^\delta)\,\deg^{\mathrm{fl}}(F).$$

Proof. Fix $c \in \mathrm{Cp}$; here $[F : \mathbb{Q}]$ refers to the field F of $\mathfrak{u}$, and F as an argument is the polynomial.

Step 1 (the geometric degrees as one integral). By Lemma D.81, for every integer $B \geq 2$ and every $F \in \mathcal{R}$ homogeneous of degree D,

$$\mathrm{gi}(\mathcal{X}_{B,c}; F) = \frac{1}{[F : \mathbb{Q}]} \int_{X_c^{\mathrm{an}}} F(\xi); \tag{1}$$

the right-hand side does not depend on B.

Step 2 (proportionality). By Proposition D.82 there is a real v_c with $\int_{X_c^{\mathrm{an}}} F(\xi) = v_c \deg^{\mathrm{fl}}(F)$ for all $F \in \mathcal{R}$ homogeneous of degree D. With (1), $\mathrm{gi}(\mathcal{X}_{B,c}; F) = \frac{v_c}{[F:\mathbb{Q}]}\deg^{\mathrm{fl}}(F)$ for every $B \geq 2$ and every such F. For $F = x^\delta$: in (21.1) the coefficient $[x^{\delta\circ w}]x^\delta$ is 1 if $\delta \circ w = \delta$

and 0 otherwise, and $\delta \circ w = \delta$ forces $w = \mathrm{id}$ because $i \mapsto n-i$ is injective; so $\deg^{\mathrm{fl}}(x^\delta) = (-1)^D$, and $B = 2$ gives $\frac{v_c}{[F:\mathbb{Q}]} = (-1)^D \mathrm{gi}(\mathcal{X}_{2,c}; x^\delta)$. Substituting,

$$\mathrm{gi}(\mathcal{X}_{B,c}; F) = (-1)^D \, \mathrm{gi}(\mathcal{X}_{2,c}; x^\delta) \, \deg^{\mathrm{fl}}(F)$$

for every $B \geq 2$, every $c = Z \in \mathrm{Cp}$ and every $F \in \mathcal{R}$ homogeneous of degree D. □

Lemma D.81. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. For $c \in \mathrm{Cp}$ and $1 \leq i \leq n$, $\tau \in \Phi$, put*

$$\xi_{\tau,i} = c_1(\mathcal{M}_{(\tau,i-1)}, h^{\mathrm{ref}}_{(\tau,i-1)}) - c_1(\mathcal{M}_{(\tau,i)}, h^{\mathrm{ref}}_{(\tau,i)})$$

on the compact complex manifold X_c^{an} of Sys_c (the first term omitted for $i = 1$), and let $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})$ be the complex manifold of all complex points of the F-scheme $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}$ (over every embedding of F into $\mathbb{C}$), with the forms $\omega_r = c_1(\mathcal{L}_r, \|\cdot\|^{\mathrm{Hdg}})$ for the roots r. For $G \in \mathcal{R}$ homogeneous of degree D let $G(\xi)$ and $G(\omega)$ be the $2D$-forms obtained by substituting $\xi_{\tau,i}$, respectively $\omega_{(\tau,i)}$, for $x_{\tau,i}$. Then for every integer $B \geq 2$, every $c \in \mathrm{Cp}$ and every $G \in \mathcal{R}$ homogeneous of degree D, the form $G(\omega)$ is integrable on $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})$ and

$$[F : \mathbb{Q}] \, \mathrm{gi}(\mathcal{X}_{B,c}; G) = \int_{X_c^{\mathrm{an}}} G(\xi) = \int_{\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})} G(\omega).$$

In particular, with $x^\delta = \prod_{\tau \in \Phi} \prod_{i=1}^{n} x_{\tau,i}^{n-i}$ and $\nu = (-1)^D x^\delta(\omega)$, one has

$$[F : \mathbb{Q}] \, (-1)^D \, \mathrm{gi}(\mathcal{X}_{2,c}; x^\delta) = \int_{\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})} \nu$$

for every $c \in \mathrm{Cp}$.

Proof. Let $B \geq 2$ and $c \in \mathrm{Cp}$. The good origin contains a root origin of $\mathcal{X}_{B,c}$ on the fixed data at c for S_B, and by its clause (smooth) $\mathcal{X}_{B,c} \otimes \mathbb{Q}$ is smooth. So Lemma 33.29, applied to the fixed data at c, $S = S_B$ and $\mathcal{X} = \mathcal{X}_{B,c}$, gives for every $G \in \mathcal{R}$ homogeneous of degree D the integrability of $G(\omega)$ on $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})$ and the displayed identity. The last assertion is the case $B = 2$, $G = (-1)^D x^\delta$, by linearity of gi. □

Proposition D.82. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. For $c \in \mathrm{Cp}$ let Sys_c be the analytic system of the fixed data at c, with compact complex manifold X_c^{an} of dimension D and, for each fundamental index a, the smooth hermitian line bundle $(\mathcal{M}_a, h_a^{\mathrm{ref}})$. For $\tau \in \Phi$ and $1 \leq i \leq n$ put*

$$\xi_{\tau,i} = c_1(\mathcal{M}_{(\tau,i-1)}, h^{\mathrm{ref}}_{(\tau,i-1)}) - c_1(\mathcal{M}_{(\tau,i)}, h^{\mathrm{ref}}_{(\tau,i)}),$$

a real closed $(1,1)$-form on X_c^{an}, where the first term is omitted for $i = 1$; for $G \in \mathcal{R}$ homogeneous of degree D let $G(\xi)$ be the $2D$-form obtained by substituting $\xi_{\tau,i}$ for $x_{\tau,i}$ (well defined, since forms of even degree commute). Then for every $c \in \mathrm{Cp}$ there is a real number v_c such that

$$\int_{X_c^{\mathrm{an}}} G(\xi) = v_c \, \deg^{\mathrm{fl}}(G) \qquad \text{for every } G \in \mathcal{R} \text{ homogeneous of degree } D.$$

Proof. Fix $c \in \mathrm{Cp}$ and write $M_c = \mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})$, the complex manifold of all complex points of $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}$ (over every embedding of F), an open and closed subset of the manifold M of Proposition 33.31.

Step 1 (the integral over the flag bundle). By Lemma D.81 (with $B = 2$), for every $G \in \mathcal{R}$ homogeneous of degree D the form $G(\omega)$ is integrable on M_c and

$$\int_{X_c^{\mathrm{an}}} G(\xi) = \int_{M_c} G(\omega). \tag{1}$$

Step 2 (pointwise proportionality). Let $G_0 = (-1)^D x^\delta$ with $x^\delta = \prod_{\tau\in\Phi}\prod_{i=1}^n x_{\tau,i}^{n-i}$. In (21.1) only $w = \mathrm{id}$ contributes to $\deg^{\mathrm{fl}}(x^\delta)$, since $i \mapsto n-i$ is injective, so $\deg^{\mathrm{fl}}(G_0) = 1$. By Proposition 33.31, at every $y \in M_c$, $G(\omega)_y = \deg^{\mathrm{fl}}(G)\,\nu_y$ for every G; with $G = G_0$ this gives $\nu_y = G_0(\omega)_y$, hence

$$G(\omega) = \deg^{\mathrm{fl}}(G)\, G_0(\omega) \quad \text{on } M_c.$$

Step 3 (conclusion). Integrating over M_c (both sides integrable by Step 1) and using (1) for G and for G_0,

$$\int_{X_c^{\mathrm{an}}} G(\xi) = \deg^{\mathrm{fl}}(G)\int_{M_c} G_0(\omega) = \deg^{\mathrm{fl}}(G)\int_{X_c^{\mathrm{an}}} G_0(\xi),$$

so $v_c = \int_{X_c^{\mathrm{an}}} G_0(\xi)$ has the required property. □

Proposition D.83. *Let $\mathfrak{u} = (E, \mathsf{H}, \mathsf{U}, p, q, \mathsf{L}, \mathrm{Sh}, F)$, $\mathcal{Y}$ and the root tower $\mathcal{T}$ of type (Φ, n, p) with its geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ and its good origin immersive at level four satisfy the hypotheses of Proposition 34.2. Then there are a hermitian space (V, h) over E of dimension n, indefinite at some real place of E^+, a neat non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_E$, an integer $m \geq 1$, a real number $c_0 > 0$ and, for every $Z \in \mathrm{Cp}$, lattices $\Lambda_{Z,1}, \dots, \Lambda_{Z,m}$ of V, all of them (over all Z and i) in one genus, such that for every $Z \in \mathrm{Cp}$*

$$c_0\,(-1)^D\,\mathrm{gi}(\mathcal{X}_{2,Z}; x^\delta) = \sum_{i=1}^m \mu_\infty(\Gamma(\Lambda_{Z,i}, \mathfrak{a})\backslash G_\infty^1(V)).$$

Proof. Put $\Phi'_\kappa = \{\rho\colon E \to \mathbb{C} :\ \rho(r) = \kappa(r_{\mathbb{C}})\}$ for a field embedding $\kappa\colon F \to \mathbb{C}$; it is Φ if $\varepsilon_\kappa = 1$ and $\overline{\Phi}$ if $\varepsilon_\kappa = -1$, and $\kappa \circ \tau \in \Phi'_\kappa$ for $\tau \in \Phi$ (as $\tau(r) = r_{\mathbb{C}}$). Let $\nu^{\mathrm{Fl}} = (-1)^D x^\delta(\omega)$ on $\mathrm{Fl}_{\mathfrak{u}}(\mathbb{C})$, ω_r the Hodge Chern forms of the root lines.

Step 1 (reduction to the flag bundle). By Lemma D.81, for every $c \in \mathrm{Cp}$,

$$[F:\mathbb{Q}]\,(-1)^D\,\mathrm{gi}(\mathcal{X}_{2,c}; x^\delta) = \int_{\mathrm{Fl}_{\mathfrak{u}}|_{Z_c}(\mathbb{C})} \nu^{\mathrm{Fl}} = \sum_{\kappa\colon F\to\mathbb{C}} \int_{\mathrm{Fl}_{\mathfrak{u}}|_{Z_c,\kappa}} \nu^{\mathrm{Fl}}, \tag{1}$$

where $\mathrm{Fl}_{\mathfrak{u}}|_{Z_c,\kappa}$ is the open and closed set of complex points over κ, i.e. the flag bundle over the complex points of $Z_c \otimes_{F,\kappa} \mathbb{C}$.

Step 2 (one piece). Fix c and κ, and let $k = k(c,\kappa) \in I_\kappa$ be the index with $u_k(\mathrm{Per}_\sigma(L_k))$ the complex points of $Z_c \otimes_{F,\kappa} \mathbb{C}$ (SU6). By Lemma 33.34 (for $Z = Z_c$; the level is good at every prime with $A = 1$ and the components split),

$$\int_{\mathrm{Fl}_{\mathfrak{u}}|_{Z_c,\kappa}} \nu^{\mathrm{Fl}} = \mu_{\nu,k}(\Gamma_k\backslash G_k^1), \tag{2}$$

with $\Gamma_k = \Gamma(L_k, \mathfrak{n})$, and G_k^1 and $\mu_{\nu,k}$ the group and Haar measure of Lemma 33.38 for $(L_{k,\mathbb{R}}, \psi_k, \Phi'_\kappa)$.

Step 3 (one space, one genus, one measure). Fix an embedding κ_0 and $c_1 \in \mathrm{Cp}$, put $k_0 = k(c_1, \kappa_0)$, $V = L_{k_0} \otimes \mathbb{Q}$ and $h = \varepsilon_{\kappa_0} h_{k_0}$. By (SU4) the form $\varepsilon_\kappa h_k$ has, at every real place of E^+, the same signature (q and p, or p and q, depending only on the sign of $\operatorname{Im} r_{\mathbb{C}}$) for all κ and k; as $p, q \geq 1$, (V, h) is indefinite at every real place. Since $\psi_k = \operatorname{Tr}_{E/\mathbb{Q}}(r\, h_k) = \operatorname{Tr}_{E/\mathbb{Q}}\big((\varepsilon_\kappa r)(\varepsilon_\kappa h_k)\big)$ is perfect on L_k, $L_k = \{x : \operatorname{Tr}(\varepsilon_\kappa r\, \varepsilon_\kappa h_k(x, L_k)) \subseteq \mathbb{Z}\}$. The element $\xi = \varepsilon_\kappa r$ satisfies $\bar\xi = -\xi$ ($r^2 = -d < 0$ at every embedding) and $\xi \mathcal{O}_E = r\mathcal{O}_E$, and E/E^+ is unramified at every finite place (by the definition of unitary Shimura data). By Lemma D.84 (with $\xi' = \varepsilon_{\kappa_0} r$, $\Lambda' = L_{k_0}$) there is an E-linear isometry $g_{\kappa,k} \colon (L_k \otimes \mathbb{Q}, \varepsilon_\kappa h_k) \to (V, h)$, and all the lattices $g_{\kappa,k} L_k$ lie in one genus with L_{k_0}, hence in one genus. For $c \in \mathrm{Cp}$ and κ put $\Lambda_{c,\kappa} = g_{\kappa,k(c,\kappa)} L_{k(c,\kappa)}$. As $\mathrm{U}(\varepsilon h) = \mathrm{U}(h)$ and an E-linear map preserves ψ_k if and only if it preserves h_k ((SU1); over $\mathbb{R}$ likewise, the trace form being non-degenerate), conjugation by $g = g_{\kappa,k}$ maps $\Gamma(L_k, \mathfrak{n})$ onto $\Gamma(\Lambda_{c,\kappa}, \mathfrak{n})$ (31.2) and G^1_k onto $G^1_\infty(V)$, which is the group G^1 of the lemma for $(V_{\mathbb{R}}, \psi_V, \Phi'_{\kappa_0})$, $\psi_V = \operatorname{Tr}(\varepsilon_{\kappa_0} r\, h)$. Now $\psi_V(gx, gy) = \varepsilon_{\kappa_0}\varepsilon_\kappa\, \psi_k(x, y)$, and $\Phi'_{\kappa_0} = \Phi'_\kappa$ if $\varepsilon_{\kappa_0}\varepsilon_\kappa = 1$, $= \overline{\Phi'_\kappa}$ otherwise; so by Lemma 33.38 (3) conjugation by g carries $\mu_{\nu,k}$ to the one Haar measure μ_ν of $(V_{\mathbb{R}}, \psi_V, \Phi'_{\kappa_0})$ on $G^1_\infty(V)$. Write $\mu_\nu = c'\mu_\infty$ with $c' > 0$. With (2),

$$\int_{\mathrm{Fl_u}|_{Z_{c,\kappa}}} \nu^{\mathrm{Fl}} = c'\, \mu_\infty\big(\Gamma(\Lambda_{c,\kappa}, \mathfrak{n}) \backslash G^1_\infty(V)\big).$$

Step 4 (conclusion). Put $\mathfrak{a} = \mathfrak{n}$ (neat, (SU5), non-zero), $m = [F : \mathbb{Q}]$, index the embeddings κ by $1 \leq i \leq m$, put $\Lambda_{c,i} = \Lambda_{c,\kappa_i}$ and $c_0 = [F : \mathbb{Q}]/c' > 0$. Summing over κ in (1) and dividing by c',

$$c_0\, (-1)^D\, \mathrm{gi}(\mathcal{X}_{2,c}; x^\delta) = \sum_{i=1}^{m} \mu_\infty\big(\Gamma(\Lambda_{c,i}, \mathfrak{a}) \backslash G^1_\infty(V)\big) \qquad (c \in \mathrm{Cp}),$$

with (V, h) of dimension n, indefinite at some real place, and all $\Lambda_{c,i}$ lattices of V in one genus; none of $V, h, \mathfrak{a}, m, c_0$ depends on c. □

Lemma D.84. *Let E be a CM field with maximal totally real subfield E^+ such that E/E^+ is unramified at every finite place. Let $\xi, \xi' \in E^\times$ with $\bar\xi = -\xi$, $\bar\xi' = -\xi'$ and $\xi\mathcal{O}_E = \xi'\mathcal{O}_E$. Let (V, h) and (V', h') be hermitian spaces over E of the same dimension such that at every real place w of E^+ the hermitian forms $h \otimes_{E^+,w} \mathbb{R}$ and $h' \otimes_{E^+,w} \mathbb{R}$ have the same signature, and let $\Lambda \subseteq V$, $\Lambda' \subseteq V'$ be lattices with*

$$\Lambda = \{x \in V :\ \operatorname{Tr}_{E/\mathbb{Q}}(\xi h(x, \Lambda)) \subseteq \mathbb{Z}\}, \qquad \Lambda' = \{x \in V' :\ \operatorname{Tr}_{E/\mathbb{Q}}(\xi' h'(x, \Lambda')) \subseteq \mathbb{Z}\}.$$

Then there is an E-linear isometry $g \colon (V, h) \to (V', h')$, and for every such g the lattices $g\Lambda$ and Λ' of V' lie in one genus.

Proof. **The trace condition is a scaled hermitian duality.** For $x \in V$, $h(x, \Lambda)$ is an $\mathcal{O}_E$-submodule of E (as $h(x, ay) = \bar a h(x, y)$), so by the definition of the inverse different

$$\operatorname{Tr}_{E/\mathbb{Q}}(\xi h(x, \Lambda)) \subseteq \mathbb{Z} \quad \text{if and only if} \quad h(x, \Lambda) \subseteq \mathfrak{c} := \xi^{-1}\mathfrak{D}^{-1}.$$

The same holds for Λ' with the same $\mathfrak{c}$, since $\xi'\mathcal{O}_E = \xi\mathcal{O}_E$. Thus

$$\Lambda = \{x \in V :\ h(x, \Lambda) \subseteq \mathfrak{c}\}, \qquad \Lambda' = \{x \in V' :\ h'(x, \Lambda') \subseteq \mathfrak{c}\}.$$

The ideal $\mathfrak{c}$ is stable under complex conjugation ($\bar\xi = -\xi$, and $\mathfrak{D}$ is stable under every automorphism of E).

Local form at a finite place. Let v be a finite place of E^+. As E/E^+ is unramified at v, either E_v is the unramified quadratic extension of F_v, and then a uniformiser ϖ_v of F_v is one of E_v, or $E_v \cong F_v \times F_v$ with conjugation exchanging the factors. In both cases the conjugation-stable fractional ideal $\mathfrak{c}R_v$ equals $c_v R_v$ for some $c_v \in F_v^\times$ (a power of ϖ_v; in the split case the two exponents agree by stability). The form $k_v = c_v^{-1} h_v$ is hermitian since $c_v \in F_v$. Taking the completion at v of the description of Λ above (the dual of a finitely generated module commutes with the flat base change $\mathcal{O}_{E^+} \to \mathcal{O}_v$, and $\mathfrak{c} \otimes \mathcal{O}_v = c_v R_v$) gives

$$\Lambda_v = \{x \in V_v : \ k_v(x, \Lambda_v) \subseteq R_v\},$$

that is, Λ_v is a self-dual R_v-lattice for k_v in the sense of Lemma 16.27, and likewise Λ'_v for $k'_v = c_v^{-1} h'_v$.

Determinants are local norms. R_v is a semilocal Dedekind domain, hence a principal ideal domain, so Λ_v is free over R_v; in a basis of Λ_v the Gram matrix of k_v is invertible over R_v (self-duality), and its determinant is conjugation-invariant, hence lies in $R_v^\times \cap F_v = \mathcal{O}_v^\times$. Fix E-bases (e_i) of V and (e'_i) of V' and put $G = (h(e_j, e_i))_{i,j}$, $G' = (h'(e'_j, e'_i))_{i,j}$, so that $\bar{x}^\top G y = h(\sum_j y_j e_j, \sum_i x_i e_i)$; G and G' are non-degenerate hermitian matrices, $\det G, \det G' \in E^{+\times}$, and a change of basis multiplies the determinant by an element $z\bar{z}$, $z \in E^\times$. Put $\delta = \det G' / \det G \in E^{+\times}$. Comparing with bases of Λ_v and Λ'_v,

$$\det G \in c_v^n \, u_v \, N(E_v^\times), \qquad \det G' \in c_v^n \, u'_v \, N(E_v^\times), \qquad u_v, u'_v \in \mathcal{O}_v^\times,$$

($n = \dim V$, N the norm of E_v/F_v), so $\delta \in \mathcal{O}_v^\times N(E_v^\times) = N(E_v^\times)$: every unit of F_v is a norm from E_v (unramified: Serre, *Local fields*, Ch. V, §2; split: $N(a,b) = ab$). At a real place w the sign of $w(\det G)$ is $(-1)^{s_w}$ with s_w the number of negative eigenvalues of $h \otimes_w \mathbb{R}$, and likewise for G'; equal signatures give $w(\delta) > 0$, a norm from $\mathbb{C}$. Write $E = E^+(\sqrt{\delta_0})$ with $\delta_0 \in E^+$; the norms from E (resp. E_v, $E \otimes_w \mathbb{R}$) are the elements $a^2 - \delta_0 b^2$. By the Hasse norm theorem (Theorem B.27, applied to δ_0 and $x = \delta$) $\delta = z\bar{z}$ with $z \in E^\times$.

Isometry. For every embedding $\tau\colon E \to \mathbb{C}$ the matrices $\tau(G)$ and $\tau(G')$ have the signature of h, resp. h', at the real place below τ (for the conjugate embedding the matrix is conjugated, with the same signature), so they agree. By Landherr's theorem (Theorem B.28) there is $\gamma \in \mathrm{GL}_n(E)$ with $\bar{x}^\top \bar{\gamma}^\top G \gamma y = \bar{x}^\top G' y$; then the E-linear map g with $g(\sum_i y_i e'_i) = \sum_i (\gamma y)_i e_i$ is an isometry $(V', h') \to (V, h)$, and its inverse is an isometry $(V, h) \to (V', h')$.

One genus. Let $g\colon (V,h) \to (V',h')$ be any isometry. Then $g\Lambda = \{x \in V' : \ h'(x, g\Lambda) \subseteq \mathfrak{c}\}$, so by the local form above both $(g\Lambda)_v = g\Lambda_v$ and Λ'_v are self-dual R_v-lattices for the hermitian form $k'_v = c_v^{-1} h'_v$ on V'_v, for every finite place v. By Lemma 16.27 there is an E_v-linear g_v preserving k'_v, hence h'_v, with $g_v(g\Lambda)_v = \Lambda'_v$; that is, $g_v \in \mathrm{U}(V')(F_v)$. So $g\Lambda$ and Λ' lie in one genus. □

D.4.3 The fibre-square identities

Theorem D.85 derives the fibre-square identities (Fq1) of the root models from proportional square systems. The compact-dual side is Theorem D.87: the Fubini–Study height of the flag scheme satisfies the fibre-square identities with the sphere moments $\kappa_k = \frac{1}{2}\sum_{i<k} \frac{1}{2i+1}$ (Fq2). The model side passes to the limit of the square integrals (Theorem D.92), using the square convergence (Fx8) from an analytic square system (Theorem D.95) and the convergence of mixed Monge–Ampère integrals.

Theorem D.85. *Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with a geometric realisation whose root models have convergent regularisation and which has proportional analytic square systems. Then*

for every integer $B \geq 2$, every $Z \in \mathrm{Cp}$ and every Levi reflection s, the pair

$$(\mathrm{pp}(\mathcal{X}_{B,Z};\cdot)|_{\mathcal{R}}, \mathrm{gi}(\mathcal{X}_{B,Z};\cdot)|_{\mathcal{R}})$$

satisfies the fibre-square identities for s with the sphere moments $\mathrm{sph} = (\mathrm{sph}_k)_{k\geq 1}$, $\mathrm{sph}_k = \frac{1}{2}\sum_{j=0}^{k-1} \frac{1}{2j+1}$*; that is, the realised tower has (Fq1). If moreover $\mathcal{T}$ has the Fubini–Study compact dual (Fs), then the realised tower has the fibre-square identities (Fq1), (Fq2).*

Proof. Fix $B \geq 2$, $Z \in \mathrm{Cp}$ and a Levi reflection s. By hypothesis $\mathcal{X}_{B,Z}$ has a proportional square model for s: a root model $\mathcal{X}'$ of type (Φ, n) with convergent regularisation, a real number $m > 0$ with $\mathrm{pp}(\mathcal{X}';F) = m\,\mathrm{pp}(\mathcal{X}_{B,Z};F)$ and $\mathrm{gi}(\mathcal{X}';F) = m\,\mathrm{gi}(\mathcal{X}_{B,Z};F)$ for $F \in \mathcal{R}$, and a square system $(k_0, \mathcal{X}'^{(2)}, g, h)$ for s on $\mathcal{X}'$ with (Fx5), (Fx6), (Fx7) which carries an analytic square system. By Theorem D.95 this square system has analytic square data (Fx5)–(Fx8), and by Theorem D.92 it has the square limit (Fx4).

The identities on $\mathcal{X}'$. Since $\mathcal{X}'$ has convergent regularisation, $\mathrm{ai}_t(\mathcal{X}';x^a) \to \mathrm{pp}(\mathcal{X}';x^a)$ for every monomial $x^a \in \mathcal{R}^+$, and by linearity $\mathrm{ai}_t(\mathcal{X}';F) \to \mathrm{pp}(\mathcal{X}';F)$ as $t \to 0^+$ for every $F \in \mathcal{R}$. Let c be an s-invariant weight.

For $F = \mathrm{lr}(c)^{D+1}$: by Lemma D.86, applied to the square system on $\mathcal{X}'$, $\mathrm{ai}_t(\mathcal{X}';F) = 0$ for every $t > 0$, hence $\mathrm{pp}(\mathcal{X}';F) = 0$.

For $1 \leq k \leq (D+1)/2$ and $F = \xi_s^{2k}\mathrm{lr}(c)^{D+1-2k}$: by Lemma D.86, $\mathrm{ai}_t(\mathcal{X}';F) = I_t(k,c)/\delta_{\mathcal{X}'}$ for every $t > 0$, where $I_t(k,c)$ is the square integral of the square system on $\mathcal{X}'$; by (Fx4) the right side tends to $-2\,\mathrm{sph}_k\,\mathrm{gi}(\mathcal{X}';\xi_s^{2k-1}\mathrm{lr}(c)^{D+1-2k})$ ($\delta_{\mathcal{X}'} \geq 1$). Hence

$$\mathrm{pp}(\mathcal{X}';\xi_s^{2k}\mathrm{lr}(c)^{D+1-2k}) = -2\,\mathrm{sph}_k\,\mathrm{gi}(\mathcal{X}';\xi_s^{2k-1}\mathrm{lr}(c)^{D+1-2k}).$$

So the pair $(\mathrm{pp}(\mathcal{X}';\cdot)|_{\mathcal{R}}, \mathrm{gi}(\mathcal{X}';\cdot)|_{\mathcal{R}})$ satisfies the identities (30.1) for s with moments sph.

Carrying over. $\mathcal{X}_{B,Z}$ and $\mathcal{X}'$ are root models of type (Φ, n) with convergent regularisation, proportional with the factor $m > 0$ on $\mathcal{R}$; by Lemma D.96 the pair $(\mathrm{pp}(\mathcal{X}_{B,Z};\cdot)|_{\mathcal{R}}, \mathrm{gi}(\mathcal{X}_{B,Z};\cdot)|_{\mathcal{R}})$ satisfies the fibre-square identities for s with moments sph. As B, Z and s were arbitrary, this is (Fq1).

If $\mathcal{T}$ has (Fs), then $h^{\mathrm{cd}} = h^{\mathrm{FS}}_{\Phi,n}$ and Theorem D.87 gives the identities for $(h^{\mathrm{cd}}, \deg^{\mathrm{fl}})$ with its numbers κ_k, which are the sph_k; this is (Fq2), and with (Fq1) the second assertion. □

Lemma D.86. *Let $\mathcal{X}$ be a root model of type (Φ, n), s a Levi reflection and $(k_0, \mathcal{X}^{(2)}, g, h)$ a square system for s on $\mathcal{X}$. Then for every $t > 0$, every s-invariant weight c and every integer $1 \leq k \leq (D+1)/2$,*

$$\mathrm{ai}_t(\mathcal{X};\mathrm{lr}(c)^{D+1}) = 0, \qquad \delta_{\mathcal{X}}\,\mathrm{ai}_t(\mathcal{X};\xi_s^{2k}\,\mathrm{lr}(c)^{D+1-2k}) = I_t(k,c).$$

Proof. Fix $t > 0$ and k, write $m = D+1-2k$, $\mathbb{P} = \prod_{k' \in K} \mathbb{P}(\mathbb{Z}^{I_{k'}})$ for the ambient space of $\mathcal{X}$, $r_1 = (\tau, i)$, $r_2 = (\tau, i+1)$ for $s = s_{\tau,i}$, and $\overline{\Xi} = \overline{\Xi}(t)$. We use heights of cycles of the regular arithmetic varieties $\mathbb{P}$, $\mathbb{P}^{(2)}$ and $\mathbb{P}' = \prod_{k' \neq k_0} \mathbb{P}(\mathbb{Z}^{I_{k'}})$ with respect to Hermitian line bundles with smooth metrics on them (Section 14, Lemma 29.26); all cycles $\mathcal{Z}$ below $(\mathcal{X}, \mathcal{X}^{(2)}, \Delta)$ are integral and flat over $\mathbb{Z}$. The numbers $\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_0)\cdots\widehat{c}_1(\overline{\mathcal{M}}_p) \mid \mathcal{Z})$ for an integral closed subscheme $\mathcal{Z}$ of absolute dimension $p+1$ are symmetric and multilinear; for a non-zero rational section σ of $\mathcal{M}_0|_{\mathcal{Z}}$ whose Weil divisor $\mathrm{div}\,\sigma$ has no component in a fibre over a prime, Lemma 30.17 gives the *inductive formula*

$$\begin{aligned}\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_0)\cdots\widehat{c}_1(\overline{\mathcal{M}}_p) \mid \mathcal{Z}) &= \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_1)\cdots\widehat{c}_1(\overline{\mathcal{M}}_p) \mid \mathrm{div}\,\sigma) \\ &\quad - \int_{\mathcal{Z}(\mathbb{C})} \log\|\sigma\|\, c_1(\overline{\mathcal{M}}_1)\wedge\cdots\wedge c_1(\overline{\mathcal{M}}_p)\end{aligned} \tag{D.15}$$

with $\log\|\sigma\|$ integrable (the vertical sum of Lemma 30.17 is empty); and for a morphism f of such ambient varieties, Lemma 30.18 with $d = 0$ (part (1) if $\dim f(\mathcal{Z}) = \dim \mathcal{Z}$, with $c = \deg(\mathcal{Z}/f(\mathcal{Z}))$, part (2) otherwise) gives the *projection formula* $\widehat{\deg}(\prod \widehat{c}_1(f^*\overline{\mathcal{M}}_j) \mid \mathcal{Z}) = \widehat{\deg}(\prod \widehat{c}_1(\overline{\mathcal{M}}_j) \mid f_*[\mathcal{Z}])$, where $f_*[\mathcal{Z}] = \deg(\mathcal{Z}/f(\mathcal{Z}))[f(\mathcal{Z})]$ if $\dim f(\mathcal{Z}) = \dim \mathcal{Z}$ and $f_*[\mathcal{Z}] = 0$ otherwise. By (29.2) and multilinearity, $\delta_{\mathcal{X}}\mathrm{ai}_t(\mathcal{X}; F) = \widehat{\deg}(F(\widehat{c}_1(\overline{\mathcal{L}}(t))) \mid \mathcal{X})$ for $F \in \mathcal{R}$ homogeneous of degree $D+1$, where x_r is replaced by $\widehat{c}_1(\overline{\mathcal{L}}_r(t))$; in particular ξ_s is replaced by $\widehat{c}_1(\overline{\Xi})$.

Step 1: reduction to integral weights. The s-invariant weights form a real vector space V_s with the lattice $V_s \cap \mathbb{Z}^{\mathrm{Rt}}$ of full rank. Both sides of each identity are polynomial functions of $c \in V_s$: the left sides by multilinearity, the right side because $\omega_{c,t}$ is linear in c and each integral in (31.5) is a finite multilinear expression in the forms $c_1(\overline{\mathcal{L}}_r(t))$. A polynomial vanishing on a lattice of full rank vanishes identically, so it suffices to treat $c \in V_s \cap \mathbb{Z}^{\mathrm{Rt}}$. For such c let $\overline{\mathcal{A}} = \bigotimes_r \overline{\mathcal{L}}_r(t)^{\otimes(-c_r)}$, of multidegree $e_A = -\sum_r c_r e_r$ and cone function $h_A = \prod_r h_r(t)^{-c_r}$; then $c_1(\overline{\mathcal{A}}) = \omega_{c,t}$ and $\mathrm{lr}(c)$ is replaced by $\widehat{c}_1(\overline{\mathcal{A}})$.

Step 2: $\overline{\mathcal{A}}$ comes from the base. Since $c_{r_1} = c_{r_2}$, $h_A = (h_{r_1}(t)h_{r_2}(t))^{-c_{r_1}} \prod_{r \neq r_1, r_2} h_r(t)^{-c_r}$ is a product of powers of functions that come from the base at k_0 by (Fx1), so h_A comes from the base at k_0. So $(e_A)_{k_0} = 0$, $\overline{\mathcal{A}} = \mathrm{pr}^*\overline{\mathcal{A}}'$ for the projection $\mathrm{pr}\colon \mathbb{P} \to \mathbb{P}'$ and the Hermitian line bundle $\overline{\mathcal{A}}'$ on $\mathbb{P}'$ with the induced multidegree and cone function, and, since $\mathrm{pr} \circ q_0 = \mathrm{pr} \circ q_1$, $q_0^*\overline{\mathcal{A}} = q_1^*\overline{\mathcal{A}}$ on $\mathbb{P}^{(2)}$.

Step 3: $\mathrm{pr}(\mathcal{X})$ has absolute dimension at most D, and the first identity. Let $\mathcal{X}' = \mathrm{pr}(\mathcal{X})$, an integral closed subscheme of $\mathbb{P}'$, and $e = \dim \mathcal{X} - \dim \mathcal{X}'$. There is a dense open $V' \subseteq \mathcal{X}'$ over which all fibres of $\mathcal{X} \to \mathcal{X}'$ have dimension e. A point (z, z') of $\mathcal{X}^{(2)}$ has $q_0(z,z') \in \mathcal{X}$, $q_1(z,z') \in \mathcal{X}$ and $\mathrm{pr}\, q_0(z,z') = \mathrm{pr}\, q_1(z,z')$, so the fibre of $\mathcal{X}^{(2)} \to \mathcal{X}'$ over $y \in V'$ embeds into $\mathcal{X}_y \times \mathcal{X}_y$ and has dimension $\leq 2e$. As $\mathcal{X}^{(2)} \to \mathcal{X} \to \mathcal{X}'$ is surjective (Fx2), $\mathcal{X}^{(2)}$ dominates $\mathcal{X}'$, and the dimension of an integral scheme of finite type over $\mathbb{Z}$ dominating $\mathcal{X}'$ is at most $\dim \mathcal{X}'$ plus the dimension of its generic fibre. Hence $D + 2 \leq \dim \mathcal{X}' + 2e = 2(D+1) - \dim \mathcal{X}'$, i.e. $\dim \mathcal{X}' \leq D < \dim \mathcal{X}$. By the projection formula for pr, $\delta_{\mathcal{X}}\mathrm{ai}_t(\mathcal{X}; \mathrm{lr}(c)^{D+1}) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{A}}')^{D+1} \mid \mathrm{pr}_*[\mathcal{X}]) = 0$. This is the first identity.

Step 4: the diagonal maps birationally onto $\mathcal{X}$. Let Δ be as in (Fx3). On Δ the coordinates z_{k_0} and z' are proportional, so $q_0|_\Delta = q_1|_\Delta =: q_\Delta$. For every point x of $\mathcal{X}$ the fibre $q_\Delta^{-1}(x)$ lies in $\{(x, z') : z' \text{ proportional to } x_{k_0}\}$, a single point; so q_Δ is quasi-finite and proper, its image is closed of dimension $\dim \Delta = D+1$, hence equal to $\mathcal{X}$. Over the generic point η of $\mathcal{X}$, the fibre Δ_η is a non-empty integral closed subscheme of $(\mathcal{X}^{(2)})_\eta \cap \{z' \wedge z_{k_0} = 0\}$; by (Fx2) the first is a linear subspace of $\mathbb{P}(\mathbf{k}(\eta)^{I_{k_0}})$, and the minors with z_{k_0} fixed are linear forms in z' cutting out the reduced $\mathbf{k}(\eta)$-point $[z_{k_0}]$. The intersection is a linear subspace contained in a point, hence equals $\operatorname{Spec}\mathbf{k}(\eta)$, and so does Δ_η. Thus $\mathbf{k}(\Delta) = \mathbf{k}(\mathcal{X})$, q_Δ is birational, and $(q_0)_*[\Delta] = [\mathcal{X}]$.

Step 5: the inductive formula on the fibre square. On $\mathcal{X}^{(2)}$ put $\overline{\mathcal{X}}_0 = q_0^*\overline{\Xi}$, $\overline{\mathcal{X}}_1 = q_1^*\overline{\Xi}$, $\overline{\mathcal{A}}_0 = q_0^*\overline{\mathcal{A}} = q_1^*\overline{\mathcal{A}}$ (Step 2), and let $\overline{\mathcal{N}}$ be $\mathcal{O}(\deg g - \deg h)$ with cone function $(h_\Xi(t) \circ q_0 \cdot h_\Xi(t) \circ q_1)^{-1/2}$, smooth and positive. By (Fx3), $\overline{\mathcal{N}}^{\otimes 2} \cong (\overline{\mathcal{X}}_0 \otimes \overline{\mathcal{X}}_1)^{\otimes -1}$ isometrically (equal multidegrees and cone functions), so $2\widehat{c}_1(\overline{\mathcal{N}}) = -\widehat{c}_1(\overline{\mathcal{X}}_0) - \widehat{c}_1(\overline{\mathcal{X}}_1)$. The rational section $\sigma = g/h$ of $\mathcal{N}$ on $\mathcal{X}^{(2)}$ has $\operatorname{div}\sigma = [\Delta]$ (Fx3) and $\log\|\sigma\| = G_t$ (31.4). For $\alpha + \beta = 2k+1$ put

$$U(\alpha, \beta) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{X}}_0)^\alpha \widehat{c}_1(\overline{\mathcal{X}}_1)^\beta \widehat{c}_1(\overline{\mathcal{A}}_0)^m \mid \mathcal{X}^{(2)}),$$

and for $0 \leq a \leq 2k$ let $J_a = \int_{\mathcal{X}^{(2)}(\mathbb{C})} G_t\, q_0^*\omega_{\Xi,t}^a \wedge q_1^*\omega_{\Xi,t}^{2k-a} \wedge q_0^*\omega_{c,t}^m$, so that $I_t(k,c) = \sum_a (-1)^a J_a$. The inductive formula (D.15) with $\overline{\mathcal{M}}_0 = \overline{\mathcal{N}}$, the section σ, a copies of $\overline{\mathcal{X}}_0$, $2k-a$ copies of $\overline{\mathcal{X}}_1$ and m copies of $\overline{\mathcal{A}}_0$ gives

$$-\tfrac{1}{2}(U(a+1, 2k-a) + U(a, 2k-a+1)) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{X}}_0)^a \widehat{c}_1(\overline{\mathcal{X}}_1)^{2k-a} \widehat{c}_1(\overline{\mathcal{A}}_0)^m \mid \Delta) - J_a.$$

On Δ the three bundles are $q_\Delta^*\overline{\Xi}$, $q_\Delta^*\overline{\Xi}$, $q_\Delta^*\overline{\mathcal{A}}$ (Step 4), so by the projection formula and $(q_0)_*[\Delta] = [\mathcal{X}]$ the degree on Δ equals $\widehat{\deg}(\widehat{c}_1(\overline{\Xi})^{2k}\widehat{c}_1(\overline{\mathcal{A}})^m \mid \mathcal{X}) = \delta_{\mathcal{X}} A$ with $A = \mathrm{ai}_t(\mathcal{X}; \xi_s^{2k}\mathrm{lr}(c)^m)$, independently of a. Hence

$$U(a+1, 2k-a) + U(a, 2k-a+1) = -2\delta_{\mathcal{X}} A + 2J_a \qquad (0 \le a \le 2k). \tag{D.16}$$

Step 6: the end terms vanish. $U(2k+1, 0) = \widehat{\deg}(q_0^*(\widehat{c}_1(\overline{\Xi})^{2k+1}\widehat{c}_1(\overline{\mathcal{A}})^m) \mid \mathcal{X}^{(2)}) = 0$ by the projection formula for q_0, since $\dim q_0(\mathcal{X}^{(2)}) = \dim \mathcal{X} < \dim \mathcal{X}^{(2)}$; likewise $U(0, 2k+1) = 0$ for q_1, using $\overline{\mathcal{A}}_0 = q_1^*\overline{\mathcal{A}}$.

Step 7: alternating summation. Write $U_a = U(a, 2k+1-a)$ for $0 \le a \le 2k+1$, so that the left side of (D.16) is $U_{a+1} + U_a$. Multiplying (D.16) by $(-1)^a$ and summing over $0 \le a \le 2k$, the left side telescopes to $U_0 + U_{2k+1} = 0$ (Step 6), while $\sum_{a=0}^{2k}(-1)^a = 1$. So $0 = -2\delta_{\mathcal{X}} A + 2\sum_a (-1)^a J_a = -2\delta_{\mathcal{X}} A + 2I_t(k, c)$, which is the second identity. □

Theorem D.87. *Let Φ be a non-empty finite set, $n \ge 2$, and put $\kappa_k = \frac{1}{2}\sum_{i=0}^{k-1} \frac{1}{2i+1}$ for $k \ge 1$ (the **sphere moments**; $\kappa_1 = \frac{1}{2}$, $\kappa_2 = \frac{2}{3}$). For every $\tau \in \Phi$ and every $1 \le i \le n-1$, with $s = s_{\tau,i}$ and $\xi_s = x_{\tau,i} - x_{\tau,i+1}$, and for every weight c with $c_{\tau,i} = c_{\tau,i+1}$,*

$$h^{\mathrm{FS}}_{\Phi,n}(\mathrm{lr}(c)^{D+1}) = 0,$$
$$h^{\mathrm{FS}}_{\Phi,n}(\xi_s^{2k}\,\mathrm{lr}(c)^{D+1-2k}) = -2\,\kappa_k\,\deg^{\mathrm{fl}}(\xi_s^{2k-1}\,\mathrm{lr}(c)^{D+1-2k}) \quad (1 \le k \le (D+1)/2).$$

In particular the pair $(h^{\mathrm{FS}}_{\Phi,n}, \deg^{\mathrm{fl}})$ satisfies the fibre-square identities for every Levi reflection with moments $(\kappa_k)_{k\ge1}$, and every root tower with the Fubini–Study compact dual (Fs) has (Fq2) with these moments.

Proof. Fix τ, i, $s = s_{\tau,i}$, an s-invariant weight c, and write $\mathrm{Fl} = \mathrm{Fl}_{\Phi,n}$, $m_k = D + 1 - 2k$.

Step 1: the $\mathbb{P}^1$-bundle. By Lemma 30.26(2), Fl is the smooth projective $\mathbb{Z}$-scheme of Φ-tuples of complete flags $0 = \mathcal{U}_{a,0} \subset \mathcal{U}_{a,1} \subset \cdots \subset \mathcal{U}_{a,n} = \mathcal{O}^n$ of locally split subbundles ($\mathrm{rk}\,\mathcal{U}_{a,j} = j$), embedded by the Plücker coordinates of its steps. On $\mathcal{O}^n$ put the standard Hermitian metric and on every $\mathcal{U}_{a,j}$ the induced one. The Plücker embedding of the factor $k = (a, j)$ restricts $\mathcal{O}(-1)$ to $\wedge^j\mathcal{U}_{a,j} \subseteq \wedge^j\mathcal{O}^n$, and the Fubini–Study metric of $\mathcal{O}(-1)$ is the restriction of the standard norm of $\wedge^j\mathbb{C}^n$, i.e. the determinant metric of $\mathcal{U}_{a,j}$. So, with $\hat{s}_k = \widehat{c}_1(\mathcal{O}(1_k), h^{\mathrm{FS}}_{1_k})|_{\mathrm{Fl}}$ (the class of the line $\mathcal{O}(1)$ of the factor k with its Fubini–Study metric, restricted to Fl),

$$\hat{s}_{(a,j)} = -\widehat{c}_1(\det \overline{\mathcal{U}}_{a,j}) \quad (1 \le j \le n-1), \qquad \hat{s}_{(a,0)} = \hat{s}_{(a,n)} = 0$$

($\det\mathcal{U}_{a,n} = \det\mathcal{O}^n$ has the section $e_1 \wedge \cdots \wedge e_n$ of norm 1). Let $\mathcal{Y}$ be the smooth projective $\mathbb{Z}$-scheme of Φ-tuples of flags of the same kind with the step (τ, i) omitted, embedded by the Plücker coordinates of the remaining steps $k \ne (\tau, i)$; it is a regular arithmetic variety of absolute dimension D, and the forgetful morphism $p\colon \mathrm{Fl} \to \mathcal{Y}$ is the restriction of the projection forgetting the factor (τ, i), so that $\hat{s}_k = p^*\hat{s}'_k$ for $k \ne (\tau, i)$, $\hat{s}'_k$ the Fubini–Study class of the factor k on $\mathcal{Y}$. Let $W = \mathcal{U}_{\tau,i+1}/\mathcal{U}_{\tau,i-1}$ on $\mathcal{Y}$, of rank 2, with the quotient metric h. A step $\mathcal{U}_{\tau,i}$ between $\mathcal{U}_{\tau,i-1}$ and $\mathcal{U}_{\tau,i+1}$ with locally free quotients is the same as a locally split line $\mathcal{S} = \mathcal{U}_{\tau,i}/\mathcal{U}_{\tau,i-1} \subseteq W$; hence p identifies Fl with $\pi\colon \mathbb{P}(W) \to \mathcal{Y}$, the tautological line being $\mathcal{S}$. Its metric induced from W is the quotient metric of $\mathcal{U}_{\tau,i}/\mathcal{U}_{\tau,i-1}$ (both are the metric of $\mathcal{U}_{\tau,i} \cap \mathcal{U}_{\tau,i-1}^{\perp}$ at complex points).

Step 2: the classes. For Hermitian bundles $V' \subseteq V$ with induced and quotient metrics, $\det V \cong \det V' \otimes \det(V/V')$ isometrically; $\widehat{c}_1$ is additive on tensor products of Hermitian line bundles and vanishes on $(\mathcal{O}, |\cdot|)$ ((GC3) of Lemma D.90), so $\widehat{c}_1(\overline{L}^\vee) = -\widehat{c}_1(\overline{L})$. Moreover $\widehat{c}_1(\overline{W}) = \widehat{c}_1(\det\overline{W})$ by Lemma D.91: $\mathcal{Y}$ is a regular arithmetic variety, $\mathcal{Y}(\mathbb{C})$ is a product of partial flag manifolds, hence connected, and the section $P\colon \mathrm{Spec}\,\mathbb{Z} \to \mathcal{Y}$ given by the standard

flags ($\mathcal{U}_{a,j} = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_j$ for the steps $(a,j) \neq (\tau,i)$) has $P^*\overline{W} = (\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{i+1})/(\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{i-1})$ with the quotient metric, the orthogonal sum of the images of $\mathbb{Z}e_i$ and $\mathbb{Z}e_{i+1}$, each of norm 1. Hence $\hat{\zeta} = \widehat{c}_1(\overline{\mathcal{S}}^\vee) = \hat{s}_{(\tau,i)} - \hat{s}_{(\tau,i-1)}$ and $\widehat{c}_1(\overline{W}) = \hat{s}'_{(\tau,i-1)} - \hat{s}'_{(\tau,i+1)}$ (pulled back), so

$$\hat{\theta} = 2\hat{\zeta} + p^*\widehat{c}_1(\overline{W}) = 2\hat{s}_{(\tau,i)} - \hat{s}_{(\tau,i-1)} - \hat{s}_{(\tau,i+1)}.$$

Put $\hat{y}_r = -\widehat{c}_1(\overline{\mathcal{X}}_r^{\mathrm{FS}}) = \hat{s}_{(a,j-1)} - \hat{s}_{(a,j)}$ for $r = (a,j)$ ($\overline{\mathcal{X}}_r^{\mathrm{FS}} = (\mathcal{O}(e_r^{\mathrm{Fl}}), h_{e_r^{\mathrm{Fl}}}^{\mathrm{FS}})|_{\mathrm{Fl}}$ with $e^{\mathrm{Fl}}_{(a,j)} = 1_{(a,j)} - 1_{(a,j-1)}$, the indices 0 and n omitted, and $h_e^{\mathrm{FS}} = \prod_k h_{1_k}^{\mathrm{FS}e_k}$, Definition 29.11). By (29.7) and multilinearity, $h^{\mathrm{FS}}_{\Phi,n}(F) = \widehat{\deg}(F(\hat{y}))$ for F homogeneous of degree $D+1$, $F(\hat{y})$ obtained by substituting $\hat{y}_r$ for x_r. Now $\xi_s(\hat{y}) = \hat{y}_{\tau,i} - \hat{y}_{\tau,i+1} = -\hat{\theta}$, and in $\mathrm{lr}(c)(\hat{y}) = -\sum_r c_r \hat{y}_r$ the two roots $(\tau,i),(\tau,i+1)$ contribute $-c_{\tau,i}(\hat{s}_{(\tau,i-1)} - \hat{s}_{(\tau,i+1)})$ while every other $\hat{y}_r$ involves only classes $\hat{s}_k$ with $k \neq (\tau,i)$. Hence $\mathrm{lr}(c)(\hat{y}) = p^*\hat{\eta}_c$ with $\hat{\eta}_c = \sum_{k\neq(\tau,i)} b_k \hat{s}'_k$ for real numbers b_k depending on c, and $\omega(\hat{\eta}_c) = \omega_c := \sum_k b_k \omega'_k$ with ω'_k the Fubini–Study forms. Expanding $\hat{\eta}_c^m$ multinomially into monomials x in the $\hat{s}'_k$ (rational classes) with real coefficients, and applying linearity to each, the numbers below are real-multilinear in the b_k.

Step 3: the arithmetic side. For $k=0$: $h^{\mathrm{FS}}_{\Phi,n}(\mathrm{lr}(c)^{D+1}) = \widehat{\deg}((p^*\hat{\eta}_c)^{D+1}) = 0$. Indeed, by the expansion above it suffices that $\widehat{\deg}(p^*\hat{s}'_{k_1}\cdots p^*\hat{s}'_{k_{D+1}}) = 0$ for all indices $k_1,\dots,k_{D+1} \neq (\tau,i)$. Since p is a morphism of regular arithmetic varieties with smooth generic fibre (Step 1), p^* is a ring homomorphism on $\widehat{\mathrm{CH}}^*(\cdot)_{\mathbb{Q}}$ ((GA1) of Lemma D.89), so this product is $p^*(x)$ with $x = \hat{s}'_{k_1}\cdots\hat{s}'_{k_{D+1}} \in \widehat{\mathrm{CH}}^{D+1}(\mathcal{Y})_{\mathbb{Q}}$. This group is 0: $\widehat{\mathrm{CH}}^{D+1}(\mathcal{Y})$ is a quotient of the group of pairs (Z,g) of a cycle Z of codimension $D+1$ on $\mathcal{Y}$ and a real current g of type (D,D) on $\mathcal{Y}(\mathbb{C})$ ([110], 3.3.3 and 3.3.4); $Z=0$ because the integral scheme $\mathcal{Y}$ has dimension D, so that no local ring of $\mathcal{Y}$ has Krull dimension $D+1$, and $g=0$ because $\mathcal{Y}(\mathbb{C})$ is a complex manifold of dimension $D-1$. Hence $x=0$ and $p^*(x)=0$. For $1 \le k \le (D+1)/2$, by (D.18) applied to each monomial ($\mathcal{Y}$ has absolute dimension $d+1$ with $d=D-1$, and $m_k = d+2-2k$):

$$h^{\mathrm{FS}}_{\Phi,n}(\xi_s^{2k}\mathrm{lr}(c)^{m_k}) = \widehat{\deg}(\hat{\theta}^{2k}\cdot p^*\hat{\eta}_c^{m_k}) = \frac{c_k}{2}\,I_k, \qquad I_k := \int_{\mathcal{Y}(\mathbb{C})} \Delta_h^{k-1} \wedge \omega_c^{m_k}.$$

Step 4: the geometric side. By Lemma 30.22, $\deg^{\mathrm{fl}}(G) = \deg(G(y)\cap[\mathrm{Fl}_{\mathbb{Q}}])$ for G of degree D, with $y_r = c_1(\mathcal{O}(-e_r^{\mathrm{Fl}}))$ the images of $\hat{y}_r$ in $\mathrm{CH}^1(\mathrm{Fl}_{\mathbb{Q}})$. The identities of Step 2 hold for these images: $\xi_s(y) = -\theta$ and $\mathrm{lr}(c)(y) = p^*\eta_c$. By the projection formula and the last assertion of Lemma D.88,

$$\deg^{\mathrm{fl}}(\xi_s^{2k-1}\mathrm{lr}(c)^{m_k}) = -\deg(p_*(\theta^{2k-1})\,\eta_c^{m_k}\cap[\mathcal{Y}_{\mathbb{Q}}]) = -2\deg(\Delta^{k-1}\eta_c^{m_k}\cap[\mathcal{Y}_{\mathbb{Q}}]) = -2\,I_k,$$

the last equality by Chern–Weil theory on the compact Kähler manifold $\mathcal{Y}(\mathbb{C})$: $c_i(W,h)$ and ω'_k represent $c_i(W_{\mathbb{C}})$ and $c_1(\mathcal{O}(1_k))$, and the degree of a zero-cycle class does not change under the extension $\mathbb{Q}\subseteq\mathbb{C}$.

Step 5. Since $c_k = 8\kappa_k$, $h^{\mathrm{FS}}_{\Phi,n}(\xi_s^{2k}\mathrm{lr}(c)^{m_k}) = 4\kappa_k I_k = -2\kappa_k \deg^{\mathrm{fl}}(\xi_s^{2k-1}\mathrm{lr}(c)^{m_k})$. For a root tower with (Fs), $h^{\mathrm{cd}} = h^{\mathrm{FS}}_{\Phi,n}$, so this is (Fq2) with moments (κ_k). The moments coincide with the closed form of the sphere-moment integrals of the fibre square computed in computer check C7, listed in Appendix B. □

Lemma D.88. *Let $\mathcal{Y}$ be a regular arithmetic variety over $\mathbb{Z}$ of absolute dimension $d+1$, $\overline{W} = (W,h)$ a Hermitian vector bundle of rank 2 on $\mathcal{Y}$, $\pi\colon \mathbb{P}(W) \to \mathcal{Y}$ the bundle of lines, $\overline{\mathcal{S}} \subseteq \pi^*\overline{W}$ the tautological line with the induced metric and $\hat{\zeta} = \widehat{c}_1(\overline{\mathcal{S}}^\vee)$. Put*

$$\hat{\theta} = 2\hat{\zeta} + \pi^*\widehat{c}_1(\overline{W}), \qquad \widehat{\Delta} = \widehat{c}_1(\overline{W})^2 - 4\widehat{c}_2(\overline{W}),$$

$$\Delta_h = c_1(W,h)^2 - 4c_2(W,h), \qquad c_k = 4\sum_{i=0}^{k-1}\frac{1}{2i+1}.$$

Then, in $\widehat{\mathrm{CH}}^*(\mathcal{Y})_{\mathbb{Q}}$,

$$\pi_*(\hat\theta^{\,0}) = 0, \qquad \pi_*(\hat\theta^{\,2k}) = a(c_k\,\Delta_h^{k-1}) \quad (k\ge 1), \qquad \pi_*(\hat\theta^{\,2k+1}) = 2\,\widehat{\Delta}^{\,k} \quad (k\ge 0). \tag{D.17}$$

Consequently, for $k\ge 1$ *and every* $x\in \widehat{\mathrm{CH}}^{\,d+2-2k}(\mathcal{Y})_{\mathbb{Q}}$,

$$\widehat{\deg}(\hat\theta^{\,2k}\cdot\pi^*x) = \frac{c_k}{2}\int_{\mathcal{Y}(\mathbb{C})}\Delta_h^{k-1}\wedge\omega(x), \tag{D.18}$$

and on the generic fibres, with $\theta = 2c_1(\mathcal{S}_{\mathbb{Q}}^\vee)+\pi^*c_1(W_{\mathbb{Q}})$ *and* $\Delta = c_1(W_{\mathbb{Q}})^2-4c_2(W_{\mathbb{Q}})$, $\pi_*(\theta^{2k-1}) = 2\Delta^{k-1}$ *in* $\mathrm{CH}^{2k-2}(\mathcal{Y}_{\mathbb{Q}})_{\mathbb{Q}}$ *for* $k\ge 1$.

Proof. Let $A=\mathbb{Q}[\mathsf{c}_1,\mathsf{c}_2]$ be the polynomial ring in two indeterminates and define $\mathsf{s}_m\in A$ by $\sum_m \mathsf{s}_m t^m = (1+\mathsf{c}_1t+\mathsf{c}_2t^2)^{-1}$ in $A[[t]]$; for $N\ge 0$ put

$$P_N=\sum_{j=1}^{N}\binom{N}{j}2^j\,\mathsf{s}_{j-1}\,\mathsf{c}_1^{N-j}, \qquad M_n=\sum_{p+q=n}\frac{\mathsf{s}_p\mathsf{s}_q}{q+1}, \qquad \Psi_N=\sum_{j=2}^{N}\binom{N}{j}2^j\,\mathsf{c}_1^{N-j}M_{j-2}.$$

Then $\hat s_m=\mathsf{s}_m(\widehat{c}_1(\overline{W}),\widehat{c}_2(\overline{W}))$ in the commutative ring $\widehat{\mathrm{CH}}^*(\mathcal{Y})_{\mathbb{Q}}$, $s'_m=\mathsf{s}_m(c_1(W,h),c_2(W,h))$ in the commutative algebra of even forms, and $-R_m=M_{m-1}(c_1(W,h),c_2(W,h))$ for $m\ge 1$.

The arithmetic intersection theory used along π and along the structure morphisms to $\operatorname{Spec}\mathbb{Z}$ is Lemma D.89: $\mathcal{Y}$ and $\mathbb{P}(W)$ are regular, flat and projective over $\mathbb{Z}$, and π is smooth of relative dimension 1. The numerical projection formula Proposition B.129 would not suffice: in (D.18) the $2k\ge 2$ factors $\hat\theta$ are not pulled back from $\mathcal{Y}$. The arithmetic Chern classes $\widehat{c}_1(\overline{W})$, $\widehat{c}_2(\overline{W})$ and their forms and cycles are those of Lemma D.90. *Step 1: reduction to two polynomial identities.* Since $\widehat{\mathrm{CH}}^*(\mathbb{P}(W))_{\mathbb{Q}}$ is commutative (GA1), $\hat\theta^N=\sum_{j=0}^N\binom{N}{j}2^j\hat\zeta^{\,j}\cdot\pi^*\widehat{c}_1(\overline{W})^{N-j}$. By the projection formula (GA2), $\pi_*(\hat\zeta^{\,j}\cdot\pi^*y)=\pi_*(\hat\zeta^{\,j})\cdot y$; $\pi_*(1)=0$ because π has relative dimension one and $\widehat{\mathrm{CH}}^{-1}=0$; for $j\ge 1$, $\pi_*(\hat\zeta^{\,j})=\hat s_{j-1}-a(R_{j-1})$ by Theorem B.139; and $a(R)\cdot\widehat{c}_1(\overline{W})^{N-j}=a(R\wedge c_1(W,h)^{N-j})$ by (GA3), since $\omega(\widehat{c}_1(\overline{W}))=c_1(W,h)$ by (GC2). Hence

$$\pi_*(\hat\theta^N)=P_N\big(\widehat{c}_1(\overline{W}),\widehat{c}_2(\overline{W})\big)+a\Big(\Psi_N\big(c_1(W,h),c_2(W,h)\big)\Big) \tag{D.19}$$

($R_0=0$ removes $j=1$ from the secondary sum). It remains to show, in A,

$$P_{2k}=0,\quad P_{2k+1}=2(\mathsf{c}_1^2-4\mathsf{c}_2)^k,\quad \Psi_{2k+1}=0\ (k\ge 0),\qquad \Psi_{2k}=c_k(\mathsf{c}_1^2-4\mathsf{c}_2)^{k-1}\ (k\ge 1),\quad \Psi_0=0. \tag{D.20}$$

Step 2: generating functions. Embed A in $L=\mathbb{Q}(\alpha,\beta)$ by $\mathsf{c}_1=-(\alpha+\beta)$, $\mathsf{c}_2=\alpha\beta$, and put $\delta=\alpha-\beta$, so $\mathsf{c}_1^2-4\mathsf{c}_2=\delta^2$ and $S(t):=\sum\mathsf{s}_mt^m=((1-\alpha t)(1-\beta t))^{-1}$. In $L[[y]]$, for any sequence (A_j) and $B\in L$, exchanging the order of summation gives $\sum_N y^N\sum_j\binom{N}{j}A_jB^{N-j}=\frac{1}{1-By}\sum_j A_jY^j$ with $Y=\frac{y}{1-By}$; take $B=\mathsf{c}_1$. Then

$$1-2\alpha Y=\frac{1-\delta y}{1-\mathsf{c}_1y}, \qquad 1-2\beta Y=\frac{1+\delta y}{1-\mathsf{c}_1y}.$$

For P: $A_j=2^j\mathsf{s}_{j-1}$ $(j\ge 1)$, $A_0=0$, so $\sum_N P_Ny^N=\frac{2Y\,S(2Y)}{1-\mathsf{c}_1y}=\frac{2y}{1-\delta^2y^2}$. For Ψ: $\sum_n M_nt^n=S(t)\sum_q\mathsf{s}_q\frac{t^q}{q+1}$, and since $\mathsf{s}_{m-1}=\frac{\alpha^m-\beta^m}{\alpha-\beta}$, $\sum_q\mathsf{s}_q\frac{t^q}{q+1}=\frac{1}{\delta t}\sum_{m\ge1}\frac{(\alpha^m-\beta^m)t^m}{m}=\frac{1}{\delta t}\Lambda(t)$ with $\Lambda(t)=\log\frac{1-\beta t}{1-\alpha t}$ (formal logarithm). With $A_j=2^jM_{j-2}$ $(j\ge 2)$,

$$\sum_N\Psi_Ny^N=\frac{4Y^2}{1-\mathsf{c}_1y}\cdot\frac{\Lambda(2Y)}{2\delta Y(1-2\alpha Y)(1-2\beta Y)}=\frac{2y}{\delta(1-\delta^2y^2)}\log\frac{1+\delta y}{1-\delta y}.$$

Since $\log\frac{1+x}{1-x} = 2\sum_{i\geq 0}\frac{x^{2i+1}}{2i+1}$ and $\frac{1}{1-x^2} = \sum_i x^{2i}$, the Cauchy product with $x = \delta y$ gives $\sum_N \Psi_N y^N = \sum_{k\geq 1} c_k \delta^{2k-2} y^{2k}$, and $\sum_N P_N y^N = \sum_k 2\delta^{2k} y^{2k+1}$. Comparing coefficients gives (D.20) in L, hence in $A \subseteq L$. (The symbolic steps and the coefficients for $N \leq 60$ have been checked independently by computer: computer check C8, listed in Appendix B.) With (D.19) this proves (D.17).

Step 3: the consequences. By (GA5) for π, the projection formula (GA2), (D.17), (GA3) and (GA5) for $\mathcal{Y}$, $\widehat{\deg}(\hat\theta^{2k}\cdot\pi^*x) = \widehat{\deg}(\pi_*(\hat\theta^{2k})\cdot x) = \widehat{\deg}(a(c_k\Delta_h^{k-1})\cdot x) = \widehat{\deg}\, a(c_k\Delta_h^{k-1}\wedge\omega(x)) = \frac{c_k}{2}\int_{\mathcal{Y}(\mathbb{C})}\Delta_h^{k-1}\wedge\omega(x)$, which is (D.18). The map $\zeta\colon \widehat{\mathrm{CH}}^* \to \mathrm{CH}^*$ forgetting the Green current is a ring homomorphism commuting with π_* and kills $a(\cdot)$ ((GA1), (GA2)), and it sends $\widehat{c}_i(\overline{W})$ to $c_i(W)$ and $\hat\zeta$ to $c_1(\mathcal{S}^\vee)$ ((GC2), in ranks 2 and 1); followed by restriction to the open generic fibres (flat pull-back, which commutes with proper push-forward), it turns $\pi_*(\hat\theta^{2k-1}) = 2\widehat{\Delta}^{k-1}$ into $\pi_*(\theta^{2k-1}) = 2\Delta^{k-1}$. □

Lemma D.89. *Let X and Y be regular arithmetic varieties over $\mathbb{Z}$ and $f\colon X\to Y$ a morphism of relative dimension e whose generic fibre $f_\mathbb{Q}$ is smooth. Then:*

(GA1) *$\widehat{\mathrm{CH}}^*(X)_\mathbb{Q}$ is a commutative and associative $\mathbb{Q}$-algebra with unit; $\zeta\colon \widehat{\mathrm{CH}}^*(X)_\mathbb{Q} \to \mathrm{CH}^*(X)_\mathbb{Q}$ is a ring homomorphism with $\zeta\circ a = 0$; f^* is a ring homomorphism with $\zeta\circ f^* = f^*\circ\zeta$.*

(GA2) *The push-forward $f_*\colon \widehat{\mathrm{CH}}^p(X)_\mathbb{Q} \to \widehat{\mathrm{CH}}^{p-e}(Y)_\mathbb{Q}$ of Gillet and Soulé satisfies $(g\circ f)_* = g_*\circ f_*$ for every further such morphism $g\colon Y\to Y'$, $\zeta\circ f_* = f_*\circ\zeta$, $\omega(f_*x) = \int_f\omega(x)$, and $f_*(f^*(y)\cdot x) = y\cdot f_*(x)$ for $x\in\widehat{\mathrm{CH}}^*(X)_\mathbb{Q}$, $y\in\widehat{\mathrm{CH}}^*(Y)_\mathbb{Q}$.*

(GA3) *$a(\eta)\cdot x = a(\eta\wedge\omega(x))$ for η as above and $x\in\widehat{\mathrm{CH}}^q(X)_\mathbb{Q}$.*

(GA4) *$f_*(a(\eta)) = a\big(\int_f\eta\big)$, which is 0 if $p-1<e$.*

(GA5) *If X has absolute dimension $d+1$ and $\widehat{\deg}\colon \widehat{\mathrm{CH}}^{d+1}(X)_\mathbb{Q}\to\mathbb{R}$ is the push-forward along the structure morphism $X\to\operatorname{Spec}\mathbb{Z}$ followed by the arithmetic degree on $\widehat{\mathrm{CH}}^1(\operatorname{Spec}\mathbb{Z})$, then $\widehat{\deg}(f_*x) = \widehat{\deg}(x)$ for $x\in\widehat{\mathrm{CH}}^{d+1}(X)_\mathbb{Q}$, and $\widehat{\deg}\, a(\eta) = \frac12\int_{X(\mathbb{C})}\eta$ for η of type (d,d).*

Proof. In this proof a and ω without further comment on a class of $\widehat{\mathrm{CH}}^p(\cdot,\mathcal{D}_{\log})$ are those of [106], and a, ω on $\widehat{\mathrm{CH}}^p(\cdot)$ those of Gillet and Soulé.

Step 0: the hypotheses of Proposition B.140 hold, and $f_\# = f_!$ on smooth forms. X and Y are regular, flat and projective over $\mathbb{Z}$, so they are arithmetic varieties in the sense of Proposition B.140, and $X_\mathbb{R}$, $Y_\mathbb{R}$ are projective. f is proper (X is proper over $\mathbb{Z}$ and Y is separated), of relative dimension e between integral schemes, and $f_\mathbb{R}$ is smooth (base change of $f_\mathbb{Q}$) and proper between proper varieties; so (BK3) applies to f. The structure morphism $X\to\operatorname{Spec}\mathbb{Z}$ also satisfies the hypotheses of (BK3): its generic fibre $X_\mathbb{Q}$ is a regular scheme of finite type over the perfect field $\mathbb{Q}$, hence smooth. Since $X(\mathbb{C})$ is compact, it is its only compactification with normal crossing boundary (empty boundary); hence the forms with logarithmic singularities at infinity on $X(\mathbb{C})$ are all smooth forms, and in the construction of (BK3) over $U = Y(\mathbb{C})$ the only $\overline{Y}$ is $Y(\mathbb{C})$ with $D=\emptyset$, so that $\mathcal{F}(Y,p) = \mathcal{D}(E_{X(\mathbb{C})},p)$ and u is the identity on global sections. Therefore $f_\# = v\circ u^{-1} = f_! = (2\pi i)^{-e}\int_f$ on global smooth forms.

Step 1: the dictionary. For η as in the statement put $x_\eta = (2\pi i)^{p-1}\eta$. It has type $(p-1,p-1)$, lies in $(2\pi i)^{p-1}$ times the real forms, and $F_\infty^* x_\eta = (-1)^{p-1}x_\eta = \overline{x_\eta}$; so x_η defines a class $\widetilde{x}_\eta\in\widetilde{\mathcal{D}}^{2p-1}_{\log}(X,p)$, and $\eta\mapsto\widetilde{x}_\eta$ respects $\operatorname{Im}\partial+\operatorname{Im}\bar\partial$. By the formula for a, $\mathrm{a}(\widetilde{x}_\eta) = [0,(2\partial\bar\partial x_\eta, -\widetilde{x}_\eta)]$,

so by (BK4) $\Psi(\mathrm{a}(\widetilde{x}_\eta))$ is the class of $(0,g)$ with $g=-2(2\pi i)^{d-p+1}[x_\eta]_X$, d the relative dimension of X, i.e.

$$g(\varphi)=-2(2\pi i)^{d-p+1}(2\pi i)^{-d}\int_{X(\mathbb{C})}\varphi\wedge(2\pi i)^{p-1}\eta=-2\int_{X(\mathbb{C})}\varphi\wedge\eta,$$

the current of the form -2η. Hence, for every p,

$$a(\eta)=\Psi\big(\mathrm{a}(-\tfrac{1}{2}\widetilde{x}_\eta)\big),\qquad \omega(\Psi(\alpha))=(2\pi i)^{-p}\,\omega(\alpha),\qquad \zeta(\Psi(\alpha))=\zeta(\alpha) \tag{D.21}$$

for $\alpha\in\widehat{\mathrm{CH}}^p(X,\mathcal{D}_{\log})$: the second is the last assertion of (BK4) (the Green current of $\Psi(\alpha)$ satisfies $dd^c g+\delta=[(2\pi i)^{-p}\omega_y]$), the third holds because Ψ does not change the cycle.

Step 2: (GA1) and (GA2). By (BK4) and Step 0, Ψ_X and Ψ_Y are isomorphisms compatible with products, pull-backs and push-forwards; in particular the push-forward of Gillet and Soulé is $f_*=\Psi_Y\circ f_*\circ\Psi_X^{-1}$ with the f_* of (BK3). Transporting (BK1) and (BK2) by Ψ, with the third identity of (D.21), gives (GA1) except $\zeta\circ a=0$, which holds because $a(\eta)$ is represented by the zero cycle. Transporting (BK3) gives functoriality, $\zeta\circ f_*=f_*\circ\zeta$, and $f_*(f^*(y)\cdot x)=y\cdot f_*(x)$ (with commutativity from (GA1)). For $\alpha\in\widehat{\mathrm{CH}}^p(X,\mathcal{D}_{\log})$, by (D.21), (BK3) and Step 0,

$$\omega(f_*\Psi\alpha)=(2\pi i)^{-(p-e)}\omega(f_*\alpha)=(2\pi i)^{-(p-e)}(2\pi i)^{-e}\int_f\omega(\alpha)=\int_f\omega(\Psi\alpha).$$

Step 3: (GA3). By linearity it suffices to take $x=\Psi[z,\mathfrak{g}_z]$ with z a q-codimensional cycle and $\mathfrak{g}_z=(\omega_z,\widetilde{g}_z)$ a Green form. Put $\xi=-\frac{1}{2}\widetilde{x}_\eta$. The zero cycle and z intersect properly in the generic fibre, so by (BK1) $\mathrm{a}(\xi)\cdot[z,\mathfrak{g}_z]$ is the class of $((0,\mathrm{a}(\xi)*\mathfrak{g}_z),0)$ with $\mathrm{a}(\xi)*\mathfrak{g}_z=\widetilde{\mathrm{a}(\xi\wedge\omega_z)}$; that is, $\mathrm{a}(\xi)\cdot[z,\mathfrak{g}_z]=\widetilde{\mathrm{a}(\xi\wedge\omega_z)}$. By (D.21), $\omega_z=(2\pi i)^q\omega(x)$, so $\xi\wedge\omega_z=-\frac{1}{2}(2\pi i)^{p+q-1}\eta\wedge\omega(x)=-\frac{1}{2}x_{\eta\wedge\omega(x)}$, where $\eta\wedge\omega(x)$ is real of type $(p+q-1,p+q-1)$ with $F^*_\infty=(-1)^{p+q-1}$ (as $F^*_\infty\omega(x)=(-1)^q\omega(x)$). Applying Ψ, compatible with products, and (D.21) in degree $p+q$:

$$a(\eta)\cdot x=\Psi\big(\mathrm{a}(\xi)\cdot[z,\mathfrak{g}_z]\big)=\Psi\big(\mathrm{a}(-\tfrac{1}{2}\widetilde{x}_{\eta\wedge\omega(x)})\big)=a(\eta\wedge\omega(x)).$$

Step 4: (GA4). By (D.21), Step 2, (BK3) and Step 0,

$$f_*a(\eta)=\Psi_Y\big(f_*\mathrm{a}(-\tfrac{1}{2}\widetilde{x}_\eta)\big)=\Psi_Y\big(\mathrm{a}(f_\#(-\tfrac{1}{2}\widetilde{x}_\eta))\big)=\Psi_Y\big(\mathrm{a}(-\tfrac{1}{2}(2\pi i)^{p-1-e}\textstyle\int_f\eta)\big)=a\big(\int_f\eta\big),$$

the last equality by (D.21) on Y in degree $p-e$, since $\int_f\eta$ is real of type $(p-1-e,p-1-e)$ and $(2\pi i)^{p-1-e}\int_f\eta=x_{\int_f\eta}$. A form of negative type is 0.

Step 5: (GA5). The structure morphism of X is that of Y composed with f, so (GA2) gives $\widehat{\deg}(f_*x)=\widehat{\deg}(x)$. Let η be of type (d,d), so $a(\eta)\in\widehat{\mathrm{CH}}^{d+1}(X)$, and let $c=\int_{X(\mathbb{C})}\eta\in\mathbb{R}$. By (GA4) for the structure morphism ($e=d$), the push-forward of $a(\eta)$ to $S=\operatorname{Spec}\mathbb{Z}$ is $a(c)$. On S ($p=1$, relative dimension 0) (D.21) gives $a(c)=\Psi(\mathrm{a}(-\frac{c}{2}))$, and $\mathrm{a}(-\frac{c}{2})=[0,(0,\frac{c}{2})]$ because $S(\mathbb{C})$ is a point, which carries no forms of type $(1,1)$. By (BK5), $\widehat{\deg}\,a(c)=\widehat{\deg}[0,(0,\frac{c}{2})]=\frac{c}{2}$. Hence $\widehat{\deg}\,a(\eta)=\frac{1}{2}\int_{X(\mathbb{C})}\eta$. (On S, (D.21) reads $\Psi\circ\mathrm{a}=a\circ(-2)$, in accordance with the sign and the factor 2 in (BK5).) □

Lemma D.90. *Let Y be a regular arithmetic variety over $\mathbb{Z}$ and $\overline{W}=(W,h)$ a hermitian vector bundle of rank n on Y (smooth metric on W_∞ over $Y(\mathbb{C})$, invariant under complex conjugation). For $k\geq 0$ let $\widehat{c}_k(\overline{W})\in\widehat{\mathrm{CH}}^k(Y)$ be the k-th arithmetic Chern class of Gillet and Soulé, and let $c_k(W,h)$ be the Chern forms: in a local frame with matrix H of h and curvature matrix $K=\bar\partial(\partial H\cdot H^{-1})$,*

$$\sum_{k\geq 0}c_k(W,h)\,t^k=\det\Big(1+t\,\tfrac{i}{2\pi}K\Big).$$

Then:

(GC1) *$f^*\widehat{c}_k(\overline{W}) = \widehat{c}_k(f^*\overline{W})$ for every morphism $f\colon X \to Y$ of regular arithmetic varieties over $\mathbb{Z}$.*

(GC2) *$\omega(\widehat{c}_k(\overline{W})) = c_k(W,h)$ and $\zeta(\widehat{c}_k(\overline{W})) = c_k(W) \in \mathrm{CH}^k(Y)$.*

(GC3) *If $\overline{W}$ is an orthogonal direct sum of hermitian line bundles $\overline{L}_1,\dots,\overline{L}_n$, then $\widehat{c}_k(\overline{W}) = e_k(\widehat{c}_1(\overline{L}_1),\dots,\widehat{c}_1(\overline{L}_n))$ in $\widehat{\mathrm{CH}}^*(Y)_{\mathbb{Q}}$, e_k the elementary symmetric polynomial. For hermitian line bundles, $\widehat{c}_1(\overline{L}\otimes\overline{M}) = \widehat{c}_1(\overline{L}) + \widehat{c}_1(\overline{M})$, and $\widehat{c}_1(\mathcal{O}_Y, |\cdot|) = 0$.*

(GC4) *If $x \in \widehat{\mathrm{CH}}^1(Y)$ satisfies $\zeta(x) = 0$ and $\omega(x) = 0$, then $x = a(c)$ for a real function c on $Y(\mathbb{C})$ that is constant on each connected component and invariant under complex conjugation.*

(GC5) *$f^*a(\eta) = a(f^*\eta)$ for f as in (GC1) and η as in Lemma D.89.*

Proof. Step 0: hypotheses. Y and X are regular, flat and projective over $\mathbb{Z}$, so Proposition B.141 and Proposition B.140 apply to them, $Y_{\mathbb{R}}$ is projective, and the map $\Psi_0\colon \widehat{\mathrm{CH}}^p(Y,\mathcal{D}_{\log}) \to \widehat{\mathrm{CH}}^p(Y)$ of (BK4) is an isomorphism compatible with products and pull-backs.

Step 1: the dictionary for Ψ. First for Ψ_0. By the last assertion of (BK4), for $\alpha = [y,(\omega_y,\widetilde{g}_y)]$ the Green current of $\Psi_0(\alpha)$ satisfies $dd^c g + \delta_y = (2\pi i)^{-p}\omega_y$ and the cycle is unchanged, so $\omega(\Psi_0\alpha) = (2\pi i)^{-p}\omega(\alpha)$ and $\zeta(\Psi_0\alpha) = \zeta(\alpha)$. Since $\mathrm{a}(\widetilde{x}_\eta) = [0,(-\mathrm{d}_{\mathcal{D}}x_\eta, -\widetilde{x}_\eta)]$, $\Psi_0(\mathrm{a}(\widetilde{x}_\eta))$ is the class of $(0,g)$ with $g(\varphi) = -2(2\pi i)^{d-p+1}(2\pi i)^{-d}\int_{Y(\mathbb{C})}\varphi\wedge(2\pi i)^{p-1}\eta = -2\int_{Y(\mathbb{C})}\varphi\wedge\eta$, i.e. $a(\eta) = \Psi_0(\mathrm{a}(-\frac{1}{2}\widetilde{x}_\eta))$ (d the relative dimension). Now the map ι of (BC1) is induced by an inclusion of complexes, so it sends the class of $[y,(\omega_y,\widetilde{g}_y)]$ to the class of the same pair; hence $\omega\circ\iota = \omega$, $\zeta\circ\iota = \zeta$, $\iota\circ\mathrm{a} = \mathrm{a}$, and, since Ψ of (BC2) and Ψ_0 are given by the same formula on pairs, $\Psi\circ\iota = \Psi_0$. As ι is surjective, the same identities hold for Ψ: for $\beta \in \widehat{\mathrm{CH}}^p(Y,\mathcal{D}_{\mathrm{ls}})$,

$$\omega(\Psi\beta) = (2\pi i)^{-p}\omega(\beta), \qquad \zeta(\Psi\beta) = \zeta(\beta), \qquad a(\eta) = \Psi\big(\mathrm{a}(-\tfrac{1}{2}\widetilde{x}_\eta)\big).$$

By (BC3)(5), $\widehat{c}_k(\overline{W}) = \Psi(\widehat{c}_k^{\mathrm{BKK}}(\overline{W}))$.

Step 2: (GC1), (GC3). Ψ is compatible with pull-backs and products (BC2), so (BC3)(1) gives (GC1), and (BC3)(2), (3) give the first two assertions of (GC3) (in $\widehat{\mathrm{CH}}^*_{\mathbb{Q}}$, where products of $\widehat{c}_1$ live). Since $\mathcal{O}_Y\otimes\mathcal{O}_Y \cong \mathcal{O}_Y$, $1\otimes 1 \mapsto 1$, is an isometry for the standard metrics, additivity gives $\widehat{c}_1(\mathcal{O}_Y,|\cdot|) = 2\,\widehat{c}_1(\mathcal{O}_Y,|\cdot|)$, hence 0.

Step 3: (GC2). By Step 1 and (BC3)(4), $\omega(\widehat{c}_k(\overline{W})) = (2\pi i)^{-k}e_k(-K) = e_k(-\frac{1}{2\pi i}K) = e_k(\frac{i}{2\pi}K) = c_k(W,h)$, because e_k (evaluated on a matrix of even forms as the invariant polynomial) is homogeneous of degree k and $-\frac{1}{2\pi i} = \frac{i}{2\pi}$. By Step 1 and (BC4), $\zeta(\widehat{c}_k(\overline{W})) = \zeta(\widehat{c}_k^{\mathrm{BKK}}(\overline{W})) = c_k(W)$.

Step 4: (GC4). Put $\beta = \Psi^{-1}(x) \in \widehat{\mathrm{CH}}^1(Y,\mathcal{D}_{\mathrm{l,ll}})$. By Step 1, $\zeta(\beta) = 0$ and $\omega(\beta) = 2\pi i\,\omega(x) = 0$, so by (BC5) $\beta = \mathrm{a}(\widetilde{u})$ with $\widetilde{u}$ the class of a $\mathrm{d}_{\mathcal{D}}$-closed element u of degree $2p-1 = 1$ for $p = 1$. As the boundary is empty and $Y(\mathbb{C})$ is compact, the elements of this degree are the smooth real functions u on $Y(\mathbb{C})$ with $F_\infty^* u = u$ (the description of $\widetilde{\mathcal{D}}^{2p-1}(X,p)$ in Proposition B.140 with $p = 1$; nothing is divided out in degree $(0,0)$), and $\mathrm{d}_{\mathcal{D}}u = -2\partial\bar\partial u$. So $\partial\bar\partial u = 0$: u is pluriharmonic, hence harmonic in local holomorphic coordinates, and by the maximum principle constant on each connected component of the compact manifold $Y(\mathbb{C})$. By Step 1 with $p = 1$ ($x_\eta = \eta$), $x = \Psi(\mathrm{a}(\widetilde{u})) = a(-2u)$; take $c = -2u$. (This agrees with [110], Theorem 3.3.5 (ii), where the kernel of $(\zeta,-\omega)$ in degree 1 is $a(H^{0,0}(Y_{\mathbb{R}}))$.)

Step 5: (GC5). Let η be of type $(p-1,p-1)$. By Step 1 on Y, $a(\eta) = \Psi_0(\mathrm{a}(-\frac{1}{2}\widetilde{x}_\eta))$; Ψ_0 commutes with pull-backs and $f^*\circ\mathrm{a} = \mathrm{a}\circ f^*$ (BK2), and $f^*x_\eta = x_{f^*\eta}$. Hence $f^*a(\eta) =$

$\Psi_0(\mathrm{a}(-\frac{1}{2}\widetilde{x}_{f^*\eta})) = a(f^*\eta)$ by Step 1 on X (Step 1 holds for every regular arithmetic variety over $\mathbb{Z}$). $\square$

Lemma D.91. *Let Y be a regular arithmetic variety over $\mathbb{Z}$ with $Y(\mathbb{C})$ connected, and $\overline{W}$ a hermitian vector bundle of rank n on Y. Suppose there is a section $P\colon \operatorname{Spec}\mathbb{Z} \to Y$ such that $P^*\overline{W}$ is isometric to an orthogonal direct sum of hermitian line bundles. Then $\widehat{c}_1(\overline{W}) = \widehat{c}_1(\det\overline{W})$ in $\widehat{\mathrm{CH}}^1(Y)_{\mathbb{Q}}$.*

Proof. Put $\delta = \widehat{c}_1(\det\overline{W}) - \widehat{c}_1(\overline{W})$.

Step 1: $\delta = a(c)$ for a real constant c. By (GC2), $\zeta(\delta) = c_1(\det W) - c_1(W) = 0$ in $\mathrm{CH}^1(Y)$, the first Chern class of a vector bundle being that of its determinant. In a local frame of W_∞ with matrix H, the induced frame of $\det W_\infty$ has norm squared $\det H$, and by Jacobi's formula $\partial \log\det H = \operatorname{tr}(\partial H \cdot H^{-1})$; hence the curvature of $\det\overline{W}$ is $\bar\partial\operatorname{tr}(\partial H\cdot H^{-1}) = \operatorname{tr} K$ and, by (GC2), $\omega(\delta) = \frac{i}{2\pi}\operatorname{tr} K - \frac{i}{2\pi}\operatorname{tr} K = 0$. By (GC4), $\delta = a(c)$ with c locally constant on $Y(\mathbb{C})$, hence a real constant since $Y(\mathbb{C})$ is connected.

Step 2: $P^*\delta = 0$. Write $P^*\overline{W} \cong \overline{L}_1 \oplus \cdots \oplus \overline{L}_n$ orthogonally. Then $\det P^*\overline{W} \cong \overline{L}_1 \otimes \cdots \otimes \overline{L}_n$ isometrically (the determinant of an orthonormal basis is a unit vector), and $P^*\det\overline{W} = \det P^*\overline{W}$ with the determinant metrics. By (GC1) and (GC3),

$$P^*\delta = \widehat{c}_1(\det P^*\overline{W}) - \widehat{c}_1(P^*\overline{W}) = \sum_j \widehat{c}_1(\overline{L}_j) - \sum_j \widehat{c}_1(\overline{L}_j) = 0.$$

Step 3: $c = 0$. By (GC5), $P^*a(c) = a(c)$ in $\widehat{\mathrm{CH}}^1(\operatorname{Spec}\mathbb{Z})$, the pull-back of a constant function to the point $\operatorname{Spec}\mathbb{Z}(\mathbb{C})$ being the same constant; by (GA5) with $d = 0$, $\widehat{\deg}\, a(c) = \frac{c}{2}$. By Step 2, $\frac{c}{2} = \widehat{\deg}(P^*\delta) = 0$. Hence $\delta = a(0) = 0$. $\square$

Theorem D.92. *Let $(k_0, \mathcal{X}^{(2)}, g, h)$ be a square system for a Levi reflection s on a root model $\mathcal{X}$ of type (Φ, n) which has analytic square data (Fx5)–(Fx8). Then the square system has the square limit (Fx4): for every s-invariant weight c and every integer $1 \le k \le (D+1)/2$,*

$$\lim_{t\to 0^+} I_t(k,c) = -2\operatorname{sph}_k \delta_{\mathcal{X}}\, \mathrm{gi}(\mathcal{X}; \xi_s^{2k-1}\mathrm{lr}(c)^{D+1-2k}).$$

Proof. Fix c and k, put $m = D+1-2k$ and $d = D-1$, so that $m = d+2-2k$ and $U_n \subseteq \mathbb{C}^d$. By (Fx8)(1), $\lim_{t\to0^+} I_t(k,c) = I_0(k,c)$; we compute $I_0(k,c)$.

Step 1: the integrands in the fibre charts. Fix n. For $\beta \in \mathrm{Rt}_s^{\mathrm{b}}$ the pluriharmonic differences in (Fx6) are killed by $\partial\bar\partial$, so on $\mathbb{C}\times U_n$

$$\Phi_n^*\omega_{\Xi,0} = \xi := \tfrac{i}{\pi}\partial\bar\partial\operatorname{xp}_{G_n(y)}(z), \qquad \Phi_n^*\omega_{c,0} = \eta_n := \tfrac{i}{\pi}\partial\bar\partial\alpha_{n,c}(y),$$
$$\alpha_{n,c} = -c_{r_1}\alpha_{n,r_{12}} - \sum_{r\neq r_1,r_2} c_r\alpha_{n,r},$$

using $c_{r_1} = c_{r_2}$ (s-invariance), so that $-\sum_r c_r\log h_r(0) = -c_{r_1}\log h_{r_{12}}(0) - \sum_{r\neq r_1,r_2} c_r\log h_r(0)$. By the compatibilities $q_j\circ\Phi_n^{(2)}(z_0,z_1,y) = \Phi_n(z_j,y)$, $(\Phi_n^{(2)})^*q_j^*\omega_{\Xi,0} = \xi_j := \frac{i}{\pi}\partial\bar\partial\operatorname{xp}_{G_n(y)}(z_j)$ and $(\Phi_n^{(2)})^*q_0^*\omega_{c,0} = \eta_n$ (pulled back to $\mathbb{C}^2\times U_n$). By (Fx7), $G_0\circ\Phi_n^{(2)} = \mathrm{ls}_{G_n(y)}(z_0,z_1)$ off the null set where $z_0 = z_1$ or $gh = 0$. These are the data of Lemma D.93 for $(U_n, G_n, \alpha_{n,c})$; write $\Omega_k^{(n)}$, $\Omega_k'^{(n)}$ for its forms.

Step 2: I_0 as a sum over fibre charts. For $0 \le a \le 2k$ let F_a be the integrand of (Fx8)(1), absolutely integrable on W°. By (Fx6) W° is, up to a null set, the disjoint union of the open-chart images $\Phi_n^{(2)}(\mathbb{C}^2 \times B_n)$, and a biholomorphism preserves the complex orientation, so by countable additivity and the change of variables formula

$$\int_{W^\circ} F_a = \sum_{n\ge1} \int_{\mathbb{C}^2\times B_n} (\Phi_n^{(2)})^* F_a = \sum_{n\ge1} \int_{\mathbb{C}^2\times B_n} \mathrm{ls}_{G_n(y)}(z_0,z_1)\, \xi_0^a \wedge \xi_1^{2k-a} \wedge \eta_n^m,$$

with $\sum_n \int_{\mathbb{C}^2\times B_n} |(\Phi_n^{(2)})^* F_a| = \int_{W^\circ} |F_a| < \infty$. Hence $I_0(k,c) = \sum_n \int_{\mathbb{C}^2\times B_n} \mathrm{ls}\,\Omega_k^{(n)}$, the series converging absolutely. Likewise, by (Fx8)(2) and (Fx6) for X°,

$$\int_{X^\circ} \omega_{\Xi,0}^{2k-1} \wedge \omega_{c,0}^m = \sum_{n\ge1} \int_{\mathbb{C}\times B_n} \Omega_k'^{(n)}, \qquad \text{absolutely convergent.}$$

Step 3: fibre-local evaluation. Exhaust U_n by compact sets $K_{n,1} \subseteq K_{n,2} \subseteq \cdots$ and put $B_{n,j} = B_n \cap (K_{n,j} \smallsetminus K_{n,j-1})$ ($K_{n,0} = \varnothing$): Borel sets with compact closure in U_n, disjoint, with union B_n. By Lemma D.93, $\int_{\mathbb{C}^2\times B_{n,j}} \mathrm{ls}\,\Omega_k^{(n)} = -2\,\mathrm{sph}_k \int_{\mathbb{C}\times B_{n,j}} \Omega_k'^{(n)}$ for all n, j. Summing over j and n (both sides absolutely convergent by Step 2, since $|\mathrm{ls}\,\Omega_k^{(n)}| \le \sum_a |(\Phi_n^{(2)})^* F_a|$) gives

$$I_0(k,c) = -2\,\mathrm{sph}_k \int_{X^\circ} \omega_{\Xi,0}^{2k-1} \wedge \omega_{c,0}^m = -2\,\mathrm{sph}_k\, \delta_{\mathcal{X}}\, \mathrm{gi}(\mathcal{X}; \xi_s^{2k-1} \mathrm{lr}(c)^m)$$

by (Fx8)(2). With $\lim_{t\to0^+} I_t(k,c) = I_0(k,c)$ this is (Fx4). □

Lemma D.93. *Let $d \ge 0$, $U \subseteq \mathbb{C}^d$ open, $G\colon U \to \mathrm{M}_2(\mathbb{C})$ of class C^2 with $G(y)$ positive definite Hermitian for all y, and $\alpha\colon U \to \mathbb{R}$ of class C^2. Write $u(z) = (1,z)^{\mathsf T}$, $\mathrm{xp}_G(y,z) = \mathrm{xp}_{G(y)}(z)$ and $\mathrm{ls}_G(y,z_0,z_1) = \mathrm{ls}_{G(y)}(z_0,z_1)$ for $y \in U$, $z, z_0, z_1 \in \mathbb{C}$, $z_0 \ne z_1$. On $\mathbb{C}\times U$ with coordinates (z,y) let $\xi = \frac{i}{\pi}\partial\bar\partial\,\mathrm{xp}_G$; on $\mathbb{C}^2\times U$ with coordinates (z_0,z_1,y) let $\xi_j = \frac{i}{\pi}\partial\bar\partial(\mathrm{xp}_G(y,z_j))$ ($j = 0,1$); on both let $\beta = \frac{i}{\pi}\partial\bar\partial\alpha(y)$. Let $k \ge 1$ with $m = d+2-2k \ge 0$, and*

$$\Omega_k = \sum_{a=0}^{2k} (-1)^a\, \xi_0^a \wedge \xi_1^{2k-a} \wedge \beta^m, \qquad \Omega_k' = \xi^{2k-1} \wedge \beta^m,$$

continuous forms of top degree on $\mathbb{C}^2\times U$ and on $\mathbb{C}\times U$. Then for every Borel set $B \subseteq U$ with compact closure in U, $\mathrm{ls}_G\,\Omega_k$ is absolutely integrable on $\mathbb{C}^2\times B$, Ω_k' is absolutely integrable on $\mathbb{C}\times B$, and

$$\int_{\mathbb{C}^2\times B} \mathrm{ls}_G\,\Omega_k = -2\,\mathrm{sph}_k \int_{\mathbb{C}\times B} \Omega_k'. \tag{D.22}$$

Moreover the same relation holds between the fibre integrals over $\mathbb{C}^2\times\{y\}$ and $\mathbb{C}\times\{y\}$ (integration of the coefficient of the fibre volume), as continuous (d,d)-forms in $y \in U$.

Proof. Step 1: extension to $\mathbb{P}^1$, integrability, reduction to fibres. Let $w = 1/z$ near $z = \infty$; since $u(z) = z\,v(w)$ with $v(w) = (w,1)^{\mathsf T}$, $\mathrm{xp}_G(y,z) = -\log|z|^2 - \log\big(v(w)^* G(y) v(w)\big) + \frac12 \log\det G(y)$, where $-\log|z|^2$ is pluriharmonic and the rest is C^2 in (w,y). So ξ, ξ_j extend to continuous forms on $\mathbb{P}^1\times U$, $\mathbb{P}^1\times\mathbb{P}^1\times U$, and $\Omega_k = f\,\mathrm{v}$, $\Omega_k' = f'\,\mathrm{v}'$ with f, f' continuous and v, v' the products of the Fubini–Study area forms $\omega_1 = \frac{i}{2\pi}\frac{dz\wedge d\bar z}{(1+|z|^2)^2}$ (of total mass 1) of the $\mathbb{P}^1$ factors with Lebesgue measure on U. The quantity under the logarithm in ls_G depends only on the lines $[v_0], [v_1]$ through $v_j = u(z_j)$ and equals $\det G(y)\,|\det(v_0,v_1)|^2 / \big((v_0^* G v_0)(v_1^* G v_1)\big)$ (Lagrange's identity for the Hermitian form $G(y)$). For y in a compact $K \subset U$ the eigenvalues of $G(y)$ lie in a fixed interval $[c_K, C_K] \subset (0,\infty)$, so $|\mathrm{ls}_G(y,z_0,z_1) - \ell(z_0,z_1)| \le C_K'$ with $\ell = \frac12 \log \frac{|\det(v_0,v_1)|^2}{|v_0|^2|v_1|^2}$, which is

$L(\langle n(z_0), n(z_1)\rangle)$ for the Bloch map n of Step 4 (by (D.23) below) and hence $\omega_1 \otimes \omega_1$-integrable on $\mathbb{P}^1 \times \mathbb{P}^1$ (Lemma D.94, as $n_*\omega_1 = \sigma$). Hence $\mathrm{ls}_G\Omega_k$ and Ω'_k are absolutely integrable on $\mathbb{C}^2 \times B$ and $\mathbb{C} \times B$ (the points at infinity and the diagonal are null sets), and by Fubini (D.22) follows from the fibrewise relation. It remains to prove the latter at a fixed $y_0 \in U$.

Step 2: holomorphic change of frame. Let $V \subseteq U$ be an open neighbourhood of y_0 and $M\colon V \to \mathrm{GL}_2(\mathbb{C})$ holomorphic; put $G' = M^*GM$ (C^2, positive definite). Where $\lambda(z,y) := (M(y)^{-1}u(z))_1 \neq 0$ put $T(z,y) = (M(y)^{-1}u(z))_2/\lambda(z,y)$, so that $M(y)^{-1}u(z) = \lambda(z,y)\,u(T(z,y))$. Then $u^*Gu = |\lambda|^2\,u(T)^*G'u(T)$ and $\det G = |\det M|^{-2}\det G'$, so

$$\mathrm{xp}_G(y,z) = \mathrm{xp}_{G'}(y, T(z,y)) - \log|\lambda(z,y)|^2 - \log|\det M(y)|,$$
$$\mathrm{ls}_G(y,z_0,z_1) = \mathrm{ls}_{G'}(y, T(z_0,y), T(z_1,y)),$$

the correction being pluriharmonic. For each y, $\lambda(\cdot,y)$ is a non-zero affine function, and $z \mapsto T(z,y)$ is a Möbius transformation (biholomorphic between complements of at most one point). The maps $\Psi(z_0,z_1,y) = (T(z_0,y), T(z_1,y), y)$ and $\psi(z,y) = (T(z,y),y)$ are holomorphic and fibre-preserving, so, writing $\xi[G']$, $\xi_j[G']$, $\Omega_k[G']$, $\Omega'_k[G']$ for the forms built from G' on V, $\xi_j = \Psi^*\xi_j[G']$, $\xi = \psi^*\xi[G']$, $\beta = \Psi^*\beta = \psi^*\beta$, hence $\mathrm{ls}_G\Omega_k = \Psi^*(\mathrm{ls}_{G'}\Omega_k[G'])$ and $\Omega'_k = \psi^*(\Omega'_k[G'])$. The real Jacobian of Ψ at (z,y) is $|\det \partial_{(z_0,z_1)}(T(z_0,y),T(z_1,y))|^2$ (block triangular with the identity on the base), so by the change of variables formula for the fibre map the fibre integrals at y of a top form and of its pull-back by Ψ (resp. ψ) agree. Hence the fibrewise relation at y_0 for G is equivalent to that for G'.

Step 3: a normal frame. Choose $C \in \mathrm{GL}_2(\mathbb{C})$ with $C^*G(y_0)C = I$, let $G_1 = C^*GC$, $H_j = \frac{\partial G_1}{\partial y_j}(y_0)$, and $M(y) = C(I - \sum_j H_j(y_j - y_{0,j}))$, invertible near y_0. As G_1 is Hermitian, $\frac{\partial G_1}{\partial \bar y_j}(y_0) = H_j^*$, and $G' = M^*GM = I + o(|y-y_0|)$:

By Step 2 we may assume $y_0 = 0$, $G(0) = I$ and $dG(0) = 0$.

Step 4: the forms over $y_0 = 0$. Let $h(z,y) = u(z)^*G(y)u(z)$, so $h(z,0) = 1 + |z|^2$. Fibre part: $\frac{i}{\pi}\partial_z\bar\partial_z(-\log h)(z,0) = -2\omega_1(z)$. Mixed part: $\partial_{\bar y_l}h(z,0) = u^*\partial_{\bar y_l}G(0)u = 0$ for all z, hence $\partial_z\partial_{\bar y_l}h(z,0) = 0$ and $\partial_z\partial_{\bar y_l}\log h = h^{-1}\partial_z\partial_{\bar y_l}h - h^{-2}\partial_z h\,\partial_{\bar y_l}h$ vanishes at $y=0$; likewise $\partial_{\bar z}\partial_{y_l}$; and $\log\det G$ does not depend on z. Base part: $\partial_{y_j}\partial_{\bar y_l}\log h(z,0) = u^*\partial_{y_j}\partial_{\bar y_l}G(0)u/|u|^2$ and $\partial_{y_j}\partial_{\bar y_l}\log\det G(0) = \mathrm{tr}\,\partial_{y_j}\partial_{\bar y_l}G(0)$ (Jacobi's formula, $G(0)=I$, $dG(0)=0$). Let $\Theta = \frac{i}{\pi}\sum_{j,l}\partial_{y_j}\partial_{\bar y_l}G(0)\,dy_j\wedge d\bar y_l$, a 2×2 matrix of $(1,1)$-forms at 0, and $b_i = \frac12\mathrm{tr}(\sigma_i\Theta)$ for the Pauli matrices $\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$; these are real $(1,1)$-forms (G is Hermitian) and $\Theta - \frac12(\mathrm{tr}\,\Theta)I = \sum_i b_i\sigma_i$. With the Bloch map $n(z) = (u^*\sigma_i u/|u|^2)_i = \frac{(2\,\mathrm{Re}\,z,\,2\,\mathrm{Im}\,z,\,1-|z|^2)}{1+|z|^2} \in S^2$ (inverse stereographic projection, so $n_*\omega_1 = \sigma$) we get, at every point over 0,

$$\xi = -2\,\omega_1(z) - \langle n(z), b\rangle, \qquad \xi_j = -2\,\omega_1(z_j) - \langle n(z_j), b\rangle, \qquad \beta = \beta(0),$$

all of even degree, hence commuting. Moreover, for $G(0) = I$,

$$1 - \frac{|u(z_0)^*u(z_1)|^2}{|u(z_0)|^2|u(z_1)|^2} = \frac{1 - \langle n(z_0), n(z_1)\rangle}{2}, \qquad \text{so} \qquad \mathrm{ls}_G(0,z_0,z_1) = L(\langle n(z_0), n(z_1)\rangle). \quad \text{(D.23)}$$

Step 5: evaluation over 0. Since $\omega_1(z_j)\wedge\omega_1(z_j) = 0$ and the fibre integral over $\mathbb{C}^2$ only sees the components containing $\omega_1(z_0)\wedge\omega_1(z_1)$, the fibre integral of $\xi_0^a\xi_1^{2k-a}\beta^m$ against a function $F(z_0,z_1)$ is that of $4\,a(2k-a)\,F\,\omega_1(z_0)\wedge\omega_1(z_1)\wedge\langle n(z_0),b\rangle^{a-1}\langle n(z_1),b\rangle^{2k-1-a}\wedge\beta^m$ (the sign is $(-2)^2(-1)^{2k-2} = 4$; it is 0 for $a \in \{0, 2k\}$). With (D.23), $n_*\omega_1 = \sigma$ and (D.24), applied in the commutative algebra of even-degree forms at 0, the fibre integral of $\mathrm{ls}_G\Omega_k$ over 0 is

$$4\iint L(\langle n_0, n_1\rangle)\,Q_k\big(\langle n_0, b\rangle, \langle n_1, b\rangle\big)\,d\sigma\,d\sigma\wedge\beta^m = 4\,\mathrm{sph}_k\,(b\cdot b)^{k-1}\wedge\beta^m.$$

Likewise the fibre integral of $\Omega'_k = \xi^{2k-1}\beta^m$ over 0 is $(2k-1)(-2)\int\langle n,b\rangle^{2k-2}d\sigma\wedge\beta^m = -2\,(b\cdot b)^{k-1}\wedge\beta^m$ by (D.25). So the first is $-2\,\mathrm{sph}_k$ times the second, which is the fibrewise relation. Both fibre integrals are continuous in y (Step 1 and dominated convergence).

□

Lemma D.94. *Let σ be the rotation-invariant probability measure on the unit sphere $S^2\subset\mathbb{R}^3$, let $L(t)=\frac12\log\frac{1-t}{2}$ for $-1\le t<1$, and for an integer $k\ge1$ let*

$$P_k(u,v)=\sum_{a=0}^{2k}(-1)^a\,u^a v^{2k-a},\qquad Q_k=\frac{\partial^2 P_k}{\partial u\,\partial v}=\sum_{a=1}^{2k-1}(-1)^a\,a(2k-a)\,u^{a-1}v^{2k-1-a}.$$

Let A be a commutative $\mathbb{R}$-algebra, $b=(b_1,b_2,b_3)\in A^3$, $\langle n,b\rangle=\sum_i n_ib_i$ for $n\in S^2$ and $b\cdot b=\sum_i b_i^2$; integrals of A-valued polynomial expressions in n are taken coefficientwise in the monomials of b. Then $(n_1,n_2)\mapsto L(\langle n_1,n_2\rangle)$ is integrable on $S^2\times S^2$, and

$$\int_{S^2}\int_{S^2}L(\langle n_1,n_2\rangle)\,Q_k(\langle n_1,b\rangle,\langle n_2,b\rangle)\,d\sigma(n_1)\,d\sigma(n_2)=\mathrm{sph}_k\,(b\cdot b)^{k-1},\tag{D.24}$$

$$\int_{S^2}\langle n,b\rangle^{2j}\,d\sigma(n)=\frac{(b\cdot b)^j}{2j+1},\qquad\int_{S^2}\langle n,b\rangle^{2j+1}\,d\sigma(n)=0\qquad(j\ge0).\tag{D.25}$$

Proof. Step 0: Archimedes, and integrability. For a unit vector e, the push-forward of σ under $n\mapsto\langle n,e\rangle$ is the uniform probability measure $\frac12\,dx$ on $[-1,1]$, and in the cylindrical coordinates $n=(\sqrt{1-x^2}\cos\theta,\sqrt{1-x^2}\sin\theta,x)$ around $e=(0,0,1)$ one has $d\sigma=\frac{1}{4\pi}\,dx\,d\theta$ (Archimedes). Hence, for fixed n_1, $\int|L(\langle n_1,n_2\rangle)|\,d\sigma(n_2)=\frac12\int_{-1}^1|L(t)|\,dt<\infty$, independently of n_1, and $L(\langle n_1,n_2\rangle)$ is integrable on $S^2\times S^2$ by Tonelli; the factors $Q_k(\cdots)$ are bounded on $S^2\times S^2$ coefficientwise. Also $\int\langle n,e\rangle^j d\sigma=\frac12\int_{-1}^1x^j dx$, which is $\frac{1}{j+1}$ for j even and 0 for j odd.

Step 1: reduction to $A=\mathbb{R}$ and $b=e$. For integers $j,l\ge0$ let

$$F_{j,l}(b)=\iint L(\langle n_1,n_2\rangle)\langle n_1,b\rangle^j\langle n_2,b\rangle^l\,d\sigma\,d\sigma,\qquad H_j(b)=\int\langle n,b\rangle^j d\sigma;$$

expanding the powers, these are homogeneous polynomials in $\mathbb{R}[b_1,b_2,b_3]$ of degrees $j+l$ and j, and the left sides of (D.24), (D.25) are their images under the ring homomorphism $\mathbb{R}[b_1,b_2,b_3]\to A$ sending b_i to the given elements. Since $\sigma\otimes\sigma$ and $\langle n_1,n_2\rangle$ are invariant under $(n_1,n_2)\mapsto(Rn_1,Rn_2)$ for $R\in\mathrm{O}(3)$, $F_{j,l}(R^{\mathsf T}b)=F_{j,l}(b)$, and likewise for H_j. An $\mathrm{O}(3)$-invariant homogeneous polynomial f of degree N satisfies $f(b)=|b|^Nf(e)$ for real $b\neq0$; if N is odd then $f(-b)=f(b)=-f(b)$, so $f=0$; if N is even then $f-f(e)\,(b\cdot b)^{N/2}$ vanishes on $\mathbb{R}^3$, hence is the zero polynomial. Applied to H_j with the values of Step 0 this gives (D.25); applied to $\sum_a(-1)^a a(2k-a)F_{a-1,2k-1-a}$ (degree $2k-2$) it reduces (D.24) to the real number

$$S_k:=\iint L(\langle n_1,n_2\rangle)\,Q_k(x_1,x_2)\,d\sigma(n_1)\,d\sigma(n_2),\qquad x_i=\langle n_i,e\rangle,$$

and we must show $S_k=\mathrm{sph}_k$.

Step 2: the azimuthal average. In the coordinates of Step 0, $\langle n_1,n_2\rangle=x_1x_2+B\cos(\theta_1-\theta_2)$ with $B=\sqrt{(1-x_1^2)(1-x_2^2)}$. By Fubini (Step 0), $S_k=\frac14\int_{-1}^1\int_{-1}^1K(x_1,x_2)\,Q_k(x_1,x_2)\,dx_1\,dx_2$ with $K(x,y)=\frac{1}{2\pi}\int_0^{2\pi}L(xy+B\cos\varphi)\,d\varphi$ (replacing φ by $\varphi+\pi$ changes the sign of the cosine). For $x,y\in(-1,1)$ put $A_0=1-xy>0$; then $A_0\ge B\ge0$ and $A_0^2-B^2=(x-y)^2$, and the classical mean value $\frac{1}{2\pi}\int_0^{2\pi}\log(A_0-B\cos\varphi)\,d\varphi=\log\frac{A_0+\sqrt{A_0^2-B^2}}{2}$ (Jensen's formula for

$|\alpha-\beta e^{i\varphi}|^2$ with $A_0=|\alpha|^2+|\beta|^2$, $B=2|\alpha\beta|$) gives, with $M=\max(x,y)$, $m=\min(x,y)$ and $A_0+|x-y|=(1+M)(1-m)$,

$$K(x,y)=\tfrac12\Big(\log\tfrac{(1+M)(1-m)}{2}-\log 2\Big)=\tfrac12\big(a(M)+c(m)\big),\qquad a(s)=\log\tfrac{1+s}{2},\quad c(s)=\log\tfrac{1-s}{2}.$$

Step 3: integration by parts. Let P be a real polynomial in two variables with $P(-x,-y)=P(x,y)$ and write $P_{xy}=\partial_x\partial_y P$. Since $a(1)=0$, $a(M)=-\int_{-1}^{1}\frac{\mathbf{1}[s>M]}{1+s}\,ds$, and since $c(-1)=0$, $c(m)=-\int_{-1}^{1}\frac{\mathbf{1}[s<m]}{1-s}\,ds$. Fubini applies, because $\iint_{x,y<s}|P_{xy}|\le\|P_{xy}\|_\infty(1+s)^2$ and $\iint_{x,y>s}|P_{xy}|\le\|P_{xy}\|_\infty(1-s)^2$; and $\iint_{x,y<s}P_{xy}=\Delta(s):=P(s,s)-P(s,-1)-P(-1,s)+P(-1,-1)$. Hence

$$\iint a(M)P_{xy}=-\int_{-1}^{1}\frac{\Delta(s)}{1+s}\,ds,$$
$$\iint c(m)P_{xy}=-\int_{-1}^{1}\frac{P(1,1)-P(1,s)-P(s,1)+P(s,s)}{1-s}\,ds=-\int_{-1}^{1}\frac{\Delta(s')}{1+s'}\,ds',$$

the last by $s=-s'$ and $P(-x,-y)=P(x,y)$. So $\iint K\,P_{xy}=-\int_{-1}^{1}\Delta(s)\,(1+s)^{-1}\,ds$.

Step 4: the bracket for P_k. P_k is homogeneous of degree $2k$, so the symmetry holds, and $P_k(s,s)=s^{2k}\sum_a(-1)^a=s^{2k}$, $P_k(s,-1)=P_k(-1,s)=\sum_{i=0}^{2k}s^i$, $P_k(-1,-1)=1$. Hence

$$\Delta(s)=1+s^{2k}-2\sum_{i=0}^{2k}s^i=-(1+s)\sum_{i=0}^{2k-1}s^i$$

and $\iint K\,Q_k=\int_{-1}^{1}\sum_{i=0}^{2k-1}s^i\,ds=\sum_{j=0}^{k-1}\frac{2}{2j+1}$. Therefore $S_k=\frac14\sum_{j<k}\frac{2}{2j+1}=\mathrm{sph}_k$.

The case $k=1$ directly. $Q_1=-1$, so

$$S_1=-\iint L\,d\sigma\,d\sigma=-\frac12\int_{-1}^{1}\frac12\log\frac{1-t}{2}\,dt=-\frac12\int_0^1\log w\,dw=\frac12=\mathrm{sph}_1.$$

Computer checks. Computer check C9, listed in Appendix B, evaluates S_k exactly in rational arithmetic for $1\le k\le 12$ by two methods sharing no step with each other: the double integral of Step 2 against Q_k monomial by monomial (without Steps 3, 4), and the Funk–Hecke expansion of L in Legendre polynomials on $S^2\times S^2$ (without Step 2), plus a direct quadrature of the original integral for $k\le 3$; all agree with sph_k. Computer check C7, also listed in Appendix B, proves the same values by generating functions in the grouping $y(x^{2k}-y^{2k})/(x+y)$, which differs from P_k by $-x^{2k}$ and has the same mixed derivative up to sign. □

Theorem D.95. *Let $(k_0,\mathcal{X}^{(2)},g,h)$ be a square system for a Levi reflection s on a root model $\mathcal{X}$ of type (Φ,n), with open parts X°, W° and positive functions $h_r(0)$ satisfying (Fx5), which carries an analytic square system. Then the square convergence (Fx8) holds. Consequently, if fibre charts with (Fx6) and (Fx7) are also given, the square system has analytic square data (Fx5)–(Fx8); in particular, the square system of a proportional square model (as in Definition 31.9) has analytic square data (Fx5)–(Fx8).*

Proof. Fix an s-invariant weight c and $1\le k\le(D+1)/2$, put $m=D+1-2k$, and let $(\widetilde X,\dots,d_X)$, $(\widetilde W,\dots,d_W)$, p_0,p_1, v, κ_v be the analytic square system, with Λ, $b_{r,\lambda}$, γ_λ, f_λ, $f_{\lambda,t}$, $\theta_r,u_r,u_{r,t}$, C,δ,R,t_0 as in (Sw1), $\bar b=\max_{r\in\mathrm{Rt}}\sum_{\lambda\in\Lambda}|b_{r,\lambda}|$, and N_X,N_X',N_W,N_W' the null sets of the two compact models.

Step 1: the pulled-back families on $\widetilde W$. For $\lambda\in\Lambda$ and $j\in\{0,1\}$ put $\gamma_{\lambda,j}=p_j^*\gamma_\lambda$, $f_{\lambda,j}=f_\lambda\circ p_j$ on $\widetilde W^\circ$ (defined as $p_j(\widetilde W^\circ)\subseteq\widetilde X^\circ$ by (Sw2)) and $f_{\lambda,j,t}=f_{\lambda,t}\circ p_j$ on $\widetilde W$. These families, indexed by

$\Lambda\times\{0,1\}$, satisfy (R1)–(R5) on $(\widetilde{W},\Theta_W,\varsigma_W)$ with the constant $C_W = C\,c_p$, where $c_p\ge 1$ is such that $p_j^*\Theta_X \le c_p\Theta_W$ for $j=0,1$ ($\widetilde{W}$ compact), and the same δ, R: (R1) $\gamma_{\lambda,j}+dd^c f_{\lambda,j,t}+C_W\Theta_W \ge p_j^*(\gamma_\lambda + dd^c f_{\lambda,t} + C\Theta_X)\ge 0$; (R2) on $\widetilde{W}^\circ$, $-C_W\log\mathrm{rW}_W \le -C\log(\mathrm{rW}_X\circ p_j)\le f_{\lambda,j}\le f_{\lambda,j,t}\le C\le C_W$ by (Sw2); (R3) $\{y\in\widetilde{W}^\circ : \mathrm{rW}_W(y)\le R(t)\}\subseteq p_j^{-1}\{x\in\widetilde{X}^\circ : \mathrm{rW}_X(x)\le R(t)\}$ by (Sw2); (R4) is pulled back; (R5) a compact $K\subseteq\widetilde{W}^\circ$ has compact image $p_j(K)\subseteq\widetilde{X}^\circ$, and by the chain rule the derivatives of order ≤ 2 of $f_{\lambda,t}\circ p_j$ on K are polynomials in those of $f_{\lambda,t}$ on $p_j(K)$ with coefficients the derivatives of p_j, bounded on K. For $r\in\mathrm{Rt}$ put $\theta_{r,j}=p_j^*\theta_r=\sum_\lambda b_{r,\lambda}\gamma_{\lambda,j}$, $u_{r,j}=u_r\circ p_j=\sum_\lambda b_{r,\lambda}f_{\lambda,j}$ on $\widetilde{W}^\circ$ and $u_{r,j,t}=u_{r,t}\circ p_j=\sum_\lambda b_{r,\lambda}f_{\lambda,j,t}$ on $\widetilde{W}$. By (R2) and $\log\mathrm{rW}_W\ge 1$, $|f_{\lambda,j,t}|\le C_W\log\mathrm{rW}_W$ on $\widetilde{W}^\circ$, hence $|u_{r,j,t}|\le\bar b\,C_W\log\mathrm{rW}_W$ there.

Step 2: the integrands. Write $\theta_{\Xi,j}=\theta_{r_1,j}-\theta_{r_2,j}$, $u_{\Xi,j,t}=u_{r_1,j,t}-u_{r_2,j,t}$, $\theta_c=-\sum_r c_r\theta_{r,0}$, $u_{c,t}=-\sum_r c_r u_{r,0,t}$, and the same without t for the limits. By definition $\omega_{\Xi,t}=\omega_{r_1,t}-\omega_{r_2,t}$ and $\omega_{c,t}=-\sum_r c_r\omega_{r,t}$, also for $t=0$ (with the notation $\omega_{r,t}$ of Definition 31.9). Since $q_j\circ\varrho_W=\varrho_X\circ p_j$ (Sw2), (Sw1) gives on $\widetilde{W}$, for $0<t\le t_0$,

$$\varrho_W^*q_j^*\omega_{\Xi,t}=\theta_{\Xi,j}+dd^cu_{\Xi,j,t},\qquad \varrho_W^*q_0^*\omega_{c,t}=\theta_c+dd^cu_{c,t},$$

and the same identities at $t=0$ on $\widetilde{W}^\circ\setminus N_W$: there $\varrho_W(y)\in W^\circ$ (compact model), $q_j(\varrho_W(y))\in X^\circ$ (Fx5) and $p_j(y)\in\widetilde{X}^\circ\cap\varrho_X^{-1}(X^\circ)$. For $0\le a\le 2k$ let $F_{a,t}=G_t\,q_0^*\omega_{\Xi,t}^a\wedge q_1^*\omega_{\Xi,t}^{2k-a}\wedge q_0^*\omega_{c,t}^m$ ($0\le t\le t_0$) and

$$\mu_t^{(a)}=(\theta_{\Xi,0}+dd^cu_{\Xi,0,t})^a\wedge(\theta_{\Xi,1}+dd^cu_{\Xi,1,t})^{2k-a}\wedge(\theta_c+dd^cu_{c,t})^m,$$

and $\mu^{(a)}$ the same with the limits $u_\cdot$ on $\widetilde{W}^\circ$. Expanding multilinearly, $\mu_t^{(a)}$ and $\mu^{(a)}$ are the same finite real combination of products $\bigwedge_{l=1}^{D+1}(\gamma_{i_l}+dd^cf_{i_l,t})$, resp. of their limits $\bigwedge_{l=1}^{D+1}(\gamma_{i_l}+dd^cf_{i_l})$, with $i_l\in\Lambda\times\{0,1\}$: each factor $\theta_{r,j}+dd^cu_{r,j,t}$ equals $\sum_\lambda b_{r,\lambda}(\gamma_{\lambda,j}+dd^cf_{\lambda,j,t})$, and likewise for the limits. By (Sw3),

$$\varrho_W^*F_{a,t}=\big(\tfrac12\log\|v\|^2+\kappa_v+g_t\big)\mu_t^{(a)},\qquad g_t=\tfrac12\sum_{j=0,1}u_{\Xi,j,t},$$

on $\widetilde{W}\setminus(D_v\cup\varrho_W^{-1}\{gh=0\})$ for $0<t\le t_0$, and with $g_0=\frac12\sum_j u_{\Xi,j}$ and $\mu^{(a)}$ on $\widetilde{W}^\circ\setminus(N_W\cup D_v\cup\varrho_W^{-1}\{gh=0\})$ for $t=0$. The removed sets are null: D_v is a proper analytic subset; g and h do not vanish identically on the integral scheme $\mathcal{X}^{(2)}$ (Fx3), so $\{gh=0\}$ is a proper analytic subset of $\mathcal{X}^{(2)}(\mathbb{C})$, null, and its preimage under the covering $\widetilde{W}\setminus N_W\to\mathcal{X}^{(2)}(\mathbb{C})\setminus N'_W$ is null.

Step 3: (Fx8)(1). By Step 1, $|\kappa_v+g_t|\le(|\kappa_v|+2\bar b\,C_W)\log\mathrm{rW}_W$ on $\widetilde{W}^\circ$ (g_t is half a sum of four terms $\pm u_{r,j,t}$, each of absolute value at most $\bar b\,C_W\log\mathrm{rW}_W$, and $\log\mathrm{rW}_W\ge 1$), and $\kappa_v+g_t\to\kappa_v+g_0$ uniformly on compact subsets of $\widetilde{W}^\circ$ by (R5). Lemma 31.35, applied on $\widetilde{W}$ (dimension $D+1$) with Θ_W, ς_W, the section v and the families of Step 1, gives for each product in the expansion of $\mu_t^{(a)}$: (1) absolute integrability of $(\log\mathrm{rW}_W-\log\|v\|^2)$ times the limit product on $\widetilde{W}^\circ\setminus D_v$, (2) convergence of the integrals against κ_v+g_t, (3) convergence of the integrals against $-\log\|v\|^2$. By linearity, $\varrho_W^*F_{a,0}$ is absolutely integrable on $\widetilde{W}^\circ$ and $\int_{\widetilde{W}}\varrho_W^*F_{a,t}\to\int_{\widetilde{W}^\circ}\varrho_W^*F_{a,0}$ as $t\to0^+$. By the change of variables along the d_W-sheeted covering $\widetilde{W}\setminus N_W\to\mathcal{X}^{(2)}(\mathbb{C})\setminus N'_W$, which maps $\widetilde{W}^\circ\setminus N_W=\varrho_W^{-1}(W^\circ\setminus N'_W)$ onto $W^\circ\setminus N'_W$: $F_{a,0}$ is absolutely integrable on W° with $\int_{W^\circ}F_{a,0}=d_W^{-1}\int_{\widetilde{W}^\circ}\varrho_W^*F_{a,0}$, and $\int_{\mathcal{X}^{(2)}(\mathbb{C})}F_{a,t}=d_W^{-1}\int_{\widetilde{W}}\varrho_W^*F_{a,t}$ for $t>0$ (the integrals in (31.5) are over the regular locus off $\{gh=0\}$; N'_W is null). Taking the alternating sum over a, $I_t(k,c)\to I_0(k,c)$. This is (Fx8)(1).

Step 4: (Fx8)(2). On $\widetilde{X}$ put $\theta_\Xi=\theta_{r_1}-\theta_{r_2}$, $\theta'_c=-\sum_r c_r\theta_r$ and similarly $u_{\Xi,t}, u'_{c,t}$ and the limits. By (Sw1), $\varrho_X^*(\omega_{\Xi,t}^{2k-1}\wedge\omega_{c,t}^m)=\mu_{X,t}:=(\theta_\Xi+dd^cu_{\Xi,t})^{2k-1}\wedge(\theta'_c+dd^cu'_{c,t})^m$ on $\widetilde{X}$

and $\varrho_X^*(\omega_{\Xi,0}^{2k-1}\wedge\omega_{c,0}^m)=\mu_X$ (the limit) on $\widetilde{X}^\circ\smallsetminus N_X$. Lemma 31.35(1) on $\widetilde{X}$ (dimension D, $v=1$, $D_v=\varnothing$), applied to each product of the multilinear expansion in the factors $\gamma_\lambda+dd^cf_{\lambda,t}$ ($\lambda\in\Lambda$), resp. their limits, with the families $(\gamma_\lambda,f_\lambda,(f_{\lambda,t})_t)_{\lambda\in\Lambda}$ of (Sw1), shows that μ_X is absolutely integrable on $\widetilde{X}^\circ$ with $\int_{\widetilde{X}^\circ}\mu_X=\int_{\widetilde{X}}\theta_\Xi^{2k-1}\wedge\theta_c'^m$, and by Stokes' theorem on $\widetilde{X}$ this equals $\int_{\widetilde{X}}\mu_{X,t}$ for every $0<t\le t_0$. By the change of variables along the d_X-sheeted covering $\widetilde{X}\smallsetminus N_X\to\mathcal{X}(\mathbb{C})\smallsetminus N_X'$ (with $\varrho_X^{-1}(X^\circ\smallsetminus N_X')=\widetilde{X}^\circ\smallsetminus N_X$),

$$\int_{X^\circ}\omega_{\Xi,0}^{2k-1}\wedge\omega_{c,0}^m=\frac{1}{d_X}\int_{\widetilde{X}^\circ}\mu_X=\frac{1}{d_X}\int_{\widetilde{X}}\mu_{X,t}=\int_{\mathcal{X}(\mathbb{C})_{\mathrm{reg}}}\omega_{\Xi,t}^{2k-1}\wedge\omega_{c,t}^m,$$

the left side absolutely convergent. Expanding

$$\omega_{\Xi,t}=c_1(\overline{\mathcal{L}}_{r_1}(t))-c_1(\overline{\mathcal{L}}_{r_2}(t))\quad\text{and}\quad\omega_{c,t}=-\sum_r c_rc_1(\overline{\mathcal{L}}_r(t)),$$

Lelong's theorem with Chern–Weil theory on each irreducible component of the projective variety $\mathcal{X}(\mathbb{C})$ (the complex points of $\mathcal{X}_{\mathbb{Q}}$; degrees are invariant under the flat base change $\mathbb{Q}\to\mathbb{C}$) gives for every monomial $\int_{\mathcal{X}(\mathbb{C})_{\mathrm{reg}}}c_1(\overline{\mathcal{L}}_{r_1'}(t))\wedge\cdots\wedge c_1(\overline{\mathcal{L}}_{r_D'}(t))=\deg\bigl(c_1(\mathcal{O}(e_{r_1'}))\cdots c_1(\mathcal{O}(e_{r_D'}))\cap[\mathcal{X}_{\mathbb{Q}}]\bigr)=\delta_{\mathcal{X}}\,\mathrm{gi}(\mathcal{X};x_{r_1'}\cdots x_{r_D'})$ ($r_l'\in\mathrm{Rt}$). As $\xi_s=x_{r_1}-x_{r_2}$ and $\mathrm{lr}(c)=-\sum_r c_rx_r$, summing gives $\int_{X^\circ}\omega_{\Xi,0}^{2k-1}\wedge\omega_{c,0}^m=\delta_{\mathcal{X}}\,\mathrm{gi}(\mathcal{X};\xi_s^{2k-1}\mathrm{lr}(c)^m)$. This is (Fx8)(2).

The consequences are the definitions of analytic square data and of a proportional square model, whose square system has (Fx5), (Fx6), (Fx7) and carries an analytic square system. □

Lemma D.96. *Let $\mathcal{X}$ and $\mathcal{X}'$ be root models of type (Φ,n) with convergent regularisation, and let $m>0$ be a real number with*

$$\mathrm{pp}(\mathcal{X}';F)=m\,\mathrm{pp}(\mathcal{X};F)\quad\textit{and}\quad\mathrm{gi}(\mathcal{X}';F)=m\,\mathrm{gi}(\mathcal{X};F)\qquad\textit{for every } F\in\mathcal{R}.$$

Let s be a Levi reflection and $\mu=(\mu_k)_{k\ge1}$ real numbers. If the pair $\bigl(\mathrm{pp}(\mathcal{X}';\cdot)|_{\mathcal{R}},\mathrm{gi}(\mathcal{X}';\cdot)|_{\mathcal{R}}\bigr)$ satisfies the fibre-square identities for s with moments μ, then so does the pair $\bigl(\mathrm{pp}(\mathcal{X};\cdot)|_{\mathcal{R}},\mathrm{gi}(\mathcal{X};\cdot)|_{\mathcal{R}}\bigr)$.

Proof. Let c be an s-invariant weight. The elements $\mathrm{lr}(c)^{D+1}$, $\xi_s^{2k}\mathrm{lr}(c)^{D+1-2k}$ and $\xi_s^{2k-1}\mathrm{lr}(c)^{D+1-2k}$ ($1\le k\le(D+1)/2$) lie in $\mathcal{R}$. By the identities for $\mathcal{X}'$ and the hypothesis,

$$m\,\mathrm{pp}(\mathcal{X};\mathrm{lr}(c)^{D+1})=\mathrm{pp}(\mathcal{X}';\mathrm{lr}(c)^{D+1})=0,$$

and for $1\le k\le(D+1)/2$

$$\begin{aligned}m\,\mathrm{pp}(\mathcal{X};\xi_s^{2k}\mathrm{lr}(c)^{D+1-2k})&=\mathrm{pp}(\mathcal{X}';\xi_s^{2k}\mathrm{lr}(c)^{D+1-2k})\\&=-2\mu_k\,\mathrm{gi}(\mathcal{X}';\xi_s^{2k-1}\mathrm{lr}(c)^{D+1-2k})\\&=-2\mu_k\,m\,\mathrm{gi}(\mathcal{X};\xi_s^{2k-1}\mathrm{lr}(c)^{D+1-2k}).\end{aligned}$$

Dividing by $m\neq0$ gives the identities (30.1) for the pair of $\mathcal{X}$. □

D.5 Torsion systems: the test-point facts

This section proves the test-point facts Proposition D.97 for torsion-regular systems with Hodge–Tate lattice families, used in Section 35 for the existence of pre-admissible torsion systems with test-point conditions (Proposition 35.23). Subsection D.5.1 shows that at a test point the Hopf orders are stable and tight, with the abelian-factor target, and that the reduction is good or Mumford-degenerate (Proposition D.99); Subsection D.5.2 proves the Hodge–Tate facts at test points (Proposition D.110).

D.5.1 Stable tight Hopf orders at test points

The subsection states Proposition D.97 and proves its algebraic part. At a test point the torsion function orders are base changes of a complete local order, and the reduction is good or Mumford-degenerate (Proposition D.99); the orders are stable tight Hopf orders and abelian factors (Proposition D.98); and the cotangent maps kill products, are equivariant, satisfy the kernel clause, and detect the square of the augmentation ideal (Lemma D.107, Lemma D.108, Proposition D.109).

Proposition D.97. *Let $\mathcal{T}$, $\mathcal{Y}$, E, c, $2 \le B_0 \le B$, $\mathcal{X} \subseteq \mathbb{P}$, $p \in S(B_0, B)$ and λ be as in Proposition 35.11, $\mathcal{T}$ of type (Φ, n, κ). Let $\mathfrak{T} = (\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_{\mathfrak{P}}, \theta))$ be a torsion-regular pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$, $R \ge 4$ an integer as in Definition 35.22, $w \in \mathcal{O}'$ with $w^{p-1} = u\,p$ for a unit u, and for every $a \in \Phi$ let $\beta, \bar{\beta}$ be bases and $(U_\nu, \iota_\nu, h_\nu)_\nu$ a Hodge–Tate atlas for $(\mathfrak{T}, (a, +), \beta, \bar{\beta})$ whose Hodge–Tate lattice family satisfies (Hm1), (Hm2), (Hm4) and (Hm5) of Proposition 35.24. Then every point $\mathfrak{q}$ of $\mathcal{X}'$ of codimension one at which p is not a unit is the first entry of a test point $(\mathfrak{q}, C, \iota_C)$ such that:*

1. *(Ox3) holds at $(\mathfrak{q}, C, \iota_C)$;*
2. *for every $a \in \Phi$, with $s = (a, +)$, $m = m_{\mathfrak{P}_s}$ and the eigen flag H_i, J_i of H as in (Hm2), (F0)–(F2) hold for the $\mathcal{O}_C$-spans of the stalks at $\mathfrak{q}$ of the Hodge–Tate lattice family of a, and $\mathcal{O}_C J_{i,\mathfrak{q}} = \mathcal{O}_C\,\mathcal{O}_{\mathcal{X}'}(-\mathrm{Ec}_{s,i})_{\mathfrak{q}}$ in C for $1 \le i \le m$.*

Proof. Let $\mathfrak{q}$ be a point of $\mathcal{X}'$ of codimension one at which p is not a unit, and let $(\mathfrak{q}, C, \iota_C)$ be the test point given by Proposition D.110. By Proposition D.98 (which holds at every test point of the torsion-regular pre-admissible system $\mathfrak{T}$), for every prime $\mathfrak{P} \mid p$ the order $\mathcal{A}_{\mathfrak{P},C}$ is an $\mathbb{F}_{\mathfrak{P}}$-stable tight Hopf order on $\Gamma_{\mathfrak{P}}$ and an abelian factor; by Lemma D.107 and Lemma D.108 (both at every test point) the three cotangent conditions of (Ox3) hold for every label u. Together these are (Ox3) at $(\mathfrak{q}, C, \iota_C)$, which is (1). Part (2) is the conclusion of Proposition D.110 at the same test point. □

Proposition D.98. *Let $\mathfrak{T}$ be a torsion-regular pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$ as in Proposition D.97 and $(\mathfrak{q}, C, \iota_C)$ a test point. Then for every prime $\mathfrak{P} \mid p$ of $\mathcal{O}_E$, $\mathcal{A}_{\mathfrak{P},C}$ is an $\mathbb{F}_{\mathfrak{P}}$-stable tight Hopf order on $\Gamma_{\mathfrak{P}}$ and an abelian factor.*

Proof. Let $h\colon \widehat{R} \to \mathcal{O}_C$, $\widehat{G}$ and $N = \prod_{\mathfrak{P}'} \Gamma_{\mathfrak{P}'}$ be as in Proposition D.99, and $M = \mathrm{Ord}(\widehat{G}[p])_h = \boxtimes_{\mathfrak{P}'} \mathcal{A}_{\mathfrak{P}',C}$ (its part (1)).

Step 1: M is a tight Hopf order on N and an abelian factor. If $\widehat{G}$ carries degeneration data as in Proposition D.99(2), Proposition D.101 applies ($\widehat{R}$ is a complete discrete valuation ring of characteristic 0, h is injective, $\widehat{G}_{\widehat{K}}$ is principally polarised, $\widehat{G}[p]$ is flat, quasi-finite, separated, of finite presentation, with constant generic fibre of points N): M is a tight abelian factor (tightness includes being a Hopf order). Otherwise $\widehat{G}$ is an abelian scheme with a principal polarisation, so $\mathcal{B} = \widehat{G} \otimes_{\widehat{R},h} \mathcal{O}_C$ is a principally polarised abelian scheme over $\mathcal{O}_C$. The group scheme $\widehat{G}[p]$ is finite and locally free, hence affine; its $\widehat{K}$-points are $\widehat{R}$-points (their coordinates are integral over the normal domain $\widehat{R}$), so translations by points of N are automorphisms and the functions given by global sections of $\mathcal{O}_{\widehat{G}[p]}$ lie in $\mathrm{Ord}(\widehat{G}[p])$; conversely local sections are determined by their values at the points of the reduced constant generic fibre ($\mathcal{O}_{\widehat{G}[p]}$ is $\widehat{R}$-torsion free), so the local sections defining a regular function glue. Hence $\mathrm{Ord}(\widehat{G}[p])$ is the image of $\Gamma(\widehat{G}[p], \mathcal{O})$, and by base change of affine schemes $M = M_{\mathcal{B}}$ with $N = N_{\mathcal{B}}$. By Lemma D.105 M is a tight Hopf order, and it is an abelian factor with $\iota = \rho = \mathrm{id}$.

Step 2: the factor at $\mathfrak{P}$ is a Hopf order and an abelian factor. Fix $\mathfrak{P}$, write $N = \Gamma_{\mathfrak{P}} \times N'$, $M = \mathcal{A}_{\mathfrak{P},C} \boxtimes M'$, and let $\mathcal{A}'$, $\iota\colon N \to N_{\mathcal{A}'}$, ρ be data exhibiting M as an abelian factor. Let $\iota_{\mathfrak{P}}\colon \Gamma_{\mathfrak{P}} \to N$, $x \mapsto (x,0)$, and $\rho_{\mathfrak{P}}\colon N \to \Gamma_{\mathfrak{P}}$ the projection. Every element of $\mathcal{A}_{\mathfrak{P},C}$ and of M' takes values in $\mathcal{O}_C$ ($\mathcal{A}_{\mathfrak{P}} \subseteq \mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O})$), and $1 \in M'$; for a pure tensor $f \boxtimes f'$, $(f \boxtimes f') \circ \iota_{\mathfrak{P}} = f'(0) f$ and $(f \boxtimes f') \circ \iota_{\mathfrak{P}} \rho_{\mathfrak{P}} = f \boxtimes f'(0) 1 \in M$. Hence $\{g \circ \iota_{\mathfrak{P}} :\ g \in M\} = \mathcal{A}_{\mathfrak{P},C}$ and $M \circ \iota_{\mathfrak{P}} \rho_{\mathfrak{P}} \subseteq M$. Put $\iota'' = \iota \circ \iota_{\mathfrak{P}}$, $\rho'' = \rho_{\mathfrak{P}} \circ \rho$; then $\rho'' \iota'' = \mathrm{id}$, $\{g \circ \iota'' :\ g \in M_{\mathcal{A}'}\} = \{m \circ \iota_{\mathfrak{P}} :\ m \in M\} = \mathcal{A}_{\mathfrak{P},C}$, and for $g \in M_{\mathcal{A}'}$, $g \circ \iota'' \rho'' = ((g \circ \iota) \circ \iota_{\mathfrak{P}} \rho_{\mathfrak{P}}) \circ \rho$; here $(g \circ \iota) \circ \iota_{\mathfrak{P}} \rho_{\mathfrak{P}} \in M$, so it equals $g' \circ \iota$ for some $g' \in M_{\mathcal{A}'}$, and $g' \circ \iota\rho \in M_{\mathcal{A}'}$. So $\mathcal{A}_{\mathfrak{P},C}$ is an abelian factor. It is a Hopf order: it is the $\mathcal{O}_C$-span of the free module $\mathcal{A}_{\mathfrak{P},\mathfrak{q}}$ whose basis is a C-basis of $\mathrm{Map}(\Gamma_{\mathfrak{P}}, C)$ (Ox2), it is an algebra containing 1, and for $f = g \circ \iota''$ the functions $f(x+y) = g(\iota''x + \iota''y)$ and $f(-x)$ are expressed through the comultiplication and antipode of the Hopf order $M_{\mathcal{A}'}$ (Lemma D.105 is not needed here: $M_{\mathcal{A}'}$ is the function algebra of the finite group scheme $\mathcal{A}'[p]$) as sums of products $g_1(\iota''x) g_2(\iota''y)$ with $g_i \circ \iota'' \in \mathcal{A}_{\mathfrak{P},C}$. The same argument applied to N' shows that M' is a Hopf order.

Step 3: tightness. By Steps 1 and 2 and Lemma D.106 (with $B_1 = \mathcal{A}_{\mathfrak{P},C}$, $B_2 = M'$), $\mathcal{A}_{\mathfrak{P},C}$ is tight.

Step 4: $\mathbb{F}_{\mathfrak{P}}$-stability. Let $a \in \mathbb{F}_{\mathfrak{P}}$ and $\tilde{a} \in \mathcal{O}_E$ a lift. The endomorphism $[\tilde{a}]$ of $G'[\mathfrak{P}]$ acts on $\Gamma_{\mathfrak{P}}$ as multiplication by a. If f is regular on $G'[\mathfrak{P}] \times U$, given near a point by a section s, then $f(a\,\cdot)$ is given near any preimage point by $[\tilde{a}]^* s$; and the translates of $f(a\,\cdot)$ are $f(t' + a\,\cdot)$, again of this form. So the generators of $\mathrm{Ord}(G'[\mathfrak{P}] \times U)$ and hence $\mathcal{A}_{\mathfrak{P}}(U)$ (Definition 35.22(Hf1)) are stable under $f \mapsto f(a\,\cdot)$, and so are the stalk at $\mathfrak{q}$ and its $\mathcal{O}_C$-span $\mathcal{A}_{\mathfrak{P},C}$. □

Proposition D.99. *Let $\mathfrak{T}$ be a torsion-regular pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$ as in Proposition D.97, with function field Ω', and $(\mathfrak{q}, C, \iota_C)$ a test point. Put $R_{\mathfrak{q}} = \mathcal{O}_{\mathcal{X}',\mathfrak{q}}$. Then there are a complete discrete valuation ring $\widehat{R}$ of characteristic 0 with fraction field $\widehat{K}$, a ring homomorphism $g\colon R_{\mathfrak{q}} \to \widehat{R}$ and an injective homomorphism $h\colon \widehat{R} \to \mathcal{O}_C$ with $h \circ g = \iota_C|_{R_{\mathfrak{q}}}$ such that, with $\widehat{G} = G' \times_{\mathcal{X}'} \mathrm{Spec}\, \widehat{R}$ (through g): $\widehat{G}_{\widehat{K}}$ is a principally polarised abelian variety; $\widehat{G}[p]$ is flat, quasi-finite, separated and of finite presentation over $\widehat{R}$ with constant generic fibre whose group of points is $N = G'[p](\Omega') = \bigoplus_{\mathfrak{P}|p} \Gamma_{\mathfrak{P}}$;*

1. *under $N = \prod_{\mathfrak{P}|p} \Gamma_{\mathfrak{P}}$, $\mathrm{Ord}(\widehat{G}[p])_h = \boxtimes_{\mathfrak{P}|p} \mathcal{A}_{\mathfrak{P},C}$;*
2. *either $\widehat{G}$ is an abelian scheme over $\widehat{R}$ to which the principal polarisation of $\widehat{G}_{\widehat{K}}$ extends, or $\widehat{G}$ carries principally polarised split degeneration data satisfying (D1)–(D4) for which it is a uniformisation.*

Proof. As in the setting of Proposition D.97, p is unramified in E. Write $\iota = \iota_C$.

Step 1: the ring $\widehat{R}$ and the maps g, h. $R_{\mathfrak{q}}$ is a discrete valuation ring ($\mathcal{X}'$ is normal, $\mathfrak{q}$ has codimension one) with fraction field Ω'; let π_0 be a uniformiser and R_0 its completion, a complete discrete valuation ring, faithfully flat over $R_{\mathfrak{q}}$, so $R_{\mathfrak{q}} \subseteq R_0$. Since $\iota^{-1}(\mathcal{O}_C) = R_{\mathfrak{q}}$, every $x \in \mathfrak{m}_{\mathfrak{q}}$ has $x^{-1} \notin R_{\mathfrak{q}}$, hence $|\iota(x)| < 1$; with $\rho = |\iota(\pi_0)| < 1$, $|\iota(y)| \le \rho^n$ for $y \in \mathfrak{m}_{\mathfrak{q}}^n$, and units u of $R_{\mathfrak{q}}$ have $|\iota(u)| = 1$. So $\iota|_{R_{\mathfrak{q}}}$ is uniformly continuous for the $\mathfrak{m}_{\mathfrak{q}}$-adic topology, and as $\mathcal{O}_C$ is complete it extends uniquely to a continuous ring homomorphism $h_0\colon R_0 \to \mathcal{O}_C$. It is injective: a non-zero $x \in R_0$ is $u\pi_0^n$ with $u \in R_0^\times$, and $u \equiv u_0 \bmod \pi_0 R_0$ for some $u_0 \in R_{\mathfrak{q}}^\times$ (equal residue fields), so $|h_0(u)| = |\iota(u_0)| = 1$ and $h_0(x) \neq 0$. In particular R_0 has characteristic 0. Let κ be the residue field of R_0. The fibre of G' at the closed point of $\mathrm{Spec}\, R_0$ is a semiabelian variety over κ (Step 2 below), an extension of an abelian variety by a torus; a torus over a field splits over a finite separable extension, so there is a finite separable extension $\kappa' = \kappa(\bar{a})$ over which this torus splits, $\bar{a}$ a root of a separable irreducible monic $\bar{f} \in \kappa[x]$. Let $f \in R_0[x]$ be a monic lift

and $\widehat{R} = R_0[x]/(f)$: since $\bar{f}$ is irreducible and separable, $\widehat{R}$ is a complete discrete valuation ring, finite and étale over R_0, with uniformiser π_0, residue field κ' and fraction field $\widehat{K} = \widehat{K_0}[x]/(f)$ (unramified extensions of complete discrete valuation rings; f is irreducible over R_0 as $\bar{f}$ is, hence over $\mathrm{Frac}(R_0) = \widehat{K_0}$ by Gauss's lemma). As C is algebraically closed, f^{h_0} has a root $\alpha \in C$, integral over $h_0(R_0) \subseteq \mathcal{O}_C$, so $\alpha \in \mathcal{O}_C$ ($\mathcal{O}_C$ is integrally closed). Let $h\colon \widehat{R} \to \mathcal{O}_C$ extend h_0 by $x \mapsto \alpha$, and $g\colon R_{\mathfrak{q}} \subseteq R_0 \subseteq \widehat{R}$. Then $h \circ g = \iota|_{R_{\mathfrak{q}}}$; $\ker h$ is a prime of $\widehat{R}$ over $\ker h_0 = 0$, so $\ker h = 0$ by incomparability for the integral extension $R_0 \subseteq \widehat{R}$. $\widehat{R}$ is a complete noetherian normal local domain of characteristic 0, and g is injective, so it extends to a field embedding $\Omega' \to \widehat{K}$ through which we regard the Ω'-points x_t of G' as $\widehat{K}$-points.

Step 2: the group $\widehat{G}$. By (Fh5) the generic point of $\mathcal{X}'$ (a cover through $f \circ \pi$ in the notation there) maps to a point of $j(\mathrm{Sh}_F)$. By (Ud7), $(G_{\mathcal{Y}}, \iota_{\mathcal{Y}}, \mathcal{L}_{\mathcal{Y}})$ is a semiabelian structure whose restriction along j is the triple of the universal object, so (Ud4) it is an abelian scheme there with $\mathcal{L}_{\mathcal{Y}}$ relatively ample and $\phi_{\mathcal{L}_{\mathcal{Y}}}$ a principal polarisation. Its pull-back $(\widehat{G}, \iota_{\widehat{G}}, \widehat{\mathcal{L}})$ to $\operatorname{Spec}\widehat{R}$ (through $G' = \pi^* G_{\mathcal{X}}$ and g) is a semiabelian structure over $\widehat{R}$; as g is injective, the generic point of $\operatorname{Spec}\widehat{R}$ maps to the generic point of $\mathcal{X}'$, so $\widehat{G}_{\widehat{K}}$ is an abelian variety, $\widehat{\mathcal{L}}|_{\widehat{G}_{\widehat{K}}}$ is ample and $\phi_{\widehat{\mathcal{L}}_{\widehat{K}}}$ is a principal polarisation. By Lemma 35.30(1) (over the complete noetherian normal local domain $\widehat{R}$, whose maximal ideal contains p as $p \in \mathfrak{m}_{\mathfrak{q}}$), $\widehat{G}[p] = \prod_{\mathfrak{P}|p} \widehat{G}[\mathfrak{P}]$ with each factor quasi-finite and flat; $\widehat{G}[p]$ is a closed subgroup scheme of the separated $\widehat{R}$-scheme of finite type $\widehat{G}$, hence separated and of finite presentation. By (Ts3), $G'[\mathfrak{P}]_{\Omega'} = G_{\mathcal{X}}[\mathfrak{P}] \times_{\Omega} \Omega'$ is constant with points $\Gamma_{\mathfrak{P}}$, so $\widehat{G}[p]_{\widehat{K}} = G'[p]_{\Omega'} \otimes_{\Omega'} \widehat{K}$ is reduced and constant with group of points $N = \bigoplus_{\mathfrak{P}} \Gamma_{\mathfrak{P}} = \prod_{\mathfrak{P}} \Gamma_{\mathfrak{P}}$. So $\widehat{G}[p]$ and each $\widehat{G}[\mathfrak{P}]$ are as in Definition 35.21 over $\widehat{R}$, with $T = N$, resp. $T = \Gamma_{\mathfrak{P}}$.

Step 3: clause (1). Let $U = \operatorname{Spec} R_U$ be an affine open neighbourhood of $\mathfrak{q}$ in $\mathcal{X}'$ and $h_1\colon R_U \to R_{\mathfrak{q}} \to \widehat{R}$. By (Hf1), $\mathcal{A}_{\mathfrak{P}}(U) = \mathrm{Ord}(G'[\mathfrak{P}]_U)$. Apply Lemma 35.28(3) to h_1 (R_U is a noetherian integrally closed domain with fraction field Ω', $\widehat{R}$ a noetherian domain), with $B = \Omega'$, $h_B\colon \Omega' \to \widehat{K}$ the embedding of Step 1 and $\sigma_t = x_t$: the base change $\widehat{G}[\mathfrak{P}]$ with the induced points is as in Definition 35.21 with the same $T = \Gamma_{\mathfrak{P}}$ (Step 2). Hence $\mathrm{Ord}(\widehat{G}[\mathfrak{P}])$ is the $\widehat{R}$-span of $\{h_1 \circ f :\ f \in \mathcal{A}_{\mathfrak{P}}(U)\}$. By Lemma D.100(2) over the discrete valuation ring $\widehat{R}$, applied to the product $\widehat{G}[p] = \prod_{\mathfrak{P}} \widehat{G}[\mathfrak{P}]$ (its points are the tuples of points, matching $N = \prod_{\mathfrak{P}} \Gamma_{\mathfrak{P}}$), $\mathrm{Ord}(\widehat{G}[p])$ is the $\widehat{R}$-span of the functions $(z_{\mathfrak{P}}) \mapsto \prod_{\mathfrak{P}} a_{\mathfrak{P}}(z_{\mathfrak{P}})$, $a_{\mathfrak{P}} \in \mathrm{Ord}(\widehat{G}[\mathfrak{P}])$, hence the $\widehat{R}$-span of the products of the functions $h_1 \circ f_{\mathfrak{P}}$, $f_{\mathfrak{P}} \in \mathcal{A}_{\mathfrak{P}}(U)$. As $h \circ h_1 = \iota|_{R_U}$, $\mathrm{Ord}(\widehat{G}[p])_h$ is the $\mathcal{O}_C$-span of the products $\prod_{\mathfrak{P}} \iota \circ f_{\mathfrak{P}}(z_{\mathfrak{P}})$, $f_{\mathfrak{P}} \in \mathcal{A}_{\mathfrak{P}}(U)$. On the other hand $\mathcal{A}_{\mathfrak{P},\mathfrak{q}}$ is the $R_{\mathfrak{q}}$-span of $\mathcal{A}_{\mathfrak{P}}(U)$ ($\mathcal{A}_{\mathfrak{P}}$ is quasi-coherent, (Ts3)), so by (Ox2) $\mathcal{A}_{\mathfrak{P},C}$ is the $\mathcal{O}_C$-span of $\{\iota \circ f :\ f \in \mathcal{A}_{\mathfrak{P}}(U)\}$, and $\boxtimes_{\mathfrak{P}} \mathcal{A}_{\mathfrak{P},C}$, the $\mathcal{O}_C$-span of the products of elements of the factors, is the $\mathcal{O}_C$-span of the same products (expand the products of $\mathcal{O}_C$-combinations). This is (1).

Step 4: clause (2). The closed fibre of $\widehat{G}$ is the base change to κ' of the closed fibre of G'_{R_0}, an extension of an abelian variety by a torus that is split over κ' (Step 1). (a) If this torus is trivial, all fibres of $\widehat{G}$ ($\operatorname{Spec}\widehat{R}$ has two points) are abelian varieties, so the semiabelian scheme $\widehat{G}$ is an abelian scheme (Faltings–Chai, Ch. I, Remark 2.8; Lan, Remark 3.3.1.8). Let $\widehat{G}^\vee$ be its dual and $\lambda = \phi_{\widehat{\mathcal{L}}}\colon \widehat{G} \to \widehat{G}^\vee$, a symmetric homomorphism whose generic fibre is the principal polarisation $\phi_{\widehat{\mathcal{L}}_{\widehat{K}}}$. By Theorem B.104 ($\operatorname{Spec}\widehat{R}$ is noetherian and normal, its generic point is a dense open subscheme, abelian schemes are semiabelian) the inverse of $\lambda_{\widehat{K}}$ extends to a homomorphism $\mu\colon \widehat{G}^\vee \to \widehat{G}$, and $\mu\lambda$, $\lambda\mu$ extend identities, so they are identities by the uniqueness in Theorem B.104: λ is an isomorphism. By Theorem B.126 ($\operatorname{Spec}\widehat{R}$ is connected and λ is a polarisation at the generic point) λ is a polarisation, hence a principal polarisation of $\widehat{G}$ extending that of $\widehat{G}_{\widehat{K}}$: the first alternative of (2). (b) Otherwise, Proposition 32.15 applies

to the semiabelian structure $(\widehat{G}, \iota_{\widehat{G}}, \widehat{\mathcal{L}})$ over $\widehat{R}$: its hypothesis (1) holds by Step 2 and (2) by the choice of κ'. So $(\widehat{G}, \iota_{\widehat{G}}, \widehat{\mathcal{L}})$ is uniformised by degeneration data $\mathfrak{D} = (B, \mathcal{L}_B, \iota_B, X, c, \psi)$ with $\mathcal{O}_E$-structure over $(\widehat{R}, \mathfrak{m})$. Forgetting the $\mathcal{O}_E$-structure, these are principally polarised split degeneration data $(P, \mathcal{L}, X, c, \psi) = (B, \mathcal{L}_B, X, c, \psi)$ satisfying (D1)–(D4) of Proposition D.101, clause by clause: (D1) B is an abelian scheme and $\mathcal{L}_B$ a rigidified symmetric cubical relatively ample line bundle of principal polarisation; (D2) X is free of finite rank and $c\colon X \to B(\widehat{R})$ a homomorphism; (D3) the $\psi(\chi)$ are nowhere-zero sections, i.e. trivialisations, of $c(\chi)^*\mathcal{L}_B^{-1}$ over $\widehat{K}$, with $\psi(-\chi) = \psi(\chi)$ under the symmetry and τ bimultiplicative; (D4) for $\chi \neq 0$, $\psi(\chi)$ is a section over $\widehat{R}$ of $\mathfrak{m} \cdot c(\chi)^*\mathcal{L}_B^{-1}$, i.e. has values in $\mathfrak{m}$ in every local trivialisation, and for every n only finitely many $\psi(\chi)$ fail to lie in $\mathfrak{m}^n c(\chi)^*\mathcal{L}_B^{-1}$. $\widehat{G}$ is a uniformisation as there: for every $e \geq 0$ the identification of $\widehat{G} \otimes \widehat{R}/\mathfrak{m}^{e+1}$ with $\widetilde{G} \otimes \widehat{R}/\mathfrak{m}^{e+1}$, $\widetilde{G}$ the extension of B by the split torus T_X with character group X given by c, compatible in e, gives the exact sequences $1 \to T_X \to \widehat{G} \otimes \widehat{R}/\mathfrak{m}^{e+1} \to B \otimes \widehat{R}/\mathfrak{m}^{e+1} \to 0$; the identification of $\widehat{\mathcal{L}} \otimes \widehat{R}/\mathfrak{m}^{e+1}$ with the pull-back of $\mathcal{L}_B$, compatible with rigidifications, symmetries and cubical structures, gives the compatible cubical rigidified line bundles; and $\widehat{G}_{\widehat{K}}$ carries the principal polarisation of $\widehat{\mathcal{L}}_{\widehat{K}}$. Finally $\widehat{G}$ is smooth, separated and commutative with $\widehat{G}_{\widehat{K}}$ the principally polarised abelian variety of Step 2: the second alternative of (2). □

Lemma D.100. *Let R be a noetherian domain with fraction field K.*

1. ***Characterisation.*** *Let Γ, T and $(x_t)_{t\in T}$ be as in Definition 35.21, and let $A \subseteq \operatorname{Map}(T, R)$ be an R-subalgebra with $\varphi_t(A) \subseteq A$ for every $t \in T$. Suppose there is an R-morphism $j\colon \Gamma \to \operatorname{Spec} A$ which is an open immersion, satisfies $j \circ x_t = \operatorname{ev}_t$ for every $t \in T$, and whose image has covering translates:* $\operatorname{Spec} A = \bigcup_{t\in T} \tau_t(j(\Gamma))$. *Then $A = \operatorname{Ord}(\Gamma)$.*

2. ***Products.*** *Let R be in addition integrally closed, and let Γ_1, Γ_2 with groups T_1, T_2 and points $(x_t^{(1)})_{t\in T_1}$, $(x_t^{(2)})_{t\in T_2}$ be as in Definition 35.21. Then $\Gamma = \Gamma_1 \times_R \Gamma_2$ with $T = T_1 \times T_2$ and $x_{(t_1,t_2)} = (x_{t_1}^{(1)}, x_{t_2}^{(2)})$ is as in Definition 35.21,* $\operatorname{Ord}(\Gamma)$ *is the R-span of the functions $(t_1, t_2) \mapsto a(t_1)\, b(t_2)$ with $a \in \operatorname{Ord}(\Gamma_1)$, $b \in \operatorname{Ord}(\Gamma_2)$, and the map $\operatorname{Ord}(\Gamma_1) \otimes_R \operatorname{Ord}(\Gamma_2) \to \operatorname{Ord}(\Gamma)$ sending $a \otimes b$ to this function is an isomorphism of R-algebras. By induction the same holds for finite products $\Gamma_1 \times_R \cdots \times_R \Gamma_k$.*

Proof. **Proof of (1).** Put $X = \operatorname{Spec} A$ and $V = j(\Gamma)$, an open subset. Since $\varphi_t \circ \varphi_{-t} = \mathrm{id}$, $\varphi_t(A) = A$ and τ_t is an automorphism; as $\varphi_s \circ \varphi_t = \varphi_{s+t}$, $\tau_t \circ \tau_s = \tau_{s+t}$, so $\tau_{-t} = \tau_t^{-1}$. The composite of $\operatorname{ev}_s\colon \operatorname{Spec} K \to X$ with τ_t corresponds to the ring map $\operatorname{ev}_s \circ \varphi_t = \operatorname{ev}_{s+t}$, so $\tau_t \circ \operatorname{ev}_s = \operatorname{ev}_{s+t}$. For $a \in A$ its value at ev_s is $a(s)$.

Step 1 (sections are determined by their values). Let $W \subseteq X$ be open, $\sigma \in \mathcal{O}_X(W)$ with $\sigma(\operatorname{ev}_t) = 0$ for every t with $\operatorname{ev}_t \in W$. We show $\sigma = 0$. Cover W by basic open sets $D(a)$, $a \in A$. For $a \in A$, $\operatorname{ev}_t \in D(a)$ if and only if $a(t) \neq 0$, and localisation at a is exact, so $\mathcal{O}_X(D(a)) = A_a \to \operatorname{Map}(T, K)_a$ is injective; since a is a unit in the factors t with $a(t) \neq 0$ and zero in the others, $\operatorname{Map}(T, K)_a = \prod_{t:\, a(t)\neq 0} K$, and the t-th coordinate of the image of an element is its value at ev_t. Hence $\sigma|_{D(a)} = 0$ for every a with $D(a) \subseteq W$, and $\sigma = 0$.

Step 2 ($A \subseteq \operatorname{Ord}(\Gamma)$). Let $f \in A$ and $t' \in T$. Then $\varphi_{t'} f \in A$, and $s = j^*(\varphi_{t'} f) \in \mathcal{O}_\Gamma(\Gamma)$ has $s(x_t) = (\varphi_{t'} f)(\operatorname{ev}_t) = f(t' + t)$ for every t, because $j \circ x_t = \operatorname{ev}_t$. So every translate of $f\colon T \to R$ is regular on Γ (with the neighbourhood $U = \Gamma$ at every point), and f is one of the generators of $\operatorname{Ord}(\Gamma)$.

Step 3 ($\operatorname{Ord}(\Gamma) \subseteq A$). As A is an R-subalgebra, it suffices to show that every $g\colon T \to R$ all of whose translates are regular on Γ lies in A. Let $t' \in T$ and $P \in \tau_{t'}(V)$; then $\tau_{-t'}(P) = j(\gamma)$

for a point γ of Γ. As $\varphi_{t'}g$ is regular on Γ, there are an open $U \ni \gamma$ and $s \in \mathcal{O}_\Gamma(U)$ with $s(x_u) = g(t'+u)$ whenever $x_u \in U$. Since j is an open immersion, $W' = j(U)$ is open in X and $\sigma' = (j|_U^{-1})^* s \in \mathcal{O}_X(W')$; as j is injective and $j(x_u) = \mathrm{ev}_u$, we have $\mathrm{ev}_u \in W'$ if and only if $x_u \in U$, and then $\sigma'(\mathrm{ev}_u) = s(x_u) = g(t'+u)$. Put $W = \tau_{-t'}^{-1}(W') = \tau_{t'}(W')$, an open neighbourhood of P, and $\sigma = \tau_{-t'}^* \sigma' \in \mathcal{O}_X(W)$. For $\mathrm{ev}_v \in W$ we have $\tau_{-t'} \circ \mathrm{ev}_v = \mathrm{ev}_{v-t'} \in W'$ and $\sigma(\mathrm{ev}_v) = \sigma'(\mathrm{ev}_{v-t'}) = g(v)$. Since the sets $\tau_{t'}(V)$, $t' \in T$, cover X, we obtain an open cover $X = \bigcup_i W_i$ with sections $\sigma_i \in \mathcal{O}_X(W_i)$ satisfying $\sigma_i(\mathrm{ev}_v) = g(v)$ whenever $\mathrm{ev}_v \in W_i$. On $W_i \cap W_k$ the difference $\sigma_i - \sigma_k$ has all values 0, so vanishes by Step 1. The σ_i glue to $a \in \mathcal{O}_X(X) = A$, and every ev_v lies in some W_i, so $a(v) = g(v)$ for all v: $g = a \in A$. With Step 2, $A = \mathrm{Ord}(\Gamma)$.

Proof of (2). Γ is a commutative group scheme which is flat, quasi-finite, separated and of finite presentation over R (these properties are stable under base change and composition). Its generic fibre is $\Gamma_{1,K} \times_K \Gamma_{2,K} = (\bigsqcup_{T_1} \operatorname{Spec} K) \times_K (\bigsqcup_{T_2} \operatorname{Spec} K) = \bigsqcup_{T_1 \times T_2} \operatorname{Spec} K$, reduced, with points exactly the $x_{(t_1,t_2)}$, and $(t_1,t_2) \mapsto x_{(t_1,t_2)}$ is an injective homomorphism. So Γ is as in Definition 35.21.

Put $A_i = \mathrm{Ord}(\Gamma_i)$. By Lemma 35.28(1), A_i is a finitely generated projective R-module, stable under every φ_t, $t \in T_i$, with $d_i A_i^{\mathrm{full}} \subseteq A_i \subseteq A_i^{\mathrm{full}}$ for some $d_i \neq 0$, where $A_i^{\mathrm{full}} = \mathrm{Map}(T_i, R)$; hence $A_i \otimes_R K = \mathrm{Map}(T_i, K)$. By Lemma 35.28(2) there is an open immersion $j_i \colon \Gamma_i \to \operatorname{Spec} A_i$ with $j_i \circ x_t^{(i)} = \mathrm{ev}_t$ and $\operatorname{Spec} A_i = \bigcup_{t \in T_i} \tau_t(j_i(\Gamma_i))$.

Let $\mu \colon A_1 \otimes_R A_2 \to \mathrm{Map}(T, R)$, $\mu(a \otimes b)(t_1,t_2) = a(t_1)b(t_2)$, a homomorphism of R-algebras, and $A = \mu(A_1 \otimes_R A_2)$, the span in the statement. The module $A_1 \otimes_R A_2$ is projective, hence torsion free, so it embeds into $(A_1 \otimes_R A_2) \otimes_R K = \mathrm{Map}(T_1, K) \otimes_K \mathrm{Map}(T_2, K)$, and the latter maps isomorphically onto $\mathrm{Map}(T, K)$ (bases of indicator functions); so μ is injective and $\mu \colon A_1 \otimes_R A_2 \to A$ is an isomorphism of R-algebras. For $t = (t_1,t_2)$, $\varphi_t \mu(a \otimes b) = \mu(\varphi_{t_1} a \otimes \varphi_{t_2} b)$, so $\varphi_t(A) \subseteq A$, and under $\operatorname{Spec} \mu$, which identifies $\operatorname{Spec} A$ with $\operatorname{Spec} A_1 \times_R \operatorname{Spec} A_2$, τ_t corresponds to $\tau_{t_1} \times \tau_{t_2}$ and ev_t to $(\mathrm{ev}_{t_1}, \mathrm{ev}_{t_2})$ (as $\mathrm{ev}_t \circ \mu(a \otimes b) = a(t_1)b(t_2)$).

Let $j = j_1 \times j_2 \colon \Gamma \to \operatorname{Spec} A_1 \times_R \operatorname{Spec} A_2 = \operatorname{Spec} A$. It is an open immersion (the composite of the base changes $j_1 \times \mathrm{id}$ and $\mathrm{id} \times j_2$ of open immersions), $j \circ x_t = (\mathrm{ev}_{t_1}, \mathrm{ev}_{t_2}) = \mathrm{ev}_t$, and $j(\Gamma) = \mathrm{pr}_1^{-1}(V_1) \cap \mathrm{pr}_2^{-1}(V_2)$ with $V_i = j_i(\Gamma_i)$. Covering translates: for a point P of $\operatorname{Spec} A$ choose t_i with $\mathrm{pr}_i(P) \in \tau_{t_i}(V_i)$; then $P \in \mathrm{pr}_1^{-1}(\tau_{t_1} V_1) \cap \mathrm{pr}_2^{-1}(\tau_{t_2} V_2) = (\tau_{t_1} \times \tau_{t_2})(\mathrm{pr}_1^{-1}(V_1) \cap \mathrm{pr}_2^{-1}(V_2)) = \tau_{(t_1,t_2)}(j(\Gamma))$, because $\mathrm{pr}_i \circ (\tau_{t_1} \times \tau_{t_2}) = \tau_{t_i} \circ \mathrm{pr}_i$ and $\tau_{t_1} \times \tau_{t_2}$ is bijective. By (1), $\mathrm{Ord}(\Gamma) = A$, which is the assertion. For k factors apply this to $(\Gamma_1 \times \cdots \times \Gamma_{k-1}) \times \Gamma_k$, the first factor being as in Definition 35.21 by the first paragraph. □

Proposition D.101. *Let p be a prime and C, $\mathcal{O}_C$, v as in Definition 35.18. Let R be a complete discrete valuation ring of characteristic 0 with maximal ideal $\mathfrak{m}$ and fraction field K, and $h \colon R \to \mathcal{O}_C$ an injective homomorphism. Let A be a principally polarised abelian variety over K and G a smooth separated commutative group scheme over R with $G_K \cong A$, given together with principally polarised split degeneration data $(P, \mathcal{L}, X, c, \psi)$ in the sense of [87, Ch. II–III]:*

(D1) *a principally polarised abelian scheme P over R with a symmetric, rigidified, cubical relatively ample line bundle $\mathcal{L}$;*

(D2) *a free abelian group X of finite rank and a homomorphism $c \colon X \to P(R)$;*

(D3) *trivialisations $\psi(\chi)$ of $c(\chi)^* \mathcal{L}^{-1}$ over K ($\chi \in X$) with $\psi(-\chi) = \psi(\chi)$ under the symmetry of $\mathcal{L}$, whose associated trivialisations $\tau(\chi,\chi') = \psi(\chi+\chi')\psi(\chi)^{-1}\psi(\chi')^{-1}$ of the biextension are bimultiplicative;*

(D4) *positivity: for $\chi \neq 0$, $\psi(\chi)$ is a section with values in $\mathfrak{m}$ (in every local trivialisation of $c(\chi)^* \mathcal{L}^{-1}$ over R), and for every n only finitely many $\psi(\chi)$ fail to have values in $\mathfrak{m}^n$;*

and for which G is a uniformisation: for every $e \geq 0$, an exact sequence $1 \to T_X \to G \otimes R/\mathfrak{m}^{e+1} \to P \otimes R/\mathfrak{m}^{e+1} \to 0$ with T_X the split torus with character group X, compatible in e, with cubical rigidified line bundles on G and P compatible with these sequences and with the polarisation of A. Assume that $G[p]$ is flat, quasi-finite and separated over R, and that $A[p]$ is constant over K with group of points N. Then $M = \mathrm{Ord}(G[p])_h \subseteq \mathrm{Map}(N, C)$, the base change along h of the regular-translate order of $G[p]$ (Definition 35.21, with $T = N$), is an abelian factor, and it is tight.

Proof. Let N_t, N_a, N_e, e and $\mu(N_t)$ be as given by Proposition D.102, $r_t = \dim_{\mathbb{F}_p} N_t$, $r_e = \dim_{\mathbb{F}_p} N_e$, $r = \max(r_t, r_e)$, and $P_C = P \otimes_{R,h} \mathcal{O}_C$. We transport functions along e and write $M \circ e = \{f \circ e : \ f \in M\}$.

Step 1: $M \circ e = \mu_C(N_t) \boxtimes M_{P_C} \boxtimes \mathrm{Map}(N_e, \mathcal{O}_C)$, where $\mu_C(N_t)$ is the $\mathcal{O}_C$-span of the homomorphisms $N_t \to \mathcal{O}_C^\times$. By Proposition D.102(3), $M \circ e$ is the $\mathcal{O}_C$-span of the products $(h \circ a) \boxtimes (h \circ b) \boxtimes (h \circ c')$, so it suffices to identify the three spans. (a) By Proposition D.102(2) the homomorphisms $N_t \to R^\times$ span $\mathrm{Map}(N_t, K)$ over K; homomorphisms being linearly independent (Dedekind), there are at least $|N_t|$ of them; composed with the injective h they are distinct homomorphisms $N_t \to \mathcal{O}_C^\times$, of which there are exactly $|N_t|$ ($N_t \cong (\mathbb{Z}/p)^{r_t}$ and $\mu_p(C)$ is cyclic of order p). Hence the $\mathcal{O}_C$-span of $h \circ \mu(N_t)$ is $\mu_C(N_t)$ (a product of characters is a character). (b) $P[p]$ is finite and locally free over R, so affine; its K-points are R-points (their coordinates are integral over the normal domain R), so translations by points of N_a are automorphisms of $P[p]$ and every function given by a global section of $\mathcal{O}_{P[p]}$ lies in $\mathrm{Ord}(P[p])$. Conversely, sections of $\mathcal{O}_{P[p]}$ over an open set are determined by their values at the points of the generic fibre ($\mathcal{O}_{P[p]}$ is R-torsion free and the generic fibre is reduced and constant by Proposition D.102(1)), so the local sections defining a function regular on $P[p]$ glue to a global section. Hence $\mathrm{Ord}(P[p])$ is the image of $\Gamma(P[p], \mathcal{O})$ under evaluation; as $\Gamma(P[p]_{\mathcal{O}_C}, \mathcal{O}) = \Gamma(P[p], \mathcal{O}) \otimes_{R,h} \mathcal{O}_C$ and $N_a = P[p](K) = P_C[p](\mathcal{O}_C)$ (both of order $|N_a|$), the $\mathcal{O}_C$-span of $h \circ \mathrm{Ord}(P[p])$ is M_{P_C}. (c) The span of $h \circ \mathrm{Map}(N_e, R)$ is $\mathrm{Map}(N_e, \mathcal{O}_C)$.

Step 2: the abelian scheme. Let E and $x \colon \mathbb{Z}/p \times \mathbb{Z}/p \to N_E$ be as in Lemma D.104, and $\mathcal{A} = P_C \times E^r$, principally polarised by the product of the principal polarisations of P_C (D1) and of E, and admitting a relatively ample line bundle. Then $N_{\mathcal{A}} = N_{P_C} \times N_E^r$ and, as $\mathcal{A}[p] = P_C[p] \times E[p]^r$ and the ring of global sections of a product of affine schemes is the tensor product of the rings of global sections, $M_{\mathcal{A}} = M_{P_C} \boxtimes M_E^{\boxtimes r}$. Fix isomorphisms $N_t \cong (\mathbb{Z}/p)^{r_t}$, $N_e \cong (\mathbb{Z}/p)^{r_e}$ and extend coordinate vectors by zeros to length r. Define $\iota \colon N \to N_{\mathcal{A}}$ by $\iota(e(z_t, z_a, z_e)) = (z_a, (x(u_k, w_k))_{k \leq r})$, u, w the extended coordinates of z_t, z_e, and $\rho \colon N_{\mathcal{A}} \to N$ by $\rho(y, (x(u_k, w_k))_k) = e((u_k)_{k \leq r_t}, y, (w_k)_{k \leq r_e})$. These are homomorphisms with $\rho \circ \iota = \mathrm{id}_N$.

Step 3: M is an abelian factor. By Lemma D.104, $M_E \circ x = \mu_C(\mathbb{Z}/p) \boxtimes \mathrm{Map}(\mathbb{Z}/p, \mathcal{O}_C)$. For a pure tensor $g = g_0 \boxtimes \bigotimes_k (\alpha_k \boxtimes \gamma_k)$ ($g_0 \in M_{P_C}$, $\alpha_k \in \mu_C(\mathbb{Z}/p)$, $\gamma_k \in \mathrm{Map}(\mathbb{Z}/p, \mathcal{O}_C)$, transported along x), $g \circ \iota \circ e$ is the product of $g_0(z_a)$, of $\prod_{k \leq r_t} \alpha_k(u_k)$ (a character of N_t), of $\prod_{k \leq r_e} \gamma_k(w_k)$, and of the constants $\alpha_k(0), \gamma_k(0) \in \mathcal{O}_C$ for the remaining indices; since the characters of N_t are exactly the products of characters of the coordinates, Step 1 gives $\{g \circ \iota : \ g \in M_{\mathcal{A}}\} = M$. Likewise $g \circ \iota \circ \rho$ is the pure tensor in which the factors at the unused coordinates are replaced by their values at 0, which lies in $M_{\mathcal{A}}$. So M is an abelian factor (Af3).

Step 4: M is tight. Reorder the coordinates of $N_{\mathcal{A}}$ as $N \times N_c$, $N_c = (\mathbb{Z}/p)^{r-r_t} \times (\mathbb{Z}/p)^{r-r_e}$ (the unused coordinates), via ι and the inclusion of N_c. By Steps 1–3, $M_{\mathcal{A}}$ becomes $M \boxtimes M_c$ with $M_c = \mu_C(\mathbb{Z}/p)^{\boxtimes(r-r_t)} \boxtimes \mathrm{Map}(\mathbb{Z}/p, \mathcal{O}_C)^{\boxtimes(r-r_e)}$. Here M is a Hopf order on N (Step 1 and Lemma D.106: $\mu_C(N_t)$ is spanned by the columns of the invertible character table, closed under products, with $\chi(x+y) = \chi(x)\chi(y)$ and $\chi(-x) = \chi(x)^{-1}$; M_{P_C} by Lemma D.105; $\mathrm{Map}(N_e, \mathcal{O}_C)$ trivially), and so is M_c. By Lemma D.105, $M_{\mathcal{A}}$ is tight on $N_{\mathcal{A}}$; tightness is preserved by the group isomorphism reordering the coordinates (all quantities of (Ho3) are transported), so $M \boxtimes M_c$ is

tight, and by Lemma D.106 M is tight. □

Proposition D.102. *Let p be a prime, R a complete discrete valuation ring of characteristic 0 with maximal ideal $\mathfrak{m}$ and fraction field K, A a principally polarised abelian variety over K, and G a smooth separated commutative group scheme over R with $G_K \cong A$, given with degeneration data $(P, \mathcal{L}, X, c, \psi)$ satisfying (D1)–(D4) of Proposition D.101 for which G is a uniformisation (as there). Assume that $G[p]$ is flat, quasi-finite, separated and of finite presentation over R and that $A[p]$ is constant over K, with group of points N. Then there are finite abelian groups N_t, N_a, N_e killed by p and a group isomorphism $e\colon N_t \times N_a \times N_e \to N$ such that:*

1. *$P[p]_K$ is constant and $N_a = P[p](K)$;*
2. *the R-subalgebra $\mu(N_t) \subseteq \mathrm{Map}(N_t, R)$ generated by the homomorphisms $N_t \to R^\times$ contains $d\,\mathrm{Map}(N_t, R)$ for some $d \in R \smallsetminus \{0\}$;*
3. *$\{f \circ e :\ f \in \mathrm{Ord}(G[p])\}$ is the R-span of the functions $(z_t, z_a, z_e) \mapsto a(z_t)\, b(z_a)\, c'(z_e)$ with $a \in \mu(N_t)$, $b \in \mathrm{Ord}(P[p])$ (formed with $T = N_a$) and $c' \in \mathrm{Map}(N_e, R)$.*

Proof. Let π_R be a uniformiser of R, κ its residue field, $\Gamma = G[p]$, and let $t \mapsto x_t$, $t \in N$, be the points of Γ_K; N is killed by p.

Step 1: G is semiabelian and its Raynaud extension is an extension of P by T_X. $\operatorname{Spec} R$ has two points. $G_K \cong A$ is an abelian variety, and by the uniformisation with $e = 0$, G_κ is an extension of the abelian variety P_κ by the split torus $T_{X,\kappa}$; so the smooth separated commutative group scheme G is semiabelian, with abelian generic fibre, and its closed fibre is an extension of an abelian variety by a split torus. Let $0 \to T \to \widetilde{G} \to A' \to 0$ be its Raynaud extension (Theorem B.109(1)). The compatible system of the sequences $1 \to T_X \to G \otimes R/\mathfrak{m}^{e+1} \to P \otimes R/\mathfrak{m}^{e+1} \to 0$ gives a formal subtorus T_X^{for} of G^{for} lifting the torus of G_κ with quotient P^{for}; by uniqueness of formal tori lifting a torus it is T^{for}, so $T \cong T_X$, and $A'^{\mathrm{for}} = G^{\mathrm{for}}/T^{\mathrm{for}} \cong P^{\mathrm{for}}$ algebraises to an isomorphism $A' \cong P$ of abelian schemes. So $\widetilde{G}$ is an extension $0 \to T_X \to \widetilde{G} \xrightarrow{\varpi} P \to 0$ over R, defined by some homomorphism $c_0\colon X \to P^\vee(R)$.

Step 2: the finite part. R is complete, hence henselian, and Γ is quasi-finite and separated, so $\Gamma = \Gamma^f \sqcup \Gamma^r$ with Γ^f the finite part, an open and closed subgroup scheme, finite and flat, and Γ^r open and closed with empty closed fibre. So Γ^r lies in the generic fibre $\Gamma_K = \bigsqcup_t \{x_t\}$: it is the open subscheme $\{x_t :\ t \notin N^f\}$, each point $\operatorname{Spec} K$, where $N^f = \{t :\ x_t \in \Gamma^f\}$, a subgroup of N. By Theorem B.119(1) (with $m = p$), $\Gamma^f = G[p]^f \cong \widetilde{G}[p]$ as group schemes. Hence $\widetilde{G}[p]_K \cong \Gamma^f_K$ is constant (an open and closed subgroup of the constant Γ_K) with group of points N^f. By Lemma D.103 (R is a discrete valuation ring, $\widetilde{G}$ the extension defined by c_0) there is an isomorphism $\widetilde{G}[p] \cong T_X[p] \times_R P[p]$, the identity on $T_X[p]$ and compatible with the projections to $P[p]$. Composing, we obtain an isomorphism of group schemes $\phi\colon T_X[p] \times_R P[p] \to \Gamma^f$. Its generic fibre identifies the constant group Γ^f_K with $T_X[p]_K \times P[p]_K$, so both factors are constant over K.

Step 3: the groups and the isomorphism e; clause (1). Put $N_t = T_X[p](K)$ and $N_a = P[p](K)$, finite abelian groups killed by p; $P[p]_K$ is constant (Step 2), which is (1). ϕ induces a group isomorphism $N_t \times N_a \to N^f$, $(z_t, z_a) \mapsto \phi(z_t, z_a)$. Choose a complement N_e of the $\mathbb{F}_p$-subspace N^f of N and put $e(z_t, z_a, z_e) = \phi(z_t, z_a) + z_e$; $e\colon N_t \times N_a \times N_e \to N$ is a group isomorphism.

Step 4: clause (2). If $X = 0$ then $N_t = 0$ and (2) holds with $d = 1$. Otherwise $T_X[p] = \mathrm{Hom}(X, \mu_p)$ is constant over K, so $\mu_p(\bar{K}) \subseteq K$, and these roots of unity are integral, so lie in $R^\times$. $T_X[p] = \operatorname{Spec} R[X/pX]$, and $N_t = \mathrm{Hom}(X/pX, \mu_p(K))$; for $\chi \in X/pX$ the function $z \mapsto z(\chi)$ is a homomorphism $N_t \to R^\times$, and these are $|N_t|$ distinct characters, i.e. all characters

of $N_t \cong (\mathbb{Z}/p)^r$ with values in $\mu_p(K)$. By orthogonality, $\sum_\chi \chi(z) = |N_t|$ if $z = 0$ and 0 otherwise, so for $z_0 \in N_t$ the function $|N_t|\,\delta_{z_0} = \sum_\chi \chi(z_0)^{-1}\chi$ lies in $\mu(N_t)$. Hence $d\,\mathrm{Map}(N_t, R) \subseteq \mu(N_t)$ with $d = |N_t| \neq 0$ (R has characteristic 0). Moreover, since a product of characters is a character, $\mu(N_t)$ is the R-span of the characters, and it is the image of $\mathcal{O}(T_X[p]) = R[X/pX]$ under evaluation at the points $z \in N_t$.

Step 5: functions on Γ^f. The scheme $P[p]$ is finite and flat over R with reduced constant generic fibre; $\mathcal{O}(P[p])$ is torsion free, so evaluation at the points of N_a embeds it into $\mathrm{Map}(N_a, K)$, and its values are integral over R, hence in R. Let $A_P \subseteq \mathrm{Map}(N_a, R)$ be its image. Each K-point of $P[p]$ extends to an R-point (finite over the normal R), so translations by N_a are automorphisms of $P[p]$ and A_P is translation stable; taking for j the isomorphism $P[p] \to \operatorname{Spec} A_P$ (an open immersion with $j \circ x_z = \mathrm{ev}_z$, whose single translate covers), Lemma D.100(1) gives $A_P = \mathrm{Ord}(P[p])$. As $\Gamma^f \cong T_X[p] \times_R P[p]$ via ϕ and the ring of functions of a fibre product of affine schemes is the tensor product of the rings of functions, evaluation at the points $\phi(z_t, z_a)$ maps $\mathcal{O}(\Gamma^f)$ onto

$$B^f = \mu(N_t) \boxtimes \mathrm{Ord}(P[p]) \subseteq \mathrm{Map}(N_t \times N_a, R),$$

the R-span of the functions $a(z_t)b(z_a)$ (Steps 4 and 5); it is injective (Γ^f is flat with reduced constant generic fibre), so $\mathcal{O}(\Gamma^f) \cong B^f$ and $\Gamma^f \cong \operatorname{Spec} B^f$, with $x_{\phi(z_t,z_a)}$ corresponding to evaluation at (z_t, z_a).

Step 6: clause (3). Let $\mathcal{A} \subseteq \mathrm{Map}(N, R)$ be the set of f with $f \circ e$ in the R-span of the functions $a(z_t)b(z_a)c'(z_e)$, $a \in \mu(N_t)$, $b \in \mathrm{Ord}(P[p])$, $c' \in \mathrm{Map}(N_e, R)$; equivalently (expand c' in indicator functions of points of N_e), $f \in \mathcal{A}$ iff $f_{z_e}\colon (z_t, z_a) \mapsto f(e(z_t, z_a, z_e))$ lies in B^f for every $z_e \in N_e$. So $f \mapsto (f_{z_e})_{z_e}$ is an isomorphism of R-algebras $\mathcal{A} \to \prod_{z_e \in N_e} B^f$, and $\operatorname{Spec}\mathcal{A}$ is the disjoint union of the open and closed subschemes $V_{z_e} \cong \operatorname{Spec} B^f$. We verify the hypotheses of Lemma D.100(1) for Γ and $\mathcal{A}$.

(a) $\mathcal{A}$ is an R-subalgebra of $\mathrm{Map}(N, R)$, stable under every φ_t: for $t = e(w_t, w_a, w_e)$, $(\varphi_t f) \circ e(z_t, z_a, z_e) = f \circ e(w_t + z_t, w_a + z_a, w_e + z_e)$, and $\mu(N_t)$, $\mathrm{Ord}(P[p])$ (Step 5: the image of $\mathcal{O}(P[p])$, stable under translations) and $\mathrm{Map}(N_e, R)$ are translation stable ($\chi(w + z) = \chi(w)\chi(z)$ for characters).

(b) The morphism j. On Γ^f let j be $\Gamma^f \cong \operatorname{Spec} B^f = V_0 \subseteq \operatorname{Spec}\mathcal{A}$ (Step 5), induced by $f \mapsto f_0$; for $t = \phi(z_t, z_a) \in N^f$, $j \circ x_t$ is the K-point $f \mapsto f_0(z_t, z_a) = f(t)$, i.e. ev_t. On Γ^r let j send the open point $x_t = \operatorname{Spec} K$ ($t \notin N^f$) to ev_t. Since $d\,\mathrm{Map}(N_t, R) \subseteq \mu(N_t)$ and $\mathrm{Ord}(P[p]) \otimes K = \mathrm{Map}(N_a, K)$ ($\mathcal{O}(P[p]) \otimes K = \mathcal{O}(P[p]_K)$), $\mathcal{A} \otimes_R K = \mathrm{Map}(N, K)$; so the generic fibre of $\operatorname{Spec}\mathcal{A}$, the open subset where π_R is invertible, is the disjoint union of the points ev_t, $t \in N$, each open with residue field K. For $t \notin N^f$, $t = e(z_t, z_a, z_e)$ with $z_e \neq 0$, so $\mathrm{ev}_t \in V_{z_e}$ is disjoint from $V_0 = j(\Gamma^f)$, and $x_t \to \{\mathrm{ev}_t\}$ is an isomorphism onto an open subscheme. Thus j is an R-morphism and an open immersion (a disjoint union of open immersions with pairwise disjoint open images), with $j \circ x_t = \mathrm{ev}_t$ for all $t \in N$.

(c) Covering translates. For $w_e \in N_e$ and $t = e(0, 0, w_e)$, $(\varphi_t f)_{z_e} = f_{z_e + w_e}$, so τ_t maps V_{z_e} onto $V_{z_e + w_e}$ (a prime $\{f : f_{z_e} \in Q\}$ of V_{z_e} goes to φ_t^{-1} of it, $\{f : f_{z_e + w_e} \in Q\}$); hence $\tau_t(V_0) = V_{w_e}$, and the translates $\tau_t(V_0)$, $t \in e(0, 0, N_e)$, are all the V_{z_e}, and they cover $\operatorname{Spec}\mathcal{A}$.

By Lemma D.100(1), $\mathrm{Ord}(G[p]) = \mathcal{A}$, which is (3). □

Lemma D.103. *Let R be a discrete valuation ring with fraction field K, p a prime, P an abelian scheme over R, X a free abelian group of finite rank, T_X the split torus over R with character group X, and $0 \to T_X \to \widetilde{G} \xrightarrow{\varpi} P \to 0$ the extension defined by a homomorphism $c\colon X \to P^\vee(R)$ (for $\chi \in X$, the push-out along $\chi\colon T_X \to \mathbb{G}_m$ is the extension of P by $\mathbb{G}_m$ with class $c(\chi)$). Suppose that $\widetilde{G}[p]_K$ is a constant group scheme. Then there is a homomorphism*

$c'\colon X \to P^\vee(R)$ *with* $c = pc'$*, and an isomorphism of group schemes over* R

$$\widetilde{G}[p] \;\cong\; T_X[p] \times_R P[p]$$

which restricts to the identity on $T_X[p]$ *and under which* $\varpi|_{\widetilde{G}[p]}$ *is the second projection. In particular there is a homomorphism of group schemes* $s\colon P[p] \to \widetilde{G}[p]$ *with* $\varpi \circ s = \mathrm{id}$.

Proof. Step 1: $c(\chi) \in pP^\vee(R)$ *for every* $\chi \in X$. Since $[p]$ is an fppf epimorphism on T_X and on P, the snake lemma gives an exact sequence $0 \to T_X[p] \to \widetilde{G}[p] \to P[p] \to 0$. Over K the three terms are constant ($\widetilde{G}[p]_K$ is, hence so are its subgroup and quotient), so $\widetilde{G}[p](K) \to P[p](K)$ is a surjection of $\mathbb{F}_p$-vector spaces and has a linear section, which defines a homomorphism of constant group schemes $\sigma\colon P[p]_K \to \widetilde{G}[p]_K$ with $\varpi\sigma = \mathrm{id}$. Fix $\chi \in X$ and let $u\colon \widetilde{G} \to E_\chi$ be the push-out along χ, an extension $0 \to \mathbb{G}_m \to E_\chi \to P \to 0$ of class $c(\chi)$. Then $u \circ \sigma$ splits the restriction $E_\chi|_{P[p]} = E_\chi \times_P P[p]$ of E_χ to $P[p]$ over K. Apply $\mathrm{Hom}_K(-, \mathbb{G}_m)$ to $0 \to P[p]_K \to P_K \xrightarrow{p} P_K \to 0$: the exact sequence

$$\mathrm{Ext}^1_K(P_K, \mathbb{G}_m) \xrightarrow{[p]^*} \mathrm{Ext}^1_K(P_K, \mathbb{G}_m) \xrightarrow{\mathrm{res}} \mathrm{Ext}^1_K(P[p]_K, \mathbb{G}_m)$$

with $[p]^*$ multiplication by p shows that $c(\chi)_K \in pP^\vee(K)$. As $P^\vee$ is proper and separated over the discrete valuation ring R, $P^\vee(R) \to P^\vee(K)$ is bijective, so $c(\chi) = p\, d_\chi$ with $d_\chi \in P^\vee(R)$.

Step 2: c' *and a second description of* $\widetilde{G}$. Choose a basis $\chi_1, \dots, \chi_r$ of X and let $c'\colon X \to P^\vee(R)$ be the homomorphism with $c'(\chi_i) = d_{\chi_i}$; then $pc' = c$ on the basis, hence on X. Let $\widetilde{G}'$ be the extension of P by T_X defined by c', and $\widetilde{G}'' = \widetilde{G}' \times_{P,[p]} P$ its pull-back along $[p]_P$, an extension of P by T_X with class $[p]^*c' = pc' = c$. Two extensions with the same class are isomorphic, and the isomorphism is unique because $\mathrm{Hom}_R(P, T_X) = 0$; so $\widetilde{G} \cong \widetilde{G}''$ as extensions.

Step 3: the p*-torsion of* $\widetilde{G}''$. A point of $\widetilde{G}''$ (with values in any R-scheme) is a pair (g', z) with $g' \in \widetilde{G}'$, $z \in P$, $\varpi'(g') = pz$; its image in P is z, and T_X embeds as $t \mapsto (t, 0)$. Such a pair is killed by p if and only if $pg' = 0$ and $pz = 0$; then $\varpi'(g') = pz = 0$, so $g' \in T_X$, and $pg' = 0$ gives $g' \in T_X[p]$. Conversely every (t, z) with $t \in T_X[p]$, $z \in P[p]$ is a point of $\widetilde{G}''[p]$ ($\varpi'(t) = 0 = pz$). Hence $(t, z) \mapsto (t, z)$ is an isomorphism of group schemes $T_X[p] \times_R P[p] \to \widetilde{G}''[p]$, the identity on $T_X[p]$ and compatible with the projections to $P[p]$. Transporting along $\widetilde{G} \cong \widetilde{G}''$ gives the assertion, and $s(z) = (0, z)$. □

Lemma D.104. *Let* p *be a prime,* C *and* $\mathcal{O}_C$ *as in Definition 35.18, and* $\zeta \in \mathcal{O}_C$ *a primitive* p*-th root of unity. There are an elliptic curve* E *over* $\mathcal{O}_C$ *(a principally polarised abelian scheme of relative dimension one) and a group isomorphism* $x\colon \mathbb{Z}/p \times \mathbb{Z}/p \to N_E$ *such that* $\{f \circ x :\ f \in M_E\}$ *is the* $\mathcal{O}_C$*-span of the functions* $(a, b) \mapsto \zeta^{ka}\, c'(b)$*,* $k \in \mathbb{Z}/p$*,* $c' \in \mathrm{Map}(\mathbb{Z}/p, \mathcal{O}_C)$.

Proof. Step 1: an ordinary curve. Since $\lfloor p/12 \rfloor + 2 < p$ for $p \geq 5$, and $j = 1 \neq 0$ for $p = 2, 3$, there is $j_0 \in \mathbb{F}_p$ which is not supersingular; let E_0 be an elliptic curve over $\mathbb{F}_p$ with $j(E_0) = j_0$, given by a Weierstrass equation. Lifting its coefficients to $\mathbb{Z}_p$ gives a Weierstrass equation whose discriminant is a unit, i.e. an elliptic curve E_1 over $\mathbb{Z}_p$ with special fibre E_0. As $\mathcal{O}_C$ is complete with $|p| < 1$, it is a $\mathbb{Z}_p$-algebra; put $E = E_1 \otimes_{\mathbb{Z}_p} \mathcal{O}_C$, an elliptic curve over $\mathcal{O}_C$, principally polarised by its origin.

Step 2: the connected–étale sequence. $E[p]$ is finite flat of order p^2 over $\mathcal{O}_C$, a henselian local ring with algebraically closed residue field k (as C is algebraically closed). In the sequence (ii), $E[p]^{\mathrm{et}}$ has order $|E_0[p](k)| = p$ (ordinary), so it is the constant group $\mathbb{Z}/p$, and $E[p]^0$ has order p. By (iii), Cartier duality turns the sequence into $0 \to (\mathbb{Z}/p)^D \to E[p] \to (E[p]^0)^D \to 0$; $(\mathbb{Z}/p)^D = \mu_p$ is connected (its special fibre is infinitesimal), so $\mu_p \subseteq E[p]^0$, and both have order p: $E[p]^0 = \mu_p$.

Step 3: the splitting. $E[p](\mathcal{O}_C) = E[p](C)$ has p^2 elements (valuative criterion, $E[p]$ finite), and $\mu_p(\mathcal{O}_C)$ has p elements; so the image of $E[p](\mathcal{O}_C) \to (\mathbb{Z}/p)(\mathcal{O}_C) = \mathbb{Z}/p$ has p elements and contains 1. A point $Q \in E[p](\mathcal{O}_C)$ over 1 defines a homomorphism $\mathbb{Z}/p \to E[p]$, $1 \mapsto Q$, splitting the sequence: $E[p] \cong \mu_p \times \mathbb{Z}/p$.

Step 4: the function module. Hence $\Gamma(E[p], \mathcal{O}) \cong \mathcal{O}_C[t]/(t^p - 1) \otimes_{\mathcal{O}_C} \mathrm{Map}(\mathbb{Z}/p, \mathcal{O}_C)$, and with $x(a, b) = (\zeta^a, b)$ the evaluation sends $t^k \otimes c'$ to $(a, b) \mapsto \zeta^{ka} c'(b)$. So $\{f \circ x : \ f \in M_E\}$ is the stated span, and x is a group isomorphism onto $N_E = \mu_p(\mathcal{O}_C) \times \mathbb{Z}/p$. □

Lemma D.105. *Let C and $\mathcal{O}_C$ be as in Definition 35.18 and P a principally polarised abelian scheme over $\mathcal{O}_C$. Then M_P is a tight Hopf order on N_P.*

Proof. Let $B = M_P$ and $N = N_P$. *B is a Hopf order on N.* $\Gamma(P[p], \mathcal{O})$ is free of rank p^{2g} over the local ring $\mathcal{O}_C$ and its generic fibre is the constant group scheme on N, so evaluation at the points is injective and identifies $\Gamma(P[p], \mathcal{O}) \otimes C$ with $\mathrm{Map}(N, C)$: B is spanned by the columns of a matrix in $\mathrm{GL}_N(C)$. It contains 1 and is closed under products, and the group law and the inverse of $P[p]$ give the comultiplication and antipode axioms of (Ho2).

The four quantities. The augmentation ideal I_B is the ideal of the unit section, so $I_B/I_B^2 = \omega_{P[p]} \cong (\mathcal{O}_C/p)^g$ has normalised length g. By Lemma 35.47, Q_B has a normalised length and $2\,\mathrm{colen}(B) \le (p-1)|N|\,\mathrm{len}_C(Q_B)$; by Proposition 35.54, $(p-1)\mathrm{len}_C(Q_B) \le g$; by Lemma 35.49, $|N|\,\mathrm{len}_C(I_B/I_B^2) \le 2\,\mathrm{colen}(B)$. Hence

$$2\,\mathrm{colen}(B) \le (p-1)|N|\,\mathrm{len}_C(Q_B) \le |N|\,g = |N|\,\mathrm{len}_C(I_B/I_B^2) \le 2\,\mathrm{colen}(B),$$

all are equalities, and the two identities of (Ho3) hold: B is tight. □

Lemma D.106. *Let N_1, N_2 be finite abelian groups with $pN_1 = pN_2 = 0$, B_i a Hopf order on N_i ($i = 1, 2$), and $B_1 \boxtimes B_2 \subseteq \mathrm{Map}(N_1 \times N_2, C)$ the $\mathcal{O}_C$-span of the functions $(x_1, x_2) \mapsto f_1(x_1) f_2(x_2)$, $f_i \in B_i$. Then $B = B_1 \boxtimes B_2$ is a Hopf order on $N = N_1 \times N_2$,*

$$\mathrm{colen}(B) = |N_2|\,\mathrm{colen}(B_1) + |N_1|\,\mathrm{colen}(B_2),$$
$$\mathrm{len}_C(I_B/I_B^2) = \sum_i \mathrm{len}_C(I_{B_i}/I_{B_i}^2), \qquad \mathrm{len}_C(Q_B) = \sum_i \mathrm{len}_C(Q_{B_i}),$$

and if B is tight, then B_1 and B_2 are tight.

Proof. Step 1: B is a Hopf order and $B \cong B_1 \otimes_{\mathcal{O}_C} B_2$. Let $E_i \in \mathrm{GL}_{N_i}(C)$ have columns $f_{i,1}, \dots$ spanning B_i; the functions $f_{1,k} \boxtimes f_{2,l}$ are the columns of the Kronecker product $E_1 \otimes E_2 \in \mathrm{GL}_N(C)$, so they are linearly independent and span B, and $f_1 \otimes f_2 \mapsto f_1 \boxtimes f_2$ is an isomorphism $B_1 \otimes B_2 \to B$. B contains $1 = 1 \boxtimes 1$, is closed under products (products of pure tensors are pure tensors), and for $g = f_1 \boxtimes f_2$ the function $((x_1, x_2), (y_1, y_2)) \mapsto f_1(x_1 + y_1) f_2(x_2 + y_2)$ is a sum of products $(g_1 \boxtimes g_2)(x_1, x_2)\,(g_1' \boxtimes g_2')(y_1, y_2)$ by the comultiplication axiom for B_1 and B_2; $x \mapsto g(-x)$ is $f_1(-\cdot) \boxtimes f_2(-\cdot) \in B$. Since $\det(E_1 \otimes E_2) = \det(E_1)^{|N_2|} \det(E_2)^{|N_1|}$, the colength formula holds.

Step 2: the cotangent modules. Each $f \in B_i$ is integral over $\mathcal{O}_C$ (B_i is a finite $\mathcal{O}_C$-algebra), so its values are integral over $\mathcal{O}_C$ and lie in $\mathcal{O}_C$; hence $B_i = \mathcal{O}_C 1 \oplus I_i$ with $I_i = I_{B_i}$ ($f = f(0)1 + (f - f(0)1)$). By Step 1,

$$B = \mathcal{O}_C \oplus (I_1 \boxtimes 1) \oplus (1 \boxtimes I_2) \oplus (I_1 \boxtimes I_2), \qquad I_B = (I_1 \boxtimes 1) \oplus (1 \boxtimes I_2) \oplus (I_1 \boxtimes I_2).$$

Multiplying the summands pairwise, $I_B^2 = (I_1^2 \boxtimes 1) \oplus (1 \boxtimes I_2^2) \oplus (I_1 \boxtimes I_2)$ (the cross products $(a \boxtimes 1)(1 \boxtimes b) = a \boxtimes b$ give all of $I_1 \boxtimes I_2$, and every product involving $I_1 \boxtimes I_2$ lies in it). Hence $I_B/I_B^2 \cong I_1/I_1^2 \oplus I_2/I_2^2$ and the normalised lengths add.

Step 3: the almost-additive cokernels. For $f\colon N \to C$ almost additive, $f(x_1,x_2) - f(x_1,0) - f(0,x_2) \in p\mathcal{O}_C$, $f(\cdot,0)$ and $f(0,\cdot)$ are almost additive, and $f(0) \in p\mathcal{O}_C$ (from $x = y = 0$). So $\mathrm{Lin}(N) = \mathrm{Lin}(N_1)\circ\mathrm{pr}_1 + \mathrm{Lin}(N_2)\circ\mathrm{pr}_2 + p\,\mathrm{Map}(N,\mathcal{O}_C)$. If moreover $f \in B$, write $f = \sum_k f_{1,k} \boxtimes f_{2,k}$; then $f(\cdot,0) = \sum_k f_{2,k}(0) f_{1,k} \in B_1$ and likewise $f(0,\cdot) \in B_2$; conversely $a_1\circ\mathrm{pr}_1 = a_1 \boxtimes 1 \in B$. Hence $\mathrm{AF}(B) + p\,\mathrm{Map}(N,\mathcal{O}_C) = \mathrm{AF}(B_1)\circ\mathrm{pr}_1 + \mathrm{AF}(B_2)\circ\mathrm{pr}_2 + p\,\mathrm{Map}(N,\mathcal{O}_C)$. The map $\mathrm{Lin}(N_1)\oplus\mathrm{Lin}(N_2) \to Q_B$, $(f_1,f_2) \mapsto f_1\circ\mathrm{pr}_1 + f_2\circ\mathrm{pr}_2$, is therefore onto; if $f_1\circ\mathrm{pr}_1 + f_2\circ\mathrm{pr}_2 = a_1\circ\mathrm{pr}_1 + a_2\circ\mathrm{pr}_2 + ph$ with $a_i \in \mathrm{AF}(B_i)$, evaluation at $(x_1,0)$ gives $f_1 - a_1 \in p\,\mathrm{Map}(N_1,\mathcal{O}_C)$ (as $f_2(0), a_2(0) \in p\mathcal{O}_C$), and symmetrically. So $Q_B \cong Q_{B_1} \oplus Q_{B_2}$ and the normalised lengths add.

Step 4: tightness passes to the factors. By Lemma 35.47, $2\,\mathrm{colen}(B_i) \le (p-1)|N_i|\,\mathrm{len}_C(Q_{B_i})$; multiplying by $|N_{3-i}|$ and adding, with Steps 1 and 3, $2\,\mathrm{colen}(B) \le (p-1)|N|\,\mathrm{len}_C(Q_B)$, with equality only if both inequalities are equalities. By Lemma 35.49, $|N_i|\,\mathrm{len}_C(I_{B_i}/I_{B_i}^2) \le 2\,\mathrm{colen}(B_i)$; multiplying by $|N_{3-i}|$ and adding, with Steps 1 and 2, $|N|\,\mathrm{len}_C(I_B/I_B^2) \le 2\,\mathrm{colen}(B)$, with equality only if both are equalities. If B is tight, both outer relations are equalities, hence so are the four inequalities for B_1, B_2, which are the identities of (Ho3) for B_1 and B_2. □

Lemma D.107. *Let $\mathfrak{T}$ be a torsion-regular pre-admissible torsion system for $(\mathcal{X},p,\lambda)$ as in Proposition D.97, $(\mathfrak{q},C,\iota_C)$ a test point, u a label and $\mathfrak{P} = \mathfrak{P}_u$. Then $\theta_{u,C}(gh) = 0$ for $g,h \in \mathcal{I}_{\mathfrak{P},C}$, and $\theta_{u,C}(x \mapsto g(ax)) = \chi_u(a)\,\theta_{u,C}(g)$ for $a \in \mathbb{F}_{\mathfrak{P}}$ and $g \in \mathcal{I}_{\mathfrak{P},C}$.*

Proof. The prime p is unramified in E ($p \notin \mathrm{Bd}$). Let $\widehat{W}$ be the completion of $\mathcal{O}_{E,(\mathfrak{P})}$ (maximal ideal $p\widehat{W}$), $\omega\colon \mathbb{F}_{\mathfrak{P}} \to \widehat{W}$ the Teichmüller map and $\hat{\sigma}_u\colon \widehat{W} \to \widehat{\mathcal{O}'}$ the continuous extension of σ_u ($\sigma_u^{-1}(\mathfrak{m}_{\mathcal{O}'}) = \mathfrak{P}$). Then $\hat{\sigma}_u \circ \omega = \chi_u$: both vanish at 0, take values in the roots of unity of order dividing $|\mathbb{F}_{\mathfrak{P}}| - 1$ (which lie in $\mathcal{O}'$ by (Ts1)) elsewhere, and agree modulo $\mathfrak{m}_{\mathcal{O}'}$, where reduction is injective on such roots of unity. Both identities are $\mathcal{O}_C$-(bi)linear in g,h, and $\mathcal{I}_{\mathfrak{P},C}$ is the $\mathcal{O}_C$-span of the stalk $\mathcal{I}_{\mathfrak{P},\mathfrak{q}}$ (Ox2); so it suffices to prove them for g,h in $\mathcal{I}_{\mathfrak{P}}(U)$, U an affine open neighbourhood of $\mathfrak{q}$, with θ_u in place of $\theta_{u,C}$. Let $\tilde{g},\tilde{h}$ be regular functions near the unit section e of $G'[\mathfrak{P}]\times U$ with values g,h, as in (Hf1).

Products. $\tilde{g}\tilde{h}$ has values gh, and $d(\tilde{g}\tilde{h}) = \tilde{g}\,d\tilde{h} + \tilde{h}\,d\tilde{g}$. The function $\tilde{g}\circ e$ on U has value $g(0) = 0$ at the generic point, so it is 0 ($\mathcal{O}(U) \subseteq \Omega'$); likewise $\tilde{h}\circ e = 0$. Hence $e^*d(\tilde{g}\tilde{h}) = 0$ and $\theta_u(gh) = 0$.

Equivariance. Let $a \in \mathbb{F}_{\mathfrak{P}}$. As $\mathcal{O}_E$ is dense in $\widehat{W}$, choose $\tilde{a} \in \mathcal{O}_E$ with $\tilde{a} - \omega(a) \in p\widehat{W}$; then $\tilde{a} \equiv a$ modulo $\mathfrak{P}$, so the endomorphism $[\tilde{a}]$ of $G'[\mathfrak{P}]$ acts on $\Gamma_{\mathfrak{P}}$ as multiplication by a, and $\sigma_u(\tilde{a}) - \chi_u(a) = \hat{\sigma}_u(\tilde{a} - \omega(a)) \in p\widehat{\mathcal{O}'} \cap \mathcal{O}' = p\,\mathcal{O}'$. The function $[\tilde{a}]^*\tilde{g}$ is regular near e ($[\tilde{a}]\circ e = e$) with values $x \mapsto g(ax)$, and $e^*d([\tilde{a}]^*\tilde{g}) = [\tilde{a}]^*(e^*d\tilde{g})$, where $[\tilde{a}]^*$ on $e^*\Omega^1_{G'[\mathfrak{P}]/\mathcal{X}'} = \omega_{G'}/\mathfrak{P}\,\omega_{G'}$ is induced by the action of $\tilde{a}$ on $\omega_{G'}$, which is multiplication by $\sigma_u(\tilde{a})$ on ω_u. Taking σ_u-isotypic components, $\theta_u(g(a\,\cdot)) = \sigma_u(\tilde{a})\,\theta_u(g) = \chi_u(a)\,\theta_u(g)$ in $\omega_u/p\,\omega_u$. □

Lemma D.108. *Let $\mathfrak{T}$, $(\mathfrak{q},C,\iota_C)$ be as in Proposition D.109, u a label and $\mathfrak{P} = \mathfrak{P}_u$. Every $g \in \mathcal{I}_{\mathfrak{P},C}$ with $g(ax) = \chi_u(a)g(x)$ for all $a \in \mathbb{F}_{\mathfrak{P}}$, $x \in \Gamma_{\mathfrak{P}}$, and $\theta_{u,C}(g) = 0$ lies in the $\mathcal{O}_C$-span of the products of two elements of $\mathcal{I}_{\mathfrak{P},C}$.*

Proof. Let u' be a label with $\mathfrak{P}_{u'} = \mathfrak{P}$, $u' \neq u$. The embeddings σ with $\sigma^{-1}(\mathfrak{m}_{\mathcal{O}'}) = \mathfrak{P}$ number $[E_{\mathfrak{P}} : \mathbb{Q}_p] = f = [\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_p]$ (p is unramified in E), and $\sigma_u \circ \mathrm{Frob}_p^k$, $0 \le k < f$, are f distinct such embeddings; so $\sigma_{u'} = \sigma_u \circ \mathrm{Frob}_p^k$ with $0 < k < f$, and $\chi_{u'} = \chi_u^{p^k}$ (reduce modulo $\mathfrak{m}_{\mathcal{O}'}$, as in Definition 35.7). As χ_u is injective on $\mathbb{F}_{\mathfrak{P}}^\times$ (its reduction is $a \mapsto \sigma_u(\tilde{a})$ modulo $\mathfrak{m}_{\mathcal{O}'}$, an embedding of fields), and $a \mapsto a^{p^k}$ is not the identity of $\mathbb{F}_{\mathfrak{P}}$, there is $a \in \mathbb{F}_{\mathfrak{P}}^\times$ with $\chi_{u'}(a) \neq \chi_u(a)$; both are roots of unity of order prime to p, distinct modulo $\mathfrak{m}_{\mathcal{O}'}$, so $\chi_u(a) - \chi_{u'}(a)$ is a unit. By

Lemma D.107 and the eigen property,

$$\chi_u(a)\,\theta_{u',C}(g) = \theta_{u',C}(x \mapsto g(ax)) = \chi_{u'}(a)\,\theta_{u',C}(g),$$

hence $\theta_{u',C}(g) = 0$. With $\theta_{u,C}(g) = 0$, the hypothesis of Proposition D.109 holds, and g lies in the span of products. □

Proposition D.109. *Let $\mathfrak{T}$ be a torsion-regular pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$ as in Proposition D.97, $(\mathfrak{q}, C, \iota_C)$ a test point and $\mathfrak{P} \mid p$ a prime of $\mathcal{O}_E$. If $g \in \mathcal{I}_{\mathfrak{P},C}$ and $\theta_{u,C}(g) = 0$ for every label u with $\mathfrak{P}_u = \mathfrak{P}$, then g lies in the $\mathcal{O}_C$-span of the products of two elements of $\mathcal{I}_{\mathfrak{P},C}$.*

Proof. As in the setting of Proposition D.97, p is unramified in E. Let $U_{\mathfrak{P}}$ be the set of labels u with $\mathfrak{P}_u = \mathfrak{P}$, $q = q_{\mathfrak{P}}$, $N = \Gamma_{\mathfrak{P}}$, and $R = R_{\mathfrak{q}}$: a discrete valuation ring ($\mathcal{X}'$ is normal and $\mathfrak{q}$ has codimension one) whose maximal ideal contains p, with fraction field Ω'.

Step 0: ranks. By (Ud6) F contains $\tau(E)$ for every embedding τ, and $F \subseteq F'$. The map $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{O}' \to \prod_{\sigma \in \mathrm{Hom}(E,F)} \mathcal{O}'$, $a \otimes x \mapsto (\sigma(a)x)_\sigma$, is a map of free $\mathcal{O}'$-modules of rank $[E : \mathbb{Q}]$ whose determinant (in an integral basis of $\mathcal{O}_E$) has square the discriminant of E, a unit in $\mathcal{O}'$ because p is unramified in E and every other prime is invertible in $\mathcal{O}'$; so it is an isomorphism. Hence the locally free $\mathcal{O}_E \otimes \mathcal{O}_{\mathcal{X}'}$-module $\omega_{G'} = \pi^*\omega_{G_{\mathcal{X}}}$ is the direct sum of its eigensubsheaves ω_u over all labels u (the labels correspond bijectively to $\mathrm{Hom}(E, F)$), and each ω_u is locally free. Put $d_{\mathfrak{P}} = \sum_{u \in U_{\mathfrak{P}}} \mathrm{rk}\,\omega_u$. By (Hf1), $\omega_{G'}/\mathfrak{P}\omega_{G'} = \bigoplus_{u \in U_{\mathfrak{P}}} \omega_u/p\,\omega_u$, so its stalk at $\mathfrak{q}$ is a free R/pR-module of rank $d_{\mathfrak{P}}$.

Step 1: the order commutes with flat base change. Let $S \to S'$ be a flat homomorphism of noetherian domains with fraction fields L, L' such that $L \otimes_S S' = L'$, and let H over S, the finite group T and the points $x_t \in H(L)$ be as in Definition 35.21; then $H_{S'}$ satisfies the same hypotheses with the same T and points. For an open $V \subseteq H$ the map $\mathcal{O}(V) \to \mathcal{O}(V_L)$ is injective (H is flat over the domain S), and V_L is the disjoint union of the points $x_t \in V$, each $\mathrm{Spec}\,L$; so sections over V are determined by their values at these points, local sections with prescribed values glue, and $f\colon T \to S$ is regular on H if and only if $f \in \mathrm{Reg}(H) = \mathrm{ev}(\mathcal{O}(H)) \cap \mathrm{Map}(T, S)$, where $\mathrm{ev}\colon \mathcal{O}(H) \to \mathrm{Map}(T, L)$ is the injective evaluation map. Flat base change gives $\mathcal{O}(H_{S'}) = \mathcal{O}(H) \otimes_S S'$ (H is quasi-compact and separated), and $\mathrm{Map}(T, L) \otimes_S S' = \mathrm{Map}(T, L')$. As S' is flat, intersections of submodules of $\mathrm{Map}(T, L)$ commute with $- \otimes_S S'$, so $\mathrm{Reg}(H_{S'}) = S'\,\mathrm{Reg}(H)$ (the S'-span in $\mathrm{Map}(T, L')$). The set of f all of whose translates are regular is the finite intersection $\bigcap_{t'} \tau_{t'}^{-1}\mathrm{Reg}(H)$, $\tau_{t'} f = f(t' + \cdot)$; it is an S-subalgebra (sums, products and translates of regular functions are regular, constants are regular), hence equal to $\mathrm{Ord}(H)$, and the same argument gives $\mathrm{Ord}(H_{S'}) = S'\,\mathrm{Ord}(H) \cong S' \otimes_S \mathrm{Ord}(H)$.

Let U be an affine open neighbourhood of $\mathfrak{q}$. By (Hf1), $\mathcal{A}_{\mathfrak{P}}(U) = \mathrm{Ord}(G'[\mathfrak{P}] \times_{\mathcal{X}'} U)$, and $\mathcal{A}_{\mathfrak{P}}$ is quasi-coherent (locally free by (Ts3)); by Step 1 for $\mathcal{O}(U) \to R$, $\mathcal{A}_{\mathfrak{P},\mathfrak{q}} = \mathrm{Ord}(G'[\mathfrak{P}]_R)$. Let $\widehat{R}$ be the completion of R, a complete discrete valuation ring with fraction field $\widehat{K} = \Omega' \otimes_R \widehat{R}$, faithfully flat over R. By Step 1, $\mathcal{O} := \mathrm{Ord}(\Gamma) \cong \widehat{R} \otimes_R \mathcal{A}_{\mathfrak{P},\mathfrak{q}}$ for $\Gamma = G'[\mathfrak{P}]_{\widehat{R}}$.

Step 2: the order over $\widehat{R}$ and its cotangent module. By (Fh5) the generic point of $\mathcal{X}'$ (a cover through $f \circ \pi$) maps to a point of $j(\mathrm{Sh}_F)$, over which $G_{\mathcal{Y}}$ is abelian (Ud7); so $G'_{\widehat{R}}$ is a semiabelian scheme with abelian generic fibre and $\mathcal{O}_E$-action over the complete noetherian normal local domain $\widehat{R}$, whose maximal ideal contains p. By Lemma 35.30, Γ is quasi-finite and flat, its finite part $W = \Gamma^f$ is a finite locally free subgroup scheme, and $\omega_W \cong \omega_{G'_{\widehat{R}}}/\mathfrak{P}\,\omega_{G'_{\widehat{R}}} = (\omega_{G'}/\mathfrak{P}\omega_{G'})_{\mathfrak{q}} \otimes_R \widehat{R}$, free of rank $d_{\mathfrak{P}}$ over $\widehat{R}/p\widehat{R}$ (Step 0). By definition of the finite part over the henselian ring $\widehat{R}$ (EGA IV$_4$ 18.5.11; Stacks Project, Tag 04GG), $\Gamma = W \sqcup \Gamma'$ with Γ' open and closed and without points over the closed point. As $\mathrm{Spec}\,\widehat{R}$ has two points, Γ' is an open

subscheme of $\Gamma_{\widehat{K}}$, the constant scheme with points x_t, $t \in N$ (Ts3). Let $N^f = \{t : \ x_t \in W\}$, a subgroup, and $B_W = \mathcal{O}(W)$, finite free over $\widehat{R}$.

1. Each x_t, $t \in N^f$, extends to an $\widehat{R}$-point y_t of W: the image of B_W under x_t^* is integral over the normal ring $\widehat{R}$. So evaluation $\rho\colon B_W \to \mathrm{Map}(N^f, \widehat{R})$ is defined, injective (as in Step 1), and $\rho(B_W)$ is stable under $b \mapsto b(t' + \cdot)$ for $t' \in N^f$ (translation by $y_{t'}$ is an automorphism of W).

2. A function $f\colon N \to \widehat{R}$ is regular on Γ if and only if $f|_{N^f} \in \rho(B_W)$: near points of W use W itself (open) and gluing as in Step 1; the points of Γ' are open points $\operatorname{Spec}\widehat{K}$, on which every value is regular.

3. Hence $\mathcal{O} = \{f : \ f(t' + \cdot)|_{N^f} \in \rho(B_W) \text{ for all } t' \in N\}$. With coset representatives $t_1 = 0, t_2, \ldots, t_k$ of N^f in N, $f \mapsto (f(t_j + \cdot)|_{N^f})_j$ is an isomorphism of $\widehat{R}$-algebras $\mathcal{O} \to \rho(B_W)^k$ (surjective: by (1) the condition for $t' \in t_j + N^f$ follows from that for t_j), under which $f \mapsto f(0)$ is evaluation at 0 in the first factor.

4. So $I_{\mathcal{O}} = \{f \in \mathcal{O} : \ f(0) = 0\}$ corresponds to $I_W \times B_W^{k-1}$, I_W the augmentation ideal of the unit section, and $I_{\mathcal{O}}^2$ to $I_W^2 \times B_W^{k-1}$ (each factor $j \geq 2$ contains its unit idempotent, which lies in $I_{\mathcal{O}}$). Therefore $I_{\mathcal{O}}/I_{\mathcal{O}}^2 \cong I_W/I_W^2 \cong e^*\Omega^1_{W/\widehat{R}} = \omega_W \cong (\widehat{R}/p\widehat{R})^{d_{\mathfrak{P}}}$.

Step 3: descent to R and base change to $\mathcal{O}_C$. Write $\mathcal{A}_{\mathfrak{P},\mathfrak{q}} = R \cdot 1 \oplus \mathcal{I}_{\mathfrak{P},\mathfrak{q}}$ and $M = \mathcal{I}_{\mathfrak{P},\mathfrak{q}}/\mathcal{I}^2_{\mathfrak{P},\mathfrak{q}}$. Under $\mathcal{O} \cong \widehat{R} \otimes_R \mathcal{A}_{\mathfrak{P},\mathfrak{q}}$, $I_{\mathcal{O}} = \widehat{R} \otimes \mathcal{I}_{\mathfrak{P},\mathfrak{q}}$ and $I_{\mathcal{O}}^2$ is the image of $\widehat{R} \otimes \mathcal{I}^2_{\mathfrak{P},\mathfrak{q}}$; by right exactness $\widehat{R} \otimes_R M \cong (\widehat{R}/p\widehat{R})^{d_{\mathfrak{P}}}$. By faithful flatness $pM = 0$, and as $R/pR \to \widehat{R}/p\widehat{R}$ is an isomorphism, $M \cong \widehat{R} \otimes_R M \cong (R/pR)^{d_{\mathfrak{P}}}$. By (Ox2), $\mathcal{A}_{\mathfrak{P},C} = \mathcal{O}_C \otimes_R \mathcal{A}_{\mathfrak{P},\mathfrak{q}}$ and $\mathcal{I}_{\mathfrak{P},C} = \mathcal{O}_C \otimes_R \mathcal{I}_{\mathfrak{P},\mathfrak{q}}$; the span of products is the image of $\mathcal{O}_C \otimes \mathcal{I}^2_{\mathfrak{P},\mathfrak{q}}$, so $\mathcal{I}_{\mathfrak{P},C}/\mathcal{I}^2_{\mathfrak{P},C} \cong \mathcal{O}_C \otimes_R M \cong (\mathcal{O}_C/p\mathcal{O}_C)^{d_{\mathfrak{P}}}$, where $\mathcal{I}^2_{\mathfrak{P},C}$ denotes the $\mathcal{O}_C$-span of the products of two elements of $\mathcal{I}_{\mathfrak{P},C}$.

Step 4: the cotangent maps at the labels over $\mathfrak{P}$ are jointly onto. Let $\Omega_{\mathfrak{P}} = \bigoplus_{u \in U_{\mathfrak{P}}} \mathcal{O}_C \otimes_R (\omega_u/p\,\omega_u)_{\mathfrak{q}} \cong (\mathcal{O}_C/p\mathcal{O}_C)^{d_{\mathfrak{P}}}$ (Step 0) and $\theta_{\mathfrak{P}} = (\theta_{u,C})_{u \in U_{\mathfrak{P}}}\colon \mathcal{I}_{\mathfrak{P},C} \to \Omega_{\mathfrak{P}}$. By Lemma D.107, $\theta_{\mathfrak{P}}$ vanishes on $\mathcal{I}^2_{\mathfrak{P},C}$. Each $\theta_{u,C}$ is onto: θ_u is onto by (Ts3), hence so is its stalk at $\mathfrak{q}$, and $\mathcal{O}_C \otimes_R -$ is right exact. Let $u \neq u'$ in $U_{\mathfrak{P}}$. The embeddings σ with $\sigma^{-1}(\mathfrak{m}_{\mathcal{O}'}) = \mathfrak{P}$ number $[E_{\mathfrak{P}} : \mathbb{Q}_p] = f = [\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_p]$ (p unramified), and $\sigma_u \circ \mathrm{Frob}_p^k$, $0 \leq k < f$, are f distinct ones, so $\sigma_{u'} = \sigma_u \circ \mathrm{Frob}_p^k$ with $0 < k < f$ and $\chi_{u'} = \chi_u^{p^k}$; as χ_u is injective on $\mathbb{F}_{\mathfrak{P}}^\times$ (its reduction is a field embedding) and $a \mapsto a^{p^k}$ is not the identity of $\mathbb{F}_{\mathfrak{P}}$, the homomorphism $\psi = \chi_u^{-1}\chi_{u'}\colon \mathbb{F}_{\mathfrak{P}}^\times \to \mathcal{O}'^\times$ is non-trivial, and $\sum_{a \in \mathbb{F}_{\mathfrak{P}}^\times} \psi(a) = 0$ (multiply by $\psi(b) \neq 1$; $\mathcal{O}'$ is a domain). For $g \in \mathcal{I}_{\mathfrak{P},C}$ put

$$e_u(g)(x) = \frac{1}{q-1} \sum_{a \in \mathbb{F}_{\mathfrak{P}}^\times} \chi_u(a)^{-1} g(ax);$$

$q - 1$ is prime to p, and $e_u(g) \in \mathcal{I}_{\mathfrak{P},C}$ because $\mathcal{A}_{\mathfrak{P},C}$ is $\mathbb{F}_{\mathfrak{P}}$-stable (Proposition D.98). By Lemma D.107, $\theta_{u',C}(e_u(g)) = \frac{1}{q-1} \sum_a \chi_u(a)^{-1}\chi_{u'}(a)\, \theta_{u',C}(g)$, which is $\theta_{u,C}(g)$ for $u' = u$ and 0 for $u' \neq u$. Given $(w_u)_u \in \Omega_{\mathfrak{P}}$, choose g_u with $\theta_{u,C}(g_u) = w_u$; then $\theta_{\mathfrak{P}}\big(\sum_u e_u(g_u)\big) = (w_u)_u$.

Step 5: conclusion. By Steps 3 and 4, $\theta_{\mathfrak{P}}$ induces a surjective $\mathcal{O}_C/p\mathcal{O}_C$-linear map $\bar{\theta}$ from $\mathcal{I}_{\mathfrak{P},C}/\mathcal{I}^2_{\mathfrak{P},C}$ to $\Omega_{\mathfrak{P}}$, between free modules of the same rank $d_{\mathfrak{P}}$ over the local ring $\mathcal{O}_C/p\mathcal{O}_C$. It splits, as the target is free; its kernel K is a direct summand, hence finitely generated, and comparing dimensions after tensoring with the residue field gives $K \otimes k = 0$, so $K = 0$ by Nakayama's lemma. Hence $\bar{\theta}$ is injective: if $g \in \mathcal{I}_{\mathfrak{P},C}$ and $\theta_{u,C}(g) = 0$ for every $u \in U_{\mathfrak{P}}$, then $g \in \mathcal{I}^2_{\mathfrak{P},C}$, the $\mathcal{O}_C$-span of the products of two elements of $\mathcal{I}_{\mathfrak{P},C}$. □

D.5.2 Hodge–Tate comparison at test points

Proposition D.110 gives the Hodge–Tate facts at test points: every point of codimension one at which p is not a unit is the first entry of a test point at which the Hodge–Tate conditions of Proposition D.97 hold. The lemmas compute the Hodge–Tate and Weil matrices of the p-torsion at such a point (Lemma D.114), place the Weil pairing in the tensor square of the test-point order (Lemma D.115), and derive the flag conditions from the flag theorem for tight Hopf orders of Section 35 (Lemma D.122).

Proposition D.110. *Let $\mathcal{T}, \dots, \lambda$, $\mathfrak{T}$, R, w and, for every $a \in \Phi$, $\beta, \bar{\beta}$ and the Hodge–Tate atlas $(U_\nu, \iota_\nu, h_\nu)_\nu$ be as in the hypotheses of Proposition D.97. Then every point $\mathfrak{q}$ of $\mathcal{X}'$ of codimension one at which p is not a unit is the first entry of a test point $(\mathfrak{q}, C, \iota_C)$ such that:*

1. *for every $a \in \Phi$, with $s = (a, +)$, (F0)–(F2) hold for the $\mathcal{O}_C$-spans of the stalks at $\mathfrak{q}$ of the Hodge–Tate lattice family of a, and $\mathcal{O}_C J_{i,\mathfrak{q}} = \mathcal{O}_C\, \mathcal{O}_{\mathcal{X}'}(-\mathrm{Ec}_{s,i})_{\mathfrak{q}}$ for $1 \le i \le m_{\mathfrak{P}_s}$.*

Proof. Let $\mathfrak{q}$ be a point of $\mathcal{X}'$ of codimension one at which p is not a unit, and let $(\mathfrak{q}, C, \iota_C)$ be the test point given by Proposition D.111. For every $a \in \Phi$, with $s = (a, +)$, that proposition gives (F1), (F2) for the $\mathcal{O}_C$-spans of the stalks at $\mathfrak{q}$ of the Hodge–Tate lattice family of a, and Lemma D.122 (at every test point, with the same hypotheses) gives $\mathcal{O}_C J_{i,\mathfrak{q}} = \mathcal{O}_C\, \mathcal{O}_{\mathcal{X}'}(-\mathrm{Ec}_{s,i})_{\mathfrak{q}}$ for $1 \le i \le m_{\mathfrak{P}_s}$. (F0) for these spans, $\mathcal{O}_C H_{\mathfrak{q}} \subseteq w^{-1}\mathcal{O}_C T_{\mathfrak{q}}$, holds at every test point by Lemma D.121, whose hypotheses are those of Proposition D.97 ($\mathfrak{T}$ torsion-regular and pre-admissible, $w^{p-1} = u\,p$ with $u \in \mathcal{O}'^{\times}$). □

Proposition D.111. *Let $\mathcal{T}, \dots, \lambda$, $\mathfrak{T}$, R, w and, for every $a \in \Phi$, $\beta, \bar{\beta}$ and the Hodge–Tate atlas $(U_\nu, \iota_\nu, h_\nu)_\nu$ be as in the hypotheses of Proposition D.97. Then every point $\mathfrak{q}$ of $\mathcal{X}'$ of codimension one at which p is not a unit is the first entry of a test point $(\mathfrak{q}, C, \iota_C)$ such that for every $a \in \Phi$, with $s = (a, +)$: (F1) and (F2) hold for the $\mathcal{O}_C$-spans of the stalks at $\mathfrak{q}$ of the Hodge–Tate lattice family of a (with K, I formed from the spans of $\Gamma_{\mathfrak{q}}$, $\Delta_{\mathfrak{q}}$, $H_{\mathfrak{q}}$).*

Proof. Let $\mathfrak{q}$ be a point of $\mathcal{X}'$ of codimension one with $p \notin \mathcal{O}_{\mathfrak{q}}^{\times}$, and $R_{\mathfrak{q}} = \mathcal{O}_{\mathcal{X}',\mathfrak{q}}$.

Step 0: the test point. Let C, $\mathcal{O}_C$ be as in Definition 35.37 for the cover $\mathcal{X}'$ at $\mathfrak{q}$ (with $\ell = p$, $N = p^R$): C is the completion of an algebraic closure of $\widehat{K} = \mathrm{Frac}(\widehat{\mathcal{O}}_{\mathfrak{q}})$, with the fixed embedding of an algebraic closure of Ω' over $\Omega' \subseteq \widehat{K}$, and let $\iota_C \colon \Omega' \to C$ be its restriction. C is algebraically closed of characteristic 0, complete, and $|p| < 1$ as $p \in \mathfrak{m}_{\mathfrak{q}}$. The absolute value of C restricts to that of the complete discretely valued field $\widehat{K}$, so $\mathcal{O}_C \cap \widehat{K} = \widehat{\mathcal{O}}_{\mathfrak{q}}$, and $\widehat{\mathcal{O}}_{\mathfrak{q}} \cap \Omega' = R_{\mathfrak{q}}$. Hence $\iota_C^{-1}(\mathcal{O}_C) = R_{\mathfrak{q}}$ and $(\mathfrak{q}, C, \iota_C)$ is a test point (Ox1). We prove (F1), (F2) at it for every $a \in \Phi$.

Step 1: data. Fix $a \in \Phi$, $s = (a, +)$, $\bar{s} = (a, -)$, $\kappa = \mathrm{rk}(s)$, $\mathfrak{P} = \mathfrak{P}_s$, $m = m_{\mathfrak{P}}$, $\sigma = \sigma_s$, the atlas $(U_\nu, \iota_\nu, h_\nu)_\nu$ of the hypothesis with its points y, and ν with $\mathfrak{q} \in U_\nu$; write $(\iota, h) = (\iota_\nu, h_\nu)$, with forms $\eta_1, \dots, \eta_m$, and also for the C-linear extensions. Fix (ϖ_k), (ϖ_k^*), I, $x_{(l,k)}$, $\mathbf{P}$, $\mathbf{J}$ as in Lemma D.117. Let $\zeta \in \Omega'$ be the primitive p^R-th root of unity of (Hm5) and ε, g_{lj}, $\mathbf{E}_\varepsilon$ as in Lemma D.119; choose the compatible system of Definition 35.37 with $\zeta_{p^R} = \zeta$ (successive p-th roots above level R, $\zeta_{p^k} = \zeta^{p^{R-k}}$ below), so $\zeta_p = \zeta^{p^{R-1}}$. Let $g = \dim \mathrm{Lie}(G'_{\Omega'})$, $w_1, \dots, w_g$ an $R_{\mathfrak{q}}$-basis of $\omega_{\mathfrak{q}}$, and Λ, $\bar{\Lambda}$ as in Lemma D.119.

Step 2: the Hodge–Tate and Weil matrices. By Lemma D.114 (cover $\mathcal{X}'$, point $\mathfrak{q}$, the basis $(x_j)_{j \in I}$ of N from Lemma D.117, the basis (w_i)) there are $\mathbf{h} \in M_{g \times I}(\mathcal{O}_C)$, $\mathbf{e} \in \mathrm{GL}_I(\mathbb{Z}_p)$, $\mathbf{e}_0 \in M_I(\mathbb{Z})$ with (1)–(3) there. Moreover $\mathbf{e} - \mathbf{E}_\varepsilon \in p\, M_I(\mathbb{Z}_p)$: for $j = (l, k)$, j' in I put $\tilde{x}_j = \varpi_k y_l$,

so $x_j = p^{R-1}\tilde{x}_j$; then

$$\zeta_p^{(\mathbf{e}_0)_{jj'}} = e_p(x_j, x_{j'}) = e_{p^R}(\tilde{x}_j, x_{j'}) = e_{p^R}(\tilde{x}_j, \tilde{x}_{j'})^{p^{R-1}} = \zeta_p^{\varepsilon(\tilde{x}_j, \tilde{x}_{j'})},$$

so $(\mathbf{e}_0)_{jj'} \equiv (\mathbf{E}_\varepsilon)_{jj'}$ modulo p, and $\mathbf{e} \equiv \mathbf{e}_0$ modulo p.

Step 3: B is a tight Hopf order on N. By Proposition D.98 each $\mathcal{A}_{\mathfrak{P}',C}$ is a tight Hopf order on $\Gamma_{\mathfrak{P}'}$. Applying Lemma D.106 to the factors one at a time, B is a Hopf order on $N = \bigoplus_{\mathfrak{P}'} \Gamma_{\mathfrak{P}'}$, and tightness passes from two tight factors B_1, B_2 on N_1, N_2 to $B_1 \boxtimes B_2$ by its additivity formulas:

$$\begin{aligned} |N|(p-1)\mathrm{len}_C(Q_B) &= |N_2| \cdot |N_1|(p-1)\mathrm{len}_C(Q_{B_1}) + |N_1| \cdot |N_2|(p-1)\mathrm{len}_C(Q_{B_2}) \\ &= 2|N_2|\mathrm{colen}(B_1) + 2|N_1|\mathrm{colen}(B_2) = 2\,\mathrm{colen}(B), \end{aligned}$$

and in the same way $|N|\,\mathrm{len}_C(I_B/I_B^2) = 2\,\mathrm{colen}(B)$.

Step 4: the almost-additive lattice of B. Identify N with $\mathbb{F}_p^I$ by $x \mapsto \hat{x}$. We apply Lemma D.112 with B (Step 3), the primitive p-th root of unity $\zeta_C = \iota_C(\zeta_p)$, the matrices $\mathbf{h}, \mathbf{e}, \mathbf{e}_0$ (Step 2: $\mathbf{e}^\mathsf{T} = -\mathbf{e}$, $\mathbf{h}\mathbf{e}^{-1}\mathbf{h}^\mathsf{T} = 0$, $\mathbf{e} - \mathbf{e}_0 \in pM_I(\mathbb{Z}_p)$, $\mathbf{e} \in \mathrm{GL}_I(\mathcal{O}_C)$), and D the map of Lemma D.116(1) followed by the isomorphism $\mathcal{O}_C \otimes (\omega_\mathfrak{q}/p\,\omega_\mathfrak{q}) \cong (\mathcal{O}_C/p)^g$ of the basis (w_i). (Wb1): by Lemma D.115(1) for the polarisation $\phi_\mathcal{L}$ of $G'_{\Omega'}$, $(z, z') \mapsto \iota_C(e_p(z, z'))$ lies in the $\mathcal{O}_C$-span of the $f \otimes f'$, $f, f' \in B$, and $\iota_C(e_p(z, z')) = \zeta_C^{\hat{z}^\mathsf{T}\mathbf{e}_0\hat{z}'}$ by Lemma D.114(1). (Wb2) is Lemma D.116(1). (Wb3): for $x \in N$ choose $y' \in G'(\Omega')[p^R]$ with $p^{R-1}y' = x$ ($G'(\Omega')[p^R] \cong (\mathbb{Z}/p^R)^{2g}$) and a Hodge–Tate value v' of y' at $\mathfrak{q}$ (Lemma 31.30(1)); the character $\chi_x = \iota_C(e_p(x, \cdot))$ is that of Lemma D.116, and by Lemma D.116(2) and Lemma D.114(3), $D(\chi_x) = [v'] \bmod p = \mathbf{h}\hat{x} \bmod p$. The lattice $\mathrm{af}(B)$ of Lemma D.112 equals $\mathrm{af}(B)$ of Lemma D.117: if $\lambda_l = f + pf'$ with $f \in B$, then f is almost additive because λ_l is $(\widehat{x + x'} - \hat{x} - \hat{x}' \in p\mathbb{Z}^I)$, so $f \in \mathrm{AF}(B)$. Hence Lemma D.112(1)–(6) hold for $\mathrm{af}(B)$, with $c = \sum_{k=1}^{p-1}(-1)^{k-1}(\zeta_C - 1)^k/k$, $v(c) = \frac{1}{p-1} = v(w)$.

Step 5: the lattices at $\mathfrak{q}$. Every integer prime to p is a unit of $R_\mathfrak{q}$, so the discriminant of E is, and $\sigma_t(\mathcal{O}_E) \subseteq \mathcal{O}' \subseteq R_\mathfrak{q}$ for every label t; the idempotents of $\mathcal{O}_E \otimes R_\mathfrak{q} \cong \prod_t R_\mathfrak{q}$ (Fh1) split the free module $\mathrm{Lie}_\mathfrak{q}$ as $\bigoplus_t(\mathrm{Lie}_\mathfrak{q} \cap \mathrm{Li}_t)$, so $\mathrm{dR}_{s,\mathfrak{q}}$ and $\mathrm{dR}_{\bar{s},\mathfrak{q}}$ are free direct summands of $\mathrm{Lie}_\mathfrak{q}$; the same holds over a neighbourhood of $\mathfrak{q}$ on which the discriminant is invertible. A germ in $\mathrm{Lie}_\mathfrak{q}$ is a section of $\mathrm{Lie}(G')$ near $\mathfrak{q}$, so the stalks of the sheaves of (Hf2) are $\mathrm{lie}_\mathfrak{q} = \{c : \sum_k c_k\beta_k \in \mathrm{dR}_{s,\mathfrak{q}}\}$ and $\bar{\mathrm{l}}_\mathfrak{q} = \{c : \sum_k c_k\bar{\beta}_k \in \mathrm{dR}_{\bar{s},\mathfrak{q}}\}$, and, $\bar{\mathrm{l}}$ being locally free near $\mathfrak{q}$, $\omega_\mathfrak{q}$ is the dual lattice of $\bar{\mathrm{l}}_\mathfrak{q}$. Write X_C for the $\mathcal{O}_C$-span of a stalk $X_\mathfrak{q}$.

1. $\mathrm{lie}_C = \{a \in C^\kappa : \Lambda a \in \mathcal{O}_C^g\}$. For $a \in C^\kappa$ and $A = \sum_k a_k\beta_k$: as $\mathrm{dR}_{s,\mathfrak{q}}$ is a direct summand of $\mathrm{Lie}_\mathfrak{q}$, $A \in \mathcal{O}_C\mathrm{dR}_{s,\mathfrak{q}}$ iff $A \in \mathcal{O}_C\mathrm{Lie}_\mathfrak{q}$ iff $w_i(A) \in \mathcal{O}_C$ for all i ((w_i) is the dual basis of a basis of $\mathrm{Lie}_\mathfrak{q}$), i.e. iff $\Lambda a \in \mathcal{O}_C^g$.

2. $\omega_C = \bar{\Lambda}^\mathsf{T}\mathcal{O}_C^g$. The dual lattice of $\bar{\mathrm{l}}_\mathfrak{q}$ is $\{(\varphi(\bar{\beta}_r))_r : \varphi \in \mathrm{Hom}_{R_\mathfrak{q}}(\mathrm{dR}_{\bar{s},\mathfrak{q}}, R_\mathfrak{q})\}$ (to c corresponds $\varphi(\sum_r c'_r\bar{\beta}_r) = c \cdot c'$), every such φ is the restriction of some $\eta = \sum_i u_i w_i \in \omega_\mathfrak{q}$ (direct summand), and $(\eta(\bar{\beta}_r))_r = \bar{\Lambda}^\mathsf{T}u$. So $\omega_\mathfrak{q} = \bar{\Lambda}^\mathsf{T}R_\mathfrak{q}^g$.

3. By (Hm1), $\mathrm{lie}_C \subseteq W_C$, $wW_C \subseteq \mathrm{lie}_C$, $\omega'_C \subseteq \omega_C$ and $\mathcal{O}_C^m = T_C \subseteq H_C$; by Lemma D.121, $H_C \subseteq w^{-1}\mathcal{O}_C^m$ (F0); by Lemma D.117(4), $H_C = \{u \in C^m :\ l\mathbf{J}u \in \mathcal{O}_C \text{ for all } l \in \mathrm{af}(B)\}$.

Step 6: $\iota^{-1}(\mathcal{O}_C^m) = W_C$. By (Hm4) at $\mathfrak{q}$, $\iota(W_\mathfrak{q}) \subseteq R_\mathfrak{q}^m$ and $W_\mathfrak{q}/wW_\mathfrak{q} \to R_\mathfrak{q}^m/wR_\mathfrak{q}^m$ is injective. $W_\mathfrak{q}$ is free (Hm1), so $W_C = \mathcal{O}_C \otimes W_\mathfrak{q}$, and by flatness $\iota(W_C) \subseteq \mathcal{O}_C^m$ and $W_C \cap \iota^{-1}(w\mathcal{O}_C^m) = wW_C$. Let $a \in C^\kappa$ with $\iota(a) \in \mathcal{O}_C^m$. As W_C has a C-basis of C^κ as $\mathcal{O}_C$-basis and $v(w) > 0$, $w^k a \in W_C$ for some $k \geq 0$; take k minimal. If $k \geq 1$, then $\iota(w^k a) = w^k\iota(a) \in w\mathcal{O}_C^m$, so $w^k a \in wW_C$ and $w^{k-1}a \in W_C$, against minimality. So $a \in W_C$.

Step 7: the images of H_C. We apply Lemma D.120 with AF = af(B), $\mathbf{h}$, $\mathbf{e}$, c, w, $\mathbf{P}$, $\mathbf{J}$, $r=\kappa$, $r'=n-\kappa$, Λ, $\bar\Lambda$, $W=W_C$ and the maps h, ι. Its hypotheses: (1)–(5) are Lemma D.112(1), (2), (3), (4), (6) (Step 4); $\mathbf{PJ}=\mathbf{1}$ and the stability of af(B) are Lemma D.117(1), (3); its H is H_C and its lie is lie$_C$ (Step 5); lie$_C\subseteq W_C$, $wW_C\subseteq$ lie$_C$ (Step 5). (6): the column of $\mathbf{P}$ of index (l,k) is $\sigma(\varpi_k)e_l$, so this is Lemma D.119(1), whose hypothesis on $\mathbf{h}$ is Lemma D.114(3). (7) is Step 6. (8): let $a\in W_\mathfrak{q}$. Then $w\sum_k a_k\beta_k\in \mathrm{dR}_{s,\mathfrak{q}}\subseteq \mathrm{Lie}_\mathfrak{q}$ ($W\subseteq w^{-1}$lie, Step 5), $\iota(a)\in R^m_\mathfrak{q}$ (Hm4), and a is a section of W over an open $U'\subseteq U_\nu$ containing $\mathfrak{q}$, so (Hm5) gives $\sum_l \iota(a)_l g_{lj}-\eta_j(\sum_k a_k\beta_k)\in p^{R-2}R_\mathfrak{q}$; as $R\geq 4$, Lemma D.119(2) gives $\mathbf{E}_\varepsilon\mathbf{J}\iota(a)+\mathbf{h}^\mathsf{T}\Lambda a\in w\mathcal{O}_C^I$, and $(\mathbf{e}-\mathbf{E}_\varepsilon)\mathbf{J}\iota(a)\in p\,\mathcal{O}_C^I\subseteq w\mathcal{O}_C^I$ by Step 2. Both sides being $\mathcal{O}_C$-linear in a, (8) holds on W_C. Therefore

$$h(H_C)=\bar\Lambda^\mathsf{T}\mathcal{O}_C^g=\omega_C,\qquad \iota^{-1}(H_C)=w^{-1}\mathrm{lie}_C.$$

Step 8: (F1) and (F2). By (Hm4) the stalks of the graphs at $\mathfrak{q}$ are $\Gamma_\mathfrak{q}=\{(c,d): wc\in W_\mathfrak{q},\ d-\iota(c)\in R^m_\mathfrak{q}\}$ and $\Delta_\mathfrak{q}=\{(d,e): wd\in R^m_\mathfrak{q},\ e-h(d)\in\omega'_\mathfrak{q}\}$. The C-linear automorphism $(c,d)\mapsto(wc,d-\iota(c))$ maps $\Gamma_\mathfrak{q}$ onto $W_\mathfrak{q}\oplus R^m_\mathfrak{q}$, so $\mathcal{O}_C\Gamma_\mathfrak{q}=\{(c,d): wc\in W_C,\ d-\iota(c)\in\mathcal{O}_C^m\}$, and likewise $\mathcal{O}_C\Delta_\mathfrak{q}=\{(d,e): wd\in\mathcal{O}_C^m,\ e-h(d)\in\omega'_C\}$. With K and I formed from these spans and H_C: since $\mathcal{O}_C^m\subseteq H_C$, $K=\{c:\ wc\in W_C,\ \iota(c)\in H_C\}$; and if $\iota(c)\in H_C\subseteq w^{-1}\mathcal{O}_C^m$, then $\iota(wc)\in\mathcal{O}_C^m$ and $wc\in W_C$ by Step 6. So $K=\iota^{-1}(H_C)=w^{-1}\mathrm{lie}_C$ by Step 7: this is (F1). By (F0), $wd\in\mathcal{O}_C^m$ for every $d\in H_C$, so $I=h(H_C)+\omega'_C=\omega_C+\omega'_C=\omega_C$ by Step 7 and $\omega'_C\subseteq\omega_C$: this is (F2). □

Lemma D.112. *Let p be an odd prime, $g\geq 1$ and $N=\mathbb{F}_p^{2g}$. For $z\in N$ let $\hat z\in\{0,\dots,p-1\}^{2g}\subseteq\mathcal{O}_C^{2g}$ be its representative; for a row vector $l\in\mathcal{O}_C^{2g}$ let $\lambda_l(z)=l\hat z$, and put*

$$\mathrm{af}(B)=\{l:\ \lambda_l\in\mathrm{AF}(B)+p\,\mathrm{Map}(N,\mathcal{O}_C)\}.$$

Let B be a tight Hopf order on N, $\zeta\in C$ a primitive p-th root of unity and $c=\sum_{k=1}^{p-1}(-1)^{k-1}(\zeta-1)^k/k$. Let $\mathbf{h}\in M_{g\times 2g}(\mathcal{O}_C)$, $\mathbf{e}\in\mathrm{GL}_{2g}(\mathcal{O}_C)$ with $\mathbf{e}^\mathsf{T}=-\mathbf{e}$ and $\mathbf{h}\,\mathbf{e}^{-1}\mathbf{h}^\mathsf{T}=0$, and $\mathbf{e}_0\in M_{2g}(\mathbb{Z})$ with $\mathbf{e}-\mathbf{e}_0\in pM_{2g}(\mathcal{O}_C)$. Assume:

(Wb1) *the function $K(z,z')=\zeta^{\hat z^\mathsf{T}\mathbf{e}_0\hat z'}$ on $N\times N$ lies in the $\mathcal{O}_C$-span of the functions $(z,z')\mapsto f(z)f'(z')$, $f,f'\in B$ (so each $\chi_x=K(x,\cdot)$, $x\in N$, lies in B: if $K=\sum_i f_i\otimes f'_i$ then $\chi_x=\sum_i f_i(x)f'_i$ with $f_i(x)\in\mathcal{O}_C$);*

(Wb2) *$D\colon B\to(\mathcal{O}_C/p)^g$ is a surjective $\mathcal{O}_C$-linear derivation at the origin: $D(ff')=f(0)D(f')+f'(0)D(f)$;*

(Wb3) *$D(\chi_x)=\mathbf{h}\hat x \bmod p$ for every $x\in N$.*

Then $v(c)=\frac{1}{p-1}$, and:

1. *$c\,l\in\mathrm{af}(B)$ for every row vector $l\in\mathcal{O}_C^{2g}$;*
2. *every row of $\mathbf{h}$ lies in af(B);*
3. *if $y\in\mathcal{O}_C^{2g}$ and $y^\mathsf{T}\mathbf{e}\in\mathrm{af}(B)$, then $\mathbf{h}y\in c\,\mathcal{O}_C^g$;*
4. *for every $u\in\mathcal{O}_C^g$ there is $y\in\mathcal{O}_C^{2g}$ with $y^\mathsf{T}\mathbf{e}\in\mathrm{af}(B)$ and $\mathbf{h}y-cu\in p\,\mathcal{O}_C^g$;*
5. *if $y_1,\dots,y_g$ are as in (4) for the unit vectors $u=e_1,\dots,e_g$, then af(B) is the $\mathcal{O}_C$-span of the rows of $\mathbf{h}$ and of $y_1^\mathsf{T}\mathbf{e},\dots,y_g^\mathsf{T}\mathbf{e}$;*
6. *for every $u\in\mathcal{O}_C^g$ there is $x\in C^{2g}$ with $lx\in\mathcal{O}_C$ for all $l\in\mathrm{af}(B)$ and $\mathbf{h}x=u$.*

Proof. The elements of B take values in $\mathcal{O}_C$ by Lemma 35.48 (B is a Hopf order). Write $L(X) = \sum_{k=1}^{p-1}(-1)^{k-1}X^k/k \in \mathbb{Z}_{(p)}[X]$, so $c = L(\zeta - 1)$, and put $\kappa = 1 + \frac{1}{p-1}$. Since p is odd, $\frac{1}{p-1} < 1$.

Step 0: $v(c) = \frac{1}{p-1}$, and $p = cs$ with $v(s) > 0$. $v(\zeta - 1) = \frac{1}{p-1}$ and for $2 \le k \le p-1$, $v((\zeta-1)^k/k) = \frac{k}{p-1} > \frac{1}{p-1}$. So $v(c) = \frac{1}{p-1}$ and $s = p/c$ has $v(s) = 1 - \frac{1}{p-1} > 0$.

Step 1: the truncated logarithm. (a) If $a, b \in \mathcal{O}_C$ have valuation $\ge \frac{1}{p-1}$, then $v\big(L(a+b+ab) - L(a) - L(b)\big) \ge \kappa$. Indeed $P(S,T) = L(S+T+ST) - L(S) - L(T) \in \mathbb{Z}_{(p)}[S,T]$, and in $\mathbb{Q}[[S,T]]$ its part of total degree $< p$ equals that of $\log((1+S)(1+T)) - \log(1+S) - \log(1+T) = 0$, because $L(X)$ and $\log(1+X)$ agree up to degree $p-1$ and $(S+T+ST)^k$ has no monomial of degree $< k$; so every monomial of P has degree $\ge p$ and $v(P(a,b)) \ge \frac{p}{p-1} = \kappa$. (b) By induction on $a \ge 0$, applying (a) to $\zeta^a - 1$ and $\zeta - 1$: $v\big(L(\zeta^a - 1) - ac\big) \ge \kappa$ for every integer $a \ge 0$; as ζ^a and $ac \bmod pc$ depend only on $a \bmod p$ ($v(pc) = \kappa$), this holds for all $a \in \mathbb{Z}$. (c) For $f \in B$, $L(f-1) = \sum_{k=1}^{p-1}(-1)^{k-1}(f-1)^k/k \in B$ ($1/k \in \mathcal{O}_C$, B is an algebra with 1). (d) $D(1) = D(1 \cdot 1) = 2D(1)$, so $D(1) = 0$; if $f(0) = 1$ then $D((f-1)^k) = 0$ for $k \ge 2$ (Leibniz, $(f-1)(0) = 0$), so $D(L(f-1)) = D(f)$.

Step 2: the functions $T(y)$. For $1 \le m \le 2g$ let e_m be the m-th unit vector of N; $\chi_{e_m}(z) = \zeta^{(\mathbf{e}_0\hat{z})_m}$ lies in B by (Wb1), with $\chi_{e_m}(0) = 1$. For $y \in \mathcal{O}_C^{2g}$ put $T(y) = \sum_m y_m\, L(\chi_{e_m} - 1) \in B$ (Step 1(c)). By Step 1(d) and (Wb3), $D(T(y)) = \sum_m y_m \mathbf{h} e_m = \mathbf{h}y \bmod p$. By Step 1(b) with $a = (\mathbf{e}_0\hat{z})_m$, and $y^{\mathsf{T}}\mathbf{e}_0\hat{z} - y^{\mathsf{T}}\mathbf{e}\hat{z} \in p\,\mathcal{O}_C$,

$$v\big(T(y)(z) - c\,\lambda_{y^{\mathsf{T}}\mathbf{e}}(z)\big) \ge \kappa \qquad \text{for all } z \in N.$$

Step 3: proof of (1). Let l be a row vector and $y = (l\mathbf{e}^{-1})^{\mathsf{T}}$, so $y^{\mathsf{T}}\mathbf{e} = l$. By Step 2, $T(y) - c\lambda_l \in p\,\mathrm{Map}(N, \mathcal{O}_C)$ ($\kappa \ge 1$). λ_l is almost additive ($\widehat{z+z'} - \hat{z} - \hat{z}' \in p\mathbb{Z}^{2g}$), hence so is $T(y)$, and $T(y) \in \mathrm{AF}(B)$; so $cl \in \mathrm{af}(B)$.

Step 4: the hypothesis of Lemma D.113. By (Ho3) equality holds in its displayed hypothesis; $v(c) = \frac{1}{p-1}$ (Step 0); and $c\,\mathrm{Lin}(N) \subseteq \mathrm{AF}(B) + p\,\mathrm{Map}(N, \mathcal{O}_C)$: an almost-additive λ satisfies $\lambda - \lambda_l \in p\,\mathrm{Map}(N, \mathcal{O}_C)$ for $l = (\lambda(e_1), \dots, \lambda(e_{2g}))$ (almost additivity, $\lambda(\hat{z}_1 e_1 + \dots)$ expanded), and $c\lambda_l \in \mathrm{AF}(B) + p\,\mathrm{Map}$ by Step 3. With $W = (\mathcal{O}_C/p)^g$, Lemma D.113 applies to D.

Step 5: the key congruence. If $f \in \mathrm{AF}(B)$, $y \in \mathcal{O}_C^{2g}$ and $f - \lambda_{y^{\mathsf{T}}\mathbf{e}} \in p\,\mathrm{Map}(N, \mathcal{O}_C)$, then $\mathbf{h}y \equiv c\,D(f) \pmod p$. Indeed $(T(y) - cf)(z) = \big(T(y)(z) - c\lambda_{y^{\mathsf{T}}\mathbf{e}}(z)\big) - c\big(f(z) - \lambda_{y^{\mathsf{T}}\mathbf{e}}(z)\big)$ has valuation $\ge \kappa$ (Step 2; $v(cp) = \kappa$), so $D(T(y) - cf) = 0$ by Lemma D.113(ii), and $D(T(y)) = \mathbf{h}y \bmod p$ (Step 2).

Step 6: proof of (2). Fix $k \le g$ and write $K = \sum_i f_i \otimes f_i'$ with $f_i, f_i' \in B$ (Wb1). Choose $r_i \in \mathcal{O}_C$ lifting the k-th coordinate of $D(f_i')$ and put $F = \sum_i r_i f_i \in B$. For $x \in N$, $\chi_x = \sum_i f_i(x) f_i'$, so by (Wb3) $F(x) = \sum_i r_i f_i(x) \equiv D(\chi_x)_k \equiv (\mathbf{h}\hat{x})_k = \lambda_{\mathbf{h}_k}(x) \pmod p$, $\mathbf{h}_k$ the k-th row. Hence F is almost additive, $F \in \mathrm{AF}(B)$ and $\mathbf{h}_k \in \mathrm{af}(B)$.

Step 7: proof of (3). If $y^{\mathsf{T}}\mathbf{e} \in \mathrm{af}(B)$, there is $f \in \mathrm{AF}(B)$ with $f - \lambda_{y^{\mathsf{T}}\mathbf{e}} \in p\,\mathrm{Map}$. By Step 5, $\mathbf{h}y = cu + pw$ with $u, w \in \mathcal{O}_C^g$ (u a lift of $D(f)$), so $\mathbf{h}y = c(u + sw) \in c\,\mathcal{O}_C^g$ (Step 0).

Step 8: proof of (4). By (Wb2) there is $f \in B$ with $D(f) = u \bmod p$, and by Lemma D.113(i) there are $g_j \in \mathrm{AF}(B)$, $a_j \in \mathcal{O}_C$ with $D(f) = \sum_j a_j D(g_j)$. Then $F = \sum_j a_j g_j \in \mathrm{AF}(B)$, $D(F) = u \bmod p$, and $F - \lambda_l \in p\,\mathrm{Map}$ for $l = (F(e_1), \dots, F(e_{2g}))$ (as in Step 4). Put $y = (l\mathbf{e}^{-1})^{\mathsf{T}}$: $y^{\mathsf{T}}\mathbf{e} = l \in \mathrm{af}(B)$, and Step 5 gives $\mathbf{h}y \equiv cu \pmod p$.

Step 9: proof of (5). Let Y be the $2g \times g$ matrix with columns y_k and $\Phi \in M_{2g}(\mathcal{O}_C)$ the matrix with rows those of $\mathbf{h}$ followed by those of $Y^{\mathsf{T}}\mathbf{e}$; Λ its row span. By (2) and the choice of the y_k, $\Lambda \subseteq \mathrm{af}(B)$. By (4), $\mathbf{h}Y = c\,\mathbf{1}_g + pU = c(\mathbf{1}_g + sU)$ with $U \in M_g(\mathcal{O}_C)$. Using $\mathbf{e}^{\mathsf{T}} = -\mathbf{e}$

and $\mathbf{h}\mathbf{e}^{-1}\mathbf{h}^{\mathsf{T}} = 0$,

$$\Phi\,\mathbf{e}^{-1}\Phi^{\mathsf{T}} = \begin{pmatrix} \mathbf{h}\mathbf{e}^{-1}\mathbf{h}^{\mathsf{T}} & \mathbf{h}\mathbf{e}^{-1}\mathbf{e}^{\mathsf{T}}Y \\ Y^{\mathsf{T}}\mathbf{e}\mathbf{e}^{-1}\mathbf{h}^{\mathsf{T}} & Y^{\mathsf{T}}\mathbf{e}\mathbf{e}^{-1}\mathbf{e}^{\mathsf{T}}Y \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{h}Y \\ (\mathbf{h}Y)^{\mathsf{T}} & -Y^{\mathsf{T}}\mathbf{e}Y \end{pmatrix},$$

whose determinant is $\pm\det(\mathbf{h}Y)^2$ (block anti-triangular). As $\det\mathbf{e} \in \mathcal{O}_C^\times$ and $\det(\mathbf{1}_g + sU) \equiv 1 \pmod s$ is a unit ($v(s) > 0$), $2v(\det\Phi) = 2v(\det\mathbf{h}Y) = 2g\,v(c)$, so $v(\det\Phi) = \frac{g}{p-1}$ and $\mathrm{len}_C(\mathcal{O}_C^{2g}/\Lambda) = \frac{g}{p-1}$.

On the other side, $l \mapsto \lambda_l \bmod p$ maps $\mathcal{O}_C^{2g}$ onto the module V of additive maps $N \to \mathcal{O}_C/p$ with kernel $p\,\mathcal{O}_C^{2g}$, and $\mathrm{af}(B)$ is the preimage of the image M of $\mathrm{AF}(B)$; so $\mathcal{O}_C^{2g}/\mathrm{af}(B) \cong V/M \cong Q_B$, which has a normalised length (Lemma 35.47). The map $f \mapsto D(f - f(0)) = D(f)$ (Step 1(d)) shows $D(I_B) = D(B) = (\mathcal{O}_C/p)^g$, and D vanishes on I_B^2 (Leibniz), so I_B/I_B^2 maps onto $(\mathcal{O}_C/p)^g$ and $\mathrm{len}_C(I_B/I_B^2) \geq g$. The two equations of (Ho3) give $(p-1)\mathrm{len}_C(Q_B) = \mathrm{len}_C(I_B/I_B^2) \geq g$. Since $\mathcal{O}_C^{2g}/\Lambda \to \mathcal{O}_C^{2g}/\mathrm{af}(B)$ is onto with kernel $\mathrm{af}(B)/\Lambda$,

$$\tfrac{g}{p-1} = \mathrm{len}_C(\mathcal{O}_C^{2g}/\Lambda) = \mathrm{len}_C(Q_B) + \mathrm{len}_C(\mathrm{af}(B)/\Lambda) \geq \tfrac{g}{p-1} + \mathrm{len}_C(\mathrm{af}(B)/\Lambda),$$

so $\mathrm{af}(B)/\Lambda$ has length 0 and is zero: $\mathrm{af}(B) = \Lambda$.

Step 10: proof of (6). $\det\Phi \neq 0$ (Step 9). Given u, let $b \in \mathcal{O}_C^{2g}$ be u followed by g zeros and $x = \Phi^{-1}b \in C^{2g}$. The first g entries of $\Phi x = b$ give $\mathbf{h}x = u$. Every $l \in \mathrm{af}(B) = \Lambda$ is $l = a\Phi$ with a a row vector over $\mathcal{O}_C$, so $lx = ab \in \mathcal{O}_C$. □

Lemma D.113. *Let N be a finite-dimensional $\mathbb{F}_p$-vector space, $n = \dim N$, and B a Hopf order on N with*

$$(p-1)\,|N|\,\mathrm{len}_C(Q_B) = 2\,\mathrm{colen}(B)$$

(for instance a tight Hopf order). Let $c \in \mathcal{O}_C$ with $v(c) = 1/(p-1)$ and $c\,\mathrm{Lin}(N) \subseteq \mathrm{AF}(B) + p\,\mathrm{Map}(N, \mathcal{O}_C)$. Let W be an $\mathcal{O}_C$-module with $pW = 0$ and $D\colon B \to W$ an $\mathcal{O}_C$-linear derivation at the origin, that is $D(fh) = f(0)\,D(h) + h(0)\,D(f)$ for all $f, h \in B$. Then:

(i) there are $g_1, \ldots, g_n \in \mathrm{AF}(B)$ such that $D(B)$ is the $\mathcal{O}_C$-span of $D(g_1), \ldots, D(g_n)$;

(ii) $D(f) = 0$ for every $f \in B$ with $v(f(x)) \geq 1 + \frac{1}{p-1}$ for all $x \in N$.

Proof. The elements of B take values in $\mathcal{O}_C$ (Lemma 35.48; B is a Hopf order), so $f(0) \in \mathcal{O}_C$ for $f \in B$. By hypothesis equality holds in Lemma 35.47; take $g_j = \varepsilon_j\tilde{\lambda}_j \in \mathrm{AF}(B)$ ($\varepsilon_j \mid p$, $1 \leq j \leq n$) as there, so that the $\tilde{\lambda}^d$ form a basis of $\mathrm{Map}(N, \mathcal{O}_C)$ (its (i)), the image M of $\mathrm{AF}(B)$ in $V = \mathrm{Lin}(N)/p\,\mathrm{Map}(N, \mathcal{O}_C)$ is $\bigoplus_j \bar{\varepsilon}_j(\mathcal{O}_C/p)\lambda_j$ with (λ_j) a basis of V (its (ii)), and the $g^d = \prod_j g_j^{d_j}$, $d \in \{0, \ldots, p-1\}^n$, form a basis of B (its (iii)). Write $e_j \in \{0, \ldots, p-1\}^n$ for the j-th unit vector, $|d| = \sum_j d_j$.

Step 1: $g_j(0) \in p\,\mathcal{O}_C$. Almost additivity at $x = y = 0$ gives $g_j(0) - 2g_j(0) \in p\,\mathcal{O}_C$.

Step 2: $D(g^d) = 0$ unless $d = e_j$ for some j. For $d = 0$: $D(1) = D(1 \cdot 1) = 2D(1)$, so $D(1) = 0$. If $|d| \geq 2$, choose j with $d_j \geq 1$ and put $d' = d - e_j$, so $|d'| \geq 1$ and $g^d = g_j\,g^{d'}$. Then $g^{d'}(0)$ is a product of values $g_i(0)$, at least one factor, all in $\mathcal{O}_C$ and in $p\,\mathcal{O}_C$ by Step 1, so $g^{d'}(0) \in p\,\mathcal{O}_C$. Hence $D(g^d) = g_j(0)D(g^{d'}) + g^{d'}(0)D(g_j) \in pW = 0$.

Proof of (i). Every $f \in B$ is $f = \sum_d a_d g^d$ with $a_d \in \mathcal{O}_C$ (basis (iii)); by Step 2 and linearity $D(f) = \sum_j a_{e_j} D(g_j)$, and conversely each $D(g_j) \in D(B)$.

Step 3: $v(\varepsilon_j) \leq \frac{1}{p-1}$. $\tilde{\lambda}_j \in \mathrm{Lin}(N)$ (its reduction λ_j is additive, and any lift of an additive map to $\mathcal{O}_C$ is almost additive), so $c\tilde{\lambda}_j \in \mathrm{AF}(B) + p\,\mathrm{Map}(N, \mathcal{O}_C)$ and $\bar{c}\lambda_j \in M$. Comparing

coefficients of λ_j in the basis (λ_k) of V: $c = \varepsilon_j a + pb$ with $a, b \in \mathcal{O}_C$. If $p = 2$ then $v(\varepsilon_j) \le v(p) = 1 = v(c)$. If $p > 2$ then $v(c) < 1$, and $v(\varepsilon_j) > v(c)$ would give $v(c) \ge \min(v(\varepsilon_j), 1) > v(c)$; so $v(\varepsilon_j) \le v(c) = \frac{1}{p-1}$.

Proof of (ii). Let $f = \sum_d a_d g^d \in B$ with $v(f(x)) \ge 1 + \frac{1}{p-1}$ for all x. Then $f = \sum_d b_d \tilde{\lambda}^d$ with $b_d = a_d \prod_j \varepsilon_j^{d_j}$. The matrix $U = (\tilde{\lambda}^d(x))_{x,d}$ lies in $\mathrm{GL}_{|N|}(\mathcal{O}_C)$ by (i) of Lemma 35.47, so the vector $(b_d)_d = U^{-1}(f(x))_x$ has entries of valuation $\ge 1 + \frac{1}{p-1}$. For $d = e_j$, $b_{e_j} = a_{e_j}\varepsilon_j$ and Step 3 give $v(a_{e_j}) \ge 1 + \frac{1}{p-1} - v(\varepsilon_j) \ge 1$, so $a_{e_j} \in p\,\mathcal{O}_C$. By the formula in the proof of (i), $D(f) = \sum_j a_{e_j} D(g_j) \in pW = 0$. □

Lemma D.114. *Let $\mathfrak{T} = (\mathcal{O}', \mathcal{X}', \pi, (\mathcal{A}_{\mathfrak{P}}, \theta))$ be a pre-admissible torsion system, $G' = \pi^* G_{\mathcal{X}}$ with function field Ω' of $\mathcal{X}'$, R as in Definition 35.22, $g = \dim_{\Omega'} \mathrm{Lie}(G'_{\Omega'})$, and q a point of $\mathcal{X}'$ of codimension one with $p \notin \mathcal{O}_q^\times$; let C, $\mathcal{O}_C$, T_p, e be as in Definition 35.37 for the cover $\mathcal{X}'$ at q. Let $x_1, \dots, x_{2g}$ be an $\mathbb{F}_p$-basis of $N_p = G'(\Omega')[p]$ and $w_1, \dots, w_g$ an $\mathcal{O}_q$-basis of ω_q; for $x \in N_p$ let $\hat{x} \in \{0, \dots, p-1\}^{2g}$ be its coordinate vector, and for $w \in \omega_q \otimes_{\mathcal{O}_q} \mathcal{O}_C$ let $[w] \in \mathcal{O}_C^g$ be its coordinate vector. Then there are $\mathbf{h} \in M_{g \times 2g}(\mathcal{O}_C)$, $\mathbf{e} \in \mathrm{GL}_{2g}(\mathbb{Z}_p)$ and $\mathbf{e}_0 \in M_{2g}(\mathbb{Z})$ such that:*

1. *$\mathbf{e}^{\mathsf{T}} = -\mathbf{e}$, $\mathbf{e} - \mathbf{e}_0 \in p\,M_{2g}(\mathbb{Z}_p)$, and $e_p(x, x') = \zeta_p^{\hat{x}^{\mathsf{T}} \mathbf{e}_0 \hat{x}'}$ for all $x, x' \in N_p$;*
2. *$\mathbf{h}\,\mathbf{e}^{-1}\mathbf{h}^{\mathsf{T}} = 0$;*
3. *for every $x \in N_p$, every $y \in G'(\Omega')[p^R]$ with $p^{R-1}y = x$ and every Hodge–Tate value w of y at q: $[w] - \mathbf{h}\hat{x} \in p\,\mathcal{O}_C^g$.*

Proof. Step 1: a lift. By Lemma 35.39 if $G' \times_{\mathcal{X}'} \operatorname{Spec} \mathcal{O}_q$ is an abelian scheme and by Proposition 35.41 otherwise (their hypotheses are those of Definition 35.37, which hold for the cover $\mathcal{X}'$ at q), there is a Hodge–Tate isotropic lift $\beta\colon T_p \to \omega_q \otimes_{\mathcal{O}_q} \mathcal{O}_C$ at q, with sign ϵ as in (Iz2).

Step 2: the Weil matrix. T_p is free of rank $2g$ over $\mathbb{Z}_p$ and $Y \mapsto Y_{(1)}$ induces $T_p/pT_p \cong N_p$; choose $Y_m \in T_p$ with $(Y_m)_{(1)} = x_m$, a $\mathbb{Z}_p$-basis by Nakayama. Put $\mathbf{e} = (e(Y_m, Y_{m'}))_{m,m'} \in M_{2g}(\mathbb{Z}_p)$, alternating as e is, and let $\mathbf{e}_0 \in M_{2g}(\mathbb{Z})$ have entries in $\{0, \dots, p-1\}$ congruent to those of $\mathbf{e}$ modulo p. For $x, x' \in N_p$, bilinearity and $e_p(x_m, x_{m'}) = \zeta_p^{e(Y_m, Y_{m'})}$ give $e_p(x, x') = \zeta_p^{\hat{x}^{\mathsf{T}} \mathbf{e}_0 \hat{x}'}$; this is (1) once $\mathbf{e}$ is invertible. As e_p is perfect, $\mathbf{e} \bmod p$ is invertible, so $\det \mathbf{e} \in \mathbb{Z}_p^\times$ and $\mathbf{e} \in \mathrm{GL}_{2g}(\mathbb{Z}_p)$.

Step 3: the Hodge–Tate matrix and (2). Let $\mathbf{h}$ be the matrix whose m-th column is $[\epsilon\beta(Y_m)]$. The basis (y'_k) with $e(Y_j, y'_k) = \delta_{jk}$ is $y'_k = \sum_l (\mathbf{e}^{-1})_{lk} Y_l$ (indeed $\sum_l \mathbf{e}_{jl}(\mathbf{e}^{-1})_{lk} = \delta_{jk}$). By (Iz3) with the basis (Y_j) and $\mathbb{Z}_p$-linearity of β,

$$0 = \sum_j \beta(Y_j) \otimes \beta(y'_j) = \sum_{j,l} (\mathbf{e}^{-1})_{lj}\, \beta(Y_j) \otimes \beta(Y_l) \quad \text{in } (\omega_q \otimes \omega_q) \otimes \mathcal{O}_C.$$

In the basis $w_k \otimes w_{k'}$ the (k, k') coefficient, multiplied by $\epsilon^2 = 1$, is $\sum_{j,l} \mathbf{h}_{kj}(\mathbf{e}^{-1})_{lj}\mathbf{h}_{k'l} = \left(\mathbf{h}(\mathbf{e}^{-1})^{\mathsf{T}}\mathbf{h}^{\mathsf{T}}\right)_{kk'}$. As $(\mathbf{e}^{-1})^{\mathsf{T}} = (\mathbf{e}^{\mathsf{T}})^{-1} = -\mathbf{e}^{-1}$, this is (2).

Step 4: additivity of Hodge–Tate values. If w_1, w_2 are Hodge–Tate values at q of $y_1, y_2 \in G'(\Omega')[p^R]$, then $w_1 + w_2$ is one of $y_1 + y_2$. Let ϕ_1, ϕ_2 be regular functions as in (Fh5); on the intersection of their domains $\phi_1\phi_2(g) = e_{p^R}(y_1, g)e_{p^R}(y_2, g) = e_{p^R}(y_1 + y_2, g)$. The function ϕ_i pulled back along the unit section is regular near q and equals $e_{p^R}(y_i, 0) = 1$ at the generic point, so it is 1; hence $d(\phi_1\phi_2) = d\phi_1 + d\phi_2$ on Lie_q, and $w_1 + w_2$ satisfies the congruence of (Fh5) for $y_1 + y_2$.

Step 5: (3). Let x, y, w be as in (3) and $Y = \sum_m \hat{x}_m Y_m \in T_p$, so $Y_{(1)} = x$ and $\epsilon\beta(Y)$ has coordinates $\mathbf{h}\hat{x}$. As $p^{R-1}Y_{(R)} = Y_{(1)} = x = p^{R-1}y$, $t = y - Y_{(R)}$ is killed by p^{R-1}, hence

$t = pt'$ with $t' \in G'(\Omega')[p^R]$ ($G'(\Omega')[p^R] \cong (\mathbb{Z}/p^R)^{2g}$). By Lemma 31.30(1) there are Hodge–Tate values w' of $Y_{(R)}$ and w'' of t' at q, and by Step 4 $w' + pw''$ is a Hodge–Tate value of y; by Lemma 31.30(1) again $w - w' - pw'' \in p^R\omega_q$. By (Iz2), $\epsilon\beta(Y) - w' \in p^R(\omega_q \otimes \mathcal{O}_C)$. Hence $[w] - \mathbf{h}\hat{x} = [w - w' - pw''] + [w' - \epsilon\beta(Y)] + p[w''] \in p\,\mathcal{O}_C^g$. □

Lemma D.115. *Let $\mathfrak{T}$ be a torsion-regular pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$ as in Proposition D.97, with p odd and function field Ω' of $\mathcal{X}'$, and let $(\mathfrak{q}, C, \iota_C)$ be a test point. Put $N = G'(\Omega')[p] = \bigoplus_{\mathfrak{P}|p} \Gamma_{\mathfrak{P}}$ and*

$$B = \boxtimes_{\mathfrak{P}|p} \mathcal{A}_{\mathfrak{P},C} \subseteq \mathrm{Map}(N, C),$$

the $\mathcal{O}_C$-span of the functions $z \mapsto \prod_{\mathfrak{P}|p} f_{\mathfrak{P}}(z_{\mathfrak{P}})$, $f_{\mathfrak{P}} \in \mathcal{A}_{\mathfrak{P},C}$, where $z_{\mathfrak{P}}$ is the $\Gamma_{\mathfrak{P}}$-component of z. Let ϕ be a polarisation of the abelian variety $G'_{\Omega'}$ and $e_p = e_p^{\phi}\colon N \times N \to \mu_p(\Omega')$ its Weil pairing. Then:

1. *the function $K(z, z') = \iota_C(e_p(z, z'))$ on $N \times N$ lies in the $\mathcal{O}_C$-span of the functions $(z, z') \mapsto f(z)f'(z')$ with $f, f' \in B$;*
2. *for every $x \in N$ the function $\chi_x = K(x, \cdot)$ lies in B, and for every prime $\mathfrak{P} \mid p$ of $\mathcal{O}_E$ its restriction to $\Gamma_{\mathfrak{P}}$ lies in $\mathcal{A}_{\mathfrak{P},C}$.*

Proof. **Setting.** Let $\widehat{R}$, $\widehat{K}$, $g\colon R_{\mathfrak{q}} \to \widehat{R}$, $h\colon \widehat{R} \to \mathcal{O}_C$ and $\widehat{G} = G' \times_{\mathcal{X}'} \operatorname{Spec} \widehat{R}$ be as in Proposition D.99. Since $h \circ g = \iota_C|_{R_{\mathfrak{q}}}$ is injective, g is injective and extends to a field embedding $g\colon \Omega' \to \widehat{K}$; so $A := \widehat{G}_{\widehat{K}} = G'_{\Omega'} \otimes_{\Omega'} \widehat{K}$ is an abelian variety, polarised by the base change $\phi_{\widehat{K}}$ of ϕ. $\widehat{G}$ is a semiabelian scheme over $\widehat{R}$ (base change of the semiabelian scheme G'), with abelian generic fibre. By Proposition D.99, $\widehat{G}[p]$ is flat, quasi-finite, separated and of finite presentation over $\widehat{R}$, with constant generic fibre whose points x_t ($t \in N$) are the images of the points of N; this is the situation of Definition 35.21 over the noetherian integrally closed domain $\widehat{R}$ with $T = N$. Put $\mathrm{Ord}_1 = \mathrm{Ord}(\widehat{G}[p]) \subseteq \mathrm{Map}(N, \widehat{R})$. The Weil pairing commutes with base change, so $e_p^{\phi_{\widehat{K}}}(x_t, x_{t'}) = g(e_p(t, t'))$ for $t, t' \in N$. These values are p-th roots of unity in $\widehat{K}$, integral over $\mathbb{Z}$, hence in the normal ring $\widehat{R}$. Define $F\colon N \times N \to \widehat{R}$ by $F(t, t') = g(e_p(t, t'))$. As e_p is bilinear, for $a \in N$ the maps $\psi_a = F(\cdot, a)$ and $\psi'_a = F(a, \cdot)$ are homomorphisms $N \to \mu_p(\widehat{K})$.

Step 1: F is the restriction of a global unit on $\widehat{G}[p] \times \widehat{G}[p]$. Let $\mathcal{P}$ be the Poincaré bundle on $A \times A^\vee$, $\mathcal{M} = (1, \phi_{\widehat{K}})^*\mathcal{P}$, rigidified at the origin, and $\mathcal{L} = \mathcal{M}^{\otimes p}$ with its cubical structure. Then $\phi_{\mathcal{L}} = p\,\phi_{\mathcal{M}} = 2p\,\phi_{\widehat{K}}$, so $K(\mathcal{L}) = \ker(2p\,\phi_{\widehat{K}})$ is finite and contains $A[p]$. $\widehat{R}$ is a complete discrete valuation ring, hence a complete noetherian normal local domain and regular. By Theorem B.119(2), applied to $S = \operatorname{Spec} \widehat{R}$, the semiabelian scheme $\widehat{G}$ and $\mathcal{L}$, the schematic closure K_S of $K(\mathcal{L})$ in $\widehat{G}$ is a closed subgroup scheme, flat and quasi-finite over S, and there is a pairing $e_S^{\mathcal{L}}\colon K_S \times_S K_S \to \mathbb{G}_{m,S}$ extending $e^{\mathcal{L}}$. Now $\widehat{G}[p]$ is closed in $\widehat{G}$ (the kernel of $[p]$ on a separated group scheme) and flat over $\widehat{R}$; so its structure sheaf has no $\widehat{R}$-torsion and $\widehat{G}[p]$ is the schematic closure in $\widehat{G}$ of its generic fibre $A[p]$. Since K_S is a closed subscheme of $\widehat{G}$ containing $A[p] \subseteq K(\mathcal{L})$, we get $\widehat{G}[p] \subseteq K_S$. Let $\Gamma = \widehat{G}[p] \times_S \widehat{G}[p]$ and $u \in \mathcal{O}(\Gamma)^\times$ the restriction of $e_S^{\mathcal{L}}$; then $u(x_t, x_{t'}) = e^{\mathcal{L}}(x_t, x_{t'})$. For $x, y \in A[p]$,

$$e^{\mathcal{L}}(x, y) = e_p(x, \phi_{\mathcal{M}}(y))^{\epsilon} = e_p(x, 2\phi_{\widehat{K}}(y))^{\epsilon} = e_p^{\phi_{\widehat{K}}}(x, y)^{2\epsilon}.$$

As p is odd, $s = u^{\epsilon(p+1)/2} \in \mathcal{O}(\Gamma)^\times$ satisfies $s(x_t, x_{t'}) = e_p^{\phi_{\widehat{K}}}(x_t, x_{t'})^{p+1} = F(t, t')$ for all $t, t' \in N$.

Step 2: homomorphisms $N \to \mu_p(\widehat{K})$ lie in Ord_1. By Lemma 35.28(1), Ord_1 is a finitely generated projective $\widehat{R}$-module with $d\,\mathrm{Map}(N, \widehat{R}) \subseteq \mathrm{Ord}_1$ for some $d \neq 0$, and $\operatorname{Spec} \mathrm{Ord}_1$

is a finite flat commutative group scheme with group law induced by the addition of N: its comultiplication sends f to $(t,t')\mapsto f(t+t')$, an element of $\mathrm{Ord}_1\otimes_{\widehat R}\mathrm{Ord}_1\subseteq\mathrm{Map}(N\times N,\widehat K)$ (the inclusion because Ord_1 is flat and $\widehat K\otimes\mathrm{Ord}_1=\mathrm{Map}(N,\widehat K)$), and its counit is $f\mapsto f(0)$. The dual $\mathrm{Ord}_1^\vee=\mathrm{Hom}_{\widehat R}(\mathrm{Ord}_1,\widehat R)$, with the product dual to the comultiplication and the counit as unit, is a commutative $\widehat R$-algebra, finitely generated as a module (Cartier duality). Let $\psi\colon N\to\mu_p(\widehat K)$ be a homomorphism. In $\mathrm{Map}(N,\widehat K)=\widehat K\otimes\mathrm{Ord}_1$ it satisfies $\psi(t+t')=\psi(t)\psi(t')$ and $\psi(0)=1$, so $\ell\mapsto\ell(\psi)$ is a homomorphism of $\widehat K$-algebras $\widehat K\otimes\mathrm{Ord}_1^\vee\to\widehat K$: $(\ell\ell')(\psi)=(\ell\otimes\ell')((t,t')\mapsto\psi(t)\psi(t'))=\ell(\psi)\ell'(\psi)$ and the unit takes the value $\psi(0)=1$. Every $\ell\in\mathrm{Ord}_1^\vee$ is integral over $\widehat R$, so $\ell(\psi)\in\widehat K$ is integral over $\widehat R$ and lies in $\widehat R$. Hence $\ell\mapsto\ell(\psi)$ restricts to an element of $\mathrm{Hom}_{\widehat R}(\mathrm{Ord}_1^\vee,\widehat R)$, which is evaluation at some $f\in\mathrm{Ord}_1$ (a finitely generated projective module is reflexive). Then $\ell(f)=\ell(\psi)$ for every $\ell\in\mathrm{Ord}_1^\vee$; these ℓ span the $\widehat K$-dual of $\mathrm{Map}(N,\widehat K)$, so $\psi=f\in\mathrm{Ord}_1$.

Step 3: $F\in\mathrm{Ord}(\Gamma)$ and its splitting. By Lemma D.100(2), Γ with $T=N\times N$ and the points $(x_t,x_{t'})$ is as in Definition 35.21. The functions regular on a group scheme as in Definition 35.21 form an $\widehat R$-subalgebra of the maps on its group of points (near a point, intersect the two neighbourhoods and add or multiply the two sections); hence every element of Ord_1, a product of sums of regular generators, is regular on $\widehat G[p]$, and for $f\in\mathrm{Ord}_1$ the functions $(t,t')\mapsto f(t)$ and $(t,t')\mapsto f(t')$ are regular on Γ (pull back the local sections along the two projections $\Gamma\to\widehat G[p]$). By Step 1, F is regular on Γ (the global section s). For $(a,b)\in N\times N$, bilinearity gives

$$F(t+a,t'+b)=F(a,b)\,F(t,t')\,\psi_b(t)\,\psi'_a(t'),$$

and $\psi_b,\psi'_a\in\mathrm{Ord}_1$ by Step 2. So every translate of F is a product of functions regular on Γ, hence regular, and $F\in\mathrm{Ord}(\Gamma)$ by definition. By Lemma D.100(2) there are $a_i,b_i\in\mathrm{Ord}_1$ $(1\le i\le r)$ with $F(t,t')=\sum_i a_i(t)b_i(t')$ for all $t,t'\in N$.

Step 4: transport to C. Applying h and using $h\circ g=\iota_C$ on Ω' (both are field embeddings extending $h\circ g=\iota_C|_{R_\mathfrak q}$, and Ω' is the fraction field of $R_\mathfrak q$),

$$K(z,z')=\iota_C(e_p(z,z'))=h(F(z,z'))=\sum_i(h\circ a_i)(z)\,(h\circ b_i)(z'),$$

and $h\circ a_i,h\circ b_i\in\mathrm{Ord}(\widehat G[p])_h=B$ by Proposition D.99(1), under the identification $N=\prod_{\mathfrak P|p}\Gamma_\mathfrak P$ used there. This is (1).

(2) With the decomposition of Step 4, $\chi_x=\sum_i(h\circ a_i)(x)\,(h\circ b_i)$ with coefficients $(h\circ a_i)(x)\in h(\widehat R)\subseteq\mathcal O_C$, so $\chi_x\in B$. Every element of $\mathcal A_{\mathfrak P',C}$ takes values in $\mathcal O_C$ (it is an $\mathcal O_C$-combination of functions with values in $\iota_C(R_\mathfrak q)\subseteq\mathcal O_C$, (Ox2)). The restriction to $\Gamma_\mathfrak P$ (all other components 0) of a generator $z\mapsto\prod_{\mathfrak P'}f_{\mathfrak P'}(z_{\mathfrak P'})$ of B is $\big(\prod_{\mathfrak P'\ne\mathfrak P}f_{\mathfrak P'}(0)\big)f_\mathfrak P\in\mathcal A_{\mathfrak P,C}$; so restriction maps B into $\mathcal A_{\mathfrak P,C}$, and $\chi_x|_{\Gamma_\mathfrak P}\in\mathcal A_{\mathfrak P,C}$.

The argument does not distinguish good and degenerate reduction at $\mathfrak q$: it uses only that $\widehat G$ is semiabelian with abelian generic fibre and part (1) of Proposition D.99. □

Lemma D.116. *Let $\mathfrak T$ be a torsion-regular pre-admissible torsion system for $(\mathcal X,p,\lambda)$ as in Proposition D.97, with p odd, function field Ω' of $\mathcal X'$ and R as in Definition 35.22, let $(\mathfrak q,C,\iota_C)$ be a test point, and let $N=G'(\Omega')[p]$ and $B\subseteq\mathrm{Map}(N,C)$ be as in Lemma D.115. Let e_p, e_{p^R} be the Weil pairings of the polarisation of the cubical line bundle of G' (as in (Fh5)), and for $x\in N$ let $\chi_x\colon N\to C$, $z\mapsto\iota_C(e_p(x,z))$. Put $\Omega_C=\mathcal O_C\otimes_{R_\mathfrak q}(\omega_\mathfrak q/p\,\omega_\mathfrak q)$, and for $f\in B$ and a prime $\mathfrak P\mid p$ of $\mathcal O_E$ let $f|_\mathfrak P$ be the restriction of f to $\Gamma_\mathfrak P\subseteq N$. Then:*

1. *$\omega_\mathfrak q=\bigoplus_u\omega_{u,\mathfrak q}$ (all labels u), so $\Omega_C=\bigoplus_u\mathcal O_C\otimes_{R_\mathfrak q}(\omega_u/p\,\omega_u)_\mathfrak q$; for $f\in B$, $f|_\mathfrak P-f(0)\in\mathcal I_{\mathfrak P,C}$;*

and

$$D(f)=\sum_{\mathfrak{P}\mid p}\ \sum_{u\colon\,\mathfrak{P}_u=\mathfrak{P}}\theta_{u,C}(f|_{\mathfrak{P}}-f(0))$$

defines a surjective $\mathcal{O}_C$-linear map $D\colon B\to\Omega_C$ with $D(ff')=f(0)D(f')+f'(0)D(f)$ for $f,f'\in B$;

2. *for every $x\in N$, every $y\in G'(\Omega')[p^R]$ with $p^{R-1}y=x$ and every Hodge–Tate value $w\in\omega_{\mathfrak{q}}$ of y at $\mathfrak{q}$: $\chi_x\in B$ and $D(\chi_x)=1\otimes(w \bmod p\,\omega_{\mathfrak{q}})$.*

Proof. **Step 0: eigen decomposition of $\omega_{\mathfrak{q}}$.** G' is the pull-back of $G_{\mathcal{X}}$, $\mathrm{Lie}(G')$ is locally free, so $\mathrm{Lie}_{\mathfrak{q}}=\mathrm{Lie}(G')_{\mathfrak{q}}$ and $\omega_{\mathfrak{q}}=\mathrm{Hom}(\mathrm{Lie}_{\mathfrak{q}},R_{\mathfrak{q}})$ is the stalk of $\omega_{G'}=\pi^*\omega_{G_{\mathcal{X}}}$. Since the discriminant of E is a unit in $R_{\mathfrak{q}}$, $\mathcal{O}_E\otimes R_{\mathfrak{q}}\to\prod_{\sigma\in\mathrm{Hom}(E,F)}R_{\mathfrak{q}}$, $a\otimes r\mapsto(\sigma(a)r)_\sigma$, is an isomorphism, and as $u\mapsto\sigma_u$ is a bijection onto $\mathrm{Hom}(E,F)$, $\omega_{\mathfrak{q}}=\bigoplus_u\omega_{u,\mathfrak{q}}$. On $\omega_{u,\mathfrak{q}}$ the ideal $\mathfrak{P}$ acts through $\sigma_u(\mathfrak{P})R_{\mathfrak{q}}$, which is $pR_{\mathfrak{q}}$ if $\mathfrak{P}_u=\mathfrak{P}$ ($\sigma_u(\mathfrak{P})\mathcal{O}'=p\mathcal{O}'$, as $p\mathcal{O}_E=\mathfrak{P}\mathfrak{P}''$ with $\sigma_u(\mathfrak{P}'')\not\subseteq\mathfrak{m}_{\mathcal{O}'}$) and $R_{\mathfrak{q}}$ otherwise ($\sigma_u(\mathfrak{P})\not\subseteq\mathfrak{m}_{\mathcal{O}'}$). Hence $\omega_{\mathfrak{q}}/\mathfrak{P}\omega_{\mathfrak{q}}=\bigoplus_{\mathfrak{P}_u=\mathfrak{P}}(\omega_u/p\,\omega_u)_{\mathfrak{q}}$, the decomposition into σ_u-isotypic components used in (Hf1) of Definition 35.22.

Step 1: D is well defined. Every element of $\mathcal{A}_{\mathfrak{P}',C}$ has values in $\mathcal{O}_C$ ((Ox2): an $\mathcal{O}_C$-combination of functions with values in $\iota_C(R_{\mathfrak{q}})$). For a generator $f(z)=\prod_{\mathfrak{P}'}f_{\mathfrak{P}'}(z_{\mathfrak{P}'})$ of B, $f|_{\mathfrak{P}}=(\prod_{\mathfrak{P}'\neq\mathfrak{P}}f_{\mathfrak{P}'}(0))f_{\mathfrak{P}}\in\mathcal{A}_{\mathfrak{P},C}$; by linearity $f|_{\mathfrak{P}}\in\mathcal{A}_{\mathfrak{P},C}$ for all $f\in B$, and since $1\in\mathcal{A}_{\mathfrak{P},C}$ and $f(0)\in\mathcal{O}_C$, $f|_{\mathfrak{P}}-f(0)\in\mathcal{I}_{\mathfrak{P},C}$. So D is an $\mathcal{O}_C$-linear map $B\to\Omega_C$.

Step 2: Leibniz rule at the origin. For $f,f'\in B$ put $g=f|_{\mathfrak{P}}-f(0)$, $g'=f'|_{\mathfrak{P}}-f'(0)$, both in $\mathcal{I}_{\mathfrak{P},C}$. Then

$$(ff')|_{\mathfrak{P}}-(ff')(0)=gg'+f(0)\,g'+f'(0)\,g,$$

and $\theta_{u,C}(gg')=0$ by Lemma D.107. Summing over $\mathfrak{P}$ and u gives $D(ff')=f(0)D(f')+f'(0)D(f)$.

Step 3: D is onto. Fix $\mathfrak{P}$. For $g\in\mathcal{A}_{\mathfrak{P},C}$ the function $f(z)=g(z_{\mathfrak{P}})$ lies in B (take $f_{\mathfrak{P}'}=1$ for $\mathfrak{P}'\neq\mathfrak{P}$), with $f|_{\mathfrak{P}'}-f(0)=0$ for $\mathfrak{P}'\neq\mathfrak{P}$ and $f|_{\mathfrak{P}}-f(0)=g-g(0)$; so $D(f)=\sum_{\mathfrak{P}_u=\mathfrak{P}}\theta_{u,C}(g-g(0))$. It therefore suffices to show that $(\theta_{u,C})_{\mathfrak{P}_u=\mathfrak{P}}\colon\mathcal{I}_{\mathfrak{P},C}\to\bigoplus_{\mathfrak{P}_u=\mathfrak{P}}\mathcal{O}_C\otimes(\omega_u/p\,\omega_u)_{\mathfrak{q}}$ is onto. Each θ_u is onto by (Ts3), hence so is its stalk at $\mathfrak{q}$ and its $\mathcal{O}_C$-linear extension $\theta_{u,C}$ (Ox2). The characters $\chi_u|_{\mathbb{F}_{\mathfrak{P}}^\times}$ ($\mathfrak{P}_u=\mathfrak{P}$) are homomorphisms to $\mathcal{O}'^\times$ and pairwise distinct: if $\chi_u=\chi_{u'}$, then $\sigma_u(a)\equiv\sigma_{u'}(a)$ modulo $\mathfrak{m}_{\mathcal{O}'}$ for all $a\in\mathcal{O}_E$; writing $\sigma_{u'}=\sigma_u\circ\gamma$ with $\gamma\in\mathrm{Gal}(E/\mathbb{Q})$, $\gamma(a)-a\in\sigma_u^{-1}(\mathfrak{m}_{\mathcal{O}'})=\mathfrak{P}$ for all a, so γ lies in the inertia group of $\mathfrak{P}$, which is trivial, and $u=u'$. By Proposition D.98, $\mathcal{A}_{\mathfrak{P},C}$ is $\mathbb{F}_{\mathfrak{P}}$-stable; $q_{\mathfrak{P}}-1$ is prime to p, a unit of $\mathcal{O}_C$. For $g\in\mathcal{I}_{\mathfrak{P},C}$ put

$$\varepsilon_u(g)=\frac{1}{q_{\mathfrak{P}}-1}\sum_{a\in\mathbb{F}_{\mathfrak{P}}^\times}\chi_u(a)^{-1}\,g(a\,\cdot)\in\mathcal{I}_{\mathfrak{P},C}.$$

By Lemma D.107, $\theta_{u',C}(\varepsilon_u(g))=\frac{1}{q_{\mathfrak{P}}-1}\sum_a\chi_u(a)^{-1}\chi_{u'}(a)\,\theta_{u',C}(g)$, which is $\theta_{u,C}(g)$ if $u'=u$ and 0 otherwise. Given $t_u\in\mathcal{O}_C\otimes(\omega_u/p\,\omega_u)_{\mathfrak{q}}$, choose $g_u\in\mathcal{I}_{\mathfrak{P},C}$ with $\theta_{u,C}(g_u)=t_u$; then $g=\sum_u\varepsilon_u(g_u)$ has $\theta_{u,C}(g)=t_u$ for every u with $\mathfrak{P}_u=\mathfrak{P}$. With Step 0 ($\Omega_C$ is the direct sum over $\mathfrak{P}$ of these targets), D is onto.

Step 4: the value on Weil characters. By Lemma D.115(2) (with the polarisation of the cubical line bundle), $\chi_x\in B$ and $\chi_x|_{\mathfrak{P}}\in\mathcal{A}_{\mathfrak{P},C}$; also $\chi_x(0)=1$. The values of $\chi_x|_{\mathfrak{P}}$ lie in $\iota_C(\Omega')$. By the remark in (Ox2), an $R_{\mathfrak{q}}$-basis (a_j) of $\mathcal{A}_{\mathfrak{P},\mathfrak{q}}$ is an Ω'-basis of $\mathrm{Map}(\Gamma_{\mathfrak{P}},\Omega')$ and a C-basis of $\mathrm{Map}(\Gamma_{\mathfrak{P}},C)$; so the coordinates of $\chi_x|_{\mathfrak{P}}$ lie in $\mathcal{O}_C\cap\iota_C(\Omega')=\iota_C(R_{\mathfrak{q}})$ (Ox1), and $g_{\mathfrak{P}}:=\chi_x|_{\mathfrak{P}}-1$, read in Ω', lies in $\mathcal{I}_{\mathfrak{P},\mathfrak{q}}$. Then $\theta_{u,C}(g_{\mathfrak{P}})=1\otimes\theta_u(g_{\mathfrak{P}})$ with the stalk of θ_u at $\mathfrak{q}$.

Let ϕ be the regular function of (Fh5) for the Hodge–Tate value w of y: defined on an open neighbourhood V of the unit point over $\mathfrak{q}$ in G', with $\phi(z) = e_{p^R}(y,z)$ for all $z \in G'(k)[p^R]$ ($k \supseteq \Omega'$ algebraically closed) and $w(v) \equiv d\phi(v)$ modulo $p^R R_{\mathfrak{q}}$ for $v \in \mathrm{Lie}_{\mathfrak{q}}$. Choose an affine open $U \ni \mathfrak{q}$ with $e(U) \subseteq V$ (e the unit section) and $g_{\mathfrak{P}} \in \mathcal{I}_{\mathfrak{P}}(U)$. Then $W = V \cap (G'[\mathfrak{P}] \times_{\mathcal{X}'} U)$ is an open neighbourhood of the unit section of $G'[\mathfrak{P}] \times_{\mathcal{X}'} U$, and $\tilde{g} = (\phi - 1)|_W$ is regular on W. At a point $z \in \Gamma_{\mathfrak{P}} \subseteq G'(\Omega')[p]$ lying in W,

$$\tilde{g}(z) = e_{p^R}(y,z) - 1 = e_p(p^{R-1}y, z) - 1 = e_p(x,z) - 1 = g_{\mathfrak{P}}(z).$$

By (Hf1), $\theta_u(g_{\mathfrak{P}})$ is the σ_u-component of the class of $e^*d\tilde{g}$ in $e^*\Omega^1_{G'[\mathfrak{P}]/\mathcal{X}'} = \omega_{G'}/\mathfrak{P}\,\omega_{G'}$, and $e^*d\tilde{g}$ is the image of $e^*d\phi \in \omega_{G'}(U)$ under restriction of differentials along $G'[\mathfrak{P}] \subseteq G'$. At $\mathfrak{q}$, $e^*d\phi$ is the form $v \mapsto d\phi(v)$ on $\mathrm{Lie}_{\mathfrak{q}}$, so $e^*d\phi - w \in p^R\omega_{\mathfrak{q}}$ ($\mathrm{Lie}_{\mathfrak{q}}$ is free). Writing $w = \sum_u w_u$ with $w_u \in \omega_{u,\mathfrak{q}}$ (Step 0), we get $\theta_u(g_{\mathfrak{P}}) = w_u \bmod p\,\omega_{u,\mathfrak{q}}$ for every u with $\mathfrak{P}_u = \mathfrak{P}$. Summing,

$$D(\chi_x) = \sum_{\mathfrak{P}} \sum_{\mathfrak{P}_u = \mathfrak{P}} 1 \otimes (w_u \bmod p) = 1 \otimes (w \bmod p\,\omega_{\mathfrak{q}}).$$

□

Lemma D.117. *Let $\mathfrak{T}$ be a torsion-regular pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$ as in Proposition D.97, with p odd, function field Ω' of $\mathcal{X}'$ and R as in Definition 35.22; let $(\mathfrak{q}, C, \iota_C)$ be a test point and $N = G'(\Omega')[p]$, $B = \boxtimes_{\mathfrak{P}'|p}\mathcal{A}_{\mathfrak{P}',C}$ as in Lemma D.115. Let $a \in \Phi$, $s = (a,+)$, $\mathfrak{P} = \mathfrak{P}_s$, $m = m_{\mathfrak{P}}$, $\sigma = \sigma_s$, let $y = (y_1, \dots, y_m)$ be points for $(\mathfrak{T}, s)$, $(\varpi_k)_{1 \le k \le d}$ a $\mathbb{Z}$-basis of $\mathcal{O}_E$ and $(\varpi_k^*)_k$ its dual basis for the trace form, $\mathrm{Tr}(\varpi_k \varpi_{k'}^*) = \delta_{kk'}$. Put $I = \{1, \dots, m\} \times \{1, \dots, d\}$ and $x_{(l,k)} = \varpi_k\, p^{R-1} y_l$. Then $(x_j)_{j \in I}$ is an $\mathbb{F}_p$-basis of N; with coordinate vectors $\hat{x}$ and $\mathrm{af}(B)$ formed for this basis, and the matrices $\mathbf{P} \in M_{m \times I}(C)$, $\mathbf{J} \in M_{I \times m}(C)$,*

$$\mathbf{P}_{i,(l,k)} = \delta_{il}\,\sigma(\varpi_k), \qquad \mathbf{J}_{(l,k),i} = \delta_{li}\,\sigma(\varpi_k^*),$$

the following hold.

1. *The entries of $\mathbf{P}$ and $\mathbf{J}$ lie in $\mathcal{O}_C$, and $\mathbf{PJ} = \mathbf{1}_m$.*
2. *For $\alpha \in \mathcal{O}_E$ with image ξ in $\mathbb{F}_{\mathfrak{P}}$, $\chi_s(\xi) - \sigma(\alpha) \in p\,\mathcal{O}_C$. For $x \in N$ with $\Gamma_{\mathfrak{P}}$-component $\sum_i \xi_i b_i$ and $1 \le i \le m$, $(\mathbf{P}\hat{x})_i - \chi_s(\xi_i) \in p\,\mathcal{O}_C$.*
3. *$l\mathbf{JP} \in \mathrm{af}(B)$ for every $l \in \mathrm{af}(B)$.*
4. *With $\mathrm{af}_C = \{c \in C^m : \ell_c \in \mathcal{A}_{\mathfrak{P},C} + p\,\mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}_C)\}$, the $\mathcal{O}_C$-spans in C^m of the stalks at $\mathfrak{q}$ satisfy $\mathcal{O}_C\,\mathrm{af}_{\mathfrak{q}} = \mathrm{af}_C$ and*

   $$\mathcal{O}_C H_{\mathfrak{q}} = \{u \in C^m : c \cdot u \in \mathcal{O}_C \text{ for all } c \in \mathrm{af}_C\} = \{u \in C^m : l\mathbf{J}u \in \mathcal{O}_C \text{ for all } l \in \mathrm{af}(B)\}.$$

Proof. **The basis.** By (Hf4), $G'(\Omega')[p^R]$ is free over $\mathcal{O}_E/p^R$ with basis $y_1, \dots, y_m$. Its p-torsion is $N = p^{R-1}G'(\Omega')[p^R]$, free over $\mathcal{O}_E/p$ with basis $p^{R-1}y_1, \dots, p^{R-1}y_m$ (multiplication by p^{R-1} induces $\mathcal{O}_E/p \cong p^{R-1}\mathcal{O}_E/p^R\mathcal{O}_E$). As the ϖ_k reduce to an $\mathbb{F}_p$-basis of $\mathcal{O}_E/p$, the $x_{(l,k)}$ form an $\mathbb{F}_p$-basis of N.

Proof of (1). $\sigma(\varpi_k) \in \mathcal{O}' \subseteq \mathcal{O}_C$. The ϖ_k^* span the inverse different $\mathfrak{d}^{-1}$; since $d_E = \pm N(\mathfrak{d}) \in \mathfrak{d}$, $d_E\varpi_k^* \in \mathcal{O}_E$, so $\sigma(\varpi_k^*) \in d_E^{-1}\mathcal{O}' \subseteq \mathcal{O}_C$. Let $\tau_1, \dots, \tau_d$ be the embeddings of E into an algebraic closure and $A = (\tau_i(\varpi_k))_{i,k}$, $A^* = (\tau_i(\varpi_k^*))_{i,k}$. Then $(A^{\mathsf{T}}A^*)_{kk'} = \mathrm{Tr}(\varpi_k\varpi_{k'}^*) = \delta_{kk'}$, so $A^*A^{\mathsf{T}} = \mathbf{1}$, and its diagonal entries give $\tau_i(\sum_k \varpi_k\varpi_k^*) = 1$, i.e. $\sum_k \varpi_k\varpi_k^* = 1$. Hence $(\mathbf{PJ})_{ij} = \sum_{(l,k)} \delta_{il}\delta_{lj}\sigma(\varpi_k)\sigma(\varpi_k^*) = \delta_{ij}\,\sigma(\sum_k \varpi_k\varpi_k^*) = \delta_{ij}$.

Proof of (2). Write $q = q_{\mathfrak{P}}$. As p is unramified, $p\mathcal{O}_E = \mathfrak{P}\mathfrak{P}''$ with $\mathfrak{P}''$ prime to $\mathfrak{P}$, and $\sigma(\mathfrak{P}'') \not\subseteq \mathfrak{m}_{\mathcal{O}'}$ (as $\sigma^{-1}(\mathfrak{m}_{\mathcal{O}'}) = \mathfrak{P}$), so $\sigma(\mathfrak{P})\mathcal{O}' = p\mathcal{O}'$. If $\xi = 0$, then $\alpha \in \mathfrak{P}$ and $\sigma(\alpha) \in p\mathcal{O}'$, while $\chi_s(0) = 0$. If $\xi \neq 0$, put $v = \sigma(\alpha)$; from $\alpha^q - \alpha \in \mathfrak{P}$ we get $v^q - v \in p\mathcal{O}'$, and by induction $v^{q^{k+1}} - v^{q^k} \in p^{k+1}\mathcal{O}'$ (if $v^{q^k} = v^{q^{k-1}} + p^k r$, raise to the q-th power: all binomial terms but the first are divisible by p^{k+1}). So $\tau = \lim_k v^{q^k}$ exists in the complete ring $\mathcal{O}_C$, $\tau - v \in p\mathcal{O}_C$ ($p\mathcal{O}_C$ is closed), $\tau^q = \tau$ and $\tau \not\equiv 0$ modulo $\mathfrak{m}_C$, so τ is a $(q-1)$-th root of unity congruent to $\sigma(\alpha)$ modulo $\mathfrak{m}_C$. Distinct $(q-1)$-th roots of unity are distinct modulo $\mathfrak{m}_C$ ($p \nmid q-1$), so $\tau = \chi_s(\xi)$, which gives the first claim.

For $x \in N$ put $\alpha_l = \sum_k \hat{x}_{(l,k)}\varpi_k \in \mathcal{O}_E$; then $x = \sum_l \alpha_l\, p^{R-1} y_l$ and $(\mathbf{P}\hat{x})_i = \sum_k \sigma(\varpi_k)\hat{x}_{(i,k)} = \sigma(\alpha_i)$. The decomposition $N = \bigoplus_{\mathfrak{P}'} \Gamma_{\mathfrak{P}'}$ is stable under $\mathcal{O}_E$, which acts on $\Gamma_{\mathfrak{P}}$ through $\mathbb{F}_{\mathfrak{P}} = \mathcal{O}_E/\mathfrak{P}$, and the $\Gamma_{\mathfrak{P}}$-component of $p^{R-1}y_l$ is b_l (Hf4). So ξ_i is the image of α_i, and $(\mathbf{P}\hat{x})_i - \chi_s(\xi_i) = \sigma(\alpha_i) - \chi_s(\xi_i) \in p\,\mathcal{O}_C$.

Proof of (3). First, $f \circ [\alpha] \in B$ for $f \in B$ and $\alpha \in \mathcal{O}_E$ ($[\alpha]$ the action on N): on a generator $z \mapsto \prod_{\mathfrak{P}'} f_{\mathfrak{P}'}(z_{\mathfrak{P}'})$ of B, $(\alpha z)_{\mathfrak{P}'} = \bar{\alpha} z_{\mathfrak{P}'}$ with $\bar{\alpha}$ the image of α in $\mathbb{F}_{\mathfrak{P}'}$, and $\mathcal{A}_{\mathfrak{P}',C}$ is $\mathbb{F}_{\mathfrak{P}'}$-stable (Proposition D.98). For $\alpha \in \mathcal{O}_E$ let $\mathbf{M}(\alpha) \in M_I(\mathbb{Z})$, $\mathbf{M}(\alpha)_{(l',t),(l,k)} = \delta_{ll'} c_t(\alpha\varpi_k)$. Since $\alpha x = \sum_{l,k} \hat{x}_{(l,k)}\alpha\varpi_k\, p^{R-1} y_l = \sum_{l,t}\big(\sum_k c_t(\alpha\varpi_k)\hat{x}_{(l,k)}\big)x_{(l,t)}$, we have $\widehat{\alpha x} \equiv \mathbf{M}(\alpha)\hat{x}$ modulo $p\mathbb{Z}^I$. Let $l \in \mathrm{af}(B)$, $\lambda_l = f + pg$ with $f \in B$. For $x \in N$, l being integral,

$$\lambda_{l\mathbf{M}(\alpha)}(x) = l\,\mathbf{M}(\alpha)\hat{x} \equiv l\,\widehat{\alpha x} = f(\alpha x) + p\,g(\alpha x) \equiv f(\alpha x) \pmod{p\,\mathcal{O}_C},$$

so $l\mathbf{M}(\alpha) \in \mathrm{af}(B)$. Finally $\mathbf{JP} = \sum_t \sigma(\varpi_t^*)\mathbf{M}(\varpi_t)$: the $((l,k),(l',k'))$ entry of the right side is $\delta_{ll'}\sigma\big(\sum_t \mathrm{Tr}(\varpi_t\varpi_{k'}\varpi_k^*)\varpi_t^*\big) = \delta_{ll'}\sigma(\varpi_{k'}\varpi_k^*)$, which is the entry of $\mathbf{JP}$. As $\mathrm{af}(B)$ is an $\mathcal{O}_C$-module and $\sigma(\varpi_t^*) \in \mathcal{O}_C$ by (1), $l\mathbf{JP} = \sum_t \sigma(\varpi_t^*)\, l\mathbf{M}(\varpi_t) \in \mathrm{af}(B)$.

Proof of (4). Let $\mathcal{M} = \mathcal{A}_{\mathfrak{P}} + p\,\mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}_{\mathcal{X}'})$, a quasi-coherent subsheaf of $\mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}_{\mathcal{X}'})$, and $\ell\colon c \mapsto \ell_c$, an injective Ω'-linear map $V_H \to \mathrm{Map}(\Gamma_{\mathfrak{P}}, \Omega')$ ($\ell_c(b_i) = c_i$) with $\ell(\mathcal{O}'^m) \subseteq \mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O})$. By definition $\mathrm{af} = \ell^{-1}(\mathcal{M})$ sectionwise, so $\mathrm{af}_{\mathfrak{q}} = \ell^{-1}(\mathcal{M}_{\mathfrak{q}})$; evaluation at the b_i gives $p\,\mathcal{O}'^m \subseteq \mathrm{af} \subseteq \mathcal{O}'^m$. Hence af is the kernel of $\mathcal{O}'^m \to \mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O})/\mathcal{M}$, a coherent sheaf, and $\mathrm{af}_{\mathfrak{q}}$ is free of rank m over the discrete valuation ring $R_{\mathfrak{q}}$. The dual lattice H is $\mathcal{H}om(\mathrm{af}, \mathcal{O})$ realised in V_H by the dot product (as $\Omega'\mathrm{af} = V_H$), and $\mathcal{H}om$ of a coherent sheaf commutes with stalks; so $H_{\mathfrak{q}}$ is the dual lattice of $\mathrm{af}_{\mathfrak{q}}$.

$\mathcal{O}_C$ is torsion-free, hence flat, over $R_{\mathfrak{q}}$. Tensoring the exact sequence $0 \to \mathrm{af}_{\mathfrak{q}} \xrightarrow{\ell} \mathcal{M}_{\mathfrak{q}} \to \mathrm{Map}(\Gamma_{\mathfrak{P}}, \Omega')/\ell(V_H)$ with $\mathcal{O}_C$, and using that an $R_{\mathfrak{q}}$-basis of $\mathcal{A}_{\mathfrak{P},\mathfrak{q}}$ is a C-basis of $\mathrm{Map}(\Gamma_{\mathfrak{P}}, C)$ (remark in (Ox2)), gives $\mathcal{O}_C\mathrm{af}_{\mathfrak{q}} = \{c \in C'^m :\ \ell_c \in \mathcal{O}_C\mathcal{M}_{\mathfrak{q}}\}$ with $\mathcal{O}_C\mathcal{M}_{\mathfrak{q}} = \mathcal{A}_{\mathfrak{P},C} + p\,\mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}_C)$; this is af_C. A basis of $\mathrm{af}_{\mathfrak{q}}$ is an $\mathcal{O}_C$-basis of af_C, and its dual basis is an $R_{\mathfrak{q}}$-basis of $H_{\mathfrak{q}}$ and an $\mathcal{O}_C$-basis of the dual lattice of af_C; so $\mathcal{O}_C H_{\mathfrak{q}}$ is the dual lattice of af_C. This is the first equality.

For the second apply Lemma D.118 with $M = B$ (whose elements take values in $\mathcal{O}_C$, being $\mathcal{O}_C$-combinations of products of elements of the $\mathcal{A}_{\mathfrak{P}',C}$, which do by (Ox2)), the basis (x_j), $\mathbb{F} = \mathbb{F}_{\mathfrak{P}}$, $\Gamma = \Gamma_{\mathfrak{P}}$ with the basis (b_i), i the inclusion, π the projection onto the $\Gamma_{\mathfrak{P}}$-component, $\chi = \chi_s$, $P = \mathbf{P}$, $J = \mathbf{J}$. Its hypotheses: $\pi \circ i = \mathrm{id}$; for $f \in B$, $f \circ i \circ \pi\colon z \mapsto f(z_{\mathfrak{P}})$ lies in B (on a generator it is $\prod_{\mathfrak{P}' \neq \mathfrak{P}} f_{\mathfrak{P}'}(0) \cdot f_{\mathfrak{P}}(z_{\mathfrak{P}})$); $M_\Gamma = \{f|_{\Gamma_{\mathfrak{P}}} : f \in B\} = \mathcal{A}_{\mathfrak{P},C}$ (a generator restricts to $\prod_{\mathfrak{P}' \neq \mathfrak{P}} f_{\mathfrak{P}'}(0) \cdot f_{\mathfrak{P}}$, and $g \in \mathcal{A}_{\mathfrak{P},C}$ is the restriction of $z \mapsto g(z_{\mathfrak{P}}) \in B$), so its af_Γ is af_C and its H_Γ is the dual lattice of af_C; $\chi_s(0) = 0$, $\chi_s(1) = 1$; $\mathbf{PJ} = \mathbf{1}$ by (1); the congruence by (2); the stability by (3). Lemma D.118(2) is the second equality. □

Lemma D.118. *Let p be a prime, N a finite-dimensional $\mathbb{F}_p$-vector space with a basis $(x_j)_{j\in I}$, and for $x \in N$ let $\hat{x} \in \{0,\dots,p-1\}^I$ be its coordinate vector. Let $M \subseteq \mathrm{Map}(N, \mathcal{O}_C)$ be an $\mathcal{O}_C$-submodule and, with $\lambda_l(x) = l\hat{x}$ for $l \in \mathcal{O}_C^I$,*

$$\mathrm{af}(M) = \{l \in \mathcal{O}_C^I :\ \lambda_l \in M + p\,\mathrm{Map}(N, \mathcal{O}_C)\}.$$

Let $\mathbb{F}$ be a finite field of characteristic p, Γ an $\mathbb{F}$-vector space with a basis $b_1,\dots,b_m$, and $i\colon \Gamma \to N$, $\pi\colon N \to \Gamma$ additive maps with $\pi \circ i = \mathrm{id}_\Gamma$ such that $f \circ i \circ \pi \in M$ for every $f \in M$. Put $M_\Gamma = \{f \circ i :\ f \in M\}$. Let $\chi\colon \mathbb{F} \to \mathcal{O}_C$ be a map with $\chi(0) = 0$, $\chi(1) = 1$; for $c \in C^m$ let $\ell_c(y) = \sum_k c_k \chi(y_k)$ for $y = \sum_k y_k b_k \in \Gamma$, and put

$$\mathrm{af}_\Gamma = \{c \in C^m :\ \ell_c \in M_\Gamma + p\,\mathrm{Map}(\Gamma, \mathcal{O}_C)\}, \qquad H_\Gamma = \{u \in C^m :\ c \cdot u \in \mathcal{O}_C \text{ for all } c \in \mathrm{af}_\Gamma\}.$$

Let $P \in M_{m\times I}(\mathcal{O}_C)$ and $J \in M_{I\times m}(\mathcal{O}_C)$ with $PJ = \mathbf{1}_m$, such that

$$(P\hat{x})_k - \chi(\pi(x)_k) \in p\,\mathcal{O}_C \quad (x \in N,\ 1 \le k \le m), \qquad lJP \in \mathrm{af}(M) \quad (l \in \mathrm{af}(M)),$$

where $\pi(x) = \sum_k \pi(x)_k b_k$. Then:

1. *$\mathrm{af}_\Gamma \subseteq \mathcal{O}_C^m$, and for $l \in \mathcal{O}_C^m$: $l \in \mathrm{af}_\Gamma$ if and only if $lP \in \mathrm{af}(M)$;*
2. *$H_\Gamma = \{u \in C^m :\ lJu \in \mathcal{O}_C \text{ for all } l \in \mathrm{af}(M)\}$.*

Proof. A congruence. For $l \in \mathcal{O}_C^m$ and $x \in N$,

$$\lambda_{lP}(x) - \ell_l(\pi(x)) = \sum_k l_k((P\hat{x})_k - \chi(\pi(x)_k)) \in p\,\mathcal{O}_C,$$

since $\lambda_{lP}(x) = lP\hat{x}$ and the l_k are integral.

Proof of (1). Let $c \in \mathrm{af}_\Gamma$, $\ell_c = g \circ i + pf$ with $g \in M$, $f \in \mathrm{Map}(\Gamma, \mathcal{O}_C)$. Since $\chi(0) = 0$ and $\chi(1) = 1$, $\ell_c(b_k) = c_k$, and $g(i(b_k)) \in \mathcal{O}_C$; so $c_k \in \mathcal{O}_C$. This proves $\mathrm{af}_\Gamma \subseteq \mathcal{O}_C^m$.

Let $l \in \mathrm{af}_\Gamma$ and write $\ell_l = g \circ i + pf$ as above. Then $G = g \circ i \circ \pi \in M$, and for $x \in N$ the congruence gives $\lambda_{lP}(x) \equiv \ell_l(\pi(x)) = G(x) + pf(\pi(x)) \equiv G(x)$ modulo $p\,\mathcal{O}_C$. So $\lambda_{lP} \in M + p\,\mathrm{Map}(N, \mathcal{O}_C)$ and $lP \in \mathrm{af}(M)$.

Conversely let $l \in \mathcal{O}_C^m$ with $lP \in \mathrm{af}(M)$, $\lambda_{lP} = g + pf$ with $g \in M$. For $y \in \Gamma$, $\pi(i(y)) = y$, so the congruence at $x = i(y)$ gives $\ell_l(y) \equiv \lambda_{lP}(i(y)) = g(i(y)) + pf(i(y)) \equiv g(i(y))$ modulo $p\,\mathcal{O}_C$. Hence $\ell_l - g \circ i \in p\,\mathrm{Map}(\Gamma, \mathcal{O}_C)$, $g \circ i \in M_\Gamma$, and $l \in \mathrm{af}_\Gamma$.

Proof of (2). Let $u \in H_\Gamma$ and $l \in \mathrm{af}(M)$. Then $lJ \in \mathcal{O}_C^m$ and $(lJ)P = lJP \in \mathrm{af}(M)$, so $lJ \in \mathrm{af}_\Gamma$ by (1), and $lJu = (lJ) \cdot u \in \mathcal{O}_C$. Conversely let $u \in C^m$ with $lJu \in \mathcal{O}_C$ for all $l \in \mathrm{af}(M)$, and let $c \in \mathrm{af}_\Gamma$. By (1), $c \in \mathcal{O}_C^m$ and $cP \in \mathrm{af}(M)$, so $c \cdot u = cPJu = (cP)Ju \in \mathcal{O}_C$. Hence $u \in H_\Gamma$. □

Lemma D.119. *Let $\mathfrak{T}$, p (odd), Ω', R, the test point $(\mathfrak{q}, C, \iota_C)$, $a \in \Phi$, $s = (a,+)$, $\bar{s} = (a,-)$, $\kappa = \mathrm{rk}(s)$, $\mathfrak{P} = \mathfrak{P}_s$, $m = m_{\mathfrak{P}}$, $\sigma = \sigma_s$, the points $y = (y_1,\dots,y_m)$, (ϖ_k), (ϖ_k^*), I, $x_{(l,k)}$ and $\mathbf{J}$ be as in Lemma D.117, and assume $R \ge 3$. Let $w \in \mathcal{O}'$ with $w^{p-1} = up$, $u \in \mathcal{O}'^\times$, bases $\beta, \bar{\beta}$ as in Definition 35.22, and (ι, h) a Hodge–Tate chart for $(\mathfrak{T}, s, \beta, \bar{\beta})$ over an open $U \ni \mathfrak{q}$ with points y and forms $\eta_1,\dots,\eta_m$; ι, h also denote their C-linear extensions, and $e_1,\dots,e_m$ is the standard basis of C^m. Let $g = \dim_{\Omega'} \mathrm{Lie}(G'_{\Omega'})$, $w_1,\dots,w_g$ an $R_{\mathfrak{q}}$-basis of $\omega_{\mathfrak{q}}$ (extended Ω'-linearly to $\mathrm{Lie}(G'_{\Omega'})$), $[v] \in C^g$ the coordinate vector of $v \in \omega_{\mathfrak{q}} \otimes C$, and $\Lambda \in M_{g\times\kappa}(\Omega')$, $\bar{\Lambda} \in M_{g\times(n-\kappa)}(\Omega')$ the matrices with entries $\Lambda_{ik} = w_i(\beta_k)$, $\bar{\Lambda}_{ik} = w_i(\bar{\beta}_k)$. Let $\mathbf{h} \in M_{g\times I}(\mathcal{O}_C)$ be such that $[v] - \mathbf{h}\hat{x} \in p\,\mathcal{O}_C^g$ for every $x \in N$, every $y' \in G'(\Omega')[p^R]$ with $p^{R-1}y' = x$ and every Hodge–Tate value v of y' at $\mathfrak{q}$. Let $\zeta \in \Omega'$ be a primitive p^R-th root of unity, $e_{p^R}(y', y'') = \zeta^{\varepsilon(y',y'')}$ the Weil pairing of the polarisation on $G'(\Omega')[p^R]$ with $\varepsilon(y', y'') \in \{0,\dots,p^R - 1\}$, and put*

$$\mathbf{E}_\varepsilon = \big(\varepsilon(\varpi_k y_l, \varpi_{k'} y_{l'})\big)_{(l,k),(l',k')\in I} \in M_I(\mathbb{Z}), \qquad g_{lj} = \sum_k \sigma(\varpi_k^*)\,\varepsilon(\varpi_k y_l, y_j).$$

Then:

1. *for every* $(l,k)\in I$, *with* $\mathbf{h}_{(l,k)}$ *the column of* $\mathbf{h}$ *of index* (l,k): $\bar\Lambda^{\mathsf T}\mathbf{h}_{(l,k)}-\sigma(\varpi_k)\,h(e_l)\in p\,\bar\Lambda^{\mathsf T}\mathcal{O}_C^{g}$;
2. *for every* $c\in V_L$ *with* $w\sum_k c_k\beta_k\in\mathrm{Lie}_{\mathfrak q}$, $\iota(c)\in R_{\mathfrak q}^{m}$ *and* $\sum_l \iota(c)_l\,g_{lj}-\eta_j\big(\sum_k c_k\beta_k\big)\in p^{R-2}R_{\mathfrak q}$ *for* $1\le j\le m$:
$$\mathbf{E}_\varepsilon\mathbf{J}\,\iota(c)+\mathbf{h}^{\mathsf T}\Lambda c\in w\,\mathcal{O}_C^{I}.$$

Proof. **Step 1: Hodge–Tate values of ϖy_j.** Let $\alpha\in\mathcal{O}_E$ and $1\le j\le m$. By (Hf4) the restriction of η_j to $\mathrm{Lie}_{\mathfrak q}$ is a Hodge–Tate value of y_j at $\mathfrak q$: there is a regular function ϕ on an open neighbourhood V of the unit point over $\mathfrak q$ with $\phi(z)=e_{p^R}(y_j,z)$ for $z\in G'(k)[p^R]$ ($k\supseteq\Omega'$ algebraically closed) and $d\phi(t)\equiv\eta_j(t)$ modulo $p^RR_{\mathfrak q}$ for $t\in\mathrm{Lie}_{\mathfrak q}$. Then $\phi\circ\iota(\bar\alpha)$ is regular on the open neighbourhood $\iota(\bar\alpha)^{-1}(V)$ of the unit point over $\mathfrak q$, and by the Rosati condition $\phi(\iota(\bar\alpha)z)=e_{p^R}(y_j,\bar\alpha z)=e_{p^R}(\alpha y_j,z)$; its differential is $t\mapsto d\phi(\mathrm{Lie}(\iota(\bar\alpha))t)\equiv\eta_j(\mathrm{Lie}(\iota(\bar\alpha))t)$ modulo $p^RR_{\mathfrak q}$. So $v_{j,\alpha}:=\eta_j\circ\mathrm{Lie}(\iota(\bar\alpha))$, restricted to $\mathrm{Lie}_{\mathfrak q}$, is a Hodge–Tate value of αy_j at $\mathfrak q$. Since $p^{R-1}(\varpi_k y_j)=x_{(j,k)}$, whose coordinate vector is the unit vector of index (j,k), the hypothesis on $\mathbf{h}$ gives
$$[v_{j,\varpi_k}]-\mathbf{h}_{(j,k)}\in p\,\mathcal{O}_C^{g}. \tag{D.26}$$

Step 2: proof of (1). For $v\in\omega_{\mathfrak q}\otimes C$, $v=\sum_i[v]_iw_i$ and so $\bar\Lambda^{\mathsf T}[v]=(v(\bar\beta_r))_r$. By Step 1 with $\alpha=\varpi_k$, and as $\mathrm{Lie}(\iota(\bar\varpi_k))$ acts on $\mathrm{Li}_{\bar s}$ by $\sigma_{\bar s}(\bar\varpi_k)=\sigma(\varpi_k)$,
$$\bar\Lambda^{\mathsf T}[v_{l,\varpi_k}]=\big(\eta_l(\sigma(\varpi_k)\bar\beta_r)\big)_r=\sigma(\varpi_k)\,(\eta_l(\bar\beta_r))_r=\sigma(\varpi_k)\,h(e_l)$$
by the definition of the forms of a chart. Applying $\bar\Lambda^{\mathsf T}$ to (D.26) (with $j=l$) gives (1).

Step 3: the Hodge–Tate side of (2). Let c be as in (2), $A=\sum_k c_k\beta_k\in\mathrm{Li}_s$, so $wA\in\mathrm{Lie}_{\mathfrak q}$, and fix $(j,k')\in I$. By (D.26), $\mathbf{h}_{(j,k')}=[v_{j,\varpi_{k'}}]-p\,r$ with $r\in\mathcal{O}_C^{g}$, hence
$$(\mathbf{h}^{\mathsf T}\Lambda c)_{(j,k')}=\sum_i(\mathbf{h}_{(j,k')})_i\,w_i(A)=v_{j,\varpi_{k'}}(A)-\frac{p}{w}\sum_i r_i\,w_i(wA).$$
The last sum lies in $\mathcal{O}_C$ ($w_i(wA)\in R_{\mathfrak q}$), so the last term lies in $w\mathcal{O}_C$. As $\mathrm{Lie}(\iota(\bar\varpi_{k'}))$ acts on Li_s by $\sigma(\bar\varpi_{k'})$, $v_{j,\varpi_{k'}}(A)=\sigma(\bar\varpi_{k'})\eta_j(A)$, and by the hypothesis on c, $\eta_j(A)-\sum_l\iota(c)_lg_{lj}\in p^{R-2}R_{\mathfrak q}\subseteq p\,\mathcal{O}_C$ ($R\ge3$). Therefore
$$(\mathbf{h}^{\mathsf T}\Lambda c)_{(j,k')}\equiv\sigma(\bar\varpi_{k'})\sum_l\iota(c)_l\,g_{lj}\pmod{w\,\mathcal{O}_C}.$$

Step 4: the Weil side of (2). The Weil pairing e_{p^R} is bilinear and alternating and satisfies the Rosati condition $e_{p^R}(\alpha y',y'')=e_{p^R}(y',\bar\alpha y'')$; so modulo p^R, ε is biadditive, $\varepsilon(y',y'')\equiv-\varepsilon(y'',y')$ and $\varepsilon(y',\alpha y'')\equiv\varepsilon(\bar\alpha y',y'')$. Hence
$$\varepsilon(\varpi_{k'}y_j,\varpi_ky_l)\equiv-\varepsilon(\varpi_ky_l,\varpi_{k'}y_j)\equiv-\varepsilon(\bar\varpi_{k'}\varpi_ky_l,y_j)\equiv-\sum_t c_t(\bar\varpi_{k'}\varpi_k)\,\varepsilon(\varpi_ty_l,y_j)\pmod{p^R}.$$
Moreover $\sum_k\sigma(\varpi_k^*)c_t(\bar\varpi_{k'}\varpi_k)=\sigma\big(\sum_k\mathrm{Tr}(\bar\varpi_{k'}\varpi_t^*\varpi_k)\varpi_k^*\big)=\sigma(\bar\varpi_{k'}\varpi_t^*)$. Since $(\mathbf{J}\iota(c))_{(l,k)}=\sigma(\varpi_k^*)\iota(c)_l$ with $\iota(c)_l\in R_{\mathfrak q}$,
$$\begin{aligned}(\mathbf{E}_\varepsilon\mathbf{J}\iota(c))_{(j,k')}&=\sum_{l,k}\varepsilon(\varpi_{k'}y_j,\varpi_ky_l)\,\sigma(\varpi_k^*)\,\iota(c)_l\\&\equiv-\sigma(\bar\varpi_{k'})\sum_l\iota(c)_l\sum_t\sigma(\varpi_t^*)\varepsilon(\varpi_ty_l,y_j)=-\sigma(\bar\varpi_{k'})\sum_l\iota(c)_l\,g_{lj}\end{aligned}$$
modulo $p^R\mathcal{O}_C$. Adding Step 3 gives (2). □

Lemma D.120. *Let p be an odd prime and $w, c \in \mathcal{O}_C$ with $v(w) = v(c) = \frac{1}{p-1}$. Let I be a finite set, $g, m, r, r' \geq 0$ integers, $\mathrm{AF} \subseteq \mathcal{O}_C^I$ an $\mathcal{O}_C$-submodule of row vectors, $\mathbf{h} \in M_{g\times I}(\mathcal{O}_C)$ and $\mathbf{e} \in \mathrm{GL}_I(\mathcal{O}_C)$, such that:*

1. *$c\,l \in \mathrm{AF}$ for every $l \in \mathcal{O}_C^I$;*
2. *every row of $\mathbf{h}$ lies in AF;*
3. *if $y \in \mathcal{O}_C^I$ and $y^{\mathsf{T}}\mathbf{e} \in \mathrm{AF}$, then $\mathbf{h}y \in c\,\mathcal{O}_C^g$;*
4. *for every $u \in \mathcal{O}_C^g$ there is $y \in \mathcal{O}_C^I$ with $y^{\mathsf{T}}\mathbf{e} \in \mathrm{AF}$ and $\mathbf{h}y - cu \in p\,\mathcal{O}_C^g$;*
5. *for every $u \in \mathcal{O}_C^g$ there is $x \in C^I$ with $lx \in \mathcal{O}_C$ for all $l \in \mathrm{AF}$ and $\mathbf{h}x = u$.*

Let $\mathbf{P} \in M_{m\times I}(\mathcal{O}_C)$, $\mathbf{J} \in M_{I\times m}(\mathcal{O}_C)$ with $\mathbf{PJ} = \mathbf{1}_m$ and $l\mathbf{JP} \in \mathrm{AF}$ for every $l \in \mathrm{AF}$, and put $H = \{u \in C^m : \ l\mathbf{J}u \in \mathcal{O}_C \text{ for all } l \in \mathrm{AF}\}$. Let $\Lambda \in M_{g\times r}(C)$, $\bar{\Lambda} \in M_{g\times r'}(C)$, $\mathrm{lie} = \{a \in C^r : \Lambda a \in \mathcal{O}_C^g\}$, and $W \subseteq C^r$ an $\mathcal{O}_C$-submodule with $\mathrm{lie} \subseteq W$ and $wW \subseteq \mathrm{lie}$. Let $h\colon C^m \to C^{r'}$ and $\iota\colon C^r \to C^m$ be C-linear maps such that:

6. *$\bar{\Lambda}^{\mathsf{T}}\mathbf{h} - h(\mathbf{P}) \in p\,\bar{\Lambda}^{\mathsf{T}} M_{g\times I}(\mathcal{O}_C)$;*
7. *$\iota^{-1}(\mathcal{O}_C^m) = W$;*
8. *$\mathbf{eJ}\iota(a) + \mathbf{h}^{\mathsf{T}}\Lambda a \in w\,\mathcal{O}_C^I$ for every $a \in W$.*

Then $h(H) = \bar{\Lambda}^{\mathsf{T}}\mathcal{O}_C^g$ and $\iota^{-1}(H) = w^{-1}\mathrm{lie}$.

Proof. Put $t = p/w$. Then $v(t) = 1 - \frac{1}{p-1}$, and as $p \geq 3$, $\frac{1}{p-1} \leq v(t) < 1$: so $t \in w\,\mathcal{O}_C \cap \mathfrak{m}_C$. H is an $\mathcal{O}_C$-submodule of C^m.

Step 0: denominators of the dual. If $x \in C^I$ and $lx \in \mathcal{O}_C$ for all $l \in \mathrm{AF}$, then $wx \in \mathcal{O}_C^I$. Indeed, by (1) the row c times the unit row of index j lies in AF, so $cx_j \in \mathcal{O}_C$ and $v(x_j) \geq -v(c) = -v(w)$.

Step 1: the key identity. If $x \in C^I$ with $wx \in \mathcal{O}_C^I$, then $h(\mathbf{P}x) = \bar{\Lambda}^{\mathsf{T}}(\mathbf{h}x - tu')$ for some $u' \in \mathcal{O}_C^g$. Indeed, by (6), $h(\mathbf{P}_j) = \bar{\Lambda}^{\mathsf{T}}(\mathbf{h}_j - pU_j)$ with $U_j \in \mathcal{O}_C^g$ for every column index $j \in I$, so

$$h(\mathbf{P}x) = \sum_j x_j\, h(\mathbf{P}_j) = \bar{\Lambda}^{\mathsf{T}}\Big(\mathbf{h}x - t\sum_j (wx_j)U_j\Big),$$

and $u' = \sum_j (wx_j)U_j \in \mathcal{O}_C^g$.

Step 2: $h(H) \subseteq \bar{\Lambda}^{\mathsf{T}}\mathcal{O}_C^g$. Let $u_0 \in H$ and $x = \mathbf{J}u_0$. Then $lx \in \mathcal{O}_C$ for $l \in \mathrm{AF}$, so $wx \in \mathcal{O}_C^I$ (Step 0), and $\mathbf{P}x = u_0$. By Step 1, $h(u_0) = \bar{\Lambda}^{\mathsf{T}}(\mathbf{h}x - tu')$, and $(\mathbf{h}x)_k = \mathbf{h}_k\mathbf{J}u_0 \in \mathcal{O}_C$ for each row $\mathbf{h}_k$ of $\mathbf{h}$, by (2) and $u_0 \in H$. So $h(u_0) \in \bar{\Lambda}^{\mathsf{T}}\mathcal{O}_C^g$.

Step 3: $\bar{\Lambda}^{\mathsf{T}}\mathcal{O}_C^g \subseteq h(H)$. Let $N' = \{u \in \mathcal{O}_C^g : \ \bar{\Lambda}^{\mathsf{T}}u \in h(H)\}$, an $\mathcal{O}_C$-submodule. Let $u \in \mathcal{O}_C^g$ and x as in (5). By Step 0, $wx \in \mathcal{O}_C^I$; and $\mathbf{P}x \in H$, since $l\mathbf{J}(\mathbf{P}x) = (l\mathbf{JP})x \in \mathcal{O}_C$ for $l \in \mathrm{AF}$. By Step 1, $h(\mathbf{P}x) = \bar{\Lambda}^{\mathsf{T}}(u - tu')$ with $u' \in \mathcal{O}_C^g$, so $u - tu' \in N'$. Hence $\mathcal{O}_C^g = N' + t\,\mathcal{O}_C^g$ with $t \in \mathfrak{m}_C$, and $N' = \mathcal{O}_C^g$ by Nakayama's lemma. So $\bar{\Lambda}^{\mathsf{T}}u \in h(H)$ for every $u \in \mathcal{O}_C^g$.

Step 4: an equivalence. Let $a \in C^r$ with $wa \in W$ and $y \in \mathcal{O}_C^I$. Then

$$y^{\mathsf{T}}\mathbf{eJ}\iota(a) \in \mathcal{O}_C \iff (\mathbf{h}y)^{\mathsf{T}}\Lambda(wa) \in w\,\mathcal{O}_C.$$

Indeed, by (8) for wa, $\mathbf{eJ}\iota(wa) = -\mathbf{h}^{\mathsf{T}}\Lambda(wa) + wz$ with $z \in \mathcal{O}_C^I$, so $w\,y^{\mathsf{T}}\mathbf{eJ}\iota(a) = -(\mathbf{h}y)^{\mathsf{T}}\Lambda(wa) + w\,y^{\mathsf{T}}z$ with $y^{\mathsf{T}}z \in \mathcal{O}_C$.

Step 5: $\iota^{-1}(H) \subseteq w^{-1}\mathrm{lie}$. Let $a \in C^r$ with $\iota(a) \in H$. For $1 \le k \le m$ the row $c\,\mathbf{P}_k$ ($\mathbf{P}_k$ the k-th row of $\mathbf{P}$) lies in AF by (1), and $c\,\mathbf{P}_k\mathbf{J}\iota(a) = c\,\iota(a)_k$ as $\mathbf{PJ} = \mathbf{1}$; so $c\,\iota(a)_k \in \mathcal{O}_C$, $w\iota(a) \in \mathcal{O}_C^m$, and $wa \in W$ by (7). Let $u \in \mathcal{O}_C^g$ and y as in (4), $\mathbf{h}y = cu + pu''$ with $u'' \in \mathcal{O}_C^g$. As $y^{\mathsf{T}}\mathbf{e} \in \mathrm{AF}$ and $\iota(a) \in H$, $y^{\mathsf{T}}\mathbf{e}\mathbf{J}\iota(a) \in \mathcal{O}_C$, so by Step 4

$$(\mathbf{h}y)^{\mathsf{T}}\Lambda(wa) = c\,u^{\mathsf{T}}\Lambda(wa) + t\,u''^{\mathsf{T}}\Lambda(w\cdot wa) \in w\,\mathcal{O}_C.$$

Since $w \cdot wa \in wW \subseteq \mathrm{lie}$, $\Lambda(w\cdot wa) \in \mathcal{O}_C^g$ and the second term lies in $t\,\mathcal{O}_C \subseteq w\,\mathcal{O}_C$. So $c\,u^{\mathsf{T}}\Lambda(wa) \in w\,\mathcal{O}_C$, i.e. $u^{\mathsf{T}}\Lambda(wa) \in \mathcal{O}_C$. As u is arbitrary, $\Lambda(wa) \in \mathcal{O}_C^g$: $wa \in \mathrm{lie}$.

Step 6: $w^{-1}\mathrm{lie} \subseteq \iota^{-1}(H)$. Let $wa \in \mathrm{lie}$; then $wa \in W$. Let $l \in \mathrm{AF}$ and $y = (l\mathbf{e}^{-1})^{\mathsf{T}} \in \mathcal{O}_C^I$, so $y^{\mathsf{T}}\mathbf{e} = l \in \mathrm{AF}$ and $\mathbf{h}y = cu$ with $u \in \mathcal{O}_C^g$ by (3). Then $(\mathbf{h}y)^{\mathsf{T}}\Lambda(wa) = c\,u^{\mathsf{T}}\Lambda(wa) \in c\,\mathcal{O}_C = w\,\mathcal{O}_C$, and Step 4 gives $l\mathbf{J}\iota(a) = y^{\mathsf{T}}\mathbf{e}\mathbf{J}\iota(a) \in \mathcal{O}_C$. So $\iota(a) \in H$. □

Lemma D.121. *Let $\mathfrak{T}$ be a torsion-regular pre-admissible torsion system for $(\mathcal{X}, p, \lambda)$ as in Proposition D.97, $w \in \mathcal{O}'$ with $w^{p-1} = u\,p$, $u \in \mathcal{O}'^{\times}$, $a \in \Phi$, $s = (a, +)$, $\mathfrak{P} = \mathfrak{P}_s$, $m = m_{\mathfrak{P}}$, and let $(\mathfrak{q}, C, \iota_C)$ be a test point. Then the $\mathcal{O}_C$-spans in C^m of the stalks at $\mathfrak{q}$ satisfy $\mathcal{O}_C H_{\mathfrak{q}} \subseteq w^{-1}\mathcal{O}_C T_{\mathfrak{q}} = w^{-1}\mathcal{O}_C^m$.*

Proof. **Step 1: reduction to $we_i \in \mathrm{af}_C$.** Put $\mathcal{M} = \mathcal{A}_{\mathfrak{P}} + p\,\mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O})$; then $\mathrm{af} = \ell^{-1}(\mathcal{M})$, where $\ell\colon V_H \to \mathrm{Map}(\Gamma_{\mathfrak{P}}, \Omega')$, $c \mapsto \ell_c$, is linear and injective ($\ell_c(b_i) = c_i$ as $\chi_s(1) = 1$, $\chi_s(0) = 0$). af is the kernel of $\mathcal{M} \to \mathrm{Map}/\ell(V_H)$ pulled back by ℓ, a coherent sheaf, so its stalk is $\mathrm{af}_{\mathfrak{q}} = \ell^{-1}(\mathcal{M}_{\mathfrak{q}})$ and $H_{\mathfrak{q}}$ is the dual lattice of $\mathrm{af}_{\mathfrak{q}}$; both are free of rank m ($p\,\mathcal{O}^m \subseteq \mathrm{af} \subseteq \mathcal{O}^m$, as $\ell(p\mathcal{O}^m) \subseteq p\,\mathrm{Map}$ and $\ell_c(b_i) = c_i$). By flatness of $\mathcal{O}_C$ over $R_{\mathfrak{q}}$, tensoring the exact sequence $0 \to \mathcal{M}_{\mathfrak{q}} \cap \ell(V_H) \to \mathcal{M}_{\mathfrak{q}} \to \mathrm{Map}(\Gamma_{\mathfrak{P}}, \Omega')/\ell(V_H)$ gives $\mathcal{O}_C\mathrm{af}_{\mathfrak{q}} = \mathrm{af}_C := \{c \in C^m : \ell_c \in \mathcal{M}_C\}$ with $\mathcal{M}_C = \mathcal{O}_C\mathcal{M}_{\mathfrak{q}} = \mathcal{A}_{\mathfrak{P},C} + p\,\mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}_C)$, and $\mathcal{O}_C H_{\mathfrak{q}}$ is the dual lattice of af_C (dual bases). Dualising, it suffices to show $w\,\mathcal{O}_C^m \subseteq \mathrm{af}_C$, i.e. that the function $x \mapsto \ell_{we_i}(x) = w\,\chi_s(x_i)$ ($x = \sum_j x_j b_j$) lies in $\mathcal{M}_C$ for every i.

Step 2: the Teichmüller character modulo p. Let $a \in \mathbb{F}_{\mathfrak{P}}$ and $\tilde{a} \in \mathcal{O}_E$ any lift. Then $\chi_s(a) - \sigma_s(\tilde{a}) \in p\,\mathcal{O}_C$. Indeed, $p\mathcal{O}_E = \mathfrak{P}\mathfrak{P}''$ with $\mathfrak{P}''$ prime to $\mathfrak{P}$, $\sigma_s(\mathfrak{P}'') \not\subseteq \mathfrak{m}_{\mathcal{O}'}$, so $\sigma_s(\mathfrak{P})\mathcal{O}' = p\mathcal{O}'$. If $a = 0$, $\tilde{a} \in \mathfrak{P}$ and $\sigma_s(\tilde{a}) \in p\mathcal{O}'$. If $a \neq 0$, $v = \sigma_s(\tilde{a})$ satisfies $v^q - v \in \sigma_s(\mathfrak{P})\mathcal{O}' = p\mathcal{O}'$ ($q = q_{\mathfrak{P}}$, $\tilde{a}^q \equiv \tilde{a}$ modulo $\mathfrak{P}$); by induction $v^{q^{k+1}} - v^{q^k} \in p^{k+1}\mathcal{O}'$, so $\tau = \lim_k v^{q^k}$ exists in the complete ring $\mathcal{O}_C$, $\tau - v \in p\mathcal{O}_C$ ($p\mathcal{O}_C$ is closed) and $\tau^q = \tau$, $\tau \neq 0$. Thus τ is a $(q-1)$-th root of unity congruent to $\sigma_s(\tilde{a})$ modulo $\mathfrak{m}_C$; as $(q-1)$-th roots of unity are distinct modulo $\mathfrak{m}_C$ ($p \nmid q-1$), $\tau = \chi_s(a)$.

Step 3: characters lie in $\mathcal{A}_{\mathfrak{P},C}$. By Proposition D.98, $\mathcal{A}_{\mathfrak{P},C}$ is an abelian factor: there are an abelian scheme A over $\mathcal{O}_C$ and homomorphisms $\iota\colon \Gamma_{\mathfrak{P}} \to N_A$, $\rho\colon N_A \to \Gamma_{\mathfrak{P}}$ with $\rho\iota = \mathrm{id}$ and $\mathcal{A}_{\mathfrak{P},C} = \{g \circ \iota : g \in M_A\}$ (Definition 35.17). Let $\psi\colon \Gamma_{\mathfrak{P}} \to \mu_p(C)$ be a homomorphism. $A[p]$ is finite locally free over $\mathcal{O}_C$, and $N_A = A[p](\mathcal{O}_C) = A[p](C)$ (valuative criterion). The homomorphism $\psi \circ \rho\colon A[p](C) \to \mu_p(C) \subseteq C^{\times}$ is a C-point of the Cartier dual $A[p]^D$, which is finite over $\mathcal{O}_C$, so it extends to an $\mathcal{O}_C$-point, i.e. to a homomorphism $A[p] \to \mathbb{G}_{m,\mathcal{O}_C}$; its coordinate is a function $a_\psi \in \Gamma(A[p], \mathcal{O})$ with $x^* a_\psi = \psi(\rho(x))$ for $x \in N_A$. So $\psi \circ \rho \in M_A$ and $\psi = (\psi \circ \rho) \circ \iota \in \mathcal{A}_{\mathfrak{P},C}$.

Step 4: $w\chi_s(x_i) \in \mathcal{M}_C$. Let $\theta_1, \dots, \theta_f \in \mathcal{O}_E$ lift an $\mathbb{F}_p$-basis $\bar{\theta}_1, \dots, \bar{\theta}_f$ of $\mathbb{F}_{\mathfrak{P}}$, and $r_k\colon \mathbb{F}_{\mathfrak{P}} \to \mathbb{F}_p$ the coordinate functionals. For $a \in \mathbb{F}_{\mathfrak{P}}$, $\sum_k \widetilde{r_k(a)}\theta_k$ is a lift of a ($\tilde{}$ the representative in $\{0, \dots, p-1\}$), so by Step 2

$$w\,\chi_s(x_i) \equiv \sum_{k=1}^{f} \sigma_s(\theta_k)\,w\,\widetilde{r_k(x_i)} \pmod{p\,\mathcal{O}_C} \qquad (x \in \Gamma_{\mathfrak{P}}).$$

Let $\zeta \in \mathcal{O}'$ be a primitive p-th root of unity; since $(\zeta-1)^{p-1} \in \widetilde{p\,\mathbb{Z}[\zeta]^\times}$ and $w^{p-1} = up$, $w/(\zeta-1)$ is a unit of $\mathcal{O}_C$. By Lemma 35.56 (with $K = C$, $\mathcal{O} = \mathcal{O}_C$), $w\, r_k(x_i) = \sum_j c_j \zeta^{j\, r_k(x_i)}$ with $c_j \in \mathcal{O}_C$, and $x \mapsto \zeta^{j\, r_k(x_i)}$ is a homomorphism $\Gamma_{\mathfrak{P}} \to \mu_p$ ($x \mapsto r_k(x_i)$ is additive), which lies in $\mathcal{A}_{\mathfrak{P},C}$ by Step 3. As $\sigma_s(\theta_k) \in \mathcal{O}' \subseteq \mathcal{O}_C$, the function $x \mapsto w\chi_s(x_i)$ lies in $\mathcal{A}_{\mathfrak{P},C} + p\,\mathrm{Map}(\Gamma_{\mathfrak{P}}, \mathcal{O}_C) = \mathcal{M}_C$. By Step 1 this proves the claim. □

Lemma D.122. *Let $\mathcal{T}, \dots, \lambda$, $\mathfrak{T}$, R, w and, for every $a \in \Phi$, $\beta, \bar\beta$ and the Hodge–Tate atlas be as in the hypotheses of Proposition D.97. Then at every test point $(\mathfrak{q}, C, \iota_C)$ of $\mathfrak{T}$, for every $a \in \Phi$, with $s = (a, +)$,*

$$\mathcal{O}_C J_{i,\mathfrak{q}} = \mathcal{O}_C\, \mathcal{O}_{\mathcal{X}'}(-\mathrm{Ec}_{s,i})_{\mathfrak{q}} \quad \text{in } C, \qquad 1 \le i \le m_{\mathfrak{P}_s}.$$

Proof. Fix a, $s = (a,+)$, $\mathfrak{P} = \mathfrak{P}_s$, $m = m_{\mathfrak{P}}$, $N = \Gamma_{\mathfrak{P}}$ with the basis $b_1, \dots, b_m$ of (Ts4), $N_i = \Gamma_{\mathfrak{P},i}$, $\chi = \chi_s$ and $B = \mathcal{A}_{\mathfrak{P},C}$; regard $\Omega' \subseteq C$ through ι_C. $R_{\mathfrak{q}}$ is a discrete valuation ring with fraction field Ω', and $\mathcal{O}_C$ is torsion free, hence flat, over it, with $\Omega' \otimes_{R_{\mathfrak{q}}} \mathcal{O}_C = C$. All stalks below are finitely generated torsion-free, hence free, $R_{\mathfrak{q}}$-modules, and for submodules of a finite-dimensional Ω'-vector space the formation of intersections, kernels and images commutes with $- \otimes_{R_{\mathfrak{q}}} \mathcal{O}_C$ (flatness).

Step 1: χ is fundamental. Let $q = |\mathbb{F}_{\mathfrak{P}}|$, $a', b' \in \mathbb{F}_{\mathfrak{P}}$, $y = \chi(a'+b')$ and $z = \chi(a') + \chi(b')$. As $\chi(x)^q = \chi(x)$ for all x and the binomial coefficients $\binom{q}{k}$, $0 < k < q$, are divisible by p, $z^q \equiv z \pmod{p\mathcal{O}_C}$; and $z \equiv y \pmod{\mathfrak{m}_C}$ by (Ts1). With $e = z - y \in \mathfrak{m}_C$, $z^q - z \equiv y^q - y + e^q - e = e(e^{q-1}-1) \pmod{p\mathcal{O}_C}$, and $e^{q-1}-1$ is a unit; so $e \in p\mathcal{O}_C$. Hence χ is a fundamental character.

Step 2: the flag conditions. By Proposition D.98, B is an $\mathbb{F}_{\mathfrak{P}}$-stable tight Hopf order on N and an abelian factor; by Proposition 35.43 it has the generation property at every N_i. By Theorem 35.58 and Step 1, $(B, (b_i))$ satisfies the flag conditions for χ; in particular there are $c_i \in \mathcal{O}_C \smallsetminus \{0\}$ with $\mathrm{Xr}_{\chi,i}(B) = \mathrm{Ad}_{\chi,i}(B) = c_i\mathcal{O}_C$ $(1 \le i \le m)$.

Step 3: the divisor side. The stalk $\mathcal{O}(\mathrm{Ec}_{s,i})_{\mathfrak{q}}$ is the kernel of the $R_{\mathfrak{q}}$-linear map $\Omega' \to \mathrm{Map}(N_i, \Omega')/\mathcal{A}_{\mathfrak{P},i,\mathfrak{q}}$, $c \mapsto c\,\xi_{s,i}$, where $\mathcal{A}_{\mathfrak{P},i,\mathfrak{q}}$ is the restriction of $\mathcal{A}_{\mathfrak{P},\mathfrak{q}}$ to N_i; it is invertible, $= e\,R_{\mathfrak{q}}$ with $e \in \Omega'^{\times}$. A basis of the free module $\mathcal{A}_{\mathfrak{P},i,\mathfrak{q}}$ is an Ω'-basis of $\mathrm{Map}(N_i, \Omega')$ (by (Ts3), $p^N\mathrm{Map} \subseteq \mathcal{A}_{\mathfrak{P}}$), hence a C-basis of $\mathrm{Map}(N_i, C)$, so $\mathcal{A}_{\mathfrak{P},i,\mathfrak{q}} \otimes \mathcal{O}_C$ is its $\mathcal{O}_C$-span, which is $B|_{N_i}$ (Ox2). Tensoring the kernel with $\mathcal{O}_C$: $e\,\mathcal{O}_C = \{c \in C : c\,\xi_{s,i} \in B|_{N_i}\}$. Since $\xi_{s,i}(\alpha b_i + k) = \chi(\alpha)$ for $\alpha \in \mathbb{F}_{\mathfrak{P}}$, $k \in N_{i-1}$, the right side is $\mathrm{Xr}_{\chi,i}(B) = c_i\mathcal{O}_C$. Hence $\mathcal{O}_C\,\mathcal{O}(-\mathrm{Ec}_{s,i})_{\mathfrak{q}} = e^{-1}\mathcal{O}_C = c_i^{-1}\mathcal{O}_C$.

Step 4: the lattice af *at the test point.* Since $\mathcal{A}_{\mathfrak{P}} + p\,\mathrm{Map}(N, \mathcal{O}) \subseteq \mathrm{Map}(N, \mathcal{O})$ and $\ell_c(b_j) = c_j$, $\mathrm{af}_{\mathfrak{q}}$ is the kernel of $R_{\mathfrak{q}}^m \to \mathrm{Map}(N, R_{\mathfrak{q}})/(\mathcal{A}_{\mathfrak{P},\mathfrak{q}} + p\,\mathrm{Map}(N, R_{\mathfrak{q}}))$, $c \mapsto \ell_c$; so $L := \mathcal{O}_C\,\mathrm{af}_{\mathfrak{q}} = \{c \in \mathcal{O}_C^m : \ell_c \in B + p\,\mathrm{Map}(N, \mathcal{O}_C)\}$, a free $\mathcal{O}_C$-module of rank m with $p\mathcal{O}_C^m \subseteq L \subseteq \mathcal{O}_C^m$. Let A_i be the image of $L \cap (0^{i-1} \oplus C^{m-i+1})$ under the i-th coordinate. Claim: $A_i = \mathrm{Ad}_{\chi,i}(B)$. If $c \in L$ with $c_1 = \dots = c_{i-1} = 0$, write $\ell_c = g + ph$ with $g \in B$, $h \in \mathrm{Map}(N, \mathcal{O}_C)$; ℓ_c is additive modulo p (Step 1) and $\ell_c(\alpha x) = \chi(\alpha)\ell_c(x)$, so g is almost additive with $g(\alpha x) - \chi(\alpha)g(x) \in p\mathcal{O}_C$, $g(N_{i-1}) \subseteq p\mathcal{O}_C$ (ℓ_c vanishes on N_{i-1}) and $g(b_i) - c_i \in p\mathcal{O}_C$: so $c_i \in \mathrm{Ad}_{\chi,i}(B)$. Conversely, for $c \in \mathrm{Ad}_{\chi,i}(B)$ with its g, put $c'_j = g(b_j) \in \mathcal{O}_C$; then $g(\sum_j x_j b_j) \equiv \sum_j \chi(x_j)c'_j = \ell_{c'}(x) \pmod p$, so $\ell_{c'} \in B + p\,\mathrm{Map}$ and $c' \in L$, with $c'_j \in p\mathcal{O}_C$ for $j < i$ and $c'_i \equiv c \pmod p$; as $p\mathcal{O}_C^m \subseteq L$, $c' - \sum_{j<i} c'_j e_j + (c - c'_i)e_i \in L \cap (0^{i-1} \oplus C^{m-i+1})$ has i-th coordinate c. So $A_i = c_i\mathcal{O}_C$ (Step 2).

Step 5: duality. Choose $v_i \in L \cap (0^{i-1} \oplus C^{m-i+1})$ with i-th coordinate c_i. By descending induction on i, every $x \in L \cap (0^{i-1} \oplus C^{m-i+1})$ is an $\mathcal{O}_C$-combination of $v_i, \dots, v_m$ (subtract $(x_i/c_i)v_i$, $x_i/c_i \in \mathcal{O}_C$); so the matrix V with columns $v_1, \dots, v_m$, lower triangular with diagonal $c_1, \dots, c_m$, has columns forming a basis of L. H is the dual of af and $\mathrm{af}_{\mathfrak{q}}$ is free, so $\mathcal{O}_C H_{\mathfrak{q}} = \{d \in$

$C^m : d\cdot c \in \mathcal{O}_C\ \forall c \in L\}$, whose basis is the rows $r_1, \dots, r_m$ of V^{-1}; V^{-1} is lower triangular with diagonal c_i^{-1}, so $r_i \in C^i \oplus 0$ with i-th coordinate c_i^{-1}. Hence $\mathcal{O}_C H_{i,\mathfrak{q}} = \mathcal{O}_C H_{\mathfrak{q}} \cap (C^i \oplus 0)$ (flat base change of the intersection) is spanned by $r_1, \dots, r_i$, and its image under the i-th coordinate is $\mathcal{O}_C J_{i,\mathfrak{q}} = c_i^{-1}\mathcal{O}_C$. With Step 3 this is the assertion. □

D.6 Reference comparisons

This section proves five results used in the proof of Proposition 31.14 for (Gt4). A realised tower with a good origin, (Gi2) and (Gi4) is compared with fixed reference models: it has the reference link (Vl) (Theorem D.126) and the comparison of heights (Rh1) against the frozen reference models (Theorem D.123), both through joint reference models (Lemma D.124). Lie fibre data of the reference models give the reference lower bound (Rh2) (Theorem D.127), through the maximal slope of the Lie lattice of Section D.2. From these, the wall inputs follow: the monomial bound (Wd) from (Vl) (Theorem D.129), and the heights of points (Wp) from (Rh1) and (Rh2) (Theorem D.131).

Theorem D.123. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p) with a geometric realisation and a good origin over an integral model of a unitary Shimura datum, and assume (Gi2) and (Gi4). For $c \in \mathrm{Cp}$ let $\mathcal{X}_c^{\mathrm{fr}}$ be the frozen reference model of Definition 34.1. Then the realised tower satisfies (Rh1) with the reference models $\mathcal{X}_c^{\mathrm{fr}}$: there are $C_{\mathrm{Rh}} \geq 1$ and $s_{\mathrm{Rh}} \in (0,1]$ such that for every $B \geq 2$ and $c \in \mathrm{Cp}$ there are an arithmetic variety $\mathcal{J}$ of absolute dimension $D+1$ and surjective morphisms $q\colon \mathcal{J} \to \mathcal{X}_{B,c}$, $q^{\mathrm{ref}}\colon \mathcal{J} \to \mathcal{X}_c^{\mathrm{fr}}$ with*

$$-C_{\mathrm{Rh}}\big(\log B + \log(1/s) + 1\big) \leq h_{c(\mu,s),t(s)}\big(q(W)\big) - h^{\mathrm{fr}}_{c(\mu,s),1}\big(q^{\mathrm{ref}}(W)\big) \leq C_{\mathrm{Rh}}(\log B + 1)$$

for every point W of $\mathcal{J}$, every dominant weight μ with $\|\mu\|_\infty \leq 2$ and every $0 < s \leq s_{\mathrm{Rh}}$.

Proof. Write $k_c = k_c^A$; all root models below carry the normalising integer k_c, so $c(\mu,s)$ is the same vector on each. For $c \in \mathrm{Cp}$ every connected component of X_c^{an} meets the dense open set X_c°; choose $R_c \geq e$ such that every component contains a point of X_c° with $\mathrm{rW} \leq R_c$, and put $t_1 = \min_c \min(t_0, e^{-R_c})$ (finitely many c; the constants of the systems are fixed) and $s_{\mathrm{Rh}} \in (0,1]$ with $t(s_{\mathrm{Rh}}) \leq t_1$. As $\log(1/t) \geq R_c \geq e$ on $(0,t_1]$, ε is continuous there and tends to 0, so $C_\varepsilon = \sup_{0<t\leq t_1} \varepsilon(t)$ is finite.

Step 1: the correspondence. Fix B, c and let $\mathcal{J} = \widetilde{\mathcal{X}}$, $q = \mathrm{pr}_1$, $q^{\mathrm{ref}} = \mathrm{pr}_0$ from Lemma D.124; on $\mathcal{J}$ consider the pulled-back root models $\widetilde{\mathcal{X}}$ (stage lines $\overline{\mathcal{L}}_r(t)$) and $\widetilde{\mathcal{X}}'$ (reference lines $\overline{\mathcal{L}}'_r(t)$). For a point W of $\mathcal{J}$, $q(W)$ and $q^{\mathrm{ref}}(W)$ are points, $\deg W_{\mathbb{Q}} = \deg(W/q(W)) \deg q(W)_{\mathbb{Q}}$, and by Lemma 30.13 (with $b=1$, $d=0$ and β the surjections $W \to q(W)$ and $W \to q^{\mathrm{ref}}(W)$ of arithmetic varieties of absolute dimension 1, whose generic fibres, spectra of number fields, are smooth over $\mathbb{Q}$; the metrics on the finite sets $q(W)(\mathbb{C})$ and $q^{\mathrm{ref}}(W)(\mathbb{C})$ are smooth; $c = \deg(W/q(W))$, resp. $\deg(W/q^{\mathrm{ref}}(W))$) $h_{c,t}(q(W))$ equals the height of W for the lines $\overline{\mathcal{L}}_r(t)$, and $h^{\mathrm{fr}}_{c,1}(q^{\mathrm{ref}}(W))$ the height of W for the lines $\overline{\mathcal{L}}'_r(t_0)$. Write $\widehat{h}_W(\cdot) = \widehat{\deg}(\widehat{c}_1(\cdot)\cdot[W])/\deg W_{\mathbb{Q}}$. Then, for $c = c(\mu,s)$ and $t = t(s) \leq t_1$, the difference to be bounded is $\Sigma_{\mathrm{fin}} + \Sigma_\infty$ with

$$\Sigma_{\mathrm{fin}} = \sum_{r\in \mathrm{Rt}^+} c_r\big(\widehat{h}_W(\overline{\mathcal{L}}_r(t)) - \widehat{h}_W(\overline{\mathcal{L}}'_r(t))\big), \qquad \Sigma_\infty = \sum_{r\in \mathrm{Rt}^+} c_r\big(\widehat{h}_W(\overline{\mathcal{L}}'_r(t)) - \widehat{h}_W(\overline{\mathcal{L}}'_r(t_0))\big).$$

Step 2: the finite places. Let $r \in \mathrm{Rt}^+$ and let f_r, g_r be the polynomials of the vertical closeness (Jr3), (Jr4), with exponents α_ℓ ($\alpha = 0$ for $r = *$). The rational section $\sigma = f_r/g_r$ of $\overline{\mathcal{L}}_r(t) \otimes \overline{\mathcal{L}}'_r(t)^{-1}$ has norm 1 on $\mathcal{J}(\mathbb{C})$ and vertical divisor, so it is regular and non-zero at the generic point of W. Let $\nu_W\colon \operatorname{Spec} \mathcal{O}_{\mathbf{k}(W)} \to W$ be the normalisation of W, a surjection of arithmetic

varieties of absolute dimension 1 with smooth generic fibres and $c = 1$; by Lemma 30.13 ($b = 1$, $d = 0$, $\beta = \nu_W$; the metrics on the finite set $W(\mathbb{C})$ are smooth) the height of W for a Hermitian line bundle $\overline{N}$ on $\mathcal{J}$ is the Arakelov degree of $\nu_W^*\overline{N}$ (this is also used in Step 3). For $\overline{N} = \overline{\mathcal{L}}_r(t) \otimes \overline{\mathcal{L}}'_r(t)^{-1}$, computed with σ (of norm 1), this gives $\deg W_{\mathbb{Q}}(\widehat{h}_W(\overline{\mathcal{L}}_r(t)) - \widehat{h}_W(\overline{\mathcal{L}}'_r(t))) = \sum_w \operatorname{ord}_w(\sigma|_W)\log \#\kappa(w)$, the sum over the closed points w of $\operatorname{Spec}\mathcal{O}_{\mathbf{k}(W)}$, the normalisation of W. Let w be a closed point of the normalisation of W, over the point $x \in \mathcal{J}$ and the prime ℓ. Choose monomials m_f, m_g of the multidegrees of f_r, g_r in coordinates non-zero at x and put $\phi = (f_r/m_f)/(g_r/m_g) \in \Omega^\times$; m_f/m_g generates the line at x, so $\operatorname{ord}_w(\sigma|_W) = \operatorname{ord}_w(\phi)$, and $\operatorname{ord}_{W'}(\phi) = \operatorname{ord}_{W'}(f_r) - \operatorname{ord}_{W'}(g_r)$ for every prime divisor W' through x; these are horizontal or lie over ℓ. If $\ell \notin S_B$, all these orders vanish, so $\phi^{\pm 1} \in \mathcal{O}_x$ by normality and $\operatorname{ord}_w(\phi) = 0$. If $\ell \in S_B$, take an integer $N \geq 1$ with $N\alpha_\ell \in \mathbb{Z}$: $\phi^{\pm N}\ell^{N\alpha_\ell}$ has non-negative order along every prime divisor through x, so lies in $\mathcal{O}_x$, and its image in the valuation ring of w gives $|\operatorname{ord}_w(\phi)| \leq \alpha_\ell \operatorname{ord}_w(\ell)$. Summing,

$$|\widehat{h}_W(\overline{\mathcal{L}}_r(t)) - \widehat{h}_W(\overline{\mathcal{L}}'_r(t))| \leq \frac{1}{\deg W_{\mathbb{Q}}} \sum_{\ell \in S_B} \alpha_\ell \sum_{w|\ell} \operatorname{ord}_w(\ell)\log \#\kappa(w) = \sum_{\ell \in S_B} \alpha_\ell \log \ell \leq C_{\mathrm{Vl}} \log B$$

by (Vl5) in Theorem D.126 (which has the same S_B and α), and the difference is 0 for $r = *$. As $|c_r| \leq 2$ for roots, $|\Sigma_{\mathrm{fin}}| \leq 2|\Phi| n\, C_{\mathrm{Vl}} \log B$.

Step 3: the archimedean places. Changing a metric changes the height by the logarithm of the ratio of cone functions at the $\deg W_{\mathbb{Q}}$ complex points of W, so Σ_∞ is the average over $x \in W(\mathbb{C})$ of $\Psi(x) = \sum_r c_r \log(h_r^0(t)/h_r^0(t_0))(z_x)$, z_x a cone point over $\mathrm{pr}_0(x)$; the auxiliary metric does not depend on t. By (Go4), for a fundamental index $a = (\tau, i)$, $\sum_{j \leq i} \log(h^0_{(\tau,j)}(t)/h^0_{(\tau,j)}(t_0)) = -\Delta_a \circ \pi_0$ with $\Delta_a = \varphi_a^T(t) - \varphi_a^T(t_0)$, so with $\Delta_{(\tau,0)} = 0$ and summation by parts ($c_{\tau,i} = -\mu_{\tau,i}$)

$$\Psi = \sum_{\tau \in \Phi} \Big(\sum_{i<n} (\mu_{\tau,i} - \mu_{\tau,i+1})\, \Delta_{(\tau,i)} + \mu_{\tau,n}\, \Delta_{(\tau,n)} \Big) \circ \pi_0.$$

By (Go1)(2), $\Delta_a \leq C\varepsilon(t_0)$ and $\Delta_a \geq -C\log\log(1/t) - 2C$. For $a = (\tau, n)$, the clause (flat) makes the metrics $h_a^{\mathrm{ref}} e^{\varphi_a^T(t)}$ and $h_a^{\mathrm{ref}} e^{\varphi_a^T(t_1)}$ both of pluriharmonic local potentials, so $\varphi_a^T(t) - \varphi_a^T(t_1)$ is pluriharmonic on the compact manifold X_c^{an}, hence constant on each component; at a point of X_c° with $\mathrm{rW} \leq R_c \leq \log(1/t_1) \leq \log(1/t)$, (Go1)(2) gives $0 \leq \varphi_a^T(t') - \varphi_a \leq t'^{1/3} + \varepsilon(t')$ for $t' \in \{t, t_1\}$, so $|\varphi_a^T(t) - \varphi_a^T(t_1)| \leq 2 + 2C_\varepsilon$ everywhere. With C_7 the maximum of the continuous function $|\varphi_a^T(t_1) - \varphi_a^T(t_0)|$ on X_c^{an}, $|\Delta_{(\tau,n)}| \leq 2 + 2C_\varepsilon + C_7 =: C_{\mathrm{det}}$. Since $\|\mu\|_\infty \leq 2$, the gaps lie in $[0,4]$ and $|\mu_{\tau,n}| \leq 2$; with $\log\log(1/t(s)) = 2\log(1/s)$,

$$-|\Phi|\big(4(n-1)(2C\log(1/s) + 2C) + 2C_{\mathrm{det}}\big) \leq \Psi \leq |\Phi|\big(4(n-1)C|\varepsilon(t_0)| + 2C_{\mathrm{det}}\big).$$

Conclusion. Taking for C_{Rh} the maximum over $c \in \mathrm{Cp}$ of 1, $2|\Phi|nC_{\mathrm{Vl}} + |\Phi|(8(n-1)C + 8(n-1)C + 2C_{\mathrm{det}})$ and $|\Phi|(4(n-1)C|\varepsilon(t_0)| + 2C_{\mathrm{det}}) + 2|\Phi|nC_{\mathrm{Vl}}$ (constants of Sys_c), both inequalities of (Rh1) follow from $\Sigma_{\mathrm{fin}} + \Sigma_\infty$, Steps 2 and 3. □

Lemma D.124. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p) with a geometric realisation and a good origin over an integral model $\mathcal{Y}$ of a unitary Shimura datum with CM field E. Let $B \geq 2$ and $c \in \mathrm{Cp}$; write $\mathcal{X}_1 = \mathcal{X}_{B,c} \subseteq \mathbb{P}_1 = \prod_{k \in K_1} \mathbb{P}(\mathbb{Z}^{I_k})$ and $\mathcal{X}_0 = \mathcal{X}_c^{\mathrm{ref}} \subseteq \mathbb{P}_0 = \prod_{k \in K_0} \mathbb{P}(\mathbb{Z}^{I_k})$ ($K_0 \cap K_1 = \varnothing$), with root origins on the fixed data at c (function fields Ω_i, embeddings g_i, root sections ϱ_r^i, maps f_i, multidegrees e_r^i, e_*^i, metrics $h_r^i(t), h_*^i$, realisations π_i), and let t_0 be the constant of Sys_c. Put $\alpha_\ell = e_\ell + 2v_\ell(\operatorname{disc} E)$ for $\ell \in S_B$. Then there are a finite set I_2 and a closed integral subscheme $\widetilde{\mathcal{X}} \subseteq \widetilde{\mathbb{P}} = \mathbb{P}(\mathbb{Z}^{I_2}) \times \mathbb{P}_1 \times \mathbb{P}_0$, flat over $\mathbb{Z}$ of absolute dimension $D + 1$, such that, writing a point of the affine cone as $z = (z_2, z_1, z_0)$:*

(Jr1) *$\widetilde{\mathcal{X}}$ is normal; the projections restrict to surjective morphisms $\mathrm{pr}_i\colon \widetilde{\mathcal{X}} \to \mathcal{X}_i$; pr_1 is birational, pr_0 is generically finite of degree $\deg_{B,c}$; $f_1 \circ \mathrm{pr}_1 = f_0 \circ \mathrm{pr}_0$, and the induced field embeddings satisfy $\iota_1 g_1 = \iota_0 g_0$;*

(Jr2) *$\pi_1 \circ \mathrm{pr}_1 = \pi_0 \circ \mathrm{pr}_0$ on $\widetilde{\mathcal{X}}(\mathbb{C})$;*

(Jr3) *for every root r and every $0 < t \le t_0$, the Hermitian line bundles $(\mathcal{O}(0, e_r^1, 0), h_r^1(t)(z_1))|_{\widetilde{\mathcal{X}}}$ and $(\mathcal{O}(0, 0, e_r^0), h_r^0(t)(z_0))|_{\widetilde{\mathcal{X}}}$ are vertically close over (S_B, α);*

(Jr4) *the auxiliary lines $(\mathcal{O}(0, e_*^1, 0), h_*^1(z_1))|_{\widetilde{\mathcal{X}}}$ and $(\mathcal{O}(0, 0, e_*^0), h_*^0(z_0))|_{\widetilde{\mathcal{X}}}$ are vertically close over $(S_B, 0)$.*

Proof. Step 1: the common field and the closure. As $g_0\colon \Omega_c \to \Omega_0$ is an isomorphism, $\iota = g_1 \circ g_0^{-1}\colon \Omega_0 \to \Omega_1 =: \Omega$ is a field embedding with $\iota g_0 = g_1$. Put $y = (y^1, \iota(y^0))$ (coordinates of $\mathbb{P}_1 \times \mathbb{P}_0$). The multihomogeneous G with $G(y) = 0$ generate a multihomogeneous prime $\mathfrak{q}$ (kernel of a ring map to a field on multihomogeneous elements) containing no coordinate that is non-zero at y; let $\mathcal{X}^{\mathrm{cl}} \subseteq \mathbb{P}_1 \times \mathbb{P}_0$ be the integral closed subscheme it defines. Its function field is generated by the coordinate ratios; the ratios of y^1 already generate Ω and those of $\iota(y^0)$ lie in Ω, so it is Ω. If G vanishes on $\mathcal{X}_1$ (resp. $\mathcal{X}_0$) then $G(y^1) = 0$ (resp. $G(\iota y^0) = \iota G(y^0) = 0$), so $\mathcal{X}^{\mathrm{cl}} \subseteq \mathcal{X}_1 \times \mathcal{X}_0$; the projection to $\mathcal{X}_1$ induces the identity of Ω (birational), the projection to $\mathcal{X}_0$ induces ι, of degree $[\Omega : \iota\Omega_0] = [\Omega : g_1\Omega_c] = \deg_{B,c}$. As Ω has characteristic 0, $\mathcal{X}^{\mathrm{cl}}$ is flat over $\mathbb{Z}$, of absolute dimension $\mathrm{trdeg}_{\mathbb{Q}}\Omega + 1 = D + 1$. The maps $f_1 \circ \mathrm{pr}_1$, $f_0 \circ \mathrm{pr}_0$ to the separated scheme $\mathcal{Y}$ both induce $\mathbf{k}(\mathcal{Y}_c) \to \Omega$ through $g_1 = \iota g_0$, so they agree at the generic point, hence on the reduced scheme $\mathcal{X}^{\mathrm{cl}}$.

Step 2: normalisation; (Jr1). The normalisation $\widetilde{\mathcal{X}} \to \mathcal{X}^{\mathrm{cl}}$ is finite, so $\widetilde{\mathcal{X}}$ embeds as a closed subscheme of $\mathbb{P}(\mathbb{Z}^{I_2}) \times \mathcal{X}^{\mathrm{cl}} \subseteq \widetilde{\mathbb{P}}$ over $\mathcal{X}^{\mathrm{cl}}$ for a finite set I_2; it is normal, integral, flat over $\mathbb{Z}$, of absolute dimension $D + 1$, with function field Ω. The projections pr_i are projective, hence proper, and dominant, hence surjective; pr_1 is birational, pr_0 generically finite of degree $\deg_{B,c}$, $f_1\mathrm{pr}_1 = f_0\mathrm{pr}_0$ by Step 1, and $\iota_1 = \mathrm{id}$, $\iota_0 = \iota$ satisfy $\iota_1 g_1 = \iota_0 g_0$. Being normal, integral, locally noetherian and mapping to $\mathcal{Y}$ through $\mathcal{X}_1$, $\widetilde{\mathcal{X}}$ is a cover in the same sense as $\mathcal{X}_1$, with $G_{\widetilde{\mathcal{X}}} = \mathrm{pr}_i^* G_{\mathcal{X}_i}$.

Step 3: (Jr2). Identify $I_{k_0^1}$ and $I_{k_0^0}$ with $\{0, \dots, m\}$ by the bijections of the root origins. The two morphisms $\widetilde{\mathcal{X}} \to \mathbb{P}^m_{\mathbb{Z}}$ given by the factors k_0^1 and k_0^0 have coordinate ratios $g_1(y_j^c/y_{j'}^c)$ and $\iota g_0(y_j^c/y_{j'}^c)$, which are equal; so they agree on a dense open set, hence everywhere. In particular the multihomogeneous polynomials $Y_{k_0^1,a}Y_{k_0^0,b} - Y_{k_0^1,b}Y_{k_0^0,a}$ vanish on $\widetilde{\mathcal{X}}$, and at every cone point z the vectors z_{1,k_0^1} and z_{0,k_0^0} are proportional. For $x \in \widetilde{\mathcal{X}}(\mathbb{C})$, $\pi_1(\mathrm{pr}_1 x)$ and $\pi_0(\mathrm{pr}_0 x)$ therefore have proportional images under pt_c, so they are equal.

Step 4: (Jr4). Choose j with $y_j^c \neq 0$ and put $f_* = Y_{k_0^1,j}$, $g_* = Y_{k_0^0,j}$; then $\deg f_* - \deg g_* = (0, e_*^1, 0) - (0, 0, e_*^0)$ (e_*^i is the unit vector at k_0^i), and neither vanishes on $\widetilde{\mathcal{X}}$. At a cone point z write $z_{1,k_0^1} = \lambda z_{0,k_0^0}$ ($\lambda \in \mathbb{C}^\times$, Step 3); as $h_*^i(z) = h_c(z_{i,k_0^i})$ and h_c is homogeneous of degree one in absolute value, $|f_*(z)|h_*^0(z_0) = |\lambda||z_{0,k_0^0,j}|h_c(z_{0,k_0^0}) = |g_*(z)|h_*^1(z_1)$. For a prime divisor W choose j' with $Y_{k_0^1,j'}$ not identically zero on W; by Step 3 neither is $Y_{k_0^0,j'}$, and $\mathrm{ord}_W(f_*) = \mathrm{ord}_W(g_1(y_j^c/y_{j'}^c)) = \mathrm{ord}_W(g_*)$.

Step 5: the polynomials for a root r. Put $d = (0, e_r^1, -e_r^0)$. For each $k \in K_1 \sqcup K_0$ choose j_k with $y_{k,j_k} \neq 0$, let $M_1 = \prod_k Y_{k,j_k}^{\max(d_k,0)}$, $M_2 = \prod_k Y_{k,j_k}^{\max(-d_k,0)}$; the element $\iota(\varrho_r^0)M_2(y)/(\varrho_r^1 M_1(y)) \in \Omega^\times$ is a rational function in the coordinate ratios, so it equals $G(y)/H(y)$ with $G, H \in \mathbb{Z}[Y_1, Y_0]$ multihomogeneous of equal multidegree, $H(y) \neq 0$. Put $f_r = GM_1$, $g_r = HM_2$: $\deg f_r - \deg g_r =$

d and

$$f_r(y)/g_r(y) = \iota(\varrho_r^0)/\varrho_r^1 \neq 0.$$

Step 6: orders. Let W be a prime divisor of $\widetilde{\mathcal{X}}$, $x_i = \mathrm{pr}_i(W)$. For each $k \in K_i$ choose c_k' with $Y_{k,c_k'}$ non-zero at x_i; by (Go3)(3) the stalk of $\mathcal{L}_r^i$ at x_i is generated by $\varrho_r^i m^i(y^i)$, $m^i = \prod_{k\in K_i} Y_{k,c_k'}^{e^i_{r,k}}$, and the coordinates $Y_{k,c_k'}$ do not vanish identically on W. With the monomials m_f, m_g in these coordinates of multidegrees $\deg f_r, \deg g_r$ (so $m_f/m_g = m^1/m^0$), the definition of ord_W gives

$$\mathrm{ord}_W(f_r) - \mathrm{ord}_W(g_r) = \mathrm{ord}_W\Big(\frac{f_r(y)}{g_r(y)} \cdot \frac{\iota(m^0(y^0))}{m^1(y^1)}\Big) = -\mathrm{ord}_W(u),$$

where $\varrho_r^1 m^1(y^1) = u \cdot \iota(\varrho_r^0 m^0(y^0))$ compares generators of $\Lambda^1_{r,W}$ and $\Lambda^0_{r,W}$ in the notation of Lemma D.125. That lemma applies to $\widetilde{\mathcal{X}} \to \mathcal{X}_i$: $\mathcal{X}_1$ has Hodge bundles at (S_B, R) with rational torsion and free Hodge–Tate spans over S_B ((Go3)(2)); $\mathcal{X}_0$ has the de Rham Hodge bundles ($S = \varnothing$); the generic flags are g_1 and g_0 of the universal flag ((Go3)(3)), so they coincide after $\iota_1 g_1 = \iota_0 g_0$. Hence $\mathrm{ord}_W(f_r) = \mathrm{ord}_W(g_r)$ if W is horizontal; if W lies over ℓ, $|\mathrm{ord}_W(f_r) - \mathrm{ord}_W(g_r)| \le (e_\ell[\ell \in S_B] + 2v_\ell(\mathrm{disc}E))\mathrm{ord}_W(\ell)$, which is $\alpha_\ell \mathrm{ord}_W(\ell)$ for $\ell \in S_B$ and 0 for $\ell \notin S_B$, since every prime dividing $\mathrm{disc}\, E$ lies in $\mathrm{Bd} \subseteq S_B$ ((Go5)).

Step 7: metrics. Let $r = (\tau, k)$ and put $a_j = (\tau, j)$, with a_0 the trivial index ($h^{\mathrm{ref}}_{a_0} = 1$, $\varphi_{a_0} = \varphi^T_{a_0} = 0$). By (Go4), on $\mathcal{X}_i$ and $0 < t \le t_0$, $\prod_{j\le k} h^i_{(\tau,j)}(t) = (h^{\mathrm{ref}}_{a_k} e^{\varphi^T_{a_k}(t)\circ\pi_i})^{-1}$ on the whole cone and $\prod_{j\le k} h^{i,\mathrm{sing}}_{(\tau,j)} = (h^{\mathrm{ref}}_{a_k} e^{\varphi_{a_k}\circ\pi_i})^{-1}$ on the cone over $\pi_i^{-1}(X_c^\circ)$. Dividing the identities for k and $k-1$, on the cone over $\pi_i^{-1}(X_c^\circ)$

$$h_r^i(t) = h_r^{i,\mathrm{sing}} \cdot \exp\big(-(\psi_{a_k}(t) - \psi_{a_{k-1}}(t)) \circ \pi_i\big), \qquad \psi_a(t) = \varphi_a^T(t) - \varphi_a \text{ on } X_c^\circ.$$

By (Jr2) the exponential factor is the same function Ξ_t on the cone over $U = (\pi_1 \mathrm{pr}_1)^{-1}(X_c^\circ)$ for $i = 0, 1$; U is the abelian locus of $\widetilde{\mathcal{X}}$ (through either $\mathcal{X}_i$), the complement of a proper Zariski closed subset defined over $\mathbb{Q}$, hence dense. Let $x \in U$ and z a cone point over x with $f_r(z)g_r(z) \neq 0$. Choose a Laurent monomial Q of multidegree $(0, e_r^1, 0) - \deg f_r$ in coordinates non-zero at z, and put $\xi = \varrho_r^1 f_r(y)Q(y) = \iota(\varrho_r^0) g_r(y)Q(y)$, an element of the root line Λ_r regular at x. Its Hodge norm at x (that of the root line of $G_{\widetilde{\mathcal{X}},x}$ for the common flag) is computed through $\mathcal{X}_1$ as follows: let m^1 be a Laurent monomial of multidegree e_r^1 in coordinates of $\mathbb{P}_1$ non-zero at z_1; then $\xi = (f_r Q/m^1)(y) \cdot \varrho_r^1 m^1(y^1)$, the first factor is a rational function of multidegree zero, regular at x with value $f_r(z)Q(z)/m^1(z_1)$, and (Go3)(4) gives $\|\varrho_r^1 m^1(y^1)\|^{\mathrm{Hdg}} = |m^1(z_1)|/h_r^{1,\mathrm{sing}}(z_1)$ at $\mathrm{pr}_1(x)$; so $\|\xi\|^{\mathrm{Hdg}}(x) = |f_r(z)Q(z)|/h_r^{1,\mathrm{sing}}(z_1)$. In the same way, through $\mathcal{X}_0$ and a Laurent monomial of multidegree e_r^0 in coordinates of $\mathbb{P}_0$, $\|\xi\|^{\mathrm{Hdg}}(x) = |g_r(z)Q(z)|/h_r^{0,\mathrm{sing}}(z_0)$. Hence $|f_r(z)|h_r^{0,\mathrm{sing}}(z_0) = |g_r(z)|h_r^{1,\mathrm{sing}}(z_1)$, and multiplying by $\Xi_t(z)$, $|f_r(z)|h_r^0(t)(z_0) = |g_r(z)|h_r^1(t)(z_1)$. Both sides are continuous on the affine cone over $\widetilde{\mathcal{X}}(\mathbb{C})$ and the points used are dense there, so the identity holds everywhere. With Step 6 this is (Jr3). □

Lemma D.125. *Let $\mathcal{Y}$ be an integral model of a unitary Shimura datum with CM field E. For $i \in \{0,1\}$ let $\mathcal{X}_i$ be a locally noetherian cover of $\mathcal{Y}$ with function field Ω_i, structure map f_i, a finite set S_i of primes with integers $R_i(\ell) \ge 1$ ($\ell \in S_i$) such that every $\ell^{R_i(\ell)}$-torsion point of $G_{\mathcal{X}_i}$ over an algebraic closure of Ω_i is defined over Ω_i, Hodge bundles $\mathcal{V}_s^i$ at (S_i, R_i) and an integral flag of them with root lines $\mathcal{L}_r^i$. Assume that* ht_x *is a free $\mathcal{O}_x$-module of rank* $\dim \mathrm{Li}$ *at every point x of $\mathcal{X}_i$ over every $\ell \in S_i$. Let $\widetilde{\mathcal{X}}$ be a locally noetherian cover of $\mathcal{Y}$ with function field $\widetilde{\Omega}$ and dominant morphisms $\mathrm{pr}_i\colon \widetilde{\mathcal{X}} \to \mathcal{X}_i$ with $f_0 \circ \mathrm{pr}_0 = f_1 \circ \mathrm{pr}_1$, inducing $\iota_i\colon \Omega_i \to \widetilde{\Omega}$, such that for every label s the flags ι_0(flag of $\mathcal{V}_s^0$)$\otimes\widetilde{\Omega}$ and ι_1(flag of $\mathcal{V}_s^1$)$\otimes\widetilde{\Omega}$ of $\mathrm{Li}_s(\widetilde{\Omega})$ coincide. For a root*

r let Λ_r be the corresponding root line of this flag (a one-dimensional $\widetilde{\Omega}$-space) and, for a point w of $\widetilde{\mathcal{X}}$, $\Lambda^i_{r,w} = \mathcal{O}_w \cdot \iota_i(\mathcal{L}^i_{r,\mathrm{pr}_i(w)}) \subseteq \Lambda_r$. For p prime put $e_p = 2$ if $p = 2$ and $e_p = \frac{1}{p-1}$ otherwise. Let W be a point of $\widetilde{\mathcal{X}}$ whose local ring is a discrete valuation ring, with normalised valuation ord_W. Then $\Lambda^0_{r,W}$ and $\Lambda^1_{r,W}$ are free of rank one, $\Lambda^1_{r,W} = u\,\Lambda^0_{r,W}$ for some $u \in \widetilde{\Omega}^\times$, and:

1. *if W lies over no prime, then $\mathrm{ord}_W(u) = 0$;*
2. *if W lies over the prime p, then*

$$|\mathrm{ord}_W(u)| \le \Big(e_p\,([p \in S_0] + [p \in S_1]) + 2\,v_p(\mathrm{disc}\,E)\Big)\,\mathrm{ord}_W(p),$$

where $[p \in S_i]$ is 1 if $p \in S_i$ and 0 otherwise.

Proof. Write $\mathcal{O} = \mathcal{O}_W$, $v = \mathrm{ord}_W$, $x_i = \mathrm{pr}_i(W)$, and identify Ω_i with its image under ι_i. Fix the label $s = s(r)$ and $k = \mathrm{st}(r)$; let $(\Lambda_{s,j})_j$ be the common flag of $\mathrm{Li}_s(\widetilde{\Omega})$.

Step 1: idempotents with denominators in the different. Let $\Sigma = \mathrm{Hom}(E,F)$ and $\varepsilon_\sigma = \sum_j \omega_j^\vee \otimes \sigma(\omega_j) \in E \otimes_{\mathbb{Q}} F$ for $\sigma \in \Sigma$. Under $E \otimes F \cong \prod_{\tau\in\Sigma} F$, $a \otimes b \mapsto (\tau(a)b)_\tau$, the τ-component of ε_σ is $\sum_j \tau(\omega_j^\vee)\sigma(\omega_j) = \delta_{\tau\sigma}$ (the matrices $(\tau(\omega_j))$ and $(\tau(\omega_j^\vee))$ are inverse transposes of each other, by $\mathrm{Tr}(\omega_j\omega_{j'}^\vee) = \delta_{jj'}$). So ε_σ is the idempotent cutting out the σ-eigenspace, and $(d\otimes 1)\varepsilon_\sigma \in \mathcal{O}_E \otimes \mathcal{O}_F$ for every $d \in \mathfrak{d}_E$, since $\omega_j^\vee \in \mathfrak{d}_E^{-1}$. Let y be a point of a cover $\mathcal{X}'$ with a dominant map $\mathrm{pr}'\colon \widetilde{\mathcal{X}} \to \mathcal{X}'$ through which $W \mapsto y$ ($\mathcal{X}'$ one of $\mathcal{X}_0, \mathcal{X}_1$). Then $\mathrm{Lie}_W = \mathcal{O}\,\mathrm{Lie}_y$ (the Lie algebra of $G_{\widetilde{\mathcal{X}}} = \mathrm{pr}'^* G_{\mathcal{X}'}$ is the pull-back). Let δ be an element of least valuation among the $\sigma_s(d)$, $d \in \mathfrak{d}_E$ (a finite set of generators suffices). We claim

$$(*) \qquad \mathcal{O}\,\mathrm{dR}_{s,y} \subseteq \mathrm{dR}_{s,W} \subseteq \delta^{-1}\mathcal{O}\,\mathrm{dR}_{s,y}.$$

The first inclusion is clear. If $e \in \mathrm{dR}_{s,W} = \mathrm{Lie}_W \cap \mathrm{Li}_s$, write $e = \sum_j a_j l_j$ with $a_j \in \mathcal{O}$, $l_j \in \mathrm{Lie}_y$; for $d \in \mathfrak{d}_E$, $(d\otimes 1)\varepsilon_{\sigma_s}$ acts on Li through ι and the scalars, preserves Lie_y (an $\mathcal{O}_E \otimes \mathcal{O}_y$-module), maps Li into Li_s, and acts on Li_s as $\sigma_s(d)$. Hence $\sigma_s(d)e = \sum_j a_j (d\otimes 1)\varepsilon_{\sigma_s} l_j \in \mathcal{O}\,\mathrm{dR}_{s,y}$, and with $\sigma_s(d) = \delta$ the claim follows. If W lies over no prime, $\mathcal{O} \supseteq \mathbb{Q}$ and δ divides the non-zero integer $\mathrm{N}(d)$ in $\mathcal{O}$, so $v(\delta) = 0$. If W lies over p, then $v \circ \sigma_s$ restricted to E is $v(p)\,v_{\mathfrak{p}}/e_{\mathfrak{p}}$ for the prime $\mathfrak{p} \mid p$ of E it defines (ramification index $e_{\mathfrak{p}}$, residue degree $f_{\mathfrak{p}}$), so $v(\delta) = v(p)\,v_{\mathfrak{p}}(\mathfrak{d}_E)/e_{\mathfrak{p}} \le v(p)\,f_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{d}_E) \le v(p)\,v_p(\mathrm{disc}\,E)$, as $\mathfrak{p}^{f_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{d}_E)}$ divides the norm $|\mathrm{disc}\,E|$.

Step 2: the Hodge lattices against $\mathrm{dR}_{s,W}$. Put $c_i = \varpi_p\delta$ if W lies over $p \in S_i$ and $c_i = 1$ otherwise. We claim

$$(**) \qquad \mathcal{O}\,\mathrm{dR}_{s,x_i} \subseteq \mathcal{O}\,V^{S_i}_{s,x_i} \subseteq c_i^{-1}\mathcal{O}\,\mathrm{dR}_{s,x_i}.$$

If x_i lies over no prime of S_i (in particular if W lies over no prime of S_i: x_i lies over p whenever W does, $\mathcal{O}_{x_i} \to \mathcal{O}$ being local), $V^{S_i}_{s,x_i} = \mathrm{dR}_{s,x_i}$. If x_i lies over $p \in S_i$ (then so does W, as p is a non-unit of $\mathcal{O}_{x_i}$ mapping into the maximal ideal of $\mathcal{O}$), $V^{S_i}_{s,x_i} = \mathrm{HT}_{s,x_i}(p^{R_i(p)})$ and Lemma 35.32(3) gives $\mathrm{dR} \subseteq \mathrm{HT}$ and $\varpi_p\sigma_s(d)\mathrm{HT} \subseteq \mathrm{dR}$ at x_i for all $d \in \mathfrak{d}_E$ (for $s = (\tau, -)$ even without $\sigma_s(d)$, and $\sigma_s(d) \in \mathcal{O}_{x_i}$), in particular for d with $\sigma_s(d) = \delta$; apply $\mathcal{O}\cdot$. Combining with $(*)$ for $y = x_i$: with $N_i = \mathcal{O}V^{S_i}_{s,x_i}$ and $D = \mathrm{dR}_{s,W}$,

$$\delta\,D \subseteq N_i \subseteq c_i^{-1}D.$$

Step 3: root lines. The flag of $\mathcal{V}^i_s$ at x_i consists of direct summands with free quotients: $V^{S_i}_{s,x_i} = V^i_k \oplus Q_k$ with $V^i_k = V^{S_i}_{s,x_i} \cap \Lambda_{s,k}$ (saturation) for each k. Hence $N_i \cap \Lambda_{s,k} = \mathcal{O}V^i_k$, and the image $\ell_k(N_i)$ of $N_i \cap \Lambda_{s,k}$ in $\Lambda_{s,k}/\Lambda_{s,k-1}$ is $\mathcal{O}\cdot(V^i_k/V^i_{k-1})$. So $\Lambda^i_{r,W} = \ell_k(N_i)^{\otimes \mathrm{sg}(r)}$, a finitely generated torsion-free, hence free, $\mathcal{O}$-module of rank one. Inclusions $M \subseteq aM'$ of lattices in

$\mathrm{Li}_s(\widetilde{\Omega})$ give $\ell_k(M) \subseteq a\,\ell_k(M')$. From Step 2, $\ell_k(N_1) \subseteq c_1^{-1}\ell_k(D) \subseteq c_1^{-1}\delta^{-1}\ell_k(N_0)$ and $\ell_k(N_0) \subseteq c_0^{-1}\ell_k(D) \subseteq c_0^{-1}\delta^{-1}\ell_k(N_1)$. Writing $\ell_k(N_1) = u'\ell_k(N_0)$, this says $-v(c_1\delta) \le v(u') \le v(c_0\delta)$, and $u = u'^{\,\mathrm{sg}(r)}$, so $|v(u)| \le v(\delta) + \max(v(c_0), v(c_1)) \le v(\delta) + v(c_0) + v(c_1)$.

Conclusion. If W lies over no prime, $v(\delta) = 0$ and $c_0 = c_1 = 1$, so $v(u) = 0$. If W lies over p: $v(\delta) \le v_p(\mathrm{disc}E)\,v(p)$, and $v(c_i) = v(\varpi_p) + v(\delta) \le e_p\,v(p) + v_p(\mathrm{disc}E)\,v(p)$ if $p \in S_i$ ($v(\zeta - 1) = v(p)/(p-1)$, $v(4) = 2v(2)$), $v(c_i) = 0$ otherwise. Hence $|v(u)| \le v(\delta) + \max(v(c_0), v(c_1)) \le 2v(\delta) + e_p\,v(p)\,\max([p \in S_0], [p \in S_1])$, which is at most $(e_p([p \in S_0] + [p \in S_1]) + 2v_p(\mathrm{disc}E))v(p)$. □

Theorem D.126. *Let $\mathcal{T}$ be a root tower of type (Φ, n, p) with a geometric realisation and a good origin over an integral model of a unitary Shimura datum with CM field E, and assume that the realised tower has the geometric inputs (Gi2) and (Gi4). Then every reference model $\mathcal{X}_c^{\mathrm{ref}}$ ($c \in \mathrm{Cp}$) has convergent regularisation, and the realised tower has the reference link (Vl) with $\mathrm{pp}_c^{\mathrm{ref}} = \mathrm{pp}(\mathcal{X}_c^{\mathrm{ref}}; \cdot)$, with $S = S_B$ and $\alpha_\ell = e_\ell + 2v_\ell(\mathrm{disc}\,E)$ ($\ell \in S_B$) at every stage B, with t_0 the constant of Sys_c, and with*

$$C_{\mathrm{Vl}} = 1 + \frac{1}{\log 2}\Big(3 + 3\log 2 + \sum_{\ell \in \mathrm{Bd}} e_\ell \log \ell + 2\log|\mathrm{disc}\,E|\Big).$$

Proof. Fix $c \in \mathrm{Cp}$ and write $\mathcal{X}_0 = \mathcal{X}_c^{\mathrm{ref}} \subseteq \mathbb{P}_0 = \prod_{k \in K_0} \mathbb{P}(\mathbb{Z}^{I_k})$ with multidegrees e_r^0 ($r \in \mathrm{Rt}^+$), $\delta_0 = [F : \mathbb{Q}]$ and $k_0 = k_c^A$ ((Go3)(1), (Go3)(5), degree one). For a fundamental index $a = (\tau, i)$ put $f_a^0 = -\sum_{j \le i} e^0_{(\tau,j)}$, and let $n_c \ge 0$ be an integer with $e^0_{r,k} + n_c \ge 0$ for all $r \in \mathrm{Rt}^+$, $k \in K_0$; let $\mathbf{1} \in \mathbb{Z}^{K_0}$ be the vector of ones and $e_H^0 = \sum_a f_a^0 + n_c\mathbf{1}$. Let $M_c^{\mathrm{ref}} \ge 0$ be the maximum of 0 and of the numbers $\deg(c_1(\mathcal{O}(\epsilon_1)) \cdots c_1(\mathcal{O}(\epsilon_D)) \cap [\mathcal{X}_{0,\mathbb{Q}}])/\delta_0$ over $\epsilon_1, \dots, \epsilon_D \in \{e_H^0\} \cup \{e_r^0 + e_H^0 : r \in \mathrm{Rt}^+\}$. These depend on c only.

Step 1: the two root models at a stage. Let $B \ge 2$, $\mathcal{X}_1 = \mathcal{X}_{B,c} \subseteq \mathbb{P}_1$ with multidegrees e_r^1, metrics $h_r^1(t), h_*^1$, $\delta_1 = [F : \mathbb{Q}]\deg_{B,c}$, $k_1 = k_c^A$, and let $\widetilde{\mathcal{X}} \subseteq \widetilde{\mathbb{P}}$ be as in Lemma D.124. Define root models on $\widetilde{\mathcal{X}}$ with $k = k_c^A$, $\delta = \delta_1$: $\widetilde{\mathcal{X}}$ with multidegrees $(0, e_r^1, 0)$, metrics $h_r^1(t)(z_1)$ for all $t > 0$ and $h_*^1(z_1)$; $\widetilde{\mathcal{X}}'$ with multidegrees $(0, 0, e_r^0)$, metrics $h_r^0(t)(z_0)$, $h_*^0(z_0)$. Each such function is smooth, positive, conjugation invariant and multihomogeneous of the stated multidegree on the punctured product, hence a cone metric. Put $f_a^1 = -\sum_{j \le i} e^1_{(\tau,j)}$ and

$$e_H = \sum_a (0, f_a^1, 0) + n_c(0, 0, \mathbf{1}) \in \mathbb{Z}^{\widetilde{K}}.$$

By Lemma 30.18 (1) with $d = 0$ for the projections of $\widetilde{\mathbb{P}}$ to $\mathbb{P}_1$ and to $\mathbb{P}_0$ and $\mathcal{Z} = \widetilde{\mathcal{X}}$, as pr_1 is surjective and birational ($c = 1$) and pr_0 is surjective of degree $\deg_{B,c} = \delta_1/\delta_0$ ($c = \deg_{B,c}$) (Lemma D.124 (Jr1)),

$$\mathrm{ai}_t(\widetilde{\mathcal{X}}; \cdot) = \mathrm{ai}_t(\mathcal{X}_1; \cdot), \qquad \mathrm{ai}_t(\widetilde{\mathcal{X}}'; \cdot) = \mathrm{ai}_t(\mathcal{X}_0; \cdot) \qquad (t > 0),$$

and likewise for the geometric degrees of pulled-back line bundles on $\widetilde{\mathcal{X}}_{\mathbb{Q}}$ (factor $\deg_{B,c}$ for pr_0).

Step 2: (Vl2) and the vertical classes. (Vl2) is (Jr3) and (Jr4) of Lemma D.124 (closeness over $(S_B, 0)$ implies closeness over (S_B, α)). For $r \in \mathrm{Rt}^+$ let f_r, g_r be the polynomials of vertical closeness; f_r/g_r is a rational section of $\mathcal{O}(e_r - e_r')$ on $\widetilde{\mathcal{X}}$ whose divisor $\mathcal{D}_r$ has order 0 along every horizontal prime divisor, i.e. is vertical, and whose norm is 1 on $\widetilde{\mathcal{X}}(\mathbb{C})$ for $0 < t \le t_0$. Hence, for these t, $\overline{\mathcal{N}}_r(t) = \overline{\mathcal{L}}_r(t) \otimes \overline{\mathcal{L}}_r'(t)^{\otimes -1}$, the restriction of a Hermitian line bundle with smooth metric on $\widetilde{\mathbb{P}}$, has the rational section f_r/g_r of norm 1 with vertical Weil divisor $\mathcal{D}_r =$

$\sum_W \mathrm{ord}_W(f_r/g_r)[W]$. On $\widetilde{\mathcal{X}}_{\mathbb{Q}}$, which is normal, f_r/g_r is a nowhere vanishing regular section, so $\mathcal{O}(e_r) \cong \mathcal{O}(e'_r)$ there.

Step 3: (Vl1). By (Gi2) $\mathcal{X}_1$, hence $\widetilde{\mathcal{X}}$ (Step 1), has convergent regularisation. For a monomial $x_{r_1} \cdots x_{r_{D+1}}$ and $0 < t \le t_0$, replacing the lines one at a time and using multilinearity,

$$\delta k^{\#}\big(\mathrm{ai}_t(\widetilde{\mathcal{X}}; x^m) - \mathrm{ai}_t(\widetilde{\mathcal{X}}'; x^m)\big) = \sum_{j=1}^{D+1} \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}'_{r_1}(t)) \cdots \widehat{c}_1(\overline{\mathcal{L}}'_{r_{j-1}}(t))\, \widehat{c}_1(\overline{\mathcal{N}}_{r_j}(t)) \\ \widehat{c}_1(\overline{\mathcal{L}}_{r_{j+1}}(t)) \cdots \widehat{c}_1(\overline{\mathcal{L}}_{r_{D+1}}(t))),$$

and by Lemma 30.17 (on the regular $\widetilde{\mathbb{P}}$, $\mathcal{Z} = \widetilde{\mathcal{X}}$, $\overline{\mathcal{M}}_0 = \overline{\mathcal{N}}_{r_j}(t)$, $\sigma = f_{r_j}/g_{r_j}$: no horizontal components, $\log\|\sigma\| = 0$) each term equals $\sum_p \log p \sum_{W \subseteq \widetilde{\mathcal{X}}_{\mathbb{F}_p}} \mathrm{ord}_W(f_{r_j}/g_{r_j})$ times the degree on W of the first Chern classes of the other D line bundles, a sum of geometric intersection numbers independent of t. Hence $\widetilde{\mathcal{X}}'$, and by Step 1 $\mathcal{X}_0$, has convergent regularisation, and $\mathrm{pp}(\widetilde{\mathcal{X}}; \cdot) = \mathrm{pp}(\mathcal{X}_{B,c}; \cdot)$, $\mathrm{pp}(\widetilde{\mathcal{X}}'; \cdot) = \mathrm{pp}(\mathcal{X}_c^{\mathrm{ref}}; \cdot) = \mathrm{pp}_c^{\mathrm{ref}}$, which does not depend on B.

Step 4: (Vl3). Let $\ell \in S_B$. For a fundamental index $a = (\tau, i)$, the weight μ with $\mu_{\tau,j} = 1$ for $j \le i$ and 0 otherwise is dominant, and $c(\mu, 0)$ has entries $-\mu$ on the roots and 0 at $*$; as $\ell \le B$ or $\ell \in \mathrm{Bd}$, (Gi4) gives $\deg(\mathcal{O}(f_a^1)|_{W'}) \ge 0$ for every integral curve $W' \subseteq \mathcal{X}_{1,\mathbb{F}_\ell}$. For an integral curve $W \subseteq \widetilde{\mathcal{X}}_{\mathbb{F}_\ell}$, $\mathrm{pr}_1(W)$ is a point or a curve W', and $\deg(\mathrm{pr}_1^*\mathcal{O}(f_a^1)|_W)$ is 0 or $\deg(W \to W')\deg(\mathcal{O}(f_a^1)|_{W'}) \ge 0$. A multidegree with non-negative entries has non-negative degree on every curve. Now, with $f^1_{(\tau,0)} = 0$, the root multidegree is $e^1_{(\tau,i)} = f^1_{(\tau,i-1)} - f^1_{(\tau,i)}$, so

$$(0, e^1_{(\tau,i)}, 0) + e_H = (0, f^1_{(\tau,i-1)}, 0) + \sum_{a \ne (\tau,i)} (0, f_a^1, 0) + n_c(0, 0, \mathbf{1}),$$

a sum of classes of non-negative degree on W; likewise e_H, $(0, e^1_*, 0) + e_H$ ($e^1_* \ge 0$) and $(0, 0, e^0_r + n_c\mathbf{1}) + \sum_a (0, f^1_a, 0) = (0, 0, e^0_r) + e_H$ ($e^0_r + n_c\mathbf{1} \ge 0$). This is (Vl3).

Step 5: (Vl4). By Step 2, on $\widetilde{\mathcal{X}}_{\mathbb{Q}}$ the line bundles $\mathcal{O}(0, e^1_r, 0)$ and $\mathcal{O}(0, 0, e^0_r)$ are isomorphic for every $r \in \mathrm{Rt}^+$, hence so are $\mathcal{O}(0, f^1_a, 0)$ and $\mathcal{O}(0, 0, f^0_a)$, $\mathcal{O}(e_H)$ and $\mathcal{O}(0, 0, e^0_H)$, and $\mathcal{O}(e_r + e_H)$ and $\mathcal{O}(0, 0, e^0_r + e^0_H)$. These are pull-backs along pr_0, which is generically finite of degree $\deg_{B,c}$ on generic fibres; so every intersection number in (Vl4) equals $\deg_{B,c}$ times the corresponding number on $\mathcal{X}_{0,\mathbb{Q}}$, which is at most $\delta_0 M_c^{\mathrm{ref}}$. Hence it is at most $\deg_{B,c}\delta_0 M_c^{\mathrm{ref}} = M_c^{\mathrm{ref}}\delta$.

Step 6: (Vl5). As $2 \in S_B$, $e_2 = 2$, and Bd contains every prime dividing $\mathrm{disc}\, E$,

$$\sum_{\ell \in S_B} \alpha_\ell \log \ell \le 2\log 2 + \sum_{3 \le \ell \le B} \frac{\log \ell}{\ell - 1} + \sum_{\ell \in \mathrm{Bd}} e_\ell \log \ell + 2\log|\mathrm{disc}\, E|.$$

Since $\frac{\log \ell}{\ell-1} = \frac{\log \ell}{\ell} + \frac{\log \ell}{\ell(\ell-1)} \le \frac{\log \ell}{\ell} + \frac{2\log \ell}{\ell^2}$ and $x \mapsto \log x/x^2$ is decreasing for $x \ge 2$, $\sum_{m \ge 3} \frac{2\log m}{m^2} \le 2\int_2^\infty \frac{\log x}{x^2}\, dx = 1 + \log 2$; with Mertens' bound, $\sum_{3 \le \ell \le B} \frac{\log \ell}{\ell-1} \le \log B + 3 + \log 2$. So the sum is at most $\log B + 3 + 3\log 2 + \sum_{\mathrm{Bd}} e_\ell \log \ell + 2\log|\mathrm{disc} E| \le C_{\mathrm{Vl}} \log B$, using $1 \le \log B/\log 2$. □

Theorem D.127. *Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with $1 \le \kappa \le n-1$ and a geometric realisation, and let $\mathcal{X}_Z^{\mathrm{ref}}$ ($Z \in \mathrm{Cp}$) be root models of type (Φ, n) that have Lie fibre data. Then there is $C_0 \ge 1$ such that for every $C \ge C_0$, every $Z \in \mathrm{Cp}$ and every point W of $\mathcal{X}_Z^{\mathrm{ref}}$, with $x = h^{\mathrm{ref}}_{c(0,1),1}(W)$:*

$$x \ge -C \qquad \text{and}$$
$$h^{\mathrm{ref}}_{c(\mu,0),1}(W) \ge -C\log(2 + C + x) - C \quad \text{for every dominant weight } \mu \text{ with } \|\mu\|_\infty \le 2.$$

In particular the clause (Rh2) holds for these reference models with any constant $C_{\mathrm{Rh}} \ge C_0$.

Proof. Since Cp is finite it suffices to find C_0 for one Z; fix Z and write $\mathcal{X} = \mathcal{X}_Z^{\mathrm{ref}} \subseteq \mathbb{P} = \prod_{k\in K}\mathbb{P}(\mathbb{Z}^{I_k})$, $k_{\mathcal{X}}$ its normalising integer, $\mathbf{1} = (1,\dots,1) \in \mathbb{Z}^K$, and $h_{\mathbf{1}}(z) = \prod_k \|z_k\|$ (Euclidean norms), a cone metric of multidegree $\mathbf{1}$. For a point W, $x = \widehat{h}_W(\overline{\mathcal{L}}_*)/k_{\mathcal{X}}$, and $h^{\mathrm{ref}}_{c(\mu,0),1}(W) = -\sum_{a,i}\mu_{a,i}\,\widehat{h}_W(\overline{\mathcal{L}}_{(a,i)}(1))$ because $c(\mu,0)_* = 0$. For $f \in \mathbb{Z}^K$ and a cone metric h of multidegree f write $\widehat{h}_W(f,h) = \widehat{\deg}(\widehat{c}_1((\mathcal{O}(f),h)|_{\mathcal{X}})\cdot[W])/\deg W_{\mathbb{Q}}$; it is additive under tensor products (products of cone functions).

Step 1: heights of sections. Let $s \in H^0(\mathcal{X},\mathcal{O}(f)|_{\mathcal{X}})$ not vanish at the closed point $W_{\mathbb{Q}}$. The point W is integral, finite and flat over $\mathbb{Z}$, so $W = \operatorname{Spec} R$ for an order R in the residue field of $W_{\mathbb{Q}}$, with $\deg W_{\mathbb{Q}}$ complex points; by Lemma 30.17 with $m = 0$ (ambient the regular $\mathbb{P}$, $\mathcal{Z} = W$, $\sigma = s|_W$; W has no horizontal prime divisors) $\widehat{\deg}(\overline{\mathcal{O}(f)}\cdot[W]) = \sum_p \log p \sum_{w\in W_{\mathbb{F}_p}} \operatorname{ord}_w(s|_W)\,[\mathbf{k}(w):\mathbb{F}_p] - \sum_{w\in W(\mathbb{C})}\log\|s\|(w)$. The first term is ≥ 0, as $s|_W$ is a regular section, so all its orders are ≥ 0, and $\|s\|$ is continuous on the compact space $\mathcal{X}(\mathbb{C})$. Hence

$$\widehat{h}_W(f,h) \geq -\log\sup_{\mathcal{X}(\mathbb{C})}\|s\|_h. \tag{D.27}$$

Consequences. (a) If $f \geq 0$ componentwise, choose for each k an index j_k with z_{k,j_k} non-zero at $W_{\mathbb{Q}}$; the monomial $\prod_k z_{k,j_k}^{f_k}$ is a section of $\mathcal{O}(f)$ non-zero at $W_{\mathbb{Q}}$. As there are finitely many monomials of multidegree f, (D.27) gives $\widehat{h}_W(f,h) \geq -c(f,h)$ with $c(f,h)$ independent of W. (b) By (Lx1) and Serre's theorem there is $m \geq 1$ such that $\mathcal{O}(me_* - \mathbf{1})|_{\mathcal{X}_{\mathbb{Q}}}$ is globally generated; since $\mathcal{X}$ is flat over $\mathbb{Z}$, $H^0(\mathcal{X},\mathcal{O}(me_*-\mathbf{1})|_{\mathcal{X}})\otimes\mathbb{Q} = H^0(\mathcal{X}_{\mathbb{Q}},\mathcal{O}(me_*-\mathbf{1}))$, so integer multiples of finitely many generating sections are sections $s_1,\dots,s_l$ over $\mathcal{X}$ without common zero on $\mathcal{X}_{\mathbb{Q}}$. Applying (D.27) to the metric $h_*^m/h_{\mathbf{1}}$ and an s_i non-zero at $W_{\mathbb{Q}}$: $mk_{\mathcal{X}}x - \widehat{h}_W(\mathbf{1},h_{\mathbf{1}}) \geq -c_1$ with c_1 independent of W. With (a) for $f = \mathbf{1}$: $mk_{\mathcal{X}}x \geq -c_1 - c(\mathbf{1},h_{\mathbf{1}})$, so

$$x \geq -c_2, \qquad \widehat{h}_W(\mathbf{1},h_{\mathbf{1}}) \leq mk_{\mathcal{X}}x + c_1. \tag{D.28}$$

Step 2: root heights are dominated by x. Let $r \in \mathrm{Rt}$. Write $e_r = f^+ - f^-$ with $f^\pm \geq 0$ and choose an integer $N \geq 1$ with $f^\pm \leq N\mathbf{1}$. Put $\eta^- = \prod_k \|z_k\|^{f_k^-}$ and $\eta^+ = h_r(1)\,\eta^-$, cone metrics of multidegrees f^- and $e_r + f^- = f^+$, so that $\widehat{h}_W(\overline{\mathcal{L}}_r(1)) = \widehat{h}_W(f^+,\eta^+) - \widehat{h}_W(f^-,\eta^-)$. For $0 \leq f \leq N\mathbf{1}$ and a cone metric h_f of multidegree f, additivity, (a) for $N\mathbf{1}-f$ and (D.28) give $-c(f,h_f) \leq \widehat{h}_W(f,h_f) = N\widehat{h}_W(\mathbf{1},h_{\mathbf{1}}) - \widehat{h}_W(N\mathbf{1}-f, h_{\mathbf{1}}^N/h_f) \leq N(mk_{\mathcal{X}}x + c_1) + c(N\mathbf{1}-f, h_{\mathbf{1}}^N/h_f)$. Since $x + c_2 \geq 0$, there is $C_5 \geq c_2 + 1$, independent of W, with

$$|\widehat{h}_W(\overline{\mathcal{L}}_r(1))| \leq C_5\,y \quad (r \in \mathrm{Rt}), \qquad y := x + C_5 \geq 1. \tag{D.29}$$

Step 3: the Lie fibre. Fix a point W and take L, the Lie fibre $\overline{V}$ and the data (a), (b) of (Lx2); write $g_{a,i} = \widehat{h}_W(\overline{\mathcal{L}}_{(a,i)}(1))$ and $C' = C_{\mathrm{Lx}} \geq 1$. By (Lx2)(a), $|\sum_{i\leq\kappa} g_{a,i} - d_{a,\kappa}| \leq \kappa C'$ and $|\sum_{j\leq n-\kappa} g_{a,n+1-j} + d'_{a,n-\kappa}| \leq (n-\kappa)C'$ (telescoping); with (Lx3),

$$|d_{a,\kappa} - d'_{a,n-\kappa}| \leq (n+1)C', \qquad -d_{a,\kappa} - d'_{a,n-\kappa} \leq \sum_{i=1}^n |g_{a,i}| + nC' \leq n(C_5 y + C'). \tag{D.30}$$

The $2|\Phi|$ pieces $F_{a,\kappa}\otimes L$, $S_{a,n-\kappa}\otimes L$ are pairwise orthogonal and span $V\otimes_{\mathcal{O}_L} L$, so they are independent and $N = \bigoplus_a (F_{a,\kappa}\oplus S_{a,n-\kappa})$ is a submodule of finite index in V, of rank $g = |\Phi|n$, whose induced norms are the orthogonal sums. The Arakelov degree is additive on orthogonal direct sums and $\widehat{\deg}\,\overline{V} = \widehat{\deg}\,\overline{N} + \log\#(V/N) \geq \widehat{\deg}\,\overline{N}$; with (D.30),

$$\max\Big(-\frac{\widehat{\deg}\,\overline{V}}{[L:\mathbb{Q}]},0\Big) \leq \max\Big(0,\ -\sum_a (d_{a,\kappa} + d'_{a,n-\kappa})\Big) \leq |\Phi|\,n\,(C_5 y + C').$$

By (Lx2)(b), the hypotheses (1), (2) of Lemma D.128 hold for $\overline{V}$ with the constant $C' \geq 1$. With $C_{\mathrm{sb}} = C_{\mathrm{ll}} + \frac{1}{2}$ and $C_6 = 2 + |\Phi| n\big(1 + C_5 + C' + \frac{1}{2}\log C'\big)$, so that $2 + g + \frac{g}{2}\log C' + \max(-\widehat{\deg}\,\overline{V}/[L:\mathbb{Q}], 0) \leq C_6 y$ as $y \geq 1$, Lemma D.128 gives

$$M := \max(\widehat{\mu}_{\max}(\overline{V}), 0) \leq C_{\mathrm{sb}}\big(\log C_6 + \log y\big) + \tfrac{1}{2}\log C'. \tag{D.31}$$

Step 4: summation by parts. Let μ be dominant with $\|\mu\|_\infty \leq 2$ and fix a; drop the index a. Write $g_i = (d_i - d_{i-1}) + \epsilon_i$ $(i \leq \kappa)$ and $g_{n+1-j} = -(d'_j - d'_{j-1}) + \epsilon'_j$ $(j \leq n - \kappa)$ with $|\epsilon_i|, |\epsilon'_j| \leq C'$. Abel summation with $d_0 = d'_0 = 0$ gives

$$-\sum_{i=1}^{n} \mu_i g_i = -\sum_{i=1}^{\kappa-1} (\mu_i - \mu_{i+1}) d_i - \sum_{j=1}^{n-\kappa-1} (\mu_{n-j} - \mu_{n+1-j}) d'_j - (\mu_\kappa - \mu_{\kappa+1}) d_\kappa - \mu_{\kappa+1}(d_\kappa - d'_{n-\kappa}) + R,$$

with $|R| \leq \sum_i |\mu_i| C' \leq 2nC'$. The coefficients $\mu_i - \mu_{i+1}$ that occur are distinct gaps of a non-increasing sequence, so they are ≥ 0 and sum to at most $\mu_1 - \mu_n \leq 4$. Each $d_i = i\,\widehat{\mu}(\overline{F}_i) \leq nM$ and $d'_j \leq nM$, by the definition of the maximal slope (F_i, S_j are non-zero submodules). With (D.30),

$$-\sum_{i=1}^{n} \mu_{a,i} g_{a,i} \geq -4nM - 2(n+1)C' - 2nC'.$$

Summing over $a \in \Phi$ and inserting (D.31):

$$h^{\mathrm{ref}}_{c(\mu,0),1}(W) \geq -4|\Phi| n C_{\mathrm{sb}} \log(x + C_5) - \big(4|\Phi| n C_{\mathrm{sb}} \log C_6 + 2|\Phi| n \log C' + |\Phi|(4n+2)C'\big).$$

Step 5: the constant. Let $C_0 \geq 1$ be at least c_2, $4|\Phi| n C_{\mathrm{sb}}$, $4|\Phi| n C_{\mathrm{sb}} \log C_6 + 2|\Phi| n \log C' + |\Phi|(4n+2)C'$ and $C_5 - 2$ (all independent of W). For $C \geq C_0$: $x \geq -c_2 \geq -C$ by (D.28); $x + C_5 \leq 2 + C + x$ and $2 + C + x \geq 2$, so $\log(x + C_5) \leq \log(2 + C + x)$ with both logarithms positive; hence $h^{\mathrm{ref}}_{c(\mu,0),1}(W) \geq -C\log(2 + C + x) - C$. Taking the maximum of C_0 over the finitely many Z proves the theorem. □

Lemma D.128. *Let L be a number field, $\overline{V} = (V, (\|\cdot\|_{V,\sigma})_\sigma)$ a Hermitian lattice over $\mathcal{O}_L$ of rank $g \geq 1$, and $C \geq 1$. For $\sigma\colon L \to \mathbb{C}$ write $V_\sigma = V \otimes_{\mathcal{O}_L,\sigma} \mathbb{C}$. Assume:*

1. *there are $0 \leq r \leq g$ and elements $\varepsilon_1, \dots, \varepsilon_r \in V$, linearly independent over $\mathcal{O}_L$, spanning a submodule U, with*

$$C^{-1}\sum_{b=1}^{r} |w_b|^2 \;\leq\; \Big\|\sum_{b=1}^{r} w_b \varepsilon_b\Big\|^2_{V,\sigma} \;\leq\; C\sum_{b=1}^{r} |w_b|^2 \qquad (\sigma\colon L \to \mathbb{C},\ w \in \mathbb{C}^r);$$

2. *either $U = V$, or there are a CM field E, a principally polarised abelian scheme $\mathcal{A}$ with $\mathcal{O}_E$-action over $\mathcal{O}_L$ and an $\mathcal{O}_L$-linear map $\phi\colon V \to \mathrm{Lie}(\mathcal{A})$ with $\ker\phi = U$, $\mathrm{Lie}(\mathcal{A})/\phi(V)$ finite, and $\|\phi_\sigma(x)\|^2_\sigma \leq C\,\|x\|^2_{V,\sigma}$ for every σ and $x \in V_\sigma$, where $\phi_\sigma = \phi \otimes_\sigma \mathbb{C}$.*

Then

$$\widehat{\mu}_{\max}(\overline{V}) \;\leq\; \big(C_{\mathrm{ll}} + \tfrac{1}{2}\big)\log\Big(2 + g + \tfrac{g}{2}\log C + \max\Big(-\frac{\widehat{\deg}\,\overline{V}}{[L:\mathbb{Q}]}, 0\Big)\Big) + \tfrac{1}{2}\log C.$$

Proof. Write $n_L = [L:\mathbb{Q}]$. We use four standard facts on Hermitian lattices over $\mathcal{O}_L$ (Bost's normalisation, as in [48]). (F1) If M has rank s and $m_1, \dots, m_s \in M$ are linearly independent, then $\widehat{\deg}\,\overline{M} = \log\#(M/(\mathcal{O}_L m_1 + \cdots + \mathcal{O}_L m_s)) - \sum_\sigma \log\|m_1 \wedge \cdots \wedge m_s\|_\sigma$, the norm on $\bigwedge^s M_\sigma$ being the Gram determinant: $\|m_1 \wedge \cdots \wedge m_s\|^2_\sigma = \det(\langle m_a, m_b\rangle_\sigma)$. (F2) If $M' \subseteq M$ has the same

rank and the induced norms, $\widehat{\deg}\,\overline{M} = \widehat{\deg}\,\overline{M'} + \log\#(M/M')$. (F3) If $N \subseteq M$ is saturated, with the induced norms on N and the quotient norms $\|y\|_\sigma = \min\{\|x\|_\sigma : x \mapsto y\}$ on $(M/N)_\sigma = M_\sigma/N_\sigma$, then $\widehat{\deg}\,\overline{M} = \widehat{\deg}\,\overline{N} + \widehat{\deg}\,\overline{M/N}$. (F4) If $\psi\colon M \to M''$ is an injective $\mathcal{O}_L$-linear map of Hermitian lattices, M of rank s, with $\|\psi_\sigma(x)\|_\sigma \le \lambda\|x\|_\sigma$ for all σ, x ($\lambda > 0$), then $\widehat{\deg}\,\overline{\psi(M)} \ge \widehat{\deg}\,\overline{M} - s\,n_L\log\lambda$, $\psi(M)$ with the norms induced from $\overline{M''}$. Indeed, by (F1) applied to m_a and $\psi(m_a)$ (the index term is the same, ψ being an isomorphism $M \to \psi(M)$), it suffices that the Gram matrices satisfy $G'' \le \lambda^2 G$ as Hermitian forms, which gives $\det G'' \le \lambda^{2s}\det G$ by monotonicity of the determinant on positive definite matrices.

Step 1: the torus lattice. Give U the norms induced from $\overline{V}$, and let $\overline{U}_0$ be U with the norms $\|\sum_b w_b\varepsilon_b\|_{0,\sigma}^2 = \sum_b |w_b|^2$. Every non-zero submodule $M \subseteq U$ satisfies $\widehat{\deg}_{\overline{U}_0}(M) \le 0$: let $s = \operatorname{rk} M$, choose linearly independent $m_1,\dots,m_s \in M$ and write $\omega = m_1\wedge\cdots\wedge m_s = \sum_{|I|=s} c_I\varepsilon_I$ in $\bigwedge^s U = \bigoplus_I \mathcal{O}_L\varepsilon_I$, with $c_I \in \mathcal{O}_L$ the $s\times s$ minors; the ε_I are orthonormal for the norm induced by $\|\cdot\|_{0,\sigma}$ (Cauchy–Binet), so $\|\omega\|_{0,\sigma} \ge |\sigma(c_J)|$ for a fixed J with $c_J \ne 0$. The map $\bigwedge^s M \to \mathcal{O}_L$, coefficient of ε_J, is injective (rank one, torsion free, non-zero on ω), and $\#(M/\sum_a \mathcal{O}_L m_a) = \#(\bigwedge^s M/\mathcal{O}_L\omega)$ (the index ideal of $\sum_a \mathcal{O}_L m_a$ in M is that of $\mathcal{O}_L\omega$ in $\bigwedge^s M$), so this index is at most $\#(\mathcal{O}_L/c_J\mathcal{O}_L) = |N_{L/\mathbb{Q}}(c_J)|$. By (F1), $\widehat{\deg}_{\overline{U}_0}(M) \le \log|N_{L/\mathbb{Q}}(c_J)| - \sum_\sigma \log|\sigma(c_J)| = 0$. By (1), $\|x\|_{0,\sigma} \le C^{1/2}\|x\|_{V,\sigma}$ and $\|x\|_{V,\sigma} \le C^{1/2}\|x\|_{0,\sigma}$; (F4) for the identity maps gives $\widehat{\deg}_{\overline{U}}(M) \le \widehat{\deg}_{\overline{U}_0}(M) + \frac{s}{2}n_L\log C \le \frac{s}{2}n_L\log C$, and, as $\widehat{\deg}\,\overline{U}_0 = 0$ by (F1) for the basis ε, $\widehat{\deg}\,\overline{U} \ge -\frac{r}{2}n_L\log C$. Hence

$$\widehat{\mu}_{\max}(\overline{U}) \le \tfrac{1}{2}\log C \quad (r \ge 1), \qquad \widehat{\deg}\,\overline{U} \le \tfrac{r}{2}\,n_L\log C. \tag{D.32}$$

If $U = V$, then $\widehat{\mu}_{\max}(\overline{V}) \le \frac{1}{2}\log C$, which is at most the right-hand side because $C_{\mathrm{ll}} \ge 0$ and the logarithm there is at least $\log 2 > 0$. From now on $U \ne V$, and $\mathcal{A}$, ϕ are as in (2).

Step 2: the abelian quotient. Put $Q = V/U$ with projection p. Since $\ker\phi = U$, ϕ induces an injection $\bar\phi\colon Q \to \operatorname{Lie}(\mathcal{A})$ with image $\phi(V)$ of finite index; so Q is torsion free, hence projective, U is saturated in V, and $g' := \operatorname{rk} Q = \operatorname{rk}\operatorname{Lie}(\mathcal{A})$ is the relative dimension of $\mathcal{A}$, with $r + g' = g$. Give Q the quotient norms of $\overline{V}$. As Q is projective, $U_\sigma = \ker(V_\sigma \to Q_\sigma)$, and ϕ_σ vanishes on U_σ; so for $x \in V_\sigma$ and every $u \in U_\sigma$, $\|\bar\phi_\sigma(p_\sigma x)\|_\sigma = \|\phi_\sigma(x+u)\|_\sigma \le C^{1/2}\|x+u\|_{V,\sigma}$, and minimising over u,

$$\|\bar\phi_\sigma(y)\|_\sigma \le C^{1/2}\|y\|_{Q,\sigma} \qquad (y \in Q_\sigma).$$

For a non-zero submodule $M \subseteq Q$ of rank s, (F4) for $\bar\phi|_M$ gives $\widehat{\mu}(\overline{M}) \le \widehat{\mu}(\overline{\bar\phi(M)}) + \frac{1}{2}\log C \le \widehat{\mu}_{\max}(\overline{\operatorname{Lie}}(\mathcal{A})) + \frac{1}{2}\log C$; and (F4) for $\bar\phi$ with (F2) for $\phi(V) \subseteq \operatorname{Lie}(\mathcal{A})$ gives $\widehat{\deg}\,\overline{Q} \le \widehat{\deg}\,\overline{\phi(V)} + \frac{g'}{2}n_L\log C \le \widehat{\deg}\,\overline{\operatorname{Lie}}(\mathcal{A}) + \frac{g'}{2}n_L\log C$. Thus

$$\widehat{\mu}_{\max}(\overline{Q}) \le \widehat{\mu}_{\max}(\overline{\operatorname{Lie}}(\mathcal{A})) + \tfrac{1}{2}\log C, \qquad \widehat{\deg}\,\overline{Q} \le \widehat{\deg}\,\overline{\operatorname{Lie}}(\mathcal{A}) + \tfrac{g'}{2}\,n_L\log C. \tag{D.33}$$

Step 3: the maximal slope of an extension. We show

$$\widehat{\mu}_{\max}(\overline{V}) \le \max\bigl(\widehat{\mu}_{\max}(\overline{U}), \widehat{\mu}_{\max}(\overline{Q})\bigr),$$

the first term omitted if $r = 0$; this holds exactly, for the induced norms on U and the quotient norms on Q. Let $M \subseteq V$ be a non-zero submodule of rank s, $M_1 = M \cap U$ of rank s_1, and $M_2 = p(M) \cong M/M_1$ of rank $s_2 = s - s_1$; M_1 is saturated in M since M/M_1 embeds in the torsion free Q. Give M the norms induced from $\overline{V}$. By (F3), $\widehat{\deg}\,\overline{M} = \widehat{\deg}\,\overline{M_1} + \widehat{\deg}\,\overline{M/M_1}$, where M_1 carries the norms induced from $\overline{U}$ and M/M_1 the quotient norms of $\overline{M}$. For $x \in M_\sigma$, the quotient norm of its class is $\min_{y\in(M_1)_\sigma}\|x+y\|_{V,\sigma} \ge \min_{u\in U_\sigma}\|x+u\|_{V,\sigma} = \|p_\sigma x\|_{Q,\sigma}$, since $(M_1)_\sigma \subseteq U_\sigma$. So the isomorphism $M/M_1 \to M_2$ does not increase norms, and (F4)

with $\lambda = 1$ gives $\widehat{\deg}\,\overline{M/M_1} \le \widehat{\deg}\,\overline{M_2}$, M_2 with the norms induced from $\overline{Q}$. Hence $\widehat{\deg}\,\overline{M} \le n_L\big(s_1\widehat{\mu}_{\max}(\overline{U}) + s_2\widehat{\mu}_{\max}(\overline{Q})\big)$ (a term with $s_i = 0$ is 0), and dividing by $n_L s$ proves the claim.

Step 4: the height of the abelian part. By (F3) for $U \subseteq V$, (D.32) and (D.33), $\widehat{\deg}\,\overline{V} = \widehat{\deg}\,\overline{U} + \widehat{\deg}\,\overline{Q} \le \widehat{\deg}\,\overline{\mathrm{Lie}}(\mathcal{A}) + \frac{g}{2} n_L \log C$. Hence $-\widehat{\deg}\,\overline{\mathrm{Lie}}(\mathcal{A})/n_L \le -\widehat{\deg}\,\overline{V}/n_L + \frac{g}{2}\log C$ and, as $\log C \ge 0$,

$$h^+(\mathcal{A}) \le \max\Big(-\frac{\widehat{\deg}\,\overline{V}}{n_L}, 0\Big) + \frac{g}{2}\log C.$$

Step 5: conclusion. By Proposition D.27 for $\mathcal{A}$ (relative dimension $g' \le g$) and Step 4,

$$\begin{aligned}\widehat{\mu}_{\max}(\overline{\mathrm{Lie}}(\mathcal{A})) &\le (C_{\mathrm{ll}} + \tfrac{1}{2})\log(2 + g' + h^+(\mathcal{A})) \\ &\le (C_{\mathrm{ll}} + \tfrac{1}{2})\log\Big(2 + g + \tfrac{g}{2}\log C + \max\Big(-\frac{\widehat{\deg}\,\overline{V}}{n_L}, 0\Big)\Big) =: \Lambda.\end{aligned}$$

By Step 3, (D.32) and (D.33), $\widehat{\mu}_{\max}(\overline{V}) \le \max(\frac{1}{2}\log C,\ \Lambda + \frac{1}{2}\log C) = \Lambda + \frac{1}{2}\log C$, since $\Lambda \ge 0$. □

Theorem D.129. *Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ whose root models have convergent regularisation, and assume that the realised tower has the reference link (Vl). For $Z \in \mathrm{Cp}$ let A_Z be the maximum of $|\mathrm{pp}_Z^{\mathrm{ref}}(x^m)|$ over the monomials $x^m \in \mathcal{R}^+$ of degree $D+1$. Then the realised tower has the monomial bound (Wd), with the constant*

$$C = \max_{Z\in\mathrm{Cp}}\Big(\frac{A_Z}{\log 2} + (D+1)\,2^D M_Z^{\mathrm{ref}}\,C_{\mathrm{Vl}}\Big).$$

Proof. Fix an integer $B \ge 2$, $Z \in \mathrm{Cp}$ and a monomial $x^m \in \mathcal{R}^+$. If x^m has degree $\ne D+1$, then $\mathrm{ai}_t(\mathcal{X}_{B,Z}; x^m) = 0$ for every t, hence $\mathrm{pp}(\mathcal{X}_{B,Z}; x^m) = 0$ and there is nothing to prove. Let $x^m = x_{r_1}\cdots x_{r_{D+1}}$ with $r_j \in \mathrm{Rt}^+$, and take $\widetilde{\mathcal{X}}$, $\widetilde{\mathcal{X}}'$, k, δ, S, α, e_H, t_0 as in (Vl) for (B, Z).

Comparison at a fixed regularisation parameter. Let $0 < t \le t_0$. Apply Lemma D.130 on $\widetilde{\mathcal{X}}$ to $\overline{\mathcal{L}}_j = \overline{\mathcal{L}}_{r_j}(t)$, the lines of $\widetilde{\mathcal{X}}$, and $\overline{\mathcal{L}}'_j = \overline{\mathcal{L}}'_{r_j}(t)$, those of $\widetilde{\mathcal{X}}'$ (for $r_j = *$ the auxiliary lines), with $M = M_Z^{\mathrm{ref}}\delta$. Its hypotheses hold: $\overline{\mathcal{L}}_j$ and $\overline{\mathcal{L}}'_j$ are vertically close over (S, α) by (Vl2); hypothesis 1 holds by (Vl3), since $e_{r_j} + e_H$ and $e'_{r_j} + e_H$ belong to the sets listed there; hypothesis 2 holds by (Vl4). Dividing the resulting inequality by $\delta k^{\#\{j:\, r_j = *\}} \ge \delta$ and using (29.2) for both root models (they have the same normalising integers) and (Vl5),

$$|\mathrm{ai}_t(\widetilde{\mathcal{X}}; x^m) - \mathrm{ai}_t(\widetilde{\mathcal{X}}'; x^m)| \le (D+1)\,2^D M_Z^{\mathrm{ref}} \sum_{p\in S}\alpha_p \log p \le (D+1)\,2^D M_Z^{\mathrm{ref}}\,C_{\mathrm{Vl}}\log B.$$

Limit. Both models have convergent regularisation, so letting $t \to 0^+$ and using (Vl1),

$$|\mathrm{pp}(\mathcal{X}_{B,Z}; x^m) - \mathrm{pp}_Z^{\mathrm{ref}}(x^m)| = |\mathrm{pp}(\widetilde{\mathcal{X}}; x^m) - \mathrm{pp}(\widetilde{\mathcal{X}}'; x^m)| \le (D+1)\,2^D M_Z^{\mathrm{ref}}\,C_{\mathrm{Vl}}\log B.$$

Conclusion. Since $B \ge 2$, $A_Z \le (A_Z/\log 2)\log B$, so

$$|\mathrm{pp}(\mathcal{X}_{B,Z}; x^m)| \le A_Z + (D+1)\,2^D M_Z^{\mathrm{ref}}\,C_{\mathrm{Vl}}\log B \le C\log B,$$

which is (Wd); C is finite because Cp is finite. □

Lemma D.130. *Let $\mathcal{X} \subseteq \mathbb{P} = \prod_{k\in K} \mathbb{P}(\mathbb{Z}^{I_k})$ be a closed integral subscheme, flat over $\mathbb{Z}$, of absolute dimension $D+1$ with $D \geq 0$. Let S be a finite set of primes, $\alpha_p \geq 0$ for $p \in S$, and $M \geq 0$. For $1 \leq j \leq D+1$ let $\overline{\mathcal{L}}_j = (\mathcal{O}(e_j), h_j)|_{\mathcal{X}}$ and $\overline{\mathcal{L}}'_j = (\mathcal{O}(e'_j), h'_j)|_{\mathcal{X}}$ be Hermitian line bundles given by cone metrics such that $\overline{\mathcal{L}}_j$ and $\overline{\mathcal{L}}'_j$ are vertically close over (S, α). Let $e_H \in \mathbb{Z}^K$ be such that*

1. *for every $p \in S$, every integral closed curve $W \subseteq \mathcal{X}_{\mathbb{F}_p}$ and every $f \in \{e_H\} \cup \{e_j + e_H,\ e'_j + e_H :\ 1 \leq j \leq D+1\}$, $\deg(\mathcal{O}(f)|_W) \geq 0$;*
2. *$\deg(c_1(\mathcal{O}(f_1)) \cdots c_1(\mathcal{O}(f_D)) \cap [\mathcal{X}_{\mathbb{Q}}]) \leq M$ for all $f_1, \ldots, f_D \in \{e_H\} \cup \{e_j + e_H :\ 1 \leq j \leq D+1\}$.*

Then

$$\left|\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_1) \cdots \widehat{c}_1(\overline{\mathcal{L}}_{D+1})) - \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}'_1) \cdots \widehat{c}_1(\overline{\mathcal{L}}'_{D+1}))\right| \leq (D+1)\, 2^D M \sum_{p\in S} \alpha_p \log p.$$

Proof. **Step 1: the comparison sections.** For each j choose f_j, g_j as in the definition of vertical closeness and put $\sigma_j = f_j/g_j$, a non-zero rational section of $\mathcal{L}_j \otimes \mathcal{L}_j'^{-1} = \mathcal{O}(e_j - e'_j)|_{\mathcal{X}}$ (by (31.3), $d_{f_j} - d_{g_j} = e_j - e'_j$). The metric of $\overline{\mathcal{L}}_j \otimes \overline{\mathcal{L}}_j'^{-1}$ has cone function h_j/h'_j, so on the affine cone over $\mathcal{X}(\mathbb{C})$, wherever $g_j \neq 0$,

$$\|\sigma_j\| = \frac{|f_j|}{|g_j|} \cdot \frac{h'_j}{h_j} = 1$$

by (31.3). The Weil divisor of σ_j is $[\operatorname{div}\sigma_j] = \sum_W \big(\operatorname{ord}_W(f_j) - \operatorname{ord}_W(g_j)\big)[W]$, the sum over prime divisors W of $\mathcal{X}$ (Fulton, *Intersection theory*, §1.2, §2.1). By vertical closeness only prime divisors $W \subseteq \mathcal{X}_{\mathbb{F}_p}$ with $p \in S$ occur, with coefficients $a_{j,W}$ satisfying

$$|a_{j,W}| \leq \alpha_p \operatorname{ord}_W(p) \qquad (W \subseteq \mathcal{X}_{\mathbb{F}_p},\ p \in S). \tag{D.34}$$

Step 2: telescoping. For $1 \leq j \leq D+1$ let $\overline{\mathcal{A}}^{(j)} = (\overline{\mathcal{L}}'_1, \ldots, \overline{\mathcal{L}}'_{j-1}, \overline{\mathcal{L}}_{j+1}, \ldots, \overline{\mathcal{L}}_{D+1})$, a list of D Hermitian line bundles, and write $\widehat{c}_1(\overline{\mathcal{A}}^{(j)})$ for the product of their arithmetic first Chern classes. Since $\widehat{\deg}$ of a product of $D+1$ arithmetic first Chern classes is symmetric and $\widehat{c}_1$ is additive in tensor products,

$$\Delta := \widehat{\deg}\Big(\prod_j \widehat{c}_1(\overline{\mathcal{L}}_j)\Big) - \widehat{\deg}\Big(\prod_j \widehat{c}_1(\overline{\mathcal{L}}'_j)\Big) = \sum_{j=1}^{D+1} \widehat{\deg}\Big(\widehat{c}_1(\overline{\mathcal{A}}^{(j)}) \cdot \widehat{c}_1(\overline{\mathcal{L}}_j \otimes \overline{\mathcal{L}}_j'^{-1})\Big).$$

Step 3: each term is a sum of vertical degrees. By Lemma 30.17 (ambient $\mathbb{P}$, $\mathcal{Z} = \mathcal{X}$, integral and flat over $\mathbb{Z}$): for restrictions $\overline{\mathcal{M}}_1, \ldots, \overline{\mathcal{M}}_D, \overline{\mathcal{N}}$ to $\mathcal{X}$ of Hermitian line bundles with smooth metrics on $\mathbb{P}$ and a non-zero rational section σ of $\mathcal{N}$ on $\mathcal{X}$,

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{M}}_1) \cdots \widehat{c}_1(\overline{\mathcal{M}}_D) \cdot \widehat{c}_1(\overline{\mathcal{N}})) = h_{\overline{\mathcal{M}}_1, \ldots, \overline{\mathcal{M}}_D}([\operatorname{div}\sigma]) - \int_{\mathcal{X}(\mathbb{C})} \log\|\sigma\|\, c_1(\overline{\mathcal{M}}_1) \wedge \cdots \wedge c_1(\overline{\mathcal{M}}_D),$$

where the horizontal components of $[\operatorname{div}\sigma]$ contribute their heights and a vertical prime divisor $W \subseteq \mathcal{X}_{\mathbb{F}_p}$ contributes $\log p \cdot \deg(c_1(\mathcal{M}_1|_W) \cdots c_1(\mathcal{M}_D|_W) \cap [W])$ times its multiplicity (the two

sums of Lemma 30.17). Apply this with $\overline{\mathcal{N}} = \overline{\mathcal{L}}_j \otimes \overline{\mathcal{L}}_j'^{-1}$ and $\sigma = \sigma_j$: the integral vanishes because $\log\|\sigma_j\| = 0$, and Step 1 gives

$$\widehat{\deg}\Big(\widehat{c}_1(\overline{\mathcal{A}}^{(j)}) \cdot \widehat{c}_1(\overline{\mathcal{L}}_j \otimes \overline{\mathcal{L}}_j'^{-1})\Big) = \sum_{p \in S} \log p \sum_{W \subseteq \mathcal{X}_{\mathbb{F}_p}} a_{j,W}\, \deg\Big(\prod_{i=1}^{D} c_1(\mathcal{A}_i^{(j)}) \cap [W]\Big). \tag{D.35}$$

Step 4: the degree on one vertical prime divisor. Fix j, $p \in S$ and a prime divisor $W \subseteq \mathcal{X}_{\mathbb{F}_p}$; W is a projective scheme over $\mathbb{F}_p$ of dimension D. Each $\mathcal{A}_i^{(j)}$ is $\mathcal{O}(g_i)|_{\mathcal{X}}$ with $g_i \in \{e_l, e_l'\}$; put $N_i = g_i + e_H$, so $\mathcal{O}(g_i) = \mathcal{O}(N_i) \otimes \mathcal{O}(e_H)^{-1}$ and $N_i \in \{e_l + e_H,\ e_l' + e_H\}$. Expanding multilinearly,

$$\deg\Big(\prod_{i=1}^{D} c_1(\mathcal{O}(g_i)) \cap [W]\Big) = \sum_{I \subseteq \{1,\dots,D\}} (-1)^{D-|I|}\, d_I(W),$$
$$d_I(W) = \deg\Big(\prod_{i \in I} c_1(\mathcal{O}(N_i))\, c_1(\mathcal{O}(e_H))^{D-|I|} \cap [W]\Big).$$

By hypothesis 1 the restrictions of $\mathcal{O}(N_i)$ and $\mathcal{O}(e_H)$ to W have non-negative degree on every integral curve in W, i.e. they are nef on W, so by Kleiman's theorem every $d_I(W) \geq 0$. Hence

$$\Big|\deg\Big(\prod_{i} c_1(\mathcal{A}_i^{(j)}) \cap [W]\Big)\Big| \leq \sum_{I} d_I(W). \tag{D.36}$$

Step 5: summing over the fibre. Since $\mathcal{X}$ is integral and flat over $\mathbb{Z}$, p is a non-zero-divisor on $\mathcal{X}$, every irreducible component of $\mathcal{X}_{\mathbb{F}_p}$ is a prime divisor of $\mathcal{X}$ (Krull), and the cycle of the fibre is $[\mathcal{X}_{\mathbb{F}_p}] = \sum_{W \subseteq \mathcal{X}_{\mathbb{F}_p}} \mathrm{ord}_W(p)[W]$. The scheme $\mathcal{X} \times_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is flat and projective over the discrete valuation ring $\mathbb{Z}_{(p)}$, so by conservation of number (Fulton, *Intersection theory*, Proposition 10.2)

$$\sum_{W \subseteq \mathcal{X}_{\mathbb{F}_p}} \mathrm{ord}_W(p)\, d_I(W) = \deg\Big(\prod_{i \in I} c_1(\mathcal{O}(N_i))\, c_1(\mathcal{O}(e_H))^{D-|I|} \cap [\mathcal{X}_{\mathbb{Q}}]\Big).$$

On the generic fibre, $c_1(\mathcal{O}(e_l')) \cap \gamma = c_1(\mathcal{O}(e_l)) \cap \gamma$ for every cycle class γ of $\mathcal{X}_{\mathbb{Q}}$: the section σ_l of $\mathcal{O}(e_l - e_l')$ has order $\mathrm{ord}_W(f_l) - \mathrm{ord}_W(g_l) = 0$ along every prime divisor W of $\mathcal{X}$ not contained in a fibre, and these are exactly the closures of the prime divisors of $\mathcal{X}_{\mathbb{Q}}$, with the same local rings; so $c_1(\mathcal{O}(e_l - e_l')) \cap [\mathcal{X}_{\mathbb{Q}}] = [\mathrm{div}(\sigma_l|_{\mathcal{X}_{\mathbb{Q}}})] = 0$, and the same holds after capping with further first Chern classes since these commute (Fulton, Proposition 2.5). Replacing each $e_l' + e_H$ by $e_l + e_H$, hypothesis 2 bounds the right side by M. With (D.34) and (D.36),

$$\sum_{W \subseteq \mathcal{X}_{\mathbb{F}_p}} |a_{j,W}| \,\Big|\deg\Big(\prod_{i} c_1(\mathcal{A}_i^{(j)}) \cap [W]\Big)\Big| \leq \alpha_p \sum_{I} \sum_{W} \mathrm{ord}_W(p)\, d_I(W) \leq \alpha_p\, 2^D M.$$

Step 6: conclusion. By Step 2, (D.35) and Step 5,

$$|\Delta| \leq \sum_{j=1}^{D+1} \sum_{p \in S} \log p \cdot \alpha_p\, 2^D M = (D+1)\, 2^D M \sum_{p \in S} \alpha_p \log p.$$

□

Theorem D.131. *Let $\mathcal{T}$ be a root tower of type (Φ, n, κ) with a geometric realisation $(\nu_{\mathcal{T}}, (\mathcal{X}_{B,Z}))$ whose root models have convergent regularisation, and assume that the realised*

tower has the reference comparison (Rh1), (Rh2). Then it has the heights of points (Wp). More precisely, for $c_0 > 0$ one may take

$$\theta_0 = \min\Big(s_{\mathrm{Rh}},\ \frac{1}{2\|\nu_{\mathcal{T}}\|_\infty + 1},\ \Big(\frac{c_0}{2\|\nu_{\mathcal{T}}\|_\infty + 1}\Big)^{1/2}\Big),$$

$C = \max(C_a, C_b)$ *with* $C_a = 2C_{\mathrm{Rh}} + (C_{\mathrm{Rh}} \log C_{\mathrm{Rh}} + 2C_{\mathrm{Rh}} + 2)/\log 2$ *and* $C_b = C_{\mathrm{Rh}} + \big((2|\Phi|n + 1)\max_Z D_Z + C_{\mathrm{Rh}}\big)/\log 2$, *where* D_Z *is a constant of Lemma D.132 for* $\mathcal{X}_Z^{\mathrm{ref}}$ *and its lines* $\overline{\mathcal{L}}_r(1)$, $r \in \mathrm{Rt}^+$.

Proof. Fix $c_0 > 0$, θ_0 as stated, an integer $B \geq 2$, $Z \in \mathrm{Cp}$, $\theta \in (0, \theta_0]$ with $s = \theta^2$ rational, and a (c_0, θ)-spaced weight λ with rational entries; write $\mathcal{X} = \mathcal{X}_{B,Z}$ and $\lambda_s = \lambda - s\nu$. Take $\mathcal{J}$, q, q^{ref} as in (Rh1) for (B, Z).

Step 0: the perturbed weight is admissible in (Rh). Since $\theta \leq \theta_0 \leq s_{\mathrm{Rh}} \leq 1$, $0 < s \leq \theta \leq s_{\mathrm{Rh}} \leq 1$. Next $\|\lambda_s\|_\infty \leq 1 + \theta^2\|\nu\|_\infty \leq 1 + \theta\|\nu\|_\infty \leq \frac{3}{2}$. For $i \neq \kappa$, $(\lambda_s)_{a,i} - (\lambda_s)_{a,i+1} \geq c_0 - 2\theta^2\|\nu\|_\infty \geq c_0 - \theta^2(2\|\nu\|_\infty + 1) \geq 0$; at the cut, $(\lambda_s)_{a,\kappa} - (\lambda_s)_{a,\kappa+1} \geq \theta - 2\theta^2\|\nu\|_\infty = \theta(1 - 2\theta\|\nu\|_\infty) > 0$. So λ_s is dominant with $\|\lambda_s\|_\infty \leq 2$, and (Rh1) applies to $\mu = \lambda_s$ and s. Also $c(\lambda_s, s) = c(\lambda_s, 0) + s\,c(0, 1)$ on each model, and $h_{c,t}(W)$ is linear in c by (30.2).

Step 1: points lift along surjective morphisms. Let $g\colon \mathcal{J} \to \mathcal{Y}$ be a surjective morphism of arithmetic varieties and W a point of $\mathcal{Y}$. The generic point η of W is a closed point of $\mathcal{Y}_{\mathbb{Q}}$. The fibre $g^{-1}(\eta)$ is non-empty and of finite type over $\mathbf{k}(\eta)$; let y be a closed point of it. Then $\mathbf{k}(y)$ is finite over $\mathbf{k}(\eta)$, hence over $\mathbb{Q}$, so y is a closed point of $\mathcal{J}_{\mathbb{Q}}$; its closure W' in $\mathcal{J}$ is integral, flat over $\mathbb{Z}$ (its generic point lies over $\mathbb{Q}$), of absolute dimension $1 + \mathrm{trdeg}_{\mathbb{Q}}\mathbf{k}(y) = 1$, i.e. a point of $\mathcal{J}$. As g is proper, $g(W')$ is closed and irreducible with generic point $g(y) = \eta$, so $g(W') = W$.

Step 2: the lower bound (a). Let W be a point of $\mathcal{X}$. By Step 1 there is a point W' of $\mathcal{J}$ with $q(W') = W$; put $W^{\mathrm{ref}} = q^{\mathrm{ref}}(W')$ and $x = h^{\mathrm{ref}}_{c(0,1),1}(W^{\mathrm{ref}})$. By (Rh1), Step 0 and (Rh2),

$$\begin{aligned} h_{c(\lambda_s,s),t(s)}(W) &\geq h^{\mathrm{ref}}_{c(\lambda_s,0),1}(W^{\mathrm{ref}}) + s\,x - C(\log B + \log(1/s) + 1) \\ &\geq -C\log y - C + s(y - 2 - C) - C(\log B + \log(1/s) + 1), \end{aligned}$$

with $y = 2 + C + x \geq 2$ (as $x \geq -C$). The function $y \mapsto sy - C\log y$ on $(0, \infty)$ has its minimum at $y = C/s$, so $sy - C\log y \geq C - C\log C - C\log(1/s)$; also $s(2 + C) \leq 2 + C$. Hence

$$\begin{aligned} h_{c(\lambda_s,s),t(s)}(W) &\geq -C\log B - 2C\log(1/s) - (C\log C + 2C + 2) \\ &\geq -C_a\big(\log B + \log(1/s)\big) = -C_a\log(B/s), \end{aligned}$$

using $C \geq 1$ (so $C\log C \geq 0$), $\log B \geq \log 2$ and $\log(1/s) \geq 0$.

Step 3: the dense set (b). By Lemma D.132 applied to $\mathcal{X}_Z^{\mathrm{ref}}$ and the lines $\overline{\mathcal{L}}_r(1)$, $r \in \mathrm{Rt}^+$, of the reference model, there is a Zariski dense set F^{ref} of points of $\mathcal{X}_Z^{\mathrm{ref}}$ on which all these heights are at most D_Z in absolute value. For $W^{\mathrm{ref}} \in F^{\mathrm{ref}}$, by (30.2),

$$h^{\mathrm{ref}}_{c(\lambda_s,s),1}(W^{\mathrm{ref}}) \leq \|c(\lambda_s, s)\|_1 D_Z \leq \big(\|\lambda_s\|_1 + s\big)D_Z \leq (2|\Phi|n + 1)D_Z.$$

For each $W^{\mathrm{ref}} \in F^{\mathrm{ref}}$ choose, by Step 1, a point W' of $\mathcal{J}$ with $q^{\mathrm{ref}}(W') = W^{\mathrm{ref}}$; let $F_{\mathcal{J}}$ be the set of these W' and $F = \{q(W') : W' \in F_{\mathcal{J}}\}$, a set of points of $\mathcal{X}$. By (Rh1), for $W' \in F_{\mathcal{J}}$,

$$h_{c(\lambda_s,s),t(s)}(q(W')) \leq (2|\Phi|n + 1)D_Z + C(\log B + 1) \leq C_b \log B.$$

$F_{\mathcal{J}}$ is dense in $\mathcal{J}$. Let $\mathcal{Y} \subseteq \mathcal{J}$ be closed and contain every $W' \in F_{\mathcal{J}}$. Then $q^{\mathrm{ref}}(\mathcal{Y})$ is closed (q^{ref} is proper) and contains every member of F^{ref}, so it is $\mathcal{X}_Z^{\mathrm{ref}}$. Some irreducible component

$\mathcal{Y}_0$ of $\mathcal{Y}$ maps onto $\mathcal{X}_Z^{\mathrm{ref}}$; it dominates $\operatorname{Spec}\mathbb{Z}$, and its function field contains that of $\mathcal{X}_Z^{\mathrm{ref}}$, so $\dim\mathcal{Y}_0 = 1+\operatorname{trdeg}_{\mathbb{Q}}\mathbf{k}(\mathcal{Y}_0) \geq 1+\operatorname{trdeg}_{\mathbb{Q}}\mathbf{k}(\mathcal{X}_Z^{\mathrm{ref}}) = D+1 = \dim\mathcal{J}$. As $\mathcal{J}$ is irreducible, $\mathcal{Y}_0 = \mathcal{J}$ as sets. *F is Zariski dense in $\mathcal{X}$.* Let $\mathcal{Y}' \subseteq \mathcal{X}$ be a closed subscheme containing every member of F. Then $q^{-1}(\mathcal{Y}')$ contains every $W' \in F_{\mathcal{J}}$, hence equals $\mathcal{J}$ as a set, so the underlying set of $\mathcal{Y}'$ contains $q(\mathcal{J}) = \mathcal{X}$; as $\mathcal{X}$ is reduced, $\mathcal{Y}' = \mathcal{X}$.

With $C = \max(C_a, C_b)$, Steps 2 and 3 are (a) and (b) of (Wp). □

Lemma D.132. *Let $\mathcal{X} \subseteq \mathbb{P} = \prod_{k\in K}\mathbb{P}(\mathbb{Z}^{I_k})$ be a closed integral subscheme, flat over $\mathbb{Z}$, of absolute dimension $d+1$, and let $\overline{\mathcal{L}}_j = (\mathcal{O}(e_j), h_j)|_{\mathcal{X}}$, $j \in J$, be finitely many Hermitian line bundles given by cone metrics. For a point W of $\mathcal{X}$ put $h_{\overline{\mathcal{L}}_j}(W) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_j)\cdot[W])/\deg W_{\mathbb{Q}}$. Then there are $C \geq 0$ and a Zariski dense set F of points of $\mathcal{X}$ with $|h_{\overline{\mathcal{L}}_j}(W)| \leq C$ for every $W \in F$ and every $j \in J$.*

Proof. **Step 1: heights of points by local terms.** Let W be a point of $\mathcal{X}$. Its generic point u is a closed point of $\mathcal{X}_{\mathbb{Q}}$ ($W_{\mathbb{Q}}$ is non-empty and of dimension 0), and $\deg W_{\mathbb{Q}} = [\mathbf{k}(u):\mathbb{Q}]$. Let $L \supseteq \mathbf{k}(u)$ be a number field and $z = (z_k)_k$, $z_k \in L^{I_k}\smallsetminus\{0\}$, multihomogeneous coordinates of u. For a cone metric h of multidegree e, the definition of the height of a point for the Hermitian line bundle $(\mathcal{O}(e),h)$ (whose integral structure at the finite places is that of $\mathcal{O}(e)$ on $\mathbb{P}$) together with the product formula gives the standard expression ([48], §3)

$$h_{(\mathcal{O}(e),h)}(W) = \frac{1}{[L:\mathbb{Q}]}\Big(\sum_{v\text{ finite}}\sum_k e_k \log\max_i |z_{k,i}|_v + \sum_{\sigma\colon L\to\mathbb{C}} \log h(\sigma z)\Big), \tag{D.37}$$

independent of L and of the coordinates.

Step 2: reduction to standard heights. For $k \in K$ put

$$h_k^{\mathrm{st}}(W) = \frac{1}{[L:\mathbb{Q}]}\Big(\sum_{v\text{ finite}} \log\max_i |z_{k,i}|_v + \sum_{\sigma}\log\max_i|\sigma(z_{k,i})|\Big) \geq 0,$$

the inequality by the product formula applied to a non-zero coordinate z_{k,i_0}. For a cone metric h_j of multidegree e_j, the function $\psi_j(z) = \log h_j(z) - \sum_k e_{j,k}\log\max_i|z_{k,i}|$ is continuous on $\widehat{\mathbb{C}} = \prod_k(\mathbb{C}^{I_k}\smallsetminus\{0\})$ and invariant under $z_k \mapsto \lambda_k z_k$, $\lambda_k \in \mathbb{C}^\times$; so it is bounded, $|\psi_j| \leq m_j$, by its values on the compact set $\{\max_i|z_{k,i}| = 1\ \forall k\}$. By (D.37),

$$\Big|h_{\overline{\mathcal{L}}_j}(W) - \sum_k e_{j,k}h_k^{\mathrm{st}}(W)\Big| \leq m_j, \qquad \text{so} \qquad |h_{\overline{\mathcal{L}}_j}(W)| \leq \Big(\sum_k|e_{j,k}|\Big)\max_k h_k^{\mathrm{st}}(W) + m_j. \tag{D.38}$$

It therefore suffices to find C_0 and a Zariski dense set F of points with $h_k^{\mathrm{st}}(W) \leq C_0$ for all k; then $C = \max_j((\sum_k|e_{j,k}|)C_0 + m_j)$.

Step 3: a finite projection. The scheme $\mathcal{X}_{\mathbb{Q}}$ is integral, of dimension d, and dense in $\mathcal{X}$ (flatness). For each k choose $i_k \in I_k$ such that z_{k,i_k} does not vanish identically on $\mathcal{X}_{\mathbb{Q}}$ (if all coordinates of a block vanished identically, $\mathcal{X}_{\mathbb{Q}}$ would be empty). Then $U = \mathcal{X}_{\mathbb{Q}} \cap \{\prod_k z_{k,i_k} \neq 0\}$ is a non-empty open affine subscheme of $\mathcal{X}_{\mathbb{Q}}$ with coordinate ring A, an integral domain of dimension d generated over $\mathbb{Q}$ by the functions $y_{k,i} = z_{k,i}/z_{k,i_k}$. By Noether normalisation there are $y_1',\dots,y_d' \in A$, algebraically independent over $\mathbb{Q}$, such that A is finite over $R = \mathbb{Q}[y_1',\dots,y_d']$. Every generator $y_{k,i}$ is then a root of a monic polynomial

$$P_{k,i}(T) = T^{m_{k,i}} + \sum_{l<m_{k,i}} a_{k,i,l}(y_1',\dots,y_d')\,T^l, \qquad a_{k,i,l} \in \mathbb{Q}[Y_1,\dots,Y_d].$$

Let $N \geq 1$ be a common denominator of all coefficients of all $a_{k,i,l}$, and B_0 the maximum over (k,i,l) of the sum of the absolute values of the coefficients of $a_{k,i,l}$. The morphism $\varphi =$

$(y'_1,\dots,y'_d)\colon U\to \mathbb{A}^d_{\mathbb{Q}}$ is finite, hence closed, and surjective, since $R\to A$ is an injective integral extension (lying over).

Step 4: the points. Let $\mathcal{T}\subseteq \mathbb{A}^d_{\mathbb{Q}}$ be the set of closed points whose coordinates (in $\overline{\mathbb{Q}}$, up to conjugation) are all roots of unity. The set $\mathcal{T}$ is Zariski dense in $\mathbb{A}^d_{\mathbb{Q}}$: a polynomial vanishing on $\mu_\infty^d\subseteq\overline{\mathbb{Q}}^d$ is zero, by induction on d, since μ_∞ is infinite. For each $\tau\in\mathcal{T}$ choose a closed point $u_\tau\in U$ with $\varphi(u_\tau)=\tau$; it is a closed point of $\mathcal{X}_{\mathbb{Q}}$ (its residue field is finite over $\mathbb{Q}$), and its closure W_τ in $\mathcal{X}$ is a point of $\mathcal{X}$. Put $F=\{W_\tau:\tau\in\mathcal{T}\}$.

F is Zariski dense. Let $\mathcal{Y}\subseteq\mathcal{X}$ be a closed subscheme containing every W_τ. Then $\mathcal{Y}\cap U$ is closed in U and contains every u_τ, so the closed set $\varphi(\mathcal{Y}\cap U)$ contains $\mathcal{T}$, hence equals $\mathbb{A}^d_{\mathbb{Q}}$; thus $\dim(\mathcal{Y}\cap U)\geq d=\dim U$ (a finite morphism does not raise dimension), and since U is irreducible, $\mathcal{Y}\cap U$ contains U as a set. So the underlying set of $\mathcal{Y}$ contains the closure of U, which is $\mathcal{X}$; as $\mathcal{X}$ is reduced, $\mathcal{Y}=\mathcal{X}$.

Heights. Let $\tau\in\mathcal{T}$, L a number field containing $\mathbf{k}(u_\tau)$, and take the coordinates $z_{k,i}=y_{k,i}(u_\tau)$ (so $z_{k,i_k}=1$). Each $\eta=z_{k,i}$ is a root of $T^m+\sum_{l<m}a_l(\zeta)T^l$, $a_l=a_{k,i,l}$, where $\zeta=(y'_1(u_\tau),\dots,y'_d(u_\tau))$ consists of roots of unity.

- At a finite place v: $|\zeta_i|_v=1$ and the coefficients of a_l lie in $N^{-1}\mathbb{Z}$, so $|a_l(\zeta)|_v\leq|N|_v^{-1}$ by the ultrametric inequality. If $|\eta|_v>|N|_v^{-1}$ (≥ 1), then $|a_l(\zeta)\eta^l|_v<|\eta|_v^{l+1}\leq|\eta^m|_v$ for every $l<m$, contradicting $\eta^m=-\sum_l a_l(\zeta)\eta^l$. So $\max_i|z_{k,i}|_v\leq|N|_v^{-1}$.
- At an embedding σ: $|\sigma(a_l(\zeta))|\leq B_0$. If $|\sigma\eta|>1+B_0$, then $|\sigma\eta|^m\leq B_0\sum_{l<m}|\sigma\eta|^l<B_0|\sigma\eta|^m/(|\sigma\eta|-1)<|\sigma\eta|^m$, a contradiction. So $\max_i|\sigma(z_{k,i})|\leq 1+B_0$.

By the product formula $\sum_{v\text{ finite}}\log|N|_v^{-1}=[L:\mathbb{Q}]\log N$, hence $h_k^{\mathrm{st}}(W_\tau)\leq\log N+\log(1+B_0)=:C_0$ for every k. Step 2 finishes the proof. □

E Hecke ideals, patching and Taylor–Wiles levels

This appendix contains the automorphic side of the Hecke realisation: Hida theory for ordinary forms on definite unitary groups with a type at an auxiliary place, and patching in the manner of Taylor and Wiles [254] at the one-dimensional points of the polarised ordinary determinant ring. It is written for a rank-n Hecke datum (Definition 50.1), so that it serves both the auxiliary fields of Part IX, in ranks two and three, and the links of the symplectic case, in rank four. It proves five statements of the main text. Proposition E.1, the existence of a Hecke ideal of zero-monodromy type with classical level modules at every Taylor–Wiles level; Proposition E.25, the automorphy, with the witness conditions, of the irreducible points of the classical ideal with a parallel inertial weight; and Proposition E.32, $R=T$ for such ideals at patchable curve points and along thick chains. These three are used in Proposition 50.11 (Section 50) and in Theorem 53.10 (Section 53). The other two, Lemma E.27 on twist stability and Lemma E.29 on purity at arithmetic points, are used only in Theorem 53.10. The proofs cite from the main text only four lemmas: Lemma 49.25, Lemma 50.13, Lemma 50.24 and Lemma 53.25.

The appendix has five sections. Section E.1 constructs the Hecke ideals of zero-monodromy type from Geraghty's ordinary Hecke algebras [105], through the Galois representations of their points, local–global compatibility, base change from the definite unitary group and the control of the level, and proves automorphy and purity at their classical points. Section E.2 proves Proposition E.32 from abstract patching and from the existence of patching systems (Proposition E.39); the latter rests on the choice of Taylor–Wiles places and elements, on a map from patched tuples to the eigenvalue algebra, and on the results of the next three sections. Section E.3 realises a

determinant over a Taylor–Wiles level as a representation, through Cayley–Hamilton quotients in the sense of Chenevier [66], and proves the Galois clause of a patching system. Section E.4 proves the Selmer bounds at the levels: the first-order dual Selmer bound (B1), by Selmer counts at the levels with the Greenberg–Wiles formula (Proposition E.100), and the obstruction bound (B2). Section E.5 constructs the patched ring and shows that classical levels give automorphic patching data. The arguments follow the standard pattern of patching. Two features shape the details: patching is done at curve points of a residually reducible determinant ring, and the level modules are cut out by types of zero monodromy.

E.1 Hecke ideals of zero-monodromy type

This section proves the Hida-theoretic statements of the appendix. The first, Proposition E.1, gives for every rank-n Hecke datum with trivial wild values a Hecke ideal of zero-monodromy type of the universal polarised ordinary determinant ring which, if proper, has classical levels; it is used in Proposition 50.11 and in Theorem 53.10. The second, Proposition E.25, says that the irreducible points of the classical ideal with a parallel inertial weight are automorphic of conjugate self-dual type with the witness conditions, and has the same two consumers. The section also proves the twist stability and the purity of the ideals cut out by an ordinary Hecke algebra of a type (Lemma E.27, Lemma E.29), which Theorem 53.10 uses. The ideals come from the ordinary Hecke algebras of definite unitary groups of Geraghty [105], taken with a type of zero monodromy at the places of St; the determinant ring maps onto them through a Galois determinant (Lemma E.16).

E.1.1 Existence of Hecke ideals of zero-monodromy type

This subsection proves Proposition E.1 from Proposition E.2. When the determinant ring has a classical ordinary point, the ordinary Hecke algebra of a zero-monodromy type at St receives the universal determinant ring, and the kernel of the surjection, taken at fixed torsion values, is a Hecke ideal of zero-monodromy type. Its classical level modules at the Taylor–Wiles levels come from the Λ-adic typed forms of Subsection E.1.4. Integral zero-monodromy types with prescribed inertial characters exist for a finite product of groups GL_n of p-adic fields (Lemma E.3).

Proposition E.1. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}'$ whose wild values are trivial ($\xi_{v,a} = 1$ for all $v \in S_\ell$ and $1 \le a \le n$), κ a twist for $\mathscr{H}$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$. Then B_n^{po} has a Hecke ideal J of zero-monodromy type for ι and κ which, if $J \neq B_n^{\mathrm{po}}$, has classical levels. Moreover, if B_n^{po} has a classical ordinary point for ι and κ, J may be taken of the form given in Proposition E.2 (for $B = B_n^{\mathrm{po}}$): $J = \ker\theta$ with $\theta\colon B_n^{\mathrm{po}} \to \mathbb{T}_{\tau,\mathfrak{m}}/\mathfrak{a}_0$ the surjection of (Gd2) of Lemma E.16 for a type datum $((V_v),\tau)$, $\mathfrak{m}$, χ_0, $\mathfrak{a}_0$ and $\mathcal{C}_{\chi_0}$ as described there.*

Proof. By Proposition E.2 (its hypotheses are those of this statement), B has a Hecke ideal of zero-monodromy type. If B has no classical ordinary point for ι and κ, take $J = B$; there is nothing more to show. Otherwise take J of the form given in Proposition E.2: a type datum $((V_v)_{v\in\mathrm{St}},\tau)$ over $\mathscr{H}$ of principal congruence level c satisfying (Zt), a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_\tau$ with $\mathbb{T}_\tau/\mathfrak{m} = \mathbb{F}$ satisfying (Mx), the homomorphism $\chi_0\colon \Delta_\ell \to \mathcal{O}'^{\times}$ with $\chi_0(d_j(\zeta)) = \kappa(\mathrm{Art}(\zeta))$, $\mathbb{T} = \mathbb{T}_{\tau,\mathfrak{m}}$, $\mathfrak{a}_0 = \mathfrak{a}_{\chi_0} \subseteq \mathbb{T}$, $T = \mathbb{T}/\mathfrak{a}_0$, $M = (e_\mathfrak{m} N_\tau)^{\chi_0} = e_\mathfrak{m} N_\tau / \sum_{\delta\in\Delta_\ell}(\langle\delta\rangle - \chi_0(\delta)) e_\mathfrak{m} N_\tau$ ($e_\mathfrak{m}$ the idempotent of $\mathbb{T}_\tau$ for $\mathfrak{m}$), free of finite rank over Λ_1 and faithful over T by (Zm1), and the surjection $\theta\colon B \to T$ of (Gd2) of Lemma E.16 ($Q = \varnothing$, $\mathfrak{a} = \mathfrak{a}_0$, $\mathcal{C} = \mathcal{C}_{\chi_0}$, which is non-empty with $\bigcap_{y\in\mathcal{C}_{\chi_0}} \ker y = \mathfrak{a}_0$ by (Zp4), as $\mathfrak{a}_0 \neq \mathbb{T}$), with $J = \ker\theta$; so the hypotheses of Lemma E.16

hold for $\mathscr{H}$, κ, B, c and this type datum. By (Gd2) ($Q = \varnothing$), for all w, $v \in S_\ell$, $1 \le j \le n$ and $z \in \mathcal{O}_{L_{\tilde{v}}}^\times$, with $\chi(\mathrm{Frob}_w) = (\mathrm{N}w)^{-1}$:

$$\begin{gathered} T_w^{(j)} = f_j(\mathrm{Frob}_w), \quad f_j(g) = \chi(g)^{j(j-1)/2}\kappa(g)^j\,\theta(\Lambda_j(D^{\mathrm{u}})(\pi_{\mathscr{H}}(g)))\ (g \in G_L); \\ \langle d_j(z)\rangle = (\chi^{j-1}\kappa)(\mathrm{Art}(z))\,\theta(\psi_{v,j}^{\mathrm{d}}(\mathrm{Art}(z))), \end{gathered} \tag{$*$}$$

and the identity for $U_{\tilde{v}}^{(j)}$ stated there. Each $f_j\colon G_L \to T$ is continuous (D^{u}, θ, χ, κ are continuous) and constant on conjugacy classes.

Step 1: the Taylor–Wiles base. Let $v \in S_\ell$ and $d_v = [L_v^+ : \mathbb{Q}_\ell]$. The principal units U^1 of L_v^+ are isomorphic to $\mu_{\ell^\infty}(L_v^+) \times \mathbb{Z}_\ell^{d_v}$ ([190], Chapter II, (5.7)); their open subgroup $1 + \ell\mathcal{O}_{L_v^+}$ contains no non-trivial root of unity ((Wf1) of Lemma E.20; ℓ is odd), so it is a torsion-free finitely generated $\mathbb{Z}_\ell$-module of rank d_v, hence free of rank d_v. Choose a $\mathbb{Z}_\ell$-basis of each $1+\ell\mathcal{O}_{L_v^+}$ and enumerate the triples (v, a, z) with $v \in S_\ell$, $1 \le a \le n$ and z in the chosen basis at v as (v_k, a_k, z_k), $1 \le k \le \sum_v nd_v = n[L^+ : \mathbb{Q}] = D$. Then the $u_k = d_{a_k}(z_k)$ form a $\mathbb{Z}_\ell$-basis of $T_n(\ell^1) = \prod_{v\in S_\ell}(1 + \ell\mathcal{O}_{L_v^+})^n$, so $\varepsilon\colon \Lambda_D \to \Lambda_1$, $X_k \mapsto [u_k] - 1$, is an isomorphism of $\mathcal{O}'$-algebras. Put $\gamma_k = \mathrm{Art}(z_k) \in I_{v_k}$ and $c_k = (\chi^{a_k-1}\kappa)(\gamma_k) \in \mathcal{O}'^\times$. As γ_k lies in the image of the pro-ℓ group $1 + \ell\mathcal{O}_{L_{v_k}^+}$, every character of finite order prime to ℓ kills it, so $\bar{c}_k = 1 = \bar{\chi}_{a_k}(\gamma_k)$ and $\mathbf{w} = (v_k, a_k, \gamma_k, c_k)_k$ is a family of weight variables. By ($*$), $\theta(\lambda_B(X_k)) = c_k\theta(\psi_{v_k,a_k}^{\mathrm{d}}(\gamma_k)) - 1 = \langle u_k\rangle - 1$, i.e. $\theta \circ \lambda_B = \langle\cdot\rangle \circ \varepsilon$, $\langle\cdot\rangle\colon \Lambda_1 \to T$ the diamond map.

Let $M^* = \mathrm{Hom}_{\Lambda_1}(M, \Lambda_1)$ with $t \in T$ acting by $\phi \mapsto \phi \circ t$, a B-module through θ. It is free of finite rank over Λ_1, and the transposed diamond operator $\langle u\rangle$ is multiplication by $[u]$ (ϕ is Λ_1-linear); so through λ_B the Λ_D-module M^* is M^* with its Λ_1-structure transported by ε, free of finite rank. T acts faithfully on M^*: if $\phi \circ t = 0$ for all ϕ, then $tm = 0$ for all $m \in M$ (M is free), so $t = 0$ by (Zm1). Hence $\mathrm{Ann}_B(M^*) = \theta^{-1}(0) = J$, and $(\mathbf{w}, M^*)$ satisfies (Hl1): it is a Taylor–Wiles base for J.

Fix now $Q = \{v_1, \dots, v_q\}$ and $\bar{w}_j$, w_j as in Definition 53.9; Q is disjoint from T_L and its places split in L. Through ι_{w_j}, G_Q, K_Q, Iw_Q, $\mathrm{Iw}_{1,Q}$ are identified with subgroups of $\prod_{v\in Q}\mathbb{G}_L(L_v^+)$, and Lemma E.22 applies to $((V_v), \tau)$ and Q; write $N(K') = N_\tau(K')$, $\mathbb{T}(K') = \mathbb{T}_\tau(K')$ and $\mathbb{T}_1 = \mathbb{T}(\mathrm{Iw}_{1,Q})$. Since $\mathbb{T}^{\mathrm{ord}}(V(K'))$ is the Λ^+-algebra generated by the operators and $\Lambda^+ = \bigoplus_u \Lambda_1[u]$ (u over representatives of $T_n(\mathcal{O}_{L^+} \otimes \mathbb{Z}_\ell)/T_n(\ell^1)$, $[u]$ acting as $\langle u\rangle$), every element of $\mathbb{T}(K')$ is a finite Λ_1-linear combination of monomials in the operators. $N(K_Q) = N_\tau$ (the level at K_Q is V), and $\mathbb{T}(K_Q) \subseteq \mathbb{T}_\tau$ is the Λ^+-subalgebra generated by the operators away from $T_L \cup Q$.

Step 2: $\mathbb{T}(K_Q)$ maps onto T. Let $T'' \subseteq T$ be the image of $e_{\mathfrak{m}}\mathbb{T}(K_Q) \subseteq \mathbb{T}$, a Λ_1-submodule of the finite Λ_1-module T, hence closed. It contains the images of the topological generators of T ((Gd0)) except possibly $T_w^{(j)}$, $(T_w^{(n)})^{-1}$ for w above Q. For such w, by the Chebotarev density theorem there are places w_i above split places outside $T_L \cup Q$ and Frobenius elements $\mathrm{Frob}_{w_i} \to \mathrm{Frob}_w$ in G_L (G_L is metrisable, and every conjugacy class of every finite quotient contains Frobenius elements of such places); by ($*$) and continuity, $e_{\mathfrak{m}}T_w^{(j)} = \lim_i f_j(\mathrm{Frob}_{w_i}) = \lim_i e_{\mathfrak{m}}T_{w_i}^{(j)} \in T''$. T is integral over T'' (it is finite over the image of Λ_1, which lies in T''), so a unit of T lying in T'' is a unit of T'', and $(T_w^{(n)})^{-1} \in T''$. Hence $T'' = T$. Let $\mathfrak{m}' = \{t \in \mathbb{T}(K_Q) : e_{\mathfrak{m}}t \in \mathfrak{m}_{\mathbb{T}}\} = \mathfrak{m} \cap \mathbb{T}(K_Q)$, a maximal ideal with residue field $\mathbb{F}$.

Step 3: $\mathfrak{m}$ is the only maximal ideal of $\mathbb{T}_\tau$ above $\mathfrak{m}'$. Let $\mathfrak{n}$ be one. $\mathbb{T}_{\tau,\mathfrak{n}}$ is a non-zero direct factor of $\mathbb{T}_\tau$, finite and torsion-free over Λ_1 ((Gd0)), so $\Lambda_1 \to \mathbb{T}_{\tau,\mathfrak{n}}$ is injective and integral; for a dominant weight λ with $\lambda_{\tau',j} > \lambda_{\tau',j+1}$ there is a prime of $\mathbb{T}_{\tau,\mathfrak{n}}$ above $\mathfrak{P}_{\lambda,1}$ (lying over), with quotient a domain finite over $\Lambda_1/\mathfrak{P}_{\lambda,1} \subseteq \overline{\mathbb{Q}}_\ell$, hence a point y of $\mathbb{T}_\tau$ through $\mathbb{T}_{\tau,\mathfrak{n}}$ with $y(\mathfrak{P}_{\lambda,1}) = 0$. By Lemma E.11 ($Q = \varnothing$, $\alpha = 1$, $r = 1$) and (Gy5)(i) there is a continuous semisimple $r_\pi\colon G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, unramified above every w up to semisimplification, with $r_\pi(\mathrm{Frob}_w)$ of characteristic

polynomial $P_{y,w}$. Its image is compact, so it stabilises a lattice over the ring of integers of a finite extension of $\mathbb{Q}_\ell$; let $\bar r$ be the semisimplified reduction, a continuous representation over $\overline{\mathbb{F}}_\ell$ whose characteristic polynomial at Frob_w is the reduction of $P_{y,w}$, i.e. is computed from the images of the $T_w^{(j)}$ in $\mathbb{T}_\tau/\mathfrak{n}$. For w above a place outside $T_L \cup Q$, $T_w^{(j)} \in \mathbb{T}(K_Q)$ and $\mathfrak{n} \cap \mathbb{T}(K_Q) = \mathfrak{m}' \subseteq \mathfrak{m}$, so by (Mx) this polynomial is $\prod_a (X - (\bar\chi_a \bar\kappa)(\mathrm{Frob}_w))$. The characteristic polynomials of $\bar r$ and of $\bigoplus_a \bar\chi_a\bar\kappa$ are locally constant functions on G_L agreeing on a dense set, hence everywhere; at Frob_w for w above Q this gives $(\mathrm{N}w)^{j(j-1)/2} T_w^{(j)} \equiv e_j((\bar\chi_a\bar\kappa)(\mathrm{Frob}_w))_a$ modulo $\mathfrak{n}$, the same value as modulo $\mathfrak{m}$ ((Mx)). So the residue maps of $\mathfrak{n}$ and $\mathfrak{m}$ agree on all topological generators of $\mathbb{T}_\tau$, with values in $\mathbb{F}$; being continuous with discrete image, they are equal, and $\mathfrak{n} = \mathfrak{m}$. Consequently the idempotent $e_{\mathfrak{m}'}$ of the semilocal ring $\mathbb{T}(K_Q)$ for $\mathfrak{m}'$, an idempotent of $\mathbb{T}_\tau$ that is 1 exactly modulo the maximal ideals above $\mathfrak{m}'$, equals $e_{\mathfrak{m}}$.

Step 4: the Hecke algebra at level $\mathrm{Iw}_{1,Q}$ *and its determinant.* By (Sm1), the map $p = p_{\mathrm{Iw}_{1,Q}K_Q} \colon N(\mathrm{Iw}_{1,Q}) \to N_\tau$ is surjective and commutes with the operators, so an element of $\mathbb{T}_1$ (a Λ_1-combination of monomials acting as 0 on $N(\mathrm{Iw}_{1,Q})$ acts as 0 on N_τ) induces an endomorphism of N_τ: a surjective homomorphism $s \colon \mathbb{T}_1 \to \mathbb{T}(K_Q)$ sending each operator to itself. Let $\mathfrak{m}_Q = s^{-1}(\mathfrak{m}')$, e_Q its idempotent, $T_Q = e_Q\mathbb{T}_1 = \mathbb{T}_{1,\mathfrak{m}_Q}$. Then $s(e_Q) = e_{\mathfrak{m}'} = e_{\mathfrak{m}}$ (Step 3), and $s_Q \colon T_Q \to T$, $t \mapsto e_{\mathfrak{m}} s(t) \bmod \mathfrak{a}_0$, is a surjective local homomorphism (Step 2) sending each operator to its image in T. $\mathbb{T}_1/\mathfrak{m}_Q = \mathbb{F}$, and $\mathfrak{m}_Q$ satisfies (Mx) (the congruences of (Mx) involve only operators, and hold in T). Apply Lemma E.17 at level $K' = \mathrm{Iw}_{1,Q}$ (with (Zt)) to $\mathfrak{m}_Q$ and χ_0, and let $\mathfrak{a}_0^Q \subseteq T_Q$ and $\mathcal{C}_{\chi_0}^Q$ be the ideal and the set of (Zp4) there. For $y \in \mathcal{C}_{\chi_0}$ (a point of $\mathbb{T}$ killing $\mathfrak{a}_0$, i.e. of T), $y \circ s_Q$ is a point of T_Q with $y(s_Q(\langle\delta\rangle)) = y(\langle\delta\rangle) = \chi_0(\delta)$ for $\delta \in \Delta_\ell$, so it kills $\mathfrak{a}_0^Q$ by (Zp4). As $\mathcal{C}_{\chi_0}$ is non-empty with $\bigcap_{y\in\mathcal{C}_{\chi_0}} \ker y = \mathfrak{a}_0$, this gives $\mathfrak{a}_0^Q \neq T_Q$ and $s_Q(\mathfrak{a}_0^Q) = 0$; so s_Q induces a surjective local homomorphism $T_Q/\mathfrak{a}_0^Q \to T$. From now on T_Q denotes $e_Q\mathbb{T}_1/\mathfrak{a}_0^Q$ and s_Q this surjection; it still sends each operator to its image in T. By (Zp4), $\mathcal{C}_{\chi_0}^Q$ is non-empty with $\bigcap_{y\in\mathcal{C}_{\chi_0}^Q} \ker y = \mathfrak{a}_0^Q$; its points satisfy (i), (ii) of (Gd2) ((Zp1), (Zp2)) and (iii): $y(\langle d_j(\zeta)\rangle) = \chi_0(d_j(\zeta)) = \kappa(\mathrm{Art}(\zeta))\,\xi_{v,j}(g_\zeta)$, the wild values being trivial. By (Gd2) of Lemma E.16 at level $\mathrm{Iw}_{1,Q}$ with $\mathfrak{a} = \mathfrak{a}_0^Q$ and $\mathcal{C} = \mathcal{C}_{\chi_0}^Q$, there are a continuous T_Q-valued determinant D^Q of dimension n on $\Gamma^{0,Q}$ and continuous characters $\psi_{v,j}^\flat \colon D_v \to T_Q^\times$ with (Od1)–(Od3) for π^Q, the reductions $\bar D_n$, $\bar\chi_j$, and $\chi_{D^Q}(\pi^Q(\mathrm{Frob}_w)) = \sum_j (-1)^j \kappa(\mathrm{Frob}_w)^{-j} (\mathrm{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j}$ for w outside $T_L \cup Q$, $\psi_{v,j}^\flat(\mathrm{Art}(z)) = (\chi^{j-1}\kappa)(\mathrm{Art}(z))^{-1}\langle d_j(z)\rangle$ and the analogous identity at $\mathrm{Art}(\varpi_{\tilde v})$.

Step 5: compatibility over T. For w outside $T_L \cup Q$, applying s_Q to $\chi_{D^Q}(\pi^Q(\mathrm{Frob}_w))$ gives, by $(*)$ and $\pi_{\mathscr{H}} = h^Q \circ \pi^Q$, the polynomial $\theta(\chi_{D^{\mathrm{u}}}(h^Q(\pi^Q(\mathrm{Frob}_w))))$. The determinants $s_Q \circ D^Q$ and $\theta \circ D^{\mathrm{u}} \circ h^Q$ on $\Gamma^{0,Q}$ are continuous and the $\pi^Q(\mathrm{Frob}_w)$ are dense in $\Gamma^{0,Q} = \pi^Q(G_L)$ (Chebotarev), so their characteristic polynomials agree everywhere, and $s_Q \circ D^Q = \theta \circ D^{\mathrm{u}} \circ h^Q$ by (Ch1). Likewise $s_Q \circ \psi_{v,j}^\flat$ and $\theta \circ \psi_{v,j}^{\mathrm{d}}$ agree on $\mathrm{Art}(\mathcal{O}_{L_{\tilde v}}^\times)$ and at $\mathrm{Art}(\varpi_{\tilde v})$ ($(*)$ and the identity for $U_{\tilde v}^{(j)}$), hence on D_v (both continuous, trivial on the closed commutator subgroup, and the image of Art is dense in D_v^{ab}).

Step 6: the ring R *and the map* β. Let $R = \{(a,b) \in T_Q \times B : s_Q(a) = \theta(b)\}$. Since s_Q and θ are surjective, $T_Q \times B = R \cdot (1,0) + R \cdot (0,1)$ (for (a,b) choose b' with $\theta(b') = s_Q(a)$ and a'' with $s_Q(a'') = \theta(b)$; then $(a,b) = (a,b')(1,0) + (a'',b)(0,1)$), so the Noetherian ring $T_Q \times B$ is finite over R, and R is Noetherian (Eakin–Nagata). $R \to T \to \mathbb{F}$ is surjective, and an element (a,b) outside its kernel has a, b units, with $(a^{-1}, b^{-1}) \in R$; so R is local with residue field $\mathbb{F}$. $(T_Q \times B)/\mathfrak{m}_R(T_Q \times B)$ is finite over $\mathbb{F}$, so $\mathfrak{m}_R(T_Q \times B)$ is an ideal of definition of the complete semilocal ring $T_Q \times B$; by Artin–Rees the subspace topology on R is the $\mathfrak{m}_R$-adic one, and R, closed in $T_Q \times B$ (compact), is complete: $R \in \mathrm{CNL}_{\mathcal{O}'}$. By Step 5, for every $\gamma \in \Gamma^{0,Q}$ the pair $(\chi_{D^Q}(\gamma), \chi_{D^{\mathrm{u}}}(h^Q(\gamma)))$ lies in $R[X]$; by (Ch4) (with $R \subseteq T_Q \times B$ and the determinants D^Q,

$D^{\mathrm{u}}\circ h^Q$) there is a continuous R-valued determinant D_R of dimension n on $\Gamma^{0,Q}$ with base changes D^Q and $D^{\mathrm{u}}\circ h^Q$. Put $\psi_{R,v,a}=(\psi^\flat_{v,a},\psi^{\mathrm{d}}_{v,a})\colon D_v\to R^\times$ (Step 5), continuous. Both components reduce to $\bar D_n$, resp. $\bar\chi_a$. (Od1) and (Od2) for $(D_R,(\psi_{R,v,a}))$ and π^Q: the relevant elements of R (coefficients of $\chi_{D_R}(\pi^Q(cg^{-1}c))-\chi_{D_R}(\pi^Q(g))$, traces $\mathrm{tr}_{D_R}(\gamma\prod_a(\pi^Q(g_a)-\psi_{R,v,a}(g_a)))$) map to 0 in T_Q by Step 4 and in B by (Od1), (Od2) for D^{u} composed with h^Q ($h^Q\circ\pi^Q=\pi_{\mathscr{H}}$; traces commute with base change), and $R\to T_Q\times B$ is injective. (Od3): for $g\in\mu_v$, $\psi^\flat_{v,a}(g)=1$ in T_Q ((Od3) in Step 4) and $\psi^{\mathrm{d}}_{v,a}(g)=1$ in B ((Od3) for B), the wild values being trivial; so $\psi_{R,v,a}(g)=1=\xi_{v,a}(g)$. Let $\beta\colon R\to B$ be the second projection: $\beta\circ D_R=D^{\mathrm{u}}\circ h^Q$ and $\beta\circ\psi_{R,v,a}=\psi^{\mathrm{d}}_{v,a}$. The first projection $\mathrm{pr}_1\colon R\to T_Q$ satisfies $s_Q\circ\mathrm{pr}_1=\theta\circ\beta$.

Step 7: the module P. Let $P'=\sum_{g\in G_Q}\rho(g)\,j_{\mathrm{Iw}_{1,Q}}(N^*(\mathrm{Iw}_{1,Q}))\subseteq P_\infty$. For $t\in\mathbb{T}_1$ written as $\sum_i\mu_i m_i$ ($\mu_i\in\Lambda_1$, m_i monomials in the operators) let t act on P' as $\sum_i\mu_i m_i$ acting on P_∞. This is well defined: if $\sum_i\mu_i m_i=0$ in $\mathbb{T}_1$, it kills $N(\mathrm{Iw}_{1,Q})$, hence (transposing) $j_{\mathrm{Iw}_{1,Q}}(N^*(\mathrm{Iw}_{1,Q}))$, hence each $\rho(g)j_{\mathrm{Iw}_{1,Q}}(N^*(\mathrm{Iw}_{1,Q}))$, as the operators and Λ_1 commute with ρ ((Sm2)). So P' is a $\mathbb{T}_1$-module on which $\mathbb{T}_1$ commutes with ρ. Let P be the submodule of e_QP' killed by the ideal $\mathfrak{a}_0^Q$ of $e_Q\mathbb{T}_1$: a module over $e_Q\mathbb{T}_1/\mathfrak{a}_0^Q=T_Q$, an R-module through pr_1, and stable under ρ, which commutes with $\mathbb{T}_1$. Let $\kappa_Q\colon G_Q\to\mathcal{O}'^\times$, $(g_j)_j\mapsto\prod_j\kappa(\mathrm{Art}_{w_j}(\det g_j))^{-1}$, with κ restricted to $D_{\bar w_j}\subseteq G_L$ (v_j splits) and Art_{w_j} the local Artin map of L_{w_j} normalised as in Theorem B.91; κ is unramified above w_j (Definition 50.1, (Rd7)), so κ_Q is trivial on $K_Q\supseteq\mathrm{Iw}_Q$. Let G_Q act on P by $\rho_P(g)=\kappa_Q(g)\rho(g)$, an R-linear action (e_Q commutes with ρ) with the same invariants under subgroups of K_Q as ρ. Through λ_R, Λ_D acts on P as Λ_1 does, transported by ε: $\mathrm{pr}_1(\lambda_R(X_k))=c_k\psi^\flat_{v_k,a_k}(\gamma_k)-1=\langle u_k\rangle-1$ (Step 4), and $\langle u\rangle$ acts on P_∞ as $[u]$.

(Hl2). $P'^{\mathrm{Iw}_{1,Q}}=j_{\mathrm{Iw}_{1,Q}}(N^*(\mathrm{Iw}_{1,Q}))=P_\infty^{\mathrm{Iw}_{1,Q}}$ by (Sm2), so $(e_QP')^{\mathrm{Iw}_{1,Q}}=e_QP_\infty^{\mathrm{Iw}_{1,Q}}$ (e_Q is an idempotent commuting with ρ), which $j_{\mathrm{Iw}_{1,Q}}$ identifies with the Λ_1-dual of $X=e_QN(\mathrm{Iw}_{1,Q})$ (a linear form on the direct summand X extends uniquely by 0), $t\in e_Q\mathbb{T}_1$ acting by $\phi\mapsto\phi\circ t$. A form is killed by $\mathfrak{a}_0^Q$ if and only if it vanishes on $\mathfrak{a}_0^QX$, and $\mathfrak{a}_0^QX=\sum_\delta(\langle\delta\rangle-\chi_0(\delta))X$: it contains the right side as $\langle\delta\rangle-\chi_0(\delta)\in\mathfrak{a}_0^Q$, and is contained in it as $\mathfrak{a}_0^Q$ kills X^{χ_0} ((Zp5) at level $\mathrm{Iw}_{1,Q}$). So $P^{\mathrm{Iw}_{1,Q}}\cong\mathrm{Hom}_{\Lambda_1}(X^{\chi_0},\Lambda_1)$, compatibly with Δ_Q. By (Wf3) of Lemma E.20 for $K''=\mathrm{Iw}_{1,Q}$ normal in $K'=\mathrm{Iw}_Q$, with quotient the ℓ-group Δ_Q, $\mathrm{Hom}_{\Lambda_1}(N(\mathrm{Iw}_{1,Q})^{\chi_0},\Lambda_1)$ is a free $\Lambda_1[\Delta_Q]$-module of finite rank; e_Q commutes with Δ_Q and with the diamond operators, so $P^{\mathrm{Iw}_{1,Q}}$ is isomorphic to a direct summand of it, finitely generated and projective over $\Lambda_1[\Delta_Q]\cong\Lambda_D[\Delta_Q]$ (through ε), and finitely generated over Λ_D.

(Hl3). $j_{K_Q}(N^*(K_Q))=j_{\mathrm{Iw}_{1,Q}}(p^*(N^*(K_Q)))\subseteq P'$, so $P'^{K_Q}=j_{K_Q}(N_\tau^*)$ by (Sm2), $N_\tau^*=\mathrm{Hom}_{\Lambda_1}(N_\tau,\Lambda_1)$. As p commutes with the operators, $t\in\mathbb{T}_1$ acts on N_τ^* by $\phi\mapsto\phi\circ s(t)$; hence $(e_QP')^{K_Q}=e_QP'^{K_Q}=j_{K_Q}(\{\phi\circ e_{\mathfrak{m}}:\phi\in N_\tau^*\})$ (Step 4: $s(e_Q)=e_{\mathfrak{m}}$), which restriction identifies with $\mathrm{Hom}_{\Lambda_1}(e_{\mathfrak{m}}N_\tau,\Lambda_1)$ (a linear form on the direct summand $e_{\mathfrak{m}}N_\tau$ extends uniquely by 0 on $(1-e_{\mathfrak{m}})N_\tau$), $t\in e_Q\mathbb{T}_1$ acting through $e_{\mathfrak{m}}s(t)$. The forms killed by $\mathfrak{a}_0^Q$ are those vanishing on $e_{\mathfrak{m}}s(\mathfrak{a}_0^Q)N_\tau$, a module containing $\sum_\delta(\langle\delta\rangle-\chi_0(\delta))e_{\mathfrak{m}}N_\tau$ (s fixes the operators) and contained in $\mathfrak{a}_0e_{\mathfrak{m}}N_\tau$ ($e_{\mathfrak{m}}s(\mathfrak{a}_0^Q)\subseteq\mathfrak{a}_0$, Step 4), which equals the former as $\mathfrak{a}_0$ kills M ((Zm1)). So restriction gives a bijective additive map $e\colon P^{K_Q}\to M^*$. For $t\in R$ and $x=j_{K_Q}(\phi\circ e_{\mathfrak{m}})$, tx corresponds to $\phi\circ e_{\mathfrak{m}}s(\tilde t)$, $\tilde t\in e_Q\mathbb{T}_1$ a lift of $\mathrm{pr}_1(t)$, which on the T-module M is $\phi\circ s_Q(\mathrm{pr}_1(t))=\phi\circ\theta(\beta(t))$, so $e(tx)=\beta(t)e(x)$.

Step 8: (Hl4). Let $z\in P$, $z\neq0$. Apply Lemma E.23 to $x=z$: it gives λ, π, y, Y, π_Y and sp with (Ls1)–(Ls4). For $t=\sum_i\mu_im_i\in\mathbb{T}_1$ put $\theta_0(t)=\sum_i y(\mu_i)\prod y(\text{factors of }m_i)$. By (Ls2), $\mathrm{sp}(tz')=\theta_0(t)\mathrm{sp}(z')$ for $z'\in P'$; in particular $\theta_0(t)$ does not depend on the expression of t (a vanishing expression acts as 0 on $P'\ni z$, and $\mathrm{sp}(z)\neq0$ by (Ls1)), and $\theta_0\colon\mathbb{T}_1\to\overline{\mathbb{Q}}_\ell$ is a homomorphism of $\mathcal{O}'$-algebras (compare $\mathrm{sp}(tt'z)$, $\mathrm{sp}((t+t')z)$ with $\mathrm{sp}(z)$). From $z=e_Qz$, $\theta_0(e_Q)=1$, and from $\mathfrak{a}_0^Qz=0$ ($z\in P$) and $\mathrm{sp}(z)\neq0$, $\theta_0(\mathfrak{a}_0^Q)=0$; so θ_0 factors through

T_Q; and $\theta_0(\langle u\rangle) = (w_0\lambda)(u)^{-1}$ on $T_n(\ell^1)$, so $\theta_0(\mathfrak{P}_{\lambda,1}) = 0$ (renormalised structure) and θ_0 is a regular point of T_Q. Put $\vartheta = \theta_0 \circ \mathrm{pr}_1 \colon R \to \overline{\mathbb{Q}}_\ell$. By (Zp1) at level $\mathrm{Iw}_{1,Q}$, $\tilde{r}_{\theta_0} = r_{\theta_0} \otimes \kappa^{-1}$ is trivial on N^Q; let r be the continuous representation of $\Gamma^{0,Q}$ with $r \circ \pi^Q = \tilde{r}_{\theta_0}$. The determinants $\det \circ r$ and $\vartheta \circ D_R = \theta_0 \circ D^Q$ are continuous with the same characteristic polynomial $\theta_0\big(\sum_j (-1)^j \kappa(\mathrm{Frob}_w)^{-j} (\mathrm{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j}\big)$ at the dense elements $\pi^Q(\mathrm{Frob}_w)$, w outside $T_L \cup Q$; so they are equal ((Ch1)). By (Ls2) and (Gy5)(i), r_π is continuous and semisimple with characteristic polynomial $P_{\theta_0,w}$ at these Frob_w, as is r_{θ_0}; so $r_\pi \cong r_{\theta_0}$ (Chebotarev, Brauer–Nesbitt) and $r \circ \pi^Q \cong r_\pi \otimes \kappa^{-1}$.

Let $Y_{\mathbb{C}} = Y \otimes_{\overline{\mathbb{Q}}_\ell, \iota} \mathbb{C}$ with G_Q acting by $\iota(\kappa_Q(g))\,\pi_Y(g) \otimes 1$, $\mathrm{sp}_{\mathbb{C}}(m) = \mathrm{sp}(m) \otimes 1$ for $m \in P$, and $\sigma_j = (\iota\pi_{v_j} \circ \iota_{w_j}^{-1}) \otimes (\iota \circ \kappa^{-1} \circ \mathrm{Art}_{w_j} \circ \det)$, an irreducible smooth representation of $\mathrm{GL}_n(L_{w_j})$. Then $\mathrm{sp}_{\mathbb{C}}(tm) = \iota(\vartheta(t))\mathrm{sp}_{\mathbb{C}}(m)$ for $t \in R$ (Step 7: R acts through pr_1), $\mathrm{sp}_{\mathbb{C}}(\rho_P(g)m) = \iota(\kappa_Q(g))\pi_Y(g)\mathrm{sp}(m) \otimes 1$ by (Ls3), and by (Ls4), twisting source and target by the same character, $Y_{\mathbb{C}}$ is spanned by the images of the $\mathrm{GL}_n(L_{w_j})$-equivariant maps $\sigma_j \to Y_{\mathbb{C}}$. By (Ug2) of Lemma E.8 at v_j (split, not above ℓ; $[L^+ : \mathbb{Q}] \geq 2$ by (Gd0)) and the compatibility of rec with twists by characters ($\mathrm{rec}(\sigma \otimes (\eta \circ \det)) = \mathrm{rec}(\sigma) \otimes \mathrm{rec}(\eta)$, $\mathrm{rec}(\eta) = \eta \circ \mathrm{Art}^{-1}$ on GL_1, Theorem B.92),

$$\mathrm{rec}\big(\sigma_j \otimes |\det|^{(1-n)/2}\big) \cong \iota\,\mathrm{WD}\big(r_\pi|_{G_{L_{w_j}}}\big)^{F\text{-}ss} \otimes \iota\kappa^{-1} \cong \iota\,\mathrm{WD}\big(r \circ \pi^Q|_{D_{\bar{w}_j}}\big)^{F\text{-}ss},$$

unconditionally. So $(\vartheta, r, Y_{\mathbb{C}}, \cdot, \mathrm{sp}_{\mathbb{C}}, (\sigma_j)_j)$ is a level point with $\mathrm{sp}_{\mathbb{C}}(z) \neq 0$: the level points for ι separate P.

Hence $(R, D_R, (\psi_{R,v,a}), \beta, P, \rho_P)$ is a classical level module at Q for the Taylor–Wiles base $(\mathbf{w}, M^*)$; as Q was arbitrary, J has classical levels. □

Proposition E.2. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ whose wild values are trivial ($\xi_{v,a} = 1$ for all $v \in S_\ell$ and $1 \leq a \leq n$), κ a twist for $\mathscr{H}$, $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and $(B, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$. Then B has a Hecke ideal of zero-monodromy type for ι and κ. Moreover, if B has a classical ordinary point for ι and κ, it may be taken of the following form: $J = \ker\theta$, with $T = \mathbb{T}_{\tau,\mathfrak{m}}/\mathfrak{a}_0$, λ_T the diamond map $[u] \mapsto \langle u\rangle$ followed by the projection, $M = N_{\tau,\mathfrak{m}}/\sum_{\delta\in\Delta_\ell}(\langle\delta\rangle - \chi_0(\delta))N_{\tau,\mathfrak{m}}$ and $f = \theta$, where $((V_v)_{v\in\mathrm{St}}, \tau)$ is a type datum over $\mathscr{H}$ with V_v the kernel of reduction modulo ϖ_v^c for an integer c as in Proposition E.5, satisfying (Zt) for ι and κ; $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}_\tau$ with $\mathbb{T}_\tau/\mathfrak{m}$ the residue field of $\mathcal{O}$, satisfying (Mx); $\chi_0 \colon \Delta_\ell \to \mathcal{O}^\times$ is the homomorphism with $\chi_0(d_j(\zeta)) = \kappa(\mathrm{Art}(\zeta))$ for $v \in S_\ell$, $1 \leq j \leq n$ and every ℓ-power root of unity $\zeta \in L_{\tilde{v}}$; $\mathfrak{a}_0 = \mathfrak{a}_{\chi_0}$ and $\mathcal{C}_{\chi_0}$ are as in (Zp4) for $\mathbb{T}_{\tau,\mathfrak{m}}$; and $\theta \colon B \to \mathbb{T}_{\tau,\mathfrak{m}}/\mathfrak{a}_0$ is the surjection of (Gd2) ($Q = \varnothing$, $\mathfrak{a} = \mathfrak{a}_0$, $\mathcal{C} = \mathcal{C}_{\chi_0}$). If moreover the reduction $\bar{\omega}$ of χ_{cyc} is non-trivial on D_v for every $v \in S_\ell$, then $\Delta_\ell = \{1\}$, $\mathfrak{a}_0 = 0$, $T = \mathbb{T}_{\tau,\mathfrak{m}}$ and $M = N_{\tau,\mathfrak{m}}$.*

Proof. If B has no classical ordinary point for ι and κ, then $J = B$ is a Hecke ideal of zero-monodromy type by definition. Assume from now on that there is one.

Step 1: the type. For $v \in \mathrm{St}$ let $\tilde{v}$ be the fixed place of L above v; $L_{\tilde{v}} = L_v^+$ is a finite extension of $\mathbb{Q}_p$ for the prime p below v, $p \neq \ell$. For $1 \leq a \leq n$ let $\chi_{v,a} = ([\bar{\chi}_a]\kappa)|_{I_{L_{\tilde{v}}}}$: a character of finite order with values in $\mathcal{O}^\times$ ($\bar{\chi}_a$ has finite order, and $\kappa^{m'}$ is trivial on $I_{L_{\tilde{v}}}$ by (Rd7)), the restriction of the continuous character $[\bar{\chi}_a]\kappa$ of $G_{L_{\tilde{v}}} \supseteq W_{L_{\tilde{v}}}$. By Lemma E.3 (with $F_v = L_{\tilde{v}}$, $v \in \mathrm{St}$) there are open normal subgroups $V_v' \subseteq \mathrm{GL}_n(\mathcal{O}_{L_{\tilde{v}}})$ and a finite free $\mathcal{O}$-module τ_0 with an action of $\prod_v \mathrm{GL}_n(\mathcal{O}_{L_{\tilde{v}}})/V_v'$ such that $\mathrm{Hom}_K(\tau_0 \otimes_{\mathcal{O},\iota} \mathbb{C}, \bigotimes_v \sigma_v) \neq 0$ if and only if every σ_v has zero monodromy and inertia $(\chi_{v,a})_a$ ($K = \prod_{v\in\mathrm{St}} \mathrm{GL}_n(\mathcal{O}_{L_{\tilde{v}}})$). Let c be an integer as in Proposition E.5, so large that the kernel of reduction modulo ϖ_v^c of $\mathbb{G}_L(\mathcal{O}_{L_v^+}) \cong \mathrm{GL}_n(\mathcal{O}_{L_{\tilde{v}}})$ (through $\iota_{\tilde{v}}$) lies in V_v' for every $v \in \mathrm{St}$ (the V_v' are open; the condition on c in Proposition E.5

persists when c grows). Let V_v be that kernel, $H = \prod_v \mathbb{G}_L(\mathcal{O}_{L_v^+})/V_v$, through which τ_0 is a representation, and $\tau = \tau_0^\vee$. Then $((V_v), \tau)$ is a type datum over $\mathscr{H}$ with principal congruence level c; (Zc) holds for τ_0 (the direction "if"), and (Zt) holds for τ: if $\mathrm{Hom}_H(\tau^\vee \otimes \overline{\mathbb{Q}}_\ell, \bigotimes_v \pi_v^{V_v}) \neq 0$, then, as $\tau^\vee \cong \tau_0$, transporting by ι gives $\mathrm{Hom}_K(\tau_0 \otimes_{\mathcal{O},\iota} \mathbb{C}, \bigotimes_v \iota\pi_v \circ \iota_{\tilde{v}}^{-1}) \neq 0$, and the direction "only if" gives zero monodromy and inertia $\bigoplus_a \iota \circ \chi_{v,a}$ for every $\mathrm{rec}(\iota\pi_v \circ \iota_{\tilde{v}}^{-1})$.

Step 2: the maximal ideal. For a classical ordinary point x, with $\tilde{r}_x$ as in (Cy1), let y_x be the point of $\mathbb{T}_\tau$ given by Lemma E.4. By (Gd0), y_x takes values in the ring of integers of a finite extension of $\mathbb{Q}_\ell$; let $\bar{y}_x$ be its reduction, a continuous homomorphism to $\overline{\mathbb{F}}_\ell$. On the topological generators of $\mathbb{T}_\tau$ ((Gd0)) its values lie in $\mathbb{F}$ and do not depend on x: $(\mathrm{N}w)^{j(j-1)/2}\bar{y}_x(T_w^{(j)})$ is the reduction of the j-th elementary symmetric function of the eigenvalues of $(\tilde{r}_x \circ \pi_{\mathscr{H}} \otimes \kappa)(\mathrm{Frob}_w)$, i.e. $e_j((\bar{\chi}_a\bar{\kappa})(\mathrm{Frob}_w))_a$, because $x \circ D^{\mathrm{u}} = \det \circ \tilde{r}_x$ reduces to $\bar{D}_n$ and $\mathrm{N}w$ is prime to ℓ; $\bar{y}_x(\langle d_j(z)\rangle)$ and $\bar{y}_x(U_{\tilde{v}}^{(j)})/\bar{y}_x(U_{\tilde{v}}^{(j-1)})$ are the reductions of $\phi_{v,j} = \chi^{j-1}\kappa(x \circ \psi_{v,j}^{\mathrm{d}})$ at $\mathrm{Art}(z)$ and $\mathrm{Art}(\varpi_{\tilde{v}})$, i.e. the values of $\bar{\chi}^{j-1}\bar{\kappa}\bar{\chi}_j$, as $x \circ \psi_{v,j}^{\mathrm{d}} \equiv \bar{\chi}_j$. Hence $\bar{y}_x = \bar{y}$ is independent of x, its image is $\mathbb{F}$, $\mathfrak{m} = \ker \bar{y}$ is a maximal ideal with $\mathbb{T}_\tau/\mathfrak{m} = \mathbb{F}$ satisfying (Mx), and every y_x factors through $\mathbb{T}_{\tau,\mathfrak{m}}$ (it is local onto its image).

Step 3: T, M and (Zm1). Write $\mathbb{T} = \mathbb{T}_{\tau,\mathfrak{m}}$ and $N = N_{\tau,\mathfrak{m}}$. By (Gd0), $\mathbb{T}$ is a complete Noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$, finite and torsion-free over Λ_1 (diamond structure); it is reduced (Lemma E.18). $\mathbb{T}_\tau$ is the product of its localisations at its finitely many maximal ideals; with the corresponding idempotent $e_\mathfrak{m}$, which commutes with the diamond operators, $N = e_\mathfrak{m}N_\tau$ is a direct summand of N_τ, and $\mathbb{T} = e_\mathfrak{m}\mathbb{T}_\tau$ acts faithfully on N (if $tN = 0$ for $t = e_\mathfrak{m}t$, then $tN_\tau = tN = 0$, so $t = 0$, $\mathbb{T}_\tau$ acting faithfully on N_τ). By (Zp5) of Lemma E.17 ($Q = \varnothing$, $\chi = \chi_0$; it applies by Steps 1 and 2), $M = N/\sum_\delta(\langle\delta\rangle - \chi_0(\delta))N$ is free of finite rank over Λ_1, and its annihilator in $\mathbb{T}$ is $\mathfrak{a}_0$. Put $T = \mathbb{T}/\mathfrak{a}_0$ and let λ_T be the diamond map followed by the projection. By Step 4, $\mathfrak{a}_0 \neq \mathbb{T}$, so $T \in \mathrm{CNL}_\mathcal{O}$. So M is a faithful T-module, free of finite rank over Λ_1 through λ_T: (Zm1).

Step 4: the surjection f and (Zm2). By Step 1 and Step 2, Lemma E.17 applies to $((V_v), \tau)$ and $\mathfrak{m}$. Let x be a classical ordinary point and y_x the point of $\mathbb{T}$ of Step 2. For $v \in S_\ell$, $1 \leq j \leq n$ and an ℓ-power root of unity $\zeta \in L_{\tilde{v}}$, Lemma E.4 gives $y_x(\langle d_j(\zeta)\rangle) = (\chi^{j-1}\kappa)(g_\zeta)\, x(\psi_{v,j}^{\mathrm{d}}(g_\zeta)) = \kappa(\mathrm{Art}(\zeta)) = \chi_0(d_j(\zeta))$: the values of these continuous characters of D_v at $\mathrm{Art}(\zeta)$ are their values at g_ζ; $g_\zeta \in \mu_v$ and $\chi(g_\zeta) = 1$ by (Wv1) of Lemma 50.24; and $x(\psi_{v,j}^{\mathrm{d}}(g_\zeta)) = \xi_{v,j}(g_\zeta) = 1$ by (Od3), the wild values being trivial. As Δ_ℓ is generated by the $d_j(\zeta)$, the first assertion of (Zp4) gives $y_x(\mathfrak{a}_0) = 0$; so $\mathfrak{a}_0 \neq \mathbb{T}$, and by the second assertion $\mathcal{C}_{\chi_0}$ is non-empty with $\bigcap_{y \in \mathcal{C}_{\chi_0}} \ker y = \mathfrak{a}_0$. Every $y \in \mathcal{C}_{\chi_0} \subseteq \mathcal{C}$ ($\mathcal{C}$ the regular points of trivial nebentypus) satisfies (i) and (ii) of (Gd2) ((Zp1), (Zp2)), and (iii): $y(\langle d_j(\zeta)\rangle) = \chi_0(d_j(\zeta)) = \kappa(\mathrm{Art}(\zeta))\,\xi_{v,j}(g_\zeta)$. By (Gd2) of Lemma E.16 with $\mathfrak{a} = \mathfrak{a}_0$, $\mathcal{C} = \mathcal{C}_{\chi_0}$, $A = T$, there is a surjective homomorphism of $\mathcal{O}$-algebras $f = \theta\colon B \to T$ with the identities stated there. Put $J = \ker f$. For $v \in S_\ell$, $1 \leq j \leq n$ and $z \in 1 + \ell\mathcal{O}_{L_v^+}$, with $g_z = \mathrm{Art}(z) \in I_v$, these identities give $\lambda_T([d_j(z)]) = \langle d_j(z)\rangle = (\chi^{j-1}\kappa)(g_z) f(\psi_{v,j}^{\mathrm{d}}(g_z))$: (Zm2).

Step 5: (Zm3). Let $y\colon T \to \overline{\mathbb{Q}}_\ell$ be a homomorphism of $\mathcal{O}$-algebras with $y(\lambda_T([u])) = (w_0\lambda)(u)^{-1}$ on $T_n(\ell^1)$ for a dominant λ with $\lambda_{\tau',j} > \lambda_{\tau',j+1}$, and with ρ_y absolutely irreducible. By (Gd0) y is continuous, i.e. a point (identified with the point $\mathbb{T} \to T \to \overline{\mathbb{Q}}_\ell$ of $\mathbb{T}$, which kills $\mathfrak{a}_0$), and $y(\mathfrak{P}_{\lambda,1}) = 0$ (renormalised structure), so y is regular. Put $\rho' = \rho_y \circ \pi_{\mathscr{H}} \otimes \kappa$, absolutely irreducible as $\pi_{\mathscr{H}}(G_L) = \Gamma^0_{\mathscr{H}}$. For every w, the inertia groups above w lie in $N_{\mathscr{H}}$ (Rd4) and κ is unramified above w (Rd7), so ρ' is unramified above w; applying y to the first identity of (Gd2), the characteristic polynomial of $\rho'(\mathrm{Frob}_w) = \kappa(\mathrm{Frob}_w)\rho_y(\pi_{\mathscr{H}}(\mathrm{Frob}_w))$ is $P_{y,w}$. By (Hx2) of Proposition E.5 (for points of $\mathbb{T}_\tau$ through Lemma E.15), ρ' is automorphic of conjugate self-dual type for ι. Let r_y be as in (Hx1); it is semisimple with characteristic polynomials $P_{y,w}$, so $r_y \cong \rho'$ (Chebotarev, Brauer–Nesbitt), and r_y is absolutely irreducible; by (Hx1)(iii), $r_y|_{D_v}$ is triangular with diagonal $(\chi^{1-j}\psi_{v,j}^y)_j$ for $v \in S_\ell$. Applying y to the second and third identities of

(Gd2), the continuous characters $y\circ f\circ\psi^{\mathrm{d}}_{v,j}$ and $\chi^{1-j}\kappa^{-1}\psi^{y}_{v,j}$ agree on $\mathrm{Art}(\mathcal{O}^\times_{L_{\tilde v}})$ and at $\mathrm{Art}(\varpi_{\tilde v})$, hence on D_v (dense image of Art). So $\rho_y\circ\pi_{\mathscr{H}}\cong r_y\otimes\kappa^{-1}$ is triangular on D_v with diagonal $(y\circ f\circ\psi^{\mathrm{d}}_{v,a})_a$.

Step 6: (Zm4). Let x be a classical ordinary point and y_x the point of $\mathbb{T}$ of Step 2; it kills $\mathfrak{a}_0$ (Step 4), so it is a point of T. The tuples $(x\circ D^{\mathrm{u}},(x\circ\psi^{\mathrm{d}}_{v,a}))$ and $(y_x\circ f\circ D^{\mathrm{u}},(y_x\circ f\circ\psi^{\mathrm{d}}_{v,a}))$ coincide: by the first identity of (Gd2) and Lemma E.4, $y_x(f(\Lambda_j(D^{\mathrm{u}})(\pi_{\mathscr{H}}(\mathrm{Frob}_w))))=\kappa(\mathrm{Frob}_w)^{-j}(\mathrm{N}w)^{j(j-1)/2}y_x(T^{(j)}_w)$ is the j-th elementary symmetric function of the eigenvalues of $\tilde r_x(\pi_{\mathscr{H}}(\mathrm{Frob}_w))$, i.e. $x(\Lambda_j(D^{\mathrm{u}})(\pi_{\mathscr{H}}(\mathrm{Frob}_w)))$ (Cy1); both determinants are continuous, the $\pi_{\mathscr{H}}(\mathrm{Frob}_w)$ are dense in $\Gamma^0_{\mathscr{H}}$ (Chebotarev), so their characteristic polynomials agree everywhere and the determinants are equal by (Ch1); and by the other identities of (Gd2) and Lemma E.4, $y_x\circ f\circ\psi^{\mathrm{d}}_{v,j}$ and $x\circ\psi^{\mathrm{d}}_{v,j}=(\chi^{j-1}\kappa)^{-1}\phi_{v,j}$ agree on $\mathrm{Art}(\mathcal{O}^\times_{L_{\tilde v}})$ and at $\mathrm{Art}(\varpi_{\tilde v})$, hence everywhere. By (Gd0), $x(B)$ and $y_x(T)$ are $\mathcal{O}$-orders in finite extensions of $\mathbb{Q}_\ell$, with residue field $\mathbb{F}$; the $\mathcal{O}$-algebra $R\subseteq\overline{\mathbb{Q}}_\ell$ they generate is finite over $\mathcal{O}$, a domain, hence local and complete, with residue field generated by elements of $\mathbb{F}$, so $R\in\mathrm{CNL}_{\mathcal{O}}$. Both x and $y_x\circ f$ are morphisms $B\to R$ classifying the same polarised ordinary determinant, so $x=y_x\circ f$ by the uniqueness in the universal property, and $x(J)=0$.

Hence $J=\ker f$, with T, λ_T, M, f, is a Hecke ideal of zero-monodromy type for ι and κ. It has the form described in the statement: the type datum satisfies (Zt) (Step 1), $\mathfrak{m}$ satisfies (Mx) (Step 2), T, λ_T and M are as stated (Step 3), and $f=\theta$ for $\mathfrak{a}=\mathfrak{a}_0$ and $\mathcal{C}=\mathcal{C}_{\chi_0}$ (Step 4).

Finally, suppose $\bar\omega|_{D_v}\neq1$ for every $v\in S_\ell$. As $\bar\omega$ is the character by which G_{L^+} acts on the ℓ-th roots of unity, L^+_v contains no primitive ℓ-th root of unity, so $\mu_{\ell^\infty}(L^+_v)=1$ and $\Delta_\ell=\{1\}$; then $M=N$, and $\mathfrak{a}_0$ is the intersection of all minimal primes of the reduced ring $\mathbb{T}$, which is 0. □

Lemma E.3. *Let $\mathcal{O}$ be a coefficient ring, $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$ an isomorphism, $n\geq1$, and $F_1,\dots,F_s$ finite extensions of $\mathbb{Q}_{p_1},\dots,\mathbb{Q}_{p_s}$ (primes p_k). For $1\leq k\leq s$ and $1\leq a\leq n$ let $\chi_{k,a}\colon I_{F_k}\to\mathcal{O}^\times$ be a character of finite order which is the restriction of a continuous character of W_{F_k}. Put $K_k=\mathrm{GL}_n(\mathcal{O}_{F_k})$ and $K=\prod_k K_k$. For an irreducible smooth representation π_k of $\mathrm{GL}_n(F_k)$ over $\mathbb{C}$ say that π_k **has zero monodromy and inertia** $(\chi_{k,a})_a$ if $\mathrm{rec}(\pi_k)|_{I_{F_k}}\cong\bigoplus_a\iota\circ\chi_{k,a}$ and $N=0$ on $\mathrm{rec}(\pi_k)$. Then there are open normal subgroups $V_k\subseteq K_k$ and a finite free $\mathcal{O}$-module τ_0 with an $\mathcal{O}$-linear action of $H=\prod_k K_k/V_k$ such that for all irreducible smooth representations π_k of $\mathrm{GL}_n(F_k)$ over $\mathbb{C}$ $(1\leq k\leq s)$:*

$$\mathrm{Hom}_K\Big(\tau_0\otimes_{\mathcal{O},\iota}\mathbb{C},\ \bigotimes_k\pi_k\Big)\neq0\iff\text{every }\pi_k\text{ has zero monodromy and inertia }(\chi_{k,a})_a.$$

Proof. Write $\mathrm{Z}_k(\pi_k)$ for the condition "π_k has zero monodromy and inertia $(\chi_{k,a})_a$".

Step 1: one factor over $\mathbb{C}$. Each $\iota\circ\chi_{k,a}$ has finite image, so its kernel is open in I_{F_k}, and it extends to W_{F_k}; hence $\bigoplus_a\iota\circ\chi_{k,a}$ is an inertial type. By Theorem B.166 there is an irreducible smooth finite-dimensional representation ξ_k of K_k over $\mathbb{C}$ such that for every irreducible smooth π_k: $\mathrm{Hom}_{K_k}(\xi_k,\pi_k)\neq0$ if and only if $\mathrm{Z}_k(\pi_k)$. As K_k is compact and ξ_k smooth and finite-dimensional, $V_k=\ker\xi_k$ is an open normal subgroup; put $Q_k=K_k/V_k$, a finite group. For $\sigma\in\mathrm{Aut}(\mathbb{C})$, ξ_k^σ is irreducible with the same kernel V_k (its matrix entries are the σ-images of those of ξ_k in a basis).

Step 2: conjugation by $\mathrm{Aut}(\mathbb{C}/\iota(E))$ preserves the condition. Let $\sigma\in\mathrm{Aut}(\mathbb{C})$ fix $\iota(E)$ pointwise and let π_k be irreducible smooth. Twisting by $|\det|^s$ $(s\in\frac12\mathbb{Z})$ or by any unramified character changes rec by an unramified character, which is trivial on I_{F_k} and does not change N; and $(\pi_k\otimes|\det|^{(1-n)/2})^\sigma=\pi_k^\sigma\otimes\chi$ with $\chi=\sigma\circ|\det|^{(1-n)/2}$ unramified. Hence, by Theorem B.93,

$\mathrm{rec}(\pi_k^\sigma)|_{I_{F_k}} \cong \sigma(\mathrm{rec}(\pi_k))|_{I_{F_k}}$ and the monodromy of $\mathrm{rec}(\pi_k^\sigma)$ is 0 if and only if that of $\mathrm{rec}(\pi_k)$ is. Since the values of $\iota\circ\chi_{k,a}$ lie in $\iota(E)$, $\sigma(\bigoplus_a \iota\circ\chi_{k,a}) \cong \bigoplus_a \iota\circ\chi_{k,a}$. So $\mathrm{Z}_k(\pi_k)$ implies $\mathrm{Z}_k(\pi_k^\sigma)$, and applying this to σ^{-1} and π_k^σ gives the converse. Moreover $\mathrm{Hom}_{K_k}(\xi_k^\sigma, \pi_k) \neq 0$ if and only if $\mathrm{Hom}_{K_k}(\xi_k, \pi_k^{\sigma^{-1}}) \neq 0$ (apply $-\otimes_{\mathbb{C},\sigma^{-1}}\mathbb{C}$ to a K_k-map), so by Step 1 and the above, $\mathrm{Hom}_{K_k}(\xi_k^\sigma, \pi_k) \neq 0$ if and only if $\mathrm{Z}_k(\pi_k)$.

Step 3: an E-rational model. Let ψ_k be the character of ξ_k and $e_k = \frac{\dim\xi_k}{|Q_k|}\sum_{g\in Q_k}\psi_k(g^{-1})g \in \mathbb{C}[Q_k]$ its central primitive idempotent. The values of ψ_k lie in $\mathbb{Q}(\zeta_{h_k})$, $h_k = |Q_k|$; let $E' = \iota(E)(\zeta_{h_k}) \subseteq \mathbb{C}$, a finite Galois extension of $\iota(E)$ with group Γ. Every $\gamma\in\Gamma$ extends to some $\sigma \in \mathrm{Aut}(\mathbb{C}/\iota(E))$ ($\mathbb{C}$ is algebraically closed), and $\gamma(e_k)$ is the central primitive idempotent of ξ_k^σ, whose character is $\sigma\circ\psi_k$. Let f_k be the sum of the distinct elements of $\{\gamma(e_k) : \gamma\in\Gamma\}$. Distinct central primitive idempotents are orthogonal, so f_k is a central idempotent; it is Γ-invariant, so its coefficients lie in the fixed field $\iota(E)$ of Γ. Let $f_k' \in E[Q_k]$ be its preimage under ι. By Wedderburn theory, $\mathbb{C}[Q_k]f_k \cong \bigoplus_{\xi'} \xi'^{\oplus\dim\xi_k}$ as left $\mathbb{C}[Q_k]$-modules, ξ' running over the distinct representations ξ_k^σ, σ as above (finitely many: one for each element of the orbit); and $(E[Q_k]f_k')\otimes_{E,\iota}\mathbb{C} = \mathbb{C}[Q_k]f_k$.

Step 4: a lattice. Let $\tau_k = \mathcal{O}[Q_k]f_k' \subseteq E[Q_k]f_k'$. It is an $\mathcal{O}$-submodule stable under left multiplication by Q_k, finitely generated (the image of $\mathcal{O}[Q_k]$ under $x\mapsto xf_k'$) and torsion-free, hence free of finite rank over the discrete valuation ring $\mathcal{O}$; it spans $E[Q_k]f_k'$ over E (as $\ell^m f_k' \in \mathcal{O}[Q_k]$ for large m and $E[Q_k]f_k' = E\cdot\mathcal{O}[Q_k]f_k'$). So $\tau_k\otimes_{\mathcal{O},\iota}\mathbb{C} \cong \bigoplus_{\xi'}\xi'^{\oplus\dim\xi_k}$ as K_k-representations, ξ' as in Step 3.

Step 5: the product. Let $\tau_0 = \bigotimes_k \tau_k$ (tensor product over $\mathcal{O}$), finite free over $\mathcal{O}$, with $H = \prod_k Q_k$ acting factorwise. Then $\tau_0\otimes_{\mathcal{O},\iota}\mathbb{C}$ is a direct sum of copies of the representations $\bigotimes_k \xi_k'$ of K, ξ_k' running over the conjugates of Step 3, each tuple occurring. For smooth representations A_k of the compact groups K_k over $\mathbb{C}$, $(\bigotimes_k A_k)^K = \bigotimes_k A_k^{K_k}$: the averaging projector of K_k onto $A_k^{K_k}$ (averaging over a finite quotient through which K_k acts on a given vector) acts on the tensor product as the identity on the other factors, and $(A\otimes B)^{K_1\times K_2} = ((A\otimes B)^{K_1})^{K_2} = (A^{K_1}\otimes B)^{K_2} = A^{K_1}\otimes B^{K_2}$. With $A_k = \mathrm{Hom}_{\mathbb{C}}(\xi_k', \pi_k) = \xi_k'^\vee\otimes\pi_k$ (ξ_k' finite-dimensional) this gives $\mathrm{Hom}_K(\bigotimes_k\xi_k', \bigotimes_k\pi_k) \cong \bigotimes_k \mathrm{Hom}_{K_k}(\xi_k', \pi_k)$, which is non-zero if and only if every factor is non-zero. By Step 2 the factor for $\xi_k' = \xi_k^\sigma$ is non-zero if and only if $\mathrm{Z}_k(\pi_k)$, independently of σ. Hence $\mathrm{Hom}_K(\tau_0\otimes_{\mathcal{O},\iota}\mathbb{C}, \bigotimes_k\pi_k) \neq 0$ if and only if $\mathrm{Z}_k(\pi_k)$ for every k; and τ_0 is a representation of $H = \prod_k K_k/V_k$ by construction. □

E.1.2 The ordinary Hecke algebra at principal congruence level

This subsection proves Proposition E.5: at principal congruence level at St the points of the ordinary Hecke algebra carry Galois representations, the irreducible points are automorphic, and the ordinary lines at the places above ℓ are read off from the eigenvalues of the U- and diamond operators (Lemma E.6). The inputs are Hida's control theorem in Geraghty's form, local–global compatibility, including Thorne's for ordinary representations [256], and base change from the definite unitary group with its cuspidality and descent statements (Proposition E.7, Proposition E.9, Lemma E.8). Classical ordinary points of the determinant ring give points of the Hecke algebra of zero-monodromy type (Lemma E.4), and Geraghty's torus algebra is finite free over the rational weight algebra (Lemma E.10).

Lemma E.4. *Let $\mathscr{H}$ over $\mathcal{O}'$, κ, $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ and c be as in Lemma E.16, $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$ an isomorphism, V_v ($v\in\mathrm{St}$) the kernel of reduction modulo ϖ_v^c, $H = \prod_{v\in\mathrm{St}}\mathbb{G}_L(\mathcal{O}_{L_v^+})/V_v$, identified through $\iota_{\tilde v}$ with a quotient of $K = \prod_{v\in\mathrm{St}}\mathrm{GL}_n(\mathcal{O}_{L_{\tilde v}})$, and τ_0 a finite free $\mathcal{O}'$-module with an $\mathcal{O}'$-linear action of H such that*

(Zc) $\mathrm{Hom}_K(\tau_0 \otimes_{\mathcal{O}',\iota} \mathbb{C}, \bigotimes_{v\in \mathrm{St}} \sigma_v) \neq 0$ *for all irreducible smooth representations* σ_v *of* $\mathrm{GL}_n(L_{\tilde{v}})$ *over* $\mathbb{C}$ *such that every* σ_v *has zero monodromy and inertia* $((\bar{\chi}_a]\kappa)|_{I_{L_{\tilde{v}}}})_{1\le a\le n}$.

Let $\tau = \tau_0^\vee = \mathrm{Hom}_{\mathcal{O}'}(\tau_0, \mathcal{O}')$ *with the contragredient action. Then for every classical ordinary point* x *of* B_n^{po} *for* ι *and* κ*, with* $\tilde{r}$ *as in (Cy1), there is a point* y *of* $\mathbb{T}_\tau$ *such that:* $y(\mathfrak{P}_{\lambda,\alpha}) = 0$ *for a dominant weight* λ *and a character* α *of finite order; for every* w*,* $P_{y,w}$ *is the characteristic polynomial of* $(\tilde{r}\circ\pi_{\mathscr{H}}\otimes\kappa)(\mathrm{Frob}_w)$*; and for every* $v\in S_\ell$*,* $1\le j\le n$ *and* $z\in\mathcal{O}_{L_{\tilde{v}}}^\times$*,* $y(\langle d_j(z)\rangle) = \phi_{v,j}(\mathrm{Art}(z))$ *and* $y(U_{\tilde{v}}^{(j)}) = y(U_{\tilde{v}}^{(j-1)})\,\phi_{v,j}(\mathrm{Art}(\varpi_{\tilde{v}}))$*, with* $\phi_{v,j} = \chi_{\mathrm{cyc}}^{j-1}\kappa\,(x\circ\psi_{v,j}^{\mathrm{d}})$ *and* $U_{\tilde{v}}^{(0)} = 1$.

Proof. Step 1: the point of $\mathbb{T}^{\mathrm{ord}}(V)$ *and its weight.* By (Hx3) of Proposition E.5 there is a point y_0 of $\mathbb{T}^{\mathrm{ord}}(V)$ with the stated values of $P_{y_0,w}$, $y_0(\langle d_j(z)\rangle)$ and $y_0(U_{\tilde{v}}^{(j)})$. Fix $v\in S_\ell$ and j. By (Cy4), $x\circ\psi_{v,j}^{\mathrm{d}}$ is de Rham with τ'-Hodge–Tate weights $h_{\tau',j}$ (integers, strictly decreasing in j); a de Rham character of D_v is Hodge–Tate, hence locally algebraic ([220]): $x\circ\psi_{v,j}^{\mathrm{d}}(\mathrm{Art}(z)) = \prod_{\tau'}\tau'(z)^{h_{\tau',j}}$ for z in an open subgroup of $\mathcal{O}_{L_{\tilde{v}}}^\times$ (with Art sending uniformisers to geometric Frobenius, a character equal to $\mathrm{Art}(z)\mapsto\prod\tau'(z)^{a_{\tau'}}$ near 1 has τ'-weight $a_{\tau'}$; $\chi(\mathrm{Art}(z)) = \prod_{\tau'}\tau'(z)$ on units); likewise $\kappa(\mathrm{Art}(z)) = \prod_{\tau'}\tau'(z)^{m_{\tau'}}$ near 1 (κ is algebraic). Put $\lambda_{\tau',n+1-j} = -(j-1) - m_{\tau'} - h_{\tau',j}$. Then $u\mapsto y_0(\langle u\rangle)(w_0\lambda)(u)$ is a continuous character of $T_n(\ell^1)$ trivial on an open subgroup , i.e. a character α of finite order, and $y_0(\mathfrak{P}_{\lambda,\alpha}) = 0$ for the renormalised structure, $u\mapsto(w_0\nu)^{-1}(u)\langle u\rangle$. λ is dominant: $\lambda_{\tau',n-j} - \lambda_{\tau',n+1-j} = h_{\tau',j} - h_{\tau',j+1} - 1 \ge 0$. Choose $b\ge 1$ with $T_n(\ell^b)\subseteq\ker\alpha$; by construction $y_0(\langle u\rangle) = \alpha(u)(w_0\lambda)(u)^{-1}$ for $u\in T_n(\ell^1)$.

Step 2: an automorphic representation with the type. By Lemma E.11 for the type datum $((V_v),\mathcal{O}'[H])$ and y_0, there are an irreducible constituent π of $S_\lambda(\overline{\mathbb{Q}}_\ell)$, unramified with $T_w^{(j)}$-eigenvalues $y_0(T_w^{(j)})$ at every split $v\notin T_L$, and a non-zero $f\in\pi\cap S_\lambda^{\mathrm{ord}}(V(\ell^{Eb,Eb}),\alpha,\overline{\mathbb{Q}}_\ell)$, E the maximal ramification index of $L^+/\mathbb{Q}$ at the places above ℓ, which is a joint eigenvector, with eigencharacter y_0, of the operators assigned by (Gy7)(c). By (Gy5)(i), r_π is semisimple with characteristic polynomial $P_{y_0,w}$ at Frob_w for all w, as is $r\otimes\kappa$, which is absolutely irreducible (Cy1); by the Chebotarev density theorem and the Brauer–Nesbitt theorem, $r_\pi\cong r\otimes\kappa$. Let $v\in\mathrm{St}$, $I = I_{\bar{w}}\cap G_L$ for a place $\bar{w}$ of $\overline{\mathbb{Q}}$ above $\tilde{v}$, the inertia group of $L_{\tilde{v}}$, and $I' = \{s\in I : \bar{\chi}_a(s) = 1\ (1\le a\le n)\}\subseteq N_{\mathscr{H}}$. Then $r(I') = 1$, so $r|_I$ factors through the finite group I/I', which embeds into $(\mathbb{F}^\times)^n$ through the $\bar{\chi}_a$ and so has order prime to ℓ; hence $r|_I\cong\bigoplus_a\psi_a$ with characters ψ_a of order prime to ℓ. The reduction of r has semisimplification with the characteristic polynomials of $\bigoplus_a\bar{\chi}_a$ ($x\circ D^{\mathrm{u}}$ reduces to $\bar{D}_n$), so the reductions of the ψ_a are the $\bar{\chi}_a|_I$ up to order, and a character of order prime to ℓ is the Teichmüller lift of its reduction: $r|_I\cong\bigoplus_a[\bar{\chi}_a]|_I$. As $\kappa|_I$ has finite order (Rd7), $(r\otimes\kappa)|_I\cong\bigoplus_a([\bar{\chi}_a]\kappa)|_I$ has finite image, so by (Wd1) $\mathrm{WD}(r_\pi|_{G_{L_{\tilde{v}}}})$ has monodromy 0 ($\exp(t_\ell(\sigma)N) = 1$ on an open subgroup forces $N = 0$) and inertia $r_\pi|_I$; Frobenius-semisimplification changes neither. By (Ug2) (applicable by (Gd0)), $\mathrm{rec}(\iota\pi_v\circ\iota_{\tilde{v}}^{-1}\otimes|\det|^{(1-n)/2})$, and hence its unramified twist $\mathrm{rec}(\iota\pi_v\circ\iota_{\tilde{v}}^{-1})$, has zero monodromy and inertia $(([\bar{\chi}_a]\kappa)|_I)_a$. By (Zc), $\mathrm{Hom}_K(\tau_0\otimes_{\mathcal{O}',\iota}\mathbb{C},\bigotimes_{v\in\mathrm{St}}\iota\pi_v\circ\iota_{\tilde{v}}^{-1})\neq 0$; as τ_0 is trivial on $\prod_v V_v$, the image of such a map lies in the $\prod_v V_v$-invariants, and transporting by ι^{-1}, $\mathrm{Hom}_H(\tau_0\otimes\overline{\mathbb{Q}}_\ell,\bigotimes_{v\in\mathrm{St}}\pi_v^{V_v})\neq 0$.

Step 3: y_0 factors through $\mathbb{T}_\tau$. By Lemma E.14 (with τ, λ, α, y_0, π and f of Step 2 and Eb in place of its r, which is allowed as $T_n(\ell^{Eb})\subseteq T_n(\ell^b)\subseteq\ker\alpha$; $\tau^\vee = \tau_0$), $y_0 = y\circ q$ for a point y of $\mathbb{T}_\tau$. As q sends each operator to the operator of the same name and respects the Λ_1-structures, y has the values of y_0 on $T_w^{(j)}$, $U_{\tilde{v}}^{(j)}$, $\langle u\rangle$, and $y(\mathfrak{P}_{\lambda,\alpha}) = 0$. □

Proposition E.5. *Let* $\mathscr{H} = (n, L, c, (\bar{v}), \mathrm{St}, v_0, \bar{v}_0, (\bar{\chi}_a))$ *over* $\mathcal{O}'$*,* κ *and* $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ *be as in the hypotheses of Proposition E.26. For* $v\mid\ell$*,* $1\le j\le n$ *and* $z\in\mathcal{O}_{L_{\tilde{v}}}^\times$ *let* $d_j(z)\in T_n(\mathcal{O}_{L_v^+})$ *correspond under* $\iota_{\tilde{v}}$ *to the diagonal matrix with* z *in position* j *and* 1 *elsewhere, and identify* D_v *with the decomposition group of* $G_{L_{\tilde{v}}}$ *through* $\bar{v}$*. Give* $\mathbb{T}_\tau$ *the renormalised* Λ_1*-structure*

$u \mapsto (w_0\nu)^{-1}(u)\langle u\rangle$ *and, for a dominant weight* λ *and a character* α *of* $T_n(\ell^1)$ *of finite order, let* $\mathfrak{P}_{\lambda,\alpha} \subsetneq \Lambda_1$ *be as in (Gy3). A* point *of* $\mathbb{T}_\tau$ *is a continuous homomorphism of* $\mathcal{O}'$*-algebras* $y\colon \mathbb{T}_\tau \to \overline{\mathbb{Q}}_\ell$*; its* Hecke polynomial *at a place* w *of* L *above a place* $v \notin T_L$ *of* L^+ *split in* L *is* $P_{y,w}(X) = \sum_{j=0}^n (-1)^j (\mathrm{N}w)^{j(j-1)/2} y(T_w^{(j)}) X^{n-j}$ *(*$T_w^{(0)} = 1$*). Fix an integer* $c \geq 1$ *such that* $\bar\chi_1(\mathrm{Art}_{L_w}(z)) = \cdots = \bar\chi_n(\mathrm{Art}_{L_w}(z)) = \kappa(\mathrm{Art}_{L_w}(z)) = 1$ *for every place* w *of* L *above a place of* St *and every* $z \in 1 + \varpi_w^c \mathcal{O}_{L_w}$ *(it exists: the* $\bar\chi_a$ *have finite order and* κ *has finite order on these inertia groups by (Rd7)). Let* $((V_v)_{v\in \mathrm{St}}, \tau)$ *be the type datum with* V_v *the kernel of reduction* $\mathbb{G}_L(\mathcal{O}_{L_v^+}) \to \mathbb{G}_L(\mathcal{O}_{L_v^+}/\varpi_v^c)$ *and* $\tau = \mathcal{O}'[H]$ *with* H *acting by left multiplication, and* V *its level. Then:*

(Hx1) *for every point* y *of* $\mathbb{T}_\tau$ *with* $y(\mathfrak{P}_{\lambda,1}) = 0$ *for a regular dominant weight* λ *(as in (Gy6)) there is a continuous semisimple* $r_y\colon G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ *such that (i) for every* w *as above, the semisimplification of the restriction of* r_y *to a decomposition group above* w *is unramified and* $r_y(\mathrm{Frob}_w)$ *has characteristic polynomial* $P_{y,w}$*; (ii)* $r_y^c \cong r_y^\vee \chi_{\mathrm{cyc}}^{1-n}$*; (iii) if* r_y *is absolutely irreducible, then for every* $v \mid \ell$*,* $r_y|_{D_v}$ *is triangular with diagonal* $(\chi_{\mathrm{cyc}}^{1-j}\psi_{v,j}^y)_{1\leq j\leq n}$*, where* $\psi_{v,j}^y$ *is the character of* D_v *with* $\psi_{v,j}^y(\mathrm{Art}_{L_{\tilde v}}(z)) = y(\langle d_j(z)\rangle)$ *for* $z \in \mathcal{O}_{L_{\tilde v}}^\times$ *and* $\psi_{v,j}^y(\mathrm{Art}_{L_{\tilde v}}(\varpi_{\tilde v})) = y(U_{\tilde v}^{(j)})/y(U_{\tilde v}^{(j-1)})$ *(*$U_{\tilde v}^{(0)} = 1$*);*

(Hx2) *for every point* y *of* $\mathbb{T}_\tau$ *with* $y(\mathfrak{P}_{\lambda,\alpha}) = 0$ *for a dominant weight* λ *and a character* α *of finite order, every absolutely irreducible continuous* $\rho\colon G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ *such that, for every* w *as above, the restriction of* ρ *to a decomposition group above* w *has unramified semisimplification and* $\rho(\mathrm{Frob}_w)$ *has characteristic polynomial* $P_{y,w}$*, is automorphic of conjugate self-dual type for* ι*;*

(Hx3) *for every classical ordinary point* x *of* B_n^{po} *for* ι *and* κ*, with* $\tilde r$ *as in (Cy1), there is a point* y *of* $\mathbb{T}_\tau$ *such that: for every* w *as above,* $P_{y,w}$ *is the characteristic polynomial of* $(\tilde r \circ \pi_{\mathscr{H}} \otimes \kappa)(\mathrm{Frob}_w)$*; and for every* $v \mid \ell$*,* $1 \leq j \leq n$ *and* $z \in \mathcal{O}_{L_{\tilde v}}^\times$*,*

$$y(\langle d_j(z)\rangle) = \phi_{v,j}(\mathrm{Art}_{L_{\tilde v}}(z)), \qquad y(U_{\tilde v}^{(j)}) = y(U_{\tilde v}^{(j-1)})\, \phi_{v,j}(\mathrm{Art}_{L_{\tilde v}}(\varpi_{\tilde v})),$$
$$\phi_{v,j} = \chi_{\mathrm{cyc}}^{j-1}\, \kappa\, (x \circ \psi_{v,j}^{\mathrm{d}}),$$

with $U_{\tilde v}^{(0)} = 1$*.*

Proof. Step 0: the setting. V_v is open and normal in $\mathbb{G}_L(\mathcal{O}_{L_v^+})$ and τ is finite free, so $((V_v), \tau)$ is a type datum. Evaluation at 1 is an isomorphism $N_\tau = \mathrm{Hom}_{\mathcal{O}'[H]}(\mathcal{O}'[H], \mathcal{S}_V) \to \mathcal{S}_V$ compatible with $\mathbb{T}^{\mathrm{ord}}(V)$, so $\mathbb{T} = \mathbb{T}^{\mathrm{ord}}(V)$. The hypotheses of Theorem B.91 hold for $F = L$, $\mathbb{G} = \mathbb{G}_L$, $T = T_L$, $U = V$, $b_0 = 1$: L is imaginary CM with the places above ℓ split (Rd2); $\mathcal{O}' \supseteq \tau'(\mathcal{O}_L)$ for all embeddings τ'; $\mathbb{G}_L$ comes from $X \mapsto {}^tX^c$ and the order $\mathrm{M}_n(\mathcal{O}_L)$, is compact at the real places, and is quasi-split at every finite place (split places: GL_n; the other places are unramified by (Rd2), where the Hermitian form $\sum x_i\bar x_i$ of discriminant 1 is quasi-split, for n even because $(-1)^{n/2}$ is a unit, hence a norm), and $\mathbb{G}_L(\mathcal{O}_{L_v^+})$ is hyperspecial at the inert places (unimodular form over the unramified extension); the places of T_L split (Rd2); $V_v = \mathbb{G}_L(\mathcal{O}_{L_v^+})$ at $v \in S_\ell$ and at the split $v \notin T_L$; $V(\ell^{1,1})$ is sufficiently small, its component at v_0 being $1 + p\,\mathrm{M}_n(\mathcal{O}_{L_{\tilde v_0}})$, p odd and unramified in L, which has no non-trivial element of finite order . For (Gy4) and Proposition E.7: L/L^+ is unramified at all finite places (Rd2), and hence $[L^+ : \mathbb{Q}] \geq 2$: if $L^+ = \mathbb{Q}$, then L is an imaginary quadratic field, whose discriminant has absolute value at least 3, so some prime ramifies in $L/\mathbb{Q} = L/L^+$. The operators $U_{\tilde v}^{(j)}$ are units of $\mathbb{T}$: $\mathbb{T}$ is finite over Λ_1 (Qh2), so semilocal, and an element of a maximal ideal $\mathfrak{n}$ acts nilpotently on $\mathcal{S}_V/\mathfrak{n}\mathcal{S}_V \neq 0$

(Nakayama, faithfulness), while $U_{\tilde v}^{(j)}$ is invertible on the ordinary module. A point y has values in the ring of integers of a finite extension of $\mathbb{Q}_\ell$ ($\mathbb{T}/\ker y$ is finite over $\mathcal{O}'$), so $y(U_{\tilde v}^{(j)})$ is an ℓ-adic unit.

Step 1: a point gives an ordinary eigenform. Let y be a point with $y(\mathfrak{P}_{\lambda,\alpha}) = 0$, λ dominant, and $r \geq 1$ with $T_n(\mathfrak{l}^r) \subseteq \ker\alpha$ (such r exist by (Qh1)). As y kills $\mathfrak{P}_{\lambda,\alpha}\mathbb{T}$ and $\overline{\mathbb{Q}}_\ell$ is reduced, (Qh3) factors y as $y_\lambda \circ s$, with s the surjection of (Qh3) and y_λ a homomorphism from the finite commutative $\mathcal{K}'$-algebra $\mathbb{T}_\lambda = \mathbb{T}_\lambda^{\mathrm{ord}}(V(\ell^{r,r}), \alpha, \mathcal{K}')$, which acts faithfully on the finite-dimensional space $S = S_\lambda^{\mathrm{ord}}(V(\ell^{r,r}), \alpha, \mathcal{K}')$. The localisation of $S \otimes_{\mathcal{K}'} \overline{\mathbb{Q}}_\ell$ at $\ker(y_\lambda \otimes 1)$ is non-zero (faithfulness), so it contains a common eigenvector $f \neq 0$ of the commuting operators of $\mathbb{T}_\lambda$ with eigencharacter y_λ; f is a form of weight λ and level $V(\ell^{r,r})$ with coefficients in $\overline{\mathbb{Q}}_\ell$, i.e. $f \in S_\lambda(\overline{\mathbb{Q}}_\ell)^{V(\ell^{r,r})}$. Let W be the $\mathbb{G}_L(\mathbb{A}_{L^+}^\infty)$-subrepresentation generated by f and π an irreducible quotient of W (W is finitely generated); π is an irreducible constituent of $S_\lambda(\overline{\mathbb{Q}}_\ell)$, the image $\bar f$ of f in $\pi^{V(\ell^{r,r})}$ is non-zero, and $\bar f$ is an eigenvector of the Hecke operators of level $V(\ell^{r,r})$ with the same eigenvalues as f (they are defined on the invariants of every smooth representation, compatibly with equivariant maps). By (Gy7)(c): $T_w^{(j)}\bar f = y(T_w^{(j)})\bar f$, $U_{\lambda,\varpi_{\tilde v}}^{(j)}\bar f = y(U_{\tilde v}^{(j)})\bar f$ and $\langle u\rangle \bar f = (w_0\lambda)(u)\, y(\langle u\rangle)\bar f$; in particular $\langle d_j(z)\rangle \bar f = \prod_{\tau'} \tau'(z)^{\lambda_{\tau',n+1-j}}\, y(\langle d_j(z)\rangle)\bar f$, τ' running over the embeddings inducing $\tilde v$. Write r_π for the representation of (Gy5).

Step 2: (Hx1)(i), (ii). Let y be as in (Hx1); apply Step 1 with $\alpha = 1$ and $r = r_0$, the maximal ramification index of $L^+/\mathbb{Q}$ at the places above ℓ ($T_n(\mathfrak{l}^{r_0}) \subseteq T_n(\ell^1)$ by (Qh1)), and put $r_y = r_\pi$, continuous and semisimple. For $w \mid v$ as above, $V_v = \mathbb{G}_L(\mathcal{O}_{L_v^+})$ and $\pi = \otimes' \pi_{v'}$ (Flath) has $\pi^{V(\ell^{r_0,r_0})} \ni \bar f \neq 0$, so π_v is unramified, $\pi_v^{\mathbb{G}_L(\mathcal{O}_{L_v^+})}$ is a line, and $T_w^{(j)}$ acts on it by $y(T_w^{(j)})$; (Gy5)(i) gives (i), and (Gy5)(ii) gives (ii) ($\varepsilon = \chi$).

Step 3: (Hx2). Let y, λ, α, ρ be as in (Hx2) and π as in Step 1; as in Step 2, r_π is semisimple and $r_\pi(\mathrm{Frob}_w)$ has characteristic polynomial $P_{y,w}$ for every w. The Frobenius elements at the places w (all but finitely many places of degree one over $\mathbb{Q}$ of L lie above split places of L^+) are dense in G_L (Chebotarev), so ρ and r_π have the same characteristic polynomials everywhere (continuity) and $\rho \cong r_\pi$ (Brauer–Nesbitt, both semisimple). So r_π is absolutely irreducible and (Bc1) gives a regular algebraic conjugate self-dual cuspidal Π of weight $\iota\lambda$ with $r_\iota(\Pi) \cong r_\pi \cong \rho$. At every place w' of L not above ℓ where Π is unramified, (To1) gives $\mathrm{WD}(\rho|_{G_{L_{w'}}})^{\text{F-ss}} \cong \mathrm{rec}^T(\iota^{-1}\Pi_{w'})$, unramified with $N = 0$; if $\Pi_{w'}$ is a subquotient of the unramified principal series of $\chi_1 \otimes \cdots \otimes \chi_n$, then ρ is unramified at w' and the eigenvalues of $\rho(\mathrm{Frob}_{w'})$ are $\chi_i(\varpi_{w'})(\mathrm{N}w')^{(n-1)/2}$; $\mathrm{Frob}_{w'}$ is the inverse of an arithmetic Frobenius, so $\rho \leftrightarrow_\iota \Pi$ in the sense fixed in Section 13.

Step 4: (Hx1)(iii). Let y, λ be as in (Hx1) with r_y absolutely irreducible, and π, $\bar f$ as in Step 2. By Step 3 (with $\rho = r_y$, $\alpha = 1$) and (Bc1), Π is cuspidal of weight $\iota\lambda$, $r_\iota(\Pi) \cong r_y$, and $\pi_v \cong \iota^{-1}\Pi_{\tilde v} \circ \iota_{\tilde v}$ for $v \mid \ell$ (v splits, (Rd2)). Fix $v \mid \ell$. Under $\pi^{V(\ell^{r_0,r_0})} = \pi_v^{\mathrm{Iw}(\tilde v^{r_0,r_0})} \otimes (\pi^v)^{V^v(\ell^{r_0,r_0})}$ the operators $U_{\lambda,\varpi_{\tilde v}}^{(j)}$ and $\langle d_j(z)\rangle$ act on the first factor; as $\bar f$ is a common eigenvector, the joint eigenspace $E \subseteq \pi_v^{\mathrm{Iw}(\tilde v^{r_0,r_0})}$ with the eigenvalues of Step 1 satisfies $\bar f \in E \otimes (\pi^v)^{V^v(\ell^{r_0,r_0})}$, so $E \neq 0$. A non-zero $x \in E \subseteq (\iota^{-1}\Pi_{\tilde v})^{\mathrm{Iw}(\varpi_{\tilde v}^{r_0,r_0})}$ ($b = r_0$ in (Ol1)) satisfies the hypotheses of (Ol1) with $u_j = y(U_{\tilde v}^{(j)})$ (units, Step 0) and $e_j(z) = \prod_{\tau'} \tau'(z)^{\lambda_{\tau',n+1-j}} y(\langle d_j(z)\rangle)$. By (Ol1), $r_y|_{D_v}$ is triangular with diagonal θ_j, $\theta_j(\mathrm{Art}(z)) = \chi(\mathrm{Art}(z))^{1-j} y(\langle d_j(z)\rangle)$ and $\theta_j(\mathrm{Art}(\varpi_{\tilde v})) = \chi(\mathrm{Art}(\varpi_{\tilde v}))^{1-j} y(U_{\tilde v}^{(j)})/y(U_{\tilde v}^{(j-1)})$; as $L_{\tilde v}^\times = \mathcal{O}^\times \times \varpi_{\tilde v}^{\mathbb{Z}}$ and Art has dense image, $\theta_j = \chi^{1-j}\psi_{v,j}^y$.

Step 5: (Hx3). Let x be classical, with $\tilde r$ and Π as in (Cy1), (Cy2), and $r = \tilde r \circ \pi_{\mathscr{H}}$. (a) $r \otimes \kappa$ is absolutely irreducible (Cy1) and $r \otimes \kappa \leftrightarrow_\iota \Pi$; by (To1) and the computation of Step 3, $r_\iota(\Pi)$ has the same Frobenius characteristic polynomials at almost all places, so $r_\iota(\Pi) \cong r \otimes \kappa$ (Chebotarev, Brauer–Nesbitt). Let λ be the weight of Π (Π of weight $\iota\lambda$). (b) Inertia away

from ℓ: for a place $w' \nmid \ell$ of L and $s \in I_{w'}$ with $\bar\chi_a(s) = 1$ for all a, $s \in N_{\mathscr{H}}$, so $r(s) = 1$; with (Rd7), $(r \otimes \kappa)|_{I_{w'}}$ factors through a finite abelian quotient, so $\mathrm{WD}((r \otimes \kappa)|_{G_{L_{w'}}})$ has $N = 0$ and its inertia representation is $(r \otimes \kappa)|_{I_{w'}}$, a sum of characters; if w' lies above a place of L^+ not in $S_\ell \cup \mathrm{St}$, the $\bar\chi_a$ are trivial on $I_{w'}$ (Rd4), so $I_{w'} \subseteq N_{\mathscr{H}}$, and with (Rd7) $r \otimes \kappa$ is unramified at w'; if w' lies above St, these characters are trivial on $\mathrm{Art}(1 + \varpi_{w'}^c \mathcal{O})$ by the choice of c. (c) Unramified places: for $w' \nmid \ell$ above a place of L^+ not in $S_\ell \cup \mathrm{St}$ (in particular above v_0, above the split $v \notin T_L$ and above the inert places, as the places of $S_\ell \cup \mathrm{St}$ split by (Rd2)), $\mathrm{rec}^T(\iota^{-1}\Pi_{w'})$ is unramified with $N = 0$ by (b) and (To1), so $\Pi_{w'}$ is unramified (the local Langlands correspondence matches unramified representations with unramified Weil–Deligne representations with $N = 0$). (d) The places of St: for w' above $v \in \mathrm{St}$, (Bg3) (for the polarizable Π, $r_{\ell,\iota}(\Pi) \cong r_\iota(\Pi)$, $U = 1 + \varpi_{w'}^c \mathcal{O}$) shows that $\sigma = \iota^{-1}\Pi_{w'}$ is a subquotient of a principal series of characters trivial on U; its Jacquet module σ_{N_n} is non-zero with semisimplification a sum of characters $\xi^s \delta_{B_n}^{1/2}$, all trivial on $T_n(\varpi_{w'}^c)$ (Bernstein–Zelevinsky; δ_{B_n} is trivial on units), so the compact group $T_n(\varpi_{w'}^c)$ acts trivially on σ_{N_n} (its smooth representations are semisimple); by (Gy7)(a) with $b = c$, $\sigma^{\mathrm{Iw}(\varpi_{w'}^{c,c})} \neq 0$, and $\mathrm{Iw}(\varpi_{w'}^{c,c})$ contains the kernel of reduction modulo $\varpi_{w'}^c$, so σ has non-zero vectors fixed by $\iota_{\tilde v}(V_v)$. (e) The places above ℓ: for $v \mid \ell$, $r \otimes \kappa$ is triangular on D_v with diagonal $\theta_j = \kappa\,(x \circ \psi_{v,j}^{\mathrm{d}})$ (Cy3), and the Hodge–Tate weights of the θ_j are strictly decreasing in j in the normalisation fixed in Section 13 (Cy4) (κ shifts all of them equally), i.e. strictly increasing in that of (To1). By (Ol2) at $\tilde v$, Π is ι-ordinary at $\tilde v$, and a non-zero $x_v \in (\iota^{-1}\Pi_{\tilde v})^{\mathrm{ord}}$, which lies in $(\iota^{-1}\Pi_{\tilde v})^{\mathrm{Iw}(\varpi_{\tilde v}^{b,b})}$ for some b, is an eigenvector of $U_{\lambda,\varpi_{\tilde v}}^{(j)}$ with eigenvalue $\prod_{a \le j} \phi_{v,a}(\mathrm{Art}(\varpi_{\tilde v}))$ and of $d_j(z)$ with eigenvalue $\phi_{v,j}(\mathrm{Art}(z)) \prod_{\tau'} \tau'(z)^{\lambda_{\tau',n-j+1}}$ ($\varepsilon^{j-1}\theta_j = \phi_{v,j}$). (f) Descent and the point: by (Gy4) and (Bc2) there are a dominant λ' and an irreducible constituent π of $S_{\lambda'}(\overline{\mathbb{Q}}_\ell)$ with $\pi_v \cong \iota^{-1}\Pi_w \circ \iota_w$ at the split v and $\pi_v^{\mathbb{G}_L(\mathcal{O}_{L_v^+})} \neq 0$ at the inert v (where Π is unramified by (c)). By (Gy5)(i) and (c), r_π and $r_\iota(\Pi)$ have the same Frobenius characteristic polynomials at almost all split places, so $r_\pi \cong r_\iota(\Pi)$, and comparing Hodge–Tate weights ((Gy5)(iv), (To1)) gives $\lambda' = \lambda$. Choose b (and $r_1 \ge b$) such that the x_v of (e) are fixed by $\mathrm{Iw}(\varpi_{\tilde v}^{b,b})$ for all $v \mid \ell$, and let $f = \otimes_{v'} f_{v'} \in \pi$ with $f_v = x_v$ ($v \mid \ell$), f_v fixed by V_v for $v \in \mathrm{St}$ (by (d)), f_{v_0} and f_v for the split $v \notin T_L$ spherical (by (c)), and f_v fixed by $\mathbb{G}_L(\mathcal{O}_{L_v^+})$ at the inert v. Then $f \in \pi^{V(\ell^{b,b})}$ is a common eigenvector of the $T_w^{(j)}$ (with the eigenvalues determined by the unramified Π_w), of the $U_{\lambda,\varpi_{\tilde v}}^{(j)}$ (unit eigenvalues) and of the diamond operators; the diamond eigenvalues on $T_n(\ell^1)$ form a character α of finite order (the e_j are smooth), trivial on $T_n(\ell^{r_1})$ for r_1 large. Taking $V(\ell^{r_1,r_1})$-invariants is exact, so the eigencharacter of f on the commutative Hecke algebra of level $V(\ell^{r_1,r_1})$ occurs in $S_\lambda(\overline{\mathbb{Q}}_\ell)^{V(\ell^{r_1,r_1})}$, in its ordinary part with $T_n(\ell^1)$ acting by α, and gives a homomorphism $\mathbb{T}_\lambda^{\mathrm{ord}}(V(\ell^{r_1,r_1}), \alpha, \mathcal{K}') \to \overline{\mathbb{Q}}_\ell$; composed with the surjection of (Qh3) it is a point y of $\mathbb{T}$ with $y(\mathfrak{P}_{\lambda,\alpha}) = 0$. By (Gy7)(c): $y(T_w^{(j)})$ is the eigenvalue of $T_w^{(j)}$ on f, so $P_{y,w}$ is the characteristic polynomial of $r_\pi(\mathrm{Frob}_w)$ (Gy5)(i), i.e. of $(r \otimes \kappa)(\mathrm{Frob}_w)$; $y(U_{\tilde v}^{(j)}) = \prod_{a \le j} \phi_{v,a}(\mathrm{Art}(\varpi_{\tilde v}))$, which is the second identity; and $y(\langle d_j(z)\rangle) = (w_0\lambda)(d_j(z))^{-1}\phi_{v,j}(\mathrm{Art}(z)) \prod_{\tau'} \tau'(z)^{\lambda_{\tau',n-j+1}} = \phi_{v,j}(\mathrm{Art}(z))$, as $(w_0\lambda)(d_j(z)) = \prod_{\tau'} \tau'(z)^{\lambda_{\tau',n+1-j}}$. □

Lemma E.6. *Let F be an imaginary CM field, $n \ge 2$, ℓ a prime, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, $\lambda = (\lambda_\tau)$ dominant (as in Theorem B.94) and Π a regular algebraic conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\iota\lambda$. Let $w \mid \ell$ be a place of F, ϖ a uniformiser of F_w, τ run over the embeddings $F_w \to \overline{\mathbb{Q}}_\ell$, $\sigma = \iota^{-1}\Pi_w$, $r = r_\iota(\Pi)|_{G_{F_w}}$, and for $z \in \mathcal{O}_{F_w}^\times$ let $d_j(z)$ be the diagonal matrix with z in position j and 1 elsewhere.*

(Ol1) *Suppose that there are $b \ge 1$ and $0 \neq x \in \sigma^{\mathrm{Iw}(\varpi^{b,b})}$ which is an eigenvector of every $U_{\lambda,\varpi}^{(j)}$ with an ℓ-adic unit eigenvalue u_j and of every $d_j(z)$ with eigenvalue $e_j(z)$. Put*

$u_0 = 1$. *Then r is conjugate to an upper triangular representation with diagonal characters $\theta_1, \dots, \theta_n$ (in this order), where θ_j has τ-Hodge–Tate weight $\lambda_{\tau,n-j+1} + j - 1$ and*

$$\theta_j(\mathrm{Art}(z)) = \varepsilon(\mathrm{Art}(z))^{1-j}\, e_j(z) \prod_\tau \tau(z)^{-\lambda_{\tau,n-j+1}} \quad (z \in \mathcal{O}_{F_w}^\times),$$
$$\theta_j(\mathrm{Art}(\varpi)) = \varepsilon(\mathrm{Art}(\varpi))^{1-j}\, \frac{u_j}{u_{j-1}}.$$

(Ol2) *Conversely, suppose that r is conjugate to an upper triangular representation with diagonal characters $\theta_1, \dots, \theta_n$ (in this order) whose τ-Hodge–Tate weights are strictly increasing in j for every τ. Then Π is ι-ordinary at w, and every non-zero x in $(\iota^{-1}\Pi_w)^{\mathrm{ord}}$ is an eigenvector of $U^{(j)}_{\lambda,\varpi}$ with eigenvalue $\prod_{a=1}^j (\varepsilon^{a-1}\theta_a)(\mathrm{Art}(\varpi))$ and of $d_j(z)$ with eigenvalue $(\varepsilon^{j-1}\theta_j)(\mathrm{Art}(z)) \prod_\tau \tau(z)^{\lambda_{\tau,n-j+1}}$, for all j and $z \in \mathcal{O}_{F_w}^\times$.*

Proof. Three classical facts. (F1) ([27, 2.12], the geometric lemma; exactness of the Jacquet functor) $I(\chi)_{N_n}$ has semisimplification $\bigoplus_{s\in S_n} \chi^s\delta^{1/2}$, $\chi^s(t) = \chi(s^{-1}ts)$; if σ is irreducible and μ is an irreducible quotient of $\sigma_{N_n} \neq 0$ (a character, $T_n(F_w)$ being abelian), Frobenius reciprocity embeds σ into $I(\mu\delta^{-1/2})$; for $s \in S_n$, $I(\chi)$ and $I(\chi^s)$ have the same irreducible subquotients. (F2) (Fontaine; as used in the proof of Corollary 2.7.8 of [105]) a de Rham character θ of G_{F_w} with τ-Hodge–Tate weights h_τ satisfies: $x \mapsto \theta(\mathrm{Art}(x)) \prod_\tau \tau(x)^{h_\tau}$ is a smooth character of $F_w^\times$, equal to $\mathrm{WD}(\theta)$; conversely a character equal to $\prod_\tau \tau^{-h_\tau} \circ \mathrm{Art}^{-1}$ on an open subgroup of inertia and de Rham has these weights. (F3) (Lubin–Tate) $\varepsilon(\mathrm{Art}(x)) = \mathrm{N}_{F_w/\mathbb{Q}_\ell}(x)\,|x|_w$, so $\varepsilon(\mathrm{Art}(z)) = \prod_\tau \tau(z)$ for units and $\varepsilon(\mathrm{Art}(\varpi)) = q^{-1}\prod_\tau \tau(\varpi)$.

Step 1: the Weil–Deligne representation of a triangular r. If r is triangular with diagonal (θ_j), the θ_j are de Rham (subquotients of the de Rham r, (To1)), D_{pst} is exact, and the semisimplification of the Weil group representation of $\mathrm{WD}(r)$ is $\bigoplus_j \mathrm{WD}(\theta_j)$. Moreover Π is polarizable (conjugate self-dual) and $r \cong r_{\ell,\iota}(\Pi)|_{G_{F_w}}$ (both global representations are semisimple with the same Frobenius characteristic polynomials at almost all places by (To1) and (Bg1)), so (Bg3) with $U = \{1\}$ and (Bg5) apply: σ is a subquotient of $I(\xi)$ for characters ξ_i with $\iota^{-1}\xi_i = \lambda_i\,|\cdot|_w^{(n-1)/2}$, where $\lambda_1, \dots, \lambda_n$ are the characters of the semisimplified $\mathrm{WD}(r)$.

Step 2: (Ol1). Let x, u_j, e_j be as in (Ol1) and $u^{(j)} = \prod_\tau \prod_{i\le j} \tau(\varpi)^{\lambda_{\tau,n-i+1}} u_j$, the eigenvalue of $U^{(j)}_\varpi$ on x, non-zero. By (Gy7)(a) with $c = b$, the image $\bar{x}$ of x in σ_{N_n} is non-zero; it is an eigenvector of $T_n(\mathcal{O}_{F_w})$ (generated by the $d_j(z)$) with eigenvalues $e_j(z)$ and of $\sigma_{N_n}(\alpha^{(j)}_\varpi)$ with eigenvalue $\delta(\alpha^{(j)}_\varpi)u^{(j)}$; as $T_n(F_w)$ is generated by $T_n(\mathcal{O}_{F_w})$ and the $\alpha^{(j)}_\varpi$, $\bar{x}$ spans a $T_n(F_w)$-stable line with character μ, which is a constituent of σ_{N_n}. Since $\sigma_{N_n} \neq 0$, (F1) gives χ with σ a subquotient of $I(\chi)$, then $\mu = \chi^s\delta^{1/2}$ for some s, and σ is a subquotient of $I(\chi')$ with $\chi' = \mu\delta^{-1/2}$. Here $\chi'_j|_{\mathcal{O}^\times} = e_j$ (δ is trivial on units) and, as $\delta^{1/2}(\alpha^{(j)}_\varpi) = q^{-j(n-j)/2}$,

$$\chi'_j(\varpi) = \frac{\mu(\alpha^{(j)}_\varpi)\,\delta^{-1/2}(\alpha^{(j)}_\varpi)}{\mu(\alpha^{(j-1)}_\varpi)\,\delta^{-1/2}(\alpha^{(j-1)}_\varpi)} = q^{-\frac{n-1}{2}}\, q^{j-1} \prod_\tau \tau(\varpi)^{\lambda_{\tau,n-j+1}}\, \frac{u_j}{u_{j-1}}.$$

As the u_j are units,

$$v_\ell(\chi'_j(\varpi)) = \frac{1}{e}\sum_\tau (\lambda_{\tau,n+1-j} - \frac{n-1}{2} + j - 1).$$

By (To2), Π is ι-ordinary at w with tuple (χ'_j), and by (To3) r is triangular with diagonal (θ_j), $\theta_j(\mathrm{Art}(z)) = e_j(z)\prod_\tau \tau(z)^{-(\lambda_{\tau,n-j+1}+j-1)}$, which is the first formula by (F3); by (F2) θ_j has

weights $h_{\tau,j} = \lambda_{\tau,n-j+1} + j - 1$, strictly increasing in j. By Step 1 the multisets $\{\mathrm{WD}(\theta_j)\}$ and $\{\lambda_i\}$ coincide, and σ is a subquotient of both $I(\xi)$ and $I(\chi')$; two principal series sharing an irreducible subquotient have the same cuspidal support ([27, Theorem 2.9]), so $\{\iota^{-1}\xi_i\} = \{\chi'_j\}$ and $\{\mathrm{WD}(\theta_j)\} = \{\chi'_j|\cdot|_w^{(1-n)/2}\}$. At ϖ: $v_\ell(\mathrm{WD}(\theta_j)(\varpi)) = \frac{1}{e}\sum_\tau h_{\tau,j}$ by (F2) ($\theta_j(\mathrm{Art}(\varpi))$ is a unit, G_{F_w} being compact), and $v_\ell(\chi'_i(\varpi)q^{(n-1)/2}) = \frac{1}{e}\sum_\tau h_{\tau,i}$; both are strictly increasing, so the bijection between the two multisets matches j with j: $\mathrm{WD}(\theta_j) = \chi'_j|\cdot|_w^{(1-n)/2}$. Evaluating at ϖ with (F2): $\theta_j(\mathrm{Art}(\varpi))\prod_\tau \tau(\varpi)^{h_{\tau,j}} = \chi'_j(\varpi)q^{(n-1)/2} = q^{j-1}\prod_\tau \tau(\varpi)^{\lambda_{\tau,n-j+1}}u_j/u_{j-1}$, i.e. $\theta_j(\mathrm{Art}(\varpi)) = (q^{-1}\prod_\tau \tau(\varpi))^{1-j}u_j/u_{j-1} = \varepsilon(\mathrm{Art}(\varpi))^{1-j}u_j/u_{j-1}$ by (F3).

Step 3: (Ol2). The multiset of τ-Hodge–Tate weights of r is $\{\lambda_{\tau,n}, \lambda_{\tau,n-1}+1, \dots, \lambda_{\tau,1}+n-1\}$ (To1), listed increasingly; as the weights of the θ_j are strictly increasing in j, θ_j has weight $h_{\tau,j} = \lambda_{\tau,n-j+1} + j - 1$. By Step 1, σ is a subquotient of $I(\chi'')$ with $\chi''_j = \mathrm{WD}(\theta_j)|\cdot|_w^{(n-1)/2}$ (order the ξ_i by the bijection with the $\mathrm{WD}(\theta_j)$; reordering does not change the subquotients, (F1)), and $v_\ell(\chi''_j(\varpi)) = \frac{1}{e}\sum_\tau h_{\tau,j} - \frac{(n-1)f}{2} = \frac{1}{e}\sum_\tau(\lambda_{\tau,n+1-j} - \frac{n-1}{2} + j - 1)$. By (To2), Π is ι-ordinary at w, so by (Gy7)(b) $(\iota^{-1}\Pi_w)^{\mathrm{ord}}$ is a line, contained in $\sigma^{\mathrm{Iw}(\varpi^{b,b})}$ for some b. The $U^{(j)}_{\lambda,\varpi}$ commute with each other and the $d_j(z)$ commute with them ($d_j(z)$ normalises $N_n(\mathcal{O})$, $\mathrm{Iw}(\varpi^{b,b})$ and commutes with $\alpha^{(i)}_\varpi$), so all of them preserve the ordinary part, which is a line: a non-zero x in it is a common eigenvector, the U-eigenvalues u_j being units. By Step 2, r is triangular with a diagonal (θ'_j) given by the formulas of (Ol1) and with weights $h_{\tau,j}$. The diagonal characters of a triangular representation are its Jordan–Hölder factors, so $\{\theta'_j\} = \{\theta_j\}$ as multisets; as θ'_j and θ_j both have weights $h_{\tau,j}$, pairwise distinct in j, $\theta'_j = \theta_j$. Solving the formulas of (Ol1) for $e_j(z)$ and u_j/u_{j-1} gives the eigenvalues claimed in (Ol2). □

Proposition E.7. *Let ℓ, $n \geq 2$, F, F^+, $\mathcal{O}$, $\mathbb{G}$ be as in Theorem B.91, with $[F^+ : \mathbb{Q}] \geq 2$, and let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism.*

(Bc1) *Let λ be a dominant weight for $\mathbb{G}$ and π an irreducible constituent of $S_\lambda(\overline{\mathbb{Q}}_\ell)$ such that r_π is absolutely irreducible. Then there is a regular algebraic conjugate self-dual cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\iota\lambda$ such that $\pi_v \cong \iota^{-1}\Pi_w \circ \iota_w$ for every finite place v of F^+ which splits as ww^c in F, and $r_\pi \cong r_\iota(\Pi)$.*

Proof. Step 1: the decomposition. The hypotheses of Lemma E.8 hold (ℓ, n, F, F^+, $\mathbb{G}$ as in Theorem B.91, $[F^+ : \mathbb{Q}] \geq 2$, ι, λ dominant, π an irreducible constituent of $S_\lambda(\overline{\mathbb{Q}}_\ell)$). By its clause (Ug1) there are $r \geq 1$, integers $k_i, m_i \geq 1$ with $\sum_i k_i m_i = n$ and unitary cuspidal automorphic representations τ_i of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$ with $\tau_i^c \cong \tau_i^\vee$ such that, with $s_i = \frac{m_i-1}{2} - \frac{n-k_i}{2}$, each $\tau'_i = \tau_i \otimes |\det|^{s_i}$ is regular algebraic, cuspidal and polarizable, and, for $\Pi = \mathrm{Sp}(\tau_1, k_1)\boxplus\cdots\boxplus\mathrm{Sp}(\tau_r, k_r)$: (i) $\Pi_w \cong \iota\pi_v \circ \iota_w^{-1}$ at every finite place v of F^+ split as ww^c in F; (ii) $r_\pi \cong \bigoplus_{i=1}^r \bigoplus_{a=0}^{k_i-1} r_{\ell,\iota}(\tau'_i)\otimes\varepsilon^{-a}$.

Step 2: only one summand. Every summand in (ii) is non-zero (of dimension $m_i \geq 1$) and r_π is absolutely irreducible, so the double sum has exactly one term: $r = 1$ and $k_1 = 1$. Then $m_1 = n$, $s_1 = \frac{n-1}{2} - \frac{n-1}{2} = 0$, $\tau'_1 = \tau_1$, and $\Pi = \mathrm{Sp}(\tau_1, 1) = \tau_1$ (for $k = 1$ the parabolic subgroup of type (m_1) is the whole group, and the induction in (Mw1) is τ_1 itself). Hence Π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which is regular algebraic (as $\tau'_1 = \Pi$) and conjugate self-dual ($\Pi^c \cong \Pi^\vee$), and $r_\pi \cong \rho_1$ with $\rho_1 = r_{\ell,\iota}(\Pi)$.

Step 3: $r_\pi \cong r_\iota(\Pi)$. Let $w \nmid \ell$ be a place of F at which Π is unramified. By (Bg1) for $\Pi = \tau'_1$ ($m_1 = n$, $s_1 = 0$), ι carries the characteristic polynomial of $\rho_1(\mathrm{Frob}_w)$ to $\prod_{\alpha\in\alpha_w(\Pi)}(X - q_w^{(n-1)/2}\alpha)$. By (To1), $\mathrm{WD}(r_\iota(\Pi)|_{G_{F_w}})^{\text{F-ss}} \cong \mathrm{rec}(\iota^{-1}\Pi_w \otimes |\det|^{(1-n)/2})$, an unramified representation on which Frob_w has the eigenvalues $q_w^{(n-1)/2}\alpha$, $\alpha \in \alpha_w(\Pi)$ (after ι); Frobenius-

semisimplification keeps the characteristic polynomial, so $r_\iota(\Pi)(\mathrm{Frob}_w)$ has the same characteristic polynomial as $\rho_1(\mathrm{Frob}_w)$. These Frobenius elements are dense in G_F (Chebotarev), so the characteristic polynomials of $r_\iota(\Pi)$ and $\rho_1 \cong r_\pi$ agree on G_F by continuity, and by the Brauer–Nesbitt theorem the semisimplification of $r_\iota(\Pi)$ is isomorphic to r_π. As r_π is irreducible, $r_\iota(\Pi)$ has a single composition factor, so $r_\iota(\Pi) \cong r_\pi$.

Step 4: the weight. Let $\iota\mu$ be the weight of Π (it exists as Π is regular algebraic). For every embedding $\tau\colon F \to \overline{\mathbb{Q}}_\ell$, (To1) gives the τ-Hodge–Tate weights $\mu_{\tau,n-j+1} + j - 1$ $(1 \le j \le n)$ of $r_\iota(\Pi)$ and (Gy5)(iv) gives those of r_π as $\lambda_{\tau,n-j+1} + j - 1$, both with ε of weight -1; by Step 3 the two multisets agree, and both lists are strictly increasing in j, so $\mu_\tau = \lambda_\tau$ and Π has weight $\iota\lambda$.

Step 5: the local components. For v split as ww^c, Step 1(i) gives $\Pi_w \cong \iota\pi_v \circ \iota_w^{-1}$, i.e. $\pi_v \cong \iota^{-1}\Pi_w \circ \iota_w$. □

Lemma E.8. *Let ℓ, $n \ge 2$, F, F^+, $\mathcal{O}$, $\mathbb{G}$ be as in Theorem B.91, with $[F^+ : \mathbb{Q}] \ge 2$; let $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an isomorphism, λ a dominant weight for $\mathbb{G}$ and π an irreducible constituent of $S_\lambda(\overline{\mathbb{Q}}_\ell)$.*

(Ug1) *There are $r \ge 1$, integers $k_i, m_i \ge 1$ $(1 \le i \le r)$ with $\sum_i k_i m_i = n$, and unitary cuspidal automorphic representations τ_i of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$ with $\tau_i^c \cong \tau_i^\vee$, such that, with $s_i = \frac{m_i-1}{2} - \frac{n-k_i}{2}$, each $\tau_i' = \tau_i \otimes |\det|^{s_i}$ is regular algebraic, cuspidal and polarizable, and, with $\Pi = \mathrm{Sp}(\tau_1, k_1) \boxplus \cdots \boxplus \mathrm{Sp}(\tau_r, k_r)$:*

1. *$\Pi_w \cong \iota\pi_v \circ \iota_w^{-1}$ for every finite place v of F^+ which splits as ww^c in F;*
2. *$r_\pi \cong \bigoplus_{i=1}^r \bigoplus_{a=0}^{k_i-1} r_{\ell,\iota}(\tau_i') \otimes \varepsilon^{-a}$.*

(Ug2) *For every finite place $v \nmid \ell$ of F^+ which splits as ww^c in F,*

$$\iota\,\mathrm{WD}\big(r_\pi|_{G_{F_w}}\big)^{F\text{-}ss} \cong \mathrm{rec}_{F_w}\big(\iota\pi_v \circ \iota_w^{-1} \otimes |\det|^{(1-n)/2}\big).$$

(Ug3) *In particular, for such v and w: if $\mathrm{rec}_{F_w}(\iota\pi_v \circ \iota_w^{-1})$ has monodromy 0, then $\mathrm{WD}(r_\pi|_{G_{F_w}})$ has monodromy 0.*

Proof. Step 1: the automorphic representation of $\mathbb{G}$. By [105], Lemma 2.2.5 (with no places carrying a character), $S_\lambda(\overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_{\ell,\iota}} \mathbb{C} \cong \mathrm{Hom}_{\mathbb{G}(F^+\otimes\mathbb{R})}(\xi_{\lambda,\iota}^\vee, \mathcal{A})$ as $\mathbb{G}(\mathbb{A}_{F^+}^\infty)$-modules, where $\mathcal{A}$ is the space of automorphic forms on $\mathbb{G}(F^+)\backslash\mathbb{G}(\mathbb{A}_{F^+})$ and $\xi_{\lambda,\iota}$ the irreducible representation of the compact group $\mathbb{G}(F^+ \otimes \mathbb{R})$ defined by λ through ι. Hence $\sigma = \xi_{\lambda,\iota}^\vee \otimes \iota\pi$ is an irreducible automorphic representation of $\mathbb{G}(\mathbb{A}_{F^+})$ with $\sigma_v \cong \iota\pi_v$ at the finite places. As $\mathbb{G}(F^+)\backslash\mathbb{G}(\mathbb{A}_{F^+})$ is compact, σ lies in the discrete spectrum; σ_∞, an irreducible representation of a compact group, is a discrete series representation; and Labesse's property $(*)$ holds for σ because $\mathbb{G}(F^+ \otimes \mathbb{R})$ is compact (the last sentence of Theorem B.96; [153], the remark before Corollaire 5.3).

Step 2: weak base change. $\mathbb{G}$ is a unitary group in n variables for F/F^+, an inner form of the quasi-split one, F^+ is totally real of degree ≥ 2 and F is totally imaginary. By Theorem B.96 ([153], Corollaire 5.3; its F, E, U are our F^+, F, $\mathbb{G}$, the infinity component is a discrete series as $\mathbb{G}(F^+ \otimes \mathbb{R})$ is compact, and property $(*)$ holds for the same reason), there are a partition $n = n_1 + \cdots + n_r$ and automorphic representations $\Pi_1, \ldots, \Pi_r$ of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ in the discrete spectrum which are conjugate self-dual, $\Pi_i^\vee \cong \Pi_i^c$, such that the isobaric sum $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$ is a weak base change of σ and: (1) at every finite place v of F^+ which splits as ww^c in F, $\Pi_w \cong \sigma_v \circ \iota_w^{-1}$; (2) Π_∞ is cohomological for an algebraic coefficient system W, so by Wigner's

lemma Π_∞ has the infinitesimal character of $W^\vee$, and at every complex place of F the exponents (p_j), (q_j) of Π are each n pairwise distinct elements of $\frac{n-1}{2}+\mathbb{Z}$.

Step 3: the discrete summands. By (Mw1), $\Pi_i \cong \mathrm{Sp}(\tau_i, k_i)$ for some $k_i \geq 1$ with $m_i = n_i/k_i \in \mathbb{Z}$ and a unitary cuspidal automorphic representation τ_i of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$; so Π is the isobaric sum of (Ug1), and (Ug1)(i) is Step 2(1) with $\sigma_v \cong \iota\pi_v$. At every finite place w at which τ_i is unramified, $\Pi_{i,w}$ is unramified with

$$\alpha_w(\Pi_i) = \bigsqcup_{a=0}^{k_i-1} q_w^{a-(k_i-1)/2}\,\alpha_w(\tau_i), \tag{E.1}$$

(by (Mw2) for $r=1$, $\mathrm{rec}(\Pi_{i,w}) = \bigoplus_a \mathrm{rec}(\tau_{i,w}) \otimes \mathrm{rec}(|\cdot|^{(k_i-1)/2-a})$ is a sum of unramified characters with monodromy 0; by (Ln6) and (Ln1) of Theorem B.92 and the bijectivity of rec, the representation with this parameter is the unramified Langlands quotient of the corresponding unramified characters, whose Satake parameters are their values at a uniformiser), and at every complex place the exponents of Π_i are those of τ_i shifted by $\frac{k_i-1}{2}-a$, $0 \leq a \leq k_i-1$ (both p and q, since $|z|_{\mathbb{C}} = z\bar{z}$). The exponents and Satake parameters of the isobaric sum Π are the unions of those of the Π_i.

Step 4: the twisted cuspidal pieces have Galois representations. (i) $\tau_i^c \cong \tau_i^\vee$: at almost all w, $\alpha_w(\Pi_i^\vee) = \alpha_w(\Pi_i)^{-1}$ and $\alpha_w(\Pi_i^c) = \alpha_{w^c}(\Pi_i)$, so by (E.1), with $t_a = a - \frac{k_i-1}{2}$ running over a set symmetric about 0, the families $(\tau_i^\vee|\det|^{t_a})_a$ and $(\tau_i^c|\det|^{t_a})_a$ have the same unions of Satake parameters at almost all places. By Theorem B.97 there is a bijection s with $\tau_i^\vee|\det|^{t_a} \cong \tau_i^c|\det|^{t_{s(a)}}$; comparing absolute values of the (unitary) central characters gives $t_a = t_{s(a)}$, so $\tau_i^\vee \cong \tau_i^c$. (ii) By Steps 2(2) and 3, the numbers $p + \frac{k_i-1}{2} - a$, for p an exponent of τ_i at a complex place and $0 \leq a < k_i$, lie in $\frac{n-1}{2}+\mathbb{Z}$ and are pairwise distinct; so the exponents of τ_i lie in $\frac{n-k_i}{2}+\mathbb{Z}$ and are pairwise distinct, and those of τ_i', shifted by s_i, are pairwise distinct elements of $\frac{m_i-1}{2}+\mathbb{Z}$ (likewise for q): τ_i' is regular algebraic and cuspidal. (iii) $\tau_i'^c \cong \tau_i^\vee|\det|^{s_i} = \tau_i'^\vee \otimes (\chi_i \circ \mathrm{N}_{F/F^+} \circ \det)$ with $\chi_i = |\cdot|_{\mathbb{A}_{F^+}}^{2s_i}$, trivial on -1 at every infinite place; so τ_i' is polarizable. Let $\rho_i = r_{\ell,\iota}(\tau_i')\colon G_F \to \mathrm{GL}_{m_i}(\overline{\mathbb{Q}}_\ell)$ (Theorem B.62); by (Bg1), at every $w \nmid \ell$ where τ_i is unramified, ι carries the characteristic polynomial of $\rho_i(\mathrm{Frob}_w)$ to $\prod_{\beta\in\alpha_w(\tau_i)}(X - q_w^{(m_i-1)/2-s_i}\beta)$.

Step 5: (Ug1)(ii). Let Σ be the set of places $w \nmid \ell$ of F above places v of F^+ split in F such that π_v is unramified and all $\tau_{i,w}$ are unramified; it contains all but finitely many places of F of degree one over F^+. For $w \in \Sigma$, (Ug1)(i) and (Gy5)(i) show that the characteristic polynomial of $r_\pi(\mathrm{Frob}_w)$ is carried by ι to $\prod_{\alpha\in\alpha_w(\Pi)}(X - q_w^{(n-1)/2}\alpha)$.

By (E.1), $q_w^{(n-1)/2}\alpha_w(\Pi) = \bigsqcup_i \bigsqcup_{a=0}^{k_i-1} q_w^{(n-1)/2+a-(k_i-1)/2}\alpha_w(\tau_i)$, and $\frac{n-1}{2} + a - \frac{k_i-1}{2} = \frac{m_i-1}{2} - s_i + a$. Since $\varepsilon^{-a}(\mathrm{Frob}_w) = q_w^a$, the semisimple representations r_π and $\rho = \bigoplus_i \bigoplus_{a=0}^{k_i-1} \rho_i \otimes \varepsilon^{-a}$ have the same characteristic polynomials at the Frobenius elements of all $w \in \Sigma$. These are dense in G_F (Chebotarev; the places of degree two over F^+ have density 0), so the characteristic polynomials agree on G_F by continuity, and $r_\pi \cong \rho$ by the Brauer–Nesbitt theorem.

Step 6: (Ug2). Let $v \nmid \ell$ split as ww^c and $K = F_w$, with W_K its Weil group. Isomorphic representations have isomorphic Weil–Deligne representations, and WD and Frobenius-semisimplification commute with direct sums and with twisting by a character of G_K trivial on inertia. As $w \nmid \ell$, $\varepsilon|_{G_K}$ is unramified with $\varepsilon(\mathrm{Art}_K(\varpi)) = q_w^{-1} = |\varpi|$ for a uniformiser ϖ ($\mathrm{Art}_K(\varpi)$ a geometric Frobenius), so $\varepsilon|_{W_K} = \mathrm{rec}_K(|\cdot|)$ by (Ln1). By (Ug1)(ii), (Bg7) applied

to τ'_i (of rank m_i) and (Ln2),

$$\iota\,\mathrm{WD}(r_\pi|_{G_K})^{F\text{-}ss} \cong \bigoplus_{i=1}^{r}\bigoplus_{a=0}^{k_i-1} \mathrm{rec}_K\big(\tau'_{i,w}\otimes|\det|^{(1-m_i)/2}\big)\otimes\mathrm{rec}_K(|\cdot|^{-a})$$
$$\cong \bigoplus_{i=1}^{r}\bigoplus_{a=0}^{k_i-1} \mathrm{rec}_K(\tau_{i,w})\otimes\mathrm{rec}_K\big(|\cdot|^{s_i+(1-m_i)/2-a}\big).$$

So the right side is $\big(\bigoplus_{i,a}\mathrm{rec}_K(\tau_{i,w})\otimes\mathrm{rec}_K(|\cdot|^{(k_i-1)/2-a})\big)\otimes\mathrm{rec}_K(|\cdot|^{(1-n)/2})$, which by (B.1) of (Mw2) is $\mathrm{rec}_K(\Pi_w)\otimes\mathrm{rec}_K(|\cdot|^{(1-n)/2}) = \mathrm{rec}_K(\Pi_w\otimes|\det|^{(1-n)/2})$ (Ln2), and $\Pi_w\cong\iota\pi_v\circ\iota_w^{-1}$ by (Ug1)(i).

Step 7: (Ug3). Frobenius-semisimplification does not change the monodromy operator, and by (Ln2) $\mathrm{rec}_K(\sigma'\otimes|\det|^{(1-n)/2}) = \mathrm{rec}_K(\sigma')\otimes\mathrm{rec}_K(|\cdot|^{(1-n)/2})$ has the monodromy operator of $\mathrm{rec}_K(\sigma')$. So by (Ug2) the monodromy of $\mathrm{WD}(r_\pi|_{G_K})$ is conjugate (after ι) to that of $\mathrm{rec}_K(\iota\pi_v\circ\iota_w^{-1})$, which is 0 by hypothesis. □

Proposition E.9. *Let ℓ, $n\geq 2$, F, F^+, $\mathcal{O}$, $\mathbb{G}$ be as in Theorem B.91, with F/F^+ unramified at all finite places, $[F^+:\mathbb{Q}]\geq 2$, and $\mathbb{G}(\mathcal{O}_{F_v^+})$ a hyperspecial maximal compact subgroup of $\mathbb{G}(F_v^+)$ for every finite place v of F^+ inert in F; let $\iota\colon\overline{\mathbb{Q}}_\ell\to\mathbb{C}$ be an isomorphism.*

(Bc2) *For every regular algebraic conjugate self-dual cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$, the irreducible constituent π of (Gy4) can be chosen such that, moreover, $\pi_v^{\mathbb{G}(\mathcal{O}_{F_v^+})}\neq 0$ for every place v of F^+ inert in F at which Π is unramified.*

Proof. Let Π be a regular algebraic conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and let $\mathcal{V}$ be the set of finite places v of F^+ inert in F at which Π is unramified; for $v\in\mathcal{V}$ let w be the place of F above v and $K_v=\mathbb{G}(\mathcal{O}_{F_v^+})$, hyperspecial by hypothesis. We follow the proof of [153], Théorème 5.4, and [105], Proposition 2.2.7, which deduces (Gy4) from it, keeping track of the test function at the places of $\mathcal{V}$.

Step 1: the setting of Labesse, §5. F^+ is totally real of degree ≥ 2, F/F^+ is totally imaginary and unramified at all finite places, and $U=\mathbb{G}_{F^+}$ is an inner form of the quasi-split unitary group U^*, quasi-split at every finite place (hypothesis of Theorem B.91), with $U(F^+\otimes\mathbb{R})$ compact. So the standing hypotheses of [153], §5.2, hold, and property $(*)$ holds for every discrete automorphic σ of U (remark before Corollaire 5.3: U_∞ compact). Π is cuspidal, θ-invariant ($\Pi^c\cong\Pi^\vee$) and cohomological at infinity (regular algebraic).

Step 2: the refined identity for associated pairs. Let $f=\bigotimes_v f_v\in C_c^\infty(U(\mathbb{A}_{F^+}))$ with f_∞ a pseudo-coefficient of the discrete series σ_∞ of the proof of Théorème 5.4 (for compact U_∞: a multiple of the conjugate of the character of an irreducible representation σ_∞, so $\mathrm{tr}\,\sigma'_\infty(f_\infty)\neq 0$ if and only if $\sigma'_\infty\cong\sigma_\infty$), and let $\varphi=\bigotimes_v\varphi_v$ be a function on $\tilde{G}(\mathbb{A}_{F^+})$ such that f_v and φ_v are associated (stable base change transfer) at every place, φ_∞ the Lefschetz function of that proof. Théorème 5.1 of loc. cit. (whose proof uses of $\tilde{f}^{U^*}=\varphi$ only that it is a stable base change transfer of f: §4.8, formula 4.12) and the separation of Hecke eigensystems in the proof of Théorème 5.4, where the classification of Jacquet–Shalika and Proposition 4.9 show that for the cuspidal Π no endoscopic group $H\neq U^*$ contributes, give the refined identity

$$\sum_{\sigma'} m(\sigma')\,\mathrm{tr}\,\sigma'(f)=\mathrm{tr}\,\tilde{\Pi}(\varphi),$$

the sum over the discrete automorphic representations σ' of U whose Hecke eigenvalues agree with those determined by Π at almost all places, $m(\sigma')\geq 1$ their multiplicities. Writing the

global intertwining operator of $\tilde{\Pi}$ as a non-zero multiple of the tensor product of local ones, normalised at the places where Π is unramified to fix the vectors invariant under the hyperspecial subgroup used ([153], before Lemme 4.8), $\operatorname{tr}\tilde{\Pi}(\varphi) = C\prod_v \operatorname{tr}\tilde{\Pi}_v(\varphi_v)$ with $C \neq 0$.

Step 3: the unit at the places of $\mathcal{V}$. Let $v \in \mathcal{V}$. U_v is unramified (quasi-split and split by the unramified extension F_w), K_v is the stabiliser of a hyperspecial point x of its building, and the stabiliser K'_w of x in $U(F_w) \cong \mathrm{GL}_n(F_w)$ is a θ-stable hyperspecial maximal compact subgroup. By Kottwitz's base change fundamental lemma for unit elements [149] (unramified group, unramified extension), 1_{K_v} and $\varphi_v^\circ = 1_{K'_w} \rtimes \theta$ are associated. Π_w is unramified; K'_w is $\mathrm{GL}_n(F_w)$-conjugate to $\mathrm{GL}_n(\mathcal{O}_{F_w})$, so $\Pi_w^{K'_w}$ is a line, preserved by the normalised intertwining operator A_w (as $\theta(K'_w) = K'_w$), on which A_w acts by a sign ($A_w^2 = 1$). Hence $\operatorname{tr}\tilde{\Pi}_w(\varphi_v^\circ) = \pm\operatorname{vol}(K'_w) \neq 0$.

Step 4: a test function with non-zero right side. By the proof of Théorème 5.4 the right side of Step 2 is not identically zero: there are associated f^0, φ^0 as in Step 2 with $\prod_v \operatorname{tr}\tilde{\Pi}_v(\varphi_v^0) \neq 0$. As $f^0 \in C_c^\infty(U(\mathbb{A}_{F^+}))$, $f_v^0 = 1_{K_v}$ for all but finitely many $v \in \mathcal{V}$; and we may take $\varphi_v^0 = \varphi_v^\circ$ there (Step 3). Let f, φ be obtained by replacing f_v^0, φ_v^0 by $1_{K_v}, \varphi_v^\circ$ at the finitely many remaining $v \in \mathcal{V}$; they are associated, and by Step 3 the product of local traces stays non-zero. So $\sum_{\sigma'} m(\sigma')\operatorname{tr}\sigma'(f) \neq 0$, and some σ' in the sum has $\operatorname{tr}\sigma'_\infty(f_\infty) \neq 0$, i.e. $\sigma'_\infty \cong \sigma_\infty$, and $\operatorname{tr}\sigma'_v(1_{K_v}) = \operatorname{vol}(K_v)\dim\sigma_v'^{K_v} \neq 0$ for every $v \in \mathcal{V}$.

Step 5: σ' has the properties of (Gy4). σ' satisfies $(*)$ (Step 1), so by [153], Corollaire 5.3, it has a weak base change $\pi' = \pi_1 \boxplus \cdots \boxplus \pi_r$, the correspondence $\sigma'_v \mapsto \pi'_v$ being local base change at the split places, compatible with infinitesimal characters at the archimedean places. As σ' has the Hecke eigenvalues of Π at almost all places, π' and Π agree at almost all places, so $\pi' = \Pi$ by the classification of Jacquet–Shalika (Π cuspidal). Hence, as in the deduction of [105], Proposition 2.2.7, from Labesse's Corollaire 5.3: $\sigma'_v \cong \Pi_w \circ \iota_w$ at every finite v split as ww^c, and σ'_∞ is the irreducible algebraic representation $\xi_\lambda^\vee$ with the infinitesimal character of Π_∞ (for the compact U_∞ an irreducible representation is determined by its infinitesimal character), for the dominant weight λ of (Gy4). By Lemma 2.2.5 of [105], $\iota^{-1}\sigma'_f$ is an irreducible constituent π of $S_\lambda(\overline{\mathbb{Q}}_\ell)$; it has the properties of (Gy4), and $\pi_v^{\mathbb{G}(\mathcal{O}_{F_v^+})} \neq 0$ for every $v \in \mathcal{V}$ by Step 4. This is (Bc2). □

Lemma E.10. *Let the data be as in Theorem B.91. For $v \in S_\ell$ let e_v be the ramification index of $F_v^+/\mathbb{Q}_\ell$, and $e_{\max} = \max_{v\in S_\ell} e_v$.*

(Qh1) *For every $b \geq 1$, $T_n(\mathfrak{l}^{be_{\max}}) \subseteq T_n(\ell^b) \subseteq T_n(\mathfrak{l}^b)$; the index $[T_n(\mathfrak{l}^b) : T_n(\ell^b)]$ is a power of ℓ, equal to 1 for every b if ℓ is unramified in F^+; and $\Lambda_b^{\mathfrak{l}}$ is a free Λ_b-module with basis any set of representatives of $T_n(\mathfrak{l}^b)/T_n(\ell^b)$.*

Suppose now that $b_0 \geq 1$ and $U(\ell^{b_0,b_0})$ is sufficiently small.

(Qh2) *$\mathcal{S}$ is a free Λ_{b_0}-module of finite rank, $\mathbb{T}^{\mathrm{ord}}(U)$ is commutative and finite as a Λ_{b_0}-module, and, if $\mathcal{S} \neq 0$, $\Lambda_{b_0} \to \mathbb{T}^{\mathrm{ord}}(U)$ is injective.*

(Qh3) *Let λ be a dominant weight, $\alpha\colon T_n(\ell^1) \to \overline{\mathbb{Q}}_\ell^\times$ a character of finite order and $r \geq 1$ with $T_n(\mathfrak{l}^r) \subseteq \ker\alpha$ (so $T_n(\mathfrak{l}^r) \subseteq T_n(\ell^1)$; if $T_n(\ell^{r_0}) \subseteq \ker\alpha$, then $r = e_{\max}r_0$ will do). The characters $\alpha'\colon T_n(\mathfrak{l}) \to \overline{\mathbb{Q}}_\ell^\times$ with $\alpha'|_{T_n(\ell^1)} = \alpha$ are $[T_n(\mathfrak{l}) : T_n(\ell^1)]$ in number, of finite order and trivial on $T_n(\mathfrak{l}^r)$, and the primes $\mathfrak{P}_{\lambda,\alpha'}^{\mathfrak{l}}$ of $\Lambda_1^{\mathfrak{l}}$ lie over $\mathfrak{P} = \mathfrak{P}_{\lambda,\alpha}$. Let $\mathcal{K}'$ be the fraction field of $\Lambda_1/\mathfrak{P}$. Then the action on $S_\lambda^{\mathrm{ord}}(U(\ell^{r,r}), \alpha, \mathcal{K}')$ induces a surjection of finite $\mathcal{K}'$-algebras*

$$\mathbb{T}^{\mathrm{ord}}(U) \otimes_{\Lambda_1} (\Lambda_1)_{\mathfrak{P}}/\mathfrak{P} \longrightarrow \mathbb{T}_\lambda^{\mathrm{ord}}(U(\ell^{r,r}), \alpha, \mathcal{K}'),$$

and its kernel is nilpotent.

(Qh4) *In (Qh3), suppose moreover that $\mathcal{O}$ contains the values of every character α' of $T_n(\mathfrak{l})$ with $\alpha'|_{T_n(\ell^1)} = \alpha$ (if ℓ is unramified in F^+: the values of α). Then $\Lambda_1/\mathfrak{P} = \mathcal{O}$, $\mathcal{K}' = \mathcal{K}$, and there is an isomorphism of $\mathcal{K}$-vector spaces*

$$\mathcal{S}_{\mathfrak{P}}/\mathfrak{P}\mathcal{S}_{\mathfrak{P}} \cong \mathrm{Hom}_{\mathcal{K}}(S_\lambda^{\mathrm{ord}}(U(\ell^{r,r}), \alpha, \mathcal{K}), \mathcal{K}) \qquad (\mathcal{S}_{\mathfrak{P}} = \mathcal{S} \otimes_{\Lambda_1} (\Lambda_1)_{\mathfrak{P}})$$

which carries the action of $T_w^{(j)}$, $U_{\tilde{v}}^{(j)}$, $\langle u \rangle$ on the left to the transposes of the operators assigned to them by (Gy7)(c) on the right, and which is equivariant for right translation by every element g as in (Gk3).

Proof. Write $G = T_n(\mathfrak{l})$, $H = T_n(\ell^1)$, $N_r = T_n(\mathfrak{l}^r)$, $\mathbb{T} = \mathbb{T}^{\mathrm{ord}}(U)$, and for $v \in S_\ell$ let ϖ_v be a uniformiser of $\mathcal{O}_v = \mathcal{O}_{F_v^+}$, so that $\ell\mathcal{O}_v = \varpi_v^{e_v}\mathcal{O}_v$.

Step 1: (Qh1). $T_n(\mathcal{O}_{F^+} \otimes \mathbb{Z}_\ell) = \prod_{v\in S_\ell} T_n(\mathcal{O}_v)$, and by the Chinese remainder theorem $\mathcal{O}_{F^+}/\mathfrak{l}^b = \prod_v \mathcal{O}_v/\varpi_v^b$ and $\mathcal{O}_{F^+}/\ell^b = \prod_v \mathcal{O}_v/\varpi_v^{be_v}$. Hence

$$T_n(\mathfrak{l}^b) = \prod_{v\in S_\ell} (1 + \varpi_v^b\mathcal{O}_v)^n, \qquad T_n(\ell^b) = \prod_{v\in S_\ell} (1 + \varpi_v^{be_v}\mathcal{O}_v)^n,$$

and $b \le be_v \le be_{\max}$ gives the inclusions; if all $e_v = 1$ the two groups coincide. The quotient is finite and filtered by the groups $(1 + \varpi_v^i\mathcal{O}_v)/(1 + \varpi_v^{i+1}\mathcal{O}_v) \cong \mathcal{O}_v/\varpi_v$ $(i \ge 1)$, of characteristic ℓ, so its order is a power of ℓ. Let $G_b = T_n(\mathfrak{l}^b)$, $H_b = T_n(\ell^b)$ and R a set of representatives of G_b/H_b. The open subgroups $N \subseteq H_b$ are cofinal among the open subgroups of G_b and normal (G_b is abelian); for each, $\mathcal{O}[G_b/N]$ is a free $\mathcal{O}[H_b/N]$-module with basis the image of R, compatibly with the transition maps. Finite direct sums commute with inverse limits, so $\Lambda_b^{\mathfrak{l}} = \varprojlim_N \mathcal{O}[G_b/N] = \bigoplus_{g\in R} g\,\Lambda_b$.

Step 2: (Qh2). By (Gy1), $\mathcal{S} \cong (\Lambda_{b_0}^{\mathfrak{l}})^m$, which is free of rank $m[G_{b_0} : H_{b_0}]$ over Λ_{b_0} by (Qh1). By (Gy2), $\mathbb{T}$ is commutative and finite over $\Lambda_{b_0}^{\mathfrak{l}}$, which is finite over Λ_{b_0} by (Qh1), and, if $\mathcal{S} \neq 0$, $\Lambda_{b_0}^{\mathfrak{l}} \to \mathbb{T}$ is injective; its restriction to the subring Λ_{b_0} (a direct summand, taking $1 \in R$) is the structure map of Λ_{b_0}, hence injective. In particular $\mathbb{T}$ is finite over $\Lambda_1 \supseteq \Lambda_{b_0}$.

Step 3: the extensions of α. The characters $w_0\nu$ and $w_0\lambda$ are defined on all of $T_n(\mathcal{O}_{F^+} \otimes \mathbb{Z}_\ell)$ with values in $\mathcal{O}^\times$. Put $\chi = \alpha\,(w_0\nu)^{-1}(w_0\lambda)^{-1}$ on H and let $\psi\colon \Lambda_1 \to \overline{\mathbb{Q}}_\ell$ be the induced homomorphism, with kernel $\mathfrak{P}$; so $\mathcal{K}' = \mathrm{Frac}\,\psi(\Lambda_1) \subseteq \overline{\mathbb{Q}}_\ell$, and $\mathcal{K}'$ contains the values of α. Let X be the set of characters $\alpha'\colon G \to \overline{\mathbb{Q}}_\ell^\times$ with $\alpha'|_H = \alpha$. Each $\alpha' \in X$ is trivial on $N_r \subseteq \ker\alpha$, so it factors through the finite group G/N_r and has finite order. As $\overline{\mathbb{Q}}_\ell^\times$ is divisible, α (a character of H/N_r) extends to G/N_r, and the extensions form a principal homogeneous set under the character group of G/H, which has $[G : H]$ elements (G/H finite abelian, $\overline{\mathbb{Q}}_\ell$ algebraically closed of characteristic 0); so $|X| = [G : H]$. For $\alpha' \in X$ let $\chi_{\alpha'} = \alpha'(w_0\nu)^{-1}(w_0\lambda)^{-1}$ on G and $\psi_{\alpha'}\colon \Lambda_1^{\mathfrak{l}} \to \overline{\mathbb{Q}}_\ell$ the induced homomorphism; its kernel is $\mathfrak{P}_{\lambda,\alpha'}^{\mathfrak{l}}$ and $\psi_{\alpha'}|_{\Lambda_1} = \psi$, so $\mathfrak{P}_{\lambda,\alpha'}^{\mathfrak{l}}$ lies over $\mathfrak{P}$.

Step 4: the fibre of $\Lambda_1^{\mathfrak{l}}$ over $\mathfrak{P}$ splits. Let $L \subseteq \overline{\mathbb{Q}}_\ell$ be a finite extension of $\mathcal{K}'$ containing the values of every $\alpha' \in X$ (finitely many roots of unity), a Λ_1-algebra through ψ. Let R be a set of representatives of G/H. By (Qh1) ($b = 1$), $\psi_{\alpha'}(\Lambda_1^{\mathfrak{l}}) = \sum_{g\in R} \chi_{\alpha'}(g)\psi(\Lambda_1) \subseteq L$, and $\Lambda_1^{\mathfrak{l}} \otimes_{\Lambda_1} L$ has L-basis R. The homomorphism of L-algebras

$$\Phi\colon \Lambda_1^{\mathfrak{l}} \otimes_{\Lambda_1} L \to L^X, \qquad x \otimes c \mapsto \big(c\,\psi_{\alpha'}(x)\big)_{\alpha'\in X},$$

is an isomorphism: fixing $\alpha'_0 \in X$, every $\alpha' \in X$ is $\alpha'_0\theta$ with θ a character of G/H, so the matrix of Φ in the basis R is $\big(\theta(g)\big)_{\theta,g} \cdot \mathrm{diag}\big(\chi_{\alpha'_0}(g)\big)_{g\in R}$, a product of the character table of

G/H (invertible by the orthogonality relations) and an invertible diagonal matrix, and both sides have dimension $|X| = [G : H]$. Consequently

$$\mathbb{T} \otimes_{\Lambda_1} L \cong \mathbb{T} \otimes_{\Lambda_1^\mathfrak{t}} (\Lambda_1^\mathfrak{t} \otimes_{\Lambda_1} L) \cong \prod_{\alpha' \in X} \mathbb{T} \otimes_{\Lambda_1^\mathfrak{t}, \psi_{\alpha'}} L, \qquad (*)$$

the factor of α' being the quotient on which $\Lambda_1^\mathfrak{t}$ acts through $\psi_{\alpha'}$; and $\mathbb{T} \otimes_{\Lambda_1^\mathfrak{t}, \psi_{\alpha'}} L = (\mathbb{T} \otimes_{\Lambda_1^\mathfrak{t}} \mathcal{K}'_{\alpha'}) \otimes_{\mathcal{K}'_{\alpha'}} L$, where $\mathcal{K}'_{\alpha'} = \operatorname{Frac} \psi_{\alpha'}(\Lambda_1^\mathfrak{t}) = (\Lambda_1^\mathfrak{t})_{\mathfrak{P}'}/\mathfrak{P}'$, $\mathfrak{P}' = \mathfrak{P}^\mathfrak{t}_{\lambda,\alpha'}$.

Step 5: the forms split in the same way. Let $W = S^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \alpha, L)$. For $u \in N_r$ the components of u at S_ℓ are diagonal and $\equiv 1$ modulo ϖ_v^r, so $u \in U(\ell^{r,r})$ and the operator $\langle u \rangle = [U(\ell^{r,r}) u U(\ell^{r,r})]$ of weight λ is $f \mapsto u \cdot f = f$ (the invariance defining $S_\lambda(U(\ell^{r,r}), L)$ in (Gk4)). The operators $\langle u \rangle$ commute with all operators of weight λ ([105], Lemma 2.3.3), so they preserve W, on which H acts through α. Thus W is a module over $A = L[G/N_r] \otimes_{L[H/N_r],\alpha} L$, and $A \cong L^X$ by $x \otimes c \mapsto (c\,\alpha'(x))_{\alpha'}$ (the argument of Step 4, with α' in place of $\chi_{\alpha'}$). Its primitive idempotents $e_{\alpha'}$ are L-linear combinations of the $\langle u \rangle$, and

$$W = \bigoplus_{\alpha' \in X} e_{\alpha'} W, \qquad e_{\alpha'} W = S^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \alpha', L).$$

As the $e_{\alpha'}$ lie in the commutative algebra $\mathbb{T}^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \alpha, L)$, restriction to the summands gives $\mathbb{T}^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \alpha, L) = \prod_{\alpha' \in X} \mathbb{T}^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \alpha', L)$. By (Gy7)(c), $u \in G$, as an element of $\Lambda_1^\mathfrak{t}$ with the renormalised structure, acts on $S^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \alpha', L)$ by $(w_0\nu)^{-1}(u)(w_0\lambda)(u)^{-1}\alpha'(u) = \chi_{\alpha'}(u)$, i.e. $\Lambda_1^\mathfrak{t}$ acts through $\psi_{\alpha'}$; so the decomposition $(*)$ and this one correspond to each other.

Step 6: (Qh3) after extending scalars to L. For a field $M \supseteq \mathcal{O}$ containing the values of a character β of G or of H and an extension $L \supseteq M$, $S^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \beta, L) = S^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \beta, M) \otimes_M L$ (the forms are functions on a finite set with values in $M_\lambda \otimes M$, and the ordinary part and the β-part are cut out by operators defined over M), and $\mathbb{T}^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \beta, L) = \mathbb{T}^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \beta, M) \otimes_M L$ (the L-span of the products of the operators). For $\alpha' \in X$, (Gy3) applies (α' of finite order, $N_r \subseteq \ker \alpha'$): $\mathbb{T} \otimes_{\Lambda_1^\mathfrak{t}} \mathcal{K}'_{\alpha'} \to \mathbb{T}^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \alpha', \mathcal{K}'_{\alpha'})$ is surjective with nilpotent kernel. Extending scalars to L is exact and keeps a nilpotent ideal nilpotent, so $\mathbb{T} \otimes_{\Lambda_1^\mathfrak{t}, \psi_{\alpha'}} L \to \mathbb{T}^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \alpha', L)$ is surjective with nilpotent kernel. By $(*)$ and Step 5, the map $s_L \colon \mathbb{T} \otimes_{\Lambda_1} L \to \mathbb{T}^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \alpha, L)$ induced by the action is the product of these maps over X, hence surjective with nilpotent kernel.

Step 7: (Qh3). $(\Lambda_1)_\mathfrak{P}/\mathfrak{P} = \mathcal{K}'$, and Λ_1 acts on $S^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \alpha, \mathcal{K}')$ through ψ (by (Gy7)(c), $u \in H$ acts by $\chi(u)$), so the action induces $s \colon \mathbb{T} \otimes_{\Lambda_1} \mathcal{K}' \to \mathbb{T}^{\mathrm{ord}}_\lambda(U(\ell^{r,r}), \alpha, \mathcal{K}')$, and $s \otimes_{\mathcal{K}'} L = s_L$ by Step 6. Both sides are finite over $\mathcal{K}'$ (Step 2). As L is faithfully flat over $\mathcal{K}'$, s is surjective; $\mathbb{T} \otimes_{\Lambda_1} \mathcal{K}' \to \mathbb{T} \otimes_{\Lambda_1} L$ is injective and $\ker(s) \otimes L = \ker(s_L)$ is nilpotent, so every element of $\ker s$ is nilpotent; $\ker s$ is a finitely generated ideal of a commutative ring, hence nilpotent.

Step 8: (Qh4). By hypothesis every $\alpha' \in X$, hence also $\alpha = \alpha'|_H$, and so χ and every $\chi_{\alpha'}$, take values in $\mathcal{O}$. The image $\psi(\Lambda_1)$ is compact, contains $\mathcal{O}$ and lies in the closure of the $\mathcal{O}$-span of the values of χ; so $\psi(\Lambda_1) = \mathcal{O} = \psi_{\alpha'}(\Lambda_1^\mathfrak{t})$, $\Lambda_1/\mathfrak{P} = \mathcal{O}$, $\mathcal{K}' = \mathcal{K} = \mathcal{K}'_{\alpha'}$, and Steps 4 and 5 apply with $L = \mathcal{K}$. The isomorphism Φ (with $L = \mathcal{K}$) gives

$$\mathcal{S}_\mathfrak{P}/\mathfrak{P}\mathcal{S}_\mathfrak{P} = \mathcal{S} \otimes_{\Lambda_1} \mathcal{K} \cong \bigoplus_{\alpha' \in X} \mathcal{S} \otimes_{\Lambda_1^\mathfrak{t}, \psi_{\alpha'}} \mathcal{K} = \bigoplus_{\alpha' \in X} \mathcal{S}_{\mathfrak{P}'}/\mathfrak{P}'\mathcal{S}_{\mathfrak{P}'} \qquad (\mathfrak{P}' = \mathfrak{P}^\mathfrak{t}_{\lambda,\alpha'}),$$

a decomposition into summands stable under the $T_w^{(j)}$, $U_{\tilde{v}}^{(j)}$, $\langle u \rangle$ and the right translations g of (Gk3), all of which commute with the action of $\Lambda_1^\mathfrak{t}$ (for g: the diamond operators are right translations by elements supported at S_ℓ, where g is trivial). By (Gk3) for each α' (values in

$\mathcal{O}$, $N_r \subseteq \ker\alpha'$), the summand of α' is isomorphic to $\mathrm{Hom}_{\mathcal{K}}(S_\lambda^{\mathrm{ord}}(U(\ell^{r,r}),\alpha',\mathcal{K}),\mathcal{K})$, compatibly with these operators as stated there. Summing over X and using $S_\lambda^{\mathrm{ord}}(U(\ell^{r,r}),\alpha,\mathcal{K}) = \bigoplus_{\alpha'} S_\lambda^{\mathrm{ord}}(U(\ell^{r,r}),\alpha',\mathcal{K})$, a decomposition stable under the same operators (Step 5), gives the isomorphism of (Qh4) with its compatibilities. □

E.1.3 Points of ordinary Hecke algebras of a type

This subsection describes the points of the ordinary Hecke algebra of a type, from which the Hecke ideals of Subsection E.1.1 are cut out. Its points at classical weights come from automorphic representations whose components at St contain the dual type, and such eigenforms give points (Lemma E.11, Lemma E.14). It is a quotient of the full ordinary Hecke algebra (Lemma E.15), it is reduced (Lemma E.18), and it receives the universal determinant ring through a Galois determinant (Lemma E.16). At zero-monodromy type the Galois representations of regular points are level-free (Lemma E.17). The lemmas on ordinary forms at levels shrunk away from ℓ, projective over the weight algebra and free over the torsion diamonds (Lemma E.12, Lemma E.20, Lemma E.21), are the module-theoretic input of the Taylor–Wiles levels.

Lemma E.11. *Let $((V_v)_{v\in\mathrm{St}},\tau)$ be a type datum, and let Q and $K'=(K'_v)_{v\in Q}$ be a finite set of places and open subgroups as in the levels at auxiliary places of Definition 50.8. Here $Q=\varnothing$ is allowed; if $Q\neq\varnothing$, the type datum is over a rank-n Hecke datum. Let $V=V(K')$ be its level at K', $N=N_\tau(K')$, $\mathbb{T}=\mathbb{T}_\tau(K')$ (for $Q=\varnothing$: V, N_τ, $\mathbb{T}_\tau$) and $H=\prod_{v\in\mathrm{St}}\mathbb{G}_L(\mathcal{O}_{L_v^+})/V_v$; let $\tau^\vee=\mathrm{Hom}_{\mathcal{O}}(\tau,\mathcal{O})$ with the contragredient action of H. Let E be the maximal ramification index of $L^+/\mathbb{Q}$ at the places above ℓ, $\mathfrak{l}$ and $T_n(\mathfrak{l})\supseteq T_n(\ell^1)$ as in Theorem B.91. Let λ be a dominant weight, $\alpha\colon T_n(\ell^1)\to\mathcal{O}^\times$ a character of finite order such that $\mathcal{O}$ contains the values of every character of $T_n(\mathfrak{l})$ whose restriction to $T_n(\ell^1)$ is α (no condition if ℓ is unramified in L^+), $r\geq 1$ with $T_n(\ell^r)\subseteq\ker\alpha$, and $y\colon\mathbb{T}\to\overline{\mathbb{Q}}_\ell$ a homomorphism of $\mathcal{O}$-algebras with $y(\mathfrak{P}_{\lambda,\alpha})=0$. If the type datum is over a rank-n Hecke datum, α may more generally be a character $T_n(\ell^1)\to\overline{\mathbb{Q}}_\ell^\times$ of finite order (with r as before and $\mathfrak{P}_{\lambda,\alpha}$ the kernel of the homomorphism $\Lambda_1\to\overline{\mathbb{Q}}_\ell$ induced by $\alpha\,(w_0\nu)^{-1}(w_0\lambda)^{-1}$, as in (Gy3)), provided y satisfies, besides $y(\mathfrak{P}_{\lambda,\alpha})=0$, $y(\langle u\rangle)=\alpha(u)\,(w_0\lambda)(u)^{-1}$ for all $u\in T_n(\ell^1)$. Then there are an irreducible constituent π of $S_\lambda(\overline{\mathbb{Q}}_\ell)$ and an embedding $\pi\subseteq S_\lambda(\overline{\mathbb{Q}}_\ell)$ such that:*

1. $\mathrm{Hom}_H\big(\tau^\vee\otimes_{\mathcal{O}}\overline{\mathbb{Q}}_\ell,\ \bigotimes_{v\in\mathrm{St}}\pi_v^{V_v}\big)\neq 0$;
2. *for every place $v\notin T_L\cup Q$ of L^+ split in L, π_v is unramified, and for every place $w\mid v$ of L the eigenvalue of $T_w^{(j)}$ on $\pi_v^{\mathbb{G}_L(\mathcal{O}_{L_v^+})}$ is $y(T_w^{(j)})$;*
3. *there is a non-zero $f\in\pi\cap S_\lambda^{\mathrm{ord}}(V(\ell^{Er,Er}),\alpha,\overline{\mathbb{Q}}_\ell)$ on which the operators assigned by (Gy7)(c) to $T_w^{(j)}$, $U_{\tilde v}^{(j)}$ and $\langle u\rangle$ act by the scalars $y(T_w^{(j)})$, $y(U_{\tilde v}^{(j)})$, $y(\langle u\rangle)$.*

Proof. Put $r'=Er$; by (Qh1), $T_n(\mathfrak{l}^{r'})\subseteq T_n(\ell^r)\subseteq\ker\alpha$.

Step 0: reduction to coefficients containing the values of the extensions of α. If $\mathcal{O}$ contains the values of every character of $T_n(\mathfrak{l})$ extending α there is nothing to do. Otherwise the type datum is over a rank-n Hecke datum $\mathscr{H}$ over $\mathcal{O}$; let $\mathcal{O}''\supseteq\mathcal{O}$ be the ring of integers of a finite extension of $\mathcal{O}[1/\ell]$ in $\overline{\mathbb{Q}}_\ell$ containing the values of every character of $T_n(\mathfrak{l})$ extending α (finitely many roots of unity: they are characters of $T_n(\mathfrak{l})/T_n(\mathfrak{l}^{r'})$), free of finite rank over $\mathcal{O}$. By Lemma E.13, $\mathscr{H}''$ (the datum with coefficients $\mathcal{O}''$) is a rank-n Hecke datum, $((V_v),\tau''=\tau\otimes_{\mathcal{O}}\mathcal{O}'')$ a type datum over it, and $N''=N\otimes_{\mathcal{O}}\mathcal{O}''$, $\mathbb{T}''=\mathbb{T}\otimes_{\mathcal{O}}\mathcal{O}''$, $\Lambda_1''=\Lambda_1\otimes_{\mathcal{O}}\mathcal{O}''$, compatibly with the

operators and with H. Let $y'' = y \otimes (\mathcal{O}'' \subseteq \overline{\mathbb{Q}}_\ell)\colon \mathbb{T}'' \to \overline{\mathbb{Q}}_\ell$ and χ the homomorphism $\Lambda_1 \to \overline{\mathbb{Q}}_\ell$ induced by $\alpha(w_0\nu)^{-1}(w_0\lambda)^{-1}$, with image $R_0 \cong \Lambda_1/\mathfrak{P}_{\lambda,\alpha}$. As y kills $\mathfrak{P}_{\lambda,\alpha}$, $y|_{\Lambda_1} = \sigma \circ \chi$ for a homomorphism of $\mathcal{O}$-algebras $\sigma\colon R_0 \to \overline{\mathbb{Q}}_\ell$; by the second hypothesis on y, σ is the identity on the generators $\chi(u)$ of R_0, so $y|_{\Lambda_1} = \chi$. Hence $y''|_{\Lambda_1''} = \chi \otimes \mathrm{id}$ is the homomorphism induced by the same character on Λ_1'', and y'' kills its kernel $\mathfrak{P}''_{\lambda,\alpha}$. The case of values in the coefficient ring, applied to $\mathscr{H}''$, $((V_v), \tau'')$ and y'', gives π with (i)–(iii) for y''; these are (i)–(iii) for y, since $\tau''^\vee \otimes_{\mathcal{O}''} \overline{\mathbb{Q}}_\ell = \tau^\vee \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_\ell$, $y''(t \otimes 1) = y(t)$, and $S_\lambda(\overline{\mathbb{Q}}_\ell)$ and $S^{\mathrm{ord}}_\lambda(V(\ell^{r',r'}), \alpha, \overline{\mathbb{Q}}_\ell)$ do not depend on the coefficient ring. We assume from now on that $\mathcal{O}$ contains the values of every character of $T_n(\mathfrak{l})$ extending α, in particular those of α. Write $\Lambda = \Lambda_1$ with the renormalised structure, $\mathfrak{P} = \mathfrak{P}_{\lambda,\alpha}$, $\mathcal{K} = \mathcal{O}[1/\ell]$, and $W = S^{\mathrm{ord}}_\lambda(V(\ell^{r',r'}), \alpha, \mathcal{K})$.

Step 1: the setting of Lemma E.12. Let $U \supseteq V$ be the group with $U_v = \mathbb{G}_L(\mathcal{O}_{L_v^+})$ for $v \in \mathrm{St}$ and $U_v = V_v$ otherwise. Then U is as in Theorem B.91 ($F = L$, $T = T_L \cup Q$, all of whose places split in L), V_v is open normal in U_v for every v and equal to it for $v \in S_\ell$ and $v \notin T_L \cup Q$, and $U/V = H$; take $G_1 = H$, $G_2 = 1$, $b_0 = 1$: $U(\ell^{1,1})$ is sufficiently small, as its component at v_0 is the kernel of reduction modulo p of $\mathbb{G}_L(\mathcal{O}_{L_{v_0}^+}) \cong \mathrm{GL}_n(\mathcal{O}_{L_{\tilde{v}_0}})$, p odd and unramified in L (Definition 50.8), which contains no non-trivial element of finite order . Here $\mathcal{S}' = \mathcal{S}_V$. By (Yp2) $\mathcal{S}_V$ is a direct summand of $\Lambda[H]^m$ for some m, and by (Yp3) $N = \mathrm{Hom}_{\mathcal{O}[H]}(\tau, \mathcal{S}_V)$ is free of finite rank over Λ (for the diamond structure, hence for the renormalised one, which differs by an automorphism of Λ).

Step 2: the fibre of N at $\mathfrak{P}$. Let $\Lambda \to A$ be a homomorphism to a commutative ring. For $\Lambda[H]$ itself, $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, \Lambda[H]) \otimes_\Lambda A \cong \Lambda \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}}(\tau, \mathcal{O}) \otimes_\Lambda A \cong \mathrm{Hom}_{\mathcal{O}[H]}(\tau, A[H])$, by the isomorphism $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, R \otimes_{\mathcal{O}} \mathcal{O}[H]) \cong R \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}}(\tau, \mathcal{O})$ for every $\mathcal{O}$-algebra R, given by $f \mapsto \epsilon \circ f$ with $\epsilon\colon R \otimes \mathcal{O}[H] \to R$ the coefficient of 1, with inverse $\varphi \mapsto (t \mapsto \sum_{h \in H} \varphi(h^{-1}t)\, h)$ (a direct check; τ is finite free over $\mathcal{O}$ and $\mathcal{O}[H]$ is free over $\mathcal{O}$ with basis H). Both sides are additive, so the natural map $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, \mathcal{S}_V) \otimes_\Lambda A \to \mathrm{Hom}_{\mathcal{O}[H]}(\tau, \mathcal{S}_V \otimes_\Lambda A)$ is an isomorphism for the direct summand $\mathcal{S}_V$ of $\Lambda[H]^m$. Take $A = \Lambda_{\mathfrak{P}}/\mathfrak{P}\Lambda_{\mathfrak{P}} = \mathcal{K}$ ($\Lambda/\mathfrak{P} = \mathcal{O}$). By (Qh4) for $U = V$, $b_0 = 1$ and the level r' (applicable: $T_n(\mathfrak{l}^{r'}) \subseteq \ker \alpha$, $\mathcal{O}$ contains the values of the extensions of α, V is as in Theorem B.91, and, for the equivariance of (Gk3), the elements of $\prod_{v \in \mathrm{St}} \mathbb{G}_L(\mathcal{O}_{L_v^+})$ have trivial components at S_ℓ and normalise every $V(\ell^{c,c})$), $\mathcal{S}_V \otimes_\Lambda \mathcal{K} \cong W^* := \mathrm{Hom}_{\mathcal{K}}(W, \mathcal{K})$, compatibly with H and with the Hecke operators (acting on W^* by the transposes of the operators of (Gy7)(c)). Hence

$$N_{\mathfrak{P}}/\mathfrak{P}N_{\mathfrak{P}} \cong \mathrm{Hom}_{\mathcal{K}[H]}(\tau \otimes_{\mathcal{O}} \mathcal{K}, W^*),$$

compatibly with the action of $\mathbb{T}^{\mathrm{ord}}(V)$.

Step 3: y is an eigensystem on this fibre. $\mathbb{T}$ acts faithfully on N, which is free of rank d over Λ; localisation is exact, so $\mathbb{T}_{\mathfrak{P}} = \mathbb{T} \otimes_\Lambda \Lambda_{\mathfrak{P}}$ acts faithfully on $N_{\mathfrak{P}}$. If $t \in \mathbb{T}_{\mathfrak{P}}$ acts by 0 on $N_{\mathfrak{P}}/\mathfrak{P}N_{\mathfrak{P}}$, its matrix in a $\Lambda_{\mathfrak{P}}$-basis has entries in $\mathfrak{P}\Lambda_{\mathfrak{P}}$, so its characteristic polynomial is $X^d + \sum_{i<d} c_i X^i$ with $c_i \in \mathfrak{P}\Lambda_{\mathfrak{P}}$, and by Cayley–Hamilton $t^d \in \mathfrak{P}\mathbb{T}_{\mathfrak{P}}$. So the kernel of $\mathbb{T}_{\mathfrak{P}}/\mathfrak{P}\mathbb{T}_{\mathfrak{P}} \to \mathrm{End}_{\mathcal{K}}(N_{\mathfrak{P}}/\mathfrak{P}N_{\mathfrak{P}})$ is nil. The restriction of y to Λ has kernel exactly $\mathfrak{P}$ (it factors through $\Lambda/\mathfrak{P} = \mathcal{O} \subseteq \overline{\mathbb{Q}}_\ell$), so y extends to $\mathbb{T}_{\mathfrak{P}}$, kills $\mathfrak{P}\mathbb{T}_{\mathfrak{P}}$ and every nilpotent element, and so factors through the image $\mathcal{A}$ of $\mathbb{T}_{\mathfrak{P}}$ in $\mathrm{End}_{\mathcal{K}}(N_{\mathfrak{P}}/\mathfrak{P}N_{\mathfrak{P}})$, a finite commutative $\mathcal{K}$-algebra. Let $X = (N_{\mathfrak{P}}/\mathfrak{P}N_{\mathfrak{P}}) \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$, on which $\mathcal{A} \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$ acts faithfully; X is the direct sum of the generalised eigenspaces of $\mathcal{A} \otimes \overline{\mathbb{Q}}_\ell$ for its characters, each non-zero; the character induced by y is one of them, so there is $0 \neq \xi \in X$ with $t\xi = y(t)\xi$ for all $t \in \mathbb{T}^{\mathrm{ord}}(V)$.

Step 4: the automorphic representation. By Step 2, ξ is a non-zero H-map $\tau \otimes \overline{\mathbb{Q}}_\ell \to W^*_{\overline{\mathbb{Q}}_\ell}$, $W_{\overline{\mathbb{Q}}_\ell} = W \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell = S^{\mathrm{ord}}_\lambda(V(\ell^{r',r'}), \alpha, \overline{\mathbb{Q}}_\ell)$, with image in the subspace of $W^*_{\overline{\mathbb{Q}}_\ell}$ on which every t acts (by transpose) as $y(t)$. Let $W_y \subseteq W_{\overline{\mathbb{Q}}_\ell}$ be the joint generalised eigenspace of the commuting operators of (Gy7)(c) for the eigenvalues $y(t)$; $W_{\overline{\mathbb{Q}}_\ell} = W_y \oplus W'$ with W' the sum of the other

generalised eigenspaces, both H-stable (H commutes with the Hecke operators, (Yp1)), and the y-eigenvectors of the transposes vanish on W'. So $\mathrm{Hom}_H(\tau\otimes\overline{\mathbb{Q}}_\ell, W_y^*)\neq 0$, and as H is finite and $\overline{\mathbb{Q}}_\ell$ has characteristic 0, $\mathrm{Hom}_H(\tau\otimes\overline{\mathbb{Q}}_\ell, W_y^*)\cong \mathrm{Hom}_H(W_y, (\tau\otimes\overline{\mathbb{Q}}_\ell)^*)$, so W_y has an irreducible H-constituent occurring in $\tau^\vee\otimes\overline{\mathbb{Q}}_\ell$. The space $S_\lambda(\overline{\mathbb{Q}}_\ell)$ is a semisimple admissible $\mathbb{G}_L(\mathbb{A}_{L^+}^\infty)$-module ([105], §2.2), the sum of its isotypic components $S_\lambda[\pi]$; $W_{\overline{\mathbb{Q}}_\ell}$ and W_y are cut out of the $V(\ell^{r',r'})$-invariants by Hecke operators (the ordinary idempotent, the projection to the α-part, generalised eigenprojections), which preserve each $S_\lambda[\pi]^{V(\ell^{r',r'})}$, as does H. So $W_y=\bigoplus_\pi(W_y\cap S_\lambda[\pi])$, and some π has $W_y\cap S_\lambda[\pi]\neq 0$ with an H-constituent occurring in $\tau^\vee\otimes\overline{\mathbb{Q}}_\ell$. As an H-module $S_\lambda[\pi]^{V(\ell^{r',r'})}$ is a sum of copies of $\bigotimes_{v\in\mathrm{St}}\pi_v^{V_v}$, which gives (i). On $S_\lambda[\pi]^{V(\ell^{r',r'})}\cong \pi^{V(\ell^{r',r'})}\otimes_{\overline{\mathbb{Q}}_\ell}\overline{\mathbb{Q}}_\ell^{m_\pi}$ (m_π the multiplicity of π) the operators act through π, so $W_y\cap S_\lambda[\pi]$ contains a joint eigenvector f for the commuting operators with the eigenvalues $y(t)$, which may be taken in one copy of π; this is (iii). For $v\notin T_L\cup Q$ split, $V_v=\mathbb{G}_L(\mathcal{O}_{L_v^+})$ and $\pi^{V(\ell^{r',r'})}\neq 0$, so π_v is unramified, its spherical line is $\pi_v^{V_v}$, and $T_w^{(j)}$ (assigned to itself by (Gy7)(c)) acts on it by the eigenvalue $y(T_w^{(j)})$ of f: (ii). □

Lemma E.12. *Let ℓ, n, F, F^+, $\mathcal{O}$, $\mathbb{G}$, T be as in Theorem B.91, and let $U=\prod_v U_v\supseteq U'=\prod_v U_v'$ be compact open subgroups of $\mathbb{G}(\mathbb{A}_{F^+}^\infty)$ such that U is as in Theorem B.91, U_v' is an open normal subgroup of U_v for every v, and $U_v'=U_v$ for every $v\in S_\ell$ and every $v\notin T$. Put $\Gamma=U/U'$ and suppose $\Gamma=G_1\times G_2$ for subgroups G_1, G_2. Let $b_0\geq 1$ be such that $U(\ell^{b_0,b_0})$ is sufficiently small, $\Lambda=\Lambda_{b_0}$, and let $\mathcal{S}'$, resp. $\mathcal{S}$, be the Pontryagin dual of $S^{\mathrm{ord}}(U'(\ell^\infty),\mathcal{K}/\mathcal{O})$, resp. $S^{\mathrm{ord}}(U(\ell^\infty),\mathcal{K}/\mathcal{O})$, with Λ^+ acting through the diamond operators and Γ acting on $\mathcal{S}'$ dually to right translation. Then:*

(Yp1) *U' is as in Theorem B.91; the action of Γ on $\mathcal{S}'$ is well defined and commutes with Λ^+ and with the operators $T_w^{(j)}$, $(T_w^{(n)})^{-1}$ (w above a place $v\notin T$ split in F) and $U_{\tilde{v}}^{(j)}$ ($v\in S_\ell$);*

(Yp2) *$\mathcal{S}'$ is a finitely generated projective $\Lambda[\Gamma]$-module;*

(Yp3) *for every finite free $\mathcal{O}$-module τ with an $\mathcal{O}$-linear action of G_1, $\mathrm{Hom}_{\mathcal{O}[G_1]}(\tau,\mathcal{S}')$ is a finitely generated projective $\Lambda[G_2]$-module, and a free one if G_2 is an ℓ-group (for instance trivial);*

(Yp4) *the dual of the inclusion $S^{\mathrm{ord}}(U(\ell^\infty),\mathcal{K}/\mathcal{O})\subseteq S^{\mathrm{ord}}(U'(\ell^\infty),\mathcal{K}/\mathcal{O})$ induces an isomorphism of Λ^+-modules $\mathcal{S}'_\Gamma\to\mathcal{S}$ from the Γ-coinvariants, compatible with the operators of (Yp1).*

Proof. Write $\mathbb{F}$ for the residue field of $\mathcal{O}$, ϖ for a uniformiser, $\mathfrak{m}_\Lambda$ for the maximal ideal of Λ; for $v\in S_\ell$ let ϖ_v be a uniformiser of $\mathcal{O}_{F_v^+}$; let $\Lambda^{\mathfrak{l}}=\Lambda_{b_0}^{\mathfrak{l}}$, which contains Λ by (Qh1), $\mathfrak{a}\subseteq\Lambda^{\mathfrak{l}}$ its augmentation ideal (the kernel of $\Lambda^{\mathfrak{l}}\to\mathcal{O}$, $[u]\mapsto 1$) and $\mathfrak{m}=\mathfrak{a}+\varpi\Lambda^{\mathfrak{l}}$, the maximal ideal of the noetherian complete local ring $\Lambda^{\mathfrak{l}}$ (by the Chinese remainder theorem $T_n(\mathfrak{l}^{b_0})=\prod_{v\in S_\ell}(1+\varpi_v^{b_0}\mathcal{O}_{F_v^+})^n$, a finitely generated abelian pro-ℓ group); and for an $\mathcal{O}$-module A and $1\leq b\leq c$ put $U'(\ell^{b,c})=\prod_{v\notin S_\ell}U_v'\times\prod_{v\in S_\ell}U(\ell^{b,c})_v$.

Step 1: (Yp1) and the levels. $U_v'=U_v$ is $\mathbb{G}(\mathcal{O}_{F_v^+})$ for $v\in S_\ell$ and for $v\notin T$ split in F, so U' is as in Theorem B.91, and $U'(\ell^{b,c})$ is the group $U'(\ell^{b,c})$ defined there. As the components of U and U' agree at S_ℓ and U_v' is normal in U_v, $U'(\ell^{b,c})$ is normal in $U(\ell^{b,c})$ and the inclusion $\prod_{v\notin S_\ell}U_v\to U(\ell^{b,c})$ induces an isomorphism $\Gamma\to U(\ell^{b,c})/U'(\ell^{b,c})$ for all $b\leq c$. Hence for $f\in S(U'(\ell^{b,c}),A)$ and $g\in\prod_{v\notin S_\ell}U_v$ the function $x\mapsto f(xg)$ is again right $U'(\ell^{b,c})$-invariant and depends only on gU'; this is the action of Γ, compatible with the inclusions between levels. The

operators $\langle u\rangle$ and $U_{\tilde v}^{(j)}$ are double coset operators supported at the places of S_ℓ, and $T_w^{(j)}$ at a place $v\notin T$; since Γ acts through the components at the places of $T\setminus S_\ell$, they commute with it. So the ordinary idempotent e (the limit of $P^{m!}$, P the product of the $U_{\tilde v}^{(j)}$, on each finite level) commutes with Γ, and Γ acts on $S^{\mathrm{ord}}(U'(\ell^\infty),\mathcal{K}/\mathcal{O})=\varinjlim_c S^{\mathrm{ord}}(U'(\ell^{c,c}),\mathcal{K}/\mathcal{O})$ and dually on $\mathcal{S}'$, commuting with the operators listed. This is (Yp1).

Step 2: the fibre of $\mathcal{S}'$ at the augmentation ideal. Let $M=S^{\mathrm{ord}}(U'(\ell^\infty),\mathcal{K}/\mathcal{O})$, so $\mathcal{S}'=M^\vee$. The diamond operator $\langle u\rangle$, $u\in T_n(\mathfrak{l}^{b_0})$, acts on $S(U'(\ell^{c,c}),A)$ by right translation by u in the components at S_ℓ (u normalises $U'(\ell^{c,c})$). The group $\mathrm{Iw}(\tilde v^{b,c})$ consists of the matrices of $\mathrm{GL}_n(\mathcal{O}_{F_v^+})$ that are upper triangular modulo ϖ_v^c and unipotent modulo ϖ_v^b ([105], §2.1); reduction of the diagonal modulo ϖ_v^c is a homomorphism $\mathrm{Iw}(\tilde v^{0,c})\to T_n(\mathcal{O}_{F_v^+}/\varpi_v^c)$ with kernel $\mathrm{Iw}(\tilde v^{c,c})$, and $\mathrm{Iw}(\tilde v^{b_0,c})$ is the preimage of the image of $(1+\varpi_v^{b_0}\mathcal{O}_{F_v^+})^n$. As the components of U' at S_ℓ are those of U, $U'(\ell^{b_0,c})=U'(\ell^{c,c})\,T_n(\mathfrak{l}^{b_0})$. Since e commutes with the diamond operators ([105], Lemma 2.3.3), the $T_n(\mathfrak{l}^{b_0})$-invariants of $S^{\mathrm{ord}}(U'(\ell^{c,c}),\mathcal{K}/\mathcal{O})$ are $S^{\mathrm{ord}}(U'(\ell^{b_0,c}),\mathcal{K}/\mathcal{O})$, which is $S^{\mathrm{ord}}(U'(\ell^{b_0,b_0}),\mathcal{K}/\mathcal{O})$ by (Gk2) for U'. Taking the union over c, $M^{T_n(\mathfrak{l}^{b_0})}=S^{\mathrm{ord}}(U'(\ell^{b_0,b_0}),\mathcal{K}/\mathcal{O})$. Let $u_1,\dots,u_k$ generate $T_n(\mathfrak{l}^{b_0})$ topologically. The ideal J of $\Lambda^{\mathfrak{l}}$ generated by the $[u_i]-1$ is $\mathfrak{a}$.

If $m\in M$ is killed by the $[u_i]-1$, its stabiliser in $T_n(\mathfrak{l}^{b_0})$ is a subgroup that is open (m has a finite level $U'(\ell^{c,c})$, so it is fixed by $T_n(\mathfrak{l}^c)$), hence closed, and contains the u_i, so it is all of $T_n(\mathfrak{l}^{b_0})$; so $M[\mathfrak{a}]=M^{T_n(\mathfrak{l}^{b_0})}=S^{\mathrm{ord}}(U'(\ell^{b_0,b_0}),\mathcal{K}/\mathcal{O})$. The set $X=\mathbb{G}(F^+)\backslash\mathbb{G}(\mathbb{A}_{F^+}^\infty)/U'(\ell^{b_0,b_0})$ is finite ($\mathbb{G}(F^+\otimes\mathbb{R})$ is compact), so $S(U'(\ell^{b_0,b_0}),A)=\mathcal{O}^X\otimes_{\mathcal{O}}A$, with e acting as $e\otimes 1$; hence $M[\mathfrak{a}]=L\otimes_{\mathcal{O}}\mathcal{K}/\mathcal{O}$ with $L=S^{\mathrm{ord}}(U'(\ell^{b_0,b_0}),\mathcal{O})=e\,\mathcal{O}^X$, a finite free $\mathcal{O}$-module. As $\mathfrak{a}$ is finitely generated, $M[\mathfrak{a}]$ is the kernel of a map $M\to M^k$, and duality gives

$$\mathcal{S}'/\mathfrak{a}\mathcal{S}'\cong(M[\mathfrak{a}])^\vee\cong\mathrm{Hom}_{\mathcal{O}}(L,\mathcal{O})=:D,$$

Γ-equivariantly (Γ acting on D contragrediently).

Step 3: D is projective over $\mathcal{O}[\Gamma]$. $U(\ell^{b_0,b_0})$ is sufficiently small, its components at S_ℓ lie in $\mathbb{G}(\mathcal{O}_{F_v^+})$, and $U'(\ell^{b_0,b_0})$ is an open normal subgroup with quotient Γ (Step 1). By (Gk1), $\mathcal{O}^X=S(U'(\ell^{b_0,b_0}),\mathcal{O})$ is a finite free $\mathcal{O}[\Gamma]$-module, and $L=e\,\mathcal{O}^X$ is a direct summand as e commutes with Γ; so L is finitely generated projective over $\mathcal{O}[\Gamma]$. The $\mathcal{O}$-dual of $\mathcal{O}[\Gamma]$ with the contragredient action is isomorphic to $\mathcal{O}[\Gamma]$ (the dual basis (δ_g) is permuted simply transitively), and $\mathrm{Hom}_{\mathcal{O}}(-,\mathcal{O})$ preserves finite direct sums and summands; so D is finitely generated projective over $\mathcal{O}[\Gamma]$, and $D/\varpi D$ over $\mathbb{F}[\Gamma]$.

Step 4: (Yp2). By (Gy1) applied to U' ($U'(\ell^{b_0,b_0})\subseteq U(\ell^{b_0,b_0})$ is sufficiently small), $\mathcal{S}'$ is a free $\Lambda^{\mathfrak{l}}$-module of finite rank. Let $R^{\mathfrak{l}}=\Lambda^{\mathfrak{l}}[\Gamma]$, a free $\Lambda^{\mathfrak{l}}$-module of finite rank with $R^{\mathfrak{l}}/\mathfrak{m}R^{\mathfrak{l}}=\mathbb{F}[\Gamma]$; the actions of $\Lambda^{\mathfrak{l}}\subseteq\Lambda^+$ and of Γ on $\mathcal{S}'$ commute (Step 1), so $\mathcal{S}'$ is an $R^{\mathfrak{l}}$-module. Since $\mathfrak{m}=\mathfrak{a}+\varpi\Lambda^{\mathfrak{l}}$, Step 2 gives $\mathcal{S}'/\mathfrak{m}\mathcal{S}'\cong D/\varpi D$, a finitely generated projective $\mathbb{F}[\Gamma]$-module: $D/\varpi D\cong\mathbb{F}[\Gamma]^r\varepsilon_0$ for some r and an idempotent matrix $\varepsilon_0\in\mathrm{M}_r(\mathbb{F}[\Gamma])$ (acting on row vectors). The ring $\mathrm{M}_r(R^{\mathfrak{l}})$ is complete for the topology of the two-sided ideal $\mathrm{M}_r(\mathfrak{m}R^{\mathfrak{l}})$ ($R^{\mathfrak{l}}$ is finite over the $\mathfrak{m}$-adically complete ring $\Lambda^{\mathfrak{l}}$), so ε_0 lifts to an idempotent $\varepsilon\in\mathrm{M}_r(R^{\mathfrak{l}})$ (lifting of idempotents modulo an ideal for which the ring is complete). Then $P=(R^{\mathfrak{l}})^r\varepsilon$ is a finitely generated projective $R^{\mathfrak{l}}$-module with an isomorphism $P/\mathfrak{m}P\cong\mathcal{S}'/\mathfrak{m}\mathcal{S}'$, and as P is projective the composite $P\to P/\mathfrak{m}P\cong\mathcal{S}'/\mathfrak{m}\mathcal{S}'$ lifts to an $R^{\mathfrak{l}}$-linear $\phi\colon P\to\mathcal{S}'$. Its image I satisfies $I+\mathfrak{m}\mathcal{S}'=\mathcal{S}'$, so ϕ is surjective by Nakayama's lemma over $\Lambda^{\mathfrak{l}}$ ($\mathcal{S}'$ is finitely generated over $\Lambda^{\mathfrak{l}}$). Let $K=\ker\phi$. Since $\mathcal{S}'$ is $\Lambda^{\mathfrak{l}}$-free, $\mathrm{Tor}_1^{\Lambda^{\mathfrak{l}}}(\mathcal{S}',\mathbb{F})=0$, so $0\to K/\mathfrak{m}K\to P/\mathfrak{m}P\to\mathcal{S}'/\mathfrak{m}\mathcal{S}'\to 0$ is exact; the last map is the chosen isomorphism, so $K/\mathfrak{m}K=0$, and K, a submodule of the finitely generated module P over the noetherian ring $\Lambda^{\mathfrak{l}}$, is finitely generated, hence $K=0$. So $\mathcal{S}'\cong P$ is finitely generated projective over $R^{\mathfrak{l}}$. Finally let $R=\Lambda[\Gamma]$. By (Qh1), $\Lambda^{\mathfrak{l}}$ is a free Λ-module of finite rank $[T_n(\mathfrak{l}^{b_0}):T_n(\ell^{b_0})]$, so $R^{\mathfrak{l}}=\Lambda^{\mathfrak{l}}\otimes_\Lambda R$ is a free R-module of the same rank. $\mathcal{S}'$ is a direct

summand of some $(R^{\mathfrak{l}})^{m'}$ as an $R^{\mathfrak{l}}$-module, hence as an R-module, so it is a direct summand of a free R-module of finite rank: $\mathcal{S}'$ is finitely generated projective over $R = \Lambda[\Gamma]$. This is (Yp2).

Step 5: (Yp3). Write $R = \Lambda[G_2] \otimes_{\mathcal{O}} \mathcal{O}[G_1]$, with $\Lambda[G_2]$ acting on the first factor and G_1 by left multiplication on the second; the two actions commute. As $\Lambda[G_2]$ is flat over $\mathcal{O}$ and τ is finitely presented over $\mathcal{O}[G_1]$,

$$\mathrm{Hom}_{\mathcal{O}[G_1]}(\tau, R) \cong \Lambda[G_2] \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}[G_1]}(\tau, \mathcal{O}[G_1]) \cong \Lambda[G_2] \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}}(\tau, \mathcal{O}),$$

the second isomorphism being $f \mapsto \epsilon \circ f$ with ϵ the coefficient of 1, with inverse $\varphi \mapsto (t \mapsto \sum_{g \in G_1} \varphi(g^{-1}t)\, g)$. So $\mathrm{Hom}_{\mathcal{O}[G_1]}(\tau, R)$ is a free $\Lambda[G_2]$-module of finite rank. By Step 4, $\mathcal{S}'$ is a direct summand of R^m for some m, as R-modules, so $\mathrm{Hom}_{\mathcal{O}[G_1]}(\tau, \mathcal{S}')$ is a direct summand of $\mathrm{Hom}_{\mathcal{O}[G_1]}(\tau, R)^m$ as $\Lambda[G_2]$-modules: finitely generated projective. If G_2 is an ℓ-group, $\Lambda[G_2]$ is a local ring: it is finite over the complete local ring Λ, so each of its maximal ideals contains $\mathfrak{m}_\Lambda \Lambda[G_2]$, and $\mathbb{F}[G_2]$ is local ($\mathbb{F}$ has characteristic ℓ); a finitely generated projective module over a local ring is free.

Step 6: (Yp4). For every c, a function in $S(U'(\ell^{c,c}), \mathcal{K}/\mathcal{O})$ is right $U(\ell^{c,c})$-invariant if and only if it is Γ-invariant (Step 1), so $S(U(\ell^{c,c}), \mathcal{K}/\mathcal{O}) = S(U'(\ell^{c,c}), \mathcal{K}/\mathcal{O})^\Gamma$, and the same holds for the ordinary parts since e commutes with Γ and the two idempotents agree on $S(U(\ell^{c,c}), \mathcal{K}/\mathcal{O})$ (the $U_{\tilde{v}}^{(j)}$ act there by the same double cosets). Invariants of the finite group Γ commute with filtered direct limits, so $S^{\mathrm{ord}}(U(\ell^\infty), \mathcal{K}/\mathcal{O}) = M^\Gamma$. Dualising the exact sequence $0 \to M^\Gamma \to M \to \prod_{g \in \Gamma} M$, $m \mapsto (gm - m)_g$, gives $(M^\Gamma)^\vee = \mathcal{S}'/\sum_g (g-1)\mathcal{S}' = \mathcal{S}'_\Gamma$, the isomorphism being induced by the dual of the inclusion. It is Λ^+-linear and commutes with the operators of (Yp1), which commute with Γ and preserve M^Γ. □

Lemma E.13. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, $((V_v)_{v \in \mathrm{St}}, \tau)$ a type datum over $\mathscr{H}$, Q, K' as in the levels at auxiliary places ($Q = \varnothing$ allowed), and $\mathcal{O}'' \supseteq \mathcal{O}$ the ring of integers of a finite extension of $\mathcal{O}[1/\ell]$ in $\overline{\mathbb{Q}}_\ell$. Let $\mathscr{H}''$ be $\mathscr{H}$ with the residual characters composed with $\mathbb{F}_{\mathcal{O}} \subseteq \mathbb{F}_{\mathcal{O}''}$ and the wild values composed with $\mathcal{O}^\times \subseteq \mathcal{O}''^\times$, and $\tau'' = \tau \otimes_{\mathcal{O}} \mathcal{O}''$. Then $\mathscr{H}''$ is a rank-n Hecke datum over $\mathcal{O}''$, $((V_v), \tau'')$ is a type datum over it with the same group H and the same levels, and, writing $''$ for the objects over $\mathcal{O}''$, there are isomorphisms*

$$\mathcal{S}''_{V(K')} \cong \mathcal{S}_{V(K')} \otimes_{\mathcal{O}} \mathcal{O}'', \qquad N''_{\tau''}(K') \cong N_\tau(K') \otimes_{\mathcal{O}} \mathcal{O}'',$$
$$\mathbb{T}''_{\tau''}(K') \cong \mathbb{T}_\tau(K') \otimes_{\mathcal{O}} \mathcal{O}'', \qquad \Lambda''_1 = \Lambda_1 \otimes_{\mathcal{O}} \mathcal{O}'',$$

compatible with the action of H and carrying each operator $T_w^{(j)}$, $(T_w^{(n)})^{-1}$, $U_{\tilde{v}}^{(j)}$, $\langle u \rangle$ to the operator of the same name tensored with $\mathcal{O}''$.

Proof. $\mathcal{O}''$ is a finitely generated torsion-free module over the discrete valuation ring $\mathcal{O}$, hence free of finite rank d; so $- \otimes_{\mathcal{O}} \mathcal{O}''$ is exact and commutes with $\mathrm{Hom}_{\mathcal{O}}$ out of any module into a finite direct sum.

The datum. Of the conditions of Definition 50.1 only (Rd1) and (Rd6) involve the coefficient ring: $\tau(\mathcal{O}_L) \subseteq \mathcal{O} \subseteq \mathcal{O}''$, and the reduction of χ_{cyc} modulo $\mathfrak{m}_{\mathcal{O}''}$ is that modulo $\mathfrak{m}_{\mathcal{O}}$ followed by the injection $\mathbb{F}_{\mathcal{O}} \to \mathbb{F}_{\mathcal{O}''}$, so it is non-trivial on $D_{\bar{v}_0} \cap G_L$; (Rd10) holds as the wild values are unchanged as elements. A type datum is a family of open normal subgroups and a finite free module with an action of H; τ'' is finite free over $\mathcal{O}''$ with the induced action.

The forms. For an $\mathcal{O}$-module M and a compact open U', $S(U', M)$ is additive in M, so the natural map $S(U', M) \otimes_{\mathcal{O}} \mathcal{O}'' \to S(U', M \otimes_{\mathcal{O}} \mathcal{O}'')$ is an isomorphism (through $\mathcal{O}'' \cong \mathcal{O}^d$), compatible with right translations and hence with the Hecke and diamond operators, the operators $U_{\tilde{v}}^{(j)}$ and H; the ordinary parts correspond, being images of an idempotent which is a limit

of polynomials in these operators. Take $M = \mathcal{K}/\mathcal{O}$, for which $M \otimes_{\mathcal{O}} \mathcal{O}'' = \mathcal{K}''/\mathcal{O}''$, and pass to the direct limit over the levels $V(K')(\ell^{c,c})$: the ordinary forms X'' of level $V(K')(\ell^\infty)$ with coefficients in $\mathcal{K}''/\mathcal{O}''$ are $X \otimes_{\mathcal{O}} \mathcal{O}''$, X those with coefficients in $\mathcal{K}/\mathcal{O}$. Their Pontryagin dual over $\mathcal{O}''$ is

$$\mathrm{Hom}_{\mathcal{O}''}(X \otimes_{\mathcal{O}} \mathcal{O}'', \mathcal{K}''/\mathcal{O}'') \cong \mathrm{Hom}_{\mathcal{O}}(X, (\mathcal{K}/\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}'') \cong \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{K}/\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}''$$

(adjunction; then $(\mathcal{K}/\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}'' \cong (\mathcal{K}/\mathcal{O})^d$), i.e. $\mathcal{S}_{V(K')} \otimes_{\mathcal{O}} \mathcal{O}''$, with the dual operators. Likewise $\Lambda_1'' = \mathcal{O}''[[T_n(\ell^1)]] = \Lambda_1 \otimes_{\mathcal{O}} \mathcal{O}''$ and $\Lambda''^+ = \Lambda^+ \otimes_{\mathcal{O}} \mathcal{O}''$ ($\mathcal{O}''$ finite free), acting through the diamond operators.

N and $\mathbb{T}$. $N''_{\tau''}(K') = \mathrm{Hom}_{\mathcal{O}''[H]}(\tau \otimes_{\mathcal{O}} \mathcal{O}'', \mathcal{S}_{V(K')} \otimes_{\mathcal{O}} \mathcal{O}'') = \mathrm{Hom}_{\mathcal{O}[H]}(\tau, \mathcal{S}_{V(K')} \otimes_{\mathcal{O}} \mathcal{O}'') = N_\tau(K') \otimes_{\mathcal{O}} \mathcal{O}''$ (adjunction; $\mathcal{S}_{V(K')} \otimes_{\mathcal{O}} \mathcal{O}'' \cong \mathcal{S}^d_{V(K')}$ as $\mathcal{O}[H]$-modules). The algebra $\mathbb{T}''^{\mathrm{ord}}(V(K'))$ is the Λ''^+-subalgebra of $\mathrm{End}_{\mathcal{O}''}(\mathcal{S}''_{V(K')})$ generated by the operators of the same names, i.e. the image of $\mathbb{T}^{\mathrm{ord}}(V(K')) \otimes_{\mathcal{O}} \mathcal{O}''$; and $\mathbb{T}''_{\tau''}(K')$ is its image in $\mathrm{End}_{\mathcal{O}''}(N_\tau(K') \otimes_{\mathcal{O}} \mathcal{O}'')$, i.e. the image of $\mathbb{T}^{\mathrm{ord}}(V(K')) \otimes_{\mathcal{O}} \mathcal{O}''$ there. The map $\mathrm{End}_{\mathcal{O}}(N) \otimes_{\mathcal{O}} \mathcal{O}'' \to \mathrm{End}_{\mathcal{O}''}(N \otimes_{\mathcal{O}} \mathcal{O}'')$ ($N = N_\tau(K')$) is injective (in a basis $e_1, \dots, e_d$ of $\mathcal{O}''$, $\sum_i \phi_i \otimes e_i$ sends $m \otimes 1$ to $\sum_i \phi_i(m) \otimes e_i$), and the image of $\mathbb{T}^{\mathrm{ord}}(V(K'))$ in $\mathrm{End}_{\mathcal{O}}(N)$ is $\mathbb{T}_\tau(K')$; by exactness of $- \otimes_{\mathcal{O}} \mathcal{O}''$ the image of $\mathbb{T}^{\mathrm{ord}}(V(K')) \otimes_{\mathcal{O}} \mathcal{O}''$ is $\mathbb{T}_\tau(K') \otimes_{\mathcal{O}} \mathcal{O}''$. □

Lemma E.14. *Let $\mathscr{H}$ over $\mathcal{O}'$, κ, $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ and c be as in Lemma E.16, V_v ($v \in \mathrm{St}$) the kernel of reduction modulo ϖ_v^c, $H = \prod_{v \in \mathrm{St}} \mathbb{G}_L(\mathcal{O}_{L_v^+})/V_v$, τ a finite free $\mathcal{O}'$-module with an $\mathcal{O}'$-linear action of H, $\tau^\vee = \mathrm{Hom}_{\mathcal{O}'}(\tau, \mathcal{O}')$, and $q\colon \mathbb{T}_{\mathcal{O}'[H]} \to \mathbb{T}_\tau$ the surjection of Lemma E.15. Let λ be a dominant weight, $\alpha\colon T_n(\ell^1) \to \overline{\mathbb{Q}}_\ell^\times$ a character of finite order, $r \geq 1$ with $T_n(\ell^r) \subseteq \ker \alpha$, and y_0 a point of $\mathbb{T}_{\mathcal{O}'[H]}$ with $y_0(\mathfrak{P}_{\lambda,\alpha}) = 0$ and $y_0(\langle u \rangle) = \alpha(u)(w_0\lambda)(u)^{-1}$ for $u \in T_n(\ell^1)$. Suppose that there are an irreducible constituent π of $S_\lambda(\overline{\mathbb{Q}}_\ell)$ with $\mathrm{Hom}_H(\tau^\vee \otimes_{\mathcal{O}'} \overline{\mathbb{Q}}_\ell, \bigotimes_{v \in \mathrm{St}} \pi_v^{V_v}) \neq 0$ and a non-zero $f \in \pi \cap S_\lambda^{\mathrm{ord}}(V(\ell^{r,r}), \alpha, \overline{\mathbb{Q}}_\ell)$ on which every operator assigned by (Gy7)(c) to an element t among the $T_w^{(j)}$, $U_{\tilde{v}}^{(j)}$, $\langle u \rangle$ acts by $y_0(t)$. Then $y_0 = y \circ q$ for a point y of $\mathbb{T}_\tau$.*

Proof. Let E be the maximal ramification index of $L^+/\mathbb{Q}$ at the places above ℓ and $r' = Er$; by (Qh1), $T_n(\mathfrak{l}^{r'}) \subseteq T_n(\ell^r) \subseteq \ker \alpha$. As $V(\ell^{r',r'}) \subseteq V(\ell^{r,r})$ and the inclusion of the forms of level $V(\ell^{r,r})$ into those of level $V(\ell^{r',r'})$ commutes with the operators of (Gy7)(c) ([105], Lemma 2.3.3), hence with the ordinary and α-projectors, $f \in \pi \cap S_\lambda^{\mathrm{ord}}(V(\ell^{r',r'}), \alpha, \overline{\mathbb{Q}}_\ell)$, with the same eigenvalues.

Step 0: values of the extensions of α in $\mathcal{O}'$. If $\mathcal{O}'$ does not contain the values of every character of $T_n(\mathfrak{l})$ extending α, let $\mathcal{O}'' \supseteq \mathcal{O}'$ be the ring of integers of a finite extension of $\mathcal{O}'[1/\ell]$ containing them (finitely many roots of unity). By Lemma E.13, over $\mathcal{O}''$ the Hecke algebras are $\mathbb{T}_{\mathcal{O}'[H]} \otimes_{\mathcal{O}'} \mathcal{O}''$ and $\mathbb{T}_\tau \otimes_{\mathcal{O}'} \mathcal{O}''$ (as $\mathcal{O}''[H] = \mathcal{O}'[H] \otimes \mathcal{O}''$), the surjection of Lemma E.15 over $\mathcal{O}''$ is $q \otimes \mathrm{id}$ (both send each operator to the operator of the same name), and $\Lambda_1'' = \Lambda_1 \otimes_{\mathcal{O}'} \mathcal{O}''$; the hypotheses on π and f do not involve the coefficient ring ($\tau''^\vee \otimes_{\mathcal{O}''} \overline{\mathbb{Q}}_\ell = \tau^\vee \otimes_{\mathcal{O}'} \overline{\mathbb{Q}}_\ell$). Put $y_0'' = y_0 \otimes (\mathcal{O}'' \subseteq \overline{\mathbb{Q}}_\ell)$. As y_0 kills $\mathfrak{P}_{\lambda,\alpha}$, its restriction to Λ_1 is $\sigma \circ \chi_0$, χ_0 the homomorphism induced by $\alpha(w_0\nu)^{-1}(w_0\lambda)^{-1}$ and σ an $\mathcal{O}'$-algebra homomorphism on its image, which is the identity on the generators by the hypothesis on $y_0(\langle u \rangle)$; so $y_0|_{\Lambda_1} = \chi_0$, $y_0''|_{\Lambda_1''} = \chi_0 \otimes \mathrm{id}$, and y_0'' kills $\mathfrak{P}''_{\lambda,\alpha}$. If the case of values in the coefficient ring gives $y_0'' = y'' \circ (q \otimes \mathrm{id})$, then $y = y''|_{\mathbb{T}_\tau \otimes 1}$ is a point of $\mathbb{T}_\tau$ with $y_0 = y \circ q$. So assume that $\mathcal{O}'$ contains the values of every character of $T_n(\mathfrak{l})$ extending α, in particular those of α.

Step 1: the fibre of N_τ. Put $\Lambda = \Lambda_1$ (renormalised structure), $\mathfrak{P} = \mathfrak{P}_{\lambda,\alpha}$, $\mathcal{K} = \mathcal{O}'[1/\ell] = \Lambda_{\mathfrak{P}}/\mathfrak{P}\Lambda_{\mathfrak{P}}$ and $W = S_\lambda^{\mathrm{ord}}(V(\ell^{r',r'}), \alpha, \mathcal{K})$. By (Gd0) and Lemma E.12 for $U = V^{\mathrm{St}} \supseteq U' = V$, $\Gamma = G_1 = H$, $G_2 = 1$, $b_0 = 1$: (Yp1) the action of H on $\mathcal{S}_V$ commutes with the Hecke operators, and (Yp2) $\mathcal{S}_V$ is a direct summand of $\Lambda[H]^m$ for some m. For an $\mathcal{O}'$-algebra R and a finite

free $\mathcal{O}'$-module τ' with an action of H, $\varphi \mapsto \epsilon \circ \varphi$ (ϵ the coefficient of 1) is an isomorphism $\mathrm{Hom}_{\mathcal{O}'[H]}(\tau', R[H]) \to R \otimes_{\mathcal{O}'} \mathrm{Hom}_{\mathcal{O}'}(\tau', \mathcal{O}')$ with inverse $\psi \mapsto (t \mapsto \sum_{h\in H} \psi(h^{-1}t)h)$; so for every homomorphism $\Lambda \to A$ the natural map $\mathrm{Hom}_{\mathcal{O}'[H]}(\tau', \Lambda[H]) \otimes_\Lambda A \to \mathrm{Hom}_{\mathcal{O}'[H]}(\tau', A[H])$ is an isomorphism, and by additivity the same holds for the direct summand $\mathcal{S}_V$. With $A = \mathcal{K}$ and (Qh4) for $U = V$, $b_0 = 1$ and the level r' (applicable: $T_n(\mathfrak{l}^{r'}) \subseteq \ker \alpha$ and $\mathcal{O}'$ contains the values of the extensions of α; for the equivariance of (Gk3), the elements of $\prod_{v\in \mathrm{St}} \mathbb{G}_L(\mathcal{O}_{L_v^+})$ have trivial components at S_ℓ and normalise every $V(\ell^{c',c'})$), for $\tau' = \tau$ and $\tau' = \mathcal{O}'[H]$:

$$N_{\tau'} \otimes_\Lambda \mathcal{K} \cong \mathrm{Hom}_{\mathcal{K}[H]}(\tau' \otimes \mathcal{K}, W^*), \qquad W^* = \mathrm{Hom}_{\mathcal{K}}(W, \mathcal{K}),$$

compatibly with the Hecke operators, which act on the right through the transposes of the operators of (Gy7)(c).

Step 2: y_0 factors through q. Let $W_{\overline{\mathbb{Q}}_\ell} = W \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell = S^{\mathrm{ord}}_\lambda(V(\ell^{r',r'}), \alpha, \overline{\mathbb{Q}}_\ell)$, W_0 its joint generalised eigenspace for the eigencharacter y_0 of the operators of (Gy7)(c), and W' the sum of the others; both are H-stable (Yp1). The copy of π containing f has $\pi^{V(\ell^{r',r'})} = (\bigotimes_{v\in \mathrm{St}} \pi_v^{V_v}) \otimes X$, with H acting on the first factor and the operators of (Gy7)(c), which involve no place of St, on the second; as $f \neq 0$ is a joint eigenvector, X contains a non-zero joint eigenvector e with eigencharacter y_0, which is ordinary with $T_n(\ell^1)$ acting through α (the ordinary and α-projectors are given by these operators), and $(\bigotimes_{v\in \mathrm{St}} \pi_v^{V_v}) \otimes e \subseteq W_0$. By hypothesis W_0 has an H-constituent occurring in $\tau^\vee \otimes \overline{\mathbb{Q}}_\ell$, so ($H$ finite, characteristic 0) $Z = \mathrm{Hom}_H(\tau \otimes \overline{\mathbb{Q}}_\ell, W_0^*) \cong \mathrm{Hom}_H(W_0, \tau^\vee \otimes \overline{\mathbb{Q}}_\ell) \neq 0$. By Step 1, Z is a subspace of $N_\tau \otimes_\Lambda \overline{\mathbb{Q}}_\ell$ ($W^*_{\overline{\mathbb{Q}}_\ell} = W_0^* \oplus W'^*$), stable under the Hecke operators, on which every element t of $\mathbb{T}_{\mathcal{O}'[H]}$ acts, through q, with the single eigenvalue $y_0(t)$. If $q(t) = 0$, then t acts by 0 on N_τ, hence on Z, so $y_0(t) = 0$. Hence $y_0 = y \circ q$ for a homomorphism of $\mathcal{O}'$-algebras $y\colon \mathbb{T}_\tau \to \overline{\mathbb{Q}}_\ell$, continuous because q is a continuous surjection of compact spaces, hence a quotient map. □

Lemma E.15. *Let $\mathscr{H}$ over $\mathcal{O}'$, κ and $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ be as in the hypotheses of Proposition E.26, let c be an integer as in Proposition E.5, V_v ($v \in \mathrm{St}$) the kernel of reduction $\mathbb{G}_L(\mathcal{O}_{L_v^+}) \to \mathbb{G}_L(\mathcal{O}_{L_v^+}/\varpi_v^c)$, V the level and $H = \prod_{v\in\mathrm{St}} \mathbb{G}_L(\mathcal{O}_{L_v^+})/V_v$. Let τ_0 be a finite free $\mathcal{O}'$-module with an $\mathcal{O}'$-linear action of H, and $\tau = \mathcal{O}'[H]$ (left multiplication). Then there is a surjective homomorphism of Λ^+-algebras $q\colon \mathbb{T}_\tau \to \mathbb{T}_{\tau_0}$ sending $T_w^{(j)}$, $(T_w^{(n)})^{-1}$, $U_{\tilde{v}}^{(j)}$ and $\langle u\rangle$ to the operators of the same names; for every point y of $\mathbb{T}_{\tau_0}$, $y \circ q$ is a point of $\mathbb{T}_\tau$ with the same values on these operators; and (Hx1), (Hx2) of Proposition E.5 hold for the points of $\mathbb{T}_{\tau_0}$ (with $P_{y,w}$ and $\psi_{v,j}^y$ defined by the same formulas).*

Proof. Both type data have the same level V, hence the same $\mathcal{S}_V$ and the same $\mathbb{T}^{\mathrm{ord}}(V) \subseteq \mathrm{End}_{\mathcal{O}'}(\mathcal{S}_V)$. Evaluation at $1 \in \mathcal{O}'[H]$ is an isomorphism $N_\tau = \mathrm{Hom}_{\mathcal{O}'[H]}(\mathcal{O}'[H], \mathcal{S}_V) \to \mathcal{S}_V$, equivariant for $\mathbb{T}^{\mathrm{ord}}(V)$ (which commutes with H); so the image of $\mathbb{T}^{\mathrm{ord}}(V)$ in $\mathrm{End}_{\mathcal{O}'}(N_\tau)$ is $\mathbb{T}^{\mathrm{ord}}(V)$ itself: $\mathbb{T}_\tau = \mathbb{T}^{\mathrm{ord}}(V)$. By definition $\mathbb{T}_{\tau_0}$ is the image of $\mathbb{T}^{\mathrm{ord}}(V)$ in $\mathrm{End}_{\mathcal{O}'}(N_{\tau_0})$ under its action on $N_{\tau_0} = \mathrm{Hom}_{\mathcal{O}'[H]}(\tau_0, \mathcal{S}_V)$ through $\mathcal{S}_V$; let q be this restriction map. It is a surjective homomorphism of Λ^+-algebras sending each generator to the operator of the same name, and it respects the renormalised Λ_1-structures, which are defined from the $\langle u\rangle$. Both algebras are finite over Λ_1 ((Qh2) of Lemma E.10 for $\mathbb{T}^{\mathrm{ord}}(V)$, and $\mathbb{T}_{\tau_0}$ is a quotient), so q is continuous for their Λ_1-adic topologies, and for a point y of $\mathbb{T}_{\tau_0}$ the composite $y \circ q$ is a continuous homomorphism of $\mathcal{O}'$-algebras, i.e. a point of $\mathbb{T}_\tau$; it kills $\mathfrak{P}_{\lambda,\alpha}$ if and only if y does, and $y \circ q$ and y take the same values on $T_w^{(j)}$, $U_{\tilde{v}}^{(j)}$, $\langle u\rangle$. The statements (Hx1) and (Hx2) for a point depend only on these values ($P_{y,w}$ through $y(T_w^{(j)})$, $\psi_{v,j}^y$ through $y(\langle d_j(z)\rangle)$ and $y(U_{\tilde{v}}^{(j)})$) and on the condition $y(\mathfrak{P}_{\lambda,\alpha}) = 0$; so they hold for y because they hold for $y \circ q$ by Proposition E.5. □

Lemma E.16. *Let $\mathscr{H}$ over $\mathcal{O}'$, κ and $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ be as in the hypotheses of Proposition E.26, $\mathbb{F}$ the residue field of $\mathcal{O}'$, c an integer as in Proposition E.5, and $((V_v)_{v\in\mathrm{St}}, \tau)$ a type datum over $\mathscr{H}$ with V_v the kernel of reduction $\mathbb{G}_L(\mathcal{O}_{L_v^+}) \to \mathbb{G}_L(\mathcal{O}_{L_v^+}/\varpi_v^c)$ for $v \in \mathrm{St}$ (and τ arbitrary). Let Q and K' be as in the levels at auxiliary places of Definition 50.8 ($Q = \varnothing$ allowed); in this statement and its proof V, N_τ and $\mathbb{T}_\tau$ denote $V(K')$, $N_\tau(K')$ and $\mathbb{T}_\tau(K')$. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_\tau$ with $\mathbb{T}_\tau/\mathfrak{m} = \mathbb{F}$ such that in $\mathbb{F}$, for all w, $1 \le j \le n$, $v \in S_\ell$ and $z \in \mathcal{O}_{L_{\tilde{v}}}^\times$:*

(Mx) $(\mathrm{N}w)^{j(j-1)/2} T_w^{(j)} \equiv e_j\big((\bar{\chi}_a\bar{\kappa})(\mathrm{Frob}_w)\big)_{1\le a\le n}$, $\langle d_j(z)\rangle \equiv (\bar{\chi}^{j-1}\bar{\kappa}\bar{\chi}_j)(\mathrm{Art}(z))$ *and* $U_{\tilde{v}}^{(j)} \equiv U_{\tilde{v}}^{(j-1)}\,(\bar{\chi}^{j-1}\bar{\kappa}\bar{\chi}_j)(\mathrm{Art}(\varpi_{\tilde{v}}))$ *modulo* $\mathfrak{m}$ *(e_j the j-th elementary symmetric function, bars denoting reductions, $U_{\tilde{v}}^{(0)} = 1$).*

(Gd0) $[L^+ : \mathbb{Q}] \ge 2$*; the hypotheses of Theorem B.91 hold for $F = L$, $\mathbb{G} = \mathbb{G}_L$, $T = T_L \cup Q$, $b_0 = 1$ and $U = V$, the level of the type datum, as well as for $U = V^{\mathrm{St}}$, the group with $V_v^{\mathrm{St}} = \mathbb{G}_L(\mathcal{O}_{L_v^+})$ for $v \in \mathrm{St}$ and $V_v^{\mathrm{St}} = V_v$ otherwise; $\mathbb{T}_\tau$ is commutative, finite and torsion-free over Λ_1 (for the diamond structure and for the renormalised one); the $U_{\tilde{v}}^{(j)}$ are units of $\mathbb{T}_\tau$; and $\mathbb{T}_\tau$ is topologically generated over $\mathcal{O}'$ by the $T_w^{(j)}$, $(T_w^{(n)})^{-1}$, $U_{\tilde{v}}^{(j)}$ and $\langle u\rangle$, $u \in T_n(\mathcal{O}_{L^+} \otimes \mathbb{Z}_\ell)$. Every homomorphism of $\mathcal{O}'$-algebras from B_n^{po}, from $\mathbb{T}_\tau$, or from a quotient of either, to $\overline{\mathbb{Q}}_\ell$ has image in the ring of integers of a finite extension of $\mathbb{Q}_\ell$, is local onto its image, and is continuous; its image is an $\mathcal{O}'$-order, finite over $\mathcal{O}'$.*

Call a point y of $\mathbb{T}_{\tau,\mathfrak{m}}$ ***regular*** *if $y(\mathfrak{P}_{\lambda,1}) = 0$ for a dominant weight λ with $\lambda_{\tau',j} > \lambda_{\tau',j+1}$ for all embeddings τ' and all j; for such y let r_y be, if $Q = \varnothing$, as in (Hx1) of Proposition E.5 (through Lemma E.15), and in general the representation r_π of (Gy5) for an irreducible constituent π given by Lemma E.11 for y (level K', $\alpha = 1$, $r = 1$): a continuous semisimple representation of G_L, well defined up to isomorphism, with $r_y(\mathrm{Frob}_w)$ of characteristic polynomial $P_{y,w}$ for every w and $r_y^c \cong r_y^\vee \chi^{1-n}$ (the two definitions agree if $Q = \varnothing$); put $\tilde{r}_y = r_y \otimes \kappa^{-1}$. Let $\mathfrak{a}$ be an ideal of $\mathbb{T}_{\tau,\mathfrak{m}}$, $A = \mathbb{T}_{\tau,\mathfrak{m}}/\mathfrak{a}$, and $\mathcal{C}$ a non-empty set of regular points with $\bigcap_{y\in\mathcal{C}} \ker y = \mathfrak{a}$.*

(Gd1) *There is a continuous A-valued determinant D^{H} of dimension n on G_L with $y \circ D^{\mathrm{H}} = \det \circ r_y$ for every $y \in \mathcal{C}$; for every w, $\chi_{D^{\mathrm{H}}}(\mathrm{Frob}_w) = \sum_{j=0}^n (-1)^j (\mathrm{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j}$ in $A[X]$; and for every point y of A and every continuous $\rho\colon G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that, for every w, ρ is unramified above w and $\rho(\mathrm{Frob}_w)$ has characteristic polynomial $P_{y,w}$, $y \circ D^{\mathrm{H}} = \det \circ \rho$.*

(Gd2) *Suppose moreover that for every $y \in \mathcal{C}$: (i) $\tilde{r}_y$ is trivial on N^Q, (ii) for every $v \in S_\ell$, $r_y|_{D_v}$ is triangular with diagonal $(\chi^{1-j}\psi_{v,j}^y)_{1\le j\le n}$, and (iii) for every $v \in S_\ell$, $1 \le j \le n$ and every ℓ-power root of unity $\zeta \in L_{\tilde{v}}$, $y(\langle d_j(\zeta)\rangle) = \kappa(\mathrm{Art}(\zeta))\,\xi_{v,j}(g_\zeta)$, with $\xi_{v,j}$ the wild values of $\mathscr{H}$ and g_ζ as in Lemma 50.24 (by (Wv2) of Lemma 50.24, (iii) is empty at every v with $\bar{\omega}|_{D_v} \ne 1$: there $\zeta = 1$ only). Then $A \in \mathrm{CNL}_{\mathcal{O}'}$; there are a continuous A-valued determinant D^Q of dimension n on $\Gamma^{0,Q}$ and continuous characters $\psi_{v,j}^\flat\colon D_v \to A^\times$ ($v \in S_\ell$, $1 \le j \le n$) satisfying (Od1), (Od2) and (Od3) of Definition 50.2 with π^Q, $\Gamma^{0,Q}$ in place of π, Γ^0, with D^Q reducing to $\det \circ \mathrm{diag}(\bar{\chi}_1, \dots, \bar{\chi}_n)$ and $\psi_{v,j}^\flat \equiv \bar{\chi}_j$ modulo $\mathfrak{m}_A$, such that for all w, $v \in S_\ell$, $1 \le j \le n$, $z \in \mathcal{O}_{L_{\tilde{v}}}^\times$ and $y \in \mathcal{C}$: $\chi_{D^Q}(\pi^Q(\mathrm{Frob}_w)) = \sum_{j=0}^n (-1)^j \kappa(\mathrm{Frob}_w)^{-j} (\mathrm{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j}$, $\psi_{v,j}^\flat(\mathrm{Art}(z)) = (\chi^{j-1}\kappa)(\mathrm{Art}(z))^{-1}\langle d_j(z)\rangle$, $\psi_{v,j}^\flat(\mathrm{Art}(\varpi_{\tilde{v}})) = (\chi^{j-1}\kappa)(\mathrm{Art}(\varpi_{\tilde{v}}))^{-1} U_{\tilde{v}}^{(j)} (U_{\tilde{v}}^{(j-1)})^{-1}$, $y \circ D^Q = \det \circ \tilde{r}_y^Q$ and $y \circ \psi_{v,j}^\flat = \chi^{1-j}\kappa^{-1}\psi_{v,j}^y$, where $\tilde{r}_y^Q$ is the representation of $\Gamma^{0,Q}$ with $\tilde{r}_y^Q \circ \pi^Q = \tilde{r}_y$. If $Q = \varnothing$, there is moreover a surjective homomorphism of $\mathcal{O}'$-algebras $\theta\colon B_n^{\mathrm{po}} \to A$ such*

that, in A, for all w, $1 \le j \le n$, $v \in S_\ell$ and $z \in \mathcal{O}_{L_{\tilde v}}^\times$,

$$(\mathrm{N}w)^{j(j-1)/2} T_w^{(j)} = \kappa(\mathrm{Frob}_w)^j\, \theta\big(\Lambda_j(D^{\mathrm{u}})(\pi_{\mathscr{H}}(\mathrm{Frob}_w))\big),$$
$$\langle d_j(z)\rangle = (\chi^{j-1}\kappa)(\mathrm{Art}(z))\, \theta\big(\psi_{v,j}^{\mathrm{d}}(\mathrm{Art}(z))\big),$$
$$U_{\tilde v}^{(j)} = U_{\tilde v}^{(j-1)}\, (\chi^{j-1}\kappa)(\mathrm{Art}(\varpi_{\tilde v}))\, \theta\big(\psi_{v,j}^{\mathrm{d}}(\mathrm{Art}(\varpi_{\tilde v}))\big),$$

and, for every $y \in \mathcal{C}$, $y \circ \theta \circ D^{\mathrm{u}} = \det \circ \tilde r'_y$ and $y \circ \theta \circ \psi_{v,j}^{\mathrm{d}} = \chi^{1-j}\kappa^{-1}\psi_{v,j}^{y}$, where $\tilde r'_y$ is the representation of $\Gamma^0_{\mathscr{H}}$ with $\tilde r'_y \circ \pi_{\mathscr{H}} = \tilde r_y$.

Proof. Step 0: points are continuous. Let R be a complete Noetherian semilocal $\mathcal{O}'$-algebra, finite over a complete Noetherian local ring with finite residue field, and $y\colon R \to \overline{\mathbb{Q}}_\ell$ a homomorphism of $\mathcal{O}'$-algebras. Then $R/\ker y$ is a complete Noetherian local domain (a complete semilocal domain is local) of characteristic 0 whose fraction field is algebraic over $\mathrm{Frac}(\mathcal{O}')$, so it has dimension one and is finite over $\mathcal{O}'$ (complete Nakayama), and $y(R)$ is an order in a finite extension E of $\mathbb{Q}_\ell$, contained in $\mathcal{O}_E$; y is local onto its image, hence continuous. This applies to B (a complete Noetherian local ring), to $\mathbb{T}$ and to their quotients, by Step 1; it is the last sentence of (Gd0).

Step 1: (Gd0), and finiteness. Let V be the level of the type datum. L/L^+ is unramified at all finite places (Rd2), so $[L^+ : \mathbb{Q}] \ge 2$: if $L^+ = \mathbb{Q}$, then L is imaginary quadratic, its discriminant has absolute value at least 3, and some prime ramifies in $L/\mathbb{Q}$. The hypotheses of Theorem B.91 hold for $F = L$, $\mathbb{G} = \mathbb{G}_L$, $T = T_L \cup Q$, $b_0 = 1$ and $U = V$ or $U = V^{\mathrm{St}}$ (the two differ only at St, which plays no role below): L is imaginary CM and the places above ℓ split by (Rd2); $\mathcal{O}' \supseteq \tau'(\mathcal{O}_L)$ for every embedding τ' by (Rd1); $\mathbb{G}_L$ comes from the involution $X \mapsto {}^tX^c$ and the order $\mathrm{M}_n(\mathcal{O}_L)$, is compact at the real places (positive definite form) and quasi-split at every finite place (split places: GL_n; the non-split places are unramified in L by (Rd2), where the form $\sum x_i\bar x_i$ of discriminant 1 is quasi-split, for n even because $(-1)^{n/2}$ is a unit, hence a norm from the unramified quadratic extension); the places of T_L split (Rd2), those of Q by assumption; $V_v = \mathbb{G}_L(\mathcal{O}_{L_v^+})$ for $v \in S_\ell$ and for $v \notin T_L \cup Q$; and $V(\ell^{1,1})$ is sufficiently small, its component at v_0 being $1 + p\,\mathrm{M}_n(\mathcal{O}_{L_{\tilde v_0}})$ with p odd and $L_{\tilde v_0}/\mathbb{Q}_p$ unramified (Rd3), which has no non-trivial element of finite order . By (Qh2) (with $b_0 = 1$), $\mathbb{T}^{\mathrm{ord}}(V)$ is commutative and finite over Λ_1, and $\mathcal{S}_V$ is free of finite rank over Λ_1, so $N_\tau \subseteq \mathrm{Hom}_{\mathcal{O}'}(\tau, \mathcal{S}_V) \cong \mathcal{S}_V^{\mathrm{rk}\,\tau}$ is torsion-free and finitely generated over Λ_1, and $\mathbb{T}$, acting faithfully on N_τ, is torsion-free over Λ_1 (if $0 \ne \mu \in \Lambda_1$ and $\mu t = 0$, then $\mu(tm) = 0$, so $tm = 0$ for all $m \in N_\tau$, and $t = 0$). So $\mathbb{T}$ is a commutative, finite, torsion-free Λ_1-algebra (for the diamond structure, hence for the renormalised one, which differs by the automorphism $[u] \mapsto (w_0\nu)^{-1}(u)[u]$ of Λ_1), complete and semilocal, the product of its finitely many localisations at maximal ideals; so $\mathbb{T}_{\mathfrak{m}}$ and A are complete Noetherian local $\mathcal{O}'$-algebras, finite over Λ_1, compact. The operators $U_{\tilde v}^{(j)}$ are units of $\mathbb{T}$: they are invertible on the ordinary module $\mathcal{S}_V$ and hence on N_τ, and an element of $\mathbb{T}$ lying in a maximal ideal $\mathfrak{n}$ acts nilpotently on $N_{\tau,\mathfrak{n}}/\mathfrak{n}N_{\tau,\mathfrak{n}}$, which is non-zero by Nakayama and the faithfulness of $\mathbb{T}_{\mathfrak{n}}$ on $N_{\tau,\mathfrak{n}}$ (N_τ is finitely generated over Λ_1 as $\mathcal{S}_V$ is, by (Qh2)). Finally $\mathbb{T}$ is topologically generated over $\mathcal{O}'$ by the images of the $T_w^{(j)}$, $(T_w^{(n)})^{-1}$, $U_{\tilde v}^{(j)}$ and $\langle u\rangle$, $u \in T_n(\mathcal{O}_{L^+} \otimes \mathbb{Z}_\ell)$: the $\mathcal{O}'$-algebra they generate is dense in the Λ^+-algebra they generate, $\mathcal{O}'[T_n(\mathcal{O}_{L^+} \otimes \mathbb{Z}_\ell)]$ being dense in Λ^+ and $\mathbb{T}$ being finite over Λ_1; the same holds for the direct factor $\mathbb{T}_{\mathfrak{m}}$ and its quotient A.

Step 2: (Gd1). First the representations r_y of a regular point y, with λ. By Lemma E.11 (level K', $\alpha = 1$, $r = 1$) there is an irreducible constituent π of $S_\lambda(\overline{\mathbb{Q}}_\ell)$ with π_v unramified, with $T_w^{(j)}$-eigenvalues $y(T_w^{(j)})$, at every split $v \notin T_L \cup Q$; by (Gy5)(i),(ii), r_π is continuous and semisimple, its restriction above w has unramified semisimplification, $r_\pi(\mathrm{Frob}_w)$ has characteristic polynomial $P_{y,w}$, and $r_\pi^c \cong r_\pi^\vee \chi^{1-n}$. Two continuous semisimple representations of G_L

with the same characteristic polynomials at all Frob_w are isomorphic (the Frob_w are dense by the Chebotarev density theorem; Brauer–Nesbitt), so r_y is well defined up to isomorphism, and for $Q = \varnothing$ it agrees with the r_y of (Hx1), which is semisimple with the same characteristic polynomials ((Hx1)(i)). In every case r_y has the properties (i) and (ii) of (Hx1), the only ones used below. For $y \in \mathcal{C}$, r_y is continuous on the compact group G_L, so after conjugation it takes values in $\mathrm{GL}_n(\mathcal{O}_{E_y})$ for a finite extension E_y of $\mathbb{Q}_\ell$ (its image is a compact subgroup of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, hence by the Baire category theorem contained in $\mathrm{GL}_n(E)$ for some finite E, and it stabilises an $\mathcal{O}_E$-lattice); enlarging E_y we may assume $y(A) \subseteq \mathcal{O}_{E_y}$ (Step 0). By (Hx1)(i), for every w the characteristic polynomial of $r_y(\mathrm{Frob}_w)$ is $P_{y,w}$, the image under y of $P_w = \sum_j (-1)^j (\mathrm{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j} \in A[X]$; this does not depend on the choice of Frob_w, r_y being unramified above w ((Hx1)(i): the semisimplification of the restriction is unramified, and the characteristic polynomial of an element depends only on the semisimplification). The map $A \to \prod_{y\in\mathcal{C}} \mathcal{O}_{E_y}$ is injective (as $\bigcap_{y\in\mathcal{C}} \ker y = \mathfrak{a}$) and continuous (Step 0), A is compact (Step 1), and the Frobenius elements Frob_w are dense in G_L by the Chebotarev density theorem (the places w have density one). By (Ch4) there is a continuous A-valued determinant D^{H} of dimension n on G_L with $y \circ D^{\mathrm{H}} = \det \circ r_y$ for all $y \in \mathcal{C}$; and $\chi_{D^{\mathrm{H}}}(\mathrm{Frob}_w) = P_w$, both having the same image under the injective map $A[X] \to \prod_y \mathcal{O}_{E_y}[X]$. Let y be any point of A and ρ as in (Gd1). Then $\det \circ \rho$ and $y \circ D^{\mathrm{H}}$ are continuous determinants with the same characteristic polynomial $y(P_w)$ at every Frob_w; by continuity the characteristic polynomials agree on all of G_L, and by (Ch1) the determinants are equal.

Step 3: the determinant on $\Gamma^{0,Q}$. Assume the hypotheses of (Gd2), and write $\tilde{r}'_y = \tilde{r}^Q_y$. For $y \in \mathcal{C}$, $\tilde{r}_y = r_y \otimes \kappa^{-1}$ is trivial on N^Q by (i), so $\tilde{r}_y = \tilde{r}'_y \circ \pi^Q$ with $\tilde{r}'_y$ continuous on $\Gamma^{0,Q}$ ($\pi^Q \colon G_L \to \Gamma^{0,Q}$ is a quotient map of profinite groups), with values in $\mathrm{GL}_n(\mathcal{O}_{E_y})$ after enlarging E_y to contain the values of κ. For every w, the inertia groups above w lie in N^Q (the $\bar{\chi}_a$ are trivial on them by (Rd4), and w lies above no place of Q) and κ is unramified above w (Rd7), so $\pi^Q(\mathrm{Frob}_w)$ and $\kappa(\mathrm{Frob}_w) \in \mathcal{O}'^{\times}$ are well defined, and the characteristic polynomial of $\tilde{r}'_y(\pi^Q(\mathrm{Frob}_w)) = \kappa(\mathrm{Frob}_w)^{-1} r_y(\mathrm{Frob}_w)$ is the image under y of $\tilde{P}_w = \sum_j (-1)^j \kappa(\mathrm{Frob}_w)^{-j} (\mathrm{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j} \in A[X]$. The elements $\pi^Q(\mathrm{Frob}_w)$ are dense in $\Gamma^{0,Q} = \pi^Q(G_L)$. By (Ch4), as in Step 2, there is a continuous A-valued determinant D of dimension n on $\Gamma^{0,Q}$ with $y \circ D = \det \circ \tilde{r}'_y$ for $y \in \mathcal{C}$ and $\chi_D(\pi^Q(\mathrm{Frob}_w)) = \tilde{P}_w$.

Step 4: the characters at ℓ. For $v \in S_\ell$ and $1 \le j \le n$ let $\psi^A_{v,j} \colon D_v \to A^\times$ be the continuous character with $\psi^A_{v,j}(\mathrm{Art}(z)) = \langle d_j(z) \rangle$ for $z \in \mathcal{O}^\times_{L_{\tilde{v}}}$ and $\psi^A_{v,j}(\mathrm{Art}(\varpi_{\tilde{v}})) = U^{(j)}_{\tilde{v}} (U^{(j-1)}_{\tilde{v}})^{-1}$ (well defined: D_v is identified with $G_{L_{\tilde{v}}}$, D_v^{ab} is the profinite completion of $L^\times_{\tilde{v}} = \mathcal{O}^\times_{L_{\tilde{v}}} \times \varpi^{\mathbb{Z}}_{\tilde{v}}$, the diamond map is continuous, the $U^{(j)}_{\tilde{v}}$ are units (Step 1), and $k \mapsto t^k$ extends continuously to $\widehat{\mathbb{Z}}$ for $t \in A^\times$, $A^\times$ being profinite). Put $\psi^\flat_{v,j} = \chi^{1-j} \kappa^{-1} \psi^A_{v,j}$. For $y \in \mathcal{C}$, $y \circ \psi^A_{v,j} = \psi^y_{v,j}$, both being continuous with the same values on the dense image of Art.

Step 5: a polarised ordinary determinant to A. (Od1): for $y \in \mathcal{C}$, by (Hx1)(ii) and (Rd7) ($\kappa\kappa^c = \chi^{1-n}$), $\tilde{r}^c_y = r^c_y \otimes (\kappa^c)^{-1} \cong r^\vee_y \chi^{1-n} (\kappa^c)^{-1} = r^\vee_y \otimes \kappa = \tilde{r}^\vee_y$, so $\tilde{r}_y(cg^{-1}c)$ and ${}^t\tilde{r}_y(g)$ have the same characteristic polynomial for $g \in G_L$: the coefficients of $\chi_D(\pi^Q(cg^{-1}c)) - \chi_D(\pi^Q(g))$ vanish at every $y \in \mathcal{C}$, hence are 0. (Od2): for $y \in \mathcal{C}$ and $v \in S_\ell$, by (ii) $r_y|_{D_v}$ is triangular with diagonal $(\chi^{1-j}\psi^y_{v,j})_j$, so $\tilde{r}_y|_{D_v}$ is triangular with diagonal $(\chi^{1-j}\kappa^{-1}\psi^y_{v,j})_j = (y \circ \psi^\flat_{v,j})_j$, and by (Fg1) (with $C = \overline{\mathbb{Q}}_\ell$) $\prod_{a=1}^n \big(\tilde{r}_y(g_a) - y(\psi^\flat_{v,a}(g_a))\big) = 0$ for $g_1, \dots, g_n \in D_v$. As $y \circ \mathrm{tr}_D = \mathrm{tr} \circ \tilde{r}'_y$ on $A[\Gamma^{0,Q}]$ (traces commute with base change), y kills $\mathrm{tr}_D(\gamma \prod_a (\pi^Q(g_a) - \psi^\flat_{v,a}(g_a)))$ for all $\gamma \in \Gamma^{0,Q}$, which is therefore 0. Reductions: $A/\mathfrak{m}_A = \mathbb{F}$; by (Mx) the reduction of $\tilde{P}_w$ is $\prod_a (X - \bar{\chi}_a(\mathrm{Frob}_w))$ ($\bar{\kappa}$ cancels), the characteristic polynomial of $\bar{D}_n$ at $\pi^Q(\mathrm{Frob}_w)$; the two continuous $\mathbb{F}$-valued determinants $D \bmod \mathfrak{m}_A$ and $\bar{D}_n$ (here $\det \circ \mathrm{diag}(\bar{\chi}_1, \dots, \bar{\chi}_n)$ on $\Gamma^{0,Q}$; the

$\bar\chi_a$ factor through it as $N^Q \subseteq N_{\mathscr H}$) have the same characteristic polynomials on a dense set, hence everywhere ($\mathbb F$ is discrete), and are equal by (Ch1). By (Mx), $\psi^A_{v,j} \equiv \bar\chi^{j-1}\bar\kappa\bar\chi_j$ on the dense image of Art, so $\psi^\flat_{v,j} \equiv \bar\chi_j$ modulo $\mathfrak m_A$. A is a complete Noetherian local $\mathcal O'$-algebra with residue field $\mathbb F$ (Step 1), i.e. $A \in \mathrm{CNL}_{\mathcal O'}$. So $D^Q = D$ and the $\psi^\flat_{v,j}$ have the properties stated in (Gd2) before θ, the identities having been shown in Steps 3 and 4. (Od3): let $v \in S_\ell$, $1 \le j \le n$ and $\zeta \in L_{\tilde v}$ an ℓ-power root of unity. As ζ is a unit and g_ζ has image $\mathrm{Art}(\zeta)$ in D^{ab}_v, $\psi^\flat_{v,j}(g_\zeta) = \chi(g_\zeta)^{1-j}\kappa(\mathrm{Art}(\zeta))^{-1}\langle d_j(\zeta)\rangle = \kappa(\mathrm{Art}(\zeta))^{-1}\langle d_j(\zeta)\rangle$, since $\chi(g_\zeta) = 1$ by (Wv1) of Lemma 50.24. By (iii), every $y \in \mathcal C$ maps $\psi^\flat_{v,j}(g_\zeta) - \xi_{v,j}(g_\zeta)$ to 0, so this element of A is 0 ($\bigcap_{y\in\mathcal C}\ker y = 0$ in A). By (Wv1) of Lemma 50.24 for the continuous homomorphism $\psi^\flat_{v,j}\colon D_v \to A^\times$, $\psi^\flat_{v,j}(g)$ is the image of $\xi_{v,j}(g)$ for every $g \in \mu_v$: (Od3). If $Q = \varnothing$, $(D, (\psi^\flat_{v,j}))$ is therefore a polarised ordinary determinant of rank n over $\mathscr H$ to A ((Od1)–(Od3)); let $\theta\colon B \to A$ classify it.

Step 6: the identities ($Q = \varnothing$). $\theta \circ D^{\mathrm u} = D$ and $\theta \circ \psi^{\mathrm d}_{v,j} = \psi^\flat_{v,j}$. The coefficient of X^{n-j} in $\chi_D(\pi^Q(\mathrm{Frob}_w)) = \tilde P_w$ gives $\theta(\Lambda_j(D^{\mathrm u})(\pi^Q(\mathrm{Frob}_w))) = \kappa(\mathrm{Frob}_w)^{-j}(\mathrm N w)^{j(j-1)/2}T^{(j)}_w$; the definition of $\psi^\flat$ gives the identities for $\langle d_j(z)\rangle$ and $U^{(j)}_{\tilde v}$; and for $y \in \mathcal C$, $y\circ\theta\circ D^{\mathrm u} = y\circ D = \det\circ\tilde r'_y$ and $y\circ\theta\circ\psi^{\mathrm d}_{v,j} = \chi^{1-j}\kappa^{-1}\psi^y_{v,j}$ (Step 4).

Step 7: θ is surjective ($Q = \varnothing$). $\theta(B) \cong B/\ker\theta$ is a local subring of A, compact and hence closed. By Step 6, with $\mathrm N w$ a power of a prime $\ne \ell$ and $\kappa(\mathrm{Frob}_w) \in \mathcal O'^\times$, every $T^{(j)}_w$ lies in $\theta(B)$; $T^{(n)}_w$ is a unit of A, so it does not lie in $\theta(\mathfrak m_B) \subseteq \mathfrak m_A$, and it is a unit of the local ring $\theta(B)$, whence $(T^{(n)}_w)^{-1} \in \theta(B)$. Again by Step 6, $\langle d_j(z)\rangle \in \theta(B)$ (as $(\chi^{j-1}\kappa)(\mathrm{Art}(z)) \in \mathcal O'^\times$), so $\langle u\rangle \in \theta(B)$ for every $u \in T_n(\mathcal O_{L^+}\otimes\mathbb Z_\ell) = \prod_{v\in S_\ell} T_n(\mathcal O_{L^+_v})$, each such u being a finite product of elements $d_j(z)$; and by induction on j, $U^{(j)}_{\tilde v} \in \theta(B)$. By the last sentence of Step 1, $\theta(B) = A$. □

Lemma E.17. *Let $\mathscr H$ over $\mathcal O'$ (residue field $\mathbb F$), κ and c be as in Lemma E.16, let $\iota\colon \overline{\mathbb Q}_\ell \to \mathbb C$ be an isomorphism, let $((V_v)_{v\in\mathrm{St}}, \tau)$ be a type datum over $\mathscr H$ with V_v the kernel of reduction modulo ϖ_v^c ($v \in \mathrm{St}$), and let Q, K' be as in the levels at auxiliary places of Definition 50.8. Here $Q = \varnothing$ is allowed; as in Lemma E.16, $\mathbb T_\tau$ denotes $\mathbb T_\tau(K')$, and w denotes a place above a split place outside $T_L \cup Q$. Suppose that*

- **(Zt)** *for every dominant weight λ, every irreducible constituent π of $S_\lambda(\overline{\mathbb Q}_\ell)$ with $\mathrm{Hom}_H(\tau^\vee \otimes_{\mathcal O'} \overline{\mathbb Q}_\ell, \bigotimes_{v\in\mathrm{St}} \pi_v^{V_v}) \ne 0$ ($\tau^\vee = \mathrm{Hom}_{\mathcal O'}(\tau, \mathcal O')$) and every $v \in \mathrm{St}$, the Weil–Deligne representation $\mathrm{rec}(\iota\pi_v \circ \iota_{\tilde v}^{-1})$ has monodromy 0 and its restriction to the inertia group of $L_{\tilde v}$ is isomorphic to $\bigoplus_{a=1}^n \iota\circ([\bar\chi_a]\kappa)$.*

Let $\mathfrak m$ be a maximal ideal of $\mathbb T_\tau$ with $\mathbb T_\tau/\mathfrak m = \mathbb F$ satisfying (Mx), and let $\mathcal C$ be the set of points y of $\mathbb T_{\tau,\mathfrak m}$ for which there is a dominant weight λ with $\lambda_{\tau',j} > \lambda_{\tau',j+1}$ for all τ', j and $y(\langle u\rangle) = (w_0\lambda)(u)^{-1}$ for every $u \in T_n(\mathcal O_{L^+}\otimes\mathbb Z_\ell)$ (regular points of trivial nebentypus). Then:

- **(Zp1)** *every regular point y of $\mathbb T_{\tau,\mathfrak m}$ satisfies (i) of (Gd2): $\tilde r_y = r_y \otimes \kappa^{-1}$ is trivial on N^Q (Definition 53.8; $N^\varnothing = N_{\mathscr H}$);*
- **(Zp2)** *every $y \in \mathcal C$ is regular and satisfies (ii) of (Gd2): for $v \in S_\ell$, $r_y|_{D_v}$ is triangular with diagonal $(\chi_{\mathrm{cyc}}^{1-j}\psi^y_{v,j})_{1\le j\le n}$;*
- **(Zp3)** *$\bigcap_{y\in\mathcal C}\ker y = 0$ in $\mathbb T_{\tau,\mathfrak m}$;*
- **(Zp4)** *let Δ_ℓ be the group of elements of ℓ-power order of $T_n(\mathcal O_{L^+}\otimes\mathbb Z_\ell)$, $\chi\colon \Delta_\ell \to \mathcal O'^\times$ a homomorphism, $\mathfrak a_\chi$ the intersection of the minimal prime ideals P of $\mathbb T_{\tau,\mathfrak m}$ with $\langle\delta\rangle - \chi(\delta) \in P$ for all $\delta \in \Delta_\ell$ ($\mathfrak a_\chi = \mathbb T_{\tau,\mathfrak m}$ if there is none), and $\mathcal C_\chi = \{y \in \mathcal C : y(\langle\delta\rangle) = \chi(\delta)$ for all $\delta \in$*

$\Delta_\ell\}$. *Then every point y of $\mathbb{T}_{\tau,\mathfrak{m}}$ with $y(\langle\delta\rangle)=\chi(\delta)$ for all $\delta\in\Delta_\ell$ satisfies $y(\mathfrak{a}_\chi)=0$, and $\bigcap_{y\in\mathcal{C}_\chi}\ker y=\mathfrak{a}_\chi$;*

(Zp5) *with Δ_ℓ, χ and $\mathfrak{a}_\chi$ as in (Zp4), let $N=e_\mathfrak{m}N_\tau(K')$ ($e_\mathfrak{m}$ the idempotent of $\mathbb{T}_\tau$ for $\mathfrak{m}$) and $N^\chi=N/\sum_{\delta\in\Delta_\ell}(\langle\delta\rangle-\chi(\delta))N$. Then N^χ is a free Λ_1-module of finite rank ($\Lambda_1=\mathcal{O}'[[T_n(\ell^1)]]$, diamond structure), and its annihilator in $\mathbb{T}_{\tau,\mathfrak{m}}$ is $\mathfrak{a}_\chi$.*

Proof. Step 0: the automorphic representation of a regular point. Let y be a point of $\mathbb{T}$ with $y(\mathfrak{P}_{\lambda,1})=0$ for a dominant λ. By Lemma E.11 (level K', $\alpha=1$, $r=1$) there is an irreducible constituent π of $S_\lambda(\overline{\mathbb{Q}}_\ell)$ with $\mathrm{Hom}_H(\tau^\vee\otimes\overline{\mathbb{Q}}_\ell,\bigotimes_{v\in\mathrm{St}}\pi_v^{V_v})\neq0$, π_v unramified with $T_w^{(j)}$-eigenvalues $y(T_w^{(j)})$ at every split $v\notin T_L\cup Q$, and a non-zero $f\in\pi\cap S_\lambda^{\mathrm{ord}}(V(\ell^{E,E}),1,\overline{\mathbb{Q}}_\ell)$ (E the maximal ramification index of $L^+/\mathbb{Q}$ at the places above ℓ) on which the operators assigned by (Gy7)(c) to $U_{\tilde v}^{(j)}$ and $\langle u\rangle$ act by $y(U_{\tilde v}^{(j)})$ and $y(\langle u\rangle)$. By (Gy5)(i), r_π is semisimple with characteristic polynomial $P_{y,w}$ at Frob_w for all w; so is r_y (its definition in Lemma E.16); by the Chebotarev density theorem and the Brauer–Nesbitt theorem, $r_y\cong r_\pi$ if λ is regular. Also $[L^+:\mathbb{Q}]\geq2$ (Lemma E.16, (Gd0)), so Lemma E.8 applies to π. Let $\tilde r=r_\pi\otimes\kappa^{-1}$. Its residual semisimplification has the characteristic polynomials of $\bigoplus_a\bar\chi_a$: by (Mx) they agree at every Frob_w, hence (continuity, Chebotarev) at every $g\in G_L$.

Step 1: (Zp1). Let y be regular and π, $\tilde r$ as in Step 0; $\tilde r_y\cong\tilde r$, so $K=\ker\tilde r_y\cap G_L$ is a closed normal subgroup of G_L with $cKc=K$, as $\tilde r^c\cong\tilde r^\vee$ ((Gy5)(ii) and (Rd7): $\tilde r^c=r_\pi^c\otimes(\kappa^c)^{-1}\cong r_\pi^\vee\chi^{1-n}(\kappa^c)^{-1}=r_\pi^\vee\otimes\kappa$). As $G_{L^+}=G_L\cup cG_L$, K is normal in G_{L^+}, and it suffices to show $s\in K$ for every generator $s\in I_{\bar w}\cap G_L$ of N^Q ($\bar w$ above neither ℓ nor a place of Q, $\bar\chi_a(s)=1$ for all a). The place of L^+ below $\bar w$ is unramified in L (Rd2). Since K is normal in G_L and $cKc=K$, and $c(I_{\bar w}\cap G_L)c=I_{c\bar w}\cap G_L$ with $\bar\chi_a(csc)=\bar\chi_{n+1-a}(s)^{-1}$ (Rd5), we may replace $\bar w$ by any G_{L^+}-conjugate; for a split place we may thus assume that $\bar w$ lies above the fixed place $\tilde v$ of L (or any chosen place w above v). Let $I'=\{s\in I_{\bar w}\cap G_L:\bar\chi_a(s)=1\ (1\leq a\leq n)\}$, $D'=D_{\bar w}\cap G_L$, the absolute Galois group of the completion L_w, q its residue cardinality, and $\Phi\in D'$ a lift of geometric Frobenius. (a) $v\notin T_L$ inert in L: $V_v=\mathbb{G}_L(\mathcal{O}_{L_v^+})$ is hyperspecial ($\mathbb{G}_L$ is reductive over $\mathcal{O}_{L_v^+}$, L/L^+ being unramified at v and the Hermitian form of unit discriminant) and $\pi_v^{V_v}\neq0$, as $\pi^{V(\ell^{E,E})}\ni f\neq0$; by (Gy5)(iii) r_π is unramified above v, and κ is unramified there (Rd7). (b) $v\notin T_L\cup Q$ split: π_v is unramified, so $\mathrm{rec}(\iota\pi_v\circ\iota_w^{-1}\otimes|\det|^{(1-n)/2})$ is unramified with monodromy 0; by (Ug2), $\mathrm{WD}(r_\pi|_{D'})^{F\text{-}ss}$ has monodromy 0 and is trivial on inertia, so by (Wd1) $r_\pi(\sigma)=\rho(\sigma)=1$ for σ in inertia ($N=0$, and Frobenius-semisimplification does not change ρ on inertia); κ is unramified (Rd7). (c) $v\in\mathrm{St}$, $w=\tilde v$: by (Zt), $\mathrm{rec}(\iota\pi_v\circ\iota_{\tilde v}^{-1})$, and hence its unramified twist by $|\det|^{(1-n)/2}$, has monodromy 0 and inertia $\bigoplus_a\iota\circ([\bar\chi_a]\kappa)$; by (Ug2) and (Wd1), $r_\pi|_{I_{\bar w}}\cong\bigoplus_a[\bar\chi_a]\kappa$, so $\tilde r|_{I_{\bar w}}\cong\bigoplus_a[\bar\chi_a]$, which is trivial on I' (a Teichmüller lift is trivial where the character is). (d) $v=v_0$, $\bar w=\bar v_0$ (allowed by the reduction above): by (Rd6) all $\bar\chi_a$ agree on D', so all eigenvalues of $\tilde r(\Phi)$ have one and the same reduction $\bar\alpha$ (Step 0), and $q\not\equiv1\bmod\ell$, $\bar\omega$ being unramified at v_0 ($p\neq\ell$) and non-trivial on D'. Let $(\rho,N)=\mathrm{WD}(r_\pi|_{D'})$. If $N\neq0$, then by (38.5), with $\rho(\Phi)=r_\pi(\Phi)$ by (Wd1) (indeed $\rho(\Phi)N=q^{-1}N\rho(\Phi)$ maps the generalised β-eigenspace into the generalised $q^{-1}\beta$-eigenspace, and some generalised eigenvector is not killed by N), $r_\pi(\Phi)$, and hence $\tilde r(\Phi)$, has two eigenvalues of ratio q, whose reductions differ by the factor $q\not\equiv1$: a contradiction. So $N=0$, $\tilde r(\sigma)=\rho(\sigma)\kappa(\sigma)^{-1}$ on inertia (Wd1), and $\tilde r(I')$ is finite. Every eigenvalue of $\tilde r(s)$, $s\in I'$, is a root of unity reducing to 1 (the residual semisimplification is $\bigoplus_a\bar\chi_a$, trivial on I'), hence of ℓ-power order, and $\tilde r(s)$ is semisimple; so $\tilde r(I')$ is a finite ℓ-group. As I' is open normal in D' and its maximal pro-ℓ quotient is a quotient of $t_\ell(I')\subseteq\mathbb{Z}_\ell(1)$ (t_ℓ the ℓ-part of the tame character; wild inertia is pro-p), $\tilde r(I')=\langle\tilde r(s_0)\rangle$ for some $s_0\in I'$, and $\tilde r(\Phi s_0\Phi^{-1})=\tilde r(s_0)^{q^{-1}}$ ($t_\ell(\Phi s\Phi^{-1})=q^{-1}t_\ell(s)$). Suppose $\tilde r(s_0)$ has an

eigenvalue $\mu \neq 1$, of order ℓ^k, $k \geq 1$, with eigenspace W_μ. Then $\tilde{r}(\Phi)W_\mu = W_{\mu^q}$; the orbit $\mu, \mu^q, \mu^{q^2}, \ldots$ has length d equal to the order of q in $(\mathbb{Z}/\ell^k)^\times$, which has a prime divisor $e \neq \ell$ ($q \not\equiv 1 \bmod \ell$, and the order of q modulo ℓ divides $\ell - 1$). On $W = \bigoplus_{i<d} W_{\mu^{q^i}}$, $\tilde{r}(\Phi)$ permutes the summands cyclically, so its characteristic polynomial there is $\det(X^d - \tilde{r}(\Phi)^d|_{W_\mu})$, and with every eigenvalue γ also $\zeta_e\gamma$ is an eigenvalue (ζ_e a primitive e-th root of unity). As $\zeta_e \not\equiv 1$ modulo the maximal ideal ($e \neq \ell$), two eigenvalues of $\tilde{r}(\Phi)$ have different reductions: a contradiction. So $\tilde{r}(s_0) = 1$ and $\tilde{r}(I') = 1$. In all cases $s \in K$.

Step 2: (Zp2). Let $y \in \mathcal{C}$ with λ. Then $y(\mathfrak{P}_{\lambda,1}) = 0$, since for $u \in \mathbb{X}_1$

$$y((w_0\nu)^{-1}(u)\langle u\rangle) = (w_0\nu)^{-1}(u)(w_0\lambda)^{-1}(u),$$

so y is regular. Take π, f as in Step 0. The operator assigned by (Gy7)(c) to $\langle u\rangle$ is $(w_0\lambda)(u)^{-1}\langle u\rangle_\lambda$ ($\langle u\rangle_\lambda$ the action on forms of weight λ), so $\langle u\rangle_\lambda f = (w_0\lambda)(u)y(\langle u\rangle)f = f$ for all $u \in \mathbb{X}$. In particular $T_n(\mathfrak{l}) \subseteq \mathbb{X}$ ($\mathfrak{l}$ the product of the places above ℓ, as in Theorem B.91) acts trivially on f, so f has level $V(\ell^{1,1})$ by the last sentence of Lemma E.21 ($U = V$, $s = 1$, $c = E$, $A = \overline{\mathbb{Q}}_\ell$, α trivial). Since $V(\ell^{0,1}) = V(\ell^{1,1}) \cdot \mathbb{X}$ (at each $v \in S_\ell$, $\mathrm{Iw}(\tilde{v}^{0,1}) = \mathrm{Iw}(\tilde{v}^{1,1})\, T_n(\mathcal{O}_{L_{\tilde{v}}})$: a matrix upper triangular modulo $\varpi_{\tilde{v}}$ is a diagonal unit matrix times one that is unipotent modulo $\varpi_{\tilde{v}}$), f is fixed by $V(\ell^{0,1})$: $\pi^{V(\ell^{0,1})} \cap S^{\mathrm{ord}}_\lambda(V(\ell^{0,1}), \overline{\mathbb{Q}}_\ell) \ni f \neq 0$ (after scaling, f has coefficients in a finite extension of $\mathcal{O}'$). This is the line of (Gy6); it contains f, on which $U^{(j)}_{\lambda,\varpi_{\tilde{v}}}$ (the operator assigned to $U^{(j)}_{\tilde{v}}$ by (Gy7)(c)) acts by $y(U^{(j)}_{\tilde{v}})$, so the eigenvalues $u^{(j)}$ of (Gy6) are the $y(U^{(j)}_{\tilde{v}})$. By (Gy6) (with $U = V$, λ regular), $r_\pi|_{G_{L_{\tilde{v}}}}$ is conjugate to an upper triangular representation with diagonal $(\varepsilon^{1-j}\psi_{\tilde{v},j})_j$, where

$$\psi_{\tilde{v},j}(\mathrm{Art}(x)) = \prod_{\tau'} \tau'(x)^{-\lambda_{\tau',n-j+1}} = (w_0\lambda)(d_j(x))^{-1} = y(\langle d_j(x)\rangle) = \psi^y_{v,j}(\mathrm{Art}(x))$$

for $x \in \mathcal{O}^\times_{L_{\tilde{v}}}$, and $\psi_{\tilde{v},j}(\mathrm{Art}(\varpi_{\tilde{v}})) = y(U^{(j)}_{\tilde{v}})/y(U^{(j-1)}_{\tilde{v}}) = \psi^y_{v,j}(\mathrm{Art}(\varpi_{\tilde{v}}))$. As $\varepsilon = \chi$ and $r_y \cong r_\pi$ (Step 0), $r_y|_{D_v}$ is triangular with diagonal $(\chi^{1-j}\psi^y_{v,j})_j$.

Step 3: (Zp3), through a given minimal prime. We show: for every minimal prime P of $\mathbb{T}$ and every $t \in \mathbb{T} \setminus P$ there is $y \in \mathcal{C}$ with $y(P) = 0$ and $y(t) \neq 0$. As $\mathbb{T}$ is reduced, every $t \neq 0$ lies outside some minimal prime, so this gives (Zp3). $\mathbb{T}$ is reduced (Lemma E.18) and finite and torsion-free over Λ_1 with the diamond structure ((Gd0)). We may enlarge $\mathcal{O}'$ to contain the e-th roots of unity, e the order of the torsion subgroup Δ of $\mathbb{X}$: $\mathbb{T} \otimes_{\mathcal{O}'} \mathcal{O}''$ is again reduced (it embeds in $\mathbb{T} \otimes \mathcal{K}''$, reduced as a base change of the reduced $\mathbb{T} \otimes \mathcal{K}'$ along a separable field extension) and has the same points, and an element of $\mathbb{T}$ is 0 if it is 0 in $\mathbb{T} \otimes \mathcal{O}''$ (flatness). Let P be a minimal prime of the original $\mathbb{T}$ and $t \notin P$. The ring $\mathbb{T} \otimes \mathcal{O}''$ is integral over $\mathbb{T}$ and contains it, so some prime lies over P; one that is minimal among the primes lying over P is a minimal prime of $\mathbb{T} \otimes \mathcal{O}''$ (a prime contained in it lies over a prime contained in P, hence over P), and it does not contain t; we rename it P. Non-zero elements of Λ_1 are non-zero-divisors on $\mathbb{T}$ and minimal primes consist of zero-divisors, so $\Lambda_1 \to \mathbb{T}/P$ is injective; $\Lambda_1 \cong \mathcal{O}'[[X_1, \ldots, X_{N_0}]]$ is a normal domain ($\mathbb{X}_1 \cong \mathbb{Z}_\ell^{N_0}$: it is torsion-free as ℓ is odd), so the minimal polynomial of $t \bmod P$ over $\mathrm{Frac}(\Lambda_1)$ has coefficients in Λ_1 and constant term $c_0 \neq 0$. Each $\langle\delta\rangle$, $\delta \in \Delta$, is a root in the domain $\mathbb{T}/P \supseteq \mathcal{O}'$ of $X^e - 1 = \prod_{\zeta\in\mu_e(\mathcal{O}')}(X - \zeta)$, so $\langle\delta\rangle \equiv \varepsilon_P(\delta) \bmod P$ for a character $\varepsilon_P\colon \Delta \to \mu_e(\mathcal{O}')$. Write $\mathbb{X} = \Delta \times F'$ with $F' \cong \mathbb{Z}_\ell^{N_0}$ ($\mathbb{X} = \prod_{v\in S_\ell}(\mathcal{O}^\times_{L_v^+})^n$, a finitely generated $\mathbb{Z}_\ell$-module times a finite group); then $\Lambda^+ = \mathcal{O}'[\Delta] \otimes_{\mathcal{O}'} \mathcal{O}'[[F']]$, and the kernel $\mathfrak{q}_P$ of $\varepsilon_P \otimes \mathrm{id}\colon \Lambda^+ \to \mathcal{O}'[[F']]$ is contained in $P \cap \Lambda^+$; as $\Lambda^+/\mathfrak{q}_P \cong \mathcal{O}'[[F']]$ is a domain of the same dimension as $\Lambda^+/(P \cap \Lambda^+)$ (both finite over Λ_1 and containing it), $\mathfrak{q}_P = P \cap \Lambda^+$. For a dominant λ, the homomorphism $\phi_\lambda\colon \Lambda^+ \to \overline{\mathbb{Q}}_\ell$, $[u] \mapsto (w_0\lambda)(u)^{-1}$, kills $\mathfrak{q}_P$ if $(w_0\lambda)^{-1}|_\Delta = \varepsilon_P$.

Let $\mathcal{L}$ be the set of dominant λ with $\lambda_{\tau',j} > \lambda_{\tau',j+1}$ and $(w_0\lambda)^{-1}|_\Delta = \varepsilon_P$. For each $v \in S_\ell$ fix an embedding τ_v of L_v^+; $\Delta = \prod_v \mu(L_v^+)^n$ with $\mu(L_v^+)$ cyclic of order e_v, on which τ_v is a faithful character; so $(w_0\lambda)^{-1}|_\Delta = \varepsilon_P$ holds as soon as each $\lambda_{\tau_v,i}$ lies in a prescribed class modulo e_v and $\lambda_{\tau',i} \equiv 0 \bmod e_v$ for the other embeddings τ' of L_v^+. Put $k(\lambda)_{\tau',j} = -\lambda_{\tau',n+1-j}$, so that $\phi_\lambda|_{\Lambda_1}$ has kernel $\mathfrak{P}_{k(\lambda)}$ in the notation of Lemma 50.13 ($F^+ = L^+$; $\mathcal{O}' \supseteq \tau'(\mathcal{O}_{L^+})$ by (Rd1)). Write $e_v = \ell^{a_v} e_v'$ with $\ell \nmid e_v'$ and $M = \prod_v e_v'$. The closure of $\{k(\lambda) : \lambda \in \mathcal{L}\}$ contains the open set Ω of $k \in (\mathbb{Z}_\ell^n)^I$ satisfying the congruences modulo ℓ^{a_v} imposed by $\mathcal{L}$: given $k^* \in \Omega \cap (\mathbb{Z}^n)^I$ and $e'' \geq \max_v a_v$, choose integers $\lambda_{\tau',i} \equiv -k^*_{\tau',n+1-i} \bmod \ell^{e''}$ and in the prescribed classes modulo M (Chinese remainder theorem; the conditions modulo ℓ^{a_v} are compatible as $k^* \in \Omega$), then add $\ell^{e''} MA(n-i)$ to $\lambda_{\tau',i}$ with A large: $\lambda \in \mathcal{L}$ and $k(\lambda) \equiv k^* \bmod \ell^{e''}$. By Lemma 50.13, $c_0 \notin \mathfrak{P}_{k(\lambda)}$ for some $\lambda \in \mathcal{L}$. As $\mathbb{T}/P$ is integral over $\Lambda^+/\mathfrak{q}_P$ (finite over Λ_1), lying over gives $y\colon \mathbb{T}/P \to \overline{\mathbb{Q}}_\ell$ extending ϕ_λ (its restriction to $\mathbb{T}$ is an $\mathcal{O}'$-algebra homomorphism from a quotient of $\mathbb{T}_\tau$ to $\overline{\mathbb{Q}}_\ell$, hence continuous by the last sentence of (Gd0) of Lemma E.16: it is a point, and $y \in \mathcal{C}$); $y(t)$ is a root of the image of the minimal polynomial, whose constant term $y(c_0) \neq 0$, so $y(t) \neq 0$; and y kills the original minimal prime.

Step 4: (Zp4). Let y be a point of $\mathbb{T}$ with $y(\langle\delta\rangle) = \chi(\delta)$ for all $\delta \in \Delta_\ell$. Its kernel is a prime ideal, so it contains a minimal prime P of $\mathbb{T}$. For $\delta \in \Delta_\ell$ of order ℓ^k, $t = \langle\delta\rangle$ satisfies $t^{\ell^k} = 1$ and $y(t) = \chi(\delta) \in \mathcal{O}'$, so (Wv3) of Lemma 50.24 (with $R = \mathbb{T}$) gives $\langle\delta\rangle - \chi(\delta) \in P$. Hence P is one of the primes whose intersection is $\mathfrak{a}_\chi$, and $y(\mathfrak{a}_\chi) = 0$; in particular $\mathfrak{a}_\chi \subseteq \bigcap_{y \in \mathcal{C}_\chi} \ker y$. Conversely, let $t \notin \mathfrak{a}_\chi$. Then there is a minimal prime P with $\langle\delta\rangle - \chi(\delta) \in P$ for all $\delta \in \Delta_\ell$ and $t \notin P$, and Step 3 gives $y \in \mathcal{C}$ with $y(P) = 0$ and $y(t) \neq 0$; then $y(\langle\delta\rangle) = \chi(\delta)$ for all δ, so $y \in \mathcal{C}_\chi$ and $t \notin \bigcap_{y \in \mathcal{C}_\chi} \ker y$. (If there is no such P, then $\mathfrak{a}_\chi = \mathbb{T}$, $\mathcal{C}_\chi$ is empty by the first assertion, and both sides are $\mathbb{T}$.)

Step 5: (Zp5). $\mathbb{T} = e_\mathfrak{m}\mathbb{T}_\tau$ acts faithfully on N (if $tN = 0$ for $t = e_\mathfrak{m} t$, then $tN_\tau(K') = 0$, so $t = 0$, $\mathbb{T}_\tau$ being a ring of endomorphisms of $N_\tau(K')$). By (Wf2) of Lemma E.20 (with $K'' = K'$), $N_\tau(K')$ is free of finite rank over $\Lambda_1[\Delta_\ell]$, a local ring ((Wf1)); $e_\mathfrak{m}$ commutes with the diamond operators, so N is a direct summand, finitely generated projective, hence free: $N \cong \Lambda_1[\Delta_\ell]^k$, and $N^\chi = N \otimes_{\mathcal{O}'[\Delta_\ell]} \mathcal{O}'$ ($\mathcal{O}'[\Delta_\ell] \to \mathcal{O}'$ given by χ) is free of rank k over Λ_1. $\Lambda_1 \cong \mathcal{O}'[[X_1, \ldots, X_{N_0}]]$ is a domain ($T_n(\ell^1)$ is torsion-free by (Wf1)); let Φ be its fraction field. Non-zero elements of Λ_1 are non-zero-divisors on $\mathbb{T}$ ((Gd0)), N and N^χ, so these embed in $\mathbb{T}_\Phi = \mathbb{T} \otimes_{\Lambda_1} \Phi$, N_Φ and N^χ_Φ, and $t \in \mathbb{T}$ kills N^χ if and only if it kills N^χ_Φ. $\mathbb{T}$ is reduced (Lemma E.18), so $\mathbb{T}_\Phi$ is reduced and finite over Φ, hence a finite product of fields; its prime ideals are the $P\mathbb{T}_\Phi$ for the primes P of $\mathbb{T}$ with $P \cap \Lambda_1 = 0$, which are exactly the minimal primes of $\mathbb{T}$ (a minimal prime consists of zero-divisors; if $P' \subseteq P$ and $P \cap \Lambda_1 = 0$, then $P' = P$ by incomparability for the integral extension $\Lambda_1 \subseteq \mathbb{T}$). So $\mathbb{T}_\Phi = \prod_P k(P)$ over the minimal primes, $k(P)$ the fraction field of $\mathbb{T}/P$, with idempotents ε_P; $N_\Phi = \bigoplus_P \varepsilon_P N_\Phi$, and every $\varepsilon_P N_\Phi \neq 0$, as $\mathbb{T}_\Phi$ acts faithfully on N_Φ. For $\delta \in \Delta_\ell$, $\langle\delta\rangle - \chi(\delta)$ acts on $\varepsilon_P N_\Phi$ through its image in $k(P)$: by 0 if $\langle\delta\rangle - \chi(\delta) \in P$, invertibly otherwise. Hence $N^\chi_\Phi = \bigoplus_{P \in \mathcal{P}_\chi} \varepsilon_P N_\Phi$, $\mathcal{P}_\chi$ the set of minimal primes containing every $\langle\delta\rangle - \chi(\delta)$, and $t \in \mathbb{T}$ kills N^χ_Φ if and only if $\varepsilon_P t = 0$, i.e. $t \in P$, for every $P \in \mathcal{P}_\chi$. So the annihilator of N^χ is $\bigcap_{P \in \mathcal{P}_\chi} P = \mathfrak{a}_\chi$. □

Lemma E.18. *Let $((V_v)_{v \in \mathrm{St}}, \tau)$ be a type datum (over a rank-n Hecke datum $\mathscr{H}$ over $\mathcal{O}$, or over L), and let Q and K' be as in the levels at auxiliary places of Definition 50.8; here $Q = \varnothing$ is allowed, and $Q \neq \varnothing$ occurs only over $\mathscr{H}$. Then the ordinary Hecke algebra $\mathbb{T}_\tau(K')$ of type τ at level K' (for $Q = \varnothing$: $\mathbb{T}_\tau$) is reduced; hence so is its localisation $\mathbb{T}_{\tau,\mathfrak{m}}$ at every maximal ideal $\mathfrak{m}$.*

Proof. Write V, N_τ, $\mathbb{T}^{\mathrm{ord}}(V)$, $\mathbb{T}_\tau$ for $V(K')$, $N_\tau(K')$, $\mathbb{T}^{\mathrm{ord}}(V(K'))$, $\mathbb{T}_\tau(K')$, $\mathbb{G} = \mathbb{G}_L$, $\mathcal{K} = \mathcal{O}[1/\ell]$, and for $c \geq 1$ put $U_c = V(\ell^{c,c})$ and $X_c = \mathbb{G}(L^+)\backslash\mathbb{G}(\mathbb{A}^\infty_{L^+})/U_c$, a finite set ($\mathbb{G}(L^+ \otimes \mathbb{R})$ is compact).

Step 1: finite levels. In trivial weight a form of level U_c with coefficients in an $\mathcal{O}$-module A is a function $X_c \to A$, so $S(U_c, \mathcal{K}/\mathcal{O}) = S(U_c, \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{K}/\mathcal{O}$ with $S(U_c, \mathcal{O})$ finite free. Let P_c be the product of the operators $U_{\tilde{v}}^{(j)}$ ($v \in S_\ell$, $1 \le j \le n$) on $S(U_c, \mathcal{O})$ and $e_c = \lim_m P_c^{m!}$, the idempotent of the finite $\mathcal{O}$-algebra $\mathcal{O}[P_c]$ cutting out the part on which P_c, equivalently every $U_{\tilde{v}}^{(j)}$, is invertible. Then $M_c = e_c S(U_c, \mathcal{O})$ is finite free, $S^{\mathrm{ord}}(U_c, \mathcal{K}/\mathcal{O}) = M_c \otimes_{\mathcal{O}} \mathcal{K}/\mathcal{O}$ (the ordinary part of a torsion module $M \otimes \mathcal{K}/\mathcal{O}$ is $eM \otimes \mathcal{K}/\mathcal{O}$, since e commutes with $\otimes$ and $U^{(j)}$ is invertible on $(1-e)M \otimes \ell^{-k}\mathcal{O}/\mathcal{O}$ only if this module is 0), and $S^{\mathrm{ord}}(V(\ell^\infty), \mathcal{K}/\mathcal{O}) = \bigcup_c M_c \otimes \mathcal{K}/\mathcal{O}$. All the operators $T_w^{(j)}$, $(T_w^{(n)})^{-1}$, $U_{\tilde{v}}^{(j)}$, $\langle u \rangle$ ($u \in T_n(\mathcal{O}_{L^+} \otimes \mathbb{Z}_\ell)$, acting through the finite quotient $T_n(\mathcal{O}_{L^+}/\ell^c)$) and the action of H preserve M_c, and the operators commute with H. Let $A_c \subseteq \mathrm{End}_{\mathcal{O}}(M_c)$ be the $\mathcal{O}$-algebra they generate (commutative, [105], Lemma 2.3.3); it is a finite $\mathcal{O}$-module, hence closed, so the image of $\mathbb{T}^{\mathrm{ord}}(V)$ (the closure of the algebra generated by the same operators over $\mathcal{O}[T_n(\mathcal{O}_{L^+} \otimes \mathbb{Z}_\ell)]$) acting on M_c lies in A_c.

Step 2: A_c acts semisimply on $M_c \otimes \overline{\mathbb{Q}}_\ell$. Transport by ι to $\mathbb{C}$. By [105], Lemma 2.2.5 (trivial weight), $\varinjlim_U S(U, \mathbb{C})$ is the space of automorphic forms on $\mathbb{G}(L^+)\backslash\mathbb{G}(\mathbb{A}_{L^+})$ with trivial archimedean type, a semisimple admissible $\mathbb{G}(\mathbb{A}_{L^+}^\infty)$-module; so $S(U_c, \mathbb{C}) = \bigoplus_\sigma m(\sigma)\, \sigma^{U_c}$ over irreducible $\sigma = \bigotimes_v' \sigma_v$ with finite multiplicities, and $\sigma^{U_c} = \bigotimes_v \sigma_v^{U_{c,v}}$. On each summand $T_w^{(j)}$ acts on the line $\sigma_v^{\mathbb{G}(\mathcal{O}_{L_v^+})}$ ($w \mid v \notin T_L \cup Q$) by a scalar, and $U_{\tilde{v}}^{(j)}$, $\langle u \rangle$ act on the factor $\sigma_v^{\mathrm{Iw}(\tilde{v}^{c,c})}$, $v \in S_\ell$; the operators $U_{\tilde{v}}^{(j)}$ preserve the lattice, so their eigenvalues are ℓ-adic integers (after ι^{-1}), e_c is the projection onto the sum of the generalised eigenspaces of P_c with unit eigenvalues, and $e_c\sigma^{U_c} = \bigotimes_{v \in S_\ell} (\sigma_v^{\mathrm{Iw}(\tilde{v}^{c,c})})^{\mathrm{ord}} \otimes \bigotimes_{v \notin S_\ell} \sigma_v^{U_{c,v}}$, where for $v \in S_\ell$ the ordinary part is the sum of the joint generalised eigenspaces of the $U_{\tilde{v}}^{(j)}$ with unit eigenvalues. For $v \in S_\ell$, $K = L_{\tilde{v}}$ with uniformiser ϖ and the irreducible smooth representation $\sigma' = \sigma_v \circ \iota_{\tilde{v}}^{-1}$ of $\mathrm{GL}_n(K)$, the operators $U_\varpi^{(j)}$ and $\sigma'(u)$, $u \in T_n(\mathcal{O}_K)$, act semisimply on $Y = (\sigma'^{\mathrm{Iw}(\varpi^{c,c})})^{\mathrm{ord}}$ by Lemma E.19; so all generators of A_c are diagonalisable and commute, and $M_c \otimes \overline{\mathbb{Q}}_\ell$ is the direct sum of the joint eigenspaces of A_c.

Step 3: $\mathbb{T}_\tau$ is reduced. By Pontryagin duality ($\mathrm{Hom}_{\mathcal{O}}(M \otimes \mathcal{K}/\mathcal{O}, \mathcal{K}/\mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ for M finite free), restriction to the submodules $M_c \otimes \mathcal{K}/\mathcal{O}$, whose union is $S^{\mathrm{ord}}(V(\ell^\infty), \mathcal{K}/\mathcal{O})$, gives an injective map $\mathcal{S}_V \to \prod_c M_c^*$, $M_c^* = \mathrm{Hom}_{\mathcal{O}}(M_c, \mathcal{O})$, equivariant for H and for $\mathbb{T}^{\mathrm{ord}}(V)$ (acting on M_c^* by transposes). Applying the left exact functor $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, -)$ gives an injective $\mathbb{T}^{\mathrm{ord}}(V)$-map $N_\tau \to \prod_c N_c$, $N_c = \mathrm{Hom}_{\mathcal{O}[H]}(\tau, M_c^*)$, a finite free $\mathcal{O}$-module; let $N_c' \subseteq N_c$ be the image of N_τ, a $\mathbb{T}^{\mathrm{ord}}(V)$-stable submodule. By Step 2, $M_c^* \otimes \overline{\mathbb{Q}}_\ell$ is the direct sum of the joint eigenspaces E_χ of the transposes of A_c (dual basis), each H-stable (the actions commute), so $N_c \otimes \overline{\mathbb{Q}}_\ell = \bigoplus_\chi \mathrm{Hom}_H(\tau \otimes \overline{\mathbb{Q}}_\ell, E_\chi)$, on whose summands every $t \in \mathbb{T}^{\mathrm{ord}}(V)$ acts by the scalar $\chi(t)$ (Step 1: t acts on M_c through A_c). Now let $t \in \mathbb{T}^{\mathrm{ord}}(V)$ act nilpotently on N_τ, $t^k N_\tau = 0$. Then t^k kills N_c'; $N_c' \otimes \overline{\mathbb{Q}}_\ell$ is a sum of its intersections with the summands above (a submodule stable under a commuting family of diagonalisable operators), on each of which t acts by a scalar $\chi(t)$ with $\chi(t)^k = 0$; so t kills $N_c' \otimes \overline{\mathbb{Q}}_\ell$, hence the torsion-free N_c'. Thus the image of tN_τ in every N_c is 0, and $tN_\tau = 0$. As $\mathbb{T}_\tau$ is the image of $\mathbb{T}^{\mathrm{ord}}(V)$ in $\mathrm{End}_{\mathcal{O}}(N_\tau)$, every nilpotent element of $\mathbb{T}_\tau$ is 0. A localisation of a reduced ring is reduced. □

Lemma E.19. *Let K be a finite extension of $\mathbb{Q}_\ell$ with uniformiser ϖ, $n \ge 1$, $c \ge 1$, and σ' an irreducible smooth representation of $\mathrm{GL}_n(K)$ on a $\overline{\mathbb{Q}}_\ell$-vector space. Let $c_0 = 0, c_1, \dots, c_n$ be rational numbers such that $c_j - c_{j-1}$ is non-decreasing in j (for instance all $c_j = 0$), and let $Y \subseteq \sigma'^{\mathrm{Iw}(\varpi^{c,c})}$ be the sum of the joint generalised eigenspaces of the commuting operators $U_\varpi^{(1)}, \dots, U_\varpi^{(n)}$ whose eigenvalues $e_1, \dots, e_n$ satisfy $v_\ell(e_j) = c_j$ for all j (v_ℓ the valuation of $\overline{\mathbb{Q}}_\ell$ with $v_\ell(\ell) = 1$). For $c_j = 0$ for all j, Y is the **ordinary part**. Then the operators $U_\varpi^{(j)}$ ($1 \le j \le n$) and $\sigma'(u)$ ($u \in T_n(\mathcal{O}_K)$) act semisimply on Y, and every joint eigenspace of this*

commuting family on Y has dimension at most one.

Proof. Write $\alpha^{(j)} = \alpha_\varpi^{(j)}$ and let $j\colon \sigma'^{\mathrm{Iw}(\varpi^{c,c})} \to X = (\sigma'_{N_n})^{T_n(\varpi^c)}$ be the map of (Gy7)(a). By (Gy7)(a), j is $T_n(\mathcal{O}_K)$-equivariant with $j \circ U_\varpi^{(j)} = \delta_B(\alpha^{(j)})^{-1}\sigma'_{N_n}(\alpha^{(j)}) \circ j$, and its kernel is the part on which some $U_\varpi^{(j)}$ is not invertible. Y is stable under the $U_\varpi^{(j)}$ and under $T_n(\mathcal{O}_K)$ (which commutes with them), the $U_\varpi^{(j)}$ are invertible on Y (their eigenvalues there have finite valuation, so are non-zero), so $Y \cap \ker j = 0$ and $j|_Y$ is injective and intertwines the actions of the $U_\varpi^{(j)}$ with those of $\delta_B(\alpha^{(j)})^{-1}\sigma'_{N_n}(\alpha^{(j)})$, and of $T_n(\mathcal{O}_K)$. X is a finite-dimensional direct summand of σ'_{N_n} stable under $T_n(K)$ (averaging over $T_n(\varpi^c)$; σ'_{N_n} is admissible), hence the direct sum of its joint generalised eigenspaces X_ψ for characters ψ of $T_n(K)$. As $T_n(K)$ is generated by $T_n(\mathcal{O}_K)$ and the $\alpha^{(j)}$, $j(Y)$ is $T_n(K)$-stable, hence the sum of its intersections with the X_ψ, and only ψ with $v_\ell(\delta_B(\alpha^{(j)})^{-1}\psi(\alpha^{(j)})) = c_j$ for every j (call such ψ *of slope* c) occur. A character ψ of $T_n(K)$ is determined by its values on $T_n(\mathcal{O}_K)$ and on the $\alpha^{(j)}$. It therefore suffices to show $\dim X_\psi \le 1$ for ψ of slope c: then $T_n(K)$ acts on X_ψ by ψ, $Y \cong j(Y)$ is a direct sum of lines on which the $U_\varpi^{(j)}$ and $T_n(\mathcal{O}_K)$ act by scalars, and distinct lines carry distinct characters ψ, hence distinct joint eigencharacters.

Let ψ be of slope c with $X_\psi \neq 0$ and $\psi_i(x) = \psi(\mathrm{diag}(1,\dots,x,\dots,1))$ (x in place i). Since $\delta_B(\alpha^{(j)}) = q^{-j(n-j)}$ and $\psi(\alpha^{(j)}) = \prod_{i\le j}\psi_i(\varpi)$, the slope condition gives $v_\ell(\psi_1(\varpi)\cdots\psi_j(\varpi)) = -fj(n-j) + c_j$ for all j, so

$$v_\ell(\psi_j(\varpi)) = -f\big(j(n-j) - (j-1)(n-j+1)\big) + (c_j - c_{j-1}) = -f(n+1-2j) + (c_j - c_{j-1});$$

put $\xi = \delta_B^{-1/2}\psi$, so $v_\ell(\xi_i(\varpi)) = -\frac{f}{2}(n+1-2i) + (c_i - c_{i-1})$, strictly increasing in i as $f > 0$ and $c_i - c_{i-1}$ is non-decreasing, and the ξ_i are pairwise distinct characters of $K^\times$. As $X_\psi \neq 0$, the $T_n(K)$-module σ'_{N_n} has the quotient ψ (a non-zero linear form on X_ψ vanishing on the images of the nilpotent operators $\sigma'_{N_n}(t) - \psi(t)$, composed with the projection onto X_ψ), so by Frobenius reciprocity ([27], Proposition 2.3) σ' embeds in the normalised induction $I(\xi) = \text{n-Ind}_B^{\mathrm{GL}_n(K)}(\xi_1 \otimes \cdots \otimes \xi_n)$. The Jacquet functor is exact, and the semisimplification of $I(\xi)_{N_n}$ is $\bigoplus_{w\in S_n} \delta_B^{1/2}\, w(\xi)$ ([27, 2.12], the geometric lemma). As the ξ_i are pairwise distinct, $w(\xi) = \xi$ only for $w = 1$, so $\psi = \delta_B^{1/2}\xi$ occurs in $I(\xi)_{N_n}$, hence in σ'_{N_n}, with multiplicity at most one; thus the generalised eigenspace of $T_n(K)$ on σ'_{N_n} for ψ, which contains X_ψ, has dimension at most one. □

Lemma E.20. *Let $\mathscr{H}$ be a rank-n Hecke datum over $\mathcal{O}$, $((V_v)_{v\in\mathrm{St}}, \tau)$ a type datum over $\mathscr{H}$, Q a finite set of places of L^+ split in L and not in T_L, and $K'' = \prod_{v\in Q} K''_v \subseteq K' = \prod_{v\in Q} K'_v$ compact open subgroups of $\prod_{v\in Q}\mathbb{G}_L(L_v^+)$ with K''_v normal in K'_v for every $v \in Q$, such that $\Gamma = K'/K''$ is an ℓ-group (for instance $K'' = K'$). Let Γ act on $N_\tau(K'')$ through $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, -)$ applied to the action on $\mathcal{S}_{V(K'')}$ dual to right translation by K'. Let Δ be the set of elements of ℓ-power order of $T_n(\mathcal{O}_{L^+} \otimes \mathbb{Z}_\ell)$. Then:*

(Wf1) *Δ is a finite subgroup and $\Delta \cap T_n(\ell^1) = \{1\}$; so $(\delta, u) \mapsto \delta u$ is an isomorphism of $\Delta \times T_n(\ell^1)$ onto an open subgroup of $T_n(\mathcal{O}_{L^+} \otimes \mathbb{Z}_\ell)$, and $\Lambda_1[\Delta\times\Gamma] = \mathcal{O}[[\Delta \times T_n(\ell^1) \times \Gamma]]$ acts on $N_\tau(K'')$ through the diamond operators and the action of Γ, which commute.*

(Wf2) *$N_\tau(K'')$ is a free $\Lambda_1[\Delta\times\Gamma]$-module of finite rank.*

(Wf3) *Let $\chi\colon \Delta \to \mathcal{O}^\times$ be a homomorphism, let*

$$N^\chi = N_\tau(K'')/\sum_{\delta\in\Delta}(\langle\delta\rangle - \chi(\delta))N_\tau(K''),$$

and let Γ *act on* $\mathrm{Hom}_{\Lambda_1}(N^\chi,\Lambda_1)$ *by* $\phi\mapsto\phi\circ\gamma^{-1}$. *Then* N^χ *and* $\mathrm{Hom}_{\Lambda_1}(N^\chi,\Lambda_1)$ *are free* $\Lambda_1[\Gamma]$*-modules of the same finite rank, and composition with the projection identifies* $\mathrm{Hom}_{\Lambda_1}(N^\chi,\Lambda_1)$ *with*

$$\{\phi\in\mathrm{Hom}_{\Lambda_1}(N_\tau(K''),\Lambda_1):\phi\circ\langle\delta\rangle=\chi(\delta)\phi\ \textit{for all}\ \delta\in\Delta\}.$$

Proof. Write ϖ for a uniformiser and $\mathbb{F}$ for the residue field of $\mathcal{O}$. For $v\in S_\ell$ let $\mathcal{O}_v$ be the ring of integers of L_v^+, ϖ_v a uniformiser, k_v the residue field and e_v the ramification index over $\mathbb{Q}_\ell$; through $\tilde v$ we have $L_{\tilde v}=L_v^+$ and $T_n(\mathcal{O}_{L^+}\otimes\mathbb{Z}_\ell)=\prod_{v\in S_\ell}T_n(\mathcal{O}_v)$, diagonal in $\prod_{v\in S_\ell}\mathrm{GL}_n(\mathcal{O}_{L_{\tilde v}})$. By [105], §2.1, $\mathrm{Iw}(\tilde v^{b,c})$ $(0\le b\le c)$ is the group of matrices of $\mathrm{GL}_n(\mathcal{O}_v)$ that are upper triangular modulo ϖ_v^c and unipotent modulo ϖ_v^b. For $b\ge1$ put

$$\Sigma_b=\prod_{v\in S_\ell}(1+\varpi_v^b\mathcal{O}_v)^n\subseteq T_n(\mathcal{O}_{L^+}\otimes\mathbb{Z}_\ell),\qquad \Sigma=\Sigma_1,$$
$$\Lambda'=\mathcal{O}[[\Sigma]],\qquad \mathfrak{a}_b=\ker(\Lambda'\to\mathcal{O}[\Sigma/\Sigma_b]),$$

and let $\mathfrak{m}'=\mathfrak{a}_1+\varpi\Lambda'$, the maximal ideal of Λ'.

Step 1: (Wf1) and the rings. Let $v\in S_\ell$. As $k_v^\times$ has order prime to ℓ, an element of $\mathcal{O}_v^\times$ of ℓ-power order lies in $1+\varpi_v\mathcal{O}_v$. If $\zeta\in L_v^+$ has order ℓ^k, $k\ge1$, then $v(\zeta-1)=e_v/(\ell^{k-1}(\ell-1))\le e_v/(\ell-1)<e_v=v(\ell)$, because $\ell\ge3$ (Definition 50.1); so $\zeta\notin1+\ell\mathcal{O}_v$. Hence $\Delta=\prod_{v\in S_\ell}\mu_{\ell^\infty}(L_v^+)^n$ is a finite group (a local field has finitely many roots of unity), $\Delta\subseteq\Sigma$, and $\Delta\cap T_n(\ell^1)=\{1\}$. The group is abelian, so $(\delta,u)\mapsto\delta u$ is an injective homomorphism; its image $\Sigma^\circ=\Delta\,T_n(\ell^1)$ contains the open subgroup $T_n(\ell^1)$, so it is open, of finite index in Σ. Each $1+\varpi_v\mathcal{O}_v$ is a finitely generated $\mathbb{Z}_\ell$-module (the logarithm maps $1+\varpi_v^b\mathcal{O}_v$ isomorphically onto $\varpi_v^b\mathcal{O}_v$ for large b), so $\Sigma\cong\Sigma_{\mathrm{tors}}\times\mathbb{Z}_\ell^d$ and $\Lambda'\cong\mathcal{O}[\Sigma_{\mathrm{tors}}][[X_1,\dots,X_d]]$ is noetherian; Σ is pro-ℓ, since $(1+\varpi_v^b\mathcal{O}_v)/(1+\varpi_v^{b+1}\mathcal{O}_v)\cong k_v$. So $\Sigma\times\Gamma$ is pro-ℓ and $\Lambda'[\Gamma]=\mathcal{O}[[\Sigma\times\Gamma]]$ is a local ring: modulo ϖ each $\mathcal{O}[\Sigma/\Sigma_b\times\Gamma]$ is the group algebra of a finite ℓ-group over a field of characteristic ℓ. Finally $\Lambda'[\Gamma]$ is a free $\Lambda_1[\Delta\times\Gamma]=\mathcal{O}[[\Sigma^\circ\times\Gamma]]$-module with basis a set of representatives of Σ/Σ°: this holds for $\mathcal{O}[\Sigma/\Sigma_b\times\Gamma]$ over $\mathcal{O}[\Sigma^\circ/\Sigma_b\times\Gamma]$ for every b with $\Sigma_b\subseteq\Sigma^\circ$, compatibly in b, and passes to the inverse limit. The diamond operators commute with each other and with the action of Γ (supported at Q, while $\langle u\rangle$ is supported at S_ℓ), which gives (Wf1).

Step 2: the levels. Let U be the group with components $\mathbb{G}_L(\mathcal{O}_{L_v^+})$ at $v\in\mathrm{St}$, K_v' at $v\in Q$, and those of the level V elsewhere, and $U'=V(K'')$. Both are as in Theorem B.91 with $F=L$, $\mathbb{G}=\mathbb{G}_L$, $T=T_L\cup Q$ (all places of T split in L; the components at S_ℓ and at the split places outside T are hyperspecial). U_v' is normal in U_v for every v, and $U_v=U_v'$ for $v\notin\mathrm{St}\cup Q$. For $1\le b\le c$, $U(\ell^{b,c})$ and $U'(\ell^{b,c})$ are sufficiently small: their component at $v_0\notin\mathrm{St}\cup Q\cup S_\ell$ is the kernel of reduction modulo p of $\mathrm{GL}_n(\mathcal{O}_{L_{\tilde v_0}})$, p odd and unramified in L (Definition 50.8), which has no element of finite order other than 1.

Reduction of the diagonal modulo ϖ_v^c is a homomorphism $\mathrm{Iw}(\tilde v^{0,c})\to T_n(\mathcal{O}_v/\varpi_v^c)$ with kernel $\mathrm{Iw}(\tilde v^{c,c})$, and $\mathrm{Iw}(\tilde v^{b,c})$ is the preimage of the image of $(1+\varpi_v^b\mathcal{O}_v)^n$. Hence $U'(\ell^{c,c})$ is normal in $U(\ell^{1,c})$, $U'(\ell^{b,c})=U'(\ell^{c,c})\,\Sigma_b$, and the inclusions of Σ (at the places of S_ℓ) and of $\prod_{v\notin S_\ell}U_v$ induce an isomorphism

$$G_c:=(\Sigma/\Sigma_c)\times H\times\Gamma\ \xrightarrow{\sim}\ U(\ell^{1,c})/U'(\ell^{c,c}).$$

For $u\in T_n(\mathcal{O}_{L^+}\otimes\mathbb{Z}_\ell)$, which normalises every $\mathrm{Iw}(\tilde v^{c,c})$, the diamond operator

$$\langle u\rangle=[U'(\ell^{c,c})\,u\,U'(\ell^{c,c})]$$

of [105], §2.3, acts on $S(U'(\ell^{c,c}),A)$ by $f\mapsto f(\cdot\,u)$. So, with Σ_b acting through diamond operators, $S(U'(\ell^{c,c}),A)^{\Sigma_b}=S(U'(\ell^{b,c}),A)$.

Step 3: the invariants of the ordinary forms. Let $1 \le b \le c$. The operators $U_{\tilde{v}}^{(j)}$ on the levels $U'(\ell^{b,c})$ and $U'(\ell^{c,c})$ are the double coset operators of the same elements $\alpha_{\varpi_{\tilde{v}}}^{(j)}$, for which one set of coset representatives serves all levels $U'(\ell^{b',c'})$ with $c' > 0$ ([105], proof of Lemma 2.5.2); so the operators of level (b,c) are the restrictions of those of level (c,c), and so are the ordinary idempotents. The Hecke algebra of a finite level is commutative ([105], Lemma 2.3.3), so e commutes with the diamond operators, and Step 2 gives $S^{\mathrm{ord}}(U'(\ell^{c,c}),A)^{\Sigma_b} = S^{\mathrm{ord}}(U'(\ell^{b,c}),A)$, which is $S^{\mathrm{ord}}(U'(\ell^{b,b}),A)$ by (Gk2) for U'. Let $M = S^{\mathrm{ord}}(U'(\ell^{\infty}),\mathcal{K}/\mathcal{O}) = \bigcup_c S^{\mathrm{ord}}(U'(\ell^{c,c}),\mathcal{K}/\mathcal{O})$, so that $\mathcal{S}' := \mathcal{S}_{V(K'')} = M^{\vee}$. Taking the union over $c \ge b$,

$$M^{\Sigma_b} = S^{\mathrm{ord}}(U'(\ell^{b,b}),\mathcal{K}/\mathcal{O}).$$

Every element of M is fixed by some Σ_c, so M is a discrete Λ'-module. Let $u_1,\dots,u_k$ generate Σ_b topologically. The ideal J of Λ' generated by the $[u_i]-1$ equals $\mathfrak{a}_b$.

So $M[\mathfrak{a}_b] = M^{\Sigma_b}$, the kernel of $M \to M^k$, $m \mapsto (([u_i]-1)m)_i$. Duality gives $\mathcal{S}'/\mathfrak{a}_b\mathcal{S}' \cong (M^{\Sigma_b})^{\vee}$. The set $X_b = \mathbb{G}_L(L^+)\backslash\mathbb{G}_L(\mathbb{A}_{L^+}^{\infty})/U'(\ell^{b,b})$ is finite ($\mathbb{G}_L(L^+\otimes\mathbb{R})$ is compact), so $S(U'(\ell^{b,b}),A) = \mathcal{O}^{X_b}\otimes_{\mathcal{O}} A$ with e acting as $e\otimes 1$; thus $M^{\Sigma_b} = L_b\otimes_{\mathcal{O}}\mathcal{K}/\mathcal{O}$ with $L_b = e\,\mathcal{O}^{X_b}$ finite free over $\mathcal{O}$, and

$$\mathcal{S}'/\mathfrak{a}_b\mathcal{S}' \cong D_b := \mathrm{Hom}_{\mathcal{O}}(L_b,\mathcal{O}),$$

equivariantly for G_b (Σ/Σ_b through the diamond operators, H and Γ by right translation, acting on D_b contragrediently).

Step 4: D_b is projective over $\mathcal{O}[G_b]$. By (Gk1) for the sufficiently small group $U(\ell^{1,b})$, whose components at S_ℓ lie in $\mathrm{GL}_n(\mathcal{O}_v)$, and its open normal subgroup $U'(\ell^{b,b})$ (Step 2), $\mathcal{O}^{X_b} = S(U'(\ell^{b,b}),\mathcal{O})$ is a finite free $\mathcal{O}[G_b]$-module, G_b acting by right translation, which on Σ/Σ_b is the action of the diamond operators (Step 2). The idempotent e commutes with G_b (Step 3 for Σ/Σ_b; H and Γ act at places outside S_ℓ, where the $U_{\tilde{v}}^{(j)}$ are supported), so L_b is a direct summand of $\mathcal{O}^{X_b}$, finitely generated projective over $\mathcal{O}[G_b]$. The $\mathcal{O}$-dual of $\mathcal{O}[G_b]$ with the contragredient action is isomorphic to $\mathcal{O}[G_b]$ (the dual basis (δ_g) is permuted simply transitively), and $\mathrm{Hom}_{\mathcal{O}}(-,\mathcal{O})$ preserves finite direct sums and summands; so D_b is finitely generated projective over $\mathcal{O}[G_b]$.

Step 5: $\mathcal{S}'$ is free over Λ'. $\mathcal{O}[G_b]$ is free over $\mathcal{O}[\Sigma/\Sigma_b]$, a local ring, so D_b is free over $\mathcal{O}[\Sigma/\Sigma_b]$, of some rank r_b. Since $\mathfrak{a}_b \subseteq \mathfrak{a}_1$ and $\Lambda'/\mathfrak{a}_1 = \mathcal{O}$, $D_b/\mathfrak{a}_1 D_b \cong \mathcal{S}'/\mathfrak{a}_1\mathcal{S}' \cong D_1$, so $r_b = r := \mathrm{rank}_{\mathcal{O}} D_1$ for every b. By (Gy1) for U' with $b_0 = 1$ ($U'(\ell^{1,1})$ is sufficiently small), $\mathcal{S}'$ is finitely generated over $\Lambda_1 \subseteq \Lambda'$, hence over Λ'. As $\mathcal{S}'/\mathfrak{m}'\mathcal{S}' \cong D_1/\varpi D_1$ has dimension r over $\mathbb{F}$, Nakayama's lemma gives a surjection $\varphi\colon \Lambda'^{r} \to \mathcal{S}'$. For every b, φ induces a surjection $(\Lambda'/\mathfrak{a}_b)^r \to \mathcal{S}'/\mathfrak{a}_b\mathcal{S}' \cong (\Lambda'/\mathfrak{a}_b)^r$ between free modules of the same finite rank over a commutative ring, which is therefore an isomorphism; so $\ker\varphi \subseteq \mathfrak{a}_b\Lambda'^{r}$ for every b, and $\bigcap_b \mathfrak{a}_b = 0$ ($\Lambda' = \varprojlim_b \mathcal{O}[\Sigma/\Sigma_b]$). Hence $\mathcal{S}'$ is free of rank r over Λ'.

Step 6: $\mathcal{S}'$ is projective over $R = \Lambda'[H\times\Gamma]$. R is a free Λ'-module of finite rank and $R/\mathfrak{m}'R = \mathbb{F}[H\times\Gamma]$. By Steps 3 and 4 ($b=1$, $G_1 = H\times\Gamma$), $\mathcal{S}'/\mathfrak{m}'\mathcal{S}' \cong D_1/\varpi D_1$ is a finitely generated projective $\mathbb{F}[H\times\Gamma]$-module: $D_1/\varpi D_1 \cong \mathbb{F}[H\times\Gamma]^s\varepsilon_0$ for an idempotent matrix $\varepsilon_0 \in \mathrm{M}_s(\mathbb{F}[H\times\Gamma])$ acting on row vectors. $\mathrm{M}_s(R)$ is complete for the topology of the two-sided ideal $\mathrm{M}_s(\mathfrak{m}'R)$ (R is finite over the $\mathfrak{m}'$-adically complete ring Λ'), so ε_0 lifts to an idempotent $\varepsilon \in \mathrm{M}_s(R)$. Then $P = R^s\varepsilon$ is finitely generated projective over R with $P/\mathfrak{m}'P \cong \mathcal{S}'/\mathfrak{m}'\mathcal{S}'$, and the composite $P \to P/\mathfrak{m}'P \cong \mathcal{S}'/\mathfrak{m}'\mathcal{S}'$ lifts to an R-linear $\psi\colon P \to \mathcal{S}'$. Its image I satisfies $I + \mathfrak{m}'\mathcal{S}' = \mathcal{S}'$, so ψ is surjective (Nakayama over Λ'). Let $K = \ker\psi$. As $\mathcal{S}'$ is free over Λ' (Step 5), $\mathrm{Tor}_1^{\Lambda'}(\mathcal{S}',\mathbb{F}) = 0$, so $0 \to K/\mathfrak{m}'K \to P/\mathfrak{m}'P \to \mathcal{S}'/\mathfrak{m}'\mathcal{S}' \to 0$ is exact; the last map is an isomorphism, so $K/\mathfrak{m}'K = 0$, and K, a submodule of a finitely generated module over the noetherian ring Λ', is finitely generated, hence $K = 0$. So $\mathcal{S}' \cong P$ is a direct summand of R^s as an R-module.

Step 7: (Wf2). Write $R = \Lambda'[\Gamma] \otimes_{\mathcal{O}} \mathcal{O}[H]$, with $\Lambda'[\Gamma]$ acting on the first factor and H by left multiplication on the second. As $\Lambda'[\Gamma]$ is flat over $\mathcal{O}$ and τ is finite free over $\mathcal{O}$, $f \mapsto \epsilon \circ f$ (ϵ the coefficient of $1 \in H$) is an isomorphism of $\Lambda'[\Gamma]$-modules

$$\mathrm{Hom}_{\mathcal{O}[H]}(\tau, R) \cong \Lambda'[\Gamma] \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}}(\tau, \mathcal{O}),$$

with inverse $\varphi \mapsto (t \mapsto \sum_{h \in H} \varphi(h^{-1}t)\, h)$; so $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, R)$ is free of finite rank over $\Lambda'[\Gamma]$. The functor $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, -)$ is additive, so by Step 6 $N_\tau(K'') = \mathrm{Hom}_{\mathcal{O}[H]}(\tau, \mathcal{S}')$ is a direct summand of $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, R)^s$ as a $\Lambda'[\Gamma]$-module: finitely generated projective, hence free, $\Lambda'[\Gamma]$ being local (Step 1). Since $\Lambda'[\Gamma]$ is free of finite rank over $\Lambda_1[\Delta \times \Gamma]$ (Step 1), whose action is the restriction of that of $\Lambda'[\Gamma]$, $N_\tau(K'')$ is free of finite rank over $\Lambda_1[\Delta \times \Gamma]$.

Step 8: (Wf3). Let $N = N_\tau(K'') \cong \Lambda_1[\Delta \times \Gamma]^k$ (Step 7). Then $N^\chi = N \otimes_{\mathcal{O}[\Delta]} \mathcal{O}$, $\mathcal{O}[\Delta] \to \mathcal{O}$ given by χ, is isomorphic to $(\Lambda_1[\Delta \times \Gamma] \otimes_{\mathcal{O}[\Delta]} \mathcal{O})^k = \Lambda_1[\Gamma]^k$ as a $\Lambda_1[\Gamma]$-module. A Λ_1-linear $\phi\colon N \to \Lambda_1$ factors through N^χ if and only if it kills every $(\langle\delta\rangle - \chi(\delta))x$, i.e. $\phi \circ \langle\delta\rangle = \chi(\delta)\phi$ for all $\delta \in \Delta$; this is the identification stated. Finally $\mathrm{Hom}_{\Lambda_1}(\Lambda_1[\Gamma], \Lambda_1)$, with γ acting by $\phi \mapsto \phi \circ \gamma^{-1}$, is free of rank one over $\Lambda_1[\Gamma]$, since $\delta_g \circ \gamma^{-1} = \delta_{\gamma g}$ for the dual basis $(\delta_g)_{g \in \Gamma}$; as $\mathrm{Hom}_{\Lambda_1}(-, \Lambda_1)$ preserves finite direct sums, $\mathrm{Hom}_{\Lambda_1}(N^\chi, \Lambda_1) \cong \Lambda_1[\Gamma]^k$. □

Lemma E.21. *Let the data be as in Theorem B.91, λ a dominant weight, $1 \le s \le c$ integers, $\mathcal{O}'' \supseteq \mathcal{O}$ the ring of integers of a finite extension of $\mathcal{O}[1/\ell]$ in $\overline{\mathbb{Q}}_\ell$, and $f \in S_\lambda^{\mathrm{ord}}(U(\ell^{c,c}), \mathcal{O}''[1/\ell])$ with $\langle u\rangle f = f$ for every $u \in T_n(\mathfrak{l}^s)$. Then $f \in S_\lambda^{\mathrm{ord}}(U(\ell^{s,s}), \mathcal{O}''[1/\ell])$. Consequently, for a field $A \supseteq \mathcal{O}$ of characteristic 0 and a character α of finite order of $T_n(\mathfrak{l})$ with values in A and $T_n(\mathfrak{l}^s) \subseteq \ker\alpha$, the inclusion $S_\lambda^{\mathrm{ord}}(U(\ell^{s,s}), \alpha, A) \subseteq S_\lambda^{\mathrm{ord}}(U(\ell^{c,c}), \alpha, A)$ is an equality.*

Proof. Write $\mathcal{K}'' = \mathcal{O}''[1/\ell]$ and ϖ'' for a uniformiser of $\mathcal{O}''$. The space $S_\lambda(U(\ell^{c,c}), \mathcal{K}'')$ consists of functions on the finite set $\mathbb{G}(F^+)\backslash\mathbb{G}(\mathbb{A}_{F^+}^\infty)/U(\ell^{c,c})$ (with values in $M_\lambda \otimes \mathcal{K}''$, subject to the invariance of (Gk4)), so after multiplying f by a power of ϖ'' we may assume $f \in S_\lambda(U(\ell^{c,c}), \mathcal{O}'')$; the hypotheses and the conclusion are unchanged.

Step 1: f lies in the ordinary part over $\mathcal{O}''$. The ordinary idempotent e of (Gk4) is an idempotent endomorphism of the finite free $\mathcal{O}''$-module $S = S_\lambda(U(\ell^{c,c}), \mathcal{O}'')$, so $S = eS \oplus (1-e)S$ and $S_\lambda^{\mathrm{ord}}(U(\ell^{c,c}), \mathcal{K}'') = eS \otimes \mathcal{K}''$. As $f \in S$ and $(1-e)f = 0$ in $S \otimes \mathcal{K}''$, the component of f in the torsion-free module $(1-e)S$ vanishes, so $f \in eS = S_\lambda^{\mathrm{ord}}(U(\ell^{c,c}), \mathcal{O}'')$.

Step 2: reduction modulo ϖ''^k. Let $k \ge 1$ and f_k the image of $\varpi''^{-k} f$ in $S_\lambda(U(\ell^{c,c}), \varpi''^{-k}\mathcal{O}''/\mathcal{O}'')$ (reduce the values of $\varpi''^{-k}f$ modulo $M_\lambda \otimes \mathcal{O}''$; the invariance under $U(\ell^{c,c})$ is preserved). The reduction map commutes with the operators of weight λ (they are given by the same double coset formulas), hence with $e = \lim_r U(\ell)^{r!}$, so $f_k = ef_k$ is ordinary, and $\langle u\rangle f_k = f_k$ for $u \in T_n(\mathfrak{l}^s)$. Its image in $S_\lambda^{\mathrm{ord}}(U(\ell^\infty), \mathcal{K}''/\mathcal{O}'')$ is therefore fixed by the $\langle u\rangle$, $u \in T_n(\mathfrak{l}^s)$, and by (Gk4)(b) (with $b = s$) it lies in $S_\lambda^{\mathrm{ord}}(U(\ell^{s,s}), \mathcal{K}''/\mathcal{O}'')$: f_k is invariant under $U(\ell^{s,s})$, i.e. $\xi_\lambda(g_{S_\ell}) f_k(xg) = f_k(x)$ for all $g \in U(\ell^{s,s})$ and x.

Step 3: conclusion. For $g \in U(\ell^{s,s})$ and x, the element $\xi_\lambda(g_{S_\ell}) f(xg) - f(x)$ of $M_\lambda \otimes \mathcal{O}''$ lies in $\varpi''^k(M_\lambda \otimes \mathcal{O}'')$ for every k by Step 2, hence is 0 ($M_\lambda \otimes \mathcal{O}''$ is finite free over $\mathcal{O}''$). So $f \in S_\lambda(U(\ell^{s,s}), \mathcal{O}'')$, and it is ordinary there: the inclusion of levels commutes with the operators $U^{(j)}_{\lambda, \varpi_{\bar v}}$ ([105], Lemma 2.3.3), hence with e. For the last sentence: if $f \in S_\lambda^{\mathrm{ord}}(U(\ell^{c,c}), \alpha, A)$, then f has coefficients in a finitely generated subfield, and writing f in a basis of A over $\mathcal{K}''$, where $\mathcal{K}''$ is a finite extension of $\mathcal{K}$ containing the values of α such that f is a finite sum $\sum a_i f_i$ with $f_i \in S_\lambda^{\mathrm{ord}}(U(\ell^{c,c}), \alpha, \mathcal{K}'')$ and $a_i \in A$ linearly independent over $\mathcal{K}''$ (the ordinary part and the α-part are cut out by operators defined over $\mathcal{K}''$), each f_i satisfies $\langle u\rangle f_i = \alpha(u) f_i = f_i$ for $u \in T_n(\mathfrak{l}^s)$, so f_i, hence f, has level $U(\ell^{s,s})$. □

E.1.4 Λ-adic typed forms of all levels

This subsection constructs the Λ-adic typed forms of all levels at a finite set of split places, a smooth module whose invariants are the forms of finite level (Lemma E.22), and shows that the classical points of weight λ separate them (Lemma E.23). With module-level classicality, natural in the level away from ℓ (Lemma E.24), they give the classical level modules of Proposition E.1. The separation uses the density of algebraic weights from the main text (Lemma 50.13).

Lemma E.22. *Let $\mathscr{H}$ be a rank-n Hecke datum over $\mathcal{O}$, $((V_v)_{v\in\mathrm{St}},\tau)$ a type datum over $\mathscr{H}$ and Q a finite set of places of L^+ split in L and not in T_L. Put $G_Q=\prod_{v\in Q}\mathbb{G}_L(L_v^+)$, let $\mathcal{U}$ be the set of compact open subgroups $K'=\prod_{v\in Q}K'_v$ of G_Q, $\Lambda=\Lambda_1$ (diamond structure), and for $K'\in\mathcal{U}$ put $N(K')=N_\tau(K')$ and $N^*(K')=\mathrm{Hom}_\Lambda(N(K'),\Lambda)$. The* operators *are the $T_w^{(j)}$, $(T_w^{(n)})^{-1}$ (w above a place $v\notin T_L\cup Q$ split in L), $U_{\tilde v}^{(j)}$ ($v\in S_\ell$) and $\langle u\rangle$ ($u\in T_n(\mathcal{O}_{L^+}\otimes\mathbb{Z}_\ell)$); they act on every $N(K')$, and on $N^*(K')$ by transposition. Then:*

(Sm1) *$N(K')$ is free of finite rank over Λ. For $K''\subseteq K'$ in $\mathcal{U}$, the map $p_{K''K'}\colon N(K'')\to N(K')$ obtained by applying $\mathrm{Hom}_{\mathcal{O}[H]}(\tau,-)$ to the Pontryagin dual of the inclusion*

$$S^{\mathrm{ord}}(V(K')(\ell^\infty),\mathcal{K}/\mathcal{O})\subseteq S^{\mathrm{ord}}(V(K'')(\ell^\infty),\mathcal{K}/\mathcal{O})$$

is surjective and commutes with the operators; if K'' is normal in K', then $p_{K''K'}$ induces an isomorphism $N(K'')_{K'/K''}\to N(K')$ from the coinvariants for the action of K'/K'' dual to right translation.

(Sm2) *There are a Λ-module P_∞, injective Λ-linear maps $j_{K'}\colon N^*(K')\to P_\infty$ ($K'\in\mathcal{U}$) with $j_{K''}\circ p^*_{K''K'}=j_{K'}$ for $K''\subseteq K'$ and $P_\infty=\bigcup_{K'}j_{K'}(N^*(K'))$, a Λ-linear action ρ of G_Q on P_∞ and an action of the operators on P_∞, commuting with ρ and compatible with every $j_{K'}$, such that $P_\infty^{K'}=j_{K'}(N^*(K'))$ for every $K'\in\mathcal{U}$, and such that $\rho(g)\circ j_{K'}=j_{gK'g^{-1}}\circ a_g^*$ for $g\in G_Q$ and $K'\in\mathcal{U}$, where right translation $r_g\colon f\mapsto f(\cdot\,g)$ maps $S^{\mathrm{ord}}(V(K')(\ell^\infty),\mathcal{K}/\mathcal{O})$ onto $S^{\mathrm{ord}}(V(gK'g^{-1})(\ell^\infty),\mathcal{K}/\mathcal{O})$ and $a_g\colon N(gK'g^{-1})\to N(K')$ is obtained by applying $\mathrm{Hom}_{\mathcal{O}[H]}(\tau,-)$ to the Pontryagin dual of r_g.*

(Sm3) *If $K''\subseteq K'$ in $\mathcal{U}$ and K'' is normal in K', then $P_\infty^{K''}$, with the action of K'/K'' induced by ρ, is a finitely generated projective $\Lambda[K'/K'']$-module, and a free one if K'/K'' is an ℓ-group.*

Proof. For $K'\in\mathcal{U}$ let $V^{\mathrm{St}}(K')\supseteq V(K')$ be the group with components $\mathbb{G}_L(\mathcal{O}_{L_v^+})$ at $v\in\mathrm{St}$ and those of $V(K')$ elsewhere. Both are as in Theorem B.91 with $F=L$, $\mathbb{G}=\mathbb{G}_L$, $T=T_L\cup Q$ (all places of T split in L; the components at S_ℓ and at split places outside T are hyperspecial; any compact open components are allowed at T), and $V^{\mathrm{St}}(K')(\ell^{1,1})$ is sufficiently small, its component at v_0 being the kernel of reduction modulo p of $\mathrm{GL}_n(\mathcal{O}_{L_{\tilde v_0}})$, p odd and unramified in L (Definition 50.8), which has no non-trivial element of finite order .

Step 1: (Sm1). By (Yp3) for $U=V^{\mathrm{St}}(K')\supseteq U'=V(K')$, $\Gamma=G_1=H$, $G_2=1$, $b_0=1$, $N(K')$ is free of finite rank over Λ. Let $K''\subseteq K'$ be normal and $\Gamma=K'/K''$. Apply Lemma E.12 to $U=V^{\mathrm{St}}(K')\supseteq U'=V(K'')$ (componentwise normal, equal outside $\mathrm{St}\cup Q$), with $U/U'=H\times\Gamma$, $G_1=H$, $G_2=\Gamma$: by (Yp1) the action of $H\times\Gamma$ on $\mathcal{S}'=\mathcal{S}_{V(K'')}$ commutes with the operators, and by (Yp2) $\mathcal{S}'$ is a direct summand of R^m for some m, $R=\Lambda[H\times\Gamma]$. Apply Lemma E.12 also to $U=V(K')\supseteq U'=V(K'')$, with quotient Γ: by (Yp4) the dual of the inclusion induces an isomorphism $\mathcal{S}'_\Gamma\to\mathcal{S}_{V(K')}$ compatible with the operators; it is H-equivariant, the inclusion of forms commuting with right translation at St. For R itself,

write $R = \Lambda[\Gamma] \otimes_{\mathcal{O}} \mathcal{O}[H]$; the map $f \mapsto \epsilon \circ f$, ϵ the coefficient of $1 \in H$, is an isomorphism $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, R) \cong \Lambda[\Gamma] \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}}(\tau, \mathcal{O})$ with inverse $\varphi \mapsto (t \mapsto \sum_{h \in H} \varphi(h^{-1}t)\, h)$ (a direct check; τ is finite free over $\mathcal{O}$), Γ-equivariant for Γ acting on the factor $\Lambda[\Gamma]$; this shows that the natural map $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, R)_\Gamma \to \mathrm{Hom}_{\mathcal{O}[H]}(\tau, R_\Gamma)$ is an isomorphism: both sides are $\Lambda \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}}(\tau, \mathcal{O})$, $R_\Gamma = \Lambda[H]$. Both functors are additive, so the natural map is an isomorphism for the direct summand $\mathcal{S}'$ of R^m; composing with $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, -)$ of the isomorphism of (Yp4) gives $N(K'')_\Gamma \cong N(K')$, induced by $p_{K''K'}$, which is therefore surjective. For arbitrary $K'' \subseteq K'$ in $\mathcal{U}$, the intersection K''' of the finitely many K'-conjugates of K'' (K'' has finite index in K') lies in $\mathcal{U}$ and is normal in K'; as inclusions of forms compose, $p_{K''K'} \circ p_{K'''K''} = p_{K'''K'}$, which is surjective, so $p_{K''K'}$ is surjective. The operators are the same double coset operators at all levels at Q (they involve no place of Q), so they commute with the inclusions of forms, hence with every $p_{K''K'}$.

Step 2: the limit. Order $\mathcal{U}$ by reverse inclusion; it is directed ($K' \cap K'' \in \mathcal{U}$). The transposes $p^*_{K''K'} \colon N^*(K') \to N^*(K'')$ are injective ($p_{K''K'}$ is surjective) and compatible ($p_{K'''K'} = p_{K''K'} \circ p_{K'''K''}$). Let P_∞ be the direct limit and $j_{K'}$ the canonical maps, which are injective; P_∞ is their union. The operators act on each $N^*(K')$ by transposition, compatibly with the p^* (Step 1), hence on P_∞.

Step 3: the action of G_Q. For $g \in G_Q$ and $K' \in \mathcal{U}$, right translation $f \mapsto f(\cdot\, g)$ maps forms of level $V(K')(\ell^{c,c})$ bijectively to forms of level $V(gK'g^{-1})(\ell^{c,c})$ (g has trivial components outside Q); it commutes with the operators and the ordinary idempotent (supported outside Q), with right translation at St and with the inclusions of forms between levels at Q. Dualising and applying $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, -)$ gives Λ-linear isomorphisms $a_g \colon N(gK'g^{-1}) \to N(K')$, compatible with the operators and with the maps p; from $(f(\cdot\, g))(\cdot\, h) = f(\cdot\, hg)$ one gets $a_{hg} = a_g \circ a_h$, so the transposes $a_g^* \colon N^*(K') \to N^*(gK'g^{-1})$ satisfy $a^*_{hg} = a^*_h \circ a^*_g$. Put $\rho(g) = a^*_g$ on $j_{K'}(N^*(K'))$; by the compatibility with the p^* this defines a Λ-linear map of P_∞, and $\rho(h)\rho(g) = \rho(hg)$, $\rho(1) = 1$: an action of G_Q commuting with the operators. If $g \in K'$, right translation by g is the identity on forms of level $V(K')$, so $\rho(g)$ fixes $j_{K'}(N^*(K'))$: $j_{K'}(N^*(K')) \subseteq P_\infty^{K'}$. Conversely let $x \in P_\infty^{K'}$. Then $x \in j_{K'''}(N^*(K'''))$ for some $K''' \in \mathcal{U}$, and replacing K''' by the intersection of the K'-conjugates of $K''' \cap K'$ we may take $K'' = K'''$ normal in K'. For $\gamma \in K'$, $\rho(\gamma)$ acts on $N^*(K'')$ as the transpose of a_γ, which is the action of $\Gamma = K'/K''$ dual to right translation in (Sm1); so $x = j_{K''}(\phi)$ with $\phi \circ a_\gamma = \phi$ for all γ, i.e. ϕ factors through $N(K'')_\Gamma \cong N(K')$ (Step 1): $\phi = p^*_{K''K'}(\psi)$ with $\psi \in N^*(K')$, and $x = j_{K'}(\psi)$. This proves (Sm2).

Step 4: (Sm3). By (Yp3) for $U = V^{\mathrm{St}}(K') \supseteq U' = V(K'')$, $G_1 = H$, $G_2 = \Gamma = K'/K''$ (Step 1), $X = N(K'')$ is a finitely generated projective $\Lambda[\Gamma]$-module, free if Γ is an ℓ-group, Γ acting through the a_γ. By (Sm2) and Step 3, $P_\infty^{K''} \cong N^*(K'') = \mathrm{Hom}_\Lambda(X, \Lambda)$ with γ acting by $\phi \mapsto \phi \circ a_\gamma$. For $X = \Lambda[\Gamma]$ this module is free of rank one: the dual basis (δ_γ) of the basis Γ is permuted simply transitively. The functor $\mathrm{Hom}_\Lambda(-, \Lambda)$ preserves finite direct sums and direct summands, so $N^*(K'')$ is finitely generated projective over $\Lambda[\Gamma]$, and free if X is. □

Lemma E.23. *Let $\mathscr{H}$ be a rank-n Hecke datum over $\mathcal{O}$, $((V_v)_{v \in \mathrm{St}}, \tau)$ a type datum over $\mathscr{H}$, Q a finite set of places of L^+ split in L and not in T_L, and P_∞, ρ as in Lemma E.22. Let $x \in P_\infty$, $x \neq 0$. Then there are a dominant weight λ with $\lambda_{\tau',j} > \lambda_{\tau',j+1}$ for all embeddings τ' and all j, an irreducible constituent π of $S_\lambda(\overline{\mathbb{Q}}_\ell)$, a homomorphism of $\mathcal{O}$-algebras $y \colon \Lambda \to \overline{\mathbb{Q}}_\ell$, values $y(t) \in \overline{\mathbb{Q}}_\ell$ for the operators t, a $\overline{\mathbb{Q}}_\ell$-vector space Y with a representation π_Y of G_Q and an additive map $\mathrm{sp} \colon P_\infty \to Y$ such that:*

(Ls1) $\mathrm{sp}(x) \neq 0$;

(Ls2) $\mathrm{sp}(\mu z) = y(\mu)\,\mathrm{sp}(z)$ *and* $\mathrm{sp}(tz) = y(t)\,\mathrm{sp}(z)$ *for $\mu \in \Lambda$, every operator t and $z \in P_\infty$;* $y([u]) = y(\langle u \rangle) = (w_0\lambda)(u)^{-1}$ *for $u \in T_n(\ell^1)$; and for every place $v \notin T_L \cup Q$ of L^+ split*

in L, π_v is unramified and $y(T_w^{(j)})$ is the eigenvalue of $T_w^{(j)}$ on $\pi_v^{\mathbb{G}_L(\mathcal{O}_{L_v^+})}$ for every place $w \mid v$ of L;

(Ls3) $\mathrm{sp}(\rho(g)z) = \pi_Y(g)\,\mathrm{sp}(z)$ *for $g \in G_Q$ and $z \in P_\infty$;*

(Ls4) *for every $v \in Q$, Y is spanned by the images of the $\mathbb{G}_L(L_v^+)$-equivariant linear maps $\pi_v \to Y$, $\mathbb{G}_L(L_v^+)$ acting on Y through the factor of G_Q at v.*

Proof. Step 0: the levels. For $K' \in \mathcal{U}$ the group $V(K')$ is as in Theorem B.91 with $T = T_L \cup Q$ (all places of T split in L by (Rd2) and the choice of Q; the components at S_ℓ and at the split places outside T are hyperspecial), and $V(K')(\ell^{1,1})$ is sufficiently small: its component at v_0 is the kernel of reduction modulo p of $\mathrm{GL}_n(\mathcal{O}_{L_{\tilde{v}_0}})$, p odd and unramified in L (Rd3), which has no non-trivial element of finite order . Write $S_{K'}$, $W_{K'}$, $\Phi_{K'}$ for the objects of Lemma E.24 for $U = V(K')$ (with λ chosen in Step 1); all $V(K')$ have the same components at S_ℓ. The group $V^{\mathrm{St}}(K')$ with components $\mathbb{G}_L(\mathcal{O}_{L_v^+})$ at $v \in \mathrm{St}$ and those of $V(K')$ elsewhere is as in Theorem B.91, contains $V(K')$ as a componentwise normal subgroup with quotient H, equal at S_ℓ and outside T; so by (Yp2) ($b_0 = 1$) $S_{K'}$ is a finitely generated projective $\Lambda[H]$-module.

Step 1: the weight. By (Sm2), $x = j_K(\phi)$ for some $K \in \mathcal{U}$ and $\phi \in N^*(K)$, $\phi \neq 0$; by (Sm1) $N(K)$ is Λ-free, so $\phi(m) \neq 0$ for some $m \in N(K)$. Let I be the set of embeddings $L^+ \to \overline{\mathbb{Q}}_\ell$; restriction is a bijection from the embeddings τ of L inducing the places $\tilde{v}$ onto I ($L_{\tilde{v}} = L_v^+$), and for $u \in T_n(\ell^1)$, $(w_0\lambda)(u)^{-1} = \prod_\tau \prod_j \tau(u_j)^{-\lambda_{\tau,n+1-j}} = \chi_k(u)$ with $k_{\tau|_{L^+},j} = -\lambda_{\tau,n+1-j}$; so $\mathfrak{p}_\lambda = \mathfrak{P}_k$, and $\lambda \mapsto k$ is a bijection from the dominant weights with $\lambda_{\tau,j} > \lambda_{\tau,j+1}$ for all τ, j onto the set $\mathcal{K}_0$ of $k \in (\mathbb{Z}^n)^I$ strictly decreasing in j for every $\tau' \in I$. The closure of $\mathcal{K}_0$ in $(\mathbb{Z}_\ell^n)^I$ is everything: for $a \in (\mathbb{Z}^n)^I$ and $e \geq 1$, $a + M\ell^e(n+1-j)_{\tau',j} \in \mathcal{K}_0$ for every large integer M. As ℓ is odd and $\mathcal{O} \supseteq \tau(\mathcal{O}_L)$ (Rd1), Lemma 50.13 gives $\bigcap_{k\in\mathcal{K}_0} \mathfrak{P}_k = 0$, so there is such a λ with $\phi(m) \notin \mathfrak{p}_\lambda$. Fix it, write $\mathfrak{p} = \mathfrak{p}_\lambda$ and $y_0 \colon \Lambda \to \Lambda/\mathfrak{p} = \mathcal{O}$.

Step 2: one level. Let $K' \in \mathcal{U}$. The natural map $N(K') \otimes_\Lambda \Lambda/\mathfrak{p} \to \mathrm{Hom}_{\mathcal{O}[H]}(\tau, S_{K'}/\mathfrak{p}S_{K'})$ is an isomorphism: for $R = \Lambda[H]$, $f \mapsto \epsilon \circ f$ (ϵ the coefficient of $1 \in H$) gives isomorphisms $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, \Lambda[H]) \cong \Lambda \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}}(\tau, \mathcal{O})$ and $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, \mathcal{O}[H]) \cong \mathrm{Hom}_{\mathcal{O}}(\tau, \mathcal{O})$ (inverse $\varphi \mapsto (t \mapsto \sum_h \varphi(h^{-1}t)h)$), compatible with $R \to R/\mathfrak{p}R = \mathcal{O}[H]$; so the map is an isomorphism for R, hence for finite sums and direct summands (both sides are additive), hence for $S_{K'}$ (Step 0). H acts on $S_{K'}$ dually to right translation by elements h with components in $\prod_{v\in\mathrm{St}} \mathbb{G}_L(\mathcal{O}_{L_v^+})$, which normalise $V(K')$ and have trivial components at S_ℓ and outside T; so by (Wn3) with $g = h$, $\Phi_{K'}$ is H-equivariant, H acting on $W_{K'}^*$ by the transposes of right translation. Tensoring with $\mathcal{K}$ (τ is finite free) and applying $\Phi_{K'}$ gives an isomorphism

$$\beta_{K'} \colon N(K') \otimes_\Lambda \Lambda/\mathfrak{p} \otimes_{\mathcal{O}} \mathcal{K} \cong \mathrm{Hom}_{\mathcal{K}[H]}(\tau_{\mathcal{K}}, W_{K'}^*) = \left((\tau_{\mathcal{K}} \otimes_{\mathcal{K}} W_{K'})_H\right)^*,$$

$\tau_{\mathcal{K}} = \tau \otimes \mathcal{K}$, H acting diagonally; the last identification is natural for H-equivariant maps of W. Put $Y_{K'} = (\tau_{\mathcal{K}} \otimes W_{K'})_H$, finite-dimensional. Since $N(K')$ is free of finite rank, $N^*(K') \otimes_\Lambda \Lambda/\mathfrak{p} \otimes \mathcal{K}$ is the $\mathcal{K}$-dual of the left side, hence $\cong Y_{K'}^{**} = Y_{K'}$ by the transpose of $\beta_{K'}$. Let $\mathrm{sp}_{K'} \colon N^*(K') \to Y_{K'}$ be reduction followed by this isomorphism: $\mathrm{sp}_{K'}(\psi)$ is the element of $Y_{K'}$ whose pairing with $\beta_{K'}(n \otimes 1)$ is $y_0(\psi(n))$, for $n \in N(K')$.

Step 3: all levels. Let $K'' \subseteq K'$ in $\mathcal{U}$; forms of level $V(K')$ are forms of level $V(K'')$, giving an H-equivariant inclusion $\iota \colon W_{K'} \subseteq W_{K''}$. By (Wn2), and since $p_{K''K'}$ is $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, -)$ of the dual of the inclusion of forms of trivial weight, $\beta_{K'} \circ (p_{K''K'} \otimes 1) = \mathrm{Hom}_H(\tau_{\mathcal{K}}, \iota^t) \circ \beta_{K''}$, and $\mathrm{Hom}_H(\tau_{\mathcal{K}}, \iota^t)$ is the transpose of $(1 \otimes \iota)_H \colon Y_{K'} \to Y_{K''}$. Transposing, $\mathrm{sp}_{K''} \circ p_{K''K'}^* = (1 \otimes \iota)_H \circ \mathrm{sp}_{K'}$. Let $W_\infty = \bigcup_{K'} W_{K'}$ (a directed union: $K' \cap K'' \in \mathcal{U}$) and $Y_\infty = (\tau_{\mathcal{K}} \otimes W_\infty)_H = \varinjlim Y_{K'}$; the maps $Y_{K'} \to Y_\infty$ are injective, H-coinvariants being exact in characteristic 0 (averaging

identifies them with invariants). By (Sm2) ($P_\infty = \bigcup j_{K'}(N^*(K'))$, $j_{K''} \circ p^*_{K''K'} = j_{K'}$, the j injective) there is a unique additive map $\mathrm{sp}_\infty \colon P_\infty \to Y_\infty$ with $\mathrm{sp}_\infty(j_{K'}(\psi)) = \mathrm{sp}_{K'}(\psi)$: if $j_{K'}(\psi) = j_{K''}(\psi'')$, both agree in $N^*(K''')$ for $K''' = K' \cap K''$, and the compatibility gives the same image.

Step 4: the operators. For an operator t let t_λ be $T^{(j)}_w$, $(T^{(n)}_w)^{-1}$, $U^{(j)}_{\lambda,\varpi_{\tilde v}}$, resp. $(w_0\lambda)(u)^{-1}\langle u\rangle$ of weight λ, for $t = T^{(j)}_w$, $(T^{(n)}_w)^{-1}$, $U^{(j)}_{\tilde v}$, $\langle u\rangle$. These commute with H and with ι (double cosets at places outside $\mathrm{St} \cup Q$, with the same components for all $V(K')$) and preserve every $W_{K'}$. By (Wn1), $\beta_{K'}$ carries the action of t on $N(K')$ to composition with t^t_λ, i.e. to the transpose of $(1 \otimes t_\lambda)_H$ on $Y_{K'}$; as t acts on $N^*(K')$ by $\psi \mapsto \psi \circ t$, $\mathrm{sp}_{K'}(t\psi) = (1 \otimes t_\lambda)\mathrm{sp}_{K'}(\psi)$. As $\mathrm{sp}_{K'}$ factors through $- \otimes_\Lambda \Lambda/\mathfrak{p}$, $\mathrm{sp}_{K'}(\mu\psi) = y_0(\mu)\mathrm{sp}_{K'}(\psi)$ for $\mu \in \Lambda$. Hence $\mathrm{sp}_\infty(tz) = (1 \otimes t_\lambda)\mathrm{sp}_\infty(z)$ and $\mathrm{sp}_\infty(\mu z) = y_0(\mu)\mathrm{sp}_\infty(z)$.

Step 5: the group G_Q. Let $g \in G_Q$, $K' \in \mathcal{U}$. g has trivial components outside $Q \subseteq T$, so by (Wn3) right translation r_g maps $W_{K'}$ onto $W_{gK'g^{-1}}$, commutes with H, and $\Phi_{K'} \circ (a'_g \otimes 1) = r^t_g \circ \Phi_{gK'g^{-1}}$, a'_g the dual of r_g on forms of trivial weight; applying $\mathrm{Hom}_{\mathcal{O}[H]}(\tau, -)$, $\beta_{K'} \circ (a_g \otimes 1)$ is $\beta_{gK'g^{-1}}$ followed by composition with r^t_g, the transpose of $(1 \otimes r_g)_H \colon Y_{K'} \to Y_{gK'g^{-1}}$. Transposing, $\mathrm{sp}_{gK'g^{-1}} \circ a^*_g = (1 \otimes r_g)_H \circ \mathrm{sp}_{K'}$. By (Sm2), $\rho(g) \circ j_{K'} = j_{gK'g^{-1}} \circ a^*_g$; so $\mathrm{sp}_\infty(\rho(g)z) = \pi_\infty(g)\mathrm{sp}_\infty(z)$ with $\pi_\infty(g) = (1 \otimes r_g)_H$ on Y_∞, a representation of G_Q ($r_h r_g = r_{hg}$).

Step 6: decomposition. Put $\overline{W} = W_\infty \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell$, $\overline{Y} = Y_\infty \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_\ell = (\tau_{\overline{\mathbb{Q}}_\ell} \otimes \overline{W})_H$, into which Y_∞ injects. Let E be as in Lemma E.24. By (Gk4)(a) ($\overline{\mathbb{Q}}_\ell$ is flat over $\mathcal{O}$, and invariants under $T_n(\ell^1)$, which acts on the forms of level $V(K')(\ell^{E,E})$ through a finite quotient, are the kernel of a finite family of differences $\langle u\rangle - 1$, so they commute with flat base change), $W_{K'} \otimes \overline{\mathbb{Q}}_\ell$ is the subspace of $e\,S_\lambda(V(K')(\ell^{E,E}), \overline{\mathbb{Q}}_\ell)$ fixed by the $\langle u\rangle$, $u \in T_n(\ell^1)$, and $e\,S_\lambda(V(K')(\ell^{E,E}), \overline{\mathbb{Q}}_\ell)$ is the sum of the joint generalised eigenspaces of the $U^{(j)}_{\lambda,\varpi_{\tilde v}}$ on $S_\lambda(V(K')(\ell^{E,E}), \overline{\mathbb{Q}}_\ell)$ with ℓ-adic unit eigenvalues . Let $K^Q = \prod_{v \notin Q} V(\ell^{E,E})_v$, the components of every $V(K')(\ell^{E,E})$ outside Q. By (Gk4)(d), $S_\lambda(\overline{\mathbb{Q}}_\ell)$ is a semisimple admissible representation of $\mathbb{G}_L(\mathbb{A}^\infty_{L^+})$; its K^Q-invariants are

$$\bigcup_{K'} S_\lambda(V(K')(\ell^{E,E}), \overline{\mathbb{Q}}_\ell)$$

(a vector is fixed by some compact open subgroup, which contains $K' \times K''$ with $K' \in \mathcal{U}$ and K'' open in $\prod_{v \notin Q} \mathbb{G}_L(L^+_v)$; so a K^Q-invariant vector is fixed by $V(K')(\ell^{E,E}) = K' \times K^Q$), G_Q acts on them by r_g, and the operators act as double coset operators of this representation at the places of S_ℓ and outside $T_L \cup Q$. Hence $\overline{W}$ is the subspace fixed by the $\langle u\rangle$, $u \in T_n(\ell^1)$, of the sum of the generalised eigenspaces with unit eigenvalues of the $U^{(j)}_{\lambda,\varpi_{\tilde v}}$ on $S_\lambda(\overline{\mathbb{Q}}_\ell)^{K^Q}$.

Write $S_\lambda(\overline{\mathbb{Q}}_\ell) = \bigoplus_\pi S[\pi]$ with $S[\pi] \cong \pi \otimes M_\pi$, π over the isomorphism classes of irreducible constituents and M_π a multiplicity space (finite-dimensional by admissibility). By Flath's tensor product theorem, $\pi \cong \bigotimes'_v \pi_v$, so $\pi^{K^Q} \cong \pi_Q \otimes X_\pi \otimes R_\pi$ with $\pi_Q = \bigotimes_{v \in Q} \pi_v$ (on which G_Q acts), $X_\pi = \bigotimes_{v \in S_\ell} \pi_v^{\mathrm{Iw}(\tilde v^{E,E})}$ and $R_\pi = \bigotimes_{v \notin Q \cup S_\ell} \pi_v^{V_v}$ (finite-dimensional, H acting through the factors at St). For $v \notin T_L \cup Q$ split, $T^{(j)}_w$ acts on R_π through $\pi_v^{\mathbb{G}_L(\mathcal{O}_{L^+_v})}$, which is 0 or a line (the spherical vectors of an irreducible admissible representation of $\mathrm{GL}_n(L_w)$); so if $R_\pi \neq 0$, π_v is unramified and $T^{(j)}_w$ acts on R_π by its eigenvalue $t^{(j)}_w(\pi)$ on $\pi_v^{\mathbb{G}_L(\mathcal{O}_{L^+_v})}$. The $U^{(j)}_{\lambda,\varpi_{\tilde v}} = (w_0\lambda)(\alpha^{(j)}_{\varpi_{\tilde v}})^{-1} U^{(j)}_{\varpi_{\tilde v}}$ ((Gy7)(c)) and $\langle u\rangle$ act on X_π factorwise. For $v \in S_\ell$ let $c_{v,j} = v_\ell\big((w_0\lambda)(\alpha^{(j)}_{\varpi_{\tilde v}})\big) = v_\ell(\varpi_{\tilde v}) \sum_{i \le j} \sum_\tau \lambda_{\tau,n+1-i}$ (τ over the embeddings inducing $\tilde v$); $c_{v,j} - c_{v,j-1} = v_\ell(\varpi_{\tilde v}) \sum_\tau \lambda_{\tau,n+1-j}$ is non-decreasing in j (λ_τ is decreasing). The unit-eigenvalue part of $\pi_v^{\mathrm{Iw}(\tilde v^{E,E})}$ for the $U^{(j)}_{\lambda,\varpi_{\tilde v}}$ is the part X_v of Lemma E.19 for $K = L_{\tilde v}$, $c = E$, slopes $c_{v,j}$, $\sigma' = \pi_v$ (through $\iota_{\tilde v}$); by that lemma the $U^{(j)}_{\varpi_{\tilde v}}$ and $T_n(\mathcal{O}_{L_{\tilde v}})$ act semisimply on X_v with joint eigenspaces

of dimension at most one. The unit-eigenvalue part of X_π is $\bigotimes_v X_v$, hence the direct sum of lines X_χ on which the operators $U^{(j)}_{\lambda,\varpi_{\tilde v}}$ and $\langle u\rangle$ act through a character χ, distinct lines carrying distinct χ; its subspace fixed by the $\langle u\rangle$, $u \in T_n(\ell^1)$, is the sum of the lines X_χ with $\chi(\langle u\rangle) = 1$ for all $u \in T_n(\ell^1)$, and only these χ occur below. Therefore

$$\overline{W} = \bigoplus_{\pi,\chi} W[\pi,\chi], \qquad W[\pi,\chi] \cong \pi_Q \otimes X_\chi \otimes R_\pi \otimes M_\pi,$$

each part stable under G_Q and H, every t_λ acting on $W[\pi,\chi]$ by a scalar $y_{\pi,\chi}(t_\lambda)$ ($t^{(j)}_w(\pi)$ for $T^{(j)}_w$, and χ for the operators at S_ℓ); and $\overline{Y} = \bigoplus_{\pi,\chi}(\tau_{\overline{\mathbb{Q}}_\ell} \otimes W[\pi,\chi])_H$.

Step 7: the specialisation. $\mathrm{sp}_\infty(x) = \mathrm{sp}_K(\phi)$ pairs with $\beta_K(m \otimes 1)$ to $y_0(\phi(m)) \neq 0$ (Step 1), so it is non-zero, and so is its image in $\overline{Y}$. Choose (π,χ) for which the component of this image is non-zero, put $Y = (\tau_{\overline{\mathbb{Q}}_\ell} \otimes W[\pi,\chi])_H$, sp the composite of sp_∞, $Y_\infty \to \overline{Y}$ and the projection to Y, $\pi_Y(g) = (1 \otimes r_g)_H|_Y$, $y = y_0$ on Λ (followed by $\mathcal{O} \subseteq \overline{\mathbb{Q}}_\ell$) and $y(t) = y_{\pi,\chi}(t_\lambda)$ for the operators. (Ls1) holds by the choice. (Ls2): by Step 4 and Step 6, $\mathrm{sp}(\mu z) = y(\mu)\mathrm{sp}(z)$ and $\mathrm{sp}(tz) = y(t)\mathrm{sp}(z)$ (the projection commutes with the t_λ); for $u \in T_n(\ell^1)$, $y([u]) = y_0([u]) = (w_0\lambda)(u)^{-1}$, and $\langle u\rangle$ of weight λ is trivial on $W_{K'}$ (definition of $W_{K'}$ in Lemma E.24), hence on $\overline{W}$ and on every $W[\pi,\chi]$, so $y(\langle u\rangle) = (w_0\lambda)(u)^{-1}$; $Y \neq 0$ forces $R_\pi \neq 0$, so for $v \notin T_L \cup Q$ split, π_v is unramified and $y(T^{(j)}_w) = t^{(j)}_w(\pi)$. (Ls3): Step 5, the parts being G_Q-stable. (Ls4): H acts trivially on π_Q, so $Y \cong \pi_Q \otimes (\tau_{\overline{\mathbb{Q}}_\ell} \otimes X_\chi \otimes R_\pi \otimes M_\pi)_H$ with G_Q acting on π_Q; for $v \in Q$, $\pi_Q = \pi_v \otimes \pi_{Q\smallsetminus\{v\}}$, and the maps $a \mapsto a \otimes b$ (b in the remaining factors) are $\mathbb{G}_L(L^+_v)$-equivariant from π_v to Y and their images span Y. □

Lemma E.24. *Let ℓ, n, F, F^+, $\mathcal{O}$, $\mathbb{G}$ and T be as in Theorem B.91, $\mathcal{K} = \mathcal{O}[1/\ell]$, λ a dominant weight for $\mathbb{G}$, and $\mathfrak{p}_\lambda \subseteq \Lambda_1$ the kernel of the homomorphism of $\mathcal{O}$-algebras $\Lambda_1 \to \mathcal{O}$, $[u] \mapsto (w_0\lambda)(u)^{-1}$. For every U as in Theorem B.91 such that $U(\ell^{1,1})$ is sufficiently small, let S_U be the Pontryagin dual of $S^{\mathrm{ord}}(U(\ell^\infty),\mathcal{K}/\mathcal{O})$, with Λ_1 acting through the diamond operators and the operators acting dually, let E be the largest ramification index of $F^+_v/\mathbb{Q}_\ell$ over the places $v \in S_\ell$, and*

$$W_U = \{f \in S^{\mathrm{ord}}_\lambda(U(\ell^{E,E}),\mathcal{K}) : \langle u\rangle f = f \text{ for all } u \in T_n(\ell^1)\},$$

$\langle u\rangle$ the operator of weight λ (if ℓ is unramified in F^+, then $E = 1$ and $W_U = S^{\mathrm{ord}}_\lambda(U(\ell^{1,1}),\mathcal{K})$). Then there are $\mathcal{K}$-linear isomorphisms

$$\Phi_U \colon (S_U/\mathfrak{p}_\lambda S_U) \otimes_{\mathcal{O}} \mathcal{K} \longrightarrow \mathrm{Hom}_{\mathcal{K}}(W_U,\mathcal{K})$$

such that:

(Wn1) *Φ_U carries $T^{(j)}_w$, $(T^{(n)}_w)^{-1}$, $U^{(j)}_{\tilde v}$ and $\langle u\rangle$ ($u \in T_n(\mathcal{O}_{F^+} \otimes \mathbb{Z}_\ell)$) to the transposes of the operators $T^{(j)}_w$, $(T^{(n)}_w)^{-1}$, $U^{(j)}_{\lambda,\varpi_{\tilde v}}$ and $(w_0\lambda)(u)^{-1}\langle u\rangle$ of weight λ on W_U;*

(Wn2) *if $U' \subseteq U$ are two such groups with $U'_v = U_v$ for every $v \in S_\ell$, then $W_U \subseteq W_{U'}$ and $\Phi_U \circ (p \otimes 1) = r \circ \Phi_{U'}$, where $p \colon S_{U'}/\mathfrak{p}_\lambda S_{U'} \to S_U/\mathfrak{p}_\lambda S_U$ is induced by the dual of the inclusion $S^{\mathrm{ord}}(U(\ell^\infty),\mathcal{K}/\mathcal{O}) \subseteq S^{\mathrm{ord}}(U'(\ell^\infty),\mathcal{K}/\mathcal{O})$ and r is the restriction of linear forms to W_U;*

(Wn3) *if $g \in \mathbb{G}(\mathbb{A}^\infty_{F^+})$ has $g_v = 1$ for every $v \in S_\ell$ and every $v \notin T$ split in F, and $U_g = gUg^{-1}$, then U_g is again such a group, right translation $r_g \colon f \mapsto f(\cdot\, g)$ maps $S^{\mathrm{ord}}(U(\ell^\infty),\mathcal{K}/\mathcal{O})$ onto $S^{\mathrm{ord}}(U_g(\ell^\infty),\mathcal{K}/\mathcal{O})$ and W_U onto W_{U_g}, and $\Phi_U \circ (a_g \otimes 1) = r^t_g \circ \Phi_{U_g}$, where $a_g \colon S_{U_g}/\mathfrak{p}_\lambda S_{U_g} \to S_U/\mathfrak{p}_\lambda S_U$ is induced by the dual of r_g and $r^t_g(\phi) = \phi \circ r_g|_{W_U}$.*

Proof. Fix U as in the statement and write $M_U = S^{\mathrm{ord}}(U(\ell^\infty), \mathcal{K}/\mathcal{O})$, $M_{\lambda,U} = S^{\mathrm{ord}}_\lambda(U(\ell^\infty), \mathcal{K}/\mathcal{O})$, $Z_U = S_\lambda(U(\ell^{E,E}), \mathcal{O})$ (E as in the statement); $\chi = w_0\lambda$ is a continuous character $T_n(\mathcal{O}_{F^+} \otimes \mathbb{Z}_\ell) \to \mathcal{O}^\times$ ($\mathcal{O}$ contains the images of all embeddings of F).

Step 1: the fibre at $\mathfrak{p}_\lambda$. The continuous $\mathcal{O}$-algebra automorphism θ of Λ_1 with $\theta([u]) = \chi(u)[u]$ ($u \in T_n(\ell^1)$) satisfies $\mathrm{aug} \circ \theta^{-1}([u]) = \chi(u)^{-1}$, so $\mathfrak{p}_\lambda = \theta(\mathfrak{a})$, $\mathfrak{a}$ the augmentation ideal, which is generated by the elements $[u] - 1$; hence $\mathfrak{p}_\lambda$ is generated by the elements $[u] - \chi(u)^{-1}$, and, Λ_1 being noetherian, by finitely many of them, $x_1, \dots, x_k$. For a discrete Λ_1-module M let $M[\mathfrak{p}_\lambda]$ be the submodule killed by $\mathfrak{p}_\lambda$, i.e. the m with $\langle u\rangle m = \chi(u)^{-1}m$ for all $u \in T_n(\ell^1)$. It is the kernel of $M \to M^k$, $m \mapsto (x_i m)_i$, so dualising gives an isomorphism $M^\vee/\mathfrak{p}_\lambda M^\vee \cong (M[\mathfrak{p}_\lambda])^\vee$, natural for Λ_1-linear maps $M \to M'$. For $M = M_U$: $S_U/\mathfrak{p}_\lambda S_U \cong (M_U[\mathfrak{p}_\lambda])^\vee$.

Step 2: passage to weight λ. Let $M^\flat_{\lambda,U}$ be the submodule of $M_{\lambda,U}$ fixed by all $\langle u\rangle$, $u \in T_n(\ell^1)$. By (Gk4)(c), λ_* maps $M^\flat_{\lambda,U}$ isomorphically onto $M_U[\mathfrak{p}_\lambda]$: if $\langle u\rangle f = f$ then $\langle u\rangle\lambda_*(f) = \chi(u)^{-1}\lambda_*(\langle u\rangle f) = \chi(u)^{-1}\lambda_*(f)$; conversely if $\lambda_*(f) \in M_U[\mathfrak{p}_\lambda]$ then $\lambda_*(\langle u\rangle f) = \chi(u)\langle u\rangle\lambda_*(f) = \lambda_*(f)$, so $\langle u\rangle f = f$ as λ_* is injective. Let $\mathfrak{l}$ and $T_n(\mathfrak{l}^b)$ be as in (Gk4)(b). By (Qh1) of Lemma E.10 (with $b = 1$), $T_n(\mathfrak{l}^E) \subseteq T_n(\ell^1)$. By (Gk4)(b) with $b = E$, the submodule of $M_{\lambda,U}$ fixed by the $\langle u\rangle$, $u \in T_n(\mathfrak{l}^E)$, is $S^{\mathrm{ord}}_\lambda(U(\ell^{E,E}), \mathcal{K}/\mathcal{O})$, compatibly with the operators; hence $M^\flat_{\lambda,U}$ is the submodule of $S^{\mathrm{ord}}_\lambda(U(\ell^{E,E}), \mathcal{K}/\mathcal{O})$ fixed by the $\langle u\rangle$, $u \in T_n(\ell^1)$. On forms of level $U(\ell^{E,E})$ the operator $\langle u\rangle$ of weight λ, $u \in T_n(\mathcal{O}_{F^+} \otimes \mathbb{Z}_\ell)$ (placed at the components of S_ℓ; it normalises $U(\ell^{E,E})$), is $f \mapsto \xi_\lambda(u)f(\cdot\, u)$, which is the identity for $u \in T_n(\mathfrak{l}^E) \subseteq U(\ell^{E,E})$ by the invariance defining $S_\lambda(U(\ell^{E,E}), A)$. So $T_n(\ell^1)$ acts on the forms of level $U(\ell^{E,E})$ through the finite abelian group $\Delta = T_n(\ell^1)/T_n(\mathfrak{l}^E)$, and $M^\flat_{\lambda,U} = S^{\mathrm{ord}}_\lambda(U(\ell^{E,E}), \mathcal{K}/\mathcal{O})^\Delta$.

Step 3: integral structure. Z_U is a finitely generated torsion-free, hence free, $\mathcal{O}$-module (evaluation at representatives of the finite set $\mathbb{G}(F^+)\backslash\mathbb{G}(\mathbb{A}^\infty_{F^+})/U(\ell^{E,E})$ embeds it in a finite sum of copies of M_λ). By (Gk4)(a) ($U(\ell^{E,E}) \subseteq U(\ell^{1,1})$ is sufficiently small), $S_\lambda(U(\ell^{E,E}), \mathcal{K}/\mathcal{O}) = Z_U \otimes_{\mathcal{O}} \mathcal{K}/\mathcal{O}$; the operators, given on both sides by the same finite sums of translates, act as $t \otimes 1$, so the ordinary idempotent acts as $e \otimes 1$, and $S^{\mathrm{ord}}_\lambda(U(\ell^{E,E}), \mathcal{K}/\mathcal{O}) = L'_U \otimes_{\mathcal{O}} \mathcal{K}/\mathcal{O}$ with $L'_U = eZ_U$ a direct summand of Z_U, finite free. The group Δ acts on Z_U through the operators $\langle u\rangle$, which commute with e ([105], Lemma 2.3.3), so it preserves L'_U. Let $L_U = (L'_U)^\Delta$. It is saturated in L'_U (if $x \in L'_U$ and $\varpi x \in L_U$, then $\varpi(\delta x - x) = 0$, so $\delta x = x$ for $\delta \in \Delta$), so L'_U/L_U is finitely generated and torsion-free, hence free, and L_U is a direct summand of L'_U, finite free. Invariants under the finite group Δ are the kernel of $x \mapsto (\delta x - x)_{\delta\in\Delta}$ and $\mathcal{K}$ is flat over $\mathcal{O}$, so $W_U = (L'_U \otimes_{\mathcal{O}} \mathcal{K})^\Delta = L_U \otimes_{\mathcal{O}} \mathcal{K}$. As L_U is a direct summand of L'_U, the map $L_U \otimes \mathcal{K}/\mathcal{O} \to L'_U \otimes \mathcal{K}/\mathcal{O}$ is injective, with image in $M^\flat_{\lambda,U} = (L'_U \otimes \mathcal{K}/\mathcal{O})^\Delta$ (Step 2); its cokernel C_U in $M^\flat_{\lambda,U}$ is finite: the Δ-invariants of $0 \to L'_U \to L'_U \otimes \mathcal{K} \to L'_U \otimes \mathcal{K}/\mathcal{O} \to 0$ give an exact sequence $L_U \otimes \mathcal{K} \to (L'_U \otimes \mathcal{K}/\mathcal{O})^\Delta \to H^1(\Delta, L'_U)$ in which the image of the first map is $L_U \otimes \mathcal{K}/\mathcal{O}$, and $H^1(\Delta, L'_U)$ is a subquotient of the finitely generated $\mathcal{O}$-module of maps $\Delta \to L'_U$ and is killed by $|\Delta|$, hence finite. Dualising $0 \to L_U \otimes \mathcal{K}/\mathcal{O} \to M^\flat_{\lambda,U} \to C_U \to 0$ and tensoring with $\mathcal{K}$, which kills the finite module $C_U^\vee$, the restriction $(M^\flat_{\lambda,U})^\vee \otimes_{\mathcal{O}} \mathcal{K} \to (L_U \otimes \mathcal{K}/\mathcal{O})^\vee \otimes_{\mathcal{O}} \mathcal{K}$ is an isomorphism. For a finite free $\mathcal{O}$-module L there are isomorphisms, natural in L,

$$(L \otimes_{\mathcal{O}} \mathcal{K}/\mathcal{O})^\vee \cong \mathrm{Hom}_{\mathcal{O}}(L, \mathrm{End}_{\mathcal{O}}(\mathcal{K}/\mathcal{O})) = \mathrm{Hom}_{\mathcal{O}}(L, \mathcal{O}),$$
$$\mathrm{Hom}_{\mathcal{O}}(L, \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{K} \cong \mathrm{Hom}_{\mathcal{K}}(L \otimes_{\mathcal{O}} \mathcal{K}, \mathcal{K})$$

($\mathrm{End}_{\mathcal{O}}(\mathcal{K}/\mathcal{O}) = \mathcal{O}$, $\mathcal{O}$ being complete; the second is the base change of linear forms on a free module of finite rank).

Definition of Φ_U. Φ_U is the composite of the isomorphism of Step 1 tensored with $\mathcal{K}$, the dual of the inverse of the isomorphism $\lambda_*\colon M^\flat_{\lambda,U} \to M_U[\mathfrak{p}_\lambda]$ of Step 2 tensored with $\mathcal{K}$, the restriction isomorphism $(M^\flat_{\lambda,U})^\vee \otimes \mathcal{K} \to (L_U \otimes \mathcal{K}/\mathcal{O})^\vee \otimes \mathcal{K}$ of Step 3, and the isomorphisms of

Step 3 for $L = L_U$. It is a $\mathcal{K}$-linear isomorphism.

(Wn1). By (Gy2) the operators on M_U commute with each other and with Λ_1, so they preserve $M_U[\mathfrak{p}_\lambda]$, and Step 1 transports the dual action on $S_U/\mathfrak{p}_\lambda S_U$ to the dual of the action on $M_U[\mathfrak{p}_\lambda]$. By (Gk4)(c), λ_* intertwines $T_w^{(j)}$ with $T_w^{(j)}$, $U_{\lambda,\varpi_{\tilde{v}}}^{(j)}$ with $U_{\tilde{v}}^{(j)}$, and $\chi(u)^{-1}\langle u\rangle$ (weight λ) with $\langle u\rangle$ (trivial weight); these operators of weight λ commute with the $\langle u\rangle$ ([105], Lemma 2.3.3), so preserve $M_{\lambda,U}^\flat$, L'_U and L_U; by (Gk4)(b) they act on $M_{\lambda,U}^\flat \subseteq S_\lambda^{\mathrm{ord}}(U(\ell^{E,E}), \mathcal{K}/\mathcal{O})$ as on the forms of level $U(\ell^{E,E})$, so the inclusion $L_U \otimes \mathcal{K}/\mathcal{O} \subseteq M_{\lambda,U}^\flat$ intertwines $t|_{L_U} \otimes 1$ with them, and the restriction isomorphism of Step 3 carries the dual of their action to the dual of $t|_{L_U} \otimes 1$. The isomorphisms of Step 3 are natural in L, so they carry the dual of $t|_{L_U} \otimes 1$ to the transpose of $t|_{L_U} \otimes 1$ on W_U. Inverses correspond to inverses. This is (Wn1).

(Wn2). U' and U have the same components at S_ℓ and at the split places outside T, and $U'(\ell^{b,c}) \subseteq U(\ell^{b,c})$; so forms of level $U(\ell^{b,c})$ are forms of level $U'(\ell^{b,c})$, giving inclusions ι of all the spaces above, for every coefficient module. Each operator $T_w^{(j)}$, $U_{\lambda,\varpi_{\tilde{v}}}^{(j)}$, $\langle u\rangle$ is a double coset operator $[U''\alpha U'']$ with α supported at one place v_0 (a split place outside T, or a place of S_ℓ), where $U''_{v_0} = U'_{v_0}$; writing $U''_{v_0}\alpha U''_{v_0} = \coprod_i x_i \alpha U''_{v_0}$ with $x_i \in U''_{v_0}$, we get $U''\alpha U'' = \coprod_i x_i\alpha U''$ for both levels with the same x_i, and the operator is $f \mapsto \sum_i \xi_\lambda((x_i\alpha)_{S_\ell}) f(\cdot\, x_i\alpha)$ (up to the same scalar) on both; hence ι commutes with all operators, with the ordinary idempotents, with direct limits over c, and with λ_*, which is composition with ϵ_λ on the values. So ι maps M_U into $M_{U'}$ Λ_1-linearly, $M_{\lambda,U}^\flat$ into $M_{\lambda,U'}^\flat$, L'_U into $L'_{U'}$, L_U into $L_{U'}$ and W_U into $W_{U'}$ (the groups Δ for U and U' are the same, and ι commutes with the $\langle u\rangle$), and all squares formed with the maps of Steps 1–3 commute. Each map in the definition of Φ is natural for these maps (Step 1 for Λ_1-linear maps, Step 2 because $\iota \circ \lambda_* = \lambda_* \circ \iota$, the restriction of Step 3 because ι maps $L_U \otimes \mathcal{K}/\mathcal{O}$ into $L_{U'} \otimes \mathcal{K}/\mathcal{O}$, Step 3 for $\mathcal{O}$-linear maps $L_U \to L_{U'}$); dualising reverses the arrows, the dual of $\iota\colon M_U \to M_{U'}$ induces p and the transpose of $\iota|_{W_U}$ is r. Hence $\Phi_U \circ (p \otimes 1) = r \circ \Phi_{U'}$.

(Wn3). $(U_g)_v = U_v$ for $v \in S_\ell$ and for $v \notin T$ split in F, so U_g is as in Theorem B.91, $U_g(\ell^{b,c}) = gU(\ell^{b,c})g^{-1}$, and $U_g(\ell^{1,1})$ is sufficiently small (its projection to $\mathbb{G}(F_v^+)$ is conjugate to that of $U(\ell^{1,1})$). If f has level $U(\ell^{b,c})$ and $u = gu'g^{-1}$ with $u' \in U(\ell^{b,c})$, then $u_{S_\ell} = u'_{S_\ell}$ and $\xi_\lambda(u_{S_\ell})(r_g f)(xu) = \xi_\lambda(u'_{S_\ell}) f(xgu') = f(xg)$; so r_g maps forms of level $U(\ell^{b,c})$ bijectively (inverse $r_{g^{-1}}$) onto forms of level $U_g(\ell^{b,c})$, for every coefficient module. For a double coset operator as in the proof of (Wn2), supported at v_0 with $g_{v_0} = 1$, the elements $x_i\alpha$ commute with g and the decomposition at v_0 is the same for both levels, so r_g commutes with it: $\sum_i \xi_\lambda((x_i\alpha)_{S_\ell}) f(xgx_i\alpha) = \sum_i \xi_\lambda((x_i\alpha)_{S_\ell}) f(xx_i\alpha g)$. Also r_g commutes with λ_* (values against arguments) and with direct limits. The argument of (Wn2), with r_g in place of ι, gives $\Phi_U \circ (a_g \otimes 1) = r_g^t \circ \Phi_{U_g}$. □

E.1.5 Automorphy and purity at points of the classical ideal

This subsection proves Proposition E.25, the automorphy with the witness conditions of the irreducible points of the classical ideal with a parallel inertial weight, which Proposition 50.11 and Theorem 53.10 use. The classical ideal is the kernel of a Galois determinant onto the irreducible-type quotient of an ordinary Hecke algebra of a type (Proposition E.26), and the witness conditions come from Lemma 53.25. The subsection also proves the two properties of ideals cut out by such Hecke algebras that Theorem 53.10 needs: stability under twists (Lemma E.27, with the criterion Lemma E.28) and purity at arithmetic points (Lemma E.29).

Proposition E.25. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}'$, κ a twist for $\mathscr{H}$, $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, and $(\psi_{v,a}^\circ)$ an inertial weight for $\mathscr{H}$ such that for every $v \in S_\ell$ and $1 \le a, b \le n$ there is an integer h_{ab} with $\psi_{v,a}^\circ(\psi_{v,b}^\circ)^{-1}\chi_{\mathrm{cyc}}^{-h_{ab}}$ trivial on an open subgroup of I_v. Let A be an integral test ring over $\mathcal{O}'$,*

$\rho\colon \Gamma^0_{\mathscr{H}} \to \mathrm{GL}_n(A)$ *a homomorphism such that* $\rho_L = \rho \circ \pi_{\mathscr{H}}|_{G_L}$ *is continuous and irreducible over* $\overline{\mathbb{Q}}_\ell$*, and* $y\colon B_n^{\mathrm{po}} \to A$ *a homomorphism of* $\mathcal{O}'$*-algebras with* $y \circ D^{\mathrm{u}} = \det \circ \rho$*,* $y(\psi^{\mathrm{d}}_{v,a}(g)) = \psi^{\circ}_{v,a}(g)$ *for all* $v \in S_\ell$*,* $1 \le a \le n$ *and* $g \in I_v$*, and* $y(\mathcal{J}_\kappa^{\mathrm{cl}}) = 0$*. Then* $\rho_L \otimes \kappa$ *is automorphic of conjugate self-dual type for* ι *and satisfies the witness conditions.*

Proof. The integral test ring A is a subring of $\overline{\mathbb{Q}}_\ell$; let $y'\colon B \to \overline{\mathbb{Q}}_\ell$ be y followed by the inclusion. As $y'(1) = 1$ and $y'(\mathcal{J}_\kappa^{\mathrm{cl}}) = 0$, $\mathcal{J}_\kappa^{\mathrm{cl}} \neq B$, so Proposition E.26 provides (Hz3); its hypotheses hold for the inertial weight $(\psi^{\circ}_{v,a})$: $y'(\psi^{\mathrm{d}}_{v,a}(g)) = \psi^{\circ}_{v,a}(g)$ for $g \in I_v$, and $y' \circ D^{\mathrm{u}} = \det \circ \rho$ with ρ, viewed over $\overline{\mathbb{Q}}_\ell$, continuous ($\Gamma$ carries the quotient topology of $\pi|_{G_L}$) and absolutely irreducible (because ρ_L is irreducible over $\overline{\mathbb{Q}}_\ell$ and $\pi(G_L) = \Gamma$). So y' is a classical ordinary point: there are $\tilde{r}\colon \Gamma \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, $r = \tilde{r} \circ \pi$ continuous, and a regular algebraic conjugate self-dual cuspidal Π with (Cy1)–(Cy3).

Step 1: $\rho_L \cong r$*, hence automorphy.* By (Cy1), $\det \circ \rho = y' \circ D^{\mathrm{u}} = \det \circ \tilde{r}$ as $\overline{\mathbb{Q}}_\ell$-valued determinants on Γ. $\rho \otimes \overline{\mathbb{Q}}_\ell$ is irreducible because ρ_L is and $\pi(G_L) = \Gamma$. The semisimplification $\tilde{r}^{\mathrm{ss}}$ has the same determinant as $\tilde{r}$ (characteristic polynomials only depend on it), so by the uniqueness in (Ch3), $\tilde{r}^{\mathrm{ss}} \cong \rho$ is irreducible; then $\tilde{r}$ has no proper invariant subspace, $\tilde{r} = \tilde{r}^{\mathrm{ss}} \cong \rho$, and $\rho_L \cong r$. By (Cy2), $\rho_L \otimes \kappa \cong r \otimes \kappa \leftrightarrow_\iota \Pi$, so $\rho_L \otimes \kappa$ is automorphic of conjugate self-dual type for ι.

Step 2: (Qw1) for $r' = \rho_L \otimes \kappa$*.* Let $w \nmid \ell$ be a finite place of L, $\bar{w} \mid w$, $D = D_{\bar{w}} \cap G_L$, $I = I_{\bar{w}} \cap G_L$, and $\bar{\chi} = (\bar{\chi}_1, \dots, \bar{\chi}_n)\colon G_L \to (\mathbb{F}^\times_{\mathcal{O}'})^n$. Every $s \in I$ with $\bar{\chi}(s) = 1$ lies in $N_{\mathscr{H}}$, so $\rho_L|_I$ factors through the finite abelian group $A_I = \bar{\chi}(I) \subseteq A_D = \bar{\chi}(D)$; hence $\rho_L|_I$ has finite image and is conjugate to $\mathrm{diag}(\phi_i \circ \bar{\chi})$ for characters ϕ_i of A_I. Extend each ϕ_i to a character ϕ_i' of A_D (characters of finite abelian groups extend from subgroups) and put $\Psi_i' = \phi_i' \circ \bar{\chi}|_D$. The continuous character κ has finite image on I (on an open subgroup of I it is $\exp(t_\ell(\cdot)N)$ with N a nilpotent 1×1 matrix, i.e. trivial). So $r'(I)$ is finite and $r'|_I$ is conjugate to $\mathrm{diag}(\Psi_i'\kappa|_I)$ with characters $\Psi_i'\kappa|_D$ of D.

Step 3: the polarisation of ρ*.* The map $\rho^\sharp\colon \Gamma \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, $\gamma \mapsto {}^t\rho(\pi(c)\gamma^{-1}\pi(c))$, is a homomorphism, irreducible because ρ is. For $g \in G_L$, (Od1) for the polarised ordinary determinant $(y' \circ D^{\mathrm{u}}, (y' \circ \psi^{\mathrm{d}}_{v,a}))$ says that $\rho(\pi(cg^{-1}c))$, hence $\rho^\sharp(\pi(g))$, has the same characteristic polynomial as $\rho(\pi(g))$. Two irreducible representations of Γ over the algebraically closed field $\overline{\mathbb{Q}}_\ell$ of characteristic 0 with the same characteristic polynomials are isomorphic (Brauer–Nesbitt), so there is $P \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ with $\rho(\pi(cg^{-1}c)) = {}^t(P\rho(\pi(g))P^{-1})$ for all $g \in G_L$.

Step 4: (Qw2) for r'*.* Let $u \in S_\ell$, with the place $\bar{u}$ of $\mathscr{H}$ and $D_u = D_{\bar{u}} \cap G_{L^+} \subseteq G_L$ ((Rd2): u splits in L). By (Cy3) and Step 1, ρ_L is triangular on D_u with diagonal $(y' \circ \psi^{\mathrm{d}}_{u,a})_a$, whose restrictions to I_u are the $\psi^{\circ}_{u,a}$. Every place of $\overline{\mathbb{Q}}$ above a place of L above u is $h\bar{u}$ or $ch\bar{u}$ with $h \in G_L$ (as $G_{L^+} = G_L \cup cG_L$), with decomposition groups in G_L equal to hD_uh^{-1}, resp. $chD_uh^{-1}c$, and inertia groups hI_uh^{-1}, resp. $chI_uh^{-1}c$. On $D' = hD_uh^{-1}$, ρ_L is triangular with diagonal $(\kappa_a)_a$, $\kappa_a(s) = y'(\psi^{\mathrm{d}}_{u,a}(h^{-1}sh))$. Let $Z \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ conjugate $\rho_L|_{D'}$ to upper triangular form. For $s \in D'$, $X = Z\rho_L(s^{-1})Z^{-1}$ is upper triangular with diagonal $(\kappa_a(s)^{-1})_a$, and by Step 3 $\rho_L(csc) = {}^t(P\rho_L(s^{-1})P^{-1})$ is conjugate to ${}^tX$ by a matrix independent of s, and $J_n\,{}^tX\,J_n$ (J_n the antidiagonal matrix of ones) is upper triangular with diagonal $(\kappa_{n+1-a}(s)^{-1})_a$; so ρ_L is triangular on $cD'c$ with diagonal $(csc \mapsto \kappa_{n+1-a}(s)^{-1})_a$. Twisting by κ multiplies every diagonal character by κ and does not change their ratios. By hypothesis, $\psi^{\circ}_{u,a}(\psi^{\circ}_{u,b})^{-1}\chi_{\mathrm{cyc}}^{-h_{ab}}$ is trivial on an open subgroup U_{ab} of I_u; since $\chi_{\mathrm{cyc}}(h^{-1}sh) = \chi_{\mathrm{cyc}}(s) = \chi_{\mathrm{cyc}}(csc)$, on D' the character $\kappa_a\kappa_b^{-1}\chi_{\mathrm{cyc}}^{-h_{ab}}$ is trivial on $hU_{ab}h^{-1}$, and on $cD'c$ the ratio of the a-th and b-th diagonal characters times $\chi_{\mathrm{cyc}}^{-h_{n+1-b,n+1-a}}$ is trivial on $chU_{n+1-b,n+1-a}h^{-1}c$, open subgroups of the respective inertia groups. This is (Qw2).

Step 5. r' is irreducible, $n \ge 2$, automorphic of conjugate self-dual type (Step 1), and satisfies

(Qw1), (Qw2); by Lemma 53.25 it satisfies the witness conditions. □

Proposition E.26. *Let $\mathscr{H} = (n, L, c, (\bar v)_{v\in S_\ell}, \mathrm{St}, v_0, \bar v_0, (\bar\chi_a)_{1\le a\le n})$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}'$, κ a twist for $\mathscr{H}$, and $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$. Then $\mathcal{J}_\kappa^{\mathrm{cl}} = B_n^{\mathrm{po}}$, or there are a type datum $((V_v)_{v\in\mathrm{St}}, \tau)$ over $\mathscr{H}$, a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_\tau$, an ideal $\mathfrak{a}$ of $\mathbb{T}_{\tau,\mathfrak{m}}$ which is the intersection of a non-empty set of minimal prime ideals of $\mathbb{T}_{\tau,\mathfrak{m}}$, and a surjective homomorphism of $\mathcal{O}'$-algebras $\theta\colon B_n^{\mathrm{po}} \to \mathbb{T}_{\tau,\mathfrak{m}}/\mathfrak{a}$ such that*

(Hz1) $\ker\theta = \mathcal{J}_\kappa^{\mathrm{cl}}$;

(Hz2) *for every $u \in T_n(\ell^1)$, $\langle u\rangle - 1$ lies in the ideal of $\mathbb{T}_{\tau,\mathfrak{m}}/\mathfrak{a}$ generated by $\mathfrak{m}_{\mathcal{O}'}$ and the elements $\theta(\psi^{\mathrm{d}}_{v,a}(g)) - 1$ with $v \in S_\ell$, $1 \le a \le n$, $g \in I_v$ and $\bar\chi_b(g) = 1$ for $1 \le b \le n$;*

(Hz3) *for every inertial weight $(\psi^\circ_{v,a})$ for $\mathscr{H}$, every homomorphism of $\mathcal{O}'$-algebras $y\colon B_n^{\mathrm{po}} \to \overline{\mathbb{Q}}_\ell$ with $y(\mathcal{J}_\kappa^{\mathrm{cl}}) = 0$, $y(\psi^{\mathrm{d}}_{v,a}(g)) = \psi^\circ_{v,a}(g)$ for all $v \in S_\ell$, $1 \le a \le n$ and $g \in I_v$, and $y \circ D^{\mathrm{u}} = \det\circ\rho$ for a continuous absolutely irreducible representation $\rho\colon \Gamma^0_{\mathscr{H}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, is a classical ordinary point for ι and κ.*

Proof. Step 1: the setting. Let $((V_v), \tau)$ be the type datum of Proposition E.5, V its level, $\mathbb{T} = \mathbb{T}_\tau$. By (Gd0) of Lemma E.16, $\mathbb{T}$ is commutative, finite and torsion-free over Λ_1 (for the diamond and the renormalised structures), the product of its localisations at its finitely many maximal ideals, topologically generated by the $T_w^{(j)}$, $(T_w^{(n)})^{-1}$, $U_{\tilde v}^{(j)}$ and $\langle u\rangle$; the $U_{\tilde v}^{(j)}$ are units; and every homomorphism of $\mathcal{O}'$-algebras from B, from $\mathbb{T}$ or from a quotient of either to $\overline{\mathbb{Q}}_\ell$ is continuous, local onto its image, with image an $\mathcal{O}'$-order in a finite extension of $\mathbb{Q}_\ell$. Also $[L^+ : \mathbb{Q}] \ge 2$ ((Gd0)).

Step 2: the maximal ideal. If there is no classical ordinary point, $\mathcal{J} = B$ and we are done. Otherwise, for every classical ordinary point x let $\tilde r_x$ be as in (Cy1) and y_x the point of $\mathbb{T}$ of (Hx3). The reduction $\bar y_x$ of y_x (Step 1) takes on the topological generators of $\mathbb{T}$ over $\mathcal{O}'$ values independent of x and lying in $\mathbb{F}$: on $T_w^{(j)}$ the reductions of $(\mathrm{N}w)^{-j(j-1)/2}$ times the coefficients of the characteristic polynomial of $(\tilde r_x \otimes \kappa)(\mathrm{Frob}_w)$, i.e. those of $\bar D_n \otimes \bar\kappa$ ($x\circ D^{\mathrm{u}}$ reduces to $\bar D_n$, and $\kappa(G_L) \subseteq \mathcal{O}'^\times$); on $\langle d_j(z)\rangle$ and $U_{\tilde v}^{(j)}$ the reductions of the $\phi_{v,j}$, i.e. of $\chi^{j-1}\kappa\bar\chi_j$. Hence $\mathfrak{m} = \ker\bar y_x$ is one maximal ideal for all x, $\mathbb{T}/\mathfrak{m} = \mathbb{F}$, and every y_x factors through $\mathbb{T}_\mathfrak{m} = \mathbb{T}_{\tau,\mathfrak{m}}$.

Step 3: regular points are dense on every component. Call a point y of $\mathbb{T}_\mathfrak{m}$ *regular* if $y(\mathfrak{P}_{\lambda,1}) = 0$ for a regular dominant λ with $\lambda_{\tau,j} > \lambda_{\tau,j+1}$ for all τ, j. Claim: for every minimal prime P of $\mathbb{T}_\mathfrak{m}$ and every $f \notin P$ there is a regular y with $y(P) = 0 \ne y(f)$. Indeed, $\Lambda_1 \to \mathbb{T}_\mathfrak{m}/P$ is injective (non-zero elements of Λ_1 are non-zero-divisors by Step 1, and a minimal prime consists of zero-divisors), and $\mathbb{T}_\mathfrak{m}/P$ is a domain finite over the normal domain $\Lambda_1 \cong \mathcal{O}'[[X_1, \dots, X_N]]$; the minimal polynomial of f mod P over $\mathrm{Frac}(\Lambda_1)$ has coefficients in Λ_1 and a non-zero constant term c_0. The kernel of $\Lambda_1 \to \overline{\mathbb{Q}}_\ell$ induced by $(w_0\nu)^{-1}(w_0\lambda)^{-1}$ is $\mathfrak{P}_{k(\lambda)}$ in the notation of Lemma 50.13 with $k(\lambda)_{\tau,j} = -(j-1) - \lambda_{\tau,n+1-j}$; the set of $k(\lambda)$, λ as above, is dense in $(\mathbb{Z}_\ell^n)^I$ (given k^* and e', choose λ with $k(\lambda) \equiv k^* \bmod \ell^{e'}$ and add $\ell^{e'}A(n-i)$ to $\lambda_{\tau,i}$ for large A). By Lemma 50.13 ($F^+ = L^+$; $\mathcal{O}' \supseteq \tau(\mathcal{O}_{L^+})$ by (Rd1)), $c_0 \notin \mathfrak{P}_{k(\lambda)}$ for some such λ. By lying over there is $y\colon \mathbb{T}_\mathfrak{m}/P \to \overline{\mathbb{Q}}_\ell$ extending $\Lambda_1/\mathfrak{P}_{k(\lambda)} \to \overline{\mathbb{Q}}_\ell$; $y(f)$ is a root of the image of the minimal polynomial, whose constant term $y(c_0) \ne 0$, so $y(f) \ne 0$. Consequently the intersection of the kernels of the regular points is the nilradical $\mathfrak{n}$ of $\mathbb{T}_\mathfrak{m}$.

Step 4: the Galois determinant and the components of irreducible type. For a regular y let r_y be as in (Hx1). The maximal ideal $\mathfrak{m}$ satisfies (Mx) of Lemma E.16 (Step 2), and by Step 3 the regular points separate $\mathbb{T}_\mathfrak{m}/\mathfrak{n}$. By (Gd1) of Lemma E.16 (with $\mathfrak{a} = \mathfrak{n}$ and $\mathcal{C}$ the set of

regular points) there is a continuous $\mathbb{T}_{\mathfrak{m}}/\mathfrak{n}$-valued determinant D^{H} of dimension n on G_L with $y \circ D^{\mathrm{H}} = \det \circ r_y$ for all regular y, and $y \circ D^{\mathrm{H}} = \det \circ \rho$ for every point y of $\mathbb{T}_{\mathfrak{m}}$ (it kills $\mathfrak{n}$) and every continuous $\rho\colon G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ unramified above every w with characteristic polynomial $P_{y,w}$ at Frob_w. For a minimal prime P let ρ_P be the semisimple representation over an algebraic closure Ω_P of $\mathrm{Frac}(\mathbb{T}_{\mathfrak{m}}/P)$ with determinant D^{H} mod P (Ch3), and let $\mathcal{P}$ be the set of minimal primes P contained in $\ker y_x$ for some classical ordinary point x. By (P2) below, ρ_P is absolutely irreducible for $P \in \mathcal{P}$, because $y_x \circ D^{\mathrm{H}} = \det(\tilde{r}_x \circ \pi_{\mathscr{H}} \otimes \kappa)$ (by (Hx3) and the remark above) is the determinant of an absolutely irreducible representation (Cy1).

(P1) For a minimal prime P with ρ_P absolutely irreducible (in particular for $P \in \mathcal{P}$) there is $g_P \notin P$ such that r_y is absolutely irreducible for every regular y with $y(P) = 0 \neq y(g_P)$: by Burnside choose $\gamma_1, \dots, \gamma_{n^2} \in G_L$ with $\rho_P(\gamma_i)$ a basis of $\mathrm{M}_n(\Omega_P)$ and let g_P lift $\det(\mathrm{tr}_{D^{\mathrm{H}}}(\gamma_i\gamma_j)) \bmod P = \det(\mathrm{tr}\,\rho_P(\gamma_i\gamma_j)) \neq 0$; if $y(g_P) \neq 0$, the $r_y(\gamma_i)$ are linearly independent, span $\mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$, and r_y has no invariant proper subspace.

(P2) For a minimal prime P with ρ_P not absolutely irreducible and every point y with $y(P) = 0$, the semisimple representation with determinant $y \circ D^{\mathrm{H}}$ is reducible: $\rho_P = \rho_1 \oplus \rho_2$ with $\dim \rho_i \geq 1$; the coefficients of the characteristic polynomials of ρ_i are roots-sums of monic polynomials over $\mathbb{T}_{\mathfrak{m}}/P$, hence lie in the integral closure R_P of $\mathbb{T}_{\mathfrak{m}}/P$ in Ω_P, so by (Ch1) $\det \rho_i$ is the base change of an R_P-valued determinant D_i; y extends to $\tilde{y}\colon R_P \to \overline{\mathbb{Q}}_\ell$ (lying over), and $y \circ D^{\mathrm{H}} = (\tilde{y} \circ D_1)(\tilde{y} \circ D_2)$ is, by (Ch3), the determinant of a direct sum of representations of dimensions $\dim \rho_1$, $\dim \rho_2$.

Let $\mathfrak{a} = \bigcap_{P \in \mathcal{P}} P$ and $A = \mathbb{T}_{\mathfrak{m}}/\mathfrak{a}$. For a classical x, $\ker y_x$ contains a minimal prime, which lies in $\mathcal{P}$. So $\mathcal{P} \neq \varnothing$ and $y_x(\mathfrak{a}) = 0$ for every classical x.

(P3) (Invariance under $N_{\mathscr{H}}$ on the components through classical points.) Let $P \in \mathcal{P}$, $P \subseteq \ker y_x$, $R = \mathbb{T}_{\mathfrak{m}}/P$, D_P the determinant D^{H} mod P twisted by κ^{-1} (as for $D^\flat$ in Step 5), w' a place of L not above ℓ, $I_{w'} \subseteq D_{w'}$ inertia and decomposition groups at a place of $\overline{\mathbb{Q}}$ above it, $I' = \{s \in I_{w'} : \bar{\chi}_a(s) = 1 \ (1 \leq a \leq n)\}$, $q = \mathrm{N}w'$ and $\Phi \in D_{w'}$ a lift of a Frobenius. Then $\chi_{D_P}(gs) = \chi_{D_P}(g)$ in $R[t]$ for all $g \in G_L$, $s \in I'$. Proof. (a) For every regular y with $y(P) = 0$, $\tilde{r}_y = r_y \otimes \kappa^{-1}$ has residual semisimplification with determinant $\bar{D}_n$ (Step 2), i.e. $\bigoplus_a \bar{\chi}_a$, trivial on I'; so $\tilde{r}_y(I')$ lies in a pro-ℓ group (the preimage of a unipotent group for a stable lattice), $\tilde{r}_y|_{I'}$ factors through the maximal pro-ℓ quotient of I', which is a quotient of $t_\ell(I') \subseteq \mathbb{Z}_\ell(1)$ (t_ℓ the ℓ-part of the tame character; wild inertia is pro-p), and I' is normal in $D_{w'}$, with $t_\ell(\Phi s \Phi^{-1}) = q^{\pm 1} t_\ell(s)$. Hence for $s \in I'$ the multiset of eigenvalues of $\tilde{r}_y(s)$ is stable under $\mu \mapsto \mu^{q^{\pm 1}}$, and every eigenvalue satisfies $\mu^Q = 1$, $Q = \prod_{k=1}^n (q^k - 1)$. (b) Let $c_i \in R$ be the coefficients of $\chi_{D_P}(s)$ and $F_i \in \mathbb{Z}[T]$ the monic polynomial whose roots are the i-th coefficients of the polynomials $\prod_{k=1}^n (X - \mu_k)$, $\mu_k^Q = 1$ (a Galois-stable finite set of algebraic integers). By (a), $F_i(c_i)$ vanishes at all regular y with $y(P) = 0$, which separate R (Step 3), so $F_i(c_i) = 0$ and c_i is algebraic over $\mathbb{Q}$; let m_i be its minimal polynomial. As $s \in N_{\mathscr{H}}$, $y_x(\chi_{D_P}(s)) = \det(X - \tilde{r}_x(\pi_{\mathscr{H}}(s))) = (X-1)^n$, so m_i has the rational root $y_x(c_i)$ and $c_i = y_x(c_i)$: $\chi_{D_P}(s) = (X-1)^n$, and $\tilde{r}_y(s)$ is unipotent at every regular y with $y(P) = 0$. (c) Let $\delta \in R$ be the symmetric expression $\prod_{i \neq j}(\alpha_i - q\alpha_j)$ in the roots α_i of $\chi_{D_P}(\Phi)$, a polynomial in its coefficients. The representation $r_{y_x} = \tilde{r}_x \circ \pi_{\mathscr{H}} \otimes \kappa$ is absolutely irreducible and attached to the cuspidal Π of (Cy2), hence isomorphic to $r_{\ell,\iota}(\Pi)$ (Theorem B.62, (Bg1), Chebotarev); it has finite image on $I_{w'}$ (it kills $I' \subseteq N_{\mathscr{H}}$ up to κ, which has finite order there by (Rd7)), so its Weil–Deligne representation at w' has $N = 0$ and, being pure (Theorem B.90: by (Pv2) $N = 0$ forces $V = \mathrm{gr}_k$), all eigenvalues of $r_{y_x}(\Phi)$ have the same absolute value under every embedding into $\mathbb{C}$; as $q > 1$, $y_x(\delta) \neq 0$

(the twist by $\kappa(\Phi)$ does not change ratios). (d) Let y be regular with $y(P) = 0 \neq y(g_P\delta)$; such y separate R (Step 3, applied to $fg_P\delta$). Then r_y is absolutely irreducible (P1), so by (Hx2) and (Hx1)(i) it is automorphic of conjugate self-dual type, $r_y \cong r_{\ell,\iota}(\Pi_y)$ for a regular algebraic conjugate self-dual cuspidal Π_y, and its Weil–Deligne representation (ρ, N) at w' is pure. If $N \neq 0$, then, N mapping each generalised eigenspace of $\rho(\Phi)$ for an eigenvalue α into the one for $q^{\pm1}\alpha$ (the monodromy relation), $\rho(\Phi)$ has eigenvalues α and $q^{\pm1}\alpha$; for an ordering of the eigenvalues along these chains N is block nilpotent, so $r_y(\Phi)$, which differs from $\rho(\Phi)$ by a factor $\exp(cN)$, has the same eigenvalues, and $y(\delta) = 0$, a contradiction. So $N = 0$, r_y is trivial on an open subgroup of $I_{w'}$, and $\tilde{r}_y(s)$ ($s \in I'$) is unipotent of finite order, i.e. 1. Hence $\chi_{D_P}(gs) - \chi_{D_P}(g)$ vanishes at all such y, so it is 0.

Step 5: the Hecke determinant over A, and θ. Let $\mathcal{C}$ be the set of regular y with $y(P) = 0 \neq y(g_P)$ for some $P \in \mathcal{P}$. The points in $\mathcal{C}$ separate A: if $f \notin \mathfrak{a}$, then $f \notin P$ for some $P \in \mathcal{P}$, $fg_P \notin P$, and Step 3 gives a regular y with $y(P) = 0 \neq y(f)y(g_P)$. Let $D^\flat$ be D^{H} mod $\mathfrak{a}$ twisted by κ^{-1} (composed with the ring automorphism $g \mapsto \kappa(g)^{-1}g$ of $A[G_L]$), so $y \circ D^\flat = \det \circ \tilde{r}_y$ with $\tilde{r}_y := r_y \otimes \kappa^{-1}$. The elements $s \in I_{\bar{v}} \cap G_L$, $\bar{v} \nmid \ell$, with $\bar{\chi}_a(s) = 1$ for all a form a set stable under conjugation by G_{L^+} (by G_L, as the $\bar{\chi}_a$ are characters; by c, as $\bar{\chi}_a(csc) = \bar{\chi}_{n+1-a}(s)^{-1}$ by (Rd5)), which therefore generates $N_{\mathscr{H}}$ as a closed subgroup. By (P3), $\chi_{D^\flat}(gs) = \chi_{D^\flat}(g)$ for $g \in G_L$ and such s, in $\prod_{P\in\mathcal{P}}(\mathbb{T}_{\mathfrak{m}}/P)[t]$, into which $A[t]$ embeds; the set of $s \in G_L$ with this property for all g is a closed subgroup (continuity; $\chi(gs_1s_2) = \chi(gs_1) = \chi(g)$, and $\chi(gs^{-1}) = \chi(gs^{-1}s)$), so it contains $N_{\mathscr{H}}$. For $y \in \mathcal{C}$ and $s \in N_{\mathscr{H}}$, then $\mathrm{tr}(\tilde{r}_y(g)(\tilde{r}_y(s) - 1)) = 0$ for all g; $\tilde{r}_y$ is absolutely irreducible (P1), so the $\tilde{r}_y(g)$ span $\mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ (Burnside) and $\tilde{r}_y(s) = 1$: $\tilde{r}_y = \tilde{r}'_y \circ \pi_{\mathscr{H}}$ with $\tilde{r}'_y$ continuous on $\Gamma^0_{\mathscr{H}}$. So $\tilde{r}_y$ is trivial on $N_{\mathscr{H}}$ for every $y \in \mathcal{C}$; and, r_y being absolutely irreducible (P1), by (Hx1)(iii) $r_y|_{D_v}$ is triangular with diagonal $(\chi^{1-j}\psi^y_{v,j})_j$ for $v \mid \ell$. Finally (iii) of (Gd2) of Lemma E.16 holds for every $y \in \mathcal{C}$: let $v \in S_\ell$, $1 \leq j \leq n$, $\zeta \in L_{\tilde{v}}$ an ℓ-power root of unity, say $\zeta^{\ell^k} = 1$, and $t = \langle d_j(\zeta)\rangle \in \mathbb{T}_{\mathfrak{m}}$; as $u \mapsto \langle u\rangle$ is a homomorphism, $t^{\ell^k} = 1$. Let $P \in \mathcal{P}$ with $y(P) = 0$ and x classical with $P \subseteq \ker y_x$. By (Hx3), $y_x(\langle d_j(z)\rangle) = \phi_{v,j}(\mathrm{Art}(z)) = (\chi^{j-1}\kappa)(\mathrm{Art}(z))\, x(\psi^{\mathrm{d}}_{v,j}(\mathrm{Art}(z)))$ for units z (as in Step 6). The universal tuple satisfies (Od3), so $x(\psi^{\mathrm{d}}_{v,j}(g_\zeta)) = \xi_{v,j}(g_\zeta)$, and $\chi(g_\zeta) = 1$ by (Wv1) of Lemma 50.24; hence $y_x(t) = \kappa(\mathrm{Art}(\zeta))\,\xi_{v,j}(g_\zeta) \in \mathcal{O}'$. By (Wv3) of Lemma 50.24 (with $R = \mathbb{T}_{\mathfrak{m}}$ and the points y_x, y, which both kill the prime P), $y(t) = y_x(t) = \kappa(\mathrm{Art}(\zeta))\,\xi_{v,j}(g_\zeta)$. Thus (i), (ii) and (iii) of (Gd2) of Lemma E.16 hold for $\mathfrak{a}$ and $\mathcal{C}$, and $\mathfrak{m}$ satisfies (Mx) (Step 2): $A \in \mathrm{CNL}_{\mathcal{O}'}$, and there is a surjective homomorphism of $\mathcal{O}'$-algebras $\theta\colon B \to A$ with the identities of (Gd2); in particular $y \circ \theta \circ D^{\mathrm{u}} = \det \circ \tilde{r}'_y$ and $y \circ \theta \circ \psi^{\mathrm{d}}_{v,j} = \chi^{1-j}\kappa^{-1}\psi^y_{v,j}$ for $y \in \mathcal{C}$.

Step 6: (Hz1). ($\mathcal{J} \subseteq \ker\theta$.) Let $y \in \mathcal{C}$ and $x_y = y \circ \theta$. With $\tilde{r}'_y$ as in (Gd2) (Step 5): (Cy1) $x_y \circ D^{\mathrm{u}} = \det \circ \tilde{r}'_y$, $\tilde{r}'_y$ absolutely irreducible; (Cy2) $\tilde{r}'_y \circ \pi_{\mathscr{H}} \otimes \kappa = r_y$ is absolutely irreducible, unramified with characteristic polynomial $P_{y,w}$ at Frob_w, and $y(\mathfrak{P}_{\lambda,1}) = 0$ with λ dominant, so by (Hx2) $r_y \leftrightarrow_\iota \Pi$ for a regular algebraic conjugate self-dual cuspidal Π; (Cy3) by (Gd2), $x_y \circ \psi^{\mathrm{d}}_{v,j} = \chi^{1-j}\kappa^{-1}\psi^y_{v,j}$, the diagonal of the triangular $\tilde{r}'_y \circ \pi_{\mathscr{H}}|_{D_v} = (r_y \otimes \kappa^{-1})|_{D_v}$ (Step 5); (Cy4) for z in $1 + \ell\mathcal{O}_{L_{\tilde{v}}}$, $y(\langle d_j(z)\rangle) = (w_0\lambda)^{-1}(d_j(z)) = \prod_\tau \tau(z)^{-\lambda_{\tau,n+1-j}}$ (as $(w_0\nu)^{-1}(u)y(\langle u\rangle) = (w_0\nu)^{-1}(w_0\lambda)^{-1}(u)$ on $T_n(\ell^1)$), and $\kappa(\mathrm{Art}(z)) = \prod_\tau \tau(z)^{m_\tau}$ on an open subgroup (κ algebraic), so $x_y \circ \psi^{\mathrm{d}}_{v,j}$ has τ-Hodge–Tate weight $h_{\tau,j} = (1-j) - m_\tau - \lambda_{\tau,n+1-j}$ and $h_{\tau,j} - h_{\tau,j+1} = 1 + \lambda_{\tau,n-j} - \lambda_{\tau,n+1-j} \geq 1$. So x_y is a classical ordinary point, and for $b \in \mathcal{J}$, $y(\theta(b)) = x_y(b) = 0$ for all $y \in \mathcal{C}$, whence $\theta(b) = 0$. ($\ker\theta \subseteq \mathcal{J}$.) Let x be classical. By Step 4, y_x induces $y'_x\colon A \to \overline{\mathbb{Q}}_\ell$, and $y'_x \circ \theta$ and x classify the same polarised ordinary determinant: by the first identity of (Gd2) and (Hx3), $y'_x \circ \theta \circ D^{\mathrm{u}}$ and $x \circ D^{\mathrm{u}} = \det \circ \tilde{r}_x$ have the same characteristic polynomial at every $\pi_{\mathscr{H}}(\mathrm{Frob}_w)$, a dense set (Chebotarev), so they are equal (continuity, (Ch1)); and by the other identities of (Gd2) and (Hx3), $y'_x \circ \theta \circ \psi^{\mathrm{d}}_{v,j}$ and $\chi^{1-j}\kappa^{-1}\phi_{v,j} = x \circ \psi^{\mathrm{d}}_{v,j}$ agree on the dense image of Art, hence everywhere. By Step 1 both $x(B)$ and $y'_x(A)$ are orders in a finite extension E; the

$\mathcal{O}'$-algebra $R \subseteq \mathcal{O}_E$ they generate is finite over $\mathcal{O}'$, a local domain with residue field $\mathbb{F}$ (generated by residues in $\mathbb{F}$), so $R \in \mathrm{CNL}_{\mathcal{O}'}$, and uniqueness in the universal property gives $x = y'_x \circ \theta$. Hence $x(\ker\theta) = 0$ for every classical x.

Step 7: (Hz2). Let $v \mid \ell$, $z \in 1 + \ell\mathcal{O}_{L_{\tilde{v}}}$, $g = \mathrm{Art}(z) \in I_v$. As $\mathrm{Art}(1+\ell\mathcal{O})$ is pro-ℓ and the $\bar{\chi}_b$ have order prime to ℓ, $\bar{\chi}_b(g) = 1$ for all b, and $(\chi^{j-1}\kappa)(g) = 1 + \mu$ with $\mu \in \mathfrak{m}_{\mathcal{O}'}$ (a continuous character of a pro-ℓ group into $\mathcal{O}'^{\times}$ lands in $1+\mathfrak{m}_{\mathcal{O}'}$). By (Gd2), $\langle d_j(z)\rangle - 1 = (1+\mu)\big(\theta(\psi^{\mathrm{d}}_{v,j}(g)) - 1\big) + \mu$. Every $u \in T_n(\ell^1)$ is a finite product of such $d_j(z)$, and $\langle uu'\rangle - 1 = \langle u\rangle(\langle u'\rangle - 1) + (\langle u\rangle - 1)$.

Step 8: (Hz3), and the weight normalisation. Let $(\psi^{\circ}_{v,a})$ be an inertial weight, $\phi_{v,a}$ de Rham characters for it as in (Rd8), $h_{\tau,a}$ their τ-Hodge–Tate weights (integers, strictly decreasing in a), and y, ρ as in (Hz3). By (Hz1), $y = y' \circ \theta$ with $y' \colon A \to \overline{\mathbb{Q}}_\ell$; let y'' be the point $\mathbb{T} \to \mathbb{T}_{\mathfrak{m}} \to A \to \overline{\mathbb{Q}}_\ell$. For $v \mid \ell$: a de Rham character of D_v is Hodge–Tate, hence locally algebraic [220]: there are integers b_τ with $\phi_{v,j}(\mathrm{Art}(z)) = \prod_\tau \tau(z)^{b_\tau}$ for z in an open subgroup of $\mathcal{O}^{\times}_{L_{\tilde{v}}}$, and $b_\tau = h_{\tau,j}$ by the remark on Hodge–Tate weights at the start of this proof. So for z in an open subgroup of $1 + \ell\mathcal{O}_{L_{\tilde{v}}}$: $y(\psi^{\mathrm{d}}_{v,j}(\mathrm{Art}(z))) = \psi^{\circ}_{v,j}(\mathrm{Art}(z)) = \prod_\tau \tau(z)^{h_{\tau,j}}$, $\kappa(\mathrm{Art}(z)) = \prod_\tau \tau(z)^{m_\tau}$, and by (Gd2)

$$y''(\langle d_j(z)\rangle) = (\chi^{j-1}\kappa)(\mathrm{Art}(z))\, y(\psi^{\mathrm{d}}_{v,j}(\mathrm{Art}(z))) = \prod_\tau \tau(z)^{(j-1)+m_\tau+h_{\tau,j}}.$$

Put $\lambda_{\tau,n+1-j} = -(j-1) - m_\tau - h_{\tau,j}$. Then $u \mapsto (w_0\nu)^{-1}(u)\, y''(\langle u\rangle)$ and $(w_0\nu)^{-1}(w_0\lambda)^{-1}$ agree on an open subgroup of $T_n(\ell^1)$, so their ratio α has finite order and $y''(\mathfrak{P}_{\lambda,\alpha}) = 0$. *Dominance.* $\lambda_{\tau,n-j} - \lambda_{\tau,n+1-j} = h_{\tau,j} - h_{\tau,j+1} - 1 \geq 0$, as the $h_{\tau,a}$ are strictly decreasing integers; so $\lambda_{\tau,1} \geq \cdots \geq \lambda_{\tau,n}$. Let $\rho' = \rho \circ \pi_{\mathscr{H}} \otimes \kappa$, continuous and absolutely irreducible ($\pi_{\mathscr{H}}(G_L) = \Gamma^0_{\mathscr{H}}$). It is unramified above every w (w lies above a place of L^+ not in $T_L \supseteq S_\ell \cup \mathrm{St}$, so inertia there lies in $N_{\mathscr{H}}$ by (Rd4); κ by (Rd7)), and, by the first identity of (Gd2) under y' and $y \circ D^{\mathrm{u}} = \det\rho$, $\rho'(\mathrm{Frob}_w)$ has characteristic polynomial $P_{y'',w}$. By (Hx2), $\rho' \leftrightarrow_\iota \Pi$ for a regular algebraic conjugate self-dual cuspidal Π: (Cy2) with $\tilde{r} = \rho$; (Cy1) holds by hypothesis. (Cy3): for $\gamma \in \Gamma^0_{\mathscr{H}}$ and $g_1, \ldots, g_n \in D_v$, (Od2) under y gives $\mathrm{tr}\big(\rho(\gamma)X\big) = 0$ with $X = \prod_a (\rho(\pi_{\mathscr{H}}(g_a)) - y(\psi^{\mathrm{d}}_{v,a}(g_a)))$; $\rho(\Gamma^0_{\mathscr{H}})$ spans $\mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ (Burnside) and the trace form is non-degenerate, so $X = 0$. The flag is obtained from this vanishing and the regularity of Π, without an element of D_v separating the residual characters. Write $V = \overline{\mathbb{Q}}_\ell^n$, $r = \rho \circ \pi_{\mathscr{H}}$ and $\psi_a = y \circ \psi^{\mathrm{d}}_{v,a} \colon D_v \to \overline{\mathbb{Q}}_\ell^{\times}$, and let $S_a \subseteq \mathrm{M}_n(\overline{\mathbb{Q}}_\ell)$ be the linear span of the $r(g) - \psi_a(g)$, $g \in D_v$. As $X = 0$ for all $g_1, \ldots, g_n \in D_v$, multilinearity gives $S_1 S_2 \cdots S_n = 0$. Each S_a is stable under left multiplication by $r(D_v)$, as $r(g)(r(g') - \psi_a(g')) = (r(gg') - \psi_a(gg')) - \psi_a(g')(r(g) - \psi_a(g))$ (ψ_a is a character). For $0 \leq k \leq n$ let $W_k = S_{k+1} \cdots S_n V$ be the span of the images (so $W_n = V$, $W_0 = 0$ and $W_{k-1} = S_k W_k$). By the stability, each W_k is $r(D_v)$-stable; hence $S_k W_k \subseteq W_k$, i.e. $W_{k-1} \subseteq W_k$, and $(r(g) - \psi_k(g))W_k \subseteq W_{k-1}$, so D_v acts on W_k/W_{k-1} by the scalar $\psi_k(g)$. Put $m_k = \dim W_k/W_{k-1}$; then $\sum_{k=1}^n m_k = n$. The W_k are also stable under $\rho'|_{D_v}$, D_v acting on W_k/W_{k-1} by $\psi_k\kappa$. Now ρ' is absolutely irreducible and $\rho' \leftrightarrow_\iota \Pi$, so $\rho' \cong r_{\ell,\iota}(\Pi)$ by (Bg1) of Theorem B.62 (Π conjugate self-dual is polarizable with $\chi = 1$), the Chebotarev density theorem and the Brauer–Nesbitt theorem. Through $\bar{v}$, $D_v = D_{\bar{v}} \cap G_L$ is the absolute Galois group of $L_{\tilde{v}} = L_v^+$, so by (Bg2) $\rho'|_{D_v}$ is de Rham, in particular Hodge–Tate, and for each embedding τ its n τ-Hodge–Tate weights are pairwise distinct, the infinitesimal character of the regular algebraic Π being regular. Subquotients of Hodge–Tate representations are Hodge–Tate, and the τ-Hodge–Tate weights, with multiplicity, are additive in short exact sequences of them (D_{HT} is exact on Hodge–Tate representations [52]). Hence the τ-Hodge–Tate weights of $\rho'|_{D_v}$ are those of the characters $\psi_k\kappa$, each counted m_k times; as they are pairwise distinct, $m_k \leq 1$ for every k, and $\sum_k m_k = n$ gives $m_k = 1$ for all k. Choosing $e_k \in W_k \smallsetminus W_{k-1}$, the basis $(e_1, \ldots, e_n)$ of V puts every $r(g)$, $g \in D_v$, in upper triangular form with diagonal $(\psi_1(g), \ldots, \psi_n(g))$: $\rho \circ \pi_{\mathscr{H}}$ is triangular on D_v with diagonal $(y \circ \psi^{\mathrm{d}}_{v,a})_a$. (Cy4): $y \circ \psi^{\mathrm{d}}_{v,a}$ and $\phi_{v,a}$ agree on I_v, so their ratio is

unramified, hence crystalline of Hodge–Tate weight 0; so $y\circ\psi^{\mathrm{d}}_{v,a}$ is de Rham with τ-Hodge–Tate weights $h_{\tau,a}$, strictly decreasing in a. So y is a classical ordinary point. □

Lemma E.27. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}'$ with trivial wild values, κ a twist for $\mathscr{H}$, $(B, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, c an integer as in Proposition E.5, $((V_v)_{v\in\mathrm{St}}, \tau)$ a type datum over $\mathscr{H}$ with V_v the kernel of reduction modulo ϖ_v^c satisfying (Zt) for ι and κ, $\mathfrak{m}$ a maximal ideal of $\mathbb{T}_\tau$ with residue field $\mathbb{F}_{\mathcal{O}'}$ satisfying (Mx), $\chi_0\colon \Delta_\ell \to \mathcal{O}'^\times$ the homomorphism with $\chi_0(d_j(\zeta)) = \kappa(\mathrm{Art}(\zeta))$ for $v\in S_\ell$, $1\le j\le n$ and ℓ-power roots of unity $\zeta\in L_{\tilde{v}}$, $\mathfrak{a}_0 = \mathfrak{a}_{\chi_0}$ and $\mathcal{C}_{\chi_0}$ as in (Zp4) for $\mathbb{T}_{\tau,\mathfrak{m}}$, and $\theta\colon B\to \mathbb{T}_{\tau,\mathfrak{m}}/\mathfrak{a}_0$ a surjective homomorphism of $\mathcal{O}'$-algebras satisfying, for all w, $1\le j\le n$, $v\in S_\ell$ and $z\in\mathcal{O}^\times_{L_{\tilde{v}}}$, the three identities of (Gd2) of Lemma E.16:*

$$(\mathrm{N}w)^{j(j-1)/2}T^{(j)}_w = \kappa(\mathrm{Frob}_w)^j\theta(\Lambda_j(D^{\mathrm{u}})(\pi_{\mathscr{H}}(\mathrm{Frob}_w))),$$
$$\langle d_j(z)\rangle = (\chi_{\mathrm{cyc}}^{j-1}\kappa)(\mathrm{Art}(z))\theta(\psi^{\mathrm{d}}_{v,j}(\mathrm{Art}(z))),$$
$$U^{(j)}_{\tilde{v}} = U^{(j-1)}_{\tilde{v}}(\chi_{\mathrm{cyc}}^{j-1}\kappa)(\mathrm{Art}(\varpi_{\tilde{v}}))\theta(\psi^{\mathrm{d}}_{v,j}(\mathrm{Art}(\varpi_{\tilde{v}}))).$$

Then $J=\ker\theta$ is stable under twists: $\tau_\beta(J)\subseteq J$ for every admissible continuous homomorphism $\beta\colon\Gamma^0_{\mathscr{H}}\to B^\times$ with $\beta\equiv 1$ modulo $\mathfrak{m}_B$.

Proof. By Lemma E.28 it suffices to give a set $\mathcal{X}$ with (Ts1), (Ts2).

Step 1: the set $\mathcal{X}$ and (Ts1). By (Zp4) every $y\in\mathcal{C}_{\chi_0}$ kills $\mathfrak{a}_0$; let $\bar{y}$ be the induced point of A and $x_y=\bar{y}\circ\theta$. Put $\mathcal{X}=\{x_y : y\in\mathcal{C}_{\chi_0}\}$; by (Gd0) each x_y is continuous with values in the ring of integers of a finite extension of $\mathbb{Q}_\ell$. Each x_y kills J; conversely, if $x_y(b)=0$ for all y, then $\theta(b)$ lies in the image in A of $\bigcap_{y\in\mathcal{C}_{\chi_0}}\ker y=\mathfrak{a}_0$ (Zp4), so $\theta(b)=0$: (Ts1).

Step 2: an eigenform for y. Fix $y\in\mathcal{C}_{\chi_0}$, with regular λ and $y(\langle u\rangle)=(w_0\lambda)(u)^{-1}$ on $T_n(\mathcal{O}_{L^+}\otimes\mathbb{Z}_\ell)$, and let $y^\natural$ be its composite with $\mathbb{T}_\tau\to\mathbb{T}_{\tau,\mathfrak{m}}$; then $y^\natural(\mathfrak{P}_{\lambda,1})=0$. By Lemma E.11 ($Q=\varnothing$, $\alpha=1$, $r=1$) there are an irreducible constituent π_y of $S_\lambda(\overline{\mathbb{Q}}_\ell)$ with $\mathrm{Hom}_H(\tau^\vee\otimes\overline{\mathbb{Q}}_\ell, \bigotimes_{v\in\mathrm{St}}\pi_{y,v}^{V_v})\neq 0$ and a non-zero $f\in\pi_y\cap S^{\mathrm{ord}}_\lambda(V(\ell^{E,E}),1,\overline{\mathbb{Q}}_\ell)$, E the maximal ramification index of $L^+/\mathbb{Q}$ at the places above ℓ, on which the operators assigned by (Gy7)(c) to $T^{(j)}_w$, $U^{(j)}_{\tilde{v}}$, $\langle u\rangle$ act by $y^\natural$ of these.

Step 3: the twisting character. Let $\beta'\colon\Gamma^0_{\mathscr{H}}\to\overline{\mathbb{Q}}_\ell^\times$ be admissible, continuous, of finite ℓ-power order, $\beta=\beta'\circ\pi|_{G_L}$, and $\xi\colon\mathbb{A}_L^\times/L^\times\to\overline{\mathbb{Q}}_\ell^\times$, $\xi(z)=\beta(\mathrm{Art}_L(z))^{-1}$, continuous of finite ℓ-power order. Anti-invariance gives $\xi(z^c)=\xi(z)^{-1}$; on $\mathbb{A}_{L^+}^\times$ ($z^c=z$) this gives $\xi^2=1$, so ξ is trivial there (ℓ odd). Its archimedean component is a finite-order character of the connected group $(L\otimes\mathbb{R})^\times$, hence trivial. Let U_1 be the norm-one torus of L/L^+. The map $z\mapsto z/z^c$ from $\mathbb{A}_L^\times$ to $U_1(\mathbb{A}_{L^+})$ has kernel $\mathbb{A}_{L^+}^\times$ and is surjective (Hilbert 90 at every place, and integrally at the finite places, all unramified in L by (Rd2)), and $U_1(L^+)=\{a/a^c : a\in L^\times\}$; so $\psi(z/z^c)=\xi(z)$ defines a continuous character ψ of $U_1(\mathbb{A}_{L^+})$ of finite order, trivial on $U_1(L^+)$ and on $U_1(L^+\otimes\mathbb{R})$; let ψ^∞ be its restriction to $U_1(\mathbb{A}_{L^+}^\infty)$, trivial on $U_1(L^+)$. At a place v of L^+ split as ww^c, $U_1(L^+_v)\cong L^\times_w$ by the projection, and $\psi_v=\xi_w$. The character β is trivial on the inertia groups above $\mathrm{St}\cup\{v_0\}$ (admissibility) and above the places of L^+ not in $S_\ell\cup\mathrm{St}$ (these inertia elements lie in $N_{\mathscr{H}}$ by (Rd4), which π kills); so ξ_w is unramified for $w\nmid\ell$, and ψ_v is trivial on $U_1(\mathcal{O}_{L^+_v})$ for $v\notin S_\ell$. For $v\in S_\ell$ and $\zeta\in L_{\tilde{v}}$ an ℓ-power root of unity, $\xi_{\tilde{v}}(\zeta)=\beta(\mathrm{Art}_{L_{\tilde{v}}}(\zeta))^{-1}=1$: the image of $\mathrm{Art}_{L_{\tilde{v}}}(\zeta)$ in D_v^{ab} is that of an element of μ_v ((Wv1) of Lemma 50.24), on which β' is trivial. Moreover $\xi_{\tilde{v}}(x)=\beta(\mathrm{Art}(x))$ for $x\in L^\times_{\tilde{v}}$, and $\xi_w(\varpi_w)=\beta(\mathrm{Frob}_w)$ (Frob_w geometric).

Step 4: the twisted eigenform. For $h\in S_\lambda(\overline{\mathbb{Q}}_\ell)$ put $h^\psi(x)=\psi^\infty(\det x)h(x)$ ($\det x\in U_1(\mathbb{A}_{L^+}^\infty)$). As $\psi^\infty\circ\det$ is trivial on $\mathbb{G}_L(L^+)$ and on an open subgroup of $\mathbb{G}_L(\mathbb{A}_{L^+}^\infty)$, $h^\psi\in S_\lambda(\overline{\mathbb{Q}}_\ell)$, and $g\cdot(h^\psi)=\psi^\infty(\det g)(g\cdot h)^\psi$; so $h\mapsto h^\psi$ carries π_y onto an irreducible constituent π'. If a

double coset operator $\sum_i \xi_\lambda(\cdot)(\cdot)g_i$ has $\psi^\infty(\det g_i) = e$ for all i, it acts on h^ψ as e times its action on h, transported. Hence, choosing $r' \geq E$ so large that $\xi_{\tilde{v}}$ is trivial on $1 + \varpi_{\tilde{v}}^{r'}\mathcal{O}_{L_{\tilde{v}}}$ for $v \in S_\ell$ (so that $\psi^\infty \circ \det$ is trivial on $V(\ell^{r',r'})$, whose matrices at S_ℓ are unipotent modulo $\varpi^{r'}$, and on V_v, $\mathbb{G}_L(\mathcal{O}_{L_v^+})$ away from S_ℓ), f^ψ is non-zero, fixed by $V(\ell^{r',r'})$, and is a joint eigenvector of the operators of (Gy7)(c) with: eigenvalue $\xi_w(\varpi_w)^j y^\natural(T_w^{(j)})$ for $T_w^{(j)}$ (its cosets have determinant in $\varpi_w^j\mathcal{O}_w^\times$); $\xi_{\tilde{v}}(\varpi_{\tilde{v}})^j y^\natural(U_{\tilde{v}}^{(j)})$ for $U_{\tilde{v}}^{(j)}$ (determinant $\varpi_{\tilde{v}}^j$ times units in $1 + \varpi^{r'}$), a unit, so f^ψ is ordinary; and $\xi_{\tilde{v}}(\det u)y^\natural(\langle u\rangle)$ for $\langle u\rangle$. So $f^\psi \in S_\lambda^{\mathrm{ord}}(V(\ell^{r',r'}), \alpha', \overline{\mathbb{Q}}_\ell)$ with $\alpha'(u) = \xi(\det u)$ on $T_n(\ell^1)$, of finite order. For $h \in \prod_{v\in \mathrm{St}} \mathbb{G}_L(\mathcal{O}_{L_v^+})$, $\psi^\infty(\det h) = 1$, so $\pi_v'^{V_v} = \pi_{y,v}^{V_v}$ as representations of $\mathbb{G}_L(\mathcal{O}_{L_v^+})$ ($v \in \mathrm{St}$), and $\mathrm{Hom}_H(\tau^\vee \otimes \overline{\mathbb{Q}}_\ell, \bigotimes_{v\in\mathrm{St}} \pi_v'^{V_v}) \neq 0$.

Step 5: the twisted point of $\mathbb{T}_\tau$. Let $\mathcal{K}'$ be the field generated over $\mathcal{O}'[1/\ell]$ by the values of the character inducing $\mathfrak{P}_{\lambda,\alpha'}$. The forms of weight λ and level $V(\ell^{r',r'})$ with coefficients in $\overline{\mathbb{Q}}_\ell$ are the base change of those with coefficients in $\mathcal{K}'$ (solutions of the same $\mathcal{K}'$-linear equations), compatibly with the operators, the ordinary projector and the α'-part; so the eigencharacter of f^ψ is a homomorphism $\mathbb{T}_\lambda^{\mathrm{ord}}(V(\ell^{r',r'}), \alpha', \mathcal{K}') \to \overline{\mathbb{Q}}_\ell$, and its composite with the surjection of (Qh3) ($U = V$, $b_0 = 1$, applicable by (Gd0); $T_n(\mathfrak{l}^{r'}) \subseteq T_n(\ell^1)$ by (Qh1) as $r' \geq E$, and $T_n(\mathfrak{l}^{r'}) \subseteq \ker\alpha'$, the components at S_ℓ of its elements being $\equiv 1$ modulo $\varpi^{r'}$, so that their determinants lie in $1 + \varpi_{\tilde{v}}^{r'}\mathcal{O}_{L_{\tilde{v}}}$, where $\xi_{\tilde{v}}$ is trivial) is a homomorphism y_0' on $\mathbb{T}^{\mathrm{ord}}(V)$, which is $\mathbb{T}_{\mathcal{O}'[H]}$ ($N_{\mathcal{O}'[H]} = \mathcal{S}_V$ by evaluation at 1), continuous by (Gd0), with the eigenvalues of Step 4 on the operators ((Gy7)(c)), $y_0'(\mathfrak{P}_{\lambda,\alpha'}) = 0$ and $y_0'(\langle u\rangle) = \alpha'(u)(w_0\lambda)(u)^{-1}$ on $T_n(\ell^1)$. By Lemma E.14 (with π', f^ψ, λ, α', r'), $y_0' = y' \circ q$ for a point y' of $\mathbb{T}_\tau$ with the same values on the operators. The values of y' and $y^\natural$ on the topological generators of $\mathbb{T}_\tau$ ((Gd0)) differ by ℓ-power roots of unity, which are $\equiv 1$ in the residue field, so $\ker\bar{y}' = \ker\bar{y}^\natural = \mathfrak{m}$ and y' factors through $\mathbb{T}_{\tau,\mathfrak{m}}$ (it is local onto its image, (Gd0)). For $\delta = d_j(\zeta) \in \Delta_\ell$, $y'(\langle\delta\rangle) = \xi_{\tilde{v}}(\zeta)y(\langle\delta\rangle) = \chi_0(\delta)$ by Step 3, and Δ_ℓ is generated by such δ; so by (Zp4) y' kills $\mathfrak{a}_0$ and induces a point $\bar{y}'$ of A.

Step 6: (Ts2). $\bar{y}'\circ\theta$ and $(x_y)_{\beta'}$ classify the same polarised ordinary determinant. By the first identity under $\bar{y}'$, the j-th coefficient of the characteristic polynomial of $\bar{y}' \circ \theta \circ D^{\mathrm{u}}$ at $\pi(\mathrm{Frob}_w)$ is $\kappa(\mathrm{Frob}_w)^{-j}(\mathrm{N}w)^{j(j-1)/2}\xi_w(\varpi_w)^j y(T_w^{(j)}) = \beta'(\pi(\mathrm{Frob}_w))^j x_y(\Lambda_j(D^{\mathrm{u}})(\pi(\mathrm{Frob}_w)))$, which is that of $(x_y \circ D^{\mathrm{u}})_{\beta'}$ ($\Lambda_j(D_{\beta'})(\gamma) = \beta'(\gamma)^j\Lambda_j(D)(\gamma)$ by homogeneity). These elements are dense in $\Gamma^0_{\mathscr{H}}$ (Chebotarev), so the characteristic polynomials agree everywhere by continuity and the determinants are equal by (Ch1). By the second and third identities, $\bar{y}'(\theta(\psi_{v,j}^{\mathrm{d}}(\mathrm{Art}(x)))) = \xi_{\tilde{v}}(x)\, x_y(\psi_{v,j}^{\mathrm{d}}(\mathrm{Art}(x))) = \beta'(\pi(\mathrm{Art}(x)))\, x_y(\psi_{v,j}^{\mathrm{d}}(\mathrm{Art}(x)))$ for $x \in \mathcal{O}_{L_{\tilde{v}}}^\times$ and $x = \varpi_{\tilde{v}}$, hence on the dense image of Art, hence on D_v. Both are morphisms to the $\mathcal{O}'$-algebra $R \subseteq \overline{\mathbb{Q}}_\ell$ generated by their images, which is finite over $\mathcal{O}'$ (Gd0), a local domain with residue field $\mathbb{F}_{\mathcal{O}'}$, so in $\mathrm{CNL}_{\mathcal{O}'}$; by uniqueness $(x_y)_{\beta'} = \bar{y}' \circ \theta$, which kills J: (Ts2). □

Lemma E.28. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}'$, $(B, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, and $\Gamma = \Gamma^0_{\mathscr{H}}$, $\pi = \pi_{\mathscr{H}}$. For a commutative ring R and a homomorphism $\beta\colon \Gamma \to R^\times$ let ϕ_β be the R-algebra automorphism of $R[\Gamma]$ with $\gamma \mapsto \beta(\gamma)\gamma$, and for an R-valued determinant D on Γ put $D_\beta = D \circ \phi_\beta$. Call β **admissible** if $\beta(\pi(csc)) = \beta(\pi(s))^{-1}$ for $s \in G_L$, β is trivial on $\pi(I_{\bar{w}} \cap G_L)$ for every place $\bar{w}$ of $\overline{\mathbb{Q}}$ above a place in $\mathrm{St} \cup \{v_0\}$, and β is trivial on μ_v for every $v \in S_\ell$. Let $J \subseteq B$ be an ideal and $\mathcal{X}$ a set of homomorphisms of $\mathcal{O}'$-algebras $x\colon B \to \overline{\mathbb{Q}}_\ell$, each continuous with image in the ring of integers of a finite extension of $\mathbb{Q}_\ell$, such that:*

(Ts1) $J = \bigcap_{x\in\mathcal{X}} \ker x$;

(Ts2) *for every $x \in \mathcal{X}$ and every admissible continuous character $\beta'\colon \Gamma \to \overline{\mathbb{Q}}_\ell^\times$ of finite ℓ-power order, the homomorphism $x_{\beta'}\colon B \to \overline{\mathbb{Q}}_\ell$ classifying $((x \circ D^{\mathrm{u}})_{\beta'}, (\beta'|_{D_v} \cdot x \circ \psi_{v,a}^{\mathrm{d}}))$ (it exists, by the first paragraph of the proof) satisfies $x_{\beta'}(J) = 0$.*

Then for every admissible continuous homomorphism $\beta\colon \Gamma \to B^\times$ with $\beta \equiv 1$ modulo $\mathfrak{m}_B$, the tuple $(D^{\mathrm{u}}_\beta, (\beta|_{D_v}\psi^{\mathrm{d}}_{v,a}))$ is a polarised ordinary determinant to B, and the morphism $\tau_\beta\colon B \to B$ of $\mathrm{CNL}_{\mathcal{O}'}$ classifying it satisfies $\tau_\beta(J) \subseteq J$.

Proof. Twisting preserves polarised ordinary determinants. Let $A \in \mathrm{CNL}_{\mathcal{O}'}$, $(D, (\psi_{v,a}))$ a polarised ordinary determinant to A and $\beta\colon \Gamma \to A^\times$ an admissible continuous homomorphism with $\beta \equiv 1$ modulo $\mathfrak{m}_A$. Then $(D_\beta, (\beta|_{D_v}\psi_{v,a}))$ is one: D_β is a continuous determinant of dimension n with the reduction of D, as $\beta \equiv 1$, and $\beta|_{D_v}\psi_{v,a} \equiv \bar{\chi}_a$. For $\gamma \in \Gamma$, $\chi_{D_\beta}(\gamma)(X) = D(X - \beta(\gamma)\gamma) = \beta(\gamma)^n\chi_D(\gamma)(\beta(\gamma)^{-1}X)$ (homogeneity of degree n). (Od1): for $g \in G_L$, anti-invariance with $s = g^{-1}$ gives $\beta(\pi(cg^{-1}c)) = \beta(\pi(g))$, and (Od1) for D gives $\chi_D(\pi(cg^{-1}c)) = \chi_D(\pi(g))$; so $\chi_{D_\beta}(\pi(cg^{-1}c)) = \chi_{D_\beta}(\pi(g))$. (Od2): $\mathrm{tr}_{D_\beta} = \mathrm{tr}_D \circ \phi_\beta$, and $\phi_\beta(\gamma\prod_a(\pi(g_a) - \beta(\pi(g_a))\psi_{v,a}(g_a))) = \beta(\gamma)\prod_a \beta(\pi(g_a)) \cdot \gamma\prod_a(\pi(g_a) - \psi_{v,a}(g_a))$, whose D-trace vanishes by (Od2) for D. (Od3): β is trivial on μ_v. If $f\colon A \to A'$ is a morphism of $\mathrm{CNL}_{\mathcal{O}'}$, then $f \circ (D_\beta) = (f \circ D)_{f\circ\beta}$.

In particular τ_β exists ($A = B$). For $x \in \mathcal{X}$ and β' as in (Ts2), let E be a finite extension of $\mathbb{Q}_\ell$ containing $x(B)$ and the values of β', and $R \subseteq \mathcal{O}_E$ the $\mathcal{O}'$-algebra generated by them: finite over $\mathcal{O}'$, a local domain, complete, with residue field $\mathbb{F}_{\mathcal{O}'}$ (generated by the residues of $x(B)$, which lie in $\mathbb{F}_{\mathcal{O}'}$ as x is local onto its image, and of ℓ-power roots of unity, which are 1); so $R \in \mathrm{CNL}_{\mathcal{O}'}$, $\beta' \equiv 1$ modulo $\mathfrak{m}_R$, and the twisted tuple of x is a polarised ordinary determinant to R, classified by $x_{\beta'}$.

Fix β as in the conclusion, $b \in J$ and $N \geq 0$; put $I = \mathfrak{m}_B^{N+1}$ and $\bar{B} = B/I$, a finite ring in $\mathrm{CNL}_{\mathcal{O}'}$.

Step 1: a finite ℓ-group. The reduction $\bar{\beta}\colon \Gamma \to \bar{B}^\times$ is continuous into a finite group; let $Q = \bar{\beta}(\Gamma)$, $q\colon \Gamma \to Q$ the induced surjection (open kernel) and $j\colon Q \to \bar{B}^\times$ the inclusion. Q is an ℓ-group: if $t \in \mathfrak{m}_B$ then $(1+t)^{\ell^k} - 1 \in \mathfrak{m}_B^{k+1}$ for all k, as $\ell \in \mathfrak{m}_B$, so every element of Q has order dividing ℓ^N. As β is admissible, so is q (the conditions are on kernels and on $q(\pi(csc)) = q(\pi(s))^{-1}$).

Step 2: the universal finite twist. $B[Q]$ is finite free over B; its quotient by $\mathfrak{m}_B B[Q]$ is $\mathbb{F}[Q]$ ($\mathbb{F} = \mathbb{F}_{\mathcal{O}'}$), local because Q is an ℓ-group, so $B[Q]$ is a complete Noetherian local $\mathcal{O}'$-algebra with residue field $\mathbb{F}$, in $\mathrm{CNL}_{\mathcal{O}'}$. The homomorphism $\gamma \mapsto [q(\gamma)] \in B[Q]^\times$ is continuous (locally constant), admissible, and $\equiv 1$ modulo the maximal ideal. By the first paragraph (with $A = B[Q]$ and the base change of the universal tuple along $B \to B[Q]$) there is a morphism $\mathrm{tw}\colon B \to B[Q]$ of $\mathrm{CNL}_{\mathcal{O}'}$ classifying the twisted tuple. Write $\mathrm{tw}(b) = \sum_{g\in Q} c_g[g]$ with $c_g \in B$.

Step 3: Fourier inversion. Let $x \in \mathcal{X}$ and $\chi\colon Q \to \overline{\mathbb{Q}}_\ell^\times$ a character; $\beta' = \chi \circ q$ is continuous, admissible, of finite ℓ-power order. Let R be as in the first paragraph for x and β', and $e\colon B[Q] \to R$, $\sum_g b_g[g] \mapsto \sum_g x(b_g)\chi(g)$, a morphism of $\mathrm{CNL}_{\mathcal{O}'}$. By the base change rule, $e \circ \mathrm{tw}$ classifies $((x \circ D^{\mathrm{u}})_{\beta'}, (\beta'|_{D_v} \cdot x \circ \psi^{\mathrm{d}}_{v,a}))$, so $e \circ \mathrm{tw} = x_{\beta'}$ by uniqueness, and (Ts2) gives $\sum_g x(c_g)\chi(g) = x_{\beta'}(b) = 0$. This holds for all characters χ of the finite abelian group Q, which span the space of functions $Q \to \overline{\mathbb{Q}}_\ell$ (there are $|Q|$ of them, linearly independent by Dedekind–Artin, as $\overline{\mathbb{Q}}_\ell$ is algebraically closed of characteristic 0); so $x(c_g) = 0$ for every g. As $x \in \mathcal{X}$ was arbitrary, (Ts1) gives $c_g \in J$ for all g.

Step 4: back to β. Let $s\colon B[Q] \to \bar{B}$, $\sum_g b_g[g] \mapsto \sum_g \bar{b}_g\, j(g)$, a morphism of $\mathrm{CNL}_{\mathcal{O}'}$. By the base change rule, $s \circ \mathrm{tw}$ and the composite of τ_β with $B \to \bar{B}$ both classify $((\bar{D}^{\mathrm{u}})_{\bar{\beta}}, (\bar{\beta}|_{D_v}\bar{\psi}^{\mathrm{d}}_{v,a}))$, bars denoting reduction modulo I ($j \circ q = \bar{\beta}$). By uniqueness they are equal, so $\tau_\beta(b) \equiv \sum_g c_g\, \tilde{j}(g)$ modulo I for lifts $\tilde{j}(g) \in B$, and $\tau_\beta(b) \in J + \mathfrak{m}_B^{N+1}$.

Step 5: Krull. N was arbitrary, and $\bigcap_N (J + \mathfrak{m}_B^{N+1}) = J$ by the Krull intersection theorem in the Noetherian local ring B/J. So $\tau_\beta(b) \in J$. □

Lemma E.29. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}'$ with residue field $\mathbb{F}$, κ a twist for $\mathscr{H}$, $(B, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism, c an integer as in Proposition E.5, $((V_v)_{v\in \mathrm{St}}, \tau)$ a type datum over $\mathscr{H}$ with V_v the kernel of reduction $\mathbb{G}_L(\mathcal{O}_{L_v^+}) \to \mathbb{G}_L(\mathcal{O}_{L_v^+}/\varpi_v^c)$, $\mathfrak{m}$ a maximal ideal of $\mathbb{T}_\tau$ with $\mathbb{T}_\tau/\mathfrak{m} = \mathbb{F}$ satisfying (Mx), $\mathfrak{a}$ an ideal of $\mathbb{T}_{\tau,\mathfrak{m}}$, and $\theta\colon B \to \mathbb{T}_{\tau,\mathfrak{m}}/\mathfrak{a}$ a surjective homomorphism of $\mathcal{O}'$-algebras such that in $\mathbb{T}_{\tau,\mathfrak{m}}/\mathfrak{a}$, for all w, $1 \le j \le n$, $v \in S_\ell$ and $z \in \mathcal{O}^\times_{L_{\tilde{v}}}$,*

$$(\mathrm{N}w)^{j(j-1)/2} T_w^{(j)} = \kappa(\mathrm{Frob}_w)^j\, \theta(\Lambda_j(D^{\mathrm{u}})(\pi_{\mathscr{H}}(\mathrm{Frob}_w))),$$
$$\langle d_j(z)\rangle = (\chi_{\mathrm{cyc}}^{j-1}\kappa)(\mathrm{Art}(z))\, \theta(\psi^{\mathrm{d}}_{v,j}(\mathrm{Art}(z))).$$

Let $J = \ker\theta$. Let $y\colon B \to \overline{\mathbb{Q}}_\ell$ be a homomorphism of $\mathcal{O}'$-algebras with $y(J) = 0$ for which there are integers $h_{v,a,\varsigma}$ ($v \in S_\ell$, $1 \le a \le n$, ς a continuous field embedding $L_v^+ \to \overline{\mathbb{Q}}_\ell$), strictly increasing in a for every v and ς, such that $y(\psi^{\mathrm{d}}_{v,a}(\sigma)) = \prod_\varsigma \varsigma(u_\sigma)^{h_{v,a,\varsigma}}$ for all σ in an open subgroup of I_v, where $u_\sigma \in \mathcal{O}^\times_{L_v^+}$ is the unit with $\mathrm{Art}_{L_v^+}(u_\sigma)$ equal to the image of σ in D_v^{ab}. Then there are a continuous semisimple representation $\rho\colon G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ with $\det\circ\rho = y\circ D^{\mathrm{u}}\circ\pi_{\mathscr{H}}|_{G_L}$, an integer $k \ge 1$, integers $d_i, m_i \ge 1$ and continuous representations $\rho_i'\colon G_L \to \mathrm{GL}_{d_i}(\overline{\mathbb{Q}}_\ell)$ ($1 \le i \le k$) such that

$$\det(X - \rho(g)) = \prod_{i=1}^{k}\prod_{j=0}^{m_i-1} \det(X - \chi_{\mathrm{cyc}}(g)^{-j}\rho_i'(g)) \qquad (g \in G_L),$$

and, if $m_i = 1$ for all i, then for all but finitely many finite places w' of L the representation ρ is unramified at w' and any two eigenvalues α, α' of $\rho(\mathrm{Frob}_{w'})$ satisfy $|\iota(\alpha)| < \mathrm{N}w'\,|\iota(\alpha')|$.

Proof. Step 1: the point of the Hecke algebra. As θ is surjective and $y(\ker\theta) = 0$, $y = y'\circ\theta$ for a homomorphism of $\mathcal{O}'$-algebras $y'\colon A \to \overline{\mathbb{Q}}_\ell$. Let $y''\colon \mathbb{T}_\tau \to \overline{\mathbb{Q}}_\ell$ be the composite of $\mathbb{T}_\tau \to \mathbb{T}_{\tau,\mathfrak{m}} \to A$ with y', and $y_0\colon \mathbb{T}^{\mathrm{ord}}(V) \to \overline{\mathbb{Q}}_\ell$ its composite with the surjection $\mathbb{T}^{\mathrm{ord}}(V) \to \mathbb{T}_\tau$ (Definition 50.8). By (Gd0) of Lemma E.16 (whose hypotheses are those made here), $[L^+ : \mathbb{Q}] \ge 2$, the hypotheses of Theorem B.91 hold for $U = V$ and $b_0 = 1$, and y, y', y'' are continuous with values in the ring of integers of a finite extension of $\mathbb{Q}_\ell$.

Step 2: the weight. Let $v \in S_\ell$. The map Art of Proposition E.5 and $\mathrm{Art}_{L_v^+}$ differ by inversion: $\mathrm{Art}(x) = \mathrm{Art}_{L_v^+}(x^{-1})$ in D_v^{ab}. $\mathrm{Art}_{L_v^+}$ maps $\mathcal{O}^\times_{L_v^+}$ homeomorphically onto the image of I_v in D_v^{ab}, so there is an open subgroup $U_v \subseteq 1 + \ell\mathcal{O}_{L_v^+}$ such that, for $z \in U_v$, $\mathrm{Art}(z)$ is the image of an element σ of the open subgroup of the hypothesis, with $u_\sigma = z^{-1}$; hence $y(\psi^{\mathrm{d}}_{v,j}(\mathrm{Art}(z))) = \prod_\varsigma \varsigma(z)^{-h_{v,j,\varsigma}}$. With this normalisation $\chi(\mathrm{Art}(z)) = \prod_\varsigma \varsigma(z)$ for units z (Lubin–Tate theory), and, as κ corresponds to an algebraic Hecke character, there are integers $b_{v,\varsigma}$ with $\kappa(\mathrm{Art}(z)) = \prod_\varsigma \varsigma(z)^{b_{v,\varsigma}}$ for z in an open subgroup of $\mathcal{O}^\times_{L_v^+}$ ([220], Chapter II); shrink U_v accordingly. The second identity of the hypothesis, under y', gives for $z \in U_v$ and $1 \le j \le n$

$$y''(\langle d_j(z)\rangle) = (\chi^{j-1}\kappa)(\mathrm{Art}(z))\, y(\psi^{\mathrm{d}}_{v,j}(\mathrm{Art}(z))) = \prod_\varsigma \varsigma(z)^{(j-1)+b_{v,\varsigma}-h_{v,j,\varsigma}}.$$

Let λ be the weight with $\lambda_{\varsigma,n+1-j} = h_{v,j,\varsigma} - (j-1) - b_{v,\varsigma}$ at the embeddings ς inducing $\tilde{v}$, $v \in S_\ell$ (the components at the conjugate embeddings being determined by these, as for every weight of $\mathbb{G}_L$). It is dominant: $\lambda_{\varsigma,n-j} - \lambda_{\varsigma,n+1-j} = h_{v,j+1,\varsigma} - h_{v,j,\varsigma} - 1 \ge 0$, the $h_{v,a,\varsigma}$ being strictly increasing integers in a. Put $\alpha(u) = y''(\langle u\rangle)\,(w_0\lambda)(u)$ for $u \in T_n(\ell^1)$, a continuous character ($u \mapsto \langle u\rangle$ and $w_0\lambda$ are homomorphisms). So α is trivial on the subgroup generated by the $d_j(z)$, $z \in U_v$, $v \in S_\ell$, which is open in $T_n(\ell^1) = \prod_{v\in S_\ell}\prod_j d_j(1 + \ell\mathcal{O}_{L_v^+})$; hence α has finite order, and $T_n(\ell^r) \subseteq \ker\alpha$ for some $r \ge 1$. In the renormalised structure, $y_0([u]) = (w_0\nu)^{-1}(u)\, y''(\langle u\rangle) =$

$\alpha(u)(w_0\nu)^{-1}(u)(w_0\lambda)^{-1}(u)$ for $u \in T_n(\ell^1)$. The restriction of y_0 to Λ_1 and the homomorphism $\Lambda_1 \to \overline{\mathbb{Q}}_\ell$ induced by the character $\alpha(w_0\nu)^{-1}(w_0\lambda)^{-1}$ are continuous homomorphisms of $\mathcal{O}'$-algebras (Step 1) which agree on the group elements, whose $\mathcal{O}'$-span is dense in Λ_1; so they are equal, and $y_0(\mathfrak{P}_{\lambda,\alpha}) = 0$.

Step 3: an automorphic representation. By Step 2, y'' kills $\mathfrak{P}_{\lambda,\alpha}$ (the renormalised structure of $\mathbb{T}^{\mathrm{ord}}(V)$ maps onto that of $\mathbb{T}_\tau$) and $y''(\langle u\rangle) = \alpha(u)(w_0\lambda)(u)^{-1}$ for $u \in T_n(\ell^1)$, with α of finite order, $T_n(\ell^r) \subseteq \ker\alpha$ and λ dominant. The type datum is over the rank-n Hecke datum $\mathscr{H}$, so Lemma E.11 ($Q = \varnothing$; α with values in $\overline{\mathbb{Q}}_\ell^\times$) gives an irreducible constituent π of $S_\lambda(\overline{\mathbb{Q}}_\ell)$ such that, for every split $v \notin T_L$, π_v is unramified and $T_w^{(j)}$ acts on $\pi_v^{\mathbb{G}_L(\mathcal{O}_{L_v^+})}$ by $t_w^{(j)} = y''(T_w^{(j)})$ (its clause (ii)).

Step 4: ρ. Let r_π be as in (Gy5) and $\rho = r_\pi \otimes \kappa^{-1}$, a continuous semisimple representation of G_L. For every w, (Gy5)(i) and Step 3 give that r_π is unramified above w with $\det(X - r_\pi(\mathrm{Frob}_w)) = \sum_j (-1)^j (\mathrm{N}w)^{j(j-1)/2} y''(T_w^{(j)}) X^{n-j}$; by the first identity of the hypothesis under y' this is $\sum_j (-1)^j \kappa(\mathrm{Frob}_w)^j y(\Lambda_j(D^{\mathrm{u}})(\pi_{\mathscr{H}}(\mathrm{Frob}_w)))X^{n-j}$, so $\rho(\mathrm{Frob}_w)$ has the characteristic polynomial of the $\overline{\mathbb{Q}}_\ell$-valued determinant $y \circ D^{\mathrm{u}} \circ \pi_{\mathscr{H}}|_{G_L}$ at Frob_w. Both $\det\circ\rho$ and this determinant are continuous determinants of dimension n on G_L; the Frob_w are dense in G_L (Chebotarev: the places of L of degree one over L^+ have density one), so their characteristic polynomials agree on G_L by continuity, and they are equal by (Ch1) of Theorem B.83.

Step 5: the decomposition. By (Ug1) of Lemma E.8 ($[L^+ : \mathbb{Q}] \geq 2$ by Step 1; λ dominant; π a constituent of $S_\lambda(\overline{\mathbb{Q}}_\ell)$) there are $k \geq 1$, integers $k_i, d_i \geq 1$ with $\sum_i k_i d_i = n$ and unitary cuspidal automorphic representations σ_i of $\mathrm{GL}_{d_i}(\mathbb{A}_L)$ (called τ_i, with m_i in place of d_i, there) such that $\sigma_i' = \sigma_i \otimes |\det|^{s_i}$, $s_i = \frac{d_i-1}{2} - \frac{n-k_i}{2}$, is regular algebraic, cuspidal and polarizable and $r_\pi \cong \bigoplus_i \bigoplus_{a=0}^{k_i-1} r_{\ell,\iota}(\sigma_i') \otimes \varepsilon^{-a}$, with $\varepsilon = \chi$. Put $m_i = k_i$ and $\rho_i' = r_{\ell,\iota}(\sigma_i') \otimes \kappa^{-1}$, continuous of dimension d_i. Then $\rho \cong \bigoplus_i \bigoplus_{j=0}^{m_i-1} \rho_i' \otimes \chi^{-j}$, and the characteristic polynomial of a direct sum is the product of those of the summands: the displayed identity holds.

Step 6: the eigenvalues when all $m_i = 1$. Then $s_i = \frac{d_i-n}{2}$ and $\rho \cong \bigoplus_i r_{\ell,\iota}(\sigma_i') \otimes \kappa^{-1}$. Let w' be a finite place of L not above ℓ, not above a place of St, at which every σ_i is unramified; all but finitely many places are such. Then κ is unramified at w' ((Rd7) of Definition 50.1), $\sigma_{i,w'}'$ is unramified with Satake parameters $q_{w'}^{-s_i}\beta$, β running over those of $\sigma_{i,w'}$, and by (Bg1) of Theorem B.62 (for the polarizable σ_i' of rank d_i), $r_{\ell,\iota}(\sigma_i')$ is unramified at w' and the eigenvalues of a geometric Frobenius on it are carried by ι to the numbers $q_{w'}^{(d_i-1)/2-s_i}\beta = q_{w'}^{(n-1)/2}\beta$. The central character of the unitary σ_i is unitary, so (Js2) of Theorem B.60 gives $q_{w'}^{-1/2} < |\beta| < q_{w'}^{1/2}$. So ρ is unramified at w', and every eigenvalue α of $\rho(\Phi)$, Φ a geometric Frobenius at w', satisfies $C\, q_{w'}^{(n-2)/2} < |\iota(\alpha)| < C\, q_{w'}^{n/2}$ with $C = |\iota\kappa(\Phi)|^{-1} > 0$ independent of α. For two eigenvalues, $|\iota(\alpha)| < Cq_{w'}^{n/2} = q_{w'} \cdot Cq_{w'}^{(n-2)/2} < q_{w'}|\iota(\alpha')|$. The eigenvalues of an arithmetic Frobenius are the inverses of these, and the condition, required for all ordered pairs, is unchanged under inversion ($|\alpha^{-1}| < q|\alpha'^{-1}|$ is $|\alpha'| < q|\alpha|$). □

E.2 Patching at curve points

This section proves Proposition E.32: a Hecke ideal of zero-monodromy type which, if proper, has classical levels has $R = T$ at patchable curve points and along thick chains; Proposition 50.11 and Theorem 53.10 use it. The method is patching in the manner of Taylor and Wiles [254], carried out at one-dimensional points of the determinant ring. The proof combines abstract patching (Proposition E.33) with the existence of patching systems (Proposition E.39); the latter takes the Galois clause from Section E.3, the dual Selmer bound from Section E.4 and the patched ring and the automorphic patching data from Section E.5.

E.2.1 Patching at curve points and abstract patching

This subsection defines abstract patching towers and patching systems along curve points (Definition E.30, Definition E.31) and proves Proposition E.32 from the existence of patching systems, Proposition E.39. The abstract input is Proposition E.33: it describes the patched module over the patching ring and its fibre, and controls the dimension of its support at the primes that avoid the isogeny element.

Definition E.30. Let Λ be a complete Noetherian local ring with maximal ideal $\mathfrak{m}_\Lambda$ and finite residue field, let $s \geq 0$ be an integer and $S = \Lambda[[y_1, \dots, y_s]]$. For integers $k, j \geq 1$ put

$$\mathfrak{c}_{k,j} = (y_1, \dots, y_s)^k + \mathfrak{m}_\Lambda^j S \subseteq S. \tag{E.2}$$

An **abstract patching tower** over S is a tuple $(R, \beta, X, f, I, e, (M_N)_{N\geq 1}, \mathcal{U})$ consisting of a Noetherian local ring R with maximal ideal $\mathfrak{m}_R$, a ring homomorphism $\beta\colon S \to R$, an R-module X, an element $f \in R$, an ideal $I \subseteq R$, an integer $e \geq 1$, R-modules M_N ($N \geq 1$) and a non-principal ultrafilter $\mathcal{U}$ on the set of positive integers (one containing no finite set), such that the following hold, where every R-module is regarded as an S-module through β:

(Tow0) X is a finitely generated free Λ-module, and $\beta(y_i)X = 0$ for $1 \leq i \leq s$;

(Tow1) for all $k, j \geq 1$, the set of N such that $M_N/\mathfrak{c}_{k,j}M_N$ is a free $S/\mathfrak{c}_{k,j}$-module of finite rank belongs to $\mathcal{U}$ (the rank may depend on N, k and j);

(Tow2) for every $j \geq 1$, the set of N for which there are R-linear maps $\varphi\colon X/\mathfrak{m}_\Lambda^j X \to M_N/\mathfrak{c}_{1,j}M_N$ and $\psi\colon M_N/\mathfrak{c}_{1,j}M_N \to X/\mathfrak{m}_\Lambda^j X$ with $\psi\circ\varphi$ and $\varphi\circ\psi$ both equal to multiplication by f belongs to $\mathcal{U}$;

(Tow3) I is generated by a finite set G such that for every $g \in G$ and every $j \geq 1$, the set of N for which there is $a \in R$ with $(fg)^e - a \in \mathfrak{m}_R^j$ and $aM_N = 0$ belongs to $\mathcal{U}$.

Here $\mathfrak{c}_{k,j}M_N$ means $\beta(\mathfrak{c}_{k,j})M_N$; by (Tow0), $\mathfrak{c}_{1,j}X = \mathfrak{m}_\Lambda^j X$. The integer s is the **number of patching variables** of the tower, f its **isogeny element** and I its **relation ideal**. The ideal of R generated by $\beta(y_1), \dots, \beta(y_s)$ is denoted $\mathfrak{y}$, and the $\mathfrak{y}$-adic completion of the localisation $R_f = R[1/f]$ is called the **patching ring** R' of the tower:

$$R' = \varprojlim_k R_f/\mathfrak{y}^k R_f, \qquad \iota\colon R \to R_f \to R' \text{ the natural homomorphism.} \tag{E.3}$$

Definition E.31. Let $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ be a universal polarised ordinary determinant of rank n over a residual setting $(n, L, c, \Gamma, \Gamma^0, \dots)$ (over a rank-n Hecke datum, or over a CM field L), with v_0 as in Definition 50.9. Write $B = B_n^{\mathrm{po}}$, and let $\mathcal{O}$ be the coefficient ring over which B is an algebra. Let $J \subseteq B$ be an ideal, $h \geq 0$, and $x_0, \dots, x_h$ curve points of B with homomorphisms $f_i\colon B \to A_{k_i} = k_i[[T]]$.

Eigenvalue algebras. Let $q \geq 0$, $\sigma_1, \dots, \sigma_q \in \Gamma^0$ and $\tilde{u}_{j,i} \in \mathcal{O}$ ($1 \leq j \leq q$, $1 \leq i \leq n$) with $\chi_{D^{\mathrm{u}}}(\sigma_j) \equiv \prod_{i=1}^n (X - \tilde{u}_{j,i})$ modulo $\mathfrak{m}_B$ for every j, and let $\mathcal{O}_1$ be a complete discrete valuation ring that is a free $\mathcal{O}$-module of finite rank, unramified over $\mathcal{O}$ (that is, $\mathfrak{m}_{\mathcal{O}}\mathcal{O}_1 = \mathfrak{m}_{\mathcal{O}_1}$). With a variable z put

$$C_0 = B[[z]][[Z_{j,i} : 1 \leq j \leq q,\ 1 \leq i \leq n]] \Big/ \Big(\text{the coefficients of } \textstyle\prod_{i=1}^n (X - \tilde{u}_{j,i} - Z_{j,i}) - \chi_{D^{\mathrm{u}}}(\sigma_j)(X),\ 1 \leq j \leq q\Big),$$

and let $u_{j,i} \in C_0$ be the class of $\tilde{u}_{j,i} + Z_{j,i}$, so that $\prod_{i=1}^{n}(X - u_{j,i}) = \chi_{D^{\mathrm{u}}}(\sigma_j)(X)$ in $C_0[X]$. The **eigenvalue algebra** of $(\mathcal{O}_1, (\sigma_j), (\tilde{u}_{j,i}))$ is $C = \mathcal{O}_1 \otimes_{\mathcal{O}} C_0$, with $u\colon B \to C$, $u(b) = 1 \otimes b$. Through $\mathcal{O}_1 \otimes_{\mathcal{O}} (B[[z]] \to C_0)$ it is an algebra over $C_1 = (\mathcal{O}_1 \otimes_{\mathcal{O}} B)[[z]]$, which equals $\mathcal{O}_1 \otimes_{\mathcal{O}} B[[z]]$ as $\mathcal{O}_1$ is a free $\mathcal{O}$-module of finite rank; z and $u_{j,i}$ also denote their images in C. For $q = 0$, $C = C_1$.

A **patching system** for J along $(x_0, \dots, x_h)$ is a tuple $(\mathcal{T}, d, C, u, \pi)$ of an abstract patching tower $\mathcal{T} = (R, \beta, X, f_{\mathcal{T}}, I, e, (M_N)_{N \geq 1}, \mathcal{U})$ over some $S = \Lambda[[y_1, \dots, y_s]]$, an integer $d \geq 0$, the eigenvalue algebra C, with u, of some $(\mathcal{O}_1, (\sigma_j)_{1 \leq j \leq q}, (\tilde{u}_{j,i}))$ as above such that the residue field of $\mathcal{O}_1$ has cardinality $|k_i|$ for every i (so the k_i have one common cardinality), and a surjective ring homomorphism $\pi\colon R \to C$. A **surjective lift** of f_i is a surjective ring homomorphism $f'\colon C \to A_{k_i}$ with $f' \circ u = f_i$ and $f'(z) = T$; its kernel is a prime ideal of C (as $C/\ker f' \cong A_{k_i}$ is a domain) with $u^{-1}(\ker f') = \ker f_i$. The tuple is required to satisfy the following, where a prime ideal of R is called **stable for the system** if it is contained in $\pi^{-1}(P'')$ for some prime ideal P'' of C with $u^{-1}(P'')$ a stable prime of B:

(Sys0) for every i and every j, the polynomial obtained from $\chi_{D^{\mathrm{u}}}(\sigma_j)$ by applying f_i to its coefficients is a product of n monic polynomials of degree 1 in $A_{k_i}[X]$;

(Sys1) $\pi(\beta(y_i)) = 0$ for $1 \leq i \leq s$;

(Sys2) for every prime ideal P of C: $\mathrm{Ann}_R(X) \subseteq \pi^{-1}(P)$ if and only if $J \subseteq u^{-1}(P)$;

(Sys3) for every i and every surjective lift f' of f_i, the prime $\mathfrak{q} = \pi^{-1}(\ker f')$ satisfies $f_{\mathcal{T}} \notin \mathfrak{q}$, and $\Lambda_{\mathfrak{p}}$ is a regular local ring of dimension d for $\mathfrak{p} = \{a \in \Lambda : \beta(a) \in \mathfrak{q}\}$;

(Sys4) for every i, every surjective lift f' of f_i, $\mathfrak{q} = \pi^{-1}(\ker f')$ and $\hat{A}$ the completion of $R_{\mathfrak{q}}$ at its maximal ideal: if $\hat{Z}, \hat{Q}$ are prime ideals of $\hat{A}$ such that $I\hat{A} \subseteq \hat{Z}$, the preimage of $\hat{Z}$ in R is stable for the system, $\dim \hat{A}/\hat{Z} \geq d + s$, and the preimage of $\hat{Q}$ in R is $\pi^{-1}(P')$ for a prime ideal P' of C with $u^{-1}(P')$ stable or thick, then $\hat{Z} \subseteq \hat{Q}$.

Proposition E.32. *Let $\mathscr{H} = (n, L, c, (\bar{v})_{v \in S_\ell}, \mathrm{St}, v_0, \bar{v}_0, (\bar{\chi}_a)_{1 \leq a \leq n})$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}'$, κ a twist for $\mathscr{H}$, $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ an isomorphism and J a Hecke ideal of zero-monodromy type for ι and κ which, if $J \neq B_n^{\mathrm{po}}$, has classical levels. Then J has $R = T$ at patchable curve points and $R = T$ along thick chains (for v_0, with the place $\bar{v}_0$).*

Proof. Write $B = B_n^{\mathrm{po}}$ and $\mathcal{J} = J$. If $\mathcal{J} = B$ both properties hold vacuously: no curve point f (a homomorphism onto a subring of $A_k \neq 0$) has $B \subseteq \ker f$, and no prime contains B. Assume $\mathcal{J} \neq B$, so $\mathcal{J}$ has classical levels and Proposition E.39 applies to it. For $\gamma \in \Gamma^0_{\mathscr{H}}$ let $\mathrm{Tr}(\gamma) \in B$ be the coefficient $\Lambda_1(\gamma)$ of $\chi_{D^{\mathrm{u}}}(\gamma)$ in the notation of (Ch1); its image under a homomorphism to a field F over which the base change of D^{u} is $\det \circ r$ is $\mathrm{tr}\, r(\gamma)$.

Step 1: stability passes to smaller primes. Let $\mathfrak{q} \subseteq \mathfrak{q}'$ be primes of B with $\mathfrak{q}'$ stable, and r' irreducible over $K' = \bar{\kappa}(\mathfrak{q}')$ with $D_{\mathfrak{q}'} = \det \circ r'$. By Burnside's theorem the K'-span of $r'(\Gamma^0_{\mathscr{H}})$ is $\mathrm{M}_n(K')$; choose $g_1, \dots, g_{n^2} \in \Gamma^0_{\mathscr{H}}$ with $(r'(g_a))_a$ a basis, and put $\delta = \det\big(\mathrm{Tr}(g_a g_b)\big)_{a,b} \in B$. Its image in K' is the Gram determinant of the basis $(r'(g_a))$ for the trace form $(Y, Y') \mapsto \mathrm{tr}(YY')$, which is non-degenerate on $\mathrm{M}_n(K')$ (its matrix in the basis of elementary matrices is a permutation matrix); so $\delta \notin \mathfrak{q}'$, hence $\delta \notin \mathfrak{q}$. By (Ch3), $D_{\mathfrak{q}} = \det \circ r$ for a semisimple r over $K = \bar{\kappa}(\mathfrak{q})$; the image of δ in K is $\det(\mathrm{tr}(r(g_a)r(g_b))) \neq 0$, so the $r(g_a)$ are linearly independent and span $\mathrm{M}_n(K)$. A proper non-zero r-stable subspace of K^n would be stable under all of $\mathrm{M}_n(K)$; so r is irreducible and $\mathfrak{q}$ is stable.

Step 2: the kernel at a point of irreducible type is stable. Let x be a curve point of irreducible type, with finite field k, representation ρ and homomorphism f. f embeds $B/\ker f$ into A_k,

hence $\kappa(\ker f)$ into F_k; fix an algebraic closure $\bar\kappa(\ker f) \subseteq \bar F_k$. By (Ch3), $D_{\ker f} = \det \circ r$ with r semisimple over $\bar\kappa(\ker f)$. Its base change to $\bar F_k$ is $\det \circ (r \otimes \bar F_k)$, and also the base change of $f \circ D^{\mathrm{u}} = \det \circ \rho$, i.e. $\det \circ (\rho \otimes \bar F_k)$. Both representations are semisimple (r is a sum of irreducibles over an algebraically closed field, which stay irreducible over $\bar F_k$; $\rho \otimes \bar F_k$ is irreducible, already on Γ'). By the uniqueness in (Ch3), $r \otimes \bar F_k \cong \rho \otimes \bar F_k$ is irreducible, hence so is r: $\ker f$ is stable.

Step 3: the chain lemma. Let $J \subseteq B$ be an ideal, $x_0, \dots, x_h$ curve points with homomorphisms $f_0, \dots, f_h$, and $(\mathcal{T}, d, C, u, \pi)$ a patching system for J along $(x_0, \dots, x_h)$, $\mathcal{T} = (R, \beta, X, f_{\mathcal{T}}, I, \dots)$ over $S = \Lambda[[y_1, \dots, y_s]]$, with patching ring R', $\iota\colon R \to R'$, and M_∞ the finitely generated R'-module of Proposition E.33; let $\mathcal{P}$ be the set of primes Q of R with $f_{\mathcal{T}} \notin Q$ and $\beta(y_1), \dots, \beta(y_s) \in Q$, and for $Q \in \mathcal{P}$ recall from (Pat0) that QR' is a prime of R' with $\iota^{-1}(QR') = Q$. Let $\mathfrak{e}_1, \dots, \mathfrak{e}_h$ be thick primes with $\mathfrak{e}_i \subseteq \ker f_{i-1} \cap \ker f_i$ and $\mathfrak{t}$ a stable prime with $J \subseteq \mathfrak{t} \subseteq \ker f_0$. We show: every prime $\mathfrak{q} \subseteq \ker f_h$ of B that is stable or thick contains J. (a) *Primes of C.* By Definition E.31, $C = \mathcal{O}_1 \otimes_{\mathcal{O}} C_0$ is the eigenvalue algebra of some $(\mathcal{O}_1, (\sigma_j)_{1 \le j \le q}, (\tilde u_{j,i}))$, where $\mathcal{O}_1$ is a complete discrete valuation ring, finite free and unramified over $\mathcal{O}$, with residue field of cardinality $|k_i|$ for all i; it is an algebra over $C_1 = B_1[[z]]$, $B_1 = \mathcal{O}_1 \otimes_{\mathcal{O}} B$, through a homomorphism $\iota\colon C_1 \to C$ with $\iota(z) = z$, and $u = \iota \circ u_1$ for $u_1\colon B \to C_1$, $u_1(b) = 1 \otimes b$. *Primes of the base C_1.* As $\mathcal{O}_1$ is a free $\mathcal{O}$-module of finite positive rank, B_1 is a free B-module of finite positive rank; so B_1 is integral over B (lying over and going-up hold) and flat over B (going-down holds). Choose a prime $\tilde P_0$ of B_1 over $\ker f_0$; for $i = 1, \dots, h$ choose by going-down $\tilde E_i \subseteq \tilde P_{i-1}$ over $\mathfrak{e}_i$ (as $\mathfrak{e}_i \subseteq \ker f_{i-1}$) and by going-up $\tilde P_i \supseteq \tilde E_i$ over $\ker f_i$ (as $\mathfrak{e}_i \subseteq \ker f_i$). By going-down choose $\tilde P_{\mathfrak{t}} \subseteq \tilde P_0$ over $\mathfrak{t}$ and $\tilde Q \subseteq \tilde P_h$ over $\mathfrak{q}$. For an ideal $\mathfrak{a}$ of B_1 let $\mathfrak{a}[[z]] \subseteq C_1$ be the set of power series with all coefficients in $\mathfrak{a}$, the kernel of $B_1[[z]] \to (B_1/\mathfrak{a})[[z]]$; for $\mathfrak{a}$ prime, $(B_1/\mathfrak{a})[[z]]$ is a domain, so $\mathfrak{a}[[z]]$ is a prime of C_1 with $\mathfrak{a}[[z]] \cap B_1 = \mathfrak{a}$, hence $u_1^{-1}(\mathfrak{a}[[z]])$ is the contraction of $\mathfrak{a}$ to B; and $\mathfrak{a} \subseteq \mathfrak{b}$ implies $\mathfrak{a}[[z]] \subseteq \mathfrak{b}[[z]]$. By Lemma E.64 (for f_i, $\mathcal{O}' = \mathcal{O}_1$, and the prime $\tilde P_i$, which contracts to $\ker f_i$) there is a surjective ring homomorphism $g_i\colon C_1 \to A_{k_i}$ with $g_i \circ u_1 = f_i$, $g_i(z) = T$ and $\ker g_i \cap B_1 = \tilde P_i$. Every power series $g = \sum_m c_m z^m$ with all $c_m \in \tilde P_i$ lies in $\ker g_i$: for each N, $g = \sum_{m<N} c_m z^m + z^N g_N$ with $g_N \in C_1$, so $g_i(g) = T^N g_i(g_N) \in T^N A_{k_i}$, and $\bigcap_N T^N A_{k_i} = 0$. So $\tilde P_i[[z]] \subseteq \ker g_i$. Put $P_i^1 = \ker g_i$, $E_i^1 = \tilde E_i[[z]]$, $P_{\mathfrak{t}}^1 = \tilde P_{\mathfrak{t}}[[z]]$, $Q^1 = \tilde Q[[z]]$, primes of C_1. Then $E_i^1 \subseteq \tilde P_{i-1}[[z]] \subseteq P_{i-1}^1$, $E_i^1 \subseteq \tilde P_i[[z]] \subseteq P_i^1$, $P_{\mathfrak{t}}^1 \subseteq P_0^1$, $Q^1 \subseteq P_h^1$, and $u_1^{-1}(P_i^1) = \ker f_i$, $u_1^{-1}(E_i^1) = \mathfrak{e}_i$, $u_1^{-1}(P_{\mathfrak{t}}^1) = \mathfrak{t}$, $u_1^{-1}(Q^1) = \mathfrak{q}$. *Passage to C.* We show: $(*)$ ι is injective, C is integral and flat over C_1, and every prime P of C with $\iota^{-1}(P) = P_i^1$ is $\ker f'$ for a surjective lift f' of f_i with $f' \circ \iota = g_i$. If $q = 0$, then $C = C_1$, ι is the identity and $f' = g_i$ does it ($g_i \circ u_1 = f_i$, $g_i(z) = T$). Let $q \ge 1$. The ring $B[[z]]$ is a complete Noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}_{\mathcal{O}}$, and $\chi_{D^{\mathrm{u}}}(\sigma_j) \equiv \prod_i (X - \tilde u_{j,i})$ modulo $\mathfrak{m}_B \subseteq \mathfrak{m}_{B[[z]]}$; by Lemma E.59 (with $A = B[[z]]$, $P_j = \chi_{D^{\mathrm{u}}}(\sigma_j)$), C_0 is a local ring and a free $B[[z]]$-module of finite rank $r \ge 1$. Hence C is a free C_1-module of rank r; so ι is injective (an element of C_1 killing a non-zero free module is 0), C is integral over C_1 (lying over and going-up hold) and flat over C_1 (going-down holds). The maximal ideal of C_0 is generated by $\mathfrak{m}_{B[[z]]}$ and the $Z_{j,i}$ (C_0 is a quotient of $B[[z]][[Z_{j,i}]]$), so $\bar C = C_0/\mathfrak{m}_{B[[z]]}C_0$ is a finite-dimensional local $\mathbb{F}_{\mathcal{O}}$-algebra whose maximal ideal, generated by the images of the $Z_{j,i}$, is nilpotent, say of exponent t; so $\bar C$ is spanned by the images of the monomials in the $Z_{j,i}$ of degree less than t, and by Nakayama's lemma (C_0 is a finite module over the local ring $B[[z]]$) these finitely many monomials generate C_0 as a $B[[z]]$-module. As $Z_{j,i} = u_{j,i} - \tilde u_{j,i}$ with $\tilde u_{j,i} \in \mathcal{O}$, C is generated as a C_1-module by finitely many products of the $u_{j,i}$. Each $u_{j,i}$ is a root of ιp_j, where $p_j \in C_1[X]$ is the image of $\chi_{D^{\mathrm{u}}}(\sigma_j)$ under u_1, since $\prod_i (X - u_{j,i}) = \chi_{D^{\mathrm{u}}}(\sigma_j)(X)$ in $C[X]$; and $g_i p_j$ is the image of $\chi_{D^{\mathrm{u}}}(\sigma_j)$ under $g_i \circ u_1 = f_i$, a product of monic polynomials of degree 1 in $A_{k_i}[X]$ by (Sys0). By Lemma E.65 (with $w = (u_{j,i})_{j,i}$, $A = A_{k_i}$, the domain, and $g = g_i$, surjective), every prime P of C with $\iota^{-1}(P) = P_i^1$ is the kernel of a surjective ring homomorphism $f'\colon C \to A_{k_i}$ with $f' \circ \iota = g_i$; then $f' \circ u = g_i \circ u_1 = f_i$ and $f'(z) = g_i(z) = T$, so f' is a surjective lift of f_i. This

proves $(*)$. Now choose by lying over a prime P_0 of C with $\iota^{-1}(P_0) = P_0^1$; for $i = 1, \dots, h$ choose by going-down $E_i \subseteq P_{i-1}$ with $\iota^{-1}(E_i) = E_i^1$ (as $E_i^1 \subseteq P_{i-1}^1$) and by going-up $P_i \supseteq E_i$ with $\iota^{-1}(P_i) = P_i^1$ (as $E_i^1 \subseteq P_i^1$); by going-down choose $P_{\mathfrak{t}} \subseteq P_0$ over $P_{\mathfrak{t}}^1$ and $Q \subseteq P_h$ over Q^1. By $(*)$, $P_i = \ker f_i'$ for a surjective lift f_i' of f_i. Then $E_i \subseteq P_{i-1} \cap P_i$, $P_{\mathfrak{t}} \subseteq P_0$, $Q \subseteq P_h$, and, as $u = \iota \circ u_1$, $u^{-1}(P_i) = \ker f_i$, $u^{-1}(E_i) = \mathfrak{e}_i$, $u^{-1}(P_{\mathfrak{t}}) = \mathfrak{t}$, $u^{-1}(Q) = \mathfrak{q}$. Put $\mathfrak{q}_i = \pi^{-1}(P_i)$. As π is surjective, preimages of primes of C are primes of R, and they contain $\beta(y_1), \dots, \beta(y_s)$ by (Sys1). By (Sys3) for the surjective lift f_i', $f_{\mathcal{T}} \notin \mathfrak{q}_i$; hence $\pi^{-1}(P') \in \mathcal{P}$ for every prime P' of C contained in some P_i. (b) *A minimal prime of the support.* Put $\tau = \pi^{-1}(P_{\mathfrak{t}}) \in \mathcal{P}$. As $J \subseteq \mathfrak{t} = u^{-1}(P_{\mathfrak{t}})$, (Sys2) gives $\mathrm{Ann}_R(X) \subseteq \tau$, so $\tau R' \in \mathrm{Supp}_{R'}(M_\infty)$ by (Pat3). As M_∞ is finitely generated over the Noetherian ring R', $\mathrm{Supp}_{R'}(M_\infty) = V(\mathrm{Ann}_{R'}(M_\infty))$; let $Z \subseteq \tau R'$ be a prime of R' minimal in it. Then $\iota^{-1}(Z) \subseteq \iota^{-1}(\tau R') = \tau = \pi^{-1}(P_{\mathfrak{t}})$ with $u^{-1}(P_{\mathfrak{t}}) = \mathfrak{t}$ stable, so every prime of R contained in $\iota^{-1}(Z)$ is stable for the system. Also $Z \subseteq \mathfrak{q}_0 R'$, as $\tau \subseteq \mathfrak{q}_0$. (c) *The key inclusion.* Suppose $Z \subseteq \mathfrak{q}_i R'$ for some i, and let $P' \subseteq P_i$ be a prime of C with $u^{-1}(P')$ stable or thick; put $\varphi = \pi^{-1}(P') \in \mathcal{P}$, so $\varphi R' \subseteq \mathfrak{q}_i R'$. We show $Z \subseteq \varphi R'$. Since $\mathfrak{q}_i R' \supseteq Z \supseteq \mathrm{Ann}_{R'}(M_\infty)$, (Pat3) gives $\mathrm{Ann}_R(X) \subseteq \mathfrak{q}_i$; with (a) and (Sys3), the hypotheses of (Pat4) hold at $\mathfrak{q}_i$ with the integer d of the system. Let $\hat{A}$ be the completion of $R'_{\mathfrak{q}_i R'}$, identified by (Pat0) with the completion of $R_{\mathfrak{q}_i}$ compatibly with the maps from R, and let $\hat{Z}$ be a prime of $\hat{A}$ minimal over $Z\hat{A}$. By (Pat4), $\hat{Z}$ lies over Z and is minimal in $\mathrm{Supp}(M_\infty \otimes_{R'} \hat{A})$, hence $I\hat{A} \subseteq \hat{Z}$ and $\dim \hat{A}/\hat{Z} \geq d + s$; its preimage in R is $\iota^{-1}(Z)$, stable for the system by (b). As $\hat{A}$ is faithfully flat over $R'_{\mathfrak{q}_i R'}$, there is a prime $\hat{Q}$ of $\hat{A}$ lying over $\varphi R'_{\mathfrak{q}_i R'}$; its preimage in R' is $\varphi R'$ and in R is $\varphi = \pi^{-1}(P')$. By (Sys4) for the point x_i and the surjective lift f_i' (so $\mathfrak{q} = \pi^{-1}(\ker f_i') = \mathfrak{q}_i$), $\hat{Z} \subseteq \hat{Q}$; taking preimages in R', $Z \subseteq \varphi R'$. (d) *Along the chain.* By induction $Z \subseteq \mathfrak{q}_i R'$ for all i: the case $i = 0$ is (b); if $Z \subseteq \mathfrak{q}_{i-1} R'$, apply (c) with $P' = E_i \subseteq P_{i-1}$, $u^{-1}(E_i) = \mathfrak{e}_i$ thick: $Z \subseteq \pi^{-1}(E_i)R' \subseteq \pi^{-1}(P_i)R' = \mathfrak{q}_i R'$. (e) *The conclusion.* Apply (c) with $i = h$ and $P' = Q \subseteq P_h$, $u^{-1}(Q) = \mathfrak{q}$ stable or thick: $Z \subseteq \varphi R'$ for $\varphi = \pi^{-1}(Q) \in \mathcal{P}$. Then $\varphi R' \in \mathrm{Supp}_{R'}(M_\infty)$, so (Pat3) gives $\mathrm{Ann}_R(X) \subseteq \varphi$, and (Sys2) gives $J \subseteq u^{-1}(Q) = \mathfrak{q}$.

Step 4: $R = T$ at patchable curve points. Let x be a patchable curve point of irreducible type, with homomorphism f, with $\mathcal{J} \subseteq \ker f$. By Step 2, $\ker f$ is stable, and by Step 1 every prime $\mathfrak{q} \subseteq \ker f$ is stable. By Proposition E.39 (with $h = 0$, $x_0 = x$; its hypotheses are among those of the present statement) there are a curve point x' with homomorphism f' such that $\ker f' = \ker f$ and a patching system for $\mathcal{J}$ along (x'). Step 3 with $h = 0$, the point x' and $\mathfrak{t} = \ker f = \ker f'$ (stable, and $\mathcal{J} \subseteq \ker f$) gives $\mathcal{J} \subseteq \mathfrak{q}$ for every prime $\mathfrak{q} \subseteq \ker f' = \ker f$ that is stable; by the above, these are all primes $\mathfrak{q} \subseteq \ker f$.

Step 5: $R = T$ along thick chains. Let $x_0, \dots, x_h$ be patchable curve points of endoscopic type with homomorphisms $f_0, \dots, f_h$, thick primes $\mathfrak{e}_i \subseteq \ker f_{i-1} \cap \ker f_i$ and a stable prime $\mathfrak{t}$ with $\mathcal{J} \subseteq \mathfrak{t} \subseteq \ker f_0$. By Proposition E.39 there are curve points $x_0', \dots, x_h'$ with homomorphisms $f_0', \dots, f_h'$ such that $\ker f_i' = \ker f_i$ for every i, and a patching system for $\mathcal{J}$ along $(x_0', \dots, x_h')$. The hypotheses of Step 3 involve only the kernels: $\mathfrak{e}_i \subseteq \ker f_{i-1}' \cap \ker f_i'$ and $\mathcal{J} \subseteq \mathfrak{t} \subseteq \ker f_0'$. So Step 3, applied to $x_0', \dots, x_h'$, shows that every prime $\mathfrak{q} \subseteq \ker f_h' = \ker f_h$ that is stable or thick contains $\mathcal{J}$. □

Proposition E.33. *Let $(R, \beta, X, f, I, e, (M_N)_{N \geq 1}, \mathcal{U})$ be an abstract patching tower over $S = \Lambda[[y_1, \dots, y_s]]$, with patching ring R' and $\iota\colon R \to R'$. For a multi-index $\alpha \in \mathbb{Z}_{\geq 0}^s$ write $|\alpha| = \sum_i \alpha_i$ and $y^\alpha = \prod_i \iota(\beta(y_i))^{\alpha_i} \in R'$, and let $\mathcal{P}$ be the set of prime ideals Q of R with $f \notin Q$ and $\mathfrak{y} \subseteq Q$. Then R' is a Noetherian ring, and there is a finitely generated R'-module M_∞ with the following properties.*

(Pat0) *For $Q \in \mathcal{P}$, QR' is a prime ideal of R' with $\iota^{-1}(QR') = Q$, and $Q \mapsto QR'$ is a bijection from $\mathcal{P}$ onto the set of prime ideals of R' containing $\mathfrak{y}R'$. For $Q \in \mathcal{P}$, the homomorphism*

$R_Q \to R'_{QR'}$ induced by ι induces an isomorphism of their completions at the maximal ideals.

(Pat1) *$R'/\mathfrak{y}R' \cong R_f/\mathfrak{y}R_f$, and $M_\infty/\mathfrak{y}M_\infty \cong X \otimes_R R_f$ as modules over this ring. For every $k \geq 0$ the map*

$$\bigoplus_{|\alpha|=k} M_\infty/\mathfrak{y}M_\infty \longrightarrow \mathfrak{y}^k M_\infty/\mathfrak{y}^{k+1}M_\infty, \qquad (w_\alpha)_\alpha \longmapsto \textstyle\sum_\alpha y^\alpha w_\alpha,$$

is an isomorphism.

(Pat2) *For every $g \in I$ there is $t \geq 1$ with $\iota(g)^t M_\infty = 0$.*

(Pat3) *For every $Q \in \mathcal{P}$: $QR' \in \operatorname{Supp}_{R'}(M_\infty)$ if and only if $\operatorname{Ann}_R(X) \subseteq Q$.*

(Pat4) *Let $\mathfrak{q} \in \mathcal{P}$ with $\operatorname{Ann}_R(X) \subseteq \mathfrak{q}$, such that for $\mathfrak{p} = \{a \in \Lambda : \beta(a) \in \mathfrak{q}\}$ the local ring $\Lambda_\mathfrak{p}$ is regular of dimension d. Let A be either $R'_{\mathfrak{q}R'}$ or its completion $\hat{A}$ at its maximal ideal (by (Pat0), $\hat{A}$ is the completion of $R_\mathfrak{q}$), and $M_A = M_\infty \otimes_{R'} A$. Then $M_A \neq 0$, $\operatorname{depth}_A(M_A) \geq d + s$, and every prime ideal P of A that is minimal in $\operatorname{Supp}_A(M_A)$ satisfies $\iota(I)A \subseteq P$ and $\dim A/P \geq d + s$. Moreover, if $Z \subseteq \mathfrak{q}R'$ is a prime ideal of R' minimal in $\operatorname{Supp}_{R'}(M_\infty)$, then every prime ideal of $\hat{A}$ minimal over $Z\hat{A}$ lies over Z and is minimal in $\operatorname{Supp}_{\hat{A}}(M_\infty \otimes_{R'} \hat{A})$.*

Proof. Here AM refers to [12] and Mat to [166]. Write $S_{k,j} = S/\mathfrak{c}_{k,j}$ and, for a family $(Z_N)_{N\geq 1}$ of sets, $\prod_\mathcal{U} Z_N$ for its ultraproduct (sequences, defined on a member of $\mathcal{U}$, modulo agreement on a member of $\mathcal{U}$). For α a multi-index, y^α also denotes $\prod_i y_i^{\alpha_i} \in S$; its action on an R-module is through β.

Step 0: finiteness. Each $\Lambda/\mathfrak{m}_\Lambda^j$ is finite (the residue field is finite). The ring

$$S_{k,j} \cong (\Lambda/\mathfrak{m}_\Lambda^j)[y_1, \ldots, y_s]/(y_1, \ldots, y_s)^k$$

is a finite local ring, free over $\Lambda/\mathfrak{m}_\Lambda^j$ on the monomials of degree $< k$, and $X/\mathfrak{m}_\Lambda^j X$ is finite by (Tow0). A finitely generated R-module that is a finite set has finite length, and a module of length λ over the local ring R is killed by $\mathfrak{m}_R^\lambda$; so for every j there is $c_j \geq 1$ with

$$\mathfrak{m}_R^{c_j}\,(X/\mathfrak{m}_\Lambda^j X) = 0. \tag{E.4}$$

(X is a finitely generated R-module: a Λ-basis generates it, Λ acting through β.)

Step 1: facts on ultraproducts. Let (Z_N) be R-modules. (U1) $\prod_\mathcal{U} Z_N$ is an R-module componentwise; ultraproducts of R-linear maps are R-linear, and an ultraproduct of sequences $Z'_N \to Z_N \to Z''_N$ that are exact for $\mathcal{U}$-almost all N is exact. Ultraproducts commute with finite direct sums. (U2) If Z is a finite R-module and $Z_N = Z$ for $\mathcal{U}$-almost all N, the diagonal map $Z \to \prod_\mathcal{U} Z_N$ is bijective: a sequence with values in a finite set is constant on a member of $\mathcal{U}$. (U3) If $J = (g_1, \ldots, g_m)$ is a finitely generated ideal of R, or of S acting through β, then $\prod_\mathcal{U} Z_N \to \prod_\mathcal{U}(Z_N/JZ_N)$ is surjective with kernel $J\prod_\mathcal{U} Z_N$, and $\prod_\mathcal{U}(JZ_N) = J\prod_\mathcal{U} Z_N$: a class (z_N) has $z_N \in JZ_N$ for $\mathcal{U}$-almost all N iff $z_N = \sum_i g_i w_{i,N}$ there, i.e. iff $(z_N) = \sum_i g_i(w_{i,N})$.

Step 2: the finite levels. Put $P_{k,j} = \prod_\mathcal{U} M_N/\mathfrak{c}_{k,j}M_N$, an R-module killed by $\mathfrak{c}_{k,j}$. (a) For $k \leq k'$, $j \leq j'$ the reductions induce an R-linear surjection $P_{k',j'} \to P_{k,j}$ with kernel $\mathfrak{c}_{k,j}P_{k',j'}$, by (U3) applied to $Z_N = M_N/\mathfrak{c}_{k',j'}M_N$. (b) *The monomial isomorphism.* Let $k, j \geq 1$. For an R-module Y put $\bar{Y} = Y/\mathfrak{c}_{k+1,j}Y$; the map $\mu_Y\colon \bigoplus_{|\alpha|=k} Y/\mathfrak{c}_{1,j}Y \to \bar{Y}$, $(w_\alpha) \mapsto \sum_\alpha y^\alpha \tilde{w}_\alpha$ ($\tilde{w}_\alpha$ any lift) is well defined, since $y^\alpha \mathfrak{c}_{1,j} \subseteq (y)^{k+1} + \mathfrak{m}_\Lambda^j S \subseteq \mathfrak{c}_{k+1,j}$, and its image is $\mathfrak{y}^k\bar{Y}$. If $\bar{Y}$ is

a free $S_{k+1,j}$-module, μ_Y is injective: for $\bar{Y} = S_{k+1,j}$ it is the map $\bigoplus_{|\alpha|=k} \Lambda/\mathfrak{m}_\Lambda^j \to S_{k+1,j}$, $(a_\alpha) \mapsto \sum a_\alpha y^\alpha$, injective by Step 0, and the free case is a direct sum of copies. By (Tow1) and (U1), (U3), the ultraproduct $\mu_{k,j}\colon \bigoplus_{|\alpha|=k} P_{1,j} \to P_{k+1,j}$ of the μ_{M_N} is injective with image $\mathfrak{y}^k P_{k+1,j}$. By (a), the kernel of $P_{k+1,j} \to P_{k,j}$ is $\mathfrak{c}_{k,j} P_{k+1,j} = \mathfrak{y}^k P_{k+1,j}$, as $\mathfrak{m}_\Lambda^j S \subseteq \mathfrak{c}_{k+1,j}$ kills $P_{k+1,j}$. So

$$0 \to \bigoplus_{|\alpha|=k} P_{1,j} \xrightarrow{\mu_{k,j}} P_{k+1,j} \to P_{k,j} \to 0 \tag{E.5}$$

is exact, compatibly with the reductions in j. (c) *Uniform annihilation.* Let $k, j \geq 1$ and $g \in G$ (the generating set of (Tow3)). For N in the member of $\mathcal{U}$ given by (Tow2) at level j, $f(M_N/\mathfrak{c}_{1,j}M_N) = \varphi\psi(M_N/\mathfrak{c}_{1,j}M_N) \subseteq \varphi(X/\mathfrak{m}_\Lambda^j X)$ is killed by $\mathfrak{m}_R^{c_j}$ (E.4), φ being R-linear. Let $\bar{M} = M_N/\mathfrak{c}_{k,j}M_N$. Then $\mathfrak{y}^k\bar{M} = 0$, and for $0 \leq i < k$ the quotient $\mathfrak{y}^i\bar{M}/\mathfrak{y}^{i+1}\bar{M}$ is the image of $\bigoplus_{|\alpha|=i} M_N/\mathfrak{c}_{1,j}M_N$ under $(w_\alpha) \mapsto \sum y^\alpha \tilde{w}_\alpha$ (well defined because $\mathfrak{m}_\Lambda^j \bar{M} = 0$); so $f\mathfrak{m}_R^{c_j}$ kills each of these k quotients and $f^k\mathfrak{m}_R^{kc_j}\bar{M} = 0$. By (Tow3) with exponent kc_j, for $\mathcal{U}$-almost all N there is a_N with $(fg)^e - a_N \in \mathfrak{m}_R^{kc_j}$ and $a_N M_N = 0$; then $f^k(fg)^e z = f^k((fg)^e - a_N)z = 0$ for $z \in \bar{M}$. Hence

$$f^k(fg)^e P_{k,j} = 0 \qquad (g \in G,\ k, j \geq 1). \tag{E.6}$$

Step 3: compatible comparison maps. For $j \geq 1$ let $W_j \in \mathcal{U}$ be the set of (Tow2) at level j, and $W_j' = W_1 \cap \cdots \cap W_j \cap \{N : N \geq j\} \in \mathcal{U}$. For $N \in W_1'$ put $j(N) = \max\{j : N \in W_j'\}$ (finite, as $N \in W_j'$ forces $j \leq N$), and choose (φ_N, ψ_N) as in (Tow2) at level $j(N)$. For $j \leq j(N)$, φ_N maps $\mathfrak{m}_\Lambda^j(X/\mathfrak{m}_\Lambda^{j(N)}X)$ into $\mathfrak{m}_\Lambda^j(M_N/\mathfrak{c}_{1,j(N)}M_N)$ and ψ_N maps $\mathfrak{c}_{1,j}(M_N/\mathfrak{c}_{1,j(N)}M_N)$ into $\mathfrak{c}_{1,j}(X/\mathfrak{m}_\Lambda^{j(N)}X) = \mathfrak{m}_\Lambda^j(X/\mathfrak{m}_\Lambda^{j(N)}X)$ (Tow0); since $\mathfrak{c}_{1,j} = \mathfrak{c}_{1,j(N)} + \mathfrak{m}_\Lambda^j S$, they induce R-linear maps $\varphi_{N,j}\colon X/\mathfrak{m}_\Lambda^j X \to M_N/\mathfrak{c}_{1,j}M_N$ and $\psi_{N,j}$ back, with both composites f and compatible with reduction in j. Since $N \in W_j'$ implies $j(N) \geq j$, these are defined for $N \in W_j'$. Let $\Phi_j\colon X/\mathfrak{m}_\Lambda^j X \to P_{1,j}$ be the diagonal bijection of (U2) followed by $\prod_{\mathcal{U}} \varphi_{N,j}$, and $\Psi_j\colon P_{1,j} \to X/\mathfrak{m}_\Lambda^j X$ be $\prod_{\mathcal{U}} \psi_{N,j}$ followed by the inverse of that bijection. They are R-linear (U1), $\Psi_j\Phi_j = f$, $\Phi_j\Psi_j = f$, and they are compatible with the maps of Step 2(a) and the reductions of X.

Step 4: limits over j. Put $U_k = \varprojlim_j P_{k,j}$ along the surjections of Step 2(a). The exact sequences (E.5) form an inverse system in j whose left terms have surjective transition maps, so the limit is exact (AM, Proposition 10.2):

$$0 \to \bigoplus_{|\alpha|=k} U_1 \xrightarrow{\mu_k} U_{k+1} \to U_k \to 0,$$

where $\mu_k((w_\alpha)) = \sum y^\alpha \tilde{w}_\alpha$ with lifts $\tilde{w}_\alpha \in U_{k+1}$ (these exist: by induction on k, $U_{k+1} \to U_1$ is surjective). Hence $U_{k+1} \to U_k$ is surjective with kernel $\mathfrak{y}^k U_{k+1} = \mu_k(\bigoplus U_1)$, μ_k is injective, $\mathfrak{y}^k U_k = 0$, and by (E.6) $f^k(fg)^e U_k = 0$ for $g \in G$. As $X \cong \Lambda^\rho$ with Λ complete, $X = \varprojlim_j X/\mathfrak{m}_\Lambda^j X$, and the limits of Step 3 are R-linear maps

$$\Phi\colon X \to U_1, \qquad \Psi\colon U_1 \to X, \qquad \Psi\Phi = f, \quad \Phi\Psi = f. \tag{E.7}$$

Step 5: inverting f; the patched module, (Pat1) and (Pat2). Put $V_k = U_k \otimes_R R_f$. Localisation is exact, so $0 \to \bigoplus_{|\alpha|=k} V_1 \to V_{k+1} \to V_k \to 0$ is exact with first map the monomial map and image $\mathfrak{y}^k V_{k+1}$; thus $V_{k+1}/\mathfrak{y}^k V_{k+1} \cong V_k$, and $\mathfrak{y}^k V_k = 0$. By (E.7), $\Phi \otimes R_f\colon X \otimes_R R_f \to V_1$ is an isomorphism with inverse $f^{-1}(\Psi \otimes R_f)$. Put $M_\infty = \varprojlim_k V_k$; it is a module over $\varprojlim_k R_f/\mathfrak{y}^k R_f = R'$. By [242, Lemma 10.98.2], applied over R_f to the finitely generated ideal $\mathfrak{y}R_f$, $M_\infty/\mathfrak{y}^k M_\infty \cong V_k$ for all k and M_∞ is $\mathfrak{y}$-adically complete. The ring R_f is Noetherian, so R' is Noetherian, $\mathfrak{y}R'$-adically complete, and $R'/\mathfrak{y}^k R' \cong R_f/\mathfrak{y}^k R_f$ (AM, Proposition 10.15, Theorem 10.26). Hence $M_\infty/\mathfrak{y}M_\infty \cong V_1 \cong X \otimes_R R_f$, and $\mathfrak{y}^k M_\infty/\mathfrak{y}^{k+1}M_\infty = \ker(V_{k+1} \to V_k)$ is

the injective image of $\bigoplus_{|\alpha|=k} V_1$ under the monomial map: this is (Pat1). The module $X \otimes_R R_f$ is finitely generated over R_f; choose $g_1, \dots, g_t \in M_\infty$ whose images generate $M_\infty/\mathfrak{y}M_\infty$. As R' is $\mathfrak{y}R'$-adically complete and M_∞ is $\mathfrak{y}$-adically separated, the g_i generate M_∞ (Mat, Theorem 8.4): M_∞ is a finitely generated R'-module. For $g \in G$, $f^{k+e} g^e U_k = 0$ and f is a unit of R_f, so $\iota(g)^e V_k = 0$ for all k and $\iota(g)^e M_\infty = 0$. For $g = \sum_{h \in G} c_h h \in I$ and $t = |G|(e-1)+1$, every monomial of $(\sum_h c_h h)^t$ is divisible by some h^e, so $\iota(g)^t M_\infty = 0$: this is (Pat2).

Step 6: primes and completions, (Pat0). By $R'/\mathfrak{y}R' \cong R_f/\mathfrak{y}R_f$, the prime ideals of R' containing $\mathfrak{y}R'$ correspond bijectively to the prime ideals of R_f containing $\mathfrak{y}R_f$, hence to $\mathcal{P}$. The prime corresponding to $Q \in \mathcal{P}$ is the preimage of $QR_f/\mathfrak{y}R_f$, i.e. $QR' + \mathfrak{y}R' = QR'$, and its preimage in R is the preimage of QR_f, which is Q. For $t \geq 1$, $\mathfrak{y}^t \subseteq Q^t$, so $R'/Q^tR' \cong R_f/Q^tR_f$ compatibly with ι; localising at QR', resp. QR_f, gives $R'_{QR'}/Q^tR'_{QR'} \cong R_Q/Q^tR_Q$. The maximal ideal of $R'_{QR'}$ is $QR'_{QR'}$, so passing to the limit over t gives the isomorphism of completions.

Step 7: (Pat3). For a finitely generated module M over a Noetherian ring and an ideal J, $\mathrm{Supp}(M/JM) = \mathrm{Supp}(M) \cap V(J)$. With $J = \mathfrak{y}R' \subseteq QR'$: $QR' \in \mathrm{Supp}_{R'}(M_\infty)$ iff $(M_\infty/\mathfrak{y}M_\infty)_{QR'} \neq 0$. By (Pat1) and Step 6 this localisation is the localisation of $X \otimes_R R_f$ at QR_f, i.e. X_Q. As X is a finitely generated R-module, $X_Q \neq 0$ iff $\mathrm{Ann}_R(X) \subseteq Q$.

Step 8: depth, (Pat4). Let $\mathfrak{q}, \mathfrak{p}, d$ be as in (Pat4), $\mathfrak{q}' = \mathfrak{q}R'$, $A_0 = R'_{\mathfrak{q}'}$ and $M_0 = (M_\infty)_{\mathfrak{q}'}$, a finitely generated module over the Noetherian local ring A_0; the elements $\iota\beta(y_i)$ lie in its maximal ideal. Localisation is exact, so the isomorphisms of (Pat1) localise: the sequence $\iota\beta(y_1), \dots, \iota\beta(y_s)$ is M_0-quasi-regular. The modules $M_0/(\iota\beta(y_1), \dots, \iota\beta(y_i))M_0$ are finitely generated over A_0, hence separated for the $\mathfrak{y}$-adic topology (Krull's intersection theorem), so the sequence is M_0-regular (Mat, Theorem 16.3). By Step 7, $M_0/\mathfrak{y}M_0 \cong X_{\mathfrak{q}}$, a non-zero module as $\mathrm{Ann}_R(X) \subseteq \mathfrak{q}$. Choose $z_1, \dots, z_d \in \mathfrak{p}$ whose images form a regular system of parameters of $\Lambda_{\mathfrak{p}}$; a regular local ring is Cohen–Macaulay, so $z_1, \dots, z_d$ is regular on $\Lambda_{\mathfrak{p}}$, hence on $X \otimes_\Lambda \Lambda_{\mathfrak{p}} \cong \Lambda_{\mathfrak{p}}^\rho$. As $\beta(\Lambda \smallsetminus \mathfrak{p}) \subseteq R \smallsetminus \mathfrak{q}$, the module $X_{\mathfrak{q}}$ is a localisation of $X \otimes_\Lambda \Lambda_{\mathfrak{p}}$, so multiplication by z_i is injective on $X_{\mathfrak{q}}/(z_1, \dots, z_{i-1})X_{\mathfrak{q}}$; and $X_{\mathfrak{q}}/(z)X_{\mathfrak{q}} \neq 0$ by Nakayama's lemma, as $\beta(z_i) \in \mathfrak{q}$. So $\iota\beta(y_1), \dots, \iota\beta(y_s), \iota\beta(z_1), \dots, \iota\beta(z_d)$ is an M_0-regular sequence in the maximal ideal of A_0, $M_0 \neq 0$ and $\mathrm{depth}_{A_0}(M_0) \geq d + s$. The completion $\hat{A}$ is faithfully flat over A_0, so tensoring preserves the injectivity of each multiplication map and the non-vanishing of the last quotient; the same holds for $A = \hat{A}$, $M_{\hat{A}} = M_0 \otimes_{A_0} \hat{A}$. Let P be minimal in $\mathrm{Supp}_A(M_A)$. Then P is an associated prime of M_A, and $\dim A/P \geq \mathrm{depth}_A(M_A) \geq d + s$ (Mat, Theorem 17.2). For $g \in I$, (Pat2) gives $\iota(g)^t \in \mathrm{Ann}_A(M_A) \subseteq P$, so $\iota(g) \in P$. Finally let $Z \subseteq \mathfrak{q}'$ be minimal in $\mathrm{Supp}_{R'}(M_\infty)$; then ZA_0 is minimal in $\mathrm{Supp}(M_0)$. Let $\hat{Z}$ be minimal over $Z\hat{A}$. If $\hat{Z} \cap A_0 \neq ZA_0$, going-down for the flat extension $A_0 \to \hat{A}$ gives $\hat{Z}' \subsetneq \hat{Z}$ lying over ZA_0, so $Z\hat{A} \subseteq \hat{Z}' \subsetneq \hat{Z}$, a contradiction; so $\hat{Z}$ lies over Z. For a finitely generated A_0-module M and a prime P' of $\hat{A}$ over P'', $(M \otimes \hat{A})_{P'} = M_{P''} \otimes \hat{A}_{P'}$ with $\hat{A}_{P'}$ faithfully flat over $(A_0)_{P''}$; so $\mathrm{Supp}(M \otimes \hat{A})$ is the preimage of $\mathrm{Supp}(M)$. Hence $\hat{Z} \in \mathrm{Supp}(M_{\hat{A}})$, and a prime $P' \subsetneq \hat{Z}$ of this support would lie over a prime of $\mathrm{Supp}(M_0)$ contained in ZA_0, i.e. over ZA_0, so $P' \supseteq Z\hat{A}$, contradicting the minimality of $\hat{Z}$ over $Z\hat{A}$. □

E.2.2 Existence of patching systems

This subsection proves Proposition E.39: patching systems exist for ideals with classical levels along extensions of patchable curve points. It defines the Selmer data of a framed curve point, the Selmer sets at the levels with the bounds (B1), (B2) and the conditions (K1)–(K3), automorphic patching data, their patched tower data, and realisations over a level (Definition E.34, Definition E.35, Definition E.36, Definition E.37, Definition E.38). The limit over the Taylor–Wiles levels is taken through ultrafilter limits of quotients of a finitely generated profinite group

(Lemma E.43, Lemma E.44, Lemma E.42). Cocycles with values in finite ℓ-groups vanish on the pro-ℓ kernel (Lemma E.45), and at the primes of the patched ring that are stable for the system the reducibility ideal of a realisation is not contained in the prime (Lemma E.46, Lemma E.47).

Definition E.34. Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, write $\Gamma = \Gamma_{\mathscr{H}}$, $\Gamma^0 = \Gamma^0_{\mathscr{H}}$, $\pi = \pi_{\mathscr{H}}$, $\bar{c} = \pi(c)$, and let $x = (k, \rho, 1_n, (\psi_{v,a}), f)$ be a curve point of B_n^{po} that is normalised of shape $(a, n-a)$, $A = A_k$. For a commutative A-algebra X let $\mathrm{Ev}_a(X) \subseteq \mathrm{M}_n(X)$ be the matrices whose (i,j) entry vanishes unless i, j lie in the same block $\{1,\dots,a\}$ or $\{a{+}1,\dots,n\}$, and $\mathrm{Od}_a(X)$ those whose (i,j) entry vanishes unless i, j lie in different blocks (so $\mathrm{Od}_n(X) = 0$). For $v \in S_\ell$, $\mu_v \subseteq D_v$ is as in Definition 50.1.

(Sd1) Framed points. A **frame** of x is a pair $((w_v)_{v\in S_\ell}, (I_u^x)_{u\in \mathrm{St}})$ of matrices $w_v \in \mathrm{GL}_n(A)$, each of whose columns has its non-zero entries in one block, such that $w_v^{-1}\rho(\pi(g))w_v$ is upper triangular with diagonal $(\psi_{v,1}(g),\dots,\psi_{v,n}(g))$ for $g \in D_v$, and of closed subgroups $I_u^x \subseteq \pi(I_{\bar{u}} \cap G_L)$ ($\bar{u}$ a place of $\overline{\mathbb{Q}}$ above u) on which ρ is trivial. Together with a frame, x is a **framed point**; it satisfies **(Np3)** if it satisfies (Pj2) of Definition 50.9, and **(Np4)** if 0 is the only $Z \in \mathrm{M}_n(A)$ with $\chi_{\mathrm{cyc}}(\gamma)\rho(\gamma)Z = Z\rho(\gamma)$ for all $\gamma \in \Delta_0$.

(Sd2) Ordinary Selmer cocycles. The A-module $Z^1_{\mathrm{ord}}(x)$ consists of the maps $z\colon \Gamma^0 \to \mathrm{Od}_a(A)$ such that: every entry of z is continuous for the T-adic topology; $z(\gamma\gamma') = z(\gamma) + \rho(\gamma)z(\gamma')\rho(\gamma)^{-1}$; $z(\bar{c}\gamma\bar{c}) = -{}^t z(\gamma)$; z vanishes on every I_u^x; and for every $v \in S_\ell$ there is $Y_v \in \mathrm{Od}_a(A)$ such that, for $g \in D_v$ and $z_v(g) = w_v^{-1}(z(\pi g) - \rho(\pi g)Y_v\rho(\pi g)^{-1} + Y_v)w_v$, the matrix $z_v(g)$ is upper triangular, with zero diagonal when $g \in \mu_v$. The **coboundaries** are $\mathrm{Bd}(x) = \{\gamma \mapsto \rho(\gamma)Y\rho(\gamma)^{-1} - Y :\ Y \in \mathrm{Od}_a(A),\ {}^tY = -Y\} \subseteq Z^1_{\mathrm{ord}}(x)$. A **Selmer basis** of x is a family $\kappa = (\kappa_1,\dots,\kappa_g)$ in $Z^1_{\mathrm{ord}}(x)$ such that $\sum_j c_j\kappa_j \in \mathrm{Bd}(x)$ with $c \in A^g$ forces $c = 0$, and for every $z \in Z^1_{\mathrm{ord}}(x)$ there are $s \geq 0$ and $c \in A^g$ with $T^s z - \sum_j c_j\kappa_j \in \mathrm{Bd}(x)$.

(Sd3) Paired dual numbers and the first-order deformation. For $g \geq 0$ and a commutative A-algebra X, the ring of **paired dual numbers** is $\mathrm{PD}_g(X) = X \oplus \bigoplus_{j=1}^g X\epsilon_j \oplus \bigoplus_{j=1}^g X\epsilon'_j \oplus X\eta$ with $\epsilon_i\epsilon'_j = \delta_{ij}\eta$ and all other products of two of $\epsilon_1,\dots,\epsilon_g,\epsilon'_1,\dots,\epsilon'_g,\eta$ equal to 0. Let $\mathrm{PB}_a \subseteq \mathrm{M}_n(\mathrm{PD}_g(A))$ be the matrices whose same-block entries lie in $A\eta$ and whose different-block entries lie in $\bigoplus_j A\epsilon'_j$. For a Selmer basis κ put $\rho_1(\gamma) = \big(1 + \sum_j \epsilon_j\kappa_j(\gamma)\big)\rho(\gamma) \in \mathrm{GL}_n(\mathrm{PD}_g(A))$.

(Sd4) Twisted dual cocycles and detecting sets. $\mathrm{TwZ}(x,\kappa)$ is the set of maps $z\colon \Gamma^0 \to \mathrm{PB}_a$ with T-adically continuous entries such that $z(\gamma\gamma') = z(\gamma) + \chi_{\mathrm{cyc}}(\gamma)\rho_1(\gamma)z(\gamma')\rho_1(\gamma)^{-1}$ and $z(\bar{c}\gamma\bar{c}) = -\chi_{\mathrm{cyc}}(\bar{c})\,{}^t z(\gamma)$; $\mathrm{TwB}(x,\kappa)$ is the set of the maps $\gamma \mapsto \chi_{\mathrm{cyc}}(\gamma)\rho_1(\gamma)Y\rho_1(\gamma)^{-1} - Y$ with $Y \in \mathrm{PB}_a$, $-\chi_{\mathrm{cyc}}(\bar{c})\,{}^tY = Y$. For $\sigma \in \Gamma^0$ and $z \in \mathrm{TwZ}(x,\kappa)$ let $\ell_\sigma(z) \in A$ be the η-coefficient of $\mathrm{tr}\big(z(\sigma)\rho_1(\sigma)\big)$. A subset $\Sigma \subseteq \Gamma^0$ **detects** (x,κ) if every $z \in \mathrm{TwZ}(x,\kappa)$ with $\ell_\sigma(z) = 0$ for all $\sigma \in \Sigma$ satisfies $T^s z \in \mathrm{TwB}(x,\kappa)$ for some $s \geq 0$.

Definition E.35. Let $\mathscr{H}$ be a rank-n Hecke datum over $\mathcal{O}$, $\bar{c} = \pi(c)$, and x a framed normalised curve point of shape $(a, n-a)$ with frame $((w_v)_{v\in S_\ell}, (I_u^x)_{u\in\mathrm{St}})$.

(Lv) Levels. A **level** over $\mathscr{H}$ is a tuple $\mathbf{V} = (\Gamma_1, h, c_1, (\phi_{1,v})_{v\in S_\ell}, (I_{1,u})_{u\in\mathrm{St}})$ of a profinite group Γ_1, a continuous surjective homomorphism $h\colon \Gamma_1 \to \Gamma$, an element $c_1 \in \Gamma_1$ with $c_1^2 = 1$ and $h(c_1) = \bar{c}$, continuous homomorphisms $\phi_{1,v}\colon D_v \to \Gamma_1^0 := h^{-1}(\Gamma^0)$ with $h \circ \phi_{1,v} = \pi|_{D_v}$, and closed subgroups $I_{1,u} \subseteq \Gamma_1^0$ with $h(I_{1,u}) = I_u^x$. The **base level** is $(\Gamma, \mathrm{id}, \bar{c}, (\pi|_{D_v}), (I_u^x))$. For a Taylor–Wiles datum and $N \geq 1$, the **level** N is $\mathbf{V}_N = (\Gamma_N, h_N, \pi^{Q_N}(c), (\pi^{Q_N}|_{D_v}), (I_{N,u})_u)$ with $I_{N,u} = \pi^{Q_N}\big(I_{\bar{u}} \cap G_L \cap \pi^{-1}(I_u^x)\big)$, $\bar{u}$ the place of the frame; then $h_N(I_{N,u}) = I_u^x$ since $h_N \circ \pi^{Q_N} = \pi$ and $I_u^x \subseteq \pi(I_{\bar{u}} \cap G_L)$. (If $I_u^x = \{1\}$, then $I_{\bar{u}} \cap G_L \cap N_{\mathscr{H}}$ consists of inertia elements killed by all $\bar{\chi}_a$, which are trivial in Γ^{Q_N}, so $I_{N,u} = \{1\}$.)

(Sc) Selmer cocycles at a level. Let $\mathbf{V}$ be a level, R a commutative A-algebra, $V \subseteq \mathrm{M}_n(R)$ an A-submodule, $\varrho\colon \Gamma_1^0 \to \mathrm{GL}_n(R)$ a homomorphism and $u_v \in \mathrm{GL}_n(R)$ ($v \in S_\ell$). The

set $\mathrm{Sel}_{\mathbf{V}}(\varrho,u,V)$ consists of the maps $z\colon \Gamma_1^0\to V$ such that: every entry of z is continuous for the topology of R defined by the ideal TR; $z(\gamma\gamma')=z(\gamma)+\varrho(\gamma)z(\gamma')\varrho(\gamma)^{-1}$; $z(c_1\gamma c_1)=-{}^t z(\gamma)$; z vanishes on every $I_{1,u}$; and for every $v\in S_\ell$ there is $Y_v\in V$ such that for all $g\in D_v$ the matrix $u_v^{-1}(z(\phi_{1,v}g)-\varrho(\phi_{1,v}g)Y_v\varrho(\phi_{1,v}g)^{-1}+Y_v)u_v$ has zero entries below the diagonal, and zero diagonal entries if $g\in\mu_v$. The set $\mathrm{Bd}_{\mathbf{V}}(\varrho,V)$ consists of the maps $\gamma\mapsto\varrho(\gamma)Y\varrho(\gamma)^{-1}-Y$ with $Y\in V$, ${}^tY=-Y$.

(Ss) The six Selmer sets. Let κ be a Selmer basis of x together with **local witnesses** $Y_{j,v}\in\mathrm{Od}_a(A)$ for κ_j at v (as in (Sd2) of Definition E.34). For a commutative A-algebra X, let $\mathrm{PW}_a^1(X)$ (resp. $\mathrm{PW}_a^2(X)$) $\subseteq \mathrm{M}_n(\mathrm{PD}_g(X))$ be the matrices whose same-block entries lie in $X\cdot 1$ (resp. $\bigoplus_j X\epsilon'_j$) and whose different-block entries lie in $\bigoplus_j X\epsilon_j$ (resp. $X\eta$); put $w_v^1=(1-\sum_j\epsilon_jY_{j,v})w_v\in\mathrm{GL}_n(\mathrm{PD}_g(X))$. With $\rho_X=\rho\circ h$ (entries mapped to X) and $\rho_{1,X}=\rho_1\circ h$ (entries mapped to $\mathrm{PD}_g(X)$):

$$\mathrm{Sel}^{\mathrm{ev}}_{\mathbf{V}}(X)=\mathrm{Sel}_{\mathbf{V}}(\rho_X,w,\mathrm{Ev}_a(X)),\quad \mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}}(X)=\mathrm{Sel}_{\mathbf{V}}(\rho_X,w,\mathrm{Od}_a(X)),$$
$$\mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}}(X)=\mathrm{Bd}_{\mathbf{V}}(\rho_X,\mathrm{Od}_a(X)),\quad \mathrm{Sel}^1_{\mathbf{V}}(X)=\mathrm{Sel}_{\mathbf{V}}(\rho_{1,X},w^1,\mathrm{PW}_a^1(X)),$$
$$\mathrm{Bd}^1_{\mathbf{V}}(X)=\mathrm{Bd}_{\mathbf{V}}(\rho_{1,X},\mathrm{PW}_a^1(X)),\quad \mathrm{Sel}^2_{\mathbf{V}}(X)=\mathrm{Sel}_{\mathbf{V}}(\rho_{1,X},w^1,\mathrm{PW}_a^2(X)).$$

Write $A_m=A/T^mA$ ($m\geq 1$).

(Bd) Bounds along a Taylor–Wiles datum (levels $\mathbf{V}_N$), for an integer $D_0\geq 0$:

(B1) there is $C\geq 0$ such that for every $m\geq 1$ there is N_0 with: for $N\geq N_0$, $\mathrm{Sel}^1_{\mathbf{V}_N}(A_m)$ is finite and $\#\mathrm{Sel}^1_{\mathbf{V}_N}(A_m)\leq\#\mathrm{Bd}^1_{\mathbf{V}_N}(A_m)\cdot(\#k)^{mD_0+C}$;

(B2) there is $C\geq 0$ such that for every $m\geq 1$ there is N_0 with: for $N\geq N_0$ and every family $(z_{i,j})_{0\leq i\leq g,\,1\leq j\leq g}$ in $\mathrm{Sel}^{\mathrm{ev}}_{\mathbf{V}_N}(A_m)$ there are $c_0,\dots,c_g\in A$, one of them a unit, and $\Psi\in\mathrm{Sel}^2_{\mathbf{V}_N}(A_m)$ such that, for all j and γ, the ϵ'_j-coefficient of $\Psi(\gamma)$ is $T^C\sum_i c_iz_{i,j}(\gamma)$.

(Kd) Conditions at one level V (with $X=A$), for an integer $D_0\geq 0$:

(K1) if $z_1,\dots,z_e\in\mathrm{Sel}^1_{\mathbf{V}}(A)$ and $\sum_i c_iz_i\in\mathrm{Bd}^1_{\mathbf{V}}(A)$ with $c\in A^e$ forces $c=0$, then $e\leq D_0$;

(K2) for every family $(z_{i,j})_{0\leq i\leq g,\,1\leq j\leq g}$ in $\mathrm{Sel}^{\mathrm{ev}}_{\mathbf{V}}(A)$ there are $c\in A^{g+1}\smallsetminus\{0\}$ and $\Psi\in\mathrm{Sel}^2_{\mathbf{V}}(A)$ such that the ϵ'_j-coefficient of $\Psi(\gamma)$ is $\sum_i c_iz_{i,j}(\gamma)$ for all j, γ;

(K3) for every $z\in\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}}(A)$ there are $s\geq 0$ and $c\in A^g$ with $T^sz-\sum_j c_j(\kappa_j\circ h)\in\mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}}(A)$.

Definition E.36. Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, $(B,D^{\mathrm{u}},(\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, $J\subseteq B$ an ideal and $\mathcal{D}$ a Taylor–Wiles datum of size q for $\mathscr{H}$ (notation of Definition 53.8; $\Gamma_N^0=\pi^{Q_N}(G_L)$). Put $D=n[L^+:\mathbb{Q}]$.

Auxiliary rings and elements. Put $S_\infty=\Lambda_D[[Y_{j,i}:1\leq j\leq q,\ 1\leq i\leq n]]$; for $a=(a_j)\in\mathbb{N}^q$, $\mathfrak{d}_a\subseteq S_\infty$ is the ideal generated by the $(1+Y_{j,i})^{\ell^{a_j}}-1$. For $N\geq 1$ and $\tilde u=(\tilde u_{j,i})\in\mathcal{O}^{q\times n}$, the **eigenvalue algebra** is

$$\mathrm{Sp}_N=B[[Z_{j,i}]]/(\text{the coefficients of }\textstyle\prod_{i=1}^n(X-\tilde u_{j,i}-Z_{j,i})-\chi_{D^{\mathrm{u}}}(h_N(\Phi_{N,j}))(X),\ 1\leq j\leq q),$$

a B-algebra, with $u_{j,i}\in\mathrm{Sp}_N$ the class of $\tilde u_{j,i}+Z_{j,i}$. For a commutative ring R, a group G, $\phi,t\in G$, $U=(U_i)\in R^n$, $1\leq i\leq n$ and $y\in R$, put in $R[G]$

$$q_1(\phi,U)=\prod_{k=1}^n(\phi-U_k),\qquad q_2(\phi,t,U,i,y)=\Big(\prod_{k\neq i}(\phi-U_k)\Big)(t-1-y)$$

(ordered products, k increasing), and for $U=(U_{j,i})\in R^{q\times n}$ and integers (m_j)

$$f_m(U)=\prod_{j=1}^{q}\prod_{i=1}^{n}\prod_{i'\neq i}(U_{j,i}-U_{j,i'})(U_{j,i}-m_jU_{j,i'})\in R.$$

Automorphic patching data. Automorphic patching data for J along $\mathcal{D}$ consist of a Taylor–Wiles base $(\mathbf{w},M)$ for J with weight map λ_B; elements $\tilde{u}_{j,i}\in\mathcal{O}$ with $\tilde{u}_{j,i}\equiv\bar{\chi}_i(\sigma_j)$ modulo $\mathfrak{m}_{\mathcal{O}}$; an integer $e\geq 1$; and for every $N\geq 1$: $R_N\in\mathrm{CNL}_{\mathcal{O}}$ with a polarised ordinary determinant $(D_N,(\psi_{N,v,a}))$ of rank n on Γ_N^0 to R_N (as in Definition 53.9); a homomorphism of $\mathcal{O}$-algebras $s_N\colon S_\infty\to R_N$; elements $U_{N,j,i}\in R_N$; an R_N-module M_N with integers $a_{N,j}\geq N$ and $\mathrm{rk}_N\geq 0$; a homomorphism of $\mathcal{O}$-algebras $\theta_N\colon R_N\to\mathrm{Sp}_N$; and additive maps $\varphi_N\colon\mathrm{Sp}_N\otimes_B M\to X_N$, $\psi_N\colon X_N\to\mathrm{Sp}_N\otimes_B M$, where $X_N=M_N/(s_N(Y_{j,i}))_{j,i}M_N$; such that, writing $f=f_{(q_{N,j})_j}$:

(Ha1) $s_N(X_k)=c_k\psi_{N,v_k,a_k}(\gamma_k)-1$ for the weight variables $\mathbf{w}=(v_k,a_k,\gamma_k,c_k)$, and $U_{N,j,i}-\tilde{u}_{j,i}\in\mathfrak{m}_{R_N}$;

(Ha2) there is an additive bijection $M_N\to(S_\infty/\mathfrak{d}_{a_N})^{\mathrm{rk}_N}$ carrying $s_N(z)m$ to z times the image of m;

(Ha3) $\theta_N\circ D_N$ is the base change to Sp_N of $D^{\mathrm{u}}\circ h_N$, $\theta_N\circ\psi_{N,v,a}=\psi^{\mathrm{d}}_{v,a}$ (in Sp_N), $\theta_N(s_N(z))=\lambda_B(z_0)$ where $z_0\in\Lambda_D$ is z modulo the $Y_{j,i}$, and $\theta_N(U_{N,j,i})=u_{j,i}$;

(Ha4) $\varphi_N(\theta_N(t)z)=t\varphi_N(z)$, $\psi_N(tx)=\theta_N(t)\psi_N(x)$, $\varphi_N\psi_N(x)=f(U_N)x$ and $\psi_N\varphi_N(z)=\theta_N(f(U_N))z$;

(Ha5) for all j, i and $g\in\Gamma_N^0$,

$$\big(f(U_N)\,\mathrm{tr}_{D_N}(g\,q_1(\Phi_{N,j},U_{N,j}))\big)^eM_N=0,$$
$$\big(f(U_N)\,\mathrm{tr}_{D_N}(g\,q_2(\Phi_{N,j},\mathrm{T}_{N,j},U_{N,j},i,s_N(Y_{j,i})))\big)^eM_N=0.$$

Definition E.37. Setting. Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, $(B,D^{\mathrm{u}},(\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, $J\subseteq B$ an ideal, $D=n[L^+:\mathbb{Q}]$, $\mathcal{D}$ a Taylor–Wiles datum of size q for $\mathscr{H}$ with H and $\sigma_1,\dots,\sigma_q$, and automorphic patching data for J along $\mathcal{D}$ with Taylor–Wiles base $(\mathbf{w},M)$, $\mathbf{w}=(v_k,a_k,\gamma_k,c_k)_{1\leq k\leq D}$, elements $\tilde{u}_{j,i}$, exponent e and, for $N\geq 1$, $(R_N,D_N,(\psi_{N,v,a}),s_N,(U_{N,j,i}),M_N,\dots)$. Let $\mathcal{O}_1$ be a complete discrete valuation ring which is a free $\mathcal{O}$-module of finite rank with $\mathfrak{m}_{\mathcal{O}}\mathcal{O}_1=\mathfrak{m}_{\mathcal{O}_1}$. Let $(\Pi,(\pi_N),\mathcal{U},\bar{\theta},c_\infty,(\Phi_j),(\mathrm{T}_j),(F_w))$ be as in the conclusion of Lemma E.42 for $\mathcal{D}$, $\bar{\Gamma}^0\subseteq\bar{\Gamma}$ the image of $\Gamma^0_{\mathscr{H}}$, $\bar{H}\subseteq\bar{\Gamma}$ the image of H, $P^0=\pi_\infty^{-1}(\bar{\theta}^{-1}(\bar{\Gamma}^0))$, and fix $\tilde{c}\in\Pi$ with $\pi_\infty(\tilde{c})=c_\infty$. By Lemma E.62 (the $\bar{\chi}_a$ are trivial on H by (Tg1)) D^{u} is the composite of a unique determinant $\bar{D}^{\mathrm{u}}$ on $\bar{\Gamma}^0$ with the projection.

Eigenvalue algebra. Recall from Definition E.31 (where C_0^{sp} is written C_0): with a variable z,

$$C_0^{\mathrm{sp}}=B[[z]][[Z_{j,i}:1\leq j\leq q,\ 1\leq i\leq n]]\Big/\Big(\text{the coefficients of }\textstyle\prod_{i=1}^n(X-\tilde{u}_{j,i}-Z_{j,i})-\chi_{D^{\mathrm{u}}}(\sigma_j)(X),\ 1\leq j\leq q\Big),$$

with $u_{j,i}\in C_0^{\mathrm{sp}}$ the class of $\tilde{u}_{j,i}+Z_{j,i}$; it lies in $\mathrm{CNL}_{\mathcal{O}}$ (the relations reduce to 0 modulo $(\mathfrak{m}_B,z,Z_{j,i})$, since $\tilde{u}_{j,i}\equiv\bar{\chi}_i(\sigma_j)$). Then $C^{\mathrm{sp}}=\mathcal{O}_1\otimes_{\mathcal{O}}C_0^{\mathrm{sp}}$, with $u\colon B\to C^{\mathrm{sp}}$, $u(b)=1\otimes b$, is the eigenvalue algebra of $(\mathcal{O}_1,(\sigma_j)_{1\leq j\leq q},(\tilde{u}_{j,i}))$ in the sense of Definition E.31, the **eigenvalue**

algebra of the data (the congruence $\chi_{D^{\mathrm{u}}}(\sigma_j) \equiv \prod_i (X - \tilde{u}_{j,i})$ modulo $\mathfrak{m}_B$ required there holds, as D^{u} reduces to $\bar{D}_n$ and $\tilde{u}_{j,i} \equiv \bar{\chi}_i(\sigma_j)$).

The ring and the Iwasawa map. Let $t^{\mathrm{u}} = (D^{\mathrm{t}}, (\psi^{\mathrm{t}}_{v,a}), s^{\mathrm{t}}, (U^{\mathrm{t}}_{j,i}), (y^{\mathrm{t}}_{j,i}))$ be the universal patched tuple to R_{tup} (formed with $\mathcal{D}$, the limit objects above and the $\tilde{u}_{j,i}$), and $R = \mathcal{O}_1 \otimes_{\mathcal{O}} R_{\mathrm{tup}}$. Let $\Lambda = \mathcal{O}_1[[X_0, X_1, \dots, X_D]]$, $S = \Lambda[[y_{j,i} : 1 \le j \le q,\ 1 \le i \le n]]$, and $\beta \colon S \to R$ the continuous homomorphism of $\mathcal{O}_1$-algebras with

$$\beta(X_0) = s^{\mathrm{t}}, \qquad \beta(X_k) = c_k \psi^{\mathrm{t}}_{v_k,a_k}(\gamma_k) - 1 \ (1 \le k \le D), \qquad \beta(y_{j,i}) = y^{\mathrm{t}}_{j,i}.$$

The map to the eigenvalue algebra. The map $\pi \colon R \to C^{\mathrm{sp}}$ is $\mathrm{id}_{\mathcal{O}_1} \otimes \pi_0$, where $\pi_0 \colon R_{\mathrm{tup}} \to C^{\mathrm{sp}}_0$ is the morphism of $\mathrm{CNL}_{\mathcal{O}}$ carrying t^{u} to the patched tuple $(\bar{D}^{\mathrm{u}} \circ \bar{\theta} \circ \pi_\infty|_{P^0}, (\psi^{\mathrm{d}}_{v,a}), z, (u_{j,i}), (0))$ (determinant and characters base changed along $B \to C^{\mathrm{sp}}_0$; the characters take the wild values on μ_v by (Od3) of Definition 50.2 for $(D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$).

Relation ideal. With D^{t} extended to R, the **relations** are the following elements of R, for $h \in P^0$:

(Wv1) (kernel) $\mathrm{tr}_{D^{\mathrm{t}}}(h(k-1))$ for $k \in \ker \pi_\infty$;

(Wv2) (polarisation) $\Lambda_i(D^{\mathrm{t}})(\tilde{c} g^{-1} \tilde{c}^{-1}) - \Lambda_i(D^{\mathrm{t}})(g)$ for $g \in P^0$ and $1 \le i \le n$;

(Wv3) (ordinary condition) $\mathrm{tr}_{D^{\mathrm{t}}}(h \prod_{a=1}^{n} (g'_a - \psi^{\mathrm{t}}_{v,a}(g_a)))$ (product with a increasing) for $v \in S_\ell$, $g_1, \dots, g_n \in D_v$ and $g'_a \in P^0$ with $\pi_\infty(g'_a) = F_v(g_a)$;

(Wv4) (inertia at the Steinberg places) $\mathrm{tr}_{D^{\mathrm{t}}}(h(g' - 1))$ for $u \in \mathrm{St}$, $g \in D_u \cap I_{\bar{u}} \cap G_L$ with $\bar{\chi}_a(g) = 1$ for $1 \le a \le n$, and $g' \in P^0$ with $\pi_\infty(g') = F_u(g)$;

(Wv5) (first Taylor–Wiles relation) $\mathrm{tr}_{D^{\mathrm{t}}}(h\, q_1(\phi, U^{\mathrm{t}}_j))$ for $1 \le j \le q$ and $\phi \in P^0$ with $\pi_\infty(\phi) = \Phi_j$, where $U^{\mathrm{t}}_j = (U^{\mathrm{t}}_{j,i})_i$;

(Wv6) (second Taylor–Wiles relation) $\mathrm{tr}_{D^{\mathrm{t}}}(h\, q_2(\phi, \tau, U^{\mathrm{t}}_j, i, y^{\mathrm{t}}_{j,i}))$ for $1 \le j \le q$, $1 \le i \le n$ and $\phi, \tau \in P^0$ with $\pi_\infty(\phi) = \Phi_j$, $\pi_\infty(\tau) = \mathrm{T}_j$.

The **relation ideal** $I \subseteq R$ is the ideal generated by all relations, and the **isogeny element** is $f = f_{(1,\dots,1)}(U^{\mathrm{t}}) = \prod_{j=1}^{q} \prod_{i \ne i'} (U^{\mathrm{t}}_{j,i} - U^{\mathrm{t}}_{j,i'})^2 \in R$.

Modules. Put $X = C^{\mathrm{sp}} \otimes_B M$, an R-module through π. Let $\mathcal{N}_0$ be the set of $N \ge 1$ such that $\bar{h}_N(\pi_N(g))\bar{H} = \bar{\theta}(\pi_\infty(g))\bar{H}$ for all $g \in \Pi$; it belongs to $\mathcal{U}$ by (L1)(ii) of Lemma E.42. For $N \in \mathcal{N}_0$, π_N maps P^0 into the image $\bar{\Gamma}^0_N$ of Γ^0_N in $\bar{\Gamma}_N$, and $\rho_N \colon R \to \mathcal{O}_1 \otimes_{\mathcal{O}} R_N[[z]]$ is $\mathrm{id}_{\mathcal{O}_1} \otimes \rho^0_N$, where $\rho^0_N \colon R_{\mathrm{tup}} \to R_N[[z]]$ is the morphism of $\mathrm{CNL}_{\mathcal{O}}$ carrying t^{u} to the patched tuple $(\bar{D}_N \circ \pi_N|_{P^0}, (\psi_{N,v,a}), z, (U_{N,j,i}), (s_N(Y_{j,i})))$ (base changed along $R_N \to R_N[[z]]$; the characters take the wild values on μ_v by (Od3) for $(D_N, (\psi_{N,v,a}))$, as in Definition 53.9); the **level module** at N is $M^\natural_N = \mathcal{O}_1 \otimes_{\mathcal{O}} (R_N[[z]] \otimes_{R_N} M_N)$, an R-module through ρ_N. For $N \notin \mathcal{N}_0$ put $M^\natural_N = 0$.

The tuple $(R, \beta, X, f, I, e + 1, (M^\natural_N)_{N \ge 1}, \mathcal{U})$ over S, together with π, C^{sp} and $u \colon B \to C^{\mathrm{sp}}$, are the **patched tower data** of the automorphic patching data (for $\mathcal{O}_1$, the limit objects and $\tilde{c}$).

Definition E.38. Let $\mathscr{H}$ be a rank-n Hecke datum over $\mathcal{O}$ with $\ell \ne 2$, $x = (k, \rho, 1_n, (\psi_{v,a}), f)$ a framed normalised curve point of shape $(a, n-a)$ with frame $((w_v), (I^x_u))$, $A = k[[T]]$, $F = F_k$ its fraction field, $\kappa = (\kappa_1, \dots, \kappa_g)$ a Selmer basis of x with local witnesses, and $\mathbf{V} = (\Gamma_1, h, c_1, (\phi_{1,v}), (I_{1,u}))$ a level over $\mathscr{H}$. Write $\gamma^* = c_1 \gamma^{-1} c_1$ for $\gamma \in \Gamma^0_1$, and extend $\gamma \mapsto \gamma^*$ to a linear anti-involution of every group ring $R[\Gamma^0_1]$.

(Bf) Bounded maps over F. A map z from Γ^0_1 to $\mathrm{M}_n(F)$ or to $\mathrm{M}_n(\mathrm{PD}_g(F))$ is **bounded** if there is $s \ge 0$ such that all coordinates of all entries of $T^s z$ lie in A and are continuous for

the T-adic topology of A. For $\bullet \in \{\mathrm{ev}, \mathrm{od}, 1, 2\}$ let $\mathrm{Sel}^{\bullet,\mathrm{b}}_{\mathbf{V}}(F)$ be the set of bounded elements of $\mathrm{Sel}^{\bullet}_{\mathbf{V}}(F)$ (over F, the topology defined by $TF = F$ is the indiscrete one, so boundedness is the only continuity condition). A g-tuple $(z_1, \dots, z_g)$ of maps $\Gamma_1^0 \to \mathrm{M}_n(F)$ **lifts** if there is $\Psi \in \mathrm{Sel}^{2,\mathrm{b}}_{\mathbf{V}}(F)$ whose ϵ'_j-coefficient is z_j for every j.

(Vr) Realisations. Let $\mathcal{O}_1$ be a complete discrete valuation ring which is an $\mathcal{O}$-algebra, free of finite rank as an $\mathcal{O}$-module, with $\mathfrak{m}_{\mathcal{O}}\mathcal{O}_1 = \mathfrak{m}_{\mathcal{O}_1}$ (for instance $\mathcal{O}_1 = \mathcal{O}$). A **realisation of x over V with coefficient ring** $\mathcal{O}_1$ is a tuple $\mathfrak{R} = (B, D, (\hat{\psi}_{v,i}), (\phi_j, \tau_j, U_{j,i}, y_{j,i}), s, f_1, \hat{B}, \lambda, \mathcal{O}', \mathcal{A}, N, \nu)$ ($v \in S_\ell$, $1 \le i \le n$, $1 \le j \le q$ for some $q \ge 0$) such that:

(Vr1) *Global ring.* Γ_1^0 satisfies Mazur's condition Φ_ℓ (Theorem B.71); B is a complete Noetherian local $\mathcal{O}_1$-algebra with residue field finite over that of $\mathcal{O}_1$; D is a continuous determinant of dimension n on $B[\Gamma_1^0]$; $\hat{\psi}_{v,i}\colon D_v \to B^\times$ are continuous characters whose values on μ_v (Definition E.34) lie in the image of $\mathcal{O} \to B$; $f_1\colon B \to A$ is a surjective homomorphism of $\mathcal{O}$-algebras with $f_1 \circ D = \det \circ (\rho \circ h)$ and $f_1 \circ \hat{\psi}_{v,i} = \psi_{v,i}$; $s \in B$ with $f_1(s) = T$.

(Vr2) *Taylor–Wiles elements.* $\phi_j, \tau_j \in \Gamma_1^0$ with $\phi_j\tau_j = \tau_j\phi_j$, $h(\tau_j) = 1$; $U_{j,i}, y_{j,i} \in B$ with $f_1(y_{j,i}) = 0$, the $f_1(U_{j,1}), \dots, f_1(U_{j,n})$ pairwise distinct and $\det(X - \rho(h(\phi_j))) = \prod_i (X - f_1(U_{j,i}))$; and B is topologically generated as an $\mathcal{O}_1$-algebra by s, the $U_{j,i}$, $y_{j,i}$, the values $\hat{\psi}_{v,i}(g)$ and the coefficients of the characteristic polynomials $\chi_D(\gamma)$, $\gamma \in \Gamma_1^0$.

(Vr3) *Relation ideal.* $\mathcal{I}_1 \subseteq B[\Gamma_1^0]$ is the intersection over $k \ge 1$ and the open normal subgroups $H' \subseteq \Gamma_1^0$ of the two-sided ideals generated by: the Cayley–Hamilton ideal of D; $\mathfrak{m}_B^k \cdot 1$; $h' - 1$ ($h' \in H'$); and the following elements together with their images under *: $\prod_{i=1}^n (\phi_{1,v}(g_i) - \hat{\psi}_{v,i}(g_i))$ ($v \in S_\ell$, $g_i \in D_v$, $i = 1$ on the left); $\gamma - 1$ ($\gamma \in I_{1,u}$, $u \in \mathrm{St}$); $\prod_{i=1}^n (\phi_j - U_{j,i})$; $\prod_{k \ne i} (\phi_j - U_{j,k}) \cdot (\tau_j - 1 - y_{j,i})$ (products in increasing order of the index).

(Vr4) *Completion and coefficients.* $\hat{B}$ is the completion of $B_{\ker f_1}$ at its maximal ideal $\hat{\mathfrak{m}}$, and $\lambda\colon \hat{B} \to F$ is the ring homomorphism extending f_1, with $\ker\lambda = \hat{\mathfrak{m}}$. The ring $\mathcal{O}'$ is a complete discrete valuation ring with maximal ideal $\mathfrak{m}_{\mathcal{O}'}$ and a ring homomorphism $\iota\colon \mathcal{O}' \to \hat{B}$ making $\hat{B}$ a complete Noetherian local $\mathcal{O}'$-algebra, such that $\lambda \circ \iota$ is surjective onto F, the image of $\mathcal{O}_1 \to \hat{B}$ (hence that of $\mathcal{O}$) lies in $\iota(\mathcal{O}')$, the image of s lies in $\iota(\mathcal{O}')$, and every $a \in A$ is $f_1(b)$ for some $b \in B$ whose image in $\hat{B}$ lies in $\iota(\mathcal{O}')$.

(Vr5) *Coefficient algebra.* $\mathcal{A}$ is a commutative ring containing $\hat{B}$ as a subring, finitely generated as a $\hat{B}$-module, with a $\hat{B}$-submodule N such that $\mathcal{A} = \hat{B} \cdot 1 \oplus N$, $yy' \in \hat{\mathfrak{m}} \cdot 1$ for $y, y' \in N$, and $N = 0$ if $a = n$.

(Vr6) *The realisation.* $\nu\colon \Gamma_1^0 \to \mathrm{M}_n(\mathcal{A})$ is multiplicative with $\nu(1) = 1$; its B-linear extension to $B[\Gamma_1^0]$ (through $B \to \hat{B} \subseteq \mathcal{A}$) vanishes on $\mathcal{I}_1$; the $\hat{B}$-span of $\nu(\Gamma_1^0)$ is the set of matrices with entries in $\hat{B} \cdot 1$ at the positions in one block and in N at the other positions; $\nu(\gamma^*) = {}^t\nu(\gamma)$; the base change of D to $\mathcal{A}$ is $\det \circ \nu$; and if $\nu(\gamma)_{ij} - b \cdot 1 \in N$ with $b \in \hat{B}$, then $\lambda(b) = \rho(h(\gamma))_{ij}$.

(Vr7) *Thickness.* Every prime ideal Q of $\hat{B}$ such that $\kappa[\Gamma_1^0]$ modulo the two-sided ideal generated by the image of $\mathcal{I}_1$ is not semisimple, for some homomorphism $\hat{B} \to \kappa$ to a field with kernel Q, contains $\mathrm{Ann}_{\hat{B}}(N)$.

(Vr8) *Continuity.* There is a finitely generated B-submodule $\mathcal{P}_0 \subseteq \mathrm{M}_n(\mathcal{A})$ containing $\nu(\Gamma_1^0)$ such that for every $j \ge 1$ there is an open normal subgroup $V_j \subseteq \Gamma_1^0$ with $\nu(\gamma v) - \nu(\gamma) \in \mathfrak{m}_B^j \mathcal{P}_0$ for $\gamma \in \Gamma_1^0$, $v \in V_j$.

A **realisation of x over $\mathbf{V}$** is a realisation of x over $\mathbf{V}$ with coefficient ring some $\mathcal{O}_1$ as above.

(Tg) Tangent and test spaces. For a realisation put $\mathfrak{a} = \iota(\mathfrak{m}_{\mathcal{O}'})\hat{B} + \hat{\mathfrak{m}}^2 + \mathrm{ired}(N)$, where $\mathrm{ired}(N) \subseteq \hat{B}$ is the ideal generated by the b with $yy' = b \cdot 1$ for some $y, y' \in N$. For $y \in \mathcal{A}$ let $\mathrm{pr}_0(y) \in \hat{B}$ and $\mathrm{pr}_1(y) \in N$ be its components. Let

- Tp be the F-vector space of additive maps $\alpha\colon \hat{B} \to F$ with $\alpha(bb') = \lambda(b)\alpha(b') + \lambda(b')\alpha(b)$, $\alpha \circ \iota = 0$ and $\alpha(\mathrm{ired}(N)) = 0$;
- La be the F-vector space of additive maps $\mu\colon N \to F$ with $\mu(by) = \lambda(b)\mu(y)$ ($b \in \hat{B}$, $y \in N$);
- Ph be the F-vector space of additive maps $\Phi\colon N \to F$ with $\Phi(\iota(o)y) = \lambda(\iota(o))\Phi(y)$ ($o \in \mathcal{O}'$) and $\Phi(\mathfrak{a}N) = 0$;

and, for additive $\alpha\colon \hat{B} \to F$ and $\mu\colon N \to F$, the **first-order maps** $\Gamma_1^0 \to \mathrm{M}_n(F)$

$$\mathrm{al}(\alpha)(\gamma) = \big(\alpha(\mathrm{pr}_0(\nu(\gamma)_{ij}))\big)_{ij}\,\rho(h(\gamma))^{-1}, \qquad \mathrm{la}(\mu)(\gamma) = \big(\mu(\mathrm{pr}_1(\nu(\gamma)_{ij}))\big)_{ij}\,\rho(h(\gamma))^{-1}.$$

A family $(m_1, \dots, m_g)$ in N is **dual to** κ if $\mathrm{la}(\mu) - \sum_j \mu(m_j)\,(\kappa_j \circ h) \in \mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}}(F)$ for every $\mu \in \mathrm{La}$.

Proposition E.39. *Let $\mathscr{H} = (n, L, c, (\bar{v})_{v\in S_\ell}, \mathrm{St}, v_0, \bar{v}_0, (\bar{\chi}_a)_{1\le a\le n})$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}'$, $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, and $J \neq B_n^{\mathrm{po}}$ an ideal of B_n^{po} which has classical levels. Let $h \ge 0$, and $x_0, \dots, x_h$ patchable curve points of B_n^{po} (for v_0, with the place $\bar{v}_0$) with homomorphisms $f_i\colon B_n^{\mathrm{po}} \to A_{k_i}$, either all of irreducible type or all of endoscopic type. Then there are curve points $x'_0, \dots, x'_h$ of B_n^{po} with homomorphisms $f'_i\colon B_n^{\mathrm{po}} \to A_{k'_i}$, over finite fields $k'_0, \dots, k'_h$ of one common cardinality, such that $\ker f'_i = \ker f_i$ for $0 \le i \le h$, and a patching system for J along $(x'_0, \dots, x'_h)$.*

Proof. Write $\mathcal{O} = \mathcal{O}'$, $B = B_n^{\mathrm{po}}$, $\mathbb{F}_{\mathcal{O}}$ for the residue field of $\mathcal{O}$, $\delta = n[L^+ : \mathbb{Q}]$ (written D in Definition E.37), and $x_i = (k_i, \rho_i, P_i, (\psi_{i,v,b}), f_i)$. Recall ℓ is odd.

Step 1: normalised points. Fix i. If x_i is of irreducible type, Proposition E.54 gives a finite field $k''_i \supseteq k_i$, an injective local homomorphism of $\mathcal{O}$-algebras $j_i\colon A_{k_i} \to A_{k''_i}$ making $A_{k''_i}$ a finite A_{k_i}-module, and a framed curve point $y_i = (k''_i, \rho''_i, 1_n, (\psi''_{i,v,b}), g_i)$, normalised of shape $(a_i, n - a_i)$ with $a_i = n$ by (Nm4); if x_i is of endoscopic type, Proposition E.53 gives the same with $1 \le a_i < n$. In both cases (Nm1)–(Nm3) hold: $g_i = j_i \circ f_i$ and $\det \circ \rho''_i = j_i \circ \det \circ \rho_i$; y_i satisfies (Pj1), (Np3) and (Np4); and y_i is spanning.

Step 2: Taylor–Wiles elements, the datum, and the extended points. By Proposition E.48, $\Gamma_{\mathscr{H}}$ satisfies Φ_ℓ. The points $y_0, \dots, y_h$ are spanning framed curve points, each normalised of some shape and satisfying (Np3); so Proposition E.50 gives $q \ge 0$, $\sigma_1, \dots, \sigma_q \in \Gamma'$ and, for each i, an extension $x'_i = (k'_i, \rho'_i, 1_n, (\psi'_{i,v,b}), f'_i)$ of y_i along some $j'_i\colon A_{k''_i} \to A_{k'_i}$, with $|k'_0| = \cdots = |k'_h|$, a Selmer basis κ'_i of x'_i such that $\{\sigma_1, \dots, \sigma_q\}$ detects (x'_i, κ'_i), and pairwise distinct $\alpha_{i,j,1}, \dots, \alpha_{i,j,n} \in A_{k'_i}$ with $\det(X - \rho'_i(\sigma_j)) = \prod_b (X - \alpha_{i,j,b})$. By Proposition E.48 there is a Taylor–Wiles datum $\mathcal{D}$ of size q for $\mathscr{H}$ with elements $\sigma_1, \dots, \sigma_q$; let H be its open normal subgroup. Then $(\alpha_{i,j,b})$ are eigenvalue data of x'_i along $\mathcal{D}$. Put $\lambda_i = j'_i \circ j_i\colon A_{k_i} \to A_{k'_i}$, an injective local homomorphism of $\mathcal{O}$-algebras making $A_{k'_i}$ a finite A_{k_i}-module. Then $f'_i = j'_i \circ g_i = \lambda_i \circ f_i$, so $\ker f'_i = \ker f_i$ as λ_i is injective. By the definition of an extension, x'_i is a spanning framed curve point normalised of shape $(a_i, n - a_i)$, and by (Et2) of Lemma E.41 it satisfies (Pj1) and (Np4), with $\det \circ \rho'_i = j'_i \circ \det \circ \rho''_i = \lambda_i \circ \det \circ \rho_i$. By (Et1) of Lemma E.41 (applied to x_i, λ_i and x'_i; x_i is patchable, and $a_i = n$ for irreducible type, $a_i < n$ for endoscopic type), x'_i is a patchable curve point of irreducible type if $a_i = n$ and of endoscopic type if $a_i < n$. Each $\kappa'_{i,j} \in Z^1_{\mathrm{ord}}(x'_i)$

has, for every $v \in S_\ell$, a matrix Y_v as in (Sd2); choosing these, κ'_i is a Selmer basis with local witnesses.

Step 3: $\rho'_i(\gamma) = 1$ for $\gamma \in \Gamma^0_{\mathscr{H}} \cap \mathrm{PL}(H)$. Fix i, write $A = A_{k'_i}$ and $\bar{c} = \pi_{\mathscr{H}}(c)$, so $\bar{c}^2 = 1$. The preimage under f'_i of the maximal ideal TA is a prime ideal of B with $B/f_i'^{-1}(TA)$ a subring of the finite field k'_i, hence a field; so it is the maximal ideal $\mathfrak{m}_B$, and $f'_i(\mathfrak{m}_B) \subseteq TA$. As $f'_i \circ D^{\mathrm{u}} = \det \circ \rho'_i$ and D^{u} reduces to $\det \circ \mathrm{diag}(\bar{\chi}_1, \dots, \bar{\chi}_n)$ modulo $\mathfrak{m}_B$, the characteristic polynomial of $\rho'_i(\gamma)$ modulo T is $\prod_a (X - \bar{\chi}_a(\gamma))$. For $h \in H$, (Tg1) gives $h \in \Gamma^0_{\mathscr{H}}$ and $\bar{\chi}_a(h) = 1$, so $\rho'_i(h)$ modulo T is unipotent; with $\ell^w \geq n$, $(\rho'_i(h) \bmod T)^{\ell^w} - 1 = (\rho'_i(h) \bmod T - 1)^{\ell^w} = 0$ in characteristic ℓ. For $m \geq 1$ let K_m be the set of $\gamma \in \Gamma^0_{\mathscr{H}}$ with $\rho'_i(\gamma) \equiv 1$ modulo T^m: an open normal subgroup of $\Gamma^0_{\mathscr{H}}$ (ρ'_i is continuous and $\mathrm{GL}_n(A/T^m A)$ is finite), and normal in $\Gamma_{\mathscr{H}} = \Gamma^0_{\mathscr{H}} \cup \bar{c}\Gamma^0_{\mathscr{H}}$, since for $\gamma \in K_m$, $\bar{c}\gamma\bar{c} = (\gamma^{-1})^*$ and $\rho'_i((\gamma^{-1})^*) = {}^t\rho'_i(\gamma^{-1}) \equiv 1$ (x'_i is normalised). For $h \in H$, $\rho'_i(h)^{\ell^w}$ lies modulo T^m in the kernel of $\mathrm{GL}_n(A/T^m A) \to \mathrm{GL}_n(k'_i)$, a group of order $|k'_i|^{n^2(m-1)}$, a power of ℓ; so $h^{\ell^{w'}} \in K_m$ for some w'. By the definition of PL (Definition 53.8, with $G = \Gamma_{\mathscr{H}}$), $\mathrm{PL}(H) \subseteq K_m$ for every m; so $\rho'_i(\gamma) \equiv 1$ modulo every T^m, i.e. $\rho'_i(\gamma) = 1$, for $\gamma \in \Gamma^0_{\mathscr{H}} \cap \mathrm{PL}(H)$.

Step 4: the patched tower. J has classical levels, so Proposition E.136 gives automorphic patching data for J along $\mathcal{D}$ with exponent $e = n!$, with elements $\tilde{u}_{j,i} \in \mathcal{O}$. As k'_0 contains $\mathbb{F}_{\mathcal{O}}$, $|k'_0| = |\mathbb{F}_{\mathcal{O}}|^r$ for some $r \geq 1$; let $\mathcal{O}_1$ be an unramified extension of $\mathcal{O}$ of degree r: a complete discrete valuation ring, free of rank r over $\mathcal{O}$, with $\mathfrak{m}_{\mathcal{O}}\mathcal{O}_1 = \mathfrak{m}_{\mathcal{O}_1}$ and residue field of cardinality $|k'_i|$ for every i. Let $(\Pi, (\pi_N), \mathcal{U}, \bar{\theta}, c_\infty, (\Phi_j), (\mathrm{T}_j), (F_w))$ be limit objects for $\mathcal{D}$ as in Lemma E.42, fix $\tilde{c} \in \Pi$ with $\pi_\infty(\tilde{c}) = c_\infty$, and let $(R, \beta, X, f, I, e+1, (M_N^\natural), \mathcal{U})$ over $S = \Lambda[[y_{j,i}]]$, C^{sp}, u and π be the patched tower data (Definition E.37). By Proposition E.57, $\mathcal{T} = (R, \beta, X, f, I, e+1, (M_N^\natural), \mathcal{U})$ is an abstract patching tower over S with $s = nq$ patching variables $y_{j,i}$. By Definition E.37, C^{sp} with u is the eigenvalue algebra of $(\mathcal{O}_1, (\sigma_j), (\tilde{u}_{j,i}))$, and the residue field of $\mathcal{O}_1$ has cardinality $|k'_i|$ for every i. By Lemma E.40, B is topologically generated as an $\mathcal{O}$-algebra by the coefficients of the $\chi_{D^{\mathrm{u}}}(\gamma)$, $\gamma \in \Gamma^0_{\mathscr{H}}$, and the values $\psi^{\mathrm{d}}_{v,a}(g)$; so Proposition E.56 and Proposition E.134 apply to these data. Put $d = 1 + \delta$. We show that $(\mathcal{T}, d, C^{\mathrm{sp}}, u, \pi)$ is a patching system for J along $(x'_0, \dots, x'_h)$; the curve points x'_i have homomorphisms $f'_i \colon B \to A_{k'_i}$ over fields of one cardinality with $\ker f'_i = \ker f_i$ (Step 2), which then proves the statement.

Step 5: (Sys0)–(Sys3). π is surjective and $\pi(\beta(y_{j,i})) = 0$ for all j, i by (Wz1) of Proposition E.56: this is (Sys1). (Sys2) is (Wz2). (Sys0): for every i and j, applying f'_i to the coefficients of $\chi_{D^{\mathrm{u}}}(\sigma_j)$ gives $\chi_{f'_i \circ D^{\mathrm{u}}}(\sigma_j) = \chi_{\det \circ \rho'_i}(\sigma_j) = \det(X - \rho'_i(\sigma_j)) = \prod_b (X - \alpha_{i,j,b})$, a product of monic polynomials of degree 1 in $A_{k'_i}[X]$. (Sys3): let f' be a surjective lift of f'_i, i.e. a surjective ring homomorphism $f' \colon C^{\mathrm{sp}} \to A_{k'_i}$ with $f' \circ u = f'_i$ and $f'(z) = T$. The curve point x'_i has eigenvalue data along $\mathcal{D}$ and $|k'_i|$ is the residue cardinality of $\mathcal{O}_1$; so (Wz3) gives, for $\mathfrak{q} = \pi^{-1}(\ker f')$, $f \notin \mathfrak{q}$ and $\Lambda_{\mathfrak{p}}$ regular local of dimension $1 + \delta = d$ for $\mathfrak{p} = \{a \in \Lambda : \beta(a) \in \mathfrak{q}\}$.

Step 6: (Sys4). Fix i and a surjective lift f' of f'_i. We apply Corollary E.66 to the tuple $(\mathcal{T}, d, C^{\mathrm{sp}}, u, \pi)$ (which has the form required there), i, f', the point $x = x'_i$ of shape $(a_i, n - a_i)$ over $A = A_{k'_i}$, $\kappa = \kappa'_i$ and $\mathcal{D}$, and check its hypotheses. (i) $\pi(I) = 0$ by (Wz1); the $\bar{\chi}_a$ are pairwise distinct by (Rd4). (ii) x'_i is of irreducible type if $a_i = n$ and of endoscopic type if $a_i < n$ (Step 2). (iii) The hypotheses of Proposition E.97 (those of Proposition E.119 and more): $\mathcal{O}$ is a discrete valuation ring; x'_i is a framed curve point normalised of shape $(a_i, n - a_i)$, κ'_i a Selmer basis with local witnesses, and x'_i has eigenvalue data along $\mathcal{D}$ (Step 2); $\rho'_i(\gamma) = 1$ for $\gamma \in \Gamma^0_{\mathscr{H}} \cap \mathrm{PL}(H)$ (Step 3); x'_i satisfies (Pj1) and (Np4) and is spanning (Step 2); $\{\sigma_1, \dots, \sigma_q\}$ detects (x'_i, κ'_i) (Step 2). (iv) $\delta + nq \leq d + s - 1$: here $d + s - 1 = \delta + nq$. (v) The limit objects of Step 4 are as in the conclusion of Lemma E.42; let $\mathbf{V}_\infty$ be the limit level formed from them. (vi) Data on R/I: by Proposition E.134 (its hypotheses: $\ell \neq 2$; the data of Step 4; B topologically generated; x'_i

framed normalised with $|k_i'|$ the residue cardinality of $\mathcal{O}_1$ and with eigenvalue data along $\mathcal{D}$; κ_i' with local witnesses satisfying all hypotheses of Proposition E.97 by (iii); the surjective lift f'), R/I carries D, $(\hat{\psi}_{v,i})$, $(\phi_j, \tau_j, U_{j,i}, y_{j,i})$ and s such that, with $f_1 = f' \circ \bar{\pi}$, the clauses of (Vr1), (Vr2) concerning them hold over $\mathbf{V}_\infty$ for $\mathcal{O}_1$ (In2), and (Lv1), (Lv2) hold over $\mathbf{V}_\infty$ (In3). (vii) (Gl1), (Gl2) for every realisation $\mathfrak{R}$ of x_i' over $\mathbf{V}_\infty$ with coefficient ring $\mathcal{O}_1$ and these data (global ring R/I, determinant D, characters, Taylor–Wiles elements and s of (vi)): if $a_i < n$ and $\hat{Z}'$ is a prime of $\hat{B}$ whose preimage in R is stable for the system, i.e. contained in $\pi^{-1}(P'')$ for a prime P'' of C^{sp} with $u^{-1}(P'')$ stable, then $\mathrm{ired}(N) \not\subseteq \hat{Z}'$ by Lemma E.46: this is (Gl1). For a prime $\hat{Q}'$ of $\hat{B}$ whose preimage in R is $\pi^{-1}(P')$, Lemma E.133 gives (Tk1), which is the first half of (Gl2), and (Tk2), which gives the second half with κ an algebraic closure of the fraction field of $\hat{B}/\hat{Q}'$ and the natural homomorphism $\hat{B} \to \kappa$ with kernel $\hat{Q}'$. So Corollary E.66 gives the conclusion of (Sys4) for i and f'. As i and f' were arbitrary, (Sys4) holds, and $(\mathcal{T}, d, C^{\mathrm{sp}}, u, \pi)$ is a patching system for J along $(x_0', \dots, x_h')$. □

Lemma E.40. *Let $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ be a universal polarised ordinary determinant of rank n over a residual setting $(n, L, c, \Gamma, \Gamma^0, \pi, (D_v \supseteq I_v)_{v\in S_\ell}, (\bar{\chi}_a))$ over a coefficient ring $\mathcal{O}$, $B = B_n^{\mathrm{po}}$, and let $A \subseteq B$ be the $\mathcal{O}$-subalgebra generated by the elements $\Lambda_i(D^{\mathrm{u}})(\gamma)$ ($\gamma \in \Gamma^0$, $1 \le i \le n$) and $\psi_{v,a}^{\mathrm{d}}(g)$ ($v \in S_\ell$, $1 \le a \le n$, $g \in D_v$). Then $B = A + \mathfrak{m}_B^N$ for every $N \ge 1$; that is, B is topologically generated as an $\mathcal{O}$-algebra by these elements.*

Proof. Write $F = \mathbb{F}_{\mathcal{O}}$, $\mathfrak{m} = \mathfrak{m}_B$, $r\colon B \to F$ the residue map, and ϖ a uniformiser of $\mathcal{O}$. Every $b \in B$ has $r(b) = r(\tilde{b})$ for some $\tilde{b} \in \mathcal{O}$ (the residue field of B is F); so $B = \mathcal{O} + \mathfrak{m}$, and in particular $B = A + \mathfrak{m}$. Let $T \subseteq \mathfrak{m}$ be the set of elements $a - \tilde{a}$ with a one of the generators of A listed in the statement and $\tilde{a} \in \mathcal{O}$ with $r(a) = r(\tilde{a})$, together with $\varpi \cdot 1$. Then $T \subseteq A \cap \mathfrak{m}$.

Step 1: $\mathfrak{m} = TB + \mathfrak{m}^2$. Put $\mathfrak{n} = TB + \mathfrak{m}^2 \subseteq \mathfrak{m}$, an ideal containing ϖB, so $\mathfrak{m}/\mathfrak{n}$ is an F-vector space. Suppose $\mathfrak{n} \neq \mathfrak{m}$ and choose an F-linear $\delta_0\colon \mathfrak{m}/\mathfrak{n} \to F$ which is not zero; let $\delta\colon \mathfrak{m} \to F$ be its composite with the projection. Let $F[\epsilon] = F \oplus F\epsilon$, $\epsilon^2 = 0$, an object of $\mathrm{CNL}_{\mathcal{O}}$. Define $\phi_0, \phi_\delta\colon B \to F[\epsilon]$ by $\phi_0(b) = r(b)$ and $\phi_\delta(b) = r(b) + \delta(b - \tilde{b})\epsilon$, where $\tilde{b} \in \mathcal{O}$ is any element with $r(\tilde{b}) = r(b)$. This does not depend on $\tilde{b}$: two choices differ by an element of $\mathfrak{m}_{\mathcal{O}} = \varpi\mathcal{O}$, whose image lies in $\varpi B \subseteq \mathfrak{n}$. The map ϕ_δ is additive and $\mathcal{O}$-linear (δ is $\mathcal{O}$-linear, $\mathcal{O}$ acting on F through its residue field, since $\varpi\mathfrak{m} \subseteq \mathfrak{n}$), $\phi_\delta(1) = 1$, and it is multiplicative:

Both are local homomorphisms of $\mathcal{O}$-algebras, i.e. morphisms of $\mathrm{CNL}_{\mathcal{O}}$, and they differ: for $b \in \mathfrak{m}$ with $\delta(b) \neq 0$ (take $\tilde{b} = 0$), $\phi_\delta(b) = \delta(b)\epsilon \neq 0 = \phi_0(b)$. For $\phi \in \{\phi_0, \phi_\delta\}$ the base change $(\phi \circ D^{\mathrm{u}}, (\phi \circ \psi_{v,a}^{\mathrm{d}}))$ is a polarised ordinary determinant of rank n to $F[\epsilon]$: continuity, the residual conditions and the identities (Od1)–(Od3) are preserved by base change along a morphism of $\mathrm{CNL}_{\mathcal{O}}$ (the characteristic polynomial and the trace of the base change are the images of those of D^{u}, and ϕ is an $\mathcal{O}$-algebra map). On the generators of A the two maps agree: for a a generator, $\phi_0(a) = r(a)$ and $\phi_\delta(a) = r(a) + \delta(a - \tilde{a})\epsilon = r(a)$, as $a - \tilde{a} \in T \subseteq \ker\delta$. Hence $\chi_{\phi_0 \circ D^{\mathrm{u}}}(\gamma) = \chi_{\phi_\delta \circ D^{\mathrm{u}}}(\gamma)$ for all $\gamma \in \Gamma^0$ (their coefficients are the images of the $\Lambda_i(D^{\mathrm{u}})(\gamma)$), so $\phi_0 \circ D^{\mathrm{u}} = \phi_\delta \circ D^{\mathrm{u}}$ by (Ch1) of Theorem B.83; and $\phi_0 \circ \psi_{v,a}^{\mathrm{d}} = \phi_\delta \circ \psi_{v,a}^{\mathrm{d}}$. By the uniqueness in the universal property of B (Definition 50.2), $\phi_0 = \phi_\delta$, a contradiction. So $\mathfrak{m} = TB + \mathfrak{m}^2$.

Step 2: approximation. We show $B = A + \mathfrak{m}^N$ by induction on N; $N = 1$ was shown. Assume $B = A + \mathfrak{m}^N$. The ideal $\mathfrak{m}^N$ is generated by the products $b_1 \cdots b_N$ with $b_k \in \mathfrak{m}$; writing each $b_k \in TB + \mathfrak{m}^2$ and expanding, $\mathfrak{m}^N \subseteq T^N B + \mathfrak{m}^{N+1}$, where T^N is the set of products of N elements of T. For $t \in T^N$ and $b \in B = A + \mathfrak{m}$, write $b = a + m$; then $tb = ta + tm$ with $ta \in A$ (as $T \subseteq A$) and $tm \in \mathfrak{m}^{N+1}$ (as $T \subseteq \mathfrak{m}$). So $\mathfrak{m}^N \subseteq A + \mathfrak{m}^{N+1}$ and $B = A + \mathfrak{m}^N \subseteq A + \mathfrak{m}^{N+1}$. □

Lemma E.41. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ and B the ring of a universal polarised ordinary determinant over $\mathscr{H}$.*

(Et1) *Let $x = (k, \rho, P, (\psi_{v,b}), f)$ be a patchable curve point of B, $j\colon A_k \to A_{k'}$ an injective local homomorphism of $\mathcal{O}$-algebras making $A_{k'}$ a finite A_k-module, and $x' = (k', \rho', 1_n, (\psi'_{v,b}), f')$ a curve point of B, normalised of shape $(a, n-a)$, satisfying (Pj1) and (Np4), with $\det\circ\rho' = j\circ\det\circ\rho$ as determinants on $\Gamma^0_{\mathscr{H}}$. If x is of irreducible type and $a = n$, then x' is a patchable curve point of irreducible type. If x is of endoscopic type and $a < n$, then x' is a patchable curve point of endoscopic type.*

(Et2) *Let x' be a framed curve point of B, normalised of some shape, satisfying (Pj1) and (Np4). Then every extension of x' satisfies (Pj1) and (Np4), and $\det\circ\rho'' = j'\circ\det\circ\rho'$ for the extension x'' along j', with representation ρ''.*

Proof. (Et1), patchability. x' is a curve point satisfying (Pj1) by hypothesis. As j is injective and local between the domains A_k, $A_{k'}$, it extends to an embedding $F_k \to F_{k'}$, again written j; as $A_{k'}$ is a finite A_k-module, $F_{k'}$ is a finite extension of $j(F_k)$. For $\gamma \in \Gamma^0_{\mathscr{H}}$ the characteristic polynomial of $\rho'(\gamma)$ is the image under j of that of $\rho(\gamma)$, as $\det\circ\rho' = j\circ\det\circ\rho$. (Pj2): an open subset of $\Gamma^0_{\mathscr{H}}$ meeting Γ' contains, by (Pj2) for x, some $\sigma \in \Gamma'$ whose characteristic polynomial for ρ is separable over F_k, i.e. has non-zero discriminant; its image under the field embedding j has the image discriminant, non-zero, so the characteristic polynomial of $\rho'(\sigma)$ is separable over $F_{k'}$. (Pj3): if $X \in \mathrm{M}_n(F_{k'})$ satisfies $\chi_{\mathrm{cyc}}(\gamma)\rho'(\gamma)X = X\rho'(\gamma)$ for $\gamma \in \Delta_0$, then for s large $T^sX \in \mathrm{M}_n(A_{k'})$ satisfies the same identities, so $T^sX = 0$ by (Np4) and $X = 0$.

(Et1), types. Let Ω be an algebraic closure of $F_{k'}$; since $F_{k'}$ is algebraic over $j(F_k)$, Ω is also an algebraic closure of F_k (through j), and we use it as $\bar{F}_k$ and $\bar{F}_{k'}$. Put $V = \rho \otimes \Omega$ and $V' = \rho' \otimes \Omega$, restricted to Γ'. Their determinants on Γ' agree. The direct sum of the Jordan–Hölder constituents of a finite-dimensional representation (its semisimplification) has the same determinant (in a basis adapted to a composition series the matrices are block upper triangular); by the uniqueness in (Ch3), applied to the group Γ', the semisimplifications of V and V' are isomorphic, i.e. V and V' have the same Jordan–Hölder constituents with multiplicities. Irreducible type ($a = n$): V is irreducible, so V' has exactly one constituent and is irreducible; x' is of irreducible type. Endoscopic type ($a < n$): let Q, r_1, r_2, P_1, P_2 be as in the definition for x. Then $V \cong (r_1 \oplus r_2) \otimes \Omega|_{\Gamma'}$ has exactly two constituents $r_1 \otimes \Omega|_{\Gamma'}$, $r_2 \otimes \Omega|_{\Gamma'}$, and these are not isomorphic: an isomorphism would be a non-zero $Y \in \mathrm{M}_{b\times a}(\Omega)$ (with $a = b$) such that $Yr_1(\sigma) = r_2(\sigma)Y$ for $\sigma \in \Gamma'$, which the definition excludes. As x' is normalised of shape $(a, n-a)$ with $a < n$, $\rho' = \mathrm{diag}(r'_1, r'_2)$ with blocks r'_1, r'_2 of sizes $a, n-a \geq 1$, and the constituents of V' are those of $r'_1 \otimes \Omega|_{\Gamma'}$ together with those of $r'_2 \otimes \Omega|_{\Gamma'}$. Each of the two contributes at least one and together there are exactly two; so $r'_1 \otimes \Omega|_{\Gamma'}$ and $r'_2 \otimes \Omega|_{\Gamma'}$ are irreducible and are the two non-isomorphic constituents. By Schur's lemma, 0 is the only $Y \in \mathrm{M}_{(n-a)\times a}(\Omega)$ with $Yr'_1(\sigma) = r'_2(\sigma)Y$ for all $\sigma \in \Gamma'$. Since x' is normalised, $\rho'(\gamma^*) = {}^t\rho'(\gamma)$, so $r'_i(\gamma^*) = {}^t r'_i(\gamma)$, which is the polarisation condition with $P_i = 1$; with $Q = 1_n$ the definition of endoscopic type holds for x' (over $F_{k'}$).

(Et2). Let x'' be the extension of $x' = (k', \rho', 1_n, (\psi'_{v,b}), f')$ along $j'\colon A_{k'} \to A_{k''}$, so $\rho'' = j'\circ\rho'$, $\psi''_{v,b} = j' \circ \psi'_{v,b}$, and $\det\circ\rho'' = j' \circ \det\circ\rho'$. (Pj1): j' is an injective homomorphism of $\mathcal{O}$-algebras and $\chi_{\mathrm{cyc}}|_{D_v}$ takes values in $\mathcal{O}^\times$, so $j' \circ \psi'_{v,a} \neq j' \circ \psi'_{v,b}$ and $j' \circ \psi'_{v,a} \neq (j' \circ \psi'_{v,b})\chi_{\mathrm{cyc}}|_{D_v}$ whenever the same holds without j'. (Np4): $A_{k''}$ is a finitely generated module over the discrete valuation ring $A_{k'}$ (through j'), torsion-free as j' is injective and $A_{k''}$ is a domain; so it is free, with a basis $e_1, \ldots, e_m$. Write $Z \in \mathrm{M}_n(A_{k''})$ as $Z = \sum_r j'(Z_r)e_r$ with $Z_r \in \mathrm{M}_n(A_{k'})$. Since $\rho''(\gamma) = j'(\rho'(\gamma))$ and $\chi_{\mathrm{cyc}}(\gamma) \in \mathcal{O}^\times$, the identity $\chi_{\mathrm{cyc}}(\gamma)\rho''(\gamma)Z = Z\rho''(\gamma)$ for $\gamma \in \Delta_0$ holds if and only if $\chi_{\mathrm{cyc}}(\gamma)\rho'(\gamma)Z_r = Z_r\rho'(\gamma)$ for every r; by (Np4) for x' all $Z_r = 0$, so $Z = 0$. □

Lemma E.42. *Let a Taylor–Wiles datum of size q for a rank-n Hecke datum $\mathscr{H}$ be given.*

Put $\bar\Gamma = \Gamma_{\mathscr H}/\mathrm{PL}(H)$, $\bar\Gamma_N = \Gamma_N/\mathrm{PL}(H_N)$, *let* $\bar h_N\colon \bar\Gamma_N \to \bar\Gamma$ *be induced by* h_N, *and write* $\bar x$ *for images in these quotients. For* $w \in S_\ell \cup \mathrm{St}$ *let* D_w *be the decomposition group* ($D_{\bar u} \cap G_{L^+}$ *for* $u \in \mathrm{St}$) *and* $\bar\phi_{N,w}\colon D_w \to \bar\Gamma_N$ *the map induced by* π^{Q_N}. *Then there are a topologically finitely generated profinite group* Π, *continuous surjections* $\pi_N\colon \Pi \to \bar\Gamma_N$, *a non-principal ultrafilter* $\mathcal U$ *on* $\mathbb N$ *(giving* $\bar\Gamma_\infty = \Pi/K_\infty$ *and* π_∞ *as in Lemma E.43), a continuous surjection* $\bar\theta\colon \bar\Gamma_\infty \to \bar\Gamma$, *elements* $c_\infty, \Phi_j, \mathrm{T}_j \in \bar\Gamma_\infty$ *and continuous homomorphisms* $F_w\colon D_w \to \bar\Gamma_\infty$ *such that:*

(L1) *for every open normal subgroup* V *of* Π *and every open normal subgroup* Y *of* $\bar\Gamma$, *for* $\mathcal U$*-many* N: *(i)* $(\ker\pi_N)V = (\ker\pi_\infty)V$; *(ii)* $\bar h_N(\pi_N(g)) \equiv \bar\theta(\pi_\infty(g)) \bmod Y$ *for all* $g \in \Pi$; *(iii) there are* $\tilde c, \tilde\phi_j, \tilde t_j \in \Pi$ *with images* $c_\infty, \Phi_j, \mathrm{T}_j$ *under* π_∞ *and images in* $\bar c_N \pi_N(V)$, $\bar\Phi_{N,j}\pi_N(V)$, $\bar{\mathrm{T}}_{N,j}\pi_N(V)$ *under* π_N, *where* $c_N = \pi^{Q_N}(c)$; *(iv) for every* w *and* $y \in D_w$ *there are* $g, g' \in \Pi$ *with* $\pi_N(g) = \bar\phi_{N,w}(y)$, $\pi_\infty(g') = F_w(y)$ *and* $g^{-1}g' \in V$;

(L2) $c_\infty^2 = 1$, $\bar\theta(c_\infty) = \bar c$, $\bar\theta(\Phi_j) = \bar\sigma_j$, $\bar\theta(\mathrm{T}_j) = 1$, $\Phi_j\mathrm{T}_j = \mathrm{T}_j\Phi_j$, $\bar\theta \circ F_w = \overline{\pi_{\mathscr H}}|_{D_w}$, *and* $\ker\bar\theta$ *is the closed normal subgroup of* $\bar\Gamma_\infty$ *generated by* $\mathrm{T}_1, \dots, \mathrm{T}_q$.

Proof. Construction. By Lemma E.44 every $\bar\Gamma_N$ is topologically generated by $d = r+s$ elements; let Π be the free profinite group on d generators and π_N a continuous homomorphism sending the free generators to such generators (onto: its image is closed and dense). Let $\mathcal U$ be a non-principal ultrafilter, and K_∞, π_∞ as in Lemma E.43. The maps $\theta_N = \bar h_N \circ \pi_N\colon \Pi \to \bar\Gamma$ are onto and kill $\ker\pi_N$; by (U3) they have a $\mathcal U$-limit θ, onto and killing K_∞, which induces $\bar\theta$. In particular $\bar\Gamma$ is topologically finitely generated. Choose lifts $\tilde c_N, \tilde\phi_{N,j}, \tilde t_{N,j} \in \Pi$ of $\bar c_N, \bar\Phi_{N,j}, \bar{\mathrm{T}}_{N,j}$, let $\tilde c, \tilde\phi_j, \tilde t_j$ be their $\mathcal U$-limits (U2), and $c_\infty, \Phi_j, \mathrm{T}_j$ their images. For w, let $D'_w = D_w \cap \pi_{\mathscr H}^{-1}(H)$ (open normal in D_w) and P_w the kernel of the map from D'_w onto its maximal pro-ℓ quotient (closed, characteristic in D'_w, so normal in D_w). Since $\bar\phi_{N,w}(D'_w) \subseteq \bar H_N$, a pro-$\ell$ group (Lemma E.45 (Pl1)), $\bar\phi_{N,w}(P_w) = 1$. $D_w \subseteq G_L$ satisfies Φ_ℓ (Tg6), so D'_w/P_w is a pro-ℓ group with finitely many continuous homomorphisms to $\mathbb Z/\ell$, hence topologically finitely generated (Burnside basis theorem, as in Lemma E.44), and so is D_w/P_w (add coset representatives). (U4) gives F_w.

(L1). (i) is (U1), (ii) is (U3), (iii) is (U2) and (iv) is (U4); a finite intersection of sets in $\mathcal U$ is in $\mathcal U$.

(L2). Fix an open normal subgroup Y of $\bar\Gamma$ and put $V = \theta^{-1}(Y)$. For $\mathcal U$-many N, $\theta \equiv \theta_N \bmod Y$ and $\tilde c \in \tilde c_N V$, so $\theta(\tilde c) \equiv \theta(\tilde c_N) \equiv \theta_N(\tilde c_N) = \bar c \bmod Y$; similarly $\theta(\tilde t_j) \equiv \bar h_N(\bar{\mathrm{T}}_{N,j}) = 1$ (Tg4) and $\theta(\tilde\phi_j) \equiv \bar h_N(\bar\Phi_{N,j}) \bmod Y$, while $\bar h_N(\bar\Phi_{N,j}) \equiv \bar\sigma_j \bmod Y$ for all large N by (Tg3) (the preimage of Y in $\Gamma_{\mathscr H}$ is open normal and contains $\mathrm{PL}(H)$), hence for $\mathcal U$-many N ($\mathcal U$ is non-principal). As Y is arbitrary, $\bar\theta(c_\infty) = \bar c$, $\bar\theta(\mathrm{T}_j) = 1$, $\bar\theta(\Phi_j) = \bar\sigma_j$. The $\mathcal U$-limit of $\tilde c_N^2 \in \ker\pi_N$ is $\tilde c^2$, so $\tilde c^2 \in K_\infty$ by (U2): $c_\infty^2 = 1$. For $y \in D_w$ and g, g' as in (iv) for V: $\theta(g') \equiv \theta(g) \equiv \theta_N(g) = \bar h_N\bar\phi_{N,w}(y) = \overline{\pi_{\mathscr H}(y)} \bmod Y$; so $\bar\theta F_w(y) = \overline{\pi_{\mathscr H}(y)}$.

Commuting. Fix an open normal subgroup V of Π, and let ℓ^a be the largest power of ℓ dividing $\#(\Pi/V)$. In $\bar\Gamma_N$, (Tg4) gives $\bar\Phi_{N,j}^{-1}\bar{\mathrm{T}}_{N,j}\bar\Phi_{N,j} = \bar{\mathrm{T}}_{N,j}^{q_{N,j}}$, and $\bar{\mathrm{T}}_{N,j} \in \bar H_N$ (as $h_N(\mathrm{T}_{N,j}) = 1 \in H$). The image of $\bar H_N$ in $\bar\Gamma_N/\pi_N(V)$, a quotient of Π/V, is an ℓ-group of order dividing ℓ^a; as $\ell^N \mid q_{N,j} - 1$ (Tg2), for $N \ge a$ we get $\bar{\mathrm{T}}_{N,j}^{q_{N,j}-1} \in \pi_N(V)$, so $\tilde\phi_{N,j}^{-1}\tilde t_{N,j}\tilde\phi_{N,j}\tilde t_{N,j}^{-1} \in (\ker\pi_N)V$. For $\mathcal U$-many N ($N \ge a$, (i), and $\tilde\phi_j \in \tilde\phi_{N,j}V$, $\tilde t_j \in \tilde t_{N,j}V$) this gives $\tilde\phi_j^{-1}\tilde t_j\tilde\phi_j\tilde t_j^{-1} \in K_\infty V$. As V is arbitrary and K_∞ is closed, $\Phi_j\mathrm{T}_j = \mathrm{T}_j\Phi_j$.

The kernel. Let $R \subseteq \Pi$ be the closure of $\langle\langle \tilde t_1, \dots, \tilde t_q\rangle\rangle K_\infty$; we show $\ker\theta = R$, which gives the last claim. $R \subseteq \ker\theta$ by the above. In $\bar\Gamma_N$, $\ker\bar h_N$ is the closed normal subgroup generated by the $\bar{\mathrm{T}}_{N,j}$ ((Tg4); the generators from $\mathrm{PL}(H_N)$ are trivial there), so for every open normal V, $(\ker\theta_N)V = \langle\langle \tilde t_{N,j}\rangle\rangle(\ker\pi_N)V$, which for $\mathcal U$-many N equals $\langle\langle\tilde t_j\rangle\rangle K_\infty V = RV$. Hence $[\bar\Gamma : \theta_N(V)] = [\Pi : (\ker\theta_N)V] = [\Pi : RV]$ for $\mathcal U$-many N; as $\bar\Gamma$ is topologically finitely generated,

it has finitely many open subgroups of this index, so $\theta_N(V) = Y_V$ for one Y_V and $\mathcal{U}$-many N. Let $x \in \ker\theta$. For $\mathcal{U}$-many N, $\theta_N(x) \in Y_V = \theta_N(V)$ (by (ii) with $Y = Y_V$), so $x \in (\ker\theta_N)V = RV$. Hence $x \in \bigcap_V RV = R$. □

Lemma E.43. *Let Π be a topologically finitely generated profinite group, $(\Gamma_N)_{N\geq 1}$ profinite groups with continuous surjections $\pi_N\colon \Pi \to \Gamma_N$, and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. For an open normal subgroup V of Π, the subgroup $(\ker\pi_N)V$ takes only finitely many values; let W_V be the one with $\{N :\ (\ker\pi_N)V = W_V\} \in \mathcal{U}$, and put $K_\infty = \bigcap_V W_V$, $\bar\Gamma_\infty = \Pi/K_\infty$, $\pi_\infty\colon \Pi \to \bar\Gamma_\infty$. Then:*

(U1) *K_∞ is a closed normal subgroup, and $K_\infty V = W_V$ for every open normal V; so $\{N : (\ker\pi_N)V = (\ker\pi_\infty)V\} \in \mathcal{U}$;*

(U2) *for every sequence (g_N) in Π there is $g \in \Pi$ with $\{N :\ g_N \in gV\} \in \mathcal{U}$ for every open normal V (a $\mathcal{U}$-limit); if $g_N \in \ker\pi_N$ for $\mathcal{U}$-many N, then $g \in K_\infty$;*

(U3) *for every profinite group Q and continuous homomorphisms $\theta_N\colon \Pi \to Q$ there is a continuous homomorphism $\theta\colon \Pi \to Q$ with $\{N :\ \theta_N \equiv \theta \bmod Y\} \in \mathcal{U}$ for every open normal Y of Q; if every θ_N is onto, so is θ; if $\theta_N(\ker\pi_N) = 1$ for $\mathcal{U}$-many N, then $\theta(K_\infty) = 1$;*

(U4) *for every profinite group Y, closed normal subgroup $P \subseteq Y$ with Y/P topologically finitely generated, and continuous homomorphisms $f_N\colon Y \to \Gamma_N$ with $f_N(P) = 1$, there is a continuous homomorphism $F\colon Y \to \bar\Gamma_\infty$ with $F(P) = 1$ such that for every open normal V of Π, for $\mathcal{U}$-many N: for every $y \in Y$ there are $g, g' \in \Pi$ with $\pi_N(g) = f_N(y)$, $\pi_\infty(g') = F(y)$ and $g^{-1}g' \in V$.*

Proof. A topologically finitely generated profinite group has only finitely many continuous homomorphisms to a given finite group (they are determined by the images of finitely many generators). In particular Π/V is finite and the subgroups $(\ker\pi_N)V \supseteq V$ correspond to subgroups of Π/V; as $\mathcal{U}$ is an ultrafilter, exactly one value occurs on a set in $\mathcal{U}$. If $V' \subseteq V$, then $(\ker\pi_N)V'V = (\ker\pi_N)V$, so $W_{V'}V = W_V$ (intersect the two sets in $\mathcal{U}$).

(U1) K_∞ is an intersection of closed normal subgroups. We have $K_\infty V \subseteq W_V$. Let $x \in W_V$. For open normal $V' \subseteq V$, $x \in W_{V'}V$, so the closed set $xV \cap W_{V'}$ is non-empty; these sets decrease as V' decreases and every finite family of them contains the one for the intersection of the V'; by compactness some w lies in all of them, i.e. $w \in xV \cap \bigcap_{V'\subseteq V} W_{V'}$, and $\bigcap_{V'\subseteq V} W_{V'} = K_\infty$ (for arbitrary V'', $W_{V''} \supseteq W_{V''\cap V}$). So $x \in K_\infty V$.

(U2) The $\mathcal{U}$-limit exists in the compact Hausdorff space Π: the images of g_N in each finite Π/V have a $\mathcal{U}$-limit, compatible in V. If $g_N \in \ker\pi_N$ for $\mathcal{U}$-many N, then for each V and $\mathcal{U}$-many N, $g \in g_N V \subseteq (\ker\pi_N)V = W_V$; so $g \in K_\infty$.

(U3) For each open normal Y of Q the composites $\Pi \to Q/Y$ lie in a finite set, so one of them, θ_Y, occurs for $\mathcal{U}$-many N; the θ_Y are compatible and define $\theta\colon \Pi \to \varprojlim Q/Y = Q$. If all θ_N are onto, every θ_Y is onto, hence $\theta(\Pi)$, compact and dense, is Q. If $\theta_N(\ker\pi_N) = 1$ for $\mathcal{U}$-many N: given Y, let $V = \theta^{-1}(Y)$ (open normal); for $\mathcal{U}$-many N we have $\theta_N \equiv \theta \bmod Y$, $\theta_N(\ker\pi_N) = 1$ and $(\ker\pi_N)V = K_\infty V$, so for $k \in K_\infty$, $k = k'v$ with $\pi_N(k') = 1$, and $\theta(k) \equiv \theta_N(k')\theta_N(v) \equiv \theta(v) \equiv 1 \bmod Y$. Hence $\theta(K_\infty) = 1$.

(U4) Let $y_1, \dots, y_t \in Y$ have images generating Y/P topologically, let $\hat F_t$ be the free profinite group on $x_1, \dots, x_t$ and $\epsilon\colon \hat F_t \to Y/P$, $x_i \mapsto y_i P$, which is onto; let $R = \ker\epsilon$. Choose $g_{i,N} \in \Pi$ with $\pi_N(g_{i,N}) = f_N(y_i)$, let g_i be their $\mathcal{U}$-limits (U2), and let $\Psi_N, \Psi\colon \hat F_t \to \Pi$ be the homomorphisms with $x_i \mapsto g_{i,N}$, resp. g_i. For each open normal V, $\Psi_N \equiv \Psi \bmod V$ for $\mathcal{U}$-many N (the generators agree modulo V). For $w \in R$, $\pi_N\Psi_N(w) = \bar f_N(\epsilon(w)) = 1$ ($\bar f_N$ induced on Y/P), so

$\Psi(w) \in (\ker \pi_N)V = W_V$ for $\mathcal{U}$-many N, hence $\Psi(w) \in K_\infty$. So $\pi_\infty \circ \Psi$ kills R and induces a continuous $\bar{F}\colon Y/P \to \bar{\Gamma}_\infty$; put $F(y) = \bar{F}(yP)$. Given V, for $\mathcal{U}$-many N with $\Psi_N \equiv \Psi \bmod V$: for $y \in Y$ choose w with $\epsilon(w) = yP$ and take $g = \Psi_N(w)$, $g' = \Psi(w)$; then $\pi_N(g) = f_N(y)$, $\pi_\infty(g') = F(y)$ and $g^{-1}g' \in V$. □

Lemma E.44. *Let a Taylor–Wiles datum for a rank-n Hecke datum $\mathscr{H}$ be given, with H, b and the levels Γ_N, $H_N = h_N^{-1}(H)$. Put $\bar{\Gamma}_N = \Gamma_N/\mathrm{PL}(H_N)$ and $\bar{H}_N = H_N/\mathrm{PL}(H_N)$, let r be the largest integer with $\ell^r \le b$, and put $s = [\Gamma_{\mathscr{H}} : H]$. Then for every N, $\bar{H}_N$ is a pro-ℓ group topologically generated by r elements, and $\bar{\Gamma}_N$ is topologically generated by $r + s$ elements.*

Proof. By Lemma E.45 (Pl1) (with $G = \Gamma_N$ and the open normal subgroup H_N), $\mathrm{PL}(H_N)$ is a closed normal subgroup of Γ_N contained in H_N and $\bar{H}_N$ is pro-ℓ. Every continuous homomorphism $\bar{H}_N \to \mathbb{Z}/\ell$ comes from one of H_N, and there are at most b of these (Tg5). The Frattini quotient $\bar{H}_N/\Phi(\bar{H}_N)$ of the pro-ℓ group $\bar{H}_N$ is an $\mathbb{F}_\ell$-vector space whose dual is $\mathrm{Hom}(\bar{H}_N, \mathbb{Z}/\ell)$, so it has dimension at most r; by the Burnside basis theorem for pro-ℓ groups (a subset generates topologically iff its image generates the Frattini quotient), $\bar{H}_N$ is topologically generated by r elements. Since h_N is onto and $H_N = h_N^{-1}(H)$, $\bar{\Gamma}_N/\bar{H}_N \cong \Gamma_N/H_N \cong \Gamma_{\mathscr{H}}/H$ has order s; adding representatives of the s cosets to r generators of $\bar{H}_N$ gives $r+s$ topological generators of $\bar{\Gamma}_N$. □

Lemma E.45. *Let ℓ be a prime, G a profinite group and $H \subseteq G$ an open normal subgroup. Then:*

(Pl1) $\mathrm{PL}(H)$ *is a closed normal subgroup of G contained in H, equal to the intersection of the open normal subgroups of H of ℓ-power index; in particular $H/\mathrm{PL}(H)$ is a pro-ℓ group;*

(Pl2) *if $\Gamma' \subseteq G$ is a subgroup containing H, M a finite abelian group of ℓ-power order with an action of Γ' that is trivial on $\mathrm{PL}(H)$, and $z\colon \Gamma' \to M$ a continuous map with $z(\gamma\gamma') = z(\gamma) + \gamma z(\gamma')$, then z vanishes on $\mathrm{PL}(H)$;*

(Pl3) *if $h\colon G' \to G$ is a continuous surjective homomorphism of profinite groups, then*

$$h(\mathrm{PL}(h^{-1}(H))) \subseteq \mathrm{PL}(H).$$

Proof. Let $\mathcal{U}$ be the set of open normal subgroups U of G such that every $x \in H$ has some $x^{\ell^e} \in U$; thus $\mathrm{PL}(H) = \bigcap \mathcal{U}$, a closed normal subgroup of G. For $U \in \mathcal{U}$ every element of the finite group $H/(H \cap U)$ has ℓ-power order, so (Cauchy) $[H : H \cap U]$ is a power of ℓ.

(Pl1). Let N be an open normal subgroup of H of ℓ-power index. As H is open and normal in G, the conjugates gNg^{-1} ($g \in G$) are finitely many open normal subgroups of H of the same index; their intersection N_G is open, normal in G, contained in N, and H/N_G embeds into $\prod H/gNg^{-1}$, an ℓ-group; so $N_G \in \mathcal{U}$. Conversely, for $U \in \mathcal{U}$, $U \cap H$ is an open normal subgroup of H of ℓ-power index. Hence $\mathcal{U}$ contains members inside H, so $\mathrm{PL}(H) \subseteq H$, and $\mathrm{PL}(H) = \bigcap_{U\in\mathcal{U}}(U \cap H)$ is the intersection of the open normal subgroups of H of ℓ-power index. The quotients of $H/\mathrm{PL}(H)$ by the images of these subgroups are ℓ-groups and these images form a basis of open neighbourhoods of 1, so $H/\mathrm{PL}(H)$ is pro-ℓ.

(Pl2). Write $P = \mathrm{PL}(H) \subseteq H \subseteq \Gamma'$. As P acts trivially on M, $z|_P$ is a homomorphism. For $x \in H$, $y \in P$, since $xyx^{-1} \in P$ acts trivially and $z(x) + x\,z(x^{-1}) = z(1) = 0$,

$$z(xyx^{-1}) = z(x) + x\,z(y) + xy\,z(x^{-1}) = z(x) + x\,z(y) + (xyx^{-1})\,x\,z(x^{-1}) = x\,z(y).$$

So $K = \ker(z|_P)$ is normal in H; it is closed (z continuous, M discrete), and P/K embeds into M, a finite ℓ-group. We show that every open normal subgroup $W \supseteq K$ of H has ℓ-power index.

Let $\bar H = H/K$, $\bar P = P/K$ (a finite normal ℓ-subgroup), $\bar W = W/K$, and choose an open normal subgroup $\bar N$ of $\bar H$ with $\bar N \cap \bar P = \{1\}$ (the open normal subgroups of the profinite group $\bar H$ form a basis at 1, and $\bar P$ is finite). Then

$$[\bar H : \bar N \cap \bar W] = [\bar H : (\bar N \cap \bar W)\bar P] \cdot [(\bar N \cap \bar W)\bar P : \bar N \cap \bar W],$$

where the first factor is the index of an open subgroup of $\bar H/\bar P \cong H/P$, a pro-ℓ group by (Pl1), hence a power of ℓ (an open subgroup of a pro-ℓ group contains an open normal subgroup of ℓ-power index), and the second factor equals $\#\bar P$ because $\bar P \cap \bar N \cap \bar W = \{1\}$. So $[H : W] = [\bar H : \bar W]$ divides a power of ℓ. As K is closed and normal in the profinite group H, K is the intersection of the open normal subgroups W of H containing it; each has ℓ-power index and therefore contains P by (Pl1). Hence $K = P$, i.e. z vanishes on P.

(Pl3). Let $H' = h^{-1}(H)$, open and normal in G'. For $U \in \mathcal{U}$, $h^{-1}(U)$ is open and normal in G', and every $x \in H'$ has $h(x)^{\ell^e} \in U$ for some e, i.e. $x^{\ell^e} \in h^{-1}(U)$. So $\mathrm{PL}(H') \subseteq \bigcap_{U\in\mathcal{U}} h^{-1}(U) = h^{-1}(\mathrm{PL}(H))$. □

Lemma E.46. *In the situation of Proposition E.134, let x have shape $(a, n-a)$ with $a < n$, and let $\mathfrak{R}$ be a realisation of x over $\mathbf{V}_\infty$ whose global ring and determinant are R/I and the determinant D of Proposition E.134. Let $\hat Z'$ be a prime ideal of $\hat B$ whose preimage $\mathfrak{z}$ in R (along $R \to R/I \to \hat B$) is contained in $\pi^{-1}(P'')$ for a prime ideal P'' of C^{sp} such that $u^{-1}(P'')$ is a stable prime of B. Then* $\mathrm{ired}(N) \not\subseteq \hat Z'$.

Proof. Step 1: a discriminant criterion. Let A be a commutative ring, G a group, D' an A-valued determinant of dimension n on G, $\mathfrak{p}$ a prime ideal of A and $\bar\kappa$ an algebraic closure of the fraction field of $A/\mathfrak{p}$. We show: the base change $D'_{\bar\kappa}$ of D' to $\bar\kappa$ is $\det\circ r$ for an irreducible $r\colon G \to \mathrm{GL}_n(\bar\kappa)$ if and only if there are $g_1, \dots, g_{n^2} \in G$ with $\delta = \det\big(\mathrm{tr}_{D'}(g_i g_j)\big)_{i,j} \notin \mathfrak{p}$. By (Ch3) of Theorem B.83, $D'_{\bar\kappa} = \det\circ r$ for some representation r, and the image of $\mathrm{tr}_{D'}(g)$ in $\bar\kappa$ is $\mathrm{tr}\, r(g)$; so the image of δ is $\det\big(\mathrm{tr}(r(g_i)r(g_j))\big)$. If r is irreducible, the $\bar\kappa$-span of $r(G)$ is $\mathrm{M}_n(\bar\kappa)$ (Burnside), so some $r(g_1), \dots, r(g_{n^2})$ form a basis, and as the trace form $(X, Y) \mapsto \mathrm{tr}(XY)$ on $\mathrm{M}_n(\bar\kappa)$ is non-degenerate, the image of δ is non-zero, i.e. $\delta \notin \mathfrak{p}$. Conversely, if the image of δ is non-zero, the $r(g_i)$ are linearly independent (a linear relation among them is a relation among the rows of the Gram matrix), so they span $\mathrm{M}_n(\bar\kappa)$; a non-zero proper $r(G)$-stable subspace would be stable under $\mathrm{M}_n(\bar\kappa)$, which is impossible, so r is irreducible.

Step 2: irreducibility at $\hat Z'$. Put $\mathfrak{Q} = u^{-1}(P'')$, a stable prime of B: the base change of D^{u} to an algebraic closure of the residue field of $\mathfrak{Q}$ is $\det\circ r$ with r irreducible on $\Gamma^0_{\mathscr{H}}$. By Step 1 there are $\gamma_1, \dots, \gamma_{n^2} \in \Gamma^0_{\mathscr{H}}$ with $\delta_0 = \det(\mathrm{tr}_{D^{\mathrm{u}}}(\gamma_i\gamma_j)) \notin \mathfrak{Q}$. As h maps $\Gamma^0_\infty = h^{-1}(\Gamma^0_{\mathscr{H}})$ onto $\Gamma^0_{\mathscr{H}}$, choose $e_i \in \Gamma^0_\infty$ with $h(e_i) = \gamma_i$, and put $\delta = \det(\mathrm{tr}_D(e_ie_j)) \in R/I$. By (In1) of Proposition E.134, applied to e_ie_j (whose image under h is $\gamma_i\gamma_j$), $\bar\pi$ carries the coefficients of the characteristic polynomial of D at e_ie_j to those of D^{u} at $\gamma_i\gamma_j$, base changed along u; in particular $\bar\pi(\mathrm{tr}_D(e_ie_j)) = u(\mathrm{tr}_{D^{\mathrm{u}}}(\gamma_i\gamma_j))$, hence $\bar\pi(\delta) = u(\delta_0) \notin P''$, since $u^{-1}(P'') = \mathfrak{Q} \not\ni \delta_0$. So $\delta \notin \bar\pi^{-1}(P'') = \pi^{-1}(P'')/I$, and a fortiori $\delta \notin \mathfrak{z}/I$, which is the preimage of $\hat Z'$ in R/I. Hence the image of δ in $\hat B$ is not in $\hat Z'$. The base change $D_{\hat B}$ of D along $R/I \to \hat B$ has $\mathrm{tr}_{D_{\hat B}}(e_ie_j)$ equal to the image of $\mathrm{tr}_D(e_ie_j)$; by Step 1, applied to $D_{\hat B}$ and $\hat Z'$, the base change of D along $R/I \to \hat B \to \bar\kappa$, with $\bar\kappa$ an algebraic closure of the fraction field of $\hat B/\hat Z'$, is $\det\circ r'$ for an irreducible $r'\colon \Gamma^0_\infty \to \mathrm{GL}_n(\bar\kappa)$.

Step 3. By Lemma E.47, applied to $\mathfrak{R}$ (of shape $(a, n-a)$ with $a < n$) and $\hat Q = \hat Z'$, $\mathrm{ired}(N) \not\subseteq \hat Z'$. □

Lemma E.47. *Let $\mathfrak{R}$ be a realisation of a framed normalised curve point x of shape $(a, n-a)$ with $a < n$ over a level $\mathbf{V}$, $\hat Q$ a prime ideal of $\hat B$ and $\bar\kappa$ an algebraic closure of the fraction field*

of $\hat{B}/\hat{Q}$. If the base change of D along $B \to \hat{B} \to \bar{\kappa}$ is $\det \circ r$ for an irreducible homomorphism $r\colon \Gamma_1^0 \to \mathrm{GL}_n(\bar{\kappa})$, then $\mathrm{ired}(N) \not\subseteq \hat{Q}$.

Proof. Suppose $\mathrm{ired}(N) \subseteq \hat{Q}$; let $\varphi\colon \hat{B} \to \bar{\kappa}$ be the homomorphism with kernel $\hat{Q}$, $\mathcal{A}_{\bar{\kappa}} = \mathcal{A} \otimes_{\hat{B}} \bar{\kappa} = \bar{\kappa} \cdot 1 \oplus N_{\bar{\kappa}}$ with $N_{\bar{\kappa}}$ the image of $N \otimes_{\hat{B}} \bar{\kappa}$ (the decomposition $\mathcal{A} = \hat{B} \cdot 1 \oplus N$ of (Vr5) is one of $\hat{B}$-modules, so it survives base change and $\bar{\kappa} \to \mathcal{A}_{\bar{\kappa}}$ is injective). For $y, y' \in N$ we have $yy' = b \cdot 1$ with $b \in \mathrm{ired}(N) \subseteq \hat{Q}$, so the product of any two elements of $N_{\bar{\kappa}}$ is 0, and the same holds for the image N_T of $N_{\bar{\kappa}} \otimes T$ in $\mathcal{A}_{\bar{\kappa}} \otimes_{\bar{\kappa}} T$ for every commutative $\bar{\kappa}$-algebra T.

By (Vr6) every $\nu(\gamma)$ has entries in $\hat{B} \cdot 1$ at the positions (i,j) in one block and in N at the others, so its image $\nu_{\bar{\kappa}}(\gamma)$ in $\mathrm{M}_n(\mathcal{A}_{\bar{\kappa}})$ is $\begin{pmatrix} P(\gamma) & X(\gamma) \\ Y(\gamma) & S(\gamma) \end{pmatrix}$ with $P(\gamma) \in \mathrm{M}_a(\bar{\kappa})$, $S(\gamma) \in \mathrm{M}_{n-a}(\bar{\kappa})$ and $X(\gamma), Y(\gamma)$ with entries in $N_{\bar{\kappa}}$. Since ν is multiplicative, $P(\gamma\gamma') = P(\gamma)P(\gamma') + X(\gamma)Y(\gamma') = P(\gamma)P(\gamma')$ (the entries of $X(\gamma)Y(\gamma')$ are sums of products of two elements of $N_{\bar{\kappa}}$), likewise $S(\gamma\gamma') = S(\gamma)S(\gamma')$, and $P(1) = 1$, $S(1) = 1$; so P, S are homomorphisms into $\mathrm{GL}_a(\bar{\kappa})$, $\mathrm{GL}_{n-a}(\bar{\kappa})$.

For a commutative $\bar{\kappa}$-algebra T, $t_b \in T$ and $\gamma_b \in \Gamma_1^0$, the matrix $M = \sum_b t_b \nu_{\bar{\kappa}}(\gamma_b)$ has the same block shape, with off-diagonal blocks in N_T. In the Leibniz expansion of $\det M$, a permutation sending some index of the first block to the second also sends some index of the second block to the first, so its term contains at least two factors from N_T and vanishes. Hence $\det M = \det(\sum_b t_b P(\gamma_b)) \det(\sum_b t_b S(\gamma_b))$. By (Vr6) the base change of D to $\mathcal{A}$ is $\det \circ \nu$, so its further base change to $\mathcal{A}_{\bar{\kappa}}$ is $\det \circ \nu_{\bar{\kappa}} = \det \circ (P \oplus S)$. In particular, for every γ the characteristic polynomials of the base change $D_{\bar{\kappa}}$ of D to $\bar{\kappa}$ and of $(P \oplus S)(\gamma)$ coincide after the injection $\bar{\kappa}[X] \to \mathcal{A}_{\bar{\kappa}}[X]$, hence in $\bar{\kappa}[X]$, and by Theorem B.83 (Ch1) $D_{\bar{\kappa}} = \det \circ (P \oplus S)$. Let $P^{\mathrm{ss}}, S^{\mathrm{ss}}$ be the semisimplifications; the characteristic polynomial of a representation is the product of those of its Jordan–Hölder factors, so by (Ch1) again $\det \circ (P \oplus S) = \det \circ (P^{\mathrm{ss}} \oplus S^{\mathrm{ss}})$. Thus $\det \circ r = \det \circ (P^{\mathrm{ss}} \oplus S^{\mathrm{ss}})$ with both representations semisimple (r is irreducible), and (Ch3) gives $r \cong P^{\mathrm{ss}} \oplus S^{\mathrm{ss}}$. But $a \geq 1$ and $n - a \geq 1$, so the right side has at least two irreducible summands, while r is irreducible: a contradiction. Hence $\mathrm{ired}(N) \not\subseteq \hat{Q}$. □

E.2.3 Choice of Taylor–Wiles places and elements

This subsection chooses the auxiliary data of the patching argument, used in Proposition E.39. Taylor–Wiles places exist whose Frobenius elements approximate prescribed elements of the cyclotomic kernel (Proposition E.48), by a Chebotarev argument with places of absolute degree one (Lemma E.49). Taylor–Wiles elements that detect all points of a chain exist after a common finite extension (Proposition E.50), because dense subsets of the cyclotomic kernel detect the twisted dual cocycles (Proposition E.51) and Selmer bases with finite detection exist (Proposition E.52). Patchable curve points of endoscopic and of irreducible type can be normalised over finite extensions of the curve (Proposition E.53, Proposition E.54, Lemma E.55).

Proposition E.48. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$. Then $\Gamma_{\mathscr{H}}$ satisfies Φ_ℓ, and for every $q \geq 0$ and all $\sigma_1, \dots, \sigma_q \in \Gamma'$ there is a Taylor–Wiles datum of size q for $\mathscr{H}$ whose elements are $\sigma_1, \dots, \sigma_q$.*

Proof. Write $\Gamma = \Gamma_{\mathscr{H}}$, $\pi = \pi_{\mathscr{H}}$, $S = S_\ell \cup \mathrm{St} \cup \{v_0\}$.

Step 0: the levels are quotients of Galois groups with restricted ramification. Let Q be a finite set of places as in Definition 53.8. For a finite place $\bar{w} \nmid \ell$ of $\overline{\mathbb{Q}}$ above a place of L^+ outside $\mathrm{St} \cup Q$ (and outside S_ℓ), $I_{\bar{w}} \subseteq G_L$ (L/L^+ is unramified at finite places, (Rd2)) and every $\bar{\chi}_a$ is trivial on $I_{\bar{w}}$ by (Rd4); so $I_{\bar{w}} \subseteq N^Q$. Hence Γ^Q is a quotient of the Galois group $G_{L^+, S_\ell \cup \mathrm{St} \cup Q}$ of the

maximal extension of L^+ unramified outside $S_\ell \cup \mathrm{St} \cup Q$ and the infinite places, which satisfies Φ_ℓ by (2) of Theorem B.71. A quotient of a group satisfying Φ_ℓ satisfies it (open subgroups pull back to open subgroups, and homomorphisms to $\mathbb{Z}/\ell$ inflate injectively). So every Γ^Q, in particular $\Gamma = \Gamma^\varnothing$, satisfies Φ_ℓ. This is the first assertion.

Step 1: H and a basis at $\mathrm{PL}(H)$. Define H by (Tg1). $\Gamma^0_{\mathscr{H}}$ has index 2; the $\bar\chi_a$ and $\chi_{\mathrm{cyc}} \bmod \ell$ are continuous with finite image; and $\bigcap_a \ker \bar\chi_a$ is stable under conjugation by $\pi(c)$ by (Rd5). So H is an open normal subgroup of Γ. Γ, a quotient of G_{L^+}, has only countably many open subgroups (they correspond to finite extensions of L^+ inside $\overline{\mathbb{Q}}$); let $U_1, U_2, \dots$ enumerate the open normal subgroups U of Γ such that every $h \in H$ has some $h^{\ell^e} \in U$ (repeat the last one if there are finitely many; $U = H$ is one of them), and put $W_N = H \cap U_1 \cap \cdots \cap U_N$. The W_N are open, normal, decreasing, contained in H, and $\bigcap_N W_N = H \cap \mathrm{PL}(H) = \mathrm{PL}(H)$ (as $\mathrm{PL}(H) \subseteq H$). If $V \supseteq \mathrm{PL}(H)$ is open, then $\Gamma \smallsetminus V$ is compact and covered by the increasing open sets $\Gamma \smallsetminus W_N$, so $W_N \subseteq V$ for N large.

Step 2: the places. Fix $N \geq 1$ and lifts $g_j \in G_L$ of σ_j^{-1} ($\sigma_j \in \Gamma' \subseteq \pi(G_L)$). Let $U'_N = \pi^{-1}(W_N) \cap \ker(\chi_{\mathrm{cyc}} \bmod \ell^N) \cap G_L$, an open normal subgroup of G_L. For $j = 1, \dots, q$ in turn, apply Lemma E.49 with $F = L$, $U = U'_N$, $g = g_j$ and the finite set of places of L above S, above ℓ, or above one of $v_{N,1}, \dots, v_{N,j-1}$: it gives a place w of L of absolute degree one, unramified over $\mathbb{Q}$, outside that set, a place $\bar w_{N,j}$ above it and an arithmetic Frobenius $\sigma \in D_{\bar w_{N,j}}$ with $I_{\bar w_{N,j}} \subseteq U'_N$ and $\sigma g_j^{-1} \in U'_N$. Put $w_{N,j} = w$, $v_{N,j} = w|_{L^+}$, $\phi_{N,j} = \sigma^{-1} \in D_{\bar w_{N,j}} \cap G_L$. As $f(w \mid p) = 1$ and L/L^+ is unramified at w, $v_{N,j}$ splits in L; it is not in S, not above ℓ, and different from the earlier $v_{N,j'}$. $q_{N,j} = \mathrm{N}w$, and $\chi_{\mathrm{cyc}}(\sigma) = \mathrm{N}w$ in $\mathbb{Z}_\ell^\times$ (σ acts on ℓ-power roots of unity as an arithmetic Frobenius); since $\sigma \in g_j U'_N$ and $\chi_{\mathrm{cyc}}(g_j) = \chi_{\mathrm{cyc}}(\sigma_j)^{-1} = 1$, $q_{N,j} \equiv 1 \bmod \ell^N$. Choose $\tau_{N,j} \in I_{\bar w_{N,j}}$ mapping to a topological generator of the maximal pro-ℓ quotient $I_{\bar w_{N,j}}/K_{N,j} \cong \mathbb{Z}_\ell(1)$; then $I_{\bar w_{N,j}}$ is the closed subgroup generated by $\tau_{N,j}$ and $K_{N,j}$. This is (Tg2).

Step 3: (Tg3). $\pi(\sigma) \in \pi(g_j) W_N = \sigma_j^{-1} W_N$, so $\pi(\phi_{N,j}) = \eta^{-1}\sigma_j$ for some $\eta \in W_N$ and $h_N(\Phi_{N,j})\sigma_j^{-1} = \pi(\phi_{N,j})\sigma_j^{-1} = \eta^{-1} \in W_N \subseteq H$. By Step 1, for $V \supseteq \mathrm{PL}(H)$ open normal, $W_N \subseteq V$ for N large.

Step 4: inertia at the new places lies in H_N, and its ℓ'-part in $\mathrm{PL}(H_N)$. Every place $\bar w$ above some $v_{N,j}$ is $g\bar w_{N,j}$ with $g \in G_{L^+}$, and $I_{\bar w} = g I_{\bar w_{N,j}} g^{-1} \subseteq G_L$ is in $N_{\mathscr{H}}$ by (Rd4) (the $\bar\chi_a$ are trivial on it, $v_{N,j} \notin S_\ell \cup \mathrm{St}$). So $h_N(\pi^{Q_N}(I_{\bar w})) = 1$, in particular $\pi^{Q_N}(I_{\bar w}) \subseteq H_N$ and $h_N(\mathrm{T}_{N,j}) = 1$. Let $x \in I_{\bar w}$ map to 1 in the maximal pro-ℓ quotient of $I_{\bar w}$, and U an open normal subgroup of Γ_N such that every element of H_N has an ℓ-power in U. The image of $I_{\bar w}$ in the finite group $H_N/(H_N \cap U)$ is an ℓ-group, hence a quotient of the maximal pro-ℓ quotient of $I_{\bar w}$, so $\pi^{Q_N}(x) \in U$. Hence $\pi^{Q_N}(x) \in \mathrm{PL}(H_N)$.

Step 5: (Tg4). In $I_{\bar w_{N,j}}/K_{N,j} \cong \mathbb{Z}_\ell(1)$ the arithmetic Frobenius $\sigma = \phi_{N,j}^{-1}$ acts by multiplication by $q_{N,j}$, so $\tau_{N,j}^{-q_{N,j}} \phi_{N,j}^{-1} \tau_{N,j} \phi_{N,j} \in K_{N,j}$ (an element of $I_{\bar w_{N,j}}$, as σ normalises inertia), and by Step 4 its image $\mathrm{T}_{N,j}^{-q_{N,j}} \Phi_{N,j}^{-1} \mathrm{T}_{N,j} \Phi_{N,j}$ lies in $\mathrm{PL}(H_N)$. $\ker h_N$ is the image of $N_{\mathscr{H}}$ in Γ_N; the generators of $N_{\mathscr{H}}$ at places outside Q_N lie in N^{Q_N}, so $\ker h_N$ is the closed normal subgroup generated by the images of $I_{\bar w}$, $\bar w$ above Q_N (on which all $\bar\chi_a$ are trivial). Each such image is $\pi^{Q_N}(g)\, \pi^{Q_N}(I_{\bar w_{N,j}})\, \pi^{Q_N}(g)^{-1}$, contained in the closure of the subgroup generated by the conjugate of $\mathrm{T}_{N,j}$ and $\pi^{Q_N}(g K_{N,j} g^{-1}) \subseteq \mathrm{PL}(H_N)$ (Step 4). $\mathrm{PL}(H_N)$ is normal (an intersection of normal subgroups); so $\ker h_N$ lies in the closure of the subgroup generated by $\mathrm{PL}(H_N)$ and the conjugates of $\mathrm{T}_{N,1}, \dots, \mathrm{T}_{N,q}$.

Step 6: (Tg5). H and H_N are open in groups satisfying Φ_ℓ (Step 0), so their sets of homomorphisms to $\mathbb{Z}/\ell$ are finite. Let $m = [\Gamma : H] = [\Gamma_N : H_N]$ and $r_1, \dots, r_m$ coset representatives of H_N in Γ_N. Let $\psi, \psi' \colon H_N \to \mathbb{Z}/\ell$ agree on the mq elements $r_i \mathrm{T}_{N,j} r_i^{-1}$, and

put $\delta = \psi - \psi'$. (i) δ vanishes on $\mathrm{PL}(H_N)$: the intersection U of the Γ_N-conjugates of $\ker\delta$ (finitely many, as H_N is open and normal) is open and normal in Γ_N, and H_N/U embeds into a product of copies of $\mathbb{Z}/\ell$, so every element of H_N has its ℓ-th power in U; thus $\mathrm{PL}(H_N) \subseteq U \subseteq \ker\delta$. (ii) δ vanishes on all conjugates $g\mathrm{T}_{N,j}g^{-1}$, $g = r_i h$ with $h \in H_N$: $\delta(r_i h \mathrm{T} h^{-1} r_i^{-1}) = \delta\big((r_i h r_i^{-1})(r_i \mathrm{T} r_i^{-1})(r_i h r_i^{-1})^{-1}\big) = \delta(r_i \mathrm{T} r_i^{-1}) = 0$, as $\mathbb{Z}/\ell$ is abelian and $\mathrm{T} \in \ker h_N \subseteq H_N$. As $\ker\delta$ is closed, (i), (ii) and Step 5 give $\ker h_N \subseteq \ker\delta$, so δ is inflated from a homomorphism $H = H_N/\ker h_N \to \mathbb{Z}/\ell$. Hence the number of homomorphisms $H_N \to \mathbb{Z}/\ell$ is at most $b = \#\mathrm{Hom}(H, \mathbb{Z}/\ell) \cdot \ell^{mq}$, independent of N.

Step 7: (Tg6). $D_v \cap G_L$ ($v \in S_\ell$) and $D_{\bar{u}} \cap G_L$ ($u \in \mathrm{St}$; u splits in L by (Rd2)) are absolute Galois groups of local fields, which satisfy Φ_ℓ by the classical fact recalled above.

So H, the σ_j, b and the places, Frobenius and tame elements of Steps 2–3 form a Taylor–Wiles datum of size q with elements $\sigma_1, \ldots, \sigma_q$. □

Lemma E.49. *Let F be a number field, $\overline{F}$ an algebraic closure, $G_F = \mathrm{Gal}(\overline{F}/F)$, $U \subseteq G_F$ an open normal subgroup, $g \in G_F$ and S a finite set of finite places of F. Then the set of finite places $v \notin S$ of F such that*

1. *v has absolute degree one and is unramified over $\mathbb{Q}$: if p is the prime below v, then $f(v \mid p) = e(v \mid p) = 1$;*
2. *there are a place $\bar{v}$ of $\overline{F}$ above v and an element σ of the decomposition group $D_{\bar{v}} \subseteq G_F$ which induces the arithmetic Frobenius $x \mapsto x^{\mathrm{N}v}$ on the residue field of $\bar{v}$, such that $I_{\bar{v}} \subseteq U$ and $\sigma g^{-1} \in U$*

has Dirichlet density $\#C/[G_F : U]$, where C is the conjugacy class of gU in G_F/U. In particular it is infinite. Equivalently, for a finite Galois extension L/F inside $\overline{F}$ and $h \in \mathrm{Gal}(L/F)$, there are infinitely many finite places $v \notin S$ with $f(v \mid p) = e(v \mid p) = 1$, unramified in L, with a place w of L above v whose Frobenius Frob_w is h.

Proof. The two forms are equivalent. Put $L = \overline{F}^U$, a finite Galois extension of F, so that restriction identifies G_F/U with $\mathrm{Gal}(L/F)$. For a place $\bar{v}$ of $\overline{F}$ with restriction w to L, restriction maps $D_{\bar{v}}$ onto D_w and $I_{\bar{v}}$ onto I_w, and an element of $D_{\bar{v}}$ inducing the arithmetic Frobenius on the residue field of $\bar{v}$ maps to an element inducing it on the residue field of w. So $I_{\bar{v}} \subseteq U$ says exactly that v is unramified in L, and then $\sigma g^{-1} \in U$ says exactly that $\mathrm{Frob}_w = gU$. Conversely, given w with $\mathrm{Frob}_w = h$ and v unramified in L, any $\bar{v}$ above w and any $\sigma \in D_{\bar{v}}$ inducing the arithmetic Frobenius satisfy 2 with g any lift of h. It therefore suffices to prove the density statement for L/F and $h = gU$, with C the conjugacy class of h.

The places with Frobenius conjugate to h have density $\#C/[L : F]$. Let P_C be the set of finite places of F, unramified in L, at which the Frobenius conjugacy class is C. By Chebotarev's density theorem, P_C has Dirichlet density $\#C/[L : F] > 0$. At a place $v \in P_C$ some place w of L above v has $\mathrm{Frob}_w = h$: if $\mathrm{Frob}_{w_0} = \tau h \tau^{-1}$ for one place w_0 above v and some $\tau \in \mathrm{Gal}(L/F)$, then $w = \tau^{-1} w_0$ has

$$\mathrm{Frob}_{\tau^{-1} w_0} = \tau^{-1} \mathrm{Frob}_{w_0} \tau = h.$$

The excluded places have density zero. Let Z be the union of three sets: S; the finitely many places that ramify in L or lie above a prime dividing the discriminant of F (these contain all v with $e(v \mid p) > 1$); and the set Z_2 of places with $f(v \mid p) \geq 2$. A finite set has density zero. For Z_2: every $v \in Z_2$ has $\mathrm{N}v \geq p^2$, and at most $[F : \mathbb{Q}]$ places lie above each prime p. Hence, for real $s > 1$,

$$\sum_{v \in Z_2} \mathrm{N}v^{-s} \leq [F : \mathbb{Q}] \sum_p p^{-2s} \leq [F : \mathbb{Q}]\, \zeta(2),$$

which stays bounded as $s \to 1^+$, while $\log \frac{1}{s-1} \to \infty$. So Z_2, and hence Z, has Dirichlet density zero.

Conclusion. Let M be the set in the statement. By the equivalence and the previous step, every place of $P_C \setminus Z$ lies in M. Conversely, condition 2 forces $v \in P_C$. So $P_C \setminus Z \subseteq M \subseteq P_C$, and M has the density of $P_C \setminus Z$ once the latter exists. For $s > 1$,

$$\sum_{v \in P_C \setminus Z} \mathrm{N}v^{-s} = \sum_{v \in P_C} \mathrm{N}v^{-s} - \sum_{v \in P_C \cap Z} \mathrm{N}v^{-s},$$

and the last sum is at most $\sum_{v \in Z} \mathrm{N}v^{-s} = o(\log \frac{1}{s-1})$. So $P_C \setminus Z$ has Dirichlet density $\#C/[L : F] > 0$. Since $M \setminus (P_C \setminus Z) \subseteq P_C \cap Z$, the same estimate shows that M has this density too. A set of positive Dirichlet density is infinite, because every finite set has density zero. □

Proposition E.50. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ with $\ell \neq 2$ and $\Gamma_{\mathscr{H}}$ satisfying Φ_ℓ, and let $x_0, \dots, x_h$ be spanning framed curve points of a universal polarised ordinary determinant over $\mathscr{H}$, each normalised of some shape, each satisfying (Np3). Then there are $q \geq 0$, elements $\sigma_1, \dots, \sigma_q \in \Gamma'$ and, for $0 \leq i \leq h$, an extension $x_i' = (k_i', \rho_i', 1_n, (\psi_{i,v,b}'), f_i')$ of x_i, with $|k_0'| = \cdots = |k_h'|$, a Selmer basis κ_i' of x_i' such that $\{\sigma_1, \dots, \sigma_q\}$ detects (x_i', κ_i'), and pairwise distinct $\alpha_{i,j,1}, \dots, \alpha_{i,j,n} \in A_{k_i'}$ with $\det(X - \rho_i'(\sigma_j)) = \prod_{b=1}^n (X - \alpha_{i,j,b})$ for $1 \leq j \leq q$.*

Proof. Write $x_i = (k_i, \rho_i, 1_n, (\psi_{i,v,b}), f_i)$ and $A_i = A_{k_i}$, $F_i = F_{k_i}$.

Step 1: separable elements are dense in Γ'. For $\sigma \in \Gamma^0_{\mathscr{H}}$ let $\delta_i(\sigma) \in A_i$ be the discriminant of the monic polynomial $\det(X - \rho_i(\sigma))$; it is a polynomial with integer coefficients in the entries of $\rho_i(\sigma)$, and $\det(X - \rho_i(\sigma))$ is separable over F_i if and only if $\delta_i(\sigma) \neq 0$. The entries of ρ_i are continuous for the T-adic topology of A_i (Definition 50.9), so δ_i is continuous; as A_i is T-adically separated, $\{0\}$ is closed and $S_i = \{\sigma : \delta_i(\sigma) \neq 0\}$ is open in $\Gamma^0_{\mathscr{H}}$. Put $S = \Gamma' \cap S_0 \cap \cdots \cap S_h$. Let $\sigma \in \Gamma'$ and U an open neighbourhood of σ. Put $U_0 = U$; if U_i is open and meets Γ', (Np3) (that is, (Pj2)) for x_i gives $\tau \in U_i \cap \Gamma' \cap S_i$, and $U_{i+1} = U_i \cap S_i$ is open and meets Γ' (it contains τ). Then $U_{h+1} \cap \Gamma' \subseteq U \cap S$ is non-empty. So the closure of S contains Γ', and $S \subseteq \Gamma'$.

Step 2: a finite detecting set. By (Cf1) of Proposition E.52 each x_i has a Selmer basis κ_i. By Proposition E.51, S detects (x_i, κ_i); by (Cf2) some finite $T_i \subseteq S$ detects (x_i, κ_i). Let $\{\sigma_1, \dots, \sigma_q\} = T_0 \cup \cdots \cup T_h \subseteq S \subseteq \Gamma'$. Detection is inherited by larger sets (a cocycle with $\ell_\sigma(z) = 0$ on the larger set has it on the smaller one), so $\{\sigma_1, \dots, \sigma_q\}$ detects every (x_i, κ_i).

Step 3: a common extension. Every σ_j lies in S, so $\det(X - \rho_i(\sigma_j))$ is separable over F_i for all i, j. (Cf3) applied to $(x_i, \kappa_i, \sigma_1, \dots, \sigma_q)$ gives $d_i \geq 1$; put $D = d_0 d_1 \cdots d_h$, a multiple of every d_i. (Cf3) with this D gives, for each i, an extension x_i' of x_i with $|k_i'| = \ell^D$, a Selmer basis κ_i' of x_i' detected by $\{\sigma_1, \dots, \sigma_q\}$, and the pairwise distinct roots $\alpha_{i,j,b}$; the residue fields all have ℓ^D elements. □

Proposition E.51. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ with $\ell \neq 2$, and $x = (k, \rho, 1_n, (\psi_{v,b}), f)$ a framed curve point of a universal polarised ordinary determinant over $\mathscr{H}$, normalised of shape $(a, n-a)$, such that for every $Z \in \mathrm{Ev}_a(A_k)$ some $T^s Z$ lies in the A_k-span of $\rho(\Gamma')$. Then for every Selmer basis κ of x and every subset $\Sigma \subseteq \Gamma'$ whose closure contains Γ', Σ detects (x, κ).*

Proof. Write $A = A_k$, $F = \mathrm{Frac}(A)$, $G = \Gamma^0_{\mathscr{H}}$, $G' = \Gamma'$, $\bar{c} = \pi_{\mathscr{H}}(c)$, $b = n - a$, $\chi = \chi_{\mathrm{cyc}}$ with values in $\mathbb{F}_\ell^\times \subseteq A^\times$ (A has characteristic ℓ). Let $z \in \mathrm{TwZ}(x, \kappa)$ with $\ell_\sigma(z) = 0$ for $\sigma \in \Sigma$. Put $E_0 = \mathrm{Ev}_a(F) \cong \mathrm{M}_a(F) \times \mathrm{M}_b(F)$, $N = \bigoplus_j \epsilon_j \mathrm{Od}_a(F)$, $E_1 = E_0 \oplus N$ and $P = \mathrm{PB}_a \otimes_A F = \eta E_0 \oplus \bigoplus_k \epsilon_k' \mathrm{Od}_a(F)$, inside $\mathrm{M}_n(\mathrm{PD}_g(F))$. As $\epsilon_i \epsilon_j = 0$, $\epsilon_i \epsilon_k' = \delta_{ik}\eta$ and $\eta\epsilon = \eta\epsilon' = 0$, and $\mathrm{Ev} \cdot \mathrm{Od}, \mathrm{Od} \cdot \mathrm{Ev} \subseteq \mathrm{Od}$, $\mathrm{Od} \cdot \mathrm{Od} \subseteq \mathrm{Ev}$: E_1 is an algebra with $N^2 = 0$, P is an E_1-bimodule,

$\rho_1(G) \subseteq E_1$, and $\langle p, e\rangle = (\eta\text{-coefficient of } \mathrm{tr}(pe))$ is a perfect pairing $P \times E_1 \to F$ (for $p = \eta P_0 + \sum_k \epsilon'_k P_k$, $e = e_0 + \sum_j \epsilon_j Y_j$ it is $\mathrm{tr}(P_0 e_0) + \sum_k \mathrm{tr}(P_k Y_k)$). Write $\gamma \star p = \chi(\gamma)\rho_1(\gamma) p \rho_1(\gamma)^{-1}$, so $z(\gamma\gamma') = z(\gamma) + \gamma \star z(\gamma')$, $z(\gamma^{-1}) = -\gamma^{-1} \star z(\gamma)$.

Step 0: vanishing on G'. $\sigma \mapsto \ell_\sigma(z)$ is a polynomial in the entries of $z(\sigma)$ and $\rho_1(\sigma)$, hence T-adically continuous; its zero set is closed and contains Σ, hence G'.

Step 1: a derivation on the group algebra. For $\sigma \in G'$ put $\zeta(\sigma) = z(\sigma)\rho_1(\sigma) \in P$. As $\chi = 1$ on G', $\zeta(\sigma\tau) = \zeta(\sigma)\rho_1(\tau) + \rho_1(\sigma)\zeta(\tau)$. Extend ρ_1 to an algebra homomorphism $F[G'] \to E_1$ and ζ linearly: $\zeta(uv) = \zeta(u)\rho_1(v) + \rho_1(u)\zeta(v)$, and $\langle \zeta(u), 1\rangle = 0$ for all u by Step 0.

Step 2: $\rho_1(F[G']) = E_1$. Let $R = \rho_1(F[G'])$. Its image under $e \mapsto e_0$ is $\rho(F[G']) = E_0$ (the spanning hypothesis, after inverting T), with kernel $R \cap N$; as $N^2 = 0$, $R \cap N$ is an E_0-sub-bimodule of $N \cong \mathrm{M}_{a\times b}(F)^g \oplus \mathrm{M}_{b\times a}(F)^g$, whose two corner bimodules are simple, non-isomorphic, with endomorphisms F; so $R \cap N = U \otimes \mathrm{M}_{a\times b} \oplus U' \otimes \mathrm{M}_{b\times a}$ for subspaces $U, U' \subseteq F^g$. Suppose $U \neq F^g$ and choose $c \in F^g \smallsetminus \{0\}$ with $\sum_j c_j u_j = 0$ for $u \in U$. Let $p\colon E_1 \to \mathrm{M}_{a\times b}(F)$ send $e_0 + \sum_j \epsilon_j Y_j$ to the (a,b)-corner of $\sum_j c_j Y_j$; it vanishes on $R \cap N$, so $p|_R = L(e_0)$ for a linear $L\colon E_0 \to \mathrm{M}_{a\times b}(F)$, and multiplicativity of R gives $L(ee') = e_{aa}L(e') + L(e)e'_{bb}$. So L is a derivation into a bimodule of the separable algebra E_0, hence inner: $L(e) = e_{aa}X - Xe_{bb}$ with $X \in \mathrm{M}_{a\times b}(F)$. At $e = \rho_1(\sigma)$, $\sigma \in G'$: the (a,b)-corner of $\kappa_c = \sum_j c_j \kappa_j$ is $\rho_{aa}(\sigma) X \rho_{bb}(\sigma)^{-1} - X$. By (Sd2), $\kappa_c(\bar c \gamma \bar c) = -{}^t\kappa_c(\gamma)$, and $\rho(\bar c\gamma\bar c) = {}^t\rho(\gamma)^{-1}$ (normalised point); $\bar c$ normalises G'; transposing exchanges the corners, so the (b,a)-corner of $\kappa_c(\sigma)$ is $\rho_{bb}(\sigma)(-{}^tX)\rho_{aa}(\sigma)^{-1} + {}^tX$. Hence $\delta = \kappa_c - dY$, with $Y \in \mathrm{Od}_a(F)$ having corners X, $-{}^tX$ (so ${}^tY = -Y$) and $dY(\gamma) = \rho(\gamma)Y\rho(\gamma)^{-1} - Y$, is a cocycle of G vanishing on the normal subgroup G'; then $\delta(\gamma) = \delta(\gamma \cdot \gamma^{-1}\sigma\gamma) = \delta(\sigma\gamma) = \rho(\sigma)\delta(\gamma)\rho(\sigma)^{-1}$ for $\sigma \in G'$, so $\delta(\gamma)$ commutes with E_0, and an element of $\mathrm{Od}_a(F)$ commuting with E_0 is 0. So $\sum_j (T^s c_j)\kappa_j = d(T^s Y) \in \mathrm{Bd}(x)$ for s with $T^s c \in A^g$, $T^s Y \in \mathrm{Od}_a(A)$, contradicting the independence of a Selmer basis. So $U = F^g$, and likewise (exchanging the corners) $U' = F^g$: $R \supseteq N$ and $R = E_1$.

Step 3: ζ factors through ρ_1. If $\rho_1(u) = 0$, then for all v, $\langle \zeta(u), \rho_1(v)\rangle = \langle \zeta(u)\rho_1(v), 1\rangle = \langle \zeta(uv), 1\rangle = 0$; by Step 2 and perfectness, $\zeta(u) = 0$. So $\zeta = D \circ \rho_1$ for a linear $D\colon E_1 \to P$ with $D(ee') = D(e)e' + eD(e')$. For $\gamma \in G$ and $\sigma \in G'$, $\sigma' = \gamma\sigma\gamma^{-1} \in G'$ and the cocycle identities give $z(\sigma') = \gamma \star z(\sigma) + z(\gamma) - \rho_1(\sigma')z(\gamma)\rho_1(\sigma')^{-1}$, hence, with $\mathrm{Ad}_\gamma(e) = \rho_1(\gamma)e\rho_1(\gamma)^{-1}$ and by linearity,

$$D(\mathrm{Ad}_\gamma e) = \chi(\gamma)\mathrm{Ad}_\gamma(D(e)) + z(\gamma)\mathrm{Ad}_\gamma(e) - \mathrm{Ad}_\gamma(e)z(\gamma) \qquad (e \in E_1). \tag{$*$}$$

Step 4: z is a coboundary over F. E_0 is separable, so there is $Y_1 \in P$ with $D(e) = eY_1 - Y_1 e$ for $e \in E_0$. Put $z'(\gamma) = z(\gamma) - (\gamma \star Y_1 - Y_1)$, a cocycle, with $z'(\sigma)\rho_1(\sigma) = D'(\rho_1(\sigma))$ on G' for $D'(e) = D(e) - (eY_1 - Y_1 e)$; D' satisfies Step 3 and $(*)$ with z', and $D'|_{E_0} = 0$. Then $D'|_N$ is a map of E_0-bimodules $N \to P$; it has no component in ηE_0 (no non-zero bimodule map from a corner $\mathrm{M}_{a\times b}$ or $\mathrm{M}_{b\times a}$ to the diagonal blocks), so by Schur $D'(\epsilon_j Y) = \sum_k \epsilon'_k M_{jk}(Y)$ with M_{jk} acting by a scalar on each corner. By (Rd6) there is $g \in G$ with $\chi(g) \neq 1$ in $\mathbb{F}_\ell$. Apply $(*)$ to $e = \epsilon_j Y$: $\mathrm{Ad}_g(\epsilon_j Y) = \epsilon_j \rho(g) Y \rho(g)^{-1}$; the commutator term lies in ηE_0; comparing ϵ'-parts, $\rho(g)M_{jk}(Y)\rho(g)^{-1} = \chi(g)\rho(g)M_{jk}(Y)\rho(g)^{-1}$ (M_{jk} commutes with conjugation by the block diagonal $\rho(g)$), so $M_{jk} = 0$. Thus $D' = 0$ and $z' = 0$ on G'. As in Step 2, $z'(\gamma)$ then commutes with $\rho_1(G')$, i.e. with E_1; the elements of P commuting with E_1 are ηw with w in the centre Z_0 of E_0, on which Ad is trivial. So $z'(\gamma) = \eta s(\gamma)$, $s(\gamma\gamma') = s(\gamma) + \chi(\gamma)s(\gamma')$, and comparing $s(\gamma g) = s(g\gamma)$ gives $s(\gamma) = \chi(\gamma)w - w$ with $w = -(1 - \chi(g))^{-1}s(g) \in Z_0$. Hence $z = dY_2$ for $Y_2 = Y_1 + \eta w \in P$, where $dY_2(\gamma) = \gamma \star Y_2 - Y_2$.

Step 5: symmetry and integrality. Let $\theta(p) = -\chi(\bar c)\,{}^t p$ on P (an involution, as $\chi(\bar c)^2 = 1$). By (Sd2) and normalisation, $\rho_1(\bar c\gamma\bar c) = {}^t\rho_1(\gamma)^{-1}$, and a direct computation gives $\theta(dY_2(\gamma)) = (d\,\theta(Y_2))(\bar c\gamma\bar c)$; as $z(\bar c\gamma\bar c) = \theta(z(\gamma))$ by (Sd4), $z = d\,\theta(Y_2)$ too, so $z = dY_3$ with $Y_3 = \frac{1}{2}(Y_2 + \theta(Y_2))$ ($\ell \neq 2$), $\theta(Y_3) = Y_3$. Choose s with $T^s Y_3 \in \mathrm{PB}_a$; then $T^s z = d(T^s Y_3) \in \mathrm{TwB}(x, \kappa)$. □

Proposition E.52. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ with $\ell \neq 2$ and $\Gamma_{\mathscr{H}}$ satisfying Φ_ℓ. Call a framed curve point $x = (k, \rho, 1_n, (\psi_{v,b}), f)$ of a universal polarised ordinary determinant over $\mathscr{H}$, normalised of shape $(a, n-a)$, **spanning** if for every $Z \in \mathrm{Ev}_a(A_k)$ some $T^s Z$ lies in the A_k-span of $\rho(\Gamma')$; an **extension** of x along an injective local homomorphism $j\colon A_k \to A_{k'}$ of $\mathcal{O}$-algebras making $A_{k'}$ a finite A_k-module is a spanning framed point $(k', j\circ\rho, 1_n, (j\circ\psi_{v,b}), j\circ f)$, normalised of shape $(a, n-a)$. Let x be spanning. Then:*

(Cf1) *x has a Selmer basis;*

(Cf2) *for every Selmer basis κ of x and every $\Sigma \subseteq \Gamma^0_{\mathscr{H}}$ that detects (x,κ), some finite subset of Σ detects (x,κ);*

(Cf3) *for every Selmer basis κ of x and all $\sigma_1, \dots, \sigma_q \in \Gamma^0_{\mathscr{H}}$ such that $\{\sigma_1, \dots, \sigma_q\}$ detects (x,κ) and every $\det(X - \rho(\sigma_j))$ is separable over F_k, there is an integer $d \geq 1$ such that for every multiple $D \geq 1$ of d there are an extension x' of x along some $j\colon A_k \to A_{k'}$ with $|k'| = \ell^D$, a Selmer basis κ' of x' such that $\{\sigma_1, \dots, \sigma_q\}$ detects (x', κ'), and pairwise distinct $\alpha_{j',1}, \dots, \alpha_{j',n} \in A_{k'}$ with $\det(X - j(\rho(\sigma_{j'}))) = \prod_i (X - \alpha_{j',i})$ for $1 \leq j' \leq q$.*

Proof. Write $A = A_k$, $F = F_k$, $\Gamma^0 = \Gamma^0_{\mathscr{H}}$; Γ^0 is open in $\Gamma_{\mathscr{H}}$, so it satisfies Φ_ℓ.

Step 1: continuous cocycles form a finitely generated module. Let V be a free A-module of finite rank with an A-linear action of Γ^0, continuous for the T-adic topology, and $Z^1(V)$ the A-module of T-adically continuous maps $z\colon \Gamma^0 \to V$ with $z(\gamma\gamma') = z(\gamma) + \gamma z(\gamma')$. The kernel Γ_1 of the action on V/TV is open. A cocycle with values in the finite group V/TV is, on Γ_1, a continuous homomorphism to $V/TV \cong (\mathbb{Z}/\ell)^N$ ($\ell = \mathrm{char}\, k$), of which there are finitely many by Φ_ℓ; and two cocycles with the same restriction to Γ_1 differ by a map that is constant on the finitely many cosets of Γ_1. So the cocycles with values in V/TV form a finite set. The kernel of reduction $Z^1(V) \to Z^1(V/TV)$ is $TZ^1(V)$ (T is a non-zero-divisor on V, and z/T is again a continuous cocycle); so $Z^1(V)/TZ^1(V)$ is finite-dimensional. $Z^1(V)$ is T-adically separated ($T^m Z^1(V)$ consists of cocycles with values in $T^m V$, and $\bigcap_m T^m V = 0$) and complete (a T-adic Cauchy sequence converges uniformly on Γ^0; the limit is a continuous cocycle). By topological Nakayama, $Z^1(V)$ is finitely generated; so is every A-submodule. Moreover $Z^1(V)$ is torsion-free.

Step 2: (Cf1). Take $V = \mathrm{Od}_a(A)$ with $\gamma \cdot Y = \rho(\gamma) Y \rho(\gamma)^{-1}$ (ρ is block diagonal, so V is stable). $Z^1_{\mathrm{ord}}(x)$ and $\mathrm{Bd}(x)$ are A-submodules of $Z^1(V)$ (all conditions of (Sd2) are A-linear in $(z, (Y_v))$), so $M = Z^1_{\mathrm{ord}}(x)/\mathrm{Bd}(x)$ is finitely generated and $M \otimes_A F$ is finite-dimensional. Choose $\kappa_1, \dots, \kappa_g \in Z^1_{\mathrm{ord}}(x)$ whose classes form an F-basis of $M \otimes_A F$ (multiply representatives by powers of T). If $\sum_j c_j \kappa_j \in \mathrm{Bd}(x)$ with $c \in A^g$, then $\sum_j c_j [\kappa_j] = 0$ in $M \otimes F$, so $c = 0$. For $z \in Z^1_{\mathrm{ord}}(x)$ write $[z] = \sum_j c'_j [\kappa_j]$ in $M \otimes F$, choose s with $c = T^s c' \in A^g$; then $T^s z - \sum_j c_j \kappa_j$ has torsion class in M, so $T^{s'}(T^s z - \sum_j c_j \kappa_j) \in \mathrm{Bd}(x)$ for some s'. So κ is a Selmer basis.

Step 3: (Cf2). Let $W = \mathrm{PB}_a$ with $\gamma \cdot Y = \chi_{\mathrm{cyc}}(\gamma) \rho_1(\gamma) Y \rho_1(\gamma)^{-1}$; W is stable (ρ is block diagonal, the κ_j have values in Od_a, and $\epsilon_j \epsilon'_j = \eta$ while all other products of $\epsilon, \epsilon', \eta$ vanish, so the conjugate of an element of PB_a has its same-block part in $A\eta$ and its different-block part in $\bigoplus_j A\epsilon'_j$), free of finite rank over A, with continuous action. $\mathrm{TwZ}(x,\kappa)$ is an A-submodule of $Z^1(W)$, hence finitely generated and torsion-free (Step 1), and the ℓ_σ are A-linear. For $\Sigma' \subseteq \Sigma$ put $K_{\Sigma'} = \{z \in \mathrm{TwZ} : \ell_\sigma(z) = 0 \ \forall \sigma \in \Sigma'\}$. Among finite $\Sigma_0 \subseteq \Sigma$ choose one with $\dim_F(K_{\Sigma_0} \otimes F)$ minimal. Let $z \in K_{\Sigma_0}$ and $\sigma \in \Sigma$. $K_{\Sigma_0 \cup \{\sigma\}} \otimes F \subseteq K_{\Sigma_0} \otimes F$ (localisation is exact) has the same dimension, so they are equal; hence $T^s z \in K_{\Sigma_0 \cup \{\sigma\}}$ for some s (TwZ is torsion-free), so $T^s \ell_\sigma(z) = 0$ and $\ell_\sigma(z) = 0$. So $K_{\Sigma_0} = K_\Sigma$, and as Σ detects, every $z \in K_{\Sigma_0}$ has $T^s z \in \mathrm{TwB}(x,\kappa)$: Σ_0 detects (x,κ).

Step 4: (Cf3). Let E/F be a splitting field of $\prod_j \det(X-\rho(\sigma_j))$, a finite separable extension. The integral closure A_E of A in E is a complete discrete valuation ring, finite and free over A, with finite residue field $k_E \supseteq k$; put $d = \log_\ell |k_E|$. Let $D \geq 1$ be a multiple of d, $k' \supseteq k_E$ the field with ℓ^D elements and $A' = k' \otimes_{k_E} A_E$, again a complete discrete valuation ring (unramified extension), finite free over A, with residue field k'; by Cohen, $A' \cong k'[[T']]$ for a uniformiser T', and $j\colon A \to A'$ is injective, local and finite. Fix an A-basis $e_1, \dots, e_r$ of A'. The roots of each $\det(X-\rho(\sigma_j))$ are integral over A and lie in E, hence in A', and are pairwise distinct by separability. Put $x' = (k', j\circ\rho, 1_n, (j\circ\psi_{v,b}), j\circ f)$ with frame $(j(w_v)), (I_u^x)$: it is a curve point normalised of shape $(a, n-a)$, framed, and spanning: $Z' \in \mathrm{Ev}_a(A')$ is $\sum_i e_i Z_i$ with $Z_i \in \mathrm{Ev}_a(A)$, and $j(T)$ is a unit times a power of T'. For every module V of Step 1 and its base change $V' = V \otimes_A A' = \bigoplus_i e_i V$ (with the action through j), a map $z' = \sum_i e_i z_i$ is a continuous cocycle (resp. satisfies the linear conditions of (Sd2), resp. of (Sd4)) if and only if every z_i is; so $Z^1_{\mathrm{ord}}(x') = \bigoplus_i e_i Z^1_{\mathrm{ord}}(x)$, $\mathrm{Bd}(x') = \bigoplus_i e_i \mathrm{Bd}(x)$, and with $\kappa'_j = j\circ\kappa_j$ (so $\rho'_1 = j\circ\rho_1$): $\mathrm{TwZ}(x',\kappa') = \bigoplus_i e_i \mathrm{TwZ}(x,\kappa)$, $\mathrm{TwB}(x',\kappa') = \bigoplus_i e_i \mathrm{TwB}(x,\kappa)$ and $\ell'_\sigma(\sum_i e_i z_i) = \sum_i e_i \ell_\sigma(z_i)$. Expanding coefficients in the basis (e_i), the two conditions of a Selmer basis for κ' follow componentwise from those for κ, and if $\ell'_{\sigma_j}(z') = 0$ for all j, then $\ell_{\sigma_j}(z_i) = 0$ for all i, j, so $T^s z_i \in \mathrm{TwB}(x,\kappa)$ for a common s, and $T^s z' \in \mathrm{TwB}(x',\kappa')$ (and T^s is a unit times a power of T'). So x', κ' and the roots have the properties of (Cf3), with $|k'| = \ell^D$. □

Proposition E.53. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ with $\ell \neq 2$, and $x = (k, \rho, P, (\psi_{v,b}), f)$ a patchable curve point of endoscopic type of a universal polarised ordinary determinant over $\mathscr{H}$. Then there are a finite field $k' \supseteq k$, an injective local homomorphism of $\mathcal{O}$-algebras $j\colon A_k \to A_{k'}$ making $A_{k'}$ a finite A_k-module, an integer $1 \leq a < n$ and a framed curve point $x' = (k', \rho', 1_n, (\psi'_{v,b}), f')$, normalised of shape $(a, n-a)$, with (Nm1), (Nm2) and (Nm3) of Proposition E.54.*

Proof. Write $A = A_k$, $F = F_k$, $\varpi = T \in A$, and give $V = F^n$ the symmetric bilinear form $\langle v, w\rangle = {}^t v P w$; it is non-degenerate, and $\langle A^n, A^n\rangle \subseteq A$ as $P \in \mathrm{GL}_n(A)$. From $\rho(\gamma^*) = P^{-1}\,{}^t\rho(\gamma) P$ we get ${}^t\rho(\gamma)P = P\rho(\gamma^*)$, that is, the adjunction identity

$$\langle \rho(\gamma)v, w\rangle = \langle v, \rho(\gamma^*)w\rangle \qquad (\gamma \in \Gamma^0,\ v, w \in V).$$

Step 1: the two blocks are orthogonal. Let $a, b \geq 1$, $a+b = n$, Q, r_1, r_2, P_1, P_2 be as in the definition of endoscopic type. Put $\rho_Q(\gamma) = Q\rho(\gamma)Q^{-1} = \mathrm{diag}(r_1(\gamma), r_2(\gamma))$ and $P_Q = {}^t Q^{-1} P Q^{-1}$, a symmetric matrix in $\mathrm{GL}_n(F)$. Then $P_Q \rho_Q(\gamma^*) = {}^t\rho_Q(\gamma) P_Q$ Write $P_Q = \begin{pmatrix} X & Y \\ {}^t Y & W \end{pmatrix}$ with X of size $a\times a$ and Y of size $a\times b$. The upper right block of the identity reads $Y r_2(\gamma^*) = {}^t r_1(\gamma) Y$. Inserting $r_2(\gamma^*) = P_2^{-1}\,{}^t r_2(\gamma) P_2$ and multiplying by P_2^{-1} on the right gives $(YP_2^{-1})\,{}^t r_2(\gamma) = {}^t r_1(\gamma)(YP_2^{-1})$; transposing, $Y_0 = {}^t(YP_2^{-1}) \in \mathrm{M}_{b\times a}(F)$ satisfies $Y_0 r_1(\gamma) = r_2(\gamma) Y_0$ for all $\gamma \in \Gamma^0$, in particular for $\gamma \in \Gamma'$. By the last condition of endoscopic type $Y_0 = 0$, so $Y = 0$ and $P_Q = \mathrm{diag}(X, W)$ with X, W invertible. Put $V_1 = Q^{-1}(F^a \oplus 0)$ and $V_2 = Q^{-1}(0 \oplus F^b)$. They are $\rho(\Gamma^0)$-stable, $V = V_1 \oplus V_2$, and as $\langle Q^{-1}x, Q^{-1}y\rangle = {}^t x P_Q y$, $\langle V_1, V_2\rangle = 0$ and the form is non-degenerate on each V_i.

Step 2: an almost self-dual stable lattice in each block. Fix $i \in \{1, 2\}$. A lattice in V_i is a finitely generated A-submodule spanning V_i; it is stable if $\rho(\gamma)M \subseteq M$ for all $\gamma \in \Gamma^0$; its dual is $M^\vee = \{v \in V_i : \langle v, M\rangle \subseteq A\}$, again a lattice (the dual basis of a basis of M is a basis of $M^\vee$, by non-degeneracy). If M is stable so is $M^\vee$: for $v \in M^\vee$, $m \in M$, $\langle \rho(\gamma)v, m\rangle = \langle v, \rho(\gamma^*)m\rangle \in A$ as $\gamma^* \in \Gamma^0$. If $M \subseteq M'$ then $M'^\vee \subseteq M^\vee$. The module $M_0 = A^n \cap V_i$ is a stable lattice (finitely generated as a submodule of A^n; spanning, as $\varpi^N v \in A^n$ for $v \in V_i$ and N large) with $\langle M_0, M_0\rangle \subseteq A$, i.e. $M_0 \subseteq M_0^\vee$. Let $\mathcal{M}$ be the set of stable lattices M with $M_0 \subseteq M \subseteq M^\vee$.

It contains M_0, and each of its members satisfies $M_0 \subseteq M \subseteq M^\vee \subseteq M_0^\vee$; as $M_0^\vee/M_0$ has finite length and k is finite, $M_0^\vee/M_0$ is finite, so $\mathcal{M}$ is finite and has a member M_i maximal for inclusion. Let $r \geq 0$ be least with $\varpi^r M_i^\vee \subseteq M_i$, and suppose $r \geq 2$. Then $N = M_i + \varpi^{r-1} M_i^\vee$ is a stable lattice containing M_i, strictly by minimality of r. For $x, y \in M_i^\vee$ we have $\langle \varpi^r x, y\rangle \in \langle M_i, M_i^\vee\rangle \subseteq A$, so $\langle \varpi^{r-1}x, \varpi^{r-1}y\rangle \in \varpi^{r-2}A \subseteq A$; together with $\langle M_i, M_i\rangle \subseteq A$ and $\langle M_i, \varpi^{r-1}M_i^\vee\rangle \subseteq A$ this gives $N \subseteq N^\vee$, so $N \in \mathcal{M}$, contradicting maximality. Hence

$$\varpi M_i^\vee \subseteq M_i \subseteq M_i^\vee.$$

Step 3: orthogonal bases. Let U be a finite-dimensional F-vector space with a non-degenerate symmetric bilinear form and $M \subseteq U$ a lattice with $\langle M, M\rangle \subseteq A$. Then M has an A-basis of pairwise orthogonal vectors. Induction on $\dim U$, the case 0 being empty: let μ be the least valuation of $\langle x, y\rangle$, $x, y \in M$ (finite, as the form is non-zero on the spanning set M). As $2\langle x, y\rangle = \langle x+y, x+y\rangle - \langle x, x\rangle - \langle y, y\rangle$ and $2 \in A^\times$ ($\ell \neq 2$), there is $e \in M$ with $\langle e, e\rangle$ of valuation μ. For $x \in M$, $c_x = \langle x, e\rangle/\langle e, e\rangle \in A$ by minimality of μ, and $x - c_x e \in M_e = \{y \in M : \langle y, e\rangle = 0\}$; so $M = Ae \oplus M_e$, and M_e is a lattice in $e^\perp$ (finitely generated as the image of M under $x \mapsto x - c_x e$; spanning, as $\varpi^N u \in M \cap e^\perp = M_e$ for $u \in e^\perp$ and N large), on which the form is non-degenerate since $U = Fe \oplus e^\perp$ orthogonally. Induction applies to M_e. Apply this to $M_i \subseteq V_i$: there is a basis $e_1, \ldots, e_m$ of M_i with $\langle e_p, e_q\rangle = 0$ for $p \neq q$ and $\langle e_p, e_p\rangle = u_p \varpi^{m_p}$, $u_p \in A^\times$, $m_p \geq 0$. The dual basis is $(u_p^{-1}\varpi^{-m_p} e_p)_p$, so $M_i^\vee$ has basis $(\varpi^{-m_p} e_p)_p$, and $\varpi M_i^\vee \subseteq M_i$ gives $m_p \in \{0, 1\}$.

Step 4: a unimodular stable lattice after adjoining a square root of ϖ. Let k' be the quadratic extension of k; every element of k is a square in k' (the squares of the cyclic group $k'^\times$ form its subgroup of index 2, of order $(|k|-1)(|k|+1)/2$, divisible by $|k|-1$ as $|k|$ is odd, so it contains $k^\times$). Let $A' = A_{k'} = k'[[T]]$, $\varpi' = T \in A'$, $F' = F_{k'}$, and $j\colon A \to A'$ the homomorphism of k-algebras with $j(\varpi) = \varpi'^2$. It is an injective local homomorphism of $\mathcal{O}$-algebras ($\mathcal{O}$ acts through $\mathbb{F}_\mathcal{O} \subseteq k$), and A' is a free $j(A)$-module with basis $(\epsilon \varpi'^e)$, ϵ in a k-basis of k', $e \in \{0, 1\}$; so A' is a finite A-module. Extend j to $F \to F'$, regard $V \subseteq F'^n = V'$ through j, extend the form F'-bilinearly (its matrix is $j(P)$), and put $V_i' = F'V_i$; then $V' = V_1' \oplus V_2'$ orthogonally. Let M_i' be the A'-span of $\{e_p : m_p = 0\} \cup \{\varpi'^{-1} e_p : m_p = 1\}$. As $\varpi'^{-1} e_p = \varpi' \cdot \varpi^{-1} e_p$, $M_i' = A'M_i + \varpi' A' M_i^\vee$, which is stable under $\rho(\Gamma^0)$ (the entries of $\rho(\gamma)$ lie in $j(A) \subseteq A'$, and M_i, $M_i^\vee$ are stable). Its basis vectors are pairwise orthogonal, with $\langle e_p, e_p\rangle = j(u_p)$ if $m_p = 0$ and $\langle \varpi'^{-1} e_p, \varpi'^{-1} e_p\rangle = \varpi'^{-2} j(u_p \varpi) = j(u_p)$ if $m_p = 1$. Write $j(u_p) = c(1 + \varpi' y)$ with $c \in k^\times$, $y \in A'$; $c = d^2$ with $d \in k'$, and $1 + \varpi' y = h^2$ with $h \in A'$ by Hensel's lemma ($X^2 - (1 + \varpi' y)$ has the simple root 1 modulo ϖ', as $2 \in k'^\times$). Dividing each basis vector by $s_p = dh \in A'^\times$ gives an A'-basis $f_1^{(i)}, \ldots, f_{\dim V_i}^{(i)}$ of M_i' with $\langle f_p^{(i)}, f_q^{(i)}\rangle = \delta_{pq}$.

Step 5: the point x'. Let $S \in \mathrm{GL}_n(F')$ be the inverse of the matrix whose columns are $f_1^{(1)}, \ldots, f_a^{(1)}, f_1^{(2)}, \ldots, f_b^{(2)}$, and $\rho'(\gamma) = S\, j(\rho(\gamma))\, S^{-1}$, the matrix of $\rho(\gamma)$ in this basis of V'. As V_1', V_2' are stable, $\rho'(\gamma)$ is block diagonal with blocks of sizes a, $n-a$ ($a = \dim V_1$, $1 \leq a < n$); as M_1', M_2' are stable, its entries lie in A', and so do those of $\rho'(\gamma)^{-1} = \rho'(\gamma^{-1})$: $\rho'(\gamma) \in \mathrm{GL}_n(A')$. By Steps 1 and 4 the Gram matrix of the basis is 1_n, i.e. ${}^tS^{-1} j(P) S^{-1} = 1_n$, so $j(P) = {}^tSS$ and

$$\rho'(\gamma^*) = S\, j(P)^{-1}\, {}^t j(\rho(\gamma))\, j(P)\, S^{-1} = S\, S^{-1}\, {}^tS^{-1}\, {}^t j(\rho(\gamma))\, {}^tS\, S\, S^{-1} = {}^t\rho'(\gamma).$$

The entries of ρ' are fixed F'-linear combinations of those of $j \circ \rho$, hence continuous for the ϖ'-adic topology ($j(\varpi^N A) \subseteq \varpi'^{2N} A'$). Put $\psi'_{v,b} = j \circ \psi_{v,b}$ and $f' = j \circ f$. For $\gamma_1, \ldots, \gamma_m \in \Gamma^0$ and indeterminates $y_1, \ldots, y_m$, $\det(\sum_p y_p \rho'(\gamma_p)) = j(\det(\sum_p y_p \rho(\gamma_p)))$ in $F'[y]$, both sides lying in $A'[y]$; specialising the y_p in any A'-algebra shows that the determinant of ρ' is the base change along j of the determinant of ρ, so $f' \circ D^{\mathrm{u}} = j \circ (f \circ D^{\mathrm{u}}) = \det \circ \rho'$.

Step 6: frames. Fix $v \in S_\ell$. There is $h \in \mathrm{GL}_n(F)$ with $h\rho(\pi(g))h^{-1}$ upper triangular with diagonal $(\psi_{v,1}(g), \ldots, \psi_{v,n}(g))$ for $g \in D_v$. Let $U_c \subseteq V'$ ($0 \leq c \leq n$) be the F'-span of the first

c columns of $j(h)^{-1}$: a flag of $\rho(\pi(D_v))$-stable subspaces on whose quotient U_c/U_{c-1} the group D_v acts by $\psi'_{v,c}$. By (Pj1) for x the characters $\psi_{v,1},\dots,\psi_{v,n}$ are pairwise distinct, hence so are the $\psi'_{v,c}$ (j is injective); by the Jordan–Hölder theorem each occurs exactly once among the composition factors of the D_v-module V', and these are the composition factors of V'_1 together with those of V'_2. (a) Every D_v-stable subspace $U\subseteq V'$ satisfies $U=(U\cap V'_1)\oplus(U\cap V'_2)$. Let p_1,p_2 be the projections of $V'=V'_1\oplus V'_2$ (they commute with D_v). The map $p_1(U)\to p_2(U)/(U\cap V'_2)$, $p_1(u)\mapsto p_2(u)$ mod $U\cap V'_2$, is well defined (if $p_1(u)=p_1(u')$ then $u-u'=p_2(u)-p_2(u')\in U\cap V'_2$), D_v-equivariant and surjective, with kernel $U\cap V'_1$ (if $p_2(u)\in U$ then $p_1(u)=u-p_2(u)\in U$). So $p_1(U)/(U\cap V'_1)\cong p_2(U)/(U\cap V'_2)$ is a subquotient both of V'_1 and of V'_2; if it were non-zero, one of its composition factors would occur twice in V'. Hence $p_1(U)\subseteq U$, so also $p_2(U)\subseteq U$, and $U=p_1(U)\oplus p_2(U)$ with $p_i(U)=U\cap V'_i$. (b) Let $d_{i,c}=\dim(U_c\cap V'_i)$. By (a) $d_{1,c}+d_{2,c}=c$, and both are non-decreasing in c, so for each $1\le c\le n$ exactly one $\beta(c)\in\{1,2\}$ has $d_{\beta(c),c}=d_{\beta(c),c-1}+1$. The modules $N_{i,c}=U_c\cap M'_i$ satisfy $N_{i,c-1}=N_{i,c}\cap U_{c-1}$, so $N_{i,c}/N_{i,c-1}$ is finitely generated and torsion-free of rank $d_{i,c}-d_{i,c-1}$, hence free of rank 0 or 1. Choose $u_c\in N_{\beta(c),c}$ mapping to a generator of $N_{\beta(c),c}/N_{\beta(c),c-1}$. By induction on c, the $u_{c'}$ with $c'\le c$ and $\beta(c')=i$ form a basis of $N_{i,c}$; as $N_{i,n}=M'_i$, $(u_1,\dots,u_n)$ is an A'-basis of $M'_1\oplus M'_2$. Let $w_v\in\mathrm{M}_n(A')$ have as c-th column the coordinates of u_c in the basis $(f^{(1)},f^{(2)})$; it lies in $\mathrm{GL}_n(A')$ (as $M'_1\oplus M'_2$ has the basis $(f^{(1)},f^{(2)})$), and its c-th column has its non-zero entries in the block $\beta(c)$. For $g\in D_v$, $\rho(\pi(g))u_c-\psi'_{v,c}(g)u_c$ lies in U_{c-1} and in $M'_1\oplus M'_2$, hence by (a) in $(U_{c-1}\cap M'_1)\oplus(U_{c-1}\cap M'_2)=N_{1,c-1}\oplus N_{2,c-1}$, the span of $u_1,\dots,u_{c-1}$. So $w_v^{-1}\rho'(\pi(g))w_v$ is upper triangular with diagonal $(\psi'_{v,1}(g),\dots,\psi'_{v,n}(g))$. With $I_u^{x'}=\{1\}$ for $u\in\mathrm{St}$, $((w_v)_v,(I_u^{x'})_u)$ is a frame. In particular $\rho'\circ\pi$ is triangular over F' on each D_v with diagonal $(\psi'_{v,c})_c$, so by Step 5 $x'=(k',\rho',1_n,(\psi'_{v,b}),f')$ is a curve point, normalised of shape $(a,n-a)$, and framed.

Step 7: (Nm1)–(Nm3). (Nm1) holds by construction and Step 5. (Pj1) for x': $\psi'=j\circ\psi$ with j injective, and $j\circ\chi_{\mathrm{cyc}}|_{D_v}=\chi_{\mathrm{cyc}}|_{D_v}$ as χ_{cyc} takes values in the image of $\mathcal{O}^\times$ and j is a map of $\mathcal{O}$-algebras. (Np3): the characteristic polynomial of $\rho'(\sigma)$ is the image under j of that of $\rho(\sigma)$, and a polynomial separable over F stays separable over F'; so (Pj2) for x gives (Pj2) for x'. (Np4): let $Z\in\mathrm{M}_n(A')$ with $\chi_{\mathrm{cyc}}(\gamma)\rho'(\gamma)Z=Z\rho'(\gamma)$ for $\gamma\in\Delta_0$. Then $Z_S=S^{-1}ZS$ satisfies $\chi_{\mathrm{cyc}}(\gamma)j(\rho(\gamma))Z_S=Z_Sj(\rho(\gamma))$; writing $Z_S=\sum_q\epsilon_q j(Z_q)$ with (ϵ_q) a basis of F' over $j(F)$ and $Z_q\in\mathrm{M}_n(F)$, each Z_q satisfies the same relation for ρ (the entries of $\rho(\gamma)$ and $\chi_{\mathrm{cyc}}(\gamma)$ lie in F), so $Z_q=0$ by (Pj3) for x, and $Z=0$. (Nm3): $\rho'(\gamma)=(Sj(Q)^{-1})\,\mathrm{diag}(j(r_1(\gamma)),j(r_2(\gamma)))\,(Sj(Q)^{-1})^{-1}$, and $Sj(Q)^{-1}$ maps $F'^a\oplus 0=j(Q)(V'_1)$ onto $S(V'_1)=F'^a\oplus 0$ and $0\oplus F'^{n-a}=j(Q)(V'_2)$ onto $S(V'_2)=0\oplus F'^{n-a}$; so $Sj(Q)^{-1}=\mathrm{diag}(S_1,S_2)$ and $\rho'(\gamma)=\mathrm{diag}(r'_1(\gamma),r'_2(\gamma))$ with $r'_i=S_i\,j(r_i)\,S_i^{-1}$. Let $\bar F$ be an algebraic closure of F'. Then r'_1, r'_2 are irreducible over $\bar F$ on Γ', and 0 is the only $Y\in\mathrm{M}_{(n-a)\times a}(\bar F)$ with $Yr'_1(\sigma)=r'_2(\sigma)Y$ on Γ' (for such Y, $S_2^{-1}YS_1$ intertwines r_1 and r_2). Let $E\subseteq\mathrm{M}_a(\bar F)\times\mathrm{M}_{n-a}(\bar F)$ be the $\bar F$-span of $\{(r'_1(\sigma),r'_2(\sigma)):\sigma\in\Gamma'\}$, a subalgebra whose two projections p_1, p_2 are onto by Burnside's theorem. $K=E\cap(\mathrm{M}_a\times 0)$ is stable under left multiplication by E, so $p_1(K)$ is a two-sided ideal of $\mathrm{M}_a(\bar F)$. If $p_1(K)=\mathrm{M}_a(\bar F)$, then $\mathrm{M}_a\times 0\subseteq E$ and, p_2 being onto, $E=\mathrm{M}_a(\bar F)\times\mathrm{M}_{n-a}(\bar F)$. If $p_1(K)=0$, then $p_2|_E$ is injective, $p_1\circ(p_2|_E)^{-1}\colon\mathrm{M}_{n-a}(\bar F)\to\mathrm{M}_a(\bar F)$ is a surjective algebra homomorphism, injective as its source is simple, so $a=n-a$ and by Skolem–Noether it is $Z\mapsto Y^{-1}ZY$ for some invertible Y; then $Yr'_1(\sigma)=r'_2(\sigma)Y$ on Γ' with $Y\neq 0$, which is excluded. So $\dim_{\bar F}E=a^2+(n-a)^2$. The F'-span of $\rho'(\Gamma')$ lies in $\mathrm{Ev}_a(F')$ and has the same dimension as E (the rank of a family of vectors does not change under field extension), so it is $\mathrm{Ev}_a(F')$. Choose $\sigma_1,\dots,\sigma_N\in\Gamma'$ with $\rho'(\sigma_q)$ an F'-basis of $\mathrm{Ev}_a(F')$; each elementary matrix in Ev_a is an F'-combination of them, and clearing denominators gives $s\ge 0$ with $\varpi'^s\mathrm{Ev}_a(A')$ contained in the A'-span of $\rho'(\Gamma')$. This is (Nm3), ϖ' being the variable T of $A_{k'}$. □

Proposition E.54. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ with $\ell\neq 2$,*

$(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ *a universal polarised ordinary determinant over* $\mathscr{H}$*, and* $x = (k, \rho, P, (\psi_{v,a}), f)$ *a patchable curve point of* B_n^{po} *of irreducible or of endoscopic type. Then there are a finite field* $k' \supseteq k$*, an injective local homomorphism of* $\mathcal{O}$*-algebras* $j\colon A_k \to A_{k'}$ *making* $A_{k'}$ *a finite* A_k*-module, an integer* $1 \le a \le n$ *and a framed curve point* $x' = (k', \rho', 1_n, (\psi'_{v,a}), f')$*, normalised of shape* $(a, n-a)$*, such that:*

(Nm1) $f' = j \circ f$ *and* $\psi'_{v,b} = j \circ \psi_{v,b}$ *for all* $v \in S_\ell$*,* $1 \le b \le n$*; in particular* $\det \circ \rho' = j \circ \det \circ \rho$*;*

(Nm2) x' *satisfies (Pj1), (Np3) and (Np4);*

(Nm3) *for every* $Z \in \mathrm{Ev}_a(A_{k'})$ *there is* $s \ge 0$ *such that* $T^s Z$ *lies in the* $A_{k'}$*-span of* $\rho'(\Gamma')$*;*

(Nm4) *if* x *is of irreducible type, then* $a = n$*.*

Proof. If x is of irreducible type, Lemma E.55 gives k', j (the inclusion $k[[T]] \subseteq k'[[T]]$, injective, local, and finite as $[k' : k] < \infty$) and x' with (Nm1)–(Nm4) and $a = n$. If x is of endoscopic type, Proposition E.53 gives k', j, $1 \le a < n$ and x' with (Nm1)–(Nm3); (Nm4) is void, as x is not of irreducible type (the two types exclude each other: at a point of endoscopic type $\rho \otimes \bar{F}_k$ restricted to Γ' is conjugate to a block-diagonal representation with blocks of sizes $a, b \ge 1$, so it is reducible). □

Lemma E.55. *Let* $\mathscr{H}$ *be a rank-n Hecke datum over a coefficient ring* $\mathcal{O}$ *with* $\ell \ne 2$*, and* $x = (k, \rho, P, (\psi_{v,a}), f)$ *a patchable curve point of irreducible type of a universal polarised ordinary determinant over* $\mathscr{H}$*. Then there are a finite field* $k' \supseteq k$ *with* $[k' : k] \le 2^n$*, and, with* $j\colon k[[T]] \to k'[[T]]$ *the inclusion, a framed curve point* $x' = (k', \rho', 1_n, (j \circ \psi_{v,a}), j \circ f)$*, normalised of shape* $(n, 0)$*, with (Nm1)–(Nm4) of Proposition E.54; moreover* $\rho' = S\rho S^{-1}$ *for some* $S \in \mathrm{GL}_n(k'[[T]])$*.*

Proof. Write $A = k[[T]]$, $F = k((T))$. $P \in \mathrm{GL}_n(A)$ is symmetric and $2 \in A^\times$, so $P = {}^t S_0 \,\mathrm{diag}(u_1, \dots, u_n)\, S_0$ with $S_0 \in \mathrm{GL}_n(A)$, $u_i \in A^\times$. Write $u_i = c_i(1 + Tm_i)$, $c_i \in k^\times$; $1 + Tm_i$ has a square root in A (Hensel, $\ell \ne 2$), and c_i has one in the field k' obtained by adjoining square roots of $c_1, \dots, c_n$ ($[k' : k] \le 2^n$). So $\mathrm{diag}(u_i) = {}^t R\, R$ with R diagonal in $\mathrm{GL}_n(A')$, $A' = k'[[T]]$; put $S = RS_0$, so $P = {}^t S S$, and $\rho'(\gamma) = S\rho(\gamma)S^{-1} \in \mathrm{GL}_n(A')$. Then $\rho'(\gamma^*) = SP^{-1}\,{}^t\rho(\gamma) P S^{-1} = SS^{-1}\,{}^t S^{-1}\,{}^t\rho(\gamma)\,{}^t S S\, S^{-1} = {}^t\rho'(\gamma)$; the entries of ρ' are T-adically continuous; $\det \circ \rho' = \det \circ \rho$, so $j \circ f$ and $j \circ \psi_{v,a}$ make x' a curve point once it is triangular on each D_v, which follows from the frame below; it is normalised of shape $(n, 0)$ with $P' = 1_n$.

Frames. For $v \in S_\ell$ there is $h \in \mathrm{GL}_n(F)$ with $h\rho(\pi g)h^{-1}$ upper triangular with diagonal $(\psi_{v,1}(g), \dots, \psi_{v,n}(g))$ for $g \in D_v$; the subspaces $V_i \subseteq F'^n$ ($F' = k'((T))$) spanned by the first i columns of Sh^{-1} form a $\rho'(\pi(D_v))$-stable flag on whose i-th graded piece D_v acts by $\psi_{v,i}$. The modules $M_i = V_i \cap A'^n$ are saturated, hence direct summands, and $\rho'(\pi(D_v))$-stable (ρ' has entries in A'); choosing a basis $w_1, \dots, w_n$ of A'^n with $M_i = \langle w_1, \dots, w_i \rangle$, $w_v = (w_1 | \cdots | w_n) \in \mathrm{GL}_n(A')$ has $w_v^{-1}\rho'(\pi g) w_v$ upper triangular with diagonal $(\psi_{v,i}(g))_i$. With $I_u^{x'} = \{1\}$ this is a frame (the column condition is empty for $a = n$).

(Nm1)–(Nm4). (Nm1) holds by construction and (Nm4) with $a = n$. (Pj1) for x': j is injective. (Np3): $\det(X - \rho'(\sigma)) = \det(X - \rho(\sigma))$, separable over F hence over F', so (Pj2) for x gives it for x'. (Np4): if $Z \in \mathrm{M}_n(A')$ satisfies $\chi_{\mathrm{cyc}}(\gamma)\rho'(\gamma)Z = Z\rho'(\gamma)$ on Δ_0, write $S^{-1}ZS = \sum_i e_i Z_i$ with (e_i) a k-basis of k' (an F-basis of F') and $Z_i \in \mathrm{M}_n(F)$; each Z_i satisfies the same relation for ρ, so $Z_i = 0$ by (Pj3) and $Z = 0$. (Nm3): $\rho \otimes \bar{F}$ is irreducible on Γ', so by Burnside the $\bar{F}$-span of $\rho(\Gamma')$ is $\mathrm{M}_n(\bar{F})$, hence the F'-span of $\rho'(\Gamma')$ is $\mathrm{M}_n(F')$; the A'-span Λ of $\rho'(\Gamma')$ is a submodule of $\mathrm{M}_n(A')$ of full rank, finitely generated (A' is Noetherian), so $T^s \mathrm{M}_n(A') \subseteq \Lambda$ for some s. □

E.2.4 The map from patched tuples to the eigenvalue algebra

This subsection proves Proposition E.56: the ring of patched tuples maps onto the eigenvalue algebra, the map kills the patching variables and the relations, and it detects the support. The patched tuples carry an abstract patching tower with nq patching variables (Proposition E.57) and form a universal ring on the patched group (Proposition E.60). The determinants of automorphic patching data factor through the pro-ℓ reduced levels (Corollary E.61, Lemma E.62), and the comparison with the level modules holds up to the isogeny element (Lemma E.58, Lemma E.63). Proposition E.56 is used in Proposition E.39 and in the construction of the patched ring, Proposition E.134.

Proposition E.56. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, and let automorphic patching data for an ideal $J \subseteq B$ along a Taylor–Wiles datum $\mathcal{D}$ of size q, $\mathcal{O}_1$, limit objects and $\tilde{c}$ be given as in Definition E.37, with patched tower data $(R, \beta, X, f, I, e+1, (M_N^\natural), \mathcal{U})$, eigenvalue algebra C^{sp}, $u\colon B \to C^{\mathrm{sp}}$ and $\pi\colon R \to C^{\mathrm{sp}}$. Assume that B is topologically generated as an $\mathcal{O}$-algebra by the coefficients of the polynomials $\chi_{D^{\mathrm{u}}}(\gamma)$, $\gamma \in \Gamma^0_{\mathscr{H}}$, and the values $\psi^{\mathrm{d}}_{v,a}(g)$, $v \in S_\ell$, $g \in D_v$. Then:*

(Wz1) *π is surjective, $\pi(\beta(y_{j,i})) = 0$ for all j, i, and $\pi(I) = 0$;*

(Wz2) *for every prime ideal P of C^{sp}: $\mathrm{Ann}_R(X) \subseteq \pi^{-1}(P)$ if and only if $J \subseteq u^{-1}(P)$;*

(Wz3) *let x be a curve point of B with homomorphism $f_x\colon B \to A_k$, $|k| = |\mathcal{O}_1/\mathfrak{m}_{\mathcal{O}_1}|$, which has eigenvalue data along $\mathcal{D}$, and let $f'\colon C^{\mathrm{sp}} \to A_k$ be a surjective ring homomorphism with $f' \circ u = f_x$ and $f'(z) = T$. Then $\mathfrak{q} = \pi^{-1}(\ker f')$ is a prime ideal of R with $f \notin \mathfrak{q}$, and for $\mathfrak{p} = \{a \in \Lambda : \beta(a) \in \mathfrak{q}\}$ the ring $\Lambda_{\mathfrak{p}}$ is regular local of dimension $1 + n[L^+ : \mathbb{Q}]$.*

Proof. Write $A = C_0^{\mathrm{sp}}$. By definition of π_0, for $x \in R_{\mathrm{tup}}[P^0]$, $\pi_0(\mathrm{tr}_{D^{\mathrm{t}}}(x)) = \mathrm{tr}_{\bar{D}^{\mathrm{u}}_A}(\bar{x})$ and $\pi_0(\Lambda_i(D^{\mathrm{t}})(g)) = \Lambda_i(\bar{D}^{\mathrm{u}}_A)(\bar{\theta}\pi_\infty(g))$, where $\bar{D}^{\mathrm{u}}_A$ is the base change of $\bar{D}^{\mathrm{u}}$ to A and $\bar{x}$ is the image of x under π_0 on coefficients and $\bar{\theta} \circ \pi_\infty$ on P^0; also $\pi_0(\psi^{\mathrm{t}}_{v,a}) = \psi^{\mathrm{d}}_{v,a}$, $\pi_0(s^{\mathrm{t}}) = z$, $\pi_0(U^{\mathrm{t}}_{j,i}) = u_{j,i}$, $\pi_0(y^{\mathrm{t}}_{j,i}) = 0$.

(Wz1). $\pi(\beta(y_{j,i})) = \pi(y^{\mathrm{t}}_{j,i}) = 0$. *Surjectivity.* $\bar{\theta} \circ \pi_\infty$ maps P^0 onto $\bar{\Gamma}^0$ (π_∞, $\bar{\theta}$ are onto and P^0 is a full preimage), so the image of π_0 contains the coefficients of all $\chi_{D^{\mathrm{u}}}(\gamma)$, $\gamma \in \Gamma^0_{\mathscr{H}}$, the values $\psi^{\mathrm{d}}_{v,a}(g)$, z and $Z_{j,i} = u_{j,i} - \tilde{u}_{j,i}$. By hypothesis and continuity of $B \to A$, the closure of the $\mathcal{O}$-algebra they generate contains B, z and the $Z_{j,i}$, hence is A: the image of π_0 is dense. As $\mathfrak{m}_A^2$ is open, $R_{\mathrm{tup}} \to A/\mathfrak{m}_A^2$ is onto, so $\mathfrak{m}_A = \mathfrak{m}_{R_{\mathrm{tup}}}A + \mathfrak{m}_A^2$ and, by Nakayama, $\mathfrak{m}_A = \mathfrak{m}_{R_{\mathrm{tup}}}A$. So A is separated for the $\mathfrak{m}_{R_{\mathrm{tup}}}$-adic topology and $A/\mathfrak{m}_{R_{\mathrm{tup}}}A = \mathbb{F}$ is generated by 1; by [166, Theorem 8.4], A is generated by 1 as an R_{tup}-module, i.e. π_0 is onto, and so is $\pi = \mathrm{id}_{\mathcal{O}_1} \otimes \pi_0$. *The relations.* It suffices that π_0 kills each relation of (Wv1)–(Wv6) (formed in R_{tup}). Let $\bar{h}$ be the image of h. (Wv1): $k \in \ker \pi_\infty$ maps to 1, so $h(k-1) \mapsto \bar{h}(1-1) = 0$. (Wv2): choose $g_0 \in G_L$ whose image in $\bar{\Gamma}^0$ is $\bar{\theta}\pi_\infty(g)$. By (L2), $\bar{\theta}(c_\infty) = \bar{c}$, so $\tilde{c}g^{-1}\tilde{c}^{-1}$ maps to the image of $cg_0^{-1}c^{-1} = cg_0^{-1}c$ ($c^2 = 1$); (Od1) for D^{u} gives equal characteristic polynomials, hence $\Lambda_i(\bar{D}^{\mathrm{u}}_A)$ agrees on the two images. (Wv3): by (L2), $\bar{\theta} \circ F_v$ is the map $D_v \to \bar{\Gamma}$ induced by $\pi_{\mathscr{H}}$, so g'_a maps to the image of g_a, and the image of the relation is $\mathrm{tr}(\bar{h}\prod_a(\pi(g_a) - \psi^{\mathrm{d}}_{v,a}(g_a)))$, which is 0 by (Od2). (Wv4): $g \in I_{\bar{u}} \cap G_L$ with all $\bar{\chi}_a(g) = 1$ at $u \in \mathrm{St}$ (not above ℓ) lies in $N_{\mathscr{H}}$, so $\pi_{\mathscr{H}}(g) = 1$ and, by (L2), g' maps to 1; the relation maps to 0 as in (Wv1). (Wv5): ϕ maps to $\bar{\theta}(\Phi_j) = \bar{\sigma}_j$ by (L2), and in $A[X]$, $\prod_k(X - u_{j,k}) = \chi_{D^{\mathrm{u}}}(\sigma_j)(X)$ by definition of A. So the relation maps to $\mathrm{tr}(\bar{h}\,\chi(\bar{\sigma}_j)(\bar{\sigma}_j))$ with $\chi = \chi_{\bar{D}^{\mathrm{u}}_A}$; $\chi(\bar{\sigma}_j)(\bar{\sigma}_j)$ lies in the Cayley–Hamilton ideal, contained in $\ker(\bar{D}^{\mathrm{u}}_A)$ by Theorem B.88, and the trace vanishes on the two-sided ideal $\ker(\bar{D}^{\mathrm{u}}_A)$. (Wv6): τ maps to $\bar{\theta}(\mathrm{T}_j) = 1$ by (L2) and $y^{\mathrm{t}}_{j,i} \mapsto 0$, so the factor $\tau - 1 - y^{\mathrm{t}}_{j,i}$ maps to 0.

(Wz2). As R acts on X through the surjection π, $\mathrm{Ann}_R(X) = \pi^{-1}(\mathrm{Ann}_{C^{\mathrm{sp}}}(X))$, so $\mathrm{Ann}_R(X) \subseteq \pi^{-1}(P)$ if and only if $\mathrm{Ann}_{C^{\mathrm{sp}}}(X) \subseteq P$, i.e. $P \in \mathrm{Supp}(X)$ ($X = C^{\mathrm{sp}} \otimes_B M$ is finite over C^{sp}, M being finite over B as in Proposition E.57). Put $\mathfrak{p}_B = u^{-1}(P)$. Then $X_P = C^{\mathrm{sp}}_P \otimes_{B_{\mathfrak{p}_B}} M_{\mathfrak{p}_B}$; if $M_{\mathfrak{p}_B} \neq 0$, Nakayama gives $M_{\mathfrak{p}_B} \otimes \kappa(\mathfrak{p}_B) \neq 0$ and $X_P \otimes \kappa(P) = (M_{\mathfrak{p}_B} \otimes \kappa(\mathfrak{p}_B)) \otimes_{\kappa(\mathfrak{p}_B)} \kappa(P) \neq 0$; if $M_{\mathfrak{p}_B} = 0$ then $X_P = 0$. So $P \in \mathrm{Supp}(X)$ if and only if $\mathfrak{p}_B \in \mathrm{Supp}_B(M) = V(\mathrm{Ann}_B(M))$, which by (Hl1) holds if and only if $J \subseteq \mathfrak{p}_B$.

(Wz3). $\ker f'$ is prime (A_k is a domain) and π is onto, so $\mathfrak{q}$ is prime. Let $(\alpha_{j,i})$ be the eigenvalue data of x: $\det(X - \rho(\sigma_j)) = \prod_i (X - \alpha_{j,i})$ with $\alpha_{j,1}, \dots, \alpha_{j,n}$ pairwise distinct. As $f_x \circ D^{\mathrm{u}} = \det \circ \rho$ and $f' \circ u = f_x$, applying f' to $\prod_i (X - u_{j,i}) = \chi_{D^{\mathrm{u}}}(\sigma_j)(X)$ gives $\prod_i (X - f'(u_{j,i})) = \prod_i (X - \alpha_{j,i})$ in $A_k[X]$; A_k being a domain, $(f'(u_{j,i}))_i$ is a permutation of $(\alpha_{j,i})_i$. Hence $f'(\pi(f)) = \prod_j \prod_{i \neq i'} (f'(u_{j,i}) - f'(u_{j,i'}))^2 \neq 0$, i.e. $f \notin \mathfrak{q}$. Next, $\phi = f' \circ \pi \circ \beta|_\Lambda \colon \Lambda \to A_k$ is a local homomorphism (composite of continuous local maps and a surjection of local rings) with $\phi(X_0) = f'(z) = T$; its residue map $\mathcal{O}_1/\mathfrak{m}_{\mathcal{O}_1} \to k$ is an injection of finite fields of the same cardinality, hence bijective. So $\mathfrak{m}_\Lambda A_k \supseteq T A_k = \mathfrak{m}_{A_k}$, hence $\mathfrak{m}_\Lambda A_k = \mathfrak{m}_{A_k}$ and $A_k/\mathfrak{m}_\Lambda A_k = k$ is generated by 1 over Λ; A_k is $\mathfrak{m}_\Lambda$-adically separated, so by [166, Theorem 8.4] ϕ is onto, and $\Lambda/\mathfrak{p} \cong A_k$ ($\mathfrak{p} = \ker \phi$) has dimension 1. $\Lambda = \mathcal{O}_1[[X_0, \dots, X_D]]$ is regular local of dimension $D + 2$, so $\Lambda_{\mathfrak{p}}$ is regular of dimension $\mathrm{ht}\,\mathfrak{p} = (D+2) - 1 = 1 + D = 1 + n[L^+ : \mathbb{Q}]$. □

Proposition E.57. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, and let automorphic patching data with exponent e for an ideal J along a Taylor–Wiles datum of size q, $\mathcal{O}_1$, limit objects and $\tilde{c}$ be given as in Definition E.37, with patched tower data $(R, \beta, X, f, I, e+1, (M_N^\natural)_{N \geq 1}, \mathcal{U})$ over $S = \Lambda[[y_{j,i} : 1 \leq j \leq q,\ 1 \leq i \leq n]]$. Then this tuple is an abstract patching tower over S, with nq patching variables. Moreover the eigenvalue algebra C_0^{sp} is a free $B[[z]]$-module of finite positive rank.*

Proof. Step 0: the ring. $R_{\mathrm{tup}} \in \mathrm{CNL}_{\mathcal{O}}$ by Proposition E.60, with residue field $\mathbb{F} = \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$. $R = \mathcal{O}_1 \otimes_{\mathcal{O}} R_{\mathrm{tup}}$ is a free R_{tup}-module of finite rank, hence Noetherian and integral over R_{tup}; every maximal ideal of R therefore contracts to $\mathfrak{m}_{R_{\mathrm{tup}}}$ and contains $\mathfrak{m}_{R_{\mathrm{tup}}} R$, and $R/\mathfrak{m}_{R_{\mathrm{tup}}} R = \mathcal{O}_1/\mathfrak{m}_{\mathcal{O}}\mathcal{O}_1$ is a field because $\mathfrak{m}_{\mathcal{O}}\mathcal{O}_1 = \mathfrak{m}_{\mathcal{O}_1}$. So R is a Noetherian local ring with $\mathfrak{m}_R = \mathfrak{m}_{R_{\mathrm{tup}}} R$. Λ is a complete Noetherian local ring with finite residue field $\mathcal{O}_1/\mathfrak{m}_{\mathcal{O}_1}$, $e + 1 \geq 1$, and $\mathcal{U}$ is a non-principal ultrafilter by Lemma E.42; it contains the complement of every finite set.

Step 1: C_0^{sp} is free of finite positive rank over $A = B[[z]]$. A is a complete Noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$, $C_0^{\mathrm{sp}} = \mathrm{Sp}_A(P)$ for $P_j = \chi_{D^{\mathrm{u}}}(\sigma_j)$ in the notation of Lemma E.59, and $P_j \equiv \prod_i (X - \bar{\chi}_i(\sigma_j)) \equiv \prod_i (X - \tilde{u}_{j,i})$ modulo $\mathfrak{m}_A$ (the reduction of D^{u} is $\det \circ \mathrm{diag}(\bar{\chi}_1, \dots, \bar{\chi}_n)$). By Lemma E.59, C_0^{sp} is free over A of some rank $r_0 \geq 1$.

Step 2: (Tow0). The Λ-module structure of X is through $\pi \circ \beta$, and by the definition of π_0: $\pi(\beta(X_0)) = z$, $\pi(\beta(X_k)) = c_k \psi^{\mathrm{d}}_{v_k, a_k}(\gamma_k) - 1 = \lambda_B(X_k) \in B$ (the weight map of $(\psi^{\mathrm{d}}_{v,a})$), and $\pi(\beta(y_{j,i})) = 0$. So $\beta(y_{j,i}) X = 0$. By (Hl1), $M \cong \Lambda_D^m$ as Λ_D-modules through λ_B; M is finitely generated over Λ_D, hence over B, hence finitely presented over the Noetherian ring B, and $B[[z]] \otimes_B M \cong M[[z]]$ (true for finite free M; both sides are right exact in M, as $M \mapsto M[[z]]$ is exact). Hence, with a basis of C_0^{sp} over A from Step 1,

$$X = \mathcal{O}_1 \otimes_{\mathcal{O}} (C_0^{\mathrm{sp}} \otimes_A M[[z]]) \cong \mathcal{O}_1 \otimes_{\mathcal{O}} M[[z]]^{r_0} \cong (\mathcal{O}_1 \otimes_{\mathcal{O}} \Lambda_D[[z]])^{r_0 m} = \Lambda^{r_0 m},$$

the isomorphisms being A-linear, so Λ_D-linear through λ_B and with X_0 acting as z; X is a free Λ-module of rank $r_0 m$.

Step 3: (Tow1). Let $k, j \geq 1$ and $N \in \mathcal{N}_0$. By (Ha2), M_N is finitely generated over S_∞ through s_N, hence over R_N, and $R_N[[z]] \otimes_{R_N} M_N \cong M_N[[z]]$ as in Step 2. S acts on $M_N^\natural$ through $\rho_N \circ \beta$: $X_0 \mapsto z$, $X_k \mapsto c_k \psi_{N, v_k, a_k}(\gamma_k) - 1 = s_N(X_k)$ by (Ha1), $y_{j,i} \mapsto s_N(Y_{j,i})$. So

(Ha2) gives an isomorphism of S-modules $M_N^\natural \cong \mathcal{O}_1 \otimes_{\mathcal{O}} ((S_\infty/\mathfrak{d}_{a_N})[[z]])^{\mathrm{rk}_N} = (S/\mathfrak{d}_{a_N}S)^{\mathrm{rk}_N}$, where $S_\infty[[z]]$ is identified with $\mathcal{O}[[X_0,\dots,X_D]][[y_{j,i}]]$ by $z = X_0$, $Y_{j,i} = y_{j,i}$. If $a_N \geq k+j$, then $\mathfrak{d}_{a_N}S \subseteq \mathfrak{c}_{k,j}$: $(1+y_{j,i})^{\ell^a} - 1 = \sum_{m\geq 1}\binom{\ell^a}{m} y_{j,i}^m$, the terms with $m \geq k$ lie in $(y)^k$, and for $1 \leq m < k$, $v_\ell\binom{\ell^a}{m} = a - v_\ell(m) \geq a-k \geq j$, so these terms lie in $\ell^j S \subseteq \mathfrak{m}_\Lambda^j S$. Hence $M_N^\natural/\mathfrak{c}_{k,j}M_N^\natural \cong (S/\mathfrak{c}_{k,j})^{\mathrm{rk}_N}$ is free of finite rank for every $N \in \mathcal{N}_0$ with $N \geq k+j$ (as $a_N \geq N$). This set belongs to $\mathcal{U}$ ($\mathcal{N}_0 \in \mathcal{U}$, and the finite set $\{N < k+j\}$ does not).

Step 4: (Tow2). This is Lemma E.58.

Step 5: (Tow3). I is generated by the set of relations (Wv1)–(Wv6); as R is Noetherian, a finite subset G of relations generates I. For each $g \in G$ and $j \geq 1$, Lemma E.63 gives the required set in $\mathcal{U}$ with exponent $e+1$.

The patching variables are the $y_{j,i}$, $1 \leq j \leq q$, $1 \leq i \leq n$: there are nq of them. □

Lemma E.58. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ and let automorphic patching data, $\mathcal{O}_1$, limit objects and $\tilde{c}$ be given as in Definition E.37, with patched tower data $(R,\beta,X,f,I,e+1,(M_N^\natural)_{N\geq 1},\mathcal{U})$ over $S = \Lambda[[y_{j,i}]]$. Then for every $j \geq 1$ the set of N for which there are R-linear maps*

$$\varphi\colon X/\mathfrak{m}_\Lambda^j X \to M_N^\natural/\mathfrak{c}_{1,j}M_N^\natural, \qquad \psi\colon M_N^\natural/\mathfrak{c}_{1,j}M_N^\natural \to X/\mathfrak{m}_\Lambda^j X$$

with $\psi\circ\varphi$ and $\varphi\circ\psi$ both equal to multiplication by f belongs to $\mathcal{U}$.

Proof. Fix $j \geq 1$. Put $P_{j'}^\infty = \chi_{D^{\mathrm{u}}}(\sigma_{j'})$ and $P_{j'}^N = \chi_{D^{\mathrm{u}}}(h_N(\Phi_{N,j'}))$; then $\mathrm{Sp}_N = \mathrm{Sp}_B(P^N)$, and we put $\mathrm{Sp}_\infty = \mathrm{Sp}_B(P^\infty)$, so that $C_0^{\mathrm{sp}} = \mathrm{Sp}_\infty[[z]]$. By (Tg3), $h_N(\Phi_{N,j'})\sigma_{j'}^{-1} \in H$, on which the $\bar{\chi}_a$ are trivial; so all P^N and P^∞ have the same reduction modulo $\mathfrak{m}_B$, and by Lemma E.59 Sp_N and Sp_∞ are free B-modules of one rank $r_0 \geq 1$. Let $M \cong \Lambda_D^m$ by (Hl1), $\Lambda' = \Lambda_D[[z]]$, and for a B-algebra Sp free of rank r_0 let $W(\mathrm{Sp}) = (\mathrm{Sp}\otimes_B M)[[z]]/\mathfrak{m}_{\Lambda'}^j(\mathrm{Sp}\otimes_B M)[[z]]$, Λ_D acting through λ_B and z by multiplication. Then $X/\mathfrak{m}_\Lambda^j X = \mathcal{O}_1 \otimes_{\mathcal{O}} W(\mathrm{Sp}_\infty)$ (as $\mathfrak{m}_\Lambda = \mathfrak{m}_{\Lambda'}\Lambda$, $\mathcal{O}_1$ being flat and unramified over $\mathcal{O}$), and $M_N^\natural/\mathfrak{c}_{1,j}M_N^\natural = \mathcal{O}_1 \otimes_{\mathcal{O}} W_N$ with $W_N = X_N[[z]]/\mathfrak{m}_{\Lambda'}^j X_N[[z]]$, Λ_D acting through s_N ($N \in \mathcal{N}_0$).

Step 1: a uniform exponent. Let $e_j = |\Lambda_D/\mathfrak{m}_{\Lambda_D}^j|^{r_0 m}$. For Sp as above, $Q = (\mathrm{Sp}\otimes_B M)/\mathfrak{m}_{\Lambda_D}^j(\mathrm{Sp}\otimes_B M)$ is a B-module (Λ_D acts through B) with $\mathrm{Sp}\otimes_B M \cong M^{r_0} \cong \Lambda_D^{r_0 m}$ over Λ_D, so $|Q| = e_j$. In the chain $Q \supseteq \mathfrak{m}_B Q \supseteq \mathfrak{m}_B^2 Q \supseteq \dots$, equality $\mathfrak{m}_B^k Q = \mathfrak{m}_B^{k+1}Q$ forces $\mathfrak{m}_B^k Q = 0$ (Nakayama), so the chain reaches 0 after at most $|Q|$ steps: $\mathfrak{m}_B^{e_j}(\mathrm{Sp}\otimes_B M) \subseteq \mathfrak{m}_{\Lambda_D}^j(\mathrm{Sp}\otimes_B M)$. Hence $\mathfrak{m}_B^{e_j}$ kills $W(\mathrm{Sp})$, and $W(\mathrm{Sp})$ depends only on $\mathrm{Sp}/\mathfrak{m}_B^{e_j}\mathrm{Sp}$.

Step 2: the set of N. $B/\mathfrak{m}_B^{e_j}$ is finite and $\bar{D}^{\mathrm{u}}$ is continuous on the compact group $\bar{\Gamma}^0$, so there is an open normal subgroup $\bar{V}$ of $\bar{\Gamma}$ with $\Lambda_i(\bar{D}^{\mathrm{u}})(\gamma v) - \Lambda_i(\bar{D}^{\mathrm{u}})(\gamma) \in \mathfrak{m}_B^{e_j}$ for all $\gamma \in \bar{\Gamma}^0$, $v \in \bar{V}\cap\bar{\Gamma}^0$, $1 \leq i \leq n$; let $V \subseteq \Gamma_{\mathscr{H}}$ be its preimage (open normal, containing $\mathrm{PL}(H)$). Let $\mathcal{N}_2$ be the set of $N \in \mathcal{N}_0$ with $N \geq j$, $h_N(\Phi_{N,j'})\sigma_{j'}^{-1} \in V$ for all j' (true for large N by (Tg3)), and $\bar{h}_N(\pi_N(g))\bar{V} = \bar{\theta}(\pi_\infty(g))\bar{V}$ for all $g \in \Pi$ ((L1)(ii)). Then $\mathcal{N}_2 \in \mathcal{U}$. Fix $N \in \mathcal{N}_2$.

Step 3: identification of the eigenvalue algebras. The coefficients of $P_{j'}^N$ and $P_{j'}^\infty$ are $\Lambda_i(\bar{D}^{\mathrm{u}})$ at elements of $\bar{\Gamma}^0$ differing by an element of $\bar{V}$, so $P^N \equiv P^\infty$ modulo $\mathfrak{m}_B^{e_j}$. Hence the identity of $B[[Z_{j',i}]]$ induces a B-algebra isomorphism $\iota\colon \mathrm{Sp}_N/\mathfrak{m}_B^{e_j} \to \mathrm{Sp}_\infty/\mathfrak{m}_B^{e_j}$ with $\iota(u_{j',i}) = u_{j',i}$, and (Step 1) a ι-semilinear bijection $W(\mathrm{Sp}_N) \to W(\mathrm{Sp}_\infty)$, also denoted ι.

Step 4: π_0 through level N. Let $\theta_N^z\colon R_N[[z]] \to \mathrm{Sp}_N[[z]]$ be θ_N on coefficients, and $\vartheta = \iota\circ\theta_N^z\circ\rho_N^0\colon R_{\mathrm{tup}} \to (\mathrm{Sp}_\infty/\mathfrak{m}_B^{e_j})[[z]]$, a morphism of $\mathrm{CNL}_{\mathcal{O}}$, as is the reduction $\bar{\pi}_0$ of π_0. We compare the images of t^{u}. By (Ha3): $s^{\mathrm{t}} \mapsto z$ under both; $U_{j',i}^{\mathrm{t}} \mapsto \iota(\theta_N(U_{N,j',i})) = u_{j',i}$; $y_{j',i}^{\mathrm{t}} \mapsto \theta_N(s_N(Y_{j',i})) = \lambda_B(0) = 0$; $\psi_{v,a}^{\mathrm{t}} \mapsto \theta_N\circ\psi_{N,v,a} = \psi_{v,a}^{\mathrm{d}}$. For the determinants it suffices

by (Ch1) to compare characteristic polynomials at $g \in P^0$: write $\pi_N(g) = p_N(\gamma)$, $\gamma \in \Gamma_N^0$; by (Ha3), $\theta_N(\Lambda_i(\bar{D}_N)(\pi_N(g))) = \theta_N(\Lambda_i(D_N)(\gamma)) = \Lambda_i(D^{\mathrm{u}})(h_N(\gamma)) = \Lambda_i(\bar{D}^{\mathrm{u}})(\bar{h}_N(\pi_N(g)))$, while π_0 gives $\Lambda_i(\bar{D}^{\mathrm{u}})(\bar{\theta}(\pi_\infty(g)))$; these agree modulo $\mathfrak{m}_B^{e_j}$ by the choice of $\mathcal{N}_2$. By the uniqueness in Proposition E.60, $\vartheta = \bar{\pi}_0$.

Step 5: the maps. Extend φ_N, ψ_N coefficientwise in z to $\Phi\colon (\mathrm{Sp}_N \otimes_B M)[[z]] \to X_N[[z]]$ and Ψ back; by (Ha4), $\Phi(\theta_N^z(t)\xi) = t\Phi(\xi)$, $\Psi(tx) = \theta_N^z(t)\Psi(x)$ for $t \in R_N[[z]]$ (each coefficient is a finite sum), $\Phi\Psi = f_{(q_{N,j'})}(U_N)$ and $\Psi\Phi = \theta_N(f_{(q_{N,j'})}(U_N))$. For $\lambda \in \Lambda_D$, $\lambda_B(\lambda) = \theta_N(s_N(\lambda))$ by (Ha3), so $\Phi(\lambda\xi) = s_N(\lambda)\Phi(\xi)$ and $\Psi(s_N(\lambda)x) = \lambda\Psi(x)$; with z-linearity, Φ and Ψ respect the submodules $\mathfrak{m}_{\Lambda'}^j(\cdot)$ and induce $\bar{\Phi}\colon W(\mathrm{Sp}_N) \to W_N$, $\bar{\Psi}\colon W_N \to W(\mathrm{Sp}_N)$. Put $\varphi = \mathrm{id}_{\mathcal{O}_1} \otimes (\bar{\Phi} \circ \iota^{-1})$ and $\psi = \mathrm{id}_{\mathcal{O}_1} \otimes (\iota \circ \bar{\Psi})$. R-linearity: for $r \in R_{\mathrm{tup}}$, $\pi_0(r)$ acts on $W(\mathrm{Sp}_\infty)$ through $\bar{\pi}_0(r) = \vartheta(r) = \iota(\theta_N^z(\rho_N^0(r)))$ (Step 1, Step 4), so $\varphi(\pi_0(r)\xi) = \bar{\Phi}(\theta_N^z(\rho_N^0(r))\iota^{-1}\xi) = \rho_N^0(r)\varphi(\xi)$, and likewise $\psi(\rho_N^0(r)x) = \vartheta(r)\psi(x) = \pi_0(r)\psi(x)$; both extend $\mathcal{O}_1$-linearly to R. Composites: $\varphi\psi$ is multiplication by $f_{(q_{N,j'})}(U_N) = \rho_N^0(f_{(q_{N,j'})}(U^{\mathrm{t}}))$, which differs from $\rho_N^0(f)$ by an element of $\ell^N R_N$ (each factor changes by $(q_{N,j'} - 1)U_{N,j',i'}$, (Tg2)); as $N \geq j$ and $\ell^j \in \mathfrak{m}_{\Lambda_D}^j$ kills W_N, $\varphi\psi = f$. Similarly $\psi\varphi$ is multiplication by $\iota(\theta_N(f_{(q_{N,j'})}(U_N))) = f_{(q_{N,j'})}(u) \equiv f_{(1)}(u) = \pi_0(f)$ modulo ℓ^N, and ℓ^j kills $W(\mathrm{Sp}_\infty)$; so $\psi\varphi = f$. This holds for every $N \in \mathcal{N}_2 \in \mathcal{U}$. □

Lemma E.59. *Let $\mathcal{O}$ be a complete discrete valuation ring with finite residue field $\mathbb{F}$, A a complete Noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$, $n, q \geq 1$, $P_1, \dots, P_q \in A[X]$ monic of degree n, and $\tilde{u}_{j,i} \in \mathcal{O}$ ($1 \leq j \leq q$, $1 \leq i \leq n$) with $P_j \equiv \prod_{i=1}^n (X - \tilde{u}_{j,i})$ modulo $\mathfrak{m}_A$. Put*

$$\mathrm{Sp}_A(P) = A[[Z_{j,i}]] \Big/ \Big(\text{the coefficients of } \textstyle\prod_{i=1}^n (X - \tilde{u}_{j,i} - Z_{j,i}) - P_j(X),\ 1 \leq j \leq q\Big).$$

Then $\mathrm{Sp}_A(P)$ is a local ring with residue field $\mathbb{F}$ and a free A-module of finite rank $r \geq 1$, and r depends only on the reductions $\bar{P}_j \in \mathbb{F}[X]$ and the $\tilde{u}_{j,i}$ modulo $\mathfrak{m}_{\mathcal{O}}$.

Proof. Let $\mathrm{Spl} = A[V_{j,i}]/(\text{coefficients of } \prod_i (X - V_{j,i}) - P_j(X),\ 1 \leq j \leq q)$, the tensor product over A of the universal splitting algebras of the P_j; it is free of rank $(n!)^q$ over A. By hypothesis $V_{j,i} \mapsto \tilde{u}_{j,i} \bmod \mathfrak{m}_{\mathcal{O}}$ defines a surjection $\mathrm{Spl} \to \mathbb{F}$ (it kills the relations, which reduce to those of $\bar{P}_j = \prod_i (X - \bar{u}_{j,i})$); let $\mathfrak{n}_0$ be its kernel, a maximal ideal. As A is complete local and Spl finite over A, $\mathrm{Spl} = \prod_{\mathfrak{n}} \mathrm{Spl}_{\mathfrak{n}}$; so $\mathrm{Spl}_{\mathfrak{n}_0} \neq 0$ is a direct summand of a free A-module of finite rank, hence free of finite rank $r \geq 1$, and $r = \dim_{\mathbb{F}}(\mathrm{Spl}/\mathfrak{m}_A\mathrm{Spl})_{\bar{\mathfrak{n}}_0}$ depends only on $\mathrm{Spl}/\mathfrak{m}_A\mathrm{Spl}$ (the splitting algebra of the $\bar{P}_j$ over $\mathbb{F}$) and the point $\bar{u}$.

$\mathrm{Spl}_{\mathfrak{n}_0}$ is finite over A, hence $\mathfrak{m}_A$-adically complete; for its maximal ideal $\mathfrak{n}$, $\mathfrak{n}^s \subseteq \mathfrak{m}_A\mathrm{Spl}_{\mathfrak{n}_0} \subseteq \mathfrak{n}$ for some s (the quotient by $\mathfrak{m}_A$ is Artinian local), so it is $\mathfrak{n}$-adically complete. Hence $A[Z_{j,i}] \to \mathrm{Spl}_{\mathfrak{n}_0}$, $Z_{j,i} \mapsto V_{j,i} - \tilde{u}_{j,i} \in \mathfrak{n}$, extends continuously to $A[[Z_{j,i}]]$, kills the defining relations of $\mathrm{Sp}_A(P)$ and gives $\alpha\colon \mathrm{Sp}_A(P) \to \mathrm{Spl}_{\mathfrak{n}_0}$. $\mathrm{Sp}_A(P)$ is a quotient of the local ring $A[[Z_{j,i}]]$ by an ideal contained in $(\mathfrak{m}_A, Z_{j,i})$ (the relations reduce to 0 modulo it), so it is local with residue field $\mathbb{F}$ and complete. $V_{j,i} \mapsto \tilde{u}_{j,i} + Z_{j,i}$ defines $\mathrm{Spl} \to \mathrm{Sp}_A(P)$, whose composite with the residue map has kernel $\mathfrak{n}_0$; elements outside $\mathfrak{n}_0$ go to units, so it factors through $\gamma\colon \mathrm{Spl}_{\mathfrak{n}_0} \to \mathrm{Sp}_A(P)$. $\gamma \circ \alpha$ is a local homomorphism of complete Noetherian local rings fixing A and the $Z_{j,i}$, hence continuous and the identity on a dense subring, hence the identity; $\alpha \circ \gamma$ is the identity on the image of Spl, hence on its localisation. So $\mathrm{Sp}_A(P) \cong \mathrm{Spl}_{\mathfrak{n}_0}$. □

Proposition E.60. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, $\mathcal{D}$ a Taylor–Wiles datum of size q for $\mathscr{H}$ with $H, \sigma_1, \dots, \sigma_q$, $(\Pi, (\pi_N), \mathcal{U}, \bar{\theta}, c_\infty, (\Phi_j), (\mathrm{T}_j), (F_w))$ as in the conclusion of Lemma E.42, $\bar{\Gamma}_\infty^0 = \bar{\theta}^{-1}(\overline{\Gamma_{\mathscr{H}}^0})$ ($\overline{\Gamma_{\mathscr{H}}^0}$ the image of $\Gamma_{\mathscr{H}}^0$ in $\bar{\Gamma}$) and $P^0 = \pi_\infty^{-1}(\bar{\Gamma}_\infty^0)$, and $\tilde{u}_{j,i} \in \mathcal{O}$ ($1 \leq j \leq q$, $1 \leq i \leq n$) with $\tilde{u}_{j,i} \equiv \bar{\chi}_i(\sigma_j)$ modulo $\mathfrak{m}_{\mathcal{O}}$. Since the $\bar{\chi}_a$ are trivial on $H \supseteq \mathrm{PL}(H)$, $\bar{D}_n$ defines a determinant $\bar{D}_n \circ \bar{\theta} \circ \pi_\infty$ on P^0. For $A \in \mathrm{CNL}_{\mathcal{O}}$, a* ***patched***

tuple *to A is $(D, (\psi_{v,a}), s, (U_{j,i}), (y_{j,i}))$ with D a continuous A-valued determinant of dimension n on P^0 whose reduction modulo $\mathfrak{m}_A$ is $\bar{D}_n \circ \bar{\theta} \circ \pi_\infty$, continuous homomorphisms $\psi_{v,a} \colon D_v \to A^\times$ with $\psi_{v,a} \equiv \bar{\chi}_a$ modulo $\mathfrak{m}_A$ and $\psi_{v,a}(g)$ equal to the image of $\xi_{v,a}(g)$ in A for $g \in \mu_v$ ($v \in S_\ell$, $1 \le a \le n$), $s \in \mathfrak{m}_A$, $U_{j,i} \in A$ with $U_{j,i} - \tilde{u}_{j,i} \in \mathfrak{m}_A$, and $y_{j,i} \in \mathfrak{m}_A$. Then there are $R_{\mathrm{tup}} \in \mathrm{CNL}_{\mathcal{O}}$ and a patched tuple t^{u} to R_{tup} such that for every $A \in \mathrm{CNL}_{\mathcal{O}}$ and every patched tuple t to A there is exactly one morphism $R_{\mathrm{tup}} \to A$ of $\mathrm{CNL}_{\mathcal{O}}$ carrying t^{u} to t (componentwise base change).*

Proof. Step 1: finiteness. P^0 is open in Π ($\bar{\Gamma}^0_\infty$ has index at most 2 in $\bar{\Gamma}_\infty$, and π_∞, $\bar{\theta}$ are continuous), and Π is topologically finitely generated (Lemma E.42). An open subgroup of a topologically finitely generated profinite group is topologically finitely generated, and a topologically finitely generated profinite group has only finitely many open subgroups of each index; so every open subgroup of P^0 has only finitely many continuous homomorphisms to $\mathbb{Z}/\ell$, i.e. P^0 satisfies Φ_ℓ. Each D_v ($v \in S_\ell$) lies in G_L and satisfies Φ_ℓ by (Tg6) of Definition 53.8.

Step 2: the factors. By (Ch2) applied to P^0 and $\bar{D} = \bar{D}_n \circ \bar{\theta} \circ \pi_\infty$ (continuous, as π_∞ and $\bar{\theta}$ are), there is a universal continuous determinant $D(\bar{D})$ on P^0 with values in a profinite local $W(k)$-algebra $A(\bar{D})$ with residue field k, Noetherian by Step 1. Put $A_{\det} = A(\bar{D}) \otimes_{W(k)} \mathcal{O}$, finite over $A(\bar{D})$, hence a complete Noetherian local $\mathcal{O}$-algebra with residue field k, with the base-changed determinant. For $A \in \mathrm{CNL}_{\mathcal{O}}$, morphisms $A_{\det} \to A$ of $\mathrm{CNL}_{\mathcal{O}}$ correspond to $W(k)$-algebra homomorphisms $A(\bar{D}) \to A$ that are local, and these are continuous (a local homomorphism of complete Noetherian local rings is continuous for the adic topologies, and $A(\bar{D})$ carries its profinite, here adic, topology); so $A_{\det}$ represents the functor $A \mapsto \{$continuous A-valued determinants of dimension n on P^0 lifting $\bar{D}\}$ on $\mathrm{CNL}_{\mathcal{O}}$. In the same way, (Ch2) with $n = 1$ applied to D_v and $\bar{\chi}_a|_{D_v}$ gives $A_{v,a} \in \mathrm{CNL}_{\mathcal{O}}$ representing continuous determinants of dimension 1 on D_v lifting $\bar{\chi}_a$; a determinant of dimension 1 on a group G' is the same as a homomorphism $G' \to A^\times$ (a multiplicative homogeneous polynomial law of degree 1 on $A[G']$ is an A-algebra homomorphism $A[G'] \to A$), and continuity corresponds, so $A_{v,a}$ represents the continuous lifts $\psi_{v,a} \colon D_v \to A^\times$ of $\bar{\chi}_a$. Let $\psi^{\mathrm{u}}_{v,a}$ be the universal lift and $\mathfrak{a}_{v,a} \subseteq A_{v,a}$ the ideal generated by the $\psi^{\mathrm{u}}_{v,a}(g) - \xi_{v,a}(g)$, $g \in \mu_v$. It lies in $\mathfrak{m}_{A_{v,a}}$: for $g \in \mu_v$ there is e with $g^{\ell^e} \in \overline{[D_v, D_v]}$, so $\xi_{v,a}(g)^{\ell^e} = 1$ by (Rd10) and $\bar{\chi}_a(g)^{\ell^e} = 1$ ($\bar{\chi}_a$ is a continuous character to an abelian group, trivial on $\overline{[D_v, D_v]}$); as $\#\mathbb{F}^\times_{\mathcal{O}}$ is prime to ℓ, both $\bar{\chi}_a(g)$ and the reduction of $\xi_{v,a}(g)$ equal 1, so $\psi^{\mathrm{u}}_{v,a}(g) - \xi_{v,a}(g) \in \mathfrak{m}_{A_{v,a}}$. Hence $A'_{v,a} = A_{v,a}/\mathfrak{a}_{v,a} \in \mathrm{CNL}_{\mathcal{O}}$ (every ideal of a Noetherian complete local ring is closed), and a morphism $A'_{v,a} \to A$ is a morphism $A_{v,a} \to A$ killing $\mathfrak{a}_{v,a}$, i.e. the classified lift takes the values $\xi_{v,a}(g)$ on μ_v. So $A'_{v,a}$ represents the continuous lifts of $\bar{\chi}_a$ that agree with $\xi_{v,a}$ on μ_v. Finally $\mathcal{O}[[s, Z_{j,i}, y_{j,i}]]$ represents the triples $(s, (Z_{j,i}), (y_{j,i}))$ of elements of $\mathfrak{m}_A$, and $U_{j,i} = \tilde{u}_{j,i} + Z_{j,i}$ is a bijection onto the families with $U_{j,i} - \tilde{u}_{j,i} \in \mathfrak{m}_A$.

Step 3: the product. The completed tensor product over $\mathcal{O}$ of finitely many objects of $\mathrm{CNL}_{\mathcal{O}}$ is in $\mathrm{CNL}_{\mathcal{O}}$ and is their coproduct there. So $R_{\mathrm{tup}} = A_{\det} \widehat{\otimes}_{\mathcal{O}} \widehat{\bigotimes}_{v,a} A'_{v,a} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[s, Z_{j,i}, y_{j,i}]]$ with the tuple t^{u} formed by the images of the universal objects represents patched tuples: a morphism $R_{\mathrm{tup}} \to A$ is the same as a family of morphisms from the factors, i.e. a patched tuple to A, and it carries t^{u} to that tuple by construction; uniqueness holds because the images of the universal objects generate the factors topologically and morphisms of $\mathrm{CNL}_{\mathcal{O}}$ are continuous. □

Corollary E.61. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, $(B, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, $J \subseteq B$ an ideal, $\mathcal{D}$ a Taylor–Wiles datum for $\mathscr{H}$ with H, and automorphic patching data for J along $\mathcal{D}$ with determinants D_N on Γ^0_N to R_N. Then for every N, $\mathrm{PL}(H_N)$ is a closed subgroup of Γ^0_N, normal in Γ_N, and there is a unique continuous R_N-valued determinant $\bar{D}_N$ of dimension n on $\Gamma^0_N/\mathrm{PL}(H_N)$ with $D_N = \bar{D}_N \circ p_N$, $p_N \colon R_N[\Gamma^0_N] \to R_N[\Gamma^0_N/\mathrm{PL}(H_N)]$ the natural map; in particular $\mathrm{tr}_{D_N}(x) =$*

$\mathrm{tr}_{\bar{D}_N}(p_N(x))$ *for all* $x \in R_N[\Gamma_N^0]$.

Proof. Apply Lemma E.62 with $\Gamma = \Gamma_N$, $\Gamma^0 = \Gamma_N^0$, $H = H_N$, $A = R_N$, $D = D_N$ and $\bar{\rho} = \mathrm{diag}(\bar{\chi}_1, \dots, \bar{\chi}_n) \circ h_N$ on Γ_N^0 (with values in $\mathrm{GL}_n(\mathbb{F}_{\mathcal{O}})$, continuous and semisimple, a sum of characters). The hypotheses hold: (a) $\Gamma_N^0 = \pi^{Q_N}(G_L)$ is open of index at most 2, hence normal, in Γ_N. (b) $H_N = h_N^{-1}(H)$ is open and normal in Γ_N (H is open and normal in $\Gamma_{\mathscr{H}}$). (c) $H_N \subseteq \Gamma_N^0$: by (Tg1) $H \subseteq \Gamma_{\mathscr{H}}^0 = h_N(\Gamma_N^0)$, so $H_N \subseteq \Gamma_N^0 \ker h_N$; and $\ker h_N = \pi^{Q_N}(N_{\mathscr{H}}) \subseteq \pi^{Q_N}(G_L) = \Gamma_N^0$, since $N_{\mathscr{H}}$ is the closed normal subgroup of G_{L^+} generated by elements of the closed normal subgroup G_L. (d) $H_N \subseteq \ker \bar{\rho}$: $\bar{\chi}_a(h_N(H_N)) = \bar{\chi}_a(H) = 1$ by (Tg1). (e) By Definition E.36 (polarised ordinary determinants on Γ_N^0 as in Definition 53.9) the reduction of D_N is $\det \circ \mathrm{diag}(\bar{\chi}_1, \dots, \bar{\chi}_n)$, i.e. $\det \circ \bar{\rho}$. The lemma gives $\bar{D}_N$, unique as p_N is surjective. The trace is a coefficient of the characteristic polynomial $D_N(X - x) = \bar{D}_N(X - p_N(x))$. □

Lemma E.62. *Let $\mathcal{O}$ be a coefficient ring with residue field k, Γ a profinite group, $\Gamma^0 \subseteq \Gamma$ an open normal subgroup, $H \subseteq \Gamma^0$ a subgroup that is open and normal in Γ, $\bar{\rho}\colon \Gamma^0 \to \mathrm{GL}_n(k)$ a continuous semisimple representation with $H \subseteq \ker \bar{\rho}$, $A \in \mathrm{CNL}_{\mathcal{O}}$, and D a continuous A-valued determinant of dimension n on Γ^0 whose reduction modulo $\mathfrak{m}_A$ is $\det \circ \bar{\rho}$. Then $\mathrm{PL}(H)$ (taken in Γ) is a closed subgroup of H, normal in Γ, and there is a unique continuous A-valued determinant D' of dimension n on $\Gamma^0/\mathrm{PL}(H)$ such that D is the composite of D' with the map $A[\Gamma^0] \to A[\Gamma^0/\mathrm{PL}(H)]$.*

Proof. Step 1: $\mathrm{PL}(H)$. It is an intersection of open normal subgroups of Γ, hence closed and normal in Γ; $U = H$ is one of them (every $h \in H$ lies in H), so $\mathrm{PL}(H) \subseteq H$. Let $N \subseteq H$ be a closed subgroup, normal in Γ, with H/N pro-ℓ. For every open normal subgroup $U \supseteq N$ of Γ, HU/U is a finite quotient of H/N, hence an ℓ-group, so every $h \in H$ has an ℓ-power in U; therefore $\mathrm{PL}(H) \subseteq \bigcap_U U = N$ (in a profinite group a closed normal subgroup is the intersection of the open normal subgroups containing it).

Step 2: comparison with Chenevier's subgroup. Let H_0 be the smallest closed normal subgroup of Γ^0 contained in $\ker \bar{\rho}$ with $\ker \bar{\rho}/H_0$ pro-ℓ. $H \cap H_0$ is closed and normal in Γ^0, and $H/(H \cap H_0)$ embeds into $\ker \bar{\rho}/H_0$, so it is pro-ℓ. Let $N = \bigcap_g g(H \cap H_0)g^{-1}$, g running over representatives of the finite group Γ/Γ^0: a closed subgroup of H, normal in Γ (H is normal in Γ), and H/N embeds into $\prod_g H/g(H \cap H_0)g^{-1} \cong \prod_g H/(H \cap H_0)$, so it is pro-$\ell$. By Step 1, $\mathrm{PL}(H) \subseteq N \subseteq H_0$.

Step 3: factorisation. A is a profinite local $W(k)$-algebra with residue field k (an object of $\mathrm{CNL}_{\mathcal{O}}$, through $W(k) \to \mathcal{O}$), and $\bar{\rho}$ is defined over k, so Theorem B.99 applies to $G = \Gamma^0$: $D = D'' \circ p_0$ for a determinant D'' on Γ^0/H_0, $p_0\colon A[\Gamma^0] \to A[\Gamma^0/H_0]$. As $\mathrm{PL}(H) \subseteq H_0$, p_0 factors as $A[\Gamma^0] \xrightarrow{p} A[\Gamma^0/\mathrm{PL}(H)] \xrightarrow{p'} A[\Gamma^0/H_0]$; put $D' = D'' \circ p'$, a determinant of dimension n with $D' \circ p = D$. Continuity: continuity of a determinant in the sense of [66] is the continuity of the finitely many coefficient maps $(g_1, \dots, g_m) \mapsto$ (coefficients of $D(t_1 g_1 + \cdots + t_m g_m)$) on G^m, for every m. For D' these maps on $(\Gamma^0/\mathrm{PL}(H))^m$, composed with the quotient map $(\Gamma^0)^m \to (\Gamma^0/\mathrm{PL}(H))^m$, are those of D, hence continuous; as this quotient map is a continuous open surjection, they are continuous. Uniqueness: p is surjective. □

Lemma E.63. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ and let automorphic patching data with exponent e, $\mathcal{O}_1$, limit objects and $\tilde{c}$ be given as in Definition E.37, with patched tower data $(R, \beta, X, f, I, e+1, (M_N^\natural)_{N \geq 1}, \mathcal{U})$. Then for every relation $g \in R$ (any element listed in (Wv1)–(Wv6)) and every $j \geq 1$, the set of N for which there is $a \in R$ with*

$$(fg)^{e+1} - a \in \mathfrak{m}_R^j \qquad \text{and} \qquad aM_N^\natural = 0$$

belongs to $\mathcal{U}$.

Proof. Fix $j \geq 1$. Write $\mathfrak{m} = \mathfrak{m}_R$; then $\ell \in \mathfrak{m}$. For $N \in \mathcal{N}_0$, by definition of ρ_N: $\rho_N(\Lambda_i(D^{\mathrm{t}})(g)) = \Lambda_i(\bar{D}_N)(\pi_N(g))$ and $\rho_N(\mathrm{tr}_{D^{\mathrm{t}}}(x)) = \mathrm{tr}_{\bar{D}_N}(\pi_N(x))$ for $g \in P^0$, $x \in R_{\mathrm{tup}}[P^0]$ (π_N applied to group elements, ρ_N^0 to coefficients), $\rho_N(\psi^{\mathrm{t}}_{v,a}) = \psi_{N,v,a}$, $\rho_N(U^{\mathrm{t}}_{j',i}) = U_{N,j',i}$, $\rho_N(y^{\mathrm{t}}_{j',i}) = s_N(Y_{j',i})$; and $\mathrm{tr}_{D_N} = \mathrm{tr}_{\bar{D}_N} \circ p_N$, $\Lambda_i(D_N) = \Lambda_i(\bar{D}_N) \circ p_N$ by Corollary E.61. All elements below are taken in R_{tup} and then in R.

Step 1: uniform continuity. P^0 is an open normal subgroup of Π (the preimage of the subgroup $\bar{\Gamma}^0$ of index at most 2). D^{t} is continuous, $R_{\mathrm{tup}}/\mathfrak{m}^j$ is finite and discrete, and P^0 is compact, so the maps $g \mapsto \Lambda_i(D^{\mathrm{t}})(g) \bmod \mathfrak{m}^j$ ($1 \leq i \leq n$) are uniformly locally constant: there is an open normal subgroup V of Π with $V \subseteq P^0$ and $\Lambda_i(D^{\mathrm{t}})(gv) - \Lambda_i(D^{\mathrm{t}})(g) \in \mathfrak{m}^j$ for all $g \in P^0$, $v \in V$, i. As $\mathrm{tr} = \Lambda_1$ is linear, $\mathrm{tr}_{D^{\mathrm{t}}}(x) - \mathrm{tr}_{D^{\mathrm{t}}}(x') \in \mathfrak{m}^j$ whenever x' arises from $x \in R_{\mathrm{tup}}[P^0]$ by replacing, in a product expression, factors $g \in P^0$ by elements of gV: indeed $agvb = agb\,(b^{-1}vb)$ with $b^{-1}vb \in V$ for $a, b \in P^0$, and one expands multilinearly.

Step 2: lifting to level N. Let $\mathcal{N}_1$ be the set of $N \in \mathcal{N}_0$ with $N \geq j$ for which (L1)(i), (iii), (iv) hold for V (and $Y = \bar{\Gamma}$); it belongs to $\mathcal{U}$. Claim: for $N \in \mathcal{N}_1$, if $g_0, g'' \in \Pi$ with $g_0^{-1}g'' \in V$ and $g' \in g'' \ker \pi_\infty$, there is $g_N \in g_0 \ker \pi_N$ with $g_N^{-1}g' \in V$. Indeed $g' = g''k$ with $k \in \ker \pi_\infty \subseteq (\ker \pi_N)V$ by (L1)(i); write $k = k_N v'$ and put $g_N = g_0 k_N$; then $g_N^{-1}g' = k_N^{-1}(g_0^{-1}g'')k_N\, v' \in V$. Applied with (L1)(iii), (iv) this gives, for $N \in \mathcal{N}_1$: (a) for $g' \in \ker \pi_\infty$ (take $g_0 = g'' = 1$) some $g_N \in \ker \pi_N \cap g'V$; (b) for g' with $\pi_\infty(g') = F_w(y)$, $w \in S_\ell \cup \mathrm{St}$, $y \in D_w$, some $g_N \in g'V$ with $\pi_N(g_N) = \bar{\phi}_{N,w}(y)$ (take g_0, g'' from (L1)(iv)); (c) for g' with $\pi_\infty(g') = \Phi_j$, resp. T_j, resp. c_∞, some $g_N \in g'V$ with $\pi_N(g_N) = \bar{\Phi}_{N,j}$, resp. $\bar{\mathrm{T}}_{N,j}$, resp. $\bar{c}_N$ (take $g'' = \tilde{\phi}_j, \tilde{t}_j, \tilde{c}'$ from (L1)(iii), whose image is $\bar{\Phi}_{N,j}\pi_N(v)$ etc., and $g_0 = g''v^{-1}$). Elements of $g'V$ lie in P^0 when g' does.

Step 3: each relation. Let r be a relation and $N \in \mathcal{N}_1$. Replace each element g' of P^0 (or $\tilde{c}$) occurring in r with a prescribed image under π_∞ by the element g_N of Step 2; call the result r_N. By Step 1, $r - r_N \in \mathfrak{m}^j$ (for (Wv2) note $c'g^{-1}c'^{-1} = \tilde{c}g^{-1}\tilde{c}^{-1} \cdot v''$ with $v'' \in V$ when $c' \in \tilde{c}V$, V being normal). Then: (Wv1), (Wv4): the replaced element lies in $\ker \pi_N$ (in (Wv4) because $g \in I_{\bar{u}} \cap G_L$ with all $\bar{\chi}_a(g) = 1$ lies in N^{Q_N} ($u \in \mathrm{St}$ is not above ℓ and not in Q_N), so $\bar{\phi}_{N,u}(g) = 1$); hence $\rho_N(r_N) = \mathrm{tr}_{\bar{D}_N}(\pi_N(h)(1-1)) = 0$. (Wv2): $\rho_N(r_N) = \Lambda_i(\bar{D}_N)(\bar{c}_N\bar{g}^{-1}\bar{c}_N^{-1}) - \Lambda_i(\bar{D}_N)(\bar{g})$ with $\bar{g} = \pi_N(g) \in \bar{\Gamma}_N^0$; choosing $g_1 \in G_L$ with $p_N(\pi^{Q_N}(g_1)) = \bar{g}$ and using $c^2 = 1$, this is 0 by (Od1) for D_N. (Wv3): $\rho_N(r_N) = \mathrm{tr}_{\bar{D}_N}(\pi_N(h)\prod_a(\bar{\phi}_{N,v}(g_a) - \psi_{N,v,a}(g_a))) = 0$ by (Od2) for D_N (lift $\pi_N(h)$ to Γ_N^0). In these cases put $a = (fr_N)^{e+1}$: $\rho_N(a) = 0$, so $aM_N^\natural = 0$, and $(fr)^{e+1} - a \in \mathfrak{m}^j$ as $fr - fr_N \in \mathfrak{m}^j$. (Wv5), (Wv6): let $f_N = f_{(q_{N,j'})_{j'}}(U^{\mathrm{t}})$. Each factor of f differs from the corresponding factor of f_N by $(q_{N,j'} - 1)U^{\mathrm{t}}_{j',i'} \in \ell^N R$ ((Tg2)), so $f - f_N \in \ell^N R \subseteq \mathfrak{m}^j$. $\rho_N(f_N r_N) = f(U_N)\,\mathrm{tr}_{\bar{D}_N}(\pi_N(h)\, q_1(\bar{\Phi}_{N,j}, U_{N,j}))$, resp. with $q_2(\bar{\Phi}_{N,j}, \bar{\mathrm{T}}_{N,j}, U_{N,j}, i, s_N(Y_{j,i}))$, an element of R_N whose e-th power kills M_N by (Ha5) (applied to a lift of $\pi_N(h)$ to Γ_N^0). Put $a = (f_N r_N)^{e+1}$: $\rho_N(a)$ kills $R_N[[z]] \otimes_{R_N} M_N$, so $aM_N^\natural = 0$, and $fr - f_N r_N = (f - f_N)r + f_N(r - r_N) \in \mathfrak{m}^j$, whence $(fr)^{e+1} - a \in \mathfrak{m}^j$. In all cases the required a exists for every $N \in \mathcal{N}_1 \in \mathcal{U}$. □

Lemma E.64. *Let $\mathcal{O}$ be a complete discrete valuation ring with finite residue field $\mathbb{F}_\mathcal{O}$, B a commutative $\mathcal{O}$-algebra, $k \supseteq \mathbb{F}_\mathcal{O}$ a finite field and $f\colon B \to k[[T]]$ a homomorphism of $\mathcal{O}$-algebras (with $\mathcal{O} \to \mathbb{F}_\mathcal{O} \subseteq k \subseteq k[[T]]$). Let $\mathcal{O}'$ be a complete discrete valuation ring that is a finite free $\mathcal{O}$-algebra, unramified over $\mathcal{O}$ (a uniformiser of $\mathcal{O}$ generates its maximal ideal), with residue field of cardinality $|k|$. Then for every prime $\mathfrak{Q}$ of $\mathcal{O}' \otimes_\mathcal{O} B$ with $\{b \in B : 1 \otimes b \in \mathfrak{Q}\} = \ker f$ there is a homomorphism of rings*

$$f'\colon \tilde{B} = (\mathcal{O}' \otimes_\mathcal{O} B)[[z]] \longrightarrow k[[T]]$$

with: $f'(1 \otimes b) = f(b)$ for $b \in B$; $f'(z) = T$; $\{y \in \mathcal{O}' \otimes_\mathcal{O} B : f'(y) = 0\} = \mathfrak{Q}$; f' surjective; and $f'|_{\mathcal{O}'}$ followed by reduction modulo T is a surjection $\mathcal{O}' \to k$.

Proof. Let $d = [k : \mathbb{F}_{\mathcal{O}}]$ and $\bar{P} \in \mathbb{F}_{\mathcal{O}}[X]$ the minimal polynomial of a generator $\bar{\beta}$ of k over $\mathbb{F}_{\mathcal{O}}$; it is irreducible of degree d and separable ($\mathbb{F}_{\mathcal{O}}$ is perfect). Let $P \in \mathcal{O}[X]$ be a monic lift of $\bar{P}$.

Step 0: $\mathcal{O}' \cong \mathcal{O}[X]/(P)$ as $\mathcal{O}$-algebras. Let k' be the residue field of $\mathcal{O}'$. As $\mathfrak{m}_{\mathcal{O}}\mathcal{O}'$ is the maximal ideal of $\mathcal{O}'$, $k' = \mathcal{O}'/\mathfrak{m}_{\mathcal{O}}\mathcal{O}'$; if $\mathcal{O}'$ is free of rank r over $\mathcal{O}$, then $\dim_{\mathbb{F}_{\mathcal{O}}} k' = r$, and $|k'| = |k|$ gives $r = d$. The fields k and k' contain $\mathbb{F}_{\mathcal{O}}$ and have the same cardinality q; each is a splitting field of $X^q - X$ over $\mathbb{F}_{\mathcal{O}}$, so they are isomorphic over $\mathbb{F}_{\mathcal{O}}$. Hence $\bar{P}$ has a root $\bar{\beta}' \in k'$, simple as $\bar{P}$ is separable, and $k' = \mathbb{F}_{\mathcal{O}}(\bar{\beta}')$ (both have degree d over $\mathbb{F}_{\mathcal{O}}$). By Hensel's lemma in the complete local ring $\mathcal{O}'$ there is $\beta' \in \mathcal{O}'$ with $P(\beta') = 0$ reducing to $\bar{\beta}'$. The $\mathcal{O}$-algebra map $\iota_0 \colon \mathcal{O}[X]/(P) \to \mathcal{O}'$, $X \mapsto \beta'$, is surjective modulo $\mathfrak{m}_{\mathcal{O}}\mathcal{O}'$ (its image there contains $\mathbb{F}_{\mathcal{O}}[\bar{\beta}'] = k'$), hence surjective by Nakayama's lemma, $\mathcal{O}'$ being a finitely generated $\mathcal{O}$-module. As $\mathcal{O}'$ is free, ι_0 splits as a map of $\mathcal{O}$-modules: $\mathcal{O}[X]/(P) \cong \mathcal{O}' \oplus K$ with $K = \ker \iota_0$ finitely generated. Both $\mathcal{O}[X]/(P)$ (basis $1, X, \dots, X^{d-1}$, as P is monic) and $\mathcal{O}'$ are free of rank d, so reduction modulo $\mathfrak{m}_{\mathcal{O}}$ gives $K/\mathfrak{m}_{\mathcal{O}}K = 0$, and $K = 0$ by Nakayama's lemma. So we may and do assume $\mathcal{O}' = \mathcal{O}[X]/(P)$.

Step 1: the extensions of f. $\mathcal{O}' \otimes_{\mathcal{O}} B = B[X]/(P)$. Let $P^f \in k[[T]][X]$ be the image of P under $\mathcal{O} \to k[[T]]$; modulo T it is $\bar{P}$ (viewed in $k[X]$), which splits over k into d distinct linear factors (the conjugates of $\bar{\beta}$). By Hensel's lemma over the complete local ring $k[[T]]$, $P^f = \prod_{i=1}^{d}(X - \beta_i)$ with $\beta_i \in k[[T]]$ lifting the distinct roots $\bar{\beta}_i$ of $\bar{P}$. The ring homomorphisms $\varphi \colon B[X]/(P) \to k[[T]]$ with $\varphi(b) = f(b)$ are exactly $\varphi_i \colon X \mapsto \beta_i$, $1 \le i \le d$.

Step 2: every prime over $\ker f$ is a kernel $\ker \varphi_i$. Let $R_0 = f(B) \cong B/\ker f$, a domain inside $k[[T]]$, with fraction field $K_0 \subseteq F_k = k((T))$. Primes of $B[X]/(P)$ contracting to $\ker f$ correspond to primes of $R_0[X]/(P)$ contracting to 0, hence (localising at $R_0 \setminus \{0\}$) to primes of $K_0[X]/(P)$, i.e. to the monic irreducible factors g of P over K_0, the prime being the contraction of (g). Over F_k, $P = \prod_i (X - \beta_i)$, so $g(\beta_i) = 0$ for some i. The kernel of φ_i on $R_0[X]/(P)$ meets R_0 only in 0 (as $R_0 \subseteq k[[T]]$), and its extension to $K_0[X]/(P)$ is the kernel of $X \mapsto \beta_i$, $K_0[X]/(P) \to F_k$, which is (g) since g is the minimal polynomial of β_i over K_0. Hence $\ker \varphi_i$ is the prime attached to g. So for the given $\mathfrak{Q}$ there is i with $\ker \varphi_i = \mathfrak{Q}$.

Step 3: f'. Define $f'(\sum_{m \ge 0} c_m z^m) = \sum_{m \ge 0} \varphi_i(c_m) T^m$; the right side is a well-defined element of $k[[T]]$ (the coefficient of T^N involves only $\varphi_i(c_0), \dots, \varphi_i(c_N)$), and f' is a ring homomorphism (multiplication of power series is compatible term by term). Then $f'(1 \otimes b) = f(b)$, $f'(z) = T$, and on constants $f' = \varphi_i$, whose kernel is $\mathfrak{Q}$.

Step 4: surjectivity. Let $R_1 = \varphi_i(\mathcal{O}') \subseteq k[[T]]$. Its image modulo T contains $\bar{\beta}_i$ and $\mathbb{F}_{\mathcal{O}}$, hence is $\mathbb{F}_{\mathcal{O}}(\bar{\beta}_i) = k$; so $\mathcal{O}' \to k$ is onto. Given $y \in k[[T]]$, choose inductively $o_m \in \mathcal{O}'$ with $y - \sum_{m<N} \varphi_i(o_m) T^m \in T^N k[[T]]$ for all N (if $y_N = T^{-N}(y - \sum_{m<N} \varphi_i(o_m) T^m)$, take o_N with $\varphi_i(o_N) \equiv y_N \bmod T$). Then $y = f'(\sum_m (o_m \otimes 1) z^m)$. □

Lemma E.65. *Let $\iota \colon C_1 \to C$ be a homomorphism of commutative rings, $w_1, \dots, w_m \in C$ and $p_1, \dots, p_m \in C_1[X]$ with $(\iota p_k)(w_k) = 0$ for every k, where $\iota p_k \in C[X]$ is obtained by applying ι to the coefficients, and suppose that C is generated as a C_1-module by finitely many products of the w_k (the empty product 1 included). Let $g \colon C_1 \to A$ be a surjective ring homomorphism onto a domain A such that for every k the polynomial $gp_k \in A[X]$ obtained by applying g to the coefficients is $\prod_l (X - a_{k,l})$ for some $a_{k,l} \in A$. Then for every prime ideal P of C with $\iota^{-1}(P) = \ker g$ there is a surjective ring homomorphism $f' \colon C \to A$ with $f' \circ \iota = g$ and $\ker f' = P$.*

Proof. Let $\varphi \colon C \to C/P$ be the quotient map. As g is surjective and $\ker g = \iota^{-1}(P) = \ker(\varphi \circ \iota)$, there is a unique ring homomorphism $\psi \colon A \to C/P$ with $\psi \circ g = \varphi \circ \iota$, and it is injective: $\psi(g(c)) = 0$ if and only if $\iota(c) \in P$, if and only if $c \in \ker g$.

For each k, applying φ to $(\iota p_k)(w_k) = 0$ gives $((\varphi \circ \iota)p_k)(\varphi(w_k)) = 0$, and $(\varphi \circ \iota)p_k = (\psi \circ g)p_k = \prod_l (X - \psi(a_{k,l}))$. So $\prod_l (\varphi(w_k) - \psi(a_{k,l})) = 0$ in the domain C/P, and $\varphi(w_k) = \psi(a_{k,l})$ for some l; thus $\varphi(w_k) \in \psi(A)$ for every k, and so $\varphi(\mu) \in \psi(A)$ for every product μ of the w_k, $\psi(A)$ being a subring. By hypothesis every $c \in C$ is $c = \sum_t \iota(c_t)\mu_t$ with $c_t \in C_1$ and μ_t products of the w_k; hence $\varphi(c) = \sum_t \psi(g(c_t))\varphi(\mu_t) \in \psi(A)$. So ψ is bijective (φ is onto). Put $f' = \psi^{-1} \circ \varphi$: it is a surjective ring homomorphism with $\ker f' = \ker \varphi = P$ and $f' \circ \iota = \psi^{-1} \circ \psi \circ g = g$. □

E.3 Realisations over Taylor–Wiles levels and the Galois clause

This section proves the Galois clause (Sys4) of a patching system at a surjective lift from the patched ring's determinant on the limit level, Corollary E.66, which Proposition E.39 uses. A determinant over a level is first realised as a representation, through the Cayley–Hamilton quotient of the group algebra in the sense of Chenevier [66] (Proposition E.67); the Selmer conditions (K1)–(K3) at the level then give the Galois clause (Proposition E.78). Three lemmas on Cayley–Hamilton quotients are used again for the patched ring in Section E.5 (Lemma E.69, Lemma E.70, Lemma E.71).

E.3.1 Realisations over a level and the Galois clause

This subsection proves Corollary E.66 from Proposition E.78, the dual Selmer bound Proposition E.97, the patched level Proposition E.94 and the existence of realisations Proposition E.67. A realisation rests on the finiteness of the closed Cayley–Hamilton quotient of a determinant that is residually a sum of distinct characters (Lemma E.69), on its anti-involution (Lemma E.70), and on its realisation in matrices over a nilpotent extension at points of block type (Lemma E.72). For an orthogonal, block-diagonally full residual representation the quotient has symmetric matrix units and is realised in matrices (Proposition E.73, Proposition E.74, Lemma E.75).

Corollary E.66. *Let $\mathscr{H}$, $B = B_n^{\mathrm{po}}$, J, $x_0, \dots, x_h$, a tuple $(\mathcal{T}, d, C, u, \pi)$ for J along them as in Proposition E.78 (as in the definition of a patching system, not required to satisfy (Sys0)–(Sys4)) with ring R, relation ideal I and $S = \Lambda[[y_1, \dots, y_s]]$, where C is the eigenvalue algebra of $(\mathcal{O}_1, (\sigma_j), (\tilde{u}_{j,i}))$, i, a surjective lift f' of f_i, $\mathfrak{q} = \pi^{-1}(\ker f')$ and $\hat{A}$ be as in Proposition E.78; assume $\pi(I) = 0$, let $\bar{\pi}\colon R/I \to C$ be the induced homomorphism, and let the $\bar{\chi}_a$ be pairwise distinct. Let x be a framed normalised curve point of shape $(a, n-a)$ with $A = A_{k_i}$, of irreducible type if $a = n$ and of endoscopic type if $a < n$, κ a Selmer basis of x with local witnesses, and a Taylor–Wiles datum of size q such that all hypotheses of Proposition E.97 hold (among them (Pj1), (Np4) and the span condition, as supplied by Proposition E.54), and assume $n[L^+ : \mathbb{Q}] + nq \le d + s - 1$. Let limit objects for this Taylor–Wiles datum be given as in the conclusion of Lemma E.42, and let $\mathbf{V}_\infty$ be the limit level formed from them (Proposition E.94). Suppose that R/I carries $(D, (\hat{\psi}_{v,i}), (\phi'_j, \tau'_j, U_{j,i}, y_{j,i}), s')$ such that, with $f_1 = f' \circ \bar{\pi}$, the clauses of (Vr1), (Vr2) concerning them hold over $\mathbf{V}_\infty$ for the coefficient ring $\mathcal{O}_1$, and (Lv1), (Lv2) hold over $\mathbf{V}_\infty$, and that (Gl1), (Gl2) hold for every realisation of x over $\mathbf{V}_\infty$ with coefficient ring $\mathcal{O}_1$ and these data. Then the conclusion of (Sys4) holds for i and f'.*

Proof. By Proposition E.94, $\mathbf{V}_\infty$ is a level over $\mathscr{H}$; by its (P3), (K1) holds at $\mathbf{V}_\infty$ with $D_0 = n[L^+ : \mathbb{Q}] + nq \le d + s - 1$, and (K2), (K3) hold; by its (P4), Γ_∞^0 satisfies Φ_ℓ. The point x satisfies (Pj1) (a hypothesis of Proposition E.97). The residue field of $\mathcal{O}_1$ has exactly $|k_i|$ elements, as required of a patching system in Definition E.31, and $\mathcal{O}_1$ is a complete discrete valuation ring, free of finite rank over $\mathcal{O}$ with $\mathfrak{m}_{\mathcal{O}}\mathcal{O}_1 = \mathfrak{m}_{\mathcal{O}_1}$, as in (Vr) of Definition E.38. So Proposition E.67 applies to x, $\mathbf{V}_\infty$, $\mathcal{O}_1$ and the data on R/I, and gives a realisation of x over $\mathbf{V}_\infty$ with coefficient ring $\mathcal{O}_1$, global ring R/I and $f_1 = f' \circ \bar{\pi}$. By hypothesis it satisfies (Gl1) and (Gl2); x satisfies

(Pj1) and the span condition (for every $Z \in \mathrm{Ev}_a(A)$ some $T^r Z$ lies in the A-span of $\rho(\Gamma')$), both hypotheses of Proposition E.97; and $\pi(I) = 0$. So Proposition E.78 gives the conclusion of (Sys4) for i and f'. □

Proposition E.67. *Let $\mathscr{H}$ be a rank-n Hecke datum over $\mathcal{O}$ with $\ell \neq 2$ and the $\bar{\chi}_a$ pairwise distinct, $x = (k, \rho, 1_n, (\psi_{v,a}), f)$ a framed normalised curve point of shape $(a, n-a)$ satisfying (Pj1), of irreducible type if $a = n$ and of endoscopic type if $a < n$, $\mathbf{V}$ a level over $\mathscr{H}$ with Γ_1^0 satisfying Φ_ℓ, and $\mathcal{O}_1$ a complete discrete valuation ring as in (Vr) of Definition E.38 whose residue field has exactly $|k|$ elements. Let $(B, D, (\hat{\psi}_{v,i}), (\phi_j, \tau_j, U_{j,i}, y_{j,i}), s, f_1)$ satisfy the clauses of (Vr1) and (Vr2) that concern them, for the coefficient ring $\mathcal{O}_1$, and assume:*

(Lv1) *$\chi_D(\gamma^*) = \chi_D(\gamma)$ for all $\gamma \in \Gamma_1^0$;*

(Lv2) *$\mathrm{tr}_D(\gamma z) = 0$ for all $\gamma \in \Gamma_1^0$ and every element z listed in the last item of (Vr3) (the ordinary products at $v \in S_\ell$, the elements $\gamma' - 1$ with $\gamma' \in I_{1,u}$, the two kinds of Taylor–Wiles elements, and their images under *).*

Then there are $\hat{B}, \lambda, \mathcal{O}', \iota, \mathcal{A}, N, \nu$ such that $(B, D, (\hat{\psi}_{v,i}), (\phi_j, \tau_j, U_{j,i}, y_{j,i}), s, f_1, \hat{B}, \lambda, \mathcal{O}', \mathcal{A}, N, \nu)$ is a realisation of x over $\mathbf{V}$ with coefficient ring $\mathcal{O}_1$.

Proof. The proof applies Lemma E.72 to B, the ring $\hat{B}$ and λ of Step 0, $f = f_1$, the group Γ_1^0 with $\gamma^* = c_1 \gamma^{-1} c_1$, the determinant D, the set $\mathcal{S}$ of Step 0 and $\rho_1 = \rho \circ h$; Steps 0–4 check its hypotheses. Write $\Gamma_1' = h^{-1}(\Gamma') \cap \Gamma_1^0$ and $\mathfrak{p} = \ker f_1$.

Step 0: the rings and the residual determinant. B is complete Noetherian local with finite residue field k_B of characteristic $\ell \neq 2$ (Vr1). The composite $\mathcal{O}_1 \to B \xrightarrow{f_1} A$ kills $\mathfrak{m}_{\mathcal{O}_1} = \mathfrak{m}_{\mathcal{O}} \mathcal{O}_1$, since f_1 is a homomorphism of $\mathcal{O}$-algebras and $\mathcal{O}$ acts on A through $\mathbb{F}_{\mathcal{O}} \subseteq k$; so it factors through the finite field $\mathbb{F}_{\mathcal{O}_1}$, whose non-zero elements are roots of unity of order prime to ℓ. Such a root of unity $\zeta \in A^\times$ lies in k: write $\zeta = \zeta_0(1 + Tg)$ with $\zeta_0 \in k^\times$, $g \in A$; if $g \neq 0$ and T^d is the lowest power of T in Tg, then $(1 + Tg)^m = 1$ with $\ell \nmid m$ (for m a multiple of the orders of ζ and ζ_0) has lowest term $mT^d(\cdot) \neq 0$, a contradiction. Hence f_1 makes k an $\mathcal{O}_1$-algebra through a field embedding $\mathbb{F}_{\mathcal{O}_1} \to k$, which is onto as both fields have $|k|$ elements, and f_1 is a homomorphism of $\mathcal{O}_1$-algebras for this structure on A. By Lemma E.68, applied with $\mathcal{O}_1$ in place of $\mathcal{O}$, there are $\hat{B}$, λ with $\ker \lambda = \hat{\mathfrak{m}}$ and $\lambda(\hat{B}) = F$, and $\mathcal{O}'$, ι with all requirements of (Vr4) for the coefficient ring $\mathcal{O}_1$; 2 is invertible in $\hat{B}$. The base change of D along $B \to k_B \to k$ equals that of $f_1 \circ D = \det \circ \rho_1$ modulo T, i.e. $\det \circ \mathrm{diag}(\bar{\chi}_1 \circ h, \ldots, \bar{\chi}_n \circ h)$ (the reduction of ρ modulo T has this determinant, as x is a curve point over $\mathscr{H}$); since $k_B \to k$ is injective and the $\bar{\chi}_a$ take values in the residue field of $\mathcal{O}$, (Ch1) gives that the base change of D to k_B is $\det \circ \mathrm{diag}(\bar{\chi}_a \circ h)$. The characters $\bar{\chi}_a \circ h$ of Γ_1^0 are continuous and pairwise distinct ($h(\Gamma_1^0) = \Gamma^0$), and Γ_1^0 satisfies Φ_ℓ, so by (F0) the spaces $Z^1(\bar{\chi}_a \circ h, \bar{\chi}_b \circ h)$ are finite-dimensional. Let $\mathcal{S} \subseteq B[\Gamma_1^0]$ be the set of elements listed in the last item of (Vr3) (with their images under *); then $\mathcal{I}_1$ is the ideal $\bar{J}$ of Lemma E.69 formed from $(B, \Gamma_1^0, D, \mathcal{S})$, and $\mathcal{S}^* = \mathcal{S}$.

Step 1: $\mathcal{S} \subseteq \ker(D)$ and the involution. $\Gamma_1^0 = h^{-1}(\Gamma^0)$ is normal in Γ_1 and $c_1^2 = 1$, so $\gamma \mapsto \gamma^* = c_1 \gamma^{-1} c_1$ is a continuous anti-automorphism of Γ_1^0 of order 2, and $\chi_D(\gamma^*) = \chi_D(\gamma)$ by (Lv1). By (Q2) (with $\mathcal{S} = \varnothing$, $\eta = 1$) $\mathrm{tr}_D(y^*) = \mathrm{tr}_D(y)$ for $y \in B[\Gamma_1^0]$. By (Lv2) and (Kt2), every element of $\mathcal{S}$ lies in $\ker(D)$.

Step 2: the setting of Lemma E.72. B is complete Noetherian local with finite residue field of characteristic $\ell \neq 2$; $\hat{B}$ is complete Noetherian local with $\ker \lambda = \hat{\mathfrak{m}}$, $\lambda(\hat{B}) = F$ and $\lambda|_B = f_1$ (Step 0); Γ_1^0 satisfies Φ_ℓ (hypothesis) and * is a continuous anti-automorphism of order 2 (Step 1); D is continuous (Vr1), with base change to k_B equal to $\det \circ \mathrm{diag}(\bar{\chi}_a \circ h)$, pairwise distinct (Step 0), and $\chi_D(\gamma^*) = \chi_D(\gamma)$ (Lv1); $\mathcal{S} \subseteq \ker(D)$, $\mathcal{S}^* = \mathcal{S}$ (Steps 0, 1), and $\mathcal{I}_1$ is the ideal $\bar{J}$

formed from $(B, \Gamma_1^0, D, \mathcal{S})$ (Step 0). The homomorphism $\rho_1 = \rho \circ h \colon \Gamma_1^0 \to \mathrm{GL}_n(A)$ is T-adically continuous (ρ is, and h is continuous), block diagonal with blocks of sizes a, $n-a$ if $a < n$ (x is normalised), and satisfies $f_1 \circ D = \det \circ \rho_1$ (Step 0) and $\rho_1(\gamma^*) = {}^t\rho_1(\gamma)$ (as $\rho(h(\gamma^*)) = {}^t\rho(h(\gamma))$, x being normalised and $h(c_1) = \bar{c}$). By (Ab1) of Lemma E.72, with J the two-sided ideal of $\hat{B}[\Gamma_1^0]$ generated by the image of $\mathcal{I}_1$, $E = \hat{B}[\Gamma_1^0]/J$ and ι the projection: E is a finitely generated $\hat{B}$-module with a Cayley–Hamilton determinant D_E, $D_E \circ \iota$ the base change of D; $(yy')^* = y'^* y^*$, $J^* = J$, $\mathrm{tr}_{D_E} \circ \iota \circ {}^* = \mathrm{tr}_{D_E} \circ \iota$; and the image P_0 of $B[\Gamma_1^0]$ in E is a finitely generated B-module with $\iota(\gamma v) - \iota(\gamma) \in \mathfrak{m}_B^j P_0$ for v in an open normal V_j.

Step 3: $\rho_f(\mathcal{S}) = 0$. Let $\rho_f \colon B[\Gamma_1^0] \to \mathrm{M}_n(A)$, $\sum b_\gamma \gamma \mapsto \sum f_1(b_\gamma)\rho_1(\gamma)$. It kills the ordinary products: $\rho_1 \circ \phi_{1,v} = \rho \circ \pi|_{D_v}$ is triangular over F with diagonal $(\psi_{v,1}, \dots, \psi_{v,n}) = (f_1 \circ \hat{\psi}_{v,i})_i$, so in a basis of F^n, with flag $V_0 \subset V_1 \subset \cdots \subset V_n$, in which every $\rho_1(\phi_{1,v}(g))$ is upper triangular with that diagonal, $N_b = \rho_1(\phi_{1,v}(g_b)) - f_1(\hat{\psi}_{v,b}(g_b))$ maps V_b into V_{b-1}, and the image $N_1 N_2 \cdots N_n$ of an ordinary product maps V_n to $V_0 = 0$. It kills $\gamma' - 1$ for $\gamma' \in I_{1,u}$, since $\rho_1(\gamma') = \rho(h(\gamma'))$ with $h(\gamma') \in I_u^x$, on which ρ is trivial. It kills $\prod_i(\phi_j - U_{j,i})$, whose image $\prod_i(\rho(h(\phi_j)) - f_1(U_{j,i}))$ is the characteristic polynomial of $\rho(h(\phi_j))$ evaluated at that matrix (Vr2), hence 0. It kills $\prod_{k \neq i}(\phi_j - U_{j,k})(\tau_j - 1 - y_{j,i})$, whose last factor maps to $\rho(1) - 1 - 0 = 0$ ($h(\tau_j) = 1$, $f_1(y_{j,i}) = 0$). As $\rho_1(\gamma^*) = {}^t\rho_1(\gamma)$ (Step 2), $\rho_f(y^*) = {}^t\rho_f(y)$, so ρ_f also kills the images under * of these elements. Hence $\rho_f(\mathcal{S}) = 0$.

Step 4: the span, and the residual representation of E. Let $\bar{F}$ be an algebraic closure of F; recall $h(\Gamma_1') = \Gamma'$ and $\rho_1 = \rho \circ h$. If $a = n$, $\rho \otimes \bar{F}$ is irreducible on Γ' (x of irreducible type), hence $\rho_1 \otimes \bar{F}$ is irreducible on Γ_1'. If $a < n$, let Q, r_1, r_2 be as in the definition of endoscopic type for x; then $Q\rho_1(\sigma)Q^{-1} = \mathrm{diag}(r_1(h(\sigma)), r_2(h(\sigma)))$ for $\sigma \in \Gamma_1'$, the $r_i \circ h$ are irreducible over $\bar{F}$ on Γ_1', and a matrix Y with $Yr_1(h(\sigma)) = r_2(h(\sigma))Y$ for all $\sigma \in \Gamma_1'$ satisfies the same relation on Γ', hence is 0. So (Ab0) of Lemma E.72 with $\Gamma'' = \Gamma_1'$ gives that the F-span of $\rho_1(\Gamma_1^0)$ is $\mathrm{M}_n(F)$ if $a = n$ and the block-diagonal matrices $\mathrm{M}_n^{(a)}(F)$ if $a < n$. By (Ab2) and Step 3, there is a homomorphism of $\hat{B}$-algebras $r \colon E \to \mathrm{M}_n(F)$ with $r(\iota(\gamma)) = \rho_1(\gamma)$, the image of $D_E(t - z)$ the characteristic polynomial of $r(z)$, $r(\iota(y^*)) = r(\iota(y))^T$ and $r(E) = \mathrm{M}_n^{(a)}(F)$.

Step 5: the realisation. By (Ab3) of Lemma E.72 (2 is invertible in $\hat{B}$ by Step 0; the span by Step 4) there are $\mathcal{A}$, N, ν with the properties listed there, and these give: (Vr5) ($\mathcal{A} = \hat{B} \cdot 1 \oplus N$, $NN \subseteq \hat{\mathfrak{m}} \cdot 1$, $N = 0$ if $a = n$); (Vr6) (ν vanishes on J, so its B-linear extension vanishes on $\mathcal{I}_1$; the span; $\nu(\gamma^*) = {}^t\nu(\gamma)$; $D_{\mathcal{A}} = \det \circ \nu$; and $\lambda(b) = r(\iota(\gamma))_{ij} = \rho(h(\gamma))_{ij}$ when $\nu(\gamma)_{ij} - b \cdot 1 \in N$); (Vr7) from its clause on the primes with a non-semisimple quotient; and (Vr8) with $\mathcal{P}_0 = \bar{\nu}(P_0)$ from its last clause. (Vr1)–(Vr4) hold by hypothesis and Step 0. So the tuple is a realisation of x over $\mathbf{V}$ with coefficient ring $\mathcal{O}_1$. □

Lemma E.68. *Let $\mathcal{O}$ be a complete discrete valuation ring with uniformiser ϖ and residue field $k_{\mathcal{O}}$, B a complete Noetherian local $\mathcal{O}$-algebra (the structure map local), k a field with a surjective ring homomorphism $\mathcal{O} \to k$, $A = k[[T]]$ with fraction field F (an $\mathcal{O}$-algebra through $\mathcal{O} \to k$), $f_1 \colon B \to A$ a surjective homomorphism of $\mathcal{O}$-algebras and $s \in B$ with $f_1(s) = T$. Let $\hat{B}$ be the completion of $B_{\ker f_1}$ at its maximal ideal $\hat{\mathfrak{m}}$. Then:*

(Cd1) *there is a unique ring homomorphism $\lambda \colon \hat{B} \to F$ whose composite with $B \to \hat{B}$ is f_1 followed by $A \subseteq F$, and $\ker \lambda = \hat{\mathfrak{m}}$;*

(Cd2) *the ϖ-adic completion $\mathcal{O}'$ of the localisation $\mathcal{O}[[X]]_{(\varpi)}$ is a complete discrete valuation ring with maximal ideal $\varpi\mathcal{O}'$, and there is a local ring homomorphism $\iota \colon \mathcal{O}' \to \hat{B}$ extending $\mathcal{O} \to \hat{B}$ with $\iota(X) = s$, such that $\lambda \circ \iota$ is surjective onto F and every $a \in A$ is $f_1(b)$ for some $b \in B$ whose image in $\hat{B}$ lies in $\iota(\mathcal{O}')$.*

In particular $(\hat{B}, \lambda, \mathcal{O}', \iota)$ satisfy (Vr4) of Definition E.38 when $\mathcal{O}$ is the coefficient ring and B, f_1, s are the global ring, point and element of a realisation and the residue field of $\mathcal{O}$ maps onto k.

Proof. (Cd1). $\mathfrak{p} = \ker f_1$ is prime and $B/\mathfrak{p} \cong A$, so $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \cong F$; since $\hat{B}/\hat{\mathfrak{m}} = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$, the projection $\hat{B} \to \hat{B}/\hat{\mathfrak{m}} \cong F$ is a homomorphism λ with kernel $\hat{\mathfrak{m}}$ whose composite with $B \to \hat{B}$ is f_1. A homomorphism λ' with this property agrees with λ on the image of $B_\mathfrak{p}$ (the elements of $B \setminus \mathfrak{p}$ go to units), hence kills $\hat{\mathfrak{m}} = \mathfrak{p}\hat{B}$ (as $f_1(\mathfrak{p}) = 0$) and agrees with λ on $\hat{B} = \mathrm{im}(B_\mathfrak{p}) + \hat{\mathfrak{m}}$.

(Cd2). $\mathcal{O}[[X]]$ is a regular local ring of dimension 2, hence a unique factorisation domain, and ϖ is a prime element; so $\mathcal{O}[[X]]_{(\varpi)}$ is a discrete valuation ring with uniformiser ϖ and residue field $\mathrm{Frac}(k_\mathcal{O}[[X]]) = k_\mathcal{O}((X))$, and its ϖ-adic completion $\mathcal{O}'$ is a complete discrete valuation ring with maximal ideal $\varpi\mathcal{O}'$ and the same residue field. Since $f_1(s) = T$ lies in the maximal ideal of A and $f_1^{-1}(\mathfrak{m}_A)$ is the maximal ideal $\mathfrak{m}_B$, $s \in \mathfrak{m}_B$; as B is complete and $\mathcal{O} \to B$ is local, there is a continuous $\mathcal{O}$-algebra homomorphism $\varphi_0 \colon \mathcal{O}[[X]] \to B$ with $\varphi_0(X) = s$, $\varphi_0(\sum o_i X^i) = \sum o_i s^i$. If $g \in \mathcal{O}[[X]] \setminus \varpi\mathcal{O}[[X]]$, then $f_1(\varphi_0(g)) = \bar{g}(T)$, where $\bar{g} \in k[[X]]$ is the image of g under the reduction $\mathcal{O} \to k$ (continuity of f_1: $f_1(\mathfrak{m}_B^j) \subseteq \mathfrak{m}_A^j$). Because $\mathcal{O} \to k$ is surjective with kernel $\varpi\mathcal{O}$ (its kernel is a maximal ideal), $k = k_\mathcal{O}$ and $\bar{g} \neq 0$; so $\varphi_0(g) \notin \mathfrak{p}$ and its image in $\hat{B}$ is a unit. Thus φ_0 extends to $\mathcal{O}[[X]]_{(\varpi)} \to \hat{B}$; it maps $\varpi^m\mathcal{O}[[X]]_{(\varpi)}$ into $\hat{\mathfrak{m}}^m$ ($\varpi \in \mathfrak{p}$ since $f_1(\varpi) = 0$), and $\hat{B}$ is $\hat{\mathfrak{m}}$-adically complete and separated, so it extends continuously to $\iota \colon \mathcal{O}' \to \hat{B}$, which is local and extends $\mathcal{O} \to \hat{B}$, with $\iota(X) = s$. $\lambda \circ \iota$ kills ϖ and sends X to T, so it induces $k((X)) \to F = k((T))$, $X \mapsto T$, which is surjective (an isomorphism). Finally let $a = \sum_i a_i T^i \in A$; choose $o_i \in \mathcal{O}$ lifting a_i and put $b = \varphi_0(\sum_i o_i X^i) = \sum_i o_i s^i \in B$. Then $f_1(b) = \sum_i a_i T^i = a$ (continuity of f_1), and the image of b in $\hat{B}$ is $\iota(\sum_i o_i X^i) \in \iota(\mathcal{O}')$.

For the last sentence: (Vr4) asks for λ with kernel $\hat{\mathfrak{m}}$ extending f_1 (Cd1), a complete discrete valuation ring $\mathcal{O}'$ with ι making $\hat{B}$ a complete Noetherian local $\mathcal{O}'$-algebra ($\hat{B}$ is the completion of a Noetherian local ring and ι is local), $\lambda \circ \iota$ onto F, the image of $\mathcal{O}$ and of s in $\iota(\mathcal{O}')$, and the lifting property of A: all given by (Cd2). □

Lemma E.69. *Let B be a complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic $\ell \neq 2$, Γ a profinite group, $n \geq 1$, and D a B-valued determinant of dimension n on Γ. Let $\bar{\chi}_1, \dots, \bar{\chi}_n \colon \Gamma \to k^\times$ be pairwise distinct continuous characters such that the base change of D to k is $\det \circ \mathrm{diag}(\bar{\chi}_1, \dots, \bar{\chi}_n)$. For $a \neq b$ let $Z^1(\bar{\chi}_a, \bar{\chi}_b)$ be the k-vector space of locally constant maps $\alpha \colon \Gamma \to k$ with*

$$\alpha(gg') = \bar{\chi}_a(g)\,\alpha(g') + \alpha(g)\,\bar{\chi}_b(g') \qquad (g, g' \in \Gamma).$$

Let $\mathrm{CH}(D) \subseteq B[\Gamma]$ be the set of coefficients, in $B[\Gamma]$, of the elements $\chi_{D'}(y)(y) \in B[t_1, \dots, t_m][\Gamma]$, where $m \geq 1$, $y = \sum_{i=1}^m t_i y_i$ with $y_i \in B[\Gamma]$, and D' is the base change of D to $B[t_1, \dots, t_m]$. Let $\mathcal{S} \subseteq B[\Gamma]$ be any subset. For $N \geq 1$ and an open normal subgroup H of Γ let $J_{N,H}$ be the two-sided ideal of $B[\Gamma]$ generated by $\mathrm{CH}(D) \cup \mathcal{S} \cup \mathfrak{m}^N \cdot 1 \cup \{h - 1 : h \in H\}$; put

$$\bar{J} = \bigcap_{N \geq 1,\ H} J_{N,H}, \qquad E = B[\Gamma]/\bar{J},$$

and let $\iota \colon B[\Gamma] \to E$ be the projection.

(F0) *If Γ satisfies Φ_ℓ, then $Z^1(\bar{\chi}_a, \bar{\chi}_b)$ is finite-dimensional for all $a \neq b$.*

(F1) *Let $h \geq 0$ be an integer with $\dim_k Z^1(\bar{\chi}_a, \bar{\chi}_b) \leq h$ for all $a \neq b$, and $c = n + \sum_{p=1}^{n-1} \binom{n(n-1)h}{p}$. There are $\gamma_1, \dots, \gamma_c \in \Gamma$ such that $\iota(\gamma_1), \dots, \iota(\gamma_c)$ generate E as a B-module; in particular E is a finitely generated B-module.*

(F2) *Under the hypothesis of (F1), for every $N \geq 1$ there is an open normal subgroup V_N of Γ with $\iota(\gamma v) - \iota(\gamma) \in \mathfrak{m}^N E$ for all $\gamma \in \Gamma$ and $v \in V_N$.*

(F3) *Under the hypothesis of (F1), the natural map $E \to \varprojlim_{N,H} B[\Gamma]/J_{N,H}$ is bijective.*

Proof. Write $E_{N,H} = B[\Gamma]/J_{N,H}$. As $\mathfrak{m}^N \cdot 1$ and $h - 1$ ($h \in H$) lie in $J_{N,H}$, $E_{N,H}$ is spanned over $B/\mathfrak{m}^N$ by the images of a set of representatives of the finite group Γ/H; $B/\mathfrak{m}^N$ is finite (each $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is a finite-dimensional k-vector space), so $E_{N,H}$ is a finite set. If $N' \geq N$ and $H' \subseteq H$ then $J_{N',H'} \subseteq J_{N,H}$, so $E_{N',H'} \to E_{N,H}$ is surjective. Let $H_\chi = \bigcap_a \ker \bar{\chi}_a$, an open normal subgroup ($k^\times$ is finite and discrete), and let $\bar{D}$ be the base change of D to k.

Step 1: (F0). Let $\alpha \in Z^1(\bar{\chi}_a, \bar{\chi}_b)$. On H_χ the cocycle relation reads $\alpha(gg') = \alpha(g') + \alpha(g)$, so $\alpha|_{H_\chi}$ is a continuous homomorphism $H_\chi \to (k, +)$; and $\alpha(g\gamma) = \bar{\chi}_a(g)\alpha(\gamma) + \alpha(g)$ for $g \in \Gamma$, $\gamma \in H_\chi$. Hence, for a set $\mathcal{R}$ of representatives of Γ/H_χ, the k-linear map $\alpha \mapsto (\alpha|_{H_\chi}, (\alpha(r))_{r\in\mathcal{R}})$ into $\mathrm{Hom}_{\mathrm{cont}}(H_\chi, k) \times k^{\mathcal{R}}$ is injective. As $(k, +) \cong \mathbb{F}_\ell^f$, $\mathrm{Hom}_{\mathrm{cont}}(H_\chi, k) \cong \mathrm{Hom}_{\mathrm{cont}}(H_\chi, \mathbb{F}_\ell)^f$ is finite under Φ_ℓ; so $Z^1(\bar{\chi}_a, \bar{\chi}_b)$ is finite-dimensional.

Step 2: the residual algebra has dimension at most c. Fix an open normal subgroup $H \subseteq H_\chi$, let $\mathrm{CH}(\bar{D}) \subseteq k[\Gamma]$ be defined as $\mathrm{CH}(D)$ with $\bar{D}$ in place of D, I_H the two-sided ideal generated by $\mathrm{CH}(\bar{D}) \cup \{h - 1 : h \in H\}$, $C = k[\Gamma]/I_H$, and $\bar{g}$ the image of $g \in \Gamma$ in C. We show $\dim_k C \leq c$.

(a) Let $\bar{\rho}\colon k[\Gamma] \to k^n$ be the k-algebra homomorphism (k^n with componentwise product) with $\bar{\rho}(g) = (\bar{\chi}_1(g), \ldots, \bar{\chi}_n(g))$, with components $\bar{\rho}_a$, extended linearly over every commutative k-algebra T. Since $\bar{D} = \det \circ \mathrm{diag}(\bar{\chi}_1, \ldots, \bar{\chi}_n)$, the base change $\bar{D}_T$ is the polynomial law $y \mapsto \prod_a \bar{\rho}_a(y)$, so

$$\chi_{\bar{D}_T}(y)(X) = \prod_{a=1}^{n} (X - \bar{\rho}_a(y)) \qquad (y \in T[\Gamma]).$$

For $y = \sum_i t_i y_i$ the a-th component of $\bar{\rho}(\chi_{\bar{D}'}(y)(y))$ is $\prod_b(\bar{\rho}_a(y) - \bar{\rho}_b(y)) = 0$; comparing coefficients, $\bar{\rho}$ kills $\mathrm{CH}(\bar{D})$. It kills $h - 1$ for $h \in H \subseteq H_\chi$. So $\bar{\rho}$ induces $\bar{\rho}_C\colon C \to k^n$, which is surjective: pairwise distinct characters $\Gamma \to k^\times$ are linearly independent over k (Dedekind), so the vectors $\bar{\rho}(g)$ span k^n. Let $K = \ker \bar{\rho}_C$ and $\varepsilon_1, \ldots, \varepsilon_n$ the standard idempotents of k^n.

(b) Every $x \in K$ satisfies $x^n = 0$: for a lift $\tilde{x} \in k[\Gamma]$, $\bar{\rho}(t\tilde{x}) = 0$, so $\chi_{\bar{D}'}(t\tilde{x})(X) = X^n$ over $k[t]$ and $\chi_{\bar{D}'}(t\tilde{x})(t\tilde{x}) = t^n\tilde{x}^n$; its coefficient $\tilde{x}^n$ lies in $\mathrm{CH}(\bar{D})$. Consequently $K \subseteq \mathrm{rad}(C)$: for $x \in K$ and $y \in C$, $yx \in K$ is nilpotent, so $1 - yx$ is invertible.

(c) C is finite-dimensional, hence integral over the henselian local ring k, in which 2 is invertible. By Theorem B.100 (with $\tau = \mathrm{id}$, $I = K$, and the orthogonal idempotents of $C/K \cong k^n$ corresponding to $\varepsilon_1, \ldots, \varepsilon_n$) there are orthogonal idempotents $e_1, \ldots, e_n \in C$ with $\bar{\rho}_C(e_a) = \varepsilon_a$. The idempotent $f = 1 - \sum_a e_a$ lies in K, so $f = f^n = 0$ by (b): $\sum_a e_a = 1$.

(d) $e_aCe_a = ke_a$. Let first $z \in e_aCe_a \cap K$, with lifts $\tilde{e}, \tilde{z} \in k[\Gamma]$ of e_a, z, and $y = t\tilde{e} + s\tilde{z} \in k[t, s][\Gamma]$. Then $\bar{\rho}(y) = t\varepsilon_a$, so $\chi_{\bar{D}'}(y)(X) = X^{n-1}(X - t)$, and the coefficients of $\chi_{\bar{D}'}(y)(y)$ lie in $\mathrm{CH}(\bar{D})$: with $u = te_a + sz \in C[t, s]$, $u^{n-1}(u - t) = 0$. If $n \geq 2$, $u^{n-1}e_a = u^{n-1}$ gives $u^{n-1}(u - t) = s\,u^{n-1}z$, whose coefficient of $t^{n-1}s$ is $e_az = z$; if $n = 1$, $e_a = 1$ and $u - t = sz$. In both cases $z = 0$. Now for $x \in e_aCe_a$, $\bar{\rho}_C(x) = \varepsilon_a\bar{\rho}_C(x)\varepsilon_a = \lambda\varepsilon_a$ for some $\lambda \in k$, and $x - \lambda e_a \in e_aCe_a \cap K = 0$. Applying $\bar{\rho}_C$ also gives $e_a\bar{g}e_a = \bar{\chi}_a(g)e_a$ for $g \in \Gamma$.

(e) Put $A_{ab} = e_aCe_b$, so $C = \bigoplus_{a,b} A_{ab}$ by (c). For $a \neq b$, $\bar{\rho}_C(A_{ab}) = \varepsilon_a k^n \varepsilon_b = 0$, so $A_{ab} \subseteq K$, while $\bar{\rho}_C$ is injective on $\bigoplus_a A_{aa} = \bigoplus_a ke_a$. Hence $K = \mathcal{N} := \bigoplus_{a\neq b} A_{ab}$, a two-sided ideal with $\mathcal{N} \cap A_{aa} = 0$. As C is spanned by the $\bar{g}$, A_{ab} is spanned by the $e_a\bar{g}e_b$, $g \in \Gamma$.

(f) For $a \neq b$ put $A'_{ab} = \sum_{d\notin\{a,b\}} A_{ad}A_{db}$. A product $x_1 \cdots x_r$ with $x_q \in A_{a_qb_q}$, $a_q \neq b_q$, vanishes unless $b_q = a_{q+1}$ for all $q < r$, and then lies in $\mathcal{N} \cap A_{a_1b_r}$. For $r = 2$ this gives $\mathcal{N}^2 = \bigoplus_{a\neq b} A'_{ab}$, hence $\mathcal{N}/\mathcal{N}^2 \cong \bigoplus_{a\neq b} A_{ab}/A'_{ab}$. For $r = n$ the $n + 1$ indices $a_1, \ldots, a_n, b_n$

take at most n values, so $a_q = a_{q'}$ for some $q < q'$ or $a_q = b_n$ for some q; the corresponding subproduct $x_q \cdots x_{q'-1}$, resp. $x_q \cdots x_n$, lies in $\mathcal{N} \cap A_{a_q a_q} = 0$. Hence $\mathcal{N}^n = 0$.

(g) $\dim_k A_{ab}/A'_{ab} \le h$ for $a \ne b$. For a k-linear form λ on A_{ab} vanishing on A'_{ab} put $\alpha_\lambda(g) = \lambda(e_a \bar{g} e_b)$. Using $\sum_d e_d = 1$ and (d),

$$e_a \overline{gg'} e_b = \sum_d e_a \bar{g} e_d \, \bar{g}' e_b \in \bar{\chi}_a(g)\, e_a \bar{g}' e_b + \bar{\chi}_b(g')\, e_a \bar{g} e_b + A'_{ab},$$

so $\alpha_\lambda(gg') = \bar{\chi}_a(g)\alpha_\lambda(g') + \alpha_\lambda(g)\bar{\chi}_b(g')$; α_λ is constant on cosets of H, hence locally constant. Thus $\lambda \mapsto \alpha_\lambda$ is a k-linear map $\mathrm{Hom}_k(A_{ab}/A'_{ab}, k) \to Z^1(\bar{\chi}_a, \bar{\chi}_b)$, injective because the $e_a \bar{g} e_b$ span A_{ab}.

(h) By (f) and (g), $\dim_k \mathcal{N}/\mathcal{N}^2 \le n(n-1)h$. Choose a subspace $V \subseteq \mathcal{N}$ of dimension at most $n(n-1)h$ with $\mathcal{N} = V + \mathcal{N}^2$; by induction $\mathcal{N}^p = V^p + \mathcal{N}^{p+1}$, so $\dim_k \mathcal{N}^p/\mathcal{N}^{p+1} \le (n(n-1)h)^p$, and $\mathcal{N}^n = 0$ gives $\dim_k C = n + \dim_k \mathcal{N} \le c$. As C is spanned by the $\bar{g}$, there are $g_1, \dots, g_c \in \Gamma$ (repetitions allowed) whose images span C.

Step 3: generators at each level. Let $N \ge 1$, H open normal, and $H' = H \cap H_\chi$. Formation of $\chi_{D'}(y)(y)$ commutes with the base change $B \to k$, and every element of $k[\Gamma]$ lifts to $B[\Gamma]$; so the image of $\mathrm{CH}(D)$ in $k[\Gamma]$ is $\mathrm{CH}(\bar{D})$, and $E_{N,H'}/\mathfrak{m}E_{N,H'} = k[\Gamma]/(\text{image of } J_{N,H'})$ is a quotient of the algebra C of Step 2 for H'. The images of $g_1, \dots, g_c$ of Step 2 (for H') span it; by Nakayama's lemma over $B/\mathfrak{m}^N$ they generate the finitely generated module $E_{N,H'}$, hence its quotient $E_{N,H}$. So the set $T_{N,H} \subseteq \Gamma^c$ of tuples whose images generate $E_{N,H}$ as a B-module is non-empty.

Step 4: one tuple for all levels. The image of γ in $E_{N,H}$ depends only on γH, so $T_{N,H}$ is a union of cosets of the open subgroup H^c of Γ^c, hence closed. If $N' \ge N$ and $H' \subseteq H$, then $T_{N',H'} \subseteq T_{N,H}$; so finitely many of the sets T_{N_i,H_i} contain $T_{\max_i N_i, \bigcap_i H_i} \ne \varnothing$. By compactness of Γ^c there is $(\gamma_1, \dots, \gamma_c) \in \bigcap_{N,H} T_{N,H}$.

Step 5: (F1) and (F3). Let $\hat{E} = \varprojlim E_{N,H}$ over the pairs (N, H), directed by $N' \ge N$, $H' \subseteq H$, with the inverse limit topology of the finite discrete sets $E_{N,H}$: a compact Hausdorff space in which the sets $\{x' : x'_{N,H} = x_{N,H}\}$ form a basis of neighbourhoods of x. The map $\hat{\iota}\colon B[\Gamma] \to \hat{E}$ has kernel $\bar{J}$, so $E \to \hat{E}$ is injective. $B \cong \varprojlim B/\mathfrak{m}^N$ is compact. The map $\theta\colon B^c \to \hat{E}$, $\theta(b) = \hat{\iota}(\sum_i b_i \gamma_i)$, is continuous (its component in $E_{N,H}$ depends only on b modulo $\mathfrak{m}^N$), so $\theta(B^c)$ is compact, hence closed; by Step 4 its image in every $E_{N,H}$ is everything, so it meets every basic neighbourhood of every point: $\theta(B^c) = \hat{E}$. As $\theta(B^c) \subseteq \hat{\iota}(B[\Gamma]) \subseteq \hat{E}$, $E \to \hat{E}$ is bijective, which is (F3), and every element of E is $\sum_i b_i \iota(\gamma_i)$, which is (F1).

Step 6: (F2). Give E the topology of $\hat{E}$. The map $\psi\colon B^c \to E$, $b \mapsto \sum_i b_i \iota(\gamma_i)$, is a continuous surjective homomorphism of compact Hausdorff groups; the induced continuous bijection $B^c/\ker\psi \to E$ from a compact space onto a Hausdorff space is a homeomorphism, so ψ is open and $\mathfrak{m}^N E = \psi((\mathfrak{m}^N)^c)$ is open in E. The map $\Gamma \to E$, $\gamma \mapsto \iota(\gamma)$, is continuous (its components are constant on cosets of H). Since $\mathfrak{m}^N E$, the image of $\mathfrak{m}^N B[\Gamma]$, is a two-sided ideal, $U = \{v \in \Gamma : \iota(v) - 1 \in \mathfrak{m}^N E\}$ is a subgroup (the elements whose image in $E/\mathfrak{m}^N E$ is 1), and it is open as the preimage of the open set $1 + \mathfrak{m}^N E$. Being open in the compact group Γ, U has finite index, and the intersection V_N of its finitely many conjugates is an open normal subgroup. For $v \in V_N$, $\iota(\gamma v) - \iota(\gamma) = \iota(\gamma)(\iota(v) - 1) \in \mathfrak{m}^N E$. □

Lemma E.70. *Let B be a complete Noetherian local ring with maximal ideal $\mathfrak{m}$, Γ a profinite group, $n \ge 1$ and D a B-valued determinant of dimension n on Γ such that the maps $\Gamma \to B$, $g \mapsto \Lambda_i(g)$ $(1 \le i \le n)$, are continuous for the $\mathfrak{m}$-adic topology. Let $\mathcal{S} \subseteq \ker(D)$, and let $J_{N,H}$, $\bar{J}$, $E = B[\Gamma]/\bar{J}$ and ι be formed from $(B, \Gamma, D, \mathcal{S})$ as in Lemma E.69. For a commutative B-algebra B' let $J' \subseteq B'[\Gamma]$ be the two-sided ideal generated by the image of $\bar{J}$, $E' = B'[\Gamma]/J'$ and*

$\iota'\colon B'[\Gamma] \to E'$ *the projection.*

(Q1) *$\bar{J} \subseteq \ker(D)$, and there is a determinant D_E of dimension n on E with $D = D_E \circ \iota$. For every commutative B-algebra B', $E' = E \otimes_B B'$, and the base change $D_{E'}$ of D_E to E' is a Cayley–Hamilton determinant of dimension n with $D_{E'} \circ \iota'$ the base change of D to B'; if E is a finitely generated B-module, E' is a finitely generated B'-module.*

(Q2) *Let $\gamma \mapsto \gamma^*$ be a continuous anti-automorphism of Γ with $\gamma^{**} = \gamma$, and $\eta\colon \Gamma \to B^\times$ a character, continuous for the $\mathfrak{m}$-adic topology, with $\eta(\gamma^*)\eta(\gamma) = 1$ and $\chi_D(\eta(\gamma)\gamma^*) = \chi_D(\gamma)$ for all $\gamma \in \Gamma$. For a commutative B-algebra B' let $y \mapsto y^*$ be the B'-linear map $B'[\Gamma] \to B'[\Gamma]$ with $\gamma \mapsto \eta(\gamma)\gamma^*$. If $\mathcal{S}^* \subseteq \bar{J}$, then $(yy')^* = y'^* y^*$, $y^{**} = y$, $J'^* = J'$ and $\operatorname{tr}_{D_{E'}}(\iota'(y^*)) = \operatorname{tr}_{D_{E'}}(\iota'(y))$ for all $y, y' \in B'[\Gamma]$.*

Proof. For $N \geq 1$ let D_N be the base change of D to $B/\mathfrak{m}^N$. For a commutative $B/\mathfrak{m}^N$-algebra T, $(D_N)_T = D_T$, so the image of $\ker(D)$ in $(B/\mathfrak{m}^N)[\Gamma]$ lies in $\ker(D_N)$.

Step 1: a level for each N. Each Λ_i followed by $B \to B/\mathfrak{m}^N$ is a continuous map from the profinite group Γ to a discrete set, hence constant on the cosets of some open normal subgroup; let H_N be an open normal subgroup with $\Lambda_i(gh) \equiv \Lambda_i(g) \bmod \mathfrak{m}^N$ for all $g \in \Gamma$, $h \in H_N$, $1 \leq i \leq n$.

Step 2: $h - 1 \in \ker(D_N)$ for $h \in H_N$. Let T be a commutative $B/\mathfrak{m}^N$-algebra and $r' = \sum_{k=1}^m c_k g_k \in T[\Gamma]$. Over $P = B[u, x_1, \ldots, x_m, y_1, \ldots, y_m]$ consider

$$X = u \cdot 1 + \sum_k x_k\, h g_k + \sum_k y_k\, g_k, \qquad X' = u \cdot 1 + \sum_k x_k\, g_k + \sum_k y_k\, g_k.$$

By (Ch1), $D_P(X) = \Lambda_n(X)$ and $D_P(X')$ are the same integer polynomial, depending only on n and $2m+1$, evaluated at the elements $\Lambda_j(w)$ for w running over the finite products of the families $(1, hg_1, \ldots, hg_m, g_1, \ldots, g_m)$ and $(1, g_1, \ldots, g_m, g_1, \ldots, g_m)$ respectively. Corresponding products w, w' satisfy $w \in w' H_N$ (H_N is normal), so $\Lambda_j(w) \equiv \Lambda_j(w') \bmod \mathfrak{m}^N$ and $D_P(X) \equiv D_P(X')$ coefficientwise modulo $\mathfrak{m}^N$. Specialising $u \mapsto 1$, $x_k \mapsto c_k$, $y_k \mapsto -c_k$ into T gives $D_T(1 + (h-1)r') = D_T(1) = 1$. By the definition of $\ker(D_N)$ and (Ck1) (the right-hand version), $h - 1 \in \ker(D_N)$.

Step 3: $\bar{J} \subseteq \ker(D)$. The generators of J_{N,H_N} map into $\ker(D_N)$: $\mathrm{CH}(D) \subseteq \ker(D)$ by (Ck2) and $\mathcal{S} \subseteq \ker(D)$; $\mathfrak{m}^N \cdot 1$ maps to 0; and the remaining generators by Step 2. As $\ker(D_N)$ is a two-sided ideal (Ck1), the image of $J_{N,H_N} \supseteq \bar{J}$ lies in $\ker(D_N)$. Let $y \in \bar{J}$, T a commutative B-algebra and $r' = \sum_k c_k g_k \in T[\Gamma]$. Put $P = B[x_1, \ldots, x_m]$ and $z = \sum_k x_k g_k$. The image of $D_P(1 + yz)$ in $(B/\mathfrak{m}^N)[x_1, \ldots, x_m]$ is $(D_N)(1 + \bar{y}\bar{z}) = 1$ for every N, so every coefficient of $D_P(1 + yz) - 1$ lies in $\bigcap_N \mathfrak{m}^N = 0$ (Krull's intersection theorem). Specialising $x_k \mapsto c_k$ gives $D_T(1 + yr') = 1$; so $y \in \ker(D)$.

Step 4: (Q1). By (Ck1), $D = \tilde{D} \circ \pi$ for the projection $\pi\colon B[\Gamma] \to B[\Gamma]/\ker(D)$; as $\bar{J} \subseteq \ker(D)$, $D_E = \tilde{D} \circ (E \to B[\Gamma]/\ker(D))$ is a determinant of dimension n with $D = D_E \circ \iota$. Right exactness of $\otimes_B B'$ gives $E \otimes_B B' = B'[\Gamma]/J'$, and finite generation passes to E'. Let A' be a commutative B'-algebra and $x \in E' \otimes_{B'} A' = E \otimes_B A'$, the image of $\sum_{i=1}^m a_i \otimes y_i$ with $a_i \in A'$, $y_i \in B[\Gamma]$. Over $P = B[t_1, \ldots, t_m]$ put $y = \sum_i t_i y_i$ and write $\chi_{D_P}(y)(y) = \sum_\beta t^\beta c_\beta$ with $c_\beta \in \mathrm{CH}(D) \subseteq \bar{J}$. Since $D = D_E \circ \iota$ and polynomial laws are compatible with the specialisation $P \to A'$, $t_i \mapsto a_i$, the element $\chi_{D_{E'}}(x)(x)$ is the image of $\chi_{D_P}(y)(y)$, i.e. $\sum_\beta a^\beta \iota(c_\beta) = 0$. So $D_{E'}$ is Cayley–Hamilton; $D_{E'} \circ \iota'$ is the base change of $D_E \circ \iota = D$.

Step 5: the involution and D. On Γ, $(\gamma\gamma')^* \mapsto \eta(\gamma\gamma')\gamma'^*\gamma^* = (\eta(\gamma')\gamma'^*)(\eta(\gamma)\gamma^*)$ and $\gamma \mapsto \eta(\gamma)\eta(\gamma^*)\gamma^{**} = \gamma$ under two applications; by linearity $(yy')^* = y'^* y^*$ and $y^{**} = y$ on every $B'[\Gamma]$.

The polynomial law $y \mapsto D(y^*)$ (extended linearly over every T) is multiplicative ($D((yy')^*) = D(y'^*)D(y^*)$, T being commutative) and homogeneous of degree n, so it is a determinant of dimension n on Γ, with characteristic polynomial $D(X - \eta(\gamma)\gamma^*) = \chi_D(\eta(\gamma)\gamma^*) = \chi_D(\gamma)$ at $\gamma \in \Gamma$. By (Ch1) it equals D. Hence, for generic $y = \sum_i t_i y_i$, $\chi_{D_P}(y^*) = \chi_{D_P}(y)$ and, $*$ being P-linear and anti-multiplicative, $(\chi_{D_P}(y)(y))^* = \chi_{D_P}(y^*)(y^*)$ with $y^* = \sum_i t_i y_i^*$ generic again: $\mathrm{CH}(D)^* \subseteq \mathrm{CH}(D)$.

Step 6: $\bar{J}^* = \bar{J}$ and (Q2). Let $N \geq 1$ and H open normal. As $\gamma \mapsto \gamma^*$ is a homeomorphism and an anti-automorphism, H^* is an open normal subgroup; η followed by $B^\times \to (B/\mathfrak{m}^N)^\times$ has open kernel. Choose an open normal $H' \subseteq H^*$ inside that kernel. For $h \in H'$, $(h-1)^* = (\eta(h)-1)h^* + (h^*-1) \in \mathfrak{m}^N B[\Gamma] + \{h''-1 : h'' \in H\}$; $(\mathfrak{m}^N \cdot 1)^* = \mathfrak{m}^N \cdot 1$; $\mathrm{CH}(D)^* \subseteq \mathrm{CH}(D)$ (Step 5); $\mathcal{S}^* \subseteq \bar{J} \subseteq J_{N,H}$. An anti-automorphism maps the two-sided ideal generated by a set onto the two-sided ideal generated by its image, so $J^*_{N,H'} \subseteq J_{N,H}$, and $\bar{J}^* \subseteq \bigcap_{N,H} J_{N,H} = \bar{J}$; with $** = \mathrm{id}$, $\bar{J}^* = \bar{J}$. The B'-linear extension maps the image of $\bar{J}$ onto itself, hence $J'^* = J'$. Finally $\mathrm{tr}_{D_{E'}}(\iota'(y))$ is minus the coefficient of X^{n-1} in $D_{B'}(X-y)$, and $D_{B'}(X-y^*) = D_{B'}((X-y)^*) = D_{B'}(X-y)$ by Step 5; so $\mathrm{tr}_{D_{E'}}(\iota'(y^*)) = \mathrm{tr}_{D_{E'}}(\iota'(y))$. □

Lemma E.71. (Kt1) *Let A be a commutative ring with $\mathrm{Spec}(A)$ connected, R an A-algebra, $D\colon R \to A$ a Cayley–Hamilton determinant of dimension $n \geq 1$, and $e_1, \ldots, e_n \in R$ orthogonal idempotents with $\sum_a e_a = 1$ and $D(1 + Xe_a) = 1 + X$ in $A[X]$ for every a. Then $\ker(D) = \{x \in R : \mathrm{tr}_D(yx) = 0$ for all $y \in R\}$.*

(Kt2) *Let B be a complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic $\neq 2$, Γ a profinite group, $n \geq 1$, and D a B-valued determinant of dimension n on Γ such that $g \mapsto \Lambda_i(g)$ is continuous for the $\mathfrak{m}$-adic topology ($1 \leq i \leq n$) and the base change of D to k is $\det \circ \mathrm{diag}(\bar{\chi}_1, \ldots, \bar{\chi}_n)$ for pairwise distinct continuous characters $\bar{\chi}_a\colon \Gamma \to k^\times$ with $Z^1(\bar{\chi}_a, \bar{\chi}_b)$ finite-dimensional for all $a \neq b$. Then for $y \in B[\Gamma]$: $y \in \ker(D)$ if and only if $\mathrm{tr}_D(gy) = 0$ for all $g \in \Gamma$.*

Proof. Two general facts. For a determinant D of dimension n and $z \in R \otimes_A T$, $D_T(1+tz) = \sum_{i=0}^n \Lambda_i(z)t^i$, so the coefficient of t is $\mathrm{tr}_D(z)$.

By (Am1), $D(1+tuv) = D(1+tvu)$, hence $\mathrm{tr}_D(uv) = \mathrm{tr}_D(vu)$.

(Kt1), the inclusion $\subseteq$. If $x \in \ker(D)$ and $y \in R$, then $yx \in \ker(D)$ (Ck1), so $D(1+tyx) = 1$ and $\mathrm{tr}_D(yx) = 0$.

(Kt1), corners. Let T be a polynomial ring $A[x_1, \ldots, x_m]$; $\mathrm{Spec}(T)$ is connected (the idempotents of $A[x_1, \ldots, x_m]$ are those of A), and D_T is Cayley–Hamilton. Fix a and $e = e_a$. By (Cd1) for $(R \otimes_A T, D_T)$, $D_T((1-e)+Ye) = D_{1-e}(1-e)\, D_e(Ye) = Y^{r(e)}$, and with $Y = 1+X$ the hypothesis gives $(1+X)^{r(e)} = 1+X$, so $r(e) = 1$. By (Cd1) and (Cd2), D_e is a Cayley–Hamilton determinant of dimension 1 on $e(R \otimes T)e$, so every $w \in e(R \otimes T)e$ equals λe with $\lambda \in T$; then $D_T(X-w) = D_T((X-\lambda)e + X(1-e)) = (X-\lambda)X^{n-1}$, so $\lambda = \mathrm{tr}_{D_T}(w)$. Hence, for $z \in R \otimes_A T$,

$$eze = \mathrm{tr}_{D_T}(eze)\, e = \mathrm{tr}_{D_T}(ze)\, e.$$

(Kt1), the inclusion $\supseteq$. Let $x \in R$ with $\mathrm{tr}_D(yx) = 0$ for all $y \in R$; by T-linearity of tr_{D_T}, $\mathrm{tr}_{D_T}(hx) = 0$ for $h \in R \otimes_A T$. For $h \in R \otimes_A T$, (Am1) and the corner formula give

$$D_T(1 + hxe_a) = D_T(1 + (hxe_a)e_a) = D_T(1 + e_a hxe_a) = D_T\big(1 + \mathrm{tr}_{D_T}(e_a hx)\, e_a\big) = D_T(1) = 1.$$

Every $h' \in R \otimes_A T'$, for any commutative A-algebra T', is the image of some $h = \sum_k x_k \otimes r_k$ under a homomorphism $A[x_1, \ldots, x_m] \to T'$, and $D_{T'}(1 + h'xe_a)$ is the image of $D_T(1 + hxe_a) = 1$. So $xe_a \in \ker(D)$ for every a, and $x = \sum_a xe_a \in \ker(D)$ (Ck1).

(Kt2). If $y \in \ker(D)$ then $\mathrm{tr}_D(gy) = 0$ by the first inclusion. Conversely, let $E_0 = B[\Gamma]/\bar{J}_0$ be the quotient of Lemma E.69 for $\mathcal{S} = \varnothing$, with projection ι_0. By (Q1) of Lemma E.70, $\bar{J}_0 \subseteq \ker(D)$ and $D = D_0 \circ \iota_0$ for a Cayley–Hamilton determinant D_0 on E_0; by (F1) E_0 is a finitely generated B-module.

Residual map. Let $\bar{\rho}\colon B[\Gamma] \to k^n$ be the ring homomorphism with $\bar{\rho}(bg) = \bar{b}\,(\bar{\chi}_1(g), \dots, \bar{\chi}_n(g))$. If $x \in \ker(D)$, its image $\bar{x} \in k[\Gamma]$ lies in the kernel of the base change $\bar{D} = \det \circ \mathrm{diag}(\bar{\chi}_a)$, so $1 = \bar{D}(1 + t\bar{x}) = \prod_a (1 + t\bar{\rho}_a(x))$ in $k[t]$, i.e. $\bar{\rho}(x) = 0$. Hence $\bar{\rho}$ factors through a surjection $\bar{\rho}_0\colon E_0 \to k^n$ (surjective since distinct characters are linearly independent); its kernel K_0 contains $\mathfrak{m}E_0$.

The inclusion $K_0 \subseteq \mathrm{rad}(E_0)$. For $u \in K_0$ the image of $\chi_{D_0}(u)(X)$ in $k[X]$ is $\prod_a (X - \bar{\rho}_{0,a}(u)) = X^n$, so by the Cayley–Hamilton property $u^n = -\sum_{i \geq 1} (-1)^i \Lambda_i(u) u^{n-i} \in \mathfrak{m}E_0$. As E_0 is a finitely generated module over the complete ring B, it is $\mathfrak{m}$-adically complete, and $1 - w$ is invertible for $w \in \mathfrak{m}E_0$ (inverse $\sum_j w^j$, $w^j \in \mathfrak{m}^j E_0$). So $1 - u^n = (1 - u)(1 + u + \cdots + u^{n-1})$ shows $1 - u$ invertible. Applied to $u = yx$ ($x \in K_0$, $y \in E_0$, K_0 an ideal), $K_0 \subseteq \mathrm{rad}(E_0)$.

Idempotents. The ring B is henselian with 2 invertible and E_0 is integral over B; by Theorem B.100 (with $\tau = \mathrm{id}$, $I = K_0$) there are orthogonal idempotents $e_1, \dots, e_n \in E_0$ with $\bar{\rho}_0(e_a)$ the a-th standard idempotent. The idempotent $f = 1 - \sum_a e_a$ lies in $\mathrm{rad}(E_0)$, so $1 - f$ is invertible and $f = 0$ from $f(1 - f) = 0$. By (Cd1) (as in the corner step, $\mathrm{Spec}(B)$ connected), $D_0(1 + Xe_a) = (1 + X)^{r(e_a)}$, whose image in $k[X]$ is $\bar{D}(1 + X\bar{e}_a) = \prod_b (1 + X\bar{\rho}_{0,b}(e_a)) = 1 + X$; so $D_0(1 + Xe_a) = 1 + X$.

Conclusion. By (Kt1) for $(E_0, D_0, (e_a))$, $\ker(D_0) = \{z : \mathrm{tr}_{D_0}(wz) = 0 \ \forall w \in E_0\}$. If $\mathrm{tr}_D(gy) = 0$ for all $g \in \Gamma$, then $\mathrm{tr}_{D_0}(\iota_0(w)\iota_0(y)) = \mathrm{tr}_D(wy) = 0$ for all $w \in B[\Gamma]$ by linearity, so $\iota_0(y) \in \ker(D_0)$; and $D_T(1 + yr') = D_{0,T}(1 + \iota_0(y)\iota_0(r')) = 1$ for all T and r', i.e. $y \in \ker(D)$. □

Lemma E.72. *Let B be a complete Noetherian local ring with maximal ideal $\mathfrak{m}_B$ and finite residue field k_B of characteristic $\ell \neq 2$; k a finite field, $A = k[[T]]$, $F = \mathrm{Frac}(A)$ and $f\colon B \to A$ a ring homomorphism; $\hat{B}$ a complete Noetherian local ring with maximal ideal $\hat{\mathfrak{m}}$, a ring homomorphism $B \to \hat{B}$, and a ring homomorphism $\lambda\colon \hat{B} \to F$ with kernel $\hat{\mathfrak{m}}$, image F, and restriction to B equal to f followed by $A \subseteq F$. Let Γ^0 be a profinite group satisfying Φ_ℓ, $\gamma \mapsto \gamma^*$ a continuous anti-automorphism of Γ^0 with $\gamma^{**} = \gamma$, extended B-linearly (and $\hat{B}$-linearly) to the group rings, $n \geq 1$, $1 \leq a \leq n$, and D a B-valued determinant of dimension n on Γ^0 with $g \mapsto \Lambda_i(g)$ continuous for the $\mathfrak{m}_B$-adic topology, whose base change to k_B is $\det \circ \mathrm{diag}(\bar{\chi}_1, \dots, \bar{\chi}_n)$ for pairwise distinct continuous characters $\bar{\chi}_b\colon \Gamma^0 \to k_B^\times$, and with $\chi_D(\gamma^*) = \chi_D(\gamma)$ for $\gamma \in \Gamma^0$. Let $\mathcal{S} \subseteq \ker(D)$ with $\mathcal{S}^* = \mathcal{S}$, $\mathcal{I} = \bar{J}$ the ideal formed from $(B, \Gamma^0, D, \mathcal{S})$, J the two-sided ideal of $\hat{B}[\Gamma^0]$ generated by the image of $\mathcal{I}$, $E = \hat{B}[\Gamma^0]/J$ and $\iota\colon \hat{B}[\Gamma^0] \to E$ the projection. Let $\rho\colon \Gamma^0 \to \mathrm{GL}_n(A)$ be a homomorphism, continuous for the T-adic topology, and $\rho_f\colon B[\Gamma^0] \to \mathrm{M}_n(A)$, $\sum b_\gamma \gamma \mapsto \sum f(b_\gamma)\rho(\gamma)$, such that $f \circ D = \det \circ \rho$, $\rho(\gamma^*) = {}^t\rho(\gamma)$ for $\gamma \in \Gamma^0$, $\rho_f(\mathcal{S}) = 0$, and, if $a < n$, every $\rho(\gamma)$ is block diagonal with diagonal blocks of sizes a and $n - a$. Let $\mathrm{M}_n^{(a)}(F)$ be the set of matrices in $\mathrm{M}_n(F)$ whose entry at (i, j) vanishes unless i, j are in the same block. Then:*

(Ab0) *(a criterion for the span) let $\bar{F}$ be an algebraic closure of F and $\Gamma'' \subseteq \Gamma^0$ a subgroup. If $a = n$ and $\rho \otimes \bar{F}$ restricted to Γ'' is irreducible, the F-span of $\rho(\Gamma^0)$ is $\mathrm{M}_n(F)$. If $a < n$ and there are $Q \in \mathrm{GL}_n(\bar{F})$, integers $a', b' \geq 1$ with $a' + b' = n$ and homomorphisms $r_1\colon \Gamma'' \to \mathrm{GL}_{a'}(\bar{F})$, $r_2\colon \Gamma'' \to \mathrm{GL}_{b'}(\bar{F})$ with $Q\rho(\sigma)Q^{-1} = \mathrm{diag}(r_1(\sigma), r_2(\sigma))$ for $\sigma \in \Gamma''$, r_1 and r_2 irreducible, and 0 the only $Y \in \mathrm{M}_{b' \times a'}(\bar{F})$ with $Yr_1(\sigma) = r_2(\sigma)Y$ for all $\sigma \in \Gamma''$, then the F-span of $\rho(\Gamma^0)$ is $\mathrm{M}_n^{(a)}(F)$;*

(Ab1) *E is a finitely generated $\hat{B}$-module carrying a Cayley–Hamilton determinant D_E of*

dimension n with $D_E \circ \iota$ the base change of D to $\hat{B}$; $(yy')^* = y'^* y^*$, $J^* = J$ and $\mathrm{tr}_{D_E}(\iota(y^*)) = \mathrm{tr}_{D_E}(\iota(y))$ for $y, y' \in \hat{B}[\Gamma^0]$; the image P_0 of $B[\Gamma^0]$ in E is a finitely generated B-module, and for every $j \geq 1$ there is an open normal subgroup V_j of Γ^0 with $\iota(\gamma v) - \iota(\gamma) \in \mathfrak{m}_B^j P_0$ for $\gamma \in \Gamma^0$, $v \in V_j$;

(Ab2) *if the F-span of $\rho(\Gamma^0)$ is $\mathrm{M}_n^{(a)}(F)$ (for instance by (Ab0)), there is a homomorphism of $\hat{B}$-algebras $r\colon E \to \mathrm{M}_n(F)$ ($\hat{B}$ acting through λ) with $r(\iota(\gamma)) = \rho(\gamma)$, such that the image of $D_E(t - z)$ in $F[t]$ is the characteristic polynomial of $r(z)$ for $z \in E$, $r(\iota(y^*)) = r(\iota(y))^T$, and $r(E) = \mathrm{M}_n^{(a)}(F)$;*

(Ab3) *if the F-span of $\rho(\Gamma^0)$ is $\mathrm{M}_n^{(a)}(F)$ and 2 is invertible in $\hat{B}$, there are a commutative ring $\mathcal{A}$ containing $\hat{B}$ as a subring and finitely generated as a $\hat{B}$-module, a $\hat{B}$-submodule $N \subseteq \mathcal{A}$ with $\mathcal{A} = \hat{B} \cdot 1 \oplus N$, $xy \in \hat{\mathfrak{m}} \cdot 1$ for $x, y \in N$ and $N = 0$ if $a = n$, and a homomorphism of $\hat{B}$-algebras $\nu\colon \hat{B}[\Gamma^0] \to \mathrm{M}_n(\mathcal{A})$ with: $\nu(J) = 0$, and the $\hat{B}$-span of $\nu(\Gamma^0)$ is the set of matrices with entries in $\hat{B} \cdot 1$ at positions in the same block and in N elsewhere; $\nu(y^*) = \nu(y)^T$; the base change of D to $\mathcal{A}$ is $\det \circ \nu$ (for every commutative $\mathcal{A}$-algebra T', $D_{T'}(\sum_b t_b \gamma_b) = \det(\sum_b t_b \nu(\gamma_b))$); if $\gamma \in \Gamma^0$, $b \in \hat{B}$ and $\nu(\gamma)_{ij} - b \cdot 1 \in N$, then $\lambda(b) = \rho(\gamma)_{ij}$; every prime ideal Q of $\hat{B}$ such that $\kappa[\Gamma^0]$ modulo the two-sided ideal generated by the image of $\mathcal{I}$ is not semisimple, for some homomorphism $\hat{B} \to \kappa$ to a field with kernel Q, contains $\mathrm{Ann}_{\hat{B}}(N)$; and $\nu = \bar{\nu} \circ \iota$ for a homomorphism of $\hat{B}$-algebras $\bar{\nu}\colon E \to \mathrm{M}_n(\mathcal{A})$, so that $\mathcal{P}_0 = \bar{\nu}(P_0)$ is a finitely generated B-module containing $\nu(\Gamma^0)$ with $\nu(\gamma v) - \nu(\gamma) \in \mathfrak{m}_B^j \mathcal{P}_0$ for $\gamma \in \Gamma^0$, $v \in V_j$.*

Proof. Step 0: (Ab0). The F-span of $\rho(\Gamma^0)$ contains that of $\rho(\Gamma'')$, and the dimension of a span does not change under extension of the field to $\bar{F}$. If $a = n$, Burnside's theorem gives that the $\bar{F}$-span of $\rho(\Gamma'')$ is $\mathrm{M}_n(\bar{F})$, so the F-span of $\rho(\Gamma^0)$ is $\mathrm{M}_n(F)$. Let $a < n$ and $V = \bar{F}^n$ with the action of Γ'' through ρ. Through Q, V is the direct sum of simple $\bar{F}[\Gamma'']$-modules of dimensions a' and b' (afforded by r_1, r_2), which are not isomorphic (an isomorphism would be a non-zero Y with $Y r_1(\sigma) = r_2(\sigma) Y$, after transposing the roles if $a' = b'$; if $a' \neq b'$ they differ in dimension). The submodules $\bar{F}^a \oplus 0$ and $0 \oplus \bar{F}^{n-a}$ (ρ is block diagonal) are direct sums of simple submodules isomorphic to these two, of dimension $< n$, hence simple; so the blocks ρ_1, ρ_2 of ρ are irreducible on Γ'' over $\bar{F}$ and not isomorphic. By Schur's lemma the commutant of the $\bar{F}$-span $\mathcal{B}$ of $\{\mathrm{diag}(\rho_1(\sigma), \rho_2(\sigma)) : \sigma \in \Gamma''\}$ in $\mathrm{End}(V)$ is $\bar{F} \times \bar{F}$, and the density theorem gives $\mathcal{B} = \mathrm{M}_a(\bar{F}) \times \mathrm{M}_{n-a}(\bar{F})$, of dimension $a^2 + (n-a)^2$. Hence the F-span of $\rho(\Gamma^0)$, contained in $\mathrm{M}_n^{(a)}(F)$ (ρ is block diagonal), has dimension $a^2 + (n-a)^2$ and equals it.

Step 1: (Ab1). Since Γ^0 satisfies Φ_ℓ, (F0) shows that all $Z^1(\bar{\chi}_b, \bar{\chi}_{b'})$ ($b \neq b'$) are finite-dimensional. Let $E_B = B[\Gamma^0]/\mathcal{I}$ with projection ι_B. By (F1) and (F2), E_B is a finitely generated B-module and for every j there is an open normal subgroup V_j with $\iota_B(\gamma v) - \iota_B(\gamma) \in \mathfrak{m}_B^j E_B$. By (Q1) with $B' = \hat{B}$ (here $\mathcal{S} \subseteq \ker(D)$ and D is continuous), $E = E_B \otimes_B \hat{B}$ is a finitely generated $\hat{B}$-module with a Cayley–Hamilton determinant D_E such that $D_E \circ \iota$ is the base change of D. By (Q2) with $\eta = 1$ (the hypothesis $\chi_D(\gamma^*) = \chi_D(\gamma)$ is given, and $\mathcal{S}^* = \mathcal{S} \subseteq \mathcal{I}$), $(yy')^* = y'^* y^*$, $J^* = J$ and $\mathrm{tr}_{D_E} \circ \iota \circ {}^* = \mathrm{tr}_{D_E} \circ \iota$. The image P_0 of $B[\Gamma^0]$ in E is the image of $E_B \to E$, $e \mapsto e \otimes 1$, a finitely generated B-module, and $\iota(\gamma v) - \iota(\gamma) = (\iota_B(\gamma v) - \iota_B(\gamma)) \otimes 1 \in \mathfrak{m}_B^j P_0$.

Step 2: ρ_f kills $\mathcal{I}$. (i) ρ_f kills $\mathrm{CH}(D)$: for $y = \sum_i t_i y_i$ the image of $\chi_{D'}(y)(y)$ is $\chi(\rho_f(y))(\rho_f(y))$, with χ the characteristic polynomial of the matrix $\rho_f(y)$ over $A[t_1, \ldots, t_m]$ (as $f \circ D = \det \circ \rho$), which is 0 by the Cayley–Hamilton theorem; so ρ_f kills all its coefficients. (ii) $\rho_f(\mathcal{S}) = 0$ by hypothesis. (iii) A ring homomorphism from the local ring B to A followed by $A \to k$ has maximal kernel, so $f(\mathfrak{m}_B) \subseteq TA$ and $f(\mathfrak{m}_B^j) \subseteq T^j A$. For $j \geq 1$,

$H_j = \{\gamma \in \Gamma^0 : \rho(\gamma) \equiv 1 \bmod T^j\}$ is an open normal subgroup (ρ is T-adically continuous). Every $y \in \mathcal{I}$ lies in the two-sided ideal generated by $\mathrm{CH}(D) \cup \mathcal{S} \cup \mathfrak{m}_B^j \cdot 1 \cup \{h - 1 : h \in H_j\}$, whose generators ρ_f maps into the two-sided ideal $T^j \mathrm{M}_n(A)$; so $\rho_f(y) \in \bigcap_j T^j \mathrm{M}_n(A) = 0$.

Step 3: (Ab2). By Step 2, ρ_f factors through E_B, and $r(e \otimes \hat{b}) = \lambda(\hat{b})\rho_f(e)$ is a well-defined homomorphism of $\hat{B}$-algebras $E = E_B \otimes_B \hat{B} \to \mathrm{M}_n(F)$, since λ restricted to B is f; $r(\iota(\gamma)) = \rho(\gamma)$. The base change of D along $B \to \hat{B} \xrightarrow{\lambda} F$ is $\det \circ \rho$ (base-changed to F), so for $z = \iota(y)$ the image of $D_E(t - z)$ is $\det(t - r(z))$. From $\rho(\gamma^*) = {}^t\rho(\gamma)$, $r(\iota(y^*)) = r(\iota(y))^T$. As E is generated over $\hat{B}$ by $\iota(\Gamma^0)$ and $\lambda(\hat{B}) = F$, $r(E)$ is the F-span of $\rho(\Gamma^0)$, which is $\mathrm{M}_n^{(a)}(F)$ by the hypothesis of (Ab2).

Step 4: (Ab3). Apply Proposition E.73 to $(\hat{B}, F, \lambda)$ ($\hat{B}$ complete Noetherian local with 2 invertible, $\ker\lambda = \hat{\mathfrak{m}}$), the group Γ^0, n, a, the image of $\mathcal{I}$ in $\hat{B}[\Gamma^0]$ as generating set (so that the two-sided ideal it generates is J), the map * and r: (R1), (R2) are (Ab1) and (R3) is (Ab2), whose hypothesis is part of that of (Ab3). This gives $\mathcal{A}$, N, ν with (G1)–(G5). (G1) is the first clause of (Ab3), (G2) the second, (G3) (for the base change $D_E \circ \iota$ of D to $\hat{B}$) the third, (G4) with $y = \gamma$ and $r(\iota(\gamma)) = \rho(\gamma)$ the fourth, and (G5) the fifth ($\kappa[\Gamma^0]$ modulo the ideal generated by the image of $\mathcal{I}$ is the ring of (G5)). By (G1), ν vanishes on J, so $\nu = \bar{\nu} \circ \iota$ with $\bar{\nu}$ a homomorphism of $\hat{B}$-algebras, in particular B-linear; $\mathcal{P}_0 = \bar{\nu}(P_0)$ is a finitely generated B-module containing $\bar{\nu}(\iota(\Gamma^0)) = \nu(\Gamma^0)$, and $\nu(\gamma v) - \nu(\gamma) = \bar{\nu}(\iota(\gamma v) - \iota(\gamma)) \in \bar{\nu}(\mathfrak{m}_B^j P_0) = \mathfrak{m}_B^j \mathcal{P}_0$ by (Ab1). □

Proposition E.73. *Let B be a complete Noetherian local ring with maximal ideal $\mathfrak{m}$ in which 2 is invertible, F a field and $\lambda\colon B \to F$ a ring homomorphism with kernel $\mathfrak{m}$. Let Γ be a group, $n \geq 1$ and $1 \leq a \leq n$; we say that $i, j \in \{1, \dots, n\}$ are in the same block if $i, j \leq a$ or $i, j > a$. Let $\mathcal{I} \subseteq B[\Gamma]$ be a subset, J the two-sided ideal generated by $\mathcal{I}$, $E = B[\Gamma]/J$ and $\iota\colon B[\Gamma] \to E$ the projection. Assume:*

(R1) *E is a finitely generated B-module carrying a Cayley–Hamilton determinant $D_E\colon E \to B$ of dimension n; write* tr *for its trace and $D = D_E \circ \iota$;*

(R2) *$y \mapsto y^*$ is a B-linear map $B[\Gamma] \to B[\Gamma]$ with $(yy')^* = y'^* y^*$, $y^{**} = y$, $J^* = J$ and $\mathrm{tr}(\iota(y^*)) = \mathrm{tr}(\iota(y))$;*

(R3) *$r\colon E \to \mathrm{M}_n(F)$ is a homomorphism of B-algebras such that the image of $D_E(t - z)$ in $F[t]$ is the characteristic polynomial of $r(z)$ for all $z \in E$, $r(\iota(y^*)) = r(\iota(y))^T$ for all y, and $r(E)$ is the set of matrices whose entry at (i, j) vanishes unless i, j are in the same block;*

(R4) *(only for (G6)) Γ is a topological group and for every $j \geq 1$ there is an open normal subgroup V_j of Γ with $\iota(\gamma v) - \iota(\gamma) \in \mathfrak{m}^j E$ for all $\gamma \in \Gamma$, $v \in V_j$.*

Then there are a commutative ring $\mathcal{A}$ containing B as a subring and finitely generated as a B-module, a B-submodule $N \subseteq \mathcal{A}$ with $\mathcal{A} = B \cdot 1 \oplus N$, $xy \in \mathfrak{m} \cdot 1$ for all $x, y \in N$ and $N = 0$ if $a = n$, and a homomorphism of B-algebras $\nu\colon B[\Gamma] \to \mathrm{M}_n(\mathcal{A})$ such that:

(G1) *$\nu(J) = 0$, and the B-span of $\nu(\Gamma)$ is the set of matrices with entries in $B \cdot 1$ at the positions (i, j) in the same block and in N at the other positions;*

(G2) *$\nu(y^*) = \nu(y)^T$ for all $y \in B[\Gamma]$;*

(G3) *D followed by $B \to \mathcal{A}$ equals $\det \circ \nu$: for every commutative $\mathcal{A}$-algebra T, $t_1, \dots, t_m \in T$ and $\gamma_1, \dots, \gamma_m \in \Gamma$, $D_T(\sum_s t_s \gamma_s) = \det(\sum_s t_s \nu(\gamma_s))$;*

(G4) *if $y \in B[\Gamma]$, $b \in B$ and $\nu(y)_{ij} - b \cdot 1 \in N$, then $\lambda(b) = r(\iota(y))_{ij}$;*

(G5) *every prime ideal Q of B such that $\kappa[\Gamma]/\langle f(\mathcal{I})\rangle$ is not semisimple for some homomorphism $f\colon B \to \kappa$ to a field with $\ker f = Q$ contains $\mathrm{Ann}_B(N)$;*

(G6) *under (R4), with $\mathcal{P} \subseteq \mathrm{M}_n(\mathcal{A})$ the finitely generated B-module of matrices described in (G1): $\nu(\Gamma) \subseteq \mathcal{P}$, and $\nu(\gamma v) - \nu(\gamma) \in \mathfrak{m}^j\mathcal{P}$ for all $j \geq 1$, $\gamma \in \Gamma$, $v \in V_j$.*

Proof. Step 1: the involution and the trace on E. Since $J^* = J$, $*$ induces a B-linear map on E, again written $*$, with $(zz')^* = z'^*z^*$, $z^{**} = z$ and $\mathrm{tr}(z^*) = \mathrm{tr}(z)$ by (R2). For $z, z' \in E$, (Am1) over $B[t]$ gives $D_E(1-tzz') = D_E(1-tz'z)$; comparing the coefficients of t gives $\mathrm{tr}(zz') = \mathrm{tr}(z'z)$. Also $r(z^*) = r(z)^T$ by (R3), as ι is surjective.

Step 2: symmetric matrix units. B is henselian, $2 \in B^\times$, E is finite over B and D_E is Cayley–Hamilton, and r satisfies the hypotheses of Proposition E.74 (its image contains the block-diagonal matrices by (R3)). So there are $e_i, x_i \in E$ $(1 \leq i \leq n)$ satisfying (U1)–(U3) for tr, $*$ and the map o with $o(i) = 1$ for $i \leq a$, $o(i) = n$ for $i > a$, with $r(e_i) = E_{ii}$, $r(x_i) = E_{o(i)i}$ and $D_E(1 + te_i) = 1 + t$. Here $o(i) = o(j)$ if and only if i, j are in the same block, and $o(\{1, \dots, n\}) = \{1\}$ if $a = n$, $= \{1, n\}$ with $1 \neq n$ if $a < n$.

Step 3: the realisation. Lemma E.75 with $I = \{1, \dots, n\}$, $p = 1$ and (if $a < n$) $q = n$ gives the commutative ring $\mathcal{A} = B \oplus N$ (with $N = 0$ if $a = n$), finite over B, containing B via $b \mapsto (b, 0)$, with $\mathcal{A} = B \cdot 1 \oplus N$ and $nn' = \langle n, n'\rangle \cdot 1$, and the injective B-algebra homomorphism $\varphi\colon E \to \mathrm{M}_n(\mathcal{A})$. Its part (3) applies to r: $r(z^*) = r(z)^T$, $r(x_i) = E_{o(i)i}$, and $r(z)_{ij} = 0$ when $o(i) \neq o(j)$ by (R3). Hence $\lambda(\langle n, n'\rangle) = 0$, i.e. $nn' \in \mathfrak{m} \cdot 1$ for $n, n' \in N$. Put $\nu = \varphi \circ \iota$.

Step 4: (G1), (G2), (G4). $\nu(J) = \varphi(\iota(J)) = 0$. The B-span of $\nu(\Gamma)$ is φ of the B-span of $\iota(\Gamma)$, which is $\varphi(E)$ since Γ spans $B[\Gamma]$ and ι is onto; by Lemma E.75 (2) this is the set of matrices described in (G1). (G2): $\nu(y^*) = \varphi(\iota(y)^*) = \varphi(\iota(y))^T$ by Lemma E.75 (2). (G4): if $\nu(y)_{ij} = (b', n)$ with $(b', n) - (b, 0) \in N$, then $b' = b$, and $\lambda(b) = r(\iota(y))_{ij}$ by Lemma E.75 (3).

Step 5: (G3). By (U1), (U2) ($e_iEe_i = Be_i$) and $D_E(1 + te_i) = 1 + t$, Lemma E.76 applies to D_E, the e_i, $S = \mathcal{A}$ and φ (with $\varphi(e_i) = E_{ii}$ by Lemma E.75 (2)): $D_{E,T}(\sum_a t_a \otimes z_a) = \det(\sum_a t_a \varphi(z_a))$ for every commutative $\mathcal{A}$-algebra T. With $z_a = \iota(\gamma_a)$, and $D_T = D_{E,T} \circ (\mathrm{id}_T \otimes \iota)$ by definition of D, this is (G3).

Step 6: (G5). $E \cong B[\Gamma]/\langle\mathcal{I}\rangle$, so Lemma E.77 gives $\mathrm{Ann}_B(N) \subseteq Q$ for every such Q.

Step 7: (G6). By Lemma E.75 (2), $\varphi(E) = \mathcal{P}$, which is finitely generated over B as E is. Hence $\nu(\Gamma) \subseteq \mathcal{P}$, and, φ being B-linear, $\nu(\gamma v) - \nu(\gamma) = \varphi(\iota(\gamma v) - \iota(\gamma)) \in \varphi(\mathfrak{m}^j E) = \mathfrak{m}^j\mathcal{P}$ for $v \in V_j$. □

Proposition E.74. *Let B be a henselian local ring with maximal ideal $\mathfrak{m}_B$ in which 2 is invertible, E a B-algebra which is finitely generated as a B-module, $D\colon E \to B$ a Cayley–Hamilton determinant of dimension $n \geq 1$ with trace* tr*, and $y \mapsto y^*$ a B-linear map $E \to E$ with $(yy')^* = y'^*y^*$, $y^{**} = y$, $\mathrm{tr}(y^*) = \mathrm{tr}(y)$ and $\mathrm{tr}(yy') = \mathrm{tr}(y'y)$ for all $y, y' \in E$. Let F be a field, $\lambda\colon B \to F$ a ring homomorphism with kernel $\mathfrak{m}_B$, and $r\colon E \to \mathrm{M}_n(F)$ a homomorphism of B-algebras such that for all $y \in E$ the image of $D(t - y)$ in $F[t]$ is the characteristic polynomial of $r(y)$ and $r(y^*) = r(y)^T$. Let $1 \leq a \leq n$, assume that every matrix in $\mathrm{M}_n(F)$ whose entries at (i, j) vanish whenever exactly one of i, j is $\leq a$ lies in $r(E)$, and let $o\colon \{1, \dots, n\} \to \{1, \dots, n\}$ be $o(i) = 1$ for $i \leq a$ and $o(i) = n$ for $i > a$. Then there are $e_1, \dots, e_n, x_1, \dots, x_n \in E$ satisfying (U1)–(U3) for* tr*, $*$ and o, with $r(e_i) = E_{ii}$, $r(x_i) = E_{o(i)i}$ and $D(1 + te_i) = 1 + t$ in $B[t]$ for every i.*

Proof. Step 1: $\ker r \subseteq \mathrm{rad}(E)$. Let $u \in \ker r$. The image of $D(t - u) = t^n + \sum_{k<n} c_k t^k$ in $F[t]$ is the characteristic polynomial of 0, so all $c_k \in \ker\lambda = \mathfrak{m}_B$, and the Cayley–Hamilton identity

$u^n + \sum_k c_k u^k = 0$ gives $u^n \in \mathfrak{m}_B E$. Now let $y \in \ker r$ and $z \in E$; then $u = zy \in \ker r$, so $1+u$ is invertible modulo $\mathfrak{m}_B E$ (its inverse there is $\sum_{k<n}(-u)^k$), hence invertible in E because $\mathfrak{m}_B E \subseteq \operatorname{rad}(E)$. As $1+zy$ is invertible for all z, $y \in \operatorname{rad}(E)$.

Step 2: orthogonal idempotents fixed by $*$. Since $r(y^*) = r(y)^T$, the two-sided ideal $\ker r$ is stable under $*$, and r induces an injective homomorphism $\bar{r}\colon E/\ker r \to \mathrm{M}_n(F)$. The diagonal matrix units lie in $r(E)$ by hypothesis; let $\epsilon_i \in E/\ker r$ with $\bar{r}(\epsilon_i) = E_{ii}$. The ϵ_i are orthogonal idempotents ($\bar{r}$ is injective), and $\epsilon_i^* = \epsilon_i$ since $\bar{r}(\epsilon_i^*) = E_{ii}^T = E_{ii}$. As E is finite, hence integral, over B, Theorem B.100 (with $S = E$, $\tau = *$, $I = \ker r$, Step 1) gives orthogonal idempotents e_i lifting ϵ_i with $\{e_1, \dots, e_n\}$ stable under $*$. If $e_i^* = e_j$, then $E_{jj} = r(e_j) = r(e_i)^T = E_{ii}$, so $j = i$: $e_i^* = e_i$. The idempotent $f = 1 - \sum_i e_i$ satisfies $r(f) = 0$, so $f \in \operatorname{rad}(E)$; then $1-f$ is invertible and $(1-f)f = 0$, so $f = 0$. Thus (U1), $e_i^* = e_i$ and $r(e_i) = E_{ii}$ hold.

Step 3: rank-one corners. By (Cd1) of Theorem B.102 ($\operatorname{Spec}(B)$ is connected as B is local), D_{e_i} is a Cayley–Hamilton determinant on $e_i E e_i$ of some dimension s_i, and D restricted to $e_iEe_i \oplus (1-e_i)E(1-e_i)$ is $D_{e_i}D_{1-e_i}$, with $\dim D_{1-e_i} = n - s_i$. Since $t - e_i = (t-1)e_i + t(1-e_i)$ and $D_{e_i}(e_i) = D(1) = 1$, homogeneity gives

$$D(t-e_i) = (t-1)^{s_i}\, t^{n-s_i}, \qquad D(1+te_i) = D_{e_i}((1+t)e_i)\, D_{1-e_i}(1-e_i) = (1+t)^{s_i}.$$

The image of $D(t-e_i)$ in $F[t]$ is the characteristic polynomial $(t-1)t^{n-1}$ of E_{ii}, so $s_i = 1$ by unique factorisation in $F[t]$. Hence $D(1+te_i) = 1+t$, and $\operatorname{tr}(e_i) = 1$ (minus the coefficient of t^{n-1} in $(t-1)t^{n-1}$). By (Cd2), $B \to e_iEe_i$, $b \mapsto be_i$, is an isomorphism, so $e_iEe_i = Be_i$, and every $w \in e_iEe_i$ equals $\operatorname{tr}(w)e_i$ (write $w = be_i$; then $\operatorname{tr}(w) = b$). This proves (U2).

Step 4: the elements x_i. For i with $o(i) = i$ (that is, $i \in \{1,n\} \cap o(\{1,\dots,n\})$) put $x_i = e_i$. For the other i, the indices $o(i)$ and i lie in the same block, so $E_{o(i)i} \in r(E)$; choose y_i with $r(y_i) = E_{o(i)i}$ and put $z_i = e_{o(i)} y_i e_i$, so that $r(z_i) = E_{o(i)o(i)}E_{o(i)i}E_{ii} = E_{o(i)i}$. As $z_i^* = e_i y_i^* e_{o(i)}$, we have $z_i^* z_i \in e_iEe_i$ and $z_i z_i^* \in e_{o(i)}Ee_{o(i)}$, so by Step 3

$$z_i^* z_i = c_i e_i, \qquad z_i z_i^* = \operatorname{tr}(z_i z_i^*)e_{o(i)} = c_i e_{o(i)}, \qquad c_i = \operatorname{tr}(z_i^* z_i) \in B.$$

Applying r to the first identity, $\lambda(c_i)E_{ii} = E_{i\,o(i)}E_{o(i)i} = E_{ii}$, so $\lambda(c_i) = 1$. The polynomial $X^2 - c_i$ has the simple root 1 modulo $\mathfrak{m}_B$ (its derivative there is $2 \neq 0$), so by Hensel's lemma there is $u_i \in B$ with $u_i^2 = c_i$ and $\lambda(u_i) = 1$; u_i is a unit. Put $x_i = u_i^{-1} z_i$. Then $x_i^* x_i = c_i^{-1} z_i^* z_i = e_i$, $x_i x_i^* = e_{o(i)}$, $e_{o(i)} x_i e_i = x_i$ and $r(x_i) = E_{o(i)i}$. For $i = o(i)$ these hold trivially for $x_i = e_i$, and $x_{o(i)} = e_{o(i)}$ for every i since $o(o(i)) = o(i)$. This proves (U3). □

Lemma E.75. *Let B be a commutative ring and E a B-algebra, with a B-linear map $y \mapsto y^*$ of E to itself such that $(yy')^* = y'^* y^*$ and $y^{**} = y$, and a B-linear map $\operatorname{tr}\colon E \to B$ such that $\operatorname{tr}(yy') = \operatorname{tr}(y'y)$ and $\operatorname{tr}(y^*) = \operatorname{tr}(y)$. Let I be a finite set, $o\colon I \to I$ a map with $o \circ o = o$, and $(e_i)_{i\in I}$, $(x_i)_{i\in I}$ elements of E such that for all $i, j \in I$:*

(U1) *$e_i e_j = \delta_{ij} e_i$ and $\sum_{i\in I} e_i = 1$;*

(U2) *$e_i^* = e_i$, $\operatorname{tr}(e_i) = 1$, and $e_iEe_i = Be_i$;*

(U3) *$e_{o(i)} x_i e_i = x_i$, $x_{o(i)} = e_{o(i)}$, $x_i x_i^* = e_{o(i)}$ and $x_i^* x_i = e_i$.*

Assume that $o(I) = \{p\}$ or $o(I) = \{p,q\}$ with $p \neq q$. Put $N = 0$ in the first case and $N = e_pEe_q$ in the second, and $\langle n, n'\rangle = \operatorname{tr}(n\, n'^*)$ for $n, n' \in N$. Then:

1. *The B-module $\mathcal{A} = B \oplus N$ with the product*

$$(b,n)(b',n') = (bb' + \langle n, n'\rangle,\ bn' + b'n) \tag{E.8}$$

is a commutative ring with unit $(1,0)$*;* $b \mapsto (b,0)$ *is an injective ring homomorphism* $B \to \mathcal{A}$*, through which* $\mathcal{A} = B \cdot 1 \oplus N$ *as* B*-modules, and* $nn' = \langle n, n' \rangle \cdot 1$ *for* $n, n' \in N$*. If* E *is a finitely generated* B*-module, so is* $\mathcal{A}$*.*

2. *For* $z \in E$ *put* $\theta(z) = (\mathrm{tr}(z),\ e_p(z + z^*)e_q) \in \mathcal{A}$ *(with second component* 0 *if* $N = 0$*). The map* $\varphi\colon E \to \mathrm{M}_I(\mathcal{A})$*,*

$$\varphi(y)_{ij} = \theta(x_i\, y\, x_j^*) \qquad (i, j \in I), \tag{E.9}$$

is an injective homomorphism of B*-algebras with* $\varphi(y^*) = \varphi(y)^T$*,* $\varphi(e_i)$ *the matrix unit at* (i,i)*, and* $\sum_i \varphi(y)_{ii} = \mathrm{tr}(y) \cdot 1$*; its image is the set of matrices* (a_{ij}) *with* $a_{ij} \in B \cdot 1$ *if* $o(i) = o(j)$ *and* $a_{ij} \in N$ *if* $o(i) \neq o(j)$*.*

3. *Let* F *be a commutative* B*-algebra with structure map* $\lambda\colon B \to F$ *and* $r\colon E \to \mathrm{M}_I(F)$ *a homomorphism of* B*-algebras with* $r(y^*) = r(y)^T$*,* $r(x_i)$ *the matrix unit at* $(o(i), i)$*, and* $r(y)_{ij} = 0$ *whenever* $o(i) \neq o(j)$*. If* $\varphi(y)_{ij} = (b, n)$*, then* $\lambda(b) = r(y)_{ij}$*. Moreover* $\lambda(\langle n, n' \rangle) = 0$ *for all* $n, n' \in N$*.*

Proof. Step 0: elementary facts. (a) For $z \in e_i E e_i$ we have $z = \mathrm{tr}(z)e_i$: by (U2), $z = se_i$ with $s \in B$, and $\mathrm{tr}(z) = s\,\mathrm{tr}(e_i) = s$. (b) For $i \neq j$ and $z \in e_i E e_j$, $\mathrm{tr}(z) = \mathrm{tr}(e_i \cdot z e_j) = \mathrm{tr}(z e_j e_i) = 0$ by (U1). (c) $(e_a z e_b)^* = e_b z^* e_a$ by (U2), so $*$ maps C_{ab} onto C_{ba}. (d) By (U3), $x_i = x_i e_i$ and $x_j^* = e_j x_j^*$; hence $x_i x_j^* = x_i e_i e_j x_j^* = 0$ for $i \neq j$, and $x_i x_i^* = e_{o(i)}$. (e) For $y \in E$ put $w_{ij}(y) = x_i y x_j^*$; by (U3) and (c), $w_{ij}(y) \in C_{o(i)o(j)}$.

Step 1: part (1). If $N = 0$ there is nothing to prove ($\mathcal{A} = B$). Let $N = C_{pq}$. The form $\langle \cdot, \cdot \rangle$ is B-bilinear, and symmetric: $\langle n, n' \rangle = \mathrm{tr}((nn'^*)^*) = \mathrm{tr}(n'n^*) = \langle n', n \rangle$. For $n, n', n'' \in N$,

$$\langle n, n' \rangle n'' = \langle n', n'' \rangle n. \tag{E.10}$$

Indeed $nn'^* \in C_{pq}C_{qp} \subseteq C_{pp}$, so $nn'^* = \langle n, n' \rangle e_p$ by (a), and $n'^* n'' \in C_{qq}$, so $n'^* n'' = \mathrm{tr}(n'^* n'')e_q$ with $\mathrm{tr}(n'^* n'') = \mathrm{tr}(n'' n'^*) = \langle n'', n' \rangle = \langle n', n'' \rangle$; hence $\langle n, n' \rangle n'' = \langle n, n' \rangle e_p n'' = nn'^* n'' = n\langle n', n'' \rangle e_q = \langle n', n'' \rangle n$. The product (E.8) is B-bilinear, commutative by symmetry, and $(1,0)$ is a unit. Associativity: expanding both sides,

the first components agree and the second agree by (E.10). So $\mathcal{A}$ is a commutative ring; $b \mapsto (b,0)$ is an injective ring homomorphism, the B-module structure of $\mathcal{A}$ is $b(b', n) = (bb', bn)$, so $\mathcal{A} = B \cdot 1 \oplus N$, and $(0,n)(0,n') = (\langle n, n' \rangle, 0)$. Finally N is the image of E under $z \mapsto e_p z e_q$, so it is finitely generated when E is.

Step 2: values of θ *on the corners.* For $z \in C_{aa}$: $\theta(z) = (\mathrm{tr}(z), 0)$, since $e_p z e_q = e_p e_a z e_a e_q = 0$ and likewise for $z^* \in C_{aa}$ ($p \neq q$). For $z \in C_{pq}$: $\theta(z) = (0, z)$, by (b) and $e_p z^* e_q \in e_p C_{qp} e_q = 0$. For $z \in C_{qp}$: $\theta(z) = (0, z^*)$ similarly. (If $N = 0$, only the first case occurs.) In particular θ restricted to C_{ab} is injective: on C_{aa} by (a), on C_{pq} trivially, on C_{qp} since $*$ is injective. Also $\theta(z^*) = \theta(z)$ for all $z \in E$, as $\mathrm{tr}(z^*) = \mathrm{tr}(z)$ and $z^* + z^{**} = z + z^*$.

Step 3: $\theta(zz') = \theta(z)\theta(z')$ *for* $z \in C_{ab}$*,* $z' \in C_{bc}$*,* $a, b, c \in o(I)$*.* If $a = b$, then $z = se_a$ with $s = \mathrm{tr}(z)$, $zz' = sz'$ and $\theta(z) = (s, 0)$, so both sides are $s\theta(z')$; similarly if $b = c$. Otherwise $a \neq b$ and $b \neq c$, so $a = c$ and $\{a, b\} = \{p, q\}$, and $zz' \in C_{aa}$, $\theta(zz') = (\mathrm{tr}(zz'), 0)$. If $(a,b) = (p,q)$: $\theta(z)\theta(z') = (0,z)(0,z'^*) = (\mathrm{tr}(zz'), 0)$. If $(a,b) = (q,p)$: $\theta(z)\theta(z') = (0,z^*)(0,z') = (\mathrm{tr}(z^* z'^*), 0) = (\mathrm{tr}((z'z)^*), 0) = (\mathrm{tr}(z'z), 0) = (\mathrm{tr}(zz'), 0)$.

Step 4: φ *is a* B*-algebra homomorphism.* B-linearity is clear. By (U1) and (U3), $w_{ij}(yy') = x_i y(\sum_k e_k) y' x_j^* = \sum_k x_i y x_k^* x_k y' x_j^* = \sum_k w_{ik}(y) w_{kj}(y')$, with $w_{ik}(y) \in C_{o(i)o(k)}$, $w_{kj}(y') \in C_{o(k)o(j)}$. By additivity of θ and Step 3, $\varphi(yy')_{ij} = \sum_k \varphi(y)_{ik}\varphi(y')_{kj}$. By (d), $w_{ij}(1) = \delta_{ij} e_{o(i)}$, so $\varphi(1)$ is the identity matrix; likewise $w_{kl}(e_i) = x_k e_i x_l^* = \delta_{ki}\delta_{li} e_{o(i)}$ gives $\varphi(e_i)$ the matrix unit at (i,i).

Step 5: transpose and trace. $w_{ij}(y^*) = x_i y^* x_j^* = (x_j y x_i^*)^* = w_{ji}(y)^*$, so $\varphi(y^*)_{ij} = \theta(w_{ji}(y)^*) = \theta(w_{ji}(y)) = \varphi(y)_{ji}$ by Step 2. For the trace, $w_{ii}(y) \in C_{o(i)o(i)}$, so $\varphi(y)_{ii} = (\mathrm{tr}(x_i y x_i^*), 0)$ and $\mathrm{tr}(x_i y x_i^*) = \mathrm{tr}(y x_i^* x_i) = \mathrm{tr}(y e_i)$; summing over i gives $\mathrm{tr}(y)$ by (U1).

Step 6: injectivity and image. By (U3), $x_i^* w_{ij}(y) x_j = e_i y e_j$, so

$$y = \sum_{i,j} x_i^*\, w_{ij}(y)\, x_j. \tag{E.11}$$

If $\varphi(y) = 0$, then $w_{ij}(y) = 0$ for all i, j by the injectivity in Step 2, and $y = 0$ by (E.11). By Step 2, $\varphi(y)_{ij} \in \theta(C_{o(i)o(j)})$, which is $B \cdot 1$ if $o(i) = o(j)$ and N otherwise. Conversely let (a_{ij}) be a matrix of that shape. Choose $w_{ij} \in C_{o(i)o(j)}$ with $\theta(w_{ij}) = a_{ij}$: $w_{ij} = s e_{o(i)}$ if $a_{ij} = (s, 0)$ and $o(i) = o(j)$; $w_{ij} = n$ if $a_{ij} = (0, n)$ and $(o(i), o(j)) = (p, q)$; $w_{ij} = n^*$ if $a_{ij} = (0, n)$ and $(o(i), o(j)) = (q, p)$. Put $y = \sum_{i,j} x_i^* w_{ij} x_j$. By (d), $w_{kl}(y) = \sum_{i,j} (x_k x_i^*) w_{ij} (x_j x_l^*) = e_{o(k)} w_{kl} e_{o(l)} = w_{kl}$, so $\varphi(y) = (a_{ij})$.

Step 7: part (3). Write E_{ab} for the matrix units of $\mathrm{M}_I(F)$. By hypothesis $r(e_{o(k)}) = r(x_{o(k)}) = E_{o(k)o(k)}$ (as $o(o(k)) = o(k)$) and $r(x_j^*) = r(x_j)^T = E_{j\,o(j)}$. Let $\varphi(y)_{ij} = (b, n)$, so $b = \mathrm{tr}(w)$ for $w = w_{ij}(y) \in C_{o(i)o(j)}$. If $o(i) \neq o(j)$, then $b = 0$ by (b) and $r(y)_{ij} = 0$ by hypothesis. If $o(i) = o(j) = a$, then $w = b e_a$ by (a), so $r(w) = \lambda(b) E_{aa}$, while $r(w) = E_{ai}\, r(y)\, E_{ja} = r(y)_{ij} E_{aa}$; comparing the entries at (a, a) gives $\lambda(b) = r(y)_{ij}$. Finally let $N \neq 0$ and $n, n' \in N = C_{pq}$. Then $r(n) = r(e_p) r(n) r(e_q) = E_{pp} r(n) E_{qq} = r(n)_{pq} E_{pq} = 0$, since $o(p) = p \neq q = o(q)$. As $n n'^* = \langle n, n' \rangle e_p$ (Step 1), $\lambda(\langle n, n' \rangle) E_{pp} = r(n) r(n')^T = 0$, so $\lambda(\langle n, n' \rangle) = 0$. □

Lemma E.76. *Let A be a commutative ring, E an A-algebra, $D\colon E \to A$ a determinant of dimension $n \geq 1$, and $e_1, \dots, e_n \in E$ orthogonal idempotents with $\sum_i e_i = 1$, $e_i E e_i = A e_i$ and $D(1 + t e_i) = 1 + t$ in $A[t]$ for every i. Let S be a commutative A-algebra and $\varphi\colon E \to \mathrm{M}_n(S)$ a homomorphism of A-algebras with $\varphi(e_i) = E_{ii}$ for every i. Then for every commutative S-algebra T, all $m \geq 1$, $t_1, \dots, t_m \in T$ and $x_1, \dots, x_m \in E$,*

$$D_T\Big(\sum_{a=1}^{m} t_a \otimes x_a\Big) = \det\Big(\sum_{a=1}^{m} t_a\, \varphi(x_a)\Big) \quad \text{in } T,$$

where D_T is the value of D on $T \otimes_A E$ and $\varphi(x_a)$ is mapped to $\mathrm{M}_n(T)$; that is, D followed by $A \to S$ equals $\det \circ \varphi$ as S-valued determinants on E.

Proof. Let P_1 be D followed by $A \to S$ and $P_2 = \det \circ \varphi$; both are S-valued determinants of dimension n on E (multiplicative, homogeneous of degree n), and we write $\Lambda_l^{(1)}, \Lambda_l^{(2)}$ for their coefficients as in Theorem B.101. By homogeneity, $P(u - tw) = \sum_l (-1)^l \Lambda_l(w) t^l u^{n-l}$ in $S[t, u]$ for either law P, so $P(1 - tw) = \sum_l (-1)^l \Lambda_l(w) t^l$ determines all $\Lambda_l(w)$, and $P(w) = \Lambda_n(w)$.

Step 1: a spanning monoid. Let $\mathcal{M} = \{1\} \cup \bigcup_{i,j} e_i E e_j$. It is closed under multiplication ($e_i E e_j \cdot e_k E e_l$ is $0 \in e_i E e_l$ if $j \neq k$ and lies in $e_i E e_l$ if $j = k$), and it spans E as an A-module, since $y = \sum_{i,j} e_i y e_j$.

Step 2: the coefficients agree on $\mathcal{M}$. For $w = 1$: $P(1 - t) = (1 - t)^n P(1) = (1 - t)^n$ for both laws. For $w \in e_i E e_j$ with $i \neq j$: $w = (e_i w) e_j$, so by (Am1) over $A[t]$, $P(1 - tw) = P(1 - e_j\, t e_i w) = P(1 - t\, e_j e_i w) = P(1) = 1$ for both laws. For $w \in e_i E e_i$: $w = a e_i$ with $a \in A$; applying the A-algebra map $A[t] \to A[t]$, $t \mapsto -at$, to $D(1 + t e_i) = 1 + t$ gives $P_1(1 - tw) = 1 - at$, while $P_2(1 - tw) = \det(1 - ta E_{ii}) = 1 - at$. In all cases $\Lambda_l^{(1)}(w) = \Lambda_l^{(2)}(w)$ for every l.

Step 3: conclusion. Let T, t_a, x_a be as in the statement. By Step 1 write $x_a = \sum_{k=1}^{K} c_{ak} w_k$ with $c_{ak} \in A$, $w_k \in \mathcal{M}$; then $\sum_a t_a \otimes x_a = \sum_k s_k \otimes w_k$ with $s_k = \sum_a c_{ak} t_a \in T$. By naturality under $S[u_1, \dots, u_K] \to T$, $u_k \mapsto s_k$, it suffices to prove $P_1(\sum_k u_k w_k) = P_2(\sum_k u_k w_k)$ in $S[u_1, \dots, u_K]$. Both sides are $\Lambda_n(\sum_k u_k w_k)$, which by (Am2) is the same integral polynomial

$P_{K,n}$ evaluated at the u_k and at the $\Lambda_j(w)$ for the finite products w of the w_k. These products lie in $\mathcal{M}$ by Step 1, so the arguments coincide by Step 2, and so do the values. □

Lemma E.77. *In the setting of Lemma E.75, let $f\colon B \to \kappa$ be a ring homomorphism to a field and assume that $f(s) \neq 0$ for some $s \in \mathrm{Ann}_B(N)$. Then $\kappa \otimes_B E$ is isomorphic, as a κ-algebra, to $\prod_{c\in o(I)} \mathrm{M}_{|o^{-1}(c)|}(\kappa)$; in particular it is semisimple. Consequently, if Γ is a group, $\mathcal{I} \subseteq B[\Gamma]$ a subset, $E \cong B[\Gamma]/\langle\mathcal{I}\rangle$ as B-algebras ($\langle\mathcal{I}\rangle$ the two-sided ideal generated by $\mathcal{I}$), and Q a prime ideal of B such that $\kappa[\Gamma]/\langle f(\mathcal{I})\rangle$ is not semisimple for some homomorphism $f\colon B \to \kappa$ to a field with $\ker f = Q$, then $\mathrm{Ann}_B(N) \subseteq Q$.*

Proof. Step 1: $\kappa \otimes_B N = 0$ and $\kappa \otimes_B \mathcal{A} \cong \kappa$. For $c \in \kappa$ and $n \in N$, $c \otimes n = cf(s)^{-1}f(s) \otimes n = cf(s)^{-1} \otimes sn = 0$. Hence $\kappa \otimes_B \mathcal{A} = \kappa \otimes_B (B \cdot 1 \oplus N) = \kappa \otimes 1$, and the ring homomorphism $\kappa \to \kappa \otimes_B \mathcal{A}$ is an isomorphism.

Step 2: the splitting. Let $\mathrm{Blk} \subseteq \mathrm{M}_I(\mathcal{A})$ be the set of matrices with entries in $B \cdot 1$ at (i,j) with $o(i) = o(j)$ and in N at the other positions; by Lemma E.75 (2), $\varphi\colon E \to \mathrm{Blk}$ is a B-algebra isomorphism. As a B-module, $\mathrm{M}_I(\mathcal{A}) = \mathrm{Blk} \oplus \mathrm{Blk}'$, where Blk' consists of the matrices with entries in N at (i,j) with $o(i) = o(j)$ and in $B \cdot 1$ elsewhere; so $\kappa \otimes \varphi\colon \kappa \otimes_B E \to \kappa \otimes_B \mathrm{M}_I(\mathcal{A}) = \mathrm{M}_I(\kappa \otimes_B \mathcal{A}) = \mathrm{M}_I(\kappa)$ is an injective homomorphism of κ-algebras. Its image is $\kappa \otimes_B \mathrm{Blk}$, which by Step 1 is the set of matrices in $\mathrm{M}_I(\kappa)$ vanishing at the positions (i,j) with $o(i) \neq o(j)$, i.e. $\prod_{c\in o(I)} \mathrm{M}_{|o^{-1}(c)|}(\kappa)$ after ordering I by blocks. A finite product of matrix algebras over a field is semisimple.

Step 3: the consequence. Suppose $s \in \mathrm{Ann}_B(N) \setminus Q$ and $\ker f = Q$; then $f(s) \neq 0$ and $\kappa \otimes_B E$ is semisimple by Step 2. Now $\kappa \otimes_B E \cong \kappa \otimes_B B[\Gamma]/\langle\mathcal{I}\rangle \cong \kappa[\Gamma]/\langle f(\mathcal{I})\rangle$ (right exactness of $\kappa \otimes_B -$ applied to $\langle\mathcal{I}\rangle \to B[\Gamma] \to E \to 0$, the image of $\kappa \otimes \langle\mathcal{I}\rangle$ in $\kappa[\Gamma]$ being the two-sided ideal generated by $f(\mathcal{I})$). So $\kappa[\Gamma]/\langle f(\mathcal{I})\rangle$ is semisimple, a contradiction. Hence $\mathrm{Ann}_B(N) \subseteq Q$. □

Proposition E.78. *Let $\mathscr{H}$ be a rank-n Hecke datum with $\ell \neq 2$, $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant over $\mathscr{H}$, $B = B_n^{\mathrm{po}}$, $J \subseteq B$ an ideal, $x_0, \dots, x_h$ curve points of B with homomorphisms $f_i\colon B \to A_{k_i}$, and $(\mathcal{T}, d, C, u, \pi)$ a tuple as in the definition of a patching system for J along $(x_0, \dots, x_h)$ in Definition E.31 (an abstract patching tower $\mathcal{T}$, an integer $d \geq 0$, an eigenvalue algebra C with u whose coefficient ring $\mathcal{O}_1$ has residue field of cardinality $|k_i|$ for every i, and a surjective ring homomorphism $\pi\colon R \to C$), which is not required to satisfy (Sys0)–(Sys4); surjective lifts and primes stable for the system are defined for such a tuple as there. Fix i, a surjective lift f' of f_i, $\mathfrak{q} = \pi^{-1}(\ker f')$ and the completion $\hat{A}$ of $R_{\mathfrak{q}}$ at its maximal ideal. Let C be the eigenvalue algebra of $(\mathcal{O}_1, (\sigma_j), (\tilde{u}_{j,i}))$. Assume $\pi(I) = 0$, and let $\bar{\pi}\colon R/I \to C$ be the induced homomorphism. Suppose there are a framed normalised curve point x of shape $(a, n-a)$ with $A = A_{k_i}$ and representation ρ that satisfies (Pj1) and such that for every $Z \in \mathrm{Ev}_a(A)$ there is $r \geq 0$ with $T^r Z$ in the A-span of $\rho(\Gamma')$, a Selmer basis κ of x with local witnesses, a level $\mathbf{V}$ over $\mathscr{H}$ at which (K1) holds with an integer $D_0 \leq d + s - 1$ and (K2), (K3) hold, and a realisation $\mathfrak{R}$ of x over $\mathbf{V}$ with coefficient ring $\mathcal{O}_1$ whose global ring is R/I, with $f_1 = f' \circ \bar{\pi}$, such that (prime ideals of $\hat{B}$ are pulled back to R along $R \to R/I \to \hat{B}$):*

(Gl1) *if $a < n$, then $\mathrm{ired}(N) \not\subseteq \hat{Z}'$ for every prime $\hat{Z}'$ of $\hat{B}$ whose preimage in R is stable for the system;*

(Gl2) *let $\hat{Q}'$ be a prime of $\hat{B}$ whose preimage in R is $\pi^{-1}(P')$ for a prime P' of C; if $a < n$ and $u^{-1}(P')$ is stable, then $\mathrm{ired}(N) \not\subseteq \hat{Q}'$; if $u^{-1}(P')$ is thick, then $\kappa[\Gamma_1^0]$ modulo the two-sided ideal generated by the image of $\mathcal{I}_1$ is not semisimple for some homomorphism $\hat{B} \to \kappa$ to a field with kernel $\hat{Q}'$.*

Then the conclusion of (Sys4) holds for i and f': if $\hat{Z},\hat{Q}$ are prime ideals of $\hat{A}$ such that $I\hat{A}\subseteq\hat{Z}$, the preimage of $\hat{Z}$ in R is stable for the system, $\dim\hat{A}/\hat{Z}\geq d+s$, and the preimage of $\hat{Q}$ in R is $\pi^{-1}(P')$ for a prime ideal P' of C with $u^{-1}(P')$ stable or thick, then $\hat{Z}\subseteq\hat{Q}$.

Proof. Reduction to $\hat{B}$. As $\pi(I)=0$, every ideal $\pi^{-1}(P)$, P a prime ideal of C, contains I; in particular $I\subseteq\mathfrak{q}$, and $\ker f_1=\bar{\pi}^{-1}(\ker f')=\mathfrak{q}/I$. The ring R of an abstract patching tower is a Noetherian local ring (Definition E.30), so $R_{\mathfrak{q}}$ is a Noetherian local ring and completion at its maximal ideal is exact on finitely generated $R_{\mathfrak{q}}$-modules. Since $(R/I)_{\mathfrak{q}/I}=R_{\mathfrak{q}}/IR_{\mathfrak{q}}$, the completion $\hat{B}$ of $(R/I)_{\ker f_1}$ at its maximal ideal is $\hat{A}/I\hat{A}$, compatibly with the maps from R. Let $\hat{Z},\hat{Q}$ be as in the conclusion. Then $I\hat{A}\subseteq\hat{Z}$ by hypothesis, and $I\hat{A}\subseteq\hat{Q}$ because the preimage $\pi^{-1}(P')$ of $\hat{Q}$ in R contains I. So $\hat{Z}'=\hat{Z}/I\hat{A}$ and $\hat{Q}'=\hat{Q}/I\hat{A}$ are prime ideals of $\hat{B}$ whose preimages in R are those of $\hat{Z}$ and $\hat{Q}$, and $\dim\hat{B}/\hat{Z}'=\dim\hat{A}/\hat{Z}\geq d+s\geq 1+D_0$. It suffices to show $\hat{Z}'\subseteq\hat{Q}'$: then $\hat{Z}$, the preimage of $\hat{Z}'$ in $\hat{A}$, lies in $\hat{Q}$, the preimage of $\hat{Q}'$.

Presentation. By Proposition E.79 (its hypotheses (Pj1) and the span condition on x are hypotheses here) there are $X_1,\dots,X_{D_0}\in\hat{\mathfrak{m}}$ (the maximal ideal of $\hat{B}$), $m_1,\dots,m_g\in N$ generating N and $c_{ij,a}\in\mathcal{O}'$ such that $(\mathcal{O}',\hat{B},\mathcal{A},N)$ and these elements satisfy all hypotheses of Lemma E.80 except those on the primes; in particular $\hat{B}$ is a complete Noetherian local ring, $\iota\colon\mathcal{O}'\to\hat{B}$ is a local homomorphism inducing a surjection of residue fields, and $\hat{\mathfrak{m}}=(X_1,\dots,X_{D_0})\hat{B}+\mathfrak{a}$ with $\mathfrak{a}=\iota(\mathfrak{m}_{\mathcal{O}'})\hat{B}+\hat{\mathfrak{m}}^2+\operatorname{ired}(N)$.

Case $a=n$. Then $N=0$ by (Vr5), so $\operatorname{ired}(N)=0$ and $\hat{\mathfrak{m}}=(X_1,\dots,X_{D_0})\hat{B}+\iota(\mathfrak{m}_{\mathcal{O}'})\hat{B}+\hat{\mathfrak{m}}^2$. This is the generation hypothesis of Lemma E.81 with $D=D_0$ (for $\mathcal{O}'\to\hat{B}$), and $\hat{Z}'$ is a prime with $\dim\hat{B}/\hat{Z}'\geq 1+D_0$; so Lemma E.81 gives $\hat{Z}'=0\subseteq\hat{Q}'$.

Case $a<n$. We apply Lemma E.80 to $(\mathcal{O}',\hat{B},\mathcal{A},N)$, the elements above, $Z=\hat{Z}'$ and $Q=\hat{Q}'$. $\dim\hat{B}/\hat{Z}'\geq 1+D_0$, and $\operatorname{ired}(N)\not\subseteq\hat{Z}'$ by (Gl1), as the preimage of $\hat{Z}'$ in R is stable for the system. For $\hat{Q}'$, whose preimage in R is $\pi^{-1}(P')$: if $u^{-1}(P')$ is stable, $\operatorname{ired}(N)\not\subseteq\hat{Q}'$ by (Gl2); if it is thick, (Gl2) gives a homomorphism $\hat{B}\to\kappa$ with kernel $\hat{Q}'$ for which $\kappa[\Gamma_1^0]$ modulo the ideal generated by $\mathcal{I}_1$ is not semisimple, and (Vr7) of the realisation gives $\operatorname{Ann}_{\hat{B}}(N)\subseteq\hat{Q}'$. So all hypotheses of Lemma E.80 hold and $\hat{Z}'\subseteq\hat{Q}'$. □

E.3.2 Presentations at realised points and deformation steps

This subsection proves Proposition E.79: at a realised point the Selmer conditions (K1)–(K3) at a level give the hypotheses of the cotangent lemma E.80, from which Proposition E.78 concludes. Its inputs come from a deformation analysis of a realisation at first and second order (Lemma E.85): the first-order maps are polarised cocycles with values in the even or odd block matrices (Lemma E.87), they are bounded (Lemma E.88), a tangent vector whose first-order map is a coboundary vanishes (Lemma E.91), and the first-order maps of the derivations of a functional lift to second order (Lemma E.93).

Proposition E.79. *Let $\mathscr{H}$ be a rank-n Hecke datum with $\ell\neq 2$, x a framed normalised curve point of shape $(a,n-a)$ satisfying (Pj1) and such that for every $Z\in\mathrm{Ev}_a(A)$ there is $r\geq 0$ with T^rZ in the A-span of $\rho(\Gamma')$, $\kappa=(\kappa_1,\dots,\kappa_g)$ a Selmer basis of x with local witnesses, $\mathbf{V}$ a level over $\mathscr{H}$ at which (K1) (with an integer $D_0\geq 0$), (K2) and (K3) hold, and $\mathfrak{R}$ a realisation of x over $\mathbf{V}$. Then there are $X_1,\dots,X_{D_0}\in\hat{\mathfrak{m}}$, $m_1,\dots,m_g\in N$ generating N as a $\hat{B}$-module, and $c_{ij,a}\in\mathcal{O}'$ $(1\leq i,j\leq g,\ 1\leq a\leq D_0)$ such that the g^2 vectors $(c_{ij,a}\bmod\mathfrak{m}_{\mathcal{O}'})_a\in(\mathcal{O}'/\mathfrak{m}_{\mathcal{O}'})^{D_0}$ are linearly independent, and, with $\ell_{ij}=\sum_a\iota(c_{ij,a})X_a$,*

$$\hat{\mathfrak{m}}=(X_1,\dots,X_{D_0})\hat{B}+\mathfrak{a},\qquad\sum_{j=1}^{g}\ell_{ij}\,m_j\in\mathfrak{a}N\quad(1\leq i\leq g).$$

In particular $(\mathcal{O}', \hat{B}, \mathcal{A}, N)$ *with these elements satisfy all hypotheses of Lemma E.80 except those on the primes* Z, Q.

Proof. The point x satisfies (Pj1) and the span condition, so Lemma E.85 applies to x, κ, $\mathbf{V}$ and $\mathfrak{R}$. Write $\bar{o} = \lambda(\iota(o))$ for $o \in \mathcal{O}'$. Since $\lambda \circ \iota \colon \mathcal{O}' \to F$ is a surjective ring homomorphism onto a field, its kernel is the maximal ideal $\mathfrak{m}_{\mathcal{O}'}$, so $o \mapsto \bar{o}$ induces an isomorphism $\mathcal{O}'/\mathfrak{m}_{\mathcal{O}'} \cong F$; and $\hat{B} = \iota(\mathcal{O}') + \hat{\mathfrak{m}}$ because $\ker\lambda = \hat{\mathfrak{m}}$. As $yy' \in \hat{\mathfrak{m}} \cdot 1$ for $y, y' \in N$, $\mathrm{ired}(N) \subseteq \hat{\mathfrak{m}}$, hence $\mathfrak{a} \subseteq \hat{\mathfrak{m}}$; and $\hat{\mathfrak{m}}^2 \subseteq \mathfrak{a}$, so $\hat{\mathfrak{m}}/\mathfrak{a}$ is a vector space over $\hat{B}/\hat{\mathfrak{m}} \cong F$ (via λ), finite-dimensional since $\hat{\mathfrak{m}}$ is a finitely generated ideal of the Noetherian ring $\hat{B}$.

Step 1: bounded forms of (K1)–(K3). Let z be a bounded element of one of the Selmer sets over F with local witnesses Y_v. Choosing s so large that $T^s z$ has T-adically continuous entries (coordinates) in A and $T^s Y_v$ has entries in A for the finitely many $v \in S_\ell$, $T^s z$ lies in the corresponding Selmer set over A (all defining conditions of Definition E.35 (Sc) are F-linear in $(z, (Y_v))$, and ρ_A, w_v, w_v^1 have entries in A, resp. $\mathrm{PD}_g(A)$); conversely the Selmer sets over A lie in those over F, and all sets over F are F-vector spaces (each condition is F-linear and boundedness is preserved, as $c = c'T^{-r}$ with $c' \in A$). Finitely many bounded elements admit a common s. Hence: ($\mathrm{K1^b}$) if $z_1, \dots, z_e \in \mathrm{Sel}^{1,\mathrm{b}}_{\mathbf{V}}(F)$ are independent modulo $\mathrm{Bd}^1_{\mathbf{V}}(F)$ then $e \le D_0$: the $T^s z_i \in \mathrm{Sel}^1_{\mathbf{V}}(A)$ satisfy the hypothesis of (K1), since $\sum_i c_i T^s z_i \in \mathrm{Bd}^1_{\mathbf{V}}(A) \subseteq \mathrm{Bd}^1_{\mathbf{V}}(F)$ with $c \in A^e$ forces $c_i T^s = 0$. ($\mathrm{K2^b}$) for every family $(z_{i,j})_{0\le i\le g, 1\le j\le g}$ in $\mathrm{Sel}^{\mathrm{ev},\mathrm{b}}_{\mathbf{V}}(F)$ there is $c \in F^{g+1} \smallsetminus \{0\}$ such that $(\sum_i c_i z_{i,j})_j$ lifts: apply (K2) to $(T^s z_{i,j})$ to get $c \in A^{g+1} \smallsetminus \{0\}$ and $\Psi \in \mathrm{Sel}^2_{\mathbf{V}}(A)$, and take $T^s c$ and Ψ (bounded). ($\mathrm{K3^b}$) for $z \in \mathrm{Sel}^{\mathrm{od},\mathrm{b}}_{\mathbf{V}}(F)$ there is $c \in F^g$ with $z - \sum_j c_j(\kappa_j \circ h) \in \mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}}(F)$: by (K3) $T^{s'}T^s z - \sum_j c'_j(\kappa_j \circ h) \in \mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}}(A)$, and we divide by $T^{s+s'}$.

Step 2: cotangent duality. Every $\alpha \in \mathrm{Tp}$ vanishes on $\hat{\mathfrak{m}}^2$ (as $\lambda(\hat{\mathfrak{m}}) = 0$), on $\iota(\mathfrak{m}_{\mathcal{O}'})\hat{B}$ (as $\alpha(\iota(o)b) = \bar{o}\,\alpha(b) + \lambda(b)\alpha(\iota(o)) = \bar{o}\,\alpha(b)$) and on $\mathrm{ired}(N)$, hence on $\mathfrak{a}$, and $\alpha(bx) = \lambda(b)\alpha(x)$ for $x \in \hat{\mathfrak{m}}$. So restriction gives an F-linear map $r \colon \mathrm{Tp} \to \mathrm{Hom}_F(\hat{\mathfrak{m}}/\mathfrak{a}, F)$; it is injective since $\hat{B} = \iota(\mathcal{O}') + \hat{\mathfrak{m}}$ and $\alpha \circ \iota = 0$. It is surjective: for ξ put $\alpha(\iota(o) + x) = \xi(x \bmod \mathfrak{a})$; this is well defined, since $\iota(o)+x = \iota(o')+x'$ gives $\iota(o-o') \in \hat{\mathfrak{m}}$, so $o - o' \in \mathfrak{m}_{\mathcal{O}'}$ and $x' - x = \iota(o-o') \in \mathfrak{a}$; it is additive, kills $\iota(\mathcal{O}')$ and $\mathrm{ired}(N) \subseteq \mathfrak{a}$, and from $(\iota(o)+x)(\iota(o')+x') = \iota(oo') + \iota(o)x' + \iota(o')x + xx'$ with $xx' \in \mathfrak{a}$ it satisfies the derivation rule. Hence Tp is finite-dimensional; by duality of finite-dimensional spaces: (T1) if $b \in \hat{\mathfrak{m}}$ and $\alpha(b) = 0$ for all $\alpha \in \mathrm{Tp}$, then $b \in \mathfrak{a}$; (T2) every F-linear functional on Tp is $\alpha \mapsto \alpha(b)$ for some $b \in \hat{\mathfrak{m}}$.

Step 3: separation by Ph. N is a finitely generated $\hat{B}$-module (a direct summand of the finite $\hat{B}$-module $\mathcal{A}$), so $N/\mathfrak{a}N$ is a finitely generated module over $\hat{B}/\mathfrak{a}$, which is finite-dimensional over F by the filtration $\hat{B} \supseteq \hat{\mathfrak{m}} \supseteq \mathfrak{a}$; through ι, $\mathfrak{m}_{\mathcal{O}'}$ acts by 0, so $N/\mathfrak{a}N$ is a finite-dimensional vector space over $\mathcal{O}'/\mathfrak{m}_{\mathcal{O}'} \cong F$. If $y \in N \smallsetminus \mathfrak{a}N$, a linear functional on it, nonzero on the class of y and transported along $\mathcal{O}'/\mathfrak{m}_{\mathcal{O}'} \cong F$, gives $\Phi \in \mathrm{Ph}$ with $\Phi(y) \neq 0$. So: (T3) $y \in N$ with $\Phi(y) = 0$ for all $\Phi \in \mathrm{Ph}$ lies in $\mathfrak{a}N$.

Step 4: the space of lifting tuples. The map $\alpha \mapsto \mathrm{al}(\alpha)$ is F-linear in α (it is defined entrywise by α, then multiplied by $\rho(h(\gamma))^{-1}$). The set of lifting g-tuples is an F-subspace: $\mathrm{Sel}^{2,\mathrm{b}}_{\mathbf{V}}(F)$ is one (Step 1) and the ϵ'_j-coefficient is F-linear. Hence $U = \{(\alpha_1, \dots, \alpha_g) \in \mathrm{Tp}^g : (\mathrm{al}(\alpha_1), \dots, \mathrm{al}(\alpha_g)) \text{ lifts}\}$ is an F-subspace of Tp^g. (A) Any $g+1$ elements $\alpha^{(0)}, \dots, \alpha^{(g)}$ of Tp^g have a nontrivial combination in U: by Lemma E.85 (Df1) the $\mathrm{al}(\alpha_j^{(i)})$ lie in $\mathrm{Sel}^{\mathrm{ev},\mathrm{b}}_{\mathbf{V}}(F)$, ($\mathrm{K2^b}$) gives $c \neq 0$ with $(\sum_i c_i \mathrm{al}(\alpha_j^{(i)}))_j = (\mathrm{al}(\sum_i c_i \alpha_j^{(i)}))_j$ lifting. Consequently $\dim_F \mathrm{Tp}^g/U \le g$. (B) Let $W = \{\alpha \in \mathrm{Tp} : \ \iota_j(\alpha) \in U \text{ for all } j\}$, $\iota_j(\alpha)$ the tuple with j-th entry α and all others 0, and $e = \dim W$. Applying Lemma E.85 (Df2), whose hypothesis is ($\mathrm{K1^b}$), to a basis of W gives $e + g^2 \le D_0$.

Step 5: linear algebra. Let $U^\perp \subseteq (\mathrm{Tp}^*)^g$ be the annihilator of U (a tuple $(\omega_1, \dots, \omega_g)$

acting by $\sum_j \omega_j(\alpha_j)$); $\dim U^\perp = \dim \mathrm{Tp}^g/U \le g$ by (A). Choose $\omega^{(1)},\dots,\omega^{(g)}$ spanning $U^\perp$ (repetitions or zeros allowed). As $U = (U^\perp)^\perp$, $\iota_j(\alpha) \in U$ iff $\omega_j^{(i)}(\alpha) = 0$ for all i; so W is the common kernel of the g^2 functionals $\omega_j^{(i)}$ and its annihilator $W^\perp \subseteq \mathrm{Tp}^*$ is their span. Choose $\beta_1,\dots,\beta_e \in \mathrm{Tp}^*$ whose restrictions form a basis of W^*; then $\{\omega_j^{(i)}\} \cup \{\beta_b\}$ spans Tp^* (for $\omega \in \mathrm{Tp}^*$ write $\omega|_W = \sum r_b\beta_b|_W$; then $\omega - \sum r_b\beta_b \in W^\perp$). Index $\{1,\dots,D_0\}$ so that its first g^2 elements are the pairs (i,j) and the next e are $1,\dots,e$ (possible by (B)); put $\bar X_{(i,j)} = \omega_j^{(i)}$, $\bar X_{g^2+b} = \beta_b$, $\bar X_a = 0$ for the remaining a, and let $\bar c_{ij} \in F^{D_0}$ be the unit vector at the index (i,j). Then the $\bar X_a$ span Tp^*, the $\bar c_{ij}$ are linearly independent, and for $(\alpha_j) \in U$ and every i: $\sum_j \sum_a \bar c_{ij,a}\bar X_a(\alpha_j) = \sum_j \omega_j^{(i)}(\alpha_j) = 0$.

Step 6: lifting to $\hat B$ and $\mathcal{O}'$. By (T2) choose $X_a \in \hat{\mathfrak{m}}$ with $\alpha(X_a) = \bar X_a(\alpha)$ for all $\alpha \in \mathrm{Tp}$, and choose $c_{ij,a} \in \mathcal{O}'$ with $\overline{c_{ij,a}} = \bar c_{ij,a}$. Under $\mathcal{O}'/\mathfrak{m}_{\mathcal{O}'} \cong F$ the vectors $(c_{ij,a} \bmod \mathfrak{m}_{\mathcal{O}'})_a$ correspond to the $\bar c_{ij}$, so they are independent. For $\alpha \in \mathrm{Tp}$, $o \in \mathcal{O}'$, $x \in \hat{\mathfrak{m}}$ we have $\alpha(\iota(o)x) = \bar o\,\alpha(x)$ (derivation rule, $\lambda(x) = 0$, $\alpha \circ \iota = 0$).

Step 7: $\hat{\mathfrak{m}} = (X_1,\dots,X_{D_0})\hat B + \mathfrak{a}$. The inclusion $\supseteq$ is clear. Let $b \in \hat{\mathfrak{m}}$. The functional $\alpha \mapsto \alpha(b)$ is $\sum_a r_a\bar X_a$ for some $r_a \in F$ (the $\bar X_a$ span); choose $o_a \in \mathcal{O}'$ with $\bar o_a = r_a$. Then $b' = b - \sum_a \iota(o_a)X_a \in \hat{\mathfrak{m}}$ and $\alpha(b') = \alpha(b) - \sum_a r_a\bar X_a(\alpha) = 0$ for all α by Step 6, so $b' \in \mathfrak{a}$ by (T1).

Step 8: generators of N and the relations. By Lemma E.83, whose hypotheses (Dg1), (Dg2), (Dg3) are Lemma E.85 (Df1), (K3^{b}) and Lemma E.84, there is a family $(m_1,\dots,m_g)$ in N dual to κ that generates N. Fix i and put $y_i = \sum_j \ell_{ij}m_j$; by (T3) it suffices that $\Phi(y_i) = 0$ for every $\Phi \in \mathrm{Ph}$. By Lemma E.85 (Df3) there are $\alpha_1,\dots,\alpha_g \in \mathrm{Tp}$ with $\alpha_j(b) = \Phi(bm_j) - \lambda(b)\Phi(m_j)$ and $(\alpha_j)_j \in U$. Since $\ell_{ij} \in \hat{\mathfrak{m}}$, $\Phi(\ell_{ij}m_j) = \alpha_j(\ell_{ij}) = \sum_a \bar c_{ij,a}\,\alpha_j(X_a) = \sum_a \bar c_{ij,a}\bar X_a(\alpha_j)$ by Step 6, and so $\Phi(y_i) = \sum_j \sum_a \bar c_{ij,a}\bar X_a(\alpha_j) = 0$ by Step 5.

Finally, $\hat B$ is a complete Noetherian local ring and ι is a local homomorphism inducing a surjection of residue fields ((Vr4); $\iota(\mathfrak{m}_{\mathcal{O}'}) \subseteq \hat{\mathfrak{m}}$ because $\lambda \circ \iota$ kills $\mathfrak{m}_{\mathcal{O}'}$), $\mathcal{A} = \hat B \cdot 1 \oplus N$ with $yy' \in \hat{\mathfrak{m}} \cdot 1$ (Vr5), and $\mathfrak{a}$ is the ideal of Lemma E.80 for $(\mathcal{O}', \hat B, N)$; so all hypotheses of that lemma other than those on Z, Q hold. □

Lemma E.80. *Let $\mathcal{O}$ be a complete discrete valuation ring with maximal ideal $\mathfrak{m}_{\mathcal{O}}$ and residue field k, and B a complete Noetherian local ring with maximal ideal $\mathfrak{m}_B$ together with a local ring homomorphism $\mathcal{O} \to B$ inducing a surjection $k \to B/\mathfrak{m}_B$. Let A be a ring containing B as a subring and finitely generated as a B-module, and $N \subseteq A$ a B-submodule with $A = B \oplus N$ as B-modules and $xy \in \mathfrak{m}_B$ for all $x, y \in N$. Let* $\mathrm{ired}(N) \subseteq B$ *be the ideal generated by the products xy with $x, y \in N$, and put*

$$\mathfrak{a} = \mathfrak{m}_{\mathcal{O}}B + \mathfrak{m}_B^2 + \mathrm{ired}(N).$$

Let $D_0, g \ge 0$, $X_1,\dots,X_{D_0} \in \mathfrak{m}_B$, $m_1,\dots,m_g \in N$ generating N as a B-module, and $c_{ij,a} \in \mathcal{O}$ ($1 \le i,j \le g$, $1 \le a \le D_0$) such that the g^2 vectors $(c_{ij,a} \bmod \mathfrak{m}_{\mathcal{O}})_{1\le a\le D_0} \in k^{D_0}$ are linearly independent over k. Put $\ell_{ij} = \sum_a c_{ij,a}X_a$ and assume

$$\mathfrak{m}_B = (X_1,\dots,X_{D_0})B + \mathfrak{a}, \qquad \sum_{j=1}^{g} \ell_{ij}\, m_j \in \mathfrak{a}N \quad (1 \le i \le g). \tag{E.12}$$

Let Z, Q be prime ideals of B with $\dim B/Z \ge 1 + D_0$ *and* $\mathrm{ired}(N) \not\subseteq Z$, *and such that* $\mathrm{ired}(N) \not\subseteq Q$ *or* $\mathrm{Ann}_B(N) \subseteq Q$. *Then $Z \subseteq Q$.*

Proof. Since $g = 0$ forces $N = 0$ and $\mathrm{ired}(N) = 0 \subseteq Z$, we have $g \ge 1$.

Step 1: A is a complete Noetherian local ring with maximal ideal $\mathfrak{m}_A = \mathfrak{m}_B + N$ and residue field $B/\mathfrak{m}_B$, and $\operatorname{ired}(N)A \subseteq \mathfrak{m}_A^2$. Write $a \in A$ as $a = b + n$ ($b \in B$, $n \in N$). The set $\mathfrak{m}_B + N$ is an ideal of A: it is additive, and $(b' + n')(b + n) = (b'b + n'n) + (b'n + bn')$ with $n'n \in \mathfrak{m}_B$ whenever $b \in \mathfrak{m}_B$. It does not contain 1: $1 = b + n$ with $b \in \mathfrak{m}_B$ gives $n = 1 - b \in B \cap N = 0$ and $b = 1$. If $b \notin \mathfrak{m}_B$, then $(b + n)(b - n) = b^2 - n^2$ with $n^2 \in \mathfrak{m}_B$ is a unit of B, so $b + n$ is a unit. Hence A is local with $\mathfrak{m}_A = \mathfrak{m}_B + N$, and $A/\mathfrak{m}_A = B/\mathfrak{m}_B$. A is Noetherian, being finite over B. As $N \cdot N \subseteq \mathfrak{m}_B$, $\mathfrak{m}_A^2 \subseteq \mathfrak{m}_B A \subseteq \mathfrak{m}_A$, so the $\mathfrak{m}_A$-adic and $\mathfrak{m}_B A$-adic topologies agree and A is complete ([166, Theorem 8.7]). Finally each generator xy of $\operatorname{ired}(N)$ is a product of two elements of $N \subseteq \mathfrak{m}_A$.

Step 2: a surjection from a power series ring. Let $P = \mathcal{O}[[x_1, \dots, x_{D_0}, y_1, \dots, y_g]]$, a regular local ring of dimension $1 + D_0 + g$ whose cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ has the k-basis $\bar{\varpi}, \bar{x}_a, \bar{y}_j$. By (E.12) and Step 1, $\mathfrak{m}_B \subseteq (X_1, \dots, X_{D_0})A + \mathfrak{m}_{\mathcal{O}} A + \mathfrak{m}_A^2$ and $N \subseteq (m_1, \dots, m_g)A$, so

$$\mathfrak{m}_A = (X_1, \dots, X_{D_0}, m_1, \dots, m_g)A + \mathfrak{m}_{\mathcal{O}} A + \mathfrak{m}_A^2.$$

By Lemma E.81 (its first assertion, for A and these $D_0 + g$ elements), the $\mathcal{O}$-algebra homomorphism $\Phi\colon P \to A$ with $\Phi(x_a) = X_a$, $\Phi(y_j) = m_j$ exists and is surjective. Hence $\Phi(\mathfrak{m}_P) = \mathfrak{m}_A$ and $\Phi(\varpi P + \mathfrak{m}_P^2) = \varpi A + \mathfrak{m}_A^2$.

Step 3: lifting the relations. The module $\mathfrak{a}N$ consists of the sums $\sum_j w_j m_j$ with $w_j \in \mathfrak{a}$ (write each element of N in terms of the m_j). So by (E.12) there are $w_{ij} \in \mathfrak{a}$ with $\sum_j \ell_{ij} m_j = \sum_j w_{ij} m_j$; put $\zeta_{ij} = \ell_{ij} - w_{ij} \in B$ and $\zeta = (\zeta_{ij})$, so that

$$\sum_{j=1}^{g} \zeta_{ij}\, m_j = 0 \quad (1 \le i \le g). \tag{E.13}$$

By Step 1, $\mathfrak{a}A \subseteq \varpi A + \mathfrak{m}_A^2$, so there are $v_{ij} \in \varpi P + \mathfrak{m}_P^2$ with $\Phi(v_{ij}) = w_{ij}$. Put $z_{ij} = \sum_a c_{ij,a} x_a - v_{ij} \in \mathfrak{m}_P$ and $z = (z_{ij})$. Then $\Phi(z_{ij}) = \zeta_{ij}$, $\Phi(\det z) = \det \zeta$, and by (E.13) the ideal $I \subseteq P$ generated by the entries of zy, $y = (y_1, \dots, y_g)^T$, lies in $\ker \Phi$.

Step 4: the z_{ij} and y_j are independent in the cotangent space of P. Writing $v_{ij} = \varpi u_{ij} + s_{ij}$ with $s_{ij} \in \mathfrak{m}_P^2$, the image of z_{ij} in $\mathfrak{m}_P/\mathfrak{m}_P^2$ is $\sum_a \bar{c}_{ij,a} \bar{x}_a - \bar{u}_{ij} \bar{\varpi}$, where bars denote residues in k. In a vanishing combination $\sum \alpha_{ij} \bar{z}_{ij} + \sum \beta_j \bar{y}_j = 0$, the coefficient of $\bar{x}_a$ gives $\sum_{i,j} \alpha_{ij} \bar{c}_{ij,a} = 0$ for every a, so all $\alpha_{ij} = 0$ by the linear independence of the g^2 vectors, and then all $\beta_j = 0$.

Step 5: $\det \zeta \in Q$. By Cramer's rule and (E.13), $\det(\zeta) m_k = \sum_i \operatorname{adj}(\zeta)_{ki} \sum_j \zeta_{ij} m_j = 0$ for every k, so $\det \zeta \in \operatorname{Ann}_B(N)$. If $\operatorname{Ann}_B(N) \subseteq Q$ we are done. Otherwise $\operatorname{ired}(N) \not\subseteq Q$, so $xy \notin Q$ for some $x, y \in N$; then $\det(\zeta)\, xy = (\det(\zeta)x)y = 0$ in B, whence $\det \zeta \in Q$.

Step 6: primes of A. The extension $B \subseteq A$ is integral. Choose primes Z', Q' of A with $Z' \cap B = Z$ and $Q' \cap B = Q$ (lying over, [166, Theorem 9.3]). Then $\dim A/Z' = \dim B/Z \ge 1 + D_0$. Some m_j is not in Z': otherwise $N \subseteq Z'$, so $xy \in Z' \cap B = Z$ for all $x, y \in N$ and $\operatorname{ired}(N) \subseteq Z$. And $\det \zeta \in Q \subseteq Q'$ by Step 5.

Step 7: conclusion. Let $\hat{Z} = \Phi^{-1}(Z')$ and $\hat{Q} = \Phi^{-1}(Q')$, primes of P. Since Φ is surjective, $P/\hat{Z} \cong A/Z'$, so $\dim P/\hat{Z} \ge 1 + D_0 = \dim P - g$; $I \subseteq \ker \Phi \subseteq \hat{Z}$; and $y_j \notin \hat{Z}$ for the j of Step 6, as $\Phi(y_j) = m_j \notin Z'$. Also $I \subseteq \hat{Q}$ and $\det z \in \hat{Q}$, since $\Phi(\det z) = \det \zeta \in Q'$. By Step 4, Lemma E.82 applies to the regular local ring P, the matrix z and the vector y; its part (3) gives $\hat{Z} \subseteq \hat{Q}$. Applying the surjection Φ, $Z' = \Phi(\hat{Z}) \subseteq \Phi(\hat{Q}) = Q'$, and $Z = Z' \cap B \subseteq Q' \cap B = Q$. □

Lemma E.81. *Let $\mathcal{O}$ be a complete discrete valuation ring with maximal ideal $\mathfrak{m}_{\mathcal{O}}$, and B a complete Noetherian local ring with maximal ideal $\mathfrak{m}_B$ together with a local ring homomorphism $\mathcal{O} \to B$ inducing a surjection $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} \to B/\mathfrak{m}_B$. Let $D \ge 0$ and $X_1, \dots, X_D \in \mathfrak{m}_B$ with*

$$\mathfrak{m}_B = (X_1, \dots, X_D)B + \mathfrak{m}_{\mathcal{O}} B + \mathfrak{m}_B^2. \tag{E.14}$$

Then the $\mathcal{O}$-algebra homomorphism $F\colon \mathcal{O}[[x_1,\dots,x_D]] \to B$ with $F(x_a) = X_a$ exists and is surjective. If moreover Z is a prime ideal of B with $\dim B/Z \geq 1+D$, then F is an isomorphism, B is a domain and $Z = 0$; in particular $Z \subseteq Q$ for every prime ideal Q of B.

Proof. Put $P = \mathcal{O}[[x_1,\dots,x_D]]$, a Noetherian local domain with maximal ideal $\mathfrak{m}_P = \mathfrak{m}_{\mathcal{O}}P + (x_1,\dots,x_D)$ and $\dim P = 1 + D$: with ϖ a uniformiser of $\mathcal{O}$, $\mathfrak{m}_P = (\varpi, x_1,\dots,x_D)$ gives $\dim P \leq 1+D$ by Krull's height theorem, and $(0) \subsetneq (x_1) \subsetneq (x_1,x_2) \subsetneq \cdots \subsetneq (x_1,\dots,x_D) \subsetneq \mathfrak{m}_P$ is a chain of primes of length $1+D$ (the quotients $\mathcal{O}[[x_{i+1},\dots,x_D]]$ are domains).

Existence of F. For a power series $\sum_\alpha c_\alpha x^\alpha$ ($c_\alpha \in \mathcal{O}$) the terms $c_\alpha X^\alpha$ lie in $\mathfrak{m}_B^{|\alpha|}$, so the series $\sum_\alpha c_\alpha X^\alpha$ converges in the $\mathfrak{m}_B$-adically complete ring B; this defines an $\mathcal{O}$-algebra homomorphism F with $F(x_a) = X_a$, multiplicativity holding on the truncations modulo $\mathfrak{m}_B^k$ for every k.

Surjectivity. Put $I = (X_1,\dots,X_D)B + \mathfrak{m}_{\mathcal{O}}B = F(\mathfrak{m}_P)B$. By (E.14), $\mathfrak{m}_B = I + \mathfrak{m}_B^2$, so the finitely generated B-module $\mathfrak{m}_B/I$ satisfies $\mathfrak{m}_B(\mathfrak{m}_B/I) = \mathfrak{m}_B/I$ and vanishes by Nakayama's lemma: $\mathfrak{m}_P B = I = \mathfrak{m}_B$. Regard B as a P-module through F. Then $\mathfrak{m}_P^k B = \mathfrak{m}_B^k$ for all k, so B is separated for the $\mathfrak{m}_P$-adic topology (Krull's intersection theorem), and $B/\mathfrak{m}_P B = B/\mathfrak{m}_B$ is generated over $P/\mathfrak{m}_P = \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$ by the image of 1, as $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} \to B/\mathfrak{m}_B$ is surjective. As P is $\mathfrak{m}_P$-adically complete, [166, Theorem 8.4] shows that 1 generates B as a P-module, i.e. F is surjective.

Dimension. Let $K = \ker F$, so $B \cong P/K$. If $K \neq 0$, every chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ of primes of B is the image of a chain of primes of P containing K, and $(0) \subsetneq F^{-1}(\mathfrak{p}_0)$ since P is a domain; so $\dim B \leq \dim P - 1 = D$. Likewise, if B is a domain and $Z \neq 0$, a chain of primes of B/Z extends at the bottom by $(0) \subsetneq Z$, so $\dim B/Z \leq \dim B - 1$. Now let Z be a prime of B with $\dim B/Z \geq 1+D$. Since $\dim B/Z \leq \dim B$, the first observation gives $K = 0$: F is an isomorphism and B is a domain of dimension $1+D$. The second gives $Z = 0$, and then $Z \subseteq Q$ for every prime Q. □

Lemma E.82. *Let P be a regular local ring with maximal ideal $\mathfrak{m}_P$, let $g \geq 1$, and let $z = (z_{ij})_{1\leq i,j\leq g}$ be a $g \times g$ matrix and $m = (m_1,\dots,m_g)^T$ a column vector with entries in $\mathfrak{m}_P$ such that the g^2+g elements z_{ij}, m_j have linearly independent images in $\mathfrak{m}_P/\mathfrak{m}_P^2$. Let $I \subseteq P$ be the ideal generated by the g entries $\sum_j z_{ij}m_j$ of zm, and put $J = I + (\det z)P$. Then:*

1. *there is a prime ideal K of P with $J \subseteq K$ such that $K \subseteq Q$ for every prime ideal Q of P containing J;*
2. *$\dim P/I \leq \dim P - g$; in particular $\dim P/K \leq \dim P - g$;*
3. *if Z is a prime ideal of P with $I \subseteq Z$, $m_j \notin Z$ for some j, and $\dim P/Z \geq \dim P - g$, then $Z = K$; in particular $Z \subseteq Q$ for every prime ideal Q of P containing J.*

Proof. Theorem numbers in this proof refer to [166].

Step 0: a prime containing I and missing some m_j contains $\det z$. By Cramer's rule $\operatorname{adj}(z)(zm) = \det(z)\,m$, so $\det(z)m_j \in I$ for every j; if $Z \supseteq I$ is prime and $m_j \notin Z$, then $\det z \in Z$.

Step 1: the chart ideals and the primes K_c. Fix $c \in \{1,\dots,g\}$. Let $\Lambda_c = P[\lambda_j : j \neq c]$ be the polynomial ring in $g-1$ variables over P, put $\lambda_c = 1$ and $\lambda = (\lambda_1,\dots,\lambda_g)^T$, and let $C_c \subseteq \Lambda_c$ be the ideal generated by

$$r_i = \sum_{j=1}^{g} z_{ij}\lambda_j \quad (1 \leq i \leq g), \qquad s_j = m_j - \lambda_j m_c \quad (j \neq c). \tag{E.15}$$

The ideal $M_c = \mathfrak{m}_P \Lambda_c$ is prime, as $\Lambda_c/M_c \cong k[\lambda_j : j \neq c]$, and $C_c \subseteq M_c$ since all z_{ij}, m_j lie in $\mathfrak{m}_P$. Put

$$K_c = \{f \in P : \ sf \in C_c \text{ for some } s \in \Lambda_c \setminus M_c\},$$

the preimage of $C_c(\Lambda_c)_{M_c}$ under $P \to (\Lambda_c)_{M_c}$.

Claim A: for every prime $\mathfrak{Q}$ of Λ_c with $M_c \subseteq \mathfrak{Q}$, the ideal $C_c(\Lambda_c)_{\mathfrak{Q}}$ is prime. Put $L = (\Lambda_c)_{\mathfrak{Q}}$, a Noetherian local ring with residue field κ. The homomorphism $P \to L$ is flat (Λ_c is free over P) and local, and its closed fibre $L/\mathfrak{m}_P L$ is the localisation of $k[\lambda_j : j \neq c]$ at the prime $\mathfrak{Q}/M_c$, a regular local ring (Theorem 19.5) of some dimension e. Let $x_1, \dots, x_d$ generate $\mathfrak{m}_P$ and let $h_1, \dots, h_e \in \mathfrak{Q}L$ lift a regular system of parameters of the fibre. Then $\mathfrak{Q}L = \mathfrak{m}_P L + (h_1, \dots, h_e)$ is generated by $d+e$ elements, while $\dim L = d + e$ by Theorem 15.1. Hence L is regular and the images of $x_1, \dots, x_d, h_1, \dots, h_e$ form a κ-basis of $\mathfrak{Q}L/\mathfrak{Q}^2 L$; in particular the κ-linear map $\iota\colon (\mathfrak{m}_P/\mathfrak{m}_P^2) \otimes_k \kappa \to \mathfrak{Q}L/\mathfrak{Q}^2 L$ induced by inclusion is injective. For $a \in \Lambda_c$ and $x \in \mathfrak{m}_P$ the image of ax is $\bar{a}\,\iota(\bar{x})$, with $\bar{a}$ the image of a in κ; so the images of the generators (E.15) are

$$\iota(\bar{z}_{ic}) + \sum_{j \neq c} \bar{\lambda}_j \, \iota(\bar{z}_{ij}) \quad (1 \leq i \leq g), \qquad \iota(\bar{m}_j) - \bar{\lambda}_j \, \iota(\bar{m}_c) \quad (j \neq c).$$

The vectors $\iota(\bar{z}_{ij})$, $\iota(\bar{m}_j)$ are κ-linearly independent, and in a vanishing linear combination of these $2g-1$ images the coefficient of the image of r_i is the coefficient of $\iota(\bar{z}_{ic})$, and that of the image of s_j is the coefficient of $\iota(\bar{m}_j)$; so the $2g-1$ images are linearly independent. By Theorems 14.2 and 14.3, $L/C_c L$ is a regular local ring, hence a domain, and Claim A follows.

Taking $\mathfrak{Q} = M_c$ in Claim A, K_c is the preimage of a prime ideal, hence prime. Moreover $J \subseteq K_c$, indeed $J \subseteq C_c \cap P$: since the c-th entry of $m - m_c\lambda$ is 0,

$$\sum_j z_{ij} m_j = m_c \, r_i + \sum_{j \neq c} z_{ij} s_j \in C_c, \qquad \det z = \det(z)\lambda_c = \sum_k \operatorname{adj}(z)_{ck} \, r_k \in C_c,$$

the second by Cramer's rule $\operatorname{adj}(z)\, z\lambda = \det(z)\lambda$.

Step 2: a transfer criterion. Let Q be a prime ideal of P, V a local ring with maximal ideal $\mathfrak{m}_V$, and $\pi\colon P \to V$ a ring homomorphism with $\ker \pi = Q$ and $\pi(\mathfrak{m}_P) \subseteq \mathfrak{m}_V$. Suppose that for some c there is $l \in V^g$ with $l_c = 1$, $\pi(z)\, l = 0$ and $\pi(m_j) = l_j \, \pi(m_c)$ for all j. Then $K_c \subseteq Q$. Indeed, let $\psi\colon \Lambda_c \to V$ be the extension of π with $\psi(\lambda_j) = l_j$ $(j \neq c)$. Then $\psi(r_i) = (\pi(z)l)_i = 0$ and $\psi(s_j) = \pi(m_j) - l_j \pi(m_c) = 0$, so $C_c \subseteq \ker \psi$; and $\mathfrak{Q} = \psi^{-1}(\mathfrak{m}_V)$ is a prime containing M_c. Let $f \in K_c$, with $s \in \Lambda_c \setminus M_c$ and $sf \in C_c$. By Claim A, $C_c(\Lambda_c)_{\mathfrak{Q}}$ is prime and contains sf. If it contained s, then $ts \in C_c \subseteq M_c$ for some $t \in \Lambda_c \setminus \mathfrak{Q}$, and $t \notin M_c$, $s \notin M_c$ contradict the primality of M_c. So $tf \in C_c$ for some $t \in \Lambda_c \setminus \mathfrak{Q}$; then $\psi(t)\pi(f) = \psi(tf) = 0$ with $\psi(t) \notin \mathfrak{m}_V$ a unit, so $\pi(f) = 0$ and $f \in Q$.

Step 3: the primes K_c do not depend on c. Let $c, c' \in \{1, \dots, g\}$. By Claim A, $D = (\Lambda_{c'})_{M_{c'}}/C_{c'}(\Lambda_{c'})_{M_{c'}}$ is a local domain, and the natural map $\pi\colon P \to D$ has kernel $K_{c'}$ and maps $\mathfrak{m}_P$ into the maximal ideal. The residue field of D is $k(\lambda_j : j \neq c')$, in which the images of the λ_j are non-zero; so every λ_j is a unit of D. Put $l = \lambda_c^{-1}\lambda \in D^g$. Then $l_c = 1$; $\pi(z)\, l = \lambda_c^{-1}(r_1, \dots, r_g)^T = 0$ in D; and since $s_j \in C_{c'}$, $\pi(m_j) = \lambda_j \pi(m_{c'})$ for all j, so $\pi(m_j) = l_j \lambda_c \pi(m_{c'}) = l_j \pi(m_c)$. Step 2 gives $K_c \subseteq K_{c'}$. By symmetry $K_c = K_{c'}$; put $K = K_1$.

Step 4: proof of (1). K is prime and contains J by Step 1. Let $Q \supseteq J$ be prime, F the fraction field of P/Q, and V a valuation ring of F with $P/Q \subseteq V$ and $\mathfrak{m}_V \cap P/Q = \mathfrak{m}_P/Q$ (Theorem 10.2); let $\pi\colon P \to P/Q \subseteq V$ and v the valuation of V. Then $\ker \pi = Q$, $\pi(\mathfrak{m}_P) \subseteq \mathfrak{m}_V$, and $\bar{z} = \pi(z)$, $\bar{m} = \pi(m)$ satisfy $\bar{z}\bar{m} = 0$ and $\det \bar{z} = 0$, as $J \subseteq Q$. If $\bar{m} \neq 0$, choose c with $\bar{m}_c \neq 0$ and $v(\bar{m}_c) \leq v(\bar{m}_j)$ for all j with $\bar{m}_j \neq 0$; then $l = \bar{m}_c^{-1}\bar{m} \in V^g$ has $l_c = 1$, $\bar{z}l = 0$ and $\bar{m}_j = l_j \bar{m}_c$. If $\bar{m} = 0$, choose $u \in F^g \setminus \{0\}$ with $\bar{z}u = 0$ (possible as $\det \bar{z} = 0$), and c with

$u_c \neq 0$ of least valuation among the non-zero entries; then $l = u_c^{-1}u \in V^g$ has $l_c = 1$, $\bar{z}l = 0$ and $\bar{m}_j = 0 = l_j\bar{m}_c$. In both cases Step 2 and Step 3 give $K = K_c \subseteq Q$.

Step 5: proof of (2). As the images of the z_{ij} and m_j are linearly independent in the d-dimensional k-space $\mathfrak{m}_P/\mathfrak{m}_P^2$, there are $t_1, \dots, t_e \in \mathfrak{m}_P$, $e = d - g^2 - g$, such that the images of the z_{ij}, m_j, t_a form a basis; by Nakayama's lemma these d elements generate $\mathfrak{m}_P$. Let σ be the family of the $d - g$ elements $t_1, \dots, t_e$, z_{ij} $(i \neq j)$ and $z_{ii} - m_i$ $(1 \leq i \leq g)$. Together with $m_1, \dots, m_g$ they generate the same ideal $\mathfrak{m}_P$, so they form a regular system of parameters, and by Theorem 14.2 $P' = P/(\sigma)$ is a regular local ring of dimension g whose maximal ideal is generated by the images $m_1', \dots, m_g'$ of the m_j. In P' the matrix z becomes the diagonal matrix with entries $m_1', \dots, m_g'$, so $IP' = (m_1'^2, \dots, m_g'^2)$; this ideal contains every monomial of degree $2g - 1$ in the m_j', hence a power of the maximal ideal of P', and $\dim P/(I + (\sigma)) = \dim P'/IP' = 0$. Theorem 13.6, applied to the local ring P/I and the images of the $d - g$ elements of σ, gives $\dim P/I \leq d - g$. As $I \subseteq K$, $\dim P/K \leq \dim P/I \leq d - g$.

Step 6: proof of (3). By Step 0, $\det z \in Z$, so $J \subseteq Z$ and $K \subseteq Z$ by (1). If $K \neq Z$, a chain of primes of length $\dim P/Z$ in P/Z lifts to a chain of primes of P starting at Z, and prefixing $K \subsetneq Z$ gives $\dim P/K \geq \dim P/Z + 1 \geq d - g + 1$, contradicting (2) (all dimensions are finite as P is Noetherian local). Hence $Z = K$, and $Z \subseteq Q$ for every prime $Q \supseteq J$ by (1). □

Lemma E.83. *Let $\mathfrak{R}$ be a realisation of a framed normalised curve point x over a level $\mathbf{V}$, and $\kappa = (\kappa_1, \dots, \kappa_g)$ a Selmer basis of x with local witnesses. Assume:*

(Dg1) *for every $\mu \in \mathrm{La}$, $\mathrm{la}(\mu) \in \mathrm{Sel}_{\mathbf{V}}^{\mathrm{od,b}}(F)$;*

(Dg2) *for every $z \in \mathrm{Sel}_{\mathbf{V}}^{\mathrm{od,b}}(F)$ there is $c \in F^g$ with $z - \sum_j c_j(\kappa_j \circ h) \in \mathrm{Bd}_{\mathbf{V}}^{\mathrm{od}}(F)$;*

(Dg3) *if $\mu \in \mathrm{La}$ and $\mathrm{la}(\mu) \in \mathrm{Bd}_{\mathbf{V}}^{\mathrm{od}}(F)$, then $\mu = 0$.*

Then there is a family $(m_1, \dots, m_g)$ in N that is dual to κ and generates N as a $\hat{B}$-module.

Proof. Step 1: La *is the dual of $N/\hat{\mathfrak{m}}N$.* N is a finitely generated $\hat{B}$-module (a direct summand of $\mathcal{A}$, (Vr5)), so $\bar{N} = N/\hat{\mathfrak{m}}N$ is a finite-dimensional vector space over $\hat{B}/\hat{\mathfrak{m}}$, which λ identifies with F. An additive $\mu\colon N \to F$ with $\mu(by) = \lambda(b)\mu(y)$ kills $\hat{\mathfrak{m}}N$ and induces an F-linear functional on $\bar{N}$, and conversely; so $\mathrm{La} \cong \mathrm{Hom}_F(\bar{N}, F)$. Hence La is finite-dimensional, and every F-linear functional on La is $\mu \mapsto \mu(y)$ for some $y \in N$.

Step 2: the coefficient map. The maps $\kappa_j \circ h$ are linearly independent over F modulo $\mathrm{Bd}_{\mathbf{V}}^{\mathrm{od}}(F)$: if $\sum_j c_j(\kappa_j \circ h)(\gamma) = \rho(h\gamma)Y\rho(h\gamma)^{-1} - Y$ with $Y \in \mathrm{Od}_a(F)$, ${}^tY = -Y$, choose s with $T^s c_j \in A$ and $T^sY \in \mathrm{Od}_a(A)$; as h is onto, $\sum_j T^s c_j \kappa_j \in \mathrm{Bd}(x)$, so $T^s c = 0$ by the definition of a Selmer basis, and $c = 0$. Hence by (Dg1) and (Dg2), for each $\mu \in \mathrm{La}$ there is exactly one $c(\mu) \in F^g$ with $\mathrm{la}(\mu) - \sum_j c_j(\mu)(\kappa_j \circ h) \in \mathrm{Bd}_{\mathbf{V}}^{\mathrm{od}}(F)$. As $\mu \mapsto \mathrm{la}(\mu)$ is F-linear and $\mathrm{Bd}_{\mathbf{V}}^{\mathrm{od}}(F)$ is an F-subspace, uniqueness makes $c\colon \mathrm{La} \to F^g$ linear; it is injective, since $c(\mu) = 0$ means $\mathrm{la}(\mu) \in \mathrm{Bd}_{\mathbf{V}}^{\mathrm{od}}(F)$, so $\mu = 0$ by (Dg3).

Step 3: the generators. By Step 1, for each j there is $m_j \in N$ with $\mu(m_j) = c_j(\mu)$ for all $\mu \in \mathrm{La}$; then $\mathrm{la}(\mu) - \sum_j \mu(m_j)(\kappa_j \circ h) \in \mathrm{Bd}_{\mathbf{V}}^{\mathrm{od}}(F)$, i.e. (m_j) is dual to κ. If the images of the m_j did not span $\bar{N}$, some nonzero $\mu \in \mathrm{La} \cong \bar{N}^*$ would vanish on all m_j, so $c(\mu) = 0$ and $\mu = 0$ by Step 2, a contradiction. So the m_j span $N/\hat{\mathfrak{m}}N$, and by Nakayama's lemma (N finitely generated over the local ring $\hat{B}$) they generate N. □

Lemma E.84. *Let $\mathfrak{R}$ be a realisation of a framed normalised curve point x over a level $\mathbf{V}$. If $\mu \in \mathrm{La}$ and $\mathrm{la}(\mu) \in \mathrm{Bd}_{\mathbf{V}}^{\mathrm{od}}(F)$, then $\mu = 0$.*

Proof. Let $F[\varepsilon]=F[X]/(X^2)$ and define $\theta\colon \mathcal{A}=\hat{B}\cdot 1\oplus N\to F[\varepsilon]$ by $\theta(b\cdot 1+y)=\lambda(b)+\varepsilon\mu(y)$. It is additive, and multiplicative: for $y,y'\in N$ write $yy'=c\cdot 1$ with $c\in \mathrm{ired}(N)\subseteq\hat{\mathfrak{m}}$; then $(b+y)(b'+y')=(bb'+c)\cdot 1+(by'+b'y)$, and θ of this is $\lambda(b)\lambda(b')+\varepsilon(\lambda(b)\mu(y')+\lambda(b')\mu(y))=\theta(b+y)\theta(b'+y')$, using $\lambda(c)=0$ and $\mu\in\mathrm{La}$. Apply θ entrywise to matrices. Write $\rho_1=\rho\circ h$; by (Vr6), $\lambda(\mathrm{pr}_0(\nu(\gamma)_{ij}))=\rho_1(\gamma)_{ij}$, so

$$\theta(\nu(\gamma))=\rho_1(\gamma)+\varepsilon\,\mathrm{la}(\mu)(\gamma)\rho_1(\gamma)\qquad(\gamma\in\Gamma_1^0).$$

Suppose $\mathrm{la}(\mu)(\gamma)=\rho_1(\gamma)Y\rho_1(\gamma)^{-1}-Y$ for all γ, with $Y\in\mathrm{Od}_a(F)$. Then $\theta(\nu(\gamma))=\rho_1(\gamma)+\varepsilon(\rho_1(\gamma)Y-Y\rho_1(\gamma))=U\rho_1(\gamma)U^{-1}$ with $U=1-\varepsilon Y$, $U^{-1}=1+\varepsilon Y$. Let $y\in N$ and (i,j) a position in different blocks. By (Vr6) the matrix yE_{ij} (entry y at (i,j), zero elsewhere) is $\sum_k b_k\nu(\gamma_k)$ for some $b_k\in\hat{B}$, $\gamma_k\in\Gamma_1^0$. Applying θ: $\varepsilon\mu(y)E_{ij}=\sum_k\lambda(b_k)U\rho_1(\gamma_k)U^{-1}=UMU^{-1}$ with $M=\sum_k\lambda(b_k)\rho_1(\gamma_k)\in\mathrm{M}_n(F)$. Comparing the components free of ε (that of $UMU^{-1}=M+\varepsilon(\dots)$ is M, that of the left side is 0) gives $M=0$, so $\varepsilon\mu(y)E_{ij}=0$ and $\mu(y)=0$. As y was arbitrary, $\mu=0$. (If $a=n$, then $N=0$ and there is nothing to prove.) □

Lemma E.85. *Let $\mathscr{H}$ be a rank-n Hecke datum with $\ell\neq 2$, $x=(k,\rho,1_n,(\psi_{v,b}),f)$ a framed normalised curve point of shape $(a,n-a)$ with frame $((w_v),(I_u^x))$, satisfying (Pj1) and such that for every $Z\in\mathrm{Ev}_a(A)$ there is $r\geq 0$ with T^rZ in the A-span of $\rho(\Gamma')$, $\kappa=(\kappa_1,\dots,\kappa_g)$ a Selmer basis of x with local witnesses, $\mathbf{V}$ a level over $\mathscr{H}$ and $\mathfrak{R}$ a realisation of x over $\mathbf{V}$. Then:*

(Df1) *for every $\alpha\in\mathrm{Tp}$ the map $\mathrm{al}(\alpha)$ satisfies, at every $v\in S_\ell$, the local condition of Definition E.35 (Sc) for $(\rho_F,w,\mathrm{Ev}_a(F))$ (with a witness $Y_v\in\mathrm{Ev}_a(F)$); for every $\mu\in\mathrm{La}$ the map $\mathrm{la}(\mu)$ satisfies the local conditions for $(\rho_F,w,\mathrm{Od}_a(F))$. Together with Lemma E.87 and Lemma E.88, this gives $\mathrm{al}(\alpha)\in\mathrm{Sel}_{\mathbf{V}}^{\mathrm{ev,b}}(F)$ and $\mathrm{la}(\mu)\in\mathrm{Sel}_{\mathbf{V}}^{\mathrm{od,b}}(F)$.*

(Df2) *Let $D_0\geq 0$ be such that $e\leq D_0$ whenever $z_1,\dots,z_e\in\mathrm{Sel}_{\mathbf{V}}^{1,\mathrm{b}}(F)$ and $\sum_i c_iz_i\in\mathrm{Bd}_{\mathbf{V}}^1(F)$ with $c\in F^e$ forces $c=0$. If $\alpha_1,\dots,\alpha_e\in\mathrm{Tp}$ are linearly independent over F and, for all i and j, the g-tuple with j-th entry $\mathrm{al}(\alpha_i)$ and all other entries 0 lifts, then $e+g^2\leq D_0$.*

(Df3) *If $(m_1,\dots,m_g)$ in N is dual to κ, then for every $\Phi\in\mathrm{Ph}$ there are $\alpha_1,\dots,\alpha_g\in\mathrm{Tp}$ with $\alpha_j(b)=\Phi(bm_j)-\lambda(b)\Phi(m_j)$ for all $b\in\hat{B}$ and j, such that $(\mathrm{al}(\alpha_1),\dots,\mathrm{al}(\alpha_g))$ lifts.*

Proof. Notation. $\rho_F=\rho\circ h$, a homomorphism $\Gamma_1^0\to\mathrm{GL}_n(A)\subseteq\mathrm{GL}_n(F)$. For $v\in S_\ell$ and $g\in D_v$ write $\phi g=\phi_{1,v}(g)$; as $h\circ\phi_{1,v}=\pi|_{D_v}$, $\rho_F(\phi g)=\rho(\pi g)$, and by the frame $T_g=w_v^{-1}\rho(\pi g)w_v$ is upper triangular with diagonal $(\psi_{v,1}(g),\dots,\psi_{v,n}(g))$. Let $J=\mathrm{diag}(1_a,-1_{n-a})$, so $JXJ=X$ for $X\in\mathrm{Ev}_a(F)$ and $JXJ=-X$ for $X\in\mathrm{Od}_a(F)$. As x is normalised of shape $(a,n-a)$, $\rho(\gamma)\in\mathrm{Ev}_a(A)$, hence $J\rho_F(\gamma)J=\rho_F(\gamma)$. Since each column of w_v has its non-zero entries in one block, $Jw_v=w_vE_v$ for a diagonal matrix E_v with entries ±1. The field F has characteristic $\ell\neq 2$, so 2 is invertible in F.

(Df1). Let $\alpha\in\mathrm{Tp}$, $\mu\in\mathrm{La}$, and $F[\varepsilon]=F\oplus F\varepsilon$ with $\varepsilon^2=0$. Using $\mathcal{A}=\hat{B}\cdot 1\oplus N$ ((Vr5)) define $\theta_\alpha,\theta_\mu\colon\mathcal{A}\to F[\varepsilon]$ by

$$\theta_\alpha(b\cdot 1+y)=\lambda(b)+\varepsilon\,\alpha(b),\qquad\theta_\mu(b\cdot 1+y)=\lambda(b)+\varepsilon\,\mu(y)\qquad(b\in\hat{B},\ y\in N).$$

Both are ring homomorphisms. They are additive and send 1 to 1. For $y,y'\in N$, (Vr5) gives $yy'=c\cdot 1$ with $c\in\mathrm{ired}(N)\subseteq\hat{\mathfrak{m}}$, so $\lambda(c)=0$, and $\alpha(c)=0$ by the definition of Tp. Hence $(b\cdot1+y)(b'\cdot1+y')=(bb'+c)\cdot1+(by'+b'y)$ is sent by θ_α to $\lambda(b)\lambda(b')+\varepsilon(\lambda(b)\alpha(b')+\lambda(b')\alpha(b))$ (derivation rule), and by θ_μ to $\lambda(b)\lambda(b')+\varepsilon(\lambda(b)\mu(y')+\lambda(b')\mu(y))$ ($\mu(by')=\lambda(b)\mu(y')$); these are the products of the images.

Let (θ, z) be $(\theta_\alpha, \mathrm{al}(\alpha))$ or $(\theta_\mu, \mathrm{la}(\mu))$, and $u(\gamma) = \theta(\nu(\gamma))$ (entrywise). By the last clause of (Vr6), $\lambda(\mathrm{pr}_0(\nu(\gamma)_{pq})) = \rho_F(\gamma)_{pq}$, so by the definition of al and la in Definition E.38

$$u(\gamma) = \big(1 + \varepsilon\, z(\gamma)\big)\rho_F(\gamma) \qquad (\gamma \in \Gamma_1^0),$$

and u is a homomorphism $\Gamma_1^0 \to \mathrm{GL}_n(F[\varepsilon])$, as ν is multiplicative with $\nu(1) = 1$ and θ is a ring homomorphism.

Fix $v \in S_\ell$. For $g \in D_v$ put $r_0(g) = T_g$, $r_1(g) = w_v^{-1} u(\phi g) w_v$, $\vartheta_{0,i} = \psi_{v,i}$ and $\vartheta_{1,i} = \theta \circ \hat{\psi}_{v,i}$ (with $\hat{\psi}_{v,i}(g) \in B$ mapped to $\hat{B} \cdot 1 \subseteq \mathcal{A}$). These are homomorphisms on D_v, and $r_1(g) - r_0(g) = \varepsilon\, w_v^{-1} z(\phi g) \rho_F(\phi g) w_v$ has entries in $F\varepsilon$. Since λ extends f_1 ((Vr4)) and $f_1 \circ \hat{\psi}_{v,i} = \psi_{v,i}$ ((Vr1)),

$$\vartheta_{1,i}(g) = \psi_{v,i}(g) + \varepsilon\, \alpha(\hat{\psi}_{v,i}(g)) \ \text{ for } \theta = \theta_\alpha, \qquad \vartheta_{1,i}(g) = \psi_{v,i}(g) \ \text{ for } \theta = \theta_\mu.$$

We apply Lemma E.86 with $R = F[\varepsilon]$, $R \to F$ the constant part (its kernel $F\varepsilon$ consists of nilpotent elements), $\mathfrak{n} = F\varepsilon$ (so $\mathfrak{n}^2 = 0$), $H = D_v$ and the data above; $r_1 - r_0$ and $\vartheta_{1,i} - \vartheta_{0,i}$ take values in $\mathfrak{n}$. Hypothesis 1 is the triangularity of T_g. Hypothesis 2: the residues of the $\vartheta_{0,i}$ are the characters $\psi_{v,i} \colon D_v \to A^\times \subseteq F^\times$, pairwise distinct by (Pj1). Hypothesis 3: for $g_1, \dots, g_n \in D_v$ the element $\prod_{i=1}^n (\phi_{1,v}(g_i) - \hat{\psi}_{v,i}(g_i))$ (increasing order) lies in $\mathcal{I}_1$ by (Vr3); the B-linear extension of ν to $B[\Gamma_1^0]$ is a ring homomorphism ($\mathcal{A}$ is commutative) vanishing on $\mathcal{I}_1$ ((Vr6)), so $\prod_i (\nu(\phi g_i) - \hat{\psi}_{v,i}(g_i)) = 0$ in $\mathrm{M}_n(\mathcal{A})$; applying θ entrywise and conjugating by w_v gives $\prod_i (r_1(g_i) - \vartheta_{1,i}(g_i)) = 0$. Hence there is $Y \in \mathrm{M}_n(F\varepsilon)$, which we write $Y = \varepsilon\, w_v^{-1} Y' w_v$ with $Y' \in \mathrm{M}_n(F)$, such that $r_1(g) r_0(g)^{-1} - 1 - (r_0(g) Y r_0(g)^{-1} - Y)$ is upper triangular with i-th diagonal entry $(\vartheta_{1,i}(g) - \psi_{v,i}(g))\psi_{v,i}(g)^{-1}$ for every $g \in D_v$. As $r_1(g) r_0(g)^{-1} - 1 = \varepsilon\, w_v^{-1} z(\phi g) w_v$ and $r_0(g) Y r_0(g)^{-1} - Y = \varepsilon\, w_v^{-1} (\rho_F(\phi g) Y' \rho_F(\phi g)^{-1} - Y') w_v$, comparing ε-coefficients gives: for every $g \in D_v$ the matrix

$$L_g(Y') = w_v^{-1} (z(\phi g) - \rho_F(\phi g) Y' \rho_F(\phi g)^{-1} + Y') w_v$$

is upper triangular with i-th diagonal entry $\delta_i(g)$, where $\delta_i(g) = \alpha(\hat{\psi}_{v,i}(g))\psi_{v,i}(g)^{-1}$ if $z = \mathrm{al}(\alpha)$ and $\delta_i(g) = 0$ if $z = \mathrm{la}(\mu)$. If $g \in \mu_v$, then $\hat{\psi}_{v,i}(g)$ lies in the image of $\mathcal{O} \to B$ ((Vr1)), whose image in $\hat{B}$ lies in $\iota(\mathcal{O}')$ ((Vr4)), and $\alpha \circ \iota = 0$; so $\delta_i(g) = 0$ for $g \in \mu_v$ in both cases.

It remains to replace Y' by a witness of the right block shape. Let $\mathcal{Z}$ be the set of $Y'' \in \mathrm{M}_n(F)$ such that for every $g \in D_v$ the matrix $L_g(Y'')$ (same formula) is upper triangular with diagonal $(\delta_i(g))_i$; it is an affine subspace of $\mathrm{M}_n(F)$ (L_g is affine in Y'') containing Y'. Put $\varsigma = 1$ if $z = \mathrm{al}(\alpha)$ and $\varsigma = -1$ if $z = \mathrm{la}(\mu)$; by Lemma E.87 z takes values in $\mathrm{Ev}_a(F)$, resp. $\mathrm{Od}_a(F)$, so $Jz(\gamma)J = \varsigma z(\gamma)$. Using $J\rho_F J = \rho_F$ and $w_v^{-1} J = E_v w_v^{-1}$, $J w_v = w_v E_v$ (recall $J^2 = 1$, $E_v^2 = 1$),

$$L_g(\varsigma J Y'' J) = \varsigma\, w_v^{-1} J (z(\phi g) - \rho_F(\phi g) Y'' \rho_F(\phi g)^{-1} + Y'') J w_v = \varsigma\, E_v\, L_g(Y'')\, E_v.$$

Conjugation by the diagonal sign matrix E_v preserves upper triangularity and the diagonal entries, and for $\varsigma = -1$ the prescribed diagonal is 0. Hence $\mathcal{Z}$ is stable under the affine involution $Y'' \mapsto \varsigma J Y'' J$, and $Y_v = \frac{1}{2}(Y' + \varsigma J Y' J) \in \mathcal{Z}$. Since $J Y_v J = \varsigma Y_v$, $Y_v \in \mathrm{Ev}_a(F)$ if $z = \mathrm{al}(\alpha)$ and $Y_v \in \mathrm{Od}_a(F)$ if $z = \mathrm{la}(\mu)$. The statement $Y_v \in \mathcal{Z}$ is exactly the local condition at v of Definition E.35 (Sc) for $(\rho_F, w, \mathrm{Ev}_a(F))$, resp. $(\rho_F, w, \mathrm{Od}_a(F))$, with witness Y_v. This proves the first two assertions of (Df1).

For the last sentence of (Df1): by Lemma E.87, $\mathrm{al}(\alpha)$ and $\mathrm{la}(\mu)$ take values in $\mathrm{Ev}_a(F)$, resp. $\mathrm{Od}_a(F)$, satisfy the cocycle rule for ρ_F and $z(c_1 \gamma c_1) = -{}^t z(\gamma)$, and vanish on every $I_{1,u}$; continuity for the topology of F defined by $TF = F$ is automatic (that topology is indiscrete); with the local conditions just proved, $\mathrm{al}(\alpha) \in \mathrm{Sel}_{\mathbf{V}}^{\mathrm{ev}}(F)$ and $\mathrm{la}(\mu) \in \mathrm{Sel}_{\mathbf{V}}^{\mathrm{od}}(F)$; both are bounded by Lemma E.88.

(Df2). Let D_0 and $\alpha_1,\dots,\alpha_e$ be as in (Df2). By (Df1), $\mathrm{al}(\alpha_i)\in \mathrm{Sel}^{\mathrm{ev,b}}_{\mathbf{V}}(F)$ for every i. So all hypotheses of Lemma E.89 hold ($\ell\neq 2$; x satisfies (Pj1) and the span condition; κ, $\mathbf{V}$, $\mathfrak{R}$, D_0 as there; the α_i independent, with $\mathrm{al}(\alpha_i)$ even bounded Selmer maps whose single-entry g-tuples lift), and it gives $e+g^2\le D_0$.

(Df3). Let $(m_1,\dots,m_g)$ be dual to κ and $\Phi\in\mathrm{Ph}$, and put $\alpha_j(b)=\Phi(bm_j)-\lambda(b)\Phi(m_j)$. By Lemma E.93 (whose hypotheses are $\ell\neq 2$, (Pj1) for x, and κ, $\mathbf{V}$, $\mathfrak{R}$, (m_j) as here) $\alpha_j\in\mathrm{Tp}$ for every j and $(\mathrm{al}(\alpha_1),\dots,\mathrm{al}(\alpha_g))$ lifts. □

Lemma E.86. *Let R be a commutative ring with a surjective ring homomorphism $R\to K$, $t\mapsto \bar t$, onto a field K whose kernel consists of nilpotent elements, $\mathfrak{n}\subseteq R$ an ideal with $\mathfrak{n}^2=0$, H a group, $n\ge 1$, $r_0,r_1\colon H\to \mathrm{GL}_n(R)$ homomorphisms with $r_1(g)-r_0(g)\in \mathrm{M}_n(\mathfrak{n})$, and $\vartheta_{0,i},\vartheta_{1,i}\colon H\to R^\times$ $(1\le i\le n)$ homomorphisms with $\vartheta_{1,i}(g)-\vartheta_{0,i}(g)\in\mathfrak{n}$, for all $g\in H$. Suppose:*

1. *every $r_0(g)$ is upper triangular with diagonal $(\vartheta_{0,1}(g),\dots,\vartheta_{0,n}(g))$;*
2. *for $i\neq j$ the characters $g\mapsto\overline{\vartheta_{0,i}(g)}$ and $g\mapsto\overline{\vartheta_{0,j}(g)}$ of H are distinct;*
3. *for all $g_1,\dots,g_n\in H$,*

$$\big(r_1(g_1)-\vartheta_{1,1}(g_1)\big)\big(r_1(g_2)-\vartheta_{1,2}(g_2)\big)\cdots\big(r_1(g_n)-\vartheta_{1,n}(g_n)\big)=0.$$

Then there is $Y\in\mathrm{M}_n(\mathfrak{n})$ such that for every $g\in H$ the matrix

$$r_1(g)r_0(g)^{-1}-1-\big(r_0(g)Yr_0(g)^{-1}-Y\big)$$

is upper triangular with i-th diagonal entry $(\vartheta_{1,i}(g)-\vartheta_{0,i}(g))\vartheta_{0,i}(g)^{-1}$ $(1\le i\le n)$.

Proof. Write $e_1,\dots,e_n$ for the standard basis of R^n and extend $r_0,r_1,\vartheta_{0,i},\vartheta_{1,i}$ R-linearly to the group ring $R[H]$. Since $r_1(x)-\vartheta_{1,i}(x)=\sum_g a_g(r_1(g)-\vartheta_{1,i}(g))$ for $x=\sum_g a_g g$, hypothesis 3 extends by multilinearity to all $x_1,\dots,x_n\in R[H]$. Likewise 1 gives that $r_0(x)$ is upper triangular with diagonal $(\vartheta_{0,1}(x),\dots,\vartheta_{0,n}(x))$ for $x\in R[H]$.

Step 1: separating elements. The R-linear map $R[H]\to R^n$, $x\mapsto(\vartheta_{0,k}(x))_k$, is surjective. Indeed, by 2 and Artin's theorem on the linear independence of distinct characters $H\to K^\times$, the map $K[H]\to K^n$, $x\mapsto(\overline{\vartheta_{0,k}}(x))_k$, is surjective (a non-zero linear form on K^n vanishing on its image would give a linear relation among the characters), so the image M' of $R[H]$ satisfies $M'+\mathfrak{m}R^n=R^n$, $\mathfrak{m}$ the kernel of $R\to K$; $\mathfrak{m}$ lies in the Jacobson radical of R ($1+t$ is a unit for nilpotent t), and Nakayama's lemma gives $M'=R^n$. Fix $x^{(i)}\in R[H]$ with $\vartheta_{0,k}(x^{(i)})=\delta_{ik}$; then also $\vartheta_{1,k}(x^{(i)})-\delta_{ik}\in\mathfrak{n}$.

Step 2: uniqueness of eigenvectors. We show: if $n\ge 2$ and $u\in R^n$ has $u_1=0$ and $r_1(g)u=\vartheta_{1,1}(g)u$ for all $g\in H$, then $u=0$. Let Z be an R-module and $u\in Z^n$ with $u_1=0$ and $r_0(g)u=\vartheta_{0,1}(g)u$ for all g, where matrices over R act on Z^n in the obvious way. If $u\neq 0$, let p be the largest index with $u_p\neq 0$; then $p\ge 2$, and the p-th coordinate of $r_0(x^{(p)})u-\vartheta_{0,1}(x^{(p)})u=0$ is, as $r_0(x^{(p)})$ is upper triangular, $(\vartheta_{0,p}(x^{(p)})-\vartheta_{0,1}(x^{(p)}))u_p=u_p$, a contradiction; so $u=0$. Apply this first to $Z=R/\mathfrak{n}$ and the image of the given u (modulo $\mathfrak{n}$, $r_1\equiv r_0$ and $\vartheta_{1,1}\equiv\vartheta_{0,1}$): so $u\in\mathfrak{n}^n$. Then $(r_1(g)-r_0(g))u=0$ and $(\vartheta_{1,1}(g)-\vartheta_{0,1}(g))u=0$ as $\mathfrak{n}^2=0$, and the same argument with $Z=R$ gives $u=0$.

Step 3: a triangularising basis. We show by induction on n: there is $V\in\mathrm{GL}_n(R)$ with $V-1\in\mathrm{M}_n(\mathfrak{n})$ such that $V^{-1}r_1(g)V$ is upper triangular with diagonal $(\vartheta_{1,1}(g),\dots,\vartheta_{1,n}(g))$ for all g. For $n=1$, hypothesis 3 says $r_1=\vartheta_{1,1}$; take $V=1$. Let $n\ge 2$. For $x_2,\dots,x_n\in R[H]$ put $P(x_2,\dots,x_n)=\prod_{k=2}^{n}(r_1(x_k)-\vartheta_{1,k}(x_k))$ (ordered product). By 3 (with $x_1=g$),

the image of $P(x_2,\dots,x_n)$ lies in $E_1=\{m\in R^n : r_1(g)m=\vartheta_{1,1}(g)m \text{ for all } g\}$. Let $v_1=P(x^{(1)},\dots,x^{(1)})e_1\in E_1$. Modulo $\mathfrak{n}$, $P(x^{(1)},\dots,x^{(1)})\equiv\prod_{k\geq 2}(r_0(x^{(1)})-\vartheta_{0,k}(x^{(1)}))=r_0(x^{(1)})^{n-1}$, and $r_0(x^{(1)})e_1=\vartheta_{0,1}(x^{(1)})e_1=e_1$; hence $v_1-e_1\in\mathfrak{n}^n$. In particular the first coordinate of v_1 is a unit, and $E_1=Rv_1$: for $m\in E_1$, $m-tv_1$ with $t=m_1(v_1)_1^{-1}$ lies in E_1 and has first coordinate 0, so it vanishes by Step 2. Let V_1 be the matrix with columns $v_1,e_2,\dots,e_n$; $V_1-1\in \mathrm{M}_n(\mathfrak{n})$, so V_1 is invertible. As $v_1\in E_1$, $V_1^{-1}r_1(g)V_1=\begin{pmatrix}\vartheta_{1,1}(g) & *\\ 0 & s_1(g)\end{pmatrix}$ with a homomorphism $s_1\colon H\to \mathrm{GL}_{n-1}(R)$. Let $s_0(g)$ be the lower right $(n-1)\times(n-1)$ block of $r_0(g)$, a homomorphism by 1, upper triangular with diagonal $(\vartheta_{0,2}(g),\dots,\vartheta_{0,n}(g))$; since $V_1\equiv 1$ modulo $\mathfrak{n}$, $s_1(g)-s_0(g)\in\mathrm{M}_{n-1}(\mathfrak{n})$. For $x_2,\dots,x_n\in R[H]$ the matrix $V_1^{-1}P(x_2,\dots,x_n)V_1$ is block upper triangular with lower right block $\prod_{k\geq 2}(s_1(x_k)-\vartheta_{1,k}(x_k))$, and its image lies in $V_1^{-1}E_1=Re_1$, so its rows $2,\dots,n$ vanish and so does that block. Hence $(s_0,s_1,(\vartheta_{0,k},\vartheta_{1,k})_{k\geq 2})$ satisfies the hypotheses with $n-1$, and induction gives V' for it. Then $V=V_1\,\mathrm{diag}(1,V')$ has the required properties.

Step 4: the matrix Y. Write $V=1+Y_0$ with $Y_0\in\mathrm{M}_n(\mathfrak{n})$; then $V^{-1}=1-Y_0$, and, using $\mathfrak{n}^2=0$ and $r_1(g)-r_0(g)\in\mathrm{M}_n(\mathfrak{n})$,

$$V^{-1}r_1(g)V=r_1(g)+r_0(g)Y_0-Y_0r_0(g)=(1+c(g)+r_0(g)Y_0r_0(g)^{-1}-Y_0)r_0(g),$$

where $c(g)=r_1(g)r_0(g)^{-1}-1$. Put $Y=-Y_0$ and $c'(g)=c(g)-(r_0(g)Yr_0(g)^{-1}-Y)$. Then $1+c'(g)=(V^{-1}r_1(g)V)\,r_0(g)^{-1}$ is a product of upper triangular matrices with diagonals $(\vartheta_{1,i}(g))_i$ and $(\vartheta_{0,i}(g)^{-1})_i$, so $c'(g)$ is upper triangular with i-th diagonal entry $\vartheta_{1,i}(g)\vartheta_{0,i}(g)^{-1}-1$. □

Lemma E.87. *Let $\mathfrak{R}$ be a realisation of x over $\mathbf{V}$, $\alpha\in\mathrm{Tp}$, $\mu\in\mathrm{La}$, and write $\rho_1(\gamma)=\rho(h(\gamma))$. Then $z=\mathrm{al}(\alpha)$ takes values in $\mathrm{Ev}_a(F)$, $z'=\mathrm{la}(\mu)$ takes values in $\mathrm{Od}_a(F)$, and each $w\in\{z,z'\}$ satisfies: $w(\gamma\gamma')=w(\gamma)+\rho_1(\gamma)w(\gamma')\rho_1(\gamma)^{-1}$ and $w(c_1\gamma c_1)=-{}^tw(\gamma)$ for $\gamma,\gamma'\in\Gamma_1^0$, and w vanishes on every $I_{1,u}$. Moreover $\alpha\mapsto\mathrm{al}(\alpha)$ and $\mu\mapsto\mathrm{la}(\mu)$ are F-linear.*

Proof. Write $\nu(\gamma)_{ij}=b_{ij}(\gamma)\cdot 1+n_{ij}(\gamma)$ with $b_{ij}(\gamma)=\mathrm{pr}_0(\nu(\gamma)_{ij})\in\hat{B}$ and $n_{ij}(\gamma)=\mathrm{pr}_1(\nu(\gamma)_{ij})\in N$. By (Vr6), $n_{ij}=0$ when i,j lie in one block and $b_{ij}=0$ otherwise, and $\lambda(b_{ij}(\gamma))=\rho_1(\gamma)_{ij}$ (for i,j in different blocks, $\nu(\gamma)_{ij}-0\cdot 1\in N$, so $\rho_1(\gamma)_{ij}=0$, as ρ is block diagonal). Put $\beta(\gamma)=(\alpha(b_{ij}(\gamma)))_{ij}$ and $\beta'(\gamma)=(\mu(n_{ij}(\gamma)))_{ij}$; so $z=\beta\rho_1^{-1}$, $z'=\beta'\rho_1^{-1}$. As $\rho_1(\gamma)$ is block diagonal, z has the block shape of β (in $\mathrm{Ev}_a(F)$) and z' that of β' (in $\mathrm{Od}_a(F)$). Linearity follows from the definitions.

Product rule. For γ,γ' and i,j in one block, $\nu(\gamma\gamma')_{ij}=\sum_k b_{ik}(\gamma)b_{kj}(\gamma')\cdot 1+\sum_k n_{ik}(\gamma)n_{kj}(\gamma')$, the first sum over k in that block and the second over k in the other; the second sum lies in $\mathrm{ired}(N)\cdot 1$. Since α kills $\mathrm{ired}(N)$ and is a derivation over λ, $\alpha(b_{ij}(\gamma\gamma'))=\sum_k(\rho_1(\gamma)_{ik}\alpha(b_{kj}(\gamma'))+\alpha(b_{ik}(\gamma))\rho_1(\gamma')_{kj})$, i.e. $\beta(\gamma\gamma')=\rho_1(\gamma)\beta(\gamma')+\beta(\gamma)\rho_1(\gamma')$ (both sides vanish at positions in different blocks). For i,j in different blocks, $n_{ij}(\gamma\gamma')=\sum_k b_{ik}(\gamma)n_{kj}(\gamma')+\sum_k n_{ik}(\gamma)b_{kj}(\gamma')$, and $\mu(by)=\lambda(b)\mu(y)$ gives $\beta'(\gamma\gamma')=\rho_1(\gamma)\beta'(\gamma')+\beta'(\gamma)\rho_1(\gamma')$. Multiplying on the right by $\rho_1(\gamma\gamma')^{-1}=\rho_1(\gamma')^{-1}\rho_1(\gamma)^{-1}$ gives the cocycle identity for $w=z,z'$. Also $\nu(1)=1$ gives $\beta(1)=(\alpha(\delta_{ij}))=0$ ($\alpha(1)=\alpha(1\cdot 1)=2\alpha(1)$) and $\beta'(1)=0$, so $w(1)=0$ and $\rho_1(\gamma)w(\gamma^{-1})\rho_1(\gamma)^{-1}=-w(\gamma)$.

Polarisation. By (Vr6), $\nu(\gamma^*)={}^t\nu(\gamma)$ with $\gamma^*=c_1\gamma^{-1}c_1$, so $\beta(\gamma^*)={}^t\beta(\gamma)$, $\beta'(\gamma^*)={}^t\beta'(\gamma)$; and $\rho_1(\gamma^*)=\rho(\bar{c}\,h(\gamma)^{-1}\bar{c})={}^t\rho_1(\gamma)$ (x normalised, $h(c_1)=\bar{c}$). Hence

$$w(\gamma^*)={}^t(\rho_1(\gamma)^{-1}w(\gamma)\rho_1(\gamma)).$$

Applying this to γ^{-1}, for which $(\gamma^{-1})^*=c_1\gamma c_1$, and using the last identity of the previous paragraph: $w(c_1\gamma c_1)={}^t(\rho_1(\gamma)w(\gamma^{-1})\rho_1(\gamma)^{-1})=-{}^tw(\gamma)$.

Frame inertia. For $\gamma' \in I_{1,u}$, $\gamma' - 1 \in \mathcal{I}_1$ by (Vr3), so $\nu(\gamma') = 1$ by (Vr6); hence $\beta(\gamma') = \beta'(\gamma') = 0$ and $w(\gamma') = 0$. □

Lemma E.88. *Let $\mathfrak{R}$ be a realisation of x over a level $\mathbf{V}$. For every $\alpha \in \mathrm{Tp}$ and $\mu \in \mathrm{La}$ the maps $\mathrm{al}(\alpha)$ and $\mathrm{la}(\mu)$ are bounded.*

Proof. We use Cohen's structure theorem: a complete Noetherian local ring of characteristic ℓ with residue field k_0 is a quotient of $k_0[[x_1, \dots, x_m]]$ for some m.

Write $\rho_1 = \rho \circ h$; its entries and those of ρ_1^{-1} lie in A and are T-adically continuous. Let $\beta(\gamma) = (\alpha(\mathrm{pr}_0(\nu(\gamma)_{ij})))_{ij}$ and $\beta'(\gamma) = (\mu(\mathrm{pr}_1(\nu(\gamma)_{ij})))_{ij}$, so $\mathrm{al}(\alpha) = \beta\rho_1^{-1}$, $\mathrm{la}(\mu) = \beta'\rho_1^{-1}$; it suffices to find s such that the entries of $T^s\beta$, $T^s\beta'$ lie in A and are continuous. Let $\alpha_B = \alpha \circ (B \to \hat{B})$; it is additive, kills the image of $\mathcal{O}$, and $\alpha_B(bb') = f_1(b)\alpha_B(b') + f_1(b')\alpha_B(b)$.

Step 1: $\alpha_B(B) \subseteq T^{-s_0}A$ for some s_0, and $\alpha_B(\mathfrak{m}_B^j) \subseteq T^{j-1-s_0}A$ for $j \geq 1$. As $\ell = 0$ in F, $\alpha_B(\ell b) = f_1(\ell)\alpha_B(b) + f_1(b)\alpha_B(\ell) = 0$, and $\alpha_B(c^\ell) = \ell f_1(c)^{\ell-1}\alpha_B(c) = 0$. The ring $\bar{B} = B/\ell B$ is complete Noetherian local of characteristic ℓ with finite (hence perfect) residue field k_0; by Cohen's theorem it is a quotient of $k_0[[x_1, \dots, x_m]]$, which is generated as a module over its subring of ℓ-th powers $k_0[[x_1^\ell, \dots, x_m^\ell]]$ by the monomials x^e with $0 \leq e_i < \ell$. Choose $u_e \in B$ mapping to the images of these monomials. Then every $b \in B$ is $\sum_e c_e^\ell u_e + \ell b'$ with $c_e, b' \in B$, and $\alpha_B(b) = \sum_e f_1(c_e)^\ell \alpha_B(u_e)$. So $\alpha_B(B)$ lies in the A-module generated by the finitely many $\alpha_B(u_e) \in F$, hence in $T^{-s_0}A$ for some s_0. For $x_1, \dots, x_j \in \mathfrak{m}_B$, $\alpha_B(x_1 \cdots x_j) = \sum_i (\prod_{k \neq i} f_1(x_k))\alpha_B(x_i) \in T^{j-1}T^{-s_0}A$, as $f_1(\mathfrak{m}_B) \subseteq TA$; sums of such products give the second claim.

Step 2: bounds. By (Vr8), $\nu(\Gamma_1^0) \subseteq \mathcal{P}_0 = \sum_{k=1}^r BP_k$ for some $P_k \in \mathrm{M}_n(\mathcal{A})$. Let $p_{k,ij} = \mathrm{pr}_0((P_k)_{ij}) \in \hat{B}$ and $q_{k,ij} = \mathrm{pr}_1((P_k)_{ij}) \in N$, and choose s_1 with $T^{s_1}\lambda(p_{k,ij})$, $T^{s_1}\alpha(p_{k,ij})$, $T^{s_1}\mu(q_{k,ij})$ in A for all k, i, j (finitely many elements of F). For $P = \sum_k b_k P_k \in \mathcal{P}_0$ ($b_k \in B$), as $\mathrm{pr}_0, \mathrm{pr}_1$ are $\hat{B}$-linear,

$$\alpha(\mathrm{pr}_0(P_{ij})) = \sum_k (f_1(b_k)\alpha(p_{k,ij}) + \lambda(p_{k,ij})\alpha_B(b_k)), \qquad \mu(\mathrm{pr}_1(P_{ij})) = \sum_k f_1(b_k)\mu(q_{k,ij}).$$

With $s = s_0 + s_1 + 1$, Step 1 shows that these lie in $T^{-s}A$ for all $P \in \mathcal{P}_0$, and in $T^{j-1-s}A$ if all $b_k \in \mathfrak{m}_B^j$ (then $f_1(b_k) \in T^j A$). Hence the entries of $T^s\beta(\gamma)$, $T^s\beta'(\gamma)$ lie in A, and for $v \in V_j$ of (Vr8), $\nu(\gamma v) - \nu(\gamma) \in \mathfrak{m}_B^j \mathcal{P}_0$ gives $T^s(\beta(\gamma v) - \beta(\gamma))$, $T^s(\beta'(\gamma v) - \beta'(\gamma)) \in T^{j-1}\mathrm{M}_n(A)$: the entries are T-adically continuous. □

Lemma E.89. *Let $\mathscr{H}$ be a rank-n Hecke datum with $\ell \neq 2$ and x a framed normalised curve point of shape $(a, n-a)$ satisfying (Pj1) and such that for every $Z \in \mathrm{Ev}_a(A)$ there is $s \geq 0$ with $T^s Z$ in the A-span of $\rho(\Gamma')$. Let $\kappa = (\kappa_1, \dots, \kappa_g)$ be a Selmer basis of x with local witnesses, $\mathbf{V}$ a level over $\mathscr{H}$, $\mathfrak{R}$ a realisation of x over $\mathbf{V}$, and $D_0 \geq 0$ an integer such that $e \leq D_0$ whenever $z_1, \dots, z_e \in \mathrm{Sel}_{\mathbf{V}}^{1,\mathrm{b}}(F)$ and $\sum_i c_i z_i \in \mathrm{Bd}_{\mathbf{V}}^1(F)$ with $c \in F^e$ forces $c = 0$. If $\alpha_1, \dots, \alpha_e \in \mathrm{Tp}$ are linearly independent over F, $\mathrm{al}(\alpha_i) \in \mathrm{Sel}_{\mathbf{V}}^{\mathrm{ev},\mathrm{b}}(F)$ for every i, and for all i and j the g-tuple with j-th entry $\mathrm{al}(\alpha_i)$ and all other entries 0 lifts, then $e + g^2 \leq D_0$.*

Proof. Write $\rho_F = \rho \circ h$, $\rho_{1,F} = (1 + \sum_k \epsilon_k(\kappa_k \circ h))\rho_F$, and for a matrix M over $\mathrm{PD}_g(F)$ let M_0 and M_{ϵ_k} be its matrices of components in $F \cdot 1$ and $F\epsilon_k$. The map $M \mapsto M_0$ is a ring homomorphism on matrices over $F \oplus \bigoplus_k F\epsilon_k$, and $(\rho_{1,F})_0 = \rho_F$.

The $e + g^2$ maps. For $1 \leq i \leq e$, Lemma E.90 applied to $z = \mathrm{al}(\alpha_i)$ gives $y_i \in \mathrm{Sel}_{\mathbf{V}}^{1,\mathrm{b}}(F)$ with $(y_i)_0 = \mathrm{al}(\alpha_i)$. For $1 \leq p, m \leq g$ put $\omega_{p,m}(\gamma) = \epsilon_m \kappa_p(h(\gamma))$. Then $\omega_{p,m} \in \mathrm{Sel}_{\mathbf{V}}^{1,\mathrm{b}}(F)$: its values lie in $\mathrm{PW}_a^1(F)$ as κ_p takes values in $\mathrm{Od}_a(A)$; since $\epsilon_m\epsilon_k = 0$, conjugation by $\rho_{1,F}(\gamma)$ acts on $\epsilon_m \mathrm{M}_n(F)$ as conjugation by $\rho_F(\gamma)$, so the cocycle identity for $\rho_{1,F}$ is that of κ_p for

ρ composed with the homomorphism h; polarisation follows from $h(c_1\gamma c_1) = \bar{c}\,h(\gamma)\bar{c}$ and that of κ_p; $\omega_{p,m}$ vanishes on $I_{1,u}$ as $h(I_{1,u}) = I_u^x$; at $v \in S_\ell$ the element $\epsilon_m Y_{p,v} \in \mathrm{PW}_a^1(F)$ ($Y_{p,v}$ the local witness of κ_p) is a witness, because, again as $\epsilon_m\epsilon_k = 0$, the local matrix equals $\epsilon_m\, w_v^{-1}(\kappa_p(\pi g) - \rho(\pi g)Y_{p,v}\rho(\pi g)^{-1} + Y_{p,v})w_v$, which is upper triangular with zero diagonal for $g \in \mu_v$ by (Sd2) of Definition E.34; and it is bounded with exponent 0, its coordinates being T-adically continuous entries of $\kappa_p \circ h$.

Independence modulo $\mathrm{Bd}^1_{\mathbf{V}}(F)$. Let $c \in F^e$, $d \in F^{g\times g}$ and $Y \in \mathrm{PW}_a^1(F)$ with ${}^tY = -Y$ be such that

$$\sum_i c_i y_i(\gamma) + \sum_{p,m} d_{p,m}\omega_{p,m}(\gamma) = \rho_{1,F}(\gamma)Y\rho_{1,F}(\gamma)^{-1} - Y \qquad (\gamma \in \Gamma_1^0).$$

Taking constant parts, $\sum_i c_i\,\mathrm{al}(\alpha_i)(\gamma) = \rho_F(\gamma)Y_0\rho_F(\gamma)^{-1} - Y_0$. By F-linearity of al (Lemma E.87) the left side is $\mathrm{al}(\sum_i c_i\alpha_i)$, so Lemma E.91 gives $\sum_i c_i\alpha_i = 0$, and $c = 0$ by independence. Hence Y_0 commutes with $\rho_F(\Gamma_1^0) = \rho(\Gamma^0)$ (h maps $\Gamma_1^0 = h^{-1}(\Gamma^0)$ onto Γ^0). Moreover $Y_0 \in \mathrm{Ev}_a(F)$ (same-block entries of PW_a^1 lie in $F\cdot 1$, different-block entries in $\bigoplus_k F\epsilon_k$) and ${}^tY_0 = -Y_0$. We show $Y_0 = 0$. Its diagonal entries satisfy $2(Y_0)_{pp} = 0$, and $2 \neq 0$ in F as $\ell \neq 2$. Let $p \neq q$ lie in the same block. The elementary matrix E_{pp} lies in $\mathrm{Ev}_a(A)$, so for some s, T^sE_{pp} is an A-linear combination of elements of $\rho(\Gamma') \subseteq \rho(\Gamma^0)$ and commutes with Y_0; the (p,q) entry of $T^s(E_{pp}Y_0 - Y_0E_{pp}) = 0$ is $T^s(Y_0)_{pq}$, so $(Y_0)_{pq} = 0$. Entries in different blocks vanish as $Y_0 \in \mathrm{Ev}_a(F)$.

Now $Y = \sum_k \epsilon_k Y_{\epsilon_k}$ with $Y_{\epsilon_k} \in \mathrm{Od}_a(F)$ and ${}^tY_{\epsilon_k} = -Y_{\epsilon_k}$, and as $\epsilon_k\epsilon_{k'} = 0$, $\rho_{1,F}(\gamma)Y\rho_{1,F}(\gamma)^{-1} = \sum_k \epsilon_k\rho_F(\gamma)Y_{\epsilon_k}\rho_F(\gamma)^{-1}$. Since $c = 0$ and the $\omega_{p,m}$ have constant part 0, comparing ϵ_m-components gives, for each m,

$$\sum_p d_{p,m}\,\kappa_p(h(\gamma)) = \rho_F(\gamma)Y_{\epsilon_m}\rho_F(\gamma)^{-1} - Y_{\epsilon_m} \qquad (\gamma \in \Gamma_1^0).$$

Choose $0 \neq t \in A$ with $td_{p,m} \in A$ for all p and $tY_{\epsilon_m} \in \mathrm{Od}_a(A)$. As $h(\Gamma_1^0) = \Gamma^0$, $\sum_p td_{p,m}\kappa_p$ is the map $\gamma' \mapsto \rho(\gamma')(tY_{\epsilon_m})\rho(\gamma')^{-1} - tY_{\epsilon_m}$ on Γ^0, an element of $\mathrm{Bd}(x)$ (as ${}^t(tY_{\epsilon_m}) = -tY_{\epsilon_m}$); by the definition of a Selmer basis $td_{p,m} = 0$ for all p, so $d = 0$.

Thus the $e+g^2$ elements $y_1,\dots,y_e$, $\omega_{p,m}$ of $\mathrm{Sel}^{1,\mathrm{b}}_{\mathbf{V}}(F)$ satisfy the hypothesis defining D_0, and $e + g^2 \leq D_0$. □

Lemma E.90. *Let $\mathscr{H}$ be a rank-n Hecke datum with $\ell \neq 2$, x a framed normalised curve point of shape $(a, n-a)$ satisfying (Pj1), $\kappa = (\kappa_1,\dots,\kappa_g)$ a Selmer basis of x with local witnesses, and $\mathbf{V}$ a level over $\mathscr{H}$. Let $z \in \mathrm{Sel}^{\mathrm{ev},\mathrm{b}}_{\mathbf{V}}(F)$ be such that, for every $1 \leq j \leq g$, the g-tuple with j-th entry z and all other entries 0 lifts. Then there is $y \in \mathrm{Sel}^{1,\mathrm{b}}_{\mathbf{V}}(F)$ such that, for every $\gamma \in \Gamma_1^0$, the component in $F\cdot 1$ of each entry of $y(\gamma)$ (for the decomposition $\mathrm{PD}_g(F) = F \oplus \bigoplus_j F\epsilon_j \oplus \bigoplus_j F\epsilon'_j \oplus F\eta$) is the corresponding entry of $z(\gamma)$.*

Proof. Write $\mathrm{PD} = \mathrm{PD}_g(F)$ and, for $u \in \mathrm{PD}$, $u = u_0 + \sum_i u_{\epsilon_i}\epsilon_i + \sum_i u_{\epsilon'_i}\epsilon'_i + u_\eta\eta$; the same subscripts on a matrix over PD denote the matrices of components. Let $\mathrm{S} = F \oplus \bigoplus_i F\epsilon_i$, a subring of PD (as $\epsilon_i\epsilon_k = 0$), and $\mathrm{E} = \bigoplus_i F\epsilon'_i \oplus F\eta$, an ideal of S-modules with $\mathrm{S\,E} \subseteq \mathrm{E}$; $u \mapsto u_0$ is a ring homomorphism $\mathrm{S} \to F$. Two facts on these rings are used throughout.

(i) For $u \in \mathrm{S}$, $\epsilon'_j u = u_0\epsilon'_j + u_{\epsilon_j}\eta$. Hence $u \in \mathrm{S}$ is 0 if and only if $u_0 = 0$ and $\epsilon'_j u = 0$ for all j.

(ii) For $S_1, S_2 \in \mathrm{M}_n(\mathrm{S})$ and $M \in \mathrm{M}_n(\mathrm{E})$, $(S_1MS_2)_{\epsilon'_i} = (S_1)_0 M_{\epsilon'_i}(S_2)_0$, since $\epsilon_k\epsilon'_i \in F\eta$ and $\eta\,\mathrm{E} = 0$.

Write $\rho_F = \rho \circ h$ and $\rho_{1,F} = \rho_1 \circ h = (1 + \sum_k \epsilon_k(\kappa_k \circ h))\rho_F$, with values in $\mathrm{GL}_n(\mathrm{S})$, inverse $\rho_F^{-1}(1 - \sum_k \epsilon_k(\kappa_k \circ h))$ and constant part ρ_F. For $v \in S_\ell$ let $Y_{k,v} \in \mathrm{Od}_a(A)$ be the given local witness of κ_k; then $w_v^1 = (1 - \sum_k \epsilon_k Y_{k,v})w_v \in \mathrm{GL}_n(\mathrm{S})$ with inverse $w_v^{-1}(1 + \sum_k \epsilon_k Y_{k,v})$ and constant part w_v. For $g \in D_v$ write $\phi g = \phi_{1,v}(g)$; as $h \circ \phi_{1,v} = \pi|_{D_v}$, $\rho_F(\phi g) = \rho(\pi g)$, and $T_g = w_v^{-1}\rho(\pi g)w_v$ is upper triangular with diagonal $(\psi_{v,1}(g), \dots, \psi_{v,n}(g))$ (the frame). On F and PD the ideal generated by T is the whole ring, so the continuity condition in the Selmer sets over F is void.

Construction of y. For each j choose $\Psi_j \in \mathrm{Sel}_{\mathbf{V}}^{2,\mathrm{b}}(F)$ whose ϵ_i'-coefficient is z for $i = j$ and 0 for $i \neq j$. Its values lie in $\mathrm{PW}_a^2(F) \subseteq \mathrm{M}_n(\mathrm{E})$, so $\Psi_j(\gamma) = \epsilon_j' z(\gamma) + \eta\, \Psi_j(\gamma)_\eta$ with $\Psi_j(\gamma)_\eta \in \mathrm{Od}_a(F)$. Put

$$y(\gamma) = z(\gamma) + \sum_{j=1}^{g} \epsilon_j\, \Psi_j(\gamma)_\eta \qquad (\gamma \in \Gamma_1^0).$$

As $z(\gamma) \in \mathrm{Ev}_a(F)$, $y(\gamma) \in \mathrm{PW}_a^1(F) \subseteq \mathrm{M}_n(\mathrm{S})$, $y(\gamma)_0 = z(\gamma)$, and by (i)

$$\epsilon_j'\, y(\gamma) = \Psi_j(\gamma) \qquad (1 \le j \le g,\ \gamma \in \Gamma_1^0). \qquad (*)$$

y is bounded: its coordinates are those of z and of the Ψ_j, all bounded, and finitely many bounded maps admit a common exponent.

Cocycle, polarisation and inertia conditions. The two sides of

$$y(\gamma\gamma') = y(\gamma) + \rho_{1,F}(\gamma) y(\gamma') \rho_{1,F}(\gamma)^{-1}$$

lie in $\mathrm{M}_n(\mathrm{S})$. Their constant parts agree because z is a cocycle for ρ_F, and their products with ϵ_j' agree by $(*)$ because Ψ_j is a cocycle for $\rho_{1,F}$ (ϵ_j' is central). By (i), applied entrywise, they are equal. In the same way $y(c_1\gamma c_1) = -{}^t y(\gamma)$ and $y|_{I_{1,u}} = 0$ follow from the corresponding properties of z and of the Ψ_j.

Local condition at $v \in S_\ell$: the discrepancy of the witnesses is triangular. Let $Y_v \in \mathrm{Ev}_a(F)$ be a witness for z and $Y^{(j)} \in \mathrm{PW}_a^2(F)$ one for Ψ_j. The ϵ_i'-components $P_{j,i} = Y^{(j)}_{\epsilon_i'}$ lie in $\mathrm{Ev}_a(F)$, and $Y_\eta^{(j)} \in \mathrm{Od}_a(F)$. Put $Q_{j,i} = P_{j,i} - \delta_{ij} Y_v \in \mathrm{Ev}_a(F)$. By (ii), the ϵ_i'-component of the local matrix $(w_v^1)^{-1}(\Psi_j(\phi g) - \rho_{1,F}(\phi g) Y^{(j)} \rho_{1,F}(\phi g)^{-1} + Y^{(j)}) w_v^1$ is $w_v^{-1}(\delta_{ij} z(\phi g) - \rho_F(\phi g) P_{j,i} \rho_F(\phi g)^{-1} + P_{j,i}) w_v$, which therefore has zero entries below the diagonal; subtracting δ_{ij} times the local matrix of z, the matrix $T_g Q' T_g^{-1} - Q'$ with $Q' = w_v^{-1} Q_{j,i} w_v$ is upper triangular for all $g \in D_v$, hence so is $T_g Q' - Q' T_g$ (multiply on the right by T_g). We show $Q'_{pq} = 0$ for $p > q$ by induction on $(n - p) + q$. If $p > q$ and the claim holds for smaller values, then $(T_g Q')_{pq} = \sum_{k \ge p} (T_g)_{pk} Q'_{kq} = \psi_{v,p}(g) Q'_{pq}$ and $(Q' T_g)_{pq} = \sum_{k \le q} Q'_{pk} (T_g)_{kq} = Q'_{pq} \psi_{v,q}(g)$, so $(\psi_{v,p}(g) - \psi_{v,q}(g)) Q'_{pq} = 0$ for all g; by (Pj1) some g has $\psi_{v,p}(g) \neq \psi_{v,q}(g)$, and $Q'_{pq} = 0$. Thus $C_j' = \sum_i \epsilon_i' w_v^{-1} Q_{j,i} w_v$ is upper triangular.

Local condition at v: the witness of y. Put $C_j = w_v^1 C_j' (w_v^1)^{-1}$. Expanding, $C_j = \sum_i \epsilon_i' Q_{j,i} + \eta \sum_k (Q_{j,k} Y_{k,v} - Y_{k,v} Q_{j,k})$; as $\mathrm{Ev}_a \cdot \mathrm{Od}_a$ and $\mathrm{Od}_a \cdot \mathrm{Ev}_a$ lie in Od_a, $C_j \in \mathrm{PW}_a^2(F)$ with $(C_j)_{\epsilon_i'} = Q_{j,i}$. Let

$$Y' = Y_v + \sum_j \epsilon_j\, Y_\eta^{(j)} \in \mathrm{PW}_a^1(F); \qquad \text{then} \qquad \epsilon_j' Y' = Y^{(j)} - C_j$$

by (i) (the ϵ_i'-components of both sides are $\delta_{ij} Y_v = P_{j,i} - Q_{j,i}$, the η-components $Y_\eta^{(j)}$). For $g \in D_v$ let $M_g = (w_v^1)^{-1}(y(\phi g) - \rho_{1,F}(\phi g) Y' \rho_{1,F}(\phi g)^{-1} + Y') w_v^1 \in \mathrm{M}_n(\mathrm{S})$ and $B_g = (w_v^1)^{-1} \rho_{1,F}(\phi g) w_v^1$. Its constant part is the local matrix $w_v^{-1}(z(\phi g) - \rho_F(\phi g) Y_v \rho_F(\phi g)^{-1} + Y_v) w_v$ of z, which vanishes below the diagonal, and on the diagonal when $g \in \mu_v$. By $(*)$,

$$\epsilon_j' M_g = (w_v^1)^{-1}(\Psi_j(\phi g) - \rho_{1,F}(\phi g) Y^{(j)} \rho_{1,F}(\phi g)^{-1} + Y^{(j)}) w_v^1 + (B_g C_j' B_g^{-1} - C_j').$$

The first summand is the local matrix of Ψ_j: zero below the diagonal, and on it when $g \in \mu_v$. B_g is upper triangular: its constant part is T_g, and its ϵ_k-component is $w_v^{-1}(\kappa_k(\pi g) - \rho(\pi g)Y_{k,v}\rho(\pi g)^{-1} + Y_{k,v})w_v \cdot T_g$, a product of upper triangular matrices by the local condition on κ_k with witness $Y_{k,v}$ (Definition E.34 (Sd2)). Likewise $B_g^{-1} = B_{g^{-1}}$ is upper triangular, and $(B_g)_{pp}(B_g^{-1})_{pp} = 1$ for all p (the entries commute). Hence $B_g C_j' B_g^{-1}$ is upper triangular with diagonal entries $(B_g)_{pp}(C_j')_{pp}(B_g^{-1})_{pp} = (C_j')_{pp}$, and the second summand is upper triangular with zero diagonal. By (i), applied entrywise, M_g has zero entries below the diagonal for every $g \in D_v$, and zero diagonal entries for $g \in \mu_v$. So Y' is a witness for y at v.

Hence $y \in \mathrm{Sel}_{\mathbf{V}}(\rho_{1,F}, w^1, \mathrm{PW}_a^1(F)) = \mathrm{Sel}_{\mathbf{V}}^1(F)$, y is bounded, and its constant part is z. $\square$

Lemma E.91. *Let $\mathscr{H}$ be a rank-n Hecke datum with $\ell \neq 2$, x a framed normalised curve point of shape $(a, n-a)$ satisfying (Pj1), $\mathbf{V}$ a level over $\mathscr{H}$ and $\mathfrak{R}$ a realisation of x over $\mathbf{V}$. Let $\alpha \in \mathrm{Tp}$, and suppose there is $Y \in \mathrm{M}_n(F)$ with*

$$\mathrm{al}(\alpha)(\gamma) = \rho(h(\gamma))\, Y\, \rho(h(\gamma))^{-1} - Y \qquad \textit{for all } \gamma \in \Gamma_1^0.$$

Then $\alpha = 0$.

Proof. Write $\rho_1(\gamma) = \rho(h(\gamma))$ for $\gamma \in \Gamma_1^0$, $\hat{b}$ for the image of $b \in B$ under $B \to B_{\ker f_1} \to \hat{B}$, and $P = \ker f_1$; by (Vr4), $\lambda(\hat{b}) = f_1(b) \in A \subseteq F$. Of (Pj1) only the first half is used: $\psi_{v,b} \neq \psi_{v,b'}$ for $b \neq b'$.

Step 1: a dual-number specialisation of the realisation. Let $F[\varepsilon] = F[X]/(X^2)$ and $\theta\colon \mathcal{A} = \hat{B}\cdot 1 \oplus N \to F[\varepsilon]$, $\theta(c\cdot 1 + y) = \lambda(c) + \varepsilon\alpha(c)$. It is a ring homomorphism: it is additive with $\theta(1) = 1$; for $y, y' \in N$ we have $yy' = c_0 \cdot 1$ with $c_0 \in \mathrm{ired}(N) \subseteq \hat{\mathfrak{m}}$ (by (Vr5)), so $\lambda(c_0) = 0$ and $\alpha(c_0) = 0$ (as $\alpha \in \mathrm{Tp}$), and the derivation rule of α gives $\theta((c+y)(c'+y')) = \lambda(cc') + \varepsilon\alpha(cc') = \theta(c+y)\theta(c'+y')$. Apply θ entrywise to matrices. By (Vr6), $\lambda(\mathrm{pr}_0(\nu(\gamma)_{ij})) = \rho_1(\gamma)_{ij}$, so by the definition of al and the hypothesis

$$\begin{aligned}\theta(\nu(\gamma)) = \rho_1(\gamma) + \varepsilon\,\mathrm{al}(\alpha)(\gamma)\rho_1(\gamma) &= \rho_1(\gamma) + \varepsilon(\rho_1(\gamma)Y - Y\rho_1(\gamma)) \\ &= W\rho_1(\gamma)W^{-1}, \qquad W = 1 - \varepsilon Y,\ W^{-1} = 1 + \varepsilon Y.\end{aligned}$$

Put $\delta(b) = \alpha(\hat{b})$ for $b \in B$. Then δ is additive, $\delta(bb') = f_1(b)\delta(b') + f_1(b')\delta(b)$, and δ vanishes on the image of the coefficient ring $\mathcal{O}_1$ of the realisation (by (Vr4) this image lies in $\iota(\mathcal{O}')$, and $\alpha \circ \iota = 0$). Let $\Theta\colon B[\Gamma_1^0] \to \mathrm{M}_n(F[\varepsilon])$ be θ composed with the B-linear extension of ν; it is a ring homomorphism (ν is multiplicative with $\nu(1) = 1$), $\Theta(b) = f_1(b) + \varepsilon\delta(b)$ for $b \in B$, $\Theta(\gamma) = W\rho_1(\gamma)W^{-1}$, and $\Theta(\mathcal{I}_1) = 0$ by (Vr6). Each element listed in (Vr3) lies in every ideal of the intersection defining $\mathcal{I}_1$, hence in $\mathcal{I}_1$, and is killed by Θ.

Step 2: δ vanishes on the generators of B listed in (Vr2).

- s: $\hat{s} \in \iota(\mathcal{O}')$ by (Vr4), so $\delta(s) = 0$.
- Coefficients of $\chi_D(\gamma)$, $\gamma \in \Gamma_1^0$: by (Vr6) the base change of D to $\mathcal{A}$ is $\det \circ \nu$, so the images in $\hat{B} \cdot 1$ of the coefficients c of $\chi_D(\gamma)$ are the coefficients of $\det(X - \nu(\gamma))$. Applying the ring homomorphism θ coefficientwise, the elements $f_1(c) + \varepsilon\delta(c)$ are the coefficients of $\det(X - W\rho_1(\gamma)W^{-1}) = \det(X - \rho_1(\gamma)) \in F[X]$; comparing ε-components, $\delta(c) = 0$.
- $\hat{\psi}_{v,b_0}(g)$, $v \in S_\ell$, $1 \leq b_0 \leq n$, $g \in D_v$: write $\tilde{\psi}_b(g') = \psi_{v,b}(g') + \varepsilon\beta_b(g')$ with $\beta_b(g') = \delta(\hat{\psi}_{v,b}(g'))$ (by (Vr1), $f_1 \circ \hat{\psi}_{v,b} = \psi_{v,b}$). For $g_1, \ldots, g_n \in D_v$, Θ kills $\prod_{b=1}^n(\phi_{1,v}(g_b) - \hat{\psi}_{v,b}(g_b))$; conjugating by W, $\prod_b(\rho_1(\phi_{1,v}g_b) - \tilde{\psi}_b(g_b)) = 0$. Since $h \circ \phi_{1,v} = \pi|_{D_v}$, $\rho_1(\phi_{1,v}g') = \rho(\pi g')$, and by the frame $w_v^{-1}\rho(\pi g')w_v$ is upper triangular with diagonal $(\psi_{v,1}(g'), \ldots, \psi_{v,n}(g'))$.

Conjugating by w_v, a product of upper triangular matrices vanishes; its (b_0,b_0) entry is $\prod_b(\psi_{v,b_0}(g_b)-\psi_{v,b}(g_b)-\varepsilon\beta_b(g_b))=0$. Take $g_{b_0}=g$ and, for $b\neq b_0$, $g_b\in D_v$ with $\psi_{v,b_0}(g_b)\neq\psi_{v,b}(g_b)$ (by (Pj1)). The factor $b=b_0$ is $-\varepsilon\beta_{b_0}(g)$, the others are $u_b+\varepsilon(\cdot)$ with $u_b\in F^\times$, so as $\varepsilon^2=0$ the entry is $-\varepsilon\beta_{b_0}(g)\prod_{b\neq b_0}u_b$, and $\beta_{b_0}(g)=0$.

- $U_{j,i}$ and $y_{j,i}$: fix j and put $u_i=f_1(U_{j,i})$. By (Vr2) the u_i are pairwise distinct and are the roots of $\det(X-\rho_1(\phi_j))$, so there is $Q\in\mathrm{GL}_n(F)$ with $Q^{-1}\rho_1(\phi_j)Q=\mathrm{diag}(u_1,\dots,u_n)$. Θ kills $\prod_i(\phi_j-U_{j,i})$; conjugating by W and then by Q, the diagonal matrix

$$\prod_i(\mathrm{diag}(u_1,\dots,u_n)-u_i-\varepsilon\delta(U_{j,i}))$$

vanishes, and its i_0-th entry is $-\varepsilon\delta(U_{j,i_0})\prod_{i\neq i_0}(u_{i_0}-u_i)$; so $\delta(U_{j,i_0})=0$ for all i_0, i.e. $\Theta(U_{j,i})=u_i$. Next, $\Theta(\tau_j)=W\rho_1(\tau_j)W^{-1}=1$ as $h(\tau_j)=1$, and $\Theta(y_{j,i})=\varepsilon\delta(y_{j,i})$ as $f_1(y_{j,i})=0$. So Θ applied to $\prod_{k\neq i}(\phi_j-U_{j,k})\cdot(\tau_j-1-y_{j,i})$ gives $-\varepsilon\delta(y_{j,i})\,W\prod_{k\neq i}(\rho_1(\phi_j)-u_k)W^{-1}=0$. Conjugating by W and Q, the i-th diagonal entry is $-\varepsilon\delta(y_{j,i})\prod_{k\neq i}(u_i-u_k)$, so $\delta(y_{j,i})=0$.

Step 3: $\delta=0$ on B. Let $B_0\subseteq B$ be the $\mathcal{O}_1$-subalgebra generated by the elements of (Vr2). The set $\{b:\delta(b)=0\}$ contains the image of $\mathcal{O}_1$ and, by the derivation rule, is closed under sums and products; by Step 2, $\delta(B_0)=0$. As B_0 is dense, $B=B_0+\mathfrak{m}_B^r$ for every $r\geq 1$. By Lemma E.92 there is $s_0\geq 0$ with $\delta(\mathfrak{m}_B^r)\subseteq T^{r-1-s_0}A$ for all $r\geq 1$. Given $b\in B$ and $r\geq 1$, write $b=b_0+b_r$ with $b_0\in B_0$, $b_r\in\mathfrak{m}_B^r$; then $\delta(b)=\delta(b_r)\in T^{r-1-s_0}A$. As $\bigcap_r T^{r-1-s_0}A=0$, $\delta(b)=0$.

Step 4: $\alpha=0$. For $b\in B$ and $t\in B\smallsetminus P$, the image e of b/t in $\hat{B}$ satisfies $\hat{b}=\hat{t}\,e$, so $0=\alpha(\hat{b})=\lambda(\hat{t})\alpha(e)+\lambda(e)\alpha(\hat{t})=f_1(t)\alpha(e)$, and $f_1(t)\neq 0$; thus α vanishes on the image of B_P. Since $\hat{B}$ is the completion of B_P at its maximal ideal, $\hat{B}=(\text{image of }B_P)+\hat{\mathfrak{m}}^2$; and $\alpha(\hat{\mathfrak{m}}^2)=0$, because $\alpha(xx')=\lambda(x)\alpha(x')+\lambda(x')\alpha(x)=0$ for $x,x'\in\hat{\mathfrak{m}}=\ker\lambda$. Hence $\alpha=0$. □

Lemma E.92. *Let $\mathfrak{R}$ be a realisation of a framed normalised curve point x over a level $\mathbf{V}$, and write $\hat{b}$ for the image of $b\in B$ in $\hat{B}$. For every $\alpha\in\mathrm{Tp}$ there is $s_0\geq 0$ such that $\alpha(\hat{b})\in T^{-s_0}A$ for all $b\in B$ and $\alpha(\hat{b})\in T^{j-1-s_0}A$ for all $j\geq 1$ and $b\in\mathfrak{m}_B^j$.*

Proof. Put $\delta(b)=\alpha(\hat{b})$ ($b\in B$). Since $\lambda(\hat{b})=f_1(b)$ (by (Vr4)) and α satisfies the derivation rule, δ is additive with $\delta(bb')=f_1(b)\delta(b')+f_1(b')\delta(b)$. The kernel $P=\ker f_1$ is finitely generated, $P=(p_1,\dots,p_t)$, as B is Noetherian. Given $b\in B$, (Vr4) gives $b'\in B$ with $f_1(b')=f_1(b)$ and $\hat{b}'\in\iota(\mathcal{O}')$, so $\delta(b')=0$ as $\alpha\circ\iota=0$; writing $b-b'=\sum_i c_ip_i$ with $c_i\in B$, $\delta(b)=\sum_i(f_1(c_i)\delta(p_i)+f_1(p_i)\delta(c_i))=\sum_i f_1(c_i)\delta(p_i)$. So $\delta(B)$ lies in the finitely generated A-submodule $\sum_i A\,\delta(p_i)$ of F, hence in $T^{-s_0}A$ for some $s_0\geq 0$. Next, $f_1(\mathfrak{m}_B)\subseteq TA$: $f_1(\mathfrak{m}_B)$ is an ideal of A (f_1 is onto), and it is proper, since $f_1(x)=1$ with $x\in\mathfrak{m}_B$ would give $f_1(1-x)=0$ for the unit $1-x$. For $x_1,\dots,x_j\in\mathfrak{m}_B$, $\delta(x_1\cdots x_j)=\sum_i(\prod_{l\neq i}f_1(x_l))\delta(x_i)\in T^{j-1}T^{-s_0}A$, and every element of $\mathfrak{m}_B^j$ is a finite sum of such products. □

Lemma E.93. *Let $\mathscr{H}$ be a rank-n Hecke datum with $\ell\neq 2$, x a framed normalised curve point of shape $(a,n-a)$ satisfying (Pj1), $\kappa=(\kappa_1,\dots,\kappa_g)$ a Selmer basis of x with local witnesses, $\mathbf{V}$ a level over $\mathscr{H}$, $\mathfrak{R}$ a realisation of x over $\mathbf{V}$, and $(m_1,\dots,m_g)$ a family in N dual to κ. Let $\Phi\in\mathrm{Ph}$ and, for $1\leq j\leq g$, $\alpha_j\colon\hat{B}\to F$, $\alpha_j(b)=\Phi(bm_j)-\lambda(b)\Phi(m_j)$. Then $\alpha_j\in\mathrm{Tp}$ for every j, and $(\mathrm{al}(\alpha_1),\dots,\mathrm{al}(\alpha_g))$ lifts.*

Proof. Notation. $\rho_F=\rho\circ h$; $\rho_{1,F}=\rho_1\circ h=(1+\sum_j\epsilon_j(\kappa_j\circ h))\rho_F$, a homomorphism into $\mathrm{GL}_n(\mathrm{PD}_g(F))$ (κ_j are cocycles and $\epsilon_i\epsilon_j=0$); $\gamma^*=c_1\gamma^{-1}c_1$; $\phi g=\phi_{1,v}(g)$ for $g\in D_v$, so

$\rho_F(\phi g) = \rho(\pi g)$, and $T_g = w_v^{-1}\rho(\pi g)w_v$ is upper triangular with diagonal $(\psi_{v,i}(g))_i$; $\hat{b}$ is the image of $b \in B$ in $\hat{B}$; $Y_{j,v} \in \mathrm{Od}_a(A)$ the local witnesses of κ_j; $J = \mathrm{diag}(1_a, -1_{n-a})$, so $JXJ = X$ for $X \in \mathrm{Ev}_a$ and $JXJ = -X$ for $X \in \mathrm{Od}_a$. Two facts on matrices: (F1) since each column of w_v has its non-zero entries in one block, $Jw_v = w_vE_v$ for a diagonal matrix E_v with entries ± 1, and $w_v^{-1}Xw_v$ has zero diagonal for $X \in \mathrm{Od}_a(F)$ (Xw_ve_p lies in the block other than that of w_ve_p, which is spanned by the w_ve_q with $q \neq p$); (F2) $\rho(\gamma^*) = {}^t\rho(\gamma)$ (normalised point), and for each j, from $\kappa_j(\bar{c}\delta\bar{c}) = -{}^t\kappa_j(\delta)$ and the cocycle rule, $\kappa_j(\delta^*) = -{}^t\kappa_j(\delta^{-1}) = {}^t\rho(\delta)\,{}^t\kappa_j(\delta)\,{}^t\rho(\delta)^{-1}$; hence $\rho_{1,F}(\gamma^*) = {}^t\rho_{1,F}(\gamma)$.

Step 1: derivations attached to Φ. For $y \in N$ let $D(y)(b) = \Phi(by) - \lambda(b)\Phi(y)$ ($b \in \hat{B}$), so $\alpha_j = D(m_j)$. We show $D(y) \in \mathrm{Tp}$. Additivity is clear, and $D(y)(\iota(o)) = 0$ as $\Phi \in \mathrm{Ph}$. Every $b \in \hat{B}$ is $\iota(o) + q$ with $q \in \hat{\mathfrak{m}}$ ((Vr4): $\lambda \circ \iota$ is onto, $\ker\lambda = \hat{\mathfrak{m}}$), and $\Phi(\iota(o)y') = \lambda(\iota(o))\Phi(y')$, $\Phi(\hat{\mathfrak{m}}^2N) = 0$ (as $\hat{\mathfrak{m}}^2 \subseteq \mathfrak{a}$). For $b = \iota(o) + q$, $b' = \iota(o') + q'$ this gives $D(y)(bb') = \lambda(\iota o)\Phi(q'y) + \lambda(\iota o')\Phi(qy) = \lambda(b)D(y)(b') + \lambda(b')D(y)(b)$. For $c \in \mathrm{ired}(N) \subseteq \mathfrak{a} \cap \hat{\mathfrak{m}}$, $D(y)(c) = 0$. So $D(y) \in \mathrm{Tp}$. Moreover D is additive in y, and $D(cy) = \lambda(c)D(y)$ for $c \in \hat{B}$ (rewrite the derivation rule for $D(y)$ at (b, c)). Hence D vanishes on $\hat{\mathfrak{m}}N$; as N is a finitely generated $\hat{B}$-module ((Vr5)), the F-span W of $D(N)$ is finite-dimensional. Let $\beta_1, \dots, \beta_e$ be a basis of W and define $\mu_k \colon N \to F$ by $D(y) = \sum_k \mu_k(y)\beta_k$; then $\mu_k \in \mathrm{La}$, and

$$\Phi(by) = \lambda(b)\Phi(y) + \sum_{k=1}^{e} \beta_k(b)\mu_k(y) \qquad (b \in \hat{B},\ y \in N). \tag{$\star$}$$

Put $c_{kj} = \mu_k(m_j)$; then $\alpha_j = \sum_k c_{kj}\beta_k$, so $\mathrm{al}(\alpha_j) = \sum_k c_{kj}A_k$ with $A_k = \mathrm{al}(\beta_k)$ (Lemma E.87).

Step 2: the test deformation. Let $\theta \colon \mathcal{A} \to \mathrm{PD}_e(F)$, $\theta(b \cdot 1 + y) = \lambda(b) + \sum_k \epsilon'_k\beta_k(b) + \sum_k \epsilon_k\mu_k(y) + \eta\,\Phi(y)$. It is a ring homomorphism: for $y, y' \in N$, $yy' = c \cdot 1$ with $c \in \mathrm{ired}(N)$, killed by λ and the β_k; $\theta(y)\theta(y') = 0$; and the η-component of $\theta(b)\theta(y')$ is $\sum_k \beta_k(b)\mu_k(y')$, which by $(\star)$ is the η-component $\Phi(by') - \lambda(b)\Phi(y')$ needed, the other components matching by the derivation rule and $\mu_k \in \mathrm{La}$. Let $u(\gamma) = \theta(\nu(\gamma))$. By (Vr6), u is a homomorphism $\Gamma_1^0 \to \mathrm{GL}_n(\mathrm{PD}_e(F))$ and

$$u(\gamma) = \Big(1 + \sum_k \epsilon_k\,\mathrm{la}(\mu_k)(\gamma) + \sum_k \epsilon'_kA_k(\gamma) + \eta\,\Xi(\gamma)\Big)\rho_F(\gamma), \qquad \Xi(\gamma) = \big(\Phi(\mathrm{pr}_1(\nu(\gamma)_{pq}))\big)_{pq}\rho_F(\gamma)^{-1}.$$

Further: (U1) $u(\gamma^*) = {}^tu(\gamma)$ ((Vr6)); (U2) $u(\gamma) = 1$ for $\gamma \in I_{1,u}$ ($\gamma - 1 \in \mathcal{I}_1$); (U3) for $v \in S_\ell$ and $g_1, \dots, g_n \in D_v$, $\prod_i(u(\phi g_i) - \vartheta_{v,i}(g_i)) = 0$ with the homomorphisms $\vartheta_{v,i} = \theta \circ \hat{\psi}_{v,i} \colon D_v \to \mathrm{PD}_e(F)^\times$ (the ordinary elements of (Vr3) lie in $\mathcal{I}_1$); here $\vartheta_{v,i}(g) = \psi_{v,i}(g) + \sum_k \epsilon'_k\beta_k(\hat{\psi}_{v,i}(g))$, and $\vartheta_{v,i}(g) = \psi_{v,i}(g)$ for $g \in \mu_v$, since then $\hat{\psi}_{v,i}(g)$ lies in the image of $\mathcal{O}$ ((Vr1)), hence in $\iota(\mathcal{O}')$ ((Vr4)), where the β_k vanish; (U4) let σ_e be the ring automorphism of $\mathrm{PD}_e(F)$ with $\epsilon_k \mapsto -\epsilon_k$, $\epsilon'_k \mapsto \epsilon'_k$, $\eta \mapsto -\eta$, and $\tau_e(X) = J\sigma_e(X)J$ on matrices; by the shape clause of (Vr6) (entries in $\hat{B} \cdot 1$ at same-block positions, in N at the others), $\tau_e(u(\gamma)) = u(\gamma)$.

Step 3: conjugation. As (m_j) is dual to κ, $\mathrm{la}(\mu_k) = K_k + \rho_FZ_k\rho_F^{-1} - Z_k$ with $K_k = \sum_j c_{kj}(\kappa_j \circ h)$, $Z_k \in \mathrm{Od}_a(F)$, ${}^tZ_k = -Z_k$. Let $G = 1 + \sum_k \epsilon_kZ_k$; then $G^{-1} = 1 - \sum_k \epsilon_kZ_k = {}^tG$ and $\tau_e(G) = G$. Put $u'(\gamma) = Gu(\gamma)G^{-1}$. Expanding (only $\epsilon_k\epsilon'_k = \eta$ survives among products),

$$u'(\gamma) = \Big(1 + \sum_k \epsilon_kK_k(\gamma) + \sum_k \epsilon'_kA_k(\gamma) + \eta\,\Xi'(\gamma)\Big)\rho_F(\gamma), \qquad \Xi' = \Xi + \sum_k \big(Z_kA_k - A_k\rho_FZ_k\rho_F^{-1}\big).$$

u' is a homomorphism satisfying (U1) (as ${}^tG = G^{-1}$), (U2), (U3) (the $\vartheta_{v,i}$ are scalars) and $\tau_e(u') = u'$.

Step 4: transfer to $\mathrm{PD}_g(F)$. Let $\mathrm{PD}' = F \oplus \bigoplus_{j=1}^{g} F\tilde{\epsilon}_j \oplus \bigoplus_{k=1}^{e} F\tilde{\epsilon}'_k \oplus F\tilde{\eta}$ with $\tilde{\epsilon}_j\tilde{\epsilon}'_k = c_{kj}\tilde{\eta}$ and all other products of two of $\tilde{\epsilon}_j, \tilde{\epsilon}'_k, \tilde{\eta}$ zero, a commutative F-algebra. The F-linear maps

$\omega_e\colon \mathrm{PD}' \to \mathrm{PD}_e(F)$, $\tilde\epsilon_j \mapsto \sum_k c_{kj}\epsilon_k$, $\tilde\epsilon'_k \mapsto \epsilon'_k$, $\tilde\eta \mapsto \eta$, and $\omega_g\colon \mathrm{PD}' \to \mathrm{PD}_g(F)$, $\tilde\epsilon_j \mapsto \epsilon_j$, $\tilde\epsilon'_k \mapsto \sum_j c_{kj}\epsilon'_j$, $\tilde\eta \mapsto \eta$, fixing F, are ring homomorphisms (both send $\tilde\epsilon_j\tilde\epsilon'_k$ to $c_{kj}\eta$). Let $\mathfrak{n}' = \bigoplus_k F\tilde\epsilon'_k \oplus F\tilde\eta$; ω_e maps $\mathfrak{n}'$ isomorphically onto $\mathfrak{n}_e = \bigoplus_k F\epsilon'_k \oplus F\eta$. Hence:

(P) a matrix over PD' whose components in F and in the $F\tilde\epsilon_j$ vanish and whose image under ω_e is 0 is 0.

Let σ' be the automorphism of PD' with $\tilde\epsilon_j \mapsto -\tilde\epsilon_j$, $\tilde\epsilon'_k \mapsto \tilde\epsilon'_k$, $\tilde\eta \mapsto -\tilde\eta$, $\tau'(X) = J\sigma'(X)J$, and σ_g, τ_g likewise on $\mathrm{PD}_g(F)$; ω_e, ω_g intertwine them. Put

$$\tilde u(\gamma) = \Big(1 + \sum_j \tilde\epsilon_j \kappa_j(h\gamma) + \sum_k \tilde\epsilon'_k A_k(\gamma) + \tilde\eta\, \Xi'(\gamma)\Big)\rho_F(\gamma), \qquad \text{so } \omega_e(\tilde u) = u'.$$

$\tilde u$ is a homomorphism: in $\tilde u(\gamma\gamma') - \tilde u(\gamma)\tilde u(\gamma')$ the F-component vanishes (ρ_F is a homomorphism), the $\tilde\epsilon_j$-component is $(\kappa_j(h\gamma\gamma') - \kappa_j(h\gamma) - \rho(h\gamma)\kappa_j(h\gamma')\rho(h\gamma)^{-1})\rho_F(\gamma\gamma') = 0$, and the image under ω_e is 0 as u' is a homomorphism; apply (P). In the same way (P), (F2), (U1)–(U2) for u', $\kappa_j \in \mathrm{Od}_a$ and $\tau_e(u') = u'$ give $\tilde u(\gamma^*) = {}^t\tilde u(\gamma)$, $\tilde u = 1$ on $I_{1,u}$ (ρ is trivial on $I_u^x = h(I_{1,u})$ and κ_j vanishes there), and $\tau'(\tilde u) = \tilde u$. Let $u'' = \omega_g(\tilde u)$. Then u'' is a homomorphism with $u''(\gamma^*) = {}^t u''(\gamma)$, $u'' = 1$ on $I_{1,u}$, $\tau_g(u'') = u''$, and

$$u''(\gamma) = \Big(1 + \sum_j \epsilon_j \kappa_j(h\gamma) + \sum_j \epsilon'_j\, \mathrm{al}(\alpha_j)(\gamma) + \eta\, \Xi'(\gamma)\Big)\rho_F(\gamma),$$

using $\sum_k c_{kj}A_k = \mathrm{al}(\alpha_j)$ (Step 1).

Step 5: the lift Ψ. Let $\mathrm{E} = \bigoplus_j F\epsilon'_j \oplus F\eta \subseteq \mathrm{PD}_g(F)$, an ideal with $\mathrm{E}^2 = 0$, and $\Psi(\gamma) = u''(\gamma)\rho_{1,F}(\gamma)^{-1} - 1$. Since $\rho_{1,F}^{-1} = \rho_F^{-1}(1 - \sum_j \epsilon_j\kappa_j \circ h)$,

$$\Psi(\gamma) = \sum_j \epsilon'_j\, \mathrm{al}(\alpha_j)(\gamma) + \eta\Big(\Xi'(\gamma) - \sum_j \mathrm{al}(\alpha_j)(\gamma)\,\kappa_j(h\gamma)\Big) \in \mathrm{M}_n(\mathrm{E}),$$

so the ϵ'_j-coefficient of Ψ is $\mathrm{al}(\alpha_j)$. From $u''(\gamma\gamma') = u''(\gamma)u''(\gamma')$ and $\mathrm{E}^2 = 0$: $\Psi(\gamma\gamma') = \Psi(\gamma) + \rho_{1,F}(\gamma)\Psi(\gamma')\rho_{1,F}(\gamma)^{-1}$. From $u''(\gamma^*) = {}^t u''(\gamma)$ and (F2): $\Psi(\gamma^*) = {}^t\rho_{1,F}(\gamma)\,{}^t\Psi(\gamma)\,{}^t\rho_{1,F}(\gamma)^{-1}$; applied to γ^{-1} (note $c_1\gamma c_1 = (\gamma^{-1})^*$) together with $\Psi(\gamma^{-1}) = -\rho_{1,F}(\gamma)^{-1}\Psi(\gamma)\rho_{1,F}(\gamma)$, this gives $\Psi(c_1\gamma c_1) = -{}^t\Psi(\gamma)$. Ψ vanishes on $I_{1,u}$. As τ_g fixes u'' and $\rho_{1,F}$, it fixes Ψ; the τ_g-fixed elements of $\mathrm{M}_n(\mathrm{E})$ are those with same-block entries in $\bigoplus_j F\epsilon'_j$ and different-block entries in $F\eta$, i.e. $\mathrm{PW}_a^2(F)$. Over $\mathrm{PD}_g(F)$ the ideal generated by T is the whole ring, so continuity is void.

Step 6: Ψ is bounded. The maps $\mathrm{al}(\alpha_j)$ are bounded (Lemma E.88), and so are $\kappa_j \circ h$, $\rho_F^{\pm1}$ (entries in A, continuous) and the constants Z_k; products and sums of bounded maps are bounded. It remains to bound $\gamma \mapsto (\Phi(\mathrm{pr}_1(\nu(\gamma)_{pq})))_{pq}$. By (Vr8), $\nu(\Gamma_1^0) \subseteq \mathcal{P}_0 = \sum_r BP_r$ (finitely many P_r). For $P = \sum_r b_r P_r$ with $b_r \in B$ and $q_r = \mathrm{pr}_1((P_r)_{pq})$, $(\star)$ gives $\Phi(\mathrm{pr}_1(P_{pq})) = \sum_r (f_1(b_r)\Phi(q_r) + \sum_k \beta_k(\hat b_r)\mu_k(q_r))$. By Lemma E.92 (for each β_k) there is s such that these values lie in $T^{-s}A$ for all $P \in \mathcal{P}_0$, and in $T^{j-1-s}A$ when all $b_r \in \mathfrak{m}_B^j$ (then $f_1(b_r) \in T^jA$). As $\nu(\gamma\upsilon) - \nu(\gamma) \in \mathfrak{m}_B^j\mathcal{P}_0$ for $\upsilon \in V_j$ ((Vr8)), T^s times these entries lie in A and are T-adically continuous.

Step 7: the local condition at $v \in S_\ell$. Let $\tilde w = (1 - \sum_j \tilde\epsilon_j Y_{j,v})w_v$, so $\omega_g(\tilde w) = w_v^1$ and $w^e := \omega_e(\tilde w) = (1 - \sum_k \epsilon_k Y_k^e)w_v$ with $Y_k^e = \sum_j c_{kj}Y_{j,v}$. For $g \in D_v$ let $r_0(g) = (w^e)^{-1}(1 + \sum_k \epsilon_k K_k(\phi g))\rho_F(\phi g)w^e$ and $r_1(g) = (w^e)^{-1}u'(\phi g)w^e$, homomorphisms $D_v \to \mathrm{GL}_n(\mathrm{PD}_e(F))$ with $r_1 - r_0 \in \mathrm{M}_n(\mathfrak{n}_e)$. Expanding, $r_0(g) = T_g + \sum_k \epsilon_k M_k(g)T_g$ with $M_k(g) = \sum_j c_{kj}\, w_v^{-1}(\kappa_j(\pi g) - \rho(\pi g)Y_{j,v}\rho(\pi g)^{-1} + Y_{j,v})w_v$, upper triangular by (Sd2) of Definition E.34 and with zero diagonal by (F1) (the bracket lies in $\mathrm{Od}_a(F)$); so $r_0(g)$ is upper

triangular with diagonal $(\psi_{v,i}(g))_i$. Apply Lemma E.86 with $R = \mathrm{PD}_e(F)$, $R \to F$ the constant part (kernel nilpotent), $\mathfrak{n} = \mathfrak{n}_e$, the group D_v, r_0, r_1, $\vartheta_{0,i} = \psi_{v,i}$ and $\vartheta_{1,i} = \vartheta_{v,i}$: hypothesis 1 was just shown, 2 is (Pj1), 3 is (U3) for u' conjugated by w^e. It gives $Y_e \in \mathrm{M}_n(\mathfrak{n}_e)$ such that $X(g) = r_1(g)r_0(g)^{-1} - 1 - (r_0(g)Y_e r_0(g)^{-1} - Y_e)$ is upper triangular with diagonal entries $d_i(g) = (\vartheta_{v,i}(g) - \psi_{v,i}(g))\psi_{v,i}(g)^{-1} \in \mathfrak{n}_e$, and $d_i(g) = 0$ for $g \in \mu_v$ by (U3). Let $\tilde{Y} \in \mathrm{M}_n(\mathfrak{n}')$ with $\omega_e(\tilde{Y}) = Y_e$, $\tilde{r}_0(g) = \tilde{w}^{-1}(1+\sum_j \tilde{\epsilon}_j\kappa_j(\pi g))\rho_F(\phi g)\tilde{w}$, $\tilde{r}_1(g) = \tilde{w}^{-1}\tilde{u}(\phi g)\tilde{w}$, and $\tilde{X}(g)$ the analogue of $X(g)$; its entries lie in $\mathfrak{n}'$ (as $\tilde{u} \equiv (1 + \sum_j \tilde{\epsilon}_j\kappa_j \circ h)\rho_F$ modulo $\mathfrak{n}'$) and $\omega_e(\tilde{X}(g)) = X(g)$, so $\tilde{X}(g)$ is upper triangular with diagonal entries $\tilde{d}_i(g) = (\omega_e|_{\mathfrak{n}'})^{-1}(d_i(g))$, zero for $g \in \mu_v$. Applying ω_g and putting $Y^\flat = w_v^1\,\omega_g(\tilde{Y})\,(w_v^1)^{-1} \in \mathrm{M}_n(\mathrm{E})$,

$$\omega_g(\tilde{X}(g)) = (w_v^1)^{-1}\big(\Psi(\phi g) - \rho_{1,F}(\phi g)Y^\flat\rho_{1,F}(\phi g)^{-1} + Y^\flat\big)w_v^1$$

is upper triangular with diagonal entries $\omega_g(\tilde{d}_i(g)) \in \bigoplus_j F\epsilon'_j$, zero for $g \in \mu_v$ (indeed $d_i(g) = \sum_k \epsilon'_k\beta_k(\hat{\psi}_{v,i}(g))\psi_{v,i}(g)^{-1}$ has no η-component, so $\tilde{d}_i(g) \in \bigoplus_k F\bar{\epsilon}'_k$, which ω_g maps into $\bigoplus_j F\epsilon'_j$; this is what makes the prescribed diagonal entries fixed by σ_g below). The set $\mathcal{Z}$ of $Y \in \mathrm{M}_n(\mathrm{E})$ for which, for all $g \in D_v$, $(w_v^1)^{-1}(\Psi(\phi g) - \rho_{1,F}(\phi g)Y\rho_{1,F}(\phi g)^{-1} + Y)w_v^1$ is upper triangular with these diagonal entries is an affine subspace containing $Y^\flat$. It is τ_g-stable: τ_g fixes Ψ, $\rho_{1,F}$ and the elements of $\bigoplus_j F\epsilon'_j$, and $\tau_g(w_v^1) = (1-\sum_j \epsilon_j Y_{j,v})Jw_vJ = w_v^1E_vJ$ by (F1) and $JY_{j,v}J = -Y_{j,v}$, so $\tau_g((w_v^1)^{-1}Mw_v^1) = (E_vJ)(w_v^1)^{-1}\tau_g(M)w_v^1(E_vJ)$ with E_vJ diagonal of entries ±1, which preserves triangularity and diagonal entries. As 2 is invertible, $\frac{1}{2}(Y^\flat + \tau_g(Y^\flat)) \in \mathcal{Z}$ is τ_g-fixed, hence lies in $\mathrm{PW}_a^2(F)$; it is a witness for Ψ at v.

Hence $\Psi \in \mathrm{Sel}_{\mathbf{V}}^{2,\mathrm{b}}(F)$ with ϵ'_j-coefficient $\mathrm{al}(\alpha_j)$ for every j: the tuple $(\mathrm{al}(\alpha_1), \dots, \mathrm{al}(\alpha_g))$ lifts. □

E.4 Selmer bounds at the Taylor–Wiles levels

This section proves the two Selmer bounds along a Taylor–Wiles datum, the first-order dual Selmer bound (B1) (Proposition E.97) and the obstruction bound (B2) (Proposition E.119), and from them the conditions (K1)–(K3) at the patched level (Proposition E.94). Proposition E.97 is used in Proposition E.39 and Corollary E.66; Proposition E.94 is used in Corollary E.66 and Proposition E.134. The bounds come from Selmer counts at the levels, with the Greenberg–Wiles formula (Proposition E.100) and Mazur's finiteness condition for the Galois groups of a Hecke datum (Lemma E.116).

E.4.1 The patched level and the first-order dual Selmer bound

This subsection proves Proposition E.94 and Proposition E.97. The dual Selmer bound compares the c-symmetric cocycles at a level (Proposition E.98) with the local and global bounds for the modules of first-order matrices (Proposition E.105), using local Tate duality; the Greenberg–Wiles inequality at the levels is Proposition E.99, the ordinary local condition above ℓ has the expected size (Lemma E.111), and local conditions of Steinberg type are the unramified ones and are self-dual (Lemma E.103). The patched level is a limit of the levels, on whose pro-ℓ kernels the Selmer cocycles vanish (Lemma E.95, Lemma E.96); its conditions (K1)–(K3) follow from (B1), (B2) and the lemmas of Subsection E.4.2. The lemma E.116 is used again in Section E.5.

Proposition E.94. *Let $\mathscr{H}$, $\mathcal{O}$, $x = (k, \rho, 1_n, (\psi_{v,a}), f)$ with frame $((w_v), (I_u^x))$, κ with local witnesses, and a Taylor–Wiles datum of size q (with H, $\sigma_1, \dots, \sigma_q$) satisfy all hypotheses of Proposition E.97, and let $(\Pi, (\pi_N), \mathcal{U}, \bar{\theta}, c_\infty, (\Phi_j), (\mathrm{T}_j), (F_w))$ be limit objects for this Taylor–Wiles datum as in the conclusion of Lemma E.42. Put*

$$\Gamma_\infty = \{(g,\gamma) \in \bar{\Gamma}_\infty \times \Gamma_{\mathscr{H}} :\ \bar{\theta}(g) = \bar{\gamma}\}, \qquad h(g,\gamma) = \gamma, \qquad c'_\infty = (c_\infty, \pi_{\mathscr{H}}(c)),$$

*($\bar\gamma$ the image of γ in $\bar\Gamma$), $\phi_{\infty,v}(y) = (F_v(y), \pi_{\mathscr H}(y))$ for $v \in S_\ell$ and $y \in D_v$, $I_{\infty,u} = \{(F_u(y), \pi_{\mathscr H}(y)) : \ y \in I_{\bar u} \cap G_L \cap \pi_{\mathscr H}^{-1}(I_u^x)\}$ for $u \in \mathrm{St}$, $\phi_j = (\Phi_j, \sigma_j)$, $\tau_j = (\mathrm{T}_j, 1)$ ($1 \le j \le q$) and $P_1 = \{1\} \times \mathrm{PL}(H)$. Then $\mathbf{V}_\infty = (\Gamma_\infty, h, c'_\infty, (\phi_{\infty,v}), (I_{\infty,u}))$ is a level over $\mathscr H$, the **limit level formed from these limit objects**; $\phi_j, \tau_j \in \Gamma_\infty^0$; P_1 is a closed normal subgroup of Γ_∞ contained in Γ_∞^0; and:*

(P1) *$h(\phi_j) = \sigma_j$, $h(\tau_j) = 1$, $\phi_j\tau_j = \tau_j\phi_j$, and $\ker h$ is the closure of the subgroup generated by the conjugates of $\tau_1, \dots, \tau_q$;*

(P2) *Γ_∞/P_1 is topologically finitely generated, $h(P_1) = \mathrm{PL}(H)$, and every element of each of the six Selmer sets of $\mathbf{V}_\infty$ over A and over every A_m vanishes on P_1;*

(P3) *(K1) with $D_0 = n[L^+ : \mathbb{Q}] + nq$, (K2) and (K3) hold at $\mathbf{V}_\infty$;*

(P4) *Γ_∞ and Γ_∞^0 satisfy Φ_ℓ.*

Proof. Let Π, π_N, $\mathcal{U}$, $\bar\Gamma_\infty = \Pi/K_\infty$, π_∞, $\bar\theta$, c_∞, Φ_j, T_j, F_w be the given limit objects, $\theta = \bar\theta \circ \pi_\infty$, and let the notation $\bar\Gamma, \bar\Gamma_N, \bar h_N, \theta_N = \bar h_N \circ \pi_N, \bar\phi_{N,w}$ be as in Lemma E.42; $\bar\Gamma^0$ is the image of $\Gamma^0_{\mathscr H}$.

The level. Γ_∞ is a closed subgroup of $\bar\Gamma_\infty \times \Gamma_{\mathscr H}$, and h is onto as $\bar\theta$ is. $c'_\infty \in \Gamma_\infty$ is an involution with $h(c'_\infty) = \pi_{\mathscr H}(c)$, by (L2) ($c_\infty^2 = 1$, $\bar\theta(c_\infty) = \bar c$). $\phi_{\infty,v}$ ($v \in S_\ell$) is a continuous homomorphism into Γ_∞^0 with $h \circ \phi_{\infty,v} = \pi_{\mathscr H}|_{D_v}$, by (L2) ($\bar\theta \circ F_v = \overline{\pi_{\mathscr H}}|_{D_v}$). For $u \in \mathrm{St}$ put $J'_u = I_{\bar u} \cap G_L \cap \pi_{\mathscr H}^{-1}(I_u^x)$, a closed subgroup of D_u; then $I_{\infty,u} = \{(F_u(y), \pi_{\mathscr H}(y)) : \ y \in J'_u\}$ is a closed subgroup of Γ_∞^0 (it lies in Γ_∞ by (L2)) with $h(I_{\infty,u}) = I_u^x$ (as $I_u^x \subseteq \pi_{\mathscr H}(I_{\bar u} \cap G_L)$). So $\mathbf{V}_\infty$ is a level. $P_1 = \{1\} \times \mathrm{PL}(H)$ lies in Γ_∞ (as $\bar\theta(1) = 1$ is the image of $\mathrm{PL}(H)$ in $\bar\Gamma$), in Γ_∞^0 since $\mathrm{PL}(H) \subseteq H \subseteq \Gamma^0_{\mathscr H}$, and is closed and normal in Γ_∞ as $\mathrm{PL}(H)$ is closed and normal in $\Gamma_{\mathscr H}$.

(P1). By (L2), $\phi_j, \tau_j \in \Gamma_\infty$, $h(\phi_j) = \sigma_j$, $h(\tau_j) = 1$, and they commute. $\ker h = \ker\bar\theta \times \{1\}$, and conjugation by (g, γ) maps $(\mathrm{T}_j, 1)$ to $(g\mathrm{T}_j g^{-1}, 1)$; so by (L2) $\ker h$ is the closure of the subgroup generated by the conjugates of the τ_j.

(P2). The first projection $\Gamma_\infty \to \bar\Gamma_\infty$ is onto with kernel P_1, and $\bar\Gamma_\infty$ is a quotient of Π. ρ and the κ_j are trivial on $\mathrm{PL}(H)$ (hypothesis and Lemma E.96 (Pv1)), and $\chi_{\mathrm{cyc}} \equiv 1 \bmod \ell$ on H (Tg1); so P_1 acts trivially on all coefficient modules. Over A_m the coefficient modules are finite and killed by ℓ, so Lemma E.95 (with $E = \Gamma_\infty$, $G = \Gamma_{\mathscr H}$, $E' = \Gamma_\infty^0$) shows that every Selmer cocycle vanishes on P_1; over A all reductions modulo T^m vanish there, hence so does the cocycle.

Transfer. Fix m. Let $\mathcal{M}$ be the finitely many modules $\mathrm{PW}^1_a(A_m)$, $\mathrm{PW}^2_a(A_m)$, $\mathrm{Ev}_a(A_m)$, $\mathrm{Od}_a(A_m)$; $\bar\Gamma^0$ acts on each through $(\rho_1, \chi_{\mathrm{cyc}})$ modulo (T^m, ℓ), which factors through $\bar\Gamma^0$ by the above; let $Y_0 \subseteq \bar\Gamma^0$ be an open normal subgroup of $\bar\Gamma$ acting trivially on all of them and contained in $\bar H = H/\mathrm{PL}(H)$. Let $X = \theta^{-1}(Y_0)$, open normal in Π, hence topologically finitely generated, and let V be the closure of $X^\ell[X, X]$: open (as X/V is a finitely generated elementary abelian ℓ-group) and normal in Π (characteristic in X). Let $\Pi^0 = \theta^{-1}(\bar\Gamma^0)$. By (L1) and (L2), for $\mathcal{U}$-many N: $(\ker\pi_N)V = K_\infty V$; $\theta_N \equiv \theta \bmod Y_0$, so $\pi_N^{-1}(\bar\Gamma_N^0) = \Pi^0$ and Π^0 acts on each $M \in \mathcal{M}$ in the same way through θ_N and through θ, with X acting trivially; the lifts of $\bar c_N$ and of c_∞ can be chosen congruent modulo V; and (iv) of (L1) holds for V. Fix such N. By Lemma E.96 (Pv2), every element of a Selmer set of $\mathbf{V}_N$ over A_m vanishes on $\mathrm{PL}(H_N)$, hence is $\bar z \circ$ (projection) for a cocycle $\bar z$ on $\bar\Gamma_N^0$; by (P2) every element of a Selmer set of $\mathbf{V}_\infty$ over A_m is $\bar z \circ$ (first projection) for a cocycle $\bar z$ on $\bar\Gamma_\infty^0$. In both cases $\tilde z = \bar z \circ \pi_N$, resp. $\bar z \circ \pi_\infty$, is a cocycle of Π^0 for the common action; its restriction to X is a homomorphism into a group of exponent ℓ, so it vanishes on V, and as V acts trivially, $\tilde z$ is constant on cosets of $(\ker\pi_N)V = K_\infty V$. Thus both Selmer sets are

identified, via $\bar z \mapsto \tilde z$, with sets of cocycles of the finite group $\Pi^0/K_\infty V$. These sets coincide: the polarisation conditions use conjugation by lifts of $\bar c_N$, resp. c_∞, which agree modulo $K_\infty V$; the conditions at $u \in \mathrm{St}$ (vanishing on $\bar\phi_{N,u}(J'_u)$, resp. $F_u(J'_u)$) and at $v \in S_\ell$ (the witness conditions on the values at $\bar\phi_{N,v}(y)$, resp. $F_v(y)$, with the same matrix $\rho_1(\pi_{\mathscr H} y)$) agree because by (iv) these values are values of $\tilde z$ at elements congruent modulo V. The coboundaries Bd^1 correspond as well (same action). Hence, for $\mathcal U$-many N, there are bijections $\mathrm{Sel}^*_{\mathbf V_N}(A_m) \to \mathrm{Sel}^*_{\mathbf V_\infty}(A_m)$ $(* = 1, \mathrm{ev}, 2)$ mapping Bd^1 onto Bd^1 and compatible with taking ϵ'_j-coefficients and with A-linear combinations.

(P3). By Proposition E.97, (B1) holds with $D_0 = n[L^+ : \mathbb Q] + nq$ and some C; for given m choose N in the $\mathcal U$-set above with $N \geq N_0(m)$ ($\mathcal U$ is non-principal): $\mathrm{Sel}^1_{\mathbf V_\infty}(A_m)$ is finite with $\#\mathrm{Sel}^1_{\mathbf V_\infty}(A_m) \leq \#\mathrm{Bd}^1_{\mathbf V_\infty}(A_m)(\#k)^{mD_0+C}$. By Proposition E.119 (whose hypotheses are among those of Proposition E.97), (B2) holds with some C; transporting a family $(z_{i,j})$ in $\mathrm{Sel}^{\mathrm{ev}}_{\mathbf V_\infty}(A_m)$ to such a level N, applying (B2) and transporting Ψ back gives the hypothesis of Lemma E.130 at $\mathbf V_\infty$. With (P2), Lemma E.129 gives (K1) and Lemma E.130 gives (K2). The characteristic polynomial of $\rho(\sigma_j)$ has the pairwise distinct roots $\alpha_{j,i} \in A$ (eigenvalue data), so (P1) and Lemma E.131 give (K3).

(P4). We apply Lemma E.132 to $G_1 = \bar\Gamma_\infty$ (a quotient of Π, hence topologically finitely generated), $G_2 = \Gamma_{\mathscr H}$ (which satisfies Φ_ℓ by Lemma E.116), $\bar G = \bar\Gamma = \Gamma_{\mathscr H}/\mathrm{PL}(H)$, $\theta = \bar\theta$, q the projection, and $\bar G_0 = \bar H = H/\mathrm{PL}(H)$, open in $\bar\Gamma$. It remains to see that $H^2(V, \mathbb Z/\ell)$ is finite for every open subgroup V of $\bar H$. Let K, K_*, E be the fixed fields in $\overline{\mathbb Q}$ of $N_{\mathscr H}$, $\pi^{-1}_{\mathscr H}(H)$, $\pi^{-1}_{\mathscr H}(\mathrm{PL}(H))$, so $K_* \subseteq E \subseteq K$, $K_* \supseteq L$ (as $H \subseteq \Gamma^0_{\mathscr H}$) and $\bar H = \mathrm{Gal}(E/K_*)$. (i) *$E$ is the maximal pro-ℓ extension of K_* inside K.* For an open normal U of $\Gamma_{\mathscr H}$ as in the definition of $\mathrm{PL}(H)$, $H/(H \cap U)$ is a finite group of ℓ-power exponent, hence an ℓ-group, and $U = H$ is allowed, so $\mathrm{PL}(H) \subseteq H$ and $\bar H$ is pro-ℓ. Conversely, if $W \subseteq H$ is open normal in H with H/W an ℓ-group, its core $W' = \bigcap_g gWg^{-1}$ (finitely many conjugates, all in H as H is normal) is open normal in $\Gamma_{\mathscr H}$ and H/W' embeds into a product of ℓ-groups, so $\mathrm{PL}(H) \subseteq W' \subseteq W$. (ii) *$E/K_*$ is unramified at the finite places not above ℓ.* Let $\bar w \nmid \ell$ and $s \in I_{\bar w} \cap G_{K_*}$; then $s \in G_L$, each $\bar\chi_a(s)$ lies in the multiplicative group of a finite field of characteristic ℓ, so for some m prime to ℓ all $\bar\chi_a(s^m) = 1$ and $s^m \in N_{\mathscr H}$ acts trivially on $K \supseteq E$; the image of s in the pro-ℓ group $\mathrm{Gal}(E/K_*)$ has order dividing m, hence is trivial. (iii) *$E = M$, the maximal pro-ℓ extension of K_* unramified at the finite places not above ℓ.* M is Galois over L^+ (K_* is, H being normal). A generator $s \in I_{\bar w} \cap G_L$ ($\bar w \nmid \ell$, $\bar\chi_a(s) = 1$) of $N_{\mathscr H}$ fixes $K \supseteq K_*$, so its image in $\mathrm{Gal}(M/L^+)$ lies in the inertia group of $\bar w$ in $\mathrm{Gal}(M/K_*)$, which is trivial; hence $N_{\mathscr H}$ fixes M, $M \subseteq K$, and $M \subseteq E$ by (i); $E \subseteq M$ by (i) and (ii). (iv) Let $V = \mathrm{Gal}(E/K'')$ be open in $\bar H$, K''/K_* finite inside E, and S the set of places of K'' above ℓ. E/K'' is pro-ℓ and unramified outside S, so $E \subseteq K''_S(\ell)$. Conversely the Galois closure $\tilde N$ of $K''_S(\ell)$ over K_* is the compositum of the fields $\sigma(K''_S(\ell)) = \sigma(K'')_S(\ell)$ with $\sigma(K'') \subseteq E$; each $\sigma(K'')_S(\ell)E/E$ is Galois pro-ℓ, so $\tilde N/E$ is pro-ℓ, $\tilde N/K_*$ is pro-ℓ (an extension of pro-ℓ groups) and unramified at the finite places not above ℓ; so $\tilde N \subseteq M = E$. Hence $V = G_S(\ell)$ for K'', and $H^2(V, \mathbb Z/\ell)$ is finite by Theorem B.38. So Lemma E.132 applies: Γ_∞ satisfies Φ_ℓ, and so does its open subgroup $\Gamma^0_\infty = h^{-1}(\Gamma^0_{\mathscr H})$ (open subgroups of Γ^0_∞ are open in Γ_∞). □

Lemma E.95. *Let ℓ be a prime, G a profinite group, $H \subseteq G$ an open normal subgroup, E a profinite group with a continuous surjective homomorphism $h\colon E \to G$, and $P_1 \subseteq h^{-1}(H)$ a closed normal subgroup of E such that $h|_{P_1}$ is injective and $h(P_1) = \mathrm{PL}(H)$. Let $E' \subseteq E$ be a subgroup containing $h^{-1}(H)$, M a finite abelian group of ℓ-power order with an action of E' that is trivial on P_1, and $z\colon E' \to M$ a continuous map with $z(\gamma\gamma') = z(\gamma) + \gamma z(\gamma')$. Then z vanishes on P_1. In particular, for $E = E_0 \times_{G/\mathrm{PL}(H)} G$ (fibre product over a continuous surjection $E_0 \to G/\mathrm{PL}(H)$, with h the second projection), this applies to $P_1 = \{1\} \times \mathrm{PL}(H)$.*

Proof. Write $P = \mathrm{PL}(H)$. As P_1 acts trivially, $f = z|_{P_1}$ is a continuous homomorphism, and for $g \in E'$, $p \in P_1$: $z(gpg^{-1}) = z(g) + g\,z(p) + gp\,z(g^{-1}) = g\,z(p)$ (using $z(g^{-1}) = -g^{-1}z(g)$ and $gp\,z(g^{-1}) = g\,z(g^{-1})$). Let $f' = f \circ (h|_{P_1})^{-1} \colon P \to M$, a continuous homomorphism ($h|_{P_1}$ is a continuous bijection of compact spaces, hence a homeomorphism). For $\eta \in H$ choose $g \in h^{-1}(\eta) \subseteq E'$; then $gP_1g^{-1} = P_1$ and $f'(\eta q \eta^{-1}) = g \cdot f'(q)$ for $q \in P$, so $N = \ker f'$ is a closed subgroup of P normalised by H, i.e. normal in H. The group H/N is an extension of H/P by P/N; H/P is pro-ℓ (Lemma E.45 (Pl1)) and P/N embeds in M, a finite ℓ-group; hence H/N is pro-ℓ. So N is the intersection of the open normal subgroups of H containing it, each of ℓ-power index, and therefore $N \supseteq P$ by (Pl1). So $f' = 0$ and z vanishes on P_1. For the fibre product, $\{1\} \times P$ is closed and normal, lies in $h^{-1}(H)$, and the second projection maps it bijectively onto P. □

Lemma E.96. *Let $\mathscr{H}$ be a rank-n Hecke datum, $x = (k, \rho, 1_n, (\psi_{v,a}), f)$ a framed normalised curve point of shape $(a, n-a)$ and $\kappa = (\kappa_1, \dots, \kappa_g)$ a Selmer basis of x with local witnesses. Fix a Taylor–Wiles datum with open normal subgroup $H \subseteq \Gamma_{\mathscr{H}}$, and assume that $\rho(\gamma) = 1$ for $\gamma \in \Gamma^0_{\mathscr{H}} \cap \mathrm{PL}(H)$. Put $H_N = h_N^{-1}(H)$. Then:*

(Pv1) $\mathrm{PL}(H) \subseteq H \subseteq \Gamma^0_{\mathscr{H}}$, *and $\kappa_j(\gamma) = 0$ and $\rho_1(\gamma) = 1$ for $\gamma \in \mathrm{PL}(H)$;*

(Pv2) *for every $N \geq 1$, $h_N(\mathrm{PL}(H_N)) \subseteq \mathrm{PL}(H)$, and for $X = A$ and for $X = A_m$ ($m \geq 1$) every element of $\mathrm{Sel}^{\mathrm{ev}}_{\mathbf{V}_N}(X)$, $\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_N}(X)$, $\mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}_N}(X)$, $\mathrm{Sel}^{1}_{\mathbf{V}_N}(X)$, $\mathrm{Bd}^{1}_{\mathbf{V}_N}(X)$, $\mathrm{Sel}^{2}_{\mathbf{V}_N}(X)$ vanishes on $\mathrm{PL}(H_N)$; the same holds at the base level with $\mathrm{PL}(H)$.*

Proof. (Pv1). $H \subseteq \Gamma^0_{\mathscr{H}}$ by (Tg1), and $\mathrm{PL}(H) \subseteq H$ by Lemma E.45 (Pl1); so $\rho = 1$ on $\mathrm{PL}(H)$. Fix j and $m \geq 1$. The map $\kappa_j \bmod T^m \colon \Gamma^0_{\mathscr{H}} \to \mathrm{Od}_a(A_m)$ is continuous (its entries are T-adically continuous and A_m is discrete) and a cocycle for the action $\gamma \cdot Z = \rho(\gamma) Z \rho(\gamma)^{-1}$; $\mathrm{Od}_a(A_m)$ is finite of order a power of $\#k$, hence of ℓ, and $\mathrm{PL}(H)$ acts trivially. By Lemma E.45 (Pl2) (with $G = \Gamma_{\mathscr{H}}$, $\Gamma' = \Gamma^0_{\mathscr{H}}$), $\kappa_j \equiv 0 \bmod T^m$ on $\mathrm{PL}(H)$ for every m, so $\kappa_j = 0$ there, and $\rho_1 = (1 + \sum_j \epsilon_j \kappa_j)\rho = 1$ there.

(Pv2). H is open and normal in $\Gamma_{\mathscr{H}}$ and h_N is a continuous surjection, so $h_N(\mathrm{PL}(H_N)) \subseteq \mathrm{PL}(H)$ by (Pl3); also $H_N \subseteq h_N^{-1}(\Gamma^0_{\mathscr{H}}) = \Gamma^0_N$ is open and normal in Γ_N. Hence on $\mathrm{PL}(H_N)$ the maps $\rho \circ h_N$ and $\rho_1 \circ h_N$ are trivial, so $\mathrm{PL}(H_N)$ acts trivially on the matrix modules $\mathrm{Ev}_a(X)$, $\mathrm{Od}_a(X)$, $\mathrm{PW}^1_a(X)$, $\mathrm{PW}^2_a(X)$ by conjugation. A coboundary $\gamma \mapsto \varrho(\gamma) Y \varrho(\gamma)^{-1} - Y$ vanishes where ϱ is trivial; this treats $\mathrm{Bd}^{\mathrm{od}}$ and Bd^1. Let z be an element of one of the four sets of cocycles. For $X = A_m$ the module is finite of ℓ-power order and z is continuous (discrete topology), so (Pl2) (with $G = \Gamma_N$, $\Gamma' = \Gamma^0_N$, H_N) gives $z = 0$ on $\mathrm{PL}(H_N)$. For $X = A$, $z \bmod T^m$ is such a cocycle for $X = A_m$ (entries continuous, values reduced), so it vanishes on $\mathrm{PL}(H_N)$ for every m, hence so does z. The base level is the case $h = \mathrm{id}$, $\Gamma_1 = \Gamma_{\mathscr{H}}$, with (Pv1). □

Proposition E.97. *Let $\mathscr{H}$, $\mathcal{O}$ (a discrete valuation ring), $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$, $x = (k, \rho, 1_n, (\psi_{v,a}), f)$, κ with local witnesses, and a Taylor–Wiles datum of size q (with H and $\sigma_1, \dots, \sigma_q$) along which x has eigenvalue data be as in Proposition E.119, with $\rho(\gamma) = 1$ for $\gamma \in \Gamma^0_{\mathscr{H}} \cap \mathrm{PL}(H)$. Assume moreover that x satisfies (Np4) and (Pj1) of Definition 50.9, that for every $Z \in \mathrm{Ev}_a(A_k)$ some $T^s Z$ lies in the A_k-span of $\rho(\Gamma')$ (Γ' as in Definition 50.9), and that $\{\sigma_1, \dots, \sigma_q\}$ detects (x, κ). Then (B1) holds for x, κ and the levels $\mathbf{V}_N$ of the datum, with $D_0 = n[L^+ : \mathbb{Q}] + nq$.*

Proof. Write $K = \#k$, $d = [L^+ : \mathbb{Q}]$, and let $M_{N,m}$, $M'_{N,m}$, $b_{N,m}$, Zb_v, ν and the constants c_1', C_v, C_q, c_0, c_2 be as in Proposition E.105 (the action on $M_{N,m}$ is that of Lemma E.107 (Rf2)). Put $C = c_1' + c_2 + \sum_{v \in S_\ell} C_v + qC_q + c_0$; it does not depend on m or N. Fix $m \geq 1$ and let N_0

be an integer at least as large as each of the integers N_0 given by (Bs3) for $j = 1, \dots, q$ and by (Bs6) for this m. Let $N \geq N_0$ and write $M = M_{N,m}$, $M' = M'_{N,m}$.

Step 1: the setting of Proposition E.98. Take $Q = Q_N$ and $T = S_\ell \cup \mathrm{St} \cup Q_N \cup \{v_0\}$, so $T^\circ = \{v_0\}$: by (Tg2) the $v_{N,j}$ lie outside $S_\ell \cup \mathrm{St} \cup \{v_0\}$, are not above ℓ and split in L; v_0 splits in L by (Rd2). Then $\Gamma^{Q_N} = \Gamma_N$ and $\Gamma^{0,Q_N} = \pi^{Q_N}(G_L) = h_N^{-1}(\Gamma^0_{\mathscr{H}})$ (as $\ker h_N = \pi^{Q_N}(N_{\mathscr{H}})$ and $N_{\mathscr{H}} \subseteq G_L$), and $\iota_v = \pi^{Q_N}|_{D_v}$ is the map $\phi_{N,v}$ of the level $\mathbf{V}_N$. By Lemma E.107 (Rf2) and (Bs0), M, M', $b_{N,m}$ satisfy the hypotheses of Proposition E.98 (M is killed by ℓ, as $\mathrm{char}\, k = \ell$). Take the local conditions $\mathcal{L}_v = B^1(D_v, M) + \mathrm{Zb}_v$ ($v \in S_\ell$), $\mathcal{L}_u = Z^1_{J_{\bar{u}}}(M)$ ($u \in \mathrm{St}$), $\mathcal{L}_w = Z^1(D_w, M)$ ($w \in Q_N \cup \{v_0\}$).

Step 2: $\mathrm{Sel}^1_{\mathbf{V}_N}(A_m) \subseteq \mathrm{Sel}$ *and* $\mathrm{Bd}^1_{\mathbf{V}_N}(A_m) = \mathrm{Bd}$ (the sets of Proposition E.98). Let $z \in \mathrm{Sel}^1_{\mathbf{V}_N}(A_m)$. The T-adic topology of A_m is discrete, so z is continuous; it is a cocycle for the action on M restricted to Γ^0_N, and $z(c_N \gamma c_N) = -{}^t z(\gamma) = c_N \cdot z(\gamma)$. For $v \in S_\ell$, with the witness $Y_v \in \mathrm{PW}^1_a(A_m)$ of (Sc) in Definition E.35, the cocycle $g \mapsto z(\iota_v g) - (\iota_v(g) \cdot Y_v - Y_v)$ lies in Zb_v, so $z \circ \iota_v \in \mathcal{L}_v$. For $u \in \mathrm{St}$, $J_{\bar{u}} \subseteq N^{Q_N}$, so $\pi^{Q_N}(J_{\bar{u}}) = \{1\}$ and $z \circ \iota_u$ vanishes on $J_{\bar{u}}$. Hence $z \in \mathrm{Sel}$. The coboundaries agree since ${}^t Y = -Y$ means $c_N \cdot Y = Y$.

Step 3: the local factors. By (Bs2), $\#\mathcal{L}_v \leq \#M \cdot K^{m(n+\nu)d_v + C_v}$. For $u \in \mathrm{St}$, Lemma E.102 (Lb2), (Lb3) give the hypotheses of Lemma E.104, and (Ur1) gives $\#\mathcal{L}_u = \#M$. For $w \in Q_N \cup \{v_0\}$ ($w \nmid \ell$), Theorem B.29 (4) and (2) give

$$\begin{aligned} \#Z^1(D_w, M) &= \#H^1(D_w, M) \cdot \frac{\#M}{\#H^0(D_w, M)} = \#M \cdot \#H^2(D_w, M) \\ &= \#M \cdot \#H^0(D_w, M^\vee) = \#M \cdot \#H^0(D_w, M'), \end{aligned}$$

using the isomorphism $\phi\colon M' \to M^\vee$, $\phi(m') = b_{N,m}(\cdot, m')$: it is injective by non-degeneracy in M', so $\#M' \leq \#M^\vee = \#M$, symmetrically $\#M \leq \#M'$, so ϕ is bijective; and it is Γ_N-equivariant, $(\gamma\phi(m'))(m) = \gamma\, b_{N,m}(\gamma^{-1}m, m') = b_{N,m}(m, \gamma m')$, hence D_w-equivariant through ι_w; so $\#\mathcal{L}_{v_{N,j}} \leq \#M \cdot K^{mn+C_q}$ by (Bs3) and $\#\mathcal{L}_{v_0} \leq \#M \cdot K^{c_0}$ by (Bs4). As $\sum_{v \in S_\ell} d_v = d$,

$$\prod_{w \in T} \#\mathcal{L}_w \leq (\#M)^{\#T} \cdot K^{m(n+\nu)d + mnq + \sum_v C_v + qC_q + c_0}.$$

Step 4: conclusion. The places with full condition include Q_N, so $Z^1_{\mathcal{L},*}$ is contained in the set of (Bs6) and $\#Z^1_{\mathcal{L},*} \leq \#B^1(\Gamma_N, M') \cdot K^{c_2}$. By (Bs1) $\#H^0(\Gamma_N, M) \leq K^{c_1'}$, and by Lemma E.107 (Rf3) (each c_i is an involution outside Γ^0_N, as complex conjugation acts non-trivially on the CM field L) $\prod_{i=1}^d \#M^{c_i=1} \geq K^{m\nu d}$. Inserting these and Step 3 into Proposition E.98 and cancelling $\#B^1(\Gamma_N, M') \cdot (\#M)^{\#T}$:

$$\#\mathrm{Sel} \cdot K^{m\nu d} \leq \#\mathrm{Bd} \cdot K^{m(n+\nu)d + mnq + C}, \qquad \text{i.e.} \qquad \#\mathrm{Sel} \leq \#\mathrm{Bd} \cdot K^{m(nd+nq)+C}.$$

With Step 2, $\mathrm{Sel}^1_{\mathbf{V}_N}(A_m)$ is finite and $\#\mathrm{Sel}^1_{\mathbf{V}_N}(A_m) \leq \#\mathrm{Bd}^1_{\mathbf{V}_N}(A_m) \cdot (\#k)^{mD_0 + C}$ with $D_0 = n[L^+ : \mathbb{Q}] + nq$, for all $N \geq N_0$. This is (B1). □

Proposition E.98. *Let $\mathscr{H}$, Q, T, $(D_w, \iota_w)_{w \in T}$, $c_1, \dots, c_d$ be as in Proposition E.99, and assume that every place of T° splits in L. Put $\bar{c} = \pi^Q(c)$. Let M be a finite abelian group killed by ℓ with a continuous action of Γ^Q, M' a finite abelian group with a continuous action of Γ^Q, and $b\colon M \times M' \to \mu_\ell$ a bi-additive map with $b(\gamma m, \gamma m') = \gamma\, b(m, m')$ (Γ^Q acting on μ_ℓ through χ_{cyc}) that is non-degenerate on both sides. Let $(\mathcal{L}_w)_{w \in T}$ be local conditions for M as in Proposition E.99, and define $Z^1_{\mathcal{L},*}$ there with M' in place of $M^\vee$. Put*

$$\begin{aligned} \mathrm{Sel} = \{ z\colon \Gamma^{0,Q} \to M \text{ continuous} :\ & z(\gamma\gamma') = z(\gamma) + \gamma z(\gamma'), \\ & z(\bar{c}\gamma\bar{c}) = \bar{c}\, z(\gamma),\ z \circ \iota_w \in \mathcal{L}_w\ (w \in T)\}, \\ \mathrm{Bd} = \{ \gamma \mapsto \gamma Y - Y\ (\gamma \in \Gamma^{0,Q}) :\ & Y \in M,\ \bar{c}Y = Y \}. \end{aligned}$$

Then the maps ι_w take values in $\Gamma^{0,Q}$, Sel *is finite,* $\mathrm{Bd} \subseteq \mathrm{Sel}$*, and*

$$\#\mathrm{Sel}\cdot\#B^1(\Gamma^Q,M')\cdot(\#M)^{\#T}\cdot\prod_{i=1}^{d}\#\{m\in M:\ c_i m=m\}\ \le\ \#\mathrm{Bd}\cdot\#H^0(\Gamma^Q,M)\cdot\#Z^1_{\mathcal{L},*}\cdot\prod_{w\in T}\#\mathcal{L}_w.$$

Proof. Write $\Gamma=\Gamma^Q$, $\Gamma^0=\Gamma^{0,Q}$.

Step 0: the groups. $N^Q\subseteq G_L$, so $\Gamma/\Gamma^0\cong G_{L^+}/G_L$ has order 2, Γ^0 is open, $\bar c\notin\Gamma^0$ and $\bar c^2=1$; so $\Gamma=\Gamma^0\sqcup\Gamma^0\bar c$. Every $w\in T$ splits in L (by (Rd2) for $S_\ell\cup\mathrm{St}$, by assumption for Q and T°), so $D_w\subseteq G_L$ and $\iota_w(D_w)\subseteq\Gamma^0$.

Step 1: $M'\cong M^\vee$. The map $\phi\colon M'\to M^\vee$, $\phi(m')=b(\cdot,m')$, is injective (non-degeneracy in M'), so $\#M'\le\#M^\vee=\#M$; symmetrically $\#M\le\#M'$; hence ϕ is bijective. It is Γ-equivariant: $(\gamma\phi(m'))(m)=\gamma\,b(\gamma^{-1}m,m')=b(m,\gamma m')$. So ϕ carries $B^1(\Gamma,M')$, $B^1(D_w,M')$ and $Z^1_{\mathcal{L},*}$ (for M') bijectively onto the same objects for $M^\vee$.

Step 2: Sel *is the set of cocycles in $Z^1_{\mathcal{L}}$ vanishing at $\bar c$.* For $z\in\mathrm{Sel}$ define $\hat z\colon\Gamma\to M$ by $\hat z(\gamma)=\hat z(\gamma\bar c)=z(\gamma)$ $(\gamma\in\Gamma^0)$. It is continuous, and a cocycle: for $\gamma,\gamma'\in\Gamma^0$,

$$\hat z(\gamma\bar c\,\gamma')=z(\gamma\cdot\bar c\gamma'\bar c)=z(\gamma)+\gamma\bar c\,z(\gamma')=\hat z(\gamma\bar c)+\gamma\bar c\,\hat z(\gamma'),$$

and the cases $\hat z(\gamma\gamma'\bar c)$, $\hat z(\gamma\bar c\gamma'\bar c)$ reduce to the same identities. Its restrictions through ι_w are those of z. Conversely the restriction to Γ^0 of a cocycle $\hat z\in Z^1_{\mathcal{L}}$ (of Proposition E.99) with $\hat z(\bar c)=0$ lies in Sel, since $\hat z(\bar c\gamma\bar c)=\hat z(\bar c)+\bar c\,\hat z(\gamma)+\bar c\gamma\,\hat z(\bar c)=\bar c\,\hat z(\gamma)$. So $\mathrm{Sel}\cong\ker\bigl(e\colon Z^1_{\mathcal{L}}\to M,\ \hat z\mapsto\hat z(\bar c)\bigr)$, which is finite by Proposition E.99.

Step 3: the image of e. From $\hat z(\bar c^2)=0$, e takes values in $M^-=\{m:\bar cm=-m\}$. For $Y\in M$ the coboundary $\gamma\mapsto\gamma Y-Y$ lies in $Z^1_{\mathcal{L}}$ (each $\mathcal{L}_w\supseteq B^1(D_w,M)$) and has e-value $\bar cY-Y$; for $m\in M^-$ and $Y=-m/2$ (ℓ odd) this is m. So e maps onto M^-. As ℓ is odd, $M=M^+\oplus M^-$ with $M^+=\{m:\bar cm=m\}$, hence

$$\#\mathrm{Sel}\cdot\#M=\#Z^1_{\mathcal{L}}\cdot\#M^+.$$

Step 4: the coboundaries. $Y\mapsto(\gamma\mapsto\gamma Y-Y)$ maps M^+ onto Bd with kernel $\{Y\in M^+:\gamma Y=Y\ (\gamma\in\Gamma^0)\}=H^0(\Gamma,M)$, so $\#\mathrm{Bd}\cdot\#H^0(\Gamma,M)=\#M^+$. By Step 2 (for $\hat z$ a coboundary of $Y\in M^+$, $e(\hat z)=0$) $\mathrm{Bd}\subseteq\mathrm{Sel}$.

Step 5: conclusion. By Proposition E.99 and Step 1,

$$\#Z^1_{\mathcal{L}}\cdot\#B^1(\Gamma,M')\cdot(\#M)^{\#T}\cdot\prod_i\#M^{c_i=1}\le\#M\cdot\#Z^1_{\mathcal{L},*}\cdot\prod_w\#\mathcal{L}_w.$$

Multiply by $\#M^+$. On the left insert $\#Z^1_{\mathcal{L}}\#M^+=\#\mathrm{Sel}\,\#M$ (Step 3), on the right insert $\#M^+=\#\mathrm{Bd}\,\#H^0(\Gamma,M)$ (Step 4), and cancel $\#M$. □

Proposition E.99. *Let $\mathscr{H}$ be a rank-n Hecke datum over $\mathcal{O}$, Q and $T\supseteq S_\ell\cup\mathrm{St}\cup Q$ as in Lemma E.102, $T^\circ=T\smallsetminus(S_\ell\cup\mathrm{St}\cup Q)$, and $d=[L^+:\mathbb{Q}]$. For each $w\in T$ fix a place $\bar w$ of $\overline{\mathbb{Q}}$ above w, put $D_w=D_{\bar w}\cap G_{L^+}$ and $\iota_w=\pi^Q|_{D_w}\colon D_w\to\Gamma^Q$; let $c_1,\dots,c_d\in\Gamma^Q$ be the images of complex conjugations at the d real places of L^+, one for each place. Let M be a finite abelian group killed by ℓ with a continuous action of Γ^Q, and $M^\vee=\mathrm{Hom}(M,\mu_\ell)$ with $\gamma\phi=\gamma\circ\phi\circ\gamma^{-1}$, Γ^Q acting on μ_ℓ through χ_{cyc}. Let $(\mathcal{L}_w)_{w\in T}$ be subgroups $B^1(D_w,M)\subseteq\mathcal{L}_w\subseteq Z^1(D_w,M)$ (for the action through ι_w) with $\mathcal{L}_u=Z^1_{J_{\bar u}}(M)$ for $u\in\mathrm{St}$ and $\mathcal{L}_t=Z^1(D_t,M)$ for $t\in T^\circ$. Put*

$$Z^1_{\mathcal{L}}=\{z\in Z^1(\Gamma^Q,M):\ z\circ\iota_w\in\mathcal{L}_w\ (w\in T)\},$$
$$Z^1_{\mathcal{L},*}=\{y\in Z^1(\Gamma^Q,M^\vee):\ y\circ\iota_w\in B^1(D_w,M^\vee)\ \text{whenever}\ \mathcal{L}_w=Z^1(D_w,M)\}.$$

Then $Z^1(\Gamma^Q, M)$ and $Z^1(\Gamma^Q, M^\vee)$ are finite and

$$\#Z^1_{\mathcal{L}} \cdot \#B^1(\Gamma^Q, M^\vee) \cdot (\#M)^{\#T} \cdot \prod_{i=1}^{d} \#\{m \in M : \ c_i m = m\} \ \le \ \#M \cdot \#Z^1_{\mathcal{L},*} \cdot \prod_{w \in T} \#\mathcal{L}_w.$$

Proof. Write $F = L^+$ (totally real), $G = G_{F,T_\infty}$ with $T_\infty = T \cup \{\text{archimedean places}\}$, $p\colon G \to \Gamma^Q$ the surjection of Lemma E.102 (Lb1), and regard M, $M^\vee$ as G-modules through p; this is the module $M^\vee$ of Proposition E.100 with $r = 1$, since G acts on μ_ℓ through χ_{cyc}. The map $D_w \to \Gamma^Q$ factors through G, so the local groups $H^i(F_w, \cdot)$ of Proposition E.100 are those of ι_w. Recall ℓ is odd (Definition 50.1).

Step 1: finiteness and passage to G. By Proposition E.101 (Pk1), $H^1(G, M)$ and $H^1(G, M^\vee)$ are finite, and B^1 is a quotient of the finite module, so $Z^1(G, M)$, $Z^1(G, M^\vee)$ are finite; by Lemma E.102 (Lb3) $z \mapsto z \circ p$ embeds $Z^1(\Gamma^Q, \cdot)$ into them, with $B^1(\Gamma^Q, \cdot) \cong B^1(G, \cdot)$ and $H^0(\Gamma^Q, \cdot) = H^0(G, \cdot)$. Let $Z^1_{\mathcal{L}}(G) = \{z \in Z^1(G, M) : \ z|_{D_w} \in \mathcal{L}_w \ (w \in T)\}$; then $z \mapsto z \circ p$ maps $Z^1_{\mathcal{L}}$ injectively into $Z^1_{\mathcal{L}}(G)$, so $\#Z^1_{\mathcal{L}} \le \#Z^1_{\mathcal{L}}(G)$.

Step 2: the local conditions. Put $L_w = \mathcal{L}_w / B^1(D_w, M) \subseteq H^1(F_w, M)$, so $\#L_w = \#\mathcal{L}_w \cdot \#H^0(F_w, M)/\#M$ (as $\#B^1(D_w, M) = \#M/\#H^0(F_w, M)$). For $u \in \mathrm{St}$, Lemma E.102 (Lb2), (Lb3) give the hypotheses of Lemma E.103 for $J = J_{\bar{u}}$, so $L_u = H^1_{J_{\bar{u}}}(M)$ and $L_u^\perp = H^1_{J_{\bar{u}}}(M^\vee)$ by (St2). For w with $\mathcal{L}_w = Z^1(D_w, M)$, $L_w = H^1(F_w, M)$ and $L_w^\perp = 0$ by Theorem B.29 (2).

Step 3: Greenberg–Wiles. Since every $\mathcal{L}_w$ contains $B^1(D_w, M)$, $Z^1_{\mathcal{L}}(G)$ is the full preimage of the Selmer group $H^1_L(M)$ of Proposition E.100 and contains $B^1(G, M)$; so $\#Z^1_{\mathcal{L}}(G) = \#H^1_L(M) \cdot \#M/\#H^0(G, M)$. Let $Z^1_{L^\perp}(G) \subseteq Z^1(G, M^\vee)$ be the preimage of $H^1_{L^\perp}(M^\vee)$; $\#Z^1_{L^\perp}(G) = \#H^1_{L^\perp}(M^\vee) \cdot \#B^1(G, M^\vee)$. Proposition E.100 for F, $S = T_\infty$, $r = 1$ (the archimedean places are the d real places, with $H^0 = M^{c_i=1}$) reads, after inserting $\#L_w$,

$$\#H^1_L(M) \cdot \#H^0(G, M^\vee) \cdot \prod_{i=1}^{d} \#M^{c_i=1} \cdot (\#M)^{\#T} = \#H^1_{L^\perp}(M^\vee) \cdot \#H^0(G, M) \cdot \prod_{w \in T} \#\mathcal{L}_w.$$

Multiplying by $\#M \cdot \#B^1(G, M^\vee)/\#H^0(G, M)$ and dropping the factor $\#H^0(G, M^\vee) \ge 1$:

$$\#Z^1_{\mathcal{L}}(G) \cdot \#B^1(G, M^\vee) \cdot (\#M)^{\#T} \cdot \prod_{i} \#M^{c_i=1} \le \#M \cdot \#Z^1_{L^\perp}(G) \cdot \prod_{w \in T} \#\mathcal{L}_w.$$

Step 4: the dual cocycles come from Γ^Q. Let $y \in Z^1_{L^\perp}(G)$. For $u \in \mathrm{St}$, $[y|_{D_u}] \in L_u^\perp = H^1_{J_{\bar{u}}}(M^\vee)$, so y vanishes on $J_{\bar{u}}$ by Lemma E.103 (St1) for $M^\vee$. For $t \in T^\circ$, $L_t^\perp = 0$, so $y|_{D_t}$ is a coboundary $g \mapsto gm - m$; it vanishes on $I_{\bar{t}} \cap G_{L^+}$, which acts trivially on M (Lb3) and on μ_ℓ ($t \nmid \ell$), hence on $M^\vee$. By (Lb3) for $M^\vee$, $y = y_0 \circ p$ with $y_0 \in Z^1(\Gamma^Q, M^\vee)$, and where $\mathcal{L}_w = Z^1(D_w, M)$, $y_0 \circ \iota_w = y|_{D_w} \in B^1(D_w, M^\vee)$ as $L_w^\perp = 0$. So $y \mapsto y_0$ embeds $Z^1_{L^\perp}(G)$ into $Z^1_{\mathcal{L},*}$, and $\#Z^1_{L^\perp}(G) \le \#Z^1_{\mathcal{L},*}$.

Combining Steps 1, 3, 4 with $\#B^1(G, M^\vee) = \#B^1(\Gamma^Q, M^\vee)$ gives the inequality. □

Proposition E.100. *Let F be a number field and ℓ a prime; assume that ℓ is odd or that F has no real place. Let S be a finite set of places of F containing the places above ℓ and the archimedean places, $S_f \subseteq S$ its finite places, $r \ge 1$, and M a finite $\mathbb{Z}/\ell^r\mathbb{Z}$-module with a continuous action of $G_{F,S}$. For $v \in S_f$ let*

$$\langle\ ,\ \rangle_v\colon H^1(F_v, M) \times H^1(F_v, M^\vee) \to H^2(F_v, \mu_{\ell^r}) \xrightarrow{\mathrm{inv}_v} \mathbb{Q}/\mathbb{Z}$$

be the cup product pairing. A ***family of local conditions*** *for M is a family $\mathcal{L} = (L_v)_{v\in S_f}$ of subgroups $L_v \subseteq H^1(F_v, M)$; its* ***dual family*** *is $\mathcal{L}^\perp = (L_v^\perp)_{v\in S_f}$, $L_v^\perp \subseteq H^1(F_v, M^\vee)$ the annihilator of L_v under $\langle\ ,\ \rangle_v$; the* ***Selmer groups*** *are*

$$H^1_{\mathcal{L}}(M) = \{x \in H^1(G_{F,S}, M) :\ \mathrm{res}_v(x) \in L_v \text{ for all } v \in S_f\},$$
$$H^1_{\mathcal{L}^\perp}(M^\vee) = \{y \in H^1(G_{F,S}, M^\vee) :\ \mathrm{res}_v(y) \in L_v^\perp \text{ for all } v \in S_f\}.$$

Then $H^1_{\mathcal{L}}(M)$ and $H^1_{\mathcal{L}^\perp}(M^\vee)$ are finite and

$$\#H^1_{\mathcal{L}}(M)\cdot\#H^0(G_{F,S}, M^\vee)\cdot\prod_{v\in S}\#H^0(F_v, M) = \#H^1_{\mathcal{L}^\perp}(M^\vee)\cdot\#H^0(G_{F,S}, M)\cdot\prod_{v\in S_f}\#L_v,$$

where the product on the left runs over all places of S, the archimedean ones included ($H^0(F_v, M)$ is the subgroup fixed by a complex conjugation at a real place v, and M at a complex place).

Proof. Write $H^i(M) = H^i(G_{F,S}, M)$, $P^i(M) = \prod_{v\in S_f} H^i(F_v, M)$, and let $\lambda^i_M\colon H^i(M) \to P^i(M)$ be the product of the restriction maps. Since S contains the places above ℓ and the archimedean places, the extension $F(\mu_{\ell^r})/F$ is unramified outside S, so μ_{ℓ^r} is a $G_{F,S}$-module, and so is $M^\vee$, a finite $\mathbb{Z}/\ell^r\mathbb{Z}$-module with continuous action; $(M^\vee)^\vee = M$ canonically. The hypothesis that ℓ is odd or F has no real place is the standing hypothesis of Theorem B.37 and of Theorem B.39, so both apply to M and to $M^\vee$; Theorem B.29 holds for every prime ℓ. All groups below are finite: $H^i(M)$, $H^i(M^\vee)$ by Theorem B.37, item 1, and the local groups by Theorem B.29, item 1.

Step 1: the archimedean places drop out. Let v be an archimedean place; $\mathrm{Gal}(\mathbb{C}/F_v)$ has order 1 or 2. If ℓ is odd, M and $M^\vee$ have odd order, so every Tate cohomology group $\hat H^i(F_v, M)$, $\hat H^i(F_v, M^\vee)$ is killed by the order of the group, which divides 2, and by ℓ^r, hence vanishes. If F has no real place, v is complex, $\mathrm{Gal}(\mathbb{C}/F_v)$ is trivial, and every Tate cohomology group of the trivial group vanishes. In either case $H^i(F_v, M) = \hat H^i(F_v, M) = 0$ and $H^i(F_v, M^\vee) = 0$ for $i \geq 1$; hence $\mathrm{Sha}^i_S(F, M) = \ker\lambda^i_M$ and $\mathrm{Sha}^i_S(F, M^\vee) = \ker\lambda^i_{M^\vee}$ for $i = 1, 2$. The Poitou–Tate nine-term sequence of Theorem B.39, item 3, for M has its local terms the products over S_f. By Theorem B.29, item 2, the sum of the pairings $\langle\ ,\ \rangle_v$ is a perfect pairing $P^1(M)\times P^1(M^\vee) \to \mathbb{Q}/\mathbb{Z}$. The part

$$H^1(M) \xrightarrow{\lambda^1_M} P^1(M) \xrightarrow{\beta} H^1(M^\vee)^* \to H^2(M) \xrightarrow{\lambda^2_M} P^2(M) \to H^0(M^\vee)^* \to 0$$

of that sequence is exact, where $\beta(x)(y) = \sum_{v\in S_f}\langle x_v, \mathrm{res}_v(y)\rangle_v$ (the dual of $\lambda^1_{M^\vee}$ composed with local duality). Exactness at $P^1(M)$ says $\mathrm{im}\,\lambda^1_M = \ker\beta = (\mathrm{im}\,\lambda^1_{M^\vee})^\perp$; as the pairing is perfect, also $(\mathrm{im}\,\lambda^1_M)^\perp = \mathrm{im}\,\lambda^1_{M^\vee}$. Exactness at $P^2(M)$ gives $\#\mathrm{im}\,\lambda^2_M = \#P^2(M)/\#H^0(M^\vee)$, hence

$$\#\mathrm{Sha}^2_S(F, M) = \#\ker\lambda^2_M = \frac{\#H^2(M)\,\#H^0(M^\vee)}{\#P^2(M)}. \tag{E.16}$$

Step 2: the local part of the Selmer groups. Put $A = \mathrm{im}\,\lambda^1_M$, $A' = \mathrm{im}\,\lambda^1_{M^\vee} = A^\perp$, $\Lambda = \prod_v L_v \subseteq P^1(M)$, so that $\Lambda^\perp = \prod_v L_v^\perp$. By definition $H^1_{\mathcal{L}}(M) = (\lambda^1_M)^{-1}(\Lambda)$ and $H^1_{\mathcal{L}^\perp}(M^\vee) = (\lambda^1_{M^\vee})^{-1}(\Lambda^\perp)$, so, by Step 1,

$$\#H^1_{\mathcal{L}}(M) = \#\mathrm{Sha}^1_S(F, M)\,\#(A\cap\Lambda),$$
$$\#H^1_{\mathcal{L}^\perp}(M^\vee) = \#\mathrm{Sha}^1_S(F, M^\vee)\,\#(A^\perp\cap\Lambda^\perp).$$

For subgroups of a finite group with a perfect pairing, $A^\perp \cap \Lambda^\perp = (A+\Lambda)^\perp$ has order

$$\#P^1(M)/\#(A+\Lambda) = \#P^1(M)\,\#(A\cap\Lambda)/(\#A\,\#\Lambda).$$

With $\#A = \#H^1(M)/\#\mathrm{Sha}^1_S(F,M)$ this gives

$$\frac{\#H^1_{\mathcal{L}}(M)}{\#H^1_{\mathcal{L}^\perp}(M^\vee)} = \frac{\#H^1(M)\,\#\Lambda}{\#\mathrm{Sha}^1_S(F,M^\vee)\,\#P^1(M)}.$$

Step 3: Sha duality and the Euler characteristics. By Theorem B.39, item 2, applied to $M^\vee$ (whose dual is M), $\#\mathrm{Sha}^1_S(F,M^\vee) = \#\mathrm{Sha}^2_S(F,M)$, given by (E.16). By Theorem B.29, item 4, $\#H^1(F_v,M) = \#H^0(F_v,M)\,\#H^2(F_v,M)\,(\#M)^{[F_v:\mathbb{Q}_\ell]}$ for $v \mid \ell$ and without the last factor otherwise; since $\sum_{v|\ell}[F_v:\mathbb{Q}_\ell] = [F:\mathbb{Q}]$,

$$\#P^1(M) = \#P^0(M)\,\#P^2(M)\,(\#M)^{[F:\mathbb{Q}]}.$$

By Theorem B.37, item 2,

$$\#H^1(M)/\#H^2(M) = \#H^0(M)\prod_{v|\infty}|\#M|_v/\#H^0(F_v,M),$$

and $\prod_{v|\infty}|\#M|_v = (\#M)^{r_1+2r_2} = (\#M)^{[F:\mathbb{Q}]}$, where r_1, r_2 are the numbers of real and complex places ($r_1 = 0$ when F has no real place). Substituting,

$$\begin{aligned}\frac{\#H^1_{\mathcal{L}}(M)}{\#H^1_{\mathcal{L}^\perp}(M^\vee)} &= \frac{\#H^1(M)\,\#P^2(M)\,\#\Lambda}{\#H^2(M)\,\#H^0(M^\vee)\,\#P^0(M)\,\#P^2(M)\,(\#M)^{[F:\mathbb{Q}]}}\\ &= \frac{\#H^0(M)\,\prod_{v\in S_f}\#L_v}{\#H^0(M^\vee)\,\#P^0(M)\,\prod_{v|\infty}\#H^0(F_v,M)},\end{aligned}$$

and $\#P^0(M)\prod_{v|\infty}\#H^0(F_v,M) = \prod_{v\in S}\#H^0(F_v,M)$. Clearing denominators gives the assertion. □

Proposition E.101. *Let F be a totally real number field of degree $d=[F:\mathbb{Q}]$, ℓ an odd prime, S a finite set of places of F containing the places above ℓ and the archimedean places, S_f its finite places, and $G = G_{F,S}$. For each real place u of F let $c_u \in G$ be the image of a complex conjugation at u. By an ℓ-**module** of a profinite group we mean a finite abelian group killed by ℓ with a continuous action. Then:*

(Pk1) *for every ℓ-module M of G, $H^i(G,M)$ is finite for all i and zero for $i\geq 3$; and for every surjective homomorphism $M\to M'$ of ℓ-modules of G, the induced map $H^2(G,M)\to H^2(G,M')$ is surjective;*

(Pk2) *for every $v\in S_f$ and every ℓ-module M of G_{F_v}, $H^i(F_v,M)$ is finite for all i and zero for $i\geq 3$, $H^2(F_v,\cdot)$ is surjective on surjections of ℓ-modules, and $\#H^1(F_v,M) = \#H^0(F_v,M)\,\#H^2(F_v,M)\,(\#M)^{d_v}$ with $d_v=[F_v:\mathbb{Q}_\ell]$ if $v\mid\ell$ and $d_v=0$ otherwise;*

(Pk3) *$c_u^2=1$; if $L\subseteq\overline{\mathbb{Q}}$ is a CM field with $L^+=F$, unramified over F outside S, then c_u acts non-trivially on L; and for every ℓ-module M of G,*

$$\#H^0(G,M)\,\#H^2(G,M)\,(\#M)^d = \#H^1(G,M)\prod_{u\ \text{real}}\#M^{c_u=1};$$

(Pk4) *for every ℓ-module M of G, every subset $T \subseteq S_f$ and every family of surjective G_{F_v}-equivariant homomorphisms $M \to Q_v$ onto ℓ-modules Q_v of G_{F_v} ($v \in T$), the cokernel of*

$$H^2(G, M) \longrightarrow \bigoplus_{v \in T} H^2(F_v, Q_v)$$

(restriction followed by push-forward) has order at most $\#\mathrm{Hom}_G(M, \mu_\ell)$.

Proof. An ℓ-module is a finite $\mathbb{Z}/\ell\mathbb{Z}$-module, so the results used below apply with $r = 1$.

(Pk1). Finiteness is Theorem B.37, item 1. Let $F' = F(\mu_\ell)$. It is totally imaginary (it contains a primitive ℓ-th root of unity, which is not real as ℓ is odd), Galois over F of degree e dividing $\ell - 1$, and unramified outside the places above ℓ; so F' lies in the maximal extension F_S of F unramified outside S and $G' = \mathrm{Gal}(F_S/F')$ is an open normal subgroup of G of index e, equal to $G_{F',S'}$ for the set S' of places of F' above S (every extension of F' inside F_S is unramified outside S', and conversely an extension of F' unramified outside S' is unramified outside S over F). By Theorem B.34, item 3, $H^i(G', M) = 0$ for $i \geq 3$. Corestriction after restriction is multiplication by e on $H^i(G, M)$, which is bijective since e is prime to ℓ and $H^i(G, M)$ is killed by ℓ; hence restriction $H^i(G, M) \to H^i(G', M) = 0$ is injective and $H^i(G, M) = 0$ for $i \geq 3$. For a surjection $M \to M'$ with kernel K (an ℓ-module), the long exact sequence gives $H^2(G, M) \to H^2(G, M') \to H^3(G, K) = 0$.

(Pk2). Finiteness, vanishing for $i \geq 3$ and the Euler characteristic are Theorem B.29, items 1 and 4 (with $[F_v : \mathbb{Q}_\ell] = d_v$ for $v \mid \ell$). Right exactness of H^2 follows from the vanishing of H^3 as in (Pk1).

(Pk3). c_u is the image of an element of order 2, so $c_u^2 = 1$. A CM field L is totally imaginary, so a complex conjugation at any embedding acts non-trivially on it. By Theorem B.37, item 2 (ℓ odd), with $H^0(F_u, M) = M^{c_u=1}$ and $|\#M|_u = \#M$ at the d real places u (and no complex places), $\#H^0(G, M)\,\#H^2(G, M)/\#H^1(G, M) = \prod_u \#M^{c_u=1}/(\#M)^d$, which is the assertion.

(Pk4). Let $\lambda\colon H^2(G, M) \to P^2 = \bigoplus_{v \in S_f} H^2(F_v, M)$ be the restriction and $q\colon P^2 \to \bigoplus_{v \in T} H^2(F_v, Q_v)$ the projection to the components in T followed by the push-forwards; q is surjective by (Pk2). The map of the statement is $q \circ \lambda$, so its cokernel is a quotient of $P^2/\mathrm{im}\,\lambda$ (as q is onto). By Theorem B.37, item 3 (ℓ odd), $P^2/\mathrm{im}\,\lambda \cong H^0(G, M^\vee)^*$, of order $\#H^0(G, M^\vee) = \#\mathrm{Hom}_G(M, \mu_\ell)$. □

Lemma E.102. *Let $\mathscr{H}$ be a rank-n Hecke datum over $\mathcal{O}$, Q a finite set of places of L^+ outside $S_\ell \cup \mathrm{St} \cup \{v_0\}$, not above ℓ and split in L, and T a finite set of finite places of L^+ containing $S_\ell \cup \mathrm{St} \cup Q$; write $T_\infty = T \cup \{\text{archimedean places}\}$ and $T^\circ = T \smallsetminus (S_\ell \cup \mathrm{St} \cup Q)$. For a place $\bar{w}$ of $\overline{\mathbb{Q}}$ above a finite place of L^+ put $J_{\bar{w}} = \{s \in I_{\bar{w}} \cap G_L :\ \bar{\chi}_a(s) = 1\ (1 \leq a \leq n)\}$. Then:*

(Lb1) *π^Q factors as $G_{L^+} \to G_{L^+,T_\infty} \xrightarrow{p} \Gamma^Q$ with p a continuous surjection, and $\ker p$ is the closed normal subgroup of G_{L^+,T_∞} generated by the images of $J_{\bar{u}}$ ($u \in \mathrm{St}$) and of $I_{\bar{t}} \cap G_{L^+}$ ($t \in T^\circ$), for any one choice of places $\bar{u}$ above u and $\bar{t}$ above t;*

(Lb2) *for $u \in \mathrm{St}$ and $\bar{u}$ above u, the decomposition group $D_{\bar{u}} \cap G_{L^+}$ of L^+ at u lies in G_L, and $J_{\bar{u}}$ is a closed subgroup of its inertia group $I_{\bar{u}} \cap G_{L^+}$, normal in $D_{\bar{u}} \cap G_{L^+}$, of finite index prime to ℓ; and for $t \in T^\circ$, $I_{\bar{t}} \cap G_{L^+} = I_{\bar{t}} \cap G_L = J_{\bar{t}}$;*

(Lb3) *for every discrete abelian group M with a continuous action of Γ^Q, regarded as a G_{L^+,T_∞}-module through p, the subgroups $J_{\bar{u}}$ ($u \in \mathrm{St}$) and $I_{\bar{t}}$ ($t \in T^\circ$) act trivially on M, $H^0(\Gamma^Q, M) = H^0(G_{L^+,T_\infty}, M)$, and $z \mapsto z \circ p$ is a bijection from $Z^1(\Gamma^Q, M)$ onto*

$$\{z \in Z^1(G_{L^+,T_\infty}, M) :\ z \text{ vanishes on the images of } J_{\bar{u}}\ (u \in \mathrm{St}) \text{ and of } I_{\bar{t}} \cap G_{L^+}\ (t \in T^\circ)\},$$

mapping $B^1(\Gamma^Q, M)$ onto $B^1(G_{L^+,T_\infty}, M)$.

Proof. Write $G = G_{L^+}$, $\bar{G} = G_{L^+,T_\infty}$ and $\Sigma = \{s \in G_L : \bar{\chi}_a(s) = 1 \ (1 \le a \le n)\}$. By (Rd5) and the remark in Definition 50.1 that the kernels of the $\bar{\chi}_a$ jointly are stable under G_L- and c-conjugation, Σ is a closed normal subgroup of $G = G_L \cup cG_L$. For $g \in G$ and a place $\bar{w}$, $gI_{\bar{w}}g^{-1} = I_{g\bar{w}}$, hence $gJ_{\bar{w}}g^{-1} = J_{g\bar{w}}$; and G acts transitively on the places of $\overline{\mathbb{Q}}$ above a given place of L^+.

Inertia outside $S_\ell \cup \mathrm{St} \cup Q$ lies in N^Q. Let $\bar{w}$ lie above a finite place $w \notin S_\ell \cup \mathrm{St} \cup Q$ of L^+; $w \nmid \ell$ as S_ℓ contains all places above ℓ. By (Rd2) L/L^+ is unramified at w, so $I_{\bar{w}} \cap G$ acts trivially on L, i.e. $I_{\bar{w}} \cap G = I_{\bar{w}} \cap G_L$; by (Rd4) every $\bar{\chi}_a$ is trivial on it. So $I_{\bar{w}} \cap G = J_{\bar{w}}$ consists of generators of N^Q (the place $\bar{w}$ is above neither ℓ nor Q). This proves the last assertion of (Lb2).

(Lb1). The kernel of $G \to \bar{G}$ is the closed normal subgroup generated by the inertia groups $I_{\bar{w}} \cap G$ at places $\bar{w}$ above finite places outside T (its fixed field is the maximal extension of L^+ unramified outside T_∞). By the previous paragraph it lies in N^Q, so π^Q factors through a continuous surjection $p\colon \bar{G} \to \Gamma^Q$, and $\ker p$ is the image of N^Q in $\bar{G}$. The image of the closed normal subgroup generated by a set X under a continuous surjection of profinite groups is the closed normal subgroup generated by the image of X (the image of a closure is closed by compactness and contained in the closure of the image). The generators of N^Q are the elements of $J_{\bar{w}}$ for $\bar{w}$ above neither ℓ nor Q; those above places outside T map to 1; for $\bar{w}$ above $t \in T^\circ$, $J_{\bar{w}} = I_{\bar{w}} \cap G$ by the previous paragraph; the remaining ones lie above St. By the conjugation rule and transitivity, one place above each u and each t suffices.

(Lb2). By (Rd2) u splits in L, so $D_{\bar{u}} \cap G$ fixes L, i.e. lies in G_L. Hence $J_{\bar{u}} = (I_{\bar{u}} \cap G) \cap \Sigma$ is closed and normal in $D_{\bar{u}} \cap G$ (both $I_{\bar{u}} \cap G$ and $\Sigma \cap D_{\bar{u}}$ are normal there). The map $s \mapsto (\bar{\chi}_a(s))_a$ embeds $(I_{\bar{u}} \cap G)/J_{\bar{u}}$ into $(\mathbb{F}_{\mathcal{O}}^\times)^n$, a finite group of order prime to ℓ.

(Lb3). $J_{\bar{u}}$ and $I_{\bar{t}} \cap G = J_{\bar{t}}$ map into $\ker p$ (they generate part of N^Q), so they act trivially on M; and H^0 agrees because p is surjective. Let $K = \ker p$. For $z \in Z^1(\bar{G}, M)$, K acts trivially, so $z|_K$ is a homomorphism, and for $g \in \bar{G}$, $x \in K$:

$$z(gxg^{-1}) = z(g) + g\,z(x) + gx\,z(g^{-1}) = z(g) + g\,z(x) + (gxg^{-1})\,g\,z(g^{-1}) = g\,z(x),$$

since gxg^{-1} acts trivially and $z(g) + g\,z(g^{-1}) = z(1) = 0$. Hence if z vanishes on the images of the $J_{\bar{u}}$ and $I_{\bar{t}} \cap G$, it vanishes on their conjugates, on the subgroup they generate and, being continuous with values in a discrete group, on its closure K (by (Lb1)). A cocycle vanishing on K satisfies $z(gx) = z(g) + g\,z(x) = z(g)$ for $x \in K$, so $z = z_0 \circ p$ for a unique map z_0 on Γ^Q, which is a continuous cocycle (Γ^Q carries the quotient topology). Conversely $z_0 \circ p$ vanishes on K. Coboundaries $g \mapsto gm - m$ correspond to each other. □

Lemma E.103. *Let F be a number field, v a finite place of F whose residue characteristic differs from the prime ℓ, $D = G_{F_v}$, $I = I_v \subseteq D$, and $J \subseteq I$ a closed subgroup, normal in D, of finite index in I prime to ℓ. Let M be a finite abelian group killed by ℓ with a continuous action of D that is trivial on J. Then:*

(St1) *for $z \in Z^1(D, M)$, the class $[z]$ lies in $H^1_J(M)$ if and only if $z|_J = 0$; and $H^1_J(M) = H^1_{\mathrm{ur}}(F_v, M)$ as subgroups of $H^1(F_v, M)$, of order $\#H^0(F_v, M)$;*

(St2) *J acts trivially on $M^\vee$, and $H^1_J(M)$ and $H^1_J(M^\vee)$ are exact annihilators of each other under the cup product pairing $H^1(F_v, M) \times H^1(F_v, M^\vee) \to H^2(F_v, \mu_\ell) \cong \mathbb{Z}/\ell\mathbb{Z}$.*

Proof. (St1). By Lemma E.104 (Ur1), $B^1(D,M) \subseteq Z^1_J(M)$, so $H^1_J(M)$ is a subgroup of $H^1(F_v,M)$ of order $\#H^0(F_v,M)$. A coboundary $g \mapsto gm - m$ vanishes on J since J acts trivially on M; hence if $z = z' + b$ with $z' \in Z^1_J(M)$ and $b \in B^1(D,M)$, then $z|_J = 0$; the converse is the definition. As J acts trivially, $H^1(J,M) = \mathrm{Hom}_{\mathrm{cont}}(J,M)$ and the restriction of $[z]$ is $z|_J$; so

$$H^1_J(M) = \ker\bigl(\mathrm{res}^D_J \colon H^1(D,M) \to H^1(J,M)\bigr).$$

Now $\mathrm{res}^D_J = \mathrm{res}^I_J \circ \mathrm{res}^D_I$, and res^I_J is injective on $H^1(I,M)$: J is normal in I, so by the inflation–restriction sequence its kernel is $H^1(I/J, M^J) = H^1(I/J,M)$, which is killed by the order $[I:J]$ of the finite group I/J and by ℓ, hence is zero as $\ell \nmid [I:J]$. Therefore $\ker \mathrm{res}^D_J = \ker \mathrm{res}^D_I = H^1_{\mathrm{ur}}(F_v,M)$.

(St2), triviality on $M^\vee$. As the residue characteristic of v is not ℓ, $F_v(\mu_\ell)/F_v$ is unramified, so I acts trivially on μ_ℓ. For $g \in J$ and $\phi \in M^\vee$, $(g\phi)(m) = g\,\phi(g^{-1}m) = \phi(m)$. So (St1) applies to the finite group $M^\vee$, killed by ℓ: $H^1_J(M^\vee) = H^1_{\mathrm{ur}}(F_v, M^\vee)$, of order $\#H^0(F_v,M^\vee)$.

Orthogonality. Let $x \in H^1_{\mathrm{ur}}(F_v,M)$, $y \in H^1_{\mathrm{ur}}(F_v,M^\vee)$. By the inflation–restriction sequence for $I \trianglelefteq D$, $x = \inf(x_0)$ and $y = \inf(y_0)$ with $x_0 \in H^1(D/I, M^I)$, $y_0 \in H^1(D/I, (M^\vee)^I)$. Cup products commute with inflation, so $x \cup y = \inf(x_0 \cup y_0)$ with $x_0 \cup y_0 \in H^2(D/I, \mu_\ell)$ for the pairing $M^I \times (M^\vee)^I \to \mu_\ell$. The group $D/I \cong \hat{\mathbb{Z}}$ has cohomological dimension 1 ([222], I.3.2, II.1), so $H^2(D/I,\mu_\ell) = 0$ and $x \cup y = 0$. Hence $H^1_{\mathrm{ur}}(F_v,M^\vee) \subseteq H^1_{\mathrm{ur}}(F_v,M)^\perp$.

Equality by counting. By Theorem B.29 (2) the pairing on H^1 is perfect, so $\#H^1(F_v,M^\vee) = \#H^1(F_v,M)$ and

$$\begin{aligned}\#H^1_{\mathrm{ur}}(F_v,M)^\perp &= \frac{\#H^1(F_v,M)}{\#H^1_{\mathrm{ur}}(F_v,M)} = \frac{\#H^1(F_v,M)}{\#H^0(F_v,M)} \\ &= \#H^2(F_v,M) = \#H^0(F_v,M^\vee) = \#H^1_{\mathrm{ur}}(F_v,M^\vee),\end{aligned}$$

using (St1) for M, Theorem B.29 (4) for $v \nmid \ell$, Theorem B.29 (2) with $i = 0$, and (St1) for $M^\vee$. So $H^1_{\mathrm{ur}}(F_v,M)^\perp = H^1_{\mathrm{ur}}(F_v,M^\vee)$; as the pairing is perfect, also $H^1_{\mathrm{ur}}(F_v,M^\vee)^\perp = H^1_{\mathrm{ur}}(F_v,M)$. With (St1) for M and $M^\vee$ this is (St2). □

Lemma E.104. *Let K be a non-archimedean local field whose residue characteristic p differs from the prime ℓ, $D = G_K$, $I \subseteq D$ its inertia subgroup, and $J \subseteq I$ a closed subgroup, normal in D, of finite index in I prime to ℓ. For a finite abelian group M killed by ℓ with a continuous action of D that is trivial on J, put $Z^1_J(M) = \{z \in Z^1(D,M) : z|_J = 0\}$ and $H^1_J(M) = Z^1_J(M)/B^1(D,M)$. Then:*

(Ur1) *$B^1(D,M) \subseteq Z^1_J(M)$, $\#Z^1_J(M) = \#M$ and $\#H^1_J(M) = \#H^0(D,M)$;*

(Ur2) *for every surjective D-equivariant homomorphism $M \to M'$ onto a module M' of the same kind, every $z' \in Z^1_J(M')$ is the image of some $z \in Z^1_J(M)$.*

Proof. Let $\Phi \in D$ be a lift of Frobenius; $D/I \cong \widehat{\mathbb{Z}}$, topologically generated by the image of Φ. Put $\bar{D} = D/J$ and $P = I/J$, a finite normal subgroup of $\bar{D}$ of order prime to ℓ with $\bar{D}/P = D/I$.

(Ur1). Coboundaries $g \mapsto gx - x$ vanish on J, which acts trivially. Let $z \in Z^1_J(M)$ and $m = \#P$. Since z vanishes on J, $z|_I$ factors through P; summing $z(gh) = z(g) + gz(h)$ over $h \in P$ gives $mz(g) = \sum_h z(h) - g\sum_h z(h)$ for $g \in I$, so $z(g) = gx - x$ on I with $x = -m^{-1}\sum_{h\in P} z(h)$ (m is invertible on M). Hence restriction maps $Z^1_J(M)$ onto $B^1(I,M)$ (onto because $B^1(D,M) \subseteq Z^1_J(M)$ restricts onto $B^1(I,M)$), and $\#B^1(I,M) = \#M/\#M^I$. The kernel consists of the cocycles vanishing on I; for such z and $h \in I$, $z(hg) = hz(g)$ and $z(hg) = z(g \cdot g^{-1}hg) = z(g)$, so z takes values in M^I and is a cocycle of D/I. The map $z \mapsto z(\Phi)$ from these cocycles to M^I is

injective (a continuous cocycle vanishing on I and at Φ vanishes on the dense subgroup generated by I and Φ) and onto: for $x \in M^I$, if Φ acts on M^I with order e, the function $k \mapsto \sum_{0\le i<k} \Phi^i x$ on $k \ge 0$ depends only on k modulo $e\ell$ (the sum of $e\ell$ consecutive terms is $\ell \sum_{i<e} \Phi^i x = 0$), hence extends to a continuous cocycle of $\widehat{\mathbb{Z}} = D/I$ with value x at Φ. So $\#Z^1_J(M) = \#M^I \cdot \#M/\#M^I = \#M$, and $\#H^1_J(M) = \#M/\#B^1(D,M) = \#M/(\#M/\#M^D) = \#M^D$.

(Ur2). Let K' be the kernel of $M \to M'$, an ℓ-module of $\bar{D}$. By standard facts of the cohomology of profinite groups [222, I.2.2, I.3.3, I.3.4], $Z^1_J(\cdot) = Z^1(\bar{D}, \cdot)$, and $H^2(\bar{D}, K') \cong H^2(\bar{D}/P, K'^P) = H^2(\widehat{\mathbb{Z}}, K'^P) = 0$. The long exact sequence of $\bar{D}$-cohomology shows that $H^1(\bar{D}, M) \to H^1(\bar{D}, M')$ is onto. So for $z' \in Z^1(\bar{D}, M')$ there are $z_0 \in Z^1(\bar{D}, M)$ and $x' \in M'$ with $z'(g) = \bar{z}_0(g) + gx' - x'$, $\bar{z}_0$ the image of z_0; lifting x' to $x \in M$, $z = z_0 + (g \mapsto gx - x)$ lies in $Z^1(\bar{D}, M) = Z^1_J(M)$ and maps to z'. □

Proposition E.105. *Let $\mathscr{H}$, $\mathcal{O}$ (a discrete valuation ring), x, κ with local witnesses $Y_{j,v}$, and a Taylor–Wiles datum of size q (with H, $\sigma_1, \dots, \sigma_q$) along which x has eigenvalue data be as in Proposition E.97, with $\rho(\gamma) = 1$ for $\gamma \in \Gamma^0_{\mathscr{H}} \cap \mathrm{PL}(H)$, x satisfying (Np4) and (Pj1) of Definition 50.9, for every $Z \in \mathrm{Ev}_a(A)$ some $T^s Z$ in the A-span of $\rho(\Gamma')$ (Γ' as in Definition 50.9), and $\{\sigma_1, \dots, \sigma_q\}$ detecting (x, κ). Put $K = \#k$, $d = [L^+ : \mathbb{Q}]$, $\nu = \binom{a}{2} + \binom{n-a}{2} + g\,a(n-a)$, $A_m = A/T^m A$, $c_N = \pi^{Q_N}(c)$, $\Gamma^0_N = h_N^{-1}(\Gamma^0_{\mathscr{H}})$. Let $\mathrm{PB}_a(A_m) \subseteq \mathrm{M}_n(\mathrm{PD}_g(A_m))$ be the matrices whose same-block entries lie in $A_m\eta$ and whose different-block entries lie in $\bigoplus_j A_m\epsilon'_j$. For $N, m \ge 1$ let $M_{N,m} = \mathrm{PW}^1_a(A_m)$ and $M'_{N,m} = \mathrm{PB}_a(A_m)$ with*

$$\gamma \cdot Z = \rho_1(h_N\gamma) Z \rho_1(h_N\gamma)^{-1}, \quad c_N \cdot Z = -{}^t Z;$$
$$\gamma \cdot Y = \chi_{\mathrm{cyc}}(\gamma)\rho_1(h_N\gamma) Y \rho_1(h_N\gamma)^{-1}, \quad c_N \cdot Y = -\chi_{\mathrm{cyc}}(c)\,{}^t Y$$

($\gamma \in \Gamma^0_N$; entries of ρ_1 and values of χ_{cyc} mapped to $\mathrm{PD}_g(A_m)$). For $v \in S_\ell$ let Zb_v be the set of cocycles $\beta \in Z^1(D_v, M_{N,m})$ (action through ι_v) such that $(w^1_v)^{-1}\beta(g)w^1_v$ is upper triangular for all $g \in D_v$, with zero diagonal for $g \in \mu_v$. Then:

(Bs0) *the formulas for $M'_{N,m}$ define a continuous action of Γ_N on the finite group $M'_{N,m}$ (for $M_{N,m}$ this is Lemma E.107 (Rf2)), and there is a bi-additive map $b_{N,m}\colon M_{N,m} \times M'_{N,m} \to \mu_\ell$, non-degenerate on both sides, with $b_{N,m}(\gamma Z, \gamma Y) = \gamma\, b_{N,m}(Z, Y)$ for $\gamma \in \Gamma_N$;*

(Bs1) *there is $c'_1 \ge 0$ with $\#H^0(\Gamma_N, M_{N,m}) \le K^{c'_1}$ for all N, m;*

(Bs2) *for $v \in S_\ell$ there is $C_v \ge 0$ with $\#(B^1(D_v, M_{N,m}) + \mathrm{Zb}_v) \le \#M_{N,m} \cdot K^{m(n+\nu)d_v + C_v}$ for all N, m;*

(Bs3) *there is $C_q \ge 0$ such that for all m and j there is N_0 with $\#H^0(D_{v_{N,j}}, M'_{N,m}) \le K^{mn+C_q}$ for $N \ge N_0$ (local groups at $\bar{w}_{N,j}$);*

(Bs4) *there is $c_0 \ge 0$ with $\#H^0(D_{v_0}, M'_{N,m}) \le K^{c_0}$ for all N, m (local group at $\bar{v}_0$);*

(Bs6) *there is $c_2 \ge 0$ such that for every m there is N_0 such that for $N \ge N_0$ the set of $y \in Z^1(\Gamma_N, M'_{N,m})$ with $y \circ \iota_{v_{N,j}} \in B^1(D_{v_{N,j}}, M'_{N,m})$ for $1 \le j \le q$ has at most $\#B^1(\Gamma_N, M'_{N,m}) \cdot K^{c_2}$ elements.*

Proof. Each item is a lemma proved separately. (Bs0) is Lemma E.106. (Bs1) is (H1) of Lemma E.109 ($H^0(\Gamma_N, M_{N,m})$ is the set of elements fixed by Γ_N), which uses the spanning hypothesis. (Bs2) is the second inequality of Lemma E.111, which uses (Pj1). (Bs3) and (Bs4) are (H3) and (H4) of Lemma E.109 (the local invariants are the elements fixed by $\iota(D_{v_{N,j}})$, resp. $\iota(D_{v_0})$); (H4) uses (Np4). (Bs6) is Lemma E.113, which uses that $\{\sigma_1, \dots, \sigma_q\}$ detects (x, κ) and that ρ is trivial on $\Gamma^0_{\mathscr{H}} \cap \mathrm{PL}(H)$. □

Lemma E.106. *Let $x = (k, \rho, 1_n, (\psi_{v,a}), f)$ be a curve point normalised of shape $(a, n-a)$ over a rank-n Hecke datum $\mathscr{H}$, κ a Selmer basis of x, and a Taylor–Wiles datum; let $N, m \geq 1$. Then the formulas $\gamma \cdot Y = \chi_{\mathrm{cyc}}(\gamma)\rho_1(h_N\gamma)Y\rho_1(h_N\gamma)^{-1}$ ($\gamma \in \Gamma_N^0$) and $c_N \cdot Y = -\chi_{\mathrm{cyc}}(c)\,{}^tY$ define a continuous action of Γ_N on the finite group $M'_{N,m}$. Fix a primitive ℓ-th root of unity ζ and put $\tau(y) = \mathrm{Tr}_{k/\mathbb{F}_\ell}(y_{m-1})$ for $y = \sum_{i<m} y_i T^i \in A_m$. Then*

$$b_{N,m}(Z,Y) = \zeta^{\tau(e(Z,Y))}, \qquad e(Z,Y) = \textit{the } \eta\textit{-coefficient of } \mathrm{tr}(ZY) \in A_m,$$

is a bi-additive map $M_{N,m} \times M'_{N,m} \to \mu_\ell$, non-degenerate on both sides, with $b_{N,m}(\gamma Z, \gamma Y) = \gamma\, b_{N,m}(Z,Y)$ for $\gamma \in \Gamma_N$ (Γ_N acting on $M_{N,m}$ as in Lemma E.107 (Rf2) and on μ_ℓ through χ_{cyc}). In particular (Bs0) of Proposition E.105 holds.

Proof. Write $M = M_{N,m}$, $M' = M'_{N,m}$; values of χ_{cyc} are taken in $\mathbb{F}_\ell \subseteq A_m$. Every $Y \in M'$ is $Y = \eta Y_0 + \sum_j \epsilon'_j Y_j$ with $Y_0 \in \mathrm{Ev}_a(A_m)$, $Y_j \in \mathrm{Od}_a(A_m)$.

The action. For $\gamma \in \Gamma^0_{\mathscr{H}}$ Lemma E.108 gives $\rho_1(\gamma)Y\rho_1(\gamma)^{-1} = \eta\big(\rho Y_0\rho^{-1} + \sum_j [\kappa_j(\gamma), \rho Y_j \rho^{-1}]\big) + \sum_j \epsilon'_j \rho Y_j \rho^{-1}$ ($\rho = \rho(\gamma)$), which lies in M': ρ is block diagonal, so $\rho Y_0 \rho^{-1} \in \mathrm{Ev}_a$, $\rho Y_j \rho^{-1} \in \mathrm{Od}_a$, and $[\mathrm{Od}_a, \mathrm{Od}_a] \subseteq \mathrm{Ev}_a$. Also ${}^tM' = M'$. As ρ_1 and χ_{cyc} are homomorphisms (Lemma E.107 (Rf1)), the formula defines an action of Γ_N^0. For $\gamma \in \Gamma_N^0$, $\gamma' = h_N\gamma$, using $\chi_{\mathrm{cyc}}(c)^2 = 1$, $\chi_{\mathrm{cyc}}(c_N\gamma c_N) = \chi_{\mathrm{cyc}}(\gamma)$ and $\rho_1(\bar{c}\gamma'\bar{c}) = {}^t\rho_1(\gamma')^{-1}$ ((Rf1)):

$$c_N \cdot (\gamma \cdot (c_N \cdot Y)) = \chi_{\mathrm{cyc}}(\gamma)\,{}^t\big(\rho_1(\gamma')\,{}^tY\rho_1(\gamma')^{-1}\big) = \chi_{\mathrm{cyc}}(\gamma)\,{}^t\rho_1(\gamma')^{-1}Y\,{}^t\rho_1(\gamma') = (c_N\gamma c_N)\cdot Y,$$

and $c_N \cdot (c_N \cdot Y) = Y$. Since Γ_N^0 has index 2 in Γ_N and $c_N^2 = 1$, $c_N \notin \Gamma_N^0$, these identities say exactly that $\gamma \mapsto (\cdot)$ on Γ_N^0 and $c_N\gamma \mapsto c_N \cdot (\gamma\cdot\,)$ define an action of Γ_N (the same argument as for (Rf2)). It is continuous: on Γ_N^0 it factors through h_N and the reduction of $(\rho_1, \chi_{\mathrm{cyc}})$ modulo (T^m, ℓ), a continuous homomorphism to a finite group (ρ and the κ_j have T-adically continuous entries), and Γ_N^0 is open.

The form τ. $\tau\colon A_m \to \mathbb{F}_\ell$ is $\mathbb{F}_\ell$-linear, and if $y \neq 0$ there is z with $\tau(yz) \neq 0$: if i is minimal with $y_i \neq 0$, take $z = cT^{m-1-i}$ with $c \in k$ and $\mathrm{Tr}_{k/\mathbb{F}_\ell}(y_i c) \neq 0$ (the trace form of the separable extension $k/\mathbb{F}_\ell$ is non-degenerate). The same holds for A_m^g with $(y,z) \mapsto \tau(\sum_t y_t z_t)$.

Bi-additivity and non-degeneracy. e is bi-additive, so $b_{N,m}$ is. For a matrix unit E_{ji} and $y \in \mathrm{PD}_g(A_m)$, $\mathrm{tr}(ZE_{ji}y) = Z_{ij}y$. Let $Z \in M$ with $b_{N,m}(Z,\cdot) = 1$. If i,j lie in the same block, $Z_{ij} \in A_m$ and $E_{ji}\eta y' \in M'$ for $y' \in A_m$, with $e(Z, E_{ji}\eta y') = Z_{ij}y'$; so $\tau(Z_{ij}y') = 0$ for all y' and $Z_{ij} = 0$. If they lie in different blocks, $Z_{ij} = \sum_t z_t\epsilon_t$ and $E_{ji}\sum_t y'_t\epsilon'_t \in M'$ with $e = \sum_t z_t y'_t$ ($\epsilon_t\epsilon'_s = \delta_{ts}\eta$); so all $z_t = 0$. Hence $Z = 0$. Symmetrically (using $\mathrm{tr}(ZY) = \mathrm{tr}(YZ)$ and the matrix units E_{ji} in M with entries in A_m or $\bigoplus_t A_m\epsilon_t$), $b_{N,m}(\cdot, Y) = 1$ forces $Y = 0$.

Equivariance. For $\gamma \in \Gamma_N^0$ with $r = \rho_1(h_N\gamma)$: $\mathrm{tr}\big(rZr^{-1} \cdot \chi_{\mathrm{cyc}}(\gamma) rYr^{-1}\big) = \chi_{\mathrm{cyc}}(\gamma)\mathrm{tr}(ZY)$, so $e(\gamma Z, \gamma Y) = \chi_{\mathrm{cyc}}(\gamma)e(Z,Y)$ and $b_{N,m}(\gamma Z, \gamma Y) = \zeta^{\chi_{\mathrm{cyc}}(\gamma)\tau(e(Z,Y))} = \gamma\, b_{N,m}(Z,Y)$ (τ is $\mathbb{F}_\ell$-linear). For c_N: $\mathrm{tr}\big((-{}^tZ)(-\chi_{\mathrm{cyc}}(c)\,{}^tY)\big) = \chi_{\mathrm{cyc}}(c)\mathrm{tr}({}^t(YZ)) = \chi_{\mathrm{cyc}}(c)\mathrm{tr}(ZY)$, so again $b_{N,m}(c_N Z, c_N Y) = c_N\, b_{N,m}(Z,Y)$. As Γ_N is generated by Γ_N^0 and c_N, equivariance holds for all of Γ_N. □

Lemma E.107. *Let $x = (k, \rho, 1_n, (\psi_{v,a}), f)$ be a curve point normalised of shape $(a, n-a)$ over a rank-n Hecke datum $\mathscr{H}$, $\kappa = (\kappa_1, \ldots, \kappa_g)$ a Selmer basis of x and $\bar{c} = \pi_{\mathscr{H}}(c)$. Fix a Taylor–Wiles datum and let $N, m \geq 1$, $c_N = \pi^{Q_N}(c)$, $\Gamma_N^0 = h_N^{-1}(\Gamma^0_{\mathscr{H}})$, $M = \mathrm{PW}^1_a(A_m)$, and $\nu = \binom{a}{2} + \binom{n-a}{2} + g\,a(n-a)$. Then:*

(Rf1) *$\rho_1(\gamma\gamma') = \rho_1(\gamma)\rho_1(\gamma')$ and $\rho_1(\bar{c}\gamma\bar{c}) = {}^t\rho_1(\gamma)^{-1}$ for $\gamma, \gamma' \in \Gamma^0_{\mathscr{H}}$, and $\rho_1(\gamma)M\rho_1(\gamma)^{-1} = M$ (entries mapped to $\mathrm{PD}_g(A_m)$);*

(Rf2) *$\gamma \cdot Z = \rho_1(h_N\gamma) Z \rho_1(h_N\gamma)^{-1}$ ($\gamma \in \Gamma_N^0$) and $c_N \cdot Z = -{}^tZ$ define a continuous action of Γ_N on the finite group M;*

(Rf3) *for every $c' \in \Gamma_N \smallsetminus \Gamma_N^0$ with $c'^2 = 1$, $\#\{Z \in M : \ c' \cdot Z = Z\} \geq (\#k)^{m\nu}$.*

Proof. (Rf1). Write $\kappa_\epsilon = \sum_j \epsilon_j \kappa_j$; products of two ϵ's vanish. The cocycle rule $\kappa_\epsilon(\gamma\gamma') = \kappa_\epsilon(\gamma) + \rho(\gamma)\kappa_\epsilon(\gamma')\rho(\gamma)^{-1}$ gives

$$(1+\kappa_\epsilon(\gamma))\rho(\gamma)(1+\kappa_\epsilon(\gamma'))\rho(\gamma') = (1+\kappa_\epsilon(\gamma)+\rho(\gamma)\kappa_\epsilon(\gamma')\rho(\gamma)^{-1})\rho(\gamma\gamma') = \rho_1(\gamma\gamma').$$

As x is normalised, $\rho(\gamma^*) = {}^t\rho(\gamma)$, i.e. $\rho(\bar c\gamma\bar c) = {}^t\rho(\gamma^{-1}) = {}^t\rho(\gamma)^{-1}$; with $\kappa_j(\bar c\gamma\bar c) = -{}^t\kappa_j(\gamma)$ (Definition E.34 (Sd2)),

$$\rho_1(\bar c\gamma\bar c) = (1 - {}^t\kappa_\epsilon(\gamma))\,{}^t\rho(\gamma)^{-1} = \big({}^t\rho(\gamma)(1+{}^t\kappa_\epsilon(\gamma))\big)^{-1} = {}^t\rho_1(\gamma)^{-1}.$$

For the last claim: if $a < n$, $\rho(\gamma)$ is block diagonal and $\kappa_j(\gamma) \in \mathrm{Od}_a(A)$, so in $\rho_1(\gamma) = \rho(\gamma) + \sum_j \epsilon_j\kappa_j(\gamma)\rho(\gamma)$ the second term has different-block entries in $\bigoplus_j A\epsilon_j$ and zero same-block entries. For $Z \in M$ (same-block entries in A_m, different-block entries in $\bigoplus_j A_m\epsilon_j$), $\rho(\gamma)Z$ and $Z\rho(\gamma)$ have the same pattern, and $\epsilon_j\kappa_j(\gamma)\rho(\gamma)Z$, $Z\epsilon_j\kappa_j(\gamma)\rho(\gamma)$ have different-block entries in $A_m\epsilon_j$ and same-block entries in the span of products $\epsilon_j\epsilon_{j'} = 0$. So $\rho_1(\gamma)M \subseteq M$, $M\rho_1(\gamma) \subseteq M$, and the same for $\rho_1(\gamma)^{-1} = \rho_1(\gamma^{-1})$. If $a = n$, $\mathrm{Od}_n = 0$ and $\rho_1 = \rho$. Hence $\rho_1(\gamma)M\rho_1(\gamma)^{-1} = M$.

(Rf2). $\Gamma_N = \Gamma_N^0 \sqcup \Gamma_N^0 c_N$ with $c_N^2 = 1$ and $h_N(c_N) = \bar c$. By (Rf1), $\gamma \mapsto \rho_1(h_N\gamma)$ is a homomorphism and conjugation by it preserves M; transposition preserves M (its pattern is symmetric). The rule $(\gamma c_N^\varepsilon)\cdot Z = \gamma\cdot(c_N^\varepsilon \cdot Z)$ therefore defines an action provided $c_N\cdot(c_N\cdot Z) = Z$, which is clear, and $c_N \cdot (\gamma \cdot Z) = (c_N\gamma c_N)\cdot(c_N\cdot Z)$, which reads, with $S = \rho_1(h_N\gamma)$ and $\rho_1(\bar c\, h_N(\gamma)\, \bar c) = {}^tS^{-1}$ from (Rf1),

$$-{}^t(SZS^{-1}) = -{}^tS^{-1}\,{}^tZ\,{}^tS = {}^tS^{-1}(-{}^tZ)\,{}^tS.$$

The action is continuous: $\rho_1 \bmod T^m$ is locally constant (the entries of ρ and κ_j are T-adically continuous), h_N is continuous and Γ_N^0 is open.

(Rf3). Put $g = c'c_N \in \Gamma_N^0$ (both c' and c_N lie in the non-trivial coset of Γ_N^0). From $c'^2 = c_N^2 = 1$, $c_Ngc_N = c_Nc' = g^{-1}$, so by (Rf1) $S := \rho_1(h_Ng)$ satisfies $S^{-1} = \rho_1(h_Ng^{-1}) = \rho_1(\bar c\, h_N(g)\, \bar c) = {}^tS^{-1}$, i.e. ${}^tS = S$. Since $c' = gc_N$, $c'\cdot Z = S(-{}^tZ)S^{-1}$. For $Y \in M$ with ${}^tY = -Y$, $Z = SY$ lies in M by (Rf1) and

$$c'\cdot Z = -S\,{}^tY\,{}^tS\,S^{-1} = SY = Z.$$

$Y \mapsto SY$ is injective, so $\#M^{c'=1}$ is at least the number of $Y \in M$ with ${}^tY = -Y$. Such Y has zero diagonal (ℓ odd) and is determined by its entries above the diagonal, which are arbitrary in A_m for the $\binom{a}{2} + \binom{n-a}{2}$ same-block positions and arbitrary in $\bigoplus_{j=1}^g A_m\epsilon_j$ for the $a(n-a)$ different-block positions; as $\#A_m = (\#k)^m$, their number is $(\#k)^{m\nu}$. □

Lemma E.108. *Let x be a normalised curve point, $\kappa = (\kappa_1, \dots, \kappa_g)$ a Selmer basis of x, X a commutative A-algebra and $\gamma \in \Gamma^0$; write $\rho = \rho(\gamma)$, $\kappa_j = \kappa_j(\gamma)$ (entries mapped to X). For $Z_0, Z_1, \dots, Z_g, Y_0, Y_1, \dots, Y_g \in \mathrm{M}_n(X)$,*

$$\rho_1(\gamma)\Big(Z_0 + \sum_j \epsilon_j Z_j\Big)\rho_1(\gamma)^{-1} = \rho Z_0\rho^{-1} + \sum_j \epsilon_j(\rho Z_j\rho^{-1} + [\kappa_j, \rho Z_0\rho^{-1}]),$$

$$\rho_1(\gamma)\Big(\eta Y_0 + \sum_j \epsilon'_j Y_j\Big)\rho_1(\gamma)^{-1} = \eta\Big(\rho Y_0\rho^{-1} + \sum_j [\kappa_j, \rho Y_j\rho^{-1}]\Big) + \sum_j \epsilon'_j \rho Y_j\rho^{-1}.$$

Proof. Put $E=\sum_j \epsilon_j\kappa_j$, so $\rho_1(\gamma)=(1+E)\rho$ and, as $\epsilon_i\epsilon_j=0$ gives $E^2=0$, $\rho_1(\gamma)^{-1}=\rho^{-1}(1-E)$. For $W\in \mathrm{M}_n(\mathrm{PD}_g(X))$, $\rho_1 W\rho_1^{-1}=(1+E)W'(1-E)=W'+EW'-W'E-EW'E$ with $W'=\rho W\rho^{-1}$; the scalars $\epsilon_j,\epsilon'_j,\eta$ are central. In PD_g, $\epsilon_i\epsilon_j=0$, $\epsilon_i\epsilon'_j=\delta_{ij}\eta$ and $\epsilon_i\eta=0$. For $W=Z_0+\sum_j\epsilon_jZ_j$: $EW'=\sum_j\epsilon_j\kappa_j\rho Z_0\rho^{-1}$, $W'E=\sum_j\epsilon_j\rho Z_0\rho^{-1}\kappa_j$, $EW'E=0$, which gives the first identity. For $W=\eta Y_0+\sum_j\epsilon'_jY_j$: $EW'=\eta\sum_j\kappa_j\rho Y_j\rho^{-1}$, $W'E=\eta\sum_j\rho Y_j\rho^{-1}\kappa_j$, and $EW'E=0$ (every term contains $\epsilon_i\eta$ or $\epsilon_i\epsilon_k$ with a further factor ϵ), which gives the second. □

Lemma E.109. *Let $\mathscr{H}$ be a rank-n Hecke datum, x a framed normalised curve point of shape $(a,n-a)$, $\kappa=(\kappa_1,\dots,\kappa_g)$ a Selmer basis of x, and a Taylor–Wiles datum of size q (with H, $\sigma_1,\dots,\sigma_q$) along which x has eigenvalue data; assume $\rho(\gamma)=1$ for $\gamma\in\Gamma^0_{\mathscr{H}}\cap\mathrm{PL}(H)$. For a subset S of Γ_N write M^S for the elements of $M=M_{N,m}$ or $M=M'_{N,m}$ fixed by every element of S under the formulas of Proposition E.105. Take $D_{v_0}=D_{\bar v_0}\cap G_{L^+}$ and $D_{v_{N,j}}=D_{\bar w_{N,j}}\cap G_{L^+}$, with $\iota=\pi^{Q_N}$ on them.*

(H1) *If for every $Z\in\mathrm{Ev}_a(A)$ some T^sZ lies in the A-span of $\rho(\Gamma')$, then there is $c'_1\ge0$ with $\#M_{N,m}^{\Gamma_N}\le K^{c'_1}$ for all $N,m\ge1$.*

(H3) *There is $C_q\ge0$ such that for all $m\ge1$ and $1\le j\le q$ there is N_0 with $\#(M'_{N,m})^{\iota(D_{v_{N,j}})}\le K^{mn+C_q}$ for all $N\ge N_0$.*

(H4) *If x satisfies (Np4), there is $c_0\ge0$ with $\#(M'_{N,m})^{\iota(D_{v_0})}\le K^{c_0}$ for all $N,m\ge1$.*

Proof. Write $\Gamma^0=\Gamma^0_{\mathscr{H}}$ and, for $\gamma\in\Gamma^0$, $E(\gamma)=\sum_j\epsilon_j\kappa_j(\gamma)$, so $\rho_1(\gamma)=(1+E(\gamma))\rho(\gamma)$ and, as $\epsilon_i\epsilon_j=0$, $\rho_1(\gamma)^{-1}=\rho(\gamma)^{-1}(1-E(\gamma))$. For a commutative A-algebra X every $Z\in\mathrm{PW}^1_a(X)$ is uniquely $Z=Z_0+\sum_j\epsilon_jZ_j$ with $Z_0\in\mathrm{Ev}_a(X)$, $Z_j\in\mathrm{Od}_a(X)$, and every $Y\in\mathrm{PB}_a(X)$ is uniquely $Y=\eta Y_0+\sum_j\epsilon'_jY_j$ with $Y_0\in\mathrm{Ev}_a(X)$, $Y_j\in\mathrm{Od}_a(X)$. By Lemma E.108, writing $\rho=\rho(\gamma)$, $\kappa_j=\kappa_j(\gamma)$:

$$\begin{aligned}\rho_1Z\rho_1^{-1}&=\rho Z_0\rho^{-1}+\sum_j\epsilon_j\big(\rho Z_j\rho^{-1}+[\kappa_j,\rho Z_0\rho^{-1}]\big),\\ \rho_1Y\rho_1^{-1}&=\eta\Big(\rho Y_0\rho^{-1}+\sum_j[\kappa_j,\rho Y_j\rho^{-1}]\Big)+\sum_j\epsilon'_j\rho Y_j\rho^{-1}.\end{aligned}\tag{1}$$

The modules $P_1=\mathrm{PW}^1_a(A)$ and $P_2=\mathrm{PB}_a(A)$ are free of finite rank over A, with $P_1/T^mP_1=M_{N,m}$, $P_2/T^mP_2=M'_{N,m}$ and $P_i\otimes_AF=\mathrm{PW}^1_a(F)$, $\mathrm{PB}_a(F)$. For $\gamma\in\Gamma^0$ and $e\in\mathbb{Z}$ the maps $Z\mapsto e\rho_1(\gamma)Z\rho_1(\gamma)^{-1}-Z$ and $Z\mapsto-{}^tZ-Z$ are A-linear maps into the free module $\mathrm{M}_n(\mathrm{PD}_g(A))$, and the action formulas of Proposition E.105 on $M_{N,m}$, $M'_{N,m}$ are their reductions modulo T^m (with γ replaced by $h_N\gamma$ and e by the image of χ_{cyc}). Each item is proved by applying Lemma E.110 to a family of such maps that does not depend on N (in (H3): for $N\ge N_0$).

(H1). $\Gamma^0_N=h_N^{-1}(\Gamma^0)$ has index 2 in Γ_N and $c_N\notin\Gamma^0_N$, so Γ_N is generated by Γ^0_N and c_N, and since h_N is surjective, $M_{N,m}^{\Gamma_N}$ is the common kernel modulo T^m of the maps $\varphi_\gamma(Z)=\rho_1(\gamma)Z\rho_1(\gamma)^{-1}-Z$ ($\gamma\in\Gamma^0$) and $\varphi_c(Z)=-{}^tZ-Z$ on P_1; this family does not depend on N. By Lemma E.110 it suffices to show that 0 is the only $Z\in P_1$ killed by all of them. Let Z be such. By (1), $\rho(\gamma)Z_0=Z_0\rho(\gamma)$ for $\gamma\in\Gamma^0$, so Z_0 commutes with the A-span of $\rho(\Gamma')$. For each i the elementary matrix E_{ii} lies in $\mathrm{Ev}_a(A)$, so some T^sE_{ii} lies in that span, $T^s(E_{ii}Z_0-Z_0E_{ii})=0$, and as A is a domain, Z_0 commutes with every E_{ii}: Z_0 is diagonal. From $\varphi_c(Z)=0$, ${}^tZ_0=-Z_0$, so $2Z_0=0$ and $Z_0=0$ (ℓ is odd, so $2\in A^\times$). Then (1) gives $\rho(\gamma)Z_j=Z_j\rho(\gamma)$ for all $\gamma\in\Gamma^0$, and the same argument shows that Z_j is diagonal; as $Z_j\in\mathrm{Od}_a(A)$ has zero diagonal, $Z_j=0$. So $Z=0$, and Lemma E.110 gives c'_1.

(H4). By (Rd2) v_0 splits in L, so $D_{v_0} \subseteq G_L$, $\iota(D_{v_0}) \subseteq \pi^{Q_N}(G_L) = \Gamma_N^0$, and $h_N(\iota(g)) = \pi_{\mathscr{H}}(g)$ for $g \in D_{v_0}$; moreover $\pi_{\mathscr{H}}(D_{v_0}) = \Delta_0$. Hence $(M'_{N,m})^{\iota(D_{v_0})}$ is the common kernel modulo T^m of the maps $\psi_\delta(Y) = \chi_{\mathrm{cyc}}(\delta)\rho_1(\delta)Y\rho_1(\delta)^{-1} - Y$ ($\delta \in \Delta_0$) on P_2, independent of N. Let $Y \in P_2$ be killed by all ψ_δ. By (1), the ϵ'_j-component gives $\chi_{\mathrm{cyc}}(\delta)\rho(\delta)Y_j = Y_j\rho(\delta)$ for all $\delta \in \Delta_0$, so $Y_j = 0$ by (Np4); then the η-component gives $\chi_{\mathrm{cyc}}(\delta)\rho(\delta)Y_0 = Y_0\rho(\delta)$, so $Y_0 = 0$ by (Np4). By Lemma E.110 (with $r = 0$) there is c_0 as stated.

(H3). Fix j and m. By (Tg2) $v_{N,j}$ splits in L, so $D_{v_{N,j}} \subseteq G_L$; it contains $\phi_{N,j}$, and $\iota(\phi_{N,j}) = \Phi_{N,j} \in \Gamma_N^0$. Since $\phi_{N,j}^{-1}$ is a Frobenius element for the norm $q_{N,j} \equiv 1 \bmod \ell^N$, $\chi_{\mathrm{cyc}}(\phi_{N,j}) \equiv 1 \bmod \ell$, so its image in k is 1. Hence every element of $(M'_{N,m})^{\iota(D_{v_{N,j}})}$ is fixed by $Y \mapsto r_m(h_N\Phi_{N,j})Y r_m(h_N\Phi_{N,j})^{-1}$, where $r_m(\gamma)$ is $\rho_1(\gamma)$ with entries reduced modulo T^m.

Approximation. $r_m\colon \Gamma^0 \to \mathrm{GL}_n(\mathrm{PD}_g(A_m))$ is a homomorphism (Lemma E.107 (Rf1)) into a finite group, and it is continuous because the entries of ρ and of the κ_i are T-adically continuous (the definition of curve points and (Sd2)). So $U_m = \ker r_m$ is open in Γ^0, hence in $\Gamma_{\mathscr{H}}$, and $U_m \supseteq \mathrm{PL}(H)$ by Lemma E.96 (Pv1). The intersection V of the finitely many conjugates of U_m in $\Gamma_{\mathscr{H}}$ is an open normal subgroup, and $V \supseteq \mathrm{PL}(H)$ since $\mathrm{PL}(H)$, an intersection of normal subgroups, is normal. By (Tg3) there is N_0 with $h_N(\Phi_{N,j})\sigma_j^{-1} \in V \subseteq U_m$ for $N \geq N_0$; as $h_N(\Phi_{N,j}) \in \Gamma^0$ and $\sigma_j \in \Gamma' \subseteq \Gamma^0$, $r_m(h_N\Phi_{N,j}) = r_m(\sigma_j)$ for $N \geq N_0$. So for $N \geq N_0$, $(M'_{N,m})^{\iota(D_{v_{N,j}})}$ lies in the kernel modulo T^m of the single map $\psi(Y) = \rho_1(\sigma_j)Y\rho_1(\sigma_j)^{-1} - Y$ on P_2, which depends neither on N nor on m.

The rank. Let $W \subseteq \mathrm{PB}_a(F)$ be the kernel of $\psi \otimes F$ and $P = \rho(\sigma_j) = \mathrm{diag}(P'_1, P'_2)$ (block diagonal, x being normalised). The characteristic polynomials f_1, f_2 of P'_1, P'_2 have product $\prod_i (X - \alpha_{j,i})$ with the $\alpha_{j,i} \in A$ pairwise distinct, so they have no common root, and $f_1(P'_2) = \prod_{f_1(\alpha)=0}(P'_2 - \alpha)$ is invertible over F. If $B \in \mathrm{M}_{a\times(n-a)}(F)$ and $P'_1B = BP'_2$, then $0 = f_1(P'_1)B = Bf_1(P'_2)$ (Cayley–Hamilton), so $B = 0$; likewise $P'_2C = CP'_1$ forces $C = 0$. Hence every matrix over F commuting with P lies in $\mathrm{Ev}_a(F)$, and no non-zero element of $\mathrm{Od}_a(F)$ commutes with P. Now let $Y \in W$. By (1) the ϵ'_j-components give $PY_iP^{-1} = Y_i$ with $Y_i \in \mathrm{Od}_a(F)$, so $Y_i = 0$; then the η-component gives $PY_0 = Y_0P$. Conversely every ηY_0 with Y_0 in the centraliser of P in $\mathrm{M}_n(F)$ lies in W. As P has n distinct eigenvalues in F, it is diagonalisable over F and its centraliser has dimension n. So $\dim_F W = n$, and Lemma E.110 gives $C_{(j)}$ with $\#(M'_{N,m})^{\iota(D_{v_{N,j}})} \leq K^{mn+C_{(j)}}$ for $N \geq N_0$. Take $C_q = \max_j C_{(j)}$. □

Lemma E.110. *Let k be a finite field, $K = \#k$, $A = k[[T]]$ with fraction field F, and $A_m = A/T^mA$ ($m \geq 1$). Let P be a free A-module of finite rank and $(\varphi_i)_{i\in I}$ a family, over an arbitrary index set I, of A-linear maps $\varphi_i\colon P \to P_i$ into free A-modules of finite rank. Let $W = \bigcap_{i\in I} \ker(\varphi_i \otimes F) \subseteq P \otimes_A F$ and $r = \dim_F W$. Then there is $C \geq 0$ such that for every $m \geq 1$*

$$\#\{x \in P/T^mP :\ (\varphi_i \otimes A_m)(x) = 0 \text{ for all } i \in I\} \ \leq\ K^{mr+C}.$$

Moreover $r = 0$ if and only if 0 is the only $x \in P$ with $\varphi_i(x) = 0$ for all $i \in I$.

Proof. The subspaces $\bigcap_{i\in I_0} \ker(\varphi_i \otimes F)$, $I_0 \subseteq I$ finite, of the finite-dimensional space $P \otimes_A F$ have a minimal element; it is contained in each $\ker(\varphi_i \otimes F)$ (otherwise adding i makes it smaller), so it equals W. Fix such a finite $I_0 = \{i_1, \dots, i_s\}$ and put $\Phi = (\varphi_{i_1}, \dots, \varphi_{i_s})\colon P \to P' = \bigoplus_{t=1}^s P_{i_t}$, an A-linear map between free modules of finite ranks p and p' with $\ker(\Phi \otimes F) = W$, so $\Phi \otimes F$ has rank $p - r$.

A is a discrete valuation ring with uniformiser T. By the Smith normal form over a principal ideal domain there are bases $e_1, \dots, e_p$ of P and $f_1, \dots, f_{p'}$ of P' and integers $b_1, \dots, b_{p-r} \geq 0$ with $\Phi(e_i) = T^{b_i} f_i$ for $i \leq p - r$ and $\Phi(e_i) = 0$ for $i > p - r$ (the number of non-zero invariant

factors is the rank of $\Phi\otimes F$, and every non-zero element of A is a unit times a power of T). Put $C=\sum_{i\le p-r} b_i$.

Let $m\ge 1$. Reducing modulo T^m, we have $P/T^mP=\bigoplus_i A_m\bar e_i$ and

$$(\Phi\otimes A_m)\Big(\sum_i x_i\bar e_i\Big)=\sum_{i\le p-r} T^{b_i}x_i\bar f_i,$$

with the $\bar f_i$ a basis of P'/T^mP'. So $\ker(\Phi\otimes A_m)$ consists of the $\sum_i x_i\bar e_i$ with $T^{b_i}x_i=0$ in A_m for $i\le p-r$, i.e. $x_i\in T^{\max(m-b_i,0)}A_m$, a group of order $K^{\min(b_i,m)}$, and $x_i\in A_m$ arbitrary for $i>p-r$. Hence

$$\#\ker(\Phi\otimes A_m)=K^{mr}\prod_{i\le p-r}K^{\min(b_i,m)}\le K^{mr+C}.$$

The set of the statement is contained in $\ker(\Phi\otimes A_m)$, which proves the bound.

For the last assertion: $\bigcap_i\ker\varphi_i=W\cap P$ (an element of P is killed by φ_i if and only if it is killed by $\varphi_i\otimes F$, as P_i is torsion free). If $w\in W$ is non-zero, then $T^sw\in P$ for some $s\ge 0$ and $T^sw\in W\cap P$ is non-zero. So $W=0$ if and only if $W\cap P=0$. □

Lemma E.111. *Let $\mathscr{H}$ be a rank-n Hecke datum, x a framed normalised curve point of shape $(a,n-a)$ satisfying (Pj1), $\kappa=(\kappa_1,\dots,\kappa_g)$ a Selmer basis of x with local witnesses $Y_{j,v}$, and a Taylor–Wiles datum. Let $v\in S_\ell$, with D_v acting on $M_{N,m}$ through $\pi^{Q_N}|_{D_v}$. Then there is $C_v\ge 0$ such that for all $N,m\ge 1$*

$$\#\mathrm{Zb}_v\le K^{m(n+\nu)(d_v+1)+C_v}\qquad\text{and}\qquad\#\big(B^1(D_v,M_{N,m})+\mathrm{Zb}_v\big)\le\#M_{N,m}\cdot K^{m(n+\nu)d_v+C_v}.$$

Proof. Fix N,m, write $M=M_{N,m}$, $w=w_v$, $w^1=w_v^1$, $\psi_i=\psi_{v,i}$, $\pi=\pi_{\mathscr{H}}$. As $D_v\subseteq G_L$ and $h_N\circ\pi^{Q_N}=\pi$, $g\in D_v$ acts on M by $Z\mapsto\rho_1(\pi g)Z\rho_1(\pi g)^{-1}$ (entries reduced modulo T^m).

Step 1: the frame turns the blocks into types. With the types $\mathrm{ty}(i)$ of the columns of w, Lemma E.112 gives $w^{-1}\mathrm{Ev}_a(X)w=\mathrm{Ev}_{\mathrm{ty}}(X)$ and $w^{-1}\mathrm{Od}_a(X)w=\mathrm{Od}_{\mathrm{ty}}(X)$ for every commutative A-algebra X, exactly a indices having type 1. Conjugation by $1-\sum_j\epsilon_jY_{j,v}$ ($Y_{j,v}\in\mathrm{Od}_a(A)$) maps $Z_0+\sum_j\epsilon_jZ_j$ to $Z_0+\sum_j\epsilon_j(Z_j-[Y_{j,v},Z_0])$, an automorphism of $\mathrm{PW}^1_a(A_m)$ since $[\mathrm{Od}_a,\mathrm{Ev}_a]\subseteq\mathrm{Od}_a$. Hence $Z\mapsto Z'=(w^1)^{-1}Zw^1$ is an isomorphism of M onto $M^w=\{Z_0+\sum_j\epsilon_jZ_j:\ Z_0\in\mathrm{Ev}_{\mathrm{ty}}(A_m),\ Z_j\in\mathrm{Od}_{\mathrm{ty}}(A_m)\}$, carrying the action to $Z'\mapsto R(g)Z'R(g)^{-1}$ with $R(g)=(w^1)^{-1}\rho_1(\pi g)w^1$.

Step 2: $R(g)$ is upper triangular with diagonal $(\psi_1(g),\dots,\psi_n(g))$. Using $\epsilon_i\epsilon_j=0$,

$$R(g)=w^{-1}\Big(1+\sum_j\epsilon_j\big(\kappa_j(\pi g)+Y_{j,v}-\rho(\pi g)Y_{j,v}\rho(\pi g)^{-1}\big)\Big)\rho(\pi g)w=\Big(1+\sum_j\epsilon_ju_j(g)\Big)b(g),$$

with $b(g)=w^{-1}\rho(\pi g)w$ upper triangular with diagonal $(\psi_i(g))$ (the frame, (Sd1)) and $u_j(g)=w^{-1}(\kappa_j(\pi g)+Y_{j,v}-\rho(\pi g)Y_{j,v}\rho(\pi g)^{-1})w$, upper triangular because $Y_{j,v}$ is a local witness for κ_j ((Sd2)), and in $\mathrm{Od}_{\mathrm{ty}}(A)$ by Step 1 ($\rho(\pi g)$ is block diagonal), so with zero diagonal. Hence $R(g)$ is upper triangular with $R(g)_{ii}=\psi_i(g)$.

Step 3: the upper triangular part. Let $U\subseteq M^w$ be the upper triangular elements. Its entries are: on the diagonal, arbitrary elements of A_m (n entries; the Z_j have zero diagonal); at (i,j), $i<j$, arbitrary elements of A_m if $\mathrm{ty}(i)=\mathrm{ty}(j)$ ($\binom{a}{2}+\binom{n-a}{2}$ pairs) and of $\bigoplus_t A_m\epsilon_t$ otherwise ($a(n-a)$ pairs). So $\#U=K^{m(n+\nu)}$. U is stable under the action. For $h\ge 0$ let $U_h\subseteq U$ be the elements whose entries of height $j-i<h$ vanish. For upper triangular R and Z', $(RZ'R^{-1})_{ij}=\sum_{i\le i'\le j'\le j}R_{ii'}Z'_{i'j'}(R^{-1})_{j'j}$ involves only entries of height $\le j-i$, and its term

with $(i',j')=(i,j)$ is $R_{ii}Z'_{ij}R_{jj}^{-1}$. Hence each U_h is stable, and $U_h/U_{h+1}\cong\bigoplus_{j-i=h}X_{ij}$, where X_{ij} ($=A_m$ or A_m^g) carries the action of g by multiplication by $\theta_{ij}(g)=\psi_i(g)\psi_j(g)^{-1}$.

Step 4: counting Zb_v. The isomorphism of Step 1 maps Zb_v onto the set Zb' of cocycles $\beta'\in Z^1(D_v,U)$ whose diagonal vanishes on μ_v. From $0\to U_{h+1}\to U_h\to U_h/U_{h+1}\to 0$ we get $\#Z^1(D_v,U_h)\le\#Z^1(D_v,U_{h+1})\cdot\#Z^1(D_v,U_h/U_{h+1})$, and the image of Zb' in $Z^1(D_v,U_0/U_1)$ (trivial action, $\theta_{ii}=1$) consists of n-tuples of continuous homomorphisms $D_v\to A_m$ vanishing on μ_v. Hence

$$\#\mathrm{Zb}_v\le\left(\#\mathrm{Hom}_{\mu_v}(D_v,A_m)\right)^n\prod_{i<j}\#Z^1(D_v,X_{ij}).$$

(a) A continuous homomorphism $D_v\to A_m$ (A_m is killed by ℓ) vanishing on μ_v factors through $D_v^{\mathrm{ab}}/\mu_v^{\mathrm{ab}}\otimes\mathbb{F}_\ell$, with μ_v^{ab} the elements of ℓ-power order. By local class field theory D_v^{ab} is the profinite completion of $(L_v^+)^\times\cong\mathbb{Z}\times\mu_{q_v-1}\times\mu_{\ell^\infty}(L_v^+)\times\mathbb{Z}_\ell^{d_v}$ (structure of the units of a local field), whose elements of ℓ-power order form the finite group $\mu_{\ell^\infty}(L_v^+)$; the quotient is $\hat{\mathbb{Z}}\times\mu_{q_v-1}\times\mathbb{Z}_\ell^{d_v}$, with maximal elementary abelian ℓ-quotient $(\mathbb{Z}/\ell)^{1+d_v}$. So $\#\mathrm{Hom}_{\mu_v}(D_v,A_m)=(\#A_m)^{1+d_v}=K^{m(1+d_v)}$.

(b) Let $i<j$ and $X=X_{ij}$, of order K^{mr} with $r\in\{1,g\}$. By Theorem B.29 (4) (with $v\mid\ell$, $[L_v^+:\mathbb{Q}_\ell]=d_v$) and $\#B^1=\#X/\#H^0$, $\#Z^1(D_v,X)=\#H^2(D_v,X)\cdot(\#X)^{d_v+1}$, and by Theorem B.29 (2) $\#H^2(D_v,X)=\#H^0(D_v,X^\vee)$, the number of additive maps $f\colon X\to\mu_\ell$ with $f(\theta_{ij}(g)y)=\chi_{\mathrm{cyc}}(g)f(y)$, i.e. vanishing on $I\cdot X$ with $I\subseteq A$ the ideal generated by the $\theta_{ij}(g)-\chi_{\mathrm{cyc}}(g)$, $g\in D_v$ (values of χ_{cyc} in $\mathbb{F}_\ell\subseteq A$). By (Pj1), $\psi_i\ne\psi_j\chi_{\mathrm{cyc}}|_{D_v}$, so $I\ne 0$ and $I=T^{e_{ij}}A$ for some $e_{ij}\ge 0$ independent of N,m; f factors through $X/T^{e_{ij}}X$, of order at most $K^{re_{ij}}$, and $\#H^2\le K^{ge_{ij}}$. So $\#Z^1(D_v,X_{ij})\le K^{mr(d_v+1)+ge_{ij}}$.

Multiplying, with $\sum_{i<j}r_{ij}=\nu$ and $C_v=g\sum_{i<j}e_{ij}$: $\#\mathrm{Zb}_v\le K^{m(n+\nu)(d_v+1)+C_v}$.

Step 5: adding the coboundaries. For $Y'\in U$ the coboundary $g\mapsto R(g)Y'R(g)^{-1}-Y'$ is upper triangular with diagonal entries $\psi_iY'_{ii}\psi_i^{-1}-Y'_{ii}=0$; so the coboundaries of the elements of $(w^1)U(w^1)^{-1}\subseteq M$ lie in $B^1(D_v,M)\cap\mathrm{Zb}_v$, and they form a group of order at least $\#U/\#H^0(D_v,M)$. As $\#B^1(D_v,M)=\#M/\#H^0(D_v,M)$,

$$\begin{aligned}\#(B^1+\mathrm{Zb}_v)&=\frac{\#B^1\cdot\#\mathrm{Zb}_v}{\#(B^1\cap\mathrm{Zb}_v)}\le\frac{\#M}{\#H^0}\cdot\#\mathrm{Zb}_v\cdot\frac{\#H^0}{\#U}\\&\le\#M\cdot K^{m(n+\nu)(d_v+1)+C_v-m(n+\nu)}=\#M\cdot K^{m(n+\nu)d_v+C_v}.\end{aligned}$$

□

Lemma E.112. *Let x be a framed normalised curve point of shape $(a,n-a)$, $v\in S_\ell$ and $w=w_v$. For $1\le i\le n$ let $\mathrm{ty}(i)\in\{1,2\}$ be the block (1 for $\{1,\dots,a\}$, 2 for $\{a+1,\dots,n\}$) containing the non-zero entries of the i-th column of w. Then exactly a indices have type 1, and for every commutative A-algebra X*

$$w^{-1}\mathrm{Ev}_a(X)w=\mathrm{Ev}_{\mathrm{ty}}(X),\qquad w^{-1}\mathrm{Od}_a(X)w=\mathrm{Od}_{\mathrm{ty}}(X),$$

where $\mathrm{Ev}_{\mathrm{ty}}(X)$, resp. $\mathrm{Od}_{\mathrm{ty}}(X)$, is the set of matrices in $\mathrm{M}_n(X)$ whose (i,j) entry vanishes unless $\mathrm{ty}(i)=\mathrm{ty}(j)$, resp. unless $\mathrm{ty}(i)\ne\mathrm{ty}(j)$. In particular every element of $w^{-1}\mathrm{Od}_a(X)w$ has zero diagonal.

Proof. Every column of w is non-zero ($w\in\mathrm{GL}_n(A)$) and has its non-zero entries in one block by (Sd1), so ty is well defined. Let $B_1=\bigoplus_{i\le a}Ae_i$, $B_2=\bigoplus_{i>a}Ae_i$ and $E_t=\bigoplus_{\mathrm{ty}(i)=t}Ae_i$. Then $w(E_t)\subseteq B_t$, and since w is invertible and $A^n=E_1\oplus E_2=B_1\oplus B_2$, $w(E_t)=B_t$; comparing ranks, exactly a indices have type 1. The same holds after base change to X. A

matrix $Z \in \mathrm{M}_n(X)$ lies in $\mathrm{Ev}_a(X)$ iff it maps $B_1 \otimes X$ and $B_2 \otimes X$ into themselves, iff $w^{-1}Zw$ maps $E_1 \otimes X$ and $E_2 \otimes X$ into themselves, iff $w^{-1}Zw \in \mathrm{Ev}_{\mathrm{ty}}(X)$; and $Z \in \mathrm{Od}_a(X)$ iff Z maps $B_1 \otimes X$ into $B_2 \otimes X$ and $B_2 \otimes X$ into $B_1 \otimes X$, iff $w^{-1}Zw$ exchanges $E_1 \otimes X$ and $E_2 \otimes X$, iff $w^{-1}Zw \in \mathrm{Od}_{\mathrm{ty}}(X)$. A diagonal position (i,i) has $\mathrm{ty}(i) = \mathrm{ty}(i)$, so elements of $\mathrm{Od}_{\mathrm{ty}}(X)$ have zero diagonal. □

Lemma E.113. *Let $\mathscr{H}$ be a rank-n Hecke datum, $x = (k, \rho, 1_n, (\psi_{v,a}), f)$ a framed curve point normalised of shape $(a, n-a)$, κ a Selmer basis of x, and a Taylor–Wiles datum of size q (with H and $\sigma_1, \dots, \sigma_q$) such that $\rho(\gamma) = 1$ for $\gamma \in \Gamma^0_{\mathscr{H}} \cap \mathrm{PL}(H)$ and $\{\sigma_1, \dots, \sigma_q\}$ detects (x, κ). Take $D_{v_{N,j}} = D_{\bar{w}_{N,j}} \cap G_{L^+}$ acting through $\iota_{N,j} = \pi^{Q_N}|_{D_{v_{N,j}}}$. Then there is $c_2 \geq 0$ such that for every $m \geq 1$ there is N_0 with: for $N \geq N_0$,*

$$\#\{y \in Z^1(\Gamma_N, M'_{N,m}) :\ y \circ \iota_{N,j} \in B^1(D_{v_{N,j}}, M'_{N,m})\ (1 \leq j \leq q)\} \ \leq\ \#B^1(\Gamma_N, M'_{N,m}) \cdot K^{c_2}.$$

This is (Bs6) of Proposition E.105.

Proof. Let TwZ_m, TwB_m, ℓ_σ be as in Lemma E.114, and let c_2 be the constant C given there for $\Sigma = \{\sigma_1, \dots, \sigma_q\} \subseteq \Gamma'$. Fix m. Write $\Gamma = \Gamma_{\mathscr{H}}$, $\Gamma^0 = \Gamma^0_{\mathscr{H}}$, $r_m(\gamma)$ for $\rho_1(\gamma)$ with entries reduced modulo T^m, and $\bar{\chi}(\gamma) \in \mathbb{F}_\ell$ for the reduction of $\chi_{\mathrm{cyc}}(\gamma)$.

Step 0: choice of N_0. Γ^0 acts continuously on the finite group $\mathrm{PB}_a(A_m)$ by $\gamma \cdot Y = \bar{\chi}(\gamma) r_m(\gamma) Y r_m(\gamma)^{-1}$; let $U' \subseteq \Gamma^0$ be the kernel of this action and of r_m, an open subgroup. It contains $\mathrm{PL}(H)$: $\mathrm{PL}(H) \subseteq H$, $\rho_1 = 1$ on $\mathrm{PL}(H)$ (Lemma E.96 (Pv1)) and $\bar{\chi} = 1$ on H (Tg1). TwZ_m is finite by Lemma E.118: its elements are locally constant maps satisfying the cocycle identity for this continuous action of Γ^0 by group automorphisms on the finite group $\mathrm{PB}_a(A_m)$, which is killed by ℓ (A_m is a k-algebra and k has characteristic ℓ), and Γ^0 satisfies Φ_ℓ by Lemma E.116. Each $z \in \mathrm{TwZ}_m$ restricts to a continuous homomorphism on U', with open kernel U_z, and z vanishes on $\mathrm{PL}(H)$ by Lemma E.45 (Pl2) (with $G = \Gamma$, the subgroup $\Gamma^0 \supseteq H$, and $\mathrm{PL}(H)$ acting trivially). Let V be the intersection of the conjugates in Γ of $U' \cap \bigcap_z U_z$: an open normal subgroup of Γ containing the normal subgroup $\mathrm{PL}(H)$. By (Tg3) there is N_0 with $h_N(\Phi_{N,j})\sigma_j^{-1} \in V$ for all j and $N \geq N_0$. For such N, $u = h_N(\Phi_{N,j})\sigma_j^{-1}$ lies in $V \subseteq U'$, so

$$r_m(h_N \Phi_{N,j}) = r_m(\sigma_j), \qquad z(h_N \Phi_{N,j}) = z(u) + u \cdot z(\sigma_j) = z(\sigma_j) \quad (z \in \mathrm{TwZ}_m). \tag{1}$$

Moreover $\bar{\chi}(\Phi_{N,j}) = 1$, since $\phi_{N,j}^{-1}$ is a Frobenius element for the norm $q_{N,j} \equiv 1 \bmod \ell^N$ (Tg2).

Now let $N \geq N_0$, $M' = M'_{N,m}$, let Z be the set in the statement, $B^1 = B^1(\Gamma_N, M')$, $Z^c = \{y \in Z :\ y(c_N) = 0\}$, $B^c = \{b \in B^1 :\ b(c_N) = 0\}$.

Step 1: removing c_N. Z is a group containing B^1 (a coboundary restricts to a coboundary). For $y \in Z^1(\Gamma_N, M')$, $0 = y(c_N^2) = y(c_N) + c_N \cdot y(c_N)$, so $y(c_N) \in M'^{-} = \{Y :\ c_N \cdot Y = -Y\}$. As ℓ is odd, for $Y \in M'^{-}$ the coboundary b_Y of Y has $b_Y(c_N) = -2Y$, so $Y \mapsto b_Y(c_N)$ maps M'^{-} onto M'^{-}. Hence $y \mapsto y(c_N)$ maps both Z and B^1 onto M'^{-}, with kernels Z^c and B^c, and

$$\#Z \cdot \#B^c = \#Z^c \cdot \#B^1. \tag{2}$$

Step 2: descent to Γ^0. Let $y \in Z^c$. Elements of $\ker h_N$ act trivially on M' (the action of Γ^0_N factors through h_N, and χ_{cyc} factors through Γ), and so do the elements of $\mathrm{PL}(H_N)$ (as $h_N(\mathrm{PL}(H_N)) \subseteq \mathrm{PL}(H)$ by Lemma E.45 (Pl3), where $\rho_1 = 1$ and $\bar{\chi} = 1$). Let W be the closure of the subgroup generated by $\mathrm{PL}(H_N)$ and the conjugates of $\mathrm{T}_{N,1}, \dots, \mathrm{T}_{N,q}$; it acts trivially, so $y|_W$ is a continuous homomorphism, and $y(g\gamma g^{-1}) = y(g) + g \cdot y(\gamma) + g\gamma \cdot y(g^{-1}) = g \cdot y(\gamma)$ for γ acting trivially (using $y(g^{-1}) = -g^{-1}y(g)$). By hypothesis there is Y_j with $y(\iota_{N,j}g) = \iota_{N,j}(g) \cdot Y_j - Y_j$ for $g \in D_{v_{N,j}}$; with $g = \tau_{N,j}$, $\mathrm{T}_{N,j} \in \ker h_N$ (Tg4) acts trivially, so $y(\mathrm{T}_{N,j}) = 0$, and y vanishes on

all conjugates of the $\mathrm{T}_{N,j}$. By Lemma E.45 (Pl2) (with $G = \Gamma_N$ and the open normal subgroup H_N), y vanishes on $\mathrm{PL}(H_N)$. So the homomorphism $y|_W$ vanishes on a set of topological generators of W, hence on W, and $W \supseteq \ker h_N$ by (Tg4). Then $y(g\gamma) = y(g) + g \cdot y(\gamma) = y(g)$ for $\gamma \in \ker h_N$, and $\Psi(y)\colon \Gamma^0 \to \mathrm{PB}_a(A_m)$, $\Psi(y)(h_N g) = y(g)$ ($g \in \Gamma^0_N$), is well defined. It is locally constant (h_N is a continuous surjection of profinite groups, hence open), it satisfies the twisted cocycle identity because y does and $h_N\gamma$ acts through $\bar\chi(h_N\gamma) r_m(h_N\gamma)$, and, as $y(c_N) = 0$, $y(c_N g c_N) = c_N \cdot (y(g) + g \cdot y(c_N)) = -\chi_{\mathrm{cyc}}(c)\, {}^t y(g)$ with $h_N(c_N g c_N) = \bar c\, h_N(g)\, \bar c$. So $\Psi(y) \in \mathrm{TwZ}_m$.

Step 3: $\ell_{\sigma_j}(\Psi(y)) = 0$. By (1), the definition of Ψ and $\bar\chi(\Phi_{N,j}) = 1$,

$$\Psi(y)(\sigma_j) = \Psi(y)(h_N \Phi_{N,j}) = y(\Phi_{N,j}) = \Phi_{N,j} \cdot Y_j - Y_j = r_m(\sigma_j) Y_j r_m(\sigma_j)^{-1} - Y_j,$$

so $\mathrm{tr}(\Psi(y)(\sigma_j) r_m(\sigma_j)) = \mathrm{tr}(r_m(\sigma_j) Y_j) - \mathrm{tr}(Y_j r_m(\sigma_j)) = 0$.

Step 4: Ψ is injective on Z^c. $\Psi(y)$ determines y on Γ^0_N, and $y(c_N g) = y(c_N) + c_N \cdot y(g) = c_N \cdot y(g)$; as $\Gamma_N = \Gamma^0_N \cup c_N \Gamma^0_N$, y is determined.

Step 5: $\#\mathrm{TwB}_m \le \#B^c$. If $Y \in \mathrm{PB}_a(A_m)$ and $-\chi_{\mathrm{cyc}}(c)\, {}^t Y = Y$, i.e. $c_N \cdot Y = Y$, its coboundary b_Y over Γ_N lies in $B^c \subseteq Z^c$ and $\Psi(b_Y)$ is the element $\gamma \mapsto \bar\chi(\gamma) r_m(\gamma) Y r_m(\gamma)^{-1} - Y$ of TwB_m. So $\mathrm{TwB}_m \subseteq \Psi(B^c)$.

Conclusion. By Steps 2–4 and Lemma E.114, $\#Z^c \le \#\{z \in \mathrm{TwZ}_m :\ \ell_{\sigma_j}(z) = 0\ (j \le q)\} \le \#\mathrm{TwB}_m \cdot K^{c_2} \le \#B^c \cdot K^{c_2}$ (Step 5), and by (2), $\#Z = \#B^1 \cdot \#Z^c / \#B^c \le \#B^1 \cdot K^{c_2}$. □

Lemma E.114. *Let $\mathscr{H}$ be a rank-n Hecke datum, x a curve point normalised of shape $(a, n-a)$, κ a Selmer basis of x, $\bar c = \pi_{\mathscr{H}}(c)$, $K = \#k$, and $\Sigma \subseteq \Gamma'$ a finite set detecting (x, κ). For $m \ge 1$ let TwZ_m be the set of locally constant maps $z\colon \Gamma^0_{\mathscr{H}} \to \mathrm{PB}_a(A_m)$ (the matrices in $\mathrm{M}_n(\mathrm{PD}_g(A_m))$ with same-block entries in $A_m\eta$ and different-block entries in $\bigoplus_j A_m \epsilon'_j$) such that $z(\gamma\gamma') = z(\gamma) + \chi_{\mathrm{cyc}}(\gamma)\rho_1(\gamma) z(\gamma') \rho_1(\gamma)^{-1}$ and $z(\bar c \gamma \bar c) = -\chi_{\mathrm{cyc}}(\bar c)\, {}^t z(\gamma)$ (entries of ρ_1 and values of χ_{cyc} mapped to $\mathrm{PD}_g(A_m)$); let TwB_m be the set of the maps $\gamma \mapsto \chi_{\mathrm{cyc}}(\gamma)\rho_1(\gamma) Y \rho_1(\gamma)^{-1} - Y$ with $Y \in \mathrm{PB}_a(A_m)$, $-\chi_{\mathrm{cyc}}(\bar c)\, {}^t Y = Y$; and for $\sigma \in \Gamma^0_{\mathscr{H}}$ let $\ell_\sigma(z) \in A_m$ be the η-coefficient of $\mathrm{tr}(z(\sigma)\rho_1(\sigma))$. Then there is $C \ge 0$ such that for all $m \ge 1$*

$$\#\{z \in \mathrm{TwZ}_m :\ \ell_\sigma(z) = 0 \text{ for all } \sigma \in \Sigma\} \ \le\ \#\mathrm{TwB}_m \cdot K^C.$$

Proof. Write $\Gamma^0 = \Gamma^0_{\mathscr{H}}$, $P = \mathrm{M}_n(\mathrm{PD}_g(A))$, a free A-module of finite rank, and $\mathrm{PB}_a = \mathrm{PB}_a(A) \subseteq P$, a direct summand spanned by coordinates, so P/PB_a is free and

$$(P/\mathrm{PB}_a) \otimes A_m = \mathrm{M}_n(\mathrm{PD}_g(A_m))/\mathrm{PB}_a(A_m).$$

We apply Lemma E.115.

The modules. Let $\mathcal{C}$ be the A-module of maps $f\colon \Gamma^0 \to P$ whose reductions modulo every T^N are locally constant (T-adically continuous maps). By Lemma E.117, item 1, $\mathcal{C}$ has no T-torsion and is T-adically separated and complete. Let $Y = \prod_{(\gamma,\gamma')} P \times \prod_\gamma P \times \prod_\gamma P/\mathrm{PB}_a \times A^\Sigma$, a product of free modules of finite rank, hence without T-torsion and separated, and

$$\Phi(f) = \Big(\big(f(\gamma\gamma') - f(\gamma) - \chi_{\mathrm{cyc}}(\gamma)\rho_1(\gamma) f(\gamma') \rho_1(\gamma)^{-1}\big),\ \big(f(\bar c\gamma\bar c) + \chi_{\mathrm{cyc}}(\bar c)\, {}^t f(\gamma)\big),\ \big(f(\gamma) \bmod \mathrm{PB}_a\big),\ \big(\ell_\sigma(f)\big)_{\sigma\in\Sigma} \Big),$$

with $\ell_\sigma(f)$ the η-coefficient of $\mathrm{tr}(f(\sigma)\rho_1(\sigma))$. Φ is A-linear and $\ker\Phi = \{z \in \mathrm{TwZ}(x,\kappa) :\ \ell_\sigma(z) = 0\ (\sigma \in \Sigma)\}$.

B. Let $B = \mathrm{TwB}(x,\kappa)$. Each $b_Y(\gamma) = \chi_{\mathrm{cyc}}(\gamma)\rho_1(\gamma)Y\rho_1(\gamma)^{-1} - Y$ with $Y \in \mathrm{PB}_a$, $-\chi_{\mathrm{cyc}}(\bar{c})^t Y = Y$, is T-adically continuous, takes values in PB_a (Lemma E.106), satisfies the cocycle identity, and $b_Y(\bar{c}\gamma\bar{c}) = \chi_{\mathrm{cyc}}(\gamma)\,{}^t\rho_1(\gamma)^{-1}Y\,{}^t\rho_1(\gamma) - Y = -\chi_{\mathrm{cyc}}(\bar{c})\,{}^t b_Y(\gamma)$ by Lemma E.107 (Rf1) and ${}^tY = -\chi_{\mathrm{cyc}}(\bar{c})Y$. For $\sigma \in \Sigma \subseteq \Gamma'$, $\chi_{\mathrm{cyc}}(\sigma) = 1$ and $\mathrm{tr}(b_Y(\sigma)\rho_1(\sigma)) = \mathrm{tr}(\rho_1(\sigma)Y) - \mathrm{tr}(Y\rho_1(\sigma)) = 0$. So $B \subseteq \ker\Phi$, and since Σ detects (x,κ), every element of $\ker\Phi$ has a multiple $T^s z$ in B.

Reduction modulo T^m. Let $X_m = \Phi^{-1}(T^mY)$ and $Z_m = \{z \in \mathrm{TwZ}_m : \ell_\sigma(z) = 0\ (\sigma \in \Sigma)\}$. The map $f \mapsto \bar{f}$ (reduction of values modulo T^m) sends X_m into Z_m, since $\Phi(f) \in T^mY$ says exactly that $\bar{f}$ satisfies the defining conditions of Z_m. It is onto: by Lemma E.117, item 2, every $z \in Z_m$ has a lift $f \in \mathcal{C}$, and $\Phi(f) \in T^mY$ because $\Phi(f)$ modulo T^mY depends only on $\bar{f} = z$. Its kernel is $X_m \cap T^m\mathcal{C} = T^m\mathcal{C}$ (Lemma E.117, item 2). Hence $\#(X_m/T^m\mathcal{C}) = \#Z_m$. Likewise $(B + T^m\mathcal{C})/T^m\mathcal{C}$ is the set of reductions of elements of B, contained in TwB_m (reduce Y).

Finiteness of $X_1/T\mathcal{C} \cong Z_1$. The formula $\gamma \cdot Y = \chi_{\mathrm{cyc}}(\gamma)\rho_1(\gamma)Y\rho_1(\gamma)^{-1}$ is a continuous action of Γ^0 by group automorphisms on the finite group $\mathrm{PB}_a(k)$ (killed by ℓ), the elements of Z_1 are locally constant maps satisfying the cocycle identity for this action, and Γ^0 satisfies Φ_ℓ by Lemma E.116; so Z_1 is finite by Lemma E.118.

By Lemma E.115 there is e with $\#Z_m = \#(X_m/T^m\mathcal{C}) \le \#((B + T^m\mathcal{C})/T^m\mathcal{C}) \cdot K^e \le \#\mathrm{TwB}_m \cdot K^e$ for all m. Take $C = e$. □

Lemma E.115. *Let k be a finite field, $K = \#k$, $A = k[[T]]$. Let C be an A-module without T-torsion that is T-adically separated and complete, Y an A-module without T-torsion that is T-adically separated, $\Phi\colon C \to Y$ an A-linear map, and $B \subseteq \ker\Phi$ a submodule such that every $f \in \ker\Phi$ has some $T^s f \in B$. For $m \ge 1$ put $X_m = \Phi^{-1}(T^mY)$. If X_1/TC is finite, there is $e \ge 0$ such that for all $m \ge 1$*

$$\#(X_m/T^mC) \;\le\; \#((B + T^mC)/T^mC) \cdot K^e.$$

Proof. Write $\mathcal{K} = \ker\Phi$ and $I = \Phi(C) \subseteq Y$.

Step 1: two intersections. $\mathcal{K} \cap T^mC = T^m\mathcal{K}$ and $X_m \cap TC = TX_{m-1}$ ($X_0 = C$): if $f = T^m g$ with $\Phi(f) = 0$, then $T^m\Phi(g) = 0$ and $\Phi(g) = 0$; if $f = Tg$ with $\Phi(f) \in T^mY$, then $\Phi(g) \in T^{m-1}Y$ (Y has no T-torsion). Since C has no T-torsion, $TX_{m-1}/T^mC \cong X_{m-1}/T^{m-1}C$, so the exact sequence $0 \to TX_{m-1}/T^mC \to X_m/T^mC \to X_1/TC$ gives $\#(X_m/T^mC) \le \#(X_1/TC)^m$ by induction.

Step 2: $\mathcal{K}$ is free of finite rank r and $T^s\mathcal{K} \subseteq B$ for some s. $\mathcal{K}/T\mathcal{K} = \mathcal{K}/(\mathcal{K} \cap TC)$ embeds in X_1/TC, so it is finite. $\mathcal{K}$ is complete for its T-adic filtration, which by Step 1 is induced from C: a Cauchy sequence (f_i) in $\mathcal{K}$ has a limit $f \in C$, and $\Phi(f) = \Phi(f - f_i) \in T^iY$ for all large i, so $\Phi(f) \in \bigcap_i T^iY = 0$. Let $k_1, \dots, k_r \in \mathcal{K}$ lift a k-basis of $\mathcal{K}/T\mathcal{K}$. For $f \in \mathcal{K}$, write successively $f = \sum_i a_i^{(0)}k_i + Tf_1$, $f_1 = \sum_i a_i^{(1)}k_i + Tf_2$, … with $a_i^{(j)} \in A$ and $f_j \in \mathcal{K}$; then $f - \sum_i(\sum_{j<J} T^j a_i^{(j)})k_i \in T^J\mathcal{K}$ for every J, and as A is complete and $\mathcal{K}$ separated, $f = \sum_i(\sum_j T^j a_i^{(j)})k_i$. So $\mathcal{K}$ is finitely generated, and being torsion free over the discrete valuation ring A, it is free; its rank is $r = \dim_k \mathcal{K}/T\mathcal{K}$. Choosing s_i with $T^{s_i}k_i \in B$ and $s = \max_i s_i$, $T^s\mathcal{K} \subseteq B$.

Step 3: the image is closed with bounded defect. Let $S = \{y \in Y : T^j y \in I \text{ for some } j\}/I$ and $S[T^j]$ its elements killed by T^j. The map $X_1 \to S[T]$, $f \mapsto [y]$ where $\Phi(f) = Ty$, is well defined (y is unique), onto, and kills TC; so $S[T]$ is finite. Multiplication by T^{j-1} embeds $S[T^j]/S[T^{j-1}]$ in $S[T]$, so every $S[T^j]$ is finite. We claim $S[T^{c+1}] = S[T^c]$ for some c. Otherwise the subgroups $D_j = T^{j-1}S[T^j] \subseteq S[T]$ are all non-zero; they decrease, so they are eventually

equal to some $D \neq 0$; fix $0 \neq s_1 \in D$. The sets $E_j = \{x \in S[T^j] : T^{j-1}x = s_1\}$ are finite and, for large j, non-empty, and $x \mapsto Tx$ maps E_{j+1} to E_j; by König's lemma there are $s_j \in E_j$ with $Ts_{j+1} = s_j$ for all j. Choose $y_j \in Y$ representing s_j; then $Ty_{j+1} = y_j + i_j$ with $i_j \in I$, and $u_j = T^j y_j \in I$ satisfies $u_{j+1} = u_j + T^j i_j$. Choose $f_1, g_j \in C$ with $\Phi(f_1) = u_1$ and $\Phi(g_j) = i_j$, and put $F_J = f_1 + \sum_{j<J} T^j g_j$, so $\Phi(F_J) = u_J$. By completeness the series $\sum_{j \geq J} T^{j-J} g_j$ converge in C; so (F_J) has a limit f with $f - F_J \in T^J C$, and $\Phi(f) - u_J = \Phi(f - F_J) \in T^J I$. Hence $\Phi(f) = T^J(y_J + h_J)$ with $h_J \in I$, for every J, so $\Phi(f) \in \bigcap_J T^J Y = 0$; then $T^J(y_J + h_J) = 0$ gives $y_J = -h_J \in I$, i.e. $s_J = 0$, and $s_1 = T^{J-1} s_J = 0$, a contradiction. So $S[T^{c+1}] = S[T^c]$, and then $S[T^j] = S[T^c]$ for all $j \geq c$ (if $T^{c+2}x = 0$ then $Tx \in S[T^{c+1}] = S[T^c]$, so $x \in S[T^{c+1}]$; induct). As S is the union of the $S[T^j]$, $T^c S = 0$. Consequently $I \cap T^m Y \subseteq T^{m-c} I$ for $m \geq c$: if $u = T^m y \in I$, then $[y] \in S$, so $T^c y \in I$ and $u = T^{m-c}(T^c y)$.

Step 4: the count. Let $m \geq c$ and $f \in X_m$. By Step 3, $\Phi(f) = T^{m-c}\Phi(g)$ with $g \in C$, and $\Phi(g) \in T^c Y$ (Y has no T-torsion), so $g \in X_c$ and $f - T^{m-c} g \in \mathcal{K}$. Hence $X_m = \mathcal{K} + T^{m-c} X_c$ (the inclusion $\supseteq$ is clear), and

$$\#(X_m/T^m C) \leq \#((\mathcal{K}+T^m C)/T^m C) \cdot \#((T^{m-c}X_c + T^m C)/T^m C) \leq \#(\mathcal{K}/T^m\mathcal{K}) \cdot \#(X_c/T^c C),$$

using $(\mathcal{K} + T^m C)/T^m C \cong \mathcal{K}/(\mathcal{K} \cap T^m C) = \mathcal{K}/T^m\mathcal{K}$ (Step 1) and the surjection $X_c/T^c C \to (T^{m-c}X_c + T^m C)/T^m C$, $g \mapsto T^{m-c} g$. By Step 2, $\#(\mathcal{K}/T^m\mathcal{K}) = K^{mr}$, while $(B + T^m C)/T^m C \cong B/(B \cap T^m\mathcal{K})$ contains the image of $T^s\mathcal{K}$, of order $\#(T^s\mathcal{K}/T^m\mathcal{K}) = K^{(m-s)r}$ when $m \geq s$; so in all cases $\#(\mathcal{K}/T^m\mathcal{K}) \leq \#((B + T^m C)/T^m C) \cdot K^{sr}$. With Step 1, $\#(X_c/T^c C) \leq \#(X_1/TC)^c$, and for $m < c$, $\#(X_m/T^m C) \leq \#(X_1/TC)^c$. Choosing e with $K^e \geq K^{sr} \cdot \#(X_1/TC)^c$ proves the bound for all $m \geq 1$. □

Lemma E.116. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$. Then $\Gamma_{\mathscr{H}}$ and $\Gamma^0_{\mathscr{H}}$ satisfy Mazur's condition Φ_ℓ.*

Proof. Let $K \in \{L^+, L\}$, so that $\Gamma_K := \pi_{\mathscr{H}}(G_K)$ is $\Gamma_{\mathscr{H}}$ or $\Gamma^0_{\mathscr{H}}$; since $N_{\mathscr{H}} \subseteq G_L \subseteq G_K$ (Definition 50.1), $\Gamma_K = G_K/N_{\mathscr{H}}$ as profinite groups. Let $\Gamma' \subseteq \Gamma_K$ be open and $G' \subseteq G_K$ its preimage, an open subgroup containing $N_{\mathscr{H}}$; so $G' = G_{K'}$ for a finite extension K' of K inside $\overline{\mathbb{Q}}$, and $\Gamma' = G_{K'}/N_{\mathscr{H}}$. Composition with the projection identifies $\mathrm{Hom}_{\mathrm{cont}}(\Gamma', \mathbb{F}_\ell)$ with the continuous homomorphisms $\phi\colon G_{K'} \to \mathbb{F}_\ell$ trivial on $N_{\mathscr{H}}$.

Let S' be the set of places of K' above the places of L^+ in $S_\ell \cup \mathrm{St}$ and above ∞; it is finite. Let w be a finite place of K' not in S' and $\bar{w}$ a place of $\overline{\mathbb{Q}}$ above w; the inertia group of $G_{K'}$ at $\bar{w}$ is $I_{\bar{w}} \cap G_{K'}$. The place of L^+ below w is finite and not in $S_\ell \cup \mathrm{St}$, so $\bar{w} \nmid \ell$ and, by (Rd4), every $\bar{\chi}_a$ is trivial on $I_{\bar{w}} \cap G_L$; hence every element of $I_{\bar{w}} \cap G_L$ is one of the generators of $N_{\mathscr{H}}$, and $I_{\bar{w}} \cap G_{K'} \subseteq I_{\bar{w}} \cap G_L \subseteq N_{\mathscr{H}}$; here $I_{\bar{w}} \cap G_{K'} \subseteq I_{\bar{w}} \cap G_L$ because $G_{K'} \subseteq G_L$ if $K = L$, and because $I_{\bar{w}} \cap G_{L^+} \subseteq G_L$ if $K = L^+$ (L/L^+ is unramified at every finite place, (Rd2)). So ϕ is trivial on the inertia groups of $G_{K'}$ at all finite places outside S', i.e. ϕ factors through $G_{K',S'}$, which satisfies Φ_ℓ by Theorem B.71, item 2; hence the ϕ form a subset of the finite set $\mathrm{Hom}_{\mathrm{cont}}(G_{K',S'}, \mathbb{F}_\ell)$. So $\mathrm{Hom}_{\mathrm{cont}}(\Gamma', \mathbb{F}_\ell)$ is finite for every open Γ', for both groups. □

Lemma E.117. *Let k be a finite field, $A = k[[T]]$, P a free A-module of finite rank and G a profinite group. Let $C(G, P)$ be the A-module of maps $f\colon G \to P$ whose reduction modulo T^N is locally constant for every $N \geq 1$. Then:*

1. *$C(G, P)$ has no T-torsion and is T-adically separated and complete;*
2. *for every $m \geq 1$, reduction of values modulo T^m is an A-linear map from $C(G, P)$ onto the set of locally constant maps $G \to P/T^m P$, with kernel $T^m C(G, P)$.*

Proof. Write $C = C(G,P)$. As $P \cong A^r$, P has no T-torsion, $\bigcap_N T^N P = 0$, and P is T-adically complete.

Item 1. If $Tf = 0$, every value $f(\gamma)$ is killed by T, so $f = 0$. If $f \in \bigcap_N T^N C$, its values lie in $\bigcap_N T^N P = 0$. First a remark used twice: if $f \in C$ has all values in $T^i P$, then $g = f/T^i$ (defined value by value, P having no T-torsion) lies in C, since the reduction of g modulo T^N is determined by the reduction of f modulo T^{N+i}, which is locally constant; so $f = T^i g \in T^i C$. Now let (f_i) be a sequence in C with $f_{i+1} - f_i \in T^i C$ for all i. For each γ, $(f_i(\gamma))$ is Cauchy in P; let $f(\gamma)$ be its limit. For $i \geq N$, $f(\gamma) \equiv f_N(\gamma) \bmod T^N$, so the reduction of f modulo T^N equals that of f_N and is locally constant: $f \in C$. Moreover $f - f_i$ has values in $T^i P$, so $f - f_i \in T^i C$ by the remark. Hence C is complete.

Item 2. The reduction $f \mapsto \bar{f}$ is A-linear and $\bar{f}$ is locally constant by definition of C. It is onto: given a locally constant $z\colon G \to P/T^m P$ and a set-theoretic section $s\colon P/T^m P \to P$ of the reduction, $f = s \circ z$ is locally constant (it takes the finitely many values of s on the image of z, on the open fibres of z), hence lies in C, and $\bar{f} = z$. Its kernel contains $T^m C$; conversely, if $\bar{f} = 0$, f has values in $T^m P$, so $f \in T^m C$ by the remark in Item 1. □

Lemma E.118. *Let ℓ be a prime, G a profinite group satisfying Mazur's condition Φ_ℓ, and M a finite abelian group with $\ell M = 0$ on which G acts by group automorphisms, continuously for the discrete topology of M. Then the set of locally constant maps $z\colon G \to M$ with*

$$z(\gamma\gamma') = z(\gamma) + \gamma \cdot z(\gamma') \qquad (\gamma, \gamma' \in G)$$

is finite.

Proof. As M is finite and discrete and the action is continuous, the kernel U of $G \to \mathrm{Aut}(M)$ is the intersection of the finitely many open stabilisers of the elements of M, an open normal subgroup of G. Let z be a map as in the statement. For $u, u' \in U$, $z(uu') = z(u) + u \cdot z(u') = z(u) + z(u')$, so $z|_U$ is a continuous homomorphism $U \to M$. Fix an isomorphism $M \cong (\mathbb{Z}/\ell)^d$ of groups ($\ell M = 0$); then $z|_U$ is given by d continuous homomorphisms $U \to \mathbb{F}_\ell$, and as U is open, $\mathrm{Hom}_{\mathrm{cont}}(U, \mathbb{F}_\ell)$ is finite by Φ_ℓ. So $z|_U$ takes at most $(\#\mathrm{Hom}_{\mathrm{cont}}(U, \mathbb{F}_\ell))^d$ values. Let $\gamma_1, \dots, \gamma_r$ be representatives of the finitely many cosets $U\gamma_i$ of U in G. For $u \in U$, $z(u\gamma_i) = z(u) + u \cdot z(\gamma_i) = z(u) + z(\gamma_i)$, so z is determined by $z|_U$ and $z(\gamma_1), \dots, z(\gamma_r) \in M$. Hence there are at most $(\#\mathrm{Hom}_{\mathrm{cont}}(U, \mathbb{F}_\ell))^d \, (\#M)^r$ such maps. □

E.4.2 The obstruction bound

This subsection proves the obstruction bound (B2), Proposition E.119, from a bound for the number of generators of the obstruction modules at the levels (Proposition E.120). The obstruction classes of first-order even Selmer families lie in the odd-block obstruction module (Lemma E.122), which has at most g generators up to a bounded factor at every Taylor–Wiles level (Lemma E.123); the reason is that odd-block Selmer cocycles at the levels are base-level cocycles up to a bounded factor (Lemma E.126, Lemma E.127). The subsection ends with the conditions (K1)–(K3) at a level, from the truncated Selmer and span bounds and from tame generators of the kernel (Lemma E.129, Lemma E.130, Lemma E.131), which Proposition E.94 uses.

Proposition E.119. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ that is a discrete valuation ring, $(B_n^{\mathrm{po}}, D^{\mathrm{u}}, (\psi_{v,a}^{\mathrm{d}}))$ a universal polarised ordinary determinant over $\mathscr{H}$, $x = (k, \rho, 1_n, (\psi_{v,a}), f)$ a framed curve point of B_n^{po} normalised of shape $(a, n-a)$, κ a Selmer basis of x with local witnesses, and a Taylor–Wiles datum for $\mathscr{H}$ (with open subgroup H*

and elements $\sigma_1,\dots,\sigma_q$) along which x has eigenvalue data. Assume $\rho(\gamma)=1$ for every $\gamma\in\Gamma^0_{\mathscr{H}}\cap \mathrm{PL}(H)$, that x satisfies (Pj1) of Definition 50.9, and that for every $Z\in \mathrm{Ev}_a(A_k)$ some T^sZ lies in the A_k-span of $\rho(\Gamma')$ (Γ' as in Definition 50.9). Then (B2) holds for x, κ and the levels $\mathbf{V}_N$ of the datum.

Proof. Let C be the constant of Proposition E.120; we show (B2) with this C. Fix $m\ge 1$, let N_0 be as in Proposition E.120, let $N\ge N_0$, and let $(z_{i,j})_{0\le i\le g,\,1\le j\le g}$ be a family in $\mathrm{Sel}^{\mathrm{ev}}_{\mathbf{V}_N}(A_m)$. Let $H=H_{N,m}$ and $\xi_0,\dots,\xi_g\in H$ be as in (Ob1), (Ob2) of Proposition E.120.

$A=k[[T]]$ is a discrete valuation ring with uniformiser T and residue field k, and H is a finite A-module with $T^mH=0$ and $\#(H/T^eH)\le(\#k)^{ge+C}$ for $0\le e\le m$ (Ob1). By Lemma E.128 there are $c_0,\dots,c_g\in A$, one of them a unit, with $\sum_i(T^Cc_i)\xi_i=T^C\sum_i c_i\xi_i=0$. By (Ob2) applied to $(T^Cc_i)_i$ there is $\Psi\in\mathrm{Sel}^2_{\mathbf{V}_N}(A_m)$ whose ϵ'_j-coefficient of $\Psi(\gamma)$ is $\sum_i T^Cc_iz_{i,j}(\gamma)=T^C\sum_i c_iz_{i,j}(\gamma)$ for all j and γ. This is (B2). □

Proposition E.120. *Let $\mathscr{H}$, $\mathcal{O}$, $(B_n^{\mathrm{po}},D^{\mathrm{u}},(\psi^{\mathrm{d}}_{v,a}))$, $x=(k,\rho,1_n,(\psi_{v,a}),f)$, $\kappa=(\kappa_1,\dots,\kappa_g)$ with local witnesses, and a Taylor–Wiles datum (with H and $\sigma_1,\dots,\sigma_q$) along which x has eigenvalue data be as in Proposition E.119, with $\rho(\gamma)=1$ for $\gamma\in\Gamma^0_{\mathscr{H}}\cap\mathrm{PL}(H)$, x satisfying (Pj1) of Definition 50.9, and for every $Z\in\mathrm{Ev}_a(A)$ some T^sZ in the A-span of $\rho(\Gamma')$ (Γ' as in Definition 50.9). Then there is $C\ge0$ such that for every $m\ge1$ there is N_0 such that for every $N\ge N_0$ there is a finite A-module $H_{N,m}$ with:*

(Ob1) *$T^mH_{N,m}=0$ and $\#(H_{N,m}/T^eH_{N,m})\le(\#k)^{ge+C}$ for $0\le e\le m$;*

(Ob2) *for every family $(z_{i,j})_{0\le i\le g,\,1\le j\le g}$ in $\mathrm{Sel}^{\mathrm{ev}}_{\mathbf{V}_N}(A_m)$ there are $\xi_0,\dots,\xi_g\in H_{N,m}$ such that for every $c\in A^{g+1}$ with $\sum_i c_i\xi_i=0$ there is $\Psi\in\mathrm{Sel}^2_{\mathbf{V}_N}(A_m)$ whose ϵ'_j-coefficient of $\Psi(\gamma)$ is $\sum_i c_iz_{i,j}(\gamma)$ for all j and γ.*

Proof. Take $H_{N,m}$ to be the obstruction module of the Selmer data $\mathcal{D}_{N,m}$ of Lemma E.122 (the odd-block module $\mathrm{Od}_a(A_m)$ over G_{L^+,T_∞}, $T=S_\ell\cup\mathrm{St}\cup Q_N$, with the ordinary quotients at S_ℓ and vanishing on J_u at St), and C, N_0 as in Lemma E.123. For $N\ge N_0$, Lemma E.123 gives (Ob1): $H_{N,m}$ is a finite A-module, $T^mH_{N,m}=0$ and $\#(H_{N,m}/T^eH_{N,m})\le(\#k)^{ge+C}$ for $0\le e\le m$; and Lemma E.122 gives (Ob2). □

Definition E.121. Let A be a commutative ring and G a profinite group. **Selmer data** $\mathcal{D}$ over G consist of a finite A-module M with a continuous A-linear action of G (M discrete) and a finite family, indexed by $w\in W$, of: a profinite group G_w with a continuous homomorphism $\phi_w\colon G_w\to G$; a finite A-module Q_w with a continuous A-linear action of G_w; a surjective A-linear map $\pi_w\colon M\to Q_w$ with $\pi_w(\phi_w(g)m)=g\,\pi_w(m)$; and an A-submodule L_w with $B^1(G_w,Q_w)\subseteq L_w\subseteq Z^1(G_w,Q_w)$ (the **local condition**). For a cochain x of G with values in M write $\pi_w\phi_w^*x$ for the cochain $(g_1,\dots,g_i)\mapsto\pi_w(x(\phi_wg_1,\dots,\phi_wg_i))$ of G_w.

The **Selmer cocycles** are $Z^1_{\mathcal{D}}=\{z\in Z^1(G,M):\ \pi_w\phi_w^*z\in L_w\text{ for all }w\}$, and the **Selmer group** is $H^1_{\mathcal{D}}=Z^1_{\mathcal{D}}/B^1(G,M)$. The **obstruction module** is $H^2_{\mathcal{D}}=Z^2_{\mathcal{D}}/B^2_{\mathcal{D}}$, where

$$Z^2_{\mathcal{D}}=\Big\{(a,(b_w)_{w\in W}):\ a\in Z^2(G,M),\ b_w\in C^1(G_w,Q_w),\ db_w=\pi_w\phi_w^*a\text{ for all }w\Big\},$$

$$B^2_{\mathcal{D}}=\Big\{(dx,(\pi_w\phi_w^*x+\lambda_w)_w):\ x\in C^1(G,M),\ \lambda_w\in L_w\Big\}.$$

($B^2_{\mathcal{D}}\subseteq Z^2_{\mathcal{D}}$ since $d(\pi_w\phi_w^*x)=\pi_w\phi_w^*dx$ and $d\lambda_w=0$.) Both $H^1_{\mathcal{D}}$ and $H^2_{\mathcal{D}}$ are A-modules; they are the first and second cohomology of the Selmer complex of M with the local conditions (L_w).

Lemma E.122. *Let $\mathscr{H}$ be a rank-n Hecke datum, $x=(k,\rho,1_n,(\psi_{v,a}),f)$ a framed curve point normalised of shape $(a,n-a)$, $\kappa=(\kappa_1,\dots,\kappa_g)$ a Selmer basis with local witnesses $X_{j,v}=Y_{j,v}\in\mathrm{Od}_a(A)$, and a Taylor–Wiles datum; let $N,m\geq 1$, $c_N=\pi^{Q_N}(c)$, $\Gamma_N^0=h_N^{-1}(\Gamma^0_{\mathscr{H}})$. Let $T=S_\ell\cup\mathrm{St}\cup Q_N$, $G=G_{L^+,T_\infty}$ with the continuous surjection $p\colon G\to\Gamma_N$ of Lemma E.102 (Lb1) (for $Q=Q_N$; here $T^\circ=\varnothing$), and for a decomposition group $D\subseteq G_{L^+}$ let $\phi_D\colon D\to G$ be the restriction of $G_{L^+}\to G$, so $p\circ\phi_D=\pi^{Q_N}|_D$. Let $\mathcal{D}_{N,m}$ be the Selmer data over G (with A acting through A_m) given by: $M=\mathrm{Od}_a(A_m)$, on which G acts through p by $\gamma\cdot P=\rho(h_N\gamma)P\rho(h_N\gamma)^{-1}$ ($\gamma\in\Gamma_N^0$) and $c_N\cdot P=-{}^tP$; for $v\in S_\ell$: $G_v=D_v$, $\phi_v=\phi_{D_v}$, $Q_v=M/F_v$ with $F_v=\{P: w_v^{-1}Pw_v$ upper triangular$\}$, π_v the projection, $L_v=B^1(D_v,Q_v)$; for $u\in\mathrm{St}$, with $\bar u$ the place of the frame: $G_u=D_{\bar u}\cap G_{L^+}$, $\phi_u=\phi_{G_u}$, $Q_u=M$, $\pi_u=\mathrm{id}$, and L_u the cocycles vanishing on J_u, the closed normal subgroup of G_u generated by $I_{\bar u}\cap G_L\cap\pi_{\mathscr{H}}^{-1}(I_u^x)$. Put $H_{N,m}=H^2_{\mathcal{D}_{N,m}}$. Then for every family $(z_{i,j})_{0\leq i\leq g,\,1\leq j\leq g}$ in $\mathrm{Sel}^{\mathrm{ev}}_{\mathbf{V}_N}(A_m)$ there are $\xi_0,\dots,\xi_g\in H_{N,m}$ such that for every $c\in A^{g+1}$ with $\sum_i c_i\xi_i=0$ there is $\Psi\in\mathrm{Sel}^2_{\mathbf{V}_N}(A_m)$ such that the ϵ'_j-coefficient of $\Psi(\gamma)$ is $\sum_i c_iz_{i,j}(\gamma)$ for all j and γ. This is (Ob2) of Proposition E.120 for $H_{N,m}$.*

Proof. Throughout, Γ_N acts on $\mathrm{Ev}_a(A_m)$ and $\mathrm{Od}_a(A_m)$ by the same formulas as on M (an action as in Lemma E.107 (Rf2), with ρ in place of ρ_1), ρ and κ_j are evaluated at $h_N\gamma$, and $\pi=\pi_{\mathscr{H}}$. For $X\in\mathrm{Od}_a$, $Y\in\mathrm{Ev}_a$ the bracket $[X,Y]=XY-YX$ lies in Od_a and is Γ_N-equivariant: for Γ_N^0 this is conjugation, and $[-{}^tX,-{}^tY]={}^tX{}^tY-{}^tY{}^tX=-{}^t[X,Y]$.

Step 0: extension to Γ_N. A cocycle z on Γ_N^0 with $z(c_N\gamma c_N)=c_N\cdot z(\gamma)$ extends uniquely to a cocycle $\tilde z$ on Γ_N with $\tilde z(c_N)=0$, namely $\tilde z(c_N\gamma)=c_N\cdot z(\gamma)$ (direct check of the cocycle identity in the four cases). This applies to $\kappa_j\circ h_N$ (into $\mathrm{Od}_a(A_m)$) and to the $z_{i,j}$ (into $\mathrm{Ev}_a(A_m)$), by (Sd2) and (Sc). For cocycles κ,P (into Od_a, Ev_a) put $(\kappa\cup P)(\gamma,\gamma')=[\kappa(\gamma),\gamma\cdot P(\gamma')]$; by equivariance of the bracket this is a continuous 2-cocycle. Cochains of Γ_N are regarded as cochains of G through p (inflation, which preserves cocycles and the identities below), and for g in a decomposition group we write ϕ_wg also for $p(\phi_wg)=\pi^{Q_N}(g)$. For X, Y fixed and cocycles P, κ one checks directly (with d as in Section 13, $(dy)(\gamma,\gamma')=\gamma y(\gamma')-y(\gamma\gamma')+y(\gamma)$):

$$d\big([X,P(\cdot)]\big)=(dX)\cup P,\qquad d\big([\kappa(\cdot),(\cdot)\,Y]\big)=-\kappa\cup dY,\qquad d\big([X,Y]\big)(\gamma)=[\gamma X,\gamma Y]-[X,Y]. \tag{1}$$

Step 1: the classes. Fix i, write $P_{i,j}=\tilde z_{i,j}$, $a_i=\sum_j\tilde\kappa_j\cup P_{i,j}$. For $v\in S_\ell$ and each j choose a witness $Y_{i,j,v}\in\mathrm{Ev}_a(A_m)$ for $z_{i,j}$ at v, and put, for $g\in D_v$ (acting through ϕ_v), $\kappa'_j(g)=\kappa_j(\pi g)-(gX_{j,v}-X_{j,v})$ and $Q_{i,j}(g)=z_{i,j}(\phi_vg)-(gY_{i,j,v}-Y_{i,j,v})$; both are cocycles of D_v with values in F_v (the witness conditions), and D_v preserves F_v and $[F_v\cap\mathrm{Od}_a,\{w_v\text{-upper triangular}\}\cap\mathrm{Ev}_a]\subseteq F_v$, because $w_v^{-1}\rho(\pi g)w_v$ is upper triangular ((Sd1)). Using (1) twice, on D_v:

$$\phi_v^*a_i=\sum_j\kappa'_j\cup Q_{i,j}+d\,\beta_{i,v},\qquad \beta_{i,v}(g)=\sum_j\big([X_{j,v},z_{i,j}(\phi_vg)]-[\kappa'_j(g),\,gY_{i,j,v}]\big),$$

and $\kappa'_j\cup Q_{i,j}$ has values in F_v. Put $b_{i,v}=\pi_v\circ\beta_{i,v}$; then $db_{i,v}=\pi_v\phi_v^*a_i$. For $u\in\mathrm{St}$: $p(\phi_u(J_u))$ lies in the closed normal subgroup of Γ_N^0 generated by $I_{N,u}$; its elements act trivially on M and $\mathrm{Ev}_a(A_m)$ (ρ is trivial on $I_u^x=h_N(I_{N,u})$), and $\tilde\kappa_j$, $P_{i,j}$ vanish on them (they vanish on $I_{N,u}$, by (Sd2) and (Sc), and a cocycle vanishing on a trivially acting element vanishes on its conjugates, $z(huh^{-1})=h\cdot z(u)$, and by continuity on the closure). Hence $\phi_u^*a_i(\gamma j,\gamma' j')=\phi_u^*a_i(\gamma,\gamma')$ for $j,j'\in J_u$, i.e. $\phi_u^*a_i$ is inflated from a 2-cocycle $\bar a_i$ of G_u/J_u with values in M (J_u acting trivially). $G_u\subseteq G_L$ by (Rd2); $G_u/(I_{\bar u}\cap G_u)\cong\hat{\mathbb{Z}}$, and $(I_{\bar u}\cap G_u)/J_u$ is a quotient of $\pi(I_{\bar u}\cap G_L)/I_u^x$, a finite group of order prime to ℓ (inertia elements on which all $\bar\chi_a$ are trivial lie in $N_{\mathscr{H}}$, so $\pi(I_{\bar u}\cap G_L)$ embeds in a product of copies of $\mathbb{F}_{\mathcal{O}}^\times$). By the Hochschild–Serre spectral sequence (the finite normal subgroup has order prime to ℓ, M is an ℓ-group) and $H^2(\hat{\mathbb{Z}},M')=0$ for finite M',

$H^2(G_u/J_u, M) = 0$. So $\bar a_i = d\bar b_i$ with $\bar b_i(1) = \bar a_i(1,1) = 0$; its inflation $b_{i,u}$ satisfies $db_{i,u} = \phi_u^* a_i$ and vanishes on J_u. Let ξ_i be the class of $(a_i, (b_{i,w})_w) \in Z^2_{\mathcal{D}_{N,m}}$.

Step 2: a relation gives Ψ. Let $c \in A^{g+1}$ with $\sum_i c_i\xi_i = 0$; put $P_j = \sum_i c_i P_{i,j}$, $Y^\flat_{j,v} = \sum_i c_i Y_{i,j,v}$ (a witness for P_j, the witness conditions being linear), $a = \sum_i c_i a_i = \sum_j \tilde\kappa_j \cup P_j$, $b_w = \sum_i c_i b_{i,w}$. By definition of $B^2_{\mathcal D}$ there are $x \in C^1(G, M)$ and $\lambda_w \in L_w$ with $a = dx$ (a inflated to G) and $b_w = \pi_w\phi_w^* x + \lambda_w$. *Descent to Γ_N.* Elements of $\ker p$ act trivially on M, and a vanishes when one argument lies in $\ker p$; so $dx = a$ gives $x(\gamma k) = x(\gamma) + \gamma\cdot x(k)$ and $x(k\gamma) = x(k) + x(\gamma)$, hence $x(\gamma k\gamma^{-1}) = \gamma\cdot x(k)$, for $k \in \ker p$: $x|_{\ker p}$ is a continuous homomorphism compatible with conjugation. At $u \in \mathrm{St}$, b_u and λ_u vanish on J_u, so x vanishes on $\phi_u(J_u)$, which contains the image of $J_{\bar u} = \{s \in I_{\bar u} \cap G_L :\ \bar\chi_a(s) = 1\ (1 \le a \le n)\}$ (as $J_{\bar u} \subseteq N_{\mathscr H}$, $\pi_{\mathscr H}(J_{\bar u}) = 1 \subseteq I^x_u$). By Lemma E.102 (Lb1) (with $T^\circ = \varnothing$), $\ker p$ is the closed normal subgroup generated by these images; so x vanishes on $\ker p$, $x(\gamma k) = x(\gamma)$, and $x = \bar x \circ p$ with $\bar x \in C^1(\Gamma_N, M)$ continuous (p is open), $d\bar x = a$ on Γ_N and $b_w = \pi_w\phi_w^*\bar x + \lambda_w$. Let $y = -\bar x$, so $dy = -a$ on Γ_N. As $\tilde\kappa_j(c_N) = P_j(c_N) = 0$, $a(c_N, \cdot) = a(\cdot, c_N) = 0$ and $a(1,1) = 0$; so $y(1) = 0$ and $c_N\cdot y(c_N) = -y(c_N)$. With $Y^* = y(c_N)/2$ (ℓ odd) let $y' = y + dY^*$; then $y'(c_N) = 0$, $dy' = -a$, and evaluating $dy' = -a$ at (γ, c_N) and $(c_N, \gamma c_N)$ gives $y'(c_N\gamma c_N) = c_N\cdot y'(\gamma) = -{}^t y'(\gamma)$. Moreover $\pi_w\phi_w^* y' + b_w = \lambda_w + d(\pi_w Y^*) \in L_w$. Define $\Psi(\gamma) = \sum_j \epsilon'_j P_j(\gamma) + \eta\, y'(\gamma)$ for $\gamma \in \Gamma^0_N$, a continuous map into $\mathrm{PW}^2_a(A_m)$.

Cocycle and c. By Lemma E.108 (second identity; an element of PW^2_a is $\eta Y_0 + \sum_j \epsilon'_j Y_j$ with $Y_0 \in \mathrm{Od}_a$, $Y_j \in \mathrm{Ev}_a$), $\rho_1\Psi(\gamma')\rho_1^{-1} = \sum_j \epsilon'_j\gamma\cdot P_j(\gamma') + \eta\big(\gamma\cdot y'(\gamma') + \sum_j[\kappa_j(\gamma), \gamma\cdot P_j(\gamma')]\big)$ ($\rho_1 = \rho_1(h_N\gamma)$); so the cocycle identity for Ψ is the cocycle identity for the P_j together with $dy' = -a$. The c_N-condition holds for the P_j and for y'.

Steinberg places. On J_u: $b_u = 0$ and $\lambda_u = 0$, and dY^* vanishes on the trivially acting $p(\phi_u(J_u))$; so y' vanishes on $p(\phi_u(J_u)) \supseteq \pi^{Q_N}(I_{\bar u} \cap G_L \cap \pi^{-1}(I^x_u)) = I_{N,u}$. The P_j vanish on $I_{N,u}$. So Ψ vanishes on $I_{N,u}$.

Places above ℓ. Let $v \in S_\ell$, $X_j = X_{j,v}$, $Y_j = Y^\flat_{j,v}$. As $\pi_v\phi_v^* y' + b_v \in B^1(D_v, Q_v)$, there is $Y' \in M$ with $\pi_v\big(y'(\phi_v g) + \beta_v(g) - (gY' - Y')\big) = 0$, where $\beta_v = \sum_i c_i\beta_{i,v}$, i.e. $\beta_v(g) = \sum_j([X_j, P_j(\phi_v g)] - [\kappa'_j(g), gY_j])$. Put $Y_0 = Y' - \sum_j[X_j, Y_j] \in \mathrm{Od}_a(A_m)$ and $Y_v = \sum_j \epsilon'_j Y_j + \eta Y_0 \in \mathrm{PW}^2_a(A_m)$. With $w^1_v = (1 - \sum_j \epsilon_j X_j)w_v$ and $(w^1_v)^{-1} = w_v^{-1}(1 + \sum_j \epsilon_j X_j)$, for $R = \Psi(\phi_v g) - \rho_1 Y_v \rho_1^{-1} + Y_v$ one finds

$$(w^1_v)^{-1} R w^1_v = w_v^{-1}\Big(\sum_j \epsilon'_j R_j + \eta\big(R_0 + \sum_j [X_j, R_j]\big)\Big)w_v,$$

with $R_j = P_j(\phi_v g) - (gY_j - Y_j)$ and $R_0 = y'(\phi_v g) - (gY_0 - Y_0) - \sum_j[\kappa_j(\pi g), gY_j]$. Each $w_v^{-1}R_j w_v$ is upper triangular with zero diagonal for $g \in \mu_v$ (Y_j is a witness for P_j). By (1) and $\kappa_j - \kappa'_j = dX_j$,

$$\begin{aligned} R_0 + \sum_j [X_j, R_j] &= y'(\phi_v g) + \beta_v(g) - (gY' - Y') + \Big(d\sum_j [X_j, Y_j]\Big)(g) \\ &\quad - \sum_j \big([gX_j - X_j, gY_j] + [X_j, gY_j - Y_j]\big), \end{aligned}$$

and the last two terms cancel by the third identity in (1). So $R_0 + \sum_j[X_j, R_j] \in F_v$: its w_v-conjugate is upper triangular, and it lies in Od_a, whose w_v-conjugates have zero diagonal (Lemma E.112). Hence Y_v is a local witness for Ψ at v in the sense of (Sc) of Definition E.35.

Therefore $\Psi \in \mathrm{Sel}^2_{\mathbf V_N}(A_m)$, and its ϵ'_j-coefficient is $P_j = \sum_i c_i z_{i,j}$ on Γ^0_N. □

Lemma E.123. *Let $\mathscr H$, $x = (k, \rho, 1_n, (\psi_{v,a}), f)$, κ with local witnesses and a Taylor–Wiles datum of size q (with H, $\sigma_1, \dots, \sigma_q$) along which x has eigenvalue data be as in Lemma E.122. Assume that $\rho(\gamma) = 1$ for $\gamma \in \Gamma^0_{\mathscr H} \cap \mathrm{PL}(H)$, that x satisfies (Pj1), and that for every $Z \in \mathrm{Ev}_a(A)$*

some T^sZ lies in the A-span of $\rho(\Gamma')$. Let $K = \#k$. Then there is $C \geq 0$ such that for every $m \geq 1$ there is N_0 such that for all $N \geq N_0$: $H_{N,m}$ is a finite A-module, $T^m H_{N,m} = 0$, and $\#(H_{N,m}/T^e H_{N,m}) \leq K^{ge+C}$ for $0 \leq e \leq m$. This is (Ob1) of Proposition E.120 for $H_{N,m}$.

Proof. Let C_0 (Lemma E.127), C_n (Lemma E.126), C_d, C_h (Lemma E.125 (Od1), (Od2)) be the constants given there, and $C = C_0 + C_n + \#S_\ell \cdot C_h + 2C_d$. Fix m and let N_0 be at least the integers N_0 of Lemma E.126 for $e = 1, \dots, m$. Let $N \geq N_0$. For $1 \leq e \leq m$ let $\mathcal{D}_e$ be the Selmer data of Lemma E.122 with A_e in place of A_m (so $\mathcal{D}_m = \mathcal{D}_{N,m}$), $M_e = \mathrm{Od}_a(A_e)$, over $G = G_{L^+,T_\infty}$, $T = S_\ell \cup \mathrm{St} \cup Q_N$. They are admissible in the sense of Lemma E.124 with $F = L^+$, $S = T_\infty$, $W = S_\ell \cup \mathrm{St}$: M_e is killed by ℓ; the places $v \in S_\ell$ are of type (i); the places $u \in \mathrm{St}$ are of type (ii) with J_u: J_u contains $J_{\bar{u}}$ of Lemma E.102, lies in the inertia group of G_u, is normal in G_u, and has finite index prime to ℓ there because $J_{\bar{u}}$ has (Lemma E.102 (Lb2)); it acts trivially on M_e.

Step 1: $\#H^2_{\mathcal{D}_e} \leq K^{ge+C-C_d}$. We evaluate the factors of Lemma E.124 (Sc1). (a) In the w_v-frame (Lemma E.112: $w_v^{-1}\mathrm{Od}_a(A_e)w_v = \mathrm{Od}_{\mathrm{ty}}(A_e)$), M_e is the set of matrices with entries in A_e at the positions (i,j) of different types, and F_v its upper triangular part; positions of different types come in pairs $(i,j),(j,i)$, so $\#Q_v = \#M_e/\#F_v = (\#M_e)^{1/2}$. (b) Let c_u be a complex conjugation; its image c' in Γ_N is an involution outside Γ_N^0 (Proposition E.101 (Pk3)), $c' = c_N\gamma$ with $\gamma \in \Gamma_N^0$, acting by $P \mapsto -{}^t(RPR^{-1})$, $R = \rho(h_N\gamma)$. From $c'^2 = 1$ and $\rho(\bar{c}\gamma'\bar{c}) = {}^t\rho(\gamma')^{-1}$ (Lemma E.107 (Rf1)) we get $R = {}^tR$; then $P \mapsto RP$ is a bijection of M_e (R is block diagonal) carrying the action of c' to $Q \mapsto -{}^tQ$, whose fixed points $\left(\begin{smallmatrix} 0 & B \\ -{}^tB & 0 \end{smallmatrix}\right)$ number $(\#M_e)^{1/2}$. With $\sum_v d_v = d$, (a) and (b) give $\prod_v (\#Q_v)^{d_v} \prod_u \#M_e^{c_u=1} = (\#M_e)^d$. (c) $\#H^0(G_v, Q_v) \leq K^{C_h}$ by (Od2), and $\#\mathrm{Hom}_G(M_e, \mu_\ell) \leq K^{C_d}$ by (Od1) (G acts through p and h_N; an equivariant map is invariant under the elements mapping into Γ', where $\chi_{\mathrm{cyc}} = 1$). (d) $H^1_{\mathcal{D}_e}$: a Selmer cocycle vanishes on $\phi_u(J_u) \supseteq$ the images of $J_{\bar{u}}$, so by Lemma E.102 (Lb3) it is inflated from a unique cocycle $\bar{z}$ of Γ_N, coboundaries corresponding to coboundaries; the local conditions say that $\bar{z}(\pi^{Q_N} g) - (gY - Y) \in F_v$ for some $Y \in M_e$ ($g \in D_v$; lift a primitive in Q_v) and that $\bar{z}$ vanishes on $I_{N,u}$ (equivalently on $p(\phi_u(J_u))$, generated as a closed normal subgroup by $I_{N,u}$, which acts trivially). Every class has a representative with $\bar{z}(c_N) = 0$: as $c_N^2 = 1$, $0 = \bar{z}(c_N^2) = \bar{z}(c_N) + c_N \cdot \bar{z}(c_N)$, so $X = \bar{z}(c_N)$ satisfies $c_N \cdot X = -X$; as ℓ is odd and $\ell M_e = 0$, $Y = -X/2 \in M_e$ is defined, and its coboundary $b_Y(g) = g \cdot Y - Y$ has $b_Y(c_N) = -2Y = X$; b_Y satisfies the local conditions (with witness Y at $v \in S_\ell$; it vanishes on $I_{N,u}$, which acts trivially), so $\bar{z} - b_Y$ represents the same class and vanishes at c_N. Two such representatives differ by a coboundary b_Y with $0 = b_Y(c_N) = c_N \cdot Y - Y$, and c_N acts by $Y \mapsto -{}^tY$ (the action of c' in (b) with $\gamma = 1$), so the representative is unique up to coboundaries of Y with ${}^tY = -Y$; and the restrictions to Γ_N^0 of such $\bar{z}$ are exactly the elements of $\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_N}(A_e)$. Hence $\#H^1_{\mathcal{D}_e} = \#\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_N}(A_e)/\#\mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}_N}(A_e)$. The coboundaries at level N are those of the base level composed with h_N (onto), so $\#\mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}_N}(A_e) = \#\mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}_0}(A_e)$, and by Lemma E.126 and Lemma E.127, $\#H^1_{\mathcal{D}_e} \leq K^{ge+C_0+C_n}$. Inserting (a)–(d) and $\#H^0(G, M_e) \geq 1$ in (Sc1): $\#H^2_{\mathcal{D}_e} \leq K^{ge+C_0+C_n+\#S_\ell C_h+C_d}$.

Step 2: (Ob1). $H = H_{N,m} = H^2_{\mathcal{D}_m}$ is finite (Sc1) and an A_m-module, so $T^m H = 0$. For $e = 0$ the bound is trivial and for $e = m$ it is Step 1. Let $1 \leq e < m$. Reduction $g\colon M_m \to M_e$ and $r\colon M_m \to M_{m-e}$, and $u\colon M_{m-e} \to M_m$, $y \mapsto T^e\tilde{y}$ ($\tilde{y}$ any lift), are maps of Selmer data (all maps are entrywise in the w_v-frames, so they respect the F_v and the Q_v), with $u \circ r = T^e$, and the sequences $0 \to M_{m-e} \xrightarrow{u} M_m \xrightarrow{g} M_e \to 0$ and $0 \to T^{m-e}M_m \to M_m \xrightarrow{r} M_{m-e} \to 0$, together with the induced sequences of the Q_v, are exact (entrywise from $0 \to A_{m-e} \to A_m \to A_e \to 0$). So $T^e H = u_* r_*(H) \subseteq u_* H^2_{\mathcal{D}_{m-e}} = \ker g_*$ by (Sc2), and

$$\#(H/T^eH) = \#(H/\ker g_*) \cdot \#(\ker g_*/T^eH) \leq \#H^2_{\mathcal{D}_e} \cdot \#\mathrm{coker}(r_*),$$

as $\ker g_*/T^eH$ is a quotient of $H^2_{\mathcal{D}_{m-e}}/r_*(H)$. By (Sc2), $\#\mathrm{coker}(r_*) \le \#\mathrm{Hom}_G(T^{m-e}M_m, \mu_\ell)$, and $T^{m-e}M_m \cong M_e$ as G-modules (multiplication by T^{m-e}), so this is at most K^{C_d} by (Od1). With Step 1, $\#(H/T^eH) \le K^{ge+C}$. □

Lemma E.124. *Let F be a totally real number field, $d = [F:\mathbb{Q}]$, ℓ an odd prime, S a finite set of places of F containing those above ℓ and the archimedean ones, $G = G_{F,S}$, and $W \subseteq S$ a set of finite places. Call Selmer data $\mathcal{D}$ over G **admissible** if M is an ℓ-module of G, the index set is W, $G_w = G_{F_w}$ with ϕ_w the natural map, each Q_w is an ℓ-module, and for each w either (i) $L_w = B^1(G_w, Q_w)$, or (ii) $w \nmid \ell$, $Q_w = M$, $\pi_w = \mathrm{id}$ and $L_w = Z^1_{J_w}(M)$ for a closed subgroup J_w of the inertia group of G_w, normal in G_w, of finite index prime to ℓ there, acting trivially on M.*

(Sc1) *For admissible $\mathcal{D}$, $H^1_{\mathcal{D}}$ and $H^2_{\mathcal{D}}$ are finite and*

$$\#H^2_{\mathcal{D}} \le \#H^1_{\mathcal{D}} \cdot \prod_{w \text{ of type (i)}} \#H^0(G_w, Q_w)\,(\#Q_w)^{d_w} \cdot \frac{\prod_{u \text{ real}} \#M^{c_u=1}}{\#H^0(G,M)\,(\#M)^d} \cdot \#\mathrm{Hom}_G(M, \mu_\ell).$$

(Sc2) *Let $\mathcal{D}_K, \mathcal{D}_M, \mathcal{D}_{M'}$ be admissible Selmer data with the same W, types and J_w, and A-linear maps $K \to M \to M'$ and $Q_w(K) \to Q_w(M) \to Q_w(M')$ compatible with the actions and the π_w, forming short exact sequences $0 \to K \to M \to M' \to 0$ and $0 \to Q_w(K) \to Q_w(M) \to Q_w(M') \to 0$. Then the induced sequence $H^2_{\mathcal{D}_K} \to H^2_{\mathcal{D}_M} \to H^2_{\mathcal{D}_{M'}}$ is exact, and the cokernel of $H^2_{\mathcal{D}_M} \to H^2_{\mathcal{D}_{M'}}$ has order at most $\#\mathrm{Hom}_G(K, \mu_\ell)$.*

Proof. Maps of Selmer data (compatible maps $M \to M'$, $Q_w \to Q'_w$) map $Z^2_{\mathcal{D}}$, $B^2_{\mathcal{D}}$ to the corresponding sets, so they induce maps on $H^1_{\mathcal{D}}$, $H^2_{\mathcal{D}}$. Write $\bar{L}_w = L_w/B^1(G_w, Q_w) \subseteq H^1(G_w, Q_w)$.

(Sc1). Consider

$$H^1(G,M) \xrightarrow{\mathrm{loc}^1} \bigoplus_w H^1(G_w, Q_w)/\bar{L}_w \xrightarrow{\delta} H^2_{\mathcal{D}} \xrightarrow{\alpha} H^2(G,M) \xrightarrow{\mathrm{loc}^2} \bigoplus_w H^2(G_w, Q_w),$$

with loc^i induced by $\pi_w\phi_w^*$ (restriction followed by push-forward), $\delta((y_w))$ the class of $(0,(y_w))$ for cocycles y_w, and α the class of the first component. These are well defined ($B^1 \subseteq L_w$; $(0,(\lambda_w)) \in B^2_{\mathcal{D}}$ for $\lambda_w \in L_w$; $(dx,\dots) \mapsto 0$). Exactness: $\delta(\mathrm{loc}^1 z)$ is the class of $(0, \pi_w\phi_w^* z) = (dz, \pi_w\phi_w^* z) \in B^2_{\mathcal{D}}$; conversely $(0,(y_w)) = (dx, (\pi_w\phi_w^* x + \lambda_w))$ forces $x \in Z^1(G,M)$ and $y_w \equiv \pi_w\phi_w^* x \bmod L_w$. If α of the class of $(a,(b_w))$ vanishes, $a = dx$, and $(a,(b_w)) - (dx, (\pi_w\phi_w^* x)) = (0, (b_w - \pi_w\phi_w^* x))$ with $b_w - \pi_w\phi_w^* x$ a cocycle; and $\alpha\delta = 0$. Finally $[a] \in \ker \mathrm{loc}^2$ iff $\pi_w\phi_w^* a = db_w$ for some b_w, i.e. iff a is the first component of an element of $Z^2_{\mathcal{D}}$. All groups in the sequence are finite by Proposition E.101 (Pk1), (Pk2), hence $H^2_{\mathcal{D}}$ is finite, and so is $H^1_{\mathcal{D}} = \ker \mathrm{loc}^1$. Counting orders,

$$\begin{aligned}\#H^2_{\mathcal{D}} &= \#\mathrm{coker}\,\mathrm{loc}^1 \cdot \#\ker \mathrm{loc}^2 \\ &= \frac{\#H^1_{\mathcal{D}} \prod_w \#(H^1(G_w,Q_w)/\bar{L}_w)}{\#H^1(G,M)} \\ &\quad \cdot \frac{\#H^2(G,M) \cdot \#\mathrm{coker}\,\mathrm{loc}^2}{\prod_w \#H^2(G_w,Q_w)}.\end{aligned}$$

By (Pk3), $\#H^2(G,M)/\#H^1(G,M) = \prod_u \#M^{c_u=1}/(\#H^0(G,M)(\#M)^d)$; by (Pk4) (with the subset W), $\#\mathrm{coker}\,\mathrm{loc}^2 \le \#\mathrm{Hom}_G(M, \mu_\ell)$. For w of type (i), $\bar{L}_w = 0$ and, by (Pk2),

$$\#H^1(G_w,Q_w)/\#H^2(G_w,Q_w) = \#H^0(G_w,Q_w)(\#Q_w)^{d_w}.$$

For w of type (ii), $\#\bar{L}_w = \#H^0(G_w, M)$ by Lemma E.104 (Ur1) (applied to the local field F_w, $D = G_w$, $J = J_w$), and $\#H^1(G_w, M) = \#H^0(G_w, M)\#H^2(G_w, M)$ by (Pk2) ($d_w = 0$); so the factor of w is 1. This gives the bound.

(Sc2). Write $f\colon M \to M'$, and f also for the induced maps. *Lifting local conditions:* every element of $L_w(M')$ is the image of an element of $L_w(M)$: for type (i) because $Q_w(M) \to Q_w(M')$ is onto, for type (ii) by Lemma E.104 (Ur2). Continuous cochains with values in M' or $Q_w(M')$ lift to continuous cochains (compose with a set-theoretic section of the finite surjection).

Exactness in the middle. The composite $H^2_{\mathcal{D}_K} \to H^2_{\mathcal{D}_{M'}}$ is zero. If $(a, (b_w)) \in Z^2_{\mathcal{D}_M}$ maps into $B^2_{\mathcal{D}_{M'}}$, say to $(dx', (\pi_w\phi_w^* x' + \lambda'_w))$, lift x' to x and λ'_w to $\lambda_w \in L_w(M)$; then $(a, (b_w)) - (dx, (\pi_w\phi_w^* x + \lambda_w))$ has components with values in K and $Q_w(K)$, so it lies in $Z^2_{\mathcal{D}_K}$.

The cokernel. Let

$$\mathrm{loc}^2_K\colon H^2(G, K) \to \bigoplus_w H^2(G_w, Q_w(K))$$

be the map defined as in (Sc1); by (Pk4), $\#\mathrm{coker}\,\mathrm{loc}^2_K \le \#\mathrm{Hom}_G(K, \mu_\ell)$. We define a homomorphism $\Theta\colon Z^2_{\mathcal{D}_{M'}} \to \mathrm{coker}\,\mathrm{loc}^2_K$. Given $(a', (b'_w))$: by (Pk1) $H^2(G, M) \to H^2(G, M')$ is onto, so $a' = f(a_0) + dx'$ with $a_0 \in Z^2(G, M)$; let $\beta_w \in C^1(G_w, Q_w(M))$ lift $b'_w - \pi_w\phi_w^* x'$; then $e_w = d\beta_w - \pi_w\phi_w^* a_0$ is a continuous 2-cocycle with values in $Q_w(K)$ (its image is $db'_w - \pi_w\phi_w^*(a' - dx') - d\pi_w\phi_w^* x' = 0$). Let $\Theta(a', (b'_w))$ be the class of $([e_w])_w$. Independence of the choices: if $a_0^\sharp, x'^\sharp, \beta_w^\sharp$ are other choices, then $f(a_0 - a_0^\sharp) = d(x'^\sharp - x')$; lifting $x'^\sharp - x'$ to x'', $k = a_0 - a_0^\sharp - dx'' \in Z^2(G, K)$, and $\beta_w^\sharp = \beta_w - \pi_w\phi_w^* x'' + \gamma_w$ with γ_w a 1-cochain into $Q_w(K)$; so $e_w^\sharp = e_w + d\gamma_w + \pi_w\phi_w^* k$, and the classes differ by $\mathrm{loc}^2_K[k]$. Using sums of choices for sums, Θ is additive. Θ vanishes on $B^2_{\mathcal{D}_{M'}}$: for $(dx', (\pi_w\phi_w^* x' + \lambda'_w))$ take $a_0 = 0$ and β_w a lift of λ'_w in $L_w(M)$, a cocycle, so $e_w = 0$. It vanishes on the image of $Z^2_{\mathcal{D}_M}$ (take $x' = 0$ and the given lifts: $e_w = 0$). Conversely if $\Theta(a', (b'_w)) = 0$, then $e_w = \pi_w\phi_w^* k + d\gamma_w$ with $k \in Z^2(G, K)$, $\gamma_w \in C^1(G_w, Q_w(K))$, and $(a_0 + k, (\beta_w - \gamma_w)) \in Z^2_{\mathcal{D}_M}$ (as $d(\beta_w - \gamma_w) = \pi_w\phi_w^*(a_0 + k)$) maps to $(a' - dx', (b'_w - \pi_w\phi_w^* x'))$, which is in the class of $(a', (b'_w))$. So Θ induces an injection of the cokernel of $H^2_{\mathcal{D}_M} \to H^2_{\mathcal{D}_{M'}}$ into $\mathrm{coker}\,\mathrm{loc}^2_K$. □

Lemma E.125. *Let $\mathscr{H}$ be a rank-n Hecke datum, x a framed curve point normalised of shape $(a, n-a)$, $\pi = \pi_{\mathscr{H}}$ and $K = \#k$. For $e \ge 1$ let $\Gamma^0_{\mathscr{H}}$ act on $\mathrm{Od}_a(A_e)$ by $\gamma \cdot P = \rho(\gamma)P\rho(\gamma)^{-1}$ and D_v ($v \in S_\ell$) by $g \cdot P = \rho(\pi g)P\rho(\pi g)^{-1}$; let $F_v(A_e) = \{P \in \mathrm{Od}_a(A_e) : w_v^{-1}Pw_v$ upper triangular$\}$.*

(Od1) *If for every $Z \in \mathrm{Ev}_a(A)$ some $T^s Z$ lies in the A-span of $\rho(\Gamma')$, then there is $C_d \ge 0$ such that for every $e \ge 1$ the number of additive maps $f\colon \mathrm{Od}_a(A_e) \to \mu_\ell$ with $f(\gamma\cdot P) = f(P)$ for all $\gamma \in \Gamma'$ and P is at most K^{C_d}. (In particular this bounds the maps with $f(\gamma\cdot P) = \gamma f(P)$ for γ in any group acting through $\Gamma^0_{\mathscr{H}} \supseteq \Gamma'$ and on μ_ℓ through χ_{cyc}, as $\chi_{\mathrm{cyc}} = 1$ on Γ'.)*

(Od2) *If x satisfies (Pj1), then $F_v(A_e)$ is D_v-stable and there is $C_h \ge 0$ with*

$$\#H^0(D_v, \mathrm{Od}_a(A_e)/F_v(A_e)) \le K^{C_h}$$

for all $e \ge 1$ and $v \in S_\ell$.

Proof. (Od1). Let $\tau\colon A_e \to \mathbb{F}_\ell$ be the non-degenerate form of Lemma E.106 (trace of the top coefficient) and ζ a primitive ℓ-th root of unity. For $Y \in \mathrm{Od}_a(A_e)$ put $f_Y(P) = \zeta^{\tau(\mathrm{tr}(PY))}$. Every additive $f\colon \mathrm{Od}_a(A_e) \to \mu_\ell$ is f_Y for exactly one Y: for the matrix units, $\mathrm{tr}(P \cdot E_{ji}y) = P_{ij}y$, and $E_{ji} \in \mathrm{Od}_a$ exactly when P_{ij} is an entry of Od_a; so $Y \mapsto f_Y$ is injective by non-degeneracy of τ, and both sides have $\#\mathrm{Od}_a(A_e)$ elements ($\mathrm{Od}_a(A_e)$ is an $\mathbb{F}_\ell$-vector space and $\mu_\ell \cong \mathbb{F}_\ell$).

Now $f_Y(\gamma\cdot P)=f_Y(P)$ for all P iff $\tau(\mathrm{tr}(P(\rho^{-1}Y\rho-Y)))=0$ for all $P\in \mathrm{Od}_a(A_e)$ ($\rho=\rho(\gamma)$), iff $\rho^{-1}Y\rho=Y$, because $\rho^{-1}Y\rho-Y\in\mathrm{Od}_a(A_e)$ (ρ is block diagonal) and the pairing is non-degenerate. So the maps in question correspond to the $Y\in\mathrm{Od}_a(A_e)$ with $\rho(\gamma)Y=Y\rho(\gamma)$ for all $\gamma\in\Gamma'$: the common kernel modulo T^e of the A-linear maps $Y\mapsto\rho(\gamma)Y-Y\rho(\gamma)$ on $\mathrm{Od}_a(A)$. Over A this common kernel is 0: such a Y commutes with the A-span of $\rho(\Gamma')$; for each i the matrix unit E_{ii} lies in $\mathrm{Ev}_a(A)$, so by the hypothesis of (Od1) some T^sE_{ii} lies in that span, $T^s(E_{ii}Y-YE_{ii})=0$, and, A being a domain, Y commutes with every E_{ii}, so it is diagonal, and a diagonal element of $\mathrm{Od}_a(A)$ is 0. By Lemma E.110 there is C_d as stated.

(Od2). Write $w=w_v$, $b(g)=w^{-1}\rho(\pi g)w$, upper triangular with diagonal $(\psi_{v,1}(g),\dots,\psi_{v,n}(g))$ by (Sd1). By Lemma E.112, $P\mapsto w^{-1}Pw$ maps $\mathrm{Od}_a(A_e)$ onto $\mathrm{Od}_{\mathrm{ty}}(A_e)$ (entries (i,j) with $\mathrm{ty}(i)\neq\mathrm{ty}(j)$; zero diagonal), carrying $F_v(A_e)$ onto its upper triangular elements and the action to $P'\mapsto b(g)P'b(g)^{-1}$. As $b(g)$ is upper triangular, F_v is stable, and the quotient is identified with the strictly lower triangular part Λ of $\mathrm{Od}_{\mathrm{ty}}(A_e)$ with the induced action. For upper triangular b, $(bP'b^{-1})_{ij}=\sum_{i'\geq i,\ j'\leq j}b_{ii'}P'_{i'j'}(b^{-1})_{j'j}$ involves only entries of depth $i'-j'\geq i-j$, and its term $(i',j')=(i,j)$ is $\psi_i\psi_j^{-1}P'_{ij}$. Hence the subgroups $\Lambda_h\subseteq\Lambda$ of classes with all entries of depth $>h$ zero are stable, and $\Lambda_h/\Lambda_{h-1}\cong\bigoplus_{i-j=h,\ \mathrm{ty}(i)\neq\mathrm{ty}(j)}A_e$ with g acting on the (i,j) summand by $\theta_{ij}(g)=\psi_i(g)\psi_j(g)^{-1}$. By left exactness of invariants, $\#H^0(D_v,\Lambda)\leq\prod_{i>j}\#H^0(D_v,A_e(\theta_{ij}))$. For $i\neq j$, (Pj1) gives $\theta_{ij}\neq1$, so the ideal of A generated by the $\theta_{ij}(g)-1$ is $T^{e_{ij}}A$ for some $e_{ij}\geq0$ independent of e, and $H^0(D_v,A_e(\theta_{ij}))=\{y\in A_e:\ T^{e_{ij}}y=0\}$ has at most $K^{e_{ij}}$ elements. Take $C_h=\max_v\sum_{i>j}e_{ij}$. □

Lemma E.126. *Let $\mathscr{H}$ be a rank-n Hecke datum, $x=(k,\rho,1_n,(\psi_{v,a}),f)$ a framed curve point normalised of shape $(a,n-a)$ and $K=\#k$. Fix a Taylor–Wiles datum of size q (with H, $\sigma_1,\dots,\sigma_q$) along which x has eigenvalue data, and assume that $\rho(\gamma)=1$ for $\gamma\in\Gamma^0_{\mathscr{H}}\cap\mathrm{PL}(H)$. Let $\mathbf{V}_0$ be the base level. Then there is $C_n\geq0$ such that for every $e\geq1$ there is N_0 with*

$$\#\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_N}(A_e)\ \leq\ \#\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_0}(A_e)\cdot K^{C_n}\qquad(N\geq N_0),$$

and every $z\in\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_N}(A_e)$ vanishing at $\mathrm{T}_{N,1},\dots,\mathrm{T}_{N,q}$ is $z_0\circ h_N$ for a unique $z_0\in\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_0}(A_e)$.

Proof. Write $\Gamma^0=\Gamma^0_{\mathscr{H}}$, $\Gamma^0_N=h_N^{-1}(\Gamma^0)$, and let Γ^0_N act on $\mathrm{Od}_a(A_e)$ by $\gamma\cdot P=\rho(h_N\gamma)P\rho(h_N\gamma)^{-1}$; elements of Γ^0_N in $\ker h_N$ or in $\mathrm{PL}(H_N)$ act trivially (for the latter, $h_N(\mathrm{PL}(H_N))\subseteq\mathrm{PL}(H)\subseteq\Gamma^0$ by Lemma E.45 (Pl1), (Pl3), and $\rho=1$ there). $\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_N}(A_e)$ is a group of cocycles $z\colon\Gamma^0_N\to\mathrm{Od}_a(A_e)$, killed by ℓ, with $z(c_N\gamma c_N)=-{}^tz(\gamma)$, $c_N=\pi^{Q_N}(c)$.

Step 1: the bounded part. For each j let $P_j=\rho(\sigma_j)=\mathrm{diag}(P'_j,P''_j)$. The characteristic polynomials f'_j, f''_j of P'_j, P''_j have product $\prod_i(X-\alpha_{j,i})$ with the $\alpha_{j,i}\in A$ pairwise distinct (eigenvalue data), so they have no common root and $f'_j(P''_j)=\prod_{f'_j(\alpha)=0}(P''_j-\alpha)$ is invertible over F. If $B\in\mathrm{M}_{a\times(n-a)}(F)$ and $P'_jB=BP''_j$, then $0=f'_j(P'_j)B=Bf'_j(P''_j)$ (Cayley–Hamilton), so $B=0$; likewise $P''_jC=CP'_j$ with $C\in\mathrm{M}_{(n-a)\times a}(F)$ forces $C=0$. Hence no non-zero element of $\mathrm{Od}_a(F)$ commutes with P_j; so by Lemma E.110 there is b_j with $\#\{P\in\mathrm{Od}_a(A_e):\ P_jP=PP_j\}\leq K^{b_j}$ for all e. Put $C_n=\sum_jb_j$. Fix e. We choose N_0. Let $U\subseteq\Gamma^0$ be the kernel of $\gamma\mapsto\rho(\gamma)\bmod T^e$, an open normal subgroup of Γ^0 (ρ is continuous for the T-adic topology), and $V=U\cap\bar cU\bar c^{-1}$ with $\bar c=\pi_{\mathscr{H}}(c)$; as $\Gamma_{\mathscr{H}}=\Gamma^0\cup\bar c\,\Gamma^0$, V is an open normal subgroup of $\Gamma_{\mathscr{H}}$. It contains $\mathrm{PL}(H)$: H is one of the open normal subgroups whose intersection defines $\mathrm{PL}(H)$, so $\mathrm{PL}(H)\subseteq H\subseteq\Gamma^0$ (Tg1), hence $\rho=1$ on $\mathrm{PL}(H)$ by hypothesis and $\mathrm{PL}(H)\subseteq U$; and $\mathrm{PL}(H)$, an intersection of normal subgroups, is normal in $\Gamma_{\mathscr{H}}$, so $\mathrm{PL}(H)=\bar c\,\mathrm{PL}(H)\bar c^{-1}\subseteq\bar cU\bar c^{-1}$. By (Tg3), applied to V for each of the finitely many j, there is N_0 with $u_{N,j}=h_N(\Phi_{N,j})\sigma_j^{-1}\in V$ for all j and $N\geq N_0$. As $\sigma_j\in\Gamma'\subseteq\Gamma^0$ and $u_{N,j}\in U$, $\rho(h_N\Phi_{N,j})=\rho(u_{N,j})\rho(\sigma_j)\equiv\rho(\sigma_j)\bmod T^e$ for all j and $N\geq N_0$. Let $N\geq N_0$, $z\in\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_N}(A_e)$, $\mathrm{T}=\mathrm{T}_{N,j}$, $\Phi=\Phi_{N,j}$. As T acts trivially,

$z(\mathrm{T}^k) = kz(\mathrm{T})$ and $z(\Phi^{-1}\mathrm{T}\Phi) = \Phi^{-1}\cdot z(\mathrm{T})$ (for u acting trivially, $z(gug^{-1}) = g\cdot z(u)$). By (Tg4), $\mathrm{T}^{-q_{N,j}}\Phi^{-1}\mathrm{T}\Phi \in \mathrm{PL}(H_N)$, and z vanishes on $\mathrm{PL}(H_N)$ by Lemma E.45 (Pl2) (applied to $G = \Gamma_N$, the open normal subgroup $H_N \subseteq \Gamma_N^0$); so $0 = -q_{N,j}z(\mathrm{T}) + \Phi^{-1}\cdot z(\mathrm{T})$, and as $q_{N,j} \equiv 1 \bmod \ell$ (Tg2) and $\ell z(\mathrm{T}) = 0$, $\Phi\cdot z(\mathrm{T}) = z(\mathrm{T})$, i.e. $z(\mathrm{T}_{N,j})$ commutes with $\rho(\sigma_j)$ modulo T^e. Hence $z \mapsto (z(\mathrm{T}_{N,j}))_j$ is a homomorphism from $\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_N}(A_e)$ to a group of order at most K^{C_n}.

Step 2: the kernel descends. Let z vanish at all $\mathrm{T}_{N,j}$. Then z vanishes on every conjugate $g\mathrm{T}_{N,j}g^{-1}$: for $g \in \Gamma_N^0$ it equals $g\cdot z(\mathrm{T}_{N,j}) = 0$, and for $g = c_N g_0$ it is $-{}^t z(g_0\mathrm{T}_{N,j}g_0^{-1}) = 0$. It vanishes on $\mathrm{PL}(H_N)$ (Step 1). The closed subgroup W generated by these elements lies in Γ_N^0 (normal, containing H_N) and acts trivially, so $z|_W$ is a continuous homomorphism vanishing on generators, hence $z|_W = 0$; by (Tg4) $\ker h_N \subseteq W$. So $z(g\gamma) = z(g)$ for $\gamma \in \ker h_N$, and $z = z_0 \circ h_N$ for a unique map $z_0\colon \Gamma^0 \to \mathrm{Od}_a(A_e)$ (h_N is onto), locally constant since h_N is open. z_0 lies in $\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_0}(A_e)$: the cocycle and $\bar c$-identities descend ($h_N(c_N) = \bar c$); z_0 vanishes on $I_u^x = h_N(I_{N,u})$; and for $v \in S_\ell$, $z(\pi^{Q_N}g) = z_0(\pi_{\mathscr{H}}g)$ for $g \in D_v$ (as $h_N \circ \pi^{Q_N} = \pi_{\mathscr{H}}$), so the local witnesses of z serve for z_0. Conversely $z_0 \mapsto z_0 \circ h_N$ is injective. Hence the kernel of the homomorphism of Step 1 has at most $\#\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_0}(A_e)$ elements, and the bound follows. □

Lemma E.127. *Let $\mathscr{H}$ be a rank-n Hecke datum, $x = (k, \rho, 1_n, (\psi_{v,a}), f)$ a framed curve point normalised of shape $(a, n-a)$ with a Selmer basis $\kappa = (\kappa_1, \dots, \kappa_g)$, $\mathbf{V}_0$ the base level and $K = \#k$. Then there is $C_0 \geq 0$ such that for all $e \geq 1$*

$$\#\mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_0}(A_e) \;\leq\; \#\mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}_0}(A_e)\cdot K^{ge+C_0}.$$

Proof. Write $\Gamma^0 = \Gamma^0_{\mathscr{H}}$, $\pi = \pi_{\mathscr{H}}$, $\bar c = \pi(c)$, $\mathrm{Sel}_e = \mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}_0}(A_e)$, $\mathrm{Bd}_e = \mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}_0}(A_e)$. We apply Lemma E.115, carrying the local witnesses along as extra coordinates.

The modules. Let $C' = C \times \mathrm{Od}_a(A)^{S_\ell}$, where C is the A-module of T-adically continuous maps $f\colon \Gamma^0 \to \mathrm{Od}_a(A)$; by Lemma E.117, item 1 ($\mathrm{Od}_a(A)$ is a free A-module of finite rank), C has no T-torsion and is T-adically separated and complete, hence so is C', its product with a free A-module of finite rank. For $v \in S_\ell$ and $X \in \mathrm{M}_n(A)$ let $\lambda_v(X)$ be the family of the entries of $w_v^{-1}Xw_v$ strictly below the diagonal. Let Y be the product, over pairs (γ, γ'), elements γ, places $u \in \mathrm{St}$ with $\gamma \in I_u^x$, and pairs (v, g) with $v \in S_\ell$, $g \in D_v$, of copies of $\mathrm{M}_n(A)$ or of $A^{n(n-1)/2}$, and let $\Phi(f, (Y_v))$ have the components

$$f(\gamma\gamma') - f(\gamma) - \rho(\gamma)f(\gamma')\rho(\gamma)^{-1}, \quad f(\bar c\gamma\bar c) + {}^t f(\gamma), \quad f(\gamma)\ (\gamma \in I_u^x),$$
$$\lambda_v(f(\pi g) - \rho(\pi g)Y_v\rho(\pi g)^{-1} + Y_v).$$

Y has no T-torsion and is separated, Φ is A-linear, and, since $w_v^{-1}\mathrm{Od}_a w_v$ has zero diagonal (Lemma E.112), $\ker\Phi$ is the set of pairs $(z, (Y_v))$ with $z \in Z^1_{\mathrm{ord}}(x)$ and (Y_v) a family of local witnesses for z; likewise $X_e = \Phi^{-1}(T^eY)$ reduces onto the set of pairs $(\bar z, (\bar Y_v))$ with $\bar z \in \mathrm{Sel}_e$ and $(\bar Y_v)$ a family of witnesses for $\bar z$ in $\mathrm{Od}_a(A_e)$ (lift $\bar z$ to $f \in C$ by Lemma E.117, item 2, and each $\bar Y_v$ to any $Y_v \in \mathrm{Od}_a(A)$; $\Phi(f, (Y_v))$ modulo T^eY depends only on the reductions), with kernel T^eC' (Lemma E.117, item 2, for the first component).

The count of pairs. Let W_e be the set of families $(\bar Y_v)$ with $\lambda_v(\bar Y_v - \rho(\pi g)\bar Y_v\rho(\pi g)^{-1}) = 0$ for all v, g (the witnesses of $\bar z = 0$), a group. The witnesses of any $\bar z \in \mathrm{Sel}_e$ form a coset of W_e. Hence $\#(X_e/T^eC') = \#\mathrm{Sel}_e \cdot \#W_e$.

B. Let $R \subseteq Z^1_{\mathrm{ord}}(x)$ be $\sum_j A\kappa_j + \mathrm{Bd}(x)$ and $B = \{(z, (Y_v)) \in \ker\Phi :\ z \in R\}$. By the definition of a Selmer basis ((Sd2)), every $z \in Z^1_{\mathrm{ord}}(x)$ has $T^s z \in R$, so every element of $\ker\Phi$ has a multiple $T^s(z, (Y_v)) \in B$. The reduction of B modulo T^e consists of pairs $(\bar z, (\bar Y_v))$ with $\bar z \in \bar R_e$, the reduction of R, and $(\bar Y_v)$ a witness family of $\bar z$; so $\#((B + T^eC')/T^eC') \leq \#\bar R_e \cdot \#W_e$.

Moreover $\bar{R}_e \subseteq \sum_j A_e\bar{\kappa}_j + \mathrm{Bd}_e$ (a coboundary of a skew $Y \in \mathrm{Od}_a(A)$ reduces to the coboundary of a skew element of $\mathrm{Od}_a(A_e)$), so $\#\bar{R}_e \le K^{ge}\cdot\#\mathrm{Bd}_e$.

Finiteness of X_1/TC'. $\#(X_1/TC') = \#\mathrm{Sel}_1\cdot\#W_1$; $W_1 \subseteq \mathrm{Od}_a(k)^{S_\ell}$ is finite, and Sel_1 is a set of locally constant maps $\Gamma^0 \to \mathrm{Od}_a(k) \subseteq \mathrm{M}_n(k)$ satisfying the cocycle identity for the continuous action $\gamma\cdot X = \bar{\rho}(\gamma)X\bar{\rho}(\gamma)^{-1}$ ($\bar{\rho}$ the reduction of ρ modulo T) of Γ^0 on the finite group $\mathrm{M}_n(k)$, killed by ℓ; as Γ^0 satisfies Φ_ℓ (Lemma E.116), Sel_1 is finite by Lemma E.118.

By Lemma E.115 there is C_0 with $\#\mathrm{Sel}_e\cdot\#W_e \le \#\bar{R}_e\cdot\#W_e\cdot K^{C_0} \le K^{ge}\cdot\#\mathrm{Bd}_e\cdot\#W_e\cdot K^{C_0}$ for all e; cancelling $\#W_e$ gives the claim. □

Lemma E.128. *Let A be a discrete valuation ring with uniformiser t and finite residue field of cardinality q_A, let $g, C, m \ge 0$ be integers, and let H be a finite A-module with $t^m H = 0$ and*

$$\#(H/t^eH) \le q_A^{ge+C} \qquad \text{for all integers } 0 \le e \le m.$$

Then for all $\xi_0,\dots,\xi_g \in H$ there are $c_0,\dots,c_g \in A$, at least one of them a unit, with $t^C\sum_{i=0}^g c_i\xi_i = 0$.

Proof. If $m \le C$, take all $c_i = 1$: $t^CH = 0$. Let $m \ge C+1$, put $e = C+1 \le m$, and suppose, for a contradiction, that $t^C\sum_i c_i\xi_i \ne 0$ whenever some c_i is a unit. Let $\varphi\colon A^{g+1} \to H$, $\varphi(c) = \sum_i c_i\xi_i$, and $N = \varphi(A^{g+1})$.

Step 1: if $\varphi(d) = t^e\varphi(d_0)$ with $d, d_0 \in A^{g+1}$, then $d \in t^eA^{g+1}$. We show $d \in t^kA^{g+1}$ by induction on $k \le e$; $k = 0$ is clear. Let $k < e$ and $d = t^kd'$. If some d'_{i_0} is a unit, put $d'' = d' - t^{e-k}d_0$; then d''_{i_0} is a unit ($e - k \ge 1$) and $t^k\varphi(d'') = \varphi(d) - t^e\varphi(d_0) = 0$, so $t^C\varphi(d'') = t^{C-k}t^k\varphi(d'') = 0$ (as $k \le C$), contrary to the assumption. So t divides every d'_i and $d \in t^{k+1}A^{g+1}$.

Step 2: counting. By Step 1, the map $(A/t^eA)^{g+1} \to N/t^eN$ induced by φ is injective, so $\#(N/t^eN) \ge (\#A/t^eA)^{g+1} = q_A^{e(g+1)}$ (the successive quotients $t^iA/t^{i+1}A$ have q_A elements). For a finite A-module P, multiplication by t^e has kernel $P[t^e]$ and cokernel P/t^eP of the same cardinality. Hence

$$q_A^{e(g+1)} \le \#(N/t^eN) = \#N[t^e] \le \#H[t^e] = \#(H/t^eH) \le q_A^{ge+C} < q_A^{ge+e},$$

as $e > C$ and $q_A \ge 2$: a contradiction. □

Lemma E.129. *Let x be a framed normalised curve point of shape $(a, n-a)$ over a rank-n Hecke datum, and let a Selmer basis $\kappa = (\kappa_1,\dots,\kappa_g)$ of x with local witnesses and a level $\mathbf{V} = (\Gamma_1, h, c_1, (\phi_{1,v}), (I_{1,u}))$ be given. Let $P_1 \subseteq \Gamma_1^0$ be a closed normal subgroup of Γ_1 such that Γ_1/P_1 is topologically finitely generated, $\rho(h(P_1)) = 1$, $\kappa_j(h(P_1)) = 0$ for all j, and every element of $\mathrm{Sel}^1_{\mathbf{V}}(A)$ and of every $\mathrm{Sel}^1_{\mathbf{V}}(A_m)$ vanishes on P_1 (for instance $P_1 = \{1\}$ when Γ_1 is topologically finitely generated). Let $D_0 \ge 0$, and assume that there is $C' \ge 0$ such that for every $m \ge 1$ the set $\mathrm{Sel}^1_{\mathbf{V}}(A_m)$ is finite and $\#\mathrm{Sel}^1_{\mathbf{V}}(A_m) \le \#\mathrm{Bd}^1_{\mathbf{V}}(A_m)\cdot(\#k)^{mD_0+C'}$. Then (K1) holds at $\mathbf{V}$: if $z_1,\dots,z_e \in \mathrm{Sel}^1_{\mathbf{V}}(A)$ are such that $\sum_i c_iz_i \in \mathrm{Bd}^1_{\mathbf{V}}(A)$ with $c \in A^e$ forces $c = 0$, then $e \le D_0$.*

Proof. Write $\varrho_1 = \rho_1\circ h$ on Γ_1^0 (entries in $\mathrm{PD}_g(A)$) and, for $R = A$ or A_m, $d_R(Y)$ for the map $\gamma \mapsto \varrho_1(\gamma)Y\varrho_1(\gamma)^{-1} - Y$. Recall that 2 is invertible in A (ℓ odd), and $\rho(\bar{c}\gamma\bar{c}) = {}^t\rho(\gamma)^{-1}$ (normalised point, $\gamma^* = \bar{c}\gamma^{-1}\bar{c}$).

Step 0: the sets are A-modules and $\mathrm{Bd}^1 \subseteq \mathrm{Sel}^1$. The conditions defining $\mathrm{Sel}_{\mathbf{V}}(\varrho, u, V)$ in Definition E.35 are A-linear in z (the local witnesses add), so $\mathrm{Sel}^1_{\mathbf{V}}(R)$ and $\mathrm{Bd}^1_{\mathbf{V}}(R)$ are A-submodules. Let $Y \in \mathrm{PW}^1_a(R)$ with ${}^tY = -Y$. $d_R(Y)$ has values in $\mathrm{PW}^1_a(R)$ (conjugation

by $\varrho_1(\gamma)$, block diagonal plus ϵ-terms in the off-diagonal blocks, preserves PW^1_a), is continuous and a cocycle; the local condition holds with witness $Y_v = Y$; and since $\kappa_j(\bar{c}\gamma\bar{c}) = -{}^t\kappa_j(\gamma)$, $\varrho_1(c_1\gamma c_1) = (1 - \sum_j \epsilon_j {}^t\kappa_j(h\gamma))^t\rho(h\gamma)^{-1} = {}^t\varrho_1(\gamma)^{-1}$, whence $d_R(Y)(c_1\gamma c_1) = {}^t\varrho_1(\gamma)^{-1}Y^t\varrho_1(\gamma) - Y = -{}^t(d_R(Y)(\gamma))$. So $\mathrm{Bd}^1_{\mathbf{V}}(R) \subseteq \mathrm{Sel}^1_{\mathbf{V}}(R)$. Reduction modulo T^m (entrywise, with the same local witnesses reduced) maps $\mathrm{Sel}^1_{\mathbf{V}}(A)$ into $\mathrm{Sel}^1_{\mathbf{V}}(A_m)$; write $\bar{z}$ for the reduction.

Step 1: a uniform bound on coefficients. Let $z_1, \dots, z_e$ be as in the statement. We show: there is $C \geq 0$ such that for all $m \geq 0$ and $c \in A^e$ with $\sum_i c_i\bar{z}_i \in \mathrm{Bd}^1_{\mathbf{V}}(A_m)$, every c_i lies in $T^{m-C}A$ (read as A for $m \leq C$). The open subgroup Γ_1^0/P_1 of index 2 of the topologically finitely generated group Γ_1/P_1 is topologically finitely generated (Schreier's lemma: if $s_1, \dots, s_r$ generate a dense subgroup, the Schreier generators for the cosets $1, c_1$ generate a dense subgroup of the index-2 subgroup); let $t_1, \dots, t_p \in \Gamma_1^0$ map to such generators. P_1 acts trivially ($\varrho_1 = 1$ on P_1), so a cocycle vanishing on P_1 satisfies $z(\gamma p) = z(\gamma)$ and induces a cocycle of Γ_1^0/P_1; coboundaries vanish on P_1 too. Hence a cocycle with T-adically continuous entries vanishing on P_1 is determined by its values at $t_1, \dots, t_p$: two such cocycles agreeing there agree on the subgroup of Γ_1^0/P_1 generated by the images (cocycle rule) and, being continuous into a Hausdorff space, on its closure. Let $M' = \mathrm{M}_n(\mathrm{PD}_g(A))^p$, a finitely generated free A-module, $V_0 = \{Y \in \mathrm{PW}^1_a(A) : {}^tY = -Y\}$, and

$$\Phi\colon A^e \oplus V_0 \to M', \qquad (c, Y) \mapsto \Big(\sum_i c_i z_i(t_k) - d_A(Y)(t_k)\Big)_{1 \leq k \leq p}.$$

If $\Phi(c, Y) = 0$, then $\sum_i c_i z_i = d_A(Y) \in \mathrm{Bd}^1_{\mathbf{V}}(A)$ by the remark just made, hence $c = 0$: each coordinate function c_i vanishes on $\ker\Phi$. By the Artin–Rees lemma [166, Thm. 8.5] for the submodule $P = \Phi(A^e \oplus V_0)$ of M' and $I = TA$ there is C with $P \cap T^{m+C}M' \subseteq T^mP$. Now let $\sum_i c_i\bar{z}_i = d_{A_m}(\bar{Y})$ with $\bar{Y} \in \mathrm{PW}^1_a(A_m)$, ${}^t\bar{Y} = -\bar{Y}$, $m \geq C$. Lift $\bar{Y}$ to $Y \in V_0$: lift each coordinate (in A_m) of the entries $\bar{Y}_{ij}$, $i < j$, to A, put $Y_{ji} = -Y_{ij}$ for $i < j$ (the positions (i, j) and (j, i) are both same-block or both different-block), and $Y_{ii} = 0$ (the diagonal entries of $\bar{Y}$ vanish since $2\bar{Y}_{ii} = 0$ and 2 is invertible). Then $\Phi(c, Y) \in P \cap T^mM' \subseteq T^{m-C}P$, so $\Phi(c, Y) = \Phi(c', Y')$ with $c' \in T^{m-C}A^e$; $(c - c', Y - Y') \in \ker\Phi$ gives $c = c' \in T^{m-C}A^e$.

Step 2: counting. Let $m \geq C$, $S_m = \mathrm{Sel}^1_{\mathbf{V}}(A_m)$, $B_m = \mathrm{Bd}^1_{\mathbf{V}}(A_m) \subseteq S_m$ (finite, nonempty), and fix a set-theoretic section σ of $A \to A/T^{m-C}A$. The map $B_m \times (A/T^{m-C}A)^e \to S_m$, $(b, \bar{c}) \mapsto b + \sum_i \sigma(\bar{c}_i)\bar{z}_i$, is injective: equal images give $\sum_i(\sigma(\bar{c}_i) - \sigma(\bar{c}'_i))\bar{z}_i = b' - b \in B_m$, so $\sigma(\bar{c}_i) - \sigma(\bar{c}'_i) \in T^{m-C}A$ by Step 1, i.e. $\bar{c} = \bar{c}'$, and then $b = b'$. As $\#(A/T^{m-C}A) = (\#k)^{m-C}$,

$$\#B_m\,(\#k)^{(m-C)e} \leq \#S_m \leq \#B_m\,(\#k)^{mD_0+C'}, \qquad \text{so} \qquad (m-C)e \leq mD_0 + C' \quad (m \geq C).$$

If $e \geq D_0 + 1$, then $(m - C)(D_0 + 1) \leq mD_0 + C'$, i.e. $m \leq C(D_0 + 1) + C'$, which fails for large m. Hence $e \leq D_0$. □

Lemma E.130. *Let x be a framed normalised curve point over a rank-n Hecke datum, and let a Selmer basis $\kappa = (\kappa_1, \dots, \kappa_g)$ of x with local witnesses and a level $\mathbf{V}$ be given. Let $P_1 \subseteq \Gamma_1^0$ be a closed normal subgroup of Γ_1 such that Γ_1/P_1 is topologically finitely generated, $\rho(h(P_1)) = 1$, $\kappa_j(h(P_1)) = 0$ for all j, and every element of $\mathrm{Sel}^{\mathrm{ev}}_{\mathbf{V}}(R)$ and $\mathrm{Sel}^2_{\mathbf{V}}(R)$, $R = A$ or A_m, vanishes on P_1 (for instance $P_1 = \{1\}$ when Γ_1 is topologically finitely generated). Assume that there is $C \geq 0$ such that for every $m \geq 1$ and every family $(z_{i,j})_{0 \leq i \leq g,\, 1 \leq j \leq g}$ in $\mathrm{Sel}^{\mathrm{ev}}_{\mathbf{V}}(A_m)$ there are $c \in A^{g+1}$ with some $c_i \in A^\times$ and $\Psi \in \mathrm{Sel}^2_{\mathbf{V}}(A_m)$ such that the ϵ'_j-coefficient of $\Psi(\gamma)$ is $T^C \sum_i c_i z_{i,j}(\gamma)$ for all j, γ. Then (K2) holds at $\mathbf{V}$.*

Proof. Reduction modulo T^m (entrywise, with reduced local witnesses) maps $\mathrm{Sel}^{\mathrm{ev}}_{\mathbf{V}}(A_{m'})$ to $\mathrm{Sel}^{\mathrm{ev}}_{\mathbf{V}}(A_m)$ and $\mathrm{Sel}^2_{\mathbf{V}}(A_{m'})$ to $\mathrm{Sel}^2_{\mathbf{V}}(A_m)$ for $m' \geq m$ (with $A_{m'} = A$ allowed); write $\bar{\Psi}^{(m)}$ for reductions.

Step 1: finiteness. The subgroup $\Gamma^0 = \pi(G_L)$ has index at most 2 in Γ, so $\Gamma_1^0 = h^{-1}(\Gamma^0)$ has index at most 2 in Γ_1, and $H = \Gamma_1^0/P_1$ is a closed, hence open, subgroup of finite index of the topologically finitely generated profinite group Γ_1/P_1. If D is a finitely generated dense subgroup of Γ_1/P_1, then $D \cap H$ has finite index in D, so is finitely generated (Schreier's lemma), and is dense in H as H is open. Let $t_1, \dots, t_p \in \Gamma_1^0$ map to generators of $D \cap H$. Since $\rho(h(P_1)) = 1$ and $\kappa_j(h(P_1)) = 0$, the homomorphism $\rho_1 \circ h$, $\rho_1(\gamma) = (1 + \sum_j \epsilon_j \kappa_j(\gamma))\rho(\gamma)$, is trivial on P_1. Let $z \in \mathrm{Sel}^2_{\mathbf{V}}(A_m)$; it vanishes on P_1, so the cocycle rule gives $z(\gamma p) = z(\gamma) = z(p\gamma)$ for $p \in P_1$, and z factors through H. By the cocycle rule its values on the subgroup generated by the images of the t_k are determined by $z(t_1), \dots, z(t_p)$, and since the entries of z are continuous (locally constant, T being nilpotent in A_m), z is determined on the closure H. The values $z(t_k)$ lie in the finite set $\mathrm{M}_n(\mathrm{PD}_g(A_m))$ (k is finite). So $\mathrm{Sel}^2_{\mathbf{V}}(A_m)$ is finite.

Step 2: closedness. Let $\Psi\colon \Gamma_1^0 \to \mathrm{PW}^2_a(A)$ be a map with $\bar\Psi^{(m)} \in \mathrm{Sel}^2_{\mathbf{V}}(A_m)$ for every m. Each entry coordinate of $\bar\Psi^{(m)}$ is locally constant (T is nilpotent in A_m), so the entries of Ψ are continuous into $A = \varprojlim A_m$. The cocycle rule, the polarisation rule and the vanishing on the $I_{1,u}$ hold modulo every T^m, hence hold, as $\bigcap_m T^m A = 0$. For $v \in S_\ell$ let $W_m \subseteq \mathrm{PW}^2_a(A_m)$ be the set of local witnesses of $\bar\Psi^{(m)}$ at v: it is non-empty and finite, and reduction maps W_{m+1} into W_m. An inverse system of non-empty finite sets has non-empty inverse limit; an element $(Y_m)_m$ defines $Y_v \in \mathrm{PW}^2_a(A)$ whose reductions are witnesses, and the conditions on Y_v (vanishing of certain entries of a matrix depending polynomially on Y_v) hold modulo every T^m, hence over A. So $\Psi \in \mathrm{Sel}^2_{\mathbf{V}}(A)$.

Step 3: the limit. Let $(z_{i,j})$ be a family in $\mathrm{Sel}^{\mathrm{ev}}_{\mathbf{V}}(A)$. For every $m \geq 1$ apply the hypothesis to the reductions $\bar z^{(m)}_{i,j}$: there are $c^{(m)} \in A^{g+1}$ with $c^{(m)}_{i(m)} \in A^\times$ and $\Psi_m \in \mathrm{Sel}^2_{\mathbf{V}}(A_m)$ with ϵ'_j-coefficient $T^C \sum_i c^{(m)}_i \bar z^{(m)}_{i,j}$. Choose i_0 with $i(m) = i_0$ for m in an infinite set M_0. As A^{g+1} is compact and metrisable (k is finite), there are an infinite $M_1 \subseteq M_0$ and $c \in A^{g+1}$ with $c^{(m)} \to c$ along M_1; since $A^\times = A \smallsetminus TA$ is closed, $c_{i_0} \in A^\times$, so $c \neq 0$. By Step 1 and induction on $j \geq 1$ there are infinite sets $M_1 \supseteq M_2 \supseteq \cdots$ and $\psi_j \in \mathrm{Sel}^2_{\mathbf{V}}(A_j)$ with $\bar\Psi^{(j)}_m = \psi_j$ for all $m \in M_j$, $m \geq j$ (finitely many values: some value is taken infinitely often). Then $\bar\psi^{(j)}_{j+1} = \psi_j$, so the ψ_j define a map $\Psi\colon \Gamma_1^0 \to \mathrm{PW}^2_a(A)$ with $\bar\Psi^{(j)} = \psi_j$, and $\Psi \in \mathrm{Sel}^2_{\mathbf{V}}(A)$ by Step 2. For $j \geq 1$ choose $m \in M_j$, $m \geq j$, with $c^{(m)} \equiv c$ modulo T^j (possible as $M_j \subseteq M_1$ is infinite). Modulo T^j, the $\epsilon'_{j'}$-coefficient of $\Psi(\gamma)$ is that of $\Psi_m(\gamma)$, i.e. $T^C \sum_i c^{(m)}_i z_{i,j'}(\gamma) \equiv T^C \sum_i c_i z_{i,j'}(\gamma)$. As this holds for every j, the $\epsilon'_{j'}$-coefficient of $\Psi(\gamma)$ equals $\sum_i c'_i z_{i,j'}(\gamma)$ with $c' = T^C c$, and $c' \neq 0$ since A is a domain. This is (K2). □

Lemma E.131. *Let x be a framed normalised curve point of shape $(a, n-a)$ over a rank-n Hecke datum, κ a Selmer basis of x with local witnesses, and $\mathbf{V}$ a level. Let $\sigma_1, \dots, \sigma_q \in \Gamma^0$ be such that the characteristic polynomial of $\rho(\sigma_j)$ is $\prod_{i=1}^n (X - \alpha_{j,i})$ with $\alpha_{j,1}, \dots, \alpha_{j,n} \in A$ pairwise distinct, and let $\phi_j, \tau_j \in \Gamma_1^0$ with $h(\phi_j) = \sigma_j$, $h(\tau_j) = 1$ and $\phi_j\tau_j = \tau_j\phi_j$ $(1 \leq j \leq q)$. Assume that $\ker h$ is the closure of the subgroup of Γ_1 generated by the conjugates $g\tau_j g^{-1}$, $g \in \Gamma_1$, $1 \leq j \leq q$. Then (K3) holds at $\mathbf{V}$: for every $z \in \mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}}(A)$ there are $s \geq 0$ and $c \in A^g$ with $T^s z - \sum_j c_j(\kappa_j \circ h) \in \mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}}(A)$.*

Proof. Write $\varrho = \rho \circ h$ on Γ_1^0; $\rho(\gamma)$ is block diagonal for $\gamma \in \Gamma^0$ (shape $(a, n-a)$). If $a = n$ then $\mathrm{Od}_a = 0$, $z = 0$ and there is nothing to prove; let $a < n$. Fix $z \in \mathrm{Sel}^{\mathrm{od}}_{\mathbf{V}}(A)$ (the frame w as local matrices).

Step 1: $z(\tau_j) = 0$. By the cocycle rule and $\varrho(\tau_j) = 1$, $z(\phi_j\tau_j) = z(\phi_j) + \rho(\sigma_j) z(\tau_j) \rho(\sigma_j)^{-1}$ and $z(\tau_j\phi_j) = z(\tau_j) + z(\phi_j)$; as $\phi_j\tau_j = \tau_j\phi_j$, $Z = z(\tau_j) \in \mathrm{Od}_a(A)$ commutes with $M = \rho(\sigma_j) =$

$\mathrm{diag}(M_1, M_2)$, M_1 of size a. Writing $Z = \begin{pmatrix} 0 & Z_{12} \\ Z_{21} & 0 \end{pmatrix}$, this says $M_1 Z_{12} = Z_{12} M_2$ and $M_2 Z_{21} = Z_{21} M_1$, hence $P(M_1) Z_{12} = Z_{12} P(M_2)$ and $P(M_2) Z_{21} = Z_{21} P(M_1)$ for every polynomial P. The characteristic polynomials P_1, P_2 of M_1, M_2 have product $\prod_i (X - \alpha_{j,i})$ with distinct roots, so they have no common root and are coprime in $F_k[X]$, $F_k = \mathrm{Frac}(A)$: $uP_1 + vP_2 = 1$ with $u, v \in F_k[X]$. By Cayley–Hamilton, $0 = P_1(M_1) Z_{12} = Z_{12} P_1(M_2)$, and $u(M_2) P_1(M_2) = 1 - v(M_2) P_2(M_2) = 1$, so $P_1(M_2)$ is invertible over F_k and $Z_{12} = 0$; symmetrically $Z_{21} = 0$ (use P_2). So $z(\tau_j) = 0$.

Step 2: z vanishes on $\ker h$. We have $\ker h \subseteq \Gamma_1^0$. For $\gamma, \gamma' \in \ker h$, $z(\gamma\gamma') = z(\gamma) + z(\gamma')$ and $z(\gamma^{-1}) = -z(\gamma)$ (cocycle rule with $\varrho(\gamma) = 1$), so $K_z = \{\gamma \in \ker h : z(\gamma) = 0\}$ is a subgroup; it is closed, since the entries of z are continuous for the T-adic topology of A, which is Hausdorff. For $g \in \Gamma_1^0$, $z(g \tau_j g^{-1}) = z(g) + \varrho(g)\big(z(\tau_j) + z(g^{-1})\big)\varrho(g)^{-1} = z(g) + \varrho(g) z(g^{-1}) \varrho(g)^{-1} = z(gg^{-1}) = 0$ by Step 1. For $g \notin \Gamma_1^0$ write $g = c_1 g'$ with $g' \in \Gamma_1^0$ (as $h(c_1) = \bar{c} \notin \Gamma^0$, Γ_1^0 has index 2 and $c_1 \notin \Gamma_1^0$); then $g \tau_j g^{-1} = c_1 (g' \tau_j g'^{-1}) c_1$ and $z(c_1 \gamma c_1) = -{}^t z(\gamma)$ gives 0. So K_z contains all conjugates of the τ_j, hence their closed span $\ker h$.

Step 3: descent to the base level. For $\gamma \in \Gamma_1^0$ and $\kappa' \in \ker h$, $z(\gamma\kappa') = z(\gamma) + \varrho(\gamma) z(\kappa') \varrho(\gamma)^{-1} = z(\gamma)$; as h maps Γ_1^0 onto Γ^0, there is a unique map $y \colon \Gamma^0 \to \mathrm{Od}_a(A)$ with $z = y \circ h$. Since $h \colon \Gamma_1^0 \to \Gamma^0$ is a continuous surjection of compact Hausdorff spaces, it is a quotient map, so the entries of y are T-adically continuous. The cocycle rule for y and ρ follows from that of z; $y(\bar{c}\gamma\bar{c}) = z(c_1 \gamma' c_1) = -{}^t z(\gamma') = -{}^t y(\gamma)$ for $h(\gamma') = \gamma$; y vanishes on $I_u^x = h(I_{1,u})$; and for $v \in S_\ell$ and $g \in D_v$, $z(\phi_{1,v} g) = y(\pi g)$ and $\varrho(\phi_{1,v} g) = \rho(\pi g)$ because $h \circ \phi_{1,v} = \pi|_{D_v}$, so the local witness Y_v of z is one for y. Hence $y \in Z^1_{\mathrm{ord}}(x)$.

Step 4. As κ is a Selmer basis, there are $s \geq 0$, $c \in A^g$ and $Y \in \mathrm{Od}_a(A)$ with ${}^t Y = -Y$ such that $T^s y(\gamma) - \sum_j c_j \kappa_j(\gamma) = \rho(\gamma) Y \rho(\gamma)^{-1} - Y$ for $\gamma \in \Gamma^0$. Composing with h, $T^s z - \sum_j c_j (\kappa_j \circ h)$ is $\gamma \mapsto \varrho(\gamma) Y \varrho(\gamma)^{-1} - Y$, an element of $\mathrm{Bd}^{\mathrm{od}}_{\mathbf{V}}(A)$. □

Lemma E.132. *Let ℓ be a prime, G_1, G_2, $\bar{G}$ profinite groups and $\theta \colon G_1 \to \bar{G}$, $q \colon G_2 \to \bar{G}$ surjective homomorphisms, and let $\Gamma = G_1 \times_{\bar{G}} G_2 = \{(a, b) : \theta(a) = q(b)\}$. Assume that G_1 and G_2 satisfy Φ_ℓ and that there is an open subgroup $\bar{G}_0$ of $\bar{G}$ such that $H^2(V, \mathbb{Z}/\ell)$ is finite for every open subgroup V of $\bar{G}$ contained in $\bar{G}_0$. Then Γ satisfies Φ_ℓ. The hypothesis Φ_ℓ on G_1 holds in particular if G_1 is topologically finitely generated.*

Proof. We use the following classical facts. A surjective homomorphism of profinite groups is open. For a closed normal subgroup P of a profinite group E the five-term inflation–restriction sequence

$$0 \to H^1(E/P, \mathbb{Z}/\ell) \to H^1(E, \mathbb{Z}/\ell) \to H^1(P, \mathbb{Z}/\ell)^E \to H^2(E/P, \mathbb{Z}/\ell)$$

is exact (e.g. Neukirch–Schmidt–Wingberg, Cohomology of Number Fields, Prop. 1.6.7). Moreover $H^1(E, \mathbb{Z}/\ell) = \mathrm{Hom}(E, \mathbb{Z}/\ell)$, and an open subgroup of a topologically finitely generated profinite group is topologically finitely generated.

Let $U \subseteq \Gamma$ be open. There are open normal subgroups $W_1 \subseteq G_1$, $W_2 \subseteq G_2$ with $U_0 = \Gamma \cap (W_1 \times W_2) \subseteq U$, and shrinking W_2 we may assume $W_2 \subseteq q^{-1}(\bar{G}_0)$ (an open subgroup of G_2 contains an open normal one); U_0 is open and normal in Γ. A homomorphism $U \to \mathbb{Z}/\ell$ vanishing on U_0 factors through the finite group U/U_0, so the restriction map $\mathrm{Hom}(U, \mathbb{Z}/\ell) \to \mathrm{Hom}(U_0, \mathbb{Z}/\ell)$ has finite fibres, and it suffices to show that $\mathrm{Hom}(U_0, \mathbb{Z}/\ell)$ is finite.

The first projection maps U_0 onto $V_1 = W_1 \cap \theta^{-1}(q(W_2))$, which is open in G_1 (q is open), with kernel $\{1\} \times P'$, $P' = W_2 \cap \ker q$. The second projection maps U_0 onto $L = W_2 \cap q^{-1}(\theta(W_1))$, open in G_2; $P' = L \cap \ker q$ is normal in L and $L/P' \cong q(L)$, an open subgroup of $\bar{G}$. Conjugation by $(a, b) \in U_0$ acts on $\{1\} \times P'$ as conjugation by b on P', so $H^1(P', \mathbb{Z}/\ell)^{U_0} = H^1(P', \mathbb{Z}/\ell)^L$.

The five-term sequence for $\{1\} \times P' \subseteq U_0$ gives an exact sequence

$$0 \to \mathrm{Hom}(V_1, \mathbb{Z}/\ell) \to \mathrm{Hom}(U_0, \mathbb{Z}/\ell) \to H^1(P', \mathbb{Z}/\ell)^L,$$

and the one for $P' \subseteq L$ gives an exact sequence $\mathrm{Hom}(L, \mathbb{Z}/\ell) \to H^1(P', \mathbb{Z}/\ell)^L \to H^2(q(L), \mathbb{Z}/\ell)$. Here $\mathrm{Hom}(V_1, \mathbb{Z}/\ell)$ and $\mathrm{Hom}(L, \mathbb{Z}/\ell)$ are finite by Φ_ℓ for G_1 and G_2, and $H^2(q(L), \mathbb{Z}/\ell)$ is finite by hypothesis, as $q(L) \subseteq q(W_2) \subseteq \bar{G}_0$ is open in $\bar{G}$; hence $H^1(P', \mathbb{Z}/\ell)^L$ is finite, and so is $\mathrm{Hom}(U_0, \mathbb{Z}/\ell)$.

If G_1 is topologically finitely generated, so is each of its open subgroups V, and $\mathrm{Hom}(V, \mathbb{Z}/\ell)$ is finite (a homomorphism is determined by its values on finitely many generators); so G_1 satisfies Φ_ℓ. □

E.5 The patched ring and automorphic patching data

This section proves the remaining inputs of Proposition E.39. Proposition E.134 says that the patched ring modulo its relation ideal carries a polarised ordinary determinant on the limit level group with the Taylor–Wiles data; with it, at thick primes under the patched ring the relation quotient of a realisation is not semisimple (Lemma E.133). Proposition E.136 says that classical levels give automorphic patching data along every Taylor–Wiles datum. It is proved through operator modules with the local relations at every Taylor–Wiles level (Definition E.135, Proposition E.137, Proposition E.142): at a Taylor–Wiles place the U-eigencharacters on Iwahori invariants determine the Frobenius eigenvalues and the tame action (Lemma E.140), up to a Vandermonde-type element (Lemma E.139, Lemma E.141).

Lemma E.133. *In the situation of Proposition E.134, let $\mathfrak{R}$ be a realisation of x (of shape $(a, n-a)$) over $\mathbf{V}_\infty$ whose global ring, determinant, characters, Taylor–Wiles elements and s are those of Proposition E.134, and let $\hat{Q}'$ be a prime ideal of $\hat{B}$ whose preimage in R is $\pi^{-1}(P')$ for a prime ideal P' of C^{sp}. Then (Gl2) of Proposition E.78 holds for $\hat{Q}'$:*

(Tk1) *if $a < n$ and $u^{-1}(P')$ is stable, then* $\mathrm{ired}(N) \not\subseteq \hat{Q}'$*;*

(Tk2) *if $u^{-1}(P')$ is thick, then for an algebraic closure κ of the fraction field of $\hat{B}/\hat{Q}'$ and the natural homomorphism $\hat{B} \to \kappa$ (with kernel $\hat{Q}'$), $\kappa[\Gamma_\infty^0]$ modulo the two-sided ideal generated by the image of $\mathcal{I}_1$ is not semisimple.*

Proof. (Tk1). The preimage $\pi^{-1}(P')$ of $\hat{Q}'$ in R is contained in $\pi^{-1}(P')$, and $u^{-1}(P')$ is stable; so Lemma E.46 (with $P'' = P'$) gives $\mathrm{ired}(N) \not\subseteq \hat{Q}'$.

(Tk2). Put $\mathfrak{Q} = u^{-1}(P')$, $\bar{C} = C^{\mathrm{sp}}/P'$ and let $\mathfrak{m}_{\bar{C}}$ be its maximal ideal; $\bar{C}$ is a Noetherian local domain (C^{sp} is a complete Noetherian local ring). Let $\varrho\colon R/I \to \bar{C}$ be $\bar{\pi}$ followed by the projection; it is onto with kernel $\pi^{-1}(P')/I$, which is the preimage of $\hat{Q}'$ in R/I. So ϱ induces an embedding $\bar{C} \to \hat{B}/\hat{Q}' \subseteq \kappa$ through which $R/I \to \hat{B} \to \kappa$ factors, and u induces $B/\mathfrak{Q} \to \bar{C}$; let $K \subseteq \kappa$ be the algebraic closure of the fraction field of $B/\mathfrak{Q}$ in κ, an algebraic closure of it.

Step 1: the residual algebra. $\mathcal{I}$ is the ideal $\bar{J}$ of Lemma E.69 formed from $(B, \Gamma^0_{\mathscr{H}}, D^{\mathrm{u}}, \mathcal{S})$, $\mathcal{S}$ the ordinary products and their images under *. B is a complete Noetherian local ring with finite residue field of characteristic $\ell \neq 2$, the base change of D^{u} to the residue field is $\det \circ \mathrm{diag}(\bar{\chi}_a)$ with pairwise distinct $\bar{\chi}_a$, and $\Gamma^0_{\mathscr{H}}$ satisfies Φ_ℓ, being open in $\Gamma_{\mathscr{H}}$, which does by Lemma E.116. By (F0)–(F2) of Lemma E.69, $E_B = B[\Gamma^0_{\mathscr{H}}]/\mathcal{I}$ is a finitely generated B-module, and for every $j \geq 1$ there is an open normal $V_j \subseteq \Gamma^0_{\mathscr{H}}$ with $\iota_B(\gamma v) - \iota_B(\gamma) \in \mathfrak{m}_B^j E_B$ (ι_B the projection). By the definition of thickness, $E_K = K[\Gamma^0_{\mathscr{H}}]/(\text{ideal generated by the image of } \mathcal{I}) = E_B \otimes_B K$ is not semisimple; it is finite-dimensional over K.

Step 2: a map on the realisation's group algebra. Let $M = E_B \otimes_B \bar{C}$ (through $B \to \bar{C}$), a $\bar{C}$-algebra, finitely generated as a $\bar{C}$-module, and $\Psi\colon (R/I)[\Gamma^0_\infty] \to M$ the ring homomorphism $\sum b_g g \mapsto \sum \varrho(b_g)\,\iota_B(h(g)) \otimes 1$. We show $\Psi(\mathcal{I}_1) = 0$. Recall that $\mathcal{I}_1$ is the intersection over $j \geq 1$ and open normal $H' \subseteq \Gamma^0_\infty$ of the two-sided ideals $J^1_{j,H'}$ generated by $\mathrm{CH}(D)$, $\mathfrak{m}^j_{R/I} \cdot 1$, $h' - 1$ ($h' \in H'$) and the elements listed in (Vr3) with their images under *. Fix j and take $H' = h^{-1}(V_j) \cap \Gamma^0_\infty$. Images of the generators:

- $h' - 1$, $h' \in H'$: $\iota_B(h(h')) - 1 \in \mathfrak{m}^j_B E_B$, so $\Psi(h' - 1) \in \mathfrak{m}^j_{\bar{C}} M$; and $\Psi(\mathfrak{m}^j_{R/I}) \subseteq \mathfrak{m}^j_{\bar{C}} M$, as ϱ is a local homomorphism.
- Ordinary products: $h \circ \phi_{\infty,v} = \pi_{\mathscr{H}}|_{D_v}$ and $\varrho(\hat{\psi}_{v,a}(g))$ is the image of $\psi^{\mathrm{d}}_{v,a}(g)$ (definition of π_0 in Definition E.37), so Ψ maps $\prod_a(\phi_{\infty,v}(g_a) - \hat{\psi}_{v,a}(g_a))$ to the image of $\prod_a(\pi_{\mathscr{H}}(g_a) - \psi^{\mathrm{d}}_{v,a}(g_a)) \in \mathcal{S} \subseteq \mathcal{I}$, i.e. to 0. As $h(c'_\infty) = \pi_{\mathscr{H}}(c)$, $h(\gamma^*) = h(\gamma)^*$, so Ψ commutes with * and the images under * also go to 0 ($\mathcal{S}^* \subseteq \mathcal{S}$).
- Frame inertia: let $\gamma' = (F_u(y), \pi_{\mathscr{H}}(y)) \in I_{\infty,u}$ with $y \in I_{\bar{u}} \cap G_L$ and $\pi_{\mathscr{H}}(y) \in I^x_u$. As ρ is trivial on I^x_u (frame), $\det(X - \rho(\pi_{\mathscr{H}} y)) = (X-1)^n$; its reduction modulo T is $\prod_a(X - \bar{\chi}_a(y))$, since $f_x \circ D^{\mathrm{u}} = \det \circ \rho$ and D^{u} reduces to $\det \circ \mathrm{diag}(\bar{\chi}_a)$. So all $\bar{\chi}_a(y) = 1$, y is one of the generators of $N_{\mathscr{H}}$ ($u \nmid \ell$), and $h(\gamma') = \pi_{\mathscr{H}}(y) = 1$: $\Psi(\gamma' - 1) = 0$; likewise for the images under *.
- Taylor–Wiles elements: $h(\phi_j) = \sigma_j$ and $\varrho(U_{j,i})$ is the image of $u_{j,i}$, so $\Psi(\prod_i(\phi_j - U_{j,i}))$ is the image of $\prod_i(\sigma_j - u_{j,i}) = \chi_{D^{\mathrm{u}}}(\sigma_j)(\sigma_j)$ (in $\bar{C}[\Gamma^0_{\mathscr{H}}]$, by the definition of the eigenvalue algebra), which is the image of an element of $\mathrm{CH}(D^{\mathrm{u}}) \subseteq \mathcal{I}$ ($m = 1$, $y_1 = \sigma_j$); so it is 0. $h(\tau_j) = 1$ and $\varrho(y_{j,i}) = 0$, so the factor $\tau_j - 1 - y_{j,i}$, and with it the second Taylor–Wiles element, maps to 0. The images under * follow as above.
- $\mathrm{CH}(D)$: let $y_1, \ldots, y_m \in (R/I)[\Gamma^0_\infty]$ and write $\Psi_0(y_i) = \sum_g a_{ig} g \in \bar{C}[\Gamma^0_{\mathscr{H}}]$ for the image under ϱ on coefficients and h on the group. By (In1) of Proposition E.134 and (Ch1) of Theorem B.83, the base change of D along ϱ is the base change of D^{u} along $B \to \bar{C}$ composed with h; so Ψ maps the coefficients of $\chi_{D'}(\sum t_i y_i)(\sum t_i y_i)$ to the images in M of the coefficients of $\chi(\sum_g s_g g)(\sum_g s_g g)$, χ the characteristic polynomial of D^{u} over $B[s_g]$, specialised at $s_g = \sum_i t_i a_{ig}$; those coefficients lie in $\mathrm{CH}(D^{\mathrm{u}}) \subseteq \mathcal{I}$, so the images are $\bar{C}$-linear combinations of images of elements of $\mathcal{I}$, i.e. 0.

Since $\mathfrak{m}^j_{\bar{C}} M$ is a two-sided ideal of M ($\bar{C}$ is central), $\Psi(J^1_{j,H'}) \subseteq \mathfrak{m}^j_{\bar{C}} M$, hence $\Psi(\mathcal{I}_1) \subseteq \bigcap_j \mathfrak{m}^j_{\bar{C}} M = 0$ by Krull's intersection theorem for the finitely generated module M over the Noetherian local ring $\bar{C}$.

Step 3: non-semisimplicity. Tensoring Ψ with κ over $\bar{C}$ gives a ring homomorphism $\kappa[\Gamma^0_\infty] \to M \otimes_{\bar{C}} \kappa = E_B \otimes_B \kappa = E_K \otimes_K \kappa$ (coefficients through $R/I \to \hat{B} \to \kappa$, which factors through ϱ), which kills the image of $\mathcal{I}_1$ by Step 2 and is onto, since h maps Γ^0_∞ onto $\Gamma^0_{\mathscr{H}}$ and E_B is spanned by the image of $\Gamma^0_{\mathscr{H}}$. The finite-dimensional K-algebra E_K is not semisimple, so its Jacobson radical $\mathfrak{r}$ is a non-zero nilpotent two-sided ideal; $\mathfrak{r} \otimes_K \kappa$ is a non-zero nilpotent two-sided ideal of $E_K \otimes_K \kappa$, which is therefore not semisimple. A quotient of a semisimple ring is semisimple; so $\kappa[\Gamma^0_\infty]$ modulo the two-sided ideal generated by the image of $\mathcal{I}_1$, which maps onto $E_K \otimes_K \kappa$, is not semisimple. □

Proposition E.134. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$ with $\ell \neq 2$, and let automorphic patching data for an ideal $J \subseteq B$ along a Taylor–Wiles datum $\mathcal{D}$ of size q (with H and $\sigma_1, \ldots, \sigma_q$), $\mathcal{O}_1$, limit objects $(\Pi, (\pi_N), \mathcal{U}, \bar{\theta}, c_\infty, (\Phi_j), (\mathrm{T}_j), (F_w))$ and $\tilde{c}$ be given as in Definition E.37, with patched tower data $(R, \beta, X, f, I, e+1, (M^\natural_N), \mathcal{U})$, C^{sp}, u and π. Assume*

that B is topologically generated as an $\mathcal{O}$-algebra by the coefficients of the polynomials $\chi_{D^{\mathrm{u}}}(\gamma)$, $\gamma \in \Gamma^0_{\mathscr{H}}$, and the values $\psi^{\mathrm{d}}_{v,a}(g)$, $v \in S_\ell$, $g \in D_v$. Let $x = (k, \rho, 1_n, (\psi_{v,a}), f_x)$ be a framed normalised curve point of B with $|k| = |\mathcal{O}_1/\mathfrak{m}_{\mathcal{O}_1}|$ which has eigenvalue data along $\mathcal{D}$, κ a Selmer basis of x with local witnesses such that x, κ and $\mathcal{D}$ satisfy all hypotheses of Proposition E.97, $f' \colon C^{\mathrm{sp}} \to A_k$ a surjective ring homomorphism with $f' \circ u = f_x$ and $f'(z) = T$, and $\mathbf{V}_\infty$ the limit level formed from the given limit objects. Then $\pi(I) = 0$; let $\bar\pi \colon R/I \to C^{\mathrm{sp}}$ be the induced homomorphism, $f_1 = f' \circ \bar\pi$, and write b_I for the image of $b \in R$ in R/I. Then:

(In1) *the base change D_I of D^{t} to R/I is $\bar D \circ \pi_\infty$ for a unique continuous R/I-valued determinant $\bar D$ of dimension n on $\bar\Gamma^0_\infty = \pi_\infty(P^0)$; put $D = \bar D \circ p_1$ on Γ^0_∞, where $p_1(g,\gamma) = g$; then $\bar\pi(\Lambda_i(D)(g,\gamma)) = u(\Lambda_i(D^{\mathrm{u}})(\gamma))$ for $(g,\gamma) \in \Gamma^0_\infty$ and $1 \le i \le n$;*

(In2) *with $\hat\psi_{v,i} = (\psi^{\mathrm{t}}_{v,i})_I$, $U_{j,i} = (U^{\mathrm{t}}_{j,i})_I$, $y_{j,i} = (y^{\mathrm{t}}_{j,i})_I$ and $s = (s^{\mathrm{t}})_I$, the tuple $(R/I, D, (\hat\psi_{v,i}), (\phi_j, \tau_j, U_{j,i}, y_{j,i}), s, f_1)$ satisfies, for the coefficient ring $\mathcal{O}_1$ and the level $\mathbf{V}_\infty$, all clauses of (Vr1) and (Vr2) that concern it;*

(In3) *(Lv1) and (Lv2) hold for D over $\mathbf{V}_\infty$.*

Proof. By (Wz1) of Proposition E.56, π is onto and $\pi(I) = 0$; so $\bar\pi$ is defined and onto, and f_1 is onto. By its definition in Definition E.37 (π_0 carries the universal patched tuple $t^{\mathrm{u}} = (D^{\mathrm{t}}, (\psi^{\mathrm{t}}_{v,a}), s^{\mathrm{t}}, (U^{\mathrm{t}}_{j,i}), (y^{\mathrm{t}}_{j,i}))$ to $(\bar D^{\mathrm{u}} \circ \bar\theta \circ \pi_\infty|_{P^0}, (\psi^{\mathrm{d}}_{v,a}), z, (u_{j,i}), (0)))$, π_0 carries $\Lambda_i(D^{\mathrm{t}})(g)$ to $\Lambda_i(\bar D^{\mathrm{u}})(\bar\theta\pi_\infty(g))$ (base changed to C^{sp}_0), $\psi^{\mathrm{t}}_{v,a}$ to $\psi^{\mathrm{d}}_{v,a}$, s^{t} to z, $U^{\mathrm{t}}_{j,i}$ to $u_{j,i}$ and $y^{\mathrm{t}}_{j,i}$ to 0.

Step 0: the ring. $R_{\mathrm{tup}} \in \mathrm{CNL}_{\mathcal{O}}$ and $\mathcal{O}_1$ is free of finite rank over $\mathcal{O}$, so $R = \mathcal{O}_1 \otimes_{\mathcal{O}} R_{\mathrm{tup}}$ is a finite R_{tup}-algebra, hence complete and Noetherian, and $R/\mathfrak{m}_{R_{\mathrm{tup}}}R = \mathcal{O}_1/\mathfrak{m}_{\mathcal{O}}\mathcal{O}_1 = \mathcal{O}_1/\mathfrak{m}_{\mathcal{O}_1}$ is a field; so R, and with it R/I, is a complete Noetherian local $\mathcal{O}_1$-algebra with residue field $\mathcal{O}_1/\mathfrak{m}_{\mathcal{O}_1}$. f_1 is a homomorphism of $\mathcal{O}$-algebras: π is $\mathcal{O}_1$-linear and f' is $\mathcal{O}$-linear, as $f' \circ u = f_x$ is.

Step 1: (In1). $\bar\Gamma^0$ (the image of $\Gamma^0_{\mathscr{H}}$) has index at most 2 in $\bar\Gamma$, so P^0 is open in the topologically finitely generated group Π and is itself topologically finitely generated. D_I is continuous, and its reduction modulo the maximal ideal is $\det \circ \mathrm{diag}(\bar\chi_a \circ \bar\theta \circ \pi_\infty)$ (patched tuple; $\bar D_n = \det \circ \mathrm{diag}(\bar\chi_a)$), with the characters $\bar\chi_a \circ \bar\theta \circ \pi_\infty$ of P^0 pairwise distinct ($\bar\theta \circ \pi_\infty$ maps P^0 onto $\bar\Gamma^0$, through which the pairwise distinct $\bar\chi_a$ factor). For continuous characters χ, χ' of a topologically finitely generated group into the residue field, a continuous cocycle in $Z^1(\chi, \chi')$ is determined by its values on a finite set of topological generators, so $Z^1(\chi, \chi')$ is finite-dimensional. As $\ell \ne 2$, (Kt2) applies to D_I on P^0: an element $y \in (R/I)[P^0]$ lies in $\ker(D_I)$ as soon as $\mathrm{tr}_{D_I}(gy) = 0$ for all $g \in P^0$. For $k \in \ker \pi_\infty$ (contained in P^0), the relations (Wv1) lie in I, so $\mathrm{tr}_{D_I}(g(k-1)) = 0$ for all $g \in P^0$, and $k - 1 \in \ker(D_I)$. The two-sided ideal generated by these $k-1$ is the kernel of $(R/I)[P^0] \to (R/I)[\pi_\infty(P^0)]$, and it lies in the two-sided ideal $\ker(D_I)$; by (Ck1), D_I factors through $(R/I)[P^0]/\ker(D_I)$, hence through $(R/I)[\pi_\infty(P^0)]$: $D_I = \bar D \circ \pi_\infty$ for a determinant $\bar D$ of dimension n, unique as π_∞ is onto. Its coefficient maps $\Lambda_i(\bar D)$ are continuous, because $\Lambda_i(\bar D) \circ \pi_\infty = \Lambda_i(D_I)$ is and $\pi_\infty|_{P^0}$ is a continuous surjection of compact groups (a quotient map). Since π_∞ is onto, $\pi_\infty(P^0) = \bar\theta^{-1}(\bar\Gamma^0) = \bar\Gamma^0_\infty$. Finally $p_1 \colon \Gamma^0_\infty \to \bar\Gamma^0_\infty$ is a continuous homomorphism, onto: for $g \in \bar\Gamma^0_\infty$ choose $\gamma \in \Gamma^0_{\mathscr{H}}$ with image $\bar\theta(g)$; then $(g,\gamma) \in \Gamma^0_\infty$. So $D = \bar D \circ p_1$ is a continuous determinant of dimension n on Γ^0_∞, and for $(g,\gamma) \in \Gamma^0_\infty$ and any $\tilde g \in P^0$ with $\pi_\infty(\tilde g) = g$, $\Lambda_i(D)(g,\gamma) = \Lambda_i(D^{\mathrm{t}})(\tilde g)_I$.

For $(g,\gamma) \in \Gamma^0_\infty$, $\bar\theta(g)$ is the image $\bar\gamma$ of γ in $\bar\Gamma$ and $D^{\mathrm{u}} = \bar D^{\mathrm{u}} \circ (\text{projection})$, so by the definition of π_0, $\bar\pi(\Lambda_i(D)(g,\gamma)) = \pi(\Lambda_i(D^{\mathrm{t}})(\tilde g)) = u(\Lambda_i(\bar D^{\mathrm{u}})(\bar\gamma)) = u(\Lambda_i(D^{\mathrm{u}})(\gamma))$.

Step 2: (Vr1). R/I is as required by Step 0; D is a continuous determinant of dimension n on $(R/I)[\Gamma^0_\infty]$ by Step 1; the $\hat\psi_{v,i}$ are continuous characters $D_v \to (R/I)^\times$; f_1 is a surjective

homomorphism of $\mathcal{O}$-algebras (Step 0). $f_1(s) = f'(\pi(s^{\mathrm{t}})) = f'(z) = T$. $f_1 \circ \hat{\psi}_{v,i} = f' \circ u \circ \psi^{\mathrm{d}}_{v,i} = f_x \circ \psi^{\mathrm{d}}_{v,i} = \psi_{v,i}$, as x is a curve point. For $(g,\gamma) \in \Gamma^0_\infty$ and $\tilde{g}$ as above, $\bar{\theta}(g)$ is the image $\bar{\gamma}$ of γ, and $D^{\mathrm{u}} = \bar{D}^{\mathrm{u}} \circ$ (projection); so

$$f_1(\Lambda_i(D)(g,\gamma)) = f'(\pi(\Lambda_i(D^{\mathrm{t}})(\tilde{g}))) = f'(u(\Lambda_i(D^{\mathrm{u}})(\gamma))) = f_x(\Lambda_i(D^{\mathrm{u}})(\gamma)) = \Lambda_i(\det \circ \rho)(\gamma),$$

the last because $f_x \circ D^{\mathrm{u}} = \det \circ \rho$. As $h(g,\gamma) = \gamma$, the determinants $f_1 \circ D$ and $\det \circ (\rho \circ h)$ on Γ^0_∞ have the same characteristic polynomials at every element, hence are equal by (Ch1). The clause Φ_ℓ of (Vr1) concerns the level: it is (P4) of Proposition E.94. Finally, for $g \in \mu_v$, $\psi^{\mathrm{t}}_{v,i}(g)$ is the image in R_{tup} of the wild value $\xi_{v,i}(g) \in \mathcal{O}$ (t^{u} is a patched tuple, Proposition E.60), so $\hat{\psi}_{v,i}(g)$ lies in the image of $\mathcal{O} \to R/I$.

Step 3: (Vr2). By (P1) of Proposition E.94, $\phi_j, \tau_j \in \Gamma^0_\infty$ commute and $h(\tau_j) = 1$, $h(\phi_j) = \sigma_j$. $f_1(y_{j,i}) = f'(0) = 0$. Applying f' to the identity $\prod_i (X - u_{j,i}) = \chi_{D^{\mathrm{u}}}(\sigma_j)(X)$ of $C^{\mathrm{sp}}[X]$ (definition of the eigenvalue algebra) and using $f' \circ u = f_x$ and $f_x \circ D^{\mathrm{u}} = \det \circ \rho$ gives $\prod_i (X - f_1(U_{j,i})) = \det(X - \rho(\sigma_j)) = \det(X - \rho(h(\phi_j)))$. By the eigenvalue data, this polynomial is $\prod_i (X - \alpha_{j,i})$ with pairwise distinct $\alpha_{j,i} \in A_k$; A_k being a domain, $(f_1(U_{j,i}))_i$ is a permutation of $(\alpha_{j,i})_i$, so pairwise distinct. *Generation.* Let $R' \subseteq R_{\mathrm{tup}}$ be the closure of the $\mathcal{O}$-subalgebra generated by the $\Lambda_i(D^{\mathrm{t}})(g)$ ($g \in P^0$), the $\psi^{\mathrm{t}}_{v,a}(g)$, s^{t}, the $U^{\mathrm{t}}_{j,i}$ and the $y^{\mathrm{t}}_{j,i}$. Fix $m \geq 1$, let $\varpi_m \colon R_{\mathrm{tup}} \to R_{\mathrm{tup}}/\mathfrak{m}^m$ ($\mathfrak{m}$ the maximal ideal) and $R'_m = \varpi_m(R')$, a finite local $\mathcal{O}$-algebra with residue field $\mathcal{O}/\mathfrak{m}_{\mathcal{O}}$, hence in $\mathrm{CNL}_{\mathcal{O}}$. By (Ch1), $\varpi_m \circ D^{\mathrm{t}}$ is the base change of a determinant D'_m with values in the subring generated by its coefficients, which lies in R'_m; D'_m is continuous (R'_m is finite and the Λ_i are locally constant) and lifts $\bar{D}_n \circ \bar{\theta} \circ \pi_\infty$, and the other components of $\varpi_m(t^{\mathrm{u}})$ lie in R'_m with the required congruences. So $\varpi_m(t^{\mathrm{u}})$ is (the base change of) a patched tuple to R'_m, and Proposition E.60 gives a morphism $\varphi \colon R_{\mathrm{tup}} \to R'_m$ carrying t^{u} to it. The composite of φ with $R'_m \subseteq R_{\mathrm{tup}}/\mathfrak{m}^m$ and the projection ϖ_m both carry t^{u} to $\varpi_m(t^{\mathrm{u}})$ (for the determinants: equal characteristic polynomials, (Ch1)); by uniqueness they coincide, so $\varpi_m(R_{\mathrm{tup}}) \subseteq R'_m$, i.e. $R' + \mathfrak{m}^m = R_{\mathrm{tup}}$. As R' is closed, $R' = R_{\mathrm{tup}}$. Hence $R = \mathcal{O}_1 \otimes_{\mathcal{O}} R_{\mathrm{tup}}$, and R/I, are topologically generated as $\mathcal{O}_1$-algebras by the images of these elements. By Step 1 the images of the $\Lambda_i(D^{\mathrm{t}})(\tilde{g})$, $\tilde{g} \in P^0$, are the coefficients of the $\chi_D(\gamma)$, $\gamma \in \Gamma^0_\infty$ (every $\tilde{g}$ maps to $\pi_\infty(\tilde{g}) = p_1(\gamma)$ for some $\gamma \in \Gamma^0_\infty$, p_1 being onto); the other images are s, $U_{j,i}$, $y_{j,i}$ and the values $\hat{\psi}_{v,i}(g)$. This is the generation clause of (Vr2).

Step 4: (In3). Write $\gamma^* = c'_\infty \gamma^{-1} c'_\infty$ on Γ^0_∞ (a normal subgroup of Γ_∞, as $h^{-1}(\Gamma^0_{\mathscr{H}})$). For $\gamma \in \Gamma^0_\infty$ the pushforward of $y \in (R/I)[\Gamma^0_\infty]$ along p_1 is written $p_{1*}y$; $\mathrm{tr}_D(y) = \mathrm{tr}_{\bar{D}}(p_{1*}y)$, and for $w \in R[P^0]$ with image w_I in $(R/I)[P^0]$, $\mathrm{tr}_{\bar{D}}(\pi_{\infty *} w_I) = \mathrm{tr}_{D^{\mathrm{t}}}(w)_I$. (Lv1). Let $\gamma \in \Gamma^0_\infty$ and $g \in P^0$ with $\pi_\infty(g) = p_1(\gamma)$. As $p_1(c'_\infty) = c_\infty = \pi_\infty(\tilde{c})$ and $c'^2_\infty = 1$, $p_1(\gamma^*) = \pi_\infty(\tilde{c} g^{-1} \tilde{c}^{-1})$, and $\tilde{c} g^{-1} \tilde{c}^{-1} \in P^0$ (P^0 is normal in Π, as $\bar{\Gamma}^0$ is normal in $\bar{\Gamma}$). By Step 1 and (Wv2), $\Lambda_i(D)(\gamma^*) = \Lambda_i(D^{\mathrm{t}})(\tilde{c} g^{-1} \tilde{c}^{-1})_I = \Lambda_i(D^{\mathrm{t}})(g)_I = \Lambda_i(D)(\gamma)$ for all i; so $\chi_D(\gamma^*) = \chi_D(\gamma)$. (Lv2) for the listed elements themselves. Let $\gamma \in \Gamma^0_\infty$ and $h_0 \in P^0$ with $\pi_\infty(h_0) = p_1(\gamma)$.

- Ordinary products, $v \in S_\ell$, $g_1, \ldots, g_n \in D_v$: $p_1(\phi_{\infty,v}(g_a)) = F_v(g_a)$; choose $g'_a \in P^0$ with $\pi_\infty(g'_a) = F_v(g_a)$ ($F_v(g_a) \in \bar{\Gamma}^0_\infty$ by (L2), as $D_v \subseteq G_L$). Then $p_{1*}(\gamma \prod_a (\phi_{\infty,v}(g_a) - \hat{\psi}_{v,a}(g_a))) = \pi_{\infty *}(h_0 \prod_a (g'_a - \psi^{\mathrm{t}}_{v,a}(g_a)))_I$, whose trace is the image of a relation (Wv3), hence 0.

- Inertia, $u \in \mathrm{St}$, $\gamma' = (F_u(y), \pi_{\mathscr{H}}(y)) \in I_{\infty,u}$ with $y \in I_{\bar{u}} \cap G_L$, $\pi_{\mathscr{H}}(y) \in I^x_u$. Then $y \in D_u$, and $\bar{\chi}_a(y) = 1$ for all a: ρ is trivial on I^x_u (frame), so $\det(X - \rho(\pi_{\mathscr{H}} y)) = (X-1)^n$, whose reduction modulo T is $\prod_a (X - \bar{\chi}_a(y))$ (the reduction of $f_x \circ D^{\mathrm{u}}$ is $\bar{D}_n$). With $g' \in P^0$, $\pi_\infty(g') = F_u(y)$, the trace of $\gamma(\gamma' - 1)$ is the image of the relation (Wv4) for h_0, g, g' with $g = y$, hence 0.

- Taylor–Wiles elements: $p_1(\phi_j) = \Phi_j$ and $p_1(\tau_j) = \mathrm{T}_j$ lie in $\bar{\Gamma}^0_\infty$ by (L2); with $\phi, \tau \in P^0$ lifting them, the traces of $\gamma \prod_k (\phi_j - U_{j,k})$ and $\gamma \prod_{k \neq i} (\phi_j - U_{j,k})(\tau_j - 1 - y_{j,i})$ are the images of the relations (Wv5) and (Wv6) (q_1, q_2 of Definition E.36 are these products), hence 0.

(Lv2) for the images under *. By (Lv1), (Q2) of Lemma E.70 applies to $(R/I, \Gamma^0_\infty, D)$ with $\mathcal{S} = \varnothing$, $\eta = 1$ and $B' = R/I$, and with (Q1) gives $\mathrm{tr}_D(y^*) = \mathrm{tr}_D(y)$ and $(yy')^* = y'^* y^*$, $y^{**} = y$ for $y, y' \in (R/I)[\Gamma^0_\infty]$. Also $\mathrm{tr}_D(gy) = \mathrm{tr}_D(yg)$ for $g \in \Gamma^0_\infty$: by multiplicativity $D(g^{-1}y'g) = D(y')$ for every y' after any base change, so $\chi_D(g^{-1}(gy)g) = \chi_D(gy)$. For a listed z and $\gamma \in \Gamma^0_\infty$: $\mathrm{tr}_D(\gamma z^*) = \mathrm{tr}_D((\gamma z^*)^*) = \mathrm{tr}_D(z\gamma^*) = \mathrm{tr}_D(\gamma^* z) = 0$ by the case already proved. □

Definition E.135. Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, $(B, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, $J \subseteq B$ an ideal, $(\mathbf{w}, M)$ a Taylor–Wiles base for J with weight variables $\mathbf{w} = (v_k, a_k, \gamma_k, c_k)_k$, $\mathcal{D}$ a Taylor–Wiles datum of size q for $\mathscr{H}$, $N \geq 1$ and $e \geq 1$. Put $D = n[L^+ : \mathbb{Q}]$. For a commutative ring R write $R[U] = R[U_{j,i}]$ and $R[U, Y] = R[U_{j,i}, Y_{j,i}]$ ($1 \leq j \leq q$, $1 \leq i \leq n$), $f = f_{(q_{N,j})_j}(U) \in R[U]$, and let $\mathrm{Sp}^{\mathrm{pol}}_N = B[U]/I_N$, where I_N is the ideal generated by the coefficients of $\prod_{i=1}^n (X - U_{j,i}) - \chi_{D^{\mathrm{u}}}(h_N(\Phi_{N,j}))(X)$, $1 \leq j \leq q$.

An **operator module at level** N (for J, the base and the datum) consists of $R \in \mathrm{CNL}_{\mathcal{O}}$ with a polarised ordinary determinant $(D_R, (\psi_{R,v,a}))$ of rank n on Γ^0_N to R; R-modules M_1, M_{w}; homomorphisms of R-algebras $\mathrm{act}_1 \colon R[U, Y] \to \mathrm{End}_R(M_1)$ and $\mathrm{act}_{\mathrm{w}} \colon R[U] \to \mathrm{End}_R(M_{\mathrm{w}})$; integers $a_j \geq N$; a homomorphism of $\mathcal{O}$-algebras $\beta \colon R \to B$; and an R-linear map $\mathrm{tr} \colon M_1 \to M_{\mathrm{w}}$, such that, with Λ_D acting on R through $X_k \mapsto c_k \psi_{R,v_k,a_k}(\gamma_k) - 1$ and $\Delta_a = \prod_{j,i} \mathbb{Z}/\ell^{a_j}$:

(Op1) $\beta \circ D_R$ is the base change of $D^{\mathrm{u}} \circ h_N$ and $\beta \circ \psi_{R,v,a} = \psi^{\mathrm{d}}_{v,a}$;

(Op2) M_1 is finitely generated over Λ_D;

(Op3) there is an action of Δ_a on M_1 by Λ_D-linear maps in which the generator of the factor (j, i) acts as $1 + \mathrm{act}_1(Y_{j,i})$, and M_1 is a projective $\Lambda_D[\Delta_a]$-module;

(Op4) tr is surjective with kernel $\sum_{j,i} \mathrm{act}_1(Y_{j,i}) M_1$, and $\mathrm{tr} \circ \mathrm{act}_1(U_{j,i}) = \mathrm{act}_{\mathrm{w}}(U_{j,i}) \circ \mathrm{tr}$.

It **satisfies the relations with exponent** e if moreover:

(Op5) there are additive maps $\varphi \colon \mathrm{Sp}^{\mathrm{pol}}_N \otimes_B M \to M_{\mathrm{w}}$ and $\psi \colon M_{\mathrm{w}} \to \mathrm{Sp}^{\mathrm{pol}}_N \otimes_B M$ with $\varphi(\beta(t)z) = \mathrm{act}_{\mathrm{w}}(t)\varphi(z)$ and $\psi(\mathrm{act}_{\mathrm{w}}(t)x) = \beta(t)\psi(x)$ for $t \in R[U]$ ($\beta(t)$ the image in $\mathrm{Sp}^{\mathrm{pol}}_N$ of t with β applied to its coefficients), $\varphi\psi = \mathrm{act}_{\mathrm{w}}(f)$ and $\psi\varphi = \beta(f)$;

(Op6) for all j and $g \in \Gamma^0_N$: $\mathrm{act}_1\big((f \cdot \mathrm{tr}_{D_R}(g\, q_1(\Phi_{N,j}, U_j)))^e\big) = 0$, the trace of D_R base-changed to $R[U, Y]$;

(Op7) for all j, i and $g \in \Gamma^0_N$: $\mathrm{act}_1\big((f \cdot \mathrm{tr}_{D_R}(g\, q_2(\Phi_{N,j}, \mathrm{T}_{N,j}, U_j, i, Y_{j,i})))^e\big) = 0$.

Here $U_j = (U_{j,1}, \ldots, U_{j,n})$.

Proposition E.136. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, $(B^{\mathrm{po}}_n, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, and $J \subseteq B^{\mathrm{po}}_n$ an ideal with classical levels. Then for every Taylor–Wiles datum $\mathcal{D}$ for $\mathscr{H}$ there are automorphic patching data for J along $\mathcal{D}$, with exponent $e = n!$.*

Proof. Let $\mathcal{D}$ be a Taylor–Wiles datum of size q for $\mathscr{H}$, with $\sigma_1, \ldots, \sigma_q$. As J has classical levels, there is a Taylor–Wiles base $(\mathbf{w}, M)$ for J, with weight map λ_B, as in the definition of

classical levels (Definition 53.9). The values $\bar{\chi}_i(\sigma_j)$ lie in the residue field $\mathbb{F}_{\mathcal{O}}$, and $\mathcal{O} \to \mathbb{F}_{\mathcal{O}}$ is surjective, so there are $\tilde{u}_{j,i} \in \mathcal{O}$ with $\tilde{u}_{j,i} \equiv \bar{\chi}_i(\sigma_j)$ modulo $\mathfrak{m}_{\mathcal{O}}$. Put $e = n!$. For every $N \geq 1$, Proposition E.137 gives an operator module at level N (for J, $(\mathbf{w}, M)$ and $\mathcal{D}$) satisfying the relations with exponent $n!$, and Proposition E.142 turns it into R_N, $(D_N, (\psi_{N,v,a}))$, s_N, $(U_{N,j,i})$, M_N, $(a_{N,j})$, rk_N, θ_N, φ_N, ψ_N satisfying (Ha1)–(Ha5) at N for these $\tilde{u}$ and e. Together with $(\mathbf{w}, M)$, $\tilde{u}$ and e these are automorphic patching data for J along $\mathcal{D}$ with exponent $n!$ (Definition E.36). □

Proposition E.137. *Let $\mathscr{H}$ be a rank-n Hecke datum over a coefficient ring $\mathcal{O}$, $(B, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$ a universal polarised ordinary determinant of rank n over $\mathscr{H}$, $J \subseteq B$ an ideal with classical levels, and $(\mathbf{w}, M)$ a Taylor–Wiles base for J as in the definition of classical levels. Then for every Taylor–Wiles datum $\mathcal{D}$ for $\mathscr{H}$ and every $N \geq 1$ there is an operator module at level N (for J, $(\mathbf{w}, M)$ and $\mathcal{D}$) which satisfies the relations with exponent $n!$.*

Proof. Fix the datum $\mathcal{D}$ of size q and $N \geq 1$; write $Q = Q_N$, $w_j = w_{N,j}$, $K_j = L_{w_j}$ with residue field k_j of order $q_j = q_{N,j}$, $G_j = \mathrm{GL}_n(K_j)$, $\Phi_j = \Phi_{N,j}$, $\mathrm{T}_j = \mathrm{T}_{N,j}$. By (Tg2) the places of Q satisfy the conditions of the level groups in Definition 53.9, so, since $(\mathbf{w}, M)$ is a Taylor–Wiles base for J as in the definition of classical levels, there is a classical level module $(R, D_R, (\psi_{R,v,a}), \beta, P, \rho)$ at Q (with the places $\bar{w}_{N,j}$), and an isomorphism ι for which the level points separate P (Hl4). The operator module is built from it.

Local choices. Identify G_{K_j} with the decomposition group $D_{\bar{w}_{N,j}} \cap G_L$. By (Tg2) $\phi_{N,j}$ is a lift of geometric Frobenius, so (as recalled in Lemma E.140) there is a unique uniformiser ϖ_j of K_j with $\mathrm{Art}_{K_j}(\varpi_j) = \phi_{N,j}|_{K_j^{\mathrm{ab}}}$; $\tau_{N,j}$ lies in inertia and its image generates the maximal pro-ℓ quotient of inertia, and $\delta_j \in \mathcal{O}_{K_j}^{\times}$ is the element with $\mathrm{Art}_{K_j}(\delta_j) = \tau_{N,j}|_{K_j^{\mathrm{ab}}}$; its residue generates $k_j^{\times}$ together with the subgroup $C_{0,j}$ of elements of order prime to ℓ. Each q_j is prime to ℓ ($w_j \nmid \ell$), hence a unit of $\mathcal{O}$ and of R.

The modules and operators ((Op1)–(Op4)). Take R, $(D_R, (\psi_{R,v,a}))$ and β from the level module: (Op1) is the condition $\beta \circ D_R = D^{\mathrm{u}} \circ h^Q$, $h^Q = h_N$, $\beta \circ \psi_{R,v,a} = \psi^{\mathrm{d}}_{v,a}$ of Definition 53.9. Put $M_1 = P^{\mathrm{Iw}_{1,Q}}$ and $M_{\mathrm{w}} = P^{\mathrm{Iw}_Q}$; (Op2) is part of (Hl2). For each j apply (Si1) of Lemma E.138 with $\Lambda = R$ to the $R[G_j]$-module $P^{(j)} = P^{\prod_{j' \neq j} \mathrm{Iw}_{1,j'}}$ (on which G_j acts, as G_j commutes with $G_{j'}$), for $C = C_{0,j}$ and for $C = k_j^{\times}$: this gives $\theta_{1,j}$ on $(P^{(j)})^{\mathrm{Iw}(C_{0,j})} = M_1$ and, by restriction (last clause of (Si1)), $\theta_{\mathrm{w},j}$ on M_{w}. The $\theta_{1,j}(t)$, $\theta_{1,j'}(t')$ commute: for $j = j'$ by (Si1); for $j \neq j'$ because the action of $G_{j'}$ commutes with the double coset operators at j, hence with their inverses. Define $\mathrm{act}_1(U_{j,i}) = q_j^{i-1}\theta_{1,j}(e_i(\varpi_j))$, $\mathrm{act}_1(Y_{j,i}) = \theta_{1,j}(e_i(\delta_j)) - 1$ and $\mathrm{act}_{\mathrm{w}}(U_{j,i}) = q_j^{i-1}\theta_{\mathrm{w},j}(e_i(\varpi_j))$; these are commuting R-linear maps, so act_1, $\mathrm{act}_{\mathrm{w}}$ are homomorphisms of R-algebras. Let $a_j = v_\ell(q_j - 1)$; by (Tg2) $q_j \equiv 1 \bmod \ell^N$, so $a_j \geq N$. As $k_j^{\times}$ is cyclic of order $q_j - 1$, $k_j^{\times}(\ell)$ is cyclic of order ℓ^{a_j} and is generated by the image of the residue of δ_j; hence $\Delta_Q \cong \prod_j (k_j^{\times}(\ell))^n \cong \Delta_a$, the generator of the factor (j,i) corresponding to the class of $\mathrm{diag}(e_i(\delta_j)) \in \mathrm{Iw}_Q$ at j. By (Si1) this class acts on M_1 as $\theta_{1,j}(e_i(\delta_j)) = 1 + \mathrm{act}_1(Y_{j,i})$; the action is R-linear, hence Λ_D-linear, and M_1 is a projective $\Lambda_D[\Delta_a]$-module by (Hl2): (Op3). Let $\mathrm{tr}\colon M_1 \to M_{\mathrm{w}}$, $\mathrm{tr}(m) = \sum_{\gamma \in \Delta_Q} \gamma m$, the norm, with values in $M_1^{\Delta_Q} = M_{\mathrm{w}}$. For a free $\Lambda_D[\Delta_a]$-module the norm is surjective onto the invariants with kernel the augmentation ideal times the module (for $\Lambda_D[\Delta_a]$ itself: invariants $\Lambda_D \cdot \sum_\gamma \gamma$, kernel of multiplication by $\sum_\gamma \gamma$ the augmentation ideal); both properties pass to direct summands, so to the projective module M_1. The augmentation ideal is generated by the $\gamma - 1$ for the generators γ of Δ_a, so $\ker \mathrm{tr} = \sum_{j,i} \mathrm{act}_1(Y_{j,i}) M_1$. Finally, $\theta_{1,j}(t)$ commutes with the action of the diagonal units (commutative image in (Si1)), which represent Δ_Q, and restricts to $\theta_{\mathrm{w},j}(t)$ on M_{w}; so $\mathrm{tr}(\mathrm{act}_1(U_{j,i})m) = \mathrm{act}_1(U_{j,i})\mathrm{tr}(m) = \mathrm{act}_{\mathrm{w}}(U_{j,i})\mathrm{tr}(m)$: (Op4).

Transport to level points. Let $(\theta, r, Y, \pi, \mathrm{sp}, (\sigma_j))$ be a level point for ι and $c = \iota \circ \theta\colon R \to \mathbb{C}$.

Since sp is additive, c-semilinear and G_Q-equivariant, it maps P^H into Y^H for every subgroup H and commutes with every double coset operator; applying (Si1) with $\Lambda = \mathbb{C}$ to the $\mathbb{C}[G_j]$-module Y (for $C = C_{0,j}$, $k_j^\times$), and writing $\theta(t) = [t_1][t_2]^{-1}$ with t_1, t_2 dominant, sp intertwines $\theta_{1,j}(t)$, $\theta_{\mathrm{w},j}(t)$ with the corresponding operators $\theta^Y_{1,j}(t)$, $\theta^Y_{\mathrm{w},j}(t)$. Hence for $H \in R[U,Y]$ and $m \in M_1$: $\mathrm{sp}(\mathrm{act}_1(H)m) = H^c(U^Y, Y^Y)\,\mathrm{sp}(m)$, where H^c has c applied to its coefficients and $U^Y_{j,i}$, $Y^Y_{j,i}$ are the analogous operators on Y; likewise for act_w. Let $r_j = r \circ \pi^Q|_{G_{K_j}}$; the level point condition at j is the hypothesis of Lemma E.140 and of Lemma E.139 on (σ_j, r_j), with $\Phi = \phi_{N,j}$, $\tau = \tau_{N,j}$, $\varpi = \varpi_j$, $\delta = \delta_j$. Since $\theta \circ D_R = \det \circ r$, $c(\chi_{D_R}(x))$ is the characteristic polynomial of $\imath r(x)$ and $c(\mathrm{tr}_{D_R}(x)) = \mathrm{tr}(\imath r(x))$ for x in the group algebra, also after base change to $R[U,Y]$ (the trace is linear). Finally, for $y \in Y^{H_j}$ with $H_j \subseteq G_j$ compact open, y is a finite sum $\sum_s h_s(x_s)$ with G_j-maps $h_s\colon \sigma_j \to Y$ (the images span Y), and averaging over H_j (each x_s is fixed by an open subgroup) gives $y = \sum_s h_s(x'_s)$ with $x'_s \in \sigma_j^{H_j}$; the h_s commute with the operators at j.

(Op6), (Op7) with exponent $n!$. Let $g \in \Gamma^0_N$, $f = f_{(q_j)_j}(U)$, and $\Xi = \mathrm{tr}_{D_R}(g\, q_1(\Phi_j, U_j))$ or $\mathrm{tr}_{D_R}(g\, q_2(\Phi_j, \mathrm{T}_j, U_j, i, Y_{j,i}))$. Write $f = f_j \cdot f'$ with $f_j = \prod_i \prod_{i' \neq i}(U_{j,i} - U_{j,i'})(U_{j,i} - q_j U_{j,i'})$; then $(f\Xi)^{n!} = f'^{n!}(f_j\Xi)^{n!}$ in the commutative ring $R[U,Y]$. For $m \in M_1$ and a level point, $y = \mathrm{sp}(m) \in Y^{\mathrm{Iw}(C_{0,j})}$, and the operator $((f_j\Xi)^{n!})^c$ (in the $U^Y_{j,\bullet}$, $Y^Y_{j,\bullet}$) acting on y is, on each $h_s(x'_s)$ ($x'_s \in \sigma_j^{\mathrm{Iw}(C_{0,j})}$), h_s of the operator $(f\,\Xi_1(A))^{n!}$, resp. $(f\,\Xi_{2,i}(A))^{n!}$, of Lemma E.139 for σ_j, r_j and $A = \imath r(g)$ (by the previous paragraph, c turns Ξ into $\mathrm{tr}(\imath r(g)\prod_k(\imath r_j(\phi_{N,j}) - U^Y_{j,k}))$, resp. the corresponding $\Xi_{2,i}$, as $\imath r(\Phi_j) = \imath r_j(\phi_{N,j})$, $\imath r(\mathrm{T}_j) = \imath r_j(\tau_{N,j})$); by (Ln1) it is 0. So $\mathrm{sp}(\mathrm{act}_1((f\Xi)^{n!})m) = (f'^{n!})^c\,((f_j\Xi)^{n!})^c\,\mathrm{sp}(m) = 0$ for every level point, and $\mathrm{act}_1((f\Xi)^{n!})m = 0$ by (Hl4).

(Op5). Apply Lemma E.141 with $h = q$, the K_j, $\Lambda = R$, $V = P$, B, β, the base module M, $\iota_M = e^{-1}\colon M \to P^{K_Q}$ for the bijection e of (Hl3) (so $\iota_M(\beta(t)m) = t\iota_M(m)$), $P_j = \chi_{D^\mathrm{u}}(h_N(\Phi_j))$ (monic of degree n), $g \in \mathbb{Z}[U]$ with $f_{(q_j)} = g\prod_j N_{q_j}(U_{j,\bullet})$ (each factor $U_{j,i'} - q_j U_{j,i}$, $i < i'$, of N_{q_j} is one of the factors of $f_{(q_j)}$). Its hypotheses hold: the q_j are units of R; multiplication by a non-zero integer k is injective on P, since $k\,m = 0$ gives $k\,\mathrm{sp}(m) = 0$ in the complex space Y for every level point, so $m = 0$ by (Hl4); its operators $U_{j,i}$ on P^{Iw_Q} are the $\mathrm{act}_\mathrm{w}(U_{j,i})$; $\mathrm{Sp} = \mathrm{Sp}^{\mathrm{pol}}_N$; and the symmetric-function hypothesis holds: for $v \in P^{K_Q}$ and a level point, $\mathrm{sp}(v) \in Y^{\mathrm{GL}_n(\mathcal{O}_{K_j})}$ is a sum of $h_s(x'_s)$ with x'_s spherical in σ_j, and by (Ln2) of Lemma E.139 $e_k(U^Y_{j,\bullet})$ acts on it by $(-1)^k$ times the coefficient of X^{n-k} in $\det(X - \imath r_j(\phi_{N,j})) = c(\chi_{D_R}(\Phi_j))$; by separation, $e_k(\mathrm{act}_\mathrm{w}(U_{j,\bullet}))v = (-1)^k\chi_{D_R}(\Phi_j)_{n-k}\,v$, and applying e, as $\beta(\chi_{D_R}(\Phi_j)) = \chi_{D^\mathrm{u}}(h_N(\Phi_j))$ by (Op1), $e_k(U_{j,\bullet})\iota_M(m) = \iota_M((-1)^k c_{j,n-k}m)$ with $P_j = \sum_r c_{j,r}X^r$. The maps φ, ψ of Lemma E.141 (with $u = \mathrm{act}_\mathrm{w}$, $\bar\beta(t)$ the image of t with β applied to its coefficients) are those of (Op5) for $f = f_{(q_{N,j})_j}$.

Thus $(R, \ldots, \mathrm{tr})$ is an operator module at level N satisfying the relations with exponent $n!$. □

Lemma E.138. *Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, uniformiser ϖ and residue field k of order q; let $G = \mathrm{GL}_n(K)$, $K_0 = \mathrm{GL}_n(\mathcal{O}_K)$, and for a subgroup $C \subseteq k^\times$ let $\mathrm{Iw}(C) \subseteq K_0$ be the group of matrices that are upper triangular modulo ϖ with diagonal entries modulo ϖ in C; $\mathrm{Iw} = \mathrm{Iw}(k^\times)$. Identify $T = (K^\times)^n$ with the diagonal matrices, call $t \in T$ dominant if $v(t_1) \geq \cdots \geq v(t_n)$, and let $e_i(x) \in T$ have x in place i and 1 elsewhere. Let Λ be a commutative ring and V a Λ-module with a Λ-linear action of G. Assume $q \in \Lambda^\times$. For dominant t let $[t]_C$ be the endomorphism $v \mapsto \sum_{g \in \mathrm{Iw}(C)t\mathrm{Iw}(C)/\mathrm{Iw}(C)} gv$ of $V^{\mathrm{Iw}(C)}$, and let $E\colon V^{\mathrm{Iw}} \to V^{\mathrm{Iw}}$, $Ev = \sum_{g \in K_0/\mathrm{Iw}} gv$.*

(Si1) *For every C there is a unique homomorphism θ_C from T to the group of Λ-linear automorphisms of $V^{\mathrm{Iw}(C)}$ with $\theta_C(t) = [t]_C$ for every dominant t. Its image is commutative; $\theta_C(t)$ is the action of t for $t \in (\mathcal{O}_K^\times)^n$; and for $C \subseteq C'$ the restriction of $\theta_C(t)$ to $V^{\mathrm{Iw}(C')}$*

is $\theta_{C'}(t)$.

Let $U_i = q^{i-1}\theta_{k^\times}(e_i(\varpi))$ $(1 \le i \le n)$, *let* $u\colon \Lambda[X_1,\dots,X_n] \to \mathrm{End}_\Lambda(V^{\mathrm{Iw}})$ *be the homomorphism with* $u(X_i) = U_i$, *and in* $\Lambda[X_1,\dots,X_n]$ *put*

$$\Delta = \prod_{i<j}(X_j - X_i), \qquad N_q = \prod_{i<j}(X_j - qX_i), \qquad \mathrm{dd}(x) = \Delta^{-1}\sum_{w\in S_n}\mathrm{sgn}(w)\,w(x),$$

S_n *permuting the variables* ($\sum_w \mathrm{sgn}(w)w(x)$ *is divisible by* Δ, *and* $\mathrm{dd}(x)$ *is symmetric).*

(Si2) $E \circ u(z) = u(z) \circ E$ *for every symmetric* $z \in \Lambda[X_1,\dots,X_n]$.

(Si3) *There are* $m \ge 1$ *and* $a_1,\dots,a_m, c_1,\dots,c_m \in \mathbb{Z}[X_1,\dots,X_n]$ *with* $\sum_{k=1}^m a_k\, w(c_k) = \Delta$ *for* $w = 1$ *and* $= 0$ *for* $w \in S_n \setminus \{1\}$; *and for any such families* $\sum_{k=1}^m u(a_k)\circ E \circ u(c_k) = u(N_q)$.

(Si4) *If multiplication by the integer* $[K_0 : \mathrm{Iw}]$ *is injective on* V, *then* $u(z)V^{K_0} \subseteq V^{K_0}$ *for every symmetric* z, *and* $E(u(y)v) = u(\mathrm{dd}(N_q y))v$ *for every* $y \in \Lambda[X_1,\dots,X_n]$ *and* $v \in V^{K_0}$.

Proof. **Step 1 (dominant double cosets).** Let $t \in T^+$. Gaussian elimination (all pivots are units) gives $\mathrm{Iw}(C) = N^+ T_C N_1^-$. For $n^- \in N_1^-$ the entry (i,j), $i > j$, of $t^{-1}n^-t$ is that of n^- times $t_i^{-1}t_j$, of valuation $\ge v(t_j) - v(t_i) \ge 0$, so $t^{-1}N_1^- t \subseteq N_1^-$; and $T_C t = tT_C$. Hence $\mathrm{Iw}(C)t\mathrm{Iw}(C) = N^+ t\,\mathrm{Iw}(C)$. Likewise $tN^+t^{-1} \subseteq N^+$. For $x, x' \in N^+$, $xt\mathrm{Iw}(C) = x't\mathrm{Iw}(C)$ iff $t^{-1}x^{-1}x't \in \mathrm{Iw}(C)$, i.e. (this matrix being upper unitriangular, and $\mathrm{Iw}(C)$ meeting the upper unitriangular group of G in N^+) iff $x^{-1}x' \in tN^+t^{-1}$. Therefore, for $v \in V^{\mathrm{Iw}(C)}$,

$$(1) \qquad [t]_C v = \sum_{x \in N^+/tN^+t^{-1}} x\,t\,v,$$

each summand depending only on the coset of x (as $tN^+t^{-1}tv = tN^+v = tv$). The right side of (1) does not involve C, and $|N^+/tN^+t^{-1}| = q^{\sum_{i<j}(v(t_i)-v(t_j))}$.

Step 2 (products). Let $t, t' \in T^+$. Then $N^+ \supseteq tN^+t^{-1} \supseteq tt'N^+(tt')^{-1}$, and $(x,x') \mapsto x \cdot tx't^{-1}$ induces a bijection $N^+/tN^+t^{-1} \times N^+/t'N^+t'^{-1} \to N^+/tt'N^+(tt')^{-1}$ (representatives of a subgroup chain; conjugation by t maps $N^+/t'N^+t'^{-1}$ onto $tN^+t^{-1}/tt'N^+(tt')^{-1}$). By (1), $[t]_C[t']_C v = \sum_{x,x'} x\,(tx't^{-1})\,tt'v = [tt']_C v$. So $t \mapsto [t]_C$ is a homomorphism of monoids $T^+ \to \mathrm{End}_\Lambda(V^{\mathrm{Iw}(C)})$ with commuting values. For $\tau \in T(\mathcal{O}_K) \subseteq T^+$ the index in (1) is 1 and $[\tau]_C v = \tau v$; for the scalar matrix $\varpi^r 1_n$ ($r \in \mathbb{Z}$, dominant) likewise $[\varpi^r 1_n]_C v = \varpi^r v$. Both are invertible.

Step 3 (quadratic relation on V^{Iw_1}). Fix $1 \le a < n$, let s be the permutation matrix of $s_a = (a\ a{+}1)$, $u_y = 1 + yE_{a,a+1}$ ($y \in \mathcal{O}_K$), and for $y \in \mathcal{O}_K^\times$ let $d_y \in T(\mathcal{O}_K)$ have entries $-y^{-1}, y$ in places $a, a+1$ and 1 elsewhere. Let $[s]$ be the endomorphism $v \mapsto \sum_{g \in \mathrm{Iw}_1 s \mathrm{Iw}_1/\mathrm{Iw}_1} gv$ of V^{Iw_1}. Since $s^{-1}N_1^- s \subseteq 1 + \varpi M_n(\mathcal{O}_K) \subseteq \mathrm{Iw}_1$, $s^{-1}T_{\{1\}}s = T_{\{1\}}$, and $N^+ = \{u_y\} \cdot N'$ with $N' = \{x \in N^+ : x_{a,a+1} = 0\}$ (the kernel of the homomorphism $x \mapsto x_{a,a+1}$) and $s^{-1}N's \subseteq N^+$, we get $\mathrm{Iw}_1 s \mathrm{Iw}_1 = \bigsqcup_{y \in R_k} u_y s \mathrm{Iw}_1$ (and $u_y s\mathrm{Iw}_1 = u_{y'}s\mathrm{Iw}_1$ iff $s^{-1}u_{y'-y}s = 1 + (y'-y)E_{a+1,a} \in \mathrm{Iw}_1$ iff $y \equiv y'$). Thus $[s]v = \sum_{y\in R_k} u_y s v$, and $u_z s v$ depends only on z modulo ϖ. Now

$$[s]^2 v = \sum_{y,y' \in R_k} u_y s u_{y'} s v.$$

For $y' = 0$, $s^2 = 1$ and the terms sum to $\sum_y u_y v = qv$. For $y' \ne 0$, in the block $(a, a+1)$, $s u_{y'} s = u_{y'^{-1}} d_{y'} s u_{y'^{-1}}$, and $u_{y'^{-1}} \in \mathrm{Iw}_1$ fixes v, so $u_y s u_{y'} s v = u_{y + y'^{-1}} d_{y'} s v$.

As y runs over R_k, $y+y'^{-1}$ runs over representatives of k; and $u_z d_{y'} = d_{y'} u_{-y'^2 z}$, $-y'^2 z$ again running over representatives. Hence $\sum_y u_y s u_{y'} s v = d_{y'}[s]v$, and

$$[s]^2 = q + c\,[s], \qquad c = \sum_{y \in R_k \setminus \{0\}} d_y$$

as endomorphisms of V^{Iw_1} ($T(\mathcal{O}_K)$ normalises Iw_1, and d_y acts on V^{Iw_1} through y modulo ϖ, as $T_{\{1\}} \subseteq \mathrm{Iw}_1$). Further $s d_y s^{-1} = d_{-y^{-1}}$, so by the same reindexing $[s] d_y v = d_{-y^{-1}}[s]v$; as $y \mapsto -y^{-1}$ permutes $k^\times$, $[s]c = c[s]$. Therefore $([s]-c)[s] = [s]([s]-c) = q$, and $[s]$ is invertible with inverse $q^{-1}([s]-c)$, q being a unit of Λ.

Step 4 (invertibility of $[t]_C$). The ring $\mathcal{H}^{(1)}$ of Theorem B.168 acts on V^{Iw_1} by $h \cdot v = \sum_{g \in G/\mathrm{Iw}_1} h(g)\, gv$. The action is through a ring homomorphism, by the convolution formula

$$(h * h')(g) = \sum_{g' \in G/\mathrm{Iw}_1} h(g') h'(g'^{-1} g),$$

and T_g acts by $v \mapsto \sum_{g' \in \mathrm{Iw}_1 g \mathrm{Iw}_1/\mathrm{Iw}_1} g' v$; so T_s acts as $[s]$ and T_t as $[t]_{\{1\}}$ for $t \in T^+$. Let $1 \le m < n$, $\alpha_m = \mathrm{diag}(\varpi 1_m, 1_{n-m})$, $\Pi = \begin{pmatrix} 0 & 1_{n-1} \\ \varpi & 0 \end{pmatrix}$ and $w_m = \begin{pmatrix} 0 & 1_m \\ 1_{n-m} & 0 \end{pmatrix}$, the permutation matrix of the permutation $j \mapsto j+m$ ($j \le n-m$), $j \mapsto j-n+m$ ($j > n-m$), which has $m(n-m)$ inversions. Then $\alpha_m = w_m \Pi^m$, and Π normalises Iw and every $\mathrm{Iw}(C)$.

Lengths in the sense of Theorem B.168: $\ell(\alpha_m) = m(n-m)$ by Step 1 (for $C = k^\times$); $\ell(\Pi^m) = 0$; and $\ell(w)$ is the number of inversions for a permutation matrix w, since Iw is the inverse image of the upper triangular group $B(k)$ under reduction $K_0 \to \mathrm{GL}_n(k)$, so that $\mathrm{Iw} w \mathrm{Iw}/\mathrm{Iw} \cong B(k) w B(k)/B(k)$, of order $q^{\#\mathrm{inversions}(w)}$ (Bruhat decomposition of $\mathrm{GL}_n(k)$). Choose a reduced word $w_m = s_{a_1} \cdots s_{a_r}$, $r = m(n-m)$. Applying Theorem B.168 repeatedly (the lengths add at each step), $T_{\alpha_m} = T_{s_{a_1}} \cdots T_{s_{a_r}} T_{\Pi^m}$, so on V^{Iw_1}

$$[\alpha_m]_{\{1\}} = [s_{a_1}] \cdots [s_{a_r}] \circ (v \mapsto \Pi^m v),$$

a composite of automorphisms (Step 3). Every $t \in T^+$ is $t = \tau \cdot \varpi^{\lambda_n} 1_n \cdot \prod_{m<n} \alpha_m^{\lambda_m - \lambda_{m+1}}$ with $\tau \in T(\mathcal{O}_K)$, $\lambda_i = v(t_i)$, all factors in T^+; by Step 2, $[t]_{\{1\}}$ is invertible. Now $\mathrm{Iw}(C) = \mathrm{Iw}_1 T_C$ (divide $g \in \mathrm{Iw}(C)$ by its diagonal), so $V^{\mathrm{Iw}(C)}$ is the set of T_C-fixed vectors of V^{Iw_1}, and $[t]_C$ is the restriction of $[t]_{\{1\}}$ by (1). By Step 2, $[t]_{\{1\}}$ commutes with the action of $T(\mathcal{O}_K) = \{[\tau]_{\{1\}}\}$, hence so does its inverse, which therefore maps $V^{\mathrm{Iw}(C)}$ into itself. So $[t]_C$ is an automorphism of $V^{\mathrm{Iw}(C)}$, and for $C \subseteq C'$ the inverse of $[t]_C$ restricts to that of $[t]_{C'}$.

Step 5 (proof of (Si1)). Every $t \in T$ is $t_1 t_2^{-1}$ with $t_1, t_2 \in T^+$ (take $t_2 = \mathrm{diag}(\varpi^{M(n-1)}, \varpi^{M(n-2)}, \ldots, 1)$ with $M \ge \max_i |v(t_{i+1}) - v(t_i)|$ and $t_1 = t t_2$). Put $\theta_C(t_1 t_2^{-1}) = [t_1]_C [t_2]_C^{-1}$. This is well defined: $t_1 t_2^{-1} = t_1' t_2'^{-1}$ gives $t_1 t_2' = t_1' t_2$, so $[t_1]_C [t_2']_C = [t_1']_C [t_2]_C$ (Step 2), and the operators commute. It is a homomorphism with commutative image extending $t \mapsto [t]_C$ on T^+, and it is the only one, since any such homomorphism takes these values on $t_1 t_2^{-1}$. On $T(\mathcal{O}_K)$ it is the action (Step 2). For $C \subseteq C'$, $V^{\mathrm{Iw}(C')} \subseteq V^{\mathrm{Iw}(C)}$ and $[t_i]_C$ and $[t_i]_C^{-1}$ restrict to $[t_i]_{C'}$ and $[t_i]_{C'}^{-1}$ (Steps 1, 4), so $\theta_C(t)$ restricts to $\theta_{C'}(t)$.

Step 6 (the integral Iwahori–Hecke ring). Let $\mathcal{H}$ be the algebra of Theorem B.167, $R = \mathbb{Z}[1/q] \subseteq \mathbb{C}$, and $\mathcal{H}_R \subseteq \mathcal{H}$ the R-span of the T_g; it is a subring, the structure constants $(T_g * T_{g'})(x) = \#\{y \in \mathrm{Iw} g \mathrm{Iw}/\mathrm{Iw} : y^{-1} x \in \mathrm{Iw} g' \mathrm{Iw}\}$ being integers. As $q \in \Lambda^\times$, Λ is an R-algebra and $\rho_V(h)v = \sum_{x \in G/\mathrm{Iw}} h(x)\, xv$ defines a ring homomorphism $\rho_V \colon \mathcal{H}_R \to \mathrm{End}_\Lambda(V^{\mathrm{Iw}})$, with $\rho_V(T_t) = [t]_{k^\times}$ for $t \in T^+$. By the Bruhat decomposition $\mathrm{GL}_n(k) = \bigsqcup_{w \in S_n} B(k) w B(k)$, $K_0 = \bigsqcup_w \mathrm{Iw} w \mathrm{Iw}$, so $\mathbf{1}_{K_0} = \sum_{w \in S_n} T_w \in \mathcal{H}_R$ and $\rho_V(\mathbf{1}_{K_0}) = E$.

Apply Steps 1–5 to the module $V' = C_c(G/\mathrm{Iw}, R)$ (finitely supported functions on G/Iw, G acting by $(gf)(x) = f(g^{-1}x)$). Then $V'^{\mathrm{Iw}} = \mathcal{H}_R$ and $[t]_{k^\times}$ is left multiplication by T_t.

Its inverse L commutes with right multiplications (as T_t* does), so $y = L(\mathbf{1}_{\mathrm{Iw}})$ satisfies $T_t * y = \mathbf{1}_{\mathrm{Iw}}$ and $y * T_t = L(T_t) = \mathbf{1}_{\mathrm{Iw}}$: $T_t \in \mathcal{H}_R^\times$. As in Step 5 we get a homomorphism $\theta\colon T \to \mathcal{H}_R^\times$, $t_1 t_2^{-1} \mapsto T_{t_1} T_{t_2}^{-1}$, and $\rho_V \circ \theta = \theta_{k^\times}$ by the uniqueness in (Si1). In $\mathcal{H}$, (IH2) gives $\theta(\varpi^\lambda) = q^{\langle\rho,\lambda\rangle}\Theta_\lambda$ for dominant λ, hence for all $\lambda \in \mathbb{Z}^n$ (both sides are homomorphisms in λ). Put $\tilde{X}_i = q^{i-1}\theta(\varpi^{e_i}) \in \mathcal{H}_R$; since $\langle\rho, e_i\rangle = \frac{n+1-2i}{2}$, $\tilde{X}_i = \kappa\,\Theta_{e_i}$ with $\kappa = q^{(n-1)/2}$. The $\tilde{X}_i$ commute, $\rho_V(\tilde{X}_i) = U_i$, and for $y \in \mathbb{Z}[X_1, \dots, X_n]$, $\rho_V(y(\tilde{X})) = u(y)$. Let $A = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the group algebra of $\mathbb{Z}^n$ (x^μ for μ), with S_n acting by $w(x_i) = x_{w(i)}$, as on exponents in (IH3); for a polynomial y put $y^\flat = y(\kappa x_1, \dots, \kappa x_n) \in A$. Then $y(\tilde{X}) = \Theta(y^\flat)$.

Step 7 (the polynomial representation). By (IH1), $T_{s_a}\mathbf{1}_{K_0} = q\mathbf{1}_{K_0}$, hence $T_w\mathbf{1}_{K_0} = q^{\ell(w)}\mathbf{1}_{K_0}$.

So $\iota\colon A \to \mathcal{H}$, $x \mapsto \Theta(x)\mathbf{1}_{K_0} = \sum_{\mu,w} x_\mu \Theta_\mu T_w$, is injective (basis in (IH2)) with image $\mathcal{H}\mathbf{1}_{K_0}$ ($\mathcal{H}$ is spanned by the $\Theta_\mu T_w$). Thus $\mathcal{H}$ acts on A by $h\cdot x = \iota^{-1}(h\iota(x))$: $\Theta(f)$ acts as multiplication by f, $\mathbf{1}_{K_0}\Theta(x)\mathbf{1}_{K_0} = \iota(\mathbf{1}_{K_0} \cdot x)$, and by (IH3) and $T_{s_a}\mathbf{1}_{K_0} = q\mathbf{1}_{K_0}$, T_{s_a} acts as

$$D_a x = q\, s_a(x) + (1-q)\frac{s_a(x) - x}{1 - x_{a+1}x_a^{-1}} = \frac{q-1}{1 - x_{a+1}x_a^{-1}}\, x + \frac{1 - q x_{a+1}x_a^{-1}}{1 - x_{a+1}x_a^{-1}}\, s_a(x)$$

(here $x^{-\alpha^\vee} = x_{a+1}x_a^{-1}$). Let $\mathcal{K}$ be the fraction field of A and $\mathcal{K}[S_n]$ the twisted group ring (elements $\sum_w f_w w$, acting on $\mathcal{K}$ by $x \mapsto \sum_w f_w w(x)$, with $wf = w(f)w$). An element of $\mathcal{K}[S_n]$ vanishing on A vanishes on $\mathcal{K}$ (write an element of $\mathcal{K}$ as x/d with $x \in A$ and d an S_n-invariant element of A, by multiplying a denominator by its conjugates), hence is 0 by Dedekind's independence of the distinct automorphisms w of $\mathcal{K}$. So $D_a = A_a + B_a s_a$ in $\mathcal{K}[S_n]$ with $B_a = (1 - qx_{a+1}x_a^{-1})/(1 - x_{a+1}x_a^{-1}) \neq 0$, and by (IH1) T_w acts as $D_w = D_{a_1} \cdots D_{a_r}$ for a reduced word $w = s_{a_1} \cdots s_{a_r}$. Expanding the product, the group elements that occur are products of subwords, of length $\le r$, and only the full subword has length r; so

$$D_w = \beta_w\, w + \sum_{\ell(v) < \ell(w)} f_{w,v}\, v, \qquad \beta_w = B_{a_1}\, s_{a_1}(B_{a_2}) \cdots (s_{a_1} \cdots s_{a_{r-1}})(B_{a_r}) \neq 0. \tag{2}$$

For the longest element w_0 ($w_0(i) = n+1-i$, the only element of length $n(n-1)/2$) the coroots $s_{a_1} \cdots s_{a_{j-1}}(e_{a_j} - e_{a_j+1})$, $1 \le j \le r$, are the $e_i - e_{i'}$, $i < i'$, each once ([49, Ch. VI, §1.6, Cor. 2]), so $\beta_{w_0} = \prod_{i<i'}(x_i - qx_{i'})/(x_i - x_{i'})$.

Step 8 (faithfulness). If $h \in \mathcal{H}$ acts as 0 on A then $h = 0$. Indeed $h = \sum_w \Theta(f_w)T_w$ by (IH2) and acts as $\sum_w f_w D_w$, which is 0 in $\mathcal{K}[S_n]$ (Step 7). If some $f_w \neq 0$, choose such w of maximal length; by (2) the coefficient of w in $\sum_v f_v D_v$ is $f_w\beta_w \neq 0$, a contradiction.

Step 9 (the symmetriser). $\mathbf{1}_{K_0}$ acts as $\mathcal{S} = \sum_w D_w$, and $T_{s_a}\mathbf{1}_{K_0} = q\mathbf{1}_{K_0}$ gives $D_a\mathcal{S}x = q\mathcal{S}x$ for $x \in A$. For $z \in A$, $D_a z - qz = (s_a(z) - z)\,\frac{1 - qx_{a+1}x_a^{-1}}{1 - x_{a+1}x_a^{-1}}$, so $D_a z = qz$ iff $s_a(z) = z$.

Hence $\mathcal{S}x$ is S_n-invariant for every $x \in A$, i.e. $s_a\mathcal{S} = \mathcal{S}$ in $\mathcal{K}[S_n]$. Writing $\mathcal{S} = \sum_v g_v v$, this reads $g_{s_a v} = s_a(g_v)$, so $g_v = v(g_1)$. By (2) only D_{w_0} contributes to the coefficient of w_0: $g_{w_0} = \beta_{w_0}$, so $g_1 = w_0(\beta_{w_0}) = \prod_{i<i'}(x_{i'} - qx_i)/(x_{i'} - x_i) = N_q/\Delta$ (in the variables x). Thus $\mathcal{S}x = \sum_v v(xN_q/\Delta) = \Delta^{-1}\sum_v \mathrm{sgn}(v)v(N_q x)$, as $v(\Delta) = \mathrm{sgn}(v)\Delta$; for a polynomial x this is $\mathrm{dd}(N_q x)$. Since N_q and Δ are homogeneous of the same degree, $\mathrm{dd}(N_q y^\flat) = (\mathrm{dd}(N_q y))^\flat$. With Step 6 this gives, for every $y \in \mathbb{C}[X_1, \dots, X_n]$ (in particular in $\mathcal{H}_R$ for $y \in \mathbb{Z}[X_1, \dots, X_n]$),

$$\mathbf{1}_{K_0}\, y(\tilde{X})\, \mathbf{1}_{K_0} = (\mathrm{dd}(N_q y))(\tilde{X})\, \mathbf{1}_{K_0}. \tag{3}$$

Step 10 (proof of (Si2)). A symmetric $z \in \Lambda[X_1, \dots, X_n]$ is a Λ-combination of orbit sums m_μ of monomials, so it suffices to take $z = m_\mu$, homogeneous of degree d with integer coefficients.

Then $z(\tilde{X}) = \Theta(\kappa^d m_\mu(x))$ is central in $\mathcal{H}$ by (IH4), so commutes with $\mathbf{1}_{K_0}$; applying ρ_V gives $u(z)E = Eu(z)$.

Step 11 (proof of (Si3)). Existence: expand $F = \prod_{i<j}(X_j - Y_i) = \sum_{k=1}^{m} a_k(X)c_k(Y)$ as a sum of products of monomials. Then $\sum_k a_k w(c_k) = \prod_{i<j}(X_j - X_{w(i)})$; for $w = 1$ this is Δ; for $w \neq 1$ let i be least with $w(i) \neq i$: then $w(i) > i$ and the factor with $j = w(i)$ vanishes. Now let a_k, c_k be any such families. For every polynomial y (over any coefficient ring),

$$\sum_k a_k \,\mathrm{dd}(c_k y) = y, \tag{4}$$

since $\Delta \sum_k a_k \mathrm{dd}(c_k y) = \sum_w \mathrm{sgn}(w) w(y) \sum_k a_k w(c_k) = \Delta y$ and Δ is a non-zero-divisor. Let $h = \sum_k a_k(\tilde{X})\mathbf{1}_{K_0} c_k(\tilde{X}) - N_q(\tilde{X}) \in \mathcal{H}_R$. For $y \in \mathbb{C}[X_1, \dots, X_n]$, by (3) and (4) with $N_q y$ in place of y,

$$h\,\iota(y^\flat) = h\,y(\tilde{X})\mathbf{1}_{K_0} = \Big(\sum_k a_k \,\mathrm{dd}(c_k N_q y) - N_q y\Big)(\tilde{X})\,\mathbf{1}_{K_0} = 0.$$

As $y \mapsto y^\flat$ is onto the polynomials of A, h kills $\iota(x)$ for every polynomial x; a general $x \in A$ is $(x_1 \cdots x_n)^{-M} x'$ with x' a polynomial, and $\Theta((x_1 \cdots x_n)^{-M})$ is central (IH4), so $h\iota(x) = \Theta((x_1 \cdots x_n)^{-M}) h\iota(x') = 0$. By Step 8, $h = 0$; applying ρ_V gives $\sum_k u(a_k)Eu(c_k) = u(N_q)$.

Step 12 (proof of (Si4)). For $v \in V^{K_0}$, $Ev = [K_0 : \mathrm{Iw}]v$, and $Ew \in V^{K_0}$ for every $w \in V^{\mathrm{Iw}}$. Let z be symmetric; by (Si2), $[K_0 : \mathrm{Iw}]u(z)v = u(z)Ev = Eu(z)v \in V^{K_0}$, so for $g \in K_0$, $[K_0 : \mathrm{Iw}](gu(z)v - u(z)v) = 0$, whence $gu(z)v = u(z)v$ by the injectivity hypothesis. For $y \in \mathbb{Z}[X_1, \dots, X_n]$, applying ρ_V to (3) gives $Eu(y)E = u(\mathrm{dd}(N_q y))E$, so $[K_0 : \mathrm{Iw}]Eu(y)v = [K_0 : \mathrm{Iw}]u(\mathrm{dd}(N_q y))v$, and cancelling, $E(u(y)v) = u(\mathrm{dd}(N_q y))v$. Both sides are Λ-linear in y (dd is, by uniqueness of the quotient by the non-zero-divisor Δ), so the identity holds for all $y \in \Lambda[X_1, \dots, X_n]$. □

Lemma E.139. *Let ℓ, K, q, ϖ, C_0, Φ, τ, δ be as in Lemma E.140, σ an irreducible smooth representation of $\mathrm{GL}_n(K)$ on a complex vector space, and $r\colon G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ continuous such that, if $\mathrm{rec}(\sigma \otimes |\det|^{(1-n)/2})$ has monodromy 0 and is trivial on the commutator subgroup of the Weil group, it is isomorphic to the Frobenius-semisimplification $\mathrm{WD}(r)^{F\text{-}ss}$ (as in Lemma E.140) of the Weil–Deligne representation of r transported by ι. Put $F = \iota r(\Phi)$, $T = \iota r(\tau)$, $\theta_1 = \theta_{C_0}$ and $\theta = \theta_{k^\times}$ (for $\Lambda = \mathbb{C}$, $V = \sigma$), and on $\sigma^{\mathrm{Iw}(C_0)}$ the commuting operators $U_i = q^{i-1}\theta_1(e_i(\varpi))$, $Y_i = \theta_1(e_i(\delta)) - 1$ ($1 \le i \le n$), and $f = \prod_i \prod_{i' \neq i}(U_i - U_{i'})(U_i - qU_{i'})$. For a matrix $A \in \mathrm{M}_n(\mathbb{C})$ and $1 \le i \le n$ let $\Xi_1(A)$ and $\Xi_{2,i}(A)$ be the operators on $\sigma^{\mathrm{Iw}(C_0)}$ obtained by substituting the U_k, Y_k into the polynomials*

$$\mathrm{tr}\Big(A \prod_{k=1}^{n}(F - U_k)\Big), \qquad \mathrm{tr}\Big(A\Big(\prod_{k \neq i}(F - U_k)\Big)(T - 1 - Y_i)\Big) \in \mathbb{C}[U_1, \dots, U_n, Y_1, \dots, Y_n]$$

(the U_k, Y_i being scalars for the matrix product). Then:

(Ln1) $(f\,\Xi_1(A))^{n!} = 0$ *and* $(f\,\Xi_{2,i}(A))^{n!} = 0$ *for every A and i;*

(Ln2) *for every $v \in \sigma^{\mathrm{GL}_n(\mathcal{O}_K)}$ and $0 \le k \le n$, $e_k(q^0\theta(e_1(\varpi)), \dots, q^{n-1}\theta(e_n(\varpi)))v = (-1)^k c_{n-k} v$, where $\det(X - F) = \sum_r c_r X^r$ and e_k is the k-th elementary symmetric polynomial.*

Proof. The dimension of $W = \sigma^{\mathrm{Iw}(C_0)}$ is at most $n!$. Let $\mathrm{Iw}_p \subseteq \mathrm{Iw}(C_0)$ be the matrices upper triangular and unipotent modulo ϖ and $j\colon \sigma^{\mathrm{Iw}_p} \to (\sigma_N)^{T(\varpi)}$ the map of (Gy7)(a) (σ is

admissible, being irreducible smooth). For $t = \alpha^{(m)} = \mathrm{diag}(\varpi 1_m, 1_{n-m})$ the Iwahori factorisation $\mathrm{Iw}(C_0) = N(\mathcal{O}_K)T_{C_0}\bar{N}(\varpi\mathcal{O}_K)$ and $t^{-1}\bar{N}(\varpi\mathcal{O}_K)t \subseteq \bar{N}(\varpi\mathcal{O}_K)$ give $\mathrm{Iw}(C_0)t\,\mathrm{Iw}(C_0) = N(\mathcal{O}_K)t\,\mathrm{Iw}(C_0)$, so $\theta_1(t)$ is the restriction to W of $U_\varpi^{(m)} = [N(\mathcal{O}_K)tN(\mathcal{O}_K)]$ on σ^{Iw_p}. These restrictions are invertible on W by (Si1), so $W \cap \ker j$, a subspace stable under the $U_\varpi^{(m)}$ on which they are invertible, is 0 by (Gy7)(a). Hence $\dim W \leq \dim(\sigma_N)^{T(\varpi)} < \infty$. If $W \neq 0$, then $X = (\sigma_N)^{T(\varpi)}$ is a non-zero finite-dimensional T-stable direct summand of σ_N (averaging over $T(\varpi)$), so the commutative family T has a common eigenvector of the dual of X, i.e. a non-zero T-map $\sigma_N \to X \to \chi'$ to a character; by Frobenius reciprocity ([27], Proposition 2.3) σ embeds into $I(\xi)$ for some characters $\xi = (\xi_1, \ldots, \xi_n)$, so $\dim W \leq \dim I(\xi)^{\mathrm{Iw}(C_0)}$. A function in $I(\xi)^{\mathrm{Iw}(C_0)}$ is determined by its values on representatives of $B\backslash \mathrm{GL}_n(K)/\mathrm{Iw}(C_0)$, and there are $n!$ of them: by the Iwasawa decomposition $\mathrm{GL}_n(K) = B\,\mathrm{GL}_n(\mathcal{O}_K)$ these double cosets are those of $B(k)\backslash \mathrm{GL}_n(k)/B_{C_0}(k)$ ($B_{C_0}(k)$ the upper triangular matrices with diagonal in C_0), and $B(k)wB(k) = B(k)w\,N(k) \subseteq B(k)wB_{C_0}(k)$ (as $wT(k)w^{-1} = T(k)$) with the Bruhat decomposition show that these are the $n!$ cosets $B(k)wB_{C_0}(k)$, $w \in S_n$. So $\dim W \leq n!$.

Generalised eigenspaces. The group $T(1 + \varpi\mathcal{O}_K)$ of diagonal matrices $\equiv 1$ modulo ϖ lies in $\mathrm{Iw}(C_0)$ and acts trivially on W, so θ_1 factors through the finitely generated abelian group $T/T(1 + \varpi\mathcal{O}_K)$, and $W = \bigoplus_\chi W_\chi$, the sum over the characters χ of T for which the joint generalised eigenspace W_χ of the $\theta_1(t)$ for the eigenvalues $\chi(t)$ is non-zero. Fix such χ, put $\alpha_i = q^{i-1}\chi(e_i(\varpi))$ and $d_\chi = \dim W_\chi \leq n!$. The operators $\theta_1(t) - \chi(t)$ on W_χ commute and are nilpotent; in a common basis they are strictly upper triangular, so the ideal $\mathfrak{n}$ they generate in the commutative algebra $\mathcal{A}$ generated by $\theta_1(T)$ on W_χ satisfies $\mathfrak{n}^{d_\chi} = 0$. Every polynomial $P(U_1, \ldots, U_n, Y_1, \ldots, Y_n)$ acts on W_χ as $P(\alpha_1, \ldots, \alpha_n, \chi(e_1(\delta)) - 1, \ldots, \chi(e_n(\delta)) - 1)$ plus an element of $\mathfrak{n}$, since $U_i - \alpha_i = q^{i-1}(\theta_1(e_i(\varpi)) - \chi(e_i(\varpi)))$ and $Y_i - (\chi(e_i(\delta)) - 1)$ lie in $\mathfrak{n}$.

Proof of (Ln1). It suffices to prove it on each W_χ. Suppose first that α is generic ($\alpha_i \neq \alpha_{i'}$, $\alpha_i \neq q\alpha_{i'}$ for $i \neq i'$). By Lemma E.140 (1), applied to σ, $C = C_0$, θ_1 (which equals $[t]$ on dominant t by (Si1)) and χ, $\det(X - F) = \prod_k(X - \alpha_k)$, so $\prod_k(F - \alpha_k) = 0$ by Cayley–Hamilton and the scalar value of $\Xi_1(A)$ is $\mathrm{tr}(A \cdot 0) = 0$; by Lemma E.140 (2), $\big(\prod_{k\neq i}(F - \alpha_k)\big)(T - \chi(e_i(\delta))) = 0$, so the scalar value of $\Xi_{2,i}(A)$ is 0. Hence $\Xi_1(A)$, $\Xi_{2,i}(A)$ lie in $\mathfrak{n}$ on W_χ. If α is not generic, the scalar value of f, namely $\prod_i \prod_{i'\neq i}(\alpha_i - \alpha_{i'})(\alpha_i - q\alpha_{i'})$, is 0, so $f \in \mathfrak{n}$ on W_χ. In either case $f\,\Xi$ ($\Xi = \Xi_1(A)$ or $\Xi_{2,i}(A)$) lies in $\mathfrak{n}$, so $(f\,\Xi)^{n!} = 0$ on W_χ as $n! \geq d_\chi$.

Proof of (Ln2). Let $v \in \sigma^{\mathrm{GL}_n(\mathcal{O}_K)}$, $v \neq 0$; then $\sigma^{\mathrm{GL}_n(\mathcal{O}_K)}$ is the line $\mathbb{C}v$ (irreducible spherical representations have a one-dimensional space of spherical vectors, by the commutativity of the spherical Hecke algebra). Put $U_i^\circ = q^{i-1}\theta(e_i(\varpi))$ on $W^\circ = \sigma^{\mathrm{Iw}}$ and $z = e_k(U_1^\circ, \ldots, U_n^\circ)$. By (Si4) of Lemma E.138 (σ is a complex vector space, so multiplication by $[\mathrm{GL}_n(\mathcal{O}_K) : \mathrm{Iw}]$ is injective) z preserves $\mathbb{C}v$, so $zv = \lambda v$ for some $\lambda \in \mathbb{C}$. Decompose $W^\circ = \bigoplus_\chi W_\chi^\circ$ into joint generalised eigenspaces of θ as above and choose χ with non-zero component v_χ of v; as z preserves each W_χ°, $zv_\chi = \lambda v_\chi$, and z acts on W_χ° as $e_k(\alpha_1, \ldots, \alpha_n)$ plus a nilpotent operator ($\alpha_i = q^{i-1}\chi(e_i(\varpi))$), so $\lambda = e_k(\alpha_1, \ldots, \alpha_n)$. By Lemma E.140 (1) (with $C = k^\times$; σ has a non-zero $\mathrm{GL}_n(\mathcal{O}_K)$-fixed vector), $\prod_i(X - \alpha_i) = \det(X - F)$, whence $e_k(\alpha) = (-1)^k c_{n-k}$. □

Lemma E.140. *Let ℓ be a prime, K a finite extension of $\mathbb{Q}_p$ with $p \neq \ell$, $\mathcal{O}_K$, k, $q = |k|$ and a uniformiser ϖ; let $C_0 \subseteq k^\times$ be the subgroup of elements of order prime to ℓ. For a subgroup $C_0 \subseteq C \subseteq k^\times$ let $\mathrm{Iw}(C) \subseteq \mathrm{GL}_n(\mathcal{O}_K)$ be the group of matrices that are upper triangular modulo ϖ with diagonal entries modulo ϖ in C. For $t = (t_1, \ldots, t_n) \in (K^\times)^n$ with $v(t_1) \geq \cdots \geq v(t_n)$ let $[t]$ be the double coset operator $[\mathrm{Iw}(C)\,\mathrm{diag}(t)\,\mathrm{Iw}(C)]$ on $\sigma^{\mathrm{Iw}(C)}$, for an irreducible smooth representation σ of $\mathrm{GL}_n(K)$ on a complex vector space. For $x \in K^\times$ and $1 \leq i \leq n$ let $e_i(x) \in (K^\times)^n$ have x in place i and 1 elsewhere.*

Let σ be such a representation, θ a homomorphism from $(K^\times)^n$ to the group of units of

$\mathrm{End}(\sigma^{\mathrm{Iw}(C)})$ *with* $\theta(t)=[t]$ *whenever* $v(t_1)\geq\cdots\geq v(t_n)$, *and* $\chi\colon (K^\times)^n\to\mathbb{C}^\times$ *a character such that the joint generalised eigenspace of the* $\theta(t)$ *for the eigenvalues* $\chi(t)$ *is non-zero. Put* $\alpha_i=q^{i-1}\chi(e_i(\varpi))$. *Let* $r\colon G_K\to\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ *be continuous such that, if* $\mathrm{rec}(\sigma\otimes|\det|^{(1-n)/2})$ *has monodromy* 0 *and is trivial on the commutator subgroup of the Weil group, it is isomorphic to the Frobenius-semisimplification* $\mathrm{WD}(r)^{F\text{-}ss}$ *of the Weil–Deligne representation of* r *transported by* ι. *Let* $\Phi\in G_K$ *be an element with* $\Phi|_{K^{\mathrm{ab}}}=\mathrm{Art}_K(\varpi)$ *(a lift of geometric Frobenius; conversely, every lift of geometric Frobenius lies in the Weil group and so has this form for exactly one uniformiser* ϖ, *as* Art_K *identifies* $K^\times$ *with the image of the Weil group in* $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, *uniformisers corresponding to geometric Frobenius lifts),* $\tau\in G_K$ *an element of inertia whose image generates the maximal pro-*ℓ *quotient of tame inertia, and* $\delta\in\mathcal{O}_K^\times$ *the element with* $\mathrm{Art}_K(\delta)=\tau|_{K^{\mathrm{ab}}}$ *(it exists as* τ *lies in inertia; its residue generates* $k^\times$ *together with* C_0*). Call* α generic *if* $\alpha_i\neq\alpha_j$ *and* $\alpha_i\neq q\alpha_j$ *for all* $i\neq j$. *Then:*

1. *if* σ *has a non-zero* $\mathrm{GL}_n(\mathcal{O}_K)$*-fixed vector or* α *is generic, the characteristic polynomial of* $\iota(r(\Phi))$ *is* $\prod_{i=1}^n(X-\alpha_i)$*;*

2. *if* α *is generic, then for every* i: $\Big(\prod_{i'\neq i}(\iota r(\Phi)-\alpha_{i'})\Big)(\iota r(\tau)-\chi(e_i(\delta)))=0$.

Proof. Write $G=\mathrm{GL}_n(K)$, $\mathrm{Iw}_1\subseteq\mathrm{GL}_n(\mathcal{O}_K)$ for the matrices upper triangular and unipotent modulo ϖ, $W=\sigma^{\mathrm{Iw}(C)}\subseteq\sigma^{\mathrm{Iw}_1}$, $\chi_i(x)=\chi(e_i(x))$ for $x\in K^\times$, and $W_\chi\neq 0$ for the joint generalised eigenspace of the hypothesis. As σ is irreducible smooth it is admissible [27], so σ^{Iw_1} is finite-dimensional. Call $t\in T$ dominant if $v(t_1)\geq\cdots\geq v(t_n)$, and put $\alpha^{(j)}=\mathrm{diag}(\varpi 1_j,1_{n-j})$.

Step 1: the Hecke operators on W. For dominant t, $tN(\mathcal{O}_K)t^{-1}\subseteq N(\mathcal{O}_K)$ and $t^{-1}\bar{N}(\varpi\mathcal{O}_K)t\subseteq\bar{N}(\varpi\mathcal{O}_K)$ (the entry (i,j) is multiplied by t_i/t_j, resp. $t_i^{-1}t_j$, of valuation ≥ 0 for $i<j$, resp. $i>j$). With the Iwahori decomposition $\mathrm{Iw}(C)=N(\mathcal{O}_K)T_C\bar{N}(\varpi\mathcal{O}_K)$ (T_C the diagonal matrices of $\mathrm{Iw}(C)$, which commute with t) this gives $\mathrm{Iw}(C)t\,\mathrm{Iw}(C)=N(\mathcal{O}_K)t\,\mathrm{Iw}(C)$, so $[t]w=\sum_{u\in N(\mathcal{O}_K)/tN(\mathcal{O}_K)t^{-1}}\sigma(ut)w$ for $w\in W$; the same computation for Iw_1 shows that $\theta(\alpha^{(j)})=[\alpha^{(j)}]$ is the restriction to W of the operator $U_\varpi^{(j)}=[N(\mathcal{O}_K)\alpha^{(j)}N(\mathcal{O}_K)]$ of (Gy7)(a) on σ^{Iw_1}. For $u\in T(\mathcal{O}_K)$ (dominant, normalising $\mathrm{Iw}(C)$), $\theta(u)=\sigma(u)|_W$. Every $t\in T$ is $u\prod_{j<n}(\alpha^{(j)})^{m_j}\cdot(\alpha^{(n)})^{m_n}$ with $u\in T(\mathcal{O}_K)$, $m_j\geq 0$ for $j<n$ and $m_n\in\mathbb{Z}$, and θ is a homomorphism.

Step 2: an exponent of the Jacquet module. Let $j\colon\sigma^{\mathrm{Iw}_1}\to(\sigma_N)^{T(\varpi)}$ be the map of (Gy7)(a) ($T(\varpi)$ the diagonal matrices $\equiv 1 \bmod\varpi$). It is $T(\mathcal{O}_K)$-equivariant and

$$j\circ U_\varpi^{(j)}=\delta_B(\alpha^{(j)})^{-1}\sigma_N(\alpha^{(j)})\circ j.$$

By Step 1 and multiplicativity (δ_B is trivial on $T(\mathcal{O}_K)$ and on the central $\alpha^{(n)}$; for $m_n<0$ invert the relation for $\alpha^{(n)}$, both sides being invertible), $j\circ\theta(t)=\delta_B(t)^{-1}\sigma_N(t)\circ j$ on W for all $t\in T$. The restriction of j to W is injective: W is stable under the $U_\varpi^{(j)}$, which act on it invertibly (as $\theta(\alpha^{(j)})$, units of $\mathrm{End}(W)$), while by (Gy7)(a) the kernel of j is the part of σ^{Iw_1} on which some $U_\varpi^{(j)}$ acts non-invertibly; so $W\cap\ker j$, a subspace stable under the $U_\varpi^{(j)}$ on which they are invertible, is 0. Let $X=(\sigma_N)^{T(\varpi)}$, the image of j, finite-dimensional and T-stable (T is commutative); averaging over the compact open subgroup $T(\varpi)$ is a T-equivariant projection of σ_N onto X, so X is a T-equivariant direct summand of σ_N. If $w\in W_\chi$ and $(\theta(t)-\chi(t))^m w=0$, then $(\sigma_N(t)-\delta_B(t)\chi(t))^m j(w)=0$; hence the joint generalised eigenspace $X_{\delta_B\chi}$ of T on X for the character $\delta_B\chi$ contains $j(W_\chi)\neq 0$. It is a direct summand of X (decomposition of a finite-dimensional space under a commuting family into joint generalised eigenspaces), and on it the operators $\sigma_N(t)-\delta_B(t)\chi(t)$ are commuting nilpotents, so their

images span a proper subspace Y'; any non-zero linear form on $X_{\delta_{B\chi}}/Y'$ gives a non-zero T-equivariant map $\sigma_N \to X \to X_{\delta_{B\chi}} \to \delta_{B\chi}$. By Frobenius reciprocity for normalised induction ([27, Proposition 2.3]: $\mathrm{Hom}_G(\sigma, I(\xi)) \cong \mathrm{Hom}_T(\sigma_N, \delta_B^{1/2}\xi)$), with $\xi = \delta_B^{1/2}\chi$, there is a non-zero, hence injective (σ irreducible), G-map $\sigma \to I(\xi_1, \dots, \xi_n)$. Since $\delta_B^{1/2}(t) = \prod_i |t_i|^{(n+1-2i)/2}$,

$$\xi_i = \chi_i\,|\cdot|^{(n+1-2i)/2}, \qquad \xi_i(\varpi)\,q^{(n-1)/2} = q^{\,i-1}\chi_i(\varpi) = \alpha_i, \qquad \xi_i(\delta) = \chi_i(\delta) = \chi(e_i(\delta)).$$

In particular the cuspidal support of σ is the multiset $\{\xi_1, \dots, \xi_n\}$ (it is unique: [27, Theorem 2.9]).

Step 3: rec *of* σ. (Generic case.) Suppose α generic. If $\xi_j = \xi_i|\cdot|$ for some $i \neq j$, then $\xi_j(\varpi) = q^{-1}\xi_i(\varpi)$, and since $\xi_m(\varpi) = q^{-(n-1)/2}\alpha_m$ for every m by the display, $\alpha_i = q\alpha_j$; this is excluded by genericity. So no two members of the cuspidal support are linked. By Zelevinsky's classification [275] σ is the Langlands quotient of a multisegment whose segments have union the cuspidal support; a segment of length ≥ 2 contains ρ and $\rho|\cdot|$, two linked members, so all segments are the singletons $\{\xi_i\}$, and by (Ln6), (Ln1) of Theorem B.92 (the singleton segment $\{\xi_i\}$ has $Q(\{\xi_i\}) = \xi_i$), $\mathrm{rec}(\sigma) = \bigoplus_i \xi_i \circ \mathrm{Art}_K^{-1}$, a sum of characters, with $N = 0$. (Spherical case.) If $\sigma^{\mathrm{GL}_n(\mathcal{O}_K)} \neq 0$, then $I(\xi)^{\mathrm{GL}_n(\mathcal{O}_K)} \neq 0$; by the Iwasawa decomposition $G = B\,\mathrm{GL}_n(\mathcal{O}_K)$ a non-zero fixed function is determined by its value at 1 and forces $\delta_B^{1/2}\xi$ to be trivial on $T(\mathcal{O}_K)$, i.e. every ξ_i is unramified. Then σ is the spherical constituent of the unramified principal series $I(\xi)$, and $\mathrm{rec}(\sigma) = \bigoplus_i \xi_i \circ \mathrm{Art}_K^{-1}$ with $N = 0$ (compatibility of rec with the Satake parametrisation of spherical representations, [121]). In both cases, by (Ln2) and (Ln1) for $\chi = |\cdot|^{(1-n)/2}$, $\mathrm{rec}(\sigma \otimes |\det|^{(1-n)/2}) = \bigoplus_i \psi_i$ with $\psi_i = (\xi_i|\cdot|^{(1-n)/2}) \circ \mathrm{Art}_K^{-1}$, which has monodromy 0 and is trivial on the commutator subgroup of the Weil group.

Step 4: the Galois side. By the hypothesis on r, $\mathrm{WD}(r)^{F\text{-}ss}$ transported by ι is $\bigoplus_i \psi_i$, with monodromy 0. Frobenius-semisimplification does not change the monodromy, so $\mathrm{WD}(r)$ has monodromy 0 and is $(\overline{\mathbb{Q}}_\ell^{\,n}, r|_{W_K}, 0)$ by (Wd1) of Definition 38.12; computing its Frobenius-semisimplification with the Frobenius lift Φ, it is $(\overline{\mathbb{Q}}_\ell^{\,n}, \rho_s, 0)$ with $\rho_s(\Phi^m\sigma) = r(\Phi)_s^m r(\sigma)$ ($m \in \mathbb{Z}$, $\sigma \in I_K$), $r(\Phi)_s$ the semisimple part of $r(\Phi)$. So there is a basis in which

$$\iota\rho_s(g) = \mathrm{diag}(\psi_1(g), \dots, \psi_n(g))$$

for g in the Weil group; in particular $\iota r(\Phi)_s = \mathrm{diag}(\psi_i(\Phi))$ and, as $\tau \in I_K$, $\iota r(\tau) = \mathrm{diag}(\psi_i(\tau))$. As $\mathrm{Art}_K^{-1}(\Phi|_{K^{\mathrm{ab}}}) = \varpi$ and $\mathrm{Art}_K^{-1}(\tau|_{K^{\mathrm{ab}}}) = \delta$, the display gives $\psi_i(\Phi) = \xi_i(\varpi)q^{(n-1)/2} = \alpha_i$ and $\psi_i(\tau) = \xi_i(\delta) = \chi(e_i(\delta))$. Hence (1): $r(\Phi)$ and its semisimple part have the same characteristic polynomial, so that of $\iota r(\Phi)$ is $\prod_i(X - \alpha_i)$. For (2), α is generic, so the α_i are pairwise distinct; $\iota r(\Phi)$ has n distinct eigenvalues, hence is semisimple and equals $\iota r(\Phi)_s = \mathrm{diag}(\alpha_i)$ in this basis, while $\iota r(\tau) = \mathrm{diag}(\chi(e_i(\delta)))$. In this basis the matrix $\big(\prod_{i' \neq i}(\iota r(\Phi) - \alpha_{i'})\big)(\iota r(\tau) - \chi(e_i(\delta)))$ is diagonal with m-th entry $\prod_{i' \neq i}(\alpha_m - \alpha_{i'}) \cdot (\chi(e_m(\delta)) - \chi(e_i(\delta)))$, which vanishes for $m \neq i$ (the factor $i' = m$) and for $m = i$ (the last factor). □

Lemma E.141. *Let $h \geq 0$ and $n \geq 1$. For $1 \leq j \leq h$ let K_j be a finite extension of some $\mathbb{Q}_{p_j}$ with residue field of order q_j, $G_j = \mathrm{GL}_n(K_j) \supseteq K_{0,j} = \mathrm{GL}_n(\mathcal{O}_{K_j}) \supseteq \mathrm{Iw}_j$ (upper triangular modulo the maximal ideal), and $G = \prod_j G_j \supseteq K_0 = \prod_j K_{0,j} \supseteq \mathrm{Iw} = \prod_j \mathrm{Iw}_j$. Let Λ be a commutative ring in which every q_j is a unit, and V a Λ-module with a Λ-linear action of G on which multiplication by every non-zero integer is injective. For each j let $U_{j,1}, \dots, U_{j,n}$ be the operators U_i of Lemma E.138 for K_j and the $\Lambda[G_j]$-module $V^{\prod_{j' \neq j} \mathrm{Iw}_{j'}}$; they act on V^{Iw}, and all $U_{j,i}$ commute, so that $u\colon \Lambda[U] = \Lambda[U_{j,i}]_{j,i} \to \mathrm{End}_\Lambda(V^{\mathrm{Iw}})$ is defined. Let B be a commutative ring, $\beta\colon \Lambda \to B$ a ring homomorphism, M a B-module, $\iota\colon M \to V^{K_0}$ an additive bijection with $\iota(\beta(t)m) = t\,\iota(m)$, and $P_1, \dots, P_h \in B[X]$ monic of degree n such that, for all j, $0 \leq k \leq n$ and*

$m \in M$,

$$e_k(U_{j,1},\dots,U_{j,n})\,\iota(m) = \iota((-1)^k c_{j,n-k}\,m), \qquad P_j = \textstyle\sum_r c_{j,r}X^r,$$

e_k the k-th elementary symmetric polynomial. Let $\mathrm{Sp} = B[U]/I$, *I the ideal generated by the coefficients of* $\prod_{i=1}^n (X - U_{j,i}) - P_j(X)$, $1 \le j \le h$, *and for* $t \in \Lambda[U]$ *let* $\bar\beta(t) \in \mathrm{Sp}$ *be the class of t with β applied to its coefficients. Let* $g \in \mathbb{Z}[U]$ *and* $f = g \cdot \prod_{j=1}^h N_{q_j}(U_{j,1},\dots,U_{j,n})$. *Then there are additive maps* $\varphi\colon \mathrm{Sp} \otimes_B M \to V^{\mathrm{Iw}}$ *and* $\psi\colon V^{\mathrm{Iw}} \to \mathrm{Sp} \otimes_B M$ *such that for all* $t \in \Lambda[U]$, $z \in \mathrm{Sp} \otimes_B M$, $x \in V^{\mathrm{Iw}}$*:*

$$\varphi(\bar\beta(t)z) = u(t)\varphi(z), \quad \psi(u(t)x) = \bar\beta(t)\psi(x), \quad \varphi(\psi(x)) = u(f)x, \quad \psi(\varphi(z)) = \bar\beta(f)z.$$

Proof. Operators at different places commute. For j fixed write $V_j = V^{\prod_{j'\neq j} \mathrm{Iw}_{j'}}$; it is G_j-stable, $V_j^{\mathrm{Iw}_j} = V^{\mathrm{Iw}}$, and $U_{j,i}$, $E_j\colon v \mapsto \sum_{g \in K_{0,j}/\mathrm{Iw}_j} gv$ are endomorphisms of V^{Iw}. The action of $G_{j'}$, $j' \neq j$, commutes with that of G_j, hence with E_j and with the double coset operators at j; it therefore commutes with $\theta_{k_j^\times}(t)$ for dominant t and with their inverses, so with all $U_{j,i}$. As $U_{j',i'}$ and $E_{j'}$ are built from the action of $G_{j'}$ in the same way, all $U_{j,i}$, E_j ($1 \le j \le h$) commute with each other on V^{Iw}, except that E_j need not commute with $U_{j,i}$. Moreover $E_1 \cdots E_h$ maps V^{Iw} into V^{K_0}. Write $u_j\colon \Lambda[X_1,\dots,X_n] \to \mathrm{End}_\Lambda(V^{\mathrm{Iw}})$, $X_i \mapsto U_{j,i}$. Since multiplication by $[K_{0,j} : \mathrm{Iw}_j]$ is injective on $V_j \subseteq V$, (Si4) applies to V_j: for symmetric z, $u_j(z)$ preserves $V_j^{K_{0,j}}$, and $E_j u_j(y)v = u_j(\mathrm{dd}(N_{q_j}y))v$ for $v \in V_j^{K_{0,j}}$. For $v \in V^{K_0}$ the vector $u_j(z)v$ (z symmetric) is again fixed by $K_{0,j'}$ for $j' \neq j$ (commutation), so $u_j(z)V^{K_0} \subseteq V^{K_0}$.

Symmetric functions. Every symmetric $z \in \Lambda[X_1,\dots,X_n]$ is $Z(e_1,\dots,e_n)$ for a polynomial Z over Λ; put $\sigma_j(z) = Z^\beta(-c_{j,n-1}, c_{j,n-2},\dots,(-1)^n c_{j,0}) \in B$, Z^β having β applied to its coefficients. By the hypothesis and the semilinearity of ι, $u_j(z)\iota(m) = \iota(\sigma_j(z)m)$; and in Sp, where $\prod_i (X - U_{j,i}) = P_j$, i.e. $e_k(U_{j,\bullet}) = (-1)^k c_{j,n-k}$, the class $\bar\beta(z(U_{j,\bullet}))$ equals $\sigma_j(z)\cdot 1$. Let $\mathrm{tr} = \iota^{-1} \circ E_1 \cdots E_h\colon V^{\mathrm{Iw}} \to M$; it is additive with $\mathrm{tr}(tx) = \beta(t)\mathrm{tr}(x)$ for $t \in \Lambda$, and by (Si2) for the V_j and the commutation above, $\mathrm{tr}(u_j(z)x) = \sigma_j(z)\mathrm{tr}(x)$ for symmetric z.

Dual families. Fix $a_1,\dots,a_r, c_1,\dots,c_r \in \mathbb{Z}[X_1,\dots,X_n]$ as in (Si3). Applying w to the relation for w^{-1} gives $\sum_k c_k\, w(a_k) = \Delta$ for $w = 1$ and 0 otherwise, so the two families play symmetric roles. For every $y \in \Lambda[X_1,\dots,X_n]$,

$$\text{(E)} \qquad \sum_k a_k\,\mathrm{dd}(c_k y) = y, \qquad \sum_k c_k\,\mathrm{dd}(a_k y) = y:$$

indeed $\Delta \sum_k a_k \mathrm{dd}(c_k y) = \sum_w \mathrm{sgn}(w) w(y) \sum_k a_k w(c_k) = \Delta y$, and Δ, a product of the non-zero-divisors $X_{i'} - X_i$, can be cancelled; the second identity is the first for the exchanged families. For a multi-index $\kappa = (\kappa_1,\dots,\kappa_h) \in \{1,\dots,r\}^h$ put $a_\kappa = \prod_j a_{\kappa_j}(U_{j,\bullet})$ and $c_\kappa = \prod_j c_{\kappa_j}(U_{j,\bullet})$ in $\mathbb{Z}[U]$.

Construction of φ. The map $B[U] \times M \to V^{\mathrm{Iw}}$, $(\sum_\epsilon b_\epsilon U^\epsilon, m) \mapsto \sum_\epsilon u(U^\epsilon)\iota(b_\epsilon m)$ (monomials U^ϵ) is biadditive and B-balanced, and sends $(\bar t\, x, m)$ to $u(t)$ of the image of (x,m) for $t \in \Lambda[U]$, $\bar t$ its image in $B[U]$ (as $\iota(\beta(\lambda)bm) = \lambda\iota(bm)$). It vanishes on $(U^\epsilon \cdot r, m)$ for $r = (-1)^k e_k(U_{j,\bullet}) - c_{j,n-k}$, the coefficient of X^{n-k} in $\prod_i (X - U_{j,i}) - P_j$: the image is $u(U^\epsilon)\big((-1)^k u(e_k(U_{j,\bullet}))\iota(m) - \iota(c_{j,n-k}m)\big) = 0$ by the hypothesis. Hence it vanishes on $I \times M$ and induces φ on $\mathrm{Sp} \otimes_B M$ with $\varphi(1 \otimes m) = \iota(m)$ and $\varphi(\bar\beta(t)z) = u(t)\varphi(z)$.

Construction of ψ and its semilinearity. Put $\psi(x) = \sum_\kappa \bar\beta(a_\kappa) \otimes \mathrm{tr}(u(c_\kappa)x)$. It is additive and $\psi(\lambda x) = \beta(\lambda)\psi(x)$ for $\lambda \in \Lambda$. Since $\psi(u(tt')x) = \psi(u(t)u(t')x)$, it suffices to show $\psi(u_j(y)x) = \bar\beta(y(U_{j,\bullet}))\psi(x)$ for one j and $y \in \Lambda[X_1,\dots,X_n]$. By the second identity of (E) with $c_{\kappa_j} y$ in place

of y, $u(c_\kappa)u_j(y) = \sum_l u(c_{\kappa[l]})\,u_j(\mathrm{dd}(a_l c_{\kappa_j} y))$, where $\kappa[l]$ is κ with κ_j replaced by l. As $\mathrm{dd}(\cdot)$ is symmetric, tr turns $u_j(\mathrm{dd}(\cdot))$ into $\sigma_j(\mathrm{dd}(\cdot)) \in B$, which passes through $\otimes_B$ and becomes the class of $\mathrm{dd}(\cdot)(U_{j,\bullet})$ in Sp. Thus

$$\psi(u_j(y)x) = \sum_{\kappa'} \Big(\sum_k \bar\beta(a_{\kappa'[k]})\,\bar\beta\big(\mathrm{dd}(a_{\kappa'_j} c_k y)(U_{j,\bullet})\big)\Big) \otimes \mathrm{tr}(u(c_{\kappa'})x),$$

and, writing $a' = \prod_{j'' \neq j} a_{\kappa'_{j''}}(U_{j'',\bullet})$ (so that $a_{\kappa'[k]} = a'\,a_k(U_{j,\bullet})$), the inner sum is

$$\bar\beta(a') \cdot \bar\beta\big(\sum_k a_k \mathrm{dd}(c_k\, y\, a_{\kappa'_j})(U_{j,\bullet})\big) = \bar\beta\big(y(U_{j,\bullet})\, a_{\kappa'}\big)$$

by the first identity of (E). So $\psi(u_j(y)x) = \bar\beta(y(U_{j,\bullet}))\psi(x)$.

The two composites for $g = 1$. For $x \in V^{\mathrm{Iw}}$, using commutation of the operators at different places and (Si3) at each place (for the module V_j),

$$\varphi(\psi(x)) = \sum_\kappa u(a_\kappa)\,E_1 \cdots E_h\, u(c_\kappa)x = \prod_{j=1}^{h} \Big(\sum_k u_j(a_k) E_j u_j(c_k) \Big) x = \prod_j u_j(N_{q_j})x.$$

For $z = \bar\beta(y) \otimes m$ with $y = \prod_j y_j(U_{j,\bullet})$, $y_j \in \Lambda[X_1, \ldots, X_n]$ (such elements span $\mathrm{Sp} \otimes_B M$ additively, together with B-multiples which pass to m), $\varphi(z) = u(y)\iota(m)$, and by (Si4) applied successively at $j = h, h-1, \ldots, 1$ (each step leaves a vector of V^{K_0}, by the first paragraph),

$$E_1 \cdots E_h\, u(c_\kappa y)\iota(m) = \prod_j u_j\big(\mathrm{dd}(N_{q_j} c_{\kappa_j} y_j)\big)\iota(m) = \iota\Big(\prod_j \sigma_j\big(\mathrm{dd}(N_{q_j} c_{\kappa_j} y_j)\big) m \Big).$$

Hence $\psi(\varphi(z)) = \sum_\kappa \bar\beta(a_\kappa) \prod_j \bar\beta\big(\mathrm{dd}(N_{q_j} c_{\kappa_j} y_j)(U_{j,\bullet})\big) \otimes m = \prod_j \bar\beta\Big(\sum_k a_k\, \mathrm{dd}(c_k N_{q_j} y_j)\Big)(U_{j,\bullet}) \otimes m = \bar\beta(\prod_j N_{q_j})z$ by (E).

General g. Replace φ by $\varphi'(z) = \varphi(\bar\beta(g)z)$: it has the same semilinearity (Sp is commutative), $\varphi'\psi = u(g)\varphi\psi = u(f)$ and $\psi\varphi' = \bar\beta(g)\bar\beta(\prod_j N_{q_j}) = \bar\beta(f)$. □

Proposition E.142. *Let $\mathscr{H}$, $\mathcal{O}$, $(B, D^{\mathrm{u}}, (\psi^{\mathrm{d}}_{v,a}))$, J, a Taylor–Wiles base $(\mathbf{w}, M)$ for J with weight map λ_B, a Taylor–Wiles datum $\mathcal{D}$ of size q, $N \geq 1$ and $e \geq 1$ be as in Definition E.135, and let $\tilde u_{j,i} \in \mathcal{O}$ with $\tilde u_{j,i} \equiv \bar\chi_i(\sigma_j)$ modulo $\mathfrak{m}_{\mathcal{O}}$. If there is an operator module at level N satisfying the relations with exponent e, then there are R_N, $(D_N, (\psi_{N,v,a}))$, s_N, $(U_{N,j,i})$, M_N, $(a_{N,j})$, rk_N, θ_N, φ_N, ψ_N as in the clause "for every $N \geq 1$" of Definition E.36 (for these $\tilde u$ and e) satisfying (Ha1)–(Ha5) at this N.*

Proof. Step 0: Λ_D-linearity. Λ_D acts on R through $\lambda_R\colon X_k \mapsto c_k\psi_{R,v_k,a_k}(\gamma_k) - 1$, and $\beta \circ \lambda_R = \lambda_B$ by (Op1) (both are continuous and agree on the X_k). Hence, taking $t \in \lambda_R(\Lambda_D) \subseteq R[U]$ in (Op5), φ and ψ are Λ_D-linear, Λ_D acting on $\mathrm{Sp}^{\mathrm{pol}}_N \otimes_B M$ through B and on M_{w} through R. All modules below are finite over Λ_D: M_1 by (Op2), M_{w} as a quotient of it (Op4), $\mathrm{Sp}^{\mathrm{pol}}_N \otimes_B M$ as $\mathrm{Sp}^{\mathrm{pol}}_N$ is a finite free B-module (generic splitting algebra of monic polynomials) and M is finite over Λ_D. A Λ_D-linear map sends $\mathfrak{m}^k_{\Lambda_D}$ times its source into $\mathfrak{m}^k_{\Lambda_D}$ times its target, and in a finite Λ_D-module $\bigcap_k \mathfrak{m}^k_{\Lambda_D}(\cdot) = 0$.

Step 1: the ring. Let $R_N = R[[Z_{j,i}, Y_{j,i}]] \in \mathrm{CNL}_{\mathcal{O}}$, $U_{N,j,i} = \tilde u_{j,i} + Z_{j,i}$, $(D_N, (\psi_{N,v,a}))$ the base change of $(D_R, (\psi_{R,v,a}))$ (base change preserves (Od1), (Od2), (Od3) and the residual conditions), $s_N\colon S_\infty \to R_N$ the continuous $\mathcal{O}$-algebra map with $X_k \mapsto c_k\psi_{N,v_k,a_k}(\gamma_k) - 1$ and $Y_{j,i} \mapsto Y_{j,i}$, and $a_{N,j} = a_j \geq N$. Then (Ha1) holds.

Step 2: the module and (Ha2). Let T_1 be the image of $R[U,Y]$ in $\mathrm{End}_{\Lambda_D}(M_1)$, a finite commutative Λ_D-algebra, hence the product of its localisations at its finitely many maximal ideals. By (Op3), $(1+Y_{j,i})^{\ell^{a_j}} = 1$ in T_1, so $Y_{j,i}$ lies in every maximal ideal (in a field of characteristic ℓ this equation forces $Y_{j,i}=0$). Let $e_1 \in T_1$ be the idempotent of the maximal ideal generated by $\mathfrak{m}_R$, the $U_{j,i}-\tilde{u}_{j,i}$ and the $Y_{j,i}$ ($e_1=0$ if they generate T_1), and $M_N = e_1 M_1$. On M_N these generators act through the maximal ideal of the finite local Λ_D-algebra e_1T_1, hence topologically nilpotently, and M_N is $\mathfrak{m}_{\Lambda_D}$-adically complete; so the action of $R[U,Y]$ on M_N extends uniquely to a continuous action of R_N (power series in $Z=U-\tilde{u}$ and Y). The action of Δ_a commutes with e_1, so M_N is a direct summand of the projective $\Lambda_D[\Delta_a]$-module M_1, hence projective and finitely generated. The map $\Lambda_D[\Delta_a] \to S_\infty/\mathfrak{d}_a$ sending the generator of the factor (j,i) to $1+Y_{j,i}$ is an isomorphism: it is well defined, and by Weierstrass division in each variable $S_\infty/\mathfrak{d}_a$ is free over Λ_D on the monomials $\prod Y_{j,i}^{b_{j,i}}$, $b_{j,i} < \ell^{a_j}$, which are triangular combinations of the images of the group elements. This ring is local (Δ_a is an ℓ-group, Λ_D local of residue characteristic ℓ), so M_N is free of some rank rk_N; through s_N the S_∞-action on M_N is this one. This is (Ha2).

Step 3: X_N. For $t \in R[U,Y]$, $\mathrm{tr}\circ \mathrm{act}_1(t) = \mathrm{act}_{\mathrm{w}}(\bar{t})\circ \mathrm{tr}$, $\bar{t}$ the image under $Y \mapsto 0$ (Op4). Let T_{w} be the image of $R[U]$ in $\mathrm{End}_{\Lambda_D}(M_{\mathrm{w}})$ and e_{w} its idempotent for the maximal ideal generated by $\mathfrak{m}_R$ and the $U_{j,i}-\tilde{u}_{j,i}$. The decompositions of M_1 and M_{w} by the maximal ideals of $R[U,Y]$, resp. $R[U]$, correspond under $Y\mapsto 0$ (all maximal ideals of T_1 contain the Y), and tr respects them; as tr is surjective with kernel $\sum \mathrm{act}_1(Y_{j,i})M_1$, it induces an isomorphism of R_N-modules $X_N = M_N/(s_N(Y_{j,i}))M_N \to e_{\mathrm{w}}M_{\mathrm{w}}$, R_N acting on the target through $Y\mapsto 0$.

Step 4: θ_N and (Ha3). $\mathrm{Sp}_N^{\mathrm{pol}}$ is finite over the complete local ring B, so it is the product of its localisations at maximal ideals; the factor at the maximal ideal $\mathfrak{n}_B$ generated by $\mathfrak{m}_B$ and the $U_{j,i}-\tilde{u}_{j,i}$ is the completion there, $B[[Z]]/I_N B[[Z]] = \mathrm{Sp}_N$ (with $U=\tilde{u}+Z$; completion is exact on finitely generated $B[U]$-modules). Let e_{s} be its idempotent, so $\mathrm{Sp}_N \otimes_B M = e_{\mathrm{s}}(\mathrm{Sp}_N^{\mathrm{pol}} \otimes_B M)$. Let $\theta_N \colon R_N \to \mathrm{Sp}_N$ be β on R, $Z_{j,i}\mapsto u_{j,i}-\tilde{u}_{j,i}$, $Y_{j,i}\mapsto 0$. By (Op1), $\theta_N \circ D_N$ is the base change of $D^{\mathrm{u}}\circ h_N$ and $\theta_N \circ \psi_{N,v,a} = \psi^{\mathrm{d}}_{v,a}$; by Step 0, $\theta_N(s_N(z)) = \lambda_B(z_0)$; and $\theta_N(U_{N,j,i}) = u_{j,i}$. This is (Ha3).

Step 5: the maps respect the parts. Let $\mathfrak{n}' \neq \mathfrak{n}_B$ be a maximal ideal of $\mathrm{Sp}_N^{\mathrm{pol}}$ and z in the corresponding factor of $\mathrm{Sp}_N^{\mathrm{pol}} \otimes_B M$. Both maximal ideals contain $\mathfrak{m}_B$ and correspond to distinct points of the variables U over the residue field; so there is $t \in \mathcal{O}[U]$ with $t \in \mathfrak{n}'$ and $t \notin \mathfrak{n}_B$ (interpolation over $\mathbb{F}_{\mathcal{O}}$, lifted). Then t acts topologically nilpotently on the $\mathfrak{n}'$-factor, and on $e_{\mathrm{w}}M_{\mathrm{w}}$ it acts as a unit plus a topologically nilpotent operator, hence invertibly, its inverse being Λ_D-linear. As $\beta(t)=t$, (Op5) gives $e_{\mathrm{w}}\varphi(z) = \mathrm{act}_{\mathrm{w}}(t)^{-k} e_{\mathrm{w}}\varphi(t^k z)$ for all k, and $t^k z \in \mathfrak{m}_{\Lambda_D}^{j_k}(\cdot)$ with $j_k \to \infty$; by Step 0 the right side lies in $\mathfrak{m}_{\Lambda_D}^{j_k} e_{\mathrm{w}}M_{\mathrm{w}}$, so $e_{\mathrm{w}}\varphi(z)=0$. Symmetrically (using $t\in\mathcal{O}[U]$ with t in a maximal ideal $\mathfrak{n}''$ of T_{w} other than that of e_{w}, and $t\notin\mathfrak{n}_B$), $e_{\mathrm{s}}\psi$ vanishes on the parts of M_{w} other than $e_{\mathrm{w}}M_{\mathrm{w}}$. Hence $\varphi(\mathrm{Sp}_N \otimes_B M) \subseteq e_{\mathrm{w}}M_{\mathrm{w}} \oplus$ (other parts) with $e_{\mathrm{w}}\varphi = e_{\mathrm{w}}\varphi e_{\mathrm{s}}$, and $e_{\mathrm{s}}\psi = e_{\mathrm{s}}\psi e_{\mathrm{w}}$.

Step 6: (Ha4), (Ha5). Put $\varphi_N = \iota^{-1}\circ e_{\mathrm{w}}\varphi|_{\mathrm{Sp}_N\otimes_B M}$ and $\psi_N = e_{\mathrm{s}}\psi\circ\iota$, $\iota\colon X_N \to e_{\mathrm{w}}M_{\mathrm{w}}$ the isomorphism of Step 3. For $t\in R[U]$ the identities $\varphi_N(\theta_N(t)z) = t\varphi_N(z)$ and $\psi_N(tx) = \theta_N(t)\psi_N(x)$ follow from (Op5) (the idempotents commute with the actions); Y acts by 0 on X_N and $\theta_N(Y)=0$; for a power series $t = t_{<k}+t_{\geq k}$ (terms of total degree $<k$, resp. $\geq k$, in Z,Y) both sides of each identity change by elements of $\mathfrak{m}_{\Lambda_D}^{j_k}(\cdot)$ with $j_k\to\infty$ (Step 2 and the topological nilpotence of $u-\tilde{u}$ on $\mathrm{Sp}_N\otimes_B M$, with Step 0), so they hold for all $t\in R_N$. By Step 5, $\varphi_N\psi_N(x) = \iota^{-1}e_{\mathrm{w}}\varphi(e_{\mathrm{s}}\psi(\iota x)) = \iota^{-1}e_{\mathrm{w}}\varphi\psi(\iota x) = \iota^{-1}e_{\mathrm{w}}\mathrm{act}_{\mathrm{w}}(f)\iota x = f(U_N)x$, and $\psi_N\varphi_N(z) = e_{\mathrm{s}}\psi(e_{\mathrm{w}}\varphi(z)) = e_{\mathrm{s}}\psi\varphi(z) = e_{\mathrm{s}}\beta(f)z = \theta_N(f(U_N))z$ for $z\in\mathrm{Sp}_N\otimes_B M$. This is (Ha4). Finally (Op6), (Op7) are identities in $\mathrm{End}(M_1)$, hence on the summand M_N, where the trace

of D_R base-changed to $R[U,Y]$ acts as tr_{D_N} with $U = U_N$ and $Y = s_N(Y)$: this is (Ha5) with exponent e. $\square$

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References

[1] A. Abbes and T. Bouche, *Théorème de Hilbert–Samuel "arithmétique"*, Ann. Inst. Fourier **45** (1995), 375–401.

[2] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, preprint, GitHub repository RH-Alpoge-Reconstruction, version of 24 November 2026.

[3] L. Alpöge, *Lean 4 formalization of the Riemann hypothesis and of the generalized Riemann hypothesis for Dirichlet L-functions*, alpo.ge (2026).

[4] L. Alpöge, *Remarks accompanying the formal development*, unpublished note (2026).

[5] Z. Amir-Khosravi, *Serre's tensor construction and moduli of abelian schemes*, Manuscr. Math. **156** (2018), 409-456.

[6] S. A. Amitsur, *On the characteristic polynomial of a sum of matrices*, Linear Multilinear Algebra **8** (1980), no.~3, 177–182.

[7] A. Ash, D. Mumford, M. Rapoport et al., *Smooth compactification of locally symmetric varieties*, Math. Sci. Press (1975).

[8] F. Andreatta, E. Goren, B. Howard et al., *Faltings heights of abelian varieties with complex multiplication*, Ann. Math. (2) **187** (2018), 391-531.

[9] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Ann. of Math. Stud. **120**, Princeton University Press (1989).

[10] E. Artin, *Galois theory*, Notre Dame Mathematical Lectures **2**, University of Notre Dame, Notre Dame, IN (1942).

[11] E. Artin and J. Tate, *Class field theory*, W. A. Benjamin (1968).

[12] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley (1969).

[13] A. Atkin and J. Lehner, *Hecke operators on* $\Gamma_0(m)$, Math. Ann. **185** (1970), 134-160.

[14] P. A. (IMB), *Un lemme matriciel effectif*, arXiv preprint arXiv:1104.5644 (2011).

[15] J. Ax, *Zeros of polynomials over local fields. The Galois action*, J. Algebra **15** (1970), 417-428.

[16] W. Banaszczyk, *New bounds in some transference theorems in the geometry of numbers*, Math. Ann. **296** (1993), 625-635.

[17] T. Barnet-Lamb, T. Gee, D. Geraghty et al., *Potential automorphy and change of weight*, Ann. Math. (2) **179** (2014), 501-609.

[18] T. Barnet-Lamb, T. Gee, D. Geraghty et al., *Local-global compatibility for $l = p$. II*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), 165-179.

[19] H. Bass, *K-theory and stable algebra*, Publ. Math., Inst. Hautes Étud. Sci. **22** (1964), 489-544.

[20] F. A. Behrend, *On sets of integers which contain no three terms in arithmetical progression*, Proc. Nat. Acad. Sci. U.S.A. **32** (1946), 331–332.

[21] J. Bellaiche and G. Chenevier, *p-adic families of Galois representations and higher rank Selmer groups*, arXiv preprint arXiv:math/0602340 (2006).

[22] J. Bellaïche and G. Chenevier, *Formes non tempérées pour* U(3) *et conjectures de Bloch–Kato*, Ann. Sci. Éc. Norm. Supér. (4) **37** (2004), 611–662.

[23] L. Berger, *Représentations p-adiques et équations différentielles*, Invent. Math. **148** (2002), 219–284.

[24] L. Berger, *Limites de représentations cristallines*, arXiv preprint arXiv:math/0201262 (2002).

[25] L. Berger, *Bloch and Kato's exponential map: three explicit formulas*, arXiv preprint arXiv:math/0209283 (2002).

[26] L. Berger, *Limites de représentations cristallines*, Compos. Math. **140** (2004), 1473–1498.

[27] I. Bernstein and A. Zelevinsky, *Induced representations of reductive p-adic groups. I*, Ann. Sci. Éc. Norm. Supér. (4) **10** (1977), 441-472.

[28] P. Berthelot, L. Breen, and W. Messing, *Théorie de Dieudonné cristalline II*, Lecture Notes in Math. **930**, Springer, Berlin (1982).

[29] P. Berthelot and A. Ogus, *Notes on crystalline cohomology*, Princeton University Press (1978).

[30] B. Bhatt, M. Morrow, and P. Scholze, *Integral p-adic Hodge theory*, Publ. Math., Inst. Hautes Étud. Sci. **128** (2018), 219-397.

[31] B. Bhatt and P. Scholze, *Prismatic F-crystals and crystalline Galois representations*, Camb. J. Math. **11** (2023), 507-562.

[32] C. Birkenhake and H. Lange, *Complex abelian varieties*, 2nd ed., Grundlehren Math. Wiss. **302**, Springer (2004).

[33] D. Blasius and J. Rogawski, *Motives for Hilbert modular forms*, Invent. Math. **114** (1993), 55–87.

[34] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, in: The Grothendieck Festschrift I, Progr. Math. **86**, Birkhäuser (1990), 333–400.

[35] E. Bombieri, *On the large sieve*, Mathematika **12** (1965), 201–225.

[36] E. Bombieri, *Counting points on curves over finite fields (d'après S. A. Stepanov)*, Séminaire Bourbaki 1972/73, Exp. 430, Lecture Notes in Math. **383** (1974), 234–241.

[37] E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, Cambridge: Cambridge University Press (2006).

[38] A. Borel, *Some finiteness properties of adele groups over number fields*, Publ. Math., Inst. Hautes Étud. Sci. **16** (1963), 101-126.

[39] A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. Éc. Norm. Supér. (4) **7** (1974), 235-272.

[40] A. Borel, *Regularization theorems in Lie algebra cohomology. Applications*, Duke Math. J. **50** (1983), 605-623.

[41] A. Borel, *Linear algebraic groups*, 2nd ed., Grad. Texts in Math. **126**, Springer (1991).

[42] A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. Math. (2) **75** (1962), 485-535.

[43] A. Borel and J. Serre, *Corners and arithmetic groups (Appendice: Arrondissement des varietes a coins par A. Douady et L. Herault)*, Comment. Math. Helv. **48** (1973), 436-491.

[44] K. Borsuk, *On the imbedding of systems of compacta in simplicial complexes*, Fundam. Math. **35** (1948), 217-234.

[45] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Berlin etc.: Springer-Verlag (1990).

[46] J.-B. Bost, *Périodes et isogénies des variétés abéliennes sur les corps de nombres (d'après D. Masser et G. Wüstholz)*, Séminaire Bourbaki, Exp. 795, Astérisque **237** (1996), 115–161.

[47] J. Bost, *Algebraic leaves of algebraic foliations over number fields.*, Publ. Math., Inst. Hautes Étud. Sci. **93** (2001), 161-221.

[48] J. Bost, H. Gillet, and C. Soulé, *Heights of projective varieties and positive Green forms*, J. Am. Math. Soc. **7** (1994), 903-1027.

[49] N. Bourbaki, *Groupes et algèbres de Lie. Chapitres 4, 5 et 6*, Hermann, Paris (1968).

[50] N. Bourbaki, *Groupes et algèbres de Lie. Chapitres 2 et 3*, Hermann, Paris (1972).

[51] L. Breen, *Fonctions thêta et théorème du cube*, Lecture Notes in Math. **980**, Springer, Berlin (1983).

[52] O. Brinon and B. Conrad, *CMI summer school notes on p-adic Hodge theory*, lecture notes, Clay Mathematics Institute (2009).

[53] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Stud. Adv. Math. **60**, Cambridge University Press (1998).

[54] J. Brox, *Distilling the generalized Riemann hypothesis*, github.com/JoseBrox/GRH_distillation (2026).

[55] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge: Cambridge University Press (1993).

[56] A. Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles*, Duke Math. J. **161** (2012), 2311-2413.

[57] A. Caraiani, M. Emerton, T. Gee, D. Geraghty, V. Paskunas and S. W. Shin, *Patching and the p-adic local Langlands correspondence*, arXiv preprint arXiv:1310.0831 (2013).

[58] A. Caraiani, *Monodromy and local-global compatibility for $l = p$*, Algebra Number Theory **8** (2014), 1597-1646.

[59] H. Carayol, *Sur les représentations ℓ-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. École Norm. Sup. (4) **19** (1986), 409–468.

[60] P. Cartier, *Representations of $\mathfrak{p}$-adic groups: a survey*, Proc. Sympos. Pure Math. **33**, Part 1 (1979), 111–155.

[61] W. Casselman, *The unramified principal series of p-adic groups. I: The spherical function*, Compos. Math. **40** (1980), 387-406.

[62] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, unpublished notes (1974, revised 1995).

[63] J. W. S. Cassels and A. Fröhlich (eds.), *Algebraic number theory*, Academic Press (1967).

[64] E. Cattani, A. Kaplan, and W. Schmid, *Degeneration of Hodge structures*, Ann. Math. (2) **123** (1986), 457-535.

[65] N. Tschebotareff, *Die Bestimmung der Dichtigkeit einer Menge von Primzahlen, welche zu einer gegebenen Substitutionsklasse gehören*, Math. Ann. **95** (1926), 191–228.

[66] G. Chenevier, *The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings*, Cambridge: Cambridge University Press (2014), 221-285.

[67] G. Chenevier and M. Harris, *Construction of automorphic Galois representations. II*, Camb. J. Math. **1** (2013), 53-73.

[68] L. Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité*, in: Automorphic Forms, Shimura Varieties, and L-functions I (Ann Arbor, 1988), Perspect. Math. **10**, Academic Press (1990), 77–159.

[69] L. Clozel, M. Harris, and R. Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations. With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras.*, Publ. Math., Inst. Hautes Étud. Sci. **108** (2008), 1-181.

[70] P. Colmez and J.-M. Fontaine, *Construction des représentations p-adiques semi-stables*, Invent. Math. **140** (2000), 1–43.

[71] C. W. Curtis and I. Reiner, *Methods of representation theory, with applications to finite groups and orders. Vol. I*, John Wiley & Sons, New York (1981).

[72] H. Darmon, F. Diamond and R. Taylor, *Fermat's last theorem*, in: Elliptic Curves, Modular Forms and Fermat's Last Theorem (Hong Kong, 1993), 2nd ed., International Press (1997), 2–140.

[73] S. Dasgupta, H. Darmon, and R. Pollack, *Hilbert modular forms and the Gross-Stark conjecture*, Ann. Math. (2) **174** (2011), 439-484.

[74] J.-F. Dat, *Caractères à valeurs dans le centre de Bernstein*, J. Reine Angew. Math. **508** (1999), 61–83.

[75] H. Davenport, *Multiplicative number theory. Revised and with a preface by Hugh L. Montgomery.*, New York, NY: Springer (2000).

[76] P. Deligne, *Formes modulaires et représentations ℓ-adiques*, Séminaire Bourbaki 1968/69, Exp. 355, Lecture Notes in Math. **179** (1971), 139–172.

[77] P. Deligne, *La conjecture de Weil. I*, Publ. Math. IHÉS **43** (1974), 273–307.

[78] P. Deligne and J. Serre, *Formes modulaires de poids 1*, Ann. Sci. Éc. Norm. Supér. (4) **7** (1974), 507-530.

[79] J. Demailly, *Analytic methods in algebraic geometry*, Somerville, MA: International Press; Beijing: Higher Education Press (2012).

[80] F. Diamond and J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics **228**, Springer, New York (2005).

[81] A. Grothendieck, *Éléments de géométrie algébrique. II: Étude globale élémentaire de quelques classes de morphismes*, Publ. Math., Inst. Hautes Étud. Sci. **8** (1961), 1-222.

[82] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents*, Publ. Math. IHÉS **11** (1961), **17** (1963).

[83] A. Grothendieck, *Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas. (Séconde partie)*, Publ. Math., Inst. Hautes Étud. Sci. **24** (1965), 1-231.

[84] A. Grothendieck, *Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas (Quatrième partie). Rédigé avec la colloboration de J. Dieudonné*, Publ. Math., Inst. Hautes Étud. Sci. **32** (1967), 1-361.

[85] J. Epstein, *Human-readable blueprint of the Alpöge RH Lean formalization*, github.com/jae-63/rh-alpoge-blueprint (2026).

[86] G. Faltings, *Hodge-Tate structures and modular forms*, Math. Ann. **278** (1987), 133-149.

[87] G. Faltings and C. Chai, *Degeneration of abelian varieties*, Berlin etc.: Springer-Verlag (1990).

[88] L. Fargues, *L'isomorphisme entre les tours de Lubin–Tate et de Drinfeld et applications cohomologiques*, in: L'isomorphisme entre les tours de Lubin–Tate et de Drinfeld, Progr. Math. **262**, Birkhäuser (2008), 1–325.

[89] L. Fargues, *La filtration canonique des points de torsion des groupes p-divisibles*, Ann. Sci. Éc. Norm. Supér. (4) **44** (2011), 905–961.

[90] B. Ferrero and L. Washington, *The Iwasawa invariant μ_p vanishes for abelian number fields*, Ann. Math. (2) **109** (1979), 377-395.

[91] D. Flath, *Decomposition of representations into tensor products*, Proc. Sympos. Pure Math. **33**, Part 1 (1979), 179–183.

[92] D. Foata and D. Zeilberger, *A Combinatorial Proof of Bass's Evaluations of the Ihara-Selberg Zeta Function for Graphs*, arXiv preprint arXiv:math/9806037 (1998).

[93] J.-M. Fontaine, *Représentations p-adiques des corps locaux. I*, in: The Grothendieck Festschrift II, Progr. Math. **87**, Birkhäuser (1990), 249–309.

[94] J.-M. Fontaine, *Le corps des périodes p-adiques*, Astérisque **223** (1994), 59–111.

[95] J. Fontaine, *Potentially semi-stable ℓ-adic representations*, Paris: Société Mathématique de France (1994), 321-347.

[96] J.-M. Fontaine, *Représentations p-adiques semi-stables*, Astérisque **223** (1994), 113–184.

[97] J. Fontaine and J. Wintenberger, *Le "corps des normes" de certaines extensions algébriques de corps locaux*, C. R. Acad. Sci., Paris, Sér. A **288** (1979), 367-370.

[98] W. Fulton, *Intersection theory.*, Berlin: Springer (1998).

[99] W. Fulton and J. Harris, *Representation theory. A first course*, New York etc.: Springer-Verlag (1991).

[100] P. Gallagher, *A large sieve density estimate near $\sigma = 1$*, Invent. Math. **11** (1970), 329-339.

[101] W. T. Gan, Y. Qiu and S. Takeda, *The Regularized Siegel-Weil Formula (The Second Term Identity) and the Rallis Inner Product Formula*, arXiv preprint arXiv:1207.4709 (2012).

[102] Ġ. (IF) and G. R. (IF), *Théorème des périodes et degrés minimaux d'isogénies*, arXiv preprint arXiv:1105.1230 (2011).

[103] S. Gelbart, *Automorphic forms on adele groups*, Ann. of Math. Stud. **83**, Princeton University Press (1975).

[104] S. Gelbart and H. Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Ann. Sci. Éc. Norm. Supér. (4) **11** (1978), 471–542.

[105] D. Geraghty, *Modularity lifting theorems for ordinary Galois representations*, Math. Ann. **373** (2019), 1341-1427.

[106] J. Burgos Gil, J. Kramer, and U. Kühn, *Cohomological arithmetic Chow rings*, J. Inst. Math. Jussieu **6** (2007), 1-172.

[107] J. Burgos Gil, J. Kramer, and U. Kühn, *Arithmetic characteristic classes of automorphic vector bundles*, Doc. Math. **10** (2005), 619-716.

[108] J. I. B. Gil, P. Philippon and M. Sombra, *Arithmetic geometry of toric varieties. Metrics, measures and heights*, arXiv preprint arXiv:1105.5584 (2011).

[109] J. Gil, P. Philippon, and M. Sombra, *Height of varieties over finitely generated fields*, Kyoto J. Math. **56** (2016), 13-32.

[110] H. Gillet and C. Soulé, *Arithmetic intersection theory*, Publ. Math., Inst. Hautes Étud. Sci. **72** (1990), 93-174.

[111] H. Gillet and C. Soulé, *An arithmetic Riemann-Roch theorem*, Invent. Math. **110** (1992), 473-543.

[112] H. Grauert and R. Remmert, *Coherent analytic sheaves*, Grundlehren Math. Wiss. **265**, Springer, Berlin (1984).

[113] D. Grayson, *Reduction theory using semistability*, Comment. Math. Helv. **59** (1984), 600-634.

[114] P. Griffiths, *Periods of integrals on algebraic manifolds. III: Some global differential-geometric properties of the period mapping*, Publ. Math., Inst. Hautes Étud. Sci. **38** (1970), 125-180.

[115] B. Gross, *On the Satake isomorphism*, Cambridge: Cambridge University Press (1998), 223-237.

[116] T. Haines, R. Kottwitz, and A. Prasad, *Iwahori-Hecke algebras*, J. Ramanujan Math. Soc. **25** (2010), 113-145.

[117] G. H. Hardy and M. Riesz, *The general theory of Dirichlet's series*, Cambridge Tracts in Math. and Math. Phys. **18**, Cambridge University Press (1915).

[118] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 6th ed., revised by D. R. Heath-Brown and J. H. Silverman, Oxford University Press, Oxford (2008).

[119] Harish-Chandra, *Discrete series for semisimple Lie groups. II: Explicit determination of the characters*, Acta Math. **116** (1966), 1-111.

[120] M. Harris, S. Kudla, and W. Sweet, *Theta dichotomy for unitary groups*, J. Am. Math. Soc. **9** (1996), 941-1004.

[121] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties. With an appendix by Vladimir G. Berkovich*, Princeton, NJ: Princeton University Press (2001).

[122] R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math. **52**, Springer (1977).

[123] H. Hasse, *Beweis eines Satzes und Widerlegung einer Vermutung über das allgemeine Normenrestsymbol.*, Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. **1931** (1931), 64-69.

[124] M. Hazewinkel, *Formal groups and applications*, Pure Appl. Math. **78**, Academic Press (1978).

[125] G. Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique*, Invent. Math. **139** (2000), 439–455.

[126] L. Herr, *Sur la cohomologie galoisienne des corps p-adiques*, Bull. Soc. Math. France **126** (1998), 563–600.

[127] H. Hida, *On nearly ordinary Hecke algebras for* $GL(2)$ *over totally real fields*, in: Algebraic number theory, Adv. Stud. Pure Math. **17** (1989), 139-169.

[128] H. Hida, *Nearly ordinary Hecke algebras and Galois representations of several variables*, Baltimore: Johns Hopkins University Press (1989), 115-134.

[129] H. Hida, *On p-adic L-functions of $GL(2)\times GL(2)$ over totally real fields*, Ann. Inst. Fourier **41** (1991), 311-391.

[130] H. Hida, *Control of nearly ordinary Hecke algebras*, lecture notes, Luminy summer school, 2007.

[131] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. Math. (2) **79** (1964), 109-203, 205-326.

[132] A. Ichino, *A regularized Siegel–Weil formula for unitary groups*, Math. Z. **247** (2004), 241–277.

[133] A. E. Ingham, *The distribution of prime numbers*, Cambridge Tracts in Math. **30**, Cambridge University Press (1932).

[134] H. Iwaniec and E. Kowalski, *Analytic number theory*, Providence, RI: American Mathematical Society (AMS) (2004).

[135] H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Math. **114** (1970).

[136] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations. I*, Amer. J. Math. **103** (1981), 499–558; *II*, ibid., 777–815.

[137] U. Jannsen, *Continuous étale cohomology*, Math. Ann. **280** (1988), 207-245.

[138] M. Jaswal, *The Riemann hypothesis for Dirichlet L-functions [after Levent Alpöge]*, preprint, Zenodo, doi:10.5281/zenodo.24410057, version 1 (21 November 2026).

[139] D. Jiang, B. Sun, and C. Zhu, *Uniqueness of Bessel models: the archimedean case*, Geom. Funct. Anal. **20** (2010), 690-709.

[140] M. Kashiwara, *The asymptotic behavior of a variation of polarized Hodge structure*, Publ. Res. Inst. Math. Sci. **21** (1985), 853-875.

[141] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, Astérisque **295** (2004), 117–290.

[142] N. Katz, *Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin*, Publ. Math., Inst. Hautes Étud. Sci. **39** (1970), 175-232.

[143] Y. Kawamata, *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), 43-46.

[144] M. Kisin, *Potentially semi-stable deformation rings*, J. Am. Math. Soc. **21** (2008), 513-546.

[145] M. Klimek, *Pluripotential theory*, Oxford etc.: Clarendon Press (1991).

[146] M. Kneser, *Strong approximation*, in: Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math. **9** (1966), 187–196.

[147] H. Koch, *Galois theory of p-extensions. Transl. from the German by F. Lemmermeyer*, Berlin: Springer (2002).

[148] J. Kollár, *Lectures on resolution of singularities*, Princeton, NJ: Princeton University Press (2007).

[149] R. Kottwitz, *Base change for unit elements of Hecke algebras*, Compos. Math. **60** (1986), 237-250.

[150] R. Kottwitz, *Tamagawa numbers*, Ann. of Math. (2) **127** (1988), 629–646.

[151] S. Kudla, *Splitting metaplectic covers of dual reductive pairs*, Isr. J. Math. **87** (1994), 361-401.

[152] S. Kudla and S. Rallis, *A regularized Siegel-Weil formula: The first term identity*, Ann. Math. (2) **140** (1994), 1-80.

[153] J.-P. Labesse, *Changement de base CM et séries discrètes*, in: On the Stabilization of the Trace Formula, Stab. Trace Formula Shimura Var. Arith. Appl. **1**, International Press (2011), 429–470.

[154] K. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, Princeton, NJ: Princeton University Press (2013).

[155] K. Lan, *Compactifications of PEL-type Shimura varieties in ramified characteristics*, Forum Math. Sigma **4** (2016), Paper No. e1, 98 p.

[156] K. Lan, *Integral models of toroidal compactifications with projective cone decompositions*, Int. Math. Res. Not. **2017** (2017), 3237-3280.

[157] E. Landau, *Über einen Satz von Tschebyschef.*, Math. Ann. **61** (1905), 527-550.

[158] W. Landherr, *Äquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper*, Abh. Math. Sem. Univ. Hamburg **11** (1936), 245–248.

[159] R. P. Langlands, *Base change for* GL(2), Ann. of Math. Stud. **96**, Princeton University Press (1980).

[160] R. Lazarsfeld, *Positivity in algebraic geometry I: Classical setting: Line bundles and linear series. II: Positivity for vector bundles, and multiplier ideals*, Berlin: Springer (2004).

[161] W. Li, *Newforms and functional equations*, Math. Ann. **212** (1975), 285-315.

[162] J. Li, *Theta lifting for unitary representations with nonzero cohomology*, Duke Math. J. **61** (1990), 913-937.

[163] U. V. Linnik, *On the least prime in an arithmetic progression. I. The basic theorem*, Mat. Sb. **15(57)** (1944), 139–178; *II. The Deuring–Heilbronn phenomenon*, ibid., 347–368.

[164] J. Loday, *K-théorie algébrique et représentations de groupes*, Ann. Sci. Éc. Norm. Supér. (4) **9** (1976), 309-377.

[165] I. Macdonald, *Symmetric functions and Hall polynomials.*, Oxford: Clarendon Press (1995).

[166] H. Matsumura, *Commutative ring theory*, Cambridge Stud. Adv. Math. **8**, Cambridge University Press (1986).

[167] B. Mazur, *Deforming Galois representations*, in: Galois Groups over $\mathbb{Q}$ (Berkeley, 1987), Math. Sci. Res. Inst. Publ. **16**, Springer (1989), 385–437.

[168] B. Mazur and W. Messing, *Universal extensions and one dimensional crystalline cohomology*, Lecture Notes in Math. **370** (1974).

[169] B. Mazur and A. Wiles, *Class fields of abelian extensions of* $\mathbb{Q}$, Invent. Math. **76** (1984), 179-330.

[170] J. McCullough, *Introduction to Commutative Algebra*, lecture notes (2019).

[171] W. Messing, *The crystals associated to Barsotti–Tate groups: with applications to abelian schemes*, Lecture Notes in Math. **264** (1972).

[172] J. Milne, *Arithmetic duality theorems*, Charleston, SC: BookSurge, LLC (2006).

[173] J. S. Milne, *Class field theory*, lecture notes, jmilne.org (2020).

[174] J. Milne, *Arithmetic duality theorems*, arXiv preprint arXiv:2509.13016 (1986).

[175] J. S. Milne, *Algebraic number theory*, version 3.08, course notes, jmilne.org (2020).

[176] J. Milnor, *Introduction to algebraic K-theory*, Ann. of Math. Stud. **72**, Princeton University Press, Princeton, NJ (1971).

[177] J. Milnor and J. Moore, *On the structure of Hopf algebras*, Ann. Math. (2) **81** (1965), 211-264.

[178] C. Mœglin and J.-L. Waldspurger, *Le spectre résiduel de* $\mathrm{GL}(n)$, Ann. Sci. Éc. Norm. Supér. (4) **22** (1989), 605–674.

[179] H. L. Montgomery, *Topics in multiplicative number theory*, Lecture Notes in Math. **227**, Springer, Berlin (1971).

[180] H. Montgomery and R. Vaughan, *Multiplicative number theory. I. Classical theory*, Cambridge: Cambridge University Press (2007).

[181] L. Moret-Bailly, *Pinceaux de variétés abéliennes*, Astérisque **129** (1985).

[182] A. Moriwaki, *Continuity of volumes on arithmetic varieties*, J. Algebr. Geom. **18** (2009), 407-457.

[183] A. Moriwaki, *Arakelov geometry*, Providence, RI: American Mathematical Society (AMS) (2014).

[184] C. Mourougane, *Computations of Bott-Chern classes on P(E)*, arXiv preprint arXiv:math/0305236 (2003).

[185] D. Mumford, *On the equations defining abelian varieties. I*, Invent. Math. **1** (1966), 287–354.

[186] D. Mumford, *Abelian varieties*, London: Oxford University Press (1970).

[187] D. Mumford, *An analytic construction of degenerating Abelian varieties over complete rings*, Compos. Math. **24** (1972), 239-272.

[188] D. Mumford, *Hirzebruch's proportionality theorem in the non-compact case*, Invent. Math. **42** (1977), 239-272.

[189] B. Naskręcki, *The announced proof of the Riemann hypothesis: map, guide and record of checks (riemann-rev), with the atlas of the proof*, github.com/nasqret/riemann-rev (2026).

[190] J. Neukirch, *Algebraic number theory. Transl. from the German by Norbert Schappacher*, Berlin: Springer (1999).

[191] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, Berlin: Springer (2008).

[192] J. Newton and J. Thorne, *Adjoint Selmer groups of automorphic Galois representations of unitary type*, J. Eur. Math. Soc. (JEMS) **25** (2023), 1919-1967.

[193] T. Oda, *The first de Rham cohomology group and Dieudonné modules*, Ann. Sci. Éc. Norm. Supér. (4) **2** (1969), 63-135.

[194] T. Ohsawa and K. Takegoshi, *On the extension of L^2 holomorphic functions*, Math. Z. **195** (1987), 197–204.

[195] T. Ono, *On the relative theory of Tamagawa numbers*, Ann. of Math. (2) **82** (1965), 88–111.

[196] C. Peters, *A criterion for flatness of Hodge bundles over curves and geometric applications*, Math. Ann. **268** (1984), 1-19.

[197] P. Philippon, *Sur des hauteurs alternatives. III*, J. Math. Pures Appl. (9) **74** (1995), 345–365.

[198] I. I. Piatetski-Shapiro, *Multiplicity one theorems*, Proc. Sympos. Pure Math. **33**, Part 1 (1979), 209–212.

[199] I. Piatetski-Shapiro and S. Rallis, *L-functions for the classical groups*, in: Explicit Constructions of Automorphic *L*-functions, Lecture Notes in Math. **1254** (1987), 1–52.

[200] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory. Transl. from the Russian by Rachel Rowen*, Boston, MA: Academic Press (1994).

[201] A. Postnikov and R. Stanley, *Chains in the Bruhat order*, J. Algebr. Comb. **29** (2009), 133-174.

[202] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. Math. (2) **96** (1972), 552-586.

[203] S. Rallis, *On the Howe duality conjecture*, Compos. Math. **51** (1984), 333-399.

[204] S. Rallis, *L-functions and the oscillator representation*, Lecture Notes in Math. **1245** (1987).

[205] M. Rapoport, *Compactifications de l'espace de modules de Hilbert-Blumenthal*, Compos. Math. **36** (1978), 255-335.

[206] M. Raynaud and L. Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module. (Criterial of flatness and projectivity. Technics of "flatification of a module.)*, Invent. Math. **13** (1971), 1-89.

[207] C. Reutenauer and M. Schützenberger, *A formula for the determinant of a sum of matrices*, Lett. Math. Phys. **13** (1987), 299-302.

[208] K. A. Ribet, *A modular construction of unramified p-extensions of* $\mathbb{Q}(\mu_p)$, Invent. Math. **34** (1976), 151–162.

[209] K. Rubin, *Euler systems. (Hermann Weyl lectures)*, Princeton, NJ: Princeton University Press (2000).

[210] M. Rubinstein, *Alpöge/Claude proof of the generalized Riemann hypothesis*, www.math.uwaterloo.ca/~mrubinst/alpoge_claude_grh (2026).

[211] M. Rubinstein, *The Alpöge–Claude proof of the Riemann hypothesis explained for the Riemann zeta function*, working draft (2026).

[212] C. Sabbah and C. Schnell, *Degenerating complex variations of Hodge structure in dimension one*, arXiv preprint arXiv:2206.08166 (2022).

[213] T. Saito, *Modular forms and p-adic Hodge theory*, Invent. Math. **129** (1997), 607-620.

[214] W. Scharlau, *Quadratic and Hermitian forms*, Grundlehren Math. Wiss. **270**, Springer (1985).

[215] M. Schlichting, *Euler class groups, and the homology of elementary and special linear groups*, arXiv preprint arXiv:1502.05424 (2015).

[216] W. Schmid, *Variation of Hodge structure: The singularities of the period mapping*, Invent. Math. **22** (1973), 211-319.

[217] P. Schneider and E. Zink, *K-types for the tempered components of a p-adic general linear group. With an Appendix: The definition of the tempered category by P. Schneider and U. Stuhler*, J. Reine Angew. Math. **517** (1999), 161-208.

[218] P. Scholze and J. Weinstein, *Moduli of p-divisible groups*, Camb. J. Math. **1** (2013), 145-237.

[219] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6** (1956), 1–42.

[220] J.-P. Serre, *Abelian ℓ-adic representations and elliptic curves*, W. A. Benjamin (1968).

[221] J.-P. Serre, *Local fields*, Grad. Texts in Math. **67**, Springer (1979).

[222] J. Serre, *Galois cohomology. Transl. from the French by Patrick Ion*, Berlin: Springer (1997).

[223] J. Serre, *Bounds for the orders of the finite subgroups of* $G(k)$., Boca Raton, FL: CRC Press; Lausanne: EPFL Press (2007), 405-450.

[224] J. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. Math. (2) **88** (1968), 492-517.

[225] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, Adv. Stud. Pure Math. **2**, North-Holland (1968).

[226] M. Demazure and A. Grothendieck (eds.), *Schémas en groupes. I: Propriétés générales des schémas en groupes (SGA 3)*, Lecture Notes in Math. **151**, Springer (1970).

[227] A. Grothendieck (with M. Raynaud and D. S. Rim), *Groupes de monodromie en géométrie algébrique (SGA 7 I)*, Lecture Notes in Math. **288** (1972).

[228] J. Shalika, *The multiplicity one theorem for* GL_n, Ann. Math. (2) **100** (1974), 171-193.

[229] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publ. Math. Soc. Japan **11**, Iwanami Shoten and Princeton University Press (1971).

[230] G. Shimura, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math. J. **45** (1978), 637-679.

[231] S. Shin, *Galois representations arising from some compact Shimura varieties*, Ann. Math. (2) **173** (2011), 1645-1741.

[232] C. L. Siegel, *Über die Classenzahl quadratischer Zahlkörper*, Acta Arith. **1** (1935), 83–86.

[233] J. Silverman, *Advanced topics in the arithmetic of elliptic curves*, New York, NY: Springer-Verlag (1994).

[234] J. Silverman, *The arithmetic of elliptic curves*, New York, NY: Springer (2009).

[235] W. Sinnott, *On the* μ*-invariant of the* Γ*-transform of a rational function*, Invent. Math. **75** (1984), 273-282.

[236] C. Skinner and A. Wiles, *Residually reducible representations and modular forms*, Publ. Math., Inst. Hautes Étud. Sci. **89** (1999), 5-126.

[237] C. Soulé, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. **55** (1979), 251–295.

[238] C. Soulé, *A bound for the torsion in the K-theory of algebraic integers*, Doc. Math. **Extra Vol.** (2003), 761-788.

[239] The Stacks Project Authors: The Stacks Project. Lemma 30.5.2, Tag 02KH. `https://stacks.math.columbia.edu/tag/02KH` (accessed 2026-11-24).

[240] The Stacks Project Authors: The Stacks Project. Lemma 35.23.21, Tag 02L6. `https://stacks.math.columbia.edu/tag/02L6` (accessed 2026-11-24).

[241] The Stacks Project Authors: The Stacks Project. Lemma 66.14.2, Tag 07S7. `https://stacks.math.columbia.edu/tag/07S7` (accessed 2026-11-23).

[242] The Stacks Project Authors: The Stacks Project. Lemma 10.98.2, Tag 09B7. `https://stacks.math.columbia.edu/tag/09B7`.

[243] The Stacks Project Authors: The Stacks Project. Theorem 38.30.7, Tag 0815. `https://stacks.math.columbia.edu/tag/0815` (accessed 2026-11-24).

[244] The Stacks Project Authors: The Stacks Project, Chapter 42, Chow Homology. `https://stacks.math.columbia.edu/chapter/42` (accessed 2026-11-25).

[245] E. Steinitz, *Rechteckige Systeme und Moduln in algebraischen Zahlkörpern. I.*, Math. Ann. **71** (1912), 328-354.

[246] E. Steinitz, *Rechteckige Systeme und Moduln in algebraischen Zahlkörpern. II.*, Math. Ann. **72** (1912), 297-345.

[247] S. Stepanov, *On the number of points of a hyperelliptic curve over a finite prime field*, Izv. Akad. Nauk SSSR, Ser. Mat. **33** (1969), 1171-1181.

[248] U. Stuhler, *Eine Bemerkung zur Reduktionstheorie quadratischer Formen*, Arch. Math. **27** (1976), 604-610.

[249] J. Tate, *Duality theorems in Galois cohomology over number fields*, in: Proc. Internat. Congr. Mathematicians (Stockholm, 1962), Inst. Mittag-Leffler (1963), 288–295.

[250] J. T. Tate, *Fourier analysis in number fields, and Hecke's zeta-functions*, in: Algebraic Number Theory (J. W. S. Cassels and A. Fröhlich, eds.), Academic Press (1967), 305–347.

[251] J. T. Tate, *p-divisible groups*, in: Proc. Conf. Local Fields (Driebergen, 1966), Springer (1967), 158–183.

[252] J. Tate, *Relations between K_2 and Galois cohomology*, Invent. Math. **36** (1976), 257–274.

[253] R. Taylor, *On Galois representations associated to Hilbert modular forms*, Invent. Math. **98** (1989), 265–280.

[254] R. Taylor and A. Wiles, *Ring-theoretic properties of certain Hecke algebras*, Ann. Math. (2) **141** (1995), 553-572.

[255] R. Taylor and T. Yoshida, *Compatibility of local and global Langlands correspondences*, J. Am. Math. Soc. **20** (2007), 467-493.

[256] J. Thorne, *Automorphy lifting for residually reducible l-adic Galois representations*, J. Am. Math. Soc. **28** (2015), 785-870.

[257] J. Thorne, *On the vanishing of adjoint Bloch-Kato Selmer groups of irreducible automorphic Galois representations*, Pure Appl. Math. Q. **18** (2022), 2159-2202.

[258] J. Thorner and A. Zaman, *A unified and improved Chebotarev density theorem*, Algebra Number Theory **13** (2019), 1039-1068.

[259] J. Thorner and A. Zaman, *Refinements to the prime number theorem for arithmetic progressions*, Math. Z. **306** (2024), Paper No. 54, 14 p.

[260] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., revised by D. R. Heath-Brown, Oxford University Press (1986).

[261] twinheterodox, *Three simplifications in the Alpöge–Claude argument for Dirichlet GRH*, preprint, Hexagon hexagon:2611.00828v1 (21 November 2026), 10 pp.

[262] W. van der Kallen, *Homology stability for linear groups*, Invent. Math. **60** (1980), 269-295.

[263] E. Viehweg, *Vanishing theorems*, J. Reine Angew. Math. **335** (1982), 1-8.

[264] M. Vignéras, *Pro-p-Iwahori Hecke ring and supersingular $\bar{F}_p$-representations*, Math. Ann. **331** (2005), 523-556.

[265] N. Wach, *Représentations p-adiques potentiellement cristallines*, Bull. Soc. Math. France **124** (1996), 375–400.

[266] L. Washington, *Introduction to cyclotomic fields.*, New York, NY: Springer (1997).

[267] C. Weibel, *The K-book. An introduction to algebraic K-theory*, Providence, RI: American Mathematical Society (AMS) (2013).

[268] A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1-87.

[269] A. Weil, *Adeles and algebraic groups*, Progr. Math. **23**, Birkhäuser, Boston (1982).

[270] A. Wiles, *On ordinary λ-adic representations associated to modular forms*, Invent. Math. **94** (1988), 529-573.

[271] A. Wiles, *The Iwasawa conjecture for totally real fields*, Ann. Math. (2) **131** (1990), 493-540.

[272] X. Yuan, *Big line bundles over arithmetic varieties*, Invent. Math. **173** (2008), 603-649.

[273] Y. Zarkhin, *Endomorphisms of Abelian varieties over fields of finite characteristic*, Math. USSR, Izv. **9** (1976), 255-260.

[274] O. Zariski and P. Samuel, *Commutative Algebra*, Vol.~II, Van Nostrand, Princeton (1960).

[275] A. Zelevinsky, *Induced representations of reductive p-adic groups. II: On irreducible representations of $GL(n)$*, Ann. Sci. Éc. Norm. Supér. (4) **13** (1980), 165-210.

[276] S. Zhang, *Positive line bundles on arithmetic varieties*, J. Am. Math. Soc. **8** (1995), 187-221.

[277] R. Zhang, *A preliminary note on Alpöge's proposed proof of the generalized Riemann hypothesis*, preprint, hexagon:2611.00809v2 (2026).

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